The Sierpiński product of graphs

Jurij Kovič, Tomaž Pisanski, Sara Sabrina Zemljič and Arjana Žitnik

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Abstract

In this paper we introduce a product-like operation that generalizes the construction of generalized Sierpiński graphs. Let $G, H$ be graphs and let $f : V(G) \rightarrow V(H)$ be a function. Then the Sierpiński product of $G$ and $H$ with respect to $f$ is defined as a pair $(K, \varphi)$, where $K$ is a graph on the vertex set $V(G) \times V(H)$ with two types of edges:

- $\{(g, h), (g, h')\}$ is an edge in $K$ for every $g \in V(G)$ and every $\{h, h'\} \in E(H)$,
- $\{(g, f(g'), (g', f(g))\}$ is an edge in $K$ for every edge $\{g, g'\} \in E(G)$;

and $\varphi : V(G) \rightarrow V(K)$ is a function that maps every vertex $g \in V(G)$ to the vertex $(g, f(g)) \in V(K)$. Graph $K$ will be denoted by $G \otimes_f H$. Function $\varphi$ is needed to define the product of more than two factors. By applying this operation $n$ times to the same graph we obtain the $n$-th generalized Sierpiński graph.

Some basic properties of the Sierpiński product are presented. In particular, we show that $G \otimes_f H$ is connected if and only if both $G$ and $H$ are connected and we present some necessary and sufficient conditions that $G, H$ must fulfill in order for $G \otimes_f H$ to be planar. As for symmetry properties, we show which automorphisms of $G$ and $H$ extend to automorphisms of $G \otimes_f H$. In many cases we can also describe the whole automorphism group of $G \otimes_f H$.

KEYWORDS: Sierpiński graphs, graph products, connectivity, planarity, symmetry.

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1 Introduction

The family of Sierpiński graphs $S^n_p$ was first introduced by Klavžar and Milutinović in [13] as a variant of the Tower of Hanoi problem. They can be defined recursively as follows: $S^1_p$ is isomorphic to the complete graph $K_p$ and $S^{n+1}_p$ is constructed from $p$ copies of $S^n_p$ by adding exactly one edge between every pair of copies of $S^n_p$+1. Sierpiński graphs $S^1_3$, $S^2_3$, and $S^3_3$ are depicted in Figure 1. In the “classical” case, when $p = 3$, the Sierpiński graphs are isomorphic to Hanoi graphs. More about Sierpiński graphs and their connections to the Hanoi graphs can be found in the recent second edition of the book about the Tower of Hanoi puzzle by Hinz et al. [7].

Sierpiński graphs have been extensively studied in most graph-theoretical aspects as well as in other areas of mathematics and even psychology. Some notable papers are [8] [10] [12] [14] [16] [19] [23] [24]. An extensive summary of topics studied on and around...
Sierpiński graphs is available in the survey paper by Hinz, Klavžar and Zemljič [9]. In that paper the authors introduced Sierpiński-type graphs as graphs that are derived from or lead to the Sierpiński triangle fractal.

![Figure 1: Sierpiński graphs $S^1_3$, $S^2_3$, and $S^3_3$.](image)

Recently these families of graphs have been generalized by Gravier, Kovše and Parreau to a family called generalized Sierpiński graphs [5]. Instead of iterating a complete graph they start with an arbitrary graph $G$ and form a self-similar graph in the same way as Sierpiński graphs are derived from a complete graph. See Figure 2 for an example of the second iteration of a generalized Sierpiński graph, where the base graph is a house. For a given graph $G$, $S^n_G$ denotes the $n$-th iteration generalized Sierpiński graph.

![Figure 2: Generalized Sierpiński graphs $S^1_G$ and $S^2_G$ when $G$ is the house graph.](image)

The generalized Sierpiński graphs have been extensively studied in the past few years. A few years after they were introduced in 2011 the first two papers appeared at the same time. Geetha and Somasundaram [4] studied their total chromatic number while Rodríguez-Velázquez and Tomás-Andreu [22] examined their Randić index. Shortly afterwards several papers followed on similar topics, but also on the chromatic number, vertex cover number, clique number, and domination number, see [21].

Metric properties have always presented intriguing problems in the family of Sierpiński-type graphs mostly due to their connection to the Hanoi graphs. Namely a solution to the Tower of Hanoi problem may be modelled as a shortest path problem on the corresponding Hanoi graph. Therefore it is not surprising that metric properties of generalized Sierpiński graphs have been studied as well. In [3] Estrada-Moreno, Rodríguez-Bazan and Rodríguez-Velázquez investigate distances and present, among other results, an algorithm for determining the distance between an extreme vertex and an arbitrary vertex of a generalized Sierpiński graph. In the recent paper [1] Alizadeh et al. investigate metric properties for generalized Sierpiński graphs where the base graph is a star graph.
At this point we would like to mention another approach towards the Sierpiński graphs. The graphs $S^n_3$ appear naturally locally by applying a series of truncations of maps; see Pisanski and Tucker [20] and Alspach and Dobson [2]. For a cubic graph $G$ this is equivalent to applying a series of truncations to $G$, where the truncation of $G$ is the line graph of the subdivision graph of $G$. For any graph and the neighbourhood of vertex of valence $d$ the repeated truncation looks like $S^n_d$. A related construction, called the clone cover, is considered by Malnič, Pisanski and Žitnik in [17].

In this paper we generalize the generalized Sierpiński graphs even further. Instead of taking just one graph, we take two (or multiple) graphs and present the operation that yields $S^n_G$ from $S^{n-1}_G$ and $G$ as a product. If we take two graphs $G$ and $H$, the resulting product locally has the structure of $H$, but globally it is similar to $G$. We call such a product operation the Sierpiński product.

The Sierpiński product shows some features of classical graph products [6], the most important being that the vertex set of the Sierpiński product of graphs $G$ and $H$ is $V(G) \times V(H)$. However, one needs some extra information outside $G$ and $H$ to complete the definition of the Sierpiński product of graphs $G$ and $H$. This information can be encoded as a function $f : V(G) \rightarrow V(H)$. Furthermore, the product is defined so that we can extend it to multiple factors. We will see that by definition the Sierpiński product of two graphs is always a subgraph of their lexicographic product.

The paper is organized as follows. In Section 2 we give a formal definition of the Sierpiński product of graphs $G$ and $H$ with respect to $f : V(G) \rightarrow V(H)$, this product is denoted by $G \otimes_f H$. We explore some graph-theoretical properties such as connectedness and planarity of the Sierpiński product. In particular, we show that $G \otimes_f H$ is connected if and only if both $G$ and $H$ are connected and we present some necessary and sufficient conditions that $G$, $H$ must fulfill in order for $G \otimes_f H$ to be planar. In Section 3 we study symmetries of the Sierpiński product of two graphs. We focus on the automorphisms of $G \otimes_f H$ that arise from the automorphisms of its factors and study the group, generated by these automorphisms. In many cases we can also describe the whole automorphism group of $G \otimes_f H$. Finally in Section 4 we consider the Sierpiński product of more than two graphs. In the special case when we have $n$ equal factors, say equal to $G$, and $f : V(G) \rightarrow V(G)$ is the identity function, their Sierpiński product is equal to $S^n_G$.

2 Definition of the Sierpiński product and basic properties

Let us first review some necessary notions. All the graphs we consider are undirected and simple. Let $G$ be a graph and $x$ be a vertex of $G$. By $N(x)$ we denote the set of vertices of $G$ that are adjacent to $x$, i.e., the neighborhood of $x$. Vertices in this paper will usually be tuples, but instead of writing them in vector form $(x_m, \ldots, x_1)$, we will usually write them as words $x_m \ldots x_1$. More precisely, vertices $(0,0,0)$ or $(0,(0,0))$ will simply be denoted by 000, except in the case when we will emphasize their origins. The number of vertices of a graph $G$, i.e., the order of $G$, will be denoted by $|G|$, and the number of edges of $G$, i.e., the size of $G$, will be denoted by $||G||$.

**Definition 2.1.** Let $G,H$ be graphs and let $f : V(G) \rightarrow V(H)$ be a function. Then the Sierpiński product of $G$ and $H$ with respect to $f$ is defined as a pair $(K, \varphi)$, where $K$ is
a graph on the vertex set \( V(K) = V(G) \times V(H) \) with two types of edges:

\[ \bullet \quad \{(g, h), (g, h')\} \text{ is an edge in } K \text{ for every vertex } g \in V(G) \text{ and every edge } \{h, h'\} \in E(H), \]

\[ \bullet \quad \{(g, f(g'), (g', f(g))\} \text{ is an edge in } K \text{ for every edge } \{g, g'\} \in E(G); \]

and \( \varphi : V(G) \rightarrow V(K) \) is a function that maps every vertex \( g \in V(G) \) to the vertex \( (g, f(g)) \in V(K) \). We will denote such Sierpiński product by \( G \otimes_f H \).

Often when we will have only two factors, we will be interested only in the graph \( K \) and not the embedding \( \varphi \) of \( G \) into \( K \). In such cases we will use the notation \( K = G \otimes_f H \). If \( V(G) \subseteq V(H) \) and \( f \) is the identity function on its domain, we will skip the index \( f \) and denote the Sierpiński product of \( G \) and \( H \) simply by \( G \otimes H \). The role of function \( \varphi \) will become clear in Section 4. Note that there are no restrictions on function \( f : V(G) \rightarrow V(H) \). However, sometimes it is convenient that for every \( g, g_1, g_2 \in V(G) \) the following property holds: if \( g_1, g_2 \in N(g) \), then \( f(g_1) \neq f(g_2) \). In this case we say that \( f \) is locally injective.

The Sierpiński product can be defined in a similar way also for graphs with loops and multiple edges. In this case, a loop in \( G \), say \( \{g, g\} \), would correspond to a loop \( \{(g, f(g)), (g, f(g))\} \) in \( G \otimes_f H \) and a multiple edge in \( G \) would correspond to a multiple edge in \( G \otimes_f H \), but all our graphs will be simple.

Figure 3, left, shows the Sierpiński product of \( C_3 \) and \( K_4 \) with respect to function \( f_1 \). Vertices of \( C_3 \) are labeled with numbers 0, 1, 2, vertices of \( K_4 \) are labeled with numbers 0, 1, 2, 3 and \( f_1 : V(C_3) \rightarrow V(K_4) \) is the identity function on its domain. Figure 3, right, shows the Sierpiński product of \( K_4 \) and \( C_3 \) with respect to \( f_2 : V(K_4) \rightarrow V(C_3) \) defined as \( f_2(4) = 3 \) and \( f_2(i) = i \) otherwise. This shows that the Sierpiński product is not commutative.

We now state some simple lemmas regarding the structure of the Sierpiński product of two graphs. We omit most of the proofs, since they follow straight from the definition.

**Lemma 2.2.** Let \( G, H \) be graphs and let \( f : V(G) \rightarrow V(H) \) be a function. Then the following statements hold.
(i) If $|G| = 1$, then $G \otimes_f H$ is isomorphic to $H$.

(ii) If $|H| = 1$, then $G \otimes_f H$ is isomorphic to $G$.

**Lemma 2.3.** Let $G, H$ be graphs and let $f : V(G) \to V(H)$ be a function. Let $G', H'$ be subgraphs of $G, H$, respectively, and let $f'$ be the restriction of $f$ to $V(G')$ such that $\text{Im}(f') \subseteq V(H')$. Then $G' \otimes_f H'$ is a subgraph of $G \otimes_f H$.

**Lemma 2.4.** Let $G, H$ be graphs and let $f : V(G) \to V(H)$ be a function. Then the following statements hold.

(i) Let $g$ be a vertex of $G$. Then the subgraph of $G \otimes_f H$, induced by the set $\{(g, h) | h \in V(H)\}$ is isomorphic to $H$.

(ii) Graph $G$ is a minor of $G \otimes_f H$.

We say that the subgraph of $G \otimes_f H$ from Lemma 2.4 (i) is associated with $g$ and denote it by $gH$. We may view $G \otimes_f H$ as obtained from identical copies of $H$, one for each vertex of $G$, and attaching for every edge $\{g, g'\} \in E(G)$ the corresponding vertex $f(g)$ in $g' H$ to the vertex $f(g')$ in $gH$. The edges of $G \otimes_f H$ naturally fall into two classes. All edges connecting different copies $gH$ are called connecting edges, while the edges inside some subgraph $gH$ are called inner edges.

**Lemma 2.5.** Let $G, H$ be graphs and let $f : V(G) \to V(H)$ be a function. Then $G \otimes_f H$ has $|G| \cdot |H|$ vertices and $||H|| \cdot |G| + ||G||$ edges. In particular, $G \otimes_f H$ has $||H|| \cdot |G|$ inner edges and $||G||$ connecting edges.

**Lemma 2.6.** Let $G$ and $H$ be graphs and let $f : V(G) \to V(H)$ be any mapping. Then the following holds.

(i) There is at most one edge connecting $gH$ and $g'H$ for every $g, g' \in V(G)$.

(ii) Suppose that $f$ is locally injective. Then any vertex of $G \otimes_f H$ is an end-vertex of at most one connecting edge.

**Proof.** (i) Only $\{(g, f(g')), (g', f(g))\}$ can connect $gH$ and $g'H$ and since $G$ is simple there is only one such edge.

(ii) Let $(g, h)$ be a vertex of $G \otimes_f H$. Since $f$ is locally injective, there exists at most one vertex $g' \in N(g)$ such that $h = f(g')$. If such a vertex exists, then $(g, h)$ is an end-vertex of the edge $\{(g, h), (g', f(g))\}$, otherwise it is not contained in any connecting edge.

The lexicographic product of graphs $G$ and $H$ is a graph $G \circ H$ with vertex set $V(G) \times V(H)$ and two vertices $(g, h)$ and $(g', h')$ are adjacent in $G \circ H$ if and only if either $g$ is adjacent with $g'$ in $G$ or $g = g'$ and $h$ is adjacent with $h'$ in $H$. In other words, $G \circ H$ consists of $|G|$ copies of $H$ and for every edge $\{g, g'\}$ in $G$, every vertex of $gH$ is connected to every vertex in $g'H$. Therefore the next result follows directly from Definition 2.1.

**Proposition 2.7.** Let $G$ and $H$ be graphs and let $f : V(G) \to V(H)$ be any mapping. Then $G \otimes_f H$ is a subgraph of $G \circ H$.

Note that for different functions $f, f'$ graphs $G \otimes_f H$ and $G \otimes_{f'} H$ may be isomorphic or nonisomorphic.
Theorem 2.8. Let $G, H$ be graphs and let $f : V(G) \to V(H)$ be a function. Let $\alpha \in \text{Aut}(G)$, $\beta \in \text{Aut}(H)$ and $f' = \beta \circ f \circ \alpha$. Then $G \otimes f H$ is isomorphic to $G \otimes f' H$.

Proof. Define a function $\gamma : V(G \otimes f H) \to V(G \otimes f' H)$ by $\gamma(g, h) = (\alpha^{-1}(g), \beta(h))$ for $g \in V(G)$ and $h \in V(H)$. Since $\alpha, \beta$ are bijections, also $\gamma$ is a bijection. Since $\beta$ is an automorphism, $\gamma$ maps inner edges to inner edges.

Take a connecting edge in $G \otimes f H$, say $\{(g, f(g')), (g', f(g))\}$, where $\{g, g'\} \in E(G)$. Then $\gamma(g, f(g')) = (\alpha^{-1}(g), \beta(f(g'))) = (\alpha(g'), \beta(f(g))) = f'(\alpha^{-1}(g')) = f'(\alpha(g')) = (\alpha^{-1}(g')) = \gamma(g, f(g'))$, and $f'(\alpha^{-1}(g')) = \beta(f(\alpha^{-1}(g')))) = \beta(f(g'))$, we see that $\gamma$ also maps a connecting edge to a connecting edge. Therefore $\gamma$ is an isomorphism. □

Corollary 2.9. Let $G$ be a graph and let $f \in \text{Aut}(G)$. Then $G \otimes f G$ is isomorphic to $G \otimes f G$.

In the remainder of this section we consider two other basic graph-theoretic properties of the Sierpiński product with respect to its factors: connectedness and planarity.

Proposition 2.10. Let $G$ and $H$ be graphs and let $f : V(G) \to V(H)$ be a function. Then $G \otimes f H$ is connected if and only if $G$ and $H$ are connected.

Proof. Suppose $G$ and $H$ are connected. Pick two vertices $(g, h), (g', h')$ in $G \otimes f H$. Then there exists a path from $g$ to $g'$ in $G$, say $g = g_0, g_1, g_2, \ldots, g_k, g_k = g'$. We construct a path from $(g, h)$ to $(g', h')$ in $G \otimes f H$, passing through $g_1 H, g_2 H, \ldots, g_k H$ in this order as follows. Let $P_0$ be a path from $(g_0, h)$ to $(g_0, f(g_1))$ in $g_0 H$, let $P_i$ be a path from $(g_i, f(g_{i-1}))$ to $(g_i, f(g_{i+1}))$ in $g_i H$ for $i = 1, \ldots, k-1$ and let $P_k$ be a path from $(g_k, f(g_{k-1}))$ to $(g_k, h')$ in $g_k H$. Such paths exist since every subgraph $g_i H$ is connected. Then $P_0 P_1 \ldots P_k$ is a path between $(g, h), (g', h')$ in $G \otimes f H$.

Conversely, suppose $G \otimes f H$ is connected. Pick two vertices $g$ and $g'$ from $G$. Then a path from $g H$ to $g' H$ in $G \otimes f H$ corresponds to a path in $G$ from $g$ to $g'$. Therefore also $G$ is connected. Suppose now that $H$ is not connected. We will show that in this case $G \otimes f H$ is not connected. Denote by $H_1$ a connected component of $H$ such that $V_i = \{g \in G \mid f(g) \in V(H_1)\}$ is nonempty. For $g \in V_i$ and $h \in V(H_1)$, all the neighbours of $(g, h)$ belong to $g_i H_1$ or to $g' H_1$ for some $g' \in V_i$. Therefore there are no edges between the set of vertices $\{(g, h) \in G \otimes f H \mid g \in V_i \text{ and } h \in V(H_1)\}$ and the rest of the vertices of $G \otimes f H$. So $G \otimes f H$ is not connected. This finishes the proof. □

We will denote by $H + g$ the graph obtained from $H$ by adding a copy of vertex $g \in V(G)$ to it and connecting it to all vertices $f(g')$, where $g' \in N(g)$. We will denote this new vertex by $g_H$.

The next Theorem characterises when a Sierpiński product $G \otimes f H$ is planar.

Theorem 2.11. Let $G, H$ be connected graphs and let $f : V(G) \to V(H)$ be any mapping. Then $G \otimes f H$ is planar if and only the following three conditions are fulfilled:

(i) graph $G$ is planar,

(ii) for every $g \in V(G)$ the graph $H + g$ is planar,

(iii) there exists an embedding of $G$ in the plane with the following property: for every $g \in V(G)$, with $g_1, g_2, \ldots, g_k$ being the cyclic order of vertices around $g$, there exists an embedding of $H + g$ in the plane such that the cyclic order of vertices around $g_H$ in $H + g$ is $f(g_k), f(g_{k-1}), \ldots, f(g_1)$. 

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Proof. All three conditions are necessary. Suppose $G \otimes_f H$ is embedded in the plane. Then a planar embedding of $G$ is obtained by contracting $gH$ to a single vertex for every $g \in V(G)$. Hence $G$ must be planar. Suppose $G$ is embedded in the plane as above. Let $g \in V(G)$ and let $g_1, g_2, \ldots, g_k$, be the cyclic order of vertices around $g$ in this embedding. Let $N(gH) = \{g'H|g' \in N(g)\}$ denote the collection of graphs $g'H$ that are adjacent to $gH$ for some $g \in V(g)$. We contract every member $g'H$ from $N(gH)$ to a single vertex. Then we keep $gH$, all the new vertices all the new edges and delete the rest of the graph. The graph obtained in this way is still embedded in the plane. Now we identify all the new vertices; we call the vertex obtained in this way $gH$. Therefore the embedding of $gH$ fulfills (iii). The converse goes by construction. We first embed $G$ in the plane as in (iii) and then expand every vertex $g$ of $G$ to $gH$, embedded in the plane as in (iii). By (iii) it is possible to connect the copies of $H$ such that the resulting graph is a plane embedding of $G \otimes_f H$.

Next result follows directly from Theorem 2.11 (ii).

Corollary 2.12. Let $G, H$ be connected graphs and let $f : V(G) \rightarrow V(H)$ be any mapping. If $G \otimes_f H$ is planar, then for every $g \in G$ there exists an embedding of $H$ in the plane such that the vertices $\{f(g')|g' \in N(g)\}$ lie on the boundary of the same face.

Using Theorem 2.11 and Corollary 2.12 we can determine when $G \otimes G$ is planar for a connected graph $G$. We also give a sufficient condition for $G \otimes_f H$ to be planar when $G \neq H$.

Corollary 2.13. Let $G$ be a connected graph and let $f : V(G) \rightarrow V(G)$ be the identity mapping. Then $G \otimes G$ is planar if and only if $G$ is outerplanar or $G = K_4$.

Proof. If $G$ is outerplanar or $K_4$, then conditions (i), (ii), (iii) from Theorem 2.11 are fulfilled, so $G \otimes G$ is planar.

Suppose now that $G$ is planar but not outerplanar. Then it contains a subdivision of $K_{2,3}$ or a subdivision of $K_4$ (with at least one additional vertex) as a subgraph. Such a graph $G$ always contains a vertex such that in every plane embedding of $G$ not all of its neighbours will be on the boundary of the same face. Therefore $G \otimes G$ is not planar by Corollary 2.12.

Theorem 2.14. Let $G, H$ be connected graphs and let $f : V(G) \rightarrow V(H)$ be any mapping. Assume that $G$ is planar, $\Delta(G) \leq 3$ and $H$ is outerplanar. Then $G \otimes_f H$ is planar.

Proof. Denote $K = G \otimes_f H$. Suppose $K$ is not planar. Then it contains a subdivision of $K_{3,3}$ or $K_5$ as a subgraph. First assume that $K$ contains a subdivision of $K_{3,3}$. There are four cases to consider, depending on how many vertices of degree 3 of the subdivision of $K_{3,3}$ are in the same copy of $H$.

1. If every vertex is in separate copy of $H$ in $K$, then by contracting $gH$ to a single vertex for every $g \in G$ we see that $K_{3,3}$ is a minor in $G$, so $G$ is not planar.
2. If there are between two and four vertices in some $gH$, then we need at least four edges connecting $gH$ to other copies of $H$ in $K$. This is not possible, since maximal degree in $G$ is at most three.

3. There are five vertices in some $gH$ and one vertex in some $g'H$ for $g \neq g'$. Since $H$ is outerplanar, $gH$ cannot contain a subdivision of $K_{2,3}$. Therefore we need at least two edges going out of $gH$ to obtain a subdivision of $K_{2,3}$ from the five vertices in $gH$. We also need three edges going out of $gH$ to connect $gH$ to the vertex of $K_{3,3}$ in $g'H$. This is again not possible, since the maximal degree of $G$ is at most 3.

4. The only remaining possibility is that all six vertices are in the same copy $gH$ of $H$. Since $H$ is outerplanar, there can be at most seven edges (or paths) between pairs of vertices of $K_{3,3}$ in $gH$. The remaining two paths must go through other copies of $H$, which means that we again need at least four edges connecting $gH$ to other copies of $H$ in $K$. A contradiction.

Therefore $K$ does not contain a subdivision of $K_{3,3}$. Next assume that $K$ contains a subdivision of $K_5$. There are three cases to consider, depending on how many vertices of degree 4 of the subdivision of $K_5$ are in the same copy of $H$.

1. If every vertex is in separate copy of $H$ in $K$, then by contracting $gH$ to a single vertex for every $g \in G$ we see that $K_5$ is a minor in $G$, so $G$ is not planar.

2. If there are between two and four vertices in some $gH$, then we need at least four edges connecting $gH$ to other copies of $H$ in $K$. This is not possible, since maximal degree in $G$ is at most three.

3. The only remaining possibility is that all five vertices of $K_5$ are in the same copy of $H$. Since $H$ is outerplanar, it doesn’t contain a subdivision of $K_4$ or $K_{2,3}$. Therefore there can be at most eight edges (or paths) between pairs of these vertices in $gH$ (in fact, there can be at most six such paths). The remaining two paths must go through other copies of $H$, which means that we need at least four edges connecting $gH$ to other copies of $H$ in $K$. A contradiction.

It follows that $K$ doesn’t contain a subdivision of $K_{3,3}$ or $K_5$, so it is planar.

If a connected graph is not planar it is natural to consider its genus. The genus of a graph $G$ is denoted by $\gamma(G)$. Recall that by Lemma 2.3, graph $G$ is a minor of $G \otimes_f H$ for any function $f : V(G) \to V(H)$, and $G \otimes_f H$ contains $|G|$ copies of $H$ as induced subgraphs. Suppose $G, H$ are connected and $f$ is arbitrary. Then it is easy to see, cf. [18, Theorem 4.4.2], that

$$\gamma(G \otimes_f H) \geq \gamma(G) + |G| \cdot \gamma(H).$$

(1)

Note that the bound is not sharp even if the factors are planar. In the case of planar Sierpiński product we were able to settle the case in Theorem 2.11. It would be interesting to find some sufficient condition for the equality in (1) to hold also for non-planar Sierpiński products.
3 Symmetry

Throughout this section let \( G, H \) be connected graphs and let \( f : V(G) \to V(H) \) be any mapping. Recall that the edge set of \( G \otimes_f H \) can be naturally partitioned into two subsets:

- **inner edges** \( \{(g, h), (g, h')\} \) for every vertex \( g \in V(G) \) and every edge \( \{h, h'\} \in E(H) \), and
- **connecting edges** \( \{(g, f(g')), (g', f(g))\} \) for every edge \( \{g, g'\} \in E(G) \).

We call this partition of the edge set the fundamental edge partition. We will say that an automorphism of \( G \otimes_f H \) respects the fundamental edge partition if it takes inner edges to inner edges, and connecting edges to connecting edges. We denote the set of all automorphisms of \( G \otimes_f H \) that respect the fundamental edge partition by \( \tilde{\text{Aut}}(G, H, f) \). It is easy to see that this set is a subgroup of the whole automorphism group of \( G \otimes_f H \). If \( G, H \) are connected graphs, the automorphisms that respect the fundamental edge partition have the following useful property.

**Proposition 3.1.** Let \( G \) and \( H \) be connected graphs. Then every automorphism \( \tilde{\gamma} \in \tilde{\text{Aut}}(G, H, f) \) permutes the subgraphs \( gH, g \in G \). In particular, the restriction \( \tilde{\gamma}|_{V(gH)} : V(gH) \to V(g'H) \), where \( g' \in V(G) \), is a graph isomorphism.

In this section we first show that any automorphism of \( G \otimes_f H \) that respects the fundamental edge partition induces automorphisms of \( G \) and \( H \). And conversely, we define two families of automorphisms of \( G \otimes_f H \) that respect the fundamental edge partition using automorphisms of \( G \) and \( H \). Then we show that in many cases all the automorphisms of \( G \otimes_f H \) respect the fundamental edge partition. Finally, we focus on the case when \( G = H \) and \( f \) is an automorphism. In this case we can completely describe the group of automorphisms that respect the fundamental edge partition.

### 3.1 Automorphisms that respect the fundamental edge partition

Let \( \tilde{\gamma} \) be an automorphism of \( G \otimes_f H \) that respects the fundamental edge partition. Then it permutes the subgraphs \( gH, g \in G \). Define a mapping \( \gamma \) such that \( \gamma(g) = g' \) if \( \tilde{\gamma} \) maps \( gH \) to \( g'H \). Obviously, \( \gamma \) is a bijection. Let \( \{g, g_1\} \) be an edge of \( G \). Then \( \{(g, f(g_1)), (g_1, f(g))\} \) is a connecting edge of \( G \otimes_f H \). Since \( \tilde{\gamma} \) respects the fundamental edge partition, it maps this edge to another connecting edge, say \( \{(g', f(g'_1)), (g'_1, f(g'))\} \), where \( g' \) and \( g'_1 \) are adjacent in \( G \). But then \( \gamma \) maps the edge \( \{g, g_1\} \) to an edge (i.e. to \( \{g', g'_1\} \)) and \( \gamma \) is an automorphism. We will say that \( \gamma \) is the projection of \( \tilde{\gamma} \) on \( G \). Conversely, \( \tilde{\gamma} \) is a lift of \( \gamma \). Note that projection of \( \tilde{\gamma} \in \text{Aut}(G \otimes_f H) \) on \( G \) is uniquely defined. However, given an automorphism of \( G \), it can have a unique lift, more than one lift or none at all.

On the other hand, the action of \( \tilde{\gamma} \) on every copy of \( gH \) in \( G \otimes_f H \) induces an automorphism \( \gamma_g \) of \( H \), defined by \( \gamma_g(h) = h' \) if \( \tilde{\gamma} \) sends \( (g, h) \) to \( (g_1, h') \) for some \( g_1 \in V(G) \) and \( h' \in V(H) \).

We will now introduce two families of automorphism of \( G \otimes_f H \) that can be obtained from automorphisms of \( G \) and \( H \). All such automorphisms respect the fundamental edge partition.
Definition 3.2. Let $G, H$ be connected graphs and let $f : V(G) \to V(H)$ be any function. Let $\alpha \in \text{Aut}(G)$ and let $\mathcal{B} : V(G) \to \text{Aut}(H)$ be any mapping. For simplicity we will write $\beta_g$ instead of $\mathcal{B}(g)$ for $g \in V(G)$. Define a mapping $\Psi(\alpha, \mathcal{B}) : V(G \otimes_f H) \to V(G \otimes_f H)$ by

$$\Psi(\alpha, \mathcal{B}) : (g, h) \mapsto (\alpha(g), \beta_g(h)).$$

If $\mathcal{B}$ is a constant function, say $\beta_g = \beta$ for all $g \in V(G)$, we denote $\Psi(\alpha, \mathcal{B})$ by $\Psi(\alpha, \beta)$.

By the discussion at the beginning of this section, we see that the following holds.

Theorem 3.3. Let $G, H$ be connected graphs and let $f : V(G) \to V(H)$ any function. Every automorphism of $G \otimes_f H$ that respects the fundamental edge partition is of form $\Psi(\alpha, \mathcal{B})$ for some $\alpha \in \text{Aut}(G)$ and some mapping $\mathcal{B} : V(G) \to \text{Aut}(H)$.

We now determine when the mapping $\Psi(\alpha, \mathcal{B})$ from Definition 3.2 is an automorphism.

Proposition 3.4. The mapping $\Psi(\alpha, \mathcal{B})$ is always a bijection.

Proof. It is enough to prove that $\Psi(\alpha, \mathcal{B})$ is injective. This is straightforward since $\alpha$ and $\beta_g$, $g \in V(G)$, are all injective.

Proposition 3.5. The mapping $\Psi(\alpha, \mathcal{B})$ is an automorphism if and only if for every $g \in V(G)$ we have $f \circ \alpha = \beta_g \circ f$ on $N(g)$. Moreover, in this case $\Psi(\alpha, \mathcal{B})$ respects the fundamental edge partition.

Proof. We first show that $\Psi(\alpha, \mathcal{B})$ always maps an inner edge to an inner edge. To see this, let $e = \{(g, h_1), (g, h_2)\}$ be an inner edge. Then $\Psi(\alpha, \mathcal{B})$ maps edge $e$ to edge $\{(\alpha(g), \beta_g(h_1)), (\alpha(g), \beta_g(h_2))\}$, which is an inner edge since $\beta_g$ is an automorphism of $H$.

Suppose now that $\Psi(\alpha, \mathcal{B})$ is an automorphism. Since $\Psi(\alpha, \mathcal{B})$ maps inner edges to inner edges, it must map connecting edges to connecting edges. Let $e = \{(g, f(g_1)), (g_1, f(g))\}$ be a connecting edge. Then $\Psi(\alpha, \mathcal{B})(e) = \{(\alpha(g), \beta_g(f(g_1)), (\alpha(g_1), \beta_{g_1}(f(g)))\}$ is also a connecting edge. Therefore $f(\alpha(g_1)) = \beta_g(f(g_1))$. Since $g_1$ can be any neighbour of $g$ in $G$, we have $f \circ \alpha = \beta_g \circ f$ on $N(g)$.

Conversely, let $f \circ \alpha = \beta_g \circ f$ on $N(g)$ for every $g \in V(G)$. Let $e = \{(g, f(g_1)), (g_1, f(g))\}$ be a connecting edge in $G \otimes_f H$. Then $\Psi(\alpha, \mathcal{B})(e) = \{(\alpha(g), \beta_g(f(g_1)), (\alpha(g_1), \beta_{g_1}(f(g)))\}$. Since $f(\alpha(g)) = \beta_{g_1}(f(g))$ and $f(\alpha(g_1)) = \beta_g(f(g_1))$, $\Psi(\alpha, \mathcal{B})(e)$ is a connecting edge. Therefore $\Psi(\alpha, \mathcal{B})$ is an automorphism.

Proposition 3.6. The mapping $\Psi(\alpha, \beta)$ is an automorphism if and only if $f \circ \alpha = \beta \circ f$.

Proof. Let $f \circ \alpha = \beta \circ f$ on $N(G)$ for every $g \in V(G)$. Since $G$ is connected it has no isolated points and so $f \circ \alpha = \beta \circ f$ on $V(G)$. The claim then follows from Proposition 3.5.

A few special cases now follow as simple corollaries.

Corollary 3.7. Suppose $G = H$ and $f$ is an automorphism. Then the mapping $\Psi(\alpha, \beta)$ is an automorphism if and only if $\beta = f \circ \alpha \circ f^{-1}$.

Corollary 3.8. Suppose $G = H$ and $f$ is the identity mapping. Then the mapping $\Psi(\alpha, \beta)$ is an automorphism if and only if $\alpha = \beta$. 
Corollary 3.9. Suppose $V(G) \subseteq V(H)$, $f$ is the identity mapping on its domain and $\beta|_{V(G)} = \alpha$. Then the mapping $\Psi(\alpha, \beta)$ is an automorphism.

Remark 3.10. If $f$ is injective and $G \neq H$, we can always relabel the vertices of $G, H$ such that $f$ is the identity on its domain.

We now give some examples showing that $f$ need not be injective or surjective and we can still have automorphisms of type $\Psi(\alpha, \beta)$. Also, if $G = H$, the mapping $f$ need not be an automorphism.

Example 3.11. Let $G = K_3$ and $H = K_{3,3}$ with edge set $\{\{1,2\}, \{2,3\}, \{2,4\}\}$, and let $f : V(G) \to V(H)$ map $1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3$. Let $\alpha = (1 2 3), \beta_1 = (1 3 5)(2 4 6), \beta_2 = (1 3 5)(2 6 4), \beta_3 = (1 3 5)$ and let $B : V(G) \to \text{Aut}(G), B(g) = \beta_g$. Then $f \circ \alpha = \beta_1 \circ f = \beta_2 \circ f = \beta_3 \circ f$ and

$$\Psi(\alpha, B) = (11 23 35)(12 24 32)(13 25 31)(14 26 34)(15 21 33)(16 22 36)$$

is an automorphism of $G \otimes_f H$ that cyclically permutes the subgraphs $gH$, see Figure 4.

Example 3.12. Let $G = H = K_{1,3}$ with edge set $\{\{1,2\}, \{2,3\}, \{2,4\}\}$, and let $f : V(G) \to V(G)$ be defined as $f = (1 2 3 4)$. Note that $f$ is a bijection that is not an automorphism of $G$. If $\alpha = (3 4)$ and $\beta = f \circ \alpha \circ f^{-1} = (1 4)$, then $f \circ \alpha = \beta \circ f$ and

$$\Psi(\alpha, \beta) = (11 14)(21 24)(31 44)(32 42)(33 43)(34 41)$$

is an automorphism of $G \otimes_f G$, that swaps copies $3G$ and $4G$, see Figure 5.

Example 3.13. Let $G = C_4$ with $V(G) = \{1,2,3,4\}$ and let $H$ be a star, with edge set $\{\{1,2\}, \{2,3\}, \{2,4\}\}$. Let $f : V(G) \to V(H)$ map $1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4$ and $4 \mapsto 3$. Note that the mapping $f$ is neither injective nor surjective. If $\alpha = (1 2)(3 4)$ and $\beta = (3 4)$, then $f \circ \alpha = \beta \circ f$ and

$$\Psi(\alpha, \beta) = (11 21)(12 22)(13 24)(14 23)(31 41)(32 42)(33 44)(34 43)$$

is a reflection automorphism of $G \otimes_f H$, swapping copies $1H, 2H$ and $3H, 4H$, see Figure 6.

Figure 4: Graphs $K_3$, $K_{3,3}$ and their Sierpiński product with respect to $f : V(K_3) \to V(K_{3,3})$, $f : 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5$. 

Figure 5: Graphs $K_3, K_{1,3}$ and their Sierpiński product with respect to $f : V(K_3) \to V(K_{1,3})$, $f : 1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3$.
Figure 5: Graph $G = K_{1,3}$ and the Sierpiński product $G \otimes_f G$ with respect to $f = (1 \ 2 \ 3 \ 4)$.

Figure 6: Graphs $C_4$, $K_{1,3}$ and their Sierpiński product with respect to $f : V(C_4) \to V(K_{1,3})$, $f : 1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 3$.

Now let us introduce the second family of automorphisms. Let $g \in V(G)$ and $\beta \in \text{Aut}(H)$. Define a mapping $\Phi(g, \beta) : V(G \otimes_f H) \to V(G \otimes_f H)$ given by

$$\Phi(g, \beta) : (g_1, h_1) \mapsto \begin{cases} (g_1, h_1) & \text{if } g_1 \neq g, \\ (g_1, \beta(h_1)) & \text{if } g_1 = g. \end{cases}$$

**Proposition 3.14.** The mapping $\Phi(g, \beta)$ is an automorphism of $G \otimes_f H$ if and only if $\beta$ is in the stabilizer of $f(N(g))$. Moreover, in this case $\Phi(g, \beta)$ respects the fundamental edge partition.

**Proof.** The mapping $\Phi(g, \beta)$ is obviously a bijection since it fixes all the vertices of $G \otimes_f H$ not in $gH$ and it permutes the vertices in $gH$. It also fixes inner edges and connecting edges that do not have any endvertex in $gH$ and it permutes inner edges in $gH$.

Take a vertex $g' \in N(G)$. Then $\{(g, f(g')), (g', f(g))\}$ is a connecting edge. The mapping $\Phi(g, \beta)$ maps $\{(g, f(g')), (g', f(g))\}$ to the set $\{(g, \beta(f(g')), (g', f(g))\}$, which is an edge if and only if $\beta(f(g')) = f(g')$. So $\Phi(g, \beta)$ is an automorphism if and only if $\beta$ is in the stabilizer of $f(g')$ for every $g' \in N(G)$.  

\[\square\]
Remark 3.15. Note that by Theorem 3.3, a mapping $\Phi(g, \beta)$ is the same as $\Psi(\alpha, B)$ for some $\alpha \in \text{Aut}(G)$ and $B : V(G) \to \text{Aut}(H)$. Indeed, it is easy to verify that for $\alpha = \text{id}$ and $B$ defined by the rules $B : g_1 \to \text{id}$ if $g_1 \neq g$ and $B : g \to \beta$, we have $\Phi(g, \beta) = \Psi(\alpha, B)$.

Given a group $X$ acting on set $Y$, we denote by $X_Y$ the stabilizer of $Y$, i.e., the subgroup of $X$ that fixes every element of $Y$. For $g \in G$ denote by $\hat{B}_g(G, H, f)$ the group generated by $\{\Phi(g, \beta_g) | \beta_g \in \text{Aut}(H)_{f(N(g))}\}$. Denote by $\hat{B}(G, H, f)$ the group generated by $\{\hat{B}_g(G, H, f) | g \in V(G)\}$.

Proposition 3.16. Let $g, g'$ be distinct vertices of $G$ and let $\beta_g \in \text{Aut}(H)_{f(N(g))}$, $\beta_{g'} \in \text{Aut}(H)_{f(N(g'))}$. Then $\Phi(g, \beta_g)$ and $\Phi(g', \beta_{g'})$ commute.

Proof. Mappings $\Phi(g, \beta_g)$ and $\Phi(g', \beta_{g'})$ commute since as permutations they have disjoint supports.

Theorem 3.17. Group $\hat{B}(G, H, f)$ is a subgroup of group $\hat{A}(G, H, f)$ and is a direct product

$$\hat{B}(G, H, f) = \prod_{g \in V(G)} \hat{B}_g(G, H, f).$$

Moreover, the group $\hat{B}(G, H, f)$ is isomorphic to the group $\prod_{g \in V(G)} \text{Aut}(H)_{f(N(g))}$.

Proof. Group $\hat{B}(G, H, f)$ is a subgroup of $\hat{A}(G, H, f)$ by the definition and Proposition 3.14. Since the groups $\hat{B}_g(G, H, f), g \in V(G)$, have pairwise only the identity in common, they generate $\hat{B}(G, H, f)$, and the elements of two distinct groups commute, equation 2 holds. The last claim is true since for every $g \in G$ the groups $\hat{B}_g(G, H, f)$ and $\text{Aut}(H)_{f(N(g))}$ are isomorphic in the obvious way.

3.2 When do all the automorphisms respect the fundamental edge partition

Given connected graphs $G, H$ and a mapping $f : V(G) \to V(H)$, in general there can exist automorphisms of $G \otimes_f H$ that do not respect the fundamental edge partition. Figure 7 shows such an example. There $G = C_4$, $H = 2K_3 + e$ and $f : V(G) \to V(H)$ is the identity function on its domain. One can easily observe that cyclic rotation of $G \otimes_f H$ maps inner edge $\{16, 15\}$ to connecting edge $\{14, 41\}$.

Note that in the example above, graph $H$ is not 2-connected. When graphs $G, H$ are both 2-connected, we have so far not been able to find an automorphism of $G \otimes_f H$ that does not respect the fundamental edge partition. Therefore we propose the following Conjecture.

Conjecture 3.18. Let $G, H$ be 2-connected graphs and let $f : V(G) \to V(H)$ be any mapping. Then

$$\hat{A}(G, H, f) = \text{Aut}(G \otimes_f H).$$

In this section we prove this conjecture for two special cases. In the first case $G = H$ and $G$ is a regular triangle-free graph. In the second case every edge of $H$ is contained in a short cycle. Note that in these two cases the assumption that $G, H$ are 2-connected is not needed.
Proposition 3.19. Let \( G \) be a connected regular triangle-free graph and let \( f : V(G) \to V(G) \) be an automorphism of \( G \). Then every automorphism of \( G \otimes_f G \) respects the fundamental edge partition.

Proof. Let \( k \) denote the valency of \( G \). Then the endvertices of every connecting edge in \( G \otimes G_f \) have valency \( k + 1 \) by Lemma 2.6 (ii). An endvertex of an inner edge may have valency \( k \) or \( k + 1 \). Clearly, if at least one endvertex of an inner edge has valency \( k \), this edge cannot be mapped to a connecting edge by any automorphism.

Suppose now that both endvertices of an inner edge \( \{ (g, g_1), (g, g_2) \} \) have degree \( k + 1 \). This is only possible if \( (g, g_1) \) and \( (g, g_2) \) are endvertices of some connecting edges, say \( \{(g, g_1), (g_1', f(g))\} \) and \( \{(g, g_2), (g_2', f(g))\} \) where \( g_1 = f(g_1') \) and \( g_2 = f(g_2') \). But then \( g_1' \) and \( g_2' \) are adjacent to \( g \) in \( G \). Since \( g_1 \) and \( g_2 \) are adjacent in \( G \) and \( f \) is an automorphism, also \( g_1' \) and \( g_2' \) are adjacent. But then \( g, g_1', g_2' \) form a triangle in \( G \), a contradiction. Therefore no inner edge can be mapped to a connecting edge, so every automorphism of \( G \otimes_f G \) respects the fundamental edge partition. \( \square \)

Lemma 3.20. Let \( G \) and \( H \) be graphs and let \( f : V(G) \to V(H) \) be any mapping. Let \( \{g, g'\} \) be an edge of \( G \).

(i) If \( \{g, g'\} \) is not contained in any cycle of \( G \), then the edge \( \{(g, f(g')), (g', f(g))\} \) is not contained in any cycle of \( G \otimes_f H \).

(ii) Let \( c \) be the length of the shortest cycle that contains \( \{g, g'\} \). Then the shortest cycle that contains the edge \( \{(g, f(g')), (g', f(g))\} \) in \( G \otimes_f H \) has length at least \( 2c \).

(iii) Suppose that \( f \) is locally injective and let \( c \) be the length of the shortest cycle that contains \( \{g, g'\} \). Then the shortest cycle that contains the edge \( \{(g, f(g')), (g', f(g))\} \) in \( G \otimes_f H \) has length at least \( 2c \).
Proof. Let \( C \) be a cycle in \( G \otimes f H \) that contains \( \{(g, f(g'), (g', f(g)))\} \). Suppose that \( \{(g, f(g'), (g', f(g))\}, \{(g', f(g)), (g, f(g))\}, \ldots, \{(g_k, f(g), (g, f(g_k)))\} \) are the connecting edges in \( C \) in that order. Then \( gg'g_1g_2\ldots g_kg \) is a cycle of length \( k \) in \( G \) that contains the edge \( \{g, g'\} \), so \( k \geq c \). Furthermore, if \( \{g, g'\} \) is not contained in any cycle of \( G \), then the edge \( \{(g, f(g'), (g', f(g))\} \) can not be contained in any cycle of \( G \otimes f H \). Recall that if \( f \) is locally injective, any vertex of \( G \otimes f H \) is an end-vertex of at most one connecting edge by Lemma 2.6. Therefore in this case the shortest cycle that contains \( \{(g, f(g'), (g', f(g))\} \) has length at least \( 2c \).

Proposition 3.21. Let \( G \) and \( H \) be connected graphs, let \( f : V(G) \rightarrow V(H) \) be any mapping and let the girth of \( G \) be equal to \( c \). In any of the following cases every automorphism of \( G \otimes f H \) respects the fundamental edge partition:

(i) \( G \) is a tree and \( H \) is a bridgeless graph;

(ii) every edge of \( H \) is contained in a cycle of length at most \( c - 1 \);

(iii) mapping \( f \) is locally injective and every edge of \( H \) is contained in a cycle of length at most \( 2c - 1 \).

Proof. By Lemma 3.20, the shortest cycle that contains a connecting edge has length at least \( c \) in case (ii), length at least \( 2c \) in case (iii) and is not contained in any cycle in case (i). Since every inner edge is contained in a cycle, in a cycle of length at most \( c - 1 \), in a cycle of length at most \( 2c - 1 \) in cases (i), (ii), (iii), respectively, a connecting edge cannot be mapped to an inner edge by any automorphism.

Using Propositions 3.19 and 3.21, we see that in some cases the group of automorphisms that respect the fundamental edge partition is in fact the whole automorphism group of \( G \otimes f H \).

Theorem 3.22. Let \( G \) be a connected regular triangle-free graph and let \( f : V(G) \rightarrow V(G) \) be an automorphism of \( G \). Then

\[ \tilde{A}(G, G, f) = Aut(G \otimes f G) \]

Theorem 3.23. Let \( G \) and \( H \) be connected graphs, let \( f : V(G) \rightarrow V(H) \) be any mapping and let the girth of \( G \) be equal to \( c \). In any of the following cases

(i) \( G \) is a tree and \( H \) is a bridgeless graph:

(ii) every edge of \( H \) is contained in a cycle of length at most \( c - 1 \);

(iii) mapping \( f \) is locally injective and every edge of \( H \) is contained in a cycle of length at most \( 2c - 1 \);

the group \( \tilde{A}(G, H, f) \) is equal to \( Aut(G \otimes f H) \).
3.3 Group of automorphisms of \( G \otimes_f G \)

We now consider the group of automorphisms that respect the fundamental edge partition in the special case when \( G = H \) and \( f : V(G) \rightarrow V(G) \) is an automorphism. Since in this case \( G \otimes_f G \) is isomorphic to \( G \otimes G \) we could restrict ourselves to the case where \( f \) is the identity. Note that in that case the structure of the automorphism group was sketched in the paper [5], but the proofs were never published.

Recall that by Corollary 3.7 every automorphism \( \alpha \) of \( G \) has a lift, \( \Psi(\alpha, f \circ \alpha \circ f^{-1}) \).

We call this automorphism the diagonal automorphism of \( G \otimes_f G \) corresponding to \( \alpha \) and denote it by \( \bar{\alpha} \). Denote by \( \hat{A}(G, f) \) the set of all diagonal automorphisms. The following proposition is straightforward to prove.

**Proposition 3.24.** The set \( \hat{A}(G, f) \) is a subgroup of \( \hat{A}(G, G, f) \), isomorphic to \( \text{Aut}(G) \).

To determine the structure of the group \( \hat{A}(G, G, f) \), we first show that every element of \( \hat{A}(G, G, f) \) can be written as a product of an element from \( \hat{B}(G, G, f) \) and an element of \( \hat{A}(G, f) \). Furthermore, we show that \( \hat{B}(G, G, f) \) is normal in \( \hat{A}(G, G, f) \).

**Theorem 3.25.** Let \( G \) be a connected graph and let \( f : V(G) \rightarrow V(G) \) be an automorphism. Let \( \tilde{\gamma} \) be an automorphism of \( G \otimes_f G \). Then there exist \( \alpha \in \text{Aut}(G) \) and \( \beta_g \in \text{Aut}(G)_{f|N(g)} \) for every \( g \in V(G) \) such that \( \tilde{\gamma} = \bar{\alpha} \left( \prod_{g \in V(G)} \Phi(\alpha, \beta_g) \right) \).

**Proof.** Let \( \alpha \) be the projection of \( \tilde{\gamma} \) to \( \text{Aut}(G) \). Then \( \bar{\alpha} = \Psi(\alpha, f \circ \alpha \circ f^{-1}) \) permutes the copies \( gG \) in the right way. Observe that \( \bar{\alpha} \) already agrees with \( \tilde{\gamma} \) on the endvertices of all the connecting edges. To obtain \( \tilde{\gamma} \) from \( \bar{\alpha} \), we only need to adjust the action of \( \bar{\alpha} \) on the vertices that are not endvertices of connecting edges. We can do this on every copy \( gG \) separately, by acting with \( \Phi(\alpha, \beta_g) \), where \( \beta_g \in \text{Aut}(G) \) is induced by \( \bar{\alpha}^{-1} \tilde{\gamma} \).

Also \( \beta_g \in \text{Aut}(G)_{f|N(g)} \) since the vertices \( f(N(g)) \) have the right image already and are fixed.

**Theorem 3.26.** Let \( f : V(G) \rightarrow V(G) \) be an automorphism. Then \( \hat{B}(G, G, f) \) is a normal subgroup of group \( \hat{A}(G, G, f) \).

**Proof.** Observe that the mapping \( \lambda : \hat{A}(G, G, f) \rightarrow \hat{A}(G, G, f) \) defined by \( \lambda : \Psi(\alpha, \mathcal{B}) \rightarrow \Psi(\alpha, f \circ \alpha \circ f^{-1}) \) is a homomorphism of groups with \( \hat{B}(G, G, f) \) being its kernel. Therefore \( \hat{B}(G, f) \) is a normal subgroup of \( \hat{A}(G, G, f) \).

**Theorem 3.27.** Let \( G \) be a connected graph and let \( f : V(G) \rightarrow V(G) \) be an automorphism. Then the group \( \hat{A}(G, G, f) \) is a semidirect product,

\[
\hat{A}(G, G, f) = \hat{A}(G, f) \ltimes \hat{B}(G, G, f).
\]

**Proof.** Group \( \hat{B}(G, G, f) \) is a normal subgroup of \( \hat{A}(G, G, f) \) by Theorem 3.26. By Theorem 3.25 every element of \( \hat{A}(G, G, f) \) can be written as a product of a diagonal automorphism and an element from \( \hat{B}(G, G, f) \). Moreover, only identity is in both \( \hat{A}(G, f) \) and \( \hat{B}(G, G, f) \). This proves that \( \hat{A}(G, G, f) \) is a semidirect product of \( \hat{A}(G, f) \) and \( \hat{B}(G, G, f) \).

Note that in general not every automorphism of \( G \) has a lift. Diagonal mappings are well defined only if \( f \) is a bijection. Define \( \hat{A}(G, H, f) \) to be the set of all diagonal mappings that are also automorphisms. This is a subgroup of \( \hat{A}(G, H, f) \). We believe that the following conjecture holds.
Conjecture 3.28. Let $G$ and $H$ be connected graphs on the same vertex set and let $f : V(G) \to V(H)$ be a bijection. Then the group $\widetilde{A}(G,H,f)$ is a semidirect product,

$$\widetilde{A}(G,H,f) = \widetilde{A}(G,H,f) \rtimes \widetilde{B}(G,H,f).$$

4 Sierpiński product with multiple factors

When extending the Sierpiński product to more than two factors we first need to specify how the graph $G$ embeds into the product $G \otimes_f H$ in order to be able to multiply it with the next graph. This is exactly the role of the function $\varphi$ from Definition 2.1. Let $G_1, G_2$ and $G_3$ be graphs and $f : V(G_2) \to V(G_1)$, $f' : V(G_3) \to V(G_2)$ be functions. Then the Sierpiński product of these graphs is constructed so that we first build $(K, \varphi) = G_2 \otimes_f G_1$ and then form the product $(K', \varphi') = G_3 \otimes_{\varphi \circ f'} K$. Note that with given functions $f$ and $f'$ we cannot form this product in any other way, therefore Sierpiński product is not associative. We will denote such Sierpiński product by $G_3 \otimes_f G_2 \otimes_f G_1$.

In Figure 8 it is shown how the product $G_2 \otimes_f G_3 \otimes_f G_1$ is formed in two steps (with $f : V(G_4) \to V(G_3)$, $f : i \mapsto i \pmod{3}$ and $f' : V(G_3) \to V(G_4)$ being the identity function on its domain).

It is now easy to see that Sierpiński products possess a nice recursive structure, similar to Sierpiński graphs and generalized Sierpiński graphs. By the same reasoning as above, the product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$, where $V(G_\ell) = \{0,1,\ldots,|G_\ell| - 1\}$, and $f_\ell : V(G_{\ell+1}) \to V(G_\ell)$, $\ell = 1, \ldots, m - 1$, are arbitrary functions, can be constructed as follows.

- First, take $|G_2|$ copies of the graph $G_1$ and label them $iG_1$, $i \in \{0, \ldots, |G_2| - 1\}$. Vertices of these graphs have labels $g_2g_1$.

- Connect any two copies $iG_1$ and $jG_1$ if there is an edge $\{i,j\}$ in $G_2$. More precisely, if $\{i,j\} \in E(G_2)$, we add an edge $\{i f_1(j), j f_1(i)\}$ between $iG_1$ and $jG_1$. The resulting graph is then indeed the Sierpiński product $G_2 \otimes_f G_1$ and the corresponding function $\varphi_1 : V(G_2) \to V(G_2 \otimes G_1)$ maps $i$ to $i f_1(i)$ for every $i \in \{0, \ldots, |G_2| - 1\}$.

- Next we form the Sierpiński product of graphs $G_2$ and $K(2) := G_2 \otimes_{f_1} G_1$. To do so we take $|G_3|$ copies of graph $K(2)$, label them $iK(2)$, $i \in \{0, \ldots, |G_3| - 1\}$, and connect $iK(2)$ and $jK(2)$ whenever $\{i,j\}$ is an edge in $G_3$. Such an edge then has the form $\{i f_2(j) f_1(f_2(j)), j f_2(i) f_1(f_2(i))\}$.

- The final step is to form the Sierpiński product of graphs $G_m$ and $K(m-1)$ in the same way as we formed all the products so far: make $|G_m|$ copies of $K(m - 1)$ and label them $iK(m - 1)$; then for every edge $\{i,j\}$ in $G_m$ we add an edge between copies $iK(m - 1)$ and $jK(m - 1)$. Such an edge then has the following form $\{i f_{m-1}(j) \cdots f_1(f_2(\ldots(f_{m-1}(j))) \cdots), j f_{m-1}(i) \cdots f_1(f_2(\ldots(f_{m-1}(i))) \cdots)\}$.

The resulting graph is the product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$.

If $G_1 = \cdots = G_m = G$ and functions $f_1, \ldots, f_{m-1}$ are all the identity function, then $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_2} G_2 \otimes_{f_1} G_1$ is the generalized Sierpiński graph $S^n_G$; see also [5].
We can calculate the order and the size of the Sierpiński product of multiple factors directly from the above construction.

**Proposition 4.1.** Let $m \geq 2$, and let $G_1, \ldots, G_m$ be arbitrary graphs. Further let $f_1 : V(G_2) \to V(G_1), \ldots, f_{m-1} : V(G_m) \to V(G_{m-1})$ be arbitrary functions. Then the order and size of Sierpiński product $G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1$ are as follows

$$|G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1| = \prod_{\ell=1}^{m} |G_\ell|,$$

$$||G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1|| = \sum_{\ell=1}^{m} \left( \prod_{j=\ell+1}^{m} |G_j| \right) ||G_\ell||.$$

Note that neither the order nor the size of the Sierpiński product depends on the functions $f_\ell$.

If $K := G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1$, $m \geq 2$, is a Sierpiński product, then the vertices of $K$ with some common prefix $g_m \ldots g_{\ell+1}$ ($\ell \geq 0$) belong to the same copy of $H := G_\ell \otimes_{f_{\ell-1}} \cdots \otimes_{f_1} G_1$. 

Figure 8: Construction of graph $C_3 \otimes_{f'} C_4 \otimes_{f} C_3$, where $f : i \mapsto i \pmod{3}$ and $f' = \text{id}$. 

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We generalize the notation from Section 2 and denote such copy by \(g_m \ldots g_t H\), where \(H = G_t \otimes_{f_{t-1}} \ldots \otimes_{f_1} G_1\). We will use this notation to state an upper bound on the diameter of a Sierpiński product. But first let us prove an auxiliary result which we will require in the result about the diameter.

**Lemma 4.2.** Let \(\{a_m\}_{m \in \mathbb{N}}\) and \(\{d_m\}_{m \in \mathbb{N}}\) be integer sequences satisfying the following recursion

\[
a_m = (d_m + 1) a_{m-1} + d_m, \quad a_1 = d_1.
\]

Denote \([m] := \{1, \ldots, m\}\). Then the closed formula for \(\{a_m\}_{m \in \mathbb{N}}\) is given by

\[
a_m = \sum_{\ell=1}^{m} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m]} d_{i_1} \cdots d_{i_\ell}.
\]

**Proof.** If \(m = 1\), the closed formula above gives us \(a_1 = \sum_{\ell=1}^{1} \sum_{\{i_1\} \subseteq \{1\}} d_j = d_1\). For \(m > 1\) we have

\[
a_m = (d_m + 1) a_{m-1} + d_m = (d_m + 1) \left( \sum_{\ell=1}^{m-1} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m-1]} d_{i_1} \cdots d_{i_\ell} \right) + d_m
\]

\[
= \sum_{\ell=1}^{m-1} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m-1]} d_{i_1} \cdots d_{i_\ell} \cdot a_{m-1} + \sum_{\ell=1}^{m-1} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m-1]} d_{i_1} \cdots d_{i_\ell} \cdot d_m
\]

\[
= \sum_{\ell=1}^{m} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m]} d_{i_1} \cdots d_{i_\ell},
\]

which completes the proof. \(\square\)

When dealing with distances it is also useful to note that the following observation holds.

**Proposition 4.3.** Let \(m \geq 2\), and let \(G_1, \ldots, G_m\) be arbitrary graphs. Let \(f_1 : V(G_2) \to V(G_1), \ldots, f_{m-1} : V(G_m) \to V(G_{m-1})\) be arbitrary functions. Finally, let \(g, g'\) be vertices in \(G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1\) and let \(\ell \in \{1, \ldots, m\}\) be the greatest index of coordinates in which \(g\) and \(g'\) differ, i.e., \(g = (g_m, \ldots, g_{\ell+1}, g_{\ell}, \ldots, g_1) =: g g_{\ell+1} \ldots g_1\) and \(g' = (g_m, \ldots, g_{\ell+1}, g'_\ell, \ldots, g'_1) =: g g'_{\ell+1} \ldots g'_1\). Then \(g\) and \(g'\) belong to the same copy of \(H := G_\ell \otimes_{f_{\ell-1}} \cdots \otimes_{f_1} G_1\), and

\[
d_G(g, g') = d_H(g_g_{\ell+1} \ldots g_1, g' g'_{\ell+1} \ldots g'_1).
\]

**Proposition 4.4.** Let \(m \geq 2\), and let \(G_1, \ldots, G_m\) be arbitrary graphs. Further let \(f_1 : V(G_2) \to V(G_1), \ldots, f_{m-1} : V(G_m) \to V(G_{m-1})\) be arbitrary functions. Then

\[
\text{diam} \left( G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1 \right) \leq \sum_{\ell=1}^{m} \sum_{\{i_1, \ldots, i_\ell\} \subseteq [m]} \text{diam} \left( G_{i_1} \right) \cdots \text{diam} \left( G_{i_\ell} \right).
\]

**Proof.** Denote \(G = G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1\) and let \(g = g_m \ldots g_1, g' = g'_m \ldots g'_1\) be vertices of \(G\). Due to Proposition \(\text{[XXX]}\) let us assume that \(g\) and \(g'\) differ in the \(m\)-th coordinate.
Because the vertices are in different copies of \( H := G_{m-1} \otimes_{f_{m-2}} \cdots \otimes_{f_1} G_1 \), we have to find a shortest path from \( g_m H \) to \( g'_m H \), and such path has length at most

\[
\text{diam}(H)(\text{diam}(G_m) + 1) + \text{diam}(G_m),
\]

because in worst case we have to cross both graphs \( g_m H \) and \( g'_m H \), but also some other copies isomorphic to \( H \). Note that on such shortest path we cannot cross more subgraphs isomorphic to \( H \) than \( \text{diam}(G_m) + 1 \), otherwise we would have a path in \( G_m \) that is longer than its diameter \( \text{diam}(G_m) \). Every time we cross between the subgraphs we add another edge to our shortest path, and this happens in at most \( \text{diam}(G_m) \) cases.

The result now follows from the fact that \( \text{(3)} \) satisfies the recursion in Lemma 4.2 with 
\[
a_m := \text{diam} \left( G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1 \right),
\]
and 
\[
d_m := \text{diam} \left( G_m \right)
\]
for \( m \in \mathbb{N} \).

In the next examples the above bound is tight.

**Example 4.5.** Let \( n, m \geq 2 \) and \( f : V(P_n) = \{1, \ldots, n\} \rightarrow V(P_m) = \{1, \ldots, m\} \) be defined as
\[
f(i) = \begin{cases} 
1 & \text{if } i \equiv 1, 2 \mod 4, \\
m & \text{if } i \equiv 0, 3 \mod 4.
\end{cases}
\]
Then, \( \text{diam}(P_n \otimes_f P_m) = nm - 1 = \text{diam}(P_n) + \text{diam}(P_m) + \text{diam}(P_m) \), which equals the upper bound from Proposition 4.4 (cf. Figure 9).

![Figure 9: A visualization of Example 4.5 for \( n = 5 \) and \( m = 6 \).](image)

**Example 4.6.** Let \( p \geq 2, \ n \geq 1 \). Then \( K_p \otimes \cdots \otimes K_p = S_p^n \). It is a well-known fact that \( \text{diam}(K_p) = 1 \), and it is also known (cf. [9, Proposition 2.12]) that \( \text{diam}(S_p^n) = 2^n - 1 \). The upper bound in Proposition 4.4 for the case of \( S_p^n \) equals \( \sum_{i=1}^n \sum_{\{i_1, \ldots, i_r\} \subseteq [n]} 1 \), which is just the number of non-empty subsets of \( [n] = \{1, \ldots, n\} \) and there are exactly \( 2^n - 1 \) such subsets, so the bound is tight for \( S_p^n \).

Distances are also important for the following open problem.

**Problem 4.1.** Suppose all graphs \( G_\ell \) are connected. What can we say about the girth of \( G_m \otimes_{f_{m-1}} \cdots \otimes_{f_1} G_1 \)? How is it related to the girths of its factors?

## 5 Conclusion

This paper generalizes Sierpiński graphs even further than generalized Sierpiński graphs, where the whole structure is based only on one graph. Here we create a product like structure of two (or more) factors. Some basic graph theoretical properties are studied
in detail, and planar Sierpiński products are completely characterized. Apart from this
the symmetries of Sierpiński products are studied as well. In general, these are not fully
understood. In many cases we are able to determine the automorphism group of Sierpiński
product of two graphs exactly.

In [11] an algorithm is given for recognizing generalized Sierpiński graphs. Given a
graph it is also natural to ask whether it can be represented as a Sierpiński product of
two or more graphs. Moreover, one can ask if such a representation is unique. The latter
question has a negative answer. Consider the Sierpiński product of $C_4$ and $2K_3 + e$
with function $f$ as in Figure 7. It can be easily verified that it is isomorphic to $C_8 \otimes f' C_3$
where $f': V(C_8) \rightarrow V(C_3)$ is defined by $f'(1) = f'(2) = f'(5) = f'(6) = 1$ and $f'(3) =
f'(4) = f'(7) = f'(8) = 2$. However, in this case not all the factors are prime with respect
to the Sierpiński product: $C_8$ can be represented as a Sierpiński product of $C_4$ and $K_2$
while $2K_3 + e$ can be represented as a Sierpiński product of $K_2$ and $C_3$. It would be
interesting to see whether there exist prime graphs with respect to the Sierpiński product
$G, H, G', H'$ and functions $f : V(G) \rightarrow V(H), f' : V(G') \rightarrow V(H')$ such that $G, H$
are not isomorphic to $G', H'$ while $G \otimes f H$ is isomorphic to $G' \otimes f' H'$.

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References

[1] Y. Alizadeh, E. Estaji, S. Klavžar, M. Petkovšek. Metric properties of
generalized Sierpiński graphs over stars, Discrete Appl. Math., to appear,
https://doi.org/10.1016/j.dam.2018.07.008.

[2] Alspach, Brian; Dobson, Edward. On automorphism groups of graph truncations.
Ars Math. Contemp. 8 (2015), no. 1, 215–223.

[3] A. Estrada-Moreno, E.D. Rodríguez-Bazan, J.A. Rodríguez-Velázquez. On distances
in generalized Sierpiński graphs. Appl. Anal. Discrete Math. 12 (2018) 49–69.

[4] J. Geetha, K. Somasundaram. Total coloring of generalized Sierpiński graphs. Aus-
tralas. J. Combin. 63 (2015) 58–69.

[5] S. Gravier, M. Kovše, A. Parreau. Generalized Sierpiński graphs. In: Posters at
EuroComb’11, Budapest,
http://www.renyi.hu/conferences/ec11/posters/parreau.pdf.

[6] R. Hammack, W. Imrich, S. Klavžar. Handbook of Product Graphs, Second Edition,
CRC Press, Boca Raton, FL, 2011.

[7] A. M. Hinz, S. Klavžar, U. Milutinović, C. Petr. The Tower of Hanoi – Myths and
Maths, Birkhäuser, Basel, 2018.
[8] A. M. Hinz, S. Klavžar, S. S. Zemljič. Sierpiński graphs as spanning subgraphs of Hanoi graphs. Cent. Eur. J. Math. 11 (2013), no. 6, 1153–1157.

[9] A. M. Hinz, S. Klavžar, S. S. Zemljič. A survey and classification of Sierpiński-type graphs. Discrete Appl. Math. 217 (2017), part 3, 565–600.

[10] C. H. Huang, J. F. Fang, C. Y. Yang. Edge-disjoint Hamiltonian cycles of WK-recursive networks. Lecture Notes in Comput. Sci. 3732 (2006) 1099–1104.

[11] W. Imrich, I. Peterin. Recognizing generalized Sierpiński graphs. Manuscript, available on https://mp.feri.um.si/osebne/peterin/clanki/recogSierpSubmit.pdf.

[12] M. Jakovac, S. Klavžar. Vertex-, edge-, and total-colorings of Sierpiński-like graphs. Discrete Math. 309 (2009) 1548–1556.

[13] S. Klavžar, U. Milutinović. Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem. Czechoslovak Math. J., 47(122) (1997), 95–104.

[14] S. Klavžar, U. Milutinović, C. Petr. 1-perfect codes in Sierpiński graphs. Bull. Aust. Math. Soc. 66 (2002) 369–384.

[15] S. Klavžar, I. Peterin, S. S. Zemljič. Hamming dimension of a graph – the case of Sierpiński graphs. European J. Combin. 34 (2013), no. 2, 460–473.

[16] S. Klavžar, S. S. Zemljič. On distances in Sierpiński graphs: almost-extreme vertices and metric dimension. Appl. Anal. Discrete Math. 7 (2013), no. 1, 72–82.

[17] A. Malnič, T. Pisanski, A. Žitnik. The clone cover. Ars Math. Contemp. 8 (2015), no. 1, 95–113.

[18] B. Mohar and C. Thomassen, Graphs on surfaces, Johns Hopkins University Press, 2001.

[19] D. Parisse. On some metric properties of the Sierpiński graphs $S(n, k)$. Ars Combin. 90 (2009) 145–160.

[20] Pisanski, Tomaž; Tucker, W. Thomas. Growth in repeated truncations of maps. Atti Sem. Mat. Fis. Univ. Modena 49 (2001), suppl., 167–176.

[21] J. A. Rodríguez-Velázquez, E.D. Rodríguez-Bazan, A. Estrada-Moreno. On generalized Sierpiński graphs. Discuss. Math. Graph Theory 37 (2017) 547–560.

[22] J.A. Rodríguez-Velázquez, J. Tomás-Andreu. On the Randić index of polymeric networks modelled by generalized Sierpiński graphs. MATCH Commun. Math. Comput. Chem. 74 (2015) 145–160.

[23] B. Xue, L. Zuo, G. Li. The Hamiltonicity and path $t$-coloring of Sierpiński-like graphs. Discrete Appl. Math. 160 (2012) 1822–1836.

[24] B. Xue, L. Zuo, G. Wang, G. Li. Shortest paths in Sierpiński graphs. Discrete Appl. Math. 162 (2014) 314–321.
Jurij Kovič
Institute of Mathematics, Physics, and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia
and
University of Primorska, FAMNIT
Glagoljaška 8, 6000 Koper, Slovenia
jurij.kovic@siol.net

Tomaž Pisanski
University of Primorska, FAMNIT
Glagoljaška 8, 6000 Koper, Slovenia
and
Institute of Mathematics, Physics, and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia
tomaz.pisanski@upr.si

Sara Sabrina Zemljic
Comenius University, Bratislava, Slovakia
and
Institute of Mathematics, Physics, and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia
sara.zemljic@gmail.com

Arjana Žitnik
University of Ljubljana, Faculty of Mathematics and Physics
Jadranska 19, 1000 Ljubljana, Slovenia
and
Institute of Mathematics, Physics, and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia
arjana.zitnik@fmf.uni-lj.si