Classical and quantum-mechanical state reconstruction

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Abstract

The aim of this paper is to present the subject of state reconstruction in classical and in quantum physics, a subject that deals with the experimentally acquired information that allows the determination of the physical state of a system. Our first purpose is to explain a method for retrieving a classical state in phase space, similar to that used in medical imaging known as computer-aided tomography. It is remarkable that this method can be taken over to quantum mechanics, where it leads to a description of the quantum state in terms of the Wigner function which, although it may take on negative values, plays the role of the probability density in phase space in classical physics. We then present another approach to quantum state reconstruction based on the notion of mutually unbiased bases—a notion of current research interest, for which we give explanatory remarks—and indicate the relation between these two approaches. Since the subject of state reconstruction is rarely considered at the level of textbooks, the presentation in this paper is aimed at graduate-level readers.

1. Introduction

The retrieval of the state of a physical system from measurable information is an important problem in classical as well as in quantum physics. The quantum-mechanical problem is seldom discussed in textbooks on quantum mechanics (exceptions are quantum optics texts, e.g., [1–4]), although the issue is of considerable interest, as it specifies the experimental data required to determine the state of the system completely; this problem is of some intricacy within quantum mechanics, where one is faced with complementary observables. Knowledge of the system state implies having the complete information that allows making predictions about any possible measurement performed on the system.

In classical Hamiltonian mechanics [5], the state of a one-particle system with one degree of freedom is specified by giving its position $q$ and its canonically conjugate momentum $p$ at
Historically, this question may be traced back to the Pauli query [13] of whether one can determine the quantum state. What are the measurable quantities whose values suffice to determine the quantum state? Obviously, this statement can be extended to mixed states as well.

The problem of state reconstruction is thus a very important one. It involves the question: What are the measurable quantities whose values suffice to determine the quantum state? Historically, this question may be traced back to the Pauli query [13] of whether one can reconstruct the wavefunction, amplitude and phase, for a one-particle system, from the probability of its position, i.e. $|\psi(x)|^2$, and that of its momentum, i.e. $|\tilde{\psi}(p)|^2$; here, $\tilde{\psi}(p)$ is the wavefunction in the momentum representation, the tilde indicating the Fourier transform. We now know that, in general, this is not possible: we need more information than these two distributions. The reader may find an interesting review on the Pauli problem in [14]. Then, the general question is: What needs to be known to determine a quantum-mechanical state?
The literature on this subject, which is still of current interest, has grown enormously ever since. Here, we have made a selection out of these approaches, with the idea of providing a link with the classical reconstruction scheme.

The classical approach based on \( P(q|p) \) is, of course, untenable in quantum physics, where a fixed momentum precludes a well-defined position probability. A similar observation is applicable to the direct approach of measuring the joint probability of \( q \) and \( p \). However, it is remarkable that the alternative method based on measuring \( X_\theta \) defined in phase space (see equation (2)) can be taken over to QM (see [2, p 143] and [1, p 101]). But then the question arises: How can that be, if there is no such thing as a joint probability density \( \rho(q, p) \) in QM? It turns out that the answer one obtains by following this procedure is a function defined in phase space, which, although is not a bona-fide probability density (it is real, but not necessarily positive definite, and has sometimes been named ‘quasi-probability’), contains all the information needed to compute any quantum mechanical expectation value we please, just as if we were given the complex wavefunction, or the density operator. This concept of quasi-probability was invented by Wigner [15] in the early days of quantum mechanics, with the purpose of finding the quantum-mechanical corrections to thermodynamic functions, and is known as the Wigner function. Several articles present the Wigner function in a pedagogical way (see, e.g., [16–19]) and are recommended as interesting reading for the student. Retrieving the Wigner function using this tomographic method is a true quantum state reconstruction, and explaining how this is achieved, and its relation with the classical tomographic approach, constitutes one of the main goals of this paper. The main results for this approach are to be found in equation (11) for the classical case and in equation (23) for the quantum-mechanical one.

The interesting feature of equation (23) is that it relates the Wigner function of the state to the probabilities appearing on its right-hand side. An important application is in the field of quantum optics, where these probabilities can be measured experimentally by means of optical interferometry, using the method known as ‘homodyne detection’. The experiments use a beam splitter that mixes the electromagnetic-field mode to be measured with that provided by a local oscillator with the same frequency. By changing the phase of the latter, one can find, in
a large number of slices, the probabilities alluded to above, which then permit the reproduction of the full quantum-mechanical state of the electromagnetic field. This procedure can thus be considered as an application to quantum mechanics of the CAT scan and is explained in detail in [2, 3].

There is another concept that has been very useful in the task of reconstructing a quantum state. To give a trivial example, consider the eigenvectors of position and momentum: if the state vector of a system is an eigenstate of momentum, the system is equally likely to be found in any of the eigenstates of position. Pairs of bases with a similar property have been extensively studied [20, 21] and are known as mutually unbiased bases (MUB). It turns out that MUB constitute a powerful tool for state reconstruction, since it is possible to express the density operator that defines the state of the system in terms of a complete orthonormal set of operators [22–25] whose eigenfunctions are MUB. We will explain the MUB approach to the problem of state reconstruction and exhibit in equation (43) how the density operator $\hat{\rho}$ itself can be expressed in terms of the measurable probabilities referred to in the previous paragraph. The slices indicated there actually correspond to MUB states. We shall also compute the Wigner transform of the state and demonstrate explicitly that the result (see equation (44)) is identical to that found with the method—explained above—based on tomography in phase space and the Wigner function. Even more important, we shall find that the two approaches correspond, essentially, to employing two ways of handling the same complete set of operators, thus providing a unified description of both methods.

We wish to emphasize that the main goal of writing this paper is to give a pedagogical presentation of a subject which has been studied for many years and is still of current interest. This paper gives, we believe, a full reply to Pauli’s query, as is also summarized in the conclusion. One other didactical presentation of this subject is given in [26, section VI] to which the reader is also referred. With this motivation, we use language that, we hope, can be followed by a physics graduate student.

This paper is organized as follows. In section 2, we review the CAT scan method, as employed for the reconstruction of a classical 2D density. In section 3.1, we present a scheme for classical state reconstruction in phase space similar to the CAT method employed in configuration space. In section 3.2, we explain how the classical scheme can be taken over to QM and explain the role played by the Wigner function. We then present, in section 4, the alternative method for quantum state reconstruction based on the notion of MUB. Finally, we give our conclusions in section 5. To avoid cluttering of the main text, we include some details of the mathematical derivations in appendices A–D.

2. The classical reconstruction scheme

First, we review briefly the method mentioned in the introduction, the CAT scan that is used for the reconstruction of a 2D configuration density $\rho(x, y)$. The mathematical procedure can be translated directly to retrieve a classical 2D phase-space density $\rho(q, p)$ and, even more interesting for us, it can be taken over to QM.

In a 2D CAT scan [7–9], a fine pencil beam of x-rays passes through a sample, shown as the shaded area in figure 1, along the ‘line of sight’ defined by $\mathbf{r} \cdot \mathbf{n} = x'_0$; here, $\mathbf{r}$ is the position vector of a point on the line of sight and $\mathbf{n}$ is a unit vector perpendicular to the line of sight, forming an angle $\theta$ with the $x$ axis, so that $\mathbf{r}$ and $\mathbf{n}$ can be written as $\mathbf{r} = x \mathbf{i} + y \mathbf{j}$, $\mathbf{n} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, with $\mathbf{i}$ and $\mathbf{j}$ being unit vectors along the $x$ and $y$ axes, respectively. Then, the equation for the line of sight becomes

$$x'_0 = x \cos \theta + y \sin \theta.$$ (3)
The line of sight is offset by the amount $x'_0$ from the rotated $y'$ axis. The beam is attenuated by scattering and absorption produced by the various parts of the sample encountered along the path. Assuming that the attenuation at $(x, y)$ is proportional to the sample density $\rho(x, y)$, the total attenuation will be proportional to

$$\rho_0(x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, \delta(x' - Cx - Sy) \rho(x, y), \quad (4)$$

where we have used an arbitrary offset value designated by $x'$, and $C$ and $S$ are defined in equation (2). Now, it is important to remark that knowing the response of the sample given by $\rho_0(x')$ for all $x'$ and directions $\theta$, we can reconstruct the density $\rho(x, y)$ of the sample. The mathematics of this problem was actually developed by J. Radon at the beginning of the 20th century [27] for the study of astronomical data. In fact, the function $\rho_0(x')$ of equation (4) is known in the literature as the Radon transform of the density $\rho(x, y)$. Thus, the task is to invert the Radon transform to find the sample density.

It is shown in appendix A that the sample density $\rho(x, y)$ can be expressed in terms of the response of the sample $\rho_0(x')$, for all $x'$ and directions $\theta$ defined above (see [2, p. 144]), as

$$\rho(x, y) = -\frac{1}{2\pi} \int_{0}^{\pi} d\theta \, \mathcal{P} \int_{-\infty}^{\infty} dx' \frac{\partial \rho_0(x')/\partial x'}{x' - (x \cos \theta + y \sin \theta)}, \quad (5)$$

where $\mathcal{P}$ stands for the Cauchy principal value of the integral. Indeed, equation (5) is the inverse Radon transform of $\rho_0(x')$.

To gain some insight into the structure of the sample response $\rho_0(x')$, it is illustrative to consider the particular case in which the sample density $\rho(x, y)$ is isotropic, i.e. dependent only on the distance $r = \sqrt{x^2 + y^2}$ from the origin and independent of the angle. If we write $x$ and $y$ in polar coordinates as $x = r \cos \phi$, $y = r \sin \phi$, equation (4) for the response $\rho_0(x')$ takes the form

$$\rho_0(x') = \int_{0}^{2\pi} d\phi \int_{0}^{\infty} dr \, r \, \delta(x' - r \cos(\phi - \theta)) \rho(r), \quad (6)$$

showing that $\rho_0(x')$ is independent of $\theta$ for the isotropic case. By direct substitution, one may also observe that in this case $\rho_0(x')$ is symmetric, i.e. $\rho_0(-x') = \rho_0(x')$.

The behaviour of $\rho_0(x')$ given in equation (4) must ensure that the sample density $\rho(x, y)$ should be positive, although this fact is not explicitly manifest in equation (5). It is thus useful to verify this property in some particular examples. For this purpose, we choose the isotropic case studied in the last paragraph. For example, at the origin of coordinates, $x = y = 0$, equation (5) gives

$$\rho(0, 0) = -\frac{1}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} dx' \frac{\partial \rho_0(x')/\partial x'}{x'} \quad (7)$$

Since $\rho_0(x')$ is a symmetric function of $x'$, $\partial \rho_0(x')/\partial x'$ is antisymmetric; the quantity $-1/x'$ appearing in equation (7) has precisely this same property, as illustrated in figure 2, so that the resulting density $\rho(0, 0)$ at the origin of coordinates is positive.

3. Classical–quantum physics state-reconstruction analogy

3.1. Classical state reconstruction

A state in classical statistical physics is determined by a probability density in phase space. In this paper, we shall always consider, for simplicity, one-particle systems with one degree of
We write the probability density in phase space as \( \rho(q, p) \), which, for convenience in our comparison with quantum mechanics, will be normalized as
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(q, p) \frac{dq \, dp}{2\pi} = 1. \tag{8}
\]
After the discussion given in the previous section on CAT in 2D configuration space \((x, y)\), it is clear that a similar method can be applied in 2D phase space \((q, p)\): if we consider the linear combination of position and momentum given in equation (2), the probabilities for the new variable \(X_\theta\) for all values of \(\theta\) can then be used to reconstruct \(\rho(q, p)\) [7].

Before proceeding, we indicate our choice for the constants \(\alpha\) and \(\beta\), which were introduced to fix dimensions. We choose \(\alpha = 1/q_0, \beta = 1/p_0\), where \(q_0\) and \(p_0\) represent any convenient scales for position and momentum. Subsequently renaming the dimensionless quantities \(q/q_0\) and \(p/p_0\) again as \(q\) and \(p\), respectively, the transformation of equation (2) reads
\[
X_\theta = Cq + Sp. \tag{9}
\]
If we designate the probability density of the variable \(X_\theta\) as \(\rho_\theta(x')\), where \(x'\) represents an arbitrary value of \(X_\theta\), we have, just as in equation (4),
\[
\rho_\theta(x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x' - Cq - Sp) \rho(q, p) \frac{dq \, dp}{2\pi}. \tag{10}
\]
The goal is to find \(\rho(q, p)\) in terms of \(\rho_\theta(x')\) by inverting equation (10). Proceeding as in the previous section and appendix A, we find the equivalent of equation (5) as
\[
\rho(q, p) = -\frac{1}{\pi} \int_{0}^{\pi} d\theta \rho \int_{-\infty}^{\infty} dx' \frac{\partial \rho_\theta(x')/\partial x'}{x' - (q\cos \theta + p\sin \theta)}. \tag{11}
\]
3.2. Quantum state reconstruction

As mentioned in the introduction, the above method based on measuring $X_θ$ defined in phase space can be taken over to QM; see, e.g., [2, p 143]. This leads to a quasi-probability density, known as the Wigner function, defined in phase space.

In what follows, our units will be such that $\hbar = 1$. Consider an arbitrary Hermitian operator $\hat{A}$. We define its Wigner transform as [1, 2, 15]

$$W_\hat{A}(q, p) = \int_{-\infty}^{\infty} e^{-ipy} \langle q + \frac{y}{2} | \hat{A} | q - \frac{y}{2} \rangle dy.$$  \hspace{1cm} (12)

For the case where the operator $\hat{A}$ is the density operator $\hat{\rho}$ defining the state of the system, we speak of the Wigner function of the state, which has the normalization property

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\hat{\rho}(q, p) \frac{dq dp}{2\pi} = 1,$$ \hspace{1cm} (13)

similar to the normalization of equation (8) adopted for the classical distribution.

It is well known [2] that the Wigner function for a state may be negative in some parts of phase space. Thus, it does not qualify as a true probability density and is referred to as a quasi-probability density. An illustration of the fact that it plays in QM a role analogous to that played by the classical probability density $\rho(q, p)$ is the similarity of equations (22) and (23) given below with equations (10) and (11), respectively.

An important property of the Wigner function, obtained from definition (12), is [1, 2]

$$\text{Tr}(\hat{A}\hat{B}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\hat{A}(q, p)W_\hat{B}(q, p) \frac{dq dp}{2\pi},$$ \hspace{1cm} (14)

for any two operators $\hat{A}$ and $\hat{B}$. We remind the reader that the trace of an operator $\hat{C}$ is the sum of its diagonal matrix elements, i.e. $\text{Tr}(\hat{C}) = \sum_i C_{ii}$, where $i$ labels the states of a complete basis. Equation (14) states that the trace of the product of two operators in Hilbert space can be evaluated as an integral in phase space of the corresponding Wigner transforms. The normalization of equation (13) is consistent with the property (14), taking $\hat{A} = \hat{\rho}$ and $\hat{B} = 1$.

The statistical expectation value of an observable $\hat{A}$, obtained by using equation (14), can be expressed as

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_\hat{\rho}(q, p)W_\hat{A}(q, p) \frac{dq dp}{2\pi},$$ \hspace{1cm} (15)

i.e. as an integral in phase space of the Wigner function for the state times the Wigner transform of the observable. With these results, the Wigner function of the state and the Wigner transform of observables can be employed to ‘do QM in phase space’.

It is also a simple exercise to show that the above definition of the Wigner function of the state $\hat{\rho}$ is equivalent to the inverse Fourier transform of the characteristic function of the density operator [1, equations (3.12) and (3.16)]:

$$W_\hat{\rho}(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{W}(u, v) e^{i(uq + vp)} du dv,$$  \hspace{1cm} (16a)

$$\tilde{W}(u, v) \equiv \text{Tr}[\hat{\rho} \ e^{-iu\hat{q} - iv\hat{p}}].$$  \hspace{1cm} (16b)

This alternative expression for the Wigner function will be relevant for the discussion presented at the end of section 4.

Now consider the observable

$$\hat{X}_\theta = C\hat{q} + S\hat{p},$$ \hspace{1cm} (17)
which is the QM counterpart of the classical quantity of equation (9). We define this observable
eigenbasis and eigenvalues by
\[ \hat{X}_0 |x'; \theta \rangle = x' |x'; \theta \rangle. \] (18)

Our approach is as follows. If the system is prepared in the state defined by the density operator \( \hat{\rho} \), we first consider the probability density \( \rho_{0QM}(x') \) that a measurement of the observable \( \hat{X}_0 \) will give the value \( x' \): this probability density will be initially expressed in terms of \( \hat{\rho} \), equation (19), and then in terms of the Wigner function \( W_{\hat{\rho}}(q, p) \) in phase space, equation (22). The final goal is to ‘invert’ this relation and show that we can retrieve the Wigner function in terms of \( \rho_{0QM}(x') \).

The probability density \( \rho_{0QM}(x') \) is given by the standard QM expression
\[ \rho_{0QM}(x') = \text{Tr}(\hat{\rho} \ P^0_{x'}), \] where \( P^0_{x'} = |x'; \theta \rangle \langle x'; \theta| \). (19)

Using equation (15), we write
\[ \rho_{0QM}(x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\hat{\rho}}(q, p) W_{\hat{P}^0_{x'}}(q, p) \frac{dq dp}{2\pi}. \] (20)

In this expression, \( W_{\hat{P}^0_{x'}}(q, p) \) is the Wigner transform of the projector \( P^0_{x'} \), which is calculated in appendix B, with the result
\[ W_{\hat{P}^0_{x'}}(q, p) = \delta(x' - (Cq + Sp)). \] (21)

Then, equation (20) takes the form (see also [3])
\[ \rho_{0QM}(x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\hat{\rho}}(q, p) \delta(x' - (Cq + Sp)) \frac{dq dp}{2\pi}. \] (22)

This last equation is the QM counterpart of equation (10) for the classical probability density \( \rho(x') \). It shows explicitly that what plays the role of the classical probability density \( \rho(q, p) \) in phase space is now the quasi-probability density given by the Wigner function \( W_{\hat{\rho}}(q, p) \). Thus, in order to invert equation (22), we just copy the result in equation (11) and write \( W_{\hat{\rho}}(q, p) \) in terms of \( \rho_{0QM}(x') \) as
\[ W_{\hat{\rho}}(q, p) = -\frac{1}{\pi} \int_{0}^{\pi} d\theta \int_{-\infty}^{\infty} dx' \frac{\partial \rho_{0QM}(x')/\partial x'}{x' - (q \cos \theta + p \sin \theta)}. \] (23)

This equation allows the reconstruction of the QM state, in the sense that from the observable probability density \( \rho_{0QM}(x') \) the Wigner function of the density operator can be extracted; its knowledge, in turn, is equivalent to that of the state itself.

As pointed out in the introduction, equation (23) relates the Wigner function of the state to the probabilities \( \rho_{0QM}(x') \) appearing on its right-hand side: the latter, as explained, for instance, in [3], can be measured in experiments in quantum optics.

This completes our analysis that shows a close analogy between the classical and the quantum state reconstruction: both require the use of the inverse Radon transform. We now turn to an alternative quantum state reconstruction scheme that does not require the use of the Radon transform.

4. Mutually unbiased bases and state reconstruction

MUB in concept were introduced by Schwinger [20] in his studies of vectorial bases for Hilbert spaces that exhibit ‘maximal degree of incompatibility’. The eigenvectors of \( \hat{x} \) and \( \hat{p} \), \( |x \rangle \) and \( |p \rangle \), respectively, are example of such bases. The information-theoretical oriented appellation ‘mutual unbiased bases’ was introduced by Wootters [21].
Consider two complete and orthonormal vectorial bases, $B_1$ and $B_2$, whose vectors will be designated by $|u; B_1\rangle$ and $|v; B_2\rangle$, respectively. The two bases are said to be MUB if and only if, for $B_1 \neq B_2$,

$$\|\langle u; B_1 | v; B_2 \rangle\|^2 = K \quad \forall u, v,$$

(24)

where $K$ is a constant independent of $u$ and $v$ (see [23]). This property means that the absolute value of the scalar product of vectors from different bases is independent of the vectorial label within either basis. This implies that if a system is measured to be in one of the states, say $|u; B_1\rangle$, of $B_1$, it is equally likely to be found in any of the states $|v; B_2\rangle$ of any other basis $B_2$, when $B_1$ and $B_2$ are MUB. The value of $K$ may depend on the bases $B_1$ and $B_2$, which indeed is the case for a continuous Hilbert space. For a Hilbert space with a finite dimensionality $d$, $K = 1/d$.

The concept of MUB is found to be of interest in several fields. For instance, the ideas are useful in a variety of cryptographic protocols [28] and signal analysis [29].

In what follows, we outline a scheme for state reconstruction based on MUB [30], which is an alternative to the one presented in the previous section.

4.1. Some properties of the operator $\hat{X}_\theta$ and its eigenstates

Our link with the previous section is the operator $\hat{X}_\theta$ and its eigenstates $|x', \theta\rangle$ defined in equations (17) and (18), as we shall show that the bases $\{|x_1; \theta_1\rangle\}$ and $\{|x_2; \theta_2\rangle\}$ ($\theta_1 \neq \theta_2$, fixed) are MUB.

It will be useful to review first the properties of $\hat{X}_\theta$ and its eigenstates. We repeat definition (17) of the operator $\hat{X}_\theta$ and introduce the new operator $\hat{P}_\theta$ as

$$\hat{X}_\theta = C \hat{x} + S \hat{p}, \quad \hat{P}_\theta = -S \hat{x} + C \hat{p}.$$  

(25)

Here, $\hat{X}_\theta$ and $\hat{P}_\theta$ are canonically conjugate, i.e. $[\hat{X}_\theta, \hat{P}_\theta] = i$, just as the original operators $\hat{x}$ and $\hat{p}$.

As a first step, we solve the eigenvalue equation (18) in the coordinate representation. In this representation, we define the wavefunction

$$\psi_{x', \theta}(x) = \langle x | x'; \theta \rangle,$$

which satisfies the equation

$$\left( x \cos \theta - i \sin \theta \frac{\partial}{\partial x} \right) \psi_{x', \theta}(x) = x' \psi_{x', \theta}(x).$$

(27)

The solution of this equation is

$$\psi_{x', \theta}(x) = F(x', \theta) e^{-ix'x} e^{i \pi (x^2 \cos \theta - 2x')},$$

(28)

where $F(x', \theta)$ is an arbitrary function of $x'$ and $\theta$. It is shown in appendix C that $F(x', \theta)$ can be completely determined, up to an arbitrary overall phase, by imposing on the states $|x'; \theta\rangle$ the requirements [32]

$$\langle x_1, \theta | x_2, \theta \rangle = \delta(x_1 - x_2),$$

(29a)

$$\langle x_1, \theta | \hat{X}_\theta | x_2, \theta \rangle = x_2 \delta(x_1 - x_2); \quad \langle x_1, \theta | \hat{P}_\theta | x_2, \theta \rangle = -i \delta'(x_1 - x_2),$$

(29b)

$$\psi_{x', \theta}(x) = \langle x | x'; \theta \rangle \to \delta(x - x'), \quad \text{as } \theta \to 0; \quad \psi_{x', \theta=\pi/2}(x) = \frac{e^{ix'x}}{\sqrt{2\pi}}.$$  

(29c)

Here, equation (29a) expresses the orthonormalization (in the sense of the Dirac delta function) of the states $|x'; \theta\rangle$. Equation (29b) requires that the matrix elements of the new canonically
conjugate operators $\hat{X}_\theta$ and $\hat{P}_\theta$ with respect to the new states $|x'; \theta\rangle$ should be equal to the matrix elements of the old canonically conjugate operators $\hat{x}$ and $\hat{p}$ with respect to the old states $|x\rangle$, as demanded by a canonical transformation. The first equation in (29c) requires that in the limit $\theta \to 0$, the overlap between the new state $|x', \theta\rangle$ and the old one $|x\rangle$ be a Dirac delta function. The second equation in (29c) requires that for $\theta = \pi/2$, i.e. when $\hat{X}_{\theta=\pi/2}$ is the momentum $\hat{p}$, the wavefunction $\psi_{x', \theta = \pi/2}(x)$ should be a plane wave with no extra phases.

The final result for the wavefunction $\psi_{x', \theta}(x)$, up to an overall constant phase, is

$$\psi_{x', \theta}(x) = \frac{e^{i\left(\frac{\pi}{4}\text{sgn}(\sin\theta) - \frac{\pi}{4}\right)}}{\sqrt{2\pi|\sin\theta|}} e^{-\frac{i}{2\sin\theta}(x^2 + x'^2)\cos\theta - 2x'x},$$

(30)

Note the symmetry of this expression under the interchange $x \leftrightarrow x'$. Since $| - x', \theta + \pi \rangle = |x', \theta\rangle$, it suffices to consider state vectors in the ranges $-\infty < x' < \infty$ and $0 \leq \theta \leq \pi$, and other values of $\theta$ repeating the eigenvectors in this range (see [2], p 144).

Of course, we can relate the new state $|x', \theta\rangle$ to the old one $|x'\rangle$ through a unitary transformation as

$$|x', \theta\rangle = \hat{U}^\dagger(\theta)|x'\rangle.$$  

(31)

For the reader’s convenience, we mention that the operator $\hat{U}$ used in this paper coincides with the one designated by $\hat{V}$ in [31] and that called $\hat{U}$ in [32]. Using equations (31) and (30), we find, for the matrix elements of the unitary operator $\hat{U}^\dagger(\theta)$ in the old basis,

$$\langle x|\hat{U}^\dagger(\theta)|x'\rangle = \frac{e^{i\left(\frac{\pi}{4}\text{sgn}(\sin\theta) - \frac{\pi}{4}\right)}}{\sqrt{2\pi|\sin\theta|}} e^{-\frac{i}{2\sin\theta}(x^2 + x'^2)\cos\theta - 2x'x}.$$  

(32)

Using the unitary operator $\hat{U}(\theta)$, we write the eigenvalue equation (18) as $\hat{U}(\theta)\hat{X}_\theta\hat{U}^\dagger(\theta)|x'\rangle = x'|x'\rangle$, implying $\hat{x} = \hat{U}(\theta)\hat{X}_\theta\hat{U}^\dagger(\theta)$. Thus, the operator $\hat{x}$ and, similarly, its canonically conjugate $\hat{p}$ transform as

$$\hat{X}_\theta = \hat{U}^\dagger(\theta)\hat{x}\hat{U}(\theta) \quad \text{and} \quad \hat{P}_\theta = \hat{U}^\dagger(\theta)\hat{p}\hat{U}(\theta).$$

(33)

The unitary transformation that fulfills equations (32) and (33) is given by the operator

$$\hat{U}(\theta) = e^{-i\theta\hat{a}}$$

(34)

where $\hat{a} = a^\dagger a$ is the number operator, and $a$ and $a^\dagger$ are the annihilation and creation operators, respectively, given by $a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$, $a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$. Indeed, we readily find that the operator (34) gives the transformation properties of $a$ and $a^\dagger$, i.e. $\hat{U}^\dagger(\theta)a\hat{U}(\theta) = e^{i\theta}\hat{a}$, $\hat{U}^\dagger(\theta)a^\dagger\hat{U}(\theta) = e^{-i\theta}\hat{a}^\dagger$, that lead to the transformation properties of $\hat{x}$ and $\hat{p}$, equation (33) (with equation (25)). One can also show that the matrix elements of the operator $e^{i\theta\hat{a}}$ are identical to those of equation (32).

Finally, as we promised at the beginning of this subsection, we verify that the bases $\{|x_1; \theta_1\rangle\}$ and $\{|x_2; \theta_2\rangle\}$ (with fixed $\theta_1 \neq \theta_2$) that we studied above are MUB. Using equation (32) and the relation $U(\theta_2)U^\dagger(\theta_1) = U^\dagger(\theta_1 - \theta_2)$, which follows from (34), we find

$$\langle x_2; \theta_2|x_1; \theta_1\rangle^2 = |\langle x_2|U^\dagger(\theta_1 - \theta_2)|x_1\rangle|^2 = \frac{1}{2\pi|S(\theta_1, \theta_2)|},$$

(35)

where $S(\theta_1, \theta_2) = \sin(\theta_1 - \theta_2)$. The number $|\langle x_2; \theta_2|x_1; \theta_1\rangle|^2$ is thus independent of $x_1$ and $x_2$, so that, according to definition (24), the two bases are MUB. As an example, for $\theta = \pi/2$, $|x'; \theta = \frac{\pi}{2}\rangle$ is an eigenfunction of $\hat{p}$ with the eigenvalue $x'$, whose projection in the $x$ representation is $e^{ix'}/\sqrt{2\pi}$ (see the second equation in (29c)), its absolute value squared being consistent with equation (35).

In appendix D, we present a simple way to derive the result of equation (35), which is an application of the idea of doing QM in phase space using the Wigner transforms, mentioned right below equation (15).
4.2. State reconstruction based on MUB

Now, we show that the MUB introduced above can be used to perform a quantum-mechanical state reconstruction. We first introduce the set of operators
\[
\hat{Z}(a, b) = e^{ia} e^{ib\hat{\rho}} = e^{-\frac{1}{2}ab} e^{i(a\hat{x} + b\hat{p})}, \quad -\infty < a, b < +\infty,
\]
where we have used the Baker–Campbell–Hausdorff identity, equation (B.4). These operators form a complete and orthogonal operator basis \[1, 2\]. They satisfy the orthogonality property
\[
\frac{1}{2\pi} \text{Tr}[\hat{Z}^\dagger(a', b')\hat{Z}(a, b)] = \delta(a' - a)\delta(b' - b).
\]
Thus, we express the density operator as a linear combination of the operators \(\hat{Z}(a, b)\) as (see also \[22, 23\])
\[
\hat{\rho} = \frac{1}{2\pi} \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db \ c(a, b) \hat{Z}(a, b) \frac{db}{2\pi}, \quad c(a, b) = \text{Tr}[\hat{\rho}\hat{Z}(a, b)].
\]
Here, \(a\) and \(b\) play the role of Cartesian coordinates. We go over to polar coordinates, defining \(a = r\cos \theta, \ b = r\sin \theta\), so that equation (38) takes the form
\[
\hat{\rho} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dr \int_{0}^{\pi} d\theta \ \text{Tr}[\hat{\rho} e^{-ir(C\hat{x} + S\hat{p})}] e^{ir(C\hat{x} + S\hat{p})}.
\]
(We use the abbreviations \(C\) and \(S\) from equation (2)). Using similar arguments to those that led from equation (A.4) to (A.5), we rewrite the above equation as
\[
\hat{\rho} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{0}^{\pi} d\theta \ \text{Tr}[\hat{\rho} e^{-it(C\hat{x} + S\hat{p})}] e^{it(C\hat{x} + S\hat{p})}.
\]
Since in the exponent of the last equation we have the operator \(\hat{X}\) (see equations (17) and (18)), the exponential can be written in its spectral representation as
\[
e^{it(C\hat{x} + S\hat{p})} = e^{\hat{\rho} \hat{X}} = \int_{-\infty}^{\infty} e^{ir\hat{\rho}} \hat{\rho} \hat{X} dr', \quad (41)
\]
where the projection operator \(\hat{\rho} \hat{X}\) is defined in equation (19). Similarly,
\[
\text{Tr}[\hat{\rho} e^{-it(C\hat{x} + S\hat{p})}] = \int_{-\infty}^{\infty} e^{-ix} \text{Tr}(\hat{\rho} e^{it\hat{X}}) dx' = \int_{-\infty}^{\infty} e^{-ix} \rho_{0}^{QM}(x') dx', \quad (42)
\]
where we have used the definition of the QM probability density \(\rho_{0}^{QM}(x')\), equation (19). Then, equation (39) for \(\hat{\rho}\), using equations (41) and (42), becomes
\[
\hat{\rho} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{0}^{\pi} d\theta \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' e^{-i(x-x')} \rho_{0}^{QM}(x') \rho_{0}^{QM}(x'') dx' dx''. \quad (43)
\]
In the last line, we have performed the radial integral and used definition (A.7). It is important to note that in this context we have been able to express the density operator \(\hat{\rho}\) directly in terms of the probability \(\rho_{0}^{QM}(x')\), with no need to use the Radon transform, thanks to the expansion of \(\hat{\rho}\), equations (39) and (41), in terms of MUB, together with equation (42), which relates the trace on its left-hand side with \(\rho_{0}^{QM}(x')\).

As we mentioned in the introduction, equation (43) relates the density operator \(\hat{\rho}\) to the probabilities \(\rho_{0}^{QM}(x')\) appearing on its right-hand side, the latter being measurable in experiments in quantum optics \[3\].
The Wigner function for the state $\hat{\rho}$ of equation (43) is identical to the result found in equation (23), which we reproduce here for completeness

$$W_\rho(q, p) = \frac{1}{\pi^2} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \frac{\partial \rho^{\text{QM}}(x')}{\partial x'} \frac{\partial \rho^{\text{QM}}(x')}{\partial x'}.$$  \hspace{1cm} (44)

This result can be proved as follows. Application of equation (12)—defining the Wigner function—for the density operator $\hat{\rho}$, equation (43), gives

$$W_\rho(q, p) = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'' \hat{P}_\epsilon(x') \hat{P}_\epsilon(x'') \hat{\rho}_p^{\text{QM}}(x') \hat{\rho}_p^{\text{OM}}(x'').$$

We evaluate the matrix element of the projector $\hat{P}_{x', \theta}$ by using its definition in equation (19), the unitary transformation, equation (31), and its explicit expression, equation (32), to find

$$\langle q + \frac{y}{2} | \hat{P}_{x', \theta} | q - \frac{y}{2} \rangle = \frac{e^{i(x' - y \cos \theta)}}{2\pi \sin \theta}.$$  \hspace{1cm} (45)

Substituting this result into equation (45) and performing the integration over $y$, we have

$$W_\rho(q, p) = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'' \hat{P}_\epsilon(x') \hat{P}_\epsilon(x'') \hat{\rho}_p^{\text{QM}}(x') \delta(x'' - (q \cos \theta + p \sin \theta))$$

$$= \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'' \hat{P}_\epsilon(x') \hat{P}_\epsilon(x'') \hat{\rho}_p^{\text{QM}}(x').$$  \hspace{1cm} (47)

Then, just as shown in appendix A that the right-hand side of (A.8) is equal to the right-hand side of (5), it follows that the right-hand side of (47) is equal to the right-hand side of (44).

Finally, we calculate the matrix elements of the density operator (43) in the coordinate representation, $\langle x_1 | \hat{\rho} | x_2 \rangle$, which is the counterpart in Hilbert space of equation (44). For the matrix elements of the projector $\hat{P}_{x', \theta}$, we find, just as in equation (46),

$$\langle x_1 | \hat{P}_{x', \theta} | x_2 \rangle = \frac{e^{i(x_1 - x_2)(\epsilon' - \frac{x'' + y}{2} \cos \theta)}}{2\pi \sin \theta},$$  \hspace{1cm} (48)

so that

$$\langle x_1 | \hat{\rho} | x_2 \rangle = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \int_{-\infty}^\infty dy \int_{-\infty}^\infty dy'' \hat{P}_\epsilon(x') \hat{P}_\epsilon(x'') \hat{\rho}_p^{\text{QM}}(x') \hat{\rho}_p^{\text{QM}}(x'') \frac{e^{i(x_1 - x_2)(\epsilon' - \frac{x'' + y}{2} \cos \theta)}}{2\pi \sin \theta}.$$  \hspace{1cm} (49)

We compare this last equation with equation (A.8) and use the result of equation (5) to obtain

$$\langle x_1 | \hat{\rho} | x_2 \rangle = -\frac{1}{\pi} \int_0^\pi d\theta \int_{-\infty}^\infty dx' \int_{-\infty}^\infty dx'' \hat{\rho}_p^{\text{QM}}(x') \hat{\rho}_p^{\text{QM}}(x'') \frac{e^{i(x_1 - x_2)(\epsilon' - \frac{x'' + y}{2} \cos \theta)}}{2\pi \sin \theta},$$  \hspace{1cm} (50)

which shows explicitly how $\hat{\rho}_p^{\text{QM}}(x')$, which is a probability density, and hence a measurable quantity, can be used to find the matrix elements of the density operator.

This completes our demonstration of the consistency of the two approaches to the problem of quantum-mechanical state reconstruction that we have considered in this paper: on the one hand, the approach presented in the previous section based on tomography in phase space and the Wigner function and, on the other, the one given in this section based on the expansion of the density operator in terms of operators defined via MUB. The difference in the strategies of these two approaches involves, essentially, two ways of handling the complete orthonormal operators $\hat{Z}(a, b)$ of equation (36). The results of section 3.2 could be regarded as using the operators $\hat{Z}(a, b)$ to construct the characteristic function of the state $\hat{\rho}$ (equation (16b)) whose inverse Fourier transform gives the Wigner function (equation (16a)). This same characteristic function plays the role of the expansion coefficient (equation (42)) of $\hat{\rho}$ as a linear combination (equations (38) and (39)) of the operators $\hat{Z}(a, b)$, thus leading to the MUB approach of this section.
5. Conclusions and remarks

In this paper, we reviewed the approach to the quantum state reconstruction problem based on the Wigner function and the Radon transform, pointing out its close analogy with classical tomography. We put emphasis on the role played by the Wigner function, which was shown to be analogous to that of the probability density in phase space for the classical problem.

Then, we analysed an alternative route for the state reconstruction, which is based on the notion of MUB and does not make use of the Radon transform. We described its connection with the method based on the Wigner function.

One important outcome of the analysis is the fact that to reconstruct a quantum state we require the probabilities of all the phase-space plane and not merely the probabilities along the position and momentum axes: this provides a negative answer to Pauli’s query posed in the introduction. Indeed, equations (23) and (43) make it clear that, from an experimental point of view, the measurement of the probability density $\rho_{QM}^{\theta}(x')$ for all angles $\theta$ in the interval $[0, \pi]$ is needed in order to reconstruct the state.

For the sake of clarity, we included in the appendices a careful analysis of various topics, like the inversion of the Radon transform, the Wigner transform of projection operators and the solution of the eigenvalue equation $\hat{X}_\theta|x'; \theta\rangle = x'|x'; \theta\rangle$ in the coordinate representation (equation (30)). Finally, in appendix D, we used a geometrical interpretation, which we think is very intuitive, to account for the overlap of two states belonging to MUB in the continuum case (equation (35)).

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Appendix A. Inverting the Radon transform: proof of equation (5)

Multiplying both sides of equation (4) by $e^{-ikx'}$ and integrating over $x'$, we find

$$\int_{-\infty}^{\infty} e^{-ikx'} \rho_0(x') \, dx' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik(Cx+y)} \rho(x, y) \, dx \, dy. \quad (A.1a)$$

We identify the two sides of this equation with the Fourier transform $\tilde{\rho}_0(k)$ of $\rho_0(x')$, and the Fourier transform $\tilde{\rho}(k_x, k_y)$ of $\rho(x, y)$, respectively, so that

$$\tilde{\rho}_0(k) = \tilde{\rho}(k_x = Ck, k_y = Sk), \quad k \in (-\infty, \infty). \quad (A.1b)$$

We recover $\rho(x, y)$ as the inverse Fourier transform of $\tilde{\rho}(k_x, k_y)$:

$$\rho(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x \, dk_y \, e^{i(k_x x + k_y y)} \tilde{\rho}(k_x, k_y), \quad (A.2)$$

where $k_x$ and $k_y$ are the Cartesian components of a wave vector $k$; using the polar coordinates

$$k_x = K \cos \phi, \quad k_y = K \sin \phi, \quad (A.3a)$$

$$K = |k| > 0, \quad (A.3b)$$
While the variable $k$ in equation (A.1b) is defined in the interval $(-\infty, \infty)$, the radial variable $K$ in equation (A.4) is defined to be non-negative and in the interval $(0, \infty)$. The range of integration of the variable $K$ can be extended to the full real axis by first splitting the interval of integration of $\phi$ into the intervals $(0, \pi)$ and $(\pi, 2\pi)$ and then making the change of variables $\phi = \phi' + \pi$. We identify the last factor with the quantity $\tilde{\rho}_0(k)$, equation (A.1b), and substitute $\tilde{\rho}_0(k)$ from the left-hand side of equation (A.1a) to write (see also [3])

$$
\rho(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{0}^{\pi} d\theta \ e^{ik(x\cos \phi + y \sin \phi)} \tilde{\rho}(k \cos \phi, K \sin \phi). \tag{A.4}
$$

We define the integral $$I_{\epsilon} \equiv \int_{-\infty}^{\infty} \rho_{\epsilon}(x') f_{\epsilon}(x' - x) \, dx' = -\int_{-\infty}^{\infty} \frac{\partial \rho_{\epsilon}(x')}{\partial x'} g_{\epsilon}(x' - \alpha) \, dx', \tag{A.11}$$

where

$$g_{\epsilon}(\xi) = 2 \frac{\xi}{\xi^2 + \epsilon^2}. \tag{A.10}$$

Using the abbreviation $\alpha = x \cos \theta + y \sin \theta$, we write the last integral in equation (A.8) as

$$I_{\epsilon} \equiv \int_{-\infty}^{\infty} \rho_{\epsilon}(x') f_{\epsilon}(x' - \alpha) \, dx' = -\int_{-\infty}^{\infty} \frac{\partial \rho_{\epsilon}(x')}{\partial x'} g_{\epsilon}(x' - \alpha) \, dx', \tag{A.11}$$

where we have used definition (A.9) and we have integrated by parts, assuming the integrated term to vanish for sufficiently large values of the argument.

The function $g_{\epsilon}(\xi)$ is shown schematically in figure A1.

As $\epsilon \to 0$, the integral of equation (A.11) tends to the principal-value integral

$$\lim_{\epsilon \to 0} I_{\epsilon} = -2P \int_{-\infty}^{\infty} \frac{\partial \rho_{\epsilon}(x')}{\partial x'} \frac{1}{x' - \alpha} \, dx'. \tag{A.12}$$

Substituting this result into equation (A.8), we then find equation (5) in the text.
Appendix B. Proof of equation (21)

We first remark that it is easy to prove the operator identity

$$W^\theta_{x'} = \delta(x' - \hat{X} \theta). \quad (B.1)$$

Therefore, we compute the required Wigner transform of the projection operator (B.1) as

$$W_{P_{x'}}(q, p) = W_{\delta(x' - \hat{X} \theta)}(q, p) \quad (B.2a)$$

$$= \int_{-\infty}^{\infty} \left( q + \frac{y}{2} \delta([x' - (C\hat{q} + S\hat{p})] \right) q - \frac{y}{2} \right) e^{-ipy} dy. \quad (B.2b)$$

It is convenient to work with the Fourier transform of this last expression with respect to the variable $x'$:

$$\int_{-\infty}^{\infty} e^{ikx'} W_{P_{x'}}(q, p) dx' \quad (B.3a)$$

$$= \int_{-\infty}^{\infty} \left( q + \frac{y}{2} \right) e^{ik(C\hat{q} + S\hat{p})} \left( q - \frac{y}{2} \right) e^{-ipy} dy \quad (B.3b)$$

$$= e^{ikC} \int_{-\infty}^{\infty} \left( q + \frac{y}{2} \right) e^{ik\hat{q}e^{i\hat{p}}} \left( q - \frac{y}{2} \right) e^{-ipy} dy, \quad (B.3c)$$

where the Baker–Campbell–Hausdorff (BCH) identity ([11, p 323] and [10, p 442]) was used:

$$e^{i\hat{A}} e^{i\hat{B}} = e^{i\hat{A}} e^{i\hat{B}} e^{-i\frac{1}{2} [\hat{A}, \hat{B}]}, \quad (B.4)$$

which is valid for any two Hermitian operators $\hat{A}$ and $\hat{B}$, whose commutator commutes with each of them, i.e. $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$. Introducing inside the matrix element of equation (B.3c) a complete set of eigenstates of position and of momentum right after the first and second exponentials, respectively, we find

$$\int_{-\infty}^{\infty} e^{ikx'} W_{P_{x'}}(q, p) dx' = e^{ik(C\hat{q} + S\hat{p})}. \quad (B.5)$$

The inverse Fourier transform of this last expression gives the result of equation (21).
Appendix C. Proof of equation (30)

To prove equation (30), we impose the requirements of equations (29a)–(29c) on the solution (equation (28)).

1) The orthonormalization condition (equation (29a)) imposed on the wavefunction \( \psi_{x, \theta}(x) \) of equation (28) gives, for the function \( F(x', \theta) \), the expression

\[
F(x', \theta) = \frac{e^{i\phi_0(x')}}{\sqrt{2\pi |\sin \theta|}}.
\]

Here, \( \phi_0(x') \) is an arbitrary phase, dependent on \( x' \) and \( \theta \). The wavefunction \( \psi_{x', \theta}(x) \) becomes

\[
\psi_{x', \theta}(x) = \frac{1}{\sqrt{2\pi |\sin \theta|}} e^{-\frac{i}{\sin \theta} \left( x^2 \cos \theta - 2x x' \right) + i\phi_0(x')},
\]

(2) The first requirement in equation (29b) for the matrix elements of \( \hat{P}_\theta \) with the wavefunction of equation (2.2) is automatically fulfilled, since (i) our starting point was the eigenvalue equation (18) and (ii) the wavefunction of equation (2.2) satisfies the orthonormalization condition (equation (29a)).

3) We first compute the matrix element of \( \hat{P}_\theta \), which appears on the left-hand side of the second requirement in equation (29b). Using equation (2.2) and the definition of \( \hat{P}_\theta \) given in equation (25), we have, in the coordinate representation,

\[
\langle x_1, \theta | \hat{P}_\theta | x_2, \theta \rangle = \int \psi_{x_1, \theta}^*(x) \left[ -\sin \theta \ x - i \cos \theta \frac{\partial}{\partial x} \right] \psi_{x_2, \theta}(x) \, dx
\]

\[
= \int \psi_{x_1, \theta}^*(x) \left[ -\frac{1}{\sin \theta} \ x + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial x} \right] \psi_{x_2, \theta}(x) \, dx
\]

\[
= -ie^{i[\phi_0(x_2) - \phi_0(x_1)]} \delta'(x_1 - x_2) + \frac{\cos \theta}{\sin \theta} x_2 \delta(x_1 - x_2).
\]

We expand the above exponential in a Taylor series and use the \( \delta \)-function identities \( x\delta'(x) = -\delta(x), x^n\delta'(x) = 0, n \geq 2 \), to write the matrix element of equation (C.3) as

\[
\langle x_1, \theta | \hat{P}_\theta | x_2, \theta \rangle = -i\delta'(x_1 - x_2) + \phi_0(x_1) \delta(x_1 - x_2) + \frac{\cos \theta}{\sin \theta} x_2 \delta(x_1 - x_2).
\]

In order to satisfy the second requirement in equation (29b), we thus need

\[
\phi_0'(x) = -\frac{\cos \theta}{\sin \theta} x',
\]

(5) which has the solution

\[
\phi_0(x') = -\frac{\cos \theta}{\sin \theta} x'^2 + \varphi(\theta),
\]

(6) where \( \varphi(\theta) \) is an arbitrary function of \( \theta \).

The wavefunction \( \psi_{x', \theta}(x) \) of equation (C.2) then becomes

\[
\psi_{x', \theta}(x) = \frac{e^{i\varphi(\theta)}}{\sqrt{2\pi |\sin \theta|}} e^{-\frac{\pi i}{4} x^2 \cos \theta - 2x x'}.
\]

(7) Choosing, for the phase \( \varphi(\theta) \),

\[
\varphi(\theta) = \frac{\pi}{4} \text{sgn}(\sin \theta) - \frac{\theta}{2},
\]

(8) we satisfy the requirements of equation (29c).

We finally find the wavefunction of equation (30).
Appendix D. Geometric derivation of equation (35)

It will suffice to evaluate the quantity $|\langle x|x'; \theta \rangle|^2$, which is the probability of finding $x$ in a unit interval around the value $x$ when the system has been prepared in the state $|x'; \theta \rangle$. We find

$$|\langle x|x'; \theta \rangle|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - (xC + pS)) |S| \frac{dp}{2\pi}.$$  \hfill (D.3)

just as in equation (35) with $\theta_1 = \theta$ and $\theta_2 = 0$. On the other hand, the appearance of the factor $|S|$ in the denominator of the result (D.3) can be understood by using an intuitive geometrical argument starting from (D.2b) as follows. We regard the delta function occurring in equation
\[ \delta(xC + pS - \chi') \approx u_e (xC + pS - \chi') \equiv u_e (x, p) \]  
(\text{D.4a})

\[ \equiv \begin{cases} 
1/\epsilon, & \text{if } -\epsilon/2 < xC + pS - \chi' < \epsilon/2, \\
\text{i.e. } xC + pS \in (\chi' - \epsilon/2, \chi' + \epsilon/2), \\
0, & \text{if } xC + pS \notin (\chi' - \epsilon/2, \chi' + \epsilon/2), 
\end{cases} \]  
(\text{D.4b})

the delta function being attained in the limit \( \epsilon \rightarrow 0 \). The non-zero region is indicated as the shaded area in figure D1. The integral in equation (D.2b) is performed along the \( p \) axis, for a fixed \( x \). The segment along the \( p \) axis contained inside the shaded area is \[ \delta p = \epsilon / \sin \theta, \]  
so that the integral in question is given by

\[ \int_{-\infty}^{\infty} \delta(xC + pS - \chi') \, dp \approx \frac{\delta p}{\epsilon} = \frac{1}{\epsilon} \frac{\epsilon}{\sin \theta} = \frac{1}{\sin \theta}, \]  
(\text{D.5})

thus reproducing the formal result of equation (D.3).

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