DECOHERENCE, CHAOS, QUANTUM-CLASSICAL CORRESPONDENCE, AND
THE ALGORITHMIC ARROW OF TIME

Wojciech H. Zurek
Theoretical Astrophysics
T-6, Mail Stop B288, LANL
Los Alamos, New Mexico 87545

Abstract

The environment—external or internal degrees of freedom coupled to the system—can, in effect, monitor some of its observables. As a result, the eigenstates of these observables decohere and behave like classical states: Continuous destruction of superpositions leads to environment-induced superselection (einselection). Here I investigate it in the context of quantum chaos (i.e., quantum dynamics of systems which are classically chaotic). I show that the evolution of a chaotic macroscopic (but, ultimately, quantum) system is not just difficult to predict (requiring accuracy exponentially increasing with time) but quickly ceases to be deterministic in principle as a result of the Heisenberg indeterminacy (which limits the resolution available in the initial conditions). This happens after a time $t_h$ which is only logarithmic in the Planck constant. A definitely macroscopic, if somewhat outrageous example is afforded by various components of the solar system which are chaotic, with the Lyapunov timescales ranging from a bit more than a month (Hyperion) to millions of years (planetary system as a whole). On the timescale $t_h$ the initial minimum uncertainty wavepackets corresponding to celestial bodies would be smeared over distances of the order of radii of their orbits into “Schrödinger cat-like” states, and the concept of a trajectory would cease to apply. In reality, such paradoxical states are eliminated by decoherence which helps restore quantum-classical correspondence. The price for the recovery of classicality is the loss of predictability: In the classical limit (associated with effective decoherence, and not just with the smallness of $\hbar$) the rate of increase of the von Neumann entropy of the decohering system is independent of the strength of the coupling to the environment, and equal to the sum of the positive Lyapunov exponents. Algorithmic aspects of entropy production are briefly explored to illustrate the effect of decoherence from the point of view of the observer. We show that “decoherence strikes twice”, introducing unpredictability into the system and extracting quantum coherence from the observer’s memory, where it enters as a price for the classicality of his records.
1. Introduction

Movements of planets have served as a paradigm of order and predictability since ancient times. This view was not seriously questioned until the time of Poincaré, who has initiated the enquiry into the stability of the solar system\(^1\) and thus laid foundations of the subject of dynamical chaos. However, only recently and as a result of sophisticated numerical experiments are the questions originally posed by Poincaré being answered. Two groups, using very different numerical approaches, have reported that the solar system is chaotically unstable\(^2,3\). The characteristic Lyapunov exponent which determines the rate of divergence of neighboring trajectories in the phase space is estimated to be \(\lambda = (4 \times 10^6)^{-1} \ [\text{year}^{-1}]\). Fortunately (and in accord with the overwhelming experimental evidence) it is likely that this instability will not alter crucial characteristics of the orbits of planets such as their average distance from the sun (although eccentricities of the orbits may not be equally safe\(^3\)). Rather, it is the location of the planet along its orbit which is exponentially susceptible to minute perturbations. Even trajectories of the massive outer planets alone appear to be exponentially unstable, although the Lyapunov exponent for that subsystem of the solar system is harder to estimate\(^2\), and may correspond to a timescale as short as few million years, or as long as 30 Myr.

While the instability of the planetary system takes place on a relatively long timescale, there are celestial bodies which become chaotically unpredictable much more rapidly. Perhaps the best studied example is Hyperion, one of the moons of Saturn. Hyperion is shaped as an elongated ellipsoid. The interaction between its quadrupole moment and the gravitational field of Saturn leads to chaotic tumbling, which results in an exponential divergence on a timescale approximately equal to twice its 21 day orbital period\(^4\). There are also numerous examples of chaos in the asteroid belt (such as Chiron) with exponential instability timescales of few hundred thousand years\(^5\).

In spite of its obviously macroscopic characteristics the solar system is, ultimately, undeniably quantum. This is simply because its constituents are subject to quantum laws. The action associated with the solar system is, of course, enormous:

\[
I \simeq \frac{GM_\odot M_J}{R_J} \times \tau_J \simeq 1.2 \times 10^{51} [\text{erg s}],
\]

where the mass \(M_J\) and period \(\tau_J\) of Jupiter were used in the estimate. Given this order of magnitude of \(I\) and the smallness of the Planck constant \((\hbar = 1.055 \times 10^{-27} \text{ erg s})\), one might have anticipated that the dynamics of the solar system is a safe distance away from the quantum regime. However, and as a consequence of the chaotic character of its evolution, this is not the case.

I will begin by showing that the macroscopic size of a system – any system – does not suffice to guarantee its classicality. Thus, quantum theory implies that even the solar system – and every other chaotic system – is in principle indeterministic, and not just “deterministically chaotic”: Classical predictability in the chaotic context would require an ever increasing accuracy of its initial conditions. This is possible in principle in classical physics. But, according to quantum mechanics, simultaneously increasing the accuracy of position and momentum would eventually violate the Heisenberg principle. This time defines the quantum predictability horizon. It is surprisingly short, and – at least for some of its components – definitely less than the age of the solar system.
Classicality is restored with the help of environment-induced decoherence, which continuously destroys the purity of the wavepackets. The resulting loss of predictability can be quantified through the rate of entropy production. For a decohering chaotic system we shall see that this rate is: (i) independent of the strength of the coupling to the environment, and (ii) given by sum of the positive Lyapunov exponents. That is, the quantum entropy production rate coincides with the Kolmogorov-Sinai entropy in open systems, even though its ultimate cause is the loss of the information to the correlations with the environment.

The observer’s own point of view of this process is briefly explored, and the relation between the algorithmic randomness of the observer’s records and the von Neumann entropy of the evolving system is noted. We point out that when the observer is monitoring a decohering quantum system, entropy can increase as a result of the information loss to the environment caused by; (a) “reduction of the state vector”, i. e., decoherence of the observer’s record states, and; (b) through the more usual channel – decoherence in the monitored system.

2. Quantum predictability horizon: How the correspondence is lost.

As a result of chaotic evolution, a patch in the phase space which corresponds to some regular (and classically “reasonable”) initial condition becomes drastically deformed: Classical chaotic dynamics is characterized by the exponential divergence of trajectories. Moreover, conservation of the volume in the phase space in the course of Hamiltonian evolution (which is initially a good approximation for sufficiently regular initial conditions even in cases which are ultimately quantum) implies that the exponential divergence in some of the directions must be balanced by the exponential squeezing – convergence of trajectories – in the other directions. It is that squeezing which forces a chaotic system to explore the quantum regime: As the wavepacket becomes narrow in the direction corresponding to momentum:

\[ \Delta p(t) = \Delta p_0 \exp(-\lambda t) \]  \hspace{1cm} (2.1)

(where \( \Delta p_0 \) is its initial extent in momentum, and \( \lambda \) is the relevant Lyapunov exponent) the position becomes delocalized: The wavepacket becomes coherent over the distance \( \ell(t) \) which can be inferred from Heisenberg’s principle:

\[ \ell(t) \geq (\hbar/\Delta p_0) \exp(\lambda t) . \]  \hspace{1cm} (2.2)

Coherent spreading of the wavepacket over large domains of space is disturbing in its own right. Moreover, it may lead to a breakdown of the correspondence principle at an even more serious level: Predictions of the classical and quantum dynamics concerning some of the expectation values no longer coincide after a time \( t_\chi \) when \( \ell(t) \) reaches the scale on which the potential is nonlinear.

Such a scale \( \chi \) can usually be defined by comparing the classical force (given by the gradient of the potential \( \partial_x V \)) with the leading order nonlinear contribution \( \sim \partial_x^3 V \):

\[ \chi \simeq \sqrt{\frac{\partial_x V}{\partial_x^3 V}} . \]  \hspace{1cm} (2.3)
For the gravitational potential $\chi \simeq R/\sqrt{2}$, where $R$ is a size of the system (i.e., a size of the orbit of the planet). The reason for the breakdown of the correspondence is that when the coherence length of the wavepacket reaches the scale of the nonlinearity,

$$\ell(t) \simeq \chi,$$

(2.4)

the effect of the potential energy on the motion can be no longer represented by the classical expression for the force$^8$, $F(x) = \partial_x V(x)$, since it is not even clear where the gradient is to be evaluated for a delocalized wavepacket. As a consequence, after a time given by:

$$t_\hbar = \chi^{-1} \ln \frac{\Delta p_0 \chi}{\hbar},$$

(2.5)

the expectation value of some of the observables of the system may even begin to exhibit noticeable deviations from the classical evolution.

This is also close to the time beyond which the combination of classical chaos and Heisenberg’s indeterminacy makes it impossible in principle to employ the concept of a trajectory. Over the time $\sim t_\hbar$ a chaotic system will spread from a regular Planck-sized volume in the phase space into a (possibly quite complicated) wavepacket with the dimensions of its envelope comparable to the range of the system. This timescale defines the quantum predictability horizon – a time beyond which the combination of classical chaos and quantum indeterminacy makes predictions not just exponentially difficult, but impossible in principle. The shift of the origin of the loss of predictability from classical deterministic chaos to quantum indeterminacy amplified by the exponential instabilities is just one of the symptoms of the inability of classical evolutions to track the underlying quantum dynamics.*

This breakdown of correspondence can be investigated more rigorously by following the evolution generated for the possibly macroscopic, yet ultimately quantum system by the Moyal bracket (that is, a Wigner transform of the quantum von Neumann equation). The Moyal bracket can be expressed through the familiar classical Poisson bracket as:

$$\{H, W\}_{MB} = -i\sin(i\hbar\{H, W\}_{PB})/\hbar.$$  

(2.6)

Above, $H$ is the Hamiltonian of the system, and $W$ is an object in the phase space (i.e., a probability distribution). In our quantum case, $W$ will denote a Wigner function – a Wigner transform of the density matrix.

* One may be concerned that the argument sketched out above may be inconsistent – that the requirements of quantum Heisenberg’s indeterminacy cannot be applied to systems which are classical. Below, I shall carry out a completely quantum analysis of this issue. But I believe that the above sketch of an argument is essentially correct. For, even ostensibly classical systems must respect the Heisenberg’s principle: The Einstein-Bohr double slit experiment debate was settled, after all, with an appeal to the constraints imposed by the Heisenberg’s principle on the accuracy of simultaneous measurements of position and momentum of the movable slit introduced by Einstein!
When the potential \( V \) in \( H \) is analytic, the Moyal bracket can be expanded in powers of the Planck constant. Consequently, the evolution of \( W \) is given by:

\[
\dot{W} = \{H, W\}_P + \sum_{n \geq 1} \frac{\hbar^{2n} (-)^n}{2^{2n} (2n + 1)!} \partial_x^{2n+1} V(x) \partial_p^{2n+1} W(x, p) .
\] (2.7)

Correction terms above will be negligible when \( W(x, p) \) is a reasonably smooth function of \( p \), that is when the higher derivatives of \( W \) with respect to momentum are small. However, the Poisson bracket alone predicts that, in the chaotic system, they will increase exponentially quickly as a result of the “squeezing” of \( W \) in momentum, Eq. (2.1). Hence, after \( \hbar \) quantum “corrections” will become comparable to the first classical term on the right hand side of Eq. (2.7). At that point the Poisson bracket will no longer suffice as an approximate generator of evolution. The phase space distribution will be coherently extended over macroscopic distances, and interference between the fragments of \( W \) will play a crucial role.

The timescale on which the quantum - classical correspondence is lost in a chaotic system can also be estimated (or, rather, bounded from above) by the formula\(^{6,7}\):

\[
t_r = \lambda^{-1} \ln(I/\hbar) ,
\] (2.8)

where \( I \) is the action which – for the solar system – we have already estimated, Eq. (1.1). It follows that, for the planetary system, quantum-classical correspondence should be lost after approximately:

\[
t_r \simeq 711 \text{ [Myr]} .
\]

This is less than a fifth of the modest estimates of the age of Earth, and, presumably, a still smaller fraction of the actual age of the solar system. When we compute instead the value of \( \hbar \), setting initial uncertainty in the momentum to \( \Delta p_0 = \hbar / \Lambda_{dB}(T) \simeq 10^9 \text{ [g cm/s]} \), where we take \( \Lambda_{dB}(T) \) for definiteness to be the thermal de Broglie wavelength of Jupiter at its present surface temperature of \( \sim 100 \text{ K} \), we estimate almost identically:

\[
t_\hbar \simeq 682 \text{ [Myr]} .
\]

A similar calculation for Hyperion results in a much smaller (and, therefore, so much more disturbing):

\[
t_\hbar \simeq 20 \text{ [yr]} .
\]

Moreover, it should be pointed out that – in the macroscopic regime considered here – the above estimates are exceedingly insensitive to either the action or the initial momentum uncertainty: Both of these quantities appear inside the logarithm.

3. The solar system as a Schrödinger cat

We have seen above that a seemingly very secure prediction of quantum physics as applied to the solar system fails: According to the Schrödinger equation, less than a billion years after its formation the behavior of the solar system should be flagrantly non-classical, with the quantum states of celestial bodies spread over dimensions comparable with the
sizes of their orbits, and with the planetary dynamics no longer in accord with the laws of Newton! Somehow, this does not seem to be the case. The source of the paradox is obvious: Chaotic dynamics increases the size of the coherent wavepacket with \( \exp(\lambda t) \), so that it becomes comparable with the dimensions of the solar system after a time \( t_h \), which is but a fraction of its age. Similarly, and after only \( \sim 20 \) years the quantum state of Hyperion would be a coherent superposition involving macroscopically distinct orientations of its major axes.

There are parallels between our discussion above and the famous argument due to Schrödinger\(^{10}\) in that a very macroscopic object (planet here, cat there) is forced, through the strict compliance with the laws of quantum mechanics, into a very non-local state, never encountered as an ingredient of our familiar “classical reality”. The main difference between the two examples – the Schrödinger cat and the classically chaotic but ultimately quantum planet – is in the manner in which they are forced into the final superposition: Schrödinger cat either lives (or dies) as a result of decay of an unstable nucleus: An intermediate step in which a quantum state of the nucleus is measured (to determine the fate of the cat) is essential. Thus, in the case of the cat it was possible to entertain the notion that the (admittedly preposterous) final superposition of dead and alive cat could be avoided if the process of measurement was properly understood. This “way out” is no longer available in the case of celestial bodies we are discussing. They evolve into states which are nonlocal and flagrantly quantum simply as a result of dynamical evolution – the measurement plays absolutely no role in setting up the paradox. And the systems involved are certainly even more macroscopic than the cat. Moreover, if the reader considers the idea of putting a living being (cat) in a superposition especially tantalizing, this is certainly occurring also in the case considered here. For, in accord with the quantum arguments presented above, after a time \( t_h \) Earth would evolve into a state corresponding to a coherent superposition of all the seasons, as well as all of the hours of the day!

So what is the resolution of the above paradox? Let us start with a few possibilities which may be tempting at the first sight, but which ultimately lead nowhere. To begin with, one might be worried that in the arguments above we have cut corners by considering just one spatial dimension and one momentum, while the solar system is inhabiting a multidimensional phase space. This is certainly true, but the squeezing in momentum and the resulting delocalization is unlikely to be alleviated by considering multidimensional phase space of all the celestial bodies for which it occurs. This is especially true for systems such as Hyperion or Chiron, which have a rather large Lyapunov exponent. Another more contrived possibility of avoiding the difficulty with quantum-classical correspondence at present would be to design an initial state which evolves into a “classical looking” state by the present epoch. This can in principle be done, but – as the reader is invited to verify – it requires initial states which are as flagrantly quantum as those we were forced to consider above.

Finally and in desperation one might consider abandoning quantum theory for some other theory which is almost exactly like quantum theory (to pass all of the experimental tests) but contains either nonlinear corrections, or allows for underlying “hidden variable” dynamics, or, perhaps, introduces “collapse of the wavepacket” \textit{ad hoc} at some fundamental level in order to get rid of the quantum nonlocality. All of these ideas face either serious
experimental constraints (which render them useless for the purpose of making quantum theory look classical) or profound theoretical difficulties (such as a conflict with the Lorentz invariance), or both.

4. Decoherence, the quantum, and the classical

I shall instead contend that quantum theory is rigorously correct, but that the superposition principle cannot be applied naively, especially to the macroscopic objects. The failure of such a simple-minded application of quantum principles to the classical domain is, however, itself a consequence of the unitarity of quantum evolution: Macroscopic objects are all but impossible to insulate from their environments. Consequently, external and internal degrees of freedom continuously “monitor” – that is, become correlated – with their state. This is the process of decoherence.

We have no room here to develop the theory of decoherence systematically and completely. A sketch with a few leads to the existing (and rapidly expanding) literature will have to suffice. The key point is the observation that in quantum mechanics information matters much more than in classical mechanics, where it can be acquired without influencing in any way the actual state of the system, which exists and evolves independently of what is known about it. Such a neat division between information and “physical reality” is impossible to implement in the quantum realm: Acquisition of information is equivalent to the establishment of a correlation, which in turn is reflected in the loss of the capacity for interference. The double slit experiment is a classic example. As soon as it is known through which slit the photon has passed, the possibility for interference is lost.

The information transfer which accompanies decoherence has the same nature as that encountered in quantum measurements, or for that matter, in quantum computation. In either case it is useful to represent it with an elementary logical gate – so-called “controlled not” or a “c-not” – which reversibly copies a single bit of information between two (two-state) quantum systems, known respectively as a “control” and a “target”:

\[
(a|0> + b|1>)c |0> \rightarrow a|0>_c |0>_t + b|1>_c |1>_t .
\] (4.1)

In short, a quantum c-not is an obvious generalization of a classical c-not gate (also occasionally known as an “exclusive or” or “xor”), which flips the state of the target bit when the control is in a state “1”, and does nothing otherwise. The analogy between Eq. (4.1) and the von Neumann model of a quantum measurement is obvious: The control bit acts as a measured system, forcing the apparatus (target bit) to measure its state.

The state on the right hand side of Eq. (4.1) is, however, not the resolution of the measurement problem, but, rather, its cause: When \(a = -b = 1/\sqrt{2}\) it is in fact identical to the states encountered in the Bohm version of the Einstein-Podolsky-Rosen experiment. The correlation established is a quantum entanglement, with all its seemingly paradoxical consequences, which deny to each of the two systems involved the right to possess “a state of their own” prior to a measurement. Decoherence converts entanglement into classical correlation by allowing the environment to carry out such additional “measurements” on the to-be-classical system. Its consequence is then a disentanglement – correlations between the system and the apparatus weaken to their classically allowed strength.

Decoherence can then be conveniently “caricatured” (if not quite “represented”) by means of the c-not like gates transferring information about the to-be-classical observable
to the environment. This is shown in Fig. 1, where a c-not symbolic representation of the measurement carried out by the target-apparatus on the control-system is also illustrated. In the process of decoherence information flows from the quantum correlations between the memory (of the observer or of the apparatus) with to-be-classical observable of the decohering entity to the correlation with the environment. Decoherence is a purely quantum effect — information does not matter in classical dynamics. In the symbolism of Fig. 1 decoherence can be conveniently contrasted with the more familiar consequence of the coupling to the environment — noise — in which the state of the environment becomes inscribed on the observable of interest.

The direction of the information flow depicted in the c-nots depends on the observables involved. This is an important consequence of the quantum nature of the information transfer. The reader can verify this by re-writing the action of the c-not in the complementary basis $|\pm\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2}$, and checking that in the new basis the roles of the control and target are reversed. This illustrates the connection between the loss of phase coherence and “reduction of the state vector” — while the environment is “measuring” a certain observable $A$, its conjugate (Fourier) complement is busy “measuring” the state of the environment, and storing the information in the phases between the eigenstates of $A$ — in the correlations with the states $|+\rangle$ and $|-\rangle$.

In idealized examples of decoherence (i.e., in absence of the self-hamiltonian) a preferred observable is selected by the interaction Hamiltonian with the environment — it satisfies (or, at least, approximates) the commutation relation$^{11,12}$:

$$[H_{int}, A] = 0 . \quad (4.2)$$

However, in more realistic circumstances involving dynamical evolution Eq. (4.2) defines only the “instantaneous” preferred (pointer) observable. Long-term predictability (which is a convenient and natural criterion$^{20}$ of the more elusive “classicality”) is optimized by the states which are least perturbed by the environment in spite of the incessant rotation between the observables and their complements caused by the dynamics$^{20,26,27}$.

A system — such as a harmonic oscillator or, for that matter, a chaotic quantum system — is then described by an effective master equation$^{28–31}$, which continuously transforms pure states into mixtures. The rate at which this happens is set in part by the coupling, but the nature of the initial state plays the decisive role. States which become least mixed are then most predictable and can be regarded as most classical. The high-temperature master equation for a particle interacting with the thermal excitations of the environment composed of harmonic oscillators is a convenient and often studied example$^{29–31}$. The reduced density matrix $\rho(x,x')$ of the system (obtained by tracing out the state of the environment) in the position representation evolves in this case according to:

$$\dot{\rho} = \begin{cases} \text{von Neumann eq.} & \frac{i}{\hbar} [H, \rho] \\ \text{relaxation} & \gamma(x-x') \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \rho \\ \text{decoherence} & \frac{2m\gamma k_B T}{\hbar^2} (x - x')^2 \rho . \end{cases} \quad (4.3)$$

Above, $H$ is the effective Hamiltonian of the system (i.e., with the potential renormalized to recognize the influence of the environment), and $\gamma$ is the relaxation rate. The interaction
Hamiltonian was assumed to couple the coordinate $x$ of the system with the coordinates of the environment oscillators. When the oscillators are collectively represented by a field $\phi(q)$, the coupling can be taken to have a form

$$H_{\text{int}} = \epsilon x \dot{\phi}(q,t),$$

(4.4) in which case the effective viscosity is $\eta = 2m\gamma = \epsilon^2/4m$.

Arbitrary superpositions of localized wavepackets can in principle exist in the Hilbert space, but, as a result of environmental monitoring, they are exceedingly unstable in practice: Continuous monitoring enforces environment-induced superselection: Only some – relatively few – of the quantum states which can exist in principle are capable of surviving the interaction with the environment more or less intact. Which states can survive depends on the form of the interaction with the environment. The general rule is that the states which are localized in the monitored observables are most stable. Moreover, when the Hamiltonian of interaction is a function of some observable, then the environment is most effective at monitoring it. This singles out the preferred pointer states, Eq. (4.2). They usually turn out to be localized in position, since the interactions tend to depend on the distance $\Lambda dB(T)/\delta x$.

The tendency towards localization in position can be characterized by the time it takes for the two fragments of the wavepacket separated in space by the distance $\delta x$ to lose quantum coherence. The decoherence time:

$$\tau_D(\delta x) = \gamma^{-1}(\Lambda dB(T)/\delta x)^2$$

(4.5) can be computed from the third term of Eq. (4.3). It is proportional to the relaxation time $\tau_R = 1/\gamma$ (which determines the rate at which the system loses energy due to the interaction with the environment), but in the macroscopic realm it is much faster: For a one gram object at room temperature the ratio of the thermal de Broglie wavelength $\Lambda dB(T) = \hbar/\sqrt{2mk_BT}$ to the separation $\delta x = 1$ cm is approximately $10^{-20}$. Hence, in the above example, and under the circumstances in which thermal excitations dominate the process of decoherence (an assumption which allows one to derive Eq. (4.3) and the simple expression, Eq. (4.5), but which does not effect the conclusion about the nearly instantaneous onset of decoherence for macroscopic objects) $\tau_D \approx 10^{-40}/\gamma$.

It follows that quantum coherence may be (and, for macroscopic objects, it is) lost exceedingly rapidly even when the relaxation time is very large. Even when the assumptions leading to the simple master equation (4.3) are not valid, and Eq. (4.5) overestimates the decoherence rate, the conclusions about the relaxation being very slow in comparison with decoherence is unlikely to change in the macroscopic realm. Preliminary experimental indication which corroborates these theoretical expectations is just at hand, following beautiful microwave cavity experiments reported here by Haroche. Further studies of various aspects of decoherence are likely to follow in the wake of the ion trap “Schrödinger cat” experiments, either as a byproduct of the quantum computation research described here by Wineland, or, more directly, as a consequence of the “reservoir engineering” proposal.

For microscopic objects (such as an electron) and/or for microscopic separations, the estimate of $\tau_D$ may become comparable to $\tau_R$, and – when the isolation from the environment is sufficient – can be much larger than the characteristic dynamical timescales. The
quantum nature of the evolution would then manifest itself unimpeided. But for Jupiter or Hyperion the opposite – the reversible classical limit\textsuperscript{17,20} with $\tau_D <<< t_{D Y N A M I C A L} <<< \tau_R$ – is going to be enforced.

5. Exponential instability vs. decoherence

In a quantum chaotic system weakly coupled to the environment the process of decoherence briefly sketched above will compete with the tendency for coherent delocalization, which occurs on the characteristic timescale given by the Lyapunov exponent $\lambda$. Exponential instability would spread the wavepacket to the “paradoxical” size, while monitoring by the environment will attempt to limit its coherent extent by smoothing out interference fringes. The two processes shall reach status quo when their rates are comparable:

$$\tau_D(\delta x) \lambda \simeq 1.$$ \hspace{1cm} (5.1)

As the decoherence rate depends on $\delta x$, this equation can be solved for the critical, steady state coherence length, which yields $\ell_c \sim \Lambda_{d B}(T) \times \sqrt{\lambda/\gamma}$.

A more careful analysis can be based on the combination of the Moyal bracket and the master equation approach to decoherence we have just sketched. In many cases (including the situation of large bodies immersed in the typical environment of photons, rarefied gases, etc.) an effective approximate equation can be derived and translated into the phase space by performing a Wigner transform on Eq. (4.3). Then:

$$\dot{W} = \{H, W\}_{PB} + 2\gamma \partial_p W + D \partial_p^2 W + \sum_{n \geq 1} \frac{\hbar^{2n} (-)^n}{2^n (2n + 1)!} \partial_x^{2n+1} V(x) \partial_p^{2n+1} W(x, p).$$ \hspace{1cm} (5.2)

The second term causes relaxation, and, in the macroscopic limit, it can be made very small without decreasing the effect of decoherence caused by the third, diffusive term. Its role is to destroy quantum coherence of the fragments of the wavefunction between spatially separated regions. Thus, in effect, this decoherence term can assure that the Poisson bracket is always accurate: Diffusion prevents the wavepacket from becoming too finely structured in momentum, which – as we have seen early on in the paper, would have caused the failure of the correspondence principle. In case of the thermal environment the diffusion coefficient $D = \eta k_B T$, where $\eta$ is viscosity. The competition between the squeezing due to the chaotic instability and spreading due to diffusion leads to a standoff when the Wigner function becomes coherently spread over:

$$\ell_c = \hbar \sqrt{\frac{\lambda}{2D}} = \Lambda_{d B}(T) \times \sqrt{\lambda/2\gamma},$$ \hspace{1cm} (5.3)

This translates into the critical (spatial) momentum scale of:

$$\sigma_c = \sqrt{\frac{2D}{\lambda}}.$$ \hspace{1cm} (5.4)

which nearly coincides with the quick estimate, Eq. (5.1).
Returning to our outrageous example, for a planet of the size of Jupiter a chaotic instability on the four million year timescale and the consequent delocalization would be easily halted even by a very rarefied medium (0.1 atoms/cm$^3$, comparable to the density of interplanetary gas in the vicinity of massive outer planets) at a temperature of 100K (comparable to their surface temperature): The resulting $\ell_c$ is of the order of $10^{-29}$ cm! Thus, decoherence is exceedingly effective in preventing the packet from spreading; $\ell_c << < \chi$, by an enormous margin. Hence, the paradox we have described in the first part of the paper has no chance of materializing.

The example of quantum chaos in the solar system is a dramatic illustration of the effectiveness of decoherence, but its consequences are, obviously, not restricted to celestial bodies: Schrödinger cats, Wigner’s friends, and, generally, all of the systems which are in principle quantum but sufficiently macroscopic will be forced to behave in accord with classical mechanics as a result of the environment-induced superselection\textsuperscript{11,12}. This will be the case whenever:

$$\ell_c \ll \chi, \quad (5.5)$$

since $\ell_c$ is a measure of the resolution of “measurements” carried out by the environment.

This incredible efficiency of the environment in monitoring (and, therefore, localizing) states of quantum objects is actually not all that surprising. We know (through direct experience) that photons are capable of maintaining an excellent record of the location of Jupiter (or any other macroscopic body). This must be the case, since we obtain our visual information about the Universe by intercepting a minute fraction of the reflected (or emitted) radiation with our eyes.

Our discussion extends and complements development which goes back more than a decade\textsuperscript{35}. We have established a simple criterion for the recovery of the correspondence, Eq. (5.5), which is generously met in the macroscopic examples discussed above. And, above all, we have demonstrated that the \textit{very same} process of decoherence which delivers “pointer basis” in the measuring apparatus can guard against violation of the quantum-classical correspondence in dynamics.

\section{The arrow of time: a price of classicality?}

Decoherence is caused by the continuous measurement-like interactions between the system and the environment. Measurements involve transfer of information, and decoherence is no exception: The state of the environment acquires information about the system. For an observer who has measured the state of the system at some initial instant the information he will still have at some later time will be influenced (and, in general, diminished) by the subsequent interaction between the system and the environment. When the observer and the environment monitor the same set of observables, information losses will be minimized. This is in fact the idea behind the \textit{predictability sieve}\textsuperscript{20} – an information-based tool which allows one to look for the einselected, effectively classical states under quite general circumstances. When, however, the state implied by the information acquired by the observer either differs right away from the preferred basis selected by the environment, or – as will be the case here – evolves dynamically into such a “discordant” state, environment will proceed to measure it in the preferred basis, and, from the observers point of view, information loss will ensue.
This information loss can be analyzed in several ways. The simplest is to compute
the (von Neumann) entropy increase in the system. This will be our objective in this
section. However, it is enlightening to complement this “external” view by looking at the
consequences of decoherence from the point of view of the observer, who is repeatedly
monitoring the system and updating his records. We shall sketch such an approach (which
utilizes algorithmic information content as a measure of randomness) in Section 7.

The loss of information can be quantified by the increase of the von Neumann entropy:

\[ H = -\text{Tr} \rho \ln \rho \]  

where \( \rho \) is the reduced density matrix of the system. We shall now focus on the rate of
increase of the von Neumann entropy in a dynamically evolving system subject to deco-
herence. As we have seen before, decoherence restricts the spatial extent of the quantum
cohort patches to the critical coherence length \( \ell_c \), Eq. (5.3). A coherent wavepacket
which overlaps a region larger than \( \ell_c \) will decohere rapidly, on a timescale \( \tau_D \) shorter
than the one associated with the classical predictability loss rate given by the Lyapunov
exponent \( \lambda \). Such a wavepacket will deteriorate into a mixture of states each of which is
coherent over a domain of size \( \ell_c \) by \( \sigma_c = \hbar / \ell_c \). Consequently, the density matrix can be
approximated by an incoherent sum of reasonably localized and approximately pure states.
When \( N \) such states contribute more or less equally to the density matrix, the resulting
entropy is \( H \approx \ln N \).

The coherence length \( \ell_c \) determines the resolution with which the environment is
monitoring the state of a chaotic quantum system. That is, by making an appropriate
measurement on the environment one could in principle localize the system to within \( \ell_c \).
As time goes on, the initial phase space patch characterizing the observer’s information
about the state of the system will be smeared over an exponentially increasing range of the
coordinate, Eq. (2.2). When the evolution is reversible, such stretching does not matter,
at least in principle: It is matched by the squeezing of the probability density in the
complementary directions (corresponding to negative Lyapunov exponents). Moreover, in
the quantum case folding will result in the interference – a telltale signature of the long
range quantum coherence, best visible in the structure of the Wigner functions.

Narrow wavepackets, and, especially, small-scale interference fringes are exceedingly
susceptible to the monitoring by the environment. Thus, the situation changes dramatically
as a result of decoherence: In a chaotic quantum system the number of independent
eigenstates of the density matrix will increase as:

\[ N \approx \ell(t) / \ell_c \approx \frac{\hbar}{\Delta p_0 \ell_c} \exp(\lambda t) \]  

Consequently, the von Neumann entropy will grow at the rate:

\[ \dot{H} \approx \frac{d}{dt} \ln(\ell(t) / \ell_c) \approx \lambda \]  

This is a “corollary” of our discussion, and perhaps even its key result: Decoherence will
help restore the quantum-classical correspondence. But we have now seen that this will
happen at a price: Loss of information is an inevitable consequence of the eradication of the “Schrödinger cat” states which were otherwise induced by the chaotic dynamics. They disappear because the environment is “keeping an eye” on the phase space, monitoring the location of the system with an accuracy set by \( \ell_c \).

Throughout this paper we have “saved” on notation, using “\( \lambda \)” to denote (somewhat vaguely) the rate of divergence of the trajectories of the hypothetical chaotic system. It is now useful to become a bit more precise. A Hamiltonian system with \( D \) degrees of freedom will have in general many (\( D \)) pairs of Lyapunov exponents with the same absolute value but with the opposite signs. These global Lyapunov exponents are obtained by averaging local Lyapunov exponents, which are the eigenvalues of the local transformation, and which describe the rates at which a small patch centered on a trajectory passing through a certain location in the phase space is being deformed. The averaging of the local exponents is achieved by following the trajectory of the system for a sufficiently long time.

The evolution of the Wigner function in the phase space is governed by the local dynamics. However, over the long haul, and in the macroscopic case, the patch which supports the probability density of the system will be exponentially stretched. The stretching and folding will produce a phase space structure which differs from the classical probability distribution because of the presence of the interference fringes, with the fine structure on the (momentum) scale of the order \( \hbar/\ell_c^{(i)}(t) \). In an isolated system this fine structure will saturate only when the envelope of the Wigner function will fill in the available phase space volume. Monitoring by the environment destroys these small scale interference fringes and keeps \( W \) from becoming narrower than \( \sigma_c \) in momentum. As a result – and in accord with Eq. (6.3) above – the entropy production will asymptotically approach the rate given by the sum of the positive Lyapunov exponents:

\[
\mathcal{H} = \sum_{i=1}^{D} \lambda^{(i)}_+ .
\] (6.4)

This result is at the same time familiar and quite surprising. It is familiar because it coincides with the Kolmogorov-Sinai formula for the entropy production rate for a classical chaotic system. Here we have seen underpinnings of its quantum counterpart. It is surprising because it is independent of the strength of the coupling between the system and the environment, even though the process of decoherence (caused by the coupling to the environment) is the ultimate source of entropy increase.

This independence is indeed remarkable, and leads one to suspect that the cause of the arrow of time may be traced to the same phenomena which are responsible for the emergence of classicality in chaotic dynamics, and elsewhere (i.e., in quantum measurements). In a sense, this is of course not a complete surprise: Von Neumann knew that measurements are irreversible\(^2\). And Zeh\(^3\) emphasized the close kinship between the irreversibility of the “collapse” in quantum measurements, and in the second law, and cautioned against circularity of using one to solve the other. What is however surprising is both that the classical-looking result has ultimately quantum roots, and that these roots are so well hidden from view that the entropy production rate depends solely on the classical Lyapunov exponents.
The environment may not enter explicitly into the entropy production rate, Eq. (6.4), but it will help determine when this asymptotic formula becomes valid. The Lyapunov exponents will “kick in” as the dimensions of the patch begin to exceed the critical sizes in the corresponding directions, $\ell^{(i)}(t)/\ell^{(i)}_{c} > 1$. The instant when that happens will be set by the strength of the interaction with the environment, which determines $\ell_{c}$. This “border territory” may be ultimately the best place to test the transition from quantum to classical. One may, for example, imagine a situation where the above inequality is comfortably satisfied in some directions in the phase space, but not in the others. In that case the rate of the entropy production will be lowered to include only these Lyapunov exponents for which decoherence is effective.

7. Decoherence, einselection, and the observer: Algorithmic view of entropy production.

The significance of the efficiency of decoherence goes beyond the example of the solar system or the task of reconciling quantum and classical predictions for classically chaotic systems. Every degree of freedom coupled to the environment will suffer loss of quantum coherence. Objects which are more macroscopic are generally more susceptible. In particular, the “hardware” responsible for our perceptions of the external Universe and for keeping records of the information acquired in course of our observations is obviously very susceptible to decoherence: Neurons are strongly coupled to the environment, and are definitely macroscopic enough to behave in an effectively classical fashion. That is, they have a decoherence timescale many orders of magnitude smaller than the relatively sluggish timescale on which they can exchange and process information. As a result, in spite of the undeniably quantum nature of the fundamental physics involved, perception and memory have to rely on the information stored in the decohered (and, therefore, effectively classical) degrees of freedom.

An excellent illustration of the constraint imposed on the information processing by decoherence comes from the recent discussions of the possibility of implementation of quantum computers: Decoherence is viewed as perhaps the most serious threat to the ability of a quantum information processing system to carry out a superposition of computations\cite{37,38}. Yet, precisely such an ability to “compute” in an arbitrary superposition would be necessary for an observer to be able to “perceive” an arbitrary quantum state. Moreover, in the external Universe only these observables which are resistant to decoherence and which correspond to “pointer states” are worth recording: Records are valuable because they allow for predictions, and resistance to decoherence is a precondition to predictability.\cite{17,20}

It is instructive to look at the evolving quantum system through the eyes of an effectively classical, but ultimately quantum observer. In what follows, we shall focus on the memory, which we shall assume consists of a large number of two-state systems (“memory cells”). We shall assume that the effective classicality is a consequence of decoherence – memory cells are immersed in an appropriate environment, which does not perturb selected pointer states, but destroys superpositions between them. Information processing would proceed through appropriate (i.e., $\text{c-not}$ like) interactions between such memory cells. There is no fundamental distinction between this memory device and a classical computer.

The measurement is initiated with an (again $\text{c-not}$ like) coupling between a subset of initially “empty” memory cells and an outside system $\mathcal{S}$. As a result of such an interaction,
a one-to-one correspondence between the states $|s_i>$ of the monitored quantum system $S$ and the record state $|r_i>$ of a subset of memory $\mathcal{M}$ cells shall be set up. Decoherence will play a role already during measurements. Thus; (a) unless the observer stores the information in the pointer states of memory, and; (b) unless he measures einselected pointer states of the system, a rapid (decoherence timescale) loss of information will ensue. In a simplest bit-by-bit c-not like example, Eq. (4.1), a perfect correlation of two sets of orthogonal states $\{ |\bar{0}>, |\bar{1}>\}$ – Schmidt basis states which may in general differ from the stable pointer states $\{ |0>, |1>\}$ for both the system and the memory (subscripts $S$ and $\mathcal{M}$, respectively) would lead from the entangled state:

$$|\psi_{SM}\rangle = a|\bar{0}>_S |\bar{0}>_\mathcal{M} + b|\bar{1}>_S |\bar{1}>_\mathcal{M} , \quad (7.1)$$

to a density matrix diagonal in the common pointer basis, given by the products of the respective pointer states of both the system and the memory:

$$\rho_{SM} = p_{00}|00><00| + p_{11}|11><11| + p_{01}|01><01| + p_{10}|10><10| . \quad (7.2)$$

Here $|01><01| = |0>_S |1>_\mathcal{M}<0>_S |1>_\mathcal{M}$, etc., and;

$$p_{00} = |a < 0|\bar{0}>_S<0|\bar{0}>_\mathcal{M} + b < 0|\bar{1}>_S<0|\bar{1}>_\mathcal{M}|^2 , \quad (7.3a)$$

$$p_{01} = |a < 0|\bar{0}>_s<1|\bar{0}>_m + b < 0|\bar{1}>_s<1|\bar{1}>_m|^2 , \quad (7.3b)$$

with the analogous formulae holding for the other probabilities.

The loss of purity and the simultaneous loss of information is caused by the decoherence which is eliminating entanglement (as it should, to bring about semblance of classicality) but which is exacerbated by the mismatch between the Schmidt basis in the post-entanglement $|\psi_{SM}\rangle$, and the pointer states of $S$ and $\mathcal{M}$. It is reflected in the presence of the “error terms”, such as $|01><01|$ in $\rho_{SM}$, Eq. (7.2). They appear with probabilities $p_{01}$, $p_{10}$, which tend to zero as the pointer states align in both the system (control) and in the memory (target). Generalization of this c-not example to a more complicated case of many states of the system and corresponding states of memory is as straightforward conceptually as it is notationally cumbersome. We shall leave it to the imagination of the reader.

In this simple bit-by-bit case\textsuperscript{11} as well as in the more general measurement situations the information can be lost to the environment through “leaks” located at; (A) the memory, and; (B), the system. In case (A), the loss of information is associated with disentaglement and reduction of the wavepacket, and is the price paid for the classicality of the records\textsuperscript{39}. When the system is isolated and this is the only reason for the information loss, the record states will be mutually orthogonal. Each of them will correspond to a unique (but not necessarily pure or orthogonal\textsuperscript{11}) conditional state of the system:

$$\rho_{S|r_i} = < r_i |\rho_{SM} | r_i > / Tr < r_i |\rho_{SM} | r_i > , \quad (7.4a)$$

which is implied by the record $|r_i>$. When $\rho_{S|r_i} = |s_i><s_i|$ (the measurement is exhaustive) “implication” is readily defined through the use of the conditional probability:

$$p(s_i|r_i) = p(s_i, r_i)/p(r_i) = 1 . \quad (7.4b)$$
This is the “collapse”. In case (B) the system leaks information, as its state will decohere when it was initially entangled with the memory in a Schmidt basis misaligned with the system pointer states, or because it dynamically evolves into such a misaligned state.

Decoherence in memory (A) is associated with the reduction of the state vector. Decoherence in the system (B) will be principally responsible for the increase of entropy in an evolving, open quantum system. Minimizing information loss in case (B) is the motivation behind the predictability sieve\textsuperscript{20,26,27}: A wise selection of what is measured will allow for the optimal preservation of the correlation with the state of the system. Optimal choice of the measured observable is obviously desirable, as it provides initial conditions most useful for making predictions.

We have seen previously how chaotic dynamics spreads regular initial states into the Schrödinger cat-like superpositions. Moreover, and as a result of the environment-induced decoherence, the von Neumann entropy of the system will increase at a rate given by the sum of the Lyapunov exponents, Eq. (6.4). How will these two effects be reflected in the memory of our observer?

When the system is isolated, any measurement which distinguishes between a complete set of states (and corresponds to a complete set of observables \( \hat{O} \)) will be equally good from the point of view of the observer. Such a measurement carried out at a time \( t_0 \) results in a record state \( |r_i(t_0)\rangle \) of some initial state which could be in principle mixed as in Eq. (7.4a), although we shall assume for simplicity that it is a pure \( |s_i(t_0)\rangle \):

\[
\rho_{SM} = \sum_i p_{ii} |r_i(t_0)\rangle \langle s_i(t_0)| .
\]

Hence, at least in principle, an observer who has a record \( r_i \) should be able to perform a measurement of a complete set of observables \( \hat{O}(t) = U(t-t_0)\hat{O}U^{-1}(t-t_0) \), where \( U(t) \) is a unitary evolution operator, related in an obvious manner to the evolution operator of the system) which at any later time \( t \) have a predictable outcome \( |s_i(t)\rangle = U(t-t_0)|s_i(t_0)\rangle \).

A sequence of records of such a succession of measurements of \( \hat{O}(t) \) occurring at subsequent instants is predictable – the corresponding set of records \( R_i = \{r_i(t_0), r_i(t_1), ... \} = \{r_i^{(0)}, r_i^{(1)}, ... \} \) is algorithmically simple. That is, the whole sequence can be generated from one of the records (say, \( r_i^{(0)} \)) by a program containing the Hamiltonian of the system and a sequence of time intervals between the consecutive measurements. The size of such a minimal program in bits will be typically much less than the size of the raw, uncompressed \( R_i \) itself.

The \textit{algorithmic information content}\textsuperscript{40–43} \( K(s) \) of a binary sequence \( s \) is a measure which can be used to characterize the predictability of the data set. It is defined as the size, in bits, of the smallest program \( p^*(s) \) which can generate the string \( s \) on universal classical** computer:

\[
K(s) = |p^*(s)| .
\]

** Note that it does not matter for the definition of the algorithmic information content whether the computer is quantum or classical. This is because a quantum computer can be in principle simulated on the classical computer. The fact that for some operations quantum computer may be exponentially more efficient is irrelevant – neither the time used
Here “|...|” is size in bits. (It should be clear from the context below when “|...|” means ‘size in bits’ or ‘absolute value of a complex number’.) We shall not discuss here the technical details of this definition. Suffice it to say that – using these ideas – it is possible to develop a theory of information content which is formally analogous to the Shannon information theory, but which measures algorithmic randomness (algorithmic information content, Kolmogorov complexity – these terms are often used interchangeably) of specific records, and makes no appeal to probabilities \(^44,45\).

In the example of the record sequences \(R_i\) discussed above:

\[
K(R_i) \leq K(\hat{H}) + K(t_0, t_1, ...) + K(r(t_0)) .
\]

That is, the size of a program required to predict the whole sequence \(R_i\) of measurement records can be reduced to the Hamiltonian, the sequence of the measurement instants, and the initial condition. This disparity between the size of the record \(R_i\) and the size of the minimal program necessary to generate it \(K(R_i)\) – the fact that \(|R_i| \gg K(R_i)\) – is the defining feature of the algorithmic simplicity.

A sequence of repeated measurements of an energy eigenstate is a common (if somewhat trivial and atypical) example of such a perfectly predictable “evolution” resulting in an algorithmically simple record set \(R_i\). In this case \(\hat{U}(t - t_0)\) is an identity (up to an overall phase), and, hence \(K(R_i) = K(r_i^{(0)})\), give or take a few bits. In more typical examples of non-trivial evolution \(|s(t)\rangle\) would really change with time, and the inequality (7.6) would be saturated. But, in any case, for a perfectly isolated quantum system, observer can in principle accomplish \(K(R_i) \ll |R_i|\) by electing to measure observables that can be predicted on the basis of his initial measurement record.

This last proviso – the ability to measure evolved initial condition – is not at all trivial. In many situations – including chaotic quantum systems – the states \(|s_i(t)\rangle\) will be wildly exotic “Schrödinger cat-like” superpositions. And a typical observer will have his disposal a rather limited set of measurable observables, often resulting in a single fixed set of potential outcome states \(|\sigma_i\rangle\). When such an observer attempts to re-measure an evolving quantum system he “knows to be” in a state \(|s_i(t)\rangle\) (which may have evolved from one of the \(|\sigma_i\rangle\)'s), his memory will be first entangled with the state of the system, and then – after a memory decoherence time – it will suffer “reduction” to end up in a correlated mixture of different (memory) pointer states:

\[
\rho_{SM|r_i^{(t)}}^{(t+1)} = |r_i^{(t)}\rangle < r_i^{(t)} | \sum_j | < \sigma_j | s_i(t+1) > |^2 | \sigma_j r_j^{(t+1)} > < \sigma_j r_j^{(t+1)} | .
\]

Here we have pulled out a common factor – observer’s record of the previous state of the system – and renormalized the conditional \(\rho_{SM|r_i}\) at the time \(t+1\) right after a new measurement. This illustrates a single step in the reduction of the state vector caused by the coupling between the “apparatus pointer” – records in the memory of the observer – and the environment in the case of exhaustive measurement.

nor the memory required matter for the definition of \(K(s)\). It is nevertheless intriguing to enquire about a quantum algorithmic information content of a state defined by the least number of \textit{qubits} needed to produce a certain quantum state.
A succession of such reductions will be represented by a sequence of branching records and a similarly co-diverging Hilbert space “trajectories” of the system punctuated by the recorded states. Two record sequences which originate from the same initial condition may initially coincide for a while, but will eventually diverge. Seen from the outside, the memory of the observer will be described by a density matrix $\rho_M$, a mixture of all the possible records $\{R\}$, each weighted with the probability $p(R)$. The associated von Neumann entropy will be:

$$H_M = -Tr \rho_M \log \rho_M = -\sum_R p(R) \log p(R) ,$$

(7.8)

where $\log = \log_2$, so that $H$ is measured in bits. The entropy of the records will increase at the rate set by the nature of measurements as well as by the dynamics of the system, which will decide the distribution of the record probabilities. For instance, in a chaotic system monitored on a time scale of the order of the dynamical time or less, entropy will typically increase at the rate given by the Lyapunov exponents, even in the absence of the environment. However, that rate is really set by the relation between the states which can be predicted on the basis of the past records, and the states the observer actually measures. Thus, when isolated, even a chaotic system can be made perfectly predictable providing that the measured observable is energy. Conversely, a trivial evolution – a rotating spin $\frac{1}{2}$ – can produce entropy when measured at time intervals equal to a quarter of its period of rotation. Indeed, even in the absence of any evolution of the system, alternating measurements of spin $\frac{1}{2}$ along any two orthogonal axes will result in the same linear rate of entropy increase (linearity is characteristic of chaos in usual dynamical context – see Eq. (6.4)). In all of these examples the increase of entropy is subjective – it depends on the choices of measurements carried out by the observer. Yet, it is real: Unpredictable sequences of records which cannot be compressed to a single initial condition will remain as evidence. The information is ultimately lost to the environment through the process (A).

We note that even when the entropy of the system is bounded (as would be the case for the spin $\frac{1}{2}$) the entropy of the records can grow without a bound – the memory is idealized here as an infinite, initially blank tape of a Turing machine.

A whole sequence of such measurements viewed from the vantage point of the observer will result in a specific set of records – a particular record state $|R> = |r_1 r_2 r_3 ... >$ in his memory, and on the diagonal of $\rho_M$. Each specific $|R>$ will appear with the probability $p(R)$ given by the product of consecutive conditional probabilities computed from the states corresponding to individual record sequences in a manner illustrated in Eq. (7.7). This probability cannot increase with addition of new events to the “recorded history”. It can however stay constant when the initial record sets the initial condition which allows for prediction of all subsequent records with certainty. Typically, however, the evolution will not be so perfectly predictable. Moreover, it can be shown that optimal record compression strategies (which have to be applied to the set of all record sequences $\{R\}$) will have $K(R) \simeq -\log_2 p(R)$.

The examples we have discussed above illustrate loss of information caused by decoherence of the records – the “collapse” process (A). The observer could in principle avoid
such “subjective” decreases of predictability by measuring only these observables he can predict with certainty. Unfortunately, in practice this is not an option – the observers can only measure certain observables of the system (and the corresponding set of their eigenstates), while the environment invariably imposes the same set of pointer states in the memory of the observer, thus eliminating entanglement. The increase of entropy is an inevitable consequence of that last step, which guarantees classicality of records. In absence of decoherence quantum records could be used by an isolated quantum observer to enhance predictability.

The most realistic (and most intriguing) case involves an evolving, decohering quantum system which is also occasionally monitored by the observer. Let us first suppose that the observer is skillful, always able to “match” the basis which diagonalizes the instantaneous conditional density matrix of the system based on his records with the choice of measurements. Then the measurement itself will not lead to a collapse – rather, it will “reveal” to the observer the system in one of the “preexisting” einselected pointer states. The observers record $R$ will simply reflect the evolution of the system under the influence of the environment. Its algorithmic information content will grow, on the average, with the increase of the entropy of the system which is now caused by the process (B) – the decohering influence of the environment on the system. Hence, the average increase of the algorithmic information $\Delta K(R)$ per measurement will be given by the conditional algorithmic information defined as the number of “extra bits” required to extend the old record:

$$\Delta < K(R) > = < K(R_{n+1}|R_n) > = \Delta \mathcal{H}_M \simeq \Delta t < \dot{\mathcal{H}}_S > .$$  

(7.9)

For a chaotic quantum system $< \mathcal{H}_S >$ is, on the average (over the phase space or – equivalently – over the ensemble of observer records weighted by their probabilities) given by the sum of positive Lyapunov exponents, Eq. (6.4).

We have thus arrived at the “observer’s own” version of the second law: The observer will be affected by the unpredictability directly, as the algorithmic randomness of his record $K(R)$ of the enfolding history of the system will increase with the number of measurements. Moreover, that entropy increase will be now more objective in that observer will not be able to prevent it solely by adjusting the measured observables of the system. But what if the observer does not measure? In that case the entropy will initially increase at essentially the same rate, although the unpredictability will be reflected in the ignorance of the observer, measured by $\mathcal{H}(\rho_S)$, which will grow with time, rather than in the clutter of the record of the unfolding history. Both effects can be taken into account simultaneously by introducing physical entropy, a measure of disorder which includes both the statistical entropy $\mathcal{H}(\rho_{S|R})$ attributed to the system by the observer who has the record $R$, and the algorithmic entropy $K(R)$:

$$Z(R) = K(R) + \mathcal{H}(\rho_{S|R}) .$$  

(7.10)

*** In simple examples (such as the quantum c-not repeated many times) one can see how retention of entanglement is reflected in an increase of predictability, resulting in recurrences of the initial state. However, in this purely quantum context concepts such as predictability are harder to define, so these remarks should be taken with a grain of salt.
This quantity combines a measure of the observers cost of storing the record $R$ and the ignorance remaining – the fact that the system is described by $\rho_{S|R}$ conditioned upon this specific record – in spite of the past measurements.

Physical entropy was introduced to remove the illusion that the observer can violate the second law by measuring: It shows that what happens in measurements is a replacement of ignorance (and simplicity) with information (and clutter), and that the sum of these two contributions for equilibrium cases yields a conservation law involving $Z$. As such, it helps put the second law on a firmer footing by introducing a quantity which the intelligent being – attempting to act, for instance, in the Maxwell’s demon capacity – can apply to itself.

Here the role of $Z$ is to connect the irreducible size of the record $R$ (which increases suddenly upon measurement, but is constant in between) with the von Neumann entropy of a system which “shrinks” upon a measurement. Physical entropy – when averaged over an ensemble of all records consistent with the initial state of the system with the appropriate weights – does not change when measurements carried out by the observers are reversible, i.e., adjusted so that they commute with the density matrix describing the system. However, imperfect measurements will add to the entropy production. Thus, when the system is kept far from equilibrium by the measurements of the observer:

\[
< Z(R) > = Tr_{M}[\rho_{M}(K(R) + \mathcal{H}(\rho_{M|R}))] \geq \mathcal{H}_S . \tag{7.11}
\]

$K(R)$ for a specific sequence of skillful measurements carried out in time intervals small compared to the time it takes for the system to reach equilibrium will increase at the rate set by the dynamics. This is ultimately due to $\dot{\mathcal{H}}_{S|R}$ of an evolving system in between observer’s measurements (but in the presence of environmental monitoring resulting in decoherence). Increase at the same dynamically determined rate is then a good description of the behavior of $Z(R)$ (their sum), the physical entropy relevant for a particular observer with a specific measurement record. There can be of course “atypical” branches, characterized by a relative simplicity of the records. However, on the average (that is, when contributions of individual branches labelled by distinct records are added with the appropriate weights, Eq. (7.11)), the increase of physical entropy far from equilibrium will be almost exactly equal to the increase of entropy caused by the combination of dynamics and decoherence. Thus, $< \dot{Z} > \simeq < \dot{\mathcal{H}}_S >$, which in a chaotic system will be give by Eq (6.4), while in a regular system $< \dot{Z} > \sim 1/t$, and the physical entropy will increase slowly (logarithmically) with time.

These estimates are contingent upon an assumption of a skillful observer – an observer who is able to measure the system using pointer observables – that is, without causing additional decoherence (and consequent entropy increases) by his measurements. This may not be a realistic assumption, especially when the observer is attempting to resolve the density matrix of the system into the individual pure states. Indeed, especially in the case of such high-resolution measurements, the optimal basis may be somewhat dependent on the outcomes of the preceding measurements – on the observer’s record. Thus, the increase of physical entropy $\dot{Z}$ will most likely be more rapid than the increase of the von Neumann entropy of the same system in the absence of measurements.

There is another reason why the physical entropy may increase not only faster, but to larger values than the ordinary von Neumann entropy: The size of the observer’s record...
will continue to increase beyond the time after which the system left on its own would have reached equilibrium. Thus, even after the density matrix of the system averaged over all of the measurement records has reached its equilibrium distribution, the conditional density matrix implied by a particular record \( R \) (and perceived by an observer in possession of that record) may be very far from equilibrium, and the system will continue to increase its physical entropy at a rate no less than \( \dot{H}(\rho_{\mathcal{S}|R}) \). For example, in a chaotic quantum system physical entropy will not saturate at the equilibrium value: Rather, the record size can continue increasing forever (or at least until the memory shall run out of empty space) at the rate given by Eq. (6.4).

We have already noted that the increase of resolution will typically increase the rate of entropy production. This is because quantum measurements involve reduction of the state vector – a process associated with the entropy increase. So, an observer may gain short-term predictability by making a high-resolution measurement, but unless he is a skillful observer, he will lose a certain amount of information through process (B), at the instant of measurement. This additional source of entropy increase will obviously depend on the frequency and the resolution of measurements. The extra penalty will be usually less when the resolution is lower.

The above discussion illustrates two complementary points of view of the entropy increase. “Outsiders” viewpoint limits the information loss to the process (B) – outsider does not monitor the system, and lets his ignorance grow (but his memory remains uncluttered). “Insider” measures the system repeatedly. He knows more, but uses up his memory to record data which may soon become obsolete (i.e., in a chaotic system obsolescence sets in on a Lyapunov time). Decoherence has allowed us to carry out this quasi-classical analysis in a completely quantum setting.

It is too early to claim that all the issues arising in the context of the transition from quantum to classical have been settled with the help of decoherence. Decoherence and ein-selection are, however, rapidly becoming a part of the standard lore. Where expected, they deliver classical states, and – as we have seen above – guard against violations of the correspondence principle. The answers which emerge may not be to everyone’s liking, and do not really discriminate between the Copenhagen Interpretation and the Many Worlds approach. Rather, they fit within either mold, providing effectively the missing elements – delineating the quantum-classical border postulated by Bohr (decoherence time fast or slow compared to the dynamical timescales on the two sides of the “border”), and supplying the scheme for defining distinct branches required by Everett (overlap of the branches is eliminated by decoherence).

Acknowledgments

John Archibald Wheeler has a legendary ability to anticipate and stimulate exciting developments in physics. I had the good luck to be at the University of Texas at the right time, where he convinced me (and a few others) that quantum physics should be understood (rather than just applied as a calculational tool). I would also like to thank Juan Pablo Paz, who has been my close collaborator in the study of the competition between the quantum superposition principle and decoherence in the chaotic setting. I am grateful to Chris Jarzynski, Raymond Laflamme, and Michael Nielsen for useful comments
on the manuscript. Last not least, I am grateful to the Nobel Foundation for a stimulating, enjoyable, and most memorable meeting.
References

1. Poincaré, H., *Les Methodes Nouvelles de la Méchanique Céleste*, (Gauthier-Villars, Paris, 1892).
2. Laskar, J., *Nature* **338**, 237 (1989).
3. Sussman, G. J., and Wisdom, J., *Science* **257**, 56-62 (1992).
4. Wisdom, J., Peale, S. J., and Maignard, F. *Icarus* **58**, 137 (1984).
5. Wisdom, J., *Icarus* **63**, 272 (1985).
6. Berman, G. P., and Zaslavsky, G. M., *Physica* (Amsterdam) **91A**, 450 (1978).
7. Berry, M. V., and Balzas N. L., *J. Phys. A* **12**, 625 (1979).
8. Zurek, W. H., and Paz, J. P., *Phys. Rev. Lett.* **72**, 2508-2511 (1994); *ibid.* **75**, 351 (1995).
9. These results were announced already some time ago, but appear to be still somewhat controversial; see selected papers in Casati, G., and Chrikov, B., *Quantum Chaos* (Cambridge University Press, Cambridge, 1995).
10. Schrödinger, E., *Naturwissenschaften* **23**, pp. 807-812, 823-828, 844-849 (1935); English translation in Wheeler, J. A. and Zurek, W. H., eds., *Quantum Theory and Measurement*, pp. 152-167 (Princeton University Press, Princeton, NJ, 1983).
11. Zurek, W. H., *Phys. Rev. D* **24**, 1516-1524 (1981).
12. Zurek, W. H., *Phys. Rev. D* **26**, 1862-1880 (1982).
13. Joos, E., and Zeh, H. D., *Zeits. Phys. B* **59**, 229 (1985).
14. Zurek, W. H., pp. 145-149 in *Frontiers of Nonequilibrium Statistical Mechanics*, G. T. Moore and M. O. Scully, eds. (Plenum, New York, 1986).
15. Milburn, G. J., and Holmes, C. A., *Phys. Rev. Lett.* **56**, 2237-2240 (1986).
16. Haake, F., and Walls, D. F., in *Quantum Optics IV*, J. D. Harvey and D. F. Walls, eds. (Springer, Berlin, 1986).
17. Zurek, W. H., *Physics Today* **44**, 36-46 (1991).
18. Gell-Mann, M., and Hartle, J. B., *Phys. Rev. D* **47**, 3345-3382 (1993).
19. Albrecht, A., *Phys. Rev. D* **48**, 3768 (1993).
20. Zurek, W. H., *Progr. Theor. Phys. B* **89**, 281-302 (1993).
21. von Neumann, J., “Measurement and reversibility” and “The measuring process”, chapters V and VI if *Mathematische Grundlagen der Quantenmechanik*, (Springer, Berlin, 1932); English translation by R. T. Beyer *Mathematical Foundations of Quantum Mechanics*, (Princeton Univ. Press, Princeton, 1955).
22. Einstein, A., Podolsky, B., and Rosen, N., *Phys. Rev* **47**, 777-780 (1935).
23. Bohm, D., *Quantum Theory*, (Prentice-Hall, Engelwood Cliffs, 1951).
24. Bell, J. S., *Physics 1*, 195-200 (1964).
25. Aspect, A., Dalibard, J., and Roger, G., *Phys. Rev. Lett.* **49**, 1804-1807 (1982).
26. Zurek, W. H., Habib, S., and Paz, J. P., *Phys. Rev. Lett.* **70**, 1187-1190 (1993); Anglin, J. R., and Zurek, W. H., *Phys Rev. D* **53**, 7327-7335 (1996).
27. Gallis, M. R., *Phys. Rev. A* **53**, 655-660 (1996); Tegmark, M., and Shapiro, H. S., *Phys. Rev. E* **50**, 2538-2547 (1994).
28. Lindblad, G., *Comm. Math. Phys. 40*, 119-130 (1976).
29. Caldeira, A. O., and Leggett, A. J., *Physica 121A*, 587-616 (1983); *Phys. Rev. A* **31**, 1059 (1985).
30. Unruh, W. G., and Zurek, W. H., *Phys. Rev.* **D40**, 1071-1094 (1989).
31. Brune, M., Hagley, E., Dreyer, J., Maître, X., Maali, A., Wunderlich, C., Raimond, J.-M., and Haroche, S., *Phys. Rev. Lett.* **77**, 4887-4890 (1996).
32. Monroe, C., Meekhof, D. M., King, B. E., and Wineland, D. J., *Science* **272**, 1131-1136 (1996).
33. Poyatos, J. F., Cirac, J. I., and Zoller, P., *Phys. Rev. Lett.* **77**, 4728-4731 (1996).
34. Habib, S., Shizume, K., and Zurek, W. H., *Phys. Rev. Lett.* in press.
35. Ott, E., Antonsen, T. M., and Hanson, J., *Phys. Rev. Lett.* **35**, 2187-2190 (1984); Dittrich, T., and Graham, R., *Phys. Rev. A42*, 4647-4660 (1990), and references therein.
36. H. D. Zeh, *The Physical Basis of the Direction of Time*, Springer, Berlin, 1989.
37. Landauer, R., *Phil. Trans. R. Soc.* **353** 367 (1995); also, in Proc. of the Drexel-4 Symposium on Quantum Nonintegrability: Quantum – Classical Correspondence, D. H. Feng and B.-L. Hu, eds. (World Scientific, Singapore, 1998); W. G. Unruh, *Phys. Rev. A51*, 992 (1995) Chuang, I. L., Laflamme, R., Shor, P., and Zurek, W. H., *Science* **270**, 1633-1635 (1995).
38. C. H. Bennett, *Physics Today* **48**, No. 10 (1995).
39. Zurek, W. H., “Information transfer in quantum measurements”, pp. 87-116 in *Quantum Optics, Experimental Gravity, and the Measurement Theory*, P. Meystre and M. O. Scully, eds. (Plenum, New York, 1983).
40. Solomonoff, R. J., *Inform. Control* **7**, 1-22 and 224-254 (1964).
41. Kolmogorov, A. N., *Problems of Information Transmission* **1**, 4-7 (1965).
42. Chaitin, G. J., *J. Assoc. Comp. Mach.* **16** 145-159 (1966).
43. Martin-Löf, P., “Algorithmen und zufällige Folgen”, lecture notes, University of Erlangen, 1966.
44. Li, M., and Vitanyi, P., *An Introduction to Kolmogorov Complexity and its Applications* (Springer, New York, 1993).
45. Cover, T. M., and Thomas, J. A., *Elements of Information Theory*, (Wiley, New York, 1991).
46. Zurek, W. H., *Phys. Rev. A40* 4731-4751 (1989); *Nature* **341** 119-124 (1989).
47. Caves, C. M., pp. 47-89 in *Physical Origins of Time Asymmetry*, J. J. Halliwell, J. Pérez-Mercader, and W. H. Zurek, eds. (Cambridge Univ. Press, Cambridge, 1994); C. M. Caves and R. Schack, *Complexity* **3**, 46-57 (1997).
48. Paz, J. P., and Zurek, W. H., *Phys. Rev. D48*, 2728-2738 (1993).
49. Gell-Mann, M., and Hartle, J. B., in *Complexity, Entropy, and the Physics of Information*, W. H. Zurek, ed. (Addison-Wesley, Reading, 1990).
50. Omnès, R., *Rev. Mod. Phys.* **64**, 339-382 (1992), and *The Interpretation of Quantum Mechanics*, (Princeton, 1994).
51. Griffiths, R. B., *Phys. Rev. A54*, 2759-2774, (1996).
52. Giulini, D., Joos, E., Kiefer, C., Kupsch, J., Stamatescu, I.-O., and Zeh, H. D., *Decoherence and the Appearance of a Classical World in Quantum Theory*, (Springer, Berlin, 1996).
Figures

Fig. 1. Information transfer in measurements and in decoherence.

a) Controlled not (c-not) as an elementary bit-by-bit measurement. Its action is described by the “truth table” according to which the state of the target bit (apparatus memory in the quantum measurement vocabulary) is “flipped” when the control bit (measured system) is $|1\rangle$ and untouched when it is $|0\rangle$ (Eq. (4.1)). This can be accomplished by the unitary Schrödinger evolution (see, e. g., Ref. 11).

b) Decoherence process “caricatured” by means of c-nots. Pointer state of the apparatus (and, formerly, target bit in the pre-measurement, Fig. 1a) now acts as a control in the continuous monitoring by the c-nots of the environment. This continuous monitoring process is symbolically “discretized” here into a sequence of c-nots, with the state of the environment assuming the role of the multi-bit target. Monitored observable of the apparatus – its pointer observable – is in the end no longer entangled with the system, but the classical correlation remains. Decoherence is associated with the transfer of information about the to-be-classical observables to the environment. Classically, such information transfer is of no consequence. In quantum physics it is absolutely crucial, as it is responsible for the effective classicality of certain quantum observables.

c) Noise is a process in which a pointer observable of the apparatus is perturbed by the environment. Noise differs from the purely quantum decoherence – now the environment acts as a control, and the c-nots which represent it act in the direction opposite to the decoherence c-nots. Usually, both decoherence and noise are present. Preferred pointer observables and the associated pointer states are selected so that the noise is minimized.
