Measuring Bayesian Robustness Using Rényi’s Divergence and Relationship with Prior-Data Conflict

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Abstract

This paper deals with measuring the Bayesian robustness of classes of contaminated priors. Two different classes of priors in the neighborhood of the elicited prior are considered. The first one is the well-known \( \epsilon \)-contaminated class, while the second one is the geometric mixing class. The proposed measure of robustness is based on computing the curvature of Rényi’s divergence between posterior distributions. The relationship between robustness and prior-data conflict has been studied. Through two examples, a strong connection between robustness and prior-data conflict has been found.

KEYWORDS: Bayesian Robustness, \( \epsilon \)-contamination, Prior-data conflict, Rényi’s divergence.

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1 Introduction

Bayesian inferences require the specification of a prior, which contains a priori knowledge about the parameter(s). In most cases, it becomes very challenging to come up with only a sole prior distribution. As a result, if the selected prior, for instance, is flawed, this may yield erroneous inferences.

In this paper, we attempt to address two issues related to priors: the sensitivity of inferences to a chosen prior (known as robustness) and the suitability of a chosen prior in the light of the data (known as prior-data conflict).

To address the first problem, it seems rational to consider a class, \( \Gamma \), of all possible priors over the parameter space. Usually a preliminary prior \( \pi_0 \) is elicited. Then robustness for all priors \( \pi \) in a neighborhood of \( \pi_0 \) is intended. A common accepted way to construct neighborhoods around the elicited prior \( \pi_0 \) is through contamination. Specifically, we will consider two different classes of contaminated or mixture of priors, which are given by

\[
\Gamma_\epsilon = \{ \pi(\theta) : \pi(\theta) = (1 - \epsilon)\pi_0(\theta) + \epsilon q(\theta), q \in Q \} \tag{1}
\]

and

\[
\Gamma_g = \{ \pi(\theta) : \pi(\theta) = c(\epsilon)\pi_0^{1-\epsilon}(\theta)q^*(\theta), q \in Q \}, \tag{2}
\]

where \( \pi_0 \) is the elicited prior, \( Q \) is a class of distributions, and \( 0 \leq \epsilon \leq 1 \) is a small given number denoting the amount of contamination. For other possible classes of priors, see of instance, De Robertis and Hartigan (1981) and Das Gupta and Studden (1988a, 1988b).

The class (1) is known as the \( \epsilon \)-contaminated class of priors. Many papers about the class (1) are found in the literature. See, for instance, Berger (1984, 1990), Berger and Berliner (1986), Sivaganesan and Berger (1989), Wasserman (1989), Dey and Birmiwal (1994) and Al-Labadi and Evans (2017).
On the other hand, the class (2) will be referred as geometric contamination or mixture class. This class was first studied, in the context of Bayesian Robustness, by Gelfand and Dey (1991), where the posterior robustness was measured using Kullback-Leibler divergence. Dey and Birmiwal (1994) generalized the results of Gelfand and Dey (1991) by using the $\phi$-divergence under (1) and (2).

In this paper, we extend the results of Gelfand and Dey (1991) and Dey and Birmiwal (1994) by applying Rényi’s divergence on both classes (1) and (2). This will give local sensitivity analysis on the effect of small perturbation to the prior.

Another problem of recent interest, known as a prior-data conflict, is when a chosen prior is strongly contradicted by the data (Evans and Moshonov, 2006; Nott, Xueou, Evans, and Engler, 2016; Nottt, Seah, AL-Labadi, Evans, Khoon and Englert, 2019). In such a situation, one has to be concerned about what the effect of the prior on the analysis. For example, it is shown in Al-Labadi and Evans (2017) that when there is prior-data conflict, then the statistical analysis that is based on relative belief inferences (Evans, 2015) is not robust with respect to the prior. In this paper, we address a possible general connection between robustness and prior-data conflict.

In Section 2, we give definitions, notations and some properties of Rényi’s divergence. In Section 3, we discuss how to check for the existence of a prior-data conflict. In Section 4, we develop curvature formulas for measuring robustness based on Rényi’s divergence. In Section 5, two examples are studied, where a clear relationship between robustness and prior-data conflict has been found. Section 6 ends with a brief summary of the results.
2 Definitions and Notations

Suppose we have a statistical model that is given by the density function \( f_\theta(x) \) (with respect to some measure), where \( \theta \) is an unknown parameter that belongs to the parameter space \( \Theta \). Let \( \pi(\theta) \) be the prior distribution of \( \theta \). After observing the data \( x \), by Bayes’ theorem, the posterior distribution of \( \theta \) is given by the density
\[
\pi(\theta|x) = \frac{f_\theta(x)\pi(\theta)}{m(x|\pi)},
\]
where
\[
m(x|\pi) = \int f_\theta(x)\pi(\theta)d\theta
\]
is the prior predictive density of the data.

To measure the divergence between two posterior distributions, we consider Rényi’s divergence (Rényi, 1961). Rényi’s divergence of order \( a \) between two posterior densities \( \pi(\theta|x) \) and \( \pi_0(\theta|x) \) is defined as:
\[
d = d(\pi(\theta|x), \pi_0(\theta|x)) = \frac{1}{a - 1} \ln \left( \int (\pi(\theta|x))^a(\pi_0(\theta|x))^{1-a}d\theta \right)
\]
\[
= \frac{1}{a - 1} \ln \left( \mathbb{E}_{\pi(\theta|x)} \left[ \frac{\pi(\theta|x)}{\pi_0(\theta|x)} \right] \right),
\]
where \( a > 0 \). Note that, the case \( a = 1 \) is defined by letting \( a \to 1 \). This leads to the Kullbak-Leibler divergence, which plays a crucial role in machine learning and information theory. For more detail about further properties of Rényi’s divergences consult, for example, Li and Turner (2016). It is also possible to calibrate Rényi’s divergence. The idea is similar to that proposed in McCulloch (1989) for the Kullback-Leibler distance and to Dey and Birmiwal (1994) for \( \phi \) divergence. Consider a biased coin that has occurred with probability \( p \). Then
Rényi’s distance between an unbiased and a biased coin is
\[
d(f_0, f_1) = \frac{1}{a-1} \ln \left[ 2^{a-1} (p^a + (1-p)^a) \right],
\]
where, for \( x = 0, 1 \), \( f_0(x) = 0.5 \) and \( f_1(x) = p^x(1-p)^{1-x} \). Now, assume that \( d(f_0, f_1) = d_0 \). It follows that
\[
d_0 = \frac{1}{a-1} \ln \left[ 2^{a-1} (p^a + (1-p)^{1-a}) \right],
\]
which simplifies
\[
2^{1-a} e^{(a-1)d_0} = p^a + (1-p)^{1-a}.
\]
Then the number \( p \) is the calibration of \( d \). In particular, for \( a = 1 \) (i.e. the Kullback-Liebler divergence), due to McCulloch (1989), it follows that
\[
p = 0.5 + 0.5 \left( 1 - 2e^{-2d_0} \right)^{1/2}.
\]
Values of \( p \) close to 1 indicate that \( f_0 \) and \( f_1 \) are quiet different, while values of \( p \) close to 0.5 implies that they are similar.

An motivating key fact about Rényi’s divergence follows from Taylor expansion. Let \( f(\epsilon) = d(\pi(\theta|x), \pi_0(\theta|x)) \), where \( \pi(\theta|x) \) is the posterior distribution of \( \theta \) given the data \( x \) under the prior \( \pi \) defined in (1) and (2). Assuming differentiability with respect to \( \epsilon \), the Taylor expansion of \( f(\epsilon) \) about \( \epsilon = 0 \) is given by
\[
f(\epsilon) = f(0) + \epsilon \frac{\partial f(\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} + \epsilon^2 \frac{\partial^2 f(\epsilon)}{\partial^2 \epsilon} \bigg|_{\epsilon=0} + \cdots.
\]
Clearly, \( f(0) = 0 \). If integration and differentiation are interchangeable,
\[
\frac{\partial f(\epsilon)}{\partial \epsilon} = \frac{a}{1-a} \frac{\int (\pi_0(\theta|x))^{1-a} (\pi(\theta|x))^{a-1} \frac{\partial \pi(\theta|x)}{\partial \epsilon} d\theta}{\int (\pi_0(\theta|x))^{1-a} (\pi(\theta|x))^a d\theta}.
\]
Hence,

$$\frac{\partial f(\epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} = \frac{a}{1-a} \int \frac{\partial \pi(\theta|x)}{\partial \epsilon} d\theta$$

$$= \frac{a}{1-a} \frac{\partial}{\partial \epsilon} \left( \int \pi(\theta|x) d\theta \right) = \frac{a}{1-a} \frac{\partial}{\partial \epsilon} (1) = 0. \quad (1)$$

On the other hand,

$$\frac{\partial^2 f(\epsilon)}{\partial^2 \epsilon} = \frac{\partial}{\partial \epsilon} \left( \frac{a}{1-a} \int \frac{\partial \pi(\theta|x)}{\partial \epsilon} \left( \pi(\theta|x)^{1-a} \left( \pi(\theta|x) \right)^{a-1} \right) d\theta \right),$$

which, at $\epsilon = 0$, reduces to

$$\frac{\partial^2 f(\epsilon)}{\partial^2 \epsilon} \bigg|_{\epsilon=0} = -a \int \left( \frac{\partial \pi(\theta|x)}{\partial \epsilon} \right)^2 \pi(\theta|x) d\theta \bigg|_{\epsilon=0}$$

$$= -a \int \left( \frac{\partial \pi(\theta|x)}{\partial \epsilon} \right)^2 \pi(\theta|x) d\theta \bigg|_{\epsilon=0}$$

$$= -a \exp(\pi(\theta|x)|\epsilon=0) \right) \bigg|_{\epsilon=0}$$

$$= -a \exp(\pi(\theta|x)|\epsilon=0),$$

where $I_{\pi(\theta|x)}(\epsilon) = E_{\pi(\theta|x)} \left[ \left( \frac{\partial \log \pi(\theta|x)}{\partial \epsilon} \right)^2 \right] \bigg|_{\epsilon=0}$ is the Fisher information function for $\pi(\theta|x)$. Thus, for $\epsilon \approx 0$, we have

$$d(\pi(\theta|x), \pi_0(\theta|x)) = \frac{\epsilon^2}{2} I_{\pi(\theta|x)}(\epsilon). \quad (3)$$

Note that, $\frac{\partial^2 f(\epsilon)}{\partial^2 \epsilon} \bigg|_{\epsilon=0} = \partial^2 d/\partial^2 \epsilon \bigg|_{\epsilon=0}$ is known as the local curvature at $\epsilon = 0$ of Rényi’s divergence. Formula (3) justifies the use of the curvature to measure the Bayesian robustness of the two classes of priors $\Gamma_\alpha$ and $\Gamma_g$ as defined in (1).
and (2), respectively. Also this formula provide a direct relationship between Fisher’s information and the curvature of Rényi’s divergence.

3 Checking for Prior-Data Conflict

A chosen prior may be incorrect by being strongly contradicted by the data (Evans, 2015). A possible contradiction between the data and the prior is referred to as a prior-data conflict. If the prior primarily places its mass in a region of the parameter space where the data suggest the true value does not lie, then there is a prior-data conflict (Evans and Moshonov, 2006). That is, prior-data conflict will occur whenever there is only a tiny overlap between the effective support regions of the model and the prior. In such a situation, we have to be concerned about what the effect of the prior is on the analysis (Evans, 2015).

Methods for checking the prior in previous sense are developed in Evans and Moshonov (2006). See also Nott, Xueou, Evans, and Engler (2016) and Nottt, Seah, AL-Labadi, Evans, Khoon and Englert (2019). A fundamental for checking the prior involves computing the probability

\[ M_T (m_T(t) \leq m_T(t_0)) , \]

where \( T \) is a minimal sufficient statistic of the model and \( M_T \) is the prior predictive probability measure of \( T \) with density \( m_T \). The value of (4) simply serves to locate the observed value \( T(x) \) in its prior distribution. If (4) is small, then \( T(x) \) lies in a region of low prior probability, such as a tail or anti-mode, which indicates a conflict. The consistency of this check follows from Evans and Jang (2011) where it is proven that, under quite general conditions, (4)
converges to

\[ \Pi_T (\pi_0(\theta) \leq \pi_0(\theta_{\text{true}})), \quad (5) \]

as the amount of data increases, where \( \theta_{\text{true}} \) is the true value of the parameter. If (5) is small, then \( \theta_{\text{true}} \) lies in a region of low prior probability which implies that the prior is not appropriate.

4 Measuring Robustness Using Rényi’s Measure

In this section, we explicitly obtain the local curvature at \( \epsilon = 0 \) of Rényi’s divergence (i.e. \( \frac{\partial^2 d}{\partial^2 \epsilon} \bigg|_{\epsilon=0} \)), to measure the Bayesian robustness of the two classes of priors \( \Gamma_\alpha \) and \( \Gamma_g \) as defined in (1) and (2), respectively.

**Theorem 1** For the \( \epsilon \)-contaminated class defined in (1), the local curvature of Rényi’s divergence at \( \epsilon = 0 \) is

\[
C_\alpha = \frac{\partial^2 d}{\partial^2 \epsilon} \bigg|_{\epsilon=0} = a \text{Var}_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right].
\]

**Proof.** Under the prior \( \pi \) defined in (1), the marginal \( m(\theta|x) \) and the posterior distribution \( \pi(\theta|x) \) can be written as

\[
m(x|\pi) = (1 - \epsilon)m(x|\pi_0) + \epsilon m(x|q)
\]

and

\[
\pi(\theta|x) = \frac{f_\theta(x)\pi(\theta)}{m(x|\pi)} = \lambda(x)\pi_0(\theta|x) + (1 - \lambda(x))\pi(x|q),
\]
where
\[
\lambda(x) = (1 - \epsilon) \frac{m(x|\pi_0)}{m(x|\pi)}.
\]

Define
\[
\gamma = (\pi(\theta|x))^a (\pi_0(\theta|x))^{1-a}.
\] (6)

Clearly,
\[
\gamma \bigg|_{\epsilon=0} = \pi_0(\theta|x) \text{ and } \int \gamma \bigg|_{\epsilon=0} \, d\theta = 1. \quad (7)
\]

Note that,
\[
d(\pi(\theta|x), \pi_0(\theta|x)) = \frac{1}{a-1} \ln \left[ \int \gamma \, d\theta \right]. \quad (8)
\]

We have
\[
\frac{\partial \gamma}{\partial \epsilon} \bigg|_{\epsilon=0} = a \frac{m(x|q) (q(\theta|x) - \pi_0(\theta|x))}{m(x|\pi_0)}.
\]

Thus,
\[
\int \frac{\partial \gamma}{\partial \epsilon} \bigg|_{\epsilon=0} \, d\theta = 0.
\]

Hence, by (8),
\[
\frac{\partial d}{\partial \epsilon} \bigg|_{\epsilon=0} = 0.
\]

Now,
\[
\frac{\partial^2 d}{\partial^2 \epsilon} = \frac{1}{a-1} \left[ \int \gamma \, d\theta \right] \left[ \int \frac{\partial^2 \gamma}{\partial \epsilon^2} \, d\theta \right] - \left[ \int \frac{\partial \gamma}{\partial \epsilon} \, d\theta \right]^2.
\]

Thus, by (7),
\[
\frac{\partial^2 d}{\partial^2 \epsilon} \bigg|_{\epsilon=0} = \frac{1}{a-1} \int \frac{\partial^2 \gamma}{\partial \epsilon^2} \bigg|_{\epsilon=0} \, d\theta.
\]
We have

$$\frac{\partial^2 \gamma}{\partial \epsilon^2} \bigg|_{\epsilon=0} = a \left( 2 - E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] \right) E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] (q(\theta|x) - \pi_0(\theta|x))
- aE_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] \left( \frac{q(\theta|x)}{\pi_0(\theta|x)} \right) (q(\theta|x) - \pi_0(\theta|x))
+ a^2 E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] (q(\theta|x) - \pi_0(\theta|x))^2. \tag{9}$$

Therefore,

$$\frac{\partial^2 d}{\partial \epsilon^2} \bigg|_{\epsilon=0} = a \left( E_{\pi_0(\theta|x)}^2 \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] E_{\pi_0(\theta|x)} \left[ \frac{q(\theta|x)}{\pi_0(\theta|x)} \right]^2 \right)
- E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] \frac{1}{E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right]}. \tag{9}$$

Since

$$E_{\pi_0(\theta|x)} \left[ \frac{q(\theta|x)}{\pi_0(\theta|x)} \right]^2 = E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right]^2 \frac{E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right]}{E_{\pi_0(\theta|x)}},$$
we have
\[
\left. \frac{\partial^2 d}{\partial \epsilon^2} \right|_{\epsilon=0} = a \left( E_{\pi_0(\theta|x)} \left[ \left( \frac{q(\theta)}{\pi_0(\theta)} \right)^2 \right] - E_{\pi_0(\theta|x)}^2 \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] \right) \\
= a \text{Var}_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right].
\]

\[\blacksquare\]

**Theorem 2** For the geometric contaminated class defined in (2), the local curvature of Rényi’s divergence at \(\epsilon = 0\) is
\[
C_\alpha = \left. \frac{\partial^2 d}{\partial \epsilon^2} \right|_{\epsilon=0} = a \text{Var}_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]
\]

**Proof.** Define \(\gamma\) as in (6). Having
\[
\pi_0(\theta|x) = \frac{f_\theta(x)\pi_0(\theta)}{m(x|\pi_0)}
\]
and
\[
\pi(\theta|x) = \frac{f_\theta(x)c(\epsilon)\pi_0^{1-\epsilon}(\theta)q'(\theta)(\theta)}{m(x|\pi)},
\]
makes
\[
\gamma = \frac{f_\theta(x)c(\epsilon)\pi_0^{1-\alpha}(\theta)q'(\theta)(\theta)}{m^{1-\alpha}(x|\pi_0)m^\alpha(x|\pi)}.
\]

From the proof of Theorem 1, we have
\[
\left. \frac{\partial^2 d}{\partial \epsilon^2} \right|_{\epsilon=0} = \left. \int \frac{\partial^2 \gamma}{\partial \epsilon^2} d\theta \right|_{\epsilon=0} - \left. \left( \int \frac{\partial \gamma}{\partial \epsilon} d\theta \right)^2 \right|_{\epsilon=0}.
\]

Since
\[
\frac{\partial}{\partial \epsilon} \ln \left( \frac{c(\epsilon)}{m(x|\pi)} \right) = E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]
\]
(Dey and Birmiwal, 1993) and
\[
\frac{\partial \gamma}{\partial \epsilon} = a \gamma \left( \frac{\partial}{\partial \epsilon} \ln \left( \frac{c(\epsilon)}{m(x|\pi)} \right) - E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \right),
\]
we obtain
\[
\int \frac{\partial \gamma}{\partial \epsilon} d\theta \bigg|_{\epsilon=0} = 0.
\]
Thus,
\[
\frac{\partial^2 d}{\partial \epsilon^2} \bigg|_{\epsilon=0} = \int \frac{\partial^2 \gamma}{\partial \epsilon^2} \bigg|_{\epsilon=0} d\theta = -a \pi_0(\theta|x) \left( -E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]^2 + E^2_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \right) + a^2 \pi_0(\theta|x) \left( \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) - E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \right)^2
\]
and
\[
\int \frac{\partial^2 \gamma}{\partial \epsilon^2} \bigg|_{\epsilon=0} d\theta = a(a - 1) Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right].
\]
Therefore, by (10),
\[
\frac{\partial^2 d}{\partial \epsilon^2} \bigg|_{\epsilon=0} = a Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right].
\]

5 Examples

In this section, the derived results are explained through the Bernoulli model and the location normal model. The connection with prior-data conflict is also investigated in these two examples.
Example 1 (Bernoulli Model). Suppose $x = (x_1, \ldots, x_n)$ is a sample from a Bernoulli distribution with parameter $\theta$. Then the minimal sufficient statistic of $\theta$ is $T(x) = \sum_{i=1}^{n} x_i \sim \text{Binomial}(n, \theta)$. Let the prior $\pi_0(\theta)$ be $\text{Beta}(\alpha, \beta)$. Thus, $\pi_0(\theta|x_1, \ldots, x_n)$ is

$$\text{Beta} \left( \alpha + \sum_{i=1}^{n} x_i, \beta + n - \sum_{i=1}^{n} x_i \right).$$  \hspace{1cm} (11)

Consider the class $\Gamma_\alpha$ as defined in (1), and let $q(\theta)$ be $\text{Beta}(c\alpha, c\beta)$ for $c > 0$. We have:

$$q(\theta) \pi_0(\theta) = A \theta^{(c-1)\alpha} (1 - \theta)^{(c-1)\beta},$$

where

$$A = \frac{B(\alpha, \beta)}{B(c\alpha, c\beta)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)\Gamma(c\alpha)\Gamma(c\beta)}. $$

Since

$$E_{\pi_0|T}(\theta^n(1 - \theta)^v) = \frac{\Gamma(\alpha + \beta + n)\Gamma(t + \alpha + u)\Gamma(n + \beta - t + v)}{\Gamma(t + \alpha)\Gamma(n + \beta - t)\Gamma(\alpha + \beta + u + v)},$$

we have

$$\text{Var}_{\pi_0|T}(\frac{q(\theta)}{\pi_0(\theta)}) = E_{\pi_0|T} [A^2 \theta^{2\alpha(c-1)} (1 - \theta)^{2\beta(c-1)}] - E_{\pi_0|T} [A \theta^{(c-1)\alpha} (1 - \theta)^{(c-1)\beta}]$$

$$= A^2 \left[ \frac{\Gamma(\alpha + \beta + n)\Gamma(t + 2c\alpha - \alpha)\Gamma(n - \beta - t + 2c\beta)}{\Gamma(t + \alpha)\Gamma(n + \beta - t)\Gamma(2c\alpha + 2c\beta + n - \alpha - \beta)} - \left( \frac{\Gamma(t + c\alpha)\Gamma(n + c\beta - t)\Gamma(\alpha + \beta + n)}{\Gamma(n + c\alpha + c\beta)\Gamma(t + \alpha)\Gamma(n + \beta - t)} \right)^2 \right]. \hspace{1cm} (12)$$

Clearly, when $c = 1$, the curvature is 0.
An important case arises when $\alpha = \beta = 1$. In this case,

$$\text{Var}_{\pi_0(\theta|\mathcal{T})} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] = \frac{(\Gamma(2c))^2 \Gamma(n + 2)\Gamma(t + 2c - 1)\Gamma(n + 2c - 1 - t)}{(\Gamma(c))^4 \Gamma(t + 1)\Gamma(n + 1 - t)\Gamma(4c + n - 2)} - \left[ \frac{\Gamma(t + c)\Gamma(n + c - t)\Gamma(2 + n)}{\Gamma(n + 2c)\Gamma(t + 1)\Gamma(n + 1 - t)} \right]^2.$$ 

More details about this case are explored in Table 1.

Next we consider the class $\Gamma_g$ defined in (2). Let $q(\theta)$ be defined as before. We have

$$\ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) = \ln(A) + (c - 1)\alpha \ln(\theta) + (c - 1)\beta \ln(1 - \theta).$$

Note that for a random variable $y \sim \text{Beta}(\alpha, \beta)$, we have

$$E[(\ln(y))^k(\ln(1 - y))^l] = \frac{1}{B(\alpha, \beta)} \int_0^1 (\ln(y))^k(\ln(1 - y))^l y^{\alpha - 1}(1 - y)^{\beta - 1} dy$$

For $k = 1, l = 0$,

$$E[\ln(y)] = \psi_0(\alpha) - \psi_0(\alpha + \beta)$$

and for $k = 2, l = 0$,

$$E \left[ (\ln(y))^2 \right] = \psi_1(\alpha) - \psi_1(\alpha + \beta),$$
where

\[ \psi_n(y) = \frac{d^{n+1}}{dy^{n+1}} \ln(\Gamma(y)) = \frac{d^n \psi_0(y)}{dx^n} \]

is the polygamma function with order \( n \), i.e. the \( n \)th derivative of the digamma function \( \psi_0(y) \).

For \( k = 0, l = 1 \),

\[ E[\ln(y)] = \psi_0(\beta) - \psi_0(\alpha + \beta). \]

and for \( k = 0, l = 2 \),

\[ E \left[ (\ln(y))^2 \right] = \psi_1(\beta) - \psi_1(\alpha + \beta). \]

For \( k = 1, l = 1 \),

\[
E[\ln(y) \ln(1 - y)] = \frac{1}{B(\alpha, \beta)} \frac{\partial^2 B(\alpha, \beta)}{\partial \alpha \partial \beta} \\
= \frac{1}{B(\alpha, \beta)} \frac{\partial}{\partial \alpha} (B(\alpha, \beta)\psi_0(\alpha) - B(\alpha, \beta)\psi_0(\alpha + \beta)) \\
= \psi_0(\alpha)\psi_0(\beta) - \psi_0(\beta)\psi_0(\alpha + \beta) - \psi_1(\alpha + \beta).
\]

Now,

\[
\text{Var}_{\pi_0(\theta|T)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] = E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right)^2 \right] \\
- \left( E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \right)^2. \tag{13}
\]
Therefore, in (13), we have

\[
E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] = \ln(A) + \alpha(c - 1) [\psi_0(\alpha + t) - \psi_0(\alpha + \beta + n)]
+ \beta(c - 1) [\psi_0(\beta + n - t) - \psi_0(\alpha + \beta + n)] \text{Fsuing}
\]

and

\[
E_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]^2 = (\ln(A))^2 + 2\alpha(c - 1) \ln(A) \left[ \psi_0(\alpha + t)
- \psi_0(\alpha + \beta + n) \right] + 2\beta(c - 1) \ln(A) \left[ \psi_0(\beta + n - t) - \psi_0(\alpha + \beta + n) \right]
+ \alpha^2(c - 1)^2 [\psi_1(\alpha + t) - \psi_1(\alpha + \beta + n)]
+ \beta^2(c - 1)^2 [\psi_1(\beta + n - t) - \psi_1(\alpha + \beta + n)]
+ 2\alpha\beta(c - 1)^2 \left( (\psi_0(\alpha + t) - \psi_0(\alpha + \beta + n))
- \psi_1(\alpha + \beta + n) \right).
\]

Now we consider checking for prior-data conflict. This requires the computation of the tail probability in (4), where

\[
m_T(t) = \int_0^1 \binom{n}{t} \theta^t (1 - \theta)^{n-t} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1} d\theta
= \binom{n}{t} \frac{\Gamma(\alpha + \beta) \Gamma(t + \alpha) \Gamma(n - t + \beta)}{\Gamma(n + \alpha + \beta) \Gamma(\alpha) \Gamma(\beta)}.
\]

Hence,

\[
M_T \left( \binom{n}{t} \frac{\Gamma(\alpha + \beta) \Gamma(t + \alpha) \Gamma(n - t + \beta)}{\Gamma(n + \alpha + \beta) \Gamma(\alpha) \Gamma(\beta)} \right) \leq \binom{n}{t} \frac{\Gamma(t_0 + \alpha) \Gamma(n - t_0 + \beta)}{\Gamma(n + \alpha + \beta) \Gamma(\alpha) \Gamma(\beta)}.
\]
This simplifies to
\[
M_T \left( \frac{\Gamma(t + \alpha)\Gamma(n - t + \beta)}{\Gamma(t + 1)\Gamma(n - t + 1)} \right) \leq \frac{\Gamma(t_0 + \alpha)\Gamma(n - t_0 + \beta)}{\Gamma(t_0 + \alpha)\Gamma(n - t_0 + 1)}.
\]

(14)

Note that, when $\alpha = \beta = 1$, (14) = 1. Hence, there is no prior-data conflict. This makes sense, because when the sampling model is correct then there can be no conflict between the data and a noninformative prior. On other hand, selecting always a noninformative prior in an attempt to avoid prior-data conflict should not be considered as this will induce bias into the analysis (Evans, 2015).

Now we consider a numerical example. Two samples of size $n = 20$ from $Bernoulli(0.25)$ and $Bernoulli(0.9)$ are generated. The obtained samples are, respectively, $x_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1)$ (with $T(x_0) = t_0 = 3$) and $x_0 = (0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1)$ (with $T(x_0) = t_0 = 17$). We consider several values of $\alpha, \beta$, and $c$. To compute the curvature, rather than working with the exact formulas, Monte Carlo approach has been used. First, we sample $\theta^{(s)}, s = 1, \ldots, 10^6$, from the posterior distribution (11). Then we compute the variance of $\frac{q(\theta^{(s)})}{\pi_0(\theta^{(s)})}$ and the variance of $\ln \left( \frac{q(\theta^{(s)})}{\pi_0(\theta^{(s)})} \right)$. This can be implemented straightforwardly in R. The values of (12), (13) and (14) are summarized in Table 1 and table 2. For instance, using the first sample with $\alpha = 5, \beta = 10$, we have (14)=0.9175. Thus there is no prior data conflict. On the other hand, for the second sample, we have (14)= 0.0013. Hence, there is a prior data conflict. Observe that, for $Beta(\alpha = 5, \beta = 10)$, from Figure 1, the prior in this case puts most of its mass on values of $\theta$ between 0 and 0.6. That is, one should expect to have a conflict with the second sample but not with the first sample.

It follows clearly from Table 1 and Table 2 that, for the same values of $\alpha, \beta$ and $c$, when there is there is no prior-data conflict, the value of the curvature is smaller than that when there is a prior-data conflict and vice versa. Thus,
Measuring Robustness Using Rényi’s Divergence

| α | β | c  | $\text{Var}_{\pi_0(\theta|T)}$ | $\frac{q(\theta)}{\pi_0(\theta)}$ | $\text{Var}_{\pi_0(\theta|T)}$ | $\ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right)$ | (14) |
|---|---|----|--------------------------------|--------------------------------|--------------------------------|--------------------------------|------|
| 5 | 10| 0.5| 0.4976                         | 0.1450                         |                                 |                                 | 0.9175 (no conflict) |
|   |   | 0.7| 0.0983                         | 0.0522                         |                                 |                                 |                  |
|   |   | 0.9| 0.0070                         | 0.0058                         |                                 |                                 |                  |
|   |   | 1  | 0                              | 0                              |                                 |                                 |                  |
|   |   | 1.1| 0.0049                         | 0.0058                         |                                 |                                 |                  |
|   |   | 1.3| 0.0328                         | 0.0522                         |                                 |                                 |                  |
|   |   | 1.5| 0.0709                         | 0.1450                         |                                 |                                 |                  |
|   |   | 1.7| 0.1117                         | 0.2841                         |                                 |                                 |                  |
|   |   | 2.0| 0.1720                         | 0.5799                         |                                 |                                 |                  |
|   |   | 3.0| 0.3439                         | 2.3196                         |                                 |                                 |                  |
|   |   | 5.0| 0.5880                         | 9.2784                         |                                 |                                 |                  |
| 0.01| 20| 0.5| 0.7720                         | 0.2064                         |                                 |                                 | 0.0007 (conflict) |
|   |   | 0.7| 0.1532                         | 0.0743                         |                                 |                                 |                  |
|   |   | 0.9| 0.0103                         | 0.0083                         |                                 |                                 |                  |
|   |   | 1  | 0                              | 0                              |                                 |                                 |                  |
|   |   | 1.1| 0.0067                         | 0.0083                         |                                 |                                 |                  |
|   |   | 1.3| 0.0405                         | 0.0743                         |                                 |                                 |                  |
|   |   | 1.5| 0.0791                         | 0.2064                         |                                 |                                 |                  |
|   |   | 1.7| 0.1122                         | 0.4046                         |                                 |                                 |                  |
|   |   | 2.0| 0.1480                         | 0.8257                         |                                 |                                 |                  |
|   |   | 3.0| 0.1861                         | 3.30300                        |                                 |                                 |                  |
|   |   | 5.0| 0.1590                         | 13.2100                        |                                 |                                 |                  |
| 1  | 1 | 0.5| 0.0397                         | 0.0387                         |                                 |                                 | 0.8571 (no conflict) |
|   |   | 0.7| 0.0152                         | 0.0139                         |                                 |                                 |                  |
|   |   | 0.9| 0.0016                         | 0.0015                         |                                 |                                 |                  |
|   |   | 1.0| 0                              | 0                              |                                 |                                 |                  |
|   |   | 1.1| 0.0015                         | 0.0015                         |                                 |                                 |                  |
|   |   | 1.3| 0.0118                         | 0.0139                         |                                 |                                 |                  |
|   |   | 1.5| 0.0288                         | 0.0387                         |                                 |                                 |                  |
|   |   | 1.7| 0.0494                         | 0.0758                         |                                 |                                 |                  |
|   |   | 2  | 0.0822                         | 0.1547                         |                                 |                                 |                  |
|   |   | 3  | 0.1707                         | 0.6188                         |                                 |                                 |                  |
|   |   | 5  | 0.2347                         | 2.4752                         |                                 |                                 |                  |

Table 1: Values of the curvature for the two classes (1) and (2) for the sample $x_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1)$. 
Table 2: Values of the curvature for the two classes (1) and (2) for the sample \( x_0 = (0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1). \)

| \( \alpha \) | \( \beta \) | \( c \) | \( \text{Var}_{\tau(\theta|T)} \) | \( \frac{q(\theta)}{\pi_0(\theta)} \) | \( \text{Var}_{\tau(\theta|T)} \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \) | (14) |
|---|---|---|---|---|---|---|
| 5 | 10 | 0.5 | 110.5019 | 0.6610 | 0.0013 (conflict) | |
| | | | 0.7 | 2.0218 | 0.2380 | |
| | | | 0.9 | 0.0519 | 0.0264 | |
| | | | 1 | 0 | 0 | |
| | | | 1.1 | 0.0145 | 0.0264 | |
| | | | 1.3 | 0.0478 | 0.2380 | |
| | | | 1.5 | 0.0583 | 0.6610 | |
| | | | 1.7 | 0.0574 | 1.2955 | |
| | | | 2.0 | 0.0501 | 2.6440 | |
| | | | 3.0 | 0.0306 | 10.5759 | |
| | | | 5.0 | 0.0183 | 42.3034 | |
| 10 | 0.5 | 0.7 | 0.0174 | 0.0164 | 0.9263 (no conflict) | |
| | | | 0.9 | 0.0019 | 0.0018 | |
| | | | 1 | 0 | 0 | |
| | | | 1.1 | 0.0018 | 0.00182 | |
| | | | 1.3 | 0.0146 | 0.0164 | |
| | | | 1.5 | 0.0374 | 0.0454 | |
| | | | 1.7 | 0.0673 | 0.0890 | |
| | | | 2.0 | 0.1214 | 0.1817 | |
| | | | 3.0 | 0.3330 | 0.7268 | |
| | | | 5.0 | 0.7474 | 2.9070 | |
| 1 | 1 | 0.5 | 0.0397 | 0.0387 | 0.1905 (no conflict) | |
| | | | 0.7 | 0.0151 | 0.0139 | |
| | | | 0.9 | 0.0016 | 0.0015 | |
| | | | 1 | 0 | 0 | |
| | | | 1.1 | 0.0015 | 0.0015 | |
| | | | 1.3 | 0.0118 | 0.0139 | |
| | | | 1.5 | 0.0288 | 0.0387 | |
| | | | 1.7 | 0.0494 | 0.0758 | |
| | | | 2.0 | 0.0822 | 0.1548 | |
| | | | 3.0 | 0.1707 | 0.6190 | |
| | | | 5.0 | 0.2348 | 2.4761 | |
it looks sensible to conclude that robustness and prior-data conflict are directly connected. For example, for $\alpha = 5, \beta = 10$ and $c = 0.5$, we have, in Table 1, $Var_{\pi_0(\theta|T)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] = 0.4976$ and $Var_{\pi_0(\theta|T)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] = 0.1450$. On the other hand, in Table 2, the counterpart values equal 110.5019 and 0.6610, respectively. Remarkably, for the case when $\alpha = \beta = 1$ (no prior-data conflict despite of the data), the curvature values (for both classes of priors (1) and (2)) are prominently small and very close. For example, for $c = 0.5$, the values of $Var_{\pi_0(\theta|T)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right]$ and $Var_{\pi_0(\theta|T)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]$ in Table 1 and Table 2 are, respectively, 0.0397 and 0.0387. Furthermore, if we fix the data (i.e., consider only Table 1 or only Table 2) but change the parameters of the prior and $c$, then the values $Var_{\pi_0(\theta|T)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right]$ and $Var_{\pi_0(\theta|T)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right]$ are smaller when there is no prior-data conflict.

**Example 2 (Location normal model).** Suppose that $x = (x_1, x_2, \ldots, x_n)$ is a sample from $N(\theta, 1)$ distribution with $\theta \in \mathbb{R}^1$. Let $N(\theta_0, \sigma_0^2)$ be the prior.
distribution of $\theta$. Then

$$\pi(\theta|x) \sim N(\mu_x, \sigma_x^2).$$

$$\mu_x = \left(\frac{\theta_0}{\sigma_0^2} + n\bar{x}\right)\left(\frac{1}{\sigma_0^2} + n\right)^{-1} \text{ and } \sigma_x^2 = \left(\frac{1}{\sigma_0^2} + n\right)^{-1}.$$ 

Let $q(\theta) \sim N(c\theta_0, 1), c > 0$. We have

$$\frac{q(\theta)}{\pi_0(\theta)} = \exp\left\{\frac{\theta_0\theta(c-1) + 0.5\theta_0^2(1-c^2)}{\sigma_0^2}\right\}.$$

Therefore, for the class (1), we have

$$\text{Var}_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] = E_{\pi_0(\theta|x)} \left[ \left( \frac{q(\theta)}{\pi_0(\theta)} \right)^2 \right] - \left( E_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] \right)^2$$

$$= \exp\left\{\frac{\theta_0^2(1-c^2)}{\sigma_0^2}\right\} \left[ M_{\pi_0(\theta|x)} \left( \frac{2\theta_0(c-1)}{\sigma_0^2} \right) \right] - \left( M_{\pi_0(\theta|x)} \left( \frac{\theta_0(c-1)}{\sigma_0^2} \right) \right)^2,$$

where $M_{\pi_0(\theta|x)}(t)$ is the moment generating function with respect to the density $\pi_0(\theta|x)$. Thus,

$$\text{Var}_{\pi_0(\theta|x)} \left[ \frac{q(\theta)}{\pi_0(\theta)} \right] = \exp\left\{\frac{\theta_0^2(1-c^2)}{\sigma_0^2}\right\} \left[ \exp\left\{\frac{2\theta_0(c-1)\mu_x}{\sigma_0^2} + \frac{2\theta_0^2(c-1)^2\sigma_x^2}{\sigma_0^4}\right\} - \exp\left\{\frac{2\theta_0(c-1)\mu_x}{\sigma_0^2} + \frac{\theta_0^2(c-1)^2\sigma_x^2}{\sigma_0^4}\right\} \right].$$

On the other hand, for the geometric contaminated class, we have

$$\ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) = \frac{1}{2} \theta_0^2(1-c^2) + \theta_0\theta(c-1).$$
Thus, by (15), we get

\[
Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] = \frac{\theta_0^2(c - 1)^2}{\sigma_0^4} Var_{\pi_0(\theta|x)} [\theta] = \frac{\theta_0^2(c - 1)^2}{\sigma_0^4} \sigma_x^2
\]

\[
= \frac{\theta_0^2(c - 1)^2}{\sigma_0^4} \left( \frac{1}{\sigma_0^2 + n} \right)^{-1}.
\] (16)

Interestingly, from (16), \( Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \) depends on the sample only through its size \( n \). As \( n \to \infty \) or \( \sigma_0 \to \infty \), \( Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \to 0 \), which indicates robustness. Also, as \( \theta \to \infty \), \( Var_{\pi_0(\theta|x)} \left[ \ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \right] \to \infty \) and no robustness will be found.

Now we study the prior-data conflict. A sufficient statistic is \( T(x) = \bar{x} \sim N(\theta, 1/n) \). So the prior predictive distribution is \( N(\theta_0, 1 + 1/n) \) (Evans and Moshonov, 2006). Hence, the prior predictive probability

\[
M_T(m_T(\bar{x}) \leq m_T(\bar{x}_0)) = 2 \left( 1 - \Phi(|\bar{x}_0 - \theta_0|/(\sigma_0^2 + 1/n)^{1/2}) \right),
\] (17)

where \( \Phi \) is cumulative distribution function of the standard normal distribution. This shows that when \( \bar{x}_0 \) lies in the tails of its prior distribution, we have evidence of a prior-data conflict. Note that, when \( \sigma_0 \to \infty \), (17) converges to 1 and no evidence of prior-data conflict will be found. Also, as \( \sigma_0 \to 0 \), (17) converges to 0 for large \( n \), unless \( \theta_0 \) is indeed the true value. Thus, for large and small values of \( \sigma_0 \), robustness and no prior-data conflict are grasped.

Now we consider a numerical example. Generating samples of size \( n = 20 \) from \( N(0, 1) \) distribution and \( N(4, 1) \) distribution, we obtain

\[
x_0 = (0.23, 1.188, -0.78, 1.28, -1.90, -0.51, 1.23, 0.17, -1.22, -0.73, -0.71, -0.29, 0.81, 1.72, 2.08, 0.84, -0.049, -0.12, -1.04, 0.79)
\]

(with \( t_0 = \bar{x}_0 = 0.1494 \) and)
\[x_0 = (4.27, 3.08, 4.04, 2.73, 4.01, 4.87, 3.02, 4.11, 4.14, 3.80, 4.89, 4.02, 4.07, 5.13, 5.00, 6.52, 2.98, 5.35, 3.26, 3.74)\]

(with \(t_0 = \bar{x}_0 = 4.1515\)), respectively. Table 3 and Table 4 below report the values of the curvature for different values of \(\theta_0, \sigma_0\) and \(c\).

As in Example 1, it follows clearly from Table 1 and Table 2 that, for the same values of \(\theta_0, \sigma_0\) and \(c\), when there is no prior-data conflict, the value of the curvature is smaller than when there is a prior-data conflict and vice versa. For example, in Table 3, for \(\theta_0 = 0.5, \sigma_0 = 0.1\) and \(c = 1.3\), we have
\[\text{Var}_{\pi_0(\theta|T)} \left[ q(\theta) / \pi_0(\theta) \right] = 0.0333\]
and
\[\text{Var}_{\pi_0(\theta|T)} \left[ \ln \left( q(\theta) / \pi_0(\theta) \right) \right] = 0.0750.\]
On the other hand, the equivalent values in Table 3 are 99.5308 and 0.0750, respectively. As pointed out earlier, from (15), the value \(\text{Var}_{\pi_0(\theta|T)} \left[ \ln \left( q(\theta) / \pi_0(\theta) \right) \right]\) is independent from the sample (only depends on \(n\)). This explains the identical values in Table 3 and Table 4. Additionally, if we fix the data (i.e., consider only Table 3 or only Table 4) but change the parameters of the prior and \(c\), then the values of \(\text{Var}_{\pi_0(\theta|T)} \left[ q(\theta) / \pi_0(\theta) \right]\) and \(\text{Var}_{\pi_0(\theta|T)} \left[ \ln \left( q(\theta) / \pi_0(\theta) \right) \right]\) are distinctly larger in the presence of prior-data conflict. For instance, in Table 3, compare the case when \(\theta_0 = 0.5, \sigma_0 = 1, c = 0.9\) with the case when \(\theta_0 = 5, \sigma_0 = 0.5, c = 0.9\).

6 Conclusions

Measuring Bayesian robustness of two classes of contaminated priors is studied. The approach is based on computing the curvature of Rényi’s divergence between posterior distributions. The method does not require specifying values for \(\epsilon\) and its computation is straightforward. In addition, a solid relationship between robustness and prior data conflict has been initiated.
$\theta_0 \quad \sigma^2_0 \quad c \quad Var_{\pi_0(\theta|T)} \quad \frac{q(\theta)}{\pi_0(\theta)} \quad Var_{\pi_0(\theta|T)} \quad ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right) \quad (17)$

| $\theta_0$ | $\sigma^2_0$ | $c$ | $\text{Var}_{\pi_0(\theta|T)}$ | $\frac{q(\theta)}{\pi_0(\theta)}$ | $\text{Var}_{\pi_0(\theta|T)}$ | $\ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right)$ |  |
|-----|-----|-----|-----------------|-----------------|-----------------|-----------------|-----|
| 0.5 | 0.1 | 0.5 | 0.4912          | 0.2083          | 0.6364 (no conflict) |  |
| 0.7 | 0.1351 | 0.0750 |  |
| 0.9 | 0.0104 | 0.0083 |  |
| 1 | 0 | 0 |  |
| 1.1 | 0.0065 | 0.0083 |  |
| 1.3 | 0.0333 | 0.0750 |  |
| 1.5 | 0.0475 | 0.2083 |  |
| 1.7 | 0.0434 | 0.4083 |  |
| 2 | 0.0237 | 0.8333 |  |
| 3 | 0.0003 | 3.3333 |  |
| 5 | 0.0000 | 13.3333 |  |
| 0.5 | 1 | 0.5 | 0.0033 | 0.0030 | 0.7323 (no conflict) |  |
| 0.7 | 0.0012 | 0.0011 |  |
| 0.9 | 0.0001 | 0.0001 |  |
| 1 | 0 | 0 |  |
| 1.1 | 0.0001 | 0.0001 |  |
| 1.3 | 0.0009 | 0.0011 |  |
| 1.5 | 0.0024 | 0.0030 |  |
| 1.7 | 0.0041 | 0.0058 |  |
| 2 | 0.0068 | 0.0119 |  |
| 3 | 0.0097 | 0.0476 |  |
| 5 | 0.0012 | 0.1905 |  |
| 0.5 | 5 | 0.5 | 0.0001 | 0.0001 | 0.8760 (no conflict) |  |
| 0.7 | 0.0000 | 0.0000 |  |
| 0.9 | 0.0000 | 0.0000 |  |
| 1 | 0 | 0 |  |
| 1.1 | 0.0000 | 0.0000 |  |
| 1.3 | 0.0000 | 0.0000 |  |
| 1.5 | 0.0001 | 0.0001 |  |
| 1.7 | 0.0002 | 0.0002 |  |
| 2 | 0.0004 | 0.0005 |  |
| 3 | 0.0014 | 0.0020 |  |
| 5 | 0.0027 | 0.0079 |  |
| 5 | 0.5 | 0.5 | $3.4740 \times 10^{14}$ | 1.136 | 0.0000 (conflict) |  |
| 0.7 | $2.6140 \times 10^9$ | 0.4091 |  |
| 0.9 | 199.6000 | 0.0455 |  |
| 1 | 0 | 0 |  |
| 1.1 | 0.0000 | 0.0455 |  |
| 1.3 | 0.0000 | 0.4091 |  |
| 1.5 | 0.0000 | 1.1360 |  |
| 1.7 | 0.0000 | 2.2270 |  |
| 2 | 0.0000 | 4.5450 |  |
| 3 | 0.0000 | 18.1800 |  |
| 5 | 0.0000 | 72.7300 |  |

Table 3: Values of the curvature for the classes (1) and (2), where the sample $x_0 = (0.23, 1.188, -0.78, 1.28, -1.90, -0.51, 1.23, 0.17, -1.22, -0.73, -0.71, -0.29, 0.81, 1.72, 2.08, 0.84, -0.049, -0.12, -1.04, 0.79)$. 

Table 4: Values of the curvature for the classes (1) and (2), where the sample $x_0 = (4.27, 3.08, 4.04, 2.73, 4.01, 4.87, 3.02, 4.11, 4.14, 3.80, 4.89, 4.02, 4.07, 5.13, 5.00, 6.52, 2.98, 5.35, 3.26, 3.74)$. 

| $\theta_0$ | $\sigma_0^2$ | $c$ | $\text{Var}_{\pi_0}(\theta|T)$ | $\frac{q(\theta)}{\pi_0(\theta)}$ | $\text{Var}_{\pi_0}(\theta|T)$ | $\ln \left( \frac{q(\theta)}{\pi_0(\theta)} \right)$ | (17) |
|-----------|-------------|-----|-------------------------------|-----------------------------|-------------------------------|--------------------------------|-----|
| 0.5       | 0.1         | 0.5 | 0.0000                        | 0.2083                      | 0.0000                        | 0.0000 (conflict)            |     |
| 0.7       |             |     | 0.0000                        | 0.0750                      |                               |                               |     |
| 0.9       |             |     | 0.0007                        | 0.0083                      |                               |                               |     |
| 1         |             |     | 0                           | 0                           |                               |                               |     |
| 1.1       |             |     | 0.0939                       | 0.0083                      |                               |                               |     |
| 1.3       |             |     | 99.5308                      | 0.0750                      |                               |                               |     |
| 1.5       |             |     | 29506.2800                   | 0.2083                      |                               |                               |     |
| 1.7       |             |     | 5604608                      | 0.4083                      |                               |                               |     |
| 2         |             |     | 9175303734                   | 0.8333                      |                               |                               |     |
| 3         |             |     | $4.7965 \times 10^{19}$      | 3.3333                      |                               |                               |     |
| 5         |             |     | $3.1485 \times 10^{36}$      | 13.3333                     |                               |                               |     |
| 0.5       | 1           | 0.5 | 0.0005                        | 0.0030                      | 0.0005                        | 0.0005 (conflict)            |     |
| 0.7       |             |     | 0.0004                        | 0.0011                      |                               |                               |     |
| 0.9       |             |     | 0.0000                        | 0.0001                      |                               |                               |     |
| 1         |             |     | 0                           | 0                           |                               |                               |     |
| 1.1       |             |     | 0.0002                       | 0.0001                      |                               |                               |     |
| 1.3       |             |     | 0.0030                       | 0.0011                      |                               |                               |     |
| 1.5       |             |     | 0.0160                       | 0.0030                      |                               |                               |     |
| 1.7       |             |     | 0.0594                       | 0.0058                      |                               |                               |     |
| 2         |             |     | 0.3056                       | 0.0119                      |                               |                               |     |
| 3         |             |     | 19.7319                      | 0.0476                      |                               |                               |     |
| 5         |             |     | 5112.6530                    | 0.1905                      |                               |                               |     |
| 0.5       | 5           | 0.5 | 0.0000                        | 0.0001                      |                               | 0.1042 (?)                    |     |
| 0.7       |             |     | 0.0000                        | 0.0000                      |                               |                               |     |
| 0.9       |             |     | 0.0000                        | 0.0000                      |                               |                               |     |
| 1         |             |     | 0                           | 0                           |                               |                               |     |
| 1.1       |             |     | 0.0000                        | 0.0000                      |                               |                               |     |
| 1.3       |             |     | 0.0000                        | 0.0000                      |                               |                               |     |
| 1.5       |             |     | 0.0002                       | 0.0001                      |                               |                               |     |
| 1.7       |             |     | 0.0004                       | 0.0002                      |                               |                               |     |
| 2         |             |     | 0.0010                       | 0.0005                      |                               |                               |     |
| 3         |             |     | 0.0069                       | 0.0020                      |                               |                               |     |
| 5         |             |     | 0.0650                       | 0.0079                      |                               |                               |     |
| 4.5       | 0.5         | 0.5 | 0.0026                        | 0.9205                      |                               | 0.6384 (no conflict)         |     |
| 0.7       |             |     | 0.0791                        | 0.3314                      |                               |                               |     |
| 0.9       |             |     | 0.0459                        | 0.0368                      |                               |                               |     |
| 1         |             |     | 0                           | 0                           |                               |                               |     |
| 1.1       |             |     | 0.0147                        | 0.0368                      |                               |                               |     |
| 1.3       |             |     | 0.0026                       | 0.3314                      |                               |                               |     |
| 1.5       |             |     | 0.0000                        | 0.9205                      |                               |                               |     |
| 1.7       |             |     | 0.0000                        | 1.8040                      |                               |                               |     |
| 2         |             |     | 0.0000                        | 3.6820                      |                               |                               |     |
| 3         |             |     | 0.0000                        | 14.7300                     |                               |                               |     |
| 5         |             |     | 0.0000                        | 58.9100                     |                               |                               |     |
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