Strong coupling in brane-induced gravity in five dimensions

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Abstract

Brane-induced gravity in five dimensions (Dvali–Gabadadze–Porrati model) exhibits modification of gravity at ultra-large distances, $r \gg r_c = M_5^2/M^3$ where $M$ is the five-dimensional gravity scale. This makes the model potentially interesting for explaining the observed acceleration of the Universe. We argue, however, that it has an intrinsic intermediate energy scale $(M^9/M_5^4)^{1/5}$. At higher energies, the model is strongly coupled. For $r_c$ of order of the present Hubble size, the strong coupling regime occurs at distances below tens of metres.

1 Introduction

Whether there exists a consistent and phenomenologically acceptable theory of gravity in which gravitational interactions get modified at ultra-large distances is an interesting problem. Indeed, the large-scale modification of gravity might become an appealing way of explaining the accelerated expansion of the Universe at the present epoch. At first sight, modification of gravity at ultra-large distances may naturally occur in theories with large or infinite extra dimensions. However, several models of this sort [1, 2, 3], in which linearized gravity experienced by brane matter has purely tensor structure, have been shown to have ghosts [4, 5]. Because of the van Dam–Veltman–Zakharov phenomenon, an alternative is that gravity linearized about flat
background has a scalar component. This feature is similar to 4d gravity with massive graviton, and it is indeed inherent in the five-dimensional induced gravity model of Dvali, Gabadadze and Porrati (DGP) [6], as well as in other no-ghost brane-world models [7].

Four-dimensional gravity with massive graviton has been shown to possess a strong-interaction energy scale, which is intermediate between the Planck mass and graviton mass [8]. This property, and other arguments [9] suggest that similar strong interaction scale may be present in no-ghost brane-world models with gravity modified at ultra-large scales. In this note we address this issue in the framework of DGP model, whose action is a sum

$$S_{tot} = S_{bulk} + S_{brane}$$

(1)

where the 5d bulk piece is

$$S_{bulk} = \frac{M^3}{2\pi^2} \int d^5 X \sqrt{g^{(5)}} R^{(5)} + \text{total divergence}$$

(2)

and the brane term is

$$S_{brane} = \frac{M_P^2}{4\pi} \int_{brane} d^4 x \sqrt{g^{(4)}} R^{(4)}$$

(3)

At distances

$$M^{-1} \ll r \ll r_c$$

(4)

where

$$r_c = \frac{M_P^2}{M^2}$$

(5)

linearized gravity with sources on the brane is four-dimensional, while it becomes five-dimensional at ultra-large distances, $$r \gg r_c$$.

In this paper we argue that DGP model has an inherent energy scale

$$E_{strong} = \left( \frac{M^9}{M_P^4} \right)^\frac{1}{2}$$

(6)

At energies above this scale, scalar (in 4d sense) degrees of freedom become strongly interacting, so the model does not admit classical treatment. It is worth noting that these degrees of freedom do not decouple. This makes DGP model an unlikely candidate for explaining the acceleration of the Universe:
if \( r_c \) is of the order of the present Hubble size, the theory is strongly coupled at distances below \( E_{\text{strong}}^{-1} \sim (r_c^3 P_L^2)^{1/5} \sim 30 \text{ m} \).

This issue has been addressed independently by Luty, Porrati and Rattazzi [10]. Their conclusion is similar in spirit, but not in detail: their energy scale of strong interaction is \( M^2 / M_{Pl} \), which is lower than our scale (6). The origin of this disagreement is unclear at the moment.

This paper is organized as follows. To get an idea of “large” degrees of freedom, we study DGP model in Section 2 at linearized level (quadratic action). We first find that the full propagator in de Donder–Fock gauge has large terms with scalar structure (in 4d sense). These terms are pure gauge in the bulk but not on the brane. We then discuss the quadratic action in Gaussian normal gauge, and introduce a change of variables after which the degrees of freedom decouple into tensor part, suppressed near the brane, and unsuppressed scalar (and vector) part. This change of variables induces large brane bending term into the metric, which is again pure gauge in the bulk but not on the brane. This term is enhanced by \( M_{Pl}^2 \), signalizing strong interaction between the scalar modes. In Section 3 we proceed to study the action at cubic order to see that naive power counting does not work, due to cancellations, but there remain unavoidable cubic terms enhanced by \( M_{Pl}^2 \). These are precisely the terms that make the theory strongly coupled at the energy scale (6). We conclude in Section 4.

2 Linearized theory

Let us begin with the linearized theory about flat background in de Donder–Fock gauge. It is straightforward to calculate the full propagator in this gauge. Let \( D_5(p; y, y') \) denote the free 5-dimensional propagator in mixed momentum-coordinate representation,

\[
D_5 = \frac{e^{-p|y-y'|}}{p}
\]

(7)

Hereafter \( X^A = (x^\mu, y) \), \( A = 0, 1, 2, 3, 5 \) and \( p \) is the four-momentum (we work with Euclidean version of DGP model). Let us denote

\[
D_0 = D_0(p, y) \equiv D_5(p; y, 0) = \frac{e^{-p|y|}}{p}
\]

(8)
and

$$D'_0 = D'_0(p, y') \equiv D_5(p; 0, y') = D_0(p, y')$$  \hspace{1cm} (9)$$

We also use the notation

$$D_{00} = D_{00}(p) \equiv D_5(p; 0, 0) = \frac{1}{p^2}$$  \hspace{1cm} (10)$$

Then the graviton propagator in DGP model in de Donder–Fock gauge has the following non-zero components

$$D_{55}^{\lambda\lambda}(p, y, y') = \frac{1}{M^2} \frac{2}{3} D_5 + \frac{M^2_{Pl}}{M^3} \frac{1}{p^2} D_0 D'_0$$  \hspace{1cm} (11)$$

$$D_{\mu5\nu}^{\lambda} = \frac{1}{M^2} D_5 \eta_{\mu}^\nu$$  \hspace{1cm} (12)$$

and

$$G_{\mu\lambda}(y = y' = 0) = \frac{D_{00}}{M^3 + M^2_{Pl}p^2 D_{00}} \left[ \frac{1}{2} (\eta^\nu_{\rho} \eta_{\mu\lambda} + \eta_{\kappa}^\nu \eta_{\mu}^\rho) - \frac{1}{3} \eta_{\mu}^\nu \eta_{\lambda}^\rho \right]$$

$$+ \frac{M^2_{Pl}}{M^3} \frac{1}{p^2} \frac{1}{M^3 + M^2_{Pl}p^2 D_{00}} \left[ -p^2 \left( \frac{1}{2} \eta^\nu_{\rho} \eta_{\mu\lambda} + \frac{1}{2} \eta_{\kappa}^\nu \eta_{\mu}^\rho - \frac{1}{3} \eta_{\mu}^\nu \eta_{\lambda}^\rho \right) + \frac{1}{2} (p_{\mu} p^\rho \eta_{\lambda}^\nu + p^\nu p^\rho \eta_{\mu\lambda} + (\lambda \leftrightarrow \rho)) 

- \frac{1}{3} p_{\mu} p^\rho \eta_{\lambda}^\nu - \frac{1}{3} \eta_{\mu}^\nu \eta_{\lambda}^\rho - \frac{2}{3} \frac{p_{\mu} p^\nu p^\rho p_{\lambda}}{p^2} \right] + \frac{M^2_{Pl}}{M^6} \frac{2 p_{\mu} p^\nu p^\rho p_{\lambda}}{p^2} \cdot D_0 D'_0$$  \hspace{1cm} (13)$$

To make contact with Ref. [6], one notices that the brane-to-brane propagator may be written as follows,

$$G_{\mu\lambda}^{y = y' = 0} = \frac{D_{00}}{M^3 + M^2_{Pl}p^2 D_{00}} \left[ \frac{1}{2} (\eta^\nu_{\rho} \eta_{\mu\lambda} + \eta_{\kappa}^\nu \eta_{\mu}^\rho) - \frac{1}{3} \eta_{\mu}^\nu \eta_{\lambda}^\rho \right]$$

$$+ \frac{M^2_{Pl} D_{00}}{M^3 (M^3 + M^2_{Pl}p^2 D_{00})} \left[ \frac{1}{2} (p_{\mu} p^\rho \eta_{\lambda}^\nu + p^\nu p^\rho \eta_{\mu\lambda} + (\lambda \leftrightarrow \rho)) 

- \frac{1}{3} p_{\mu} p^\rho \eta_{\lambda}^\nu - \frac{1}{3} \eta_{\mu}^\nu \eta_{\lambda}^\rho - \frac{2}{3} \frac{p_{\mu} p^\nu p^\rho p_{\lambda}}{p^2} \right] + \frac{M^2_{Pl}}{M^6} \frac{2 p_{\mu} p^\nu p^\rho p_{\lambda}}{p^2} \cdot D_{00}$$  \hspace{1cm} (14)$$
The first term here determines the interaction between conserved sources on the brane, at linearized level. At intermediate distances (5) one has $M_{Pl}^2 p^2 D_{00} \gg M^3$, so this interaction has 4d form. Note that the second term does not vanish when contracted with conserved $T^\lambda_\mu$, so matter on the brane couples to scalar degrees of freedom at strength set by the 5d mass $M$.

Let us come back to the full propagator. It has large parts, the last terms in (11) and (13). These terms may be gauged away everywhere in the bulk, but not on the brane. Indeed, they may be parametrized by introducing a 4d scalar “field” $\varphi(x)$ whose propagator equals $1/p^2$, and whose contribution to metric is

$$ h_{\mu\nu}(p, y) = \sqrt{2 M_{Pl}^2 / M^3} p_\mu p_\nu D_0(p, y) \cdot \varphi(p) $$

$$ h_{55}(p, y) = \sqrt{2 M_{Pl}^2 / M^3} p^2 D_0(p, y) \cdot \varphi(p) \quad (15) $$

Outside the brane, the “field” $\varphi$ may indeed be gauged away. Thus, we see that 4d scalars in DGP model have fairly peculiar properties; in particular, they appear enhanced by $M_{Pl}$.

To study the model in more detail, let us move to the Gaussian normal gauge,

$$ h_{55} = h_{5\mu} = 0 \quad (16) $$

and calculate the quadratic action. Let us decompose the metric

$$ h_{\mu\nu} = h_{\mu\nu}^{TT} + (p_\mu u_\nu + p_\nu u_\mu) + p_\mu p_\nu v + \frac{1}{2} \eta_{\mu\nu} \phi \quad (17) $$

where $h_{\mu\nu}^{TT}$ is transverse traceless (in 4d sense),

$$ p_\mu h_{\nu}^{TT} \mu = h_{\mu}^{TT} \mu = 0 \quad (18) $$

and $u_\mu$ is transverse

$$ p_\mu u^\mu = 0 \quad (19) $$

Then one finds, at quadratic order,

$$ S = \int dy \, d^4 x \, L_{GN} \quad (20) $$
where

$$L_{GN} = h^{TT} \mu \nu [M^3(\partial_y^2 - p^2) - M_{Pl}^2 p^2 \delta(y)] h^{TT}_{\mu \nu}$$

$$+ 2M^3 u^\mu p^2 \partial_y^2 u_{\mu}$$

$$+ 3[M^3 \delta^2 \partial_y^2 v + M^3 \phi \partial_y^2 \phi - \frac{M^3}{2} \phi p^2 \phi - \frac{M_{Pl}^2}{2} \delta(y) \phi p^2 \phi]$$

(21)

We are interested in the part that contains 4d scalars \(v\) and \(w\), last line in eq. (21). We can get rid of the very last term, which is proportional to \(M_{Pl}^2\), by defining a new field \(\hat{v}\), such that

$$v(p, y) = \phi(p, 0) \cdot \frac{M_{Pl}^2}{2M^3} |y| + \hat{v}(p, y)$$

(22)

Then in terms of \(w\) and \(\hat{v}\) the last line in (21) is precisely the part of quadratic 5d action (in the gauge (16)) that contains 4d scalars,

$$L_{scalar} = 3M^3(\phi p^2 \partial_y^2 \hat{v} + \phi \partial_y^2 \phi - \frac{1}{2} \phi p^2 \phi)$$

(23)

Hence, “canonically normalized” scalars and vectors are

$$\hat{v}_{can}^{\mu}, \phi_{can}^{\mu}, u_{\mu}^{can} = \frac{\hat{v}}{M^{3/2}}, \frac{\phi}{M^{3/2}}, \frac{u_{\mu}}{M^{3/2}}$$

(24)

where we made use of the fact that the quadratic term with \(u_{\mu}\), second line in (21), also has the 5d form.

But the metric contains large piece

$$h^{\text{large}}_{\mu \nu} = \partial_{\mu} \partial_{\nu} \frac{M_{Pl}^2}{2M^3} |y| \cdot \phi(p, 0)$$

(25)

This piece is pure gauge everywhere outside the brane. Say, at \(y > 0\) the large piece may be gauged away by the gauge transformation with

$$\xi_5 = \frac{M_{Pl}^2}{M^3} \cdot \phi(p, 0)$$

$$\xi_{\mu} = -\frac{M_{Pl}^2}{M^3} \cdot y \cdot \phi(p, 0)$$

(26)

On the other hand, the large piece (25) is not pure gauge on the brane.

Most naively, the cubic and higher order terms in the action appear to be enhanced by high powers of \(M_{Pl}^2\), because of the presence of the large piece (25) in the metric. However, since this piece is longitudinal in 4d sense, and pure gauge outside the brane, one expects strong cancellations. Let us see that large terms in the cubic action indeed cancel, but not completely.
3  Cubic order

We still work in Gaussian normal gauge,

\[ g_{55} = 1, \quad g_{5\mu} = 0 \] (27)

The brane is placed at \( y = 0 \), and we consider metric \( g_{\mu\nu} \) symmetric with respect to the brane,

\[ g_{\mu\nu}(x, -y) = g_{\mu\nu}(x, y) \] (28)

The bulk Lagrangian may be conveniently written in the form

\[ L_{\text{bulk}} = M^3 \sqrt{g} R^{(4)} + M^3 \sqrt{\hat{g}} \left( \frac{1}{4} g^{\mu\nu} \partial_5 g_{\mu\nu} g^{\lambda\rho} \partial_5 g_{\lambda\rho} - \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} \partial_5 g_{\mu\lambda} \partial_5 g_{\nu\rho} \right) \]

\[ \equiv M^3 \sqrt{\hat{g}} R^{(4)} + \Delta L \] (29)

We pursue the idea of making the change of variables from the metric \( g_{\mu\nu} \) to another “metric” \( \hat{g}_{\mu\nu} \) in such a way that the largest pieces of the brane action are cancelled by the contribution due to the bulk action. The metric \( \hat{g}_{\mu\nu} \) is related to \( g_{\mu\nu} \) by a gauge transformation on the right of the brane, and another gauge transformation on the left of the brane, so new contribution to the action appears at the brane only. The transformation from metric \( g_{\mu\nu} \) to \( \hat{g}_{\mu\nu} \) is single-valued on the brane, so metric induced on the brane is uniquely defined in terms of \( \hat{g}_{\mu\nu} \). Metric \( \hat{g}_{AB} \) still has to obey the gauge conditions (27) and symmetry property (28).

To this end, let us study what are the 5d gauge transformations that leave the conditions (27) satisfied. We write

\[ g_{AB}(X) = \frac{\partial \hat{X}^C}{\partial X^A} \frac{\partial \hat{X}^D}{\partial X^B} \hat{g}_{CD}(\hat{X}(X)) \] (30)

where \( X^A = (x^\mu, y) \), and in perturbation theory

\[ \hat{X}^A = X^A + \xi^A(X) \] (31)

We will need the relation between metric perturbations \( h_{\mu\nu} \) and \( \hat{h}_{\mu\nu} \) at quadratic order,

\[ h_{AB} = \partial_A \xi_B + \partial_B \xi_A + \partial_C \hat{h}_{AB} \xi^C + \partial_A \xi^C \hat{h}_{BC} + \partial_B \xi^C \hat{h}_{AC} + \partial_A \xi^C \partial_B \xi_C \] (32)
where all functions are functions of $X$ and indices are raised and lowered by Euclidean metric. We require

$$\hat{h}_{55} = 0$$

(33)
on the left of the brane, and on the right of the brane separately. This gives the following equation

$$2\partial_5\xi_5 + \partial_5\xi^5 \cdot \partial_5\xi_5 + \partial_5\xi^\mu \cdot \partial_5\xi_\mu = 0$$

(34)

The requirement

$$\hat{h}_{5\mu} = 0$$

(35)
gives

$$\partial_5\xi_\mu + \partial_\mu\xi_5 + \partial_5\xi^5 \cdot \partial_\mu\xi_5 + \partial_5\xi^\nu \cdot \partial_\mu\xi_\nu + \partial_5\xi^\nu \cdot \hat{h}_{\mu\nu} = 0$$

(36)

We stress that these equations should be satisfied on the left of the brane and on the right of the brane separately. To ensure the symmetry property (28) for both $h_{\mu\nu}$ and $\hat{h}_{\mu\nu}$, we take $\hat{X}^\mu$ symmetric, and $\hat{X}^5$ anti-symmetric in $y \equiv X^5$, that is

$$\xi^\mu(x, -y) = \xi^\mu(x, y)$$

$$\xi^5(x, -y) = -\xi^5(x, y)$$

(37)

Solving eqs. (34) and (36) on the left and on the right of the brane, and imposing (37), we find

$$\xi_5 = \epsilon \cdot \text{sign}(y) - \frac{1}{2} \partial_\mu\epsilon \partial^\mu\epsilon \cdot y$$

(38)

$$\xi_\mu = -\partial_\mu\epsilon \cdot |y| + \partial_\nu\epsilon \cdot \text{sign}(y) \cdot \int_0^y dy' \hat{h}^\nu_{\mu}$$

(39)

where $\epsilon$ is an arbitrary function of 4d coordinates only,

$$\epsilon = \epsilon(x^\mu)$$

(40)

Physically, $\epsilon(x)$ is a brane bending function, at least to linear order in metric perturbations. We will use the freedom parametrized by $\epsilon(x)$ to get rid of certain large terms in the total action.

The 4d components of the metric are then

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \partial_\mu\xi^5 \partial_\nu\xi_5 + \partial_\mu\xi^\lambda \partial_\nu\xi_\lambda + \partial_\mu\xi^5 \hat{h}_{\nu\lambda} + \partial_\nu\xi^5 \hat{h}_{\mu\lambda} + \partial_5\hat{h}_{\mu\nu}\xi^5 + \partial_\lambda\hat{h}_{\mu\nu}\xi^\lambda$$

(41)
Note that in spite of the jump of $\xi^5$, the latter relation is well defined, i.e., $h_{\mu\nu}$ is uniquely defined on the brane in terms of $\hat{h}_{\mu\nu}$ and $\epsilon$ (provided that $\hat{h}_{\mu\nu}$ is symmetric).

Let us clarify the logic again. We may forget about previous steps, and merely consider eq. (41) as the definition of the change of variables from $h_{\mu\nu}$ to $\hat{h}_{\mu\nu}$ in the gauge (27), with $\xi^5$ and $\xi^\mu$ defined by eqs. (38) and (39) in terms of yet arbitrary function $\epsilon(x)$. The further procedure is to calculate the total action, up to cubic order, and then choose $\epsilon(x)$ to simplify this action.

We will need the expression for $h_{\mu\nu}$ on the brane:

$$h_{\mu\nu}(y = 0) = \hat{h}_{\mu\nu} + \partial_\mu \epsilon \partial_\nu \epsilon$$  \hspace{1cm} (42)

Let us now calculate the action in terms of $\hat{h}_{\mu\nu}$ and $\epsilon$. We begin with the bulk term. Since $h_{\mu\nu}$ does not jump across the brane, the action is the sum of integrals of the Lagrangian (29) over regions left and right of the brane. We write in each of these regions

$$(\sqrt{g} R^{(5)}[g])(X) = \det \left( \frac{\partial \hat{X}^A}{\partial X^B} \right) (\sqrt{\hat{g}} \hat{R}^{(5)}[\hat{g}])(\hat{X}(X))$$  \hspace{1cm} (43)

This gives, up to cubic order,

$$\sqrt{g} R^{(5)}[g] = \sqrt{\hat{g}} \hat{R}^{(5)} + \partial_A (\sqrt{\hat{g}} \hat{R}^{(5)} \xi^A) + \frac{1}{2} \partial_A [\partial_B (\sqrt{\hat{g}} \hat{R}^{(5)}) \xi^A \xi^B + \sqrt{\hat{g}} \hat{R}^{(5)} \xi^A \partial_B \xi^B] - \sqrt{\hat{g}} \hat{R}^{(5)} \xi^B \partial_B \xi^A$$  \hspace{1cm} (44)

where

$$\hat{R}^{(5)} = R^{(5)}[\hat{g}]$$  \hspace{1cm} (45)

and all quantities are functions of $X$. Since $\hat{g}$ and $\hat{R}^{(5)}$ are symmetric, and $\xi_\mu$ vanishes at the brane, one finds that the integration of (44) over regions left and right of the brane gives the following additional contribution to the action

$$-2M^3 \int_{brane} d^4x \sqrt{\hat{g}} \hat{R}^{(5)} \epsilon$$  \hspace{1cm} (46)

Let us now consider the second term in eq.(29). It is equal to

$$2M^3 \partial_5^2 \sqrt{g}$$  \hspace{1cm} (47)
and hence contributes to the action as

$$-2M^3 [\partial_5 \sqrt{g}]_{y \rightarrow +0}$$  (48)

Now, we have

$$\sqrt{g} = \det(\delta_{\mu}^{\nu} + \partial_{\nu} \xi_{\mu}) \cdot \sqrt{\hat{g}(x^\mu + \xi^\mu, y + \xi^5)}$$  (49)

For $\xi^5 = 0$ the right hand side is a total 4d divergence, so we have to evaluate the determinant here to the first order only, $\det(\delta_{\mu}^{\nu} + \partial_{\nu} \xi_{\mu}) = 1 + \partial_{\mu} \xi_{\mu}$.

Then modulo total 4d divergence, one has up to cubic order

$$\sqrt{g} = \sqrt{\hat{g}} + \det(\delta_{\mu}^{\nu} + \partial_{\nu} \xi_{\mu}) \cdot \left( \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^5 + \partial_{\sigma} \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^\sigma \xi^5 + \frac{1}{2} \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^5 \xi^5 \right)$$  (50)

which gives, again up to total 4d divergence,

$$\sqrt{g} = \sqrt{\hat{g}} + \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^5 - \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^\mu \partial_{\mu} \xi^5 + \frac{1}{2} \partial_{\nu} \sqrt{\hat{g}} \cdot \xi^5 \xi^5$$  (51)

Now, both $\partial_{\nu} \sqrt{\hat{g}}$ and $\xi^\mu$ vanish on the brane, so the third term here does not contribute to (48). The fourth term does not contribute at cubic level too, because $(\xi^5)^2 = \epsilon^2$ is continuous at quadratic level. The contribution from the second term is entirely due to the first term in (38), since the second term in (38) is smooth across the brane. Thus, the additional contribution to the action is

$$-4M^3 \int_{brane} d^4x \partial_{\nu}^2 \sqrt{\hat{g}} \cdot \epsilon$$  (52)

This contribution, together with the contribution (46) adds to

$$-2 \int_{brane} d^4x \epsilon \cdot L_{bulk}[\hat{g}] = -2 \int_{brane} d^4x \epsilon(x) \cdot (M^3 \sqrt{\hat{g}} \hat{R}^{(4)} + \Delta L[\hat{g}_{\mu\nu}])$$  (53)

where $\Delta L$ is defined in eq. (29).

Let us turn to the brane action. The contribution due to $\epsilon$ into the brane action comes from the second term in eq. (42). To cubic order, it is

$$M_{Pl}^2 \int d^4x \sqrt{\hat{g}} \left( \hat{R}^{(4)} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R}^{(4)} \right) \partial_{\mu} \epsilon \partial_{\nu} \epsilon$$  (54)

Thus, the total action $S_{tot}[\hat{g}]$ equals $S_{tot}[\hat{g}]$ plus the sum of (53) and (54).
Now, let us discuss scalar and vector modes (in 4d sense). These are parametrized as follows,
\[
\hat{g}_{\mu\nu}(x, y) = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\rho}}{\partial x^{\nu}} \hat{g}_{\lambda\rho}(\tilde{x}(x, y), y)
\]
where
\[
\tilde{x}^\mu = x^\mu + \pi^\mu(x, y)
\]
(at linear level, \(\pi_\mu = u_\mu + \partial_\mu v\) in notations of section 2) and
\[
\tilde{g}_{\mu\nu} = e^{2\phi(x,y)} \eta_{\mu\nu}
\]
The brane action is then
\[
S_{brane}[\hat{g}] = 6M_P^2 \int d^4x \ e^{2\phi} \partial_\mu \phi \partial^\mu \phi
\]
Thus, the total action, up to cubic level, is
\[
S_{tot} = S_{bulk}[\hat{g}] + S_{brane}[\hat{g}] + S_{\epsilon}
\]
where \(S_{\epsilon}\) is the additional brane term, the sum of (53) and (54). We write
\[
\hat{R}^{(4)}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R}^{(4)} = -2\partial_\mu \partial_\nu \phi + 2\eta_{\mu\nu} \Box^{(4)} \phi + \text{(higher orders in } \phi, \pi) \quad (60)
\]
and
\[
\sqrt{\hat{g}} \hat{R}^{(4)} = -6\Box^{(4)} \phi + \text{(higher orders in } \phi, \pi) \quad (61)
\]
and obtain explicitly
\[
S_{\epsilon} = \int d^4x \ \left[ 12M^3 \epsilon \cdot \Box^{(4)} \phi + M_P^2 \cdot \partial_\mu \epsilon \partial_\nu \epsilon \cdot (-2\partial_\mu \phi + 2\eta_{\mu\nu} \Box^{(4)} \phi) + 2M^3 \epsilon \cdot L_2(\phi, \pi^\mu) \right]
\]
where \(L_2\) is quadratic in fields \(\phi\) and \(\pi^\mu\),
\[
L_2 = -12\phi \Box^{(4)} \phi - 6\partial_\mu \phi \partial^\mu \phi - 6\partial_\mu (\Box^{(4)} \phi \cdot \pi^\mu) + 12(\partial_5 \phi)^2 + 6\partial_5 \phi \partial_5 \pi^\mu + (\partial_5 \partial_\mu \pi^\mu)^2 - \frac{1}{2} \partial_\mu \partial_5 \pi_\nu \partial^\nu \partial_5 \pi^\mu - \frac{1}{2} \partial_\mu \partial_5 \pi_\nu \partial^\nu \partial_5 \pi^\nu \quad (63)
\]
At quadratic order, the relevant terms in the action are

\[
6 \int d^4 x \left( 2M^3 \epsilon^{(1)} \Box^{(4)} \phi + M_{Pl}^2 \partial_\mu \phi \partial^\mu \phi \right)
\] (64)

where \( \epsilon^{(1)} \) is linear in metric perturbations. To get rid of these terms, we choose (cf. eq. (22))

\[
\epsilon^{(1)}(x) = \frac{M_{Pl}^2}{2M^3} \phi(x, y = 0)
\] (65)

then the contribution (64) vanishes.

At cubic level, the largest terms are

\[
\int d^4 x \left[ 12M^3 \epsilon^{(2)} \Box^{(4)} \phi + M_{Pl}^2 \partial_\mu \epsilon^{(1)} \cdot \partial^\nu \epsilon^{(1)} \cdot \left( -2\partial_\mu \partial_\nu \phi + 2\eta_{\mu\nu} \Box^{(4)} \phi \right) \right]
\]

\[
\int d^4 x \left[ 12M^3 \epsilon^{(2)} \cdot \Box^{(4)} \phi 
- \frac{M_{Pl}^6}{4M^6} \partial_\mu \phi \partial^\nu \phi \cdot \left( -2\partial_\mu \partial_\nu \phi + 2\eta_{\mu\nu} \Box^{(4)} \phi \right) \right]
\] (66)

These are cancelled out by choosing

\[
\epsilon^{(2)}(x) = \frac{M_{Pl}^6}{12M^9} \cdot (\partial_\mu \phi \partial^\mu \phi)(x, y = 0)
\] (67)

After this choice is made, the action for 4d scalars and vectors becomes, up to cubic level,

\[
S_{tot} = S_{bulk}(\phi, \pi^\mu) + M_{Pl}^2 \int_{brane} d^4 x \cdot L_2(\phi, \pi^\mu)
\] (68)

Thus, we see that due to the cancellations, the largest possible terms disappear, and the cubic action is enhanced by \( M_{Pl}^2 \) only.

In fact, we can do better by choosing

\[
\epsilon^{(2)}(x) = \frac{M_{Pl}^6}{12M^9} \cdot (\partial_\mu \phi \partial^\mu \phi)(x, y = 0) + \frac{M_{Pl}^2}{4M^3} (3\phi^2 - 2\partial_\mu \phi \pi^\mu)
\] (69)

Then those terms in eq. (63) that do not contain transverse derivatives, cancel out, and we obtain finally

\[
S_{tot} = S_{bulk}(\phi, \pi^\mu) + M_{Pl}^2 \int_{brane} d^4 x \cdot \left[ \frac{1}{2} \partial_\mu \partial_\nu \pi^\mu \partial_\rho \partial_\sigma \pi^\nu \left( \partial_5 \phi \right)^2 
+ \frac{1}{2} \partial_\mu \partial_\nu \pi^\mu \partial_\rho \partial_\sigma \pi^\nu - \frac{1}{2} \partial_\mu \partial_\nu \pi^\mu \partial_\rho \partial_\sigma \pi^\nu \right]
\] (70)
Further reduction of the cubic action is impossible, as the remaining terms are not proportional to $\Box^{(4)}\phi$.

With “canonicaly normalized” scalars and vectors (24), the largest interaction term is proportional to

$$\frac{M_{Pl}^2}{M^{9/2}}$$

On dimensional grounds we conclude that the energy scale of strong interaction between the scalar and vector modes is indeed given by eq. (6).

4 Discussion

The approach taken in this paper is not quite satisfactory. The problems with this approach are twofold. First, it is not at all obvious that the conclusion that the strong interaction scale is (6) will not be different when higher (quartic, etc.) orders are included. Indeed, the powers of $M_{Pl}/M$ proliferate, as is seen in eq. (67). Second, the geometrical meaning of the whole procedure of cancelling large terms is obscure. On the other hand, our calculation does show that DGP model becomes strongly coupled at low energy scale. It remains to be understood whether this feature is inevitable in no-ghost models with modification of gravity at ultra-large scales.

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References

[1] C. Charmousis, R. Gregory and V. A. Rubakov, Phys. Rev. D 62 (2000) 067505 [arXiv:hep-th/9912160];
R. Gregory, V. A. Rubakov and S. M. Sibiryakov, Phys. Rev. Lett. 84 (2000) 5928 [arXiv:hep-th/0002072].

[2] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross and J. Santiago, Nucl. Phys. B 584 (2000) 313 [arXiv:hep-ph/9912552].

[3] G. R. Dvali and G. Gabadadze, Phys. Rev. D 63 (2001) 065007 [arXiv:hep-th/0008054];
G. Dvali, G. Gabadadze, X. r. Hou and E. Sefusatti, arXiv:hep-th/0111266.

[4] L. Pilo, R. Rattazzi and A. Zaffaroni, JHEP 0007 (2000) 056 [arXiv:hep-th/0004028].

[5] S. L. Dubovsky and V. A. Rubakov, arXiv:hep-th/0212222.

[6] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208 [arXiv:hep-th/0005016].

[7] I. I. Kogan, S. Mouslopoulos, A. Papazoglou and G. G. Ross, Phys. Rev. D 64 (2001) 124014 [arXiv:hep-th/0107086].

[8] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, arXiv:hep-th/0210184.

[9] M. Porrati, Phys. Lett. B 534 (2002) 209 [arXiv:hep-th/0203014].

[10] M. A. Luty, M. Porrati and R. Rattazzi, hep-th/0303116.