Typical structure of hereditary graph families. II. Exotic examples

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Abstract

A graph $G$ is $H$-free if it does not contain an induced subgraph isomorphic to $H$. The study of the typical structure of $H$-free graphs was initiated by Erdős, Kleitman and Rothschild [EKR76], who have shown that almost all $C_3$-free graphs are bipartite. Since then the typical structure of $H$-free graphs has been determined for several families of graphs $H$, including complete graphs, trees and cycles. Recently, Reed and Scott [RS] proposed a conjectural description of the typical structure of $H$-free graphs for all graphs $H$, which extends all previously known results in the area.

We construct an infinite family of graphs for which the Reed-Scott conjecture fails, and use the methods we developed in the prequel paper [NY20] to describe the typical structure of $H$-free graphs for graphs $H$ in this family.

Using similar techniques, we construct an infinite family of graphs $H$ for which the maximum size of a homogenous set in a typical $H$-free graph is sublinear in the number of vertices, answering a question of Loebl et al. [LRS+10] and Kang et al. [KMRS14].

1 Introduction

Let $F$ be a family of graphs. We say that $F$ is hereditary if $F$ is closed under isomorphism and taking induced subgraphs. Let $H$ be a graph, we say that a graph $G$ is $H$-free if it does not contain an induced subgraph isomorphic to $H$, and we denote by $\text{Forb}(H)$ the family of all $H$-free graphs. Clearly, $\text{Forb}(H)$ is hereditary. More generally, if $\mathcal{H}$ is a collection of graphs we say that a graph $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$, and we denote by $\text{Forb}(\mathcal{H})$ the family of all $\mathcal{H}$-free graphs.

In this paper we study the typical structure of graphs in $\text{Forb}(H)$. Our main results are constructions of graphs $H$ for which this structure is more complex than in the previously known examples.

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For a family of graphs $\mathcal{F}$, let $\mathcal{F}^n$ denote the set of graphs in $\mathcal{F}$ with vertex set $[n] = \{1, 2, \ldots, n\}$. We say that a property $\mathcal{P}$ holds for almost all graphs in $\mathcal{F}$, if

$$\lim_{n \to \infty} \frac{|\mathcal{F}^n \cap \mathcal{P}|}{|\mathcal{F}^n|} = 1.$$ 

The study of the typical structure of graphs in $\text{Forb}(H)$ was initiated by Erdős, Kleitman and Rothschild [EKR76]. They have shown that almost all $C_3$-free graphs are bipartite. Prömel and Steger [PS91, PS92] obtained a structural characterization of typical $C_4$ and $C_5$-free graphs. They have shown that vertices of almost all $C_4$-free graphs can be partitioned into a clique and a stable set, and that for almost all $C_5$-free graph $G$ either the vertices of $G$ can be partitioned into a clique and a set inducing a disjoint union of cliques, or the vertices of $G$ can be partitioned into a stable set and a set inducing a complete multipartite graph.

Recently, Reed and Scott [RS] proposed a conjecture which informally states that a similar description of almost all $H$-free graphs is possible for any $H$. To state it precisely we need a few definitions. A pattern $\mathfrak{F} = (F_1, \ldots, F_k)$ is a finite collection of hereditary families of graphs. We say that a partition $\mathcal{P} = (P_1, P_2, \ldots, P_k)$ of the vertex set of a graph $G$ is an $\mathfrak{F}$-partition for a pattern $\mathfrak{F}$ as above if $G[P_i] \in F_i$ for every $i \in [k]$. Let $\mathcal{P}(\mathfrak{F})$ denote the family of all graphs admitting an $\mathfrak{F}$-partition. If $\mathfrak{F}$ is a pattern with $|\mathfrak{F}| = l$ such that every element of $\mathfrak{F}$ is the same family $\mathcal{F}$ then we refer to an $\mathfrak{F}$-partition as an $(\mathcal{F}, l)$-partition and denote $\mathcal{P}(\mathfrak{F})$ by $\mathcal{P}(\mathcal{F}, l)$.

A pattern $\mathfrak{F}$ is $H$-free if $H \not\in \mathcal{P}(\mathfrak{F})$. We say that $\mathfrak{F}$ is sharply $H$-free if it is $H$-free and for every $\mathcal{F} \in \mathfrak{F}$ every minimal graph $J$ which is not in $\mathcal{F}$ is isomorphic to a subgraph of $H$. It is easy to see that every maximal $H$-free pattern is sharply $H$-free, and so we restrict our attention to sharply $H$-free patterns. Note that, conveniently, for any graph $H$ and any integer $k$ there are finitely many sharply $H$-free patterns of size $k$, as every element of such pattern is completely determined by the collection of the subgraphs of $H$ which belong to it.

A structural description of $\text{Forb}(H)$ along the lines of the results of [EKR76, PS91, PS92] in the language we have just introduced can now be stated as follows. For almost every $G \in \text{Forb}(H)$ there exists a pattern $\mathfrak{F}$ such that

1. $\mathfrak{F}$ is sharply $H$-free,
2. $G \in \mathcal{P}(\mathfrak{F})$,
3. elements of $\mathfrak{F}$ are “structured”.

Let $\mathcal{C} = \text{Forb}(K_2)$ denote the family of all complete graphs and let $\mathcal{S} = \text{Forb}(K_2)$ denote the family of all edgeless graphs. Then the result of [EKR76] shows that $\mathfrak{F} = (\mathcal{S}, \mathcal{S})$ satisfies the above conditions for $H = C_3$, while for $H = C_5$ we need to take to $\mathfrak{F} = (\mathcal{C}, \mathcal{C})$ or $\mathfrak{F} = (\mathcal{S}, \mathcal{C})$, depending on $G$.

Note that $\mathfrak{F} = (\mathcal{F}(H))$ trivially satisfies conditions (S1) and (S2), but does not give any insight in the structure of $\text{Forb}(H)$. Thus we need to formalize condition (S3). We say

1. I.e. almost all $C_4$-free graphs are split graphs.
2. That is every $J \not\in \mathcal{F}$ such that $J \setminus v \in \mathcal{F}$ for every $v \in \mathcal{F}$.
that a pattern $\mathcal{F}$ is proper if $\mathcal{F}^n \neq \emptyset$ for every $\mathcal{F} \in \mathcal{F}$ and every positive integer $n$. By Ramsey’s theorem the above condition is equivalent to the requirement that for every $\mathcal{F} \in \mathcal{F}$ either $\mathcal{C} \subseteq \mathcal{F}$ or $\mathcal{S} \subseteq \mathcal{F}$.

For a pair of non-negative integers $s$ and $t$ let $\mathcal{F}(s,t) = (\mathcal{S}, \ldots, \mathcal{S}, \mathcal{C}, \ldots, \mathcal{C})$ denote the proper pattern consisting of $s$ families of edgeless graphs and $t$ families of complete graphs, and let $\mathcal{H}(s,t)$ denote $P(\mathcal{F}(s,t))$. Thus $\mathcal{H}(s,t)$ is a family of all graphs whose vertex set can be partitioned into $s$ stable sets and $t$ cliques. For every proper pattern $\mathcal{F}$ we have $\mathcal{H}(s,t) \subseteq \mathcal{F}$ for some $s, t$ such that $s + t = |\mathcal{F}|$, and thus the following observation holds.

**Observation 1.1.** For a graph $H$ and a positive integer $l$ the following are equivalent.

- there exist non-negative integers $s$ and $t$ with $s + t = l$ such that $H \not\in \mathcal{H}(s,t)$,
- there exists a proper $H$-free pattern $\mathcal{F}$ such that $|\mathcal{F}| = l$.

The maximum integer $l$ satisfying the conditions of Observation 1.1 for a graph $H$ is called the witnessing partition number of $H$ and is denoted by $\chi_c(H)$. One can impose meaningful structure on the elements of a proper $H$-free pattern $\mathcal{F}$ by insisting simply that it has maximum possible size, i.e. $|\mathcal{F}| = \chi_c(H)$. Combining conditions (S1),(S2) and (S3), we say that a pattern $\mathcal{F}$ is a clean $H$-free pattern if $\mathcal{F}$ is proper, sharply $H$-free, and $|\mathcal{F}| = \chi_c(H)$.

We say that a clean $H$-free pattern is a clean $H$-free profile for a graph $G$ if $G \in P(\mathcal{F})$. We can now precisely state the Reed-Scott’s conjecture mentioned above.

**Conjecture 1.2.** For every graph $H$, almost every $H$-free graph has a clean $H$-free profile.

Conjecture 1.2 has been verified for cliques [EKR76], cycles [PS91, PS92, BB11], trees [RY] and critical graphs [BB11].

Our first main result shows that Conjecture 1.2 is false in general.

**Theorem 1.3.** There exists infinitely many graphs $H$ such that almost every $H$-free graph has no clean $H$-free profile.

The proof of Theorem 1.3 can be vaguely outlined as follows. The graphs $H$ satisfying the theorem are constructed so that clean $H$-free patterns are severely restricted. To do this we ensure that $H$ admits a variety of partitions into graphs with simple structure. The vertex sets of parts of these partitions are chosen at random to further guarantee that $H$ does not admit the partitions into “simple” graphs, except for the ones we specifically prescribed.

Using variants of this technique, we are able to generate examples of graphs $H$ such that almost all $H$-free graphs have given structure for a fairly wide variety of specifications. Our second class of examples constructed this way answers a question from [LRS+10, KMRS14] related to the famous Erdős-Hajnal conjecture, which we now state.

A homogenous set in a graph is either an independent set or a clique. We denote by $h(G)$ the size of the largest homogenous set in a graph $G$. Erdős and Hajnal made the following conjecture.

**Conjecture 1.4 (Erdős-Hajnal Conjecture [EH89]).** For every graph $H$, there exists an $\varepsilon = \varepsilon(H) > 0$ such that all $H$-free graphs $G$ have $h(G) \geq |V(G)|^\varepsilon$. 

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The conjecture appears to be very hard and is known to hold only for a few graphs \( H \), see [Chu14] for a survey. A way to relax the conjecture in line with the subject of this paper is to consider almost all \( H \)-free graphs for a given graph \( H \). Loebl et al. [LRS+10] did just this proving the following.

**Theorem 1.5** (Loebl et al. [LRS+10]). For every graph \( H \), there exists an \( \varepsilon = \varepsilon(H) > 0 \) such that almost all \( H \)-free graphs \( G \) have \( h(G) \geq |V(G)|^\varepsilon \).

Kang et al. [KMRS14] have shown that a stronger conclusion holds for almost all graphs \( H \). We say that a graph \( H \) has the asymptotic linear Erdős-Hajnal property if there exists \( b > 0 \) such that almost all \( H \)-free graphs \( G \) satisfy \( h(G) \geq b|V(G)| \).

**Theorem 1.6** (Kang et al [KMRS14]). Almost all graphs have the asymptotic linear Erdős-Hajnal property.

It is mentioned in [LRS+10] and [KMRS14] that \( P_3 \), the path on 3 vertices, does not have the asymptotic linear Erdős-Hajnal property. More precisely, as a direct corollary to a result Aleksandrovskii (cf. [Yak95]), the authors of [LRS+10, KMRS14] observed the following.

**Observation 1.7.** Almost all \( P_3 \)-free graphs \( G \) have \( h(G) = \Theta\left(\frac{|V(G)|}{\log |V(G)|}\right) \).

The authors in [KMRS14] and [LRS+10] asked if there exists graphs other than \( P_3 \) and, possibly, \( P_4 \) that do not have the asymptotic linear Erdős-Hajnal property. We answer this question affirmatively.

**Theorem 1.8.** There exist infinitely many graphs which do not have the asymptotic linear Erdős-Hajnal property.

We also present a third class of examples of similar nature to the classes appearing in Theorems 1.3 and 1.8 but we postpone its description to the next section, as it requires more preparation to motivate.

Showing that the graphs \( H \) in the families that we construct have the claimed properties requires us to analyze the structure of typical \( H \)-free graphs. In Section 2 we present the tools for such analysis, which we developed in [NY20]. In Section 3 we explicitly describe our families of exotic examples, including the families satisfying Theorems 1.3 and 1.8 and use the results from Section 2 to analyze the structure typical \( H \)-free graphs for graphs \( H \) in these families. Finally, in Section 4 we present constructions of infinite families of graphs with the properties specified in Section 3.

## 2 Tools from [NY20]

In this section we present the results from [NY20], which allow us to analyze the typical structure of typical graphs in \( \text{Forb}(H) \) for graphs \( H \) constructed in the later sections.

We start by extending the definition of the witnessing partition number to general hereditary families. The **coloring number** \( \chi_c(\mathcal{F}) \) of a graph family \( \mathcal{F} \) is the maximum integer \( l \) such that \( \mathcal{H}(s, l-s) \subseteq \mathcal{F} \) for some \( 0 \leq s \leq l \). Clearly, \( \chi_c(H) = \chi_c(\text{Forb}(H)) \) for every graph \( H \). We say that \( \mathcal{F} \) is **thin** if \( \chi_c(\mathcal{F}) \leq 1 \).
The following is a key definition in our structural results. Let $F$ be a hereditary graph family, and let $l = \chi_c(F)$. Let $\iota(J)$ denote the hereditary family of graphs isomorphic to induced subgraphs of a graph $J$. We say that a graph $J$ is $F$-reduced if there exists an integer $0 \leq s \leq l - 1$ such that

$$\mathcal{P}(\iota(J), \mathcal{H}(s, l - 1 - s)) \subseteq F^s$$

We say that $J$ is $F$-dangerous if $J$ is not $F$-reduced. Let red($F$) and dang($F$) denote the families of all $F$-reduced and $F$-dangerous graphs, respectively. For brevity we write red($H$) and dang($H$) instead of red(Forb($H$)) and dang(Forb($H$)), respectively.

Note that if $\mathcal{F}$ is a proper pattern such that $\mathcal{P}(\mathcal{F}) \subseteq F$ and $|\mathcal{F}| = l$ then $T \subseteq \text{red}(F)$ for every family $T \in \mathcal{F}$. In particular, we have $\mathcal{P}(\mathcal{F}) \subseteq \mathcal{P}(\text{red}(F), l)$ for every such pattern $\mathcal{F}$. The description of typical structure of $F$ given in Theorem 2.1 below relaxes Conjecture 1.2 in the direction suggested by this observation: Under several significant technical restrictions on $F$ we show that almost all graphs in $F$ belong to $\mathcal{P}(\text{red}(F), l)$.

Let us now present these restrictions. A substar is a subgraph of a star, and an antistar is a complement of a substar. We say that a hereditary family $F$ is Apex-free if it is not a substar and an antistar. It turns out that assuming that the family $F$ is apex-free significantly simplifies analysis of its structure. We say that $F$ is Meager if it is thin and apex-free.

We say that a hereditary family $F$ with $l = \chi_c(F) \geq 2$ is smooth if for every $\delta > 0$ there exists $n_0$ such that

$$|F^n| \geq 2^{((l-1)/l-\delta)n}\left|F^{n-1}\right|$$

for all integers $n \geq n_0$. As $|F^n| \geq 2^{(l-1)n^2/2l-o(n^2)}$ for every hereditary family as above, we expect “reasonable” hereditary families to be smooth, yet it appears difficult to prove that a given hereditary family is smooth without first understanding its structure.

**Theorem 2.1 (NY20 Theorem 2.6).** Let $F$ be an Apex-free hereditary family, let $l = \chi_c(F) \geq 2$, let $\mathcal{K} \subseteq \text{dang}(F)$ be a finite set of graphs, and let $T = \text{Forb}(\mathcal{K})$. If $\mathcal{P}(T, l) \cap F$ is smooth then almost all graphs $G \in F$ admit a $(T, l)$-partition.

In addition to Theorem 2.1 we will use an easy lemma which is helpful in verifying that conditions of Theorem 2.1 are satisfied. We say that a family $F$ is extendable if there exists $n_0 \geq 0$ such that for every $G \in F$ with $|V(G)| \geq n_0$ we have $G = G' \setminus v$ for some $G' \in F$ and $v \in V(G')$.

**Lemma 2.2 (NY20 Lemma 2.9).** Let $\mathcal{F}$ be a proper pattern such that every $T \in \mathcal{F}$ is extendable and thin. Then the family $\mathcal{P}(\mathcal{F})$ is smooth.

Our applications of Theorem 2.1 use not only existence of a structured partition of a typical graph, but the facts that such a partition is unique and essentially balanced in the following sense. We say that a partition $\mathcal{X}$ of an $n$ element set is $\varepsilon$-balanced if $|X - n/|\mathcal{X}|| \leq n^{1-\varepsilon}$ for all $X \in \mathcal{X}$.

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3I.e. $F$ contains all graphs which admit a vertex partition into $l$ parts, such that the first part induces a subgraph of $J$, $s$ of the remaining parts are stable sets, and the rest are cliques.
Lemma 2.3 ([NY20, Corollary 2.11]). Let $\mathcal{F}$ be a proper pattern such that every $T \in \mathcal{F}$ is meager and extendable. Then there exists $\varepsilon > 0$ such that almost all graphs in $\mathcal{P}(\mathcal{F})$ admit a unique $\mathcal{F}$-partition, and such a partition is $\varepsilon$-balanced.

Let $(G_1, \ldots, G_l)$ be a collection of vertex disjoint graphs, and let $X = \bigcup_{i \in [l]} V(G_i)$. We say that a graph $G$ is an extension of $(G_1, \ldots, G_l)$ if $V(G) = X$, and $G_i$ is an induced subgraph of $G$ for every $i \in [l]$.

Lemma 2.4 ([NY20, Lemma 2.12]). Let $T$ be a meager hereditary family, let $l$ be an integer, and let $\varepsilon > 0$ be real. Let $(G_1, \ldots, G_l)$ be a collection of graphs such that $G_i \in T$ for every $i \in [l]$ and $X = (V(G_1), \ldots, V(G_l))$ is an $\varepsilon$-balanced partition of $[n]$. Then $X$ is the unique $(T, l)$-partition of $G$ for almost every extension $G$ of $(G_1, \ldots, G_l)$.

The main application of Theorem 2.1 in [NY20], which we will also need here, is a generalization of the following result of Balogh and Butterfield [BB11].

In [BB11] a graph $H$ is defined to be critical if there exists an integer $n_0$ such that every $K \in \text{red}(H)$ with $|V(K)| \geq n_0$ is either complete or edgeless. Thus a clean $H$-free profile of a graph $G$ corresponds to a partition of $V(G)$ in to cliques and stable sets (and potentially bounded size graphs), such that $H$ does not admit a partition with the same structure. The following characterization of critical graphs $H$ given in [BB11] shows that one indeed can find such a partition for almost all $H$-free graphs. It implies, in particular, that critical graphs satisfy Conjecture 1.2.

Theorem 2.5 ([BB11]). A graph $H$ is critical if and only if for almost every $G \in \text{Forb}(H)$ we have $G \in \mathcal{H}(s, t)$ for some pair of non-negative integers $s$ and $t$ such that $s + t = \chi_c(H)$ and $\mathcal{H}(s, t) \subseteq \text{Forb}(H)$.

To describe and motivate our generalization of Theorem 2.5 we need several additional definitions. We say that a set $S \subseteq V(G)$ is a core of a graph $G$ if for every $v \in V(G)$ either $v$ is adjacent to every vertex of $V(G) - S$ or $v$ is not adjacent to any vertex in $S$. We say that a graph $G$ is an $s$-star for an integer $s \geq 0$ if $G$ has a center of size at most $s$. Thus 0-stars are exactly complete or edgeless graphs, and 1-stars are induced subgraphs of stars or antistars.

We say that a hereditary family $\mathcal{F}$ is $s$-critical for an integer $s \geq 0$ if there exists $n_0$ such that every $K \in \text{red}(\mathcal{F})$ with $|V(K)| \geq n_0$ is an $s$-star. We say that a graph $H$ is $s$-critical if $\text{Forb}(H)$ is $s$-critical. Thus 0-critical graphs are exactly critical graphs.

Prömel and Steger [PS93] have shown that 1-critical graphs are exactly the extremal graphs according to a certain metric related to the structure of $\text{Forb}(H)$.

Clearly, every 0-critical graph is 1-critical, but, as noted in [BB11], it is not obvious whether the converse holds. Our final main result in the vein of Theorems 1.3 and 1.8 show that it does not.

Theorem 2.6. There exist infinitely many graphs which are 1-critical, but not 0-critical.

In the proof of Theorem 1.3 we construct a family of counterexamples to Conjecture 1.2 that are 2-critical. We suspect that the conjecture does not hold even for 1-critical graphs.

\footnote{The definition of critical graphs considered in [PS93] differs from our definition of 1-critical graphs, but as we show in Section 3.3 the definition we give here is equivalent.}
In spite of this in [NY20] we obtained a structural description of typical graphs in \( \mathcal{F} \) for any \( s \)-critical family \( \mathcal{F} \), which we now present.

We define an \((l, s)\)-constellation (or simply a constellation) to be a quadruple \( \mathcal{J} = (J, \phi, \alpha, \beta) \), where \( J \) is a (possibly empty) graph, and \( \phi, \alpha, \beta \) are functions such that \( \phi : V(J) \rightarrow [l] \) satisfies \( |\phi^{-1}(i)| \leq s \) for every \( i \in [l] \), \( \alpha : V(J) \rightarrow \{0, 1\} \) and \( \beta : [l] \rightarrow \{0, 1\} \). We say that a constellation \( \mathcal{J} \) is irreducible if for every \( v \in V(J) \) if \( \beta(\phi(v)) = \alpha(v) \) then there exists \( u \in V(J) \setminus \{v\} \) such that \( \phi(u) = \phi(v) \) and either \( uv \in E(G) \) and \( \alpha(u) = 0 \), or \( uv \notin E(G) \) and \( \alpha(v) = 1 \).

Given an \((l, s)\)-constellation \( \mathcal{J} = (J, \phi, \alpha, \beta) \), define a \( \mathcal{J} \)-template in a graph \( G \) to be a tuple \((\psi, X_1, X_2, \ldots, X_l)\) such that

\begin{itemize}
  \item \((X_1, X_2, \ldots, X_l)\) is a partition of \( V(G) \),
  \item \( \psi : V(J) \rightarrow V(G) \) is an embedding satisfying \( \psi(v) \in X_{\phi(v)} \) for every \( v \in V(J) \),
\end{itemize}

and, denoting the image of \( \psi \) by \( Z \), we have

\begin{itemize}
  \item if \( \alpha(v) = 1 \) then \( \psi(v) \) is adjacent to every vertex in \( X_{\phi(v)} - Z \) in \( G \), and, otherwise, \( \psi(v) \) is adjacent to no vertex in \( X_{\phi(v)} - Z \), and,
  \item if \( \beta(i) = 1 \) then \( X_i - Z \) is a clique in \( G \), and, otherwise, \( X_i - Z \) is an independent set.
\end{itemize}

Thus, in particular, \( Z \cap X_i \) is a core of \( G[X_i] \) for every \( i \in [l] \) and thus \( X_i \) induces an \( s \)-star in \( G \) for every \( i \in [l] \) and if \( |X_i - Z| \geq 2 \) and \( \mathcal{J} \) is irreducible then \( Z \cap X_i \) is a minimal core of \( G[X_i] \).

Let \( \mathcal{P}(\mathcal{J}) \) denote the family of induced subgraphs of all graphs which admit a \( \mathcal{J} \)-template.

**Theorem 2.7** ([NY20] Theorem 2.17). Let \( \mathcal{F} \) be an \( s \)-critical hereditary family with \( \chi_c(\mathcal{F}) = l \). Then for almost every graph in \( G \in \mathcal{F} \) there exists an irreducible \((l, s)\)-constellation \( \mathcal{J} \) such that \( G \in \mathcal{P}(\mathcal{J}) \subseteq \mathcal{F} \).

Note that if \( \mathcal{J} \) is an \((l, 0)\)-constellation then a \( \mathcal{J} \)-template in a graph \( G \) is a partition of \( V(G) \) into \( l \) homogenous sets, the fixed number of which are cliques. Thus Theorem 2.7 does indeed generalize one of the directions of Theorem 2.5.

A \( \mathcal{J} \)-template is similar to the structure proposed by Conjecture 1.2 but in addition to prescribing the structure on the parts of the partition given by the template, we prescribe the behavior of a finite number of additional edges. As Theorem 1.3 shows this additional restriction is sometimes necessary. It is tempting to attempt to formulate a common generalization of Conjecture 1.2 and Theorem 2.7 but we were unable to find a plausible one.

Finally, we need a bound on the number of graphs admitting an \( \mathcal{J} \) template.

**Lemma 2.8** ([NY20] Lemma 2.16]). Let \( \mathcal{J} = (J, \phi, \alpha, \beta) \) be an irreducible \((l, s)\)-constellation then

\[
|\mathcal{P}^n(\mathcal{J})| = \Theta(n^{|V(J)|} |\mathcal{H}^n(l, 0)|).
\]
3 Exotic examples

3.1 A family of counterexamples to the Reed-Scott’s conjecture

In this subsection we explicitly define a family of graphs, which satisfy Theorem 1.3.

First, an extra notation. Given two families of graphs $F_1$ and $F_2$, let $F_1 \lor F_2$ denote the family of all graphs which are disjoint unions of a graph in $F_1$ and a graph in $F_2$. Similarly, let $F_1 \land F_2$ denote the family of joins of graphs in $F_1$ with graphs in $F_2$. Clearly, if $F_1$ and $F_2$ are hereditary then so are $F_1 \lor F_2$ and $F_1 \land F_2$.

Given a positive integer $l$, we say that a graph $H$ is an $l$-ARS-graph or simply an ARS-graph if $H$ satisfies the following conditions:

(ARS1) For every $1 \leq s \leq l$, $V(H)$ can be partitioned into $s$ stable sets and $l - s$ cliques;

(ARS2) For each graph class $G \in \{ \iota(K_1) \lor C, \iota(S_3) \land C, \iota(C_4) \land C, \iota(\overline{P}_3) \land C \}$, $V(H)$ can be partitioned into $l - 1$ cliques and a set inducing a graph in $G$;

(ARS3) There exists a partition $X_0 = \{X_1, X_2, \ldots, X_l\}$ of $V(H)$, such that $X_1, X_2, \ldots, X_{l-2}$ are cliques, each of $X_{l-1}$ and $X_l$ induce a subgraph of $H$ with exactly one non-edge, and the vertices of these two non-edges form an independent set in $H$;

(ARS4) For every partition $\mathcal{X} \neq X_0$ of $V(H)$ with $|\mathcal{X}| = l$ there exists $X \in \mathcal{X}$ such that $H[X]$ contains at least two non-edges.

In Section 4.1 we prove the following.

Theorem 3.1. For infinitely many integers $l$ there exists a $l$-ARS-graph.

Meanwhile, we will show that every $l$-ARS-graph satisfies Theorem 1.3 thus proving Theorem 1.3 modulo Theorem 3.1.

We start with a few of easy lemmas.

Lemma 3.2. For every positive integer $h$ there exists $N > 0$ satisfying the following. Let $G$ be a graph with $|V(G)| \geq N$ and at least two non-edges then $G$ contains an induced subgraph $J$ with $|V(J)| = h$ such that $J$ is either edgeless, or an antistar, or a join of one of the graphs in $\{ S_3, C_4, P_3 \}$ with a complete graph.

Proof. Let $n$ be a positive integer such that every graph on $n$ vertices contains a homogenous set on $h$ vertices. We show that $N = 5n$ satisfies the lemma.

Let $G$ be as in the lemma statement. We suppose that $G$ contains no stable set on $h$ vertices, as otherwise the lemma holds. Thus every set of $n$ vertices of $G$ contains a clique on $h$ vertices. Suppose first that there exists $v \in V(G)$ with at least $n$ non-neighbors, then $v$ together with a clique of size $h - 1$ chosen among its non neighbors induces a desired antistar. Thus we assume that every vertex of $G$ has at most $n$ non-neighbors. It follows

5For example, $S \lor S = S$ and $S \land S$ is the family of complete bipartite graphs.
that every set of at most four vertices of $G$ has at least $n$ common neighbors, and so there exists a clique of size $h$ among those neighbors.

Thus it suffices to show that if $G$ contains at least two non-edges, then it contains an induced subgraph isomorphic to one of $S_3, C_4$ or $P_3$, but this is clear. □

**Corollary 3.3.** Let $H$ be an $l$-ARS graph then $\chi_c(H) = l$, and there exists an integer $n_0$ such that every graph $G \in \text{red}(H)$ with $|V(G)| \geq n_0$ contains at most one non-edge. In particular, $H$ is 2-critical.

**Proof.** It follows from (ARS1) and (ARS2) that $\chi_c(H) < l + 1$, and it follows from (ARS3) and (ARS4) that $V(H)$ can not be partitioned into $l$ cliques, implying that $\chi_c(H) \geq l$. Thus $\chi_c(H) = l$.

Let $h = |V(H)|$ and let $n_0$ be such that the conclusion of Lemma 3.2 holds with $N = n_0$. Suppose for a contradiction that there exists $G \in \text{red}(H)$ with $|V(G)| \geq n_0$ such that $G$ has at least two non-edges. Then by Lemma 3.2 there exists $J \in \text{red}(H)$ such that $|V(J)| \geq |V(H)|$ and $J$ is either edgeless, or an antistar, or a join of one of the graphs in $\{S_3, C_4, P_3\}$ with a complete graph. It follows from (ARS1) and (ARS2) that for every $0 \leq s \leq l - 1$ we can partition $V(H)$ into $s$ stable sets, $l - 1 - s$ cliques and a subgraph of $J$, a contradiction. □

Let $\mathcal{I} = \mathcal{I}(S_2) \land \mathcal{C}$ denote the family of graphs with at most one non-edge, and let $\mathcal{A}(l)$ denote the family of all graphs $G$ such that there exists an $(\mathcal{I}, l)$-partition $\mathcal{X}$ of $G$ such that $X \cup X'$ does not contain an independent set of size four for all $X, X' \in \mathcal{X}$. The following theorem describes the structure of typical $H$-free graphs for an $l$-ARS graph $H$.

**Theorem 3.4.** Let $H$ be an $l$-ARS graph. Then

(i) $\mathcal{A}(l) \subseteq \text{Forb}(H),$

(ii) almost all $H$-free graphs are in $\mathcal{A}(l),$

(iii) $|\text{Forb}^n(H)| = \Theta(n^{2l}|\mathcal{H}^n(l, 0)|).

**Proof.** The condition (i) follows from (ARS3) and (ARS4).

Let $\mathcal{J} = (J, \phi, \alpha, \beta)$ be an $(l, 2)$-constellation such that $\mathcal{P}(\mathcal{J}) \subseteq \text{Forb}(H)$. We claim that $\mathcal{P}(\mathcal{J}) \subseteq \mathcal{A}(l)$. By Corollary 3.3 and Theorem 2.7 this claim implies (ii). By Lemma 2.8 it additionally implies $|\text{Forb}^n(H)| = O(n^{2l}|\mathcal{H}^n(l, 0)|)$.

Note that (ARS1)-(ARS3) imply that $\beta$ and $\alpha$ are identically one, and there does not exist an independent set of $\{u_1, v_1, u_2, v_2\}$ in $J$ such that $\phi(u_i) = \phi(v_i)$ for $i = 1, 2$. This observation immediately implies $\mathcal{P}(\mathcal{J}) \subseteq \mathcal{A}(l)$, as claimed.

Let $J$ be a graph with $V(J) = \{u_i, v_i\}_{i=1}^l$ obtained from a complete graph by deleting edges $u_iv_i$ for every $i \in [l]$. Let $\phi(u_i) = \phi(v_i) = i$ for every $i \in l$, and let $\beta$ and $\alpha$ be identically one, and let $\mathcal{J} = (J, \phi, \alpha, \beta)$. Then $\mathcal{P}(\mathcal{J})$ consists of all graphs $G$ such that there exists an $(\mathcal{I}, l)$-partition $\mathcal{X}$ of $G$ so that if $u_1, v_1 \in X_1$, $u_2, v_2 \in X_2$ are pairwise distinct vertices for some $X_1, X_2 \in \mathcal{X}$ and $u_iv_i \notin E(G)$ for $i = 1, 2$ then $u_1u_2, u_1v_2, v_1u_2, v_1v_2 \notin E(G)$. It follows that $\mathcal{P}(\mathcal{J}) \subseteq \mathcal{A}(l)$. As $\mathcal{J}$ is irreducible, we have $|\mathcal{P}^n(\mathcal{J})| = \Theta(n^{2l}|\mathcal{H}^n(l, 0)|)$ by Lemma 2.8.

Together with the upper bound established above this implies (iii) □
Proof of Theorem 1.3. We show that if $H$ is an $l$-ARS graph for some $l \geq 2$, then almost all $H$-free graphs admit no clean $H$-profile.

Let $\mathcal{F}$ be a clean $H$-free pattern. Then $\mathcal{F} \subseteq \text{red}(H)$ for every $\mathcal{F} \in \mathcal{F}$. By Corollary 3.3 and Lemma 2.3, there exists $\varepsilon > 0$ such that almost every graph in $G \in P(\mathcal{F})$ admits an $(I,l)$-partition and such a partition is unique and $\varepsilon$-balanced. By (ARS3) at most one element of $\mathcal{F}$ contains $I$, and thus if $V(G)$ is sufficiently large the above partition must be a partition of $V(G)$ into at most one non-edge and $l - 1$ cliques, corresponding to a $J$ template for an $(l,2)$-constellation $J = (J, \phi, \alpha, \beta)$ with $|V(J)| = 2$. Thus by Lemma 2.8 and Theorem 3.4 (iii) we have

$$|P_n(\mathcal{F})| = O(n^2 |H^n(l,0)|) = o(|\text{Forb}(H)|),$$

as desired. \qed

3.2 Graphs with no asymptotic linear Erdős-Hajnal property.

In this section we describe a family of graphs satisfying Theorem 1.8.

Let $l$ be an integer. We say that a graph $H$ is a $(P_3,l)$-jumble or simply a $P_3$-jumble if

(S1) for every $0 \leq s \leq l - 1$, $V(H)$ can be partitioned into $s$ stable sets, $l - 1 - s$ cliques, and a set $Z$ such that $H[Z]$ is isomorphic to $P_3$, and

(S2) for every partition $X_1, X_2, \ldots, X_l$ of $V(H)$ there exists $i$ such that $H[X_i]$ is not $P_3$-free.

In Section 4.2 we will show the following.

Theorem 3.5. There exist $(P_3,l)$-jumbles for infinitely many postive integers $l$.

In this section we show that every $P_3$-jumble has no asymptotic linear Erdős-Hajnal property, thus proving Theorem 1.8 modulo Theorem 3.5. The main step of the argument is the description of a typical $H$-free graph for a $P_3$-jumble $H$, which follows directly from Theorem 2.1.

Theorem 3.6. Let $H$ be an $(P_3,l)$-jumble, and let $\mathcal{T} = \text{Forb}(P_3)$. Then $P(\mathcal{T},l) \subseteq \text{Forb}(H)$. Conversely, there exists $\varepsilon > 0$ such that almost every graph $G \in \text{Forb}(H)$ admits a unique $\varepsilon$-balanced $(\mathcal{T},l)$-partition.

Proof. Let $\mathcal{F} = \text{Forb}(H)$. We have $P(\mathcal{T},l) \subseteq \mathcal{F}$ by (S2). Thus, $\chi_c(\mathcal{F}) \geq l$. On the other hand (S1) implies that $\chi_c(\mathcal{F}) \leq l$, and $P_3$ is dangerous for $\mathcal{F}$. As $\mathcal{T}$ is clearly extendable, Lemma 2.2 implies that $P(\mathcal{T},l)$ is smooth. Applying Theorem 2.1 and Lemma 2.3 with $\mathcal{K} = \{P_3\}$ now yields the conclusion. \qed

Corollary 3.7. Let $H$ be a $P_3$-jumble. Then almost all $H$-free graphs $G$ on $n$ vertices satisfy

$$h(G) = O\left(\frac{n}{\log n}\right).$$

Proof. Let $l = \chi_c(H)$, $\mathcal{F} = \text{Forb}(H)$, $\mathcal{T} = \text{Forb}(P_3)$ and let $\varepsilon > 0$ be as in Theorem 3.6. We say that $(G, X)$ is a good pair if $G \in \mathcal{F}$, and $X$ is the unique $\varepsilon$-balanced $(\mathcal{T},l)$-partition of $G$. By Theorem 3.6 almost every $G \in \mathcal{F}$ belongs to a (unique) good pair.
We claim that for every \( \varepsilon \)-balanced partition \( \mathcal{X} \) of \([n]\), we have \( h(G) = O\left( \frac{n}{\log n} \right) \) for almost every good pair \((G, \mathcal{X})\). Clearly this claim implies the corollary.

Let \( \mathcal{X} = (X_1, \ldots, X_t) \), and let \( \mathcal{Z} \) be the set of all possible sequences \((G_1, \ldots, G_t)\) such that \( V(G_i) = X_i \) and \( G_i \in \mathcal{T} \). Every such sequence extends to \( 2^m \) graphs \( \mathcal{F}^n \), where \( m = \sum_{1 \leq i \leq t} |X_i||X_j| \). Let \( \mathcal{Z}(C) \) be the set of all sequences in \( \mathcal{Z} \) such that \( h(G_i) \leq C \frac{n}{\log n} \) for every \( i \in [l] \). By Observation 1.7 there exists \( C > 0 \) independent on \( n \) such that \( |\mathcal{Z}(C)| = |\mathcal{Z}| - o(\mathcal{Z}) \). By Lemma 2.4 almost every extension of a given sequence in \( \mathcal{Z}(C) \) gives rise to a good pair, implying that at least \( (1 - o(1))2^m|\mathcal{Z}| \) good pairs of the form \((G, \mathcal{X})\) satisfy \( h(G) \leq lC \frac{n}{\log n} \). On the other hand the remaining sequences in \( \mathcal{Z} \) correspond to \( o(2^m|\mathcal{Z}|) \) good pairs, which finishes the proof of the claim.

### 3.3 Exotic PS-critical graphs

In this section we present a family of graphs satisfying Theorem 2.6. The description of these graphs is very similar to the description of ARS-graphs in Section 3.3 and some of the results from that section carry over after minor modifications.

First, let us show that the definition of 1-critical graphs is equivalent to the original definition given in [PS93], as promised in the introduction. We need the following analogue of Lemma 3.2.

**Lemma 3.8.** For every positive integer \( h \) there exists \( N \) satisfying the following. If \( G \) is a graph with \( |V(G)| \geq N \) and \( G \) is neither complete nor edgeless, then \( G \) contains an induced subgraph \( J \) such that \( |V(J)| = h \) and either \( J \) or \( \bar{J} \) is a star or has exactly one edge.

**Proof.** Let \( N \) be such that every graph on at least \( N \) vertices contains a homogenous set of size at least \( 2h \). Thus without loss of generality we assume that \( G \) contains a maximal independent set \( S \) such that \( |S| \geq 2h \). As \( G \) is not edgeless there exists \( v \in V(G) - S \) and \( v \) has a neighbor in \( S \). If \( v \) has at least \( h \) neighbors in \( S \) then \( G[S \cup \{v\}] \) contains a star on \( h \) vertices and, otherwise, \( G[S \cup \{v\}] \) contains an \( h \) vertex induced subgraph with exactly one edge.

**Lemma 3.9.** For every positive integer \( h \) there exists \( N \) satisfying the following. If \( G \) is a graph with \( |V(G)| \geq N \) such that neither \( G \) nor \( \bar{G} \) is edgeless or a star, then \( G \) contains an induced subgraph \( |V(J)| = h \) and either \( J \) or \( \bar{J} \) has exactly one edge, or is obtained from a star by deleting one edge, or is join of a graph on two vertices and an independent set.

**Proof.** By Lemma 3.8 there exists \( N \) such that every graph \( G \) with \( |V(G)| \geq N \) such that neither \( G \) nor \( \bar{G} \) is edgeless contains an induced subgraph \( J' \) such that \( |V(J')| \geq 2h \) and either \( J' \) or \( \bar{J}' \) is a star or has exactly one edge. We may assume without loss of generality that \( J' \) is a maximal induced subgraph of \( G \) which is a star. Let \( S \) be the set of all the leaves of \( J' \), and let \( u \) be the center of \( J' \).

Consider arbitrary \( v \in V(G) - V(J') \). The join of \( G[\{u, v\}] \) and the neighborhood of \( v \) in \( S \) is an induced subgraph of \( G \). Therefore, if \( v \) has at least \( h \) neighbors in \( S \) then the lemma holds, and so we assume that \( v \) has at least \( h \) non-neighbors in \( S \). If \( v \) has at least one neighbor in \( S \) then we can find an \( h \) vertex induced subgraph of \( G \) with exactly one edge. It remains to consider the case, when \( \{v\} \cup S \) is independent. By maximality of \( J \), we
have $uv \notin E(G)$, and so $V(J) \cup \{v\}$ induces a graph obtained from a star by deleting one edge, as desired.

In [PS93, BB11] a graph $H$ is defined to be PS-critical if every sufficiently large join of a graph on two vertices and an independent set is dangerous for $\text{Forb}(H)$, and the same is true for every sufficiently large graph obtained from a star by deleting an edge, as well as for the complements of the above graphs. The next corollary, which is an immediate consequence of Lemma 3.9, implies that this definition coincides with the definition of 1-critical given in the introduction.

**Corollary 3.10.** Let $\mathcal{F}$ be a hereditary family. Suppose that there exists graphs $J_1, J_2, J_3$ such that $J_i, \bar{J}_i \in \text{dang}(\mathcal{F})$ for every $i \in [3]$, $J_1$ is obtained from a star by deleting an edge, and $J_2$ and $J_3$ are joins of an independent set with $K_2$ and $S_2$, respectively. Then $\mathcal{F}$ is 1-critical.

**Proof.** It follows from Lemma 3.9 that there exists an integer $n_0$ such that for every graph $K \in \text{red}(\mathcal{F})$ with $|V(K)| \geq n_0$ either $K$ or $\bar{K}$ is edgeless or a star.

We are almost ready to define the family of graphs satisfying Theorem 2.6. In addition to the classes of graphs used to define ARS-graphs, we need to introduce the following graph class. We denote by $\mathcal{C}^+$ the family of all graphs $G$ such that either $G$ is complete, or $G \setminus v$ is complete for some vertex $v \in V(G)$ such that $\deg(v) \leq 1$.

Let $l$ be a positive integer. We say that a graph $H$ is an $l$-EPS-graph (or simply an EPS-graph) if $H$ satisfies the following conditions:

(EPS1) For every $1 \leq s \leq l$, $V(H)$ can be partitioned into $s$ stable sets and $l - s$ cliques;

(EPS2) For each graph class $\mathcal{G} \in \{\iota(S_2) \lor \mathcal{C}, \iota(K_2) \lor \mathcal{C}, \iota(S_2) \land \mathcal{C}, \mathcal{C}^+\}$, $V(H)$ can be partitioned into $l - 1$ cliques and a set inducing a graph in $\mathcal{G}$;

(EPS3) There exists no partition $\mathcal{X} = (X_1, X_2, \ldots, X_l)$ of $V(H)$ such that $H[X_1]$ is a clique or a complement of a star, and $X_i$ is a clique in $H$ for $2 \leq i \leq l$.

**Lemma 3.11.** Every EPS-graph is 1-critical, but not 0-critical.

**Proof.** Let $H$ be an $l$-EPS-graph. It follows from (EPS1) and (EPS2) that $\chi_c(H) \leq l$, and it follows from (EPS3) that $\chi_c(H) \geq l$. Therefore $\chi_c(H) = l$, and (EPS1) and (EPS2) further guarantee that there exists graphs $J_1, J_2$ and $J_3$ satisfying the conditions of Corollary 3.10 for $\mathcal{F} = \text{Forb}(H)$. Thus $H$ is 1-critical by Corollary 3.10.

By (EPS3) there exists no partition $(P_1, P_2, \ldots, P_l)$ of $V(H)$ such that $H[P_1]$ is a complement of a star, and $P_2, \ldots, P_l$ are cliques in $H$. Therefore $H$ is not 0-critical, as no complement of a star is $\text{Forb}(H)$-dangerous.

By Lemma 3.11 the following theorem, which is proved in Section 4.1, implies Theorem 2.6.

**Theorem 3.12.** There exists infinitely many EPS-graphs.
4 Constructions

4.1 Proof of Theorems 3.1 and 3.12

In this section we construct infinite families of ARS-graphs and of EPS-graphs.

Let us start by informally sketching our construction. We start by specifying the partitions of $V(H)$ which will satisfy properties (ARS1)-(ARS3) and (EPS1)-(EPS2), respectively. These partitions will be chosen using a randomized procedure subject to certain transversality conditions. The graph $H$ will then be constructed by specifying its structure within each part of the partitions. The transversality and randomness will be used to ensure that these specifications don’t conflict with each other, and that every large enough “structured” subgraph of $H$ is close to being a part of one of the partitions. The last condition will guarantee that the respective conditions (ARS4) and (EPS3) also holds.

We use the following definitions in the description of the properties of the constructed partitions. We say that families of finite sets $\mathcal{P}$ and $\mathcal{P}'$ are transversal if $|P \cap P'| \leq 1$ for all $P \in \mathcal{P}'$ and $P' \in \mathcal{P}$. We say that a set $S$ is covered by $\mathcal{P}$ if there exists $P \in \mathcal{P}$ such that $S \subseteq P$, and otherwise, we say that a set $S$ is uncovered by $\mathcal{P}$. We say that an element of $x \in P$ is $P$-exclusive with respect to $\mathcal{P}$ for $P \in \mathcal{P}$ if for every $P' \in \mathcal{P}$ such that $P' \neq P$ and $x \in P'$, we have $|P \cap P'| = 1$. We say that $P \in \mathcal{P}$ is distinctive with respect to $\mathcal{P}$ if there exist at least two $P$-exclusive elements. We say that a set $S$ is $\mathcal{P}$-wild if at least two distinct two element subsets of $S$ are uncovered by $\mathcal{P}$, and, otherwise we say that $S$ is $\mathcal{P}$-tame. We say that a family of sets is $\mathcal{P}$-tame if every element of it is $\mathcal{P}$-tame.

The technical part of the proof of Theorems 3.1 and 3.12 consists of establishing the following lemma.

**Lemma 4.1.** Let $l$ and $k$ be positive integers such that $l \geq 500k$, $k^{3/2} \geq 3600l$ and $(l-k)k$ is divisible by $l$. Let $X$ be a set with $|X| = (l-k)k + l$. Then there exist partitions $\mathcal{Q} = (Q_0, Q_1, Q_2, \ldots, Q_k), \mathcal{P}_0, \ldots, \mathcal{P}_4$ of $X$ and a partition $\mathcal{R}$ of $X - Q_0$ with the following properties:

(Q) $|Q_0| = l$ and $|Q_j| = l - k$ for every $j \in [k],$

(R) $|\mathcal{R}| = l - k$, and $\mathcal{R}$ and $\mathcal{Q}$ are transversal,

moreover, for every $i \in \{0, 1, 2, 3, 4\}$ we have

(P1) $|\mathcal{P}_i| = l$ and $|P| = k(l-k)/l + 1$ for every $P \in \mathcal{P}_i,$

(P2) $\mathcal{P}_i$ and $\mathcal{Q}$ are transversal,

finally, let $\mathcal{P} = \mathcal{R} \cup (\cup_{j=0}^{4} \mathcal{P}_j)$, then

(P3) at least two sets of $\mathcal{P}_i$ are distinctive with respect to $\mathcal{P},$

(P4) let $Z \subseteq X$ be such that $|Z| \leq 9$, then for every $\mathcal{P}$-tame partition $\mathcal{P}_*$ of $X - Z$ with $|\mathcal{P}_*| = l$ there exists $0 \leq i \leq 4$ such that $\mathcal{P}_*$ coincides with $\mathcal{P}_i$ on $X - Z$.

Before proceeding to the proof of Lemma 4.1 we derive Theorems 3.1 and 3.12 from it.
Proof of Theorem 3.14. Given positive integers $l, k$ satisfying the conditions of Lemma 4.1, we will construct a $l$-ARS-graph $H$. Let $X, Q, R, P_0, \ldots, P_4$ be as in Lemma 4.1.

Let $V(H) = X$. The edges of $H$ are determined as follows. Let $P'_0, P''_0$ be two sets in $\mathcal{P}_0$ distinctive with respect to $\mathcal{P}$, and let $P_i$ be a distinctive set in $\mathcal{P}_i$ for $1 \leq i \leq 4$. Let $Z = \{P'_0, P''_0, P_1, \ldots, P_4\}$. For each $P \in \mathcal{P}$ we join every pair of vertices in $P$ by an edge, and we modify $H[Z]$ for $Z \in Z$ as follows. We delete an edge from each of $P'_0$ and $P''_0$, delete all edges incident to a single vertex in $P_1$, two edges sharing an end in $P_2$, a matching of size two in $P_3$, and edges of a triangle in $P_4$. As the sets in $Z$ are distinctive, we can do the deletions so that all the deleted edges are incident to the vertices in the corresponding sets which are exclusive with respect to $\mathcal{P}$, and so do not belong to any other set in $\mathcal{P}$. This finishes the description of the construction of $H$.

It is not hard now to verify that the properties (ARS1)-(ARS4) holds. Note that $Q$ is a partition of $V(H)$ into $k + 1$ stable sets, and $\mathcal{R} \cup \{Q_0\}$ is a partition of $V(H)$ into $l - k$ cliques and a stable set. Thus (ARS1) holds. Partitions $\mathcal{P}_1, \ldots, \mathcal{P}_4$ satisfy (ARS2), and $\mathcal{P}_0$ satisfies (ARS3) by construction. Finally, suppose for a contradiction that (ARS4) does not hold. Thus there exists a partition $\mathcal{X}$ of $X$ with $|\mathcal{X}| = l$ such that $\mathcal{X} \neq \mathcal{P}_0$ and every part of $\mathcal{X}$ induces in $H$ a subgraph with at most one non-edge. Thus by construction every element of $\mathcal{X}$ is $\mathcal{P}$-tame. It follows from (P4) that $\mathcal{X} = \mathcal{P}_i$ for some $1 \leq i \leq 4$, and thus some part of $X$ induces at least two non-edges, a contradiction. \qed

Proof of Theorem 3.12. The proof is analogous to the proof of Theorem 3.1 above, the only difference is that we adapt the description of the edges of $H$ to satisfy (EPS2)-(EPS3), rather than (ARS2)-(ARS4), as follows.

As in the proof of Theorem 3.1 let $P_i$ be a distinctive set in $\mathcal{P}_i$ for $1 \leq i \leq 4$ and let $v_i, u_i \in P_i$ be $P_i$-exclusive with respect to $\mathcal{P}$. Let $Z' = \{u_1, v_1, \ldots, u_4, v_4\}$. For each $P \in \mathcal{P} - \mathcal{P}_0$ we join every pair of vertices in $P$ by an edge, and we modify $H[P_i]$ for $1 \leq i \leq 4$.

We delete the following edges: $u_1v_1$, all edges in $H[P_2]$ incident to $u_2$ or $v_2$, all edges in $H[P_3]$ incident to $u_3$ or $v_3$, except $u_3v_3$, and all edges in $H[P_4]$ incident $u_4$, except $u_4v_4$. As before restricting deletions to the edges incident to the exclusive vertices ensures that these deletions do not affect the subgraphs induced by other sets in $\mathcal{P}$. Finally, for each $z \in Z'$ we add to $H$ edges joining $z$ to all vertices $x \in X$ such that $\{x, z\}$ is uncovered by $\mathcal{P}$ and $x \in Q_i$ for some $1 \leq i \leq k/2$.

As in the proof of Theorem 3.1 note that partitions $\mathcal{Q}$ and $\mathcal{R} \cup \{Q_0\}$ satisfy (EPS1), and partitions $\mathcal{P}_1, \ldots, \mathcal{P}_4$ satisfy (EPS2), by construction.

It remains to verify (EPS3). Suppose that $\mathcal{X} = (X_1, \ldots, X_l)$ is a partition of $X$ violating (EPS3). Let $x_i \in X_i$ be such that $X_i - \{x_i\}$ is a clique in $H$, and let $Z = Z' \cup \{x_1\}$. Then the restriction $\mathcal{P}_* = \mathcal{P}_*$ of $\mathcal{X}$ to $X - Z$ is as in Lemma 4.1 (P4). Thus $\mathcal{P}_*$ coincides with $P_i$ on $X - Z$ for some $i \in [4]$. However, $P_i$ is not a subset of a part of $\mathcal{X}$. Thus there exists $v \in P_i$, $P' \in \mathcal{P}_i, P' \neq P_i$ and $j \in [i]$ such that $\{v\} \cup (P' - Z) \subseteq X_j$. But $v$ is neither complete nor anticomplete to $P' - Z$ by construction, a contradiction. Thus (EPS3) holds. \qed

Proof of Lemma 7.11. Clearly, it is possible to select $\mathcal{Q}$ and $\mathcal{R}$ as in the lemma statement satisfying (Q) and (R).

Given $\mathcal{Q}$ and $\mathcal{R}$, we select $\mathcal{P}_0, \ldots, \mathcal{P}_4$ from the set of all partitions of $X$ satisfying (P1) and (P2) uniformly and independently at random. We will show that with positive probability
both (P3) and (P4) hold. Rather than verifying (P4) directly, first, for every pair \( \{P', P''\} \subset \{R, P_0, \ldots, P_4\} \), we will require that

\[(P5) \|P' \cap P''\| \leq \sqrt{k} \text{ for all } P' \in P' \text{ and } P'' \in P''.\]

\[(P6) \text{ For all } P_* \subseteq P' \text{ and } P_*' \subseteq P'' \text{ such that } |P'_*| \geq l/10 \text{ and } |P''_*| \geq \sqrt{k}/10 \text{ we have} \]

\[\left( \bigcup_{P' \in P'_*} P' \right) \cap \left( \bigcup_{P'' \in P''_*} P'' \right) \neq \emptyset.\]

Our first goal is to show that each of (P3),(P5) and (P6) is satisfied with probability at least 3/4.

We start with (P3). We say that \( x \in X \) is \( P_0 \)-exclusive if \( x \) is \( P \)-exclusive with respect to \( P \) for \( P \in P_0 \) such that \( x \in P \). Let \( P' = R \cup P_1 \ldots \cup P_4 \). Fix arbitrary \( x \in X \) and let \( y(i) \) be the unique element of \( Q_i \) such that \( \{x, y(i)\} \) is covered by \( P_0 \), if such an element exists. It is easy to see that the probability that \( \{x, y(i)\} \) is covered by \( P' \) is at most \( 5/(l - k) \). It follows that the probability that \( x \) is not \( P_0 \)-exclusive is at most \( 5k/(l - k) \). Therefore, the expected number of not \( P_0 \)-exclusive elements is at most

\[|X| \frac{5k}{l - k} = 5k^2 + \frac{5kl}{l - k} \leq 10k^2,\]

and so the probability that there are at least 200\(k^2\) elements of \( X \) which are not \( P_0 \)-exclusive is at most 1/20. It follows that (P3) holds for \( P_0 \) with probability at least 19/20. By symmetry, (P3) holds for all \( i \) with probability at least 3/4.

Moving on to (P5), we say that \( P \) is a full \( Q \)-transversal if \( |P \cap Q| = 1 \) for all \( Q \in Q \). Let \( P_1, P_2 \) be two full \( Q \)-transversal chosen uniformly and independently at random. Then

\[\Pr[|P_1 \cap P_2| \geq \sqrt{k}] \leq \frac{(k + 1)^{\sqrt{k}}}{(l - k)^{\sqrt{k}}} \leq \frac{1}{2\sqrt{k}} < \frac{1}{60l^2}.\]

Note that \( P' \), \( P'' \) and in (P5) can be considered as subsets of two full \( Q \)-transversal chosen uniformly and independently at random. Thus the inequality above and the union bound imply that (P5) fails with probability at most 1/4.

Next, for (P6), let \( Y = \cup_{P' \in P'_*} P' \), and let \( Z = \cup_{P'' \in P''_*} P'' \). Let \( Y_i = Q_i \cap Y \) and let \( Z_i = Q_i \cap Z \) for every \( i \in [k] \). Note that \( |Y_i| \geq l/10 - k \geq l/20 \) for every \( i \), as at most \( k \) sets in \( P' \) are disjoint from \( Y_i \).

Let

\[B = \{(i, P'' \mid i \in [k], P'' \in P''_* \cap Z \cap Z_i = \emptyset\}.\]

As every \( P'' \in P''_* \) is disjoint from at most \( k - k(l - k)/l \) sets in \( Q \), we have

\[|B| \leq \frac{k^2\|P''_*\|}{l}.\]

Let \( J = \{i \in [k] \mid |Z_i| \geq \sqrt{k}/30\}. \) As every \( i \in [k] - J \) belongs to at least \( |P''_*| - \sqrt{k}/30 \geq 2\|P''_*\|/3 \) elements of \( B \) we have

\[(k - |J|) \frac{2\|P''_*\|}{3} \leq k^2\|P''_*\|/l,\]
and so $|J| \geq k/2$. We have

$$
\Pr[Y_i \cap Z_i = 0] \leq \left(1 - \frac{|Y_i|}{l-k}\right)^{|Z_i|} \leq \left(\frac{19}{20}\right)^{\frac{k}{30}} \leq \left(\frac{1}{2}\right)^{\frac{k}{600}}
$$

for every $i \in J$. The corresponding events are independent for all $i \in J$, and so we have

$$
\Pr[Y \cap Z = 0] \leq \left(\frac{k}{30}\right)^{k/600}
$$

Summing over all possible choices of subset $\{P', P''\}$ and subsets $P_* \subseteq P'$ and $P''_* \subseteq P''$, we conclude that (P6) fails with probability at most

$$
15 \cdot 2^{2t} \cdot \left(\frac{1}{2}\right)^{k/12} \leq 15 \cdot \left(\frac{1}{2}\right)^{t} \leq \frac{1}{4},
$$

as desired.

Thus there exist partitions $P_0, \ldots, P_4$ of $X$ satisfying properties (P1)-(P3),(P5) and (P6). We claim that these properties imply (P4). Clearly, this claim implies the lemma.

First, we will show that the following additional property holds.

(P7) Let $S \subseteq X$ be such that $S$ is $P$-tame and $|S| > k/2$ then $S$ is covered by $P$.

By considering pairs containing arbitrary $x \in S$, we deduce that $|S \cap P'| \geq k/12 - 1$ for some $P'' \in P$. Suppose that there exists $y \in S - P$. By considering the pairs consisting of $y$ and an element of $S \cap P'$ we deduce that there exists $P'' \in P, P'' \neq P'$ such that $|P' \cap P''| \geq k/72 - 2 > \sqrt{k}$, contradicting (P5). Thus $S \subseteq P'$, and (P7) holds.

Define a weight function $w : X \to \mathbb{R}_+$ by setting $w(x) = 1$ for $x \in X - Q_0$ and $w(x) = k^2/l$ for $x \in Q_0$. Note that $w(P) = k$ for every $P \in P$, and $w(X) = kl$.

Let $Z, P_*$ be as in (P4). Then $\sum_{P \in P_*} w(P) = kl - w(Z)$. Suppose that there exists $P \in P_*$ with $w(P) > k$. Then $|P| \leq k/2$ by (P7). Moreover, $|P \cap Q_0| \leq 2$ as $P$ is $P$-tame. Therefore $w(P) \leq 2k^2/l + k/2 < k$, a contradiction. Thus $w(P) \leq k$ for every $P \in P_*$, and so

$$
\sum_{P \in P_*} (k - w(P)) \leq w(Z) \leq \frac{9k^2}{l}
$$

(1)

for every $P_* \subseteq P_*$. In particular, we have

$$
|P| \geq w(P) - \frac{2k^2}{l} \geq k - \frac{11k^2}{l} \geq \frac{9k}{10}
$$

for every $P \in P_*$.

By (P7) there exists $P' \in \{R, P_0, \ldots, P_4\}$ such that at least $l/6$ elements of $P_*$ are covered by $P'$. If every element of $P_*$ is covered by $P'$ then (P4) holds.

Thus we assume that there exists $S \in P_*$ such that $S$ is uncovered by $P'$. Let $P_*'$ be the set of all $P' \in P'$ such that there exists $P \in P_*$ so that $P \subseteq P'$, and let

$$
\mathcal{L} = \{(P, P') \mid P \in P_*, P' \in P_*', P \subseteq P'\}.
$$

Then

$$
\sum_{P' \in P_*} |P' \cap P| \leq \sum_{(P, P') \in \mathcal{L}} |P' \cap S| \leq \sum_{(P, P') \in \mathcal{L}} |P' - P| \leq \sum_{(P, P') \in \mathcal{L}} (k - w(P)) \leq \frac{9k^2}{l} \leq \frac{k}{5},
$$

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where the penultimate inequality holds by \( \square \). Therefore at least \( 4k/5 \) elements of \( S \) are covered by an element of \( P' - P'' \).

By (P5) it follows that \( |P' - P''| \geq \sqrt{k}/2 \), implying that there exists
\[
P'' \in \{ R, P_0, \ldots, P_4 \} - \{ P' \}
\]
such that at least \( \sqrt{k}/10 \) elements of \( P_* \) are covered by \( P'' \). Define \( P_* \) to be the set of all elements of \( P'' \) which contain an element of \( P_* \). Let
\[
I = (\cup_{P' \in P_*} P') \cap (\cup_{P'' \in P_*} P'').
\]
Then \( |I| \geq |P'| - l/10 \geq l/15 \) as at most \( l/10 \) elements of \( P_* \) are disjoint from \( \cup_{P'' \in P_*} P'' \) by (P6). Let \( P_* \) be the set of elements of \( P_* \) covered by \( P'_* \cup P''_* \). Repeating the calculations in the previous paragraph with \( P_* \) instead of \( P'_* \), and \( I \) instead of \( S \), we obtain
\[
\sum_{P \in P} (k - w(P)) \geq |I| \geq l/15,
\]
in contradiction to \( \square \).

4.2 Constructing \( P_3 \)-jumbles

Let \( l \) be a positive integer. Let \( X \) be a finite set with \( |X| = l^2 \), and let \( R, C, D \) be a triple of partitions of \( X \) such that \( |R| = |C| = |D| = l \), and each part of one of these partitions has exactly one element in common with each part of every other partition. We say that the triple \( B = (R, C, D) \) is an \( l \)-square on \( X \), and we refer to elements of \( R, C, D \) as rows, columns and diagonals of \( B \), respectively, and to all elements of \( R \cup C \cup D \) as lines of \( B \). Note that \( l \)-squares exist for every \( l \), indeed they are just different representations of Latin squares.

We use \( l \)-squares as the building blocks of a more involved construction. An \( l \)-pattern \( Z \) is a tuple \( (P, B_1, \ldots, B_l) \) such that \( P \) is a graph isomorphic to \( P_3 \), and \( B_i \) is an \( l \)-square on a set \( X_i \) for every \( 1 \leq i \leq l \), such that \( V(P) \) and \( X_1, \ldots, X_l \) are pairwise vertex disjoint. Let \( V(Z) = V(P) \cup X_1 \cup \ldots \cup X_l \). In particular, \( |V(Z)| = l^3 + 3 \). The lines of \( Z \) are the lines of its squares.

Let \( G \) be a graph, and let \( B \) be an \( l \)-square on \( X \subseteq V(G) \). We say that \( B \) induces an \( (l, r) \)-square in \( G \) for an integer \( 0 \leq r \leq l \) if the rows of \( B \) and \( r \) diagonals of \( B \) induce independent sets in \( G \), while the columns of \( B \) and the remaining \( l - r \) diagonals induce cliques.

Finally, given an \( l \)-pattern \( Z = (P, B_1, \ldots, B_l) \) we construct a random graph \( G = G(Z) \) with \( V(G) = V(Z) \) as follows. We say that a pair of vertices of \( \{u, v\} \) of \( V(Z) \) is fixed if either \( \{u, v\} \subseteq V(P) \) or \( \{u, v\} \subseteq L \) for some line of \( Z \), and we say that \( \{u, v\} \) is free, otherwise. The intersection of \( E(G) \) with the set of fixed pairs is defined deterministically as follows. We require that \( G[V(P)] \) coincides with \( P \), and that \( B_i \) induces an \( (l, i) \)-square in \( G \) for every \( i \). Each free pair of vertices forms an edge with probability \( 1/2 \) independently at random. The following lemma is the main result of this section and immediately implies Theorem 3.5.

Lemma 4.2. Let \( Z = (P, B_1, \ldots, B_l) \) be an \( l \)-pattern. Then the random graph \( G = G(Z) \) is a.a.s. an \( (P_3, l^2 + 1) \)-jumble.
Proof. First, using the fixed part of $G$, we show that for every $0 \leq s \leq l^2$ there exists a partition of $G \setminus V(P)$ into $l^2$ parts, inducing $s$ stable sets and $l^2 - s$ cliques, thus verifying that the condition (S1) in the definition of $(P_3, l^2 + 1)$-jumble always holds. Indeed, let $s = q l + r$ for some $0 \leq q \leq l$ and $0 \leq r \leq l - 1$. If $q = l$ then $r = 0$ and the desired partition consists of rows of all the squares of $Z$. Otherwise, we form the partition by taking the diagonals of $B_r$, the rows of $q$ other squares of $Z$ and the columns of the remaining squares.

We say that $G$ is regular if every $X \subseteq V(Z)$ such that $G[X]$ is $P_3$-free satisfies one of the following

(X1) $X$ is a subset of a line of $Z$,

(X2) $|X| \leq 3l/5$ and $X - \{x\}$ is a subset of a line of $Z$ for some $x \in X$,

(X3) $|X| \leq 2l/5$.

Note that if $G$ is regular then $G$ satisfies the condition (S2) in the definition of $(P_3, l^2 + 1)$-jumble. Indeed, suppose for a contradiction that $X_1, X_2, \ldots, X_{l^2+1}$ is a partition of $V(G)$ such that $G[X_i]$ is $P_3$-free for every $i$. Then $V(P)$ intersects at least two distinct part of the partition, and thus we suppose without loss of generality that either $|V(P) \cap X_1| \geq 2$ and $|V(P) \cap X_2| \geq 1$, or $|V(P) \cap X_i| = 1$ for $i = 1, 2, 3$. It follows from regularity of $G$ that $|X_1| + |X_2| + |X_3| \leq 2l$ in both cases, and $|X_j| \leq l$ for every $j \geq 4$. Thus $\sum_{i=1}^{l^2+1} |X_i| \leq l^3 < |V(G)|$, a contradiction.

It remains to show that a.a.s. $G$ is regular. We say that $S \subseteq V(G)$ is $v$-diverse for $v \in V(G) - S$ if $v$ has at least two neighbors and at least two non-neighbors in $S$. A standard application of the Chernoff and union bounds implies that a.a.s. $G$ satisfies the following condition

(*) Let $X$ be a subset of a line $L$ of $Z$ and let $v_1, v_2 \in V(G) - L$ be distinct. If $|X| \geq 4l/7$ then $X$ is $v_1$-diverse, and if $|X| \geq l/3$ then $X$ is either $v_1$-diverse or $v_2$-diverse.

Thus we conclude that (*) holds for $G$. Note that if $X$ is a subset of a line $L$, and $X$ is $v$-diverse, then $G[X \cup \{v\}]$ is not $P_3$-free.

Let $r = \lceil 2l/5 \rceil$. We will show that for fixed $X \subseteq V(Z)$ with $|X| = r$ such that $|X \cap L| < l/3$ for every line $L$ the probability that $G[X]$ is $P_3$-free is at most $\left(\frac{2}{3}\right)^{l^2/500}$. The union bound implies that a.a.s. $G[X]$ is not $P_3$-free for any such $X$. Combining this with (*) we deduce that $G$ is a.a.s. regular.

It remains to verify the above bound on the probability that $G[X]$ is $P_3$-free. We say that an ordered triple $(u, v_1, v_2)$ of vertices of $X$ is a seagull with wings $\{u, v_1\}$ and $\{u, v_2\}$ if $\{u, v_1\}$ and $\{u, v_2\}$ are free. A colony $\mathcal{C}$ is a collection of seagulls, such that no wing of a seagull is a subset of the vertex set of another seagull. Suppose that for all free pairs of vertices of $Z$ except for the wings of the seagulls in $\mathcal{C}$ we determined whether they belong to $G$ or not. Conditioned on this event the probability that a given seagull in $\mathcal{C}$ induces a $P_3$ in $G$ is at least $1/4$, and the corresponding events are independent for all the seagulls in a colony. Thus it suffices to find a colony of size $l^2/500$.

Suppose first that $|X \cap L_0| \geq l/11$ for some line $L_0$ of $G$. Let $Y = X - L_0$. Then $|Y| \geq l/15$ and for every vertex $y \in Y$ there exist at most two vertices in $x \in [X \cap L_0]$ such that $\{x, y\}$ is fixed. Thus for each such $y$ we can find a colony of at least $l/25$ seagulls with
wings sharing the vertex \( y \) and otherwise disjoint from all the remaining vertices in \( X \cap L_0 \). Taking the union of such colonies over all \( y \in Y \) produces the required colony.

Suppose now that \( |X \cap L| \leq l/11 \) for every line \( L \) of \( G \). Choose arbitrary \( Y \subseteq X \) with \( |Y| = \lceil l/15 \rceil \). Then for every vertex \( y \in Y \) there exist at least \( 2l/33 \) vertices in \( X - Y \) which form a free pair with \( y \). We can now repeat the argument in the previous paragraph.

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