From braid groups to mapping class groups

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Abstract

This paper is a survey of some properties of the braid groups and related groups that lead to questions on mapping class groups.

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1 Introduction

I have to confess that I am not a leader in the area of mapping class groups and Teichmüller geometry. Nevertheless, I can ask a single question which, I believe, should be of interest for experts in the field.

Question. Which properties of the braid groups can be extended to the mapping class groups?

The present paper is a sort of survey of some (not always well-known) properties of the braid groups and related groups which lead to questions on mapping class groups. The reader will find throughout the text several more or less explicit questions, but none of them are stated as “official” problems or conjectures. This choice is based on the idea that the proof often comes together with the question (or the theorem), and, moreover, the young researchers should keep in mind that the surroundings and the applications of a given problem should be the main motivations for its study, and not the question itself (or its author).

Throughout the paper, Σ will denote a compact, connected, and oriented surface. The boundary ∂Σ, if non empty, is a finite collection of simple closed curves. Consider a finite set P = {P_1, . . . , P_n} of n distinct points, called punctures, in the interior of Σ. Define H(Σ, P) to be the group of orientation-preserving homeomorphisms h : Σ → Σ such that h is the identity on each boundary component of Σ and h(P) = P. The mapping class group M(Σ, P) = π_0(H(Σ, P)) of Σ relative to P is defined to be the set of isotopy classes of mappings in H(Σ, P), with composition as group operation. We emphasize that, throughout an isotopy, the boundary components and the punctures of P remain fixed. It is clear that, up to isomorphism, this group depends only on the genus g of Σ, on the number p of components of ∂Σ, and on the cardinality n of P. So, we may write M(g, p, n) in place of M(Σ, P), and simply M(Σ) for M(Σ, ∅).

Let D = {z ∈ ℂ; |z| ≤ 1} be the standard disk, and let P = {P_1, . . . , P_n} be a finite collection of n punctures in the interior of D. Define a n-braid to be a n-tuple β = (b_1, . . . , b_n) of disjoint smooth paths in D × [0, 1], called the strings of β, such that:
• the projection of $b_i(t)$ on the second coordinate is $t$, for all $t \in [0, 1]$ and all $i \in \{1, \ldots, n\}$;

• $b_i(0) = (P_i, 0)$ and $b_i(1) = (P_{\chi(i)}, 1)$, where $\chi$ is a permutation of $\{1, \ldots, n\}$, for all $i \in \{1, \ldots, n\}$.

An isotopy in this context is a deformation through braids which fixes the ends. Multiplication of braids is defined by concatenation. The isotopy classes of braids with this multiplication form a group, called the *braid group on n strings*, and denoted by $B_n(\mathcal{P})$. This group does not depend, up to isomorphism, on the set $\mathcal{P}$, but only on the cardinality $n$ of $\mathcal{P}$, thus we may write $B_n$ in place of $B_n(\mathcal{P})$.

The group $B_n$ has a well-known presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |j - i| > 1,
$$

$$
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |j - i| = 1.
$$

There are other equivalent descriptions of $B_n$, as a group of automorphisms of the free group $F_n$, as the fundamental group of a configuration space, or as the mapping class group of the $n$-punctured disk. This explains the importance of the braid groups in many disciplines.

Now, the equality

$$B_n(\mathcal{P}) = \mathcal{M}(\mathbb{D}, \mathcal{P}) \quad (1)$$

is the source of the several questions that the reader will find in the paper.

## 2 Garside groups and Artin groups

We start this section with a brief presentation of Garside groups. Our presentation of the subject draws in many ways from the work of Dehornoy [16] as well as [19], and, like all treatments of Garside groups, is inspired ultimately by the seminal papers of Garside [26], on braid groups, and Brieskorn and Saito [8], on Artin groups.

Let $M$ be an arbitrary monoid. We say that $M$ is *atomic* if there exists a function $\nu : M \to \mathbb{N}$ such that

• $\nu(a) = 0$ if and only if $a = 1$;

• $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Such a function on $M$ is called a *norm* on $M$.

An element $a \in M$ is called an *atom* if it is indecomposable, namely, if $a = bc$ then either $b = 1$ or $c = 1$. This definition of atomicity is taken from [19]. We refer to [19] for a list of further properties all equivalent to atomicity. In the same paper, it is shown that a subset of $M$ generates $M$ if and only if it contains the set of all atoms. In particular, $M$ is finitely generated if and only if it has finitely many atoms.

Given that a monoid $M$ is atomic, we may define left and right invariant partial orders $\leq_L$ and $\leq_R$ on $M$ as follows.
• Set \( a \leq_L b \) if there exists \( c \in M \) such that \( ac = b \).
• Set \( a \leq_R b \) if there exists \( c \in M \) such that \( ca = b \).

We shall call these left and right divisibility orders on \( M \).

A Garside monoid is a monoid \( M \) such that

• \( M \) is atomic and finitely generated;
• \( M \) is cancellative;
• \((M, \leq_L)\) and \((M, \leq_R)\) are lattices;
• there exists an element \( \Delta \in M \), called Garside element, such that the set \( L(\Delta) = \{x \in M; x \leq_L \Delta \} \) generates \( M \) and is equal to \( R(\Delta) = \{x \in M; x \leq_R \Delta \} \).

For any monoid \( M \), we can define the group \( G(M) \) which is presented by the generating set \( M \) and the relations \( ab = c \) whenever \( ab = c \) in \( M \). There is an obvious canonical homomorphism \( M \to G(M) \). This homomorphism is not injective in general. The group \( G(M) \) is known as the group of fractions of \( M \). Define a Garside group to be the group of fractions of a Garside monoid.

Remarks.

1. Recall that a monoid \( M \) satisfies Õre’s conditions if \( M \) is cancellative and if, for all \( a, b \in M \), there exist \( c, d \in M \) such that \( ac = bd \). If \( M \) satisfies Õre’s conditions then the canonical homomorphism \( M \to G(M) \) is injective. A Garside monoid obviously satisfies Õre’s conditions, thus any Garside monoid embeds in its group of fractions.

2. A Garside element is never unique. For example, if \( \Delta \) is a Garside element, then \( \Delta^k \) is also a Garside element for all \( k \geq 1 \) (see [10]).

Let \( M \) be a Garside monoid. The lattice operations of \((M, \leq_L)\) are denoted by \( \lor_L \) and \( \land_L \). These are defined as follows. For \( a, b, c \in M \), \( a \land_L b \leq_L a, a \land_L b \leq_L b \), and if \( c \leq_L a \) and \( c \leq_L b \), then \( c \leq_L a \land_L b \). For \( a, b, c \in M \), \( a \leq_L a \lor_L b, b \leq_L a \lor_L b \) and, if \( a \leq_L c \) and \( b \leq_L c \), then \( a \lor_L b \leq_L c \). Similarly, the lattice operations of \((M, \leq_R)\) are denoted by \( \lor_R \) and \( \land_R \).

Now, we briefly explain how to define a biautomatic structure on a given Garside group. By [22], such a structure furnishes solutions to the word problem and to the conjugacy problem, and it implies that the group has quadratic isoperimetric inequalities. We refer to [22] for definitions and properties of automatic groups, and to [10] for more details on the biautomatic structures on Garside groups.

Let \( M \) be a Garside monoid, and let \( \Delta \) be a Garside element of \( M \). For \( a \in M \), we write \( \pi_L(a) = \Delta \land_L a \) and denote by \( \partial_L(a) \) the unique element of \( M \) such that \( a = \pi_L(a) \cdot \partial_L(a) \).

Using the fact that \( M \) is atomic and that \( L(\Delta) = \{x \in M; x \leq_L \Delta \} \) generates \( M \), one can easily show that \( \pi_L(a) \neq 1 \) if \( a \neq 1 \), and that there exists a positive integer \( k \) such that \( \partial_L^k(a) = 1 \). Let \( k \) be the lowest integer satisfying \( \partial_L^k(a) = 1 \). Then the expression

\[
a = \pi_L(a) \cdot \pi_L(\partial_L(a)) \cdot \cdots \cdot \pi_L(\partial_L^{k-1}(a))
\]
is called the normal form of a.

Let \( G = G(M) \) be the group of fractions of \( M \). Let \( c \in G \). Since \( G \) is a lattice with positive cone \( M \), the element \( c \) can be written \( c = a^{-1}b \) with \( a, b \in M \). Obviously, \( a \) and \( b \) can be chosen so that \( a \wedge b = 1 \) and, with this extra condition, are unique. Let \( a = a_1a_2\ldots a_p \) and \( b = b_1b_2\ldots b_q \) be the normal forms of \( a \) and \( b \), respectively. Then the expression

\[ c = a_p^{-1}\ldots a_2^{-1}a_1^{-1}b_1b_2\ldots b_q \]

is called the normal form of \( c \).

**Theorem 2.1 (Dehornoy [16]).** Let \( M \) be a Garside monoid, and let \( G \) be the group of fractions of \( M \). Then the normal forms of the elements of \( G \) form a symmetric rational language on the finite set \( L(\Delta) \) which has the fellow traveler property. In particular, \( G \) is biautomatic.

The notion of a Garside group was introduced by Dehornoy and the author [19] in a slightly restricted sense, and, later, by Dehornoy [16] in the larger sense which is now generally used. As pointed out before, the theory of Garside groups is largely inspired by the papers of Garside [26], which treated the case of the braid groups, and Brieskorn and Saito [8], which generalized Garside’s work to Artin groups. The Artin groups of spherical type which include, notably, the braid groups, are motivating examples. Other interesting examples include all torus link groups (see [47]) and some generalized braid groups associated to finite complex reflection groups (see [2]).

Garside groups have many attractive properties: solutions to the word and conjugacy problems are known (see [16], [46], [25]), they are torsion-free (see [15]), they are biautomatic (see [16]), and they admit finite dimensional classifying spaces (see [18], [10]). There also exist criteria in terms of presentations which detect Garside groups (see [19], [16]).

Let \( S \) be a finite set. A Coxeter matrix over \( S \) is a square matrix \( M = (m_{st})_{s,t \in S} \) indexed by the elements of \( S \) and such that \( m_{ss} = 1 \) for all \( s \in S \), and \( m_{st} = m_{ts} \in \{2, 3, 4, \ldots, +\infty\} \) for all \( s, t \in S, s \neq t \). A Coxeter matrix \( M = (m_{st})_{s,t \in S} \) is usually represented by its Coxeter graph, \( \Gamma \), which is defined as follows. The set of vertices of \( \Gamma \) is \( S \), two vertices \( s, t \) are joined by an edge if \( m_{st} \geq 3 \), and this edge is labeled by \( m_{st} \) if \( m_{st} \geq 4 \).

Let \( \Gamma \) be a Coxeter graph with set of vertices \( S \). For two objects \( a, b \) and \( m \in \mathbb{N} \), define the word

\[ \omega(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even}, \\ (ab)^{\frac{m-1}{2}}a & \text{if } m \text{ is odd}. \end{cases} \]

Take an abstract set \( \mathcal{S} = \{\sigma_s; s \in S\} \) in one-to-one correspondence with \( S \). The Artin group of type \( \Gamma \) is the group \( A = A_\Gamma \) generated by \( S \) and subject to the relations

\[ \omega(\sigma_s, \sigma_t : m_{st}) = \omega(\sigma_t, \sigma_s : m_{st}) \quad \text{for } s, t \in S, s \neq t, \text{ and } m_{st} < +\infty, \]

where \( M = (m_{st})_{s,t \in S} \) is the Coxeter matrix of \( \Gamma \). The Coxeter group of type \( \Gamma \) is the group \( W = W_\Gamma \) generated by \( S \) and subject to the relations

\[ s^2 = 1 \quad \text{for } s \in S, \]

\[ (st)^{m_{st}} = 1 \quad \text{for } s, t \in S, s \neq t, \text{ and } m_{st} < +\infty. \]
Note that $W$ is the quotient of $A$ by the relations $\sigma_s^2 = 1$, $s \in S$, and $s$ is the image of $\sigma_s$ under the quotient epimorphism.

The number $n = |S|$ of generators is called the rank of the Artin group (and of the Coxeter group). We say that $A$ is irreducible if $\Gamma$ is connected, and that $A$ is of spherical type if $W$ is finite.

Coxeter groups have been widely studied. Basic references for them are [6] and [30]. In contrast, Artin groups are poorly understood in general. Beside the spherical type ones, which are Garside groups, the Artin groups which are better understood include right-angled Artin groups, 2-dimensional Artin groups, and FC type Artin groups. Right-angled Artin groups (also known as graph groups or free partially commutative groups) have been widely studied, and their applications extend to various domains such as parallel computation, random walks, and cohomology of groups. 2-dimensional Artin groups and FC type Artin groups have been introduced by Charney and Davis [9] in 1995 in their study of the $K(\pi,1)$-problem for complements of infinite hyperplane arrangements associated to reflection groups.

Let $G$ be an Artin group (resp. Garside group). Define a geometric representation of $G$ to be a homomorphism from $G$ to some mapping class group. Although I do not believe that mapping class groups are either Artin groups or Garside groups, the theory of geometric representations should be of interest from the perspective of both mapping class groups and Artin (resp. Garside) groups. For instance, it is not known which Artin groups (resp. Garside groups) can be embedded into a mapping class group.

First, recall that an Artin group $A$ associated to a Coxeter graph $\Gamma$ is said to be of small type if $m_{st} \in \{2, 3\}$ for all $s, t \in S$, $s \neq t$, where $M = (m_{st})_{s,t \in S}$ is the Coxeter matrix of $\Gamma$.

An essential circle in $\Sigma$ is an oriented embedding $a : S^1 \rightarrow \Sigma$ such that $a(S^1) \cap \partial \Sigma = \emptyset$, and $a(S^1)$ does not bound any disk. We shall use the description of the circle $S^1$ as $\mathbb{R}/2\pi\mathbb{Z}$. Take an (oriented) embedding $\Lambda_a : S^1 \times [0, 1] \rightarrow \Sigma$ of the annulus such that $\Lambda_a(\theta, 0) = a(\theta)$ for all $\theta \in S^1$, and define a homomorphism $T_a \in \mathcal{H}(\Sigma)$, which restricts to the identity outside the interior of the image of $\Lambda_a$, and is given by

$$(T_a \circ \Lambda_a)(\theta, x) = \Lambda_a(\theta + 2\pi x, x) \quad \text{for} \quad (\theta, x) \in S^1 \times [0, 1],$$

inside the image of $\Lambda_a$. The Dehn twist along $a$ is the element of $\mathcal{M}(\Sigma)$ represented by $T_a$. The following result is easily checked and may be found, for instance, in [4].
Figure 1: Identification of annuli.

**Proposition 2.2.** Let $a_1, a_2 : S^1 \to \Sigma$ be two essential circles, and, for $i = 1, 2$, let $\tau_i$ denote the Dehn twist along $a_i$. Then

\[
\begin{align*}
\tau_1 \tau_2 &= \tau_2 \tau_1 & \text{if } a_1 \cap a_2 = \emptyset, \\
\tau_1 \tau_2 \tau_1 &= \tau_2 \tau_1 \tau_2 & \text{if } |a_1 \cap a_2| = 1.
\end{align*}
\]

We assume now that $\Gamma$ is a small type Coxeter graph, and we associate to $\Gamma$ a surface $\Sigma = \Sigma(\Gamma)$ as follows.

Let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of $\Gamma$. Let $<$ be a total order on $S$ which can be chosen arbitrarily. For each $s \in S$, we define the set

\[St_s = \{ t \in S; m_{st} = 3 \} \cup \{ s \}.\]

We write $St_s = \{ t_1, t_2, \ldots, t_k \}$ such that $t_1 < t_2 < \cdots < t_k$, and suppose $s = t_j$. The difference $i - j$ is called the relative position of $t_i$ with respect to $s$, and is denoted by $\text{pos}(t_i : s)$. In particular, $\text{pos}(s : s) = 0$.

Let $s \in S$. Put $k = |St_s|$. We denote by $An_s$ the annulus defined by

\[An_s = \mathbb{R}/2k\mathbb{Z} \times [0, 1].\]

For each $s \in S$, write $P_s$ for the point $(0, 0)$ of $An_s$. The surface $\Sigma = \Sigma(\Gamma)$ is defined by

\[\Sigma = \left( \prod_{s \in S} An_s \right) / \approx,
\]

where $\approx$ is the relation defined as follows. Let $s, t \in S$ such that $m_{st} = 3$ and $s < t$. Put $p = \text{pos}(t : s) > 0$ and $q = \text{pos}(s : t) < 0$. For each $(x, y) \in [0, 1] \times [0, 1]$, the relation $\approx$ identifies the point $(2p + x, y)$ of $An_s$ with the point $(2q + 1 - y, x)$ of $An_t$ (see Figure 1). We identify each annulus $An_s$ and the point $P_s$ with their image in $\Sigma$, respectively.
We now define the monodromy representation $\rho : A_\Gamma \to M(\Sigma)$. Let $s \in S$, and put $k = |St_s|$. We denote by $a_s : S^1 \to \Sigma$ the essential circle of $\Sigma$ such that $a_s(\theta)$ is the point $(k\theta, \frac{1}{2})$ of $A_{n_s}$ (see Figure 2). We let $\tau_s$ denote the Dehn twist along $a_s$. One has $a_s \cap a_t = \emptyset$ if $m_{st} = 2$, and $|a_s \cap a_t| = 1$ if $m_{st} = 3$. Therefore, by Proposition 2.2, we have the following.

**Proposition 2.3.** Let $\Gamma$ be a small type Coxeter graph. There exists a well-defined group homomorphism $\rho : A_\Gamma \to M(\Sigma)$ which sends $\sigma_s$ to $\tau_s$ for each $s \in S$.

**Example.** Consider the braid group $B_n$, $n \geq 3$. If $n$ is odd, then the associated surface, $\Sigma$, is a surface of genus $\frac{n-1}{2}$ with one boundary component. If $n$ is even, then $\Sigma$ is a surface of genus $\frac{n-2}{2}$ with two boundary components (see Figure 3). For each $i = 1, \ldots, n-1$, let $a_i : S^1 \to \Sigma$ denote the essential circle pictured in Figure 3, and let $\tau_i$ denote the Dehn twist along $a_i$. Then the monodromy representation $\rho : B_n \to M(\Sigma)$ of the braid group sends $\sigma_i$ to $\tau_i$ for all $i = 1, \ldots, n-1$.

Let $O_2$ denote the ring of germs of functions at 0 of $\mathbb{C}^2$. Let $f \in O_2$ with an isolated singularity at 0, and let $\mu$ be the Milnor number of $f$. To the germ $f$ one can associate a surface $\Sigma$ with boundary, called the Milnor fiber of $f$, an analytic subvariety $D$ of $D(\eta) = \{ t \in \mathbb{C}^\mu; ||t|| < \eta \}$, where $\eta > 0$ is a small number, and a representation $\rho : \pi_1(D(\eta) \setminus D) \to M(\Sigma)$, called the
geometric monodromy of \( f \). Recall that the simple singularities are defined by the equations

\[ A_n : \quad f(x, y) = x^2 + y^{n+1}, \quad n \geq 1, \]
\[ D_n : \quad f(x, y) = x(x^{n-2} + y^2), \quad n \geq 4, \]
\[ E_6 : \quad f(x, y) = x^3 + y^4, \]
\[ E_7 : \quad f(x, y) = x(x^2 + y^3), \]
\[ E_8 : \quad f(x, y) = x^3 + y^5. \]

By [1] and [7], for each \( \Gamma \in \{ A_n; n \geq 1 \} \cup \{ D_n; n \geq 4 \} \cup \{ E_6, E_7, E_8 \} \), the group \( \pi_1(D(\eta) \setminus D) \) is isomorphic to the Artin group of type \( \Gamma \), and \( \rho : A_{\Gamma} \to M(\Sigma) \) is the monodromy representation of the group \( A_{\Gamma} \).

The monodromy representations are known to be faithful for \( \Gamma = A_n \) and \( \Gamma = D_n \) (see [45]), but are not faithful for \( \Gamma \in \{ E_6, E_7, E_8 \} \) (see [51]). Actually, the monodromy representations are not faithful in general (see [36]).

3 Dehornoy’s ordering

Call a group or a monoid \( G \) left-orderable if there exists a strict linear ordering \( < \) of the elements of \( G \) which is left invariant, namely, for \( x, y, z \in G \), \( x < y \) implies \( zx < zy \). If, in addition, the ordering is a well-ordering, then we say that \( G \) is left-well-orderable. If there is an ordering which is invariant under multiplication on both sides, we say that \( G \) is biorderable.

Recall that the braid group \( B_n \) has a presentation with generators \( \sigma_1, \ldots, \sigma_{n-1} \) and relations

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1, \tag{2} \]
\[ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for} \quad |i - j| = 1. \tag{3} \]

The submonoid generated by \( \sigma_1, \ldots, \sigma_{n-1} \) is called the positive braid monoid on \( n \) strings, and is denoted by \( B^+_n \). It has a monoid presentation with generators \( \sigma_1, \ldots, \sigma_{n-1} \) and relations (2) and (3), it is a Garside monoid, and the group of fractions of \( B^+_n \) is \( B_n \).

The starting point of the present section is the following result due to Dehornoy [13].

**Theorem 3.1 (Dehornoy [13]).** The Artin braid group \( B_n \) is left-orderable, by an ordering which is a well-ordering when restricted to \( B^+_n \).

Left-invariant orderings are important because of a long-standing conjecture in group theory: that the group ring of a torsion-free group has no zero divisors. This conjecture is true for left-orderable groups, thus Theorem 3.1 implies that \( \mathbb{Z}[B_n] \) has no zero divisors. However, the main point of interest of Theorem 3.1 is not only the mere existence of orderings on the braid groups, but the particular nature of the construction. This has been already used to prove that some representations of the braid groups by automorphisms of groups are faithful (see [50] and [12]), and to study the so-called “weak faithfulness” of the Burau representation (see [14]), and I am sure that these ideas will be exploited again in the future.

Let \( G \) be a left-orderable group, and let \( < \) be a left invariant linear ordering on \( G \). Call an element \( g \in G \) positive if \( g > 1 \), and denote by \( \mathcal{P} \) the set of positive elements. Then \( \mathcal{P} \) is a
subsemigroup \(i.e. \mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}\), and we have the disjoint union \(G = \mathcal{P} \sqcup \mathcal{P}^{-1} \sqcup \{1\}\), where \(\mathcal{P}^{-1} = \{g^{-1}: g \in \mathcal{P}\}\). Conversely, if \(\mathcal{P}\) is a subsemigroup of \(G\) such that \(G = \mathcal{P} \sqcup \mathcal{P}^{-1} \sqcup \{1\}\), then the relation \(<\) defined by \(f < g\) if \(f^{-1}g \in \mathcal{P}\) is a left invariant linear ordering on \(G\).

The description of the positive elements in Dehornoy’s ordering is based on the following result which, in my view, is more interesting than the mere existence of the ordering, because it leads to new techniques to study the braid groups.

Let \(B_{n-1}\) be the subgroup of \(B_n\) generated by \(\sigma_1, \ldots, \sigma_{n-2}\). Let \(\beta \in B_n\). We say that \(\beta\) is \(\sigma_{n-1}\)-positive if it can be written

\[
\beta = \alpha_0 \sigma_{n-1}^{-1} \alpha_1 \sigma_{n-1}^{-1} \alpha_2 \ldots \sigma_{n-1}^{-1} \alpha_l,
\]

where \(l \geq 1\) and \(\alpha_0, \alpha_1, \ldots, \alpha_l \in B_{n-1}\). We say that \(\beta\) is \(\sigma_{n-1}\)-negative if it can be written

\[
\beta = \alpha_0 \sigma_{n-1} \alpha_1 \sigma_{n-1} \alpha_2 \ldots \sigma_{n-1} \alpha_l,
\]

where \(l \geq 1\) and \(\alpha_0, \alpha_1, \ldots, \alpha_l \in B_{n-1}\). We denote by \(\mathcal{P}_{n-1}\) the set of \(\sigma_{n-1}\)-positive elements, and by \(\mathcal{P}_{n-1}^{-}\) the set of \(\sigma_{n-1}\)-negative elements. Note that \(\mathcal{P}_{n-1}^{-} = \mathcal{P}_{n-1}^{-1} = \{\beta^{-1} : \beta \in \mathcal{P}_{n-1}\}\).

**Theorem 3.2 (Dehornoy [13]).** We have the disjoint union \(B_n = \mathcal{P}_{n-1} \sqcup \mathcal{P}_{n-1}^{-} \sqcup B_{n-1}\).

Now, the set \(\mathcal{P}\) of positive elements in Dehornoy’s ordering can be described as follows. For \(k = 1, \ldots, n-1\), let \(\mathcal{P}_k\) denote the set of \(\sigma_k\)-positive elements in \(B_{k+1}\), where \(B_{k+1}\) is viewed as the subgroup of \(B_n\) generated by \(\sigma_1, \ldots, \sigma_k\). Then \(\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_{n-1}\). It is easily checked using Theorem 3.2 that \(\mathcal{P}\) is a subsemigroup of \(B_n\) and that \(B_n = \mathcal{P} \sqcup \mathcal{P}^{-1} \sqcup \{1\}\).

Dehornoy’s proof of Theorem 3.2 involves rather delicate combinatorial and algebraic constructions which were partly motivated by questions in set theory. A topological construction of Dehornoy’s ordering of the braid group \(B_n\), viewed as the mapping class group of the \(n\)-punctured disk, is given in [24]. This last construction has been extended to the mapping class groups of the punctured surfaces with non-empty boundary by Rourke and Wiest [48]. This extension involves some choices, and, therefore, is not canonical. However, it is strongly connected to Mosher’s normal forms [42] and, as with Dehornoy’s ordering, is sufficiently explicit to be used for other purposes (to study faithfulness properties of “some” representations of the mapping class group, for example. This is just an idea...). Another approach (see [49]) based on Nielsen’s ideas [43] gives rise to many other linear orderings of the braid groups, and, more generally, of the mapping class groups of punctured surfaces with non-empty boundary, but I am not convinced that these orderings are explicit enough to be used for other purposes.

The mapping class group of a (punctured) closed surface is definitively not left-orderable because it has torsion. However, it is an open question to determine whether some special torsion-free subgroups of \(\mathcal{M}(\Sigma)\) such as Ivanov’s group \(\Gamma_m(\Sigma)\) (see [32]) are orderable. The Torelli group is residually torsion-free nilpotent (see [27]), therefore is biorderable. We refer to [17] for a detailed account on Dehornoy’s ordering and related questions.
4 Thurston classification

Recall that an essential circle in $\Sigma \setminus \mathcal{P}$ is an oriented embedding $a : S^1 \to \Sigma \setminus \mathcal{P}$ such that $a(S^1) \cap \partial \Sigma = \emptyset$, and $a(S^1)$ does not bound a disk in $\Sigma \setminus \mathcal{P}$. An essential circle $a : S^1 \to \Sigma \setminus \mathcal{P}$ is called generic if it does not bound a disk in $\Sigma$ containing one puncture, and if it is not isotopic to a boundary component of $\Sigma$. Recall that two circles $a, b : S^1 \to \Sigma \setminus \mathcal{P}$ are isotopic if there exists a continuous family $a_t : S^1 \to \Sigma \setminus \mathcal{P}$, $t \in [0, 1]$, such that $a_0 = a$ and $a_1 = b$. Isotopy of circles is an equivalence relation that we denote by $a \sim b$. Observe that $h(a) \sim h'(a')$ if $a \sim a'$ and $h, h' \in \mathcal{H}(\Sigma, \mathcal{P})$ are isotopic. So, the mapping class group $\mathcal{M}(\Sigma, \mathcal{P})$ acts on the set $\mathcal{C}(\Sigma, \mathcal{P})$ of isotopy classes of generic circles.

Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$. We say that $f$ is a pseudo-Anosov element if it has no finite orbit in $\mathcal{C}(\Sigma, \mathcal{P})$, we say that $f$ is periodic if $f^m$ acts trivially on $\mathcal{C}(\Sigma, \mathcal{P})$ for some $m \geq 1$, and we say that $f$ is reducible otherwise.

The use of the action of $\mathcal{M}(\Sigma, \mathcal{P})$ on $\mathcal{C}(\Sigma, \mathcal{P})$ and the “Thurston classification” of the elements of $\mathcal{M}(\Sigma, \mathcal{P})$ play a prominent role in the study of the algebraic properties of the mapping class groups. Here are some applications.

Theorem 4.1.

1. (Ivanov [32]). Let $f$ be a pseudo-Anosov element of $\mathcal{M}(\Sigma, \mathcal{P})$. Then $f$ has infinite order, $(f) \cap Z(\mathcal{M}(\Sigma, \mathcal{P})) = \{1\}$, where $(f)$ denotes the subgroup generated by $f$, and $(f) \times Z(\mathcal{M}(\Sigma, \mathcal{P}))$ is a subgroup of finite index of the centralizer of $f$ in $\mathcal{M}(\Sigma, \mathcal{P})$.

2. (Birman, Lubotzky, McCarthy [5]). If a subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$ contains a solvable subgroup of finite index, then it contains an abelian subgroup of finite index.

3. (Ivanov [32], McCarthy [40]). Let $G$ be a subgroup of $\mathcal{M}(\Sigma, \mathcal{P})$. Then either $G$ contains a free group of rank 2, or $G$ contains an abelian subgroup of finite index.

The use of the action of $\mathcal{M}(\Sigma, \mathcal{P})$ on $\mathcal{C}(\Sigma, \mathcal{P})$ is also essential in the determination of the (maximal) free abelian subgroups of $\mathcal{M}(\Sigma, \mathcal{P})$ (see [5]), and in the calculation of the automorphism group and abstract commensurator group of $\mathcal{M}(\Sigma, \mathcal{P})$ (see [31, 33, 41, 34]).

The present section concerns the question of how to translate these techniques to other groups such as the (spherical type) Artin groups, or the Coxeter groups.

Let $\Gamma$ be a Coxeter graph, let $M = (m_{st})_{s, t \in S}$ be the Coxeter matrix of $\Gamma$, and let $A = A_\Gamma$ be the Artin group of type $\Gamma$. For $X \subset S$, we denote by $\Gamma_X$ the full subgraph of $\Gamma$ generated by $X$, and by $A_X$ the subgroup of $A$ generated by $S_X = \{ \sigma_x : x \in X \}$. By [38] (see also [41]), the group $A_X$ is the Artin group of type $\Gamma_X$. A subgroup of the form $A_X$ is called a standard parabolic subgroup, and a subgroup which is conjugate to a standard parabolic subgroup is simply called a parabolic subgroup. We shall denote by $\mathcal{AC} = \mathcal{AC}(\Gamma)$ the set of parabolic subgroups different from $A$ and from $\{1\}$.

Let $f \in A$. We say that $f$ is an (algebraic) pseudo-Anosov element if it has no finite orbit in $\mathcal{AC}$, we say that $f$ is an (algebraic) periodic element if $f^m$ acts trivially on $\mathcal{AC}$ for some $m > 0$, and we say that $f$ is an (algebraic) reducible element otherwise.
The above definitions are motivated by the following result.

**Proposition 4.2.** Let \( f \) be an element of the braid group \( B_n \) (viewed as the mapping class group \( \mathcal{M}(\mathbb{D},\mathcal{P}) \) as well as the Artin group associated to the Dynkin graph \( A_{n-1} \)).

- \( f \) is a pseudo-Anosov element if and only if \( f \) is an algebraic pseudo-Anosov element.
- \( f \) is a periodic element if and only if \( f \) is an algebraic periodic element.
- \( f \) is a reducible element if and only if \( f \) is an algebraic reducible element.

**Proof.** Recall that \( \mathbb{D} = \{ z \in \mathbb{C}; |z| \leq 1 \} \), \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) is a set of \( n \) punctures in the interior of \( \mathbb{D} \), and \( B_n = \mathcal{M}(\mathbb{D},\mathcal{P}) \). Let \( \sigma_1, \ldots, \sigma_{n-1} \) be the standard generators of \( B_n \). We can assume that each \( \sigma_i \) acts trivially outside a disk containing \( P_i \) and \( P_{i+1} \), and permutes \( P_i \) and \( P_{i+1} \).

Let \( a : S^1 \to \mathbb{D} \setminus \mathcal{P} \) be a generic circle. Then \( a(S^1) \) separates \( \mathbb{D} \) into two components: a disk \( \mathbb{D}_a \) bounded by \( a(S^1) \), and an annulus \( A_a \) bounded by \( a(S^1) \cup S^1 \), where \( S^1 = \{ z \in \mathbb{C}; |z| = 1 \} \) (see Figure 4). Let \( m = |\mathbb{D}_a \cap \mathcal{P}| \). Then the hypothesis “\( a \) is generic” implies that \( 2 \leq m \leq n - 1 \). Furthermore, the group \( \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \) is isomorphic to \( B_m \).

Let \( \tau_a \) denote the Dehn twist along \( a \). If \( m = |\mathbb{D}_a \cap \mathcal{P}| \geq 3 \), then the center \( Z(\mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P})) \) of \( \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \) is the (infinite) cyclic subgroup generated by \( \tau_a \). If \( m = 2 \), then \( \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \) is an infinite cyclic group, and \( \tau_a \) generates the (unique) index 2 subgroup of \( \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \).

Recall that, if \( \tau_{a_1} \) and \( \tau_{a_2} \) are two Dehn twists along generic circles \( a_1 \) and \( a_2 \), respectively, and if \( \tau_{a_1}^k = \tau_{a_2}^k \) for some \( k_1, k_2 \in \mathbb{Z} \setminus \{0\} \), then \( k_1 = k_2 \) and \( a_1 = a_2 \). Recall also that, if \( \tau_a \) is the Dehn twist along a generic circle \( a \), then \( g \tau_a g^{-1} \) is the Dehn twist along \( g(a) \), for any \( g \in \mathcal{M}(\mathbb{D},\mathcal{P}) \). We conclude from the above observations that, if \( g \cdot \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \cdot g^{-1} = \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \), then \( g(a) = a \). Conversely, if \( g(a) = a \), then \( g(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) = (\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \), thus \( g \cdot \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \cdot g^{-1} = \mathcal{M}(\mathbb{D}_a,\mathbb{D}_a \cap \mathcal{P}) \).

Write \( S = \{ 1, 2, \ldots, n - 1 \} \). For \( X \subset S \), we set \( S_X = \{ \sigma_x; x \in X \} \), and denote by \( (B_n)_X \) the subgroup of \( B_n \) generated by \( S_X \).
Let \( g \in M \) and the only Dehn twists which lie in the center of \((X)\) there exists \( \tau \). Recall that the only Dehn twists which lie in the center of \((X)\) thus \( a \) belongs to the center of \((X)\). In particular, \( X \) and \( H \) exist such that \( \{ \tau \} = \{ 1 \} \) and \( \mathcal{M}(D_a, D_a \cap \mathcal{P}) = (B_n)_{X(m)} \) (see Figure 5). Observe that, if \( a : S^1 \to \mathbb{D} \setminus \mathcal{P} \) is a generic circle such that \( |D_a \setminus \mathcal{P}| = m + 1 \), then there exists \( g \in \mathcal{M}(D, \mathcal{P}) = B_n \) such that \( g \mathcal{M}(D_a, D_a \cap \mathcal{P}) g^{-1} = \mathcal{M}(D_a, D_a \cap \mathcal{P}) = (B_n)_{X(m)} \). In particular, \( \mathcal{M}(D_a, D_a \cap \mathcal{P}) \) is a parabolic subgroup of \( B_n \).

Take \( X \subset S \), \( X \neq S \). Decompose \( X \) as a disjoint union \( X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_q \), where \( X_k = \{i_k, i_k + 1, \ldots, i_k + t_k\} \) for some \( i_k \in \{1, \ldots, n-1\} \) and some \( t_k \in \mathbb{N} \), and \( i_k + t_k + 1 < i_{k+1} \), for all \( k = 1, \ldots, q \). We can choose generic circles \( b_1, \ldots, b_q : S^1 \to \mathbb{D} \setminus \mathcal{P} \) such that

\[
\mathcal{D}_{b_k} \cap \mathcal{P} = \{P_{i_k}, \ldots, P_{i_k+t_k}, P_{i_k+t_k+1}\}, \quad \text{for} \ 1 \leq k \leq q,
\]

\[
\mathcal{D}_{b_l} \cap \mathcal{D}_{b_l} = \emptyset, \quad \text{for} \ 1 \leq k \neq l \leq q,
\]

\[
\mathcal{M}(\mathcal{D}_{b_k}, \mathcal{D}_{b_k} \cap \mathcal{P}) = (B_n)_{X_k}, \quad \text{for} \ 1 \leq k \leq q.
\]

(See Figure 6.) It is easily checked that

\[
(B_n)_X = (B_n)_{X_1} \times \cdots \times (B_n)_{X_q} = \mathcal{M}(\mathcal{D}_{b_1}, \mathcal{D}_{b_1} \cap \mathcal{P}) \times \cdots \times \mathcal{M}(\mathcal{D}_{b_q}, \mathcal{D}_{b_q} \cap \mathcal{P}),
\]

and the only Dehn twists which lie in the center of \((B_n)_X\) are \( \tau_{b_1}, \tau_{b_2}, \ldots, \tau_{b_q} \).

Let \( g \in \mathcal{M}(D, \mathcal{P}) \) which fixes some \( a \in \mathcal{C}(D, \mathcal{P}) \). Then \( g \mathcal{M}(D_a, D_a \cap \mathcal{P}) g^{-1} = \mathcal{M}(D_a, D_a \cap \mathcal{P}) \) and \( \mathcal{M}(D_a, D_a \cap \mathcal{P}) \) is a parabolic subgroup, thus \( g \) fixes some element in \( \mathcal{C} \).

Let \( g \in \mathcal{M}(D, \mathcal{P}) \) which fixes some element in \( \mathcal{C} \). Up to conjugation, we may suppose that there exists \( X \subset S \) such that \( g(B_n)_X g^{-1} = (B_n)_X \). We take again the notations introduced above. Recall that the only Dehn twists which lie in the center of \((B_n)_X\) are \( \tau_{b_1}, \tau_{b_2}, \ldots, \tau_{b_q} \), thus \( g \) must permute the set \( \{\tau_{b_1}, \tau_{b_2}, \ldots, \tau_{b_q}\} \). So, there exists \( m \geq 1 \) such that \( g^m \tau_{b_1} g^{-m} = \tau_{b_1} \), that is, \( g^m(b_1) = b_1 \).

Let \( g \in \mathcal{M}(D, \mathcal{P}) \) which fixes all the elements of \( \mathcal{C}(D, \mathcal{P}) \). Let \( A \in \mathcal{C} \). There exist \( h \in B_n \) and \( X \subset S \) such that \( A = h(B_n)_X h^{-1} \). We keep the notations used above. The element \( h^{-1}gh \) also fixes all the elements of \( \mathcal{C}(D, \mathcal{P}) \). In particular, we have \((h^{-1}gh)(b_k) = b_k \) for all
$k = 1, \ldots, q$. This implies that $h^{-1}gh$ fixes $\mathcal{M}(D_{b_k}, D_{b_k} \cap \mathcal{P})$ for all $k = 1, \ldots, q$, thus $h^{-1}gh$ fixes $(B_n)_X = \mathcal{M}(D_{b_1}, D_{b_1} \cap \mathcal{P}) \times \cdots \times \mathcal{M}(D_{b_q}, D_{b_q} \cap \mathcal{P})$, therefore $g$ fixes $A = h(B_n)_X h^{-1}$.

Let $g \in \mathcal{M}(D, \mathcal{P})$ which fixes all the elements of $\mathcal{AC}$. Let $a \in C(D, \mathcal{P})$. Since $\mathcal{M}(D_a, D_a \cap \mathcal{P})$ is a parabolic subgroup, we conclude that $g\mathcal{M}(D_{a}, D_{a} \cap \mathcal{P})g^{-1} = \mathcal{M}(D_{a}, D_{a} \cap \mathcal{P})$, thus $g(a) = a$. □

In spite of the fact that the normalizers of the parabolic subgroups of the spherical type Artin groups are quite well understood (see [44], [28]), it is not known, for instance, whether the centralizer of an algebraic pseudo-Anosov element $g$ in a spherical type Artin group $A$ has $\langle g \rangle \times \mathbb{Z}(A)$ as finite index subgroup. This question might be the starting point for understanding the abelian subgroups of the spherical type Artin groups.

We turn now to a part of Krammer’s Ph. D. thesis which describes a phenomenon for Coxeter groups similar to the “Thurston classification” for the mapping class groups: in a non-affine irreducible Coxeter group $W$, there are certain elements, called essential elements, which have infinite order, and have the property that the cyclic subgroups generated by them have finite index in their centralizer. Although almost all elements of $W$ are essential, it is very hard to determine whether a given element is essential.

Let $\Gamma$ be a Coxeter graph with set of vertices $S$, and let $W = W_\Gamma$ be the Coxeter group of type $\Gamma$. Let $\Pi = \{\alpha_s; s \in S\}$ be an abstract set in one-to-one correspondence with $S$. The elements of $\Pi$ are called simple roots. Let $U = \oplus_{s \in S} \mathbb{R}\alpha_s$ denote the (abstract) real vector space having $\Pi$ as a basis. Define the canonical form as the symmetric bilinear form $\langle , \rangle : U \times U \to \mathbb{R}$ determined by

$$\langle \alpha_s, \alpha_t \rangle = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} < +\infty, \\ -1 & \text{if } m_{s,t} = +\infty. \end{cases}$$

For $s \in S$, define the linear transformation $\rho_s : U \to U$ by

$$\rho_s(x) = x - 2\langle x, \alpha_s \rangle \alpha_s, \quad \text{for } x \in U.$$  

Then the map $s \mapsto \rho_s$, $s \in S$, determines a well-defined representation $W \to GL(U)$, called canonical representation. This representation is faithful and preserves the canonical form.
The set $\Phi = \{w\alpha_s; w \in W \text{ and } s \in S\}$ is called the root system of $W$. The set $\Phi^+ = \{\beta = \sum_{s \in S} \lambda_s \alpha_s \in \Phi; \lambda_s \geq 0 \text{ for all } s \in S\}$ is the set of positive roots, and $\Phi^- = -\Phi^+$ is the set of negative roots. For $w \in W$, we set

$$\Phi_w = \{\beta \in \Phi^+; w\beta \in \Phi^-\}.$$ 

The following proposition is a mixture of several well-known facts on $\Phi$. The proofs can be found in [29].

**Proposition 4.3.**

1. We have the disjoint union $\Phi = \Phi^+ \sqcup \Phi^-$. 
2. Let $\lg : W \to \mathbb{N}$ denote the word length of $W$ with respect to $S$. Then $\lg(w) = |\Phi_w|$ for all $w \in W$. In particular, $\Phi_w$ is finite.
3. Let $w \in W$ and $s \in S$. Then

$$\lg(ws) = \begin{cases} 
\lg(w) + 1 & \text{if } w\alpha_s \in \Phi^+, \\
\lg(w) - 1 & \text{if } w\alpha_s \in \Phi^-.
\end{cases}$$

4. Let $w \in W$ and $s \in S$. Write $\beta = w\alpha_s \in \Phi$, and $r_\beta = ws w^{-1} \in W$. Then $r_\beta$ acts on $U$ by

$$r_\beta(x) = x - 2(x, \beta)\beta.$$ 

Let $u, v \in W$ and $\alpha \in \Phi$. We say that $\alpha$ separates $u$ and $v$ if there exists $\varepsilon \in \{\pm 1\}$ such that $u\alpha \in \Phi^\varepsilon$ and $v\alpha \in \Phi^{-\varepsilon}$. Let $w \in W$ and $\alpha \in \Phi$. We say that $\alpha$ is $w$-periodic if there exists some $m \geq 1$ such that $w^m\alpha = \alpha$.

**Proposition 4.4.** Let $w \in W$ and $\alpha \in \Phi$. Then precisely one of the following holds.

1. $\alpha$ is $w$-periodic.
2. $\alpha$ is not $w$-periodic, and the set $\{m \in \mathbb{Z}; \alpha$ separates $w^m$ and $w^{m+1}\}$ is finite and even.
3. $\alpha$ is not $w$-periodic, and the set $\{m \in \mathbb{Z}; \alpha$ separates $w^m$ and $w^{m+1}\}$ is finite and odd.

Call $\alpha$ $w$-even in Case 2, and $w$-odd in Case 3.

**Proof.** Assume that $\alpha$ is not $w$-periodic, and put $N_w(\alpha) = \{m \in \mathbb{Z}; \alpha$ separates $w^m$ and $w^{m+1}\}$. We have to show that $N_w(\alpha)$ is finite.

If $m \in N_w(\alpha)$, then $w^m\alpha \in \Phi_w \cup -\Phi_w$. On the other hand, if $w^{m_1}\alpha = w^{m_2}\alpha$, then $w^{m_1-m_2}\alpha = \alpha$, thus $m_1 - m_2 = 0$, since $\alpha$ is not $w$-periodic. Since $\Phi_w \cup -\Phi_w$ is finite (see Proposition 4.3), we conclude that $N_w(\alpha)$ is finite.

For $X \subset S$, we denote by $W_X$ the subgroup of $W$ generated by $X$. Such a subgroup is called a standard parabolic subgroup. A subgroup of $W$ conjugated to a standard parabolic subgroup is simply called a parabolic subgroup. An element $w \in W$ is called essential if it does not lie in any
proper parabolic subgroup. Finally, recall that a Coxeter group $W$ is said to be of affine type if the canonical form $\langle \cdot, \cdot \rangle$ is non-negative (namely, $\langle x, x \rangle \geq 0$ for all $x \in U$).

Now, we can state Krammer’s result.

**Theorem 4.5 (Krammer [35]).** Assume $W$ to be an irreducible non-affine Coxeter group. Let $w \in W$.

1. $w$ is essential if and only if $W$ is generated by the set $\{r_\beta; \beta \in \Phi^+ \text{ and } \beta w\text{-odd}\}$.
2. Let $m \in \mathbb{N}$, $m \geq 1$. $w$ is essential if and only if $w^m$ is essential.
3. Suppose $w$ is essential. Then $w$ has infinite order, and $\langle w \rangle = \{w^m; m \in \mathbb{Z}\}$ is a finite index subgroup of the centralizer of $w$ in $W$.

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