CONTINUITY AND SEPARATION FOR
POINTWISE-SYMMETRIC ISOTONIC CLOSURE
FUNCTIONS

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Abstract. In this paper, we show that a pointwise-symmetric isotonic closure function is uniquely determined by the pairs of sets it separates. We then show that when the closure function of the domain is isotonic and the closure function of the codomain is isotonic and pointwise-symmetric, functions which separate only those pairs of sets which are already separated are continuous, generalizing a result in [1] which is, in turn, a generalization of a result in [2].

A generalized closure space is a pair \((X, \text{cl})\) consisting of a set \(X\) and a closure function \(\text{cl}\), a function from the power set of \(X\) to itself. The closure of a subset \(A\) of \(X\), denoted \(\text{cl}(A)\), is the image of \(A\) under \(\text{cl}\). The exterior of \(A\) is \(\text{ext}(A) = X - \text{cl}(A)\), and the interior of \(A\) is \(\text{int}(A) = X - \text{cl}(X - A)\).

We say that \(A\) is closed if \(A = \text{cl}(A)\), \(A\) is open if \(A = \text{int}(A)\), and \(N\) is a neighborhood of \(x\) if \(x \in \text{int}(N)\).

In this paper, we use terminology for generalized closure spaces found in [3], namely, we say that a closure function \(\text{cl}\) defined on \(X\) is

1. grounded, if \(\text{cl}(\emptyset) = \emptyset\).
2. isotonic, if \(\text{cl}(A) \subseteq \text{cl}(B)\) whenever \(A \subseteq B\).
3. enlarging, if \(A \subseteq \text{cl}(A)\), for each subset \(A\) of \(X\).
4. idempotent, if \(\text{cl}(A) = \text{cl}(\text{cl}(A))\), for each subset \(A\) of \(X\).
5. sub-linear, if \(\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)\), for all \(A, B \subseteq X\).

Additionally, we define a kind of separation useful for the task at hand:

Definition 1. Subsets \(A\) and \(B\) of \(X\) are said to be closure-separated in a generalized closure space \((X, \text{cl})\) (or simply, \(\text{cl}\)-separated) if \(A \cap \text{cl}(B) = \emptyset\) and \(\text{cl}(A) \cap B = \emptyset\), or, equivalently, if \(A \subseteq \text{ext}(B)\) and \(B \subseteq \text{ext}(A)\).
Before we begin, we also define a relatively weak separation axiom:

**Definition 2.** Exterior points are closure-separated in a generalized closure space $\langle X, \text{cl} \rangle$ if, for each $A \subseteq X$ and for each $x \in \text{ext}(A)$, $\{x\}$ and $A$ are cl-separated.

**Theorem 1.** Let $\langle X, \text{cl} \rangle$ be a generalized closure space in which exterior points are cl-separated, and let $S$ be the pairs of cl-separated sets in $X$. Then, for each subset $A$ of $X$, the closure of $A$ is

$$\text{cl}(A) = \{ x \in X : \{\{x\}, A\} \not\in S \}.$$ 

**Proof.** In any generalized closure space, $\text{cl}(A) \subseteq \{ x \in X : \{\{x\}, A\} \not\in S \}$: suppose that $y \notin \{ x \in X : \{\{x\}, A\} \not\in S \}$, that is, $\{\{y\}, A\} \in S$. Then $\{\{y\} \cap \text{cl}(A) = \emptyset$, and so, $y \notin \text{cl}(A)$.

Suppose now that $y \notin \text{cl}(A)$. By hypothesis, $\{\{y\}, A\} \in S$, and hence, $y \notin \{ x \in X : \{\{x\}, A\} \not\in S \}$. □

**Definition 3.** A closure function cl defined on a set $X$ is pointwise-symmetric when, for all $x, y \in X$, if $x \in \text{cl}(\{y\})$, then $y \in \text{cl}(\{x\})$.

A generalized closure space $\langle X, \text{cl} \rangle$ is $R_0$ when, for all $x, y \in X$, if $x$ is in each neighborhood of $y$, then $y$ is in each neighborhood of $x$.

In [3], a generalized closure space with a pointwise-symmetric closure function is said to be $(R0c)$, while that which we denote by $R_0$ is there denoted $(R0)$. Note that both conditions hold whenever exterior points are closure-separated:

**Corollary 1.** Let $\langle X, \text{cl} \rangle$ be a generalized closure space in which exterior points are cl-separated. Then cl is pointwise-symmetric and $\langle X, \text{cl} \rangle$ is $R_0$.

**Proof.** Suppose that exterior points are cl-separated in $X$. If $x \in \text{cl}(\{y\})$, then $\{x\}$ and $\{y\}$ are not cl-separated, and hence, $y \in \text{cl}(\{x\})$. Hence, cl is pointwise-symmetric.

Suppose that $x$ belongs to every neighborhood of $y$, that is, $x \in M$ whenever $y \in \text{int}(M)$. Letting $A = X - M$ and rewriting contrapositively, $y \in \text{cl}(A)$ whenever $x \in A$.

Suppose $x \in \text{int}(N)$. $x \notin \text{cl}(X - N)$, so $x$ is cl-separated from $X - N$. Hence, $\text{cl}(\{x\}) \subseteq N$.

$x \in \{x\}$, so $y \in \text{cl}(\{x\}) \subseteq N$. Hence, $\langle X, \text{cl} \rangle$ is $R_0$. □

While these three axioms are not equivalent in general, they are equivalent when the closure function is isotonic:
Theorem 2. Let \((X, \text{cl})\) be a generalized closure space with \(\text{cl}\) isotonic. Then the following are equivalent:

1. exterior points are \(\text{cl}\)-separated.
2. \(\text{cl}\) is pointwise-symmetric.
3. \((X, \text{cl})\) is \(R_0\).

Proof. Suppose that (2) is true. Let \(A \subseteq X\), and suppose \(x \in \text{ext}(A)\). Then, as \(\text{cl}\) is isotonic, for each \(y \in A\), \(x \notin \text{cl}(\{y\})\), and hence, \(y \notin \text{cl}(\{x\})\). Hence, \(\text{cl}(\{x\}) \cap A = \emptyset\). Hence, (2) implies (1), and by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let \(x, y \in X\) such that \(x\) is in every neighborhood of \(y\), that is, \(x \in N\) whenever \(y \in \text{int}(N)\). Then \(y \in \text{cl}(A)\) whenever \(x \in A\), and in particular, since \(x \in \{x\}\), \(y \in \text{cl}(\{x\})\). Hence, \(x \notin \text{cl}(\{y\})\). Thus, if \(y \in B\), then \(x \in \text{cl}(\{y\}) \subseteq \text{cl}(B)\), as \(\text{cl}\) is isotonic. Hence, if \(x \in \text{int}(C)\), then \(y \in C\), that is, \(y\) is in every neighborhood of \(x\). Hence, (2) implies (3).

Finally, suppose that \((X, \text{cl})\) is \(R_0\) and suppose that \(x \in \text{cl}(\{y\})\). Since \(\text{cl}\) is isotonic, \(x \in \text{cl}(B)\) whenever \(y \in B\), or, equivalently, \(y\) is in every neighborhood of \(x\). Since \((X, \text{cl})\) is \(R_0\), \(x \in N\) whenever \(y \in \text{int}(N)\). Hence, \(y \in \text{cl}(A)\) whenever \(x \in A\), and in particular, since \(x \in \{x\}\), \(y \in \text{cl}(\{x\})\). Hence, (3) implies (2).

Hence, when \(\text{cl}\) is isotonic and pointwise-symmetric, the collection of \(\text{cl}\)-separated sets uniquely determines \(\text{cl}\). In fact, such closure functions can be defined entirely in terms of the pairs of sets which they closure-separate:

Theorem 3. Let \(S\) be a set of unordered pairs of subsets of a set \(X\) such that, for all \(A, B, C \subseteq X\),

1. if \(A \subseteq B\) and \(\{B, C\} \in S\), then \(A, C \in S\), and
2. if \(\{x\}, B \in S\) for each \(x \in A\) and \(\{y\}, A \in S\) for each \(y \in B\), then \(\{A, B\} \in S\).

Then there is a unique pointwise-symmetric isotonic closure function \(\text{cl}\) on \(X\) which closure-separates the elements of \(S\).

Proof. Define \(\text{cl}\) by \(\text{cl}(A) = \{x \in X : \{x\}, A \notin S\}\), for every \(A \subseteq X\). If \(A \subseteq B \subseteq X\) and \(x \in \text{cl}(A)\), then \(\{x\}, A \notin S\). Hence, \(\{x\}, B \notin S\), that is, \(x \in \text{cl}(B)\). Hence, \(\text{cl}\) is isotonic. Also, \(x \in \text{cl}(\{y\})\) iff \(\{x\}, \{y\} \notin S\) iff \(y \in \text{cl}(\{x\})\), and thus, \(\text{cl}\) is pointwise-symmetric.
Suppose that \( \{A, B\} \in \mathcal{S} \). Then \( A \cap \text{cl}(B) = A \cap \{x \in X : \{\{x\}, B\} \notin \mathcal{S}\} = \{x \in A : \{\{x\}, B\} \notin \mathcal{S}\} = \emptyset \). Similarly, \( \text{cl}(A) \cap B = \emptyset \). Hence, if \( \{A, B\} \in \mathcal{S} \), then \( A \) and \( B \) are \text{cl}-separated.

Now suppose that \( A \) and \( B \) are \text{cl}-separated. Then \( \{x \in A : \{\{x\}, B\} \notin \mathcal{S}\} = A \cap \text{cl}(B) = \emptyset \) and \( \{x \in A : \{\{x\}, B\} \notin \mathcal{S}\} = \text{cl}(A) \cap B = \emptyset \). Hence, \( \{\{x\}, B\} \in \mathcal{S} \) for each \( x \in A \) and \( \{\{y\}, A\} \in \mathcal{S} \) for each \( y \in B \), and thus, \( \{A, B\} \in \mathcal{S} \).

Furthermore, many properties of closure functions can be expressed in terms of the sets they separate:

**Theorem 4.** Let \( \mathcal{S} \) be the pairs of \text{cl}-separated sets of a generalized closure space \((X, \text{cl})\) in which exterior points are closure-separated. Then \text{cl} is

1. grounded if, and only if, for all \( x \in X \), \( \{\{x\}, \emptyset\} \in \mathcal{S} \).

2. enlarging if, and only if, for all \( \{A, B\} \in \mathcal{S} \), \( A \) and \( B \) are disjoint.

3. sub-linear if, and only if, \( \{A, B \cup C\} \in \mathcal{S} \) whenever \( \{A, B\} \in \mathcal{S} \) and \( \{A, C\} \in \mathcal{S} \).

Moreover, if \text{cl} is enlarging and for all \( A, B \subseteq X \), \( \{\{x\}, A\} \notin \mathcal{S} \) whenever \( \{\{x\}, B\} \notin \mathcal{S} \) and \( \{\{y\}, A\} \notin \mathcal{S} \), for each \( y \in B \), then \text{cl} is idempotent. Also, if \text{cl} is isotonic and idempotent, then \( \{\{x\}, A\} \notin \mathcal{S} \) whenever \( \{\{x\}, B\} \notin \mathcal{S} \) and \( \{\{y\}, A\} \notin \mathcal{S} \), for each \( y \in B \).

**Proof.** Recall that by Theorem 1, \text{cl}(A) = \{x \in X : \{\{x\}, A\} \notin \mathcal{S}\}, for every \( A \subseteq X \).

Suppose that for all \( x \in X \), \( \{\{x\}, \emptyset\} \in \mathcal{S} \). Then \text{cl}(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin \mathcal{S}\} = \emptyset \). Hence, \text{cl} is grounded.

Conversely, if \( \emptyset = \text{cl}(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin \mathcal{S}\} \), then \( \{\{x\}, \emptyset\} \in \mathcal{S} \), for all \( x \in X \).

Suppose that for all \( \{A, B\} \in \mathcal{S} \), \( A \) and \( B \) are disjoint. Since \( \{\{a\}, A\} \notin \mathcal{S} \) if \( a \in A \), \( A \subseteq \text{cl}(A) \), for each \( A \subseteq X \). Hence, \text{cl} is enlarging.

Conversely, suppose that \text{cl} is enlarging and \( \{A, B\} \in \mathcal{S} \). Then \( A \cap B \subseteq \text{cl}(A) \cap B = \emptyset \).

Suppose that \( \{A, B \cup C\} \in \mathcal{S} \) whenever \( \{A, B\} \), \( \{A, C\} \in \mathcal{S} \). Let \( x \in X \) and \( B, C \subseteq X \) such that \( \{\{x\}, B \cup C\} \notin \mathcal{S} \). Then \( \{\{x\}, B\} \notin \mathcal{S} \) or \( \{\{x\}, C\} \notin \mathcal{S} \). Hence, \( \text{cl}(B \cup C) \subseteq \text{cl}(B) \cup \text{cl}(C) \), and therefore, \text{cl} is sub-linear.
Conversely, suppose that $\text{cl}$ is sub-linear, and let $\{A, B\}, \{A, C\} \in S$. Then $\text{cl}(A \cup B) \cap C \subseteq (\text{cl}(A) \cup \text{cl}(B)) \cap C = (\text{cl}(A) \cap C) \cup (\text{cl}(B) \cap C) = \emptyset$, and $(A \cup B) \cap \text{cl}(C) = (A \cap \text{cl}(C)) \cup (B \cap \text{cl}(C)) = \emptyset$.

Suppose that $\text{cl}$ is enlarging, and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$, for each $y \in B$. Then $\text{cl}(\text{cl}(A)) \subseteq \text{cl}(A)$: if $x \in \text{cl}(\text{cl}(A))$, then $\{\{x\}, \text{cl}(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in \text{cl}(A)$; hence, $\{\{x\}, A\} \notin S$. And since $\text{cl}$ is enlarging, $\text{cl}(A) \subseteq \text{cl}(\text{cl}(A))$. Thus, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$, for each $A \subseteq X$.

Finally, suppose that $\text{cl}$ is isotonic and idempotent. Let $x \in X$ and $A, B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and, for each $y \in B$, $\{\{y\}, A\} \notin S$. Then $x \in \text{cl}(B)$ and for each $y \in B$, $y \in \text{cl}(A)$, that is, $B \subseteq \text{cl}(A)$. Hence, $x \in \text{cl}(B) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A)$.

**Definition 4.** If $(X, \text{cl}_X)$ and $(Y, \text{cl}_Y)$ are generalized closure spaces, then a function $f : X \to Y$ is said to be

1. closure-preserving, if $f(\text{cl}_X(A)) \subseteq \text{cl}_Y(f(A))$, for each $A \subseteq X$.
2. continuous, if $\text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))$, for each $B \subseteq Y$.

In general, neither condition implies the other. However, we easily obtain the following result:

**Theorem 5.** Let $(X, \text{cl}_X)$ and $(Y, \text{cl}_Y)$ be generalized closure spaces, and let $f : X \to Y$.

1. If $f$ is closure-preserving and $\text{cl}_Y$ is isotonic, then $f$ is continuous.
2. If $f$ is continuous and $\text{cl}_X$ is isotonic, then $f$ is closure-preserving.

**Proof.** Suppose that $f$ is closure-preserving and $\text{cl}_Y$ is isotonic. Let $B \subseteq Y$. $f(\text{cl}_X(f^{-1}(B))) \subseteq \text{cl}_Y(f(f^{-1}(B))) \subseteq \text{cl}_Y(B)$, and hence, $\text{cl}_X(f^{-1}(B)) \subseteq f^{-1}(f(\text{cl}_X(f^{-1}(B)))) \subseteq f^{-1}(\text{cl}_Y(B))$.

Suppose that $f$ is continuous and $\text{cl}_X$ is isotonic. Let $A \subseteq X$. $\text{cl}_X(A) \subseteq \text{cl}_X(f^{-1}(f(A))) \subseteq f^{-1}(\text{cl}_Y(f(A)))$, and hence, $f(\text{cl}_X(A)) \subseteq f(f^{-1}(\text{cl}_Y(f(A)))) \subseteq \text{cl}_Y(f(A))$.

**Definition 5.** Let $(X, \text{cl}_X)$ and $(Y, \text{cl}_Y)$ be generalized closure spaces, and let $f : X \to Y$. If, for all $A, B \subseteq X$, $f(A)$ and $f(B)$ are not $\text{cl}_Y$-separated whenever $A$ and $B$ are not $\text{cl}_X$-separated, then we say that $f$ is nonseparating.

Note that $f$ is nonseparating if and only if $A$ and $B$ are $\text{cl}_X$-separated whenever $f(A)$ and $f(B)$ are $\text{cl}_Y$-separated.
Theorem 6. Let \((X, \text{cl}_X)\) and \((Y, \text{cl}_Y)\) be generalized closure spaces, and let \(f : X \longrightarrow Y\).

(1) If \(\text{cl}_Y\) is isotonic and \(f\) is nonseparating, then \(f^{-1}(C)\) and \(f^{-1}(D)\) are \(\text{cl}_X\)-separated whenever \(C\) and \(D\) are \(\text{cl}_Y\)-separated.

(2) If \(\text{cl}_X\) is isotonic and \(f^{-1}(C)\) and \(f^{-1}(D)\) are \(\text{cl}_X\)-separated whenever \(C\) and \(D\) are \(\text{cl}_Y\)-separated, then \(f\) is nonseparating.

\[\text{Proof.}\]
Let \(A, B \subseteq X\) such that \(C = f(A)\) and \(D = f(B)\) are \(\text{cl}_Y\)-separated. Then \(f^{-1}(C)\) and \(f^{-1}(D)\) are \(\text{cl}_X\)-separated, and since \(\text{cl}_X\) is isotonic, \(A \subseteq f^{-1}(f(A)) = f^{-1}(C)\) and \(B \subseteq f^{-1}(f(B)) = f^{-1}(D)\) are \(\text{cl}_X\)-separated as well. \(\blacksquare\)

Theorem 7. Let \((X, \text{cl}_X)\) and \((Y, \text{cl}_Y)\) be generalized closure spaces, and let \(f : X \longrightarrow Y\). If \(f\) is closure-preserving, then \(f\) is nonseparating.

\[\text{Proof.}\]
Suppose that \(f\) is closure-preserving and that \(A, B \subseteq X\) are not \(\text{cl}_X\)-separated. Suppose that \(\text{cl}_X(A) \cap B \neq \emptyset\). Then \(\emptyset \neq f(\text{cl}_X(A)) \cap B \subseteq f(\text{cl}_X(A)) \cap f(B) \subseteq \text{cl}_Y(f(A)) \cap f(B)\). Similarly, if \(A \cap \text{cl}_X(B) \neq \emptyset\), then \(f(A) \cap \text{cl}_Y(B) \neq \emptyset\). Hence, \(f(A)\) and \(f(B)\) are not \(\text{cl}_Y\)-separated. \(\blacksquare\)

Corollary 2. Let \((X, \text{cl}_X)\) and \((Y, \text{cl}_Y)\) be generalized closure spaces with \(\text{cl}_X\) isotonic, and let \(f : X \longrightarrow Y\). If \(f\) is continuous, then \(f\) is nonseparating.

\[\text{Proof.}\]
If \(f\) is continuous and \(\text{cl}_X\) is isotonic, then \(f\) is closure-preserving. Hence, by the previous result, \(f\) is nonseparating. \(\blacksquare\)

Theorem 8. Let \((X, \text{cl}_X)\) and \((Y, \text{cl}_Y)\) be generalized closure spaces with exterior points \(\text{cl}_Y\)-separated in \(Y\), and let \(f : X \longrightarrow Y\). Then \(f\) is nonseparating iff \(f\) is closure-preserving.

\[\text{Proof.}\]
By Theorem 7 if \(f\) is closure-preserving, then \(f\) is nonseparating. Suppose that \(f\) is nonseparating, and let \(A \subseteq X\). If \(\text{cl}_X(A) = \emptyset\), then \(f(\text{cl}_X(A)) = \emptyset \subseteq \text{cl}_Y(f(A))\).

Suppose \(\text{cl}_X(A) \neq \emptyset\). Let \(S_X\) and \(S_Y\) denote the pairs of \(\text{cl}_X\)-separated subsets of \(X\) and the pairs of \(\text{cl}_Y\)-separated subsets of \(Y\), respectively. Let \(y \in f(\text{cl}_X(A))\), and let \(x \in \text{cl}_X(A) \cap f^{-1}(\{y\})\). Since \(x \in
\( \text{cl}_X(A), \{\{x\}, A\} \notin \mathcal{S}_X, \) and since \( f \) is nonseparating, \( \{\{y\}, f(A)\} \notin \mathcal{S}_Y \). Since exterior points are \( \text{cl}_Y \)-separated, \( y \in \text{cl}_Y(f(A)) \). Thus, \( f(\text{cl}_X(A)) \subseteq \text{cl}_Y(f(A)) \), for each \( A \subseteq X \).

\[ \square \]

**Corollary 3.** Let \((X, \text{cl}_X)\) and \((Y, \text{cl}_Y)\) be generalized closure spaces with isotonic closure functions and with \( \text{cl}_Y \) pointwise-symmetric, and let \( f : X \rightarrow Y \). Then \( f \) is nonseparating iff \( f \) is continuous.

**Proof.** Since \( \text{cl}_Y \) is isotonic, exterior points are closure-separated in \((Y, \text{cl}_Y)\). Since both closure functions are isotonic, \( f \) is closure-preserving iff \( f \) is continuous. Hence, we can apply the previous theorem. \[ \square \]

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