DISSIPATION ENHANCEMENT BY MIXING FOR EVOLUTION
$p$–LAPLACIAN ADVECTION EQUATIONS

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Abstract. In this paper, we quantitatively consider the enhance-dissipation effect of the advection term to the $p$-Laplacian equations. More precisely, we show the mixing property of flow for the passive scalar enhances the dissipation process of the $p$-Laplacian in the sense of $L^2$ decay; that is, the $L^2$ decay can be arbitrarily fast. The main ingredient of our argument is to understand the underlying iteration structure inherited from the evolution $p$-Laplacian advection equations. This extends the well-known results of the dissipation enhancement result of the linear Laplacian by Constantin, Kiselev, Ryzhik and Zlatoš into a non-linear setting.

1. Introduction

In the study of incompressible fluid, one fundamental phenomena that arise in a wide variety of application is dissipation enhancement, whose mechanism in general comes from the following two primary sources: mixing, which induces filamentation and facilitates the formation of small scales (namely, high frequencies); and diffusion, which efficiently damps small scales (namely, high frequencies) and accelerates the dissipation process.

Without diffusion, the behavior of passive scalar advected by incompressible mixing flows has been extensively studied in recent decades. For example, in [23, 27], the authors measured mixing by multi-scale norms; another direction along this line of research is to study the mixing efficiency of incompressible flows (see, for instance [1, 13, 21, 24, 29]). However, to our best knowledge, no apriori limit to the resolution has been attained via mixing in this regime.

While in the presence of diffusion, the effects of mixing may be enhanced, balanced, or even suppressed by diffusion. One of the most famous models used to study the interaction between these two sources is the advection-diffusion equation

$$\begin{cases}
\partial_t \vartheta + u \cdot \nabla \vartheta - \nu \Delta \vartheta = 0 \\
\vartheta(0, x) = \vartheta_0 \in L^2_0,
\end{cases}$$

where $u$ stands for a time-dependent or time-independent incompressible velocity field, $\nu > 0$ presents the strength of diffusion, proportional to the inverse Péclet number and $L^2_0$ refers the collection of all $L^2(T^d)$ functions with mean zero.

The study of the dissipation enhancement for the linear advection equations is a popular topic in recent years. It dates back to the celebrated work [9] by Constantin, Kiselev, Ryzhik and Zlatoš in 2008, in which, they first found an equivalent condition for the time-independent $u$ to enhance the dissipation effect, in the sense of the $L^2$ norm faster decay of the solution, on a compact manifold like torus. Later,
there have been many other extensions along this direction (see, e.g., [28, 31]). In particular, in [10, 17], the authors studied the dissipation enhancement of (1.1) under various mixing conditions, and notably their methods lead to an explicit quantitative bound when the mixing flow $u$ is assumed to be polynomial and exponential.

As some byproducts, understanding dissipation enhancement can help us study stabilization phenomena of the singularity. We refer the interested reader [6, 7, 14–16, 18–20, 22] and the reference therein for more details.

The goal of this paper is to study the effect of mixing on dissipation enhancement to non-linear diffusion models. One natural nonlinear model which resembles (1.1) is the $p$-Laplacian dissipation. More precisely, we consider the following system: for $p > 2$, $u$ an incompressible flow on $\mathbb{T}^d$ and $\nu > 0$,

\begin{equation}
\begin{aligned}
\partial_t \theta + (u(t) \cdot \nabla) \theta - \nu \nabla \cdot \left( |\nabla \theta|^{p-2} \nabla \theta \right) &= 0; \\
\theta(x, 0) &= \theta_0(x) \in L^2_0.
\end{aligned}
\end{equation}

Here $p$ measures the level of diffusion, and it is clear that when $p = 2$, (1.2) reduces the regular Laplacian dissipation. The interest in considering such generalized equations is motivated by their applications in the mathematical modeling of various real-world processes, such as the flows of electrorheological or thermo-rheological fluids [3, 4, 26], the problem of thermistors [30], and processing of digital images [8].

As far as we know, the equation (1.2) has been studied a lot via various mathematical points of view in the absence of advection term, however the advective case and the corresponding dissipation enhancement phenomenon for (1.2) is less understood.

In this paper, we consider how the advection terms can help the $p$-Laplacian term to dissipate the solution’s $L^2$ norm. The novelty of this paper is two fold.

1. The main difference between (1.2) and the linear model (1.1) is that its solution becomes degenerate or singular at the points where $|\nabla \theta| = 0$, which prevents one from expecting the existence of classical solutions. We overcome this difficulty by working directly on the weak solutions which satisfy extra regularity properties (see, Section 3);

2. The solution operator of (1.2) is non-linear, which suggests that the semigroup approach inherited from the linear model (1.1) might fail in our case. Instead of using the semigroup structure, we explore a new iteration structure (see, Lemma 4.5) underlying the equation (1.2), which essentially plays the same role as the semigroup structure in the linear model case.

To this end, the plan of this paper is as follows. We begin by defining mixing rates, introduce the non-linear dissipation time, and state our main results, as well as some applications in Section 2. Section 3 is devoted to explore several properties for the weak solution of the equation (1.2). As a consequence, we give a priori estimate for the non-linear dissipation time. Finally, in Section 4, we prove the quantitative dissipation time bound for the $p$-Laplacian dissipation. The proof involves a study of the underlying iteration structure of the equation (1.2).

2. Main results

Let $M$ be a (smooth) closed Riemannian manifold in $\mathbb{R}^d$, and $u$ be a smooth, time dependent, divergence free vector field on $M$. Our model case for $M$ would
be the $d$-dimensional torus $\mathbb{T}^d$. Let further,  
$$0 < \lambda_1 \leq \lambda_2 \leq \ldots$$
be the eigenvalues of $-\Delta$ on $M$. Moreover, without loss of generality, we assume $\text{Vol}(M) = 1$. For any $p > 1$, we denote $L^p := L^p(M)$ the usual $L^p$ space on $M$ and for any $\alpha \in \mathbb{R}$, recall the homogeneous Sobolev space of order $\alpha$ is given by  
$$H^\alpha = H^\alpha(M) := \left\{ f = \sum_i a_i e_i : \| f \|_{H^\alpha}^2 := \sum_i \lambda_i^\alpha |a_i|^2 < \infty \right\},$$
where in the above definition, $e_i$ is the normalized eigenvector corresponding to the eigenvalue $\lambda_i$, $i \geq 1$. Note that under this formulation, $\| \nabla (\cdot) \|_2 = \| \cdot \|_{H^1}$.

The goal of this paper is to introduce and study the dissipation enhancement of advection to a nonlinear diffusivity, which plays a crucial role in the thin film type equations. Here is the setup. Let $\mathbb{T}^d := [0,1]^d$ be the standard torus in $\mathbb{R}^d$ and $\nu > 0$ be the strength of the diffusion. Now for $p > 2$ and each time $s \geq 0$, we consider following non-linear parabolic equation with gradient nonlinearity on the $d$-dimensional torus with advection of an incompressible vector field $u(t)$:  
\begin{align*}
\begin{cases}
\partial_t \theta_s + (u(t) \cdot \nabla) \theta_s - \nu \Delta_p \theta_s = 0; \\
\theta_s(t) = \theta_{s,0}, \quad t = s.
\end{cases}
\end{align*}
for $t > s$, with initial data $\theta_{s,0} = \theta_0(s)$. Here,  
1. $\nabla$ is the covariant derivative;
2. $\Delta_p \theta_s := \nabla \cdot (|\nabla \theta_s|^{p-2} \nabla \theta_s)$ is the $p$-Laplacian;
3. $\theta_{s,0} = \theta_0(s)$, where $\theta_0(\cdot)$ is the solution of (2.1) with $s = 0$ and initial data $\theta_{0,0} \in L^2_0(\mathbb{T}^d)$, which is the space of $L^2$ integrable functions on $\mathbb{T}^d$ with mean zero.

**Remark 2.1.** We make a remark that the solution of (2.1) should be understood in weak sense. Moreover, all these solutions have certain regularity (see, Theorem 3.2), which, in particular, guarantees that $\theta_{s,0}$ is a measurable function and hence (2.1) is well-defined. Furthermore, we can actually see that $\theta_{s,0}$ also belongs to $L^2_0(\mathbb{T}^d)$ (see, Corollary 3.3).

We are interested in the behavior of solutions of (2.1) for $\nu \ll 1$ and a fixed initial data $\theta_{0,0}$. The prototype of our model is the linear diffusion equation. One typical example would be  
\begin{align*}
\begin{cases}
\partial_t \vartheta_s + (u(t) \cdot \nabla) \vartheta_s - \nu (-\Delta)^\alpha \vartheta_s = 0; \\
\vartheta_s(t) = \vartheta_{s,0}, \quad t = s,
\end{cases}
\end{align*}
for $t > s$, with initial data $\vartheta_{s,0} \in L^2_0(\mathbb{T}^d)$, where $\Delta$ is the Laplace-Beltrami operator on $\mathbb{T}^d$. Observe that when $\alpha = 1$, (2.2) becomes the advection diffusion equation (see, e.g., [9]); when $\alpha = 2$, (2.2) refers to the advective hyperdiffusion equation. In the sequel, our interest will lie in the case when $\alpha = 1$, as this is exactly our main equation (2.1) with $p = 2$, which can be viewed as the “endpoint case” for the non-linear model.
One crucial concept to describe the behavior of the linear model (2.2) when \( \nu \ll 1 \) is the linear dissipation time of the flow \( u \) for the linear models (i.e. the time required for the system to dissipate a constant fraction of its initial energy) is given by

\[
\tau_d := \sup_{s \in \mathbb{R}} \left\{ \inf \left\{ t - s : t \geq s, \text{ and } \|\theta_s(t)\|_2 \leq \frac{\|\theta_s,0\|_2}{e} \text{ for all } \theta_s,0 \in L^2_0(T^d) \right\} \right\}.
\]

Note that, for example (say, \( \alpha = 1 \)), since the solution of (2.2) is strong, we are able to multiply \( \theta_s \) on both sides of the equation in (2.2) and integrate over \( T^d \) to see

\[
\frac{1}{2} \partial_t \|\theta_s(t)\|_2^2 + \nu \|\theta_s\|_{H^1} = 0,
\]

and hence

\[
\|\theta_s(t)\|_2 \leq e^{-\nu(t-s)} \|\theta_s,0\|_2.
\]

This suggests

\[
\tau_d \leq \frac{1}{\nu \lambda_1},
\]

where we recall that \( \lambda_1 \) is the principal eigenvalue of \(-\Delta \) on \( T^d \). Moreover, it turns out this is the best one can hope, that if \( u \) is only assumed to be incompressible (see, [9]).

An important feature for \( \tau_d \) is that when the flow \( u \) is assumed to be mixing, one can improve the estimate (2.6) into, heuristically,

\[
\tau_d \leq \frac{C}{\nu \lambda N},
\]

where \( C \) and \( N \) are some universals constant which only depends on the equation and the mixing condition (see, e.g. [11, 17] for a more comprehensive treatment).

Let us recall the mixing condition first.

**Definition 2.2.** Let \( h : [0, \infty) \to (0, \infty) \) be a strictly decreasing function that vanishes at infinity, and \( 0 \leq \alpha < \infty, \beta \geq 0 \). Let further, \( \varphi_{s,t} : T^d \to T^d \) be the flow map of \( u \) defined by

\[
\partial_t (\varphi_{s,t}) = -u(\varphi_{s,t}) \quad \text{and} \quad \varphi_{s,s} = \text{Id}.
\]

(1) We say that the flow \( u \) is strongly \( \alpha, \beta \) mixing with rate function \( h \) if for all \( f \in \dot{H}^\alpha, g \in \dot{H}^\beta \), we have

\[
\| (f \circ \varphi_{s,t}, g) \| \leq h(t-s) \| f \|_{\dot{H}^\alpha} \| g \|_{\dot{H}^\beta}.
\]

Or equivalently, for all \( f \in \dot{H}^\alpha \), there holds

\[
\| f \circ \varphi_{s,t} \|_{\dot{H}^{-\beta}} \leq h(t-s) \| f \|_{\dot{H}^\alpha};
\]

(2) We say that \( \varphi \) is weakly \( \alpha, \beta \) mixing with rate function \( h \) if for all \( f \in \dot{H}^\alpha, g \in \dot{H}^\beta \), we have

\[
\left( \frac{1}{t-s} \int_s^t |(f \circ \varphi_{s,r}, g)|^2 \, dr \right)^{\frac{1}{2}} \leq h(t-s) \| f \|_{\dot{H}^\alpha} \| g \|_{\dot{H}^\beta}.
\]
The phenomenon of improving from (2.6) to (2.7) under the mixing condition is referred as **dissipation enhancement**. The purpose of this paper is to explore such a phenomenon for the advection equation with $p$-Laplacian evolution (2.1). To state the main results, we first extend the definition of the linear dissipation time to its non-linear counterpart.

**Definition 2.3.** Let $\theta_{0,0} \in L^2_0(\mathbb{T}^d)$. The **non-linear dissipation time** associated to the advection equation with $p$-Laplacian diffusion (2.1) is given by

$$
\kappa_d := \sup_{s \in \mathbb{R}} \left( \inf \left\{ t - s \mid t \geq s, \text{ and } \|\theta_s(t)\|_2 \leq \frac{\|\theta_{s,0}\|_2}{\left[ (p-2) \|\theta_{s,0}\|_2^{p-2} + 1 \right]^{\frac{1}{p-2}}} \right\} \right),
$$

where $\theta_s(t)$ is the weak solution of (2.1) with initial data $\theta_{s,0}$.

We make several remarks before we proceed.

**Remark 2.4.**
1. The study of $\kappa_d$ is a little bit subtle compared to $\tau_d$. More precisely, the solutions of the linear advection equation (2.2) possess strong or even classical solution, if initial data is provided smooth enough; while, to our best knowledge, one cannot expect the existence of classical solution to (2.1), due to the degeneration or singularity at the points where $|\nabla \theta| = 0$, and hence the solutions of (2.1) are interpreted only in the weak sense (see, e.g., [5, 12]). We will overcome this difficulty and study the corresponding dissipation enhancement phenomenon to (2.1) by arguing that such weak solutions (see, Definition 3.1) have certain regularity (see, Section 3);

2. The definition of $\kappa_d$ is indeed consistent with $\tau_d$. For example, by a standard non-linear Gronwall type estimate, one can see that if $u$ is assumed to be incompressible, then

$$
\kappa_d \leq \frac{1}{\nu \lambda_{\frac{p}{2}}} \quad \text{(see, Corollary 3.3)}
$$

Moreover, if $u$ is assumed to be mixing, then our main results (see, Theorem 2.5 and Theorem 2.7) suggest that

$$
\kappa_d \leq \frac{C}{\nu \lambda_{\frac{p}{2}}} \quad \text{for some } C \text{ and } N \text{ only depending on } (2.1) \text{ and the mixing condition. These facts clearly resemble and generalize both (2.6) and (2.7)};
$$

3. It is not hard to see that the term $\|\theta_s(t)\|_2$ obeys a polynomial decay with the magnitude $\frac{1}{(t-s)^{p}}$ when $t$ is sufficiently large. Heuristically, the non-linear dissipation time $\kappa_d$ describes quantitatively how large the coefficient of $(t-s)^{\frac{p}{2}}$ is in such a decay, under various assumptions on $u$, this is of the same flavor when we consider the linear dissipation time $\tau_d$, which is used to study how large the coefficient of the term $t-s$ is in the exponential decay (see, (2.5)).

We are ready to statement our main results.
Then be a strictly decreasing function that vanishes at infinity. If $u$ is strongly $\alpha, \beta$ mixing with rate function $h$, then

\begin{equation}
\kappa_d \leq \frac{C}{\nu H_1(\nu)^{\frac{\alpha}{\beta}} H_{1,v,h}},
\end{equation}

where $C$ is an absolute constant that only depends on $h$, $\|\nabla u\|_{\infty}$, $p$, $\|\theta_{0,0}\|_2$, the strongly mixing condition, any dimension constants and any constants that depends on $T^d$,

\begin{equation}
H_{1,v,h} := \min \left\{ 1, 2^{-p-1} \cdot h^{-1} \left( \frac{H_1(\nu)^{-\frac{\alpha}{\beta}}}{2^{1-\frac{\alpha}{\beta}}} \right) ^{\frac{\alpha}{\beta}} \right\},
\end{equation}

and $H_1 : (0, \infty) \to (0, \infty)$ is defined by

\begin{equation}
H_1(\mu) := \sup \left\{ \lambda \left| \frac{\lambda^d D_p \|\theta_{0,0}\|_2^{p-2}}{h^{-1} \left( \frac{\lambda^{\frac{\alpha}{\beta}}}{2^{1-\frac{\alpha}{\beta}}} \right)^\beta} \cdot e^{4\|\nabla u\|_{\infty} h^{-1} \left( \frac{\lambda^{\frac{\alpha}{\beta}}}{2^{1-\frac{\alpha}{\beta}}} \right)^{\frac{\alpha}{\beta}}} \right| \leq \frac{\|\nabla u\|_{\infty}^2}{4\mu} \right\},
\end{equation}

Here

\begin{equation}
D_p := 48^{p-1} p^p 2^{p(p-1)}.
\end{equation}

and $h^{-1}$ is the inverse function of $h$.

**Corollary 2.6.** Let $\alpha, \beta, u, h, p$ and $\theta_{0,0}$ be as in Theorem 2.5. Let further, $\nu < 1$. Then

1. If the mixing rate function $h : (0, \infty) \to (0, \infty)$ is the power law

\begin{equation}
h(t) = \frac{c}{t^q}
\end{equation}

for some $q > 0$, then the nonlinear dissipation time is bounded by

\begin{equation}
\kappa_d \leq \frac{C}{\nu^{\frac{\alpha}{\beta} + \delta}}
\end{equation}

where $\delta := \frac{pq}{\alpha + \beta}$

and $C = C(\alpha, \beta, c, q, p, d, \|\nabla u\|_{L^\infty}, \|\theta_{0,0}\|_2)$ is a finite constant;

2. If the mixing rate function $h : [0, \infty) \to (0, \infty)$ is the exponential function

\begin{equation}
h(t) = c_1 \exp(-c_2 t),
\end{equation}

for some constant $c_1, c_2 > 0$, then the nonlinear dissipation time is bounded by

\begin{equation}
\kappa_d \leq \frac{C}{\nu^{\frac{\alpha}{\beta} + \delta}}
\end{equation}

where $\delta := \frac{4\|\nabla u\|_{L^\infty} (\alpha + \beta)}{pc_2 + 4\|\nabla u\|_{L^\infty} (\alpha + \beta)}$,

$C = C(\alpha, \beta, c_1, c_2, p, d, \|\nabla u\|_{L^\infty}, \|\theta_{0,0}\|_2)$ is a finite constant.

**Theorem 2.7.** Let $0 < \alpha \leq 1$, $\beta > 0$, $p > 2$, $\theta_{0,0} \in L_0^2(T^d)$ and $h : [0, \infty) \to (0, \infty)$ be a strictly decreasing function that vanishes at infinity. If $u$ is weakly $\alpha, \beta$ mixing with rate function $h$, then

\begin{equation}
\kappa_d \leq \frac{C}{\nu H_2(\nu)^{\frac{\alpha}{\beta}} H_{2,v,h}},
\end{equation}

and $H_2 : (0, \infty) \to (0, \infty)$ is defined by

\begin{equation}
H_2(\mu) := \sup \left\{ \lambda \left| \frac{\lambda^d \|\theta_{0,0}\|_2^{p-2}}{h^{-1} \left( \frac{\lambda^{\frac{\alpha}{\beta}}}{2^{1-\frac{\alpha}{\beta}}} \right)^\beta} \cdot e^{4\|\nabla u\|_{\infty} h^{-1} \left( \frac{\lambda^{\frac{\alpha}{\beta}}}{2^{1-\frac{\alpha}{\beta}}} \right)^{\frac{\alpha}{\beta}}} \right| \leq \frac{\|\nabla u\|_{\infty}^2}{4\mu} \right\},
\end{equation}

Here

\begin{equation}
D_p := 48^{p-1} p^p 2^{p(p-1)}.
\end{equation}

and $h^{-1}$ is the inverse function of $h$. 








where $C$ is an absolute constant that only depends on $h$, $\|\nabla u\|_{\infty}$, $p$, $\|\theta_{0,0}\|_2$, the weakly mixing condition, any dimension constants and any constants that depends on $T^d$,

\begin{equation}
H_{2,\nu,h} := \min \left\{ 1, \ 2^{-p-1} : h^{-1} \left( \frac{H_2(\nu)^{-\frac{d+2\alpha+2\beta}{4}}}{2^{1-\frac{d+2\alpha+2\beta}{4}} + \frac{p-2}{4}} \right)^\frac{p-2}{4} \right\}.
\end{equation}

Here $H_2 : (0,\infty) \to (0,\infty)$ is defined by

\begin{equation}
H_2(\mu) := \sup \left\{ \lambda \left| \frac{\lambda^2 d_{\nu}^2}{h^{-1}} D_p \|\theta_{0,0}\|_2^{2-2} \cdot e^{-4\|\nabla u\|_{\infty} h^{-1} \left( \frac{\lambda^{-d+2\alpha+2\beta}}{2\nu^2} \right)} \right| \leq \|\nabla u\|_{\infty}^2 \right\},
\end{equation}

where $D_p$ is defined as in (2.14) and $\epsilon = \epsilon(T^d)$ is a finite constant that only depends on $T^d$.

**Corollary 2.8.** Let $\alpha, \beta, u, h, p$ and $\theta_{0,0}$ be as in Theorem 2.5. Let further, $\nu \ll 1$.

1. If the mixing rate function $h : (0,\infty) \to (0,\infty)$ is the power law (2.15), then the nonlinear dissipation time is bounded by

\begin{equation}
\kappa_d \leq \frac{C}{\nu |\ln \nu|^\delta}, \quad \text{where} \quad \delta := \frac{2pq}{2\alpha + 2\beta + d}
\end{equation}

and $C = C(\alpha, \beta, c, p, q, d, \|\nabla u\|_{L^\infty}, \|\theta_{0,0}\|_2)$ is a finite constant;

2. If the mixing rate function $h : [0,\infty) \to (0,\infty)$ is the exponential (2.17), then the nonlinear dissipation time is bounded by

\begin{equation}
\kappa_d \leq \frac{C}{d^\delta}, \quad \text{where} \quad \delta := \frac{2\|\nabla u\|_{L^\infty} (d + 2\alpha + 2\beta)}{pc_2 + 2\|\nabla u\|_{L^\infty} (d + 2\alpha + 2\beta)}
\end{equation}

and $C = C(\alpha, \beta, c_2, p, d, \|\nabla u\|_{L^\infty}, \|\theta_{0,0}\|_2)$ is a finite constant.

**Remark 2.9.** We will see in the proof of Theorem 2.7 that the constant $\epsilon$ can be determined by Weyl’s formula, which states

\begin{equation}
\lambda_j \simeq \frac{4\pi \Gamma \left( \frac{d}{2} + 1 \right)^{\frac{2}{d}}}{\text{Vol}(T^d)^{\frac{d}{2}}} j^{\frac{d}{2}}
\end{equation}

asymptotically as $j \to \infty$ (see, e.g., [25]). Then, for example, we can take

\begin{equation}
\epsilon := (1 + \epsilon) \lim_{j \to \infty} \frac{j}{\lambda_j^{\frac{d}{2}}} = \frac{1 + \epsilon \text{Vol}(T^d)}{(4\pi)^\frac{d}{2} \Gamma \left( \frac{d}{2} + 1 \right)}
\end{equation}

for some fixed $\epsilon$ which is sufficiently small.

**Remark 2.10.** 1. Our main results are also true when the domain $T^d$ is replaced by any closed Riemannian manifold;

2. One crucial fact to prove the dissipation enhancement results for the linear advection equation is to make use of the iteration structure for the exponential functions. While for the proof of Theorem 2.5 and Theorem 2.7,
one novelty is that we will adapt new iteration structure given by certain rational functions (see, Lemma 4.3);

3. Note that if we let $p = 2$ in both Theorem 2.5 and Theorem 2.7,

$$H_{1,v,h} = H_{2,v,h} = \frac{1}{8}$$

and our main improvements (2.11) and (2.19), respectively will “recover” the improved dissipation time $\tau_d$ for the linear model (2.2) with $\alpha = 1$ under both strongly mixing condition and weakly mixing condition, respectively (see, e.g., [17, Theorem 2.16 and Theorem 2.19]). So are both Corollary 2.6 and Corollary 2.8 (see, e.g., [17, Corollary 2.17 and Corollary 2.20]).

4. Observe that when $\nu$ is sufficiently small, $H_{1,v,h} = H_{2,v,h} \equiv 1$. Therefore, for $\nu$ sufficiently small, the estimates (2.11) and (2.19) can be simplified as

$$\kappa_d \leq \frac{C}{\nu H_i(\nu)^2}, \quad i = 1, 2.$$

3. Preliminaries

The goal of this section is to study the regularity of the weak solutions of equation (2.1). Such regularity results have been considered under a much more general setting by Antontsev and Shamarev [2], in which, they studied the Dirichlet problem for a class of nonlinear parabolic equations with nonstandard anisotropic growth conditions. For the purpose of being self-contained, we will adapt their setting to ours. As a consequence, we show that the weak solution $\theta_s$ satisfies the energy inequality associated to (2.1), which further implies the trivial estimate (2.10) for $\kappa_d$.

We start with recalling several basic definitions. Let $\mathbf{V} (\mathbb{T}^d)$ be the Banach space given by

$$\mathbf{V} (\mathbb{T}^d) := \{ g(x) : g \in L^2(\mathbb{T}^d) \cap W^{1,p}(\mathbb{T}^d) \},$$

and for any $T > 0$, we define the Banach space $\mathbf{W} (\mathbb{T}^d \times [0,T])$ as

$$\mathbf{W} (\mathbb{T}^d \times [0,T]) := \left\{ f : [0,T] \mapsto \mathbf{V}(\mathbb{T}^d) : f \in L^2 (\mathbb{T}^d \times [0,T]), \quad |D_i f|^p \in L^1 (\mathbb{T}^d \times [0,T]), \quad f \text{ periodic on } \Gamma_T \right\},$$

with the norm

$$\|f\|_{\mathbf{W}(\mathbb{T}^d \times [0,T])} := \|f\|_{L^2(\mathbb{T}^d \times [0,T])} + \sum_{i=1}^{d} \|D_i f\|_{L^p(\mathbb{T}^d \times [0,T])}.$$ 

Here, $\Gamma_T$ is the lateral boundary of the cylinder $\mathbb{T}^d \times [0,T]$ and $p > 2$ is defined in (2.1).

Finally, we let $\mathbf{W}' (\mathbb{T}^d \times [0,T])$ be the dual space of $\mathbf{W} (\mathbb{T}^d \times [0,T])$, that is, the space of all linear functionals over $\mathbf{W} (\mathbb{T}^d \times [0,T])$. Note that $w \in \mathbf{W}' (\mathbb{T}^d \times [0,T])$
if and only if
\[
\begin{cases}
w = w_0 + \sum_{i=1}^{d} D_i w_i, & w_0 \in L^2 \left( \mathbb{T}^d \times [0, T] \right), \ w_i \in L^{p'} \left( \mathbb{T}^d \times [0, T] \right); \\
\forall \phi \in W \left( \mathbb{T}^d \times [0, T] \right), \ \langle w, \phi \rangle_W := \int_0^T \int_{\mathbb{T}^d} \left( w_0 \phi + \sum_{i=1}^{d} w_i D_i \phi \right) dx dt.
\end{cases}
\]

**Definition 3.1.** Given \( T > 0 \), a function \( \theta(x, t) \in W \left( \mathbb{T}^d \times [0, T] \right) \cap L^\infty \left( 0, T; L^2(\mathbb{T}^d) \right) \) is called a weak solution of (2.1) if for every test function \( \zeta \in Z := \{ \eta(z) : \eta \in W \left( \mathbb{T}^d \times [0, T] \right) \cap L^\infty \left( 0, T; L^2(\mathbb{T}^d) \right), \ \eta_t \in W' \left( \mathbb{T}^d \times [0, T] \right) \} \), and every \( t_1, t_2 \in [0, T] \), the following identity holds
\[
(3.1) \quad \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \left[ \partial_t \zeta \cdot \left( \nu \sum_{i=1}^{d} |D_i \theta|^{p-2} D_i \theta + u_i(t) \cdot \theta \right) D_i \zeta \right] dx dt = \int_{\mathbb{T}^d} \theta \zeta dx \bigg|_{t_1}^{t_2}.
\]

**Theorem 3.2 (2. Theorem 3.1).** For every \( s \geq 0 \) and \( \theta_{0,0} \in L^2_{0}(\mathbb{T}^d) \), the problem (2.1) has at least one weak solution \( \theta_s \in W \left( \mathbb{T}^d \times [s, T] \right) \) satisfying the estimate
\[
\| \theta_{s} \|_{L^\infty(s,T;L^2(\mathbb{T}^d))} + \nu \int_s^T \int_{\mathbb{T}^d} |D_i \theta_{s}|^p dx dt \leq C \left( \| \theta_{0,0} \|_{L^2(\mathbb{T}^d)}^2 + 1 \right),
\]
where \( C \) is an absolute constant independent of \( \theta_s \). Moreover, \( \frac{d}{ds} \theta_s \in W' \left( \mathbb{T}^d \times [s, T] \right) \).

Note that Theorem 3.2 indeed suggests that there exists at least one weak solution, which also belongs to the test function space \( Z \). As some byproducts, we collect several useful facts about (2.1).

**Corollary 3.3.**
1. If \( \theta_{0,0} \in L^2_{0}(\mathbb{T}^d) \), then so does \( \theta_{s,0} := \theta_0(s) \);
2. For any \( \zeta \in Z \), there holds that
\[
(3.2) \quad \int_{\mathbb{T}^d} \left[ \frac{\partial \theta_s}{\partial t} \cdot \zeta + \left( \nu \sum_{i=1}^{d} |D_i \theta_s|^{p-2} D_i \theta_s + u_i(t) \cdot \theta_s \right) \cdot D_i \zeta \right] dx = 0;
\]
3. The following estimate holds
\[
\kappa_d \leq \frac{1}{\nu \lambda_1};
\]
4. The weak solution of (2.1) defined in Theorem 3.2 is unique in \( L^2 \) sense.

**Proof.** 1. Recall that \( \theta_{s,0} := \theta_0(s) \). Therefore, it suffices to note that the constant function \( \zeta \equiv 1 \in Z \). The desired claim follows from letting \( t_1 = 0 \) and \( t_2 = s \) in (3.1).

2. Note that by the definitions of \( W(\mathbb{T}^d \times [0, T]) \) and \( Z \) and the fact that \( \theta_s \in Z \), it is easy to see that the function
\[
F(t) := \int_{\mathbb{T}^d} \theta_s(x, t) \frac{\partial \zeta}{\partial t}(x, t) - \left[ \nu \sum_{i=1}^{d} |D_i \theta_s|^{p-2}(x, t) D_i \theta_s(x, t) \right. \\
\left. + u_i(x, t) \cdot \theta_s(x, t) \right] D_i \zeta(x, t) dx
\]
is a continuous function with respect to $t$. The desired claim then follows by taking
the $t$ derivative on both sides of (3.1), with, for example, $t_1 = 0$ and $t_2 = t$,
respectively.

3. Since $\theta_s \in \mathbb{Z}$, we can therefore let $\zeta = \theta_s$ in (3.2). This together with the fact
that $u$ is divergence free yields the energy estimate for (2.1):

$$
\frac{1}{2} \partial_t \|\theta_s(t)\|_2^2 + \nu \|\nabla \theta_s(t)\|^p_p = 0,
$$

which implies

$$
\partial_t \|\theta_s(t)\|^2_2 \leq -2\nu \|\nabla \theta_s(t)\|^2_2 \leq -2\nu \lambda_1^\frac{p}{p-2} \|\theta_s(t)\|^2_2
$$

Therefore, an easy application of the (non-linear) Gronwall’s inequality yields

$$
\|\theta_s(t)\|_2 \leq \|\theta_s,0\|_2 \left[ \nu \lambda_1^\frac{p}{2} (p-2) (t-s) \|\theta_s,0\|^2_2 + 1 \right]^{\frac{1}{p-2}},
$$

which implies the desired claim.

4. This is clear from (3.4). □

4. Dissipation enhancement for evolution $p$-Laplacian advection
   equations

In this section we prove Theorem 2.5 and 2.7. The main idea behind the proof is
to split the analysis into two different cases. In the first case, we assume
$$
\|\nabla \theta_s(t)\|_p \|\theta_s(t)\|_2
$$
is large, and obtain decay of $\|\theta_s\|_2$ using the energy inequality (3.3); in the second
case, $\|\nabla \theta_s(t)\|_p \|\theta_s(t)\|_2$ is small, and hence the adnamics are well approximated by that of
the underlying dynamical system where there are no diffusion term. The mixing
assumption now forces the generation of high frequencies, and the rapid dissipation
of these gives an enhanced decay of $\|\theta_s\|_2$.

4.1. The Strongly Mixing Case. Let $s \geq 0$ be any time. We first consider the
case when

$$
\frac{\|\nabla \theta_s(t)\|_p}{\|\theta_s(t)\|_2}
$$
is large. More precisely, if for some $c_0 > 0$, we have

$$
\|\nabla \theta_s(t)\|_p \geq c_0^\frac{p}{2} \|\theta_s(t)\|_2,
$$

for all $s \leq t \leq t_0$, then by the energy inequality (3.3), we have

$$
\partial_t \|\theta_s(t)\|^\frac{p}{2}_2 \leq -2\nu c_0^\frac{p}{2} \|\theta_s(t)\|^\frac{p}{2}_2,
$$

which implies

$$
\|\theta_s(t)\|_2 \leq \frac{\|\theta_s,0\|_2}{\left[ \nu c_0^\frac{p}{2} (p-2) (t-s) \|\theta_s,0\|^\frac{p}{2}_2 + 1 \right]^{\frac{1}{p-2}}},
$$

for all $s \leq t \leq t_0$.

We now turn to the second case, in which, the ratio (4.1) is relatively “small”.
The idea is to use the strongly mixing assumption of $u$, to show that there exists a
$t_0 > s$, such that $\|\theta_s(t_0)\|_2$ becomes sufficiently small, in the sense of the non-linear
dissipation time. We now turn to the details.
We start with understanding the relation between our non-linear model (2.1) and the transport equation. More precisely, we have the following result.

**Lemma 4.1.** Let $\phi_s$ be the solution of the following transport equation

\[
\begin{cases}
\partial_t \phi_s + (u \cdot \nabla) \phi_s = 0; \\
\phi_s(t) = \theta_{s,0}, \quad t = s.
\end{cases}
\]

Then for all $t \geq s$,

\[\|\theta_s(t) - \phi_s(t)\|^2_2 \leq \frac{d^{p-2} D_{p} \nu}{\|\nabla u\|_\infty} \cdot e^{2\|\nabla u\|_\infty (t-s)} \cdot \|\nabla \theta_{s,0}\|^p_\nu,
\]

where $D_{p}$ is defined in (2.14).

**Proof.** Let $\omega(t) = \theta_s(t) - \phi_s(t)$. Note that $\omega(s) = 0$. Since the solution for the transport equation (4.3) exists in strong sense, this means

\[
\int_{\mathbb{T}^d} \left[ \frac{\partial \phi_s}{\partial t} \cdot \zeta + \sum_{i=1}^d u_i(t) \cdot \phi_s \cdot D_i \zeta \right] dx = 0
\]

for any $\zeta \in \mathbb{Z}$. We now subtract the above equation with (3.2) (with the same choice of $\zeta$ in both equations) to get

\[
\int_{\mathbb{T}^d} \frac{\partial \omega}{\partial t} \cdot \zeta + \left( \nu \sum_{i=1}^d |D_i \omega|^{p-2} D_i \omega \cdot D_i \zeta dx \right) - \nu \int_{\mathbb{T}^d} \sum_{i=1}^d |D_i \theta|^{p-2} D_i \theta \cdot D_i \zeta dx,
\]

for any $\zeta \in \mathbb{Z}$. Observe that $\omega \in \mathbb{Z}$. Indeed, it suffices to show that $\phi_s \in \mathbb{Z}$, which can be easily verified by Gronwall’s inequality. Therefore, we are able to take $\zeta = \omega$ in (4.4), which further gives

\[
\frac{1}{2\nu} \|\partial_t \omega\|^2_2 + \|\nabla \omega\|^p_\nu
\]

\[= \left\langle |\nabla \omega|^{p-2} \nabla \omega - |\nabla \theta_s|^{p-2} \nabla \theta_s, \nabla \omega \right\rangle
\]

\[= \left\langle |\nabla \omega|^{p-2} \nabla \omega - |\nabla (\omega + \phi_s)|^{p-2} \nabla (\omega + \phi_s), \nabla \omega \right\rangle
\]

\[= I_1 + I_2,
\]

where

$I_1 := - \left\langle |\nabla \omega|^{p-2} \nabla \phi_s, \nabla \omega \right\rangle$

and

$I_2 := \left\langle \left(|\nabla \omega|^{p-2} - |\nabla (\omega + \phi_s)|^{p-2}\right) \nabla (\omega + \phi_s), \nabla \omega \right\rangle$.

**Estimate of $I_1$.**

\[
I_1 \leq \int_{\mathbb{T}^d} |\nabla \omega|^{p-1} |\nabla \phi_s| dx \leq \|\nabla \omega\|^{p-1}_\nu \|\nabla \phi_s\|_\nu.
\]
Estimate of $I_2$. Note that

$$I_2 \leq \int_{\mathbb{T}^d} \left| |\nabla \omega|^{p-2} - |\nabla (\omega + \phi_s)|^{p-2} \right| \cdot |\nabla (\omega + \phi_s)| |\nabla \omega| \, dx$$

$$= (p - 2) \int_{\mathbb{T}^d} (\alpha_{t,x} |\nabla \omega| + (1 - \alpha_{t,x}) |\nabla (\omega + \phi_s)|)^{p-3}$$

$$\cdot |\nabla | - |\nabla (\omega + \phi_s)| \cdot |\nabla (\omega + \phi_s)| |\nabla \omega| \, dx$$

(4.6) $$\leq (p - 2) \int_{\mathbb{T}^d} (\alpha_{t,x} |\nabla \omega| + (1 - \alpha_{t,x}) |\nabla (\omega + \phi_s)|)^{p-3}$$

$$\cdot |\nabla \phi_s| \cdot |\nabla (\omega + \phi_s)| |\nabla \omega| \, dx,$$

where in the second equation above, we use the mean value theorem and $\alpha_{t,x} \in [0, 1]$ which depends on the values of $t$ and $x \in \mathbb{T}^d$. We now consider two different cases for the value of $p$.

Case I: $p \geq 3$. Recall for any $\ell \geq 0$ and $a, b > 0$, we have

$$a + b < C_{\ell}(a^\ell + b^\ell),$$

where

$$C_{\ell} = \begin{cases} 1, & 0 \leq \ell \leq 1; \\ 2^{\ell - 1}, & \ell > 1. \end{cases}$$

Therefore, by (4.7),

$$\begin{align*}
(4.6) &\leq (p - 2) C_{p-3} \cdot \int_{\mathbb{T}^d} |\nabla \omega|^{p-2} |\nabla \phi_s| |\nabla (\omega + \phi_s)| \, dx \\
&\quad + (p - 2) C_{p-3} \cdot \int_{\mathbb{T}^d} |\nabla (\omega + \phi_s)|^{p-2} |\nabla \phi_s| |\nabla \omega| \, dx
\end{align*}$$

(4.8) It is easy to check that

$$2(p - 2) C_{p-3} + 2(p - 2) C_{p-3} C_{p-2} \leq p 2^p,$$

and hence, we have

$$\text{RHS of (4.8)} \leq p 2^p \cdot \left[ \int_{\mathbb{T}^d} |\nabla \omega|^{p-1} |\nabla \phi_s| \, dx + \int_{\mathbb{T}^d} |\nabla \omega|^{p-2} |\nabla \phi_s|^2 \, dx \\
+ \int_{\mathbb{T}^d} |\nabla \phi_s|^{p-1} |\nabla \omega| \, dx \right].$$

(4.9) $$\leq p 2^p \cdot (|\nabla \omega|_p^{p-1} |\nabla \phi_s|_p + |\nabla \phi_s|_p^{p-1} |\nabla \|\omega\|_p + |\nabla \omega|_p^{p-2} |\nabla \phi_s|_p^2).$$

Case II: $2 < p < 3$. The second case is similar to the first one. The only difference is how we estimate the term

$$\begin{align*}
(4.10) &\quad (\alpha_{t,x} |\nabla \omega| + (1 - \alpha_{t,x}) |\nabla (\omega + \phi_s)|)^{p-3}.
\end{align*}$$

Note that since $p - 3 < 0$, it follows that

$$(4.10) \leq \min \left\{ (\alpha_{t,x} |\nabla \omega|)^{p-3}, (1 - \alpha_{t,x}) |\nabla (\omega + \phi_s)|^{p-3} \right\}.$$  

An easy pigeonholing yields for each $t$ and $x$, at least one of $\alpha_{t,x}$ and $1 - \alpha_{t,x}$ belongs to $[\frac{1}{2}, 1]$, this allows us to bound (4.10) further by

$$2^{3-p} \left( |\nabla \omega|^{p-3} + |\nabla (\omega + \theta_s)|^{p-3} \right) \leq 8 \left( |\nabla \omega|^{p-3} + |\nabla (\omega + \theta_s)|^{p-3} \right).$$
This implies when $2 < p < 3$, we have

$$
(4.6) \quad \leq \quad 8(p - 2) \left[ \int_{\mathbb{T}^d} |\nabla \omega|^{p-2} |\nabla \phi_0| |\nabla (\omega + \phi_0)| \, dx \\
+ \int_{\mathbb{T}^d} |\nabla \omega|^{p-2} |\nabla \phi_0| |\nabla (\omega + \phi_0)| \, dx \right]
$$

$$
(4.11) \quad \leq \quad 16 \cdot \left( \|\nabla \omega\|_{p-1} \|\nabla \phi_0\|_p + \|\nabla \phi_0\|_{p-1} \|\nabla \omega\|_p + \|\nabla \omega\|_{p-2} \|\nabla \phi_0\|^2_p \right).
$$

Therefore, combining both cases, namely (4.9) and (4.11), we have

$$
(4.12) \quad I_2 \leq 16p^2p^{p-1} \cdot \left( \|\nabla \omega\|_{p-1} \|\nabla \phi_0\|_p + \|\nabla \phi_0\|_{p-1} \|\nabla \omega\|_p + \|\nabla \omega\|_{p-2} \|\nabla \phi_0\|^2_p \right).
$$

By (4.5), (4.12) and a standard calculation by using Young’s inequality, there holds

$$
\frac{1}{2p} \partial_t \|\omega\|^2_2 \leq I_1 + I_2 \leq \|\nabla \omega\|_p^p + D_p \|\nabla \phi_0\|_p^p,
$$

where $D_p$ is defined in (2.14). Hence, we have

$$
(4.13) \quad \partial_t \|\nabla \omega\|^2_2 \leq 2D_p \nabla \phi_0 \|_p^p.
$$

Since $\phi_0$ solves the transport equation (4.3), an easy application of the Gronwall’s inequality yields

$$
(4.14) \quad \|\nabla \phi_0\|_p^p \leq d_p^{\frac{2}{p-2}} \cdot e^{2\|\nabla u\|_\infty (t - s)} \|\nabla \theta_{s,0}\|^p_p.
$$

The desired result then follows from (4.13) and (4.14).

The next lemma deals with the case when the initial data has a relatively “small” $W^{1,p}$ energy. To begin with, we highlight the following easy fact from Calculus:

**Lemma 4.2.** For $0 < x < 1$ and $p > 2$, there holds

$$
1 - \frac{2x}{p-2} \leq \frac{1}{(1 + x)^{\frac{2}{p-2}}}
$$

*Proof. The proof is obvious by the Taylor expansion.*

**Lemma 4.3.** Choose $\lambda_N$ to be the largest eigenvalue satisfying $\lambda \leq H_1(\nu)$ where $H_1(\nu)$ is defined as in (2.13). If

$$
(4.15) \quad \|\nabla \theta_{s,0}\|_p < \lambda_N^{\frac{1}{p}} \|\theta_{s,0}\|_2
$$

then we have

$$
(4.16) \quad \|\theta_{\mathbf{t}_0}\|_2 \leq \frac{\|\theta_{s,0}\|_2}{\left( \frac{\nu H_1(\nu) \frac{p}{2} - 1}{2^{1 - \frac{2}{p}}} \right) \left( \frac{p-2}{2} \right) (p-2)(t_0 - s) \|\theta_{s,0}\|_2^{p-2} + 1}^{\frac{1}{p-2}}
$$

at a time to given by

$$
\mathbf{t}_0 := s + 2h^{-1} \left( \frac{\lambda_N^{\frac{1}{p}}}{2} \right)
$$
We now establish a lower bound of the term where $N$ is the projection operator from $N \theta \in L^2(s, r)$ and $N \beta$. Therefore, it suffices to bound the last two terms in (4.17).

Proof. Integrating the energy inequality (3.3), we have

$$\int_t^s \| \nabla \theta_s(r) \|^p_p \, dr \geq \left( \frac{\lambda_N(t_0 - s)}{8} \right)^{\frac{2}{p}} \cdot \| \theta_{s,0} \|^2_2,$$

for any $t > s$.

We claim that for the choice of $\lambda_N$ and $t_0$ will guarantee

$$\int_t^s \| \nabla \theta_s(r) \|^2_2 \, dr \geq \left( \int_t^s \| \nabla \theta_s(r) \|^2_2 \, dr \right)^{\frac{2}{p}}.$$

Since $p > 2$, Hölder’s inequality yields

$$\int_s^{t_0} \| \nabla \theta_s(r) \|^2_2 \, dr \geq \left( \int_s^{t_0} \| \nabla \theta_s(r) \|^2_2 \, dr \right)^{\frac{2}{p}}.$$

We now establish a lower bound of the term

$$\int_s^{t_0} \| \nabla \theta_s(r) \|^2_2 \, dr$$

by following the argument in [17, Lemma 5.1]. More precisely,

$$\int_s^{t_0} \| \nabla \theta_s(r) \|^2_2 \, dr \geq \lambda_N \int_{t_0}^{t_0} \| (I - P_N) \theta_s(r) \|^2_2 \, dr$$

$$\geq \frac{\lambda_N}{2} \int_{t_0}^{t_0} \| (I - P_N) \theta_s(r) \|^2_2 \, dr$$

$$- \lambda_N \int_{t_0}^{t_0} \| (I - P_N) (\theta_s(r) - \phi_s(r)) \|^2_2 \, dr$$

$$\geq \frac{\lambda_N(t_0 - s)}{4} \| \theta_{s,0} \|^2_2 - \frac{\lambda_N}{2} \int_{t_0}^{t_0} \| P_N \phi_s(r) \|^2_2 \, dr$$

$$- \lambda_N \int_{t_0}^{t_0} \| \theta_s(r) - \phi_s(r) \|^2_2 \, dr,$$

where $P_N$ is the projection operator from $L^2$ to the subspaces spanned by the first $N$ eigenvectors. Therefore, it suffices to bound the last two terms in (4.19). For the second term, using the strongly mixing condition (2.8) and (4.15), we see that

$$\int_{t_0}^{t_0} \| P_N \phi_s(r) \|^2_2 \, dr \leq \lambda_N^\beta \int_{t_0}^{t_0} \| \phi_s(r) \|^2_2 \, dr \leq \lambda_N^\beta \int_{t_0}^{t_0} h(r - s)^2 \| \theta_{s,0} \|^2_{H^2} \, dr$$

$$\leq \frac{\lambda_N^\beta}{2} \left( \frac{t_0 - s}{2} \right)^2 \| \theta_{s,0} \|^2_{H^2}$$

$$\leq \frac{\lambda_N^\beta}{2} \left( \frac{t_0 - s}{2} \right)^2 \| \theta_{s,0} \|^2_{H^2} - 2^2 \| \nabla \theta_{s,0} \|^2_2$$

$$\leq \frac{\lambda_N^\beta}{2} \left( \frac{t_0 - s}{2} \right)^2 \| \theta_{s,0} \|^2_{H^2} - 2^2 \| \nabla \theta_{s,0} \|^2_2$$

$$\leq \frac{\lambda_N^\beta}{2} \left( \frac{t_0 - s}{2} \right)^2 \| \theta_{s,0} \|^2_{H^2} - 2^2 \| \nabla \theta_{s,0} \|^2_2$$

$$\leq \frac{\lambda_N^\beta}{2} \left( \frac{t_0 - s}{2} \right)^2 \| \theta_{s,0} \|^2_{H^2}$$

(4.20)
Finally, we bound the last term (4.19). By Lemma 4.1, we have

\[
\int_{t_0}^{t_0 + s} \| \theta_s(r) - \phi_s(r) \|^2 dr \leq \frac{d^{n+2} D_p \nu}{2 \| \nabla u \|_\infty^2} \cdot e^{2 \| \nabla u \|_\infty (t_0 - s)} \| \nabla \theta_{s,0} \|^p
\]

\[
\leq \frac{\lambda_N^{\frac{\beta}{2}} \cdot d^{n+2} D_p \nu}{2 \| \nabla u \|_\infty^2} e^{2 \| \nabla u \|_\infty (t_0 - s)} \| \theta_{s,0} \|^2
\]

\[
\leq \frac{\lambda_N^{\frac{\beta}{2}} \cdot d^{n+2} D_p \nu}{2 \| \nabla u \|_\infty^2} \cdot e^{2 \| \nabla u \|_\infty (t_0 - s)} \| \theta_{s,0} \|^2.
\]

(4.21)

Therefore, combining (4.19) with (4.20) and (4.21), we have

\[
\int_s^{t_0} \| \nabla \theta_s(r) \|^2 dr \geq \lambda_N (t_0 - s) \| \theta_{s,0} \|^2 \cdot \left( \frac{1}{4} - \frac{\lambda_N^{\alpha+\beta}}{4} h \left( \frac{t_0 - s}{2} \right) \right)^2
\]

\[
+ \frac{\lambda_N^{\frac{\beta}{2}} \cdot d^{n+2} D_p \nu}{2 \| \nabla u \|_\infty^2} \cdot e^{2 \| \nabla u \|_\infty (t_0 - s)} \| \theta_{s,0} \|^2.
\]

(4.22)

By our choice of \( \lambda_N \) and \( t_0 \), we have

\[
\frac{\lambda_N^{\alpha+\beta}}{4} h \left( \frac{t_0 - s}{2} \right) \leq \frac{1}{16}, \quad \text{and} \quad \frac{\lambda_N^{\frac{\beta}{2}} \cdot d^{n+2} D_p \nu}{2 \| \nabla u \|_\infty^2} \cdot e^{2 \| \nabla u \|_\infty (t_0 - s)} \leq \frac{1}{16}.
\]

Hence, (4.22) reduces to

\[
\int_s^{t_0} \| \nabla \theta_s(r) \|^2 dr \geq \lambda_N (t_0 - s) \| \theta_{s,0} \|^2 \cdot \frac{1}{8},
\]

which further implies (4.18).

We now turn to the proof of (4.16). To begin with, we collect several facts:

1. Recall that we are only interested in the case when \( \nu \) is small, this together with the definition of \( H_1(\nu) \) asserts that for \( \nu \) sufficiently small, \( H_1(\nu) \) is increasing when \( \nu \) tends to 0;

2. From the definition of \( H_2(\nu) \), together with the fact that \( h^{-1} \) is strictly decreasing (since \( h \) is assumed to be strictly deceasing), one can check that

\[
\lim_{\nu \to 0} \nu \lambda_N^{\frac{\beta}{2}} h^{-1} \left( \frac{\lambda_N^{\alpha+\beta}}{2} \right) = 0.
\]

Moreover, this together with the definition of \( t_0 \), suggests that

\[
\lim_{\nu \to 0} \nu \lambda_N^{\frac{\beta}{2}} h^{-1} \left( \frac{\lambda_N^{\alpha+\beta}}{2} \right) (t_0 - s) = 0;
\]

(4.24)

3. When \( \nu \) is sufficiently small, there holds that

\[
\frac{1}{2} H_1(\nu) \leq \lambda_N \leq H_1(\nu).
\]

This is due to Fact 1 above and the Weyl’s lemma (2.24).
Combining both (4.17) and (4.18), we have
\[
\|\theta_s(t_0)\|_2^2 \leq \|\theta_{s,0}\|_2^2 - 2\nu \left( \frac{\lambda_N(t_0 - s)}{8} \right)^\frac{2}{p} \|\theta_{s,0}\|_2^p
\]
\[
= \|\theta_{s,0}\|_2^2 \cdot \left[ 1 - 2\nu \left( \frac{\lambda_N(t_0 - s)}{8} \right)^\frac{2}{p} \|\theta_{s,0}\|_2^{p-2} \right]
\]
\[
= \|\theta_{s,0}\|_2^2 \cdot \left[ 1 - 2\nu \frac{\lambda_N^2}{8^2} \cdot (t_0 - s) \cdot (t_0 - s) \cdot \|\theta_{s,0}\|_2^{p-2} \right]
\]
(4.26)
\[
= \|\theta_{s,0}\|_2^2 \cdot \left[ 1 - \nu \cdot \frac{\lambda_N^2}{4^2} \cdot h^{-1} \left( \frac{\lambda_N^2}{2} \right)^\frac{p-2}{2} \cdot (t_0 - s) \cdot \|\theta_{s,0}\|_2^{p-2} \right].
\]

Note that \(\|\theta_{s,0}\|_2 \leq \|\theta_{0,0}\|_2\), by (4.24), we may assume that when \(\nu\) is sufficiently small,
\[
0 < \nu \cdot \frac{\lambda_N^2}{4^2} \cdot h^{-1} \left( \frac{\lambda_N^2}{2} \right)^\frac{p-2}{2} \cdot (t_0 - s) \cdot \|\theta_{s,0}\|_2^{p-2} < \frac{1}{2}.
\]

Therefore, an application of Lemma 4.2 yields that
(4.26)
\[
\frac{1}{\|\theta_{s,0}\|_2^2} \leq \frac{1}{\left( \frac{\nu \lambda_N^2}{4^2} \cdot h^{-1} \left( \frac{\lambda_N^2}{2} \right)^\frac{p-2}{2} \cdot (p-2)(t_0 - s)\|\theta_{s,0}\|_2^{p-2} + 1 \right)^\frac{p-2}{2}},
\]
which, together with (4.25), implies (4.16). The proof is complete. \(\square\)

Finally, let us turn to prove the main result Theorem 2.5. We need the following easy fact.

**Lemma 4.4.** Let \(a > 0\) and \(p > 2\), then the function
\[
F_a(x) := \frac{x}{(ax^{p-2} + 1)^{\frac{1}{p-2}}}
\]
is increasing when \(x \geq 0\).

**Proof.** It is easy to check that \(F'_a(x) = \frac{1}{(ax^{p-2} + 1)^{\frac{1}{p-2}}} > 0\), which clearly implies the desired claim. \(\square\)

The following lemma is the key ingredient to run the iteration argument when we prove the main result Theorem 2.5.

**Lemma 4.5.** Let \(b, c > 0\) and \(p > 2\). Suppose
(4.27) \[x_1 \leq F_b(t'_1 - t'_0)(x_0),\]
and
(4.28) \[x_2 \leq F_c(t'_2 - t'_1)(x_1),\]
where \(x_0, x_1, x_2 > 0\), \(t'_2 > t'_1 > t'_0\) and \(F_a(t'_1 - t'_0)\) and \(F_b(t'_2 - t'_1)\) are defined in Lemma 4.4. Then we have,
\[x_2 \leq F_d(t'_2 - t'_0)(x_0),\]
where \( d = \min\{b, c\} \).

**Proof.** By applying Lemma 4.4 and a direct computation we have
\[
x_2 \leq F_r(t_2' - t_1')(F_b(t_1' - t_0')(x_0))
\]
\[
= \frac{(c(t_2' - t_1')x_0^{p-2} + b(t_1' - t_0')x_0^{p-2} + 1)^{1/p-2}}{x_0}
\]
\[
\leq \frac{(d(t_2' - t_0')x_0^{p-2} + 1)^{1/p-2}}{x_0} = F_d(t_2' - t_0')(x_0)
\]
\[\square\]

**Proof of Theorem 2.5.** Repeatedly applying (4.2) with \( c_0 = \lambda_N \) and Lemma 4.3, together with the fact (4.25) and Lemma 4.5, we obtain an increasing sequence of times \( (t_k') \), such that for each \( k \geq 1 \), there holds
\[
\|\theta_s(t_k')\|_2 \leq \frac{\|\theta_{s,0}\|_2}{\left( \frac{\nu H_1(\nu)^{\frac{p}{2}}}{2^p} (p-2)(t_k' - s)\|\theta_{s,0}\|_2^{p-2} + 1 \right)^{\frac{1}{p-2}}},
\]
where \( H_{1,\nu,h} \) is defined in (2.12), that is
\[
H_{1,\nu,h} := \min \left\{ 1, 2^{-p-1} \cdot h^{-1} \left( \frac{H_1(\nu)^{\frac{p}{2}}}{2^{1 - \frac{p+1}{2}}} \right)^{\frac{p-2}{p-2}} \right\}.
\]
and
\[
t_k' + 1 - t_k' \leq t_0.
\]
Let us include more details for the iteration argument above. Indeed, although our estimates (4.2) and (4.16) are no longer linear, the iteration structure still works. For example, let \( t_0' := s \) and suppose
\[
\|\theta_s(t_1')\|_2 \leq \frac{\|\theta_{s,0}\|_2}{\left( \frac{\nu H_1(\nu)^{\frac{p}{2}}}{2^p} (p-2)(t_1' - s)\|\theta_{s,0}\|_2^{p-2} + 1 \right)^{\frac{1}{p-2}}},
\]
and
\[
\|\theta_s(t_2')\|_2 \leq \frac{\|\theta_{s}(t_1')\|_2}{\left( \frac{\nu H_1(\nu)^{\frac{p}{2}}}{2^p} h^{-1} \left( \frac{H_1(\nu)^{\frac{p}{2}}}{2^{1 - \frac{p+1}{2}}} \right)^{\frac{p-2}{p-2}} (p-2)(t_2' - t_1')\|\theta_s(t_1')\|_2^{p-2} + 1 \right)^{\frac{1}{p-2}}},
\]
Applying Lemma 4.4 with \( x = \|\theta_s(t_1')\|_2 \) and
\[
a = \frac{\nu H_1(\nu)^{\frac{p}{2}}}{2^p} h^{-1} \left( \frac{H_1(\nu)^{\frac{p}{2}}}{2^{1 - \frac{p+1}{2}}} \right)^{\frac{p-2}{p-2}} (p-2)(t_2' - t_1'),
\]
and Lemma 4.3 with (4.27) being replaced by (4.30), and (4.28) being replaced by (4.31), respectively, we have
\[
\|\theta_s(t_2')\|_2 \leq \frac{\|\theta_{s,0}\|_2}{\left( \frac{\nu H_1(\nu)^{\frac{p}{2}}}{2^p} (p-2)(t_2' - s)\|\theta_{s,0}\|_2^{p-2} + 1 \right)^{\frac{1}{p-2}}}.
\]
It is then clear that (4.29) follows by applying the above procedure $k$ times.

Finally, note that from (4.29), we can immediately concludes that

\[(4.33)\]
\[\kappa_d \leq \frac{2^c}{\nu H_1(\nu)^{\frac{q}{2}} H_{1,\nu,h}} + (t_0 - s).\]

By the definition of $H_1(\nu)$ and the choice of $\lambda_N$ and $t_0$, there exists some constant $C' > 0$, which only depends on $h$, $\|\nabla u\|_\infty$, $p$, $\|\theta_{s,0}\|$, the strongly mixing condition and any dimension constants, such that

\[(4.34)\]
\[t_0 - s \leq \frac{C'}{\nu H_1(\nu)^{\frac{q}{2}}} \leq \frac{C'}{\nu H_1(\nu)^{\frac{q}{2}} H_{1,\nu,h}},\]

where in the second estimate above, we have used the fact that $H_{1,\nu,h} \leq 1$. The desired estimate (2.11) then clearly follows from (4.33) and (4.34). \(\square\)

To prove Corollary 2.6, it suffices to compute the function $H_1$ explicitly for the specific rate functions of interest.

**Proof of Corollary 2.6.** We first note that when $\nu \ll 1$, $H_{1,\nu,h} \equiv 1$. When the mixing rate function $h$ is given by the power law (2.15), we compute

\[(4.35)\]
\[H_1(\nu) = C_0 |\log \nu|^{\frac{q}{2}},\]

where $C_0 = C_0(\alpha, \beta, c, p, d, \|\nabla u\|_{L_{\infty}}, \|\theta_{s,0}\|_2)$. The desired estimate (2.16) then follows by substituting (4.35) into (2.11).

When the mixing rate function $h$ is given by the exponential law (2.17), we have

\[H_1(\nu) = \nu^{-\frac{c}{2}\alpha + \beta \|\nabla u\|_{L_{\infty}(\alpha + \beta)}}.\]

This together with (2.11) yields the desired estimate (2.18). \(\square\)

### 4.2. The Weakly Mixing Case

We now turn to the proof of Theorem 2.7. The proof is similar to the proof of Theorem 2.5. The main difference is that the analog of Lemma 4.3 for the weakly mixing case is weaker. More precisely, we have the following result.

**Lemma 4.6.** Choose $\lambda_N$ to be the largest eigenvalue satisfying $\lambda \leq H_2(\nu)$ where $H_2(\nu)$ is defined as in (2.21). If

\[(4.36)\]
\[\|\nabla \theta_{s,0}\|_p < \lambda_N^\frac{q}{2}\|\theta_{s,0}\|_2\]

then we have

\[(4.37)\]
\[\|\theta_s(t)\|_2 \leq \left(\frac{\nu H_1(\nu)^{\frac{q}{2}}}{2^{\frac{q}{2} + 1}} \log h^{-1} \left(\frac{H_1(\nu)^{\frac{d+2\alpha + 2\beta}{4 - (\alpha + \beta)}}}{2^{1 - \frac{d+2\alpha + 2\beta}{4 - (\alpha + \beta)}}}\right)^{\frac{p-2}{2}} \right)^{\frac{1}{p-2}} \|\theta_{s,0}\|_2 \left(p - 2\right)(t_0 - s)\|\theta_{s,0}\|_2^{p-2} + 1 + \frac{1}{p-2} \right)\]

at a time $t_0$ given by

\[t_0 := s + 2h^{-1} \left(\frac{\lambda_N^{\frac{d+2\alpha + 2\beta}{4 - (\alpha + \beta)}}}{2\sqrt{\tau}}\right),\]

where $\tau$ is defined in (2.25).
Proof of Theorem 2.7. Given Lemma 4.6, the proof of Theorem 2.7 is the same as the proof of Theorem 2.5.

Moreover, as in the proof of Corollary 2.6, the proof of Corollary 2.8 only involves computing $H_2$ explicitly when $h$ is assumed to be power law or exponential, and hence we would like to leave the detail to the interested reader. Finally, we proof Lemma 4.6.

Proof of Lemma 4.6. The only difference between the proof of Lemma 4.6 and Lemma 4.3 is that when we estimate the term

$$\int_{t_0}^{t_0} \|P_N \phi_s(r)\|_2^2 dr,$$

instead of using the strongly mixing assumption, we need to bound it via the weakly mixing assumption (2.9). More precisely, we have

$$\int_{t_0}^{t_0} \|P_N \phi_s(r)\|_2^2 dr \leq \int_{t_0}^{t_0} \sum_{\ell=1}^{N} |\langle \phi_s(r), e_\ell \rangle|^2 dr$$

$$\leq \sum_{\ell=1}^{N} \frac{t_0-s}{2} h \left(\frac{t_0-s}{2}\right)^2 \|\phi_s(0)\|_{H^*}^2 \lambda_\ell^\beta$$

$$\leq \frac{N(t_0-s)}{2} h \left(\frac{t_0-s}{2}\right)^2 \lambda_N^\beta \|\theta_{s,0}\|_{H^*}^2$$

$$\leq \frac{N(t_0-s)}{2} h \left(\frac{t_0-s}{2}\right)^2 \cdot \lambda_N^{\alpha+\beta} \|\theta_{s,0}\|_2^2$$

$$\leq \frac{c(t_0-s)}{2} h \left(\frac{t_0-s}{2}\right)^2 \cdot \lambda_N^{\alpha+2\beta} \|\theta_{s,0}\|_2^2.$$

Here in the last estimate above, we have used the Weyl’s formula (2.24) and Remark 2.9.

Therefore, as the proof of Lemma 4.3, we substitute (4.38) and (4.21) into (4.19), we see that

$$\int_{t_0}^{t_0} \|\nabla \theta_s(r)\|_2^2 dr \geq \lambda_N (t_0-s) \|\theta_{s,0}\|_2^2 \cdot \left(1 - \frac{c(t_0-s)}{2} h \left(\frac{t_0-s}{2}\right)^2 \right)^2$$

$$- \frac{\lambda_N^\beta d^{\beta+2\beta} D_\nu \|\theta_{0,0}\|_2^2}{2 \|\nabla u\|_{L^\infty}^2 (t_0-s)} \cdot e^{2\|\nabla u\|_{L^\infty} (t_0-s)}.$$

Finally, by the choice of $\lambda_N$ and $t_0$, we have

$$\frac{\lambda_N^\beta d^{\beta+2\beta} D_\nu \|\theta_{0,0}\|_2^2}{2 \|\nabla u\|_{L^\infty}^2 (t_0-s)} \cdot e^{2\|\nabla u\|_{L^\infty} (t_0-s)} \leq \frac{1}{16},$$

hence (4.23) is still true in this case. The rest of the proof is then the same as those in the proof of Lemma 4.3.

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