A GEOMETRIC WAY TO GENERATE BLUNDON TYPE INEQUALITIES

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Abstract

We present a geometric way to generate Blundon type inequalities. Theorem 3.1 gives the formula for \( \cos \overline{POQ} \) in terms of the barycentric coordinates of the points \( P \) and \( Q \) with respect to a given triangle. This formula implies Blundon type inequalities generated by the points \( P \) and \( Q \) (Theorem 3.2). Some applications are given in the last section by choosing special points \( P \) and \( Q \).

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1 Introduction

Consider \( O \) the circumcenter, \( I \) the incenter, \( G \) the centroid, \( N \) the Nagel point, \( s \) the semiperimeter, \( R \) the circumradius, and \( r \) the inradius of triangle \( ABC \).

Blundon’s inequalities express the necessary and sufficient conditions for the existence of a triangle with elements \( s, R \) and \( r \):

\[
2R^2 + 10Rr - r^2 - 2(R - 2r) \sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r) \sqrt{R^2 - 2Rr}.
\]

(1)

Clearly these two inequalities can be written in the following equivalent form

\[
|s^2 - 2R^2 - 10Rr + r^2| \leq 2(R - 2r) \sqrt{R^2 - 2Rr},
\]

(2)

and in many references this relation is called the fundamental inequality of triangle \( ABC \).

The standard proof is an algebraic one, it was first time given by W.J. Blundon [5] and it is based on the characterization of cubic equations with the roots the length sides of a triangle. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenc [16], and to the papers of C. Niculescu [17],[18], R.A. Satnoianu [20], and S. Wu [22] have obtained some improvements of this important inequality.
The following result was obtained by D. Andrica and C. Barbu in the paper [3] and it contains a simple geometric proof of (1). Assume that the triangle $ABC$ is not equilateral. The following relation holds:

$$\cos \widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R - 2r) \sqrt{R^2 - 2Rr}}.$$  \hfill (3)

If we have $R = 2r$, then the triangle must be equilateral and we have equality in (1) and (2). If we assume that $R - 2r \neq 0$, then inequalities (1) are direct consequences of the fact that $-1 \leq \cos \widehat{ION} \leq 1$.

In this geometric argument the main idea is to consider the points $O$, $I$ and $N$, and then to get the formula (3). It is a natural question to see what is a similar formula when we keep the circumcenter $O$ and we replace the points $I$ and $N$ by other two points $P$ and $Q$. In this way we obtain Blundon type inequalities generated by the points $P$ and $Q$. Section 2 contains the basic facts about the main ingredient helping us to do all the computations, that is the barycentric coordinates. In Section 3 we present the analogous formula to (3), for the triangle $POQ$, and we derive the Blundon type inequalities generated in this way. The last section contains some applications of the results in Section 3 as follows: the classical Blundon’s inequalities, the dual Blundon’s inequalities obtained in the paper [3], the Blundon’s inequalities generated by two Cevian points of rank $(k; l; m)$.

2 Some basic results about barycentric coordinates

Let $P$ be a point situated in the plane of the triangle $ABC$. The Cevian triangle $DEF$ is defined by the intersection of the Cevian lines though the point $P$ and the sides $BC, CA, AB$ of triangle. If the point $P$ has barycentric coordinates $t_1 : t_2 : t_3$, then the vertices of the Cevian triangle $DEF$ have barycentric coordinates given by: $D(0 : t_2 : t_3), E(t_1 : 0 : t_3)$ and $F(t_1 : t_2 : 0)$. The barycentric coordinates were introduced in 1827 by Möbius (see [10]). The using of barycentric coordinates defines a distinct part of Geometry called Barycentric Geometry. More details can be found in the monographs of C. Bradley [10], C. Coandă [11], C. Coșniță [12], C. Kimberling [14], and in the papers of O. Bottema [9], J. Scott [21], and P. Yiu [23].

It is well-known ([11],[12]) that for every point $M$ in the plane of triangle $ABC$, then the following relation holds:

$$(t_1 + t_2 + t_3)\overrightarrow{MP} = t_1 \overrightarrow{MA} + t_2 \overrightarrow{MB} + t_3 \overrightarrow{MC}.$$ \hfill (4)

In the particular case when $M \equiv P$, we obtain

$$t_1 \overrightarrow{PA} + t_2 \overrightarrow{PB} + t_3 \overrightarrow{PC} = \overrightarrow{0}.$$

This last relation shows that the point $P$ is the barycenter of the system $\{A, B, C\}$ with the weights $\{t_1, t_2, t_3\}$. The following well-known result is very useful in
computing distances from the point \( M \) to the barycenter \( P \) of the system \( \{ A, B, C \} \) with the weights \( \{ t_1, t_2, t_3 \} \).

**Theorem 2.1.** If \( M \) is a point situated in the plane of triangle \( ABC \), then

\[
(t_1 + t_2 + t_3)^2 MP^2 = (t_1 MA^2 + t_2 MB^2 + t_3 MC^2)(t_1 + t_2 + t_3) - (t_2 t_3 a^2 + t_3 t_1 b^2 + t_1 t_2 c^2), \tag{5}
\]

where \( a = BC, b = CA, c = AB \), are the length sides of triangle.

**Proof.** Using the scalar product of two vectors, from (4) we obtain:

\[
(t_1 + t_2 + t_3)^2 MP^2 = t_1^2 MA^2 + t_2^2 MB^2 + t_3^2 MC^2 + 2t_1 t_2 MA \cdot MB + 2t_1 t_3 MA \cdot MC + 2t_2 t_3 MB \cdot MC,
\]

that is

\[
(t_1 + t_2 + t_3)^2 MP^2 = t_1^2 MA^2 + t_2^2 MB^2 + t_3^2 MC^2 +
\]

\[
t_1 t_2(MA^2 + MB^2 - AB^2) + t_1 t_3(MA^2 + MC^2 - AC^2) + t_2 t_3(MB^2 + MC^2 - BC^2),
\]

hence,

\[
(t_1 + t_2 + t_3)^2 MP^2 = (t_1 MA^2 + t_2 MB^2 + t_3 MC^2)(t_1 + t_2 + t_3) - (t_2 t_3 a^2 + t_3 t_1 b^2 + t_1 t_2 c^2).
\]

To get the last relation we have used the definition of the scalar product and the Cosine Law as follows

\[
2MA \cdot MB = 2MA \cdot MC \cos \hat{AB} = 2MA \cdot MB \frac{MA^2 + MB^2 - AB^2}{2MA \cdot MB} = MA^2 + MB^2 - AB^2.
\]

\[\square\]

If we consider that \( t_1, t_2, t_3 \), and \( t_1 + t_2 + t_3 \) are nonzero real numbers, then the relation (5) becomes the Lagrange's relation

\[
MP^2 = \frac{t_1 MA^2 + t_2 MB^2 + t_3 MC^2}{t_1 + t_2 + t_3} - \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right), \tag{6}
\]

If we consider in (6) \( M \equiv O \), the circumcenter of the triangle, then it follows

\[
R^2 - OP^2 = \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right), \tag{7}
\]

The following version of Cauchy-Schwarz inequality is also known in the literature as Bergström's inequality (see [6], [7], [8], [19]): If \( x_k, a_k \in \mathbb{R} \) and \( a_k > 0, k = 1, 2, \cdots, n \), then

\[
\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_n^2}{a_n} \geq \left( \frac{x_1 + x_2 + \cdots + x_n}{a_1 + a_2 + \cdots + a_n} \right)^2,
\]

with equality if and only if

\[
\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}.
\]
Using Bergström’s inequality and relation (4), we obtain
\[ R^2 - OP^2 \geq \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \cdot (a + b + c)^2 \]
that is in any triangle with semiperimeter \( s \) the following inequality holds:
\[ R^2 - OP^2 \geq \frac{4s^2 t_1 t_2 t_3}{(t_1 + t_2 + t_3)^3}, \]
where \( t_1 : t_2 : t_3 \) are the barycentric coordinates of \( P \) and \( t_1, t_2, t_3 > 0 \). Equality holds if an only if \( t_1 = a, t_2 = b, t_3 = c \), that is \( P \equiv I \), the incenter of the triangle \( ABC \).

**Theorem 2.2.** ([11], [12]). If the points \( P \) and \( Q \) have barycentric coordinates \( t_1 : t_2 : t_3, \) and \( u_1 : u_2 : u_3, \) respectively, with respect to the triangle \( ABC, \) and \( u = u_1 + u_2 + u_3, t = t_1 + t_2 + t_3, \) then
\[ PQ^2 = -\alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right), \]
where the numbers \( \alpha, \beta, \gamma \) are defined by
\[ \alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}. \]

3 **Blundon type inequalities generated by two points**

**Theorem 3.1.** Let \( P \) and \( Q \) be two points different from the circumcircle \( O, \) having the barycentric coordinates \( t_1 : t_2 : t_3 \) and \( u_1 : u_2 : u_3 \) with respect to the triangle \( ABC, \) and let \( u = u_1 + u_2 + u_3, t = t_1 + t_2 + t_3. \) If \( t_1, t_2, t_3, u_1, u_2, u_3 \neq 0, \) then the following relation holds
\[ \cos P\overline{OQ} = \frac{2R^2 - \frac{4s^2}{t^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) - \frac{u_1 u_2 u_3}{u^3} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) + \alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right)}{2 \sqrt{R^2 - \frac{4s^2}{t^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \cdot \left[ R^2 - \frac{u_1 u_2 u_3}{u^3} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \right]}} \]
where \( a, b, c \) are the length sides of the triangle and
\[ \alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}. \]

**Proof.** Applying the relation (7) for the points \( P \) and \( Q, \) we have
\[ OP^2 = R^2 - \frac{t_1 t_2 t_3}{t^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \]
and
\[ OQ^2 = R^2 - \frac{u_1 u_2 u_3}{u^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right). \]
We use the Law of Cosines in the triangle \(POQ\) to obtain
\[
\cos \angle POQ = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ},
\]
and from relations (8), (11), (12) and (13) we obtain the relation (9).

**Theorem 3.2.** Let \(P\) and \(Q\) be two points different from the circumcircle \(O\), having the barycentric coordinates \(t_1 : t_2 : t_3\) and \(u_1 : u_2 : u_3\) with respect to the triangle \(ABC\), and let \(u = u_1 + u_2 + u_3\), \(t = t_1 + t_2 + t_3\). If \(t_1, t_2, t_3, u_1, u_2, u_3 \neq 0\), then the following inequalities hold
\[
-2 \sqrt{\left[ \frac{R^2 - t_1 t_2 t_3}{t^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \right] \cdot \left[ \frac{R^2 - u_1 u_2 u_3}{u^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \right]}
\leq
\]
\[
\alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) + 2R^2 - \left[ \frac{t_1 t_2 t_3}{t^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \right] + \left[ \frac{u_1 u_2 u_3}{u^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \right]
\leq
\]
\[
2 \sqrt{\left[ \frac{R^2 - t_1 t_2 t_3}{t^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \right] \cdot \left[ \frac{R^2 - u_1 u_2 u_3}{u^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \right]}
\]
where \(a, b, c\) are the length sides of the triangle and the numbers \(\alpha, \beta, \gamma\) are defined by (10).

**Proof.** The inequalities (14) are simple direct consequences of the fact that \(-1 \leq \cos \angle POQ \leq 1\). The equality in the right inequality holds if and only if \(\angle POQ = 0\), that is the points \(O, P, Q\) are collinear in the order \(O, P, Q\) or \(O, Q, P\). The equality in the left inequality holds if and only if \(\angle POQ = \pi\), that is the points \(O, P, Q\) are collinear in the order \(P, O, Q\) or \(Q, O, P\).

From Theorem 3.1 it follows that it is a natural and important problem to construct the triangle \(ABC\) from the points \(O, P, Q\), when we know their barycentric coordinates. In the special case when \(P = I\) and \(Q = N\) we know that that points \(I, G, N\) are collinear, determining the Nagel line of triangle, and the centroid \(G\) lies on the segment \(IN\) such that \(IG = \frac{1}{2} IN\). Then, using the Euler’s line of the triangle, we get the orthocenter \(H\) on the ray \((OG\) such that \(OH = 3OG\). In this case the problem is reduced to the famous Euler’s determination problem i.e. to construct a triangle from its incenter \(I\), circumcenter \(O\), and orthocenter \(H\) (see the paper of P.Yiu [24] for details and results). This is a reason to call the problem as the general determination problem.

### 4 Applications

The formula (3) and the classical Blundon’s inequalities (1) can be obtained from (9) and (14) by considering \(P = I\), the incenter, and \(Q = N\), the Nagel point of the triangle. Indeed, the barycentric coordinates of incenter \(I\) and of Nagel’s
point $N$ are $(t_1, t_2, t_3) = (a, b, c)$, and $(u_1, u_2, u_3) = (s - a, s - b, s - c)$, respectively. We have

\[ u = u_1 + u_2 + u_3 = s - a + s - b + s - c = s, \quad u_1u_2u_3 = (s - a)(s - b)(s - c) = r^2s, \]  

(15) and

\[ t = t_1 + t_2 + t_3 = 2s, \quad t_1t_2t_3 = abc = 4Rrs. \]  

(16)

We obtain

\[ \alpha = \frac{s - a}{s} = \frac{2s - 3a}{2s}, \quad \beta = \frac{2s - 3b}{2s}, \quad \gamma = \frac{2s - 3c}{2s}. \]  

(17)

Therefore

\[ \alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) = \sum_{\text{cyc}} \beta \gamma a^2 = \sum_{\text{cyc}} \left( 1 - \frac{3b}{2s} \right) \left( 1 - \frac{3c}{2s} \right) a^2 = \]  

\[ \sum_{\text{cyc}} a^2 \left( 1 - \frac{3b}{2s} \right) \left( 1 - \frac{3c}{2s} \right) a^2 = \]  

\[ \sum_{\text{cyc}} a^2 - 3 \sum_{\text{cyc}} a^2 + \frac{3}{2s} \sum_{\text{cyc}} a^2 + \frac{9abc}{4s^2} \sum_{\text{cyc}} a = \]  

\[ \sum_{\text{cyc}} a^2 - 3 \sum_{\text{cyc}} a^2 + \frac{3}{2s} \sum_{\text{cyc}} a^2 + \frac{9abc}{2s} = \]  

\[ -2(2s^2 - 2r^2 - 8Rr) + 3(s^2 - 3r^2 - 6Rr) + 18Rr \]

that is

\[ \alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) = -s^2 - 5r^2 + 16Rr. \]  

(18)

Now, using (16) and (17) we get

\[ \frac{t_1t_2t_3}{t} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) = \frac{4Rrs}{4s^2} \cdot 2s = 2Rr, \]  

(19)

and

\[ \frac{u_1u_2u_3}{u} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) = \frac{r^2s}{s^2} \left( \frac{a^2}{s - a} + \frac{b^2}{s - b} + \frac{c^2}{s - c} \right) = \]  

\[ \frac{1}{s^2} \sum_{\text{cyc}} a^2 (s - b)(s - c) = \frac{1}{s^2} \left( \sum_{\text{cyc}} a^2 - s \sum_{\text{cyc}} [a^2(a + b + c) - a^3] + abc \sum_{\text{cyc}} a \right) = \]  

\[ \frac{1}{s^2} \left( \sum_{\text{cyc}} a^2 - s^2 \sum_{\text{cyc}} a^2 + 8Rrs^2 \right) = \]  

\[ \frac{1}{s^2} \left[ 2s^2(s^2 - 3r^2 - 6Rr) - s^3(2s^2 - 2r^2 - 8Rr) + 8Rrs^2 \right] = 4Rr - 4r^2 \]  

(20)

Using the relations (18)-(20) in (9) we obtain the relation (3). These computations are similar to those given by complex numbers in [1].
Now, consider the excenters \( I_a, I_b, I_c \) and \( N_a, N_b, N_c \), the adjoint points to the Nagel point \( N \). For the definition and some properties of the adjoint points \( N_a, N_b, N_c \) we refer to the paper of D.Andrica and K.L.Nguyen [2]. Let \( s, R, r, r_a, r_b, r_c \) be the semiperimeter, circumradius, inradius, and exradii of triangle \( ABC \), respectively. Considering the triangle \( I_aON_a \), D.Andrica and C.Barbu [3] have proved the following formula

\[
\cos \angle I_aON_a = \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R + 2r_a) \sqrt{R^2 + 2Rr_a}}.
\] (21)

where \( \alpha = \frac{a^2 + b^2 + c^2}{4} \).

Using formula (21), we get the dual form of Blundon’s inequalities given in the paper [3]

\[
0 \leq \frac{a^2 + b^2 + c^2}{4} \leq R^2 - 3Rr_a - r_a^2 + (R + 2r_a) \sqrt{R^2 + 2Rr_a}.
\] (22)

There are similar inequalities involving the exradii \( r_a \) and \( r_c \).

We known that the barycentric coordinates of the excenter \( I_a \) are \((t_1, t_2, t_3) = (-a, b, c)\), and of the adjoint Nagel point \( N_a \) are \((u_1, u_2, u_3) = (s, c - s, b - s)\). Using formula (9) we can obtain the relation (21) and then the dual form of the classical Blundon’s inequalities (22).

We have

\[
u = u_1 + u_2 + u_3 = s - a, \quad u_1u_2u_3 = s(s - b)(s - c)
\]

and

\[
t = t_1 + t_2 + t_3 = 2(s - a), \quad t_1t_2t_3 = -abc = -4Rrs.
\]

We obtain

\[
\alpha = \frac{2s + a}{2(s - a)} = 1 + \frac{3a}{2(s - a)},
\]

\[
\beta = \frac{2c - 2s - b}{2(s - a)} = 1 - \frac{3b}{2(s - a)},
\]

\[
\gamma = \frac{2b - 2s - c}{2(s - a)} = 1 - \frac{3c}{2(s - a)}.
\]

Therefore,

\[
\frac{t_1t_2t_3}{t^2} \left( \frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) = \frac{-4Rrs}{4(s - a)^2} \cdot 2(s - a) = -2Rr_a,
\] (23)

and

\[
\frac{u_1u_2u_3}{u^2} \left( \frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) = \frac{s(s - c)(s - b)}{(s - a)^2} \left( \frac{a^2}{s} - \frac{b^2}{s - c} - \frac{c^2}{s - b} \right) = \left( -\frac{-a}{s - a} \cdot \frac{s - b}{s} \cdot \frac{s - c}{s - a} + b^2 \cdot \frac{s - b}{s - a} + c^2 \cdot \frac{s - c}{s - a} \right) = \ldots
\]
\[- \left( -a^2 \cdot \frac{r_a}{r_b} \cdot \frac{r_a}{r_c} + b^2 \cdot \frac{r_a}{r_b} + c^2 \cdot \frac{r_a}{r_c} \right) = \]
\[- r_a^2 \left( \frac{-a^2}{r_b r_c} + \frac{b^2}{r_b} + \frac{c^2}{r_c} \right) = -r_a^2 \left( \frac{4R}{r_a} + 4 \right) = -4Rr_a - 4r_a^2, \tag{24} \]

where we have used the relation \( \frac{a^2}{r_b r_c} + \frac{b^2}{r_b} + \frac{c^2}{r_c} = \frac{4R}{r_a} + 4 \) (see [2], p. 134).

Now, we will calculate the expression:

\[ E = \alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) + \frac{a^2 + b^2 + c^2}{2} = \]
\[ a^2 \beta \gamma + \frac{a^2}{2} + b^2 \alpha \gamma + \frac{b^2}{2} + c^2 \alpha \beta + \frac{c^2}{2} = \]
\[ a^2 \left[ 1 - \frac{3(b + c)}{2(s - a)} + \frac{9bc}{4(s - a)^2} \right] + \frac{a^2}{2} + \]
\[ b^2 \left[ 1 + \frac{3(a - c)}{2(s - a)} - \frac{9ca}{4(s - a)^2} \right] + \frac{b^2}{2} + \]
\[ c^2 \left[ 1 + \frac{3(a - b)}{2(s - a)} - \frac{9ab}{4(s - a)^2} \right] + \frac{c^2}{2}, \]

that is

\[ E = a^2 \left[ \frac{-3s}{2(s - a)} + \frac{9bc}{4(s - a)^2} \right] + b^2 \left[ \frac{3(s - c)}{2(s - a)} - \frac{9ca}{4(s - a)^2} \right] + c^2 \left[ \frac{3(s - b)}{2(s - a)} - \frac{9ab}{4(s - a)^2} \right] = \]
\[ \frac{3}{2(s - a)} \left( -a^2 s + b^2 (s - c) + c^2 (s - b) \right) + \frac{9abc}{4(s - a)^2} (a - b - c) = \]
\[ \frac{3}{2(s - a)} \left( s(-a^2 + b^2 + c^2) - bc(b + c) \right) - 18Rr_a. \tag{25} \]

We have
\[ s(-a^2 + b^2 + c^2) - bc(b + c) = 2sbc \cos A - 2bcs + abc = \]
\[ 2sbc \cos A - 1 + abc = abc - 4s(p - b)(p - c) = abc - 4Sr_a = 4S(R - r_a), \tag{26} \]

where \( S \) denotes the area of triangle \( ABC \). From relations (25) and (26) we get

\[ E = \frac{3}{2(s - a)} \cdot 4S(R - r_a) - 18Rr_a = 6r_a(R - r_a) - 18Rr_a = -12Rr_a - 6r_a^2, \]

therefore

\[ \alpha \beta \gamma \left( \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) = -12Rr_a - 6r_a^2 - \frac{a^2 + b^2 + c^2}{2}. \tag{27} \]

Using formulas (23), (24) and (27) in the general formula (9) we obtain the relation (21).
In the paper [13], N. Minculete and C. Barbu have introduced the Cevians of rank \((k; l; m)\). The line \(AD\) is called \textit{ex-Cevian of rank} \((k; l; m)\) or \textit{exterior Cevian of rank} \((k; l; m)\), if the point \(D\) is situated on side \((BC)\) of the non-isosceles triangle \(ABC\) and the following relation holds:

\[
\frac{BD}{DC} = \left( \frac{c^k}{b} \right) \cdot \left( \frac{s - c}{s - b} \right)^l \cdot \left( \frac{a + b}{a + c} \right)^m.
\]

In the paper [13] it is proved that the Cevians of rank \((k; l; m)\) are concurrent in the point \(I(k, l, m)\) called \textit{the Cevian point of rank} \((k; l; m)\) and the barycentric coordinates of \(I(k, l, m)\) are:

\[
a^k(s - a) \cdot (b + c)^m : b^k(s - b) \cdot (a + c)^m : c^k(s - c) \cdot (a + b)^m.
\]

In the case \(l = m = 0\), we obtain the Cevian point of rank \(k\).

Let \(I_1, I_2\) be two Cevian points with barycentric coordinates:

\[
I_1[a^k(s - a)^l \cdot (b + c)^m : b^k(s - b)^l \cdot (a + c)^m : c^k(s - c)^l \cdot (a + b)^m], \quad i = 1, 2.
\]

Denote \(t_1^i = a^k(s - a)^l \cdot (b + c)^m, t_2^i = b^k(s - b)^l \cdot (a + c)^m, t_3^i = c^k(s - c)^l \cdot (a + b)^m, i = 1, 2\). From formula (9) we obtain

\[
\cos I_1\overline{O}I_2 = \frac{2R^2 - \frac{t_1^1 t_2^1 t_3^1}{(t_1^1)^2} \left( \frac{a^2}{t_1^1} + \frac{b^2}{t_1^1} + \frac{c^2}{t_1^1} \right) - \frac{t_1^2 t_2^2 t_3^2}{(t_1^2)^2} \left( \frac{a^2}{t_1^2} + \frac{b^2}{t_1^2} + \frac{c^2}{t_1^2} \right) + \alpha \beta \gamma \left( \frac{a^2}{t_1^1} + \frac{b^2}{t_1^2} + \frac{c^2}{t_1^2} \right)}{2 \sqrt{\left[ \left( \frac{a^2}{t_1^1} + \frac{b^2}{t_1^1} + \frac{c^2}{t_1^1} \right) \left( \frac{a^2}{t_1^2} + \frac{b^2}{t_1^2} + \frac{c^2}{t_1^2} \right) \right]}}
\]

(28)

where \(T_1 = t_1^1 + t_2^1 + t_3^1, T_2 = t_1^2 + t_2^2 + t_3^2\), and for \(i = 1, 2\), we have

\[
\frac{t_1^1 t_2^3 t_3^3}{(T_1)^2} \left( \frac{a^2}{t_1^1} + \frac{b^2}{t_1^1} + \frac{c^2}{t_1^1} \right) = \frac{\prod a^k(s - a)^l \cdot (b + c)^m}{\sum a^k(s - a)^l \cdot (b + c)^m} \sum a^k(s - a)^l \cdot (b + c)^m,
\]

and

\[
\alpha = \frac{\sum a^k(s - a)^l \cdot (b + c)^m}{\sum a^k(s - a)^l \cdot (b + c)^m} - \frac{\sum a^k(s - a)^l \cdot (b + c)^m}{\sum a^k(s - a)^l \cdot (b + c)^m},
\]

\[
\beta = \frac{\sum a^k(s - b)^l \cdot (a + c)^m}{\sum a^k(s - a)^l \cdot (b + c)^m} - \frac{\sum a^k(s - b)^l \cdot (a + c)^m}{\sum a^k(s - a)^l \cdot (b + c)^m},
\]

\[
\gamma = \frac{\sum a^k(s - c)^l \cdot (a + b)^m}{\sum a^k(s - a)^l \cdot (b + c)^m} - \frac{\sum a^k(s - c)^l \cdot (a + b)^m}{\sum a^k(s - a)^l \cdot (b + c)^m}.
\]
If $I_1, I_2$ are Cevian points of rank $k_1, k_2$, then formula (28) becomes
\[
\cos \widehat{I_1 \overline{O I_2}} = \frac{2R^2 - (abc)^{k_1} S_{i_1} - (abc)^{k_2} S_{i_2} + \sum_{i \neq j = 1}^{l} (a^4 - b^4)(S_i - b_i)S_j}{2 \sqrt{[R^2 - (abc)^{k_1} S_{i_1}] [R^2 - (abc)^{k_2} S_{i_2}]}}
\]
(29)
where $S_i = a^i + b^i + c^i$.

Here are few special cases of formula (29). For $k_1 = 0$ and $k_2 = 1$ we get the centroid $G$ and the incenter $I$ of barycentric coordinates $(1; 1; 1)$ and $(a; b; c)$, respectively. Formula (29) becomes
\[
\cos \widehat{GOI} = \frac{6R^2 - s^2 - r^2 + 2Rr}{2 \sqrt{9R^2 - 2s^2 + 2r^2 + 8Rr \cdot \sqrt{R^2 - 2Rr}}}
\]
(30)
where $abc = 4sRr$, $S_0 = 3$, $S_1 = 2s$, $S_2 = 2(s^2 - r^2 - 4Rr)$.

For $k_2 = 2$ we obtain the Lemoine point $L$ of triangle $ABC$, of barycentric coordinates $(a^2; b^2; c^2)$, and other two formulas are generated
\[
\cos \widehat{GOL} = \frac{6R^2 S_2 - S_2^2 + 4S_4}{2 \sqrt{9R^2 - S_2 \cdot \sqrt{R^2 - S_2^2 - 48(Rrs)^2}}}
\]
(31)
where $S_4 = S_2^2 - 2[(s^2 + r^2 + 4Rr)^2 - 16Rrs^2]$, and
\[
\cos \widehat{IOL} = \frac{R S_2 + r S_2 - 4rs^2}{2 \sqrt{R^2 - 2Rr \cdot \sqrt{S_2^2 - 48r^2s^2}}}
\]
(32)
Each of the formulas (30),(31),(32) generates a Blundon type inequality, but these inequalities have not nice geometric interpretations.

Let $I_1, I_2, I_3$ be three Cevian points of rank $(k; l; m)$ with barycentric coordinates as follows:
\[a^k(s - a)^l(b + c)^m : b^k(s - b)^l(a + c)^m : c^k(s - c)^l(a + b)^m, \ i = 1, 2, 3\]
and let $t_1^i = a^k(s - a)^l(b + c)^m$, $t_2^i = b^k(s - b)^l(a + c)^m$, $t_3^i = c^k(s - c)^l(a + b)^m$. Now, consider the numbers
\[
\alpha_{ij} = \frac{t_1^i}{t_1^i + t_2^i + t_3^i} - \frac{t_1^i}{t_1^i + t_2^i + t_3^i}
\]
and
\[
\beta_{ij} = \frac{t_2^i}{t_1^i + t_2^i + t_3^i} - \frac{t_2^i}{t_1^i + t_2^i + t_3^i}
\]
and
\[
\gamma_{ij} = \frac{t_3^i}{t_1^i + t_2^i + t_3^i} - \frac{t_3^i}{t_1^i + t_2^i + t_3^i}
\]
for all $i, j \in \{1, 2, 3\}$. Applying the relation (5) we obtain

$$I_i I_j^2 = -\alpha_{ij} \cdot \beta_{ij} \cdot \gamma_{ij} \left(\frac{a^2}{\alpha_{ij}} + \frac{b^2}{\beta_{ij}} + \frac{c^2}{\gamma_{ij}}\right),$$

for all $i, j \in \{1, 2, 3\}$. Using the Cosine Law in triangle $I_1I_2I_3$ it follows

$$\cos \angle I_1I_2I_3 = \frac{I_1^2 + I_2^2 - I_3^2}{2I_1I_2} = \frac{-a^2(\beta_{12}\gamma_{23} + \beta_{23}\gamma_{13} - \beta_{31}\gamma_{12}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31})}{2 \sqrt{-\beta_{12}\gamma_{23}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{13}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2}}$$

(33)

**Theorem 4.1.** The following inequalities hold

$$-2\sqrt{-\beta_{12}\gamma_{13}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \leq$$

$$-a^2(\beta_{12}\gamma_{12} + \beta_{23}\gamma_{23} - \beta_{31}\gamma_{13}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31}) \leq$$

$$2 \sqrt{-\beta_{12}\gamma_{13}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \leq$$

(34)

Proof. The inequalities (34) are simple direct consequences of the inequalities $-1 \leq \cos \angle I_1I_2I_3 \leq 1$. □

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