TWO-POINT CORRELATIONS IN THE COBE\(^1\) DMR FOUR-YEAR ANISOTROPY MAPS

G. HINSHAW, A. J. BANDAY, C. L. BENNETT, K. M. GÓRSKI, A. KOGUT, C. H. LINNEWEAVER, G. F. SMOOT, AND E. L. WRIGHT

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ABSTRACT

The two-point temperature correlation function is evaluated from the 4 yr COBE DMR microwave anisotropy maps. We examine the two-point function, which is the Legendre transform of the angular power spectrum, and show that the data are statistically consistent from channel to channel and frequency to frequency. The most likely quadrupole normalization is computed for a scale-invariant power-law spectrum of CMB anisotropy, using a variety of data combinations. For a given data set, the normalization inferred from the two-point data is consistent with that inferred by other methods. The smallest and largest normalizations deduced from any data combination are 16.4 and 19.6 \(\mu K\), respectively, with a value \(\sim 18 \mu K\) generally preferred.

Subject headings: cosmic microwave background — cosmology: observations

1. INTRODUCTION

The detection of large angular scale anisotropies in the cosmic microwave background (CMB) radiation was first reported by the COBE DMR experiment in 1992 (Smoot et al. 1992; Bennett et al. 1992; Wright et al. 1992; Kogut et al. 1992). The initial detection was based only on the first year of flight data. Since that time the DMR team processed and analyzed the first 2 yr of data and found the results to be consistent with the first year results (Bennett et al. 1994; Górski et al. 1994; Wright et al. 1994a). We have now processed and analyzed the full 4 yr of DMR observations; this paper is one of a series describing the results of our analysis. The maps and an overview of the scientific results are given in Bennett et al. (1996).

In this paper we analyze the anisotropy in the 4 yr DMR maps using the two-point correlation function as a measure of the angular power spectrum. The COBE DMR experiment was designed to measure the CMB anisotropy on angular scales of \(\sim 7^\circ\), corresponding to spherical harmonic multipole moments of order \(\ell \lesssim 30\). The DMR has produced full-sky maps of the CMB temperature at each of three frequencies, 31.5, 53, and 90 GHz, with two independent channels, A and B, at each frequency. In principle, one can obtain an estimate of the CMB power spectrum from an anisotropy map simply by decomposing the map, \(T(\theta, \phi)\), into spherical harmonic components and averaging them to find the mean power per mode \(\ell\): \(T(\theta, \phi) = \sum a_{\ell m} Y_{\ell m}(\theta, \phi)\) with power spectrum \(a_{\ell} = (\sum_{m=-\ell}^{\ell} |a_{\ell m}|^2)/(2\ell + 1)\). In practice, however, there are a number of complications that arise. First, the need to apply a Galactic cut to the data renders the spherical harmonics nonorthogonal, thereby coupling the \(a_{\ell m}\) coefficients and increasing their uncertainty. Moreover, since \(a_{\ell}^2\) is a quadratic form, any uncertainty in the \(a_{\ell m}\) (whether due to coupling, instrument noise, systematic effects, and/or foreground sources) produces a positive bias in the estimate of the power spectrum. However, see Górski et al. (1996) and Wright et al. (1996) for spherical harmonic–based analyses that account for these difficulties.

An alternative to estimating the power spectrum is to evaluate its Legendre transform, the two-point correlation function. For a given power spectrum with multipole amplitudes \(C_\ell = \langle |a_{\ell m}|^2 \rangle\), the predicted covariance between pairs of map pixels \(i\) and \(j\) with angular separation \(\alpha_{ij}\) is

\[
C(\alpha_{ij}) = \left\langle T_i T_j \right\rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) W_\ell F_\ell P_\ell (\cos \alpha_{ij}),
\]

where \(T_i\) is the CMB temperature in pixel \(i\), the angle brackets denote an average over an ensemble of universal observers, \(W_\ell\) is the experimental window function that includes the effects of beam smoothing and finite pixel size (\(W_\ell = G_\ell F_\ell\), where the \(G_\ell\) are the Legendre coefficients of the DMR beam pattern, tabulated by Wright et al. 1994b, and \(F_\ell\) are the Legendre coefficients for a circular top-hat function with area equal to the pixel area; the coefficients for index level 6 pixels are available upon request), and \(P_\ell (\cos \alpha_{ij})\) is the Legendre polynomial of order \(\ell\). We estimate the two-point correlation function in our sky by evaluating the average product of all map temperatures with a fixed angular separation, \(C(\alpha) = \sum_{i,j} w_i w_j T_i T_j / \sum_{i,j} w_i w_j\), where the sum is restricted to pixel pairs \((i,j)\) separated by an angle \(\alpha\), \(T_i\) is the observed temperature in pixel \(i\) after monopole and dipole (and optionally quadrupole) subtraction, and \(w_i\) is the statistical weight of pixel \(i\). This statistic is straightforward to compute and offers a quick test of the consistency of the power spectrum from map to map.

In the approximation that the two-point function can be treated as a multivariate Gaussian distribution, one can form a likelihood function with which to estimate the power spectrum normalization. In § 2 we present the two-point correla-
tion data in the 4 yr DMR maps and examine its consistency from map to map. In §3 we evaluate the Gaussian likelihood as a function of the mean expected quadrupole moment, \( Q_{\text{rms}} \), under the assumption of a scale-invariant, power-law spectrum of anisotropy.

2. TWO-POINT CORRELATION DATA

The two-point correlation function, as given above, is the average product of all pixel temperatures with a fixed angular separation. The correlation data are binned into angular separation bins of width 2\(^\circ\)/6 with the first bin reserved for all pixel pairs \((i, j)\) such that \(i = j\), the second bin for pairs with separation between 0\(^\circ\) and 2\(^\circ\)/6, and so forth; there are 71 such bins. For the present analysis we employ the maps pixelized in Galactic coordinates and use the custom Galaxy cut described by Bennett et al. (1996), for which there are 3881 surviving pixels. To minimize the total uncertainty in the two-point data, due to noise plus cosmic and sample variance, we assign equal weight to each surviving pixel, \( w_j = 1 \), instead of \( w_j = 1/\sigma_j^2 \). This has the effect of increasing the noise contribution slightly, but it decreases the sample variance by a slightly larger factor. Uniform weighting reduces the total uncertainty by a few percent for the 4 yr data.

The basic two-point functions obtained from the single-frequency maps are shown in Figure 1. We plot both the autocorrelation of the weighted sum of channels A and B at each frequency (the coefficients used to form the weighted average maps analyzed in this Letter are given in Table 1) and the cross-correlation between channels A and B. For reference, we also plot, as a solid line, the autocorrelation of the weighted average of all six DMR channel maps. The error bar

\[
\begin{align*}
31 \text{w} & \quad \text{31A} & \quad \text{31B} & \quad \text{53A} & \quad \text{53B} & \quad \text{90A} & \quad \text{90B} \\
31 \text{w} & \quad 0.611 & \quad 0.389 & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
53 \text{w} & \quad 0 & \quad 0 & \quad 0.579 & \quad 0.421 & \quad 0 & \quad 0 \\
90 \text{w} & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 0.382 & \quad 0.618 \\
31 + 90 & \quad 0 & \quad 0.412 & \quad 0.299 & \quad 0.10 & \quad 0.179 & \quad 0.164 \\
31 + 53 + 90 & \quad 0.049 & \quad 0.032 & \quad 0.378 & \quad 0.275 & \quad 0.102 & \quad 0.164 \\
\text{Correlation} & \quad 0.049 & \quad 0.032 & \quad 0.378 & \quad 0.275 & \quad 0.102 & \quad 0.164 \\
\text{Combination} & \quad -0.185 & \quad -0.117 & \quad 0.367 & \quad 0.266 & \quad 0.256 & \quad 0.413
\end{align*}
\]

at each frequency, after monopole and dipole subtraction. Left-hand panels show the autocorrelation functions obtained from a weighted average of the A and B channel maps (the point at zero lag is off-scale due to the noise contribution). Right-hand panels show the cross-correlation of the A and B channels, which are sensitive only to common structure in the maps. Error bars represent the uncertainty due to instrument noise, as described in the text. To guide the eye, the solid line is the autocorrelation of the weighted average of all six channels maps. All six two-point functions are statistically consistent with each other.
For the scale-invariant case, $C_{k}$ spectrum of CMB fluctuations, $\Delta C_{k}$ is weak, and (2) approximately traced by the DIRBE 140 GHz map after monopole and dipole subtraction. Top right: cross-correlation function of the 53 GHz weighted sum map with the 90 GHz weighted map. Bottom left: autocorrelation function of the weighted average map with best-fit Galaxy template maps subtracted from the map (Kogut et al. 1996a). Bottom right: autocorrelation function of the linear combination map designed to cancel free-free emission. This map has a best-fit model of the synchrotron and dust emission also subtracted. In all panels the error bars represent the uncertainty due to instrument noise, as described in the text. Zero-lag point in each of the autocorrelation functions is off-scale due to the noise contribution. To guide the eye, the solid line is the autocorrelation of the weighted average map (top left panel).

3. QUADRUPOLE NORMALIZATION

Given a power-law model of initial Gaussian density fluctuations, $P(k) \propto k^\alpha$, where $P(k)$ is the power spectrum of density fluctuations as a function of comoving wavenumber $k$, it is possible to derive the corresponding angular power spectrum of CMB fluctuations, $C_\ell = \langle |\ell m|^2 \rangle$ (Bond & Efstathiou 1987). The result is

$$C_\ell = C_2 \frac{T(\ell + (n - 1)/2)T(9 - n)/2}{\Gamma(\ell + (5 - n)/2)\Gamma(3 + n/2)}.$$ (2)

For the scale-invariant case, $n = 1$, this reduces to $C_\ell = 6C_2/\ell(\ell + 1)$, which has one free parameter, the mean quadrupole moment $C_2$. We customarily express the normalization in terms of $Q_{\text{rms-PS}} = \langle |\ell m|^2 \rangle^{1/2}$, the mean rms temperature fluctuation expected in the quadrupole component of the anisotropy. We determine the most likely quadrupole normalization, $Q_{\text{rms-PS}}$, from the two-point function by evaluating the Gaussian approximation to the likelihood function

$$L(Q_{\text{rms-PS}}) \propto e^{-1/2M\Delta C^T \cdot M^{-1} \cdot \Delta C}/\sqrt{\det(M)}.$$ (3)

Here $\Delta C^T$ and $\Delta C$ are row and column vectors with entries $\Delta C_{k} = C_{k} - \langle C_{k} \rangle$, $a = 0, \ldots, 70$, and $M = \langle \Delta C(\Delta C)^T \rangle$ is the covariance matrix of the correlation function. The angle brackets denote averages over both measurement errors and over the ensemble of anisotropy fields implied by cosmic variance for a given $C_\ell$. We estimate the mean correlation and covariance matrix as a function of $Q_{\text{rms-PS}}$ using Monte Carlo simulations described above. The simulations account for all important aspects of our data processing including monopole and dipole (and quadrupole) removal on the cut sky. Because of this subtraction, the bins of a given correlation function are not all independent so the covariance matrices derived from the simulations are formally singular. We invert these matrices to form $\chi^2$ using singular value decomposition, which permits an unambiguous identification of the zero modes that arise due to multipole subtraction. We then evaluate the logarithm of the likelihood, $\ln L = -\chi^2/2$, and identify the maximum.

We test the likelihood method for accuracy by feeding the

| MAP 1 | MAP 2 | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $\chi^2$ |
|-------|-------|-----------------|-----------------|-------|
| 31A | 31B | 18.2 $\pm$ 4.1 | 68.3 | 18.0 $\pm$ 4.6 | 71.0 |
| 53A | 53B | 18.3 $\pm$ 1.6 | 73.5 | 18.6 $\pm$ 1.7 | 69.6 |
| 90A | 90B | 16.4 $\pm$ 2.2 | 73.5 | 18.4 $\pm$ 2.3 | 71.0 |

Single-Frequency Autocorrelation

| MAP 1 | MAP 2 | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $\chi^2$ |
|-------|-------|-----------------|-----------------|-------|
| 31ws | 31ws | 17.1 $\pm$ 3.7 | 67.9 | 17.6 $\pm$ 4.0 | 79.6 |
| 53ws | 53ws | 18.7 $\pm$ 1.6 | 99.9 | 19.4 $\pm$ 1.6 | 97.5 |
| 90ws | 90ws | 17.5 $\pm$ 2.0 | 63.5 | 19.0 $\pm$ 2.2 | 61.5 |

Multifrequency Cross-Correlation

| MAP 1 | MAP 2 | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $\chi^2$ |
|-------|-------|-----------------|-----------------|-------|
| 53 + 90 | 53 + 90 | 18.5 $\pm$ 1.4 | 84.2 | 19.6 $\pm$ 1.5 | 83.5 |
| 31 + 53 + 90 | 31 + 53 + 90 | 18.6 $\pm$ 1.4 | 80.0 | 19.3 $\pm$ 1.4 | 78.2 |

Multifrequency Autocorrelation

| MAP 1 | MAP 2 | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $Q_{\text{rms-PS}}$ (\(\mu K\)) | $\chi^2$ |
|-------|-------|-----------------|-----------------|-------|
| Correlation | Correlation | 17.5 $\pm$ 1.4 | 76.2 | 18.5 $\pm$ 1.4 | 78.3 |
| Combination | Combination | 16.7 $\pm$ 2.0 | 89.2 | 17.8 $\pm$ 2.2 | 92.4 |

$^a$ Coefficients that comprise the map combinations in these columns are given in Table 1, except for 53ss and 90ss, which are straight sum maps: $(A + B)/2$.

$^b$ $\ell_{\text{min}}$ is the lowest order multipole remaining in the map after subtracting a best-fit multipole of order $\ell_{\text{min}} - 1$. $Q_{\text{rms-PS}}$ is the most likely quadrupole normalization, after calibrating the likelihood with Monte Carlo simulations. $\chi^2$ is tabulated, for reference, with respect to the mean of a scale-invariant model with the corresponding most likely normalization.
simulated two-point functions into the likelihood function and solving for an ensemble of $Q_{\text{rms}}$ maxima. We define the bias in our method to be $\Delta Q = \langle Q_{\text{max}} \rangle - Q_n$, where $\langle Q_{\text{max}} \rangle$ is the mean of the recovered maxima and $Q_n$ is the simulation input normalization. The resulting bias depends on the noise level in the data, but ranges from $-0.2$ to $-0.4$ $\mu$K for the all the cases except the 31 GHz data where it is $\sim 1$ $\mu$K. We correct for this bias in all reported results. The uncertainty we assign to $Q_{\text{rms}}$ is the rms scatter of the ensemble $Q_{\text{max}}$, which typically exceeds the rms of the Gaussian likelihood by about 10%.

The corrected power spectrum normalization deduced from a variety of data combinations is given in Table 2. The smallest and largest normalization deduced from any data combination are 16.4 and 19.6 $\mu$K, respectively, with values $\sim 18$ $\mu$K generally preferred. The normalization inferred from the two-point function is now in better agreement with other determinations than was the case with the 2 yr data (Bennett et al. 1994). The change is due to data selection: with the 2 yr data, we only analyzed the 53 $\times$ 90 GHz cross-correlation function; with the 4 yr data we have analyzed many more data combinations, including the autocorrelation of a weighted average multifrequency map which yields a normalization $\sim 1.5$ $\mu$K higher than the cross-correlation. The multifrequency autocorrelation is more comparable to the data analyzed by other methods, and the two-point analysis yields consistent results in that case. For a comparison, see Table 2 of Bennett et al. (1996). In general, the normalization inferred from the 4 yr data is slightly less than what we found after 2 yr, in part because of the extension of the Galaxy cut. For comparison, the 31 $+$ 53 $+$ 90 GHz autocorrelation with a straight 20° cut yields a best-fit normalization of $Q_{\text{rms}} = 19$ $\mu$K. As shown in Table 2, the effects of further modeling and subtraction of Galactic emission are less than 1 $\mu$K in the normalization.

While a likelihood analysis is capable of inferring the best-fit parameters for a given model, it does not say anything per se about the goodness of fit. For reference we have included in Table 2 the values of $\chi^2$ at the maximum likelihood value of $Q_{\text{rms}}$ (with $n = 1$). Since the two-point function is only approximately multivariate Gaussian distributed, our tabulated statistic is only approximately $\chi^2$ distributed. However, we have used our Monte Carlo simulations to compute the expected distribution of this statistic and find it to be approximately $\chi^2$ with a mean of $\sim 70$ and a standard deviation of $\sim 12$. The values computed with the DMR data are very consistent with this distribution, implying that the data are well fit by a scale-invariant power spectrum.

Analyses of $Q_{\text{rms}}$ have also been reported by Górski et al. (1996), Hinshaw et al. (1996), Kogut et al. (1996b), and Wright et al. (1996). All results lie between 16.4 and 19.6 $\mu$K, with most between 17.5 and 18.5 $\mu$K. In general, all methods for analyzing a given data combination give consistent results, while there is modest dependence on data selection. Fortunately, this dependence does not exceed the statistical uncertainty due to cosmic variance and instrument noise. Based on our judgment, a reasonable value for the normalization of a scale-invariant spectrum is $Q_{\text{rms}} = 18 \pm 1.6$ $\mu$K, where the quoted error is slightly dilated, relative to the statistical error, to encompass the range of normalizations observed in different data subsets.

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