A Perturbative Approach to the Relativistic Harmonic Oscillator

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Abstract

A quantum realization of the Relativistic Harmonic Oscillator is realized in terms of the spatial variable $x$ and $\frac{d}{dx}$ (the minimal canonical representation). The eigenstates of the Hamiltonian operator are found (at lower order) by using a perturbation expansion in the constant $c^{-1}$. Unlike the Foldy-Wouthuysen transformed version of the relativistic hydrogen atom, conventional perturbation theory cannot be applied and a perturbation of the scalar product itself is required.

1 Introduction

The Relativistic Harmonic Oscillator is probably the simplest relativistic system containing bound states, yet it exhibits the typical problems of Relativistic Quantum Mechanics. Many papers have been devoted to the solution of this relativistic system [1, 2, 3, 4, 5] although the first question is probably to define what we understand by a Relativistic Harmonic Oscillator.

In a previous paper [6], we adopted an algebraic method for both defining and solving such an oscillator equation which started with the Lie operator algebra

$$[\hat{E}, \hat{x}] = -\frac{i\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = i m \omega^2 \hbar \hat{x}, \quad [\hat{x}, \hat{p}] = i\hbar (1 + \frac{1}{mc^2} \hat{E}),$$

which is an affine version of the Lie algebra of the 1+1 anti-de Sitter group ($\approx \mathfrak{sl}(2, R)$) and reproduces the appropriate limits (in the sense of İnönü and Wigner group contractions [7]) as $\omega \rightarrow 0$ (going to the Poincaré group in 1+1D) and/or $c \rightarrow \infty$ (going to the Harmonic Oscillator group), although in this paper we will be concerned only with the

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$c \to \infty$ limit, i.e. the one leading to the Harmonic Oscillator. The solution was given through a **manifestly covariant** representation of the corresponding group (i.e. the wave functions are solutions of a Klein-Gordon-like equation). The energy eigenfunctions in configuration space consist of a general weight function (the vacuum) which converges to a Gaussian in the limit $c \to \infty$, a specific power of the vacuum (reflecting the explicit dependence on time of the manifestly covariant representation) which reduces to unity in this limit, the usual time-dependent phase factor $\exp(-iE_nt)$ and a polynomial leading to the corresponding non-relativistic Hermite polynomial as $c \to \infty$. As a consequence of the explicit dependence on time, the scalar product was defined by using the invariant measure $dx \, dt$. These representations correspond to a realization in the real variables $(x, t)$ of the Discrete series $D^+(k)$ of the $SL(2, \mathbb{R})$ group for Bargmann index $k = N \equiv \frac{mc^2}{\hbar\omega}$.

In this paper we study the problem appearing in the configuration-space representation when the time dependence is factorized out in the search for a **minimal, canonical realization** (given in terms of only $x$ and $\frac{\partial}{\partial x}$) in a way similar to that of the relativistic hydrogen atom. In this last case, a Foldy-Wouthuysen transformation leads to a Hamiltonian containing higher-order relativistic corrections to the non-relativistic Hamiltonian. Ordinary perturbation theory provides the energy levels as a power series in $c^{-1}$ of the exact values already obtained from the manifestly covariant (Dirac) equation.

In the present case, however, the situation is quite different because the terms in the Hamiltonian that are considered here as higher-order relativistic corrections to the non-relativistic Hamiltonian are not Hermitian with respect to the non-relativistic scalar product (with measure $dx$) and a perturbation of the scalar product itself is required. These higher-order relativistic corrections should be understood as a power expansion in $1/k$ of the representations of the Discrete series for large Bargmann indices $k = N \equiv \frac{mc^2}{\hbar\omega}$.

Our group quantization method essentially consists in exponentiating the abstract algebra (either Poisson or operator algebra) of basic quantities defining a given physical system, usually a $u(1)$-centrally extended algebra $\hat{G}$, and considering in the resulting group $\hat{G}$ the two mutually commuting sets of vector fields, $\chi^R(\hat{G})$ and $\chi^L(\hat{G})$ which generate the left and right action of the group on itself. One set of vector fields will constitute the operators of the theory, while the other can be used to fully reduce the action of the former. To be a bit more precise, we consider the subspace of complex functions on $\hat{G}$, $\mathcal{F}(\hat{G})$, that satisfy the $U(1)$-equivariance condition, $\Xi \Psi = i\Psi$, where $\Xi$ is the central generator $\hat{X}^L_\zeta = \hat{X}^R_\zeta$ and $\zeta \in U(1)$. The right-invariant vector fields (generating the finite left action) $\hat{X}^R \in \chi^R(\hat{G})$ act on $\mathcal{F}(\hat{G})$ as a reducible representation (corresponding to the Bohr-Sommerfeld quantization). The reduction defining the (true) quantization is achieved by imposing on $\mathcal{F}(\hat{G})$ the so-called polarization condition in terms of a subalgebra $\mathcal{P}$ of left-invariant vector fields, $\hat{X}^L \Psi = i\Psi$, which preserve the action of $\hat{X}^R$ ($[\hat{X}^R, \hat{X}^L] = 0$). The polarization $\mathcal{P}$, originally defined as a maximal left subalgebra of $\hat{G}$ containing the kernel of the Lie algebra cocycle $\Sigma$ and excluding the central generator $\Xi$, $\mathcal{P}$ can be further generalized by allowing it to contain operators in the left enveloping algebra $U\chi^L(\hat{G})$.

The paper is organized as follows: In section 2 the exact solutions for the Relativistic Harmonic Oscillator, firstly given in [3] and obtained later through a second order
polarization in \[14\], are given. A reduction to a minimal representation (in terms of \(x\) and \(d/dx\)) is tempted through a naive elimination of the time variable and the substitution \(i\hbar \frac{d}{dt} \rightarrow E_0 - mc^2\), but this leads to a theory which is not unitary. The solution is achieved by modifying the scalar product and the operators in a consistent though not quite well understood way. In section 3 a perturbative approach to the problem of the reduction is proposed. From an exact infinite-order polarization (which represents the “square root” of the Casimir leading to the Klein-Gordon equation) we obtain a perturbative expansion for a Schrödinger-like equation, where the zeroth-order Hamiltonian is the non-relativistic Harmonic Oscillator Hamiltonian. From this Schrödinger equation we immediately realize that the minimal representation is not unitary because the perturbed Hamiltonian (as opposite to the case of the hydrogen atom) is not Hermitian with respect to the measure \(dx\) of the non-perturbed theory. Thus a perturbation of the scalar product itself is proposed, and the solution coincides (at lower order in powers of \(1/c^2\)) with the exact solution proposed in section 2.

2 The Relativistic Harmonic Oscillator

To quantize the physical system characterized by the algebra (1), we must exponentiate this algebra and derive left and right vector fields. The left ones are\(\) (see \[14\] for the expression on the right-invariant vector fields):

\[
\begin{align*}
\tilde{X}^L_t &= \frac{p}{m} \frac{\partial}{\partial x} - m\omega^2 x \frac{\partial}{\partial p} + \frac{P_0}{mc^2\alpha^2} \frac{\partial}{\partial t} \\
\tilde{X}^L_p &= \frac{p}{mc \partial x} + \frac{mcx}{P_0 + mc\hbar} \Xi \\
\tilde{X}^L_x &= \frac{P_0}{mc \partial x} + \frac{p}{mc^2\alpha^2} \frac{\partial}{\partial t} - \frac{P_0}{P_0 + mc\hbar} \Xi,
\end{align*}
\]

(2)

where \(P_0 \equiv \sqrt{mc^2 + p^2 + m^2\omega^2 x^2}\) and \(\alpha \equiv \sqrt{1 + \omega^2 c^{-2} x^2}\).

As was mentioned in the introduction, the representation given by the right-invariant vector fields acting on complex wave functions on the group \(\tilde{G}\) is reducible. Thus, we need to impose conditions on the wave functions (polarization conditions) in order to reduce the representation space, and this is achieved by a polarization subalgebra of left-invariant vector fields. There is a first-order polarization, leading to the Bargmann-Fock representation \[14\], but in this paper we are interested in the configuration-space representation, and for this purpose we need a higher-order polarization \[12, 13\].

There is a second-order polarization which leads to the manifestly covariant representation:

\[
< \tilde{X}^{HO}_t \equiv (\tilde{X}^L_t)^2 - c^2(\tilde{X}^L_x)^2 + \frac{2imc^2}{\hbar} \tilde{X}^L_t + \lambda \frac{imc^2\omega}{\hbar} \Xi, \tilde{X}^L_p >,
\]

(3)

where the numerical parameter \(\lambda\) is arbitrary but can be chosen to yield the results previously obtained \[13\]. Imposing the \(U(1)\)-equivariance condition and solving the equation

\footnote{They were called in Ref. \[1\] \(L_t, L_p, L_x\), respectively}
\(\hat{X}_p^L \Psi = 0\) allows us to factor out the common \(\zeta\) and \(p\)-dependence. The remaining equation is a Klein-Gordon-like equation for \(\varphi(x,t)\):

\[
\frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{2mc^2}{\hbar \alpha^2} \frac{\partial \varphi}{\partial t} - \frac{2\omega^2 x}{\hbar \alpha^2} \frac{\partial \varphi}{\partial x} - \frac{\alpha^2 \partial^2 \varphi}{\hbar^2 \alpha^2 \partial x^2} - \frac{m^2 c^4}{\hbar^2} \varphi - \frac{\lambda}{\hbar} m c^2 \omega \varphi + \frac{m^2 c^4}{\hbar^2} \varphi = 0. \quad (4)
\]

In this equation the wave function \(\varphi(x,t)\) has the rest mass subtracted. If we restore it, defining \(\tilde{\varphi} = e^{i mc^2 t} \varphi\), we obtain the more standard expression:

\[
\left( \square + \frac{m^2 c^2}{\hbar^2} + \chi R \right) \tilde{\varphi} = 0, \quad (5)
\]

where \(\square \equiv \frac{1}{c^2 \alpha^2} \frac{\partial^2}{\partial t^2} - \frac{2\omega^2 x}{c^2 \alpha^2} - \alpha^2 \frac{\partial^2}{\partial x^2}\) is the D'Alembert operator on anti-de Sitter space-time (in 1+1D), \(R \equiv -2 \frac{\omega^2}{c^2}\) is the scalar curvature and \(\chi \equiv \frac{N\lambda}{\hbar} \) is a parameter providing the coupling of the scalar field to the gravitational field (see [14]).

The normalized (positive-energy) solutions to the Klein-Gordon-like equation are:

\[
\Psi^{(N,\lambda)}(x,t) \equiv C^{(N,\lambda)}e^{-ibc^2 t}e^{-c_n H^{(N,\lambda)}(x)}, \quad (6)
\]

where \(b_n = c_n = c_0 + n \equiv \frac{1}{2} N + N_n + n \equiv E^{(N,\lambda)}(x)/\hbar \omega\), \(N\lambda \equiv \sqrt{1 + 4N(N - \lambda)}\),

\[
C^{(N,\lambda)} = \sqrt{\frac{\omega}{2\pi}} \left( \frac{m\omega}{\hbar \pi} \right)^{\frac{1}{4}} \frac{1}{2^{n/2}\sqrt{n!}} \frac{\Gamma(2n + 1)(2N)^n}{\Gamma(2n + 1) \sqrt{N} \Gamma(N) \lambda}, \quad (7)
\]

\(N = \frac{mc^2}{\hbar \omega}\), and the Relativistic Hermite polynomials [3] [14], \(H^{(N,\lambda)}\), satisfy:

\[
(1 + \frac{\xi^2}{N}) \frac{d^2}{d\xi^2} H^{(N,\lambda)} - \frac{2}{N} (N\lambda + n - \frac{1}{2}) \xi \frac{d}{d\xi} H^{(N,\lambda)} + \frac{n}{N} (2N\lambda + n) H^{(N,\lambda)} = 0, \quad (8)
\]

where \(\xi \equiv \frac{m\omega}{\hbar} x\) (see [13] for the relation between the Relativistic Hermite Polynomials and the Gegenbauer polynomials).

The wave functions are orthonormal according to the \(t\)-\(x\) scalar product

\[
< \Psi^{(N,\lambda)}_n | \Psi^{(N,\lambda)}_m > = C^{(N,\lambda)} C^{(N,\lambda)} \int dx dt e^{-i(m-n)\omega t} \alpha^{-1+2N\lambda+n+m} \times \notag
H^{(N,\lambda)}_n H^{(N,\lambda)}_m = \delta_{nm}, \quad (9)
\]

the measure of which comes from the invariant volume \(P_0^{-1} dp dx dt\) after a simple but non-trivial regularization of the \(p\)-integration [14].

The annihilation and creation operators (see [14]), when acting on \(\varphi(x,t)\), have the form:

\[
\hat{Z} = \sqrt{\frac{\hbar}{2m\omega}} e^{i\omega t} \left[ \frac{\partial}{\partial x} + i \frac{\omega x}{c^2 \alpha} \frac{\partial}{\partial t} + \frac{m\omega x}{\hbar \alpha} \right] \notag
\]

\[
\hat{Z}^\dagger = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} \left[ -\frac{\partial}{\partial x} + i \frac{\omega x}{c^2 \alpha} \frac{\partial}{\partial t} + \frac{m\omega x}{\hbar \alpha} \right]. \quad (10)
\]
These operators are the adjoint of each other with respect to the scalar product $\langle \rangle$. Their action on (6) is:

\[
\hat{Z} \Psi_{n}^{(N,\lambda)} = \sqrt{\frac{2N_{\lambda} + n}{2N}} \Psi_{n-1}^{(N,\lambda)} \\
\hat{Z}^{\dagger} \Psi_{n}^{(N,\lambda)} = \sqrt{(n+1) \frac{2N_{\lambda} + n + 1}{2N}} \Psi_{n+1}^{(N,\lambda)}.
\] (11)

The representations here obtained belong to the Discrete series $D^{+}(k)$ of $sl(2, \mathbb{R})$ with Bargmann index $k = N = \frac{mc^2}{\hbar \omega}$. Only for half-integer values of $k > \frac{1}{2}$ they exponentiate to a univalued representation of the group $SL(2, \mathbb{R})$ (the rest of the values of $k$ provide, however, univalued representations of the universal covering group of $SL(2, \mathbb{R})$).

A problem arises, however, when one tries to factorize out the time dependence to obtain a minimal representation. The functions $\alpha^{-c_{n}} H_{n}^{(N,\lambda)}$ are no longer orthogonal unless we modify the scalar product in the form $\int d\mathbf{x} \rightarrow \int d\mathbf{x} \alpha^{-2}$ (for a discussion on the non-fully understood modified scalar product see [14]).

With this new scalar product, the normalized wave functions are:

\[
\Psi_{n}^{(N,\lambda)}(x) \equiv C_{n}^{(N,\lambda)} \alpha^{-c_{n}} H_{n}^{(N,\lambda)}(x),
\] (12)

with

\[
C_{n}^{(N,\lambda)} = \sqrt{\frac{\omega^{2}}{2\pi}} \left( \frac{m\omega}{\hbar} \right)^{\frac{1}{2}} \frac{1}{2^{n/2}n!} \frac{\Gamma(2N_{\lambda} + 1)(2N)^{n}}{\Gamma(2N_{\lambda} + n + 1)} \sqrt{\frac{N_{\lambda} + n + \frac{1}{2}}{N_{\lambda} + \frac{3}{2}}} \frac{\Gamma(N_{\lambda} + \frac{3}{2})}{\sqrt{N!}},
\] (13)

Neither the operators $\hat{Z}$ and $\hat{Z}^{\dagger}$ are adjoint to each other with respect to the new scalar product, so they must be appropriately corrected. The relation between the corrected operators and the old ones, when acting on the energy eigenfunctions, is given by:

\[
\hat{Z}' = e^{-i\omega t} \sqrt{\frac{N_{\lambda} + n - \frac{1}{2}}{N_{\lambda} + n + \frac{1}{2}}} \hat{Z} \\
\hat{Z}'^{\dagger} = e^{i\omega t} \sqrt{\frac{N_{\lambda} + n + \frac{3}{2}}{N_{\lambda} + n + \frac{1}{2}}} \hat{Z}^{\dagger}.
\] (14)

These expressions are the generalization to arbitrary $\lambda$ of the ones given in [14] for $\lambda = 1$.

These results suggest that there is a unitary transformation $\hat{U}$ relating the manifestly covariant and the minimal representations, i.e. $\Psi_{n}^{(N,\lambda)}(x) = \hat{U} \Psi_{n}^{(N,\lambda)}(x, t)$. This transformation is of the form:

\[
\hat{U} = e^{\frac{\hbar}{\hbar \omega} \left(E_{n}^{(N,\lambda)} - mc^2\right)} \sqrt{\frac{E_{n}^{(N,\lambda)}}{\hbar \omega N_{\lambda}}},
\] (15)

when acting on energy eigenfunctions. Having into account that the Hamiltonian $\hat{H} \equiv i\hbar \frac{\partial}{\partial t}$ satisfies $\hat{H} \Psi_{n}^{(N,\lambda)}(x, t) = E_{n}^{(N,\lambda)} \Psi_{n}^{(N,\lambda)}(x, t)$, we can write $\hat{U}$ as:

\[
\hat{U} = e^{\frac{i\hbar}{\hbar \omega} \hat{H} - mc^2} \sqrt{\frac{\hat{H}}{\hbar \omega N_{\lambda}}},
\] (16)
when acting on an arbitrary function. With this expression we can obtain the form of the operators $\hat{Z}'$ and $\hat{Z}''$ when acting on arbitrary functions, not only energy eigenfunctions, simply transforming them by $\hat{U}$: $\hat{Z}' = \hat{U} \hat{Z} \hat{U}^{-1}$, $\hat{Z}'' = \hat{U} \hat{Z}^\dagger \hat{U}^{-1}$.

One of the problems with this approach is the lack of a Schrödinger-like equation providing an expression of $\hat{H}$ in terms of $x$ and $\frac{d}{dx}$, since we have only at our disposal the Klein-Gordon-like equation (4) and we would need its “square root”. In the next section we shall obtain an expression for $\hat{H}$ (at low order in $1/N$) in terms of $x$ and $\frac{d}{dx}$ through a Schrödinger-like equation derived from an infinite-order polarization.

3 A Perturbative Approach Involving a Perturbed Scalar Product

Another way of approaching the $t$-factorization problem consists in taking the “square root” of the second-order polarization above, a solution to the conditions defining an (infinite-order) polarization given in a power series $<\tilde{X}_t^\infty, \tilde{X}_p^L>$, where

$$\tilde{X}_t^\infty = \tilde{X}_t^L + \frac{i}{\hbar} \left\{ \sqrt{\hbar^2 \omega^2 N (N - \lambda)} - \hbar^2 c^2 \left[ (\tilde{X}_x^L)^2 + m^2 \omega^2 (\tilde{X}_p^L)^2 \right] - mc^2 \right\}. \quad (17)$$

This infinite-order polarization can be solved order by order to obtain a perturbative expansion for the wave functions. The first-order terms in $1/N$, or, equivalently, in $1/c^2$, for $\tilde{X}_t^\infty$ are (we are taking into account the other polarization equation, $\tilde{X}_p^L \Psi = 0$):

$$\tilde{X}_t^\infty \approx \tilde{X}_t^L + \frac{i\hbar}{2m} (\tilde{X}_x^L)^2 - \frac{i}{4N} \left[ \frac{\hbar^2}{2m^2 \omega} (\tilde{X}_x^L)^4 - \omega (1 - 2\sigma) \right]. \quad (18)$$

Here we have introduced $\lambda \equiv \frac{\sigma}{N}$ since, although the parameter $\lambda$ can take any value, to obtain the correct energy for the non-relativistic Harmonic Oscillator in the limit $c \to \infty$, it has to be of order lower or equal to $1/N$. In particular, the solutions obtained in [1], characterized by $\lambda = 1$, do not satisfy this requirement since the energy eigenvalues in the limit $c \to \infty$ are $E_n = \hbar \omega n$, losing the vacuum energy $\frac{1}{2}\hbar \omega$ which characterizes the quantum fluctuations of the Harmonic Oscillator system.

Once the common $\zeta$- and $p$-dependences have been factorized out, the new polarization gives for $\phi(x, t)$:

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^4}{8m^3 c^2} \frac{\partial^4 \phi}{\partial x^4} - \frac{\hbar^2}{2m} \left( 1 + \frac{3\omega^2 x^2}{2c^2} \right) \frac{\partial^2 \phi}{\partial x^2} - \frac{\hbar^2 \omega^2 x^2}{2mc^2} \frac{\partial \phi}{\partial x} + \frac{1}{2} m \omega^2 x^2 \left( 1 - \frac{\omega^2 x^2}{4c^2} \right) \phi + \frac{\hbar^2 \omega^2 x^2}{4mc^2} (1 - 2\sigma) \phi + O(c^{-4}) \equiv (\hat{H} - mc^2) \phi. \quad (19)$$

(Note that we have substracted the rest mass from the Hamiltonian, in order to get the correct non-relativistic limit). Ordinary perturbation theory, when applied to the
Hamiltonian (19), yields the correct energy \( E_n \approx \hbar \omega (\frac{1}{2} + n) + \frac{\hbar}{8N}(1 - 4\sigma) + O(c^{-4}) \) and eigenfunctions \( \phi_n(x, t) = \exp(-\frac{i}{\hbar}E_n t)\tilde{\phi}_n(x) \), where

\[
\tilde{\phi}_n(x) = \left( \frac{m\omega}{\hbar\pi} \right)^{\frac{1}{2}} \frac{1}{2^{n/2}\sqrt{n!}} \exp(-m\omega x^2/2\hbar) \left( H_n(\xi) + \frac{1}{8N}H_{n+4}(\xi) + H_{n+2}(\xi) + 4n(n-1)H_{n-2}(\xi) - 2n(n-1)(n-2)(n-3)H_{n-4}(\xi) \right) + O(c^{-4}), \tag{20}
\]

for any value of \( \sigma \). The differences among different values of \( \sigma \) come at order \( 1/c^4 \), as far as \( \sigma \) is of order 1. If \( \sigma \) was of order \( c^2 \) (i.e., \( \lambda \) of order 1), then the corrective terms would appear at order 1 in the energy and at order \( 1/c^2 \) in the wave function, as is the case of the solutions considered in [4], as commented before. In this paper we shall restrict ourselves to the case \( \sigma \) of order 1 (\( \lambda \) of order \( 1/c^3 \)), since it reproduces completely the non-relativistic limit, not only the wave functions, but also the energy.

However, \( \int dx\tilde{\phi}_n(x)\tilde{\phi}_m(x) \neq 0 \) for \( m \neq n \), the reason being that the perturbative terms added to the non-relativistic Hamiltonian (19) are not Hermitian. It is important to stress that with this perturbative approach one can see clearly why the energy wave functions are not orthogonal to each other, since we have an explicit expression for the Hamiltonian (although at lower order in powers of \( 1/c^2 \)), and this proves to be non-Hermitian.

We could add corrective terms to the perturbed Hamiltonian in order to make it Hermitian (note that the Hermiticity of \( \hat{H} \) does not depend on the value of \( \sigma \)), but since the expression of \( \hat{H} \) is given \emph{a priori}, from a more general theory\(^2\), we are forced to associate the problem with the scalar product, which is no longer adequate for the perturbed theory. Thus, a perturbation of the scalar product itself is required, and this can be achieved by considering an arbitrary power expansion in \( c^{-2} \) as the perturbed measure, and determining the corresponding coefficients by using the condition that the Hamiltonian be Hermitian at each order. As a result, we get the expression

\[
\int dx\{1 - \omega^2 x^2 c^{-2} + O(c^{-4})\} \tag{21}
\]

for the perturbed measure, restoring in this way the unitarity of the theory. The power series in \( c^{-2} \) we have obtained corresponds to that of \( \int dx\alpha^{-2} \), in agreement with the solution proposed for the exact case in the previous section (see also [14] for a discussion).

The normalized wave functions (according to the perturbed measure) are:

\[
\Phi_n(x) \equiv \left( 1 + \frac{2n+1}{4N} \right)^{-\frac{1}{2}}\phi_n(x) = \phi_n(x) + \frac{1}{16N} \sqrt{(n+1)(n+2)(n+3)(n+4)}\phi_{n+4}(x) + 4\sqrt{(n+1)(n+2)}\phi_{n+2}(x) + 8(n+\frac{1}{2})\phi_n(x) + 4\sqrt{n(n-1)}\phi_{n-2}(x) \tag{22}
\]

\(^2\)The perturbed Hamiltonian has been obtained from a power series expansion for a higher-order polarization, but it could well have been obtained from a Foldy-Wouthuysen transformation of a Dirac equation for the Relativistic Harmonic Oscillator, as it is the case for the corrective relativistic terms of the Hydrogen atom system [10].
\[-\sqrt{n(n-1)(n-2)(n-3)}\phi_{n-4}(x),\]

where \(\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{\exp(-m\omega x^2/2\hbar)}{2^{n/4}\sqrt{n!}} H_n(\xi)\) are the normalized (non-relativistic) Harmonic Oscillator wave functions. It is easy to check that these wave functions coincide, up to \(O(1/c^4)\), with the exact wave functions [12] normalized with respect to the measure \(\int dx/\alpha^2\).

We can easily check that the relation between \(\Phi(x)\) and \(\varphi(x, t)\),

\[\Phi_n(x) = e^{\frac{\hbar}{\kappa} E_n t} \left(1 + \frac{n + \frac{1}{2}}{2N^2}\right) \varphi_n(x, t),\]

(23)
corresponds with the lower order terms of the transformation \(\hat{U}\) given in (13) for energy eigenfunctions. But now, when passing to arbitrary functions in (16), we have an explicit expression, at least at lower order, for the Hamiltonian \(\hat{H}\) in terms of \(x\) and \(\frac{d}{dx}\) given by (19).

We think that the present perturbative approach provides strong support to the idea that Perturbation Theory must be in general modified in order to incorporate "non-Hermitian" corrections, like terms of the form \(x^2 \frac{d^2}{dx^2}\), which appear as a consequence of the non-trivial curvature of the space-time (such as the Anti-de Sitter universe). Furthermore, our particular example also constitutes an important step towards the solution of the general Cauchy problem in non-globally hyperbolic spacetimes [16, 17], the Anti-de Sitter universe being one of the few exactly solvable cases.

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