Short-time dynamic in the Majority vote model: The ordered and disordered initial cases

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This work presents short-time Monte Carlo simulations for the two dimensional Majority-vote model starting from ordered and disordered states. It has been found that there are two pseudo-critical points, each one within the error-bar range of previous reported values performed using fourth order cumulant crossing method. The results show that the short-time dynamic for this model has a dependence on the initial conditions. Based on this dependence a method is proposed for the evaluation of the pseudo critical points and the extraction of the dynamical critical exponent $z$ and the static critical exponent $\beta/\nu$ for this model.

PACS numbers: 64.60.Ht,75.40.-s,05.70.Ln

Keywords:

I. INTRODUCTION

Critical phenomena in equilibrium statistical systems is one of the most important topics in physics. Much of the attention has been focused on the universality of the critical exponents, with several universality classes already characterized in equilibrium systems. On the other hand the critical behavior of non-equilibrium statistical systems has been receiving a lot of attention in recent years, but the characterization of the different universality classes is far from be complete. One of the simplest non-equilibrium models is the two dimensional majority vote model, an Ising-like system (up-down symmetry and spin-flip dynamic) with a continuous order-disorder transition with the same critical exponents that the two dimensional Ising model [1,2], as expected from the prediction of Grinstein et al. [3]: every spin system with spin-flip dynamic and up-down symmetry falls in the Ising model universality class. However, there is some controversy about the universality class for higher dimensions. A recent work claims [4] that the upper critical dimension for the majority vote model is 6 instead of 4, based on numerical calculations. Another discrepancy in the critical exponents have been found in simulations on non-regular lattices [6,7]. It must be mentioned that all of the results mentioned above were performed using standard “Monte Carlo” simulations and Finite Size Scaling approaches for the evaluation of the static critical exponents. On the other hand the time evolution can gives important information about the universality of a given system. It has been shown by Janssen et al. [8] that when systems with relaxation dynamics are quenched from high temperatures to the critical temperature there is a short critical universal behavior. Numerical simulations have confirmed this behavior in the Ising and the Pott models (see reference [9] for a review of these results). Concerning the critical dynamic in systems without detailed balance, there are some works that evaluate the critical dynamic exponent $z$ [11,12] and the fluctuation-dissipation ratio $X_\infty$ [13] for the Majority model, but always starting from a disordered state. As expected the results were compatible with the Ising ones. Given all these results one should expect that the basic assumption for the dynamic relaxation of the k-th moment of the order parameter starting from an completely ordered state will be the same as in the Ising model, but this has not been proved yet.

The aims of this work are: a) to evaluate the critical point using short time dynamic starting from ordered and disordered initial conditions and b) evaluate the dynamic critical exponent $z$ and the static critical exponent $\beta/\nu$. This will test if the Grinstein prediction holds for the short time dynamic in the majority vote model.

II. MODELS AND DEFINITIONS

In the Majority vote model each lattice site has a spin whose values are $\sigma = \pm 1$ and its dynamic can be grouped by the spin flip rule

$$W_i = \frac{1}{2}[1 - \sigma_i f(H_i)].$$

(1)

here $H_i$ is the local field $\sum_{nn} \sigma_j = 0, \pm 2, \pm 4$ produced by its nearest neighbors and $f$ is a function with up down symmetry that depends on two control parameters $f(0) = 0, f(2) = -f(-2) = x$ and $f(4) = -f(4) = y$. The parameters $x$ and $y$ can be associated with interface and bulk temperatures respectively [13] using the relations

$$x = \tanh 2\beta_2, \quad y = \tanh 4\beta_4.$$  

(2)

The Majority model is obtained setting $x = y$ ($\beta_4 < \beta_2$)and the critical point is at $x_c = 0.850(2)$ [2,3].

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The equilibrium case can be obtained along the line $y = 2x/(1 + x^2)$ (Glauber dynamic), where the temperatures are equal ($\beta_2 = \beta_4 = \beta$) and the critical point is $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$.

The order parameter is the standard for Ising-like systems, defined by

$$m = \frac{1}{N} \langle \sum_i \sigma_i \rangle$$

where $N = L^2$ is the total number of lattice sites ($L$ is the lateral size.

Starting from a disordered state the dynamic for the $k$-th moment of the order parameter was derived by Janssen et al [8], the mathematical expression is given by

$$m^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu} m^{(k)}(b^{-z \tau}, b^{1/\nu} \tau, b^{-1} L, b^{x_0} m_0),$$

here $\tau = (T - T_c)/T_c$ is the reduced temperature, or the reduced control parameter for non-equilibrium systems, $t$ is the dynamic time variable, $b$ is the re-scaling factor, $\beta$ and $\nu$ are static critical exponents, $z$ is the dynamic critical exponent, $x_0$ is the scaling dimension of the initial (small) order parameter $m_0$ (see [9] and reference therein).

At the critical point for sufficiently small $m_0$ and large systems ($L \to \infty$) the order parameter follows a power law dynamic

$$m(t) \sim m_0 t^\theta,$$

where $\theta$ is defined by $(x_0 - \beta/\nu)/z$. One must remark that there is a strong dependence on the initial value of the order parameter. This dependence does not affect the power law behavior at the critical point, what is affected is the exponent $\theta$ that tends to the real value in the limit $m_0 \to 0$.

For the dynamic starting, from the ordered state we have the assumption that the scaling dynamic form is given by

$$m^{(k)}(t, \tau, L) = b^{-k\beta/\nu} m^{(k)}(b^{-z \tau}, b^{1/\nu} \tau, b^{-1} L),$$

again it can be obtained the scaling form at the critical point taking the limit $L \to \infty$.

$$M(t) \sim t^{-\beta/\nu z}.$$ (7)

This has been proved in equilibrium systems, like the Ising or the Potts model.

For the evaluation of the critical point it has been used the fact that theoretically the order parameter evolves as a power law at the critical point. If it is evaluated the difference between a power law and the time evolution for different values of the control parameter $x$ we will expect a minimum in those differences at the critical point.

The simulations were performed choosing the lattice site randomly starting from both, a completely ordered state ($m = 1.0$) and a carefully prepared disordered state with small magnetization values $m_0 < 0.1$. The order parameter is evaluated as a function of the time $t$, with $\Delta t = 1$ corresponding to a Monte Carlo time step (MCTS). In order to avoid finite size effects we used lattices with lateral size $L = 2^8$ ($N = L^2$). The average were taken with at least $10^5$ independent simulations and 250 and 1000 MCTS for the evaluation of the critical point and dynamical exponents respectively.

### III. RESULTS

Given that the power law behavior of the order parameter as function of $t$ (decay from an ordered state, increase for a disordered one) one can performed simulations for different values of $x$ around the expected critical point. In Fig. 1 we can observe that above and below a certain value of $x$ we have dynamic that differ from a power law (illustrated by the dashed line).

![Fig. 1](image.png)

From here one must define the criterion that will measure which is the best power law curve, in this work it was used the measurement of the $\chi^2$ for each curve with respect to a power law behavior. I must remark that previous results for equilibrium systems give the same result for the critical point for both initial states. However, two different values were found for the majority vote model, see Fig. 2.

The critical point values obtained were $x_c = 0.85007(6)$ and $x_c = 0.85147(2)$ for the order and disorder state respectively, both results are in perfect agreement with the obtained in the static case (references [1–3]) and are quite close.

We have a clearly dependence on the initial condition in short time dynamic of this model, which is not the
case for equilibrium systems. There is no doubt about the critical point obtained from the ordered phase, since \( m_0 = 1 \) is one of the fixed point under renormalization group transformations. However, one must remember that the power law is valid only in the limit \( m_0 \to 0 \) (the other fixed point) for the disorder case, assuming that the "real" critical point is located at this limit, one can proceed to evaluate the critical point for another values of \( m_0 \) and with this values extrapolate the critical point for the disordered phase. The results are showed in table I.

| \( m_0 \) | 0.0375 | 0.0500 | 0.0625 | 0.0750 | 0.0875 |
|---|---|---|---|---|---|
| \( x_c \) | 0.84920(10) | 0.84972(9) | 0.85019(6) | 0.85076(5) | 0.85147(2) |

Once that each value \( x_c(m_0) \) has been evaluated, the dynamical exponent \( \theta \) can be obtained. For the evaluation of this exponent the simulations were performed with 1000 MCTS discarding the first time steps, since there is an initial time scale \( t_{mic} \) where the power law stabilizes \( t_{mic} \sim 20 \) (see Fig. 3). The results for the exponent \( \theta \) are showed in table II.

| \( m_0 \) | 0.0375 | 0.0500 | 0.0625 | 0.0750 | 0.0875 |
|---|---|---|---|---|---|
| \( \theta \) | 0.1769(8) | 0.1774(4) | 0.1782(3) | 0.1788(4) | 0.1792(3) |

With an extrapolation of these values to \( m_0 = 0 \) the value of the critical point and the \( \theta \) exponent were evaluated (see Fig. 4). The result for the critical point was \( x_c = 0.84860(10) \), which is clearly different from the ordered one, however both values are within the error bar from the obtained in the static case \( 0.848 \leq x_c \leq 0.852 \). From now on the pseudo critical point evaluated with the decay process will be denoted as \( x_c^d \) and the evaluated with the growing process with \( x_c^g \). This surprising result seems similar to the obtained for weak first-order phase transitions [10], where two pseudo-critical points exits due to the metastable states above and below the critical point. However, there is an important difference in this case: for weak order phase transitions the smaller critical point corresponds to the decay process, and the bigger corresponds to the growing process, contrary to the majority vote model case.

In the evaluation of the exponent \( \theta \) a linear extrapolation gives the result of 0.175(3), which is lower from the values of the two dimensional Ising model, \( \theta = 0.191(1) \), and from the previously evaluated in references [11, 12] for the majority vote model, \( \theta = 0.192(2) \). The difference with respect to the Ising model could be understood considering that the results obtained here seems to indicate a hole new dynamic. The differences with previous results for the majority model can be explained observing the simulations details used previously: first the critical point used was \( x = 0.850 \), which is above the result for \( x_c^d \). Second the systems sizes used previously were really small (\( L = 32 \)), at this size the growing process is not very long and is really hard to see the power law behavior.

One can obtain the dynamical exponent \( z \) evaluating the second moment of the magnetization at the critical point \( x_c^d \)

\[
m(2) \sim t^\nu, \quad y = (d - 2\beta/\nu)/z, \tag{8}
\]

and the autocorrelation

\[
A(t) = \sum_i \sigma_i(t = 0)\sigma(t), \quad A(t) \sim t^{-\lambda}, \quad \lambda = \frac{d}{z} - \theta. \tag{9}
\]
Both starting from $m_0 = 0$ and using 1000 MCTS. Again there is a $t_{mic}$ in each case (around 20 for the autocorrelation and 75 for the second moment, see Fig. 5). The results obtained are $y = 0.799(17)$ and $\lambda = 0.758(2)$. Combining both results it can be obtained the values $z = 2.143(9)$ and $\beta/\nu = 0.143(18)$. A summary of the results are showed in Table III, where it can be observed discrepancies between most of the values for the majority model and the Ising ones. It must be remark that all these results were obtained using just the growing process.

Finally the decay exponent $\beta/\nu z$ was evaluated starting from an ordered phase (at $x_c^d$), using 1000 MCTS (Fig. 6), the result was $\beta/\nu = 0.0526(5)$, that is lower compared to the Ising one, 0.0580(5). Again the first time steps were discarded for the evaluation of the exponent (Fig. 6). Theoretically it is possible to obtain the $z$ exponent using the known value of $\beta/\nu$, or knowing the $z$ value one can obtain the $\beta/\nu$ value, but in both cases the results depend on values obtained with a growing process (z) or the static simulations (\beta/\nu). The approach taken in this work is that the growing and the decay process are different and it could be possible that the dynamic exponent $z$ is different in each case, so for the decay case the reporting value is $z = 2.37(2)$.

The fact that we have two pseudo-critical points (that does not corresponds to a weak phase transition) and that the decay and growing process are slower that in the Ising model must be related to the absence of detailed balance condition. One of the consequences of this absence is that we do not have a unique thermodynamic temperature, in this case we have two, so looking at the snapshots for different initial conditions at the pseudo-critical points we can speculate about the competition between the two "temperatures" that governs the dynamic in non-equilibrium Ising systems. Figure 7 shows the time evolution with $m_0 = 0$ at the two pseudo-critical points, a) for $x_c^d$ and b) for $x_c^o$. Initially the number of sites with spin-flip probability depending on $\beta_2$ are very similar to the ones depending on $\beta_4$, the time increases from left to right and it can be observed that at $x_c^o$ the coarsening

|                | Majority vote model | Ising     |
|----------------|---------------------|-----------|
| $\theta$      | 0.175(3)            | 0.191(1)  |
| $\lambda$     | 0.758(2)            | 0.737(1)  |
| $z$            | 2.143(9)            | 2.155(3)  |
| $y$            | 0.799(17)           | 0.817(7)  |
| $\beta/\nu$   | 0.143(18)           | 1/4       |

Fig. 4: a) Evaluation of the critical point $x_c$ and b) the $\theta$ exponent.

Fig. 5: (color online) a) Evaluation of $\lambda$, the continuous line shows the autocorrelation time evolution and the dashed line shows the power law behavior with $\lambda = 0.758$. b) Evaluation of $y$, the continuous line shows the second moment order parameter and the dashed line shows the power law behavior with $y = 0.799$.

Fig. 6: (color online) Order parameter relaxation ($m_0 = 1$), here the dashed line shows the power law behavior with $\beta/\nu = 0.0526$. 

TABLE III: Summary of the results in this work and of the Ising model.
IV. CONCLUSION

In this work it has been shown that the short time dynamic in the majority vote model presents power law behavior at different control parameters for the growing, $x_g^c = 0.84860(10)$, and the decay processes, $x_d^o = 0.85007(6)$. These pseudo-critical points are compatibles with results for the critical point reported previously, $x_c = 0.850(2)$. It has been show also that the dynamic in both cases is slower that in Ising model for all the quantities calculated ($m$, $m^{(2)}$ and $A$). These results seems to be related to the competing dynamic between the interface ($\beta_2$) and bulk ($\beta_4$) temperatures associated to the dynamic, and as consequence to the absence of detailed balance in the system. In order to corroborate these results additional simulations must be carry on in systems without detailed balance. The dynamical critical exponent ($z$) and the static critical exponent ($\beta/\nu$) has been evaluated independently using a growing process, in both cases the results were close to the Ising ones. For the decay process the $z$ exponent was evaluated using results from static simulations founding that the value is different from the obtained in the growing process.

V. ACKNOWLEDGMENTS

I wish to thank G. Pérez for his useful comments. This work was supported by Conacyt México through Grant No. 61418/2007.

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