Spatial-Translation-Induced Discrete Time Crystals

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A discrete time crystal is a phase unique to nonequilibrium systems, where discrete time translation symmetry is spontaneously broken. Most of conventional time crystals proposed so far rely on spontaneous breaking of on-site symmetries and their corresponding on-site symmetry operations. In this Letter, we propose a new time crystal dubbed “spatial-translation-induced discrete time crystal (STI-DTC)”, which is realized by spatial translation and its symmetry breaking. Owing to the properties of spatial translation, in this new time crystal, various time crystal orders can emerge only by changing the filling but not changing the driving protocol. We demonstrate that local transport of charges or spins shows a nontrivial oscillation, enabling detection and applications of time crystal orders. Our proposal opens up a new avenue of realizing time crystal orders by spatial translation.

Introduction.—A system whose Hamiltonian is periodic in time is called Floquet system. The past decade has seen a tremendous growth of interest in such Floquet systems, which produce a variety of phases controlled by periodic driving \[1-8\]. It has also turned out that Floquet systems have novel phases that cannot exist in equilibrium. For example, Floquet systems can realize an intriguing topological phase which hosts chiral edge states even though all the Chern numbers are zero \[9-13\], and this phenomenon is unique to nonequilibrium systems.

One of the most striking phases realized in Floquet systems is a time crystal phase. It has been indeed proved that time crystals cannot exist in thermal equilibrium and are inherent in nonequilibrium \[14\]. Note that several time crystals have already been observed in experiments \[15-18\]. The time crystal is a phase in which time translational symmetry is spontaneously broken. In the case of Floquet systems, this means that the observables have a period of \(2T, 3T, \ldots\) in spite of the Hamiltonian having a period of \(T\), and this phase is called a discrete time crystal (DTC) \[19, 20\]. Moreover, in the time crystal phases, the oscillation frequency of the observables is stabilized by many-body effects and shows robustness against small perturbations.

In most of conventional time crystals proposed so far, symmetry operation and phases brought by many body localization (MBL) or spontaneous symmetry breaking (SSB) are utilized to realize the time crystal orders \[19-25\]. However, among them, only on-site symmetries by finite groups \(\mathbb{Z}_n\) have been focused on, thus leading to a restriction that changing the driving protocol is required to realize different types of time crystal orders.

In this Letter, focusing on spatial translation symmetry, which is a non-local but infinite group symmetry, we propose a new time crystal that is realized by spatial translation and its symmetry breaking, and we thereby provide a feasible platform to realize various kinds of time crystals. First, note that spatial translation symmetry breaking can induce various orders. For example, many kinds of spatial structures are realized in crystals depending on patterns of spontaneous breaking of continuous spatial translation symmetry. Furthermore, a variety of charge density wave (CDW) orders can be realized by changing the filling when discrete spatial translation symmetry is spontaneously broken. Therefore, by utilizing this characteristics, in the new DTC, which is dubbed “spatial-translation-induced DTC” (STI-DTC), various time crystal orders can be realized and controlled without changing the protocol in sharp contrast to the previously proposed time crystals. We further demonstrate that, in STI-DTCs, spatial translation induces local transport and the current shows nontrivial oscillation due to the time crystal orders. This property is characteristic of STI-DTCs, having merits for detection and application of time crystal orders. We also provide some platforms for STI-DTCs such as cold atoms and trapped ions, and especially demonstrate quantum circuits as one of the most promising platforms for STI-DTCs.

Definition and example of DTCs.—First of all, let us clarify the definition of time crystals. Originally, time crystals are introduced as systems where spontaneous time translational symmetry breaking (TTSB) occurs \[20\]. Since we consider only the cases when the Hamiltonian is periodic in time with period \(T\), this means that local observables have a period different from the Hamiltonian’s period \(T\) in its (quasi)steady states. However, the definition characterized only by TTSB is inadequate for time crystals because trivial examples such as Rabi oscillation are also included. In order to preclude such examples and define the time crystal as a stable phase of matter, it should be defined as a phase where not only TTSB occurs but also the period of oscillation is robust to perturbations which do not change the driving period \[20\]. When the period of the local observables is \(nT\), the phase is called \(nT\)-DTC.

Time crystals in Floquet systems are realized in sev-
After every period, the particle density at a certain site oscillates with a double period of the driving. The non-trivial \(2T\)-period oscillation is expected to be stabilized by the CDW order. Therefore, this system would be a \(2T\)-DTC if the assumption were correct.

It is also anticipated that an \(nT\)-DTC (\(n = 2, 3, 4, \ldots\)) is realized if the filling is equal to \(1/n\) since a particle localizes in every \(n\) sites in each of CDW states. From this discussion, STI-DTCs have high controllability of \(n\) unlike other conventional time crystals, so that STI-DTCs can realize various time crystal phases easily by changing the filling.

One important question is how the spatial translation operation can be realized by local Hamiltonians. If any long time can be taken for one period, spatial translation operation is possible in a one-dimensional ring by topological Thouless pumping \([30]\), which is adiabatically performed. It has already been experimentally realized in cold atoms \([31–33]\) and the combination with CDW has been theoretically suggested \([34–37]\). However, topological Thouless pumping, which requires infinite time even for one period, is not suitable for realization of time crystals. In order to overcome this difficulty, we propose below a one-dimensional ladder ring as a candidate of STI-DTCs, which is nonadiabatically realizable (See Fig. 2 (b)).

**Model in 1D ladder.**—Here, we would like to describe how to realize a STI-DTC in a one-dimensional ladder ring. In this model, as shown in Fig. 2 (b), spatial translation \(T_A\) by one site in the sublattice A and the opposite one \(T_B^{-1}\) in the sublattice B are induced after every period. Since the total amount of pumping is zero, the time-dependent Hamiltonian is nonadiabatically realizable by switching local Hamiltonians as follows \([12]\) (See Fig. 2 (a)):}

\[
H(t) = \begin{cases} 
H_1 & (0 \leq t \leq \tau/2) \\
H_2 & (\tau/2 < t \leq \tau) \\
H_{SSB} & (\tau < t < T),
\end{cases}
\]

where each Hamiltonian is defined as

\[
H_1 = -\frac{\pi}{\tau} \sum_i (c_{i,A}^\dagger c_{i,B} + h.c.),
\]

\[
H_2 = -\frac{\pi}{\tau} \sum_i (c_{i+1,A}^\dagger c_{i,B} + h.c.),
\]

\[
H_{SSB} = \sum_{\alpha=A,B} \sum_{i,j} \frac{U_{ij}}{2} n_{i,\alpha} n_{j,\alpha}.
\]

Here, \(c_{i,\alpha}\) and \(n_{i,\alpha}\) respectively represent the annihilation and the number operator of spinless fermions at site \(i\) in a sublattice \(\alpha = A, B\). \(U_{ij}\) represents strength of long-range repulsive interaction and then \(H_{SSB}\) is a Hamiltonian that induces spontaneous spatial translation symmetry breaking at low temperature. As seen below, the time evolution under the Hamiltonians \(H_1\) and \(H_2\) generates the spatial translation \(T_A \otimes T_B^{-1}\) and the time translation symmetric state.
After Step 1 or Step 2, the particle numbers are exchanged between sites linked by green or blue lines. Yellow lines in Step 3 mean repulsive interaction. (b) Dynamics of the model. The period of particle number becomes $2T$ in the case of half-filling.

### Time translation symmetry breaking

Since the explicit Hamiltonian has been constructed, let us confirm that TTSB occurs in the system. How the spatial translation symmetry is broken is prepared as the initial state, then the particle density or the current at each site oscillates with a different period from the Hamiltonian (Fig. 2 (b)). In the case of $1/n$ filling, since particles localize at every $n$ sites, TTSB occurs and $nT$-oscillation is observed.

### Robustness of TTSB

One of the nontrivial properties of time crystals is robustness, that is, the oscillation of the observables is hardly influenced by some small perturbations. In the case of $nT$-time crystals, the highest peak of the Fourier component of the oscillation at $\omega T = 2\pi/n$ does not move nor split even if there is a small perturbation [20].

Assume that a small perturbation is imposed on the duration of the Hamiltonian $H_2$. Then, the time-dependent Hamiltonian $H(t)$ changes to

$$H(t) = \begin{cases} H_1 & (0 \leq t \leq \tau/2) \\ H_2 & (\tau/2 < t \leq (1+r)\tau/2) \\ H_{SSB} & ((1+r)\tau/2 < t \leq T) \end{cases}$$

When $r = 1$, this formula is reduced to the case with no perturbation. The independent parameters of the system are $r$ and $U_{int}T_{SSB}$ where $T_{SSB}$ is equal to $T - (1+r)\tau/2$, which represents the duration of $H_{SSB}$. Since the theorem about prethermalization in [21] is not necessarily applicable to the current system [39], robustness to the perturbation is examined by the exact diagonalization for finite systems. We assume a long-range repulsive interaction $U_{i,j} = U_{int}/r_{i,j}^3$, where $r_{i,j}$ represents the distance between the site $i$ and $j$. Such a long-range interaction is realized in trapped ions and nitrogen-vacancy centers of diamond, which are platforms of DTCs [15, 16].

Figure S2 represents the results when $U_{int}T_{SSB} = 1.0$ in the case of 1. $2 \times 8$ sites at half-filling and II. $2 \times 9$ sites at 1/3-filling. The ground state of $H_{SSB}$ is prepared as the initial state. In both cases, when $r = 1.05$, the oscillation of the particle number hardly decays (See (a)), thus TTSB behavior is robust. On the other hand, when $r = 1.15$, the oscillation rapidly decays and then time crystal order is lost (See (d)). Robustness can be examined also from their Fourier spectrum described by (b) and (e). Each of the peaks at $\omega T/2\pi = 1/2$ and $\omega T/2\pi = 1/3$ corresponds to each of $2T$-oscillation and $3T$-oscillation. It is notable that these peaks do not move from their original positions when $r = 1.05$. This property is unique to time crystal phases.

Lifetime of the time crystal is also calculated for each $r$ and $U_{int}T_{SSB}$ (See Fig. 3). Here, the lifetime is defined as the time when the amplitude of the oscillations becomes

$$n_{i,A}(kT) = n_{i-k,A}, \quad n_{i,B}(kT) = n_{i+k,B}$$

Though its form is slightly different from Eq. (1), by the existence of $U_p$, TTSB can be induced since the spatial translation $\mathbb{T}_A \otimes \mathbb{T}_B^{-1}$ moves particles regardless of $U_p$. In fact, in the Heisenberg picture,

$$n_{i,A}(kT) = n_{i-k,A}, \quad n_{i,B}(kT) = n_{i+k,B}$$

is satisfied for $k \in \mathbb{N}$ [39]. Thus, when a CDW state, where spatial translation symmetry is broken, is prepared as the initial state, then the particle density or the current at each site oscillates with a different period from the Hamiltonian (Fig. 2 (b)). In the case of $1/n$ filling, since particles localize at every $n$ sites, TTSB occurs and $nT$-oscillation is observed.
Form is different from... phenomenon originates from the statistics of fermions. In interaction when the filling is half (See Fig. 4 (a)). This filling. DTCs exist and can realize stable there. From these results, we conclude that STI-DTCs are expected to be re-
alized in various systems since the essential ingredients... lifetime is more than 100^90% of the initial value. Yellow regions indicate that the lifetime is more than 100^T/T, where T represents the driving period.

Note that robustness is also observed even if there is no interaction when the filling is half (See Fig. 4 (a)). This phenomenon originates from the statistics of fermions. In the unperturbed case, where the Floquet operator is described by Eq. [S8], the effective density-density interaction appears in $U_p$ because of the commutation relation of fermion operators. Even in perturbed cases, though the form is different from $U_p$, the effective density-density interaction appears and could stabilize CDW states in the case of half-filling.

Experimental setup.—STI-DTCs are expected to be realized in various systems since the essential ingredients for them are spatial translation and its symmetry breaking. For example, spin or quasispin systems can also realize STI-DTCs by utilizing SWAP gates and antiferromagnetic order [38]. Thus it is also expected that there are various platforms for realizing STI-DTCs.

For example, in trapped ions and Rydberg atoms, long-range interactions which can realize CDW order have been experimentally observed [16, 40]. On the other hand, in cold atoms, selective hopping by $H_1$ and $H_2$ is theoretically proposed by moving optical lattices [12] and long-range interactions can be realized by dipole-dipole interactions and electric fields [41]. Thus, these are candidates for STI-DTCs.

One of the most promising platforms is a quantum circuit as a quantum simulator [42, 43]. Since dynamics of Floquet systems is described by $U_f$, STI-DTCs can be realized once the unitary gate of $U_f$ is prepared. In fact, $U_f$ described by Eq. [S8] is realizable since the time evolution operators under $H_1$ and $H_2$ can be composed of SWAP gates and CPHASE gates, and the one under $H_{SSB}$ can be realized by CPHASE gates [38]. Quantum circuits are or will be realized by various platforms such as superconducting qubits and quantum dots [43–46]. Thus STI-DTCs on quantum circuits will also be realized by various platforms.

Conclusions.—In this paper, a new type of discrete time crystals: spatial-translation-induced time crystals have been proposed. In this time crystal, spatial translation operation changes the system from a CDW state to another CDW state, thus the particle number oscillates with a different period from its Hamiltonian. Unlike almost all the conventional time crystals, time crystal

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FIG. 3. Dynamics of the particle number and its Fourier spectrum in the case of I. $2 \times 8$ sites at half-filling and in the case of II. $2 \times 9$ sites at $1/3$-filling. (a, b) when $r = 1.05$ and (d, e) when $r = 1.15$, respectively. (c) and (f) are enlarged figures of (a) and (d), respectively. In each calculation, the interaction $U_{int}T_{SSB}$ is 1.0 and the duration $r/T$ is 0.1. The peaks at $\omega T/2\pi = 0$ in the spectrum are neglected since they are not related to TTSB behavior. In each situation, the dynamics shown in the figures is the full time evolutions but not the stroboscopic ones.

FIG. 4. Lifetime for each $r$ and $U_{int}T_{SSB}$ is described by the colors. In the yellow regions, lifetime is more than 100^T, where $T$ represents the driving period.
orders in this system can be controlled by changing its filling. One-dimensional ladder ring under periodic drive has been proposed as a promising model of STI-DTCs which is realizable without adiabaticity.

One question left open is to clarify the origin of robustness. In this paper, robustness of the STI-DTCs has been confirmed by numerical calculations. In spite of inapplicability of the theorem about prethermalization in [21], robustness of TTSB behavior has been observed, as seen in other examples [13, 23, 41, 47]. Though it has been qualitatively demonstrated in this paper that the CDW order supports the robustness, its quantitative evaluation by analytical calculation is also desired.

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Supplemental Materials for
“Spatial-Translation-Induced Discrete Time Crystals”

S1. Detailed Calculation for TTSB

A. Floquet operator

Assume that the Hamiltonian is given by

\[
H(t) = \begin{cases} 
H_1 & (0 \leq t \leq \tau/2) \\
H_2 & (\tau/2 < t \leq \tau) \\
H_{SSB} & (\tau < t \leq T)
\end{cases}
\]

(S1)

And

\[
H_1 = -\frac{\pi}{\tau} \sum_i (c_{i,A}^\dagger c_{i,B} + h.c.),
\]

(S2)

\[
H_2 = -\frac{\pi}{\tau} \sum_i (c_{i+1,A}^\dagger c_{i,B} + h.c.),
\]

(S3)

\[
H_{SSB} = \sum_{\alpha=A,B} \sum_{i,j} \frac{U_{ij}}{2} n_{i,\alpha} n_{j,\alpha}.
\]

(S4)

Then, we show that the Floquet operator \(U_f\) is written as follows,

\[
U_f = (\mathbb{T}_A \otimes \mathbb{T}_B^{-1}) \exp(-i H_{SSB}(T - \tau)) U_p,
\]

(S5)

\[
U_p = \exp \left\{ -i \pi \sum_i n_{i,A} (n_{i,B} + n_{i+1,B}) \right\},
\]

(S6)

First, let us consider \(U_1 \equiv \exp(-i H_1 T/2),\)

\[
U_1 = \exp \left( \frac{\pi}{2} \sum_i (c_{i,A}^\dagger c_{i,B} + h.c.) \right)
= \prod_i \left\{ \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{\pi}{2} \right)^n (c_{i,A}^\dagger c_{i,B} + h.c.)^n \right\}.
\]

When the integer \(n\) is larger than 1 and even,

\[
(c_{i,A}^\dagger c_{i,B} + h.c.)^n = (c_{i,A}^\dagger c_{i,B} c_{i,B}^\dagger c_{i,A})^{n/2} + (c_{i,B}^\dagger c_{i,A} c_{i,A}^\dagger c_{i,B})^{n/2}
= n_{i,A}^{n/2} (1 - n_{i,B})^{n/2} + n_{i,B}^{n/2} (1 - n_{i,A})^{n/2}
= n_{i,A}(1 - n_{i,B}) + n_{i,B}(1 - n_{i,A})
= P_{i,A}^B.
\]

The operator \(P_{i,A}^B\) is the projection onto the subspace where \(n_{i,A} + n_{i,B} = 1\). On the other hand, when \(n\) is odd,

\[
(c_{i,A}^\dagger c_{i,B} + h.c.)^n = (c_{i,A}^\dagger c_{i,B} + h.c.) (c_{i,A}^\dagger c_{i,B} + h.c.)^{n-1}
= (c_{i,A}^\dagger c_{i,B} + h.c.) P_{i,A}^B.
\]

Therefore, we obtain \(U_1\) as follows,

\[
U_1 = \prod_i \left\{ Q_{i,A}^B + i P_{i,A}^B (c_{i,A}^\dagger c_{i,B} + h.c.) \right\},
\]

(S7)

where \(Q_{i,A}^B = 1 - P_{i,A}^B\) is the projection onto the subspace where \(n_{i,A} + n_{i,B} = 0\) or \(2\). If the Fock states \(\{|n\}\) is \(|n_{i,A}...n_{N,A}n_{1,B}...n_{N,B}\rangle\) chosen as the basis,

\[
U_1 |\{n\}\rangle = i^S(\{n\}) |n_{1,B}...n_{N,B}n_{1,A}...n_{N,A}\rangle,
\]

(S8)

\[
S(\{n\}) = \sum_i \{ n_{i,A}(1 - n_{i,B}) + n_{i,B}(1 - n_{i,A}) \}
\]

(S9)

where \(S(\{n\})\) represents how many times particles are transferred between sites \((i, A)\) and \((i, B)\) whose particle numbers are different. From these equations, \(U_1\) gives not only an exchange of particles between \((i, A)\) and \((i, B)\) but also the corresponding phase \(i^S(\{n\})\). Similarly, \(U_2 \equiv \exp(-i H_2 T/2)\) exchanges the particle numbers between sites \((i + 1, A)\) and \((i, B)\) for every \(i\) and gives the phase \(i^T(\{n\})\), where

\[
T(\{n\}) = \sum_i \{ n_{i+1,A}(1 - n_{i,B}) + n_{i,B}(1 - n_{i+1,A}) \}.
\]

(S10)

Therefore, \(U_2 U_1\) is calculated as follows (Note that \(U_2\) is performed after \(U_1\), thus \(T(\{n\})\) appears in the form where indices \((i, A)\) and \((i, B)\) are exchanged for every \(i\) :)

\[
U_2 U_1 = (\mathbb{T}_A \otimes \mathbb{T}_B^{-1}) U,
\]

(S11)

where \(U\) is given as

\[
U = \sum_i \{ n_{i,A}(1 - n_{i,B}) + n_{i,B}(1 - n_{i,A})
+ n_{i+1,B}(1 - n_{i,A}) + n_{i,A}(1 - n_{i+1,B}) \}
= 2 \sum_i \{ n_{i,A} + n_{i,B} \}
- 2 \sum_i n_{i,A} n_{i,B} + n_{i+1,B}.
\]

Since \(\sum_i (n_{i,A} + n_{i,B})\) is conserved, this term merely gives a global phase to the state. By removing this term by a proper gauge transformation, the phase term \(i^U\) is derived as follows:

\[
i^U \simeq \exp \left\{ -i \pi \sum_i n_{i,A} (n_{i,B} + n_{i+1,B}) \right\} = U_p.
\]

(S12)

Since the Hamiltonian \(H_{SSB}\) commutes with \(\mathbb{T}_A \otimes \mathbb{T}_B^{-1}\), we obtain the Floquet operator \(U_f\) as Eq. (S5).

B. TTSB

When STI-DTCs are composed of spinless fermions, the additional phase term \(U_p\) is included in the Floquet
operator, which is different from the previous studies. Here, we would like to confirm that the Floquet operator $U_f$ given by Eq. (S20) induces TTSB, even if it includes $U_p$.

For any integer $n (\geq 1)$, the particle number on $(i, A)$ at $t = nT$, $n_{i,A}(nT)$ is given as follows in the Heisenberg picture,

$$n_{i,A}(nT) = U^n f^{-n} n_{i,A} U^n. \quad (S13)$$

In the case of $n = 1$, since $U_{SSB} \equiv \exp(-iH_{SSB}(T - \tau))$ and $U_p$ commute with $n_{j,\alpha}$ for any $j$ and $\alpha = A, B$, $n_{i,A}(T) = U_p^{-1} U_{SSB}^{-1} n_{i-1,A} U_{SSB} U_p$.

$$n_{i,A}(T) = n_{i-1,A}. \quad (S14)$$

is satisfied. By repeating this calculation, we have

$$n_{i,A}(nT) = n_{i-n,A} \quad (S15)$$

for any integer $n \geq 1$. Similarly, the equation for the sublattice $B$ is given by

$$n_{i,B}(nT) = n_{i+n,B}. \quad (S16)$$

Therefore, if the initial state is prepared as a CDW state, the particle number at each site oscillates with the corresponding period, and then TTSB occurs.

### S2. REALIZATION IN SPIN SYSTEMS

#### A. Model and TTSB

In the main text, it has been proved that spontaneous TTSB can occur in spinless fermion systems. In order to explore experimental setups for STI-DTCs, it is important to clarify whether or not they are realizable in other systems. We would like to suggest that STI-DTCs are also realizable in spin systems. Assume that the system is a one-dimensional ladder ring and each site has a spin half. The essential point is to prepare spatial translation $T_A \otimes T_B^{-1}$ and spatial translation symmetry breaking in spin systems. Therefore, the time-dependent Hamiltonian $H(t)$ is assumed to be the form of Eq. (S1), where each Hamiltonian is defined as

$$H_1 = \frac{\pi}{2\tau} \sum_i (1 + \vec{\sigma}_{i,A} \cdot \vec{\sigma}_{i,B}), \quad (S17)$$

$$H_2 = \frac{\pi}{2\tau} \sum_i (1 + \vec{\sigma}_{i+1,A} \cdot \vec{\sigma}_{i,B}), \quad (S18)$$

$$H_{SSB} = J_{int} \sum_{\alpha = A, B} \sum_i \sigma^\alpha_i \sigma_i^{\alpha+1}. \quad (S19)$$

The time evolution operators under the Hamiltonians $H_1$ and $H_2$ are calculated as

$$e^{-iH_1 \tau/2} = \prod_i \chi^{(i,B)}_{(i,A)}, \quad e^{-iH_2 \tau/2} = \prod_i \chi^{(i,B)}_{(i+1,A)} \quad (S20)$$

The time-dependent Hamiltonian $H(t)$ is given by Eq. (S20), induces TTSB, even if it includes $U_p$. Therefore, if the initial state is prepared as a CDW state, the particle number on each site oscillates with the corresponding period, and then TTSB occurs.

#### B. Robustness

In this section, we show that the STI-DTC in spin systems is also robust to small perturbations. Assume that the perturbed Hamiltonian is described by

$$H(t) = \begin{cases} 
H_1 & (0 \leq t \leq \tau/2) \\
H_2 & (\tau/2 < t \leq (1 + r)\tau/2) \\
H_{SSB} & ((1 + r)\tau/2 < t \leq T),
\end{cases} \quad (S24)$$

FIG. S1. Dynamics of the $z$ component of spin at a certain site and its Fourier spectrum (a,b) when $r = 1.07$ and (d,e) when $r = 1.15$. (c) and (f) are enlarged figures of (a) and (d), respectively. In each calculation, the interaction $J_{int}T_{SSB}$ is 0.1.

Here, a global phase is removed by a gauge transformation. The operator $\chi^\delta_{\gamma}$ is defined by

$$\chi^\delta_{\gamma} = \frac{1}{2} (1 + \vec{\sigma}_{\gamma} \cdot \vec{\sigma}_{\delta}), \quad (S21)$$

and exchanges the states between the site $\gamma$ and the site $\delta$, that is,

$$\chi^\delta_{\gamma} |\psi_{\gamma}\rangle_{\gamma} |\psi_{\delta}\rangle_{\delta} = |\psi_{\gamma}\rangle_{\gamma} |\psi_{\delta}\rangle_{\delta}. \quad (S22)$$

Thus, $\chi^\delta_{\gamma}$ is called a SWAP gate. From the same reason as the spinless fermion systems, the time evolution under $H_1$ and $H_2$ generates spatial translation in the sublattice $A$ and opposite translation in the sublattice $B$, thus the Floquet operator $U_f$ is written as

$$U_f \equiv T \exp \left\{ -i \int_0^T H(t) dt \right\} = (T_A \otimes T_B^{-1}) e^{-iH_{SSB}(T - \tau)}. \quad (S23)$$

Therefore, if we prepare an antiferromagnetic state as the initial state, then spontaneous TTSB is induced by this Floquet operator.
where $H_1, H_2$ and $H_{SSB}$ are given by Eqs. (S17), (S18), and (S19). Since SSB under the Hamiltonian $H_{SSB}$ occurs only at zero temperature, the initial state is assumed to be the ground state of $H_{SSB}$, that is, the antiferromagnetic state. Since independent parameters of the system are $r$ and $J_{int}T_{SSB}$, the expectation value of $\sigma_{i,A}^z$ is calculated for each $r$ and $J_{int}T_{SSB}$ by exact diagonalization.

Figure S1 shows the time evolution of the expectation value of $\sigma_{i,A}^z$ (See (a),(c),(d), and (f)) and its Fourier spectrum (See (b) and (d)). All of them are calculated under $J_{int}T_{SSB} = 0.1$ and the number of sites $N = 2 \times 8$. When $r = 1.07$, as shown in Fig. S1 (a),(b), and (c), the oscillation hardly decays and the highest peak of the Fourier spectrum is pronounced and does not move from $\omega T = \pi$. Thus, the time crystal order is maintained in that case. Figure S1 (d),(e), and (f) shows the strongly perturbed case $r = 1.15$. Then, the oscillation rapidly decays and the highest peak at $\omega T = \pi$ becomes much lower. The strong perturbation $r = 1.15$ breaks time crystal order.

The lifetime for each $r$ and $J_{int}T_{SSB}$ in $2 \times 8$ sites is shown in Fig. S2 (i) (c). The definition of lifetime is the same as 2T-DTCs composed of spinless fermions, which has been described in the main text. Lifetime is identified by colors in the graph, and the yellow regions represent the cases when the lifetime is more than 100 periods. Thus, it is concluded that time crystal order is stable within these yellow regions.

S3. SIZE-DEPENDENCE OF TTSB BEHAVIOR

In the main text, the robustness of TTSB behavior has been confirmed for a certain finite system size. It is important to examine the size-dependence of this robustness. The lifetimes for different size systems are also calculated by exact diagonalization, as shown in Fig. S2. Figure S2 (i) represents the case of spin systems, and Fig. S2 (ii) represents the case of half-filled spinless fermion systems. In both cases, the lifetimes look periodic in the strength of the interaction and there are “nodes”, which are points at $r = 1$ where the lifetime dramatically decreases even when $r$ is slightly moved from 1. Moreover, the number of nodes is proportional to the number of sites in each sublattice $L$. This comes from the resonance between the energy from $H_{SSB}$ and the driving frequency $2\pi/T_{SSB}$. Let us consider the case of spin systems. From Fig. S2 (i), the nodes appear at

$$J_{int}T_{SSB} = \frac{n\pi}{2L}, \quad n \in \mathbb{Z}.$$  \hspace{1cm} (S25)

The period of the lifetime in $J_{int}T_{SSB}$ is equal to $\pi/2L$. The ground state of $H_{SSB}$ is prepared at $t = 0$, thus the effect of $\exp(-iH_{SSB}T_{SSB})$ is approximated by $\exp(-iE_GT_{SSB})$ in an early time regime, where the ground state energy $E_G$ of $H_{SSB}$ is equal to $-2LJ_{int}$. When $J_{int}T_{SSB}$ differs by $n\pi/2L$, the change brought by this difference in the Floquet operator $U_f$ is equal to $\exp(i\pi) = \pm 1$, which is expected to be trivial. Therefore, the lifetime is periodic in $J_{int}T_{SSB}$, and since there is no robustness in the case of no interaction, the nodes appear when Eq. (S25) is satisfied. Resonant behavior of STI-DTCs can also be seen from the dynamics at different nodes (Fig. S3 (a) and (b)). In an early time regime, in which the effect of perturbations is small yet, the dynamics when the interaction is fine-tuned by Eq. (S26) are similar to one another.

In the case of spinless fermion systems, since the non-trivial phase term $U_p$ appears and the interaction is long-ranged, conditions for appearance of nodes cannot be simply described as Eq. (S25). It is also notable that there is robustness even when there is no interaction because of the effective density-density interaction brought by $U_p$. However, similar resonance takes place because similarity of the dynamics at different nodes is observed.
as shown in Fig. S3 (c) and (d). Such resonance that spoils robustness when there is a long-range interaction is also observed in S1.

In any size examined here, robustness of TTSB behavior exists. Since the promising experimental resources such as cold atoms, trapped ions, and quantum circuits, have at most $O(10^1)$ sites, we can conclude from the results that robustness of STI-DTCs will be observed in these setups. As a matter of theoretical interest, the behavior in the thermodynamic limit is greatly important, but this problem is left for future work.

S4. REALIZATION BY QUANTUM CIRCUITS

In this section, we describe how to realize STI-DTCs by quantum circuits. In order to realize STI-DTCs, it is enough to prepare the unitary gate equivalent to the Floquet operator $U_f$ since Floquet systems can be simulated by the repetition of the unitary gate $U_f$. In principle, any unitary gate is constructible by unitary operations on 1 qubit and CNOT gates, which is well known as universal quantum computation S2. Here, we describe $U_1 \equiv \exp(-iH_1\tau/2)$, $U_2 \equiv \exp(-iH_2\tau/2)$, and $U_3 \equiv \exp(-iH_{SSB}T_{SSB})$ concretely in the case of spinless fermion systems.

**Unitary gates $U_1$ and $U_2$**

The action of $U_1$, which is given by Eq. (S7), is to swap particles between sites $(i,A)$ and $(i,B)$ for every $i$ and to give the corresponding phase to the state. The latter part can be completed by combining NOT gates and CPHASE gates since giving the phase $e^{i\pi/2}$ only when the total particle number in $(i,A)$ and $(i,B)$ is equal to 1 is required. Therefore, the unitary gate $U_1$ can be constructed by imposing the gates shown in Fig. S4 (a) on sites $(i,A)$ and $(i,B)$ for each $i$. Since the unitary operator $U_2$ acts on sites $(i+1, A)$ and $(i,B)$ for each $i$ in the same way as $U_1$, the unitary gate $U_2$ is also realizable by the gates shown in Fig. S4(a).

**Unitary gate $U_3$**

The unitary operator $U_3$, which is equal to \( \exp \left( -i \sum_{i,j,\alpha} U_{ij} n_{i,\alpha} n_{j,\alpha} / 2 \right) \), represents the effect of the long-range repulsive interaction. Since all of the terms in the summation commute with one another, $U_3$ is realizable by combining unitary gates $U_{ij}^\alpha$ acting on two sites $(i, \alpha)$ and $(j, \alpha)$, where $U_{ij}^\alpha$ is given by

\[
U_{ij}^\alpha = \text{diag}(1, 1, 1, e^{-iU_{ij}T_{SSB}}). \quad (S26)
\]

Here, $\{|0\rangle_{i\alpha} \otimes |0\rangle_{j\alpha}, |0\rangle_{i\alpha} \otimes |1\rangle_{j\alpha}, |1\rangle_{i\alpha} \otimes |0\rangle_{j\alpha}, |1\rangle_{i\alpha} \otimes |1\rangle_{j\alpha}\}$ is chosen as the basis. Thus, the role of the unitary gate $U_{ij}^\alpha$ is to give a certain phase dependent on the interaction $U_{ij}$ only to the component of $|1\rangle_{i\alpha} \otimes |1\rangle_{j\alpha}$ and this is realizable by a CPHASE gate (Fig. S4(b)).
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[S2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (CAMBRIDGE UNIVERSITY PRESS, 2010).