GLOBAL EXISTENCE AND DECAY OF SOLUTIONS FOR HARD POTENTIALS TO THE FOKKER-PLANCK-BOLTZMANN EQUATION WITHOUT CUT-OFF

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Abstract. In this article we study the large-time behavior of perturbative classical solutions to the Fokker-Planck-Boltzmann equation for non-cutoff hard potentials. When the initial data is a small perturbation of an equilibrium state, global existence and temporal decay estimates of classical solutions are established.

1. Introduction and main result. The Fokker-Planck-Boltzmann equation is concerned with the motion of particles in a thermal bath where the bilinear interaction is one of the main characters [2, 4]. In the present paper, we consider the initial value problem of the Fokker-Planck-Boltzmann equation

\[
\partial_t F + v \cdot \nabla_x F = Q(F,F) + \epsilon \nabla_v \cdot (vF) + \kappa \Delta_v F, \quad F|_{t=0} = F_0, \quad (1.1)
\]

where \( F(t,x,v) \), \((t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \) is the distribution function and \( \epsilon, \kappa \) are given nonnegative constants. The bilinear collision operator \( Q(F,G) \) is given by

\[
Q(G,F)(t,x,v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*^*, \sigma)(F'G'_* - FG_*) dv_* d\sigma, \quad (1.2)
\]

where \( F' = F(t,x,v') \), \( F = F(t,x,v) \), \( G'_* = G(t,x,v'_*) \) and \( G_* = G(t,x,v_*) \) for short, and the pairs \((v,v_*')\) and \((v',v'_*)\) stand respectively for the velocities of particles before and after collision, with the following momentum and energy conservation rules fulfilled,

\[
v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.
\]

From the above relations we have the so-called \( \sigma \)-representation, with \( \sigma \in \mathbb{S}^2 \),

\[
\begin{align*}
v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\
v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{align*}
\]

The cross-section \( B(v - v_*, \sigma) \) in (1.2) depends on the relative velocity \( |v - v_*| \) and the deviation angle \( \theta \) with \( \cos \theta = \sigma \cdot (v - v_*)/|v - v_*| \). Without loss of generality we may assume that \( B(v - v_*, \sigma) \) is supported on the set \( \theta \in (0, \pi/2) \) where \( (v - v_*) \cdot \sigma \geq 0 \),

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since as usual $B$ can be replaced by its symmetrized version, and furthermore we may suppose it takes the following form:

$$B(v - v_\ast, \sigma) = |v - v_\ast|^\gamma b(\cos \theta),$$

where $\gamma \in (-3, 1]$.

In this paper, we assume that $\gamma + 2s \geq 0$, $\epsilon = \kappa > 0$ and the angular function $b(\cos \theta)$ is not locally integrable: for $c_b > 0$ it satisfies

$$\frac{c_b}{\theta^{1+2s}} \leq \sin \theta b(\cos \theta) \leq \frac{1}{c_b \theta^{1+2s}}, \ s \in (0, 1).$$

In what follows, we consider the classical solutions to $(1.1)$ near a global Maxwellian $\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$. To this end, we use $f$ to denote the perturbation of $F$ around the Maxwellian $\mu$ as $F = \mu + \sqrt{\mu}f$, then the fluctuation $f$ satisfies the Cauchy problem

$$\begin{cases}
\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f) + \epsilon L_{FP}f, \\
|t| = 0\ f_0 = \mu^{-1/2}(F_0 - \mu).
\end{cases} \tag{1.4}$$

Here, the linear operator $L$, the bilinear form $\Gamma(f, g)$ and the classical linear Fokker-Planck operator $L_{FP}$ are respectively given by

$$L_f = -\mu^{-1/2}Q(\mu, \sqrt{\mu}f) - \mu^{-1/2}Q(\sqrt{\mu}f, \mu),$$

$$\Gamma(f, g) = \mu^{-1/2}Q(\sqrt{\mu}f, \sqrt{\mu}g),$$

$$L_{FP}f = \Delta_x f + \frac{1}{4}(6 - |v|^2)f.$$

It is well known that $L$ is non-negative and the null space $N$ of $L$ is given by

$$N = \text{span}\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, (|v|^2 - 3)\sqrt{\mu}\}. \tag{1.5}$$

Now we define $P$ the orthogonal projection from $L^2(\mathbb{R}^3)$ to $N$. As in [15], for any given function $f$, one can write

$$\begin{cases}
Pf = \{a(t, x) + b(t, x) \cdot v + c(t, x)(|v|^2 - 3)\}\mu^{1/2}, \\
a = \int_{\mathbb{R}^3} \mu^{1/2}f dv, \\
b = \int_{\mathbb{R}^3} v\mu^{1/2}f dv, \\
c = \frac{1}{6} \int_{\mathbb{R}^3} (|v|^2 - 3)\mu^{1/2}f dv = \frac{1}{6} \int_{\mathbb{R}^3} (|v|^2 - 3)\mu^{1/2}Pf dv.
\end{cases} \tag{1.6}$$

Therefore, we have the macro-micro decomposition introduced in [15],

$$f(t, x, v) = Pf(t, x, v) + (I - P)f(t, x, v), \tag{1.7}$$

where $I$ denotes the identity operator, $Pf$ and $(I - P)f$ are called the macroscopic component and the microscopic component of $f$ respectively. As in [9, 11], for later use, one can write $P$ as

$$\begin{cases}
Pf = P_0f \oplus P_1f, \\
P_0f = a(t, x)\mu^{1/2}, \\
P_1f = \{b(t, x) \cdot v + c(t, x)(|v|^2 - 3)\}\mu^{1/2}. \tag{1.8}
\end{cases}$$

Before stating our main results, we list some notations as follows. Throughout this paper, $c_1, c_2, \cdots$ stand for various generic positive constants. $C$ denotes some positive constant (generally large) and $\lambda$ denotes some positive constant (generally
small), where both $C$ and $\lambda$ may take different values in different places. Furthermore $A \lesssim B$ means $A \leq CB$, and $A \gtrsim B$ means $B \lesssim A$. In addition, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. The multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ will be used to record spatial and velocity derivatives respectively. Specifically,

$$\partial^\alpha = \partial_{x^1}^{\alpha_1} \partial_{x^2}^{\alpha_2} \partial_{x^3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$  

Similarly, the notation $\partial^\alpha$ will be used when $\beta = 0$, and likewise for $\partial_\beta$. The length of $\alpha$ is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \leq \alpha$ means that each component of $\alpha'$ is not greater than that of $\alpha$, and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$. Let $\langle \cdot , \cdot \rangle$ denote the standard $L^2$ inner product in $\mathbb{R}^3$, with its $L^2$ norm given by $| \cdot |_2$. $L^2_l$ denotes the weighted space with the norm, for $l \in \mathbb{R}$

$$|f|_{L^2_l}^2 = \int_{\mathbb{R}^3} \langle v \rangle \frac{l}{\langle v \rangle^l} |f(v)|^2 dv,$$

where $\langle v \rangle = \sqrt{1 + |v|^2}$. Moreover, $(\cdot , \cdot )$ is the $L^2$ inner product in $\mathbb{R}_+^3 \times \mathbb{R}_+^3$ with its $L^2$ norm denoted by $| | \cdot |$. For later use, we now list series of notations introduced in [13, 14]. Let $S'(\mathbb{R}^3)$ be the space of the tempered distribution functions. $N^{s,\gamma}$ denotes the weighted geometric fractional Sobolev space

$$N^{s,\gamma} = \{ f \in S'(\mathbb{R}^3); |f|_{N^{s,\gamma}} < +\infty \}$$

with the anisotropic norm

$$|f|^2_{N^{s,\gamma}} = |f|^2_{L^1_0} + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\langle v \rangle \langle v' \rangle)^{\frac{s+\gamma+1}{2}} \frac{|f' - f|^2}{d(v,v')^{3+2s}} 1_{d(v,v') \leq 1} dv' dv.$$  

Here the anisotropic metric $d(v,v')$ measuring the fractional differentiation effects is given by

$$d(v,v') = \sqrt{|v-v'|^2 + \frac{1}{4} |v|^2 - |v'|^2}$$

and $1_A$ is the standard indicator function of the set $A$. In $\mathbb{R}_+^3 \times \mathbb{R}_+^3$, we use $| | \cdot |_{N^{s,\gamma}} = | | \cdot |_{N^{s,\gamma}} |_{L^2(\mathbb{R}^6)}$.

For an integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, its Fourier transform is defined by

$$\hat{f}(k) = T f(k) = \int_{\mathbb{R}^3} e^{-2\pi i x \cdot k} f(x) dx, \ x \cdot k = \sum_{j=1}^3 x_j k_j,$$

for $k \in \mathbb{R}^3$, where $i = \sqrt{-1} \in \mathbb{C}$. For two complex vectors $a, b \in \mathbb{C}^3$, $(a|b) = a \cdot \bar{b}$ denotes the dot product over the complex field, where $\bar{b}$ is the ordinary complex conjugate of $b$.

For $r \geq 1$, we define the mixed Lebesgue space $Z_r = L^2(\mathbb{R}^3; L^r(\mathbb{R}_+^3))$ with the norm

$$||f||_{Z_r} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f(x,v)|^r dx dv \right)^{1/r}.$$  

As in [13], the velocity weight function $w = w(v)$ always denotes $w(v) = \langle v \rangle$. For $l \in \mathbb{R}$, the velocity-weighted $| | \cdot |_{N^{s,\gamma}}$-norm is given by

$$|f|^2_{N^{s,\gamma}} = |w f|^2_{L^2_{l+\gamma}} + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle v \rangle^{\gamma+2s+1} w^2(v) \frac{|f' - f|^2}{d(v,v')^{3+2s}} 1_{d(v,v') \leq 1} dv' dv.$$  

We also define $B_C \subset \mathbb{R}^3$ to be the Euclidean ball of radius $C$ centered at the origin and denote $L^2(\mathbb{R}^3)$ as the space $L^2$ on this ball and likewise for other spaces.
For an integer $K$ and $K \geq 4$. Fix $l \geq 0$. Given a solution $f(t,x,v)$ to the Boltzmann equation (1.4), we define an instant energy functional $E_{K,l}(t)$, which satisfies

$$E_{K,l}(t) \approx \sum_{|\alpha| + |\beta| \leq K} ||w^{l-|\beta|} \partial^\beta f(t)||^2,$$

and the dissipation rate

$$D_{K,l}(t) = \sum_{1 \leq |\alpha| \leq K} ||\partial^\alpha f(t)||^2_{N^\gamma} + \epsilon \lambda_{FP} \sum_{|\alpha| \leq K} ||\{I - P_0\} \partial^\alpha f(t)||^2$$

$$+ \sum_{|\alpha| + |\beta| \leq K} ||\partial^\alpha \{I - P\} f(t)||^2_{N^\gamma_{l-|\beta|}}.$$ (1.10)

Our main result are stated as follows.

**Theorem 1.1.** Fix $l \geq 0$ and $f_0(x,v)$. For $K \geq 4$, if $E_{K,l}(0)$ and $\epsilon$ are sufficiently small, then the Cauchy problem to the Fokker-Planck-Boltzmann equation (1.4) admits a unique classical global solution $f$. Furthermore, if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$, then $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$.

**Theorem 1.2.** Fix $l \geq 0$ and $f_0(x,v)$. For $K \geq 4$. Let $f(t,x,v)$ be the solution to the Cauchy problem (1.4) obtained in Theorem 1.1. If $\epsilon_{K,l}$ and $\epsilon$ are sufficiently small. Then for any $t \geq 0$ we have

$$E_{K,l}(t) \lesssim (1 + t)^{-3/2} \epsilon_{K,l},$$

where $\epsilon_{K,l}$ is defined by

$$\epsilon_{K,l} = E_{K,l}(0) + ||f_0||^2_{2^l}.$$ (1.12)

There has been much rigorous analysis established recently about the Fokker-Planck-Boltzmann equation (1.4). DiPerna and Lions [5] obtained the global existence of the renormalized solutions in the $L^1$ framework. Hamdache [19] proved the global existence near the vacuum state by a direct construction. In [20], Li and Matsumura showed that a strong solution exists globally in time and tends time-asymptotically to an self-similar Maxwellian with respect to time and velocity for a small perturbation of an absolute Maxwellian. Under Grad’s angular cut-off assumption, Xiong, Wang, and Wang [26] established the global classical solution existence and time decay estimates to the equation (1.4) in the perturbation framework.

Recently there have been many important results on the global existence and long time behavior for the Boltzmann equation and other kinetic equations. For the hard-sphere model with angular cutoff, the global existence of solutions to the Vlasov-Poisson-Boltzmann system was proved in [27] and [8] in different function spaces, and the corresponding large-time behavior of solutions were obtained in [28] and [9], respectively. For the hard potential $0 < \gamma < 1$ or hard sphere, we can refer to [6, 15, 16, 21, 22]. For the soft potentials $-3 < \gamma < 0$, there also has been much important progress. Guo [17] used energy methods developed in [16, 18] to generalize the results in [3] to the case $-3 < \gamma < 0$ and obtained the time decay rate of $e^{-\lambda t^p}$ for some $\lambda > 0$ and $0 < p < 1$ in [25]. For $-1 < \gamma < 0$, Ukai and Asano established the global solutions to the Boltzmann equation near a global Maxwellian and the decay rate of $t^{-\alpha}$ with $0 < \alpha < 1$, their optimal case in whole space is $\alpha = 3/4$. Without the angular cut-off assumption, Strain [23] obtained the optimal time decay rates of classical solutions to the Boltzmann equation with hard and soft potentials in the whole space, Duan and Liu [12] established the
global existence and convergence rates of classical solutions to the Vlasov-Poisson-Boltzmann system when initial data is near Maxwellians. Our approach is based on the methods in [10, 11] for the Vlasov-Poisson-Boltzmann system, although their results are in the angular cut-off case.

In this paper, we establish the global classical solution to equation (1.4) by the standard energy method, and obtain the same decay rate as theirs in [26], which is studied using Fourier analysis, the only difference is that the equation is in the context of non-cutoff cross-section. One of the main difficulty lies in the the term \( \langle v \rangle |\partial_{\beta'} (I-P)f| \) generated by the \( v \)-derivatives \( \partial_{\beta'} \) acting on the Fokker-Planck operator, for \( \beta' \in \mathbb{R}^3 \). We use the velocity weight function \( w^{1-|\beta|} \) to control the above term \( \langle v \rangle |\partial_{\beta'} (I-P)f| \), where \( 0 \leq |\beta'| < |\beta| \), with help of the smallness of \( \epsilon \), we can close our energy estimate. Detailed procedures can be illustrated by the following proofs.

The rest of this paper is arranged as follows. In Section 2, we give several lemmas for late use frequently. In Section 3 and Section 4, we give the proof of the global classical solution to the equation (1.4) and the decay rate respectively.

2. Preliminaries. In this section, we give several lemmas which will be frequently used later.

The first lemma is concerned with the lower bound for the linearized collision operator \( L \).

**Lemma 2.1** (Theorem 8.1 in [13]). There exists a constant \( \delta_0 > 0 \) such that

\[
\langle Lf, f \rangle \geq \delta_0 \| (I-P)f \|_{N^s, \gamma}^2.
\]  

The second lemma concerns the coercive interpolation inequalities for \( L \).

**Lemma 2.2** (Lemma 2.6 in [13]). For any multi-indices \( \alpha, \beta, l \geq 0 \), and any small \( \eta > 0 \) there is a positive constant \( C_\eta \) such that

\[
\langle w^{2l-|\beta|} |\partial_\beta Lf, \partial_\beta f| \rangle \geq |\partial_\beta f|_{N^l, \gamma}^2 - \eta \sum_{|\beta_1| \leq \beta} |\partial_{\beta_1} f|_{N^l, \gamma}^2 - C_\eta \| \partial^s f \|_{L^2(B_{C^2})}^2.
\]  

Furthermore, there exists some constant \( C > 0 \) such that

\[
\langle w^{2l} |Lf, f| \rangle \geq |f|_{N^l, \gamma}^2 - C \| f \|_{L^2(B_{C^2})}^2.
\]  

**Remark 1.** The estimate (2.3) indeed holds for any \( l \in \mathbb{R} \) as follows from Lemma 2.4 and Lemma 2.5 in [13].

The following trilinear estimates are the main ones we use below.

**Lemma 2.3** (Theorem 2.1 in [13]). Let \( \gamma + 2s \geq 0 \), then we have

\[
|\{ (g, h), f \} | \lesssim |g|_2 \| h \|_{N^s, \gamma} \| f \|_{N^s, \gamma}.
\]  

Leibniz formula gives

\[
\partial_\beta^2 \Gamma(g, h) = \sum_{\beta_1 + \beta_2 = \beta} \sum_{\alpha_1 \leq \alpha} C_{\alpha_\beta, \alpha_\beta_1}^{\beta_1, \beta_2} \Gamma_{\beta_2} \partial_{\beta_1}^{\alpha_\beta_1} (g, \partial_{\beta_1}^{\alpha_\beta_1} h).
\]  

Here \( C_{\alpha_\beta, \alpha_\beta_1}^{\beta_1, \beta_2} \) is a non-negative constant which is derived from the Leibniz rule. Also, \( \Gamma_{\beta} \) is the bilinear operator with derivatives on the Maxwellian \( \mu \) given by

\[
\Gamma_{\beta}(g, h) = \int_{\mathbb{R}^3} \int_{\beta^2} B(v - v_\ast, \sigma) \mu_{\beta}(v_\ast) \{ g_\ast h' - g_\ast h \} d\sigma dv_\ast.
\]

For this \( \Gamma_{\beta} \), we have
Lemma 2.4 (Proposition 6.1 in [13]). For any multi-index $\beta \in \mathbb{R}^3$, any $t \in \mathbb{R}$. Suppose that $\phi$ are the smooth rapidly decaying basis vectors in (1.7). Then
\[
|\langle w^2 \Gamma_\beta(g, \phi), f \rangle| \lesssim |w^t g|_{L_{x+y}^2} \cdot |f|_{L_{x+y}^2}.
\] (2.6)

Lemma 2.5 ((6.6) in [13]). Let $\alpha + 2s \geq 0$. For any multi-index $\alpha$, any $d^+, d^-, d' \geq 0$ with $d = d^+ - d^-$ and $d' \leq d^-$ we have
\[
|\langle w^2 \Gamma_\beta(g, h), f \rangle| \lesssim |w^{d^+ - d'} g|_{L_{x+y}^2} \cdot |h|_{N_\alpha' \gamma_\alpha} \cdot |f|_{N_\alpha' \gamma_\alpha}.
\] (2.7)

In what follows, we shall write down the macroscopic equations of equation (1.4) by applying the macro-micro decomposition (1.7). For this purpose, we define the high-order moment functions $A = (A_{jm})_{3 \times 3}$ and $B = (B_1, B_2, B_3)$ by
\[
A_{jm}(f) = \langle (v_j v_m - 1) \mu^{1/2}, f \rangle, \quad B_j(f) = \frac{1}{10}(|v|^2 - 5) v_j \mu^{1/2}, f \rangle.
\] (2.8)

Then, as in [9], one can deduce from (1.4) a fluid-type system of equations
\[
\begin{aligned}
\partial_t a + \nabla_x \cdot b &= 0, \\
\partial_t b + \nabla_x (a + 2c) + \nabla_x \cdot A(I - P)f + cb &= 0, \\
\partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{5}{3} \nabla_x \cdot B(I - P)f + 2\epsilon c &= 0,
\end{aligned}
\] (2.9)

and
\[
\begin{aligned}
2\partial_x b_j + 2\partial_c c + 4ec &= -\partial_j A_{jj}(I - P)f + A_{jj}(R + G), \\
\partial_j b_m + \partial_x b_j &= -\partial_j A_{jm}(I - P)f + A_{jm}(R + G), \quad j \neq m, \\
\partial_t B_j(I - P)f + \partial_x c &= B_j(R + G),
\end{aligned}
\] (2.10)

with
\[
R = -v \cdot \nabla_x (I - P)f + \epsilon L_{FP}(I - P)f - L(I - P)f, \quad G = \Gamma(f, f).
\] (2.11)

Now we estimate the higher order functions (2.8) in $L^2$ norm.

Lemma 2.6. Let $\alpha + 2s \geq 0$, $0 \leq j, m \leq 3$. For any $|\alpha| \leq K$, we have
\[
\max\{|\partial^\alpha A_{jm}(I - P)f||_{L^2(R^3)}, |\partial^\alpha B_j(I - P)f||_{L^2(R^3)}\}
\lesssim |\partial^\alpha (I - P)f||_{N_{\alpha' \gamma_\alpha}}.
\] (2.12)

Furthermore, for any $|\alpha| \leq K - 1$, it holds that
\[
\max\{|\partial^\alpha A_{jm}(R)||_{L^2(R^3)}, |\partial^\alpha B_j(R)||_{L^2(R^3)}\}
\lesssim (1 + \epsilon) \sum_{|\alpha_1| \leq |\alpha| + 1} |\partial^\alpha (I - P)f||_{N_{\alpha' \gamma_\alpha}}.
\] (2.13)

Proof. Since the velocity can be absorbed by the global Maxwellian $\mu$, which exponentially decays in $v$, it is straightforward to estimate (2.12).

Now we estimate (2.13). It is easy to confirm that:
\[
|\partial^\alpha A_{jm}(v \cdot \nabla_x (I - P)f)||_{L^2(R^3)} \lesssim \sum_{|\alpha_1| \leq |\alpha| + 1} |\partial^\alpha (I - P)f||_{N_{\alpha' \gamma_\alpha}},
\]
\[
|\partial^\alpha A_{jm}(\epsilon L_{FP}(I - P)f)||_{L^2(R^3)} \lesssim \epsilon \sum_{|\alpha_1| \leq |\alpha| + 1} |\partial^\alpha (I - P)f||_{N_{\alpha' \gamma_\alpha}}.
\]

Recall that $L \mu = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu})$, using (2.4), we have
\[
\partial^\alpha A_{jm}(L(I - P)f) = \langle (v_j v_m - 1) \mu^{1/2}, L(\partial^\alpha (I - P)f) \rangle \lesssim |\partial^\alpha (I - P)f||_{N_{\alpha' \gamma_\alpha}}.
\]
With the above estimates, we get
\[ \| \partial^\alpha A_{jm}(R) \|_{L^2(\mathbb{R}_3^2)} \lesssim (1 + \epsilon) \sum_{|\alpha_1| \leq |\alpha| + 1} \| \partial^{\alpha_1}(\{I - P\} f) \|_{N^{-\gamma}}. \]

The other one shares similar arguments above.

**Lemma 2.7.** Let \( \gamma + 2s \geq 0, \ 1 \leq j, m \leq 3 \). For any \( |\alpha| \leq K, K \geq 4, l \geq 0 \), it holds that
\[ \max\{\| \partial^\alpha A_{jm}(G) \|_{L^2(\mathbb{R}_3^2)}, \| \partial^\alpha B_j(G) \|_{L^2(\mathbb{R}_3^2)} \} \lesssim E_{K,t}(t) \mathcal{D}_{K,t}(t). \]  
(2.14)

**Proof.** We just need to prove one of them above. Recall that
\[ \partial^\alpha A_{jm}(G) = (v_j v_m - 1) \mu^{1/2}, \partial^{\alpha_1} f, \partial^{\alpha_1} f). \]

We know that
\[ \Gamma(f, f) = \Gamma(P f, P f) + \Gamma((I - P) f, P f) + \Gamma(f, \{I - P\} f). \]

Firstly we estimate (2.14) for the third term \( \Gamma(f, \{I - P\} f) \) above. Use Leibniz formular in (2.5) and apply Lemma 2.5 (hence we let \( d^+ = l, d^- = d' = 0 \)) to obtain, for \( l \geq 0 \),
\[ \left| \langle \partial^{\alpha_1} f, \{I - P\} f \rangle \right| \leq \int_{\mathbb{R}_3^2} \left| \partial^{\alpha_1} f \right|^2 dx \]
\[ \lesssim \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}_3^2} \left| \partial^{\alpha_1} f \right|^2 \left| \partial^{\alpha_1} \{I - P\} f \right|^2 dx. \]

Now take the supremum of either \( \| w^j \partial^{\alpha_1} f \|_{L_{2}^2} \) or \( \| \partial^{\alpha_1} \{I - P\} f \|_{N^{-\gamma}}, \) whichever contains the minimal total number of derivatives, and use the Sobolev embedding \( H^2(\mathbb{R}_3^2) \hookrightarrow L^\infty(\mathbb{R}_3^2) \) for this term to obtain (2.14).

For the second term in (2.15), we notice that
\[ \Gamma((I - P) f, P f) = \sum_{i=1}^{3} \psi_i(t, x) \Gamma((I - P) f, \phi_i), \]
where the \( \psi_i(t, x) \) are the elements from (1.6) and the \( \phi_i \) are the velocity basis vectors in (1.5). Thus from Lemma 2.4
\[ \int_{\mathbb{R}_3^2} \left| (v_j v_m - 1) \mu^{1/2}, \partial^{\alpha} \Gamma((I - P) f, P f)) \right|^2 dx \]
\[ \lesssim \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}_3^2} \left| w^j \partial^{\alpha_1} (I - P) f \right|^2 \left| \partial^{\alpha_1} [a, b, c] \right|^2 dx. \]

Here \( \| [a, b, c] \| \) is just the Euclidean square norm of the coefficients from (1.4). Use the same arguments above to get (2.14).
The last case to consider is $\Gamma(Pf, Pf)$. Applying Lemma 2.4 again
\[
\int_{\mathbb{R}^3} \left| \langle v_j v_m - 1 \rangle \mu^{1/2} \partial^\alpha \Gamma(Pf, Pf) \right|^2 dx \leq \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} |\partial^{\alpha_1} [a, b, c]|^2 |\partial^{\alpha - \alpha_1} [a, b, c]|^2 dx.
\]
Now if $|\alpha| > 0$ then we again use $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ to get (2.14).

However, when $|\alpha| = 0$, we combine the $L^{2^*}(\mathbb{R}^3)$ gradient Sobolev inequality, for $2^* = 6$ is the Sobolev conjugate of 2, with $W^{1, 2^*}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ to obtain
\[
\|f\|_{L^2(\mathbb{R}^3)}^2 \lesssim \|f\|_{W^{1, 2^*}(\mathbb{R}^3)} \lesssim \|\nabla_x f\|^2_{L^2(\mathbb{R}^3)}.
\]
(2.16)

Now take the $L^\infty(\mathbb{R}^3)$ norm of $[a, b, c]$, use (2.16) to get (2.14).

With the estimates to the higher order functions above, we state the key estimates on the macroscopic dissipation in the following Proposition.

**Proposition 1.** Suppose that $K \geq 4$. Then there is an interactive energy functional $E_{int}(t)$ such that
\[
|E_{int}(t)| \lesssim \sum_{|\alpha| \leq K} \|\partial^\alpha f(t)\|^2
\]
and
\[
\frac{d}{dt} E_{int}(t) + \lambda \sum_{|\alpha| \leq K-1} \|\partial^\alpha \nabla_x (a, b, c)\|^2_{L^2(\mathbb{R}^3)} \leq (1 + \epsilon^2) \sum_{|\alpha| \leq K} \|\{I - P\} \partial^\alpha f\|^2_{L^2(\mathbb{R}^3)} + \epsilon^2 \sum_{|\alpha| \leq K-1} \|\partial^\alpha (b, c)\|^2_{L^2(\mathbb{R}^3)} + E_{K,t}(t) \mathcal{D}_{K,t}(t),
\]
(2.18)
where $E_{int}(t)$ is the linear combination of the following terms over $|\alpha| \leq K - 1$ and $1 \leq j \leq 3$:
\[
\mathcal{J}_a^a(t) = (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3)},
\]
\[
\mathcal{J}_a^{b,j}(t) = \left( \frac{1}{2} \sum_{m \neq j} \partial_{x_m} \partial^\alpha A_{mm}(\{I - P\} f) - \sum_m \partial_{x_m} \partial^\alpha A_{jm}(\{I - P\} f), \partial^\alpha b_j \right)_{L^2(\mathbb{R}^3)} ,
\]
\[
\mathcal{J}_c^{a,j}(t) = (\partial^\alpha B_j(\{I - P\} f), \partial_{x_j} \partial^\alpha c)_{L^2(\mathbb{R}^3)}.
\]

**Proof.** Recall $a, b, c$ in (1.6), one has, for any $\zeta \in \mathbb{N}^3$,
\[
\|\partial_\zeta^c (a, b, c)\|_{L^2(\mathbb{R}^3)} \leq \|\partial_\zeta f\|.
\]
Applying (2.12) and the above estimate, we can easily deduce (2.17).

Next we begin to prove (2.18).

**step 1.** Estimate on $b$. For any $\eta > 0$, one has
\[
\frac{d}{dt} \sum_{|\alpha| \leq K-1} \sum_j \mathcal{J}_a^{b,j}(t) + \frac{1}{2} \sum_{|\alpha| \leq K-1} \|\partial^\alpha \nabla_x b\|^2_{L^2(\mathbb{R}^3)} \leq C\eta \sum_{|\alpha| \leq K-1} \|\partial^\alpha \nabla_x (a, c)\|^2_{L^2(\mathbb{R}^3)} + C\eta \sum_{|\alpha| \leq K-1} \epsilon^2 \|\partial^\alpha b\|^2_{L^2(\mathbb{R}^3)} + C\eta \|\partial^\alpha (\{I - P\} f)\|^2_{L^2(\mathbb{R}^3)} + C\eta E_{K,t}(t) \mathcal{D}_{K,t}(t).
\]
(2.19)
Indeed, for fixed $j \in \{1, 2, 3\}$, we use (2.10) to achieve

$$\Delta_x b_j - \partial_x \partial_x b_j = - \partial_t \left[ \frac{1}{2} \sum_{m \neq j} \partial_x A_{mmm} (\{I - P\} f) - \sum_m \partial_x A_{jm} (\{I - P\} f) \right]$$

(2.20)

$$+ \frac{1}{2} \sum_{m \neq j} \partial_x A_{mm} (R + G) - \sum_m \partial_x A_{jm} (R + G).$$

Let $|\alpha| \leq K - 1$. Apply $\partial^\alpha$ to (2.20), then take the $L^2(\mathbb{R}^3_x)$ inner product between it and $\partial^\alpha b_j$ to get

$$\frac{d}{dt} \eta_{\alpha,j}(t) + ||\nabla_x \partial^\alpha b_j||^2_{L^2(\mathbb{R}^3_x)} + ||\partial_x \partial^\alpha b_j||^2_{L^2(\mathbb{R}^3_x)}$$

$$= \left( \frac{1}{2} \sum_{m \neq j} \partial_x \partial^\alpha A_{mm} (\{I - P\} f) - \sum_m \partial_x \partial^\alpha A_{jm} (\{I - P\} f, \partial^\alpha \partial_t b_j) \right)_{L^2(\mathbb{R}^3_x)}$$

$$+ \left( \frac{1}{2} \sum_{m \neq j} \partial_x \partial^\alpha A_{mm} (R + G) - \sum_m \partial_x \partial^\alpha A_{jm} (R + G, \partial^\alpha b_j) \right)_{L^2(\mathbb{R}^3_x)}$$

$$= I_1^b + I_2^b.$$ (2.21)

Use (2.9)2 (the second equation of (2.9)) to replace $\partial_t b_j$ to obtain

$$I_1^b \leq \eta ||\partial^\alpha \partial_t b_j||^2_{L^2(\mathbb{R}^3_x)} + C_\eta \sum_{|\alpha| \leq K} ||\partial^\alpha \chi (\{I - P\} f)||^2_{L^2(\mathbb{R}^3_x)}$$

$$\leq 4\eta \{ ||\nabla_x (a, c)||^2_{L^2(\mathbb{R}^3_x)} + c^2 \} ||\partial^\alpha b_j||^2_{L^2(\mathbb{R}^3_x)} + C_\eta \sum_{|\alpha| \leq K} ||\partial^\alpha \chi (\{I - P\} f)||^2_{N, \gamma},$$

where we have used (2.12).

For $I_2^b$, using (2.13) and (2.14) and integrating by parts yields

$$I_2^b = \left( \frac{1}{2} \sum_{m \neq j} \partial^\alpha A_{mm} (R + G, \partial_x \partial^\alpha b_j) \right)_{L^2(\mathbb{R}^3_x)}$$

$$+ \sum_m \left( \partial^\alpha A_{jm} (R + G, \partial_x \partial^\alpha b_j) \right)_{L^2(\mathbb{R}^3_x)}$$

$$\leq \frac{1}{2} ||\nabla_x \partial^\alpha b_j||^2_{L^2(\mathbb{R}^3_x)} + C \sum_m ||\partial^\alpha A_{jm} (R, G)||^2_{L^2(\mathbb{R}^3_x)}$$

$$\leq \frac{1}{2} ||\nabla_x \partial^\alpha b_j||^2_{L^2(\mathbb{R}^3_x)} + C(1 + c^2) \sum_{|\alpha| \leq K} ||\partial^\alpha \chi (\{I - P\} f)||^2_{N, \gamma} + C \varepsilon_{KJ}(t) \mathcal{D}_{KJ}(t).$$

Thus, Combining the above two estimates for $I_1^b, I_2^b$ with (2.21) and then taking summation over $1 \leq j \leq 3$ and $|\alpha| \leq K - 1$, we can obtain (2.19).

**step 2.** Estimate on $c$. For any $\eta > 0$, it holds that

$$\frac{d}{dt} \sum_{|\alpha| \leq K - 1} \sum_j \eta_{\alpha,j}(t) + \frac{1}{2} \sum_{|\alpha| \leq K - 1} ||\partial^\alpha \nabla_x c||^2_{L^2(\mathbb{R}^3_x)}$$

$$\leq C\eta \sum_{|\alpha| \leq K - 1} ||\partial^\alpha \nabla_x c||^2_{L^2(\mathbb{R}^3_x)} + C\eta \sum_{|\alpha| \leq K - 1} c^2 ||\partial^\alpha c||^2_{L^2(\mathbb{R}^3_x)}$$

$$+ C\eta (1 + c^2) \sum_{|\alpha| \leq K} ||\partial^\alpha (\{I - P\} f)||^2_{N, \gamma} + C\eta \varepsilon_{KJ}(t) \mathcal{D}_{KJ}(t).$$ (2.22)
In fact, for $|\alpha| \leq K - 1$, applying $\partial^\alpha$ to the macroscopic equation (2.10)_3, multiplying it by $\partial_x \partial^\alpha c$ and then integrating it over $\mathbb{R}^3_x$ to get

\[
\frac{d}{dt} \sigma^\alpha_{i,j}(t) + ||\partial_x \partial^\alpha c||^2_{L^2(\mathbb{R}^3_x)} = \left( \partial^\alpha B_j(\{I - P\} f), \partial_t \partial_x \partial^\alpha c \right)_{L^2(\mathbb{R}^3_x)} + \left( \partial^\alpha B_j(\mathcal{R} + G), \partial_x \partial^\alpha c \right)_{L^2(\mathbb{R}^3_x)}
\]

= $I_1^\alpha + I_2^\alpha$.

For $I_1^\alpha$, use (2.9) to replace $\partial_t c$ to obtain

\[
I_1^\alpha = - \left( \partial_x \partial^\alpha B_j(\{I - P\} f), \partial_t \partial^\alpha c \right)_{L^2(\mathbb{R}^3_x)} - \eta ||\partial^\alpha \partial_t c||^2_{L^2(\mathbb{R}^3_x)} + C_\eta ||\partial_x \partial^\alpha B_j(\{I - P\} f)||^2_{L^2(\mathbb{R}^3_x)} - \eta ||\partial^\alpha \nabla_x b||^2_{L^2(\mathbb{R}^3_x)} + 4\epsilon^2 ||\partial^\alpha c||^2_{L^2(\mathbb{R}^3_x)} + C_\eta \sum_{|\alpha| \leq K} ||\partial^\alpha_0(\{I - P\} f)||^2_{N,\gamma}.
\]

Using (2.13) and (2.14), we compute

\[
I_2^\alpha \leq \frac{1}{2} ||\partial_x \partial^\alpha c||^2_{L^2(\mathbb{R}^3_x)} + C ||\partial^\alpha B_j(\mathcal{R} + G)||^2_{L^2(\mathbb{R}^3_x)} - \frac{1}{2} ||\partial_x \partial^\alpha c||^2_{L^2(\mathbb{R}^3_x)} + C(1 + \epsilon^2) \sum_{|\alpha| \leq K} ||\partial^\alpha_0(\{I - P\} f)||^2_{N,\gamma} + C E_{KL}(t) D_{KL}(t).
\]

Therefore, (2.22) follows by plugging (2.24) and (2.25) into (2.23), and summing it over $1 \leq j \leq 3$ and $|\alpha| \leq K - 1$.

**Step 3.** Estimate on $a$. Let $|\alpha| \leq K - 1$. Apply $\partial^\alpha$ to (2.9)_2, multiply it by $\partial^\alpha \nabla_x a$ and then integrate it over $\mathbb{R}^3_x$ to get

\[
\partial_t (\partial^\alpha b, \partial^\alpha \nabla_x a)_{L^2(\mathbb{R}^3_x)} + ||\partial^\alpha \nabla_x a||^2_{L^2(\mathbb{R}^3_x)} = -2 \partial^\alpha \nabla_x c + \partial^\alpha \nabla_x A(\{I - P\} f) + c \partial^\alpha b, \partial^\alpha \nabla_x a \right)_{L^2(\mathbb{R}^3_x)} + \left( \partial^\alpha b, \partial_t \partial^\alpha \nabla_x a \right)_{L^2(\mathbb{R}^3_x)}
\]

\[
\frac{d}{dt} \sum_{|\alpha| \leq K - 1} (\partial^\alpha b, \partial^\alpha \nabla_x a)_{L^2(\mathbb{R}^3_x)} + \frac{1}{2} \sum_{|\alpha| \leq K - 1} ||\partial^\alpha \nabla_x a||^2_{L^2(\mathbb{R}^3_x)} \lesssim \sum_{|\alpha| \leq K - 1} ||\partial^\alpha \nabla_x (b, c)||^2_{L^2(\mathbb{R}^3_x)} + \sum_{|\alpha| \leq K - 1} c^2 ||\partial^\alpha b||^2_{L^2(\mathbb{R}^3_x)} + \sum_{|\alpha| \leq K} ||\partial^\alpha(\{I - P\} f)||^2_{N,\gamma}.
\]

**Step 4.** Combination. For $M > 0$ to be determined later, a appropriate linear combination gives

\[
M \times (2.19) + M \times (2.22) + (2.26).
\]
We choose $M$ sufficiently large such that the first term on the right hand side of (2.26) can be absorbed by the dissipation of $b$ and $c$. Fixing this $M > 0$, then we choose $\eta > 0$ sufficiently small such that the first terms on the right hand side of (2.19) and (2.22) are absorbed by the full dissipation of $b$ and $c$. Thus, we have proved (2.18).

Lastly, for the macroscopic components, we have two results. Note that in (1.6), we easily get the following lemma, the details are omitted for simplicity.

**Lemma 2.8.** Let $\gamma + 2s \geq 0$. Then

$$
\sum_{|\alpha| \leq K-1} ||\nabla_x P \partial^\alpha f(t)||_{N^{s,\gamma}}^2 \approx \sum_{|\alpha| \leq K-1} ||\nabla_x P \partial^\alpha(a, b, c)(t)||_{L^2(\mathbb{R}^3)}^2.
$$

(2.27)

**Lemma 2.9.** For $\alpha \in \mathbb{N}^3$, $|\alpha| \leq K$. Then

$$
||\partial^\alpha(b, c)||_{L^2(\mathbb{R}^3)}^2 \lesssim ||(I - P) \partial^\alpha f||_{N^{s,\gamma}}^2 + ||\{I - P_0\} \partial^\alpha f||^2.
$$

(2.28)

**Proof.** Applying (1.8), direct computation shows

$$
||\partial^\alpha b||_{L^2(\mathbb{R}^3)} \lesssim ||P_1 \partial^\alpha f||, \quad ||\partial^\alpha c||_{L^2(\mathbb{R}^3)} \lesssim ||P_1 \partial^\alpha f||.
$$

Thus

$$
||\partial^\alpha(b, c)||_{L^2(\mathbb{R}^3)}^2 \lesssim ||P_1 \partial^\alpha f||^2 \lesssim ||(I - P) \partial^\alpha f||_{N^{s,\gamma}}^2 + ||\{I - P_0\} \partial^\alpha f||^2.
$$

□

**Remark 2.** Recall that $\mathcal{E}_{int}(t)$ in Proposition 1. With (2.18), (2.27) and (2.28), we get the important estimate to be used in the next section.

$$
\frac{d}{dt} \mathcal{E}_{int}(t) + \lambda \sum_{|\alpha| \leq K-1} ||\nabla_x P \partial^\alpha f(t)||_{N^{s,\gamma}}^2 \\
\lesssim (1 + c^2) \sum_{|\alpha| \leq K} ||(I - P) \partial^\alpha f||_{N^{s,\gamma}}^2 + c^2 \sum_{|\alpha| \leq K-4} ||(I - P_0) \partial^\alpha f||^2 + \mathcal{E}_{K,l}(t) \mathcal{D}_{K,l}(t).
$$

(2.29)

3. **global existence.** The following short time existence of classical solution to (1.4) can be established by performing the standard arguments as in [13, 20, 17]. Here we omit the details.

**Lemma 3.1.** For any $l \geq 0$, $K \geq 4$, there exist $\delta > 0$ and $T > 0$ such that if $\mathcal{E}_{K,l}(0) \leq \delta$, there exist a constant $C_0 > 0$ and a unique classical solution $f(t, x, v)$ to equation (1.4) in $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$
\sup_{0 \leq t \leq T} \mathcal{G}(f(t)) \leq C_0 \mathcal{E}_{K,l}(0).
$$

(3.1)

Here, $\mathcal{G}(f(t))$ is defined by

$$
\mathcal{G}(f(t)) = \mathcal{E}_{K,l}(t) + \int_0^t \mathcal{D}_{K,l}(\tau) d\tau.
$$

Moreover, if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$, then $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$ with $0 \leq t \leq T$. 

Based on the above local existence result, we can establish the global existence for equation (1.4). Now we give some estimates for the linearized Fokker–Planck operator $L_{FP}$ and the collision operators $\Gamma$.

The first one is concerned with the dissipative property of the Fokker–Planck operator $L_{FP}$ without weight, which has been proved in [7, 1].

**Lemma 3.2.** $L_{FP}$ is a linear self-adjoint operator with respect to the duality induced by the $L^2$-scalar product. Furthermore, there exists a constant $\lambda_{FP} > 0$ such that

$$-(f, L_{FP}f) \geq \lambda_{FP}||\{I - P_0\}f||_{L^2_{\gamma}}^2.$$  (3.2)

The second lemma concerns the dissipative property of the Fokker–Planck operator $L_{FP}$ with weight.

**Lemma 3.3.** Let $\gamma + 2s \geq 0$. Then

$$\langle L_{FP}\partial_\beta^\gamma f, w^{2(l-|\beta|)}\partial_\beta^\gamma f \rangle \leq -\lambda_{FP} ||\{I - P_0\}(w^{l-|\beta|}\partial_\beta^\gamma f)||^2 + C||\partial_\beta^\gamma f||_{N_{\gamma}^{l-|\beta|}}^2.$$  (3.3)

**Proof.** Integrating by parts yields

$$\left(L_{FP}\partial_\beta^\gamma f, w^{2(l-|\beta|)}\partial_\beta^\gamma f \right) - \left(L_{FP}(w^{l-|\beta|}\partial_\beta^\gamma f), w^{l-|\beta|}\partial_\beta^\gamma f \right)$$

$$= - \left(\nabla_v \cdot (\partial_\beta^\gamma f \nabla_v w^{l-|\beta|}) + \nabla_v w^{l-|\beta|} \cdot \nabla_v \partial_\beta^\gamma f, w^{l-|\beta|}\partial_\beta^\gamma f \right)$$

$$= \left(\nabla_v w^{l-|\beta|} \partial_\beta^\gamma f, \nabla_v (w^{l-|\beta|}\partial_\beta^\gamma f) \right) - \left(\nabla_v w^{l-|\beta|} \cdot \nabla_v \beta^\gamma f, w^{l-|\beta|}\partial_\beta^\gamma f \right)$$

$$= ||\nabla_v w^{l-|\beta|} \partial_\beta^\gamma f||^2 \leq C||w^{l-|\beta|}\partial_\beta^\gamma f||_{L^2_{\gamma} + 2s(\mathbb{R}^n)}^2 \leq C||\partial_\beta^\gamma f||_{N_{\gamma}^{l-|\beta|}}^2.$$  

Here, we have used $\gamma + 2s \geq 0$ and the fact that

$$||\nabla_v w^{l-|\beta|}|| = \left|(I - |\beta|)w^{l-|\beta|} \frac{v}{1 + |v|^2} \right| \leq w^{l-|\beta|}.$$  

Applying Lemma 3.2 and the above estimate, the lemma is proved.  

The following Lemma is concerned with weighted estimates on the nonlinear collision operator $\Gamma$. Using (3.1), (3.5) in [23], we have

**Lemma 3.4.** Let $\alpha, \beta \in \mathbb{R}^3$, $|\alpha| + |\beta| \leq K$, $K \geq 4$. Then

$$|\langle \partial^{\alpha} \Gamma(f, f), \partial^{\beta} f \rangle| \lesssim \sqrt{E_K(t)} D_K(t),$$  (3.4)

where $E_K(t) = E_{K,0}(t)$ and $D_K(t)$ is given by

$$D_K(t) = \sum_{1 \leq |\alpha| \leq K} ||\partial^{\alpha} f(t)||_{N_{\gamma}\gamma}^2 + \sum_{|\alpha| + |\beta| \leq K} ||\partial_\beta^\gamma \{I - P\} f(t)||_{N_{\gamma}^{l-|\beta|}}^2.$$  

Furthermore, for any $l \geq 0$,

$$|\langle w^{2l-2|\beta|}\partial_\beta^\gamma \Gamma(f, f), \partial_\beta^\gamma \{I - P\} f \rangle| \lesssim \sqrt{E_{K,1}(t)} D_{K,1}(t).$$  (3.5)

Now we start to establish the global energy estimate for (1.4). Firstly, we consider the unweighted estimate on the solution $f$ of (1.4).

**Lemma 3.5.** Let $K \geq 4$, $l \geq 0$. Then

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq K} C_{\alpha} ||\partial^{\alpha} f(t)||^2 + \delta_0 \sum_{|\alpha| \leq K} ||\{I - P\} \partial^{\alpha} f(t)||_{N_{\gamma}\gamma}^2$$
We split the solution to find
\[ \text{From (3.4), and take } \]
we set
\[ \text{Then with Lemma 2.1, Lemma 3.2 and the above estimates, we easily get (3.6).} \]

Proof. Taking the spatial derivatives of \( \partial^\alpha \) of (1.4) to obtain
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq K} \| \partial^\alpha f(t) \|^2 + \sum_{|\alpha| \leq K} (L \partial^\alpha f, \partial^\alpha f) - \epsilon \sum_{|\alpha| \leq K} (L_{FP} \partial^\alpha f, \partial^\alpha f)
\]
\[ = \sum_{|\alpha| \leq K} (\partial^\alpha \Gamma(f, f), \partial^\alpha f). \]

From (3.4),
\[ \sum_{|\alpha| \leq K} (\partial^\alpha \Gamma(f, f), \partial^\alpha f) \lesssim \sqrt{\mathcal{E}_K(t)} \mathcal{D}_K(t) \lesssim \sqrt{\mathcal{E}_{K,l}(t)} \mathcal{D}_{K,l}(t). \]

Then with Lemma 2.1, Lemma 3.2 and the above estimates, we easily get (3.6). \( \square \)

Next, we consider the weighted energy estimates. Before stating the next lemma, we set
\[ \mathcal{D}_K^l(t) = \sum_{|\alpha| \leq K} \| \nabla_x \partial^\alpha f(t) \|_{L_{x,\gamma}}^2 + \sum_{|\alpha| \leq K} \| (I - P) \partial^\alpha f(t) \|_{L_{x,\gamma}}^2. \]

Lemma 3.6. Fix \( l \geq 0 \). For \( K \geq 4 \). If \( \epsilon > 0 \) is small enough, there exists a constant \( C_1 > 0 \) such that
\[ \frac{d}{dt} \mathcal{E}_{K,l}(t) + \lambda \mathcal{D}_{K,l}(t) \leq C_1 \left( \sqrt{\mathcal{E}_{K,l}(t)} + \mathcal{E}_{K,l}(t) \right) \mathcal{D}_{K,l}(t), \quad \forall t \geq 0. \]

Here \( \lambda > 0 \) may depend on \( l \).

Proof. Step 1. We split the solution \( f \) to equation (1.4) into \( f = Pf + (I - P)f \)
and take \( \{I - P\} \) of the resulting equation, then use
\[ L_{FP} Pf = PL_{FP} f \]

to find
\[ \partial_t \{I - P\} f + v \cdot \nabla_x \{I - P\} f + L\{I - P\} f \]
\[ = \Gamma(f, f) + \epsilon L_{FP} \{I - P\} f - \{I - P\} (v \cdot \nabla_x Pf) + Pf (v \cdot \nabla_x \{I - P\} f). \]

Let \( l \geq 0 \). For \( |\alpha| \leq K - 1 \), we take \( \partial^\alpha \) to (3.8), multiply by \( w^{2l} \partial^\alpha \{I - P\} f \), and then integrate in \( x, v \) to get
\[ \frac{1}{2} \frac{d}{dt} \| w^l \partial^\alpha \{I - P\} f \|^2 + (w^{2l} L \partial^\alpha \{I - P\} f, \partial^\alpha \{I - P\} f) \]
\[ = \epsilon (L_{FP} \partial^\alpha \{I - P\} f, w^{2l} \partial^\alpha \{I - P\} f) + \Gamma_1 + \Gamma_2, \]
where
\[ \Gamma_1 = (w^{2l} \partial^\alpha \Gamma(f, f), \partial^\alpha \{I - P\} f), \]
\[ \Gamma_2 = -\{I - P\} (v \cdot \nabla_x Pf, w^{2l} \partial^\alpha \{I - P\} f) \]
\[ + (P (v \cdot \nabla_x \{I - P\} f), w^{2l} \partial^\alpha \{I - P\} f). \]

Now we estimate each of the four terms in (3.9).

Using (3.5), one has
\[ |\Gamma_1| \lesssim \sqrt{\mathcal{E}_{K,l}(t)} \mathcal{D}_{K,l}(t). \]
Then with Cauchy-Schwartz inequality, we easily get $|\Gamma_2| \lesssim \mathcal{D}_K^0(t)$. Apply Lemma 2.2, we see that, for some appropriate constant $\lambda_1 > 0$,

$$(w^{2L_2}\partial^\alpha (I - P)f, \partial^\alpha (I - P)f) \geq \lambda_1 ||(I - P)\partial^\alpha f||_{N_1}^2 - C\lambda_1 ||(I - P)\partial^\alpha f||_{L_2}^2.$$  

Furthermore, from Lemma 3.3, we get

$$(L_{FP}\partial^\alpha (I - P)f, w^{2L_2}\partial^\alpha (I - P)f)$$

$$\leq -\lambda_{FP} ||(I - P_0)(u^{t}\partial^\alpha (I - P)f)||^2 + C||\partial^\alpha (I - P)f||_{N_1}^2.$$  

Plug the above estimates into (3.9) and choose $\epsilon$ small enough to achieve

$$c_1 \frac{d}{dt} \sum_{|\alpha| \leq K-1} ||w^{t}\partial^\alpha (I - P)f||^2 + \lambda \sum_{|\alpha| \leq K-1} ||(I - P)\partial^\alpha f||_{N_1}^2$$

$$+ \epsilon \lambda_{FP} \sum_{|\alpha| \leq K-1} ||(I - P_0)(u^{t}\partial^\alpha f)||^2$$

$$\leq C\{\mathcal{D}_K^0(t) + \sqrt{E_{K,l}(t)}\mathcal{D}_K(t)\}. \tag{3.10}$$

**Step 2.** For $1 \leq |\alpha| \leq K$, we take $\partial^\alpha$ of (1.4), multiply the result by $w^{2L_2}\partial^\alpha f$ for $l \geq 0$, and then integrate in $x, v$ to achieve

$$\frac{1}{2} \frac{d}{dt} ||w^{t}\partial^\alpha f||^2 + (w^{2L_2}\partial^\alpha f, \partial^\alpha f) = \epsilon (L_{FP}\partial^\alpha f, w^{2L_2}\partial^\alpha f) + (\partial^\alpha w^{2L_2}\Gamma(f, f), \partial^\alpha f).$$

Using the similar arguments above, we easily compute

$$c_2 \frac{d}{dt} \sum_{1 \leq |\alpha| \leq K} ||w^{t}\partial^\alpha f||^2 + 2\lambda \sum_{1 \leq |\alpha| \leq K} ||\partial^\alpha f||_{N_1}^2$$

$$+ \epsilon \lambda_{FP} \sum_{1 \leq |\alpha| \leq K} ||(I - P_0)(w^{t}\partial^\alpha f)||^2$$

$$\leq C\{\mathcal{D}_K^0(t) + \sqrt{E_{K,l}(t)}\mathcal{D}_K(t)\}. \tag{3.11}$$

Note that $||\partial^\alpha f||_{N_1}^2 \gtrsim ||\partial^\alpha (I - P)f||_{N_1}^2$. Then we obtain

$$c_3 \frac{d}{dt} \sum_{1 \leq |\alpha| \leq K} ||w^{t}\partial^\alpha f||^2 + \lambda \sum_{1 \leq |\alpha| \leq K} ||\partial^\alpha f||_{N_1}^2 + \lambda \sum_{1 \leq |\alpha| \leq K} ||(I - P_0)(w^{t}\partial^\alpha f)||^2$$

$$+ \epsilon \lambda_{FP} \sum_{1 \leq |\alpha| \leq K} ||(I - P_0)(w^{t}\partial^\alpha f)||^2$$

$$\leq C\{\mathcal{D}_K^0(t) + \sqrt{E_{K,l}(t)}\mathcal{D}_K(t)\}. \tag{3.12}$$

**Step 3.** Fix $|\alpha| + |\beta| \leq K$ with $|\beta| \geq 1$. Applying $\partial^\alpha \beta$ to (3.8), we have

$$\partial_t \partial^\alpha \beta (I - P)f + \partial^\alpha \beta L(I - P)f \partial^\alpha \beta \Gamma(f, f) + \epsilon \partial^\alpha \beta L_{FP}(I - P)f + I_1 + I_2, \tag{3.13}$$

where

$$I_1 = -\partial^\alpha \beta (I - P)(v \cdot \nabla_x P/f) + \partial^\alpha \beta P(v \cdot \nabla_x (I - P)f),$$

$$I_2 = -\partial^\alpha \beta (v \cdot \nabla_x (I - P)f).$$
Then multiplying (3.13) by $w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f$ and integrate over $\mathbb{R}^3 \times \mathbb{R}^3$ to get

$$
\frac{1}{2} \frac{d}{dt} \left( \|w^{l-|\beta|} \partial_\beta^\alpha \{I-P\} f\|^2 + \|w^{2l-2|\beta|} \partial_\beta^\alpha L\{I-P\} f, \partial_\beta^\alpha \{I-P\} f \right)
= \epsilon (\partial_\beta^\alpha L_{FP}\{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f) + \sum_{j=1}^3 \tilde{I}_j,
$$

(3.14)

where

$$
\tilde{I}_j = (I_j, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f), \quad j = 1, 2,
$$

$$
\tilde{I}_3 = (w^{2l-2|\beta|} \partial_\beta^\alpha \Gamma(f, f), \partial_\beta^\alpha \{I-P\} f).
$$

From (3.5) we have $|\tilde{I}_3| \lesssim \sqrt{\mathcal{E}_{K,l}(t)\mathcal{D}_{K,l}(t)}$. Analogous to the estimates in $\Gamma_2$ in Step 1, $|\tilde{I}_1| \lesssim \mathcal{D}_{K,l}^\alpha(t)$. Apply Lemma

$$
(w^{2l-2|\beta|} \partial_\beta^\alpha L\{I-P\} f, \partial_\beta^\alpha \{I-P\} f) \gtrsim \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}} - C\|\partial_\beta^\alpha \{I-P\} f\|^2_{L^2(B_C)}.
$$

We begin to estimate $\tilde{I}_2$, for a small constant $\eta > 0$,

$$
\tilde{I}_2 = (\partial_\beta^\alpha (v \cdot \nabla_x \{I-P\} f), w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f)
\quad - (v \cdot \nabla_x \partial_\beta^\alpha \{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f)
\quad = \sum_{|\beta_1| = 1} C_{\beta_1} \partial_\beta^\alpha \{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f
\quad \leq \eta \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}} + C\eta \sum_{|\beta_1| < \eta \leq |\beta|} \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}}.
$$

Lastly considering the first term on the right side of (3.14), we have

$$
(\partial_\beta^\alpha L_{FP}\{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f) - (L_{FP} \partial_\beta^\alpha \{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f)
\quad = \frac{1}{4} \sum_{0 < \beta_1 \leq \beta} C_{\beta_1} \lambda (\partial_\beta^\alpha \{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f)
\quad \leq C \sum_{0 < \beta_1 \leq \beta} \|\partial_\beta^\alpha \{I-P\} f\|_{N_{\gamma}^{s,-\gamma}} + C\eta \sum_{|\beta_1| < \eta \leq |\beta|} \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}}.
$$

With the above estimate and Lemma 3.3, we have

$$
\epsilon (\partial_\beta^\alpha L_{FP}\{I-P\} f, w^{2l-2|\beta|} \partial_\beta^\alpha \{I-P\} f)
\quad \leq - \epsilon \lambda_{FP} \|\{I-P_0\} (w^{l-|\beta|} \partial_\beta^\alpha \{I-P\} f)\|^2 + \epsilon \eta \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}}
\quad + C\epsilon \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}} + C\eta \epsilon \sum_{|\beta_1| < \eta \leq |\beta|} \|\partial_\beta^\alpha \{I-P\} f\|^2_{N_{\gamma}^{s,-\gamma}}.
$$
We choose a small constant \( \epsilon > 0, \eta > 0 \), add together each of these estimates to obtain
\[
\frac{d}{dt} \sum_{|\alpha| + |\beta| \leq K} C_{\beta} \|w^{l-|\beta|} \partial_\beta^\alpha (I-P) f\|^2 + \lambda \sum_{|\alpha| + |\beta| \leq K} \|\partial_\beta^\alpha (I-P) f\|_{N_{l+\gamma}}^2 \\
+ \epsilon \lambda_{FP} \sum_{|\alpha| + |\beta| \leq K} \|\{I-P_0\}(w^{l-|\beta|} \partial_\beta^\alpha (I-P) f)\|^2 \\
\leq C(\mathcal{D}_K^0(t) + \sum_{|\alpha| \leq K} \|\partial^\alpha (I-P) f\|_{N_{l+\gamma}}^2 + \sqrt{E_{K,l}(t) \mathcal{D}_{K,l}(t)}).
\]

Then taking summation over \(|\beta| \geq 1, |\alpha| + |\beta| \leq K\) and taking proper linear combination of those \(K\) estimates with properly chosen constants \(C_{\beta} > 0 \) \((1 \leq |\beta| \leq K)\), we have
\[
\frac{d}{dt} \sum_{|\alpha| + |\beta| \leq K} C_{\beta} \|w^{l-|\beta|} \partial_\beta^\alpha (I-P) f\|^2 + \lambda \sum_{|\alpha| + |\beta| \leq K} \|\partial_\beta^\alpha (I-P) f\|_{N_{l+\gamma}}^2 \\
+ \epsilon \lambda_{FP} \sum_{|\alpha| + |\beta| \leq K} \|\{I-P_0\}(w^{l-|\beta|} \partial_\beta^\alpha (I-P) f)\|^2
\]
\[
\leq C(\mathcal{D}_K^0(t) + \sum_{|\alpha| \leq K} \|\partial^\alpha (I-P) f\|_{N_{l+\gamma}}^2 + \sqrt{E_{K,l}(t) \mathcal{D}_{K,l}(t)}).
\]  

**Step 4.** We are ready to define the suitable instant energy functional as
\[
E_{K,l}(t) = \frac{1}{2} \sum_{|\alpha| \leq K} C_0 \|\partial^\alpha f(t)\|^2 + \kappa_2 E_{int}(t) + c_1 \kappa_3 \sum_{|\alpha| \leq K-1} \|w^l \partial^\alpha (I-P) f\|^2 \\
+ c_3 \kappa_4 \sum_{l \leq |\alpha| \leq K} \|w^l \partial^\alpha f\|^2 + \kappa_5 \sum_{|\alpha| + |\beta| \leq K} C_{\beta} \|w^{l-|\beta|} \partial_\beta^\alpha (I-P) f\|^2,
\]
for constants \(0 < \kappa_5 \ll \kappa_4 \ll \kappa_3 \ll \kappa_2 \ll \kappa_1 \ll 1\) to be chosen small enough. With (2.17) we can easily show that (1.9) is satisfied. In the further combination
\[
\kappa_1 \times (3.6) + \kappa_2 \times (2.29) + \kappa_3 \times (3.10) + \kappa_4 \times (3.12) + \kappa_5 \times (3.15),
\]
which implies (3.7), for some small constant \(\epsilon > 0\).

Now we are ready to prove Theorem 1.1 by the usual continuation arguments.

**Proof of Theorem 1.1.** We choose the initial data \(E_{K,l}(0)\) such that \(E_{K,l}(0) \leq M\), where \(M\) is defined as \(M = \min\{\delta, \frac{1}{2C_0}, \frac{\lambda^2}{4C_0C_1}\}\). Recall that \(C_0, C_1, \lambda\) are constants in (3.1) and (3.7), and \(\delta\) appears in Lemma 3.1. Note that \(E_{K,l}(0) \leq M \leq \delta\), then from Lemma (3.1) there exists a solution \(f\) for some \(T > 0\), and from the local estimate (3.1), we have \(E_{K,l}(t) \leq C_0 M\) for \(0 < t < T\). We define
\[
T^* = \sup\{t \in \mathbb{R}^+ | E_{K,l}(t) \leq C_0 M\} > 0.
\]

Note that on \([0,t]\) for \(0 < t < T^*\), \(E_{K,l}(t) \leq C_0 M < 1\), then \(\sqrt{E_{K,l}(t)} \leq E_{K,l}(t)\).

For the choice of \(M\), \(\lambda - 2C_1 \sqrt{C_0 M} > 0\), the global energy estimate (3.7) implies
that
\[ \frac{d}{dt} \mathcal{E}_{K,l}(t) + (\lambda - 2C_1\sqrt{C_0M})D_{K,l}(t) \leq 0. \] (3.16)

Thus
\[ \mathcal{E}_{K,l}(t) + (\lambda - 2C_1\sqrt{C_0M}) \int_0^t D_{K,l}(\tau) d\tau \leq \mathcal{E}_{K,l}(0). \]

This induces that \( T^* = +\infty \). Thus we finish the proof of Theorem 1.1. \( \Box \)

4. Time decay. This section is devoted to obtaining the time decay rate of the global solution \( f \) to the Fokker-Planck-Boltzmann equation (1.4). Firstly, we consider the linearized Fokker-Planck-Boltzmann equation
\[ \begin{cases}
\partial_t f + v \cdot \nabla_x f + Lf = g + \epsilon LF_P f, \\
f|_{t=0} = f_0(x,v).
\end{cases} \] (4.1)

For the nonlinear system (1.4), the non-homogeneous source term is given by
\[ g = \Gamma(f,f). \] (4.2)

In this case \( g = (I - P)g \). Formally, the solution \( f \) to the Cauchy problem (4.1) can be written as the mild form
\[ \begin{aligned}
f(t) &= e^{-\epsilon B} f_0 + \int_0^t e^{-(t-s)\epsilon B} g(s) ds, \\
B &= L + v \cdot \nabla_x - \epsilon LF_P,
\end{aligned} \] (4.3)

where \( e^{-\epsilon B} \) is the linear solution operator for the Cauchy problem to (4.1) with \( g = 0 \).

The first lemma is concerned with the estimate on the macroscopic dissipation.

**Lemma 4.1.** For any \( t \geq 0 \) and \( k \in \mathbb{R}^3 \), there is \( M > 0 \) such that the free energy functional \( \mathcal{E}_{free}(t,k) \) defined by
\[ \begin{aligned}
\mathcal{E}_{free}(t,k) &= \frac{1}{2} \sum_{j, m \neq j} \frac{i k_j}{1 + |k|^2} A_{mm}(\{I - P\} \hat{f}) - \sum_m \frac{i k_m}{1 + |k|^2} A_{jm}(\{I - P\} \hat{f}) |b_j|^2 \\
&+ M \sum_j \left( B_j(\{I - P\} \hat{f}) \frac{i k_j \hat{a}}{1 + |k|^2} \right) + \sum_j \left( \hat{b}_j \frac{i k_j \hat{a}}{1 + |k|^2} \right)
\end{aligned} \] (4.4)

satisfies
\[ \text{Re} \mathcal{E}_{free}(t,k) \lesssim |\hat{f}|^2 \] (4.5)

and
\[ \begin{aligned}
\partial_t \text{Re} \mathcal{E}_{free}(t,k) &+ \frac{\lambda |k|^2}{1 + |k|^2} (|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2) \\
&\lesssim \epsilon^2 (|\hat{b}|^2 + |\hat{c}|^2) + \epsilon^2 |\{I - P\} \hat{f}|^2_{N,\gamma} + |\hat{g}, \epsilon|^2.
\end{aligned} \] (4.6)

Here \( \epsilon \) is a linear combination of \( \{e_m\} \), which are the smooth exponentially decaying velocity basis vectors contained in (1.6) and (2.8).

**Proof.** Our proof below will rely on the results obtained in section 2, in (2.11) we replace \( G \) to \( g \), other conclusions remain unchanged.

Estimate on \( b \). For \( 0 < \eta < 1 \), one has
\[ \begin{aligned}
\partial_t \text{Re} \sum_j \left( \frac{1}{2} \sum_{m \neq j} i k_j A_{mm}(\{I - P\} \hat{f}) - \sum_m i k_m A_{jm}(\{I - P\} \hat{f}) |b_j|^2 \right) \\
&+ (1 - \eta) |k|^2 |\hat{b}|^2
\end{aligned} \] (4.7)
\[ \leq \eta |k|^2 (|\hat{u}|^2 + |\hat{v}|^2) + c^2 |\hat{b}|^2 + C_\eta (1 + |k|^2 + c^2)(|\{I - P\} \hat{f}|_{N^{\gamma}, \gamma}^2 + |\langle \hat{g}, e \rangle|^2). \]

In fact, the Fourier transform of (2.20) gives \((G\) is replaced by \(g)\)

\[ \partial_t \left[ \frac{1}{2} \sum_{m \neq j} ik_j A_{mm}(\{I - P\} \hat{f}) - \sum_m ik_m A_{jm}(\{I - P\} \hat{f}) \right] + |k|^2 \hat{b}_j + k_j^2 \hat{b}_j \]

\[ = \frac{1}{2} \sum_{m \neq j} ik_j A_{mm}(\hat{R} + \hat{g}) - \sum_m ik_m A_{jm}(\hat{R} + \hat{g}), \]

where \(R\) is defined in (2.11).

Now we take the complex dot product with \(b_j\) to obtain

\[ \partial_t \left( \frac{1}{2} \sum_{m \neq j} ik_j A_{mm}(\{I - P\} \hat{f}) - \sum_m ik_m A_{jm}(\{I - P\} \hat{f}) \right) + (|k|^2 + k_j^2) |\hat{b}_j|^2 \]

\[ = \left( \frac{1}{2} \sum_{m \neq j} ik_j A_{mm}(\hat{R} + \hat{g}) - \sum_m ik_m A_{jm}(\hat{R} + \hat{g}) \right)^2 \]

\[ + \left( \frac{1}{2} \sum_{m \neq j} ik_j A_{mm}(\{I - P\} \hat{f}) - \sum_m ik_m A_{jm}(\{I - P\} \hat{f}) \right) |\partial_t \hat{b}_j \]

\[ = S_1 + S_2. \]

Note that

\[ \hat{R} = -v \cdot k \{I - P\} \hat{f} + e L_{EP}(\{I - P\} \hat{f}) - L(\{I - P\} \hat{f}). \]

With the same arguments in Lemma 2.6, we obtain

\[ |A_{jm}(\hat{R})|^2 \lesssim (1 + |k|^2 + c^2)(\{I - P\} \hat{f}|_{N^{\gamma}, \gamma}^2. \]

Thus \(S_1\) is bounded by

\[ S_1 \leq \eta |k|^2 |\hat{b}_j|^2 + C_\eta (|A_{jm}(\hat{R})|^2 + |A_{jm}(\hat{g})|^2) \]

\[ \leq \eta |k|^2 |\hat{b}_j|^2 + C_\eta (1 + |k|^2 + c^2)(|\{I - P\} \hat{f}|_{N^{\gamma}, \gamma}^2 + |\langle \hat{g}, e \rangle|^2). \]

For \(S_2\), using the Fourier transform of (2.9) to obtain

\[ \partial_t \hat{b}_j + ik_j (\hat{u} + 2\hat{c}) + \sum_m ik_m A_{jm}(\{I - P\} \hat{f}) + c\hat{b}_j = 0 \]

(4.10)

Replace \(\partial_t \hat{b}_j\) above to get

\[ S_2 \leq \eta |k|^2 (|\hat{u}|^2 + |\hat{v}|^2) + c^2 |\hat{b}_j|^2 + C_\eta (1 + |k|^2) \sum_{j,m} |A_{jm} \{I - P\} \hat{f}|^2 \]

\[ \leq \eta |k|^2 (|\hat{u}|^2 + |\hat{v}|^2) + c^2 |\hat{b}_j|^2 + C_\eta (1 + |k|^2) (|\{I - P\} \hat{f}|_{N^{\gamma}, \gamma}^2. \]

Therefore, one can take the real part of (4.8) and plug (4.9) and (4.10) into it, then sum over \(\{j = 1, 2, 3\}\) to obtain (4.7).

\textit{Estimate on} \(\hat{c}\). For \(0 < \eta < 1\), we have

\[ \partial_t \text{Re} \sum_{j} (B_j (\{I - P\} \hat{f}) ik_j \hat{c}) + (1 - \eta) |k|^2 |\hat{c}|^2 \]

\[ \leq \eta |k|^2 |\hat{b}_j|^2 + c^2 |\hat{b}_j|^2 + C_\eta (1 + |k|^2 + c^2)(|\{I - P\} \hat{f}|_{N^{\gamma}, \gamma}^2 + |\langle \hat{g}, e \rangle|^2). \]

(4.12)

In fact, the Fourier transform of (2.10) implies

\[ \partial_t B_j (\{I - P\} \hat{f}) + ik_j \hat{c} = B_j (\hat{R} + \hat{g}), \]
then take the complex dot product with \( ik_j \hat{c} \) to obtain
\[
\partial_t (B_j(\{I - P\} \hat{f})|ik_j \hat{c}) + |k_j|^2 |\hat{c}|^2 \\
= (B_j(\hat{R} + \hat{g})|ik_j \hat{c}) + (B_j(\{I - P\} \hat{f})|ik_j \partial_t \hat{c}) \\
= S_3 + S_4. \tag{4.13}
\]

Similar to \( S_1 \) above, \( S_3 \) is bounded by
\[
S_3 \leq \eta |k_j|^2 |\hat{c}|^2 + C_\eta (|B_j(\hat{R})|^2 + |B_j(\hat{g})|^2) \\
\leq \eta |k_j|^2 |\hat{c}|^2 + C_\eta (1 + |k|^2 + \epsilon^2)((|I - P\} \hat{f})^2_{N^*, \gamma} + |(\hat{g}, \epsilon)|^2). \tag{4.14}
\]
For \( S_4 \), using the Fourier transform of \((2.9)_3\) to obtain
\[
\partial_t \hat{c} + \frac{1}{3} k \cdot \hat{b} + \frac{5}{3} \sum_j ik_j B_j(\{I - P\} \hat{f}) + 2\epsilon \hat{c} = 0
\]
to replace \( \partial_t \hat{c} \), we get
\[
S_4 \leq \eta |k|^2 |\hat{b}|^2 + \epsilon^2 |\hat{c}|^2 + C_\eta (1 + |k|^2) \sum_j |B_j(\{I - P\} \hat{f})|^2 \\
\leq \eta |k|^2 |\hat{b}|^2 + \epsilon^2 |\hat{c}|^2 + C_\eta (1 + |k|^2)(|I - P\} \hat{f})^2_{N^*, \gamma}. \tag{4.15}
\]
Hence, \((4.12)\) follows by taking the real part of \((4.13)\) and applying \((4.14)\) and \((4.15)\), and then taking the summation over \( \{j = 1, 2, 3\} \).

Estimate on \( \hat{a} \). It holds for any \( 0 < \eta < 1 \) that
\[
\partial_t \Re \sum_j (b_j|ik_j \hat{a}) + (1 - \eta)|k|^2 |\hat{a}|^2 \\
\leq |k|^2 |\hat{b}|^2 + C_\eta (|k|^2 |\hat{c}|^2 + |k|^2)(|I - P\} \hat{f})^2_{N^*, \gamma} + \epsilon^2 |\hat{b}|^2). \tag{4.16}
\]
In fact, using \((4.10)\), and taking the complex dot product with \( ik_j \hat{a} \), and then taking the summation over \( \{j = 1, 2, 3\} \), one has
\[
\partial_t \sum_j (\hat{b}_j|ik_j \hat{a}) + |k|^2 |\hat{a}|^2 \\
= -2 \sum_j (ik_j \hat{c}|ik_j \hat{a}) - \sum_{j,m}(ik_mA_{jm}(\{I - P\} \hat{f})|ik_j \hat{a}) \\
- \sum_j (\epsilon \hat{b}_j|ik_j \hat{a}) + \sum_j (\hat{b}_j|ik_j \partial_t \hat{a}). \tag{4.17}
\]
The first three terms on the right hand side of \((4.17)\) are bounded by
\[
\eta |k|^2 |\hat{a}|^2 + C_\eta (|k|^2 |\hat{c}|^2 + |k|^2)(|I - P\} \hat{f})^2_{N^*, \gamma} + \epsilon^2 |\hat{b}|^2) \tag{4.17}
\]
For the last term in \((4.17)\), we have
\[
\sum_j (\hat{b}_j|ik_j \partial_t \hat{a}) = \sum_j (\hat{b}_j|ik_j \{-ik \cdot \hat{b}\}) = |k \cdot \hat{b}|^2 \leq |k|^2 |\hat{b}|^2.
\]
Here we have used the Fourier transform of \((2.9)_1\)
\[
\partial_t \hat{a} + ik \cdot \hat{b} = 0.
\]
Then we can deduce \((4.16)\) by putting the above estimates into \((4.17)\) and taking the real part.

Therefore, \((4.6)\) follows from the proper linear combination of \((4.7)\), \((4.12)\) and \((4.16)\) by taking \( M > 0 \) large enough and \( 0 < \eta < 1 \) small enough.
Now we begin to prove (4.5). From (4.4), one has
\[ |E_{\text{free}}(t, k)| \lesssim (|a|^2 + |b|^2 + |c|^2) + \sum_{j,m}(|A_{jm}((I - P)\hat{f})|^2 + |B_j((I - P)\hat{f})|^2) \]
\[ \lesssim |P\hat{f}|^2 + |(I - P)\hat{f}|^2 \lesssim |\hat{f}|^2. \]
Thus the proof of this lemma is completed. \( \square \)

Next we give estimates on the microscopic dissipation and microscopic weighted dissipation, which will be used below.

Consider (4.1), taking the Fourier transform in \( x \) gives
\[ \partial_t \hat{f} + iv \cdot k\hat{f} + L\hat{f} = \hat{g} + \epsilon L_{FP}\hat{f}. \]
Then we multiply the above equation with \( \epsilon > 0 \) to obtain
\[ \frac{1}{2} \frac{d}{dt} |\hat{f}(t, k)|^2 + \Re\langle L\hat{f}, \hat{f} \rangle - \epsilon \Re\langle L_{FP}\hat{f}, \hat{f} \rangle = \Re\langle \hat{g}, \hat{f} \rangle. \]

From (2.1) and (3.2), one has
\[ \frac{d}{dt} |\hat{f}(t, k)|^2 + \lambda \left\{ |(I - P)\hat{f}|^2_{N^\gamma} + \epsilon \lambda_{FP} |(I - P)\hat{f}|^2 \right\} \lesssim |\Re\langle \hat{g}, \hat{f} \rangle|. \quad (4.18) \]

Then we consider estimates on the microscopic weighted dissipation. Replace \( \Gamma(f, f) \) to \( g \) in (3.8), then take the Fourier transform in \( x \) to yield
\[ \partial_t (I - P)\hat{f} + iv \cdot k(I - P)\hat{f} + L(I - P)\hat{f} \]
\[ = \hat{g} + \epsilon L_{FP}(I - P)\hat{f} - (I - P)(iv \cdot k\hat{f}) + P(iv \cdot k(I - P)\hat{f}) \]
\[ = \hat{g} + \epsilon L_{FP}(I - P)\hat{f} - iv \cdot k\hat{f} + P(iv \cdot k\hat{f}). \]

For any \( \lambda \in \mathbb{R} \), multiply the above equation by \( w^2(I - P)\hat{f} \) and integrate over \( v \) to obtain
\[ \frac{1}{2} \frac{d}{dt} |w^2(I - P)\hat{f}(t, k)|^2 + \Re\langle w^2L(I - P)\hat{f}, (I - P)\hat{f} \rangle \]
\[ = \epsilon \Re\langle w^2L_{FP}(I - P)\hat{f}, (I - P)\hat{f} \rangle + \Re\langle w^2\hat{g}, (I - P)\hat{f} \rangle + J_1, \quad (4.19) \]
where
\[ J_1 = -\Re\langle w^2(iv \cdot kP\hat{f}) - w^2P(iv \cdot k\hat{f}), (I - P)\hat{f} \rangle. \]

Apply Remark 1 to achieve, for any \( \lambda \in \mathbb{R} \),
\[ \Re\langle w^2L(I - P)\hat{f}, (I - P)\hat{f} \rangle \geq \lambda \langle (I - P)\hat{f} \rangle_{N^\gamma}^2 - C\langle (I - P)\hat{f} \rangle_{\mathbb{H}^1}^2. \]
Using the same arguments in Lemma 3.3 yields
\[ \epsilon \Re\langle w^2L_{FP}(I - P)\hat{f}, (I - P)\hat{f} \rangle \leq -\lambda_{FP} \langle (I - P)\hat{f} \rangle_{N^\gamma}^2 \]
\[ + C \epsilon \langle (I - P)\hat{f} \rangle_{N^\gamma}^2. \]

For the last term \( J_1 \), owing to the rapid decay in the coefficients of (1.6), we get
\[ |J_1| \leq \eta \langle (I - P)\hat{f} \rangle_{N^\gamma}^2 + C \eta |k|^2 \langle (I - P)\hat{f} \rangle_{L^1}^2. \]
Plug the above three estimates into (4.19) and chose a small \( \epsilon > 0 \) to obtain
\[ \frac{d}{dt} |w^2(I - P)\hat{f}(t, k)|^2 + \lambda \left\{ |(I - P)\hat{f}|^2_{N^\gamma} \right\} \]
\[ + \epsilon \lambda_{FP}|(I - P_0)(w' (I - P) \hat{f})|^2_2 \]
\[ \leq C_\kappa |k|^2 |P \hat{f}|^2_2 + C|(I - P) \hat{f}|^2_2 + C|(w^{2l} \hat{g}, (I - P) \hat{f})|. \quad (4.20) \]

With the similar arguments above, for a small \( \epsilon > 0 \), we easily get
\[ \frac{d}{dt} |w' \hat{f}(t, k)|^2_2 + \lambda \left\{ \frac{|\hat{f}|_{N' \gamma}^2}{2} + \epsilon \lambda_{FP}|(I - P_0)(w' \hat{f})|^2_2 \right\} \]
\[ \leq C|\hat{f}|^2_2 + C|(w^{2l} \hat{g}, \hat{f})|. \quad (4.21) \]

Based on the above estimates, we give the following lemma.

**Lemma 4.2.** Fix \( l \in \mathbb{R} \). For \( t \geq 0 \) and \( k \in \mathbb{R}^3 \), consider the Cauchy problem (4.1) with \( g = 0 \). If \( \epsilon > 0 \) is small enough, there is a weighted time-frequency functional \( \mathcal{E}_l(t, k) \) such that
\[ \mathcal{E}_l(t, k) \approx |w' \hat{f}|^2_2, \quad (4.22) \]
and
\[ \partial_t \mathcal{E}_l(t, k) + \lambda (1 \wedge |k|^2) |\hat{f}|_{N' \gamma}^2 \leq 0, \quad (4.23) \]
where \( 1 \wedge |k|^2 = \min\{1, |k|^2\} \).

**Proof.** Define
\[ \mathcal{E}(t, k) = |\hat{f}|^2_2 + \kappa_1 \text{Re} \mathcal{E}_{\text{free}}(t, k), \quad (4.24) \]
for a constant \( \kappa_1 \) to be determined later, where \( \mathcal{E}_{\text{free}}(t, k) \) is given in (4.4). Recall that in (4.5), we can choose \( \kappa_1 > 0 \) small enough such that \( \mathcal{E}(t, k) \approx |\hat{f}|^2_2 \).

Note that in Lemma 2.9, \( |\hat{b}|^2 + |\hat{c}|^2 \lesssim (|I - P| \hat{f}|_{N' \gamma}^2 + |I - P_0| \hat{f})^2 \). Plug (4.6) and (4.18) into (4.24) to obtain
\[ \partial_t \mathcal{E}(t, k) + \lambda \left\{ |(I - P) \hat{f}|_{N' \gamma}^2 + \epsilon \lambda_{FP}|(I - P_0) \hat{f}|^2_2 \right\} \]
\[ + \frac{\lambda |k|^2}{1 + |k|^2} (|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2) \leq 0. \quad (4.25) \]

For the weighted estimates, we need to introduce a new energy splitting as follows. With (4.24) we define
\[ \mathcal{E}_l^1(t, k) = 1_{|k| \leq 1}(\mathcal{E}(t, k) + \kappa_2 |w' (I - P) \hat{f}|^2_2), \]
\[ \mathcal{E}_l^1(t, k) = 1_{|k| > 1}(\mathcal{E}(t, k) + \kappa_3 |w' \hat{f}|^2_2), \]
where \( \kappa_2, \kappa_3 \) will be determined later.

For \( \mathcal{E}_l^1(t, k) \), note that \( |P \hat{f}|^2_2 \lesssim |\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 \), we combine (4.25) with (4.21) for \( |k| > 1 \) to achieve, for some small \( \kappa_2 > 0 \),
\[ 1_{|k| \leq 1}(\partial_t \mathcal{E}_l^1(t, k) + \lambda |\hat{f}|_{N' \gamma}^2) \leq 0. \]
Here we have used the fact that \( \frac{|k|^2}{1 + |k|^2} \geq \frac{1}{2} \) with \( |k| > 1 \).

Furthermore, when \( |k| \leq 1 \) it holds that \( \frac{|k|^2}{1 + |k|^2} \geq \frac{|k|^2}{4} \). In this case we apply (4.25) and (4.20) to obtain for a small \( \kappa_2 > 0 \),
\[ 1_{|k| \leq 1}(\partial_t \mathcal{E}_l^0(t, k) + \lambda |k|^2 |\hat{f}|_{N' \gamma}^2) \leq 0. \]
Lastly we define \( \mathcal{E}_l(t, k) = \mathcal{E}_l^0(t, k) + \mathcal{E}_l^1(t, k) \) and we notice that (4.22) is satisfied. Then (4.23) follows from adding the above two inequalities. \( \square \)
Proposition 2. Fix \( l \in \mathbb{R} \). Let \( g = 0 \) in (4.1). If \( \epsilon > 0 \) is small enough, then the solution of this equation satisfies
\[
||w^l e^{-\mathcal{B} t} f_0|| \lesssim (1 + t)^{-3/4} (||w^l f_0|| + ||w^l f_0||_{Z_1}),
\]
for any \( t \geq 0 \).

Proof. Set \( \hat{\rho}(k) = \lambda(1 \wedge |k|^2) \). Since \( \gamma + 2s \geq 0 \) and \( g = 0 \), we can rewrite (4.23) as
\[
\partial_t \mathcal{E}_l(t, k) + \hat{\rho}(k) \mathcal{E}_l(t, k) \leq 0.
\]

The Gronwall inequality then implies
\[
\mathcal{E}_l(t, k) \leq e^{-\rho(\lambda t)} \mathcal{E}_l(0, k).
\]
Applying (4.22) to achieve, for any \( l \in \mathbb{R} \),
\[
||u^l f||^2 \approx \int_{\mathbb{R}^3} \mathcal{E}_l(t, k) dk.
\]
Notice that for \( |k| \leq 1 \), \( \hat{\rho}(k) = \lambda|k|^2 \), and for \( |k| \geq 1 \), \( \hat{\rho}(k) = \lambda \). With (4.27), we have
\[
\int_{\mathbb{R}^3} \mathcal{E}_l(t, k) dk \leq \int_{|k| \leq 1} e^{-\lambda|k|^2 t} \mathcal{E}_l(0, k) dk + e^{-\lambda t} \int_{|k| \geq 1} \mathcal{E}_l(0, k) dk.
\]
As in [23], from Appendix A, we have
\[
\int_{|k| \leq 1} e^{-\lambda|k|^2 t} \mathcal{E}_l(0, k) dk \lesssim (1 + t)^{-2/3} ||w^l f_0||_{Z_1}^2.
\]
For the integration over \( |k| \geq 1 \),
\[
\int_{|k| \geq 1} e^{-\lambda t} \mathcal{E}_l(0, k) dk \lesssim e^{-\lambda t} ||w^l f_0||^2.
\]
Collecting the above estimates as well as (4.28) gives (4.26).

Proof of Theorem 1.2. From (1.9) and (1.10), we have
\[
\mathcal{E}_{K,l}(t) \lesssim ||Pf||^2 + D_{K,l}(t).
\]
Combining this with (3.16), we conclude
\[
\frac{d}{dt} \mathcal{E}_{K,l}(t) + \lambda \mathcal{E}_{K,l}(t) \lesssim ||Pf||^2.
\]
The Gronwall inequality gives
\[
\mathcal{E}_{K,l}(t) \lesssim e^{-\lambda t} \mathcal{E}_{K,l}(0) + \int_0^t e^{-\lambda(t-s)}||Pf(s)||^2 ds.
\]
For now, we consider the pointwise time decay estimates on \( ||Pf(t)||^2 \). Formally, the solution \( f \) to the Cauchy problem (1.4) can be written as (4.3) where \( g \) is given by (4.2). We write as \( f(t) = I_0(t) + I_1(t), \)
with
\[
I_0(t) = e^{-t\mathcal{B} f_0}, \quad I_1(t) = \int_0^t e^{-(t-s)\mathcal{B}} \Gamma(f, f)(s) ds.
\]
We now apply (4.26) with \( l = -b \leq 0 \) to be determined to \( I_0(t) \) and \( I_1(t) \) respectively to obtain
\[
||w^{-b} I_0(t)|| \lesssim (1 + t)^{-3/4} (||w^{-b} f_0|| + ||w^{-b} f_0||_{Z_1}).
\]
and
\[ ||w^{-b}I_1(t)|| \leq \int_0^t ||w^{-b}e^{-(t-s)B}\Gamma(f,f)(s)||ds \]
\[ \lesssim \int_0^t (1 + t - s)^{-3/4}(||w^{-b}\Gamma(f,f)(s)|| + ||w^{-b}\Gamma(f,f)(s)||_{Z_1})ds. \]

Here, we can regard \( \Gamma(f,f) \) as \( f_0 \) in the last estimate. Define
\[ E_{K,l}(t) = \sup_{0 \leq s \leq t} (1 + s)^{3/2}\mathcal{E}_{K,l}(s). \]

Applying (3.22) in [23] to obtain
\[ ||w^{-b}I_1(t)|| \lesssim \int_0^t (1 + t - s)^{-3/4}\mathcal{E}_K(s)ds \]
\[ \lesssim \int_0^t (1 + t - s)^{-3/4}\mathcal{E}_{K,l}(s)ds \]
\[ \lesssim E_{K,l}(t) \int_0^t (1 + t - s)^{-3/4}(1 + s)^{-3/2}ds \]
\[ \lesssim (1 + t)^{-3/4}E_{K,l}(t). \]

Here we have chosen \( b > 0 \) sufficiently large and used the time decay estimates for the time integrals as in [24] proposition 4.5.

Combining with the above estimates on \( I_0(t) \) and \( I_1(t) \) to yield
\[ || Pf(t)||^2 \lesssim ||w^{-b}f(t)||^2 \]
\[ \lesssim ||w^{-b}I_0(t)||^2 + ||w^{-b}I_1(t)||^2 \tag{4.30} \]
\[ \lesssim (1 + t)^{-3/2}(||f_0||^2 + ||f_0||_{Z_1}^2) + (1 + t)^{-3/2}[E_{K,l}(t)]^2. \]

Plugging the above estimates into (4.29) to obtain
\[ E_{K,l}(t) \lesssim (1 + t)^{-3/2}(||f_0||_{Z_1}^2 + E_{K,l}(0) + [E_{K,l}(t)]^2). \]

Recall that \( \epsilon_{K,l} \) in (1.12), then
\[ E_{K,l}(t) \lesssim ||f_0||_{Z_1}^2 + E_{K,l}(0) + [E_{K,l}(t)]^2 \lesssim \epsilon_{K,l} + [E_{K,l}(t)]^2. \]

Lastly, we use a bootstrap argument with \( \epsilon_{K,l} \) sufficiently small to obtain (1.11) in Theorem 1.2. \( \square \)

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