The Low Level Modular Invariant Partition Functions
of Rank-Two Algebras

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Abstract

Using the self-dual lattice method, we make a systematic search for modular invariant partition functions of the affine algebras $g^{(1)}$ of $g = A_2$, $A_1 + A_1$, $G_2$, and $C_2$. Unlike previous computer searches, this method is necessarily complete. We succeed in finding all physical invariants for $A_2$ at levels $\leq 32$, for $G_2$ at levels $\leq 31$, for $C_2$ at levels $\leq 26$, and for $A_1 + A_1$ at levels $k_1 = k_2 \leq 21$. This work thus completes a recent $A_2$ classification proof, where the levels $k = 3, 5, 6, 9, 12, 15, 21$ had been left out. We also compute the dimension of the (Weyl-folded) commutant for these algebras and levels.
I. INTRODUCTION

The subject of this paper is the classification of the partition functions of some conformal field theories. The problem can be stated in general terms as follows. Consider a conformal field theory that has an operator algebra decomposable into a pair of commuting holomorphic and antiholomorphic chiral algebras, $\hat{g}_L$ and $\hat{g}_R$, and a Hilbert space which can be written as a finite sum of irreducible representations $(\lambda_L, \lambda_R)$ of $\hat{g}_L \times \hat{g}_R$ with multiplicity $N_{\lambda_L \lambda_R}$. With each such representation $(\lambda_L, \lambda_R)$, associate a pair of characters $\chi_{\lambda_L}$ and $\chi_{\lambda_R}$. Then define the partition functions of the theory as combinations of bi-products of characters of the form

$$Z = \sum N_{\lambda_L \lambda_R} \chi_{\lambda_L} \chi_{\lambda_R}^*.$$

The problem is to find, for a given algebra $(\hat{g}_L, \hat{g}_R)$, all combinations (1.1) such that:

(P1) $Z$ is invariant under transformations of the modular group; (P2) the coefficients $N_{\lambda_L \lambda_R}$ are non-negative integers; and (P3) $N_{11} = 1$ if $\lambda = 1$ denotes the vacuum.

Any function $Z$ satisfying the modular invariance condition (P1) will be called an invariant; if in addition it satisfies the condition that each coefficient $N_{\lambda_L \lambda_R} \geq 0$, then it is said to be a positive invariant; and finally, if the conditions (P1), (P2) and (P3) are all met, then it is considered to be a physical invariant. Clearly any conformal theory must have at least these three properties to be physically meaningful.

Such functions appear in the context of statistical physics, for example in the early work of Cardy [1] when he formulated a conformal system on a finite rectangular strip with periodic boundary conditions. They also arise in the theory of strings, for example in the classical paper by Gepner and Witten [2] where the authors discussed the compactification of closed strings on Lie group manifolds. The action for a closed string is the same as that for a nonlinear sigma model with properly chosen couplings (i.e. the Wess-Zumino-Witten model) and is known to be not only conformally invariant but also invariant under the much larger current algebra induced by the underlying group, which forms itself a subclass of the Kac-Moody algebras. In the context of the closed string theory the object of interest is the factor pertaining to the group manifold of the vacuum-to-vacuum string amplitude in the lowest-order (one-loop) approximation.

It is quite possible that, among all such invariants, some particular ones may lead to phenomenologically interesting, even viable, string models, as some recent analyses [3] have suggested. Also, it has long been a hope by many that a good understanding of the partition functions could make a positive contribution to the broader and as yet incomplete task of classifying conformal field theories. This hope is justified partly by the fact that a conformal field theory can be identified in some sense through its partition functions, and partly by some recent progress in the computing and understanding of the invariants. The characters of an algebra, considered in their full dependence on the coordinates of the space on which they are defined, are linearly independent [4]. And so are functions of the type $\chi_{\lambda_L} \chi_{\lambda_R}^*$. It means that two theories with the same partition functions are identical and, conversely, different theories must correspond to different partition functions.

Many modular invariant partition functions have already been known, and much insight has been gained through the efforts of many, in particular [5], [6], and [7], among
others. But classification proofs, which determine all the physical invariants that belong to a certain class, exist only for a few cases, namely, the untwisted Kac-Moody algebra $A^{(1)}_1$ [5,8] and the coset models based on it, such as the minimal unitary Virasoro models [5] and the $N = 1$ minimal superconformal models [9]; recently a classification proof for $A^{(1)}_2$ has also been obtained [10]. These proofs hold for an arbitrary level (a number through which affine representations can be related to finite ones). In addition, the complete list of level-one partition partitions for the simple Lie algebras $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, and the five exceptional algebras has been found [11,12].

Systematic numerical searches for invariants of algebras are useful in guiding and confirming theoretical analyses, and in discovering new exceptional invariants. Several of these searches have already been made. For example, [13,3] use a procedure based on Verlinde’s formula rephrased as an eigenvalue problem. However in general these searches are not necessarily complete – i.e. it is possible that an elusive and presently unknown exceptional invariant could escape them (e.g. [13,3] place an upper bound on the sizes of the coefficients $N_{\lambda_L \lambda_R}$). In the work reported in the present paper, we follow yet another method, proposed by Warner [14] and independently by Roberts and Terao [15]. In their approach, invariants are generated by using the Weyl-Kac formula of characters and the theta functions associated with even self-dual lattices. It has been shown [12] that this method is complete: it will yield all the invariants of a given class. It is a number-theoretic technique that treats all invariants – exceptional invariants as well as members of infinite series – on the same footing. It is quite practical, at least for low rank algebras and small levels, as in the cases we considered.

Its completeness and practicality make this lattice method ideally suited to finish off the $A^{(1)}_2$ classification. In [10] this classification was accomplished for all levels $k$, except for $k = 3, 5, 6, 9, 12, 15, 21$. These levels survived because the proof in [10] broke down for them. In this paper our analysis of $A^{(1)}_2$ includes those remaining levels, thus completing the classification for $A^{(1)}_2$.

We will implicitly assume throughout this paper that the algebras $\hat{g}_L, \hat{g}_R$ of the holomorphic and antiholomorphic sectors of the theory are identical. This is the situation most commonly considered; the problem of asymmetric (heterotic) invariants is addressed in [16].

In Sec. II we will describe the method. In Sec. III we discuss the calculations of the invariants for the affinizations of the four rank-two algebras: $A_2$ at levels $\leq 32$; $G_2$ at levels $\leq 31$; $C_2$ at levels $\leq 26$; and $A_1 + A_1$ at levels $k_1 = k_2 \leq 21$. The results of our calculations, namely the complete set of all physical invariants for those algebras and levels, are given in Sec. IV. Our concluding remarks are found in Sec. V. The Appendix describes in more detail how we find all of the desired lattices.

II. THE LATTICE APPROACH

In this section we will briefly review some basic elements of the affine Kac-Moody algebras [17,18] and describe the lattice method of Roberts-Terao-Warner [14,15] that we
used in our numerical calculations. We will consider here simple algebras \( g \), but analogous statements hold also for \( g \) semi-simple.

Let \( g \) be a finite-dimensional simple Lie algebra of rank \( r \), and let \( \alpha_1^\vee, \ldots, \alpha_r^\vee \) be its simple coroots. The fundamental weights of \( g \) are the vectors \( \beta_1, \ldots, \beta_r \), defined by the inner products \( \beta_i \cdot \alpha_j^\vee = \delta_{ij} \). Let \( \rho \) be the sum of all the fundamental weights, \( \rho = \sum \beta_i \).

The coroots span the coroot space. By coroot lattice of \( g \), denoted by \( M = M_g \), we mean the integral span of its coroots, a subspace of the coroot space. The dual \( M^* \) of the coroot lattice is called the weight lattice, the set of integral weight vectors. Any vector \( \lambda \in M^* \) can be decomposed as \( \lambda = \sum_{i=1}^r m_i \beta_i \) where \( m_i \in \mathbb{Z} \). If \( \lambda \) is a highest weight, all \( m_1, \ldots, m_r \) are nonnegative integers.

Let \( \hat{g} = g^{(1)} \) be the untwisted affine extension of \( g \) defined by a central element and a derivation. Weight vectors of \( \hat{g} \) may then be denoted \( \hat{\lambda} = (\lambda, k, n) \), where \( \lambda \) is a weight of \( g \), \( n \) an eigenvalue of the derivation, and \( k \) an eigenvalue of the central element (called the level). Each unitary highest weight representation of \( \hat{g} \) is associated with an extended Dynkin diagram with \( r+1 \) nodes labelled by \( m_0, m_1, \ldots, m_r \), where \( m_1, \ldots, m_r \) are related to a horizontal highest weight vector \( \lambda \) in the usual way, and the label of the extra node is given by \( m_0 = k - \sum_{i=1}^r m_i a_i^\vee \), where \( k \) is the level of the representation and \( a_i^\vee \) are the positive integral colabels of \( g \). The number \( 1 + \sum a_i^\vee \) is known as the dual Coxeter number and denoted by \( h^\vee \).

An integrable highest weight representation of \( \hat{g} \) is defined as one with \( m_0 \geq 0 \). From this condition it follows that the level \( k \) is also a nonnegative integer since for an irreducible highest weight representation of a simple Lie algebra \( m_1, \ldots, m_r \) are nonnegative integers. It is thus convenient to introduce for each level \( k = 0, 1, 2, \ldots \) a subset of \( M^* \) containing the horizontal highest weight, \( \lambda \), of each level-\( k \) integrable highest weight representation of \( \hat{g} \):

\[
P_+(g, k) \overset{\text{def}}{=} \{ \sum_{i=1}^r m_i \beta_i \mid m_i \in \mathbb{Z}, \ 0 \leq m_i, \ \sum_{i=1}^r m_i a_i^\vee \leq k \}.
\]

For a given level, the number of horizontal highest weights is finite. In the context of a conformally invariant current algebra theory, each \( \lambda \in P_+(g, k) \) is associated with an ‘integrable’ primary field. Only integrable fields are physically relevant; all of the other possible primary fields are nonintegrable and, as such, irrelevant in the sense that any correlation function containing one or more of these fields would vanish identically. We will usually find it more convenient to use \( \lambda + \rho \) in place of the horizontal highest weight \( \lambda \in P_+(g, k) \).

The Weyl-Kac formula, which proves a very powerful tool in studying modular invariance, expresses the character of an irreducible integrable highest weight representation of \( \hat{g} \) as a certain sum over the Weyl group of \( \hat{g} \). Since this Weyl group turns out to be a semidirect product of the finite dimensional Weyl group, \( W(g) \), and a lattice translation group, each summand can be recast in terms of a \( \theta \) series which takes care of the translation group, leaving only the summation over \( W(g) \) explicit. The result is the celebrated Kac-Peterson formula [4].

Generally, given any lattice \( \Lambda \), the translate \( v + \Lambda \) of any vector \( v \in \mathbb{Q} \otimes \Lambda \) defines a
glue class. The theta series of that glue class is given by

$$\Theta(v + \Lambda)(z|\tau) \overset{\text{def}}{=} \sum_{x \in v + \Lambda} \exp[\pi i \tau x^2 + 2\pi i z \cdot x].$$  \hspace{1cm} (2.1)$$

Here \( \tau \in \mathbb{C} \), and \( z \) is a complex vector lying in \( \mathbb{C} \otimes \Lambda \). If the lattice \( \Lambda \) is chosen to be positive definite this series converges, and in fact is analytic for all such \( z \) and any \( \tau \) in the upper half-plane. Given any Euclidean lattice \( \Lambda \) and a positive number \( \ell \), we write \( \Lambda^{(\ell)} \) for the positive definite scaled lattice \( \sqrt{\ell} \Lambda \) and \( \Lambda^{(-\ell)} \) for the corresponding negative definite scaled lattice.

The Kac-Peterson character formula for any \( \lambda \in P_+(g,k) \) can now be written as:

$$\chi_{\lambda + \rho}^{g,k}(z, \tau) = \sum_{w \in W(g)} \epsilon(w) \frac{\Theta(\frac{\lambda + \rho}{\sqrt{K}} + M^{(K)})(\sqrt{K} w(z)|\tau)}{D_g(z|\tau)}.$$  \hspace{1cm} (2.2a)

$$D_g(z|\tau) \overset{\text{def}}{=} \sum_{w \in W(g)} \epsilon(w) \Theta(\frac{\rho}{\sqrt{h^{(\nu)}}} + M^{(h^{(\nu)})})(\sqrt{h^{(\nu)}} w(z)|\tau),$$  \hspace{1cm} (2.2b)

where \( K = k + h^{(\nu)} \) is the height of the representation, and \( \epsilon(w) = \det(w) \in \{\pm 1\} \) the “parity” of the transformation \( w \in W(g) \).

Actually, the character should have been written as \( \chi_{\lambda + \rho}^{g,k}(u, z, \tau) \) in its full dependence on all three coordinates \( z \in \mathbb{C} \otimes M \) and \( u, \tau \in \mathbb{C}, \Im(\tau) > 0 \). However, the \( u \)-dependence is trivial, being an overall exponential, and will not play any role in our study; for this reason and without loss in generality we set \( u = 0 \). On the other hand, it is important to maintain the full \( z \)-dependence; otherwise the characters would cease to be linearly independent, and the partition functions of two representations that are complex conjugates of each other would become identical.

So far \( \chi_{\lambda}^{g,k} \) has been defined only for \( \lambda - \rho \in P_+(g,k) \). However, for later purposes, it is convenient to use (2.2a) to define \( \chi_{\lambda}^{g,k} \) for any \( \lambda \in M^* \). The following observation [17] permits us to evaluate these \( \chi_{\lambda}^{g,k} \). For any \( \lambda \in M^* \) and \( k \geq 0 \), there are two possibilities: either the character vanishes

$$\chi_{\lambda}^{g,k}(z, \tau) = 0$$  \hspace{1cm} (2.3a)

for all \( z \) and \( \tau \), or there exists a unique \( w' \in W(g) \) and unique \( \lambda' \in P_+(g,k) \) such that \( \lambda' + \rho = w'(\lambda) \pmod{M^{(K^2)}} \) and

$$\chi_{\lambda}^{g,k}(z, \tau) = \epsilon(w') \chi_{\lambda'}^{g,k}(z, \tau)$$  \hspace{1cm} (2.3b)

for all \( z \) and \( \tau \). We call \( \tilde{\epsilon}(\lambda) \) the “parity” of \( \lambda \) defined as \( \tilde{\epsilon}(\lambda) = 0 \) if (2.3a) holds, and \( \tilde{\epsilon}(\lambda) = \epsilon(w') \) if (2.3b) does. In the latter case \([\lambda]\) will denote the unique \( \lambda' + \rho = w'(\lambda) \pmod{M^{(K^2)}} \).

Let us now recall a few relevant facts about lattices [19,20]. An integral lattice is one in which all vectors give integral dot products. An even lattice is an integral lattice in which all vectors have even norms. An odd lattice is an integral lattice with at least one odd-norm vector. A self-dual lattice \( \Lambda \) is one which equals its dual \( \Lambda^* \). \( \Lambda \) is self-dual iff it
is integral and has determinant $|\Lambda| = 1$. One calls a gluing $\Lambda$ of some lattice $\Lambda_0$ any lattice that can be written as a finite disjoint union of glue classes of $\Lambda_0$. In other words, $\Lambda_0$ is a sublattice in $\Lambda$ of finite index, i.e., such that $\Lambda/\Lambda_0$ is a finite group.

In the approach we are following, extensive use is made of Lorentzian, or indefinite, lattices. In such a lattice, every vector $x$ may be written $(x_L; x_R)$, and the inner product of any pair of lattice vectors $x$ and $x'$ is given by $x \cdot x' = x_L \cdot x'_L - x_R \cdot x'_R$. In particular, let us consider the $2r$-dimensional indefinite lattice $M^{(K)}_g \oplus M^{(-K)}_g$ that we shall call $\Lambda^{g,k} \triangleq (M^{(K)}_g; M^{(K)}_g)$, with the height of $g$, $K = k + h^\vee$, being used as the scaling factor. Its elements are written as $x = (x_L; x_R)$, where $x_L, x_R \in M^{(K)}_g$. A gluing $\Lambda$ of $\Lambda^{g,k}$ will look like $\Lambda = \Lambda^g_k [x_1, \ldots, x_n] \triangleq \{ x + \sum_{i=1}^n x_i l_i | l_i \in \mathbb{Z}, x \in \Lambda^{g,k} \}$, where the vectors $x_1, \ldots, x_n \in \Lambda^{g,k}$ are called the glues of the lattice $\Lambda$. For example, take $x_i = (\beta_i / \sqrt{K}; \beta_i / \sqrt{K})$ where, as usual, $\beta_i$ are the fundamental weights of $g$. Then the gluing $\Lambda_D = \Lambda^g_k [x_1, \ldots, x_r]$, called the diagonal gluing, can be seen to be both even and self-dual. (In the appendix we give a practical test for determining whether or not a given gluing is self-dual.)

For any gluing $\Lambda$ of the lattice $\Lambda^{g,k}$, let us define its level $k$ partition function as

$$Z^g_{\Lambda}(g, k)(z_L z_R | \tau) \triangleq \sum_{w_L, w_R \in W(g)} \epsilon(w_L) \epsilon(w_R) \sum_{(x_L, x_R) \in \Lambda} \exp[\pi i (\tau x_L^2 - \tau^* x_R^2)]$$

$$\times \exp[2\pi i \sqrt{K} (w_L(z_L) \cdot x_L - w_R(z_R)^* \cdot x_R)] / D_g(z_L | \tau) D_g(z_R | \tau)^*.$$

Because $\Lambda$ is a gluing of $\Lambda^{g,k}$, (2.2a) tells us we can write $Z^g_{\Lambda}(g, k)(z_L z_R | \tau)$ as a sum of terms proportional to

$$\chi_{\lambda_L}^{g,k}(z_L, \tau) \chi_{\lambda_R}^{g,k}(z_R, \tau)^*$$

for $\lambda_L, \lambda_R \in M^*$. Eqs.(2.3a, b) further tell us that $Z^g_{\Lambda}(g, k)(z_L z_R | \tau)$ can be written as a linear combination (over $\mathbb{Z}$) of terms of the form (2.5), but with $\lambda_L - \rho$ and $\lambda_R - \rho$ now lying in $P_+(g, k)$, that is,

$$Z^g_{\Lambda}(g, k)(z_L z_R | \tau) \triangleq \sum_{\lambda_L, \lambda_R \in P_+ + \rho} (N_{\Lambda})_{\lambda_L, \lambda_R} \chi_{\lambda_L}^{g,k}(z_L, \tau) \chi_{\lambda_R}^{g,k}(z_R, \tau)^*,$$

where $(N_{\Lambda})_{\lambda_L, \lambda_R}$ are integral scalars. For example, the diagonal gluing $\Lambda_D$ has the coefficient matrix $(N_{\Lambda_D})_{\lambda_L, \lambda_R} = ||W(g)|| \delta_{\lambda_L \lambda_R}$. (||$W(g)||$ is the order of the Weyl group of $g$.)

If in addition the lattice $\Lambda$ is both even and self-dual, its associated partition function $Z^g_{\Lambda}$ is modular invariant. This is the key property we are looking for. The RTW lattice method suggests we generate all these functions (in the Appendix we describe how we did so). The resulting invariants are not necessarily linearly independent, nor are they immediately consistent with conditions (P2) or (P3). But it is always possible to deduce from them a linear independent set and to construct linear combinations

$$Z(z_L z_R | \tau) = \sum_{\Lambda} A_{\Lambda} Z^g_{\Lambda}(g, k)(z_L z_R | \tau)$$

(2.7)
that satisfy both (P2) and (P3) (precisely how we do this is discussed at the end of the Appendix). Such sums will certainly be physical invariants provided each \( \Lambda \) included in the sum is an even self-dual gluing of \( \Lambda^{g,k} \). The set of the partition functions \( Z_\Lambda(g,k) \) corresponding to all even self-dual gluings \( \Lambda \) of \( \Lambda^{g,k} \) is non-empty (containing at least the diagonal invariant) and finite (its content being limited by the level and the rank of the algebra). As shown in [12] this set is complete: it spans the *Weyl-folded commutant* of \( (g,k) - i.e. \) the complex space of all partition functions (1.1) invariant under the modular group. Thus the lattice method will allow us to compute all possible, and hence all physical, invariants for a given \( (g,k) \).

Incidentally, this RTW lattice method has been generalized in [16] to heterotic modular invariants, where it is also proven to be complete. It can also be used to find modular invariants for a given \( (g,k) \) group. Thus the lattice method will allow us to compute all possible, and hence all physical, invariants satisfying conditions (P2) and (P3) (precisely how we do this is discussed at the end of the Appendix).

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Incidentally, this RTW lattice method has been generalized in [16] to heterotic modular invariants, where it is also proven to be complete. It can also be used to find modular invariants for the coset models.

### III. INVARIANTS OF THE RANK-TWO ALGEBRAS

The above formalism can be applied in principle to any Lie algebra \( g \) at any level \( k \). However, since the dimension of the corresponding lattices increases with the rank \( r \) of the algebra and the members of the set \( P_+(g,k) \) grow rapidly in number with \( r \) and level \( k \), we will limit ourselves in this work to the four rank-two algebras, \( A_2 \), \( A_1 + A_1 \), \( G_2 \), and \( C_2 \), and to levels approximately less than 30. Following the approach outlined above, we have calculated the partition functions for all of the even self-dual gluings of \( (A_2^{(k+3)}, A_2^{(k+3)}) \) and \( (A_1^{(k+2)} \oplus A_1^{(k+2)}, A_1^{(k+2)} \oplus A_1^{(k+2)}) \) for various values \( k = 1, 2, \ldots \). From these results, we deduced the corresponding invariants for \( G_2 \) and \( C_2 \). In each case, a set of linear independent invariants was obtained and finally, by linear combinations, the physical invariants satisfying conditions (P2) and (P3) were determined.

In the algebra \( g = A_2 \), the two colabels are equal to 1, and so the dual Coxeter number is \( h^\vee = 3 \) and the height \( K = k + 3 \) at level \( k \). Let \( \beta_1 \) and \( \beta_2 \) stand for the fundamental weights of \( A_2 \), and \( \rho = \beta_1 + \beta_2 \), their sum. The coroot (= root) lattice is also denoted by \( A_2 \), and its dual by \( A_2^* \). As usual, any \( \lambda = m \beta_1 + n \beta_2 \in A_2^* \) is identified through its Dynkin labels as \( (m,n) \).

The set of all possible horizontal highest weights corresponding to level \( k \) is given by

\[
P_+(A_2,k) = \{ m \beta_1 + n \beta_2 \mid m, n \in \mathbb{Z}, \ 0 \leq m, n, \ m + n \leq k \}. \quad (3.1a)
\]

The trivial representation (the vacuum) at level \( k \) defined by the highest weight \( \lambda = 0 \) is associated with the character \( \chi_{\rho}^k = \chi_{11}^k \), so (P3) reads \( N_{11,11} = 1 \). We will also refer to the set of highest weights

\[
P(A_2,k) = P_+(A_2,k) + \rho = \{ m \beta_1 + n \beta_2 \mid m, n \in \mathbb{Z}, \ 0 < m, n, \ m + n < k + 3 \}. \quad (3.1b)
\]

The first step in our procedure is to find all self-dual gluings of \( (A_2^{(K)}, A_2^{(K)}) \), a step which we will describe in some detail in the Appendix. Next, to convert a lattice partition function (2.4) to the form (1.1), it suffices to write \( \chi_\lambda \) for arbitrary \( \lambda \in A_2^* \) in terms of \( \chi_{\lambda'} \) for \( \lambda' \in P(A_2,k) \). This is not difficult once the action of the six elements of \( W(A_2) \) on \( A_2^* \)
is written out explicitly. For example, \( \chi_{m,n} = 0 \iff m \equiv 0 \pmod{K} \), or \( n \equiv 0 \pmod{K} \), or \( m + n \equiv 0 \pmod{K} \).

Finally, under the outer automorphism \( h \) of the \( A_2 \) algebra, the character transforms as

\[
\chi_{h(m,n)} = \chi_{n,m}.
\]

(3.2a)

If \( Z \) is any invariant with coefficient matrix \( N \), we can construct its conjugation, \( Z^c \), by

\[
N_{\lambda L, \lambda R}^c = N_{\lambda L, h(h(\lambda_R))}.
\]

(3.2b)

The conjugation of any physical invariant is also a physical invariant. We can similarly define two other invariants, \( cZ \) and \( Z^c \), respectively by \( cN_{\lambda L, \lambda R} = N_{h(\lambda_L), \lambda R} \) and \( cN_{\lambda L, \lambda R}^c = N_{h(\lambda_L), h(\lambda_R)} \), but in this algebra one always has \( cZ = Z^c \) and \( cZ^c = Z \).

The fact that \( G_2 \) contains \( A_2 \) as a subalgebra of equal rank gives us a many-to-one mapping from the \( A_2 \) invariants to the \( G_2 \) invariants (this was also discussed in [14]). In particular, let \( \tilde{\alpha}_1 \) (\( \tilde{\alpha}_2 \)) denote the long (short) simple root of \( G_2 \); then the colabels are \( a_1^\vee = 2 \) and \( a_2^\vee = 1 \), and Kac-Peterson formula (2.2a) tells us

\[
\tilde{\chi}_{m,n} = \chi_{m,m+n} - \chi_{m+n,m},
\]

(3.3)

where the character \( \tilde{\chi} \) belongs to \( G_2^{(1)} \) at level \( k \), and the characters \( \chi \) belong to \( A_2^{(1)} \) at level \( k+1 \). Given any \((k+1)\)-level \( A_2 \) invariant \( Z \) in \((1.1)\), we can construct a \( k \)-level \( G_2 \) invariant \( \tilde{Z} \) by applying \((3.3)\) to \( Z - Z^c - cZ + cZ^c \). Unfortunately, although this mapping preserves \((P1)\), it does not necessarily preserve either \((P2)\) or \((P3)\). Nevertheless, all \( G_2 \) invariants can be obtained in this way from \( A_2 \) invariants. Thus we can read off the \( G_2 \) Weyl-folded commutant of level \( k \) from the \( A_2 \) Weyl-folded commutant of level \( k+1 \); the physical invariants are then obtained in the usual way.

Let us now turn to the case \( g = A_1 + A_1 \). The lattice \( M_g \) here is the orthogonal direct sum \( A_1 + A_1 = \mathbb{Z}^{(2)} \). The two fundamental weights of this algebra are \( \beta_1 = (1/\sqrt{2}, 0) \) and \( \beta_2 = (0, 1/\sqrt{2}) \). The vector \( \rho \) is as usual \( \beta_1 + \beta_2 \). The integrable highest weight representations of the Kac-Moody algebra \( \hat{g} = \hat{A}_1 + \hat{A}_1 \) have two levels, \( k_1 \) and \( k_2 \), one for each \( \hat{A}_1 \) factor. Define the set

\[
P_+(A_1 + A_1, k_1, k_2) = \{ m\beta_1 + n\beta_2 \mid m, n \in \mathbb{Z}, \ 0 \leq m \leq k_1, \ 0 \leq n \leq k_2 \}.
\]

(3.4)

The characters of this algebra, \( \chi_{m,n}^{k_1,k_2} \), are labelled by \((m, n)\). They are just products of the characters of \( \hat{A}_1^{(1)} \)

\[
\chi_{m,n}^{k_1,k_2} = \chi_{m}^{k_1} \chi_{n}^{k_2}.
\]

(3.5)

When \( k_1 = k_2 \) (the case we will be considering), there is a conjugation that interchanges the two \( \hat{A}_1 \) factors. It maps a weight \((m, n)\) to \((n, m)\), as in \((3.2a)\). Through this mapping, any invariant \( Z \) can be associated with three other invariants, \( Z^c \), \( cZ \), and \( cZ^c \), as for \( A_2 \). However, unlike for \( A_2 \), these four invariants will all be distinct in general.

Because \( C_2 \) contains \( A_1 + A_1 \) as an equal rank subalgebra, the same trick used for \( G_2 \) will equally work here. The colabels are \( a_1^\vee = a_2^\vee = 1 \). A formula analogous to \((3.3)\) holds.
It defines a surjective map from the \((k + 1, k + 1)\)-level \(A_1 + A_1\) Weyl-folded commutant to the \(k\)-level \(C_2\) Weyl-folded commutant as before.

There is a property already discussed in [12] (also found in [21]) that proves to be quite useful in practice. Consider any \(\lambda_L - \rho, \lambda_R - \rho \in P_+(g, k)\) and \(\Lambda^{g,k}\), the Lorentzian root lattice of \(g\) scaled up in the usual way. We call a positive integer \(L\) the order of \((\lambda_L; \lambda_R)\) in \(\Lambda^{g,k}\) when, for any integer \(m\), \((m\lambda_L; m\lambda_R) \in \Lambda_k\) iff \(L\) divides \(m\). For each \(\ell\) relatively prime to \(L\), \((\ell\lambda_L; \ell\lambda_R)\) is related nontrivially to \((\lambda_L; \lambda_R)\), that is, \(\bar{\epsilon}(\ell\lambda_L)\bar{\epsilon}(\ell\lambda_R) \neq 0\) and

\[
N_{\lambda_L\lambda_R} = \bar{\epsilon}(\ell\lambda_L)\bar{\epsilon}(\ell\lambda_R)N_{[\ell\lambda_L][\ell\lambda_R]}
\]

(\(\bar{\epsilon}\) and \([\lambda]\) are defined in (2.3a, b)). See [12] for examples.

This property has two valuable consequences. All such \([(\ell\lambda_L); (\ell\lambda_R)]\) form a family of essentially equivalent representations, so that just one representative of each family need be stored. Another practical implication of (3.6) is that if for some \(\ell\), \(\bar{\epsilon}(\ell\lambda_L)\bar{\epsilon}(\ell\lambda_R) = -1\), then \(N_{\lambda_L\lambda_R} = 0\) for any positive invariant \(N\).

Call the family \([(\ell\lambda_L); (\ell\lambda_R)]\) a positive parity family if \(\bar{\epsilon}(\ell\lambda_L)\bar{\epsilon}(\ell\lambda_R) = 1\) for all \(\ell\) relatively prime to the order \(L\). By the positive parity commutant we mean the subspace of the Weyl-folded commutant consisting of all invariants \(Z\) with the property that \(N_{\lambda_L\lambda_R} \neq 0\) only for \((\lambda_L; \lambda_R)\) in positive parity families. The positive parity commutant contains all positive invariants and so is the only part of the Weyl-folded commutant we need to consider. As we shall see, it is generally significantly smaller than the full Weyl-folded commutant.

**IV. THE PHYSICAL INVARIANTS**

Most of this section is devoted to listing all the physical invariants we found. Our results are summarized in Tables 1 and 2. We give there the dimensions of the Weyl-folded and positive parity commutants, along with the numbers of physical invariants for each algebra and level we considered. The positive parity commutant was defined at the end of the previous section; it necessarily contains all the physical invariants and was the subspace we focused on. The tables show the smallness of its dimension \(P\), especially for \(A_2, G_2\) and \(C_2\), compared with the dimension \(D\) of the Weyl-folded commutant. This strongly suggests that a key component of a classification proof for these (and probably all) algebras is eq.(3.6). Indeed that is the case with the \(A_2\) classification proof in [10].

It seems to be quite difficult to obtain a general formula for the dimension \(D\) of the Weyl-folded commutant (though much more is known [22] about a related space, the pre-Weyl-folded commutant). Tables 1 and 2 suggest the following formulae for \(D\) at prime
heights $K = k + h^\vee$:

\[
\begin{align*}
\text{for } A_2 : \quad D &= \begin{cases} (K + 5)/3 & \text{when } K \equiv 1 \pmod{3} \\
(K + 1)/3 & \text{when } K \equiv 2 \pmod{3} \end{cases}; \\
\text{for } G_2 : \quad D &= \begin{cases} (K + 5)/6 & \text{when } K \equiv 1 \pmod{3} \\
(K + 1)/6 & \text{when } K \equiv 2 \pmod{3} \end{cases}; \\
\text{for } A_1 + A_1 : \quad D &= \begin{cases} (K + 3)/2 & \text{when } K \equiv 1 \pmod{4} \\
(K + 1)/2 & \text{when } K \equiv 3 \pmod{4} \end{cases}; \\
\text{for } C_2 : \quad D &= \begin{cases} (K + 3)/4 & \text{when } K \equiv 1 \pmod{4} \\
(K + 1)/4 & \text{when } K \equiv 3 \pmod{4} \end{cases}.
\end{align*}
\]

Of course all these formulae only apply to prime $K$; formulae for composite heights will be more complicated and are presently unknown (with one exception noted below). The above formula for $A_2$ is proven in [23]. Ph. Ruelle [24] has also obtained the formula for $G_2$ given above. In addition, at the heights $K = p^2$ of $A_2$, where $p \equiv 5 \pmod{12}$ is prime, E. Thiran [24] has found the formula $D = 2(7q^2 - 3q + 1)$ where $q = (K + 1)/6$. For $k = 22$ (the only point of intersection of Thiran’s formula and our work), both this formula and Table 1 gives $D = 10$.

One curiosity we obtained was that all invariants, physical or otherwise, for all the algebras and levels we considered were symmetric – i.e. their coefficient matrices satisfied $N_{\lambda\lambda'} = N_{\lambda'\lambda}$. This relation is not true of all algebras (e.g. $g = D_{4n}$), but whenever it does hold, an easy calculation shows that the coefficient matrices of all invariants will commute with each other, so it could be a valuable theoretical property when studying the commutant. We have been able to prove, using the analysis in [23], that all invariants of $A_2$ height $K$ will be symmetric whenever $K$ is a product of distinct primes and is not divisible by 3 (a similar result will then necessarily hold for these heights of $G_2$), but the case of general $K$ seems more difficult.

Letting $P^k$ denote the set $P(A_2, k)$ in (3.1b) and dropping the superscripts ‘$A_2, k$’ from the characters, the known physical invariants of $A_2$ become:

\[
\begin{align*}
A_k \overset{\text{def}}{=} & \sum_{\lambda \in P^k} |\chi_{\lambda}|^2; \quad (4.1a) \\
D_k \overset{\text{def}}{=} & \sum_{(m,n) \in P^k} \chi_{m,n} \chi_{\omega^k(m-n),(m,n)}^*; \quad \text{for } k \not\equiv 0 \pmod{3} \text{ and } k \geq 4; \quad (4.1b) \\
D_k \overset{\text{def}}{=} & \frac{1}{3} \sum_{(m,n) \in P^k} |\chi_{m,n} + \chi_{\omega(m,n)} + \chi_{\omega^2(m,n)}|^2 \quad \text{for } k \equiv 0 \pmod{3}; \quad (4.1c) \\
E_5 \overset{\text{def}}{=} & |\chi_{1,1} + \chi_{3,3}|^2 + |\chi_{1,3} + \chi_{4,3}|^2 + |\chi_{3,1} + \chi_{3,4}|^2 \\
& + |\chi_{3,2} + \chi_{1,6}|^2 + |\chi_{4,1} + \chi_{1,4}|^2 + |\chi_{2,3} + \chi_{6,1}|^2; \quad (4.1d) \\
E_9^{(1)} \overset{\text{def}}{=} & |\chi_{1,1} + \chi_{1,10} + \chi_{10,1} + \chi_{5,5} + \chi_{5,2} + \chi_{2,5}|^2 \\
& + 2|\chi_{3,3} + \chi_{3,6} + \chi_{6,3}|^2; \quad (4.1e)
\end{align*}
\]
two invariants due to conformal embeddings at \( k \) in [27], and \( (4.3) \) invariant. Note that together with their conjugations Eq. (4.2) exhaust all known. This list was proven complete in [10] for all but finitely many levels. The remaining levels are all covered by this program, so we now know eqs. (4.1) give the complete list of \( A_2 \) physical invariants.

Now letting \( P^k \) denote the set \( \rho + P_+(G_2, k) \) and again dropping superscripts from the characters, the known \( G_2 \) physical invariants are the series \( A_k \)\( \sum_{\lambda \in P^k} |\chi_\lambda|^2, \) (4.2a)

two invariants due to conformal embeddings at \( k = 3 \) and \( k = 4 \) (found by [27])

\[
E_3 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{2,2}|^2 + 2|\chi_{1,3}|^2; \quad (4.2b)
\]

\[
E_4^{(1)} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,4}|^2 + |\chi_{2,1} + \chi_{1,5}|^2 + 2|\chi_{2,2}|^2; \quad (4.2c)
\]

and another invariant at \( k = 4 \) (found in [13])

\[
E_4^{(2)} \overset{\text{def}}{=} |\chi_{1,1}|^2 + |\chi_{2,2}|^2 + |\chi_{1,3}|^2 + |\chi_{2,3}|^2 + |\chi_{1,4}|^2 + \chi_{2,1}\chi_{*5} + \chi_{1,5}\chi_{*2,1} + \chi_{3,1}\chi_{*1,2} + \chi_{1,2}\chi_{*3,1}. \quad (4.2d)
\]

Eqs. (4.2) exhaust all known \( G_2 \) physical invariants. This list has been proven complete for all prime heights \( K = k + 4 \) satisfying \( K \equiv 5, 7 \) (mod 12) [24]; of course our program verifies its completeness for all other \( k \leq 31.\)

For the algebra \( A_1 + A_1, \) we have only investigated the case \( k_1 = k_2 = k = K - 2. \) The physical invariants belonging to infinite series are all simple current invariants [7], and have been described in [3]. We shall label them using simple current notation. These invariants may be given in terms of their coefficient matrices \( N_{ij,i'j'} \).

In the next few paragraphs, we mean by \( A_k, D_k, E_{10}, E_{16}, \) and \( E_{28} \) the various \( A_1 \) physical invariants [5]. Tensor products of their coefficient matrices are coefficient matrices of \( A_1 + A_1 \) physical invariants. These tensor products give rise to some of the series
solutions. In all, there are 4 series when \( k \) is odd and 12 series when \( k \) is even, and a number of exceptional invariants at various levels.

For \( k \) odd, the 4 series are:
(i) the diagonal invariant \( N_k(0) \) defined by 
\[
N_k(0) \overset{\text{def}}{=} A_k \otimes A_k;
\]
(ii) the invariant \( N_k(J_1 J_2) \), for \( k > 1 \), defined by 
\[
N_k(J_1 J_2)_{ij,i'j'} \overset{\text{def}}{=} \begin{cases} 
\delta_{ii'}\delta_{jj'} & \text{if } i \equiv j \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} & \text{otherwise} 
\end{cases}; \quad (4.3a)
\]
as well as their conjugations \( N_k(0)^c \) and \( N_k(J_1 J_2)^c \).

For \( k \equiv 2 \pmod{4} \), there are 12 series:
(i) the diagonal invariant \( N_k(0) \);
(ii) the invariant \( N_k(J_1) \) defined by 
\[
N_k(J_1) \overset{\text{def}}{=} D_k \otimes A_k \text{ for } k > 2;
\]
(iii) the invariant \( N_k(J_2) \) defined by 
\[
N_k(J_2) \overset{\text{def}}{=} A_k \otimes D_k \text{ for } k > 2;
\]
(iv) the invariant \( N_k(J_1 J_2) \) defined by 
\[
N_k(J_1 J_2)_{ij,i'j'} \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } i \equiv j \pmod{2} \\
\delta_{ii'}\delta_{jj'} + \delta_{i,K-i'}\delta_{j,K-j'} & \text{otherwise} 
\end{cases}; \quad (4.3b)
\]
(v) the invariant \( N_k(J_1; J_2) \) defined by 
\[
N_k(J_1; J_2) \overset{\text{def}}{=} D_k \otimes D_k \text{ for } k > 2;
\]
(vi) the invariant \( N_k(J_1 J_2) \), for \( k > 2 \), defined by 
\[
N_k(J_1 J_2)_{ij,i'j'} \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{ii'}\delta_{jj'} + \delta_{i,K-i'}\delta_{j,K-j'} & \text{if } i \equiv j \equiv 0 \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} + \delta_{ii'}\delta_{jj'} & \text{otherwise} 
\end{cases}; \quad (4.3c)
\]
together with their conjugations \( N^c \). We find \( N_2(J_1 J_2)^c = N_2(J_1 J_2) \).

For \( k \equiv 0 \pmod{4} \), the 12 series are:
(i) the diagonal invariant \( N_k(0) \);
(ii) the invariant \( N_k(J_1) \) defined by 
\[
N_k(J_1) \overset{\text{def}}{=} D_k \otimes A_k;
\]
(iii) the invariant \( N_k(J_2) \) defined by 
\[
N_k(J_2) \overset{\text{def}}{=} A_k \otimes D_k;
\]
(iv) the invariant \( N_k(J_1 J_2) \) defined by 
\[
N_k(J_1 J_2)_{ij,i'j'} \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{ii'}\delta_{jj'} & \text{if } i \equiv j \equiv 0 \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} & \text{if } i \equiv 1, j \equiv 0 \pmod{2} \\
\delta_{ii'}\delta_{jj'} & \text{if } i \equiv 0, j \equiv 1 \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} & \text{if } i \equiv j \equiv 0 \pmod{2}
\end{cases}; \quad (4.3d)
\]
(v) the invariant \( N_k(J_1; J_2) \) defined by 
\[
N_k(J_1; J_2) \overset{\text{def}}{=} D_k \otimes D_k;
\]
(vi) the invariant \( N_k(aut) \) defined by 
\[
N_k(aut)_{ij,i'j'} \overset{\text{def}}{=} \begin{cases} 
\delta_{ii'}\delta_{jj'} & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} & \text{if } i \equiv 1, j \equiv 0 \pmod{2} \\
\delta_{ii'}\delta_{jj'} & \text{if } i \equiv 0, j \equiv 1 \pmod{2} \\
\delta_{i,K-i'}\delta_{j,K-j'} & \text{if } i \equiv j \equiv 0 \pmod{2}
\end{cases}; \quad (4.3e)
\]
together with their conjugations \(N^c\).

Besides these series, we also obtain a number of \(A_1 + A_1\) exceptional physical invariants. They occur at any level \(k\) where there exists an \(A_1\) exceptional invariant; invariants of this type are given by the tensor products \(A_{10} \otimes E_{10}, D_{10} \otimes E_{10}, E_{10} \otimes A_{10}, E_{10} \otimes D_{10}, E_{10} \otimes E_{10}, A_{16} \otimes E_{16}, D_{16} \otimes E_{16}, E_{16} \otimes A_{16}, E_{16} \otimes D_{16}\) and \(E_{16} \otimes E_{16}\), and their respective conjugations \(Z^c\). There exist further sporadic exceptional physical invariants not of this type, which we denote by \(E''_k\) to distinguish them from the \(A_1\) exceptions. They are:

\[
E'_4 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,5} + \chi_{5,1} + \chi_{5,5}|^2 + 2|\chi_{1,3} + \chi_{5,3}|^2 + 2|\chi_{3,1} + \chi_{3,5}|^2 + 4|\chi_{3,3}|^2; \quad (4.3f)
\]
\[
E'_6 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{3,5} + \chi_{5,3} + \chi_{7,7}|^2 + |\chi_{3,3} + \chi_{7,1} + \chi_{1,7} + \chi_{5,5}|^2
+ |\chi_{2,4} + \chi_{4,2} + \chi_{6,4} + \chi_{4,6}|^2; \quad (4.3g)
\]
\[
E''_8 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,9} + \chi_{9,1} + \chi_{9,9}|^2 + |\chi_{3,3} + \chi_{3,7} + \chi_{7,3} + \chi_{7,7}|^2
+ (\chi_{1,3} + \chi_{1,7} + \chi_{9,3} + \chi_{9,7} + \chi_{3,1} + \chi_{3,9} + \chi_{7,1} + \chi_{7,9})\chi_{5,5}^*
+ |\chi_{3,5}|^2 + |\chi_{3,5} + \chi_{7,5} + \chi_{5,3} + \chi_{5,7}|^2 + |\chi_{1,5} + \chi_{9,5} + \chi_{5,1} + \chi_{5,9}|^2; \quad (4.3h)
\]
\[
E''_10^{(1)} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,7} + \chi_{5,5} + \chi_{5,11} + \chi_{7,1} + \chi_{7,7} + \chi_{11,5} + \chi_{11,11}|^2
+ |\chi_{1,11} + \chi_{1,5} + \chi_{7,11} + \chi_{7,5} + \chi_{5,7} + \chi_{5,1} + \chi_{11,1} + \chi_{11,7}|^2
+ 2|\chi_{4,4} + \chi_{4,8} + \chi_{8,4} + \chi_{8,8}|^2; \quad (4.3i)
\]
\[
E''_10^{(2)} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,7} + \chi_{11,5} + \chi_{11,11}|^2 + |\chi_{5,5} + \chi_{5,11} + \chi_{7,1} + \chi_{7,7}|^2
+ |\chi_{1,5} + \chi_{1,11} + \chi_{11,1} + \chi_{11,7}|^2 + |\chi_{5,1} + \chi_{5,7} + \chi_{7,5} + \chi_{7,11}|^2
+ |\chi_{2,4} + \chi_{2,8} + \chi_{10,4} + \chi_{10,8}|^2 + |\chi_{4,4} + \chi_{4,8} + \chi_{8,4} + \chi_{8,8}|^2 + 2|\chi_{6,4} + \chi_{6,8}|^2
+ |\chi_{3,1} + \chi_{3,7} + \chi_{9,5} + \chi_{9,11}|^2 + |\chi_{3,5} + \chi_{3,11} + \chi_{9,1} + \chi_{9,7}|^2; \quad (4.3j)
\]
together with the three conjugations \(E''_10^{(2) c}, cE''_10^{(2)}\) and \(cE''_10^{(2) c}\) of \((4.3j)\).

The invariant \((4.3g)\) is due to a conformal embedding. The invariant \((4.3f)\) was found in [3], while \((4.3h)\) was found in [13]. The invariant \((4.3i)\) can be understood as due to the conformal embedding of \(A_1 + A_1\) level \((10,10)\) into \(C_2 + C_2\) level \((1,1)\), applied to a non-diagonal physical invariant of the latter. The invariant \((4.3j)\) can be written as the product of the coefficient matrix of \(A_{10} \otimes E_{10}\) with \(N_{10}(J_1 J_2)\) (see \((4.3b)\)). Note that \(k = 10\) has an incredibly large number of 27 physical invariants.

Eqs. \((4.3)\) exhaust all the \(A_1 + A_1\) physical invariants for \(k_1 = k_2 \leq 21\). The only other known ones for \(k_1 = k_2\) occur at \(k = 28\).

As before let \(P^k\) denote the set \(\rho + P_+(C_2, k)\). The \(C_2\) physical invariants include two series of solutions, namely,

\[
A_k \overset{\text{def}}{=} \sum_{\chi \in P^k} |\chi\lambda|^2, \quad (4.4a)
\]
\[
D_k \overset{\text{def}}{=} \sum_{(m,n) \in P^k \atop m \text{ odd}} |\chi_{m,n}|^2 + \sum_{(m,n) \in P^k \atop m \text{ even}} \chi_{m,n}^* \chi_{m,K-n} \text{ for } k > 1 \text{ odd,} \quad (4.4b)
\]
and four exceptional solutions:
\[ E_3 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{3,2}|^2 + |\chi_{1,4} + \chi_{3,1}|^2 + 2|\chi_{2,2}|^2; \] (4.4d)
\[ E_7 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,6} + \chi_{3,3} + \chi_{7,2}|^2 \\
+ |\chi_{1,3} + \chi_{1,8} + \chi_{3,4} + \chi_{7,1}|^2 + 2|\chi_{4,2} + \chi_{4,4}|^2; \] (4.4e)
\[ E_8 \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,9}|^2 + |\chi_{3,3} + \chi_{3,5}|^2 + |\chi_{5,5} + \chi_{5,1}|^2 + |\chi_{1,7} + \chi_{3,1}|^2 + |\chi_{5,4} + \chi_{5,2}|^2 \\
+ |\chi_{9,1}|^2 + \chi_{9,1}(\chi_{1,2} + \chi_{1,8})^* + (\chi_{1,2} + \chi_{1,8})\chi_{9,1}^* + |\chi_{5,3}|^2 + \chi_{5,3}(\chi_{3,6} + \chi_{3,2})^* \\
+ (\chi_{3,6} + \chi_{3,2})\chi_{5,3}^* + |\chi_{1,5}|^2 + \chi_{1,5}(\chi_{3,7} + \chi_{3,1})^* + (\chi_{3,7} + \chi_{3,1})\chi_{1,5}^* + |\chi_{3,4}|^2 \\
+ \chi_{3,4}(\chi_{7,1} + \chi_{7,3})^* + (\chi_{7,1} + \chi_{7,3})\chi_{3,4}^* + |\chi_{7,2}|^2 \\
+ \chi_{7,2}(\chi_{1,4} + \chi_{1,6})^* + (\chi_{1,4} + \chi_{1,6})\chi_{7,2}^*; \] (4.4f)
\[ E_{12} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,13} + \chi_{3,4} + \chi_{3,8} + 2\chi_{5,5} + \chi_{7,1} + \chi_{7,7} + \chi_{9,2} + \chi_{9,4}|^2. \] (4.4g)
Eqs.(4.4b, c) are found in [26], (4.4d, e, g) are conformal embeddings found in [27], and (4.4f) is found in [13]. Eqs. (4.4) exhaust all known \( C_2 \) physical invariants, and is the complete list for \( k \leq 26 \).

V. CONCLUSION

In this paper we present a numerical search for physical invariants of the rank-two algebras \( A_2, A_1 + A_1, G_2, \) and \( C_2 \) at levels approximately less than 30. Within these limits, made necessary by the modest capacity of our personal computer, we have shown first that the self-dual lattice method is a useful and practical tool; in particular, it uncovers in these algebras and at these levels, all physical invariants, those belonging to the series as well as those lying outside the series. Secondly, we have shown that it can fruitfully complement a purely analytical approach. For example, it can identify the physical invariants at those levels of \( A_2 \) which eluded theoretical arguments. It also can uncover regularities and suggest approaches useful for these theoretical arguments. Furthermore, not only does it determine the dimensions of various commutants at specific levels but it also can suggest general formulas applicable to many other, and in some cases all, levels. It is clear that the analytical method and the present numerical method, especially when the restrictions on the ranks and levels of the algebras are relaxed, can successfully complement each other in working towards a solution of the classification of the conformal field theories.

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Table 1. Dimensions of the commutants and numbers of physical invariants for $A_2$ and $G_2$

| k | $A_2$ | | $G_2$ | |
|---|---|---|---|---|
|   | D | P | N | D | P | N |
| 1 | 2 | 2 | 2 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| 3 | 4 | 4 | 3 | 2 | 2 | 2 |
| 4 | 4 | 4 | 4 | 2 | 3 | 3 |
| 5 | 6 | 6 | 6 | 1 | 1 | 1 |
| 6 | 3 | 3 | 3 | 3 | 1 | 1 |
| 7 | 6 | 4 | 4 | 2 | 1 | 1 |
| 8 | 4 | 4 | 4 | 4 | 1 | 1 |
| 9 | 10 | 8 | 7 | 3 | 1 | 1 |
| 10 | 6 | 4 | 4 | 4 | 1 | 1 |
| 11 | 8 | 4 | 4 | 4 | 1 | 1 |
| 12 | 9 | 4 | 4 | 6 | 1 | 1 |
| 13 | 12 | 4 | 4 | 3 | 1 | 1 |
| 14 | 6 | 4 | 4 | 5 | 1 | 1 |
| 15 | 11 | 4 | 4 | 4 | 1 | 1 |
| 16 | 8 | 4 | 4 | 10 | 1 | 1 |
| 17 | 21 | 7 | 4 | 6 | 1 | 1 |
| 18 | 12 | 5 | 4 | 6 | 1 | 1 |
| 19 | 12 | 4 | 4 | 4 | 1 | 1 |
| 20 | 8 | 4 | 4 | 12 | 1 | 1 |
| 21 | 26 | 9 | 5 | 5 | 1 | 1 |
| 22 | 10 | 4 | 4 | 7 | 1 | 1 |
| 23 | 14 | 4 | 4 | 5 | 1 | 1 |
| 24 | 12 | 4 | 4 | 15 | 1 | 1 |
| 25 | 26 | 4 | 4 | 5 | 1 | 1 |
| 26 | 10 | 4 | 4 | 21 | 1 | 1 |
| 27 | 28 | 4 | 4 | 6 | 1 | 1 |
| 28 | 12 | 4 | 4 | 13 | 1 | 1 |
| 29 | 24 | 4 | 4 | 9 | 1 | 1 |
| 30 | 17 | 4 | 4 | 9 | 1 | 1 |
| 31 | 18 | 4 | 4 | 11 | 1 | 1 |
| 32 | 22 | 4 | 4 | | | |

$k =$ level; $D =$ dimension of the Weyl-folded commutant; $P =$ dimension of the positive parity commutant; and $N =$ number of physical invariants.
Table 2. Dimensions of the commutants and numbers of physical invariants for $A_1 + A_1$ and $C_2$

| $k$ | $A_1 + A_1$ | $C_2$ |
|-----|-------------|--------|
|     | $D$ | $P$ | $N$ | $D$ | $P$ | $N$ |
| 1   | 2   | 2   | 2   | 1   | 1   | 1   |
| 2   | 3   | 3   | 3   | 2   | 2   | 2   |
| 3   | 4   | 4   | 4   | 3   | 3   | 3   |
| 4   | 9   | 9   | 13  | 2   | 2   | 2   |
| 5   | 4   | 4   | 4   | 3   | 2   | 2   |
| 6   | 10  | 10  | 13  | 3   | 2   | 2   |
| 7   | 6   | 4   | 4   | 5   | 3   | 3   |
| 8   | 13  | 11  | 13  | 3   | 3   | 3   |
| 9   | 6   | 4   | 4   | 9   | 4   | 2   |
| 10  | 27  | 25  | 27  | 4   | 2   | 2   |
| 11  | 8   | 4   | 4   | 6   | 2   | 2   |
| 12  | 17  | 10  | 12  | 9   | 3   | 3   |
| 13  | 22  | 4   | 4   | 7   | 2   | 2   |
| 14  | 20  | 11  | 12  | 5   | 2   | 2   |
| 15  | 10  | 4   | 4   | 10  | 2   | 2   |
| 16  | 33  | 19  | 22  | 5   | 2   | 2   |
| 17  | 10  | 4   | 4   | 14  | 2   | 2   |
| 18  | 44  | 18  | 12  | 11  | 2   | 2   |
| 19  | 26  | 4   | 4   | 9   | 2   | 2   |
| 20  | 25  | 10  | 12  | 6   | 2   | 2   |
| 21  | 12  | 4   | 4   | 21  | 2   | 2   |
| 22  |     |     |     | 9   | 2   | 2   |
| 23  |     |     |     | 11  | 2   | 2   |
| 24  |     |     |     | 12  | 2   | 2   |
| 25  |     |     |     | 17  | 2   | 2   |
| 26  |     |     |     | 8   | 2   | 2   |

$k =$ level; $D =$ dimension of the Weyl-folded commutant; $P =$ dimension of the positive parity commutant; and $N =$ number of physical invariants.
APPENDIX

We describe an algorithm for finding all even self-dual gluings of $\Omega_k = (A_2^{(k+3)}; A_2^{(k+3)})$, where $k$ is as usual the level. The method can be generalized to other algebras.

This algorithm involves the gluing construction of lattices [19,28]. One of the powers of gluing is the ease with which e.g. self-duality can be verified. A lattice is self-dual iff it is both integral and its determinant equals one. A gluing $\Lambda = \Lambda_0[g_1, \ldots, g_n]$ is integral iff its base lattice $\Lambda_0$ is integral, its glue generators $g_i$ all lie in $\Lambda_0^*$, and the dot products $g_i \cdot g_j$ are all integers. The following fact [28] allows for straightforward computation of the determinant $|\Lambda|$ of a gluing:

$$|\Lambda| = |\Lambda_0|/\|G\|^2,$$

where $\|G\|$ is the order of the glue group $G = \Lambda/\Lambda_0$ (i.e. the additive group spanned (mod $\Lambda_0$) by the glue generators $g_i$), or equivalently the number of different glue classes $x + \Lambda_0$ in $\Lambda$.

Let $K = k + 3$ and $\ell = 3K$. Let $\alpha_1$ and $\beta_1$ be the first simple root and its corresponding fundamental weight of $A_2$. By $(a, b)$ we mean the two-dimensional (Euclidean) vector $(a, b) = a\beta_1/\sqrt{K} + b\alpha_1/\sqrt{K} \in A_2^{(K)^*}$ and, similarly, $(a, b; c, d)$ the four-dimensional (Lorentzian) vector $(a\beta_1/\sqrt{K} + b\alpha_1/\sqrt{K}; a\beta_1/\sqrt{K} + b\alpha_1/\sqrt{K}) \in \Omega_k^*$. It can be shown that any vector in $A_2^{(K)^*}$ equals $(a, b)$ for some integers $a, b$; similarly for $\Omega_k^*$ and $(a, b; c, d)$. Any even self-dual gluing $\Lambda$ of $\Omega_k$ has a maximal left-hand side sublattice $\Lambda_L$ defined by the set $\Lambda_L = \{x_L| (x_L; 0) \in \Lambda\}$. The first set, $S_1$, consists of $(\ell, 0)$ and all the vectors $(a, b)$ such that $0 < a < \ell$, $0 \leq b < K$, $a$ divides $\ell$, and $a^2/3 + ab + b^2 \equiv 0$ (mod $K$). The second set, $S_2$, consists of all vectors $(0, d)$ such that $0 < d \leq K$, $d$ divides $K$, and $d^2 \equiv 0$ (mod $K$).

The possible lattices $\Lambda_L$ and $\Lambda_R$ are then given by all the triples $a, b, d$ satisfying the following conditions: $(a, b) \in S_1$, $(0, d) \in S_2$, $b < d$, $\ell b$ is divisible by $ad$, and $(a + 2b)d \equiv 0$ (mod $K$). The three congruences (mod $K$) correspond to the conditions that the glues $(a, b)$ and $(0, d)$ have even norms and integral dot products. The remaining conditions are designed to force a unique choice of glue generators. There is an obvious one-to-one correspondence between a triple $a, b, d$ and an even gluing of $A_2^{(K)}$, given by

$$a, b, d \rightarrow A_2^{(K)}[(a, b), (0, d)] \overset{\text{def}}{=} \bigcup_{i,j \in \mathbb{Z}} \{A_2^{(K)} + i(a, b) + j(0, d)\} = \bigcup_{i=1}^{\ell/a} \bigcup_{j=1}^{K/d} \{A_2^{(K)} + i(a, b) + j(0, d)\}.$$

It turns out that $(a, b)$ and $(0, d)$ form a basis for such a lattice. We are actually generating more lattices here than we need later on: several lattices may lead to the same lattice sum.
(2.4) because of the "Weyl-folding" (summing over $W(g)$) involved in this calculation; Weyl-equivalent gluings may be eliminated at this point.

The next step in the procedure is to choose the base lattice $(\Lambda_L; \Lambda_R)$ satisfying the determinantal condition $|\Lambda_L| = |\Lambda_R|$. This is equivalent to choosing triples $a$, $b$, $d$ (associated with $\Lambda_L$) and $a'$, $b'$, $d'$ (associated with $\Lambda_R$) satisfying $ad = a'd'$.

To construct the four-dimensional even self-dual gluings $\Lambda$, we define first the integers $L_1 = \ell/a$, $L_2 = K/d$, $L'_1 = \ell/a'$, and $L'_2 = K/d'$, and then the vectors

\begin{align*}
  h_1 &= (2L_1, -L_1) \\
  h_2 &= (-3L_2 - 2L_1b/d, 2L_2 + L_1b/d) \\
  h'_1 &= (2L'_1, -L'_1) \\
  h'_2 &= (-3L'_2 - 2L'_1b'/d', 2L'_2 + L'_1b'/d')
\end{align*}

These satisfy $h_1 \cdot (a, b) = h'_2 \cdot (a', b') = \delta_{i1}$ and $h_1 \cdot (0, d) = h'_2 \cdot (0, d') = \delta_{i2}$, so $h_1$ and $h_2$ ($h'_1$ and $h'_2$) form a basis for $\Lambda_L^*$ ($\Lambda_R^*$). Next we find all integers $w$, $x$, $y$, $z$, where $0 \leq w$, $y < a'$ and $0 \leq x$, $z < d'$, such that the two Lorentzian four-dimensional vectors, $(h_1; wh'_1 + xh'_2)$ and $(h_2; yh'_1 + zh'_2)$, have even norms and integral dot products. To each such choice of $w$, $x$, $y$, $z$ corresponds the gluing

$$
\Lambda = (\Lambda_L; \Lambda_R) \left[ (h_1; wh'_1 + xh'_2), (h_2; yh'_1 + zh'_2) \right] = (A_2^{(K)}; A_2^{(K)}) \left[ ((00; a', b')), (00; 0, d'), (h_1; wh'_1 + xh'_2), (h_2; yh'_1 + zh'_2) \right] = \bigcup_{i=1}^{L_1} \bigcup_{j=1}^{L_2} \bigcup_{l=1}^{a} \bigcup_{m=1}^{d} (\Omega_k + i(00; a', b') + j(00; 0, d') + l(h_1; wh'_1 + xh'_2) + m(h_2; yh'_1 + zh'_2)).
$$

By construction, the resulting lattice is even and also necessarily self-dual: the number of different glue classes in it is given by $adL'_1L'_2 = 3K^2$. All even self-dual gluings $\Lambda$ of $\Omega_k$ correspond to some choice of $a$, $b$, $c$, $a'$, $b'$, $d'$ and $w$, $x$, $y$, $z$.

We will conclude with brief comments on how we find all invariants satisfying (P2) and (P3). Using Gaussian elimination on the set $\{Z_{\Lambda}\}$ of lattice partition functions, we construct a basis for the Weyl-folded commutant, as well as a basis $\{Z^{(1)}, \ldots, Z^{(P)}\}$ for a subspace (the positive parity commutant defined at the end of Sec. III) which necessarily contains all physical invariants. We are interested in the linear combinations $Z = \sum A_i Z^{(i)}$. Even when the dimension $P$ of the commutant is large, it is trivial to find the conditions on the $A_i$ equivalent to the demand that the coefficients $N_{\lambda_L\lambda_R}$ of $Z$ be integers: e.g. in almost all cases we have considered, this condition is simply that each $A_i \in \mathbb{Z}$. In our notation (P3) becomes the condition that $N_{\rho\rho} = 1$; for our bases, (P3) is always equivalent to the choice $A_1 = 1$. Using the remaining positivity conditions $N_{\lambda_L\lambda_R} \geq 0$, we then find upper and lower bounds for the remaining $A_i$ (the most common ones are $0 \leq A_i \leq 1$ or $-1 \leq A_i \leq 0$). We then use the computer to run through the possible values of $A_i$, checking for positivity of all coefficients. As we see in the tables, most dimensions $P$ are small, but even for large $P$ this final computer search is fast.
REFERENCES

[1] J. L. Cardy, Nucl. Phys B270, 186 (1986).
[2] D. Gepner and E. Witten, Nucl. Phys. B278, 493 (1986).
[3] J. Fuchs, A. Klemm, M. Schmidt and D. Verstegen, Int. J. Mod. Phys. A 7, 2245 (1992).
[4] V. Kac and D. Peterson, Adv. Math. 53, 125 (1984).
[5] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 [FS18], 445 (1987).
[6] G. Moore and N. Seiberg, Nucl. Phys. B313, 16 (1989).
[7] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B327, 673 (1989);
  A. N. Schellekens, Phys. Lett. B 244, 255 (1990);
  B. Gato-Rivera and A. N. Schellekens, Commun. Math. Phys. 145, 85 (1992).
[8] A. Kato, Mod. Phys. Lett. A2, 585 (1987);
  A. Cappelli, C. Itzykson and J.-B. Zuber, Commun. Math. Phys. 113, 1 (1987);
  D. Gepner and Z. Qiu, Nucl. Phys. B285, 423 (1987).
[9] D. Kastor, Nucl. Phys. B280 [FS18], 304 (1987).
[10] T. Gannon, “The classification of affine SU(3) modular invariant partition functions”,
    (Carleton preprint, 1992).
[11] C. Itzykson, Nucl. Phys. (Proc. Suppl.) 5B, 150 (1988);
    P. Degiovanni, Commun. Math. Phys. 127, 71 (1990).
[12] T. Gannon, “WZW commutants, lattices, and level-one partition functions”, Nucl.
    Phys. B (to appear).
[13] D. Verstegen, Nucl. Phys. B346, 349 (1990).
[14] N. P. Warner, Commun. Math. Phys. 130, 205 (1990).
[15] P. Roberts, Phys. Lett. B244, 429 (1990);
    P. Roberts and H. Terao, Int. J. Mod. Phys. A 7, 2207 (1992).
[16] T. Gannon, “Partition functions for heterotic WZW conformal field theories”, Nucl.
    Phys. B (to appear).
[17] V. G. Kac, Infinite Dimensional Lie Algebras, 3rd ed., (Cambridge University Press,
    Cambridge, 1990).
[18] S. Kass, R. V. Moody, J. Patera and R. Slansky, Affine Lie Algebras, Weight Multi-
    plicities, and Branching Rules Vol.1 (University of California Press, Berkeley, 1990).
[19] J. H. Conway and N. J. A. Sloane, Sphere packings, Lattices and Groups, (Springer-
    Verlag, New York, 1988).
[20] W. Lerche, A. N. Schellekens and N. P. Warner, Phys. Reports 177, 1 (1989).
[21] Ph. Ruelle, E. Thiran and J. Weyers, “Implications of an arithmetical symmetry of
    the commutant for modular invariants”, (DIAS preprint STP-92-26, 1992).
[22] M. Bauer and C. Itzykson, Commun. Math. Phys. 127, 617 (1990);
    Ph. Ruelle, Commun. Math. Phys. 133, 181 (1990);
    M. Bauer, “Aspects de l’invariance conforme”, Ph.D. thesis, Saclay.
[23] Ph. Ruelle, E. Thiran and J. Weyers, Comm. Math. Phys. 133, 305 (1990).
[24] (Ph. Ruelle, private communication);
    Ph. Ruelle, Ph.D. thesis, Louvain-la-Neuve, September 1990.
[25] D. Altschüler, J. Lacki and Ph. Zaugg, Phys. Lett. B 205, 281 (1988).
[26] D. Bernard, Nucl. Phys. B288, 628 (1987).
[27] P. Christe and F. Ravanani, Int. J. Mod. Phys. A 4, 897 (1989).
[28] T. Gannon and C. S. Lam, Rev. Math. Phys. 3, 331 (1991).