Existence of period-3 orbits and border-collision bifurcations in n-dimensional piecewise linear continuous maps

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Abstract Piecewise linear recurrent neural networks (PLRNNs) form the basis of many successful machine learning applications for time series prediction and dynamical systems identification, but rigorous mathematical analysis of their dynamics and properties is lagging behind. Here we make a contribution to this topic by investigating the existence of period-3 orbits and border-collision bifurcations in n-dimensional piecewise linear continuous maps, extending previous results for period-1 and period-2 orbits. This is particularly important as for one-dimensional maps the existence of period-3 orbits implies chaos. It is shown that these period-3 orbits collide with the switching boundary in a border-collision bifurcation, and parametric regions for the existence of both stable and unstable period-3 orbits and border-collision bifurcations will be derived theoretically. These results are graphically summarized in a classification tree. Finally, numerical simulations demonstrate the implementation of our results and are found to be in good agreement with the theoretical derivations. Our findings thus provide a basis for understanding limit cycle behavior in PLRNNs, how it emerges in bifurcations, and how it may lead into chaos.

Keywords Piecewise linear continuous maps · Period-3 orbits · Stability · Border-collision bifurcations · recurrent neural networks · limit cycle.

1 Introduction

A piecewise smooth discrete-time dynamical system is a discrete-time map whose state space is split into two or more components (subregions) by some discontinuity borders or switching manifolds, such that in each subregion there is a different functional form of the map [1][2][6][23]. Piecewise smooth (PWS) maps have received growing attention in recent years, as they have a wide range of applications in various areas such as neural dynamics, switching circuits, or impacting mechanical systems [21]. One important type of piecewise smooth maps is piecewise linear continuous maps which are continuous, but have some discontinuities in their Jacobian matrix across the switching boundaries. Piecewise linear recurrent neural networks (PLRNNs), which build on so-called Rectified Linear Units (ReLU), $\phi(z) = \max(z, 0)$, as the networks nonlinear activation function, are one
example of such maps. In general, RNNs are the standard these days in machine learning for processing sequential, time-series information, due to their success in domains rich in temporal structure like natural language processing [14, 24], prediction of consumer behavior [15], movement trajectories [18], or identification of dynamical systems from experimental data [13]. ReLU-based RNNs are particularly popular as they allow for highly efficient inference and training algorithms that exploit their piecewise linear structure [16, 19, 5, 13]. To understand the representational and computational capabilities of these systems in the various application areas indicated above, it is important to study them more systematically from a mathematical, dynamical systems perspective. In particular, periodic motion forms an integral part of many natural data domains (like speech or movement signals), and limit cycles of various orders have been extracted from brain signals recently using PLRNNs [13].

There are different types of bifurcations in piecewise smooth dynamical systems, notably bifurcations that occur because of the existence of the discontinuity boundaries. These form the class of discontinuity-induced bifurcations that only exist in piecewise smooth systems [5]. In particular, border-collision bifurcations, or C-bifurcations, have many applications in engineering, computational neuroscience, biology, economics and the social sciences, [10]. They arise when either fixed points or periodic points of a piecewise smooth map collide with one of the switching boundaries at a critical value of the bifurcation parameter [2, 3]. Feigin [7, 8, 9] was among the first to study analytical conditions for border-collision bifurcations of fixed points and period-2 orbits in piecewise linear continuous maps. In [2] some theoretical results on the existence of period-2 orbits in n-dimensional piecewise smooth continuous maps with one bifurcation parameter were presented. Subsequently, [12] provided a description of border-collision bifurcations in one-dimensional discontinuous maps, as well as some conditions for the creation and stability of different periodic motions and chaos. Later, Higham et al. [11] considered the occurrence of period-1 and period-2 fixed points in n-dimensional piecewise linear maps with a gap. In [6] the existence of period-1 and period-2 orbits during a border collision bifurcation was discussed for n-dimensional piecewise linear discontinuous maps with two parameters. Very recently, Patra [20] investigated the coexistence of a period-2 orbit, a period-3 orbit, and an unstable chaotic orbit for some parameter values of a 3D piecewise linear normal form map (see also [13] for similar numerical observations in PLRNNs inferred from data).

However, since high-dimensional PWS linear maps, of which PLRNNs are an example, with an exponentially growing number of discontinuity boundaries are hard to handle, here we consider such systems locally in the neighborhood of only one switching manifold, defined by

$$Z^{(k+1)} = F_\mu(Z^{(k)}) = \begin{cases} 
A_1 Z^{(k)} + \mu h; & z_s^{(k)} \leq 0 \\
A_2 Z^{(k)} + \mu h; & z_s^{(k)} > 0 
\end{cases} \tag{1}$$

Although (1) is a reduced system, by adding some assumptions we can extend the obtained results to PLRNNs more generally. Up to now, conditions for the existence of period-k orbits ($k \geq 3$) have been established only in one- and two-dimensional PWS maps, but not for n-dimensional PWS maps more generally [6, 20]. Here we extend these previous results for n-dimensional PWS linear maps with one discontinuity boundary and work out the conditions for the existence and stability of period-3 fixed points, which has been an open problem so far [6]. We will define parametric regions for the occurrence of stable and unstable period-3 orbits in such maps. Our theoretical results reveal that these period-3 orbits lie precisely on the switching border for a specific value of the bifurcation parameter, implying that the system undergoes a border-collision bifurcation. We note that for one-dimensional PWS linear maps the existence of period-3 orbits implies the existence of chaos [17].

This paper is organized as follows: Section 2 provides some mathematical preliminaries. In section 3 the existence of period-3 orbits, as well as the occurrence of border-collision bifurcations, for system (1) is investigated. Next, we study the stability of these orbits, provide parametric regions for the occurrence of stable and unstable period-3 orbits, and give a diagrammatic summary of the obtained results. In section 4 finally, numerical simulations are provided that indicate how to apply our findings.
2 Preliminaries

Consider the PWS map \( I \) on \( \mathbb{R}^n \), where \( z_i^{(k)} \) \((1 \leq s \leq n)\) is the \( s \)-th component of \( Z^{(k)} = (z_1^{(k)}, z_2^{(k)}, ..., z_n^{(k)})^T \in \mathbb{R}^n \). \( A_1 = [a_{ij}^{(1)}] \) and \( A_2 = [a_{ij}^{(2)}] \), \( i,j = 1, 2, ..., n \), are \( n \times n \) matrices with real entries, \( h = (h_1, h_2, ..., h_n)^T \in \mathbb{R}^n \), and \( \mu \in \mathbb{R} \) is the bifurcation parameter of the system.

Assume that matrices \( A_1 \) and \( A_2 \) are identical except for the \( s \)-th column, i.e. \( a_{ij}^{(1)} = a_{ij}^{(2)} \) if \( j \neq s \). Furthermore, let us denote the discontinuity boundary of map \( I \) by \( \Sigma \), and the two subregions separated through this boundary by \( S^- \) and \( S^+ \):

\[
S^- = \{ Z^{(k)} = (z_1^{(k)}, z_2^{(k)}, ..., z_n^{(k)})^T \in \mathbb{R}^n; \, H(Z^{(k)}) = z_s^{(k)} < 0 \},
\]

\[
\Sigma = \{ Z^{(k)} = (z_1^{(k)}, z_2^{(k)}, ..., z_n^{(k)})^T \in \mathbb{R}^n; \, H(Z^{(k)}) = z_s^{(k)} = 0 \},
\]

\[
S^+ = \{ Z^{(k)} = (z_1^{(k)}, z_2^{(k)}, ..., z_n^{(k)})^T \in \mathbb{R}^n; \, H(Z^{(k)}) = z_s^{(k)} > 0 \},
\]

where the scalar function \( H : \mathbb{R}^n \to \mathbb{R} \), with \( H(Z^{(k)}) = z_s^{(k)} \) has nonvanishing gradient. Then, we can rewrite map \( I \) as

\[
Z^{(k+1)} = F_\mu(Z^{(k)}) = \begin{cases} 
F^-_\mu(Z^{(k)}) = A_1 Z^{(k)} + \mu h; & Z^{(k)} \in S^- \\
F^+_\mu(Z^{(k)}) = A_2 Z^{(k)} + \mu h; & Z^{(k)} \in S^+ .
\end{cases}
\]

**Proposition 1** In the map \( I \), suppose that \( \{\alpha_1\}_{i=1,2,\ldots,n} \) and \( \{\beta_j\}_{j=1,2,\ldots,n} \) denote the eigenvalues of \( A_1 \) and \( A_2 \), respectively. Also, let \( \sigma_{\alpha\alpha}^+ \), be the number of real eigenvalues of \( A_1^k \) which are greater than 1, and \( \sigma_{\beta\beta}^+ \), the number of real eigenvalues of \( A_2^k A_1 \) which are greater than 1. Moreover, assume \( P_{A_1^k}(\lambda) \) and \( P_{A_2^k A_1}(\lambda) \) are the characteristic polynomials of \( A_1^k \) and \( A_2^k A_1 \), respectively. Then,

(I) \( P_{A_1^k}(1)P_{A_2^k A_1}(1) > 0 \), iff \( \sigma_{\alpha\alpha}^+ + \sigma_{\beta\beta}^+ \) is an even number \( (\sigma_{\alpha\alpha}^+ + \sigma_{\beta\beta}^+ = 2k, \, k = 0, 1, 2, \ldots) \).

(II) \( P_{A_1^k}(1)P_{A_2^k A_1}(1) < 0 \), iff \( \sigma_{\alpha\alpha}^+ + \sigma_{\beta\beta}^+ \) is an odd number \( (\sigma_{\alpha\alpha}^+ + \sigma_{\beta\beta}^+ = 2k+1, \, k = 0, 1, 2, \ldots) \).

**Proof** Proposition 1 can be proven easily based on the considerations in [2].

**Theorem 1 (Period three implies chaos)** Suppose that \( F : I \to I \) is a continuous map with \( I \subset \mathbb{R} \). If \( F \) has a period-3 orbit, then \( F \) is chaotic.

**Proof** See [17].

3 Period-3 orbits and bifurcations in the piecewise linear map \( I \)

3.1 Existence of period-3 fixed points of the map \( I \)

**Lemma 1** Let \( A_1 \) and \( A_2 \) be two \( n \times n \) matrices that differ only in their \( s \)-th column \((1 \leq s \leq n)\), i.e. \( a_{ij}^{(1)} = a_{ij}^{(2)} \) if \( j \neq s \). Then, there are special forms for the matrices \( A_1 \) and \( A_2 \) for which \( A_1^k \) and \( A_2^k \) also differ only in their \( s \)-th column, for all \( k = 1, 2, 3, \ldots \). In this case, \( A_1^k \), \( A_2^k \) and \( A_1^{k_1} A_2^{k_2} A_1^{k_3} A_2^{k_4} A_1^{k_5} A_2^{k_6} \cdots A_1^{k_{n-1}} A_2^{k_n} \), \((k_1 + k_2 + k_3 + k_4 + \ldots + k_n = k)\), are also equal except for the \( s \)-th column.

**Proof** Suppose that \( A_1 \) and \( A_2 \) are two \( n \times n \) matrices that differ in their \( s \)-th column. Without loss of generality, we assume that \( s = 1 \). Therefore, just the first columns of \( A_1 \) and \( A_2 \) are different and they can be partitioned in the following way:

\[
A_1 = \begin{pmatrix} a & \overrightarrow{c^T} \\ b & A \end{pmatrix}, \quad A_2 = \begin{pmatrix} d & \overrightarrow{c^T} \\ e & A \end{pmatrix},
\]

(6)
such that \( A \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in \mathbb{R}^{n-1} \). In this case, \( A_1^2 \) and \( A_2^2 \) can be written as

\[
A_1^2 = \begin{pmatrix} a^2 + \overrightarrow{c} \overrightarrow{T} \overrightarrow{b} & \overrightarrow{c} \overrightarrow{T} + \overrightarrow{c} \overrightarrow{T} A \\ a \overrightarrow{b} + A \overrightarrow{b} & A \overrightarrow{b} + b \overrightarrow{c} + A^2 \end{pmatrix}, \quad A_2^2 = \begin{pmatrix} d^2 + \overrightarrow{c} \overrightarrow{T} \overrightarrow{b} & d \overrightarrow{c} \overrightarrow{T} + \overrightarrow{c} \overrightarrow{T} A \\ d \overrightarrow{b} + A \overrightarrow{b} & A \overrightarrow{b} + b \overrightarrow{c} + A^2 \end{pmatrix}.
\]

Thus, \( A_1^2 \) and \( A_2^2 \) also differ only in their first columns iff

\[
\left\{ \begin{array}{l}
a \overrightarrow{c} \overrightarrow{T} + \overrightarrow{c} \overrightarrow{T} A = d \overrightarrow{c} \overrightarrow{T} + \overrightarrow{c} \overrightarrow{T} A \\
\overrightarrow{b} \overrightarrow{c} + A^2 = \overrightarrow{c} \overrightarrow{c} + A^2
\end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l}
(a - d) \overrightarrow{c} \overrightarrow{T} = 0 \\
(\overrightarrow{b} - \overrightarrow{c}) \overrightarrow{T} = 0.
\end{array} \right.
\]

Since the first columns of \( A_1 \) and \( A_2 \) are different, the last assertion holds if \( \overrightarrow{c} = \overrightarrow{0} \). In this case we have:

\[
A_1 = \begin{pmatrix} a & \overrightarrow{0} \overrightarrow{T} \\ b & A \end{pmatrix}, \quad A_2 = \begin{pmatrix} d & \overrightarrow{0} \overrightarrow{T} \\ e & A \end{pmatrix}, \quad A_1^3 = \begin{pmatrix} a^3 & \overrightarrow{0} \overrightarrow{T} \\ a^2 b + A b & A_1 \end{pmatrix}, \quad A_2^3 = \begin{pmatrix} d^3 & \overrightarrow{0} \overrightarrow{T} \\ d^2 A + A A & A_2 \end{pmatrix}, \quad \vdots
\]

Thus means that \( A_1^k \) and \( A_2^k \) \((k = 1, 2, 3, \cdots)\) differ only in their first columns. Likewise, in this case \( A_1^k, A_2^k \) and \( A_1^{k_1} A_2^{k_2} A_1^{k_3} A_2^{k_4} \cdots A_1^{k_{n-1}} A_2^{k_n} \) \((k_1 + k_2 + k_3 + \cdots + k_n = k)\) are also equal except for the first column, which completes the proof. The proof can be carried out analogously for \( s \neq 1 \).

Now, to examine period-3 fixed points and border-collision bifurcations of \( F_\mu(Z^{(k)}) \), we state the following theorem:

**Theorem 2** Consider the \( n \)-dimensional discontinuous map \( S \), and let the matrices \( A_1 \) and \( A_2 \) have the special forms given in lemma 1 such that \( A_1^2 \) and \( A_2^2 \) differ only in their \( s \)-th column as well. Moreover, let \( \mu \) denote the bifurcation parameter for the map, and assume without loss of generality \( \mu > 0 \) \((\mu < 0)\) there are the following possibilities based upon the parameters \( a \) and \( d \) which are the first elements of the first columns of \( A_1 \) and \( A_2 \), respectively:

1. \( a \neq d \): In this case, let \( A^3 \) have no eigenvalue equal to 1. Then, there are two parametric regions

\[
\mathcal{R}_1 = \mathcal{R}_1(a, d) = \left\{ a, d \in \mathbb{R}; 0 < a < 1, d < -a - 1 \right\},
\]

\[
\mathcal{R}_2 = \mathcal{R}_2(a, d) = \left\{ a, d \in \mathbb{R}; a > 1, d < -a - 1 \right\} \cup \left\{ a, d \in \mathbb{R}; a < -1, d > \frac{1}{1 - a} \right\}
\]

Corresponding to the cases \( \sigma_{a a a}^+ + \sigma_{b a a}^+ = 2k + 1 \) and \( \sigma_{a a a}^+ + \sigma_{b a b}^+ = 2k \) \((k = 0, 1, 2, \cdots)\) respectively, in which a period-3 orbit based on the \( k \) points \( Z_1, Z_2 \in S^- \), and \( Z^* \in S^+ \) occurs in the map \( S \). Moreover this period-3 orbit which exists for \( \mu > 0 \) \((\mu < 0)\), collides with the border at \( \mu = 0 \). This means that the system undergoes a border-collision bifurcation at \( \mu = 0 \).

2. \( a = d \): In this case no period-3 orbit based on the three points \( Z_1, Z_2 \in S^- \) and \( Z^* \in S^+ \) exists. If \( a \neq d \), then \( \sigma_{a a a}^+ + \sigma_{b a b}^+ \) is always an even number.
We can proceed analogously for $s \neq 1$.

**Proof** Suppose that $Z_1^* = (z_{11}^*, z_{12}^*, ..., z_{1n}^*)^T \in S^-$, $Z_2^* = (z_{21}^*, z_{22}^*, ..., z_{2n}^*)^T \in \Sigma$ and $Z^{**} = (z_1^{**}, z_2^{**}, ..., z_n^{**})^T \in S^+$ are three points of a period-3 orbit for the system (5) such that

$$
\begin{align*}
F^-(Z_1^*) &= Z_2^*, \\
F^-(Z_2^*) &= Z^{**}, \\
F^+(Z^{**}) &= Z_1^*.
\end{align*}
$$

(13)

Due to the definition of map (5) we can write

$$
F^-(Z_1^*) = Z_2^* \implies A_1Z_1^* + \mu h = Z_2^*,
$$

(14)

$$
F^-(Z_2^*) = Z^{**} \implies A_2Z_2^* + \mu h = Z^{**},
$$

(15)

$$
F^+(Z^{**}) = Z_1^* \implies A_2Z^{**} + \mu h = Z_1^*.
$$

(16)

Next, we extend ideas from (10) for two points to the scenario with three points $Z_1^*, Z_2^*$ and $Z^{**}$, denoting the distance between the points $Z_1^*, Z_2^*$ and $Z^{**}$ by

$$
\begin{align*}
\triangle M_1 &= Z_2^* - Z_1^* = (z_{21}^* - z_{11}^*, z_{22}^* - z_{12}^*, ..., z_{2n}^* - z_{1n}^*)^T = (\delta m_{11}, \delta m_{12}, ..., \delta m_{1n})^T, \\
\triangle M_2 &= Z^{**} - Z_2^* = (z_1^{**} - z_{21}^{**}, z_2^{**} - z_{22}^{**}, ..., z_n^{**} - z_{2n}^{**})^T = (\delta m_{21}, \delta m_{22}, ..., \delta m_{2n})^T, \\
\triangle M_3 &= Z_1^* - Z^{**} = (z_1^* - z_1^{**}, z_2^* - z_2^{**}, ..., z_n^* - z_n^{**})^T = (\delta m_{31}, \delta m_{32}, ..., \delta m_{3n})^T.
\end{align*}
$$

(17)

(18)

(19)

Accordingly, using equations (14), (16) we get

$$
\begin{align*}
\triangle M_1 &= Z_2^* - Z_1^* = A_1Z_1^* - A_2Z^{**}, \\
\triangle M_2 &= Z^{**} - Z_2^* = A_1Z_2^* - A_2Z^{**}, \\
\triangle M_3 &= Z_1^* - Z^{**} = A_2Z^{**} - A_1Z_2^*.
\end{align*}
$$

(20)

(21)

(22)

Since for $A_1 = [a_{ij}^{(1)}]$ and $A_2 = [a_{ij}^{(2)}]$ we have: $a_{ij}^{(1)} = a_{ij}^{(2)} = a_{ij}$ if $j \neq s$, writing $\triangle M_1$ in scalar form yields

$$
\begin{align*}
\delta m_{1k} &= \sum_{j=1}^{n} a_{kj}^{(1)} z_{1j}^* - \sum_{j=1}^{n} a_{kj}^{(2)} z_{1j}^* = \sum_{j=1}^{n} a_{kj}^{(1)} z_{1j}^* + a_{ks}^{(1)} z_{1s}^* - \sum_{j=1}^{n} a_{kj}^{(2)} z_{1j}^* + a_{ks}^{(2)} z_{1s}^*, \quad k = 1, 2, \ldots, n.
\end{align*}
$$

(23)

Adding and subtracting the term $a_{ks}^{(1)} z_{1s}^*$ to equation (23) gives

$$
\begin{align*}
\delta m_{1k} &= \sum_{j=1}^{n} a_{kj}^{(1)} (z_{1j}^* - z_{1j}^*) + [a_{ks}^{(1)} - a_{ks}^{(2)}] z_{1s}^*, \quad k = 1, 2, \ldots, n.
\end{align*}
$$

(24)

On the other hand, by adding and subtracting the term $a_{ks}^{(2)} z_{1s}^*$ to equation (23) we obtain

$$
\begin{align*}
\delta m_{1k} &= \sum_{j=1}^{n} a_{kj}^{(2)} (z_{1j}^* - z_{1j}^*) + [a_{ks}^{(1)} - a_{ks}^{(2)}] z_{1s}^*, \quad k = 1, 2, \ldots, n.
\end{align*}
$$

(25)

Now, using (19) and writing equations (24) and (25) in vector form, we have

$$
\begin{align*}
\triangle M_1 - \triangle M_3 &= [A_1 - A_2] z_{1s}^{**}, \\
\triangle M_1 - \triangle M_3 &= [A_1 - A_2] z_{1s}^*.
\end{align*}
$$

(26)

(27)
where \([\ldots]_s\) indicates the s-th column. Similarly, writing \(\Delta M_3\) in scalar form results in

\[
\delta m_{3k} = \sum_{j=1}^{n} a_{kj}^{(2) * *} z_j^* - \sum_{j=1}^{n} a_{kj}^{(1) * *} z_j^* = \sum_{j=1}^{n} a_{kj}(z_j^{* *} - z_j^*) + a_{kj}^{(2) * *} z_j^* - a_{kj}^{(1) * *} z_j^{* *}, \quad k = 1, 2, \ldots n. \tag{28}
\]

Again, adding and subtracting \(a_{kj}^{(1) * *} z_j^*\) to equation (28) gives

\[
\delta m_{3k} = \sum_{j=1}^{n} a_{kj}^{(1) * *} (z_j^{* *} - z_j^*) + [a_{kj}^{(2)} - a_{kj}^{(1) * *}] z_j^*, \quad k = 1, 2, \ldots n. \tag{29}
\]

Also, by adding and subtracting \(a_{kj}^{(2) * *} z_j^*\) to equation (28) we have

\[
\delta m_{3k} = \sum_{j=1}^{n} a_{kj}^{(2) * *} (z_j^* - z_j^{* *}) + [a_{kj}^{(2)} - a_{kj}^{(1) * *}] z_j^{* *}, \quad k = 1, 2, \ldots n. \tag{30}
\]

Using (18) and writing equations (29) and (30) in vector form yields

\[
\Delta M_3 - A_1 \Delta M_2 = [A_2 - A_1] \delta z_s^{* *}, \tag{31}
\]

\[
\Delta M_3 - A_2 \Delta M_2 = [A_2 - A_1] \delta z_s^*. \tag{32}
\]

From (20) and (21) we have

\[
\Delta M_2 = A_1 (Z_2^* - Z_1^*) = A_1 \Delta M_1. \tag{33}
\]

Substituting (33) into (31) we obtain

\[
\Delta M_3 - A_1^2 \Delta M_1 = [A_2 - A_1] \delta z_s^{* *}. \tag{34}
\]

According to (26) we can write

\[
\Delta M_1 = A_1 \Delta M_3 + [A_1 - A_2] \delta z_s^{* *}. \tag{35}
\]

Inserting (35) into (34) gives

\[
(I - A_1^2) \Delta M_3 = \left[(I - A_1^2) [A_2 - A_1] \delta z_s^{* *}\right] z_s^{* *}. \tag{36}
\]

Thus, if we suppose that \((I - A_1^2)\) is invertible, then

\[
\Delta M_3 = \frac{F^{***}}{P_{A_1^2}(1)} z_s^{* *}, \tag{37}
\]

where

\[
F^{***} = [f_{ij}^{***}]_{n \times 1} = \text{adj}(I - A_1^2) \left[(I - A_1^2) [A_2 - A_1] \delta z_s^{* *}\right] = \text{adj}(I - A_1^2)v_1, \tag{38}
\]

and \(P_{A_1^2}(\lambda) = |\lambda I - A_1^2|\) is the characteristic polynomial of \(A_1^2\).

Performing the same procedure for equations (27), (32) and (33) we get

\[
(I - A_2 A_1 A_2) \Delta M_3 = A_2 A_1 [A_1 - A_2] z_s^{* *} + [A_2 - A_1] z_s^{* *}. \tag{39}
\]

In this case, assuming \((I - A_2 A_1 A_2)\) to be invertible, we can rewrite equation (39) as

\[
\Delta M_3 = \frac{1}{P_{A_2 A_1 A_2}(1)} \left(F_1^* z_s^{* *} + F_2^* z_s^{* *}\right), \tag{40}
\]
where
\[ F_1^* = \text{adj} (I - A_2 A_1 A_2) \left[ (A_2 - A_1) \right] = \text{adj} (I - A_2 A_1 A_2) v_2, \] (41)
\[ F_2^* = \text{adj} (I - A_2 A_1 A_2) \left[ A_2 - A_1 \right] = \text{adj} (I - A_2 A_1 A_2) v_3, \] (42)
and \( P_{A_2 A_1 A_2} (\lambda) = |\lambda I - A_2 A_1 A_2| \).

Moreover, carrying out a similar procedure for (26), (31) and (33), it is found that
\[ (I - A_1^3) \triangle M_1 = \left[ (A_1 - I) \left[ A_2 - A_1 \right] \right] z_s^*. \] (43)
Then the assumption of invertibility of \( (I - A_1^3) \) provides
\[ \triangle M_1 = \frac{\tilde{F}^*}{P_{A_1^3} (1)} z_s^*, \] (44)
where
\[ \tilde{F}^* = \text{adj} (I - A_1^3) \left[ (A_1 - I) \left[ A_2 - A_1 \right] \right] = \text{adj} (I - A_1^3) \tilde{v}_1, \] (45)
Repeating the procedure one more time for (27), (32) and (33) yields
\[ (I - A_2^2 A_1) \triangle M_1 = \left[ A_1 - A_2 \right] z_s^* + A_2 \left[ A_2 - A_1 \right] z_s^*, \] (46)
Assuming \( (I - A_2^2 A_1) \) to be invertible, we have
\[ \triangle M_1 = \frac{1}{P_{A_2^2 A_1} (1)} \left( \tilde{F}_1^* z_s^* + \tilde{F}_2^* z_s^* \right), \] (47)
where
\[ \tilde{F}_1^* = \text{adj} (I - A_2^2 A_1) \left[ (A_1 - I) \left[ A_2 - A_1 \right] \right] = \text{adj} (I - A_2^2 A_1) \tilde{v}_2, \] (48)
\[ \tilde{F}_2^* = \text{adj} (I - A_2^2 A_1) \left[ (A_2) \left[ A_2 - A_1 \right] \right] = \text{adj} (I - A_2^2 A_1) \tilde{v}_3, \] (49)
and \( P_{A_2^2 A_1} (\lambda) = |\lambda I - A_2^2 A_1| \).

Writing (37), (40), (44) and (47) in scalar form results in
\[ \delta m_{3k} = \frac{f_{k}^{**}}{P_{A_2} (1)} z_s^*, \quad k = 1, 2, 3, \ldots, n, \] (50)
\[ \delta m_{3k} = \frac{1}{P_{A_2 A_1 A_2} (1)} \left( f_{1k}^{**} z_s^* + f_{2k}^{**} z_s^* \right), \quad k = 1, 2, 3, \ldots, n, \] (51)
\[ \delta m_{1k} = \frac{f_{k}^{**}}{P_{A_1} (1)} z_s^*, \quad k = 1, 2, 3, \ldots, n, \] (52)
\[ \delta m_{1k} = \frac{1}{P_{A_2^2 A_1} (1)} \left( f_{1k}^{**} z_s^* + f_{2k}^{**} z_s^* \right), \quad k = 1, 2, 3, \ldots, n. \] (53)
Now, without loss of generality let \( s = 1 \), such that \( A_1 \) and \( A_2 \) have the form (5) and hence, according to Lemma (1), \( A_1^3, A_2 A_1 A_2 \) and \( A_2^2 A_1 \) only differ in their first columns. Thereby, the cofactors of the first column of \( A_1^3 \) are the same as for the first column of \( A_2 A_1 A_2 \), and also the same as for the first column of \( A_2^2 A_1 \). Hence, we have
\[ \text{[first row of } \text{adj}(I - A_1^3)] = \text{[first row of } \text{adj}(I - A_2 A_1 A_2)] = \text{[first row of } \text{adj}(I - A_2^2 A_1)] \] (54)
\[ = \left[ f(A) \ 0 \ 0 \ \ldots \ 0 \right], \] (55)
where \( f(A) \in \mathbb{R} \) is a function of the elements of matrix \( A \). Since \( (I - A_1^2), (I - A_2 A_1 A_2) \) and \( (I - A_2^2 A_1) \) are assumed to be invertible, we have \( |I - A_1^2|, |I - A_2 A_1 A_2|, |I - A_2^2 A_1| \neq 0 \). Then, from the relations
\[
\begin{align*}
|\text{adj}(I - A_1^2)| &= |(I - A_1^2)|^{n-1}, \\
|\text{adj}(I - A_2 A_1 A_2)| &= |(I - A_2 A_1 A_2)|^{n-1}, \\
|\text{adj}(I - A_2^2 A_1)| &= |(I - A_2^2 A_1)|^{n-1},
\end{align*}
\]

it follows that \( f(A) \neq 0 \). Furthermore,
\[
\begin{align*}
\text{first row of } \text{adj}(I - A_1^2) v_1 &= \text{first row of } \text{adj}(I - A_1^2) v_1 = f(A)(1 - a^2)(a - d), \quad (57) \\
\text{first row of } \text{adj}(I - A_2 A_1 A_2) v_2 &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) v_2 = f(A)(-ad)(a - d), \quad (58) \\
\text{first row of } \text{adj}(I - A_2 A_1 A_2) v_3 &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) v_3 = f(A)(a - d), \quad (59)
\end{align*}
\]

where \( v_1 = (I - A_1^2)[A_2 - A_1]_1, v_2 = (- A_2 A_1)[A_2 - A_1]_1 \) and \( v_3 = [A_2 - A_1]_1 \).

In addition,
\[
\begin{align*}
\text{first row of } \text{adj}(I - A_1^2) \tilde{v}_1 &= \text{first row of } \text{adj}(I - A_1^2) \tilde{v}_1 = f(A)(a - 1)(a - d), \quad (60) \\
\text{first row of } \text{adj}(I - A_2 A_1 A_2) \tilde{v}_2 &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) \tilde{v}_2 = -f(A)(a - d), \quad (61) \\
\text{first row of } \text{adj}(I - A_2 A_1 A_2) \tilde{v}_3 &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) \tilde{v}_3 = f(A) d (a - d), \quad (62)
\end{align*}
\]

where \( \tilde{v}_1 = (A_1 - I)[A_2 - A_1]_1, \tilde{v}_2 = -[A_2 - A_1]_1 \) and \( \tilde{v}_3 = A_2 [A_2 - A_1]_1 \).

Now there are two cases:

**Case (1) \( a \neq d \):** In the case, due to \( (38), (41) \) and \( (42) \), for \( k = 1 \) and \( s = 1 \), it is concluded that
\[
\begin{align*}
f_{11}^* &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) v_2 = \frac{-ad}{1 - a^2} \text{first row of } \text{adj}(I - A_1^2) v_1 = \frac{-ad}{1 - a^2} f_1^{**}, \quad (63) \\
f_{21}^* &= \text{first row of } \text{adj}(I - A_2 A_1 A_2) v_3 = \frac{1}{1 - a^2} \text{first row of } \text{adj}(I - A_1^2) v_1 = \frac{1}{1 - a^2} f_1^{**}. \quad (64)
\end{align*}
\]

Thus, denoting \( \alpha_1 = \frac{-ad}{1 - a^2} \neq 0 \) and \( \beta_1 = \frac{1}{1 - a^2} \neq 0 \), we have
\[
\begin{align*}
f_{11}^* &= \alpha_1 f_1^{**}, \\
f_{21}^* &= \beta_1 f_1^{**}.
\end{align*}
\]

Accordingly, equations \( (50) \) and \( (51) \) for \( k = 1 \) and \( s = 1 \) result in
\[
\frac{z_1^{**}}{P_{A_1}^{(1)}} = \frac{\alpha_1 z_{11}^* + \beta_1 z_{21}^*}{P_{A_2 A_1 A_2}^{(1)}}. \quad (67)
\]

Similarly, from equations \( (60), (61) \) and \( (62) \) for \( k = 1 \) and \( s = 1 \) it is determined that
\[
\begin{align*}
f_{11}^* &= \alpha_2 f_1^{**}, \\
f_{21}^* &= \beta_2 f_1^{**},
\end{align*}
\]
where \( \alpha_2 = \frac{-1}{a - 1} \neq 0 \) and \( \beta_2 = \frac{d}{a - 1} \neq 0 \). Analogously, due to equations (52) and (53), for \( k = 1 \) and \( s = 1 \) we can write

\[
\frac{z_1^{**}}{P_{A_1^3(1)}} = \alpha_2 z_1^{11} + \beta_2 z_1^{21}.
\] (70)

On the other hand, from (50) and (22) we have

\[
z_{1k}^* = z_{k}^{**} + \frac{f_{k}^{**}}{P_{A_1^3(1)}} z_{1}^{**}, \quad k = 1, 2, 3, \ldots, n.
\] (71)

In addition, writing equation (16) in scalar form gives

\[
z_{1k}^* = \sum_{j=1}^{n} a_{kj}^{(2)} z_{j}^{**} + \mu h_k, \quad k = 1, 2, 3, \ldots, n.
\] (72)

By (71) and (72) we obtain

\[
z_{k}^{**} = \sum_{j=1}^{n} a_{kj}^{(2)} z_{j}^{**} - \frac{f_{k}^{**}}{P_{A_1^3(1)}} z_{1}^{**} + \mu h_k, \quad k = 1, 2, 3, \ldots, n.
\] (73)

Writing (73) in vector form gets

\[
Z^{**} = \hat{A}_2 Z^{**} + \mu h,
\] (74)

where

\[
\hat{A}_2 = A_2 - \frac{1}{P_{A_1^3(1)}} L.
\] (75)

such that

\[
L = \begin{pmatrix}
f_{1}^{**} & 0 & \cdots & 0 \\
f_{2}^{**} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
f_{n}^{**} & 0 & \cdots & 0
\end{pmatrix}.
\] (76)

Assuming \((I - \hat{A}_2)\) to be invertible, equation (75) can be solved for \( Z^{**} \) as

\[
Z^{**} = (I - \hat{A}_2)^{-1} \mu h.
\] (77)

Since,

\[
|I - \hat{A}_2| = \frac{P_{A_1^3(1)} I - P_{A_1^3(1)} A_2 + L}{P_{A_1^3(1)}} = \frac{g}{P_{A_1^3(1)}},
\] (78)

equation (77) becomes

\[
Z^{**} = \frac{\text{adj} (I - \hat{A}_2)}{g} P_{A_1^3(1)} \mu h = JP_{A_1^3(1)} \mu.
\] (79)

Again, by writing equation (79) in scalar form, for \( k = 1 \) we get

\[
z_{1}^{**} = J_1 P_{A_1^3(1)} \mu.
\] (80)
Then, using (67) implies
\[ \alpha_1 z_{11}^* + \beta_1 z_{21}^* = J_1 P_{A_2 A_1}(1) \mu. \] (81)
In addition, from (70) we obtain
\[ \alpha_2 z_{11}^* + \beta_2 z_{21}^* = J_1 P_{A_2^2 A_1}(1) \mu. \] (82)
Furthermore, by equations (81) and (82) we have
\[
\begin{cases}
    z_{11}^* = -\frac{P_{A_2 A_1}(1) \beta_2 - \beta_1 P_{A_2^2 A_1}(1)}{\beta_1 \alpha_2 - \alpha_1 \beta_2} J_1 \mu \\
    z_{21}^* = \frac{P_{A_2 A_1}(1) \alpha_2 - \alpha_1 P_{A_2^2 A_1}(1)}{\beta_1 \alpha_2 - \alpha_1 \beta_2} J_1 \mu
\end{cases}
\] (83)
On the other side, it follows that
\[ P_{A_2 A_1 A_2}(\lambda) = |\lambda I - A_2 A_1 A_2| = (\lambda - ad) |\lambda I_{n-1} - A^3| = (\lambda - ad) P_{A^3}(\lambda) \] (84)
\[ = (\lambda - d^2 a) P_{A^3}(\lambda) = (\lambda - d^2 a) |\lambda I_{n-1} - A^3| = |\lambda I - A_2^2 A_1| = P_{A_2^2 A_1}(\lambda). \]
Hence for \( ad \neq 0, a \neq \pm 1 \) and \( ad^2 \neq 1 \), we can rewrite (83) as
\[
\begin{cases}
    z_{11}^* = \frac{(d + ad + 1)(1 - a)}{1 - ad^2} J_1 P_{A_2^2 A_1}(1) \mu \\
    z_{21}^* = \frac{(a + ad + 1)(1 - a)}{1 - ad^2} J_1 P_{A_2^2 A_1}(1) \mu
\end{cases}
\] (85)
Therefore, a period-3 orbit will occur in the discontinuous map (5) if and only if
\[
\begin{cases}
    z_{11}^* = \frac{(d + ad + 1)(1 - a)}{1 - ad^2} J_1 P_{A_2^2 A_1}(1) \mu < 0 \\
    z_{21}^* = \frac{(a + ad + 1)(1 - a)}{1 - ad^2} J_1 P_{A_2^2 A_1}(1) \mu < 0 \\
    z_{11}^{**} = J_1 P_{A_2^3}(1) \mu > 0
\end{cases}
\] (86)
or, equivalently,
\[
\begin{cases}
    (d + ad + 1)(a + ad + 1) > 0 \\
    (d + ad + 1)(1 - a) \frac{P_{A_2^2 A_1}(1) P_{A_2^3}(1) < 0}{1 - ad^2} \\
    (a + ad + 1)(1 - a) \frac{P_{A_2^2 A_1}(1) P_{A_2^3}(1) < 0}{1 - ad^2}
\end{cases}
\] (87)
for \( \mu > 0 \) (or \( \mu < 0 \)), \( ad \neq 0, a \neq \pm 1 \) and \( ad^2 \neq 1 \).

Consequently, for \( P_{A_2^2 A_1}(1) P_{A_2^3}(1) < 0 \) a period-3 orbit will occur for \( \mu > 0 \) (or \( \mu < 0 \)) provided that
\[
\begin{cases}
    (d + ad + 1)(a + ad + 1) > 0, \\
    (d + ad + 1)(1 - a) \frac{P_{A_2^2 A_1}(1) P_{A_2^3}(1) > 0}{1 - ad^2} \\
    (a + ad + 1)(1 - a) \frac{P_{A_2^2 A_1}(1) P_{A_2^3}(1) > 0}{1 - ad^2}
\end{cases}
\] (88)
Similarly, when $P_{A_2 A_1} (1) P_{A_3} (1) > 0$, a period-3 orbit comes into existence for $\mu > 0$ (or $\mu < 0$) iff

\[
\begin{aligned}
(d + ad + 1)(a + ad + 1) &> 0, \\
\frac{(d + ad + 1)(1 - a)}{1 - ad^2} &< 0, \\
\frac{(a + ad + 1)(1 - a)}{1 - ad^2} &< 0
\end{aligned}
\]  

(89)

such that in both cases $ad \neq 0$, $a \neq \pm 1$ and $ad^2 \neq 1$.

Besides,

\[
P_{A_3} (\lambda) = |\lambda I_n - A_3| = (\lambda - a^3) |\lambda I_n - A_3| = (\lambda - a^3) P_{A_3} (\lambda),
\]

(90)

and therefore

\[
P_{A_3} (1) P_{A_2 A_1 A_2} (1) = (1 - a^3) (1 - dad) P_{A_3}^2 (1).
\]

(91)

Since $a \neq d$, $P_{A_2 A_1} (1) P_{A_3} (1) \neq 0$ if

\[
(1 - a^3) (1 - dad) \neq 0, \quad P_{A_3} (1) \neq 0.
\]

(92)

Also, the condition $P_{A_3} (1) \neq 0$ is true, if $A_3$ has no eigenvalue equal to 1.

Accordingly, from proposition 1 and relations (88), (89) and (91), it is concluded that:

For every $\mu > 0$ ($\mu < 0$) there is a period-3 orbit based on the three points $Z_1^*, Z_2^+ \in S^- \text{ and } Z^{**} \in S^+$, if

(1) $A_3$ has no eigenvalue equal to 1,

(2) for $\sigma_{\alpha \alpha}^+ + \sigma_{\beta \alpha}^+ = 2k + 1$, $k = 0, 1, 2, \cdots$, the parameters $a$ and $d (a \neq d)$ belong to the region

\[
\mathcal{R}_1 = \mathcal{R}_1 (a, d) = \left\{ a, d \in \mathbb{R}; a \neq 1, (1 - a^3) (1 - dad) < 0, (d + ad + 1) (a + ad + 1) > 0, \quad \frac{(d + ad + 1)(1 - a)}{1 - ad^2} > 0, \quad \frac{(a + ad + 1)(1 - a)}{1 - ad^2} > 0 \right\} = \left\{ a, d \in \mathbb{R}; 0 < a < 1, d < -\frac{a - 1}{a} \right\}.
\]

(93)

(3) for $\sigma_{\alpha \alpha}^+ + \sigma_{\beta \alpha}^+ = 2k$, $k = 0, 1, 2, \cdots$, the parameters $a$ and $d (a \neq d)$ belong to the region

\[
\mathcal{R}_2 = \mathcal{R}_2 (a, d) = \left\{ a, d \in \mathbb{R}; a \neq 1, (1 - a^3) (1 - dad) > 0, (d + ad + 1) (a + ad + 1) > 0, \quad \frac{(d + ad + 1)(1 - a)}{1 - ad^2} < 0, \quad \frac{(a + ad + 1)(1 - a)}{1 - ad^2} < 0 \right\} = \left\{ a, d \in \mathbb{R}; a > 1, d < -\frac{a - 1}{a} \right\}
\]

\[
\cup \left\{ a, d \in \mathbb{R}; a < -1, d > \frac{1}{-a - 1} \right\}.
\]

(94)

Case (2) $a = d$: In this situation, one can easily see that $f_1^{**} = f_1^{*} = 0$. Hence, from relations (50) and (52), it follows that $\delta m_{3k} = \delta m_{1k} = 0$. Therefore, due to (17) and (19), we have $z_1^{**} = z_1^* = z_2^*$. This means that there is not any period-3 orbit based on the three points $Z_1^*, Z_2^+ \in S^-$ and $Z^{**} \in S^+$. 

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Furthermore for \( a = d \), the relation
\[
P_{A_2}^3(1) P_{A_2 A_1 A_2}(1) < 0,
\]
is never satisfied, since in this case
\[
P_{A_2}^3(1) P_{A_2 A_1 A_2}(1) = (1 - a^3)^2 P_{A_2}^2(1) \geq 0.
\]
Moreover \( P_{A_2}^3(1) P_{A_2 A_1 A_2}(1) = 0 \) for \( a = d = 1 \), while \( P_{A_2}^3(1) P_{A_2 A_1 A_2}(1) > 0 \) for \( a = d \neq 1 \). In the last case, proposition \( \text{[1]} \) implies that \( \sigma_{\alpha_{100}}^+ + \sigma_{\alpha_{010}}^+ \) is an even number.

For \( s \neq 1 \), analogous results can easily be obtained.

**Fig. 1** The parametric regions \( R_1 \) and \( R_2 \) within which period-3 orbits exist for the map \( \text{(5)} \).

**Remark 1** Parametric regions \( R_1 \) and \( R_2 \) in the proof of theorem \( \text{[2]} \) defined by relations \( \text{(93)} \) and \( \text{(94)} \), are illustrated in Fig. 1.

**Remark 2** According to theorem \( \text{[2]} \), a period-3 orbit occurs for the discontinuous map \( \text{(5)} \) if and only if \( a, d \in R_1 \cup R_2 \). Hence, as shown in Fig. 1, for all parameters \( a \) and \( d \) with \( ad > 0 \) there are no period-3 orbits based on the three points \( Z_1^- , Z_2^- \in S^- \) and \( Z_3^+ \in S^+ \), as in this case \( a, d \notin R_1 \cup R_2 \).

**Remark 3** Suppose that \( \text{(1)} \) is a one-dimensional map on \( I \subset \mathbb{R} \). Since \( \text{(1)} \) is continuous, so by theorem \( \text{[1]} \) the existence of a period-3 orbit for \( \text{(1)} \) implies the existence of chaos.

**Corollary 1** If \( \det(A^2 + A + I) \neq 0 \), then \( A^3 \) has an eigenvalue equal to 1 iff \( A \) has an eigenvalue equal to 1.

**Proof** Let \( \det(A^2 + A + I) \neq 0 \), then due to the relation
\[
\det(A^3 - I) = \det(A^3 - I^3) = \det ((A - I)(A^2 + A + I)),
\]
det\((A^3 - I) = 0 \) if and only if \( \det(A - I) = 0 \). This implies that \( \lambda = 1 \) is an eigenvalue of \( A^3 \), iff \( \lambda = 1 \) is an eigenvalue of \( A \).
Remark 4 Suppose that matrices $A_1$ and $A_2$ satisfy the conditions of theorem 2. Furthermore, let matrix $A = A_1^T A_2^T$ be the common block of partitioned matrices $A_1$ and $A_2$. If $\det(A^2 + A + I) \neq 0$, then by corollary 1 the condition "$A_3$ has no eigenvalue equal to 1" in theorem 2 can be replaced by the condition "$A$ has no eigenvalue equal to 1".

3.2 Stability of period-3 orbits

Here, we will show that the stability of period-3 orbits of the map (5) depends on parameters $a, d$, and also on the eigenvalues of the matrix $A^3$.

**Theorem 3** Consider map (5) with matrices $A_1$ and $A_2$ satisfying the conditions of theorem 2. Suppose that there is a period-3 orbit $O_3$ for (5) which satisfies the conditions given in the first case of theorem 2. Then $O_3$ is a stable period-3 orbit based on the points $Z_1, Z_2 \in S^-$ and $Z_3 \in S^+$ for (5), if

1. $P_{A^3}(1) = |I - A^3| \neq 0$,

2. $a, d \in \mathcal{R} = \left\{ a, d \in \mathbb{R} ; 0 < a < \frac{1}{2} (\sqrt{5} - 1), \frac{-1}{a^2} < d < \frac{-a - 1}{a} \right\}$, \hspace{1cm} (97)

3. and the magnitude of each eigenvalue of $A^3$ is less than 1.

Moreover, when $\sigma_{\alpha\alpha\alpha}^+ + \sigma_{\beta\beta\beta}^+$ is even, $O_3$ cannot be stable.

**Proof** Let $O_3$ be a period-3 orbit for (5), specified by the points $Z_1, Z_2 \in S^-$ and $Z_3 \in S^+$. Then by theorem 2 we infer that $A^3$ has no eigenvalue equal to 1. Moreover, depending on whether $\sigma_{\alpha\alpha\alpha}^+ + \sigma_{\beta\beta\beta}^+$ is an odd or even number, the parameters $a, d$ belong to $\mathcal{R}_1$ or $\mathcal{R}_2$, respectively ($a \neq d$). On the other side, the characteristic polynomial of $O_3$ is given by

$$P_{A^3}(\lambda) = |A_n - A_2 A_1^T| = (\lambda - da^2) |A_{n-1} - A|^3 = (\lambda - da^2) P_{A^3}(\lambda).$$ \hspace{1cm} (98)

Therefore, $O_3$ is a stable period-3 orbit provided that the magnitude of each eigenvalue of $A^3$ is less than 1, and also $-1 < da^2 < 1$. The last assertion means that $a, d \in \mathcal{R} = (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \mathcal{R}_3$ ($a \neq d$), where $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined by (93) and (94), and

$$\mathcal{R}_3 = \mathcal{R}_3(a, d) = \left\{ a, d \in \mathbb{R} ; -1 < da^2 < 1 \right\}.$$ \hspace{1cm} (99)

Furthermore,

$$\mathcal{R} = (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_3 = \left\{ a, d \in \mathbb{R} ; 0 < a < \frac{1}{2} (\sqrt{5} - 1), \frac{-1}{a^2} < d < \frac{-a - 1}{a} \right\},$$

as $\mathcal{R}_2 \cap \mathcal{R}_3 = \varnothing$. Thus for every $a$ and $d$ in $\mathcal{R}_2$, there is no stable period-3 orbit $O_3$. This means that when $\sigma_{\alpha\alpha\alpha}^+ + \sigma_{\beta\beta\beta}^+$ is even, $O_3$ cannot be stable. This completes the proof.

**Remark 5** The parametric region $\mathcal{R}$ is displayed in red in Fig. 2.
Corollary 2 Assume that all eigenvalues of $A^3$ are real and $O_3$ agrees with the conditions of theorem 3, i.e., it is a stable period-3 orbit for (5). Then, $P_{A^3}(1) = |I - A^3| > 0$ and $tr(A^3) < n - 1$.

Proof Since $O_3$ satisfies the conditions of theorem 3, the magnitude of each eigenvalue of $A^3$ is less than 1. Thus all eigenvalues of $A^3$ are less than 1, as all of them are real. In this case for every eigenvalue $a_i (i = 1, 2, \cdots, n - 1)$ of $A^3$, it is concluded that $1 - a_i > 0$. Hence

$$P_{A^3}(1) = |I - A^3| = \prod_{i=1}^{n-1} (1 - a_i) > 0,$$

and also, $tr(A^3) = \sum_{i=1}^{n-1} a_i < n - 1$.

3.3 Summery of obtained results

Suppose that $O_3$ is a period-3 orbit of (5), specified by the points $Z_1, Z_2 \in S^-$ and $Z_3 \in S^+$. Moreover, let us denote the magnitude of each eigenvalue of $A^3$ by $|\lambda_{A^3}^i|, i = 1, 2, \cdots$. Using these definitions, the results above can be summarized in the form of a classification tree as given in Fig. 3.
Fig. 3 Classification tree for the existence and stability of stable period-3 orbits for the map \( f \).

4 Numerical simulations

Example 1 For \( n = 2 \), the discontinuous map \( f \) has the form

\[
\begin{pmatrix}
x^{(k+1)} \\
y^{(k+1)}
\end{pmatrix} = \begin{cases} 
A_1 \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \mu h; & x^{(k)} < 0 \\
A_2 \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \mu h; & x^{(k)} > 0
\end{cases},
\]

(101)

where the matrices \( A_1 \) and \( A_2 \) differ only in the first column, and

\[
A_1 = \begin{pmatrix} a & 0 \\ b & A \end{pmatrix}, \quad A_2 = \begin{pmatrix} d & 0 \\ e & A \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad a, b, d, e, A, h_1, h_2 \in \mathbb{R}.
\]

(102)

Here, \( A \in \mathbb{R} \) is the common block matrix of \( A_1 \) and \( A_2 \). Recall that \( A^3 \) has no eigenvalue equal to 1, if and only if \( A \neq 1 \). Besides, for the system \( (101) \) we have:

\[
S^- = \{ (x, y)^T \in \mathbb{R}^2; H(x, y) = x < 0 \},
\]

(103)

\[
\Sigma = \{ (x, y)^T \in \mathbb{R}^2; x = 0 \},
\]

(104)

\[
S^+ = \{ (x, y)^T \in \mathbb{R}^2; x > 0 \}.
\]

(105)

Furthermore, by \( (91) \) we have

\[
P_{A_1^3}(1) P_{A_2 A_1} A_2(1) = (1 - a^3) (1 - dad) (1 - A^3)^2.
\]

(106)

For \( A \neq 1, a \neq d \) and \( h_1 \neq 0 \), by theorems 2 and 3 it is found that:
1. For \((1 - a^3)(1 - ad)(1 - A^3)^2 > 0\), i.e., when \(\sigma_{\alpha_\alpha}^+ + \sigma_{\beta_\beta}^+\) is even, there is not any stable period-3 orbit for the system \((101)\).

2. For \((1 - a^3)(1 - ad)(1 - A^3)^2 < 0\), i.e., when \(\sigma_{\alpha_\alpha}^+ + \sigma_{\beta_\beta}^+\) is odd, there is a stable period-3 orbit for \((101)\) provided that \(a,d \in \mathbb{R}_1 \cap \mathbb{R}_3\), and \(A < 1\).

To find fixed points of a period-3 orbit of the map \((101)\), according to our theoretical results the following equations must be solved:

\[
\begin{align*}
\begin{cases}
(x_1) = A_2 A_1^2 (y_1) + [A_2 A_1 + A_2 + I] \mu h \\
(y_1) = A_1 A_2 A_1 (y_2) + [A_1 A_2 + A_1 + I] \mu h \\
(x_2) = A_1^3 A_2 (y_3) + [A_1^3 + A_1 + I] \mu h.
\end{cases}
\end{align*}
\]

(107)

for \(x_1, x_2 < 0\) and \(x_3 > 0\). To solve these equations, the matrices \((I - A_2 A_1^2)\), \((I - A_1 A_2 A_1)\) and \((I - A_1^3 A_2)\) must be invertible, that is

\[
\det(I - A_2 A_1^2) = \det(I - A_1 A_2 A_1) = \det(I - A_1^3 A_2) = (1 - a^2 d)(1 - A^3) \neq 0,
\]

or equivalently \(a^2 d \neq 1\) and \(A^3 \neq 1\). By this we have

\[
\begin{align*}
\begin{cases}
(x_1) = (I - A_2 A_1^2)^{-1} [A_2 A_1 + A_2 + I] \mu h \\
(y_1) = (I - A_1 A_2 A_1)^{-1} [A_1 A_2 + A_1 + I] \mu h \\
(x_2) = (I - A_1^3 A_2)^{-1} [A_1^3 + A_1 + I] \mu h.
\end{cases}
\end{align*}
\]

(109)

Solving the system \((109)\) we obtain

\[
\begin{align*}
x_1 &= -\frac{d + ad + 1}{da^2 - 1} \mu h_1 \\
y_1 &= \left(-A_1 [Ab(ad + d + 1) + b(ad + a + 1)] - c(a^2 + a + 1)\right) h_1 + \frac{1}{1 - A} h_2 \mu \\
x_2 &= -\frac{a + ad + 1}{da^2 - 1} \mu h_1 \\
y_2 &= \left(-A_2 [Ab(ad + a + 1) + c(a^2 + a + 1)] - b(ad + d + 1)\right) h_1 + \frac{1}{1 - A} h_2 \mu \\
x_3 &= -\frac{a + a^2 + 1}{da^2 - 1} \mu h_1 \\
y_3 &= \left(-A_3 [Ac(a^2 + a + 1) + b(ad + d + 1)] - b(ad + a + 1)\right) h_1 + \frac{1}{1 - A} h_2 \mu.
\end{align*}
\]

(110)
Now suppose that \( h_1, \mu \neq 0 \). Then, due to the relation
\[
\mathcal{R}_1 \cup \mathcal{R}_2 =
\left\{ a, d \in \mathbb{R}; 0 < a < 1, d < -\frac{a - 1}{a} \right\} \cup \left\{ a, d \in \mathbb{R}; a > 1, d < -\frac{a - 1}{a} \right\} \cup \left\{ a, d \in \mathbb{R}; a < -1, d > \frac{1}{-a - 1} \right\}
\]
\( a, d \in \mathcal{R}_1 \cup \mathcal{R}_2 \), if and only if
\[
\begin{cases}
\frac{d + ad + 1}{da^2 - 1} > 0 \\
\frac{a + ad + 1}{da^2 - 1} > 0, \\
\frac{a + a^2 + 1}{da^2 - 1} < 0
\end{cases}
\quad \text{or} \quad
\begin{cases}
\frac{d + ad + 1}{da^2 - 1} < 0 \\
\frac{a + ad + 1}{da^2 - 1} < 0, \\
\frac{a + a^2 + 1}{da^2 - 1} > 0
\end{cases}
\]
Due to (110) this implies that \( x_1 < 0, x_2 < 0 \) and \( x_3 > 0 \) hold for some \( \mu > 0 \) or \( \mu < 0 \). In this case, a period-3 orbit given by the three points \((x_1, y_1)^T, (x_2, y_2)^T \in S^- \) and \((x_3, y_3)^T \in S^+ \) exists for system (101).

Let us choose the parameters of system (101) as follows:
\[
a = 0.4, \quad d = -4, \quad b = 1, \quad e = -1, \quad A = 0.4, \quad h_1 = 1, \quad h_2 = 0, \quad \mu = 0.8
\]
In this case, \( a, d \in \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_3 \), and \( A < 1 \). Also, \((1-a^3)(1-dad)(1-A^3)^2 = -4.4281396224 < 0 \) which implies \( \sigma_{\alpha_0}^+ + \sigma_{\beta_0}^a \) is odd. Hence, by theorems 2 and 3 there is a stable period-3 orbit denoted by \( \mathcal{O}_3 = \{(x_1, y_1)^T, (x_2, y_2)^T, (x_3, y_3)^T\} \), where
\[
\begin{align*}
(x_1, y_1)^T &= (-2.24390243902, 1.05482593287)^T, \\
(x_2, y_2)^T &= (-0.0975609756098, -2.7392120075)^T, \\
(x_3, y_3)^T &= (0.760975609756, -1.19324577861)^T.
\end{align*}
\]
Moreover, the eigenvalues of the period-3 orbit are \( \lambda_1 = a^2d = -0.64 \) and \( \lambda_2 = A^3 = 0.064 \), which both have magnitude less than one. Therefore there exists a period-3 sink for system (101), as illustrated in Fig. 4.

![Fig. 4 Example of a stable period-3 orbit \( \mathcal{O}_3 \) of the system (101) for \( a = 0.4, \quad d = -4(a, d \in \mathcal{R}) \).](image)
Finally, we can deduce the existence of a border-collision bifurcation in system (101) at $\mu = 0$ by considering equations (110), from which we easily see that all three points of the orbit $O_3$ collide with the border $\Sigma$ at the $C$-bifurcation value $\mu = 0$.

5 Conclusions

Our aim in this paper was to investigate period-3 orbits and border-collision bifurcations of the piecewise linear continuous map (5) on $\mathbb{R}^n$. First, we considered a special partitioned form for the matrices $A_1$ and $A_2$ of the map (5). Under these conditions we could extend the ideas in [6,10] for period-1 and period-2 orbits to period-3 orbits in (5), for which we established conditions for their existence (or nonexistence). In addition, the stability of these periodic points was discussed. In particular, some parametric regions for both stable and unstable period-3 orbits were determined. We also proved that in these regions the system can undergo a border-collision bifurcation, as a period-3 orbit of the map (5) falls precisely onto the discontinuity boundary $\Sigma$ at a critical value of parameter $\mu$. To the best of our knowledge, this is the first time such results for the occurrence of period-3 orbits have been presented for $n$-dimensional PWS maps. Furthermore, a schematic outline of our theoretical results was given in form of a classification tree, and numerical simulations were performed to illustrate an application of the results.

The results presented here are an important step toward a more systematic study of limit cycles in PLRNNs, as they have been demonstrated, for example, in PLRNNs inferred from experimental (brain imaging) data [13]. In this context we also remark that the conditions used in Lemma 1, in particular $\vec{c} = 0$, are not too restrictive, as recent results on the application of PLRNNs to diverse challenging data sets have shown that zeroing out the inputs from other states for a subset of PLRNN variables (i.e., setting row vectors of the transition matrix to 0 except for the s-th entry) indeed leads to much improved performance [22]. Our results may thus not only help to understand and analyze the cyclic behavior of trained PLRNNs, but also bifurcations in these systems as parameters change throughout the training process [4].

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7 Conflict of Interest

The authors declare that they have no conflict of interest.

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