Solitons on Singularities

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ABSTRACT
We describe solitons that live on the world–volumes of D5 branes wrapped on deformed $A_2$ singularities fibered over $C(x)$. We show that monopoles are D3 branes wrapped on a node of the deformed singularity and stretched along $C(x)$. F and D–term strings are D3 branes wrapped on a node of a singularity that is deformed and resolved respectively. Domain walls require deformed $A_3$ singularities and correspond to D5 branes wrapped on a node and stretched along $C(x)$.

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1. Introduction

In field theory, solitons are important objects because they teach us about the semi-classical and nonperturbative physics. The same is true for supersymmetric field theories that live on the world–volumes of different D–branes. Since, most if not all field theories can be obtained by some configuration of D–branes[1], we expect to find solitons in many D–brane configurations. In fact, a large body of work already exists on solitons in supersymmetric field theories that can be obtained in intersecting brane models[2].

An alternative way to examine world–volume theories of D–branes is to locate them on singularities[3]. For example, one can wrap D5 branes on (the nodes of) $A_n$ singularities and examine the world–volume theory in the $3 + 1$ noncompact directions. The resulting world–volume gauge group and couplings, matter content and superpotential are well–understood[3,4]. By deforming and resolving the different nodes of the singularity, and taking large volume limits, one can obtain many different field theories. It has been shown that some of these lead to supersymmetry breaking[5,6].

In this paper, we describe solitons of different dimensions that live on $A_n$ singularities. In particular, we concentrate on monopoles[7], vortices[8] and domain walls[9] that can live on the (noncompact) world–volume of D5 branes wrapped on nodes of deformed $A_n$ singularities fibered on $C(x)$. We obtain the different soliton solutions by either deforming or resolving the nodes of the singularity. For example, monopoles are obtained by a particular deformation that breaks the gauge group to $U(1)$ at low energies. On the other hand, these monopoles are D3 branes that are wrapped on the same node as the D5 brane and stretched along $C(x)$. Vortices are obtained by a different deformation of the singularity which breaks the $U(1)$ spontaneously. They correspond to D3 branes that wrap the same node but stretch along one of the the noncompact world–volume directions. We show that these correspond to F–term strings[10]. D–term strings[11,12,13] are obtained by a resolution[14] of the node in addition to its deformation. This leads to an
anomalous D–term on the world–volume theory which gives rise to D–term strings. Non–Abelian vortices arise if there are multiple D5 branes wrapped on the different nodes of the singularity. Monopoles and vortices both require a singularity with at least two nodes, i.e. $A_2$. Domain walls, on the other hand, require at least three nodes and therefore the smallest singularity that gives rise to them is a deformed and fibered $A_3$ singularity. The deformations lead to isolated vacua and therefore to domain walls. If, in addition, the node is resolved, there are semi–local vortices[10] connected to the domain walls. We show that these domain walls are D5 branes wrapped on a node and stretched along $C(x)$. Our treatment is purely semi–classical and we are mainly interested in the existence of different solitons that live on singularities. In particular, in this paper, we are not concerned about interactions between solitons, their moduli spaces, world–volume theories or other quantum properties.

The paper is organized as follows. In section 2, we describe the monopole solution that lives on a deformed $A_2$ singularity. We obtain monopoles on the world–volume theory and show that their properties match those of wrapped D3 branes. In section 3, we describe F and D–term strings which are vortex solutions in the world–volume theory. We show that these correspond to wrapped D3 branes stretched along one world–volume direction. We also show that wrapping multiple D5 branes on the singularity leads to non–Abelian vortices. Section 4 contains the description of domain walls as solutions to the world–volume theory with isolated vacua and as wrapped D5 brane configurations. Section 5 contains a discussion of our results and our conclusions.

2. Monopoles

We begin with the description of monopoles[7,15,16] which are localized solitons in the $3 + 1$ dimensional theory. In order to describe monopoles that live on singularities, we consider the deformed $A_2$ singularity fibered over $C(x)$ given by

$$uv = (z - mx)(z + mx)(z - m(x - 2a))$$  \hspace{1cm} (1)
which has two nodes ($S^2_s$) located at $x = 0$ and $x = a$. We wrap one D5 brane on each node. In the noncompact 3 + 1 dimensional world–volume theory, the gauge group is $U(1)_1 \times U(1)_2$ with the couplings

$$\frac{4\pi}{g_i^2} = \frac{V_i}{(2\pi)^2 g_s \ell_s^2} \quad i = 1, 2$$

(2)

where $g_s$ and $\ell_s$ are the string coupling and length respectively. $V_i$ is the “stringy volume”[17] of the $i$th node given by $V_i = (2\pi)^4 \ell_s^4 (B_i^2 + r_i^2 + \alpha_i^2)^{1/2}$ with

$$B_i = \int_{S^2} B^{NS} \quad r_i^2 = \int_{S^2} J$$

(3)

i.e. $B_i$ is the NS-NS flux through the $i$th node and $r_i^2$ is the volume of the blown-up $S^2_s$. The deformations of the singularity are parametrized by $\alpha_i$ which are related to the singlet F–terms in field theory. For simplicity, we can decouple $U(1)_1$ by making $g_1^2$ very small. From eq. (2) we see that this can be done by taking $V_1 >> \ell_s^2$. Then $U(1)_1$ becomes a global symmetry and the remaining gauge group is simply $U(1)_2$. The matter content consists of two neutral (singlet) fields $\phi_1, \phi_2$ and a pair of (charged) fields $Q_{12}, Q_{21}$ with charges $(1, -1)[3]$. The singlet superpotential is obtained from the deformation data by[4]

$$W(\phi_i) = \int_{\phi_i} \left( z_i(x) - z_{i+1}(x) \right) dx$$

(4)

where $z_i$ are the zeros of the different factors in eq. (1). Including the Yukawa term the superpotential becomes

$$W = m\phi_1^2 - m(\phi_2 - a)^2 + Q_{12}Q_{21}(\phi_2 - \phi_1)$$

(5)

(Following the usual abuse of notation common in the literature, here $a$ is parameter with units of mass and is related to the distance $a$ in eq. (1) by a factor of $2\pi \ell_s^2$.)
which is the string tension.) The F–terms obtained from $W$ are

$$F_{\phi_1} = 2m\phi_1 - Q_{12}Q_{21}$$ (6)

$$F_{\phi_2} = -2m(\phi_2 - a) + Q_{12}Q_{21}$$ (7)

$$F_{Q_{12}} = Q_{21}(\phi_2 - \phi_1)$$ (8)

$$F_{Q_{21}} = Q_{12}(\phi_2 - \phi_1)$$ (9)

In addition, we need to include the D–term for $U(1)_2$,

$$D = |Q_{12}|^2 - |Q_{21}|^2$$ (10)

The F and D–terms vanish and supersymmetry is preserved for $\phi_1 = 0$, $\phi_2 = a$ and $Q_{12} = Q_{21} = 0$. In this vacuum all matter fields are massive with $m_{\phi_1} = m_{\phi_2} = 2m$ and $m_Q = 2a$.

Before decoupling $U(1)_1$, the gauge group was $U(1)_1 \times U(1)_2$ due to the nonzero $\phi_2$ VEV. In fact, $\phi_{1,2}$ are the diagonal entries of a scalar in the adjoint of $U(2)$ and the VEVs of the diagonal entries break the gauge group spontaneously $U(2) \to U(1)_1 \times U(1)_2$. The fields $\phi_1, \phi_2$ are heavy and decouple at low energies, $E << m$. They can be integrated out by setting their F–terms to zero. If we assume $a < E << m$, only the charged fields $Q_{12}, Q_{21}$ remain in the spectrum. The low–energy superpotential is given by

$$W = aQ_{12}Q_{21}$$ (11)

This superpotential (together with the D–term) has a supersymmetric vacuum with $Q_{12} = Q_{21} = 0$. This is exactly what we would have obtained if we assumed $E << a, m$ above and integrated out the charged fields as well. As we will later see, we assume $a << m$ so that the monopole is light enough to be in the spectrum.
at low energies, $E << m$. We are left only with the gauge group $U(1)_2$ which arose from the spontaneous breaking $U(2) \to U(1)_2$. More precisely, the original gauge group was $SU(2)$ rather than $U(2)$ since the Abelian gauge group that describes the center of mass, $U(1) = [U(1)_1 + U(1)_2]/2$, decouples from matter. Therefore, the spontaneous breaking is actually $SU(2) \to [U(1)_1 - U(1)_2]/2$. (When $U(1)_1$ is decoupled this reduces to $U(1)_2$.) Therefore, due to topological considerations, there is a monopole in this model.

In the world–volume field theory, the monopole is described by

$$\phi_2 = \hat{r}_i \sigma_i (\text{arcoth}(ar) - 1) \quad A_\mu = -\epsilon_{ijk} \hat{r}_i \sigma_i \left(1 - \frac{ar}{\sinh(ar)}\right)$$

with magnetic charge, $g_m$, and field

$$g_2 g_m = 2\pi n \quad B_i = \frac{g_m}{4\pi r^2} \hat{r}_i$$

where $n$ is the topological (magnetic) charge of the monopole. The mass of a monopole (of charge $n$) is

$$m_m = \frac{4\pi}{g_2^2} an$$

This monopole is actually a D3 brane wrapped on the second node with volume $V_2$ and stretched between the two wrapped D5 branes at $x = 0$ and $x = a$. The mass of such a brane is (with $d = a$)

$$m = T_{D3} V_2 d = \frac{V_2}{(2\pi)^3 g_s \ell_s^4} (2\pi a \ell_s^2) = \frac{4\pi}{g_2^2} a$$

which exactly matches the monopole mass in eq. (14) for $n = 1$. We see that the topological (or magnetic) charge of the monopole corresponds to the winding number of the D3 brane on $S_2^2$. Note that if we want the monopoles to remain in the low–energy theory with $E << m$, we need to assume $a < E << m$ as we discussed above.
This model also contains dyons that carry both electric and magnetic charges. In the above field theory, W bosons have mass $m_W = a$. In fact, W bosons are fundamental strings stretched between the two D5 branes wrapped on the nodes at $x = 0$ and $x = a$. Thus, a dyon with charge $(p, q)$ is a bound state of $p$ fundamental strings and $q$ wrapped D3 branes stretching from $x = 0$ to $x = a$. This bound state has a mass given by

$$m_d = \sqrt{p^2a^2 + q^2\frac{4\pi a}{g_s^2}}$$

which is precisely the dyon mass expected in field theory. It is well–known that a fundamental string can be bound to D3 brane (on a torus). This is simply a configuration T–dual to the bound state between a fundamental string and a D–string. However, in our case the D3 brane is not wrapped on a torus but on $S^2$ which does not allow T–duality. Nevertheless, the above configuration is the only candidate with a mass that matches the expected dyon mass. It would be interesting to understand this bound state from the wrapped D3 world–volume theory point of view and verify our result.

The field theory described by the superpotential in eq. (5) has $\mathcal{N} = 1$ supersymmetry. The monopoles which we described above break supersymmetry and therefore are not BPS. As a result, there are corrections to the monopole solution and mass in eqs. (12)-(14) since they are not protected by supersymmetry. Nevertheless, we expect these monopoles to be stable due to conservation of topological charge. In order to get BPS monopoles we need $\mathcal{N} = 2$ supersymmetry which, in our case, is broken due to the singlet masses in eq. (5) arising from the twisting (fibering) of the $A_2$ singularity over $C(x)$. We expect BPS monopoles to exist in D5 world–volume theories that arise from untwisted $A_2$ singularities. For example, consider the $A_2$ singularity defined by

$$uv = (z - z_0(m, a))(z - z_0(m, a))(z - z_0(m, a))$$

which means all $z_i = z_0(m, a)$ are equal (where $z_0(m, a)$ may or may not vanish).
The gauge group and matter content remain as before with a simpler superpotential

\[ W = Q_{12}Q_{21}(\phi_2 - \phi_1) \]  

(18)
i.e. without singlet mass terms. As a result, the model has \( \mathcal{N} = 2 \) supersymmetry
which is a reflection of the geometry without fibering. We can then go to the
Coulomb branch with \( Q_{12} = Q_{21} = 0 \) and nonzero \( \phi_1 \neq \phi_2 \) which are flat directions.
When \( \phi_1 \neq \phi_2 \), the gauge group is spontaneously broken \( U(2) \rightarrow U(1)_1 \times U(1)_2 \)
(where \( U(1)_1 \) can be decoupled as before by taking \( V_1 \) very large in which case \( \phi_1 \)
also decouples with a frozen VEV) and we get monopoles as above with masses

\[ m_m = \frac{4\pi}{g_2^2}(\phi_2 - \phi_1)n \]  

(19)
These are BPS monopoles and therefore their solution and mass do not get any
corrections. As before, they correspond to D3 branes that wrap the second node
and stretch between \( \phi_1 \) and \( \phi_2 \) along \( C(x) \).

We can easily generalize our results to the case of the spontaneous breaking
\( U(N) \rightarrow U(1)^N \) with \( N \) types of monopoles corresponding to each unbroken \( U(1) \).
This is described by a deformed \( A_N \) singularity fibered over \( C(x) \)

\[ uv = (z - z_1)(z - z_2) \cdots (z - z_N)(z - z_{N+1}) \]  

(20)
where each zero has the form \( z_i = m(x - a_i) \) with \( i = 1, \ldots, N \). This geometry has
\( N \) nodes at \( x = a_i \). If we wrap one D5 brane on each node we get the gauge group
\( U(1)^N \). The original \( U(N) \) is broken down spontaneously by the different VEVs
for the diagonal entries in the adjoint of \( U(N) \). These diagonal entries are the
singlet fields \( \phi_i \) that live on each node and their VEVs correspond to the locations
of the nodes (on \( C(x) \)) on which the D5 branes wrap. Each unbroken \( U(1)_i \) may
have a monopole with a mass

\[ m_i = \frac{4\pi}{g_i^2}(a_{i+1} - a_i)n \]  

(21)
where \( g_i^2 \) is fixed by the “stringy volume” of the \( i^{th} \) node as in eq. (2). Again,
each of these monopoles is described by a D3 brane wrapped on the $i^{th}$ node and stretched between the wrapped D5 branes at $x = a_i$ and $x = a_{i+1}$.

3. Vortices

In this section, we consider one dimensional solitons, namely vortices in $3 + 1$ dimensions$[8,18,19]$. We obtain F and D–term strings and describe them in terms of D3 branes wrapped on nodes of deformed and resolved $A_2$ singularities. We generalize these results to the case of non–Abelian vortices by considering multiple D5 branes wrapped on the nodes of the singularity.

3.1. F–term Strings: In order to describe vortices that live on singularities, we consider a slightly different deformed $A_2$ singularity fibered on $C(x)$ described by

$$uv = (z - mx)(z + mx)(z + m(x - 2a))$$  \hspace{5cm} (22)

Compared to the singularity in eq. (1), this leads to the same gauge group, $U(1)_1 \times U(1)_2$, and matter content, namely two bifundamentals $Q_{12}, Q_{21}$ and two singlets $\phi_{1,2}$ with a slightly different superpotential given by

$$W = m\phi_1^2 - 2ma\phi_2 + Q_{12}Q_{21}(\phi_2 - \phi_1)$$  \hspace{5cm} (23)

As before, we decouple $U(1)_1$ by taking the volume of the first node $V_1$ to be very large, i.e. $V_1 \gg \ell_s^2$. We note that, now there is an F–term for the massless field $\phi_2$, $F = -2ma$, in the superpotential. $\phi_1$ is massive and decouples at low energies, $E \ll m$. It can be integrated out by setting its F–term

$$F_{\phi_1} = 2m\phi_1 - Q_{12}Q_{21}$$  \hspace{5cm} (24)

to zero. The low–energy superpotential becomes

$$W = \phi_2(Q_{12}Q_{21} - 2ma) - \frac{(Q_{12}Q_{21})^2}{4m}$$  \hspace{5cm} (25)
and gives rise to the F–terms

\[ F_{\phi_2} = Q_{12} Q_{21} - 2ma \quad (26) \]

\[ F_{Q_{12}} = \phi_2 Q_{21} - \frac{Q_{12} Q_{21}^2}{2m} \quad (27) \]

\[ F_{Q_{21}} = \phi_2 Q_{12} - \frac{Q_{12}^2 Q_{21}}{2m} \quad (28) \]

In addition, there is the D–term for \( U(1)_2 \)

\[ D = |Q_{12}|^2 - |Q_{21}|^2 \quad (29) \]

F and D–terms vanish in the supersymmetric vacuum with \( |Q_{12}| = |Q_{21}| \) and

\[ Q_{12} Q_{21} = 2ma \quad \phi_2 = a \quad (30) \]

In this vacuum, eq. (24) gives \( \phi_1 = a \) so the singlet VEVs are equal. We see that \( U(1)_2 \) is spontaneously broken by the \( Q_{12}, Q_{21} \) VEVs since they carry charges 1, −1 respectively. As a result, the photon gets a mass of \( 2g_2 \sqrt{ma} \) whereas the matter fields have masses \( m_Q = m_{\phi_2} = \sqrt{2ma} \). As usual, topological considerations imply that the spontaneous breaking of \( U(1)_2 \) leads to vortex solutions.

Far away from the core of the vortex, at large \( r \), the solution is

\[ Q_{12} = Q_{21}^\dagger = \sqrt{2ma} e^{in\theta} \quad A_\theta = \frac{n}{gr} \quad F_{\mu\nu} = 0 \quad (31) \]

where \( n \) is the topological winding number. The vortex (along the z direction) has a metric which has a conical singularity

\[ ds^2 = -dt^2 + dz^2 + dr^2 + r^2 \left( 1 + \frac{2man}{M_P^2} \right) d\theta^2 \quad (32) \]

Near the core of the vortex, at small \( r \), the solution is

\[ Q_{12} = Q_{21} = 0 \quad A_\theta = \frac{M_P^2}{2mag} \left( 1 - \cos \left( \frac{2mag}{M_P} \right) r \right) \quad (33) \]

It is well–known that the vortex with winding number \( n \) carries a magnetic
flux of

\[ \Phi_n = \int B_z dx dy = 2\pi n \]  

(34)

i.e., the winding number is the magnetic flux. The tension of the vortex is

\[ T_n = 2\pi F n = 4\pi ma \]  

(35)

In order for the vortex to exist at low energies, \( E << m \), we need \( T << m^2 \) which means we need to assume \( a << m \). The vortex tension arises due to the nonzero F–term in eq (23). Thus, eq. (31) in fact describes an F–term string[10]. This vortex has a size (width) given by

\[ w \sim \frac{1}{g_2 \sqrt{F}} \sim \frac{1}{g_2 \sqrt{2ma}} \sim \sqrt{\pi \ell_s} \]  

(36)

where we used eqs. (2) and (38) below. We see that the vortex size is about the string length and independent of the parameters of the field theory. This is due to the identical dependence of \( g_2^{-2} \) and the F–term on \( V_2 \) and \( g_s \).

The vortex we described above is actually a D3 brane wrapped on the second node, \( S^2_2 \). We can find the relation between the F–term, \( F = 2ma \), and the “stringy volume” of the second node \( V_2 \) by equating the energy of the D5 brane wrapped on \( S^2_2 \) to the vacuum energy in the field theory for vanishing VEVs,

\[ \frac{1}{2} g_2^2 F^2 = T_{D5} V_2 = \frac{V_2}{(2\pi)^5 g_s \ell_s^6} \]  

(37)

The factor of \( g_2^2/2 \) above is due to a subtlety related to the normalization of \( \phi_2 \). The normalization that is common in the literature which we used in the above superpotentials has a hidden factor of \( g_2/\sqrt{2} \) for every factor of \( \phi_2 \). This is the reason for the absence of any coupling constant in the superpotential, e.g. in eq. (23). In order to find the relation between \( F \) and \( V_2 \) we need to restore these
factors of $g_2$. We do this only here since the normalization of $\phi_2$ does not affect any of our other results. Using eq. (2) for $g_2$ we find

$$F = 2ma = \frac{V_2}{\left(2\pi\right)^4 g_s \ell_s^4}$$

which establishes the relation between $F$ and $V_2$. Then, the tension of a D3 brane wrapped on $S_2^2$ is

$$T_F = T_{D3}V_2 = \frac{V_2}{\left(2\pi\right)^3 g_s \ell_s^4} = 2\pi F = 4\pi ma$$

which is precisely the tension of the vortex with $n = 1$. This shows that the F–term string we found in field theory is a D3 brane wrapped on $S_2^2$ and stretched along one of the noncompact world–volume directions. The topological charge $n$ is simply the number of times the D3 brane wraps $S_2^2$.

A D3 brane inside a D5 brane constitutes a generalization of a magnetic flux tube[17]. This is usually shown using the coupling between the spacetime RR potential that couples to the D3 brane and the world–volume gauge field strength on the D5 brane. Alternatively, this configuration is T–dual to a D1 brane inside a D3 brane which is known to represent a magnetic flux tube. We find that after wrapping both branes on $S_2^2$, the wrapped D3 brane carries one unit of magnetic flux as expected from a vortex string. This is a little surprising since there is no T–duality on $S^2$ and we cannot directly connect our brane configuration to that of a D1 brane inside a D3 brane. It would be interesting to resolve this problem by examining the world–volume theory of a D5 brane wrapped on an $S^2$.

Note that the superpotential in eq. (23) has $\mathcal{N} = 1$ supersymmetry and the F–term string is not BPS. Therefore, we expect the string solution and tension to receive corrections. However, we expect the string to be stable due to conservation of topological charge. As we mentioned above, the $\mathcal{N} = 1$ supersymmetry is a result of fibering $A_2$ over $C(x)$ which manifests itself through singlet masses in the
superpotential. Consider the deformed but not fibered $A_2$ singularity

$$uv = (z + a)(z + 2a)(z + a)$$

where $a_i$ are $x$ independent and satisfy $\alpha_i = z_{i+1} - z_i$ with $\alpha_i = \int_{S^2_i} B_{NS}$, i.e. they parametrize the deformation due to NS flux through the nodes. (Unlike above, here the parameter $a$ has dimension 2.) This singularity gives rise to the superpotential

$$W = Q_{12}Q_{21}(\phi_2 - \phi_1) + a\phi_1 - a\phi_2 = (Q_{12}Q_{21} - a)(\phi_2 - \phi_1)$$

Now, we can take $\phi_1 \neq \phi_2$ so that $U(2) \rightarrow U(1)_1 \times U(1)_2$. Then supersymmetry requires $Q_{12}Q_{21} = a$. Decoupling $U(1)_1$ as before, we see that $U(1)_2$ is spontaneously broken and the model has $F$–term strings with tension $T_F = 2\pi a n$. These are BPS strings since they exist in an $\mathcal{N} = 2$ supersymmetric model. As above, they are described by D3 branes wrapped on $V_2$.

The model described by eq. (23) also contains monopoles. This is not surprising since inside the vortex the Abelian group $U(1)_2$ remains unbroken. In addition, since $Q_{12} = Q_{21} = 0$ inside the vortex, the singlets, $\phi_1, \phi_2$ may have different VEVs. We find that near the core of the vortex, $\phi_1 = 0$ whereas $\phi_2$ is free. These monopoles have mass $m_m = 4\pi \phi_2 / g_2^4$. Since $U(1)_2$ is broken outside the vortices and is restored only in their core, these monopoles are not free but confined by the strings which are flux tubes. In this case, the probability for monopole–anti monopole creation determines whether the vortices are short flux tubes confining monopoles or long cosmic strings[8,20]. The probability is given by

$$P \sim \exp\left(-\pi m_m^2 / T_F\right) \sim \exp \left(-\frac{4\pi^2}{g_2^4} \frac{\phi_2^2}{ma}\right)$$

Clearly, we can make this probability as small as we want by taking $\phi_2 >> \sqrt{ma}$. This choice stabilizes the vortices against monopole–anti monopole pair creation and leads to long cosmic $F$–strings[20,21].

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3.2. D–term Strings: We now show that D–term strings[11,13] can also live on singularities. Consider another deformed $A_2$ singularity fibered on $C(x)$ given by

$$uv = (z - mx)(z + mx)(z + mx)$$  \hspace{1cm} (43)

The gauge group is again $U(1)_1 \times U(1)_2$ and the matter content consists of two singlets $\phi_1, \phi_2$ and one pair of bifundamentals $Q_{12}, Q_{21}$ with the superpotential

$$W = m\phi_1^2 + Q_{12}Q_{21}(\phi_2 - \phi_1)$$  \hspace{1cm} (44)

As before, we decouple $U(1)_1$ by taking the volume of the first node to be very large in string units. At low energies $E \ll m$, $\phi_1$ decouples (with vanishing VEV) and we are left with

$$W = \phi_2 Q_{12} Q_{21}$$  \hspace{1cm} (45)

In addition, we blow up the second node which gives rise to an anomalous D–term[14]

$$\xi = \int_{S^2} J$$  \hspace{1cm} (46)

where $J$ is the Kahler form on $S^2$. This blow–up of the second node is the main difference between D–term strings and F–term strings described in the previous section. The D–term for $U(1)_2$ becomes

$$D_2 = |Q_{12}|^2 - |Q_{21}|^2 + \xi$$  \hspace{1cm} (47)

We see that a supersymmetric vacuum now requires at least a nonzero VEV for $Q_{21}$ which breaks $U(1)_2$ spontaneously, i.e $|Q_{21}|^2 = \xi$ and $|Q_{12}| = \phi_2 = 0$. As before, the spontaneous breaking of $U(1)_2$ means that the theory has vortex solutions. These can be obtained from eqs. (31)-(35) by replacing $F = 2ma$ with $\xi$. These vortices are D–term strings that carry $n$ units of magnetic flux and have tension $T_D = 2\pi \xi n$.  

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Like the F–term string, the D–term string is also a D3 brane wrapped on the second node. The argument is identical to the one we gave above for F–term strings with the replacement of $F = 2ma$ by $\xi$. As a result, we find the relation

$$\xi = \frac{V_2}{(2\pi)^4 g_s \ell_s^4}$$

(48)

between the anomalous D–term and the blow–up volume $V_2$. The tension of a D3 brane wrapped on $V_2$ is

$$T_D = T_{D3} V_2 = \frac{V_2}{(2\pi)^3 g_s \ell_s^4} = 2\pi \xi$$

(49)

which matches that of the D–term string with $n = 1$. For these vortices to exist at low energies, $E << m$, we need to assume $\sqrt{\xi} << m$.

D–term strings are BPS even in $\mathcal{N} = 1$ supersymmetric models like the one described by eq. (44)[10,11]. Therefore, their solution and tension do not receive any corrections. We can obtain D–term strings in $\mathcal{N} = 2$ supersymmetric models by wrapping the D3 branes on untwisted $A_2$ singularities such as the one given by eq. (39) with a blown up second node.

It is easy to show that there are monopoles living inside D–term strings; in other words D–term strings confine monopoles. The argument is identical to the one for F–term strings. Inside a D–term string, $Q_{12} = Q_{21} = 0$ and therefore we can have a nonzero $\phi_2$; in fact $\phi_2$ is free. Since near the core, $\phi_1 = 0$, any nonzero $\phi_2$ gives rise to monopoles with mass $m_m = 4\pi \phi_2/g_5^2$. Modifying eq. (42) for the probability for monopole–anti monopole creation, we find that the stability of D–term strings can be guaranteed by taking $\phi_2 >> \sqrt{\xi}$.

3.3. Non–Abelian Vortices: Non–Abelian vortices[22,23] can also be obtained by wrapping multiple D5 branes on the nodes of the deformed $A_2$ singularity. In order to get a non–Abelian vortex we can wrap $N_f$ and $N_c$ D5 branes on the first and second nodes respectively resulting in a $U(N_f) \times U(N_c)$ gauge group. Then the
bifundamentals $Q_{12}$, and $Q_{21}$ are in the $(N_f, \bar{N}_c)$ and $(\bar{N}_f, N_c)$ representations of the gauge group respectively. If we take the volume of the first node, $V_1$, to be very large, i.e. $V_1 >> \ell_s^2$ the $U(N_f)$ coupling given by eq. (2) becomes very small. As a result, the non–Abelian gauge dynamics decouples and $U(N_f)$ becomes a global symmetry. Then, $Q_{12}, Q_{21}$ become $N_f$ flavors in the $N_c$ and $\bar{N}_c$ representations of the remaining gauge group $U(N_c)$. The field $\phi_2$ is an adjoint of the gauge $U(N_c)$ and a singlet of the global group $U(N_f)$. On the other hand, $\phi_1$, which decouples at low energies, $E << m$, is an adjoint of the global $U(N_f)$ and a singlet of the gauge group $U(N_c)$.

There is no vortex solution for $N_f < N_c$. In addition, for $N_f > N_c$, the strings are semi–local[24,25] (in the language of $N_c = 1$) and may have arbitrary size. This means that they can expand without limit and dissolve. (These can lead to stable vortices in the limit $g_2 \to \infty$.) Therefore, stable non–Abelian strings require $N_f = N_c$. We can realize the case with $N_f = N_c = N$ by wrapping an equal number of D5 branes on the two nodes. We end up with $N$ flavors of $Q_{12}, Q_{21}$ in the $N, \bar{N}$ representations of the $U(N)$ gauge group and an adjoint $\phi_2$ which is a singlet of the flavor group. The physics is described by the generalization of eqs. (23)-(30) for $U(N)$ with $N_f = N$ flavors which means that they are modified simply by the addition of the trace $Tr$ operator where needed. The non–Abelian vortex solutions are obtained by embedding the solution given by eqs. (31)-(33) into any one of the $U(2)$ subgroups of $U(N)$. It is easy to see that the non–Abelian vortex has the same tension as the Abelian one, i.e. $T_F = 4\pi ma$ (for vortex number $n = 1$). Clearly, by the same reasoning above, we can show that the non–Abelian vortex is also a D3 brane wrapped on the second node.

Non–Abelian D–term strings can be obtained by wrapping multiple D3 branes on the nodes of the singularity in eq. (43) and repeating the steps in section 3.2. This is a straightforward exercise which we leave to the reader. BPS properties of non–Abelian F or D–term strings are the same as those of their Abelian counterparts. Therefore in order to get non–Abelian BPS F–term strings, we can wrap multiple D3 branes on the untwisted singularity in eq. (40). On the other hand,
non–Abelian BPS D–term strings are obtained by wrapping multiple D5 branes on
the singularity in eq. (43).

As an interesting point, we note that the moduli space of a non–Abelian $(U(N))$
 vortex is $R^2 \times CP^{N-1}$ where $R^2$ and $CP^{N-1}$ parametrize the location of the vortex
 in two dimensional transverse space and the orientation of the $U(2)$ subgroup in
$U(N)$ respectively. The size of $CP^{N-1}$ is $4\pi/g_s^2$. Using eq. (2) we find that this is
equal to $V_2/(2\pi)^2 g_s \ell_s^2$ i.e. the volume of the second node in string units.

4. Domain Walls

Finally, in this section, we describe domain walls[9,2,26,27] which are two di-
 mensional solitons in $3 + 1$ dimensions. In order to obtain domain walls we need
to enlarge the singularity to a deformed $A_3$ singularity fibered over $C(x)$ defined by

$$uv = z(z + m(x-a))(z + m(x-a))(z - mx)$$  (50)

In this case, the gauge group is $U(1)_1 \times U(1)_2 \times U(1)_3$ and there are three singlets
$\phi_{1,2,3}$ arising from each of the three nodes. We can decouple $U(1)_1$ and $U(1)_3$
from matter by taking $g_1$ and $g_3$ to be very small. As before we accomplish this by
taking $V_1 >> \ell_s^2$ and $V_3 >> \ell_s^2$ respectively. In addition, there are two pairs of
bifundamentals $Q_{12}, Q_{21}, Q_{23}, Q_{32}$. Under $U(1)_2$ which is the only remaining gauge
group these have charges $1, -1, -1, 1$ respectively. The superpotential is given by

$$W = \frac{m}{2}(\phi_1 - a)^2 - \frac{m}{2}(\phi_3 - \frac{a}{2})^2 + Q_{12}Q_{21}(\phi_2 - \phi_1) + Q_{23}Q_{32}(\phi_3 - \phi_2)$$  (51)

We see that $\phi_1$ and $\phi_3$ are massive and decouple at low energies $E << m$. (The
bifundamentals also get masses of order $a$ but we keep them in the spectrum by
assuming $a < E << m$.) This is done by setting their F–terms

$$F_{\phi_1} = m(\phi_1 - a) - Q_{12}Q_{21}$$  (52)
$$F_{\phi_3} = -m(\phi_3 - \frac{a}{2}) + Q_{23}Q_{32}$$ (53)

to zero. Thus we get

$$\phi_1 = \frac{Q_{12}Q_{21}}{m} + a \quad \phi_3 = \frac{Q_{23}Q_{32}}{m} + \frac{a}{2}$$ (54)

Substituting the above VEVs into eq. (51) we get the low-energy superpotential

$$W = Q_{12}Q_{21} \left( \phi_2 - a - \frac{Q_{12}Q_{21}}{2m} \right) + Q_{23}Q_{32} \left( \frac{a}{2} + \frac{Q_{23}Q_{32}}{2m} - \phi_2 \right)$$ (55)

which leads to the F–terms

$$F_{Q_{12}} = Q_{21} \left( \phi_2 - a - \frac{Q_{12}Q_{21}}{m} \right)$$ (56)

$$F_{Q_{21}} = Q_{12} \left( \phi_2 - a - \frac{Q_{12}Q_{21}}{m} \right)$$ (57)

$$F_{Q_{23}} = Q_{32} \left( \frac{a}{2} + \frac{Q_{23}Q_{32}}{m} - \phi_2 \right)$$ (58)

$$F_{Q_{32}} = Q_{23} \left( \frac{a}{2} + \frac{Q_{23}Q_{32}}{m} - \phi_2 \right)$$ (59)

$$F_{\phi_2} = Q_{12}Q_{21} - Q_{23}Q_{32}$$ (60)

In addition, we blow up the second node in order to get an anomalous D–term for $U(1)_2$ where $\xi = \int_{S^2} J$. The D–term becomes

$$D = (|Q_{12}|^2 - |Q_{21}|^2 - |Q_{23}|^2 + |Q_{32}|^2 + \xi)$$ (61)

From the F and D–terms above, we find that the scalar potential has two isolated
supersymmetric vacua at

\[ \phi_2 = a \quad |Q_{21}|^2 = \xi \quad Q_{12} = Q_{23} = Q_{32} = 0 \quad (62) \]

and

\[ \phi_2 = \frac{a}{2} \quad |Q_{23}|^2 = \xi \quad Q_{12} = Q_{21} = Q_{23} = 0 \quad (63) \]

As a result, there is a domain wall which interpolates between these two vacua. In the world-volume field theory, the domain wall solution (in the limit \( a^2 >> g_s^2 \xi \)) is given by

\[ Q_{21} = \frac{\sqrt{\xi}}{A} e^{-az} \quad Q_{23} = \frac{\sqrt{\xi}}{A} e^{az/2} \quad \phi_2 = \frac{a}{4} (3 - \tanh z) \quad (64) \]

where \( A^2 = e^{-2az} + e^{az} \) and for simplicity, we assumed that the domain wall is normal to the \( z \) direction and located at \( z = 0 \). The tension of the wall is

\[ T_w = \xi \left( a - \frac{a}{2} \right) = \frac{1}{2} \xi a \quad (65) \]

The brane configuration that corresponds to the domain wall is a D5 brane wrapped on the second node and stretched between \( x = a/2 \) and \( x = a \), i.e. between the two D5 branes wrapped on the first and third nodes. The tension of such a brane is

\[ T = T_{D5} V_2 (a - a/2)(2\pi)\ell_s^2 = \frac{V_2}{(2\pi)^5 g_s \ell_s^6} (\pi a \ell_s^2) = \frac{1}{2} \xi a \quad (66) \]

using eq. (48) for \( \xi \). For the domain wall to exist at low energies, \( E << m \), we need to assume \( \xi a << m^3 \).

We also note that outside the domain wall, i.e. for either one of the solutions in eqs. (61) or (62), \( U(1)_2 \) is spontaneously broken by the charged field VEVs. Therefore, we expect to find vortex solutions outside the domain walls. The two vacua in eqs. (61) and (62) lead to the same vortices with tension \( T_s = 2\pi \xi \). These
are D–term strings since their tension arises from an anomalous D–term. Thus, there are vortices on either side of the domain wall. These strings are semi–local since the gauge group is $U(1)_2$ but there are effectively two flavors. Thus, we expect them to be unstable and dissolve. We can try to stabilize these strings by not decoupling either $U(1)_1$ or $U(1)_3$.

For example, if we decouple only $U(1)_3$, the remaining gauge group becomes $U(1)_1 \times U(1)_2$ and in the vacuum given by eq. (62) the gauge symmetry gets enhanced to $U(2)$ which is spontaneously broken by the VEVs of $Q_{12}, Q_{21}$. This leads to non–Abelian vortices with $N_c = N_f = 2$ which are stable. In the other vacuum given by eq. (63) which is on the other side of the domain wall, vortices are still semi–local and unstable. Clearly, if we do not decouple any of the $U(1)_s$, the vortices on both sides of the domain wall become stable non–Abelian strings with $N_c = N_f = 2$.

In addition, as in section 2, we expect to find monopoles living inside the vortices. From eq. (62) we see that $U(1)_1 \times U(1)_2$ is broken everywhere except inside the D–term strings where $Q_{12} = Q_{21} = 0$. Then, $\phi_2$ can have a VEV different than $a$ leading to two types of monopoles charged under $U(1)_1$ or $U(1)_2$ with masses $m_m = 4\pi(\phi_2 - a)/g_i^2$ where $i = 1, 2$. Thus, we expect to find different types of confined monopoles that live in the vortices which are confining magnetic flux tubes. If we do not decouple $U(1)_3$ we get two other types of monopoles charged under either $U(1)_2$ or $U(1)_3$ with masses $m_m = 4\pi(\phi_2 - a/2)/g_i^2$ where $i = 2, 3$.

We see that the physics of domain walls is quite rich even in the simplest case we considered above. Generalizing our results to $A_n$ singularities and/or the non–Abelian case with multiple D5 branes wrapped on the nodes of the singularity is expected to lead to richer and more interesting results.

**5. Conclusions and Discussion**

In this paper, we described monopoles, vortices and domain walls that live on
the world–volumes of D5 branes wrapped on deformed and/or resolved $A_2$ and $A_3$ singularities fibered over a complex plane $C(x)$. We showed how these different solitons are in fact D3 or D5 branes wrapped on the nodes of the singularity and stretched along $C(x)$. The minimum requirement for the existence of monopoles and F–term strings is a deformed singularity with two nodes, i.e. a deformed $A_2$. D–term strings require, in addition, the resolution of the node on which the D3 brane is wrapped. On the other hand, domain walls require a deformed singularity with at least three nodes, i.e. a deformed $A_3$ with a resolved node. Our results can be easily generalized to the cases of multiple D5 branes on deformed $A_N$ singularities such that between any two neighboring nodes there is a different monopole or vortex and among any three nodes there is a domain wall (with or without strings attached). Such a brane configuration would give rise to a collection of solitons on the noncompact world–volume.

At first sight, our results are somewhat surprising. For example, it is well-known that a D3 brane inside a D5 brane constitutes a magnetic flux tube[17]. One expects this to hold even when both branes are compactified on a $T^2$ since T–duality relates this to a configuration with a D1 brane inside a D3 brane. However, in our case, we compactify both branes on $S^2$ rather than on $T^2$ and T–duality does not apply. The same argument can be repeated for our description of monopoles and domain walls. Again, it is well–known that the end of a D3 brane that ends on a D5 brane behaves as a monopole on the D5 brane world–volume. This configuration describes our monopoles on the singularity even though both branes are wrapped on $S^2$ (rather than on $T^2$ which would allow T–duality). Domain walls on $A_3$ give rise to the same puzzle. Our results strongly indicate that wrapping branes on nodes of singularities or $S^2$s somehow has the same effect as wrapping them on tori, at least as far as the existence of solitons are concerned. It would be interesting to clarify this puzzle by dimensionally reducing world–volume theories of D5 branes wrapped on $S^2$ and finding out the different soliton solutions.

The world–volume theories and other quantum properties of vortices and domain walls have been obtained in the framework of intersecting branes[28-32].
These describe the low–energy physics of the solitons in terms of their moduli. For D5 and D3 branes wrapped on the deformed and resolved nodes we expect to find similar world–volume theories. Since our setup of branes on $A_n$ singularities is related to intersecting brane models this would not be very surprising. For example, the $A_2$ singularity corresponds to two NS5 branes separated along a compactified $x_6$ direction. (This can be seen as three NS5 branes where the first and third ones are identified.) The D5 branes wrapped on the two nodes correspond to D4 branes stretched between the two NS5 branes. Singlet masses are obtained by rotating one of the NS5 branes whereas F and D–terms are obtained by shifting one of the NS5 branes along the $x_8 + ix_9$ and $x_7$ directions respectively. Thus, our models can also be described in terms of intersecting branes.

However, the models we examined in this paper are not exactly dual to the intersecting brane models of refs. [28-32]. The main difference is the amount of supersymmetry; the models we examined above have $\mathcal{N} = 1$ supersymmetry whereas those in refs. [28-32] have $\mathcal{N} = 2$ (like the cases described by eqs. (17) and (40)). Another important difference seems to be the existence of D6 branes in these intersecting brane constructions. The models considered in this paper are dual to intersecting brane models that contain semi–infinite D4 branes (which correspond to the D5 branes wrapped on the decoupled nodes) rather than D6 branes. Therefore, we cannot directly use the results in refs. [28-32] especially for describing the soliton moduli spaces or world–volume theories.

It would be worthwhile to find out the world–volume theories for the solitons described in this paper. Since, our models have only $\mathcal{N} = 1$ supersymmetry, this will involve the collective coordinates of the solitons which are flat directions that are analogous to the moduli in $\mathcal{N} = 2$ supersymmetric theories. These theories should help us understand the low–energy physics of the above solitons and their interactions.

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