Design of Observer for Linear Systems with Quantized Output

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Abstract

This paper considers an observer-based control for linear systems having outputs quantized by uniform quantizers, where the state-feedback control is applied using the states estimated by the observer. The observer gain is designed to mitigate the effects of the quantization error on the system output. Specifically, we design the optimal observer gain to minimize the $H_2$ norm of the transfer function from the quantization error to the system output as a function of the observer gain. Numerical examples are provided to see the performance of our design.

1 Introduction

In a networked control system, information between the controller and the plant may be transmitted over rate limited communication channels, where continuous-valued signals are quantized into low-resolution signals. If the rate is not sufficient, quantization errors can cause serious performance degradation.

Stabilizability and observability under a communication constraint have been studied in [1]. A necessary and sufficient condition on the information rate for asymptotic stabilizability and observability has been presented. The same condition has been shown in [2] as well for a necessary and sufficient condition for exponential stabilizability of discrete-time LTI systems with random initial states. These theoretical results based on the information theory give valuable insights into control under limited data rates. However, the assumptions for these theoretical results are not satisfied with practical quantizers in general.

Since uniform quantizers are still often utilized, this paper considers a networked control system where the output of the system is quantized by a uniform quantizer and the quantized output is transmitted to the controller. At the controller, an observer is constructed and a state-feedback control is applied using the states estimated by the observer.

There have been several methods to determine the observer gain. In [3], an observer-based control for continuous-time systems with quantized outputs has been proposed where the state of the system converges to an ultimate bounded set surrounding the origin. However, it does not necessarily minimize the effects of the quantization error.

This paper designs the observer gain to mitigate the effects of the quantization error on the system output. More specifically, we design the optimal observer gain to minimize the $H_2$ norm of the transfer function from the quantization error to the system output as a function of the observer gain. Numerical examples are provided to see the performance of our design.

Notation: The $z$ transform of a sequence (or a vector) $h = \{h_k\}_{k=\ell}^\infty$ is denoted as $H[z] = \sum_{k=\ell}^\infty h_k z^{-k}$. The output sequence $y$ of a linear and time-invariant (LTI) system $H[z]$ with the input sequence $x$ (i.e. $y = h \ast x$ where $\ast$ denotes the convolution) is expressed as $y = H[z] x$.

2 Networked control system and quantization

Fig. 1 depicts a generalized plant, in which the plant is assumed to be linear and time-invariant (LTI) and the signals $w_c, z, y, u$ are vector-valued functions of time in general. The inputs to the plant are the exogenous input $w_c$ and the output $u$ of the controller. The controller generates the input $u$ to the plant, using the observation signal $y$ of the plant.

We convert the observation signal $y$ into a discrete-valued signal by using a simple yet universal uniform quantizer.

Based on the discretized observation signal, the control signal is generated by a digital controller. We express signals and systems in discrete-time and assume that the plant is a single-input and single-output sys-
of a linear observer is given by

\[ y \]

the system output

the transfer function from the quantization error to the output of the plant be

\[ H \]

where

\[ w \]

After straightforward computations, one can find that the transfer function from the quantization error \( w \) to the system output \( y \) is given by

\[ H[z] = C_p(zI - A_p + B_pK)^{-1}B_pK(zI - A_p + LC_p)^{-1}L. \]  

In (7), all the parameters \( H[z] \) are given except for the observer gain \( L \). We determine the observer gain \( L \) to mitigate the effect of the quantization error on the system output.

If the quantization error \( w \) of the static uniform quantizer is a statistically white random signal with zero mean and variance \( \sigma_w^2 \), the mean squared error (MSE) of the system output due to the quantization error is given from (6) by

\[ E\{\|\epsilon_k\|_2^2\} = \|H[z]\|_2^2 \sigma_w^2 \]  

where \( E\{\cdot\} \) denotes the expectation operator. We would like to minimize the MSE with respect to the observer gain \( L \).

3 Synthesis of observer

Let us see how the minimization of the MSE is converted into an optimization problem. The state-space equations from \( w_k \) to \( \epsilon_k \) is expressed as

\[ \begin{align*}
\tilde{x}_{k+1} &= A\tilde{x}_k + Bw_k \\
\epsilon_k &= C\tilde{x}_k
\end{align*} \]  

where \( \tilde{x}_k \) is an augmented state and

\[ A = \begin{bmatrix} A_p - B_pK & -B_pK \\ 0 & A_p - LC_p \end{bmatrix}, \]

\[ B = \begin{bmatrix} 0 \\ L \end{bmatrix}, \quad C = \begin{bmatrix} C_p & 0 \end{bmatrix}. \]

It is known that \( \|H[z]\|_2^2 < \mu_e \) if and only if there exists a positive definite matrix \( P \) such that [4]

\[ \begin{bmatrix} P & PA & PB \\ AT\,P & P & 0 \\ B'T\,P & 0 & I \end{bmatrix} > 0 \]  

Then, the problem is equivalent to the minimization of \( \mu_e \) with respect to \( L \in \mathbb{R}^{n_k \times 1} \) and a positive definite matrix \( P \) subject to (11) and (12).

We can express \( A \) as

\[ A = \begin{bmatrix} A_p - B_pK & -B_pK \\ 0 & A_p \end{bmatrix} - \begin{bmatrix} 0 \\ L \end{bmatrix} \begin{bmatrix} 0 & C_p \end{bmatrix}. \]  

Although (12) is a linear matrix equality (LMI) of \( P \), (11) is a non-convex bilinear matrix inequality, since \( A \) has \( L \) as in (13) and \( PA \) contains a multiplication between \( P \) and \( L \).

Here we take an exhaustive search to obtain the optimal observer. Since the eigenvalues of \( A'T_p - LC_p \) have to exist within a unit circle, the range of each entry of \( L \) is limited. Moreover, we can easily evaluate \( ||H[z]||_2 \). We search all possible values for \( L \) to find the optimal \( L \).
Table 1: Comparison of MSEs

|      | Theory (10^{-2}) | Simulation (10^{-2}) |
|------|------------------|-----------------------|
| Original | 9.51             | 8.90                  |
| Optimized | 7.66             | 8.13                  |

4 Numerical example

To verify the efficiency of our approach, we consider an unstable system of order 2 whose state-space matrices are given by

\[ A_p = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 1 \end{bmatrix}. \]

The gain \( K \) for the state-feedback control is determined by the linear quadratic regulator technique to minimize

\[ \sum_{k=0}^{\infty} (x_k^T Q_{qfr} x_k + r |u_k|^2) \] (14)

where the weights are

\[ Q_{qfr} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}, \quad r = 1. \] (15)

The original observer gain \( L \) is set to be \([7/6, 7/6]^T\) so that the eigenvalues of \( A_p^T L - L C_p \) are 1/2 and 1/3. The optimal observer gain is found to be \([8/9, 2]^T\).

The gain \( K \) is chosen to be \([0.7581, 0.0578]^T\) so that the eigenvalues of \( A_p^T P - L C_p \) are 1/3 and 1/3. The optimal observer gain is found to be \([0.1065, 0.0658]^T\).

The initial value \( \hat{x}_0 \) for the state-estimate is again set to be \( \hat{x}_0 = [0, 0]^T \) to remove its effects on the output. If there is no quantization error, there is no state-estimation error in the observer.

Table 1 compares theoretical and empirical MSEs of the original system and the optimized system, where the empirical MSE is obtained by averaging the differences of the outputs of the systems with and without quantization. As we expected, the system having the optimized observer gain for the uniform quantizer enjoys smaller MSE than the original system, which validate our design approach.

5 Conclusion

We have considered an observer-based control for LTI SISO systems having outputs quantized by a uniform quantizer. The optimal observer gain has been designed to minimize the \( H_2 \) norm of the transfer function from the quantization error to the system output as a function of the observer gain. Numerical examples show that the optimal observer exhibits smaller MSE than a non-optimized observer.

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