SCHRÖDINGER OPERATORS WITH COMPLEX-VALUED POTENTIALS AND NO RESONANCES

T. CHRISTIANSEN

Abstract. In dimension $d \geq 3$, we give examples of nontrivial, compactly supported, complex-valued potentials such that the associated Schrödinger operators have no resonances. If $d = 2$, we show that there are potentials with no resonances away from the origin. These Schrödinger operators are isophasal and have the same scattering phase as the Laplacian on $\mathbb{R}^d$. In odd dimensions $d \geq 3$ we study the fundamental solution of the wave equation perturbed by such a potential. If the space variables are held fixed, it is super-exponentially decaying in time.

1. Introduction

In this paper we consider compactly supported, complex-valued potentials $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, $d \geq 2$. Our first result is that for $d \geq 3$ there are many such nontrivial $V$ so that the meromorphic continuation of the resolvent $(\Delta + V - \lambda^2)^{-1}$ has no poles; for $d = 2$ we give examples with no poles except, perhaps, at the origin. These results are surprising, as there are no such nontrivial potentials with this property in one dimension [4, 10, 14, 20]. Moreover, it is known that for nontrivial real-valued, smooth, compactly supported potentials in all dimensions greater than two the resolvent must have infinitely many poles [9, 12, 13]. We also show that the Schrödinger operators with the potentials we construct, like the Laplacian, have scattering phase 0. In addition, in odd dimensions at least three, we show that for potentials without associated resonances the fundamental solution of the perturbed wave equation is, for fixed values of the space variables, super-exponentially decaying in time.

Let $\Delta$ be the non-negative Laplacian on $\mathbb{R}^d$, and let $R_0(\lambda) = (\Delta - \lambda^2)^{-1}$ be the resolvent, bounded on $L^2(\mathbb{R}^d)$ for $0 < \arg \lambda < \pi$. Then, as an operator from $L^2_{\text{comp}}(\mathbb{R}^d)$ to $H^2_{\text{loc}}(\mathbb{R}^d)$, $R_0$ has an analytic continuation to $\mathbb{C}$ if $d$ is odd. If $d$ is even, the continuation is to $\Lambda$, the logarithmic cover of the complex plane (There is a singularity at the origin if $d = 2$). If $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, $R_V(\lambda) = (\Delta + V - \lambda^2)^{-1}$ is bounded on $L^2(\mathbb{R}^d)$ for all but a finite number of $\lambda$ with $0 < \arg \lambda < \pi$. Like $R_0$, as an operator from $L^2_{\text{comp}}(\mathbb{R}^d)$ to $H^2_{\text{loc}}(\mathbb{R}^d)$, $R_V$ has a meromorphic continuation to $\mathbb{C}$ (for $d$ odd) or $\Lambda$ ($d$ even). The poles of the continuation are called resonances, or scattering poles. They behave like eigenvalues in a number of ways and may

Partially supported by NSF grant DMS 0088922.
known lower bound to hold for a general class of potentials is, for no
trivial with this order of growth [21, 22]. Lower bounds are more delicate.

The best for a particular family of complex-valued potentials.

scattering-theoretic questions for the one-dimensional Schrödinger operator with
PT
-symmetric quantum mechanics and other applications, and [7] for a study of

This holds both for real and complex potentials. If

d
[4, 10, 14, 20]. This holds both for real and complex potentials. If

d
= 1,

\begin{equation}
\lim_{r \to \infty} \frac{N_V(r)}{r} = \frac{2}{\pi} \text{diam}(\text{supp}(V))
\end{equation}

[11]. We show in this paper that it is necessary to assume that

V

is real-valued, giving evidence of the subtlety of the behaviour of resonances in dimension bigger

than one.

Upper bounds on a resonance-counting function in even dimensions can be

found in [15, 16], and some lower bounds for smooth real-valued potentials in [12].

On \mathbb{R}^d we use the “cylindrical” coordinates \((\rho, \theta, x') \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}^{d-2},

with \(x_1 = \rho \cos \theta, x_2 = \rho \sin \theta.\) When \(d = 2,\) we understand that the \(x'\) coordinates are omitted. We shall use these coordinates for the statement of the following

theorem and in Section 2.

\textbf{Theorem 1.} Let \(V_1 \in L^\infty_{\text{comp}}(\mathbb{R}^+), V_2 \in L^\infty_{\text{comp}}(\mathbb{R}^{d-2}),\) and \(m \in \mathbb{Z} \setminus \{0\}.\) If \(d \geq 3,\)

let \(V(\rho, \theta, x') = e^{im\theta}V_1(\rho)V_2(x')\), and if \(d = 2,\) let \(V(\rho, \theta) = e^{im\theta}V_1(\rho).\) Then, if

\(d \geq 3,\) there are no poles of \(R_V(\lambda),\) and if \(d = 2,\) there are no poles of \(R_V(\lambda)\) away

from the origin.

These potentials are related to ones used for infinite cylinders in [3]. We note

that \(V_1\) and \(V_2\) can be chosen so that \(V\) is smooth.

If \(V_1\) and \(V_2\) are real-valued, then \(\Delta + V\) is a \(\mathcal{PT}\)-symmetric operator [2]. Here

\(\mathcal{T}\) is complex-conjugation, \((T \psi)(x) = \overline{\psi}(x),\) and \((\mathcal{P} \psi)(x_1, x_2, x') = \psi(x_1, -x_2, x').\) See [12] for further references to studies of \(\mathcal{PT}\)-symmetric operators, including

\(\mathcal{PT}\)-symmetric quantum mechanics and other applications, and [7] for a study of

scattering-theoretic questions for the one-dimensional Schrödinger operator with

a particular family of complex-valued potentials.

For a potential \(W \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C}),\) let \(S_W(\lambda)\) be the associated scattering

matrix, \(s_W(\lambda) = \det S_W(\lambda),\) and \(\sigma_W(\lambda) = \log s_W(\lambda)\) be the scattering phase, with

\(\sigma_W(0) \in [0, 2\pi)\) to determine it uniquely. Here \(\lambda \in \mathbb{C}\) if \(d\) is odd and \(\lambda \in \Lambda\) if \(d\) is

even. Like resonances, the scattering phase (for \(\lambda \in \mathbb{R}\)) may be thought of as an

analog of discrete spectral data for our setting.

\textbf{Theorem 2.} Let \(V\) be as in Theorem [7]. \(\lambda \in \mathbb{C}\) if \(d \geq 3\) is odd, \(\lambda \in \Lambda\) if \(d\) is even.

Then \(s_V(\lambda) = 1\).
That is, these Schrödinger operators are isophasal and have the same phase as the Schrödinger operator with the trivial potential. For examples of isophasal manifolds and references to further results in that direction, see e.g. [5].

In Section 2 we give a direct proof of Theorem 2 without using the results of Theorem 1. Here we make some comments about the relationship between Theorems 1 and 2 in odd dimensions, which is the simpler case. With at most a finite number of exceptions, the poles of $R_V$ correspond, with multiplicity, to the poles of $s_V$. If $R_V$ is regular at $\lambda_0$, then so is $s_V$. See [24] for a careful discussion of these questions. Moreover $\lambda$ is a pole of $s_V$ if and only if $-\lambda$ is a zero of $s_V$, and the multiplicities are the same. Thus, using the Weierstrass factorization theorem, Theorem 1, and results of [24],

$$s_V(\lambda) = e^{g_V(\lambda)},$$

where $g_V(\lambda)$ is a polynomial of degree at most $d$. Further considerations put additional restrictions on $g_V$. On the other hand, if $s_V(\lambda) \equiv 1$, $R_V$ can have have at most a finite number of poles. A priori, one cannot rule out, for example, the possibility that $R_V$ has a pole at $\lambda_0$ and another one, of the same multiplicity, at $-\lambda_0$. In this case, $s_V$ would be holomorphic at both $\lambda_0$ and $-\lambda_0$ [24]. Thus neither Theorem 1 nor Theorem 2 immediately implies the other, even in odd dimensions.

In many settings, the imaginary parts of resonances are related to the rate of decay of solutions of a wave equation on compact sets. In the absence of resonances we consider the decay of the fundamental solution $G_V(t)$ of the perturbed wave equation

$$\begin{align*}
\left(D_t^2 - (\Delta + V)\right)G_V(t) &= 0 \\
G_V(0) &= 0 \\
(G_V)_t(0) &= I.
\end{align*}$$

Here $V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C})$ and $D_t = \frac{1}{i} \frac{\partial}{\partial t}$. These equations uniquely determine $G_V(t)$.

**Theorem 3.** Let $d \geq 3$ be odd, $V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C})$, and let $G_V(t)$ be the operator determined by (1), with $G_V(t, x, y)$ its Schwartz kernel. Then if $R_V(\lambda)$ has no poles for $\lambda \in \mathbb{C}$ and if $\chi \in C_c^\infty(\mathbb{R}^d)$, then $\chi(x)G_V(t, x, y)\chi(y)$ is super-exponentially decreasing in $t$.

As an immediate consequence of Theorems 1 and 3 we obtain

**Corollary 4.** In dimension $d \geq 3$ odd, there are nontrivial potentials $V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C})$ such that the fundamental solution $G_V(t)$ determined by (1) decays faster than any exponential when the space variables are restricted to compact sets.

It is a pleasure to thank N. Kalton and K. Shin for helpful conversations and M. Zworski for his constructive comments on an earlier version of this note.
2. Proof of Theorems 1 and 2

Recall that in this section we use the coordinates \((\rho, \theta, x')\) on \(\mathbb{R}^d\). For \(j \in \mathbb{Z}\), let \(P_j\) denote projection onto \(e^{ij\theta}\). That is,

\[
(P_j f)(\rho, \theta, x') = \frac{1}{2\pi} e^{ij\theta} \int_0^{2\pi} f(\rho, \theta', x') e^{-ij\theta'} d\theta'.
\]

**Lemma 5.** Let \(\lambda \in \mathbb{C}\) if \(d\) is odd, \(d \geq 3\) and \(\lambda \in \Lambda\) if \(d\) is even. If \(d = 2\), assume \(\lambda \neq 0\). For \(\chi \in C_\infty^\infty(\mathbb{R}^d)\) and for \(j \in \mathbb{Z}\), \(|j|\) sufficiently large (depending on \(\lambda\)),

\[
\|\chi R_0(\lambda) P_j \chi\| \leq \frac{C}{|j|^2 - C}
\]

for some constant \(C\) depending on \(\lambda\) and \(\chi\).

**Proof.** Note that \(\Delta P_j = P_j \Delta\), so that \(R_0 P_j = P_j R_0\). We may assume that \(\chi\) is independent of \(\theta\), since the general case will follow from this special one. Thus we may use \(\chi P_j = P_j \chi\).

We have

\[
\chi(\Delta - \lambda^2) R_0(\lambda) \chi = \chi^2,
\]

so that

\[
(\Delta - \lambda^2) \chi R_0(\lambda) P_j \chi = \chi^2 P_j - [\chi, \Delta] R_0(\lambda) \chi P_j.
\]

Then

\[
\|(\Delta - \lambda^2) \chi R_0(\lambda) P_j \chi\| \leq \|\chi^2 P_j\| + \|[\chi, \Delta] R_0(\lambda) \chi P_j\|
\]

\[
\leq C
\]

where the constant depends on \(\lambda\) and \(\chi\). But, for some \(c_\chi > 0\),

\[
\|(\Delta - \lambda^2) \chi P_j v\| \geq (c_\chi j^2 - \text{Re} \lambda^2) \|\chi P_j v\|.
\]

Thus, when \(|j|\) is so large that \(c_\chi j^2 - \text{Re} \lambda^2 > 0\),

\[
\|\chi R_0(\lambda) P_j \chi\| \leq \frac{C}{c_\chi j^2 - \text{Re} \lambda^2}.
\]

We shall use the following elementary lemma in our proof of the theorem.

**Lemma 6.** Let \(\{a_j\}_{j=\infty}^{\infty} \in \ell^2\) and \(m \in \mathbb{Z}\), \(m \neq 0\). Suppose for each \(j\) there is a constant \(C_j\) such that \(|a_{j+m}| \leq C_j |a_j|\). If, in addition, there is a \(J\) such that \(|a_{j+m}| \leq |a_j|\) when \(|j| \geq J\), then \(a_j = 0\) for all \(j\).

**Proof.** Suppose \(a_{j_0} \neq 0\) for some \(j_0\). Then since \(|a_{j+m}| \leq C_j |a_j|\), \(a_{j_0 - km} \neq 0\) for \(k = 1, 2, 3, \ldots\).

Since \(\{a_j\} \in \ell^2\), \(|a_{j_0-km}| \to 0\) as \(k \to \infty\). But when \(|j_0 - (k+1)m| > J\),

\[
|a_{j_0}-km| \leq |a_{j_0-(k+1)m}| \leq |a_{j_0-(k+2)m}| \leq \ldots
\]

Thus \(a_{j_0-km} = 0\), a contradiction.

Now we are able to give the proof of the first theorem.
Suppose, on the contrary, \( \lambda \in \mathbb{C} \) if \( d \) is odd or \( \lambda \in \Lambda \) if \( d \) is even is a pole of the resolvent, and \( \lambda \) is not the origin if \( d = 2 \). Then, since

\[
(\Delta + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda)
\]

and \( R_0 \) is holomorphic (if \( d \geq 3 \), or holomorphic away from the origin (if \( d = 2 \)), there is a nontrivial \( u \in L^2(\mathbb{R}^d) \) so that

\[
(I + VR_0(\lambda))u = 0.
\]

The function \( u \) is necessarily supported on \( \text{supp} \ V \). We can write

\[
u(\rho, \theta, x') = \sum_{-\infty}^{\infty} u_j(\rho, x') e^{ij\theta}\]

(where we omit the \( x' \) variables if \( d = 2 \)). Since

\[
u = -VR_0(\lambda)u,
\]

if \( \chi \in C_0^\infty(\mathbb{R}^d) \), with \( \chi \equiv 1 \) on \( \text{supp} \ V \),

\[
u_{j+m} = -V_1V_2\chi R_0(\lambda)\chi P_j u
\]

and

\[
\|u_{j+m}\|_{L^2(\mathbb{R}^d)} \leq C_j\|u_j\|_{L^2(\mathbb{R}^d)}.
\]

By Lemma when \(|j|\) is sufficiently large,

\[
\|u_{j+m}\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{j^2-C}\|u_j\|_{L^2(\mathbb{R}^d)}.
\]

Using Lemma applied to \( \{\|u_j\|_{L^2(\mathbb{R}^d)}\} \), \( \|u_j\|_{L^2(\mathbb{R}^d)} = 0 \) for all \( j \), and thus \( u \equiv 0 \). \( \square \)

The proof of Theorem does not use any of the other results of this section.

**Proof of Theorem** Fix \( \lambda \in \mathbb{C} \) if \( d \) is odd and \( \lambda \in \Lambda \) if \( d \) is even. If \( d = 2 \) assume that \( \lambda \) is not the origin. Recall that for \( W_{\text{comp}}(\mathbb{R}^d; \mathbb{C}) \), \( S_V \) denotes the scattering matrix associated with \( \Delta + W \) and \( s_V(\lambda) = \det S_V(\lambda) \).

Let \( V = V(\rho, \theta, x') \) be as defined in Theorem and, for \( z \in \mathbb{C} \), let \( W_z(\rho, \theta, x') = z^mV(\rho, \theta, x') \). If \( m > 0 \), \( W \) is holomorphic as a function of \( z \in \mathbb{C} \), and if \( m < 0 \), it is holomorphic on \( \mathbb{C} \setminus \{0\} \), and meromorphic on \( \mathbb{C} \). Then, for fixed \( \lambda \), \( S_{W_z}(\lambda) \) and

\[
h_\lambda(z) = \det S_{W_z}(\lambda)
\]

depend meromorphically on \( z \in \mathbb{C} \).

For \( \phi \in \mathbb{R} \), \( (e^{i\phi})^mV(\rho, \theta, x') = V(\rho, \theta + \phi, x') \) where we make the identification \( \theta + 2k\pi = \theta \) for \( k \in \mathbb{Z} \). That is, \( (e^{i\phi})^mV \) corresponds to \( V \) under a rotation of angle \( \phi \) in the \( x_1x_2 \) plane. Although in general the scattering matrix \( S_V(\lambda) \) is not invariant under rotations of \( \mathbb{R}^d \), its eigenvalues are, and so \( s_V(\lambda) \) is unchanged. Thus

\[
h_\lambda(e^{i\phi}) = s_{e^{i\phi}V}(\lambda) = s_V(\lambda) = h_\lambda(1).
\]

Then \( h_\lambda(z) \) is a meromorphic function of \( z \in \mathbb{C} \) which is constant on the unit circle. Thus it is a constant function \( h_\lambda(z) \equiv h_\lambda(1) \).
If \( m > 0 \),
\[
h_\lambda(1) = h_\lambda(z) = h_\lambda(0) = \det S_0(\lambda) = 1.
\]
If \( m < 0 \),
\[
h_\lambda(1) = h_\lambda(z) = \lim_{r \to \infty} h_\lambda(r) = \lim_{r \to \infty} \det S_{r,0}(\lambda) = 1.
\]
This proves the theorem except for the question of what happens at the origin if \( d = 2 \). However, the regularity properties of \( s_V \) imply that it is 1 at the origin as well. \( \square \)

3. Proof of Theorem

In this section, we show that for odd \( d \) and for \( V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C}) \) without resonances the fundamental solution of the perturbed wave equation \( D_t^2 - (\Delta + V) \) decays super-exponentially.

Lemma 7. For \( V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C}) \) such that \( \Delta + V \) has no eigenvalues,
\[
G_V(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sin(t\lambda) \left( R_V(\lambda) - R_V(-\lambda) \right) d\lambda
\]
where \( G_V(t) \) is determined by (2).

If \( V \) is real-valued, this lemma follows immediately from the spectral theorem and Stone’s formula. We are unaware of a reference that would directly imply this result for complex-valued potentials and thus include a proof below.

Proof. Let \( f \in C^\infty_c(\mathbb{R}^d) \) and
\[
u_f(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sin(t\lambda) \left( R_V(\lambda) - R_V(-\lambda) \right) f d\lambda.
\]
We shall show that
\[
(D_t^2 - (\Delta + V))\nu_f(t) = 0
\]
\[
u_f(0) = 0
\]
\[(\nu_f)_t(0) = f.
\]
This will show that the operator determined by the right-hand side of (2) agrees with \( G_V(t) \) on a dense subspace of \( L^2(\mathbb{R}^d) \), and thus, since \( G_V(t) \) is continuous, the two coincide. Here we use the uniqueness of solutions of the initial value problem for the perturbed wave equation.

We use the notation \( \langle x \rangle = (1 + |x|^2)^{1/2} \). When \( \text{Im} \lambda \geq 0 \) and \( s > 1/2 \)
\[
\| \langle x \rangle^{-s} R_0(\lambda) \langle x \rangle^{-s} \| \leq \frac{C}{|\lambda|}
\]
[13, Corollary 3.6] and, since \( \Delta + V \) has no eigenvalues,
\[
\| \langle x \rangle^{-s} R_V(\lambda) \langle x \rangle^{-s} \| \leq \frac{C}{|\lambda|}
\]
for the same $\lambda$ and $s$. Since
\[
(R_V(\lambda) - R_V(-\lambda)) f = \frac{1}{\lambda^2} (R_V(\lambda) - R_V(-\lambda)) (\Delta + V) f
\]
the integral in (3) converges absolutely as an element of $\langle x \rangle^s L^2(\mathbb{R}^d)$, for any $s > 1/2$. It is easy to see then that the first two equalities in (4) hold.

To prove the third equality in (4) we use the $H_\infty$ functional calculus [8]. Thus
\[
(5) \quad f = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\mu + 10} (\Delta + V - \mu)^{-1}((\Delta + V + 10)f) d\mu
\]
where $\gamma$ is the contour defined by the function
\[
(6) \quad g(t) = \begin{cases} -te^{-i\theta} - 1, & t \leq 0 \\ te^{i\theta} - 1, & t \geq 0 \end{cases}
\]
and $\theta$ is chosen sufficiently small that $\gamma$ does not enclose $\pm i\sqrt{10}$. Now we shall do some contour deformation which is valid because $(\Delta + V - \mu)^{-1}$ has no poles and because the integrand (as an element of $\langle x \rangle^s L^2(\mathbb{R}^d)$, $s > 1/2$) is bounded in norm by $C|\mu|^{-3/2}$ when $|\mu|$ is large. Doing a contour integration and a change of variables, we see that
\[
(7) \quad f = \frac{1}{\pi i} \int_{\text{Im } \lambda = \epsilon > 0} \frac{1}{\lambda^2 + 10} R_V(\lambda)((\Delta + V + 10)f) \lambda d\lambda.
\]
Taking the limit as $\epsilon \downarrow 0$, a simple further change of variables, and noting that $(R_V(\lambda) - R_V(-\lambda))((\Delta + V + 10)f) = (\lambda^2 + 10)(R_V(\lambda) - R_V(-\lambda))f$ finishes the proof.

Using this lemma, the proof of Theorem 5 is straightforward.

**Proof of Theorem 5** Recall the resolvent equation:
\[
(7) \quad R_V(\lambda) = R_0(\lambda) - R_V(\lambda)V R_0(\lambda).
\]
Thus, using Lemma 4
\[
G_V(t) = G_0(t) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sin(t \lambda) (R_V(\lambda)V R_0(\lambda) - R_V(-\lambda)V R_0(-\lambda)) d\lambda.
\]
Since $G_0(t, x, y)$ is supported on $|x - y| = |t|$, it is super-exponentially decaying on compact sets in $x$ and $y$.

Let $\chi' \in C_0^\infty(\mathbb{R}^d)$. Then
\[
\|\chi' R_0(\lambda)\chi\| \leq \frac{C}{1 + |\lambda|} (1 + e^{-C|\text{Im } \lambda|})
\]
[22] (16)]. Using this, for $M \in \mathbb{R}$, $\text{Im } \lambda > -M$, $|\lambda|$ sufficiently large,
\[
R_V(\lambda)\chi = R_0(\lambda) \sum_{j=0}^{\infty} (-1)^j (V R_0(\lambda))^j \chi,
\]
recalling that $\chi \in C^\infty_c(\mathbb{R}^d)$. Thus, using in addition the fact that $R_V$ has no poles,

$$
\|\chi R_V(\lambda)\chi\| \leq \frac{C}{1 + |\lambda|} (1 + e^{-C \text{Im} \lambda})
$$

when $\text{Im} \lambda > -M$. These estimates, along with the fact that $R_V$ has no poles, allows us to see by contour deformation

$$
\int_{-\infty}^{\infty} e^{it\lambda} \chi (R_V(\lambda)VR_0(\lambda) - R_V(-\lambda)VR_0(-\lambda)) \chi d\lambda \\
= \int_{-\infty}^{\infty} e^{it(\lambda+i\beta)} \chi (R_V(\lambda+i\beta)VR_0(\lambda+i\beta) - R_V(-\lambda-i\beta)VR_0(-\lambda-i\beta)) \chi d\lambda \\
= O(e^{-t\beta})
$$

for $\beta \in \mathbb{R}$. A similar argument shows that

$$
\int_{-\infty}^{\infty} e^{-it\lambda} \chi (R_V(\lambda)VR_0(\lambda) - R_V(-\lambda)VR_0(-\lambda)) \chi d\lambda = O(e^{-t\beta}).
$$

Since $\beta$ is arbitrary, $\chi(x)G_V(t, x, x')\chi(x')$ decays faster than any exponential. $\square$

References

[1] Carl Bender and Stefan Boettcher, Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, Phys. Rev. Lett. 80 (1998), no. 24, 5243–5246.

[2] E. Caliceti, S. Graffi, and J. Sjöstrand, Spectra of $\mathcal{PT}$-symmetric operators and perturbation theory, preprint.

[3] T. Christiansen, Asymptotics for a resonance-counting function for potential scattering on cylinders. To appear, J. Funct. Anal.

[4] R. Froese, Asymptotic distribution of resonances in one dimension, J. Differential Equations 137 (1997), no. 2, 251–272.

[5] C. Gordon and P. Perry, Continuous families of isophalal scattering manifolds, preprint 2002.

[6] L. Guillopé and M. Zworski, Scattering asymptotics for Riemann surfaces, Ann. of Math. 145 (1997), 597-660.

[7] G. Lévai, F. Cannata, and A. Ventura, Algebraic and scattering aspects of a $\mathcal{PT}$-symmetric solvable potential, J. Phys. A 34 (2001), no. 4, 839–844.

[8] A. McIntosh, Operators which have an $H_\infty$ functional calculus. Miniconference on operator theory and partial differential equations (North Ryde, 1986), 210–231, Proc. Centre Math. Anal. Austral. Nat. Univ., 14, Austral. Nat. Univ., Canberra, 1986.

[9] R.B. Melrose, Geometric scattering theory, Stanford lectures, Cambridge University Press, Cambridge, 1995.

[10] T. Regge, Analytic properties of the scattering matrix, Nuovo Cimento 8 (5), (1958), 671–679.

[11] A. Sá Barreto, Remarks on the distribution of resonances in odd dimensional Euclidean scattering, Asymptot. Anal. 27 (2001), no. 2, 161–170.

[12] A. Sá Barreto, Lower bounds for the number of resonances in even-dimensional potential scattering, J. Funct. Anal. 169 (1999), no. 1, 314–323.

[13] A. Sá Barreto and M. Zworski, Existence of resonances in potential scattering, Comm. Pure Appl. Math 49 (1996), 1271-1280.

[14] B. Simon, Resonances in one dimension and Fredholm determinants, J. Funct. Anal. 178 (2000), no. 2, 396–420.
[15] G. Vodev, *Sharp bounds on the number of scattering poles in the two-dimensional case*, Math. Nachr. 170 (1994), 287–297.

[16] G. Vodev, *Sharp bounds on the number of scattering poles in even-dimensional spaces*, Duke Math. J. 74 (1994), no. 1, 1–17.

[17] G. Vodev, *Resonances in the Euclidean scattering*, Cubo Matemática Educacional 3 no. 1 (2001), 317-360.

[18] D. Yafaev, *Scattering theory: some old and new problems*, Lecture Notes in Mathematics, 1735. Springer-Verlag, Berlin, 2000. xvi+169 pp.

[19] M. Znojil, *Should \( PT \) symmetric quantum mechanics be interpreted as nonlinear?*, J. Nonlinear Math. Phys 9 (2002), suppl. 2, 122-133.

[20] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. 73 (2) (1987), 277-296.

[21] M. Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials*, J. Funct. Anal. 82 (1989), 370-403.

[22] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. 59 (1989), no. 2, 311–323.

[23] M. Zworski, *Counting scattering poles*. In: Spectral and scattering theory (Sanda, 1992), 301–331, Lecture Notes in Pure and Appl. Math., 161, Dekker, New York, 1994.

[24] M. Zworski, *Poisson formulae for resonances*, Séminaire sur les Équations aux Dérivées Partielles, 1996-1997, Exp. No. XIII, 14pp., École Polytech., Palaiseau, 1997.

[25] M. Zworski, *Resonances in physics and geometry*, Notices Amer. Math. Soc. 46 (1999), no. 3, 319–328.

[26] M. Zworski, *Quantum resonances and partial differential equations*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 243–252, Higher Ed. Press, Beijing, 2002.

**Department of Mathematics**  
**University of Missouri**  
**Columbia, Missouri 65211**  
**tjc@math.missouri.edu**