Approximate Solution of 2-Dimensional VO Linear Fractional Partial Differential Equation

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Abstract. The non-polynomial spline method has been used to solving 2-dimensional variable-order (VO) fractional partial differential equations (FPDE). For VO fractional derivative, described in the sense of the Caputo. The main objective of this study and advantage of the proposed method is to investigate a public approximation for the frequency of the trigonometric functions of the non-polynomial part of the spline function. The powerful algorithm of the proposed method gives high accuracy results.

1. Introduction
The variable-order fractional is the one of most important tool in nowadays, with successful applied in mechanics [1], It can be used in the design of abnormal diffusion because it can depict the time-based diffusion process more efficiently than the partial derivative of the static arrangement means that the order of α.[2-3]. The topic is very active and of great interest due to its many applications, not only in Mathematics, but also in physics and in engineering, and it has proven that it describes complex phenomena in nature better and more broadly. [4-10]. The effect of differences between the use of fixed-order and variable-order fractional derivatives has been discussed and studied in [11].The addition and mathematical formalization of coefficients of variable-order fractions. The Evolutionary control equations for successful application have led to the modelling of complex world problems, from the study of biology and mechanics to transport processes and to many important applications. Partial variable calculus is an unknown branch of calculus that provides wonderful opportunities to simulate interdisciplinary processes. The scientific community has been extensively exploring the applications of fractional variable order in modelling and physical systems engineering. Among the goals of this work, will be starting point for researchers who interested in approaching this wonderful topic. We are interested in developing computational and analytical methods and applying complex physical systems using advanced simulation software. For more details, see [12-18].

This paper proposes non-polynomial spline functions (SF) to approximate the solution of the fractional partial differential equations of variable order (FPDEVO).
2. Definitions and properties:

2.1 Some Definitions:
In this part, some necessary and important definitions and mathematical principles of variable-order fractional derivatives and some properties of fractional derivatives will be mentioned.

Definition 2.1 [19] The Caputo and Riemann fractional derivatives of order $\alpha$ of $f(t)$ defined as:

$$\overset{\circ}{C}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

$$\alpha D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$  \hspace{1cm} ...(1)

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $n > 0$.

After folding the definition of fixed-order fractional derivatives, we show the variable-order fractional differentiation factor. Fixed-order fractional derivatives are expanded into the variable-order fractional meaning.

Definition 2.2 [19] The Riemann FPDEO of $\alpha(t)$ is defined as:

$$\overset{\circ}{D}_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha(t)-n+1}} ds$$ \hspace{1cm} ...(2)

where $n-1 < \alpha_{\text{min}} < \alpha(t) < \alpha_{\text{max}} < n$, $n \in \mathbb{N}$ for $t \in [0, \tau]$.

Other definitions of the derivative of variable arrangement have been proposed with respect to Caputo.

Definition 2.3 [19] Let $n-1 < \alpha(t) \leq n$ for all $t \in [0, \tau]$ the operator $\overset{\circ}{C}D_t^{\alpha(t)}$ defined by

$$\overset{\circ}{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha(t)-n+1}} ds$$ \hspace{1cm} ...(3)

and

$$\overset{\circ}{D}_b^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha(t)-n+1}} ds$$

is called the Caputo fractional derivative of VO of $\alpha(t)$.

If $\alpha(t)$ is a constant function, then the order of the partial variable is changed to the derivative of the constant order. The two definitions of variable-order derivatives are not generally equivalent, but are related by the following relationship:

$$\alpha D_t^{\alpha(t)} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-\alpha(t)}}{\Gamma(k+1-\alpha(t))} + \overset{\circ}{D}_t^{\alpha(t)} f(t).$$ \hspace{1cm} ...(4)

The formula for the fractional Caputo derivative, $0 < \alpha(t) \leq 1$: 

\[ c_0^\alpha D_t^\beta x^\beta = \begin{cases} 0, & \beta = 0, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(t))} x^{\beta-\alpha(t)}, & \beta = 0,1,2,\ldots \end{cases} \quad \ldots (5) \]

**Definition 2.4** [20] Let 0 < α(x, t) ≤ 1 the operator \( c_0^\alpha D_t^{\alpha(x,t)} \) defined by

\[ c_0^\alpha D_t^{\alpha(x,t)} u(x,t) = \frac{1}{\Gamma(n-\alpha(x,t))} \int_0^x \frac{1}{(x-s)^{\alpha(x,s)-n+1}} \frac{\partial^n u(x,t)}{\partial s^n} \, ds \quad \ldots (6) \]

is called the Caputo space fractional derivative of variable-order. Next, let us introduce some properties of fractional derivatives [21].

1- Linearity: if the operator \( D_t^\alpha(A g_1(t) + B g_2(t)) = A D_t^\alpha g_1(t) + B D_t^\alpha g_2(t) \), then of must be linear. where \( g_1(t) \) and \( g_2(t) \) are any two function, \( A, B \in \mathbb{R} \), \( \mathbb{R} \) be a regain s.t \( \mathbb{R} = \{(x,t): a \leq x \leq b, c \leq t \leq d\}, \alpha \in \mathbb{R}^+ \) for any type of fractional derivatives.

2- Analyticity:-if \( f(x) \) is analytic then \( D_t^\alpha f(x) \) is also analytic function of order \( \alpha \) and \( z \)

3- Law of exponents :- \( D_t^\alpha a D_t^\beta f(t) = D_t^{\alpha+\beta} f(t) \), and existence the inverse of the operator \( D_t^\alpha a D_t^{-\alpha} f(t) = f(t) \).

4- Identity: if \( D_t^\alpha f(t) = f(t) \), then \( D_t^\alpha f(t) \) is identity when \( \alpha \) is zero order.

### 2.2 Two-dimensional non-polynomial spline for solving FPDEVO

Consider the partition \( \Delta = \{t_0, t_1, t_2, \ldots, t_n\} \) of \([a, b] \in \mathbb{R} \). Let \( S(\Delta) \) indicate the set of piecewise polynomials on interval \( I_i = [t_i, t_{i+1}] \) of partition \( \Delta \), let \( u(x,t) \) be the exact solution, this new method provides an approximation. Also, \( C^\omega \) in the trigonometric portion of the non-polynomial slices it compensates for the loss of smoothness inherited by the polynomial. The non-polynomial spline function, obtained by the segment \( P_i(t) \). Each non-polynomial spline of \( n \) order \( P_i(t) \) has the form:

\[ p_i(t) = a_i \cos k(t - t_i) + b_i \sin k(t - t_i) + \cdots + y_i(t - t_i)^{n-1} + z_i \]

\[ \ldots (7) \]

Where \( a_i, b_i, y_i \), and \( z_i \) are constants and \( k \) is repeat the trigonometric functions that will be used to increase the accuracy of the method of order \( \alpha, 0 < \alpha \leq 1 \).

**Definition 3.1** [22] Let \( a_i, b_i, c_i \) and \( d_i \) are constants to be determined

\[ p_i(t) = a_i \cos k(t - t_i) + b_i \sin k(t - t_i) + c_i(t - t_i) + d_i \]

\[ \ldots (8) \]

is called Linear Non-Polynomial SF

Now, by tensor product we will construct two dimensional non-polynomial spline. Let \( \mathbb{R} \) be a regain s.t \( \mathbb{R} = \{(x,t): a \leq x \leq b, \, c \leq t \leq d\} \) the method of finding two dimensional functions \( g(x,y) \) in a tensor product space \( S_1 \otimes S_2 \), such that \( S_1 = \text{span} \{\cos(t), \sin(t), t, 1\} \) and \( S_2 = \text{span} \{\cos(x), \sin(x), x, 1\} \)

\[ S_1 \otimes S_2 = z(x,t) \quad \ldots (9) \]

Let the matrix form:

\[ A C = \Phi \quad \ldots (10) \]

where \( A = \Psi \otimes \Phi \) is tensor product of two matrices \( \Psi \) and \( \Phi \) of dimensions \( n \times m \)

\[ \Psi = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix}, \quad \Phi = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix} \]
\[ A = \Psi \otimes \Phi = \begin{bmatrix} a_{1,1} \Phi & \ldots & a_{1,m} \Phi \\ \vdots & \ddots & \vdots \\ a_{n,1} \Phi & \ldots & a_{n,m} \Phi \end{bmatrix} \]

and

\[ C = [c_{1,1} \ldots c_{m,1} c_{1,2} \ldots c_{m,2} \ldots c_{m,n}]^T \]

\[ F = [f_{1,1} \ldots f_{m,1} f_{1,2} \ldots f_{m,2} \ldots f_{m,n}]^T \]

Solve (10) to find the unknown vector.

The following theorem present the two dimensional linear non-polynomial spline functions that will be used to evaluate the approximate solution of FPDEVO which is achieved by evaluating Caputo fractional derivative of the spline basis.

**Theorem 3.1** The Caputo variable-order fractional derivative of the linear non-polynomial spline approximate the solution of FPDEVO’s of the form

\[ D^{\alpha(x,t)}u(x,t) = f(x,t,u_x,u_t,u_{xx},u_{tt}), \ 0 \leq x \leq 1, 0 \leq \alpha(x,t) \leq 1, t > 0 \ldots (11) \]

with initial condition \( u(x,0) = g_1(x), 0 \leq x \leq 1 \ldots (12) \)

and boundary condition \( u(0,t) = g_2(t), \ u(1,t) = g_3(t), t > 0 \ldots (13) \)

where \( g_1(x), g_2(t) \) and \( g_3(t) \) are arbitrary functions.

**proof:**

Let \( z(x,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} \psi_j(t) \phi_i(x) \ldots (14) \)

where \( \phi_i(x) \) and \( \psi_j(t) \) are the basis of the linear non-polynomial spline function.

From the initial condition given by equation (12), one may get:

\[ \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} \psi_j(0) \phi_i(x) = g_1(x) \ldots (15) \]

substituting the knot points for the x-axis to get an equation for each knot point, and form boundary condition given in equation (13), we have

\[ \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} \psi_j(t) \phi_i(0) = g_2(t) \ldots (16) \]

and \( \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} \psi_j(t) \phi_i(1) = g_3(t) \ldots (17) \)

similarly, substituting the knot points for the t-axis to get an equation for each knot point at \( x = 0 \) and \( x = 1 \)

the assumed solution in equation (14) is substituted in equation (11), we have

\[ \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} \psi_j(t) \phi_i(x) = f(x,t,\psi'(t),\phi'(x),\psi''(t),\phi''(x)) \ldots (18) \]

substitute the knot points \( (x_i,t_j) \) for \( i = 0,1,2, \ldots, m \), \( j = 0,1,2, \ldots, n \), to get an equation for each pair \( (i,j) \), for all \( i = 0,1,2, \ldots, m \), \( j = 0,1,2, \ldots, n \) then from equations (14-18) and a system with unknown coefficients \( c_{i,j} \) must be determined to compute equation (14).

Now, present the method in the following algorithm FPDEVO:
The Algorithm (3.1): (FPDEVO)
To find the approximate solution to (14), we choose \( n > 0 \) and follow these steps:

Step 1: Set \( h = (a - b)/n, \ x_i = x_o + i \ h, i = 0,1,2, ..., n \)
where \( x_o = a, x_n = b \)

Step 2: Set \( k = (d - c)/m, \ \ t_j = t_o + j \ k, j = 0,1,2,....m \) where \( t_o = c, t_m = d \)

Step 3: Evaluate \( a_o b_o, c_o and d_o \) in comparison with equation (8)

Step 4: Calculate the matrix \( A \) by using equations 11-14

Step 5: Evaluate the vector \( F \) by using (10)

Step 6: Calculate \( A^{-1} \) by step 4

Step 7: Evaluate the coefficient \( C_{i,j} \) by using step 5 and step 6 such that \( C_{i,j} = A^{-1} F \) by using equation (10)

Step 8: Calculate approximate solution \( z(x,t) \) using step 7 and equation (14)

3. Illustrative Example
There are two illustrative examples of variable order linear fractional partial differential equations to prove the proposed action linear non-polynomial partial spline method, where MathCad 15 program applied for computation the results.

Example 3.1

\[ D^2_x u(x,t) + 0.5 \cos \left( \frac{\alpha(x,t) \pi}{2} \right) D^\alpha_x u(x,t) - \frac{2}{t^{4+1}} u(x,t) = f(x,t) \]

for \( x \in [0,1], t \in [0,1] \)

subject to initial condition (IC): \( u(x,0) = x^2(8 - x) \)

and the boundary conditions (BC): \( u(0,t) = 0 \), \( u(1,t) = 7(t^2 + 1) \)

the exact solution [20] given by:

\[ u(x,t) = x^2(8 - x)(t^2 + 1) \]

with \( f(x,t) = -(t^2 + 1) \left( \frac{16 x^2 - \alpha(x,t)}{t(3 - \alpha(x,t))} + \frac{6 x^3 - \alpha(x,t)}{t(4 - \alpha(x,t))} \right) \]

and \( \alpha(x,t) = 1.5 + 0.5e^{-(x/t^2)} \)

by algorithm (3.1) the approximate solution is:

\[ z(x,t) = (-85.484 \cos t - 40.66 \sin t + 81.904 t + 70.891) \cos x \]

\[ + (397.705 \cos t - 15.124 \sin t - 36.852 t - 398.745) \sin x \]

\[ + (-388.014 \cos t + 6.369 \times 10^{-6} \sin t + 62.695 t + 389.181) x \]

\[ + (85.484 \cos t + 40.66 \sin t - 81.904 t - 70.891) \]

Table (3.1) illustrate the exact, approximate solution and absolute error

| X  | 0   | 0.01 | 0.02 | 0.03 | 1/3 | 0.01 | 0.02 | 0.03 | 2/3 | 0.01 | 0.02 |
|----|-----|------|------|------|-----|------|------|------|-----|------|------|
| t  | 0   | 0    | 0    | 0    | 0   | 0.852| 0.852| 0.898| 0.853| 3.259| 3.275|
| u(x,t) | 0   | 0.852| 0.852| 0.898| 0.853| 3.259| 3.267| 3.275| 3.261| 1.833×10^{-4}|
| z(x,t) | 0   | 0.852| 0.852| 0.898| 0.853| 3.259| 3.275| 3.267| 3.261| 1.833×10^{-4}|
| | | | | | | | | | | |
The approximation solution $z(x, t)$ and the exact solution $u(x, t)$ are illustrated in figure (3.1), and the absolute error $e_r(x, t)$ is given in figure (3.2).

### Table 1: Accuracy Comparison

| $t$  | $z$  | $u$  | $e_r$          |
|------|------|------|----------------|
| 0    | 7    | 7    | $2.386 \times 10^{-4}$ |
| 0.01 | 7.001| 7.001| $2.3 \times 10^{-4}$  |
| 0.02 | 7.003| 7.003| $2.214 \times 10^{-4}$ |
| 0.03 | 7.007| 7.006| $2.065 \times 10^{-4}$ |

**Figure (3.1):** the approximate surface $z(x, t)$ and the exact surface $u(x, t)$ for example (3.1)

**Figure (3.2):** the error $e_r(x, t)$ for example (3.1)
Figure (3.1) summarizes the results approximate solution $z(x,t)$ in Table (3.1) gives a more realistic approximation $az(x,t) = 1.5 + 0.5e^{-(x/t)^{2-1}}$.

Example (3.2):
\[
\begin{align*}
\frac{\partial}{\partial t}D_t^{\alpha(x,t)}u(x,t) - D_xu(x,t) + D_tu(x,t) - D_x^2u(x,t) &= f(x,t), \text{ for } x \in [0,1], t \in [0,1] \\
\text{subject to } IC: u(x,0) &= 5x(1-x) \\
\text{and BC: } u(0,t) &= u(1,t) = 0
\end{align*}
\]

the exact solution [23] given by $u(x,t) = 5x(t + 1)(1 - x)$

with $f(x,t) = -\frac{5x(1-x)}{(3-\alpha(x,t))} t^{1-\alpha(x,t)} - 5(t(2x - 3) + x^2 + x - 3)$

and $\alpha(x,t) = 0.8 + 0.005 \sin(x) \cos(tx) $

by algorithm (3.1) the approximate solution is:

\[
\begin{align*}
z(x,t) &= (-4.857 \cos t - 0.626 \sin t + 8.906 t + 13.839) \cos x \\
&\quad + (7.246 \cos t - 0.255 \sin t + 5.808 t - 2.39) \sin x \\
&\quad + (-8.33 \cos t + 7.573 \times 10^{-8} \sin t + 0.867 t + 8.33)x \\
&\quad + (4.857 \cos t + 0.626 \sin t + 8.906 t - 13.839)
\end{align*}
\]

Table (3.2) illustrate the exact approximate solution and absolute error

| X | t | $z(x,t)$ | $u(x,t)$ | $|z(x,t) - u(x,t)|$ |
|---|---|---------|---------|------------------|
| 0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.01 | 0.0 | 0.0 | 0.0 |
| 0.02 | 0.0 | 0.0 | 0.0 |
| 0.03 | 0.0 | 0.0 | 0.0 |
| 1/3 | 0 | 1.111 | 1.111 | 0.356 $\times 10^{-4}$ |
| 0.01 | 1.122 | 1.122 | 0.346 $\times 10^{-3}$ |
| 0.02 | 1.133 | 1.133 | 0.714 $\times 10^{-3}$ |
| 0.03 | 1.143 | 1.144 | 0.107 $\times 10^{-2}$ |
| 2/3 | 0 | 1.111 | 1.111 | 0.688 $\times 10^{-4}$ |
| 0.01 | 1.122 | 1.122 | 0.211 $\times 10^{-3}$ |
| 0.02 | 1.133 | 1.133 | 0.487 $\times 10^{-3}$ |
| 0.03 | 1.144 | 1.144 | 0.761 $\times 10^{-3}$ |
| 1 | 0 | 0.9343 $\times 10^{-4}$ | 0 | 0.9343 $\times 10^{-4}$ |
| 0.01 | 0.8733 $\times 10^{-4}$ | 0 | 0.8733 $\times 10^{-4}$ |
| 0.02 | 0.8116 $\times 10^{-4}$ | 0 | 0.8116 $\times 10^{-4}$ |
| 0.03 | 0.743 $\times 10^{-4}$ | 0 | 0.743 $\times 10^{-4}$ |

The approximation solution $z(x,t)$ and the exact solution $u(x,t)$ are illustrated in figure (3.3), and the absolute error $e(x,t)$ is given in figure (3.4)
Figure (3.3): the approximate surface $z(x, t)$ and the exact surface $u(x, t)$ for example (3.2)

Figure (3.4): the error $e(x, t)$ for example (3.2)

Figure (3.3) summarizes the results. The approximate solution $z(x, t)$ in Table (3.2) gives a more realistic approximation at $x = 0.8 + 0.005 \sin(x) \cos(tx)$.

**Conclusion and future work**

The linear non-polynomial spline function is presented for solving the FPDEVO. Our proposed method is based on trigonometric and polynomial, and it is completely different from the previously shown methods. The results in Table 3.1, 3.2 and figures 3.1, 3.3 show that the proposed method can be used to solve the problem with high accuracy.
Indeed, implementation of an FPDEVO algorithm may also be on our future lines of research. Also we may use the proposed method to solve system of FPDEVO.

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