Correlation functions of the \(XXZ\) spin-\(\frac{1}{2}\) Heisenberg chain at the free fermion point from their multiple integral representations

N. Kitanine\(^1\), J. M. Maillet\(^2\), N. A. Slavnov\(^3\), V. Terras\(^4\)

Abstract

Using multiple integral representations, we derive exact expressions for the correlation functions of the spin-\(\frac{1}{2}\) Heisenberg chain at the free fermion point \(\Delta = 0\).
1 Introduction

In the article [1] new multiple integral representations for the correlation functions of the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain have been obtained. In the present article, we apply the results of [1] to compute the correlation functions of the spin-$\frac{1}{2}$ Heisenberg chain at the free fermion point $\Delta = 0$.

Generically, the Hamiltonian of the finite cyclic $XXZ$ chain with $M$ sites (where $M$ is supposed to be even) has the form

$$H_{XXZ} = \sum_{m=1}^{M} \left( \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1) - \frac{h}{2} \sigma_m^z \right).$$  \hspace{1cm} (1.1)

Here, $\sigma_m^{x,y,z}$ denote the local spin operators (Pauli matrices) associated with the $m$-th site of the chain, $\Delta$ is the anisotropy parameter, and $h$ an external classical magnetic field. The particularization of this model to the case $\Delta = 0$ is known as the $XX$ chain (isotropic $XY$ model [2]):

$$H_{XX} = \sum_{m=1}^{M} \left( \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y - \frac{h}{2} \sigma_m^z \right).$$  \hspace{1cm} (1.2)

In spite of the fact that the $XX$ spin-$\frac{1}{2}$ chain is equivalent to a model of free fermions, its correlation functions are quite non-trivial. They had been studied for a long period by numerous authors [3, 4, 5, 6, 7, 8] and the key results in this field are presently known. It is worth mentioning however that the methods used in the works listed above rely essentially on the free fermion features of the $XX$ model. Therefore, they cannot be applied to the more general $XXZ$ case, at least without significant modifications. On the contrary, we expect our present approach, which relies on multiple integral representations of correlation functions, to be instructive for the study of the general case as well.

In 1992 [9], multiple integral representations for the correlation functions of the $XXZ$ chain at zero temperature, $\Delta > 1$ and $h = 0$ have been obtained from the $q$-vertex operator approach. Later, in 1996 [10] (see also [11]), similar answers were formulated for the case $|\Delta| \leq 1$. A proof of these formulas, together with their extension to non-zero magnetic field, has been obtained in 1999 [12, 13] for both regimes using algebraic Bethe ansatz and the actual resolution of the quantum inverse scattering problem [13, 14]. It results from these articles that, starting from the so-called elementary blocks, one can in principle obtain a multiple integral representation for any $n$-point correlation function of the $XXZ$ chain.

More precisely, if $|\psi_g\rangle$ denotes the ground state, and $E_{m,\epsilon}^{\epsilon',m}$ the elementary operators acting on the quantum space $\mathcal{H}_m$ at site $m$ as the $2 \times 2$ matrices $E_{\epsilon,\epsilon'} = \delta_{\epsilon,\epsilon'} \delta_{\epsilon',\epsilon}$, the elementary blocks for correlation functions are defined as

$$F_m(\{\epsilon_j, \epsilon_j'\}) = \frac{\langle \psi_g | \prod_{j=1}^{m} E_{j}^{\epsilon_j,\epsilon_j'} | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}. \hspace{1cm} (1.3)$$
The methods developed in [9]–[12] enable one to obtain the quantity (1.3) in terms of an integral with \( m \) integrations. It is clear that an arbitrary correlation function in the ground state can be expressed as a linear combination of the elementary blocks (1.3) and, hence, as a linear combination of such multiple integrals. It should be stressed however that, although these formulas are quite explicit, the actual analytic computation of these multiple integrals is missing up to now. Moreover, the evaluation of correlations of physical relevance, like spin-spin correlation functions, is a priori quite involved. Indeed, if we consider for example the correlation function \( \langle \sigma^z_1 \sigma^z_m \rangle \), the identity

\[
\langle \psi_g | \sigma^z_1 \sigma^z_m | \psi_g \rangle \equiv \langle \psi_g | (E^{11}_1 - E^{22}_1) \prod_{j=2}^{m-1} (E^{11}_j + E^{22}_j)(E^{11}_m - E^{22}_m) | \psi_g \rangle
\]

shows that the corresponding linear combination of elementary blocks is actually given as a sum of \( 2^m \) terms. This means that the number of terms to sum up grows exponentially with \( m \), which in particular makes it extremely difficult to solve the problem of asymptotic behavior at large distance.

Thus, up to recently, the situation in this field was as follows. On one hand, the free fermion limit (\( \Delta = 0 \)) of the XXZ model was well studied, but the extension of these results to the general case came up against serious problems. On the other hand, the multiple integrals approach formally provided the possibility to compute the correlation functions for arbitrary \( \Delta \); however, because of the technical reasons mentioned above, no result has up to now been reproduced via this method, even for the simplest case of free fermions\(^1\).

The goal of this paper is to study the correlation functions of the XX chain using the new multiple integral representations obtained in [1]. In fact, these new representations are nothing but re-summations of the multiple integral expressions for the elementary blocks. In particular, they enable us to present the spin-spin correlation functions of the type (1.4) as a sum of only \( m \) terms instead of \( 2^m \). We would like to point out that in this paper we do not obtain new results, but only reproduce the known answers via a new method. Moreover, we consider here the XX model only as a test for the relevance of the formulas obtained in [1]. We hope that some of the methods developed in the present publication can be applied (perhaps after certain modifications) to the general XXZ chain as well.

This article is organized as follows. In the next section, we introduce some useful notations and give the list of formulas obtained in [1] for the correlation functions of the XXZ model. In Section 3, we compute the correlation function of the third components of spin. The emptiness formation probability is considered in Section 4. In Section 5, we obtain a Fredholm determinant representation for the correlation function \( \langle \sigma^z_1 \sigma^z_{m+1} \rangle \). The asymptotic analysis of this Fredholm determinant is performed in Section 6. Some perspectives are discussed in the conclusion.

\(^1\)Recently, in [16], the probability to find in the ground state a string of particles with spin down (the emptiness formation probability) was computed at \( \Delta = 0 \) by the method of multiple integrals. Let us stress that, contrary to the spin-spin correlation functions, this quantity can be expressed as a single elementary block. We consider the emptiness formation probability in Section 4 of the present article.
2 Correlation functions of the XXZ chain

For the reader’s convenience, we gather in this section the list of results obtained in [1] for the correlation functions of the XXZ model. Since eventually we study the limit $\Delta = 0$, we hereafter restrict ourselves to the case $|\Delta| < 1$.

Let us first of all recall that the Hamiltonian (1.1) possesses the symmetries

$$UH_{XXZ}(\Delta, h)U^{-1} = -H_{XXZ}(-\Delta, h), \quad U = \prod_{m=1}^{M/2} \sigma_{2m}^z,$$

$$VH_{XXZ}(\Delta, h)V^{-1} = H_{XXZ}(\Delta, -h), \quad V = \prod_{m=1}^{M} \sigma_{m}^x.$$  \hspace{1cm} (2.1)

Due to this freedom, there is no common definition of the XXZ model. It means that the expressions of correlation functions obtained in different publications may coincide up to a common sign and/or the sign of $\Delta$ and $h$.

The standard parameterization of the anisotropy parameter is $\Delta = \cosh \eta$. In the regime $|\Delta| < 1$ the parameter $\eta$ is imaginary, and we set $\eta = -i\zeta$, $\zeta > 0$. The free fermion point $\Delta = 0$ corresponds to $\zeta = \pi/2$.

The general structure of the expressions obtained in [1] for the spin-spin correlation functions is the following:

$$\langle \sigma_1^\alpha \sigma_{m+1}^\beta \rangle = \sum_{n=0}^{m-1} \oint_{C_z} d^{n+1}z \oint_{C_\lambda} d^n \lambda \oint_{C_\mu} d^2 \mu \int f_m(\{\lambda, z\}) \Gamma_n^{\alpha\beta}(\{\lambda, \mu, z\}) S_h(\{\lambda, z\}).$$  \hspace{1cm} (2.2)

Here $\alpha, \beta = x, y, z$, and the functions $\Gamma_n^{\alpha\beta}(\{\lambda, \mu, z\})$ and $f(\{\lambda, z\})$ are purely algebraic quantities, which in particular do not depend on the regime nor on the magnetic field. Their explicit forms for specific correlation functions are given below.

The integration contour $C_z$ surrounds the point $z_j = -i\zeta/2$ ($z_j = -i\pi/4$ for $\Delta = 0$), where the function $f(\{\lambda, z\})$ has a pole. All other singularities of the integrand (2.2) are outside the contour $C_z$.

The contours $C_\lambda$ and $C_\mu$ depend on $\Delta$ and $h$. In all the examples considered below we have $C_\lambda = C_\mu$. For $|\Delta| < 1$ and $h \geq 0$, the contour $C_\lambda$ is an interval $[-\Lambda, \Lambda]$ of the real axis, where the value of $\Lambda$ is uniquely defined by $\Delta$ and $h$, although in the general case the dependency $\Lambda = \Lambda(\Delta, h)$ is rather implicit. At $\Delta = 0$, however, the integration domain can be found explicitly from the fact that $\cosh 2\Lambda = 4/h$. Note that if $h \to 0$, one has $\Lambda \to \infty$. On the other hand, if $h$ approaches its critical value $h_c = 4$, then $\Lambda \to 0$ and all the correlation functions become trivial, which comes physically from the fact that the ground state of the Hamiltonian (1.2) is then purely ferromagnetic. Therefore we consider below only the case $0 \leq h \leq h_c$. Due
to the symmetry \(2.1\), we also do not need to consider negative magnetic field, although all our results remain valid for \(h < 0\) as well\[1\].

Finally, the integrand \(2.2\) contains a function \(S_h(\{\lambda, z\})\), which also depends on the value of the magnetic field. This function is equal to the determinant of a matrix of elements \(\rho(\lambda_j, z_k)\), where \(\rho(\lambda, z)\) is the so-called ‘inhomogeneous density’, solution of the integral equation

\[
-2\pi i \rho(\lambda, z) + \int_{-\Lambda}^{\Lambda} K(\lambda - \mu) \rho(\mu, z) d\mu = t(\lambda, z),
\]

with

\[
K(\lambda) = \frac{i \sin 2\zeta}{\sinh(\lambda + i\zeta) \sinh(\lambda - i\zeta)}, \quad t(\lambda, z) = \frac{-i \sin \zeta}{\sinh(\lambda - z) \sinh(\lambda - z - i\zeta)}.
\]

(2.3)

Note that, at \(z = -i\zeta/2\), the function \(\rho(\lambda, z)\) coincides with the spectral density of the ground state. In the free fermion limit \(\Delta = 0\) \((\zeta = \pi/2)\), one has \(K(\lambda) = 0\), and thus

\[
\rho(\lambda, z) = \frac{i}{2\pi} t(\lambda, z) = \frac{i}{\pi \sin 2(\lambda - z)}.
\]

(2.5)

After this general setting, let us now be more specific and present the explicit formulas for some of the correlation functions of the \(XXZ\) chain in the domain \(|\Delta| < 1\). We consider below essentially three different cases:

a) the correlation function \(g^{zz}_m = \langle \sigma^z_1 \sigma^z_{m+1} \rangle\) in a magnetic field;

b) the emptiness formation probability \(\tau(m)\) in a magnetic field;

c) the correlation functions \(g^{+-}_m = \langle \sigma^+_m \sigma^-_{m+1} \rangle\) in zero magnetic field.

The reader can find the derivation of the corresponding multiple integral representations in \[1\].

a) The correlation function of the third components of spin can be evaluated from the generating functional \(\langle \exp(\beta Q_{1,m}) \rangle\), where \(Q_{1,m} = \frac{1}{2} \sum_{k=1}^m (1 - \sigma^3_k)\), as

\[
\langle \sigma^z_1 \sigma^z_{m+1} \rangle = \left( 2D^2_m \frac{\partial^2}{\partial^2 \beta^2} - 4D_m \frac{\partial}{\partial \beta} + 1 \right) \langle \exp(\beta Q_{1,m}) \rangle \bigg|_{\beta=0} = 2D^2_m \langle Q^2_{1,m} \rangle - 4D_m \langle Q_{1,m} \rangle + 1.
\]

(2.6)

Here, the symbols \(D_m\) and \(D^2_m\) denote respectively the first and the second lattice derivative,

\[
D_m f(m) \equiv f(m + 1) - f(m), \quad D^2_m f(m) \equiv f(m + 1) + f(m - 1) - 2f(m).
\]

(2.7)

The expectation value of the functional \(\langle \exp(\beta Q_{1,m}) \rangle\) is given by (5.8) of \[1\]:

\[
\langle \exp(\beta Q_{1,m}) \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \int d\lambda \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{-\Lambda}^{\Lambda} d\lambda \prod_{a=1}^{n} \left( \frac{\sinh(z_a - \frac{i\zeta}{2}) \sinh(\lambda_a + \frac{i\zeta}{2})}{\sinh(z_a + \frac{i\zeta}{2}) \sinh(\lambda_a - \frac{i\zeta}{2})} \right)^m
\]

\[
\times W_n(\{\lambda\}|z\}) \cdot \det_n \left[ \tilde{M}_{jk}(\{\lambda\}|z\}) \right] \cdot \det_n \left[ \rho(\lambda_j, z_k) \right].
\]

(2.8)

\[1\] Actually the restriction \(h \geq 0\) is not necessary. In the case \(h < 0\), the integration contour \(C_{\Lambda}\) becomes

\([-\Lambda + \frac{i\pi}{2}, -\infty + \frac{i\pi}{2}] \cup [\infty + \frac{i\pi}{2}, \Lambda + \frac{i\pi}{2}]\), where \(\Lambda\) is the real positive solution of the equation \(\cosh 2\Lambda = 4/|h|\).
Here and further we use the notation $\det_n$ for the determinants of $n \times n$ matrices. The function $W_n$ is defined by

$$W_n(\{\lambda\}, \{z\}) = \prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\sinh(\lambda_a - z_b - i\zeta) \sinh(\lambda_a - z_b + i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta) \sinh(\lambda_a - \lambda_b + i\zeta)},$$  \hspace{1cm} (2.9)

and the entries of the matrix $\tilde{M}_{jk}$ are

$$\tilde{M}_{jk}(\{\lambda\}|\{z\}) = t(z_k, \lambda_j) + e^{\beta} t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j - i\zeta) \sinh(\lambda_j - z_a - i\zeta)}{\sinh(\lambda_j - \lambda_a - i\zeta) \sinh(z_a - \lambda_j - i\zeta)},$$  \hspace{1cm} (2.10)

where the functions $t(\lambda, z)$ and $\rho(\lambda, z)$ are defined in (2.4) and (2.3) respectively.

For $h = 0$, it is more convenient to derive the correlation function $\langle \sigma^z_{1,m} \rangle$ from the generating functional $\langle \exp(\beta Q_{1,m})\sigma^z_{m+1} \rangle$:

$$\langle \sigma^z_{1,m} \rangle = -2D_{m-1} \frac{\partial}{\partial \beta} \langle \exp(\beta Q_{1,m})\sigma^z_{m+1} \rangle \bigg|_{\beta = 0}, \quad h = 0.$$  \hspace{1cm} (2.11)

The expectation value of the functional $\langle \exp(\beta Q_{1,m})\sigma^z_{m+1} \rangle$ is given by (6.7) of [1]:

$$\langle \exp(\beta Q_{1,m})\sigma^z_{m+1} \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \frac{1}{C_{m-n}} \int_{C_{m-n}} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{\mathbb{R}} \prod_{a=1}^{n} \left( \frac{\sinh(z_a - \frac{i\zeta}{2}) \sinh(\lambda_a + \frac{i\zeta}{2})}{\sinh(z_a + \frac{i\zeta}{2}) \sinh(\lambda_a - \frac{i\zeta}{2})} \right)^{m} \times \prod_{a=1}^{n} \left( \frac{\sinh(\lambda_a + \frac{i\zeta}{2})}{\sinh(z_a + \frac{i\zeta}{2})} \right) \times \det_n \left[ \tilde{M}_{jk}(\{\lambda\}|\{z\}) \right] \det_{n+1} \left[ \rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_n), \rho(\lambda_j, -\frac{i\zeta}{2}) \right].$$  \hspace{1cm} (2.12)

b) The emptiness formation probability (the probability to find in the ground state a string of particles with spin down in the first $m$ sites) is defined as

$$\tau(m) = \langle \prod_{k=1}^{m} \frac{1 - \sigma^z_k}{2} \rangle.$$  \hspace{1cm} (2.13)

The multiple integral representation of this correlation function can be easily obtained from the generating functional $\langle \exp(\beta Q_{1,m}) \rangle$ (see equation (C.9) of [1]):

$$\tau(m) = \lim_{\xi_1, \ldots, \xi_m \rightarrow -i\zeta} \frac{1}{m!} \int_{-\Lambda}^{\Lambda} \prod_{a<b}^{m} \frac{Z_m(\{\lambda\}, \{\xi\})}{\sinh(\xi_a - \xi_b)} \rho(\lambda_j, \xi_k) d^m \lambda,$$  \hspace{1cm} (2.14)
Let us begin our calculations with the correlation function

\[ \langle \sigma^+_{m-1} \rangle = \frac{m}{\prod_{a=1}^{m} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - \xi_b) \sinh(\lambda_a - \xi_b - i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta)}} \cdot \frac{\det_{m} t(\lambda_j, \xi_k)}{\prod_{a>b} \sinh(\xi_a - \xi_b)}. \]  

(2.15)

c) Let us consider finally the correlation functions \( g_{m}^{+1-} = \langle \sigma_{m+1}^+ \sigma_{m+1}^- \rangle \) and \( g_{m}^{-} = \langle \sigma_{m+1}^- \sigma_{m+1}^+ \rangle \).

It is easy to see that \( g_{m}^{+1-}(h) = g_{m}^{-}(h) \). For zero magnetic field, these two correlation functions coincide. In this case their multiple integral representation is given by (6.13) of [1]:

\[
\langle \sigma_{m+1}^{+} \rangle = \frac{1}{\prod_{a=1}^{n} \prod_{b=1}^{n+2} \frac{\sinh(\lambda_{n+1} - z_a - i\zeta) \sinh(\lambda_{n+2} - z_a)}{\sinh(\lambda_{n+1} - \lambda_{n+2} - i\zeta) \sinh(\lambda_{n+2} - \lambda_{n+1})}} \cdot 
\]

\[
\times \hat{W}_{n}(\{\lambda\}, \{z\}) \cdot \det_{n+1} \hat{M}_{jk} \cdot \det_{n+2} \left[ \rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_{n+1}), \rho(\lambda_j, -\frac{i\zeta}{2}) \right]. \]  

(2.16)

Here,

\[
\hat{W}_{n}(\{\lambda\}, \{z\}) = \frac{\prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\sinh(\lambda_a - z_b - i\zeta) \sinh(z_b - \lambda_a - i\zeta)}{\prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\sinh(\lambda_a - \lambda_b - i\zeta) \prod_{a=1}^{n+1} \prod_{b=1}^{n} \sinh(z_a - z_b - i\zeta)}}}{\prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\sinh(\lambda_a - \lambda_b - i\zeta) \prod_{a=1}^{n+1} \prod_{b=1}^{n} \sinh(z_a - z_b - i\zeta)}}}, \]  

(2.17)

and the entries of the \((n + 1) \times (n + 1)\) matrix \( \hat{M} \) are

\[
\hat{M}_{jk} = t(z_k, \lambda_j) - t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j - i\zeta) \prod_{b=1}^{n+1} \frac{\sinh(\lambda_j - z_b - i\zeta)}{\sinh(\lambda_j - \lambda_a - i\zeta) \prod_{b=1}^{n} \frac{\sinh(z_b - \lambda_j - i\zeta)}}}}, \quad j \leq n, \]  

\[
\hat{M}_{n+1,k} = t(z_k, -\frac{i\zeta}{2}), \quad j = n + 1. \]  

(2.18)

To conclude this section, let us recall once more that all the multiple integral representations given above hold for arbitrary \(-1 < \Delta < 1\). In order to particularize these expressions to the case of the \(XX\) model, one has to set \( \zeta = \pi/2, \Lambda = \text{arccosh}(4/h)/2 \), and to use the expression (2.15) for the inhomogeneous density \( \rho(\lambda, z) \).

3 Correlation function of the third components of spin

Let us begin our calculations with the correlation function \( \langle \sigma_{m+1}^{+} \rangle \), which is the simplest spin-spin correlation function of the \(XX\) model.
Observe that, at $\zeta = \pi/2$, the equation (2.8) simplifies drastically. First of all, the matrix $\tilde{M}_{jk}(\{\lambda\}|\{z\}) = \frac{2(e^\beta - 1)}{\sinh 2(\lambda_j - z_k)}$, $\zeta = \pi/2$, (3.1) and, hence, to compute its determinant one can use the identity

$$\det_n \frac{1}{\sinh(x_j - x_k) \sinh(y_k - y_j)} = \prod_{j>k}^{n} \sinh(x_j - y_k)$$

(3.2)

However, it is more important to notice that $\det_n \tilde{M}_{jk}$ is proportional to $(e^\beta - 1)^n$: this means that, if one takes the first (respectively the second) derivative with respect to $\beta$ and sets $\beta = 0$ as in (2.6), only the terms $n \leq 1$ (respectively $n \leq 2$) in the sum (2.8) do not vanish. Thus, after some simple computation, one obtains

$$\langle Q_1, m \rangle = \frac{1}{4\pi^2} \oint_{C_z} d\lambda \int_{-\Lambda}^{\Lambda} \varphi^m(\lambda) \varphi^{-m}(\lambda) d\lambda \frac{d\lambda}{\sinh^2(\lambda - z)}.$$

(3.3)

$$\langle Q_1^2, m \rangle = \langle Q_1, m \rangle + \frac{1}{32\pi^4} \oint_{C_z} d^2 \lambda \int_{-\Lambda}^{\Lambda} \prod_{a=1}^{2} (\varphi^m(z_a) \varphi^{-m}(\lambda_a)) \left( \frac{\det_2 \frac{1}{\sinh(\lambda_j - z_k)}}{\sinh^2(\lambda - z)} \right)^2,$$

(3.4)

where we have introduced the notation

$$\varphi(z) = \frac{\sinh(z - \frac{i\pi}{4})}{\sinh(z + \frac{i\pi}{4})}.$$

(3.5)

Let us first consider the integral (3.3). As the contour $C_z$ surrounds only the singularity $z = -i\pi/4$, where $\varphi(z)$ admits a pole of order $m$, the value of the $z$-integral in (3.3) is given by the corresponding residue at $z = -i\pi/4$. However, this way to compute the $z$-integral is not very convenient, especially for large $m$. Instead, we suggest to deform the original contour $C_z$ into an infinite horizontal strip of boundary $\Gamma$ given by $\Im z = z_0 - \pi$ and $\Im z = z_0$, where $0 < z_0 < 3\pi/4$. Then, obviously,

$$\oint_{C_z} \frac{\varphi^m(z)}{\sinh^2(\lambda - z)} dz = \oint_{\Gamma} \frac{\varphi^m(z)}{\sinh^2(\lambda - z)} dz - 2\pi i \text{Res} \left. \frac{\varphi^m(z)}{\sinh^2(\lambda - z)} \right|_{z=\lambda}.$$

(3.6)

Since the integrand is a periodic function with period $i\pi$, and since it vanishes at $z \to \pm\infty$, it is clear that the integral with respect to the new contour $\Gamma$ is equal to zero. Thus, to compute the $z$-integral in (3.3), it is enough to take the residue in the second order pole at $z = \lambda$. The remaining integral with respect to $\lambda$ is then trivially computable, and we obtain

$$\langle Q_1, m \rangle = \frac{m}{\pi} \arctan(\sinh 2\Lambda).$$

(3.7)
In the next sections, we shall deal with the change of integration variables \( \cosh 2 \lambda = (\cos p)^{-1} \).

Therefore, let us at this stage introduce \( p_0 \) such that \( \cosh 2 \Lambda = (\cos p_0)^{-1} \). Then (3.4) takes the form

\[
\langle Q_{1,m} \rangle = \frac{mp_0}{\pi},
\]

where \( p_0 = \arccos \left( \frac{4}{h} \right) \).

The integral (3.4) can be taken in the same manner. The integration with respect to \( z_1 \) and \( z_2 \) leads to

\[
\langle Q_{2,1,m} \rangle = \frac{mp_0}{\pi} + \left( \frac{mp_0}{\pi} \right)^2 + \frac{1}{4\pi^2} \int_{-\Lambda}^{\Lambda} \left( \frac{\varphi^m(\lambda_1)\varphi^{-m}(\lambda_2) - \varphi^m(\lambda_1)\varphi^{-m}(\lambda_2)}{\sinh(\lambda_1 - \lambda_2)} \right)^2 d\lambda_1 d\lambda_2. \tag{3.9}
\]

In fact, we do not need to compute \( \langle Q_{2,1,m} \rangle \) itself, but only its second lattice derivative. Differentiating (3.9) with respect to \( m \), we immediately arrive at

\[
D_m^2 \langle Q_{1,m} \rangle = 2 \left( \frac{p_0}{\pi} \right)^2 - \frac{1}{\pi^2 m^2} (1 - \cos 2mp_0). \tag{3.10}
\]

Combining (3.3) and (3.10), we finally obtain

\[
\langle \sigma_{1}^z \sigma_{m+1}^z \rangle = \left( \frac{2p_0}{\pi} - 1 \right)^2 - \frac{2}{\pi^2 m^2} (1 - \cos 2mp_0). \tag{3.11}
\]

It is worth mentioning that, in spite of the fact that we have formally restricted ourselves to the case \( h \geq 0 \), the result (3.11) remains valid for \( h < 0 \) as well. For zero magnetic field, \( p_0 = \pi/2 \), and the constant contribution to the correlation function disappears. In order to get rid of this constant term from the very beginning, it is more convenient to derive \( \langle \sigma_{1}^z \sigma_{m+1}^z \rangle \) from the generating functional \( \langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle \) (see (2.12)). Then, for \( \Delta = 0 \), the corresponding sum reduces to the single term \( n = 1 \), which gives

\[
\langle \sigma_{1}^z \sigma_{m+1}^z \rangle = \frac{2i}{\pi^3} \int_{C_+} dz \int_{\mathbb{R}} d\lambda_1 d\lambda_2 \left( \cosh 2\lambda_2 \cosh 2\lambda_1 \right) \frac{\varphi^m(z)\varphi^{-m}(\lambda_1)}{\sinh 2(\lambda_2 - z)}. \tag{3.12}
\]

Using the method of calculations described above, we find

\[
\langle \sigma_{1}^z \sigma_{m+1}^z \rangle = \frac{2}{\pi^2 m^2} (-1)^m - 1, \quad \text{at} \quad h = 0. \tag{3.13}
\]

4 Emptiness formation probability

The emptiness formation probability (2.13) constitute one of the simplest example of correlation functions. In particular, unlike the spin-spin correlation functions studied in Sections 3, 5 and 6, it can be directly expressed as a single elementary block of the form (1.3), therefore as a single (multiple) integral which can be written in the symmetric form (2.14). In this section,
we explain how to compute this integral in the case $\Delta = 0$, and how to analyze its asymptotic behavior in the limit $m \to \infty$.

Setting $\zeta = \pi/2$ in the equations (2.14), (2.15), we have

$$
\tau(m) = \lim_{\xi_1, \ldots, \xi_m \to -i\pi/4} \frac{1}{m!} \int_{-\Lambda}^{\Lambda} \frac{Z_m(\{\lambda\}, \{\xi\})}{\prod_{a<b} \sinh(\xi_a - \xi_b)} \det_m \left( \frac{2}{\sinh 2(\lambda_j - \xi_k)} \right) d^m \lambda,
$$

(4.1)

and

$$
Z_m(\{\lambda\}, \{\xi\}) = 2^{-m^2} \prod_{a=1}^{m} \prod_{b=1}^{m} \frac{\sinh(2(\lambda_a - \xi_b))}{\cosh(\lambda_a - \lambda_b)} \prod_{a>b} \sinh^{-1}(\xi_a - \xi_b) \det_m \left( \frac{2}{\sinh 2(\lambda_j - \xi_k)} \right).
$$

(4.2)

Once again, we have to deal with determinants of Cauchy matrices. Computing them via (3.2) and setting $\xi_j = -i\pi/4$, we obtain

$$
\tau(m) = \frac{2^{m^2}}{m!(2\pi)^m} \int_{-\Lambda}^{\Lambda} \frac{\prod_{a<b} \sinh^2(\lambda_a - \lambda_b)}{\prod_{a=1}^{m} \cosh^m(2\lambda_a)} d^m \lambda.
$$

(4.3)

The representation (4.3) can be reduced to a Toeplitz determinant. Indeed, the change of variables $\cosh 2\lambda_j = \cos^{-1} p_j$ leads to

$$
\tau(m) = \frac{2^{m^2-m}}{m!(2\pi)^m} \int_{-p_0}^{p_0} \prod_{a>b} \sin^2 \left( \frac{p_a - p_b}{2} \right) d^m p = \frac{1}{m!(2\pi)^m} \int_{-p_0}^{p_0} \Delta(\epsilon^{-ip}) \Delta(\epsilon^{ip}) d^m p,
$$

(4.4)

where $\Delta(\epsilon^{\pm ip})$ denote Van-der-Monde determinants of variables $\epsilon^{\pm ip_j}$. Due to the symmetry of the integrand with respect to all $p_j$, one can replace one of these Van-der-Monde determinants with the product of its diagonal elements multiplied by $m!$, which gives us

$$
\tau(m) = \frac{1}{(2\pi)^m} \int_{-p_0}^{p_0} \prod_{k=1}^{p_0} e^{-i(k-1)p_k} \det_m \left( e^{i(j-1)p_k} \right) d^m p = \det_m \left( \frac{1}{2\pi} \int_{-p_0}^{p_0} e^{i(j-k)p} dp \right).
$$

(4.5)

The representation (4.5) of $\tau(m)$ as a Toeplitz determinant has already been obtained in [16] from the multiple integral representation given in [14].

Thus, (4.5) provides an explicit expression of the emptiness formation probability, at least if $m$ is small enough. However, it is more important to be able to extract the asymptotic behavior of $\tau(m)$ at $m \to \infty$. There exist several ways to do this. Firstly, one can analyze the determinant (4.3) as in [16]. Secondly, the determinant (4.3) can be transformed to a Fredholm determinant of a linear integral operator [7], the asymptotic behavior of which can be evaluated from the matrix Riemann-Hilbert problem [15]. Here we propose a third approach, based on
the application of the saddle point method directly to (4.3). It is possible that this method can be used also for the general XXZ model.

Let us rewrite (4.3) in the following way:

\[
\tau(m) = 2m^2 \int_D e^{m^2 S(\lambda)} \, d\lambda, \quad (4.6)
\]

where

\[
S(\{\lambda\}) = \frac{1}{m^2} \sum_{a>b} \log \sinh^2(\lambda_a - \lambda_b) - \frac{1}{m} \sum_{a=1}^{m} \log \cosh 2\lambda_a, \quad (4.7)
\]

and the domain \(D\) is defined by \(-\Lambda \leq \lambda_1 \leq \ldots \leq \lambda_m \leq \Lambda\). The integrand in (4.6) is positive within the domain \(D\) and vanishes on its boundary. Moreover, it is not difficult to check that the matrix of the second derivatives \(\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_k}\) is negatively defined. Hence, the integrand has a unique maximum in \(D\), which is given by the system

\[
m \frac{\partial S}{\partial \lambda_j} = 2m \sum_{a \neq j} \coth(\lambda_j - \lambda_a) - 2 \tanh 2\lambda_j = 0. \quad (4.8)
\]

Following the standard arguments of the saddle point method, we assume that, in the limit \(m \to \infty\), the solutions of the system (4.8) are distributed on the interval \([-\Lambda, \Lambda]\) according to a certain density \(\rho_0(\lambda)\). Then, in this limit, (4.8) becomes an integral equation for this density:

\[
\tanh 2\lambda = V.P. \int_{-\Lambda}^{\Lambda} \coth(\lambda - \mu)\rho_0(\mu) \, d\mu. \quad (4.9)
\]

In its turn, the integral (4.8) can be approximated by the value of the integrand in the saddle point:

\[
\tau(m) \to 2m^2 e^{m^2 S_0}, \quad m \to \infty, \quad (4.10)
\]

where

\[
S_0 = \frac{1}{2} \int_{-\Lambda}^{\Lambda} \log \sinh^2(\lambda - \mu)\rho_0(\lambda)\rho_0(\mu) \, d\lambda d\mu - \int_{-\Lambda}^{\Lambda} \log \cosh 2\lambda \cdot \rho_0(\lambda) \, d\lambda. \quad (4.11)
\]

The analytic expression of the function \(\rho_0(\lambda)\) can be determined as follows. Setting \(x = e^{2\lambda}\), we transform (4.9) into

\[
\frac{x}{x^2 + 1} = V.P. \int_a^b \frac{dy}{x - y} \hat{\rho}_0(y), \quad (4.12)
\]

where \(\hat{\rho}_0(x) = \frac{1}{2} e^{-2\lambda} \rho_0(\lambda)\) and \(a = e^{-2\Lambda}, \ b = e^{2\Lambda}\). The solution of the singular integral equation (4.12) can be obtained in a standard way via the scalar Riemann–Hilbert problem. Let us define

\[
f_\pm(x) = \int_a^b \frac{dy}{x - y \pm i0} \hat{\rho}_0(y). \quad (4.13)
\]
Then we have
\[ 2\pi i \hat{\rho}_0(x) = f_-(x) - f_+(x), \quad \int_a^b \hat{\rho}_0(x) \, dx = 1. \] (4.14)

At the same time, \( f(x) \) satisfies the relation
\[ f_-(x) + f_+(x) = \frac{2x}{x^2 + 1}, \] (4.15)
and, hence,
\[ f(x) = f^0(x) \left\{ C + \frac{1}{\pi i} \int_a^b \frac{dy}{y-x} \cdot \frac{y}{(y^2 + 1)f^0_+(y)} \right\}, \] (4.16)
where \( f^0(x) = \left( (x-a)(x-b) \right)^{-1/2} \) and \( C \) is a constant. Substituting this expression into (4.14), we eventually obtain
\[ \hat{\rho}_0(x) = \frac{1}{\pi x^2 + 1} \sqrt{\frac{a + b}{2(x-a)(b-x)}}, \] (4.17)
which results into
\[ \rho_0(\lambda) = \frac{1}{\pi} \frac{\cosh \lambda}{\cosh 2\lambda} \sqrt{\frac{\cosh 2\Lambda}{\sinh(\Lambda - \lambda) \sinh(\Lambda + \lambda)}}. \] (4.18)

This enables us to obtain the analytic expression of \( S_0 \) in terms of \( \Lambda \):
\[ S_0 = -\log \left( 2 \frac{\sqrt{\cosh 2\Lambda}}{\sinh \Lambda} \right). \] (4.19)

Taking into account that \( \cosh 2\Lambda = 4/h \), we finally obtain the following asymptotic equivalent of \( \tau(m) \) in terms of the magnetic field \( h \):
\[ \tau(m) \to \left( \frac{4 - h}{8} \right)^{m^2/2}, \quad m \to \infty. \] (4.20)

Thus, this method provides an alternative derivation of the asymptotic behavior of the emptiness formation probability \[13\].

5 Correlation function \( \langle \sigma^+_1 \sigma^-_{m+1} \rangle \) as Fredholm determinant

Unlike the correlations of the third components of spin (see Section 3), the correlation function \( \langle \sigma^+_1 \sigma^-_{m+1} \rangle \) remains non-trivial even at the free fermion point \( \Delta = 0 \). In this section, we show how to compute it from (2.16) for zero magnetic field.

Let us first consider the part of the integrand (2.16) which depends on \( \lambda_{n+1} \) and \( \lambda_{n+2} \):
\[
\frac{\det_{n+2} \rho(\lambda_j, z_k)}{\sinh(\lambda_{n+1} - \lambda_{n+2})} \cdot \left( \prod_{a=1}^{n+1} \sinh(\lambda_{n+1} - z_a - i\zeta) \sinh(\lambda_{n+2} - z_a) \right) \cdot \left( \prod_{a=1}^{n} \sinh(\lambda_{n+1} - \lambda_a - i\zeta) \sinh(\lambda_{n+2} - \lambda_a) \right),
\]
where one should set $z_{n+2} = -i\zeta/2$ in the last column of the determinant of densities. We can shift the integration contour for $\lambda_{n+2}$ by $-i\zeta$, and then replace $\lambda_{n+2}$ by $\lambda_{n+2} - i\zeta$ in the integrand. This changes the sign of the density function, and we obtain

$$
- \det_{n+2} \rho(\lambda_j, z_k) \cdot \frac{1}{\sinh(\lambda_{n+1} - \lambda_{n+2} + i\zeta)} \frac{\prod_{a=1}^{n+1} \sinh(\lambda_{n+1} - z_a - i\zeta) \sinh(\lambda_{n+2} - z_a - i\zeta)}{\prod_{a=1}^{n} \sinh(\lambda_{n+1} - \lambda_a - i\zeta) \sinh(\lambda_{n+2} - \lambda_a - i\zeta)}.
$$

We see that for $\zeta = \pi/2$ the integrand becomes an antisymmetric function of $\lambda_{n+2}$ and $\lambda_{n+1}$ and, hence, the corresponding integral vanishes. It remains then to take into account the contribution of the poles which have been crossed during the shift of the $\lambda_{n+2}$-contour. It is easy to see that we have crossed only one singularity, which corresponds to the pole of the function $\rho(\lambda_{n+2}, -\frac{i\zeta}{2})$ at $\lambda_{n+2} = \frac{-i\zeta}{2}$. The residue in this point gives

$$
\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \oint_{C_z} \frac{dz_j}{2\pi i} \int_{\mathbb{R}} d^{n+1} \lambda \prod_{a=1}^{n+1} \left( \frac{\sinh(z_a - \frac{i\zeta}{2})}{\sinh(z_a + \frac{i\zeta}{2})} \right) \prod_{a=1}^{m} \left( \frac{\sinh(\lambda_a + \frac{i\zeta}{2})}{\sinh(\lambda_a - \frac{i\zeta}{2})} \right)
$$

$$
\times \frac{1}{\sinh(\lambda_{n+1} + \frac{i\zeta}{2})} \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a - i\zeta) \sinh(z_a + \frac{i\zeta}{2})}{\sinh(\lambda_{n+1} - \lambda_a - i\zeta) \sinh(\lambda_a + \frac{i\zeta}{2})}
$$

$$
\times \hat{W}_n(\{\lambda\}, \{z\}) \cdot \det_{n+1} \tilde{M}_{jk} \cdot \det_{n+1} [\rho(\lambda_j, z_k)].
$$

At this stage, we can again use the fact that $\tilde{M}$ and $\rho(\lambda_j, z_k)$ become Cauchy matrices at $\zeta = \pi/2$. To compute their determinants, it is convenient to use the following modification of (5.2):

$$
\det_{\mathbb{R}} \frac{1}{\sinh^2(x_j - y_k)} = \prod_{j>k} \cosh(x_j - x_k) \cosh(y_k - y_j) \cdot \frac{1}{2^n \prod_{j,k=1}^{n} \cosh(x_j - y_k)},
$$

(5.2)

Substituting the corresponding expressions of $\det_{n+1} \tilde{M}_{jk}$ and $\det_{n+1} \rho(\lambda_j, z_k)$ into (5.1), we arrive at

$$
\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \frac{1}{2i} \sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \left( \frac{i}{\pi} \right)^{n+1} \int_{\mathbb{R}} d^{n+1} \lambda \prod_{a=1}^{n} \varphi^{m+1} \varphi^{1-m} \left( \frac{1}{\sinh(\lambda_{n+1} + \frac{i\zeta}{2})} \right)
$$

$$
\times \oint_{C_z} \prod_{j=1}^{n+1} \frac{dz_j}{2\pi i} \prod_{a=1}^{n+1} \varphi^{m-1} \left( \frac{1}{\sinh(\lambda_j - z_k)} \right) \cdot \det_{n+1} \left( \begin{array}{c} 1 \\ \cdot \\ \cdot \\ \frac{1}{\sinh(\lambda_k - \lambda_1)} \\ \cdot \\ \frac{1}{\sinh(\lambda_k - \lambda_n)} \end{array} \right) \cdot \det_{n+1} \left( \begin{array}{c} 1 \\ \cdot \\ \cdot \\ \frac{1}{\sinh(\lambda_k + \frac{i\zeta}{2})} \end{array} \right).
$$

(5.3)
As we have seen above, the correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$ and the emptiness formation probability $\tau(m)$ at $\Delta = 0$ are completely described by only one or two terms at $\Delta = 0$. The peculiarity of the correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$ is that, even in the limit of free fermions, all the terms of the corresponding series survive. Nevertheless, it is possible to express (5.3) into a more compact form.

The contour integrals with respect to $z_k$ in (5.3) can be easily computed. Due to the symmetry of the integrand with respect to all the variables $z_k$, $1 \leq k \leq n + 1$, we can make the replacement

$$\det_{n+1} \left( \frac{1}{\sinh(\lambda_j - z_k)} \right) \rightarrow (n + 1)! \prod_{a=1}^{n+1} \frac{1}{\sinh(\lambda_a - z_a)}. $$

Then, inserting for each $z_k$ the factors $\sinh^{-1}(\lambda_k - z_k)$ and $\varphi^{m-1}(z_k)$ into the $k$-th column of the remaining determinant, we can integrate separately each of these columns with respect to $z_k$, using the method described in Section 3:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \frac{1}{2i} \sum_{n=0}^{m-1} \frac{1}{n!} \left( \frac{i}{\pi} \right)^{n+1} \int \mathbb{R} d^{n+1} \lambda \prod_{a=1}^{n} \varphi^{-m+1}(\lambda_a) \frac{\det_{n+1} U_{jk}}{\sinh(\lambda_{n+1} + \frac{\pi m}{4})},$$

(5.4)

where

$$U_{jk} = \begin{cases} 
\frac{1}{2\pi i} \int_{C_z} \frac{\varphi^{m-1}(z_k) dz_k}{\sinh(z_k - \lambda_j) \sinh(\lambda_k - z_k)} = \frac{\varphi^{m-1}(\lambda_j) - \varphi^{m-1}(\lambda_k)}{\sinh(\lambda_j - \lambda_k)}, & j \leq n, \ j \neq k, \\
-\frac{1}{2\pi i} \int_{C_z} \frac{\varphi^{m-1}(z_k) dz_k}{\sinh^2(z_k - \lambda_j)} = 2i(m - 1) \frac{\varphi^{m-1}(\lambda_j)}{\cosh(2\lambda_j)}, & j \leq n, \ j = k, \\
\frac{1}{2\pi i} \int_{C_z} \frac{\varphi^{m-1}(z_k) dz_k}{\sinh(z_k + \frac{i\pi}{4}) \sinh(\lambda_k - z_k)} = \frac{\varphi^{m-1}(\lambda_k)}{\sinh(\lambda_k + \frac{i\pi}{4})}, & j = n + 1. 
\end{cases}$$

(5.5)

The sum (5.4) is very similar to the expansion of a Fredholm determinant of a linear integral operator. To make it more clear, let us introduce

$$V(\lambda, \mu) = i \left( \frac{\varphi(\lambda)/\varphi(\mu)}{\pi \sinh(\lambda - \mu)} \right)^{m-1} - \left( \frac{\varphi(\mu)/\varphi(\lambda)}{\pi \sinh(\lambda - \mu)} \right)^{m-1},$$

(5.6)

where $V(\lambda, \lambda)$ is defined by continuity from (5.6). Then the equation (5.4) can be written as

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{1}{n!} \int \mathbb{R} d^{n+1} \lambda \cdot \det_{n+1} U_{jk},$$

(5.7)
with

\[
U_{jk} = \begin{cases} 
V(\lambda_j, \lambda_k) & j, k \leq n, \\
V(\lambda_j, \lambda_{n+1}) \cdot \frac{\varphi^{m-1}((\lambda_{n+1}))}{2i \sinh(\lambda_{n+1} + \frac{\pi i}{4})} & k = n + 1, j \leq n \\
\frac{i \varphi^{m-1}(\lambda_k)}{\pi \sinh(\lambda_k + \frac{\pi i}{4})} & j = n + 1, k \leq n \\
\frac{\varphi^{m-1}(\lambda_{n+1})}{2\pi \sinh^2(\lambda_{n+1} + \frac{\pi i}{4})} & j, k = n + 1,
\end{cases}
\]  

(5.8)

It is shown in Appendix A that (5.7) results into the derivative of a Fredholm determinant:

\[
\langle \sigma^+ \sigma^- \rangle = \frac{\partial}{\partial \alpha} \det \left( I + V(\lambda, \mu) + \alpha \frac{2\pi}{R(\lambda, \mu)} \right),
\]  

(5.9)

where \(I\) denotes the identity operator and

\[
R(\lambda, \mu) = \frac{\varphi(\lambda) \varphi(\mu)}{\sinh(\lambda + \frac{\pi i}{4}) \sinh(\mu + \frac{\pi i}{4})}.
\]  

(5.10)

The operator \(I + V + \frac{\alpha}{2\pi} R\) acts on the real axis, and \(\alpha\) is an auxiliary parameter. Since \(R(\lambda, \mu)\) is a one-dimensional projector, the determinant in (5.9) is a linear function of \(\alpha\). Hence, the derivative of the determinant does not depend on \(\alpha\). To compare (5.9) with the result obtained in [7] one can make the standard change of variables \(\cosh 2\lambda = \cos^{-1} p, \cosh 2\mu = \cos^{-1} q\). Then, the kernel \(V\) and the projector \(R\) become

\[
V(p, q) = -\frac{\sin \frac{m-1}{2}(p - q)}{\pi \sin \frac{1}{2}(p - q)}, \quad R(p, q) = (-1)^m e^{\frac{im}{2}(p + q)},
\]  

(5.11)

where the integral operator acts on the interval \([-\pi/2, \pi/2]\). Finally, replacing \(p\) with \(-p\) and \(q\) with \(-q\), we arrive at

\[
\langle \sigma^+_1 \sigma^-_{m+1} \rangle = (-1)^m \frac{\partial}{\partial \alpha} \det \left( I - \frac{\sin \frac{m-1}{2}(p - q)}{\pi \sin \frac{1}{2}(p - q)} + \frac{\alpha}{2\pi} e^{-\frac{im}{2}(p + q)} \right).
\]  

(5.12)

This formula coincides with the result of [7] up to the factor \((-1)^m\). The existence of this factor is due to the fact that we use a different definition for the Hamiltonian, as it was mentioned in the beginning of Section 2.

6 Long-distance asymptotics of \(\langle \sigma^+_1 \sigma^-_{m+1} \rangle\)

The leading asymptotic behavior of the correlation function \(\langle \sigma^+_1 \sigma^-_{m+1} \rangle\) was computed in [3, 4]. Later, in [7], a Fredholm determinant representation of the dynamic temperature correlation
function was obtained for an arbitrary value of the external magnetic field. The determinant (5.12) appears to be a particular case of this result. To compute its asymptotic behavior at large \( m \), one can use the methods of the matrix Riemann–Hilbert problem which were developed in [8] to study the dynamic temperature correlations. However, these methods allow to find the asymptotics of the determinant only up to a multiplicative constant, whereas the determinant (5.12) can be computed explicitly as a finite product of \( \Gamma \)-functions. In this section, we present the corresponding derivation and reproduce the results of the papers [3, 4].

Observe first that the kernel \( V(p,q) \) (5.11) is degenerated:

\[
V(p,q) = -\sin \frac{m-1}{2}(p-q) = -\frac{1}{\pi} \sum_{k=1}^{m-1} e^{i(p-q)(k-m/2)}. \tag{6.1}
\]

Thus, the corresponding Fredholm determinant can be reduced to the determinant of a matrix of finite size. Indeed, if a kernel \( K(p,q) \) has the form

\[
K(p,q) = \sum_{k=1}^{m} f_k(p) g_k(q),
\]

then

\[
det(I + K(p,q)) = det_m(\delta_{jk} + M_{jk}), \tag{6.2}
\]

with

\[
M_{jk} = \int_C f_j(p) g_k(p) \, dp. \tag{6.3}
\]

Here \( C \) is the contour where the operator \( I + K \) acts. In our case, \( C = [-\pi/2, \pi/2] \), and

\[
f_k(p) = -\frac{1}{\pi} e^{ip(k-m/2)}, \quad k = 1, \ldots, m-1,
\]

\[
f_m(p) = \frac{\alpha}{2\pi} e^{-ip\frac{m}{2}},
\]

\[
g_k(q) = e^{-iq(k-m/2)}, \quad k = 1, \ldots, m.
\]

Thus, for \( j < m \),

\[
M_{jk} = -\delta_{jk} - \frac{2}{\pi} \left\{ \begin{array}{ll}
0, & \text{for } j - k \text{ even}, \\
(-1)^{(j-k-1)/2} \cdot \frac{1}{j-k}, & \text{for } j - k \text{ odd},
\end{array} \right. \tag{6.5}
\]

whereas the elements of the last line of the matrix \( M \) are given by

\[
M_{mk} = \frac{\alpha}{\pi} \left\{ \begin{array}{ll}
0, & \text{for } k \text{ even}, \\
(-1)^{(k-1)/2} \cdot \frac{1}{\kappa}, & \text{for } k \text{ odd}.
\end{array} \right. \tag{6.6}
\]

Note that the parameter \( \alpha \) enters only the last line. Hence, taking the derivative of the determinant with respect to \( \alpha \), we need to differentiate only the elements of this line, and we obtain

\[
\langle \sigma_1^+ \sigma_{m+1}^- \rangle = -\frac{2^{m-1}}{\pi^m} \det_m(a_{jk}), \tag{6.7}
\]

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where
\[ a_{jk} = \begin{cases} 
0, & \text{for } j - k \text{ even}, \\
(-1)^{(j-k-1)/2} \cdot \frac{1}{j-k}, & \text{for } j - k \text{ odd}, 
\end{cases} \quad (6.8) \]
for \( j < m \), and
\[ a_{mk} = \begin{cases} 
0, & \text{for } k \text{ even}, \\
(-1)^{(k-1)/2} \cdot \frac{1}{k}, & \text{for } k \text{ odd}, 
\end{cases} \quad (6.9) \]
for \( j = m \).

Our goal is now to compute \( \det_m(a_{jk}) \). To do this, let us first reorder the columns and the lines of the matrix \( a_{jk} \) such that it becomes a \( 2 \times 2 \) block-matrix. One has to move:

1. the columns with the number \( 2k \) to the position \( k \) for \( k = 1, \ldots, \left[ \frac{m}{2} \right] \);
2. the lines with the number \( 2j - 1 \) to the position \( j \) for \( j = 1, \ldots, \left[ \frac{m}{2} \right] \);
3. the last line with the number \( m \) to the position \( \left[ \frac{m}{2} \right] + 1 \).

Here, \( \left[ \frac{m}{2} \right] \) denotes the integer part of \( m \). After these transformations, we arrive at
\[ \det_m(a_{jk}) = (-1)^{m-1} \det \left( \begin{array}{cc} a_{2j-1,2k} & 0 \\
0 & a_{2j-2,2k-1} \end{array} \right), \quad \text{with} \quad a_{0,2k-1} \equiv a_{m,2k-1}. \quad (6.10) \]

Hereby the sizes of the blocks are \( \left[ \frac{m}{2} \right] \times \left[ \frac{m}{2} \right] \) for \( a_{2j-1,2k} \) and \( \left[ \frac{m+1}{2} \right] \times \left[ \frac{m+1}{2} \right] \) for \( a_{2j-2,2k-1} \). Observe now that
\[ a_{2j-1,2k} = a_{2j-2,2k-1} = \frac{(-1)^{j-k-1}}{2j - 2k - 1}. \quad (6.11) \]

Hence, we obtain
\[ \det_m(a_{jk}) = -\det_{\left[ \frac{m}{2} \right]} \left( \frac{1}{2j - 2k - 1} \right) \cdot \det_{\left[ \frac{m+1}{2} \right]} \left( \frac{1}{2j - 2k - 1} \right). \quad (6.12) \]

It remains to use the analog of (3.2) for rational functions
\[ \det_n \frac{1}{x_j - y_k} = \frac{\prod_{j>k} (x_j - x_k)(y_k - y_j)}{\prod_{j,k=1}^n (x_j - y_k)}, \quad (6.13) \]
which gives
\[ \det_m(a_{jk}) = (-1)^{m-1} \prod_{k=1}^{m \left[ \frac{m}{2} \right]} \prod_{j=1}^{j \neq k \left[ \frac{m}{2} \right]} \frac{j - k}{j - k - \frac{1}{2}} \prod_{k=1}^{j \neq k \left[ \frac{m+1}{2} \right]} \prod_{j=1}^{j \neq k \left[ \frac{m+1}{2} \right]} \frac{j - k}{j - k - \frac{1}{2}}. \quad (6.14) \]
After the computation of the products with respect to $j$, we substitute the result into (6.7) and eventually obtain
\[
\langle \sigma_+ \sigma_{m+1}^- \rangle = \frac{(-1)^m}{\sqrt{2m}} \prod_{k=1}^{[m]} \frac{\Gamma^2(k)}{\Gamma(k - \frac{1}{2}) \Gamma(k + \frac{1}{2})} \prod_{k=1}^{[m+1]} \frac{\Gamma^2(k)}{\Gamma(k - \frac{1}{2}) \Gamma(k + \frac{1}{2})}.
\] (6.15)

Thus, we have computed the Fredholm determinant (5.12) as a finite product of $\Gamma$-functions. This form enables one to evaluate the large $m$ asymptotic behavior of the correlation function $\langle \sigma_+ \sigma_{m+1}^- \rangle$ in a rather simple way (see Appendix B). The result reads
\[
\langle \sigma_+ \sigma_{m+1}^- \rangle = \frac{(-1)^m}{\sqrt{2m}} \exp \left\{ \frac{1}{2} \int_0^\infty \frac{dt}{t} \left[ e^{-4t} - \frac{1}{\cosh^2 t} \right] \right\} \left( 1 - \frac{(-1)^m}{8m^2} + O(m^{-4}) \right).
\] (6.16)

**Conclusion**

To conclude this article, we would like to discuss the significance of our results and the perspectives they open concerning the computation of correlation functions in a more general case. Actually, the purpose of the work presented here was double.

Our first goal was to demonstrate that it was really possible in the free fermion limit to compute the spin-spin correlation functions using their multiple integral representations. Indeed, the ability of this method to provide effective results, even in the simplest case $\Delta = 0$, has for a long time been seriously under question. Thanks to the new formulas obtained in [1], which correspond in fact to certain re-summations of the elementary blocks [9]–[12], we were able here to solve this problem.

The second goal of this paper concerns the possible application of these new integral representations to the general $XXZ$ model. We hope that some of the technical methods presented here are not specific to the free fermion point, and that they can also be successfully adapted to study a more general case.

In particular, the method we used here to compute the $z$-integrals might be quite efficient for the evaluation of the long distance asymptotics of the spin-spin correlation functions. Recall that, in this paper, we have deformed the original contour $C_z$ into a horizontal strip $\Gamma$ of width $i\pi$. At $\Delta = 0$, the contribution coming from $\Gamma$ vanishes due to the periodicity of the integrand. In the general case this property is no longer valid since the density function $\rho(\lambda, z)$ is no longer $i\pi$-periodical (except at $\zeta = \frac{\pi}{2n}$, where $n$ is a positive integer). Nevertheless, it seems that one can control the order of the contribution coming from $\Gamma$ as $m \to \infty$. Indeed, it is possible to choose the strip such that $|\varphi(z)| < 1$ (see (6.13)) on its boundaries. Then the factor $\varphi^m(z)$ becomes exponentially small uniformly on an arbitrary finite interval of the new integration contour for $z$. Preliminary estimates show that the integrals over $\Gamma$ decrease as some negative powers of $m$ as $m \to \infty$. Thus, it is very possible that the leading long distance asymptotics of the spin-spin correlation functions are given by the residues of the integrand within the strip $\Gamma$. 

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Finally, we would also like to draw the reader’s attention on the method used for the evaluation of the long-distance asymptotics of the emptiness formation probability. In fact, the saddle point approach can be applied directly to the representation (2.14) in the general case as well. One can thus expect that the emptiness formation probability decreases as \( \exp\{-c(\Delta, h)m^2\} \) as \( m \to \infty \). The main obstacle to find the coefficient \( c(\Delta, h) \) is related to the asymptotic analysis of \( \det_m t(\lambda_j, \xi_k) \) (recall that for \( \Delta = 0 \) this determinant is explicitly computable). The problem we mention here coincides one to one with the problem of the computation of the partition function of the six-vertex model with domain wall boundary conditions \([17], [18]\) which was solved for the homogeneous case in \([19], [20]\). For our purpose, it would be desirable to extend these results to the inhomogeneous case as well, which is still an open problem. It seems nevertheless that the emptiness formation probability admits a Gaussian behavior.

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A Fredholm determinant

Let us first recall the definition of a Fredholm determinant: if an operator \( I + K \) acts on an interval \( C \) as

\[
[(I + K)\phi](\lambda) = \phi(\lambda) + \int_C K(\lambda, \mu)\phi(\mu) \, d\mu,
\]

then its Fredholm determinant is

\[
\det(I + K) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C d\lambda_1 \cdots d\lambda_n \cdot \det_n K(\lambda_j, \lambda_k).
\]

The series (5.7) has almost the same form. To make it exactly the same, we observe that \( \text{rank } \tilde{U}_{jk} \leq m - 1 \), and hence \( \det_{n+1} \tilde{U}_{jk} = 0 \) as soon as \( n > m - 1 \). Therefore the sum in (5.7) can be extended up to infinity. We can then use the identity

\[
\det_{n+1} W_{jk} = \left. \left(W_{n+1,n+1} - \frac{\partial}{\partial \alpha}\right) \det_n (W_{jk} + \alpha W_{j,n+1} W_{n+1,k}) \right|_{\alpha=0}
\]
which holds for any arbitrary matrix $W$. The sum (5.7) can thus be written as

$$
\langle \sigma_1^{+} \sigma_{m+1}^{-} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{\mathbb{R}} \frac{\varphi^{m-1}(\lambda_{n+1})}{2\pi \sinh^2(\lambda_{n+1} + \frac{\pi}{4})} - \frac{\partial}{\partial \alpha} \right) d\lambda_{n+1} + \frac{\alpha}{2\pi} \int_{\mathbb{R}} \frac{\varphi^{m-1}(\lambda_{n+1})}{2\pi \sinh^2(\lambda_{n+1} + \frac{\pi}{4})} d\lambda_{n+1}
$$

\times \int_{\mathbb{R}} d^m \lambda \cdot \det_{n} \left( V(\lambda_j, \lambda_k) + \frac{\alpha}{2\pi} \int_{\mathbb{R}} V(\lambda_j, \lambda_{n+1}) \frac{\varphi^{m-1}(\lambda_{n+1})}{\sinh(\lambda_{n+1} + \frac{\pi}{4})} \sinh(\lambda_k + \frac{\pi}{4}) \right) \left|_{\alpha=0} \right.

The series (A.4) is now exactly of the form (A.2), which means that the correlation function $\langle \sigma_1^{+} \sigma_{m+1}^{-} \rangle$ can be represented as the following Fredholm determinant:

$$
\langle \sigma_1^{+} \sigma_{m+1}^{-} \rangle = \left( \int_{\mathbb{R}} \frac{\varphi^{m-1}(\nu)}{2\pi \sinh^2(\nu + \frac{\pi}{4})} - \frac{\partial}{\partial \alpha} \right) d\nu
$$

\times \det \left( I + V(\lambda, \mu) + \frac{\alpha}{2\pi} \int_{\mathbb{R}} V(\lambda, \nu) \frac{\varphi^{m-1}(\nu)}{\sinh(\nu + \frac{\pi}{4})} \sinh(\mu + \frac{\pi}{4}) \right) \left|_{\alpha=0} \right.

To reduce this determinant to the form (5.3), we use the following lemma:

**Lemma A.1.** Let an integral operator

$$
I + K(\lambda, \mu) + \alpha \int_{C} K(\lambda, \nu) y(\nu) x(\mu) d\nu
$$

acts on an interval $C$. Hereby the kernel $K(\lambda, \mu)$ and the functions $x(\lambda), y(\lambda)$ are such that the Fredholm determinant of this operator exists. Then

$$
\left( \int_{C} x(\nu) y(\nu) d\nu - \frac{\partial}{\partial \alpha} \right) \det \left( I + K(\lambda, \mu) + \alpha \int_{C} K(\lambda, \nu) y(\nu) x(\mu) d\nu \right) \left|_{\alpha=0} \right.

= \frac{\partial}{\partial \alpha} \det \left( I + K(\lambda, \mu) + \alpha y(\lambda) x(\mu) \right).

(A.6)

**Proof.** Assume first that det$(I + K) \neq 0$, i.e. that there exists the inverse operator $I - G = (I + K)^{-1}$. Then one can extract the determinant of the operator $I + K$:

$$
\left( \int_{C} x(\nu) y(\nu) d\nu - \frac{\partial}{\partial \alpha} \right) \det \left( I + K(\lambda, \mu) + \alpha \int_{C} K(\lambda, \nu) y(\nu) x(\mu) d\nu \right) \left|_{\alpha=0} \right.

= \det(I + K) \left( \int_{C} x(\nu) y(\nu) d\nu - \frac{\partial}{\partial \alpha} \right) \det \left( I + \alpha \int_{C} G(\lambda, \nu) y(\nu) x(\mu) d\nu \right) \left|_{\alpha=0} \right.

(A.7)
The operator \( \int_{C} G(\lambda, \nu)y(\nu)x(\mu) \, d\nu \) is a one-dimensional projector, hence,

\[
\det \left( I + \alpha \int_{C} G(\lambda, \nu)y(\nu)x(\mu) \, d\nu \right) = 1 + \alpha \int_{C} G(\mu, \nu)y(\nu)x(\mu) \, d\nu \, d\mu. \tag{A.8}
\]

Substituting this into (A.7), we obtain

\[
\left( \int_{C} x(\nu)y(\nu) \, d\nu - \frac{\partial}{\partial \alpha} \right) \det \left( I + K(\lambda, \mu) + \alpha \int_{C} K(\lambda, \nu)y(\nu)x(\mu) \, d\nu \right) \bigg|_{\alpha=0} = \det(I + K) \int_{C} (\delta(\nu - \mu) - G(\mu, \nu))y(\nu)x(\mu) \, d\nu \, d\mu. \tag{A.9}
\]

In its turn the last equality is equivalent to

\[
\left( \int_{C} x(\nu)y(\nu) \, d\nu - \frac{\partial}{\partial \alpha} \right) \det \left( I + K(\lambda, \mu) + \alpha \int_{C} K(\lambda, \nu)y(\nu)x(\mu) \, d\nu \right) \bigg|_{\alpha=0} = \det(I + K) \frac{\partial}{\partial \alpha} \det \left( I + \alpha \int_{C} (I - G)(\lambda, \nu)y(\nu)x(\mu) \, d\nu \, d\mu \right). \tag{A.10}
\]

Finally, transforming the product of the two determinants into a single one, we arrive at (A.6).

If \( \det(I + K) = 0 \), we can consider the modified operator \( I + \gamma K \), where \( \gamma \) is some complex. The Fredholm determinant \( \det(I + \gamma K) \) is an entire function of \( \gamma \) and, hence, it has a finite number of isolated zeros in any closed domain of the complex plane. Thus, we can choose \( \gamma \) such that \( \det(I + \gamma K) \neq 0 \), repeat all the transformations described above and continue the result to the point \( \gamma = 1 \). Thus, the lemma is proved. \( \square \)

It remains to observe that the structure of the determinant (A.5) coincides with the one of (A.6) if we set \( K(\lambda, \mu) = V(\lambda, \mu) \) and

\[
x(\lambda) = y(\lambda) = \frac{\varphi^{m-1}(\lambda)}{\sqrt{2\pi \sinh(\lambda + \frac{i\pi}{4})}}. \tag{A.11}
\]

Thus, we arrive at (5.9).

**B  Asymptotic study of the product of \( \Gamma \)-functions**

We need to compute the asymptotic behavior of the quantity

\[
e^{-\phi N} = \prod_{k=1}^{N} \frac{\Gamma^2(k)}{\Gamma(k - \frac{1}{2})\Gamma(k + \frac{1}{2})}. \tag{B.1}
\]
where $N \to \infty$. Expanding $\phi_N$ into a Taylor series, we obtain
\[
\phi_N = 2 \sum_{k=1}^{N} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left( \frac{1}{2} \right)^{2n} \psi^{(2n-1)}(k), \tag{B.2}
\]
where
\[
\psi^{(2n-1)}(z) = \frac{d^{2n}}{dz^{2n}} \log \Gamma(z). \tag{B.3}
\]
The sum with respect to $k$ in (B.2) can be computed using
\[
\sum_{k=1}^{N} \psi^{(s)}(k) = N \psi^{(s)}(N + 1) + s \left( \psi^{(s-1)}(N + 1) - \psi^{(s-1)}(1) \right). \tag{B.4}
\]
Substituting this into (B.2), we obtain
\[
\phi_N = 2 \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left( \frac{1}{2} \right)^{2n} \left[ N \psi^{(2n-1)}(N + 1) + (2n - 1) \left( \psi^{(2n-2)}(N + 1) - \psi^{(2n-2)}(1) \right) \right]. \tag{B.5}
\]
This series is absolutely convergent, and we can make an asymptotic estimate of each term separately. To do this, let us present $\phi_N$ in the form
\[
\phi_N = \frac{1}{4} \log N + S_1 + S_2, \tag{B.6}
\]
where
\[
S_1 = \frac{1}{4} (N \psi'(N + 1) + \psi(N + 1) + C - \log N) - 2 \sum_{n=2}^{\infty} \frac{2n - 1}{(2n)!} \left( \frac{1}{2} \right)^{2n} \psi^{(2n-2)}(1). \tag{B.7}
\]
Here $C = -\psi(1)$ is the Euler constant. The last term in (B.6) is
\[
S_2 = 2 \sum_{n=2}^{\infty} \frac{1}{(2n)!} \left( \frac{1}{2} \right)^{2n} \left[ N \psi^{(2n-1)}(N + 1) + (2n - 1) \psi^{(2n-2)}(N + 1) \right]. \tag{B.8}
\]
Recall also the asymptotic expansion for the logarithm of $\Gamma$-function
\[
\log \Gamma(z) = z \log z - z - \frac{1}{2} \log 2\pi + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + O(z^{1-2n}), \quad |z| \to \infty, \tag{B.9}
\]
where $B_{2k}$ are Bernoulli numbers. Using (B.9) one can easily see that at $N \to \infty$ the term $S_1$ has a finite limit, while $S_2$ vanishes.

To compute the limiting value of $S_1$, one can use for example the integral representation for the derivatives of $\psi(z)$:
\[
\psi^{2n-1}(z) = \int_0^\infty \frac{e^{-tz}t^{2n-1}}{1 - e^{-t}} \, dt, \quad n \geq 1, \quad \Re(z) > 0. \tag{B.10}
\]
Then the series in $[B.7]$ becomes the expansion of the exponent, and we obtain

$$S_1 = -\frac{1}{4} \int_0^{\infty} \frac{dt}{t} \left[ e^{-4t} - \frac{1}{\cosh^2 t} \right], \quad N \to \infty. \quad (B.11)$$

This gives us constant the contribution to $\phi_N$.

In order to find the corrections to this quantity, we need first to take into account the higher order corrections to $\psi'(N+1)$ and $\psi(N+1)$ in $S_1$ and then to take several terms from the series $S_2$. In particular, we find

$$\phi_N = \frac{1}{4} \log N - \frac{1}{4} \int_0^{\infty} \frac{dt}{t} \left[ e^{-4t} - \frac{1}{\cosh^2 t} \right] + \frac{1}{64N^2} + O(N^{-4}), \quad N \to \infty. \quad (B.12)$$

This formula can be directly applied for the computation of the asymptotic behavior of $(6.15)$.

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