Evaluating characterizations of homomorphisms on truncated vector lattices of functions

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Dedicated to the memory of Professor Abdelmajid Triki

Abstract

Let $L$ be a (non necessarily unital) truncated vector lattice of real-valued functions on a nonempty set $X$. A nonzero linear functional $\psi$ on $L$ is called a truncation homomorphism if it preserves truncation, i.e.,

$$\psi(f \wedge 1_X) = \min\{\psi(f), 1\} \text{ for all } f \in L.$$ 

We prove that a linear functional $\psi$ on $L$ is a truncation homomorphism if and only if $\psi$ is a lattice homomorphism and

$$\sup\{\psi(f) : f \leq 1_X\} = 1.$$ 

This allows us to prove different evaluating characterizations of truncation homomorphisms. In this regard, a special attention is paid to the continuous case and various results from the existing literature are generalized.

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1 Introduction

It is well-known that a linear functional $\psi$ on the lattice-ordered algebra $C(X)$ of all real-valued continuous functions on a Tychonoff space $X$ is a unital lattice homomorphism if and only if it is an evaluation at some point $x$ of $X$, i.e.,

$$\psi(f) = f(x) \quad \text{for all } f \in C(X)$$

(see, e.g., Theorem 2.33 in [1]). In their remarkable papers [8, 9], Garrido and Jaramillo investigated the extent to which such a representation can be generalized to a wider class of unital vector sublattices of $C(X)$. In this regard, they have mainly proved that if $\psi$ is a linear functional on a unital vector sublattice $L$ of $C(X)$, then $\psi$ is a unital lattice homomorphism if and only if $\psi$ is an evaluation at some point in the Stone-Čech compactification $\beta X$ of $X$. They obtained, as consequences, some necessary and sufficient conditions on $L$ for $X$ to be $L$-realcompact, i.e., any unital lattice homomorphism on $L$ is an evaluation at some point in $X$. They also used their aforementioned representation theorem to establish the equivalence between unital lattice homomorphisms and positive algebra homomorphisms on unital lattice-ordered subalgebras of $C(X)$. Although they cover a quite large spectrum of function lattices, these results, relevant as they are, cannot deal with the non-unital case. It seems to be natural therefore to look beyond the framework of lattices containing the constant functions. From this point of view, we have thought about vector sublattices possessing the so-called Stone property. Recall here that a vector subspace $E$ of the lattice-ordered algebra $\mathbb{R}^X$ of all real-valued functions on an arbitrary non-empty $X$ is said to possess the Stone property if $E$ contains with any function $f$ the function $f \wedge 1_X$ defined by

$$(1_X \wedge f)(x) = 1 \wedge f(x) = \min\{1, f(x)\} \quad \text{for all } x \in X,$$

where $1_X$ is the indicator (or characteristic) function of $X$. We call a truncated vector sublattice of $\mathbb{R}^X$, after Fremlin in [7], any vector sublattice $L$ of $\mathbb{R}^X$ which possesses the Stone property (we do not assume that $1_X$ is present in $L$). As a matter of fact, the strength of the relationship between this structure and duality is not a new idea. This goes back to the mid-19th Century when Stone himself proved that, for every $\sigma$-order continuous positive linear functional $\psi$ on a truncated vector sublattice $L$ of $\mathbb{R}^X$, there exists a
measure $\mu$ on $X$ such that
\[
\varphi(f) = \int_X f \, d\mu \quad \text{for all } f \in L
\]
(see, e.g., Theorem 4.5.2 in [4]). This fundamental result is, by now, referred to as the Daniell-Stone Representation Theorem. It is, therefore, surprising that there has been no study of evaluating properties of homomorphisms on truncated vector sublattices of $\mathbb{R}^X$. This paper will actually try to address this omission. Against this background, a suitable concept of homomorphisms must be introduced, for manifest reasons of compatibility. To meet this need, we drew inspiration from the recent work [2] by Ball to define a truncation homomorphism on the truncated vector sublattice $L$ of $\mathbb{R}^X$ as a nonzero linear functional $\psi$ on $L$ that preserves truncation, i.e.,
\[
\psi(1_X \wedge f) = 1 \wedge \psi(1_X) = \min\{1, \psi(1_X)\} \quad \text{for all } f \in L.
\]
A short synopsis of the content of this paper seems in order.
In Section 2, the connection between truncation homomorphisms and lattice homomorphisms are considered in some details. For instance, we prove that any truncation homomorphism is automatically a lattice homomorphism, then we find the missing condition for the converse to hold. The third section contains the evaluating characterizations of truncation homomorphisms we are looking for. Indeed, it turns out that a linear functional $\psi$ on a truncated vector sublattice $L$ of $\mathbb{R}^X$ is a truncation homomorphism if and only if $\psi$ is a $\mathcal{F}$-evaluation on $L$, i.e., for every $f, g \in L$ and every $\varepsilon \in (0, \infty)$, the inequalities
\[
|f(x) - \psi(f)| \leq \varepsilon \quad \text{and} \quad |g(x) - \psi(g)| \leq \varepsilon
\]
hold for some $x \in X$ (depending on $f, g$ and $\varepsilon$). Also, we show that for any truncation homomorphism $\psi$ on a truncated vector sublattice $L$ of $\mathbb{R}^X$ there exists a net $(x_\sigma)_\sigma$ in $X$ such that
\[
\lim f(x_\sigma) = f(x) \quad \text{in } \mathbb{R} \quad \text{for all } f \in L.
\]
This brings us to the last section, in which the continuous case is investigated. We prove, among other characterizations, that any truncation homomorphism on a truncated vector sublattice of $C(X)$ is an evaluation at some point of $\beta X$ of $X$. As pointed out above, the unital case was resolved.
(in an alternative way) by Garrido and Jaramillo. We end the paper by providing sufficient (and sometimes necessary) conditions on $L$ for $X$ to be $L$-realcompact, i.e., any truncation homomorphism on $L$ is a one-point evaluation.

Efforts have been made to make this work more or less accessible to a large audience in such a way it could be understood by readers with a standard first-year graduate background on algebra and topology. In spite of that, we use the great books [1, 7] on Vector Lattices, [10, 12] on Real-Valued Functions, and [5, 13] on General Topology as sources for unexplained terminology and notation (unless otherwise stated explicitly).

2 Connection with lattice homomorphisms

Our first discussion may well not have been quite on the agenda, but we think that it is sufficiently interesting to be incorporated into the text. Recall from the introduction that a vector subspace $E$ of $\mathbb{R}^X$ is said to possess the Stone property if

$$f \wedge 1_X \in E \quad \text{for all } f \in E.$$ 

It turns out that any vector subspace $\mathbb{R}^X$ possessing the Stone property is a vector sublattice of $\mathbb{R}^X$, provided that it contains the constant functions.

**Proposition 2.1** Let $E$ be a vector subspace of $\mathbb{R}^X$ such that $1_X \in E$. Then $E$ possesses the Stone property if and only if $E$ is vector sublattice of $\mathbb{R}^X$.

**Proof.** Sufficiency being straightforward, we prove Necessity. Assume that $E$ possesses the Stone property and choose $f \in E$. Observe that

$$|1_X - f| = 1_X + f - 2(f \wedge 1_X) \in E$$

Thus, if $g$ is an arbitrary element in $E$ then $1_X - g \in E$ and so

$$|g| = |1_X - (1_X - g)| \in E.$$ 

This implies that $E$ is a vector sublattice of $\mathbb{R}^X$, as required. ■

We thought at a moment that the result should hold for any vector subspace of $\mathbb{R}^X$. However, both implications are not true in general as the following examples show.
Example 2.2  (i) A function $f \in \mathbb{R}^{[0, \infty)}$ is said to be essentially linear if there exists $x_f > 0$ and $a_f \in \mathbb{R}$ such that

$$f(x) = a_fx \quad \text{for all } x \in (x_f, \infty).$$

It is not hard to see that the subset $L$ of $\mathbb{R}^{[0, \infty)}$ is a vector sublattice of $\mathbb{R}^{[0, \infty)}$. However, if

$$i(x) = x \quad \text{for all } x \in [0, \infty)$$

then $i \in L$, while $i \land 1_{[0, \infty)} \notin L$. This means that $L$ does not possess the Stone property.

(ii) For every $f \in C(\mathbb{R})$, we define $\tau f \in C(\mathbb{R})$ by

$$(\tau f)(x) = xf(x) \quad \text{for all } x \in \mathbb{R}.$$  

Obviously, the set

$$E = \{\tau f : f \in C(\mathbb{R})\}$$

is a vector subspace of $C(\mathbb{R})$. Choose $f \in E$ and define $g \in C(\mathbb{R})$ by

$$g(x) = \frac{f(x)}{\sup \{f(x), 1\}} \quad \text{for all } x \in \mathbb{R}.$$  

It is an elementary exercise to show that

$$\tau f \land 1_{\mathbb{R}} = \tau g \in E,$$

meaning that $E$ does possess the Stone property. Nevertheless, $E$ is not a vector sublattice of $\mathbb{R}^X$ since, if $e$ is the function given by

$$e(x) = x \quad \text{for all } x \in \mathbb{R},$$

then $e \in E$ while $|e| \notin E$.

Now, let’s get to the heart of the matter. As before, we call a truncated vector sublattice of $\mathbb{R}^X$ any vector sublattice $L$ of $\mathbb{R}^X$ possessing the Stone property. We emphasize that we do not assume that truncated vector sublattices of $\mathbb{R}^X$ contain $1_X$. A vector sublattice of $\mathbb{R}^X$ containing $1_X$ is called a unital vector sublattice of $\mathbb{R}^X$. Obviously, any unital vector sublattice of $\mathbb{R}^X$
is a truncated vector sublattice of $\mathbb{R}^X$. A nonzero linear functional $\psi$ on the truncated vector sublattice $L$ of $\mathbb{R}^X$ is called a truncation homomorphism if

$$\psi (f \wedge 1_X) = \psi (f) \wedge 1 = \min \{ \psi (f), 1 \} \quad \text{for all } f \in L.$$ 

Also, recall that the linear functional $\psi$ on a vector sublattice $L$ of $\mathbb{R}^X$ is called a lattice homomorphism if

$$\psi (f \wedge g) = \psi (f) \wedge \psi (g) = \min \{ \psi (f), \psi (g) \} \quad \text{for all } f, g \in L.$$ 

Clearly, a linear functional $\psi$ on $L$ is a lattice homomorphism if and only if

$$|\psi (f)| = \psi (|f|) \quad \text{for all } f \in L.$$ 

Notice that any lattice homomorphism $\psi$ on the vector sublattice $L$ of $\mathbb{R}^X$ is positive (and thus increasing), that is to say,

$$\psi (f) \geq 0 \quad \text{for all } f \in L^+,$$

where $L^+$ denotes the set of all positive functions in $L$.

Connections between truncation homomorphisms and lattice homomorphisms on truncated vector sublattices of functions are studied next.

**Lemma 2.3** Any truncation homomorphism on a truncated vector sublattice $L$ of $\mathbb{R}^X$ is a lattice homomorphism on $L$.

**Proof.** Let $\psi$ be a truncation homomorphism on the truncated vector sublattice $L$ of $\mathbb{R}^X$. First, we claim that $\psi$ is positive. To this end, choose $f \in L$ and $n \in \{1, 2, \ldots \}$. If $f \leq 0$ then

$$n\psi (f) = \psi (nf) = \psi (1_X \wedge nf) = 1 \wedge \psi (nf) \leq 1.$$ 

It follows that $\psi (f) \leq 0$ because $n$ is arbitrary in $\{1, 2, \ldots \}$. This yields that $\psi$ is positive, as required. Now, let $f \in L$ and observe that

$$0 \leq n\psi (f^+) - n\psi (f)^+$$

because $\psi$ is positive. Moreover,

$$n\psi (f^+) \leq \psi (nf^+) - 1 \wedge \psi (nf^+) + 1 = \psi (nf^+) - \psi (1_X \wedge nf^+) + 1$$

$$= \psi (nf^+ - 1_X \wedge nf^+) + 1 = \psi (nf - 1_X \wedge nf) + 1$$

$$= \psi (nf) - 1 \wedge \psi (nf) + 1 = \psi (nf)^+ - 1 \wedge \psi (nf)^+ + 1$$

$$\leq \psi (nf)^+ + 1.$$
Therefore,

\[ 0 \leq n \left[ \psi(f^+) - \psi(f) \right] \leq 1. \]

We derive that \( \psi(f^+) = \psi(f)^+ \) and the proof is complete. \( \blacksquare \)

It is all too clear that the converse of Lemma \ref{lem:2.3} fails. It is natural therefore to ask for the missing condition for a lattice homomorphism on a truncated vector sublattice of \( \mathbb{R}^X \) to be a truncation homomorphism. The following theorem answers this question.

**Theorem 2.4** Let \( \psi \) be a linear functional on a truncated vector sublattice \( L \) of \( \mathbb{R}^X \). Then the following are equivalent.

(i) \( \psi \) is a truncation homomorphism.

(ii) \( \psi \) is a lattice homomorphism and

\[ \text{sup} \{ \psi(f) : f \in L \text{ and } f \leq 1_X \} = 1. \]

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( \psi \) is a truncation homomorphism. By Lemma \ref{lem:2.3}, \( \psi \) is a lattice homomorphism. On the other hand, if \( f \in L \) with \( f \leq 1_X \) then \( f = f \wedge 1_X \) and so

\[ \psi(f) = \psi(f \wedge 1_X) = \psi(f) \wedge 1 \leq 1. \]

Moreover, let \( \lambda \in \mathbb{R} \) and suppose that \( \lambda \geq \psi(f) \) for every \( f \in L \) with \( f \leq 1_X \). Choose \( f \in L \) such that \( \psi(f) \neq 0 \) (such a function \( f \) exists because, by definition, \( \psi \neq 0 \)) and put

\[ g = 1_X \wedge \frac{1}{|\psi(f)|} |f|. \]

Clearly, \( g \in L \) and \( g \leq 1_X \). Hence,

\[ \lambda \geq \psi(g) = \psi \left( 1_X \wedge \frac{1}{|\psi(f)|} |f| \right) = 1 \wedge \frac{\psi(|f|)}{|\psi(f)|} = 1 \wedge \frac{|\psi(f)|}{|\psi(f)|} = 1 \]

(where we use once again Lemma \ref{lem:2.3}). This means that

\[ 1 = \text{sup} \{ \psi(f) : f \in L \text{ and } f \leq 1_X \}, \]

as desired.
(ii) $\Rightarrow$ (i) Let $f \in L$ and observe that $\psi(f \wedge 1_X) \leq \psi(f)$ because $\psi$ is positive. Moreover, $f \wedge 1_X \in L$ and $f \wedge 1_X \leq 1_X$, so, $\psi(f \wedge 1_X) \leq 1$. This means that $\psi(f \wedge 1_X) \leq \psi(f) \wedge 1$. Conversely, pick $g \in L$ with $g \leq 1_X$ and observe that from $f \wedge g \leq f \wedge 1_X$ it follows that

$$\psi(f \wedge 1_X) \geq \psi(f) \wedge \psi(g) = \psi(f) \wedge \psi(g).$$

Accordingly,

$$\psi(f \wedge 1_X) \geq \sup \{ \psi(f) \wedge \psi(g) : g \in L \text{ and } g \leq 1_X \} = \psi(f) \wedge \sup \{ \psi(g) : g \in L \text{ and } g \leq 1_X \} = \psi(f) \wedge 1.$$

This ends the proof of the theorem. ■

A lattice homomorphism $\psi$ on a unital vector sublattice $L$ of $\mathbb{R}$ is said to be unital if $\psi(1_X) = 1$. Hence, we get the following as a direct inference of Theorem 2.4.

**Corollary 2.5** A linear functional $\psi$ on a unital vector sublattice of $\mathbb{R}^X$ is a truncation homomorphism if and only if $\psi$ is a unital lattice homomorphism.

### 3 Evaluating characterizations

We start this section with two technical lemmas.

**Lemma 3.1** If $f, g \in \mathbb{R}^X$ then

$$|f + 1_X| \wedge g = |f| - 2(f^\sim \wedge 1_X) + (g - |f| + 2(f^\sim \wedge 1_X)) \wedge 1_X.$$

**Proof.** First, observe that

$$|f + 1_X| = (f + 1_X) \vee (-f - 1_X)$$
$$= [(f + 1_X) \vee (f - 1_X)] \vee (-f - 1_X)$$
$$= (f + 1_X) \vee [(f - 1_X) \vee (-f - 1_X)]$$
$$= (|f| - 2f^\sim + 1_X) \vee (|f| - 1_X)$$
$$= |f| - [(2f^\sim - 1_X) \wedge 1_X]$$
$$= |f| - 2(f^\sim \wedge 1_X) + 1_X.$$
It follows that

\[ |f + 1_X| \land g = (|f| - 2 (f^- \land 1_X) + 1_X) \land g \]

\[ = |f| - 2 (f^- \land 1_X) + (g - |f| + 2 (f^- \land 1_X)) \land 1_X, \]

which is the desired equality. ■

**Lemma 3.2** Let \( L \) be a truncated vector sublattice of \( \mathbb{R}^X \) and \( \psi \) be truncation homomorphism on \( L \). Then

\[ |f + \lambda| \land g \in L \quad \text{and} \quad \psi (|f + \lambda| \land g) = |\psi (f) + \lambda| \land \psi (g) \]

hold for all \( f, g \in L \) and \( \lambda \in \mathbb{R} \).

**Proof.** Let \( f, g \in L \) and \( \lambda \in \mathbb{R} \). In view of Theorem 2.3, we have nothing to prove if \( \lambda = 0 \). So, assume that \( \lambda > 0 \). For the sake of brevity, we put

\[ u = \frac{1}{\lambda} f \quad \text{and} \quad v = \frac{1}{\lambda} g. \]

Notice that \( u, v \in L \). Using Lemma 3.1 we get

\[ |f + \lambda| \land g = \lambda (|u + 1_X| \land v) \in L. \]

Furthermore, Theorem 2.3 together with Lemma 3.1 yields that

\[ \psi (|u + 1_X| \land v) = \psi (|u| - 2 (u^- \land 1_X) + (v - |u| + 2 (u^- \land 1_X)) \land 1_X) \]

\[ = |\psi (u)| - 2 (\psi (u^-) \land 1) + 1 \land (\psi (v) - |\psi (u)| + 2 (\psi (u^-) \land 1)) \]

\[ = (|\psi (u) + 1| \land \psi (v)) = \frac{1}{\lambda} (|\psi (f) + \lambda| \land \psi (g)). \]

Thus,

\[ \psi (|f + \lambda| \land g) = |\psi (f) + \lambda| \land \psi (g). \]

Now, suppose that \( \lambda < 0 \). By the positive case, we have

\[ |f + \lambda| \land g = |f - \lambda| \land g \in L \]

and, analogously,

\[ \psi (|f + \lambda| \land g) = \psi (|f - \lambda| \land g) \]

\[ = |\psi (f) - \lambda| \land \psi (g) \]

\[ = |\psi (f) + \lambda| \land \psi (g). \]
This completes the proof of the lemma. ■

At this point, let \( n \in \{1, 2, \ldots \} \). A linear functional \( \psi \) on a vector subspace \( L \) of \( \mathbb{R}^X \) is called an \( \overline{\text{m}} \)-evaluation if, for every \( f_1, \ldots, f_n \in L \) and every \( \varepsilon \in (0, \infty) \), there exists \( x \in X \) such that

\[
|\psi(f_k) - f_k(x)| \leq \varepsilon \quad \text{for all} \ k \in \{1, \ldots, n\}.
\]

We have gathered now all the ingredients we need to prove the central theorem of this section.

**Theorem 3.3** Let \( L \) be a truncated vector sublattice of \( \mathbb{R}^X \) and \( \psi \) be a linear functional on \( L \). Then the following are equivalent.

(i) \( \psi \) is a truncation homomorphism on \( L \).

(ii) \( \psi \) is an \( \overline{\text{m}} \)-evaluation on \( L \) for all \( n \in \{1, 2, \ldots \} \).

(iii) \( \psi \) is an \( \underline{\text{m}} \)-evaluation on \( L \).

(iv) There exists a net \((x_\sigma)\) of elements of \( X \) such that

\[
\psi(f) = \lim f(x_\sigma) \text{ in } \mathbb{R} \quad \text{for all } f \in L.
\]

**Proof.** First, observe that the implications (ii) \( \Rightarrow \) (iii) and (iv) \( \Rightarrow \) (i) are obvious. The other implications are quite involved.

(i) \( \Rightarrow \) (ii) Let \( n \in \{1, 2, \ldots \} \) and choose \( e \in L \) such that \( \psi(e) \neq 0 \). Using Theorem 2.3, we get

\[
\psi(|e|) = |\psi(e)| > 0.
\]

So, by replacing \( e \) by \( |e| \wedge 1_X \) (if needed), we can assume that \( 0 \leq e \leq 1_X \) in \( \mathbb{R}^X \). Let \( \varepsilon \in (0, \infty) \) and put \( \theta = \min \{1, \varepsilon\} \). Given \( f_1, f_2, \ldots, f_n \in L \), we define

\[
h = \frac{1}{\theta} \left( e \wedge \bigvee_{k=1}^n |f_k - \psi(f_k)| \right) \in \mathbb{R}^X.
\]

From Lemma 3.2 it follows that \( h \in L \). Moreover, as easy calculation based on Theorem 2.3 and Lemma 3.2 yields that \( \psi(h) = 0 \). It follows that

\[
\psi(h - e) = -\psi(e) < 0.
\]
Therefore, there exists \( x \in X \) such that \( h(x) - e(x) < 0 \) (because \( \psi \) is positive). We derive that

\[
e(x) \wedge \bigvee_{k=1}^n |f_k(x) - \psi(f_k)| = \theta h(x) < \theta e(x).
\]

Since \( \theta \in (0, 1] \) and \( 0 < e(x) \leq 1 \), we obtain

\[
\bigvee_{k=1}^n |f_k(x) - \psi(f_k)| \leq \theta e(x) \leq \theta \leq \varepsilon
\]

and (ii) follows.

(iii) \( \Rightarrow \) (i) Let \( f \in L \) and observe that if \( \varepsilon \in (0, \infty) \) then there exists \( x \in X \) such that

\[
|(1_X \wedge f)(x) - \psi(1_X \wedge f)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |f(x) - \psi(f)| \leq \frac{\varepsilon}{2}.
\]

By the classical Birkhoff’s Inequality (see, e.g., Theorem 1.9 (ii) in [1]), we derive that

\[
|1 \wedge \psi(f) - \psi(1_X \wedge f)| = |(1_X \wedge f)(x) - 1 \wedge \psi(f)|
+ |(1_X \wedge f)(x) - \psi(1_X \wedge f)|
\leq 1 \wedge |f(x) - \psi(f)| + \frac{\varepsilon}{2} \leq 1 \wedge \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrary in \( (0, \infty) \), we conclude that \( \psi(1_X \wedge f) = 1 \wedge \psi(f) \). This means that \( \psi \) is a truncation homomorphism on \( L \), as required.

(ii) \( \Rightarrow \) (iv) First, assume that \( L \) separates the points of \( X \) and define a map \( J : X \to \mathbb{R}^L \) by

\[
J(x) = (f(x))_{f \in L} \quad \text{for all} \quad x \in X.
\]

By the separation condition, the map \( J \) is one-to-one and thus \( X \) can be considered as a subset of \( \mathbb{R}^L \). Let \( \mathbb{R}^L \) be endowed with its usual Tychonoff product topology and put \( \omega = (\psi(f))_{f \in L} \). Denote by \( \Omega \) a neighborhood of \( \omega \) in \( \mathbb{R}^L \). There exists \( \varepsilon \in (0, \infty) \) and a non-empty finite subset \( A \) of \( L \) such that

\[
\omega \in \prod_{f \in L} \Omega_f \subset \Omega.
\]
where
\[ \Omega_f = \mathbb{R} \text{ if } f \notin A \quad \text{and} \quad \Omega_f = (\psi (f) - \alpha, \psi (f) + \varepsilon) \text{ if } f \in A. \]

By Theorem 3.3, there exists \( x \in X \) such that
\[ |\psi (f) - f (x)| < \varepsilon \text{ for all } f \in A. \]

This yields that \( x \in \Omega \cap X \) and that \( \omega \in \overline{X} \), where \( \overline{X} \) is the closure of \( X \) in \( \mathbb{R}^L \). It follows that there exists a net \( (x_\sigma) \) in \( X \) converging to \( \omega \), i.e.,
\[ \lim x_\sigma = (\psi (f))_{f \in L} \text{ in } \mathbb{R}^L. \]

Choose \( f \in L \) and use the continuity of the projection \( \pi_f : \mathbb{R}^L \to \mathbb{R} \) defined by
\[ \pi_f (\psi) = \psi (f) \text{ for all } \psi \in \mathbb{R}^L \]
to write
\[ \lim f (x_\sigma) = \lim \pi_f (x_\sigma) = \pi_f (\lim x_\sigma) = \psi (f). \]

Now, we discuss the general case. An equivalence relation \( \sim \) can be defined on \( X \) by putting \( x \sim y \) if and only if \( f (x) = f (y) \) for all \( f \in L \). The set of all equivalence classes \( \tilde{x} \) is denoted by \( \tilde{X} \). Define a map \( T \) from \( L \) into \( \mathbb{R}^{\tilde{X}} \) by putting
\[ T (f) (\tilde{x}) = f (x) \text{ for all } f \in L \text{ and } x \in X. \]

Clearly, \( T \) is well-defined and it is linear. Moreover, if \( f \in L \) and \( x \in X \) then
\[ T (f \wedge 1_X) (\tilde{x}) = (f \wedge 1_X) (x) = f (x) \wedge 1 = T (f) (\tilde{x}) \wedge 1 = (T (f) \wedge 1_{X^\sim}) (\tilde{x}). \]

Define a map \( \tilde{\psi} \) from \( T (L) \) to \( \mathbb{R} \) by
\[ \tilde{\psi} (T (f)) = \psi (f) \text{ for all } f \in L. \]

This map is obviously well-defined and it is a truncation homomorphism on \( T (L) \), which is a truncated vector sublattice of \( \mathbb{R}^{\tilde{X}} \). Since \( T (L) \) separates the points of \( \tilde{X} \), the first case guaranties the existence of a net \( (x_\sigma) \) in \( X \) such that
\[ \psi (f) = \tilde{\psi} (T (f)) = \lim (T f) (\tilde{x_\sigma}) = \lim f (x_\sigma). \]

This completes the proof of theorem. \( \blacksquare \)
It should be pointed out that Theorem 3.3 provides the optimal evaluating characterization of truncation homomorphisms. Indeed, consider the linear form \( \psi \) defined on the unital vector sublattice \( C([0, 1]) \) of \( \mathbb{R}^{[0,1]} \) by

\[ \psi(f) = \int_0^1 f(x) \, dx \quad \text{for all } f \in C([0, 1]). \]

By the first Mean Value Theorem for Definite Integrals, we see that \( \psi \) is a \( T \)-evaluation. However, \( \varphi \) is far from being a truncation homomorphism since it is not a lattice homomorphism.

Recall at this point that \( \mathbb{R}^X \) is also an associative algebra with respect to the pointwise product. Also, recall that a linear functional \( \psi \) on a subalgebra \( A \) of \( \mathbb{R}^X \) is called an algebra homomorphism if

\[ \psi(fg) = \psi(f) \psi(g) \quad \text{for all } f, g \in A. \]

The last result of this section extends the equivalence (i) \( \Leftrightarrow \) (ii) of [8, Lemma 2.3] in two directions (the \( C(X) \)-case is well-known and can be found, for instance, in [3] or [6]). On the one hand, the subalgebra under consideration is not assumed to contains \( 1_X \) and, on the other hand, functions in this subalgebra need not be continuous (no topology is involved).

**Corollary 3.4** Let \( A \) be a truncated vector sublattice and a subalgebra of \( \mathbb{R}^X \). A linear functional \( \psi \) on \( A \) is a Stone homomorphism if and only if \( \psi \) is a positive algebra homomorphism.

**Proof.** The ‘only if’ part follows immediately from the implication (i) \( \Rightarrow \) (iv) in Theorem 3.3. Conversely, suppose that \( \psi \) is a positive algebra homomorphism. We claim that \( \psi \) is a truncation homomorphism. To this end, we shall use Theorem 2.4. First, pick \( f \in A \) and observe that

\[ \psi(|f|)^2 = \psi(|f|^2) = \psi(f^2) = \psi(f)^2. \]

Since \( \psi \) is positive, \( \psi(|f|) \geq 0 \) and thus \( \psi(|f|) = |\psi(f)| \). We conclude that \( \psi \) is a lattice homomorphism. Now, let \( f \in A \) such that \( f \leq 0 \). Since \( \psi \) is positive, \( \psi(f) \leq 0 \). Moreover, if \( 0 \leq f \leq 1 \) then \( f^2 \leq f \) from which it follows that

\[ 0 \leq \psi(f)^2 \leq \psi(f^2) \leq \psi(f). \]
Hence, either $\psi(f) = 0$ or $\psi(f) \leq 1$. In summary, if $f \in A$ with $f \leq 1$ then $\psi(f) \leq 1$. We derive that the supremum

$$a = \sup \{ \psi(f) : f \in A \text{ and } f \leq 1 \}$$

exists in $\mathbb{R}^+$. Clearly,

$$a = \sup \{ \psi(f) : f \in A \text{ and } 0 \leq f \leq 1 \}$$

and so

$$a^2 = \sup \{ \psi(f)^2 : f \in A \text{ and } 0 \leq f \leq 1 \}$$

$$= \sup \{ \psi(f^2) : f \in A \text{ and } 0 \leq f \leq 1 \}$$

$$= \sup \{ \psi(f) : f \in A \text{ and } 0 \leq f \leq 1 \} = a.$$ 

We derive that $a = 0$ or $a = 1$. If $a = 0$ then, obviously, $\psi = 0$ which is not the case. Thus $a = 1$ and the corollary follows.

4 Continuous case

In order to avoid unnecessary repetition we will assume throughout this section that $X$ is a Tychonoff space. Any truncated vector sublattice of $\mathbb{R}^X$ which is contained in $C(X)$ is called a truncated vector sublattice of $C(X)$. In the first result of this section, we shall prove that any truncation homomorphism on a truncated vector sublattice of $C(X)$ is an evaluation at some point of the Stone-Čech compactification $\beta X$ of $X$. The unital version of this representation theorem has been obtained with a completely different approach used by Garrido and Jaramillo in [8, 9]. Still, we need to recall that any $f \in C(X)$ can be extended uniquely to a continuous function $f^\beta$ from $\beta X$ into the one-point compactification $\mathbb{R} \cup \{ \infty \}$ of $\mathbb{R}$.

**Theorem 4.1** Let $L$ be a truncated vector sublattice of $C(X)$. A nonzero linear functional $\psi$ on $L$ is a truncation homomorphism if and only if there exists $u \in \beta X$ such that

$$\psi(f) = f^\beta(u) \quad \text{for all } f \in L.$$
Proof. The ‘if’ part being obvious, we prove the ‘only if’ part. Assume that $\psi$ is a truncation homomorphism. By Theorem 3.3, there exists a net $(x_\sigma)$ in $X$ such that

$$\lim f(x_\sigma) = \psi(f) \text{ in } \mathbb{R} \text{ for all } f \in L.$$ 

Replacing if necessary $(x_\sigma)$ by a subnet, we may suppose that $(x_\sigma)$ converges to some $u \in \beta X$. Take $f \in L$ and observe that

$$f^\beta(u) = f^\beta(\lim x_\sigma) = \lim f^\beta(x_\sigma) = \lim f(x_\sigma) = \varphi(f).$$

This completes the proof. ■

As is well known, a subset $L$ of $C(X)$ is said to separate points from closed sets if whenever $F$ is a closed set in $X$ and $x \notin F$, then $f(x) \notin f(F)$ for some $f \in L$. Here, $f(F)$ denotes the closure of $f(F)$ in $\mathbb{R}$. Such a subset $L$ determines the topology of $X$, meaning that the topology of $X$ coincides with the weak topology induced by $L$. Moreover, $X$ turns out to be completely regular and so a Tychonoff space. In particular, if $L$ separates points and closed sets in $X$, then a net $(x_\sigma)$ of elements of $X$ converges to some $x \in X$ if and only if, for every $f \in L$, the net $(f(x_\sigma))$ converges in $\mathbb{R}$ to $f(x)$. On the other hand, a truncated vector sublattice $L$ of $C(X)$ is said to be $L$-realcompact if any truncation homomorphism $\psi$ on $L$ is a point-evaluation on $L$, that is, there exists $x \in X$ such that

$$\psi(f) = f(x) \text{ for all } f \in L.$$ 

The following result is a consequence of the previous theorem.

**Corollary 4.2** Let $L$ be a truncated vector sublattice of $C(X)$ which separates points and closed sets. Then the following are equivalent.

(i) $X$ is $L$-realcompact.

(ii) A net $(x_\sigma)$ in $X$ converges in $X$ if and only if the net $(f(x_\sigma))$ converges in $\mathbb{R}$ for every $f \in L$.

(iii) For every $u \in \beta X \setminus X$ there exists $f \in L$ such that $f^\beta(u) = \infty$.

Proof. (i) $\Rightarrow$ (ii) Assume that $X$ is $L$-realcompact and pick a net $(x_\sigma)$ in $X$ such that $\lim f(x_\sigma)$ exists in $\mathbb{R}$ for every $f \in L$. Define $\psi : L \to \mathbb{R}$ by putting

$$\psi(f) = \lim f(x_\sigma) \text{ for all } f \in L.$$
A short moment’s thought reveals that $\psi$ is a truncation homomorphism on $L$. Hence, there exists $x \in X$ such that

$$\psi(f) = f(x) \quad \text{for all } f \in L.$$  

Since $L$ separates points and closed sets, the net $(x_\sigma)$ converges to $x$ in $X$.

(ii) $\Rightarrow$ (iii) Arguing by contradiction, assume that there is some $u \in \beta X \setminus X$ for which

$$f^\beta(u) \in \mathbb{R} \quad \text{for all } f \in L.$$  

By density, there exists a net $(x_\sigma)$ in $X$ which converges to $u$. Hence, if $f \in L$ then

$$\lim f(x_\sigma) = \lim f^\beta(x_\alpha) = f^\beta(u) \in \mathbb{R}.$$  

In other words, the net $(f(x_\sigma))$ converges in $\mathbb{R}$ and so, by (ii), the net converges in $X$. This yields that $u \in X$, a contradiction.

(iii) $\Rightarrow$ (i) Let $\psi$ be a truncation homomorphism on $L$. By Theorem 4.1, there exists $u \in \beta X$ such that

$$\psi(f) = f^\beta(u) \quad \text{for all } f \in L.$$  

In particular,

$$f^\beta(u) \in \mathbb{R} \quad \text{for all } f \in L.$$  

By (iii), the element $u$ must be in $X$, which leads to the conclusion. ■

Now we are close to completing the paper, again with a result that gives a sufficient condition on $L$ for $X$ to be $L$-realcompact. We need first to recall that if $\mathbb{R}^I$ is a product of real lines equipped with its Tychonoff product topology the, for every $i \in I$, the projection $\pi_i : \mathbb{R}^I \to \mathbb{R}$ defined by

$$\pi_i((x_j)) = x_i \quad \text{for all } (x_j) \in \mathbb{R}^I$$

is continuous.

**Corollary 4.3** Suppose that $X$ is a closed set in an appropriate Tychonoff product space $\mathbb{R}^I$ and let $L$ be a truncated vector sublattice of $C(X)$ such that $\pi_i \in L$ for all $i \in I$. The $X$ is $L$-realcompact.

**Proof.** Since $L$ contains the projections, then $L$ separates points and closed sets. Hence, we can apply the previous corollary. Let $(x_\sigma)$ be a net such that $(f(x_\sigma))$ converges in $\mathbb{R}$ for all $f \in L$. In particular, for every $i \in I$, the net $(\pi_i(x_\sigma))$ converges in $\mathbb{R}$, say to $\ell_i$. But then $(x_\sigma)$ converges in $\mathbb{R}^I$ to $x = (\ell_i)$. Since $X$ is closed in $\mathbb{R}^I$ we derive that $x \in X$ and (ii) in Corollary 4.2 allows us to conclude. ■
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