ON THE COMPLETENESS OF THE SYSTEM OF ROOT VECTORS FOR FIRST-ORDER SYSTEMS

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Abstract. The paper is concerned with the completeness problem of root functions of general boundary value problems for first order systems of ordinary differential equations. Namely, we introduce and investigate the class of weakly regular boundary conditions. We show that this class is much broader than the class of regular boundary conditions introduced by G.D. Birkhoff and R.E. Langer. Our main result states that the system of root functions of a boundary value problem is complete and minimal provided that the boundary conditions are weakly regular. Moreover, we show that in some cases the weak regularity of boundary conditions is also necessary for the completeness. Also we investigate the completeness for 2 × 2 Dirac and Dirac type equations subject to irregular or even to degenerate boundary conditions.

We emphasize that our results are the first results on the completeness problem for general first order systems even in the case of regular boundary conditions.

1. Introduction

Spectral theory of non-selfadjoint boundary value problems (BVP) for nth order ordinary differential equations (ODE)
\[ y^{(n)} + q_1 y^{(n-2)} + \ldots + q_{n-1} y = \lambda^n y \] (1.1)
on a finite interval \( I = (a, b) \) takes its origin in the classical papers by Birkhoff [2], [3] and Tamarkin [41], [42], [43]. They introduced the concept of regular boundary conditions (BC) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of such problems for ODE. Moreover, they proved that the system of root functions, i.e. eigenfunctions and associated functions (EAF) of the regular BVP is complete. Their results are also treated in classical monographs (see, for instance, [36, Section 2] and [14, Chapter 19]).

However, some natural and important boundary conditions are not regular. For instance, a boundary value problem with separated boundary conditions is regular if and only if \( n = 2l \), where \( l \) is the number of boundary conditions at the left (right) endpoint of the interval \( I \). Note that the completeness of EAF of boundary value problems with an arbitrary separated BC was stated (without proof) much later by M.V. Keldysh in his famous communication [20]. However, the proof of this result was first appeared in the paper by A.A. Shkalikov [38]. The completeness property of other non-regular BVP for nth order ordinary differential equations on \([0, 1]\) has been studied by A.G. Kostyuchenko and A.A. Shkalikov [23], A.P. Khromov [22], V.S. Rykhlov and many others.

On the other hand, V.P. Mihailev [34] and G.M. Keselman [21] independently proved that the system of EAF of a boundary value problem for equation (1.1) forms a Riesz basis provided that the boundary conditions are strictly regular. Similar
results are also obtained in [14, Chapter 19.4]. Moreover, for boundary conditions which are regular but not strictly regular, A.A. Shkalikov [39], [40] proved that the system of EAF forms a Riesz basis of subspaces.

In this paper we consider first order systems of ODE of the form

\[ Ly := L(Q)y := \frac{1}{i} B \frac{dy}{dx} + Q(x)y = \lambda y, \quad y = \text{col}(y_1, ..., y_n), \]

(1.2)

where \( B \) is a non-singular diagonal \( n \times n \) matrix,

\[ B = \text{diag}(b_1^{-1}I_{n_1}, ..., b_r^{-1}I_{n_r}) \in \mathbb{C}^{n \times n}, \quad n = n_1 + ... + n_r, \]

(1.3)

with complex entries satisfying \( b_j \neq b_k \) for \( j \neq k \), and \( Q(\cdot) \) is a potential matrix. We also assume that \( Q(\cdot) \in L^2([0, 1]; \mathbb{C}^{n \times n}) \). In the sequel we consider its block-matrix representation

\[ Q = (d_{jk})_{j,k=1}^{n_1 \oplus ... \oplus n_r} \]

with respect to the orthogonal decomposition \( \mathbb{C}^n = \mathbb{C}^{n_1} \oplus ... \oplus \mathbb{C}^{n_r} \). With the system (1.2) one associates, in a natural way, the maximal operator \( L = L(Q) \) acting in \( L^2([0, 1]; \mathbb{C}^n) \) on the domain \( \text{dom}(L) = W^1_2([0, 1]; \mathbb{C}^n) \).

Note that, systems form a more general object than ordinary differential equations. Namely, the \( n \)th-order differential equation (1.1) can be reduced to the system (1.2) with \( r = n \) and \( b_j = \exp(2\pi ij/n) \) (see [27]). The systems (1.2) are of significant interest in some theoretical and practical questions. For instance, if \( n = 2m \), \( B = \text{diag}(I_m, -I_m) \) and \( q_{11} = q_{22} = 0 \), the system (1.2) is equivalent to the Dirac system [25], [31]. Note also that equation (1.2) is used to integrate the problem of \( N \) waves arising in the nonlinear optics [37].

To obtain a BVP, we adjoin to equation (1.2) the following boundary conditions

\[ Cy(0) + Dy(1) = 0, \quad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n \times n}. \]

(1.4)

We denote by \( L_{C,D} := L_{C,D}(Q) \) the operator associated in \( L^2([0, 1]; \mathbb{C}^n) \) with the BVP (1.2)–(1.4). It is defined as the restriction of \( L = L(Q) \) to the domain

\[ \text{dom}(L_{C,D}) = \{ y \in W^1_2([0, 1]; \mathbb{C}^n) : Cy(0) + Dy(1) = 0 \}. \]

(1.5)

Moreover, in what follows we always impose the maximality condition

\[ \text{rank}(CD) = n, \]

(1.6)

or equivalently

\[ \ker(CC^* + DD^*) = \{0\}. \]

Apparently, the spectral problem (1.2)–(1.4) has first been investigated by G. D. Birkhoff and R. E. Langer [4]. Namely, they have extended some previous results of Birkhoff and Tamarkin on non-selfadjoint BVP for ODE to the case of BVP (1.2)–(1.4). More precisely, they introduced the concepts of regular and strictly regular boundary conditions (1.4) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding BVP (the operator \( L_{C,D} \)). Moreover, they proved a pointwise convergence result on spectral decompositions of the operator \( L_{C,D} \) corresponding to the BVP (1.2)–(1.4).

However, to the best of our knowledge the problem of the completeness of the root system of a general BVP (1.2)–(1.4) has not been investigated yet. Some results in this direction were known only for the case of Dirac systems. The present paper presents the first results in this direction. More precisely, we introduce the concept of weakly regular BC for the system (1.2) and establish the completeness of EAF for this class of BVP (note that this class contains boundary conditions which are regular in the sense of [4]).
To state the main results, we need to the following construction. Let $A = \text{diag}(a_1, \ldots, a_n)$ be a diagonal matrix with entries $a_k$ (not necessarily distinct) that are not lying on the imaginary axis, $\Re a_k \neq 0$. Starting with arbitrary matrices $C, D \in \mathbb{C}^{n \times n}$, we define the auxiliary matrix $T_A(C, D) \in \mathbb{C}^{n \times n}$ as follows:

- if $\Re a_k > 0$, then the $k$th column in the matrix $T_A(C, D)$ coincides with the $k$th column of the matrix $C$,
- if $\Re a_k < 0$, then the $k$th column in the matrix $T_A(C, D)$ coincides with the $k$th column of the matrix $D$.

It is clear that $T_A(C, D) = T_{-A}(D, C)$.

Let us recall the definition of regular boundary conditions from [4]. Consider the lines $l_j := \{ \lambda \in \mathbb{C} : \Re(ib_j, \lambda) = 0 \}, j \in \{1, 2, \ldots, r\}$, of the complex plane. The lines $l_j$ divide the complex plane in $m \leq 2r$ sectors $\sigma_1, \sigma_2, \ldots, \sigma_m$. Let $z_1, z_2, \ldots, z_m$ be complex numbers such that $iz_j$ lies in the interior of $\sigma_j$ for $j \in \{1, \ldots, m\}$. The boundary conditions (1.4) are called regular whenever

$$\det T_{z_j, B}(C, D) \neq 0, \quad j \in \{1, \ldots, m\}. \quad (1.7)$$

Note that the boundary conditions (1.4) are regular if and only if $\det T_{z, B}(C, D) \neq 0$ for every admissible $z \in \mathbb{C}$, i.e. for such $z$ that $\Re(zB)$ is non-singular.

**Definition 1.1.** The boundary conditions (1.4) are called weakly $B$-regular (or, simply, weakly regular) if there exist three complex numbers $z_1, z_2, z_3$, satisfying the following conditions:

(a) the origin is an interior point of the triangle $\triangle z_1 z_2 z_3$;
(b) $\det T_{z_j, B}(C, D) \neq 0$ for $j \in \{1, 2, 3\}$.

Now the first main result of the paper reads as follows.

**Theorem 1.2.** Let $Q \in L^2([0,1]; \mathbb{C}^{n \times n})$ and let boundary conditions (1.4) be weakly $B$-regular. Then the system of root functions of the BVP (1.2)–(1.4) (of the operator $L_{C,D}(Q)$) is complete and minimal in $L^2[0,1] \otimes \mathbb{C}^n$.

We emphasize that the class of weakly regular boundary conditions is much wider than the class of regular BC. For instance, for splitting boundary conditions (1.4) to be regular it is necessary that: (i) $n = 2k$, where $k$ is the number of conditions at zero; (ii) the matrix $\Re(zB)$ has zero signature for every admissible $z$. However, for odd $n = 2k + 1$ splitting BC with $k$ conditions at 0 are weakly $B$-regular, in general, whenever $b_j = \exp(\frac{2\pi n j}{2k+1})$ (see Example 3.6 for details). Moreover, there exist splitting irregular but weakly regular BC for $n = 2k$ too.

In the case of $B = B^*$ weak regularity of boundary conditions (1.4) is equivalent to their regularity. Moreover, denoting by $P_+$ and $P_-$ the spectral projectors onto ”positive” and ”negative” parts of the spectrum of $B = B^*$, respectively, one expresses the regularity of boundary conditions (1.4) as follows:

$$\det(CP_+ + DP_-) \neq 0 \quad \text{and} \quad \det(CP_- + DP_+) \neq 0. \quad (1.8)$$

Thus, Theorem 1.2 yields the following result.

**Corollary 1.3.** Let $Q \in L^2[0,1] \otimes \mathbb{C}^{n \times n}$, $B = B^*$ and let conditions (1.8) be satisfied. Then the system of root functions of the BVP (1.2)–(1.4) is complete and minimal in $L^2[0,1] \otimes \mathbb{C}^n$. 

In some particular cases this statement has been obtained by V.A. Marchenko [31] \((2 \times 2\) Dirac system, \(B = \text{diag}(-1, 1)\)) and V.P. Ginzburg [17] \((B = I_n, Q = 0)\) (see Remark 4.5 below).

Note that conditions (1.8) are also necessary for completeness if \(Q = 0\). However, they are no longer necessary if \(Q \neq 0\) even for \(Q = Q^*\). We demonstrate this fact in passing by stating a special case of Theorem 5.1 that gives new conditions of the completeness of irregular BVP for \(2 \times 2\) Dirac systems.

**Proposition 1.4.** Let \(B = \text{diag}(-1, 1)\), \(Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}\) and \(Q_{12}(\cdot), Q_{21}(\cdot) \in C[0, 1]\). Assume that

\[
J_{13}Q_{12}(0) - J_{42}Q_{21}(1) \neq 0, \quad J_{13}Q_{12}(1) - J_{42}Q_{21}(0) \neq 0, \quad (1.9)
\]

where \(J_{13} := \det \begin{pmatrix} c_{11} & d_{11} \\ c_{21} & d_{21} \end{pmatrix}, J_{42} := \det \begin{pmatrix} d_{12} & c_{12} \\ d_{22} & c_{22} \end{pmatrix}\). Then the system of root functions of the problem (1.2)–(1.4) is complete and minimal in \(L^2([0, 1]; C^2)\).

We emphasize that the assumptions of Proposition 1.4 depend on \(Q\) although they guarantee the completeness even if both conditions (1.8) are violated. However, these assumptions cover irregular and even degenerate BC (1.4).

In connection with Corollary 1.3 and Proposition 1.4 we mention the papers [44], [45], [19], [35] and [6], [7], [8], [9], [10], [11], [12], [13], that appeared during the last decade. Basically they are devoted to the Riesz basis property of EAF for BVP with strictly regular (and just regular) BC for \(2 \times 2\) Dirac systems. The most complete and detailed results in this direction have been obtained by P. Djakov and B. Mityagin [6], [7], [10], [11], [12], [13]. In the recent preprint [12] they proved equiconvergence and pointwise convergence of spectral decompositions of Dirac operators with regular BC. The result on pointwise convergence improves and generalizes the corresponding result from [4] for \(2 \times 2\) Dirac systems. Moreover, in [11], [13] a criterion for EAF to form a Riesz basis for periodic (resp., antiperiodic) 1D Dirac operator is established.

Let us also mention the recent papers by F. Gesztesy and V. Tkachenko [15], [16]. In particular, in [16], as well as in the recent preprint by P. Djakov and B. Mityagin [11], the authors established a criterion for eigenfunctions and associated functions to form a Riesz basis for periodic (resp., antiperiodic) Sturm–Liouville operators on \([0, 1]\). This criterion is formulated in terms of periodic (resp., antiperiodic) and Dirichlet eigenvalues.

Note also that using the approach from [33] Theorems 1.2 and 5.1 can be applied for the study of uniqueness of mixed BVP for first order systems of partial differential equations.

The paper is organized as follows. In Section 2 we present a result on asymptotic behavior of solutions of equation (1.2) as \(\lambda \to \infty\). This result generalizes the classical Birkhoff result [2] (see also [36]) and completes the result from [4].

In Section 3 we present the proof of Theorem 1.2. We also prove here (see Corollary 3.2) that if the BC are weakly regular, then the system of root functions of the adjoint operator \(L_{C,D}^*\) is complete and minimal too. Besides, we present here some examples of irregular BC that are weakly regular. In particular, we show that under very weak assumptions the splitting BC are weakly regular.

In Section 4 we investigate the problem (1.2)–(1.4) with \(B = B^*\) (Dirac type systems). We prove Corollary 1.3. We also show that for dissipative (accumulative)
operators $L_{C,D}$ the first (the second) condition in (1.8) yields completeness (see Corollary 4.3). It is also proved here that in the case $Q = 0$ conditions (1.8) of (weak) regularity are necessary for completeness.

In Section 5 we investigate boundary value problems for $2 \times 2$ Dirac type systems $(B = B^*)$ and present other sufficient conditions of the completeness in the irregular case. In the proof of the main result of the section, Theorem 5.1, we substantially exploit triangular transformation operators that were constructed for general $n \times n$ Dirac type systems in [27]. For Dirac system we also find some necessary conditions for completeness that show, in particular, the sharpness of conditions (1.9) for the validity of Proposition 1.4 (see Proposition 5.13).

Finally, in Section 6 we investigate BVP (1.2)–(1.4) for $n = 2$ with $B = \text{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$ and complete Theorem 1.2 for this case. Namely, in Theorem 6.1 we prove completeness and minimality of the root functions of the BVP (1.2), (1.4) with $C = \begin{pmatrix} 1 & -h_0 \\ 0 & -h_1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_0 h_1 \neq 0$, when the BC (1.4) are not weakly regular. In this case completeness of the adjoint operator $L_{C,D}(Q)^*$ depends on $Q$. However, we show in Corollary 6.3 that in the case $B = \text{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$ weak $B$-regularity of boundary conditions (1.4) is equivalent to the completeness of both operators $L_{C,D}(0)$ and $L_{C,D}(0)^*$ with $Q = 0$.

The main results of the paper have been announced in [29, 30].

Notation. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $C^n$. $C^{n \times n}$ stands for the set of $n \times n$ matrices with complex entries; $I_n (\in C^{n \times n})$ stands for the unit matrix; by $\kappa_+(A)$ ($\kappa_-(A)$) we denote the number of positive (negative) eigenvalues of the selfadjoint matrix $A$.

$a_n(1)$ stands for an $n \times n$ matrix function with entries of the form $o(1)$; $[f(x)]$ stands for the function of the form $f(x)(1 + o(1))$.

2. Preliminaries

2.1. The asymptotic behavior of solutions to first-order systems. Here we present a result on the asymptotical growth of solutions to first order systems of equations (1.2). This result slightly generalizes the corresponding result from [4, p.71-87] on systems (1.2) where it was obtained under a stronger assumption $Q \in C^1[0,1] \otimes C^{m \times n}$. In turn, the latter result from [4] generalizes the classical Birkhoff theorem on n-th-order ordinary differential equation (see, for instance, [2], [36]). We present the proof for the sake of completeness. Moreover, our exposition slightly differs from that in [4] and is shorter.

To this end, we need the following lemma.

Lemma 2.1. Let $a_1, a_2, \ldots, a_r$ be different complex numbers. Then the complex plane can be divided into at most $r^2 - r$ sectors $S_p$ with vertexes at the origin and such that for any $p$ the numbers $a_j$ can be renumbered so that the following inequalities hold:

$$\Re(a_{j_1} \lambda) < \Re(a_{j_2} \lambda) < \cdots < \Re(a_{j_p} \lambda), \quad \lambda \in S_p. \quad (2.1)$$

Proof. Let $l_{jk}$ be the set of $z$ satisfying $\Re(a_j z) = \Re(a_k z)$. Then $l_{jk}$ is the line on the complex plane passing through the origin. All such lines divide the complex plane into at most $r^2 - r$ sectors. Assume that $a_j$ are ordered in such a way that inequalities (2.1) hold for a certain $\lambda_0$ lying inside a sector. In this case, since $\Re(a_{jk} \lambda) \neq \Re(a_{j0} \lambda)$ for any $\lambda$ inside the sector and all the functions $\Re(a_{j} \lambda)$,
\[ j \in \{1, 2, \ldots, r\}, \text{are continuous, it follows that the inequalities (2.1) are valid for every } \lambda \text{ from the chosen sector as well.} \]

Clearly, each of the sectors \( S_p \) is of the form \( S_p = \{ z : \varphi_{1p} < \arg z < \varphi_{2p} \} \). Fix \( p \) and denote by \( S \) the sector strictly embedded into the latter, i.e.,

\[
S := \{ z : \varphi_{1p} + \varepsilon_1 < \arg z < \varphi_{2p} - \varepsilon_2 \}, \quad \text{where } \varepsilon_1, \varepsilon_2 > 0;
\]

\[
S_R := \{ z \in S : |z| > R \}. \quad (2.2)
\]

**Proposition 2.2.** Assume that \( B = \text{diag}(b_1^{-1}I_{n_1}, b_2^{-1}I_{n_2}, \ldots, b_r^{-1}I_{n_r}) \) is a nonsingular diagonal \( n \times n \) matrix with \( b_j \neq b_k \) for \( j \neq k \), and \( Q(x) = (q_{jk}(x))_{j,k=1} \) where \( q_{jk}(:) \in L^1[0,1] \otimes \mathbb{C}^{n_j \times n_k} \) and \( q_{jk}(\cdot) \equiv 0, \ j \in \{1, 2, \ldots, r\} \). Further, let \( S \) be the sector of the form (2.2). Then the numbers \( \{ib_j\}^r_1 \) can be renumbered with respect to the sector \( S \) in accordance with (2.1), i.e.,

\[
\Re(ib_j, \lambda) < \Re(ib_{j+1}, \lambda) < \cdots < \Re(ib_r, \lambda), \quad \lambda \in S. \quad (2.3)
\]

Moreover, for a sufficiently large \( R \), equation (1.2) has the fundamental system of matrix solutions

\[
Y_k(x; \lambda) = \begin{pmatrix}
y_{1k}(x; \lambda) \\
y_{2k}(x; \lambda) \\
\vdots \\
y_{nk}(x; \lambda)
\end{pmatrix}, \quad y_{jk}(\cdot; \lambda) : [0,1] \to \mathbb{C}^{n_j \times n_k}, \quad k \in \{1, 2, \ldots, r\}, \quad (2.4)
\]

which is analytic with respect to \( \lambda \in S_R \) and has the asymptotic behavior (uniformly in \( x \))

\[
y_{jk}(x; \lambda) = (I_{n_k} + o(1))e^{ib_k\lambda x}, \quad \lambda \in S_R,
\]

\[
y_{jk}(x; \lambda) = o(1)e^{ib_k\lambda x}, \quad \lambda \in S_R, \quad \text{for } j \neq k. \quad (2.5)
\]

**Proof.** The first statement is immediate from Lemma 2.1. Without loss of generality we assume that \( b_{jk} = b_k \), \( k \in \{1, 2, \ldots, r\} \). Besides for simplicity, we restrict ourselves to the case of the matrix \( B \) with simple spectrum, i.e., assume that \( n_k = 1 \) for \( k \in \{1, 2, \ldots, r\} \). In this case, \( r = n \), and \( Y_k(x; \lambda) \) is the vector column with the components \( y_{jk}(x; \lambda), \ j \in \{1, 2, \ldots, n\} \).

Denote \( \tilde{q}_{jl}(t) = -ib_jq_{jl}(t) \). It is easy to check that, for every fixed \( k \in \{1, 2, \ldots, r\} \), a solution of the system of integral equations

\[
\begin{cases}
y_{jk}(x; \lambda) = \int_0^x e^{ib_j\lambda(x-t)} \sum_{l=1}^r \tilde{q}_{jl}(t)y_{lk}(t; \lambda) \, dt \quad \text{for } j < k, \\
y_{jk}(x; \lambda) = \int_0^x e^{ib_j\lambda(x-t)} \sum_{l=1}^r \tilde{q}_{jl}(t)y_{lk}(t; \lambda) \, dt + e^{ib_j\lambda x} \quad \text{for } j = k, \quad (2.6) \\
y_{jk}(x; \lambda) = -\int_x^1 e^{ib_j\lambda(x-t)} \sum_{l=1}^r \tilde{q}_{jl}(t)y_{lk}(t; \lambda) \, dt \quad \text{for } j > k,
\end{cases}
\]

is the solution to the system (1.2) as well.

Let us verify that system (2.6) has a unique solution for sufficiently large absolute values of \( \lambda \in S \), and this solution satisfies conditions (2.5). Introduce new functions \( z_{jk}(x; \lambda) \) by setting

\[
z_{jk}(x; \lambda) := e^{-ib_k\lambda x}y_{jk}(x; \lambda), \quad j, k \in \{1, 2, \ldots, r\}. \quad (2.7)
\]
Then the $k$-th equation in the system (2.6) yields

$$z_{kk}(x; \lambda) = 1 + \int_0^x \sum_{j=1}^r \tilde{q}_{kj}(t)z_{jk}(t; \lambda) \, dt.$$  

(2.8)

By substituting expressions (2.7) and (2.8) into the system (2.6) we obtain

$$
\begin{cases}
z_{jk}(x; \lambda) = \int_0^x \tilde{q}_{jk}(t)e^{i(b_j-b_k)x}dt + \sum_{1 \leq l \leq r} t \int_0^x e^{i(b_j-b_k)x}dt \\
	imes \left( x \tilde{q}_{jl}(t)z_{lk}(t; \lambda) + \tilde{q}_{jk}(t) \int_0^t \tilde{q}_{kl}(\tau)z_{lk}(\tau; \lambda) \, d\tau \right) dt, & j < k \\
\end{cases}
$$

(2.9)

$$
\begin{cases}
z_{jk}(x; \lambda) = -\int_x^0 \tilde{q}_{jk}(t)e^{i(b_j-b_k)x}dt - \sum_{1 \leq l \leq r} t \int_0^x e^{i(b_j-b_k)x}dt \\
\times \left( x \tilde{q}_{jl}(t)z_{lk}(t; \lambda) + \tilde{q}_{jk}(t) \int_0^t \tilde{q}_{kl}(\tau)z_{lk}(\tau; \lambda) \, d\tau \right) dt, & j > k
\end{cases}
$$

where the prime over a sum means that the summation is taken over $l \neq k$.

We put

$$u_{jk}(x; \lambda) = \begin{cases}
\int_0^x e^{i(b_j-b_k)x}dt, & j < k, \\
-\int_x^0 e^{i(b_j-b_k)x}dt, & j > k.
\end{cases}$$

(2.10)

Further, let

$$A_{jk}(\lambda)f(x) :=
\begin{cases}
\int_0^x e^{i(b_j-b_k)x}dt \left( \tilde{q}_{jl}(t)f(t) + \tilde{q}_{jk}(t) \int_0^t \tilde{q}_{kl}(\tau)f(\tau) \, d\tau \right) dt, & j < k \\
-\int_x^0 e^{i(b_j-b_k)x}dt \left( \tilde{q}_{jl}(t)f(t) + \tilde{q}_{jk}(t) \int_0^t \tilde{q}_{kl}(\tau)f(\tau) \, d\tau \right) dt, & j > k.
\end{cases}$$

(2.11)

Clearly, $A_{jk}(\cdot) : C[0, 1] \rightarrow C[0, 1]$ forms the family of continuous operators depending on $\lambda$ analytically. Moreover, due to inequalities (2.3), $\|A_{jk}(\lambda)\| = o(1)$ for $\lambda \in S$, $\lambda \to \infty$.

The system (2.9) can be rewritten in the form

$$z_{jk}(x; \lambda) = u_{jk}(x; \lambda) + \sum_{1 \leq l \leq n} A_{jk}(\lambda)z_{lk}(t; \lambda), \quad j \neq k.$$  

(2.12)

Applying the method of successive approximations in the space $C[0, 1] \otimes C^r$ to system (2.12) and using the relation $\|A_{jk}(\lambda)\| = o(1)$ we conclude that, for sufficiently large $|\lambda|$, $\lambda \in S$, the system (2.12) has unique solution. Furthermore, the functions $z_{jk}(x; \lambda)$ are analytic with respect to $\lambda \in S$, and the following relations hold uniformly in $x \in [0, 1]$

$$z_{jk}(x; \lambda) = u_{jk}(x; \lambda)(1 + o(1)), \quad \lambda \in S, \quad \lambda \to \infty, \quad j \in \{1, 2, \ldots, n\}, \quad j \neq k$$  

(2.13)

The proof of this fact is similar to that of [36, Lemma 4.4.1]. Taking account of the relations $u_{jk}(x; \lambda) = o(1)$ as $\lambda \to \infty$, (2.13) can be rewritten as

$$z_{jk}(x; \lambda) = o(1), \quad \text{for} \quad \lambda \in S, \quad \lambda \to \infty, \quad j \neq k.$$  

(2.14)
By substituting (2.14) into (2.8) we obtain
\[ z_{kk}(x; \lambda) = 1 + o(1), \quad \lambda \in S, \quad \lambda \to \infty. \] (2.15)

Next by substituting both (2.14) and (2.15) in (2.7) we arrive at (2.5).

It remains to note that, due to (2.5) for \( x = 0 \), we have
\[ Y(x; \lambda) = (y_{jk}(x; \lambda))_{j,k=1}^n = I_n + o_n(1). \] (2.16)

Hence the system of solutions \( Y_k(x; \lambda) \) is linearly independent for \( \lambda \in S_R \) with sufficiently large \( R \).

**Remark 2.3.** Replacing the condition \( q_{jl} \in L^1(0,1) \) by the stronger condition \( q_{jl} \in L^\infty(0,1) \), we arrive at the stronger estimate
\[ y_{jk}(x; \lambda) = \left( \delta_j^k + O \left( \frac{1}{|\lambda|} \right) \right) e^{|\theta_k^j\lambda x|}, \quad \lambda \in S, \quad \lambda \to \infty. \] (2.17)

However, the estimate (2.17) is false in general if only \( Q \in L^1(0,1) \otimes \mathbb{C}^{n \times n} \). For instance, consider the \( 2 \times 2 \) system (1.2) with \( B = \text{diag}(i, -i) \)
\[ \begin{aligned}
  y_1'(x; \lambda) &= \lambda y_1(x; \lambda) \\
  y_2'(x; \lambda) &= -\lambda y_2(x; \lambda) + \frac{y_2(x; \lambda)}{\sqrt{1 + x}}.
\end{aligned} \] (2.18)

Here \( q_{12} \equiv 0, q_{21} = \frac{1}{\sqrt{1 + x}} \in L^1(0,1) \). Using the estimate \( \int_0^1 e^{-2\lambda x^2} d\tau \sim \frac{1}{\sqrt{\lambda}} \) it is
difficult to show that that the estimate (2.17) is false.

2.2. The minimality property. Apparently the following statement is well known for experts. We present it with the proof for completeness.

**Lemma 2.4.** Let \( T \in \mathcal{S}_\infty(\mathcal{H}) \) and \( \ker T = \{0\} \). Then the system of EAF of the operator \( T \) is minimal.

**Proof.** Let \( \{\lambda_j\}_1^\infty \) be a system of eigenvalues of \( T \) arranged in descending order of their modulus:
\[ |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| \geq |\lambda_{k+1}| \geq \ldots > 0. \] (2.19)

Denote by \( \mathcal{H}_j(T) := \mathcal{H}_{\lambda_j}(T) \) the corresponding root subspaces of \( T \). It is easily seen that
\[ \mathcal{H}_j(T) \perp \mathcal{H}_k(T^*) \quad \text{for} \quad j \neq k. \] (2.20)

Moreover, by Fredholm theorem, \( \dim \mathcal{H}_j(T) = \dim \mathcal{H}_j(T^*) \), \( j \in \mathbb{N} \), since \( \lambda_j \neq 0 \).

Further, let \( \{e_{jp}\}_{p=1}^{n_j} \) and \( \{f_{jk}\}_{k=1}^{n_j} \) be the bases in \( \mathcal{H}_j(T) \) and \( \mathcal{H}_j(T^*) \), respectively.

Then the "Gram matrix"
\[ G_j := (\langle e_{jp}, f_{jk} \rangle)_{p,k=1}^{n_j} \]
is non-singular. Assuming the contrary we find a non-zero vector \( f = \sum_{k=1}^{n_j} a_k f_{jk} \in \mathcal{H}_j(T^*) \) which is orthogonal to \( \mathcal{H}_j(T) \). Thus, due to (2.20)
\[ f \perp \mathcal{H}_j := \text{span}\{\mathcal{H}_k(T) : k \in \mathbb{N}\}, \]
i.e. \( f \in \mathcal{H}_2 := \mathcal{H}_j^1 \). Let \( P_2 \) be an orthogonal projection on the subspace \( \mathcal{H}_2 \). By [18, Lemma 1.4.2] the operator \( T_2 = P_2TP_2 \) is veltorla operator, hence so is the adjoint operator \( T_2^* = P_2T^*P_2 \).

Since \( f \in \mathcal{H}_j(T^*) \), we can find \( k < n_j \) such that \( u := (T^* - \lambda_j)k f \neq 0 \) and \( T^*u = \lambda_j u \). Since \( \mathcal{H}_2 \) is an invariant subspace for \( T_2^* \), \( u \in \mathcal{H}_2 \) and \( T_2^*u = T^*u = \lambda_j u \) where \( \lambda_j \neq 0 \). This contradiction shows that the matrix \( G_j \) is non-singular.

Thus, the basis \( \{f_{jk}\}_{k=1}^{n_j} \) in \( \mathcal{H}_j(T^*) \) can be chosen to be biorthogonal to the basis \( \{e_{jp}\}_{p=1}^{n_j} \), i.e. to satisfy \( \langle e_{jp}, f_{jk} \rangle = \delta_{pk}, \ p, k \in \{1, \ldots, n_j\} \). Consider the
union of both systems. Then using the latter identities and (2.20) we obtain two
biorthogonal systems. Thus, the system
\[ \bigcup_{p=1}^{\infty} \{ e_{jp} \}_{p=1}^{\infty} \] is minimal. □

3. Completeness of the root functions of BVP for first
order-systems

3.1. Proof of the main result. Here we present the proof of Theorem 1.1. On
the second step we use the idea of reduction of the proof of completeness of the
BVP (1.2), (1.4) to the investigation of that for solutions to the (incomplete) Cauchy
problem. The idea of such reduction goes back to the paper by A.A. Shkalikov [38]
where it was applied to BVP for nth order differential equations.

Proof of Theorem 1.1. (i) Suppose that \( \Phi(x; \lambda) \) is a fundamental \( n \times n \)
matrix solution of equation (1.2) corresponding to the initial condition
\[ \Phi(0; \lambda) = I_n. \] (3.1)
Further, denote by \( \Phi_j(x; \lambda) \) the \( j \)th vector column of the matrix \( \Phi(x; \lambda) \), i.e.,
\[ \Phi(x; \lambda) = (\Phi_1, \ldots, \Phi_n), \quad \Phi_j(x; \lambda) = \text{col}(\varphi_{1j}, \ldots, \varphi_{nj}). \] (3.2)
It is clear that the general solution of equation (1.2) is of the form
\[ U(x; \lambda) = \sum_{j=1}^{n} \alpha_j(\lambda) \Phi_j(x; \lambda), \quad \alpha_j(\lambda) \in \mathbb{C}. \] (3.3)
By substituting (3.3) into (1.4) we derive to the equation for eigenvalues and eigen-
functions of problem (1.2), (1.4):
\[ C \sum_{j=1}^{n} \alpha_j(\lambda) \Phi_j(0; \lambda) + D \sum_{j=1}^{n} \alpha_j(\lambda) \Phi_j(1; \lambda) = \]
\[ = (C\Phi(0; \lambda) + D\Phi(1; \lambda)) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = (C + D\Phi(1; \lambda)) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0. \] (3.4)
The equation (3.4) has nontrivial solution if and only if the matrix \( A_\Phi(\lambda) := (C + 
D\Phi(1; \lambda)) \) is singular, i.e., if
\[ \Delta_\Phi(\lambda) := \det A_\Phi(\lambda) := \det(C + D\Phi(1; \lambda)) = 0. \] (3.5)
It follows that the spectrum \( \sigma(L_{C,D}) \) of problem (1.2), (1.4) coincides with the
roots of the characteristic determinant \( \Delta_\Phi(\cdot) \). In what follows we will show that
the assumption (b) of the theorem yields the nondegeneracy of the \( \Delta_\Phi(\lambda) \), i.e., the
relation \( \Delta_\Phi(\lambda) \not\equiv 0 \). Therefore, the spectrum \( \sigma(L_{C,D}) \) of problem (1.2), (1.4) is
discrete, i.e., \( \sigma(L_{C,D}) =: \{ \lambda_k \}_1^{\infty} \).

Denote by \( A_\Phi(\lambda) = (\Delta_{jk}(\lambda))_{j,k=1}^{n} \) the matrix associated to \( A_\Phi(\lambda) \), and introduce
the vector functions
\[ U_j(x; \lambda) := \sum_{k=1}^{n} \Delta_{jk}(\lambda) \Phi_k(x; \lambda), \quad j \in \{ 1, 2, \ldots, n \}. \] (3.6)
Here two cases are possible: \( U_j(x; \lambda_j) \neq 0 \) and \( U_j(x; \lambda_k) = 0 \). If \( U_j(x; \lambda_k) \neq 0 \) then
relations (3.4), (3.5) and (3.6) together imply that \( U_j(x; \lambda_k) \) is an eigenfunction of
problem (1.2), (1.4) corresponding to the eigenvalue \( \lambda_k \).
Moreover, if $\lambda_k$ is an $m_k$-multiple ($m_k > 1$) zero of the function $\Delta(\lambda) := \Delta_\varphi(\lambda)$, then the vector functions
\[
\frac{1}{p!} D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k} := \frac{1}{p!} \frac{\partial^p}{\partial \lambda^p} U_j(x; \lambda) \bigg|_{\lambda=\lambda_k}, \quad p \in \{0, 1, \ldots, m_k - 1\}, \tag{3.7}
\]
form a chain of an eigenfunction and associated functions of problem (1.2), (1.4) corresponding to the eigenvalue $\lambda_k$. Indeed, we have
\[
\frac{1}{p!} LD_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k} = \frac{1}{p!} D_\lambda^p LU_j(x; \lambda)|_{\lambda=\lambda_k} = \frac{1}{p!} D_\lambda^p (\lambda U_j(x; \lambda))|_{\lambda=\lambda_k} = \frac{1}{p!} \lambda_k D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k} + \frac{1}{(p-1)!} D_\lambda^{p-1} U_j(x; \lambda)|_{\lambda=\lambda_k}. \tag{3.8}
\]
Besides, both (3.4) and (3.6) yield that $D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k}$ satisfies the boundary condition (1.4). For instance, in the case $p = 1$, this is implied by the relation
\[
(C + D\Phi(1; \lambda_k)) \begin{pmatrix} \Delta'_{11}(\lambda_k) \\ \vdots \\ \Delta'_{1n}(\lambda_k) \end{pmatrix} + (C + D\Phi(1; \lambda_k))' \begin{pmatrix} \Delta_{11}(\lambda_k) \\ \vdots \\ \Delta_{1n}(\lambda_k) \end{pmatrix} = 0, \tag{3.9}
\]
which holds for $\lambda = \lambda_k$ if $\Delta(\lambda_k) = \Delta'(\lambda_k) = 0$.

Now let $U_j(x; \lambda_k) = 0$. As above, we consider the sequence of the vector functions $D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k}, \quad p \in \{0, 1, \ldots, m_k - 1\}$. Let $s$ stand for the minimal number $p$ such that $LD_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k} \neq 0$, i.e.,
\[
\begin{cases}
D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k} = 0 & \text{for } p \in \{0, 1, \ldots, s-1\}; \\
D_\lambda^s U_j(x; \lambda)|_{\lambda=\lambda_k} \neq 0. 
\end{cases} \tag{3.10}
\]

In this case, we obtain:
\[
\frac{1}{s!} LD_\lambda^s U_j(x; \lambda)|_{\lambda=\lambda_k} = \frac{1}{s!} D_\lambda^s LU_j(x; \lambda)|_{\lambda=\lambda_k} = \frac{1}{s!} D_\lambda^s (\lambda U_j(x; \lambda))|_{\lambda=\lambda_k} = \frac{1}{s!} \lambda_k D_\lambda^s U_j(x; \lambda)|_{\lambda=\lambda_k} + \frac{1}{(s-1)!} D_\lambda^{s-1} U_j(x; \lambda)|_{\lambda=\lambda_k} = \frac{1}{s!} \lambda_k D_\lambda^s U_j(x; \lambda)|_{\lambda=\lambda_k}, \tag{3.11}
\]
since $D_\lambda^{s-1} U_j(x; \lambda)|_{\lambda=\lambda_k} = 0$. Hence for $s < m_k$ the sequence of the vector functions $D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k}, \ldots, D_\lambda^{m_k} U_j(x; \lambda)|_{\lambda=\lambda_k}$ forms a chain of an eigenfunction and associated functions of problem (1.2), (1.4) corresponding to the eigenvalue $\lambda_k$. In this case, the fulfillment of the boundary conditions is verified as above.

Thus, the system of functions $\{D_\lambda^p U_j(x; \lambda)|_{\lambda=\lambda_k}\}_{p=0}^{m_k-1}$ is either zero, or it span the root subspace of the operator $L_{C,D}$ corresponding to $\lambda_k$.

(ii) In this step we reduce the problem (1.2)–(1.4) to similar problem with a potential matrix $Q(\cdot) = (q_{jk}(\cdot))_{j,k=1}^r$ having zero diagonal, i.e. $q_{jj}(\cdot) = 0, \quad j \in \{1, \ldots, r\}$. It will allow us to apply Proposition 1.2.

To this end we denote by $W(\cdot)$ the fundamental $n \times n$ matrix solution of the Cauchy problem
\[
iBW'(x) = Q_1(x)W(x), \quad W(0) = I_n, \tag{3.12}
\]
where the $n \times n$ matrix function $Q_1(\cdot)$ is quasidiagonal with blocks $q_{jj}(\cdot)$,
\[
Q_1(x) = \text{diag}(q_{11}(x), \ldots, q_{rr}(x)). \tag{3.13}
\]
Since $BQ_1(x) = Q_1(x)B$ for any $x \in [0,1]$, the matrix functions $W_1(\cdot) = BW(\cdot)$ and $W_2(\cdot) = W(\cdot)B$ satisfy equation (3.12) and common initial conditions

$$iBW'(x) = Q_1(x)W_j(x), \quad W_j(0) = B, \quad j \in \{1, 2\}. \quad (3.14)$$

According to the Cauchy uniqueness theorem $W_1(x) = W_2(x)$ for $x \in [0,1]$, i.e.

$$W(x)B - BW(x) = 0, \quad x \in [0,1]. \quad (3.15)$$

Letting $\tilde{L} = (I \otimes W)^{-1}L(I \otimes W)$ we deduce from (1.2), (3.12) and (3.15) that for any $f \in C^1[0,1] \otimes \mathbb{C}^n$

$$\tilde{L}f - \lambda f = W^{-1}(x)(-iB)W(x)f' + W^{-1}(x)(-iB)W'(x)f$$

$$+ W^{-1}(x)Q(x)W(x)f - \lambda f = -iB\frac{d}{dx} f + \tilde{Q}(x)f - \lambda f, \quad (3.16)$$

where

$$\tilde{Q}(x) := W^{-1}(x)(Q(x) - Q_1(x)W(x)). \quad (3.17)$$

It follows from (3.15) that the matrix function $W(\cdot)$ is quasidiagonal,

$$W(x) = \text{diag}(W_{11}(x), \ldots, W_{rr}(x)), \quad (3.18)$$

with $n \times n$ nonsingular matrix blocks $W_{jj}(\cdot)$, $j \in \{1, \ldots, r\}$. It follows from (3.17) and (3.18) that $\tilde{Q}(\cdot)$ is of the form

$$\tilde{Q}(x) = (\tilde{Q}_{jk}(x))^{r}_{j,k=1}, \quad \tilde{Q}_{jj}(x) = 0, \quad x \in [0,1], \quad j \in \{1, \ldots, r\}. \quad (3.19)$$

Thus, the problem (1.2), (1.4) transforms into similar problem for equation (3.16) with $\tilde{Q}(\cdot)$ instead of $Q(\cdot)$ and the boundary conditions

$$C_1y(0) + D_1y(1) = 0 \quad (3.20)$$

in place of (1.4). Here $C_1 := CW(0) = C$ and $D_1 := DW(1)$. Due to the block structure (3.18) of $W(\cdot)$ and conditions $\det W_{jj}(\cdot) \neq 0$ the pairs $\{C, D\}$ and $\{C, DW(1)\}$ satisfy the conditions of Theorem 1.2 only simultaneously.

Thus, in what follows without loss of generality we may assume that the matrix function $Q(\cdot) = (q_{jk}(\cdot))^{r}_{j,k=1}$ has zero diagonal, i.e. $q_{jj}(\cdot) = 0, j \in \{1, \ldots, r\}$.

(iii) We prove the completeness of system (3.7) by contradiction. To this end, we assume that there exists a vector function $f = \text{col}(f_1, \ldots, f_n) \in L^2[0,1] \otimes \mathbb{C}^n$ orthogonal to this system. Consider the entire function

$$F_1(\lambda) := (U_1(x; \lambda), f(x))_{L^2[0,1] \otimes \mathbb{C}^n} = \sum_{j=1}^{n} \Delta_{1j}(\lambda) \int_{0}^{1} \langle \Phi_j(x; \lambda), f(x) \rangle \, dx. \quad (3.21)$$

Clearly, any $\lambda_k \in \sigma(L_{C,D})$ is the zero of $F_1(\cdot)$ of multiplicity at least $m_k$, i.e.,

$$F_1^{(p)}(\lambda) \bigg|_{\lambda=\lambda_k} = 0, \quad p \in \{0, 1, \ldots, m_k-1\}, \quad \lambda_k \in \sigma(L_{C,D}). \quad (3.22)$$

Thus, the ratio

$$G_1(\lambda) := \frac{F_1(\lambda)}{\Delta(\lambda)} \quad (3.23)$$

is an entire function. Let us prove that $G_1(\lambda) \equiv 0$ by estimating its growth.

To this end we obtain another representation of $G_1(\cdot)$ which is more convenient for the estimation. Moreover, to simplify the notions, we restrict ourselves to the case $r = n$, i.e., assume that the spectrum of the matrix $B$ is simple.
As in Proposition 2.2, the complex plane can be divided into the sectors \( S_p = \{ z \in \mathbb{C} : \varphi_p < \arg z < \varphi_{p+1} \} \) such that, for all \( \lambda \) inside a certain sector, the numbers \( b_j \) can be ordered as
\[
\Re(ib_1 \lambda) < \cdots < \Re(ib_p \lambda) < 0 < \Re(ib_{p+1} \lambda) < \cdots < \Re(ib_n \lambda). \tag{3.24}
\]
Moreover, for a sufficiently large \( R > 0 \), in the domain
\[
S_{p, \epsilon, R} := \{ \lambda \in \mathbb{C}_+ : \varphi_p + \epsilon < \arg \lambda < \varphi_{p+1} - \epsilon, \ |\lambda| > R \}, \tag{3.25}
\]
there exist \( n \) linearly independent solutions \( Y_j(x; \lambda) = \text{col}(y_{1j}, \ldots, y_{nj}) \) analytic with respect to \( \lambda \) and having the following asymptotic behavior
\[
y_{jk}(x; \lambda) = \left( \delta_k^j + o(1) \right) e^{ib_j \lambda x}, \quad \lambda \in S_{p, \epsilon, R}, \tag{3.26}
\]
uniform with respect to \( x \in [0, 1] \).

Since the solutions \( Y_j(; \lambda) \) \((1 \leq j \leq n)\) are linearly independent for any \( \lambda \in S_{p, \epsilon, R} \), then the fundamental \( n \times n \) matrices \( \Phi(x; \lambda) \) and \( Y(x; \lambda) := (Y_1, \ldots, Y_n) \) of the system (1.2) are related by
\[
\Phi(x; \lambda) = Y(x; \lambda)P(\lambda), \quad x \in [0, 1], \ \lambda \in S_{p, \epsilon, R}. \tag{3.27}
\]
where \( P(\lambda) = (p_{jk}(\lambda))_{k,j=1}^n \) is an analytical invertible matrix function in \( S_{p, \epsilon, R} \).

Further, apart from \( A_\Phi(\lambda) \), we introduce the matrix function
\[
A_Y(\lambda) = CY(0; \lambda) + DY(1; \lambda), \tag{3.28}
\]
and denote its determinant by \( \Delta_Y(\lambda) := \det A_Y(\lambda) \). Besides this, alongside with \( U_j(x; \lambda) \) of the form (3.6), we consider the vector functions
\[
V_j(x; \lambda) := \sum_{k=1}^n \Delta_Y^{jk}(\lambda)Y_k(x; \lambda), \quad j \in \{1, 2, \ldots, n\}, \tag{3.29}
\]
where \( \Delta_Y^{jk}(\lambda) \) is the cofactor of the \( jk \)th entry of the matrix \( A_Y(\lambda) \). Clearly, \( V_j(x; \lambda) \) are holomorphic in \( S_{p, \epsilon, R} \).

Both (3.27), (3.28) and the definition of \( A_\Phi(\lambda) \) (see (3.5)) yield the relations
\[
A_\Phi(\lambda) = A_Y(\lambda)P(\lambda), \quad \Delta_\Phi(\lambda) = \Delta_Y(\lambda) \det P(\lambda). \tag{3.30}
\]
Let \( A_\Phi(\lambda) := (a_{jk}(\lambda))_{j,k=1}^n \), \( A_Y(\lambda) := (\tilde{a}_{jk}(\lambda))_{j,k=1}^n \). Taking account of these notation, we derive from (3.27) and (3.30) the relations
\[
\begin{pmatrix}
\varphi_1 & \varphi_2 & \cdots & \varphi_n \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n
\end{pmatrix} =
\begin{pmatrix}
y_{1j} & y_{j2} & \cdots & y_{jn} \\
\alpha_{1j} & \tilde{\alpha}_{j2} & \cdots & \tilde{\alpha}_{jn} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{1j} & \tilde{\alpha}_{j2} & \cdots & \tilde{\alpha}_{jn}
\end{pmatrix}P(\lambda), \quad j \in \{1, \ldots, n\}. \tag{3.31}
\]
Note that the system (3.31) is equivalent to the formal equality that can be obtained from the first equation in (3.30) if one replaces the first lines in the matrices \( A_\Phi(\lambda) \) and \( A_Y(\lambda) \) by the "lines" \((\Phi_1, \ldots, \Phi_n)\) and \((Y_1, \ldots, Y_n)\), respectively. The desirable connection between the vector functions \( U_1(x; \lambda) \) and \( V_1(x; \lambda) \) is implied now by (3.6), (3.29) and (3.31):
\[
U_1(x; \lambda) = V_1(x; \lambda) \det P(\lambda), \quad \lambda \in S_{p, \epsilon, R}. \tag{3.32}
\]
By setting
\[
\bar{F}_1(\lambda) := \langle V_1(x; \lambda), f(x) \rangle_{L^2(\mathbb{C}^\nu)} = \sum_{j=1}^{n} \Delta^j_Y(\lambda) \int_{0}^{1} \langle Y_j(x; \lambda), f(x) \rangle \, dx
\]
\[
= \sum_{j=1}^{n} \Delta^j_Y(\lambda) \sum_{k=1}^{n} \int_{0}^{1} Y_{kj}(x; \lambda) f_k(x) \, dx
\]
and by taking into account (3.21), (3.32) and (3.33), we arrive at the relation
\[
F_1(\lambda) = \bar{F}_1(\lambda) \det P(\lambda).
\]  
Finally, combining the second equality in (3.30) with (3.34), we arrive at the second representation of the entire function \(G_1(\cdot)\):
\[
G_1(\lambda) = \bar{F}_1(\lambda) / \Delta_Y(\lambda), \quad \lambda \in S_{p, \tau, R}.
\]  
(iv) In this step we estimate \(G_1(\cdot)\) on the rays \(l_m = \{\zeta_m t : t \in \mathbb{R}_+\}, m \in \{1, 2, 3\}\), using the representation (3.35). Here \(\zeta_m = iz_m\) where \(z_m\) are taken from the condition (b) of the theorem.
Since \(C = (c_{kj})_{k,j=1}^{n}, D = (d_{kj})_{k,j=1}^{n}\), it follows from (3.28) and (3.26) that the matrix \(A_Y(\lambda)\) admits the following representation
\[
A_Y(\zeta_m t) = \begin{pmatrix}
\sum_{j=1}^{n} |c_{kj}| + |d_{kj}| e^{ib_j \zeta_m t} & \cdots & |c_{kn}| + |d_{kn}| e^{ib_n \zeta_m t} \\
\vdots & \ddots & \vdots \\
|c_{n1}| + |d_{n1}| e^{ib_1 \zeta_m t} & \cdots & |c_{nn}| + |d_{nn}| e^{ib_n \zeta_m t}
\end{pmatrix}.
\]  
Noting that
\[
|c_{kj}| + |d_{kj}| e^{ib_j \zeta_m t} \sim c_{kj} \quad \text{for} \quad \Re(b_j z_m) > 0, \quad k \in \{1, \ldots, n\},
\]
and
\[
|c_{kj}| + |d_{kj}| e^{ib_j \zeta_m t} \sim d_{kj} e^{ib_j \zeta_m t} \quad \text{for} \quad \Re(b_j z_m) < 0, \quad k \in \{1, \ldots, n\},
\]
we arrive at the asymptotic estimate for the characteristic determinant
\[
\Delta_Y(\zeta_m t) = \det A_Y(\zeta_m t) = e^{\beta_m t} (\det T_{\zeta_m t}(C, D) + o(1)) \quad \text{as} \quad t \to \infty,
\]  
along the ray \(l_m\). Here \(\beta_m := \sum_{\Re(ib_j \zeta_m) > 0} ib_j \zeta_m\) and \(T_{\zeta_m t}(C, D)\) is the matrix from the assumption (b) of the theorem.
Next we estimate \(\bar{F}_1(\cdot)\). Since \(\Delta^j_Y(\zeta_m t) = O(e^{\beta_m t})\) for \(\Re(ib_j \zeta_m) < 0\), estimates (3.26) yield
\[
\Delta^j_Y(\zeta_m t) Y_j(x; \zeta_m t) = e^{\beta_m t} O(e^{ib_j \zeta_m t x}).
\]  
If \(\Re(ib_j \zeta_m) > 0\) then \(\Delta^j_Y(\zeta_m t) Y_j(x; \zeta_m t) = O(e^{(\beta_m - ib_j \zeta_m) t})\), and in this case we obtain:
\[
\Delta^j_Y(\zeta_m t) Y_j(x; \zeta_m t) = e^{(\beta_m - ib_j \zeta_m) t} O(e^{ib_j \zeta_m t x}) = e^{\beta_m t} O(e^{ib_j \zeta_m t (x-1)}).
\]  
Denote by \(s_-\) the maximal negative number from \(\Re(ib_j \zeta_m)\), and by \(s_+\) the minimal positive number from the same set. Then we have
\[
\Delta^j_Y(\zeta_m t) Y_j(x; \zeta_m t) = e^{\beta_m t} O(\max(e^{s_- t x}, e^{s_+ t(x-1)})), \quad j \in \{1, 2, \ldots, n\}.
\]  
Hence the function \(V_1\) of the form (3.29) are estimated along the rays \(l_m = \{\lambda : \lambda = \zeta_m t\}\), as above, i.e.,
\[
V_1(x; \zeta_m t) = e^{\beta_m t} O(\max(e^{s_- t x}, e^{s_+ t(x-1)})) = e^{\beta_m t} O(e^{s_- t x} + e^{s_+ t(x-1)}).
\]
It follows that
\[\tilde{F}_1(\zeta_m) = \int_0^1 \langle V_i(x; \zeta_m), f(x) \rangle \, dx = e^{\beta_m t} O \left( \int_0^1 |f(x)|(e^{s-tx} + e^{s+t(x-1)}) \, dx \right)\]
\[\leq C e^{\beta_m t} \sqrt{\int_0^1 |f(x)|^2 \, dx} \sqrt{\int_0^1 (e^{s-tx} + e^{s+t(x-1)})^2 \, dx} = o(e^{\beta_m t}), \quad (3.42)\]
since \(\int_0^1 (e^{s-tx} + e^{s+t(x-1)})^2 \, dx \to 0\) as \(t \to \infty\).

Combining estimates (3.37) and (3.42) we get
\[G_1(\zeta_m) = \frac{\tilde{F}_1(\zeta_m)}{\Delta_Y(\zeta_m)} = \frac{o(e^{\beta_m t})}{(\det T_{zm} E(C, D) + o(1))e^{\beta_m t}} \to 0 \quad \text{as} \quad t \to \infty.\]

It follows from (3.23), (3.21), that \(G_1(\cdot)\) is the entire function of type not greater than exponential, hence it is bounded in each of the (convex) angles formed by pairs of the rays \(l_k\). Since the origin is the interior point of the triangle \(\triangle_{\zeta_1,\zeta_2,\zeta_3}\), we obtain that these angles cover the whole complex plane. Thus, \(G_1(\cdot)\) is bounded in \(\mathbb{C}\) and tends to zero along each of the rays \(l_k\). Hence \(G_1(\lambda) \equiv 0\), by the Liouville theorem.

As in (3.23), we introduce the functions
\[G_j(\lambda) := F_j(\lambda)\Delta(\lambda)^{-1}, \quad j \in \{2, 3, \ldots, n\}, \quad (3.43)\]
and show that \(G_j(\lambda) \equiv 0\) for \(j \in \{2, 3, \ldots, n\}\).

(v) Note that, for \(\lambda \notin \sigma(L_{C,D})\), the functions \(U_j(x; \lambda)\) form the fundamental systems of solutions of the system (1.2). Since \(f(x)\) is orthogonal to all the \(U_j(x; \lambda)\), \(j \in \{1, 2, \ldots, n\}\), we conclude that it is orthogonal to all solutions of the system (1.2) whenever \(\lambda \notin \sigma(L_{C,D})\). Therefore,
\[\int_0^1 \langle \Phi_j(x; \lambda), f(x) \rangle \, dx = 0, \quad \lambda \notin \sigma(L_{C,D}), \quad j \in \{1, 2, \ldots, n\}. \quad (3.44)\]

But, due to the continuity of the integral (3.44) with respect to \(\lambda\) and the discreteness of the set \(\sigma(L_{C,D})\), the following relations hold:
\[\int_0^1 \langle \Phi_j(x; \lambda), f(x) \rangle \, dx \equiv 0, \quad \lambda \in \mathbb{C}, \quad j \in \{1, 2, \ldots, n\}. \quad (3.45)\]

(vi) At this step, we show that the vector function \(f\) satisfying relations (3.45) is the zero function. To this end, consider the resolvent \(R_L(\lambda)\) of the operator \(L\) of the form (1.2) subject to the initial conditions
\[Y(0) = \text{col}(y_1(0), \ldots, y_n(0)) = 0. \quad (3.46)\]
As above, let \(\Phi(x; \lambda)\) stand for the fundamental matrix solution of the equation (1.2) satisfying the condition (3.1). It can easily be seen that the Green matrix of the Cauchy problem (1.2), (3.46) is
\[G(x, t; \lambda) = \begin{cases} \Phi(x; \lambda)\Phi^{-1}(t; \lambda)(-iB)^{-1}, & t \leq x, \\ 0, & t > x, \end{cases} \quad (3.47)\]
and is an entire function with respect to \(\lambda \in \mathbb{C}\). Hence \(R_L(\lambda)\) is a Volterra operator:
\[(R_L(\lambda)\varphi)(x) = \int_0^x G(x, t; \lambda)\varphi(t) \, dt, \quad \varphi \in L^2[0, 1].\]
Alongside with the \(\Phi(x; \lambda)\), consider the matrix function
\[Y(x; \lambda) := (Y_1(x; \lambda), \ldots, Y_n(x; \lambda))\]
consisting from the solutions \(Y_j(x; \lambda) = \text{col}(y_{1j}, ..., y_{nj})\) satisfying the asymptotic relations (3.26). Clearly, \(Y(x; \lambda)\) is the fundamental matrix of (1.2) for \(\lambda \in S_\pm := \pm S_{p.e,R}\). By (3.27) \(Y(x; \lambda) = \Phi(x; \lambda)P^{-1}(\lambda), \ \lambda \in S_\pm\), where \(P^{-1}(\lambda) \in \mathbb{C}^n \times n\) for \(\lambda \in S_\pm\). Therefore,

\[
Y(x; \lambda)Y^{-1}(t; \lambda)(-iB)^{-1} = \Phi(x; \lambda)\Phi^{-1}(t; \lambda)(-iB)^{-1}, \ \lambda \in S_\pm,
\]

and the Green matrix \(G(x, t; \lambda)\) is the analytic continuation of the matrix function \(Y(x; \lambda)Y^{-1}(t; \lambda)(-iB)^{-1}\). In particular, for \(\lambda \in S_\pm\) the operator \(R^*_L(\lambda) := (R_L(\lambda))^*\) admits the representation

\[
(R^*_L(\lambda)\varphi)(x) = (iB^{-1})^*Y^{-1}(x, \lambda)^* \int_x^1 Y^*(t, \lambda)\varphi(t)dt, \ \lambda \in S_\pm. \tag{3.48}
\]

Further, since \(f\) satisfies conditions (3.45), we have

\[
\int_0^1 Y^*(t, \lambda)f(t)dt = 0, \ \lambda \in S_\pm. \tag{3.49}
\]

From (3.26) follows that \(Y(x; \lambda)\) admits the representation

\[
Y(x; \lambda) = I_n(x; \lambda)e(x; \lambda), \ \lambda \in S_\pm, \tag{3.50}
\]

in which \(I_n(x; \lambda) = I_n + \omega_n(1)\) and

\[
e(x; \lambda) := \text{diag}(e^{ib_1\lambda x}, ..., e^{ib_n\lambda x}). \tag{3.51}
\]

By multiplying (3.49) from the left by the matrix

\[
\bar{e}(x; \lambda) := \text{diag}(e^{ib_1\lambda x}, ..., e^{ib_n\lambda x}) = e^{-1}(x; \lambda)^*\]

and by taking into account (3.50) and (3.51), we arrive at the relation

\[
\Theta(x; \lambda) := \int_x^1 \bar{e}(x - t; \lambda)I^*_n(t; \lambda)f(t)dt =
\]

\[
= -\int_0^x \bar{e}(x - t; \lambda)I^*_n(t; \lambda)f(t)dt, \ \lambda \in S_\pm. \tag{3.52}
\]

By setting

\[
g(t; \lambda) = \text{col}(g_1(t; \lambda), ..., g_n(t; \lambda)) := I^*_n(t; \lambda)f(t), \ \lambda \in S_\pm, \tag{3.53}
\]

we rewrite the matrix equality (3.52) as a system of \(n\) scalar equalities:

\[
\int_x^x e^{ib_j(x-t)}g_j(t; \lambda)dt = -\int_x^1 e^{ib_j(x-t)}g_j(t; \lambda)dt,
\]

\[
\lambda \in S_\pm, \quad j \in \{1, 2, \ldots, n\}. \tag{3.54}
\]

Since \(\Re(ib_j) = -\Re(ib_j)\) then (3.24) implies that the functions \(e^{ib_j\lambda x}\), \((x \in [0, 1])\) are bounded in the sector \(S_-\) for \(j \in \{1, \ldots, \kappa\}\) and in the sector \(S_+\) for \(j \in \{\kappa + 1, \ldots, n\}\). Due to (3.53) the functions \(g_j(\cdot; \lambda)\) have uniformly bounded norms in \(L^2[0, 1]\) for \(\lambda \in S_\pm\). Now we conclude from (3.54) that

\[
\Theta(x; \lambda) = o(1) \quad \text{for} \quad \lambda \in S_\pm, \lambda \to \infty \quad \text{(for every} \ x \in [0, 1]). \tag{3.55}
\]

Further, denote

\[
G_j(x; \lambda) := (R^*_L(\lambda)f_j)(x).
\]
By (3.48) and (3.50)–(3.52), for $\lambda \in S_\pm$, the $G_f(x;\lambda)$ admits the representation

$$G_f(x;\lambda) = (iB^{-1})^\ast \int_x^1 \mathcal{I}_n^{-1}(x;\lambda)^\ast \bar{e}(x-t;\lambda) \mathcal{I}_n^*(t;\lambda)f(t)dt = (iB^{-1})^\ast \mathcal{I}_n^{-1}(x;\lambda)^\ast \Theta(x;\lambda),$$

and hence from (3.55) we conclude that

$$G_f(x;\lambda) = o(1) \text{ for } \lambda \in S_\pm, \lambda \to \infty.$$  

But $G_f(x;\lambda)$ is the entire function of exponential type (for every $x \in [0,1]$). Moreover, since $G_f(x;\lambda)$ is bounded along the pair of rays in $S_+$ and along the pair of opposite rays in $S_-$, it is bounded in $\mathbb{C}$ due to the Phragmén-Lindelöf theorem [24]. By the Liouville theorem, $G_f(x;\lambda)$ does not depend on $\lambda$, i.e., $G_f(x;\lambda) =: c(x)$, $x \in [0,1]$. Due to (3.57) the function $c(x)$ is zero and hence $(R_l^*(\lambda)f)(x) = G_f(x;\lambda) \equiv 0$. It follows that $f = 0$.

(vii) The minimality of the system of EAF follows from Lemma 2.4 applied to the resolvent operator $R_{L_{C,D}}(\lambda)$ with $\lambda \in \rho(L_{C,D})$.

\begin{corollary}
Let $Q \in L^2[0,1]\otimes \mathbb{C}^{n\times n}$ and let the matrices $T_{\pm B}(C,D)$ and $T_{\mp B}(C,D) = T_{\mp B}(D,C)$ be nonsingular for some $z \in \mathbb{C}$. Then

(i) The boundary conditions (1.4) are weakly $B$-regular.

(ii) The system of EAF of the operator $L_{C,D}(Q)$ is complete and minimal in $L^2([0,1];\mathbb{C}^n)$.

\end{corollary}

\begin{proof}
Since all the numbers $\Re(zb_k)$ are different from zero, we get that, for sufficiently small $\delta$, the signs of $\Re((1 \pm \delta)zb_k)$ coincide with the sign of $\Re(zb_k)$. It follows that the matrices $T_{\pm B}(C,D)$, $T_{(1+\delta)B}(C,D)$ and $T_{(1-\delta)B}(C,D)$ coincide. Thus, we can apply Theorem 1.2 to the operator $L_{C,D}(Q)$ and the points $z_1 = (1+\delta)z$, $z_2 = (1-\delta)z$ and $z_3 = -z$.

\end{proof}

3.2. Completeness result for adjoint operator.

\begin{corollary}
Let boundary conditions (1.4) be weakly $B$-regular. Then

(i) The boundary conditions

$$C_\ast g(0) + D_\ast g(1) = 0$$

of the adjoint boundary value problem are weakly $B^\ast$-regular.

(ii) The system of root functions of the adjoint operator $L_{C,D}^\ast$ is complete and minimal in $L^2[0,1]\otimes \mathbb{C}^n$.

\end{corollary}

\begin{proof}
(i) The adjoint operator $L_{C,D}^\ast := (L_{C,D})^\ast$ is defined as a restriction of the maximal differential operator

$$L^\ast := \frac{1}{i}B^\ast \otimes \frac{d}{dx} + Q^\ast(x), \quad \text{dom}(L^\ast) = W_2^1([0,1];\mathbb{C}^n),$$

to the domain $\text{dom}(L_{C,D}^\ast) = \{g \in W_2^1([0,1];\mathbb{C}^n) : C_\ast g(0) + D_\ast g(1) = 0\}$. Moreover, if $Cf(0) + Df(1) = 0$ and $C_\ast g(0) + D_\ast g(1) = 0$, we have

$$\langle Bf(0), g(0) \rangle - \langle Bf(1), g(1) \rangle = 0.$$  

Put $\bar{B} := \text{diag}(B, -B)$ and consider $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^n$ as a space with bilinear form

$$w(\varphi, \psi) := \langle \bar{B}\varphi, \psi \rangle = \langle B\varphi_1, \psi_1 \rangle - \langle B\varphi_2, \psi_2 \rangle,$$
where \( \varphi = \col(\varphi_1, \varphi_2) \), \( \psi = \col(\psi_1, \psi_2) \). The condition (3.59) means that the subspace \( \ker(C, D) \) is the right \( w \)-orthogonal to \( \ker(C, D) \) in \( H \). Since \( \dim \ker(C, D) = \dim \ker(C, D) = n \), the subspace \( \ker(C, D) \) is non-degenerate and \( \{ \ker(C, D) \}_w = \ker(C, D) \), i.e. \( \ker(C, D) \) is the (right) \( w \)-orthogonal complement of \( \ker(C, D) \).

Let \( \beta_1, \beta_2, \ldots, \beta_{2n} \) be the eigenvalues of \( \tilde{B} \) and let \( e_1, e_2, \ldots, e_{2n} \) be the corresponding eigenvectors. For every admissible \( z \) (i.e. such that \( z\beta_k \not\in \mathbb{i}\mathbb{R} \) for every \( k \leq 2n \)) we put \( H_z = \text{span}\{e_k : \mathbb{R}(z\beta_k) > 0\} \). Since \( \beta_{n+k} = -\beta_k \in \sigma(\tilde{B}) \), \( k \in \{1, \ldots, n\} \), \( \dim H_z = n \) for every admissible \( z \).

Next we note that

\[
T_{zB}(C, D) = (C, D)|_{H_z}.
\] (3.61)

Therefore, \( T_{zB}(C, D) \neq 0 \) if and only if \( \ker((C, D)|_{H_z}) = \{0\} \), i.e. \( \ker((C, D) \cap H_z = \{0\} \). Since \( \dim \ker(C, D) = \dim H_z = n \), the latter identity is also valid for the right \( w \)-orthogonal complements of these subspaces, i.e. \( \ker(C, D) \cap H_{-z} = \{0\} \).

Alongside the space \( H \), we consider the same space \( H_* = \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n \) equipped with another non-degenerate bilinear form

\[
w^*(\varphi, \psi) = \langle \tilde{B}^* \varphi, \psi \rangle = \langle B^* \varphi_1, \psi_1 \rangle - \langle B^* \varphi_2, \psi_2 \rangle.
\]

Next we define the corresponding subspaces \( H_{z_2} \) with respect to the form \( w^*(\cdot, \cdot) \) (matrices \( z\tilde{B}^* \)) and note that

\[
T_{zB^*}(C_*, D_*) = (C_*, D_*)|_{H_{z_2}}.
\] (3.62)

Since \( \mathbb{R}(z\beta_k) = \mathbb{R}(z\beta_k) \), one has \( H_{z_2} = H_{z_2} \). Hence \( \ker(C, D) \cap H_{z_2} = \{0\} \) is equivalent to \( \ker(C, D) \cap H_{z_2} = \{0\} \). Combining this equivalence with relations (3.61) and (3.62) we get

\[
det T_{zB}(C, D) \neq 0 \iff det T_{zB^*}(C_*, D_*) \neq 0.
\]

Hence boundary conditions (3.58) are weakly \( B^* \)-regular and conditions of Definition 1.1 are satisfied with points \( -\infty, -\infty, -\infty \).

(ii) Combining statement (i) with Theorem 1.2 we get the result. \( \square \)

**Remark 3.3.** (i) Theorem 1.2 remains valid for the integro-differential operator

\[
-iy' + Q(x)y + \int_0^x M(x,t)y(t)dt = \lambda y, \quad y \in \col(y_1, y_2, \ldots, y_n),
\] (3.63)

with a kernel \( M(x,t) \in L^\infty(\Omega) \otimes \mathbb{C}^{n \times n} \).

(ii) If the maximality condition (1.6) is violated, i.e. \( \rank(C, D) \leq n-1 \), then the characteristic determinant (3.5) is identical zero. Indeed, in this case

\[
\rank(C + D\Phi(1; \lambda)) = \rank \left((C, D) \left( \begin{smallmatrix} I_n \\ \Phi(1; \lambda) \end{smallmatrix} \right) \right) \leq \rank(C, D) \leq n - 1.
\]

Hence \( \Delta_\Phi(\lambda) = \det(C + D\Phi(1; \lambda)) \equiv 0 \), \( \lambda \in \mathbb{C} \).

Note however that the latter might happen even whenever \( \rank(C, D) = n \).
3.3. Examples.

Example 3.4. Assume that $C \in \mathbb{C}^{n \times n}$, and $\det C \neq 0$. Let also $D = CM$, where $M \in \mathbb{C}^{n \times n}$ and all its principal minors are nonsingular. In this case, the matrix $T_A(I_n, M)$ is nonsingular for every matrix $A$. Hence the matrix $T_A(C, D) = CT_A(I, M)$ is always nonsingular.

For instance, the boundary conditions

$$y_j(0) = d_j y_j(1), \quad d_j \neq 0, \quad j \in \{1, 2, \ldots, n\},$$

(3.64)

that include the periodic ones ($d_j = 1$) have this form with $C = I_n$ and $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and hence are weakly $B$-regular for any non-singular $B$.

Note that conditions (3.64) are regular, i.e., the matrix $T_{zB}(C, D)$ is nonsingular for every admissible $z \in \mathbb{C}$.

Next we present several examples of irregular BC (1.4) that are weakly $B$-regular. To this end we prove the following fact mentioned in the Introduction.

Lemma 3.5. Assume that the boundary conditions (1.4) split in $k$ conditions at 0 and $n-k$ conditions at 1. Then

(i) If $\Re(zB)$ is invertible and $\det T_{zB}(C, D) \neq 0$, then $k = \kappa_+ (\Re(zB))$.

(ii) If the boundary conditions are regular, then $n = 2k$ and $\kappa_+ (\Re(zB)) = \kappa_- (\Re(zB))$ for every admissible $z \in \mathbb{C}$, i.e., for those $z$ that $\Re(zB)$ is invertible.

Proof. (i) Let $z \in \mathbb{C}$ be admissible, i.e., the matrix $\Re(zB)$ is nonsingular. Then the matrix $T_{zB}(C, D)$ exists and has $l$ columns from $C$ and $n-l$ columns from $D$. By the definition, $l = \kappa_+ (\Re(zB))$. Further, since the last $n-k$ rows of the matrix $C$ and the first $k$ rows of the matrix $D$ are zero, the matrix $T_{zB}(C, D)$ has at least two zero submatrices of sizes $(n-k) \times l$ and $k \times (n-l)$. Since $\det T_{zB}(C, D) \neq 0$, one has $n-k+l \leq n$ and $k+n-l \leq n$. Hence $k = l$.

(ii) Let the boundary conditions be regular and $\det (zB) \neq 0$. Then both matrices $T_{zB}(C, D)$ and $T_{-zB}(D, C)$ are well-defined and nonsingular. By the statement (i), $k = \kappa_+ (\Re(zB))$ and $k = \kappa_+ (\Re(-zB))$. Since $\kappa_+ (\Re(-zB)) = \kappa_- (\Re(zB))$, one has $\kappa_+ (\Re(zB)) = \kappa_- (\Re(zB)) + 2k = \kappa_+ (\Re(zB)) + \kappa_- (\Re(zB)) = n$. □

Example 3.6. Let $n = 2k + 1$, $B = \text{diag}(b_1, \ldots, b_n)$ with $b_j = \exp\left(\frac{\pi ij}{n}\right)$, $j \in \{1, 2, \ldots, n\}$, and let BC (1.4) split in $k$ conditions at 0 and $k+1$ conditions at 1. Then the lines $\{z \in \mathbb{C} : \Re(izb_j) = 0\}$ divides $\mathbb{C}$ in $2n$ sectors $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$ such that the point $iz_p$ belongs to the interior of $\sigma_p$, for $p \in \{1, 2, \ldots, 2n\}$, where $z_p = \exp\left(\frac{\pi ip}{n}\right)$. Note that for $p \equiv k \pmod{2}$ we have $\kappa_+(\Re(z_pB)) = k + 1$ and hence, by Lemma 3.5, the matrix $T_{z_pB}(C, D)$ is singular.

However, in general, for other values of $p$ the matrix $T_{z_pB}(C, D)$ is nonsingular. More precisely, if $p \equiv k + 1 \pmod{2}$ then $\kappa_+(\Re(z_pB)) = k$ and

$$\det T_{z_pB}(C, D) = C \begin{pmatrix} 1 & 2 & \cdots & k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \cdot D \begin{pmatrix} k+1 & \cdots & n \\ j_{k+1} & \cdots & j_n \end{pmatrix}$$

(3.65)

where $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, $1 \leq j_{k+1} < j_{k+2} < \ldots < j_n \leq n$, $\Re(z_p b_{j_k}) > 0$ for $1 \leq \nu \leq k$ and $\Re(z_p b_{j_k}) < 0$ for $k+1 \leq \nu \leq n$. Here $A \begin{pmatrix} j_1 & j_2 & \cdots & j_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$ stands for the minor of $n \times m$-matrix $A = (a_{jk})$ composed of the entries in the rows with the indices $j_1, \ldots, j_p \in \{1, \ldots, n\}$ and the columns with the indices $k_1, \ldots, k_p \in \{1, \ldots, m\}$. 

Assume that for some values \( p_1, p_2, p_3 \in \{1, 2, \ldots, 2n\} \) satisfying \( p_1 < p_2 < p_3 \), \( p_1 \equiv p_2 \equiv p_3 \equiv k + 1 \pmod{2} \), \( p_2 - p_1 < n \), \( p_3 - p_2 < n \) and \( p_3 - p_1 > n \) the corresponding minors of matrices \( C \) and \( D \) from equality (3.65) for values \( p = p_1, p_2, p_3 \) are non-zero. Then the boundary conditions (1.4) will be weakly \( B \)-regular if we put \( z_j = \exp \left( \frac{\pi i p_j}{n} \right) \), \( j \in \{1, 2, 3\} \), in Definition 1.1 of weak \( B \)-regularity. However, by Lemma 3.5, these boundary conditions are irregular.

One obtains an explicit example by setting \( n = 3 \) and

\[
\begin{align*}
  c_{11} y_1(0) + c_{12} y_2(0) + c_{13} y_3(0) &= 0 \\
  d_{21} y_1(1) + d_{23} y_3(1) &= 0 \\
  d_{32} y_2(1) + d_{33} y_3(1) &= 0
\end{align*}
\]

where all the coefficients are non-zero. Here we can take \( p_j = 2j, j \in \{1, 2, 3\} \).

We obtain another explicit example of irregular but weakly \( B \)-regular splitting boundary conditions (1.4) for system (1.2) with \( n = 2k + 1 \), by setting

\[
(C \ D) = \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
\vdots & c_2 & \ldots & c_n & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
c_1^{k-1} & c_2^{k-1} & \ldots & c_n^{k-1} & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\
0 & 0 & \ldots & 0 & d_1 & d_2 & \ldots & d_n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & d_1^k & d_2^k & \ldots & d_n^k
\end{pmatrix}.
\]

Here \( c_j \neq c_k \) and \( d_j \neq d_k \) for \( j \neq k \). Now any \( k \times k \)-minor of the matrix \( C \) that corresponds to its first \( k \) rows is the Vandermonde determinant, hence it is non-zero. The same is true for any \( (k+1) \times (k+1) \)-minor of the matrix \( D \) that corresponds to its last \( k+1 \) rows. Hence \( \det T_{zB}(C,D) \neq 0 \) for any \( p \in \{1, 2, \ldots, 2n\} \) such that \( p \equiv k + 1 \pmod{2} \). So, we can take \( p_1 = 2, p_2 = 4, p_3 = n + 3 \) for odd \( k \) and \( p_1 = 1, p_2 = 3, p_3 = n + 2 \) for even \( k \).

Next we present two examples of non-splitting boundary conditions that are irregular but weakly \( B \)-regular.

**Example 3.7.** Let \( n = 3 \), \( B = \text{diag}(b_1, b_2, b_3) \) and \( b_j = \exp \left( \frac{2\pi ij}{3} \right) \), \( j \in \{1, 2, 3\} \). Consider the boundary conditions (1.4) of the form:

\[
\begin{align*}
y_1(0) &= d_{12} y_2(1) + d_{13} y_3(1) \\
y_2(0) &= d_{21} y_1(1) + d_{23} y_3(1) \\
y_3(0) &= d_{31} y_1(1) + d_{32} y_2(1)
\end{align*}
\]

where all the coefficients \( d_{jk} \) are non-zero. In this case, the matrix \( T_{zB}(C,D) \) is nonsingular for \( z = \exp \left( \frac{2\pi ij}{3} \right) \), \( j \in \{1, 2, 3\} \), but it is singular for \( z = -\exp \left( \frac{2\pi ij}{3} \right) \), \( j \in \{1, 2, 3\} \). For instance, for \( z = 1 \) we have

\[
\det T_{zB}(C,D) = \det T_B(C,D) = \begin{vmatrix}
0 & d_{12} & 0 \\
d_{21} & 0 & 0 \\
d_{31} & d_{32} & 1
\end{vmatrix} = -d_{12}d_{21} \neq 0.
\]
At the same time, for \( z = -1 \) one has

\[
\det T_{zB}(C, D) = \det T_{-B}(C, D) = \det \begin{pmatrix} 1 & 0 & d_{13} \\ 0 & 1 & d_{23} \\ 0 & 0 & 0 \end{pmatrix} = 0.
\]

**Example 3.8.** Let \( n = 3, B = \text{diag}(b_1, b_2, b_3) \) and \( b_j = \exp \left( \frac{2\pi i j}{3} \right), \ j \in \{1, 2, 3\} \). Consider boundary conditions (1.4) of the form:

\[
c_1y_1(0) = c_2y_2(0) = c_3y_3(0) = d_1y_1(1) + d_2y_2(1) + d_3y_3(1),
\]

where all the coefficients are non-zero. In this case, the matrix \( T_{zB}(C, D) \) is nonsingular for \( z = -\exp \left( \frac{2\pi i j}{3} \right), \ j \in \{1, 2, 3\} \), but it is singular for \( z = \exp \left( \frac{2\pi i j}{3} \right), \ j \in \{1, 2, 3\} \). For instance, for \( z = -1 \) we have

\[
\det T_{zB}(C, D) = \det T_{-B}(C, D) = \det \begin{pmatrix} c_1 & 0 & d_3 \\ 0 & c_2 & d_3 \\ 0 & 0 & d_3 \end{pmatrix} = c_1c_2d_3 \neq 0.
\]

On the other hand, for \( z = 1 \)

\[
\det T_{zB}(C, D) = \det T_B(C, D) = \det \begin{pmatrix} d_1 & d_2 & 0 \\ d_1 & d_2 & 0 \\ d_1 & d_2 & c_3 \end{pmatrix} = 0.
\]

4. The case of a selfadjoint matrix \( B = B^* \)

Suppose that \( B = B^* \in \mathbb{C}^{n \times n} \) and \( \det B \neq 0 \). To state the next result, we denote by \( P_+ \) and \( P_- \) the spectral projectors onto ”positive” and ”negative” parts of the spectrum of a selfadjoint matrix \( B = B^* \), respectively, and put

\[
T_\pm := T_\pm(B; C, D) := CP_\pm + DP_\pm.
\]

**Proposition 4.1.** Assume that \( B = B^* \) and \( Q \in L^2[0, 1] \otimes \mathbb{C}^{n \times n} \). If

\[
\det T_+(B; C, D) \neq 0 \quad \text{and} \quad \det T_-(B; C, D) \neq 0,
\]

then the system of EAF of the operator \( L_{C, D} \) is complete and minimal in the space \( L^2[0, 1] \otimes \mathbb{C}^n \).

**Proof.** To prove the completeness, it suffices to note that

\[
T_+(B; C, D) = T_B(C, D) \quad \text{and} \quad T_-(B; C, D) = T_B(D, C)
\]

and to put \( z = 1 \) in Corollary 3.1. \( \square \)

Next we clarify Proposition 4.1 for accumulative (dissipative) BVP. Recall that an operator \( T \) in a Hilbert space \( \mathcal{H} \) is called accumulative (dissipative) whenever

\[
\text{Im}(Tf, f) \leq 0 \ (\geq 0), \quad f \in \text{dom}(T).
\]

**Lemma 4.2.** Let \( B = B^* \) and let the operator \( L_{C, D}(0) \) be accumulative (dissipative). Then \( \det T_+(B; C, D) \neq 0 \) (\( \det T_-(B; C, D) \neq 0 \)).

**Proof.** Since the operator \( L_{C, D}(0) \) is accumulative, one has

\[
2 \text{Im}(L_{C, D}(0)y, y) = \langle By(0), y(0) \rangle - \langle By(1), y(1) \rangle \leq 0, \quad y \in \text{dom}(L_{C, D}).
\]

As in the proof of Corollary 3.2 we let \( \tilde{B} := \text{diag}(B, -B) = \tilde{B}^* \) and equip the space \( \mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^n \) with the non-degenerate Hermitian bilinear form (3.60). Let also
Assume that the operator $L_{C,D}(0)$ is accumulative (dissipative), and $\det T_-(B; C, D) \neq 0$ (det $T_+(B; C, D) \neq 0$). Then both conditions (4.2) are satisfied and the system of root functions of the operator $L_{C,D}(Q)$ with $Q \in L^2[0,1] \otimes \mathbb{C}^{n \times n}$ is complete and minimal in $L^2[0,1] \otimes \mathbb{C}^n$.

**Proof.** Combining Lemma 4.2 with Proposition 4.1 yields the statement. □

**Corollary 4.4.** Suppose that $C, D \in \mathbb{C}^{n \times n}$ satisfy both the maximality condition (1.6) and the relation $CB^{-1}C^* - DB^{-1}D^* = 0$. Then the system of root functions of the operator $L_{C,D}(Q)$ is complete and minimal in $L^2[0,1] \otimes \mathbb{C}^n$.

**Proof.** It follows from the assumptions of the corollary that the operator $L_{C,D}(0)$ with $Q = 0$ is selfadjoint. It remains to apply Corollary 4.3. □

**Remark 4.5.** (i) In case of the $2 \times 2$ Dirac system, a close problem on completeness of matrix solutions in the space of matrix functions is studied in [31]. Moreover, conditions (4.2) are equivalent to conditions (1.3.39) from [31].

(ii) In case of the simplest operator $L_{C,D} = -iI_n \otimes \frac{d}{dx}$ ($B = I_n, Q = 0$), another (and rather complicated) proof of Corollary 4.3 was obtained in [17].

(iii) Corollary 4.4 is implied by the known M.V. Keldysh theorem ([20], [18], [32]) since the operator $L_{C,D}(0)$ of the form (1.2), (1.4) with $Q = 0$ is selfadjoint, and its resolvent has a finite (equal to 1) order.

Next we show that, in the case of zero potential matrix, $Q \equiv 0$, conditions (4.2) of Proposition 4.1 are also necessary.

**Proposition 4.6.** The system of root functions of the boundary problem

\[
-ibgy' = \lambda y, \quad B = B^*, \quad y = \text{col}(y_1, \ldots, y_n), \quad Cg(0) + Dy(1) = 0, \quad (4.4) \quad (4.5)
\]

is incomplete in $L^2[0,1] \otimes \mathbb{C}^n$ whenever $\det T_-(B; C, D) = 0$. Moreover, in this case its defect is infinite.

**Proof.** Since $\det T_-(B; C, D) = 0$, one of the boundary conditions is of the form

\[
\sum_{k=1}^{n} c_k y_k(\xi; \lambda) = 0, \quad \text{where} \quad \begin{cases} \xi = 0, & \text{for } b_k > 0, \\ \xi = 1, & \text{for } b_k < 0. \end{cases} \quad (4.6)
\]

Every solution $Y(x; \lambda) = \text{col}(y_1(x; \lambda), \ldots, y_n(x; \lambda))$ of equation (4.4) satisfying condition (4.6) admits the following representation

\[
y_k(x; \lambda) = \begin{cases} a_k e^{ib_k |\lambda| x}, & \text{for } b_k > 0, \\ a_k e^{ib_k |\lambda| (1-x)}, & \text{for } b_k < 0, \end{cases} \quad \text{where} \quad \sum_{k=1}^{n} c_k a_k = 0. \quad (4.7)
\]
Let $\alpha$ be a positive number such that $\frac{\alpha}{|b_k|} < 1$ for all $k$. We put for $b_k > 0$

$$\varphi_k(x) = \begin{cases} \frac{\alpha}{|b_k|}, & \text{for } 0 \leq x < \frac{\alpha}{|b_k|} \\ 0, & \text{for } \frac{\alpha}{|b_k|} \leq x \leq 1 \end{cases},$$

for $b_k < 0$

$$\varphi_k(x) = \begin{cases} \frac{\alpha}{|b_k|}, & \text{for } 0 \leq 1 - x < \frac{\alpha}{|b_k|} \\ 0, & \text{for } \frac{\alpha}{|b_k|} \leq 1 - x \leq 1 \end{cases},$$

and $\Phi(x) := \text{col}(\varphi_1(x), \ldots, \varphi_n(x))$. From (4.7) one gets

$$(y_k(x; \lambda), \varphi_k(x))_{L^2[0,1]} = \int_0^{\alpha} a_k e^{i \lambda |b_k|} c_k |b_k| \, dx = c_k a_k \int_0^\alpha e^{i \lambda t} \, dt.$$

Here we use the change $x \to 1 - x$ for $b_k < 0$. It follows that

$$(Y(x; \lambda), \Phi(x)) = \sum_{k=1}^n (y_k(x; \lambda), \varphi_k(x))_{L^2[0,1]} = \left( \sum_{k=1}^n c_k a_k \right) \int_0^\alpha e^{i \lambda t} \, dt = 0.$$

Thus, $\Phi(\cdot)$ is orthogonal to all the solutions of equation (4.4) satisfying condition (4.6). Hence it is orthogonal to the system of root functions of the operator $L_{C,D}$. Thus, the system of root functions of the operator $L_{C,D}$ is incomplete. \qed

**Remark 4.7.** All the results of this section including Theorem 1.2 and Propositions 4.1 and 4.6 remain valid (with the same proofs) for $Q \in L^1[0,1] \otimes \mathbb{C}^{n \times n}$. We stated them for $Q \in L^2[0,1] \otimes \mathbb{C}^{n \times n}$ because only in this case the domain $\text{dom}(L_{C,D}(Q))$ has simple description (1.5). Moreover, the results on completeness remain valid for the spaces $L^p[0,1] \otimes \mathbb{C}^n$ with $p \in [1, \infty)$.

5. **Irregular BVP for $2 \times 2$ Dirac Type Systems**

5.1. **Sufficient conditions of completeness.** Here we substantially supplement Proposition 4.1 confining ourselves to the case of the second order system ($n = 2$). We consider irregular BC and indicate other completeness conditions that depend on $Q$. In particular, we show that, as distinct from the case $Q(\cdot) \equiv 0$, conditions (4.2) of Proposition 4.1 are not necessary for the completeness of the system of root functions even in the case of $Q(\cdot) \neq Q^*(\cdot) \neq 0$ and dissipative (accumulative) boundary conditions.

Consider the $2 \times 2$ Dirac type system:

$$-iB y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1], \quad (5.1)$$

where

$$B = \text{diag}(b_1^{-1}, b_2^{-1}), \quad b_1 < 0 < b_2 \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}. \quad (5.2)$$

To the system (5.1) we join boundary conditions (1.4) rewritten for convenience in the form

$$U_j(y) := a_{j1} y_1(0) + a_{j2} y_2(0) + a_{j3} y_1(1) + a_{j4} y_2(1) = 0, \quad j \in \{1,2\}. \quad (5.3)$$

Further, let $\Phi(x; \lambda)$ be the fundamental matrix of the system (5.1) (uniquely) determined by the initial condition $\Phi(0; \lambda) = I_2$, i.e.,

$$\Phi(x; \lambda) := \begin{pmatrix} \varphi_1(x; \lambda) & \varphi_2(x; \lambda) \\ \varphi_1(x; \lambda) & \varphi_2(x; \lambda) \end{pmatrix}, \quad \varphi_j(x; \lambda) := \begin{pmatrix} \varphi_{1j}(x; \lambda) \\ \varphi_{2j}(x; \lambda) \end{pmatrix}, \quad j \in \{1,2\},$$

where $\varphi_j(x; \lambda)$ is the $j$-th component of $\varphi_j(x; \lambda)$.
Lemma 5.4. Let the representations $Q(x;\lambda) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\Phi_1(0;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Phi_2(0;\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The eigenvalues of problem (5.1)–(5.3) are the roots of the characteristic equation $\Delta(\lambda) := \det U(\lambda) = 0$, where

$$U(\lambda) := \begin{pmatrix} U_1(\Phi_1(x;\lambda)) & U_1(\Phi_2(x;\lambda)) \\ U_2(\Phi_1(x;\lambda)) & U_2(\Phi_2(x;\lambda)) \end{pmatrix} = \begin{pmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{21}(\lambda) & u_{22}(\lambda) \end{pmatrix}. \quad (5.4)$$

By putting $J_{jk} = \det \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}$, $j,k \in \{1,\ldots,4\}$, we arrive at the following expression for the characteristic determinant:

$$\Delta(\lambda) = J_{12} + J_{34} e^{(b_1 + b_2)\lambda} + J_{32} \varphi_{11}(\lambda) + J_{13} \varphi_{12}(\lambda) + J_{42} \varphi_{21}(\lambda) + J_{14} \varphi_{22}(\lambda), \quad (5.5)$$

where $\varphi_{jk}(\lambda) := \varphi_{jk}(1;\lambda)$. If $Q = 0$ then $\varphi_{12}(x;\lambda) = \varphi_{21}(x;\lambda) = 0$, and the characteristic determinant $\Delta_0(\lambda)$ has the form

$$\Delta_0(\lambda) = J_{12} + J_{34} e^{(b_1 + b_2)\lambda} + J_{32} e^{ib_1\lambda} + J_{14} e^{ib_2\lambda}. \quad (5.6)$$

For the problem (5.1)–(5.2) we have $\det(T_+) = J_{32}$ and $\det(T_-) = J_{14}$ where $T_\pm$ are defined by (4.1). Thus, condition (4.2) means that $J_{32} \cdot J_{14} = \det(T_1 \cdot T_2) \neq 0$ and presents the regularity condition of problem (5.1)–(5.3). For the Dirac system $(-b_1 = b_2 = 1)$, the regularity condition is stronger than the nondegeneracy of boundary conditions; the last one means that $\Delta_0(\lambda) \neq J_{12} + J_{34} = \text{const.}$

**Theorem 5.1.** Let $Q_{12}(\cdot), Q_{21}(\cdot) \in C[0,1]$. If

$$|J_{32}| + |b_1 J_{13} Q_{12}(0) + b_2 J_{42} Q_{21}(1)| \neq 0, \quad (5.7)$$

$$|J_{14}| + |b_1 J_{13} Q_{12}(1) + b_2 J_{42} Q_{21}(0)| \neq 0, \quad (5.8)$$

then the system of root functions of problem (5.1)–(5.3) (i.e. of the operator $L_{C,D}(Q)$) is complete and minimal in $L^2([0,1];\mathbb{C}^2)$.

**Corollary 5.2.** Let $Q_{12}(\cdot), Q_{21}(\cdot) \in C[0,1]$, and let $J_{32} = J_{14} = 0$. If

$$b_1 J_{13} Q_{12}(0) + b_2 J_{42} Q_{21}(1) \neq 0, \quad (5.9)$$

$$b_1 J_{13} Q_{12}(1) + b_2 J_{42} Q_{21}(0) \neq 0, \quad (5.10)$$

then the system of root functions of problem (5.1)–(5.3) is complete and minimal.

**Remark 5.3.** (i) In the case $b_1 = b_2 = 1$, Theorem 5.1 gives completeness even in the case of degenerated boundary conditions.

(ii) If $J_{32} = J_{14} = 0$, $Q_{12}(\cdot) = Q_{21}(\cdot)$ and $Q_{12}(0) = Q_{12}(1) \neq 0$, then conditions (5.7)–(5.8) acquire the form $b_1 J_{13} + b_2 J_{42} \neq 0$ not depending on $Q$.

(iii) If the BC are $y_1(0) = y_1(1) = 0$, then conditions (5.7)–(5.8) acquire a simple form $Q_{12}(0) : Q_{12}(1) \neq 0$ not depending on $Q_{21}$. In this case the system of root functions of the unperturbed operator $L_{C,D}(0)$ (with $Q = 0$) is incomplete.

To prove this theorem, we use the transformation operators existing for general systems of the form (1.2) with $B = B^*$ due to [27, Theorem 1].

**Lemma 5.4.** [27] Assume that $e_{\pm}(\cdot;\lambda)$ are solutions of the system (5.1) corresponding to the initial conditions $e_+(0;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_-(0;\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $e_{\pm}(\cdot;\lambda)$ admit the representations

$$e_{\pm}(x;\lambda) = (I + K_{\pm}) e_{\pm}^0(x;\lambda) = e_{\pm}^0(x;\lambda) + \int_0^x K_{\pm}(x,t) e_{\pm}^0(t;\lambda) dt, \quad (5.11)$$
and by the boundary conditions

\[ e^0_\pm(x; \lambda) = \left( e^{ib_1 \lambda x} \pm e^{ib_2 \lambda x} \right), \quad K_\pm(x, t) = \left( K^\pm_1(x, t) \right)_{j, k = 1}, \]

and \( K^\pm_{ij}(\cdot, \cdot) \in W^1_1(\Omega), \ \Omega = \{0 \leq t \leq x \leq 1\}. \) Moreover, \( K^\pm_{ij} \in C^1(\Omega) \) if \( Q \in C(\Omega) \otimes C^{2 \times 2} \).

The following lemma is the key result for proving Theorem 5.1. It is similar to the known statement for Sturm-Liouville operator (cf. [28, Lemma 6]).

**Lemma 5.5.** Let \( Q(\cdot) \in C(\Omega) \otimes C^{2 \times 2} \), and let \( K_\pm(\cdot, \cdot) \) be the kernels of the transformation operators given by (5.11). Then the following relations hold:

\[
\begin{align*}
K_1^+(1, 1) - K_1^-(1, 1) &= 2ib_1(b_2 - b_1)^{-1} \cdot b_1 Q_{12}(0), \quad (5.12) \\
K_1^+(1, 1) + K_1^-(1, 1) &= 2ib_1(b_2 - b_1)^{-1} \cdot b_2 Q_{21}(1), \quad (5.13) \\
K_1^+(1, 1) - K_1^-(1, 1) &= 0, \quad (5.14) \\
K_1^+(1, 1) - K_1^-(1, 1) &= 0, \quad (5.15) \\
K_2^+(1, 1) + K_2^-(1, 1) &= 2ib_2(b_1 - b_2)^{-1} \cdot b_1 Q_{12}(1), \quad (5.16) \\
K_2^+(1, 1) - K_2^-(1, 1) &= 2ib_2(b_1 - b_2)^{-1} \cdot b_2 Q_{21}(0). \quad (5.17)
\end{align*}
\]

**Proof.** In the case of \( Q(\cdot) \in C[0, 1] \otimes C^{2 \times 2} \), the kernels \( K^\pm(\cdot, \cdot) \) of the transformation operators are related by

\[ BD_x K^\pm(x, t) + D_x K^\pm(x, t)B = -iQ(x)K^\pm(x, t), \quad (x, t) \in \Omega, \]

and by the boundary conditions

\[
\begin{align*}
&K^\pm_1(x, x) = i \frac{b_1 b_2}{b_1 - b_2} Q_{12}(x), \\
&K^\pm_2(x, x) = i \frac{b_1 b_2}{b_2 - b_1} Q_{21}(x), \\
&2b_2 K^\pm_1(x, 0) \pm b_1 K^\pm_2(x, 0) = 0, \\
&b_2 K^\pm_2(x, 0) \pm b_1 K^\pm_2(x, 0) = 0
\end{align*}
\]

(see [27]). Relations (5.13)–(5.16) are immediately implied by (5.19).

Further, the kernels \( K^\pm(\cdot, \cdot) \) are related by

\[ K^+(x, t) = K^-(x, t) + \Psi(x - t) + \int_t^x K^-(x, s) \Psi(s - t) ds \]

(see [27, formula (1.44)]), where \( \Psi(\cdot) \) stands for the diagonal matrix function, \( \Psi(\cdot) = \text{diag}(\Psi_1(\cdot), \Psi_2(\cdot)) \in C^1[0, 1] \otimes C^{2 \times 2} \). It follows from (5.19)–(5.21) that

\[
\begin{align*}
\Psi_1(0) &= K^+_1(0, 0) - K^-_1(0, 0) = -b_1 b_2^{-1} \left( K^+_1(0, 0) + K^-_1(0, 0) \right) \\
&= 2ib_1^2(b_2 - b_1)^{-1} Q_{12}(0), \quad (5.22) \\
\Psi_2(0) &= K^+_2(0, 0) - K^-_2(0, 0) = -b_1 b_2^{-1} \left( K^+_2(0, 0) + K^-_2(0, 0) \right) \\
&= -2ib_2^2(b_2 - b_1)^{-1} Q_{21}(0). \quad (5.23)
\end{align*}
\]

On the other hand, due to (5.21) we have

\[ K^+_j(1, 1) - K^-_j(1, 1) = \Psi_j(0), \quad j \in \{1, 2\}. \quad (5.24) \]

Combining (5.22) and (5.23) with (5.24) we arrive at relations (5.12), (5.17). \( \square \)

**The proof of Theorem 5.1.** (i) The spectrum \( \sigma(L_{C,D}) \) of the operator \( L_{C,D} \) generated by problem (5.1)–(5.3) in \( L^2([0, 1]; C^2) \) coincides with the zero set of the
determinant $\Delta(\cdot)$, and the multiplicity $p_n$ of the zero $\lambda_n$ of the (entire) function $\Delta(\cdot)$ coincides with the dimension of the root subspace

$$\mathcal{H}_n := \text{span}\{\ker(L_{C,D} - \lambda_n)^k : k \in \mathbb{Z}_+\}, \quad \dim \mathcal{H}_n = p_n$$

(see [1, Sec.5.6], [32], [36]). Let us introduce solutions $w_j(x; \lambda)$ of (5.1) by setting

$$w_1(x; \lambda) := u_{22}(\lambda)\Phi_1 - u_{21}(\lambda)\Phi_2, \quad w_2(x; \lambda) := -u_{12}(\lambda)\Phi_1 + u_{11}(\lambda)\Phi_2,$$

where $u_{j1}(\cdot), u_{j2}(\cdot)$ are entries of the matrix $U(\cdot)$ of the form (5.4). Clearly, $U_j(w_j) = \Delta(\lambda)$ and $U_1(w_2) = U_2(w_1) = 0$; in particular, $U_j(w_j; \lambda_n) = \Delta(\lambda_n) = 0$. Further, the functions $w_j^{(k)}(x; \lambda) := D^k w_j(x; \lambda)$ satisfy the equations

$$L w_j^{(k)} = \lambda w_j^{(k)} + kw_j^{(k-1)}, \quad j \in \{1, 2\}. \quad (5.26)$$

Since $U_j(D^k w_j(x; \lambda)) = L^k(U_j(w_j(x; \lambda)))$ and $\lambda_n$ is the root of characteristic determinant $\Delta(\cdot)$ of multiplicity $p_n$, then the functions $D^k w_j(x; \lambda)|_{\lambda = \lambda_n}, k \in \{1, \ldots, p_n\}$, satisfy boundary conditions (5.3) as well. Hence in the case of $\dim \ker(L_{C,D} - \lambda_n) = 1$, at least one of the two systems $\{w_j^{(k)}(\cdot; \lambda)\}_{k=1}^{p_n}, j \in \{1, 2\}$, forms a chain of an eigenfunction and associated functions.

If $\dim \ker(L_{C,D} - \lambda_n) = 2$, the root subspace $\mathcal{H}_n$ has the form

$$\mathcal{H}_n = \text{span}\{D^k w_j(x; \lambda)|_{\lambda = \lambda_n}, k \in \{0, 1, \ldots, p_n - 1\}, j \in \{1, 2\}\} \quad (5.27)$$

By assuming that the system of root functions of the operator $L_{C,D}$ is incomplete in $L^2([0,1]; \mathbb{C}^2)$, we find a vector $(0 \neq f = \text{col}(f_1, f_2)$ orthogonal to this system. Hence we conclude that the entire functions

$$w_j(\lambda; f) := \int_0^1 \langle w_j(x; \lambda), f(x) \rangle \, dx \quad j \in \{1, 2\}, \quad (5.28)$$

have a zero of multiplicity $\geq p_n$ at every point $\lambda_n \in \sigma(L_{C,D})$. Thus, $G_j(\cdot; f) := w_j(\cdot; f)/\Delta(\cdot), j \in \{1, 2\}$, is the entire function. Let us estimate their growth.

(ii) First we estimate the growth of $\Delta(\cdot)$ from below. Since $\Phi(0; \lambda) = I_2$ and $e_{\pm}(0; \lambda) = (\pm 1)$ due to (5.11), we have

$$2\Phi_1(\cdot; \lambda) = e_+(\cdot; \lambda) + e_-(\cdot; \lambda), \quad 2\Phi_2(\cdot; \lambda) = e_+(\cdot; \lambda) - e_-(\cdot; \lambda).$$

By setting

$$R_{jk}^+(t) := K_{jk}^+(1,t) \pm K_{jk}^-(1,t), \quad j, k \in \{1, 2\}, \quad (5.29)$$

and by taking into account representations (5.11) for the solutions $e_{\pm}(\cdot; \lambda)$, we obtain

$$2\varphi_{11}(1; \lambda) = 2e^{ib_1\lambda} + \int_0^1 R_{11}^+(t)e^{ib_1\lambda t} \, dt + \int_0^1 R_{11}^-(t)e^{ib_1\lambda t} \, dt, \quad (5.30)$$

$$2\varphi_{12}(1; \lambda) = \int_0^1 R_{11}^+(t)e^{ib_1\lambda t} \, dt + \int_0^1 R_{12}^+(t)e^{ib_2\lambda t} \, dt, \quad (5.31)$$

$$2\varphi_{21}(1; \lambda) = \int_0^1 R_{21}^+(t)e^{ib_1\lambda t} \, dt + \int_0^1 R_{22}^+(t)e^{ib_2\lambda t} \, dt, \quad (5.32)$$

$$2\varphi_{22}(1; \lambda) = 2e^{ib_2\lambda} + \int_0^1 R_{21}^-(t)e^{ib_1\lambda t} \, dt + \int_0^1 R_{22}^-(t)e^{ib_2\lambda t} \, dt. \quad (5.33)$$

Noting that $R_{jk}^\pm(\cdot) \in C^1[0,1], j, k \in \{1, 2\}$, we integrate by parts in (5.30), (5.32), (5.31) and (5.33) and insert the expressions thus obtained
into (5.5). Then we arrive at the following expression for the characteristic determinant

$$\Delta(\lambda) = J_{12} + J_{34}e^{i(b_1 + b_2)\lambda} + (J_{32} + \frac{r_1(1)}{2ib_1\lambda}) e^{ib_1\lambda} + (J_{14} + \frac{r_2(1)}{2ib_2\lambda}) e^{ib_2\lambda}$$

$$- \frac{r_1(0)}{2ib_1\lambda} - \frac{r_2(0)}{2ib_2\lambda} - \int_0^1 r_1'(t) e^{ib_1\lambda t} dt - \int_0^1 r_2'(t) e^{ib_2\lambda t} dt$$

(5.34)

in which

$$r_1(t) := J_{32}R_{11}^+(t) + J_{13}R_{11}^-(t) + J_{42}R_{21}^+(t) + J_{14}R_{21}^-(t),$$

$$r_2(t) := J_{32}R_{12}^+(t) + J_{13}R_{12}^-(t) + J_{42}R_{22}^+(t) + J_{14}R_{22}^-(t).$$

By Lemma 5.5,

$$J_{32} + \frac{r_1(1)}{2ib_1\lambda} = J_{32} \left(1 + \frac{R_{11}^+(1)}{2ib_1\lambda}\right) + \frac{b_1J_{13}Q_{12}(0) + b_2J_{42}Q_{21}(1)}{(b_2 - b_1)\lambda},$$

$$J_{14} + \frac{r_2(1)}{2ib_2\lambda} = J_{14} \left(1 + \frac{R_{22}^+(1)}{2ib_2\lambda}\right) + \frac{b_1J_{13}Q_{12}(1) + b_2J_{42}Q_{21}(0)}{(b_1 - b_2)\lambda}.$$ (5.35)

(5.36)

Conditions (5.7)–(5.8) yield now that

$$\left|J_{32} + \frac{r_1(1)}{2ib_1\lambda}\right| \geq \frac{c}{|\lambda| + 1}, \quad \left|J_{14} + \frac{r_2(1)}{2ib_2\lambda}\right| \geq \frac{c}{|\lambda| + 1}, \quad c > 0, \quad \lambda \in \mathbb{C}\setminus\{0\}.$$ (5.37)

This implies the desired estimates for $\Delta(\cdot)$ from below:

$$|\Delta(\lambda)| \geq \frac{c}{|\lambda| + 1} \exp(|b_+\Im\lambda|), \quad \lambda \in \Omega^\pm := \{\lambda : \epsilon \leq \pm \arg\lambda \leq \pi - \epsilon\},$$

where $b_- := b_1, \ b_+ := b_2$.

(iii) In this step we estimate the growth of $w_j(\cdot; f)$ from above. We show that

$$w_j(x; \lambda) = O(\exp(|b_+\Im\lambda|)), \quad \lambda \in \Omega^\pm.$$ (5.38)

Let $Y_j := \text{col}(y_j, y_{2j}), \ j \in \{1, 2\}$, be the solution of (5.1) satisfying (3.26), i.e.

$$y_{kj}(x, \lambda) = (b_k^0 + o(1)) \exp(ib_k\lambda x), \quad \lambda \in \Omega^+_x, \quad j, k \in \{1, 2\},$$ (5.39)

and let $\overline{U}(\lambda) := (\overline{u}_{jk}(\lambda))^2_{j,k=1} := (\overline{U}_j(Y_k))_{j,k=1}$. Alongside solutions (5.25) we introduce solutions

$$V_1(x, \lambda) = \overline{u}_{22}(\lambda)Y_1 - \overline{u}_{21}(\lambda)Y_2, \quad V_2(x, \lambda) = -\overline{u}_{12}(\lambda)Y_1 + \overline{u}_{11}(\lambda)Y_2.$$ (5.40)

According to (3.27) and (3.30) the fundamental matrices $\Phi(x, \cdot)$ and $Y(x, \cdot) = (Y_1(\cdot; \lambda) \ Y_2(\cdot; \cdot))$ of equation (5.1) as well as the matrices $U(\cdot)$ and $\overline{U}(\cdot)$ are connected by

$$\Phi(x, \lambda) = Y(x, \lambda)P(\lambda) \quad \text{and} \quad U(\lambda) = \overline{U}(\lambda)P(\lambda), \quad \lambda \in \Omega^+_x,$$ (5.41)

where $P(\cdot)$ is the invertible holomorphic $2 \times 2$ matrix function. Hence (cf. (3.32))

$$w_j(x, \lambda) = V_j(x, \lambda) \det P(\lambda), \quad \lambda \in \Omega^+_x, \quad j \in \{1, 2\}.$$ (5.42)

It follows from (5.39) that

$$\overline{u}_{11}(\lambda) = O(e^{ib_1\lambda}), \quad \overline{u}_{12}(\lambda) = O(1), \quad \overline{u}_{21}(\lambda) = O(e^{ib_1\lambda}), \quad \overline{u}_{22}(\lambda) = O(1),$$

as $\lambda \to \infty, \ \lambda \in \Omega^+_x$. It follows with account of (5.39) and (5.40) that

$$V_1(x, \lambda) = O(e^{ib_1\lambda}), \quad V_2(x, \lambda) = O(e^{ib_1\lambda}) \quad \text{as} \ \lambda \to \infty, \ \lambda \in \Omega^+_x.$$ (5.43)
Moreover, substituting \( x = 0 \) in the first of equalities (5.41) and taking into account (5.39) and \( \Phi(0, \lambda) = I_2 \), we get \( P(\lambda) = I_2 + a_2(\lambda) \). Combining this relation with (5.42) and (5.43) yields (5.38) for \( \lambda \in \Omega^+ \). The second relation in (5.38) is proved similarly. In turn, combining estimates (5.38) with (5.28) yields

\[
 w_j(\lambda; f) = o(\exp(\|b\|_\infty \lambda)) \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Omega^+. \tag{5.44}
\]

Hence applying the Phragmen-Lindelöf theorem to the functions \( G_j(\cdot; f) \) in the angles \( \Omega^\pm \), we conclude that \( G_j(\cdot; f) = \text{const} \), \( j \in \{1, 2\} \). Using the same technique as in [28] one can prove that \( G_j(\cdot; f) = 0 \), \( j \in \{1, 2\} \). Now the proof is completed by applying steps (iv) and (v) of the proof of Theorem 1.2. The minimality is implied by Lemma 2.4.

\[\square\]

**Remark 5.6.** In the case of Dirac system \((-b_1 = b_2 \in \mathbb{R}_+)\) the step (iii) of the proof can be substantially simplified. To this end we set

\[
 \Phi_{jr}(x, \lambda) := \varphi_{2j}(1, \lambda) \Phi_1(x, \lambda) - \varphi_{1j}(1, \lambda) \Phi_2(x, \lambda), \quad j \in \{1, 2\}. \tag{5.45}
\]

Clearly, \( \Phi_{jr}(\cdot, \lambda) \) is the solution of equation (5.1). Moreover, since \( \text{tr} Q(x) = 0, \ x \in [0, 1] \), by Liouville theorem det \( \Phi(x, \lambda) = \det \Phi(0, \lambda) = 1, \ x \in [0, 1] \). Hence

\[
 \Phi_{1r}(1, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \Phi_{2r}(1, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.46}
\]

On the other hand,

\[
w_j(x, \lambda) = [a_{j1} \varphi_{12}(0, \lambda) + a_{j2} \varphi_{22}(0, \lambda) + a_{j3} \varphi_{12}(1, \lambda) + a_{j4} \varphi_{22}(1, \lambda)] \Phi_1(x, \lambda) \\
-\left[a_{j1} \varphi_{11}(0, \lambda) + a_{j2} \varphi_{21}(0, \lambda) + a_{j3} \varphi_{11}(1, \lambda) + a_{j4} \varphi_{21}(1, \lambda)\right] \Phi_2(x, \lambda) \\
= a_{j2} \Phi_1(x, \lambda) - a_{j1} \Phi_2(x, \lambda) + a_{j3} \Phi_{1r}(x, \lambda) + a_{j4} \Phi_{2r}(x, \lambda).
\]

Combining this representation with (5.28) we arrive at (5.38).

**Corollary 5.7.** Assume the conditions of Theorem 5.1. Then the system of root functions of the operator \( L^*_{C,D} \) is also complete and minimal in \( L^2([0, 1]; \mathbb{C}^2) \).

**Proof.** If \( J_{32} \neq 0 \) and \( J_{14} \neq 0 \) then Corollary 3.2 is applicable. Now let \( J_{32} \cdot J_{14} = 0 \). Then one of the conditions (5.9) or (5.10) holds.

Hence either \( J_{13} \neq 0 \) or \( J_{22} \neq 0 \). Without loss of generality we assume that \( J_{13} \neq 0 \). Let \( a_j := \text{col}(a_{j1}, a_{j2}), \ j \in \{1, \ldots, 4\} \). Then \( J_{13} \neq 0 \) implies \( a_1 \neq 0 \) and \( a_3 \neq 0 \). Now we consider three cases.

(i) \( J_{13} = J_{32} = 0 \). Then conditions (5.9) and (5.10) hold true. Since \( a_1 \neq 0 \) and \( a_3 \neq 0 \) then \( a_4 = a_1 a_3 \) and \( a_2 = a_3 a_4 \) with some \( a_1, a_2 \in \mathbb{C} \). Hence conditions (5.3) are equivalent to the following ones:

\[
y_1(0) = -a_1 y_2(1), \quad y_1(1) = -a_2 y_2(0), \tag{5.47}
\]

It can easily be seen that the adjoint operator \( L^*_{C,D} := (L_{C,D})^* \) is defined by the differential expression \( L^* = -iBd/dx + Q^*(x) \), where

\[
 Q^*(x) = \begin{pmatrix} 0 & Q^*_{12}(x) \\ Q^*_{21}(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q_{12}(x) \\ Q_{21}(x) & 0 \end{pmatrix},
\]

and the boundary conditions

\[
 U_{1*}(y) := \overline{a_1} b_2 y_1(0) + b_1 y_2(1) = 0, \quad U_{2*}(y) := b_1 y_2(0) + \overline{a_2} b_2 y_1(1) = 0. \tag{5.48}
\]

It follows from (5.47) and (5.48) that

\[
 J_{12*} = b_1^2 \overline{J}_{13} \quad \text{and} \quad J_{13*} = b_2^2 \overline{J}_{32} = b_2^2 \overline{J}_{21} = b_2^2 \overline{J}_{32}. \tag{5.49}
\]
Now we check conditions (5.9), (5.10) for the operator $L_{C,D}^\ast$. Due to (5.49), expressions (5.9), (5.10) for $L_{C,D}^\ast$ are of the form
\[
\begin{align*}
b_1 J_{13}, Q_{12}, (0) + b_2 J_{13}, Q_{21}, (1) &= b_1 b_2 [b_2 \mathcal{J}_{42} Q_{21}(0) + b_1 \mathcal{J}_{13} Q_{12}(1)], \\
b_2 J_{42}, Q_{21}, (1) + b_1 J_{13}, Q_{12}, (0) &= b_1 b_2 [b_2 \mathcal{J}_{42} Q_{21}(1) + b_1 \mathcal{J}_{13} Q_{12}(0)],
\end{align*}
\]
and different form zero by the assumptions of Theorem 5.1.

(ii) $J_{32} = 0, J_{14} \neq 0$. Then condition (5.9) hold true. Since $a_3 \neq 0$ the condition $J_{32} = 0$ means that $a_2 = a_3$ with some $\alpha \in \mathbb{C}$. Since $J_{14} \neq 0$ we represent boundary conditions (5.3) as
\[
\begin{pmatrix}
y_1(0) \\
y_2(1)
\end{pmatrix} = - \begin{pmatrix} a_{11} & a_{14} \\
a_{21} & a_{24}
\end{pmatrix}^{-1} \begin{pmatrix} a a_{13} & a_{13} \\
a a_{23} & a_{23}
\end{pmatrix} \begin{pmatrix} y_2(0) \\
y_1(1)
\end{pmatrix} = \begin{pmatrix} \alpha \beta_1 & \beta_1 \\
\alpha \beta_2 & \beta_2
\end{pmatrix} \begin{pmatrix} y_2(0) \\
y_1(1)
\end{pmatrix},
\]
where $\beta_1 = -J_{14}^{-1} J_{24}$ and $\beta_2 = -J_{14}^{-1} J_{12}$. Thus, conditions (5.3) take the form
\[
\begin{align*}
U_1(y) &= 1 \cdot y_1(0) - \alpha \beta_1 \cdot y_2(0) - \beta_1 \cdot y_1(1) + 0 \cdot y_2(1) = 0, \\
U_2(y) &= 0 \cdot y_1(0) - \alpha \beta_2 \cdot y_2(0) - \beta_2 \cdot y_1(1) + 1 \cdot y_2(1) = 0.
\end{align*}
\]
(5.50)

Now boundary conditions for the adjoint operator $L_{C,D}^\ast$ are rewritten as follows:
\[
\begin{align*}
U_{1*}(y) &= -b_1^{-1} \mathcal{J}_{13} y_1(0) + 0 \cdot y_2(0) + b_1^{-1} y_1(1) + b_2^{-1} \mathcal{J}_{2} y_2(1) = 0, \\
U_{2*}(y) &= b_1^{-1} \mathcal{J}_{13} y_1(0) + b_2^{-1} y_2(0) + 0 \cdot y_2(1) - b_2^{-1} \mathcal{J}_{2} y_2(1) = 0.
\end{align*}
\]
(5.51)

Both relations (5.50) and (5.51) yield that $J_{14*} = 0, J_{32*} = -b_1^{-1} b_2^{-1} \neq 0$ and
\[
b_1 J_{13*} = -b_1^{-1} \mathcal{J}_{13} = b_1^{-1} \mathcal{J}_{42}, \quad b_2 J_{42*} = -b_2^{-1} \mathcal{J}_{2} = b_2^{-1} \mathcal{J}_{13}.
\]
(5.52)

The equations thus obtained allow us to prove that the condition (5.10) for $L_{C,D}^\ast$ is equivalent to the conditions (5.9) for $L_{C,D}$. Indeed, taking account of relations $Q_{ij*}(x) = Q_{ji}(x), i \neq j$, and (5.52), we get
\[
\begin{align*}
b_1 J_{13}, Q_{12}, (1) + b_2 J_{42}, Q_{21}, (0) &= b_1^{-1} \mathcal{J}_{42} Q_{21}(1) + b_2^{-1} \mathcal{J}_{13} Q_{12}(0) \\
&= b_1^{-1} b_2^{-1} [b_1 J_{13} Q_{12}(0) + b_2 J_{42} Q_{21}(1)] \neq 0.
\end{align*}
\]
(5.53)

(iii) $J_{32} \neq 0, J_{14} = 0$. This case is similar to (ii).

Thus, in all cases the assumptions of Theorem 5.1 hold true for the adjoint operator $L_{C,D}^\ast$, and hence the system of its root functions is complete and minimal in $L^2([0,1]; \mathbb{C}^2)$.

\[\square\]

**Corollary 5.8.** Suppose that the operator $L_{C,D}(0)$ of the form (5.1)–(5.3) is dissipative. If $Q \in C^0[0,1] \otimes \mathbb{C}^{2 \times 2}$ and condition (5.9) is fulfilled, then the systems of root functions of both operators $L_{C,D}(Q)$ and $L_{C,D}^\ast(Q)$ are complete and minimal in $L^2([0,1]; \mathbb{C}^2)$.

**Proof.** Since $L_{C,D}(0)$ is dissipative, it follows from Lemma 4.2 that the condition $J_{14} = \det T_- \neq 0$ is met. It suffices to apply Theorem 5.1 and Corollary 5.7. \[\square\]

**Remark 5.9.** Dissipative boundary conditions for the equation (5.1) are always nondegenerated. But, as distinguished from the case of the Sturm-Liouville operator, they are not necessarily regular because they do not guarantee the validity of the first regularity condition in (4.2). Moreover, even in the case of $Q = Q^\ast$, the condition $\det T_1 \neq 0$ is not necessary for the completeness of the system of root functions of the dissipative operator $L_{C,D}(Q)$. \[\square\]
Note else that there exist non-Volterra dissipative operators \( L_{C,D}(Q) \) for which the system of root functions is not necessarily complete in \( L^2([0,1];\mathbb{C}^2) \).

Next we consider boundary conditions (5.3) of the special form

\[
U_1(y) := y_1(0) - \beta_1 y_2(0) = 0, \quad U_2(y) := y_2(1) - \beta_2 y_2(0) = 0.
\]

Corollary 5.10. Suppose that \( Q \in C[0,1] \otimes \mathbb{C}^{2 \times 2}, \beta_1 \in \mathbb{C} \setminus \{0\} \) and \( L_{C,D} \) is the operator of form (5.1)–(5.3), where \( U_1 \) and \( U_2 \) are defined by (5.54). Then:

(i) the operator \( L_{C,D} \) is dissipative whenever \( \text{Im} Q(x) \geq 0 \) and

\[
b^{-1}_2 |\beta_2|^2 \leq b^{-1}_1 |\beta_1|^2;
\]

(ii) if \( Q_{21}(1) \neq 0 \), then the system of root functions of the operator \( L_{C,D} \) is complete and minimal.

Proof. (ii) If \( Q_{21}(1) \neq 0 \), then Theorem 5.1 is applicable, since in this case we have \( J_{32} = 0 = J_{13} \) and \( J_{14} = 1 \) and \( J_{42} = \beta_1 \neq 0 \).

Remark 5.11. (i) We emphasize that for \( Q_{21}(1) \neq 0 \) the completeness (and the minimality) of the EAF system of the operator \( L_{C,D}(Q) \) holds in the assumptions of Corollary 5.10 with \( \beta_2 = 0 \) too. In the latter case the second of the conditions (5.54) is ”of Volterra type” and the corresponding operator \( L_{C,D}(0) \) with \( Q = 0 \) is incomplete. Moreover, for \( Q = 0 \) the operator \( L_{C,D}(0) \) has a Volterra inverse.

Remark 5.12. (i) Theorem 5.1 might be considered as an analog of a special case of the completeness result on BVP for Sturm-Liouville operators with degenerate BC (see [28, Theorem 1]). More general result even for \( n \times n \) Dirac type systems that involves considerations of derivatives of a smooth potential matrix \( Q \) is more complicated and will be considered in the forthcoming paper [26].

(ii) In connection with Theorem 5.1 and other results of this section we mention the papers [44], [45], [19] devoted to the Riesz basis property of EAF for BVP with separated (and hence strictly regular) BC for \( 2 \times 2 \) Dirac systems ([44], [45], [35]) and for \( 2 \times 2 \) Dirac type systems ([19]).

The Riesz basis property of EAF for BVP with regular but non-strictly regular (including periodic, antiperiodic and other) BC for \( 2 \times 2 \) Dirac systems have been investigated by P. Djakov and B. Mityagin [35], [7], [10]. Namely, in [35] and [7] they proved the Riesz basis property of subspaces (spectral projections) for \( 2 \times 2 \) Dirac system with periodic and antiperiodic BC. In the next publication [10] these authors extended their result to the case of arbitrary regular but not strictly regular BC. Moreover, in [10] they proved the Riesz basis property of the system of EAF for BVP with general strictly regular BC under the assumption \( Q_{12}, Q_{21} \in L^2[0,1] \).

5.2. Necessary conditions of completeness. Here we complete Theorem 5.1 by the following result on necessary conditions of completeness which demonstrate that conditions (5.7), (5.8) for the Dirac system are sharp.

Proposition 5.13. Assume that \( B = \text{diag}(-1,1), \ J_{14} = J_{32} = 0 \) but \( J_{13} J_{42} \neq 0 \). Further, let \( 0 \notin \text{supp} P_1 \cup \text{supp} P_2 \), where

\[
P_1(x) := J_{13}Q_{12}(x) - J_{42}Q_{21}(1-x), \quad P_2(x) := J_{13}Q_{12}(1-x) - J_{42}Q_{21}(x).
\]

Then the defect of the system of root functions of problem (5.1)–(5.3) in \( L^2[0,1] \otimes \mathbb{C}^2 \) is infinite.
Proof. By assumption, there exists an \( \varepsilon > 0 \) such that
\[
P_1(x) = P_2(x) = 0, \quad x \in [0, \varepsilon] \cup [1 - \varepsilon, 1].
\] (5.57)
Let \( w_j := \text{col}(w_{j1}, w_{j2}) \) be defined by (5.25). Since \( J_{14} = J_{32} = 0 \) and \( J_{13}J_{42} \neq 0 \), we conclude that the boundary conditions (5.3) are equivalent to the following ones
\[
y_1(0) = -\alpha_1 y_2(1), \quad y_2(0) = -\alpha_2 y_1(1),
\] (5.58)
where \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Denote
\[
z_j(x; \lambda) := \begin{pmatrix} z_{j1}(x; \lambda) \\ z_{j2}(x; \lambda) \end{pmatrix} := \begin{pmatrix} -\alpha_1 w_{j2}(1 - x; \lambda) \\ -\alpha_2 w_{j1}(1 - x; \lambda) \end{pmatrix}, \quad j \in \{1, 2\}.
\] (5.59)
Let us demonstrate that, for \( x \in [0, \varepsilon] \) and every \( k \in \mathbb{N} \), the functions \( z_j^{(k)}(x; \lambda) := D^k_{\alpha}z_j(x; \lambda), j \in \{1, 2\} \), alongside with the functions \( w_j^{(k)}(x; \lambda) \), satisfy the equation (5.26). Indeed, from (5.26) and with account of (5.57) and (5.59) we obtain:
\[
L z_j^{(k)} = -iB \frac{d}{dx} z_j^{(k)} + Q(x) z_j^{(k)} = i \frac{d}{dx} \left( -\alpha_1 w_{j2}^{(k)}(1 - x; \lambda) \right) + \left( Q_{12}(x) z_j^{(k)}(x; \lambda) \right) = -\lambda \alpha_2 w_{j2}^{(k)}(1 - x; \lambda) - kD_{\lambda}^{k-1} \alpha_2 w_{j2}^{(k)}(1 - x; \lambda)
\]
\[
= \lambda z_j^{(k)}(x; \lambda) + k z_j^{(k-1)}(x; \lambda), \quad x \in [0, \varepsilon], \quad j \in \{1, 2\}.
\]
Further, since \( w_j^{(k)}(x; \lambda_j) = D^k_{\alpha}w(x; \lambda)|_{\lambda=\lambda_j}, \quad k \in \{1, \ldots, p_n\} \) satisfy the boundary conditions (5.58), then from (5.58) and (5.59) we obtain that
\[
z_j^{(k)}(0; \lambda_j) = -\alpha_1 w_{j2}^{(k)}(1; \lambda_j) = w_{j1}^{(k)}(0; \lambda_j), \quad z_j^{(k)}(0; \lambda_j) = w_{j2}^{(k)}(0; \lambda_j),
\]
for \( j \in \{1, 2\} \) and \( n \in \mathbb{N} \). Therefore, by the uniqueness theorem we have
\[
D^k_{\alpha}z_j(x; \lambda)|_{\lambda=\lambda_j} = D^k_{\alpha}w_j(x; \lambda)|_{\lambda=\lambda_j}, \quad x \in [0, \varepsilon], \quad k \in \{0, 1, \ldots, p_n - 1\}.
\] (5.60)
Further, let \( f = \text{col}(f_1, f_2) \in L^2[0, 1] \otimes \mathbb{C}^2, \quad f(x) = 0 \) for \( x \in [\varepsilon, 1 - \varepsilon] \) and
\[
f_1(x) = \alpha_1^{-1} f_2(1 - x), \quad f_2(x) = \alpha_2^{-1} f_1(1 - x), \quad x \in [0, \varepsilon].
\] (5.61)
Let us show that \( f \) is orthogonal to the system of root functions of the problem. Taking account of (5.27), (5.59), (5.60) and (5.61), we obtain
\[
\int_0^\varepsilon \langle w_j^{(k)}(x; \lambda_j), f(x) \rangle dx = \int_0^\varepsilon \langle w_j^{(k)}(x; \lambda_j), f(x) \rangle dx
\]
\[
+ \int_0^\varepsilon \langle w_j^{(k)}(1 - x; \lambda_j), f(1 - x) \rangle dx = \int_0^\varepsilon \langle w_j^{(k)}(x; \lambda_j), f_1(x) - \alpha_1^{-1} f_2(1 - x) \rangle dx
\]
\[
+ \int_0^\varepsilon \langle w_j^{(k)}(x; \lambda_j), f_2(x) - \alpha_2^{-1} f_1(1 - x) \rangle dx = 0, \quad n \in \mathbb{N}.
\]
It follows that the defect of the system of root functions is infinite. \( \square \)

**Remark 5.14.** Proposition 5.13 is similar to that of [28, Proposition 9] for the Sturm-Liouville operator with degenerate boundary conditions.
6. Completeness of irregular BVP for 2 × 2 systems with \( B \neq B^* \)

Consider system (5.1) with the matrix \( B = \text{diag}(b_1^{-1}, b_2^{-1}) \neq B^* \) assuming that \( b_1/b_2 \notin \mathbb{R} \). In this case the lines \( \{ \lambda \in \mathbb{C} : \Re(ib_j\lambda) = 0 \} \), \( j \in \{1, 2\} \), divide the complex plane in two pairs of vertical sectors and Corollary 3.1 guarantees the completeness and the minimality of the root system of problem (5.1), (5.3) in the following cases:

\[
(i) \quad J_{14} J_{23} \neq 0 \quad \text{and} \quad (ii) \quad J_{12} J_{34} \neq 0.
\]  

(6.1)

Here we consider equation (5.1) subject to the boundary condition \( s \) following cases:

- The line \( \text{line} \) divides the complex plane into two half-planes. By Proposition 2.2, in each of these half-planes equation (5.1) has the fundamental system of solutions \( \{U_1(x;\lambda), U_2(x;\lambda)\} \) satisfying the asymptotics

\[
Y_1(x;\lambda) = \left( e^{ib_1 x (1 + o(1))} \right) \quad \text{and} \quad Y_2(x;\lambda) = \left( e^{ib_2 x (1 + o(1))} \right),
\]

(6.3)

as \( \lambda \to \infty \) uniformly with respect to \( x \in [0,1] \). In particular, in these half-planes,

\[
Y_1(0;\lambda) = \left( 1 + o(1) \right) \quad \text{and} \quad Y_2(0;\lambda) = \left( 0 \right) \quad \text{as} \quad \lambda \to \infty.
\]

(6.4)

Let \( \Phi_1(x;\lambda) = \left( \varphi_{11}(x;\lambda) \ \varphi_{12}(x;\lambda) \right) \) and \( \Phi_2(x;\lambda) = \left( \varphi_{21}(x;\lambda) \ \varphi_{22}(x;\lambda) \right) \) stand for the solutions of the Cauchy problem for system (5.1) satisfying the initial conditions

\[
\Phi_1(0;\lambda) = \left( 1 \right) \quad \text{and} \quad \Phi_2(0;\lambda) = \left( 0 \right).
\]

(6.5)

Then it follows from (6.4) and (6.5) that in any of the above half-planes

\[
\Phi_1(x;\lambda) = (1 + o(1))Y_1(x;\lambda) + o(1)Y_2(x;\lambda),
\]

\[
\Phi_2(x;\lambda) = o(1)Y_1(x;\lambda) + (1 + o(1))Y_2(x;\lambda).
\]

(6.6)

Hence the corresponding characteristic determinant is

\[
\Delta(\lambda) = \det \left( \begin{array}{cc}
1 & -h_0 \\
\varphi_{11}(1;\lambda) & \varphi_{12}(1;\lambda) - h_1
\end{array} \right) = -h_1 + h_0 \varphi_{11}(1;\lambda) + \varphi_{12}(1;\lambda) = -h_1 + h_0 e^{ib_1 \lambda} + o(e^{ib_1 \lambda}) + o(e^{ib_2 \lambda}).
\]

(6.7)

The vector function

\[
w(x;\lambda) = \left( \begin{array}{c}
w_1(x;\lambda) \\
w_2(x;\lambda)
\end{array} \right) = h_0 \Phi_1(x;\lambda) + \Phi_2(x;\lambda), \quad \lambda \in \mathbb{C},
\]

(6.8)

satisfies both the equation (5.1) and the first of the boundary conditions (6.2). Let the vector function \( f(x) = \text{col}(f_1(x), f_2(x)) \) be orthogonal to the system of root
functions of problem (5.1), (6.2). Then the quotient
\[ F(\lambda) = \frac{\langle w(x; \lambda), f(x) \rangle}{\Delta(\lambda)} = \frac{\int_0^1 (w_1(x; \lambda) \overline{f_1(x)} + w_2(x; \lambda) \overline{f_2(x)}) \, dx}{-h_1 + h_0 e^{ib_1 \lambda} + o(e^{ib_1 \lambda}) + o(e^{ib_2 \lambda})} \] (6.9)
is entire function of at most first growth.

Introduce the sector \( S_{b_1, b_2} \) by setting
\[ S_{b_1, b_2} := \{ \theta \in \mathbb{C} : 0 < \Re(i b_2 \theta) < \Re(i b_1 \theta) \}. \] (6.10)
Then, for \( t \to +\infty \), we obtain:
\[ \int_0^1 (w_1(x; \theta t) \overline{f_1(x)} + w_2(x; \theta t) \overline{f_2(x)}) \, dx = O \left( \int_0^1 |e^{ib_1 \theta t x}|(|f_1(x)| + |f_2(x)|) \, dx \right) = o(|e^{ib_2 \theta t}|), \quad \theta \in S_{b_1, b_2}. \] (6.11)
Similarly, we have
\[ \Delta(\lambda) = \Delta(\theta t) = -h_1 + h_0 e^{ib_1 \theta t} + o(e^{ib_1 \theta t}) + o(e^{ib_2 \theta t}) \sim |h_0 e^{ib_1 \theta t}|, \quad \theta \in S_{b_1, b_2}, \] (6.12)
as \( t \to \infty \). Combining (6.11) with (6.12) we arrive at the relation
\[ \lim_{t \to +\infty} F(\theta t) = \lim_{t \to +\infty} \frac{\int_0^1 (w_1(x; \theta t) \overline{f_1(x)} + w_2(x; \theta t) \overline{f_2(x)}) \, dx}{\Delta(\theta t)} = 0, \quad \theta \in S_{b_1, b_2}. \] (6.13)

On the other hand, for \( \theta \in S_{b_1, b_2} \) one gets
\[ \int_0^1 (w_1(x; \theta t) \overline{f_1(x)} + w_2(x; \theta t) \overline{f_2(x)}) \, dx = O \left( \int_0^1 |e^{ib_2 \theta t x}|(|f_1(x)| + |f_2(x)|) \, dx \right) \to 0 \quad \text{as} \quad t \to -\infty, \quad \theta \in S_{b_1, b_2}, \]
and
\[ \Delta(\lambda) = \Delta(\theta t) = -h_1 + h_0 e^{ib_1 \theta t} + o(e^{ib_1 \theta t}) + o(e^{ib_2 \theta t}) \to -h_1 \] as \( t \to -\infty, \theta \in S_{b_1, b_2} \).
Combining these estimates we obtain
\[ \lim_{t \to -\infty} \frac{\int_0^1 (w_1(x; \theta t) \overline{f_1(x)} + w_2(x; \theta t) \overline{f_2(x)}) \, dx}{\Delta(\theta t)} = 0, \quad \theta \in S_{b_1, b_2}. \] (6.14)

Choose numbers \( \theta_1, \theta_2 \in S_{b_1, b_2} \) not lying on the same line with the origin. Then the rays \( \theta_1 t, \theta_2 t \) \( (t > 0) \) and \( \theta_1 t, \theta_2 t \) \( (t < 0) \) divide the complex plane into four sectors with openings less than \( \pi \). It follows from estimates (6.13) and (6.14) that the function \( F(\cdot) \) is bounded on these rays. Being an entire function of order not exceeding one, the function \( F(\cdot) \) is bounded on each of these sectors, by the Phragmen-Lindelöf theorem. Thus, \( F(\cdot) \) is bounded on the whole complex plane and, by the Liouville theorem, it is a constant. It follows from (6.14) that \( F(\lambda) \equiv 0 \).

Thus, the vector function \( f(x) \) is orthogonal to \( w(x; \lambda) \) for all \( \lambda \). In particular, it is orthogonal to all solutions of the system (5.1) subject to the following boundary conditions
\[ \begin{cases} y_1(0) = h_0 y_2(0) \\ y_1(1) = y_2(1). \end{cases} \] (6.15)
In this case $J_{13}J_{34} \neq 0$, and conditions (6.15) are weakly regular. By Theorem 1.2, the system of the root functions of the problem (5.1), (6.15) is complete in $L^2([0,1];C^2)$. Hence $f(x) \equiv 0$.

The minimality property is implied by Lemma 2.4.

Theorems 1.2 and 6.1 make it possible to describe all boundary conditions for systems (5.1) with $Q = 0$ such that the root functions system of the problem (5.1), (5.3) is complete.

**Corollary 6.2.** Let $Q = 0$ in the assumptions of Theorem 6.1. Then the system of root functions of problem (5.1), (5.3) is incomplete if and only if the pair of the boundary conditions (5.3) is equivalent to that contained at least one of the "Volterra" conditions: $y_j(0) = 0$ or $y_j(1) = 0$, $j \in \{1,2\}$.

**Proof. Necessity.** Assume for simplicity that one of the boundary conditions is of the form $y_1(0) = 0$. Then the system of root functions of problem (5.1), (5.3) is either empty or has the form \( \{\text{col}(0,e^{i2\pi(n+\alpha)x})\}_{n \in \mathbb{Z}} \) for some $\alpha \in \mathbb{C}$. Clearly, it is incomplete in $L^2([0,1];C^2)$.

**Sufficiency.** Assume that the system of root functions is incomplete. Then by Theorem 1.2 condition (6.1) is violated. Without loss of generality we can assume that $J_{14} = 0$ and $J_{34} = 0$. Consider two cases.

(i) $J_{13} = 0$. Then the matrix composed of 1st, 3rd and 4th columns of the matrix $(C D)$ has rank 1. By equivalent transformations the matrix $(C D)$ of boundary conditions is reduced to the matrix with the only one non-zero entry in the second row. In other words, one of the boundary conditions is reduced to a "Volterra" condition $y_2(0) = 0$.

(ii) $J_{13} \neq 0$. Then the boundary conditions are equivalent to the following ones

$$y_1(0) = h_0y_2(0), \quad y_1(1) = h_1y_2(0),$$

that is, to conditions (6.2) with arbitrary $h_0, h_1$.

By Theorem 6.1 we have $h_0h_1 = 0$. Hence again one of the condition is of Volterra type.

We emphasize that as distinct from Theorem 5.1 the assumptions of Theorem 6.1 do not depend on $Q$. Moreover, Theorem 6.1 shows that Proposition 4.6 is no longer valid whenever $B \neq B^*$. In other words, as distinct from the case of $B = B^*$, the **weak regularity of boundary conditions (1.4) is not necessary for completeness of the operator** $L_{C,D}(0)$ **with** $Q = 0$. **However**, the following criterion takes place.

**Corollary 6.3.** Let $n = 2$ and $B = \text{diag}(b_1^{-1}, b_2^{-1})$ with $a := b_1b_2^{-1} \notin \mathbb{R}$. Then the boundary conditions (5.3) are weakly regular if and only if both operators $L_{C,D}(0)$ and $L_{C,D}(0)^*$ are complete in $L^2([0,1];C^2)$.

**Proof. Necessity** is implied by Theorem 1.2 and Corollary 3.2.

**Sufficiency.** Assume that both operators $L_{C,D}(0)$ and $L_{C,D}(0)^*$ are complete in $L^2([0,1];C^2)$ but the BC (5.3) are not weakly regular. Then, by Corollary 6.2, we can assume that BC are equivalent to conditions (6.2). In this case the adjoint operator $L_{C,D}^*$ is defined by the differential expression $L^* = -iB^*d/dx + Q^*(x)$ and the boundary conditions

$$\overline{h_0}y_1(0) + \overline{\pi}y_2(0) - \overline{h_1}y_1(1) = 0, \quad y_2(1) = 0. \quad (6.16)$$
The second condition is of Volterra type and, by Corollary 6.2, operator $L^*_{C,D}$ is incomplete. This contradicts the assumption. □

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