Corrigendum to:
Elliptic Boundary Value Problems with Fractional Regularity Data: The First Order Approach

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2020 Mathematics Subject Classification. Primary: 35J25, 42B35, 47A60. 
Secondary: 35J57, 35J46, 35J47, 42B25, 42B30, 42B37, 47D06.

Key words and phrases. Second-order elliptic systems, boundary value problems, 
tent spaces, Besov–Hardy–Sobolev spaces, bisectorial operators, functional 
calculus, off-diagonal estimates, interpolation, layer potentials

Abstract. This is a corrigendum to the monograph “Elliptic Boundary Value 
Problems with Fractional Regularity Data: The First Order Approach” by 
the first two authors. It addresses two major imprecisions in the preliminary 
material.

The preliminary material of the monograph [1] written by the first two authors 
contains two major imprecisions that necessitates a number of (in the end harmless) 
changes throughout the entire text. One is about identification of abstract and con-
crete Hardy spaces for perturbed Dirac operators, the other one about interpolation 
of quasi-Banach function spaces. Since these erroneous statements are not unlikely 
to spread, we provide a detailed corrigendum, including further background and 
corrected statements for all affected results. All other results remain unchanged.

A correction on the classification region of Section 6

In the introduction, Theorem 1.8, summarising references [4] and [3], is stated 
incorrectly. This induces some changes in later statements in the introduction and 
the definition of the classification region in Chapter 6 should be modified for the 
results in Chapter 6 and 7 to be fully correct.

As is mentioned after the statement of Theorem 1.8, there is an interval of 
exponents that has been introduced in [4] and every p in this interval is such that 
the assumption $H^p_{DB} \simeq H^p_D$ is satisfied. It was unfortunate that this interval was named 
$I_0(H, DB)$, because this notation is used in Chapter 5 with another definition (after 
Definition 5.10) as the maximal interval in $(0, \infty)$ for which $H^p_{DB} \simeq H^p_D$, causing 
the confusion. A detailed explanation of the possible difference is given [2, Chapter 
15], to which we refer.

Let us call $p_+(DB)$ the upper bound of the interval introduced [4]. It is a 
number in $(2, \infty]$ with a precise meaning. Its definition is used in the proof of 
quasinecessity obtained in [3] and therein the results are (correctly) stated with the 
constraint $p < p_+(DB)$. Consequently, the following should be corrected here.

- One should correct Theorem 1.8 by assuming that $p$ in 
  (i) satisfies $p < p_+(DB)$ and that $p'$ in (ii) satisfies $p' < 
  p_+(DB^*)$.
- Next, Theorem 1.11 should be modified by assuming $p < 
  p_+(DB)$ when $\theta = 0$ and $p > (p_+(DB^*)')$ when $\theta = -1$.
- The same modification should be applied in Theorems 1.12, 
  case (i), and 1.14, to the pairs $(\theta_0, p_0)$ and $(\theta_1, p_1)$.

Since all identification results for $\theta(p) \in \{0, 1\}$ that appear in the later chapters 
refer back to [4, 3], also the definition of the classification region $J(H, DB)$ in 
Definition 6.10 should be adapted. It should read as follows:

**Definition 6.10.** We define the classification region for DB as 
follows. For $X = B$ we let
\[ J(B, DB) := \{ p \in I_{\text{max}} : [i(p) \leq 2 \text{ and } p \in I(B, DB)] \]
\[ \quad \text{or} \quad [i(p) > 2 \text{ and } p^\circ \in I(B, DB^*)] \} \]
and for \( X = H \) we let
\[ J(H, DB) := \{ p \in I_{\text{max}} : [-1 < \theta(p) < 0], \]
\[ \quad \text{and} \quad [i(p) \leq 2 \text{ and } p \in I(H, DB)] \]
\[ \quad \text{or} \quad [i(p) > 2 \text{ and } p^\circ \in I(H, DB^*)], \]
\[ \quad [\theta(p) = 0], \]
\[ \quad \text{and} \quad [i(p) \in I_0(H, DB), i(p) < p_+(DB)], \]
\[ \quad [\theta(p) = -1], \]
\[ \quad \text{and} \quad [i(p^\circ) \in I_0(H, DB^*), i(p^\circ) < p_+(DB^*)] \}. \]

We note that there is no modification for the classification region \( J(B, DB) \) for Besov data, as it is only concerned with regularity \(-1 < \theta(p) < 0\). The correct classification region \( J(H, DB) \) is smaller than what has originally been stated in the monograph. Consequently, the results come with additional restrictions when \( \theta(p) \in \{0, 1\} \), which, again, were treated in [3], whereas proofs when \( 0 < \theta(p) < 1 \) remain correct as they stand.

Finally, let us mention the question whether there exists \( p \in [p_+(DB), \infty) \) for which \( H^p_{DB} \simeq H^p_D \). In the case where \( B \) is a block matrix, [2, Chapter 15] establishes that the answer is no. The general case remains not fully understood at this stage.

### A correction on interpolation results

The interpolation results, in particular in Section 2.3, require two main adjustments. To this end, it is instructive to consider the exponent \( p \) in the \((j, \theta)\)-plane and distinguish three regions as in the figure below: The finite range \( j(p) > 0 \), the infinite range \( j(p) \leq 0 \) and the duality range \( 0 \leq j(p) < 1 \).

The first adjustment is that interpolation of tent spaces, \( Z \)-spaces and homogeneous smoothness spaces only translates to taking convex combinations of exponents in the \((j, \theta)\)-plane, if we restrict ourselves to the right half-plane with \( j(p) \geq 0 \).

For example, real interpolation of a Besov space \( B^{\infty, \infty}_{s+\alpha} \) with \( B^{p,p}_D \) for finite \( p \) always yields a Besov space with finite exponent — no matter how far to the left the exponent \( p = (\infty, s; \alpha) \), corresponding to \((-\alpha/n, s)\), lies in the \((j, \theta)\)-plane, compare with the references in Section 2.3.

The additional assumption \( j(p) \geq 0 \) is a (harmless) restriction for results on tent and Hardy–Sobolev spaces.

For \( Z \)-spaces, we see from Definition 2.15 that \( Z^{\infty, s; \alpha} = Z^{\infty, s+\alpha; 0} \), and the same applies to Besov spaces. Hence, the same space can have different parametrisations in the \((j, \theta)\)-plane and for a correct formulation of interpolation results we have to pick the unique one with \( j(p) \geq 0 \). In other words, when \( p, q \) are infinite with \( 0 = j(p) \) and \( p \leftrightarrow q \), then we have by definition that \( Z^q = Z^p \). The distinct \( Z \)-spaces are only those with \( j(p) \geq 0 \). The same applies for the Besov scale.
A CORRECTION ON INTERPOLATION RESULTS

Exponent regions: The gray duality range described by $0 \leq j(p) < 1$ corresponds to the spaces that can be obtained as duals of Banach spaces with finite exponent $p$ (that is, $j(p) > 0$). It lies in the intersection of finite exponents and infinite exponents with $j(p) = 0$. The geometrically interpolating exponent $[p, q]_\theta$ captures the interpolation behaviour of function spaces if $j(p), j(q) \geq 0$.

The second adjustment is connected to a general strategy of proof for interpolation results in three steps: first one checks the finite range, then gets the duality range by duality methods for interpolation functors, and finally one gets all exponents by Wolff reiteration because the finite range and the duality range have a substantial overlap. The issue is that no Wolff reiteration is known for the complex interpolation method of Kalton–Mitrea!

The note [5] contains a proof of the usual Wolff reiteration theorem for four $p$-convex quasi-Banach function spaces, where only one of the ‘exterior spaces’ needs to be separable, provided that all four spaces have the so-called Fatou property. For details, we refer to [5], whereas for the purpose of this corrigendum it suffices to note that tent- and $Z$-spaces are covered if one of the ‘exterior exponents’ is finite, which is the case in the scenario described above. This is how Wolff reiteration for complex interpolation spaces has to be understood in the context of Section 2.3.

We come to the concrete changes in the text that follow from the above two main adjustments and their consequences. All other results in the monograph remain true as stated with the sole exception of Theorem 7.8, for which we describe the modification further below.
Changes in Section 2.3

The corrected statement and proof for real interpolation is as follows. We need to assume \( j(p), j(q) \geq 0 \), also in order to justify duality for real interpolation.

**Theorem 2.30 (Real interpolation of tent spaces: full range).** Suppose that \( p \) and \( q \) are exponents with \( \theta(p) \neq \theta(q) \) and \( j(p), j(q) \geq 0 \), and \( 0 < \eta < 1 \). Then

\[
(T^p, T^q)_{\eta,p_\eta} = Z^{[p,q]_\eta},
\]

where \( p_\eta = i([p,q]_\eta) \).

**Proof.** For finite exponents this is Theorem 2.18. Let now \( 1 < i(p), i(q) \leq \infty \). Since \( j(p), j(q) \geq 0 \), we have that \( T^p, T^q \) are Banach spaces. Hence, in this case the claim follows by writing

\[
(T^p, T^q)_{\eta,p_\eta} = ((T^p')', (T^q')')_{\eta,p_\eta} = (T^p', T^q')'_{\eta,p_\eta}
\]

via the duality theorem for real interpolation [22, Theorem 3.7.1], using that \( T^p \cap T^q \) is dense in both \( T^p \) and \( T^q \), and then noting that

\[
p'_\eta = i([p,q]_\eta)' = i([p',q']_\eta).
\]

The full result follows by Wolff reiteration [76, Theorem 1] which,
for the real method holds in the quasi-Banach setting. \( \Box \)

In the subsequent result on real interpolation, the same strategy has to be used. For finite exponents one uses reiteration (but the one allowing quasi-Banach spaces in [22, Theorem 3.11.4]). Then duality and Wolff reiteration complete the argument as before. The assumption \( j(p), j(q) \geq 0 \) is needed again, however, as in Theorem 2.30, they can both be infinite:

**Proposition 2.31 (Real interpolation of \( Z \)-spaces).** Let \( p \) and \( q \) be exponents with \( \theta(p) \neq \theta(q) \) and \( j(p), j(q) \geq 0 \), and let \( \eta \in (0,1) \). Then

\[
(Z^p, Z^q)_{\eta,p_\eta} = Z^{[p,q]_\eta},
\]

where \( p_\eta = i([p,q]_\eta) \).

Also in Proposition 2.33 we need \( j(p), j(q) \geq 0 \). The strategy is once again the same and the case of finite exponents is what is being deferred to Section 2.6. Duality for the complex method on Banach spaces requires one space to be reflexive, which explains why the assumption that one exponent is finite has to be maintained as in the original text.

**Proposition 2.32 (Complex interpolation of \( Z \)-spaces).** Let \( p \) and \( q \) be exponents with \( j(p), j(q) \geq 0 \) and equality for at most one of them, and let \( \eta \in (0,1) \). Then

\[
[Z^p, Z^q]_\eta = Z^{[p,q]_\eta}.
\]

The restriction to exponents with \( j(p) \geq 0 \) also requires some more care in the proof of the mixed embedding in Theorem 2.34 because in the proof \( r = [p,q]_2 \) may not satisfy \( j(r) \geq 0 \) even if \( p \) and \( q \) are finite. A three step strategy as above is once again necessary and we give a corrected proof of the unchanged statement.
Theorem 2.34 (Mixed embeddings). Let $X_0, X_1 \in \{T, Z\}$ and let $p \hookrightarrow q$ with $p \neq q$. Then we have the embedding

$\left( X_0 \right)^p \hookrightarrow \left( X_1 \right)^q$.

Proof. We first treat the case that $p$ and $q$ are finite. When $X_0 = X_1 = T$, this is Theorem 2.11.

Let $r = [p, q]_{1+\epsilon}$ with $\epsilon > 0$ small enough to guarantee that $r$ is still finite. Then $p \hookrightarrow r$ and $[p, r]_{(1+\epsilon)}^{-1} = q$ (by Lemmas 2.1 and 2.2). Then we have embeddings $T^p \hookrightarrow T^r$ (trivially) and $T^p \hookrightarrow T^q$ (Theorem 2.11). Hence

$T^p \hookrightarrow (T^p, T^r)_{(1+\epsilon)^{-1}, ([p, r]_{(1+\epsilon)}^{-1})} = Z^{[p, r]_{(1+\epsilon)}^{-1}} = Z^q$

by Theorem 2.30, using that $p \neq q$ and $p \hookrightarrow q$ imply $\theta(p) \neq \theta(q)$. Similarly, putting $s = [p, q]_{1/2}$, we have $T^s \hookrightarrow T^q$ and $T^q \hookrightarrow T^q$, so

$Z^p = (T^s, T^q)_{1/2, ([s, q]_{1/2})} \hookrightarrow T^q$.

Finally, putting $t = [p, q]_{1/2}$ and using the previous results, we have

$Z^p \hookrightarrow T^t \hookrightarrow Z^q$,

which completes the proof for finite exponents.

The next case is that $p'$ and $q'$ are both finite. Then the claim follows from the first case by duality and this covers the infinite exponents and those in the duality range, see the figure above.

The remaining case is that $q$ is infinite and $p$ is finite but not in the duality range. Then the connecting segment contains a finite exponent $r = [p, q]_\theta$ with $\theta \in (0, 1)$ in the duality range. Combining the previous two cases yields

$\left( X_0 \right)^p \hookrightarrow \left( X_0 \right)^r \hookrightarrow \left( X_1 \right)^r$.

The proof is complete. □

Changes in Section 4.3

The interpolation results of Section 2.3 directly correspond to the interpolation results for adapted spaces in Theorem 4.28. Hence, this theorem should be read as follows:

Theorem 4.28 (Interpolation of completions). Fix $0 < \eta < 1$ and $\psi \in \Psi_\infty$. Let $p_0$ and $p_1$ be exponents, and set $p_\eta := [p_0, p_1]_\eta$.

(i) Suppose $j(p_0), j(p_1) \geq 0$, with equality for at most one exponent. Then we have the identification

$[\psi H_A^{p_0}, \psi H_A^{p_1}]_\eta = \psi H_A^{p_\eta}$.

(ii) Suppose $j(p_0), j(p_1) \geq 0$, with equality for at most one exponent. Then we have the identification

$[\psi B_A^{p_0}, \psi B_A^{p_1}]_\eta = \psi B_A^{p_\eta}$. 
(iii) Suppose \( j(p_0), j(p_1) \geq 0 \) and that \( \theta(p_0) \neq \theta(p_1) \). Then we have the identification 

\[
(\psi X_A^{p_0}, \psi X_A^{p_1})_{\eta, i(p_\eta)} = \psi B_A^{p_\eta}.
\]

All of these statements have analogues for spectral subspaces.

**Changes in Section 6.5**

The interpolation results of Section 2.3 and Section 4.3 have been joined in the proof of Theorem 6.41. Hence, also in this theorem, (ii) needs the additional assumption \( j(p_1) \geq 0 \) and in (iii) it is needed that \( j(p_0), j(p_1) \geq 0 \).

**Changes in Section 7.1**

Theorem 7.8 is the only result, where infinite exponents mistakenly appear by interpolation between infinite and finite exponent. In fact, Šneĭberg’s extrapolation theorem [66] only applies to interior points of complex interpolation scales. This result only works for finite exponents. On the other hand, the restriction \( i(p) > 1 \) in the Besov case is unnecessary because complex interpolation for \( Z \)-spaces was also obtained in the quasi-Banach range in Proposition 2.32. (Section 2.6 was finalised at the very end of composing the monograph.) Theorem 7.8 and the succeeding remark should therefore be modified as follows.

**Theorem 7.28 (Extrapolation of well-posedness)** Let \( B = \hat{\mathcal{A}} \), and suppose \( p \in J(X, DB) \) is a finite exponent with \( \theta(p) \in (-1, 0) \). Suppose that \( (P_X)^p_A \) is well-posed. Then there exists a \((j, \theta)\)-neighbourhood \( O_p \) of \( p \) such that for all \( q \in O_p \) with \( \theta(q) \leq 0 \), \( (P_X)^q_A \) and \( (P_X)^p_A \) are mutually well-posed.

**Remark 7.29** With \( X = H \), a corresponding result is true for \( \theta(p) \in \{-1, 0\} \), but well-posedness is only obtained for \( q \) near \( p \) with \( \theta(q) = \theta(p) \). The proof uses the same argument.

It should be noted that the subsequent results (Corollary 7.10 and Theorem 7.11) are not affected by the modification above since they only use finite exponents anyway.

**Further typos**

- Front matter: in the Library of Congress Cataloging-in-Publication Data, the first author’s year of birth is off by 10 years. The direction of the error is left as an exercise to the reader.
- Page 4, line 2: “Holder” should read “Hölder”.
- In the line before Theorem 2.11, replace “by the same argument” with “by duality, using Theorem 2.7”.
- Page 24: In the formula (2.7) for \( \|f\|_{Z^p} \), it is implicit that the measure on \( \mathbb{R}^{1+n} \) is \( \frac{dsdt}{t} \).
- Page 25: In Proposition 2.20, change \( \ell^p \) to \( \ell^i(p) \).
- Page 30, line 5: \( \|g\|_{Z^p} \) should read \( \|g\|_{Z^p'} \).
• Page 31, line before Proposition 2.31: The correct reference is [23, Theorem 3.11.5].

• Page 36, line -7: Change $D^\alpha f(0)$ to $D^\alpha \hat{f}(0)$.

• Page 37, statement of Theorem 2.49: Change $s$ to $\alpha$.

• Page 56, Lemma 3.17: The statement should be modified to $\psi \in \Psi_\sigma^{\tau+}$, $\varphi \in \Psi_\tau^{(\sigma-\delta)+}$. In the proof, when $\sigma + \tau = 0$ there is uniform boundedness, and Theorem 3.8 applies when $\sigma + \tau > 0$.

• Page 66: In the definition of a completion, replace “continuous injective” by “isometric”.

• Page 69, Proposition 4.24: For infinite exponents, bounded means continuous for the weak* topology (obtained by duality). For the density in the proof, one can select a sequence $(F_k)$ in the proof of Corollary 4.9 with compact support in $t$, so that $f_k \in \mathcal{D}(\eta(A))$.

• Page 69, Remark 4.25: There is also a completed version of the density property in Corollary 4.9. This is necessary to perform interpolation.

• Page 76: In the second display of the proof of Theorem 5.3, one should replace $t^{-1}$ by $t^{-1} | \cdot |$.

• Pages 85-87, proof of Lemma 5.29: There are several adjustments necessary.
  (a) The value of $\text{const.}$ in line 2 and 3 should be 1. The assumptions that $\| \chi \|_{\infty} + \tau \| \nabla \chi \|_{\infty} \lesssim 1$ and $\| \chi_1 \|_{\infty} + \tau \| \nabla \chi_1 \|_{\infty} \lesssim 1$ are missing.
  (b) In the series of inequalities at the bottom of page 86, on the fourth inequality, one should have used the Cauchy–Schwarz inequality for the integral in $y$ and then argue as suggested. Moreover, $\tau^n$ should read $\tau^n$ (as on the line after).
  (c) In the series of inequalities at the top of page 87, in the third inequality a factor $\tau^{-2}$ is missing. Moreover, one should delete the subscript $\chi$ and in the fourth inequality, replace $\tau^{-2\theta}$ by $\tau^{-2\theta - 2}$.

• Page 95, Proposition 6.1: The Whitney parameter on the left-hand side should be smaller than $c_0$ and not equal to $c_0$.

• Page 98, Lemma 6.7: The limit is $t \to \infty$, not $t \to 0$.

• Page 101, line -9: The convergence below the display is only weakly in $L^2_{\text{loc}}(\mathbb{R}^{n+1})$ for infinite exponents.

• Page 117, Corollary 6.38: $X^{p+}_D$ should read $X^{p+}_{DB}$.

• Page 128, Theorem 7.5: The isomorphisms should be understood as topological isomorphisms (hence, bi-continuous) and the open mapping theorem is used implicitly in the proof. In the quasi-Banach range one reference is [6, Theorem 1.4]. In the infinite exponent range, continuity is for the weak *-topology, which in all appearing cases is a Fréchet topology.
• Page 129, line 10: In the sentence starting by “evidently”, one should mention the use of a density argument of three spaces of the considered type in the intersection of two of them, together with the continuity of the inverses proved in Theorem 7.5.
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