Exact partition functions of the Ising model on $M \times N$ planar lattices with periodic-aperiodic boundary conditions

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Abstract. The Grassmann path integral approach is used to calculate exact partition functions of the Ising model on $M \times N$ square (sq), plane triangular (pt) and honeycomb (hc) lattices with periodic-periodic (pp), periodic-antiperiodic (pa), antiperiodic-periodic (ap) and antiperiodic-antiperiodic (aa) boundary conditions. The partition functions are used to calculate and plot the specific heat, $C/k_B$, as a function of the temperature, $\theta = k_B T/J$. We find that for the $N \times N$ sq lattice, $C/k_B$ for pa and ap boundary conditions are different from those for aa boundary conditions, but for the $N \times N$ pt and hc lattices, $C/k_B$ for ap, pa, and aa boundary conditions have the same values. Our exact partition functions might also be useful for understanding the effects of lattice structures and boundary conditions on critical finite-size corrections of the Ising model.
1. Introduction

Universality and scaling are two important concepts in the theory of critical phenomena [1, 2] and the Ising model [3] has been a model widely used in such studies. Recently, exact universal amplitude ratios and finite-size corrections to scaling in critical Ising model on planar lattices have received much attention [4, 5, 6, 7, 8, 9, 10, 11, 12]. This may be due to the fact that the hypothesis of universality leads naturally to the consideration of universal critical amplitudes and amplitude combinations [13], and for the comparison between experiment and theory in relation to scaling and universality, it is often a more rigorous test to use amplitude relations rather than critical exponent values. Moreover, it is also well known that the finite-size scaling functions depend on the boundary conditions [14], and there has been considerable recent interest in studying lattice model with various boundary conditions [15, 16, 17, 18, 19, 20, 21]. The study of exact universal amplitude ratios and finite-size corrections to scaling in critical Ising model is usually based on the analytical solutions of the model on finite lattices. Although the exact solution of the Ising model on $M \times N$ square (sq) lattice had been obtained long time ago [22], and the exact expression of the partition function of the Ising model on $M \times N$ plane triangular (pt) lattice has been obtained by lattice field theories recently [23], there is still no published results for the exact solutions of the Ising model on $M \times N$ pt and honeycomb (hc) lattices with periodic-aperiodic boundary conditions. The purpose of this paper is to fill this gap. In the present paper we use the Grassmann path integral to calculate exact partition functions of the Ising model on $M \times N$ sq, pt and hc lattices with periodic-periodic (pp), periodic-antiperiodic (pa), antiperiodic-periodic (ap) and antiperiodic-antiperiodic (aa) boundary conditions. The partition functions are used to calculate and plot the specific heat, $C/k_B$, as a function of the temperature, $\theta = k_B T/|J|$. We find that for the $N \times N$ sq lattice, $C/k_B$ for pa and ap boundary conditions are different from those for aa boundary conditions, but for the $N \times N$ pt and hc lattices, $C/k_B$ for ap, pa, and aa boundary conditions have the same values. Our exact partition functions might also be useful for understanding the effects of lattice structures and boundary conditions on critical finite-size corrections of the Ising model.

Two-dimensional Ising model on the sq lattice at vanishing magnetic field was first solved by Onsager by the use of Lie algebra [3]. The exact solution he obtained was Ising model on an infinite lattice. The original method was rather complicated, and it was later improved by Kaufman [22] who obtained the exact solution of the Ising model on a finite torus by using the theory of spinor representation. The successful treatments of the two-dimensional Ising model brought the studies of phase transition into the modern era. Onsager’s solution in one hand showed the previous classical theories were unreliable in their quantitative predictions, and on the other hand provided a great stimulus to explore the true behaviour near the critical point. After Onsager’s original solution, many quite different mathematical approaches were developed, but the approaches were still complicated. Among them, Schultz, Mattis and Lieb gave explicitly
Exact partition functions of the Ising model on $M \times N$ planar lattices

the fermionic treatment in the framework of transfer-matrix formalism \[24\], and Kac and Ward developed the combinatorial method \[25, 26\]. Both methods reformulated the two-dimensional Ising model as a free-fermionic field theory in terms of anticommuting Grassmann variables, which enclosed the fact that the Ising model on two dimensional regular lattices may be viewed as free-fermionic theory. The other alternative method in literature was the Pfaffian representation, which was introduced by Kasteleyn \[27\] to translate Ising spins into dimers that can be reduced to some Pfaffian \[28\]. Stephenson has used the Pfaffian representation to solve the Ising model on the pt lattice, but the solution was restricted to $6L \times 6L$ lattice due to its $6 \times 6$ basic nonvanishing matrix elements and was exact only in the limit of $L \to \infty$ \[29\]. Recently, by using the connections between Pfaffian, dimer and Ising model, Nash and O’Connor have obtained the exact expression of the partition function of the pt lattice Ising model on a finite torus \[23\]. They first employed the lattice field theories to obtain the exact partition function of the Gaussian model, and then established the exact expression of the partition function of the pt lattice Ising model from the analysis of the appropriate lattice determinants and the parameterization according to the results in \[29\].

On the other hand, in view of the simplifying the approach, a remarkable progress was achieved by Plechko who modified the traditional fermionic interpretation and introduced a nonstandard approach \[30\]. By the use of this approach, Plechko himself has not only rederived Onsager’s and Kaufman’s results in a relatively simple way \[30\], but also obtained the partition functions of a class of triangular type decorated lattices \[31\], and a triangular lattice net with holes \[32\]. Quite recently, by using the same approach, Wu at al. have obtained the $M \times N$ sq lattice Ising model with periodic-aperiodic boundary condition \[4\], and Liaw at al. have successfully solved triangular and hexagonal lattices on a cylinder geometry ($M \times \infty$) with periodic and antiperiodic boundary condition \[33\]. This approach is based on the integration over the anticommuting Grassmann variables and the mirror-ordered factorization principle in two-dimensional density matrix \[30, 31, 32, 33\], and does not involve the traditional transfer-matrix or combinatorial considerations. The whole scheme of the method can be illustrated schematically as shown below \[30\]:

$$Z = \text{Sp} \{ Z(\sigma) \} \to \text{Sp} \{ Z(\sigma|\chi) \} \to \text{Sp} \{ Z(\chi) \} = Z,$$

where “Sp” stands for the average over spin variables ($\sigma$) or Grassmann variables ($\chi$). The original partition function $Z$ is expressed purely by spin variables ($\sigma$) at each lattice site. With a set of anticommuting Grassmann variables ($\chi$) being introduced to factorize the local bond Boltzmann weight such that spin variables are decoupled, the partition function passes to a mixed $Z(\sigma|\chi)$ representation. Then, by eliminating the spin variables in the mixed $Z(\sigma|\chi)$ representation, the fermionic interpretation $Z(\chi)$ of the two-dimensional Ising model can be obtained, and after carrying out the Grassmann integral, the analytic solution for the partition function and free energy can be achieved \[30, 31, 32, 33\].

In the present paper, we work in this framework to obtain exact partition functions
of $M \times N$ pt and hc lattices with different boundary conditions, including pp, pa, ap and aa boundary conditions. We used these results to calculate and plot the specific heat, $C/k_B$, as a function of the temperature, $\theta = k_B T/J$. Our results show that for the sq lattice, $C/k_B$ for pa and ap boundary conditions are different from those for aa boundary conditions, but for the pt and hc lattices, $C/k_B$ for ap, pa, and aa boundary conditions have the same values. Beside these analyses, our exact partition functions may also be used for understanding the effects of lattice structures and boundary conditions on critical properties and critical finite-size corrections of the Ising model.

This paper is organized as follows. In section 2, we set up a general form of the partition function for pt and hc lattices. Then, three pairs of conjugate Grassmann variables are introduced for a lattice site to factorize the Boltzmann weights, and the principle of mirror ordering are used to rearrange the Grassmann factors so we can perform the summation over Ising spins to obtain a pure fermionic expression of the partition function. In section 3, using the Fourier transform technique we complete the integrations over the Grassmann variables to obtain the exact solution of the partition function. Then, the solution is subjected to periodic-aperiodic boundary conditions, including pp, pa, ap and aa boundary conditions. We further consider the shift behaviours of the maximum of the specific heats of these systems in section 4. Finally, we discuss some problems for further studies in section 5.

2. The Partition Function

Consider Ising ferromagnets on $M \times N$ pt and hc lattices as shown in figure 1, in which the former is considered as a sq lattice with a single second-neighbor interaction, and the latter contains an inner spin in each lattice cell. The corresponding Hamiltonians, respectively, read as

$$H_t = - \sum_{m=1}^{M} \sum_{n=1}^{N} (J_1 \sigma_{mn} \sigma_{m+1n} + J_2 \sigma_{mn} \sigma_{mn+1} + J_3 \sigma_{m+1n} \sigma_{mn+1}) ,$$

(1)

and

$$H_h = - \sum_{m=1}^{M} \sum_{n=1}^{N} (J_1 \sigma_0 \sigma_{mn} + J_2 \sigma_0 \sigma_{mn+1} + J_3 \sigma_0 \sigma_{m+1n}) ,$$

(2)

where $J_i$ with $i = 1, 2, 3$ are the coupling constants ($J_i > 0$ for ferromagnetic lattices), $\sigma_{mn} = \pm 1$ is the Ising spin located at the site $(m, n)$, and $\sigma_0$ denotes the inner Ising spin in hc lattice. Using the identity of the Boltzmann weight,

$$\exp(\beta J_i \sigma_\mu \sigma_\nu) = \cosh(\beta J_i) [1 + \tanh(\beta J_i) \sigma_\mu \sigma_\nu] ,$$

(3)

$\beta = (k_B T)^{-1}$, and performing the sum over $\sigma_0$, the partition functions of two lattices can be formulated in a single three spin-polynomial representation,

$$Z = 2^{N_s} \left[ \prod_{i=1}^{n_b} \cosh(\beta J_i) \right]^{N_s}$$
Exact partition functions of the Ising model on $M \times N$ planar lattices

\[
\times \text{Sp}_{(\sigma)} \left\{ \prod_{m=1}^{M} \prod_{n=1}^{N} (\alpha_0 + \alpha_1 \sigma_{mn} \sigma_{m+1n} + \alpha_2 \sigma_{mn} \sigma_{m+n+1} + \alpha_3 \sigma_{m+1n} \sigma_{m+n+1}) \right\} ,
\]

where $N_s$ is the number of lattice sites ($N_s = MN$ for sq and pt lattice, $N_s = 2MN$ for hc lattice) and $n_b$ is the number of bonds per lattice cell ($n_b = 2$ for sq lattice, $n_b = 3$ for pt and hc lattice), symbol "Sp" stands for spin average defined by

\[
\text{Sp} \left[ \cdots \right] = \frac{1}{2} \sum_{(\sigma_i = \pm 1)} \left[ \cdots \right], \quad \text{Sp} \left[ 1 \right] = 1, \quad \text{Sp} \left[ \sigma_i \right] = 0
\]

and $\alpha_i$'s are defined as

\[
\alpha_0^T = 1 + t_1 t_3 t_3, \quad \alpha_1^T = t_1 + t_3 t_3, \quad \alpha_2^T = t_2 + t_3 t_1, \quad \alpha_3^T = t_3 + t_1 t_2 ,
\]

$t_i = \tanh (\beta J_i)$ with $i = 1, 2, 3$, for pt lattice, and

\[
\alpha_0^H = 1, \quad \alpha_1^H = t_1 t_3, \quad \alpha_2^H = t_1 t_2, \quad \alpha_3^H = t_2 t_3 ,
\]

for hc lattice.

To factorize the partition, we rewrite the partition function as

\[
Z_H = 2^{N_s} \left[ \prod_{i=1}^{n_b} \cosh (\beta J_i) \right]^{N_s} \times \text{Sp}_{(\sigma)} \left\{ \prod_{m=1}^{M} \prod_{n=1}^{N} r_0 \left( 1 + r_1 \sigma_{mn} \sigma_{m+1n} \right) \left( 1 + r_2 \sigma_{mn} \sigma_{m+n+1} \right) \left( 1 + r_3 \sigma_{m+1n} \sigma_{m+n+1} \right) \right\} ,
\]

where $r_i$ with $i = 0, 1, 2, 3$ vary from one lattice to the other, and are related to $\alpha_i$'s from

\[
\alpha_0 = r_0 \left( 1 + r_1 r_3 \right), \quad \alpha_1 = r_0 \left( r_1 + r_2 r_3 \right), \quad \alpha_2 = r_0 \left( r_2 + r_1 r_3 \right), \quad \alpha_3 = r_0 \left( r_3 + r_1 r_2 \right) .
\]

For pt lattice, the relation between $r_i$ and $t_i$ is trivial, i.e. $r_0 = 1$ and $r_i = t_i$, but for hc lattice, the relation is nontrivial and is determined by equations (7) and (9).

It is more convenient to define the generalized reduced partition function as

\[
Q = r_0^{MN} \hat{Q},
\]

with

\[
\hat{Q} = \prod_{m=1}^{M} \prod_{n=1}^{N} \text{Sp}_{(\sigma_{mn})} \left[ \left( 1 + r_1 \sigma_{mn} \sigma_{m+1n} \right) \left( 1 + r_2 \sigma_{mn} \sigma_{m+n+1} \right) \left( 1 + r_3 \sigma_{m+1n} \sigma_{m+n+1} \right) \right] .
\]

To construct the fermionic representation of the generalized partition function, we associate each lattice site $(m, n)$ with three pairs of conjugate Grassmann variables, $\{a_{mn}, a_{mn}^*; b_{mn}, b_{mn}^*; c_{mn}, c_{mn}^*\} \in \chi$. All of these Grassmann variables are anticommuting, and their square are zero. Their integral obeys the basic rules

\[
\int d\chi = 0, \quad \int d\chi \cdot \chi = 1 ,
\]

\[
\int d\chi \cdot \Omega (\chi + \eta) = \int d\chi \cdot \Omega (\chi) ,
\]
Exact partition functions of the Ising model on $M \times N$ planar lattices

for an arbitrary vector $\eta$ with anticommuting components, and there is the relation

$$1 + r_i \sigma_\mu \sigma_\nu = \int d\chi^* d\chi e^{\chi^* \chi} (1 + \chi \sigma_\mu) (1 + r_i \chi^* \sigma_\nu).$$  \hspace{1cm} (14)

Using these Grassmann variables, we can rewrite the reduced partition function as

$$\tilde{Q} = \prod_{m=1}^{M} \prod_{n=1}^{N} \Psi_{A_{mn}} \Psi_{B_{mn}} \Psi_{C_{mn}},$$  \hspace{1cm} (15)

where "Sp" stands for the averaging with Gaussian weight

$$\text{Sp}_{(\chi_i)} \left[ \cdots \right] = \int d\chi^*_i d\chi_i e^{\chi_i \chi^*_i} \left[ \cdots \right],$$  \hspace{1cm} (16)

with the rules

$$\text{Sp}_{(\chi_i)} \left[ \chi_i \chi_i^* \right] = - \text{Sp}_{(\chi_i)} \left[ \chi_i^* \chi_i \right] = 1,$$
$$\text{Sp}_{(\chi_i)} \left[ \chi_i \right] = \text{Sp}_{(\chi_i)} \left[ \chi_i^* \right] = 0,$$

and the Grassmann factors, $A, A^*, B, B^*, C, \text{ and } C^*$, are defined as

$$A_{mn} = 1 + a_{mn} \sigma_{mn}, \quad A^*_{mn} = 1 + r_1 a_{m-1n}^* \sigma_{mn};$$  \hspace{1cm} (19)
$$B_{mn} = 1 + b_{mn} \sigma_{mn}, \quad B^*_{mn} = 1 + r_2 b_{mn-1}^* \sigma_{mn};$$  \hspace{1cm} (20)
$$C_{mn} = 1 + c_{mn-1} \sigma_{mn}, \quad C^*_{mn} = 1 + r_3 c_{m-1n}^* \sigma_{mn}. \hspace{1cm} (21)$$

In this way, a Boltzmann weight is decoupled to the product of two factors of separated spins.

For simplicity, we express the reduced partition function as

$$\tilde{Q} = \text{Sp}_{(a,b,c)} \left\{ \prod_{m=1}^{M} \prod_{n=1}^{N} \Psi_{A_{mn}}^A \Psi_{B_{mn}}^B \Psi_{C_{mn}}^C \right\},$$  \hspace{1cm} (22)

where $\Psi_{A_{mn}}^A, \Psi_{B_{mn}}^B \text{ and } \Psi_{C_{mn}}^C$ are defined by

$$\Psi_{A_{mn}}^A = \text{Sp}_{(\sigma_{mn})} \left( A_{mn} A_{m+1n}^* \right),$$  \hspace{1cm} (23)
$$\Psi_{B_{mn}}^B = \text{Sp}_{(\sigma_{mn})} \left( B_{mn} B_{mn+1}^* \right),$$  \hspace{1cm} (24)
$$\Psi_{C_{mn}}^C = \text{Sp}_{(\sigma_{mn})} \left( C_{mn+1} C_{m+1n}^* \right).$$  \hspace{1cm} (25)

We first treat the boundary weight and consider periodic boundary condition in both directions:

$$\Psi_{A_{Mn}}^A = \text{Sp}_{(\sigma_{Mn})} \left[ (1 + a_{Mn} \sigma_{Mn}) (1 + r_1 a_{0n}^* \sigma_{M+1n}) \right]$$
$$= \text{Sp}_{(\sigma_{Mn})} \left[ (1 + r_1 a_{0n}^* \sigma_{1n}) (1 + a_{Mn} \sigma_{Mn}) \right]$$
$$= \text{Sp}_{(\sigma_{Mn})} \left( A_{1n}^* A_{Mn} \right),$$  \hspace{1cm} (26)
which implies
\[ a_{0n}^* = -a_{Mn}^*, \]  
(27)

Similarly, from
\[ \Psi^B_{mN} = \text{Sp}_{(\sigma_{mN})} (B_{mNB_{mN+1}}) = \text{Sp}_{(\sigma_{mN})} (B_{m1}^* B_{mN}), \]  
(28)
\[ \Psi^C_{mN} = \text{Sp}_{(\sigma_{mN})} (C_{mN+1}C_{m+1N}^*) = \text{Sp}_{(\sigma_{mN})} (C_{1m}^* C_{mN+1}), \]  
(29)
\[ \Psi^C_{mN} = \text{Sp}_{(\sigma_{mN})} (C_{mN+1}C_{m+1N}^*) = \text{Sp}_{(\sigma_{mN})} (C_{m1}C_{m+1N}^*), \]  
(30)
we have
\[ b_{m0}^* = -b_{mN}, \]  
(31)
\[ c_{0n}^* = -c_{Mn}, \]  
(32)
\[ c_{m0} = c_{mN}. \]  
(33)

Since \( c_{m0} = c_{mN} \), \( \Psi^C_{mN} \) need not to be treated as a boundary weight, and only \( \Psi^A_{mN}, \) \( \Psi^B_{mN} \) and \( \Psi^C_{mN} \) should be considered. However, this situation becomes ambiguous when we take Fourier transform of these Grassmann variables with single set of exponential factors in equations (56) and (57). Because the Fourier exponential factors are associated with directions in \( M \) and \( N \), the sign factor in front of \( b_{mN}^* \) takes effects simultaneously on \( b_{mN}^* \) and \( c_{mN}. \) Therefore, the real situation is that instead of the relation in equation (33), we must take
\[ c_{m0} = -c_{mN}. \]  
(34)

A self-consistent way to assign a minus sign to \( c_{m0} \) and obtain the relation in equation (34) is interchanging \( C_{m1} \) in equation (30) with another Grassmann factor. An equivalent but more convenient approach is to consider the rearrangement of \( B_{m1}^* \) in the boundary weight together with the rearrangement of \( C_{m1} \) in the reduced partition function. To see this, we express the reduced partition function as
\[
\tilde{Q} = \text{Sp} \left\{ \text{Sp}_{(\sigma)} \left[ \left( \prod_{m=1}^{M-1} \prod_{n=1}^{N} \Psi^A_{mn} \Psi^C_{mn} \right) \Psi_B \left( \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn}B_{mn+1}^* \right) \right] \right\}
\]  
(35)

with the boundary weight \( \Psi_B \)
\[
\Psi_B = \text{Sp}_{(a,b,c)} \left[ \left( \prod_{n=1}^{N} \Psi^A_{Mn} \right) \left( \prod_{m=1}^{M} \Psi^B_{mN} \right) \left( \prod_{n=1}^{N} \Psi^C_{Mn} \right) \right]
\]  
(36)
and
\[
\prod_{m=1}^{M-1} \prod_{n=1}^{N} \Psi^A_{mn} \Psi^C_{mn} = \prod_{m=1}^{M-1} \prod_{n=1}^{N} A_{mn}C_{mn+1}C_{m+1n}^*A_{m+1n}^*
\]  
(37)
In this way, the configurations of the reduced partition function can be further arranged of
\[ \tilde{Q} = \frac{1}{2} \left( \tilde{Q}_\gamma \big|_{\Gamma_1} + \tilde{Q}_\gamma \big|_{\Gamma_2} + \tilde{Q}_\gamma \big|_{\Gamma_3} - \tilde{Q}_\gamma \big|_{\Gamma_4} \right), \] with superscripts + and − being the sign factors in boundary Grassmann factors
\[ A_{1n}^*, B_{m1}^* \] and \( C_{1n} \), and simultaneously move \( C_{m1} \) from right of \( \prod_{n=2}^{N} C_{mm}^* \) to left of \( A_{m1} \) in equations (36) and (37). Here we note that the superscripts + and − respectively correspond to periodic and antiperiodic boundary condition imposed on the spin variables and in turn on the Grassmann variables. Hence, the reduced partition function becomes
\[ \tilde{Q} = \frac{1}{2} \left( \tilde{Q}_\gamma \big|_{\Gamma_1} + \tilde{Q}_\gamma \big|_{\Gamma_2} + \tilde{Q}_\gamma \big|_{\Gamma_3} - \tilde{Q}_\gamma \big|_{\Gamma_4} \right), \] with
\[ \tilde{Q}_\gamma = \mathcal{S}_{(a,b,c)} \left\{ \mathcal{S}_{(\sigma)} \left[ \prod_{m=1}^{M-1} \Theta_m^* \Theta_{m+1}^* \right] \cdot \Psi \cdot \left( \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn} B_{mn+1}^* \right) \right\}, \] and
\[ \Psi_\gamma = \mathcal{S}_{(a,b,c)} \left\{ \mathcal{S}_{(\sigma)} \left[ \Theta_{m}^* \left( \prod_{m=1}^{M} \bar{B}_{m1}^* \right) \Theta_M \left( \prod_{m=1}^{M} \bar{B}_{mN}^* \right) \right] \right\}. \] where we have defined
\[ \Theta_m = \prod_{n=1}^{N} C_{mn}^* A_{mn} \quad \text{and} \quad \Theta_m^* = \prod_{n=1}^{N} C_{mn}^* A_{mn}^*, \] and the boundary conditions \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) are defined as
\[ \Gamma_1 = (a_{0n}^* = -a_{Mn}^*, b_{m0}^* = -b_{mN}^*, c_{0n}^* = -c_{Mn}^*), \] \[ \Gamma_2 = (a_{0n}^* = -a_{Mn}^*, b_{m0}^* = +b_{mN}^*, c_{0n}^* = -c_{Mn}^*), \] \[ \Gamma_3 = (a_{0n}^* = +a_{Mn}^*, b_{m0}^* = -b_{mN}^*, c_{0n}^* = +c_{Mn}^*), \] \[ \Gamma_4 = (a_{0n}^* = +a_{Mn}^*, b_{m0}^* = +b_{mN}^*, c_{0n}^* = +c_{Mn}^*). \] In this way, the configurations of the reduced partition function can be further rearrangement and expressed as
\[ \tilde{Q}_\gamma = \mathcal{S}_{(a,b,c)} \mathcal{S}_{(\sigma)} \left\{ \left( \prod_{m=1}^{M-1} \Theta_m^* \Theta_{m+1}^* \right) \Theta_1^* \left( \prod_{m=1}^{M} \bar{B}_{m1}^* \right) \Theta_M \left( \prod_{m=1}^{M} \bar{B}_{mN}^* \right) \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn} B_{mn+1}^* \right\} \]
\[ \begin{align*}
\hat{Q}_\gamma &= \text{Sp} \text{ Sp}_{(a,b,c)}(\sigma) \left\{ \prod_{m=1}^{M} \Theta_m^{*} B_{m1}^{*} \Theta_m \left( \prod_{n=1}^{n} C_{mn} A_{mn} \right) \left( \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn} B_{mn+1}^{*} \right) \right\}. \\
&= \text{Sp} \text{ Sp}_{(a,b,c)}(\sigma) \left\{ \prod_{m=1}^{M} \Theta_m^{*} B_{m1}^{*} \left( \prod_{n=1}^{N-1} C_{mn} A_{mn} \right) \left( \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn}^{*} B_{mn+1}^{*} \right) \right\}. 
\end{align*} \] (47)

To have a complete mirror-ordered form, we have to rearrange the terms in the last two brackets. To achieve this, first we note that

\[ \hat{Q}_\gamma = \text{Sp} \text{ Sp}_{(a,b,c)}(\sigma) \left\{ \prod_{m=1}^{M} \Theta_m^{*} B_{m1}^{*} \left( \prod_{n=1}^{N-1} C_{mn} A_{mn} \right) \left( \prod_{m=1}^{M} \prod_{n=1}^{N-1} B_{mn}^{*} B_{mn+1}^{*} \right) \right\}. \] (48)

The boundary term of with \( m = M \), denoted by \( T \), can be formulated

\[ T = \Theta_M^{*} \left( \prod_{n=1}^{N-1} B_{Mn}^{*} C_{Mn} A_{Mn} B_{Mn} \right) B_{MN}^{*} C_{MN} A_{MN} B_{MN}. \]

\[ = \left( \prod_{n=1}^{N} C_{Mn}^{*} A_{Mn}^{*} \right) \left( \prod_{n=1}^{N} B_{Mn}^{*} C_{Mn} A_{Mn} B_{Mn} \right) \]

\[ = \prod_{n=1}^{N} C_{Mn}^{*} A_{Mn}^{*} B_{Mn}^{*} C_{Mn} A_{Mn} B_{Mn}. \] (49)

due to the fact that \( \text{Sp}_{(\sigma_m)} [C_{Mn}^{*} A_{Mn}^{*} B_{Mn}^{*} C_{Mn} A_{Mn} B_{Mn}] \) for a given \( n \) is a commutable object. By continuing such construction from \( m = M \) down to \( m = 1 \), we can obtain the expression

\[ \hat{Q}_\gamma = \text{Sp}_{(a,b,c)} \left\{ \prod_{m=1}^{M} \prod_{n=1}^{N} \text{Sp}_{(\sigma_m)} [C_{mn}^{*} A_{mn}^{*} B_{mn}^{*} C_{mn} A_{mn} B_{mn}] \right\}. \] (50)

For this partition function, the factors containing the same spin are grouped together and we can perform the average over spins. As a result, we have

\[ \hat{Q}_\gamma = \int \prod_{m=1}^{M} \prod_{n=1}^{N} da_{mn}^{*} da_{mn} db_{mn}^{*} db_{mn} dc_{mn}^{*} dc_{mn} \exp \left( \sum_{m=1}^{M} \sum_{n=1}^{N} F_{mn} \right), \] (51)

with

\[ F_{mn} = a_{mn}^{*} a_{mn} + b_{mn}^{*} b_{mn} + c_{mn} c_{mn}^{*} + r_{1} r_{3} c_{m-1n}^{*} a_{m-1n}^{*} + \left( r_{3} c_{m-1n}^{*} + r_{1} a_{m-1n}^{*} \right) r_{2} b_{mn-1}^{*} + \left( r_{3} c_{m-1n}^{*} + r_{1} a_{m-1n}^{*} + r_{2} b_{mn-1}^{*} \right) c_{mn-1} + \left( r_{3} c_{m-1n}^{*} + r_{1} a_{m-1n}^{*} + r_{2} b_{mn-1}^{*} + c_{mn-1} \right) a_{mn} + \left( r_{3} c_{m-1n}^{*} + r_{1} a_{m-1n}^{*} + r_{2} b_{mn-1}^{*} + c_{mn-1} + a_{mn} \right) b_{mn}. \] (52)
Since there is no mix on \(a_{mn}\) and \(b_{mn}\), the integral in the above expression can be simplified by integrating out the \(a_{mn}\) and \(b_{mn}\) fields by means of the identity
\[
\int db da \exp \left( \lambda a b + aL + L'b \right) = \lambda \exp \left( \lambda^{-1} LL' \right), \tag{53}
\]
where \(a, b\) are Grassmann variables, \(L, L'\) are linear fermionic forms independent of \(a, b\) and \(\lambda\) is a parameter. The result then becomes
\[
\tilde{Q}_\gamma = \int \prod_{m=1}^{M} \prod_{n=1}^{N} dg_{mn}^* dg_{mn} dc_{mn}^* dc_{mn} \exp \left( \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn} \right), \tag{54}
\]
with
\[
G_{mn} = c_{mn}c_{mn}^* + g_{mn}g_{mn}^* \\
+ r_1 r_3 c_{m-1n}^* g_{m-1n} \\
- \left( r_3 c_{m-1n}^* + r_1 g_{m-1n} \right) r_2 g_{mn-1}^* \\
+ \left( r_3 c_{m-1n}^* + r_1 g_{m-1n} - r_2 g_{mn-1}^* \right) c_{mn-1} \\
- \left( r_3 c_{m-1n}^* + r_1 g_{m-1n} - r_2 g_{mn-1}^* + c_{mn-1} \right) \left( g_{mn} + g_{mn}^* \right), \tag{55}
\]
where we have changed the notations for the fields by \((a_{mn}^*, b_{mn}^*) \rightarrow (g_{mn}, -g_{mn}^*)\). This is the pure fermionic representation of the reduced partition function.

3. Exact Solution

Next, to carry out the integration, we have to use the technique of Fourier transform to treat the Grassmann variables which mix together with the variables at different sites. The Fourier transformation is defined as
\[
X_{mn} = \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} X_{pq} e^{-i \frac{2\pi}{M} mp} e^{-i \frac{2\pi}{N} nq}, \tag{56}
\]
and
\[
X_{mn}^* = \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} X_{pq}^* e^{i \frac{2\pi}{M} mp} e^{i \frac{2\pi}{N} nq}, \tag{57}
\]
where the variables \(X_{mn}\) and \(X_{mn}^*\) denotes one of the variables \(\{c_{mn}, g_{mn}\}\) and \(\{c_{mn}^*, g_{mn}^*\}\) respectively.

After performing the Fourier transformation, the partition function becomes
\[
\tilde{Q}_\gamma = \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \int dV_{pq} \exp \left( H_{pq} \right), \tag{58}
\]
with the measure \(dV_{pq}\) defined as
\[
dV_{pq} = dg_{pq}^* dg_{pq} dc_{pq}^* dc_{pq}, \tag{59}
\]
and the function $H_{pq}$ is given by

$$H_{pq} = \left(1 - r_3 e^{-i \frac{2\pi p}{M} - i \frac{2\pi q}{N}}\right) c_{pq} c_{pq}^*$$

$$+ \left(r_2 - e^{i \frac{2\pi q}{N}}\right) c_{pq} g_{pq}^* + r_3 \left(r_1 - e^{-i \frac{2\pi p}{M}}\right) c_{pq} g_{pq}$$

$$- e^{i \frac{2\pi q}{N}} \left(1 + r_1 e^{-i \frac{2\pi p}{M}}\right) c_{pq} g_{M-pN-q} - r_3 e^{-i \frac{2\pi p}{M}} \left(1 + r_2 e^{i \frac{2\pi q}{N}}\right) c_{pq} g_{M-pN-q}^*$$

$$+ \left(1 - r_1 e^{i \frac{2\pi p}{M}} - r_2 e^{-i \frac{2\pi q}{N}} - r_1 r_2 e^{i \frac{2\pi p}{M} e^{-i \frac{2\pi q}{N}}}\right) g_{pq} g_{pq}^*$$

$$- r_1 e^{i \frac{2\pi p}{M}} g_{pq} g_{M-pN-q} + r_2 e^{-i \frac{2\pi q}{N}} g_{pq}^* g_{M-pN-q}^*.$$  

(60)

Because $H_{pq}$ contains not only the variables, $X_{pq}$ and $X_{pq}^*$, but also the variables, $X_{M-pN-q}$ and $X_{M-pN-q}^*$, instead of calculating $\tilde{Q}_\gamma$, it is easier to calculate $\tilde{Q}_\gamma^2$ given by

$$\tilde{Q}_\gamma^2 = \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \int dV_{pq} dV_{M-pN-q} \exp \left( H_{pq} + H_{M-pN-q}^* \right).$$  

(61)

Here $H_{M-pN-q}^*$ can be obtained from $H_{pq}$ by replacing $p$ by $M - p$ and $q$ by $N - q$ for the Grassmann variables and replacing the coefficient in front of Grassmann variables by its complex conjugate. Completing the integration yields

$$Q_\gamma = \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \left[ A_0 - A_1 \cos \frac{2\pi p}{M} - A_2 \cos \frac{2\pi q}{N} - A_3 \cos \left( \frac{2\pi p}{M} - \frac{2\pi q}{N} \right) \right]^{1/2},$$  

(62)

with

$$A_0 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2,$$  

(63)

$$A_1 = 2 (\alpha_0 \alpha_1 - \alpha_2 \alpha_3),$$  

(64)

$$A_2 = 2 (\alpha_0 \alpha_2 - \alpha_1 \alpha_3),$$  

(65)

$$A_3 = 2 (\alpha_0 \alpha_3 - \alpha_1 \alpha_2),$$  

(66)

where $\alpha_0$, $\alpha_1$, $\alpha_2$, and $\alpha_3$ are given by equations (63) and (64) for pt and hc lattices, respectively.

### 3.1. Periodic-periodic boundary condition

According to equation (63), the reduced partition function for ferromagnetic lattices with pp boundary condition is

$$Q^{pp} = \frac{1}{2} \left[ \Omega_{\frac{1}{2}, \frac{1}{2}} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} - \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{0, 0} \right],$$  

(67)

where the superscript $p$ refers to periodic boundary condition and

$$\Omega_{\mu \nu} = \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \left[ A_0 - A_1 \cos \frac{2\pi (p+\mu)}{M} - A_2 \cos \frac{2\pi (q+\nu)}{N} - A_3 \cos \left( \frac{2\pi (p+\mu)}{M} - \frac{2\pi (q+\nu)}{N} \right) \right]^{1/2}.$$  

(68)

The sign factor in front of the last term is a result of the standard consideration of the Grassmann integral over the zero-mode variable $p = q = 0$ for ferromagnetic couplings.
Exact partition functions of the Ising model on $M \times N$ planar lattices. When the integral of equation (61) is carried out, it is always positive, but this is not the case for equation (58). There are unpaired terms from zero-mode in equation (58) under various boundary conditions and they contribute a sign factor to $Q_4$ for $0 \leq t_i \leq 1$. The partition function for pp boundary condition then becomes

$$Z^{pp} = \frac{1}{2} 2^{N_s} \left[ \prod_{i=1}^{n_b} \cosh (\beta J_i) \right]^{N_s} \left[ \Omega_{\frac{1}{2}, \frac{1}{2}}^{1,0} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} - \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(69)

Furthermore, the free energy density per $k_B T$ of the system defined by

$$f^{pp} = -\frac{1}{N_s} \ln Z^{pp},$$

(70)

then takes the form

$$f^{pp} = -\frac{(N_s - 1)}{N_s} \ln 2 - \sum_{i=1}^{n_b} \ln [\cosh (\beta J_i)] - \frac{1}{N_s} \ln \left[ \Omega_{\frac{1}{2}, \frac{1}{2}}^{1,0} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} - \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(71)

### 3.2. Periodic-antiperiodic boundary condition

For pa boundary condition, equation (38) is replaced by

$$B^{-} (CA)^+ = \frac{1}{2} \left[ (CA)^+ B^+ + (CA)^+ B^- - (CA)^- B^+ + (CA)^- B^- \right],$$

(72)

and the partition function has the form

$$Z^{pa} = \frac{1}{2} 2^{N_s} \left[ \prod_{i=1}^{n_b} \cosh (\beta J_i) \right]^{N_s} \left[ \Omega_{\frac{1}{2}, \frac{1}{2}}^{1,0} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right],$$

(73)

where the superscript $a$ refers to antiperiodic boundary condition. The corresponding free energy density per $k_B T$ is

$$f^{pa} = -\frac{(N_s - 1)}{N_s} \ln 2 - \sum_{i=1}^{n_b} \ln [\cosh (\beta J_i)] - \frac{1}{N_s} \ln \left[ \Omega_{\frac{1}{2}, \frac{1}{2}}^{1,0} + \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(74)

### 3.3. Antiperiodic-periodic boundary condition

Similarly, for ap boundary condition, equation (38) is replaced by

$$B^+ (CA)^- = \frac{1}{2} \left[ (CA)^+ B^+ - (CA)^+ B^- + (CA)^- B^+ + (CA)^- B^- \right],$$

(75)

and

$$Z^{ap} = \frac{1}{2} 2^{N_s} \left[ \prod_{i=1}^{n_b} \cosh (\beta J_i) \right]^{N_s} \left[ \Omega_{\frac{1}{2}, \frac{1}{2}}^{1,0} - \Omega_{\frac{1}{2}, 0} + \Omega_{0, \frac{1}{2}} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(76)
Exact partition functions of the Ising model on $M \times N$ planar lattices

The corresponding free energy density per $k_B T$ is

$$f^{ap} = -\frac{(N_s - 1)}{N_s} \ln 2 - \sum_{i=1}^{n_b} \ln \cosh (\beta J_i)$$

$$- \frac{1}{N_s} \ln \left[ \Omega_{1,1} - \Omega_{1,0} + \Omega_{0,1} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(77)

3.4. Antiperiodic-antiperiodic boundary condition

For aa boundary condition, equation (38) becomes

$$B^{-} (CA)^{-} = \frac{1}{2} \left[ -(CA)^{+} B^{+} + (CA)^{+} B^{-} + (CA)^{-} B^{+} + (CA)^{-} B^{-} \right],$$

(78)

and the partition function is

$$Z^{aa} = \frac{1}{2} 2^{N_s} \prod_{i=1}^{n_b} \cosh(\beta J_i) \gamma_{N_s} \left[ -\Omega_{1,1} - \Omega_{1,0} + \Omega_{0,1} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(79)

The corresponding free energy density per $k_B T$ is

$$f^{aa} = -\frac{(N_s - 1)}{N_s} \ln 2 - \sum_{i=1}^{n_b} \ln \cosh (\beta J_i)$$

$$- \frac{1}{N_s} \ln \left[ -\Omega_{1,1} - \Omega_{1,0} + \Omega_{0,1} + \text{sgn} \left( \frac{\theta - \theta_c}{\theta_c} \right) \Omega_{00} \right].$$

(80)

Note that by taking $t_3 = 0$, $n_b = 2$ and $N_s = N_b = MN$, we have $A_3 = 0$,

$$\Omega_{\mu \nu} = \prod_{p=0}^{M-1} \prod_{q=0}^{N-1} \left[ A_0 - A_1 \cos \frac{2\pi (p+\mu)}{M} - A_2 \cos \frac{2\pi (q+\nu)}{N} \right]^{1/2}.$$ 

(81)

and all the results we obtained reduce to those of sq lattice.

Accordingly, the critical temperature can be determined in the thermodynamic limit from the zero of the free energy contributed by the zero mode,

$$A_0 - A_1 - A_2 - A_3 = 0.$$ 

(82)

It follows that for isotropic coupling, we have

$$\theta_c = \left[ \frac{1}{2} \ln \left( 1 + \sqrt{2} \right) \right]^{-1} = 2.269185..., \quad \text{for sq lattice with } \theta = k_B T / J,$$

(83)

$$\theta_c = \left[ \frac{1}{2} \ln \left( \sqrt{3} \right) \right]^{-1} = 3.640956..., \quad \text{for pt lattice and }$$

(84)

$$\theta_c = \left[ \frac{1}{2} \ln \left( 2 + \sqrt{3} \right) \right]^{-1} = 1.518651..., \quad \text{for he lattice.}$$

(85)
4. Specific heat

The specific heat per spin $C/k_B$ for the Ising model on $M \times N$ sq, pt and hc lattices with isotropic couplings are shown, respectively, in figure 2(a), 3(a), 4(a) for $M/N = 1$, and in figure 2(b), 3(b), 4(b) for $M/N = 1/2$. Figure 3(c) and 4(c) show, respectively, results for pt and hc lattices under pa and aa boundary conditions and for $M/N = 1, 1/2, 1/4$. In general, for three lattices with the same lattice size, the specific heat under pp boundary condition is always larger than those under other boundary conditions. Note that for sq lattices with $M/N = 1$, $C_{pa}$ and $C_{aa}$ are distinct in figure 2(a), but for pt and hc lattices with $M/N = 1$ in figure 3(a) and 4(a), they coincide and are non-distinguishable due to the last term in the bracket of equation (68), which is associated with the structure symmetries of pt and hc lattices. These behaviours can be violated by taking aspect ratio $\xi = M/N \neq 1$, and the results are shown in figures 3(b), (c) and 4(b), (c).

We further study the displacements of the maximum of $C_{pp}$ and $C_{pa}$. The shift behaviours of the maximum in $C_{NN}(T)$ are shown in figure 5. The slopes of the curves implies the rates of approach of $C_{pp}$ and $C_{pa}$ to their limiting behaviours. For periodic-periodic boundary condition, these lattices have linear behaviours in $N \to \infty$ and can be described by the formula (86):

$$\frac{(T_c - T_{\text{max}})}{T_c} \sim \frac{a}{N}, \quad \text{as } N \to \infty.$$  

(86)

For periodic-antiperiodic boundary condition, the corresponding formula is also provided by finite-size scaling ansatz. However, for numerical analysis, instead of equation (86), we use

$$\frac{(T_c - T_{\text{max}})}{T_c} = \frac{a}{N} + \frac{b_1}{N^2} + \frac{b_2}{N^3} + \ldots.$$  

(87)

As a result, we have $a_{pp}^s = 0.360$, $b_{1,s}^{pp} = -0.47$, $a_{pa}^s = 0.18$, $b_{1,s}^{pa} = -2.19$, for the sq lattice, $a_{pp}^t = 0.363$, $b_{1,t}^{pp} = -0.91$, $a_{pa}^t = 0.09$, $b_{1,t}^{pa} = 0.60$ for the pt lattice, $a_{pp}^h = 0.268$, $b_{1,h}^{pp} = 0.24$, $a_{pa}^h = 0.09$, $b_{1,h}^{pa} = 0.87$ for the hc lattice, and the value of $b_2$ is of the order of 1. The values of $a_{pp}$ is larger than $a_{pa}$ for three lattices, and this implies the approach to limiting behaviour for pp boundary condition is faster than pa boundary conditions. Since the logarithmic divergence of the specific heat is independent of boundary conditions and can not be used to distinguish $C_{pp}$ and $C_{pa}$ of large lattice, then the values of $a$ may be used to distinguish two boundary conditions.

5. Discussion

We have solved the exact partition functions of $M \times N$ pt and hc lattices with different boundary conditions. These results can provide the analytical background for further studies on the effects of lattice structures and boundary conditions on critical properties and critical finite-size corrections of the Ising model.
 Firstly, universal finite-size scaling functions for critical systems have received much attention in recent years [15, 16, 17, 20, 21, 37, 38], and it is well known that the finite-size scaling functions depend on the boundary conditions [14]. Hu, Lin and Chen, and Okabe and Kikuchi have discussed the difference in the finite-size scaling functions for lattice models under periodic boundary and free boundary conditions in connection with the universal finite-size scaling function for percolation problem [15] and Ising model [17], respectively. Other boundary conditions, such as the Ising model on an \( M \times N \) simple-quartic lattice embedded on a Möbius strip and Klein bottle also has been studied [18]. Kaneda and Okabe found that there is interesting aspect ratio dependence of the value of the Binder parameter at criticality for various boundary conditions [19]. It is then interesting to have a rigorous test of finite-size scaling function and critical finite-size corrections for different planar Ising model under various boundaries.

In addition, by using Monte Carlo method, Hu, Lin and Chen [15, 16], Tomita, Okabe and Hu [21] have found that the universal finite-size scaling functions of the scaled quantities for sq, pt and hc lattices depend on the aspect ratios and have very good universal finite-size scaling behaviours when the aspect ratios of these lattices have the proportions \( 1 : \sqrt{3}/2 : \sqrt{3} \). This further implies lattice-structure-dependence of the universal finite-size scaling function and it would be a rigorous test from analytical aspect. To aforementioned topics, we have found finite-size scaling behaviours for sq, pt and hc lattices under period-aperiodic boundary conditions. By selecting a very small numbers of nonuniversal metric factors, we have further found very good universal finite-size scaling behaviours for these lattices, and the results will be presented in other paper.

Finally, the discussion of the specific heat in this paper also inspire another problem. Quite recently, Izmailian and Hu have found exact amplitude ratio and finite-size corrections for the \( M \times N \) sq lattice Ising model on a torus [8], and new sets of the universal amplitude ratios of subdominant correction to scaling amplitudes [7]. The results of section 4 suggest that \( a^{pp}/a^{pa} \) for sq, pt and hc lattices are roughly 2, 4, and 3. The question is "is there exact relations between \( a^{pp} \) and \( a^{pa} \) for these lattices?" It is interesting to study this question and to have a heuristic argument for this simple relation.

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Exact partition functions of the Ising model on $M \times N$ planar lattices

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**Figure 1.** (a) The global structure of the triangular lattice used in this paper. A basic cell of the lattice site is given by \((m,n)\), and the coupling constants are \(J_1, J_2\) and \(J_3\). (b) The global structure of the honeycomb lattice used in this paper. Each basic cell contains an inner Ising spin \(\sigma_0\).

**Figure 2.** The specific heat per spin for (a) \(N \times N\) square Ising lattices with isotropic couplings under \(pp, pa, ap\) and \(aa\) boundary conditions, and (b) \(M \times N\) square Ising lattices with isotropic couplings and aspect ratio \(M/N = 1/2\) under \(pp, pa, ap\) and \(aa\) boundary conditions. The critical point \(\theta_c\) is marked by a vertical line.

**Figure 3.** The specific heat per spin for (a) \(N \times N\) plane-triangular Ising lattices with isotropic couplings under \(pp, pa, ap\) and \(aa\) boundary conditions, (b) \(M \times N\) plane-triangular Ising lattices with isotropic couplings and aspect ratio \(M/N = 1/2\) under \(pp, pa, ap\) and \(aa\) boundary conditions, and (c) \(M \times N\) plane-triangular Ising lattices with isotropic couplings and aspect ratio \(M/N = 1, 1/2, 1/4\) under \(pa\) and \(aa\) boundary conditions. The critical point \(\theta_c\) is marked by a vertical line.

**Figure 4.** The specific heat per spin for (a) \(N \times N\) honeycomb Ising lattices with isotropic couplings under \(pp, pa, ap\) and \(aa\) boundary conditions, (b) \(M \times N\) honeycomb Ising lattices with isotropic couplings and aspect ratio \(M/N = 1/2\) under \(pp, pa, ap\) and \(aa\) boundary conditions, and (c) \(M \times N\) honeycomb Ising lattices with isotropic couplings and aspect ratio \(M/N = 1, 1/2, 1/4\) under \(pa\) and \(aa\) boundary conditions. The critical point \(\theta_c\) is marked by a vertical line.

**Figure 5.** (a) Variation of \((T_{\text{max}} - T_c)\) with finite \(N\) for \(N \times N\) square Ising lattices with isotropic couplings under \(pp\) and \(pa\) boundary conditions. The broken lines are given by \((T_{\text{max}} - T_c) / T_c = a/N\) and indicate the limiting behaviour as \(N \to \infty\). (b) Variation of \((T_{\text{max}} - T_c)\) with finite \(N\) for \(N \times N\) plane-triangular Ising lattices with isotropic couplings under \(pp\) and \(pa\) boundary conditions. (c) Variation of \((T_{\text{max}} - T_c)\) with finite \(N\) for \(N \times N\) honeycomb Ising lattices with isotropic couplings under \(pp\) and \(pa\) boundary conditions.