This paper is dedicated to S.P. Novikov, on the occasion of his 65th birthday.

The initial boundary value problem on the segment for the Nonlinear Schrödinger equation; the algebro-geometric approach. I

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Abstract

This is the first of a series of papers devoted to the study of classical initial-boundary value problems of Dirichlet, Neumann and mixed type for the Nonlinear Schrödinger equation on the segment. Considering proper periodic discontinuous extensions of the profile, generated by suitable point-like sources, we show that the above boundary value problems can be rewritten as nonlinear dynamical systems for suitable sets of algebro-geometric spectral data, generalizing the classical Dubrovin equations.

In this paper we consider, as a first illustration of the above method, the case of the Dirichlet problem on the segment with zero-boundary value at one end, and we show that the corresponding dynamical system for the spectral data can be written as a system of ODEs with algebraic right-hand side.

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1
1 Introduction

In the natural phenomena, evolutionary processes very often take place in a domain bounded in space and time. At the boundary the system has a non-trivial interaction with the external world or experimental environment. The mathematical description of such situations is given by initial boundary value (IBV) problems for evolution equations. Since soliton equations are largely applicable, IBV problems for them are of large interest from both mathematical and physical points of view. It is the case, for instance, of the IBV problems for the Nonlinear Schrödinger (NLS) equation

\[ iq_t = q_{xx} \mp 2q^2\bar{q}, \quad q = q(x,t), \quad q \in \mathbb{C} \]  

which describes the complex amplitude modulation of a quasi-monochromatic wave packet in strongly dispersive and weakly nonlinear media, and then arises in many natural contexts, like in fiber optics, plasma physics and fluid dynamics.

Two IBV problems for soliton equations are well-studied and understood.

1) The initial problem on the line with rapidly decreasing boundary conditions at infinity (the Cauchy problem on the line). This problem is solved using the scattering transform, first introduced for the Korteweg-de Vries (KdV) equation in [22] and later used to solve the NLS equation in [46]. In this case the scattering data evolve linearly and can be treated as a nonlinear analog of the continuous Fourier data.

2) The IBV problem on the segment \([0,L]\) with periodic boundary conditions. Its solution is based on the so-called finite-gap approach, first introduced in [40].

More general IBV problems (for example with Dirichlet, Neumann, mixed and Robin boundary conditions) for soliton equations are believed to be non-integrable. Only for special choices of the boundary values the problems remain integrable [11,44] and the above two standard integration procedures with appropriate reductions on the spectral data can be applied (see [11] for the semi-line case and [5] for the segment one; see also [45] for the semi-line case using the approach of [44]); but these cases cover only a limited number of interesting physical applications.

Applying the scattering transform to more general IBV problems on the semi-line or on the segment, one meets the following basic
difficulty: the time evolution of the scattering matrix depends not only on prescribed boundary values, but also on unknown ones. Several strategies have been developed to overcome such difficulty (a list of them can be found at the end of this introduction). One of the possible strategies, motivated by the classical solutions of the Dirichlet and Neumann problems for second order linear PDEs by the sine and cosine Fourier transforms, consists in transforming the given IBV problem into a certain Cauchy problem on the whole line by introducing a proper extension of the profile such that the known boundary values are encoded into external point sources and the unknown ones disappear from this new formulation. Using this strategy, the Dirichlet problem on the semi-line for the NLS equation was investigated in [14].

More recently, the same strategy was applied to the Dirichlet and Neumann problems on the semi-line for the NLS equation in [7] using a different formalism. In [7] the time evolution of the scattering matrix $\tilde{S}(\lambda,t)$ of the transformed problem on the whole line is expressed in terms of the scattering matrix $S(\lambda, t)$ of the original semi-line problem and of the given boundary values. To close the system, one uses the non-local connection $	ilde{S}(\lambda,t) = S^{-1}(-\lambda,t)\sigma S(\lambda,t)$, where $\sigma = 1$ for the Dirichlet case, and $\sigma = \sigma_3$ for the Neumann case ($\sigma_3$ is the Pauli matrix). Due to the analyticity properties of the scattering matrix $S(\lambda, t)$, the above non-local relation can be resolved for $S(\lambda, t)$ using the appropriate Riemann-Hilbert problem.

Motivated by this result, we have recently begun a research activity with the purpose of applying the same methodology to the IBV problems on the segment $[0,L]$ for the NLS equation. In particular, we have found that:

a) the Dirichlet problem is conveniently reformulated as an odd $2L$-periodic Cauchy problem on the line with $\delta'$ sources located at the points $nL$ with $n \in \mathbb{Z}$.

b) The Neumann problem is reformulated as an even $2L$-periodic Cauchy problem on the line with $\delta$ sources located at the points $nL$.

c) The mixed problem with, for instance, $u(0,t)$ and $u_x(L,t)$ given, is reformulated as an odd $4L$-periodic Cauchy problem such that $q(2L-x,t) = q(x,t)$, $0 \leq x \leq L$, with $\delta'$ sources located at the points $2nL$ and $\delta$ sources at the points $(2n+1)L$. Here and later we denote by $u(x,t)$ the NLS field in the segment $[0,L]$ and by $q(x,t)$ its extension to the whole line.
Through these new formulations of the original IBV problems, we have been able to obtain the following results.

1) We have eliminated from the formalism all the unknown boundary values.

2) We have transformed the original IBV problem into a Cauchy problem for periodic profiles, for which the powerful tools of the finite-gap method are at hand.

3) We have used the algebro-geometric tools of the finite gap method to reduce the original IBV problem to a nonlinear system of ordinary differential equations describing the time evolution of the spectral data of the transformed periodic problem with forcings. This system gives a satisfactory analytic description of the IBV problem under investigation.

Since the IBV problem on the segment is equivalent to a periodic problem for non-smooth profiles produced by point-like sources, then the corresponding spectral theory presents the following novel features with respect to the smooth case.

1) The lengths of the spectral gaps decay slowly in the high-energy limit (they are of order $1/\lambda$ for the Dirichlet problem and of order $1/\lambda^2$ for the Neumann one), while, in the smooth case, the gaps lengths decay faster than any power of $\lambda$. 
2) The number of gaps is always infinite and the finite-gap approximation converges rather slowly.

3) The branch points are not conserved any more and their time evolution must be included into the set Dubrovin equations.

4) The evolution of the spectral curve is described in terms of the isospectral deformations suggested in [24].

In this paper we illustrate the above method on the following particular Dirichlet problem for the defocussing NLS equation:

\[ iu_t = u_{xx} - 2u^2 \bar{u}, \quad u = u(x,t), \quad u \in \mathbb{C}, \quad 0 \leq x \leq L, \quad t \geq 0. \tag{2} \]

\[ u(0,t) = v_0(t), \quad u(L,t) = v_L(t) = 0, \quad u(x,0) = u_0(x), \quad 0 \leq x \leq L, \tag{3} \]

where \( v_0(t) \) and \( u_0(x) \) satisfy the obvious matching conditions \( v(0) = u_0(0), \quad u_0(L) = 0 \). The choice of the defocussing case is due to its simpler analytic structure (the auxiliary spectral problem is self-adjoint) and the choice of 0-boundary at \( x = L \) simplifies the Dubrovin equations. From the point of view of the method used, these simplifications are not essential and the solution of the general Dirichlet, Neumann and mixed IBV problems, for both the focussing and defocussing NLS, will be presented in a more detailed forthcoming paper.

As it was previously indicated, to study the IBV problem (2), (3) it is convenient to introduce the odd 2\( L \)-periodic extension of \( u(x,t) \):

\[ q(x,t) = \sum_{n \in \mathbb{Z}} u(x - 2nL,t)H(x - 2nL)H((2n + 1)L - x) - u(2nL - x)H(2nL - x)H(x - (2n - 1)L), \quad x \in \mathbb{R}, \quad t > 0. \tag{4} \]

This extension satisfies the following odd 2\( L \)-periodic Cauchy problem on the line with \( \delta' \) forcings

\[ iq_t = q_{xx} - 2q^2 \bar{q} - 2v_0(t) \sum_{n \in \mathbb{Z}} \delta'(x - 2nL), \quad x \in \mathbb{R}, \quad t > 0, \tag{5} \]

\[ q(x,0) = \sum_{n \in \mathbb{Z}} u_0(x - 2nL)H(x - 2nL)H((2n + 1)L - x) - u_0(2nL - x)H(2nL - x)H(x - (2n - 1)L), \tag{6} \]

which will be the subject of investigation of the following sections. We show, in particular, that unlike the case of generic forcings, which lead to a highly non-local formalism, the point-like sources arising from the IBV problem (2), (3) lead to a system of ODEs for the spectral data which can be written in local form after a proper extension of the phase.
space. This picture is consistent with the mild non-locality found in the semi-line case [7].

The paper is organized as follows. In §2 we summarize the main results concerning the finite gap theory of the defocusing NLS equation for smooth periodic profiles. In §3 we present the spectral characterization of the discontinuous periodic profiles arising from the Dirichlet problem. In §4 we derive the nonlinear system of ordinary differential equations describing the time evolution of the spectral data.

We end this introductory section with a list of other approaches to the study of IBV problems for soliton equations, developed during the last few years. In [42] an “elbow scattering” in the (x, t)-plane is presented to deal with the semi-line problem for KdV, leading to Gel’fand - Levitan - Marchenko formulations. In [17] [18] a different approach, based on a simultaneous x-t spectral transform, has been introduced and rigorously developed in [19] [20], to solve IBV problems for soliton equations on the semi-line. We were recently informed [21] that this method was also extended to the segment case. This method allows for a rigorous asymptotics [16] and captures in a natural way the known cases of linearizable IBV problems. In §5 and §6 two alternative approaches to the study of IBV problems for soliton equations on the segment and on the semi-line have been presented. In the first method, applicable to both the semi-line and segment cases, the unknown boundary values are expressed in terms of elements of the scattering matrix $S(\lambda, t)$, thus obtaining a nonlinear integro-differential evolution equation for $S$. In the second method one applies a suitable linear operator to $S_t S^{-1}$ to eliminate the unknown boundary data. It is also shown that the resulting time evolution of $S$, nonlinear and mildly nonlocal, coincides with that obtained applying the point source strategy of this paper [7]. Also the formalism presented in [7] allows for rigorous asymptotics and captures in a natural way the case of linearizable IBV problems. Some integrable boundary conditions for soliton equations and their connection with symmetries have been investigated in [25] and in references therein quoted. In some non-generic cases of soliton equations corresponding to singular dispersion relations, like the stimulated Raman scattering (SRS) equations and the sine Gordon (SG) equation in light-cone coordinates, the evolution equation of the scattering matrix does not contain unknown boundary data. The SG equation on the semi-line has been treated using the x-t spectral transform [17]: the SRS and the SG equations on the semi-line have also been treated using a more traditional x-transform.
method respectively in \[34\] and in \[35\]; the x-spectral data used in this last approach satisfy a nonlinear evolution equation of Riccati type. Apart from the simultaneous \(x-t\) transform, all the above approaches are based on the traditional IST \[47\]. A different approach, based on the Kac-Moody representation for the SG equation in laboratory coordinates, was used in \[4\] to solve the Robin problem on the semiline; it uses critically the finite-speed character of the equation.

The spectral formalism for studying forced soliton equations has been developed by several authors, especially in connection to the theory of perturbations (see, for instance, \[39\] and \[23\]). Soliton equations subjected to one-point-like source were investigated in \[15\]. Soliton equations perturbed by slowly varying forcings were investigated by averaging procedures in \[13\] and \[31\]. We finally remark that IBV problems and forced problems for C-integrable equations have been considered in \[6\], \[2\].

## 2 Periodic spectral transform for the defocussing NLS

### 2.1 Direct periodic spectral transform

The Lax pair for the defocussing NLS equation:

\[
iq_t = q_{xx} - 2q^2\bar{q}, \quad q = q(x, t),
\]

was found in \[46\]. We use its zero-curvature form:

\[
\frac{\partial \Psi(\lambda, x, t)}{\partial x} = U(\lambda, x, t)\Psi(\lambda, x, t),
\]

\[
\frac{\partial \Psi(\lambda, x, t)}{\partial t} = V(\lambda, x, t)\Psi(\lambda, x, t),
\]

where \(\Psi(\lambda, x, t)\) is a 2-component vector:

\[
\Psi(\lambda, x, t) = \begin{bmatrix} \psi^1(\lambda, x, t) \\ \psi^2(\lambda, x, t) \end{bmatrix}
\]

and \(U(\lambda, x, t), V(\lambda, x, t)\) are the following \(2 \times 2\) matrices:

\[
U(\lambda, x, t) = \begin{bmatrix} i\lambda & iq(x, t) \\ -i\bar{q}(x, t) & -i\lambda \end{bmatrix},
\]

\[
V(\lambda, x, t) = \begin{bmatrix} i\lambda & -iq(x, t) \\ i\bar{q}(x, t) & i\lambda \end{bmatrix}.
\]
\[ V(\lambda, x, t) = 2\lambda U(\lambda, x, t) + \begin{bmatrix} iq & q_x \\ \bar{q}_x & -i\bar{q} \end{bmatrix}. \]  

Equation (8) can be rewritten as the following spectral problem

\[ \mathcal{L}\Psi(\lambda, x, t) = \lambda \Psi(\lambda, x, t), \]  

where

\[ \mathcal{L} = \begin{bmatrix} -i\partial_x & -q(x, t) \\ -\bar{q}(x, t) & i\partial_x \end{bmatrix}. \]  

The first symmetry of the NLS hierarchy is the infinitesimal gauge

\[ q_{t_0} = iq. \]  

Its zero-curvature representation has the following form:

\[ \frac{\partial \Psi}{\partial x} = U\Psi, \quad \frac{\partial \Psi}{\partial t_0} = V_0\Psi, \]  

where

\[ V_0 = \frac{i}{2}\sigma_3 = \begin{bmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{bmatrix}. \]  

Assume, that our potential is periodic with the period 2L:

\[ q(x + 2L, t) \equiv q(x, t) \]  

Denote by \( \hat{T}(\lambda, x, a, t) \) the fundamental solution of (8)

\[ \frac{\partial \hat{T}(\lambda, x, a, t)}{\partial x} = U(\lambda, x, t)\hat{T}(\lambda, x, a, t), \quad \hat{T}(\lambda, a, a, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]  

We shall use:

\[ \hat{T}(\lambda, b, a, t) = \hat{\Psi}(\lambda, b, t)\hat{\Psi}^{-1}(\lambda, a, t) \]  

where \( \hat{\Psi}(\lambda, x, t) \) is an arbitrary matrix solution of (8) with non-zero determinant.

The matrix

\[ \hat{T}(\lambda, t) = \hat{T}(\lambda, 2L, 0, t) \]  

is called monodromy matrix.

Let us associate spectral data with \( q(x, t) \) (we treat \( \mathcal{L} \) as an ordinary differential operator in \( x \) depending on the parameter \( t \)).

**Lemma 1** The following spectral problems are self-adjoint:
1. Main problem: the spectral problem for the operator $L$ on the whole line in the space $L^2(\mathbb{R})$.

2. Auxiliary problem: the spectral problem for the operator $L$ on the interval $[0, 2L]$ with the boundary conditions:

$$\psi^1(0) + \psi^2(0) = 0, \quad \psi^1(2L) + \psi^2(2L) = 0. \quad (22)$$

The spectrum of the main problem is the union of closed intervals of the real line. The complement of this spectrum is the union of open intervals called gaps. Let us denote the boundary points of the gap number $j$ by $E_{2j}$ and $E_{2j+1}$, $E_{2j} < E_{2j+1}$. The spectrum of the auxiliary problem is discrete. Each interval $[E_{2j}, E_{2j+1}]$ (the closure of the gap $j$) contains exactly one point of the auxiliary spectrum.

For generic potentials the gaps can be enumerated by integers $-\infty < j < \infty$, and for sufficiently large $j$ the gap number $j$ is located near the point $\pi j/2L$. Each point $\lambda_k$ of the auxiliary spectrum is associated with a gap, and we can assume that $E_{2k} \leq \lambda_k \leq E_{2k+1}$. The lengths of gaps tends to 0 as $|j| \to \infty$. The decay rate depends on the smoothness of $q(x, t)$ and coincides with the decay rate of the Fourier coefficients of the potential.

For non-generic potentials the spectrum of the auxiliary problem can also be enumerated so that $\lambda_k \to \pi k/2L \ |k| \to \infty$, but some of these points lie inside the spectrum of the main problem. Such potentials can be interpreted as the results of “closing” some gaps: $E_{2j+1} \to E_{2j}$. To construct the inverse transform for the self-adjoint problem discussed in our text, only the points of the auxiliary spectrum $\lambda_j$ associated with gaps are essential.

These spectra possess the following simple characterization in terms of the monodromy matrix $\hat{T}(\lambda, t)$:

**Lemma 2**  

1. A real number $\lambda$ lies in the spectrum of the main problem if and only if $|\text{tr} \hat{T}(\lambda, t)| \leq 2$.

2. A number $\lambda_k$ is a point of the auxiliary spectrum if and only if the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector of $\hat{T}(\lambda, t)$, or equivalently,

$$T_{11}(\lambda, t) + T_{21}(\lambda, t) - T_{12}(\lambda, t) - T_{22}(\lambda, t) = 0.$$
The Bloch eigenfunction $\Psi(\gamma, x)$ is, by definition, the common eigenfunction of $L$ and of the shift operator

$$L \Psi(\gamma, x, t) = \lambda \Psi(\gamma, x, t), \quad \Psi(\gamma, x + 2L, t) = e^{2iLp(\gamma)} \Psi(\gamma, x, t), \quad (23)$$

where $\gamma$ is a point of a Riemann surface $\Gamma$, which is a two-sheeted covering of the $\lambda$-plane, $P$ denotes the projection from $\Gamma$ to the $\lambda$-plane: $P\gamma = \lambda$ and $e^{2iLp(\gamma)}$ is an eigenvalue of the monodromy matrix. The branch points of $\Gamma$ are real and coincide with the ends $E_j$ of the gaps. Denote by $a_k$ the oval lying over the interval $[E_{2k}, E_{2j+k}]$ and by $\sigma$ the transposition of sheets.

The function $p(\gamma)$ is defined modulo $(\pi/L)n$, $n \in \mathbb{Z}$. It is called the quasimomentum. It is odd with respect to the transposition of sheets:

$$p(\sigma \gamma) \equiv -p(\gamma), \mod(\pi/L). \quad (24)$$

The differential of the quasimomentum $dp = \frac{dp}{d\lambda}d\lambda$ is holomorphic at the finite part of the spectral curve $\Gamma$. It has exactly 2 zeroes in each oval $a_k$: $\alpha_k^+$ and $\alpha_k^-$, $\sigma \alpha_k^- = \alpha_k^-$:

$$\left. \frac{\partial p(\gamma, t)}{\partial \lambda} \right|_{\gamma = \alpha_k^+} = 0, \quad (25)$$

$$\text{Im } p(\alpha_k^+) > 0. \quad (26)$$

Denote by $\alpha_k$ the projection of these zeroes to the $\lambda$-plane.

Equation (23) defines $\Psi(\gamma, x, t)$ up to a constant factor. In our text we use the following normalization:

$$\Phi(\gamma, 0, 0) = 1, \quad (27)$$

where

$$\Phi(\gamma, x, t) = \psi^1(\gamma, x, t) + \psi^2(\gamma, x, t). \quad (28)$$

It is also convenient to introduce the following function:

$$\Xi(\gamma, x, t) = (\psi^1(\gamma, x, t) - \psi^2(\gamma, x, t))\Phi^{-1}(\gamma, x, t). \quad (29)$$

By definition:

$$\Psi(\gamma, x, t) = \left[ \frac{1 + \Xi(\gamma, x, t)}{1 - \Xi(\gamma, x, t)} \right] \Phi(\gamma, x, t). \quad (30)$$
For each \( x \) and \( t \), function \( \Phi(\gamma, x, t) \) has exactly one zero \( \gamma_k(x, t) \) on the oval \( a_k \). The poles \( \gamma_k \) of \( \Phi(\gamma, x, t) \) do not depend on \( x, t \), they lie on the ovals over the intervals \([E_{2k}, E_{2j+k}]\) and each oval contains exactly one pole \( \gamma_k \), with \( \gamma_k = \gamma_k(0,0) \). Denote by \( \lambda_k \) the projections of the points \( \gamma_k \) to the \( \lambda \)-plane. From the definition it follows that they are the points of the auxiliary spectrum corresponding to the gaps.

Hence the following spectral data are associated with a periodic potential of the defocussing NLS equation.

1. A collection of real branch points \( E_k \) (or, equivalently, the spectral curve \( \Gamma \)).
2. The divisor of poles \( \gamma_k \) in \( \Gamma \).

These data uniquely determine the potential \( q(x, t) \).

### 2.2 The periodic inverse problem. Finite-gap potentials

A potential \( q(x, t) \) is called finite-gap if the surface \( \Gamma \) is algebraic or, equivalently, if the genus \( g \) of \( \Gamma \) is finite. In the finite-gap case we define \( \Gamma \) by the algebraic equation:

\[
\mu^2 = R(\lambda), \quad R(\lambda) = \prod_{k=0}^{2g+1} (\lambda - E_k). \tag{31}
\]

By definition a point \( \gamma \) of \( \Gamma \) is a pair of complex numbers \( \gamma = (\lambda, \mu) \) satisfying (31). Therefore the poles and the zeroes of \( \Phi(\lambda, x, t) \) are pairs of complex numbers \( \gamma_k = (\lambda_k, \mu_k) \) and complex functions \( \gamma_k(x, t) = (\lambda_k(x, t), \mu_k(x, t)) \) respectively, satisfying (31). Here \( 0 \leq k \leq g \).

The role of the finite-gap potentials in the soliton theory is analogous to the role of finite Fourier series in the linear theory.

In the finite-gap case the function \( \Xi(\gamma, x, t) \) has the following analytic properties:

1. \( \Xi(\gamma, x, t) \) is meromorphic in \( \gamma \) on the spectral curve \( \Gamma \).
2. \( \Xi(\gamma, x, t) \) has at most first-order poles at the divisor points \( \gamma_k(x, t) \) and no other singularities.
3. \( \Xi(\gamma, x, t) \to \pm 1 \) as \( \gamma \to \pm \infty \).
Lemma 3  The properties formulated above completely determine the function $\Xi(\gamma, x, t)$. The explicit formula for $\Xi(\gamma, x, t)$ is given by:

$$\Xi(\gamma, x, t) = \frac{\mu}{\prod_{k=0}^{g} (\lambda - \lambda_k(x, t))} + \sum_{k=0}^{g} \frac{r_k(x, t)}{2(\lambda - \lambda_k(x, t))},$$

(32)

where

$$r_k(x, t) = 2\prod_{j=0, \ldots, g, j \neq k}^{g} (\lambda_k(x, t) - \lambda_j(x, t)).$$

(33)

One of the possible procedures for reconstructing potentials by the spectral data is the following. The ends $E_k$ of the gaps are constants of motion, i.e. they are $x, t$-independent. The divisor of zeroes $\gamma_k(x, t)$ satisfy the following systems of first-order ordinary differential equations in $x$ and $t$ (Dubrovin equations):

$$\frac{\partial \lambda_k(x, t)}{\partial x} = -i \left[ \frac{1}{2} \sum_{j=0}^{2g+1} E_j - \sum_{j=0, \ldots, g, j \neq k} \lambda_j(x, t) \right] r_k(x, t),$$

(34)

$$\frac{\partial \lambda_k(x, t)}{\partial t} = -2i \left[ \frac{1}{2} \sum_{j=0}^{2g+1} E_j^2 - \sum_{j=0, \ldots, g, j \neq k} \lambda_j^2(x, t) \right] r_k(x, t).$$

(35)

Solving equations (34), (35) for the initial data $\lambda_k(0, 0) = \lambda_k, \mu_k(0, 0) = \mu_k$, one obtains the divisor of zeroes $\gamma_k(x, t)$ for all $x, t$. Using the reconstruction formula

$$q(x, t) = \sum_{k=0}^{g} \left[ \lambda_k(x, t) - \frac{1}{2} (E_{2k} + E_{2k+1}) - \frac{r_k(x, t)}{2} \right]$$

(36)

one gets the potential $q(x, t)$.

In the derivation of the Dubrovin equations one assumes that the spectral curve has a finite genus. These equations can also be used for the infinite-gap case, but it is necessary to be careful with the associated convergence problems.

We use also Dubrovin equations for the infinitesimal gauge (15):

$$\frac{\partial \lambda_k(x, t)}{\partial t_0} = -\frac{i}{2} \gamma_k(x, t)$$

(37)
In the finite-gap case, the Dubrovin equations can be explicitly linearized using the Abel transform and the potentials can be written in terms of Riemann Θ-functions associated with the curve Γ. But we do not use this property in our text.

In the finite-gap case the λ-derivative of the quasimomentum differential reads as:

$$\frac{\partial p(\gamma)}{\partial \lambda} = \frac{\prod_{k=0}^{g} (\lambda - \alpha_k)}{\mu}$$  \hspace{1cm} (38)

**Remark.** The algebro-geometrical (or finite-gap) solutions of the KdV equation were first constructed in [40], where the zero-curvature representation was also first introduced. The complete finite-gap theory for the periodic Schrödinger operator was developed in [9, 10, 11, 20, 33, 37] (see the survey article [12] and the books [47], [3]). A pure algebraic formulation of the finite-gap integration procedure and its generalization for 2+1 dimensional systems were obtained in [30].

Finite-gap solutions of the defocusing NLS equation were first constructed in [27]. The first θ-functional formulas for the focussing NLS were written in [28]. A detailed study of the finite-gap NLS solutions can be found in [11]. An interpretation of generic periodic 1-d Schrödinger potentials in terms of Θ-functions of infinitely many variables was suggested in [38]. A generalization of this approach to generic matrix ordinary differential operators was suggested in [13]. The spectral transform for the operator with potentials in the Hilbert space $L^2([0, 2L])$ was developed in [29].

In the finite-gap case we mark a point of $\gamma \in \Gamma$ by a pair of complex numbers $\gamma = (\lambda, \mu)$. But, to study the limit $g \to \infty$, it is convenient to renormalize $\mu$. Assume that we have a point inside the k-th gap: $\lambda \in \mathbb{R}, E_{2k} \leq \lambda \leq E_{2k+1}$. Then we define a new variable $\tilde{\mu}$ by

$$\mu = i\tilde{\mu} \left( \prod_{\substack{j \neq k \neq 2k+1 \neq \lambda \neq \lambda_j}} (\lambda - E_j) \right) \text{sgn} \left( \prod_{j \neq k} (\lambda - \lambda_j) \right),$$  \hspace{1cm} (39)

$$\tilde{\mu}^2 = (E_{2k+1} - \lambda)(\lambda - E_{2k}).$$  \hspace{1cm} (40)

($\mu$ is pure imaginary inside gaps and we choose $\tilde{\mu}$ to be real, therefore we have the constant $i$ in (39)).

Since the residues $r_k$ are pure imaginary, it is convenient to introduce the real quantities $\tilde{r}_k(x, t)$ as follows:

$$r_k(x, t) = 2i\tilde{r}_k(x, t).$$  \hspace{1cm} (41)
From (33) it follows:
\[ \tilde{r}_k = \tilde{\mu}_k \prod_{j \neq k} \sqrt{\frac{(|\lambda_k - E_{2j})|(|\lambda_k - E_{2j+1})|}{(|\lambda_k - \lambda_j|)}}. \] (42)

2.3 Isoperiodic deformations of the spectral curve

Finite-gap solutions corresponding to generic surfaces are quasiperiodic in \( x \) and \( t \). To construct \( x \)-periodic solutions with a prescribed period, it is necessary to put nontrivial additional constraints on the Riemann surface. These constraints for the KdV equation were formulated in [36] in terms of conformal maps. After minor modifications, the approach of [36] can be applied to the defocussing NLS equation but, for our purposes, an alternative approach, suggested in [24], is more convenient. The main idea of [24] is to use the so-called \textit{isoperiodic deformations}, i.e. ordinary differential equations on the moduli space of Riemann surfaces such that all \( x \) quasiperiods of the potential are conservation laws. Therefore, if we have one periodic potential, we can construct new solutions integrating these deformations. The starting point can be chosen in the neighborhood of the zero potential using the perturbation theory.

Isoperiodic deformations arose in [32], [13], [43] in the form of ODE’s with a rather complicated transcendental right-hand side. In [24] it was shown that, by extending the phase space, these flows can be transformed to a very simple form with a rational right-hand side.

Consider the following collection of \( g+1 \) flows of the branch points \( E_j \) and of the projections \( \alpha_j \) of the quasimomentum differential zeroes \( E_{2j} < \alpha_j < E_{2j+1} \):

\[ \frac{\partial E_j}{\partial \tau_k} = \frac{c_k}{E_j - \alpha_k} \text{ with } j = 0, \ldots, 2g+1, \quad k = 0, \ldots, g; \] (43)

\[ \frac{\partial \alpha_j}{\partial \tau_k} = \frac{c_k}{\alpha_j - \alpha_k} \text{ with } j = 0, \ldots, k - 1, k + 1, \ldots, g; \] (44)

\[ \frac{\partial \alpha_k}{\partial \tau_k} = \frac{1}{2} \sum_{j=0}^{2g+1} \frac{c_k}{E_j - \alpha_k} - \sum_{j \neq k} \frac{c_k}{\alpha_j - \alpha_k}. \] (45)
where
\[ c_k = \sqrt{\frac{\prod_{j=0}^{2g+1} (\alpha_k - E_j)}{\prod_{j \neq i} (\alpha_k - \alpha_j)^2}}. \] (46)

Here \( \tau_k \) denotes the parameter of the flow associated with the point \( \alpha_k \).

In [24] the following properties of equations (43)-(45) were proved:

**Lemma 4**

1. These flows preserve all \( x \)-quasiperiods of the corresponding NLS equation solutions. In particular, if the starting solution is \( x \)-periodic with the period \( 2L \), it remains \( 2L \)-periodic in \( x \).

2. These flows commute pairwise.

3. Assume that the parameters \( \tau_k \) are normalized by the following condition: they tend to 0 if the lengths of all gaps tend to 0. Then
\[ p(\alpha_k^\pm) = \mp i\tau_k, \mod (\pi/L). \] (47)

4. The variation of the quasimomentum by these flows is given by
\[ \frac{\partial p(\gamma, \vec{\tau})}{\partial \tau_k} = -\frac{c_k}{\lambda - \alpha_k} \frac{\partial p(\gamma, \vec{\tau})}{\partial \lambda}. \] (48)

### 3 Asymptotics of the spectrum for discontinuous periodic profiles

In the Fourier theory for periodic smooth functions it is well-known that the coefficients \( \hat{f}_k \) of the series decay faster than any power of \( k \). If the function \( f(x) \) is piecewise smooth and discontinuous at a finite number of points \( x_1, \ldots, x_n \), then the coefficients decay as \( 1/k \) and the leading terms are:

\[ \hat{f}_k = \frac{1}{2\pi ik} \sum_{j=1}^{n} \delta_j e^{-\frac{2\pi i k x_j}{2L}} + O\left(\frac{1}{k^2}\right). \] (49)

Here \( 2L \) is the period of the function and \( \delta_j \) is the jump at the point \( x_j \): \( \delta_j = f(x_j^+) - f(x_j^-) \), where \( f(x^\pm) = \lim f(x \pm |\epsilon|), \epsilon \to 0 \).
In addition, if the right and left derivatives of \( f(x) \) coincide at all discontinuity points, then the corrections are of order \( 1/k^3 \).

The periodic spectral transform can be treated as a nonlinear analog of the discrete Fourier transform. In the small potential limit the lengths of the gaps are \( |2\hat{f}_k| \) and the phase of \( \hat{f}_k \) is encoded in the position of the divisor.

Since the Dirichlet IBV problem is reformulated as an odd \( 2L \)-periodic Cauchy problem with jumps \( 2v_0(t) \) at the points \( 2nL \) and jumps \( -2v_L(t) \) at the points \( (2n+1)L \), then we have to find a spectral characterization of this kind of discontinuous profiles.

**Proposition 1** Let the potential \( q(x) \) be odd, \( 2L \)-periodic and sufficiently smooth outside the discontinuity points \( nL \) and have the jumps \( 2v_0(t), -2v_L(t) \) at the points \( 2nL, (2n+1)L \) respectively. Then the odd character of \( q(x) \) implies the following symmetry of the spectrum:

\[
E_k = -E_{-k}, \quad \lambda_k = -\lambda_{-k}, \quad k \in \mathbb{Z}
\]  

(50)

and the above discontinuities imply the following asymptotics:

\[
E_{2k} = \frac{\pi k}{2L} + \frac{I_1}{\pi k} + \frac{|d_k|}{\pi k} + O\left(\frac{1}{k^2}\right),
\]

\[
E_{2k+1} = \frac{\pi k}{2L} + \frac{I_1}{\pi k} + \frac{\text{Im} d_k}{\pi k} + O\left(\frac{1}{k^3}\right),
\]

\[
\lambda_k = \frac{\pi k}{2L} + \frac{I_1}{\pi k} + \text{Im} \frac{d_k}{\pi k} + O\left(\frac{1}{k^3}\right),
\]

\[
\alpha_k = \frac{\pi k}{2L} + \frac{I_1}{\pi k} + O\left(\frac{1}{k^3}\right),
\]

(51)

where

\[
d_k = d_k(t) = v_0(t) - (-1)^k v_L(t),
\]

\[
I_1(t) = \int_0^L dx |u(x,t)|^2
\]

(52)

and \( v_0(t) = u(0,t), \ v_L(t) = u(L,t) \).

The absence of \( O(1/k^2) \) corrections, due to the symmetry (50), will be essential for us.

The idea of the proof is the following: in each continuity interval equation (8) can be asymptotically diagonalized by the gauge transformation

\[
\Psi(\lambda, x) = G(\lambda, x)\Psi_1(\lambda, x),
\]

(53)

where \( G(\lambda, x) \) is an asymptotic series in \( 1/\lambda \). All coefficients of \( G(\lambda, x) \) can be computed explicitly in terms of \( q(x) \) and its derivatives. Taking
into account the discontinuities at 0, \( L \) and \( 2L \), one gets the following representation for the transition matrix in the interval \([0, 2L]\):

\[
T(\lambda) = G(\lambda, 0^-)e^{iLP_2(\lambda)\sigma_3}G^{-1}(\lambda, L^+)G(\lambda, L^-)e^{iLP_1(\lambda)\sigma_3}G^{-1}(\lambda, 0^+),
\]

(54)

where \( p_{1,2}(\lambda) \) are the averages on the intervals \([0, L]\) and \([L, 2L]\) respectively of the first component of the diagonalized \( U \) matrix.

By Lemma 2 the branch points are determined by the equation

\[
\text{tr} T(\lambda) = \pm 2
\]

and the divisor at \( x = 0 \) is determined by the equation

\[
T_{11}(\lambda) + T_{21}(\lambda) - T_{12}(\lambda) - T_{22}(\lambda) = 0.
\]

The asymptotics (51) can be obtained expanding these equations in inverse powers of \( \lambda \).

**Lemma 5**  
Let \( \Gamma \) be an infinite genus spectral curve with real branch points \( E_k \) such, that

\[
E_{2k} = \frac{\pi k}{2L} + O\left(\frac{1}{k}\right), \quad E_{2k+1} = \frac{\pi k}{2L} + O\left(\frac{1}{k}\right)
\]

and each oval \( a_k \) contains exactly one divisor point \( \gamma_k \). Then the quantities \( r_k \) and \( \tilde{r}_k \) given by (33) and (42) are well-defined.

The proof is rather straightforward.

This potential has 80 gaps (it corresponds to 40 Fourier harmonics).
4 Time evolution of the spectral data

In this section we derive the equations describing the time evolution of the spectral data – the spectral curve and the divisor – for the Dirichlet problem with zero at one end

\[ u(0, t) = v_0(t), \quad u(L, t) = v_L(t) = 0, \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq L. \] (56)

Let

\[ v_0(t) = A(t)e^{iB(t)}, \] (57)

where \( A(t) \) and \( B(t) \) are real functions.

The calculations show that the evolution of the divisor is described by the sum of two separately divergent vector fields. But if the boundary function \( v_0(t) \) is real, both divergences simultaneously vanish. This suggests to use the following gauge:

\[ q(x, t) = \tilde{q}(x, t)e^{iB(t)} \] (58)

and to calculate the Dubrovin equations for the function \( \tilde{q}(x, t) \). It satisfies the following equation:

\[ i\tilde{q}_t(x, t) = \tilde{q}_{xx}(x, t) - 2|\tilde{q}(x, t)|^2\tilde{q}(x, t) + \dot{B}(t)\tilde{q}(x, t) - 2A(t) \sum_{n \in \mathbb{Z}} \delta'(x - 2Ln). \] (59)

The right-hand side of (59) is a sum of 3 vector fields:

\[ \frac{\partial \tilde{q}}{\partial t} = D_s \tilde{q} + D_g \tilde{q} + D_f \tilde{q}, \] (60)

where

\[ D_s \tilde{q} = -i\tilde{q}(x, t)_{xx} + 2i|\tilde{q}(x, t)|^2\tilde{q}(x, t), \] (61)

\[ D_g \tilde{q} = -i\dot{B}(t)\tilde{q}(x, t), \] (62)

\[ D_f \tilde{q} = 2iA(t) \sum_{n \in \mathbb{Z}} \delta'(x - 2Ln). \] (63)

Starting from now we denote by \( \lambda_k(x, t) \) the divisor corresponding to the gauged potential \( \tilde{q}(x, t) \). It is natural to write the evolution of the branch points, of the zeroes of the quasimomentum differential and of the divisor as the sum of these 3 flows:

\[ \frac{dE_k(t)}{dt} = D_s E_k(t) + D_g E_k(t) + D_f E_k(t), \] (64)
\[
\frac{d\alpha_k(t)}{dt} = D_s\alpha_k(t) + D_g\alpha_k(t) + D_f\alpha_k(t),
\]

\[
\frac{\partial\lambda_k(0, t)}{\partial t} = D_s\lambda_k(0, t) + D_g\lambda_k(0, t) + D_f\lambda_k(0, t).
\]

The action of the vector fields \(D_s\) and \(D_g\) can be obtained as the \(g \to \infty\) limit of the finite-gap formulas, but it is necessary to check the convergence of these expressions. From the results mentioned in Section 2 and, in particular, from equations (35), (37), we have

\[
D_sE_k(t) = D_gE_k(t) = D_s\alpha_k(t) = D_g\alpha_k(t) = 0,
\]

\[
D_s\lambda_k(0, t) = 4 \left[ \frac{1}{2} \sum_j E_j^2(t) - \sum_{j \neq k} \lambda_j^2(0, t) \right] \tilde{r}_k(0, t),
\]

\[
D_g\lambda_k(0, t) = -\frac{dB(t)}{dt} \tilde{r}_k(0, t).
\]

Lemma 6 Let the spectral data have the asymptotics (61) of Proposition 1, with \(d_k(t) = A(t) \in \mathbb{R}\). Then the right-hand side of (65) is well-defined.

The proof is straightforward.

The rest of the section is devoted to the proof of the following Main Theorem, describing the effect of the forcing on the dynamics of the spectral data.

Theorem 1 The effect of the forcing on the spectral data is described by the following vector fields:

\[
D_fE_j(t) = \sum_k \frac{\partial E_j}{\partial \tau_k} D_f\tau_k(t) = \sum_k \frac{c_k(t)}{E_j(t) - \alpha_k(t)} D_f\tau_k(t),
\]

\[
D_f\alpha_j(t) = \sum_k \frac{\partial \alpha_j}{\partial \tau_k} D_f\tau_k(t) = \sum_k \frac{c_k(t)}{\alpha_j(t) - \alpha_k(t)} D_f\tau_k(t) + \frac{c_j(t)}{2} \left[ \sum_{k \neq j} \frac{c_j(t)}{E_k(t) - \alpha_j(t)} - \sum_{k \neq j} \frac{c_j(t)}{\alpha_k(t) - \alpha_j(t)} \right] D_f\tau_j(t),
\]

\[
D_f\lambda_k(0, t) = \left[ 4A(t)\lambda_k(0, t) \frac{M_k(t) + 2}{M_k(t) + 2A(t)^2} \right] \tilde{r}_k(0, t).
\]
where
\[
M_k = e^{4iLP(\gamma_k(0,t),t)} - 1, \tag{73}
\]
\[
D_f \tau_k(t) = \frac{-2\alpha_k(t)}{L} A(t) \frac{1 - \Xi(\alpha_k^+,0,t)\Xi(\alpha_k^-,0,t)}{\Xi(\alpha_k^+,0,t) - \Xi(\alpha_k^-,0,t)}, \tag{74}
\]
\[
\Xi(\gamma,x,t) = \mu \prod_{j \neq k} \frac{\sqrt{(\lambda - \lambda_j(x,t))}}{|\lambda - \lambda_j(x,t)|} + \sum_k \frac{\tilde{r}_k(x,t)}{(\lambda - \lambda_k(x,t))} \tag{75}
\]
and \(c_k\) are defined by (46).

These vector fields can be transformed into an algebraic form by introducing the following extended phase space: \(\{E_k, \alpha_k, \lambda_k, \mu_k, M_k\}\).

The equations describing the time evolution of the remaining parameters have the following form:
\[
d_t \tilde{\mu}_k(0,t) = \frac{\tilde{\mu}_k(0,t)}{2} \left[ \frac{d}{dt} \lambda_k(0,t) - \frac{d}{dt} E_{2k}(t) \right] \frac{\lambda_k(0,t) - E_{2k}(t)}{\lambda_k(0,t) - E_{2k+1}(t)} + \frac{d}{dt} \lambda_k(0,t) - \frac{d}{dt} E_{2k+1}(t) \left| \lambda_k(0,t) - \alpha_j(t) \right| \tag{76}
\]
\[
d_t M_k(t) = 4L(M_k(t)+1) \prod_{j \neq k} \frac{|(\lambda_k(0,t) - \alpha_j(t))|}{\sqrt{(\lambda_k(0,t) - E_{2j}(t))(\lambda_k(0,t) - E_{2j+1}(t))}} \times \lambda_k(0,t) - \frac{\alpha_k(t)}{\tilde{\mu}_k(t)} \left[ \frac{d}{dt} \lambda_k(0,t) - \sum_j c_j(t) \frac{\lambda_k(0,t) - \alpha_j(t)}{\lambda_k(0,t) - \alpha_j(t)} D_f \tau_j(t) \right]. \tag{77}
\]

**Remark.** The above dynamical system contains infinite sums and products. Their convergence follows from direct calculations using the asymptotic formulas (51).

The proof of Theorem 1 is based on the following formula:
\[
\delta \hat{T}(\lambda,t) = \int_0^{2L} \hat{T}(\lambda,2L,\tilde{x},t)\delta U(\lambda,\tilde{x},t)\hat{T}(\lambda,\tilde{x},0,t)d\tilde{x}. \tag{78}
\]

Let us introduce the following notations:
\[
\tilde{T}(\gamma,x,t) = \tilde{\Psi}(\gamma,x,t) \tilde{T}(\lambda,\tilde{x},t)\tilde{\Psi}(\gamma,x,t), \tag{79}
\]
where \(\tilde{\Psi}(\gamma,x,t)\) is the following \(2 \times 2\) matrix solution of (8) combined from the Bloch solutions:
\[
\tilde{\Psi}(\gamma,x,t) = [\Psi(\gamma,x,t) \Psi(\sigma \gamma,x,t)] = \begin{pmatrix} \psi_1(\gamma,x,t) & \psi_1(\sigma \gamma,x,t) \\ \psi_2(\gamma,x,t) & \psi_2(\sigma \gamma,x,t) \end{pmatrix}. \tag{80}
\]
By definition,
\[ \tilde{T}(\lambda, t_0, t_0) = \begin{pmatrix} e^{2iLP(\gamma, t_0)} & 0 \\ 0 & e^{-2iLP(\gamma, t_0)} \end{pmatrix}. \]  

(81)

A direct calculation shows that
\[ \delta \tilde{T}(\lambda, t, t_0) \bigg|_{t = t_0} = \tilde{T}(\lambda, t_0, t_0) \int_0^{2L} \tilde{\Psi}^{-1}(\lambda, \tilde{x}, t_0) \delta U(\lambda, \tilde{x}, t_0) \tilde{\Psi}(\lambda, \tilde{x}, t_0) d\tilde{x}. \]  

(82)

For the variation of the potential \( \tilde{q} \) defined by (63) we obtain
\[ \delta \tilde{T}(\gamma, t, t_0) \bigg|_{t = t_0} = \tilde{T}(\gamma, t_0, t_0) \left( \frac{1}{W} \int_0^{2L} \begin{pmatrix} \psi^2(\sigma\gamma, \tilde{x}, t_0) & -\psi^1(\sigma\gamma, \tilde{x}, t_0) \\ -\psi^2(\gamma, \tilde{x}, t_0) & \psi^1(\gamma, \tilde{x}, t_0) \end{pmatrix} \right) \times \]
\[ \times \begin{pmatrix} i\delta\lambda & -2A(t_0)[\delta'(x) + \delta'(x - 2L)]\delta_f \\ -2A(t_0)[\delta'(x) + \delta'(x - 2L)]\delta_f & -i\delta\lambda \end{pmatrix} \times \]
\[ \times \begin{pmatrix} \psi^1(\gamma, \tilde{x}, t_0) & \psi^2(\gamma, \tilde{x}, t_0) \\ \psi^2(\gamma, \tilde{x}, t_0) & \psi^1(\gamma, \tilde{x}, t_0) \end{pmatrix} \]
\[ = \frac{1}{W} \begin{pmatrix} e^{2iLP(\gamma, t_0)} & 0 \\ 0 & e^{-2iLP(\gamma, t_0)} \end{pmatrix} \left( \begin{array}{cc} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{array} \right) (\gamma, t_0), \]  

where \( W \) is the Wronskian determinant
\[ W = \psi^1(\gamma, x, t_0)\psi^2(\sigma\gamma, x, t_0) - \psi^2(\gamma, x, t_0)\psi^1(\sigma\gamma, x, t_0), \]  

(84)

\[ \Delta_{11} = i\delta\lambda \int_0^{2L} \psi^1(\gamma, x, t_0)\psi^2(\sigma\gamma, x, t_0) + \psi^2(\gamma, x, t_0)\psi^1(\sigma\gamma, x, t_0) dx + \]
\[ +2A(t_0)\delta_f \int_0^{2L} \left( \psi^1(\gamma, x, t_0)\psi^1(\sigma\gamma, x, t_0) - \psi^2(\gamma, x, t_0)\psi^2(\sigma\gamma, x, t_0) \right) [\delta'(x) + \delta'(x - 2L)] dx, \]  

(85)

\[ \Delta_{21} = -2i\delta\lambda \int_0^{2L} \psi^1(\gamma, x, t_0)\psi^2(\gamma, x, t_0) dx + \]
\[ +2A(t_0)\delta_f \int_0^{2L} \left( (\psi^2(\gamma, x, t_0))^2 - (\psi^1(\gamma, x, t_0))^2 \right) [\delta'(x) + \delta'(x - 2L)] dx \]  

(86)
(we do not use the formulas for $\Delta_{12}$, $\Delta_{22}$ in our calculations).

Let us first calculate the deformation of the spectral curve. From (25), (47) it follows that

$$
\frac{d\tau_k}{dt}\bigg|_{t=t_0} = i \frac{dp(\alpha_k^+(t), t)}{dt} \bigg|_{t=t_0} = i \frac{\partial p(\gamma, t)}{\partial t} \bigg|_{\gamma=\alpha_k^+(t_0)}.
$$

Then, from (83), it follows that

$$
\frac{d\tau_k}{dt}\bigg|_{t=t_0} = \frac{\Delta_{11}}{2LW} \bigg|_{\gamma=\alpha_k^+(t_0)}
$$

with $\delta \lambda = 0$.

The quadratic combination of eigenfunctions in (85) is periodic, therefore we have

$$
\Delta_{11} = 2A(t_0) \times
$$

$$
\times \int_{-\epsilon}^{\epsilon} \left( \psi^1(\alpha_k^+, x, t_0)\psi^1(\alpha_k^-, x, t_0) - \psi^2(\alpha_k^+, x, t_0)\psi^2(\alpha_k^-, x, t_0) \right) \delta'(x) dx =
$$

$$
= -2A(t_0) \left[ \psi^1(\alpha_k^+, x, t_0)\psi^1(\alpha_k^-, x, t_0) - \psi^2(\alpha_k^+, x, t_0)\psi^2(\alpha_k^-, x, t_0) \right]_{x=0} =
$$

$$
= -4i\alpha_k(t_0)A(t_0) \left[ \psi^1(\alpha_k^+, 0, t_0)\psi^1(\alpha_k^-, 0, t_0) + \psi^2(\alpha_k^+, 0, t_0)\psi^2(\alpha_k^-, 0, t_0) \right].
$$

Using the fact that the Wronskian does not depend on $x$, the definition (30) and the arbitrariness of $t_0$, we obtain equation (74). Using the isoperiodic deformation formulas (43)-(44), we finally obtain the evolution equations (70), (71) of the spectral curve.

Let us calculate now the evolution of the divisor with respect to the vector field $D_f$. Combining Lemma 2 with the definition of $\tilde{T}$ we obtain the following property of the divisor points $\lambda_k$: the vector

$$
\begin{pmatrix} 1 \\ 0 
\end{pmatrix}
$$

remains the eigenvector of $\tilde{T}$ after variation, or equivalently, $\Delta_{21}=0$. Therefore

$$
D_f \lambda_k(0, t) = iA(t) N D,
$$

where

$$
N = - \int_{0}^{2L} \left( (\psi^2(\gamma_k(0, t), x, t))^2 - (\psi^1(\gamma_k(0, t), x, t))^2 \right) [\delta'(x) + \delta'(x-2L)] dx =
$$

22
We used here the definition of the divisor points $\gamma_k$ by the condition
$$\psi_1(\gamma_k(x,t),0,0,t) - \psi_2(\gamma_k(x,t),0,0,t) = 2.$$  \hspace{1cm} (89)

As a corollary we obtain
$$\psi_1(\gamma_k(0,t),0,0,t) = 1,$$  \hspace{1cm} (90)
$$\psi_2(\gamma_k(0,t),0,0,t) = -1,$$  \hspace{1cm} (90)
$$\psi_1(\gamma,0,t) + \psi_2(\gamma,0,t) = \frac{2}{\Xi(\gamma,0,t)}. \hspace{1cm} (91)$$

Let us calculate $N$:
$$\left((\psi^2(\gamma,x,t))^2 - (\psi^1(\gamma,x,t))^2\right)' =$$
$$=-2i\lambda(\psi^2(\gamma,x,t))^2 + (\psi^1(\gamma,x,t))^2 - 2i(q(x,t) + \bar{q}(x,t))\psi^1(\gamma,x,t)\psi^2(\gamma,x,t);$$
therefore
$$N = -2i\lambda_k(0,t) \int_0^{2L} \left(\left((\psi^2(\gamma_k(0,t),x,t))^2 + (\psi^1(\gamma_k(0,t),x,t))^2\right) [\delta(x) + \delta(x - 2L)] dx -$$
$$-2i \int_0^{2L} \left(q(x,t) + \bar{q}(x,t)\right) \psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma_k(0,t),x,t) [\delta(x) + \delta(x - 2L)] dx.$$

If $f(x)$ is a continuous function, then
$$\int_0^\epsilon f(x)\delta(x) dx = \frac{1}{2} f(0), \hspace{1cm} (93)$$
where $\epsilon$ is any positive constant. Therefore the first integral in (92) is well-defined:
$$\int_0^{2L} \left(\left((\psi^2(\gamma_k(0,t),x,t))^2 + (\psi^1(\gamma_k(0,t),x,t))^2\right) [\delta(x) + \delta(x - 2L)] dx =$$
$$-2i \int_0^{2L} \left(q(x,t) + \bar{q}(x,t)\right) \psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma_k(0,t),x,t) [\delta(x) + \delta(x - 2L)] dx. \hspace{1cm} (92)$$
\[ = e^{4iLp(\gamma_k(0,t))} + 1.\]

But the second integral depends on the regularization of the \(\delta\)-function. In our problem the \(\delta\)-term arises as the compensation of the discontinuity in \(q(x,t)\). Therefore it is natural to write, near the point \(x = 0\),

\[ \delta(x) = \frac{1}{2A(t)} \text{Re} \tilde{q}_x(x,t) + \text{regular part.} \]

We use also the fact that the function \(\text{Im} \tilde{q}(x,t)\) is continuous and vanishes at the point \(x = 0\); therefore

\[ \int_0^{2L} 2(\text{Re} \tilde{q}(x,t))\psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma_k(0,t),x,t) [\delta(x) + \delta(x-2L)] = \]

\[ = \lim_{\epsilon \to 0+} \left[ \int_0^{\epsilon} + \int_{2L-\epsilon}^{2L} \frac{1}{A(t)} \text{Re} \tilde{q}(x,t) \text{Re} \tilde{q}_x(x,t)\psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma_k(0,t),x,t) dx \right] = \frac{1}{2} [e^{4iLp(\gamma_k(0,t))} - 1] A(t). \]

Combining (94) and (95) we obtain:

\[ N = -2i\lambda_k(0,t) \left[ e^{4iLp(\gamma_k(0,t))} + 1 \right] - iA(t)[e^{4iLp(\gamma_k(0,t))} - 1]. \]

Let us now calculate \(D\):

\[ D = \lim_{\gamma' \to \gamma_k(0,t)} \frac{1}{2} \int_0^{2L} \psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma',x,t) + \psi^1(\gamma',x,t)\psi^2(\gamma_k(0,t),x,t) dx = \]

\[ = \frac{i}{2} \lim_{\gamma' \to \gamma_k(0,t)} \left[ \psi^1(\gamma_k(0,t),x,t)\psi^2(\gamma',x,t) - \psi^1(\gamma',x,t)\psi^2(\gamma_k(0,t),x,t) \right]_{\chi' = \gamma_k(0,t)} = \]

\[ \times \frac{\partial}{\partial \lambda'} \left[ \psi^1(\gamma_k(0,t),0,t)\psi^2(\gamma',0,t) - \psi^1(\gamma',0,t)\psi^2(\gamma_k(0,t),0,t) \right]_{\gamma' = \gamma_k(0,t)} = \]

\[ = \frac{i}{2} \left[ e^{4iLp(\gamma_k(0,t))} - 1 \right] \frac{\partial}{\partial \lambda'} \left[ \psi^1(\gamma',0,t) + \psi^2(\gamma',0,t) \right]_{\gamma' = \gamma_k(0,t)}. \]
Taking into account (91) we obtain

$$\mathcal{D} = \left[ e^{4iLP(\gamma_k(0,t))} - 1 \right] \frac{i}{r_k(0,t)}$$

(97)

and, finally, equation (72).

Equation (76) can be obtained by direct differentiation of (40) and formula (77) follows immediately from (48) and (73).

5 Concluding remarks

1. In our method we study the IBV problem (2), (3) through the following steps:

(a) From the initial condition $u_0(x)$ we build its odd periodic extension $q(x,0)$ as in (6) and then we construct, through the direct periodic spectral transform summarized in Section 2.1, the initial spectral data \{\(E_k(0), \alpha_k(0), \lambda_k(0,0), \mu_k(0,0), M_k(0)\}\).

(b) Solving the system of ODEs (64) - (72), (76), (77) with the above initial conditions, we obtain the evolved spectral data \{\(E_k(t), \alpha_k(t), \lambda_k(0,t), \mu_k(0,t), M_k(t)\}\).

(c) Solving the x-Dubrovin equations (34) we obtain, from \(\lambda_k(0,t), \lambda_k(x,t)\).

(d) Using formula (36) we reconstruct the potential $q(x,t)$, whose restriction to the segment $[0,L]$ gives the solution $u(x,t)$ of the IBV problem (2), (3).

(e) It is well-known that the x-Dubrovin equations can be linearized, at fixed $t$, on the Jacobi variety constructed from the spectral data \{\(E_k(t), \gamma_k(0,t)\}\}. Correspondingly, the solution $q(x,t)$ can be written explicitly in terms of Riemann $\theta$-functions. This is an alternative procedure to the above steps (1c), (1d). However we remark that this Jacobi variety depends on $t$ in a highly non-trivial way, in contrast to the integrable cases.

2. Although the x-Dubrovin equations can be linearized on the Jacobi variety and the solutions can be written down explicitly in terms of Riemann $\theta$-functions, it turns out that, from the numerical point of view, it is often (and, in particular, in our case) more convenient to integrate the Dubrovin equations directly.
This idea was communicated to one of the authors (P.G.) by A.Osborne long times ago, and it was used by the author in his numerical experiments.

3. We believe that our approach be the proper nonlinear analog of the discrete sine-Fourier transform method, which allows one to solve the Dirichlet problem on the segment for the linear Schrödinger equation.

4. We consider this approach - in which the original IBV problem is reformulated as a forced periodic IBV problem that, without forcing, is known to be integrable - as a convenient way to treat the original IBV problem. One of the reasons is the following. It is well-known that, to analyse a perturbed (by a forcing) nonlinear equation, one has to calculate the spectral decomposition of the forcing in terms of the eigenfunctions of the linearized equation. But, for soliton equations, this decomposition can be explicitly written in terms of the auxiliary spectral problem. Therefore the calculation of the time evolution of the spectral data due to the forcing can be considered as the proper nonlinear analog of the Fourier analysis.

5. A detailed study of the system of ODEs (64) - (72), (76), (77) will be the subject of future investigation. Preliminary numerical experiments seem to show that, starting with zero initial condition and applying a smooth localized forcing, only a finite number of nonlinear modes remain essentially non-zero when the external forcing vanishes. Correspondingly, only a finite subset of the above system of ODEs is essential for gaining a correct picture of the process.

6. The method presented in this paper can obviously be applied to study the general Dirichlet, Neumann and mixed problems for the focussing and defocussing NLS equations on the segment. These general results will be presented in subsequent works.

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