A generalized meshing analysis method and its application to toroidal surface enveloping conical worm drive

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Abstract
A generalized method for the meshing analysis of conical worm drive is proposed, whose mathematical model is more general and whose application scope is expanded. A universal mathematical model, which can be conveniently applied to left-handed and right-handed conical worm pairs and their tooth flanks on different sides, is established by introducing the helical spin coefficient and tooth side coefficient of the conical worm. The pressure angle at the reference point, which is a key parameter for calculating the curvature parameters and lubrication angle, is determined based on the unit normal vector of the worm helical surface and is no longer determined by the tooth profile angle in the worm shaft section. The above improvement breaks away from the limitation of the classic meshing analysis method based on the reference-point-based meshing theory and thus expands its application scope. The toroidal surface enveloping conical worm drive is taken as an instance to illustrate the proposed method and the numerical example studies are conducted. The approaches to determine the reference point, the normal unit vector, and the curvature parameters at the reference point are all demonstrated in detail. The numerical results all manifest that the method presented in the current work is correct and practicable.

Keywords
Gear geometry, meshing analysis, reference cone, curvature parameter, conical worm drive

Introduction
Conical worm drive is a type of cross-axis offset gear transmission, which is composed of a worm and a face worm wheel whose indexing surfaces are both conical surfaces.1,2 The reasonable use of the spatial meshing field makes the conical worm drive possess a series of excellent meshing characteristics. For this, the conical worm drive has attracted many scholars to conduct research on it and some new toothed conical worm pairs have been proposed, such as Spiroid gear or Archimedes conical worm drive,1 the involute conical worm drive,3,4 and the enveloped conical worm drives,5–7 as shown in Figure 1.

As is well known, meshing analysis is an important means to evaluate the meshing performance of the gear drive. For the conical worm drive, there are two typical kinds of meshing analysis method in the existing literatures,6,7 one is the method strictly based on the tooth surface equations while the other is the method based on the classic reference-point-based meshing theory.8–12 Among them, the former method is strictly based on...
the solution of the tooth surface equations to obtain the numerical results of the meshing characteristic parameters. In the whole process, the calculation of the geometrical parameters of the worm pair is not involved. Therefore, this method usually cannot intuitively reflect the potential relationship between the meshing characteristics and the geometric design of the conical worm pair. For the latter method, in the calculation process, the key geometric parameters of the reference cone corresponding to the conical worm pair can be obtained, and based on this, the meshing characteristic parameters of the conical worm pair at the reference point are calculated.

As is well known, the main idea of the classical reference-point-based mesh theory was proposed by E. Wildhaber and was applied to the geometric design and the cutting setting calculation of the hypoid gear primarily. This is a huge advancement in the theory of gear mesh and provides a theoretical basis for the study of staggered shaft gearing. While this theory had not been elaborated detailedly and systematically in a period of time, so a lot of exploration and research works were carried out on it.

In 1961, Baxter studied the design of the reference cones of the hypoid gear.

Since the 1970s, Litvin researched the geometry and mesh principle of the hypoid gear drive. Specifically, the main idea of the classical gear mesh theory based on the reference point is expounded. Dong conducted a comprehensive study on the gear mesh theory based on the reference point. The radius design of the cutterhead was proved rigorously and a new calculating formula of the induced normal curvature at reference point was derived. These works make the classic theory suggested by E. Wildhaber more rigorous and can also be applied to the curvature analysis of Spiroid drive.

Almost at the same time, Zeng conducted a rigorous mathematical demonstration of the idea proposed by E. Wildhaber and applied it to the design and manufacture of the spiral bevel gear. More innovatively, the blank design and cutting calculation of the spiral bevel gear are attributed to the calculation of the parameters and the curvature at the reference point of an aligned hypoid gear in his work.

But for the meshing analysis method based on the classic reference-point-based meshing theory, its application objects are relatively limited at present. The reason is that the classic reference-point-based meshing theory has some problems shown as below:

(i) In the classic theory, the reference point was determined by the geometric relationship of the reference cones and some assumptions rather than by the rigorous tooth surface equations. In essence, this is an approximate method because it cannot guarantee that the reference point is actually located on the tooth flank without the strictly limited of the tooth flank equations. As a result, the meshing features of the corresponding conical worm pair at this reference point may not be investigated accurately.

(ii) The classical theory can only be applied to the conical worm with linear tooth profile because the pressure angle at the reference point always is approximately calculated with the assist of the tooth profile angle in the worm’s shaft section, which is inconvenient for some types of conical worms with non-linear tooth profile in their shaft section, such as the involute conical worm drive and the enveloped conical worm drives. Because of the above reason, the application scope of the classical theory is limited to some extent.

Lately, Ref. put forward a new method to calculate the reference point and solve the problem (i) mentioned
above. However, this work did not mention the reference cones and its related geometric parameters, and the problem (ii) is also not involved.

In the current work, a generalized meshing analysis method combining the methods based on tooth surface equations and the reference point-based meshing theory is presented, which is to solve this problem (ii) with the aid of the unit normal vector of the conical worm helicoids determined by the tooth surface equations to compute the pressure angle at the reference point, thereby the dependence of the existing method on the tooth profile angle in the shaft section of the conical worm is avoided. Besides, the rotation coefficient and tooth surface coefficient are introduced, and then a general mathematical model is established based on this. The above work effectively expand the application scope of the existing method. In order to verify the effectiveness of the method proposed in this paper, the toroidal enveloping conical worm drive has been studied as an application object, and some new characteristics of this type of conical worm drive have been discovered.

**Generalized meshing analysis method of conical worm drive**

**Frames based on reference cones**

In the reference-point-based meshing theory, a pair of reference cones can be established based on a set of rules after the reference point is determined, and then the relevant research can be carried out, such as determining the dimension of worm wheel blank based on the dimension of worm blank, and calculating the meshing characteristic parameters of conical worm pair.9,11,20

For the sake of illustration, the conical worm and the conical worm wheel in mesh are marked as gear 1 and gear 2 respectively. In order to obtain a high transmission efficiency, the rotation directions of the two gears should be different. In other words, when the gear 1 is left-handed, the gear 2 should be right-handed, and vice versa.

As shown in Figure 2, two immobile frames \( \sigma(O; \vec{i}, \vec{j}, \vec{k}) \) and \( \sigma_{o2}(O_2; \vec{i}_{o2}, \vec{j}_{o2}, \vec{k}_{o2}) \) are built to determine the installation site of the gear 1 and to label the initial location of the gear 2, respectively. Therein, the point \( O \) and \( O_2 \) are the pedals of the common perpendicular of the axes of the gears 1 and 2. The basal vectors \( \vec{k} \) and \( \vec{k}_{o2} \) are severally collinear with the axes \( O_1 \# M_1 \) and \( O_2 \# M_2 \), and the included angle between them is the shaft angle, \( \Sigma \), of the gears 1 and 2. The vectors \( \vec{i} \) and \( \vec{i}_{o2} \) are all along the common vertical line between the axes of the two gears. Besides, three auxiliary frames \( \sigma_o(P; \vec{i}_o, \vec{j}_o, \vec{k}_o) \), \( \sigma_{o2}(P; \vec{i}_{o2}, \vec{j}_{o2}, \vec{k}_{o2}) \) are built to calculate the relevant geometric parameters, as displayed in Figures 2 to 5. Therein, \( \vec{i}_o \) is along the opposite direction of the relative speed \( V_{12} \) of the gears 1 and 2, \( \vec{i}_o \) and \( \vec{i}_{o2} \) are along the extensions of the lines \( O_1 P \) and \( O_2 P \), respectively. \( \vec{k}_o \) and \( \vec{k}_{o2} \) are coincident with the line \( M_1 M_2 \) and are all point to \( M_2 \), while \( \vec{k}_o \) is in the opposite direction. The included angle formed by the vectors \( \vec{i}_o \) and \( \vec{i}_o \) is the helix angle \( \beta_1 \) of the gear 1, and the included angle between \( \vec{i}_o \) and \( \vec{i}_{o2} \) is the helical angle \( \beta_2 \) of the gear 2.

![Figure 2. Reference cones formed based on the reference point: (a) reference cones of the left-handed gear 1 and the right-handed gear 2 and (b) reference cones of the right-handed gear 1 and the left-handed gear 2.](image-url)
Basic geometric parameters

Mathematically speaking, the reference point \( P \) in the frame \( \sigma \) can be represented as

\[
(\vec{r}_P) = X_P \mathbf{i} + Y_P \mathbf{j} + Z_P \mathbf{k}
\]

where \( X_P, Y_P, \) and \( Z_P \) respectively represent the coordinate components of \( P \) in \( \sigma \).

By the coordinate conversion from the frame \( \sigma \) to \( \sigma_{o2} \), the radius vector \( \vec{r}_{o2} \) in the frame \( \sigma_{o2} \) can be obtained as

\[
(\vec{r}_{o2})_{o2} = R \left[ i_{o2}, - \Sigma \right] (\vec{r}_P) - (-1)^h \mathbf{a}_{o2}\]

where \( X_{o2} = X_P - (-1)^h a \), \( Y_{o2} = Y_P \cos \Sigma + Z_P \sin \Sigma \), and \( Z_{o2} = - Y_P \sin \Sigma + Z_P \cos \Sigma \). Besides, \( R \left[ i_{o2}, - \Sigma \right] \) is the matrix for the coordinate transformation from \( \sigma \) to \( \sigma_{o2} \). In particular, the coefficient \( h \) is used to distinguish the turning directions of the gear 1 tooth surface. When the gear 1 is left-handed, \( h = 0 \); while when the gear 1 is right-handed, \( h = 1 \). The introduction of the coefficient \( h \) allows the established mathematical model to be conveniently applied to conical worm pairs with different turning directions.

Based on equation (2), the dividing circle radius of the gear 2 at the reference point \( P \) can be obtained as

\[
r_2 = \sqrt{X_{o2}^2 + Y_{o2}^2}
\]

\[
= \sqrt{X_P - (-1)^h a}^2 + (Y_P \cos \Sigma + Z_P \sin \Sigma)^2
\]

It can be seen from equation (3) that when the values of the center distance and shaft angle of the conical worm pair are given, the value of dividing circle radius of the conical worm wheel will be determined by the coordinate values of the reference point.

See Figure 3, the closed angle \( \eta_1 \) between the forward direction of \( \mathbf{i} \) and the line \( L_1P \) is defined as the position angle of the reference point, which is utilized to denote the position of the reference point \( P \) in \( \sigma \). In line with the geometric relations shown in these figures, the angle \( \eta_1 \) can be determined as

\[
\sin \eta_1 = -\frac{Y_P}{r_1}, \quad \cos \eta_1 = \frac{X_P}{r_1}
\]

Similarly, the angle \( \eta_2 \) between the forward direction of \( \mathbf{i}_{o2} \) and the line \( L_2P \) also is the position angle of the reference point, which denote the position of the reference point in \( \sigma_{o2} \), as shown in Figure 4. In the light of the geometric relations demonstrated in these figures, the position angle \( \eta_2 \) can be acquired as

\[
\sin \eta_2 = \frac{Y_{o2}}{r_2}, \quad \cos \eta_2 = \frac{X_{o2}}{r_2}
\]

In a general way, it can be supposed that the gear 1 spins on its axial line at the palstance \( |\hat{\omega}_1| = 1 \text{ rad/s} \), accordingly, the palstance of the gear 2 can be represented as \( |\hat{\omega}_2| = 1/\eta_2 \text{ rad/s} \). In accordance with the geometric location displayed in Figure 2, the relative palstance of the gears 1 and 2 can be received in \( \sigma \) as

\[
|\hat{\omega}_{12}| = (\hat{\omega}_1) - (\hat{\omega}_2) = -\frac{1}{\eta_2} \frac{X_P \sin \Sigma \mathbf{j} + (1 + \frac{1}{\eta_2} \cos \Sigma) \mathbf{k}}{r_2}
\]

As displayed in Figure 2, \( \hat{\Omega}_2 = (\hat{\omega}_2) = \hat{\omega}_2 \). With the aid of equations (1) and (6), the relative velocity between the two gears at the reference point can be obtained in \( \sigma \) as

\[
\begin{align*}
\hat{V}_1 &= \hat{V}_2 + \hat{\omega}_2 \times \mathbf{r}_P, \\
\hat{V}_2 &= \hat{\Omega}_2 \times \mathbf{r}_{o2}.
\end{align*}
\]
\[
\begin{align*}
\left( \vec{V}_{12} \right) &= \left( \vec{\omega}_{12} \times (\vec{r}_P) - (\vec{\omega}_2) \times (\vec{O}_2\vec{O}) \right) \\
&= V_x \hat{i} + V_y \hat{j} + V_z \hat{k}
\end{align*}
\]

where \( V_x = -\left[ Y_p \left( 1 + \cos \Sigma \right) \right] \), \( V_y = X_p + \frac{X_p + (-1)^{h+1}a}{l_{12}} \cos \Sigma \), and \( V_z = \frac{X_p + (-1)^{h+1}a}{l_{12}} \sin \Sigma \).

On the grounds of the geometric positional relationship shown in Figure 3, the basal vector \( \vec{k}_b \) can be represented in \( \sigma \) as following with the help of the spherical vector function

\[
\begin{align*}
(\vec{k}_b) &= \vec{m}[-\eta_1, - (90^\circ - \delta_1)] \\
&= -\cos \delta_1 \cos \eta_1 \hat{i} + \cos \delta_1 \sin \eta_1 \hat{j} + \sin \delta_1 \hat{k}
\end{align*}
\]

As shown in Figure 3, the relative velocity vector \( \vec{V}_{12} \) and the basal vector \( \vec{k}_b \) are coplanar to each other, namely \( (\vec{V}_{12}) \cdot (\vec{k}_b) = 0 \). With the aid of equations (7) and (8), the expression of \( \delta_1 \) can be acquired as below

\[
\tan \delta_1 = \frac{X_p Z_p + (-1)^{h}a Y_p \cot \Sigma}{r_1 \left[ (-1)^{h}a - X_p \right]}.
\]

By the coordinate transformations, the relative velocity \( \vec{V}_{12} \) and the basal vector \( \vec{k}_b \) can separately be represented in \( \sigma_{o2} \) as

\[
\begin{align*}
(\vec{k}_{o2}) &= \vec{m}[-\eta_2, - (90^\circ - \delta_2)] \\
&= -\cos \delta_2 \cos \eta_2 \hat{i} - \cos \delta_2 \sin \eta_2 \hat{j} + \sin \delta_2 \hat{k}
\end{align*}
\]

\[
\begin{align*}
\left( \vec{V}_{12} \right)_{o2} &= R_{\vec{t}_o2} - \Sigma \left( \vec{V}_{12} \right) \\
&= V_x \hat{i} + V_y \hat{j} + V_z \hat{k}
\end{align*}
\]

where

\[
\begin{align*}
V_x &= X_p \left[ \cos \Sigma + \frac{1}{l_{12}} \right] + (-1)^{h+1}a, \\
V_y &= X_p \left[ \cos \Sigma + \frac{1}{l_{12}} \right] + (-1)^{h+1}a, \\
V_z &= -V_y \sin \Sigma + V_z \cos \Sigma - X_p \sin \Sigma.
\end{align*}
\]

Undoubtedly, \( (\vec{V}_{12})_{o2} \cdot (\vec{k}_{o2}) = 0 \) can be received for \( \vec{V}_{12} \bot \vec{k}_b \). On the basis of equations (10) and (11), the angle \( \delta_2 \) can be obtained from the expression shown below

\[
\tan \delta_2 = \frac{1}{r_2} \left\{ Y_p \left[ (-1)^{h+1}a \right] + \Sigma \right\} - Z_p \cos \Sigma.
\]

Equations (9) and (12) both manifest that when the position of the reference point changes, the semi-cone angles \( \delta_1 \) and \( \delta_2 \) may be also changed. Taking into account the actual physical meaning, both \( \delta_1 \) and \( \delta_2 \) should be acute angles.

As shown in Figure 3, the vectors \( \vec{V}_{12}, \vec{t}_a, \) and \( \vec{j}_a \) are coplanar. According to their mutual position, \( \vec{V}_{12} \cdot \vec{j}_a = |\vec{V}_{12}| \sin \beta_1 \) and \( \vec{V}_{12} \cdot \vec{t}_a = |\vec{V}_{12}| \cos \beta_1 \) can be obtained, where \( |\vec{V}_{12}| \) is the norm of \( \vec{V}_{12} \). Based on this, the spiral angle \( \beta_1 \) can be determined as

\[
\tan \beta_1 = \theta \left\{ \frac{(\vec{V}_{12}) \cdot (\vec{j}_a)}{(\vec{V}_{12}) \cdot (\vec{t}_a)} \right\} \\
= \cos \delta_1 \left[ Y_p Z_p \sin \Sigma - (-1)^{h}a X_p \cos \Sigma + (i_{12} + \cos \Sigma) r_1 \right] \\
\frac{|X_p + (-1)^{h}a | r_1 \sin \Sigma}{r_1 \left[ (-1)^{h}a - X_p \right]^2}
\]

where \( \vec{t}_a = \vec{m}[-\eta_1, - (90^\circ - \delta_1)] \)

\[
\vec{j}_a = \vec{g}(-\eta_1) = \sin \delta_1 \hat{i} + \cos \eta_1 \hat{j}.
\]

Similarly, the spiral angle \( \beta_2 \) in Figure 4 can be determined as

\[
\tan \beta_2 = \theta \left\{ \frac{(\vec{V}_{12})_{o2} \cdot (\vec{j}_{o2})}{(\vec{V}_{12})_{o2} \cdot (\vec{t}_{o2})} \right\} \\
= -\cos \delta_2 \frac{Y_p Z_p + \left[ r_1 - (-1)^{h}a X_p \right] \cos \Sigma + r_2 \left[ \frac{1}{l_{12}} \sin \Sigma \right]}{r_2}
\]

where \( \vec{t}_{o2} = \vec{m}[-\eta_2, - (90^\circ - \delta_2)] \)

\[
\vec{j}_{o2} = \vec{g}(-\eta_2) = \sin \delta_2 \cos \eta_2 \hat{i} + \sin \delta_2 \sin \eta_2 \hat{j} + \cos \delta_2 \hat{k}.
\]

Considering the actual physical meaning, the obtained calculated results of equations (13) and (14) should be consistent with the following geometric facts. For the left-handed gear 1 and the right-handed gear 2, the value of \( \beta_1 \) is negative but the value of \( \beta_2 \) is positive. For the right-handed gear 1 and the left-handed gear 2, the value of \( \beta_1 \) is positive but the value of \( \beta_2 \) is negative. Besides, when the position of the reference point changes, the values of \( \beta_1 \) and \( \beta_2 \) will also change.

By the coordinate transformation, \( \vec{V}_{12} \) can be expressed in \( \sigma_o \), as
After determining the geometrical parameters based on the two reference cones, the curvature relationship between the tooth surfaces of the conical worm and conical worm wheel in mesh at the reference point can be investigated, and then the related meshing characteristics can be evaluated. In the current section, the local meshing characteristic parameters at the reference point, including the induced principal curvature and the lubrication angle, will be determined on the basis of the above calculations and the related theoretical background.

As exhibited in Figure 5, a coordinate system \( \{ P; \hat{a}_o, \hat{a}_p \times \hat{a}_p, \hat{b}_o \} \) is established at the reference point. In particular, the basal vector \( \hat{a}_p \) is collinear with \( \hat{V}_{12} \), namely \( \hat{a}_p = \hat{V}_{12} / |\hat{V}_{12}| = -\hat{n}_i \). Therefore, \( \sigma_p \) may be called as the velocity coordinate frame.

Since the conical worm and the conical worm wheel are in mesh at the reference point, it can be received that \( \hat{n}_p \cdot \hat{V}_{12} = 0 \) in accordance with the conjugate condition of the surface couple. Therefore, \( \hat{n}_p \) is perpendicular to \( \hat{V}_{12} \) and it thus lies in the plane formed by the basal vectors \( \hat{j}_o \) and \( \hat{k}_o \). In general, the remainder of the inclined angle among the normal direction of the gear 1 tooth flank and the common normal line of the two reference cones at the reference point, namely the included angle between the vectors \( \hat{n}_p \) and \( \hat{j}_o \), is defined as the pressure angle of the gear 1 and usually symbolized as \( \alpha_n \). For this, the normal vector \( \hat{n}_p \) can be represented in \( \sigma_o \) as

\[
\hat{n}_p = (\hat{n}_p)_o = (\hat{n}_p)_o \hat{k}_o = -\hat{f}_o
\]

Figure 5. The coordinate frames for the curvature analysis at the reference point.

\[
\begin{align*}
(\hat{V}_{12})_o &= R[\hat{k}_n, -\beta_1](\hat{V}_{12})_a \\
&= R[\hat{k}_n, -\beta_1](\hat{J}_2 \cdot 90^\circ - \delta_1)R[\hat{k}_n, -\eta_1](\hat{V}_{12})_a \\
&= V_{ax}\hat{i}_o + V_{ay}\hat{j}_o
\end{align*}
\]

where \( V_{ax} = \frac{r_2}{i_2} \sin \beta_2 - r_1 \sin \beta_1 \), \( V_{ay} = \frac{r_2}{i_2} \cos \beta_2 - r_1 \cos \beta_1 \).

Parameters for meshing analysis

As a key parameter to determine the normal vector \( \hat{n}_p \) and calculate the induced principal curvature in the classical reference-point-based meshing theory, always determined with the help of the helical angle \( \beta_1 \) and the tooth profile angle \( \alpha_1 \) in the shaft section of the gear 1, namely \( \tan \alpha_n = \sin \beta_1 \tan \alpha_1 \). First of all, this is an approximate empirical formula, the parameters calculated by it may be inaccurate. Secondly, the value of \( \nu_n \) can be acquired readily if the tooth profile is linear in the worm shaft section since it is a constant value along the whole tooth profile. Nevertheless, for the worm with non-linear tooth profile in its axial cross-section, the value of \( \nu_n \) usually is inconstant at different points on the tooth profile, and thus is complicated to be determined. In this case, the above formula will be very inconvenient to calculate the value of \( \alpha_n \). To overcome the above defects, a new method for calculating the value of \( \alpha_n \) is proposed and its specific idea is described as follows.

From Figure 2, the normal unit vector \( \hat{n}_p \) also can be represented in \( \sigma \) as below by the coordinate transformations

\[
\begin{align*}
(\hat{n}_p)_o &= R[\hat{k}_o, -\eta_1](\hat{n}_p)_a \\
&= R[\hat{k}_o, -\eta_1](\hat{J}_2 \cdot \delta_1)R[\hat{k}_o, -(1)^b \beta_1](\hat{n}_p)_o \\
&= (N_{s_1} \cos \alpha_n + N_{s_2} \sin \alpha_n)\hat{f}_o \\
&+ (N_{p_1} \cos \alpha_n + N_{p_2} \sin \alpha_n)\hat{j}_o \\
&+ (N_{c_1} \cos \alpha_n + N_{c_2} \sin \alpha_n)\hat{k}_o
\end{align*}
\]

where \( N_{s_1} = (-1)^{s+1} \left( \sin \delta_1 \cos \eta_1 \sin \left( -1 \right)^b \beta_1 \right) + \sin \eta_1 \cos \left( -1 \right)^b \beta_1 \) \( N_{s_2} = \cos \delta_1 \cos \eta_1 \), \( N_{p_1} = (-1)^s \left( \sin \delta_1 \sin \eta_1 \sin \left( -1 \right)^b \beta_1 \right) - \cos \eta_1 \cos \left( -1 \right)^b \beta_1 \) \( N_{p_2} = -\cos \delta_1 \sin \eta_1 \), \( N_{c_1} = (-1)^{s+1} \cos \delta_1 \sin \left( -1 \right)^b \beta_1 \), \( N_{c_2} = -\sin \delta_1 \).
Since \( \mathbf{\hat{n}}_p \) is a unit vector, it can be ensured that the expression in equation (17) equals that determined by the tooth surface equations when any two components of them are several equal. Without loss of generality, it can be obtained that

\[
\begin{align*}
N_{x1} \cos \alpha_n + N_{x2} \sin \alpha_n &= n_{p_x} \\
N_{y1} \cos \alpha_n + N_{y2} \sin \alpha_n &= n_{p_y},
\end{align*}
\]

(18)

where \( n_{p_x} \) and \( n_{p_y} \) are the coordinate components of \( \mathbf{\hat{n}}_p \) that determined based on the tooth surface equations.

Since \( \alpha_n \) is an acute angle, it can be determined as

\[
\alpha_n = \arcsin \left( \frac{n_{p_y}N_{x1} - n_{p_x}N_{y1}}{\cos \beta_1 \cos \delta_1} \right)
\]

(19)

From equation (19), it can be seen that the calculation of the pressure angle \( \alpha_n \) will be no longer limited by the shape of the tooth profile of the conical worm.

Since this new method mentioned above is strictly based on the tooth surface equation, it is an accurate solution method and no longer an approximate method.

In line with the gearing mesh theory, then the meshing limit function at the reference point can be received as\(^9\)

\[
\Phi_{p_1} = \mathbf{\hat{n}}_1 \cdot \left[ \mathbf{\hat{v}}_1 \times \mathbf{\hat{\omega}}_1 + \mathbf{\hat{\omega}}_1 \times \mathbf{\hat{v}}_2 - \left( \frac{d \mathbf{\hat{\omega}}_2}{d \varphi_1} \right) \right] \cdot \mathbf{\hat{v}}_2
\]

\[
= \frac{1}{i_{12} \sqrt{a_0^2 + b_0^2}} \sin (\alpha_0 - \alpha_n)
\]

(20)

where \( a_0 = (-1)^s(r_2 \sin \beta_2 \sin \delta_1 - r_1 \sin \beta_1 \sin \delta_2) \), \( b_0 = \cos \beta_2 (r_1 \cos \delta_2 + r_2 \cos \delta_1) \), the angle \( \alpha_0 \) can be defined as the limited pressure angle and

\[
\sin \alpha_0 = \frac{a_0}{\sqrt{a_0^2 + b_0^2}} \quad \cos \alpha_0 = \frac{b_0}{\sqrt{a_0^2 + b_0^2}}.
\]

Actually, equation (20) can be used as a basis for judging whether the meshing limit line exists in the meshing area. If \( \Phi_{p_1} = 0 \), the reference point will be on the meshing limit line. In this case, the meshing area will be divided into two zones: effective meshing area and ineffective meshing area. As a result, the full length of the worm cannot be fully utilized. In order to avoid the occurrence of the above phenomenon, \( \alpha_0 \neq \alpha_n \) should be guaranteed. It can be seen that the pressure angle, \( \alpha_n \), is also a key parameter that determines whether or not there is a meshing limit line on the tooth surface of the worm pair. From this perspective, the method of accurately calculating the pressure angle proposed above is also of great significance.

By definition, the limit function of the curvature interference can be written as\(^9\)

\[
\Psi = \mathbf{\hat{v}}_{12}^2(k_{1v} - k_{ov})
\]

(21)

where \( k_{1v} \) is the normal curvature along the direction of \( \mathbf{\hat{v}}_{12} \) at the reference point, \( k_{ov} \) is the critical normal curvature at the reference point of the conical worm tooth flank and it can be acquired as

\[
N_{a} = \left( \cos \delta_1 \sin \beta_1 + \frac{1}{i_{12}} \cos \delta_2 \sin \beta_2 \right) \sin \alpha_n + \left( -1 \right)^s \sin \delta_1 \cos \alpha_n + \mathbf{\hat{v}}_{12} \left| k_{1v} \right|,
\]

\[
N_{b} = \cos \delta_1 \cos \beta_1 + \frac{1}{i_{12}} \cos \delta_2 \cos \beta_2 + \left| \mathbf{\hat{v}}_{12} \right| \tau_{1v}.
\]

(23)

Based on the above theoretical derivations, the induced principal curvature between the tooth flanks of conical worm and conical worm wheel at the reference point can be obtained as

\[
K_{12}^{\alpha_n} = \frac{N_{a}^2 + N_{b}^2}{\Psi}
\]

(24)
Then, the sliding angle of the two gears at the reference point can be obtained as

\[
\theta_{st} = \arcsin \left( \frac{\cos \beta_2 |\Psi - \Phi_{st}|}{r_1 \sin \beta_2 \sqrt{\Psi_{12}}} \right) \tag{25}
\]

**Pre-determined parameters for meshing analysis of toroidal surface enveloping conical worm drive**

The current section is intended to provide the parameters which are required in the method mentioned above when performing the meshing analysis of conical worm drive, including: the coordinate components of the reference point and the normal vector of the tooth surface of the conical worm, and the curvature parameters along the direction of the relative velocity of the conical worm pair at the reference point. In this process, the toroidal surface enveloping conical worm is taken as the gear 1 and the conical worm gear is regarded as the gear 2. Consistent with the above theoretical derivation, the following calculations are comprehensive and can be applied to both the left-handed conical worm and the right-handed conical worm.

**Determination of normal unit vector \( \vec{n}_P \) and reference point \( P \)**

As described in Ref., the toroidal surface enveloping conical worm helicoid is ground by the discoid abrasion wheel with the working torus, and the vector equation of the working torus, \( \Sigma_d \), can be obtained in \( \sigma_d \) as

\[
(\vec{r}_d)_{st} = x_d \vec{i}_d + y_d \vec{j}_d + z_d \vec{k}_d \tag{26}
\]

where \( x_d = (\rho \sin \phi + R_k - \rho \sin \alpha) \cos \theta, \quad y_d = (\rho \sin \phi + R_k - \rho \sin \alpha) \sin \theta, \quad z_d = \rho (\cos \phi - \cos \alpha) \), therein, the angles \( \theta \) and \( \phi \) are the surface parameters of \( \Sigma_d, R_k \) is the nominal radius of the grinding wheel, \( \rho \) is the radius of working arc of the grinding wheel, and \( \alpha \) is the nominal pressure angle of the grinding wheel.

In the course of generating the toroidal surface enveloping conical worm, four coordinate frames \( \sigma_{od} \{O_{od}; \vec{i}_{od}, \vec{j}_{od}, \vec{k}_{od} \}, \quad \sigma_d \{O_d; \vec{i}_d, \vec{j}_d, \vec{k}_d \}, \quad \sigma_{ol} \{O_l; \vec{i}_l, \vec{j}_l, \vec{k}_l \} \) and \( \sigma_1 \{O_1; \vec{i}_1, \vec{j}_1, \vec{k}_1 \} \) are established, as shown in Figure 6. Among them, \( \sigma_{od} \) and \( \sigma_d \) are severally denote the original location and the working location of the grinding wheel. Meanwhile, \( \sigma_{ol} \) and \( \sigma_1 \) are used to denote the initial position and the current position of the conical worm blank, respectively. The basal vectors \( \vec{k}_{od} \) and \( \vec{k}_1 \) are all collinear with the axial line of the conical worm, and they all point to the heel of worm. Besides, \( \delta_1 \) is the semi-conical angle of the conical worm and \( \gamma \) is the worm lead angle.

As displayed in Figure 6, the frames \( \sigma_{ol} \{O_l; \vec{i}_l, \vec{j}_l, \vec{k}_l \} \) and \( \sigma_1 \{O_1; \vec{i}_1, \vec{j}_1, \vec{k}_1 \} \) are used to indicate the incipient location and the working position of the worm, respectively. Meanwhile, \( \sigma_{od} \{O_d; \vec{i}_d, \vec{j}_d, \vec{k}_d \} \) and \( \sigma_2 \{O_2; \vec{i}_2, \vec{j}_2, \vec{k}_2 \} \) are applied to point the initial position and the current position of the worm wheel, respectively. To maintain the consistency of the calculation models, the setting of the above coordinate systems are all consistent with that of the frames based on the two reference cones mentioned above.

Based on the transformation of frames, the normal unit vector of the conical worm helicoid in \( \sigma_{od} \) can be expressed as

**Figure 6.** Frames in the course of cutting engagement of toroidal surface enveloping conical worm: (a) left-handed toroidal surface enveloping conical worm and (b) right-handed toroidal surface enveloping conical worm.
tooth flank toward to the heel of conical worm, which can be named mula in this paper corresponding to the tooth surface cutting engagement of toroidal surface enveloping conical worm. When Figure 7, the three components of three-dimensional Euclidian space, as displayed in Coordinate systems in the engagement course of toroidal surface enveloping conical worm drive: (a) left-handed worm /C0/C1 and (b) right-handed worm and left-handed worm gear.

where \((\mathbf{n}_g)_d\) is the normal unit vector of \(\Sigma_g\) in \(\sigma_d\), and
\[
(\mathbf{n}_g)_d = n^*_ax + n^*_oy + n^*_ozK^*_o1
\]

(27)

where \(S\) is used to distinguish the tooth flanks of conical worm. When \(S = 0\), the related formulas in this paper corresponding to the tooth surface toward to the toe of conical worm, which can be named as \(i\) flank. When \(S = 1\), all the formulas indicates the tooth flank toward to the heel of conical worm, which can be called as \(e\) flank.

Since the frames \(\sigma_{o1}^*\) and \(\sigma^*\) are translational in the three-dimensional Euclidian space, as displayed in Figure 7, the three components of \(\mathbf{n}_p\) in \(\sigma^*\) can be expressed as
\[
n_{px} = n^*_ax, \quad n_{py} = n^*_oy, \quad n_{pz} = n^*_oz
\]

(28)

By the gear geometry, the meshing function of the cutting engagement of toroidal surface enveloping conical worm can be obtained as follows 9,22

\[
\Phi_d = (\mathbf{n}_o) \cdot [\mathbf{j}_o]_1 \times R[\mathbf{f}_o, S\pi - \gamma] (\mathbf{r}_d)_d
\]

\[
- (\mathbf{r}_d)_o \times (O_1O'_o) + d(O_1O'_o) / d\varphi
\]

(29)

where
\[
A_d = (-1)^{s + b} p \rho \sin \delta_1 (\cos \gamma \cos \theta \sin \phi - \sin \gamma \cos \phi),
\]
\[
B_d = (B_{d1} \sin \theta + B_{d2} \cos \theta) \sin \phi + (B_{d3} \sin \theta + B_{d4}) \cos \phi,
\]
\[
B_{d1} = (-1)^{s + b} p \rho \cos \alpha \sin \gamma + (-1)^{s} p \rho \sin \delta_1,
\]
\[
B_{d2} = (-1)^{s} p \rho \cos \gamma + (-1)^{s + b} p \rho \sin \delta_1 \sin \gamma,
\]
\[
B_{d3} = (-1)^{s + b} (R_g - \rho \cos \alpha) \sin \gamma,
\]
\[
B_{d4} = (-1)^{s + b} p \rho \sin \delta_1 \cos \gamma.
\]

By Olivier’s second principle, the roll cutting procedure of the worm wheel and the engagement course of the toroidal surface enveloping conical worm pair can be investigated without differentiating. During the meshing process of the worm gearing, the conical worm revolution angle is marked as \(\varphi_1\), meanwhile, the corresponding angle of rotation of the worm wheel is symbolised as \(\varphi_2\) and \(\varphi_2 = \varphi_1 / i_{12}\), where \(i_{12}\) is the transmission ratio of the conical worm gearing in question.

In \(\sigma_{o1}^*\), the equation of worm helical surface \(\Sigma_1^{(s)}\) can be expressed as

\[
(\mathbf{r}_1)_o = R[\mathbf{k}_{o1}^*, \varphi_1 - \varphi] [O_1O_s] + R[\mathbf{f}_o, S\pi - \gamma] (\mathbf{r}_d)_d
\]

\[
= x_{o1} f_{o1} + y_{o1} f_{o2} + z_{o1} f_{o3}
\]

(30)
where \( x_{o1}^* = x_1 \cos \phi_1 - y_1 \sin \phi_1 \), \( y_{o1}^* = x_1 \sin \phi_1 + y_1 \cos \phi_1 \), \( z_{o1}^* = (\text{L}_w - L_{o1}) \cos \delta_1 + z_{od} \). Therein,

\[
\begin{align*}
x_1 &= x_{od} \cos \phi + \left( -1 \right)^{j} P_{d \phi} \cos \delta_1 + a_d + y_d \sin \phi, \\
y_1 &= -x_{od} \sin \phi + \left( -1 \right)^{j} P_{d \phi} \sin \delta_1 + a_d + y_d \cos \phi, \\
x_{od} &= \left( -1 \right)^{j} \left\{ \rho \left[ \sin \phi - \sin \alpha \right] + R_g \right\} \cos \theta \cos \gamma - \rho \left[ \cos \phi - \cos \alpha \right] \sin \gamma, \\
z_{od} &= \left( -1 \right)^{j} \left\{ \rho \left[ \sin \phi - \sin \alpha \right] + R_g \right\} \cos \theta \sin \gamma + \rho \left[ \cos \phi - \cos \alpha \right] \cos \gamma.
\end{align*}
\]

On the basis of equation (8), the relative velocity of the conical worm and the coupled worm wheel can be acquired in \( \sigma_{o1}^* \) as

\[
\left( \vec{V}_{12} \right)_{o1}^* = V_{12} \vec{k}_{o1}^* + V_{12} \vec{a}_{o1}^* + V_{12} \vec{a}_{o1}^* (31)
\]

where

\[
V_{12} = \left[ \left( \sigma_{o1}^* + k_d a + \frac{L_w}{2} \right) \sin \Sigma + \sigma_{o1}^* \sin \Sigma \right]/i_{12},
\]

\[
V_{12} = \left( i_{12} \sigma_{o1}^* + \sigma_{o1}^* + \left( -1 \right)^{j+1} \right) \sin \Sigma;/i_{12},
\]

\[
V_{12} = \left[ \left( \sigma_{o1}^* + k_d a + \frac{L_w}{2} - A_{1 \text{dod}} \right) \sin \Sigma \right]/i_{12},
\]

\[
\begin{align*}
A &= \left[ \left( \sigma_{o1}^* + k_d a + \frac{L_w}{2} \right) \sin \phi \sin \theta - A_{1 \text{dod}} \right] \sin \Sigma + \left( -1 \right)^{j+1} \left[ \alpha_{o1} \cos \Sigma \right], \\
B &= - \left[ A_{2} \left( \sigma_{o1}^* + k_d a + \frac{L_w}{2} \right) - A_{1 \text{dod}} \right] \sin \Sigma + \left( -1 \right)^{j} a \sin \phi \sin \theta \cos \Sigma, \\
C &= \left( -1 \right)^{j+1} \left[ \alpha_{o1} \sin \Sigma \right] \\
- (i_{12} + \cos \Sigma)(A_{2} \left( i_{12} \right) - A_{3} \sin \phi \sin \theta), \\
A_{1} &= \left( -1 \right)^{j} \left( \sin \gamma \sin \phi \cos \theta + \cos \gamma \cos \phi \right), \\
A_{2} &= \left( -1 \right)^{j} \left( \cos \gamma \sin \phi \cos \theta - \sin \gamma \cos \phi \right), \\
A_{3} &= \left( -1 \right)^{j} \left\{ \left[ \rho \left[ \sin \phi - \sin \alpha \right] + R_g \right] \cos \theta \cos \gamma - \rho \left[ \cos \phi - \cos \alpha \right] \sin \gamma \right\}.
\end{align*}
\]

As demonstrated in Figure 8, a plane coordinates \( O_1 - x_{R1}, y_{R1} \) is established in the axial cross-section of the conical worm, therein, \( x_{R1} = z_{o1}^* \) and \( y_{R1} = \sqrt{x_{o1}^* 2 + y_{o1}^* 2} \).

By the geometric site displayed in Figure 8, the reference point \( P_1 \) on the worm screw may be determined by the non-linear equations of tooth surfaces as shown below

\[
\begin{align*}
z_{o1}^* &= L_P, \\
\sqrt{x_{o1}^* 2 + y_{o1}^* 2} &= r_P, \\
\Phi(x, \theta, \phi) &= 0, \\
\Phi(x, \theta, \phi, \varphi) &= 0.
\end{align*}
\]

After solving equation (33) iteratively, the tooth surface parameters of the point \( P_1 \) will be acquired, and then its coordinates in \( \sigma_{o1}^* \) can be worked out. By the coordinate transformation from \( \sigma \) to \( \sigma_{o1}^* \), the coordinates of the reference point \( P_1 \) can be represented as

\[
X_p = x_{o1}^* \sin \varphi, \quad Y_p = y_{o1}^*, \quad Z_p = k_d a + z_{o1}^* + \frac{L_w}{2}
\]

**Figure 8.** The reference point in the worm shaft section.

**Determination of curvature parameters along the direction of \( \vec{V}_{12} \)**

A Cartesian space \( \sigma_{M} \{ M; (\hat{g}_1)_d, (\hat{g}_2)_d, (\hat{n}_d) \} \) is built at an arbitrary point on \( \Sigma_d \). Therein, the basal vectors \((\hat{g}_1)_d \) and \((\hat{g}_2)_d \) are the two principal direction vectors of \( \Sigma_d \), and they can be determined as

\[
(\hat{g}_1)_d = \frac{\partial(\hat{R})_d}{\partial \phi} \quad (\hat{g}_2)_d = \frac{\partial(\hat{R})_d}{\partial \theta}
\]

\[
\begin{align*}
(\hat{g}_1)_d &= \cos \phi \cos \theta \hat{I}_d + \cos \phi \sin \theta \hat{J}_d - \sin \phi \hat{K}_d, \\
(\hat{g}_2)_d &= (\hat{n})_d \times (\hat{g}_1)_d = - \sin \theta \hat{I}_d + \cos \theta \hat{J}_d
\end{align*}
\]

In the course of grinding the toroidal surface enveloping conical worm, the vectors \((\hat{g}_1)_d \) and \((\hat{g}_2)_d \) may be expressed in \( \sigma_{o1}^* \) as
where \( g_{w1}^{(s)} = (-1)^s (\cos \phi \cos \theta \cos \gamma + \sin \phi \sin \gamma) \),
\[ g_{w1}^{(a)} = (-1)^a (\cos \phi \cos \theta \sin \gamma - \sin \phi \cos \gamma). \]

According to the gear engagement theory, the limit function of curvature interference in the course of the grinding engagement can be obtained as
\[ \Psi_d = \lambda_d \left[ V_d^{(g)} g_{w1}^{(g)} + V_d^{(p)} \cos \phi \sin \theta + g_{w1}^{(a)} p_{a1} \sin \delta_1 \right] 
+ \mu_d \left[ (-1)^s \left( V_d^{(g)} \cos \gamma + p_{a1} \sin \delta_1 \gamma \right) \sin \theta 
+ V_d^{(p)} \cos \theta \right] + A_d. \]

where
\[ \lambda_d = - \frac{1}{\rho} \left[ V_d^{(g)} g_{w1}^{(g)} + V_d^{(p)} \cos \phi \sin \theta + p_{a1}^{(a)} \sin \delta_1 \right] \]
\[ + (-1)^s \sin \gamma \sin \theta, \]
\[ \mu_d = \frac{\sin \phi}{\rho \sin \phi + f} \left\{ (-1)^s \left( V_d^{(g)} \cos \gamma + p_{a1} \sin \delta_1 \gamma \right) \sin \theta 
- V_d^{(p)} \cos \theta \right\} + g_{w1}^{(a)} \]

On the grounds of the above results, the normal curvatures along \((\hat{g}_1)_{01}\) and \((\hat{g}_2)_{01}\), and the geodesic torsion along \((\hat{g}_1)_{01}\) at the point on the conical worm helical surface can be acquired as
\[ k_{w1}^{(1)} = - \frac{1}{\rho} \frac{\lambda_d}{\Psi_d}, \]
\[ k_{w1}^{(2)} = - \frac{\sin \phi}{\rho \sin \phi + f} - \frac{\mu_d}{\Psi_d}, \]
\[ \gamma_{w1}^{(1)} = - \frac{\lambda_d \mu_d}{\Psi_d}. \]

During the process of the engagement of the worm pair, a frame \( \sigma_P \{ P, (\hat{a}_x)_{01}, (\hat{a}_n)_{01}, (\hat{n})_{01} \} \) is built at the reference point. Therein, the two basal vectors \((\hat{a}_x)_{01}\) and \((\hat{a}_n)_{01}\) are in the dividing plane \( T \) and they can be acquired as
\[ (\hat{a}_x)_{01} = R \left[ K_{01}, \varphi_1 - \varphi \right] (\hat{g}_1)_{01} \]
\[ = \alpha_x^{(a)} \hat{l}_{01} + \alpha_x^{(l)} \hat{n}_{01} + g_{w1}^{(a)} \hat{k}_{01} \]
\[ (\hat{a}_n)_{01} = R \left[ K_{01}, \varphi_1 - \varphi \right] (\hat{g}_2)_{01} \]
\[ = \alpha_n^{(a)} \hat{l}_{01} + \alpha_n^{(l)} \hat{n}_{01} + (-1)^s + 1 \sin \gamma \sin \theta \hat{k}_{01}. \]

where \( \alpha_x^{(a)} = g_{w1}^{(a)} \cos (\varphi_1 - \varphi) - \sin \theta \cos \phi \sin (\varphi_1 - \varphi), \)
\[ \alpha_x^{(p)} = \frac{g_{w1}^{(a)} \sin (\varphi_1 - \varphi) + \sin \phi \cos \phi \cos (\varphi_1 \varphi),}{\cos} \]
\[ \alpha_n^{(a)} = (-1)^s + 1 \cos \gamma \sin \theta \sin (\varphi_1 - \varphi) - \cos \theta \cos (\varphi_1 - \varphi), \]
\[ \alpha_n^{(p)} = (-1)^s + 1 \cos \gamma \sin \theta \sin (\varphi_1 - \varphi) + \cos \theta \cos (\varphi_1 - \varphi). \]

As shown in Figure 9, the included angle \( \varphi_e \), between \((\hat{a}_x)_{01}\) and \((\hat{V}_{12})_{01}\), can be determined by the following expressions
\[ \cos \varphi_e = \frac{(\hat{a}_x)_{a1} \cdot (\hat{V}_{12})_{a1}}{|(\hat{V}_{12})_{a1}|}, \quad \sin \varphi_e = \frac{(\hat{a}_n)_{a1} \cdot (\hat{V}_{12})_{a1}}{|(\hat{V}_{12})_{a1}|} \]

According to the generalized Rodrigues formula, the normal curvature and the geodesic torsion of the conical worm helicoid along \((\hat{V}_{12})_{01}\) can severally be represented as
\[ k_{11} = k_{g1}^{(1)} \cos^2 \varphi_e + k_{g1}^{(2)} \sin^2 \varphi_e + 2 \tau_{g1}^{(1)} \sin \varphi_e \cos \varphi_e \]
\[ \tau_{11} = \left[ k_{g2}^{(1)} - k_{g1}^{(1)} \right] \sin \varphi_e \cos \varphi_e + \tau_{g1}^{(1)} (2 \cos^2 \varphi_e - 1) \]

With the aid of equations (44) and (45), the value of \( \tilde{N} \) in equation (23) will be acquired. Based on it, the value of the induced principle curvature \( K_{11}^{(2)} \) can be worked out at the reference point according to equation (24), and then the lubrication angle \( \theta_{\varphi} \) at the reference point also can be acquired with the help of equation (25).

**Numerical example investigations**

Four numerical examples are provided to verify the correctness of the method suggested in this work and to explore some characteristics of the toroidal enveloping conical worm drive. Concretely, the examples ⊙ and ⊙ both indicate the worm pair combined by the left-handed conical worm and the right-handed worm gear, ⊙ represents the f flank of the worm pair, and ⊙ represents the e flank of the worm pair. Similarly, the
Table 1. Basic parameters of conical worm pairs.

| Parameters | Numerical examples |
|------------|-------------------|
|            | ①     | ②     | ③     | ④     |
| Σ (°)      | 90     |        |        |        |
| a (mm)     | 100    |        |        |        |
| l12        | 51     |        |        |        |
| Z1         | 4      |        |        |        |
| δ1 (°)     | 5      |        |        |        |
| m1 = 2a(l12Z1) (mm) | 4 |        |        |        |
| L = 0.7a + m1 (mm) | 74 |        |        |        |
| r0 (mm)    |        | 0.5   |        |        |
| p = m1Z1/2 (mm) | 4 |        |        |        |
| ra1 (mm)   | 26     |        |        |        |
| h = h/2 m0 (mm), (h = 1) | 4 |        |        |        |
| h1 = h1/2 m0 (mm), (h1 = 1.25) | 5 |        |        |        |
| h2 = 2m0 (mm) | 8    |        |        |        |
| Rf (mm)    | 111    |        |        |        |
| p (mm)     | 40     |        |        |        |
| α (°)      | 18     |        |        |        |
| b2 = k0b (mm), k0 = 0.49 | 49 |        |        |        |
| δ2 (°)     | 83     |        |        |        |
| h          | 0      | 0     | 0     | 1     |
| s         | 0      | 0     | 1     | 1     |

Table 2. Numerical results of different methods.

| Example | Parameters at the reference point |
|---------|-----------------------------------|
|         | Three methods of meshing analysis |
|         | Method a | Method b | Method c |
| ①      | α (°)    |          |          |
|        | Φ1       | Φ2       | Ψ        |
|        |          |          |          |
| ②      |          |          |          |
| ③      |          |          |          |
| ④      |          |          |          |
| ⑤      |          |          |          |

Computed results and discussion

The numerical results provided in this section mainly include two parts: the first part is used to verify the proposed method (Tables 2 and 3), and the second part is used to study the characteristics of the toroidal enveloping conical worm drive (Table 4).

Table 2 provides the numerical results of the major parameters at the selected reference point on the tooth surface of the worm in these examples. For Examples ① and ②, the reference point at the tooth top of i flank on the toe of the conical worm. For Examples ③ and ④, the reference point at the tooth top of e flank on the toe of the conical worm. These data were obtained respectively using three different methods, namely: Method a indicates the traditional method based on the reference-point based meshing theory,8-12 Method b represents the method based on the tooth surface equations,6,7 and Method c represents the generalized method suggested in this paper.

Since Method b is strictly based on the tooth surface equations, mathematically, the calculation results obtained based on it can undoubtedly be used as a standard to measure the accuracy of the calculation. As listed in Table 2, the calculated results of the related parameters acquired by Method a are all significantly different from that by Method b, especially in the e flank of the conical worm. This means that Method a maybe inaccurate and inapplicable to the toroidal enveloping conical worm drive. At the same time, it can be seen that the numerical results calculated by Method c are completely consistent with those obtained by Method b, which can prove that Method c is effective.

In Table 3, P1 represents the reference point at the tooth top of i flank on the toe of the conical worm, and P1 represents the reference point at the tooth top of e flank on the toe of the conical worm. The numerical results provided in Table 3 reflect that the value of parameters at the reference point have nothing to do with the rotation directions of the conical worm pair, in other words, the meshing characteristics of the worm pair have nothing to do with its rotation directions, which is consistent with the recognized guideline in the gear community. The above facts also reflect that the generalized method proposed in this paper is correct and applicable.

Table 4 provided the numerical results of the parameters at the reference points P1, P2, and P3 on the tooth surface of the conical worm in Example ①, the point P1 on the tooth crest at the toe of the conical worm, the point P2 on the tooth crest at the middle of
of conical worm gear. The values of
parameters at reference points are all different from the values of $t_1$
and $t_2$ in Table 1, it means that the reference cones are
actually different from the index cones of the conical
worm, these strategies are theoretically
applicable to other types of conical worm drives. Based
on the generalized method, the values of the related
parameters at reference points.

| Parameters                  | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|-----------------------------|-------|-------|-------|-------|
| $r_1$ (mm)                  | 26    | 26    | 26    | 26    |
| $r_2$ (mm)                  | 109.2735 | 109.2735 | 100.6110 | 100.6110 |
| $\delta_1$ (°)             | 3.2135 | 3.2135 | 15.6196 | 15.6196 |
| $\delta_2$ (°)             | 82.9629 | 82.9629 | 57.1945 | 57.1945 |
| $\beta_1$ (°)              | -85.6397 | -85.6397 | 85.9563 | 85.9563 |
| $\beta_2$ (°)              | 22.6941 | 22.6941 | 21.6603 | 21.6603 |
| $\alpha_0$ (°)             | -24.5847 | -24.5847 | -13.7742 | -13.7742 |
| $\alpha_a$ (°)             | 14.8869 | 14.8869 | -8.4125 | -8.4125 |
| $\Phi_{1v}$                 | 0.7001 | 0.7001 | 0.0908 | 0.0908 |
| $k_{1v}$ (mm$^{-1}$)       | 0.0077 | 0.0077 | 0.0219 | 0.0219 |
| $\tau_{1v}$                | -0.0028 | -0.0028 | 0.0023 | 0.0023 |
| $\Psi$                     | 0.0248 | 0.0248 | 0.0061 | 0.0061 |
| $K_{12}^{(N)}$ (mm$^{-1}$) | 0.0310 | 0.0310 | 0.0737 | 0.0737 |
| $\theta_{1v}$ (°)          | 76.0048 | 76.0048 | 80.0024 | 80.0024 |

According to the data displayed in Table 4, the value
of $r_2$ decreases gradually from $P_1$ to $P_3$, which is consistent with the characteristics of the conical worm gear. The values of $\delta_1$ and $\delta_2$ at these reference points are all different from the values of $\delta_1$ and $\delta_2$ in Table 1, it means that the reference cones are actually different from the index cones of the conical worm pair. The value of the helix angles at these points are all different. This shows that the toroidal enveloping conical worm drive is a type of worm drive with variable helix angle, which is obviously different from the cylindrical worm drive. According to the value of pressure angle $\alpha_a$ and the value of $\Psi$, it can be known that the meshing limit line and the curvature interference limit line both not exist on the tooth surfaces of the worm pair in question. The numerical results of $K_{12}^{(N)}$ and $\theta_{1v}$ at these reference points can reflect the
local meshing characteristics of the worm pair, such as the contact stress level and the lubrication performance between the tooth surface couple of the worm pair, are favorable. From Table 4, it can be seen that the area near the toe maybe the weakest zone on the whole helical of conical worm from the perspective of meshing quality. Therefore, we can take the point here as the key reference point of designing the conical worm pair to pre-control and achieve the favorable meshing characteristic.

Conclusions

The meshing analysis method of the conical worm drive has been generalized, which is embodied in the following aspects:

1. The pressure angle at the reference point is computed based on the normal unit vector acquired by the tooth surface equations. Compared with the calculation method in the classical theory, the generalized method is no longer limited by the tooth profile shape of the worm and the numerical results calculated by it can be more accurate.

2. A more universal mathematical model for meshing analysis has been established by introducing the helical spin coefficient and tooth side coefficient of the conical worm. It not only can be directly applied to the conical worm gearing with different rotation directions, but also can be conveniently used to the different tooth flanks of worm pair.

3. Based on the above results, not only can the geometric parameters of reference cones and worm wheel be accurately calculated, but also the meshing analysis can be implemented precisely at any point on the tooth surface of conical worm pair. Besides, the new method also can easily determine whether the meshing limit line and the curvature interference limit line exist on the tooth surface of the conical worm pair or not. Compared with the method based on tooth surface equations, the generalized method can avoid solving the complex nonlinear equations of tooth surfaces.

The toroidal surface enveloping conical worm pair is taken as the instance to illustrate the generalized method. The approaches to compute the predetermined parameters for the meshing analysis based on the generalized method are illustrated. Since the method suggested in this paper is a purely analytical method that has nothing to do with the tooth shape of the conical worm, these strategies are theoretically applicable to other types of conical worm drives. Based on the generalized method, the values of the related
parameters are calculated. At the same time, some comparison calculations are also provided. These results manifest that the generalized method presented in the current work is correct and practicable. Besides, some characteristics of the toroidal surface enveloping conical worm drive are also discovered.

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**Appendix**

**Notations**

| Symbol | Description |
|--------|-------------|
| Σ      | shaft angle of the conical worm gearing |
| a      | center distance of the conical worm gearing |
| i₁₂    | transmission ratio of the conical worm gearing |
| Z₁     | number of the conical worm thread |
| δ₁     | semi-cone angle of the conical worm |
| δ₁     | semi-cone angle of the reference cone corresponding to the conical worm module along the pitch cone generatrix of conical worm |
| m₀     | thread length of the conical worm |
| Lₘ     | mounting distance coefficient of the conical worm |
| kₘ     | tooth pitch along pitch cone generatrix of conical worm |
| rₒ     | outside radius at toe of the conical worm |
| Rₒ     | profile radius in shaft section of grinding wheel |
| α      | nominal pressure angle of the grinding wheel |
| b₂     | tooth breadth of the conical worm gear |
| δ₂     | semi-cone angle of the conical worm gear |

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\( \delta_2 \) semi-cone angle of the reference cone corresponding to the conical worm gear

\( r_2 \) dividing circle radius of worm wheel

\( \dot{V}_{12} \) relative velocity of worm pair at the reference point

\( \beta_1 \) spiral angle of conical worm at the reference point

\( \beta_2 \) spiral angle of conical worm wheel at reference point

\( S \) coefficient to distinguish the tooth flanks of conical worm

\( h \) coefficient to distinguish the turning directions of gears

\( K_{\text{N}}^{(12)} \) induced principal curvature of worm pair

\( \theta_{\text{u}} \) sliding angle of worm pair

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