EXPONENTIALLY $S$-NUMBERS

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Abstract. Let $S$ be the set of all finite or infinite increasing sequences of positive integers. For a sequence $S = \{s(n)\}, n \geq 1$, from $S$, let us call a positive number $N$ an exponentially $S$-number ($N \in E(S)$), if all exponents in its prime power factorization are in $S$. Let us accept that $1 \in E(S)$. We prove that, for every sequence $S \in S$ with $s(1) = 1$, the exponentially $S$-numbers have a density $h = h(E(S))$ such that

$$\sum_{i \leq x, i \in E(S)} 1 = h(E(S))x + O(\sqrt{x} \log xe^{\frac{\sqrt{\log x}}{\log \log x}}),$$

where $c = 4\sqrt{2} \approx 7.4430...$ and $h(E(S)) = \prod_p (1 + \sum_{i \geq 2} \mu(\frac{n}{p^i}) - \mu(\frac{n}{p^{i-1}}))$, where $\mu(n)$ is the characteristic function of $S$.

1. Introduction

Let $S$ be the set of all finite or infinite increasing sequences of positive integers. For a sequence $S = \{s(n)\}, n \geq 1$, from $S$, let us call a positive number $N$ an exponentially $S$-number ($N \in E(S)$), if all exponents in its prime power factorization are in $S$. Let us accept that $1 \in E(S)$. For example, if $S = \{1\}$, then the exponentially 1-numbers form the sequence $B$ of square-free numbers, and, as well-known,

$$\sum_{i \leq x, i \in B} 1 = \frac{6}{\pi^2}x + O(x^{\frac{1}{2}}).$$

In case, when $S = B$, we obtain the exponentially square-free numbers (for the first time this notion was introduced by M. V. Subbarao in 1972 [6], see A209061[5]). Namely the exponentially square-free numbers were studied by many authors (for example, see [2], [6] (Theorem 6.7), [7], [8], [9]). In these papers, the authors analyzed the following asymptotic formula

$$\sum_{i \leq x, i \in E(B)} 1 = \prod_p (1 + \sum_{a=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a})x + R(x),$$

where the product is over all primes, $\mu$ is the Möbius function. The best result of type $R(x) = o(x^{\frac{1}{2}})$ was obtained by Wu (1995) without using RH (more exactly see [9]). In 2007, assuming that RH is true, Tóth [8] obtained $R(x) = O(x^{\frac{1}{3}+\epsilon})$ and in 2010, Cao and Zhai [2] more exactly found that $R(x) = Cx^{\frac{1}{5}} + O(x^{\frac{38}{191}+\epsilon})$, where $C$ is a computable constant. Besides, Tóth [7] studied also the exponentially $k$-free numbers, $k \geq 2$. 

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In this paper, without using RH, we obtain a general formula with a remainder term \(O(\sqrt{x} \log x e^{\sqrt{\log \log x}})\) (c is a constant) not depended on \(S \in \mathbb{S}\) beginning with 1. More exactly, we prove the following.

**Theorem 1.** For every sequence \(S \in \mathbb{S}\) the exponentially \(S\)-numbers have a density \(h = h(E(S))\) such that, 1) if \(s(1) > 1\), then \(h = 0\), while 2) if \(s(1) = 1\), then

\[
\sum_{i \leq x, \ i \in E(S)} 1 = h(E(S))x + O(\sqrt{x} \log x e^{\sqrt{\log \log x}}),
\]

with \(c = 4\sqrt{2}/\log 2 = 7.443083\ldots\) and

\[
h(E(S)) = \prod_p (1 + \sum_{i \geq 2} \frac{u(i) - u(i - 1)}{p^i}),
\]

where \(u(n)\) is the characteristic function of sequence \(S\): \(u(n) = 1\), if \(n \in S\) and \(u(n) = 0\) otherwise.

In particular, in case \(S = B\) we obtain (2) with a less good remainder term, but which is suitable for all sequences in \(\mathbb{S}\) beginning with 1.

2. Lemma

For proof Theorem 1, we need a lemma proved earlier (2007) by the author [4], pp.200-202. For a fixed square-free number \(r\), denote by \(B_r\) the set of square-free numbers \(n\) for which \(\gcd(n, r) = 1\), and put

\[b_r(x) = |B_r \cap \{1, 2, ..., x\}|.\]

In particular, \(B = B_1\) is the set of all square-free numbers.

**Lemma 1.**

\[b_r(x) = \frac{6r}{\pi^2} \prod_{p \mid r} (p + 1)^{-1} x + R_r(x),\]

where for every \(x \geq 1\) and every \(r \in B\)

\[|R_r(x)| \leq \begin{cases} k\sqrt{x}, & \text{if } r \leq N \\ k \cdot e^{\sqrt{\log \log r} \sqrt{x}}, & \text{if } r \geq N + 1. \end{cases}\]

where \(k = 3.5 \prod_{2 \leq p \leq 23} (1 + \frac{1}{\sqrt{p}}) = 57.682607\ldots\) (in case \(r = 1\), \(k = 3.5\)), \(c = 4\sqrt{2/\log 2} = 7.443083\ldots\), \(N = 6469693229\).

3. Proof of Theorem 1

1) Denote by \(\Upsilon\) the sequence \(\{2, 3, 4, \ldots\}\) of all natural numbers without 1. Let \(S\) do not contain 1. Then, evidently, \(E(S) \subseteq E(\Upsilon)\). Note that the sequence \(E(\Upsilon)\) is called also powerful numbers (sequence A001694 in [5]).
Bateman and Grosswald [1] proved that

\[ \sum_{i \leq x, \ i \in E(\Upsilon)} 1 = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/6}). \]

So, \( h(E(\Upsilon)) = 0 \). Then what is more \( h(E(S)) = 0 \).

Furthermore, denote by \( r(n) \) the product of all distinct prime divisors of \( n \); set \( r(1) = 1 \).

2) Now let \( 1 \in S \). Note that the set \( E(\Upsilon) \cap E(S) \) contains 1 and all numbers of \( E(S) \) whose exponents in their prime power factorizations are more than 1. Evidently, every number \( y \in E(S) \) has a unique representation as the product of some number \( a \in E(\Upsilon) \cap E(S) \) and a number \( m \in B_{r(a)}\).

In particular, if \( y \) is square-free, then \( a = 1 \), \( m = y(\in B_1) \). For a fixed \( a \in E(\Upsilon) \cap E(S) \), denote the set of \( y = am \in E(S) \) by \( E(S)^{(a)} \). Then \( E(S) = \bigcup_{a \in E(S) \cap E(\Upsilon)} E(S)^{(a)} \), where the union is disjoint. Consequently, by Lemma [1] we have

\[ \sum_{i \leq x, \ i \in E(S)} 1 = b_1(x) + \sum_{4 \leq a \leq x, \ a \in E(S) \cap E(\Upsilon)} b_{r(a)} \left( \frac{x}{a} \right) \]

\[ = \frac{6}{\pi^2} \left( 1 + \sum_{4 \leq a \leq x, \ a \in E(S) \cap E(\Upsilon)} \prod_{p \mid r(a)} \left( 1 - \frac{1}{p+1} \right) \frac{1}{a} \right) x + R(x), \]

where

\[ |R(x)| \leq 3.5 \sqrt{x} + \sum_{4 \leq a \leq x, \ a \in E(S) \cap E(\Upsilon)} \left| R_{r(a)} \left( \frac{x}{a} \right) \right| \leq 3.5 \sqrt{x} + \]

\[ + \sum_{4 \leq a \leq x: r(a) \leq N} \left| R_{r(a)} \left( \frac{x}{a} \right) \right| + \sum_{a \leq x: r(a) \geq N + 1} \left| R_{r(a)} \left( \frac{x}{a} \right) \right| \]

with \( N = 6469693229 \).

Let \( x > N \) go to infinity. Distinguish two cases: (i) \( r(a) \leq N \); (ii) \( r(a) > N \).

(i) \( r(a) \leq N \). Denote by \( E(\Upsilon)(n) \) the \( n \)-th powerful number (in increasing order). According to (5), \( E(\Upsilon)(n) = \left( \frac{\zeta(3)}{\zeta(3/2)} \right)^2 n^2 (1 + o(1)) \). So, \( \Sigma_{1 \leq n \leq x} \frac{1}{\sqrt{E(\Upsilon)(n)}} = O(\log x) \). Hence, by (7) and Lemma 1

\[ |R(x)| \leq 3.5 \sqrt{x} + k \sqrt{x} \sum_{a \leq x, \ a \in E(S) \cap E(\Upsilon)} \frac{1}{\sqrt{a}} = O(\sqrt{x} \log x). \]

(ii) \( r(a) > N \). Then, by (7) and Lemma 1
where the last sum does not exceed
\[
\sum_{\nu+1 \leq a \leq x, \, r(a) \geq \nu+1} \frac{1}{\sqrt{a}} \leq e^{c \frac{\log r(a)}{\log \log r(a)}} O(\log x).
\]

So, \( R(x) = O(\sqrt{x} \log x e^{c \frac{\log r(a)}{\log \log x}}) \) and, by (6), we have
\[
\sum_{i \leq x, \, i \in E(S)} 1 = \frac{6}{\pi^2} \left( 1 + \sum_{4 \leq a \leq x, \, a \in E(S) \cap E(T)} \prod_{p \mid r(a)} \left( 1 - \frac{1}{p+1} \right) \frac{1}{a} \right) x + O(\sqrt{x} \log x e^{c \frac{\log r(a)}{\log \log x}}).
\]

Moreover, if we replace here the sum \( \sum_{a \leq x, \, a \in E(S) \cap E(T)} \) by the sum \( \sum_{a \in E(S) \cap E(T)} \), then the error does not exceed \( \frac{6x}{\pi^2} \sum_{n > x} \frac{1}{E(T)(n)} = \frac{6x}{\pi^2} O(1/x) = O(1) \), then the result does not change. So, finally,
\[
\sum_{i \leq x, \, i \in E(S)} 1 = \frac{6}{\pi^2} \left( \sum_{a \in E(S) \cap E(T)} \prod_{p \mid r(a)} \left( 1 - \frac{1}{p+1} \right) \frac{1}{a} \right) x + O(\sqrt{x} \log x e^{c \frac{\log r(a)}{\log \log x}}).
\]

Formula (8) shows that, if \( 1 \in S \), then \( E(S) \) has a density.

4. Completion of the proof

It remains to evaluate the sum (8). For that we follow the scheme of [4], pp.203-204. For a fixed \( l \in B \), denote by \( C(l) \) the set of all \( E(S) \cap E(T) \)-numbers \( a \) with \( r(a) = l \). Recall that \( r(1) = 1 \). By (8), we have
\[
\sum_{i \leq x, \, i \in E(S)} 1 = \frac{6}{\pi^2} x \sum_{l \in B} \prod_{p \mid l} \left( 1 - \frac{1}{p+1} \right) \frac{1}{a} + R(x).
\]

Consider the function \( A : N \to R \) given by:
\[
A(l) = \begin{cases} \sum_{a \in C(l)} \frac{1}{a}, & l \in B, \\ 0, & l \notin B. \end{cases}
\]

Example 1.
\[
A(1) = \sum_{a \in C(1)} \frac{1}{a} = \sum_{r(a)=1} \frac{1}{a} = 1.
\]
Example 2. Let $p$ be prime. Since $r(p) = p$, then
\[ A(p) = \sum_{a \in C(p)} \frac{1}{a} = \sum_{i \geq 2} \frac{1}{p^{s(i)}}. \]
The sum not contains $\frac{1}{p^{s(1)}} = \frac{1}{p}$ since, by the condition, $a \in E(S) \cap E(\Upsilon)$, but the sequence $E(\Upsilon)$ not contains any prime.

Example 3. Let $p < q$ be primes. Since $r(pq) = pq$, then
\[ A(pq) = \sum_{i \geq 2, j \geq 2} \frac{1}{p^{s(i)} q^{s(j)}}. \]
It is evident that, if $l_1, l_2 \in B$ and $\gcd(l_1, l_2) = 1$, then
\[ A(l_1 l_2) = \sum_{a \in C(l_1 l_2)} \frac{1}{a} = \sum_{a \in C(l_1)} \frac{1}{a} \sum_{a \in C(l_2)} \frac{1}{a} = A(l_1) A(l_2). \]
It follows that $A(l)$ is a multiplicative function. Hence the function $f$ which is defined by
\[ f(l) = \prod_{p | l} \left( 1 - \frac{1}{p + 1} \right) A(l) \]
is also multiplicative. Evidently, by the definition of $A(n)$,
\[ \sum_{n=1}^{\infty} f(n) \leq \sum_{n=1}^{\infty} A(n) \leq \sum_{a \in E(\Upsilon)} \frac{1}{a} < \infty. \]
Consequently (3, p.103):
\[ \sum_{n=1}^{\infty} f(n) = \prod_{p} (1 + f(p) + f(p^2) + \ldots). \]
Since $f(p^k) = 0$ for $k \geq 2$, then by (2):
\[ \sum_{i \leq x, \ i \in E(S)} 1 = \frac{6}{\pi^2} x \sum_{l=1}^{\infty} f(l) + R(x) = \frac{6}{\pi^2} x \prod_{p} (1 + f(p)) + R(x) = \]
\[ = \frac{6}{\pi^2} x \prod_{p} \left( 1 + \left( 1 - \frac{1}{p + 1} \right) \left( \frac{1}{p^{s(2)}} + \frac{1}{p^{s(3)}} + \frac{1}{p^{s(4)}} + \ldots \right) \right) + R(x). \]
Now we have
\[ h(E(S)) = \frac{6}{\pi^2} \prod_{p} (1 + \left( 1 - \frac{1}{p + 1} \right) \sum_{i \geq 2} \frac{1}{p^{s(i)}}) = \]
\[ \frac{6}{\pi^2} \prod_{p} (1 + \sum_{i \geq 2} \frac{p}{(p + 1)p^{s(i)}}) = \]
\[
\prod_p \left( (1 - \frac{1}{p^2}) - (1 - \frac{1}{p}) \sum_{i \geq 2} \frac{u(i)}{p^i} \right)
\]
and, taking into account that \(u(1) = 1\), we find
\[
h(E(S)) = \prod_p \left( (1 - \frac{1}{p^2}) - (1 - \frac{1}{p}) + (1 - \frac{1}{p}) \sum_{i \geq 1} \frac{u(i)}{p^i} \right) = \prod_p \left( (1 - \frac{1}{p}) + \sum_{i \geq 1} \frac{u(i)}{p^i} - \frac{1}{p} \sum_{i \geq 1} \frac{u(i)}{p^i} \right) = \prod_p \left( (1 - \frac{1}{p}) + \sum_{i \geq 2} \frac{u(i)}{p^i} - \frac{1}{p} \sum_{i \geq 2} \frac{u(i-1)}{p^{i-1}} \right) = \prod_p \left( 1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i} \right)
\]
which gives the required evaluation of the sum in (8) and completes the proof of the theorem.

5. A question of D. Berend

Let \(p_n\) be the \(n\)-th prime. Let \(A = \{S_1, S_2, \ldots\}\) be an infinite sequence of sequences \(S_i \in S\) beginning with 1. We say that a positive number \(N\) is an exponentially \(A\)-number (\(N \in E(A)\)), if in case that \(p_n, n \geq 1\), divides \(N\), then its exponent in the prime power factorization of \(N\) belongs to \(S_n\). We accept that \(1 \in E(A)\). How will change Theorem 1 for the exponentially \(A\)-numbers?

An analysis of the proof of Theorem 1 shows that also in this more general case, for every sequence \(A\) there exists a density \(h(A)\) of the exponentially \(A\)-numbers such that
\[
\sum_{i \leq x, \ i \in E(A)} 1 = h(E(A))x + R(x),
\]
where \(R(x)\) is the same as in Theorem 1 and
\[
h(E(A)) = \prod_{n \geq 1} (1 + \sum_{i \geq 2} \frac{u_n(i) - u_n(i-1)}{p_n^i}),
\]
where \(u_n(k)\) is the characteristic function of sequence \(S_n\): \(u_n(k) = 1\), if \(k \in S_n\) and \(u_n(k) = 0\) otherwise.

Example 4. Let
\[
A = \{S_1 = \{1\}, S_2 = \{1, 2\}, ..., S_n = \{1, ..., n\}, ...\}.
\]
Then, by (13),
\[
h(E(A)) = \prod_{n \geq 1} (1 - \frac{1}{p_{n+1}}) = 0.7210233\ldots.
\]
Let 1 ∈ S. Then the density h(E(S)) is in the interval $[6/\pi^2, 1]$. Whether the set $\{h(E(S))\}$ is a dense set in this interval?

D. Berend (private communication) gave a negative answer. Indeed, consider the set $S_1$ of sequences $\{S\}$ containing 2. Then, evidently, $h(E(S)) \geq h(E(\{1, 2\}))$ such that, by Theorem 1,

\begin{equation}
(14) \quad h(E(S))|_{S \in S_1} \geq \prod_p \left(1 - \frac{1}{p^3}\right).
\end{equation}

Now consider the set $S_2$ of sequences $\{S\}$ not containing 2. Then $h(E(S)) \leq h(E(\{1, 3, 4, 5, 6, \ldots\}))$ such that, by Theorem 1

\begin{equation}
(15) \quad h(E(S))|_{S \in S_2} \leq \prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^3}\right) = \prod_p \left(1 - \frac{p-1}{p^3}\right).
\end{equation}

Thus, by (14)-(15), we have a gap in the set $\{h(E(S))\}$ in interval

\begin{equation}
\left(\prod_p \left(1 - \frac{p-1}{p^3}\right), \prod_p \left(1 - \frac{1}{p^3}\right)\right).
\end{equation}

Of course, this Berend’s idea has far-reaching effects.

7. Acknowledgement

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