Combinatorial Invariants from Four Dimensional Lattice Models

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Abstract

We study the subdivision properties of certain lattice gauge theories based on the groups $Z_2$ and $Z_3$, in four dimensions. The Boltzmann weights are shown to be invariant under all type $(k,l)$ subdivision moves, at certain discrete values of the coupling parameter. The partition function then provides a combinatorial invariant of the underlying simplicial complex, at least when there is no boundary. We also show how an extra phase factor arises when comparing Boltzmann weights under the Alexander moves, where the boundary undergoes subdivision.

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1 Introduction

The application of quantum field theory to the study of certain problems in topology has been a very fruitful one; we refer the reader to [1] for a general review of this subject. To state things quite simply, it has been possible to compute a variety of topological invariants as correlation functions in special quantum field theories. While many of these applications employ continuum field theory techniques, the lack of a precise formulation of quantum field theory is a serious handicap when one’s goal is ultimately to provide self-contained rigorous argument. These difficulties might be sidestepped if discrete lattice models can be employed in ways which avoid the continuum limit.

In [2], we presented a class of lattice gauge theories which enjoyed some novel properties under lattice subdivision. The models were defined on a triangulated 4-manifold with boundary, in terms of compact Wilson variables. While we were motivated by some discrete structures which had a formal resemblance to the pure Chern-Simons theory [3, 4], once the model is defined no further reference to any continuum theory need be made. We found that it was possible to define models based on the gauge groups \(Z_2\) and \(Z_3\) in which the Boltzmann weight was invariant under type 4 Alexander subdivision, at certain discrete values of the coupling parameter. These observations came as a result of computer studies, and we were able to present some exact calculations using Mathematica [5].

Here, we will present analytic proofs of the subdivision properties of the models defined in [2]. We will show that the Boltzmann weight of these theories is invariant under all type \((k, l)\) subdivisions of the underlying simplicial complex. This provides one with a partition function which is a combinatorial invariant of that complex, at least in the absence of a boundary. We also show how the Boltzmann weights behave under Alexander subdivision [7], where the boundary itself is subdivided. We find that there is a phase factor associated with these general subdivision moves at the level of the Boltzmann weights.

Other topological lattice models have been formulated previously [8, 9, 10], and we should say a word about them here. While these theories are also formulated in terms of a triangulation, and an analysis of subdivision
properties has been given, we are not aware of any connection with the
models considered in this paper. In particular, the Boltzmann weights of
these other models are assembled from the $6j$ symbols, and their extensions.
In [11, 12], certain Chern-Simons type theories were constructed for finite
groups. It is unclear whether a relation exists between those models and our
four dimensional theory defined on a manifold with boundary.

The next section begins with a review of simplicial complexes and lattice
gauge theory, followed by a definition of the models we consider. An overview
of subdivision moves is also given. We then present our main results regarding
$(k, l)$ subdivision invariance, and their proof. Alexander subdivision is then
examined in these models, and we remark on some issues that arise with
different groups. We close then with a few comments.

2 Definition of the Models

To begin, let us recall the basic elements of the theory of simplicial complexes;
we refer to [13], which will be the source for our definitions and notation, for
further details.

Let \{\(v_0, \ldots, v_n\)\} be a geometrically independent set of points in some
ambient euclidean space \(R^N\). An n-simplex spanned by this set of vertices,
is the set of points \(x\) of \(R^N\) which satisfies,

\[
x = \sum_{i=0}^{n} t_i v_i , \quad \text{with} \quad 1 = \sum_{i=0}^{n} t_i , \quad \text{and} \quad t_i \geq 0 \quad \text{for all} \quad i.
\]  

Pictorially, these can be regarded as points, line segments, triangles, and
tetrahedrons for \(n\) equals zero through three respectively. A simplex which
is spanned by any subset of the vertices is called a face of the original simplex.
An orientation of a simplex is a choice of ordering of its vertices, and we let

\([v_0, \ldots, v_n]\) denote the oriented simplex with the orientation class given by the ordering
\(v_0 \cdots v_n\).
A simplicial complex $K$ in $R^N$ is a collection of simplices which are glued together under two restrictions. Any face of a simplex in $K$ is also in $K$, and the intersection of any two simplices in $K$ must be a face of each of them. The picture here is that of a collection of simplices glued together under the above restriction. We will think of a spacetime manifold as being approximated by a certain simplicial complex.

One defines the boundary operator $\partial_p$ on $\sigma = [v_0, \ldots, v_p]$ by:

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p],$$

where the ‘hat’ indicates a vertex which has been omitted. It is easy to show that the composition of boundary operators is zero.

In a Wilson formulation of lattice gauge theory, the basic dynamical variables are given by group valued maps on the 1-simplices (denoted $[a, b]$) with the rule that $U_{ba} = U_{ab}^{-1}$. A configuration of the system is then specified by a collection of group elements, one for each “link”. One has, in addition, a gauge group associated to each of the vertices, and the action of that group on the link variables is defined by,

$$U_{ab} \rightarrow g_a U_{ab} g_b^{-1},$$

where $g_a$ is a group element associated with the vertex $a$. This group action is also called a gauge transformation.

Given a compact gauge group $G$, together with an invariant measure, one can define a theory with partition function

$$Z = \prod_\alpha \int dU_\alpha \exp[\beta S(U)],$$

where the action functional $S$ of the theory is taken to be a gauge invariant function of the link variables defined above, and the index $\alpha$ indicates the set of independent 1-simplices in the simplicial complex $K$. In the case of a discrete gauge group, the group integration (whose volume we normalize to unity) is a discrete sum,

$$\int dU \rightarrow \frac{1}{|G|} \sum_U,$$
where $|G|$ denotes the order of the group. One can also define correlation functions of the link variables,

$$
< U_\gamma_1 \cdots U_\gamma_p > = \prod_\alpha \int dU_\alpha \ U_\gamma_1 \cdots U_\gamma_p \ \exp[\beta S(U)] .
$$

(7)

It should be emphasized that, in general, all of these quantities depend not only on the coupling parameter $\beta$, but also on the simplicial complex $K$.

A central role in the construction of lattice gauge theory actions is played by the holonomy. Let $U_{abc} = U_{ab} U_{bc} U_{ca}$ be the holonomy based at the first vertex $a$, around the triangle determined by $a, b$ and $c$, and traversed in the order from left to right.

We take the action of our theory to be given by,

$$
S = \sum (U - U^{-1}) \ast (U - U^{-1}) ,
$$

(8)

where $U$ is the above holonomy combination, and the sum here is over all elementary 4-simplices in the simplicial complex. A matrix trace is also to be included for the case of non-Abelian Lie groups. The $\ast$-product [14], to be recalled presently, is designed to capture some of the properties of the wedge product of differential forms. In a continuum limit, the quantity $U - U^{-1}$ becomes proportional to the curvature of a connection, and (8) then goes over to the Chern form. We mention this from a purely motivational standpoint; we will not make any use of continuum theories in this paper. Let us now recall the definition of the star product [14].

The star product is a variant of the usual cup product of cochains which achieves graded commutativity at the expense of associativity. Let $c^r$ and $c^s$ be two maps from the set of oriented $r$- and $s$-simplices respectively, into a group, and let $< c^r, [v_0, \cdots, v_r] >$ represent the evaluation of this map on the particular $r$-simplex $[v_0, \cdots, v_r]$. In our applications, we have been using the notation $U_{abc}$ for the quantity $< U, [abc] >$. Denote by $P$, one of the $(r + s + 1)!$ permutations of the set of vertices $\{v_0, \cdots, v_{r+s}\}$, which span some $(r + s)$-simplex, and by $P v_i$ the value of that permutation on $v_i$. The star product of $c^r$ and $c^s$ is defined by,

$$
\begin{align*}
< c^r \ast c^s, [v_0, \cdots, v_{r+s}] > &= \\
&\frac{1}{(r + s + 1)!} \sum_P (-1)^{|P|} \ < c^r, [P v_0, \cdots, P v_r] > \cdot \ < c^s, [P v_r, \cdots, P v_{r+s}] > ,
\end{align*}
$$

(9)
when the order \( v_0 \cdots v_{r+s} \) is in the equivalence class of the orientation of the simplex \([v_0, \cdots, v_{r+s}]\) (this determines the overall sign of the product), and where the sum is over all permutations of the vertices. The actual number of independent terms in this sum is given by the number of ways one can partition the set of vertices into two parts which contain one vertex in common, and an easy counting yields
\[
\frac{(r + s + 1)!}{r! s!}.
\] (10)

As we have seen, the holonomy \( U \) is a group valued map on 2-simplices, and therefore the above action is naturally defined on a 4-simplex. One should note that the quantity \((U - U^{-1})_{abc}\) enjoys the property of antisymmetry in its last two indices; this is a simple consequence of the relation \(U_{abc} = U_{abc}^{-1}\). When the group is Abelian, one has, moreover, antisymmetry in all three indices. The star product in our action, when evaluated on a given 4-simplex, leads generically to 5! terms, however, the symmetries present in \(\mathbb{F}\) reduce that number to 15 distinct combinations. The Boltzmann weight for this theory, evaluated on the simplex \([v_0, v_1, v_2, v_3, v_4]\), can now be written in the form:
\[
W[v_0, v_1, v_2, v_3, v_4] = B[v_0, v_1, v_2, v_3, v_4] B[v_0, v_1, v_3, v_4, v_2] B[v_0, v_1, v_4, v_2, v_3] \\
B[v_1, v_0, v_2, v_4, v_3] B[v_1, v_0, v_3, v_2, v_4] B[v_1, v_0, v_4, v_3, v_2] \\
B[v_2, v_0, v_1, v_3, v_4] B[v_2, v_0, v_3, v_4, v_1] B[v_2, v_0, v_4, v_1, v_3] \\
B[v_3, v_0, v_1, v_4, v_2] B[v_3, v_0, v_2, v_1, v_4] B[v_3, v_0, v_4, v_2, v_1] \\
B[v_4, v_0, v_1, v_2, v_3] B[v_4, v_0, v_2, v_3, v_1] B[v_4, v_0, v_3, v_1, v_2],
\] (11)

where,
\[
B[v_0, v_1, v_2, v_3, v_4] = \exp[\beta (U - U^{-1})_{v_0 v_1} (U - U^{-1})_{v_0 v_2} (U - U^{-1})_{v_0 v_3} v_4].
\] (12)

One general feature worth observing is that each \(B\) factor depends on two independent holonomies which share a common vertex; the term “bowtie” seems appropriate to describe this structure. In the following, our analysis shall proceed for general complex coupling \(\beta\).

Our main concern here is to study these models for the case of the discrete Abelian groups \(\mathbb{Z}_p\). However, an action which depends on the combination
\((U - U^{-1})\) necessarily leads to a trivial theory for the case of \(Z_2\). One may then wish to consider the action

\[
S = \sum (U - 1) \ast (U - 1) .
\]  

(13)

However, for Abelian groups, the holonomy \(U_{abc}\) is invariant under cyclic permutations of the indices. In addition, for the case of \(Z_2\), we have the relation \(U = U^{-1}\), for all group elements. As a result, the holonomy combination is in fact symmetric in all indices, and the action above vanishes. Nevertheless, as shown in [2], one can simply define the Boltzmann weight for a given 4-simplex to be of the form (11), with

\[
B[v_0, v_1, v_2, v_3, v_4] = \exp[\beta(U - 1)_{v_0v_1v_2}(U - 1)_{v_0v_3v_4}] .
\]  

(14)

One can proceed and compute the partition function for these theories defined on a simplicial complex \(K\). We wish to study, however, the behavior of this function under subdivision of the complex. Let us now recall two bases of subdivision operations which accommodate an analysis of this question.

**The Alexander Moves:**

Consider a single 4-simplex \([v_0, v_1, v_2, v_3, v_4]\). The subdivision operations of Alexander type can in turn be described as follows.

**Type 1 Alexander subdivision:**

\[
[v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] ,
\]  

(15)

**Type 2 Alexander subdivision:**

\[
[v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] ,
\]  

(16)

**Type 3 Alexander subdivision:**

\[
[v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] + [v_0, v_1, v_2, x, v_4] ,
\]  

(17)

**Type 4 Alexander subdivision:**

\[
[v_0, v_1, v_2, v_3, v_4] \rightarrow [x, v_1, v_2, v_3, v_4] + [v_0, x, v_2, v_3, v_4] + [v_0, v_1, x, v_3, v_4] + [v_0, v_1, v_2, x, v_4] + [v_0, v_1, v_2, v_3, x] .
\]  

(18)
One can picture the move of type 1 as the introduction of an additional vertex \( x \), which is placed at the center of the 1-simplex \([v_0, v_1]\), and is then joined to all the remaining vertices of the 4-simplex. Moves 2 to 4 involve a similar construction, where \( x \) is placed at the center of the simplices \([v_0, v_1, v_2]\), \([v_0, v_1, v_2, v_3]\), and finally \([v_0, v_1, v_2, v_3, v_4]\). There is, in addition, a type 0 move which is effected by replacing a vertex of the simplicial complex by a new vertex. This can be considered as a degenerate case, and need not concern us in the following.

According to Alexander [7], two simplicial complexes are said to be equivalent if and only if it is possible to transform one into the other by a sequence of these moves. Hence, any function of \( K \) which is invariant under these moves yields a combinatorial invariant of the simplicial complex.

The \((k, l)\) Moves:
Another basis of subdivision operations is available, known as the \((k, l)\) moves, and these allow for a more convenient analysis. In particular, it has been shown [3] that the basis of \((k, l)\) moves is equivalent to the Alexander moves for the case of closed manifolds, for dimensions less than or equal to four. In the four dimensional case under study, we have five \((k, l)\) moves, with \( k = 1, \ldots, 5 \), and \( k + l = 6 \). It suffices to consider the first three cases; the \((4, 2)\) and \((5, 1)\) moves are inverse to the \((2, 4)\) and \((1, 5)\) moves, respectively.

The \((1, 5)\) move:

\[
[v_0, v_1, v_2, v_3, v_4] \to [x, v_1, v_2, v_3, v_4] + [v_0, x, v_3, v_4] + [v_0, v_1, x, v_3, v_4] + [v_0, v_1, v_2, x, v_4] + [v_0, v_1, v_2, v_3, x], \tag{19}
\]

The \((2, 4)\) move:

\[
[v_0, v_1, v_2, v_3, x] + [v_0, v_1, v_2, v_3, y] \to [v_0, v_1, v_2, x, y] + [v_0, v_2, v_3, x, y]
+ [v_0, v_1, v_3, y, x] + [v_1, v_2, v_3, y, x], \tag{20}
\]

The \((3, 3)\) move:

\[
[v_0, v_1, v_2, y, x] + [v_0, v_1, v_2, y, z] + [v_0, v_1, v_2, z, x] \to [x, y, z, v_0, v_1]
+ [x, y, z, v_1, v_2] + [x, y, z, v_2, v_0]. \tag{21}
\]

A simple observation is that the \((1, 5)\) move is identical to the type 4 Alexander subdivision. One notes that the simplices on the left hand side
of the (2, 4) move share a common 3-simplex $[v_0, v_1, v_2, v_3]$, while those on
the right have a common 1-simplex $[x, y]$. For the case of the (3, 3) move,
the 2-simplex $[v_0, v_1, v_2]$ is common to the left hand side, with $[x, y, z]$ being
common to the right. Furthermore, one can verify that the boundary of the
4-simplex remains unchanged as a result of these operations.

Previous Results:
Before proceeding with the general analysis, we recall the results obtained for
the 4-disk and 4-sphere, for the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. A complete calculation of
the partition function for arbitrary complex coupling $\beta$ was presented in [4].
The central observation was that subdivision invariant points were present
in both models. Indeed, it was explicitly checked that the partition functions
of the 4-disk remained invariant under all Alexander moves, at these special
points. It was further shown that the results for $S^4$ were invariant with
respect to Alexander type 4 subdivision. For each of the models studied, a
natural scale factor was present, and we denote this by $s(2) = \exp[4\beta]$ and
$s(3) = \exp[-3\beta]$ for the case of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. The corresponding
subdivision invariant points are then given when $s(2)^2 = 1$, and $s(3)^3 = 1$.
A point worth mentioning is that the Boltzmann weights themselves are
group valued at these special points. Let us quickly summarize some of
those results.

Beginning with the $\mathbb{Z}_2$ theory, we found the partition function on the
4-disk to be given by

$$Z(s(2)) = \frac{1}{2^4}(9 + 7s(2)) ,$$

when $s(2)^2 = 1$. The two roots of unity, +1 and −1 yield the values 1 and
1/8 respectively. For the case of the 4-sphere $S^4$, we find that the partition
function assumes a value of $Z = 1$, when $s(2)^2 = 1$.

Turning now to the $\mathbb{Z}_3$ theory, we found the partition function on the
4-disk to be:

$$Z(s(3)) = \frac{1}{3^4}(29 + 26(s(3) + s(3)^{-1})) ,$$

when $s(3)^3 = 1$. The trivial subdivision invariant point $s(3) = 1$ yields a
value $Z = 1$, while the other two cube roots of unity give a value $Z = 1/27$.
Again, for the case of $S^4$, one finds a value of $Z = 1$ when $s(3)^3 = 1$. 
These results were obtained through the use of Mathematica [5] to evaluate the partition functions. While we could show through exhaustive computer checks that these models had interesting subdivision invariant points, a clear analytic understanding of these special properties was generally lacking. This we supply in the following sections.

3 Main Results

Having laid the foundational material in the preceding sections, we can now state and prove the main results. The aim of this section is to establish the behavior of the Boltzmann weights under all \((k, l)\) type subdivisions. In order to treat both the \(\mathbb{Z}_2\) and \(\mathbb{Z}_3\) models uniformly, it is expedient to let \(\mathcal{X}\) denote the combinations \(U - 1\) and \(U - U^{-1}\), respectively, for those models. It will further be convenient to let \(B[0, 1, 2, 3, 4]\) represent the expression \(B[v_0, v_1, v_3, v_4]\); using subscripts to keep track of vertices should cause no confusion. We begin with a lemma.

**Lemma:** The Boltzmann weights for a given vertex ordering satisfy the conditions,

\[
B[0, 1, 2, 3, 4] B[0, 1, 2, 4, 5] B[0, 1, 2, 5, 3] = \exp[\beta X_{v_0 v_1 v_2} X_{v_3 v_4 v_5}] ,
\]

\[
B[0, 1, 2, 3, 4] B[1, 2, 0, 4, 3] = \exp[\beta X_{v_0 v_1 v_2} X_{v_0 v_1 v_3}] \exp[-\beta X_{v_0 v_1 v_2} X_{v_0 v_1 v_3}] ,
\]

at the points \(s(2)^2 = 1\) and \(s(3)^3 = 1\), in the \(\mathbb{Z}_2\) and \(\mathbb{Z}_3\) theories respectively.

Consider first the \(\mathbb{Z}_2\) case. One notices immediately that the relation (24) is trivially satisfied for \(U_{v_0 v_1 v_2} = 1\), so we only need to consider the case \(U_{v_0 v_1 v_2} = -1\). For simplicity of notation, let \(x = U_{v_0 v_3 v_4}\), \(y = U_{v_0 v_4 v_5}\), and \(z = U_{v_0 v_5 v_3}\). Noticing that \(U_{v_3 v_4 v_5} = x y z\), our assertion is then equivalent to,

\[
1 = \exp[-2 \beta ((x - 1) + (y - 1) + (z - 1) - (xyz - 1))] .
\]

Now, \(x\), \(y\), and \(z\) are independent group elements, and the image set of the function

\[
(x, y, z) \rightarrow (x - 1) + (y - 1) + (z - 1) - (xyz - 1)
\]
is easily seen to be \(\{0, -4\}\). Recalling that \(s(2) = \exp[4\beta]\), one sees then that (26) is satisfied at \(s(2)^2 = 1\).

The \(Z_3\) relation follows in the same way; here we need only check the case \(U_{v_0v_1v_2} = \exp[\pm 2\pi i/3]\). The assertion (24) is then equivalent to,

\[1 = \exp[\pm \beta i \sqrt{3} ((x - x^{-1}) + (y - y^{-1}) + (z - z^{-1}) - (xyz - x^{-1}y^{-1}z^{-1}))],\]

where \(x = U_{v_0v_3v_4}, y = U_{v_0v_4v_5}\), and \(z = U_{v_0v_5v_3}\) are independent \(Z_3\) elements. The image set of the function,

\[ (x, y, z) \to (x - x^{-1}) + (y - y^{-1}) + (z - z^{-1}) - (xyz - x^{-1}y^{-1}z^{-1}) \]

is easily seen to be \(\{0, \pm 3i\sqrt{3}\}\). With \(s(3) = \exp[-3\beta]\), one then finds that (28) is satisfied at the points \(s(3)^3 = 1\).

The proof of the second relation (25) is very similar, and we omit the details. Our main result is the following theorem.

**Theorem:** The full Boltzmann weights satisfy the relation,

\[
W[0, 1, 2, 3, 4] W[0, 1, 2, 4, 5] W[0, 1, 2, 5, 3] = W[0, 1, 3, 4, 5] W[1, 2, 3, 4, 5] W[2, 0, 3, 4, 5],
\]

at the points \(s(2)^2 = 1\) and \(s(3)^3 = 1\), in the \(Z_2\) and \(Z_3\) theories respectively.

The proof of this, while straightforward, is surprisingly tedious. Each of the \(W\) factors is itself a product of 15 factors. One can write out all 90 \(B\) factors that occur in (30), and methodically use the the identities established in the lemma to verify the claim. In our analysis, we used the identity,

\[
B[0, 1, 2, 3, 4] B[0, 1, 2, 4, 5] B[0, 1, 2, 5, 3] = B[3, 4, 5, 0, 1] B[3, 4, 5, 1, 2] B[3, 4, 5, 2, 0],
\]

which is a trivial consequence of (24), to eliminate all but 18 of the 90 terms. The identity (24) was then used to polish off the remaining factors.

Armed with this theorem, is now a simple matter to understand the subdivision properties of the Boltzmann weights under the remaining moves.

**Corollary:** The full Boltzmann weights satisfy the following two relations:

\[
W[0, 1, 2, 3, 4] W[0, 1, 2, 5, 3] = W[1, 2, 3, 4, 5] W[2, 0, 3, 4, 5] W[0, 1, 3, 4, 5] W[1, 0, 2, 4, 5],
\]

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at the points \( s(2)^2 = 1 \) and \( s(3)^3 = 1 \) for the groups \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) respectively.

The proof here is a simple application of the theorem, together with the fact that,

\[
W[0, 1, 2, 3, 4]^{-1} = W[0, 1, 2, 4, 3] \quad (34)
\]

in our theories; this relation holds at \( s(2)^2 = 1 \) in the \( \mathbb{Z}_2 \) case, and quite generally in the \( \mathbb{Z}_3 \) model. This means that \( W \) is actually invariant under a reversal of orientation at the special points in the \( \mathbb{Z}_2 \) model, and is exchanged for its inverse in all models based on the action (8).

As a consequence of these results, one can immediately establish the fact that the partition function for these models provides a combinatorial invariant of an arbitrary simplicial complex, at least for the case of zero boundary. In particular, we can now assert that the results presented previously [2] for the case of \( S^4 \) do indeed correspond to a combinatorial invariant.

4 Behavior under the Alexander Moves

The subdivision moves introduced by Alexander, and reviewed in an earlier section, are slightly more complicated. These moves generally induce subdivisions of the boundary of a given 4-simplex. Nevertheless, our understanding of the type \((k, l)\) subdivision will allow us to fully analyze these other moves.

Consider the type 3 Alexander move where we add the \( v_5 \) vertex to the center of \([v_0, v_1, v_2, v_3]\). Using subscripts once again to keep track of the vertices, this move takes the form:

\[
[0, 1, 2, 3, 4] \rightarrow [5, 1, 2, 3, 4] + [0, 5, 2, 3, 4] + [0, 1, 5, 3, 4] + [0, 1, 2, 5, 4] \quad (35)
\]

It is useful to note how the boundary transforms under this move; a simple check reveals that the boundary component \([0, 1, 2]\) undergoes a three dimensional Alexander type 3 subdivision, namely,

\[
[0, 1, 2, 3] \rightarrow [5, 1, 2, 3] + [0, 5, 2, 3] + [0, 1, 5, 3] + [0, 1, 2, 5] \quad (36)
\]
The fundamental question is how $W[0, 1, 2, 3, 4]$ is related to the weights of the four 4-simplices on the right hand side of equation (35). Again, the $(3, 3)$ identity we established in the last section proves to be the key to resolving this. It is a quick exercise to show that,

$$W[0, 1, 2, 3, 4] = W[5, 0, 1, 2, 3] (W[5, 1, 2, 3, 4] W[0, 5, 2, 3, 4] W[0, 1, 5, 3, 4] W[0, 1, 2, 5, 4]).$$

(37)

The Boltzmann weight $W$ is not invariant under this move, but it picks up what one might wish to view as a phase factor associated with adding the $v_5$ vertex to the center of $[v_0, v_1, v_2, v_3]$; the “phase” being the quantity $W[5, 0, 1, 2, 3]$. It is equally simple to write the corresponding “phases” associated with the type 1 and 2 Alexander moves, though we won’t catalogue them here. The type 4 move is identical to $(1, 5)$, and we know that there is no phase factor in that case.

5 $Z_4$ and Beyond

One rather immediate question about the results we have laid out in the previous two sections would be with regard to extending them to the general $Z_p$ case. This is not automatic, and an analysis of the group $Z_4$ already begins to show a departure from what happened for $Z_2$ and $Z_3$. If one repeats the same analysis for a $Z_4$ type theory defined by the action (8), one finds that not all fourth roots of unity yield subdivision invariant points under the $(k, l)$ moves. Defining the analogous scale factor to be $s(4) = \exp[-4\beta]$, one finds only the two points corresponding to $s(4)^2 = 1$. A similar situation arises for $Z_6$; with the scale factor denoted by $s(6) = \exp[-3\beta]$, one finds subdivision invariant points when $s(6)^3 = 1$. For $Z_5$, however, the entire structure of the theory is rather more complicated, and it is an open question as to whether one can find subdivision invariant points. Equally, we must leave extensions to other types of groups, both discrete and continuous, for future investigation.
6 Concluding Remarks

Having established the combinatorial invariance of the partition function for the $Z_2$ and $Z_3$ models, perhaps the most pressing issue is to determine the precise nature of this invariant. In particular, it is of interest to explicitly compute the invariant for a variety of closed manifolds. As we have seen, the Boltzmann weight is invariant under the Alexander moves, up to certain “phase” factors associated with the subdivision induced on the boundary. Our conclusion thus falls short of declaring the partition function to be a combinatorial invariant for a four dimensional manifold with boundary. Nevertheless, as we have seen from our computer studies, the partition function on the 4-disk is indeed invariant under all subdivision moves of Alexander type. Further work is required in this arena. One might also hope that the natural correlation functions associated with each of these models may enjoy special properties with respect to subdivision, but we leave that for the future.

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