Condensation of Handles
in the Interface of 3D Ising Model

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Abstract

We analyze the microscopic, topological structure of the interface between domains of opposite magnetization in 3D Ising model near the critical point. This interface exhibits a fractal behaviour with a high density of handles. The mean area is an almost linear function of the genus. The entropy exponent is affected by strong finite-size effects.
1. Introduction

The interface in 3D statistical systems above the roughening temperature behaves like a free, fluctuating surface with a topology determined by the boundary conditions. Recently, new computational algorithms have been applied to these systems [1-6], providing us with a powerful tool for testing our ideas on the behaviour of fluctuating surfaces. Alternatively, these surfaces can be thought of as space-time histories of closed strings which can be used to describe the infrared behaviour of the dual $Z_2$ gauge theory.

In a previous paper [7] we studied the free-energy of the interface as a function of its shape. We found rather strong shape effects which are accurately described by the gaussian limit of the Nambu-Goto action.

This observation suggests that interface configurations are mostly made of smooth surfaces subjected to long-wavelength fluctuations which account for the observed finite-size effects. On the other hand, this picture seems to strongly disagree with the process of crumpling, which should take place for random surfaces embedded in a target space of dimension $d \geq 1$. In other words, if the interface is described by the Nambu-Goto action, it should take the shape of a branched polymer, rather than that of a smooth surface.

In this letter we face this dilemma by studying the interface from a microscopic point of view. Actually, it turns out that the interface at small scales is much more similar to a sponge than to a smooth surface. In particular there is a strong, almost linear correlation between the area and the genus of the surface, indicating the formation of a large number of microscopic handles which is proportional to its area. Yet, the partition function summed over all the genera behaves like that of a smooth toroidal surface: this means that the huge number of microscopic handles, produced by the instability toward crumpling, have as the net effect a simple non-perturbative renormalization of the physical quantities associated to the surface, according to an old conjecture on string theory [8].
2. The method

The first problem to be solved in order to study the microscopic structure of the interface is to express the 3D Ising model or its dual gauge version as a gas of suitable closed surfaces. There are essentially two different ways to do it. One is based on the strong coupling expansion of the dual gauge model [9] and involves also self-intersecting and non-orientable surfaces. Here instead we identify the surfaces as the Peierls interfaces of an arbitrary Ising spin configuration on a cubic lattice [10]. By construction these are closed, orientable surfaces composed of plaquettes of the dual lattice orthogonal to each frustrated link. In this way each plaquette appears at most once in the construction of the surface $S$.

**Tab. I**

Graphs for the microscopic reconstruction of the interface. Each cube is dual to a vertex $V_q$ with $q$ edges of the interface as drawn in fig. 1. The encircled sites have a spin different from the other sites. Graphs with different vertex decompositions correspond to end-points of contact lines. The first decomposition is for positive vertices, while the other is for negative vertices.

![Graphs](image-url)

| Graph  | Description                                      |
|--------|--------------------------------------------------|
| ![V3](image-url) | $V_3$                                             |
| ![2V3 ~ V6](image-url) | $2V_3 \sim V_6$                                   |
| ![V4](image-url) | $V_4$                                             |
| ![2V3](image-url) | $2V_3$                                            |
| ![V3 + V5](image-url) | $V_3 + V_5$                                       |
| ![2V4](image-url) | $2V_4$                                            |
| ![4V3 ~ 2V6 ~ 2V3 + V6](image-url) | $4V_3 \sim 2V_6 \sim 2V_3 + V_6$                 |
| ![V6](image-url) | $V_6$                                             |
| ![3V3 ~ V6 + V3](image-url) | $3V_3 \sim V_6 + V_3$                            |
A link belonging to $S$ is said to be regular if it glues two plaquettes of $S$. The only singularities which may appear on these surfaces are links gluing four distinct plaquettes of $S$. These singular links form contact lines of the surface which are sometimes improperly called self-intersections. These contact lines are the only potential sources of ambiguities in the topological reconstruction of the surface. We shall see shortly how to remove them in a simple, consistent way. A vertex of the surface $S$ is dual to an elementary cube of the lattice of the Ising configurations. According to the distribution of spins inside the cube, there are twelve distinct vertices as listed in Table I. Some of them include singular links and may be interpreted as the coalescence of distinct, regular vertices. Only the end-points of contact lines can be decomposed into two or more inequivalent ways: for instance the first graph in the second row of the table can be thought either as the coalescence of two vertices with three edges, or as a single vertex with six edges and the singular link is splitted accordingly into two different ways as drawn in figure 1.

![Example of decodification of a graph of Table I. This cube has six frustrated links, labelled by $a,b,...,f$, which are dual to six plaquettes forming a vertex with a singular link. It can be considered either as a 6-vertex, or as two 3-vertices. This choice depends on the sign of the magnetization of the cube, which can be in this case $\pm 4$.](image)

Choosing arbitrarily this splitting procedure for each end-point may generate global obstructions in the surface reconstruction, due to a constraint which links together the splitting of two end-points connected by a contact line. In order to formulate this con-
straint, it is useful to attribute a sign to each singular vertex according to the sign of the magnetization of the corresponding elementary cube. Then, working out some explicit example, it is easy to convince oneself that there are no global obstructions if one chooses different splittings for vertices of different sign. More precisely, the only constraint to be fulfilled in the replacement of each singular link with a pair of regular ones is that two end-points connected by a contact line must be split in the same way if they have the same sign, or in the opposite way if their sign is different. The two decompositions appearing for singular vertices in table I correspond precisely to the two possible signs of the vertex. The only case in which the surface reconstruction is not immediate is the first graph of the last row of tab. I, corresponding to a vertex with six singular lines: the three possible decompositions cannot be selected by the sign of its magnetization (which is zero), but by the signs of the end-points of the six contact lines.

The main consequence of the above construction is that we have obtained a simple rule to assign unambiguously to each Ising configuration a set of self-avoiding closed random surfaces. Notice that this fact agrees with a result of David [11] who found that a gas of self-avoiding surfaces in a special three-dimensional lattice belongs to the same universality class of the Ising model.

We may now evaluate the genus \( h \) of each self-avoiding surface through the Euler relation

\[
F - E + V = \chi(S) = 2 - 2h
\]

(1)

where \( F \) is the number of faces (plaquettes), \( E \) the number of edges, \( V \) the number of vertices and \( \chi \) the Euler characteristic; the genus \( h \) gives the number of handles. Denoting by \( N_q \) the number of vertices of coordination number \( q \), we have obviously, according to table I,

\[
V = N_3 + N_4 + N_5 + N_6 + N_7 \quad ; \quad 4F = 2E = 3N_3 + 4N_4 + 5N_5 + 6N_6 + 7N_7
\]

(2)
which gives at once an even simpler expression for the genus of this kind of surfaces

\[ 4(V - F) = N_3 - N_5 - 2N_6 - 3N_7 = 8 - 8h \ . \]  

(3)

3. Results

We are interested on the topology of the interface between two macroscopic domains of opposite magnetization. For this reason we consider very elongated lattices with periodic boundary conditions in the two short directions (denoted in the following by \( L \)) and antiperiodic boundary conditions in the long direction (denoted by \( L_z \)). This forces the formation of an odd number of interfaces in the \( L_z \) direction. We then isolate one of these interfaces by reconstructing all the spin clusters of the configuration, keeping the largest one and flipping the others. Note, as a side remark, that for the values of \( \beta \), \( L \) and \( L_z \) we have studied, typical configurations contained only one macroscopic cluster, besides a huge number of microscopic ones. It is now easy to evaluate area of the interface by counting simply the number of frustrated links of this cleaned configuration.

The growth of this area as a function of the size of the lattice gives a first description of the fractal behaviour of the interface: in a set of Monte Carlo simulations near the critical point on elongated lattices \( L^2 \times L_z \) with \( L_z \geq 120 \) and \( 8 \leq L \leq 16 \) (see table II) we found that the mean area \(< F >\) of the interface is a strongly varying function of the transverse size \( L \), well parametrized by the following power law

\[ < F > = \kappa L^{d_H} \ , \]  

(4)

with \( d_H \sim 3.7 \) and \( \kappa \sim 0.47 \). As a consequence, a sizeable fraction of the lattice is invaded by the interface, and a small increasing of the transverse lattice section implies a rapid growth of the area of the interface, so we have to take very elongated lattices in order to avoid wrapping of the interface around the antiperiodic direction, which would give rise to unwanted finite volume effects.
Each interface $S$ of area $F$ of an arbitrary Ising configuration contributes to the partition function simply with a term $e^{-\beta F}$; then we can define the following generating functional $Z$

$$Z(\beta) = \sum_F \sum_{\text{surfaces of area } F} e^{-\beta F} = \sum_F \sum_h Z_h(F) \quad ,$$

where $Z_h(F)$ is the partition function for a surface of area $F$ and genus $h$. Since we are dealing with macroscopic surfaces, we can consistently assume that their multiplicity is well described by the asymptotic behaviour of the entropy of large random surfaces [12], which yields

$$Z_h(F)_{F \to \infty} \sim F^{b_\chi - 1} e^{\mu F} e^{-\beta F} = F^{b_\chi - 1} e^{-\mu F} \; ,$$

where, using the terminology of two-dimensional quantum gravity (2DQG), $\mu$ is the cosmological constant and the exponent $b_\chi$ is a function of the Euler characteristic $\chi$. In the continuum theory of 2DQG this exponent can be evaluated exactly for the coupling to conformal matter of central charge $c \leq 1$ [13]. The result is

$$b_\chi = -\frac{b}{2} \chi = b(h - 1) \; ,$$

with

$$b = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12} \; .$$

$b$ is a monotonically decreasing function of $c$ for $c \leq 1$ and becomes complex for $c > 1$, where the surface get crumpled and these formulas lose any physical meaning. It is known that for $c = 1$ there are logarithmic corrections to eq.(3), while for $c > 1$ it is only known that the number of surfaces of given area is exponentially bounded [12]. It turns out that eq.(3) fits well to our numerical data.

We can evaluate the exponent $b_\chi$ for the interface by measuring the mean area at fixed genus. Indeed defining

$$Z_h(\mu) = \sum_F Z_h(F) \; ,$$

6
we have

\[ < F >_h = -\frac{\partial}{\partial \mu} \log Z_h(\mu) \sim \frac{b \chi}{\mu} \].

In figure 2 we report a typical outcome of this analysis. Actually \(< F >_h\) is dominated by a linear term, like in eq.(7), but there is also a small, negative quadratic contribution which probably accounts for the self-avoidance constraint: when the number of handles \(h\) is very large the excluded volume effects (proportional to \(h^2\)) become important. The data are well fitted by

\[ < F >_h = \frac{a}{\mu} + \frac{b}{\mu} h + \frac{d}{\mu} h^2 \].

The slope \(b\) can be called the entropy exponent, and \(a\) is related to the strings susceptibility by \(a = \gamma_s - 2\). Comparing eq.(11) with eq.(7) we have of course, for \(c \leq 1\), \(d = 0\) and

\[ b = -a \equiv 2 - \gamma_s \].

Since we are dealing with surfaces of very high genus \(h \sim 10^2\), we can evaluate \(b/\mu\) and \(d/\mu \sim -10^{-4} b/\mu\) very accurately, while the constant \(a\) is affected by large errors and cannot used to evaluate \(\gamma_s\), nevertheless we shall argue shortly that for our surfaces eq.(12) is no longer true.

Combining eq.(10) with other physical quantities at fixed genus it is possible to evaluate the cosmological constant \(\mu\) as a function of \(\beta\) and of the lattice size. In particular, using the derivative \(\frac{\partial Z_h(\mu)}{\partial h}\), we get easily the following equation

\[ \log(1/\mu) = -\psi(b \chi) + < \log F > \],

where \(\psi\) denotes the logarithmic derivative of the Euler \(\Gamma\) function. This formula fits well the numerical data for large areas and we used it to evaluate \(\mu\). The results are reported in table II as well as the entropy exponent \(b\) defined in eq.(11). It turns out that \(b\) is affected by rather strong finite size effects, being a decreasing function of the lattice size \(L\) an hence of the area. Similar finite size effects have been observed for the string susceptibility exponent in the planar quantum gravity coupled to \(c \geq 1\) matter [14]. Note
however that, if one assumes eq.(12), one should conclude that \( \gamma_s \) exceeds the theoretical upper bound \( \gamma_s = \frac{1}{2} \) [13]. A possible way out is that eq.(12) is not fulfilled by this kind of surfaces, indeed the argument leading to such upper bound assumes \( h = 0 \) and gives no restriction on \( b \). On the other hand there is a renormalization group argument [16] showing clearly that eq.(12) is justified only for \( c \leq 1 \).

\[ \text{Tab. II} \]

**Mean area** \(< F >\), **entropy exponent** \( b \), **cosmological constant** \( \mu \) and **renormalized cosmological constant** \( \mu_R \) of the interface in a set of elongated lattices of shape \( L^2 \times L_z \) at \( \beta - \beta_c = 0.0059 \).

| \( L^2 \times L_z \) | \(< F >\) | \( b \) | \( \mu \) | \( \mu_R \) |
|---------------------|---------|-------|--------|--------|
| 8\(^2\) \times 120  | 1057(10)| 0.73(20)| 0.0141(39)| 0.00122(4) |
| 10\(^2\) \times 120  | 2434(19)| 0.61(15)| 0.0097(24)| 0.000613(15) |
| 12\(^2\) \times 180  | 4719(47)| 0.30(3)| 0.0047(5)| 0.000347(4) |
| 14\(^2\) \times 240  | 8506(103)| 0.19(3)| 0.0026(5)| 0.000208(3) |
| 16\(^2\) \times 320  | 14803(249)| 0.11(2)| 0.0015(2)| 0.000125(5) |

It is also possible to define a renormalized cosmological constant \( \mu_R \) by considering the sum over all the genera \( Z(F) = \sum_h Z_h(F) \). It turns out that for large areas the distribution of surfaces is accurately described by an exponential fall off of the type

\[
Z(F)_{F \to \infty} \sim ce^{-\mu_R F},
\]

as shown in fig.3. Comparing this equation with eq.(6), we argue that the sum over all the topologies produces as net effect an effective interface which behaves as a smooth surface with a different cosmological constant. Actually the renormalization effect is very large: \( \mu_R \) is about one order of magnitude smaller than the unrenormalized quantity \( \mu \) (see table II). This phenomenon seems the two-dimensional analogue of a quantum gravity effect described by Coleman [17] who argued that the sum over topologies has the effect of making the cosmological constant vanishing small.

In conclusion, we have found a simple way to generate high-genus self-avoiding random
surfaces. The most interesting question is of course whether their study will lead to new insights in the physics of random surfaces.

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Figure Captions

Figure 2 The mean area of the interface as a function of the genus in a lattice of size $12^2 \times 240$ at $\beta - \beta_c = 0.0044$.

Figure 3 The multiplicity of interface configurations summed over all the topologies as a function of the area in a lattice of size $14^2 \times 240$ at $\beta - \beta_c = 0.0059$. The lower set of data is the multiplicity a fixed genus $h = 70$. The stright line is the fit to eq.(14).
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9304001v1