Reconciling $\phi$ radiative decays with other data for $a_0(980)$, $f_0(980)$, $\pi\pi \to KK$ and $\pi\pi \to \eta\eta$

D.V. Bugg$^1$,  
Queen Mary, University of London, London E1 4NS, UK

Abstract

Data for $\phi \to \gamma(\eta\pi^0)$ are analysed using the $KK$ loop model and compared with parameters of $a_0(980)$ derived from Crystal Barrel data. The $\eta\pi$ mass spectrum agrees closely and the absolute normalisation lies just within errors. However, BES parameters for $f_0(980)$ predict a normalisation for $\phi \to \gamma(\pi^0\pi^0)$ at least a factor 2 lower than is observed. This discrepancy may be eliminated by including constructive interference between $f_0(980)$ and $\sigma$. The magnitude required for $\sigma \to KK$ is consistent with data on $\pi\pi \to KK$. A dispersion relation analysis by Büttiker, Descotes-Genon and Moussallam of $\pi\pi \to KK$ leads to a similar conclusion. Data on $\pi\pi \to \eta\eta$ also require decays of $\sigma$ to $\eta\eta$. Four sets of $\pi\pi \to KK$ data all require a small but definite $f_0(1370)$ signal.

PACS: 13.25.Gv, 14.40.Gx, 13.40.Hq

1 Introduction

Data for $\phi(1020) \to \gamma(\eta\pi^0)$ and $\gamma(\pi^0\pi^0)$ from Novosibirsk [1-3] and the KLOE collaboration at Daphne [4,5] may throw light on $a_0(980)$ and $f_0(980)$, as Achasov and Ivanchenko pointed out in 1989 [6]. A vigorous debate has followed the publication of data. An extensive list of references is given in a recent paper of Boglione and Pennington [7].

Fig. 1 shows the model conventionally assumed for radiative $\phi$ decays. Both the $\phi \to K^+K^-$ decay and rescattering at the $f_0$ (or $a_0$) vertex are short-ranged. In the intermediate state, kaons propagate to larger radii where they radiate a photon through an electric dipole moment.

These data provide information only on the lower sides of $a_0(980)$ and $f_0(980)$, i.e. their coupling to $\eta\pi$ and $\pi\pi$. There is no direct information on their coupling to $KK$, except by assuming the $KK$ loop model and using the absolute normalisation. For $\gamma(\eta\pi^0)$, this normalisation depends on the product $g^2[a_0(980) \to \eta\pi]\times g^2[a_0(980) \to KK]$. In order to test the model, a comparison is made with parameters of $a_0(980)$ from high statistics Crystal Barrel data on $\bar{p}p \to \pi^0(\eta\pi^0)$ [8] and $\bar{p}p \to \omega(\eta\pi^0)$ [9]. There is agreement within the errors, supporting the $KK$ loop model.

$^1$email address: D.Bugg@rl.ac.uk
Figure 1: The KK loop model for $\phi$ radiative decays.

The BES collaboration has made an accurate determination of $f_0(980)$ parameters from $J/\Psi \rightarrow \phi\pi^+\pi^-$ and $\phi K^+K^-$ data [10]. Their values of $g_\phi^2$ predict a normalisation for $\phi \rightarrow \gamma(\pi^0\pi^0)$ a factor 2 smaller than observed in both Novosibirsk and Dapline data, if $f_0(980)$ alone is responsible. This discrepancy is resolved by adding a broad component coupling to both $\pi\pi$ and $KK$ and interfering constructively with $f_0(980)$. It may be parametrised as the high mass tail of the $\sigma$ pole, although other interpretations are possible, e.g. those of Au, Morgan and Pennington [11] and of Anisovich, Anisovich and Sarantsev [12].

This broad component should also appear in $\pi\pi \rightarrow KK$. It will be shown that it can indeed be accommodated naturally there. A similar broad component is required to fit data on $\pi\pi \rightarrow \eta\eta$. A combined fit is made to data on $\pi\pi$ elastic scattering, $\pi\pi \rightarrow KK$ and $\pi\pi \rightarrow \eta\eta$. The $\pi\pi$ phase shifts and elasticities are taken from the re-analysis by Bugg, Sarantsev and Zou [13] of moments from Cern-Munich data [14].

Formulæ are discussed in Section 2. Readers interested only in results can skip this formalism. An awkward problem arises in parametrising $\pi\pi \rightarrow KK$ near 1 GeV. The broad component and $f_0(980)$ both go through 90° at $\sim 1$ GeV. It is difficult to find a parametrisation which accomodates both resonances while still satisfying unitarity. The scheme which is adopted here is driven by the features of the data and is explained in Section 2.

Sections 3 and 4 then report the fits to data on $\phi \rightarrow \gamma(\eta\pi^0)$ and $\gamma(\pi^0\pi^0)$. Section 5 concerns the fit to $\pi\pi \rightarrow KK$ and $\eta\eta$, using parameters consistent with $\phi \rightarrow \gamma(\pi^0\pi^0)$. A brief summary is given in Section 6.

2 Formulae

2.1 $\phi$ radiative decays

Close, Isgur and Kumano [15] give formulæ for the mass distribution of $\pi\pi$ or $\pi\eta$ pairs predicted by the $KK$ loop model. It is important to keep track of the absolute normalisation and also write formulæ in a way where the comparison between $\pi^0\pi^0$ and $\pi\eta$ is as simple as possible. With some trivial re-arrangement
of symbols, their eqns. (3.6) and (3.7) may be written for \( \phi \rightarrow \gamma(\pi^0\pi^0) \) as:

\[
\frac{d\Gamma}{dm} = |I(a,b)|^2 \alpha g_\phi^2 g_{K^+K^-}^2 - g_{\pi^0\pi^0}^2 \frac{\chi}{|D(s)|^2} \\
\chi = \frac{m_{\pi^\pm}^2 \rho_{\pi\pi}(s)}{96\pi^6 m_K^2}
\]

(1)

Here \( s \) is the invariant mass squared of the \( \pi\pi \) pair, \( k_\gamma \) is the photon momentum, \( m \) are masses, \( \alpha \) is the fine-structure constant \( \simeq 1/137 \) and \( I(a,b) \) is a formula for the KK loop integral given by eqns. (3.4) and (3.5) of Close et al; the quantities \( a \) and \( b \) are defined there. The cubic dependence on \( k_\gamma \) arises from gauge invariance and is the familiar \( E^3_{\gamma} \) factor for an \( E1 \) transition. The denominator \( D(s) \) in eqn. (1) is written for \( f_0(980) \) in the Flatté form:

\[
D(s) = M^2 - s - \frac{i}{16\pi} [g_{\pi\pi\pi\pi}^2 \rho_{\pi\pi}(s) + g_{KK\pi\pi}^2 \rho_{KK}(s)];
\]

(4)

the factor \( 1/16\pi \) follows the definition used by the Particle Data Group [16] in their eqn. (38.17), after allowing for a factor \( 4\pi \) from integrating the S-wave over angles. Eqn. (4) is the form used by KLOE. Note that values of \( g^2 \) in this equation refer to the sum over charge states:

\[
g_{\pi\pi}^2 = 3g_{\pi^0\pi^0}^2 = (3/2)g_{\pi^+\pi^-}^2 \\
g_{KK}^2 = 2g_{K^0\pi^0}^2 = 2g_{K^+\pi^-}^2.
\]

(5)

(6)

The BES collaboration writes the Flatté form using

\[
g'_1 = \frac{g_{\pi\pi}^2}{16\pi} \\
g'_2 = \frac{g_{KK}^2}{16\pi}
\]

(7)

(8)

Below the \( KK \) threshold, \( \rho_{KK}(s) \) in eqn. (4) needs to be continued analytically as \(+ i\sqrt{4m_K^2/s - 1}\).

Eqn. (3.1) of Close et al. defines \( g_\phi^2 \) via:

\[
\Gamma(\phi \rightarrow K^+K^-) = \frac{g_\phi^2}{48\pi m_\phi^2} (m_\phi^2 - 4m_K^2)^{3/2}.
\]

(9)

The coupling constants \( g^2 \) for \( f_0 \) decay are defined to have dimensions of \( \text{GeV}^2 \), but \( g_\phi^2 \) is dimensionless and so is \( I(a,b) \). Dimensions of eqns. (1) and (2) balance between left and right-hand sides; this was not apparent in the papers of either Close et al or Achasov and Ivanchenko, since they replace \( g_{\pi\pi}^2 g_{KK}^2 \) by a single \( g^2 \). The two powers of \( g^2 \) appear explicitly in the KLOE publication [5]. The relative normalisation of data on \( \phi \rightarrow \gamma(\pi^0\pi^0) \) and \( \pi \rightarrow \gamma(\eta\pi^0) \) will play an important role:

\[
\frac{(d\Gamma/dm)_{\pi^0\pi^0}}{(d\Gamma/dm)_{\eta\pi^0}} = \frac{g^2_{\pi K K} (g_{f_0\pi\pi}/3) |D(s)|^2_{f_0,\pi\pi}(s)}{g_{\eta K K}^2 (g_{f_0\pi\pi}/3) |D(s)|^2_{f_0,\pi\pi}(s)}
\]

(10)
2.2 Identical particles

At this point, it is necessary to pause and ask whether decays to $\pi^0\pi^0$ are affected by the fact that they are identical particles, whereas $\pi^0\eta$ are not. This is a tricky point. An $I = 0$ $f_0$ couples to the isospin combination $(\pi^+\pi^- + \pi^0\pi^0)/\sqrt{3}$, where the coefficients are isospin Clebsch-Gordan coefficients. Experimentally, the $\pi^+\pi^-$ integrated cross section may be obtained by counting $\pi^+$ over $4\pi$ solid angle. There are two amplitudes, one for producing a $\pi^+$ at angle $\theta$, say, and the other for producing a $\pi^-$ at that angle and an accompanying $\pi^+$ at angle $(\theta + \pi)$. Both are in S-waves, so at angle $\theta$ there is a total amplitude for $\pi^+$ of $2/\sqrt{3}$. Integrating over $4\pi$ solid angle, the intensity of observed $\pi^+$ is $(4/3)4\pi$. For the $\pi^0\pi^0$ case, there are likewise two $\pi^0$ at angles $\theta$ and $(\theta + \pi)$ so the total $\pi^0$ amplitude is $2\pi^0/\sqrt{3}$. In this case, the integration should be only over $2\pi$ solid angle, so as to avoid counting $\pi^0$ pairs twice; the integrated intensity is $(4/3)2\pi$. The total $\pi\pi$ intensity is $8\pi$, compared with the total $\eta\pi$ intensity of $4\pi$, where there are no effects for identical particles.

So the $\pi\pi$ amplitude is increased by a factor 2 by the identity of the two pions. However, it is essential to remember that this factor 2 appears also in the Breit-Wigner denominator. The Breit-Wigner amplitude becomes

$$f = \frac{g_{K^+K^-\rho\pi\pi}(2/\sqrt{2})(1/\sqrt{3})}{M^2 - s - i[2g^2_{\pi\pi}\rho\pi\pi + g^2_{KK}\rho_{KK}]}.$$  \hspace{1cm} (11)

In the numerator, the factor 2 allows for the doubling of the $\pi\pi$ amplitude by identity of the two pions and the factor $1/\sqrt{2}$ allows for the fact the $\pi^0$ are to be considered only over $2\pi$ solid angle. However, experimentalists do not write the Breit-Wigner amplitude in this way. They ignore identical particle effects and write it in all cases as

$$f = \frac{G_{K^+K^-\rho\pi\pi}/\sqrt{3}}{M^2 - s - i[G^2_{\pi\pi}\rho\pi\pi + G^2_{KK}\rho_{KK}]}.$$  \hspace{1cm} (12)

where $G = g\sqrt{2}$. It is essential to remember this convention when using Breit-Wigner parameters deduced in the BES experiment. The upshot is that in comparing with Kloe data, one should use eqn. (12) and there is no effect from the identity of the pions in the final state.

2.3 How to treat the $\sigma$ pole

In the KLOE analysis of $\phi \rightarrow \gamma(\pi^0\pi^0)$, a further contribution was included from the $\sigma$ pole. This requires some discussion. For elastic $\pi\pi$ scattering, the amplitude may be written

$$f_{el} = \frac{N(s)}{D(s)}$$  \hspace{1cm} (13)

and there is a zero in $N(s)$ at the Adler point $s \sim 0.5m^2$, just below threshold [17]. The Adler zero is a basic feature of Chiral Perturbation Theory and figures prominently in the series of papers on $\sigma$, $\kappa$, $f_0(980)$ and $a_0(980)$ by Oset and collaborators [18-23]. Data on $J/\Psi \rightarrow \omega\sigma$ may be fitted in both magnitude and phase using

$$f_{prod} \propto \frac{1}{D(s)}$$  \hspace{1cm} (14)
with the same denominator but a constant numerator [24]. The justification is that left-hand singularities due to coupling of $\sigma$ to $\omega J/\Psi$ are very distant. The question is whether the Adler zero is needed in the amplitude for fitting KLOE data or not.

An interesting intermediate case is the production of the $\pi\pi$ S-wave in $\Upsilon' \rightarrow \Upsilon \pi\pi$. Recent BELLE data determine $\pi\pi$ mass spectra for the transitions $4S \rightarrow 1S, 3S \rightarrow 1S$ and $2S \rightarrow 1S$ [25]. The first and last follow elastic scattering closely. The transition $3S \rightarrow 1S$ deviates from this and shows a hint towards the $\pi\pi$ threshold, reminiscent of that observed in $J/\Psi \rightarrow \omega\sigma$, where the Adler zero is absent. The combined mass of $\Upsilon(3S)$ and $2\pi$ lies close to the $BB$ threshold, and Moxhay suggests that the decay proceeds through coupling to $B\bar{B}$ intermediate states [26]. These data suggest that for small momentum transfers the Adler zero is needed in the amplitude unless there is a flavour change. From Fig. 1 for $\phi$ radiative decays, it is plausible that the right-hand vertex should be the same as that in the on-shell $KK \rightarrow \pi\pi$ process, since the kinematics are similar. It will be shown that this successfully fits the data.

2.4 $\rho \rightarrow \eta\gamma$ and $\pi^0\gamma$

There are also amplitudes due to $\phi \rightarrow \pi^0\rho$, $\rho \rightarrow \eta\gamma$ and $\rho \rightarrow \pi^0\gamma$. Formulae for their intensities and interferences with $a_0$ or $f_0$ are given by Bramon et al. [27] and Achasov and Gubin [28]; a small correction to [28] for the intensity of $\phi \rightarrow \rho\pi$, $\rho \rightarrow \eta\gamma$ is made by Achasov and Kiselev [29]. Also eqn. (20) of Ref. [28] requires an additional factor $(m_\phi^2 - s)$ in the numerator; this correction is made in a second paper of Achasov and Kiselev [30] in their eqn. (21).

The $\rho\pi$ contribution to $\gamma(\pi^0\pi^0)$ peaks at $m_{\pi\pi} = 550$ MeV and is small. Interferences with $f_0(980)$ have rather little effect on the fit to $\gamma(\pi^0\pi^0)$. However, the contribution of $\rho \rightarrow \eta\pi^0$ is quite significant in $\gamma(\eta\pi^0)$.

Branching fractions for these processes will now be discussed, starting with $\rho \rightarrow \pi^0\gamma$. Achasov et al. report $\Gamma(\rho \rightarrow \gamma\pi^0) = 73.5 \pm 11$ keV [31]. This agrees with the PDG average for $\Gamma(\rho^0 \rightarrow \gamma\pi^\pm)$ of $68 \pm 7$ keV. The weighted average is $70 \pm 6$ keV. The PDG gives an average for $\Gamma(\phi \rightarrow \rho\pi + \pi^+\pi^-\pi^0)/\Gamma_{total} = 0.151 \pm 0.009$. Dividing this by 3 for neutral final states and assuming it is all $\rho\pi$, the branching fraction of $\phi \rightarrow \rho\pi^0 \rightarrow \pi^0\pi^0\gamma$ is $2.4 \times 10^{-5}$.

The experimental groups apply cuts to remove $\phi \rightarrow \omega\pi^0$, $\omega \rightarrow \pi^0\gamma$. It is hard to estimate the fraction of $\rho\pi^0$ events which this removes; a sharp cut in the $\rho$ line-shape at the quoted mass cut leaves $\sim 40\%$ of $\rho\pi^0$ events, i.e. a branching fraction of $\phi \rightarrow \pi^0\pi^0 \sim 0.96 \times 10^{-5}$. The uncertainty is not serious, since most events lie at low $\pi^0\pi^0$ masses and have little effect on the $f_0(980)$ peak.

The arithmetic for the $\rho\pi^0$ contribution to $\gamma(\eta\pi^0)$ goes similarly. The weighted mean of SND [31] and CMD2 [32] values for $\Gamma(\rho \rightarrow \gamma\eta)/\Gamma_{total}$ is $(2.95 \pm 0.26) \times 10^{-4}$. Taking the $\phi \rightarrow \rho\pi^0$ branching fraction to be $0.151/3$, as above, and using the KLOE branching fraction $\phi \rightarrow \gamma(\eta\pi^0) = 0.83 \times 10^{-4}$, the ratio $\Gamma(\phi \rightarrow \rho\pi, \rho \rightarrow \gamma\eta)/\Gamma(\phi \rightarrow all \gamma\eta\pi^0) = 0.18 \pm 0.02$. These events do contribute significantly to the fit; interference with $\gamma a_0(980)$ is included.

2.5 Formulae for $\pi\pi \rightarrow KK$

There is a major question how to combine the amplitudes for $f_0(980)$ and the broad $\sigma$. Data on $\pi\pi$ elastic scattering from the Cern-Munich experiment and
elsewhere show that phase shifts of $f_0(980)$ and $\sigma$ add to a good approximation. Both go through $90^\circ$ near 1 GeV and the phase shift for the full amplitude passes rapidly through $180^\circ$ just below 1 GeV. A natural and successful way of describing this situation is the Dalitz-Tuan prescription [33], where S-matrices of the two components are multiplied:

$$S_{total} = S_A S_B = \eta_A \eta_B e^{i(\delta_A + \delta_B)}.$$  

(15)

However, it is not obvious that it is right to multiply elasticities. An alternative approach is to use the K-matrix. It is conventional to add K-matrix elements for the two components [12]. The penalty in this approach is that each K-matrix pole appears where the total phase goes through $90^\circ$; the $f_0(980)$ is then described as a delicate interference between two poles displaced substantially from 1 GeV. In the present work that is inconvenient, since $f_0(980)$ is the focus of attention in $\phi$ decays; a formula is needed to parametrise it directly. Furthermore, it is not obvious that Nature chooses to add K-matrix elements.

Since $K \propto \tan \delta$, an alternative choice is

$$K_{total} = \frac{K_A + K_B}{1 - K_A K_B}$$  

(16)

so that phases again add below the $KK$ threshold. However, this alternative fails to fit $\pi\pi \to KK$ data immediately above the $KK$ threshold. This is because inelasticity of the $\sigma$ amplitude grows fairly slowly above threshold, so eqn. (16) demands a rather elastic amplitude there.

One cannot add inelasticities of $f_0(980)$ and $\sigma$, since this leads to values outside the unitary circle. The approach adopted here is pragmatic. The Dalitz-Tuan prescription is simple, fits the phase of elastic scattering adequately, predicts a reasonable result for the elasticity parameter $\eta$ and successfully fits the data. The $\eta$ parameter for the BES $f_0(980)$ drops precipitously to $\eta = 0.27$ at 1.01 GeV and then rises again (see Fig. 11(a) below). Since $f_0(980)$ and $\sigma$ are approximately in phase, it seems likely that the effect of the $\sigma$ component will be to increase the inelasticity to a value somewhere between 0 and the value for $f_0(980)$ alone. In practice, multiplying the S-matrix elements for $S_{11}$ and $S_{22}$ describes the data well up to 1.2 GeV, obtaining $S_{12}$ from the exact relation

$$2|S_{12}|^2 = 1 + |S_{33}|^2 - |S_{11}|^2 - |S_{22}|^2,$$  

(17)

where indices 1,2,3 refer to $\pi\pi$, $KK$ and $\eta\eta$. Note, however, that $S_{12}$ itself does not factorise into $f_0(980)$ and $\sigma$ components. This is a symptom of extensive multiple scattering. It may well be that there is no simple formula which fully describes strong coupling of the two amplitudes.

Above 1.2 GeV, the appearance of $f_0(1370)$, $f_0(1500)$ and $f_0(1790)$ makes the treatment via the Dalitz-Tuan prescription intractable. There, amplitudes are small enough that unitarity corrections are less than normalisation errors, and it is adequate to add amplitudes. That is the procedure adopted here, making a smooth join at 1.17 GeV using Lagrange multipliers to ensure that the magnitude and phase join continuously. For the $\eta\eta$ channel, amplitudes are so small that unitarity plays no significant role, so one can add amplitudes.

At the higher masses, $f_0(1370)$, $f_0(1500)$ and $f_0(1790)$ overlap significantly. The overlap makes the phase of the background below each resonance a sensitive
parameter. In order to achieve a satisfactory fit, it is necessary to multiply each amplitude by a fitted phase factor exp(iθ) and add amplitudes.

In summary, multiplying η parameters near threshold puts a lower bound on the intensity 92% of the unitarity limit, and reduces the normalisation uncertainty from 50 to 8%. At higher masses, the amplitude is fitted freely in terms of known resonance, ignoring possible multiple scattering from one resonance to another where they overlap. This allows maximum freedom to fit σ and f_0(980), the main object of the present work.

An important point is that the f_0(980) signal appears as a sharp dip in elastic scattering, because phases add. In φ radiative decays, it appears as a peak. In this case, it is appropriate to add amplitudes, since they are weak and the unitarity limit imposes no significant constraint; in elastic scattering, the amplitude must follow the unitarity limit up to the KK threshold.

2.6 The 4π channel

Both ππ → KK and ππ → ηη cross sections drop fairly rapidly with increasing mass, and almost disappear above 1.9 GeV. To some extent, this is due to competition from the 4π channel. Unfortunately, there are almost no data on the 4π channel, so the s-dependence of this inelasticity is poorly known. As in the work of Bugg, Sarantsev and Zou in 1996 [13], 4π phase space is parametrised empirically by a formula close to a Fermi function:

\[ \rho_4(s) = \frac{\sqrt{1 - 16m_π^2/s}}{1 - \exp[\Lambda(s_0 - s)]}. \] (18)

Here Λ = 2.85 GeV^{-2} and s_0 = 2.5 GeV^2. This function approximates closely the onset of ρρ and σσ final states.

2.7 The ππ S-wave

The propagator of the broad σ amplitude is written in eqn. (20) as the sum of contributions from ππ, KK, ηη and 4π, labelled 1–4. In the first three, the Adler zero is accommodated in eqn. (21) by a factor

\[ A(s) = (s - s_A)/(M^2 - s_A), \] (19)

where s_A = (0.41 ± 0.06)m_π^2. The factor 0.41 comes from the work of Leutwyler and Colangelo, using Chiral Perturbation Theory [34]. A form factor FF_i(s) = exp(-αk_i^2) is included into channels 2 and 3 (KK and ηη) in eqns. (22) and (23). Note that FF_i(s)^2 is multiplying the intensity of the KK channel, which is the quantity appearing in the width of the σ. It is required to parametrise the rapid drop of cross sections with increasing mass. The same α is used for both channels; k_i is the centre of mass momentum in each channel. The propagator is then written:

\[
\begin{align*}
D(s) & = M^2 - s - g_1^2 z_{sub} - m(s) - i[g_1^2 \rho_1 + g_2^2 \rho_2 + g_3^2 \rho_3 + g_4^2(s)] \quad (20) \\
g_1^2 & = B_1 \exp[-(s - M^2)/B_2]A(s) \quad (21) \\
g_2^2 & = r_2 g_1^2 FF_2^2(s) \quad (22) \\
g_3^2 & = r_3 g_1^2 FF_3^2(s) \quad (23) \\
FF_4(s) & = \exp(-\alpha_i k_i^2) \quad (24)
\end{align*}
\]
\[
g^2 = B_4 \rho_{4\pi}(s)/\rho_{4\pi}(M^2) \quad (25)
\]
\[
z(s) = \frac{1}{\pi} \left[ 2 + \rho_1 \ln \frac{1 - \rho_1}{1 + \rho_1} \right] \quad (26)
\]
\[
z_{\text{sub}} = z(s) - z(M^2) \quad (27)
\]
\[
m(s) = \frac{s - M^2}{\pi} \int \frac{\Gamma_{\text{tot}}(s')ds'}{(s' - s)(s' - M^2)}. \quad (28)
\]

For other resonances, the exponential of eqn. (21) and the form factors \(FF\) are superfluous because the resonances are narrow; they are simply omitted, but the dispersive term \(m(s)\) is retained. The amplitude for \(\pi\pi\) elastic scattering is

\[
T_{11} = 1 + 2iS_{11} \quad (29)
\]
\[
S_{11} = \eta \exp(2i\delta) = g^2\rho_1/D(s). \quad (30)
\]

In fitting elastic data, S-matrices from \(\sigma, f_0(980), f_0(1370), f_0(1500)\) and \(f_0(1790)\) are multiplied together.

The function \(g_1(s)\) was introduced by Zou and Bugg [35]. It takes a very simple empirical form which succeeds in fitting the broad S-wave over the whole mass range from threshold to 1.9 GeV using just the Adler zero (taken from other work) and three fitted parameters \(B_1, B_2\) and M. At the KK and \(\eta\eta\) thresholds, \(r_2 = g_{KK}^2/g_{\pi\pi}^2\) and \(r_3 = g_{\eta\eta}^2/g_{\pi\pi}^2\).

The motivation for the formalism is to make amplitudes fully analytic. This has been achieved below the region of strong \(4\pi\) elasticity by Achasov and Kiselev [30]. Here the opening of the strong \(4\pi\) channel is treated analytically for the first time. This will be important in demonstrating the presence of \(f_0(1370)\); otherwise, the issue of analyticity turns out not to be crucial for present work. The function \(z(s)\) accounts analytically for the opening of \(\pi\pi\) phase space \(\rho_1(s)\); it is taken from the work of Leutwyler and Colangelo [34]. It has the good feature of eliminating the divergence at \(s = 0\) due to \(\sqrt{1/s}\) in \(\rho_1(s) = \sqrt{1 - 4m_s^2/s}\). In eqn. (27), a subtraction is made at \(s = M^2\), where the real part of the S-wave amplitude goes to zero.

![Figure 2: The dispersive term \(m(s)\) for the \(4\pi\) channel.](image)

The term \(m(s)\) is a dispersive contribution to the real part of the amplitude, allowing for the inelasticity in all channels. The inelasticity in \(4\pi\) is large and gives rise to a slowly varying dispersive term illustrated in Fig. 2. It is far too wide to be confused with \(f_0(1370), f_0(1500)\) or \(f_0(1790)\). The dispersive term
$m(s)$ is also included into the Breit-Wigner forms for $f_0(1370)$, $f_0(1500)$ and $f_0(1790)$, again with a subtraction on resonance. This is important for $f_0(1370)$, where $\Gamma_{4\pi}$ is dominant.

One new result arises concerning the position of the $\sigma$ pole. Fitting it with the new fully analytic formulae described above moves its pole position to $(506 \pm 30) - i(238 \pm 30)$ MeV from the BES value of $(541 \pm 39) - i(252 \pm 42)$ MeV \cite{36}. The shift arises from the elimination of the singularity at $s = 0$ and dispersive corrections to the $\pi\pi$ channel. It is within the experimental errors, which now go down because of better control over the extrapolation to the pole.

![Figure 3: The fit to KLOE data \cite{5} where (a) $\eta \to \gamma\gamma$, (b) $\eta \to 3\pi$; in (a), dashed and dotted curves show the $\rho\pi$ contribution and interference with $a_0(980)$; (c) the energy dependent factor appearing in $d\Gamma/dm$, but omitting $|I(a,b)|^2$, (d) $|I(a,b)|^2$ itself.](image)

3 Fit to $\phi \to \gamma(\eta\pi^0)$

If parameters of $a_0(980)$ are fitted directly to $\phi \to \gamma\eta\pi^0$ data of Ref. \cite{5}, the problem is that the upper side of the resonance is not visible, resulting in values of $M$, $g^2_{\eta\pi}$ and $g^2_{KK}$ which are strongly correlated.

Most determinations quoted by the PDG have been obtained by fitting simple Breit-Wigner line-shapes with constant width to $\eta\pi$ or $KK$ alone. These are of no use for present purposes, since they ignore the cusp at the $KK$ threshold.

A rather precise determination, not quoted by the PDG, was made in Ref. \cite{8}, combining data on $\bar{p}p \to \omega(\eta\pi^0)$ and $\bar{p}p \to (\eta\pi^0)\pi^0$ at rest. Both have high statistics (280K events for $\eta\pi^0\pi^0$) and very low experimental backgrounds. The first reaction is dominated by $\omega a_0(980)$. Although a simple Breit-Wigner of constant width was fitted, the data determine precisely the location of the $a_0$ peak at half-height; the error quoted on the central mass is $1.23(\text{stat}) \pm 0.34(\text{syst})$.
MeV and for the width is $0.34^{(stat)} \pm 0.12^{(syst)}$ MeV. Those constraints go a long way towards breaking the correlation between couplings to $\eta\pi$ (which dominates the lower side of the resonance) and $KK$ (which dominates the upper side). In $\eta\pi^0\pi^0$, the $a_0(980)$ appears very conspicuously in the Dalitz plot as an ‘edge’ at the $KK$ threshold; at this edge, the phase turns abruptly by $90^\circ$ in the Argand diagram. The $a_0$ interferes with the broad $\sigma$ in the $\pi\pi$ channel. This interference determines rather precisely the phase variation of the $a_0$ as a function of $\eta\pi$ mass. On the other hand, there is some uncertainty in the parametrisation of the $\sigma$ as a function of $\pi^0\pi^0$ mass. Taking into account all uncertainties, quoted parameters are $M = 999 \pm 5$ MeV, $g_{\eta\pi}^2 = 221 \pm 20$ MeV and $r = g_{KK}^2/g_{\eta\pi}^2 = 1.16 \pm 0.18$.

The fit to KLOE data with these parameters immediately reproduces the line-shape accurately. The fitted normalisation is low by 32%, which is close to the combined errors, as follows. From $g_{\eta\pi}^2$, there is an error of 19% in the magnitude predicted for KLOE data, and from $r$ there is a further error of 15%. The error quoted by the PDG for the normalisation of $\phi \rightarrow \gamma \eta\pi$ is 6%. To achieve agreement in normalisation, it is necessary to stretch all parameters by $1\sigma$ as follows. For $a_0$, $g_{\eta\pi}^2$ has been increased to 241 MeV, and $r = g_{KK}^2/g_{\eta\pi}^2$ to 1.34; this requires reoptimising $M = 991.5$ MeV. The branching fraction of $\rho \rightarrow \eta\pi$ has been increased to 0.20 and the normalisation of KLOE data has been decreased by 6%. The final fit is shown in Figs 3(a) and (b). Dashed and dotted curves show the intensity of the $\rho \rightarrow \eta\pi$ signal and its interference with $a_0(980)$. A minor detail is that a form factor $\exp(-\beta k_{\pi}^2)$ is included to allow for the charge radius of the kaon, 0.560 fm. [16].

Figs. 3(c) and (d) illustrate the strong energy-dependent factors contributing to the intensity. Fig. 3(d) shows $|I(1,b)|^2$ from the $KK$ loop. It falls dramatically as the kaons go off the mass-shell. Fig. 3(c) illustrates the remaining factors from $mk_{\rho\pi}^2$, eqn. (2). These factors inflate the lower side of $a_0(980)$. The agreement with the $a_0$ parameters of Ref. [8] for both line-shape and normalisation confirms Achasov’s $KK$ loop model at the level of $\sim 15\%$ in amplitude. That conclusion will be important in considering the $\gamma\pi^0\pi^0$ data.

![Figure 4: Fit to KLOE $\gamma(\pi^0\pi^0)$ data with $f_0(980) + \rho\pi$ when the normalisation is floated.](image)
Figure 5: The magnitude of the $\pi\pi \rightarrow KK$ amplitude from Böttiker et al [39] fitted using the Roy equations and experimental data of Cohen et al [40].

4 Fit to $\phi \rightarrow \gamma(\pi^0\pi^0)$

KLOE data on $\phi \rightarrow \gamma(\pi^0\pi^0)$ [4] can be fitted fairly well with BES parameters for $f_0(980)$ if the absolute normalisation is floated, see Fig. 4. The best fit is obtained by fitting the form factor $FF$ freely and stretching BES parameters to the limits of their quoted statistical and systematic errors as follows: $M = 0.957$ GeV, $g_1^2 = 0.190$ GeV, $g_2^2/g_1^2 = 3.75$. However, $\chi^2 = 122$, compared with 60 for the fit shown later in Fig. 6 including $\sigma \rightarrow KK$. The data are taken from Fig. 4 of the KLOE publication [4] after subtracting their estimated background; they are not taken from their Table 5, where the background has been unfolded in such a way as to accomodate a dip near 520 MeV.

There is however a serious problem with the absolute normalisation. The fit with $f_0(980)$ alone has a normalisation at least a factor 2 lower than the data. The program is identical to that used for $\gamma\eta\pi^0$ except for trivial changes in the parameters of the Breit-Wigner amplitude and the change from the mass of the $\eta$ to the $\pi^0$. A direct comparison of the relative normalisation of $\gamma\pi^0\pi^0$ and $\gamma\eta\pi^0$ has been made using eqn. (10).

A variety of explanations for the normalisation are possible, but look odd in view of the agreement in normalisation for $\gamma\eta\pi^0$ data. Firstly, Oller remarks that there may be a contact term involving two $K^0$ in the loop diagram, and a photon radiated from the $\phi$ decay vertex [37]. Secondly, Oset and collaborators have suggested that $\phi$ may decay through intermediate states involving $K^*K$, where $K^*$ may be $K^*(890)$, $K_1(1270)$ or $K_{11}(1400)$ [38].

A further alternative is proposed here. This is that the broad $\sigma$ couples to the $KK$ loop. It will be shown in Section 5 that data on $\pi\pi \rightarrow KK$ require this broad component. Independent support is available from the work of Böttiker, Descotes-Genon and Moussallam [39]. They apply the Roy equations to fitting data on $\pi K \rightarrow \pi K$ and $\pi\pi \rightarrow KK$. For the present discussion, this corresponds to calculating the $\pi\pi \rightarrow KK$ amplitude below the $KK$ threshold as a continuation of the amplitude above threshold using (i) the dispersion integral over the physical $\pi\pi \rightarrow KK$ process, (ii) nearby pole terms due to $t$ and $u$-channel exchanges of $\rho(770)$ and $K^*(890)$. Their Figure
Figure 6: (a) Fit to KLOE $\gamma\pi^0\pi^0$ data after background subtraction; (b) contributions from $f_0(980)$ (full curve), $\sigma$ (dashed) and interference (dotted); (c) an enlarged view of the contribution from $\rho\pi$, $\rho \rightarrow \gamma\pi^0$; (d) $|g_0^0|$ from my fit (full curve) and the contribution from $\sigma$ alone (dashed).

For the fit presented here, the $f_0(980)$ contribution is fixed in magnitude to BES parameters. The charge form factor of the photon to $KK$ is fitted freely. The magnitude of $\sigma \rightarrow KK$ optimises with a threshold ratio $r_2 = g_{KK}^2/g_{\pi\pi}^2 = 0.6 \pm 0.1$. Its phase with respect to $f_0(980)$ requires some comment. The production process is electromagnetic, and will produce a phase shift only of order $\alpha = 1/137$. In $\pi\pi$ elastic scattering, both $\sigma$ and $f_0(980)$ have a phase shift of $\sim 90^\circ$ at 990 MeV. Their relative phase is constrained in fitting KLOE data within $\pm 10^\circ$, the combined error from $\sigma$ phases and the mass of $f_0(980)$. It fits naturally at the edge of this band; if the constraint is removed, $\chi^2$ changes by only 2 and the fit hardly changes.

The resulting fit is shown in Fig. 6(a). Panel (b) displays the $f_0(980)$ intensity as the full curve, the $\sigma$ as the dashed curve and the interference between them as the dotted curve. Panel (c) shows the fitted $\rho\pi$ signal (full curve) compared with data at low masses. Its magnitude is fixed according to the arithmetic of Section 2.4. A marginal improvement may be obtained by fitting.
it freely, but there is no significant effect on the fitted $\sigma$ amplitude.

Fig. 6(d) shows $|g_0|$ from my fit to $\pi\pi \to KK$ as the full curve. The peak due to $f_0(980)$ is slightly lower than that in Fig. 5 because the $f_0(980)$ of BES is slightly broader than that of Büttiker et al. Secondly, the fit requires a fairly strong form factor to cut off the lower sides of both $f_0(980)$ and $\sigma$. It is parametrised as $\exp(-8.4k^2/\gamma)$, where $k$ is in GeV. Comments on the origin of this factor will be given in Section 4.1.

Figure 7: Fits to Novosibirsk data on $\phi \to \gamma(\eta\pi^0)$ [2] and $\phi(\pi^0\pi^0)$ [3].

For completeness, Fig. 7 shows fits to Novosibirsk data on $\phi \to \gamma(\eta\pi^0)$ and $\gamma^*\pi^0$. The point in $\gamma(\eta\pi^0)$ at 0.98 GeV is clearly too high and is slightly inflating the PDG average for the branching ratio $\phi \to \gamma(\eta\pi^0)$.

### 4.1 Sensitivity to fitting parameters

The heights of the peaks at 970 MeV in both $\gamma(\eta\pi^0)$ and $\gamma(\pi^0\pi^0)$ have limited sensitivity to resonance parameters. In the denominators $D(s)$, $g_{\pi\rho\pi}^2$ or $g_{\eta\rho\pi}^2$ appear in the imaginary part, while $M$ and $g_{KK}^2\sqrt{4M^2_K/s-1}$ appears in the real part. There is considerable compensation between the numerator $N(s)$ and denominator $D(s)$ of the Breit-Wigner amplitudes. For $\gamma(\eta\pi^0)$, it is necessary to constrain the location of the $a_0$ peak at half-height within, say, 2 or 2.5 standard deviations of the values quoted in Section 3, adding statistical and systematic errors linearly. With this constraint, $M$, $g_{\pi\pi}^2$ and $g_{KK}^2$ may be varied freely. Because of the feedback between $N(s)$ and $D(s)$, it is only just possible to accommodate the 32% discrepancy by including also flexibility in the $\rho\pi$ contribution and uncertainty in the KLOE normalisation.

For $f_0(980)$, the BES $\pi\pi$ peak must again be constrained in mass and the full width must be constrained to $34 \pm 4$ MeV. There is more flexibility in $g_{KK}^2$, since the BES $f_0(980) \to KK$ signal interferes with a background due to $\sigma \to KK$; the error on $g_{KK}^2$, namely $\pm0.25$(stat) $\pm 0.21$(syst) quoted by BES, represents one standard deviation. It turns out that with this flexibility, the absolute normalisation of the $f_0(980)$ peak cannot change by more than $\pm15\%$. Some fits reported in the literature have changed $g_{\pi\pi}^2$ and $g_{KK}^2/g_{\pi\pi}^2$ by factors of 2; that is unrealistic.

The peak at 970 MeV in KLOE data determines the magnitude of the background amplitude there. The fit is not unduly sensitive to the parametrisation of $\pi\pi$ phase shifts. What matters for $\gamma(\pi^0\pi^0)$ is only the way the $\sigma$ and $f_0(980)$ amplitudes go out of phase off resonance. Any form which reproduces $\pi\pi$ phase
shifts within errors of, say, ±4° is adequate. The full apparatus of eqns. (18)–(30) has been used to scrutinise the combined fit with \( \pi\pi \rightarrow \pi\pi, KK \) and \( \eta\eta \). An equivalent and more convenient form which fits all the data equally well may be obtained by (i) dropping the dispersive contribution \( m(s) \) completely, (ii) also the tiny contribution from the \( 4\pi \) channel below the \( KK \) threshold, (iii) replacing the effect of \( m(s) \) for \( KK \) and \( \eta\eta \) below their thresholds by a form factor \( \exp(-5.2|k_K^2|/\pi) \), with \( k_K \) in GeV/c. Then \( M = 0.7366 \) GeV, \( B_1 = 0.6470 \) GeV\(^2\), \( B_2 = 4.271 \) GeV\(^2\), \( B_4 = 0.00221 \) GeV\(^2\). Values of other parameters will be discussed below but will be collected here for convenience of reference: \( r_3 = 0.20, r_2 = 0.6 \).

A crucial parameter in fitting \( \sigma \rightarrow KK \) in KLOE data is the form factor of Eqn. (24), \( FF = \exp(-8.4k_s^2) \), where \( k_s \) is in GeV. Without this factor, the background amplitude is too large at low mass; one can see this in the difference between Figs. 5 and 6(d). This form factor has at least three possible origins.

Firstly, it may be due to the size of the \( KK \) cloud. This would require an RMS radius of 1.4 fm. It is well known that a source of finite size produces a form factor \( \sin(kr)/kr = 1 - k^2r^2/6 + \ldots \). The second sheet pole of \( f_0(980) \) lies at \((998\pm4) - i(17\pm4) \) MeV. The binding energy is so close to the \( KK \) threshold that the \( f_0(980) \) inevitably has a \( KK \) cloud of roughly this size, resembling the long-range tail of the deuteron. The pole for \( a_0(980) \) is much further away, at 1032 – 85 MeV, so the RMS radius of \( a_0(980) \) has conventional dimensions and a weaker form factor.

A second straightforward possibility is that \( f_0(980) \) and the broad \( \sigma \) component mix over the mass range where they are both large. This mixing will be confined naturally to their region of strong overlap.

A third possibility is that \( |g_0^0| \) fitted by Büttiker et al. has some flexibility in the range \( s = 0.2 - 0.8 \) fm. It is well constrained near 1 GeV, and also for \( s = -0.5 \) to 0 GeV\(^2\) (by data on \( \pi K \rightarrow \pi K \)). In between, Adler zeros in \( \pi\pi \) at \( s \sim 0.5m_{KK}^2 \) and in coupling to \( KK \) at \( s \sim 0.5m_{KK}^2 \) may pull \( |g_0^0| \) down. Data on \( \phi \rightarrow \gamma(\pi^0\pi^0) \) appear to be the only data directly sensitive to this possibility.

5 Fits to \( \pi\pi \rightarrow KK \) and \( \eta\eta \)

5.1 Introduction

A problem with data on \( \pi\pi \rightarrow KK \) is that cross sections from different experiments differ in normalisation by up to a factor 2. This problem is minimised here by using the BES parameters for \( f_0(980) \) to assist the absolute normalisation. All \( \pi\pi \rightarrow KK \) data are rescaled to the BES normalisation. However, one can still not rely on \( \pi\pi \rightarrow KK \) data to separate branching fractions of \( f_0(1370) \), \( f_0(1500) \) and \( f_0(1790) \) to \( KK \) and \( \eta\eta \). Vastly better determinations are available from the analysis of Crystal Barrel data on \( pp \) annihilation at rest to \( 3\pi^0, \eta\pi^0\pi^0, \eta\eta\pi^0 \) and several \( KK \pi \) charge combinations. All of these data have been fitted self-consistently by Sarantsev and collaborators; decay widths to \( \pi\pi \) and \( \eta\eta \) are taken from this analysis [41]. Their errors are used to set limits within which parameters may vary; Table 1 shows these limits and values at which they optimise. Note however, that the present data really only determine the products \( \Gamma_{\pi\pi} \times \Gamma_{KK} \) and \( \Gamma_{\pi\pi} \times \Gamma_{\eta\eta} \). Values of \( \Gamma_{4\pi} \) are determined from \( \Gamma_{4\pi} = \Gamma_{\text{total}} - \Gamma_{\pi\pi} - \Gamma_{KK} - \Gamma_{\eta\eta} \) at each resonance mass.

Some comments are needed on \( f_0(1370) \) and \( f_0(1790) \). There has been some unnecessary controversy recently concerning the existence of \( f_0(1370) \), so its
Table 1: Limits used in the fit (in MeV); optimised values are in parentheses.

| State | $f_0(1370)$ | $f_0(1500)$ | $f_0(1790)$ |
|-------|-------------|-------------|-------------|
| $M$   | 1290–1335 (1290) | 1495–1510 (1501) | 1770–1800 (1800) |
| $\Gamma_{total}$ | 200–280 (280) | 110–135 (135) | 180–280 (280) |
| $\Gamma_\pi$ | 34–58 (48) | 35–39 (35) | 30–100 (72) |
| $\Gamma_{KK}/\Gamma_\pi$ | 0.04–0.235 (0.235) | 0.22–0.272 (0.272) | 0.16–1.2 (0.47) |
| $\Gamma_{\eta\eta}/\Gamma_\pi$ | 0.08–0.18 (0.18) | 0.04–0.18 (0.11) | 0.1–1.0 (0.20) |

The properties will be outlined briefly; it will be the subject of a separate publication giving full details from latest analyses of Crystal Barrel data. It appears most clearly in $\bar{p}p \rightarrow 3\pi^0$, where it is statistically at least a 50 standard deviation signal. Statistics are 600K events, with almost no experimental background. In early analyses of these data, there was some sensitivity to the parametrisation of the $\pi\pi$ S-wave amplitude, with which it interferes. The behaviour of the broad S-wave amplitude is now known much better.

The $f_0(1370)$ decays dominantly to $4\pi$ with $\Gamma(\pi\pi)/\Gamma(4\pi) \leq 0.2$. The $4\pi$ inelasticity increases rapidly with mass, so it is essential to treat the $s$-dependence of this width fully. The result is that the $\pi\pi$ signal peaks at $1320 \pm 30$ MeV, as in the analysis of Anisovich et al. [41]. The $4\pi$ channel peaks approximately 75 MeV higher, because of the rapidly expanding phase space. The apparent width in $\pi\pi$ and $4\pi$ channels is likewise different. PDG listings assign unreasonably large errors because these facts have not been taken into account in most analyses. The $s$-dependence of the $4\pi$ width is taken into account here, including the associated dispersive correction to $m(s)$ in the Breit-Wigner amplitude.

In elastic scattering, the intensity of $f_0(1370)$ is $<4\%$ of the unitarity limit. It is swamped by the $f_2(1270)$, which has a spin multiplicity 5 times larger. Achasov and Shestakov point out a rather large error in $\Gamma_{4\pi}$ for $f_2(1270)$ [42]. This adds to the difficulty of normalising Cern-Munich moments in the region of $f_0(1370)$.

The $f_0(1790)$ is required as well as $f_0(1710)$ by recent BES data. The $f_0(1710)$ appears clearly in $J/\Psi \rightarrow \omega K^+K^-$ [43]. It is conspicuously absent in $J/\Psi \rightarrow \phi\pi^+\pi^-$ [36]. Those two sets of data require $\pi\pi/KK < 0.11$ for $f_0(1710)$ with 95% confidence. In $J/\Psi \rightarrow \phi\pi^+\pi^-$, there is a distinct $\pi\pi$ signal at 1790 MeV, but no significant signal in $\phi K^+K^-$ [10]. The branching fraction $\pi\pi/KK$ is a factor 25 larger than allowed for $f_0(1710)$, hence requiring a separate resonance $f_0(1790)$. There was previous evidence for it in $J/\Psi \rightarrow \gamma 4\pi$ [44,45] and $\bar{p}p \rightarrow \eta\eta\pi^0$ [46]. Here, both $f_0(1710)$ (with its upper limit for $\pi\pi$ coupling) and $f_0(1790)$ (with its upper limit for $KK$ coupling) are included. However, the present data do not distinguish cleanly between them.

### 5.2 $\pi\pi \rightarrow \eta\eta$

Fig. 8 shows fits to $\pi\pi \rightarrow KK$ data and $\eta\eta$. The fit to $\eta\eta$ in panel (d) will be discussed first, since it is simple and concerns only one set of data, from GAMS [47]. The normalisation is such that the unitarity limit is at an intensity of 0.25; this arises from the relation $T_{12} = S_{12}/2i$ between T and S-matrices. The full curve shows the overall fit, including contributions from $\sigma$ (dotted curve), $f_0(1370)$, $f_0(1500)$ and $f_0(1790)/f_0(1710)$. The main features are a gradually
falling $\sigma \rightarrow \eta\eta$ signal and destructive interference with $f_0(1500)$, which is mainly responsible for the dip at 1.45 GeV. The peak at higher masses comes from $f_0(1500)$ interfering with $f_0(1710)$ and $f_0(1790)$. The Argand diagram is shown in Fig. 9(b).

The fitted ratio $r_3 = g_3^2/g_1^2$ at the $\eta\eta$ threshold is $0.20 \pm 0.04$. The error arises from normalisation uncertainty and a possible weak coupling of $f_0(980)$ to $\eta\eta$; an upper limit on this coupling is obtained from BES data on $J/\psi \rightarrow \phi\pi\pi$ and $\phi K^+K^-$, from the fact that no drop is observed at the $\eta\eta$ threshold in the $\pi\pi$ mass spectrum. This upper limit is $r_3 \simeq 0.33$.

### 5.3 $\pi\pi \rightarrow KK$

It is first necessary to review the many available sets of data. It is important to realise that an analysis of moments was required in order to separate spins $J = 0, 1$ and 2 and small amounts of higher $J$. The papers make clear that there is some cross-talk with the high mass tail of $\rho(770)$ and with $f_2(1270)$, whose branching ratio to $KK$ is not particularly well known.
Figure 9: Argand diagrams for (a) $\pi\pi \rightarrow KK$ and (b) $\pi\pi \rightarrow \eta\eta$; masses are marked in GeV.

Costa et al. [50] produced data on $\pi^-p \rightarrow K^+K^-n$, but found 8 alternative solutions, because of ambiguities over the exchange process and the possibility of both $I = 1$ and 0 contributions to the final state. Cason et al. [51] produced data on $\pi^-p \rightarrow K^0_SK^0_S$ and observed a shoulder at 1300 MeV. They obtained two solutions, one with a narrow S-wave resonance (width $\sim 95$ MeV) at this mass and the second with a more slowly varying amplitude similar to today’s $f_0(1370)$. The situation clarified further with data from Pawlicki et al. [49] on both $\pi^-p \rightarrow K^+K^-n$ and $\pi^+n \rightarrow K^-K^+p$. These data favoured the second solution of Cason et al. and assigned $I = 0$. A meticulous analysis of the $t$-dependence of data was made by Martin and Ozmutlu [52]. This analysis showed that $\pi$ exchange dominates at low $t$, as expected. The paper of Polychronakos et al. [53] on $\pi^-p \rightarrow nK^0_SK^0_S$ is a full length paper on the data of Cason et al. The publication of Cohen et al. [40] gives a revised partial wave analysis of the data of Pawlicki et al. and again favours a structure close to today’s $f_0(1370)$. Etkin et al. [54] reported data on $\pi^-p \rightarrow K^0_SK^0_Sn$. Further statistics were added in the paper of Lindenbaum and Longacre [48].

It is noteworthy that all the later sets of data [48], [40] and [53] observed a definite small bump at 1300 MeV, which was christened the $\epsilon$ at the time and is now $f_0(1370)$. All also observed a threshold peak due to $f_0(980)$. However, there are discrepancies concerning the height of the $f_0(980)$ peak with respect to the 1300 MeV mass range. The analyses of Martin and Ozmutlu and Longacre et al. agree well. The revised analysis by Cohen et al. of the Pawlicki et al. data gives a distinctly lower $f_0(980)$ than that of Longacre et al. And the data of Polychronakos et al. give an even lower $f_0(980)$ peak.

To clarify the situation including today’s information about $f_0(1370)$, the four sets of data from Lindenbaum and Longacre, Cohen et al., Martin and Ozmutlu and Polychronakos et al. were included into the initial stages of the analysis reported here. This was despite the fact that those of Martin and Ozmutlu and Cohen et al. are based on the same actual data before the moments analysis. The objective is to see whether the analysis favours one or the other. This is not the case and the fit goes midway between the two, so they are both retained in the final fit, but with a weight half that of Longacre et al.

The final fit is shown in Figs. 8(a) and (b). The $f_0(980)$ produces the threshold peak, but its tail at high mass cannot account for the remaining
features. Some $\sigma \to KK$ is definitely required and the dotted curve shows the optimum fit. It requires a form factor $FF = \exp(-5.2k^2)$ with $k$ is GeV/c. However, some of this form factor may reflect the effect of competition between $KK$ and $4\pi$ channels. The $\eta\eta$ threshold contributes a drop in the intensity of 0.02 between 1.09 and 1.18 GeV. The $f_0(1370)$ helps fit the bump at 1300 MeV. Destructive interference with $f_0(1500)$ is required to fit the dip at 1550 MeV. The fit to the 1550–1900 MeV intensity is not perfect because of difficulties in reproducing the large phase variation reported by Etkin et al. [54]. The fit to their phases is shown in panel (b). However, one should be aware that other phase determinations show systematic differences with their data: the whole curve can move up or down by up to 30$^\circ$, although the trend of the phase with mass remains similar to Fig. 8(b). These differences probably arise from the way spin 2 is fitted in different analyses. Etkin et al. included $f_2'(1525)$, but at that time the existence of $f_2(1565)$ was not known, and could lead to small systematic shifts. Another possible source of systematic error is that the present fit ends at 1900 MeV; there may be some contribution from the known $f_0(2020)$ which is presently not fitted.

The optimum fit gives $r_2 = 0.6^{+0.1}_{-0.2}$, eqn. (22), in close agreement with the KLOE data. This value can depend systematically on the s-dependence fitted to $f_0(980)$ and on $4\pi$ inelasticity. The $f_0(980)$ is fitted with a conservative form factor $\exp(-2.7k^2)$, where $k$ is kaon centre of mass momentum; this corresponds to a radius of 0.8 fm for the $\pi\pi \to KK$ interaction, i.e. a conventional radius. Fig. 8(c) compares the fit with intensities derived by Cohen et al [40] (squares) and those of Polychronakos et al. [53] (triangles). The latter are systematically low below 1.15 GeV, so they are omitted from the final fit.

Fig. 10(a) shows the fit without $f_0(1370)$. It fails to fit the dip below 1300 MeV. Since that structure is observed in 3 experiments and 4 sets of data, it is hard to deny the presence of $f_0(1370)$; its fitted mass and width agree within errors with the best determinations from Crystal Barrel [41] and BES [10]. There is presently no alternative explanation of the 1300 MeV bump. The onset of $4\pi$ inelasticity is far too slow to account for structure with the width observed around 1300 MeV, and is anyway taken into account fully in the present analysis. Fig. 10(b) also shows the change in fitting $\pi\pi \to \eta\eta$ when $f_0(1370)$ is removed; however, the points in Fig. 8(d) fluctuate around the fit by more than statistics, so it is difficult to estimate the significance of $f_0(1370) \to \eta\eta$. 

Figure 10: Fits without $f_0(1370)$ for (a) $\pi\pi$, (b) $\eta\eta$. 

The optimum fit gives $r_2 = 0.6^{+0.1}_{-0.2}$, eqn. (22), in close agreement with the KLOE data. This value can depend systematically on the s-dependence fitted to $f_0(980)$ and on $4\pi$ inelasticity. The $f_0(980)$ is fitted with a conservative form factor $\exp(-2.7k^2)$, where $k$ is kaon centre of mass momentum; this corresponds to a radius of 0.8 fm for the $\pi\pi \to KK$ interaction, i.e. a conventional radius. Fig. 8(c) compares the fit with intensities derived by Cohen et al [40] (squares) and those of Polychronakos et al. [53] (triangles). The latter are systematically low below 1.15 GeV, so they are omitted from the final fit.

Fig. 10(a) shows the fit without $f_0(1370)$. It fails to fit the dip below 1300 MeV. Since that structure is observed in 3 experiments and 4 sets of data, it is hard to deny the presence of $f_0(1370)$; its fitted mass and width agree within errors with the best determinations from Crystal Barrel [41] and BES [10]. There is presently no alternative explanation of the 1300 MeV bump. The onset of $4\pi$ inelasticity is far too slow to account for structure with the width observed around 1300 MeV, and is anyway taken into account fully in the present analysis. Fig. 10(b) also shows the change in fitting $\pi\pi \to \eta\eta$ when $f_0(1370)$ is removed; however, the points in Fig. 8(d) fluctuate around the fit by more than statistics, so it is difficult to estimate the significance of $f_0(1370) \to \eta\eta$. 

18
Fig. 11(a) illustrates the way elasticities of $f_0(980)$ and $\sigma$ combine. The dotted curve shows the result obtained by multiplying $f_0(980)$ and $\sigma$ contributions, according to the model assumed here. Fig. 11(b) shows the phase shift fitted to $\pi\pi$ elastic scattering; errors above the $f_0(980)$ are typically $\pm 15^\circ$.

6 Summary

Data on $\phi \to \gamma(\eta\pi)$ agree within errors in both absolute normalisation and mass spectrum with the parameters of $a_0(980)$ from Ref. [8]; the KLOE data suggest a normalisation which can be accommodated by shifting $a_0(980)$ parameter to $M = 991.5 \pm 4$ MeV, $g_{\eta\pi}^2 = 0.241 \pm 14$ GeV$^2$, $g_{KK}^2/g_{\eta\pi}^2 = 1.34 \pm 0.13$. The errors quoted here are reduced compared with Ref. [8] after the combined fit with KLOE data. The agreement with KLOE data suggests that the $KK$ loop model is reliable. If BES parameters for $f_0(980)$ are used, data on $\phi \to \gamma(\pi^0\pi^0)$ have a normalisation higher than the fit by at least a factor 2. This discrepancy may be solved by allowing constructive interference between $f_0(980)$ and $\sigma \to KK$. The fit requires a ratio $r_2 = g^2(\sigma \to KK)/g^2(\sigma \to \pi\pi) = 0.6 \pm 0.1$ at the $KK$ threshold. Data on $\pi\pi \to KK$ also require coupling of $\sigma$ to $KK$ with the same value of $r_2$ within experimental errors. Data on $\pi\pi \to \eta\eta$ require a ratio $r_3 = g^2(\sigma \to \eta\eta)/g^2(\sigma \to \eta\eta) = 0.20 \pm 0.04$ at the $\eta\eta$ threshold.

The interpretation of branching ratios in terms of models for $f_0(980)$, $a_0(980)$, $\sigma$ and $\kappa$ is discussed in an accompanying publication.

Acknowledgements

I am grateful to Dr. S. Giovanella for explaining technical points concerning KLOE data. I am grateful to Prof. H. Leutwyler for extensive discussions of how to fit the $\pi\pi$ S-wave amplitude with analytic functions. I wish to thank Prof. A. Sarantsev for discussion about the status and parameters of $f_0(1370)$. I also wish to thank Dr. C. Hanhart for valuable discussion on formulae.
References

[1] M.N. Achasov et al., Phys. Lett. B438 (1998) 441 and B479 (2003) 53.
[2] R.R. Akhmetshin et al., Phys. Lett. B462 (1999) 380.
[3] M.N. Achasov et al., Phys. Lett. B440 (1998) 442 and B485 (2000) 349.
[4] A. Aloisio et al., Phys. Lett. B537 (2002) 21.
[5] A. Aloisio et al., Phys. Lett. B536 (2002) 209.
[6] N.N. Achasov and V.N. Ivanchenko, Nucl. Phys. B315 (1989) 465.
[7] M. Bolgione and M.R. Pennington, Euro. Phys. J C30 (2003) 503.
[8] D.V. Bugg, V.V. Anisovich, A.V. Sarantsev and B.S. Zou, Phys. Rev. D50 (1994) 4412.
[9] C. Amsler et al., Phys. Lett. B327 (1994) 425.
[10] M. Ablikim et al., Phys. Lett. B607 (2005) 243.
[11] K.L. Au, D. Morgan and M.R. Pennington, Phys. Rev. D35 (1987) 1633.
[12] A.V. Anisovich, V.V. Anisovich and A.V. Sarantsev, Zeit. Phys. A359 (1997) 173.
[13] D.V. Bugg, A.V. Sarantsev and B.S. Zou, Nucl. Phys. B 471 (1996) 59.
[14] B. Hyams et al., Nucl. Phys. B64 (1973) 134.
[15] F. Close, N. Isgur and S. Kumano, Nucl. Phys. B389 (1993) 513.
[16] Particle Data Group (PDG), Phys. Lett. B592 (2004) 1.
[17] D.V. Bugg, Phys. Lett. B572 (2003) 1.
[18] E. Marco, S. Hirenzaki, E. Oset and H. Toki, Phys. Lett. B470 (1999) 20.
[19] J.A. Oller and E. Oset, Nucl. Phys. A620 (1997) 438.
[20] J.A. Oller, E. Oset and J.R. Pelaez, Phys. Rev. D59 (1999) 074001.
[21] J.A. Oller and E. Oset, Phys. Rev. D60 (1999) 074023.
[22] M. Jamin, J.A. Oller and A. Pich, Nucl. Phys. B587 (2000) 331.
[23] A. Gomez Nicola and J.R. Pelaez, Phys. Rev. D65 (2002) 054009.
[24] D.V. Bugg, Eur. Phys. J C37 (2004) 433.
[25] K. Abe et al., (BELLE), hep-ex/0512034.
[26] P. Moxhay, Phys. Rev. D39 (1989) 3497.
[27] A. Bramon et al., Eur. Phys. J C26 (2002) 253.
[28] N.N. Achasov and V.V. Gubin, Phys. Rev. D63 (2001) 094007.
[29] N.N. Achasov and A.V. Kiselev, Phys. Rev. D68 (2003) 014006.
[30] N.N. Achasov and A.V. Kiselev, hep-ph/0512047.
[31] M.N. Achasov et al., hep-ex/0512027.
[32] R.A. Akhmetshin et al., Phys. Lett. B509 (2001) 217.
[33] R.H. Dalitz and S. Tuan, Ann. Phys. (N.Y.) 10 (1960) 307.
[34] H. Leutwyler and G. Colangelo, private communication.
[35] B.S. Zou and D.V. Bugg, Phys. Rev. D 50 (1993) 591.
[36] M. Ablikim et al., Phys. Lett. B598 (2004) 149.
[37] J. A. Oller, Nucl. Phys. A714 (2003) 161.
[38] J.E. Palomar, L. Roca, E. Oset and M.J. Vicente Vacas Nucl. Phys. A729 (2003) 743.
[39] P. Büttiker, Descotes-Genon and Moussaillam, Eur. Phys. J C33 (2004) 409.
[40] D. Cohen et al., Phys. Rev. D22 (1980) 2595.
[41] V.V. Anisovich, C.A. Nikonov and A.V. Sarantsev, Phys. Atom. Nucl. 65 (2002) 1545.
[42] N. N. Achasov and G.N. Shestakov, Phys. Rev. D67 (2003) 114018.
[43] M. Ablikim et al., Phys. Lett. B603 (2004) 138.
[44] D.V. Bugg et al., Phys. Lett. B353 (1995) 378.
[45] J.Z. Bai et al., Phys. Lett. B472 (2000) 207.
[46] A. Anisovich et al., Phys. Lett. B449 (1999) 154.
[47] F. Binon et al., Nu. Cim. A78 (1983) 313.
[48] S.J. Lindenbaum and R.S. Longacre et al., Phys. Lett. B274 (1992) 492.
[49] A.J. Pawlicki et al., Phys. Rev. D15 (1977) 3196.
[50] G. Costa et al., Nucl. Phys. B175 (1980) 402.
[51] N.M. Cason et al., Phys. Rev. Lett. 36 (1976) 1485.
[52] A.D. Martin and E.N. Ozmuthu, Nucl. Phys. B158 (1979) 520.
[53] V.A. Polychronakos et al., Phys. Rev. D19 (1979) 1317.
[54] A. Etkin et al., Phys. Rev. D25 (1982) 1786.
[55] S. Pislak et al., Phys. Rev. Lett. 87 (2001) 221801.