Exact Extreme Value Statistics and the Halo Mass Function

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ABSTRACT
Motivated by observations that suggest the presence of extremely massive clusters at uncomfortably high redshifts for the standard cosmological model to explain, we develop a theoretical framework for the study of the most massive haloes, e.g. the most massive cluster found in a given volume, based on Extreme Value Statistics (EVS). We proceed from the exact distribution of the extreme values drawn from a known underlying distribution, rather than relying on asymptotic theory (which is independent of the underlying form), arguing that the former is much more likely to furnish robust statistical results. We illustrate this argument with a discussion of the use of extreme value statistics as a probe of primordial non-Gaussianity.

Key words: methods: analytical – methods: statistical – dark matter – large-scale structure of Universe – galaxies: clusters

1 INTRODUCTION
The standard ‘concordance’ or Lambda cold dark matter (ΛCDM) cosmological model incorporates the idea that large scale structure in the universe is assembled hierarchically from Gaussian-distributed initial perturbations in the density of Cold Dark Matter. In the hierarchical models, structure in the universe forms in a ‘bottom up’ fashion, with small-scale density perturbations collapsing first before merging over time to form larger and larger CDM haloes (White & Frenk 1991; Peacock 2000). Baryonic matter falls into these haloes, becoming shocked and virialised to form galaxies (e.g. Benson 2010). The exact details of the rate and magnitude of structure formation are highly sensitive to the contents and dynamics of the universe and, as such, have the potential to constrain deviations from the minimal ΛCDM model. Indeed, the most massive collapsed object in the universe can on its own supply a definitive test of cosmological models, in that the observation of a single sufficiently massive CDM halo has the ability to rule out at high significance levels models in which such a large object is unlikely to form. In particular, the inference that extremely dense haloes must have arisen from large upward density fluctuations seems a promising way to probe possible departures from initial Gaussianity.

In accord with this line of reasoning, there has recently been considerable interest in the existence of high-mass, high-redshift galaxy clusters as a means of identifying deviations from ΛCDM cosmology. Since the discovery by Jee et al. (2009) of a cluster at $z \sim 1.4$ with a mass of $8.5 \pm 1.7 \times 10^{14} M_\odot$, and other apparently challenging objects (Brodwin et al. 2010; Santos et al. 2011; Foley et al. 2011), several authors have reported tension between the existence of such objects and concordance cosmology. Jimenez & Verde (2009), Cayón et al. (2011) and Hoyle et al. (2011) all report that this tension can be eased by the presence of primordial non-Gaussianity, parameterised by $f_{NL}$, at levels which far exceed (by a factor $\sim 10$) the limits imposed by the CMB (Komatsu et al. 2011).

Whilst models exist that predict a running of $f_{NL}$ with scale (Lo Verde et al. 2008), it is important to explore the robustness of these detections before concluding that changes to the standard model are needed. Furthermore, future surveys will only increase the observed volume in which clusters may exist, so the most massive clusters found will increase accordingly.

While the motivation for focussing on such objects is strong, in order to perform model selection with high mass clusters we need to understand the statistical properties of such objects. One way of considering this problem is through Extreme Value Statistics (EVS) (Gumbel 1958; Katz & Nadarajah 2002), which seek to make predictions for the greatest (or least) valued random variable drawn from an underlying distribution. There has recently been a resurgence of interest in applying EVS to the field of cosmology with papers by Mikelsons et al. (2009), Yamila Yaryura et al. (2010), Colombi et al. (2011), Davis et al. (2011), Waizmann et al.
(2011) and Chongchitnan & Silk (2011), the last three dealing with high-mass clusters in particular. In this paper we look more carefully at the underlying theory, derive from first principles the exact extreme value statistics of the halo mass function, and investigate their usefulness for constraining cosmology.

The paper is organised as follows. In section 2 we introduce exact extreme value statistics and show how they may be formulated for the case of the halo mass function, in both the ΛCDM case and one including amounts of primordial non-Gaussianity. Section 3 compares this theoretical prediction for the most-massive cluster with Monte-Carlo simulations. In section 4 we conclude and discuss prospects for future work in this area.

2 METHODS

2.1 Exact and Asymptotic Extreme Value Statistics

If we consider a sequence of $N$ random variates $\{M_i\}$ drawn from a cumulative distribution $F(m)$ then there will be a largest value of the sequence:

$$M_{\text{max}} \equiv \sup \{M_1, \ldots M_N\}.$$  \hspace{1cm} (1)

If these variables are mutually independent and identically distributed then the probability that all of the deviates are less than or equal to some $m$ is given by:

$$\Phi(M_{\text{max}} \leq m; N) \equiv F(M_1 \leq m) \ldots F(M_N \leq m) = F^N(m)$$  \hspace{1cm} (2)

and the probability distribution for $M_{\text{max}}$ is then found by differentiating (2):

$$\phi(M_{\text{max}} = m; N) \equiv NF'(m)[F(m)]^{N-1} = NF'(m)[F(m)]^{N-1}$$  \hspace{1cm} (3)

This gives the exact extreme value distribution for $N$ observations drawn from a known underlying distribution $f(m)$. However, it is the seminal result of extreme value statistics (Fréchet 1927, Fisher & Tippett 1928) that, in analogy with the central limit theorem for sample means, even in cases where $f(m)$ is not explicitly known, in the limit $N \to \infty$ the distribution $\phi(\hat{m}_N)$ of a suitably rescaled variable

$$\hat{m}_N = \frac{m - a_N}{b_N},$$

(where $a_N$ and $b_N$ are functions of $N$ determined by the underlying distribution) asymptotically approaches one of only three limiting forms: the Type-I, II and III (also known as Gumbel, Fréchet and Weibull respectively) extreme value distributions. The functions $a_N$ and $b_N$ may be determined via the reciprocal hazard function:

$$r(m) = \frac{1 - F(m)}{f(m)}$$  \hspace{1cm} (4)

$$b_N = F^{-1}(1 - \frac{1}{N}), a_N = r(b_N)$$  \hspace{1cm} (5)

It is possible to encapsulate all these asymptotic distributions within the Generalised Extreme Value (GEV) distribution:

$$G(\hat{m}_N; \gamma) = \exp\{-[1 + \gamma \hat{m}_N]^{-1/\gamma}\}, \hspace{1cm} \text{where values of the shape parameter } \gamma = 0, \gamma > 0 \text{ and } \gamma < 0 \text{ pick out Type-I, II and III distributions respectively. We have given this distribution the symbol } G(m) \text{ as opposed to } \phi(m) \text{ to emphasize the difference between exact and asymptotic distributions. It is possible to determine the asymptotic value of } \gamma \text{ (Gnedенко 1943, Györgyi et al. 2010), and hence the asymptotic distribution type, but this process proves to be only analytically tractable for simple distributions.}

The shape parameter describes the form of the asymptotic distribution $G(\hat{m}_N; \gamma)$, but exact distributions $\phi(m; N)$ will still have a best fitting value for $\gamma$. Measuring $\gamma$ from a finite sized sample from a distribution which is in the domain of attraction for the Type-I extreme value distribution will lead to a measurement which converges towards zero as the sample size increases. For distributions lying in the domain of attraction of types II and III, $\gamma$ will converge to an unknown value, depending on form of the underlying distribution. The rate of this convergence can be spectaculalrly slop; for the specific case of a Gaussian distribution (for which it can be analytically determined that the asymptote is the $\gamma = 0$ distribution) convergence goes as $\sqrt{\ln N}$ only. It is therefore necessary to be extremely careful that any observed value (or change in value) of the shape parameter $\gamma$ is due to changes in the underlying distribution, rather than due to the convergence of the exact distribution $\phi(m; N)$ to the asymptotic one $G(\hat{m}_N; \gamma)$.

2.2 Extreme Value Statistics of the Halo Mass Function

We now seek to determine the statistical distribution of extreme values for the masses of CDM haloes, and in particular the validity of the asymptotic form (6) for realistic cosmological volumes. Press & Schechter (1974) were the first to provide an analytic method for predicting the co-moving number density $n(M)$ of haloes of a given mass $M$, in differential form $dn/dM$, considering spherical collapse of density perturbations in the matter field. Subsequent to this, there has been much work developing the halo mass function, both analytic and by fitting functions to N-body simulations.

We choose to use the mass function from Sheth & Tormen (1999) including effects from ellipsoidal collapse:

$$\frac{dn}{dM} = A\sqrt{\frac{2a\delta_c}{\pi\sigma_M}} \exp\left(-\frac{a\delta_c^2}{2\sigma_M^2}\right) \left[1 + \left(\frac{\sigma_M^2}{a\delta_c^2}\right)^p\right] \frac{\tilde{\rho}}{M} \frac{dn(\sigma_M^2)}{dM},$$  \hspace{1cm} (7)

where $\sigma_M^2$ is the variance of the matter field smoothed with a top hat window of radius $R = (3M/4\pi \rho)^{1/3}$, with linear power spectrum $P(k)$:

$$\sigma_M^2 = \int_0^\infty \frac{dk}{2\pi} k^2 P(k)W^2(k; R),$$  \hspace{1cm} (8)

$\tilde{\rho}$ is the mean density in the Universe, $\delta_c \approx 1.686$ is the critical overdensity for collapse and $\{A, a, p\}$ are parameters fitted to an N-body simulation and here given their original values of $\{0.322, 0.707, 0.3\}$. Throughout, we use a power spectrum calculated using CAMB\(^1\) and the WMAP7+BAO+SN Maximum Likelihood parameters from Komatsu et al. (2011). Using the halo mass function as a

\(^1\) http://camb.info
predictor of number densities of haloes \( n(M) \), we can construct a probability distribution function (pdf) for halo mass to be used in the calculation of the extreme value distribution outlined above:

\[
f(m) = \frac{1}{n_{\text{tot}}} \frac{dn(m)}{dm},
\]

\[
F(m) = \frac{1}{n_{\text{tot}}} \left[ \int_{-\infty}^{m} dM \frac{dn(M)}{dM} \right],
\]

where the normalisation factor

\[
n_{\text{tot}} = \int_{-\infty}^{\infty} dM \frac{dn(M)}{dM}
\]

is the total (co-moving) number density of haloes. For a constant redshift box of volume \( V \) the total number of expected haloes \( N \) is then given by \( n_{\text{tot}} V \). These distributions can be inserted into equation (3) to predict the pdf of the highest mass dark matter halo within the volume.

The form of halo mass distribution in ΛCDM and alternative cosmologies can also be examined; as an example of deviations from ΛCDM we include the effects of primordial non-Gaussianity. The halo mass function has long been known to be sensitive to the presence of primordial non-Gaussianity [Lucchin & Matarrese 1988] and these effects have been replicated within N-body simulations [Grossi et al. 2009; Pillepich et al. 2010]. We include non-Gaussianity into the model via the non-Gaussian correction factor \( \mathcal{R}(f_{\text{NL}}) \) of Lo Verde et al. (2008) (LMSV):

\[
\mathcal{R}_{\text{LMSV}}(f_{\text{NL}}) = 1 + \frac{\sigma^2}{6M} \left( S_{\delta}(\sigma) \left( \frac{\sigma^2}{\sigma^2_\delta} - 1 \right) + \frac{dS_{\delta}}{d\ln \sigma} \left( \frac{\sigma^2}{\sigma^2_\delta} - 1 \right) \right),
\]

where \( S_{\delta} \) is the normalised skewness of the matter density field, for which we use the approximation:

\[
S_{\delta} \simeq 3 \times 10^{-4} f_{\text{NL}} \sigma^{-1}.
\]

\( f_{\text{NL}} \) given by equation (2.7) of Enqvist et al. (2011). The choice of the LMSV version is motivated by Figure 1 in which we plot three methods of including primordial non-Gaussianity in the halo mass function; the \( \mathcal{R}(f_{\text{NL}}) \) correction factors of LMSV and Matarrese et al. (2000) (MVJ) and the analytically applied non-Gaussianity of Maggiore & Riotto (2010) (MR), all applied to the \( f_{\text{NL}} = 0 \) MR mass function. As can be seen (and as observed by Enqvist et al. (2011) when applied to the Tinker et al. (2008) mass function), the MVJ correction factor leads to a divergence in the mass function in the high-mass limit, which in this analysis we are still required to integrate over. By applying non-Gaussianity to the MR mass function we can explicitly see that it is the \( \mathcal{R}(f_{\text{NL}}) \) factor which leads to this divergence, rather than the mass function itself. In order to evaluate the efficacy of this formulation of the extreme value statistics of the halo mass function, we compare the extreme value pdf calculated from (9, 11) to Monte Carlo simulations of the most massive halo in a universe with a given mass function. In each cosmology, we construct an ensemble of realisations of the halo mass function; each realisation is constructed by calculating the expected number of haloes in a bin of width \( \Delta \log m \) and drawing from a Poisson distribution with this mean. The drawn value is then taken as the number of haloes in this bin for this realisation, generating a mock catalogue of uncorrelated haloes in the volume \( V \). The largest cluster mass for the realisation is determined as the central value of the highest occupied bin (which is always singly occupied). The distribution of highest-mass cluster in each catalogue is then recorded over \( 10^5 \) realisations.

3 RESULTS AND COMPARISONS WITH OTHER WORK

Figure 2 shows the results of the above procedure for the Sheth & Tormen (1999) mass function with WMAP7 cosmological parameters. Plotted are Monte Carlo results with Poisson errors, the exact extreme value distribution calculated using (10) and asymptotic Type-I (Gumbel) and GEV distributions fitted using a maximum likelihood method. It can be seen that the predictions of the exact extreme value distribution (9) well match the results of the Monte-Carlo simulations. As can be expected, including the extra degree of freedom of the shape parameter \( \gamma \) greatly improves the fit of the GEV distribution over the Type-I.

Figure 3 shows the convergence of the shape parameter \( \gamma \) for a variety of spherical volumes and values of the non-Gaussianity parameter \( f_{\text{NL}} \). Values of \( \gamma \) are estimated with a maximum likelihood method and error bars represent 95% confidence intervals. As can be seen, whilst the shape parameter appear well converged for volumes above \( r \geq 30 h^{-1}\text{Mpc} \), there is enough statistical noise so as to wash out any potential detection of \( f_{\text{NL}} \lesssim 300 \) by using \( \gamma \) as a test statistic, even in this simple case with uncorrelated haloes. Davis et al. (2011) also consider the extreme value statistics of the halo mass function, forming the extreme value distribution as the differential of the void probability:

\[
\Phi^\text{void}(M_{\text{max}} = m) = \frac{dP_0(m)}{dm}.
\]

where, in the Poisson limit, the void probability is given by:

\[
P_0(m) = \exp( -n( > m) V).
\]

Shown in Figure 4 is the comparison between the extreme value distributions calculated using equations (14)
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Figure 2. The extreme value distributions for the Sheth-Tormen halo mass function. Shown are the exact distribution and two best-fitting asymptotic distributions: a Type-I (Gumbel, dash-dotted) distribution and a general extreme value distribution with free $\gamma$ parameter (GEV, dashed).

Figure 3. The shape parameter $\gamma$ for different volumes and values of $f_{NL}$, estimated using a maximum likelihood method and with 95% error bars. Points for $f_{NL} = 100$ and $f_{NL} = 300$ are horizontally offset by +2.5, +5 $h^{-1}$Mpc respectively. Convergence appears to be sufficient at volumes $\gtrsim 30 h^{-1}$Mpc and $\gamma$ appears to be poor at discriminating between different values of $f_{NL}$.

Figure 4. Comparison of Davis et al. (2011) (DDCSP) and this work, showing the agreement of both methods of determining the extreme value statistics of the halo mass function. The dotted line represents the DDCSP version with halo correlations included.

and (3), showing excellent agreement for the case of uncorrelated haloes, as is to be expected. The method of Davis et al. (2011) can be readily modified to account for correlated, biased haloes, primarily because of the simple form taken by effects of correlations on the void probability, but it remains a future endeavour to include these effects in the exact model. However, the agreement of extreme value distributions at the high mass end in the cases of both correlated and uncorrelated haloes means that meaningful inferences on likelihoods of most massive clusters may still be drawn from the simple uncorrelated models.

4 DISCUSSION AND CONCLUSIONS

We have explored an avenue towards the construction of the exact distribution of halo masses which does not entail the assumption that the distribution belongs to one of the asymptotic types discussed in the classical literature of extreme value statistics. Using both analytical and numerical techniques we have shown that there can be significant differences between the exact and asymptotic distributions and show in particular that the shape parameter $\gamma$ is unlikely to provide an effective statistical discriminator between Gaussian and non-Gaussian theories of structure formation.

The approach we have taken relies on accurate knowledge of the behaviour of the underlying distribution for large halo masses. Even for the case of Gaussian initial conditions (i.e. $f_{NL} = 0$) there is some theoretical uncertainty in what this behaviour actually is. There exist a number of plausible halo mass functions in the literature (e.g. Sheth & Tormen 1999, Jenkins et al. 2001, Reed et al. 2003, Tinker et al. 2008), all of which have differing tail behaviour and the level of indeterminacy worsens when we consider non-Gaussian models, as discussed in section 3.

Nevertheless, analytical approaches like those discussed in this paper will certainly play an important role in this area for some considerable time. The most massive haloes are so rare that probing them using numerical techniques will require enormous volumes to be simulated with sufficient resolution to obtain accurate halo masses whilst at the same time avoiding boundary artifacts. For example, in order to determine the probability distribution of the most massive cluster in the Hubble volume we would need an ensemble of simulations, each so large that it would comprise a large number of independent Hubble volumes. Faced with the significant computational cost of such a programme, there can be no doubt that analytical theory, calibrated by smaller scale simulations, will be the principal theoretical tool by which extreme objects will be studied. We will adopt this approach in future work.

The use of extreme value statistics as described in this work also has the advantage over studies which seek to use rare objects to constrain mass functions of clusters $n(M)$ (e.g. Vikhlinin et al. 2003, Allen et al. 2011) in that, in the EVS approach, a given object can always set a lower limit on the global extremum. This avoids the difficulty (in addition to the determination of cluster mass) of defining in...
a unbiased way precisely what volume is being probed, a process vulnerable to a posteriori selection effects.

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