The Role of Mass and External Field on the Fermionic Casimir Effect

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Abstract

The aim of this work is to investigate the role played by the fermion mass and that of an external field on the fermionic Casimir energy density under $S^1 \times R^3$ topology. Both twisted and untwisted spin connections are considered and the exact calculation is performed using a somewhat different approach based on the combination of the analytic regularization method through $\alpha$-representation and the Euler-Maclaurin summation formula.
I. INTRODUCTION

As well as the Lamb-Retherford shift of atomic energy levels and the electron’s anomalous magnetic moment, the Casimir force between two parallel perfectly conducting plates is one of the most remarkable manifestations of quantum vacuum fluctuations. It was first predicted on theoretical grounds by H. B. G. Casimir in 1948 [1] and experimentally verified on a qualitative level by Sparnaay [2] ten years later. Recently, high accuracy experiments have been performed by Lamoreaux [3] and by Mohideen and Roy [4]. For a more detailed account on the subject there are excellent reviews in the literature [5].

In 1975, employing Casimir essential ideas, Johnson [6] investigated the effects of boundaries on a massless Dirac field in the context of MIT-bag model [7] and found an energy density shift of the same order of magnitude as that obtained by Casimir for the electromagnetic field. Adopting Johnson’s extend approach to Casimir effect, in order to allow for other quantum fields, many authors have investigated the effects of different boundary conditions on the fields considered [8]. Hence, one can say that a modern view of Casimir effect might take into account those effects caused by non-trivial space topologies on the vacuum of quantum fields.

It is worth noting that in this general context some Casimir setups (field + boundary condition + external sources) present quite complicated final expressions for the Casimir energy density, which ultimately obscure any possible physical interpretation. In particular, the results obtained in [9] for the massive spinor field indicate a mass dependent energy density which calls for a deep investigation. Similar difficulties also appear in the case considered in [10] where the fermion field is also subjected to an external magnetic field.

In order to handle the above mentioned shortcomings, we present here an alternative treatment which allows us to extract new information concerning the role of mass and that of an external field to the fermionic Casimir effect. This is achieved by a suitable combination of the method of analytic regularization in the context of gamma function representation (also called $\alpha$-representation [11]) and the use of the well-known Euler-Mclaurin summation formula [12].

The main ideas of our construct are introduced in the next section, where we consider the
well-established electromagnetic Casimir effect. In section III, the same procedure is employed to the case of a massive spinor field with boundaries analogous to that considered in [8] and [9]. In section IV, the effects of an external constant and homogeneous magnetic field is also considered and the connection with the so-called Euler-Kockel-Heisenberg Effective Lagrangian density [17] is addressed. Finally, in section V, we make some concluding remarks pointing out directions of future investigations.

II. THE CASE OF ELECTROMAGNETIC FIELDS

As was originally considered by Casimir in 1948, the divergent vacuum energy density associated with the electromagnetic field is given by

$$\varepsilon_0 = \frac{1}{8\pi^2L} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{[k_x^2 + k_y^2 + (an)^2]^{-1/2}}, \quad a = \pi/L$$

(1)

where the discrete values of $k_z$ reflects the boundary condition imposed by two parallel conducting plates with area $A = l^2$ and separated by a distance $L$ in such a way that $l \gg L$ ($a = \pi/L$) [1].

In (1), the integrals are quadratically divergent quantities which claim for a consistent regularization prescription. Despite the familiar regularization methods found in the literature [11] [13], we shall consider here a quite different one. It consists in the combination of the analytic regularization scheme, using the gamma function integral representation, with the so-called Euler-Mclaurin summation formula [12]. To see how this works, we start by taking the analytic extension of the integrand in (1), which turns out to be a regular functional. This is achieved by means of the gamma function integral representation

$$\frac{1}{A^{1+\delta}} = \frac{1}{\Gamma(1+\delta)} \int_{0^+}^{\infty} d\eta \, \eta^\delta e^{-\eta},$$

(2)

valid for $\delta > -1$, which allows us to rewrite (1) as

$$\varepsilon_0^R = \frac{1}{2(2\pi)^2 L} \frac{\pi}{\Gamma(-1/2 + \delta)} \sum_{n=-\infty}^{\infty} \int_{0^+}^{\infty} d\eta \, \eta^{-5/2+\delta} e^{-(an)^2\eta},$$

(3)

where the gaussian integrals in $k_x$ and $k_y$ have already been calculated. Note that, for $\delta \to 0$, we expect to recover the original theory, i.e., expression (1). This will be done only at the end of the calculations.
The divergent sum over \( n \) appearing in (3) is performed by means of the Euler-Mclaurin summation formula \[12\]

\[
\sum_{n=M}^{N} f(n) = \int_{M}^{N} F(x)dx + \frac{1}{2}[f(N) + f(M)] + \sum_{k=1}^{K} \frac{B_{2k}}{2k!} [F^{2k-1}(N) - F^{2k-1}(M)] \\
+ \frac{1}{(2K + 1)!} \int_{N}^{M} B_{2K+1}(x - [x]) F^{2K+1}(x)dx
\]

(4)

where \( B_{m} = B_{m}(0) \) and the \( B_{m}(x) \) are the Bernoulli polynomials. We preserve here the same notation used in \[12\]. The last term in (4), also called the remainder term, vanishes if \( F(z) \) is an entire function. Further, if \( n \) is integer and \( M \leq n \leq N \), then \( F(n) = f(n) \). In the present context, identifying the entire function \( f(n) \) with \( f(n) = e^{-(an)^2} \eta \) (5)

and, since \( 0 \leq n \leq \infty \), we are allowed to rewrite (3) as

\[
(\varepsilon_{0})^{R} = \frac{1}{(2\pi)^2 L} \frac{\pi}{\Gamma(-1/2 + \delta)} \left\{ \int_{0}^{\infty} d\eta \, \eta^{-5/2 + \delta} \left[ \int_{0}^{\infty} dn \, f(n) - \frac{1}{2} f(n)|_{n\rightarrow\infty} \right. \right. \\
\left. \left. + \frac{1}{12} f'(n)|_{n\rightarrow\infty} - \frac{1}{12} f'(n)|_{n\rightarrow0} - \frac{1}{720} f'''(n)|_{n\rightarrow\infty} + \frac{1}{720} f'''(n)|_{n\rightarrow0} + \ldots \right] \right\}
\]

(6)

where (2) was used. In going from expression (6) to (7) we notice that only the \( (n \rightarrow \infty) \)-terms contribute to the energy density. Usually, the methods found in the literature extract information arising from the \( (n \rightarrow 0) \)-terms and, as we will show in the next section, this generates quite different final results. However, we must stress that, while the exponential function in (5) is an analytical function over the entire complex plane, the power function in the integrand of (1) is a multiple-valued function, which has a branch cut along the real axis \[14\].

Now, in order to obtain a consistent result with the original theory we now take the limit \( \delta \rightarrow 0 \) in (7). First, we must handle the divergent contribution arising from the second term in the curly
brackets. Such divergence is eliminated using the freedom in the choice of $\delta$ in (7). In fact, for consistency with (3), $\delta$ are constrained to be greater than $1/2$. However, to obtain the correct final result the considered region in the complex plane must be analytic continued to $\delta \geq 1$. Hence, as the limit $n \to \infty$ is performed, the second term contribution turns out to be zero. On the other hand, the contribution from the remaining terms, which becomes $n$-independent when the limit $\delta \to 0$ is taken, gives

$$\varepsilon_0 = \frac{1}{4(2\pi)^2} \int_0^\infty d\eta \eta^{-3} + \frac{\pi^2}{720L^4}.$$  \hspace{1cm} (8)

Finally, calculating the corresponding energy for the whole space (which is equivalent to taking $L \to \infty$ in the above expression) and subtracting from it expression (8), yields

$$\Delta \varepsilon = (\varepsilon_0)_{L \to \infty} - \varepsilon_0 = -\frac{\pi^2}{720L^4}$$ \hspace{1cm} (9)

which is the expected vacuum energy density derived by Casimir in 1948. Since there is no a priori reason for assuming that the vacuum energy in the presence of boundaries is greater than in their absense, we have defined $\Delta \varepsilon$ in such a manner that (9) indicates that the force between the plates is in fact attractive, as has been confirmed by precise experiments [3] - [4].

III. THE MASSIVE SPINOR FIELD: $S^1 \times R^3$

The case of noninteracting spinor fields subjected to $S^1 \times R^3$ space topology with twisted and untwisted spin connections was first considered by DeWitt, Hart and Ishan in 1979 [8] and one year later it was generalized to the massive case by Ford [9], who obtained an intricate mass dependent expression for the vacuum energy density. In this section we intend to rederive these results using the procedure developed in the last section. As will be shown, the effect of the fermion mass to the Casimir energy density is, in fact, null.

*Once we adopt this definition, it must be preserved if we consider other topologies and boundary conditions, in order to be able to distinguish attractive from repulsive effects.
A. Untwisted Case

For an untwisted spinor field, the vacuum energy density is given by

$$\varepsilon_{\text{unt}}^0 = -\frac{1}{(2\pi)^2 L} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \left[ m^2 + p_x^2 + p_y^2 + p_z^2 \right]^{1/2}. \quad (10)$$

where $p_z = a^2 n^2$, with $n = 0, \pm 1, \pm 2, \ldots$ and $a = 2\pi/L$. As in the electromagnetic case, the above quantity is divergent and must also be regularized. Using the same procedure employed in the last section, we write (10) as

$$\left( \varepsilon_{\text{unt}}^0 \right)^R = -\frac{1}{2\pi L} \frac{1}{\Gamma(-1/2 + \delta)} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d\eta \eta^{-5/2 + \delta} e^{-(m^2 + a^2 n^2)\eta},$$

where we have already performed two gaussian integrals in $p_x$ and $p_y$. Instead of use the Abel-Plana formula [15]-[16] we may invoke the Euler-McLaurin summation formula (4), in order to find an expression similar to (6), where the only difference rely upon the definition of the entire function $f(n)$. Here

$$f(n) = e^{-(m^2 + a^2 n^2)\eta}.$$

Following the same steps done in the last section, it is straightforward to obtain

$$\left( \varepsilon_{\text{unt}}^0 \right)^R = -\frac{1}{(2\pi L) \Gamma(-1/2 + \delta)} \left\{ \int_{0}^{\infty} d\eta \eta^{-3+\delta} e^{-m^2 \eta} \frac{\sqrt{\pi}}{2\alpha} + \frac{1}{12 \alpha} \left[ \frac{\Gamma(-1/2 + \delta)}{(m^2 + a^2 n^2)^{-1/2+\delta}} (-2a^2 n) \right]_{n \to \infty} \right\},$$

where $\alpha = \frac{1}{2\pi L} \frac{1}{\Gamma(-1/2 + \delta)}$. Instead of use the Abel-Plana formula we may invoke the Euler-McLaurin summation formula (4), in order to find an expression similar to (6), where the only difference rely upon the definition of the entire function $f(n)$. Here

$$f(n) = e^{-(m^2 + a^2 n^2)\eta}.$$

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where $\alpha = \frac{1}{2\pi L} \frac{1}{\Gamma(-1/2 + \delta)}$. Instead of use the Abel-Plana formula we may invoke the Euler-McLaurin summation formula (4), in order to find an expression similar to (6), where the only difference rely upon the definition of the entire function $f(n)$. Here

$$f(n) = e^{-(m^2 + a^2 n^2)\eta}.$$

Following the same steps done in the last section, it is straightforward to obtain

$$\left( \varepsilon_{\text{unt}}^0 \right)^R = -\frac{1}{(2\pi L) \Gamma(-1/2 + \delta)} \left\{ \int_{0}^{\infty} d\eta \eta^{-3+\delta} e^{-m^2 \eta} \frac{\sqrt{\pi}}{2\alpha} + \frac{1}{12 \alpha} \left[ \frac{\Gamma(-1/2 + \delta)}{(m^2 + a^2 n^2)^{-1/2+\delta}} (-2a^2 n) \right]_{n \to \infty} \right\},$$

where $\alpha = \frac{1}{2\pi L} \frac{1}{\Gamma(-1/2 + \delta)}$. Instead of use the Abel-Plana formula we may invoke the Euler-McLaurin summation formula (4), in order to find an expression similar to (6), where the only difference rely upon the definition of the entire function $f(n)$. Here

$$f(n) = e^{-(m^2 + a^2 n^2)\eta}.$$

Following the same steps done in the last section, it is straightforward to obtain

$$\left( \varepsilon_{\text{unt}}^0 \right)^R = -\frac{1}{(2\pi L) \Gamma(-1/2 + \delta)} \left\{ \int_{0}^{\infty} d\eta \eta^{-3+\delta} e^{-m^2 \eta} \frac{\sqrt{\pi}}{2\alpha} + \frac{1}{12 \alpha} \left[ \frac{\Gamma(-1/2 + \delta)}{(m^2 + a^2 n^2)^{-1/2+\delta}} (-2a^2 n) \right]_{n \to \infty} \right\},$$

where $\alpha = \frac{1}{2\pi L} \frac{1}{\Gamma(-1/2 + \delta)}$. Instead of use the Abel-Plana formula we may invoke the Euler-McLaurin summation formula (4), in order to find an expression similar to (6), where the only difference rely upon the definition of the entire function $f(n)$. Here

$$f(n) = e^{-(m^2 + a^2 n^2)\eta}.$$
\[ \varepsilon_{0}^{\text{unt}} = \frac{1}{8\pi^2} \int_{0}^{\infty} d\eta \eta^{-3} e^{-m^2\eta} - \frac{\pi^2}{720L^4}. \tag{14} \]

Subtracting (14) from the corresponding usual Minkowski vacuum energy (which correspond to taking \( L \to \infty \) in (14)) we find the fermionic Casimir energy density

\[ \Delta \varepsilon^{\text{unt}} = \frac{2\pi^2}{45L^4}, \tag{15} \]

which is in complete agreement with [8], the only difference being a factor 2, which reflects the four-component spinor field representation we are using. De Witt et al considered a two-component spinor field.

\section*{B. Twisted Case}

Expression (14) clearly shows the independence of the Casimir energy density with respect to the fermion mass. This feature also occurs when twisted boundary condition is considered. In this case \( p_z = (2n+1)a \), with \( n = 0, \pm 1, \pm 2, \ldots \) and \( a = \pi/L \). The regulated vacuum energy density is now given by

\[ (\varepsilon_0^{\text{twi}})^R = \frac{\sqrt{\pi}}{(2\pi)^2 L} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d\eta \eta^{-5/2} e^{-(m^2+(2n+1)^2a^2)\eta}. \tag{16} \]

Since the above integral is regular, we are allowed to interchange the sum and the integral, and than use the following mathematical trick

\[ \sum_{n=-\infty}^{\infty} e^{-(m^2+(2n+1)^2a^2)\eta} = \sum_{n=-\infty}^{\infty} e^{-(m^2+n^2a^2)\eta} - \sum_{n=-\infty}^{\infty} e^{-(m^2+(2n)^2a^2)\eta}. \tag{17} \]

In this way the problem of solving the twisted case reduces to that of computing two terms proportional to that in the untwisted case. In fact, we have

\[ \varepsilon_0^{\text{twi}} = \frac{1}{25}\varepsilon_0^{\text{unt}} - \varepsilon_0^{\text{unt}} = -\frac{7}{8(2\pi)^2} \int_{0}^{\infty} d\eta \eta^{-3} + 2\frac{7\pi^2}{360L^4}. \tag{18} \]

Again, subtracting this result from that where \( L \to \infty \), we obtain

\[ \Delta \varepsilon_0^{\text{twi}} = -2\frac{7\pi^2}{360L^4}, \tag{19} \]
which coincide with the familiar result found in the literature for the massless fermionic Casimir effect \[8\].

IV. THE INFLUENCE OF AN EXTERNAL FIELD

Another interesting problem to be analysed using the present construct is related to influence of an external magnetic field on the Casimir energy associated to the Dirac field. This problem has been recently proposed in the context of Effective Quantum Electrodynamics using the so-called Schwinger proper-time method \[10\]. However, a clear answer concerning the role of the external field on the fermionic Casimir energy density is yet an open problem which deserves further investigation. The purpose of this section is to implement, in the context of Weisskopf method \[19\] \[20\], the prescription presented in the previous sections in order to get a better understanding of the above mentioned problem.

We restrict our calculation to the case where an untwisted massless spinor field is subjected to an external constant uniform magnetic field. As is well known \[18\], the negative energy levels for an electron of charge \(e = -|e|\) in the presence of an uniform and constant magnetic field \(H_z = -H\) is given by

\[
-\epsilon_{p,\sigma} = -\sqrt{m^2 + (2n + 1 - \sigma)|e|H + p_z^2},
\]

where \(n = 0, 1, 2, 3...\) and \(\sigma = \pm 1\). Taking into account the density of states in the interval \(dp_z\)

\[
\frac{|e|H \, dp_z}{2\pi}
\]

and the fact that all the levels except \(n = 0, \sigma = 1\) are doubly degenerate (the levels \(n, \sigma = -1\) and \(n + 1, \sigma = 1\) coincide), we obtain the energy density of vacuum electrons,

\[
\varepsilon_0' = -\sum_{p,\sigma} \epsilon_{p,\sigma}
= -\frac{|e|H}{(2\pi)^2} \int_{-\infty}^{+\infty} \left\{ \sqrt{p_z^2} + 2 \sum_{n=1}^{\infty} \sqrt{2|e|Hn + p_z^2} \right\} dp_z.
\]

where \(p_z\) turned out to be a discrete quantity in virtue of the untwisted \(S^1 \times R^3\) space topology we are assuming. Using (2), the energy density (22) may be rewritten in the more convenient form,
corresponding derivatives and limits, we arrive at integrals in (25), namely,

\[ (\varepsilon_0')^R = -\frac{|e|H}{2\pi L} \sum_{n=0}^{\infty} \frac{1}{\Gamma(-1/2 + \delta)} \int_{-\infty}^{\infty} d\eta \eta^{-3/2 + \delta} \left( \sum_{n'=-\infty}^{\infty} e^{-2|e|Hn + a^2 n'^2} \eta \right) \]

Furthermore, assuming the weak field regime \( (H \ll n) \) and \( |e| = e \),

\[ f(n') = e^{-(an')^2} \eta, \quad (24) \]

and \( n' = 0, \pm 1, \pm 2, \ldots, a = 2\pi/L \). Applying the Euler-Maclaurin formula (4) and performing the corresponding derivatives and limits, we arrive at

\[ (\varepsilon_0')^R = -\frac{\sqrt{\pi}}{(2\pi)^2} \frac{|e|H}{\Gamma(-1/2 + \delta)} \int_{0}^{\infty} d\eta \eta^{-2 + \delta} \sum_{n=0}^{\infty} e^{-\alpha \eta} \]

\[ \times \left\{ n' \int_{0}^{\infty} d\eta \eta^{-1/2 + \delta} e^{-(an)^2 \eta} \sum_{n=0}^{\infty} e^{-\alpha \eta} \right\} n' \rightarrow \infty \]

\[ -\frac{2^4(2\pi)^5}{720L^5} \frac{|e|H}{\Gamma(-1/2 + \delta)} \left\{ n' \int_{0}^{\infty} d\eta \eta^{3/2 + \delta} e^{-(an)^2 \eta} \sum_{n=0}^{\infty} e^{-\alpha \eta} \right\} n' \rightarrow \infty \]

\[ +\frac{12(2\pi)^3}{720L^5} \frac{|e|H}{\Gamma(-1/2 + \delta)} \left\{ n' \int_{0}^{\infty} d\eta \eta^{1/2 + \delta} e^{-(an)^2 \eta} \sum_{n=0}^{\infty} e^{-\alpha \eta} \right\} n' \rightarrow \infty. \quad (25) \]

The sum in the integrands can be eliminated by noting that

\[ \sum_{n=0}^{\infty} e^{-\alpha \eta} = \sum_{n=0}^{\infty} e^{-2|e|Hn} = \coth(|e|H\eta). \quad (26) \]

Furthermore, assuming the weak field regime \( (H \ll 1) \) we are allowed to expand the kernel of the integrals in (25), namely,

\[ (\varepsilon_0')^R = -\frac{\sqrt{\pi}}{(2\pi)^2} \frac{|e|H}{\Gamma(-1/2 + \delta)} \int_{0}^{\infty} d\eta \eta^{-2 + \delta} \coth(|e|H\eta) \]

\[ \times \left\{ n' \int_{0}^{\infty} d\eta \eta^{-1/2 + \delta} e^{-(an)^2 \eta} \left( \frac{1}{|e|H\eta} + \frac{|e|H\eta}{3} + \Sigma \right) \right\} n' \rightarrow \infty \]

\[ -\frac{2^4(2\pi)^5}{720L^5} \frac{|e|H}{\Gamma(-1/2 + \delta)} \left\{ n' \int_{0}^{\infty} d\eta \eta^{3/2 + \delta} e^{-(an)^2 \eta} \left( \frac{1}{|e|H\eta} + \frac{|e|H\eta}{3} + \Sigma \right) \right\} n' \rightarrow \infty \]

\[ +\frac{12(2\pi)^3}{720L^5} \frac{|e|H}{\Gamma(-1/2 + \delta)} \left\{ n' \int_{0}^{\infty} d\eta \eta^{1/2 + \delta} e^{-(an)^2 \eta} \left( \frac{1}{|e|H\eta} + \frac{|e|H\eta}{3} + \Sigma \right) \right\} n' \rightarrow \infty. \quad (27) \]

where

\[ \Sigma \equiv \sum_{k=2}^{\infty} \frac{2^{2k}B_k}{(2k)!} |e|H\eta^{2k-1}, \quad (28) \]
and the $B_k$’s are the Bernoulli numbers.

We are now in position to perform, term by term in the expansion of integral (27), the limit $n' \to \infty$. After a straightforward calculation we obtain

$$
\varepsilon'_{0} = \frac{|e|H}{8\pi^{2}} \int_{0}^{\infty} d\eta \eta^{-2} \coth(|e|H\eta) - \frac{2\pi^{2}}{45L^{4}},
$$

where the same kind of analytic extension made in the previous sections was performed in the manipulation of the first term in the second line of (27). Again, it gives no contribution.

Finally, the energy density of the empty space may be obtained by taking the limit of zero field and infinite volume in (29). We must subtract (29) from this quantity, obtaining

$$
\Delta \varepsilon_{0} = -\frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{dn}{n^{2}} \{ |e|H\eta \coth(|e|H\eta) - 1 \} + \frac{2\pi^{2}}{45L^{4}},
$$

which clearly shows the influence of the external magnetic field to the fermionic Casimir effect.

It must be noted that the above expression recovers (15) in the limit of zero magnetic field. In addition, the first term in (30) might be recognized as the Euler-Kockel-Heisenberg correction to the effective Lagrangian density, which accounts for the nonlinear effects induced by the external field in effective quantum electrodynamics \[17\] - \[20\]. It provides exactly the same contribution obtained when the limit $L \to \infty$ is considered, i.e., the contribution from the boundaries just add a field independent amount to the E-K-H effective Lagrangian density. The independence of both effects clarify the physics governing the behaviour of quantum fields under the influence of external fields and/or boundaries conditions. The generalization of the above calculation to the twisted case is immediate as well as the inclusion of the fermion mass.

V. CONCLUDING REMARKS

Using an approach based on the combination of analytic regularization method throught $\alpha$-representation and the Euler-Maclaurin summation formula we had rederived the electromagnetic and the fermionic Casimir energy densities. The later, which comprises the main results of the present article, was considered in the case of $S^3 \times R^3$ space topology, where the role played by the fermion mass and that of an external field on the Casimir energy density were fully investigated.
As was shown in section III, the present approach provided a powerful way to deal with in each step of the calculation, the divergences inherent to the theory. It was found that the fermion mass doesn’t play any influence on the twisted and untwisted fermionic Casimir energy densities, which is in contrast with the first results obtained by Ford [9]. Experiment may provide the final answer.

We have also seen that, when an external magnetic field is considered, its effect on the Casimir energy density appears as an $L$-independent term (which was ultimately identified with the well known Euler-Kockel-Heisenberg Effective Lagrangian density) plus a term identical to that obtained when the external field is absent, expression (15). This result clearly shows the independence of the external field on the boundary conditions, although this seems to be in apparent disagreement with the results found in [10].

Finally, it must be emphasized that the present construct is a simple and easily generalizable method to reexamine many other phenomena. Among these are those related to the Effective Quantum Electrodynamics in the context of the “old fashioned” Weisskopf’s method [19], recently readdressed [20].

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