Variational formulation of the electromagnetic radiation-reaction problem

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A fundamental issue in classical electrodynamics is represented by the search of the exact equation of motion for a classical charged particle under the action of its electromagnetic (EM) self-field - the so-called radiation-reaction equation of motion (RR equation). In the past, several attempts have been made assuming that the particle electric charge is localized point-wise (point-charge). These involve the search of possible so-called "regularization" approaches able to deal with the intrinsic divergences characterizing point-particle descriptions in classical electrodynamics. In this paper we intend to propose a new solution to this problem based on the adoption of a variational approach and the treatment of finite-size spherical-shell charges. The approach is based on three key elements: 1) the adoption of the relativistic synchronous Hamilton variational principle recently pointed out (Tessarotto et al, 2006); 2) the variational treatment of the EM self-field, for finite-size charges, taking into account the exact particle dynamics; 3) the adoption of the axioms of classical mechanics and electrodynamics. The new RR equation proposed in this paper, departing significantly from previous approaches, exhibits several interesting properties. In particular: a) unlike the LAD (Lorentz-Abraham-Dirac) equation, it recovers a second-order ordinary differential equation which is fully consistent with the law of inertia, Newton principle of determinacy and Einstein causality principle and b) unlike the LL (Landau-Lifschitz) equation, it holds also in the case of sudden forces. In addition, it is found that the new equation recovers the customary LAD equation in a suitable asymptotic approximation.

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1 - INTRODUCTION

The goal of this paper is to investigate a well-known theoretical issue of classical electrodynamics. This is concerned with the solution of the so-called radiation-reaction problem (RR problem), i.e., the description of the dynamics of classical charges (charged particles) in the presence of their EM self-fields. For contemporary science the possible solution of the RR problem represents not merely an unsolved intellectual challenge, but a fundamental prerequisite for the proper formulation of all physical theories which are based on the description of relativistic dynamics of classical charged particles. These involve, for example, the consistent formulation of the relativistic kinetic theory of charged particles and of the related fluid descriptions (i.e., the relativistic magneto-hydrodynamic equations obtained by means of suitable closure conditions), both essential in plasma physics and astrophysics.

Surprisingly, until recently\textsuperscript{1} (hereon denoted Ref.A) the problem has remained substantially unsolved, despite efforts spent by the scientific community in more than one century of intensive theoretical research (see related discussion in Ref.\textsuperscript{2}; for a review see Refs.\textsuperscript{3}). In particular, still missing is the exact relativistic equation of motion for a classical charged particle in the presence of its electromagnetic (EM) self-field, also known as (exact) RR equation. For definiteness, in the following we shall consider the RR problem in the case of a flat (Minkowski) space-time, although a similar problem can be posed, in principle, also for curved space-time and in the context of a general-relativistic formulation. This requires that $g \equiv \det \{ g_{\mu\nu} \} = -1$, $g_{\mu\nu}$ denoting the Minkowski metric tensor with signature $(1, -1, -1, -1)$.

The equation, to be achieved exclusively in the framework of a classical-mechanics description, should result non-asymptotic. Namely, the exact RR equation should not rely on any asymptotic expansion (i.e., a truncated perturbative expansion), in particular for the electromagnetic field generated by the charged particle, to be performed in terms of any possible infinitesimal parameter which may characterize the particle itself (assuming that in some sense the particle has a finite "size", i.e., it is not point-like). On the other hand, by assumption, a classical particle should satisfy at least two basic properties: a) to have no "internal structure" and b) to be spherically symmetric (when seen with respect to the particle rest-frame). These hypotheses (which are manifestly satisfied by point-particles), should be fulfilled also by finite-size particles in which the mass and/or the electric charge have a finite-size distribution. Hence, these parameters should (only) be related to the radii of the mass and/or charge distributions. Following the prescription pointed out in Ref.A, here we intend to prove that such an equation can be obtained explicitly, without introducing any so-called"regularization" scheme, i.e., leaving unchanged the axioms of classical electrodynamics. The result is reached by considering classical finite-size charges, and,
In detail, the charge density - when seen with respect to each particle rest-frame - is taken as a classical Lorentzian model \([4]\)). For these particles the charge is considered spatially distributed in a bounded 3D domain (i.e., characterized by a finite-size charge distribution). In particular, the charge density acts when seen with respect to each particle rest-frame - is taken by assumption: a) \textit{spherically symmetric}, b) \textit{radially localized on a spherical surface} \(\delta \Omega_s\) having a finite radius \(\sigma > 0\); c) \textit{quasi-rigid}, i.e., to remain constant on \(\delta \Omega_s\) with respect to the same reference frame. Hypotheses a)-c) are manifestly all consistent with the above requirements for a classical particle. Instead, as far as the mass distribution is concerned, it is assumed as point-wise localized in the center of the spherical surface \(\delta \Omega_s\). This permits us to neglect the additional degrees of freedom occurring in such a case. Thus, from this viewpoint the particle is still treated as a point-particle. In this paper we intend to show - in particular - that, unlike the point-charge case, \textit{for a finite-size classical charge of this type the exact RR equation can be explicitly constructed based on the synchronous Hamilton variational principle.}

\[\text{1a - Motivations and historical background}\]

The occurrence of self-forces, in particular the electromagnetic (EM) one which is produced by the EM fields generated by the particles themselves, is an ubiquitous phenomenon which characterizes the dynamics of classical charged particles. It is well-known that the self-force acts on a (charged) particle when it is subject also to the action of an arbitrary external force (Lorentz, 1892 \([4]\); see also for example Landau and Lifschitz, 1957 \([5]\)). This phenomenon is usually called as \textit{radiation reaction (RR)} (Pauli \([6]\) or \textit{radiation damping} (see \([7]\)), although a distinction between the two terms is actually made by some authors \([8]\).

In classical mechanics the RR problem was first posed by Lorentz in his historical work (Lorentz, 1985 \([9]\); see also Abraham, 1905 \([10]\)). Traditional approaches are based either on the RR equation due to Lorentz, Abraham and Dirac (first presented by Dirac in 1938 \([11]\)), nowadays popularly known as the \textit{LAD equation} or the equation derived from it by Landau and Lifschitz \([5]\), via a suitable "reduction process", the so-called \textit{LL equation}. As recalled elsewhere (see related discussion in Ref.\([2]\)) several aspects of the RR problem - and of the LAD and LL equations - are yet to find a satisfactory formulation/solution. Common feature of all previous approaches is the adoption of an asymptotic expansion for the EM self-field (or for the corresponding EM 4-potential), rather than the exact representation of the force-field. This, in turn, implies that such methods permit to determine - at most - only an asymptotic approximation for the (still elusive) exact equation of motion for a charged particle subject to its own EM self-field.

\[\text{1b - Difficulties with previous RR equations}\]

Since Lorentz famous paper \([4]\) many textbooks and research articles have appeared on the subject of RR. Many of them have criticized aspects of the RR theory, and in particular the LAD and LL equations (for a review see \([3]\), where one can find the discussion of the related problems). More recently, another equation has been proposed by Medina \([12]\), here denoted as \textit{Medina equation}, which applies for spherically symmetric and finite-size classical particles. In these approaches, the charged particles are typically considered quasi-rigid, i.e., their charge densities are assumed stationary, when seen with respect to the corresponding particle rest-frame, and eventually also \textit{point-like}, i.e., both the radii of the mass (if larger than zero) and of charge distributions are assumed much smaller (i.e., infinitesimal) with respect to any other classical scale-length characterizing the particle dynamics.

It is often said that current formulations of the RR problem are unsatisfactory, because of their possible violation of basic principles of classical dynamics as well as for some of their properties. These include in particular:

- \textit{for the LAD equation}: 1) The violation of Newton’s principle of determinacy (NPD), because the LAD equation requires the specification of the initial acceleration, besides the initial state; 2) The existence of so-called runaway solutions, i.e., solutions which blow up in time. In fact, if a constant external force is applied one can show that the general solution of the LAD equations diverges exponentially in the future (blow-up). 3) For the same reason, the LAD equation violates also another fundamental principle of classical mechanics, the Galilei principle of inertia (GPI), according to which an isolated particle must have a constant velocity in any inertial Galilean frame.

- \textit{for the LL equation}: 1) The use of an iterative approach for its derivation (from the LAD equation) does not appear justifiable for fully relativistic particles. In such a case, in fact, the EM self-force cannot generally be considered a small perturbation of the external EM force. 2) The LL equation becomes invalid in the case of sudden forces, i.e., forces which are not smooth functions of time. 3) The neglect of the EM mass: in the original derivation of the LL equation, given by Landau and Lifschitz \([5]\), the so-called "EM mass" was ignored, which amounts to neglect all possible EM relativistic corrections to the inertial mass produced by the EM self-force. In the framework of classical electrodynamics the latter position appears unfounded (see discussion in Ref. \([2]\) and Ref.A). However, in the formulation of the LL equation given by Rohrlich \([13]\) this effect has been included.
• for both equations: the derivations of both equations (LAD and LL) are made under the implicit assumption that all the expansions in powers used near the particle trajectory are valid for the whole range of values of particle velocity, in particular, arbitrarily close to that of the light in vacuum. However, it is easy to see that this is not the case.

• for the Medina equation: the use of a perturbative approach, in particular to evaluate the RR force in the rest frame. This is, however, a non-relativistic equation. Therefore, the corresponding relativistic equation is also necessarily asymptotic in character.

In our view this clearly indicates that the route to the solution of the RR problem should be based on the search of the exact relativistic RR equation, i.e., the construction of a non-perturbative equation of motion for a particle in the presence of its EM self-field.

1c - The search of an exact RR equation

A critical aspect of the RR problem is, however, related to the search of the exact relativistic RR equation for classical charged particles, in the sense specified above. Despite previous attempts, this equation is still missing. As far as the LAD equation is concerned, this is obvious because to obtain it the EM self-field is usually evaluated by means of an asymptotic expansion. This is true, of course, also for the LL equation, which according to Rohrlich should be considered as the "exact" relativistic equation of motion for a classical point-like spherically-symmetric charge, having a charge distribution with an infinitesimal radius \( \sigma \) (in this case the equation is intrinsically asymptotic since it depends on the infinitesimal parameter \( \sigma \)).

This feature - as pointed out in Ref. A - is also reflected by the circumstance that these equations are non-variational \(^2\), i.e., they do not admit a variational formulation, at least in the customary sense of the standard Hamilton principle, used in classical mechanics and electrodynamics \(^3\), i.e., for the conventional 8-dimensional phase-space spanned by the 4-vectors \( \{r^\mu, u^\mu = g_{\mu\nu} dr^\nu/ds \} \), \( g_{\mu\nu} \) denoting the (Minkowski) metric tensor. This result is clearly in contrast to the basic principles both of classical mechanics and electrodynamics. In particular, it conflicts with Hamilton's action principle, which - under such premises (i.e., the validity of LAD and/or LL equations) - should actually hold true only in the case of inertial motion (or neglecting altogether the EM self-force)! A consequence which follows is that the dynamics of point-like charged particles described by these approximate model equations is not Hamiltonian. However, it is not clear whether this feature is only an accident, i.e., is only due to the approximations introduced so far, or is actually an intrinsic feature of the RR problem.

Another key issue is, however, related to the treatment of the RR problem for point-particles in a proper sense, and in particular to the conditions of validity of the relativistic Hamilton variational principle \(^4\) in such a case. Actually, difficulties with the treatment of point-particles in classical electrodynamics and general relativity have been known for a long time. They are due to intrinsic divergences produced by the EM self-field \(^8\). In fact one can show that this problem is ill-posed since the self-fields diverge in the neighborhood of a point-particle's world line. For this reason in the past several authors, including Born and Infeld, Dirac, Wheeler and Feynman (see discussion in Ref. \[7\]), tried to modify classical electrodynamics in an effort to eliminate all divergent contributions arising due to EM self-interactions. This is the so-called regularization problem for point-particles, based on the introduction of suitable modifications of Maxwell's electrodynamics.

There is an extensive literature devoted to possible ways to achieve this goal. These theories either directly introduce 'ad hoc' modified definitions for the EM self-force (or of the EM self 4-potential) or introduce axiomatic approaches involving modifications of classical electrodynamics. Examples of the first type is provided by Dirac \([11]\) and Dewitt and Brehme \([15]\) who determined the RR self-force for a point particle belonging respectively to the Minkowski and curved space-times by imposing local energy conservation on a tube surrounding the particle's world line and subtracting the infinite contributions to the force through a so-called mass renormalization scheme. More recently, Ori \([16]\) who suggested a regularization scheme involving averaging of multipole moments. Another attempt is based on the adoption of an axiomatic approach in order to produce the general equation of motion for a point particle coupled to a scalar field. In recent years several different methods have been proposed for calculating the motion of a point particle coupled to its EM self-fields (for a review and references on the subject see for example \([17]\)). Finally, still another possible strategy involves introducing appropriate modifications of the EM self 4-potential. Typically this is done (see for example Rohlich \([18]\)) by assuming that there exists a decomposition of the EM field, whereby each particle "feels" only the action of external particles and of a suitable part of the EM self-field. While this decomposition becomes clearly questionable for finite-size particles, its consistency with first principles - and in particular with standard quantum mechanics - seems dubious, to say the least \([7]\).

Another possible approach for the search of an exact RR equation is represented by the description of classical charges by means of finite-size extended particles. An example of this type is provided by Medina \([12]\), who investigated the dynamics of a point particle character-
ized by an arbitrary spherically-symmetric charge. In his approach a formal integral representation for the RR force in the particle rest frame is achieved. This result is used to extrapolate the same force for point-like particles and to evaluate the general form of the RR 4-force in an arbitrary reference frame, thus yielding an approximate representation of the relativistic RR equation (Medina RR equation). By doing so, however, an asymptotic approach is inevitably adopted again. Another interesting feature of the Medina’s approach is that the extended-phase space variational approach is achieved by introducing an acceleration-dependent Lagrangian function.

In this paper, we intend to follow a similar route choosing, however, to consider: 1) spherical-shell charges and 2) the adoption, from the beginning, of a relativistic spherical-shell charges choosing, however, to consider: 1) R\textsuperscript{2}\, equation and 2) the adoption, from the beginning, of a relativistic variational approach based on the customary phase-space Hamilton variational principle. As we intend to prove in the following, this permits us to obtain an exact, i.e., non-asymptotic, RR equation.

1d - Main results

In this paper we intend to pose, for classical finite-size charged particles represented by shell-charges, the problem of the construction of the exact RR equation, in the sense indicated above. We want to show that its explicit construction can be achieved in the framework of classical electrodynamics, based on a straightforward generalization of Hamilton variational principle. The approach is based on the adoption of the relativistic (hybrid) synchronous Hamilton variational principle recently pointed out. Its basic feature is that it can be expressed virtually in terms of arbitrary “hybrid” variables (i.e., generally non-Lagrangian and non-canonical variables). The traditional approach, valid for point-particles, is extended to finite-size spherical-shell charges, by taking into account the contribution of the retarded EM self-potential generated by the particles themselves. Thus, based on the construction of the Euler-Lagrange equations stemming from the variational principle, the exact relativistic equation of motion for a charged particle of this type, immersed in a prescribed EM field and subject to the simultaneous action of its EM self-field, can be achieved explicitly in this way (THM.1-THM.3). In particular, it is found that the exact RR equation in covariant form is (see THM’s.1 and 2):

\[ m_o c du_\mu(s) = \frac{q}{c} \mathbf{F}^{(ext)}(s) \cdot dr_\nu(s) + ds \overline{G}_\mu. \]  

(1)

Here \( \mathbf{F}^{(ext)}(s) \) is the surface-average Faraday tensor - acting on a point particle located at the 4-position \( r \equiv \{ r^\mu, \mu = 0,3 \} \) - which is generated by the external EM field. In particular, the surface-averaging operator acting on a smooth position-dependent function \( A \), and denoted as \( \overline{A} \), is defined according to Appendix A (see Eq.(2)).

Moreover, \( \overline{G}_\mu \) is the (surface-average) RR 4-vector produced by the EM self-field and due to the action of the particle on itself. The rest of the notation is standard. Thus, \( c \) is the speed of light in vacuum, \( m_o \) and \( q \) are respectively the inertial rest-mass and charge of the particle, \( r^\mu(s) \equiv r^\mu(t(s)) \) denotes its position 4-vector parametrized in terms of the arc lengths \( s \) and \( u^\mu(s) = \frac{dr^\mu}{ds} \) is the corresponding 4-velocity. The explicit form of \( \overline{G}_\mu \) is found to be (see THM.2)

\[ \overline{G}_\mu = 2c \left( \frac{q}{c} \right)^2 \frac{1}{[R^\alpha u_\alpha(t)]^2} \left[ \frac{dr^\mu(t - t_{ret})}{ds} - R^\nu u_\nu(t) \frac{ds}{R^\alpha u_\alpha(t)} \right]. \]  

(2)

Here \( t - t_{ret} \) is the retarded time, with \( t_{ret} \) denoting a suitable delay-time [see Eq.(2)], while \( R^\alpha = r^\alpha(t) - r^\alpha(t - t_{ret}) \) and \( r^\alpha(t - t_{ret}) \) is the 4-position vector evaluated at the retarded time \( t' \). As a consequence of Eq.(2) the properties of \( \overline{G}_\mu \) can be immediately established (see THM.3). In particular, \( \overline{G}_\mu \) depends, besides \( r^\mu(s) \) and \( u^\mu(s) \) evaluated at the local time \( t = t(s) \), also on the 4-position and 4-velocity [of the particle itself], evaluated at the retarded time \( t' \), i.e., \( r^\alpha(t - t_{ret}) \) and \( \frac{dr^\alpha(t - t_{ret})}{ds} \). It follows that \( \overline{G}_\mu \) is a smooth function which is generally defined everywhere in a suitable extended phase space. Hence, the RR equation Eq.(1) is a retarded second-order ordinary differential equation. As a main consequence, the equation, together with the initial conditions

\[ r^\mu(s_o) = r^\mu_0, \]  

(3)

\[ u^\mu(s_o) = u^\mu_0, \]  

(4)

prescribed so that \( u^\mu_0 u_{\mu 0} = 1 \), defines locally a well-posed problem (THM.1). In addition, its solution results consistent with all basic principles of classical mechanics, including the principles of Galilei inertia, Newton determinacy and Einstein causality (THM.3).

To gain deeper insight and to allow comparisons with previous approaches, various asymptotic approximations and limits are considered in the sequel. These include: 1) the proof of the non-existence of the point-particle limit for the present theory (see THM.4), i.e., that the exact RR equation is not defined in such a case; 2) the “short-time” asymptotic approximation for the RR equation obtained in the so-called “short-time” ordering (see THM.5). This is obtained by introducing a Taylor expansion in terms of the dimensionless ratio \( \xi \equiv (t - t')/t > 0 \) (delay-time ratio), to be assumed infinitesimal; 3) the weakly-relativistic approximation for the RR equation, obtained by introducing a Taylor expansion in terms of the dimensionless ratio \( \beta \equiv v(t)/c \), again to be considered infinitesimal (as appropriate for the description of non-relativistic particle dynamics; see THM.6).
The analysis is useful to assess the accuracy and limits of validity of the customary LAD equation, either in the relativistic or weakly-relativistic descriptions. In both cases it is found that the LAD equation (as well as the related LL equation) provided, at most, only an asymptotic approximation to the exact RR equation \(^\text{(THM.6)}\). This conclusion can typically be reached, however, only provided the external EM field, defined in terms of the Faraday tensor \(F^{(\text{ext})}_{\mu\nu}\), is a suitably smooth function of the particle proper time \(\tau \equiv s/c\). In particular, both LAD and LL equations may not be valid for "sudden forces", i.e., external fields which are locally discontinuous with respect to \(\tau\).

1e - Scheme of presentation

The scheme of the presentation is as follows. In Sections 1 and 2 a brief overview of previous treatments is given in order to analyze the intrinsic difficulties met by previous point-charges descriptions for the RR problem. In the subsequent sections (Sec. 3 to 7) the new treatment which applies for finite-size charges is presented. In particular:

- In Sec.3 the exact EM 4-potential generated by a finite-size spherical shell is evaluated.
- In Sec.4 the Hamilton synchronous variational principle for a finite-size charge is developed and an explicit form of the RR equation is obtained (see \text{THM.1}). In particular, it is proven that the resulting relativistic RR equation is a second-order ordinary differential equation which defines a well-posed problem, i.e., that the solution of the corresponding initial-value problem locally exists and is unique.
- As a consequence (Sec.5), the 4-vector \(\mathbf{G}_{\mu}\) is introduced which describes the generalized Lorentz force acting on the particle generated by its EM self-field (see \text{THM.2}).
- In Sec.6 the main properties of \(\mathbf{G}_{\mu}\) are investigated. As a result it is proven that the RR equation is consistent with all basic principles of classical mechanics (THM.3).
- In Sec.7 the non-existence of the point-particle limit [for the RR equation] is proven (THM.4).

In Sec.8 the short-time approximation for \(\mathbf{G}_{\mu}\) is obtained. The resulting asymptotic RR equation is found consistent with the customary relativistic LAD equation (THM.5). Finally, in Sec.9 possible weakly-relativistic approximations of the RR equation are discussed. Also in this case, the resulting RR equation can be realized, unlike the customary weakly-relativistic LAD equation, by means of a second-order differential equation (THM.6).

2 - THE IMPOSSIBILITY IN CLASSICAL ELECTRODYNAMICS OF A VARIATIONAL DESCRIPTION FOR CLASSICAL POINT-CHARGES

A corner-stone of classical mechanics is represented by the Hamilton variational principle, which permits to determine the coupled set of equations formed by the particle dynamical equations and Maxwell’s equations \([6,14]\). As a consequence, both the particle state and the EM field in which the particle is immersed are uniquely determined by means of this variational principle. The choice of the dynamical variables which define the particle state remains in principle arbitrary. Thus, they can always be represented by so-called "hybrid" variables, i.e., superabundant variables which generally do not define a Lagrangian state. This implies, thanks to Darboux theorem, that it should always be possible to identify them locally with canonical variables. As a basic consequence, classical systems of charged particles are expected to define Hamiltonian systems, i.e., their canonical states should be extrema of the corresponding Hamiltonian action, while the corresponding particle dynamics, provided by the Euler-Lagrange equations determined by the same variational principle, necessarily should coincide with Hamilton’s equations of motion.

Nevertheless, it is easy to prove that for charged point-particles the Hamilton principle fails (see Ref.A). In fact, one can show that, if the Hamilton principle is expressed via a synchronous hybrid variational principle \([21]\), the point-charge action integral can be written in the form (here the notation is given according to Ref.A)

\[
S\left(r^\mu, u^\mu, \chi\right) = \int_{-\infty}^{\infty} \left( m_c u^\mu + \frac{q}{c} A^\mu(r) \right) dr^\mu + \int_{s_1}^{s_2} ds \chi(s) \left[ u^\mu(s) u^\mu(s) - 1 \right]
\]

(Hamilton action), which is applicable if the 4-potential \(A_{\mu}\) is considered prescribed. It is immediate to prove that the functional is actually not-defined. The reason is due to the intrinsic divergences appearing in the point-particle self 4-potential \(A_{\mu}^{(\text{self})}\) (see Appendix B). In fact, due to the superposition principle the EM 4-potential \(A_{\mu}(r)\) can always be represented in terms of the fundamental decomposition

\[
A_{\mu} = A_{\mu}^{(\text{self})} + A_{\mu}^{(\text{ext})},
\]

where \(A_{\mu}^{(\text{self})}\) and \(A_{\mu}^{(\text{ext})}\) denote respectively the point-particle self 4-potential and the external 4-potential. In particular, by assumption \(A_{\mu}^{(\text{self})}(r) \equiv A_{\mu}^{(\text{self})}(r(s))\) is a solution of the Maxwell’s equations which in flat space-time are given by

\[
\partial_\nu F^{\nu\mu}_{\text{self}} = \frac{4\pi}{c} j^\mu,
\]
with \( j^\mu(r') = \int_{\partial S} ds' u^\mu(s') \delta^{(d)}(r - r(s')) \) denoting the 4-current carried by the point charge. Hence, in the functional \( S_1(r^\mu, u_\mu, \chi) \), the 4-vector function \( A_\mu \) must be considered a prescribed function of the varied 4-vector \( r^\mu(s) \). Invoking the causality principle, the explicit form of \( A_\mu^{(self)} \) for a point particle immersed in the Minkowski space-time \( M^4 \equiv \mathbb{R}^4 \) can be easily recovered (see Appendix B) and is provided by the well-known retarded EM 4-potential (in covariant form)

\[
A_\mu^{(self)}(r) = \frac{q}{c} \frac{u_\mu(t')}{R^{\alpha} u_\alpha(t')},
\]

which can be represented in the equivalent integral form given by Eq. (8). Here \( R^\mu, u_\mu(t'), u^\mu(t') \) and \( t' \) are respectively the bi-vector \( R^\mu \equiv r^\mu - r^\nu, \) with \( r^\mu \equiv r^\mu(t') \), the 4-velocity \( u^\mu(t') \equiv \frac{dr^\mu(t')}{dt} r^\mu(t') = \gamma' \frac{dr^\mu(t')}{dt} \) and its covariant components \( u_\mu(t') \), while \( t' \) is a suitable retarded time. In particular this is defined so that

\[
t - t' = \frac{|r - r'|}{c},
\]

where \( r' \equiv r(t') \). We now notice that for an arbitrary varied curve \( r(s) \), the inf of \( t - t' \) is generally not strictly positive in the case of a point-charge. As a consequence, the contributions carried by \( A_\mu^{(self)} \) in the functional \( S_1(r^\mu, u_\mu, \chi) \) contain essential divergences. This means that, when the self 4-potential \( A_\mu^{(self)} \) is properly taken into account in the point-charge action functional \( S_1(r^\mu, u_\mu, \chi) \), the functional cannot actually be defined.

### 3 - THE RETARDED EM SELF 4-POTENTIAL OF A FINITE-SIZE CHARGE

A prerequisite for the subsequent developments is the determination of the EM self-potential \( (A_\mu^{(self)}) \) produced by a prescribed charge distribution. As indicated above, in this paper we wish to consider the case of a classical particle characterized by point-particle mass and - respectively - finite-size charge distributions. For definiteness, here we shall determine the EM 4-potential generated by a finite-size spherical-shell particle immersed in the Minkowski space-time. In particular, we assume that when observed with respect to the particle rest-frame the charge density takes the form

\[
\rho(r, t) = \frac{q}{4\pi \sigma^2} \delta(|r - r(t)| - \sigma).
\]

First, let us evaluate the retarded electrostatic (ES) potential generated by \( \rho(r, t) \) and measured at a position \( r \) defined in such a frame. This is manifestly defined as

\[
\Phi^{(self)}(r, t) = \int d^3 r' \frac{1}{R} \rho(r, t - \frac{R}{c}),
\]

with \( R \equiv |R| \) and \( R = r - r' \). It is well known that \( \Phi^{(self)}(r, t) \) can be determined conveniently by introducing an expansion in Legendre polynomials for the integrand \( \frac{1}{R} \rho(r, t - \frac{R}{c}) \). As a result one can readily show that for a finite-size spherical-shell charge the retarded ES potential is (see for example [20])

\[
\Phi^{(self)}(r, t) = \begin{cases} \frac{q}{c} R \geq \sigma \\ \frac{q}{c} R < \sigma, \end{cases}
\]

where

\[
R \equiv |R|, \quad \Phi^{(self)}(r, t) = \Phi^{(self)}(r, t),
\]

Therefore, in the internal domain \((R < \sigma)\) the EM self-potential does not produce any self-field. Instead, in the external domain \((R \geq \sigma)\) its expression is the same as that produced by a point-charge. In both cases the ES potential is manifestly spherically symmetric, therefore it follows by construction that in the rest-frame:

\[
\Phi^{(self)}(r, t) = \Phi^{(self)}(r, t),
\]

where \( \Phi^{(self)}(r, t) \) is the surface-average [31]. The corresponding expression of the EM 4-potential in a moving frame can be easily obtained by applying a Lorentz transformation. In particular, since in this case the external domain is defined by the inequality \( R^\alpha R_\alpha \geq \sigma^2 \) the corresponding surface-average EM self 4-potential \( \Phi^{(self)}_\mu \) is given again by Eq. (8), namely

\[
(16) \quad \Phi^{(self)}_\mu(r) = \frac{q}{c} \frac{u_\mu(t')}{R^{\alpha} u_\alpha(t')},
\]

Instead, in the internal domain \((R^\alpha R_\alpha < \sigma^2)\) there results necessarily \( \Phi^{(self)}_\mu \equiv const., \) so that \( \Phi^{(self)}_\mu \equiv 0 \) in this subset. As a further consequence, if \( r^\alpha \) is the 4-position vector of the point-particle, there results (see Appendix C)

\[
\Phi^{(self)}_\mu(r) = \frac{2q}{c} \int d^3 r' \delta(R^\alpha R_\alpha - \sigma^2),
\]

where \( R^\alpha = r^\alpha - r'^\alpha. \)

### 4 - THE RR EQUATION FOR A FINITE-SIZE CHARGE

In this section we wish address the key issue posed in this paper, i.e., the problem of the explicit construction of the relativistic RR equation for a finite-size spherical shell charge. Here we intend to prove that, as earlier pointed out in Ref.A, this goal can be uniquely established based on a suitable formulation of the Hamilton variational principle. More precisely, we intend to prove that:
• the exact RR equation can be obtained by making use of a suitably modified form of the synchronous Hamilton variational principle appropriate for finite-size charges (see THM.1);
• the solution of the related initial-value problem exists and is unique, i.e., the RR equation defines a well-posed problem (THM.1).

4a - Treatment of finite-size particles

First, let us generalize the Hamilton action functional [Eq.(2)] to treat finite-size particles. This is obtained formally by introducing in $S_1(rμ, uμ, χ)$ the replacements

$$ds \rightarrow W(r, s) \frac{dΩ}{\sqrt{-g}},$$

$$dr^μ \rightarrow \frac{dr^μ}{ds} W(r, s) \frac{dΩ}{\sqrt{-g}},$$

where $W(r, s)$ is the so-called "wire function", generally to be identified with a suitable distribution. For a generic $W(r, s)$ the appropriate form of the variational functional (to be expressed again in synchronous form [21]) becomes

$$S_1(rμ, uμ, χ) =$$

$$= \int \frac{dΩ}{\sqrt{-g}} W(r, s) (m_α c u_μ + \frac{q}{c} A_μ(r)) \frac{dr^μ}{ds} +$$

$$+ \int_{s_1}^{s_2} \frac{dΩ}{\sqrt{-g}} W(r, s) χ(s) [u_μ u^μ - 1].$$

4b - The wire function of a spherical-shell charge

Let us now consider, in particular, the case of a spherical-shell charge, while requiring that the mass is still point-wise localized, i.e., it is a point-particle (see discussion in Sec.1a) with 4-position $rμ(s)$ and 4-velocity $u^μ(s)$. To obtain the appropriate representation of the wire function in this case, let us introduce the coordinate transformation $rμ → (s, ξ^1, ξ^2, ρ)$. Here $s$ is the arc length along the particle world line, $ξ^1$ and $ξ^2$ are two curvilinear angle-like coordinates on the surface $∂Ω_σ$, and $ρ$ is the 4-scalar defined so that $ρ^2 = R^2 R_α$, with $R^2 = r^2 - r^2(s)$ and $r^2(s)$ denoting the 4-position of the point particle. It follows that the wire-function for a spherical-shell charge can be defined as

$$W = \frac{1}{4πσ^2} δ(ρ - σ),$$

while the invariant volume element is $dρ^2 dρ ω(ρ)/\sqrt{-g}$ with $\sqrt{-g} = 1$ for flat space-time and $dσ^2/\sqrt{-g}$ denoting a suitable invariant surface element. It follows that $W$ is non-zero only if

$$r^α = r^α(s) + σ n^α(ξ^1, ξ^2),$$

where $n^α$ is a unit 4-vector ($n^α n_α = 1$) depending only on $(ξ^1, ξ^2)$. Thus, in particular, in the rest-frame of the same particle $W$ takes the form

$$W = \frac{1}{4πσ^2} δ(|r - r(s)| - σ),$$

while $dσ(n)$ can be identified with the solid angle (surface element of 3-sphere of unit radius centered at the particle position $r$), $ρ = |r - r(s)|$ and the 4-vector $n$ reads $n = (0, n)$, $n$ denoting the normal unit 3-vector to the surface $∂Ω_σ$. It follows that for an arbitrary 4-tensor $A(r^μ(s) + σn)$ evaluated at the 4-position (22) one can define an appropriate surface-average (see Appendix A).

4c - Spherical-shell charge Hamilton principle

The appropriate form of the Hamilton action functional for a spherical-shell charge is found to be given by the following Lemma.

**LEMMA 1 - Spherical-shell charge action integral**

For a finite-size spherical-shell charge the Hamilton action integral defined by Eq. (20) reads:

$$S_1(rμ, uμ, χ) =$$

$$= \frac{1}{4π} \int dσ(n) \int_1^2 \left[ m_α c u_μ(s) + \frac{q}{c} A_μ^{(ext)}(r(s) + σn) \right] dr^μ +$$

$$+ ΔS_1(rμ) +$$

$$+ \frac{1}{4π} \int dσ(n) \int_{s_1}^{s_2} χ(s) [u_μ(s) u^μ(s) - 1] ds$$

(Hamilton action integral), where $ΔS_1(rμ)$ is the functional carrying the contribution of the EM self 4-potential

$$ΔS_1(rμ) = \frac{1}{4π} \int dσ(n) \int_1^2 \frac{q}{c} A_μ^{(ext)}(r(s) + σn) dr^μ.$$

In view of Eq.(17) and the surface average [37] there results

$$ΔS_1(rμ) =$$

$$= 2 \left( \frac{q}{c} \right)^2 \int_1^2 dr^μ \int_1^2 dr_σ δ(R^α R_α - σ^2).$$

**Proof -** The proof of Eq.(24) follows from the wire-function functional [Eq.(21)] upon invoking Eq.(23) for the wire function. Instead, the specific form of the functional $ΔS_1(rμ)$ [Eq.(26)], which carries the EM self 4-potential, follows invoking the integral representation [17]. **Q.E.D.**

As a basic consequence, invoking in particular the surface-average of $A_μ^{(ext)}(r(s) + σn)$ given by Eq. (31), all
terms in the integrand of the action functional (24) become independent of the surface element \(d\Sigma(n)\). Hence, the action functional reduces simply to:

\[
S_1(r^\mu, \mu_r, \chi) = \int \left( m_o c u^\mu(s) + \frac{q}{c} A^{(ext)}_{\mu}(r(s)) \right) dr^\mu + 2 \left( \frac{q}{c} \right)^2 \int_1^2 dr^\mu \int_1^2 dr^\mu \delta(R^\alpha R_o - \sigma^2) + \int_{\Sigma} \chi(s) [\mu_\mu(s) u^\mu(s) - 1] ds.
\]

Let us now prove that the relativistic dynamics of a (finite-size) spherical shell particle is uniquely prescribed by the Hamilton variational principle defined in terms of \(S_1(r^\mu, \mu_r, \chi)\), specified according to Eq. (24). In particular, in this case, due to the assumption that the mass of the particle is point-wise localized, the extremal curve must be necessarily of the form \(r \equiv r(s)\) [see Assumption 3 in THM.1]. The following result then holds:

**THM.1 - Hamilton principle for a spherical-shell charge**

Let us assume that: 1) the real varied functions \(f(s) \equiv \{r^\mu(s), \mu_r(s), \chi(s)\}\) belong to a suitable functional class \(\{f\}\) in which end points and boundaries are kept fixed; 2) the Hamilton action integral \(S_1(r^\mu, \mu_r, \chi)\) defined by Eq. (24) is assumed to exist for all \(f(s) \in \{f\}\). Here, \(u^\mu(s) = g^{\mu\nu} u^\nu(s)\), while \(g^{\mu\nu} = g^{\mu\nu}(r(s))\) denotes the counter-varient components of the metric tensor, each one to be considered dependent on the generic varied curve \(r(s)\); furthermore, \(m_o\) and \(q\) are respectively the constant rest mass and electric charge of a point particle and \(ds\) the line element; 3) an extremal curve \(f \in \{f\}\) of \(S_1\) is assumed of the form \(f(s)\), i.e., to be independent of \(n\); 4) if \(r(s)\) is an extremal curve of \(S_1\) the line element \(ds\) satisfies the constraint \(ds^2 = g^{\mu\nu}(r(s))dr^\mu(s)dr^\nu(s)\).

Then it follows that:

**T1.1** if the synchronous variations \(\delta f(s)\) [see also Appendix D] are considered as independent, the Euler-Lagrange equations following from the synchronous variational principle

\[
\delta S_1(r^\mu, \mu_r, \chi) = 0
\]

yield identically the RR equation of motion for a finite-size spherical-shell charged particle, which reads:

\[
m_o c du^\mu(s) = \frac{q}{c} A^{(ext)}_{\mu}(r(s)) dr^\nu(s) + dr^k H_{\mu k},
\]

where \(\bar{A}^{(ext)}_{\mu}(r(s))\) is the surface-average of the Faraday 4-tensor \(\bar{A}^{(ext)}_{\mu\nu}(r(s)) = \partial_{\nu}A_{\mu}^{(ext)} - \partial_{\mu}A_{\nu}^{(ext)} \) evaluated at the 4-position \(r(s)\). In addition, \(r^\mu \equiv r^\mu(t), r^\mu(t) \equiv r^\mu(t'), u^\mu = \frac{dr^\mu}{ds}\) is the 4-velocity, \(v^\mu(t)\) denotes \(v^\mu(t) = \frac{dr^\mu}{dt}\) and \(H_{\mu k}\) is the function

\[
H_{\mu k} = 2 \left( \frac{q}{c} \right)^2 \int \left[ \frac{1}{c (t - t') - \frac{1}{c^2} \frac{dr^\nu}{dt} \cdot (r - r')} \right] dr^\nu = 0.
\]

Finally, \(r \equiv r(t)\) and \(r' \equiv r(t')\), while \(t' = t - t_{ret}\) denotes the retarded time and \(t_{ret}\) a suitable delay-time;

**T1.2** the delay-time \(t_{ret}\) is the positive root of the equation

\[
R^\alpha R_o = \sigma^2
\]

(delay-time equation) which is

\[
t_{ret}(t) \equiv t - t' = \frac{1}{c} \sqrt{(r(t) - r(t - t_{ret}))^2 + \sigma^2} > 0;
\]

**T1.3** let us require that the 4-vector-field \(\bar{A}^{(ext)}_{\mu}(r)\) is suitably smooth in the whole Minkowski space-time \(M^4\), i.e., is at least \(C^2(M^4)\); then the initial-value problem set by the Euler-Lagrange equation (24), with the initial conditions

\[
x(t_o) = x_o,
\]

(where \(x(t_o) \equiv [r^\mu(t_o), \mu_r(t_o)]\) and \(x_o \equiv [r^\mu_o, \mu_{\mu o}]\) denotes a suitable initial state), is locally well-posed.

**Proof - T1.1** It is immediate to construct explicitly the Euler-Lagrange equations of the Hamilton action \(S_1(r^\mu, \mu_r, \chi)\). In fact, first, since \(\frac{\partial}{\partial \mu_r} \delta(R^\alpha R_o - \sigma^2) = \frac{\partial}{\partial \mu_r} \delta(R^\alpha R_o - \sigma^2) = 0\), the variations with respect to \(\chi(s)\) and \(\mu_r\) deliver respectively

\[
u^\mu(s)u^\mu(s) - 1 = 0, \quad m_o c du^\mu + 2 \chi(s) u^\mu(s) ds = 0,
\]

while it must result for consistency \(2\chi(s) = -m_o c\) (as in the case in which \(A_{\mu}^{(self)}\) is assumed to vanish identically). To reach Eq. (29), instead, let us invoke Lemma 2 [see Appendix D]. Then, thanks to assumption 3), the variation with respect to \(r^\mu\) can easily be proven to yield the Euler-Lagrange equation defined by Eq. (24). Together with Eq. (35), this manifestly defines the RR equation, i.e., the exact relativistic equation of motion for a point charge subject to the simultaneous action of a prescribed external EM field and of its self-EM field.

**T1.2** Recalling that in the Minkowski metric the retarded-time equation [Eq. (31)] reads

\[
R^\alpha R_o = c^2(t - t')^2 - (r - r')^2 = \sigma^2,
\]

with \(R^\alpha = \bar{R}^\alpha(t) - \bar{R}^\alpha(t')\) and \(r' = r(t')\), the proof of Eq. (32) is straightforward.
Finally, it is immediate to show that the problem defined by Eq. (33), together with the initial conditions defined by Eq. (33), admits a local existence and uniqueness theorem (fundamental theorem). In fact it is obvious that Eq. (29) can be cast in the form of a delay-differential equation, i.e.,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{X}(\mathbf{x}(t), \mathbf{x}(t-t_{ret}), t),$$  \hfill (37)

where \( \mathbf{x}(t) \) and \( \mathbf{x}(t-t_{ret}) \) denote respectively the "instantaneous" and "retarded" states \( \mathbf{x}(t) \equiv [r^\mu(t), u_\mu(t)] \) and \( \mathbf{x}(t-t_{ret}) \equiv [r^\mu(t-t_{ret}), u_\mu(t-t_{ret})] \), while \( \mathbf{X}(\mathbf{x}(t), \mathbf{x}(t-t_{ret}), t) \) is a suitable \( C^2 \) real vector field depending smoothly on both of them. It is manifest that the fundamental theorem holds for Eqs. (33) - (37). In fact, by considering \([\mathbf{X}] \mathbf{x}(t-t_{ret})\) as a prescribed function of time, the previous equation recovers the canonical form

$$\frac{d\mathbf{x}(t)}{dt} = \check{\mathbf{X}}(\mathbf{x}(t), t),$$  \hfill (38)

with \( \check{\mathbf{X}}(\mathbf{x}(t), t) \) denoting the corresponding \( C^2 \) real vector field. This proves the statement. Q.E.D.

### 5 - Determination of the RR 4-vector \( \overline{G}_\mu \)

A basic consequence of THM.1 is that the RR equation can be expressed in covariant form. This permits us to identify the RR 4-vector \( \overline{G}_\mu \), which represents the (generalized) Lorentz force produced on a charged particle by its EM self-field. Here we intend to show, in particular, that \( \overline{G}_\mu \) can be expressed in covariant form and uniquely parameterized in terms of the proper length \( s \), defined at the point-particle 4-position vector \( r^\mu \). The main result is represented by the following theorem which provides also an explicit representation of the 4-vector \( \overline{G}_\mu \).

**THM.2 - Covariant representation of \( \overline{G}_\mu \)**

For the Minkowski metric the covariant RR equation reads

$$m_\alpha cdu_\alpha(s) = \frac{d}{c}(\overline{G}_\mu)_{\mu}(s)dr^\nu(s) + \overline{G}_\mu ds.$$  \hfill (39)

Here the 4-vector \( \overline{G}_\mu \equiv (G_\alpha, G) \) is defined as:

$$\overline{G}_\mu = 2c \left( \frac{d}{c} \right)^2 \left( \frac{1}{R^\nu u_\nu(t')} R_k^{\nu} \right) \left[ \frac{1}{R^\nu u_\nu(t')} \right] k = \mu.$$  \hfill (40)

(this covariant representation with respect to \( s' \)). Here \( s \) and \( s' \) are defined respectively by \( ds = cd\sqrt{1 - \beta^2(t)} \) and \( ds' = cd\sqrt{1 - \beta^2(t')} \), where \( t' = t - t_{ret} \) is the retarded time and \( \beta^2(t) = \frac{1}{c^2} \left( \frac{dr(t)}{dt} \right)^2 \). An equivalent representation of \( \overline{G}_\mu \) in terms of the particle arc length \( s \) is:

$$\overline{G}_\mu = 2c \left( \frac{d}{c} \right)^2 \left[ \frac{1}{R^\nu u_\nu(t') \left[ R^\nu u_\nu(t') \right]^2 \mu_m(t') \mu_m(t) \right] \left[ R^\nu u_\nu(t') \right] \left[ R^\nu u_\nu(t') \right]^2 \mu_m(t') \mu_m(t) \right] \left[ R^\nu u_\nu(t') \right]$$  \hfill (41)

of \( \overline{G}_\mu \) with respect to \( s \), where \( R^\alpha = r^\alpha(t) - r^\alpha(t - t_{ret}) \). This can be proven to yield also a parametric representation of \( \overline{G}_\mu \) in terms of \( s \).

**Proof -** The proof of the first covariant representation of \( \overline{G}_\mu \) [given by Eq. (40)] follows immediately. In fact, by definition there results \( \frac{d}{ds} \left[ R^\alpha u_\alpha(t') \right] = \frac{d}{ds} \left[ R^\alpha u_\alpha(t') R_k^{\nu} \right] u_k(t') \), where \( u_k(t') = \gamma(t') \mu_m(t') \), with \( \gamma(t') = 1/\sqrt{1 - \beta^2(t')} \) and \( \mu_m(t') \) denoting \( \mu_m(t') = \frac{d}{dt}[\mu_m(t)] \). Instead, to prove the representation (41) we first notice that by construction \( d(R^\alpha R_k) = 0 \). Hence the two differential constraints \( dr^k(t)R_k = dr^k(t)R_k \) and \( \frac{1}{R^\nu u_\nu(t)} \frac{d}{ds} = \frac{1}{R^\nu u_\nu(t)} \frac{d}{ds} \) (see also Lemma 3 in Appendix D) must be fulfilled too. This implies that the following differential identity must hold

$$\frac{d}{ds} r_k(t') R_k - \frac{d}{ds} r_k(t') R_k \right] R_{\mu} =$$  \hfill (42)

$$\frac{d}{ds} r_k(t') R_k - \frac{d}{ds} r_k(t') R_{\mu}.$$  \hfill (43)

Substituting this expression in Eq. (40) there follows

$$\overline{G}_\mu = 2c \left( \frac{d}{c} \right)^2 \left[ \frac{1}{R^\nu u_\nu(t') \left[ R^\nu u_\nu(t') \right]^2 \mu_m(t') \mu_m(t) \right] \left[ R^\nu u_\nu(t') \right].$$  \hfill (44)

Invoking Lemma 3 this delivers Eq. (41). Here we notice that the proper-time derivatives \( \frac{d}{ds} r_k(t') \) and \( \frac{d}{ds} r_k(t') \) are evaluated invoking the chain rule. This is obtained by introducing the diffeomorphism \( t \to s(t) \equiv s \) [and similarly \( t' \to s'(t') \equiv s' \)] with its inverse transformation \( s \to t(s) \). It follows \( t'(s') = t(s) - t_{ret} \). This proves that Eq. (41) delivers a parametric representation of \( \overline{G}_\mu \) in terms of the local arc length \( s \). Q.E.D.

Thus, remarkably, Eq. (41) shows that, when parameterized in terms of the local arc length \( s \), the 4-vector \( \overline{G}_\mu \) depends - at most - on first-order derivatives (with respect to \( s \)) of the 4-position, i.e., is a function only of the 4-bi-vector \( R^\alpha \) and of the derivatives \( u_k(t') = \frac{d}{ds} r_k(t') \) and \( \frac{d}{ds} r_k(t') \).
6 - PROPERTIES OF $\mathbf{G}_\mu$

In this section we intend to investigate the main properties of the 4-vector $\mathbf{G}_\mu$ (and hence of the RR equation given above). We intend to show that they are fully consistent with the basic principles of classical mechanics. In particular it is immediate to prove that $\mathbf{G}_\mu$ fulfills:

- Galilei’s principle of inertia: in fact, in the case of inertial motion it results identically $\mathbf{G}_\mu \equiv 0$;
- the characteristic property of the Lorentz force, i.e., the Lorentzian constraint
  \[ \mathbf{G}_\mu u^\mu = 0. \] (44)
- Newton’s principle of determinacy and Einstein’s causality principle.

Finally, it can be shown that:

- $\mathbf{G}_\mu$ is defined also in the case of “sudden forces”.

These results are summarized in the following theorem:

**THM.3 - Properties of $\mathbf{G}_\mu$**

In validity of THM.1 and THM.2, the vector $\mathbf{G}_\mu$ fulfills the following properties:

- $T3_1$) in case of inertial motion in a given proper-time interval $[s_1, s_2]$, there results identically $\mathbf{G}_\mu \equiv 0$;
- $T3_2$) if $F^{ext}_{\mu\nu}(r(s)) \equiv 0 \forall s$ in a given proper-time interval $[s_1, s_2]$ and with respect to an inertial frame, then there results identically $\mathbf{G}_\mu \equiv 0 \forall s \in [s_1, s_2]$ (Galilei’s inertia principle);
- $T3_3$) $\mathbf{G}_\mu$ satisfies the Lorentzian constraint condition
  \[ \mathbf{G}_\mu u^\mu = 0. \] (45)

Moreover, assuming that the RR equation (39), with (40), admits smooth solutions in the proper-time interval $[s_a, s_b]$, in such an interval:

- $T3_4$) $\mathbf{G}_\mu$ fulfills the Einstein’s causality principle, namely for any $s \in [s_a, s_b]$, $r^\mu(s)$ depends only on the past history of $r_\nu(s)$, i.e., \( \{r^\mu(s), \forall s \leq s\} \);
- $T3_5$) $\mathbf{G}_\mu$ fulfills Newton’s determinacy principle, namely for any $s_0 \in [s_a, s_b]$, the knowledge of the particle initial state \( \{r^\mu(s_0), u_\nu(s_0)\} \) determines uniquely the particle state \( \{r^\mu(s), u_\nu(s)\} \) at any $s \geq s_0$ which belongs to $[s_a, s_b]$;
- $T3_6$) $\mathbf{G}_\mu$ is defined also in the case of “sudden forces”.

For example, let us require that the external EM field has the form

\[
F^{(ext)}_{\mu\nu}(r(s)) = \begin{cases} 
0 & s \leq s_0 \\
F^{(0)}_{\mu\nu} & s > 0
\end{cases}
\] (46)

with $F^{(0)}_{\mu\nu}$ a constant 4-tensor and $s_0 \in [s_a, s_b]$. In such a case one can prove that the solution of the RR equation exists and is unique.

**Proof** - To prove propositions $T3_1$ and $T3_2$ let us assume that in the interval $[s_1, s_2]$ the motion is inertial, namely that \( \frac{d}{ds} u_\mu(s) \equiv 0, \forall s \in [s_1, s_2] \). This implies that in $[s_1, s_2]$, $u_\mu \equiv u_\mu(0)$, with $u_\mu(0)$ denoting a constant 4-vector velocity. It follows $\forall s, s' \in [s_1, s_2]$, $r_\mu(s) = r_\mu(s') + u_\mu(s')(s - s')$ and $R_\mu = u_\mu(0)(s - s')$. Hence, there results identically

\[
\frac{dr_\mu(t - t_{ret})}{ds} R_k - \frac{dr_\mu(t - t_{ret})}{ds} R_\mu = \frac{u_\mu(0)u_\mu(0)(s - s') - u_\mu(0)u_\mu(0)(s - s')}{s - s'} = 0.
\] (47)

Propositions $T3_3$, $T3_4$ and $T3_5$ follow, similarly, by direct inspection of Eqs. (39) and (40), or similarly Eq. (41). In particular, $T3_5$ is an immediate consequence of THM.1 and the fact that the RR equation defines a well-posed initial-value problem. Finally, the proof of proposition $T3_6$ can be obtained by explicit construction of the solution of the RR equation (see analogous treatment given in Ref. [2] for the weakly-relativistic LAD equation).

Q.E.D.

7 - NON-EXISTENCE OF THE POINT-CHARGE LIMIT

An important aspect of the present formulation concerns the validity of the RR equation obtained letting

\[
\sigma \to 0^+
\] (48)

(point-charge limit) in the definition of $\mathbf{G}_\mu$ [see Eq. (40) or (41)]. Here we intend to prove that:

- the exact RR equation is not defined in the limit (48) [see following THM.4]. In other words, the point-charge limit for $\mathbf{G}_\mu$ is not defined.

To establish the result let us introduce yet another representation of the 4-vector $\mathbf{G}_\mu$ which makes explicit its dependence in terms of the parameter $\sigma$, the radius of the spherical charge distribution. For definiteness let us introduce the position

\[
w \equiv v(t') + \frac{1}{(t' - t_2)} \int_0^{t_2} dt_2 a(t_2)(t - t_2). \] (49)

Eq. (50) can also be written as

\[
R^\mu R_\mu = c^2(t' - t)^2 \left( 1 - \frac{u^2}{c^2} \right) = \sigma^2,
\] (50)

so that the delay-time $t' - t = t_{ret}$, with $t_{ret} > 0$, reads

\[
t_{ret} = \frac{\sigma}{c\sqrt{1 - \frac{u^2}{c^2}}}. \] (51)
Here it is obvious that for all \( \sigma > 0 \) the following inequalities must hold

\[
\begin{align*}
1 - \frac{w^2/c^2}{c^2} & > 0, \\
t_{\text{ret}} &= \frac{\sqrt{1 - \frac{w^2}{c^2}}}{c} > \frac{1}{c}, \\
R^\alpha v_\alpha(s) &= \frac{c^2}{\sqrt{1 - \frac{w^2}{c^2}}} \left[ 1 - \frac{1}{c^2} \frac{d\tau(t)}{dt} \cdot w \right] > 0, \\
\end{align*}
\]

Thus, introducing the 4-vector \( X_k \equiv \frac{1}{c^2} \sqrt{1 - \frac{w^2}{c^2}} R_k = \{ c, w \} \) one obtains for \( \overline{\mathcal{G}}_{\mu} \) the representation

\[
\overline{\mathcal{G}}_{\mu} = -2c \left( \frac{\sigma}{\sigma c} \right)^2 \frac{1}{\sigma c} \left[ 1 - \frac{1}{c^2} \frac{d\tau(t)}{dt} \cdot w \right] \left( \frac{1}{c^2} \right),
\]

which displays explicitly its dependence in terms of \( \sigma \).

The singular limit \( \sigma \to 0^+ \)

Let us now investigate the limit \( \sigma \to 0^+ \) for \( \overline{\mathcal{G}}_{\mu} \). The following (non-existence) theorem holds:

**THM.4 - Non-existence of the point-charge limit for \( \overline{\mathcal{G}}_{\mu} \)**

The limit \( \lim_{\sigma \to 0^+} \overline{\mathcal{G}}_{\mu} \) is not defined. In other words: for spherically symmetric charges the RR 4-vector is not defined in the limit \( \lim_{\sigma \to 0^+} \).

Proof - Let us introduce the absurd hypothesis that the following limits exist:

\[
\begin{align*}
\lim_{\sigma \to 0^+} \left( 1 - \frac{w^2}{c^2} \right) & > 0, \\
\lim_{\sigma \to 0^+} \left( 1 - \frac{1}{c^2} \frac{d\tau(t)}{dt} \cdot w \right) & > 0,
\end{align*}
\]

and moreover that for non-inertial motion there results

\[
0 < \lim_{\sigma \to 0^+} \left| \frac{d}{ds} \hat{H}_{\mu k} \right| < \infty,
\]

where

\[
\hat{H}_{\mu k} \equiv \frac{d\tau_{\mu k}(t-t_{\text{ret}})}{ds} X_k - \frac{d\tau_{\mu k}(t-t_{\text{ret}})}{ds} X_{\mu} \left( 1 - \frac{1}{c^2} \frac{d\tau(t)}{dt} \cdot w \right) \sqrt{1 - \frac{v^2(t)}{c^2}}.
\]

In such a case, invoking Eq. (53) for \( \overline{\mathcal{G}}_{\mu} \), it follows necessarily

\[
\lim_{\sigma \to 0^+} \overline{\mathcal{G}}_{\mu} \propto \lim_{\sigma \to 0^+} \frac{2}{c^2 \sigma^2} = \infty.
\]

Hence, in validity of (54) - (56), the limit \( \lim_{\sigma \to 0^+} \overline{\mathcal{G}}_{\mu} \) does not exist. On the other hand if one of the inequalities (54) - (56) is violated, the motion defined by the RR equation [Eq. (59)] is non-physical, which brings again the same conclusion. Q.E.D.

8 - SHORT-TIME APPROXIMATION AND THE LAD EQUATION

A crucial point in the Dirac evaluation of the LAD equation [11] was the power-series expansion of the retarded potential in terms of a suitably defined small dimensionless parameter \( \xi \), related to the proper-time difference between emission \((t')\) and observation \((t)\) times,

\[
0 < \xi \equiv \frac{(t - t')}{t},
\]

to be assumed as infinitesimal (short-time ordering). The same approach was also adopted by DeWitt and Brehme [12] in their covariant generalization of the LAD equation valid in curved space-time.

In analogy, here we introduce a power-series expansion with respect to the dimensionless parameter \( \xi \) of the form

\[
\overline{\mathcal{G}}_{\mu} = \sum_{k=0}^{\infty} \xi^k \overline{\mathcal{G}}_{\mu}^{(k)},
\]

which is assumed to converge for

\[
\xi \ll 1
\]

(short-time asymptotic ordering). The power series expansion is actually obtained by introducing a Taylor expansion for the 4-position vector \( r^\mu(t - t_{\text{ret}}) \) in terms of the retarded time \( t' \), namely letting

\[
r^\mu(t - t_{\text{ret}}) = \sum_{k=0}^{\infty} \frac{(t' - t)^k}{k!} \frac{d^k r^\mu(t)}{dt^k}.
\]

Manifestly, for the validity (i.e., the convergence) of the series, a prerequisite is that \( r^\mu(s) \equiv r^\mu(t(s)) \) is a \( C^{(\infty)} \) function. In turn, this requires that also the Faraday tensor generated by the external EM field, \( F_{\mu}^{(ext)\nu} \), must be \( C^{(\infty)} \). The use of the expansion (59) to represent the 4-vector \( \overline{\mathcal{G}}_{\mu} \) reduces, formally, the RR equation to a local and infinite-order ordinary differential equation. In view of THM.1 and the assumed convergence of the series its (infinitely) smooth solution must still exist and be uniquely defined. As a side consequence, this means that the initial conditions for such an equation must necessarily be considered as uniquely prescribed in terms of the initial conditions defined above [see Eqs. (3) and (4)] and the same RR-equation. Nevertheless, despite these features, the full series-representation of the RR equation obtained in this way appears practically useless for actual applications.
As an alternative, however, assuming \( \xi \) as infinitesimal an asymptotic approximation for \( G_{\mu} \) [and the exact RR equation Eq. (29) or equivalent Eq. (39)] can in principle be achieved, subject again to suitable smoothness assumptions to be imposed on the external field. Here we intend to prove, in particular, that in this way:

- the relativistic LAD equation is recovered as a leading-order asymptotic approximation to the exact RR equation. In fact, provided suitable smoothness conditions are met by the external field, the 4-vector \( G_{\mu} \) recovers asymptotically - in a suitable approximation - the usual form of RR equation provided by the LAD equation. This conclusion is achieved by introducing for the 4-vector \( G_{\mu} \) an asymptotic expansion with respect to the dimensionless parameter \( \xi \ll 1 \), obtained by means of a Taylor expansion in terms of the retarded time \( t' \), i.e., of the form

\[
\tau(t - t_{\text{ret}}) = \sum_{k=0}^{N} \frac{(t' - t)^k}{k!} \frac{d^k \tau(t)}{d t^k},
\]

with \( N > 0 \) to be suitably prescribed.

In such a case the following result holds:

**THM.5 - First-order, short-time asymptotic approximation for \( G_{\mu} \)**

Let us now assume that the EM 4-potential of the external field \( A_{\mu}^{(\text{ext})}(r) \) is a smooth function of \( r \). In such a case, in validity of the asymptotic ordering (61) and neglecting corrections of order \( \xi^N \), with \( N \geq 1 \) (first-order approximation), the following asymptotic approximation holds for \( G_{\mu} \):

\[
G_{\mu} \approx \left\{ m_{\text{oEM}} \frac{d}{d s} u_{\mu} + g_{\mu} \right\} [1 + O(\xi)], \tag{64}
\]

with \( g_{\mu} \) denoting the 4-vector

\[
g_{\mu} = \frac{2}{3} q^2 c \left[ \frac{d^2}{d s^2} u_{\mu} - u_{\mu}(s) \frac{d^2}{d s^2} u_k \right], \tag{65}
\]

and

\[
m_{\text{oEM}} = \frac{q^2}{c^2 \sigma} \left[ 1 + \frac{(t - t')}{2} \frac{d}{d s} \frac{1}{\sqrt{1 - \frac{v(t)^2}{c^2}}} \right] \tag{66}
\]

the EM mass.

**Proof** - To reach the proof let us first evaluate asymptotic expansions for the 4-vectors \( R^k \), \( \frac{d_{\text{ext}}(t-t_{\text{ret}})}{d s} \), the 4-scalar \( R^{a}_{
abla} u_{\alpha}(s) \) and the time delay \( t - t' \equiv t_{\text{ret}} \). Neglecting corrections of order \( \xi^N \) with \( N > 3 \), and denoting \( \gamma \equiv \gamma(t(s)) = 1/\sqrt{1 - v^2(t(s))/c^2} \) and \( u^k \equiv u^k(t(s)) \), one obtains by Taylor expansion

\[
R^k \cong \frac{c(t - t')}{\gamma} u^k - \frac{c^2(t - t')^2}{2\gamma} \frac{d}{d s} \left( \frac{u^k}{\gamma} \right) + \tag{67}
\]

\[
+ \frac{c^3(t - t')^3}{6\gamma} \frac{d}{d s} \left( \frac{1}{\gamma} \frac{d u^k}{d s} \right)
\]

and similarly denoting \( r_{\mu} \equiv r_{\mu}(t(s)) \),

\[
\frac{d r_{\mu}(t - t_{\text{ret}})}{d s} \cong \tag{68}
\]

\[
\cong \frac{d r_{\mu}}{d s} - \frac{c(t - t')}{\gamma} \frac{d^2 r_{\mu}}{d s^2} + \frac{c^2(t - t')^2}{2\gamma^2} \frac{d^3 r_{\mu}}{d s^3}
\]

Thus, Eqs. (67) and (68) imply

\[
R^a_{\alpha}(t) \cong \frac{c(t - t')}{\gamma} - \frac{c^2(t - t')^2}{2\gamma} \frac{d}{d s} \frac{1}{\gamma} + \tag{69}
\]

\[
+ \frac{c^3(t - t')^3}{6\gamma} \frac{d}{d s} \left( \frac{1}{\gamma} \frac{d u^\alpha}{d s} \right) u_{\alpha},
\]

where \( \frac{d}{d s} \frac{1}{\gamma} = \frac{\gamma}{c} \frac{d}{d s} \sqrt{1 - \frac{v(t)^2}{c^2}} = -\gamma \frac{v(t) \cdot a(t)}{c^2} \). Finally, we notice that there results

\[
t - t' = \frac{\sigma}{c} \sqrt{1 - \frac{v(t)^2}{c^2}} \cong \tag{70}
\]

\[
\cong \frac{\sigma}{c} \gamma \left[ 1 + \frac{3}{2} \frac{\gamma \cdot v(t) \cdot a(t)}{c} \right]
\]

By substituting Eqs. (67)-(70) in Eq. (40) [or equivalent in Eq. (11)] it is immediate to recover after straightforward calculations Eq. (44). Q.E.D.

We remark that Eq. (69) for \( g_{\mu} \) coincides formally with the usual expression of the EM self-force adopted in the LAD equation (see related discussion in Ref. [2]). However, to recover the customary expression of the EM mass usually given [for the LAD equation] (see for example Ref. [13]), requires retaining only the leading-order approximation

\[
m_{\text{oEM}} \cong \frac{q^2}{c^2 \sigma} [1 + O(\xi)], \tag{71}
\]

This amounts to ignore the correction factor

\[
\frac{1}{\left[ 1 + \frac{(t - t')}{2} \frac{d}{d s} \frac{1}{\sqrt{1 - \frac{v(t)^2}{c^2}}} \right]} \cong 1 + \frac{\sigma}{c} \frac{v(t) \cdot a(t)}{c^2}, \tag{72}
\]

i.e., a term of order \( \xi^0 \) in Eq. (69). Hence this approximation is not sufficient, since the term \( g_{\mu} \) in Eq. (69) is of order \( \xi^0 \) too. We conclude that, for consistency, in place of (71), the more accurate approximation (69) should be used for the EM mass \( m_{\text{oEM}} \).
An important issue is related to the conditions of smoothness - required by THM.5 for the validity of Eqs. (64)-(66) - which must be imposed on the external EM field, i.e., on \( A^{(ext)}_\mu (r(s)) \). It is obvious, in particular, that locally discontinuous (in \( s \)) external fields must generally be excluded, since the previous expansions [see Eqs. (67)-(70)] manifestly do not hold near the discontinuities. An example is provided by so-called “sudden forces”. These occur when the corresponding Faraday tensor \( F^{(ext)}_{\mu\nu} (r(s)) \) is permitted to be locally discontinuous with respect to \( s \) (which may be achieved by turning on and off repeatedly the external EM field). For the validity of THM.5 this case must generally be excluded. In fact, it is obvious that the Taylor expansions (67)-(70) generally do not hold in the neighborhood of the discontinuities. This clearly prevents also the validity of the LL equation as well of analogous asymptotic approximation of Eq. (61) (see also related discussion in Ref. [2]).

9 - WEAKLY-RELATIVISTIC APPROXIMATION

Although the covariant representation given by Eq. (41) is of general validity, it is worth discussing here also its weakly-relativistic approximation. This enables a direct comparison with the original Lorentz approach [4] and the known result obtained by Sommerfeld, Page, Caldirola and Yaghjian [22, 23, 24, 25] in the case of a finite-size spherical-shell charge, a fact which is relevant not only for historical reasons. Indeed, as previously pointed out [2], also the weakly-relativistic LAD equation exhibits the same difficulties characteristic of the relativistic LAD equation. In particular, it yields a third-order ordinary differential equation which exhibits the known physical inconsistencies (violation of NPD and GPI, existence of runaway solutions which blow up in time, etc.). Here we intend to show how, even in the weakly-relativistic approximation, the present theory is able to overcome such difficulties. For the sake of definiteness, let us determine the asymptotic approximation for \( \sigma_\mu \), obtained by assuming

\[
\beta \equiv v(t)/c \ll 1
\]

(weakly-relativistic approximation). For this purpose let us introduce a Taylor expansion with respect to \( \beta \), while leaving unchanged the dependence in terms of the retarded time \( \tau' \). As shown in Appendix E, in such a case the following result holds:

**THM.6 - Weakly-relativistic asymptotic approximation for \( \sigma_\mu \)**

In validity of the asymptotic ordering (73) and neglecting corrections of order \( \beta^n \), with \( n \geq 3 \), the following asymptotic approximation holds for \( \sigma_\mu \):

\[
\sigma_\mu \equiv (G_0 = 0, G) \equiv G,
\]

where:

- **T6a** first asymptotic approximation: in the case of the representation (41) the 3-vector \( G \) reads:

\[
G \equiv -\frac{2}{\sigma} \frac{1}{c^2} \left( \frac{\partial}{\partial t} v(t) - \sigma \right) + \frac{c}{\sigma} \frac{\partial}{\partial t} \left( r(t) - r(t - \frac{\sigma}{c}) \right)
\]

- **T6b** second asymptotic approximation: in the case of the representation (41), instead, the 3-vector \( G \) becomes:

\[
G \equiv 2c \frac{q}{\sigma} \frac{1}{c^2} \left( \frac{\partial}{\partial t} r(t - t_{ret}) \right) - \frac{r(t) - r(t - t_{ret})}{||t - t'||}
\]

- **T6c** finally, upon invoking also the short-time ordering (59) and a suitable condition of smoothness for the external EM field, one recovers in both cases [Eqs. (75) or (76)] the usual weakly relativistic approximation:

\[
G \equiv g + m_{EM} \ddot{r}(t)
\]

where

\[
g \equiv -\frac{2q^2}{3c^3} r,
\]

\[
m_{EM} \equiv \frac{q^2}{c^2 \sigma},
\]

are respectively the well-known weakly-relativistic EM self-force 3-vector and the EM mass.

**Proof** (see Appendix E). We notice that the apparent non-uniqueness of the two representations given above [Eqs. (75) and (76)] can be resolved by noting that the \( \beta \)-expansion should be actually carried out also in terms of the delay-time \( t_{ret} \) (which should be considered itself of order \( \beta^n \), with \( n > 0 \) to be suitably defined). Indeed, if the short-time expansion is introduced, as found in Appendix E, Eqs. (77) both imply Eqs. (78) and (79). In the same sense, Eqs. (75) and (76) can also be proven to be in agreement with the well-known Sommerfeld-Page-Caldirola-Yaghjian result [22, 23, 24, 25] for weakly-relativistic spherical-shell charges. The resulting equations, (78) and (79), are manifestly consistent with the customary weakly-relativistic approximation for the LAD equation (see, for example, also related discussion in Ref. [2]).

10 - CONCLUDING REMARKS

In this paper an exact solution has been obtained for the RR problem. The result has been achieved in the case of a spherical-shell finite-size charge. As a main consequence, the exact RR equation, describing the relativistic dynamics of such a particle in the presence of its EM
self-field has been achieved (see THM.1 and THM.2).
Although its charge has been assumed as spatially distributed, we have shown that, by assuming the mass as point-wise localized, the dynamics is reduced to that of a point particle. The resulting RR equation appears free from all the difficulties met by previously classical RR equations (THM.1-THM.3). In particular, besides being fully relativistic, the new equation:

1) has been achieved via a variational formulation based on the adoption of the Hamilton variational principle.
The treatment has been made transparent by adopting a synchronous form of the variational principle;

2) unlike the LAD equation: results consistent with the Newton’s principle of determinacy, Einstein principle of causality, Galilei law of inertia and does not exhibit so-called runaway solutions;

3) unlike the LL equation: does not involve the adoption of iterative approaches for its derivation;

4) unlike the LAD and LL equations: is valid also in the case of sudden forces and does not exhibit any singular behavior (i.e., provided the radius of the charge remains strictly positive);

5) unlike all previous equations (LAD and LL and the Medina equations): it is not asymptotic.

6) unlike in the Medina approach: the variational approach is based on the Hamilton variational principle in the ordinary phase-space, which allows us to retain the customary formulation of classical mechanics and classical electrodynamics.

In addition, as a side result, we have pointed out a correction to the LAD equation, appearing in the EM mass, which is demanded by the perturbative expansion [see Eq. 66].

The conceptual simplicity of the present approach and its general applicability to arbitrary systems of charges of this type make the present results of extraordinary deep and wide-ranging implications. These are related, in particular, to the description of relativistic dynamics of systems of classical finite-size charged particles. The conceptual simplicity of the present approach and its general applicability to arbitrary systems of charges of this type make the present results of extraordinary relevance for relativistic theories (such as kinetic theory of charged particles and gyrokinetic theory for magnetoplasmas) and related applications in astro- and plasma physics.

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APPENDIX A: SURFACE-AVERAGE OPERATOR

Following the notations introduced in Sec.4b and in case of flat space-time, if $A(r + \sigma n)$ is a smooth (tensor) function of the 4-position vector $r + \sigma n$, we define its surface-average as

$$\mathcal{A}(r) = \frac{1}{4\pi} \int d\Sigma(n) A(r + \sigma n). \quad (80)$$

In particular, identifying $A$ with the Faraday tensor $F^\nu_\mu (r + \sigma n)$, its surface-average is

$$\mathcal{F}^\nu_\mu (r) = \frac{1}{4\pi} \int d\Sigma(n) F^\nu_\mu (r + \sigma n). \quad (81)$$

APPENDIX B: INTEGRAL REPRESENTATION FOR $A^\mu_{self}(r)$ (CASE OF A POINT CHARGE)

The integral representation for $A^\mu_{self}$ can also be obtained directly from Maxwell’s equations. Let us consider first the case of a point-charge. By assumption $A^\mu_{self}$ satisfies Maxwell’s equations (in flat space-time)

$$\partial_\mu F^{\mu\nu}_{\text{self}} = \frac{4\pi}{c} j^\nu, \quad (82)$$

where for a point particle:

$$j^\nu (r”) = q \int ds’ u^\mu (s’) \delta^{(4)} (r – r (s’)) \quad (83)$$

(with $\delta^{(4)} (r – r (s’))$ denoting the 4-dimensional Dirac delta). There results therefore

$$A^\mu_{\text{self}} = \frac{4\pi}{c} \int d^4 r’ G(r – r’) j^\mu (r’), \quad (85)$$

where $G(r – r)$ is the retarded Green function which satisfies the equation

$$\Box G(r – r’) = \delta^{(4)} (r – r’) \quad (86)$$

and is such that

$$G(r – r’) = 0 \quad (87)$$
for $r^0 < r_0$. It follows
\[ G(r - r') = \frac{1}{2\pi} \delta(R^\mu R_\mu) \Theta(r^0 - r'^0), \]
and hence
\[ A_{\mu}^{(self)}(r) = \frac{4\pi}{c} \int d^4 r' \frac{1}{2\pi} \delta(R^\mu R_\mu) \]
\[ \Theta(r^0 - r'^0) \]
\[ q \int ds' u_\mu(s') \delta^{(4)}(r' - r(s')), \]
\[ A_{\mu}^{(self)}(r) = \frac{2}{c} \int d^4 r' \delta(R^\mu R_\mu) \Theta(r^0 - r'^0) q \]
\[ \int ds' u_\mu(s') \delta^{(4)}(r' - r(s')). \]
This implies also
\[ A_{\mu}^{(self)}(r) = \frac{2q}{c} \int ds' u_\mu(s') \delta(R^\mu s') R_\mu(s'), \]
where $R^\mu(s') = r^\mu - r^\mu(s')$. The last integral can also be written as
\[ A_{\mu}^{(self)}(r) = \frac{2q}{c} \int dt' \delta(R^\mu R_\mu). \]
This is an integral representation for $A_{\mu}^{(self)}(r)$, by construction equivalent to Eq. (85).

**APPENDIX C: INTEGRAL REPRESENTATION FOR $A_{\mu}^{(self)}$ (CASE OF A SPHERICAL-SHELL CHARGE)**

To prove that the differential and integral representations for $A_{\mu}^{(self)}$ (10) and (17) are equivalent it is sufficient to notice that the following identity holds:
\[ \delta(R^\alpha R_\alpha - \sigma^2) = \delta(t - t' - t_{ret}) \]
\[ \frac{1}{2c^2} \left| (t - t') - \frac{1}{c^2} \frac{dr(t')}{dt} \cdot (r - r') \right| \]
In fact there follows
\[ T_{\mu}^{(self)}(r) = \frac{2q}{c} \int_{t_1}^{t_2} dt' \delta(R^\alpha R_\alpha - \sigma^2) = \]
\[ \frac{2q}{c} \left[ \frac{1}{2c^2} \left| (t - t') - \frac{1}{c^2} \frac{dr(t')}{dt} \cdot (r - r') \right| \frac{dr(t')}{dt} \right]_{t' = t - t_{ret}} \]
\[ = \frac{q}{c} \left[ \frac{u_\mu(t')}{R^\alpha u_\alpha(t')} \right]_{t' = t - t_{ret}}, \]
which recovers immediately Eq. (10).

**APPENDIX D - OTHER LEMMAS**

**LEMMA 2 - Synchronous variation of $\Delta S_1$**
The synchronous variation of $\Delta S_1(r^\mu)$ reads
\[ \delta \Delta S_1 = \delta A + \delta B, \]
where
\[ \delta A = -4 \left( \frac{q}{c} \right)^2 g_{\mu
u} \frac{1}{4\pi} \int d\Sigma(n) \int_{t_1}^{t_2} dt' \delta(R^\alpha R_\alpha - \sigma^2), \]
\[ \delta B = 4 \left( \frac{q}{c} \right)^2 g_{\alpha\beta} \frac{1}{4c^2} \int d\Sigma(n) \int_{t_1}^{t_2} dt' \delta(R^\alpha R_\alpha - \sigma^2). \]
There results respectively:
\[ \delta A = 4 \left( \frac{q}{c} \right)^2 g_{\mu\nu} \frac{1}{4\pi} \int d\Sigma(n) \int_{t_1}^{t_2} dt' \delta R^\nu R^\mu \]
\[ \delta B = -4 \left( \frac{q}{c} \right)^2 g_{\nu\mu} \frac{1}{4\pi} \int d\Sigma(n) \int_{t_1}^{t_2} dt' \delta R^\nu R^\mu. \]
where
\[ A_{\nu}^{(self)} = \frac{1}{2c^2} \left| (t' - t) - \frac{1}{c^2} \frac{dr(t')}{dt} \cdot (r - r') \right| \]
\[ \frac{d}{dt'} \left\{ u_{\nu}(t') \left[ \left( \frac{R^\nu}{c^2} \right) - \frac{dr(t')}{dt} \cdot (r - r') \right] \right\}. \]

**Proof - In fact let us assume that the metric tensor $g_{\mu\nu}$ is constant and symmetric (Minkowski space-time). In this case the synchronous variation of $\Delta S_1$ is given by Eqs. (95) and (96) where
\[ d \left[ \int_{t_1}^{t_2} dt' \delta(R^\alpha R_\alpha) \right] = dr^k \int_{t_1}^{t_2} dt' \]
\[ \sqrt{1 - \frac{1}{c^2} \frac{dr(t')}{dt}^2} u_{\nu}(t') \frac{\partial}{\partial r^k} \left[ \delta(R^\alpha R_\alpha - \sigma^2) \right]. \]
Hence it follows,
\[ \delta A = -4 \left( \frac{q}{c} \right)^2 g_{\mu\nu} \frac{1}{4\pi} \int d\Sigma(n) \int_{t_1}^{t_2} dt' \delta R^\nu R^\mu \]
\[ \int_{t_1}^{t_2} dt' \sqrt{1 - \frac{1}{c^2} \frac{dr(t')}{dt}^2} u_{\nu}(t') \]
\[ \frac{\partial}{\partial r^k} \left[ \delta(R^\alpha R_\alpha - \sigma^2) \right]. \]
while
\[
\delta B \equiv 4 \left( \frac{q}{c} \right)^2 g_{\alpha \beta} \frac{1}{4\pi} \int d\Sigma(n) \int_1^2 dr^\alpha \delta r^\mu
\]
\[
\int_{t_1}^{t_2} cdt' \sqrt{1 - \frac{1}{c^2} \frac{dr^2}{dt^2}} \frac{dr^2}{dt^2} u^\beta(t')
\]
\[
\frac{\partial}{\partial r^k} \delta(R^a R_a - \sigma^2).
\]

Hence, there follows the identity
\[
\frac{\partial}{\partial r^k} \delta(R^a R_a - \sigma^2) = \frac{\partial(R^a R_a)}{\partial r^k} \delta(R^a R_a - \sigma^2) = 2R_k \frac{d}{dt} \frac{\delta(R^a R_a - \sigma^2)}{\delta R^a R_a}.
\]

APPENDIX E - WEAKLY RELATIVISTIC APPROXIMATION

In validity of the asymptotic ordering (93) there results [from Eq.(100)] by Taylor expansion in \( \beta \), while retaining exactly all dependencies in terms of the retarded time \( t' \),
\[
\overline{G}_\mu \approx 2c^2 \left( \frac{q}{c} \right)^2 \frac{1}{c^2 (|t - t'|)^2} \frac{dh_{\beta}(t - t_r)}{dt} + \frac{R'}{c^2 (|t - t'|)^2} \frac{dh_{\beta}(t - t_r)}{dt},
\]

where \( R' \equiv \{ct_{ret}, r(t') - r(t - t_{ret}) \} \). Here the delay time \( t_{ret} \equiv t - t' \), evaluated in a similar way from Eq.(101) neglecting corrections of order \( \beta \), reads:
\[
t_{ret} \approx \frac{\sigma}{c}.
\]

It follows
\[
\overline{G}_\mu \approx (0, G).
\]

In particular, the spatial 3-vector \( G \) reads in case of Eq.(108):
\[
G \approx -2c \left( \frac{q}{c} \right)^2 \frac{1}{c^2 (|t - t'|)^2} \frac{dh_{\beta}(t - t_r)}{dt} + \frac{R'}{c^2 (|t - t'|)^2} \frac{dh_{\beta}(t - t_r)}{dt},
\]

This equations, with (110), implies Eq.(75). Finally, let us evaluate also the corresponding short-time approximation, obtained invoking also the ordering (50). By Taylor expansion in \( \xi \equiv (t - t')/t \) there results to leading order
\[
\frac{d}{dt} v(t') + \frac{v(t')}{(t - t')^2} \frac{R(t) - R(t')}{(t - t')^2} \approx \frac{1}{2} \frac{d^2}{dt^2} v(t)
\]

Therefore, one obtains finally the weakly-relativistic (and short-time) approximation
\[
G \approx \left( \frac{q}{c} \right)^2 \left[ -\frac{1}{\sigma} \frac{d}{dt} v(t) + \frac{2}{3c} \frac{d^2}{dt^2} v(t) \right],
\]
which similarly recovers Eq. (77). Instead, in the case of Eq. (109) in an analogous way there results:

\[
G \simeq 2c \left( \frac{2}{c^2} \right)^2 \frac{1}{t' - t} \left[ \frac{dr(t'-t_{ret})}{dt} \right],
\]

which implies Eq. (76). Hence it follows

\[
\frac{dr(t-t_{ret})}{dt} \simeq \frac{r(t) - r(t-t_{ret})}{(t - t')},
\]

which implies again Eq. (113) and therefore recovers the same weakly-relativistic approximation given by Eq. (77).

[1] M. Tesserotto, C. Cremaschini, M. Dorigo, P. Nicolini and A. Beklemishev, *The exact radiation-reaction equation for a classical charged particle*, contributed paper at RGD26 (Kyoto, Japan, July 2008); arXiv:0807.1819 (2008).

[2] M. Dorigo, M. Tesserotto, P. Nicolini and A. Beklemishev, *On the validity of the LAD and LL classical radiation-reaction equations*, contributed paper at RGD26 (Kyoto, Japan, July 2008); physics/arXiv.org:0806.4458 (2008).

[3] F. Rohrlich, *Classical Charged particles* (Addison-Wesley, Reading MA), Chap. VI (1965); C. Teitelboim, Phys.Rev. D1, 1572 (1970); D2, 1763 (1970); Teitelboim, D. Villaroel, Ch. G. van Weert, Riv. Nuovo Cim. 3, 1 (1980); A. Sokolov, I. Ternov, *Radiation from Relativistic Electron*, AIP, NY (1986); S. Parrott, *Relativistic Electrodynamics and Differential Geometry*, Springer-Verlag, NY (1987); S. Parrott, Found. Phys. 23, 1093 (1993).

[4] H.A. Lorentz, *Le theorie electromagnetique de Maxwell et son application aux corps mouvants*, Archives Need-