Dark matter halo merger and accretion probabilities in the excursion set formalism

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ABSTRACT

The merger and accretion probabilities of dark matter haloes have so far only been calculated for an infinitesimal time interval. This means that a Monte Carlo simulation with very small time-steps is necessary to find the merger history of a parent halo. In this paper, we use the random walk formalism to find the merger and accretion probabilities of haloes for a finite time interval. Specifically, we find the number density of haloes at an early redshift that will become part of a halo with a specified final mass at a later redshift, given that they underwent \( n \) major mergers, \( n = 0, 1, 2, \ldots \). We reduce the problem into an integral equation which we then solve numerically. To ensure the consistency of our formalism, we compare the results with Monte Carlo simulations and find very good agreement. Though we have done our calculation assuming a flat barrier, the more general case can easily be handled using our method. This derivation of finite time merger and accretion probabilities can be used to make more efficient merger trees or implemented directly into analytical models of structure formation and evolution.

Key words: galaxies: evolution – galaxies: haloes – cosmology: theory – dark matter.

1 INTRODUCTION

Since the introduction of the spherical collapse model of dark matter haloes by Press and Schechter (Press & Schechter 1974) and its generalizations such as extended Press–Schechter (EPS) (Bond et al. 1991) and ellipsoidal collapse (Sheth & Tormen 2002), it has been used extensively in the cosmological literature as a fast and accurate method to quantitate the distribution of collapsed dark matter objects in the Universe. This in turn is the backbone of semi-analytic theories of galaxy formation and evolution (Cole et al. 2000). Modified versions of the excursion set formalism underlying EPS have also found application in different contexts such as ionized bubble growth in the early Universe (Furlanetto, Zaldarriaga & Hernquist 2004). The advantages of this method compared to direct N-body simulation include its superior speed, which allows the exploration of large ranges of parameter space, and redshift range than is currently accessible to N-body simulations.

It must, however, be noted that the excursion set formalism has its own shortcomings. Sheth & Pitman (1997) pointed out the inconsistencies between the excursion set and binary merger trees, and Benson, Kamionkowski & Hassani (2005) showed that the excursion set theory is not self-consistent in the sense that the merger kernel is not symmetric, i.e. \( R(m_1, m_2) \neq R(m_2, m_1) \). Also, since the excursion set theory is based on uncorrelated random walks, the future history of a halo does not depend on its environment. Therefore, it is not possible to explain within this model the recent result from simulations that the formation history of haloes is correlated with their environments (Sheth & Tormen 2004). Nevertheless, simplicity of implementation of the excursion set model together with its surprisingly good description of halo properties has made it an indispensable tool for cosmologists.

In the EPS formalism, one finds the probability \( P(M_1, z_1|M_2, z_2) \)\( dM_1 \), which gives the probability of a point mass being part of a halo with mass in between \( M_1 \) and \( M_1 + dM_1 \) at redshift \( z_1 \), given that it was (or will be) part of a halo with mass \( M_2 \) at redshift \( z_2 \). During this redshift interval, it could merge with any number of haloes whose masses add up to \( |M_1 - M_2| \). However, from the viewpoint of galaxy formation, accretion of small haloes into the larger one do not have the ability to change the evolution of the galaxy or galaxies inside it. Indeed, it is assumed that only at major mergers in which the mass of the merged halo satisfies a condition to be large enough, have this ability. It is therefore interesting to ask the question: given the criterion for a major merger, is it possible to find an analytical result to describe the progenitor distribution of a parent halo, based on how many times they have undergone a major merger? For example, what is the number density of haloes at redshift, say, \( z = 1 \) which underwent \( n \) major mergers, \( n = 0, 1, 2, \ldots \), and eventually ended up being a galaxy halo at the present time.

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Here we present an analytical method based exclusively on excursion set assumptions to find these number densities. We reduce the problem to an integral equation which can then be solved numerically. Our derivation is given in Section 2. We show in Section 3 that Monte Carlo (MC) simulations agree with the results of our semi-analytic model.

It is also possible to cast the integral equation into a scale invariant form. This can be a great advantage since we just need to solve the integral equation once and scale the result to find the general formula. This is done in section 4. We discuss how the method presented in this paper can be generalized and conclude in section 5.

A few appendices are added for further clarification. In Appendix A we give a straightforward method to solve the integral equation, and in Appendix B we describe very briefly how we implemented our MC simulation.

2 THE ACCRETION PROBABILITY

The main task of this paper is to find an analytical result for $p_{\text{acc}}(M_{1}, t_{1}, t_{1})$, the probability that a halo of mass $M_{1}$ at time $t_{1}$ had a progenitor of mass in the range $(M_{2}, M_{2} + dM_{2})$ at an earlier time $t_{2}$ and that the progenitor halo never merged with any halo of mass larger than a specified mass resolution $M_{\text{res}}$ on its journey from $t_{2}$ to $t_{1}$. It is, however, more convenient to work with the variables $S$ and $\omega$ instead of $M$ and $t$. Here, $S$ is the rms mass fluctuation inside spheres of mass $M$:

$$S(M) = \sigma^2(R_M) = \int d\ln k \Delta^2(k) |W(k, R_M)|^2,$$

where $\Delta^2(k) = k^3P(k)/2\pi^2$ and $P(k)$ is the power spectrum of the mass fluctuations. Also, the barrier height $\omega = \delta \Delta t/D_{\text{lin}}$, where $\delta_c \approx 1.68$ is the linearly extrapolated density to the time of collapse in the spherical collapse model and $D(t)$ is the cosmological linear growth factor.

Finding $p_{\text{acc}}(M_{2}[t_{2}, t_{1}], t_{1})dM_{2}$ in the excursion set formalism is equivalent to finding $f_{\text{acc}}(S_{f}[\omega_{f}, S_{f}, \omega_{f}])$, the probability that a random walk starting from $(S_{f}, \omega_{f})$ has its first upcrossing between $S_{f} + dS_{f}$ and $S_{f}$ at the barrier height $\omega_{f}$, given that it never had a jump larger than $M_{\text{res}}$, and that $S_{f}$ and $\omega_{f}$ are approximately Gaussian (James et al. 2006).

In order to accomplish this, we let a number of random walks start from $(S_{f}, \omega_{f})$ (see Fig. 1), where in this paper $S_{f}$ and $\omega_{f}$ denote $\sigma^2(M_{f})$ and $\sigma t_{f}$, respectively, and so forth for other subscripts. Setting a barrier at $\omega_{f} > \omega_{f}$, that is at earlier times, we know the fraction of random walks that have their first upcrossing in the interval $(S_{f}, S_{f} + dS_{f})$, regardless of whether they accrete or merge, is

$$f_{\text{acc}}(S_{f}[\omega_{f}, S_{f}, \omega_{f}])dS_{f} = \frac{(\omega_{f} - \omega_{f})^{2}}{(2\pi)^{1/2}(S_{f} - S_{f})^{1/2}} \exp \left[ \frac{(\omega_{f} - \omega_{f})^{2}}{2(S_{f} - S_{f})} \right] dS_{f}. \quad (2)$$

The next step is to note that the fraction of random walks that start from $(S_{f}, \omega_{f})$ and have their first upcrossing between $S_{f}$ and $S_{f} + dS_{f}$ at the height $\omega_{f}$ and have one and only one jump from $S_{1}$ to $S_{2}$ during the interval $\omega$ and $\omega + d\omega$ is

$$f_{\text{acc}}(S_{1}[\omega_{f}, S_{f}, \omega_{f}])dS_{1} f(S_{1} \rightarrow S_{2}; \omega)dS_{2}d\omega \times f_{\text{acc}}(S_{2}[\omega_{f}, S_{f}, \omega_{f}])dS_{2},$$

$$f(S_{1} \rightarrow S_{2}; \omega)dS_{2}d\omega$$

where $f(S_{1} \rightarrow S_{2}; \omega)dS_{2}d\omega$ is the probability that a random walk will have a sudden jump from $S_{1}$ to somewhere between $S_{2}$ and $S_{2} + dS_{2}$ in an infinitesimal interval $d\omega$. Clearly, we must have

$$\omega_{f} \leq \omega \leq \omega_{f}.$$
Figure 2. Feynman diagrams illustrating equation (8) for the total probability for a random walk starting from \((S_i, \omega_f)\) to have its first upcrossing between \(S_i\) and \(S_i + dS\) at \(\omega_f\). A solid line denotes a period of time in which any sequence of accretion and merger events can take place. A dashed line denotes a period of time in which only accretion takes place and a cross indicates a merger. The first equality says that the total probability is equal to the sum of the fractions of walks which underwent no mergers, one merger, two mergers, etc. We can rearrange the sum to give the second equality, which states that the total probability is equal to the sum of the probability to have no mergers and the probability to have at least one merger.

\[
(S_i, S_i + dS), \text{ no matter what happened during their journey, is equal to the sum of the fractions of walks which underwent no mergers, one merger, two mergers, etc.}
\]

Putting the integral formulae for \(f_{\text{merger}}(S_i|\omega_f, S_f, \omega_f)\) and higher order terms into equation (8), we will find an integral equation for the unknown \(f_{\text{acc}}(S_i|\omega_f, S_f, \omega_f)\). However, this equation written in this form is not computationally tractable since higher order terms will require a prohibitive number of integrations.

Note that the conditional probability densities in equation (8) are analogous to propagators. It is instructive to visualize each term in equation (8) using a diagrammatic notation, as in Fig. 2. The figure makes it clear that this equation expands the full propagator in terms of bare propagators (the accretion probability) with interactions (mergers). We can use this insight to rearrange the above equation as in the figure, which shows that one can re-sum the terms in equation (8) to write down a tractable integral equation. Writing the second equality of Fig. 2 in the language of equation (8), we find the important result:

\[
f_{\text{tot}}(S_i|\omega_f, S_f, \omega_f) = f_{\text{acc}}(S_i|\omega_f, S_f, \omega_f) + \int_{\omega_f}^{\omega} d\omega \int_{S_i}^{S_f} dS_i \int_{S_i+\Delta(S_i)}^{S_f} dS_i \times f_{\text{acc}}(S_i|\omega_f, S_f, \omega_f) f_{\text{tot}}(S_i|\omega, S_2, \omega). \tag{9}
\]

This is a Voltera integral equation which we will solve numerically. We refer the interested reader for a discussion of solving this equation numerically to Appendix A. In the next section, we compare these semi-analytic results to MC simulations.

3 COMPARISON WITH MONTE CARLO SIMULATION

There are several methods for generating a merger tree using MC techniques (see Somerville & Kolatt 1997; Cole et al. 2000). Here we choose to use a binary merger tree with accretion method mainly due to its simplicity of implementation. In this scheme the time interval is chosen to be so small that the probability of a merger is very low. This in turn ensures that the probability of more than one merger is negligible in a given time-step. Then a halo one time-step back will have a smaller mass due to accretion or division into two progenitors. The details of the method can be found in Appendix B.

It should be noted that our MC simulation gives number weighted probabilities. However, \(f_{\text{acc}}\) in equation (9) is a mass-weighted probability. We can easily change \(f_{\text{acc}}\) to a number-weighted probability \(p_{\text{acc}}\) using \(p_{\text{acc}} = (M/M)^{f_{\text{acc}}}\).

Each panel of Fig. 3 shows \(p_{\text{acc}}\) versus \(S_i\) for a different lookback redshift \(z\), with parent mass \(M_p = 2.5 \times 10^{12} M_\odot\) and redshift \(z_f = 0\). The mass resolution is fixed at \(M_{\text{res}} = M_p/10\). The dashed line is our analytical solution, the histogram shows the result of the MC simulation and the solid line is \(f_{\text{tot}}\) given by equation (2). We see an excellent agreement between the MC simulation and our analytic result, and note some intuitively sensible trends in the figures that are worth mentioning. First, \(p_{\text{acc}} = p_{\text{tot}}\) for \(S_i < 4.1\). That is because to have a merger we need \(M_i \leq M_f - M_{\text{res}} = 1.8 \times 10^{12}\), which corresponds to \(S_i = 4.1\). For \(S_i\) smaller than this, the mass jump is always less than mass resolution and only accretion can happen; hence, \(p_{\text{acc}} = p_{\text{tot}}\) in this region. For \(S_i\) larger than 4.1 mergers are allowed, so the probability of having one or more merger is non-zero and \(p_{\text{acc}}\) will be less than \(p_{\text{tot}}\). Also, the probability of having at least one merger, \(p_{\text{tot}}\), increases monotonically with increasing redshift as more and more haloes have a chance to undergo a merger. Finally, as we look further back in time, haloes with smaller masses have a chance to reach \(M_f\) by just accreting, so \(p_{\text{acc}}\) spreads to smaller masses with increasing redshift.

Although our method gives the full distribution over the mass of the progenitor haloes, one might only be interested in the fraction of haloes that had no major merger during a given redshift interval. To find this quantity, one can easily integrate the full distribution \(p_{\text{acc}}(S|\omega, S_i, \omega_f)\) from \(S = S_i\) to \(S = \infty\). For example, in Fig. 4 this fraction is 0.75, 0.65, 0.56 and 0.43 for the panels from top left to bottom right. As expected, this fraction is a decreasing function of lookback redshift: the longer a halo evolves the more probable it is to have a major merger.

In Fig. 4 we show \(p_{\text{acc}}\) versus \(S_i\), with the same \(M_f\) and \(z_f\) as above, but here we look at the distribution with different choices of \(M_{\text{res}}\) at a fixed lookback redshift \(z = 0.5\). Again the agreement between our analytical result and the MC simulations is very good. Again, there are some intuitively reasonable trends in the figure that should be noted. As we discussed above, \(p_{\text{acc}}\) must be equal to \(p_{\text{tot}}\) for \(M_f \geq M_{\text{limit}} = M_f - M_{\text{res}}\). As \(M_{\text{res}}\) gets smaller, this limiting mass becomes larger. Therefore \(S_{\text{limit}} = S(M_{\text{limit}})\), the \(S_i\) below which \(p_{\text{acc}} = p_{\text{tot}}\) becomes smaller, which can easily be seen in the figure. Note that different panels have different scales. Also given a fixed lookback redshift, decreasing the mass resolution increases the number of events we classify as mergers thus raising the probability for a halo.

\[
\int_{S_i}^{S_f} dS_i \int_{S_i+\Delta(S_i)}^{S_f} dS_i \times f_{\text{acc}}(S_i|\omega_f, S_f, \omega_f) f_{\text{tot}}(S_i|\omega, S_2, \omega). \tag{9}
\]
The dashed lines in these figures show our semi-analytic result for \( p_{\text{acc}} \) versus \( S \) with the same \( M_f \) and \( z_f \) as Fig. 3, but here we look at the distribution at a fixed lookback redshift \( z_l = 0.5 \) for decreasing \( M_{\text{res}} \), from top left to bottom right: \( M_{\text{res}} = M_f/5, M_f/10, M_f/20 \) and \( M_f/30 \). Solid lines show \( p_{\text{tot}} \) and histograms the results of MC simulations for \( p_{\text{acc}} \).

Figure 4. The dashed lines in these figures show our semi-analytic result for \( p_{\text{acc}} \) versus \( S \) with the same \( M_f \) and \( z_f \) as Fig. 3, but here we look at the distribution at a fixed lookback redshift \( z_l = 0.5 \) for decreasing \( M_{\text{res}} \), from top left to bottom right: \( M_{\text{res}} = M_f/5, M_f/10, M_f/20 \) and \( M_f/30 \). Solid lines show \( p_{\text{tot}} \) and histograms the results of MC simulations for \( p_{\text{acc}} \).

to have at least one merger. Since \( p_{\text{tot}} \) is not affected by the choice of \( M_{\text{res}}, p_{\text{acc}} \) accordingly decreases with decreasing \( M_{\text{res}} \). The fraction of haloes that have had at least one major merger is 0.38, 0.57, 0.73 and 0.80 from top left to bottom right, respectively.

4 SCALED SOLUTION

In general, for any given prescription for \( M_{\text{res}} \) one can find the solution of integral equation (9) to obtain \( f_{\text{acc}}(S_i|\omega, S_f, \omega_f) \). Generally, this needs to be solved for each given final halo mass \( M_f \). However, if we impose a special mass resolution for a chosen cosmology, it is possible to cast the integral equation into a scale invariant form. Then we need only solve this equation once. To achieve this, we need to define \( M_{\text{res}} \) so as to satisfy the following two equations simultaneously:

\[
\Delta(S) \equiv \sigma^2[M(S) - M_{\text{res}}] - S = CS \tag{10}
\]

and

\[
\Delta'(S) \equiv S - \sigma^2[M(S) + M_{\text{res}}] = C'S, \tag{11}
\]

where \( C \) and \( C' \) are constants independent of \( S \). Recall that \( \Delta(S) \) and \( \Delta'(S) \) appear in the limits of integration of equation (9). For small \( M_{\text{res}} \) compared to \( M(S) \), we can easily see, by Taylor expansion, that to second order in \( M_{\text{res}}/M \) these equations can be satisfied if we take \( C = C' = 1 \) and the mass resolution as

\[
M_{\text{res}}(S) = -\frac{S}{dS/dM} C. \tag{12}
\]

For example, for a scale invariant matter power spectrum with power index \( n, P(k) \propto k^n \), the above equation gives

\[
M_{\text{res}} = \frac{n + 3}{3} CM_{\text{parent}}, \tag{13}
\]

i.e. a merger is defined when the mass of any progenitor of the halo is larger than a constant fraction of the parent mass.

Using equations (10) and (11), we now rewrite the integral equation (9) in a scale invariant form. To do so we define the new variables

\[
u \equiv S/f_S \quad \theta \equiv (\omega - \omega_f)/S_f^{1/2} \tag{14}
\]

and the functions \( \tilde{f}_{\text{acc}} \) and \( \tilde{f}_{\text{acc}}' \):

\[
\tilde{f}_{\text{acc}}(S_i|\omega, S_f, \omega_f) = S_f^{-1} \tilde{f}_{\text{acc}}\left(\frac{S_i}{S_f}, \frac{\omega - \omega_f}{S_f^{1/2}}\right) \tag{15}
\]

With these definitions, equation (9) can be written in the manifestly scale invariant form

\[
1 = \tilde{f}_{\text{acc}}(u_1|\theta) + \int_0^{u_1} d\theta \int_1^{u_1(1-C')} du_1 \int_{u_1(1+C)}^{u_2} du_2 \times \tilde{f}_{\text{acc}}(u_1|\theta) K(\theta, u_1, u_2, \theta), \tag{16}
\]

where the kernel for the spherical collapse model is

\[
K = \frac{(2\pi)^{3/2}(u_1 - 1)^{3/2}}{\theta_1 \theta} \exp\left(-\frac{\theta_1^2}{2(u_1 - 1)}\right) \times \frac{\theta}{(2\pi)^{3/2}(u_2 - u_1)^{3/2}} \exp\left(-\frac{\theta^2}{2(u_1 - 1)}\right) \times \frac{1}{(2\pi)^{3/2}(u_2 - u_1)^{3/2}} \exp\left(-\frac{(\theta_1 - \theta)^2}{2(u_1 - u_2)}\right). \tag{17}
\]

For a given \( C \) and \( C' \) this equation can be solved for \( \tilde{f}_{\text{acc}}(u|\theta) \). This calculation can be facilitated by noting that the integral over \( S_f \) can be done analytically. Having found \( \tilde{f}_{\text{acc}}(u|\theta) \) one can find \( f_{\text{acc}}(S_i|\omega, S_f, \omega_f) \) for arbitrary \( S_i, \omega_f, S_f \) and \( \omega_f \) using equation (15). Note that the spectral index \( n \) only enters through equation (13).

The result of this calculation for a power-law matter power spectrum with \( n = -1 \) and \( C = 0.01 \) is shown in Fig. 5. The solid
line, as usual, denotes \( p_{\text{tot}} \), the lighter histogram in each panel indicates the result of MC simulation for \( p_{\text{acc}} \) and the dashed line on top is our numerical solution of equation (16) scaled according to equation (15). The panels from top left to bottom right are for \( \omega = 0.2, 0.3, 0.4 \) and 0.5. Assuming \( \omega = 1.69(1 + z) \) and \( z_f = 0 \), these correspond to lookback redshifts 0.118, 0.178, 0.237 and 0.296, respectively. One can see that the agreement is very good for all redshifts considered.

The darker histogram in Fig. 5 shows the result of our MC simulation for \( p_{\text{merger}}(S_i, \omega_i; S_f, \omega_f) \). This is the number density of haloes in a given range \((S_i, S_i + dS_i)\) that have a parent halo of mass corresponding to \( S_i \) at time \( \omega_i \) and have merged once in their journey from \( \omega_i \) to \( \omega_f \). Now that we have found \( f_{\text{acc}} \), it is possible to calculate \( p_{\text{merger}}(S_i, \omega_i; S_f, \omega_f) \) using equation (7). \( p_{\text{merger}} \) is nothing but \( f_{\text{merger}}(S_i, \omega_i; S_f, \omega_f) \) multiplied by \( M_1/M \) to convert from mass density to number density. This result is shown by the dot–dashed line on top of the histogram. The match is very good. As expected, for small \( \Delta \omega \) the probability of one merger is much smaller than probability of accretion, which can be seen in the top-left plot. Also in this plot, we can see that the tail of the distribution \( p_{\text{merger}} \) approaches \( p_{\text{tot}} \). This says that the probability of having more than one merger is negligible for small \( \Delta \omega \), as expected.

On the other hand, when \( \Delta \omega \) gets larger more and more haloes have a chance to merge, so \( p_{\text{tot}} \) flattens and \( p_{\text{merger}} \) rises. Also, with a large \( \Delta \omega \) there is a finite probability of having more than one merger since the tail of \( p_{\text{merger}} \) is considerably below \( p_{\text{tot}} \). One can continue this calculation and find \( p_{\text{mergers}}(\theta) \) and so on, which we have not shown here. Where this hierarchy should be terminated clearly depends on how far we look back in time; a larger \( \Delta \omega \) means more chance of a merger and therefore requires higher merger terms.

Though for a general power spectrum, \( M_{\text{res}} \) must be small compared to \( M_{\text{parent}} \) for the approximate solution 12 to work, for the special case of a white noise power spectrum, i.e. \( n = 0 \), one can find an exact analytic solution for equations (10) and (11) for arbitrarily large \( M_{\text{res}} \). In this case \( M(S) = k/S \), where \( k \) is a constant, and if one defines the ratio \( r \equiv M_{\text{res}}/M_{\text{parent}} \) then it is easy to show that equations (10) and (11) are satisfied if \( C = r/(1 - r) \) and \( C' = C/(1 + 2C) \). In Fig. 6 we have shown \( f_{\text{acc}}(\theta) \) as a function of \( u \) for a few choices of \( \theta \) for the case of a white-noise power spectrum.

The left-hand panel is calculated for a major merger definition of \( M_1/M_2 = 1/3 \), which in our notation is equivalent to \( r = 1/4 \), and the right one for \( r = 1/10 \), that is \( M_1/M_2 = 1/9 \). These universal curves can be translated to the physical quantity \( f_{\text{acc}}(S_i, \omega_i; S_f, \omega_f) \) by using equation (15). One can use these curves to get a quick estimate of the fraction of haloes of a given mass that only accreted during a given time interval. To find this quantity one needs to integrate \((M_f/M)f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f)\) over all \( S < S_f \). Using equation (15),

\[
F_{\text{acc}}(\theta) = \int_S^{\infty} dS_i \frac{M_f}{M_1} f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f) \\
= \int_1^{\infty} du \frac{\theta}{2(\pi)^{1/2}(u-1)^{1/2}} \exp \left( -\frac{\theta^2}{2(u-1)} \right) f_{\text{acc}}(u_1|\theta).
\]

As an example, for a definition of major merger of \((M_1: M_2) = (1: 3)\), we find that the fraction of the galaxy sized haloes at the present time \((M_f = 10^{12} M_\odot)\) or \( S_f = 5.2 \) that have had at least one major merger after redshifts 0.2 0.4, 0.6, 0.8 are 0.04, 0.13, 0.22 and 0.27, respectively. Though this calculation, strictly speaking, is only valid for the white-noise power spectrum, previous results show that the formulae obtained for the special case of white-noise work surprisingly well for an arbitrary power spectrum (see e.g. Moreno, Giocoli & Sheth 2008). So, it is reasonable to expect this extrapolation to work for our formalism also. In any case, even if this extrapolation turns out to be wrong, one can always solve the more general integral equation (9) for an arbitrary power spectrum.

5 DISCUSSION AND CONCLUSION

We have used the random walk formalism to find the accretion probability, i.e. the probability for a parent halo to have a progenitor in a given mass interval at a given earlier time given that it has not merged with a halo of a mass larger than the mass resolution. As a concrete example, we have worked out the accretion probability in the special case where the barrier is flat, the mass resolution is constant and we look backward in time. However this method can be extended to solve more general problems.

For example, while we have used a constant barrier, it is well known that this barrier shape does not match the results of N-body simulations. However, our formalism can be generalized to the case of a moving barrier, which has proven to give a very good match to N-body simulations. One only needs to find the appropriate formulae for \( f_{\text{tot}} \) and \( f(S_i \rightarrow S_f; \omega) \) for the moving barrier and solve the integral equation (9). These functions can in general be found numerically, using for example the method of Zhang & Hui (2006). However there are analytical results for simple barriers that reproduce the results of N-body simulations, e.g. the square root barrier (Mahmood & Rajesh 2005; Giocoli et al. 2007). Since the aim of this paper is not to compare with numerical simulations, we will leave this calculation for future work.

Here we have always looked backward in time. However, in certain cases it might be more convenient to find \( f_{\text{acc}}(M_f|M) \) in the forward sense. In that case, it gives the probability of a halo of mass \( M_i \) at an early time to accrete and become a larger halo in a given mass interval. This problem can be solved with our formalism by using the forward form of equation (9), with the forward form of \( f_{\text{tot}} \) and \( f_{\text{merger}} \) (Lacey & Cole 1993). This forward in time probability of accretion is what Somerville & Kolatt (1997) call \( P_{\text{acc}} \) and is the only missing ingredient in their formalism to find the complete statistics of halo progenitors and therefore to construct a self-consistent merger tree.

Once \( f_{\text{acc}} \) in the forward sense has been found it can be used for a variety of applications. For example, given a population of objects of mass \( M \) at time \( t \), some of these objects will be destroyed in the course of their evolution by merging with other objects. To find the fraction of objects that have survived from the initial time to the observation time (see Verde et al. 2001, for the case of clusters of galaxies) we need to find the fraction of objects whose haloes have
not merged with haloes more massive than a given threshold; in other words, they have only accreted from the initial redshift to the redshift of observation. This is precisely what \( f_{\text{acc}} \) in the forward sense is.

It is important to note that our method is not equivalent to the MC method, due to the inherent inconsistencies of the excursion set formalism (Sheth & Pitman 1997). Because of these inconsistencies, the halo progenitor probability distribution computed from a large number of small time-steps will not necessarily be equivalent to the prediction of the excursion set formalism for that finite time interval, see e.g. Somerville & Kolatt (1997). For this same reason, there does not exist a unique MC method. In all versions of the MC method, there are some assumptions beyond the excursion set formalism to alleviate the above inconsistency. Therefore, it is not surprising that there is a small discrepancy between our analytic results and our MC method. However, for the special case of a white-noise power spectrum there is a self-consistent algorithm to generate merger trees (Sheth & Pitman 1997). In that case, we expect our semi-analytical result presented in Section 4 to be equivalent to the Sheth and Pitman result. We defer this comparison to future work.

Finally, this method can lead to a major improvement in the speed of merger tree generation in MC simulations. Since there was no formula for the accretion or merger probabilities in a finite time interval, past MC codes had to use infinitesimal time-steps to be able to use the known formula for merger probabilities (equation 6), making the computation very time consuming. For example, for a given time interval, our unknown coefficients method for solving the integral equation takes an order of magnitude less time to compute than the brute force MC method. It should be noted that since neither of the numerical codes we implemented in this paper are optimally efficient, both computation times are still to be minimized. However, the increase in speed is more likely for the integral equation, since one can use the already existing numerical methods to expedite the code or even find an approximate analytical solution, e.g. perturbationally. It is harder and less straightforward to design a faster MC method.

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APPENDIX A: NUMERICAL SOLUTION

The goal of this section is to solve equation (9) for \( f_{\text{acc}}(S_j|\omega_f, \omega_f) \) numerically using the method of unknown coefficients. To begin, it is more convenient to work with the new unknown function \( f_d \equiv f_{\text{tot}} - f_{\text{acc}} \) instead of \( f_{\text{acc}} \). Suppressing the limits of integration for simplicity of notation, we can then write the integral equation (9) in the form

\[
f_d(S_i|\omega_f, \omega_f) = b(S_i, \omega_f, \omega_f)
- \int d\omega \int d\omega_1 \int d\omega_2 f_d(S_i|\omega, \omega_f) \times f(S_i \rightarrow S_2; \omega) f_{\text{acc}}(S_i|\omega, \omega_2),
\]

(A1)

where

\[
b(S_i, \omega_f, \omega_f) \equiv \int d\omega \int d\omega_1 \int d\omega_2 f_d(S_i|\omega, \omega_2),
\]

(A2)

and is a known function. We are seeking a solution of \( f_d(S|\omega, \omega_f) \) for a given \( (S, \omega_f) \), so we further suppress these two arguments such that \( f_d(S, \omega) \equiv f_d(S|\omega, \omega_f) \). In the method of unknown coefficients one expands the unknown function \( f_d \) as a sum of some appropriate basis functions \( q_n(S, \omega) \),

\[
f_d(S, \omega) = \sum_{n=1}^{N} a_n q_n(S, \omega)
\]

(A3)

with the goal of finding the coefficients \( a_n \). The functions \( q_n(S, \omega) \) should go to zero when \( \omega \to \omega_f \) or \( S \to S_f + \Delta(S_f) \). The latter limit arises because by definition, for \( M - M_f < M_{\text{crit}} \) there is no merger, so \( f_{\text{tot}} = f_{\text{acc}} \) in this range. The number of terms \( N \) we need to have a convergent solution depends on the choice of the basis functions. A simple choice which we used in our calculation is the power series:

\[
q_n(S, \omega) \equiv (\omega - \omega_f)^n [S - (S_f + \Delta(S_f))]^{n^2}, \quad n 1, n 2 1.
\]

(A4)

We then plug this into the integral equation and evaluate it for a given choice of \( (S, \omega) = (S_f, \omega_f) \):

\[
\sum_{n=1}^{N} a_n q_n(S_f, \omega_f) = b(S_f, \omega_f) - \sum_{n=1}^{N} a_n w_n(S_f, \omega_f),
\]

(A5)

where \( b \) is defined in equation (A2) and the \( w_n \) are

\[
w_n(S_f, \omega_f) \equiv \int d\omega \int d\omega_1 \int d\omega_2 f_{\text{tot}}(S_f|\omega, \omega_2) \times f_{\text{acc}}(S_f|\omega, \omega_2).
\]

(A6)

Equation (A5) is a simple algebraic equation for the unknown coefficients \( a_n \). If we plug \( N \) different choices of \( (S_f, \omega_f) \) into this equation, we will have \( N \) equations for the \( N \) unknowns \( a_n \).

APPENDIX B: MONTE CARLO SIMULATION

First, we briefly describe how the binary merger with accretion works. For a small change in \( \Delta \omega \), the probability of absorbing a mass \( \Delta S \) in this time interval is given by

\[
P(\Delta S, \Delta \omega) dS = \frac{1}{(2\pi)^{1/2}} \frac{\Delta \omega}{(\Delta S)^{3/2}} \exp \left[ -\frac{(\Delta \omega)^2}{2(\Delta S)} \right].
\]

(B1)
Starting from a parent halo with mass $S_p$ at time $\omega$, we go backward in time to $\omega - \Delta \omega$. Then, if $M(\Delta S) < M_{\text{res}}$ we consider that to be accreted mass, stop tracking its history, and take $M_p - M(\Delta S)$ as the new parent halo at redshift $\omega - \Delta \omega$ assuming that this new $M_p$ is larger than $M_{\text{res}}$. If not, we consider that as accreted mass and do not continue to track its history. We choose $\Delta \omega$ small enough to ensure that the probability of having a merger in this time interval becomes small. In other words, we demand

$$\Delta \omega < \sqrt{S(M_p - M_{\text{res}}) - S(M_p)}.$$  \hfill (B2)

Then we generate a random number $\Delta S$ consistent with the distribution $B_1$. This is a very easy task to do since equation (B1) can be converted to a Gaussian distribution by a change of variable $x = \Delta \omega / (2\sqrt{\Delta S})$. This procedure is then repeated with the new haloes as the parent haloes until we reach the time $\omega_i$, where we want to compare our numerical results with the MC simulation.

While making the merger tree, we keep track of each halo to know how many times in their history they experienced a merger so we will be able to find the distribution of haloes with no merger, one merger and so forth.

There are several other methods to generate merger trees and it would be interesting to see if the small discrepancy between our results and the MC will disappear using these methods. For example, the method outlined in Moreno et al. (2008), which takes discrete steps in mass rather than time, seems to be using more of the ingredients of the random walk formalism and therefore seems to be a better method to compare to our results. We leave this comparison to future work.

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