Equivariant embeddings of Hermitian symmetric spaces

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MS received 28 November 2006; revised 3 April 2007

Abstract. We prove that equivariant, holomorphic embeddings of Hermitian symmetric spaces are totally geodesic (when the image is not of exceptional type).

Keywords. Complex variables; metric geometry (symmetric spaces).

1. Introduction

Let $H, G$ be connected semi-simple Lie groups and $X_H, X_G$ the associated symmetric spaces. We assume that they are Hermitian. An equivariant embedding is a pair $(F, f)$ where $F: H \to G$ is a homomorphism, $f: X_H \to X_G$ is a holomorphic map and

$$f(h \cdot x) = F(h)f(x), \quad x \in X_H, h \in H.$$ 

We assume that $H, G$ have no compact factors and that $f$ is injective. Then, as is easily checked, the kernel of $F$ is finite. Replacing $H$ by its image, we will also assume $F$ injective and therefore identify $H$ with its image in $G$.

Such maps have been classified by Satake [8] and Ihara [3] when $X_H$ is totally geodesic in $X_G$. The purpose of this note is to show the following theorem.

Theorem. Assume $G$ has no factors of exceptional type. Then any equivariant embedding $X_H \to X_G$ is totally geodesic.

We should emphasize the rather surprising content of this result when compared with the case of compact Hermitian symmetric spaces. If $G$ is compact, the symmetric space $X_G$ (assumed Hermitian) is a generalized Grassmanian. The natural maps of algebraic geometry between Grassmanians – in particular the Veronese and Segre embeddings – are holomorphic and equivariant with respect to natural maps of the associated groups. Very few are totally geodesic: in fact by duality between compact and non-compact symmetric spaces, the totally geodesic equivariant maps between compact spaces correspond to those between non-compact spaces, which are quite rare (see [2]). However, this result becomes more natural from the 'global' point of view, i.e., if one considers arithmetic quotients of the symmetric spaces.

Assume $H, G$ are semi-simple groups defined over $\mathbb{Q}$, $F: H \to G$ is defined over $\mathbb{Q}$ and $f: X_H \to X_G$ is an equivariant embedding. For suitable arithmetic subgroups $\Delta \subset H(\mathbb{Q})$ and $\Gamma \subset G(\mathbb{Q})$, $f$ defines a holomorphic map

$$g: S_H \to S_G,$$

where $S_H = \Delta \backslash X_H$ and $S_G = \Gamma \backslash X_G$. 

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In this situation, recall that $S_H$ and $S_G$ have a remarkable family of distinguished points, the CM-points or special points [5]. Also note that $S_H$, $S_G$ are in fact algebraic varieties over $\mathbb{C}$, and that $g(S_H)$ is an algebraic subvariety of $S_G$ by a theorem of Borel. Assume $g(S_H)$ has one CM-point. By using the action of $H(\mathbb{Q})$ on $X_H$ one easily sees that it has a dense subset of CM-points for the complex topology. A conjecture of André [11] and Oort [7] then implies that $g(S_H)$ is a totally geodesic submanifold of $S_G$ (and $X_H$ is a totally geodesic submanifold of $X_G$).

It is not obvious that $g(S_H)$ should have one CM-point; note, however, the following. The Hermitian symmetric spaces are open subspaces of their compact duals – generalized Grassmanians. An equivariant holomorphic embedding will generally be given by a natural holomorphic map between the compact duals. Given the $\mathbb{Q}$-structure, CM-points correspond to subspaces (in the Grassmanians) verifying some rationality conditions. It is natural to expect these to be preserved. The embedding of the symmetric space for $SU(p, 1)$, $X_{p, 1}$, into $X_{P, Q}$ where $P = \binom{p}{1}$, $Q = \binom{p}{k-1}$ [8] gives a very graphic example.

Another strong motivation for the theorem is given by Mok’s rigidity results. Assume for simplicity that $H$ is irreducible over $\mathbb{Q}$ and $\operatorname{rk}(H) > 1$ (this is the real rank). Then Mok (Ch. 6, Thm 4.1 of [4]) – see also the discussion at the beginning of ch. 9 – has shown that any holomorphic map $S_H \rightarrow S_G$ is totally geodesic. If $F: H(\mathbb{R}) \rightarrow G(\mathbb{R})$ (we now denote the Lie groups by $G(\mathbb{R})$, $H(\mathbb{R})$ as we will be using rationality arguments) is given and if $F$ is $G(\mathbb{R})$-conjugate to a map defined over $\mathbb{Q}$, Mok’s theorem implies our local assertion.

More generally, assume $F$ is given, and assume that there exists a totally real number field $L$ and a map $F_L: H \rightarrow G$ defined over $L$ such that, for each real prime $v$ of $L$ (thus $L_v \cong \mathbb{R}$),

$$F_{L_v}: H(\mathbb{R}) \rightarrow G(\mathbb{R})$$

is conjugate to $F$. Then, again using Mok’s results, we deduce that $F$ is totally geodesic. The set of homomorphisms $F: H \rightarrow G$, over an algebraically closed field, and modulo $G$-conjugation, is discrete (homomorphism of semi-simple groups up to conjugacy are rigid). Thus $F: H(\mathbb{R}) \rightarrow G(\mathbb{R})$ is $G(\mathbb{R})$-conjugate to a map $F_\theta$ defined over $\mathbb{Q}$; the $G$-conjugacy class of $F_\theta$ is an irreducible variety. If it is defined over $\mathbb{Q}$, a theorem of Moret–Bailly [6] implies that there is a totally real number field $L$, and a map $F_L: H \otimes_{\mathbb{Q}} L \rightarrow G \otimes_{\mathbb{Q}} L$ verifying our condition.

It is of course, difficult to compute the field of rationality of the class associated to $F$. One may, however, pose the following:

**Problem.** If $H$, $G$ be semisimple groups over $\mathbb{Q}$ and $F: H \rightarrow G$ a homomorphism defined over $\mathbb{R}$, does there exist a totally real field $L$ and $F_L: H \rightarrow G/L$ such that $F_\theta$ is $G(\mathbb{R})$-conjugate to $F$ at each real prime of $L$?

Finally, Mok has informed us that he could prove the theorem even for exceptional $G$. His proof, however, is more difficult and necessitates global geometric computations.

2. **Reductions**

Let $G$ be a connected semi-simple Lie group, with finite center and no compact factor, associated to a Hermitian symmetric space $X$. Fix a point $x \in X$. Then $x$ defines a maximal compact subgroup $K \subset G$ and a Cartan involution $\theta$ on $g = \operatorname{Lie}(G)$. Let $g = \mathfrak{k} \oplus \mathfrak{p}$.
be the Cartan decomposition. There exists an element $\zeta \in Z(K)$ such that $\text{Ad}(\zeta)$ induces on $p$ the multiplication by $i = \sqrt{-1}$ defining the complex structure. Then $\zeta^2 \in Z(K)$ induces, by the adjoint action, the Cartan involution. By construction this holomorphic structure is $G$-equivariant: if $x' = g \cdot x$ the associated data are obtained by conjugation by $g$. In particular, $\zeta' = \text{Ad}(g)\zeta \in K'$ is well-defined by $x'$ since $\zeta$ is $K$-invariant, and this family of quasi-complex structures defines the holomorphic structure on $X$.

Now assume $H \subset G, f: X_H \to X_G$ verify our conditions. Fix a base point $x \in X_H$. This defines maximal compact subgroups $K_H \subset K_G$. (We will drop indexes for the group $G$). Thus

$$g = \mathfrak{t} \oplus \mathfrak{p},$$

$$\mathfrak{h} = \mathfrak{t}_H \oplus \mathfrak{p}_H$$

and the (injective) map $F: \mathfrak{h} \to \mathfrak{g}$ has the following properties:

$$F(\mathfrak{t}_H) \subset \mathfrak{t},$$

(1)

$$F(X) = F_c(X) + F_p(X),$$

(2)

$$(X \in \mathfrak{p}_H, F_c(X) \in \mathfrak{t}, F_p(X) \in \mathfrak{p})$$

$$F_p(\mathfrak{t}_H X) = t_G F_p(X),$$

(3)

where $\mathfrak{t}_H, t_G$ are ‘multiplication by $\sqrt{-1}$’ on $\mathfrak{p}_H, \mathfrak{p}$, given by $\zeta_H, \zeta_G$. Conversely, if a morphism $F: \mathfrak{h} \to \mathfrak{g}$ verifies (1)–(3), $F$ defines a map $H/K_H \to G/K_G$, holomorphic at $x = eK_H$ and in fact at every point by a computation similar to that as above. Note that $f$ is a totally geodesic immersion if and only if,

$$F(\mathfrak{p}_H) \subset \mathfrak{p},$$

i.e., if $F_c \equiv 0$.

(see p. 47 ff. of [8])

Identifying $\mathfrak{t}_H$ with a subalgebra of $\mathfrak{t}$ by (1), we note that the two components $F_c$ and $F_p$ are $\mathfrak{t}_H$-equivariant. Moreover, let $\mathfrak{h} = \oplus \mathfrak{h}_i$ be a decomposition of $\mathfrak{h}$ in simple factors. Then $\zeta_H$ or $\zeta_H$ decomposes accordingly, so the restriction $\mathfrak{F}_i$ to $\mathfrak{h}_i$ again verifies the conditions. Thus we may assume that $\mathfrak{h}$ is simple.

In this case it is known (see S Helgason, Differential Geometry and Symmetric Spaces, ch. VIII, §5) that the (real) representation of $\mathfrak{t}_H$ on $\mathfrak{p}_H$ is irreducible. The $\mathfrak{t}_H$-map $\mathfrak{F}_c: \mathfrak{p}_H \to \mathfrak{t}$ is therefore injective or zero. Assume (changing notation) that $\mathfrak{h}_1 \subset \mathfrak{h}$ is a $\theta$-stable semi-simple subalgebra such that the injection $\mathfrak{p}_1 \subset \mathfrak{p}_H$ is holomorphic (for the choice of $\zeta_1 \in Z(K_1)$ where $K_1$ is the obvious maximal compact subgroup of $H_1 = \exp(h_1) \subset H$).

It suffices then to check that $F_c = 0$ on $\mathfrak{p}_1$. But any Hermitian Lie algebra $\mathfrak{h}$ contains a subalgebra $\mathfrak{h}_1$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, the injection being holomorphic in the obvious sense (in fact it contains $\mathfrak{sl}(2, \mathbb{R})^r$ where $r$ is the real rank (see e.g. Ch. 5 of [3]). Thus we are reduced to the case when $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R})$.

We can also replace $G$ by a larger group. By the results of Satake, $X_G$ embeds into $X_{G_1}$ where $G_1 = SU(p, p)$, as a totally geodesic subvariety, via an equivariant embedding. Finally we are reduced to the case when $H$ is locally isomorphic to $SL(2, \mathbb{R})$ or $SU(1, 1)$ and $G$ to $SU(p, p)$. (Note that this does not apply when $G$ has exceptional factors).
3. Computations

In this paragraph we consider the case, to which we are reduced, when $H = SU(1, 1)$ and $G = SU(p, p)$. We try to solve the linear algebra problem of §2 – find $F$ verifying (1)–(3).

We have

$$h = \left\{ \begin{pmatrix} a & \bar{z} \\ \bar{z} & -a \end{pmatrix} : z \in \mathbb{C}, \ a \subset i\mathbb{R} \right\},$$

$$g = \left\{ \begin{pmatrix} A & Z \\ \bar{Z} & B \end{pmatrix} : \text{Tr}(A) + \text{Tr}(B) = 0 \right\},$$

where the block matrices are of size $p \times p$, $Z$ is (complex) arbitrary and $A, B$ are skew-hermitian. Let $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} i \\ -i \end{pmatrix}$, $w = \begin{pmatrix} i \\ -i \end{pmatrix}$, a basis of $h$ (the empty entries are zero).

Let $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $h = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, a basis of $h \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$. We take $\mathfrak{k} \subset \mathfrak{g}$ given by block-diagonal matrices, so

$$g = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} Z^* \\ Z \end{pmatrix} : Z \in M_p(\mathbb{C}) \right\}.$$

If $F : h \to g$ verifies (3) we have

$$F(u) = \begin{pmatrix} A & Z \\ Z^* & B \end{pmatrix},$$

$$F(v) = \begin{pmatrix} C & iZ \\ -iZ^* & D \end{pmatrix},$$

$A, \ldots, D$ verifying of course (5). Let $X, Y, H$ be the images of $x, y, h$. Using (5), (6) and (7) we have

$$X = \begin{pmatrix} E & Z \\ F & \end{pmatrix},$$

$$Y = \begin{pmatrix} -E^* \\ Z^* \end{pmatrix},$$

$$H = [X, Y] = \begin{pmatrix} -[E, E^*] + ZZ^* & -ZF^* + E^*Z \\ FZ^* - Z^*E & -[F, F^*] - Z^*Z \end{pmatrix},$$

where $E, F$ and $Z$ are arbitrary $p \times p$-matrices (with $\text{Tr}(E) + \text{Tr}(F) = 0$). Since $h = i^{-1}w$, $H$ is block-diagonal by (1); conjugating $w$ under $K = S(U(p) \times U(p))$ we can assume that the block-diagonal entries of $H$ are diagonal matrices $H_1, H_2$. The eigenvalues of $H$ are integral, and constitute the eigenvalues of a representation of $\mathfrak{sl}(2, \mathbb{C})$. 


Let \( V \cong \mathbb{C}^{2p} \) be the space of the natural representation of \( G \), and \( V = V_+ \oplus V_- \) its decomposition into a positive and a negative subspace. Then

\[
E : V_+ \to V_+, \\
F : V_- \to V_-, \\
Z : V_- \to V_+.
\]

Let \( \lambda_1 > \cdots > \lambda_{t+1} \) be the distinct eigenvalues of \( H \) in \( V_+ \) and \( \mu_1 > \cdots > \mu_{s+1} \) the eigenvalues in \( V_- : s, t \geq 0 \). We can write \( V = V^{\text{even}} \oplus V^{\text{odd}} \), the eigenvalues being even or odd in each summand; this decomposition is preserved by \( X \) and \( Y \). The decomposition is orthogonal and compatible with \( V = V_+ \oplus V_- \). If \( v \) belongs to the \( \lambda \)-eigenspace of \( V_+ \) (resp. \( V_- \)), \( E_v \) (resp. \( Z_v, F_v \)) belongs to the \( (\lambda + 2) \)-eigenspace of \( V_+ \) (resp. \( V_+, V_- \)).

Consider first the odd part of \( V \). We can write in \( V^{\text{odd}} \):

\[
E = \begin{pmatrix}
0 & E_1 \\
0 & E_2 \\
& & \ddots \\
& & & E_t \\
0 & & & & 0
\end{pmatrix}, \\
E^* = \begin{pmatrix}
0 & E_1^* \\
& & \ddots \\
& & & E_t^* \\
& & & & 0
\end{pmatrix}.
\]

Writing \( \text{diag}(A_1, \ldots, A_{t+1}) \) for a block-diagonal matrix we have

\[
EE^* = \text{diag}(E_1E_1^*, \ldots, E_tE_t^*, 0) \\
E^*E = \text{diag}(0, E_1E_1^*, \ldots, E_tE_t^*).
\]

According to (10),

\[
-[E, E^*] + ZZ^* = \text{diag}(-E_1E_1^*, \ldots, E_tE_t^*) + ZZ^* \\
= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{t+1}), \tag{11}
\]

where the eigenvalues are now those in \( V^{\text{odd}} \), the last ‘diagonal’ matrix including of course the multiplicities. Considering the restriction of the corresponding Hermitian forms to the last summand we see that

\[
E_tE_t^* + ZZ^* = \lambda_{t+1} \geq 0;
\]

since the representation is odd, \( \lambda_1 > \cdots > \lambda_{t+1} > 0 \).

Similarly in \( V^{\text{odd}} \):

\[
F = \begin{pmatrix}
0 & F_1 \\
0 & F_2 \\
& & \ddots \\
& & & F_s \\
0 & & & & 0
\end{pmatrix}, \\
F^* = \begin{pmatrix}
0 & F_1^* \\
& & \ddots \\
& & & F_s^* \\
& & & & 0
\end{pmatrix},
\]

\[
-[F, F^*] - Z^*Z = \text{diag}(-F_1F_1^*, \ldots, F_sF_s^*) - Z^*Z \\
= \text{diag}(\mu_1, \mu_2, \ldots, \mu_{s+1}) \tag{12}
\]

whence \( 0 > \mu_1 > \cdots > \mu_{s+1} \).
Thus now a multiple of the standard representation, in conformity with Satake’s results.

By (11) and (13),

$$\text{diag}(-E_1E_1^+, E_1^+E_1 - E_2E_2^+, \ldots, E_t^+E_t + Z_tZ_t^+) = (\lambda_1, \ldots, \lambda_{t+1})$$

with positive eigenvalues. This is impossible unless

$$\begin{cases}
t = 0, \lambda_1 = 1, E = 0, \\
Z = Z_1, ZZ^* = 1.
\end{cases}$$

(The identity (15) implies that $t = 0$; since there is only one eigenvalue, the representation theory of $SL(2)$ forces it to be 1.)

This implies of course that the only eigenvalue $\mu$ is $-1$, so $s = 0$ and $F = 0$. Since $E, F$ vanish the embedding is totally geodesic; the representation of $SL(2)$ or $SU(1, 1)$ is a multiple of the standard representation, in conformity with Satake’s results.

Consider now the even part of $V$. The first part of the argument still applies, yielding now

$$\lambda_1 > \cdots > \lambda_{t+1} \geq 0,$$

$$0 \geq \mu_1 > \cdots > \mu_{s+1}.$$  \hspace{1cm} (17)  \hspace{1cm} (18)

Now $Z$ is the sum of

$$Z_1 : V_-(0) \to V_+(2),$$

$$Z_2 : V_-(2) \to V_+(0).$$

Thus

$$ZZ^* = \text{diag}(0, \ldots, 0, Z_tZ_t^*)$$

$$Z^*Z = \text{diag}(Z_t^*Z_1, Z_2^*Z_2, 0, \ldots, 0).$$

By (11) and (19),

$$- [E_1E^+] + ZZ^*$$

$$= (-E_1E_1^+, E_1^+E_1 - E_2E_2^+, \ldots, E_{t-1}^+E_{t-1} - E_tE_t^+ + Z_tZ_t^*, E_t^+E_t + Z_2Z_2^*)$$

$$= (\lambda_1, \ldots, \lambda_t, \lambda_{t+1}),$$

where we assume so far that both 2 and 0 are eigenvalues in $V_{+}^{\text{even}}$. This implies first that there are only two eigenvalues since $-E_1E_1^* = \lambda_1 > 0$ for $t > 1$. Furthermore, the last entry in (21) yields $E_1E_1^* + Z_2Z_2^* = 0$, whence $E = E_1 = 0$ and $Z_2 = 0$.

If 2 does not occur in $V_{+}^{\text{even}}$, the representation on $V_{+}^{\text{even}}$ is trivial; if 0 does not occur $Z_2$ is absent. In this case,

$$ZZ^* = \text{diag}(0, \ldots, 0, Z_tZ_t^*),$$

$$Z^*Z = \text{diag}(Z_t^*Z_1, 0 \ldots 0)$$

$$\quad$$

Finally the only non-vanishing part of $Z$ is a map $Z_1 : V_-(-1) \to V_+(1)$ (where $V(\lambda)$, $V_{+}(\lambda)$ denote the eigenspaces of $H$). Thus

$$ZZ^* = \text{diag}(0, 0, \ldots, Z_tZ_t^*),$$

$$Z^*Z = \text{diag}(Z_t^*Z_1, 0, \ldots, 0).$$

(14)
and

\[-[E, E^*] + ZZ^* = (-E_1 E_1^*, E_1^* E_1 - E_2 E_2^*, \ldots, E_t E_t^* + Z_1 Z_1^*)
= (\lambda_1, \ldots, \lambda_{t+1})\]

with \(\lambda_{t+1} = 2\). This equality \((-E_1 E_1^* = \lambda_1 > 0)\) implies that there is only one eigenvalue \((t = 0)\) and therefore \(E = 0\).

Of course, a similar computation, as in the odd case, applies to the negative part, using now (20); if there are two eigenvalues \((0, -2)\) we deduce that

\[F = F_1 = 0, \ Z_1 = 0.\]

Thus \(Z = 0\), contrary to the assumption that it represented the tangent map to an equivariant embedding.

Finally, consider the case where 0 does not occur in \(V_{+}^{\text{even}}\) or \(V_{-}^{\text{even}}\).

The computations being symmetric we can assume for instance that it is missing in \(V_{+}^{\text{even}}\); we already know that the eigenvalue 2 only occurs, so the eigenvalues in \(V_{+}^{\text{even}}\) are \((2, 0, -2)\); moreover \(E = F = 0\) by the arguments given already, so the embedding should be totally geodesic. We know that this is impossible, by Satake’s results. In fact, using (22) and (12) we see that

\[\text{diag}(-F_1 F_1^* - Z_1 Z_1^*, F_1^* F_1) = (\mu_1, \mu_2) = (0, -2)\]

which is impossible.

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