Sampling Associated with a Unitary Representation of a Semi-Direct Product of Groups: A Filter Bank Approach

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Abstract: An abstract sampling theory associated with a unitary representation of a countable discrete non abelian group $G$, which is a semi-direct product of groups, on a separable Hilbert space is studied. A suitable expression of the data samples, the use of a filter bank formalism and the corresponding frame analysis allow for fixing the mathematical problem to be solved: the search of appropriate dual frames for $\ell^2(G)$. An example involving crystallographic groups illustrates the obtained results by using either average or pointwise samples.

Keywords: semi-direct product of groups; unitary representation of a group; LCA groups; dual frames; sampling expansions

1. Statement of the Problem

In this paper, an abstract sampling theory associated with non abelian groups is derived for the specific case of a unitary representation of a semi-direct product of groups on a separable Hilbert space. Semi-direct product of groups provide important examples of non abelian groups such as dihedral groups, infinite dihedral group, Euclidean motion groups or crystallographic groups. Concretely, let $(n,h) \mapsto U(n,h)$ be a unitary representation on a separable Hilbert space $H$ of a semi-direct product $G = N \rtimes_{\phi} H$, where $N$ is a countable discrete LCA (locally compact abelian) group, $H$ is a finite group, and $\phi$ denotes the action of the group $H$ on the group $N$ (see Section 2 infra for the details); for a fixed $a \in H$ we consider the $U$-invariant subspace in $H$

$$A_a = \left\{ \sum_{(n,h) \in G} a(n,h) U(n,h) a : \{a(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\},$$

where we assume that $\{U(n,h)a\}$ is a Riesz sequence for $H$, i.e., a Riesz basis for $A_a$ (see Ref. [1] for a necessary and sufficient condition). Given $K$ elements $b_k$ in $H$, which do not belong necessarily to $A_a$, the main goal in this paper is the stable recovery of any $x \in A_a$ from the given data (generalized samples)

$$L_k x(n) := \langle x, U(n,1_H) b_k \rangle_H, \quad n \in N \text{ and } k = 1,2,\ldots,K.$$
where $1_H$ denotes the identity element in $H$. These samples are nothing but a generalization of average sampling in shift-invariant subspaces of $L^2(\mathbb{R}^d)$; see, among others, Refs. [2–9]. The case where $G$ is a discrete LCA group and the samples are taken at a uniform lattice of $G$ has been solved in Ref. [10]; this work relies on the use of the Fourier analysis in the LCA group $G$ (see also Ref. [11]). In the case involved here, a classical Fourier analysis is not available and, consequently, we need to overcome this drawback.

Having in mind the filter bank formalism in discrete LCA groups (see, for instance, Refs. [12–14]), the given data $\{L_k x(n)\}_{n \in N; k=1,2,...,K}$ can be expressed as the output of a suitable $K$-channel analysis filter bank corresponding to the input $a = \{a(n,h)\}_{(n,h) \in C}$ in $\ell^2(G)$. As a consequence, the problem consists of finding a synthesis part of the former filter bank allowing perfect reconstruction; in addition, only Fourier analysis on the LCA group $N$ is needed. Then, roughly speaking, substituting the output of the synthesis part in $x = \sum_{(n,h) \in C} a(n,h) U(n,h)a$, we will obtain the corresponding sampling formula in $A_a$.

This said, as it could be expected, the problem can be mathematically formulated as the search of dual frames for $\ell^2(G)$ having the form

$$\{ T_nh_k \}_{n \in N; k=1,2,...,K} \quad \text{and} \quad \{ T_ng_k \}_{n \in N; k=1,2,...,K}.$$

Here, $h_k, g_k \in \ell^2(G)$, $T_nh_k(m,h) = h_k(m-n,h)$ and $T_ng_k(m,h) = g_k(m-n,h)$, $(m,h) \in G$, where $n \in N$ and $k = 1,2,...,K$. In addition, for any $x \in A_a$, we have the expression for its samples

$$L_k x(n) = \langle a, T_nh_k \rangle_{\ell^2(G)}, \quad n \in N \quad \text{and} \quad k = 1,2,...,K.$$

Needless to say, frame theory plays a central role in what follows; the necessary background on Riesz bases or frame theory in a separable Hilbert space can be found, for instance, in Ref. [15]. Finally, sampling formulas in $A_a$ having the form

$$x = \sum_{k=1}^K \sum_{n \in N} L_k x(n) U(n,1_H) c_k \quad \text{in} \quad H,$$

for some $c_k \in A_a, k = 1,2,...,K$, will come out by using, for $g \in \ell^2(G)$ and $n \in N$, the shifting property $T_{U,a}(T_{n,g}) = U(n,1_H)(T_{U,a}g)$ that satisfies the natural isomorphism $T_{U,a} : \ell^2(G) \rightarrow A_a$ which maps the usual orthonormal basis $\{ \delta_{(n,h)} \}_{(n,h) \in C}$ for $\ell^2(G)$ onto the Riesz basis $\{ U(n,h)a \}_{(n,h) \in C}$ for $A_a$.

All these steps will be carried out throughout the remaining sections. For the sake of completeness, Section 2 includes some basic preliminaries on semi-direct product of groups and Fourier analysis on LCA groups. The paper ends with an illustrative example involving the quasi regular representation of a crystallographic group on $L^2(\mathbb{R}^d)$; sampling formulas involving average or pointwise samples are obtained for the corresponding $U$-invariant subspaces in $L^2(\mathbb{R}^d)$.

2. Some Mathematical Preliminaries

In this section, we introduce the basic tools in semi-direct product of groups and in harmonic analysis in a discrete LCA group that will be used in the sequel.

2.1. Preliminaries on Semi-Direct Product of Groups

Given groups $(N, \cdot)$ and $(H, \cdot)$, and a homomorphism $\phi : H \rightarrow Aut(N)$, their semi-direct product $G := N \rtimes_\phi H$ is defined as follows: The underlying set of $G$ is the set of pairs $(n,h)$ with $n \in N$ and $h \in H$, along with the multiplication rule

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1\phi_{h_1}(n_2), h_1h_2), \quad (n_1, h_1), (n_2, h_2) \in G,$$
where we denote \( \phi(h) := \phi_h \); usually, the homomorphism \( \phi \) is referred to as the action of the group \( H \) on the group \( N \). Thus, we obtain a new group with identity element \( (1_N, 1_H) \), and inverse \((n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1}) \).

In addition, we have the isomorphisms \( N \cong N \times \{1_H\} \) and \( H \cong \{1_N\} \times H \). Unless \( \phi_h \) equals the identity for all \( h \in H \), the group \( G = N \times \varphi H \) is not abelian, even for abelian \( N \) and \( H \) groups. The subgroup \( N \) is a normal subgroup in \( G \). Some examples of semi-direct product of groups:

1. The dihedral group \( D_{2N} \) is the group of symmetries of a regular \( N \)-sided polygon; it is the semi-direct product \( D_{2N} = \mathbb{Z}_N \rtimes \mathbb{Z}_2 \) where \( \phi_0 \equiv \text{Id}_{\mathbb{Z}_N} \) and \( \phi_1(n) = -n \) for each \( n \in \mathbb{Z}_N \). The infinite dihedral group \( D_\infty \) defined as \( \mathbb{Z} \rtimes \mathbb{Z}_2 \) for the similar homomorphism \( \phi \) is the group of isometries of \( \mathbb{Z} \).

2. The Euclidean motion group \( E(d) \) is the semi-direct product \( \mathbb{R}^d \rtimes \varphi O(d) \), where \( O(d) \) is the orthogonal group of order \( d \) and \( \varphi \) is the orthogonal group of order \( d \) and \( x \in \mathbb{R}^d \). It contains as a subgroup any crystallographic group \( \mathbb{MZ}^d \rtimes \varphi \Gamma \), where \( \mathbb{MZ}^d \) denotes a full rank lattice of \( \mathbb{R}^d \) and \( \Gamma \) is any finite subgroup of \( O(d) \) such that \( \varphi_\gamma(\mathbb{MZ}^d) = \mathbb{MZ}^d \) for each \( \gamma \in \Gamma \).

3. The orthogonal group \( O(d) \) of all orthogonal real \( d \times d \) matrices is isomorphic to the semi-direct product \( SO(d) \rtimes \varphi C_2 \), where \( SO(d) \) consists of all orthogonal matrices with determinant 1 and \( C_2 = \{ I, R \} \) a cyclic group of order 2; \( \varphi \) is the homomorphism given by \( \varphi(A) = A \) and \( \varphi_R(A) = RAR^{-1} \) for \( A \in SO(d) \).

Suppose that \( N \) is an LCA group with Haar measure \( \mu_N \) and \( H \) is a locally compact group with Haar measure \( \mu_H \). Then, the semi-direct product \( G = N \rtimes \varphi H \) endowed with the product topology becomes also a topological group. For the left Haar measure on \( G \), see Ref. [1].

2.2. Some Preliminaries on Harmonic Analysis on Discrete LCA Groups

The results about harmonic analysis on locally compact abelian (LCA) groups are borrowed from Ref. [16]. Notice that, in particular, a countable discrete abelian group is a second countable Hausdorff LCA group.

For a countable discrete group \( (N, \cdot) \), not necessarily abelian, the convolution of \( x, y : N \to \mathbb{C} \) is formally defined as \( (x * y)(m) := \sum_{n \in N} x(n)y(n^{-1}m), m \in N \). If, in addition, the group is abelian, therefore denoted by \( (N, +) \), the convolution reads as

\[
(x * y)(m) := \sum_{n \in N} x(n)y(m - n), \quad m \in N.
\]

Let \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) be the unidimensional torus. We said that \( \xi : N \to \mathbb{T} \) is a character of \( N \) if \( \xi(n + m) = \xi(n)\xi(m) \) for all \( n, m \in N \). We denote \( \xi(n) = \langle n, \xi \rangle \). Defining \( (\xi + \gamma)(n) = \xi(n)\gamma(n) \), the set of characters \( \hat{N} \) with the operation \( + \) is a group, called the dual group of \( N \); since \( N \) is discrete \( \hat{N} \) is compact (Ref. [16], Prop. 4.4). For \( x \in L^1(N) \), we define its Fourier transform as

\[
X(\xi) = \hat{x}(\xi) := \sum_{n \in N} x(n)\overline{\xi(n)}, \quad \xi \in \hat{N}.
\]

It is known (Ref. [16], Theorem 4.5) that \( \hat{\mathbb{Z}} \cong \mathbb{T} \), with \( \langle n, z \rangle = z^n \), and \( \hat{\mathbb{Z}} = \mathbb{Z} \) := \( \mathbb{Z}/s\mathbb{Z} \), with \( \langle n, m \rangle = W^n_m \), where \( W_n = e^{2\pi i/n} \).

There exists a unique measure, the Haar measure \( \mu \) on \( \hat{N} \) satisfying \( \mu(\xi + E) = \mu(E) \), for every Borel set \( E \subset \hat{N} \) (Ref. [16], Section 2.2), and \( \mu(\hat{N}) = 1 \). We denote \( \int_{\hat{N}} X(\xi)d\xi = \int_{\hat{N}} X(\xi)\mu(\xi) \). If \( N = \mathbb{Z} \),

\[
\int_{\hat{N}} X(\xi)d\xi = \int_{\mathbb{C}} X(z)dz = \frac{1}{2\pi} \int_0^{2\pi} X(e^{i\omega})d\omega,
\]

and, if \( N = \mathbb{Z}_s \),

\[
\int_{\hat{N}} X(\xi)d\xi = \int_{\mathbb{C}} X(n)dn = \frac{1}{s} \sum_{n \in \mathbb{Z}_s} X(n).
\]
If \( N_1, N_2, \ldots, N_d \) are abelian discrete groups, then the dual group of the product group is \( (N_1 \times N_2 \times \ldots \times N_d)^\wedge \cong \hat{N}_1 \times \hat{N}_2 \times \ldots \times \hat{N}_d \) (see ([16], Prop. 4.6)) with

\[
\langle (n_1, n_2, \ldots, n_d), (\xi_1, \xi_2, \ldots, \xi_d) \rangle = \langle n_1, \xi_1 \rangle \langle n_2, \xi_2 \rangle \cdots \langle n_d, \xi_d \rangle.
\]

The Fourier transform on \( \ell^1(N) \cap \ell^2(N) \) is an isometry on a dense subspace of \( L^2(\hat{N}) \); Plancherel theorem extends it in a unique manner to a unitary operator of \( \ell^2(\hat{N}) \) ([16], p. 99). The following lemma, giving a relationship between Fourier transform and convolution, will be used later (see Ref. [17]):

**Lemma 1.** Assume that \( a, b \in \ell^2(N) \) and \( \hat{a}(\xi) \hat{b}(\xi) \in L^2(\hat{N}) \). Then, the convolution \( a * b \) belongs to \( \ell^2(N) \) and \( a * b(\xi) = \hat{a}(\xi) \hat{b}(\xi), \) a.e. \( \xi \in \hat{N} \).

### 3. Filter Bank Formalism on Semi-Direct Product of Groups

In what follows, we will assume that \( G = N \rtimes_H \) where \( (N, +) \) is a countable discrete abelian group and \( (H, \cdot) \) is a finite group. Having in mind the operational calculus \( (n, h) \cdot (m, l) = (n + \phi_h(m), hl) \), \( (n, h)^{-1} = (\phi_h^{-1}(-n), h^{-1}) \) and \( (n, h)^{-1} \cdot (m, l) = (\phi_h^{-1}(m - n), h^{-1}l) \), the convolution \( a * h \) of \( a, h \in \ell^2(G) \) can be expressed as

\[
(a \ast h)(m, l) = \sum_{(n, h) \in G} a(n, h) h[(n, h)^{-1} \cdot (m, l)] = \sum_{(n, h) \in G} a(n, h) h(\phi_h^{-1}(m - n), h^{-1}), \quad (m, l) \in G.
\]  

(1)

For a function \( a : G \to \mathbb{C} \), its \( H \)-decimation \( \downarrow_H a : N \to \mathbb{C} \) is defined as \( (\downarrow_H a)(n) := a(n, 1_H) \) for any \( n \in N \). Thus, we have

\[
\downarrow_H (a \ast h)(m) = (a \ast h)(m, 1_H) = \sum_{(n, h) \in G} a(n, h) h(\phi_h^{-1}(m - n), h^{-1}) = \sum_{(n, h) \in G} a(n, h) h[(n - m, h)^{-1}], \quad m \in N.
\]  

\[
(2)
\]

Defining the polyphase components of \( a \) and \( h \) as \( a_h(n) := a(n, h) \) and \( h_h(n) := h\lfloor (-n, h)^{-1} \rfloor \) respectively, we write

\[
\downarrow_H (a \ast h)(m) = \sum_{n \in N} \sum_{h \in H} a_h(n) h_h(m - n) = \sum_{h \in H} (a_h \ast_N h_h)(m), \quad m \in N.
\]

For a function \( c : N \to \mathbb{C} \), its \( H \)-expander \( \uparrow_H c : G \to \mathbb{C} \) is defined as

\[
(\uparrow_H c)(n, h) = \begin{cases} 
  c(n) & \text{if } h = 1_H, \\
  0 & \text{if } h \neq 1_H.
\end{cases}
\]

In case \( \uparrow_H c \) and \( g \) belong to \( \ell^2(G) \), we have

\[
(\uparrow_H c \ast g)(m, l) = \sum_{(n, h) \in G} (\uparrow_H c)(n, h) g[(n, h)^{-1} \cdot (m, l)] = \sum_{(n, h) \in G} (\uparrow_H c)(n, h) g(\phi_h^{-1}(m - n), h^{-1}l) = \sum_{n \in N} c(n) g(m - n, l) = (c \ast_N g_l)(m), \quad m \in N, \; l \in H,
\]

where \( g_l(n) := g(n, l) \) is the polyphase component of \( g \).
From now on, we will refer to a $K$-channel filter bank with analysis filters $h_k$ and synthesis filters $g_k$, $k = 1, 2, \ldots, K$ as the one given by (see Figure 1)

$$c_k := \downarrow H (a \ast h_k), \quad k = 1, 2, \ldots, K, \quad \text{and} \quad \beta = \sum_{k=1}^{K} (\uparrow H c_k) \ast g_k,$$

where $a$ and $\beta$ denote, respectively, the input and the output of the filter bank. In polyphase notation,

$$c_k(m) = \sum_{h \in H} (a_k \ast N h_{k,h})(m), \quad m \in N, \quad k = 1, 2, \ldots, K,$$

$$\beta_l(m) = \sum_{k=1}^{K} (c_k \ast N g_{l,k})(m), \quad m \in N, \quad l \in H,$$

where $a_k(n) := a(n,h)$, $\beta_l(n) := \beta(n,l)$, $h_{k,h}(n) := h_k[(-n,h)^{-1}]$ and $g_{l,k}(n) := g_k(n,l)$ are the polyphase components of $a$, $\beta$, $h_k$ and $g_k$, $k = 1, 2, \ldots, K$, respectively. We also assume that $h_k, g_k \in \ell^2(G)$ with $\hat{h}_{k,h}, \hat{g}_{l,k} \in L^\infty(\hat{N})$ for $k = 1, 2, \ldots, K$ and $h \in H$; from Lemma 1, the filter bank (3) is well defined in $\ell^2(G)$.

![Figure 1. The K-channel filter bank scheme.](image)

The above $K$-channel filter bank (3) is said to be a perfect reconstruction filter bank if and only if it satisfies $a = \sum_{k=1}^{K} (\uparrow H c_k) \ast g_k$ for each $a \in \ell^2(G)$, or equivalently, $a_h = \sum_{k=1}^{K} (c_k \ast N g_{l,k})$ for each $h \in H$.

Since $N$ is an LCA group where a Fourier transform is available, the polyphase expression (4) of the filter bank (3) allows us to carry out its polyphase analysis.

**Polyphase Analysis: Perfect Reconstruction Condition**

For notational ease, we denote $L := |H|$, the order of the group $H$, and its elements as $H = \{h_1, h_2, \ldots, h_L\}$. Having in mind Lemma 1, the $N$-Fourier transform in $c_k(m) = \sum_{h \in H} (a_h \ast N h_{k,h})(m)$ gives $\hat{c}_k(\gamma) = \sum_{h \in H} \hat{h}_{k,h}(\gamma) \hat{a}_h(\gamma)$ a.e. $\gamma \in \hat{N}$ for each $k = 1, 2, \ldots, K$. In matrix notation,

$$C(\gamma) = H(\gamma) A(\gamma) \quad \text{a.e.} \quad \gamma \in \hat{N},$$

where $C(\gamma) = (\hat{c}_1(\gamma), \hat{c}_2(\gamma), \ldots, \hat{c}_K(\gamma))^\top$, $A(\gamma) = (\hat{a}_{h_1}(\gamma), \hat{a}_{h_2}(\gamma), \ldots, \hat{a}_{h_L}(\gamma))^\top$, and $H(\gamma)$ is the $K \times L$ matrix

$$H(\gamma) = \begin{pmatrix}
\hat{h}_{1,h_1}(\gamma) & \hat{h}_{1,h_2}(\gamma) & \cdots & \hat{h}_{1,h_L}(\gamma) \\
\hat{h}_{2,h_1}(\gamma) & \hat{h}_{2,h_2}(\gamma) & \cdots & \hat{h}_{2,h_L}(\gamma) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{h}_{K,h_1}(\gamma) & \hat{h}_{K,h_2}(\gamma) & \cdots & \hat{h}_{K,h_L}(\gamma)
\end{pmatrix},$$

where $\hat{h}_{k,h} \in \ell^2(\hat{N})$ is the Fourier transform of $h_{k,h}(n) := h_k[(-n,h)^{-1}] \in \ell^2(N)$.
The same procedure for $\beta_i(m) = \sum_{k=1}^K (c_k * g_{l,k}) (m)$ gives $\hat{\beta}_i (\gamma) = \sum_{k=1}^K \hat{g}_{l,k} (\gamma) \hat{c}_k (\gamma)$ a.e. $\gamma \in \hat{N}$. In matrix notation,
$$
B(\gamma) = G(\gamma) C(\gamma) \quad \text{a.e. } \gamma \in \hat{N},
$$
where $B(\gamma) = (\hat{\beta}_{h_1}(\gamma), \hat{\beta}_{h_2}(\gamma), \ldots, \hat{\beta}_{h_L}(\gamma))^\top$, $C(\gamma) = (\hat{c}_1(\gamma), \hat{c}_2(\gamma), \ldots, \hat{c}_K(\gamma))^\top$ and $G(\gamma)$ is the $L \times K$ matrix
\begin{equation}
G(\gamma) = \begin{pmatrix}
\hat{g}_{h_1,1}(\gamma) & \hat{g}_{h_1,2}(\gamma) & \cdots & \hat{g}_{h_1,K}(\gamma) \\
\hat{g}_{h_2,1}(\gamma) & \hat{g}_{h_2,2}(\gamma) & \cdots & \hat{g}_{h_2,K}(\gamma) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{g}_{h_1,1}(\gamma) & \hat{g}_{h_2,2}(\gamma) & \cdots & \hat{g}_{h_1,K}(\gamma)
\end{pmatrix},
\end{equation}

where $\hat{g}_{h_i,k} \in L^2(\hat{N})$ is the Fourier transform of $g_{h_i,k}(n) := g_k(n, h_i) \in \ell^2(N)$.

Thus, in terms of the polyphase matrices $G(\gamma)$ and $H(\gamma)$, the filter bank (3) can be expressed as
$$
B(\gamma) = G(\gamma) H(\gamma) A(\gamma) \quad \text{a.e. } \gamma \in \hat{N}.
$$

As a consequence of Equation (7), we have:

**Theorem 1.** The K-channel filter bank given in Equation (3), where $h_k, g_k$ belong to $\ell^2(G)$ and $\hat{g}_{h_i,k}, \hat{h}_{h_i,k}$ belong to $L^{\infty}(\hat{N})$ for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, L$, satisfies the perfect reconstruction property if and only if $G(\gamma) H(\gamma) = I_L$ a.e. $\gamma \in \hat{N}$, where $I_L$ denotes the identity matrix of order $L$.

**Proof.** First of all, note that the mapping $\alpha \in \ell^2(G) \mapsto A \in L^2_1(\hat{N})$ is a unitary operator. Indeed, for each $\alpha, \beta \in \ell^2(G)$, we have the isometry property
$$
\langle \alpha, \beta \rangle_{\ell^2(G)} = \sum_{(m,h) \in G} \alpha(m, h) \overline{\beta(m, h)} = \sum_{h \in H} \langle \alpha_h, \beta_h \rangle_{\ell^2(N)} = \sum_{h \in H} \langle \hat{\alpha}_h, \hat{\beta}_h \rangle_{L^2(\hat{N})} = \langle A, B \rangle_{L^2_1(\hat{N})},
$$

It is also surjective since the $N$-Fourier transform is a surjective isometry between $\ell^2(N)$ and $L^2(\hat{N})$. Having in mind this property, Equation (7) tells us that the filter bank satisfies the perfect reconstruction property if and only if $G(\gamma) H(\gamma) = I_L$ a.e. $\gamma \in \hat{N}$. $\square$

Notice that, in the perfect reconstruction setting, the number of channels $K$ must be necessarily bigger or equal that the order $L$ of the group $H$, i.e., $K \geq L$.

4. Frame Analysis

For $m \in N$, the translation operator $T_m : \ell^2(G) \rightarrow \ell^2(G)$ is defined as
$$
T_m \alpha(n, h) := \alpha((m, 1_H)^{-1} \cdot (n, h)) = \alpha(n - m, h), \ (n, h) \in G.
$$

The involution operator $\alpha \in \ell^2(G) \mapsto \alpha \in \ell^2(G)$ is defined as $\bar{\alpha}(n, h) := \overline{\alpha((n, h)^{-1}]}$, $(n, h) \in G$. As expected, the classical relationship between convolution and translation operators holds. Thus, for the K-channel filter bank (3), we have (see (2)):
$$
c_k(m) = \downarrow_H (\alpha * h_k)(m) = \langle \alpha, T_m \hat{h}_k \rangle_{\ell^2(G)}, \quad m \in N, \ k = 1, 2, \ldots, K.
$$

In addition,
$$
(\uparrow_H c_k * g_k)(m, h) = \sum_{n \in N} c_k(n) g_k(m - n, h) = \sum_{n \in N} \langle \alpha, T_n \hat{h}_k \rangle_{\ell^2(G)} T_n \hat{g}_k(m, h).
$$
In the perfect reconstruction setting, for any $\alpha \in \ell^2(G)$, we have
\begin{equation}
\alpha = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} \langle \alpha, T_n h_k \rangle \hat{\rho}(G) T_n g_k \quad \text{in} \quad \ell^2(G).
\end{equation}

Given $K$ sequences $f_k \in \ell^2(G)$, $k = 1, 2, \ldots, K$, our main tasks now are: (i) to characterize the sequence $\{T_n f_k\}_{n \in \mathbb{N}; k=1, 2, \ldots, K}$ as a frame for $\ell^2(G)$, and (ii) to find its dual frames having the form $\{T_n g_k\}_{n \in \mathbb{N}; k=1, 2, \ldots, K}$.

To the first end, we consider a $K$-channel analysis filter bank with analysis filters $h_k := \tilde{f}_k$, i.e., the involution of $f_k$, $k = 1, 2, \ldots, K$; let $H(\gamma)$ be its associated $K \times L$ polyphase matrix (5). First, we check that Equation (5) is:
\begin{equation}
H(\gamma) = \left( \tilde{f}_{k,h_i}(\gamma) \right)_{i=1, 2, \ldots, L},
\end{equation}
where $\tilde{f}_{k,h_i}(\gamma)$ denotes the Fourier transform in $L^2(\hat{N})$ of $f_{k,h_i}(n) = f_k(n, h_i)$ in $\ell^2(N)$. Indeed, for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, L$, having in mind that $h_{k,h_i}(n) = h_k(\{-n, h_i\}^{-1})$ for analysis filters, we have:
\begin{align*}
\tilde{f}_{k,h_i}(\gamma) &= \sum_{n \in \mathbb{N}} h_{k,h_i}(n)(-n, \gamma) = \sum_{n \in \mathbb{N}} h_k([\{-n, h_i\}^{-1}](n, \gamma) = \sum_{n \in \mathbb{N}} \tilde{f}_k([\{-n, h_i\}^{-1}](n, \gamma) \\
&= \sum_{n \in \mathbb{N}} f_k(n, h_i)(-n, \gamma) = \sum_{n \in \mathbb{N}} f_k(n, h_i)(-n, \gamma) = \tilde{f}_{k,h_i}(\gamma), \quad \gamma \in \hat{N}.
\end{align*}

Next, we consider its associated constants
\begin{align*}
A_H := \operatorname{ess inf}_{\gamma \in \hat{N}} \lambda_{\min} \left[ H^*(\gamma) H(\gamma) \right] \quad \text{and} \quad B_H := \operatorname{ess sup}_{\gamma \in \hat{N}} \lambda_{\max} \left[ H^*(\gamma) H(\gamma) \right].
\end{align*}

**Theorem 2.** For $f_k$ in $\ell^2(G)$, $k = 1, 2, \ldots, K$, consider the associated matrix $H(\gamma)$ given in Equation (10). Then,
\begin{enumerate}
\item The sequence $\{T_n f_k\}_{n \in \mathbb{N}; k=1, 2, \ldots, K}$ is a Bessel sequence for $\ell^2(G)$ if and only if $B_H < \infty$.
\item The sequence $\{T_n g_k\}_{n \in \mathbb{N}; k=1, 2, \ldots, K}$ is a frame for $\ell^2(G)$ if and only if the inequalities $0 < A_H \leq B_H < \infty$ hold.
\end{enumerate}

**Proof.** Using Plancherel theorem ([16], Theorem 4.25), for each $\alpha \in \ell^2(G)$, we get
\begin{align*}
\langle \alpha, T_n f_k \rangle_{\hat{\rho}(G)} &= \sum_{h \in H} \langle \alpha_h, f_{k,h}(\cdot - n) \rangle_{\hat{\rho}(N)} = \sum_{h \in H} \int_{\hat{N}} \hat{a}_h(\gamma) \tilde{f}_{k,h}(\gamma)(-n, \gamma) d\gamma \\
&= \int_{\hat{N}} \sum_{h \in H} \hat{a}_h(\gamma) \tilde{f}_{k,h}(\gamma)(-n, \gamma) d\gamma = \int_{\hat{N}} H_k(\gamma) A(\gamma)(-n, \gamma) d\gamma,
\end{align*}
where $A(\gamma) = (\hat{a}_{h_1}(\gamma), \hat{a}_{h_2}(\gamma), \ldots, \hat{a}_{h_L}(\gamma))^T$ and $H_k(\gamma)$ denotes the $k$-th row of $H(\gamma)$.
Since $\{\{-n, \gamma\}\}_{n \in \mathbb{N}}$ is an orthonormal basis for $L^2(\hat{N})$, in case that $H(\gamma) A(\gamma) \in L^2_{\hat{N}}$, we have
\begin{align*}
\sum_{k=1}^{K} \sum_{n \in \mathbb{N}} |\langle \alpha, T_n f_k \rangle|^2 &= \sum_{k=1}^{K} \int_{\hat{N}} |H_k(\gamma) A(\gamma)(-n, \gamma)|^2 d\gamma \\
&= \int_{\hat{N}} \|H(\gamma) A(\gamma)\|^2 d\gamma.
\end{align*}

If $B_H < \infty$, having in mind that $\|\alpha\|^2_{\hat{\rho}(G)} = \|A\|^2_{H^2(\hat{N})} = \int_{\hat{N}} \|A(\gamma)\|^2 d\gamma$, the above equality and the Rayleigh–Ritz theorem ([18], Theorem 4.2.2) prove that $\{T_n f_k\}_{n \in \mathbb{N}; k=1, 2, \ldots, K}$ is a Bessel sequence for $\ell^2(G)$ with Bessel bound less or equal than $B_H$. 
On the other hand, if $K < B_H$, then there exists a set $\Omega \subset \hat{N}$ having a strictly positive measure such that $\lambda_{\max}(H^*(\gamma)H(\gamma)) > K$ for $\gamma \in \Omega$. Consider $\alpha$ such that its associated $A(\gamma)$ is 0 if $\gamma \notin \Omega$, and $A(\gamma)$ is a unitary eigenvector corresponding to the largest eigenvalue of $H^*(\gamma)H(\gamma)$ if $\gamma \in \Omega$. Thus, we have that

$$\sum_{k=1}^{K} \sum_{n \in N} |\langle a, T_n f_k \rangle|^2 = \int_{\hat{N}} \|H(\gamma)A(\gamma)\|^2 d\gamma > K \int_{\hat{N}} \|A(\gamma)\|^2 d\gamma = K \|a\|^2_{\ell^2(G)}.$$ 

As a consequence, if $B_H = \infty$, the sequence is not Bessel, and, if $B_H < \infty$, the optimal bound is precisely $B_H$.

Similarly, by using inequality $\|H(\gamma)A(\gamma)\|^2 \geq \lambda_{\min}(H^*(\gamma)H(\gamma)) \|A(\gamma)\|^2$, and that equality holds whenever $A(\gamma)$ is a unitary eigenvector corresponding to the smallest eigenvalue of $H^*(\gamma)H(\gamma)$; one proves the other inequality in part 2. \qed

**Corollary 1.** The sequence $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ is a Bessel sequence for $\ell^2(G)$ if and only if for each $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, L$ the function $f_{k, h_i}$ belongs to $L^\infty(\hat{N})$.

**Proof.** It is a direct consequence of the equivalence between the spectral and Frobenius norms for matrices [18]. \qed

It is worth mentioning that $f_k$ in $\ell^1(G)$, $k = 1, 2, \ldots, K$, implies that the sequence $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ is always a Bessel sequence for $\ell^2(G)$ since each function $f_{k, h_i}$ is continuous and $\hat{N}$ is compact. In this case, the frame condition for $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ reduces to $\|H(\gamma)\| = 1$ for all $\gamma \in \hat{N}$ or, equivalently,

$$\min_{\gamma \in \hat{N}} \left( \det[H^*(\gamma)H(\gamma)] \right) > 0.$$ 

To the second end, a $K$-channel filter bank formalism allows, in a similar manner, to obtain properties in $\ell^2(G)$ of the sequences $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ and $\{T_n g_k\}_{n \in N; k = 1, \ldots, K}$. In case they are Bessel sequences for $\ell^2(G)$, the idea is to consider a $K$-channel filter bank (3) where the analysis filters are $h_k := f_k$ and the synthesis filters are $g_k$, $k = 1, 2, \ldots, K$. As a consequence, the corresponding polyphase matrices $H(\gamma)$ and $G(\gamma)$, given in Equations (5) and (6), are

$$H(\gamma) = \left(\hat{h}_{i,k}(\gamma)\right)_{k=1,2,\ldots,K \atop i=1,2,\ldots,L}, \quad G(\gamma) = \left(\hat{g}_{i,k}(\gamma)\right)_{k=1,2,\ldots,K \atop i=1,2,\ldots,L}, \quad \gamma \in \hat{N}. \quad (11)$$

**Theorem 3.** Let $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ and $\{T_n g_k\}_{n \in N; k = 1, \ldots, K}$ be two Bessel sequences for $\ell^2(G)$, and $H(\gamma)$ and $G(\gamma)$ their associated matrices (11). Under the above circumstances, we have:

(a) The sequences $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ and $\{T_n g_k\}_{n \in N; k = 1, \ldots, K}$ are dual frames for $\ell^2(G)$ if and only if condition $G(\gamma)H(\gamma) = I_L$ a.e. $\gamma \in \hat{N}$ holds.

(b) The sequences $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ and $\{T_n g_k\}_{n \in N; k = 1, \ldots, K}$ are biorthogonal sequences in $\ell^2(G)$ if and only if condition $H(\gamma)G(\gamma) = I_K$ a.e. $\gamma \in \hat{N}$ holds.

(c) The sequences $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ and $\{T_n g_k\}_{n \in N; k = 1, \ldots, K}$ are dual Riesz bases for $\ell^2(G)$ if and only if $K = L$ and $G(\gamma) = H(\gamma)^{-1}$ a.e. $\gamma \in \hat{N}$.

(d) The sequence $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ is an $A$-tight frame for $\ell^2(G)$ if and only if condition $H^*(\gamma)H(\gamma) = AI_L$ a.e. $\gamma \in \hat{N}$ holds.

(e) The sequence $\{T_n f_k\}_{n \in N; k = 1, \ldots, K}$ is an orthonormal basis for $\ell^2(G)$ if and only if $K = L$ and $H^*(\gamma) = H(\gamma)^{-1}$ a.e. $\gamma \in \hat{N}$.

**Proof.** Having in mind Equation (9) and Corollary 1, part (a) is nothing but Theorem 1.
The output of the analysis filter bank (3) corresponding to the input $g_k$ is a $K$-vector whose $k$-entry is

$$c_{k,L}(m) = \downarrow_H (g_k * h_k)(m) = \langle g_k, T_m h_k \rangle_{\ell^2(G)} = \langle g_k, T_m f_k \rangle_{\ell^2(G)},$$

and whose $N$-Fourier transform is $C_k(\gamma) = H(\gamma) G_k(\gamma)$ a.e. $\gamma \in \hat{N}$, where $G_k$ is the $k'$-column of the matrix $G(\gamma)$. Note that $\{T_n f_k\}_{n \in N; k=1,2,..,K}$ and $\{T_n g_k\}_{n \in N; k=1,2,..,K}$ are biorthogonal if and only if $\langle g_k, T_m f_k \rangle_{\ell^2(G)} = \delta(k-k')\delta(m)$. Therefore, the sequences $\{T_n f_k\}_{n \in N; k=1,2,..,K}$ and $\{T_n g_k\}_{n \in N; k=1,2,..,K}$ are biorthogonal if and only if $H(\gamma) G(\gamma) = I_K$. Thus, we have proved (b).

Having in mind ([15], Theorem 7.1.1), from (a) and (b), we obtain (c).

We can read the frame operator corresponding to the sequence $\{T_n f_k\}_{n \in N; k=1,2,..,K}$, i.e.,

$$S(a) = \sum_{k=1}^K \sum_{n \in N} \langle a, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad a \in \ell^2(G),$$

as the output of the filter bank (3), whenever $h_k = \tilde{f}_k$ and $g_k = f_k$, for the input $a$. For this filter bank, the $(k, h_i)$-entry of the analysis polyphase matrix $H(\gamma)$ is $\tilde{f}_{k, h_i}(\gamma)$ and the $(h_j, k)$-entry of the synthesis polyphase matrix $G(\gamma)$ is $f_{h_j, k}(\gamma)$; in other words, $G(\gamma) = H^*(\gamma)$. Hence, the sequence $\{T_n f_k\}_{n \in N; k=1,2,..,K}$ is an $A$-tight frame for $\ell^2(G)$, i.e.,

$$a = \frac{1}{A} \sum_{k=1}^K \sum_{n \in N} \langle a, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad a \in \ell^2(G),$$

if and only if $H^*(\gamma) H(\gamma) = A I_k$ for all $\gamma \in \hat{N}$. Thus, we have proved (d).

Finally, from (c) and (d), the sequence $\{T_n f_k\}_{n \in N; k=1,2,..,K}$ is an orthonormal system if and only if $H^*(\gamma) = H(\gamma)^{-1}$ a.e. $\gamma \in \hat{N}$.

5. Getting on with Sampling

Suppose that $\{U(n,h)\}_{(n,h) \in G}$ is a unitary representation of the group $G = N \times_H H$ on a separable Hilbert space $\mathcal{H}$, and assume that for a fixed $a \in \mathcal{H}$ the sequence $\{U(n,h) a\}_{(n,h) \in G}$ is a Riesz sequence for $\mathcal{H}$ (see Ref. ([1], Theorem A)). Thus, we consider the $U$-invariant subspace in $\mathcal{H}$

$$A_a = \left\{ \sum_{(n,h) \in G} a(n,h) U(n,h) a : \{a(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\}.$$

For $K$ fixed elements $b_k \in \mathcal{H}$, $k = 1, 2, \ldots, K$, not necessarily in $A_a$, we consider for each $x \in A_a$ its generalized samples defined as

$$L_k x(m) := \langle x, U(m,1_H) b_k \rangle_{\mathcal{H}}, \quad m \in N \text{ and } k = 1, 2, \ldots, K. \quad (12)$$

The task is the stable recovery of any $x \in A_a$ from the data $\{L_k x(m)\}_{m \in N; k=1,2,..,K}$.

In what follows, we propose a solution involving a perfect reconstruction $K$-channel filter bank. First, we express the samples in a more suitable manner. Namely, for each $x = \sum_{(n,h) \in G} a(n,h) U(n,h) a$ in $A_a$, we have

$$L_k x(m) = \sum_{(n,h) \in G} a(n,h) \langle U(n,h) a, U(m,1_H) b_k \rangle$$

$$= \sum_{(n,h) \in G} a(n,h) \langle a, U[(n,h)^{-1} \cdot (m,1_H)] b_k \rangle = \downarrow_H (a \ast h_k)(m), \quad m \in N,$$

where $a = \{a(n,h)\}_{(n,h) \in G} \in \ell^2(G)$, and $h_k(n,h) := \langle a, U(n,h) b_k \rangle_{\mathcal{H}}$ also belongs to $\ell^2(G)$ for each $k = 1, 2, \ldots, K$. 

Suppose also that there exists a perfect reconstruction $K$-channel filter-bank with analysis filters the above $h_k$ and synthesis filters $g_k$, $k = 1, 2, \ldots, K$, such that the sequences $\{T_n h_k\}_{n \in \mathbb{N}; k = 1, 2, \ldots, K}$ and $\{T_n g_k\}_{n \in \mathbb{N}; k = 1, 2, \ldots, K}$ are Bessel sequences for $\ell^2(G)$. Having in mind Equation (9), for each $\alpha = \{\alpha(n, h)\}_{(n, h) \in G}$ in $\ell^2(G)$, we have

$$\alpha = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} U(\alpha \ast h_{k})(n) T_n g_k = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} L_k x(n) T_n g_k \text{ in } \ell^2(G). \quad (13)$$

In order to derive a sampling formula in $A_\alpha$, we consider the natural isomorphism $T_{U, \alpha} : \ell^2(G) \rightarrow A_\alpha$ which maps the usual orthonormal basis $\{\delta_{(n, h)}\}_{(n, h) \in G}$ for $\ell^2(G)$ onto the Riesz basis $\{U(n, h) a\}_{(n, h) \in G}$ for $A_\alpha$, i.e.,

$$T_{U, \alpha} : \delta_{(n, h)} \mapsto U(n, h) a \text{ for each } (n, h) \in G.$$

This isomorphism $T_{U, \alpha}$ possesses the following shifting property:

**Lemma 2.** For each $m \in \mathbb{N}$, consider the translation operator $T_m$ operator defined in Equation (8). For each $m \in \mathbb{N}$, the following shifting property holds

$$T_{U, \alpha}(T_m f) = U(m, 1_H)(T_{U, \alpha} f), \quad f \in \ell^2(G). \quad (14)$$

**Proof.** For each $\delta_{(n, h)}$, it is easy to check that $T_m \delta_{(n, h)} = \delta_{(m+n, h)}$. Hence,

$$T_{U, \alpha}(T_m \delta_{(n, h)}) = U(m + n, h) a = U(m, 1_H)U(n, h) a = U(m, 1_H)(T_{U, \alpha} \delta_{(n, h)}).$$

A continuity argument proves the result for all $f$ in $\ell^2(G)$. \Box

Now, for each $x = T_{U, \alpha} \alpha \in A_\alpha$, applying the isomorphism $T_{U, \alpha}$ and the shifting property (14) in Equation (13), we get for each $x \in A_\alpha$ the expansion

$$x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} L_k x(n) T_{U, \alpha} (T_n g_k) = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} L_k x(n) U(n, 1_H)(T_{U, \alpha} g_k)$$

$$= \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} L_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H}, \quad (15)$$

where $c_k = T_{U, \alpha} g_k$, $k = 1, 2, \ldots, K$. In the following, the sampling theorem in the subspace $A_\alpha$ holds:

**Theorem 4.** For $K$ fixed $h_k \in H$, let $L_k : A_\alpha \rightarrow \mathbb{C}^N$ be its associated $U$-system defined in Equation (12) with corresponding $h_k \in \ell^2(G)$, $k = 1, 2, \ldots, K$. Assume that its polyphase matrix $H(\gamma)$ given in Equation (5) has all its entries in $L^\infty(\hat{N})$. The following statements are equivalent:

1. The constant $A_H = \inf \lambda_{\text{min}} [H^*(\gamma)H(\gamma)] > 0$.
2. There exist $g_k$ in $\ell^2(G)$, $k = 1, 2, \ldots, K$, such that the associated polyphase matrix $G(\gamma)$ given in (6) has all its entries in $L^\infty(\hat{N})$, and it satisfies $G(\gamma)H(\gamma) = I_K$ a.e. $\gamma \in \hat{N}$.
3. There exist $K$ elements $c_k \in A_\alpha$ such that the sequence $\{U(n, 1_H) c_k\}_{n \in \mathbb{N}; k = 1, 2, \ldots, K}$ is a frame for $A_\alpha$ and, for each $x \in A_\alpha$, the sampling formula

$$x = \sum_{k=1}^{K} \sum_{n \in \mathbb{N}} L_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H} \quad (16)$$

holds.
4. There exists a frame \( \{ C_{k,n} \}_{n \in N; k = 1, \ldots, K} \) for \( A_a \) such that for each \( x \in A_a \) the expansion

\[
x = \sum_{k=1}^{K} \sum_{n \in N} L_k x(n) C_{k,n} \quad \text{in } \mathcal{H}
\]

holds.

Proof. (1) implies (2). The \( L \times K \) Moore–Penrose pseudo-inverse \( H^*(\gamma) \) of \( H(\gamma) \) is given by \( H^*(\gamma) = \left[ H^*(\gamma) H(\gamma) \right]^{-1} H^*(\gamma) \). Its entries are essentially bounded in \( \tilde{N} \) since the entries of \( H(\gamma) \) belong to \( L^\infty(\tilde{N}) \) and \( \det^{-1} \left[ H^*(\gamma) H(\gamma) \right] \) is essentially bounded \( \tilde{N} \) since \( 0 < A_H \). In addition, \( H^*(\gamma) H(\gamma) = I_L \) a.e. \( \gamma \in \tilde{N} \). The inverse \( N \)-Fourier transform in \( L^2(\tilde{N}) \) of the \( k \)-th column of \( H^*(\gamma) \) gives \( \mathbf{g}_k, k = 1, 2, \ldots, K \).

(2) implies (3). According to Theorems 2 and 3, the sequences \( \{ T_n h_k \}_{n \in N; k = 1, 2, \ldots, K} \) and \( \{ T_n g_k \}_{n \in N; k = 1, 2, \ldots, K} \) form a pair of dual frames for \( \ell^2(G) \). We deduce the sampling expansion as in Formula (15). In addition, the sequence \( \{ U(n, 1_H) c_{k,n} \}_{n \in N; k = 1, 2, \ldots, K} \) is a frame for \( A_a \).

Obviously, (3) implies (4). Finally, (4) implies (1). Applying \( T_{U,a}^{-1} \), we get that the sequences \( \{ T_n h_k \}_{n \in N; k = 1, 2, \ldots, K} \) and \( \{ T_{U,a}^{-1} (c_{k,n}) \}_{n \in N; k = 1, 2, \ldots, K} \) form a pair of dual frames for \( \ell^2(G) \); in particular, by using Theorem 2, we obtain that \( 0 < A_H \).

All the possible solutions of \( G(\gamma) H(\gamma) = I_L \) a.e. \( \gamma \in \tilde{N} \) with entries in \( L^\infty(\tilde{N}) \) are given in terms of the Moore–Penrose pseudo inverse by the \( L \times K \) matrices \( G(\gamma) := H^*(\gamma) + U(\gamma) \left[ I_K - H(\gamma) H^*(\gamma) \right] \), where \( U(\gamma) \) denotes any \( L \times K \) matrix with entries in \( L^\infty(\tilde{N}) \).

Notice that \( K \geq L \) where \( L \) is the order of the group \( H \). In case \( K = L \), we obtain:

**Corollary 2.** In the case \( K = L \), assume that its polyphase matrix \( H(\gamma) \) given in Equation (5) has all entries in \( L^\infty(\tilde{N}) \). The following statements are equivalent:

1. The constant \( A_H = \text{ess inf}_{\gamma \in \tilde{N}} \lambda_{\text{min}} \left[ H^*(\gamma) H(\gamma) \right] > 0 \).
2. There exist \( L \) unique elements \( c_k, k = 1, 2, \ldots, L \) in \( A_a \) such that the associated sequence \( \{ U(n, 1_H) c_{k,n} \}_{n \in N; k = 1, 2, \ldots, L} \) is a Riesz basis for \( A_a \) and the sampling formula

\[
x = \sum_{k=1}^{L} \sum_{n \in N} L_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H}
\]

holds for each \( x \in A_a \).

Moreover, the interpolation property \( L_k c_{k,n} (n) = \delta_{kk'} \delta_{n,0_N}, \) where \( n \in N \) and \( k, k' = 1, 2, \ldots, L \), holds.

Proof. In this case, the square matrix \( H(\gamma) \) is invertible and the result comes out from Theorem 3. From the uniqueness of the coefficients in a Riesz basis expansion, we get the interpolation property.

Denote \( H = \{ h_1, h_2, \ldots, h_L \} \); for a fixed \( b \in \mathcal{H} \), we consider the samples

\[
L_k x(m) := \langle x, U(m, h_k) b \rangle, \quad m \in N \text{ and } k = 1, 2, \ldots, L,
\]

of any \( x \in A_a \). Since \( U(m, h_k) b = U(m, 1_H) U(0_N, h_k) b = U(m, 1_H) b_k \), where \( b_k := U(0_N, h_k) b \), \( k = 1, 2, \ldots, L \), we are in a particular case of Equation (12) with \( K = L \).

Notice also that the subspace \( A_a \) can be viewed as the multiple generated \( U \)-invariant subspace of \( \mathcal{H} \)

\[
\text{span} \{ U(n, 1_H) a_h : n \in N, h \in H \}
\]

with \( L \) generators \( a_h := U(0_N, h) a \in \mathcal{H}, h \in H \), obtained from \( a \in \mathcal{H} \) by the action of the group \( H \).
5.1. An Example Involving Crystallographic Groups

The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \times O(d)$ corresponding to the homomorphism $\phi : O(d) \to Aut(\mathbb{R}^d)$ given by $\phi_A(x) = Ax$, where $A \in O(d)$ and $x \in \mathbb{R}^d$. The composition law on $E(d) = \mathbb{R}^d \times O(d)$ reads $(x, A) \cdot (x', A') = (x + Ax', AA')$.

Let $M$ be a non-singular $d \times d$ matrix and $\Gamma$ a finite subgroup of $O(d)$ of order $L$ such that $A(MZ^d) = MZ^d$ for each $A \in \Gamma$. We consider the crystallographic group $C_{M, \Gamma} := MZ^d \rtimes \phi \Gamma$ and its quasi regular representation (see Ref. [1]) on $L^2(\mathbb{R}^d)$

$$U(n, A)f(t) = f[A^\top (t-n)], \quad n \in M\mathbb{Z}^d, A \in \Gamma, f \in L^2(\mathbb{R}^d).$$

For a fixed $\varphi \in L^2(\mathbb{R}^d)$ such that the sequence $\{U(n,A)\varphi\}_{(n,A) \in C_{M,\Gamma}}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ (see, for instance, Refs. [19,20]) we consider the $U$-invariant subspace in $L^2(\mathbb{R}^d)$

$$A_\varphi = \left\{ \sum_{(n,A) \in C_{M,\Gamma}} a(n,A) \varphi[A^\top (t-n)] : \{a(n,A)\} \in l^2(C_{M,\Gamma}) \right\}$$

$$= \left\{ \sum_{(n,A) \in C_{M,\Gamma}} a(n,A) \varphi(At-n) : \{a(n,A)\} \in l^2(C_{M,\Gamma}) \right\}.$$

Choosing $K$ functions $b_k \in L^2(\mathbb{R}^d), k = 1,2,\ldots,K$, we consider the average samples of $f \in A_\varphi$

$$E_k f(n) = \langle f, U(n,1)b_k \rangle = \langle f, b_k(\cdot-n) \rangle, \quad n \in M\mathbb{Z}^d.$$

Under the hypotheses in Theorem 4, there exist $K \geq L$ sampling functions $\psi_k \in A_\varphi$ for $k = 1,2,\ldots,K$, such that the sequence $\{\psi_k(\cdot-n)\}_{n \in M\mathbb{Z}^d, k = 1,2,\ldots,K}$ is a frame for $A_\varphi$, and the sampling expansion

$$f(t) = \sum_{k=1}^{K} \sum_{n \in M\mathbb{Z}^d} \langle f, b_k(\cdot-n) \rangle_{L^2(\mathbb{R}^d)} \psi_k(t-n) \quad \text{in } L^2(\mathbb{R}^d) \quad (17)$$

holds.

If the generator $\varphi \in C(\mathbb{R}^d)$ and the function $t \mapsto \sum_{n} |\varphi(t-n)|^2$ is bounded on $\mathbb{R}^d$, a standard argument shows that $A_\varphi$ is a reproducing kernel Hilbert space (RKHS) of bounded continuous functions in $L^2(\mathbb{R}^d)$. As a consequence, convergence in $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

Notice that the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \varphi \mathbb{Z}_2$ is a particular crystallographic group with lattice $\mathbb{Z}$ and $\Gamma = \mathbb{Z}_2$. Its quasi regular representation on $L^2(\mathbb{R})$ reads

$$U(n,0)f(t) = f(t-n) \quad \text{and} \quad U(n,1)f(t) = f(-t+n), \quad n \in \mathbb{Z} \text{ and } f \in L^2(\mathbb{R}).$$

Thus, we could obtain sampling formulas as (17) for $K \geq 2$ average functions $b_k$.

The quasi regular unitary representation of a crystallographic group $C_{M,\Gamma}$ on $L^2(\mathbb{R}^d)$ motivates the next section:

5.2. The Case of Pointwise Samples

Let $\{U(n,h)\}_{(n,h) \in G}$ be a unitary representation of the group $G = N \rtimes \varphi H$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. If the generator $\varphi \in L^2(\mathbb{R}^d)$ satisfies that, for each $(n,h) \in G$, the function $U(n,h)\varphi$ is continuous on $\mathbb{R}^d$, and the condition

$$\sup_{t \in \mathbb{R}^d} \sum_{(n,h) \in G} |U(n,h)\varphi|(t)^2 < \infty,$$
then the subspace $A_\varphi$ is an RKHS of bounded continuous functions in $L^2(\mathbb{R}^d)$; proceeding as in [21], one can prove that the above conditions are also necessary.

For $K$ fixed points $t_k \in \mathbb{R}^d$, $k = 1, 2, \ldots, K$, we consider for each $f \in A_\varphi$ the new samples given by

$$L_k f(n) := \left[ U(-n, 1_H) f \right](t_k), \quad n \in \mathbb{N} \text{ and } k = 1, 2, \ldots, K. \quad (18)$$

For each $f = \sum_{(m,h) \in G} a(m,h) U(m,h) \varphi$ in $A_\varphi$ and $k = 1, 2, \ldots, K$, we have

$$L_k f(n) = \sum_{(m,h) \in G} a(m,h) U((-n, 1_H) \cdot (m,h)) \varphi(t_k) = \sum_{(m,h) \in G} a(m,h) \left[ U(m - n, h) \varphi \right](t_k) = \langle a, T_n f_k \rangle_{\ell^2(G)}, \quad n \in \mathbb{N},$$

where $a = \{a(m,h)\}_{(m,h) \in G}$ and $f_k(m,h) := \left[ U(m,h) \varphi \right](t_k), (m,h) \in G$. Notice that $f_k$ belongs to $\ell^2(G)$, $k = 1, 2, \ldots, K$. As a consequence, under the hypotheses in Theorem 4 (on these new $f_k := f_k$, $k = 1, 2, \ldots, K$), a sampling formula such as (16) holds for the data sequence $\{L_k f(n)\}_{n \in \mathbb{N}, k = 1, 2, \ldots, K}$ defined in Equation (18).

In the particular case of the quasi regular representation of a crystallographic group $C_{M \Gamma} = \mathbb{M} \mathbb{Z}^d \rtimes_{\varphi} \Gamma$, for each $f \in A_\varphi$, the samples (18) read

$$L_k f(n) = \left[ U(-n, 1) f \right](t_k) = f(t_k + n), \quad n \in \mathbb{M} \mathbb{Z}^d \text{ and } k = 1, 2, \ldots, K.$$

Thus (under hypotheses in Theorem 4), there exist $K$ functions $\psi_k \in A_\varphi$, $k = 1, 2, \ldots, K$, such that for each $f \in A_\varphi$ the sampling formula

$$f(t) = \sum_{k = 1}^K \sum_{n \in \mathbb{M} \mathbb{Z}^d} f(t_k + n) \psi_k(t - n), \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

6. Conclusions

In this paper, we have derived an abstract regular sampling theory associated with a unitary representation $(n,h) \mapsto U(n,h)$ of a group $G$ which is a semi-direct product of two groups, $N$ countable discrete abelian group and $H$ finite, on a separable Hilbert space $\mathcal{H}$; here, regular sampling means that we are taken the samples at the group $N$. Concretely, the sampling theory is obtained in the $U$-invariant subspace of $\mathcal{H}$ generated by $a \in \mathcal{H}$ that is

$$A_a = \left\{ \sum_{(n,h) \in G} a(n,h) U(n,h) a : \{a(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\},$$

and the samples of $x \in A_a$ are given by $L_k x(n) := \langle x, U(n, 1_H) b_k \rangle_{\mathcal{H}'}$, $n \in \mathbb{N}$, where $b_k, k = 1, 2, \ldots, K$, denote $K$ fixed elements in $\mathcal{H}$ which do not belong necessarily to $A_a$. We look for $K$ elements $c_k \in A_a$ such that the sequence $\{U(n, 1_H) c_k\}_{n \in \mathbb{N}, k = 1, 2, \ldots, K}$ is a frame for $A_a$ and, for each $x \in A_a$, the sampling formula $x = \sum_{k=1}^K \sum_{n \in \mathbb{N}} L_k x(n) U(n, 1_H) c_k$ holds.

A similar problem was solved when the group $G$ is a discrete LCA group and the samples are taken at a uniform lattice of $G$ (see Ref. [10]). In the case of an abelian group, we have the Fourier transform, a basic tool in this previous work. In the present work, a classical Fourier analysis on $G$ is not available, but if $G$ is a semi-direct product of the form $G = N \rtimes_{\varphi} H$, where $N$ is a countable discrete abelian group and $H$ is a finite group, the Fourier transform on the abelian group $N$ allows us to solve the problem by means of a filter bank formalism. Recalling the filter bank formalism in discrete LCA
groups, the defined samples are expressed as the output of a suitable \( K \)-channel analysis filter bank corresponding to the input \( x \in \mathbb{A}_n \). The frame analysis of this filter bank along with the synthesis one giving perfect reconstruction allows us to obtain a pair of suitable dual frames for obtaining the desired sampling result, which is written as a list of equivalent statements (see Theorem 4).

Although the semi-direct product of groups represents, so to speak, the simplest case of non-abelian groups, this paper can be a good starting point for finding sampling theorems associated with unitary representations of non-abelian groups that are not isomorphic to a semi-direct product of groups.

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