Hessian estimates for the conjugate heat equation coupled with the Ricci flow

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Abstract

In this short note we obtain some local and global upper bounds for the Hessian of a positive solution to the conjugate heat equation coupled with the Ricci flow.

Key words: Hessian estimates; conjugate heat equation; Ricci flow

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1 Introduction

In the literature there are extensive researches on derivative estimates for solutions to the heat equation or conjugate heat equation coupled with the Ricci flow; see for example [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], and [17].

In this short note we obtain some local and global upper bounds for the Hessian of a positive solution to the conjugate heat equation coupled with the Ricci flow. I was inspired by Han-Zhang [9]. To state our results we first introduce some notations. Fix \( T > 0 \). Let \((M, g(t))\), \( t \in [0, T] \), be a (not necessarily complete) solution to Hamilton’s Ricci flow

\[
\frac{\partial g(t)}{\partial t} = -2Ric(g(t))
\]

on a manifold \( M \) (without boundary) of dimension \( n \). For any points \( x, y \in M \), let \( d(x, y, t) \) be the distance between \( x \) and \( y \) w.r.t. \( g(t) \). For \((x_0, t_0) \in M \times (0, T]\), \( r > 0 \), and \( T' > 0 \), let

\[
Q_{r,T'}(x_0, t_0) := \{(x, t) \in M \times [0, T] \mid d(x, x_0, t) \leq r, t_0 - T' \leq t \leq t_0\}
\]

be a parabolic cube.

Consider a smooth positive solution \( u \) to the conjugate heat equation \((\frac{\partial}{\partial t} + \Delta_{g(t)} - R_{g(t)})u = 0\) coupled with the Ricci flow on \( M \times [0, T]\), where \( R_{g(t)} \) is the scalar curvature of the metric \( g(t) \). We have the following global upper bound for \( \text{Hess} \ u \).
Theorem 1.1. Let \((M^n, g(t)), t \in [0, T]\), be a Ricci flow on a compact manifold, \(u\) be a positive solution to the conjugate heat equation coupled with the Ricci flow on \(M \times [0, T]\) with \(0 < u \leq A\). Then we have

\[
\nabla^2 u \leq u \left( \frac{18}{T-t} + C \right) \left( 1 + \log \frac{A}{u} \right) g(t) \quad \text{on} \quad M \times [0, T],
\]

and, in particular,

\[
\Delta u \leq u \left( \frac{18n}{T-t} + C \right) \left( 1 + \log \frac{A}{u} \right) \quad \text{on} \quad M \times [0, T],
\]

where \(C\) depends on \(n\), the upper bounds of \(|Rm|, |\nabla Ric|, \) and \(|\nabla^2 R|\) on \(M \times [0, T]\).

We also get a local upper bound for \(\text{Hess} \ u\).

Theorem 1.2. Let \((M^n, g(t)), t \in [0, T]\), be a (not necessarily complete) Ricci flow, \(u\) be a positive solution to the conjugate heat equation coupled with the Ricci flow on \(M \times [0, T]\) with \(0 < u \leq A\). Let \(x_0 \in M\). Assume that the parabolic cube \(Q_{4r,T}(x_0, T)\) is compact. Then we have

\[
\nabla^2 u \leq u \left( \frac{C_0}{T-t} + C_0 + C_2 \right) \left( 1 + \log \frac{A}{u} \right) g(t) \quad \text{on} \quad Q_{4r,T}(x_0, T),
\]

where \(C_0\) is a universal constant, and \(C_2\) depends on \(n\), the upper bounds of \(|Rm|, |\nabla Ric|, \) and \(|\nabla^2 R|\) on \(Q_{4r,T}(x_0, T)\).

In Section 2 we derive local gradient estimates for a positive solution to the conjugate heat equation coupled with the Ricci flow. In Section 3 we prove Theorems 1.1 and 1.2.

2 Gradient estimates

Let \((M^n, g(t)), t \in [0, T]\), be a (not necessarily complete) Ricci flow, and \(u\) be a positive solution to the conjugate heat equation

\[
\left( \frac{\partial}{\partial t} + \Delta_{g(t)} - R_{g(t)} \right) u = 0
\]

coupled with the Ricci flow on \(M \times [0, T]\). Sometimes we’ll write \(\partial_t\) for \(\frac{\partial}{\partial t}\), and \(u_t\) for \(\frac{\partial u}{\partial t}\). Let \(f = \log u\). Then

\[
f_t = -\Delta f - |\nabla f|^2 + R.
\]

Fix \(\alpha > 1\). Let \(\tau = T - t\), and

\[
F = \tau \left( \frac{|\nabla u|^2}{u^2} + \alpha \frac{u_t}{u} - \alpha R \right) = \tau (|\nabla f|^2 + \alpha f_t - \alpha R).
\]

Let \(x_0 \in M\). Assume that the parabolic cube \(Q_{2r,T}(x_0, T)\) is compact, and \(-K_0 \leq \text{Ric} \leq K_0, |\nabla R| \leq K_1, \) and \(\Delta R \leq K_2\) on \(Q_{2r,T}(x_0, T)\).
Lemma 2.1. With the above assumption, for any $\varepsilon > 0$ we have

\[
(\Delta + \partial_t) F \geq -2\langle \nabla f, \nabla F \rangle + \frac{2\tau}{n+\varepsilon}(f_t + |\nabla f|^2 - R)^2 - (|\nabla f|^2 + \alpha f_t - \alpha R)
\]

\[
- 2\tau|2 - \alpha|K_0|\nabla f|^2 - 2\tau(\alpha - 1)|\nabla f| - \frac{n(n+\varepsilon)}{2\varepsilon} \alpha^2 \tau K_0^2 - \alpha \tau K_2
\]
on $Q_{2\tau,T}(x_0, T)$.

Proof. We have

\[
\Delta |\nabla f|^2 = 2|\nabla^2 f|^2 + 2\langle \nabla f, \nabla \Delta f \rangle + 2\text{Ric}(\nabla f, \nabla f),
\]

\[
\Delta f_t = (\Delta f)_t - 2\langle \text{Ric}, \nabla^2 f \rangle,
\]

\[
\tau \Delta f = \tau(\alpha - 1)(f_t - R) - F,
\]

\[(|\nabla f|^2)_t = 2\langle \nabla f, \nabla f_t \rangle + 2\text{Ric}(\nabla f, \nabla f),
\]

and

\[
\Delta F = \tau[2|\nabla^2 f|^2 + 2\langle \nabla f, \nabla \Delta f \rangle + 2\text{Ric}(\nabla f, \nabla f) + \alpha((\Delta f)_t - 2\langle \text{Ric}, \nabla^2 f \rangle) - \alpha \Delta R]
\]

\[
= 2\tau|\nabla^2 f|^2 - 2\langle \nabla f, \nabla F \rangle + 2\tau(\alpha - 1)\langle \nabla f, \nabla f_t \rangle - 2\tau(\alpha - 1)\langle \nabla R, \nabla f \rangle
\]

\[
+ 2\tau \text{Ric}(\nabla f, \nabla f) + \alpha \tau f_t - (|\nabla f|^2)_t + R_t - 2\langle \text{Ric}, \nabla^2 f \rangle - \alpha \tau \Delta R
\]

\[
= 2\tau|\nabla^2 f|^2 - 2\langle \nabla f, \nabla F \rangle - F - \frac{F}{\tau} - 2\alpha \tau \langle \text{Ric}, \nabla^2 f \rangle + 2\tau(2 - \alpha)\text{Ric}(\nabla f, \nabla f)
\]

\[
- 2\tau(\alpha - 1)\langle \nabla R, \nabla f \rangle - \alpha \tau \Delta R.
\]

See p. 61 of [3].

Note that

\[
|\langle \text{Ric}, \nabla^2 f \rangle| \leq |\text{Ric}||\nabla^2 f| \leq \frac{\alpha(n+\varepsilon)}{4\varepsilon}|\text{Ric}|^2 + \frac{\varepsilon}{\alpha(n+\varepsilon)}|\nabla^2 f|^2
\]

for any $\varepsilon > 0$, and

\[
|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2 = \frac{1}{n}(f_t + |\nabla f|^2 - R)^2,
\]

then the lemma follows. \qed

Proposition 2.2. Let $(M^n, g(t))$, $t \in [0,T]$, be a (not necessarily complete) Ricci flow and $u$ be a positive solution to the conjugate heat equation coupled with the Ricci flow on $M \times [0,T]$. Let $x_0 \in M$. Suppose that $Q_{2\tau,T}(x_0, T)$ is compact and $-K_0 \leq \text{Ric} \leq K_0$, $|\nabla R| \leq K_1$, $\Delta R \leq K_2$ on $Q_{2\tau,T}(x_0, T)$. For any $\alpha > 1$ and $\varepsilon > 0$, we have

\[
\frac{|\nabla u|^2}{u^2} + \frac{u_t}{u} - \alpha R \leq \frac{(n+\varepsilon)\alpha^2}{2(T-t)} + C(r^{-2} + r^{-1} + 1) \quad \text{on } Q_{r,T}(x_0, T) \setminus \{(x,T) \mid x \in M\},
\]

where the constant $C$ depends on $n, \alpha, \varepsilon, K_0, K_1$ and $K_2$. 

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Proof. As in [12], choose a smooth cutoff function $\psi : [0, \infty) \to [0, 1]$ with $\psi = 1$ on the interval $[0, 1]$, $\psi = 0$ on $[2, \infty)$, and

$$\psi' \leq 0, \ |\psi'|^2 \leq C_0 \psi, \ \psi'' \geq -C_0,$$

where $C_0$ is a universal constant. Let $\phi(x, t) = \psi\left(\frac{d(x, x_0, t)}{r}\right)$. Suppose that the maximum of the function $\phi F$ is positive, otherwise the result follows trivially. Assume that $\phi F$ achieves its positive maximum at the point $(x_1, t_1)$. Then $(x_1, t_1) \in Q_{2r, T}(x_0, T)$ with $t_1 \neq T$. By Calabi’s trick [2] we may assume that $\phi F$ is smooth at $(x_1, t_1)$. Let $\tau_1 = T - t_1$. We compute at the point $(x_1, t_1)$ using Lemma 2.1,

$$0 \geq (\Delta + \partial_t)(\phi F) \geq \tau_1 \phi \frac{2}{n + \varepsilon} (f_t + |\nabla f|^2 - R)^2 - CF \sqrt{\phi r^{-1}} |\nabla f| - \phi F \frac{1}{\tau_1} - C \tau_1 |\nabla f|^2 - C \tau_1 \phi - CF(r^{-2} + r^{-1} + 1),$$

where the constant $C$ depends on $n, \alpha, \varepsilon, K_0, K_1$ and $K_2$; compare [3], [13], and [16]. Then we proceed as in [12] and [5].

The following corollary is a slight improvement of Lemma 4.1 in [5] in the Ricci flow case.

Corollary 2.3. Suppose that $(M^n, g(t))$, $t \in [0, T]$, is a complete Ricci flow with $-K_0 \leq \text{Ric} \leq K_0$, $|\nabla R| \leq K_1$, $\Delta R \leq K_2$ on $M \times [0, T]$, and that $u$ is a positive solution to the conjugate heat equation coupled with the Ricci flow on $M \times [0, T]$. For any $\alpha > 1$ and $\varepsilon > 0$, at $(x, t) \in M \times [0, T)$ we have

$$\frac{|\nabla u|^2}{u^2} + \frac{\alpha}{u} - \alpha R \leq \frac{(n + \varepsilon)\alpha^2}{2(T - t)} + C,$$

where the constant $C$ depends on $n, \alpha, \varepsilon, K_0, K_1$ and $K_2$.

Proposition 2.4. Let $(M^n, g(t))$, $t \in [0, T]$, be a (not necessarily complete) Ricci flow and $u$ be a positive solution to the conjugate heat equation coupled with the Ricci flow on $M \times [0, T]$. Let $x_0 \in M$. Suppose that $Q_{2r, T}(x_0, T)$ is compact and $-K_0 \leq \text{Ric} \leq K_0$, $|\nabla R| \leq K_1$, $\Delta R \leq K_2$ on $Q_{2r, T}(x_0, T)$. For any $\alpha > 1$ and $\varepsilon > 0$, at $(x, t) \in Q_{r, T}(x_0, T)$ with $t \neq T$ we have

$$\frac{|\nabla u|^2}{u^2} + \frac{\alpha}{u} - \alpha R \leq \frac{n\alpha^2}{T - t} + \frac{C\alpha^2}{r^2}(r\sqrt{K_0} + \frac{\alpha^2}{\alpha - 1}) + C\alpha^2 K_0 + \frac{n\alpha^2}{\alpha - 1} |2 - \alpha|K_0 + \frac{\alpha - 1}{2}K_1| + n\alpha^2 K_0 + \alpha \sqrt{n(\alpha - 1)K_1} + \alpha \sqrt{n\alpha K_2},$$

where the constant $C$ depends only on $n$.

Proof. In Lemma 2.1 we let $\varepsilon = n$ and get

$$(\Delta + \partial_t)F \geq -2(|\nabla f, \nabla F| + \frac{T}{n}(f_t + |\nabla f|^2 - R)^2 - (|\nabla f|^2 + f_t - \alpha R) - \tau(2 - \alpha)K_0 + (\alpha - 1)K_1)|\nabla f|^2 - n\alpha^2 \tau K_0^2 - (\alpha - 1)\tau K_1 - \alpha \tau K_2.$$
3 Hessian estimates

Let \((M^n, g(t)), t \in [0,T]\), be a Ricci flow and \(u\) be a smooth positive solution to the conjugate heat equation coupled with the Ricci flow on \(M \times [0,T]\).

**Lemma 3.1.** Let \(u\) be a positive solution to the conjugate heat equation coupled with the Ricci flow with \(0 < u \leq A\), and \(f = \log \frac{u}{A}\). Let \(u_{ij}\) denote the Hessian of \(u\) in local coordinates, and \(v_{ij} := \frac{u_{ij}}{u(1-f)}\). In local coordinates we have

\[
(\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla) v_{ij} = \frac{|\nabla f|^2 + Rf}{1-f} v_{ij} + \frac{1}{u(1-f)}[2R_{kl} u_{kl} + R_{il} u_{jl} + R_{ji} u_{il} + 2(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) \nabla_l u - u \nabla_i \nabla_j R - \nabla_i R \nabla_j u - \nabla_j R \nabla_i u - R_{ij}].
\]

Here we adopt the usual convention on summations. For example, \(R_{ijkl} u_{kl}\) means \(g^{ab} g^{pq} R_{aijp} u_{bq}\). Note also that our convention on the curvature tensor \(R_{ijkl}\) is the same as that of R. Hamilton, but is different from that of Han-Zhang [9].

**Proof.** The proof is similar to that of Lemma 3.1 in [9]. \(\square\)

**Lemma 3.2.** Let \(u\) be a positive solution to the conjugate heat equation coupled with the Ricci flow with \(0 < u \leq A\), and \(f = \log \frac{u}{A}\). For a function \(h\) we let \(h_i\) denote the 1-form \(dh\) in local coordinates. We also let \(w_{ij} := \frac{u_{ij}}{u^2(1-f)^2}\) denote the 2-tensor \(\frac{du \otimes du}{u^2(1-f)^2}\) in local coordinates. In local coordinates we have

\[
(\partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla) w_{ij} = \frac{2(|\nabla f|^2 + Rf)}{1-f} w_{ij} + \frac{(Ru)_i u_j + u_i (Ru)_j}{u^2(1-f)^2} + (v_{ki} + f w_{ki}) (v_{kj} + f w_{kj}) + R_{ik} w_{kj} + R_{jk} w_{ki},
\]

where \(v_{ij}\) is as in Lemma 3.1.

**Proof.** The proof is similar to that of Lemma 3.2 in [9]. \(\square\)

**Proof of Theorem 1.1.** Let \(V = (v_{ij})\), \(W = (w_{ij})\), \(w = \text{tr} W = g^{ij} w_{ij} = \frac{|\nabla f|^2}{(1-f)^2}\), and

\[
L = \partial_t + \Delta - \frac{2f}{1-f} \nabla f \cdot \nabla.
\]

Then by Lemmas 3.1 and 3.2 we have

\[
LV = (1-f) w V + P,
\]

\[
LW = 2(1-f) w W + 2(V + f W)^2 + Q,
\]

where \(P\) is a 2-tensor whose \((i, j)\)-th component in local coordinates is given by

\[
P_{ij} = \frac{Rf}{1-f} v_{ij} + \frac{1}{u(1-f)}[2R_{ij} u_{kl} + R_{il} u_{jl} + R_{ji} u_{il} + 2(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}) \nabla_l u - u \nabla_i \nabla_j R - \nabla_i R \nabla_j u - \nabla_j R \nabla_i u - R_{ij}].
\]
and $Q$ is a 2-tensor whose $(i, j)$-th component in local coordinates is given by

$$Q_{ij} = \frac{2Rf}{1-f}w_{ij} + \frac{(Ru)_{ij} + u_i(Ru)_j}{u^2(1-f)^2} + R_{ik}w_{kj} + R_{jk}w_{ki}.$$ 

Now with the help of Theorem 10 in [7] and Corollary 2.3 here, we can proceed as in Han-Zhang [9]. 

\[\square\]

**Proof of Theorem 1.2.** With the help of Theorem 10 in [7], Proposition 2.2 here, and a space-time cutoff function, we can proceed as in Han-Zhang [9] with minor modifications. 

\[\square\]

**Corollary 3.3.** Let $(M^n, g(t), t \in [0, T])$ be a (not necessarily complete) Ricci flow, $u$ be a positive solution to the conjugate heat equation coupled with the Ricci flow on $M \times [0, T]$ with $0 < u \leq A$. Let $x_0 \in M$. Assume that the parabolic cube $Q_{4r,T}(x_0, T)$ is compact. Then we have

$$\nabla^2 u \leq u\left(\frac{C_0}{T-t} + \frac{C_0}{r^2} + C_2\right)(1 + \log \frac{A}{u})^2 g(t) \quad \text{on} \quad Q_{r,T}(x_0, T) \setminus \{(x, T) \mid x \in M\},$$

and, in particular,

$$\Delta u \leq u\left(\frac{C_0n}{T-t} + \frac{C_0n}{r^2} + C_2\right)(1 + \log \frac{A}{u})^2 \quad \text{on} \quad Q_{r,T}(x_0, T) \setminus \{(x, T) \mid x \in M\},$$

where $C_0$ is a universal constant, and $C_2$ depends on $n$, the upper bounds of $|Rm|$, $|\nabla Ric|$, and $|\nabla^2 R|$ on $Q_{r,T}(x_0, T)$.

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