On the Bel radiative gravitational fields

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Abstract
We analyze the concept of intrinsic radiative gravitational fields defined by Bel and we show that the three radiative types, N, III and II, correspond with the three following different physical situations: pure radiation, asymptotic pure radiation and generic (non-pure, non-asymptotic pure) radiation. We introduce the concept of observer at rest with respect to the gravitational field and that of proper super-energy of the gravitational field and we show that, for non-radiative fields, the minimum value of the relative super-energy density is the proper super-energy density, which is acquired by the observers at rest with respect to the field. Several super-energy inequalities are also examined.

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1. Introduction

With the purpose of defining intrinsic states of gravitational radiation, Bel [1–3] introduced a rank 4 tensor which plays an analogous role for gravitation to that played by the Maxwell–Minkowski tensor for electromagnetism. In the vacuum case, this super-energy Bel tensor is divergence free and it coincides with the super-energy Bel–Robinson tensor $T$.

Using the tensor $T$, Bel defined the relative super-energy density and the super-Poynting vector associated with an observer. Then, following the analogy with electromagnetism, the intrinsic radiative gravitational fields are those for which the Poynting vector does not vanish for any observer [1, 3]. The Bel approach, based on super-energy concepts, leads to the same gravitational fields as the Pirani [4] one, which is based on intrinsic geometric considerations (see [3, 5] for these and other radiation criteria).

It is worth remarking that Bel super-energy quantities do not represent gravitational energy. Nevertheless, the relationship between super-energy and quasi-local gravitational energy has been largely discussed [6] (a wide list of references on this subject can be found in [7, 8]).

The interest of the Bel approach has recently been remarked by García–Parrado [9], who introduces new relative super-energy quantities and writes the full set of equations for these...
super-energy quantities. This study leads naturally to a concept of intrinsic radiation which is less restrictive than Bel’s [9]. We will analyze García–Parrado’s proposal in a forthcoming work [10] where we will also give an intrinsic characterization of the new radiative classes.

It is worth remarking that Bel and García–Parrado definitions are local, and a gravitational field radiative at a point of the spacetime could be non-radiative at another point. In this work, we give several definitions and results that are also local because they are based on algebraic considerations.

Here we revisit Bel’s ideas in depth. The analogy with electromagnetism helps us understand the already known concepts and the new ones we are introducing. For this reason, we devote section 2 to summarizing several known results on the electromagnetic field. We present them in a way that facilitates their extension to the gravitational field.

In section 3, we introduce the notation used in this work for the super-energy quantities defined by Bel and García–Parrado. In section 4, we give a set of super-energy inequalities which extend the previously known ones. These inequalities are used to prove our essential results.

In section 5, we define the pure radiative, asymptotic pure radiative and generic radiative gravitational fields, and we show that these three different physical situations correspond to the three Bel radiative cases, namely the Petrov–Bel types N, III and II, respectively. The asymptotic behavior of every radiative type is also analyzed.

Section 6 is devoted to studying non-radiative fields. Extending to the gravitational field the concepts of observer at rest and of proper energy density introduced for the electromagnetic field by Coll [11], we define the concepts of observer at rest with respect to a gravitational field and of proper super-energy density. We show that, for a non-radiative field, the minimum value of the super-energy density is the proper one and it is attained for the observers at rest. The proper super-energy density is also analyzed for radiative fields.

Finally, we present three appendices. The first one explains some notation. The second one summarizes the canonical forms of the Bel–Robinson tensor, while the third one gives the accurate proof of the main theorem stated in section 4.

2. Intrinsic radiative electromagnetic fields

The concept of radiative electromagnetic fields is well known. Electromagnetic radiative states are modeled by null electromagnetic fields. In this section, we revisit this topic summarizing several known definitions and properties. We also point out that some concepts, such as field of pure radiation and field of intrinsic radiation which are concurrent in the electromagnetic case, must be considered as conceptually different. Then, these differences could be important when analyzing super-energy radiative gravitational fields.

2.1. The Faraday and Maxwell–Minkowski tensors: relative formulation

We shall note $g$ the spacetime metric with signature convention $(-, +, +, +)$. The electromagnetic field is modeled with the Faraday 2-form $F$. The electromagnetic energy–momentum (Maxwell–Minkowski) tensor $M$ is given in terms of $F$ by $M = -\frac{1}{2}[F^2 + *F^2]$, where $*$ stands for the Hodge dual operator.

For any observer (unitary time-like vector) $u$, the relative electric and magnetic fields are given by $e = F(u)$ and $b = *F(u)$, respectively, and the relative energy density $\rho$, Poynting vector $s_\perp$ and electromagnetic stress tensor $M_\perp$ are given by

$$\rho = M(u, u) = \frac{1}{2}(e^2 + b^2), \quad s_\perp = -M(u)\perp = *(u \wedge e \wedge b), \quad M_\perp = \rho\gamma - e \otimes e - b \otimes b.$$
where, for a tensor $A$, $A \perp$ denotes the orthogonal projection defined by the projector $\gamma = u \otimes u + g$. In terms of these relative energetic variables, the Maxwell–Minkowski tensor $M$ takes the expression

$$M = M_\perp + s_\perp \sim \otimes u + \rho u \otimes u,$$

(1)

where, for the two vectors $a$ and $b$, $a \sim \otimes b = a \otimes b + b \otimes a$.

When $F$ is a Maxwell field, $dF = 0$, $\nabla \cdot F = 0$, $M$ satisfies the conservation equation $\nabla \cdot M = 0$. For any observer, the time-like component of this equation shows that the relative vector $s_\perp$ is, indeed, the flux of the relative scalar $\rho$.

### 2.2. Algebraic restrictions and Plebański energy conditions

The electromagnetic energy tensor $M$ satisfies the algebraic Rainich conditions $[12]$

$$\text{tr} M^2 = 0, \quad M^2 = \chi^2 g, \quad \chi \equiv \tfrac{1}{2} \sqrt{\text{tr} M^2}, \quad M(x, x) \geq 0,$$

(2)

where $x$ is any given time-like vector. The last one states that the weak energy condition holds, that is, the energy density is non-negative for any observer.

A significant property is that the above Rainich conditions imply $M^2(x, x) \leq 0$ for an arbitrary time-like vector $x$. This means that the energy–momentum density, $s = -M(u) = s_\perp + \rho u$, is a causal vector for any observer $u$. This is a physical requirement which expresses that the amount of radiating energy is a part of the total energy:

$$\rho^2 - s^2_\perp = s^2 = \chi^2 \geq 0.$$

(3)

Note that the scalar $\sqrt{-s^2}$ built with the relative magnitude $s$ is the electromagnetic invariant $\chi$, which has been called the proper energy density of the electromagnetic field $[11]$.

It is worth remarking that Plebański energy conditions ($M(x, x) \geq 0, M^2(x, x) \leq 0$) $[13]$ restrict any energy tensor $M$ to be (real) type I or type II with additional constraints on its eigenvalues. Then the invariant $\text{tr} M^2$ is non-negative. Thus, generically, the Plebański energy conditions for an energy tensor $M$ state $\rho \geq 0, s^2 \leq 0, \text{tr} M^2 \geq 0$. In the case of the electromagnetic field, these quantities are bounded by the proper energy $\chi$.

**Proposition 1.** (Energy conditions). Let $M$ be the Maxwell–Minkowski energy tensor and for any observer $u$ let us define the relative spacetime quantities:

$$s = -M(u), \quad \rho = M(u, u).$$

(4)

Then, the following energy conditions hold:

$$\text{tr} M^2 \equiv 4\chi^2 \geq 0, \quad s^2 = -\chi^2 \leq 0, \quad \rho \geq \chi \geq 0.$$

(5)

### 2.3. Intrinsic radiative states: null fields

The energy density $\rho$ and the stress tensor $M_\perp$ are related by $\text{tr} M_\perp = \rho$, and they vanish only when $F$ vanishes. Thus, any of these relative quantities enable an observer to know if an electromagnetic field is present or not, and they give a measure of the intensity of this field. Nevertheless, the Poynting vector $s_\perp$ relative to an observer can vanish for non-zero electromagnetic fields. This fact enables us to distinguish a special class of fields.

**Definition 1.** An energy tensor $M$ represents a state of intrinsic radiation (at a point) when the Poynting vector $s_\perp$ does not vanish for any observer.

**Proposition 2.** The intrinsic radiative electromagnetic fields are the null fields which are characterized by one of the following equivalent conditions:
(i) the invariant (proper energy density) \( \chi = \frac{1}{2} \sqrt{\text{tr} M^2} \) vanishes;
(ii) the electric and magnetic fields are orthogonal, \((e, h) = 0\), and equimodular, \(e^2 = h^2\).

For a better understanding of the intrinsic radiation states for both the electromagnetic and gravitational fields, we give the following definition.

**Definition 2.** An energy tensor \( M \) represents a state of pure radiation (at a point) when the whole energy density is radiated as Poynting energy, \( \rho = |s|_\perp \).

An energy tensor \( M \) represents a state of asymptotic pure radiation (at a point) when \( \rho \neq |s|_\perp \) and for any positive real number \( \epsilon \), one can find an observer for which the non-radiated energy \( \rho - |s|_\perp \) is smaller than \( \epsilon \).

As a consequence of proposition 2, we have the following.

**Proposition 3.** All the intrinsic radiative electromagnetic fields are of pure radiation.

The concept of asymptotic pure radiation does not give a new class in the electromagnetic case because \( \rho^2 - s^2 _\perp \) is an invariant. Nevertheless, we will see in this work that the definitions given above distinguish three different super-energy radiative gravitational fields.

The energy tensor of a null electromagnetic field takes the expression \( M = \ell \otimes \ell \), where \( \ell \) is the (light-like) fundamental vector of the null field. For any observer, the energy–momentum density \( s \) points towards the fundamental direction. More precisely, for any observer \( u \), the fundamental vector takes the expression \( \ell = \sqrt{\rho} (u + e) \), where \( \rho \) is the relative energy density and \( e \) is the unit vector pointing out the spatial direction of propagation of radiation, \( \ell _\perp = \sqrt{\rho} e \propto s _\perp \). Then, the observers \( \tilde{u} \) traveling (with respect to \( u \)) in the direction (respectively, opposite direction) of \( s _\perp \) are \( \tilde{u} = \cosh \varphi u + \sinh \varphi e \), with \( \varphi > 0 \) (respectively, \( \varphi < 0 \)), and the relative energy density is \( \tilde{\rho} = \rho e^{-2\varphi} \). Consequently, we obtain the following.

**Proposition 4.** The fundamental vector \( \ell \) of a null electromagnetic field determines the spatial direction of propagation of radiation, that is, for any observer, \( \ell _\perp \propto s _\perp \).

For a family of observers each having a spatial velocity at a point tangent and parallel (opposite) to \( \ell _\perp \), the energy density measured by an observer at the same point decreases (increases) and tends to zero (infinity) as its spatial velocity increases and approaches the speed of light. A similar conclusion holds for the radiated energy.

2.4. Non-intrinsic radiative states: observer at rest and proper energy density

For a non-intrinsic radiative electromagnetic field, at least one observer exists that sees a vanishing relative Poynting vector.

**Proposition 5.** The non-intrinsic radiative electromagnetic fields are the non-null fields which are characterized by having a non-vanishing proper energy density, \( \chi = \frac{1}{2} \sqrt{\text{tr} M^2} \neq 0 \).

The following definition naturally arises [11].

**Definition 3.** The observers that see the proper energy density \( \chi \) as their energy density, for which the Poynting vector vanishes, are said observers at rest with respect to the electromagnetic field.

The energy tensor of a non-null electromagnetic field takes the expression \( M = -\chi (v-h) \), where \( v \) (respectively, \( h \)) is the projector on the time-like (respectively, space-like) principal plane. Then, we have the following.
Proposition 6. The observers at rest with respect to a non-null electromagnetic field are those lying on the time-like principal plane.

On the other hand, from expression (3), we obtain the following.

Proposition 7. For a non-null electromagnetic field, the minimum value of the relative energy density is the proper energy density \( \chi \), which is acquired by the observers at rest with respect to the field.

3. The Weyl and Bel–Robinson tensors: relative formulation

In vacuum, the intrinsic properties of a gravitational field depend on the Weyl tensor \( W \). Then, the Bel tensor coincides with the Bel–Robinson tensor given in terms of \( W \) as \([1–3]\)

\[
T_{\mu\beta\nu} = \frac{1}{4} \left( W_{\mu}^\rho \sigma W_{\rho\sigma\nu} + \ast W_{\mu}^\rho \sigma \ast W_{\rho\sigma\nu} \right). \tag{6}
\]

For any observer \( u \), the relative electric and magnetic Weyl fields are given by \( E = W(u; u) \) and \( H = \ast W(u; u) \), respectively. The relative super-energy density \( \tau \), super-Poynting (energy flux) vector \( q_\perp \), super-stress tensor \( t_\perp \), stress flux tensor \( Q_\perp \) and stress-stress tensor \( T_\perp \) are given by

\[
\tau = T(u, u, u, u), \quad q_\perp = -T(u, u)_{\perp}, \quad t_\perp = T(u)_{\perp}, \quad Q_\perp = -T(u)_{\perp}, \quad T_\perp.
\]

Bel introduced \( \tau \) and \( q_\perp \) years ago \([1, 3]\), and recently García–Parrado \([9]\) has considered \( t_\perp \), \( Q_\perp \) and \( T_\perp \) giving their expressions in terms of the electric and magnetic Weyl fields.

In terms of these relative super-energetic variables, the Bel–Robinson tensor \( T \) takes the expression \([9]\)

\[
T = T_{\perp} + 4 \{Q_\perp \otimes u\} + 6 \{t_\perp \otimes u \otimes u\} + 4 \{q_\perp \otimes u \otimes u \otimes u\} + \tau u \otimes u \otimes u \otimes u, \tag{7}
\]

where \( \otimes \) denotes the symmetrization of a tensor \( A \).

In vacuum, the Bianchi identities imply that \( T \) satisfies the conservation equation \( \nabla \cdot T = 0 \). For any observer, this equation shows that the relative quantities \( q_\perp \) and \( Q_\perp \) play the role of fluxes of the relative quantities \( \tau \) and \( t_\perp \), respectively \([9]\).

4. Algebraic restrictions and super-energy inequalities

Elsewhere \([14, 15]\) we have studied the Bel–Robinson tensor \( T \) as an endomorphism on the nine-dimensional space of the traceless symmetric tensors. Its nine eigenvalues \( \{t_k, \tau_k, \bar{\tau}_k\} \) depend on the three complex Weyl eigenvalues \( \{\rho_k\} \) as

\[
t_k = |\rho_k|^2; \quad \tau_k = \rho_i \bar{\rho}_j; \quad (ijk) \equiv \text{even permutation of (123)}. \tag{8}
\]

We have also intrinsically characterized the algebraic classes of \( T \) \([14]\), and we have given their Segré type and their canonical form \([15]\). The part of these results needed in this work are summarized in appendix B.

Bergqvist and Lankinen \([16]\) obtained the algebraic constraints on the Bel–Robinson tensor playing a similar role to that played by the Rainich conditions for the electromagnetic energy tensor:

\[
\text{tr} T = 0, \quad T \cdot T = B(T^2), \quad T(x, x, x, x) \geq 0, \tag{9}
\]

where \( x \) is any given time-like vector. The last one implies that the weak super-energy condition holds, that is, the super-energy density is non-negative for an arbitrary observer. The second one states that the six-order tensor \( (T \cdot T)_{\alpha\beta\gamma\mu\nu} = T_{\alpha\beta\gamma\nu} T_{\mu\nu} \) depends on the four-order one.
\((T^2)_{\alpha\beta\lambda\mu} = T_{\alpha\beta\sigma} T^{\sigma}_{\lambda\mu}\). The explicit expression of the linear operator \(B\) can be found in [16]. A direct consequence of this constraint is the known relation

\[ \tau T^2 = \frac{1}{4}(T, T) g. \] (10)

Note that the quadratic scalar \((T, T) = T_{\alpha\beta\lambda\mu} T^{\alpha\beta\lambda\mu}\) associated with \(T\) is non-negative, because of \(64(T, T) = (W, W)^2 + (W, \ast W)^2\) as a consequence of the results in [14].

The Bel–Robinson tensor \(T\) also satisfies another super-energy inequality, namely \(q = -T(u, u, u) = \tau u + q_\perp\) is a causal vector for any observer \(u\),

\[ \tau^2 - q_\perp^2 = -q^2 = -(q, q) \geq 0. \] (11)

It is worth mentioning that the super-energy inequalities \(\tau \geq 0\) and \((q, q) \leq 0\) can be derived from a stronger condition which satisfies the Bel–Robinson tensor: the generalized dominant energy condition. A wide study about the dominant energy condition for super-energy tensors and general tensors can be found in [7, 17].

The condition \((q, q) \leq 0\) was shown by Bonilla and Senovilla [18] using the relative electric and magnetic Weyl tensors, and it was recovered in [19] exploiting the spinorial formalism. Here, we present a stronger inequality working with the Bel–Robinson tensor itself. Our tensorial proof is based on the following main theorem which is shown in appendix C.

**Theorem 1.** Let \(T\) be the Bel–Robinson tensor and let us define its invariant scalars:

\[ \alpha = \frac{1}{2} \sqrt{(T, T)}, \quad \xi = \frac{1}{4} \sum t_i. \] (12)

where \(t_i\) are the Bel–Robinson real eigenvalues. Then, for any observer \(u\),

\[ T(u, u, u, u) \geq \xi \geq \frac{1}{2} \alpha, \quad T^2(u, u, u, u) \geq \frac{1}{2} \alpha^2. \] (13)

From here we can show stronger constraints for \(\tau\) and \((q, q)\), and also other super-energy inequalities that we collect in the following statement.

**Theorem 2.** (Super-energy inequalities). Let \(T\) be the Bel–Robinson tensor and for any observer \(u\), let us define the relative spacetime quantities:

\[ Q = -T(u), \quad t = T(u, u), \quad q = -T(u, u, u), \quad \tau = T(u, u, u, u). \] (14)

Then, the following super-energy inequalities hold:

\[ (T, T) \equiv 4\alpha^2 \geq 0, \quad (Q, Q) = -\alpha^2 \leq 0, \quad (t, t) \geq \frac{1}{2} \alpha^2, \quad (q, q) \leq -\frac{1}{4} \alpha^2, \quad \tau \geq \frac{1}{2} \alpha \geq 0. \] (15)

The first condition in (15) is the definition of the scalar \(\alpha\). The second one follows from (10). The third and the fifth ones come from (13). Finally, the fourth one is a consequence of the following conditions:

\[ 3(t_\perp t_\perp) - 2(q_\perp q_\perp) - \tau^2 = \frac{1}{4} \alpha^2, \quad (t_\perp t_\perp) - 2(q_\perp q_\perp) + \tau^2 \geq \frac{1}{4} \alpha^2, \] (16)

which come from the Bergqvist–Lankinen constraint (9) and the third condition in (15), respectively.

From the above restrictions (16), we also recover another result by Bonilla and Senovilla used in the proof of the causal propagation of gravity in vacuum [18].

**Proposition 8.** The amount of super-stress is bounded by the amount of super-energy,

\[ (t_\perp t_\perp) \leq \tau^2. \]
5. Intrinsic super-energy radiative gravitational fields

The super-energy density $\tau$, the super-stress tensor $t_\perp$ and the stress–stress tensor $T_\perp$ are related by $\text{tr} \ T_\perp = t_\perp$, $\text{tr} \ t_\perp = \tau$, and they vanish only when the Weyl tensor $W$ vanishes [9]. Thus, any of these relative quantities enable an observer to know if the purely gravitational part of the field is present or not, and they give a measure of the intensity of this field. Nevertheless, the Poynting vector $q_\perp$ and the stress flux tensor $Q_\perp$ relative to an observer can vanish for a non-zero Weyl tensor.

If we consider $\tau$ as a measure of the gravitational field, its flux $q_\perp$ denotes the presence of gravitational radiation. This is the point of view of Bel [1, 3], who gave the following definition.

**Definition 4. (Intrinsic gravitational radiation, Bel 1958)**. In a vacuum spacetime, there exists intrinsic gravitational radiation (at a point) if the super-Poynting vector $q_\perp$ does not vanish for any observer.

But we can also consider $t_\perp$ as a measure of the gravitational field. Then its flux $Q_\perp$ denotes the presence of gravitational radiation. This fact has been pointed out by García–Parrado [9], who has given the following definition.

**Definition 5. (Intrinsic super-energy radiation, García–Parrado 2008)**. In a vacuum spacetime, there exists intrinsic super-energy radiation (at a point) if the stress flux tensor $Q_\perp$ does not vanish for any observer.

The criterion given by Bel leads to gravitational fields of Petrov–Bel type N, III and II as modeling gravitational radiative states [3] according to Pirani’s proposal [4]. For these and other radiation criteria, see [3, 5].

The definition given by García–Parrado is less restrictive than the Bel one and it allows type I radiative gravitational fields [9]. This and other satisfactory properties show the interest of this generalization which claims for a deeper study undertaken elsewhere [10]. In this paper, we focus on analyzing Bel’s radiative gravitational fields.

In his work on gravitational radiation, Lichnerowicz [20] considers type N gravitational fields as modeling pure radiation, and Bonilla and Senovilla [18] showed that this case can be characterized by the condition $(q, q) = 0$. Then, the super-energy density $\tau$ equals the amount of radiating energy $|q_\perp|$. Our definition 2 links this feature with the Lichnerowicz terminology.

**Definition 6. (Pure gravitational radiation)**. In a vacuum spacetime, there exists pure gravitational radiation (at a point) when the whole super-energy density is radiated as Poynting super-energy, $\tau = |q_\perp|$.

Note that the definition we give is Bonilla–Senovilla’s description of the Lichnerowicz concept of pure radiation. Evidently, we have the following intrinsic characterization [18].

**Proposition 9.** The pure radiative states are the type N gravitational fields.

A type N spacetime admits a quadruple-null Debever direction $\ell$ which is named the fundamental direction of the gravitational field. For any observer, the relative quantity $q$ points towards the fundamental direction. Then, from the canonical expression of a type N Bel–Robinson tensor, a similar reasoning to that we have done before proposition 4 leads to the following result.

**Proposition 10.** The fundamental direction $\ell$ of a type N gravitational field determines the spatial direction of propagation of radiation, that is, for any observer, $\ell_\perp \propto q_\perp$. 
For a family of observers each having a spatial velocity at a point tangent and parallel (opposite) to $\ell_\perp$, the super-energy density measured by an observer at the same point decreases (increases) and tends to zero (infinity) as its spatial velocity increases and approaches the speed of light. A similar conclusion holds for the radiated super-energy.

Now we analyze the non-pure radiative states. The following definition distinguishes a subclass.

**Definition 7. (Asymptotic pure gravitational radiation).** In a vacuum spacetime, there exists asymptotic pure gravitational radiation (at a point) when $\tau \neq |q_\perp|$ and for any positive real number $\epsilon$, one can find an observer for which the non-radiated energy $\tau - |q_\perp|$ is smaller than $\epsilon$.

The fourth super-energy condition given in expression (15) of theorem 2 states $(q, q) \leq -\frac{1}{4} \alpha^2$. In a type II spacetime, the Bel–Robinson real eigenvalues do not vanish (see appendix B) and consequently, $\alpha \neq 0$. Then, non-asymptotic pure radiation exists in this case.

Nevertheless, in a type III spacetime, $\alpha = 0$. Moreover, a triple-null Debever direction $\ell$ (named the fundamental direction) and a simple one $k$ exist. Then, taking into account the canonical form of a type III Bel–Robinson tensor (see expression (B.8) in appendix B), a similar reasoning to that we have done before proposition 4 leads to the following result.

**Proposition 11.** The asymptotic pure radiative states are the type III gravitational fields.

Let $\ell$ be the fundamental direction of a type III gravitational field. For a family of observers each having a spatial velocity at a point tangent and parallel (opposite) to $\ell_\perp$, the super-energy density measured by an observer at the same point decreases (increases) and tends to zero (infinity) as its spatial velocity increases and approaches the speed of light. A similar conclusion holds for the radiated super-energy.

Observers in the Debever plane $\{\ell, k\}$ are those for which $\ell_\perp \propto q_\perp$.

Finally, the type II spacetimes model the generic radiative states. Now a double-null Debever direction (named the fundamental direction) and two simple ones $k_1$ and $k_2$ exist. Then, taking into account the canonical form of a type II Bel–Robinson tensor (see appendix B), we obtain (C.4) (see appendix C). Then, a similar reasoning to that we have done before proposition 4 leads to the following result.

**Proposition 12.** The generic (non-pure, non-asymptotic pure) radiative states are the type II gravitational fields.

Let $\ell$ be the fundamental direction of a type II gravitational field. For a family of observers each having a spatial velocity at a point tangent and parallel (opposite) to $\ell_\perp$, the super-energy density measured by an observer at the same point decreases (increases) and tends to a positive value (infinity) as its spatial velocity increases and approaches the speed of light. Meanwhile, radiated super-energy decreases (increases) and tends to zero (infinity).

### 6. Non-radiative gravitational fields: observer at rest and proper super-energy density

From Bel’s point of view, non-radiative gravitational fields are those for which at least an observer exists which sees a vanishing relative super-Poynting vector. The following definition naturally arises.

**Definition 8.** The observers for which the super-Poynting vector vanishes are said observers at rest with respect to the gravitational field.
First, we have this immediate result [3].

**Proposition 13.** The non-intrinsic radiative gravitational fields are the Petrov–Bel type I or D spacetimes.

A type D spacetime admits two double-null Debever directions $\ell$ and $k$. For any observer lying on the Weyl principal plane $\{\ell, k\}$, the electric and magnetic Weyl tensors simultaneously diagonalize. On the other hand, a type I spacetime only admits an observer with this property. Then, we have the following significant and known result [3].

**Proposition 14.** The observers at rest with respect to the gravitational field are those for which

the electric and magnetic Weyl tensors simultaneously diagonalize.

In a type I spacetime, a unique observer $e_0$ at rest with respect to the gravitational field exists.

In a type D spacetime, the observers $e_0$ at rest with respect to the gravitational field are those lying on the Weyl principal plane.

The proper energy density defined in the electromagnetic case has two qualities: first, it is the energy density measured by observers at rest with respect to the field and secondly, it is the minimum value of the relative energy density. Now, in the gravitational case, we take one of these properties as a definition and we will prove the other one as a theorem.

For a type I Bel–Robinson tensor $T$, the super-energy density relative to the observer $e_0$ at rest with the field can be obtained from expressions (C.5) and (C.6) given in appendix C by taking $u = e_0$. Then, we obtain

$$\tau_0 \equiv T(e_0, e_0, e_0, e_0) = \xi,$$

(17)

where $\xi$ is defined in (12). Moreover, the above expression is also valid in a type D spacetime for any observer $e_0$ at rest with the field.

Note that the right-hand term of expression (17) is a scalar invariant which can be defined in any spacetime (radiative or not). Then, we give the following definition.

**Definition 9.** We call the proper super-energy density of a gravitational field the invariant scalar

$$\xi \equiv \frac{1}{4} \sum t_i.$$
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Appendix A. Notation

(i) Composition of two 2-tensors $A$ and $B$ as endomorphisms: $A \cdot B$, $(A \cdot B)_{\alpha}^{\beta} = A_{\alpha \mu} B^{\mu \beta}$.

(ii) In general, for the arbitrary tensors $S$ and $T$, $S \cdot T$ will be used to indicate the contraction of adjacent indexes on the tensorial product.

(iii) Square and trace of a 2-tensor $A$: $A^2 = A \cdot A$, $\text{tr} A = A^{\alpha \alpha}$.

(iv) The action on one or more vectors of an arbitrary tensor $S$ as multilinear form will be denoted by $S(x)$, $S(x, y)$, $S(x, y, z)$,... For example, the action of a 2-tensor $A$ as an endomorphism $A(x)$ and as a bilinear form $A(x, y)$:

$$A(x)^\beta = A_{\rho \beta} x^\rho, \quad A(x, y) = A_{\alpha \beta} x^\alpha y^\beta.$$ 

(v) The quadratic scalar associated with an arbitrary tensor $S$ will be denoted by $(S, S)$. For example, if $x$ is a vector and $A$ is a 2-tensor,

$$(x, x) = x^2 = g(x, x), \quad (A, A) = A^{\alpha \beta} A_{\alpha \beta}.$$ 

Appendix B. Canonical forms of the Bel–Robinson tensor

The Bel–Robinson tensor $T$ defines an endomorphism on the space of the traceless symmetric 2-tensors [14, 15]. The nine eigenvalues $\{t_k, \tau_k, \bar{\tau}_k\}$ depend on the three (complex) Weyl eigenvalues $\{\rho_i\}$ as $t_k = |\rho_k|^2$, $\tau_k = \rho_i \bar{\rho}_j$, with $(ijk)$ being an even permutation of $(123)$.

Three independent invariant scalars can be associated with $T$. In fact, the nine eigenvalues $\{t_i, \tau_i, \bar{\tau}_i\}$ can be written in terms of three scalars $\{p_i\}$ as [14]

$$t_i = -(p_j + p_k), \quad \tau_i = p_i + iq, \quad q^2 = p_1 p_2 + p_2 p_3 + p_3 p_1.$$ (B.1)

Note that the scalars $\{p_i\}$ satisfy the following inequalities:

$$p_j + p_k \leq 0 \quad (j \neq k); \quad p_1 p_2 + p_2 p_3 + p_3 p_1 \geq 0.$$ (B.2)

Conversely, the scalars $p_i$ depend on the three real Bel–Robinson eigenvalues $\{t_i\}$ as

$$2p_i = t_i - t_j - t_k \quad i \neq j \neq k \neq i.$$ (B.3)

Studying the eigentensors of the Bel–Robinson tensor $T$ allows us to obtain its canonical form for the different Petrov–Bel types [15]. Now we summarize some of these results.

Type I

Let $\{e_0, e_i\}$ be the canonical frame of a type I Weyl tensor. We can define the traceless symmetric 2-tensors:

$$\Pi_i = \frac{1}{2} (v_i - h_i), \quad \Pi_{ij} = \frac{1}{2} (e_i \otimes e_j + i e_0 \otimes e_k).$$ (B.4)
with \((ijk)\) being an even permutation of \((123)\), and where \(v_i = -e_0 \otimes e_0 + e_i \otimes e_i\), and \(h_i = g - v_i\). Then, \(\{\Pi_i, \Pi_{jk}, \Pi_{\bar{k}j}\}\) is an orthonormal frame of eigentensors of the Bel–Robinson tensor \(T\). Moreover, \(T\) takes the canonical expression \([15]\)

\[
T = \sum_{i=1}^{3} t_i \Pi_i \otimes \Pi_i + \sum_{(ijk)} t_{(ijk)} \Pi_{jk} \otimes \Pi_{\bar{k}j} + \sum_{(ijk)} \bar{t}_{(ijk)} \Pi_{\bar{k}j} \otimes \Pi_{jk}, \tag{B.5}
\]

where \((ijk)\) is an even permutation of \((123)\).

**Type D**

Let \(\{e_0, e_1\}\) be a canonical frame of a type D Weyl tensor, that is, the pairs \(\{e_0, e_1\}\) and \(\{e_2, e_3\}\) generate the Weyl principal 2-planes. Then, the Bel–Robinson tensor \(T\) takes the canonical expression \((B.5)\) with the eigenvalues restricted by \(t_2 = t_3 = t_1 \neq 0, t_1 = 4t_2\) and \(t_2 = t_3 = -2t_2\) \([15]\).

**Type II**

A type II Weyl tensor admits a double-null Debever direction \(\ell\) and two simple ones \(k_1\) and \(k_2\). Moreover, a time-like principal plane exists which contains the direction \(\ell\). Let \(k\) be the other null direction in the principal plane. Then, from an adapted null frame of vectors \(\{\ell, k, m, \bar{m}\}\), we can define the frame of 2-tensors \(\{\Pi, \Lambda, K, N, \bar{N}, \Omega, \bar{\Omega}, M, \bar{M}\}\) given by

\[
\Pi = -\frac{1}{3} (\ell \otimes k + m \otimes \bar{m}), \quad N = -\frac{1}{\sqrt{2}} \ell \otimes \bar{m}, \quad \Omega = \frac{1}{\sqrt{2}} k \otimes m, \quad M = m \otimes m, \quad \Lambda = -\ell \otimes \ell, \quad K = -k \otimes k. \tag{B.6}
\]

Then, the Bel–Robinson eigenvalues are restricted by \(t_2 = t_3 = t_1 \neq 0, t_1 = 4t_2\) and \(t_2 = t_3 = -2t_2\), and the Bel–Robinson tensor \(T\) takes the canonical expression \([15]\)

\[
T = 4t_2 \Pi \otimes \Pi - 2t_2 (\Omega \otimes N + \bar{\Omega} \otimes \bar{N} - N \otimes N - \bar{N} \otimes \bar{N}) + t_2 (\Lambda \otimes K + M \otimes \bar{M} - \Lambda \otimes \bar{M} - \Lambda \otimes M + \Lambda \otimes \Lambda). \tag{B.7}
\]

**Type III**

A type III Weyl tensor admits a triple-null Debever direction \(\ell\) and a simple one \(k\). Then, from an adapted null frame of vectors \(\{\ell, k, m, \bar{m}\}\), we can define the frame of 2-tensors \(\{\Pi, \Lambda, K, N, \bar{N}, \Omega, \bar{\Omega}, M, \bar{M}\}\) given in \((B.6)\). Then, all the eigenvalues vanish, and the Bel–Robinson tensor \(T\) takes the canonical expression \([15]\)

\[
T = \Lambda \otimes \Pi + N \otimes \bar{N}. \tag{B.8}
\]

**Type N**

A type N Weyl tensor admits a quadruple-null Debever direction \(\ell\). Then, all the eigenvalues vanish, and the Bel–Robinson tensor \(T\) takes the canonical expression \([15]\)

\[
T = \ell \otimes \ell \otimes \ell \otimes \ell. \tag{B.9}
\]

**Appendix C. Proof of theorem 1**

The proof of the theorem is based on the following two lemmas that we will prove later.

**Lemma 1.** Let \(T\) be a Bel–Robinson tensor of type II, III or N. Then, for any observer \(u\),

\[
T(u, u, u, u) > \xi = \frac{1}{2} \alpha. \tag{C.1}
\]
Lemma 2. Let $T$ be a Bel–Robinson tensor of type I or D. Then, for any observer $u$, 
\[ T(u, u, u, u) \geq \xi \geq \frac{1}{2} \alpha. \] (C.2)

**Proof of theorem**

The first inequality in expression (13) of theorem 1 is a direct consequence of the above two lemmas.

Now we prove the second inequality in expression (13) of theorem 1. In [14], we have introduced a second-order super-energy tensor $T_{(2)}$ associated with the traceless part $W_{(2)}$ of the square $W^2$ of the Weyl tensor $W$. That is, $T_{(2)}$ is defined as (6) by changing $W$ by $W_{(2)}$. It follows that $T_{(2)}$ has the same properties as $T$ [14]. Then, we can apply to it the first inequality in expression (13) already proved. Thus, for any observer $u$,
\[ T_{(2)}(u, u, u, u) \geq \frac{1}{2} \sqrt{(T_{(2)}, T_{(2)})}. \] (C.3)

From the specific expression of $T_{(2)}$ (see [14] for more details), we can compute the left-hand and the right-hand terms of the above inequality, and we obtain the second inequality in (13).

**Proof of lemma 1**

The Bel–Robinson eigenvalues vanish for types N and III and, consequently, both invariants $\xi$ and $\alpha$ vanish. Then, inequality (C.1) follows from the weak energy condition.

On the other hand, the eigenvalues and canonical form (B.7) of a type II Bel–Robinson tensor lead to $2\xi = 3\lambda = \alpha$. Moreover, an arbitrary observer $u$ can be written in the Weyl canonical frame as $u = \lambda (e^{\theta} e^l + e^{-\theta} k) + \mu (e^{\lambda} m + e^{-\lambda} \bar{m})$, $2(\lambda^2 - \mu^2) = 1$. Then, using again (B.7), we obtain
\[ T(u, u, u, u) = \xi \left(1 + 6B, \quad B \equiv 2\mu^2 + 4\mu^4 \sin^2 2\sigma + \left(\frac{1}{4} \alpha^2 - 2\mu^2 \cos^2 2\sigma\right)^2 > 0. \] (C.4)

Thus, (C.1) holds for type II spacetimes.

**Proof of lemma 2**

Taking into account the eigenvalue relation (8) and the Bel–Robinson canonical form (B.5), we have
\[ 16\xi^2 = \sum_{i=1}^{3} t_i^2 = \sum_{i,j} t_i t_j = \sum_{i=1}^{3} t_i^2 + 2 \sum_{i,j} t_i t_j = \sum_{i=1}^{3} t_i^2 + 2 \sum_{k=1}^{3} |t_k|^2 \geq \sum_{i=1}^{3} (t_i^2 + t_i^2 + \bar{t}_i^2) = (T, T) = 4\alpha^2 \]

and then the second inequality in (C.2) holds.

In order to prove the first one, let us write the observer $u$ in a Weyl canonical frame $\{e_i, e_i\}$, $u = u^i e_i$, and let us calculate the relative super-energy density by using the Bel–Robinson canonical form (B.5). Then, we obtain
\[ T(u, u, u, u) = \xi + \Omega, \quad \Omega \equiv \Phi + 6qA, \] (C.5)

where $q$ is given in (B.1) and
\[ \Phi \equiv A' r, \quad r_i \equiv -(4p_i + p_j + p_k), \quad A' \equiv (u^i)^2 (u^j)^2 - (u^l)^2 (u^k)^2, \quad A \equiv \prod_i u^i, \] (C.6)

with $(i, j, k)$ being an even permutation of $(1, 2, 3)$. Thus, we must show that $\Omega \geq 0$, an inequality which is a consequence of the following two conditions:
\[ \Phi \geq 0, \quad R \equiv \Phi^2 - (6qA)^2 \geq 0. \] (C.7)
Let us study the first one. From (C.6), we obtain
\[ \Phi = -P_i P_i, \quad P_i \equiv 4(u^0_j)^2 (u^j)^2 + (u^j)^2 + (u^j)^2 + ((u^j)^2 - (u^j)^2)^2 \geq 0, \quad (C.8) \]
with \((i, j, k)\) being an even permutation of \((1, 2, 3)\). At least one \(t_i\) (say \(t_3\)) does not vanish in spacetimes of types I and D. Then, from (B.2), we have
\[ -p_3 \geq \frac{p_1^2 p_2}{p_1 + p_2}, \]
and substituting into expression (C.8) of \(\Phi\) we obtain \(\Phi \geq -P_3 p_1 - P_2 p_2 + P_3 p_1 p_2 / (p_1 + p_2)\), and then
\[ \Phi \geq -\frac{1}{p_1 + p_2} P_{AB} p_A p_B, \quad P_{AB} \equiv P^A, \quad 2r^{12} \equiv P^1 + P^2 - P^3, \quad \alpha, \beta = 1, 2. \quad (C.9) \]
The quadratic form \(P \equiv P_{AB} p_A p_B\) has principal minors of order 1 which are non-negative, \(P^1 \geq 0\) (see (C.8)), and a straightforward calculation leads to a determinant \(\Delta_P\) which is also non-negative,
\[ \Delta_P = 9 \left[ (u^0_j)^2 \sum_{i<j} (u^j u^l)^2 + \left[ 4 \sum_{i=1}^{3} (u^i)^2 + 3 \right] \prod_{j=1}^{3} (u^j)^2 \right] \geq 0. \]
Now we can apply the following theorem (see [21], p 309): a quadratic form is non-negative if, and only if, all the principal minors are non-negative. Consequently, we have \(P \geq 0\), and from (C.9) and (B.2), we obtain \(\Phi \geq 0\).

Let us now study the second condition in (C.7). Developing the expression of \(R\), we arrive at the quadratic form
\[ R = R_{ij} r_i r_j, \quad R_{ij} \equiv [(u^0_j)^2 (u^j)^2 + (u^j)^2 (u^k)^2]^2, \quad R_{ij} \equiv A_i A_j - 4A^2, \quad i, j = 1, 2, 3. \quad (C.10) \]
where, in the expression of \(R_{ij}\), \((i, j, k)\) is an even permutation of \((1, 2, 3)\). Note that the principal minors of order 1 are non-negative, \(R^1 \geq 0\). The principal minors of order 2 \(\Delta_{ij}\), and the determinant \(\Delta\), are also non-negative:
\[ \Delta_{ij} = 4A^2 \left[ 1 + (u^ij)^2 + (u^i j)^2 \right] \left[ (u^j)^2 + (u^j)^2 \right] \geq 0, \]
\[ \Delta = 64A^4 \left[ \sum_{i<j} (u^i u^j)^2 + (u^i)^2 \right] + \sum_{i<j} (u^i u^j)^2 + 2 \prod_{j=1}^{3} (u^j)^2 \geq 0. \]
Consequently, applying again the stated theorem on non-negative quadratic forms, we obtain \(R \geq 0\).

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