Effect of a uniform magnetic field on unsteady natural convection of nanofluid

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1. Introduction

In recent years, flow and heat transfer analysis have become popular among the researchers due to their importance for many industrial areas such as electronic cooling, building insulators, solar collectors, food storage industry and nuclear reactors etc. Especially, the researchers have been focused on the mechanism of heat transfer of nanofluids which is the mixture of nanoparticles and base fluid. Adding nanoparticles into the fluid is a useful method to enhance the heat transfer rate of conventional transfer fluids with low thermal conductivity such as water, oils, ethylene glycol, etc. Generally, copper, aluminium, silver, etc. are preferred as a nanoparticle.

The first study about nanofluid was done by Choi in [1]. A number of literature reviews on nanofluid can be found in [2–7]. In these studies, the advantages of nanofluids for heat transfer mechanism, their challenges and limiting factors, and their theoretical and numerical investigations were summarized. Also, many numerical studies on nanofluids with different numerical techniques and various domain geometry have been published. Mahian et al. in [8,9] presented the review study in two parts by intended to be a comprehensive reference for researchers and practitioners interested in nanofluids and in the many applications of nanofluid flow.

Besides the classical natural convection problem, the researchers want to understand the flow behaviour and heat transfer mechanism of nanofluids under magnetic field influence. The problem of natural convection of enclosures under a magnetic field has many applications in engineering and industrial area such as the crystal growth in fluids, purification of molten metals, the metal casting, nuclear reactor cooling, microelectronic devices and geothermal energy extractions. The convection of the electrically conducting fluid can be controlled using the Lorentz force caused by magnetic field. Therefore, in the literature, there are many studies about the influence of the magnetic field on natural convection flow in the absence of nanoparticle. A study with differential quadrature method (DQM) is presented by Ece and Büyük in [10]. In this study, the domain of the problem was considered as an inclined rectangular enclosure and the results are given for different aspect ratio, inclination angles and magnetic field direction. Another study with DQM was performed by Kahveci and Öztuna in [11]. A laterally heated enclosure with an off-centred partition was considered as problem domain and results were presented for several values of Rayleigh, Hartmann and Prandtl numbers, and partition location. A numerical study about the double diffusive natural convection flow in a rectangular enclosure under a uniform magnetic field was presented by Teamah in [12]. In this study the governing equations are discretized using finite volume method (FVM). Another FVM solutions were presented by Pirmohammadi and Ghasemi in [13] and Öztop et al. in [14] to examine the influence of the magnetic field on natural and mixed convection flows, respectively. In these
studies, a tilted enclosure was considered as a problem domain by Pirmohammedi and Ghasemi whereas, a top-sided lid-driven cavity was used as a problem domain by Öztöp et al. Alsoy-Akgün and Tezer-Sezgin examined DRBEM and DQM solutions of the problem by comparing the DRBEM and DQM results in [15] and Alsoy-Akgün and Lesnic extended the analysis to inverse natural magnetoh-convection in [16].

Natural convection heat transfer performance strongly depends on the strength and direction of the magnetic field. The heat transfer decreases with an increase of the magnetic field and the direction of applied magnetic field affect the heat transfer differently [17]. In some devices, the enhanced heat transfer is desired such as magnetic field sensors. The use of nanofluids with higher thermal conductivity improve the heat transfer performance of such devices [18]. Many numerical studies about the problem in the presence of the nanoparticle can be found in the literature. In [19], Dulkirvich and Colaco showed that in order to control the electrically conducting melt flow-fields and heat transfer, externally applied magnetic field and electric field can be used. Hamad et al. discussed similarity reduction for problems of free convection flow of a nanofluid under the influence of constant magnetic field in [20]. The problem was defined in a vertical semi-infinite flat plate and numerical results were obtained using finite difference method (FDM). Lattice Boltzmann method (LBM) was applied for the solution of the problem by Kefayati in [21,22]. In [21], Al2O3-water filled an open enclosure was used as problem domain whereas in [22], Cu-water filled a linearly heated enclosure was used. Another numerical study was performed for the solution of free convection in Al2O3-water nanofluid in a concentric annulus between square and elliptic cylinder using LBM by Sheikholeslami in [23]. Also, in [24,25], Sheikholeslami used control volume finite element method (CVFEM) to show that Ferrohydrodynamic (FHD) and MHD effects on ferro fluid in a semi annulus, and to analyse that MHD natural convection in a Cu-water nanofluid in an inclined enclosure, respectively. Hamida presented a numerical study on natural convection in an ethylene glycol-copper filled square enclosure under a uniform horizontal magnetic field with finite element using COMPOL Multiphysics in [26]. Ghasemi et al. numerically examined the problem for Al2O3-water nanofluid using control volume method (CVM) in [17].

The problem of natural convection of nanofluid under an effect of magnetic field was also considered in a domain with different geometries. In [27], Mahmoudi et al. solved the steady natural convection in a 2D triangular enclosure filled with Cu-water nanofluid under the effect of magnetic field problem by using FVM. In [28], Ghasemi presented the results of a numerical study of steady natural convection in a U-shaped enclosure filled with Cu-water nanofluid under the influence of horizontal magnetic field by using CVM. In [29], Mehmood et al. examined the unsteady mixed convection in a lid-driven square cavity with an isothermally heated square blockage inside filled with Al2O3-water nanofluid with magnetic field effect. In this study, Galerkin FEM and Crank-Nicolson schema are used for governing equations and time discretization, respectively. In [30], Elshehab et al. presented the problem of unsteady natural convection in an inclined L-shaped enclosure filled with Cu-water nanofluid under differentially heated walls in the presence of an inclined magnetic field. The fully implicit FDM is used for the solution of the governing equations.

In all these studies mentioned above, the domain discretization methods such as FDM, LBM, FVM and FEM are considered as the solution procedure. Discretization of the whole domain of the problem causes very large system of algebraic equations and needs extra computational effort. Also, sometimes they give inaccurate results. On the other hand, DRBEM is boundary only discretization method and it does not necessary to discretization of whole domain of the problem. This is the main advantages of the method. DRBEM is a very suitable method for the solution of differential equations which involve the Laplacian and one or more terms. In the solution procedure, all the terms except the Laplace term are considered as a inhomogeneity and they are approximated using the radial basis functions. Thus, the nonlinear problems can be solved by using DRBEM. Tezer-Sezgin et al. presented DRBEM solution of the problem by comparing the results with FEM in [18]. In this study, both FEM and DRBEM results are given for the steady problem and DRBEM implementation is performed with Laplace equation using its fundamental solution and results are obtained up to values of $Ha = 60$ and $Ra = 10^5$.

Another idea for the DRBEM is to reduce the differential equation to an inhomogeneous modified Helmholtz equation and then to use its fundamental solution. By using this way the inhomogeneous term of the governing equation can be kept simple and this can be reduce the interpolation error to a minimum. This result is the important advantage of the method used in this study over the DRBEM with the fundamental solution of Laplace equation. In this study, DRBEM solution of unsteady natural convection in a Al2O3-water nanofluid filled square enclosure under the horizontal magnetic field effect is presented. In order to better demonstrate the performance of the method and compare the results with the previous works, Al2O3 is preferred as nanoparticles. The results are obtained for varying values of $Ra$ (Rayleigh number), $Ha$ (Hartmann number) and $\phi$ (particle volume fraction). In the solution procedure, $(1/2\pi) \ln(x)$ and $(1/2\pi) K_0(x)$ are taken as fundamental solutions for stream function, and transformed form of the vorticity transport and temperature equations, respectively. Thus, more
information about the original governing equations can be used. Also, one can eliminate the need of extra time integration scheme in DRBEM and stability analysis by using the transformed form of the governing equations.

The plan of the study can be described briefly as follows. In Section 2, the mathematical formulation of the problem is presented by giving the definitions of the physical parameters. Also, the governing equations are prepared for DRBEM solution procedure. In Section 3, the DRBEM method is expressed for both modified Helmholtz equation which enables one to eliminate the need of use an extra time integration procedure. In Section 4, the numerical results are discussed and compared with the previous works. Finally, the study is completed with the presentation of the conclusions in Section 5.

Nomenclature

- \( B_0 \) magnetic field intensity
- \( b_1 \) right hand side function of Poisson equation
- \( b_2, b_3 \) right hand side functions of modified Helmholtz equations
- \( C_p \) specific heat, \((J/kgK)\)
- \( f, \tilde{f} \) radial basis functions for Poisson and modified Helmholtz equations
- \( F, \tilde{F} \) coordinate matrices for Poisson and modified Helmholtz equations
- \( g \) gravitational acceleration, \((m/s^2)\)
- \( h \) length of the cavity, \((m)\)
- \( Ha \) Hartmann number, \((B_0 h \sqrt{\alpha_{nf}/\rho_{nf} c_p})\)
- \( H_p, G_p \) BEM matrices for Poisson equation
- \( H_{p1}, H_{p2}, G_{p1}, G_{p2}, G_{T} \) BEM matrices for modified Helmholtz equation
- \( k \) thermal conductivity, \((W/mK)\)
- \( M_{b1}, M_{l1} \) the number of boundary elements and internal nodes
- \( p \) dimensionless pressure, \((\bar{p} \rho f^2/\rho_{nf} \alpha f^2)\)
- \( p' \) fluid pressure, \((Pa)\)
- \( Pr \) Prandtl number, \((\nu f/\alpha f)\)
- \( r \) distance between source and field points
- \( Ra \) Rayleigh number, \((g \beta f h^3 (T_h - T_c)/\nu f \alpha f)\)
- \( t \) dimensionless time, \((t' \alpha f/2)\)
- \( t' \) time, \((s)\)
- \( T \) dimensionless temperature, \((T' - T_c)/(T_h - T_c)\)
- \( T' \) temperature, \((K)\)
- \( T_{cold} \) cold temperature, \((K)\)
- \( T_{hot} \) hot temperature, \((K)\)
- \( U', V' \) velocity components of fluid in \(x, y\) directions, \((m/s)\)
- \( u^*, q^* \) fundamental solution and its normal derivative
- \( \hat{u}, \hat{q} \) particular solution and its normal derivative
- \( w \) vorticity
- \( x, y \) dimensionless coordinates, \((x'/h, y'/h)\)
- \( \gamma \) cartesian coordinates, \((m)\)

Greek symbols

- \( \alpha \) thermal diffusivity, \((m^2/s)\)
- \( \beta \) thermal expansion coefficient, \((1/K)\)
- \( \gamma_1, \gamma_2, \gamma_3 \) unknown coefficients
- \( \Gamma \) boundary of the domain
- \( \Delta T \) reference temperature difference
- \( \Delta t \) time step
- \( \theta_{w}, \theta_{T} \) relaxation parameters
- \( \lambda_{w}, \lambda_{T} \) wave numbers
- \( \mu \) dynamic viscosity, \((N s/m^2)\)
- \( \nu \) kinematic viscosity, \((m^2/s)\)
- \( \rho \) density, \((kg/m^3)\)
- \( \rho c_p \) heat capacity
- \( \sigma \) electrical conductivity, \((\mu S/cm)\)
- \( \phi \) particle volume fraction
- \( \psi \) stream function
- \( \Omega \) two-dimensional domain

Superscripts

- \( ^* \) dimensional parameters
- \( ^{(c)} \) time level

Subscripts

- \( p \) particle
- \( f \) fluid
- \( nf \) nanofluid

2. Problem definition

The natural convection problem, unsteady flow which is influenced by a magnetic field is considered for nanofluid (Al2O3 -water) filled square enclosure. The non-dimensional form of the governing equations of the problem is defined with pressure \( p \), velocity \( U, V \) and temperature \( T \) as

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (1)
\]

\[
\frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\mu_{nf}}{\rho_{nf} \alpha_f} \nabla^2 U, \quad (2)
\]

\[
\frac{\partial V}{\partial t} = -\frac{\partial p}{\partial y} + \frac{\mu_{nf}}{\rho_{nf} \alpha_f} \nabla^2 V
\]
where $D/Dt = (\partial/\partial t) + U(\partial/\partial x) + V(\partial/\partial y)$. Here, $\mu$, $\rho$, $\alpha$ and $\beta$ are dynamic viscosity, density, thermal diffusivity and thermal expansion coefficient, respectively. The parameters in non-dimensional form are induced as

\[
\begin{align*}
\sigma &= \frac{\mu \varepsilon}{\rho \alpha f}, \\
\rho &= \frac{1 - \phi}{\phi}, \\
(\rho \beta) &= (1 - \phi)(\rho \beta)_{y} + \phi(\rho \beta)_{p}, \\
(\rho C_{p}) &= (1 - \phi)(\rho C_{p})_{y} + \phi(\rho C_{p})_{p}, \\
\alpha &= \frac{k_{f}}{(\rho C_{p})_{f}}, \\
\beta &= \frac{k_{f} + 2k_{r} - 2\phi(k_{r} - k_{p})}{k_{r} + 2k_{r} + \phi(k_{r} - k_{p})},
\end{align*}
\]

where $\phi$ is the volume fraction of particle, $C_{p}$ is the specific heat of particle or fluid and $k$ is the thermal conductivity. Also, “$t$”, “$p$” and “$f$” which appear in Equations (1)–(6) are used to indicate the given parameters which belong to fluid, particle and nanofluid, respectively. The governing Equations (1)–(4) are rewritten in the $\psi - w - T$ form inserting the definitions

\[
\begin{align*}
u &= \frac{\partial \psi}{\partial y}, \\
\psi &= \frac{\partial \psi}{\partial x}, \\
\psi &= \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x},
\end{align*}
\]

as

\[
\begin{align*}
\psi &= -w, \\
\frac{\mu_{nf}}{\rho_{nf} \alpha_{f}} \psi &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} \\
&- RaPr \frac{(\rho \beta)_{nf}}{\rho_{nf} \beta_{f}} \frac{\partial T}{\partial x} - Ha^{2}Pr \frac{\partial^2 \psi}{\partial x^2}, \\
\frac{\alpha_{nf}}{\alpha_{f}} \psi &= \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}.
\end{align*}
\]

where $\psi$ is stream function and $w$ is vorticity. These equations are given together with boundary conditions presented in Figure 1.

The numerical solution procedure will be used for the governing equations after transforming the Equations (9) and (10) to the modified Helmholtz equations. For this aim, first, forward finite difference approximations are used for time derivatives as

\[
\begin{align*}
\frac{\partial w}{\partial t} &= \frac{w^{(c+1)} - w^{(c)}}{\Delta t} \quad \text{and} \quad \frac{\partial T}{\partial t} = \frac{T^{(c+1)} - T^{(c)}}{\Delta t}.
\end{align*}
\]

In addition, the values $w$ and $T$ on the left side of Equations (9) and (10) are approximated with the parameters $0 < \theta_{w}, \theta_{T} < 1$

\[
\begin{align*}
\theta_{w} = \frac{\theta_{w} w^{(c+1)} + (1 - \theta_{w})w^{(c)}}{\Delta t}, \\
T^{(c+1)} = \frac{\theta_{T} T^{(c+1)} + (1 - \theta_{T})T^{(c)}}{\Delta t},
\end{align*}
\]

where $c+1$ and $c$ represent current and previous time levels, respectively. Also, $w^{(c)} = w(x, y, t_{c}), T^{(c)} = T(x, y, t_{c}), t_{c} = c\Delta t$ and $\Delta t$ is the time step. Two inhomogeneous modified Helmholtz equations are constructed by substituting the above approximations in (11) and (12) into the Equations (9) and (10). Therefore, Equations (8)–(10) can be written in the iterative form as below

\[
\begin{align*}
\nabla^{2} \psi^{(c+1)} &= -w^{(c)}, \\
\nabla^{2} w^{(c+1)} - \lambda_{w}^{2} w^{(c+1)} &= \left(\frac{\theta_{w} - 1}{\theta_{w}}\right) \nabla^{2} w^{(c)} - \lambda_{w}^{2} w^{(c)} + \frac{\rho_{nf} \alpha_{f}}{\mu_{nf} \theta_{w}} \left(\frac{\partial \psi^{(c+1)} \partial w^{(c)}}{\partial y \partial x} - \frac{\partial \psi^{(c+1)} \partial w^{(c)}}{\partial x \partial y}\right) \\
&- RaPr \frac{(\rho \beta)_{nf} \alpha_{f}}{\rho_{nf} \beta_{f} \mu_{nf} \theta_{w}} \frac{\partial T^{(c)}}{\partial x} - Ha^{2}Pr \frac{\partial^{2} \psi^{(c+1)}}{\partial x^{2}}.
\end{align*}
\]
\[ \nabla^2 T^{(c+1)} - \lambda_T^2 T^{(c+1)} = \left( \frac{\theta_T - 1}{\theta_T} \right) \nabla^2 T^{(c)} - \lambda_T^2 T^{(c)} + \alpha \frac{\alpha_{nf}}{\alpha_{nf} \theta_T} \left( \frac{\partial \psi^{(c+1)} + \partial T^{(c)}}{\partial y} - \frac{\partial \psi^{(c+1)} + \partial T^{(c)}}{\partial x} \right), \]

where \( \lambda_T^2 = \rho_{nf} \alpha_T / \mu_{nf} \Delta T_{nf} \) and \( \lambda_T^2 = \alpha_{nf} \Delta t_{\theta_T} \), and \( c \) indicates iteration number.

### 3. DRBEM implementation

The DRBEM is a numerical solution procedure which enables to convert the partial differential equations defined on the domain into the boundary integral equations defined on the boundary using the fundamental solution of the differential equation. But, generally, the fundamental solutions of differential equations are not available. In the DRBEM idea, the available fundamental solution is used only for a piece of the governing equation and remaining terms are considered as a non-homogeneous term.

In this study the governing equations (Equations (13)–(15)) are written in the forms of Poisson and modified Helmholtz equations as

\[ \nabla^2 \psi^{(c+1)} = b_1^{(c)}, \quad (16) \]
\[ \nabla^2 w^{(c+1)} - \lambda_w^2 w^{(c+1)} = b_2^{(c)}, \quad (17) \]
\[ \nabla^2 T^{(c+1)} - \lambda_T^2 T^{(c+1)} = b_3^{(c)}, \quad (18) \]

where

\[ b_1^{(c)} = -w^{(c)}, \]
\[ b_2^{(c)} = \left( \frac{\theta_w - 1}{\theta_w} \right) \nabla^2 w^{(c)} - \lambda_w^2 w^{(c)} + \frac{\rho_{nf} \alpha_T}{\mu_{nf} \theta_w} \left( \frac{\partial \psi^{(c+1)} + \partial w^{(c)}}{\partial y} - \frac{\partial \psi^{(c+1)} + \partial w^{(c)}}{\partial x} \right) - \frac{R \alpha \beta}{\rho_{nf} \mu_{nf} \theta_w} \nabla T^{(c)} - \frac{H \alpha^2}{\rho_{nf} \mu_{nf} \theta_w} \nabla^2 \psi^{(c+1)}, \]
\[ b_3^{(c)} = \left( \frac{\theta_T - 1}{\theta_T} \right) \nabla^2 T^{(c)} - \lambda_T^2 T^{(c)} + \frac{\alpha_T}{\alpha_{nf} \theta_T} \left( \frac{\partial \psi^{(c+1)} + \partial T^{(c)}}{\partial y} - \frac{\partial \psi^{(c+1)} + \partial T^{(c)}}{\partial x} \right). \]

Here, Equation (13) takes the form of Poisson equation by considering all the terms as inhomogeneity except the Laplace term \( (\nabla^2) \). Similarly, Equations (14) and (15) take the form of modified Helmholtz equation by considering all the terms as inhomogeneity except the modified Helmholtz operators \( (\nabla^2 - \lambda^2) \). Now, Equations (16) is weighted through the domain \( \Omega \) by fundamental solution \( u_{\psi}^* = (1/2\pi) \ln(1/r) \) of Laplace equation, and Equation (17) and (18) are weighted through the domain by fundamental solutions \( u_{w}^* = (1/2\pi) K_0(\lambda w r) \) and \( u_{T}^* = (1/2\pi) K_0(\lambda T r) \) of modified Helmholtz equations, respectively. Here, \( K_0(x) \) is the second kind modified Bessel function of order zero and \( r \) is the distance from an arbitrary point \( i \) to a point on the boundary. After applying the Green’s second identity to the obtained weighted residual statements, the left hand side of the Equations (16)–(18) are transformed to the boundary integrals as

\[ c_i \psi_i^{(c+1)} + \int_{\Gamma} \left( q_{\psi}^* \psi^{(c+1)} - u_{\psi}^* \frac{\partial \psi^{(c+1)}}{\partial n} \right) \, d\Gamma = \int_{\Omega} b_1^{(c)} u_{\psi}^* \, d\Omega, \quad (20) \]
\[ c_i w_i^{(c+1)} + \int_{\Gamma} \left( q_{w}^* w^{(c+1)} - u_{w}^* \frac{\partial w^{(c+1)}}{\partial n} \right) \, d\Gamma = \int_{\Omega} b_2^{(c)} u_{w}^* \, d\Omega, \quad (21) \]
\[ c_i T_i^{(c+1)} + \int_{\Gamma} \left( q_{T}^* T^{(c+1)} - u_{T}^* \frac{\partial T^{(c+1)}}{\partial n} \right) \, d\Gamma = \int_{\Omega} b_3^{(c)} u_{T}^* \, d\Omega. \quad (22) \]

Here \( \Gamma \) is the boundary of the domain \( \Omega \), and \( q_{\psi}^* \)’s are the normal derivatives of variables \( q_{\psi}^* = \partial u_{\psi}^* / \partial n, R = \psi, w, T, n \) is the outward normal derivative to the boundary, \( c_i \) is a constant defined as \( c_i = \theta_i / 2\pi, \theta_i \) is the integral angle at the source point \( i \).

In order to transform the domain integrals in the Equations (20)–(22) into the boundary integrals, \( b_i \)’s \((i = 1, 2, 3)\) are approximated using the radial basis functions \( f_i \)’s \((f_i = 1 + r_j)\) and \( f_i \)’s \((f_i = r_i^2 \log r_j)\) as

\[ b_1^{(c)} = \sum_{j=1}^{M_B+M_I} \gamma_1 f_j(x, y), \]
\[ b_2^{(c)} = \sum_{j=1}^{M_B+M_I} \gamma_2 f_j(x, y), \]
\[ b_3^{(c)} = \sum_{j=1}^{M_B+M_I} \gamma_3 f_j(x, y), \]

where \( \gamma_1 \)’s \((i = 1, 2, 3)\) are initially unknown coefficients, \( r_j \) is the distance vector from the source to the field points. Here, these approximations are obtained by discretizing the boundary using \( M_B \) constant boundary elements and using \( M_I \) selected interior points. The connections between the radial basis functions and the particular solutions can be given

\[ \nabla^2 \hat{u}_{\psi ij} = f_i, \]
\[ (\nabla^2 - \lambda_w^2) \hat{u}_{wj} = f_j, \]
\[ (\nabla^2 - \lambda_T^2) \hat{u}_{Tj} = f_j. \]
The details about the particular solutions and their normal derivatives can be found in [31–33]. Substituting the Equations (23)–(25) into the Equations (20)–(22), respectively, the right hand sides of these equations have the Laplace and modified Helmholtz operators as

\[
c_iψ_j^{(c+1)} + \int_{Γ} \left( q_ψ^∗ ψ_j^{(c+1)} - u_ψ^∗ \frac{∂ψ_j^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j \int_{Ω} (V^2 ϕ_j) u_ψ^* \, dΩ,
\]

\[
c_iw_j^{(c+1)} + \int_{Γ} \left( q_w^* w_j^{(c+1)} - u_w^* \frac{∂w_j^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j(t) \int_{Ω} (V^2 ω_{wj} - λ_ω^2 ω_{wj}) u_w^* \, dΩ,
\]

\[
c iT_i^{(c+1)} + \int_{Γ} \left( q_T^* T_i^{(c+1)} - u_T^* \frac{∂T_i^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j(t) \int_{Ω} (V^2 ρ_{iT} - λ_ρ^2 ρ_{iT}) u_T^* \, dΩ.
\]

Now, DRBEM idea can be used for the right hand sides of obtained Equations (29)–(31) and so only the boundary integrals are appeared in the governing equations

\[
c_iψ_j^{(c+1)} + \int_{Γ} \left( q_ψ^∗ ψ_j^{(c+1)} - u_ψ^* \frac{∂ψ_j^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j \left[ c_iu_ψ^{ij} + \int_{Γ} \left( q_ψ^* u_{ψj} - u_ψ^* \frac{∂ψ_j}{∂n} \right) \, dΓ \right],
\]

\[
ciw_j^{(c+1)} + \int_{Γ} \left( q_w^* w_j^{(c+1)} - u_w^* \frac{∂w_j^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j(t) \left[ c_0 w_{wj} + \int_{Γ} \left( q_w^* ω_{wj} - u_w^* \frac{∂ω_{wj}}{∂n} \right) \, dΓ \right],
\]

\[
ciT_i^{(c+1)} + \int_{Γ} \left( q_T^* T_i^{(c+1)} - u_T^* \frac{∂T_i^{(c+1)}}{∂n} \right) \, dΓ
\]

\[
= \sum_{j=1}^{M_b+M_t} \gamma_j(t) \left[ c_0 ρ_{iTj} + \int_{Γ} \left( q_T^* ρ_{iTj} - u_T^* \frac{∂ρ_{iTj}}{∂n} \right) \, dΓ \right].
\]

The ultimate form of the governing equations can be written as one global scheme in a matrix form for both boundary and interior nodes as

\[
H_ψψ^{(c+1)} - G_ψ \frac{∂ψ^{(c+1)}}{∂n} = (H_ψ̂U_ψ - G_ψ̂Q_ψ) F^{-1} b_1^{(c)},
\]

\[
H_ww^{(c+1)} + G_w \frac{∂w^{(c+1)}}{∂n} = (H_ŵU_w + G_ŵQ_w) F^{-1} b_2^{(c)},
\]

\[
H_TT^{(c+1)} + G_T \frac{∂T^{(c+1)}}{∂n} = (H_T̂U_T + G_T̂Q_T) F^{-1} b_3^{(c)}.
\]

where all the size of matrices and vectors are \((M_b + M_t) \times (M_b + M_t)\) and \((M_b + M_t) \times 1\), respectively. The entries of the boundary coefficient matrices are given in [31–33] as

\[
\frac{\partial E}{\partial x} = \frac{\partial F^{-1} E}{\partial x}, \quad \frac{\partial E}{\partial y} = \frac{\partial F^{-1} E}{\partial y}
\]

and

\[
w^{(c+1)} = \left[ \frac{\partial F^{-1} }{\partial x} \left( \frac{\partial F^{-1} }{\partial x} ψ^{(c+1)} \right) \right]
\]

\[
+ \frac{\partial F^{-1} }{\partial y} \left( \frac{\partial F^{-1} }{\partial y} ψ^{(c+1)} \right),
\]

where \(E\) represents \(ψ, w\) or \(T\). Thus, the matrix-vector form of the Equations (35)–(37) can be solved iteratively. The procedure starts with the assignment of initial condition of vorticity and temperature at the whole domain, as \(w^{(0)} = 0\) and \(T^{(0)} = 0\). The solution procedure can be described as follows
Figure 2. Grid independency for \( Ra = 10^5 \), \( Ha = 30 \) and \( \phi = 0.03 \).

**Solution Procedure:**

1. Solve the stream function equation to obtain \( \psi(c+1) \) using the value of \( w(c) \).
2. Solve vorticity transport equation using the values of \( w(c), T(c) \) and \( \psi(c+1) \).
3. Solve the temperature equation using the values of \( T(c) \) and \( \psi(c+1) \).
4. Check the stopping criteria to continue the iterative procedure.
5. Update the values of \( w(c), T(c) \) for the next time level.
6. Continue steps 1–5 at the next time level until the stopping criteria is satisfied for all variables.

**4. Numerical results**

In the present study, DRBEM solution is given for the problem of unsteady natural convection flow of \( Al_2O_3 \)-water based nanofluids which is influenced by a magnetic field. The boundary conditions of \( \psi \) and \( T \) are taken as in Figure 1 and Equation (45) is used for unknown vorticity boundary conditions. The thermophysical properties are used as considered in [17]. All the results are presented by using graphs to show that the effect of \( Ra, Ha \) and \( \phi \) on the fluid behaviour. The stopping criteria for the computations is

\[ \max_s \left| E^{(c+1)} - E^{(c)} \right| \leq 10^{-5}, \quad s = 1, \ldots, M_B + M_T. \]  

where \( E \) represents \( \psi, w \) or \( T \). The local Nusselt number of the nanofluid along the hot wall surface can be expressed as [17]

\[ Nu_j(y) = -\frac{k_n \partial T}{k_f \partial x} \bigg|_{x=0}. \]  

The average Nusselt number is calculated by integrating local Nusselt number over the heat source

\[ Nu_m = \int_0^1 Nu_j(y) \, dy. \]

**Table 1.** The time increment \( \Delta t \) for \( \phi = 0.04 \) and \( M_B = 200 \).  

| \( Ra \) | \( \Delta t \) |
|-------|------|
| \( 10^3 \) | \( 4.998 \) |
| \( 10^4 \) | \( 4.738 \) |
| \( 10^5 \) | \( 5.309 \) |
| \( 10^6 \) | \( 4.896 \) |

**Table 2.** Comparison of the present results with that of Ghasemi et al. [17]  

| \( Ha \) | \( Nu_m \) | \( Nu_m \) | \( Nu_m \) | \( Nu_m \) |
|-------|------|------|------|------|
| 0     | 4.998 | 4.738 | 5.309 | 4.896 |
| 15    | 11.233 | 11.053 | 11.667 | 11.561 |
| 30    | 4.276 | 4.143 | 4.491 | 4.211 |
| 60    | 2.308 | 2.369 | 2.415 | 2.415 |

First, grid independency test is done and results are given in Figure 2. From the figure it can be concluded that \( M_B = 200 \) constant boundary elements satisfy the grid independency. So, it is used in all analyses done in the rest of the study.

In Table 1 we present the necessary time increments for reaching steady state solutions of the problem for the values of Rayleigh number \( (10^2 \leq Ra \leq 10^6) \) and for the values of Hartman number \( (0 \leq Ha \leq 75) \) by taking fixed volume fraction \( \phi = 0.04 \). From the table it can be said that as \( Ra \) or \( Ha \) increases we need to use smaller \( \Delta t \) in computations. Code validation has been presented in Table 2 for the steady state results and results are good agreement with that of [17].

In Figure 3, in order to determine when the solutions achieve the steady-state, streamline, vorticity and temperature along the horizontal \( y = 0.5 \) \( (0 \leq x \leq 1) \) are displayed at fixed \( Ra = 10^5 \), \( Ha = 0 \) and \( \phi = 0.03 \) for several time levels \( (0.05 \leq t \leq 1) \). From the figure it can
be conclude that after $t \geq 0.4$ the values of all variables ($\psi, w, T$) do not change and which means that the state results are obtained for all variables. Thus, in the rest of the study, the computations are carried out up to the time level $t = 1$ which is enough since the steady state has been already achieved.

The effects of Hartmann and Rayleigh numbers on the streamlines, vorticity contours and isotherms are
displayed in Figures 4, 5 and 6, respectively. From the Figure 4 it is observed that the circular flow behaviour can be seen for lower values of Rayleigh and Hartmann numbers. As the Rayleigh number increases, the circular shape of the streamlines becomes an ellipse while as the Hartmann number increases, the shape of the streamlines is elongated vertically. Also, for higher values of Hartmann and Rayleigh numbers, the central vortex of streamlines tends to become diagonal and the boundary layers occur near the vertical walls. From the vorticity contours presented in Figure 5, it can be concluded that for lower values of Rayleigh number, the vorticity action occurs by forming a vortex at the centre of the cavity. As Rayleigh number increases, this primary vortex starts to split into two new vortices and these newly occur vortices move towards the left bottom and the right upper corners. As Hartmann number increases, this primary vortex is separated into two new vortices and then these vortices are located near the top and the bottom walls. The effect of Rayleigh and Hartmann numbers on isotherms are given in Figure 6. When $Ra = 10^3$ and $Ha = 0$, the behaviour of the isotherms looks like a vertical line. When Rayleigh number increases, isotherms became parallel to the top and the bottom walls while when Hartmann number increases, isotherms are almost parallel to the left and right walls. Also, the magnitude of all variables ($\psi$, $w$ and $T$) increases as Rayleigh number increases, and decreases as Hartmann number increases. All these results are to be expected because under the low Rayleigh number condition there is not
Figure 6. Isotherms for several values of Rayleigh number \(10^3 \leq Ra \leq 10^6\) and Hartmann number \(0 \leq Ha \leq 60\) for \(\phi = 0.03\).

enough convection in the system. The viscous forces are dominating the system and the buoyancy effect induces a weak flow perturbation effect. In other words, the conduction mechanism dominates the heat transfer performance. When the Rayleigh number takes the value between \(10^3 \leq Ra \leq 10^4\), both conduction and convection govern the heat transfer mechanism in the system. But for the greater Rayleigh number, because of the stronger buoyancy effect, heat transfer performance is dominated by the convection mechanism. On the other hand, the magnetic field suppress the heat transfer between the hot and cold wall. Thus, when the magnetic field intensity gets stronger, the effect of buoyancy force is reduced and the system becomes governed by the conduction.

In Figures 7 and 8, the effects of Hartmann and Rayleigh numbers on \(y\)-velocity and temperature along the \(y = 0.5\) are given, respectively. The extremum values of the \(y\)-velocity are observed for \(10^3 \leq Ra \leq 10^6\) and \(0 \leq Ha \leq 60\) in Figure 7. When the Hartmann number increases, due to the suppressive effect of the magnetic field, a decrease takes place at the magnitude of the \(y\)-velocity. Besides this, depending on the influence of the intense buoyancy force caused by the increase in the Rayleigh number an increase occurs at the magnitude of the \(y\)-velocity. From the temperature profiles depicted in Figure 8, it can be said that the most noticeable effect of Hartmann number is at \(Ra = 10^5\). This effect does not occur when the \(Ra = 10^3\), and it starts at the \(Ra = 10^4\). However, when \(Ra = 10^6\),
Hartmann number loses its influence on the temperature profile.

Figure 9 shows the influences of the Hartmann number and particle volume fraction on stream function, vorticity and temperature along \( x = 0.5 \). The variation of particle volume fraction depending on the Hartmann number shows different effects on the stream function profile. At \( Ha = 0 \), the magnitude of the stream function increases as the particle volume fraction increases. But as the Hartmann number increases 0 to 30, the particle volume fraction has an insignificant influence on the streamline. Besides this, when the Hartmann number achieve the higher values (\( Ha = 60 \)), the effect of particle volume fraction on the streamline changes. In this case, the magnitude of the stream function decreases as the particle volume fraction increases. Also, there is a reduction on the value of the streamlines as Hartmann number increase from 0 to 60.

From the vorticity profiles it can be seen that the particle volume fraction affect the magnitude of the vorticity at the absence of magnetic field, however, after increasing the Hartmann number this effect disappears. Conversely, for the lower values of Hartmann number, the temperature profile is not influenced by the changing of the particle volume fraction. But, for higher values of Hartmann number, as the value of particle volume fraction increases, the temperature increases at the left bottom corner and decreases at right upper corner.

In Figure 10, the variation of average Nusselt numbers with the Hartmann numbers (\( 0 \leq Ha \leq 75 \)) are given for several values of Rayleigh number (\( 10^3 \leq Ra \leq 10^6 \)) at \( \phi = 0.03 \). For lower values of Rayleigh number (\( Ra = 10^3 \)) the heat transfer mechanism is dominated by conduction. So, heat transfer performance is not influenced by the increased magnetic field strength. Consequently, the average Nusselt number does not affected by the increasing values of Hartmann number. Besides this, it is observed that, for higher values of Rayleigh number, there is a reduction on the values of average Nusselt number as the Hartmann number increases. This result is expected because when the Hartmann number increases although Rayleigh number takes the higher values (\( Ra = 10^6 \)) the magnetic field suppress the convection flow.

The variation of average Nusselt number ratio (\( \frac{Num}{Num, Ha=0} \)) with the Rayleigh numbers (\( 10^3 \leq Ra \leq 10^6 \)) are given for the values of Hartmann numbers between 0 to 75 by keeping the particle volume fraction fixed as \( \phi = 0.03 \) in Figure 11. From the figure it can be said that, when the \( Ra = 10^3 \) the values of Nusselt number ratio almost the same for all values of the Hartmann number. Additionally, the effect of the Rayleigh
number on the ratio can be seen very clear as Hartmann number increases. Especially, each graph takes minimum value in a different Rayleigh number and the minimum values of the ratio decreases as Hartmann number increases.

5. Conclusion

In this study, we consider the numerical solution of two dimensional, unsteady natural convection problem in a square enclosure filled with Al₂O₃-water nanofluid. DRBEM has been used as a solution procedure and results are given for a range of Rayleigh number $10^3 \leq Ra \leq 10^6$, Hartmann number $0 \leq Ha \leq 60$ and particle volume fraction $0 \leq \phi \leq 0.2$. In this study, the unsteady governing equations are obtained by adding time derivatives of velocity components to the momentum equation and by adding time derivative of temperature to the energy equation. So, the need of extra time integration scheme occurs for the solution procedure. But this problem is eliminated by transforming the governing equations to the modified Helmholtz equation. Therefore all the results are obtained with this advantage. The effects of the parameters on the flow behaviour and temperature field are presented graphically. The results of this study can be summarized as follows:

- When the Rayleigh number increases, heat transfer performance is dominated by the convection mechanism due to the stronger buoyancy effect. Conversely, when the Hartmann number increases, the effect of buoyancy force is reduced and system is governed by the conduction. So, the magnetic field has a negative effect on the flow behaviour.
- Most noticeable effect of Hartmann number over the temperature profile is observed at $Ra = 10^5$. But Hartmann number becomes lose this effect over the system when $Ra = 10^6$ because of the intense buoyancy force.
- The effect of the solid volume fraction on the fluid behaviour is strongly related with the values of the Hartman number. When the Hartmann number increases from 10 to 60, both the magnitude of streamlines and the magnitude of vorticity decrease. But the magnitude of temperature changes from left side of the cavity to the right side of the cavity. Similar with the other variables it decreases at the right side.
Figure 9. Variation of stream function, vorticity and temperature (from left to right) along $x = 0.5$ at various particle volume fraction ($0 \leq \phi \leq 0.2$) when $Ra = 10^5$, for $Ha = 0$, 30 and 60 from top to bottom.

Figure 10. Variation of average Nusselt number with several Hartmann number ($0 \leq Ha \leq 75$) for $\phi = 0.03$.

Figure 11. Variation of average Nusselt number ration with several Rayleigh number ($10^3 \leq Ra \leq 10^6$) for $\phi = 0.03$. 
of the cavity but it increases at the left side of the cavity.

In this study, since the unsteady natural convection of nanofluid equation is solved, the solution can be obtained at any required time level. The steady state solution can be obtained as $t \to \infty$. Thus, the obtained solutions can be compared with the solution of the steady state natural convection of nanofluid equation to check the accuracy of the results. The results which is obtained in this study are good agreement with the results for steady problem in [17,18].

Disclosure statement
No potential conflict of interest was reported by the author.

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