A structure theorem on non-homogeneous linear equations in Hilbert spaces

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Abstract: A very particular by-product of the result announced in the title reads as follows: Let \((X, \langle \cdot, \cdot \rangle)\) be a real Hilbert space, \(T: X \to X\) a compact and symmetric linear operator, and \(z \in X\) such that the equation \(T(x) - \|T\| x = z\) has no solution in \(X\). For each \(r > 0\), set \(\gamma(r) = \sup_{x \in S_r} J(x)\), where \(J(x) = \langle T(x) - 2z, x \rangle\) and \(S_r = \{ x \in X : \|x\|^2 = r \}\). Then, the function \(\gamma\) is \(C^1\), increasing and strictly concave in \([0, +\infty[\), with \(\gamma'(0, +\infty[) = \|T\|, +\infty\]; moreover, for each \(r > 0\), the problem of maximizing \(J\) over \(S_r\) is well-posed, and one has

\[ T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z \]

where \(\hat{x}_r\) is the only global maximum of \(J|_{S_r}\).

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Here and in the sequel, \((X, \langle \cdot, \cdot \rangle)\) is real Hilbert space. For each \(r > 0\), set

\[ S_r = \{ x \in X : \|x\|^2 = r \} \]

In [1], we established the following result (with the usual conventions \(\sup \emptyset = -\infty\), \(\inf \emptyset = +\infty\)):

THEOREM A ([1], Theorem 1). - Let \(J : X \to \mathbb{R}\) be a sequentially weakly continuous \(C^1\) functional, with \(J(0) = 0\).

Set

\[ \rho = \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \]

and

\[ \sigma = \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} \].

Let \(a, b\) satisfy

\[ \max\{0, \rho\} \leq a < b \leq \sigma \].

Assume that, for each \(\lambda \in ]a, b[\), the functional \(x \to \lambda \|x\|^2 - J(x)\) has a unique global minimum, say \(\hat{y}_\lambda\). Let \(M_a\) (resp. \(M_b\) if \(b < +\infty\) or \(M_b = \emptyset\) if \(b = +\infty\)) be the set of all global minima of the functional \(x \to a \|x\|^2 - J(x)\) (resp. \(x \to b \|x\|^2 - J(x)\) if \(b < +\infty\)). Set

\[ \alpha = \max \left\{ 0, \sup_{x \in M_b} \|x\|^2 \right\} \],
\[ \beta = \inf_{x \in M_a} \|x\|^2 \]

and, for each \( r > 0 \),

\[ \gamma(r) = \sup_{x \in S_r} J(x) \]

Finally, assume that \( J \) has no local maximum with norm less than \( \beta \).

Then, the following assertions hold:

(\( a_1 \)) the function \( \lambda \to g(\lambda) := \|\hat{y}_\lambda\|^2 \) is decreasing in \( ]a, b[ \) and its range is \( ]\alpha, \beta[ \);

(\( a_2 \)) for each \( r \in ]\alpha, \beta[ \), the point \( \hat{x}_r := \hat{y}_{g^{-1}(r)} \) is the unique global maximum of \( J_{|S_r} \) and every maximizing sequence for \( J_{|S_r} \) converges to \( \hat{x}_r \);

(\( a_3 \)) the function \( r \to \hat{x}_r \) is continuous in \( ]\alpha, \beta[ \);

(\( a_4 \)) the function \( \gamma \) is \( C^1 \), increasing and strictly concave in \( ]\alpha, \beta[ \);

(\( a_5 \)) one has

\[ J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r \]

for all \( r \in ]\alpha, \beta[ \);

(\( a_6 \)) one has

\[ \gamma'(r) = g^{-1}(r) \]

for all \( r \in ]\alpha, \beta[ \).

We want to remark that, in the original statement of [1], one assumes that \( X \) is infinite-dimensional and that \( J \) has no local maxima in \( X \setminus \{0\} \). These assumptions come from [2] whose results are applied to get \((a_3), (a_4)\) and \((a_5)\). The validity of the current formulation just comes from the proofs themselves given in [2] (see also [3]).

The aim of this very short paper is to show the impact of Theorem A in the theory of non-homogeneous linear equations in \( X \).

So, throughout the sequel, \( z \) is a non-zero point of \( X \) and \( T : X \to X \) is a continuous linear operator.

We are interested in the study of the equation

\[ T(x) - \lambda x = z \]

for \( \lambda > \|T\| \). In this case, by the contraction mapping theorem, the equation has a unique non-zero solution, say \( \hat{v}_\lambda \). Our structure result just concerns such solutions.

As usual, we say that:
- \( T \) is compact if, for each bounded set \( A \subset X \), the set \( \overline{T(A)} \) is compact;
- \( T \) is symmetric if

\[ \langle T(x), u \rangle = \langle T(u), x \rangle \]

for all \( x, u \in X \).

We also denote by \( V \) the set (possibly empty) of all solutions of the equation

\[ T(x) - \|T\| x = z \]
and set 
\[ \theta = \inf_{x \in V} \|x\|^2. \]

Of course, \( \theta > 0 \).

Our result reads as follows:

**THEOREM 1.** - Assume that \( T \) is compact and symmetric.

For each \( \lambda > \|T\| \) and \( r > 0 \), set 
\[ g(\lambda) = \|\hat{v}_\lambda\|^2 \]

and 
\[ \gamma(r) = \sup_{x \in S_r} J(x) \]

where 
\[ J(x) = \langle T(x) - 2z, x \rangle. \]

Then, the following assertions hold:

\( b_1 \) the function \( g \) is decreasing in \( \|T\|, +\infty[ \) and 
\[ g(\|T\|, +\infty[) = 0, \theta[ ; \]

\( b_2 \) for each \( r \in ]0, \theta[ \), the point \( \hat{x}_r := \hat{v}_{g^{-1}(r)} \) is the unique global maximum of \( J|_{S_r} \) and every maximizing sequence for \( J|_{S_r} \) converges to \( \hat{x}_r ; \)

\( b_3 \) the function \( r \to \hat{x}_r \) is continuous in \( ]0, \theta[ ; \)

\( b_4 \) the function \( \gamma \) is \( C^1 \), increasing and strictly concave in \( ]0, \theta[ ; \)

\( b_5 \) one has 
\[ T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z \]

for all \( r \in ]0, \theta[ ; \)

\( b_6 \) one has 
\[ \gamma'(r) = g^{-1}(r) \]

for all \( r \in ]0, \theta[ . \)

Before giving the proof of Theorem 1, we establish the following

**PROPOSITION 1.** - Let \( T \) be symmetric and let \( J \) be defined as in Theorem 1.

Then, for \( \hat{x} \in X \), the following are equivalent:

(i) \( \hat{x} \) is a local maximum of \( J \).

(ii) \( \hat{x} \) is a global maximum of \( J \).

(iii) \( T(\hat{x}) = z \) and \( \sup_{x \in X} \langle T(x), x \rangle \leq 0 \).

**PROOF.** First, observe that, since \( T \) is symmetric, the functional \( J \) is Gâteaux differentiable and its derivative, \( J' \), is given by 
\[ J'(x) = 2(T(x) - z) \]
for all $x \in X$ ([4], p. 235). By the symmetry of $T$ again, it is easy to check that, for each $x \in X$, the inequality
\[ J(\hat{x} + x) \leq J(\hat{x}) \]  
(1)
is equivalent to
\[ (2(T(\hat{x}) - z) + T(x), x) \leq 0 . \]  
(2)
Now, if (i) holds, then $J'(\hat{x}) = 0$ (that is $T(\hat{x}) = z$) and there is $\rho > 0$ such that (1) holds for all $x \in X$ with $\|x\| \leq \rho$. So, from (2), we have $(T(x), x) \leq 0$ for the same $x$ and then, by linearity, for all $x \in X$, getting $(iii)$. Vice versa, if $(iii)$ holds, then (2) is satisfied for all $x \in X$ and so, by (1), $\hat{x}$ is a global maximum of $J$, and the proof is complete. △

Proof of Theorem 1. For each $x \in X$, we clearly have
\[ J(x) \leq \|T(x) - 2z\||x| \leq \|T\||x|^2 + 2\|z\||x| \]
and so
\[ \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \leq \|T\| \]  
(3)
Moreover, if $v \in X \setminus \{0\}$ and $\mu \in \mathbb{R} \setminus \{0\}$, we have
\[ \frac{J(\mu v)}{\|\mu v\|^2} \geq -2 \frac{\langle z, v \rangle}{\mu \|v\|^2} - \|T\| \]
and so
\[ \limsup_{x \to 0} \frac{J(x)}{\|x\|^2} = +\infty . \]  
(4)
Moreover, the compactness of $T$ implies that $J$ is sequentially weakly continuous ([4], Corollary 41.9). Now, let $\lambda \geq \|T\|$. For each $x \in X$, set
\[ \Phi(x) = \|x\|^2 . \]
Then, for each $x, v \in X$, we have
\[ \langle \lambda \Phi'(x) - J'(x) - (\lambda \Phi'(v) - J'(v)), x - v \rangle = \langle 2\lambda(x - v) - 2(T(x) - T(v)), x - v \rangle \geq 2\lambda\|x - v\|^2 - 2\|T(x) - T(v)\|\|x - v\| \geq 2(\lambda - \|T\|)\|x - v\|^2 . \]  
(5)
From (5) we infer that the derivative of the functional $\lambda \Phi - J$ is monotone, and so the functional is convex. As a consequence, the critical points of $\lambda \Phi - J$ are exactly its global minima. So, $\hat{v}_\lambda$ is the only global minimum of $\lambda \Phi - J$ if $\lambda > \|T\|$ and $V$ is the set of all global minima of $\|T\|\Phi - J$. Now, assume that $J$ has a local maximum, say $w$. Then, by Proposition 1, $w$ is a global minimum of $-J$ and $\sup_{x \in X}(T(x), x) \leq 0$. Since $T$ is symmetric, this implies, in particular, that $\|T\|$ is not in the spectrum of $T$. So, $V$ is a singleton. By Proposition 1 of [1], we have
\[ \|w\|^2 \geq \theta . \]

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In other words, $J$ has no local maximum with norm less than $\theta$. At this point, taking (3) and (4) into account, we see that the assumptions of Theorem A are satisfied (with $a = \|T\|$ and $b = +\infty$, and so $\alpha = 0$ and $\beta = \theta$), and the conclusion follows directly from that result.

Some remarks on Theorem 1 are now in order.

REMARK 1. - Each of the two properties assumed on $T$ cannot be dropped. Indeed, consider the following two counter-examples.

Take $X = \mathbb{R}^2$, $z = (1, 0)$ and $T(t, s) = (t + s, s - t)$ for all $(t, s) \in \mathbb{R}^2$. So, $T$ is compact but not symmetric. In this case, we have

$$\hat{x}_r = (-\sqrt{r}, 0),$$
$$\gamma(r) = r + 2\sqrt{r}$$

for all $r > 0$. Hence, in particular, we have

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = (1, \sqrt{r}) \neq z .$$

That is, $(b_5)$ is not satisfied.

Now, take $X = l_2$, $z = \{w_n\}$, where $w_2 = 1$ and $w_n = 0$ for all $n \neq 2$, and $T(\{x_n\}) = \{v_n\}$ for all $\{x_n\} \in l_2$, where $v_1 = 0$ and $v_n = x_n$ for all $n \geq 2$.

So, $T$ is symmetric but not compact. In this case, we have $\theta = +\infty$ and

$$\gamma(r) = r - 2\sqrt{r}$$

for all $r \geq 4$. Hence, $\gamma$ is not strictly concave in $]0, +\infty[.$

REMARK 2. - Note that the compactness of $T$ serves only to guarantee that the functional $x \to \langle T(x), x \rangle$ is sequentially weakly continuous. So, Theorem 1 actually holds under such a weaker condition.

REMARK 3. - A natural question is: if assertions $(b_1) - (b_6)$ hold, must the operator $T$ be symmetric and the functional $x \to \langle T(x), x \rangle$ sequentially weakly continuous ?

REMARK 4. - Note that if $T$, besides to be compact and symmetric, is also positive (i.e. $\inf_{x \in X} \langle T(x), x \rangle \geq 0$), then, by classical results, the operator $x \to T(x) - \|T\|x$ is not surjective, and so there are $z \in X$ for which the conclusion of Theorem 1 holds with $\theta = +\infty$.

We conclude with an application Theorem 1 to a classical Dirichlet problem.

So, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\lambda_1$ be the first eigenvalue of the problem

$$\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega .
\end{cases}$$

Fix a non-zero continuous function $\varphi : \overline{\Omega} \to \mathbb{R}$. 5
For each \( \mu \in ]0, \lambda_1[ \), let \( u_\mu \) be the unique classical solution of the problem
\[
\begin{aligned}
-\Delta u &= \mu (u + \varphi(x)) \quad \text{in} \; \Omega \\
 u &= 0 \quad \text{on} \; \partial \Omega .
\end{aligned}
\]
Also, set
\[
\psi(\mu) = \int_{\Omega} |\nabla u_\mu(x)|^2 dx
\]
and
\[
\eta(r) = \sup_{u \in U_r} \Phi(u)
\]
where
\[
\Phi(u) = \int_{\Omega} |u(x)|^2 dx + 2 \int_{\Omega} \varphi(x) u(x) dx
\]
and
\[
U_r = \left\{ u \in H^1_0(\Omega) : \int_{\Omega} |\nabla u(x)|^2 dx = r \right\} .
\]
Finally, denote by \( A \) the set of all classical solutions of the problem
\[
\begin{aligned}
-\Delta u &= \lambda_1 (u + \varphi(x)) \quad \text{in} \; \Omega \\
u &= 0 \quad \text{on} \; \partial \Omega
\end{aligned}
\]
and set
\[
\delta = \inf_{u \in A} \int_{\Omega} |\nabla u(x)|^2 dx .
\]
Then, by using standard variational methods, we can directly draw the following result from Theorem 1:

**THEOREM 2.** - The following assertions hold:

(c\(_1\)) the function \( \psi \) is increasing in \( ]0, \lambda_1[ \) and one has
\[
\psi(]0, \lambda_1[) = ]0, \delta[ ;
\]

(c\(_2\)) for each \( r \in ]0, \delta[ \), the function \( w_r := u_{\psi^{-1}(r)} \) is the unique global maximum of \( \Phi_{|U_r} \) and each maximizing sequence for \( \Phi_{|U_r} \) converges to \( w_r \) with respect to the topology of \( H^1_0(\Omega) \);

(c\(_3\)) the function \( r \to w_r \) is continuous in \( ]0, \delta[ \) with respect to the topology of \( H^1_0(\Omega) \);

(c\(_4\)) the function \( \eta \) is \( C^1 \), increasing and strictly concave in \( ]0, \delta[ \);

(c\(_5\)) for each \( r \in ]0, \delta[ \), the function \( w_r \) is the unique classical solution of the problem
\[
\begin{aligned}
-\Delta u &= \frac{1}{\eta'(r)} (u + \varphi(x)) \quad \text{in} \; \Omega \\
u &= 0 \quad \text{on} \; \partial \Omega ;
\end{aligned}
\]

(c\(_6\)) one has
\[
\eta'(r) = \frac{1}{\psi^{-1}(r)}
\]
for all \( r \in ]0, \delta[ \).
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