ON THE TOPOLOGICAL FULL GROUP
OF A MINIMAL CANTOR $\mathbb{Z}^2$-SYSTEM

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Abstract. Grigorchuk and Medynets recently announced that the topological full group of a minimal Cantor $\mathbb{Z}$-action is amenable. They asked whether the statement holds for all minimal Cantor actions of general amenable groups as well. We answer in the negative by producing a minimal Cantor $\mathbb{Z}^2$-action for which the topological full group contains a non-abelian free group.

1. Introduction

Let $G$ be a group acting on a compact space $\Sigma$ by homeomorphisms. The topological full group associated to this action is the group of all homeomorphisms of $\Sigma$ that are piecewise given by elements of $G$, each piece being open. Thus there are finitely many pieces at a time, all are clopen, and this construction is most interesting when $\Sigma$ is a Cantor space. The importance of the topological full group has come to the fore in the classification results of Giordano–Putnam–Skau [2, 3].

Grigorchuk and Medynets announced that the topological full group of a minimal Cantor $\mathbb{Z}$-action is amenable [6]. This is particularly interesting in combination with the work of Matui [8], who showed that the derived subgroup is often a finitely generated simple group. Grigorchuk–Medynets further asked in [6] whether their result holds for actions of general amenable groups as well. We shall prove that it fails already for the group $\mathbb{Z}^2$:

Theorem 1. There exists a free minimal Cantor $\mathbb{Z}^2$-action whose topological full group contains a non-abelian free group.

Three comments are in order, see the end of this note:
1. There also exist free minimal Cantor $\mathbb{Z}^2$-actions whose topological full group is amenable, indeed locally virtually abelian.
2. Minimality is fundamental for the study of topological full groups. Even for $\mathbb{Z}$, it is easy to construct Cantor systems whose topological full group contains a non-abelian free group (using e.g. ideas from [9] or [11]).
3. Our example will be a minimal subshift and in this situation the topological full group is sofic by a result of [1].

2. Proof of the Theorem

We realize the Cantor space as the space $\Sigma$ of all proper edge-colourings of the “quadrille paper” two-dimensional Euclidean lattice by the letters $A, B, C, D, E, F$ (with the topology of pointwise convergence relative to the discrete topology on the finite set of letters). Recall here

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that an edge-colouring is called proper if the edges adjacent to a given vertex are coloured differently. There is a natural $\mathbb{Z}^2$-action on $\Sigma$ by homeomorphisms defined by translations.

To each letter $x \in \{A, \ldots, F\}$ corresponds a continuous involution of $\Sigma$, which we still denote by the same letter. It is defined as follows on $\sigma \in \Sigma$: if the vertex zero is connected to one of its four neighbours $v$ by an edge labelled by $x$, then $v$ is uniquely determined and $x\sigma$ will be the colouring $\sigma$ translated towards $v$ (i.e. the origin is now where $v$ was). Otherwise, $x\sigma = \sigma$. This involution is contained in the topological full group of the $\mathbb{Z}^2$-action.

We have thus a homomorphism from the free product $\langle A \rangle \ast \cdots \ast \langle F \rangle$ to the topological full group. Notice that this free product preserves any $\mathbb{Z}^2$-invariant subset of $\Sigma$. We shall establish Theorem 1 by proving that $\Sigma$ contains a minimal non-empty closed $\mathbb{Z}^2$-invariant subset $M$ on which the $\mathbb{Z}^2$-action is free and on which the action of $\Delta := \langle A \rangle \ast \langle B \rangle \ast \langle C \rangle$ is faithful. This implies the theorem indeed, for $\Delta$ has a (finite index) non-abelian free subgroup.

A pattern of a colouring $\sigma \in \Sigma$ is the isomorphism class of a finite labelled subgraph of $\sigma$. We call $\sigma$ homogeneous if for any pattern $P$ of $\sigma$ there is a number $f(P)$ such that the $f(P)$-neighbourhood of any vertex in the lattice contains the pattern $P$. The following facts are well-known and elementary (see e.g. [5]).

**Lemma 2.** The orbit closure of $\sigma \in \Sigma$ is minimal if and only if $\sigma$ is homogeneous. In that case, any $\tau$ in the orbit closure has the same patterns as $\sigma$ and is homogenous with the same function $f$. 

Now, we first enumerate the non-trivial elements of the free product $\Delta$. Then, we label the integers with the natural numbers in such a way that the following property holds: for each $i \in \mathbb{N}$ there is $g(i) \geq 1$ such that any subinterval of length $g(i)$ in $\mathbb{Z}$ contains at least one element labelled by $i$. Such a labelling exists: for instance, label an integer by the exponent of 2 in its prime factorization (with an arbitrary adjustment for 0).

We use the labelling above to construct a specific proper edge-colouring $\lambda \in \Sigma$. Let $w$ be a word in $\Delta$ that is the $i$-th in the enumeration. Consider the vertical vertex-lines $(v, \cdot)$ in the lattice such that $v$ is labelled by $i$. Colour those vertical lines the following way. Starting at the point $(v, 0)$, copy the string $w$ onto the half-line above, beginning from the right end of $w$ (i.e. write $w^{-1}$ upwards). Then colour the following edge by $D$, then copy the string $w$ again and repeat the process ad infinitum. Also, continue the process below $(v, 0)$ so as to obtain a periodic colouring of the whole vertical line. Repeating the process for all non-trivial words $w$, we have coloured all vertical lines. Finally, colour all horizontal lines periodically with $E$ and $F$.

The resulting colouring $\lambda$ has the following property. For any non-trivial $w \in \Delta$ there is a number $h(w)$ such that the $h(w)$-neighbourhood of any vertex of the lattice contains a vertical string of the form $w^{-1}D$. Let $\Omega(\lambda) \subseteq \Sigma$ be the $\mathbb{Z}^2$-orbit closure of $\lambda$. Then all the elements of $\Omega(\lambda)$ have the same property. Now, let $M$ be an arbitrary minimal subsystem of $\Omega(\lambda)$ (in fact it is easy to see that $\lambda$ is homogeneous and hence $\Omega(\lambda)$ is already minimal). Notice that the $\mathbb{Z}^2$-action on $M$ is free because $\lambda$ has no period. In order to prove the theorem, it is enough to show that for any $\sigma$ in $M$ and any non-trivial $w \in \Delta$ there exists a $\mathbb{Z}^2$-translate of $\sigma$ which is not fixed by $w$.

Pick thus any $\sigma \in M$. Then, by the above property of the orbit closure, there exists a translate $\tau$ of $\sigma$ such that the vertical half-line pointing upwards from the origin starts with
the string $w^{-1} D$. Hence if we apply $w$ to the translate we reach a point $\tau$ such that the colour of the edge pointing upwards from the origin is coloured by $D$. Thus $\tau$ is not fixed by $w$, finishing the proof. $\square$

3. Comments

Some $\mathbb{Z}^2$-systems have a completely opposite behaviour to the ones constructed for Theorem 1. We shall see this by extending the method of Proposition 2.1 in [7].

Recall that the $p$-adic odometer is the minimal Cantor system given by adding 1 in the ring $\mathbb{Z}_p$ of $p$-adic integers. Taking the direct product, we obtain a minimal Cantor $\mathbb{Z}^2$-action on $\Sigma := \mathbb{Z}_p \times \mathbb{Z}_p$. The proposition below and its proof can be immediately extended to products of more general odometers.

**Proposition 3.** The full group of this minimal Cantor $\mathbb{Z}^2$-system is an increasing union of virtually abelian groups.

**Proof (compare [7]).** Consider $\mathbb{Z}_p$ as the space of $\mathbb{Z}/p\mathbb{Z}$-valued (infinite) sequences. Given a pair of finite sequences of length $n$, we obtain an $n$-cylinder set in $\Sigma$ as the space of pairs of sequences starting with the given prefixes. Thus, $n$-cylinders determine a partition $\mathcal{P}_n$ of $\Sigma$ into $p^{2n}$ clopen subsets. Moreover, the clopen partition associated to any given element $g$ of the topological full group can be refined to $\mathcal{P}_n$ when $n$ is large enough. It remains only to observe that the collection of all such $g$, when $n$ is fixed, is a subgroup of the semi-direct product $(\mathbb{Z}^2)^{\mathcal{P}_n} \rtimes \text{Sym}(\mathcal{P}_n)$, where $\text{Sym}(\mathcal{P}_n)$ is the permutation group of the coordinates indexed by $\mathcal{P}_n$. $\square$

Regarding the second comment of the introduction, suffice it to say that a generic proper colouring of the linear graph by three letters $A, B, C$ gives a faithful non-minimal representation of the free product $\langle A \rangle \ast \langle B \rangle \ast \langle C \rangle$ into the topological full group of the associated $\mathbb{Z}$-subshift (compare [9] or [4] for generic constructions).

As for the last comment, Proposition 5.1(1) in [1] implies that the topological full group of any minimal subshift of any amenable group is a sofic group (in the notations of [1], the kernel $N$ is trivial by an application of Lemma 2). In combination with Matui’s results [8], this already shows the existence of a sofic finitely generated infinite simple group without appealing to [6].

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