Non-asymptotic tails estimations for sums of random vectors 
having moderate decreasing tails.

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Abstract.

We derive the sharp non-asymptotical uniform estimations for tails of distributions for classical normed sums of centered normed independent random vectors having a moderate decreasing individual tails of summands.

Key words and phrases: Random variable and vector (r.v.), Banach space, space of continuous functions, Monte Carlo method, ordinary and moderate tail of distribution, Lebesgue - Riesz and Grand Lebesgue Spaces, generating function, natural function and distance, slowly varying at infinity function, normed sum; probability, expectation and variance; metric, entropy and entropy integral, semi-distance, Young - Fenchel transform, non-asymptotic estimate, example.

1 Statement of problem.

Let $B = (B, \|\cdot\|_B)$ be separable Banach space equipped with the norm $\|\cdot\|_B$; we will concentrate our attention on the space of continuous numerical valued functions
defined on certain compact metric space. We will write for the case when $B$ is 
or
ordinary real line $||x||B = |x|$ – the ordinary absolute value.

Let $(\Omega, B, P)$ with expectation $E$ and variance $Var$ be certain probability
space and let also $\xi$ be a centered in the weak sense (mean zero) random variable (vector) (r.v.)

$$\forall y \in B^* \Rightarrow E_y(\xi) = 0,$$

with values in the space $B : P(\xi \in B) = 1$ and define by $\xi_i$ the independent
copies of $\xi$.

Let us assume that the r.v. $\xi$ has a finite weak second moment:

$$\forall y \in B^* \Rightarrow \text{Var} [y(\xi)] < \infty.$$

Put therefore the normed as ordinary sum

$$S_n := n^{-1/2} \sum_{i=1}^{n} \xi_i, \quad n = 1, 2, 3, \ldots$$

(1)

Further, define for arbitrary such a r.v. $\xi$ its tail function

$$T[\xi](u) \overset{\text{def}}{=} P(||\xi||B > u), \quad u > 0.$$  

(2)

The investigation of tail of distribution of the sequence of the r.v. - s $\{S_n\}$
is the classical theme for the probability theory. We devote this preprint to the
investigation of the case when the r.v.- s $\{\xi_i\}$ have a so - called moderate tails of
distribution.

**Definition 1.1.** We will say that the r.v. $\xi$ has a moderate decreasing tail of
distribution, write $\text{Law}(\xi) \in \text{MDT}$, or equally write more detail

$$\xi \in \text{MDT}(\beta, \gamma, V),$$

if for some constants $\beta = \text{const} > 2$, $\gamma = \text{const} \in \mathbb{R}$, and for some positive continuous slowly varying at infinity function $V = V(y), y \in (1, \infty)$

$$T[\xi](u) \leq u^{-\beta} \ln^\gamma u \frac{V}{V(\ln u)}, \quad u \geq e.$$  

(3)

Our claim in this preprint is an exact uniform tail estimation for nat-
ural normed sums of random vectors having a moderate decreasing tails
of distributions:

$$Q(u) = Q[\xi](u) \overset{\text{def}}{=} \sup_n T[S_n](u), \quad u \geq e,$$

(4)

where $\xi \in \text{MDT}(\beta, \gamma, V)$. 


Notice, that it follows from the condition (3), in particular, \( \beta > 2 \), that \( \text{Var}(\xi) < \infty \); therefore the norming sequence \( 1/\sqrt{n} \) in (1) is natural.

Of course, these estimations may be used as ordinary in the statistics and in the method Monte-Carlo, see e.g. [3], [4]. In detail, let \( a \) be some unknown vector in the space \( B \) such that for certain r.v. \( \zeta \) having a distribution \( \mu : P(\zeta \in A) = \mu(A) \) for any Borelian set \( A \) from the whole space \( B \) there holds
\[
a = \mathbf{E}g(\zeta) = \int_B g(w) \mu(dw).
\]

Let \( \zeta_i, i = 1, 2, \ldots \) be independent copies \( \zeta \). The consistent a.e. as \( n \to \infty \) in the norm || \cdot ||B estimate of the value \( a \) has a form
\[
a_n := n^{-1} \sum_{i=1}^{n} g(\zeta_i).
\]

Here \( \xi_i = g(\zeta_i) - a \), and the estimation of the variable \( Q[g(\zeta)](u) \) as \( u \to \infty \) may be used for the building of non-asymptotical confidence region for the value \( a \) in the norm || \cdot ||B.

Roughly speaking, we will ground that if the source random variable \( \xi \) has the moderate decreasing tail of distribution, then under appropriate conditions the r.v. - s \( S_n \) have also the moderate decreasing tail of distribution uniformly relative the parameter \( n \).

## 2 One dimensional case.

### Some facts from the theory of Grand Lebesgue Spaces (GLS).

Define as ordinary for arbitrary numerical valued r.v. \( \zeta \) its Lebesgue-Riesz \( L_p \) norm
\[
||\zeta||_p \equiv [\mathbf{E}|\zeta|^p]^{1/p}, \quad p \in [1, \infty),
\]
\[
||\zeta||_{\infty} \equiv \text{vraisup}_{\omega \in \Omega}|\eta(\omega)|.
\]

Recall, see e.g. [1], [2], [6], [7], [8], [9], [10], [14], [15], [16], [17], [18], [19], [20] and so one, that the so-called Grand Lebesgue Space (GLS) \( G\psi = G\psi(b), \quad b = \text{const} \in (2, \infty] \) equipped with the norm \( ||\zeta||G\psi \) of the r.v. \( \zeta \) is defined as follows:
\[
G\psi = \{ \zeta : \Omega \to R, ||\zeta||G\psi < \infty, \} \quad \text{where}
\]
\[
||\zeta||G\psi \equiv \sup_{p \in [2, b]} \left[ \frac{||\zeta||_p}{\psi(p)} \right].
\]
Here \( \psi = \psi(p), \ p \in [2, b) \) is bounded from below measurable function, which is names as ordinary as a generating function for this space. The set of all such a functions will be denoted by \( \Psi \) or more concrete \( \Psi(b) \).

Note that one can consider a more general case \( p \in (1, \infty] \), but in this report only \( p \in [2, b), \ b \in (2, \infty) \).

A very popular class of these spaces form the subgaussian random variables, i.e. for which \( \psi(p) = \psi_2(p) = \sqrt{p}, \ b = \infty \).

More generally, \( \psi(p) = \psi_m(p) := p^{1/m}, \ p \geq 1 \).

Suppose the r.v. \( \zeta \) belongs to the space \( G\psi_m \). The correspondent tail estimate is follow:

\[
\max \left\{ P(\zeta \geq u), \ P(\zeta \leq -u) \right\} \leq \exp \left\{ -\left(\frac{u}{K}\right)^m \right\}, \ u > 0,
\]

and correspondent inverse conclusion also holds true.

These space are used in particular for obtaining of the exponential decreasing tail estimates for sums of random variables, independent or not, see e.g. [14], [18], sections 1.6, 2.1 - 2.5.

For instance, if \( E\xi = 0 \) and for some value \( m = \text{const} > 0 \) and \( b = \infty \)

\[
\max \left\{ P(\xi \geq u), \ P(\xi \leq -u) \right\} \leq \exp \left\{ -u^m \right\}, \ u > 0,
\]

then

\[
\sup_n \max \left\{ P(S(n) \geq u), \ P(S(n) \leq -u) \right\} \leq \exp \left\{ -C(m)u^{\min(m, 2)} \right\}, \ u > 0, \ C(m) \in (0, \infty),
\]

and the last estimate is essentially non-improvable.

Another possibility; the so-called natural function. Let \( \eta \) be a random variable (r.v.) such that

\[
\exists b \in (2, \infty], \ \forall p \in [2, b) \Rightarrow ||\xi||_p < \infty.
\]

By definition, the natural function \( \psi[\xi](p) \) for the r.v. \( \xi \) is defined as follows

\[
\psi[\xi](p) \overset{def}{=} ||\xi||_p, \ p \in [2, b).
\]

Evidently, \( ||\xi|| = G\psi[\xi] = 1 \).

The belonging of some r.v. \( \xi \) to certain GLS \( G\psi, \ \psi \in \Psi \) is closely related with its tail behavior. Indeed, denote \( \nu(p) := p\ln\psi(p) \),

\[
\nu^*(y) := \sup_{p \in [2, b]} (py - \nu(p)), \ y \geq e, -\]

the so-called (regional) Young-Fenchel transform. If the r.v. $0 \neq \xi \in G_\psi$, for definiteness let $||\xi||_{G_\psi} = 1$, then

$$T[\xi](z) \leq \exp \left( -\nu^*(\ln z) \right), \quad z \geq e. \quad (6)$$

Conversely, let the tail function $T[\xi](x)$ for the one-dimensional random variable $\xi$ be given; then

$$E|\xi|^p = p \int_0^\infty x^{p-1} T[\xi](x) \, dx,$$

and we observe

$$||\xi||_{G_\psi} = \sup_p \left\{ \frac{p \int_0^\infty x^{p-1} T[\xi](x) \, dx}{\psi(p)} \right\}^{1/p}. \quad (7)$$

**Examples.**

Assume that the source one-dimensional r.v. $\xi$ has a moderate decreasing tail of distribution, see (3). We intent to evaluate its natural function as $p \in [2, \beta)$, which we will denote by $\kappa(p) = \kappa[\beta, \gamma, V](p)$. Namely,

$$\kappa^p[\beta, \gamma, V](p) - e \overset{def}{=} p^{-1} E|\xi|^p - e \leq \int_e^\infty x^{p-1} \ln^\gamma(x) \, V(\ln x) \, dx.$$

We need in this purpose to introduce the following auxiliary function $\theta[\gamma](p) = \theta[\gamma]_{\beta,V}(p)$ for the values $\gamma \in \mathbb{R}, p \in [2, \beta)$

**A.** $\gamma > -1 \Rightarrow$

$$\theta[\gamma](p) = \theta[\gamma]_{\beta,V}(p) \overset{def}{=} (\beta - p)^{-\gamma-1} V(1/(\beta - p)). \quad (8)$$

**B.** $\gamma = -1 \Rightarrow$

$$\theta[-1](p) = \theta[\gamma](p) = \theta[\gamma]_{\beta,V}(p) \overset{def}{=} |\ln(\beta - p)| \, V(1/(\beta - p)). \quad (9)$$

**C.** $\gamma < -1 \Rightarrow$

$$\theta[\gamma](p) = \theta[\gamma]_{\beta,V}(p) \overset{def}{=} V(1/(\beta - p)). \quad (10)$$

**Proposition 2.1.**

$$\kappa^p(p) = \kappa^p[\beta, \gamma, V](p) \simeq \theta[\gamma]_{\beta,V}(p), \quad p \in [2, \beta). \quad (11)$$

**Proof.** We must consider separately the three cases.
A. \( \gamma > -1 \).

Let the r.v. \( \xi \) be as in (3); we intend to estimate its (absolute) moment \( \mathbb{E}|\xi|^p, p \in [2, \beta) \). We have as \( p \to \beta - 0 \)

\[
\kappa^p(p) \leq \int_{1}^{\infty} x^{p-1-\beta} \ln^\gamma x \ V(\ln x) \ dx = \]

\[
\int_{0}^{\infty} e^{-y(\beta-p)} y^{\gamma} V(y) \ dy = (\beta - p)^{-\gamma-1} \int_{0}^{\infty} e^{-z} z^{\gamma} V(z/(\beta-p)) \ dz \sim \]

\[
\Gamma(\gamma + 1) \ (\beta - p)^{-\gamma-1} V(1/(\beta-p)) = \theta[\gamma](p). \]

where as ordinary \( \Gamma(\cdot) \) is Gamma function. See for the more detail explanation [15], [20].

B. \( \gamma = -1 \). Then as \( p \in [2, \beta), p \to \beta - 0 \)

\[
\kappa^p(p) \sim \int_{e}^{\infty} x^{p-1-\beta} \ln^{-1}(x) \ V(\ln x) \ dx = \int_{1}^{\infty} e^{-y(\beta-p)} y^{-1} V(y) \ dy \sim \]

\[
\int_{C(\beta-p)} e^{-z} z^{-1} V(z/(\beta-p)) \ dz \sim \]

\[
V(1/(\beta-p)) \int_{C(\beta-p)} e^{-z} z^{-1} \ dz \sim V(1/(\beta-p)) \int_{(\beta-p)}^{1} z^{-1} \ dz = \]

\[
V(1/(\beta-p)) |\ln(\beta-p)| = \theta[-1](p). \]

C. \( \gamma < -1 \). Then again as \( p \in [2, \beta), p \to \beta - 0 \)

\[
\kappa^p(p) \sim \int_{e}^{\infty} x^{p-1-\beta} \ln^\gamma x \ V(\ln x) \ dx = \int_{1}^{\infty} e^{-(\beta-p)y} y^{\gamma} V(y) \ dy = \]

\[
(\beta - p)^{-\gamma-1} \int_{(\beta-p)}^{\infty} e^{-z} z^{\gamma} V(z/(\beta-p)) \ dz \sim \]

\[
V(1/(\beta-p))(\beta - p)^{-\gamma-1} \int_{(\beta-p)}^{e} z^{\gamma} \ dz \sim V(1/(\beta-p)) = \theta[\gamma](p). \]

**Theorem 2.1.** Assume that the centered random variable satisfies the inequality (3). Denote \( \tau(p) := \ln \theta[\gamma]_{\beta,V}(p), 2 \leq p < \beta \). Our proposition:

\[
Q[\xi](Cu) \leq \exp \{ -\tau^*(\ln u) \}, \ u \geq e, \ C = C(\gamma, \beta, V) \in (0, \infty). \quad (12) \]

**Proof.** We apply first of all Proposition 2.1:
Further, we will use the famous moment estimations for the sums of the centered independent random variables, see e.g. [5], [12], [13], [16], [29]. Since the interval of the values $p$ is bounded: $2 \leq p < \beta$, one can write

$$\sup_n \mathbb{E}|S_n|^p \leq C_1(\beta, \gamma, V) \cdot \theta[\gamma]_{\beta, V}(p), \ p \in [2, \beta).$$  \tag{14}$$

It remains to apply the estimate (6).

**Examples.**

**I.** Suppose for example that the source one-dimensional centered r.v. $\xi$ has a moderate decreasing tail of distribution, where as above $\beta > 2$, and suppose here that $\gamma > -1$, see (3). Then

$$Q(u) = Q[\xi](u) \overset{def}{=} \sup_n T[S_n](u) \leq C_1(\beta, \gamma, L) \ u^{-\beta} \ \ln^{\gamma+1} u \ V(\ln u), \ u \geq e,$$  \tag{15}

and this estimation is essentially non-improvable.

The non-improvability may be ground by means of consideration of the following example

$$T[\xi](u) = u^{-\beta} \ \ln^{\gamma} u \ V(\ln u), \ u \geq e,$$  \tag{16}

as long as

$$Q[\xi](u) \geq T[\xi](u) = u^{-\beta} \ \ln^{\gamma} u \ V(\ln u), \ u \geq e.$$

More complicated examples may be found in [15], [20].

**II.** Suppose now that $\gamma = -1$:

$$T[\xi](u) \leq u^{-\beta} \ \ln^{-1} u \ V(\ln u), \ u \geq e,$$

then

$$Q[\xi](u) \leq C_2(\beta, V) \ \ln \ln u \ V(\ln u), \ u \geq e^e.$$

**III.** Let now $\gamma < -1$; and suppose in addition that $\lim_{u \to \infty} V(u) = 0$. We state
\[ Q[\xi](u) \leq C_3(\beta, \gamma, V) V(\ln u), \ u \geq e. \]

The possible lower bound in both of last examples is quite alike one in the first example:

\[ Q[\xi](u) \geq T[\xi](u), \ u \geq e. \]

3 Main result. Space of continuous functions.

Let \( Z = \{z\} \) be arbitrary set; the semi-distance function \( \rho = \rho(z_1, z_2), \ z_1, z_2 \in Z \) on this set will be clarified below. Recall that the semi-distance function is non-negative symmetrical function vanishing in the diagonal \( \rho(z, z) = 0 \), satisfying the triangle inequality but in general case the relation \( \rho(z_1, z_2) = 0 \) does not imply that \( z_2 = z_1 \).

Let \( \eta = \eta(z), \ z \in Z \) be separable centered: \( \mathbb{E}\eta(z) = 0 \) numerical valued random field (r.f.). Let also \( \eta_i = \eta_i(z), \ i = 1, 2, \ldots \) be independent copies of \( \eta(z) \).

Put as above

\[ Y_n(z) \overset{\text{def}}{=} n^{-1/2} \sum_{i=1}^{n} \eta_i(z) \]

and

\[ W(u) \overset{\text{def}}{=} \mathbb{P}(||Y_n(\cdot)|| > u), \ u \geq e. \]

Hereafter the symbol \( ||f(\cdot)|| \) denotes an uniform norm of the function \( f : Z \rightarrow R \) equipped with the uniform norm \( ||\cdot|| \) will be denoted as usually by \( C(Z) = C(Z, \rho) \).

Let us suppose that

\[ \sup_{z \in Z} T[\eta(z)](u) \leq u^{-\beta} \ln^\gamma u \ V(\ln u), \ u \geq e - \]

the uniform MDT condition.

Define the following natural generating function

\[ \psi\gamma(p) = \psi[\gamma]_{\beta, V}(p) \overset{\text{def}}{=} \theta^{1/p}[\gamma]_{\beta, V}(p), \ 2 \leq p < \beta, \]

then it follows from (18) that
Let us introduce then the following bounded natural distance, more precisely, semi-distance on the set $Z$

$$\rho(z_1, z_2) \overset{\text{def}}{=} ||\eta(z_1) - \eta(z_2)||_G \psi_{\gamma} \beta, V, z_1, z_2 \in Z. \quad (20)$$

Denote by $H(\epsilon) = H(Z, \rho, \epsilon)$, $0 < \epsilon \leq C_5$, the metric entropy of the whole set (space) $Z$ relative the metric $\rho$, i.e. the (natural) logarithm of the minimal amount of the closed ball in the distance $\rho$, which cover this set $Z$. Set $N(\epsilon) = N(Z, \rho, \epsilon) = \exp H(Z, \rho, \epsilon)$.

**Theorem 3.1.** Suppose that $\gamma > -1$ and that the following entropic integral convergent:

$$I(N) \overset{\text{def}}{=} \int_0^{C_5} N^{(\gamma+1)/\beta}(Z, \rho, \epsilon) \, d\epsilon < \infty. \quad (21)$$

Then the set $Z$ is a pre-compact semi-metric space relative to the distance function $\rho(\cdot, \cdot)$; the random field $\eta(z)$, as well as all the r.f. s $Y_n(z)$ are $\rho$-continuous almost everywhere:

$$P(\eta(\cdot) \in C(Z, \rho)) = P(Y_n(\cdot) \in C(Z, \rho)) = 1 \quad (22)$$

and moreover

$$\sup_n P(\sup_{z \in Z} |Y_n(z)| > u) \leq C_6(\gamma, \beta, V; I(N)) \, u^{-\beta} \ln^{\gamma+1} u \, V(\ln u), \, u \geq e. \quad (23)$$

**Proof.** It follows from the proof of proposition of Theorem 2.1., indeed, we use the inequality (14), that uniformly relative both the parameters $(z, n)$

$$\sup_n E \sup_{z \in Z} |Y_n(z)|^p \leq C_7(\beta, \gamma, V) \cdot \theta[\gamma]_{\beta, V}(p) = C_7 \psi_{\gamma}^{p}(p), \, p \in [2, \beta), \quad (24)$$

or equally

$$\sup_n \sup_{z \in Z} ||Y_n(z)||_p \leq C_8(\beta, \gamma, V) \cdot \psi_{\gamma}(p), \, p \in [2, \beta); \quad (25)$$

$$\sup_n \sup_{z \in Z} ||Y_n(z)||_G \psi_{\gamma} \leq C_8(\beta, \gamma, V) < \infty. \quad (26)$$

We find quite analogously

$$\sup_n \sup_{z_1, z_2 \in Z, \rho(z_1, z_2) > 0} \left[ \frac{||Y_n(z_1) - Y_n(z_2)||_G \psi_{\gamma}}{\rho(z_1, z_2)} \right] \leq C_8(\beta, \gamma, V) < \infty. \quad (27)$$
Both the propositions of theorem 3.1 (22) and (23) follows immediately from Theorem 3.17.1 of monograph [18], chapter 3, section 17; see also an article [28].

**Example 3.1.** Assume that $Z = D$ is bounded convex subset of the whole Euclidean space $R^d$ equipped with ordinary Euclidean norm $|z|, z \in D \subset R^d$. Suppose that the distance $\rho$ is such that

$$\rho(z_1, z_2) \leq C_9 |z_1 - z_2|^\alpha, \quad \alpha = \text{const} \in (0, 1].$$

Then

$$N(Z, \rho, \epsilon) \leq C_{10} \epsilon^{-d/\alpha}, \quad \epsilon \in (0, C).$$

The condition (21) is satisfied iff

$$\frac{\beta}{\gamma + 1} > \frac{d}{\alpha}.$$

4 Concluding remarks.

Note that under conditions of theorem 3.1 the r.f. $Y_n(\cdot)$ not only are continuous a.e., but satisfies the Central Limit Theorem in the space $C(Z, \rho)$. This implies that the distributions of the r.f. $Y_n(z)$ converges as $n \to \infty$ weakly in this space to the distribution of the centered Gaussian r.f. $Y_\infty(z)$ having at the same covariation function as r.f. $\eta(z)$. See [18], chapter 4, section 4.4.

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