QJT as a Regularization:  
Origin of the New Gauge Anomalies 

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Abstract  

QJT is considered as a regularization of QFT, where the fields are replaced by finite $p$-jets. The regularized phase space is infinite-dimensional, because not all histories are determined by initial conditions. Gauge symmetries are not fully preserved by the regularization, and gauge anomalies arise. These anomalies are of a new type, not present in QFT. They generically diverge when the regulator is removed, but can be made finite with a particular choice of field content, provided that spacetime has at most four dimensions. The field content appears to include unphysical fields that violate the spin-statistics theorem.

1 Introduction  

Quantum Jet Theory (QJT) is an approach to quantization of field theories, in which not only the fields but also the observer’s trajectory acquires quantum dynamics [4, 5, 6]. The main idea is to replace every quantum field by a jet, which is essentially the same thing as a Taylor expansion around some point $q$. A Taylor series is observer dependent in the sense that it depends on the choice of expansion point, which can be identified with the observer’s position.

There are good reasons to expect that QJT becomes important in the quantization of gravity. Namely, QJT is a deformation of QFT, where the deformation parameters are the observer’s mass $M$ and charge $e$. QFT is recovered in the limit $M \to \infty$ (so the observer’s position and velocity
commute) and $e \to 0$ (so the observer does not disturb the fields). This limit is readily taken for all interactions except gravity, where $e = M$. The physical problem with a QFT of gravity is thus that we tacitly assume both that the observer’s inert mass is infinite and that his heavy mass is zero.

Historically, QJT grew out of the projective representation theory of gauge and diffeomorphism algebras, i.e. the multi-dimensional analogues of affine and Virasoro algebras [3, 9]. Jets become essential because it is impossible to construct lowest-energy representations of these algebras starting from the fields themselves; one must pass to trajectories in the space of $p$-jets before quantization. The fact that QJT leads to new anomalies proves that it is substantially different from QFT, which is positive since QFT is incompatible with gravity.

With this history in mind, it is not surprising that the most striking feature of QJT is the appearance of new gauge and diff anomalies, which can not be formulated within QFT. These anomalies were discussed in [7], but that paper was unfortunately unusually opaque, even for this author. The purpose of the present paper is to clarify the origin of these new anomalies.

In quantum theory, a symmetry may become anomalous if there is no regularization which respects the symmetry, and the new QJT anomalies are no exception. QJT suggests a natural regularization procedure; replace $\infty$-jets by $p$-jets, $p$ finite, i.e. truncate the Taylor expansions at order $p$. The algebra of gauge transformations acts in a well-defined manner on $p$-jets, because it can only lower the order of Taylor coefficients. However, the $p$-jet regularization is still breaks the symmetry, because the equations of motions can only be implemented in the “body” and not in the “skin”. Although a $p$-jet itself only has finitely many components, the $p$-jet phase space becomes infinite-dimensional, because some histories are not determined in terms of initial conditions. The new anomalies arise in this infinite-dimensional “skin” of the $p$-jet phase space.

According to popular belief, a gauge anomaly is automatically a sign of inconsistency. However, this is not correct. E.g., the no-ghost theorem in string theory asserts that the free string can be quantized without ghosts in $d \leq 26$ dimensions [11]. In other words, also the subcritical free string defines a consistent quantum theory, despite its conformal anomaly. In general, a gauge anomaly turns a classical gauge symmetry into a quantum global symmetry, which acts on the Hilbert space instead of reducing it. This may or may not be consistent, depending on whether the unreduced Hilbert space admits a positive-definite metric preserved by the anomalous gauge symmetry.

Whereas a finite gauge anomaly may (or may not) be consistent, it seems
obvious that an infinite gauge anomaly renders the theory nonsensical. The anomalies in the \( p \)-jet regularization depend on \( p \), and diverge in the field theory limit \( p \to \infty \) if the number of spatial dimensions \( d \geq 2 \). However, it was observed in \cite{1, 5} that the divergent part could be made to cancel, provided that \( d \leq 3 \). In this sense, QJT correctly postdicts that spacetime has 3 + 1 dimensions. However, the cancellations lead to further conditions on the field content, which seem difficult to reconcile with physics. In the present paper we concentrate on gauge anomalies in theories of Yang-Mills type, and we find below that, in addition to the gauge field and matter fermions, one must add a field which violates the spin-statistics theorem: a fermion with a second-order equation of motion. It is unclear how this result should be interpreted. However, it still represents progress compared to \cite{5}, where not even unphysical solutions to the consistency conditions were known.

2 Free scalar field

2.1 Fields

The basic idea behind QJT is best described with a simple example: a free scalar field \( \phi(x) = \phi(x, t) \) in \((d + 1)\)-dimensional spacetime. As usual, spacetime indices are denoted by Greek letters \( \mu, \nu \), and spatial indices by Latin letters \( i, j \): \( x = (x^\mu) \in \mathbb{R}^{d+1} \) has the spacetime decomposition \( x = (x, x^0) \), where \( x = (x^i) \); in this section, \( t = x^0 \) denotes the time coordinate.

The action reads

\[
S = \frac{1}{2} \int d^{d+1}x \left( (\partial_0 \phi)^2 - (\nabla \phi)^2 - \omega^2 \phi^2 \right). \tag{1}
\]

The mass is denoted by \( \omega \) to avoid confusion with multi-indices introduced below. The equations of motion are of the form \( \mathcal{E}(x, t) = 0 \), where

\[
\mathcal{E} = -\frac{\delta S}{\delta \phi} = \partial_0^2 \phi - \nabla^2 \phi + \omega^2 \phi. \tag{2}
\]

The phase space may be identified with the space of solutions of the equations of motion \cite{2}. Since the equations (2) are second order, the solutions depends on \( \phi(x, 0) \) and \( \partial_0 \phi(x, 0) = \pi(x) \). Alternatively, we can coordinatize phase space by the Fourier modes \( a_k \) and \( a_k^\dagger \), because a general solution is of the form

\[
\phi(x, t) = \int d^{d+1}k \left( a_k e^{i(k_0 t + k \cdot x)} + a_k^\dagger e^{-i(k_0 t + k \cdot x)} \right), \tag{3}
\]
where $k_0 = \omega_k \equiv \sqrt{k^2 + \omega^2}$. The relation between the two phase space bases is obviously

$$
\phi(x, 0) = \int d^{d+1}k (a_k + a_k^\dagger) e^{ik \cdot x},
$$

$$
\partial_0 \phi(x, 0) = i \int d^{d+1}k k_0 (a_k - a_k^\dagger) e^{ik \cdot x}.
$$

(4)

2.2 Infinite jets

Next we reformulate dynamics of the free scalar field in jet space. By definition, a $p$-jet is an equivalence class of functions; two functions belong to the same class if their derivatives up to order $p$, evaluated at some given point $q$, are the same. Locally, a $p$-jet has a unique representative which is a polynomial of order at most $p$, namely the truncated Taylor series around the point $q$. We may and will therefore identify $p$-jets with Taylor expansions truncated at order $p$; an infinite jet is hence a Taylor series where the sum continues to infinite order.

The field $\phi(x, t)$ corresponds to the $\infty$-jet with components $\{\phi, m(t), q^j(t)\}$ via the Taylor expansion

$$
\phi(x, t) = \sum_m \frac{1}{m!} \phi_m(t) (x - q(t))^m.
$$

(5)

We employ standard multi-index notation:

$$
m = (m_1, m_2, ..., m_d), \quad \text{all } m_i \geq 0,
$$

(6)

is a multi-index with length $|m| = \sum_{j=1}^d m_j$. The factorial is defined by $m! = m_1!m_2!...m_d!$ and the power by $(x - q)^m = (x^1 - q^1)^{m_1}(x^2 - q^2)^{m_2}...(x^d - q^d)^{m_d}$. Denote by $m + \hat{j}$ the multi-index with one extra in the $j$:th position, i.e.

$$
m + \hat{j} = (m_1, m_2, ..., m_j + 1, ..., m_d).
$$

(7)

The Taylor coefficients of the partial derivative field $\partial_j \phi$ are $\phi, m + j$:

We assume that the time coordinate is given by

$$
x^0 = q^0(t) = t.
$$

(8)

The time derivative acts as

$$
\partial_0 \phi(x,t) = \partial_t \phi = \sum_m \frac{1}{m!} (\dot{\phi}, m(t) - \sum_{j=1}^d \dot{q}^j \phi, m + j) (x - q(t))^m.
$$

(9)
For convenience, we define coefficients $\phi_{m+\hat{0}}$ by

$$
\partial_0\phi(x, t) \equiv \sum_m \frac{1}{m!} \phi_{m+\hat{0}}(t) (x - q(t))^m.
$$

(10)

Comparison with the previous equation immediately yields

$$
\phi_{m+\hat{0}} = \dot{\phi}_m = \sum_{j=1}^d \dot{q}^j \phi_{m+j}.
$$

(11)

The equations of motion take the form $E_{m+\hat{0}} = 0$, where

$$
E_{m} = \sum_{j=0}^d \eta^{\mu\nu} \phi_{m+\mu+\nu} + \omega^2 \phi_{m+\hat{0}}
\equiv \phi_{m+2\hat{0}} - \sum_{j=1}^d \phi_{m+2j} + \omega^2 \phi_{m},
$$

(12)

where $\eta^{\mu\nu} = \text{diag}(1, -1, ..., -1)$ is the Minkowski metric and

$$
\phi_{m+2\hat{0}} = \ddot{\phi}_m = 2 \sum_{j=1}^d \dot{q}^j \ddot{\phi}_{m+j}
- \sum_{j=1}^d \ddot{q}^j \phi_{m+j} + \sum_{i,j=1}^d \dddot{q}^i \dddot{q}^j \phi_{m+i+j}.
$$

(13)

As the notation suggests, $E_{m}$ are the Taylor coefficients of the equation of motion field $E(x, t)$ in (2).

The equation $E_{m} = 0$ can be written as $\dddot{\phi}_m = ...$, and is thus a second-order equation for the Taylor coefficients $\phi_{m}(t)$. A solution is fully specified by the Cauchy data $\dot{\phi}_{m}(0)$ and $\ddot{\phi}_{m}(0)$. Since phase space can be identified with the space of solutions, a basis for phase space is given by $\phi_{m}(0)$ and $\pi_{m}(0) = \phi_{m+\hat{0}}(0)$. The phase space is infinite-dimensional, because the multi-index $m \in \mathbb{N}^d$ can take infinitely many values. This is expected, because an infinite jet contains essentially the same information as the field itself, modulo assumptions about convergence of the Taylor series (5). To find the relation to the Fourier basis in (3), we expand the exponential as

$$
e^{ik \cdot x} = e^{ik \cdot q} \sum_m \frac{1}{m!} (ik)^m (x - q)^m.
$$

(14)
We find
\[ \phi_m(0) = \int d^{d+1}k \left( a_k + a_k^\dagger \right) e^{ik\cdot q} (i\mathbf{k})^m, \]
\[ \pi_m(0) = \phi_{m+\delta}(0) = \int d^{d+1}k i \hbar \left( a_k - a_k^\dagger \right) e^{ik\cdot q} (i\mathbf{k})^m. \tag{15} \]

To make sense of the divergent integrals over \( k \) involves subtleties which are ignored.

## 2.3 Finite \( p \)-jets

QJT has a natural built-in regularization method: truncate from infinite jets to \( p \)-jets, \( p \) finite. This means that the Taylor series (5) is truncated at order \( p \), i.e.
\[ \phi(x, t) \approx \sum_{|m| \leq p} \frac{1}{m!} \phi_{m}(t) \left( x - q(t) \right)^{m}. \tag{16} \]

A basis for the space of all histories in \( p \)-jet space consists of \( q^i(t) \) and \( \phi_{m}(t) \) for all \( m \) such that \(|m| \leq p\). This suggests that the phase space, i.e. the space of histories which solve the equations of motion, should be spanned by \( \phi_{m}(0) \) and \( \pi_{m}(0) = \phi_{m+\delta}(0) \) with \(|m| \leq p\). Hence the regulated phase space should be finite-dimensional, and in fact the dimension should equal \( 2^{(d+p)} \).

However, except for case \( d = 0 \), i.e. ordinary quantum mechanics, this expectation is wrong. When \( d = 0 \), we have the phase space of the harmonic oscillator, which indeed is two-dimensional and spanned e.g. by the vectors \( \phi(0) \) and \( \pi(0) \). To see what goes wrong in higher dimensions, we return to the Taylor expansion of the equations of motion. The second term in (12) involves the term \( \phi_{m+2\delta} \), which is only defined for \(|m| \leq p-2 \) since \( \phi_{m} \) is only defined for \(|m| \leq p \). Therefore, the equations of motion for \( \phi_{m} \) with \(|m| = p-1 \) or \( p \) are undefined. The situation is the same for the time derivative \( \phi_{m+2\delta} \), which according to (13) also depends on spatial derivatives of order \(|m| + 2 \).

The correct equations of motion read
\[ \mathcal{E}_m = \begin{cases} \phi_{m+2\delta} - \sum_{j=1}^{d} \phi_{m+2j} + \omega^2 \phi_{m}, & |m| \leq p-2 \\ \text{undefined} & |m| = p-1, p \end{cases} \tag{17} \]

One may imagine introducing some dynamics for the top modes, e.g. \( \mathcal{E}_m \equiv 0 \) for \(|m| = p-1, p \). However, such an assumption would be incorrect, as it
would invalidate the solutions [3]. The correct treatment is to leave the
dynamics undefined for the top modes. This means that the equations of
motion do not fully specify the histories $\phi_m(t)$ in terms of data living at
t = 0.

The full $p$-jet phase space, i.e. the space of $p$-jet histories which solve
the equations of motion (17), is spanned by the basis

$$
\begin{align*}
\phi_m(0), \pi_m(0) &= \phi_{m+0}(0), & |m| \leq p - 2, \\
\phi_m(t), & \forall t \in \mathbb{R}, & |m| = p - 1, p.
\end{align*}
$$

The $p$-jet phase space is infinite-dimensional because the equations of motion
are unable to determine some histories in terms of data living at $t = 0$.

To facilitate further discussion, we define

- The **body** of a $p$-jet consists of the components $\phi_m$ such that the
  corresponding component $E_m$ is defined.

- The **skin** of a $p$-jet is the complement of the body, i.e. the components
  $\phi_m$ such that $E_m$ is undefined.

- The **thickness** of the skin is $n$, if the body consists of $\phi_m$ with $|m| \leq p - n$. For theories without gauge symmetries, the thickness is equal
to the order of the equations of motion.

- The terms body and skin are used about the $p$-jet phase space as well,
to denote the subspace spanned by the body and skin of the $p$-jet.

In particular for the scalar field, a $p$-jet component $\phi_m$ belongs to the body
if $|m| \leq p - 2$, it belongs to the skin if $|m| = p - 1$ or $p$, and the thickness
of the skin equals two.

We can now rephrase the main observation of this subsection: the body
of the $p$-jet phase space is finite-dimensional, as expected by truncation
from the $\infty$-jet phase space, but the skin is infinite-dimensional, because
the equations of motion do not fix the skin of a $p$-jet in terms of initial data.
The infinite-dimensional skin turns out to be the source of anomalies.

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1 The concepts are called body and skin rather than the perhaps more natural terms
bulk and shell, because the latter have other meanings as well, whereas the former appear
to be unused.
2.4 BV-BRST

The BV (Batalin-Vilkovisky) method [2] gives a cohomological construction of the phase space from arbitrary histories. It underlies the Manifestly Covariant Canonical Quantization (MCCQ) programme in [4, 5]. Although not without problems, it clarifies some aspects of the $p$-jet phase space; in particular, it clearly exhibits the separation between body and skin.

Return to the field formulation in subsection 2.1. In addition to the bosonic fields $\phi(x, t)$, we introduce fermionic antifields $\phi^*(x, t)$. Define the BV-BRST operator $\delta$ by

$$\delta \phi = 0,\quad \delta \phi^* = \mathcal{E} = \Box \phi + \omega^2 \phi. \tag{19}$$

The cohomology groups are

$$H^0(\delta) = C(\phi)/\mathcal{N},$$
$$H^i(\delta) = 0, \quad \text{if } k > 0, \tag{20}$$

where $C(\phi)$ is the space of smooth functionals over $\phi$, and $\mathcal{N}$ is the ideal generated by $\mathcal{E}$. The BV complex thus yields a resolution of the space of functionals over phase space, which consists of histories that solve the equations of motion $\mathcal{E} = 0$ [2].

In MCCQ, we introduce the canonical momenta in the history phase space, satisfying the canonical commutation relations (CCR)

$$[\phi(x, t), \pi(x', t')] = i\delta(x - x')\delta(t - t'),$$
$$\{\phi^*(x, t), \pi^*(x', t')\} = \delta(x - x')\delta(t - t'). \tag{21}$$

Note that the CCR in history space are instantaneous; the RHS is proportional to $\delta(t - t')$. We can now rewrite the BV-BRST operator as a bracket; for any functional $F(\phi, \phi^*)$, $\delta F = [Q, F]$, where

$$Q = \int d^4x dt \mathcal{E}(x, t)\pi^*(x, t). \tag{22}$$

The main problem with MCCQ is overcounting; the canonical momentum $\pi$ is not related to $\partial_0 \phi$. We can overcome this problem by making the identification $\pi = \partial_0 \phi$, which amounts to adding further terms to $\delta$. However, these terms necessarily break the manifest covariance which was a main motivation.
In the space of $p$-jet histories, the BV-BRST operators becomes

$$
\begin{align*}
\delta \phi, m &= 0, & |m| \leq p, \\
\delta \phi^*, m &= \mathcal{E}, m, & |m| \leq p - 2.
\end{align*}
$$

(23)

Since $\mathcal{E}, m$ is only defined for the body of the $p$-jet, so is the antifield $\phi^*, m$. Introduce canonical momenta that obey the instantaneous CCR

$$
\begin{align*}
[\phi, m(t), \pi^m(t')] &= i\delta_m^n \delta(t - t'), \\
\{\phi^*, m(t), \pi^* m(t')\} &= \delta^m_n \delta(t - t').
\end{align*}
$$

(24)

The BV-BRST operator can now be written as a bracket $\delta F = [Q, F]$, where

$$
Q = \sum_{|m| \leq p - 2} \int dt \, \mathcal{E}, m(t) \pi^* m(t).
$$

(25)

There are two things to note:

- The sum only runs over $m$ in the body, because the antifield $\phi^*, m$ and its canonical momentum $\pi^* m$ are only defined there.

- $Q$ is already normal ordered, because $\mathcal{E}, m$ is independent of the antifield. This will no longer be true in the presence of gauge symmetries.

A jet does not only consist of the Taylor coefficients $\phi, m$, but also of the expansion point $q$, which can be thought of as the observer’s position. To fully specify jet dynamics, we must thus introduce some equations of motion for this quantity as well. For simplicity, we equip $q(t)$ with the dynamics of a free relativistic point particle with mass $M$, described by the action

$$
S = -M \int dt \, \sqrt{1 - \dot{q}^2}.
$$

(26)

This leads to the equations of motion

$$
\mathcal{E}^i_q = \frac{d}{dt} \left( \frac{\dot{q}^i}{\sqrt{1 - \dot{q}^2}} \right) = 0.
$$

(27)

To implement this in cohomology, we introduce antifields $q^*_i(t)$ and posit that the BV-BRST operator acts as

$$
\begin{align*}
\delta q^i &= 0, \\
\delta q^*_i &= \mathcal{E}^i_q.
\end{align*}
$$

(28)
3 Reparametrizations

3.1 Fields

The Taylor expansion (5) is obtained from the spacetime jet
\[ \phi(x) = \sum_m \frac{1}{m!} \phi_m (x - q)^m \]
by setting \( x^0 = q^0 = t \), cf (5). In this section we relax this condition, and only assume that there is some function \( q^0(t) \) such that \( x^0 = q^0(t) \), whereas \( q^0(t) \neq t \) in general. This function is not completely arbitrary, but is assumed to be everywhere smooth and invertible. Since the condition \( x^0 = q^0 \) still holds, the \( m_0 \) dependence in the Taylor series disappears.

We now have two types of time-like coordinates: the physical time \( x^0 \), which appears directly in the equations of motion, and the time parameter \( t \), which is not observable. They are related by
\[ x^0 = q^0(t), \quad t = \tau(x^0). \]
Consequently, there are two types of time derivatives:
\[ \partial_t \phi = \dot{q} \partial_0 \phi, \quad \partial_0 \phi = \partial_0 \tau \partial_t \phi. \]
Because the functions \( q^0(t) \) and \( \tau(x^0) \) are each other’s inverses, \( \dot{q} \partial_0 \tau = 1 \).

The time parameter \( t \) is a non-observable gauge variable. The theory is invariant under infinitesimal reparametrizations \( t \mapsto t + f(t) \). The field transforms as
\[ \phi(x,t) \mapsto \phi(x,t - f(t)) \]
\[ = \phi(x,t) - f(t) \partial_t \phi(x,t) \]
\[ \equiv \phi(x,t) + L_f \phi(x,t). \]
For simplicity we will only consider reparameterizations which leave the surface \( t = 0 \) invariant. Hence we demand that
\[ f(0) = 0. \]
This is not an essential restriction, but makes preservation of the phase space spanned by \( \xi \) manifest. The gauge generators \( L_f \) satisfy the Witt algebra
\[ [L_f, L_g] = L_{[f,g]}, \quad [f, g] = f \dot{g} - g \dot{f}. \]
Reparametrizations act as

\[
\begin{align*}
[L_f, \phi] &= -f \partial_t \phi, \\
[L_f, \partial_\mu \phi] &= -f \partial_t \partial_\mu \phi, \\
[L_f, \mathcal{E}] &= -f \partial_t \mathcal{E}.
\end{align*}
\]  

(35)

In other words, the field \( \phi \) and its derivatives w.r.t. physical coordinates transform as fields with weight 0 under reparametrizations, which is necessary because otherwise the equations of motion would not transform homogeneously. The observer’s trajectory and its time derivative transform as

\[
\begin{align*}
[L_f, q^\mu] &= -f \dot{q}^\mu, \\
[L_f, \dot{q}^\mu] &= -f \ddot{q}^\mu - \dot{f} \dot{q}^\mu = -\frac{d}{dt}(f \dot{q}^\mu).
\end{align*}
\]  

(36)

Hence \( q^\mu \) also has weight 0, whereas \( \dot{q}^\mu \) is a density with weight +1. The reparametrization algebra admits the off-shell realization

\[
L_f = i \int dt dx f(t) \partial_t \phi(x,t) \pi(x,t),
\]  

(37)

where \( \pi(x,t) = -i \delta/\delta \phi(x,t) \) satisfies (21). The representation defined by (35) acts in a non-trivial way on general histories, but preserves the phase space spanned by \( \phi(x,0) \) and \( \partial_0 \phi(x,0) \), because we assumed that \( f(0) = 0 \). Reparametrizations generate a gauge symmetry under which the physical phase space is invariant.

3.2 \( p \)-jets

It follows from (35) that the reparametrization algebra acts in \( p \)-jet space as

\[
\begin{align*}
[L_f, \dot{\phi}_m] &= -f \dot{\phi}_m, \\
[L_f, \dot{\phi}_{m+\mu}] &= -f \dot{\phi}_{m+\mu}, \\
[L_f, \mathcal{E}_m] &= -f \dot{\mathcal{E}}_m,
\end{align*}
\]  

(38)

together with the relations written down in (36). Hence it admits the off-shell realization

\[
L_f = i \sum_{|m| \leq p} \int dt f(t) \dot{\phi}_m(t) \pi^m(t) + i \int dt f(t) \dot{q}^\mu(t) p_\mu(t),
\]  

(39)
where $\pi^m(t) = -i\delta/\delta \phi^m(t)$ and $p^\mu(t) = -i\delta/\delta q^\mu(t)$; they satisfy the instantaneous CCR \[24\] and
\[
[q^\mu(t), p^\nu(t')] = i\delta^\mu_\nu \delta(t - t').
\] (40)

We emphasize that (39) is an off-shell realization which is valid before the equations of motion have been taken into account. Reparametrizations act trivially on the body of the $p$-jet phase space, for the same reason that they act trivially on the field phase space: $f(0) = 0$. However, they do not act trivially on the skin, which depends on histories $\phi^m(t)$ for all $t$. In other words, the $p$-jet regularization does not preserve the reparametrization gauge invariance of the original field formulation.

The theory is quantized by introducing a Fock vacuum, which is annihilated by negative frequency modes. The body of the phase space must be polarized in some way, but exactly how this is done is not important, because this part of the phase space is finite-dimensional. In contrast, the skin of the phase space is infinite-dimensional and the choice of polarization is essential. The correct choice is to demand that negative frequency modes annihilate the vacuum $|0\rangle$. For simplicity, assume that the time parameter $t \in S^1$ takes values on the circle. This assumption is of course unphysical, because it leads to the introduction of closed time-like curves, but has some advantages. E.g., that we may expand any jet history in Fourier modes:
\[
\phi^m(t) = \sum_{\beta = -\infty}^{\infty} \phi^m(\beta)e^{i\beta t},
\] (41)
and analogously for the momenta $\phi^m(\pm \beta)$. We now posit that the vacuum is annihilated by negative-frequency modes:
\[
\phi^m(-\beta)|0\rangle = \phi^m(\pm \beta)|0\rangle = 0, \quad \text{for all } -\beta < 0.
\] (42)

There is some choice how to treat the zero modes with $\beta = 0$, but exactly how this is done is not essential since they only span a finite-dimensional subspace.

The reparametrization generators must be normal ordered to avoid infinite contributions after quantization. Hence we must e.g. replace in (39)
\[
\dot{\phi}_m(t) \pi^m(t) \mapsto \dot{\phi}_m(t) \pi^m(t): = \dot{\phi}_m(t) \pi_{<}^m(t) + \pi_{>}^m(t) \dot{\phi}_m(t),
\] (43)
where $\pi_{<}^m$ and $\pi_{>}^m$ only runs over negative and positive Fourier modes, respectively. Moreover, only the skin of the phase space contributes, because
reparametrizations act trivially on the finite-dimensional body. The normal-ordered generators thus read

\[ L_f = \sum_{p-1 \leq |m| \leq p} i \int dt \, f(t) : \dot{\phi}_m(t) \pi^m(t) :. \]  \hfill (44)

The contribution from the Taylor coefficients comes only from the skin, which is the difference between the full p-jet and the body. We can therefore write

\[ L_f = L_f^{(p)} - L_f^{(p-2)} , \]  \hfill (45)

where

\[ L_f^{(r)} = \sum_{|m| \leq r} i \int dt \, f(t) : \dot{\phi}_m(t) \pi^m(t) :. \]  \hfill (46)

There is also a contribution from the observer’s trajectory \( q^\mu(t) \) of the form

\[ L_q^\mu = i \int dt \, f(t) : \dot{q}^\mu(t) p_\mu(t) :. \]  \hfill (47)

The dynamics for \( q^i(t) \), which follows from the action (26), reduces the independent degrees of freedom to the finite-dimensional space spanned by \( q^i(0) \) and \( \dot{q}^i(0) \), on which reparameterizations act trivially. We discuss this issue further in subsection 3.4.

### 3.3 Reparametrization anomalies

After normal ordering, the Witt algebra (34) acquires an extension and is replaced by the Virasoro algebra

\[ [L_f, L_g] = L_{[f,g]} + \frac{c}{24\pi i} \int dt \left( \ddot{f}(t) \dot{g}(t) - \ddot{g}(t) \dot{f}(t) \right). \]  \hfill (48)

As is well known, the contribution from a single bosonic function of \( t \) to the central charge is \( c = 2 \). The number of different multi-indices with \( |m| \leq r \) in \( d \) dimensions is \( \binom{d+r}{d} \). In view of (45), the total central charge for the skin becomes

\[ c_{\text{Tot}} = 2 \left( \binom{d+p}{d} \right) - 2 \left( \binom{d+p-2}{d} \right). \]  \hfill (49)

There are a number of points to observe with this formula.
• The reparametrization gauge symmetry becomes anomalous. Classically, we could eliminate the time parameter $t$ by setting $t = x^0$. Such a gauge fixing is not allowed after quantization due to the nonzero central charge.

• The anomaly originates from the infinite-dimensional skin. The body of the $p$-jet phase space is finite-dimensional and can not give rise to anomalies.

• When $d = 0$, the central charge vanishes: $c = 0$.

• When $d = 1$, the central charge is independent of $p$: $c = 4$. The skin always consists of two functions $\phi_{p-1}(t)$ and $\phi_p(t)$.

• When $d \geq 2$, the central charge diverges in the limit $p \to \infty$.

• There is also an additional contribution from the observer’s trajectory $q^\mu(t)$, which we deal with in the next subsection. However, this contribution does not diverge when $p \to \infty$ and is therefore not our main concern.

The crucial observation is that the central charge diverges when $d \geq 2$. We regard this as a sign of inconsistency, which must be avoided if QJT is to make sense. The rest of this paper is devoted to finding ways to cancel the infinite parts of anomalies.

### 3.4 BV-BRST

The main benefit of MCCQ is that it facilitates the counting necessary to compute anomalies like the one in the previous subsection. When relaxing the condition $q^0(t) = t$, we turn $q^0$ into a dynamical variable, and the action (26) is replaced by

$$S_q = -M \int dt \sqrt{\dot{q}^\mu \dot{q}_\mu(t)}. \tag{50}$$

Since we have introduced an extra degree of freedom to describe the same physics, the equations of motion

$$\mathcal{E}^\mu_q = \frac{d}{dt} \left( \frac{\dot{q}^\mu}{\sqrt{q^2}} \right) = 0 \tag{51}$$

are redundant:

$$\frac{\dot{q}_\mu}{\sqrt{q^2}} \mathcal{E}^\mu_q = \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{q}_\mu \dot{q}^\mu}{q^2} \right) \equiv 0. \tag{52}$$
This can be cast in a more familiar form by noting that the momentum \( p_\mu = M \dot{q}_\mu / \sqrt{\dot{q}_\mu^2} \) satisfies \( p^2 = M^2 \), and hence \( dp^2 / dt = 0 \).

To implement this in cohomology, we introduce fermionike antifields \( q^*_\mu(t) \) for the equations of motion \( E^\mu \), bosonic second-order antifields \( \zeta(t) \) for the redundancy \((52)\), and a fermionic ghost \( c(t) \) which identifies states related by reparametrizations. This gives us the extended phase space of \( p \)-jet histories, over which the BV-BRST complex is defined. The BV-BRST operator acts as

\[
\begin{align*}
\delta c &= -c \dot{c}, \\
\delta \phi_{,m} &= -c \dot{\phi}_{,m}, \\
\delta \phi^*_{,m} &= E_{,m} - c \dot{\phi}^*_{,m}, \\
\delta q^\mu &= -c \dot{q}^\mu, \\
\delta q^*_{\mu} &= E_{,q}^\mu - \frac{d}{dt}(c q^\mu), \\
\delta \zeta &= \frac{\dot{q}^\mu}{\sqrt{\dot{q}^\mu^2}} q^*_{\mu} - \frac{d}{dt}(c \zeta).
\end{align*}
\] (53)

The reparametrization algebra acts on the extended phase space. Each quantity in \((53)\) transforms as a density with weights \((-1 \, (c), 0 \, (\phi_{,m}), 0 \, (\phi^*_{,m}), 0 \, (q^\mu), +1 \, (q^*_{\mu}), +1 \, (\zeta))\), respectively. These weights are reflected in the terms proportional to \( c \) in \((53)\).

The action of the BV-BRST operator can be written as a bracket in the same way as in subsection 2.4. For each quantity in \((53)\) (\( c, \phi_{,m}, \) etc.), we introduce the corresponding momentum in history space, which is defined by instantaneous CCR like \((24)\) or \((40)\). The BV-BRST charge \( Q \) is then defined in analogy with \((25)\); for each relation in \((53)\), we add a term that is linear in momenta.

However, there is a crucial difference compared to the situation in subsection 2.4; the BRST charge is not automatically normal ordered. E.g., to implement the relation \( \delta \phi_{,m} \) the BRST charge must contain the term

\[
Q_\phi = \sum_{|m| \leq p} \int dt \, c(t) \dot{\phi}_{,m}(t) \pi^{-m}(t).
\] (54)

After normal ordering of these \( \binom{d+p}{d} \) terms, \( Q_\phi \) is no longer nilpotent, and reparametrization may fail to be a gauge symmetry on the quantum level. However, that \( Q_\phi^2 \neq 0 \) does not necessarily imply that \( Q_\phi^2 = 0 \); the contributions from different fields to the anomaly may cancel.
The situation is summarized in the following table.

| Field  | Weight | Order | Parity | $c$   |
|--------|--------|-------|--------|-------|
| $c$    | $-1$   | 0     | $F$    | $-26$ |
| $\phi_{,m}$ | 0     | $p$   | $B$    | $2\left(\frac{d+p}{d}\right)$ |
| $\phi_{,m}^*$ | 0     | $p-2$ | $F$    | $-2\left(\frac{d+p-2}{d}\right)$ |
| $q^\mu$ | 0     | 0     | $B$    | $2(d+1)$ |
| $q^\mu_\lambda$ | 1     | 0     | $F$    | $-2(d+1)$ |
| $\zeta$ | 1     | 0     | $B$    | 2     |

The columns contain the following information: the type of field, its weight under reparametrizations, the maximal order for which is defined (only applies to the Taylor coefficients), the Grassmann parity (bosonic or fermionic), and the contribution to the central charge. Adding all contributions to the central charge, we find

$$c_{\text{Tot}} = 2\left(\frac{d+p}{d}\right) - 2\left(\frac{d+p-2}{d}\right) - 24,$$

which agrees with (49) except that the observer’s trajectory and the ghost is no long ignored.

### 3.5 Scalar densities

The field $\phi(x,t)$ does not have to transform as a field, i.e. as a density with weight zero, under reparametrizations. Instead, it may transform as a density of weight $\lambda$. The transformation law (35) is then replaced by

$$[L_f, \phi] = -f \partial_t \phi - \lambda \dot{f} \phi,$$

from which it follows that

$$[L_f, \partial_\mu \phi] = -f \partial_t \partial_\mu \phi - \lambda \dot{f} \partial_\mu \phi,$$

$$[L_f, \mathcal{E}] = -f \partial_t \mathcal{E} - \lambda \dot{f} \mathcal{E}. $$

However, the weight can only be nonzero if the equations of motion $\mathcal{E}$ are homogeneous in $\phi$, because otherwise $\mathcal{E}$ would not transform homogeneously. The discussion in this paper has been phrased for the free scalar field, but the linear equations of motion have not been used until this point, and the construction goes through also for interacting theories. However, we must now restrict ourselves to non-interacting theories, because nonzero weight is only possible if the equations of motion are homogeneous.
Alas, this is not a serious restriction, because we can introduce interactions in the presence of several types of fields. Consider e.g. a scalar field minimally coupled to an electromagnetic field $A_\mu$. The action reads
\[ S = \frac{1}{2} \int d^{d+1}x \left( ((\partial_\mu + e A_\mu) \phi)^2 - \omega^2 \phi^2 \right). \] (58)

The equations of motion
\[ \mathcal{E} = (\partial_\mu + e A_\mu)^2 \phi + \omega^2 \phi \] (59)
are homogeneous in $\phi$ but not in $A_\mu$. Thus we may consistently assume that $\phi$ transforms as a density with any weight $\lambda$, whereas $A_\mu$ must have weight $\lambda = 0$.

As is well known, the central charge of a single scalar density is
\[ c = 2(1 - 6\lambda + 6\lambda^2). \] (60)

The formula for the total central charge for the skin (49) is replaced by
\[ c_{\text{Tot}} = 2(1 - 6\lambda + 6\lambda^2)\left( \left( \frac{d + p}{d} \right) - \left( \frac{d + p - 2}{d} \right) \right). \] (61)

A nonzero weight thus modifies the value of $c_{\text{Tot}}$, but when $d \geq 2$ it still diverges when $p \to \infty$.

In the presence of additional fields, such as the gauge potential $A_\mu$, there are additional contributions to the central charge.

### 3.6 Fermions

Consider a model with a free fermionic field. The action
\[ S = \int d^{d+1}x \, \bar{\psi} (i \gamma^\mu \partial_\mu - \omega) \psi \] (62)
leads to the equations of motion
\[ \mathcal{E} = i \gamma^\mu \partial_\mu \psi - \omega \psi, \]
\[ \bar{\mathcal{E}} = i \partial_\mu \bar{\psi} \gamma^\mu + \omega \bar{\psi}. \] (63)

Upon passage to $p$-jet space, these equations become
\[ \mathcal{E}_m = \sum_{\mu=0}^d i \gamma^\mu \psi,_{m+\mu} - \omega \psi,_{m}, \]
\[ \bar{\mathcal{E}}_m = \sum_{\mu=0}^d i \bar{\psi},_{m+\mu} \gamma^\mu + \omega \bar{\psi},_{m}. \] (64)
The equations of motion have order one, so the skin has thickness one. Moreover, $E_m$ is linear in $\psi_m$ (and $\tilde{E}_m$ is linear in $\tilde{\psi}_m$), which means that we may consistently assume that $\psi_m$ transforms as a density of weight $\lambda$. The central charge for each fermionic function $\psi_m(t)$ is the same as (60), up to a sign:

$$c = -2(1 - 6\lambda + 6\lambda^2). \quad (65)$$

The total central charge comes from the skin of thickness one, and is given by

$$c_{\text{Tot}} = -2(1 - 6\lambda + 6\lambda^2) \left( \begin{pmatrix} d + p \\ d \end{pmatrix} - \begin{pmatrix} d + p - 1 \\ d \end{pmatrix} \right). \quad (66)$$

Here we used the identity

$$\begin{pmatrix} d + p \\ d \end{pmatrix} - \begin{pmatrix} d + p - 1 \\ d \end{pmatrix} = \begin{pmatrix} d + p - 1 \\ d - 1 \end{pmatrix}, \quad (67)$$

which underlies our strategy for cancelling the leading contributions to anomalies.

### 3.7 Both free bosons and fermions

We now combine the fields from the previous two subsections, and consider a model with the following field content:

- $n_b$ bosonic fields with weight $\lambda_b$.
- $n_f$ fermionic fields with weight $\lambda_f$.

Each skin degree of freedom makes the following contribution to the central charge:

$$c_b = 2n_b(1 - 6\lambda_b + 6\lambda_b^2),$$

$$c_f = -2n_f(1 - 6\lambda_f + 6\lambda_f^2). \quad (68)$$

Since the bosonic skin has thickness 2 and the fermionic skin thickness 1, the total central charge becomes

$$c_{\text{Tot}} = c_b \left( \begin{pmatrix} d + p \\ d \end{pmatrix} - \begin{pmatrix} d + p - 2 \\ d \end{pmatrix} \right) + c_f \left( \begin{pmatrix} d + p \\ d \end{pmatrix} - \begin{pmatrix} d + p - 1 \\ d \end{pmatrix} \right)$$

$$= \left( c_b + c_f \right) \begin{pmatrix} d + p \\ d \end{pmatrix} - c_f \begin{pmatrix} d + p - 1 \\ d \end{pmatrix} - c_b \begin{pmatrix} d + p - 2 \\ d \end{pmatrix}. \quad (69)$$
This expression vanishes if \( d = 0 \) and is independent of \( p \) when \( d = 1 \). In the special case that
\[
c_b = -c_0, \quad c_f = 2c_0, \quad \text{for some } c_0 > 0,
\]
we find by repeated use of the identity \( 67 \) that the central charge becomes
\[
c_{\text{Tot}} = c_0 \left( \frac{d + p}{d} \right) - 2c_0 \left( \frac{d + p - 1}{d} \right) + c_0 \left( \frac{d + p - 2}{d} \right)
\]
\[
= c_0 \left( \frac{d + p - 1}{d - 1} \right) - c_0 \left( \frac{d + p - 2}{d - 1} \right)
\]
\[
= c_0 \left( \frac{d + p - 2}{d - 2} \right).
\]

In particular, when \( d = 2 \) the central charge \( c_{\text{Tot}} = c_0 \) independent of \( p \); the anomaly does not diverge in the \( p \to \infty \) limit.

This example exhibits the main characteristics of QJT. A priori, jet quantization of free fields only makes sense in \( d \leq 1 \) dimensions, due to the appearence of infinite reparametrization anomalies. However, with a clever choice of field content, the leading divergencies can be made to cancel, leaving a finite anomaly also in \( d = 2 \) dimensions. In contrast, the anomaly is never convergent if \( d \geq 3 \); there are simply not enough terms in \( 69 \) to arrange sufficient cancellation.

Note that the condition \( c_0 > 0 \) is a necessary (but presumably not sufficient) condition for unitarity. Equation \( 70 \) leads to a finite central charge also if \( c_0 < 0 \), but we also demand that the representation of the Virasoro algebra \( 45 \) is unitary, something which is only possible if \( c_{\text{Tot}} > 0 \).

### 3.8 Green’s functions and anomalies

In this subsection we emphasize the relation between reparametrization anomalies and locality, in the sense of Green’s functions depending on separation.

Consider some scalar field \( \phi(x, x^0) \). The behaviour of the correlation function
\[
G(x - y, x^0 - y^0) = \langle 0 | \phi(x, x^0) \phi(y, y^0) | 0 \rangle
\]
when the physical points \( x \) and \( y \) coalesce is governed by the anomalous dimensions \( h \):
\[
G(x - y, x^0 - y^0) \approx \frac{A}{((x^0 - y^0)^2 - (x - y)^2)^h},
\]
for some constant $A$. In particular,

$$G(0, x^0 - y^0) \approx \frac{A}{(x^0 - y^0)^{2h}}. \quad (74)$$

The physical time coordinates are related to gauge time parameters as in (30): $x^0 = q^0(t)$, $y^0 = q^0(t')$. The correlation function expressed in terms of the time parameter thus diverges when $t \to t'$ as

$$G(0, t - t') \approx \frac{B}{(t - t')^{2h}}, \quad (75)$$

where $B = A/(q^0(t))^{2h}$. Since $q^0(t)$ relates physical time to parameter time, it must be an everywhere smooth and invertible function, which means that $q^0(t) \neq 0$ for every $t$. Hence the divergence of the Green’s function is governed by the same anomalous dimension $h$, independent of whether it is expressed in terms of physical or gauge time.

As is well known in conformal field theory, correlators of the form (75) are compatible with local diffeomorphism symmetry on the circle, but only if the central charge in the Virasoro algebra (48) is nonzero, and indeed positive. This is because all unitary, quantum representations of the Virasoro algebra with lowest $L_0$ eigenvalue $\hbar > 0$ have $c > 0$. Hence locality and unitarity together imply that the reparametrization symmetry be anomalous.

## 4 Gauge theory

### 4.1 Free Maxwell field

The action reads

$$S = \frac{1}{4} \int d^{d+1}x \ F_{\mu\nu} F^{\mu\nu}, \quad (76)$$

where the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (77)$$

The equations of motion,

$$\mathcal{E}_\mu = \partial^\nu F_{\nu\mu} = \Box A_\mu - \partial_\mu \partial^\nu A_\nu, \quad (78)$$

are redundant and do hence not completely fix the time evolution, due to the identity

$$\partial^\mu \mathcal{E}_\mu \equiv 0. \quad (79)$$
Consequently, the theory is invariant under the gauge symmetry $A_\mu \mapsto A_\mu + \partial_\mu X$, for $X(x)$ an arbitrary function over spacetime.

The gauge transformations generate the algebra $\text{map}(d+1, u(1))$ of maps from $(d+1)$-dimensional spacetime to the abelian Lie algebra $u(1)$. In terms of the smeared generators $\mathcal{J}_X = \int d^{d+1}x X(x) J(x)$, the bracket reads

$$[\mathcal{J}_X, \mathcal{J}_Y] = 0. \quad (80)$$

The action on the fields is given by

$$[\mathcal{J}_X, A_\mu] = \partial_\mu X,$$
$$[\mathcal{J}_X, F_{\mu\nu}] = 0,$$
$$[\mathcal{J}_X, E_\mu] = 0. \quad (81)$$

There are several ways to treat a gauge symmetry. For our purposes the most convenient choice is a BV-BRST formalism analogous to the one introduced in subsection 2.4. To this end, we introduce a fermionic antifield $A^*_\mu$, a bosonic second-order antifield $\zeta$, and a fermionic ghost $c$. The BV-BRST operator $\delta$ is defined by

$$\delta c = 0,$$
$$\delta A_\mu = \partial_\mu c,$$
$$\delta A^*_\mu = \partial^\nu F_{\nu\mu},$$
$$\delta \zeta = \partial^\mu A^*_\mu. \quad (82)$$

One readily checks that $\delta^2 = 0$ and that zeroth cohomology group can be identified with the space of gauge-invariant functionals of $A_\mu(x,0)$ and $F_{\mu0}(x,0)$, i.e. gauge-invariant functionals of the magnetic and electric fields at time $t = 0$.

### 4.2 Free Maxwell $p$-jets

As in the scalar field case, we pass to $p$-jet space by expanding the Maxwell field in a Taylor series:

$$A_\mu(x, t) \approx \sum_{|m| \leq p} \frac{1}{m!} A_{\mu,m}(t) (x - q(t))^m. \quad (83)$$

We can immediately translate the field concepts above to their jet space analogs. Field strength:

$$F_{\mu\nu,m} = A_{\nu,m+\bar{\nu}} - A_{\mu,m+\bar{\mu}}. \quad (84)$$
Equations of motion:

\[
\mathcal{E}_{\mu, m} = \sum_{\nu, \rho=0}^{d} \eta^{\nu\rho} F_{\nu\mu, m+\hat{\rho}} \\
= \sum_{\nu, \rho=0}^{d} \eta^{\nu\rho} \left( A_{\mu, m+\hat{\nu}} - A_{\nu, m+\hat{\rho}} - A_{\mu, m+\hat{\nu}} - A_{\rho, m+\hat{\mu}} \right). \tag{85}
\]

Redundancy:

\[
\sum_{\mu, \nu=0}^{d} \eta^{\mu\nu} \mathcal{E}_{\mu, m+\hat{\nu}} = 0. \tag{86}
\]

Gauge transformations:

\[A_{\mu, m} \mapsto A_{\mu, m} + \partial_{m+\hat{\mu}} X(q), \tag{87}\]

where we use the notation

\[
\partial_{m} X(q) = \partial_{m_1} \partial_{m_2} \ldots \partial_{m_{N}} X(q(t)). \tag{88}\]

The equations of motion are of second order, but the redundancy condition \((86)\) involves derivatives of one order higher. The thickness of the skin is thus three. The cleanest way to construct the p-jet phase space is to use the BV-BRST formalism. We can immediately read off the action of the BRST operator \(\delta\) on p-jets from \((82)\):

\[
\delta c_{m} = 0, \\
\delta A_{\mu, m} = c_{m+\hat{\mu}}, \tag{89}
\]

\[
\delta A_{\mu, m} = \sum_{\nu, \rho=0}^{d} \eta^{\nu\rho} F_{\nu\mu, m+\hat{\rho}}, \\
\delta \zeta_{m} = \sum_{\mu, \nu=0}^{d} \eta^{\mu\nu} A_{\mu, m+\hat{\nu}}.
\]

In view of the second equation in \((89)\), it might appear that we need to define the ghost \(c_{m}\) for all \(|m| \leq p + 1\). However, the p-jet BRST operator only needs to implement the gauge symmetry in the body of the p-jet \(A_{\mu, m}\), which consists of \(|m| \leq p - 3\) because the skin has thickness three. It therefore
suffices to define $c_m$ for $|m| \leq p - 2$. The maximal order and the Grassmann parity of the various fields and antifields are listed in the following table:

| Field   | Order | Parity |
|---------|-------|--------|
| $A_{\mu,m}$ | $p$   | $B$    |
| $A^{*}_{\mu,m}$ | $p - 2$ | $F$    |
| $c_m$     | $p - 2$ | $F$    |
| $\zeta_m$ | $p - 3$ | $B$    |

In the parity column, $B$ stands for bosonic and $F$ for fermionic.

The algebra of gauge transformations acts on the $p$-jet $A_{\mu,m}$ as

$$[\mathcal{J}_X, A_{\mu,m}] = \partial_{m+\hat{\mu}}X(q).$$

It follows that $\mathcal{J}_X$ commutes with the field strength $F_{\mu\nu,m}$ and with the equations of motion. Therefore, the action on the fields and anti-fields in $\mathcal{S}_9$ is given by

$$[\mathcal{J}_X, A^{*}_{\mu,m}] = [\mathcal{J}_X, c_m] = [\mathcal{J}_X, \zeta_m] = 0.$$  \hfill (92)

An explicit off-shell realization of the gauge generators, acting in the space of arbitrary $p$-jet histories, is

$$\mathcal{J}_X = \sum_{|m| \leq p} \sum_{\mu=0}^d \int dt \; i\partial_{m+\hat{\mu}}X(q(t))E^{\mu,m}(t),$$

where $E^{\mu,m} = -i\delta/\delta A_{\mu,m}$ satisfies the instantaneous CCR

$$[A_{\mu,m}(t), E^{\nu,n}(t')] = i\delta^{\nu}_{\mu}\delta_{m,n}\delta(t - t').$$

Because the expression (93) is linear in $E$, normal ordering is not possible and the gauge algebra does not acquire any extension.

### 4.3 Yang-Mills theory

The situation becomes more interesting if we consider a non-abelian gauge theory. Let $\mathfrak{g}$ denote a finite-dimensional Lie algebra with generators $J^a$, totally anti-symmetric structure constants $f^{abc}$, and Killing metric $\delta^{ab}$. The Lie brackets are given by

$$[J^a, J^b] = i f^{abc} J^c.$$  \hfill (95)
The gauge transformations generate the algebra map \((d + 1, \mathfrak{g})\) of maps from \((d + 1)\)-dimensional spacetime into \(\mathfrak{g}\). For every \(\mathfrak{g}\)-valued function \(X = X_a(x)J^a\), we define the smeared operator \(J_X\), with brackets

\[
[J_X, J_Y] = J_{[X,Y]},
\]  

(96)

where \([X,Y] = if^{abc}X_a Y_b J^c\). This algebra admits the “central” extension

\[
[J_X, J_Y] = J_{[X,Y]} - \frac{k}{2\pi i} \delta^{ab} \int \dot{q}^i(t) X_a(q(t)) \partial \mu Y_b(q(t)),
\]

(97)

which is the natural generalization of the affine Kac-Moody algebra \(\hat{\mathfrak{g}}\) to multi-dimensions. Since this extension is proportional to the second Casimir \(\text{tr} J^a J^b\) rather than to the third Casimir \(d^{abc} = \text{tr} (J^a J^b + J^b J^a) J^c\), it does not arise in QFT.

The construction of the phase space proceed in analogy with the abelian case. The covariant derivative:

\[
D_\mu = \partial_\mu + A^a J^a.
\]

(98)

Field strength:

\[
F^a_{\mu\nu} = [D_\mu, D_\nu]^a.
\]

(99)

Equations of motion:

\[
\mathcal{E}^a_\mu = (D_\nu F^a_{\nu\mu})^a = 0.
\]

(100)

Redundancy:

\[
(D^\mu \mathcal{E}_\mu)^a \equiv 0.
\]

(101)

The construction of the phase space is simplest within the BV-BRST formalism. To this end, we introduce an antifield \(A^a_\mu\), a ghost \(c^a\), and a second-order antifield \(\zeta^a\). The BV-BRST operator \(\delta\) acts in the extended phase space as

\[
\begin{align*}
\delta c^a &= if^{abc} b^c, \\
\delta A^a_\mu &= if^{abc} A^b_\mu c^c + \partial_\mu c^a, \\
\delta A^a_\mu &= (D_\nu F^a_{\nu\mu})^a + if^{abc} A^b_\mu c^c, \\
\delta \zeta^a &= (D^\mu A^a_\mu)^a + if^{abc} \zeta^b c^c.
\end{align*}
\]

(102)
The zeroth cohomology group can be identified with the space of gauge-invariant functionals over $A_\mu^a(x,0)$ and $\partial_\mu A_\mu^a(x,0)$, i.e. functionals over the physical phase space.

We now proceed to $p$-jets. The algebra of gauge transformations (96) acts on the fields and antifields:

$$\begin{align*}
[J_X, A_{\mu, m}^a] &= if^{abc}(X^b A^c_\mu)_m + \partial_{m+\mu} X^a, \\
[J_X, A_{\mu, m}^{* a}] &= if^{abc}(X^b A^{* c}_\mu)_m, \\
[J_X, c_{m}^a] &= if^{abc}(X^b c^c)_m, \\
[J_X, \zeta_{m}^a] &= if^{abc}(X^b \zeta^c)_m,
\end{align*}$$

(103)

where $(XA)_m$ denotes the $m$:th coefficient in the Taylor expansion of $X(x)A(x,t)$ around the point $x = q(t)$, i.e.

$$(XA)_m = \sum_{|n| \leq p} \frac{m}{n!} \partial_{m-n}X(q(t))A_n(t).$$

(104)

From (103) we can read off an explicit expression for the gauge generators $J_X$, in analogy with (93). However, in constrast to the abelian case, this expression contains bilinear terms which must be normal ordered after quantization. This normal ordering gives rise to a Kac-Moody-like extension of the form (97).

If a field transforms in the $g$ representation $\rho$, the “abelian charge” $k$ in (97) is $k = -Q_\rho$ (if the field is bosonic) or $k = +Q_\rho$ (if the field is fermionic), where the value of the second Casimir operator in $\rho$ is given by

$$\text{tr} \ J^a J^b = Q_\rho \delta^{ab}.$$  

(105)

In particular, the gauge potential and its antifields and ghost all transform in the adjoint representation of $g$, so $\rho = \text{ad}$ here. Moreover, $A_{\mu}^a$ and $A_{\mu}^{* a}$ are vector fields with $d+1$ components. We can therefore write down a list analogous to (90), with an extra column which denotes the contribution to the abelian charge from each jet component.

| Field | Order | Parity | $k$ |
|-------|-------|--------|-----|
| $A_{\mu, m}^a$ | $p$ | $B$ | $-(d+1)Q_{\text{ad}}$ |
| $A_{\mu, m}^{* a}$ | $p-2$ | $F$ | $(d+1)Q_{\text{ad}}$ |
| $c_{m}$ | $p-2$ | $F$ | $Q_{\text{ad}}$ |
| $\zeta_{m}$ | $p-3$ | $B$ | $-Q_{\text{ad}}$ |
The total extension is thus

\[ k_{\text{Tot}} = -(d+1)Q_{ad} \left( \frac{d+p}{d} \right) + (d+1)Q_{ad} \left( \frac{d+p-1}{d} \right) + Q_{ad} \left( \frac{d+p-2}{d} \right) - Q_{ad} \left( \frac{d+p-3}{d} \right) \]

\[ = -(d+1)Q_{ad} \left( \frac{d+p-1}{d+1} \right) + Q_{ad} \left( \frac{d+p-3}{d+1} \right) \]

(107)

In the non-abelian case, \( k_{\text{Tot}} \) vanishes if \( d = 0 \), it equals \(-Q_{ad}\) independent of \( p \) if \( d = 1 \), and it diverges with \( p \) if \( d \geq 2 \). In the abelian case, \( Q_{ad} = 0 \), and \( k_{\text{Tot}} = 0 \) in any numbers of dimensions, in agreement with our observation in the previous subsection.

### 4.4 Matter fields

We introduce fermions via the minimal coupling prescription. For simplicity, we write down formulas for the Maxwell theory only, but it is straightforward to write down the required modifications in the non-abelian case. To the free Maxwell action (76) we add the Dirac action

\[ S_\psi = \int d^{d+1}x \, \bar{\psi} \left( i\gamma^\mu (\partial_\mu - eA_\mu) - \omega \right) \psi. \]

(108)

The equations of motion become

\[ E_\mu A_\mu = \partial_\nu F_{\nu\mu} - j_\mu = 0, \]

\[ E_\psi = i\gamma^\mu (\partial_\mu - eA_\mu) \psi - \omega \psi = 0, \]

\[ \bar{E}_\psi = i(\partial_\mu + eA_\mu) \bar{\psi} \gamma^\mu + \omega \bar{\psi} = 0, \]

(109)

where the current

\[ j^\mu = ie\bar{\psi} \gamma^\mu \psi. \]

(110)

Because of current conservation, \( \partial_\mu j^\mu = 0 \), the equations of motion are redundant and do not completely fix the time evolution:

\[ \partial^\mu \mathcal{E}_\mu \equiv 0. \]

(111)

As a result, we still have an \( u(1) \) gauge symmetry \((80)\), which acts on the fields as by

\[ [\mathcal{J}_X, A_\mu] = \partial_\mu X, \]

\[ [\mathcal{J}_X, \psi] = eX \psi, \]

\[ [\mathcal{J}_X, \bar{\psi}] = -eX \bar{\psi}. \]

(112)
In $p$-jet space, the action on the fermions reads $[\mathcal{J}_X, \psi_m] = (X\psi)_m$, etc.

Bosonic matter can also be introduced, e.g. by adding a scalar electrodynamics term \([58]\) to the action. For our purposes, the main difference between bosons and fermions resides in the sign of the extension \([57]\). If we assume that there are $n_\phi$ bosonic species $\phi$ and $n_\psi$ fermionic species $\psi$ (where the conjugate $\bar{\psi}$ counts as another species), the bosonic and fermionic contributions to the abelian charge $k$ become

$$
k_\phi = -n_\phi Q_\phi, \quad k_\psi = +n_\psi Q_\psi, \quad (113)$$

respectively. We can readily generalize this to several different type of species transforming in different representations $\rho$; the abelian charge is simply the sum of the contributions from the different species, including sign.

We now turn to a general non-abelian gauge theory, with both fermionic and bosonic matter. After passage to $p$-jet space, the BV-BRST complex is built from the following content:

| Field $\ A^a_{\mu,m}$ $\ A^*_\mu,m$ $\ c_{\mu,m}$ $\ \zeta_{\mu,m}$ $\ \psi_{\mu,m}$ $\ \psi^*_{\mu,m}$ $\ \phi_{\mu,m}$ $\ \phi^*_{\mu,m}$ | Order | Parity | $k$ |
|---|---|---|---|
| $p$ $p - 2$ $p - 2$ $p - 3$ $p$ $p - 1$ $p$ $p - 2$ | $B$ $F$ $F$ $B$ $F$ $B$ $B$ $F$ | | $-(d + 1)Q_{ad}$ $(d + 1)Q_{ad}$ $Q_{ad}$ $-Q_{ad}$ $n_\psi Q_\psi$ $-n_\psi Q_\psi$ $-n_\phi Q_\phi$ $n_\phi Q_\phi$ |

The columns list the type of field, the maximal order for which the corresponding jet is defined, its Grassmann parity (bose/fermi) and the contribution to the abelian charge in \([97]\). Summing the various contributions, the total abelian charge is

$$k_{\text{Tot}} = k_0 \binom{d + p}{d} + k_1 \binom{d + p - 1}{d} + k_2 \binom{d + p - 2}{d} + k_3 \binom{d + p - 3}{d},$$

where

$$
k_0 = -(d + 1)Q_{ad} + n_\psi Q_\psi - n_\phi Q_\phi, \\
k_1 = -n_\psi Q_\psi, \\
k_2 = (d + 1)Q_{ad} + Q_{ad} + n_\phi Q_\phi, \\
k_3 = -Q_{ad}. \quad (115)$$
If we choose
\[ k_1 = -3k_0, \quad k_2 = 3k_0, \quad k_3 = -k_0, \] (116)
repeated use of the identity (67) leads to
\[ k_{\text{Tot}} = k_0 \left( \frac{d + p - 3}{d - 3} \right). \] (117)

Provided that the conditions (116) are satisfied, the total abelian charge vanishes when \( d \leq 2 \) and has a finite value if \( d = 3 \). We read off from (115) that in order for this to happen, we must have
\[ -(d + 1)Q_{\text{ad}} + n_\psi Q_\psi - n_\phi Q_\phi = k_0, \]
\[ -n_\psi Q_\psi = -3k_0, \]
\[ (d + 1)Q_{\text{ad}} + Q_{\text{ad}} + n_\phi Q_\phi = 3k_0, \]
\[ -Q_{\text{ad}} = -k_0. \] (118)

These are four equations for three unknowns, and would in general not be solvable. However, the equation system turns out to be singular, and admits the solution
\[ k_0 = Q_{\text{ad}}, \]
\[ n_\psi Q_\psi = 3Q_{\text{ad}}, \]
\[ n_\phi Q_\phi = (1 - d)Q_{\text{ad}} = -2Q_{\text{ad}}, \] if \( d = 3 \). (119)

In order for the QJT of a non-abelian gauge theory to have a finite gauge anomaly in \( 3 + 1 \) dimensions, these conditions on the matter content are necessary.

No interesting solution to the conditions (119) has been found, and in fact there is a serious problem with the negative sign in the last equation; it implies a violation of the spin-statistics problem. Since \( \phi \) is bosonic, the abelian charge \( k_\phi = -n_\phi Q_\phi \) must be negative; however, the last equation above implies that \( k_\phi \) is positive, so \( \phi \) must in fact be fermionic. On the other hand, we assumed that the equations of motion for \( \phi \) are second order, which implies that \( \phi \) has integer spin. The only solution to (119) is thus that \( \phi \) is an integer spin fermion, which violates the spin-statistics theorem if \( \phi \) is a physical field. We discuss this matter further in the conclusion.
5 Conclusion

In this paper we considered QJT as a regularization method: we replace all fields by $p$-jets, i.e. their Taylor expansions truncated at order $p$. Although the space of $p$-jets is finite-dimensional, the $p$-jet phase space is infinite-dimensional, because only histories in the body are specified by initial conditions.

The $p$-jet regularization does not preserve the gauge symmetries of the original theory. The gauge symmetries become anomalous in the regularized theory due to the infinite dimensionality of the skin, and this anomaly does not vanish when the regularization is removed by taking the $p \to \infty$ limit. Worse, the corresponding “abelian charges” diverge in more than $1 + 1$ dimensions, but with a clever choice of field content the anomalies can be rendered finite in $3 + 1$ dimension; the critical number of spatial dimensions $d = 3$ equals the thickness of the skin.

We studied conditions for cancelling the divergent parts of Yang-Mills anomalous, but no solutions were found. In fact, the solution in (119) appears to violate the spin-statistics theorem; the field $\phi$ should be fermionic but have second-order equations of motion, i.e. integer spin. It is conceivable that one could interpret $\phi$ as the ghost for some additional symmetry, perhaps having something to do with confinement or gauge symmetry breaking. If so, the apparent violation of the spin-statistics theorem is not a problem, because $\phi$ is not a physical field. This issue deserves further investigation.

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