ON \( k \)-CONNECTED \( \Gamma \)-EXTENSIONS OF BINARY MATROIDS

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Abstract. Slater introduced the point-addition operation on graphs to classify 4-connected graphs. The \( \Gamma \)-extension operation on binary matroids is a generalization of the point-addition operation. In this paper, we obtain necessary and sufficient conditions to preserve \( k \)-connectedness of a binary matroid under the \( \Gamma \)-extension operation. We also obtain a necessary and sufficient condition to get a connected matroid from a disconnected binary matroid using the \( \Gamma \)-extension operation.

Keywords: binary matroid, splitting, \( k \)-connected, \( \Gamma \)-extension

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1. Introduction

We refer to [9] for standard terminology in graphs and matroids. The matroids considered here are loopless and coloopless. Slater [12] introduced the point-addition operation on graphs and used it to classify 4-connected graphs. Azanchiler [1] extended this operation to binary matroids, which is defined by Shikare et al. [11], as follows:

Definition 1.1. [1] Let \( M \) be a binary matroid with ground set \( S \) and standard matrix representation \( A \) over \( GF(2) \). Let \( X = \{x_1, x_2, \ldots, x_m\} \subset S \) be an independent set in \( M \) and let \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \) be a set such that \( S \cap \Gamma = \emptyset \). Suppose \( A' \) is the matrix obtained from the matrix \( A \) by adjoining \( m \) columns labeled by \( \gamma_1, \gamma_2, \ldots, \gamma_m \) such that the column labeled by \( \gamma_i \) is same as the column labeled by \( x_i \) for \( i = 1, 2, \ldots, m \). Let \( A^X \) be the matrix obtained by adjoining one extra row to \( A' \) which has entry 1 in the column labeled by \( \gamma_i \) for \( i = 1, 2, \ldots, m \) and zero elsewhere. The vector matroid of the matrix \( A^X \), denoted by \( M^X \), is called as the \( \Gamma \)-extension of \( M \) and the transition from \( M \) to \( M^X \) is called as \( \Gamma \)-extension operation on \( M \).

An example given at the end of the paper illustrates the definition. Note that the ground set of the matroid \( M^X \) is \( S \cup \Gamma \) and \( M^X \setminus \Gamma = M \). Therefore \( M^X \) is an extension of \( M \). The \( \Gamma \)-extension operation is related to the splitting operation on binary matroids, which is defined by Shikare et al. [11], as follows:

Definition 1.2. [1] Let \( M \) be a binary matroid with standard matrix representation \( A \) over \( GF(2) \) and let \( Y \) be a non-empty set of elements of \( M \). Let \( A_Y \) be the matrix obtained by adjoining one extra row to the matrix \( A \) whose entries are 1 in the columns labeled by the elements of the set \( Y \) and zero otherwise. The vector matroid of the matrix \( A_Y \), denoted by \( M_Y \), is called as the splitting matroid of \( M \) with respect to \( Y \), and the transition from \( M \) to \( M_Y \) is called as the splitting operation with respect to \( Y \).

Let \( M \) be a binary matroid with ground set \( S \) and let \( X = \{x_1, x_2, \ldots, x_m\} \) be an independent set in \( M \). Obtain the extension \( M' \) of \( M \) with ground set \( S \cup \Gamma \), where \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \) is disjoint from \( S \), such that \( \{x_i, \gamma_i\} \) is a 2-circuit in \( M' \) for each \( i \). The matroid \( M'_X \) obtained from \( M' \) by splitting the set \( \Gamma \) is the \( \Gamma \)-extension matroid \( M^X \).

The splitting operation with respect to a pair of elements, which is a special case of Definition 1.2, was earlier defined by Raghunathan et al. [10] for binary matroids as an extension of the corresponding graph operation due to Fleischner [7].

Whenever we write \( M^X \), it is assumed that \( X \) is a non-empty independent set of the matroid \( M \).

Azanchiler [1] characterized the circuits and the bases of the \( \Gamma \)-extension matroid \( M^X \) in terms the circuits and bases of \( M \), respectively. Some results on preserving graphicness of \( M \) under the...
The Γ-extension operation are obtained in [2]. Borse and Mundhe [8] characterized the binary matroids $M$ for which $M^X$ is graphic for any independent set $X$ of $M$.

A $k$-separation of a matroid $M$ is a partition of its ground set $S$ into two disjoint sets $A$ and $B$ such that $\min \{ |A|, |B| \} \geq k$ and $r(A) + r(B) - r(M) \leq k - 1$. A matroid $M$ is $k$-connected if it does not have a $(k - 1)$-separation. Also, $M$ is connected if it is 2-connected.

In general, the splitting operation does not preserve the connectivity of a given matroid. Borse and Dhotre [4] provided a sufficient condition to preserve connectedness of a matroid while Borse [3] and Malwadkar et al. [5] gave a sufficient condition to get a $k$-connected binary matroid under this operation.

The Γ-extension operation also does not give $k$-connected binary matroid in general. Azanchiler [1] obtained sufficient conditions to preserve 2-connectedness and 3-connectedness of a binary matroid under this operation.

In this paper, we obtain necessary and sufficient conditions to preserve $k$-connectedness under the Γ-extension operation for any integer $k \geq 2$. We also give necessary and sufficient conditions to get a connected matroid from a disconnected binary matroid in terms of the Γ-extension operation.

2. Proofs

We need some lemmas.

Lemma 2.1. [1] Let $M$ be a binary matroid with ground set $S$ and let $X$ be an independent set in $M$. Suppose $M^X$ is the Γ-extension of $M$ with ground set $S \cup \Gamma$. Let $r$ and $r'$ be the rank functions of $M$ and $M^X$, respectively. Then

(i) $\Gamma$ is independent in $M^X$;
(ii) $r'(A) = r(A)$ if $A \subset S$;
(iii) $r'(A) \geq r(S \cap A) + 1$ if $A$ intersects $\Gamma$;
(iv) $r'(M^X) = r(M) + 1$.

Lemma 2.2. [1] Let $M$ be a binary matroid with ground set $S$ and let $X$ be an independent set in $M$. Then $Z \subset S \cup \Gamma$ is a circuit of $M^X$ if and only if one of the following conditions holds:

(i) $Z$ is a circuit of $M$;
(ii) $Z = \{x_i, x_j, \gamma_i, \gamma_j\}$ for some distinct elements $x_i, x_j$ of $X$ and the corresponding elements $\gamma_i, \gamma_j$ of $\Gamma$;
(iii) $Z = J \cup (D - X_J)$, where $J \subset \Gamma$ with $|J|$ even and $D$ is a circuit of $M$ containing the set $X_J = \{x_i \in X : \gamma_i \in J\}$.

Lemma 2.3 ([9], pp 273). Let $M$ be a $k$-connected matroid with at least $2(k - 1)$ elements. Then every circuit and every cocircuit of $M$ contains at least $k$ elements.

The next lemma is a consequence of [9, Proposition 2.1.6].

Lemma 2.4. [3] Let $M$ be a matroid with ground set $S$ and let $Y \subset S$ such that $r(M \setminus Y) = r(M) - 1$. Then $Y$ contains a cocircuit of $M$.

The following result follows immediately from Lemma 2.3 and Lemma 2.4.

Corollary 2.5. Let $M$ be a $k$-connected matroid with ground set $S$ such that $|S| \geq 2(k - 1)$. Then $r(M \setminus Y) = r(M)$ for any $Y \subset S$ with $|Y| < k$.

We now give necessary and sufficient conditions to obtain a $k$-connected matroid from the given $k$-connected binary matroid as follows.

Theorem 2.6. Let $k \geq 2$ be an integer and $M$ be a $k$-connected binary matroid with at least $2(k - 1)$ elements and $X$ be an independent set in $M$. Then the Γ-extension matroid $M^X$ is $k$-connected if and only if $|X| \geq k$ and $2 \leq k \leq 4$. 

Proof. Suppose $|X| \geq k$ and $2 \leq k \leq 4$. We prove that $M^X$ is $k$-connected. The ground set of $M^X$ is $S \cup \Gamma$, where $\Gamma$ is disjoint from the ground set $S$ of $M$. Since $|\Gamma| = |X|$, $|\Gamma| \geq k$. By Lemma 2.1(i), $\Gamma$ is independent in $M^X$. Suppose $r$ and $r'$ denote the rank functions of $M$ and $M^X$, respectively. Assume that $M^X$ is not $k$-connected. Then $M^X$ has a $(k-1)$-separation $(A, B)$. Therefore $A$ and $B$ are non-empty disjoint subsets of $S \cup \Gamma$ such that $S \cup \Gamma = A \cup B$ and further,

$$\min \{ |A|, |B| \} \geq k - 1$$
$$r'(A) + r'(B) - r'(M^X) \leq k - 2. \quad (1)$$

As $A$ and $B$ are non-empty, each of them intersects $S$ or $\Gamma$ or both. We consider the three cases depending on whether $A$ intersect only $S$ or only $\Gamma$ or both and obtain a contradiction in each of these cases.

**Case (i).** $A$ intersects only $\Gamma$.

As $A \subset \Gamma$, $B = (S - A) \cup \Gamma$. Since $\Gamma$ is independent, $A$ is independent in $M^X$. Consequently, $r'(A) = |A| \geq k - 1$. Suppose $A \neq \Gamma$. Then, by Lemma 2.1(iii) and (iv), $r'(B) \geq r(S) + 1 = r(M) + 1 = r'(M^X)$. Hence $r'(B) = r'(M^X)$. Therefore $r'(A) + r'(B) - r'(M^X) \geq k - 1$, which contradicts (1). Therefore $A = \Gamma$. Hence $B = S$ and $r'(A) = |\Gamma| \geq k$. By Lemma 2.1(ii) and (iv), $r'(B) = r'(S) = r(S) = r(M) = r'(M^X) - 1$. Therefore $r'(A) + r'(B) - r'(M^X) \geq k - 1$, which is a contradiction to (1).

**Case (ii).** $A$ intersects only $S$.

As $A \cap \Gamma = \emptyset$, $A \subset S$ and $B = (S - A) \cup \Gamma$. Therefore, by Lemma 2.1(i) and (ii), $r'(A) = r(A)$ and $r'(B) \geq r'(\Gamma) = |\Gamma| \geq k$. Suppose $|S - A| \leq k - 2$. Then, by Corollary 2.5, $r(A) = r(M)$. Consequently, by Lemma 2.1(iv),

$$r'(A) + r'(B) - r'(M^X) = r(A) + r'(B) - (r(M) + 1) \geq r'(B) - 1 \geq k - 1,$$

which is a contradiction to (1). Hence $|S - A| \geq k - 1$. By Lemma 2.1 (ii) and (iii), $r(S - A) = r'(S - A) \leq r'(A) - 1$. Therefore, by Inequality (1),

$$r(A) + r(S - A) - r(M) \leq r'(A) + r'(B) - 1 - r'(M^X) + 1 \leq k - 2.$$

This shows that $A$ and $S - A$ gives a $(k - 1)$-separation of $M$, which is a contradiction to fact that $M$ is $k$-connected.

**Case (iii).** $A$ intersects both $S$ and $\Gamma$.

Let $S_1 = A \cap S$ and $\Gamma_1 = A \cap \Gamma$. Since $B \neq \emptyset$, it intersects $S$ or $\Gamma$. If $B$ intersects only $S$ or only $\Gamma$, then we get a contradiction by interchanging roles of $A$ and $B$ in Case (i) and Case (ii). Therefore $B$ intersects both $S$ and $\Gamma$. Let $S_2 = B \cap S$ and $\Gamma_2 = B \cap \Gamma$. Then $S_1 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$ for $i = 1, 2$.

By Lemma 2.1(ii) and (iii), $r(S_1) = r'(S_1) \leq r'(A) - 1$ and $r(S_2) = r'(S_2) \leq r'(B) - 1$. By (1),

$$r(S_1) + r(S_2) - r(M) \leq r'(A) - 1 + r'(B) - 1 - r'(M^X) + 1 \leq k - 3.$$

Hence, if $|S_1| \geq k - 2$ and $|S_2| \geq k - 2$, then $(S_1, S_2)$ gives a $(k - 2)$-separation of $M$, a contradiction to fact that $M$ is $k$-connected. Consequently, $|S_1| \leq k - 3$ or $|S_2| \leq k - 3$.

Suppose $|S_1| \leq k - 3$. As $k \leq 4$ and $1 \leq |S_1|$, $k = 4$ and $|S_1| = k - 3 = 1$. Thus $A$ contains exactly one element, say $x$, of $M$. Further, $|A| \geq k - 1 = 4 - 1 = 3$. We claim that $r'(A) \geq 3$.

Suppose $r'(A) \leq 2$. Then $A$ contains a circuit $C$ of $M^X$ such that $|C| \leq 3$. Since $\Gamma$ is independent in $M^X$, $C$ is not a subset of $\Gamma$. Therefore $C$ contains $x$ and $C - \{x\} \subset A - \{x\} \subset \Gamma$. In the last row of the matrix $A^X$ which represents the matroid $M^X$, the columns corresponding to the elements of $\Gamma$ have entries 1 and rest of the entries in that row are zero. As $C$ is a circuit, the sum of the columns of the $A^X$ corresponding to the elements of $C$ is zero over GF(2). This implies that $C$ contains at least two elements of $\Gamma$. Hence $C = \{x, \gamma_1, \gamma_2\}$ for some $\gamma_1, \gamma_2 \in \Gamma$. Let $x_1$ and $x_2$ be elements of the matroid $M$ corresponding to $\gamma_1$ and $\gamma_2$, respectively. By Lemma 2.2(ii), $C_1 = \{x_1, x_2, \gamma_1, \gamma_2\}$ is a circuit in $M^X$. Since $M^X$ is a binary matroid, the symmetric difference $C \Delta C_1 = \{x, x_1, x_2\}$ of the circuits $C$ and $C_1$ contains a circuit, say $C_2$, of $M^X$. Hence $C_2$ is a circuit in $M^X \setminus \Gamma = M$ such that $|C_2| = 3 = 4 - 1 = k - 1$, a contradiction by Lemma 2.3. Hence $r'(A) \geq 3$. Since $|S_1| \leq k - 3$, by Corollary 2.5, $r(S_2) = r(S - S_1) = r(M)$. Therefore, by Lemma
Therefore, by Lemma 2.3, 

\[ r'(B) = r'(M^X). \]

Hence

\[ r'(A) + r'(B) - r'(M^X) = r'(A) \geq 3 = k - 1, \]

a contradiction to (1).

Suppose \( |S_2| \leq k - 3. \) Then, as in the above paragraph, we see that \( r'(B) \geq 3 = k - 1 \) and

\[ r'(A) = r'(M^X) \]

so \( r'(A) + r'(B) - r'(M^X) = r'(B) \geq k - 1, \) a contradiction to (1).

Thus we get contradictions in Cases (i), (ii) and (iii). Therefore \( M^X \) is \( k \)-connected.

Conversely, suppose \( M^X \) is \( k \)-connected. The last row of the matrix \( A^X \), which represents \( M^X \), has 1’s in the columns corresponding to the set \( \Gamma \) and zero elsewhere. Hence \( \Gamma \) contains a cocircuit of \( M^X \). By Lemma 2.2, \( |\Gamma| \geq k \) and so \( |X| = |\Gamma| \geq k \). Therefore \( M^X \) contains a 4-circuit. Thus \( M^X \) is \( k \)-connected.

We now give a necessary and sufficient condition to get a connected matroid \( M^X \) from the disconnected matroid \( M \). If \( X \) is disjoint from a component \( D \) of \( M \), then it follows from Lemma 2.2 that \( D \) is a component of \( M^X \) also. Therefore to get a connected matroid \( M^X \) from the disconnected matroid \( M \), it is necessary that \( X \) intersects every component of \( M \). In the following theorem, we prove that this obvious necessary condition is also sufficient.

**Theorem 2.7.** Let \( M \) be a disconnected binary matroid and let \( X \) be an independent set in \( M \). Then \( M^X \) is connected if and only if every component of \( M \) intersects \( X \).

**Proof.** Let \( M_1, M_2, \ldots, M_r \) be the components of \( M \). Suppose each \( M_i \) intersects \( X \). Let \( S \) be the ground set of \( M \). Then the ground set of \( M^X \) is \( S \cup \Gamma \), where \( S \cap \Gamma = \phi \). Since each \( M_i \) is connected in \( M \) and \( M^X \setminus \Gamma = M \), each \( M_i \) is connected in \( M^X \) too. Therefore each \( M_i \) is contained in a component of \( M^X \). We show that all \( M_i \) are contained in a single component of \( M^X \). Since \( M \) is disconnected, it has at least two components and so \( r \geq 2 \). Let \( D \) be a component of \( M^X \) containing \( M_1 \) and let \( j \in \{2, 3, \ldots, r\} \). Suppose \( X \) contains an element \( x_1 \) of \( M_1 \) and an element \( x_j \) of \( M_j \). Suppose \( \gamma_1 \) and \( \gamma_j \) are elements of \( \Gamma \) corresponding to \( x_1 \) and \( x_j \), respectively. Then, by Lemma 2.2(ii), \( C = \{x_1, x_j, \gamma_1, \gamma_j\} \) is a 4-circuit in \( M^X \). As \( C \) contains an element of the component \( D \) of \( M^X \), \( C \) is contained in \( D \). Therefore \( D \) contains the element \( x_j \) of \( M_j \). Consequently, \( M_j \) is contained in \( D \). Thus all components of \( M \) are contained in \( D \). Therefore \( S \subset D \). Let \( \gamma \) be an arbitrary member of \( \Gamma \) and let \( x \) be the member of \( X \) corresponding to \( \gamma \). Then, by Lemma 2.2(ii), \( \gamma \) and \( x \) belong to a 4-circuit, say \( Z \), of \( M^X \). As \( x \in Z \cap D \), \( Z \subset D \) and so \( \gamma \in D \). Therefore \( \Gamma \subset D \). Consequently, \( D \) is the only component of \( M^X \). Hence \( M^X \) is connected.

The converse readily follows from the discussion prior to the statement of the theorem.

**Example 2.8.** We illustrate Theorem 2.6 by using the Fano matroid \( F_7 \). The ground set of \( F_7 \) is \( \{1, 2, 3, 4, 5, 6, 7\} \) and the standard matrix representation of \( F_7 \) over \( GF(2) \) is as follows:

\[
A = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1
\end{pmatrix}.
\]

Let \( X = \{1, 2\} \) and \( Y = \{1, 2, 3\} \). Then \( X \) and \( Y \) are independent in \( F_7 \). Further,

\[
A^X = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_2 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

and

\[
A^Y = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_2 & \gamma_3 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

Let \( F_7^X \) and \( F_7^Y \) be the vector matroids of \( A^X \) and \( A^Y \), respectively. It is well known that \( F_7 \) is 3-connected. One can check that \( F_7^Y \) is 3-connected while \( F_7^X \) is 2-connected but not 3-connected.

**References**

[1] H. Azanchier, "Γ-extension of binary matroids", ISRN Discrete Mathematics 2011, Article 629707 (8 pages) (2011).

[2] H. Azanchier, "On extension of graphic matroids", Lobachevskii J. Math. 36, 38-47 (2015).

[3] Y. M. Borse, "A note on n-connected splitting-off matroids", Ars Combin. 128, 279-286 (2016).
[4] Y. M. Borse and S. B. Dhotre, "On connected splitting matroids", Southeast Asian Bull. Math. 36(1), 17-21 (2012).
[5] Y. M. Borse and G. Mundhe, "On n-connected splitting matroids", AKCE Int. J. Graphs Comb. (2017)(in press), doi:10.1016/j.akcej.2017.12.001.
[6] Y. M. Borse and G. Mundhe, "Graphic and cographic Γ-extensions of binary matroids," Discuss. Math. Graph Theory 38, 889-898 (2018).
[7] H. Fleischner, Eulerian Graphs and Related Topics Part 1, Vol. 1 (North Holland, Amsterdam, 1990).
[8] P. P. Malavadkar, M. M. Shikare and S. B. Dhotre, "A characterization of n-connected splitting matroids", Asian-European J. Combin. 7(4), Article 14500600 (7 pages) (2014).
[9] J. G. Oxley, Matroid Theory (Oxford University Press, Oxford, 1992).
[10] T. T. Raghunathan, M. M. Shikare and B. N. Waphare, "Splitting in a binary matroid", Discrete Math. 184, 267-271 (1998).
[11] M. M. Shikare, G. Azadi and B. N. Waphare, "Generalized splitting operation for binary matroids and its applications", J. Indian Math. Soc. New Ser. 78 No.1-4, 145-154 (2011).
[12] P. J. Slater, "A Classification of 4-connected graphs", J. Combin. Theory Series B 17, 281-298 (1974).

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