Mixed $H_2/H_\infty$ Control for Itô-type Stochastic Time-Delay Systems with Applications to Clothing Hanging Device

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This paper deals with the problem of mixed $H_2/H_\infty$ control for Itô-type stochastic time-delay systems. First, the $H_2/H_\infty$ control problem for stochastic time-delay systems is presented, which considers the mean square stability, $H_2$ control performance index, and the ability of disturbance attenuation of the closed-loop systems. Second, by choosing an appropriate Lyapunov–Krasoviskii functional and using matrix inequality technique, some sufficient conditions for the existence of state feedback $H_2/H_\infty$ controller for stochastic time-delay systems are obtained in the form of linear matrix inequalities. Third, two convex optimization problems with linear matrix inequality constraints are formulated to design the optimal mixed $H_2/H_\infty$ controller which minimizes the guaranteed cost of the closed-loop systems with known and unknown initial functions, and the corresponding algorithm is given to optimize $H_2/H_\infty$ performance index. Finally, a numerical example is employed to show the effectiveness and feasibility of the proposed method.

1. Introduction

Over the past decades, there has been a rapid increase of interest in the study of stochastic systems due to the importance of stochastic models in science and engineering, such as finance systems [1] and power systems [2]. And a lot of excellent results have been obtained. For example, Zhu et al. [3] investigated the tracking control issue of stochastic systems subject to time-varying full state constraints and input saturation. In [4], the stability of a class of discrete-time stochastic nonlinear systems with external disturbances was considered. The finite-time tracking control of a class of stochastic quantized nonlinear systems was studied in [5]. Furthermore, since stochasticity and time delay are the main sources resulting in the complexities of systems in reality, considerable interests have been focused on a general model of stochastic time-delay systems. For example, the problem of guaranteed cost robust stable control was considered via state feedback for a class of uncertain stochastic systems with time-varying delay in [6]. In [7], the mean square exponential stability of neutral-type linear stochastic time-delay systems with three different delays by using the Lyapunov–Krasovskii functionals was studied. In [8], the finite-time dissipative control for stochastic interval systems with time delay and Markovian switching was investigated. Some other nice results can be referred to [9–17] and the references therein.

At present, $H_\infty$ control has been receiving increased attention because it can suppress external interference, and many efforts have been devoted to extending the results for $H_\infty$ control over the last few decades. For instance, Ma and Liu [18] investigated the finite-time $H_\infty$ control problem for singular Markovian jump system with actuator fault through the sliding mode control approach. In [19], the problem of nonfragile observer-based $H_\infty$ control for stochastic time-delay systems was considered. The problems of robust stabilization and robust $H_\infty$ control with maximal decay rate were investigated for discrete-time stochastic systems with time-varying norm-bounded parameter uncertainties in [20]. Some other nice results can be referred to [21–26]. On the contrary, $H_\infty$ control is an effective way to attenuate the disturbance, while $H_2$ control can guarantee quadratic
performance cost. By combining $H_2$ control and $H_{\infty}$ control theory, the mixed $H_2/H_{\infty}$ control theory is obtained. Owing to the fact that the mixed $H_2/H_{\infty}$ control can minimize a desired control performance and eliminate the effect of disturbance, it is more attractive than the sole $H_{\infty}$ control in engineering practice. For example, Gao et al. [12] investigated the problem of $H_2/H_{\infty}$ control for nonlinear stochastic systems with time-delay and state-dependent noise. $H_2/H_{\infty}$ control problem of stochastic systems with random jumps was solved in [27]. Sathananthan et al. [28] studied the existence of state feedback $H_2/H_{\infty}$ controller for Itô-type stochastic time-delay systems. Although the problem of $H_2/H_{\infty}$ control has been investigated, there are few literature studies on Itô-type stochastic time-delay systems.

Motivated by the abovementioned discussions, in this work, we aim to investigate the mixed $H_2/H_{\infty}$ control for Itô-type stochastic time-delay systems. It is difficult to design state feedback $H_2/H_{\infty}$ controller because of the complicated structure of the system. The main contributions of this paper are as follows. (i) The definition of $H_2/H_{\infty}$ control for Itô-type stochastic time-delay systems is presented, which considers stability, $H_2$ control performance index, and $H_{\infty}$ control performance index. (ii) The new sufficient conditions for the existence of state feedback $H_2/H_{\infty}$ controller are provided in the form of linear matrix inequalities. (iii) An algorithm is given to optimize $H_2/H_{\infty}$ performance index.

The organization of this paper is as follows. Section 2 is devoted to the problem statement, preliminaries, and lemmas. Section 3 provides the sufficient conditions for the existence of state feedback $H_2/H_{\infty}$ controller for Itô-type stochastic time-delay systems. Section 4 gives an algorithm to solve the theorems. Section 5 presents a numerical example to demonstrate the effectiveness of the proposed method. Section 6 is our conclusions.

Notations: $A^\prime$ denotes the transpose of matrix $A$; $\text{tr}(A)$ indicates the trace of matrix $A$; $A \geq 0$ $(A \geq 0)$ indicates that $A$ is a positive definite (positive semidefinite) matrix; $I_{n\times n}$ represents a $n$-dimensional identity matrix; $\mathcal{R}^n$ shows $n$-dimensional Euclidean space; $E$ represents the mathematical expectation of random process; and the asterisk “*” in the matrix indicates symmetry term.

### 2. Preliminaries

Consider the following Itô-type stochastic time-delay system described by

\[
\begin{align*}
\dot{x}(t) &= [A_{11} x(t) + A_{12} x(t - \tau) + B_{11} u(t) + B_{12} v(t)] dt + [A_{21} x(t) + A_{22} x(t - \tau) + B_{21} u(t)] dw(t), \\
\tau(t) &= C_{1} x(t) + D_{1} u(t), \\
x(t) &= \phi(t), \forall t \in [-\tau, 0],
\end{align*}
\]

where $x(t)$ is the state of the system, $u(t)$ is the control input, $\tau(t)$ is the control output, $\phi(t)$ is the initial state function, and $w(t)$ is a one-dimensional standard Wiener process defined on probability space $(\Omega, \mathcal{F}, \mathcal{F}, P)$. $\mathcal{F}$ stands for the smallest $\sigma$-algebra generated by $w(s), 0 \leq s \leq t$, i.e., $\mathcal{F}_t = \sigma[w(s) | 0 \leq s \leq t]$. $\tau > 0$ is the time delay. $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, C_{1},$ and $D_{1}$ are constant matrices with appropriate dimensions.

Next, a new definition of the mean square stability for system (1) is given.

**Definition 1.** System (1) ($u(t) \equiv 0$ and $v(t) \equiv 0$) is said to be mean square stable, if

\[
\lim_{t \to \infty} E\|x(t)\|^2 = 0. \tag{2}
\]

Then, some lemmas for obtaining the main results are introduced.

**Lemma 1** (see [29]). Let $V(t, x) \in C^{1,2}(R^+_0, R^n)$ be a scalar function, and $V(t, x) > 0$, for the following stochastic system:

\[
\dot{x}(t) = a(x) dt + b(x) dw(t). \tag{3}
\]

The Itô formula of $V(t, x)$ is given as follows:

\[
dV(t, x) = LV(t, x) dt + \frac{\partial LV(t, x)}{\partial x} b(x) dw(t), \tag{4}
\]

where

\[
LV(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} a(x) + \frac{1}{2} \frac{\partial^2 V(t, x)}{\partial x^2} b(x). \tag{5}
\]

**Lemma 2** (see [30]). For given $x \in R^n$, $y \in R^n$, $N \in R^{n \times m}$, and $\rho > 0$, then we have

\[
2x'Ny \leq \rho x'x + \frac{1}{\rho} y'N'y. \tag{6}
\]

**Lemma 3** (see [6]). For some real matrices $N, M' = M$ and $R = R' > 0$, the following three conditions are equivalent:

\[
M + NR^{-1}N' \leq 0, \tag{7}
\]

\[
\begin{bmatrix} M & N \\ N' & -R \end{bmatrix} < 0,
\]

\[
\begin{bmatrix} M & -N \\ -N' & -R \end{bmatrix} < 0.
\]
3. Mixed $H_2/H_\infty$ Control for Stochastic Time-Delay Systems

In this section, a state feedback $H_2/H_\infty$ controller will be designed.

We consider a state feedback controller for system (1) is

$$\begin{align*}
\dot{x}(t) &= [A_{11}x(t) + A_{12}x(t - \tau) + B_1Kx(t) + B_1v(t)]dt + [A_{21}x(t) + A_{22}x(t - \tau) + B_2Kx(t)]d\omega(t), \\
z(t) &= C_1x(t) + D_1Kx(t), \\
x(t) &= \phi(t), \quad \forall t \in [-\tau, 0].
\end{align*}$$

For a given scalar $\gamma > 0$ and two symmetric positive definite matrices $T$ and $R$, if there are two symmetric positive definite matrices $P$ and $Q$ such that

$$\begin{bmatrix} 
\Gamma_{11} & PA_{12} + (A_{21} + B_2K)^T PA_{22} & PB_{12} \\
* & -Q + A_{12}^T PA_{22} & 0 \\
* & * & -\gamma^2I
\end{bmatrix} < 0,$$

hold, where $\Gamma_{11} = Q + (A_{21} + B_2K)^T P (A_{21} + B_2K) + 2 (A_{11} + B_1K)^T P (A_{11} + B_1K) + (C_1 + D_1K)^T (C_1 + D_1K) + T + K^T RK$, then (2) is a mixed $H_2/H_\infty$ controller of system (3), and the corresponding guaranteed cost for system (3) is

$$J^*_2 = E [x'(0)Px(0) + \int_0^{\infty} x'(s)Qx(s)ds].$$

Proof. The following proof is divided into three parts. First, it is proved that the closed-loop system (3) is mean square stable.

According to Lemma 3, condition (7) implies

$$\begin{bmatrix} 
\Sigma_{11} & PA_{12} + (A_{21} + B_2K)^T PA_{22} \\
* & -Q + A_{12}^T PA_{22}
\end{bmatrix} < 0,$$

Due to $T > 0$, $R > 0$, and $\gamma > 0$, we can obtain $(C_1 + D_1K)^T (C_1 + D_1K) > 0$, $(1/\gamma^2)PB_{12}B_{12}^T P > 0$, and $K^T RK > 0$; then, (8) implies

$$\begin{bmatrix} 
\Sigma_{11} & PA_{12} + (A_{21} + B_2K)^T PA_{22} \\
* & -Q + A_{12}^T PA_{22}
\end{bmatrix} < 0,$$

where $\Sigma_{11} = Q + (A_{21} + B_2K)^T P (A_{21} + B_2K) + 2 (A_{11} + B_1K)^T P (A_{11} + B_1K)$. Let

$$V(x(t), t) = x'(t)Px(t) + \int_0^t x'(s)Qx(t + s)ds$$

and the differential generation operator of system (3) be $L_1V(x(t))$ with $v = 0$; then,
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\[ L_1 V(x(t), t) = x'(t)Qx(t) - x'(t - \tau)Qx(t - \tau) + 2[x'(t)(A_{11} + B_{11}K)' + x'(t - \tau)A_{12}]Px(t) + [x'(t)(A_{21} + B_{21}K)'] + x'(t - \tau)A_{22}]P[(A_{11} + B_{11}K)x(t) + A_{22}x(t - \tau)] = x'(t)Qx(t) - x'(t - \tau)Qx(t - \tau) + 2[x'(t)(A_{11} + B_{11}K)' + x'(t - \tau)A_{12}]Px(t) + x'(t - \tau)A_{22}x(t - \tau) + x'(t)A_{21}K)'P(A_{21} + B_{21}K)x(t) + 2x'(t - \tau)A_{22}P(A_{21} + B_{21}K)x(t), \]

that is,

\[ L_1 V = \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \begin{bmatrix} \Omega_{11} & PA_{12} + (A_{21} + B_{21}K)'PA_{22} \\ * & -Q + A_{22}^2PA_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}, \]

where \( \Omega_{11} = Q + (A_{11} + B_{11}K)'P + P(A_{11} + B_{11}K) + (A_{21} + B_{21}K)'P(A_{21} + B_{21}K). \)

In the light of (9), we can derive that \( L_1 V(x(t), t) < 0, \) that is, the closed-loop system (3) is asymptotically stable in mean square sense.

Secondly, we prove that the control output \( z(t) \) satisfies \( H_\infty \) index for any nonzero disturbance \( v(t) \) under zero initial condition.

According to (7), \( T > 0, \) and \( K'RK > 0, \) we can obtain

\[ \Psi_{11} = Q + (A_{11} + B_{11}K)'P + P(A_{11} + B_{11}K) + (A_{21} + B_{21}K)'P(A_{21} + B_{21}K), \]

\[ \Psi_{12} = PA_{12} + (A_{21} + B_{21}K)'PA_{22} \]

\[ PB_{12} \]

where \( \Psi_{11} = Q + (A_{21} + B_{21}K)'P(A_{21} + B_{21}K) + 2(A_{11} + B_{11}K)'P + (C_1 + D_1K)'(C_1 + D_1K). \)

Notice that

\[ E \int_0^\infty (z'z - y'y)dt \leq E \int_0^\infty \begin{bmatrix} x(t) \\ x(t - \tau) \\ v(t) \\ v(t - \tau) \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} & PB_{12} & 0 \\ * & -Q + A_{22}^2PA_{22} & 0 & 0 \\ * & * & -\gamma^2I & 0 \\ * & * & * & -\gamma^2I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \\ v(t) \\ v(t - \tau) \end{bmatrix} dt, \]

where \( L_2 V(x(t), t) \) is the infinitesimal operator of system (3) for any nonzero disturbance \( v(t), \) and

\[ E \int_0^\infty (z'z - y'y)dt \leq E \int_0^\infty \begin{bmatrix} x(t) \\ x(t - \tau) \\ v(t) \\ v(t - \tau) \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} & PB_{12} & 0 \\ * & -Q + A_{22}^2PA_{22} & 0 & 0 \\ * & * & -\gamma^2I & 0 \\ * & * & * & -\gamma^2I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \\ v(t) \\ v(t - \tau) \end{bmatrix} dt, \]

Then, we can see that
where $\Psi_{12} = PA_{12} + (A_{21} + B_{21}K)'PA_{22}$.

Based on (12), we can see $E\int_{0}^{\infty} (\xi - y^*v') \, dt \leq 0$, that is, (12) implies that $E\int_{0}^{\infty} \|z(t)\|^2 \, dt < \gamma^2 E\int_{0}^{\infty} \|v(t)\|^2 \, dt$. Therefore, system (3) satisfies $H_{\infty}$ index.

Thirdly, we prove that system (3) satisfies $H_2$ index under the condition of $v(t) = 0$.

Based on (13) and (14), $(C_1 + D_1K)'(C_1 + D_1K) > 0$, and $(1/\gamma^2)PB_{12}B_{12}'P > 0$, we obtain that

$$\Theta_{11} = \Omega_{11} + T + K'RK.$$  

Due to

$$E\int_{0}^{\infty} (x'(t)Tx(t) + u'(t)Ru(t)) \, dt$$

$$= E\int_{0}^{\infty} [(x'(t)(T + K'RK)x(t) + L_1V - L_1V] \, dt$$

$$= E\int_{0}^{\infty} [x'(t)(T + K'RK)x(t) + L_1V] \, dt - E\int_{0}^{\infty} dV$$

$$= E\int_{0}^{\infty} [x'(t)(T + K'RK)x(t)] \, dt - EV (x(\infty), \infty) + EV (x(0), 0)$$

$$= E\int_{0}^{\infty} [x(t)] \, [\Theta_{11} \Psi_{12} \Theta_{14}] \, [x(t)] \, dt + EV (x(0), 0),$$

(23)

$$= J_s = E\int_{0}^{\infty} (x'(t)Tx(t) + u'(t)Ru(t)) \, dt$$

$$< EV (x(0), 0)$$

$$= E\int_{0}^{\infty} (x'(t)P \, x(t) + \int_{-\infty}^{\infty} x'(s)Q \, x(s) \, ds)$$

$$= J_s^*.$$  

The proof is completed here. $\Box$

In order to solve the complex problem to seek the solution caused by the nonlinear terms in Lemma 4, we give the following Lemma 5.

**Lemma 5.** For a given scalar $\gamma > 0$ and two symmetric positive definite matrices $\bar{T}$ and $\bar{R}$, if there are two symmetric positive definite matrices $\bar{P}$ and $\bar{Q}$ and a matrix $M$ such that

$$[\Xi_{11} \, \Xi_{12} \, \bar{P} \, \bar{Q} \, 2\bar{P}A_{21}' \, 2MtB_{21}' \, B_{12}]$$

$$= [\begin{array}{cccccc}
\Xi_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{P} & 0 & 0 & 0 \\
0 & 0 & \bar{Q} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{P} & 0 & 0 \\
0 & 0 & 0 & \bar{Q} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{P} & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma^2 I
\end{array}]$$

$$< 0$$  

(26)

Proof: According to Lemma 2 and (13), if the following inequality

$$[Y_{11} \, PA_{12} \, PB_{12}]$$

$$= [\begin{array}{cccc}
\Theta_{11} & \Psi_{12} & * & * \\
* & \Theta_{14} & * & * \\
* & * & \Theta_{11} & \Psi_{12} \\
* & * & * & \Theta_{14}
\end{array}]$$

$$< 0$$

(27)

$$\Xi_{11} = \bar{P}A_{11}' + A_{11}\bar{P} + M'B_{11}' + B_{11}M,$$

$$\Xi_{12} = \sqrt{5} PC_{11}' \bar{P} + \sqrt{5} M'D_{11}' \bar{M},$$

and $\Xi_{22} = \text{diag} \{ -I, -\bar{P}, -I, -\bar{R} \}$; then, (8) is a mixed $H_2/H_{\infty}$ controller of system (9), and the corresponding guaranteed cost for system (9) is $J_s^* = E[x(0) P \, x(0) + \int_{-\infty}^{\infty} x'(s)Q \, x(s) \, ds]$. In this case, $K = M\bar{P}^{-1}$. 

$$\begin{bmatrix}
Y_{11} & PA_{12} & PB_{12} \\
& -Q + 3A_{22}'PA_{22} & 0 \\
& * & -\gamma^2 I
\end{bmatrix}$$

< 0
hold, where $Y_{11} = Q + 4A'_2PA_{21} + 4K'B_{21}P + 3$

$PB_{21}K + 2A'_1P + 2K'B_{11}P + 3$

$C'_1C_1 + 3K'D'_1D_1K + T + K'RK$, then (13) holds.

Using $diag[P^{-1}, Q^{-1}, I]$ to premultiply and postmultiply inequality (27), we have

$$
\begin{bmatrix}
\tilde{Y}_{11}
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{12} Q^{-1} & B_{12} \\
* & -Q^{-1} + 3Q^{-1}A_{22}PA_{22}Q^{-1} & 0
\end{bmatrix} < 0,
$$

(28)

hold, where

$$
\begin{bmatrix}
\tilde{Y}_{11} = P^{-1}QP^{-1} + 4P^{-1}A_{12}P^{-1}
\end{bmatrix}^{-1}
\begin{bmatrix}
PA_{21} + 4P^{-1}K'B_{21}P + 2P^{-1}A_{21} + 2P^{-1}K'B_{11} + 3P^{-1}
\end{bmatrix}C'_1C_1 + 3P^{-1}K'D'_1D_1K + P^{-1}TP^{-1}

+ P^{-1}K'RKP^{-1}. \text{ Let } \tilde{P} = P^{-1}, M = K\tilde{P}, \tilde{Q} = Q^{-1}, \tilde{R} = R^{-1},

\text{and } \tilde{T} = T^{-1}; \text{ by Lemma 3, we obtain (26) from (28).}

Summarizing the process, the proof is completed. □

Next, in order to get the least upper bound for cost function among all the possible solutions to inequality (26), the convex optimization problem is provided as follows:

**Theorem 1.** For system (9), if the following optimization problem

$$
\min_{\gamma > 0, \alpha > 0, W > 0, P > 0, Q > 0, M}
\begin{bmatrix}
\alpha + \text{tr}(W)
\end{bmatrix}
$$

subject to (26) and

$$
\begin{bmatrix}
-\alpha & x'(0) \\
x(0) & -\tilde{P}
\end{bmatrix} < 0,
$$

(30)

$$
\begin{bmatrix}
-W & N' \\
N & -\tilde{Q}
\end{bmatrix} < 0,
$$

(31)

has a solution $\alpha, W, \tilde{P}, \tilde{Q},$ and $M$, then controller $u(t) = MP^{-1}x(t)$ is an optimal state feedback $H_2/H_{\infty}$ controller which ensures the minimization of guaranteed cost $J^*_x = E[x'(0)\tilde{P}^{-1}x(0) + \int_0^\infty x'(s)\tilde{Q}^{-1}x(s)ds]$ for system (9), where $\int_0^\infty x(s)x'(s)ds = NN'$. Proof. From Lemma 5, the controller $u(t) = MP^{-1}x(t)$ is a guaranteed cost control law of system (9). (30) is equivalent to $x'(0)\tilde{P}^{-1}x(0) < \alpha$; (31) is equivalent to $N'\tilde{Q}^{-1}N < W$.

Therefore, we can obtain

$$
\int_0^\infty x'(s)\tilde{Q}x(s)ds = \int_0^\infty \text{tr}\left[x'(s)\tilde{Q}^{-1}x(s)\right]ds
$$

$$
= \text{tr}\left(NN'\tilde{Q}^{-1}\right) = \text{tr}\left(N'\tilde{Q}^{-1}N\right) < \text{tr}(W).
$$

Thus, we can obtain $J^*_x < \alpha + \text{tr}(W)$.

Therefore, the minimization of $\alpha + \text{tr}(W)$ implies the minimization of guaranteed cost for system (9).

The proof is completed here. □

**Remark 1.** It is an ideal case that the initial function is known. However, in general, the initial function of system (1) is not known, but the guaranteed cost depends on it. In order to avoid the dependence, we assume that the initial function is a white noise process with zero expectation function and unit covariance function.

When the initial function is not known, we have

$$
J^*_x = E\left[x'(0)\tilde{P}^{-1}x(0) + \int_0^\infty x'(s)\tilde{Q}^{-1}x(s)ds\right]
$$

$$
= E\left[\text{tr}\left(x(0)x'(0)\tilde{P}^{-1}\right)\right] + \int_0^\infty E\left[\text{tr}\left(x(s)x'(s)\tilde{Q}^{-1}\right)\right]ds
$$

$$
= \text{tr}\left(E\left(x(0)x'(0)\tilde{P}^{-1}\right)\right) + \int_0^\infty \text{tr}\left(E\left(x(s)x'(s)\tilde{Q}^{-1}\right)\right)ds
$$

$$
= \text{tr}(\tilde{P}^{-1}) + \tau \times \text{tr}(\tilde{Q}^{-1}).
$$

(33)

Therefore, we have the following optimization problem:

$$
\min_{\gamma > 0, W > 0, P > 0, Q > 0, M}
\begin{bmatrix}
\text{tr}(W_1) + \tau \times \text{tr}(W_2)
\end{bmatrix},
$$

(34)

which subjects to (26) and

$$
\begin{bmatrix}
W_1 & I \\
I & \tilde{P}
\end{bmatrix} > 0,
$$

(35)

$$
\begin{bmatrix}
W_2 & I \\
I & \tilde{Q}
\end{bmatrix} > 0.
$$

(36)

**Theorem 2.** If there exist solution to (26), (34)–(36) then controller $u(t) = MP^{-1}x(t)$ is an optimal state feedback $H_2/H_{\infty}$ controller which ensures the minimization of guaranteed cost (18) for system (1).

Proof. From Lemma 5, the controller $u(t) = MP^{-1}x(t)$ is a $H_2/H_{\infty}$ controller of system (9). We can see (34) is equivalent to $0 < \tilde{P}^{-1} < W_1$ and (35) is equivalent to $0 < \tilde{Q}^{-1} < W_2$ from Lemma 3. Therefore, the minimization of $\text{tr}(W_1) + \tau \times \text{tr}(W_2)$ implies the minimization of the guaranteed cost for system (1).

The proof is complete. □

4. Numerical Algorithms

In this section, an algorithm is presented in order to find the minimum value of $\alpha + \text{tr}(W)$ in Theorem 1. The similar algorithm can also be applied to Theorem 2.

By analyzing (26), (30), (31) in Theorem 1, we find that if (26), (30), (31) have no feasible solutions when $\gamma$ takes the initial value, then (26), (30), (31) will have no feasible
solutions for all $\gamma > 0$. Next, we search for $\gamma$ from the initial value that makes (26), (30), (31) have feasible solutions to optimize $\alpha + \text{tr}(W)$ by using linear search algorithm. The specific algorithm is as follows.

5. Numerical Examples

The coefficient matrices of system (1) are given as follows:

$$A_{11} = \begin{bmatrix} -30 & 10 \\ -10 & -20 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -2.7 & 0.8 \\ 0.9 & -1.6 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -0.2 & -0.5 \\ -0.3 & -1.4 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -4 \\ 3 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 0.1 & -0.1 \\ 0.5 & 0.7 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

$$N = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix},$$

$$R = 1,$$

$$\tau = 1.$$

First case: when the initial function is known and $x(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T, t \in [-1, 0]$. In order to find the minimum value of $\alpha + \text{tr}(W)$, we obtain the relationship between $\alpha + \text{tr}(W)$ and $\gamma$ by Algorithm 1, which is shown in Figure 1.

As can be seen from Figure 1, $\alpha + \text{tr}(W)$ decreases with the increase of $\gamma$, and $\min[\alpha + \text{tr}(W)] = 38.4173$ when $\gamma = 0.4$, and $\min[\alpha + \text{tr}(W)] = 29.7250$ when $\gamma = 1.98$.

Take $\gamma = 0.8$, according to Theorem 1, we obtain that

$$\bar{P} = \begin{bmatrix} 0.8484 & 0.0215 \\ 0.0215 & 0.5896 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} 18.8352 & -4.2480 \\ -4.2480 & 1.0503 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0629 & -0.0472 \end{bmatrix},$$

$$W = 23.0449,$$

$$\alpha = 7.7981.$$

Therefore, the optimal state feedback $H_2/H_{\infty}$ controller is $u(t) = [0.0762 - 0.0828]x(t)$, and the guaranteed cost of closed-loop system is $J^* = 30.8430$.

Take external disturbance $v(t) = \sin(t)$, then we can obtain the curves of $x_1$ and $x_2$ and $E\|x(t)\|^2$ in Figure 2. From Figure 2, we can see that $E\|x(0)\|^2 = 5$ and $\lim E\|x(t)\|^2 = 0$, that is, closed-loop system (9) is mean square stable.

Second case: when the initial function is a white noise process with zero expectation function and unit covariance function, in order to find the minimum value of $\text{tr}(W_1) + \tau \times \text{tr}(W_2)$, we obtain the relationship between $\text{tr}(W_1) + \tau \times \text{tr}(W_2)$ and $\gamma$ by Algorithm 1, which is shown in Figure 3.

As can be seen from Figure 3, $\text{tr}(W_1) + \tau \times \text{tr}(W_2)$ decreases with the increase of $\gamma$, and $\min[\text{tr}(W_1) + \tau \times \text{tr}(W_2)] = 19.5450$ when $\gamma = 0.4$, $\min[\text{tr}(W_1) + \tau \times \text{tr}(W_2)] = 13.6648$ when $\gamma = 1.98$.

Take $\gamma = 0.8$, according to Theorem 2, we obtain that

$$\bar{P} = \begin{bmatrix} 0.8520 & 0.0217 \\ 0.0217 & 0.5889 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} 20.5511 & -4.6765 \\ -4.6765 & 1.1567 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0629 & -0.0472 \end{bmatrix},$$

$$\text{tr}(W_1) = 2.8746,$$

$$\text{tr}(W_2) = 11.4205.$$
Therefore, the optimal state feedback $H_2/H_\infty$ controller is $u(t) = [0.0759 - 0.0829]x(t)$, and the guaranteed cost of closed-loop system is $J^* = 14.2951$.

6. Conclusion

In this paper, the mixed $H_2/H_\infty$ control problem for Itô-type stochastic time-delay systems is presented, and the description of $H_2/H_\infty$ control problem for stochastic time-delay systems is given. On the basis of matrix transformation and convex optimization method, state feedback $H_2/H_\infty$ controller is obtained to make the system satisfy $H_\infty$ performance index and $H_2$ performance index. Moreover, an algorithm is given to solve state feedback controller and optimize $H_2/H_\infty$ performance index. Finally, a numerical example is used to show the feasibility of the results. In the future work, we will investigate mixed $H_2/H_\infty$ control for the more complex systems, such as, stochastic Markov jump systems with time delay.

Data Availability

The data used to support the findings of this study are available from the corresponding upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] C.-F. Wu, B.-S. Chen, and W. Zhang, "Multiobjective investment policy for a nonlinear stochastic financial system: a fuzzy approach," IEEE Transactions on Fuzzy Systems, vol. 25, no. 2, pp. 460–474, 2017.
[2] R. Yu and S. Gao, "Applications of probabilistic production simulation in power system," *Power System Protection and Control*, vol. 40, no. 11, pp. 149–155, 2012.

[3] Q. Zhu, Y. Liu, and G. Wen, "Adaptive neural network control for time-varying state constrained nonlinear systems with input saturation," *Information Sciences*, vol. 527, pp. 191–209, 2020.

[4] W. Xie and Q. Zhu, "Stability of discrete-time stochastic nonlinear systems with event-triggered state-feedback control," *Physica A: Statistical Mechanics and Its Applications*, vol. 547, p. 123283, 2020.

[5] F. Wang, L. Zhang, S. Zhou, and Y. Huang, "Neural network-based finite-time control of quantized stochastic nonlinear systems," *Neurocomputing*, vol. 362, pp. 195–202, 2019.

[6] P. Cui, C. Zhang, and M. Wang, "Guaranteed cost control of stochastic uncertain systems with time-varying delays," in *Proceedings of the 6th World Congress on Intelligent Control and Automation*, Dalian, China, June 2006.

[7] Z.-Y. Li, S. Shang, and J. Lam, "On stability of neutral-type linear stochastic time-delay systems with three different delays," *Applied Mathematics and Computation*, vol. 360, pp. 147–166, 2019.

[8] G. Chen, Y. Gao, and S. Zhu, "Finite-time dissipative control for stochastic interval systems with time-delay and Markovian switching," *Applied Mathematics and Computation*, vol. 310, pp. 169–181, 2017.

[9] H. Wang, B. Chen, and C. Lin, "Adaptive neural control for strict-feedback stochastic nonlinear systems with time-delay," *Neurocomputing*, vol. 77, no. 1, pp. 267–274, 2012.

[10] R. Nie, Q. Ai, S. He, Z. Yan, X. Luan, and F. Liu, "Robust finite-time control and estimation for uncertain time-delayed switched systems by observer-based sliding mode technique," *Optimal Control Applications and Methods*, Wiley, Hoboken, NJ, USA, 2020.

[11] Z. Yan, Y. Song, and J. H. Park, "Finite-time stability and stabilization for stochastic markov jump systems with mode-dependent time delays," *ISA Transactions*, vol. 68, pp. 141–149, 2017.

[12] M. Gao, L. Sheng, and W. Zhang, "Stochastic H2/H∞ control of nonlinear systems with time-delay and state-dependent noise," *Applied Mathematics and Computation*, vol. 266, pp. 429–440, 2015.

[13] S. Luo and F. Deng, "A note on delay-dependent stability of Itô-type stochastic time-delay systems," *Automatica*, vol. 105, pp. 443–447, 2019.

[14] S. He, H. Fang, M. Zhang, F. Liu, and Z. Ding, "Adaptive optimal control for a class of nonlinear systems: the online policy iteration approach," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 31, no. 2, pp. 549–558, 2020.

[15] P. Cheng, J. Wang, S. He, X. Luan, and F. Liu, "Observer-based asynchronous fault detection for conic-type nonlinear jumping systems and its application to separately excited DC motor," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 67, no. 3, pp. 951–962, 2020.

[16] H. Min, S. Xu, B. Zhang, and Q. Ma, "Globally adaptive control for stochastic nonlinear time-delay systems with perturbations and its application," *Automatica*, vol. 102, pp. 105–110, 2019.

[17] B. Zhou and W. Luo, "Improved Razumikhin and Krasovskii stability criteria for time-varying stochastic time-delay systems," *Automatica*, vol. 89, pp. 382–391, 2018.

[18] Y. Ma and Y. Liu, "Finite-time H∞ sliding mode control for uncertain singular stochastic system with actuator faults and bounded transition probabilities," *Nonlinear Analysis: Hybrid Systems*, vol. 33, pp. 52–75, 2019.

[19] J. Zhou, J. Park, and Q. Ma, "Non-fragile observer-based H∞ control for stochastic time-delay systems," *Applied Mathematics and Computation*, vol. 291, pp. 69–83, 2016.

[20] J. Feng and S. Xu, "Robust H∞ control with maximal decay rate for linear discrete-time stochastic systems," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 1, pp. 460–469, 2009.

[21] Y. Ren, W. Wang, W. Zhou, and M. Shen, "Stochastic incremental H∞ control for discrete-time switched systems with disturbance dependent noise," *Information Sciences*, vol. 513, pp. 519–535, 2020.

[22] Z. Yan, M. Zhang, Y. Song, and S. Zhong, "Finite-time H∞ control for Itô-type nonlinear time-delay stochastic systems," *IEEE Access*.

[23] S. He, H. Fang, M. Zhang, F. Liu, X. Luan, and Z. Ding, "Online policy iterative-based H∞ optimization algorithm for a class of nonlinear systems," *Information Sciences*, vol. 495, pp. 1–13, 2019.

[24] H. Liu, X. Li, X. Liu, and H. Wang, "Backstepping-based decentralized bounded-H∞ adaptive neural control for a class of large-scale stochastic nonlinear systems," *Journal of the Franklin Institute*, vol. 356, no. 15, pp. 8049–8079, 2019.

[25] D. Zhang, J. Cheng, D. Zhang, and K. Shi, "Nonfragile H∞ control for periodic stochastic systems with probabilistic measurement," *ISA Transactions*, vol. 86, pp. 39–47, 2019.

[26] X. Xing, Y. Liu, and B. Niu, "H∞ control for a class of stochastic switched nonlinear systems: an average dwell time method," *Nonlinear Analysis: Hybrid Systems*, vol. 19, pp. 198–208, 2016.

[27] Q. Zhang and Q. Sun, "A maximum principle approach to stochastic H2/H∞ control with random jumps," *Acta Mathematica Scientia*, vol. 35, no. 2, pp. 348–358, 2015.

[28] S. Sathananthan, M. J. Knap, and L. H. Keel, "Guaranteed cost H∞ control for linear stochastic markovian switching systems," *IFAC Proceedings Volumes*, vol. 44, no. 1, pp. 5459–5464, 2011.

[29] F. B. Hanson, "Applied stochastic processes and control for jump-diffusions: modelling," *Analysis and Computation*, SIAM, Philadelphia, PA, USA, 2007.

[30] Z. Yan, G. Zhang, and J. Wang, "Non-fragile robust finite-time H∞ control for nonlinear stochastic Itô systems using neural network," *International Journal of Control, Automation, and Systems*, vol. 10, no. 5, pp. 873–882, 2012.