An Affine Integral Inequality of an Arbitrary Degree for Stability Analysis of Linear Systems With Time-Varying Delays

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ABSTRACT
This paper is concerned with the stability analysis problems of linear systems with time-varying delays using integral inequalities. To reduce the conservatism of stability criteria obtained with Lyapunov-Krasovskii approach, there has been a growing tendency to utilize various integral quadratic functions in the construction of Lyapunov-Krasovskii functionals. Consequently, integral inequalities also have played key roles to derive stability criteria guaranteeing the negativity of the Lyapunov-Krasovskii functional’s derivative. Recently, by utilizing first-degree or second-degree orthogonal polynomials, new free-matrix-based integral inequalities have been proposed for integral quadratic functions containing both a system state variable and its time derivative. This paper tries to generalize these inequalities and their stability criteria with a note on the relation among the existing integral inequalities and the proposed one. In this note, it is shown that increasing a degree of the proposed integral inequality only reduces or maintains the conservatism by deriving the hierarchical stability criteria. Four numerical examples including practical systems demonstrate the effectiveness of the proposed methods in terms of allowable upper delay bounds.

INDEX TERMS
Stability criteria, time delays, time-varying delay, integral inequality, linear matrix inequalities.

I. INTRODUCTION
Time delays are common phenomena arisen in many practical systems including chemical systems, biological systems, mechanical engineering systems, and networked control systems [1]. Such phenomena often cause undesirable dynamic behaviors such as control performance degradation or even system instability. It is thus an important issue to formulate systems into time-delay systems and to analyze the system stability before implementing other control strategies. Obviously, stability analysis for a linear system with time delays is the foundation because analysis techniques for this system can be extended to controller and filter synthesis problems of various systems including switched systems, neural network systems, fuzzy systems with time delays. Thus, this paper also considers linear systems with time delays.

Time delays can be classified into two types depending on delay properties: constant time delays and time-varying delays. In the case of constant time delays, it has been noted that necessary and sufficient conditions guaranteeing system stability can be obtained using complete Lyapunov-Krasovskii functional (LKF) [2]. However, in the case of time-varying delays, there still exist open problems due to an infinite dimensional property of this class of systems. For stability analysis of time-delay systems, the Lyapunov-Krasovskii (L-K) approach has been one of popular methods to derive numerically tractable optimization problems, where stability conditions are usually formulated into linear matrix inequalities (LMIs). In the L-K approach, the negativity of LKF’s derivative guarantees asymptotic stability of relevant systems. Consequently, with use of the L-K approach, construction of LKF candidate and numerical methods to derive LMIs guaranteeing negativity of LKF’s derivative have been hot issues to reduce the conservatism.
of stability criteria [3]. In the construction of LKFs, there has been a growing tendency to utilize more integral quadratic functions made of system state variables and free matrices [1, 4–15]. Inevitably, the time derivative of LKF candidate also contains integral quadratic functions which cannot be straightforwardly utilized as LMI conditions. Thus, in the literature [16]–[21], Jensen inequality [22] has been frequently utilized to obtain a convex upper bound of LKF’s derivative. In the presence of time-varying delays, however, the Jensen inequality provides reciprocally convex upper bounds with respect to the length of an integral interval, which leads to non-convex stability conditions. To make a non-convex bound into a convex one by employing free variables, lower bound lemmas for reciprocal convexity [23], [24] have been simultaneously utilized or Moon et al.’s inequality [25] have been employed instead of Jensen inequality.

The negative convex upper bound of LKF’s derivative sufficiently guarantees an asymptotic stability of time-delay systems. Therefore, with use of orthogonal polynomials and free matrices, there have been numerous efforts to extend Jensen inequality and Moon et al.’s inequality into more generalized ones which can reduce the bounding gap between a LKF’s derivative and its convex upper bound [11], [24], [26]–[33]. According to whether or not free matrices are utilized, existing integral inequalities can be categorized as extended Moon et al.’s inequalities or extended Jensen inequalities. The first category includes a free-matrix-based integral inequality [32], Chen’s affine inequalities [34], a polynomials-based integral inequality [8], orthogonal-polynomials-based integral inequalities [11], [33], and an affine Bessel-Legendre (B-L) inequality [12]. The second category includes the Wirtinger-based integral inequality [24], [26], auxiliary-function-based integral inequalities [27], [28], Chen’s inequalities [29], and a B-L inequality [30], [31].

A common feature of these integral inequalities is that the integral quadratic function only consists of a system state or its derivative, separately. Recently, new free-matrix-based integral inequalities [35], [36] for integral quadratic functions containing both a system state and its time derivative have been proposed. In these papers, with a construction of LKF containing correlated integral terms of a system state and its derivative, the new free-matrix-based integral inequalities have effectively reduced the conservatism of stability criteria. However, since the integral inequalities proposed in [35], [36] are constructed only with use of zero-, first-, and second-degree orthogonal polynomials, there still exists a room for improvement. Such observations motivate our work.

This paper aims at deriving sufficient LMI conditions guaranteeing that a linear system with a time-varying delay is asymptotically stable with the allowable upper delay bound as large as possible. To archive this goal, the main contributions of this paper can be summarized as follows:

- An affine inequality of an arbitrary degree is proposed by fully utilizing Legendre polynomials. Further, a note on the relation among exiting inequalities and the proposed one is provided for effective comparisons.

In this note, it is shown that the proposed integral inequality is a generalized version of the free-matrix-based integral inequalities [35], [36] and the orthogonal polynomials-based integral inequalities [11], [33]. Further, it is proven that increasing a degree of the proposed integral inequality only reduces a bounding gap between an integral quadratic function and its affine upper bounds.

- Based on the proposed inequality, hierarchical stability conditions are developed. Since an arbitrary degree is an non-negative integer to be chosen, it is hard to compare numerical results of all degrees. Thus, with the hierarchical stability criteria, a note on the relation between a degree and corresponding stability criteria is provided. In this note, it is demonstrated that increasing a degree of the proposed integral inequality only reduces or maintains the conservatism.

Four numerical examples of stability analysis for linear systems with time-varying delays are given. In these examples including practical systems, the effectiveness of the proposed methods are shown in terms of allowable upper delay bounds for a given lower delay bounds.

Notations: $X > 0$ ($X \succeq 0$) means that $X$ is a real symmetric positive definite matrix (positive semidefinite). $\text{col} \{x_1, x_2, \ldots , x_n\} = \{x_1^T , x_2^T , \ldots , x_n^T \}^T$. $X \otimes Y$ means $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. $\text{He}(M)$ denotes $M + M^T$. $\mathbb{N}$ and $\mathbb{R}^n$ denote the set of positive integer and the $n$-dimensional Euclidean space, respectively. $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integer. $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent the set of symmetric matrices and the set of positive definite matrices of $\mathbb{R}^{n \times n}$, respectively. The matrix $I_n$ represents the identity matrix in $\mathbb{R}^{n \times n}$. The notation $0_{m,n}$ stands for the matrix in $\mathbb{R}^{n \times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. $\otimes$ denotes Kronecker product.

II. AN AFFINE INTEGRAL INEQUALITY OF AN ARBITRARY DEGREE

This section introduces several lemmas, a definition, and a delayed system formulation. Then, an affine integral inequality of an arbitrary degree is proposed. Consider the following linear system with a time-varying delay:

$$
\begin{cases}
\dot{x}(t) = Ax(t) + A_dx(t-h(t)) & \forall t \geq 0 \\
x(0) = \phi(0) & \forall t \in [-h_2, 0].
\end{cases}
$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $\phi(t)$ is an initial condition, $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices, $h(t)$ is a time-varying delay satisfying

$$
h(t) \in [h_1, h_2] \quad \text{s.t.} \quad 0 \leq h_1 \leq h_2, \quad h_1, h_2 \in \mathbb{R},
$$

$$
h_{12} = h_2 - h_1.
$$

Lemma 1 [8]: For scalars $i \in \mathbb{Z}_{\geq 0}, a, b \in \mathbb{R}$, let $x(r) \in \mathbb{R}^{n \times n}$ be an integrable function: $(x(r))r \in [a, b]$. Then,
we have
\[
\frac{i+1}{b-a} \int_a^b \left( \frac{r-a}{b-a} \right)^i x(r)dr = I_i(a, b),
\]
where
\[
I_i(a, b) = \frac{(i+1)!}{(b-a)^{i+1}} \times \int_a^b \int_{r_1}^{r_2} \cdots \int_{r_{i+1}}^{b} x(r_{i+1})dr_{i+1} \cdots dr_1,
\]
and
\[
r_0 = a.
\]

Definition 1 (Legendre Polynomials [30]): For scalars \(a, b \in \mathbb{R}, a < b\), Legendre polynomials over the integral interval \([a, b]\) can be defined as follows:

\[
L_i(r) = \sum_{j=0}^{i} \binom{i}{j} \left( \frac{r-a}{b-a} \right)^j \quad \forall r \in [a, b],
\]

where
\[
\binom{i}{j} = \frac{(-1)^{i+j} (i+j)!}{j! (i-j)!}.
\]

This polynomial function satisfies the following properties:

1) Boundary conditions
\[
L_i(b) = 1, \quad L_i(a) = (-1)^i.
\]

2) Orthogonality
\[
\int_a^b L_i(r) L_j(r)dr = \begin{cases} 
0 & \text{if } i \neq j, \\
\frac{b-a}{2i+1} & \text{if } i = j.
\end{cases}
\]

Before deriving stability criteria, we propose the following affine integral inequality of an arbitrary degree \(N \in \mathbb{Z}_{\geq 0}\).

Lemma 2: For scalars \(a, b \in \mathbb{R}\), let \(x(t) \in \mathbb{R}^n\) be a continuous function: \(x(t): t \in [a, b]\). Then, for matrices \(R \in \mathbb{S}^n_+\), \(X_i \in \mathbb{R}^{q \times n}\) \((i \in \{0, N\} \cap \mathbb{Z})\), \(Y_j \in \mathbb{R}^{q \times n}\) \((j \in \{0, N - 1\} \cap \mathbb{Z})\), an arbitrary vector \(\zeta(t) \in \mathbb{R}^q\), integers \(N \in \mathbb{Z}_{\geq 0}\), the following inequality holds:

\[
- \int_a^b \left[ x(t) \right]^T R \left[ x(t) \right]dr \leq \Omega_N(a, b),
\]

where
\[
\Omega_N(a, b) = (b-a)x^T(t)X_N R_N^{-1} S_N^T \zeta(t) + He \left[ \zeta^T(t) \left( \sum_{j=0}^{N} X_j \Gamma_j(a, b) + \sum_{j=1}^{N-1} Y_j \Upsilon_j(a, b) \right) \right],
\]
\[
S_N = \begin{bmatrix} [X_0 Y_0] & [X_1 Y_1] & \cdots & [X_{N-1} Y_{N-1}] & [X_N Y_N] \end{bmatrix},
\]
\[
Y_N = 0_{q,n},
\]
\[
R_N = R \oplus 3R \oplus \cdots \oplus (2N + 1)R,
\]
\[
\Gamma_j(a, b) = \int_a^b L_j(r) x(r)dr.
\]

Then, the following equalities hold.

\[
\bar{S}_N = \begin{bmatrix} [X_0 Y_0] & [X_1 Y_1] & \cdots & [X_{N-1} Y_{N-1}] & [X_N Y_N] \end{bmatrix},
\]

Then, the following equalities hold.

\[
\begin{align}
\bar{S}_N &= \begin{bmatrix} [X_0 Y_0] & [X_1 Y_1] & \cdots & [X_{N-1} Y_{N-1}] & [X_N Y_N] \end{bmatrix}, \\
\bar{S}_N^{1/2} &= \begin{bmatrix} [X_0 Y_0] & [X_1 Y_1] & \cdots & [X_{N-1} Y_{N-1}] & [X_N Y_N] \end{bmatrix}, \\
\bar{S}_N^{1/2} &= \begin{bmatrix} [X_0 Y_0] & [X_1 Y_1] & \cdots & [X_{N-1} Y_{N-1}] & [X_N Y_N] \end{bmatrix},
\end{align}
\]

Utilizing the equalities (20)-(23), the following equality holds.

\[
\int_a^b \begin{bmatrix} \bar{x}(r) \\ \bar{\zeta}(r) \end{bmatrix}^T \bar{S}_N^{-1} \bar{S}_N^{1/2} \bar{S}_N^{-1/2} \bar{S}_N \begin{bmatrix} \bar{x}(r) \\ \bar{\zeta}(r) \end{bmatrix}dr
\]
\[ \int_a^b \begin{bmatrix} \dot{x}(r) \\ x(r) \end{bmatrix}^T R \begin{bmatrix} \dot{x}(r) \\ x(r) \end{bmatrix} dr + \Omega_N(a, b). \]

Based on Schur complement, the following augmented matrix is positive semidefinite since \( R \) is a positive definite matrix.

\[
\begin{bmatrix}
S_N R^{-1} S_N^T & \tilde{S}_N \\
\bar{S}_N & R
\end{bmatrix} \geq 0.
\]

Therefore, the following inequality holds.

\[
\int_a^b \begin{bmatrix} \dot{x}(r) \\ x(r) \end{bmatrix}^T R \begin{bmatrix} \dot{x}(r) \\ x(r) \end{bmatrix} dr + \Omega_N(a, b) \geq 0.
\]

It ends the proof.

For comparisons, the orthogonal polynomials based integral inequalities [11], [33] are given as follows:

**Lemma 3:** [11], [33] For scalars \( a, b \in \mathbb{R}, \) let \( x(t) \in \mathbb{R}^n \) be a continuous function: \( x(t) : t \in [a, b] \). Then, for integers \( N \in \mathbb{Z}_{\geq 0}, \) an arbitrary vector \( \zeta(t) \in \mathbb{R}^d, \) positive definite matrices \( R_1, R_2 \in \mathbb{S}_n^+, \) and matrices \( M_1 \in \mathbb{R}^{q \times (N+1)n}, M_2 \in \mathbb{R}^{q \times Nn}, \) the following inequalities hold:

\[
- \int_a^b \dot{x}(r) R_1 \dot{x}(r) dr \leq \tilde{\Omega}_{1N}(a, b),
\]

\[
- \int_a^b x^T(r) R_2 x(r) dr \leq \tilde{\Omega}_{2N-1}(a, b),
\]

where

\[
\tilde{\Omega}_{1N}(a, b) = (b-a) \zeta^T(t) M_1 \bar{R}_N^{-1} M_1^T \zeta(t)
+ He \left\{ \zeta^T(t) M_1 \bar{\Gamma}_N(a, b) \right\},
\]

\[
\tilde{\Omega}_{2N-1}(a, b) = (b-a) \zeta^T(t) M_2 R_{N-1}^{-1} M_2^T \zeta(t)
+ He \left\{ \zeta^T(t) M_2 \bar{\Upsilon}_{N-1}^{-1}(a, b) \right\},
\]

\[
\bar{\Gamma}_N(a, b) = \text{col} \{ \bar{Y}_0, \bar{Y}_1, \ldots, \bar{F}_N \},
\]

\[
\bar{\Upsilon}_{N-1}^{-1}(a, b) = \text{col} \{ \bar{\Upsilon}_0, \bar{\Upsilon}_1, \ldots, \bar{\Upsilon}_{N-1} \}.
\]

**Remark 1:** This remark discusses the relation among the proposed inequality and the existing inequalities in the literature [11], [12], [30], [33], [35], [36]. Compared to the inequalities [35], [36], the proposed integral inequality is constructed by fully utilizing Legendre polynomials and thus provides the upper bound of an arbitrary degree \( N \). When \( N = 1, 2, \) the proposed inequality reduces to the free-matrix based integral inequalities of [35], [36], respectively. In [11], it was shown that an orthogonal-polynomials-based integral inequality [11] is more general than the B-L inequality [30] and affine B-L inequality [12]. Differently from the orthogonal-polynomials-based integral inequalities [11], [33], Lemma 2 provides the upper bound of the following augmented integral quadratic function which contains both the system state vector \( x(t) \) and its time derivative \( \dot{x}(t) \).

\[
- \int_a^b \dot{x}(r) R \dot{x}(r) dr, \quad R \in \mathbb{S}_n^+.
\]

Here, the positive definite matrix \( R \) is a free variable to be chosen. Thus, if

\[
R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \quad \text{s.t.} \quad R_1, R_2 \in \mathbb{S}_n^+, \quad (33)
\]

Lemma 2 reduces to the following summation of the orthogonal-polynomials-based integral inequalities in [11], [33].

\[
- \int_a^b \dot{x}(r) R_1 \dot{x}(r) dr - \int_a^b x^T(r) R_2 x(r) dr
\leq \tilde{\Omega}_{1N}(a, b) + \tilde{\Omega}_{2N-1}(a, b).
\]

Here, utilizing the free matrices \( x_i, (i \in [0, N] \cap \mathbb{Z}) \) and \( Y_j, (j \in [0, N-1] \cap \mathbb{Z}) \) in Lemma 2, the free matrices \( M_1 \) and \( M_2 \) in Lemma 3 can be represented as follows:

\[
M_1 = [X_0 X_1 \cdots X_N],
\]

\[
M_2 = [Y_0 Y_1 \cdots Y_{N-1}].
\]

Based on such discussions, it is clearly shown that the proposed integral inequality is more general than the existing inequalities in the literature [11], [12], [30], [33], [35].

**Remark 2:** This remark provides a note on the effectiveness of the generalized integral inequality in Lemma 2. For the same vector \( \zeta(t) \), the relation between the upper bounds of the degree \( N+1 \) and \( N \) can be obtained as follows:

\[
\Omega_{N+1}(a, b) - \Omega_N(a, b) = \zeta^T(t) \left( \frac{b-a}{2N+1} [X_N Y_N] R^{-1} [X_N Y_N]^T - \frac{b-a}{2N+3} [X_{N+1} 0] R^{-1} [X_{N+1} 0]^T \right) \zeta(t).
\]

\[
\bar{\Omega}_{1N}(a, b) - \bar{\Omega}_{2N-1}(a, b) = \zeta^T(t) [X_N Y_N] \bar{\Upsilon}_{N-1}^{-1}(a, b) \bar{\Upsilon}_{N-1}(a, b),
\]

where the matrices \( X_N, Y_N \) are free variables of the upper bound \( \Omega_{N+1}(a, b) \). If the matrices \( \bar{X}_N, \bar{Y}_N \) are defined by

\[
\bar{X}_{N+1} = \frac{-2N+3}{b-a} \gamma_{N+1} W_1^{-1},
\]

\[
\bar{Y}_N = 0,
\]

\[
\gamma_{N+1} \zeta(t) = \bar{\Gamma}_N(a, b),
\]

\[
R^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix},
\]

the equality (34) reduces to the following inequality:

\[
\Omega_{N+1}(a, b) - \Omega_N(a, b) = \frac{-2N+3}{b-a} \bar{\Gamma}_N^T(a, b) W_1^{-1} \bar{\Gamma}_N(a, b) < 0.
\]

Since \( R \) is a positive definite matrix, \( R^{-1}, W_1 \), and \( W_1^{-1} \) are also positive definite matrices. Thus, increasing a degree \( N \) of the inequality (11) only reduces a bounding gap between
III. APPLICATIONS TO STABILITY ANALYSIS OF LINEAR SYSTEMS WITH TIME-VARYING DELAYS

This section derives stability criteria for linear systems with time-varying delays by employing the proposed inequality. Before deriving main results, several vectors and matrices are defined as follows:

\[
\xi_N(t) = \text{col}\{x(t), x(t - h_1), x(t - h(t)), x(t - h_2)\},
\]

\[
I_0(t - h_1, t - h(t)), I_0(t - h(t), t - h_2),
\]

\[
\mathbb{I}_{N-1}(t - h_1, t) \in \mathbb{R}^{(N + 6)n},
\]

\[
e_i = \begin{cases} 0 & \text{if } i \in [1, N + 6] \cap \mathbb{Z}, \\ e_0 = A e_1 + A_d e_3. 
\end{cases}
\]

For \( P_N \in \mathbb{S}_{(N+2)n}^+, Z_1, Z_2 \in \mathbb{S}_n^+, R_1, R_2 \in \mathbb{S}_{2n}, N \in \mathbb{N}, \) we introduce a simple LKF:

\[
V(t) = V_{1N}(t) + \sum_{i=2}^{N} V_i(t),
\]

\[
V_{1N}(t) = \eta_{1N}(t) P_N \eta_{1N}(t),
\]

\[
V_2(t) = \int_{t-h_1}^{t} x^T(r) Z_1 x(r) dr,
\]

\[
V_3(t) = \int_{t-h_2}^{t} x^T(r) Z_2 x(r) dr,
\]

\[
V_4(t) = \int_{t-h_1}^{t} \int_{t-s}^{t} \left[ \frac{\partial}{\partial s} x(r) \right]^T R_1 \left[ \frac{\partial}{\partial s} x(r) \right] ds dr,
\]

\[
V_5(t) = \int_{t-h_2}^{t} \int_{t-s}^{t} \left[ \frac{\partial}{\partial s} x(r) \right]^T R_2 \left[ \frac{\partial}{\partial s} x(r) \right] ds dr,
\]

where

\[
\eta_{1N}(t) = \text{col}\{x(t), \int_{t-h_1}^{t} x(r) dr, h_1 \mathbb{I}_{N-1}(t - h_1, t)\}
\]

\[
= \left( \frac{h_2 - h(t) - h_1}{h_2} x_1(t) + \frac{h(t) - h_1}{h_2} x_2(n) \right) \xi_N(t).
\]

\[
\chi_{1N} = \text{col}\{e_1, e_1 e_2, e_1 e_3, \cdots, e_1 e_{N+6}\},
\]

\[
\chi_{2N} = \text{col}\{e_1, e_1 e_2, e_1 e_3, \cdots, e_1 e_{N+6}\},
\]

\[
\mathbb{I}_{N-1}(a, b) = \text{col}\{I_0(a, b), I_1(a, b), \cdots, I_{N-1}(a, b)\}.
\]

Utilizing the presented LKF (43) and Lemma 2, we have the following theorem.

Theorem 1: Given scalars \( h_1, h_2 \in \mathbb{R} \) and \( N \in \mathbb{N}, \) the system (1) is asymptotically stable if there exist matrices \( P_N \in \mathbb{S}_{(N+2)n}^+, R_1, R_2 \in \mathbb{S}_{2n}, Z_1, Z_2 \in \mathbb{S}_n^+, X_{ij}(j \in [0, N - 1] \cap \mathbb{Z}), \)

\[
Y_{ij} (j \in [0, N - 1] \cap \mathbb{Z}), X_{20}, X_{21}, X_{30}, X_{31}, Y_{20}, Y_{30} \in \mathbb{R}^{(N+6)n \times n} \]

such that

\[
\left[ \begin{array}{cccc}
\Lambda_0 + \Lambda_{1N} & \sqrt{h_1} S_{11N} & \sqrt{h_1} S_{12N} \\
\sqrt{h_1} S_{11N}^T & -\bar{R}_{1N} & 0 \\
\sqrt{h_1} S_{12N}^T & 0 & -\bar{R}_{2}
\end{array} \right] < 0,
\]

\[
\text{where}
\]

\[
\bar{R}_{1N} = R_1 \oplus 3 R_1 \oplus \cdots \oplus (2N + 1) R_1,
\]

\[
\bar{R}_2 = R_2 \oplus 3 R_2,
\]

\[
G_{1N} = e_1 - (-1)^N e_2 - \sum_{j=0}^{N-1} \eta_{j}^N e_{j+7},
\]

\[
G_{20} = e_2 - e_3,
\]

\[
G_{30} = e_3 - e_4,
\]

\[
U_{1N} = \sum_{j=0}^{N-1} \eta_{j}^N e_{j+7},
\]

\[
U_{20} = e_5, U_{30} = e_6,
\]

\[
S_{11N} = [X_{10} Y_{10}] [X_{11} Y_{11}]
\]

\[
\cdots [X_{(N-1)} Y_{(N-1)}] [X_{1N} 0, n],
\]

\[
S_2 = [X_{20} Y_{20}] [X_{21} 0, n],
\]

\[
S_3 = [X_{30} Y_{30}] [X_{31} 0, n],
\]

\[
\lambda_0 = e_1^T Z_1 e_1 - e_2^T Z_2 e_2 + e_3^T Z_2 e_2 - e_4^T Z_2 e_4
\]

\[
+ h_1 \left[ \begin{array}{c}
e_0 \\
e_1 \\
e_2 \\
e_3
\end{array} \right]^T R_1 \left[ \begin{array}{c}
e_0 \\
e_1 \\
e_2 \\
e_3
\end{array} \right] + h_1 \left[ \begin{array}{c}
e_0 \\
e_1 \\
e_2 \\
e_3
\end{array} \right]^T R_2 \left[ \begin{array}{c}
e_0 \\
e_1 \\
e_2 \\
e_3
\end{array} \right] + H e_1 \right]
\]

\[
\Lambda_{1N} = H e_1 \left[ \begin{array}{c}
\sum_{j=0}^{N-1} X_{1j} Y_{1j} + h_1 \sum_{j=0}^{N-1} Y_{1j} U_{1j}
\end{array} \right],
\]

\[
\Lambda_{2N} = H e_1 \left[ \begin{array}{c}
\sum_{j=0}^{N-1} X_{1j} Y_{1j} + h_1 \sum_{j=0}^{N-1} Y_{1j} U_{1j}
\end{array} \right],
\]

\[
\chi_{3N} = \left\{ \begin{array}{ll}
\text{col}\{e_0, e_2 - e_4, e_1 - e_2\} & \text{if } N = 1 \\
\text{col}\{e_0, e_2 - e_4, e_1 - e_2, 2(e_1 - e_2), 3(e_1 - e_2), \cdots, N(e_1 - e_{N+5})\} & \forall N \geq 2.
\end{array} \right.
\]

Proof: The time-derivative of the LKF (43) along the trajectory of (1) is obtained as follows:

\[
\dot{V}(t) = V_{1N}(t) + \sum_{i=2}^{N} \dot{V}_i(t),
\]

\[
\dot{V}_{1N}(t) = 2 \eta_{1N}(t) P_N \eta_{1N}(t),
\]

\[
\dot{V}_2(t) = x^T(t) Z_1 x(t) - x^T(t - h_1) Z_1 x(t - h_1),
\]

\[
\dot{V}_3(t) = x^T(t - h_1) Z_2 x(t - h_1) - h_2 x^T(t - h_2) Z_2 x(t - h_2),
\]

\[
\dot{V}_4(t) = h_1 \left[ \dot{x}(t) \right]^T R_1 \left[ \dot{x}(t) \right] + Q_1(t).
\]
\[\dot{V}_3(t) = h_{12} \left[ \dot{x}(t) \right]^T R_2 \left[ \dot{x}(t) \right] + Q_2(t) + Q_3(t), \quad (74)\]

where

\[Q_1(t) = -\int_{t-h_1}^{t} \left[ \dot{x}(r) \right]^T R_1 \left[ \dot{x}(r) \right] dr, \quad (75)\]

\[Q_2(t) = -\int_{t-h_2}^{t-h_1} \left[ \dot{x}(r) \right]^T R_2 \left[ \dot{x}(r) \right] dr, \quad (76)\]

\[Q_3(t) = -\int_{t-h_1}^{t-h_2} \left[ \dot{x}(r) \right]^T R_2 \left[ \dot{x}(r) \right] dr. \quad (77)\]

When \([a, b] = [t - h_1, t], \) \(i \in [0, N - 1] \cap \mathbb{Z},\)

\[h_1 \frac{d}{dt} I_i(t - h_1, t) = (i + 1) x(t) \]

\[-\frac{i(i + 1)}{h_1} \int_{t-h_1}^{t} \left( r - t - h_1 \right)^{i-1} x(r) dr = (i + 1)(x(t) - I_{i-1}(t - h_1, t)). \quad (78)\]

Then,

\[h_1 \frac{d}{dt} i_1(t - h_1, t) = \begin{bmatrix} x(t) - x(t - h_1) \\ 2(x(t) - I_0(t - h_1, t)) \\ \vdots \\ (i + 1)(x(t) - I_{i-1}(t - h_1, t)) \end{bmatrix}, \quad (79)\]

where

\[I_{-1}(a, b) = x(a). \quad (80)\]

Thus, it holds that

\[\dot{\eta}_N(t) = col\{\dot{x}(t), x(t - h_1) - x(t - h_2), \]

\[h_1 \dot{I}_{N-1}(t - h_1, t)\}

\[\dot{\xi}_N(t). \quad (81)\]

For the integral quadratic functions (75)-(77), the proposed integral inequality in Lemma 2 can be utilized. In Lemma 2, since \(\zeta(t)\) is an arbitrary function of time, let us define

\[\zeta(t) = \dot{\xi}_N(t). \quad (82)\]

where \(\dot{\xi}_N(t)\) is the augmented vector (40) containing whole system state variables. According to an integral interval \([a, b],\) the vectors \(\Gamma_i(a, b)\) and \(\Upsilon_i(a, b)\) in Lemma 2 are defined as follows:

1) If \([a, b] = [t - h_1, t],\)

\[\Gamma_i(a, b) = G_{1i}\dot{\xi}_N(t), \quad (83)\]

\[\Upsilon_i(a, b) = h_1 U_1\dot{\xi}_N(t). \quad (84)\]

2) If \([a, b] = [t - h(t), t - h_1],\)

\[\Gamma_i(a, b) = G_{2i}\dot{\xi}_N(t), \quad (85)\]

\[\Upsilon_i(a, b) = (h(t) - h_1)U_2\dot{\xi}_N(t). \quad (86)\]

3) If \([a, b] = [t - h_2, t - h(t)],\)

\[\Gamma_i(a, b) = G_{3i}\dot{\xi}_N(t), \quad (87)\]

\[\Upsilon_i(a, b) = (h_2 - h(t))U_3\dot{\xi}_N(t). \quad (88)\]

Utilizing Lemma 2 with the definitions (83)-(88), the upper bounds of integral quadratic functions (75)-(77) can be obtained as follows:

\[Q_1(t) \leq \dot{\xi}_N^T(t) \left( h_1 S_1 N \xi_N^{-1} S_1^T N \right) + H e \sum_{j=0}^{N} X_{ij} G_{ij} + h_1 \sum_{j=0}^{N} Y_{ij} S_2 \quad (89)\]

\[Q_2(t) \leq \dot{\xi}_N^T(t) \left( h(t) - h_1 S_2 R_2^{-1} S_2^T \right) + H e \left( X_{20} G_{20} + X_{21} G_{21} \right. \]

\[+ (h(t) - h_1) Y_{20} U_2) \quad \} \dot{\xi}_N(t), \quad (90)\]

\[Q_3(t) \leq \dot{\xi}_N^T(t) \left( h_2 - h(t) S_2 R_2^{-1} S_2^T \right) + H e \left( X_{30} G_{30} + X_{31} G_{31} \right. \]

\[+ (h_2 - h(t)) Y_{30} U_3) \quad \} \dot{\xi}_N(t). \quad (91)\]

By combining (1)-(91), the time derivative of the LKF (43) can be bounded as follows:

\[\dot{V}(t) \leq \dot{\xi}_N^T(t) \Xi_N(h(t)) \dot{\xi}_N(t), \quad (92)\]

where

\[\Xi_N(h(t)) = \left( \begin{array}{c} h_2 - h(t) \left( \frac{h_2}{h_{12}} \right) \left( \Lambda_0 + \Lambda_{1N} + \Pi_{1N} \right) \\
+ \left( \frac{h(t) - h_1}{h_{12}} \right) \left( \Lambda_0 + \Lambda_{2N} + \Pi_{2N} \right) \end{array} \right), \quad (93)\]

\[\Pi_{1N} = h_1 S_1 N \xi_N^{-1} S_1^T N + h_1 S_2 R_2^{-1} S_2^T, \quad (94)\]

\[\Pi_{2N} = h_1 S_1 N \xi_N^{-1} S_1^T N + h_1 S_2 R_2^{-1} S_2^T. \quad (95)\]

\(\Xi_N(h(t))\) is convex with respect to \(h(t) \in [h_1, h_2].\) Thus, based on a convex combination property, \(\Xi_N(h(t)) < 0\) if and only if \(\Xi_N(h_1) < 0\) and \(\Xi_N(h_2) < 0.\) Based on Schur complement process, \(\Xi_N(h_1) < 0\) and \(\Xi_N(h_2) < 0\) are guaranteed if conditions (53)-(54) hold. It ends the proof. ■

The stability conditions formulated into LMIs in Theorem 1 form a hierarchy. In other words, increasing a degree \(N\) in Theorem 1 can only reduce or maintain the conservatism. Based on the following theorem, it can be shown that Theorem 1 at the degree \(N + 1\) is less conservative than that of the degree \(N.

**Theorem 2:** For the time-delay system (1), a set \(S_N(h_1, h_2)\) given by

\[S_N(h_1, h_2) = \left\{ h_1, h_2 \in \mathbb{R} : \exists P_N \in S_+^{(N+3)n}, \right. \]

\[R_1, R_2 \in S_2^{+}, Z_1, Z_2 \in S_4^{+}, \]

\[X_{i1} (i \in [0, N] \cap \mathbb{Z}, Y_{ij} (j \in [0, N - 1] \cap \mathbb{Z}, \]

\[X_{20}, X_{21}, X_{30}, X_{31}, Y_{20}, Y_{30} \in \mathbb{R}^{q \times n} \]

\[s.t. \Xi_N(h_1) < 0, \Xi_N(h_2) < 0 \]

satisfies

\[S_N(h_1, h_2) \subset S_{N+1}(h_1, h_2), \quad \forall N \in \mathbb{N}. \quad (97)\]
Proof: Assume that \( S_N(h_1, h_2) \) is not an empty set since it is a trivial inclusion. The positive definite matrices in Theorem 1 are free variables to be chosen. With the LKF (43), let us define
\[
P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & \epsilon I \end{bmatrix},
\]
where \( \epsilon > 0 \) is also a scalar to be chosen. Then, it holds that
\[
V_{N+1}(t) = V_N(t) + \epsilon h_1^2 I_T^T (t - h_1, t) I_N(t - h_1, t) > 0 \text{ if } V_N(t) > 0.
\]
With the positive definite matrix \( P_{N+1} \),
\[
\dot{V}_{1(N+1)}(t) - \dot{V}_{1N}(t) = 2\eta_{N+1}(t) \begin{bmatrix} P_N & 0 \\ 0 & \epsilon I \end{bmatrix} \eta_{N+1}(t) - 2\eta_{N}(t) P_N \eta_{N}(t)
\]
\[= 2\epsilon h_1(N + 1)(x(t) - I_{N-1}(t - h_1, t))^T \times I_N(t - h_1, t).
\]
Considering (100) and Remark 2 with
\[
\xi(t) = \xi_{N+1}(t),
\]
\[
S_{11}(N+1) = \begin{bmatrix} S_{1N} & 0 \\ 0 & [X_{1(N+1)} - 0] \end{bmatrix},
\]
\[
S_{21}(\xi(t) = \xi_{N+1}(t)) = \begin{bmatrix} S_{21} & 0 \\ 0 & S_{22}(\xi(t) = \xi_{N+1}(t)) \end{bmatrix},
\]
the following equality holds:
\[
\xi_{N+1}^T(t) S_{N+1}(h(t)) \xi_{N+1}(t) - \xi_N^T(t) S_N(h(t)) \xi_N(t)
\]
\[= \xi_{N+1}(t) \left[ \begin{bmatrix} 0 \\ S_{1N} \end{bmatrix} \right] \xi_{N+1}(t)
\]
\[= 2\epsilon h_1(N + 1)(x(t) - I_{N-1}(t - h_1, t))^T I_N(t - h_1, t) - \frac{2N + 3}{h_1} \xi_{N+1}^T(t) G_{T(1+N)} W_{11}^{-1} G_{11(N+1)} \xi_{N+1}(t),
\]
where
\[
R_{11}^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{13} \end{bmatrix}
\]
(101). Then, the time derivative of the LKF (43) can be bounded as follows:
\[
\dot{V}(t) \leq \xi_{N+1}^T(t) S_{N+1}(h(t)) \xi_{N+1}(t)
\]
\[= \xi_N^T(t) S_N(h(t)) \xi_N(t)
\]
\[+ 2\epsilon h_1(N + 1)(x(t) - I_{N-1}(t - h_1, t))^T I_N(t - h_1, t) - \frac{2N + 3}{h_1} \xi_{N+1}^T(t) G_{T(1+N)} W_{11}^{-1} G_{11(N+1)} \xi_{N+1}(t)
\]
\[\leq \xi_N^T(t) S_N(h(t)) \xi_N(t),
\]
(102) where \( S_{N+1}(h(t)) \) is also convex with respect to \( h(t) \in [h_1, h_2] \). Thus, for a sufficiently small \( \epsilon > 0 \), the conditions \( \Sigma_N(h_1) < 0 \), \( \Sigma_N(h_2) < 0 \) also guarantee \( \Sigma_{N+1}(h_1) < 0 \), \( \Sigma_{N+1}(h_2) < 0 \). Consequently, a stability region verified by \( \Sigma_{N+1}(h(t)) < 0 \) is larger than \( \Sigma_N(h(t)) < 0 \), and the relation (97) holds.

Remark 3: For the effective comparison, we derive hierarchical stability criteria constructed with the Legendre polynomial function of an arbitrary degree \( N \in \mathbb{N} \). Theorem 2 shows that stability criteria in Theorem 1 with a degree \( N + 1 \) have, at least, the same conservatism obtained from Theorem 1 with a degree \( N \).

In Remark 1, it was shown that the proposed integral inequality in Lemma 2 is more general than those of Lemma 2. Since Lemma 3 is a special case of Lemma 2, the stability criteria utilizing the proposed inequality in Lemma 2 have the less or the same conservatism of the stability criteria based on Lemma 3. For a numerical comparison between Lemma 2 and Lemma 3, the following corollary can be obtained by utilizing Lemma 3 instead of Lemma 2.

Corollary 1: Given scalars \( h_1, h_2 \in \mathbb{R} \) and \( N \in \mathbb{N} \), the system (1) is asymptotically stable if there exist matrices \( P_N \in \mathbb{S}^n_{N+2} \), \( R_{pq} \in \mathbb{S}^n_0 \) \((p = 1, 2, q = 1, 2)\), \( Z_1, Z_2 \in \mathbb{S}^n_0 \), \( X_{11} \in [0, N] \cap \mathbb{Z} \), \( Y_{1j} \in [0, N - 1] \cap \mathbb{Z} \), \( X_{20}, X_{21}, X_{32}, Y_{20}, Y_{30} \in \mathbb{R}^{(N+6)n \times n} \) such that
\[
\begin{bmatrix}
\tilde{\Lambda}_0 + \Lambda_{1N} & \sqrt{h_1} S_{1N} & \sqrt{h_1} S_{2N}
\end{bmatrix}
\begin{bmatrix}
\sqrt{h_1} S_{1N}^T & \vec{R}_{1N} & 0 \\
\sqrt{h_1} S_{2N} & 0 & \vec{R}_{2N}
\end{bmatrix} < 0,
\]
(103)
\[
\begin{bmatrix}
\tilde{\Lambda}_0 + \Lambda_{2N} & \sqrt{h_1} S_{1N} & \sqrt{h_1} S_{2N}
\end{bmatrix}
\begin{bmatrix}
\sqrt{h_1} S_{1N}^T & -\vec{R}_{1N} & 0 \\
\sqrt{h_1} S_{2N} & 0 & -\vec{R}_{2N}
\end{bmatrix} < 0,
\]
(104) where
\[
\tilde{\Lambda}_0 = e_1^T Z_1 e_1 - e_1^T Z_1 e_2 + e_1^T Z_2 e_2 - \epsilon_4^T Z_2 e_4 + h_1 (e_1^T R_{11} e_0 + e_1^T R_{12} e_1) + h_2 (\epsilon_4^T R_{21} e_0 + e_1^T R_{22} e_4)
\]
\[+ H e_2 X_{20} G_{20} + X_{21} G_{21} + X_{30} G_{30} + X_{31} G_{31},
\]
(105)
\[\vec{R}_{1N} = (R_{11} + R_{12}) \oplus (3R_{11} + 3R_{12}) \oplus \cdots \oplus (2N + 1)R_{11} \oplus (2N + 1)R_{12},
\]
(106)
\[\vec{R}_{2} = (R_{21} + R_{22}) \oplus (3R_{21} + 3R_{22}),
\]
(107) and the other notations are already defined in Theorem 1.

Proof: As stated in Remark 1, Lemma 3 is a special case of Lemma 2 adopting the condition (33). Therefore, if the positive definite matrices \( R_1 \) and \( R_2 \) in Theorem 1 is modified as
\[
R_1 = R_{11} + R_{12}, \quad R_2 = R_{21} + R_{22},
\]
(108)
Theorem 1 reduces to the Corollary 1 utilizing Lemma 3 instead of Lemma 2. In this case, the vector \( \tilde{\Lambda}_0 \) (65), the matrices \( \vec{R}_{1N} \) (55), \( \vec{R}_2 \) (56) are changed into (105)-(107), respectively. Then, the time derivative of the LKF (43) can be bounded as follows:
\[
\dot{V}(t) \leq \xi_{N+1}^T(t) S_{N+1}(h(t)) \xi_{N+1}(t),
\]
(109)
where
\[
\tilde{\gamma}(h(t)) = \left(\frac{h_2 - h(t)}{h_{12}}\right)\left(\tilde{\Lambda}_0 + \tilde{A}_{1N} + \tilde{\Pi}_{1N}\right) + \left(\frac{h(t) - h_1}{h_{12}}\right)\left(\tilde{\Lambda}_0 + \tilde{A}_{2N} + \tilde{\Pi}_{2N}\right),
\]
(110)
\[
\tilde{\Pi}_{1N} = h_1 S_{1N}\tilde{R}_{1N}^{-1}S_{1N}^T + h_{12} S_{3N} \tilde{R}_2^{-1} S_{3N}^T,
\]
(111)
\[
\tilde{\Pi}_{2N} = h_1 S_{1N}\tilde{R}_{1N}^{-1}S_{1N}^T + h_{12} S_{3N} \tilde{R}_2^{-1} S_{3N}^T.
\]
(112)

The other notations are already defined in Theorem 1. \(\tilde{\gamma}(h(t))\) is convex with respect to \(h(t) \in [h_1, h_2]\). Thus, based on a convex combination property, \(\tilde{\gamma}(h(t)) < 0\) if and only if \(\tilde{\gamma}(h_1) < 0\) and \(\tilde{\gamma}(h_2) < 0\). Based on Schur complement process, \(\tilde{\gamma}(h_1) < 0\) and \(\tilde{\gamma}(h_2) < 0\) are guaranteed if conditions (53)-(54) hold. It ends the proof.

Based on Corollary 1, it can be shown that there exist trade-offs between the conservatism and computational burden, which will be shown in four numerical examples. To effectively show the performance of the proposed single integral inequality, Theorem 1, Theorem 2, and Corollary 1 have been developed with the simple LKF (43).

**IV. NUMERICAL EXAMPLES**

In this section, numerical examples are presented to demonstrate the effectiveness of the proposed approaches.

*Example 1:* Consider the system (1) with the following matrices:
\[
A = \begin{bmatrix}
0 & 1.0 \\
-10.0 & -1.0
\end{bmatrix}, \quad A_d = \begin{bmatrix}
0.0 & 0.1 \\
0.1 & 0.2
\end{bmatrix}.
\]
(113)

In the field of stability analysis of time-delay systems, the conservatism has been checked in terms of the allowable upper bound \(h_2\) guaranteeing the asymptotic stability of the system (1) for the given lower bound \(h_1\). Table 2 lists the allowable upper bound of \(h(t)\) that guarantees the asymptotic stability of the system (1) with the matrices (113). To compare computational burden, numbers of variables are listed in Table 1. It can be seen that the proposed stability criteria in Theorem 1 provides the upper bound \(h_2\) which is larger than or similar to those in [8], [23], [24], [27], [35], [37]–[39]. It shows the effectiveness of the proposed integral inequality in terms of the conservatism. Since the proposed inequality of the degree \((N = 1)\) is the same as the free-matrix-based integral inequality [35], Theorem 1 \((N = 1)\) provides the same number of variables and the upper bound \(h_2\). Also, the result of [36] can be substituted with Theorem 1 \((N = 2)\) because the proposed integral inequality of degree \((N = 2)\) is the same as the free-matrix based integral inequality in [36]. Due to the generality of the proposed integral inequality, Corollary 1 utilizing Lemma 3 can be obtained from Theorem 1 by imposing certain structure on the free matrices in Theorem 1. This corollary is more conservative than Theorem 1, whereas it has less number of variables. Thus, the comparison between Theorem 1 and Corollary 1 clearly show trade-offs between number of variables and the conservatism. The proposed approaches show better performance in terms of the conservatism. State trajectories given in Figure 1 are obtained with time-varying delay \(h(t) = 1.3\sin(5t) + 2.34\) in Figure 2 and the initial condition \(x(0) = [1.0 \ 0.5]^T\).

In Table 1, Theorem 1 \((N = 4)\) demonstrates that the system (113) is asymptotically stable in \(h(t) \in [1.0, 3.68]\). Thus, all state responses converge when \(h(t) = 1.3\sin(5t) + 2.34\).

*Example 2:* Consider the system (1) with the following matrices:
\[
A = \begin{bmatrix}
0.0 & 1.0 \\
-12.0 & -2.0
\end{bmatrix}, \quad A_d = \begin{bmatrix}
0.0 & 0.1 \\
0.1 & 0.2
\end{bmatrix}.
\]
(114)

Table 2 shows the allowable upper bound of \(h(t)\) for given various \(h_1\). From the maximum delay bound \(h_2\) and Theorem 2, it is worth mentioning that the proposed stability criteria in Theorem 1 reduces more conservatism than those obtained in [8], [23], [24], [27], [35], [37]–[39]. State trajectories given in Figure 3 are obtained with an uniformly distributed random time-varying delay \(h(t) \in [0.9, 5.97]\) in Figure 4 and the initial condition \(x(0) = [1.0 \ 0.5]^T\). In Table 3, Theorem 1 \((N = 4)\) demonstrates that the system (113) is asymptotically stable in \(h(t) \in [0.9, 5.97]\). Thus, all state responses converge.

**TABLE 1.** Number of variables \(n\) is the dimension of the state vector \(x(t) \in \mathbb{R}^n\), \(N\) is the degree of Legendre polynomials \(L_N(t)\).

| Methods       | Number of variables |
|---------------|---------------------|
| Corollary 1   | \((2.5N^2 + 21N + 46)n^2 + (0.5N + 3)n\) |
| Corollary 1 \((N = 1)\) | 69.5n^2 + 3.5n |
| Corollary 1 \((N = 2)\) | 98n^2 + 4n |
| Corollary 1 \((N = 3)\) | 131.5n^2 + 4.5n |
| Corollary 1 \((N = 4)\) | 173n^2 + 6n |
| Theorem 1 \((N = 1)\) | 72.5n^2 + 4.5n |
| Theorem 1 \((N = 2)\) | 101n^2 + 5n |
| Theorem 1 \((N = 3)\) | 134.5n^2 + 5.5n |
| Theorem 1 \((N = 4)\) | 173n^2 + 6n |

**TABLE 2.** Allowable upper bound \(h_2\) for given \(h_1\).

| \(h_1\) | 0.30 | 0.70 | 1.00 | 2.00 |
|---------|------|------|------|------|
| [37]    | 0.97 | 1.21 | 1.49 | 1.79 |
| [23]    | 1.14 | 1.28 | 1.64 | 1.96 |
| [23]    | 1.26 | 1.48 | 1.83 | 2.12 |
| [39]    | 1.44 | 1.64 | 2.02 | 2.31 |
| [27]    | 1.78 | 2.13 | 2.70 | 2.96 |
| [24]    | 1.89 | 2.18 | 2.59 | 2.79 |
| [8]     | 2.00 | 2.34 | 2.91 | 3.15 |
| [35]    | 2.62 | 3.00 | 3.21 | 3.33 |

**TABLE 3.** Allowable upper bound \(h_2\) for given \(h_1\).

| \(h_1\) | 0.30 | 0.70 | 1.00 | 2.00 |
|---------|------|------|------|------|
| [37]    | 1.99 | 2.28 | 2.69 | 2.89 |
| [23]    | 2.00 | 2.34 | 2.95 | 3.19 |
| [23]    | 2.00 | 2.34 | 2.99 | 3.22 |
| [24]    | 2.00 | 2.34 | 2.99 | 3.24 |
| [35]    | 2.62 | 3.00 | 3.21 | 3.33 |

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Example 3: Consider the system (1) with the following matrices:

\[ A = \begin{bmatrix} 0.0 & 1.0 \\ -100.0 & -1.0 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}. \]  

(115)

For various \( h_1 \), the allowable upper bounds of \( h(t) \) guaranteeing this system asymptotically stable are listed in Table 4. It is worth noting that Theorem 1 reduces more conservatism as the degree \( N \) becomes larger. It clearly shows the effectiveness of the proposed integral inequality.
Example 4: Let us consider the following system coming from the dynamics of machining chatter [40].

\[
\dot{x}(t) = A_0 x(t) + B u(t), \\
y(t) = C x(t),
\]

where

\[
A_0 = \begin{bmatrix}
0.00 & 0.00 & 1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 1.00 \\
-10.00 & 10.00 & 0.00 & 0.00 \\
5.00 & -15.00 & 0.00 & -0.25
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0.00 & 0.00 & 1.00 & 0.00
\end{bmatrix}^T,
\]

\[
C = \begin{bmatrix}
1.00 & 0.00 & 0.00 & 0.00
\end{bmatrix}.
\]

A delayed static output feedback controller is utilized:

\[
u(t) = -K y(t) + K y(t - h(t)),
\]

where \(K\) is the gain of the controller (116), and \(h(t)\) is the time-varying delay of the system (1). Thus, the resulting dynamics is modeled by the system (1) with the following matrices:

\[
A = A_0 - BCK, \quad A_d = BCK.
\]

TABLE 5. (Example 4) allowable upper bound \(h_2\) for a given \(h_1 = 1.5000\) and \(K = 0.4\).

| \(h_1\) | \(h_2\) |
|---|---|
| Corollary 1 \((N = 1)\) | 1.5552 |
| Corollary 1 \((N = 2)\) | 1.7051 |
| Corollary 1 \((N = 3)\) | 1.7093 |
| Corollary 1 \((N = 4)\) | 1.7095 |
| Theorem 1 \((N = 1)\) | 1.5667 |
| Theorem 1 \((N = 2)\) | 1.7060 |
| Theorem 1 \((N = 3)\) | 1.7101 |
| Theorem 1 \((N = 4)\) | 1.7143 |

For the given controller gain \(K = 0.4\) and \(h_1 = 1.5000\), the allowable upper bounds of \(h(t)\) guaranteeing the asymptotically stability of this system are listed in Table 5. It can be seen that the proposed integral inequality can reduce a more conservatism than Corollary 1 obtained with the orthogonal polynomials based integral inequalities [11], [33]. Further, this results clearly show that the proposed inequality reduces more conservatism as the degree \(N\) increases. State trajectories given in Figure 5 are obtained with an uniformly distributed random time-varying delay \(h(t) \in [1.5000, 1.7143]\) in Figure 6 and the initial condition \(x(0) = [0.5, 0.5, 0.5]_T\).
Since Theorem 1 ($N = 4$) demonstrates that the system (113) is asymptotically stable in $h(t) \in [1.5000, 1.7143]$, all state responses converge.

V. CONCLUSION

In this paper, we have proposed the affine integral inequality of an arbitrary degree and associated hierarchical stability criteria for linear systems with time-varying delays. Based on the proposed approaches, it was clearly shown that the proposed integral inequality is more general than several existing integral inequalities. Due to such generality, increasing the degree of the inequality only reduces or maintains the conservatism, which is also shown in this paper. Compared to the existing works, four numerical examples have shown the effectiveness of our results. In the future work, the proposed inequality can be applied to numerous systems such as switched systems, neutral systems, neural network systems, fuzzy systems, nonlinear telerobotic systems, and networked control systems with time delays.

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