Periodicity and Unbordered Words: 
A Proof of the Extended Duval Conjecture

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Abstract

The relationship between the length of a word and the maximum length of its unbordered factors is investigated in this paper. Consider a finite word \( w \) of length \( n \). We call a word bordered, if it has a proper prefix which is also a suffix of that word. Let \( \mu(w) \) denote the maximum length of all unbordered factors of \( w \), and let \( \partial(w) \) denote the period of \( w \). Clearly, \( \mu(w) \leq \partial(w) \).

We establish that \( \mu(w) = \partial(w) \), if \( w \) has an unbordered prefix of length \( \mu(w) \) and \( n \geq 2\mu(w) - 1 \). This bound is tight and solves the stronger version of a 21 years old conjecture by Duval. It follows from this result that, in general, \( n \geq 3\mu(w) - 2 \) implies \( \mu(w) = \partial(w) \) which gives an improved bound for the question asked by Ehrenfeucht and Silberger in 1979.

1 Introduction

Periodicity and borderedness are two properties of words which are investigated in this paper. These two concepts—periodicity and borderedness—are fundamental and play a rôle (explicitly or implicitly) in many areas. Just a few of those areas are string searching algorithms [15, 3, 8], data compression [23, 7], and codes [2], which are classical examples, but also computational biology, e.g., sequence assembly [19] or superstrings [4], and serial data communications systems [5] are areas among others where periodicity and borderedness of words (sequences) are important concepts. It is well known that these two word properties do not exist independently from each other. However, it is somewhat surprising that no clear relation has been established so far, despite the fact that this basic question has been around for more than 20 years.

Let us consider a finite word (a sequence of letters) \( w \). We denote the length of \( w \) by \( |w| \) and call a subsequence of consecutive letters of a word factor. The period of \( w \), denoted by \( \partial(w) \), is the smallest positive integer \( p \) such that the \( i \)-th letter equals the \( (i + p) \)-th letter for all \( 1 \leq i \leq |w| - p \). Let \( \mu(w) \) denote the maximum length of all unbordered factors of \( w \). A word is bordered, if it has a proper prefix that is also a suffix, where we call a prefix proper, if it is neither
empty nor contains the entire word. For the investigation of the relationship between $|w|$ and the maximality of $\mu(w)$, that is, $\mu(w) = \partial(w)$, we consider the special case where the longest unbordered prefix of a word is of the maximum length, that is, no unbordered factor is longer than that prefix. Let $w$ be an unbordered word. Then a word $wu$ is a Duval extension (of $w$), if every unbordered factor of $wu$ has at most length $|w|$, that is, $\mu(wu) = |w|$. We call $wu$ trivial Duval extension, if $\partial(wu) = |w|$, or with other words, if $u$ is a prefix of $w^k$ for some $k \geq 1$. For example, let $w = abaabb$ and $u = aaba$. Then $wu = abaabbaaba$ is a nontrivial Duval extension of $w$ since (i) $w$ is unbordered, (ii) all factors of $wu$ longer than $w$ are bordered, that is, $|w| = \mu(wu) = 6$, and (iii) the period of $wu$ is 7, and hence, $\partial(wu) > |w|$. Note, that this example satisfies $|u| = |w| - 2$.

In 1979 a line of research was initiated [11, 1, 10] exploring the relationship between the length of a word $w$ and $\mu(w)$. In 1982 these efforts culminated in the following result by Duval: If $|w| \geq 4\mu(w) - 6$ then $\partial(w) = \mu(w)$. However, it was conjectured [1] that $|w| \geq 3\mu(w)$ implies $\partial(w) = \mu(w)$ which follows if Duval’s conjecture [10] holds true.

**Conjecture 1.1.** Let $wu$ be a nontrivial Duval extension of $w$. Then $|u| < |w|$.

After that, no progress was recorded, to the best of our knowledge, for 20 years. However, the topic remained popular, see for example Chapter 8 in [17]. The most recent results are by Mignosi and Zamboni [20] and the authors of this article [13]. However, not Duval’s conjecture but rather its opposite is investigated in those papers, that is: Which words admit only trivial Duval extensions? It is shown [20] that unbordered, finite factors of Sturmian words allow only trivial Duval extensions, with other words, if an unbordered, finite factor of a Sturmian word of length $\mu(w)$ is a prefix of $w$, then $\partial(w) = \mu(w)$. Sturmian words are binary infinite words of minimal subword complexity, that is, a Sturmian word contains exactly $n + 1$ different factors of length $n$ for every $n \geq 1$; see [21] or Chapter 2 in [17]. That result was later improved [13] by showing that Lyndon words [18] allow only trivial Duval extensions and the fact that every unbordered, finite factor of a Sturmian word is a Lyndon word. A Lyndon word is a word that is minimal among all its conjugates with respect to some lexicographic order, where a word $uv$ is a conjugate of $vu$.

The main result in this paper is a proof of the extended version of Conjecture 1.1.

**Theorem 1.2.** Let $wu$ be a Duval nontrivial extension of $w$. Then $|u| < |w| - 1$.

The example mentioned above shows that this bound on the length of a nontrivial Duval extension is tight. Theorem 1.2 implies the truth of Duval’s conjecture, as well as, the following corollary (for any word $w$).

**Corollary 1.3.** If $|w| \geq 3\mu(w) - 2$, then $\partial(w) = \mu(w)$.

This corollary confirms the conjecture by Assous and Pouzet in [11] about a question asked by Ehrenfeucht and Silberger in [14].

Our main result, Theorem 1.2, is presented in Section 4, which uses the notations introduced in Section 2 and preliminary results from Section 3. We conclude with Section 5.
2 Notations

In this section we introduce the notations of this paper. We refer to [13] [17] for more basic and general definitions.

We consider a finite alphabet $A$ of letters. Let $A^*$ denote the monoid of all finite words over $A$ including the empty word, denoted by $\varepsilon$. Let $w = w_{(1)} w_{(2)} \cdots w_{(n)}$ where $w_{(i)}$ is a letter, for every $1 \leq i \leq n$. We denote the length $n$ of $w$ by $|w|$. An integer $1 \leq p \leq n$ is a period of $w$, if $w_{(i)} = w_{(i+p)}$ for all $1 \leq i \leq n - p$. The smallest period of $w$ is called the minimum period (or simply, the period) of $w$, denoted by $\partial(w)$. A nonempty word $u$ is called a border of a word $w$, if $w = uv = v'u$ for some suitable words $v$ and $v'$. We call $w$ bordered, if it has a border that is shorter than $w$, otherwise $w$ is called unbordered. Note, that every bordered word $w$ has a minimum border $u$ such that $w = uwv$, where $u$ is unbordered. Let $\mu(w)$ denote the maximum length of unbordered factors of $w$. Suppose $w = uv$, then $u$ is called a prefix of $w$, denoted by $u \preceq w$, and $v$ is called a suffix of $w$, denoted by $v \succeq w$. Let $u, v \neq \varepsilon$. Then we say that $u$ overlaps $v$ from the left or from the right, if there is a word $w$ such that $|w| < |u| + |v|$, and $u \preceq w$ and $v \preceq w$, or $v \succeq w$ and $u \succeq w$, respectively. We say that $u$ overlaps (intersects) with $v$, if either $v$ is a factor of $u$ or $u$ is a factor of $v$ or $u$ overlaps $v$ from the left or right.

Let us consider the following examples. Let $A = \{a, b\}$ and $u, v, w \in A^*$ such that $u = abaa$ and $v = baaba$ and $w = abaaba$. Then $|w| = 6$, and $3, 5, 6$ are periods of $w$, and $\partial(w) = 3$. We have that $a$ is the shortest border of $u$ and $w$, whereas $ba$ is the shortest border of $v$. We have $\mu(w) = 3$. We also have that $u$ and $v$ overlap since $u \preceq w$ and $v \succeq w$ and $|w| < |u| + |v|$.

We continue with some more notations. Let $w$ and $u$ be nonempty words where $w$ is also unbordered. We call $wu$ a Duval extension of $w$, if every factor of $wu$ longer than $|w|$ is bordered, that is, $\mu(wu) = |w|$. A Duval extension $wu$ of $w$ is called trivial, if $\partial(wu) = \mu(wu) = |w|$. A nontrivial Duval extension $wu$ of $w$ is called minimal, if $u$ is of minimal length, that is, $u = u'a$ and $w = u'bw'$ where $a, b \in A$ and $a \neq b$.

Example 2.1. Let $w = ababababababb$ and $u = aaba$. Then

$$w.u = ababababababb.aaba$$

(for the sake of readability, we use a dot to mark where $w$ ends) is a nontrivial Duval extension of $w$ of length $|wu| = 18$, where $\mu(wu) = |w| = 14$ and $\partial(wu) = 15$. However, $wu$ is not a minimal Duval extension, whereas

$$w.u' = ababababababb.aa$$

is minimal, with $u' = aa \preceq u$. Note, that $wu$ is not the longest nontrivial Duval extension of $w$ since

$$w.v = ababababababb.ababa$$

is longer, with $v = abaa$ and $|vw| = 20$ and $\partial(wv) = 17$. One can check that $wv$ is a nontrivial Duval extension of $w$ of maximum length, and at the same time $wv$ is also a minimal Duval extension of $w$. 

3
Let an integer \( p \) with \( 1 \leq p < |w| \) be called point in \( w \). Intuitively, a point \( p \) denotes the place between \( w(p) \) and \( w(p+1) \) in \( w \). A nonempty word \( u \) is called a repetition word at point \( p \) if \( w = xy \) with \( |x| = p \) and there exist \( x' \) and \( y' \) such that \( u \preceq x'x \) and \( u \preceq yy' \). For a point \( p \) in \( w \), let
\[
\partial(w, p) = \min\{|u| \mid u \text{ is a repetition word at } p\}
\]
denote the local period at point \( p \) in \( w \). Note, that the repetition word of length \( \partial(w, p) \) at point \( p \) is necessarily unbordered and \( \partial(w, p) \leq \partial(w) \). A factorization \( w = uv \), with \( u, v \neq \varepsilon \) and \( |u| = p \), is called critical, if \( \partial(w, p) = \partial(w) \), and, if this holds, then \( p \) is called critical point.

**Example 2.2.** The word
\[
w = ab.aa.b
\]
has the period \( \partial(w) = 3 \) and two critical points, 2 and 4, marked by dots. The shortest repetition words at the critical points are \( aab \) and \( baa \), respectively. Note, that the shortest repetition words at the remaining points 1 and 3 are \( ba \) and \( a \), respectively.

### 3 Preliminary Results

We state some auxiliary and well-known results about repetitions and borders in this section which will be used to prove Theorem 1.2 in Section 4.

**Lemma 3.1.** Let \( zf = gzh \) where \( f, g \neq \varepsilon \). Let \( az' \) be the maximum unbordered prefix of \( az \). If \( az \) does not occur in \( zf \), then \( agz' \) is unbordered.

**Proof.** Assume \( agz' \) is bordered, and let \( y \) be its shortest border. In particular, \( y \) is unbordered. If \( |z'| \geq |y| \) then \( y \) is a border of \( az' \) which is a contradiction. If \( |az'| = |y| \) or \( |az'| < |y| \) then \( az \) occurs in \( zf \) which is again a contradiction. If \( |az'| < |y| \leq |az| \) then \( az' \) is not maximum since \( y \) is unbordered; a contradiction.

The proof of the following lemma is easy.

**Lemma 3.2.** Let \( w \) be an unbordered word and \( u \preceq w \) and \( v \preceq w \). Then \( uw \) and \( wv \) are unbordered.

The critical factorization theorem is one of the main results about periodicity of words. A weak version of it was first conjectured by Schützenberger [22] and proved by Césari and Vincent [6]. It was developed into its current form by Duval [9]. We refer to [12] for a short proof of the CFT.

**Theorem 3.3 (CFT).** Every word \( w \), with \( |w| \geq 2 \), has at least one critical factorization \( w = uv \), with \( u, v \neq \varepsilon \) and \( |u| < \partial(w) \), i.e., \( \partial(w, |u|) = \partial(w) \).

We have the following two lemmas about properties of critical factorizations.
Lemma 3.7. Let $w = uv$ be unbordered and $|u|$ be a critical point of $w$. Then $u$ and $v$ do not overlap.

Proof. Note, that $\partial (w, |u|) = \partial (w) = |w|$ since $w$ is unbordered. Let $|u| \leq |v|$ without restriction of generality. Assume that $u$ and $v$ overlap. If $u = u's$ and $v = sv'$, then $\partial (w, |u|) \leq |s| < |w|$. On the other hand, if $u = sv'$ and $v = v's$, then $w$ is bordered with $s$. Finally, if $v = sut$ then $\partial (w, |u|) \leq |su| < |w|$. □

The next result follows directly from Lemma 3.1

Lemma 3.4. Let $a_0a_1$ be unbordered and $|u_0| = 1$ be a critical point of $a_0a_1$. Then for any word $z$, we have $u_i, v_{i+1}$, where the indices are modulo 2, is either unbordered or has a minimum border $g$ such that $|g| \geq |u_0| + |u_1|$.

The next theorem states a basic fact about minimal Duval extensions. See [13] for a proof of it.

Theorem 3.6. Let $wu$ be a minimal Duval extension of $w$. Then $u$ occurs in $w$.

The following Lemmas 3.7, 3.8 and 3.9 and Corollary 1.9 are given in [10]. Let $a_0, a_1 \in A$, with $a_0 \neq a_1$, and $t_0 \in A^*$. Let the sequences $(a_i), (s_i), (s_i'), (s_i''), (t_i)$, for $i \geq 1$, be defined by

- $a_i = a_i \pmod{2}$, that is, $a_i = a_0$ or $a_i = a_1$, if $i$ is even or odd, respectively,
- $s_i$ such that $a_is_i$ is the shortest border of $a_it_{i-1}$,
- $s_i'$ such that $a_{i+1}s_i'$ is the longest unbordered prefix of $a_{i+1}s_i$,
- $s_i''$ such that $s_is_i'' = s_i$,
- $t_i$ such that $t_is_i'' = t_{i-1}$.

For any parameters of the above definition, the following holds.

Lemma 3.7. For any $a_0, a_1$, and $t_0$ there exists an $m \geq 1$ such that

$$|s_1| < \cdots < |s_m| = |t_{m-1}| \leq \cdots \leq |t_0|$$

and $s_m = t_{m-1}$ and $|t_0| \leq |s_m| + |s_m-1|$.

Lemma 3.8. Let $z \leq t_0$ such that $a_0z$ and $a_1z$ do not occur in $t_0$. Let $a_0z_0$ and $a_1z_1$ be the longest unbordered prefixes of $a_0z$ and $a_1z$, respectively. Then

1. if $m = 1$ then $a_0t_0$ is unbordered,
2. if $m > 1$ is odd, then $a_1s_m$ is unbordered and $|t_0| \leq |s_m| + |z_0|$, and
3. if $m > 1$ is even, then $a_0s_m$ is unbordered and $|t_0| \leq |s_m| + |z_1|$.

Lemma 3.9. Let $v$ be an unbordered factor of $w$ of length $\mu(w)$. If $v$ occurs twice in $w$, then $\mu(w) = \partial (w)$.

Corollary 3.10. Let $wu$ be a Duval extension of $w$. If $w$ occurs twice in $wu$, then $wu$ is a trivial Duval extension.
4 Main Result

The extended Duval conjecture is proven in this section.

**Theorem 1.2.** Let $wu$ be a nontrivial Duval extension of $w$. Then $|u| < |w| - 1$.

*Proof.* Recall that every factor of $wu$ which is longer than $|w|$ is bordered since $wu$ is a Duval extension of $w$. Let $z$ be the longest suffix of $w$ that occurs twice in $zu$.

If $z = \varepsilon$ then $a \not< w$ and $u = b^j$, where $a, b \in A$ and $a \neq b$ and $j \geq 1$, but now $|u| < |w|$ since $ab^j$ is unbordered. Moreover, $w = b^k a w' a$ with $k < j$, otherwise $wu$ is a trivial Duval extension, and either $aw' a b^j$ is bordered, in this case it follows $j \leq |w'|$, or $aw' a b^j$ is unbordered. In both cases it follows $|u| < |w| - 1$.

So, assume $z \neq \varepsilon$. We have $z \neq w$ since $wu$ is otherwise trivial by Corollary 1.3.

Let $a, b \in A$ be such that

$$w = w' az \quad \text{and} \quad u = u' bz r$$

and $z$ occurs in $zr$ only once, that is, $bz$ matches the rightmost occurrence of $z$ in $u$.

Note, that $bz$ does not overlap $az$ from the right, by Lemma 3.2 and therefore $u'$ exists, although it might be empty. Naturally, $a \neq b$ by the maximality of $z$, and $w' \neq \varepsilon$, otherwise $azu bz$ has either no border or $w$ is bordered (if $azu bz$ has a border not longer than $z$) or $az$ occurs in $zu$ (if $azu bz$ has a border longer than $z$); a contradiction in any case.

Let $az_0$ and $bz_1$ denote the longest unbordered prefix of $az$ and $bz$, respectively.

Let $a_0 = a$ and $b_1 = b$ and $t_0 = zr$ and the integer $m$ be defined as in Lemma 3.8.

We have then a word $s_m$, with its properties defined by Lemma 3.8 such that

$$t_0 = s_m t'. \quad \text{(1)}$$

Consider $azu'bz_0$. We have that $az$ and $azu'bz_0$ are both prefixes of $a_0zu$, and $bz_0$ is a suffix of $azu'bz_0$ and $az$ does not occur in $zu'bz_0$. It follows from Lemma 3.1 that $azu'bz_0$ is unbordered, and hence,

$$|azu'bz_0| \leq |w|. \quad \text{(1)}$$

**Case:** Suppose that $m$ is even. Then we have $2 \leq m$ and $as_m = a_m s_m$ is unbordered and $|t_0| \leq |s_m| + |z_1|$ by Lemma 3.8.

Suppose $|t_0| = |s_m| + |z_1|$ and $z_1 = z$. Then $|s_{m-1}| = |z|$ by Lemma 3.7. Note, that $s_i \leq t_i - 1 \leq t_0$ for all $1 \leq i \leq m$, and hence, it follows that $s_i \leq z$ for all $1 \leq i < m$. In particular, $s_{m-1} = z$. We have that $bz = a_1 s_{m-1}$ is a border of $bt_{m-2} (= a_1 t_{m-2})$. But now, $bz$ occurs in $t_0$, and hence, in $u$, since $t_i \leq t_0$, for all $0 \leq i < m$, which is a contradiction.
So, assume that $|t_0| < |s_m| + |z_1|$ or $|z_1| < |z|$. Suppose $|s_m| \leq |z_0|$. Then $|azu'b_{z_0}| \leq |w|$ and

$$
|u| = |azu| - |z| - 1
= |azu'b_{z_0}| - |z_0| + |t_0| - |z| - 1
< |azu'b_{z_0}| - |z_0| + |s_m| + |z_1| - |z| - 1
\leq |w| + |z_1| - |z| - 1
\leq |w| - 1
$$

if $|t_0| < |s_m| + |z_1|$, or

$$
|u| = |azu| - |z| - 1
= |azu'b_{z_0}| - |z_0| + |t_0| - |z| - 1
\leq |azu'b_{z_0}| - |z_0| + |s_m| + |z_1| - |z| - 1
\leq |w| + |z_1| - |z| - 1
< |w| - 1
$$

if $|z_1| < |z|$. We have $|u| < |w| - 1$ in both cases.

Let then $|s_m| > |z_0|$. We have that $s_m$ is unbordered, and since $az_0$ is the longest unbordered prefix of $az$, we have $az \leq as_m$, and hence, $|z| \leq |s_m|$. Now, $azu'bs_m$ is unbordered otherwise its shortest border is longer than $az$, since no prefix of $az$ is a suffix of $as_m$, and $az$ occurs in $w$; a contradiction. So, $|azu'bs_m| \leq |w|$ and $|u| < |w| - 1$, since either $|z_1| \leq |z|$, or $|t_0| < |z_m| + |z_1|$.

**Case:** Suppose that $m$ is odd. Then $bs_m = a_m s_m$ is an unbordered word and $|t_0| \leq |s_m| + |z_0|$; see Lemma 3.8. Surely $s_m \neq \varepsilon$.

If $|s_m| \leq |z|$, then $|u| < |w| - 1$ since

$$
|u| = |azu'b_{z_0}| - |b_{z_0}| - |az|
$$

and $|azu'b_{z_0}| \leq |w|$, by \(1\), and $|t_0| \leq |s_m| + |z_0|$.

Assume thus that $|s_m| > |z|$, and hence, also $z \leq s_m$. Since $s_m \neq \varepsilon$, we have $|bs_m| \geq 2$, and therefore, by the critical factorization theorem, there exists a critical point $p$ in $bs_m$ such that $bs_m = v_0 v_1$, where $|v_0| = p$.

$$
\begin{array}{c|c|c|c|c|c|c}
& & w & & & & \\
& a & z & & u' & & \infty \\
& z_0 & & & & & \\
& & & b & z & & r \\
& & & & s_m & & t' \\
& & & & v_0 & v_1 & & \\
\end{array}
$$

In particular,

$$
|z_0| < v_0 v_1 .
$$

Note, that if $s_m = z$ then $|z_0| < |z|$ since $b \leq z_0$ and $bs_m$ does not end with $b$ because it is unbordered. We have therefore in all cases

$$
|z_0| < |v_0 v_1| - 1.
$$
Let
\[ u = u_0'v_0v_1u_1 \]
be such that \( v_0v_1 \) does not occur in \( u_0' \). Note, that \( v_0v_1 \) does not overlap with itself since it is unbordered, and \( v_0 \) and \( v_1 \) do not overlap by Lemma 3.4. Consider the prefix \( wu_0'bz \) of \( wu \) which is bordered and has a shortest border \( g \) longer than \( z \), and hence, \( bz \preceq g \), otherwise \( w \) is bordered since \( z \preceq w \). Moreover, \( g \leq w \), for otherwise \( az \) would occur in \( u_0'v_0v_1u_1 \), and hence, \( bz \) occurs in \( w \). Let
\[ w = w_0bzw_1 \]
such that \( bz \) occurs in \( w_0bz \) only once, that is, we consider the leftmost occurrence of \( bz \) in \( w \). Note, that
\[ |w_0bz| \leq |g| \leq |u_0'bz| \quad (4) \]
where the first inequality comes from the definition of \( w_0 \) above and the second inequality from the fact that \( |u_0'bz| < |g| \) implies that \( w \) is bordered. Let
\[ f = bzw_1u_0'v_0v_1 \]
If \( f \) is unbordered, then \( |f| \leq |w| \), and hence, \( |u_0'v_0v_1| \leq |w_0| \). Now, we have \( |u_0'| < |w_0| \) which contradicts (4).

Therefore, \( f \) is bordered. Let \( h \) be its shortest border.

Surely, \( |bz| < |h| \) otherwise \( v_0v_1 \) is bordered by (2). So, \( bz \preceq h \). Moreover, \( |v_0v_1| \leq |h| \) otherwise \( bz \) occurs in \( s_m \) contradicting our assumption that \( bzs \) marks the rightmost occurrence of \( bz \) in \( u \). So, \( v_0v_1 \preceq h \), and \( v_0v_1 \) occurs in \( w \) since \( w_0h \preceq w \) by (4). Let
\[ w_0bzv' = w_0h = u_0'v_0v_1 . \]

Note, that \( v_0v_1 \) does not occur in \( u_0' \) otherwise it occurs in \( u_0' \) contradicting our assumption on \( u_0' \). Moreover, we have \( h = bzv' \preceq u_0'v_0v_1 \). Let \( u_0'v_0v_1 = u_0h \). Consider
\[ f_0 = wu_0bz \]
which has a shortest border \( h_0 \).
Surely, \( bz \leq h_0 \) otherwise \( w \) is bordered with a suffix of \( z \). Moreover, \(|w_0bz| \leq |h_0|\) and \(|h_0| \leq |u_0bz|\) since \( bz \) does not occur in \( w_0 \) and \( w \) is unbordered. From that and \( w_0h = u_0'v_0v_1 \) and \( u_0h = u_0'v_0v_1 \) follows now \(|w_0'| \leq |u_0'|\) and
\[
u_0'v_0v_1 = u_0bzv' \text{ and } w_0 \text{ occurs in } u_0. \tag{5}
\]

Let now
\[
w = u_0'v_0v_1w_1' \cdots v_0v_1w_2'v_0v_1w_1'v_0v_1w_2
\]
for some word \( w_2 \) that does not contain \( v_0v_1 \), and
\[
u = u_0'v_0v_1u_1' \cdots v_0v_1u_2'v_0v_1u_1'v_0v_1t'
\]
such that \( v_0v_1 \) does not occur in \( w_k' \), for all \( 0 \leq k \leq i \), or \( v_k' \), for all \( 0 \leq \ell \leq j \). Note, that these factorizations of \( w \) and \( u \) are unique, and, moreover, \( w_2 \neq \varepsilon \). (Indeed, if \( w_2 = \varepsilon \) then \( v_0v_1 \not\subseteq w \) and \( az \not\subseteq v_0v_1 \), and \( az \) would occur in \( w \); a contradiction.)

We claim that either \( i = j \) and \( w_k' = u_k' \), for all \( 1 \leq k \leq i \) or \(|u| < |w| - 1 \).

Assume \( k = 1 \). We show that \( w_1' = u_1' \). Consider
\[
f_1 = v_1w_1'v_0v_1w_2u_0v_0v_1u_1' \cdots v_0v_1u_1'v_0.
\]
If \( f_1 \) is unbordered, then \(|u| < |w| - 1 \) since \(|f_1| \leq |w|\) and
\[
|u| = |f_1| - |v_1w_1'v_0v_1w_2| + |v_1t'|
\]
and \(|t'| \leq |z_0| \leq |z| < |bz| \leq |v_0v_1|\) and \( w_2 \neq \varepsilon \). Assume then that \( f_1 \) is bordered, and let \( h_1 \) be its shortest border. Clearly, \( h_1 = v_1g_1v_0 \) for some \( g_1 \) (possibly \( g_1 = \varepsilon \)) since \( v_0 \) and \( v_1 \) do not overlap. We show that \( h_1 \leq v_1w_1'v_0 \). Indeed, otherwise either

1. \( az \) occurs in \( u \), in case \( v_1w_1'v_0v_1w_2 \leq h_1 \), a contradiction to our assumption on \( az \), or
2. \( v_0 \) and \( v_1 \) overlap, in case \(|v_0| \leq |z|\) and
\[
|v_1w_1'v_0v_1w_2| - |az| + |v_0| < |h_1| < |v_1w_1'v_0v_1w_2|
\]
and then \( v_0 \) occurs in \( z \), contradicting Lemma 3.3 or
3. \(|u| < |w| - 1 \), in case \( v_0w_3 \leq w_2 \) and \(|az| \leq |v_0w_3|\), then \( v_0w_3u'v_0v_1 \) is unbordered and the result follows from \(|t'| < |v_0w_3| - 1 \), since \(|az| \neq |v_0w_3|\) for \( v_0 \) does not begin with \( a \).

Moreover, \( h_1 \leq v_1w_1'v_0 \) since \( v_0v_1 \) does not occur in \( v_1w_1'v_0 \). So, let
\[
v_1w_0' = g_1v_0w_1'' \text{ and } v_1u_1' = u_1''v_1g_1. \tag{6}
\]
Consider,
\[ f_2 = v_0w''v_1w_2u'_0v_0v_1u'_2 \cdots v_0v_1u'_1v_0v_1 . \]
If \( f_2 \) is unbordered, then \( |u| < |w| - 1 \) since \( |f_2| \leq |w| \) and
\[ |u| = |f_2| - |v_0w''v_1w_2| + |u'| \]
and \( |u'| \leq |z_0| \leq |z| \leq |v_0v_1| \) and \( w_2 \neq \varepsilon \). Assume then that \( f_2 \) is bordered, and let \( h_2 \) be its shortest border. Since \( v_0 \) and \( v_1 \) do not overlap, \( v_0v_1 \preceq h_2 \).
Also \( h_2 \leq v_0w''v_1 \) since \( v_0v_1 \) does not occur in \( w_2 \) (and \( v_0 \) and \( v_1 \) do not overlap) and \( az \) does not occur in \( h_2 \) (and so \( h_2 \) does not stretch beyond \( w \)). We have \( v_0w''v_1 \leq h_2 \) since \( v_0v_1 \) does not occur in \( v_0w''v_1 \) unless \( w'' = \varepsilon \). Hence, we have \( h_2 = v_0w''v_1 \) and
\[ w''v_0v_1 = g_1h_2 \quad \text{and} \quad h_2 \preceq u'_1v_0v_1 . \] (7)

Consider,
\[ f_3 = v_0v_1w'_1v_0v_1w_2u'_0v_0v_1u'_2 \cdots v_0v_1u'_2v_0u''v_1 . \]
If \( f_3 \) is unbordered, then \( |u| < |w| - 1 \) since \( |f_3| \leq |w| \) and
\[ |u| = |f_3| - |v_0v_1w'_1v_0v_1w_2| + |g_1v_0v_1u'| \]
and \( |u'| \leq |z_0| \leq |z| \leq |v_0v_1| \) and \( |g_1| \leq |u'| \) and \( w_2 \neq \varepsilon \). Assume, that \( f_3 \) is bordered. Then \( f_3 \) has a shortest border \( h_3 \) such that \( v_0v_1 \preceq h_3 \). We have \( h_3 = v_0u'_1v_1 \) by the arguments from the previous paragraph. Moreover,
\[ v_0v_1u'_1 = h_3g_1 \quad \text{and} \quad v_0v_1u'_1 \leq h_3 . \] (8)

Observe, that \( f_2 \) and \( f_3 \) imply that the number of occurrences of \( v_1 \) and \( v_0 \), respectively, is the same in \( u''_1 \) and \( u'_1 \) since \( v_0 \) and \( v_1 \) do not overlap. Now, let
\[ h_1 = v_1g_1v_0 = h''_0v_1h'_1v_0 = v_1h'_0v_0h''_0 \]
where \( v_1 \) and \( v_0 \) occur only once in \( v_1h'_1 \) and \( h''_0v_0 \), respectively.
Now, let

\[ f_2' = v_0' h_0'' w_1'' v_1 w_2 v_0 v_1 u_j' \cdots v_0 v_1 u_1' v_0 v_1 \]

and

\[ f_3' = v_0 v_1 w_1' v_0 v_1 w_2 v_0 v_1 u_j' \cdots v_0 v_1 u_1' v_0 u'' h_1'' v_1 \]

with the respective shortest borders \( h_2' \) and \( h_3' \) (which are both not empty, if \( |w| \geq |w| - 1 \); as in the case of \( f_2' \) and \( f_3' \) and \( v_0 v_1 \neq h_2' \) and \( v_0 v_1 \leq h_3' \).

We have \( h_2' \leq v_0' h_0'' w_1'' v_1 \) since \( v_0 v_1 \) does not occur in \( w_2 \) and \( az \) does not occur in \( h_2' \) (and so \( h_2' \) does not stretch beyond \( w \)). We have \( v_0' h_0'' w_1'' v_1 \leq h_2' \) since \( v_0 v_1 \) does not occur in \( w_1' \). Hence, we have \( h_2' = v_0' h_0'' w_1'' v_1 \) and

\[ w_1' v_0 v_1 = h_0' v_0 v_1 w_1'' v_1 = h_0' h_2' \quad \text{and} \quad h_2' \leq v_1' v_0 v_1 . \]

We have \( h_3' = v_0 u_1'' h_1'' v_1 \) by the arguments from the previous paragraph. Moreover,

\[ v_0 v_1 u_1' = v_0 u_1'' h_1'' v_1 h_1' = h_3' h_1' \quad \text{and} \quad v_0 v_1 u_1' \leq h_3' . \]

It is now straightforward to see that

\[ w_1'' = u_1'' = \varepsilon \]

for otherwise \( v_1 \) and \( v_0 \) occur more than once in \( v_1 h_1' \) and \( h_0' v_0 \), respectively. From (3) follows now

\[ w_1' = g_1 = u_1' . \]
Assume $1 < k \leq \min\{i,j\}$ and $w'_k = u'_k$, for all $1 \leq \ell < k$. Let us denote both $w'_k$ and $u'_k$, for all $1 \leq \ell < k$.

We show that $w'_k = u'_k$. Consider

$$f_4 = v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0' v_0 v_1 u'_{j} \cdots v_0 v_1 u'_j v_0.$$  

If $f_4$ is unbordered, then $|u| < |w| - 1$ since $|f_4| \leq |w|$ and

$$|u| = |f_4| - |v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0| + |v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 t'|$$  

and $|t'| \leq |z_0| \leq |z| < |bz| \leq |v_0 v_1|$ and $w_2 \neq \varepsilon$. Assume, $f_4$ is bordered. Then $f_4$ has a shortest border $h_4$ such that $|v_0 v_1| \leq |h_4|$. Let $h_4 = v_1 g_4 v_0$.

If $|v_1 w'_k v_0| < |h_4|$ then there exists an $\ell < k$ such that

$$h_4 = v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_{\ell+1} v_0 v_1 v'_\ell v_0$$

where $v'_{\ell} \leq v'_\ell$. That implies

$$w'_k = v'_{\ell}$$

since $v_0 v_1$ does neither occur in $v'_{\ell}$ nor in $w'_k$. Now, consider

$$f_5 = v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0' v_0 v_1 u'_j \cdots v_0 v_1 u'_j v_0 v_1 v'_{k-1} v_0 v_1 \cdots v''_{\ell} v_0.$$  

If $f_5$ is unbordered, then $|u| < |w| - 1$ since $|f_4| < |f_5|$, see above. Assume, $f_5$ is bordered. Then $f_5$ has a shortest border $h_5$ such that

$$|h_4| < |h_5|$$

for otherwise $h_4$ is not the shortest border of $f_4$, since either $h_4 \leq h_5$ or $h_5 \leq h_4$, and the latter implies that $h_4$ is bordered, and hence, not minimal. But now, we have a $\ell' < \ell$ such that

$$h_5 = v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_{\ell'+1} v_0 v_1 v'_{\ell'} v_0$$

where $v'_{\ell'} \leq v'_{\ell'}$. We have $|f_4| < |f_5| < |f_6|$ where

$$f_6 = v_1 w'_k v_0 v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0' v_0 v_1 u'_j \cdots v_0 v_1 u'_j v_0 v_1 v'_{k-1} v_0 v_1 \cdots v''_{\ell'} v_0 ,$$

which is either unbordered and $|u| < |w| - 1$ since $|f_4| < |f_5|$, or it is bordered with a shortest border $h_6$, and we have $|h_4| < |h_5| < |h_6|$ and a factor $f_7$, such that $|f_4| < |f_5| < |f_6| < |f_7|$, and so on, until eventually an unbordered factor is reached proving that $|u| < |w| - 1$.

Assume then that $h_4 \leq v_1 w'_k v_0$. We also have that $h_4 \leq v_1 u'_k v_0$ since $v_0 v_1$ does not occur in $w'_k$. So, let $w'_k v_0 = g_4 v_0 u''_k$ and $v_1 u'_k = u''_k v_1 g_4$.

Consider,

$$f_8 = v_0 u''_k v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0' v_0 v_1 u'_j v_0 v_1 \cdots u'_k v_0 v_1.$$  

If $f_8$ is unbordered, then $|u| < |w| - 1$ since $|f_8| \leq |w|$ and

$$|u| = |f_8| - |v_0 u''_k v_1 v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 w_0| + |v'_{k-1} v_0 v_1 \cdots v'_1 v_0 v_1 t'|$$
and $|t'| \leq |z_0| \leq |z| < |b| \leq |v_0v_1|$ and $w_2 \neq \varepsilon$. Assume, $f_8$ is bordered. Then $f_8$ has a shortest border $h_8$ such that $v_0v_1 \not\subseteq h_8$.

If $|h_8| > |v_0u_k'v_1|$ then the same argument as in the case $|v_1w_k'v_0| < |h_4|$ above shows that $|u| < |w| - 1$. If $|h_8| < |v_0u_k''v_1|$ then $v_0v_1$ occurs in $u_k'$; a contradiction. Hence, we have $h_8 = v_0u_k''v_1$ and

$$w_k'v_0v_1 = g_1h_8 \quad \text{and} \quad h_8 \not\subseteq u_k'v_0v_1 \quad (9)$$

Consider,

$$f_9 = v_0v_1w_k'v_0v_1v_{k-1}v_0v_1 \cdots v'_1v_0v_1w_2'v_0v_1u'_1v_0v_1u'_1 \cdots u'_{k+1}v_0u_k''v_1.$$ 

If $f_9$ is unbordered, then $|u| < |w| - 1$ since $|f_9| \leq |w|$ and

$$|u| = |f_9| - |v_0v_1w_k'v_0v_1v_{k-1}v_0v_1 \cdots v'_1v_0v_1w_2| + |g_4v_0v_1v_{k-1}v_0v_1 \cdots v'_1v_0v_1t'|$$

and $|t'| \leq |z_0| \leq |z| < |b| \leq |v_0v_1|$ and $|g_4| \leq |w_k'|$ and $w_2 \neq \varepsilon$. Assume, $f_9$ is bordered. Then $f_9$ has a shortest border $h_9$ such that $v_0v_1 \not\subseteq h_9$. We have $h_9 = v_0u_k''v_1$ by the arguments from the previous paragraph. Moreover,

$$v_0v_1u_k' = h_9g_1 \quad \text{and} \quad h_9 \subseteq v_0v_1u_k'. \quad (10)$$

Observe, that (9) and (10) imply that the number of occurrences of $v_1$ and $v_0$, respectively, is the same in $w_k'$ and $u_k'$ since $v_0$ and $v_1$ do not overlap. Now, let

$$h_4 = v_1g_4v_0 = h_0''v_1h_1'v_0 = v_1h_0''v_0h_0''$$

where $v_1$ and $v_0$ occur only once in $v_1h_1'$ and $h_0''v_0$, respectively.

Now, let

$$f_8' = v_0h_0''w_k''v_1v_{k-1}' v_0v_1w_2'v_0v_1u'_1v_0v_1u'_1 \cdots v_0v_1u_k'v_0v_1$$

and

$$f_9' = v_0v_1w_k'v_0v_1v_{k-1}' v_0v_1w_2'v_0v_1u'_1v_0v_1u'_1 \cdots v_0v_1u_k'v_0v_1$$

with the respective shortest borders $h_8'$ and $h_9'$ (which are both not empty, if $|u| \geq |w| - 1$; as in the case of $f_8$ and $f_9$). Analogously to the cases of $f_8$ and $f_9$, we have

$$w_k'v_0v_1 = h_9'h_8' \quad \text{and} \quad v_0v_1u_k' = h_9'h_1'.$$

It is now straightforward to see that

$$h_8' = h_9' = v_0v_1$$

and

$$h_4 = v_0w_k'v_1 = v_0u_k'v_1$$

and hence, $w_k' = u_k'$. In this case, we denote both $w_k'$ and $u_k'$ by $v_k'$. 

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Now, we have

\[ v = v_0v_1u'_1 \cdots v_0v_1u'_2v_0v_1w'_1 = v_0v_1u'_1 \cdots v_0v_1u'_2v_0v_1u'_1 \]

where \( \epsilon = \min\{i, j\} \).

If \( i < j \) then

\[ |w'_0| < |u'_0v_0v_1u'_j \cdots v_0v_1u'_{i+1}| \quad (11) \]

since \( |w'_0| \leq |u'_0| \) by 3. Let

\[ f_{11} = v_1w_2u'_0v_0v_1u'_j \cdots v_0v_1u'_{i+1}\bar{v}_0v_1 . \]

Then \( |w| < |f_{11}| \) by (11), and hence, \( f_{11} \) is bordered. Let \( h_{11} = v_1g_{11}v_0 \) be the shortest border of \( f_{11} \). Recall, that \( w_2 \neq \epsilon \) and either \( az \ll v_1w_2 \) or \( v_1w_2 \ll az \). If \( |v_1w_2| < |az| \) then \( v_1 \) necessarily occurs in \( z \), and hence, it overlaps with \( v_0 \) (since \( bz \leq v_0v_1 \)); a contradiction. So, we have \( az \ll v_1w_2 \). Surely, \( |h_{11}| < |v_1w_2| \) (and so \( h_{11} \leq v_1w_2 \)) for otherwise \( az \) occurs in \( u \) which contradicts our assumption that \( z \) is of maximum length. Let \( w_2 = g_1v_0w_5 \). Note, that \( |v_0w_5| \neq |az| \) since \( az \) and \( v_0 \) begin with different letters. We have \( |az| < |v_0w_5| \) since otherwise \( v_0 \) occurs in \( z \), and hence, overlaps with \( v_1 \) which is a contradiction. Consider,

\[ f_{12} = v_0w_5u'_0v_0v_1u'_j \cdots v_0v_1u'_{i+1}\bar{v}_0v_1 . \]

If \( f_{12} \) is unbordered, then \( |u| < |w| - 1 \) since \( |f_{12}| \leq |w| \) and

\[ |u| = |f_{12}| - |v_0w_5| + |t'| \]

and \( |az| < |v_0w_5| \) and \( |t'| \leq |z| \leq |z| < |bz| < |v_0w_5| \). Assume, \( f_{12} \) is bordered. Then \( f_{12} \) has a shortest border \( h_{12} = g_{12}v_0v_1 \) with \( |az| < |h_{12}| \), for otherwise \( az \) occurs in \( u \). Let \( v_0w_5 = g_{12}v_0v_1w_6 \). But, now

\[ w = w'_0\bar{v}_0v_0v_1g_{12}v_0v_1w_6 \]

where \( v_0v_1w_6 \ll w_2 \), contradicting our assumption that \( v_0v_1 \) does not occur in \( w_2 \).

If \( i > j \) then

\[ w = u'_0v_0v_1u'_1 \cdots v_0v_1w'_{j+1}\bar{v}_0v_0v_1w_2 \quad \text{and} \quad u = u'_0\bar{v}_0v_0v_1t' \]

and \( |w| \geq |u| - |t'| + |v_0v_1| \). We have \( |u| < |w| - 1 \) since \( |t'| \leq |z| < |v_0v_1| - 1 \) by 3.

Assume \( i = j \). Then

\[ w = u'_0\bar{v}_0v_0v_1w_2 \quad \text{and} \quad u = u'_0\bar{v}_0v_0v_1t' . \]

Consider

\[ f' = v_1w_2u'_0\bar{v}_0 . \]

If \( f' \) is bordered, then it has a shortest border \( h' = v_1g'v_0 \).
Recall, that $w_2 \neq \varepsilon$ and either $az \lessdot v_1 w_2$ or $v_1 w_2 \lessdot az$. If $|v_1 w_2| < |az|$ then $v_1$ occurs in $z$, and hence, overlaps with $v_0$ since $b z \leq v_0 v_1$; a contradiction. So, we have $az \lessdot v_1 w_2$. Surely, $|h'| < |v_1 w_2|$ for otherwise $az$ occurs in $u$ which contradicts our assumption. Let $w_2 = g' v_0 w_4$. Note, that $|v_0 w_4| \neq |az|$ since $az$ and $v_0$ begin with different letters. We have $|az| < |v_0 w_4|$ since otherwise $v_0$ occurs in $z$, and hence, overlaps with $v_1$ which is a contradiction. Consider now,

$$f'' = v_0 w_4 v_0 v_1 v_1 .$$

If $f''$ is unbordered, then it easily follows that $|u| < |w| - 1$ since we have $|t' | < |az|$ and $|az| < |v_0 w_4|$.

$$w = w_0 v_0 v_1 v_1 w_4$$

which contradicts our assumption that $w = w_0 v_0 v_1 v_1 w_4$ and $v_0 v_1$ does not occur in $w_2$.

If $f'$ is unbordered, then $|f'| \leq |w|$, and hence, $|w_0'| \geq |u_0'|$. But, we also have $|w_0'| \leq |u_0'|$; see 5. That implies $|w_0'| = |u_0'|$. Moreover, the factors $w_0$ and $b z v'$ have both nonoverlapping occurrences in $u_0' v_0 v_1$ by 5. Therefore, $w_0' = u_0'$. Now,

$$w = x a w_7 \quad \text{and} \quad u = x b t''$$

where $w_0' v_0 v_1 \leq x$ and $a, b \in A$ and $a \neq b$ and $w_7 \lessdot w_2$ and $t'' \lessdot t'$. We have that $xb$ occurs in $w$ by Theorem 3.6. Since $xb$ is not a prefix of $w$ and $v_0 v_1$ does not overlap with itself, we have $|xb| + |v_0 v_1| \leq |w|$. From $|t'| \leq |z_0| < |v_0 v_1| - 1$ we get $|u| < |w| - 1$ and the claim follows.

Note, that the bound $|u| < |w| - 1$ on the length of a nontrivial Duval extension $wu$ of $w$ is tight, as the following example shows.
Example 4.1. Let \( w = a^nba^{n+m}bb \) and \( u = a^{n+m}ba^n \) with \( n, m \geq 1 \). Then
\[
w.u = a^nba^{n+m}bb.a^{n+m}ba^n
\]
is a nontrivial Duval extension of \( w \) and \( |u| = |w| - 2 \).

In general, Duval [10] proved that we have \( \partial(w) = \mu(w) \), for any word \( w \), if \( |w| \geq 4\mu(w) - 6 \). Duval also noted that already \( |w| \geq 3\mu(w) \) implies \( \partial(w) = \mu(w) \), provided his conjecture holds. Corollary 1.3 follows from Theorem 1.2.

**Corollary 1.3.** If \( |w| \geq 3\mu(w) - 2 \) then \( \partial(w) = \mu(w) \).

However, this bound is unlikely to be tight. The best example for a large bound that we could find is taken from [1].

Example 4.2. Let
\[
w = a^nba^{n+1}ba^nba^{n+2}ba^nba^{n+1}ba^n.
\]
We have \( |w| = 7n + 10 \) and \( \mu(w) = 3n + 6 \) and \( \partial(w) = 4n + 7 \).

So, we have that the precise bound for the length of a word that implies \( \partial(w) = \mu(w) \) is larger than \( 7/3\mu(w) - 4 \) and not larger than \( 3\mu(w) - 2 \). The characterization of the precise bound of the length of a word as a function of its longest unbordered factor is still an open problem.

5 Conclusions

In this paper we have given a confirmative answer to a long standing conjecture [10] by proving that a Duval extension \( uu \) of \( w \) longer than \( 2|w| - 2 \) is trivial. This bound is tight and also gives a new bound on the relation between the length of an arbitrary word \( w \) and its longest unbordered factors \( \mu(w) \), namely that \( |w| \geq 3\mu(w) - 2 \) implies \( \partial(w) = \mu(w) \) as conjectured (more weakly) in [1]. We believe that the precise bound can be achieved with methods similar to those presented in this paper.

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