COMBINATORIAL ASPECTS OF CONNES’S EMBEDDING CONJECTURE
AND ASYMPTOTIC DISTRIBUTION
OF TRACES OF PRODUCTS OF UNITARIES

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ABSTRACT. In this paper we study the asymptotic distribution of the moments of (non-normalized) traces $\text{Tr}(w_1), \text{Tr}(w_2), \ldots, \text{Tr}(w_r)$, where $w_1, w_2, \ldots, w_r$ are reduced words in unitaries in the group $\mathcal{U}(N)$. We prove that as $N \to \infty$ these variables are distributed as normal gaussian variables $\sqrt{\mathbb{Z}} \mathbb{C}, \ldots, \sqrt{\mathbb{Z}_n} \mathbb{C}$, where $j_1, \ldots, j_r$ are the number of cyclic rotations of the words $w_1, \ldots, w_r$, leaving them invariant. This extends a previous result by Diaconis (10), where this it was proved, that $\text{Tr}(U), \text{Tr}(U^2), \ldots, \text{Tr}(U^p)$ are asymptotically distributed as $Z_1, \sqrt{2}Z_2, \ldots, \sqrt{p}Z_p$.

We establish a combinatorial formula for $\int [\text{Tr}(w_1)]^2 \cdots [\text{Tr}(w_p)]^2$. In our computation we reprove some results from 1.

1. INTRODUCTION

Connes’s embedding conjecture (3) for the case of discrete groups states that every discrete group $\Gamma$ can be asymptotically embedded in the algebra of $N$ by $N$ matrices, when $N$ tends to infinity. As observed in 10 (see also 7 and 5) it amounts to prove that for every finite subset $F$ of $\Gamma$, for every $\varepsilon > 0$, there exist $N$ and unitaries $\{a_f \mid f \in F\}$ in $\mathcal{U}(N)$ such that $\|a_f a_{f_2} - a_{f_1} a_{f_2}\|_{\text{HS}} \leq \varepsilon \|\text{Id}\|_{\text{HS}}$ for all $f_1, f_2 \in F$. Here by $\|\cdot\|_{\text{HS}}$ we denote the Hilbert-Schmidt norm

$$\|A\|_{\text{HS}} = \text{Tr}(A^* A)^{1/2}, \quad A \in M_N(\mathbb{C}),$$

$\text{Tr}$ being the (non-normalized) trace on $M_N(\mathbb{C})$. If $\Gamma$ is a group with presentation $\langle F_\infty \mid R \rangle$, where $R$ are the relations, it can be proved (see 10) that the Connes’s embedding conjecture is equivalent to show that for any $\varepsilon > 0$, $w_1, w_2, \ldots, w_N \in R$, and for any $w_0 \notin R$, assuming that $w_0, w_1, \ldots, w_N$ are the words on the letters $a_1, \ldots, a_M$, there exist $N$ and unitaries $U_1, U_2, \ldots, U_p$ in $\mathcal{U}(N)$ such that if $W_0, \ldots, W_s$ are the corresponding words obtained by substituting $a_1, \ldots, a_M$ with $U_1, \ldots, U_p$ we have (with $\text{Tr} = \frac{1}{p} \text{Tr}$)

$$\text{tr}(W_0) < \varepsilon, \quad \text{tr}(W_1) > 1 - \varepsilon, \quad \ldots, \quad \text{tr}(W_s) > 1 - \varepsilon.$$

Consequently, a natural object to study is the following: Let $F_M$ be the free group with $M$ generators $a_1, a_2, \ldots, a_M$. Let $w_0, w_1, \ldots, w_s$ be the reduced words in $F_M$ and let $f_{w_0}, \ldots, f_{w_s}$ be the functions on $(\mathcal{U}(N))^M$ obtained by evaluating the traces $\text{Tr}(W_0), \ldots, \text{Tr}(W_s)$ of the words $W_0, \ldots, W_s$ at an $M$-uple $(U_1, \ldots, U_M)$. Then one

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has to determine the joint moments of these functions, i.e. the quantities (for all $\alpha_0, \ldots, \alpha_s$ in $\mathbb{N}$)

$$\int_{U(N)} |f_{w_0}|^{\alpha_1} \cdots |f_{w_s}|^{\alpha_s} \, dU_1 \cdots dU_M,$$

with respect to the Haar measure.

In particular, after normalizing with the factor $\frac{1}{N}$, if one determines the measure for these moments, then one could solve the inequality (1).

2. **Computation of** $\int_{U(N)} u_{i_1,j_1} \cdots u_{i_p,j_p} u_{r_1,s_1}^* \cdots u_{r_p,s_p}^* \, dU$

The following computation was first performed by D. Weingarten, F. Xu, and B. Collins. At the time of writing the paper we were not aware of the previous literature, so we include our own proof for this computation.

Let $S_n$ be the group of $n$-permutations and let $\mathbb{C}[S_n]$ be the group algebra. As in [2], we denote by $W_\sigma^N$ the coefficient of $\sigma \in S_n$ in the inverse of the element $\Phi^N = \sum_{\sigma \in S_n} N^{\# \sigma} \sigma \in \mathbb{C}[S_n]$ ($\# \sigma$ is the number of cycles in $\sigma$; the element $\Phi^N$ is invertible as it will be proven below for $N > n$). Thus we take

$$(\Phi^N)^{-1} = \sum_{\sigma \in S_n} W_\sigma^N \cdot \sigma.$$ 

Note that $\Phi^N$ is a central element and hence so is $\sum_{\sigma \in S_n} W_\sigma^N \cdot \sigma$. With these notations we have:

**Theorem 2.1.** For $N, n \in \mathbb{N}$, $N > n$ and $dU$ the Haar measure on $U(N)$, let $i_1, \ldots, i_n, j_1, \ldots, j_n, r_1, \ldots, r_s, s_1, \ldots, s_p$ be indices from 1 to $N$. Denote the entries of a unitary by $u_{ij}$ and the entries of its adjoint by $u_{ij}^* = \overline{u_{ji}}$. Then

$$\int_{U(N)} u_{i_1,r_1} \cdots u_{i_n,r_n} u_{s_1,j_1}^* \cdots u_{s_p,j_p}^* \, dU = \sum_{\sigma \in S_n} W_{\sigma \theta}^N \cdot 1$$

with the sum in the right hand side running over all $\sigma, \theta$ in $S_n$ such that $j_\alpha = \sigma(i_\alpha)$, $a = 1, 2, \ldots, n$ and $s_\beta = r_\theta(\beta)$, $B = 1, 2, \ldots, n$.

**Proof.** Let $L^2(M_N(\mathbb{C})^n, \mu_B^N)$ be the Hilbert space obtained by endowing $M_N(\mathbb{C})^n$ with the measure $Ce^{-\operatorname{Tr}(A_1^*A_1) \cdots -\operatorname{Tr}(A_n^*A_n)}$, $(A_1, \ldots, A_n) \in M_N(\mathbb{C})^n$, where $C$ is a constant, so that the entries functions $(A_1, A_2, \ldots, A_n) \mapsto a_{ij}^{(1)} \cdots a_{ij}^{(n)}$, have norm 1. Here $a_{ij}^{(t)}$ are the $ij$-entries of the matrix $A^{(t)}$ on the $t$-th component of the product $(M_N(\mathbb{C}))^n$.

Denote, for $\sigma$ in $S_n$, by $\chi_{\sigma}$ the function

$$(?) \quad \chi_{\sigma} = \sum_{i_1, \ldots, i_n=1}^N a_{i_1,\sigma(i_1)}^{(1)} a_{i_2,\sigma(i_2)}^{(2)} \cdots a_{i_n,\sigma(i_n)}^{(n)}.$$ 

Then, from the theory of symmetric functions ([6], [9]), the functions $\chi_{\sigma}$ generate the subspace functions on $(M_N(\mathbb{C}))^n$ that are invariant to the diagonal action of $U(n)$ on $(M_N(\mathbb{C}))^n$: $(A_1, \ldots, A_n) \mapsto (UA_1U^*, \ldots, UA_nU^*)$, $U \in U(N)$. Moreover, for $n < N$ the functions $\{\chi_{\sigma} \mid \sigma \in S_n\}$ are independent ([9]) and the scalar product $\langle \chi_{\sigma}, \chi_{\mu} \rangle$ depends only on $\sigma^{-1}\mu$ and it is equal to $N^{\#(\sigma^{-1}\mu)}$. 

Consequently, \((\chi_\sigma, \chi_\mu)_{\sigma, \mu \in S_n}\) represents the matrix of the convolution with \(\Phi_N\) on \(L^2(S_n)\). Consequently, the inverse of \(\Phi_N\) (which exists since the functions are independent) is the matrix \((W_{\sigma^{-1}})_{\sigma, \mu \in S_n}\). Let \(P\) be the projection from \(L^2((M_N(\mathbb{C}))^n, \mu)\) onto the space of \(\mathcal{U}(N)\) invariant functions. Then on one hand, since \(\mu\) is an invariant measure, it follows that \(P\) is the average over \(\mathcal{U}(N)\) by integration. Hence

\[
P(\langle a_{i_1 j_1} \cdots a_{i_n j_n} \rangle) = \int_{\mathcal{U}(N)} (u a_{i_1 j_1} \cdots (u a_{i_n j_n})_{i_1 j_1} \cdots dU.\]

On the other hand, assume \(P(\langle a_{i_1 j_1} \cdots a_{i_n j_n} \rangle) = \sum_{\sigma \in S_n} c_\sigma \chi_\sigma\), where \(c_\sigma\) depends on \(i_1, \ldots, i_n, j_1, \ldots, j_n\). Then, for all \(\mu \in S_n\),

\[
\langle a_{i_1 j_1} \cdots a_{i_n j_n} - \sum_{\sigma} c_\sigma \chi_\sigma, \chi_\mu \rangle = 0,
\]

and hence

\[
\langle a_{i_1 j_1} \cdots a_{i_n j_n}, \chi_\mu \rangle = \sum_{\sigma} c_\sigma \langle \chi_\sigma, \chi_\mu \rangle, \quad \forall \mu \in S_n
\]

But \(\langle a_{i_1 j_1} \cdots a_{i_n j_n}, \chi_\mu \rangle\) is the vector (indexed) by \(\mu \in S_n\) with the property that the \(\mu\)-th component is equal to 1 if and only if \(j_a = i_{\mu(a)}\), \(a = 1, 2, \ldots, n\).

Let \(R_{\sigma, \mu}\) be the inverse of the matrix \((\langle \sigma, \mu \rangle)_{\sigma, \mu \in S_n}\). We have noted before that \(R_{\sigma, \mu} = W_{\mu^{-1}}\sigma\). From \(\mathbf{3}\), by inversion, we deduce that \(c_\mu = \sum_{\sigma} R_{\sigma, \mu}\), where the sum runs over all \(\mu\) such that \(j_a = i_{\mu(a)}\), \(a = 1, 2, \ldots, n\). Thus

\[P(a_{i_1 j_1} \cdots a_{i_n j_n}) = \sum_{\mu \in S_n} R_{\sigma, \mu} \chi_\sigma, \quad S' = \{\mu \in S_n \mid j_a = i_{\mu(a)}, a = 1, 2, \ldots, n\}\]

From \(\mathbf{2}\) we obtain that

\[P(a_{i_1 j_1} \cdots a_{i_n j_n}) = \sum_{r_1, \ldots, r_n = 1}^{s_1, \ldots, s_n = 1} \int_{\mathcal{U}(N)} u_{i_1 r_1} \cdots u_{i_n r_n} a_{r_1 s_1}^{(1)} \cdots a_{r_n s_n}^{(n)} u_{s_1 j_1}^* \cdots u_{s_n j_n}^* \cdot dU\]

\[= \sum_{r_1, \ldots, r_n = 1}^{s_1, \ldots, s_n = 1} a_{r_1 s_1}^{(1)} \cdots a_{r_n s_n}^{(n)} \int_{\mathcal{U}(N)} u_{i_1 r_1} \cdots u_{i_n r_n} u_{s_1 j_1}^* \cdots u_{s_n j_n}^* \cdot dU.\]

Identifying the coefficients from the last formula with \(\mathbf{4}\) we obtain our statement. \(\square\)

3. Formula for \(\int_{\mathcal{U}(N)^2} \text{Tr}(W_1) \cdots \text{Tr}(W_n) \cdot dU \cdot dV\)

In this section we deduce a formula for the integral of traces of words only in case of \(\mathcal{U}(N)^2\) (instead of \(\mathcal{U}(N)^M\)) for simplicity. A similar formula was derived in \(\mathbf{1}\). Since the shape of the combinatorial aspect of the formula is important for the computation of the asymptotics, we derive our formula directly from the preceding section.

Let \(w_1, w_2, \ldots, w_n\) be reduced words in \(F_2 = (a, b)\), and let \(W_1, W_2, \ldots, W_n\) be the corresponding words viewed as functions in the variables \((U, V) \in \mathcal{U}(N)^2\) obtained by substituting \((U, V)\) for \((a, b)\). We describe \(\text{Tr}(w_1) \cdots \text{Tr}(w_n)\) in terms of a permutation \(\gamma\) and write \(\text{Tr}(w_1) \cdots \text{Tr}(w_n) = \text{Tr}_\gamma(w_1 \cdots w_n)\), where \(\gamma\) is described as follows.
Let \( n \) be the total number of occurrences of the symbol \( u \) in \( W_1, W_2, \ldots, W_n \). Let \( m \) be the total number of occurrences of \( v \). For the integral \( \int_{\mathcal{U}(N)} \text{Tr}(W_1) \cdots \text{Tr}(W_p) \, dU \, dV \) to be non-zero it is necessary (8) that \( n \) equals the number of \( U^* \) and that \( m \) equals the numbers of \( V^* \). We introduce a set of symbols (indexed by the letters \( u, v, u^*, v^* \)) respectively.

\[
X = \{1_u, \ldots, n_u, 1_{u^*}, \ldots, n_{u^*}, 1_v, \ldots, m_v, 1_{v^*}, \ldots, n_{v^*}\}. 
\]

Thus \( X \) is a set with \( 2(n+m) \) elements.

**Definition 3.1.** Given \( w_1, \ldots, w_p \) and \( X \) as above we define a permutation \( \gamma \) of \( X \) by means of the formula

\[
\text{Tr}_\gamma(w_1 \cdots w_p) = \text{Tr}(w_1) \cdots \text{Tr}(w_p)
\]

\[
= \sum_{a_1 u \cdots a_n u = 1}^{a_1 u^* \cdots a_{n u} = 1} u_{a_1 u} a_\gamma(1_u) \cdots u_{a_n u} a_\gamma(n_u) u_{1_{u^*}}^* a_\gamma(1_{u^*}) \cdots u_{n_{u^*}}^* a_\gamma(n_{u^*})
\]

\[\cdots v_{1_v} a_\gamma(1_v) \cdots v_{n_{v^*}}^* a_\gamma(n_{v^*}) \, du \, dv.\]

We denote the term on right hand side by \( \Phi_\gamma(U, V) \), where \( U, V \in \mathcal{U}(N) \).

Note that since the words are reduced \( \gamma \) has no fixed points.

With these notations we have:

**Theorem 3.2.** The integral of \( \text{Tr}(w_1) \cdots \text{Tr}(w_p) \) over \( \mathcal{U}(N)^2 \) is

\[
\int_{\mathcal{U}(N)^2} \text{Tr}_\gamma(U, V) \, dU \, dV = \sum_{\substack{\sigma_u, \sigma_v \in S(1_u, \ldots, n_u) \\sigma_v \circ S(1_v, \ldots, n_v)}} W^N \sum_{\sigma_u, \sigma_v \in S(1_u, \ldots, n_u)} W^N \sigma_v \circ S(1_v, \ldots, n_v)
\]

where \( R(\gamma, \sigma_u, \sigma_v, \theta_u, \theta_v) \) is the equivalence relation on \( X \) generated by

\[
\begin{align*}
i_{u^*} &= \gamma(\sigma_u(t_u)), \quad \gamma(\sigma_u(t_u)) = \theta_u(t_u), \quad t = 1, 2, \ldots, n, \\
s_{v^*} &= \gamma(\sigma_v(t_v)), \quad \gamma(\sigma_v(t_v)) = \theta_v(t_v), \quad s = 1, 2, \ldots, m.
\end{align*}
\]

Here \( \sharp R(\gamma, \sigma_u, \theta_u, \sigma_v, \theta_v) \) is the number of classes in the equivalence relation.

**Proof.** Indeed, we have that

\[
\int_{\mathcal{U}(N)^2} \text{Tr}_\gamma(U, V) \, dU \, dV
\]

\[= \left( \sum_{a_1 u^*, \ldots, a_n u^* = 1}^{a_1 u^*, \ldots, a_{n u} = 1} \int_{\mathcal{U}(N)} u_{a_1 u} a_\gamma(1_u) \cdots u_{a_n u} a_\gamma(n_u) u_{1_{u^*}}^* a_\gamma(1_{u^*}) \cdots u_{n_{u^*}}^* a_\gamma(n_{u^*}) \, dU \right) \]

\[\cdots \int_{\mathcal{U}(N)} v_{a_1 v} a_\gamma(1_v) \cdots v_{a_n v} a_\gamma(n_v) v_{1_{v^*}}^* a_\gamma(1_{v^*}) \cdots v_{n_{v^*}}^* a_\gamma(n_{v^*}) \, dV \right) \]

...
We now apply the formula from the preceding section, interchange the summation formula with the summation after \( \sigma_u, \theta_u, \sigma_v, \theta_v \), where \( \sigma_u, \theta_u \) are the permutations that appear in the integrals for the \( u \)'s and \( \sigma_v, \theta_v \) are the permutations that appear in the summations for the \( \theta \)'s.

**Remark 3.3.** Since the words are allays reduced, the equivalence relation \( R(\gamma, \sigma_u, \theta_u, \sigma_v, \theta_v) \) has no singleton classes and hence \( |R(\gamma, \sigma_u, \theta_u, \sigma_v, \theta_v)| \leq n + m. \)

### 4. The Asymptotics for

\[
\int \int (\text{Tr}(W_1))^\alpha_1 (\text{Tr}(W_1))^\beta_1 \cdots (\text{Tr}(W_p))^\alpha_p (\text{Tr}(W_p))^\beta_p \, dU \, dV
\]

In this section we show that for all words \( w_1, \ldots, w_p \) in \( F_2 \) by taking \( W_1, \ldots, W_p \) to be the corresponding functions on \( U(N)^2 \) we have that

\[
\int \int \left( \text{Tr}(W_1) \right)^{\alpha_1} \left( \text{Tr}(W_1) \right)^{\beta_1} \cdots \left( \text{Tr}(W_p) \right)^{\alpha_p} \left( \text{Tr}(W_p) \right)^{\beta_p} \, dU \, dV
\]

is \( O\left( \frac{1}{N} \right) \) unless \( \alpha_1 = \beta_1, \ldots, \alpha_p = \beta_p \) in which case the integral is \( \alpha_1! \cdots \alpha_p! \cdot (j(w_1))^{\alpha_1} \cdots (j(w_p))^{\alpha_p} \), where \( j_i = j(w_i) \) is the numbers of cyclic rotations of the word \( w_i \) which leave \( w_i \) invariant. As in (3) this means that the asymptotic distribution of the variables \( \text{Tr}(W_1), \ldots, \text{Tr}(W_p) \) as \( N \to \infty \) is that of \( \sqrt{\gamma_1 Z_1, \ldots, \sqrt{\gamma_p Z_p}, \ldots, \gamma_p Z_p} \), where \( Z_1, \ldots, Z_p \) are independent gaussian variables.

**Theorem 4.1.** Let \( w_1, \ldots, w_p \) be the words on \( F_2 \) and \( W_1, \ldots, W_p \) be the corresponding functions on \( U(N)^2 \). An integral of the form

\[
\int \int (\text{Tr}(W_1))^\alpha_1 (\text{Tr}(W_1))^\beta_1 \cdots (\text{Tr}(W_p))^\alpha_p (\text{Tr}(W_p))^\beta_p \, dU \, dV
\]

is non-zero (modulo \( O\left( \frac{1}{N} \right) \)) if and only if it can be written in the form

\[
\int \int |\text{Tr}(W_1)|^{2\alpha_1} \cdots |\text{Tr}(W_p)|^{2\alpha_p} \, dU \, dV
\]

in which case it is equal to \( \alpha_1! \cdots \alpha_p! j_1^{\alpha_1} \cdots j_p^{\alpha_p} \), with \( j_i = j(w_i) \) the number of cyclic rotations of the word \( w_i \) that are leaving \( w_i \) invariant.

Consequently, \( \text{Tr}(W_1), \ldots, \text{Tr}(W_p) \) have the asymptotic moment distribution (as \( N \to \infty \)) of \( \sqrt{\gamma_1 Z_1, \ldots, \sqrt{\gamma_p Z_p}, \ldots, \sqrt{\gamma_p Z_p}} \), where \( Z_1, \ldots, Z_p \) are independent normal gaussian variables.

**Proof.** We rewrite the formula from the preceding section as follows.

\[
\int \int_{U(N)^2} \text{Tr}(W_1) \cdots \text{Tr}(W_p) \, dU \, dV
\]

\[
= \sum_{\beta_1 \cdots \beta_n \beta_{n+1} \cdots \beta_n = 1} \int \int a^{(1)}_{\beta_1 \beta_2} \cdots a^{(n)}_{\beta_1 \beta_1} \cdots a^{(n+1)}_{\beta_{n-1} \beta_{n+1}} \cdots
\]

\[
\cdots a^{(n_p)}_{\beta_{n_p} \beta_{n_p+1}} \, dU \, dV
\]

where the symbols \( a^{(1)} \cdots a^{(n)} \) belong to the set \( \{ U, V, U^*, V^* \} \). Here \( n_i - n_{i-1} \) is the length of the word \( w_i, i = 1, 2, \ldots, s \).
Denote by $\tilde{U}$ the set of all symbols $a^{(i)}$ that are equal to the letter $u$, and similarly for $\tilde{U}^*, \tilde{V}, \tilde{V}^*$. Because of [8], unless $\text{card} \tilde{U} = \text{card} \tilde{U}^* = n, \text{card} \tilde{V} = \text{card} \tilde{V}^* = m$, the integral is $O(\frac{1}{n})$.

According to the formula in the preceding paragraph the integral will be the summation over all bijection $\sigma_u : \tilde{U} \to \tilde{U}^*, \theta_u : \tilde{U}^* \to \tilde{U}, \sigma_v : \tilde{V} \to \tilde{V}^*, \theta_v : \tilde{V}^* \to \tilde{V}$, of

$$
\sum_{\sigma_u, \theta_u, \sigma_v, \theta_v} W_{\sigma_u, \theta_u, \sigma_v, \theta_v} \cdot W_{\sigma_v, \theta_v, \nu, \theta_v}
$$

where $R(\sigma, \theta, \sigma, \theta)$ is the equivalence relation generated by

$$
\sigma_u(i) + 1 \sim i, \quad \theta_u(i) \sim i + 1, \quad \sigma_v(i) + 1 \sim i, \quad \theta_v(i) \sim i + 1
$$

where by the operation $i + 1$, we mean successively (when $i = n_1, n_2, \ldots, n_s$)

$$
n_1 + 1 = 1, \quad n_2 + 1 = n_1 + 1, \quad \ldots, \quad n_s + 1 = n_{s-1} + 1
$$

corresponding to the cycle $(1, \ldots, n_1)(n_1 + 1, \ldots, n_2)(n_{s-1} + 1, \ldots, n_s))$.

The equivalence relation has no singletons and hence $N^2R(\sigma, \theta, \sigma, \theta)$ is at most $N^{n+m}$, while the term $W_{\sigma, \theta, \sigma, \theta} \cdot W_{\sigma, \theta, \nu, \theta}$ is of the order $N^k$, where $k \leq n + m$, with equality if and only if $\sigma_u = \theta_u^{-1}, \sigma_v = \theta_v^{-1}$.

This means that the only non-zero terms will come from equivalence relations of the form

$$
\sigma_u(i) + 1 \sim i, \quad \sigma_u^{-1}(i) \sim i + 1, \quad \sigma_v(i) + 1 \sim i, \quad \sigma_v^{-1}(i) \sim i + 1.
$$

To obtain the number of terms (6) in this case we need to determine the permutations $\sigma_u, \sigma_v$ for which this equivalence relation has all the classes of two elements.

Denote by $\gamma$ the permutation with the property that $\gamma|\tilde{U} = \sigma_u, \gamma|\tilde{U}^* = \sigma_u^{-1}, \gamma|\tilde{V} = \sigma_v, \gamma|\tilde{V}^* = \sigma_v^{-1}$. Then $\gamma$ is an involution and the equivalence relation is described as

$$
\gamma(i) + 1 \sim i, \quad \gamma(i) \sim i + 1.
$$

But $i \sim \gamma(i) + 1$ and $i = (i - 1) + 1 \sim \gamma(i - 1)$ and hence $\gamma(i - 1) = \gamma(i) + 1$ for all $i$ (since the equivalence relations have only singletons). This means that if $i$ runs over the elements in a word $w_1$, then $\gamma(i)$ must run in the opposite direction over the elements of the conjugate word $w_1^{-1}$.

In consequence, the integral in the statement is non-zero (modulo $O(\frac{1}{n})$) only if it is of the form $\int_{U(N)^2} |\text{Tr}(W_1)|^2 \cdots |\text{Tr}(W_p)|^2 dU dV$ and in this case the integral is equal to the number of possible pairings between a word and cyclic rotation of its inverse. This completes the proof.

5. A COMBINATORIAL FORMULA FOR $\int_{U(N)^2} |\text{Tr}(W_1)|^2 \cdots |\text{Tr}(W_p)|^2 dU dV$

In this section we establish a formula that is specifically adapted for integrals of products of absolute values of traces. Indeed a positive answer for the Connes’s embedding conjecture would require the joint distribution of the variables $|\text{Tr}(W_1)| \cdots |\text{Tr}(W_p)|$ as functions on $U(N)$.

Thus let $w_1, \ldots, w_p$ be reduced words in $F_2$, and let $X$ be the total set of symbols of $U, U^*, V, V^*$ in $|\text{Tr}(w_1)|^2 \cdots |\text{Tr}(w_p)|^2$, where each element in $X$ corresponds to a specific occurrence of the corresponding symbol in $|\text{Tr}(w_1)|^2 \cdots |\text{Tr}(w_p)|^2$. 
Assume there are \( n \) occurrences for the symbol \( U \), \( m \) occurrences for the symbol \( V \), and hence that the set \( X \) has \( 2(n + m) \) elements. As in the preceding section \( X \) is partitioned as \( \bar{U} \cup \bar{U}^* \cup \bar{V} \cup \bar{V}^* \), where \( \bar{U}, \bar{U}^*, \bar{V}, \bar{V}^* \) are the set of symbols of \( u, u^*, v, v^* \) respectively in \( X \).

Let \( \Psi \) be the map which associates to each symbol \( a \) in \( X \), which comes from a word \( w_1 \) its corresponding symbol \( a^* \) in \( w_1^{-1} \) and vice versa for a symbol \( a \) in \( w_1^{-1} \) it associates the corresponding symbol \( a^* \) in \( w_1 \). Then \( \Psi \) is an involution, \( \Psi \) maps \( \bar{U} \) onto \( \bar{U}^* \) and \( \bar{V} \) onto \( \bar{V}^* \). Let \( I \) be the map associating to each symbol \( a \) the successor of \( \Psi(a) \) in the inverse word.

Let \( S_{\bar{U}} \) (and respectively \( S_{\bar{U}}^*, S_{\bar{V}}, S_{\bar{V}}^* \)) be the set of permutations of the sets \( \bar{U} \) (respectively \( \bar{U}^*, \bar{V}, \bar{V}^* \)). For each \( \sigma_u \in S_{\bar{U}}, \sigma_v \in S_{\bar{V}}, \theta_v \in S_{\bar{V}}^* \) let \( (\sigma_u, \sigma_v, \theta_v) \) be the concatenation of these permutations to a permutation of \( X \).

With the above notations we have:

**Proposition 5.1.** Let \( w_1, \ldots, w_p \) be words in \( F_2 \) and let \( W_1, \ldots, W_p \), be the corresponding words as functions on \( U(N)^2 \).

Let \( R \) be the following element in

\[
\mathbb{C}(S_{\bar{U}}) \otimes \mathbb{C}(S_{\bar{U}}^*) \otimes \mathbb{C}(S_{\bar{V}}) \otimes \mathbb{C}(S_{\bar{V}}^*) :
R = \sum_{\sigma_u \in S_{\bar{U}}, \sigma_v \in S_{\bar{V}}, \theta_v \in S_{\bar{V}}^*} N^{|I|} \varphi(\sigma_u, \sigma_v, \theta_v) \sigma_u \otimes \theta_v \otimes \sigma_v \otimes \theta_v.
\]

Note that \( R \) depends only of the cardinalities of the sets \( I(\bar{U}) \cap \bar{U}^*, I(\bar{U}) \cap \bar{V}, I(\bar{U}) \cap \bar{V}^*, \ldots, I(\bar{V}^*) \cap \bar{V} \).

Let \( \Phi \) be the linear map from \( \mathbb{C}(S_{\bar{U}}) \otimes \mathbb{C}(S_{\bar{U}}^*) \otimes \mathbb{C}(S_{\bar{V}}) \otimes \mathbb{C}(S_{\bar{V}}^*) \) into \( \mathbb{C} \) which associates to \( \sigma_1 \otimes \theta_1 \otimes \sigma_2 \otimes \theta_2 \) the number \( W_{\sigma_1, \theta_1, \sigma_2, \theta_2} \). Then

\[
\int \int_{U(N)^2} |\text{Tr}(W_1)|^2 \cdots |\text{Tr}(W_p)|^2 \, dU \, dV = \Phi(R).
\]

**Proof.** Introduce an indexing of the elements in \( X \) so that the symbol corresponding to a term \( u_{\beta_1, \beta_1+1} \) in a word \( w_i \) correspond to a \( u_{\beta_1, \beta_1+1} \) for a word in \( w_i^{-1} \). Here we use the convention that the elements in \( w_i \) have indexing after \( \beta_1, \beta_2, \ldots \). Then computing the integral

\[
\int \int_{U(N)^2} |\text{Tr}(W_1)|^2 \cdots |\text{Tr}(W_p)|^2 \, dU \, dV
\]

will amount to compute integrals of the form

\[
\int_{U(N)} u_{\beta_1, \beta_1+1} \cdots u_{\beta_r, \beta_r+1} \cdots u_{\beta_1, \beta_r} \cdots u_{\beta_r, \beta_1} \, dU.
\]

Thus here

\[
U = \{s, \ldots, r, \ldots\}, \quad U^* = \{s+1, \ldots, r-1, \ldots\}.
\]

Then the map \( \Psi \) will map \( s, \tau \) onto \( s+1, r-1 \) respectively and I will map \( s, r \) into \( \pi, \pi \) respectively.

We use the formula from Section 2 and we have the sum over permutations \( \sigma_u \) of the symbols \( \{s, \ldots, r, \ldots\} \), and permutations \( \theta_v \) of the symbols \( \{s+1, \ldots, r-1, \ldots\} \).
Hence the corresponding equivalence relation corresponding to these permutations (and the similar permutations for \( \theta \)) will be exactly
\[
\bar{\sigma} \sim \sigma_u(s), \quad \bar{\tau} \sim \sigma_u(r) \quad \text{and} \quad \bar{s+1} \sim \theta_u(s+1), \quad r-1 \sim \theta_u(\bar{\tau}-1),
\]
which is exactly the equivalence relation corresponding to \( I \circ (\sigma_u, \theta_u, \sigma_x, \theta_x) \). This completes the proof. \( \Box \)

6. AN EXAMPLE FOR THE COMPUTATION OF
\[
\int_{H(N)} |\text{Tr}(U_1^a V_{s1}^1 U_2^a V_{s2}^1 \ldots U_n^a V_{s_n}^1)|^2 \ldots |\text{Tr}(U_1^a V_{s1}^a \ldots U_n^a V_{s_n}^a)|^2 \, dU \, dV
\]

We apply the algorithm in the preceding section for the calculation of a product of words in which between powers of \( u \) of degree at least 3 are intercalated powers of \( v \) of degree \( \pm 1 \). We will describe the structure of the element \( R \) in such a case since \( \Psi \) is easy to be described in this situation.

**Proposition 6.1.** For the integral, \(|p_u^n| \geq 3\), \( \varepsilon_u^n = \pm 1\),
\[
\int_{H(N)} |\text{Tr}(U_1^a V_{s1}^1 U_2^a V_{s2}^1 \ldots U_n^a V_{s_n}^1)|^2 \ldots |\text{Tr}(U_1^a V_{s1}^a \ldots U_n^a V_{s_n}^a)|^2 \, dU \, dV
\]
the element \( R \) is described as follows:

Let \( n \) be the total number of \( u \)'s and \( m \) the total number of \( v \)'s.
The structure of the element \( R \), viewed as an element of \( \mathbb{C}(S_u) \otimes \mathbb{C}(S_m) \otimes \mathbb{C}(S_n) \) is described as follows:

Since \( p_u^n \geq 1 \) we have that \( n \geq m \). Then we have a set
\[X = \{1_x, 2_x, \ldots, (n - m)_x\} \cup \{1_y, 2_y, \ldots, m_y\} = X_0 \cup Y \]
and the first factor \( S_u \) is identified with \( S(x) \) (permutations of \( X \)), while the second \( S_u \) is identified with \( S(X) \) where
\[
\bar{X} = \{1_x, 2_x, \ldots, (n - m)_x\} \cup \{1_y, 2_y, \ldots, m_y\} = \bar{X}_0 \cup \bar{Y}.
\]

We consider also two sets \( A = \{1_a, \ldots, m_a\} \) and \( \bar{A} = \{1_a, \ldots, m_a\} \). Then the first factor \( S_m \) is identified with \( S(A) \), while the second with \( S(\bar{A}) \).

The map \( I \) acts on \( X_0 \cup X \) by mapping \( i_x \) into \( \bar{i}_x \) and \( i_x \) into \( i_x \), while on the set \( Y \cup \bar{Y} \), \( I \) maps \( i_y \) into \( i_a \) and \( \bar{i}_y \) into \( \bar{i}_a \) (or \( i_y \) into \( \bar{i}_a \) and \( \bar{i}_y \) into \( i_a \)). \( I \) is an involution. Then
\[
R = \sum_{\sigma, \bar{\sigma} \in S(X), S(\bar{X}), \theta, \bar{\theta} n} N^{2(I\circ(\sigma, \bar{\sigma}, \theta, \bar{\theta}))} \sigma \otimes \bar{\sigma} \otimes \theta \otimes \bar{\theta}.
\]

**Proof.** This follows by identifying the sets \( \bar{U}, \bar{U}^* \) from the preceding proposition with the sets \( X, \bar{X} \), while \( V, \bar{V}^* \) are identified with \( Y, \bar{Y} \). \( \Box \)

**Remark 6.2.** The map \( \Psi \) can be explicitly describe as a map from \( X \) onto \( \bar{X} \), in terms of the alternating signs in \( p_1^a, \varepsilon_1^a, \ldots, p_a^a, \varepsilon_a^a \), while on the set \( Y \) it simply maps \( i_a \) into \( \bar{i}_a \).
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We have been informed that R. Speicher, P. Sniady and J. Mingo have obtained independently in a joint paper in preparation the same theorem as our result in Section 4.

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