On Rounds in Quantum Communication

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Abstract

We investigate the power of interaction in two player quantum communication protocols. Our main result is a rounds-communication hierarchy for the pointer jumping function $f_k$. We show that $f_k$ needs quantum communication $\Omega(n)$ if Bob starts the communication and the number of rounds is limited to $k$ (for any constant $k$). Trivially, if Alice starts, $O(k \log n)$ communication in $k$ rounds suffices. The lower bound employs a result relating the relative von Neumann entropy between density matrices to their trace distance and uses a new measure of information.

We also describe a classical probabilistic $k$ round protocol with communication $O(n/k \cdot (\log^{(k/2)} n) + \log k + k \log n)$ in which Bob starts the communication.

Furthermore as a consequence of the lower bound for pointer jumping we show that any $k$ round quantum protocol for the disjointness problem needs communication $\Omega(n^{1/k})$ for $k = O(1)$.

1 Introduction

Quantum mechanical computing and communication has been studied extensively during the last decade. Communication has to be a physical process, so an investigation of the properties of physically allowed communication is desirable, and the fundamental theory of physics available to us is quantum mechanics.

The theory of communication complexity deals with the question how efficient communication problems can be solved and has various applications to lower bound proofs (introduction to (classical) communication complexity can be found in [19]). The communication complexity approach to lower bounds consists of reducing a lower bound proof for some computational model to a communication complexity lower bound, where several techniques for such proofs are available, see [13] for many examples.

In a quantum protocol (as defined in [30]) two players Alice and Bob each receive an input, and have to compute some function defined on the pair
of inputs cooperatively. To this end they exchange messages consisting of qubits, until the result can be produced by some measurement done by one of the players (for overviews on quantum communication complexity see \cite{29} and \cite{16}).

A slightly different scenario proposed in \cite{8} and \cite{9} allows the players to start the protocol possessing some (input independent) qubits that are entangled with those of the other player. Due to the “superdense coding” technique of \cite{2} in this model 2 classical bits can be communicated by transmitting one qubit (and using up one EPR pair). See \cite{30, 15} for some examples of lower bounds via communication complexity in the quantum setting. Unfortunately so far only few “applicable” lower bound methods for quantum protocols are known: the rank lower bound is known to hold for exact (i.e., errorless) quantum communication \cite{1, 3}, the (usually weak) discrepancy lower bound for bounded error quantum protocols \cite{18}.

One breakthrough result of the field of quantum computing is Grover’s search algorithm that retrieves an item from an unordered list within $O(\sqrt{n})$ questions \cite{11}, outperforming every classical algorithm for the problem. By an application of this search algorithm to communication complexity in \cite{4} an upper bound of $O(\sqrt{n \log n})$ is shown for the bounded error quantum communication complexity of the disjointness problem $DISJ_n$ (both players receive an incidence vector of a subset of $\{1, \ldots, n\}$ and have to decide whether the sets are not disjoint: $\bigvee (x_i \land y_i)$), one of the most important communication problems. This yields the largest gap between quantum and classical communication complexity known so far for a total function, the probabilistic communication complexity of disjointness is $\Omega(n)$ \cite{19}. Currently no superlogarithmic lower bound on the bounded error quantum communication complexity of the disjointness problem is known.

Unfortunately the protocol for disjointness using Grover search takes $\Theta(\sqrt{n})$ rounds, an unbounded increase in interaction compared to the trivial protocol communicating $n$ bits in 1 round. Similar phenomena show up in the polynomial gaps between quantum Las Vegas and probabilistic bounded error communication complexity for total functions, see \cite{3, 15, 2, 29}.

We are interested in the question how efficient total communication problems can be solved in the quantum model when the number of rounds is restricted. The most severe restriction is one-way communication, where only a monologue is transmitted from one player to the other, who then decides the function value. This has been investigated e.g. in \cite{18}, where a lower bound method based on the so called VC-dimension is proved, which allows to prove an exponential advantage for 2 round classical compared to 1 round quantum communication complexity. Kremer \cite{18} exhibits a

\footnote{Exponential gaps between quantum communication complexity and classical probabilistic communication complexity are known only for partial functions, and are possible without interaction \cite{29, 16}.}
complete problem for the class of problems with polylogarithmic quantum one-way communication complexity (in the case of bounded error).

In a series of papers (see [10, 12, 24, 25, 14]) more general round hierarchies of the following form are given for classical protocols: A function $f_k$ (usually the so called pointer jumping function) has $k$ round communication complexity $k \log n$ if Alice starts the communication, but has a much larger $k$ round communication complexity when Bob starts.

Our main result is that $k$-round quantum protocols need communication $n/2^{O(k)} - k \log n$ to compute the pointer jumping function when Bob starts. So changing the starting player (or reducing the number of rounds by 1) may result in drastically increased communication also in the quantum case.

Nayak et. al. [22] have proved a lower bound of $\Omega(n^{1/k})$ for the quantum communication complexity of a certain subproblem of pointer jumping, in the situation when B starts and $k$ rounds are allowed.

We begin our consideration of the complexity of pointer jumping (in section 5) with the description of a classical randomized protocol for pointer jumping using communication $O(n/k \cdot (\log^{(k/2)} n + \log k) + k \log n)$ in the situation where Bob starts and only $k$ rounds are allowed. This upper bound is close to the known lower bounds for classical protocols [24, 15].

The general strategy of our new lower bound for pointer jumping is to bound the value of a certain measure of information between the qubits of one player and the “next” pointer in terms of the analogous quantity for the previous pointer plus the average information on pointers in possession of the other player.

The mentioned protocol makes clear why the usual measure of information does not work in this approach. So (after defining the main notions of quantum computing in section 2 and the model of communication complexity in section 3) we introduce a new measure of quantum information in section 4. This measure is tied to the usual, von Neumann measure of quantum information by a theorem, which connects the trace distance between density matrices to their relative von Neumann entropy.

The lower bound on pointer jumping implies via reductions lower bounds for the $k$ round bounded error quantum communication complexity of the disjointness problem of the order $\Omega(n^{1/k})$ for all constant $k$, see section 6.

We conclude from our result that quantum communication is dependent on interaction, as one should expect for a “realistic” mode of communication. We also conclude that good speedups by quantum protocols imply the use of nontrivial interaction in the case of total functions: for an asymptotic speedup by quantum Las Vegas protocols always more than one round is necessary [14]. By the results in this paper (and similar results in [22]) rounds are also crucial in quantum speedups for the disjointness problem.

The lower bound for a subproblem of pointer jumping given in [22] and the lower bound in this paper use at the basis of the proofs similar techniques. The main ingredient of the proof in [22], the “average encoding theorem”,
follows directly from our theorem 1, which states a connection between a new measure of information (based on the trace distance) and the relative von Neumann entropy. We make use of a fact from [22], namely the “local transition theorem”. A combined version of both papers appears in [7]. Our results also hold in the model, in which prior entanglement is available.

The paper is organized as follows: In the next section we give some background on quantum mechanics. Then we define the communication model in section 3. In section 4 we consider measures of information and entropy. In section 5 we prove our results on the complexity of pointer jumping. Section 6 contains the lower bound for the disjointness problem.

2 Quantum States and Transformations

Quantum mechanics is a theory of reality in terms of states and transformations of states. See [23, 26] for general information on this topic with an orientation on quantum computing.

In quantum mechanics pure states are unit norm vectors in a Hilbert space, usually $\mathbb{C}^k$. We use the Dirac notation for pure states. So a pure state is denoted $|\phi\rangle$ or $\sum_{x \in \{0, \ldots, k-1\}} \alpha_x |x\rangle$ with $\sum_{x \in \{0, \ldots, k-1\}} |\alpha_x|^2 = 1$ and with $\{ |x\rangle | x \in \{0, \ldots, k-1\}\}$ being an orthonormal basis of $\mathbb{C}^k$.

Inner products in the Hilbert space are denoted $\langle \phi | \psi \rangle$, outer (matrix valued) products $|\phi\rangle \langle \psi |$.

If $k = 2^l$ then the basis is also denoted $\{ |x\rangle | x \in \{0,1\}^l\}$. In this case the space $\mathbb{C}^{2^l}$ is the $l$-wise tensor product of the space $\mathbb{C}^2$. The latter space is called a qubit, the former space consists of $l$ qubits.

Usually also mixed states are considered.

**Definition 1** Let $\{ \rho_i, |\phi_i\rangle\} |i = 1, \ldots, k\}$ with $\sum_i p_i = 1$ and $p_i \in [0,1]$ be an ensemble of pure states of a quantum system, also called a mixed state. $\rho_i = |\phi_i\rangle \langle \phi_i |$ is the density matrix of the pure state $|\phi_i\rangle$. $\sum_i p_i \rho_i$ is the density matrix of the mixed state. For a bipartite system with density matrix $\rho_{AB}$ denote $\rho_A = \text{trace}_B \rho_{AB}$.

As usual measurements of certain observables and unitary transformations are considered as basic operations on states, see [23, 26] for definitions. For all possible measurements on a mixed state the results are determined by its density matrix. In quantum mechanics the density matrix plays an analogous role to the density function of a random variable in probability theory. Note that a density matrix is Hermitian, positive semidefinite and has trace 1. Thus it has only real, nonnegative eigenvalues that sum to 1. Linear transformations on density matrices are called superoperators.

**Definition 2** A superoperator is a linear map on density matrices. A superoperator is positive, if it sends positive semidefinite matrices to positive
semidefinite matrices. A superoperator is called completely positive, if its tensor product with the identity superoperator is positive on density matrices over each finite dimensional extension of the underlying Hilbert space.

Trace-preserving completely positive superoperators map density matrices to density matrices and capture all physically allowed transformations. These include unitary transformations, tracing out subsystems, forming a tensor product with some constant qubits, and general measurements. The following important fact characterizes the allowed superoperators in terms of unitary transformations and tracing out (see [26]). This fact is known as the Kraus representation theorem.

**Fact 1** The following statements are equivalent:

1. A superoperator $T$ sending density matrices over $H_1$ to density matrices over $H_2$ is trace preserving and completely positive.

2. There is a Hilbert space $H_3$ with $\dim(H_3) \leq \dim(H_1)$ and a unitary transformation $U$, such that for all density matrices $\rho$ over $H_1$ the following holds:

   $$T\rho = \text{trace}_{H_1 \otimes H_3}[U(\rho \otimes |0_{H_3} \otimes H_2\rangle\langle 0_{H_3} \otimes H_2|)U^\dagger].$$

So allowed superoperators can be simulated by adding some blank qubits, applying a unitary transformation and tracing out, i.e., “dropping some qubits”.

**Definition 3** A purification of a mixed state with density matrix $\rho$ over some Hilbert space $H$ is any pure state $|\phi\rangle$ over some space $H \otimes K$ such that $\text{trace}_K |\phi\rangle\langle \phi| = \rho$.

### 3 The Communication Model

In this section we provide definitions of the computational models considered in the paper. We begin with the model of classical communication complexity.

**Definition 4** Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function. In a communication protocol player Alice and Bob receive $x$ and $y$ and compute $f(x,y)$. The players exchange binary encoded messages. The communication complexity of a protocol is the worst case number of bits exchanged for any input. The deterministic communication complexity $D(f)$ of $f$ is the complexity of an optimal protocol for $f$.

In a randomized protocol both players have access to public random bits. The output is required to be correct with probability $1 - \epsilon$ for some constant $\epsilon$. 

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The randomized communication complexity of a function $R_\epsilon(f)$ is then defined analogously to the deterministic communication complexity. We define $R(f) = R_{1/3}(f)$. A protocol has $k$ rounds, if the players exchange $k$ messages with Alice and Bob alternating as speakers. In message $k + 1$ one player announces the result. Alice protocol is called one-way if only one player sends a message, and then the other player announces the result. The complexity notations are superscripted with the number of allowed rounds and eventually with the player starting, like $D_{k/3}^k$ or $D_k$ (usually Alice starts).

Now we define quantum communication protocols. For general information on quantum computation see [23] and [26].

**Definition 5** In a quantum protocol both players have a private set of qubits. Some of the qubits are initialized to the input before the start of the protocol, the other qubits are in state $|0\rangle$. In a communication round one of the players performs some unitary transformation on the qubits in his possession and then sends some of his qubits to the other player (the latter step does not change the global state but rather the possession of individual qubits). The choice of qubits to be sent and of unitary operations is fixed in advance by the protocol.

At the end of the protocol the state of some qubits belonging to one player is measured and the result is taken as the output. The communication complexity of the protocol is the number of qubits exchanged.

In a (bounded error) quantum protocol the correct answer must be given with probability $1 - \epsilon$ for some $1/2 > \epsilon > 0$. The (bounded error) quantum complexity of a function, called $Q_\epsilon(f)$, is the complexity of an optimal protocol for $f$. $Q(f) = Q_{1/3}(f)$.

In fact we will consider a more general model of communication complexity, in which the players can apply all physically allowed superoperators to their private qubits. But due to the Kraus representation theorem (see fact 1) this model can be simulated by the above model without increasing communication (with the help of additional private qubits).

We have to note that in the defined model no intermediate measurements are allowed to control the choice of qubits to be sent or the time of the final measurement. Thus for all inputs the same amount of communication and rounds is used. As a generalization one could allow intermediate measurements, whose results could be used to choose the qubits to be sent and possibly when to stop the communication protocol. One would have to make sure that the receiving player knows when a message ends.

A protocol with $k$ rounds in this more general model can be simulated while loosing a factor of at most $k$ in the communication: for each measurement the operations given by the Kraus representation theorem are used. The
measurement’s result is then stored in some ancilla qubits. Now the global
state is a superposition over the results and a superposition of the appro-
priate communications can be used as a communication. This superposition
uses as many qubits as the worst case message of that round. This may
be at most the complexity of the whole protocol, so the overall complexity
increases by at most a factor of \( k \). While this simulation may not be sat-
sfactory in general, it suffices to keep our lower bound valid in the more
general model.

In \[8\] and \[9\] a different model of quantum communication (the communi-
cation model with entanglement) is proposed. Alice and Bob may possess
an arbitrary input-independent set of (entangled) qubits in the beginning.
Then they communicate according to an ordinary quantum protocol. This
model can be simulated by allowing first an arbitrary input-independent
communication with no cost followed by a usual quantum communication
protocol in which the cost is measured. The superdense coding technique
of \[4\] allows to transmit \( n \) bits of classical information with \( \lceil n/2 \rceil \) qubits in
this model.

**Definition 6** The quantum bounded error communication complexity with
entanglement and error \( \epsilon \) is denoted \( Q^\epsilon_{\text{pub}}(f) \). Let \( Q^\epsilon_{\text{pub}}(f) = Q_{1/\epsilon}(f) \).

For surveys on quantum communication complexity see \[29\] and \[16\].

## 4 Quantum Information Theory

Our main result in the next section uses information theory arguments. First
we define the classical notions of entropy and information.

**Definition 7** Let \( X : \Omega \to S \) be a random variable on finite sets \( \Omega, S \) (as
usual the argument of \( X \) is dropped). The density function (or distribution)
of \( X \) is \( p_X : S \to [0,1] \), where \( p_X(x) \) is the probability of the event \( X = x \).
The entropy of \( X \) is \( H(X) = -\sum_{x \in S} p_X(x) \log p_X(x) \).
Let \( X,Y \) be random variables over \( \Omega \). The joint density function of \( XY \)
is \( p_{XY}(x,y) \). The information between \( X \) and \( Y \) is \( H(X : Y) = H(X) + H(Y) - H(XY) \).
We use the convention \( 0 \log 0 = 0 \).

Now we define the quantum mechanical notions of entropy and information.

**Definition 8** The von Neumann entropy of a density matrix \( \rho_X \) is de-
finite by \( S(X) = S(\rho_X) = -\text{trace}(\rho_X \log \rho_X) \). The relative von Neumann
entropy between two density matrices \( \rho, \sigma \) of the same size is \( S(\rho||\sigma) =
\text{trace}(\rho(\log \rho - \log \sigma)) \). This value may be infinite.
The von Neumann information is 
\[ S(X : Y) = S(X) + S(Y) - S(XY) \] (see also \[3\]). Here \( S(X) \) is the von Neumann entropy of the reduced density matrix \( \rho_X = \text{trace}_Y \rho_{XY} \).

The conditional von Neumann information is 
\[ S(X : Y|Z) = S(XZ) + S(YZ) - S(2Z) - S(XYZ) \].

Note that the von Neumann entropy depends only on the eigenvalues of a matrix and is thus invariant under unitary transformations. If the underlying Hilbert space has dimension \( d \) then the von Neumann entropy of the density matrix is bounded by \( \log d \).

Not all relations in classical information theory hold for von Neumann entropy. The following fact contains the so-called Araki-Lieb inequality (*) and its consequences, which describes a notable difference to classical entropy (see \[26, 7\]).

**Fact 2** For all bipartite states \( \rho_{XY} \):
\[
S(X) + S(Y) \geq S(XY) \geq |S(X) - S(Y)|,
\]
\[
S(X : Y) \leq 2S(X).
\]

Then also \( S(X : Y|Z) \leq S(X : YZ) \leq 2S(X) \) holds.

The following is an important property of the von Neumann entropy, see \[26\]. This property is known as the Lindblad-Uhlmann monotonicity of the von Neumann entropy.

**Fact 3** For all trace-preserving, completely positive superoperators \( F \) and all density matrices \( \rho, \sigma \):
\[
S(\rho||\sigma) \geq S(F\rho||F\sigma).
\]

We are going to introduce another measure of information based on the distinguishability between a bipartite state and the state described by the tensor product of its two reduced density matrices. Now we consider measures of distinguishability. One such measure is the relative entropy. For probability distributions the total variational distance is another useful measure.

**Definition 9** If \( p, q \) are probability distributions on \( \{1, \ldots, n\} \), then their distance is defined
\[
||p - q|| = \sum_{i=1}^{n} |p(i) - q(i)|.
\]

The following norm on linear operators is considered in \[1\].

**Definition 10** Let \( \rho \) be the matrix of a linear operator. Then the trace norm of \( \rho \), denoted \( ||\rho||_1 \), is the sum of the absolute values of the elements of the multiset of all eigenvalues of \( \rho \). In particular \( ||\rho||_1 = \text{Tr}(\sqrt{\rho^\dagger \cdot \rho}) \).
Note that the distance $||\rho - \sigma||_1$ is a real value for Hermitian matrices $\rho, \sigma$. The trace norm has a close relation to the measurable distance between states as shown in [1].

**Fact 4** For an observable $O$ and a density matrix $\rho$ denote $p^O_\rho$ the distribution on the outcomes of a measurement as induced by $O$ on the state $\rho$.

$$||\rho - \sigma||_1 = \max_O \{||p^O_\rho - p^O_\sigma||\}.$$ 

So two density matrices that are close in the trace distance cannot be distinguished well by any measurement.

The next lemma is related to fact 4 and follows from fact 1.

**Lemma 1** For each Hermitian matrix $\rho$ and each trace-preserving completely positive superoperator $F$:

$$||\rho||_1 \geq ||F(\rho)||_1.$$ 

We employ the following theorem to bound the trace distance in terms of relative entropy. A classical analogue of the theorem can be found in [3] and has been used e.g. in [27].

**Theorem 1** For density matrices $\rho, \sigma$ of the same size:

$$S(\rho||\sigma) \geq \frac{1}{2\ln 2}||\rho - \sigma||^2_1.$$ 

**Proof:** Since both the norm and the relative entropy are invariant under unitary transformations we assume that the basis of the density matrices diagonalizes $\rho - \sigma$. Note that in general neither $\rho$ nor $\sigma$ are diagonal now. Let $S$ be the multiset of all nonnegative eigenvalues of $\rho - \sigma$ and $R$ the multiset of all its negative eigenvalues. All eigenvalues are real since $\rho - \sigma$ is Hermitian. Now if the dimension of the space $H_S$ spanned by the eigenvectors belonging to $S$ has dimensions $k$ and the space $H_R$ spanned by the eigenvectors belonging to $R$ has dimensions $n - k$, then increase the size of the underlying Hilbert space so that both spaces have the same dimension $n' = \max\{k, n - k\}$. The density matrices have zero entries at the corresponding positions. Now we view the density matrices as density matrices over a product space $H_2 \otimes H_{n'}$, where the $H_2$ space “indicates” the space $H_S$ or $H_R$.

We trace out the space $H_{n'}$ in $\rho, \sigma, \rho - \sigma$. The obtained $2 \times 2$ matrices are $\tilde{\rho}, \tilde{\sigma}, \tilde{\rho} - \tilde{\sigma}$. Note that the matrix $\tilde{\rho} - \tilde{\sigma}$ is diagonalized and contains the sum of all nonnegative eigenvalues, and the sum of all negative eigenvalues on its diagonal. Furthermore $\rho - \sigma = \tilde{\rho} - \tilde{\sigma}$. 

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Due to Lindblad-Uhlmann monotonicity of the relative von Neumann entropy we get $S(\rho||\sigma) \geq S(\tilde{\rho}||\tilde{\sigma})$. We will bound the latter by

$$\frac{1}{(2 \ln 2)} ||\tilde{\rho} - \tilde{\sigma}||^2 = \frac{1}{(2 \ln 2)} ||\rho - \sigma||^2,$$

and then conclude the theorem since the trace norm of $\rho - \sigma$ is the sum of absolute values of its eigenvalues which is the sum of absolute values of eigenvalues of $\rho - \sigma$ by construction, i.e., $||\rho - \sigma||_1 = ||\rho - \sigma||_1$.

So we have to prove the theorem only for $2 \times 2$ density matrices. Assume that the basis is chosen so that $\sigma$ is diagonal. Then

$$\rho = \begin{pmatrix} a & b \\ b^* & 1 - a \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} c & 0 \\ 0 & 1 - c \end{pmatrix}.$$ 

The relative von Neumann entropy $S(\rho||\sigma) = -S(\rho) - \text{trace}[\rho \log \sigma]$. The second term is $-a \log c - (1 - a) \log(1 - c)$.

The first term is minus the entropy of the distribution induced by the eigenvalues of $\rho$. So we compute the eigenvalues.

The eigenvalues of $\rho$ are the zeroes of its characteristic polynomial $t^2 - t + a(1 - a) - bb^*$. These are $1/2 \pm \sqrt{1/4 - a(1 - a) + bb^*}$. Let $x = 1/2 + \sqrt{1/4 - a(1 - a) + bb^*}$. Then $S(\rho) = H(x)$.

The squared norm of $\rho - \sigma$ is the squared sum of absolute values of the eigenvalues of $\rho - \sigma$. That matrix has the characteristic polynomial $t^2 - (-a(1 - a) + a(1 - c) + (1 - a)c - c(1 - c) + bb^*)$. Thus its eigenvalues are $\pm \sqrt{-a(1 - a) + a(1 - c) + (1 - a)c - c(1 - c) + bb^*}$. The squared norm as squared sum of absolute values of the eigenvalues is

$$4(a^2 + c^2 - 2ac + bb^*).$$

First we consider the case $a = c$. To prove this case we have to show that $H(a) - H(x) \geq 2 \log(e)bb^* = 2 \log(e)\left[(x - 1/2)^2 - 1/4 + a(1 - a)\right] = 2 \log(e)\left[(x^2 - x) - (a^2 - a)\right]$.

Considering the function $H(y)/\log(e) + 2y^2 - 2y$, we find that it is nonnegative and monotone decreasing for $y$ between $1/2$ and $1$. Thus the inequality holds, when $1/2 \leq a$ and $a \leq x$. The first condition can be assumed w.l.o.g., and the second condition follows from the fact that $x \geq 1/2$ is an eigenvalue and $a \geq 1/2$ is a diagonal element.

Now we look at the case $c \geq a$. If $c < a$, we can use the same argument for $1 - c$ and $1 - a$ instead. We want to show that

$$f(c) = S(\rho||\sigma)/\log(e) - \frac{1}{2} ||\rho - \sigma||^2_1 \geq 0.$$ 

We know this is true for $a = c$, so we show that increasing $c$ cannot decrease the difference. This holds since:

$$f'(c) = -a/c + (1 - a)/(1 - c) - 2(2c - 2a)$$

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\[
\frac{(1 - a) c - a(1 - c)}{c(1 - c)} - 4(c - a) \\
\geq 4(c - a) - 4(c - a) \geq 0.
\]

This yields the theorem for the \(2 \times 2\) case and thus in general by the previous considerations. \(\Box\)

Note that for a bipartite state \(\rho_{AB}\) the following holds:

\[
S(A : B) = S(\rho_{AB}||\rho_A \otimes \rho_B) \geq \frac{1}{2\ln 2} ||\rho_{AB} - \rho_A \otimes \rho_B||_1^2.
\]

Thus the measurable distance between the tensor product state and the “real” bipartite state can be bounded in terms of the information. We will call the value \(D(A : B) = ||\rho_{AB} - \rho_A \otimes \rho_B||_1\) the informational distance.

The next lemma collects a few properties of informational distance that follow easily from the previous discussion.

**Lemma 2** For all states \(\rho_{ABC}\) the following holds:

1. \(D(A : B) = D(B : A)\).
2. \(D(AB : C) \geq D(A : C)\).
3. \(0 \leq D(A : B) \leq 2\).
4. \(D(A : B) \geq ||F(\rho_{AB}) - F(\rho_A \otimes \rho_B)||_1\) for all completely positive and trace-preserving superoperators \(F\).
5. \(D(A : B) \leq \sqrt{2S(A : B)}\).

Note that lemma 2.5 implies one of the main ingredients of the round hierarchy discovered in [22] (the “average encoding theorem”).

Consider some density matrix \(\rho_{AB}\) that is block diagonal (with classical \(\rho_A\)) in the basis composed as the tensor product of the standard basis for \(A\) and some other basis for \(B\). Then denote \(\rho_B^{(a)}\) the density matrix obtained by fixing \(A\) to some classical value \(a\) and normalizing. \(Pr(a)\) is the probability of \(a\).

The next properties of informational distance will be used later.

**Lemma 3**

1. Let \(\rho_{AB}\) be the density matrix of a state, where \(\rho_B\) corresponds to the density function of a classical random variable \(B\) on \(|0\rangle\) and \(|1\rangle\) with \(Pr(B = 1) = Pr(B = 0) = 1/2\). Let there be a measurement acting on the \(A\) system only and yielding a Boolean random variable \(X\) with \(Pr(X = B) \geq 1 - \epsilon\) and \(Pr(X \neq B) \leq \epsilon\) (while the same measurement applied to \(\rho_A \otimes \rho_B\) yields a distribution with \(Pr(X = B) = Pr(X \neq B) = 1/2\)). Then \(D(A : B) \geq 1 - 2\epsilon\).
2. For all block diagonal $\rho_{AB}$, where $\rho_A$ corresponds to a classical distribution $p_A$ on the standard basis vectors for $A$, the following holds:

$$D(A : B) = E_a||\rho_A^{(a)} - \rho_B||_1.$$  

**Proof:** For the first item observe that $D(A : B) \geq D(X : B) \geq 1 - 2\epsilon$. The second item is a consequence of $D(A : B) = ||\rho_{AB} - p_A \otimes \rho_B||_1$. 

5 Rounds in Quantum Communication

It is well known that for deterministic, probabilistic, (and even limited non-deterministic) communication complexity there are functions which can be computed much more efficiently in $k$ rounds than in $k-1$ rounds (see [10], [12], [24], [25], [14]). In most of these results the pointer jumping function is considered.

**Definition 11** Let $V_A$ and $V_B$ be disjoint sets of $n$ vertices each.

Let $F_A = \{f_A | f_A : V_A \rightarrow V_B\}$, and $F_B = \{f_B | f_B : V_B \rightarrow V_A\}$.

$f(v) = f_{f_A,f_B}(v) = \begin{cases} f_A(v) & \text{if } v \in V_A, \\ f_B(v) & \text{if } v \in V_B. \end{cases}$

Define $f^{(0)}(v) = v$ and $f^{(k)}(v) = f(f^{(k-1)}(v))$.

Then $g_k : F_A \times F_B \rightarrow (V_A \cup V_B)$ is defined by $g_k(f_A, f_B) = f^{(k+1)}_{f_A,f_B}(v_1)$, where $v_1 \in V_A$ is fixed. The function $f_k : F_A \times F_B \rightarrow \{0, 1\}$ is the XOR of all bits in the binary code of the output of $g_k$.

Nisan and Wigderson proved in [24] that $f_k$ has a randomized $k$ round communication complexity of $\Omega(n/k^2 - k \log n)$ if B starts communicating and a deterministic $k$ round communication complexity of $k \log n$ if Alice starts. The lower bound can also be improved to $\Omega(n/k + k)$, see [13]. Nisan and Wigderson also describe a randomized protocol computing $g_k$ with communication $O((n/k) \log n + k \log n)$ in the situation, where Bob starts and $k$ rounds are allowed. Ponzio et. al. show that the deterministic communication complexity of $f_k$ is $O(n)$ then, if $k = O(1)$ [25].

With techniques similar to the ones in this paper we can also show a lower bound of $\frac{(1-2\epsilon)^2 n}{k^2} - k \log n$ for the randomized $k$ round complexity of $f_k$ when B starts, which is better than the above lower bounds for small constant values of $k$.

Interaction in quantum communication complexity has also been investigated by Nayak, Ta-Shma, and Zuckerman [22]. For the pointer jumping function their results imply the following:

**Fact 5** $Q^{B,k}(f_k) = \Omega(n^{1/k}/k^4)$.

First we give a new upper bound. The next result combines ideas from [24] and [25].
Theorem 2 $R^k_{e,B}(g_k) \leq O\left(\frac{n}{k^2} \cdot (\log^{(k/2)} n + \log k) + k \log n \right)$.

Proof: First Bob guesses with public random bits $(4/\epsilon) \cdot (n/k)$ vertices. For each chosen vertex $v$ Bob communicates the first $\log^{(k/2)} n + 3 \log k$ bits of $f_B(v)$.

In round $t$ the active player communicates the pointer value $v_t = f^{(t-1)}(v_1)$. If it’s Alice’s turn, then she checks, whether $v_t$ is in Bob’s list of the first round. Then Alice knows $\log^{(k/2)} n + 3 \log k$ bits of $f_B(v_t)$. Note that this happens with probability $1 - \epsilon$ during the first $k/2$ rounds. In the following assume that this happened in round $i = k/2$, otherwise the protocol errs.

Beginning from the round $i$, when Alice gets to know the $\log^{(k/2)} n + 3 \log k$ bits of $f(v_i)$ the players communicate in round $i + t$ for all possible values of $f(v_{i+t})$ the most significant $\log^{(k/2-t)} n + 3 \log k$ bits. Since there are at most $n/\log^{(k/2-t)} n \cdot k^3$ such values $O(n/k^2)$ bits communication suffices.

In the last round $v_{k+2}$ is found. Overall the communication is at most

$$k \log n + O\left((1/\epsilon) \cdot (n/k)(\log^{(k/2)} n + 3 \log k)\right) + k \cdot O(n/k^2) .$$

\[ \square \]

Corollary 1 If $k \geq 2 \log^*(n)$ then $R^k_{e,B}(g_k) \leq O\left((\frac{n}{k^2} + k \log k)\right)$.

We can replace $k \log n$ by $k \log k$ in the above expression, because that term dominates only if $\log k = \Theta(\log n)$.

Next we are going to prove a lower bound on the quantum communication complexity of the pointer jumping function $f_k$, for the situation that $k$ rounds are allowed and Bob sends the first message. We will consider a quantity $d_t$ capturing the information the active player has in round $t$ on vertex $t + 1$ of the path. This quantity will be the informational distance between the active player’s qubits and vertex $t + 1$. Our goal will be to bound $d_t$ in terms of $d_{t-1}$ plus a term related to the average information on pointers in the other player’s input. This leads to a recursion imposing a lower bound on the communication complexity, since in the end the protocol must have reasonably large information to produce the output, and in the beginning the respective information is 0.

The informational distance $d_t$ measures the distance between the state of, say, Alice’s qubits together with the vertex $t + 1$ of the path, and the tensor product of the states of Alice’s qubits and vertex $t + 1$. In the product state Alice has no information on the vertex, so if the two states are close, Alice’s powers to say something about the vertex are very limited. We will use the triangle inequality to bound $d_t$ by the sum of three intermediate distances. In the first step we move from the state given by the protocol to a state in which the $t$-th vertex is replaced by a uniformly random vertex, independent of previous communications. The distance to such a state can be bounded in terms of $d_{t-1}$, because that quantity puts a bound on Bob’s ability to detect such a replacement. We use the local transition theorem
from 22 to conceal Alice’s ability to detect such a replacement. Once the $t$-th vertex is random, we can move in the next step to a state in which also vertex $t+1$ is random. The cost of this step corresponds to the average information a player has on a random pointer in the other player’s input. The last step is similar to the first and reverses the first one’s effect, i.e., replaces the “randomized” $t$-th vertex by its real value again. We arrive at the desired product state.

**Theorem 3** $Q^{k,B, pub}(f_k) \geq n/2^{O(k)} - k \log n$.

**Proof:** Fix a quantum protocol, of the following form. The protocol computes $f_k$ with error $1/3$, $k$ rounds, Bob starting. At any time in the protocol Alice has access to qubits containing her input, some “work” qubits and some of the qubits used in messages so far, the same holds for Bob. We assume that the players never change their inputs. Usually a protocol gets some classical $f_A$ and $f_B$ as inputs, but we will investigate what happens if the protocol is started on a superposition over all inputs, in which all inputs have the same amplitude, i.e., on

$$\sum_{f_A \in \mathcal{F}_A, f_B \in \mathcal{F}_B} \frac{1}{n^n} |f_A\rangle |f_B\rangle.$$ 

Note that $|\mathcal{F}_A| = |\mathcal{F}_B| = n^n$. The superposition over all inputs is measured after the protocol has finished, so that a uniformly random input and the result of the protocol on that input are produced.

The density matrix of the global state of the protocol is $\rho_{M_A,t M_B,t F_A F_B}$. Here $F_A, F_B$ are the qubits holding the inputs of Alice and Bob and $M_A,t$ resp. $M_B,t$ are the other qubits in the possession of Alice and Bob before the communication of round $t$. The state of the latter two systems of qubits may be entangled. In the beginning these qubits are independent of the input.

We also require that in round $t$ the vertex $v_t = f(t-1)(v_1)$ is communicated by a classical message and stored by the receiving player. This increases the communication by an additive $k \log n$ term. We demand that before round $t$ the $t$’th vertex of the path is measured (remember that we are in a super-position over $f_A, f_B$). This vertex is stored in some qubits $V_t$. $V_t$ has the fixed value $v_1$. In general, before the beginning of round $t$, we have a mixture and in each pure state in the mixture the first $t-1$ vertices are fixed and $V_t$ is either $F_A(v_{t-1})$ or $F_B(v_{t-1})$. We then measure $V_t$ in the standard basis. The measurements do not affect the correctness of the protocol.

Let us denote $d_t = D(M_B,t F_B : F_A(V_t))$ when $t$ is odd, and $d_t = D(M_A,t F_A : F_B(V_t))$ when $t$ is even. In this definition, we assume that the register $F_A(V_t)$ (or $F_B(V_t)$) has been measured, although this measurement is not part of the protocol. Note that for $t > 1$, $V_t$ is uniformly random, so (for odd $t$) the distance $d_t$ is taken as the average over $v$, of the informational distance
between the state of $M_B,F_B$ restricted to $V_t$ being equal to $v$, and $F_A(v)$, and similarly for even $t$.

We assume that the communication complexity of the protocol is $\delta n$ and prove a lower bound $\delta \geq 2^{-O(k)}$. The general strategy of the proof is induction over the rounds, to successively bound $d_t,d_{t+1},\ldots,d_{k+1}$.

Bob sends the first message. As Bob has seen no message yet, we have that $I(M_B,F_B:F_A(V_1)) = 0$, and hence $d_1 = 0$. We show that

**Lemma 4** $d_{t+1} \leq 4\sqrt{d_t} + \sqrt{4\delta}$.

We see that $d_t+1 \leq 3^t\delta^{1/2}$ for all $t \geq 0$. After round $k$ one player, say Alice, announces the result which is supposed to be the parity of $F_B(V_k+1)$ and included in $M_{A,t+1}$. On the one hand $d_{k+1} = D(M_{A,k+1}:F_B(V_k+1)) \leq 3^k\delta^{1/2}$. On the other hand, by Lemma 3.1 $D(M_{A,k+1}:\oplus F_B(V_k+1)) \geq 1 - \frac{2}{3} = \frac{1}{3}$. Together, $\frac{1}{3} \leq 3^k\delta^{1/2}$, so $\delta \geq 2^{-O(k)}$.

We now turn to proving Lemma 4.

W.l.o.g. let Alice be active in round $t + 1$. Let $M_A = M_{A,t+1}$ and $M_B = M_{B,t+1}$. Before the $t + 1$ round $V_{t+1} = F_A(V_t)$ is measured. The resulting state is a probabilistic ensemble over the possibilities to fix $V_1,\ldots,V_{t+1}$, which are then classically distributed. Alice’s reduced state is block diagonal with respect to the possible values of the vertices $V_1,\ldots,V_{t+1}$. For any value $v$ of $V_{t+1}$ let $\rho_{M_A,M_B,F_B}^{V_{t+1}=v} = \rho_{M_A,M_B,F_B}^{V_{t+1}=v}$ denote the pure state with vertex $V_{t+1}$ fixed to $v$. We are interested in the value

$$d_{t+1} = D(M_A F_A : F_B(V_{t+1})) = \mathbb{E}_v \left\| \rho_{M_AF_AF_B}^{v} - \rho_{M_AF_A} \otimes \rho_{F_B(v)} \right\|_1,$$

where the distribution on vertices $v$ (induced by the state of the system $F_A,F_B$) is uniform. Recall that in the definition of $d_{t+1}$, $F_B(v)$ is assumed to be uniformly random (i.e., measured). The above quantity measures how much Alice knows about the value $F_B$ gives to the current vertex $V_{t+1}$.

We define

$$\gamma_v \overset{\text{def}}{=} \left\| \rho_{M_B F_B}^{v} - \rho_{M_B F_B} \right\|_1. \tag{1}$$

I.e., $\gamma_v$ is the distance between the state of Bob ($\rho_{M_B F_B}^{v}$) before he receives the message in round $t + 1$, and the state $\rho_{M_B F_B}$, which is his state averaged over $v = F_A(V_t)$. We show below that these two are almost always close to each other (this reflects the fact that Bob does not know much about $F_A$).

For the purposes of the proof, we also consider a run of the protocol on the uniform superposition over inputs, where the qubits $V_1,V_2,\ldots$ are not measured during the course of the protocol. Let $\tilde{\rho}_{M_A MBF_AF_B}$ be the state before the communication in round $t + 1$ in this run of the protocol. For any $v \in U_B$, we define:

$$\beta_v \overset{\text{def}}{=} \left\| \tilde{\rho}_{M_AF_AF_B}^{v} - \tilde{\rho}_{M_AF_A} \otimes \rho_{F_B(v)} \right\|_1. \tag{2}$$
where $F_B(v)$ is assumed to have been measured. Note that $\rho_{F_B(v)} = \tilde{\rho}_{F_B(v)}$ with this measurement; both are randomly distributed over $V_A$.

Let $\rho_{M_A M_B F_A F_B R}$ (respectively, $\rho^u_{M_A M_B F_A F_B}$) be a purification of $\rho_{M_A M_B F_A F_B}$ (resp. $\rho^u_{M_A M_B F_A F_B}$), where $R$ is some additional space used to purify the random path $V_1, \ldots, V_{t+1}$ (resp. $V_1, \ldots, V_t$).

We employ the following fact from [22] (the “local transition theorem”). The fact is a variation of the impossibility result for unconditionally secure quantum bit commitment due to Mayers [21] and Lo and Chau [20].

**Fact 6** Let $\rho_1, \rho_2$ be two density matrices with support in a Hilbert space $H$, $K$ a Hilbert space of dimension at least $\dim H$, and $|\phi_1\rangle, |\phi_2\rangle$, any purifications of $\rho_1$ resp. $\rho_2$ in $H \otimes K$. Then there is a purification $|\phi_2'\rangle$ of $\rho_2$ in $H \otimes K$, that is obtained by applying a unitary transformation $I \otimes U$ to $|\phi_2\rangle$, where $U$ is acting on $K$ and $I$ is the identity operator on $H$.

$|\phi_2'\rangle$ has the property

$$|| |\phi_1\rangle \langle \phi_1 | - |\phi_2'\rangle \langle \phi_2' ||_1 \leq 2 \sqrt{|| |\phi_1 - \phi_2' ||_1}.$$  

Now, due the above fact there is a local unitary transformation $U_v$ acting only on $F_A M_A R$ such that

$$\sigma^v_{M_A M_B F_A F_B R} \overset{\text{def}}{=} U_v \rho_{M_A M_B F_A F_B R} U_v^\dagger,$$

and $\rho^v_{M_A M_B F_A F_B R}$ are close to each other. Moreover,

**Lemma 5** For all vertices $v \in V_B$,

$$\left\| \rho^v_{M_A F_A} - \sigma^v_{M_A F_A} \right\|_1 \leq \left\| \rho^v_{M_A F_A F_B(v)} - \sigma^v_{M_A F_A F_B(v)} \right\|_1 \leq 2 \sqrt{\gamma_v}, \quad (3)$$

$$\left\| \sigma^v_{M_A F_A F_B(v)} - \sigma^v_{M_A F_A} \otimes \rho_{F_B(v)} \right\|_1 \leq \beta_v. \quad (4)$$

We will also prove:

**Lemma 6** For the uniform distribution on vertices $v$ (induced by the state of the system $F_A, F_B$),

$$E_v \gamma_v \leq d_t, \text{ and}$$

$$E_v \beta_v \leq \sqrt{4} \delta. \quad (6)$$

Thus, for all $v$:

$$\left\| \rho^v_{M_A F_A F_B(v)} - \rho^v_{M_A F_A} \otimes \rho_{F_B(v)} \right\|_1
\[
\leq \left\| \rho^v_{MAF_AF_B} - \sigma^v_{MAF_AF_B} \right\|_1 + \left\| \sigma^v_{MAF_AF_B} - \sigma^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1 + \left\| \sigma^v_{MAF_A} \otimes \rho_{FB}(v) - \rho^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1
\leq 4\sqrt{\gamma_v} + \left\| \sigma^v_{MAF_AF_B} - \sigma^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1 \quad \text{From equation (3)}
\leq 4\sqrt{\gamma_v} + \beta_v \quad \text{From equation (4)}.
\]

Finally,
\[
D(M_{AF_A} : F_B(V_{t+1})) = \mathbb{E}_v \left( \left\| \rho^v_{MAF_AF_B(v)} - \rho^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1 \right)
\leq \mathbb{E}_v [4\sqrt{\gamma_v} + \beta_v]
\leq 4\sqrt{E_{\gamma_v}} + \mathbb{E}_v \beta_v \quad \text{By Jensen’s inequality}
\leq 4\sqrt{\gamma_d} + \sqrt{4\delta} \quad \text{By Lemma \ref{lemma}}.
\]

This completes the proof of Lemma \ref{lemma}.

We finish the proof of Theorem 3 by proving Lemmas \ref{lemma} and \ref{lemma}.

**Proof of Lemma \ref{lemma}**

For equation (3), notice that
\[
\left\| \rho^v_{MAF_A} - \sigma^v_{MAF_A} \right\|_1 \leq \left\| \rho^v_{MAF_AF_B} - \sigma^v_{MAF_AF_B} \right\|_1 \leq \left\| \rho^v_{MAF_AF_B} - \sigma^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1,
\]
and by fact 6 this is at most \(2\sqrt{\gamma_v}\).

For equation (4),
\[
\left\| \sigma^v_{MAF_AF_B} - \sigma^v_{MAF_A} \otimes \rho_{FB}(v) \right\|_1 \\
\leq \left\| \sigma^v_{MAF_ARF_B} - \sigma^v_{MAF_AR} \otimes \rho_{FB}(v) \right\|_1 \\
= \left\| \rho_{MAF_ARF_B} - \rho_{MAF_AR} \otimes \rho_{FB}(v) \right\|_1 \quad \text{By unitarity}
= \left\| \hat{\rho}_{MAF_AR} - \hat{\rho}_{MAF_A} \otimes \rho_{FB}(v) \right\|_1 \quad \text{(*)}
= \beta_v \quad \text{By definition (3)}.
\]

For (*) notice that \(R\) holds the path \(V_1, \ldots, V_{t+1}\), which is determined by \(M_{AF_A}\). We can apply a unitary transformation that “erases” this, to give us the state \(\hat{\rho}_{MAF_A}\). The lemma is proved.

**Proof of Lemma \ref{lemma}**
For equation (5), we have
\[ E_v \gamma_v = E_v \left( \rho_{M_B F_B F_A(u)}^{V_i = u} - \rho_{M_B F_B(u)} \otimes \rho_{F_A(u)} \right) \]
\[ = D(M_B F_B : F_A(V_i)) \]
\[ \leq D(M_{B,t} F_B : F_A(V_i)) = d_t. \]

The last step follows from the fact that Bob sends the \( t \)'th message, and this only decreases the informational distance: \( D(M_{B,t+1} F_B : F_A(V_i)) \leq D(M_{B,t} F_B : F_A(V_i)) \).

To derive equation (6), we first bound the information Alice has on Bob's input.

**Lemma 7** For all \( t \), \( I(M_A F_A : F_B) \leq 2\delta n \), irrespective of whether some registers have been measured or not.

In the beginning Alice has no information about \( F_B \), i.e., \( I(M_A F_A : F_B) = 0 \). Recall that at most \( \delta n \) qubits are communicated in the protocol. Any qubit sent from Alice to Bob does not increase her information on Bob’s input. Any local unitary transformation does not increase her information.

Now assume Bob sends a qubit \( Q \). Then \( I(M_A Q F_A : F_B) = I(M_A F_A : F_B) + I(Q : F_B | M_A F_A) \leq I(M_A F_A : F_B) + 2 \) due to the Araki-Lieb inequality (fact 2). So each qubit sent from Bob to Alice increases her information on his input by at most 2. We thus get \( I(M_A F_A : F_B) \leq 2\delta n \) at all times \( t \).

Since measurements only decrease mutual information, the bound also holds when certain registers are measured and others are not, during the course of the protocol. The lemma is proved.

Now consider the situation that \( F_B \) is distributed uniformly instead of being in the uniform superposition (in other words, when \( F_B \) has been measured). Then \( E_v I(M_A F_A : F_B(v)) \leq 2\delta \) (where \( v \) is uniformly random), using that the \( F_B(v) \) are mutually independent. Now,
\[ E_v \beta_v = E_v \left( \hat{\rho}_{M_A F_A F_B(v)} - \hat{\rho}_{M_A F_A} \otimes \rho_{F_B(v)} \right) \]
\[ = E_v D(M_A F_A : F_B(v)), \]
where \( F_A M_A M_B F_B \) are as in the protocol without measurements. Further,
\[ E_v D(M_A F_A : F_B(v)) \leq E_v \sqrt{2 I(M_A F_A : F_B(v))} \]
\[ \leq \sqrt{2 E_v I(M_A F_A : F_B(v))} \]
\[ \leq \sqrt{4\delta}, \]
by Lemma 2 and Jensen’s inequality. \( \square \)
6 The Disjointness Problem

We now investigate the bounded round complexity of the disjointness problem. Here Alice and Bob each receive the incidence vector of a subset of a size $n$ universe. They reject iff the sets are disjoint. It is known that $Q^1(DISJ_n) \geq (1 - H(\epsilon))n$ [13, 14]. Furthermore $Q(DISJ_n) = O(\sqrt{n} \log n)$ by an application of Grover search [4]. In this protocol $\Theta(\sqrt{n})$ rounds are used. By a simple reduction (see [15]) we get the following result.

**Theorem 4** $Q^{k,\text{pub}}(DISJ_n) = \Omega(n^{1/k})$ for $k = O(1)$.

**Proof:** Suppose we are given a $k$ round quantum protocol for the disjointness problem having error $1/3$ and using communication $c$. W.l.o.g. we can assume Bob starts the communication, because the problem is symmetrical. We reduce the pointer jumping function $f_k$ to disjointness.

In a bipartite graph with $2n$ vertices and outdegree 1 there are at most $n^k$ possible paths of length $k$ starting at vertex $v_1$. For each such path we use an element in our universe for the disjointness problem. Given the left resp. right side of a specific graph Alice and Bob construct an instance of $DISJ_n$. Alice checks for each possible path of length $k$ from $v_1$ whether the path is consistent with her input and whether the paths leads to a vertex $v_{k+2}$ with odd number (if the $k+1$st vertex is on the left side). In this case she takes the corresponding element of the universe into her subset. Bob does the analogous with his input. Now, if the two subsets intersect, then the element in the intersection witnesses a length $k+1$ path leading to a vertex with odd number. If the subsets do not intersect, then the length $k+1$ path from $v_1$ leads to a vertex with even number.

So we obtain a $k$ round protocol for $f_k$ in which Bob starts. The communication is $c = \Omega(n)$ for any constant $k$, the input length for the constructed instance of disjointness is $N = n^k$ and we get $Q^{k,\text{pub}}(DISJ_N) = \Omega(N^{1/k})$ for $k = O(1)$. \hfill \Box

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