NUMBER PARTITIONING WITH SPLITTING∗

SAMUEL BISMUTH †, VLADISLAV MAKAROV ‡, EREL SEGAL-HALEVI §, AND DANA SHAPIRA ¶

Abstract.
We consider a variant of the \( n \)-way number partitioning problem, in which some fixed number \( s \) of items can be split between two or more bins. We show a two-way polynomial-time reduction between this variant and a second variant, in which the maximum bin sum must be within a pre-specified interval. We prove that, for any fixed integer \( n \geq 3 \), the second variant can be solved in polynomial time if the length of the allowed interval is at least \( (n - 2)/n \) times the maximum item size, and it is \( \text{NP} \)-complete otherwise. Using the equivalence between the variants, we prove that, for any fixed integer \( n \geq 3 \), number-partitioning with \( s \) split items can be solved in polynomial time if \( s \geq n - 2 \), and it is \( \text{NP} \)-complete otherwise.

Key words. FPTAS, algorithm, complexity, combinatorial problems, partitioning problems

1. Introduction. In the classic \( n \)-way number partitioning problem (where \( n \geq 2 \) is a fixed integer), the input is a list \( X \) of \( m \in \mathbb{N} \) integers, \( X = (x_1, \ldots, x_m) \), \( x_i \in \mathbb{N} \), and the objective is to find an \( n \)-way partition (a partition into \( n \) bins) of \( X \) such that the maximum bin sum is minimized:

\[
\text{\( n \)-Way–\( \text{MinMax} \)\( (X) \): Minimize } \max(b_1, \ldots, b_n), \text{ where } b_1, \ldots, b_n \text{ are sums of bins in an } n\text{-way partition of } X.
\]

The problem is known to be \( \text{NP} \)-hard for every fixed \( n \geq 2 \) [8]. Like in many other combinatorial optimization problems, one could consider a continuous (fractional) variant of \( n \)-Way–\( \text{MinMax} \), in which each item can be split between two or more bins. This variant is very easy to solve: by splitting each item equally into \( n \) parts and putting a part in each bin, it is possible to attain a partition with \( b_1 = \cdots = b_n = \frac{1}{n} \sum_i x_i \), which is the best possible value.

While theoretical problems are usually classified as either “discrete” or “continuous”, practical partitioning problems may lie in between these two extremes: every item can be split if needed, but the splitting may be expensive or inconvenient, and therefore the number of split items should be bounded. As an example, consider two heirs who inherited three houses and have to divide them fairly. The house values are 100, 200, 400. If all houses are considered discrete, then an equal division is not possible. If all houses can be split, then an equal division is easy to attain by giving each heir 50% of every house, but it is inconvenient since it requires all houses to be jointly managed. A solution often used in practice is to decide in advance that a single house can be split. In this case, after receiving the input, we can determine that splitting the house with value 400 lets us attain a division in which each heir receives the same value of 350.\(^1\)

Motivated by this and other similar examples, we define a variant of the \( n \)-Way–\( \text{MinMax} \) problem, which accepts, in addition to the list \( X \) of integers to partition, a parameter \( s \in \mathbb{N} \), \( 0 \leq s \leq m \), which specifies that at most \( s \) numbers from \( X \) are allowed to be split between bins.

∗The paper started from discussions in the stack exchange network \[1 \text{ 2]. The case of } 3 \text{–Split–Perfect}(X, 1) \text{ was first solved by Mikhail Rudoy using case analysis. The relation to FPTAS was raised by Chao Xu. We are also grateful to John L. 3.}

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†Department of Computer Science, Ariel University, Ariel 40700, Israel (samuelbismuth101@gmail.com).
‡Department of Mathematics and Computer Science, St. Petersburg State University, St. Petersburg 199178, Russia (vm450@yandex.ru).
§Department of Computer Science, Ariel University, Ariel 40700, Israel (erelsgl@gmail.com)
¶Department of Computer Science, Ariel University, Ariel 40700, Israel (shapird@g.ariel.ac.il).

1 Split the house worth 400 such that one heir gets 7/8 of it, and the other gets 1/8 of it plus both the 100 and 200 houses.
n–Split–MinMax$(X, s)$: Minimize $\max(b_1, \ldots, b_n)$, where $b_1, \ldots, b_n$ are sums of bins in an $n$-way partition of $X$ in which at most $s$ items are split.

When $s = 0$, the problem is equivalent to $n$–Way–MinMax, which is NP-hard. In contrast, when $s \geq n - 1$ the problem is easily solvable by the following algorithm: put the items on a line, cut the line into $n$ pieces with an equal total value, and put each piece in a bin. Since $n - 1$ cuts are made, at most $n - 1$ items need to be split. So for $n = 2$, the runtime complexity of the $n$–Split–MinMax problem is well-understood (assuming $P \neq NP$): it is polynomial-time solvable if and only if $s \geq 1$. The main goal of this paper is to analyze the runtime complexity of the problem for any integers $n \geq 3$ and $1 \leq s \leq n - 2$. Our main result is:

**Theorem 1.1.** (a) For any fixed integer $n \geq 3$, $n$–Split–MinMax can be solved in polynomial time for any integer $s \geq n - 2$, and it is NP-complete for any integer $s < n - 2$.

(b) NP-completeness holds even for the decision version of $n$–Split–MinMax, which we call $n$–Split–Perfect, which only asks if there exists a perfect partition (a partition with equal bin sums).

We emphasize that the problem definitions do not determine in advance which items will be split, but only limits the number of items that can be split. This is motivated by practical settings in which splitting items is possible but undesirable. The solver may decide which items to split after receiving the input.

In order to solve $n$–Split–MinMax, we consider another variant of $n$-way number partitioning, a decision problem which asks whether the maximum bin sum is within a pre-specified interval around the average bin sum, $S := \left(\sum_i x_i\right)/n$. The runtime complexity of this problem depends on the size of the allowed interval. In general, the problem is NP-complete when the interval is “small” and in $P$ when the interval is “large”. Specifically, when $n = 2$, the runtime complexity depends on the ratio of the allowed interval to the average bin sum, while when $n \geq 3$ it depends on the ratio of the allowed interval to the largest item size. Therefore, we present two variants of the problem. The first is parametrized by a rational number $t \geq 0$:

\[ n\text{-Interval–t}(X, t): \text{ Decide if there exists a partition of } X \text{ into } n \text{ bins with sums } b_1, \ldots, b_n \text{ such that } S \leq \max(b_1, \ldots, b_n) \leq (1 + t) \cdot S, \text{ where } S \text{ is defined as the average bin size, } S := \left(\sum_i x_i\right)/n. \]

The second is parametrized by a rational number $d \geq 0$:

\[ n\text{-Interval–d}(X, d): \text{ Decide if there exists a partition of } X \text{ into } n \text{ bins with sums } b_1, \ldots, b_n \text{ such that } S \leq \max(b_1, \ldots, b_n) \leq S + d \cdot M, \text{ where } S := \left(\sum_i x_i\right)/n \text{ and } M := \left(\max_i x_i\right)/n. \]

Our secondary result is:

**Theorem 1.2.** (a) For $n = 2$ and any rational number $t > 0$, the $n$–Interval–t$(X, t)$ problem can be solved in time $O(\text{poly}(m, 1/t))$, where $m$ is the number of input items in $X$.

(b) For any fixed $n \geq 3$, the $n$–Interval–d$(X, d)$ problem can be solved in time polynomial in $m$ for every rational number $d \geq n - 2$, and it is NP-complete for every rational number $d < n - 2$.

Note that $S \leq \max(b_1, \ldots, b_n)$ always holds by the pigeonhole principle; writing this trivial lower bound explicitly in the problem statement highlights the similarity between the $n$–Interval–t problem and a Fully Polynomial-Time Approximation Scheme (FPTAS) of the $n$–Way–MinMax problem, the latter formulated as follows. For every fixed integer $n \geq 2$, the $n$–Way–MinMax problem has an FPTAS [18] that finds, for each real $\epsilon > 0$, a partition of $X$ such that $OPT \leq \max(b_1, \ldots, b_n) \leq (1 + \epsilon) \cdot OPT$, in time $O(\text{poly}(m, 1/\epsilon))$, where $OPT$ is the smallest possible value of $\max(b_1, \ldots, b_n)$.

Despite the similarity, the problems are not identical and an FPTAS for $n$–Way–MinMax does not automatically solve $n$–Interval–t. In Subsection 3.1 we show how an FPTAS for a slightly
different problem can be used to solve 2-Interval-t and prove Theorem 1.2(a). In the rest of Section 3 we analyze the n-Interval-d problem and prove Theorem 1.2(b).

In Section 4, we prove a two-way polynomial-time reduction between the partitioning with interval target and partitioning with split items problems. Using this reduction and Theorem 1.2(b), we conclude the proof of Theorem 1.1.

There is another way to control the amount of splitting. Instead of considering the number of split items, one can measure the number of times each item is split. The number of splittings is at least the number of split items, but it might be larger. For example, a single item split into 10 different bins counts as 9 splittings. We name this variant n-Split-Perfect(\(\mathcal{X}, s\)). It is similar to the n-Split-Perfect(\(\mathcal{X}, s\)) problem with the exception that \(s\) denotes the number of splittings. Here, too, when \(s \geq n - 1\) a solution can be easily found. We prove that this variant is NP-complete for all \(s < n - 1\), even when deciding whether there exists a perfect partition. Our last result is:

**Theorem 1.3.** For any fixed integer \(n \geq 2\) of bins and a fixed \(s \in \mathbb{N}\) such that \(s \leq n - 2\), the problem n-Times-Split-Perfect(\(\mathcal{X}, s\)) is NP-complete.

The full proof details are in the Appendix C. Note that a weaker result was presented in [15], that n-Times-Split-Perfect(\(\mathcal{X}, s\)) is NP-complete for a given list \(\mathcal{X}\) of \(m\) items and all \(s < n - 2\).

2. Related work. The idea of finding fair allocations with a bounded number of split items originated from Brams and Taylor [3,4]. They presented the Adjusted Winner (AW) procedure for allocating items among two agents with possibly different valuations. AW finds an allocation that is envy-free (no agent prefers the bundle of another agent), equitable (both agents receive the same subjective value), and Pareto-optimal (there is no other allocation where some agent gains and no agent loses), and in addition, at most a single item is split between the agents. Hence, AW solves a problem that is similar to 2-Split-Perfect(\(\mathcal{X}, 1\)) but more general, since AW allows the agents to have different valuations to the same items. AW was applied (at least theoretically) to division problems in divorce cases and international disputes [5,11] and was studied empirically in [7,14]. The AW procedure is designed for two agents. For \(n \geq 3\) agents, the number of splitting was studied in an unpublished manuscript of Wilson [17] using linear programming techniques. He proved the existence of an egalitarian allocation of goods (i.e., an allocation in which all agents have a largest possible equal utility [12]), with at most \(n - 1\) split items; this can be seen as a generalization of n-Split-Perfect(\(\mathcal{X}, n - 1\)).

Goldberg et al. [9] studied the problem of consensus partitioning. In this problem, there are \(n\) agents with different valuations, and the goal is to partition a list of items into some \(k\) subsets (where \(k\) and \(n\) may be different), such that each agent values each subset at exactly \(1/k\). They prove that a consensus partitioning with at most \(n(k - 1)\) split items can be found in polytime.

Most similar to our paper is the recent work of Sandormirskiy and Segal-Halevi [13]. Their goal is to find an allocation among \(n\) agents with different valuations, which is both fair and fractionally Pareto-optimal (fractional-PO or fPO, for short), a property stronger than Pareto-optimality. This is a very strong requirement: when \(n\) is fixed, and the valuations are generic (i.e., for every two agents, no two items have the same value-ratio), the number of fPO allocations is polynomial in \(m\), and it is possible to enumerate all such allocations in polynomial time. Based on this observation, they present an algorithm that finds an allocation with the smallest number of split items, among all allocations that are fair and fPO. In contrast, in our paper, we do not require fPO. Dropping the fPO requirement may allow allocations with fewer split items, but the number of potential allocations becomes exponential, so enumerating them all is no longer feasible.

Recently, Bei et al. [1] studied an allocation problem where some items are divisible and some are indivisible. In contrast to our setting, in their setting the distinction between divisible and indivisible items is given in advance, that is, the algorithm can only divide items that are predetermined as divisible. In our setting, only the number of divisible items is given in advance, but
the algorithm is free to choose which items to split after receiving the input.

Fractional relaxations of integer linear programs are very common in approximation algorithm design. However, most works do not explicitly bound the number of fractional variables. One similar notion is related to preemption in job-scheduling. In the world of machine scheduling, the problem $n$-Way-MinMax is known as “makespan-minimization on identical machines”. Preemption means that a job can be split into parts, each of which can be scheduled independently, either on the same or on a different machine, with the additional constraint that no two parts of a job are scheduled on different machines at the same time. We found two works in which the amount of preemption is bounded: Soper and Strusevich [16] allow at most one preemption, while Liu and Cheng [10] consider a variant in which there is a penalty for each preemption.

Bourjolly and Pulleyblank [2] study similar notions in the context of vertex cover in graphs. They present an algorithm for finding a minimum fractional vertex cover, in which the number of fractional vertices is as small as possible.

It is interesting to see that splitting occurs in other settings, and in particular, different computer science fields can integrate the concept of bounded splitting into their problems.

The main results of our paper are based on FPTAS-s for several number-partitioning problems. Woeginger [18] gives a general method for converting a dynamic program to an FPTAS. He shows that his method can be used to design FPTAS-s for hundreds of different combinatorial optimization problems. We apply his method for several different problems. Woeginger does not analyze the exact run-time of the FPTAS-s resulting from his method. We remark that the FPTAS-s designed by Woeginger’s method for partitioning problems are polynomial in $m$, but exponential in $n$; they are polynomial only when $n$ is considered a fixed parameter. The same is true for our algorithms.

3. Partition with Interval Target. In this section we analyze the two related problems $n$-Interval-t and $n$-Interval-d. Recall that, by the definition of these problems, they accept as input a list $X$ of positive integers with sum $n \cdot S$ and maximum element $n \cdot M$, where $S$ and $M$ are rational numbers.

3.1. The $n$-Interval-t problem. Given an instance of $n$-Interval-t($X$, $t$), we say that a partition of $X$ is $t$-feasible if $S \leq \max(b_1, \ldots, b_n) \leq (1 + t) \cdot S$, where $b_1, \ldots, b_n$ are the bin sums. The $n$-Interval-t($X$, $t$) problem is to decide whether a $t$-feasible allocation exists.

As a first (incomplete) attempt to solve $n$-Interval-t, let us apply a known FPTAS for the $n$-Way-MinMax problem [18]. Denote the largest bin sum in the solution obtained by the FPTAS for a given list $X$ and a given $\epsilon > 0$ by FPTAS($n$-Way-MinMax($X$), $\epsilon$).

If FPTAS($n$-Way-MinMax($X$), $\epsilon$) $\leq (1 + t) \cdot S$, then the partition returned by the FPTAS is $t$-feasible, so the answer to $n$-Interval-t($X$, $t$) is “yes”.

If FPTAS($n$-Way-MinMax($X$), $\epsilon$) $> (1 + t) \cdot S$, then the partition returned by the FPTAS is not $t$-feasible, but we cannot conclude that the answer to $n$-Interval-t($X$, $t$) is “no”, since there may be some other $t$-feasible partition. However, this outcome does give us valuable information about the instance: by the definition of FPTAS, it implies that the optimal value to $n$-Way-MinMax($X$) is larger than $(1 + t) \cdot S/(1 + \epsilon)$. This implies that, in any $n$-way partition of $X$, there is at least one bin with sum larger than $(1 + t) \cdot S/(1 + \epsilon)$.

Definition 3.1. Given an instance of $n$-Interval-t($X$, $t$), a real number $\epsilon > 0$, and a partition of $X$, an almost-full bin is a bin with sum larger than $(1 + t) \cdot S/(1 + \epsilon)$.

In the previous paragraph we have proved the following lemma:

Lemma 3.2. For any integer $n \geq 2$, rational $t > 0$ and real $\epsilon > 0$, if FPTAS($n$-Way-MinMax($X$), $\epsilon$) $> (1 + t) \cdot S$, then in any $n$-way partition of $X$, at least one bin is almost-full.

To gain more information on the instance, we apply an FPTAS for a constrained variant of $n$-Way-MinMax, with a Critical Coordinate (CC). For an integer $n \geq 2$, a list $X$, and a rational number $t > 0$, we define the following problem:
n-Way−CC(\(X, t\)):  Minimize \(\max(b_2, \ldots, b_n)\) subject to \(b_1 \leq (1 + t) \cdot S\), where \(b_1, \ldots, b_n\) are sums of bins in an \(n\)-way partition of \(X\).

The general technique developed by Woeginger [18] for converting a dynamic program to an FPTAS can be used to design an FPTAS for \(n\)-Way−CC; we give the details in Appendix A.1. We denote by FPTAS\((n\text{-Way−CC}(X, t, \epsilon)\) the largest bin sum in the obtained solution. The following lemma is the key for our algorithms.

**Lemma 3.3.** For any \(n \geq 2, t > 0, \epsilon > 0\), if FPTAS\((n\text{-Way−CC}(X, t, \epsilon) > (1 + t) \cdot S\), then in any \(t\)-feasible \(n\)-way partition of \(X\), at least two bins are almost-full.

**Proof.** Suppose by contradiction that there exists a \(t\)-feasible partition of \(X\) with at most one almost-full bin. It is possible to reorder the bins in the partition such that bin 1 has the largest sum; the resulting partition is still \(t\)-feasible, and still has at most one almost-full bin. Since bin 1 has the largest sum, if there is one almost-full bin, it must be bin 1. So in any case, bins 2, \ldots, \(n\) are not almost-full, so \(\max(b_2, \ldots, b_n) \leq (1 + t) \cdot S/(1 + \epsilon)\). Moreover, \(b_1 \leq (1 + t) \cdot S\) since the partition is \(t\)-feasible. Therefore, FPTAS\((n\text{-Way−CC}(X, t, \epsilon) \leq (1 + t) \cdot S\) by the definition of FPTAS. This contradicts the lemma’s assumption.

Using Lemma 3.3, we can now derive a complete algorithm for 2−Interval−t.

**Algorithm 3.1** 2–Interval−t
1. \(b_2 \leftarrow \text{FPTAS}(2\text{-Way−CC}(X, t, \epsilon = t/2)\).
2. If \(b_2 \leq (1 + t) \cdot S\), return “yes”.
3. Else, return “no”.

**Theorem 3.4 (Theorem 1.2(a)).** For any rational \(t > 0\), Algorithm 3.1 solves the problem 2−Interval−t\((X, t)\) in time \(O(\text{poly}(m, 1/t))\), where \(m\) is the number of input items in \(X\).

**Proof.** The run-time of Algorithm 3.1 is dominated by the run-time of the FPTAS for the problem 2−Way−CC, which by definition of FPTAS is in \(O(\text{poly}(m, 1/\epsilon))\), which is \(O(\text{poly}(m, 1/t))\) since \(\epsilon = t/2\). It remains to prove that Algorithm 3.1 indeed solves 2−Interval−t correctly.

If \(b_2\), the returned bin sum of \(\text{FPTAS}(2\text{-Way−CC}(X, t, \epsilon = t/2)\), is at most \((1 + t) \cdot S\), then the partition found by the FPTAS is \(t\)-feasible, so Algorithm 3.1 answers “yes” correctly.

Otherwise, by Lemma 3.3, in any \(t\)-feasible partition of \(X\) into two bins, both bins are almost-full. This means that, in any \(t\)-feasible partition, both \(b_1\) and \(b_2\) are larger than \((1 + t) \cdot S/(1 + \epsilon)\), which is larger than \(S\) since \(\epsilon = t/2\). This means that \(b_1 + b_2 > 2S\). But this is impossible, since the sum of all items is \(n \cdot S = 2S\) by assumption. Therefore, no \(t\)-feasible partition exists, and Algorithm 3.1 answers “no” correctly.

For the following sections, we will need a variant of \(n\)-Interval−t with an additional constraint, that all bins must have an equal number of items. For this problem, it is assumed w.l.o.g. that \(m\) (the number of items in \(X\)) is a multiple of \(n\) (since otherwise a solution does not exist).

\(n\text{-Interval−Eq}(X, t)\):  Decide if there exists a \(t\)-feasible partition of \(X\) into \(n\) bins such that each bin contains exactly \(m/n\) items.

We can solve this problem for \(n = 2\) in a similar way to Algorithm 3.1. This requires an FPTAS to the corresponding equal-cardinality variant of 2−Way−CC, which we call 2−Way−Eq−CC. The proof that this problem has an FPTAS is given in Appendix A.2. By inserting this FPTAS into Algorithm 3.1, we get an algorithm for solving 2−Interval−t−Eq:

**Theorem 3.5.** There is an algorithm for solving 2−Interval−t−Eq\((X, t)\) for any \(t > 0 \in \mathbb{Q}\) in time \(O(\text{poly}(m, 1/t))\), where \(m\) is the number of input items in \(X\).
Remark 3.6. The reader may wonder why we cannot use a similar algorithm for \(n \geq 3\). For example, we could have considered a variant of \(n\)-Way-CC with two critical coordinates:

\[
\text{Minimize } \max(b_3, \ldots, b_n) \text{ subject to } b_1 \leq (1 + t) \cdot S \text{ and } b_2 \leq (1 + t) \cdot S, \text{ where }
\]

If the FPTAS for this problem does not find a \(t\)-feasible partition, then any \(t\)-feasible partition must have at least three almost-full bins. Since not all bins can be almost-full, one could have concluded that there is no \(t\)-feasible partition into \(n = 3\) bins.

Unfortunately, the problem with two critical coordinates probably does not have an FPTAS even for \(n = 3\), since it is equivalent to the Multiple Subset Sum problem, which does not have an FPTAS unless \(P = NP\) [6]. In the next subsection we handle the case \(n \geq 3\) in a different way.

3.2. The \(n\)-Interval-d problem: an algorithm for \(n \geq 3\) and \(d \geq n - 2\). Given an instance of \(n\)-Interval-d(\(\mathcal{X}, d\)), where the sum of items is \(n \cdot S\) and the maximum item is \(n \cdot M\), where \(S, M \in \mathbb{Q}\), we say that a partition of \(\mathcal{X}\) is \(d\)-possible if \(S \leq \max(b_1, \ldots, b_n) \leq S + d \cdot M\), where \(b_1, \ldots, b_n\) are the bin sums. The \(n\)-Interval-d(\(\mathcal{X}, d\)) problem is to decide whether a \(d\)-possible allocation exists. Given an instance of \(n\)-Interval-d(\(\mathcal{X}, d\)), we let \(t := dM/S\), so that a partition is \(d\)-possible if-and-only-if it is \(t\)-feasible.

The algorithm starts by running FPTAS(n-\(n\)-Way-CC(\(\mathcal{X}, t\), \(\epsilon = t/4m^2\)).

If the FPTAS finds a \(t\)-feasible partition, we return “yes”. Otherwise, by Lemma 3.3, any \(t\)-feasible partition must have at least two almost-full bins. Now, we take a detour from the algorithm and prove some existential results about partitions with two or more almost-full bins. We assume that there are more items than bins, that is, \(m > n\). Then, \(\epsilon = t/4m^2 < t/4n^2\). This assumption is without loss of generality, since if \(m \leq n\) the problem is trivial. Moreover, we can assume that \(t < 1\), because otherwise each bin has a lot of extra space and the problem can be decided by a linear-time greedy algorithm (see Appendix E).

3.2.1. Structure of partitions with two or more almost-full bins. We distinguish between big, medium and small items defined as follows. A big-item is an item with size greater than \(nS(\frac{M}{n-2} - 2\epsilon)\), while a small-item is an item with size smaller than \(2nS\epsilon\). A medium (sized) item is an item with size between \(2nS\epsilon\) and \(nS(\frac{M}{n-2} - 2\epsilon)\). Our main structural Lemma is the following.

**Lemma 3.7.** Suppose that \(d \geq n - 2\), \(t = dM/S < 1\), \(\epsilon = t/4m^2\) and the following properties hold.

1. There is no \(t\)-feasible partition with at most 1 almost-full bin;
2. There is a \(t\)-feasible partition with at least 2 almost-full bins.

Then, there is a \(t\)-feasible partition with the following properties.
(a) Exactly two bins (w.l.o.g. bins 1 and 2) are almost-full.
(b) The sum of every not-almost-full bin \(i \in \{3, \ldots, n\}\) satisfies

\[
(1 - \frac{2}{n-2} t - 2\epsilon) \cdot S \leq b_i \leq (1 - \frac{2}{n-2} t + (n-1)2\epsilon) \cdot S.
\]

(c) Every almost-full bin contains only big-items.
(d) Every not-almost-full bin contains big-items that are larger than every item in bin 1,2, or small-items.
(e) There are no medium-items at all.
(f) Every not-almost-full bin contains the same number of big-items, say \(\ell\), where \(\ell\) is an integer (it may contain, in addition, any number of small-items).
(g) Every almost-full bin contains \(\ell + 1\) big-items (and no small-items).

The full proof appears in Appendix B; here we provide a sketch.
Proof Sketch. We start with an arbitrary $t$-feasible partition with some $r \geq 2$ almost-full bins $1, \ldots, r$, and convert it using a sequence of transformations to another $t$-feasible partition satisfying properties (a)–(g), as explained below. Note that the transformations are not part of our algorithm and are only used to prove the lemma.

First, we note that there must be at least one bin which is not almost-full, since the sum of an almost-full bin is larger than $S$ whereas the sum of all $n$ bins is $n \cdot S$.

For (a), if there are $r \geq 3$ almost-full bins, we move any item from one of the almost-full bins $3, \ldots, r$ to some not-almost-full bin. We prove that, as long as $r \geq 3$, the target bin remains not-almost-full. This transformation is repeated until $r = 2$ and only bins 1 and 2 remain almost-full.

For (b), for the lower bound, if there is $i \in \{3, \ldots, n\}$ for which $b_i$ is smaller than the lower bound, we move an item from bins 1, 2 to bin $i$. We prove that bin $i$ remains not-almost-full, so by assumption (1), bins 1, 2 must remain almost-full. We repeat until $b_i$ satisfies the lower bound. Once all bins satisfy the lower bound, we prove that the upper bound is satisfied too.

For (c), if bin 1 or 2 contains an item that is not big, we move it to some bin $i \in \{3, \ldots, n\}$. We prove that bin $i$ remains not-almost-full, so by assumption (1), bins 1, 2 must remain almost-full. We repeat until bins 1 and 2 contain only big-items.

For (d), if some bin $i \in \{3, \ldots, n\}$ contains an item bigger than $2nS\epsilon$ and smaller than any item in bin 1 or bin 2, we exchange it with an item from bin 1 or 2. We prove that, after the exchange, $b_i$ remains not-almost-full, so bins 1, 2 must remain almost-full. We repeat until bins 1, 2 contain only the smallest big-items. Note that transformations (b), (c), (d) increase the sum in the not-almost-full bins 3, ..., $n$, so eventually the process must end.

For (e), it follows logically from properties (d) and (c): if bins 1, 2 contain only big items and the other bins contain only big and small items, then the instance cannot contain any medium items (that are neither big nor small). For clarity and verification, we provide a stand-alone proof. We show that if a medium item is in an almost-full bin, we can move it to some not-almost-full bin and get a new allocation with only one almost-full bin. If the medium item is in some not-almost-full bin, say bin $i$, we can exchange it with some big item in one almost-full bin such that bin $i$ remains not-almost-full, and then move it to a non-almost-full bin such that both bins remain non-almost-full.

For (f), we use the fact that the difference between two not-almost-full bins is at most $2nS\epsilon$ by property (b), and show that it is too small to allow a difference of a whole big-item.

For (g), because by (d) bins 1 and 2 contain the smallest big-items, whereas their sum is larger than bins 3, ..., $n$, they must contain at least $\ell + 1$ big-items. We prove that, if they contain $\ell + 2$ big-items, then their sum is larger than $(1 + t)S$, which contradicts $t$-feasibility.

Properties (f) and (g) imply:

Corollary 3.8. Suppose that $d \geq n - 2$, $t = dM/S$ and $\epsilon = t/4m^2$. Let $B \subseteq X$ be the set of big items in $X$. If there is a $t$-feasible partition with at least two almost-full bins, and no $t$-feasible partition with at most one almost-full bin, then $|B| = n\ell + 2$ for some positive integer $\ell$.

3.2.2. Back to the algorithm. We have left the algorithm at the point when FPTAS($n$–Way–CC($X, t, \epsilon = t/4m^2$) did not find a $t$-feasible partition. Lemma 3.3 implies that if a $t$-feasible partition exists, then there exists a $t$-feasible partition satisfying all properties of Lemma 3.7 and Corollary 3.8. We can find such a partition (if it exists) in two steps:

- For bins 1, 2: Find a $t$-feasible partition of the $2\ell + 2$ smallest items in $B$ into two bins with $\ell + 1$ items in each bin.
- For bins 3, ..., $n$: Find a $t$-feasible partition of the remaining items in $X$ into $n - 2$ bins. For bins 3, ..., $n$, we use the FPTAS for the problem $(n - 2)$–Way–MinMax. If it returns a $t$-feasible partition, we are done. Otherwise, by Lemma 3.2, every partition into $(n - 2)$ bins must have at least one almost-full bin. But by Lemma 3.7(a), all bins 3, ..., $n$ are not almost-full — a contradiction. Therefore, if the FPTAS does not find a $t$-feasible partition, we answer “no”.

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Bins 1 and 2 require a more complicated algorithm. We use the following notation (where \( n \) denotes the same parameter \( n \geq 3 \) of the original \( n\text{--Interval}\text{--d} \) problem):

- \( B_{1,2} \) is the set of \( 2\ell + 2 \) smallest items in \( B \) (where \( B \) is the set of big items in \( \mathcal{X} \)).
- \( 2S_{1,2} \) is the sum of items in \( B_{1,2} \) (for some \( S_{1,2} \in \mathbb{Q} \)).
- \( nM_{1,2} \) is the largest item in \( B_{1,2} \) (for some \( M_{1,2} \in \mathbb{Q} \)).

Construct a new set, \( \overline{B}_{1,2} \), by replacing each item \( x \in B_{1,2} \) by its “inverse”, defined by \( \overline{x} := nM_{1,2} - x \). Note that, since all items in \( B_{1,2} \) are big-items, all inverses are between 0 and \( 2nS_1 \) (see the proof of Lemma 3.7(f) in Appendix B). Denote the sum of inverses by \( 2\overline{S}_{1,2} \).

Given a \( t\)-feasible partition of \( B_{1,2} \) with sums \( b_1 \) and \( b_2 \) and \( \ell + 1 \) items in each bin, denote the sums of the corresponding partition of \( \overline{B}_{1,2} \) by \( \overline{b}_1 \) and \( \overline{b}_2 \), respectively. Since both bins contain \( \ell + 1 \) items, 
\[
\overline{b}_i = (\ell + 1) \cdot nM_{1,2} - b_i, \text{ for } i \in \{1, 2\}, \text{ and } \overline{S}_{1,2} = (\ell + 1) \cdot nM_{1,2} - \overline{S}_{1,2},
\]
Now,
\[
b_1 \leq (1 + t)S \iff \overline{b}_1 \geq (\ell + 1)nM_{1,2} - (1 + t)S
\]
\[
\iff \overline{b}_2 \leq 2\overline{S}_{1,2} - (\ell + 1)nM_{1,2} + (1 + t)S
\]
\[
= \overline{S}_{1,2} + (\overline{S}_{1,2} - (\ell + 1)nM_{1,2}) + (S + tS)
\]
\[
= \overline{S}_{1,2} + (S + tS - S_{1,2}),
\]
and similarly, 
\[
\overline{b}_1 \leq \overline{S}_{1,2} + (S + tS - S_{1,2}).
\]

So the problem of finding a \( t\)-feasible partition of \( B_{1,2} \) with \( \ell + 1 \) items in each bin is equivalent to the problem of finding a \( \bar{t}\)-feasible partition of \( \overline{B}_{1,2} \) with \( \ell + 1 \) items in each bin, where \( \bar{t} := (S + tS - S_{1,2})/\overline{S}_{1,2} \). By Theorem 3.5, this problem can be solved in \( O(\text{poly}(m, 1/\bar{t})) \) time.

It remains to prove that \( 1/\bar{t} \) is polynomial in \( m \). By Lemma 3.7(d), in each of the not-almost-full bins, there are \( \ell \) items that are at least as large as \( nM_{1,2} \) (in addition to some small-items). Therefore, 
\[
\overline{S}_{1,2} \geq 2\overline{S}_{1,2} + (n - 2) \cdot \ell \cdot nM_{1,2},
\]
which implies that
\[
S - S_{1,2} \geq -\frac{n - 2}{n} S_{1,2} + \frac{n - 2}{n} \ell \cdot nM_{1,2}.
\]
Since \( d \geq n - 2 \),
\[
tS = dM \geq (n - 2)M \geq (n - 2)M_{1,2}.
\]
Summing up these two inequalities gives:
\[
S - S_{1,2} + tS \geq \frac{n - 2}{n} \left( \ell \cdot nM_{1,2} - S_{1,2} + nM_{1,2} \right)
\]
\[
= \frac{n - 2}{n} \left( (\ell + 1) \cdot nM_{1,2} - S_{1,2} \right) = \frac{n - 2}{n} \cdot \overline{S}_{1,2}.
\]
Therefore, \( \bar{t} \geq \frac{n - 2}{n} \), so \( 1/\bar{t} \in O(1) \), and the sub-problem runs in \( O(\text{poly}(m)) \) time.

3.2.3. Complete algorithm. We are now ready to present the complete algorithm for \( n\text{--Interval}\text{--d} \), presented in Algorithm 3.2.

Theorem 3.9 (Theorem 1.2(b), first part). For any fixed integer \( n \geq 3 \) and rational number \( d \geq n - 2 \), Algorithm 3.2 solves \( n\text{--Interval}\text{--d}(\mathcal{X}, d) \) in \( O(\text{poly}(m)) \) time, where \( m \) is the number of items in \( \mathcal{X} \).

Proof. If Algorithm 3.2 answers “yes”, then clearly a \( t\)-feasible partition exists. To complete the correctness proof, we have to show that the opposite is true as well.

Suppose there exists a \( t\)-feasible partition. If the partition has at most one almost-full bin, then by Lemma 3.3, it is found by the FPTAS in step 2. Otherwise, the partition must have at least two almost-full bins, and there exists a \( t\)-feasible partition satisfying the properties of Lemma 3.7.
Algorithm 3.2 $n$-Interval-d (complete algorithm)

1: $t \leftarrow dM/S$ and $\epsilon \leftarrow t/(4m^2)$.
2: If $\text{FPTAS}(n\text{-Way-CC}(X, t), \epsilon) \leq (1 + t) \cdot S$, return “yes”.
3: $B \leftarrow \{x_i \in X \ | \ x_i > nS(\frac{1}{n-2} - 2\epsilon)\}$ \hspace{1cm} $\triangleright$ big items
4: If $|B|$ is not of the form $n\ell + 2$ for some integer $\ell$, return “no”.
5: $B_{1,2} \leftarrow$ the $2\ell + 2$ smallest items in $B$. \hspace{1cm} $\triangleright$ break ties arbitrarily
6: $B_{3,n} \leftarrow X \setminus B_{1,2}$. \hspace{1cm} $\triangleright$ big and small items
7: $(b_3, \ldots, b_n) \leftarrow \text{FPTAS}((n-2)\text{-Way-MinMax}(B_{3,n}), \epsilon)$ \hspace{1cm} $\triangleright$ an $(n-2)$-partition of $B_{3,n}$
8: If $\max(b_3, \ldots, b_n) > (1 + t)S$, return “no”.
9: $B_{1,2}^{\prime} \leftarrow \{nM_{1,2} - x \ | \ x \in B_{1,2}\}$ and $t^\prime \leftarrow (S + tS - S_{1,2})/S_{1,2}$.
10: Look for a $t^\prime$-feasible partition of $B_{1,2}$ into two subsets of $\ell + 1$ items as in Theorem 3.5.
11: If a $t^\prime$-feasible partition is found, return “yes”. Otherwise return “no”.

By Corollary 3.8, the algorithm does not return “no” in step 4. By properties (a) and (b), there exists a partition of $B_{3,n}$ into $n-2$ bins $3, \ldots, n$ which are not almost-full. By Lemma 3.2, the FPTAS in step 7 finds a partition with $\max(b_3, \ldots, b_n) \leq (1 + t)S$. The final steps, regarding the partition of $B_{1,2}$, are justified by the discussion at Subsection 3.2.2.

3.3. Hardness for $n \geq 3$ bins and $d < n - 2$. The following theorem complements the results of the previous subsection.

Theorem 3.10 (Theorem 1.2(b), second part). Given a fixed integer $n \geq 3$ and a positive rational number $d < n - 2$, the problem $n$-Interval-d$(X, d)$ is NP-complete.

Proof. Given an $n$-way partition of $m$ items, summing the sizes of all elements in each bin allows us to check whether the partition is $d$-possible in linear time. So, the problem is in NP.

To prove that $n$-Interval-d is NP-Hard, we reduce from the Equal-Cardinality Partition problem, proved to be NP-hard in [8]: given a list with an even number of integers, decide if they can be partitioned into two subsets with the same sum and the same cardinality.

Given an instance $X_1$ of Equal-Cardinality Partition, denote the number of items in $X_1$ by $2m'$. Define $M$ to be the sum of numbers in $X_1$ divided by $2n(1 - \frac{d}{n-2})$, so that the sum of items in $X_1$ is $2n(1 - \frac{d}{n-2})M$ (where $n$ and $d$ are the parameters in the theorem statement). We can assume w.l.o.g. that all items in $X_1$ are at most $n(1 - \frac{d}{n-2})M$, since if some item is larger than half of the sum, the answer is necessarily “no”.

Construct an instance $X_2$ of the Equal-Cardinality Partition problem by replacing each item $x$ in $X_1$ by $nM - x$. So $X_2$ contains $2m'$ items between $n(\frac{d}{n-2})M$ and $nM$. Their sum, which we denote by $2S'$, satisfies

$$2S' = 2m' \cdot nM - 2n \left(1 - \frac{d}{n-2}\right)M = 2n \left(m' - 1 + \frac{d}{n-2}\right)M.$$ 

Clearly, $X_1$ has an equal-sum equal-cardinality partition (with sum $n \left(1 - \frac{d}{n-2}\right)M$) if and only if $X_2$ has an equal-sum equal-cardinality partition (with sum $S' = n \left(m' - 1 + \frac{d}{n-2}\right)M$).

Construct an instance $(X_3, d)$ of $n$-Interval-d by adding $(n-2)(m'-1)$ items of size $nM$. Note that $nM$ is indeed the largest item size in $X_3$. Denote the sum of item sizes in $X_3$ by $nS$. Then

$$nS = 2S' + (n-2)(m'-1) \cdot nM = n \left(2(m'-1) + \frac{2d}{n-2} + (n-2)(m'-1)\right)M = n \left(n(m'-1) + \frac{2d}{n-2}\right)M;$$

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\[ S + dM = \left( n(m' - 1) + \frac{2d}{n-2} + d \right) M = \left( n(m' - 1) + \frac{nd}{n-2} \right) M = S', \]

so a partition of \( \mathcal{X}_3 \) is \( d \)-possible if and only if the sum of each of the \( n \) bins in the partition is at most \( S + dM = S' \).

We now prove that if \( \mathcal{X}_2 \) has an equal-sum equal-cardinality partition, then the corresponding instance \((\mathcal{X}_3,d)\) has a \( d \)-possible partition, and vice versa.

If \( \mathcal{X}_2 \) has an equal-sum partition, then the items of \( \mathcal{X}_2 \) can be partitioned into two bins of sum \( S' \), and the additional \((n-2)(m'-1)\) items can be divided into \( n-2 \) bins of \( m'-1 \) items each. Note that the sum of these items is

\[(m'-1)\cdot nM = n(m'-1)M = S - \frac{2}{n-2}dM < S + dM = S',\]

so the resulting partition is a \( d \)-possible partition of \( \mathcal{X}_3 \).

Conversely, suppose \( \mathcal{X}_3 \) has a \( d \)-possible partition. Let us analyze its structure.
- Since the partition is \( d \)-possible, the sum of every two bins is at most \( 2(S + dM) \).
- So the sum of every \( n-2 \) bins is at least \( nS - 2(S + dM) = (n-2)S - 2dM \).
- Since the largest \((n-2)(m'-1)\) items in \( I_3 \) sum up to exactly \((n-2)S - 2dM\) by (3.1), every \( n-2 \) bins must contain at least \((n-2)(m'-1)\) items.
- Since \( I_3 \) has \((n-2)(m'-1)+2m'\) items overall, \( n-2 \) bins must contain exactly \((n-2)(m'-1)\) items, such that each item size must be \( nM \), and their sum must be \((n-2)S - 2dM \).
- The other two bins contain together \( 2m' \) items with sum \( 2(S + dM) \), so each of these bins must have a sum of exactly \( S + dM \). Since \((m'-1)\cdot nM < S + dM\) by (3.1), each of these two bins must contain exactly \( m' \) items.

These latter two bins are an equal-sum equal-cardinality partition for \( \mathcal{X}_2 \). This construction is done in polynomial time, completing the reduction. \( \Box \)

4. Partition with Split Items. To solve the optimization problem \( n \text{-} \text{Split} \text{-} \text{MinMax} \), we present the following decision problem. For a fixed number \( 2 \leq n \in \mathbb{N} \) of bins, given a list \( \mathcal{X} \), the number of split-items \( 0 \leq s \leq m \), \( s \in \mathbb{N} \), and a rational number \( t \geq 0 \), define:

\[ n \text{-} \text{Split} \text{-} \text{Bound}(\mathcal{X},s,t): \text{ Decide if there exists a partition of } \mathcal{X} \text{ into } n \text{ bins, with } \]

\[ \text{at most } s \text{ split items, such that } \max(b_1,\ldots,b_n) \leq (1+t)S. \]

The special case \( t = 0 \) corresponds to the \( n \text{-} \text{Split} \text{-} \text{Perfect}(\mathcal{X},s) \) problem. The following Lemma shows that, w.l.o.g., we can consider only the largest items for splitting.

**Lemma 4.1.** For every partition with \( s \in \mathbb{N} \) split items and bin sums \( b_1,\ldots,b_n \), there exists a partition with the same bin sums \( b_1,\ldots,b_n \) in which only the \( s \) largest items are split.

**Proof.** Consider a partition in which some item \( x \) is split between two or more bins, whereas some item \( y > x \) is allocated entirely to some bin \( i \). Construct a new partition as follows:
- Move item \( x \) entirely to bin \( i \);
- Split item \( y \) into two parts with sizes \( x \) and \( y - x \);
- Remove the part with size \( x \) from bin \( i \), and split it among the bins in the same proportions that the item \( x \) was split originally.

All bin sums remain the same. Repeat the argument until only the largest items are split. \( \Box \)

**Theorem 4.2.** For any fixed integers \( n \geq 2 \) and \( d \geq 0 \), there is a polynomial-time reduction from \( n \text{-} \text{Interval} \text{-} d(\mathcal{X},d) \) to \( n \text{-} \text{Split} \text{-} \text{Bound}(\mathcal{X},s=d,t=0) \).

**Proof.** Given an instance \( \mathcal{X} \) of \( n \text{-} \text{Interval} \text{-} d(\mathcal{X},d) \), construct an instance \( \mathcal{X}' \) of \( n \text{-} \text{Split} \text{-} \text{Bound}(\mathcal{X}',d,0) \) by adding \( d \) items of size \( nM \), where \( nM \) is the size of the biggest item in \( \mathcal{X} \).

First, assume that \( \mathcal{X} \) has a \( d \)-possible partition. Then there are \( n \) bins with a sum at most \( S + dM \). Take the \( d \) added items of size \( nM \) and add them to the bins, possibly splitting some items
between bins, such that the sum of each bin becomes exactly \( S + dM \). This is possible because the sum of the items in \( X' \) is \( nS + dnM = n(S + dM) \). The result is a 0-feasible partition of \( X' \) with at most \( d \) split items.

Second, assume that \( X' \) has a 0-feasible partition with at most \( d \) split items. Then there are \( n \) bins with sum exactly \( S + dM \). By Lemma 4.1, we can assume the split items are the largest ones, which are the \( d \) added items of size \( nM \). Remove these items to get a partition of \( X' \). The sum in each bin is now at most \( S + dM \), so the partition is \( d \)-possible.

This construction is done in polynomial time, which completes the proof.

**Corollary 4.3 (Theorem 1.1(b)).** For a fixed integers \( n \geq 3 \) and \( s \in \{0, 1, \ldots, n - 3\} \), the problem \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, 0) \equiv n\text{-}\text{Split}\text{-}\text{Perfect}(X', s) \) is NP-complete.

**Proof.** Theorem 3.10 and Theorem 4.2 together imply that the problem is NP-hard. The problem is in NP since given a partition, summing the sizes of the elements (or element fractions) in each bin lets us to check in linear time whether the partition has equal bin sums.

**Theorem 4.4.** For any fixed integers \( n \geq 2, s \geq 0 \) and rational \( t \geq 0 \), there is a polynomial-time reduction from \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, t) \), to \( n\text{-}\text{Interval}\text{-}\text{d}(X', d) \) for some rational \( d \geq s \).

**Proof.** Given an instance \( X' \) of \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, t) \), denote the sum of all items in \( X' \) by \( nS \) and the largest item size by \( nM \) where \( S, M \in Q \). Construct an instance \( X'' \) of \( n\text{-}\text{Interval}\text{-}\text{d}(X', d) \) by removing the \( s \) largest items from \( X' \). Denote the sum of remaining items by \( nS' \) for some \( S' \leq S \), and the largest remaining item size by \( nM' \) for some \( M' \leq M \). Note that the size of every removed item is between \( nM' \) and \( nM \), so \( sM' \leq S - S' \leq sM \). Set \( d := (S + tS - S')/M' \), so \( S' + dM' = S + tS \). Note that \( d \geq (S - S')/M' \geq s \).

First, assume that \( X' \) has a \( t \)-feasible partition with \( s \) split items. By Lemma 4.1, we can assume that only the \( s \) largest items are split. Therefore, removing the \( s \) largest items results in a partition of \( X'' \) with no split items, where the sum in each bin is at most \( S + tS = S' + dM' \). This is a \( d \)-possible partition of \( X'' \).

Second, assume that \( X'' \) has a \( d \)-possible partition. In this partition, each bin sum is at most \( S' + dM' = S + tS \), so it is a \( t \)-feasible partition of \( X'' \). To get a \( t \)-feasible partition of \( X' \), take the \( s \) previously-removed items and add them to the bins, possibly splitting some items between bins, such that the sum in each bin remains at most \( S + tS \). This is possible because the sum of all items is \( nS \leq n(S + tS) \).

This construction is done in polynomial time, which completes the proof.

Combining Theorem 4.4 with Theorem 3.9 provides the following.

**Corollary 4.5.** For any fixed integers \( n \geq 3, s \geq n - 2 \) and rational \( t \geq 0 \), \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, t) \) can be solved in polynomial time.

Finally, for any fixed integer \( n \geq 2 \) and \( s \geq n - 2 \), we can solve the \( n\text{-}\text{Split}\text{-}\text{MinMax}(X', s) \) optimization problem by using binary search on the parameter \( t \) of the \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, t) \) problem. The details are given in Appendix D. As explained there, the binary search procedure needs to solve at most \( \log_2(nS) \) instances of \( n\text{-}\text{Split}\text{-}\text{Bound}(X', s, t) \).

**Corollary 4.6 (Theorem 1.1(a)).** For any fixed integers \( n \geq 3 \) and \( s \geq n - 2 \), the problem \( n\text{-}\text{Split}\text{-}\text{MinMax}(X', s) \) can be solved in polynomial time in the size of the binary representation of its input.

5. Conclusion and Future Directions. We presented two variants of the multiway number partitioning problem. In the language of machine scheduling, \( n\text{-}\text{Split}\text{-}\text{MinMax} \) corresponds to finding a schedule that minimizes the makespan on \( n \) identical machines when \( s \) jobs can be split between the machines; \( n\text{-}\text{Interval}\text{-}\text{d} \) corresponds to finding a schedule in which the makespan is in a given interval. It may be interesting to study these two variants in the more general setting of non-identical machines.
In the language of fair item allocation, we have solved the problem of finding a fair allocation among \( n \) agents with identical valuations, when the ownership on some \( s \) items may be shared among agents. With identical valuations, "fair" simply means that each agent receives the same sum of values. When agents may have different valuations, there are various generalizations to this fairness notion, such as proportionality, envy-freeness or equitability. A future research direction is to develop algorithms for finding such allocations with a bounded number of shared items.

Our analysis shows the similarities and differences between these two variants and the more common notion of FPTAS. One may view our results as introducing an alternative kind of approximation: instead of keeping the items discrete and allowing a bounded deviation in the objective value, we keep the objective value optimal and allow a bounded number of items to become fractional. This approximation notion may be meaningful in other optimization problems.

More generally, consider any optimization problem that can be solved by a mixed-integer program, where some of the problem variables are rationals and some are integers. Instead of deciding in advance (in the problem definition) which of the variables must be integers, one can consider a variant in which only the number of the integer variables is fixed in advance, whereas the solver can decide which variables are integers after receiving the input.

APPENDIX.

Appendix A. FPTAS-s for various partitioning problems. Our algorithms use FPTAS-s for several variants of the number partitioning problem. These FPTAS-s are developed using a general technique described by Woeginger [18]. We briefly describe this technique below. The first step is to develop a dynamic programming (DP) algorithm that solves the problem exactly. The algorithm should have a specific format called simple DP, defined by the following parameters:

- The size of the input vectors, \( \alpha \in \mathbb{N} \) (in our settings, usually \( \alpha = 1 \) since each input is a single integer);
- The size of the state vectors, \( \beta \in \mathbb{N} \) (in our settings, each state represents a partition of a subset of the inputs);
- A set of initial states \( V_0 \) (in our settings, \( V_0 \) usually contains a single state — the zero vector — representing the empty partition);
- A set of transition functions \( F \); each function in \( F \) accepts a state and an input, and returns a new state (in our settings, each function in \( F \) corresponds to putting the new input in one of the bins);
- A set of filter functions \( H \); each function \( h_j \in H \) corresponds to a function \( f_j \in F \). It accepts a state and an input, and returns a positive value if the new state returned by \( f_j \) is infeasible (infeasible states are kept out of the state space).
- An objective function \( G \), that maps a state to a numeric value.

The DP algorithm processes the inputs one by one. For each input \( k \), it applies every transition function in \( F \) to every state in \( V_{k-1} \), to produce the new state-set \( V_k \). Finally, it picks the state in \( V_m \) that minimizes the objective function. Formally:

1. Let \( V_0 \) be the set of initial states.
2. For \( k := 1, \ldots, m \):
   \[ V_k := \{ f_j(s,x) | f_j \in F, \ s \in V_{k-1}, \ h_j(s,x) \leq 0 \} \]
3. Return
   \[ \min \{ G(s) | s \in V_m \} \]

Every DP in this format can be converted to an FPTAS if it satisfies a condition called critical-coordinate-benevolence (CC-benevolence) To prove that a dynamic program is CC-benevolent, we need to define a degree vector \( D \) of size \( \beta \), which determines how much each state is allowed to
deviate from the optimal state. The functions in \( F, G, H \) and the vector \( D \) should satisfy several conditions listed in Lemma 6.1 of [18]. These conditions use the term \([D, \Delta]\)-close, defined as follows.

For a real number \( \delta > 1 \) and two vectors \( V = [v_1, \ldots, v_\beta] \) and \( V' = [v'_1, \ldots, v'_\beta] \), we say that \( V \) is \([D, \Delta]\)-close to \( V' \) if

\[
\Delta^{-d_l} \cdot v_l \leq v'_l \leq \Delta^{d_l} \cdot v_l, \text{ for } l = 1, \ldots, \beta
\]

That is, each coordinate in \( V' \) deviates from the corresponding coordinate in \( V \) by a multiplicative factor determined by \( \Delta \) and by the degree vector \( D \). The \( \Delta \) is a factor determined by the required approximation accuracy \( \epsilon \). Note that if some coordinate \( l \) in \( D \) is 0, then the definition of \([D, \Delta]\)-close requires that the coordinate \( l \) in \( V' \) is equal to coordinate \( l \) in \( V \) (no deviation is allowed).

We now use Lemma 6.1 of [18] to prove that the problems \( n\text{-Way} - \text{CC} \) and \( n\text{-Interval} - t\text{-Eq} \) are CC-benevolent, and therefore there is an FPTAS for solving these problems.

**A.1. n-Way–CC.** Recall that the objective of \( n\text{-Way} - \text{CC}(\mathcal{X}, t) \) is to find an \( n \)-way partition of \( \mathcal{X} \) minimizing the largest bin sum, subject to the constraint that the sum of bin \#1 is at most \((1 + t) \cdot S\), where \( t \in \mathbb{Q} \) is given in the input and \( S = (\sum_{x \in \mathcal{X}} x)/n \) is the average bin sum.

**Proof. The dynamic program.** We define a simple DP with \( \alpha = 1 \) (the size of the input vectors) and \( \beta = n \) (the size of the state vectors). For \( k = 1, \ldots, m \) define the input vector \( U_k = [x_k] \) where \( x_k \) is the size of item \( k \). A state \( V = [b_1, b_2, \ldots, b_n] \) in \( V_k \) encodes a partial allocation for the first \( k \) items, where \( b_i \) is the sum of items in bin \( i \in \{1, 2, \ldots, n\} \) in the partial allocation. The set \( F \) contains \( n \) transition functions \( f_1, \ldots, f_n^g \):

\[
\begin{align*}
&f_1(x_k, b_1, b_2, \ldots, b_n) = [b_1 + x_k, b_2, \ldots, b_n] \\
&f_2(x_k, b_1, b_2, \ldots, b_n) = [b_1, b_2 + x_k, \ldots, b_n] \\
&\vdots \\
&f_n(x_k, b_1, b_2, \ldots, b_n) = [b_1, b_2, \ldots, b_n + x_k]
\end{align*}
\]

Intuitively, the function \( f_i \) corresponds to putting item \( k \) in bin \( i \). The set \( H \) contains a function \( h_1(x_k, b_1, b_2, \ldots, b_n) = b_1 + x_k - (1 + t) \cdot S \), and functions \( h_i(x_k, b_1, b_2, \ldots, b_n) \equiv 0 \) for \( i \in \{2, \ldots, n\} \).

These functions represent the fact that the sum of the first bin must always be at most \((1 + t) \cdot S\) (if it becomes larger than \((1 + t) \cdot S\), then \( h_1 \) will return a positive value and the new state will be filtered out). There are no constraints on the sums of bins \( 2, \ldots, n \).

The initial state space \( V_0 \) contains a single vector \([0, 0, \ldots, 0]\). The minimization objective is

\[
G(b_1, b_2, \ldots, b_n) = \max\{b_1, b_2, \ldots, b_n\}.
\]

**Benevolence.** We show that our problem is CC-benevolent, as defined at [18][Section 6]. We use the degree vector \( D = [1, 1, \ldots, 1] \), and define \( b_1 \) as the critical coordinate.

All the transition functions are polynomials of degree 1. The value of the function \( f_1(U, V) \) only depends on \( x_k \) which is on \( U \), and \( b_1 \) which is our critical-coordinate. So, C.1(i) on the function set \( F \) is fulfilled.

The functions \( h_1, h_2, \ldots, h_n \) are polynomials; the monomials do not depend on \( b_2, \ldots, b_n \), and the monomial that depends on \( b_1 \) has a positive coefficient. So, C.2(i) on the function set \( H \) is fulfilled.

If a state \([b_1, b_2, \ldots, b_n]\) is \([D, \Delta]\)-close to another state \([b'_1, b'_2, \ldots, b'_n]\), then by \([D, \Delta]\)-close definition, we have \( v'_1 \leq \Delta \cdot v_1, v'_2 \leq \Delta \cdot v_2, \ldots, v'_n \leq \Delta \cdot v_n \), so

\[
\max\{b'_1, b'_2, \ldots, b'_n\} \leq \Delta \cdot \max\{b_1, b_2, \ldots, b_n\},
\]

and \( G(V') \leq \Delta \cdot G(V) \). Therefore, C.3(i) on the function \( G \) is fulfilled (with degree \( g = 1 \)).
The definition in [18] also allows a domination relation, but we do not need it in our case (formally, our domination relation $\preceq_{\text{dom}}$ is the trivial relation). So, the statement conditions C.1(ii), C.2(ii), C.3(ii) are fulfilled.

Condition C.4 (i) holds since all functions in $F$ can be evaluated in polynomial time. C.4 (ii) holds since the cardinality of $F$ is a constant. C.4 (iii) holds since the cardinality of $V_0$ is a constant. C.4(iv) is satisfied since the value of the coordinates is upper bounded by the sum of the items $nS$, so their logarithm is bounded by the size of the input. Hence, our problem is CC-benevolent. By the main theorem of [18], it has an FPTAS.

\textbf{A.2. 2-Way-Eq-CC}. Recall that the objective is to find a 2-way partition of $\mathcal{X}$ minimizing the largest bin sum, subject the the constraint that the sum of bin #1 is at most $(1 + t) \cdot S$, and additionally, the number of items in each bin is the same.

\textbf{Proof. The dynamic program}. We define a simple DP with $\alpha = 1$ (the size of the input vectors) and $\beta = 4$ (the size of the state vectors). For $k = 1, \ldots, m$ define the input vector $U_k = [x_k]$, where $x_k$ is the size of item $k$. A state $V = [b_1, b_2, l_1, l_2]$ in $V_k$ encodes a partial allocation for the first $k$ items, where $b_i$ is the sum of items in bin $i \in \{1, 2\}$ in the partial allocation and $l_i$ is the number of items in bin $i$. The set $F$ contains two transition functions $F_1$ and $F_2$:

$$f_1(x_k, b_1, b_2, l_1, l_2) = [b_1 + x_k, b_2, l_1 + 1, l_2]$$

$$f_2(x_k, b_1, b_2, l_1, l_2) = [b_1, b_2 + x_k, l_1, l_2 + 1]$$

Intuitively, the function $f_i$ corresponds to putting item $k$ in bin $i$. Let $H$ be a set of two functions $h_1$ and $h_2$. The function $h_1(x_k, b_1, b_2, l_1, l_2) = b_1 + x_k - (1 + t)S$ corresponds to an upper bound of $(1 + t)S$ on the sum of the first bin, and a function $h_2(x_k, b_1, b_2, l_1, l_2) \equiv 0$ that corresponds to having no upper bound on the sum of the second bin. Finally, set the minimization objective to

$$G(b_1, b_2, l_1, l_2) = \begin{cases} \max\{b_1, b_2\}, & \text{if } l_1 = l_2 \\ \infty, & \text{otherwise} \end{cases}$$

The initial state space $V_0$ is set to $\{[0, 0, 0, 0]\}$.

\textbf{Benevolence}. We show that our problem is CC-benevolent, as defined at [18][Section 6].

We define the degree vector as $D = [1, 1, 0, 0]$. Note that the third and fourth coordinates correspond to the number of items in each bin, for which we need an exact number and not an approximation.

Lemma 4.1(i) is satisfied since if a state $[b_1, b_2, l_1, l_2]$ is $[D, \Delta]$-close to another state $[b_1', b_2', l_1', l_2']$, by definition of $[D, \Delta]$-close [Section 2](2.1), we must have $l_1 = l_1'$ and $l_2 = l_2'$ because the degree of coordinates 3 and 4 is 0. At coordinates 1 and 2, both transition functions are polynomials of degree 1. Furthermore, the value of the function $f_1(U, V)$ only depends on $x_k$ which is on $U$ and $b_1$ which is our critical-coordinate. So, C.1(i) on the function set $F$ is fulfilled.

The functions $h_1$ and $h_2$ are polynomials; the monomials do not depend on $b_2$, and the monomial that depends on $b_1$ has a positive coordinate. So, C.2(i) on the function set $H$ is fulfilled.

If a state $[b_1, b_2, l_1, l_2]$ is $[D, \Delta]$-close to another state $[b_1', b_2', l_1', l_2']$ (where $\Delta$ is a factor determined by the required approximation accuracy $\epsilon$), by $[D, \Delta]$-close definition, we have $l_1 = l_1'$ and $l_2 = l_2'$ because the degree of coordinates 3 and 4 is 0, also $b_1' \leq \Delta b_1$ and $b_2' \leq \Delta b_2$, so $\max\{b_1', b_2'\} \leq \Delta \max\{b_1, b_2\}$, so $G(V') \leq \Delta gG(V)$. Therefore, C.3(i) on the function $G$ is fulfilled (with degree $g = 1$).

Again we do not need a domination relation, so we use the trivial relation $\preceq_{\text{dom}}$. Then the statement conditions C.1(ii), C.2(ii), C.3(ii) are fulfilled.

Condition C.4 (i) holds since all functions in $F$ can be evaluated in polynomial time. C.4 (ii) holds since the cardinality of $F$ is a constant. C.4 (iii) holds since the cardinality of $V_0$ is a constant. C.4(iv) is satisfied for coordinates 1, 2 since their value is upper bounded by the sum of the items
nS, so their logarithm is bounded by the size of the input. For coordinates 3, 4 (whose degree is 0), the condition is satisfied since their value is upper bounded by the number of items m. Hence, our problem is CC-benevolent. By the main theorem of [18], it has an FPTAS.

Appendix B. Proof of Lemma 3.7:
Properties of instances with at least two almost-full bins.

By the lemma assumption, there exists a t-feasible partition with some \( r \geq 2 \) almost-full bins 1, \ldots, r. We take an arbitrary such partition and convert it using a sequence of transformations to another t-feasible partition satisfying the properties stated in the lemma, as explained below.

Proof of Lemma 3.7, property (a). Pick a t-feasible partition with some \( r \geq 3 \) almost-full bins. Necessarily \( r \leq n - 1 \), otherwise the sum of \( n \) almost-full bins would be larger than \( nS \). It is sufficient to show that there exists a t-feasible partition with \( r - 1 \) almost-full bins; then we can proceed by induction down to 2. Assume w.l.o.g. that the almost-full bins are 1, \ldots, r and the not-almost-full bins are \( r + 1, \ldots, n \). By definition,

\[
b_1, \ldots, b_r \geq (1 + t) \cdot S/(1 + \epsilon) > ((1 + t) \cdot S)(1 - \epsilon) > (1 + t - 2\epsilon) \cdot S,
\]

so,

\[
\sum_{i=r+1}^{n} b_i = nS - \sum_{i=1}^{r} b_i < nS - rS \cdot (1 + t - 2\epsilon) = (n - r - rt + 2r\epsilon) \cdot S.
\]

Denote by \( b_{\min} \) the bin with the smallest bin sum (breaking ties arbitrarily). By the pigeonhole principle,

\[
b_{\min} < \frac{(n - r - rt + 2r\epsilon) \cdot S}{n - r} = \left(1 - \frac{r}{n - r}t + \frac{r}{n - r}2\epsilon\right) \cdot S.
\]

By assumption, all item sizes are at most \( nM \). Suppose we take one item from bin \( r \), and move it to \( b_{\min} \). Then after the transformation,

\[
b_{\min} < \left(1 - \frac{r}{n - r}t + \frac{r}{n - r}2\epsilon\right) \cdot S + nM
\]

\[
\leq \left(1 - \frac{3}{n - 3}t + \frac{n - 1}{n - (n - 1)}2\epsilon\right) \cdot S + ntS/d \quad (3 \leq r \leq n - 1 \text{ and } dM = tS)
\]

\[
\leq \left(1 - \frac{3}{n - 3}t + 2n\epsilon - 2\epsilon\right) \cdot S + \frac{n}{n - 2}tS \quad (d \geq n - 2)
\]

\[
< \left(1 - \frac{2}{n - 2}t - 2\epsilon\right) \cdot S + \frac{n}{n - 2}tS \quad (\epsilon = t/4m^2)
\]

\[
= (1 + t - 2\epsilon) \cdot S < (1 + t) \cdot S/(1 + \epsilon),
\]

so \( b_{\min} \) remains not-almost-full.

If bin \( r \) is still almost-full, then we are in the same situation: we have exactly \( r \) almost-full bins. So we can repeat the argument, and move another item from bin \( r \) to the (possibly different) minimum-sum bin. Since the number of items in bin \( r \) decreases with each step, eventually, it becomes not-almost-full.

This process continues until we have a partition with exactly two almost-full bins, which we assume to be bins 1 and 2. The not-almost-full bins are bins 3, \ldots, \( n \). We now transform the partition so that the sum in each bin 3, \ldots, \( n \) is bounded within a small interval.

Proof of Lemma 3.7, property (b). For proving the lower bound, suppose there is some \( i \in \{3, \ldots, n\} \) with \( b_i < \left(1 - \frac{2}{n - 2}t - 2\epsilon\right) \cdot S \). Move an item from bin 1 to bin \( i \). Its new sum
satisfies
\begin{align*}
b_i < \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \cdot S + nM &= \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \cdot S + ntS/d \\
&= \left( 1 + \left( -\frac{2}{n-2} + \frac{n}{d} \right) t - 2\epsilon \right) \cdot S \\
&\leq \left( 1 + \left( -\frac{2}{n-2} + \frac{n}{n-2} \right) t - 2\epsilon \right) \cdot S \\
&= (1 + t - 2\epsilon) \cdot S,
\end{align*}
(B.1)
so bin \( i \) is still not almost-full. Assumption (1) implies that \( b_1, b_2 \) must still be almost-full. So we are in the same situation and can repeat the argument until the lower bound is satisfied.

The upper bound on \( b_i \) is proved by simply subtracting the lower bounds of the other \( n-1 \) bins from the sum of all items, \( nS \):
\begin{align*}
b_i &\leq nS - (b_1 + b_2) - (n-3) \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \cdot S \\
&< nS - 2((1 + t - 2\epsilon) \cdot S) - (n-3) \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \cdot S \\
&= \left( n - 2(1 + t - 2\epsilon) - (n-3) \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \right) \cdot S \\
&= \left( 1 - \frac{2}{n-2} t + (n-1)2\epsilon \right) \cdot S,
\end{align*}
(B.2)
so property (b) is satisfied.

A corollary of property (b) is that the difference between the sums of two not-almost-full bins is at most \( 2n\epsilon S = nSt/2m^2 \).

Next, we transform the partition such that the almost-full bins 1 and 2 contain only "big" items, which we defined as items larger than \( nS \left( \frac{t}{n-2} - 2\epsilon \right) \).

**Proof of Lemma 3.7, property (c).** Suppose bin 1 or 2 contains an item of size at most \( nS \left( \frac{t}{n-2} - 2\epsilon \right) \). We move it to some not-almost-full bin \( b_i, i \geq 3 \). By property (b), the new sum of bin \( i \) satisfies
\begin{align*}
b_i &\leq \left( 1 - \frac{2}{n-2} t + (n-1)2\epsilon \right) \cdot S + nS \left( \frac{t}{n-2} - 2\epsilon \right) \\
&= \left( 1 + \left( -\frac{2}{n-2} + \frac{n}{n-2} \right) t + (n-1-n)2\epsilon \right) \cdot S \\
&= (1 + t - 2\epsilon)S,
\end{align*}
so bin \( i \) remains not-almost-full. Assumption (1) implies that \( b_1, b_2 \) remain almost-full. So properties (a), (b) still hold, and we can repeat the argument. Each move increases the sum in the not-almost-full bins, so eventually all items of size at most \( nS \left( \frac{t}{n-2} - 2\epsilon \right) \) are in bins 3, \ldots, \( n \).

Next, we transform the partition so that the not-almost-full bins contain only the largest big-items and all the small-items.

**Proof of Lemma 3.7, property (d).** Suppose there exists some \( i \in \{3, \ldots, n\} \) for which \( b_i \) contains an item \( x \) such that \( x \geq 2nS\epsilon \), but \( x < y \) for some item \( y \) in bin 1 or 2. We exchange \( x \)
and \( y \). The sum in bin \( i \) increases, but not too much:

\[
b_i \leq \left( 1 - \frac{2}{n-2} t + (n-1)2\epsilon \right) \cdot S + nM - 2nS\epsilon
\]

\[
\leq \left( 1 - \frac{2}{n-2} t + (n-1)2\epsilon \right) \cdot S + \frac{ntS}{d} - 2nS\epsilon \quad \text{(since } dM = tS) \]

\[
\leq \left( 1 - \frac{2}{n-2} t + (n-1)2\epsilon \right) \cdot S + \frac{ntS}{n-2} - 2nS\epsilon \quad \text{(since } d \leq n-2) \]

\[
\leq (1 + t - 2\epsilon) \cdot S,
\]

so \( b_i \) is still not almost-full. So by (1), bins 1 and 2 must remain almost-full, and we can repeat the argument. Each move increases the sum in bins \( 3, \ldots, n \), so the process must end. \[\square\]

Properties (c) and (d) imply that the input list \( \mathcal{X} \) contains no medium items; for completeness and verification, we provide a stand-alone proof below.

**Proof of Lemma 3.7, property (e).** Assume that there is an allocation in which bins 1 and 2 are almost-full, and bins \( 3, \ldots, n \) are not-almost-full. Suppose for contradiction that \( \mathcal{X} \) contains a medium item \( x \). Consider two cases.

**Case 1:** \( x \) is in an almost-full bin, say bin 1. Then, we can move \( x \) to some not-almost-full bin \( i \geq 3 \). By the inequalities in the proof of (c), after the move we have \( b_i < (1 + t - 2\epsilon)S \), so bin \( i \) remains not-almost-full. Moreover, since the previous sum of bin 1 was at most \( (1 + t)S \), its new sum is: \( b_1 < (1 + t)S - (2nS\epsilon) = (1 + t - 2n\epsilon)S < (1 + t - 2\epsilon)S \), since \( n > 1 \). So bin 1 is not almost-full anymore. We now have a new allocation with only one almost-full bin (bin 2). Similarly, if \( x \) is in bin 2, we can move it to another bin, and get an allocation with only bin 1 almost-full. This contradicts the lemma assumption (1).

**Case 2:** \( x \) is in some not-almost-full bin \( i \geq 3 \). We exchange \( x \) with some big item \( y \) in bin 1. By the inequalities in the proof of (d),

\[
b_i < (1 + t - 2\epsilon) \cdot S,
\]

so bin \( i \) remains not-almost-full. The sum in bin 1 decreases due to the exchange. If bin 1 becomes not almost-full, then we are done: we have a new allocation with only bin 2 almost-full. If bin 1 remains almost-full, then we have a situation as in Case 1; by moving \( x \) to some bin \( i \geq 3 \), we get a new allocation with only bin 2 almost-full. Again, this contradicts (1). \[\square\]

**Proof of Lemma 3.7, property (f).** We first claim that, for every two big-items, \( x, y \), the difference \( |y - x| < 2nS\epsilon \). Assume w.l.o.g. that \( y > x \). By definition, \( nM \) is the largest item size, so \( y \leq nM \). Since \( x \) is a big-item, \( x > nS \left(\frac{t}{n-2} - 2\epsilon\right) \). So

\[
y - x < nM - nS \left(\frac{t}{n-2} - 2\epsilon\right)
\]

\[
= \frac{ntS}{d} - \frac{ntS}{n-2} + 2nS\epsilon \quad \text{(since } dM = tS) \]

\[
\leq 2S\epsilon n. \quad \text{(since } d \geq n-2) \]

Now, assume for contradiction that two not-almost-full bins contain a different number of big-items; say bin \( j \) has at least one big-item more than bin \( i \). We show that even if all the big-items in bin \( j \) are smaller than all big-items in bin \( i \), and all the (at most \( m \)) small-items are in bin \( i \),
the difference between them satisfies

\[ b_j - b_i > nS \left( \frac{t}{n-2} - 2\epsilon \right) - m \cdot 2n\epsilon S - m \cdot 2n\epsilon S \]
\[ = nS \left( \frac{t}{n-2} - 2\epsilon \right) - 2m \cdot 2n\epsilon S \]
\[ = nS \left( \frac{t}{n-2} - 2\epsilon(1 + 2m) \right) \]
\[ \geq nS \left( \frac{t}{m-2} - \frac{t}{m} \right) \quad \text{(since } \epsilon = t/4m^2) \]
\[ = nS \left( \frac{2t}{(m-2)m} \right) \]
\[ > 2nSt/m^2. \]

But by property (b), the difference is at most \( 2nS\epsilon = nSt/2m^2 \) — a contradiction. \( \square \)

Proof of Lemma 3.7, property (g). The size of every item is at most \( nM \leq ntS/(n-2) \). By (f), each not-almost-full bin contains \( \ell \) big-items. So by the previous conditions:

\[
B.3 \quad b_i \leq \ell \cdot ntS/(n-2) + m \cdot 2n\epsilon S
\]

We focus on bin 1; the proof for bin 2 is the same. If bin 1 contains fewer than \( \ell + 1 \) items, then \( b_1 \leq b_i \), which contradicts (1). Assume for contradiction that bin 1 contains more than \( \ell + 1 \) items. Then:

\[
b_1 \geq (\ell + 2) \cdot nS \left( \frac{t}{n-2} - 2\epsilon \right) \quad \text{(by big-item size)}
\]
\[
= \ell nSt/(n-2) + m \cdot 2n\epsilon S - m \cdot 2n\epsilon S - \ell nS2\epsilon + 2nSt/(n-2) - 4nS\epsilon
\]
\[
\geq b_1 - m \cdot 2n\epsilon S - \ell nS2\epsilon + 2nSt/(n-2) - 4nS\epsilon \quad \text{(by B.3)}
\]
\[
\geq \left( 1 - \frac{2}{n-2} t - 2\epsilon \right) \cdot S - m \cdot 2n\epsilon S - \ell nS2\epsilon + 2nSt/(n-2) - 4nS\epsilon \quad \text{(by B.1)}
\]
\[
= S + \left( \frac{2n - 2}{n-2} \right) tS - m \cdot 2n\epsilon S - \ell nS2\epsilon - 4nS\epsilon - 2\epsilon S
\]
\[
= S + tS + tS((n/n-2) - m \cdot 2n\epsilon/t - \ell n2\epsilon/t - 4n\epsilon/t - 2\epsilon/t)
\]
\[
> S + tS + tS(1 - 1/2 - \ell/2m - 1/m - 1/2nm) \quad \text{(since } \epsilon < t/4nm)\]
\[
> S + tS, \quad \text{(since } 0 < \ell < m/3)\]

which contradicts the assumption that \( (b_1, b_2, \ldots, b_n) \) represent a \( t \)-feasible partition. \( \square \)

Appendix C. Proof of Theorem 1.3: Hardness of Partitioning with Splittings.

Proof. Given a partition with \( n \) bins, \( m \) items and \( s \) splittings, summing the size of each element (or fraction of element) in each bin allows us to check whether or not the partition is equal in linear time. So, the problem is in \( \text{NP} \).

To prove that \( n \)-Times-Split-Perfect is \( \text{NP} \)-Hard, we apply a reduction from the Subset Sum problem. Given an instance \( X_1 \) of the subset sum problem with \( m \) items summing up to \( S \) with target sum \( T < S \), we build an instance \( X_2 \) of the problem \( n \)-Times-Split-Perfect, by adding two items, \( x_1, x_2 \), such that \( x_1 = S + T \) and \( x_2 = 2S(s+1) - T \) and \( n - 2 - s \) auxiliary
items of size $2S$. Notice that the sum of the items in $\mathcal{X}_2$ equals
\[
S + (S + T) + 2S(s + 1) - T + 2S(n - 2 - s) \\
= 2S + 2S(s + 1) + 2S(n - 2 - s) \\
= 2S \cdot (1 + s + 1 + n - 2 - s) \\
= 2Sn.
\]
The goal is to partition the items into $n$ bins with a sum of $2S$ per bin, and at most $s$ splittings. First, assume that there is a subset of items $W_1$ in $\mathcal{X}_1$ with sum equal to $T$. Define a set, $W_2$, of items that contains all items in $\mathcal{X}_1$ that are not in $W_1$, plus $x_1$. The sum of $W_2$ is $(S - T) + x_1 = S + T + S - T = 2S$. Assign the items of $W_2$ to the first bin. Assign each auxiliary item to a different bin. There are $n - (n - 2 - s + 1) = s + 1$ bins left. The sum of the remaining items is $2S(s + 1)$. As explained in the introduction, they can be partitioned into $s + 1$ bins of equal sum $2S$, with at most $s$ splittings. All in all, there are $n$ bins with a sum of $2S$ per bin, and the total number of splittings is $s$.

Second, assume that there exists an equal partition for $n$ bins with $s$ splittings. Since $x_2 = 2S(s + 1) - T = 2S \cdot s + (2S - T) > 2S \cdot s$, this item must be split between $s + 1$ bins, which makes the total number of splittings at least $s$. Also, the auxiliary items must be assigned without splittings into $n - 2 - s$ different bins. There is $n - s - 1 - n + 2 + s = 1$ bin remaining, say bin $i$, containing only whole items, not containing any part of $x_2$, and not containing any auxiliary item. Bin $i$ must contain $x_1$, otherwise its sum is at most $S$ (sum of items in $\mathcal{X}_1$). Let $W_1$ be the items of $\mathcal{X}_1$ that are not in bin $i$. The sum of $W_1$ is $S - (2S - x_1) = x_1 - S = T$, so it is a solution to $\mathcal{X}_1$.

**Appendix D. $n$–Split–MinMax.** We first prove a property of the optimal bin sum of the $n$–Split–MinMax$(\mathcal{X}, s)$ problem.

**Lemma D.1.** If the optimal bin sum value of the $n$–Split–MinMax$(\mathcal{X}, s)$ problem is not an integer, then the $n$–way partition of $\mathcal{X}$ is a perfect partition (that is, all bin sums are equal to the average bin size $S$).

**Proof.** Assume that the optimal bin sum value of $n$–Split–MinMax$(\mathcal{X}, s)$ is not an integer. The fractional part of the bin sum must come from a split item (since the non-split items are integers). If the partition is not perfect, then there is a bin with a smaller sum. Move a small fraction of the split item to a bin with a smaller sum. This yields a partition with a smaller maximum bin sum — a contradiction to optimality.

Now, we show how to use binary search in order to solve the $n$–Split–MinMax$(\mathcal{X}, s)$ problem. The idea is to execute the $n$–Split–Bound$(\mathcal{X}, s, t)$ algorithm several times, trimming the parameter $t$ at each execution in a binary search style. When $t = n - 1$, the answer to $n$–Split–Bound$(\mathcal{X}, s, t)$ is always “yes”, since even if we put all items in a single bin, its sum is $nS = (1 + t)S$. Moreover, by Lemma D.1, the optimal bin sum value of $n$–Split–MinMax$(\mathcal{X}, s)$ is either an integer or equal to $S$. Therefore, in addition to considering the case where $t = 0$, we only need to consider values of $t$ that differ by at least $1/S$, since if the bin sums are not equal to $S$ they are always integers, so they differ by at least 1. Therefore, at most only $nS$ different values of $t$ have to be checked, so the binary search requires at most $\log_2(nS)$ executions of $n$–Split–Bound$(\mathcal{X}, s, t)$. This is polynomial in the size of the binary representation of the input.

**Appendix E. A greedy algorithm for $n$–Interval–t$(\mathcal{X}, t)$ when $t \geq 1$.**

When $t \geq 1$, each of the $n$ bins has at least $S + tS \geq 2S$ space. Because the total size of the items is $nS$, there are at most $n$ items with size $S$ or greater. We can put each of them into its own bin (obviously, there is no solution if some item does not fit into a bin by itself).

Now, only the items with size less than $S$ are left. We can greedily put them into the bins one-by-one. Indeed, there is a total of $tS \cdot n \geq nS$ extra space in the bins (when compared to the
total size of all items). Therefore, at each moment, at least one bin always has $S$ or more free space available, which is enough for an item of size less than $S$. Therefore, when $t \geq 1$, the answer to $n-$Interval$-t(X,t)$ is “yes” if all items sizes are at most $(1 + t)S$, and “no” otherwise. This can be decided in time $O(m)$.

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