Operator algebra as an application of logarithmic representation of infinitesimal generators

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Abstract. The operator algebra is introduced based on the framework of logarithmic representation of infinitesimal generators. In conclusion a set of generally-unbounded infinitesimal generators is characterized as a module over the Banach algebra.

1. Introduction

The logarithmic representation of infinitesimal generator is introduced in Ref. [1]. Its role is further discussed in Ref. [2]. Let $X$ be a Banach space, and $Y$ be a dense Banach subspace of $X$. Let $\{U(t,s)\}_{-T \leq t, s \leq T}$ be an evolution family of operators satisfying the semigroup property (for example, see [3]):

\[ U(t,r)U(r,s) = U(t,s), \]
\[ U(s,s) = U(s,t)U(t,s) = I \]
on $X$. Let the infinitesimal generator of $U(t,s)$ be denoted by $A(t)$ from $Y$ to $X$. The logarithmic representation for the infinitesimal generator is

\[ A(t) u_s = (I + \kappa U(s,t)) \frac{\partial_t}{t} \log (U(t,s) + \kappa I) u_s, \]

where $u_s$ is an element in $Y$, $\kappa \neq 0$ is a certain complex number, and $\log$ denotes a principal branch of the logarithm. $\partial_t$ stands for a kind of weak differential defined in Ref. [1]. For this definition, operators $U(t,s)$ and $A(t)$ are assumed to commute [1]. In particular the representation plays a role of extracting the bounded part of the infinitesimal generator. In this article a formulation of operator algebra is established for generally-unbounded infinitesimal generators.

2. Basic relation

The relation between exponential and logarithm functions of operators is shown under the validity of logarithmic representation (1). According to Eq. (1) the logarithm function in this case is given by $\log (U(t,s) + \kappa I)$, and is formally or expectantly equal to the indefinite integral

\[ \log (U(t,s) + \kappa I) = \int (I + \kappa U(s,t))^{-1} A(t) \, dt. \]
Its exponential function can be calculated as
\[
\exp[\log (U(t, s) + \kappa I)] = U(t, s) + \kappa I
\]  
\[\text{(2)}\]

\[\text{Log exp[Log } (U(t, s) + \kappa I)\text{]} = \text{Log}(U(t, s) + \kappa I)\]
is always satisfied, but the above formal equality \[\int (I + \kappa U(s, t))^{-1} A(t) \, dt = \text{Log } (U(t, s) + \kappa I)\]
is not necessarily satisfied because the limited range of imaginary spectral distribution is true only for the right hand side. In this sense \[\text{Log } (U(t, s) + \kappa I)\]
corresponds to the extracted bounded part of the infinitesimal generator \[A(t)\].

A set of these operators is a candidate for bounded-operator algebra. Indeed, if \(\kappa\) is taken from the resolvent set of operator \(U(t, s), U(t, s) + \kappa I\) and \((U(t, s) + \kappa I)^{-1}\) are necessarily bounded on \(X\), and then the boundedness of \[\text{Log } (U(t, s) + \kappa I)\]
follows. That is, \[\text{Log } (U(t, s) + \kappa I)\] and \[U(t, s) + \kappa I\] are related by the exponential function of operators which can be defined by the convergent power series. Meanwhile the logarithm function \[\text{Log } (U(t, s) + \kappa I)\]
cannot necessarily be defined by the convergent power series which is too restrictive to represent the function of \[U(t, s)\] (i.e., \(|U(t, s) - 1| < 1\) is required).

3. Main results
Let \[\text{Log}(U(t, s) + \kappa I)\] be denoted by
\[
a(t, s) = \text{Log } (U(t, s) + \kappa I),
\]  
\[\text{(3)}\]
where some algebraic properties of \(a(t, s)\) can be found in Ref. [2]. Since \(a(t, s)\) is bounded on \(X\), \(e^{a(t, s)}\) is well-defined by the convergent power series. Consequently the exponentiability is reduced to \[\text{Log } (U(t, s) + \kappa I)\]. Using Eq. (2),
\[
U(t, s) = e^{a(t, s)} - \kappa I,
\]
follows for a certain complex number \(\kappa\). It is understood by this equation that, with respect to the mathematical property of infinitesimal generators represented by Eq. (1), it is sufficient to consider \(a(t, s)\) instead of \(A(t)\).

**Theorem 1.** Let \(U_i(t, s)\) be evolution operators satisfying Eq. (1), and \[\text{Log } U_i(t, s)\] be well-defined for any \(t, s \in [-T, T]\) and \(i = 1, 2, \ldots, n\). \[\text{Log } U_i(t, s)\] are assumed to commute with each other.

\[V_{Lg}(X) := \{k \text{Log } U_i(t, s); \ k \in \mathbb{C}, \ t, s \in [-T, T]\} \subset B(X)\]
is a normed vector space over the complex number field, where \(B(X)\) denotes a set of all the bounded operators on \(X\), and the operator norm is equipped with \(B(X)\).

**Theorem 2.** Let \(U_i(t, s)\) be evolution operators satisfying Eq. (1) for any \(t, s \in [-T, T]\) and \(i = 1, 2, \ldots, n\). For a certain \(K \in B(X)\), let a subset of \(B(X)\) in which each element is assumed to commute with \[\text{Log } (U_i(t, s) + K)\] be \(B_{ab}(X)\). \[\text{Log } (U_i(t, s) + K)\] are assumed to commute with each other.

\[B_{Lg}(X) := \{K \text{Log } (U_i(t, s) + K); \ K \in B_{ab}(X), \ K \in B(X), \ t, s \in [-T, T]\} \subset B(X)\]
is a module over the Banach algebra \(B(X)\).
4. Outlines of the proofs

4.1. A normed vector space

The proof of Theorem 1 is presented. In case of $\kappa = 0$ the operator $a(t, s)$ is reduced to

$$\Log U(t, s) \in B(X).$$

The operator sum is calculated using Dunford-Riesz integral

$$\Log U(t, r) + \Log U(r, s)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \Log \lambda (\lambda - U(t, r))^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma'} \Log \lambda' (\lambda' - U(r, s))^{-1} d\lambda'$$

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\Log \lambda + \Log \lambda') (\lambda - U(t, r))^{-1} (\lambda' - U(r, s))^{-1} d\lambda' d\lambda$$

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\Log \lambda' (\lambda - U(t, r))^{-1} (\lambda' - U(r, s))^{-1} d\lambda' d\lambda$$

$$= \Log [U(t, r)U(r, s)] = \Log U(t, s),$$

then the sum closedness is clear. Here $\Gamma'$ is assumed to be included in $\Gamma$, and this condition is not so restrictive in the present setting. In a different situation, when $U(t, r)$ and $V(t, r)$ commute for the same $t$ and $r$, another kind of sum is calculated as

$$\Log U_1(t, r) + \Log U_2(t, r)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \Log \lambda (\lambda - U_1(t, r))^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma'} \Log \lambda' (\lambda' - U_2(t, r))^{-1} d\lambda'$$

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\Log \lambda + \Log \lambda') (\lambda - U_1(t, r))^{-1} (\lambda' - U_2(t, r))^{-1} d\lambda' d\lambda$$

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\Log \lambda' (\lambda - U_1(t, r))^{-1} (\lambda' - U_2(t, r))^{-1} d\lambda' d\lambda$$

$$= \Log [U_1(t, r)U_2(t, r)],$$

where, for $W(t, r) = U_1(t, r)U_2(t, r)$, the semigroup property is satisfied as

$$W(t, r)W(r, s) = U_1(t, r)U_2(t, r)U_1(r, s)U_2(r, s) = U_1(t, r)U_1(r, s)U_2(t, r)U_2(r, s) = W(t, s),$$

$$W(s, s) = W(s, t)W(t, s) = I,$$

and then the sum closedness is clear. Although the logarithm function is inherently a multi-valued function, the uniqueness of sum operation is ensured by the single-valued property of the principal branch “Log”. Consequently, since the closedness for scalar product is obvious,

$$V_{\Log}(X) = \{k\Log U(t, s); \ k \in \mathbb{C}, \ t, s \in [-T, T]\} \subset B(X)$$

is a normed vector space over the complex number field. In particular the zero operator $\Log I$ is included in $V_{\Log}(X)$. Theorem 1 has been proved.

4.2. $B(X)$-module

The proof of Theorem 2 is presented. It is worth generalizing the above normed vector space. In this sense, utilizing a common operator $K \in B(X)$, components are changed to $\Log (U_1(t, r) + K)$.
The operator sum is calculated as

\[
\log (U(t, r) + K) + \log (U(r, s) + K)
\]

\[
\begin{align*}
&= \frac{1}{2\pi i} \int_{\Gamma} \log \lambda (\lambda - U(t, r) - K)^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma'} \log \lambda' (\lambda' - U(r, s) - K)^{-1} d\lambda' \\
&= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\log \lambda + \log \lambda') (\lambda - U(t, r) - K)^{-1} (\lambda' - U(r, s) - K)^{-1} d\lambda' d\lambda
\end{align*}
\]

(6)

After introducing a certain \(K\) with sufficiently large \(|K|\), it is always possible to take integral path \(\Gamma'\) to be included in \(\Gamma\). Since the part \(\text{"}KU(t, r) + KU(r, s) + K^2\text{"}\) is included in \(B(X)\), the sum-closedness is clear. In a different situation, when \(U_1(t, r)\) and \(U_2(t, r)\) commute for the same \(t\) and \(r\), another kind of sum is calculated as

\[
\log (U_1(t, r) + K) + \log (U_2(t, r) + K)
\]

\[
\begin{align*}
&= \frac{1}{2\pi i} \int_{\Gamma} \log \lambda (\lambda - U_1(t, r) - K)^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma'} \log \lambda' (\lambda' - U_2(t, r) - K)^{-1} d\lambda' \\
&= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} (\log \lambda + \log \lambda') (\lambda - U_1(t, r) - K)^{-1} (\lambda' - U_2(t, r) - K)^{-1} d\lambda' d\lambda
\end{align*}
\]

(7)

Since the part \(\text{"}KU_1(t, r) + KU_2(t, r) + K^2\text{"}\) is included in \(B(X)\), the sum-closedness is clear.

The product \(K\log (U_1(t, s) + K2) \in B(X)\) is justified by the operator product equipped with \(B(X)\). Consequently, since the closedness for operator product within \(B(X)\) is obvious,

\[
B_{Lg}(X) = \{K\log (U(t, s) + K); \ K \in B_{ab}(X), \ K \in B(X), \ t, s \in [-T, T]\}
\]

is a module over the Banach algebra \((B(X)\text{-module})\). In particular relation \(V_{Lg}(X) \subset B_{Lg}(X)\) is satisfied. Theorem 2 has been proved.

For the structure of \(B_{Lg}(X)\), a certain originally unbounded part can be classified to \(\log (U(t, s) + K) \in B(X)\), and the rest part to \(K \in B_{ab}(X)\). Here the terminology “originally unbounded” is used, because some unbounded operators are reduced to bounded operators under the validity of the logarithmic representation.

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