Scaling topological charge in the \( \text{CP}^3 \) model using a fixed point action

Rudolf Burkhalter \(^a\)

\(^a\)Institute for Theoretical Physics, University of Berne, Sidlerstrasse 5, CH-3012 Berne, Switzerland

We define a fixed point action in two-dimensional lattice \( \text{CP}^{N-1} \) models. The fixed point action is a classical perfect lattice action, which is expected to show strongly reduced cut-off effects in numerical simulations. Furthermore, the action has scale invariant instanton solutions, which enables us to define a topological charge without topological defects. We present results for the scaling of the topological susceptibility from a Monte Carlo simulation in the \( \text{CP}^3 \) model.

1. Introduction

The use of fixed point (FP) actions for asymptotically free theories has been shown to be profitable in reducing lattice artefacts in numerical simulations \([1,2]\). Furthermore, it has been shown how to define a FP topological charge in order to study topological properties \([3]\). However, a determination of the topological susceptibility in the O(3) nonlinear \( \sigma \)-model did not show a scaling behavior. The result suggests, not quite unexpectedly \([4]\), that the topological susceptibility is not a physical quantity in this model. In the \( \text{CP}^3 \) model one expects the topological susceptibility to be meaningful. However, recent determinations gave contradictory results \([5–9]\).

In order to overcome these problems, we discuss the definition of a fixed point action and a fixed point topological charge for \( \text{CP}^{N-1} \) models. We present results of numerical simulations in the \( \text{CP}^3 \) model. A more detailed discussion of the definitions and the numerical results can be found in Ref. \([10]\).

2. Fixed point action

Two-dimensional \( \text{CP}^{N-1} \) models are a class of models which in many respects are similar to QCD. They consist of \( N \)-component, complex spin fields \( z^i(x) \) of unit length. The FP action is defined as the FP of an exact renormalization group (RG) transformation. The RG transformation relates the spins \( z_n \) sitting at the lattice sites of a \( 2 \times 2 \) block \( n_B \) of a fine lattice with a block spin \( \zeta_{n_B} \) sitting on a coarse lattice. The FP action is determined by the FP equation

\[
A_{\text{FP}}(\zeta) = \min_{\{z\}} \left\{ A_{\text{FP}}(z) + T(\zeta, z) \right\},
\]

with the transformation kernel

\[
T(\zeta, z) = \kappa \sum_{n_B} \left( \lambda_{n_B} - \sum_{n \in n_B} |\zeta_{n_B} z_n|^2 \right). \tag{2}
\]

Here \( \kappa \) is a free parameter that is used to optimize the corresponding FP action and \( \lambda_{n_B} \) is the largest eigenvalue of the matrix

\[
M_{n_B} = \sum_{n \in n_B} z_n \otimes \bar{z}_n. \tag{3}
\]

One can solve the FP equation iteratively for any given input configuration \( \{\zeta\} \). In doing this, we may use a multigrid with \( \{\zeta\} \) on the coarsest level. The configurations on the finer levels are varied until the minimum is reached which yields the value of the the FP action for \( \{\zeta\} \). On the finest level the field is very smooth, so we can use any discretization of the continuum action, e.g. the standard action.

If one wants to use the FP action in a numerical simulation one has to use a parametrization. Our parametrization has the form

\[
A_{\text{FP}}^{\text{par}}(z) = -\frac{1}{2} \sum_{n,r} \rho(r) \theta^2_{n,n+r} \tag{4}
\]

\[
+ \sum_{n_i,n_j,...} \text{coupling} \times \text{products of } \theta^2_{n_i,n_j},
\]

where \( \rho(r) \) is a function of the distance between the spins on the lattice. In order to optimize the FP action, we use a parametrization that is invariant under lattice translations. The FP action is chosen to be invariant under lattice translations, rotations, and reflections. The FP action is determined by the FP equation

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where \( \theta_{n,n_j} = \arccos(|\vec{z}_{n,n_j}|) \) is the angle between two spins. We determined a parametrization for the CP\(^3\) model with 32 couplings (2 analytically calculated quadratic couplings \( \rho \) and 30 numerically determined higher order couplings which involve 2-, 3- and 4-spin interactions) which deviates only negligibly from the (minimized) FP action even for rough configurations \[10\].

3. Fixed point topological charge

In Ref. \[3\] it was shown, that the FP action admits scale invariant instanton solutions. In particular, it can be shown, that if \( \{\zeta\} \) is an instanton configuration of size \( \rho \) (in units of the lattice spacing) and hence its action has a value of 2\(\pi\), then the minimizing configuration \( \{z(\zeta)\} \) is an instanton configuration as well. Its size is \( 2\rho \) and the value of its action is also 2\(\pi\).

This property leads us to the definition of a FP topological charge \[10\]. It is obtained in the limit of infinitely many RG transformations:

\[
Q_{FP}(\zeta) = \lim_{k \to \infty} Q(z^{(k)}(\zeta)), \tag{5}
\]

where \( \{z^{(k)}\} \) is the solution of the iterated FP equation on the lowest level in a \( k \)-level multigrid. As the configurations get smoother on each successive level, one may choose for the charge \( Q \) on the lowest level any lattice discretization of the topological charge. In this work we used the geometric definition of the topological charge. The combination FP action and FP topological charge obeys for any configuration the inequality \( A_{FP} \geq 2\pi |Q_{FP}| \), i.e. there are no topological defects.

This property is well illustrated in Fig. 1. Here we plot for the CP\(^3\) model the topological charges and the actions for 2-instanton solutions on a torus.

To calculate the FP charge in numerical simulations, it is very time consuming to minimize a multigrid for every configuration. One needs a parametrization of the solution \( \{z(\zeta)\} \) of the FP equation. We construct a gauge invariant parametrization by calculating the matrix

\[
M_n = \sum_{n_B} \alpha(n, n_B) \zeta_{n_B} \otimes \bar{\zeta}_{n_B} \tag{6}
\]

and define the fine field variable \( z_n \) as the eigenvector of \( M_n \) with largest eigenvalue. In higher order terms enters the angle \( \theta_{n,n_B}^2 \) between the coarse spins at sites \( m_B \) and \( m_B' \), respectively. Similar to the FP action we can determine for the CP\(^3\) model the coefficients \( \alpha \) and \( \beta \) analytically and by a fitting procedure \[10\].

4. Numerical results

We performed MC simulations in the CP\(^3\) model using the parametrized FP action with a hybrid overrelaxation algorithm. We determined the FP charge by calculating the charge on the first finer level which was obtained by either minimizing Eq. \( (4) \) or by using the parametrization \( (6) \). We keep the volume \( L/\xi \approx 5.5 - 6 \) approximately constant. At this value we still observe finite size effects but the constant volume allows to look for a scaling behavior.

The mass gap shows an unexpected asymptotic scaling behavior even at quite small correlation lengths (Fig. \[3\]). Furthermore the ratio \( m/\Lambda_L^{(2)} = 8.1(1) \) (obtained in “infinite” volume) is remarkably small.
For the topological susceptibility

\[ \chi_t = \frac{\langle Q^2 \rangle}{V} \]

we observe a scaling behavior. In Fig. 2 we show the results for the dimensionless quantity \( \chi_t \xi^2 \). One sees a raise at small correlation lengths, which is due to lost small instantons in this region. At correlation length \( \xi \approx 10 \) this effect is already saturated and a scaling plateau is reached. In “infinite” volume we obtain the value \( \chi_t \cdot \xi^2 = 0.070(2) \).

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