THE UNIFORM ROE ALGEBRA OF AN INVERSE SEMIGROUP

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Abstract. Given a discrete and countable inverse semigroup $S$ one can study, in analogy to the group case, its geometric aspects. In particular, we can equip $S$ with a natural metric, given by the path metric in the disjoint union of its Schützenberger graphs. This graph, which we denote by $Λ_S$, inherits much of the structure of $S$. In this article we compare the C*-algebra $R_S$ generated by the left regular representation of $S$ on $ℓ^2(S)$ and $ℓ^∞(S)$, with the uniform Roe algebra over the metric space, namely $C^*_u(Λ_S)$. This yields a characterization of when $R_S = C^*_u(Λ_S)$, which generalizes finite generation of $S$.

The graph $Λ_S$, and the FL condition above, also allow to analyze large scale properties of $Λ_S$ and relate them with C*-properties of the uniform Roe algebra. In particular, we show that domain measurability of $S$ (a notion generalizing Day’s definition of amenability of a semigroup, cf., \cite{5}) is a quasi-isometric invariant of $Λ_S$. Moreover, we characterize property A of $Λ_S$ (or of its components) in terms of the nuclearity and exactness of the corresponding C*-algebras. We also treat the special classes of F-inverse and E-unitary inverse semigroups from this large scale point of view.

Dedicated to Pere Ara on the occasion of his 60\textsuperscript{th} birthday.

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1. Introduction

Given a discrete and finitely generated group $G$ there are several C*-algebras naturally associated to it. Among them, one of the most relevant one is the so-called uniform Roe algebra of $G$, which can be constructed in, at least, the following ways (see, e.g., \cite{33, 5}):

- Let $λ: G \to B(ℓ^2(G))$ be the left regular representation of $G$ and consider $ℓ^∞(G) \subset B(ℓ^2(G))$ as diagonal operators. The algebra $R_G$ is then the C*-algebra generated by these operators, i.e.,
$$R_G := C^*(ℓ^∞(G) \cdot \{λ_g \mid g \in G\}) \subset B(ℓ^2(G)).$$

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Similarly, we can consider the natural action of $G$ on $\ell^\infty(G)$ given by left translation of the argument and construct the reduced crossed product $\ell^\infty(G) \rtimes_r G \subset B(\ell^2(G))$.

- Given a finite, symmetric and generating set $K \subset G$, one may consider the (left) Cayley graph of $G$ with respect to $K$, which we denote by $\Lambda(G, K)$. Then $\Lambda(G, K)$ with the path distance is a metric space of bounded geometry (see Section 1.2) and, thus, one can construct the corresponding uniform Roe algebra $C^*_u(\Lambda(G, K))$, i.e., the $C^*$-algebra generated by finite propagation operators in $B(\ell^2(G))$ (see Section 3). This $C^*$-algebra is, in particular, generated by partial isometries representing partial bijections of the metric space $t: A \to B$, where $A, B \subset G$ and

$$
\sup \{ d(a, t(a)) \mid a \in A \} < \infty.
$$

The following well known theorem states that the approaches above generate the same $C^*$-algebra (cf., [8 Proposition 5.1.3]).

**Theorem 1.1.** Let $G$ be a countable and discrete group $G$. Then

$$\ell^\infty(G) \rtimes_r G = R_G = C^*_u(\Lambda(G, K)).$$

Each way to generate the uniform Roe algebra of a group has its advantages. For instance, using the approximations provided by $R_G$, where elements can be approximated by linear combinations of terms of the form $f \lambda_g$, where $g \in G$, $f \in \ell^\infty(G)$, is convenient to study the trace space of the algebra (see, e.g., [2, 3]). On the other hand, crossed products are instead useful to analyze structural aspects of the algebra like, e.g., the relation between the nuclearity of the $C^*$-algebra and the amenability of the action (cf., [8 Theorem 4.3.4]). Finally, the $C^*$-algebra $C^*_u(\Lambda(G, K))$ captures the large-scale geometry of $G$. Moreover, the preceding theorem shows that every partial translation $t: A \to B$ as in Eq. (1.1) can also be represented as a combination of left multiplication by elements of the group $G$. This proves to be helpful when studying the property A of $G$ (see [8, 28, 33] or Section 1.2).

In the present article we will generalize Theorem 1.1 to the context of a discrete and countable inverse semigroup $S$. In particular, we will associate with $S$ a graph $\Lambda_S$, endowed with the path length metric, and focus on the relation between the $C^*$-algebras $R_S$ and $C^*_u(\Lambda_S)$. We also analyze large scale properties of $\Lambda_S$ and relate them with $C^*$-properties of the uniform Roe algebra.

Recall that a semigroup $S$ (which we assume to be countable and discrete) is a set equipped with a binary and associative operation. We say that $S$ is an inverse semigroup if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. We denote by $E(S) := \{ s^* s \mid s \in S \}$ the set of projections, which is a commutative inverse subsemigroup of $S$ (some standard references on this topic are [22, 24, 25]). It is well known that the dynamics in $S$ given by left multiplication are locally injective, meaning that for each $s \in S$ there is some $D_{s^*s} \subset S$ (the domain of $s$) such that if $sz = sy$ then $x = y$ for any $x, y \in D_{s^*s}$. This allows to analyze notions like amenability (introduced by Day for general semigroups in [3]) in a more detailed way. For example, it was introduced in [22] the more general notion of domain measurability, which precisely captures the dynamical invariance of the probability measure on $S$ (cf., Section 3). Moreover, inverse semigroups allow for a natural connection with $C^*$-algebras via a standard generalization of the left regular representation mentioned above. The representing operators in this case are partial isometries denoted by $V_s$, where $s \in S$. The $C^*$-algebra $R_S$ is, in analogy to the group case, the $C^*$-completion of the sets $\{ V_s \mid s \in S \}$ and $\ell^\infty(S)$. We refer to [1] [28, 32, 11, 12, 25] for additional references on the relation with groupoids and $C^*$-algebras and to [18, 13] for connections to coarse geometry.

To obtain a metric space associated with $S$ we have to consider the so-called (left) Schützenberger graphs of $S$. Two elements $x, y \in S$ are $L$-related if $x^* x = y^* y$, and, thus, the corresponding equivalence classes (denoted by $L$) contain exactly one projection. The semigroup therefore decomposes into the disjoint union

$$S = \bigsqcup_{e \in E(S)} L_e.$$ 

Assuming that $S$ is generated by a fixed symmetric set $K$, each component above can be given a graph structure (and hence a path length metric), where $x, y \in L_e$ are joined by an edge if $kkx = y$ for $k \in K$. The resulting graph is known as the (left) Schützenberger graph of $e \in S$, which we denote by $\Lambda_{L_e}$. Their disjoint union $\Lambda_S := \bigsqcup_{e} \Lambda_{L_e}$ is then an undirected graph that describes the geometry
of the semigroup. Observe that \(\Lambda_S\) can also be constructed by erasing all the directed edges in the usual left Cayley graph of \(S\) (see Section 3.1). This construction allows to construct the uniform Roe algebra \(C^u(S)\), which is an analogue of the uniform Roe algebra of a group. However, in the context of inverse semigroups, the C*-algebra \(\mathcal{R}_S\) need not coincide with \(C^u(S)\) (see Example 3.23). The problem arises as the semigroup \(S\) is naturally ordered, and the C*-algebra \(\mathcal{R}_S\) inherits that order, while the graph \(\Lambda_S\) does not. To circumvent this problem we have introduced the notion of a \textit{finitely labelable} inverse semigroup (see Definition 3.15) which generalizes the class of finitely generated inverse semigroups. In fact, this notion is not only required if one wants to generalize Theorem 1.1 but it turns out that it is also necessary (cf., Theorem 3.20):

**Theorem 1.** Let \(S\) be a countable and discrete inverse semigroup generated by a symmetric set \(K \subset S\). Then following statements are equivalent:

1. \(S\) is finitely labelable (see Definition 3.15).
2. The C*-algebras satisfy \(\ell^\infty(S) \cong S = C^u_u(\Lambda_S)\).

In this article we also address several \textit{large scale properties} of the inverse semigroup in relation with, in particular, the graph \(\Lambda_S\). For example, we show that domain measurability of \(S\) is a quasi-isometric invariant of \(\Lambda_S\) (cf., Theorem 4.2), generalizing the well known statement that amenability of groups is a quasi-isometric invariant.

**Theorem 2.** Let \(S\) and \(T\) be finitely labelable inverse semigroups and suppose \(S\) and \(T\) are quasi-isometric. If \(T\) is domain measurable then so is \(S\).

Finally, we characterize the property A of the components of \(\Lambda_S\) in terms of the nuclearity and exactness of the corresponding C*-algebras (see Theorem 4.16). Indeed, since the connected components of the graph \(\Lambda_S\) are the Schützenberger graphs of \(S\) (see [35, 36, 22] and Section 3 below), we have

**Theorem 3.** Let \(S\) be a countable and discrete inverse semigroup, and let \(L \subset S\) be an \(\mathcal{L}\)-class such that the left Schützenberger graph \(\Lambda_L\) is of bounded geometry. Let \(p_L\) be the orthogonal projection from \(\ell^2(S)\) onto \(\ell^2(L)\). Suppose, moreover, that the graph \(\Lambda_L\) is finitely labelable. Then the following are equivalent:

1. The graph \(\Lambda_L\) has Yu’s property A.
2. The C*-algebra \(p_L\mathcal{R}_S p_L\) is nuclear.
3. The C*-algebra \(p_L C^u_u(\Lambda_L) p_L\) is exact.

The article is structured as follows. Section 2 recalls definitions, notation and constructions in the context of inverse semigroups. Furthermore, in Section 2.1 we prove that, for inverse semigroups, \(\mathcal{R}_S = \ell^\infty(S) \cong S = C^u_u(\Lambda_S)\). Section 3 considers the more subtle metric space scenario, constructs the graph \(\Lambda_S\) and proves Theorem 1.1. Section 4 studies two quasi-isometric invariants of the graph \(\Lambda_S\), first domain measurability and then property A. In particular, Section 4.2 focuses on the left Schützenberger graphs of \(S\) and proves Theorem 4.3. As an application, we also relate the exactness of the reduced semigroup C*-algebra, the nuclearity of the uniform Roe algebra and the property A of the graph \(\Lambda_S\). The special classes of F-inverse and E-unitary inverse semigroups are also considered.

**Conventions:** Throughout the paper, \(S\) stands for a countable and discrete inverse semigroup (not necessarily unital and not necessarily finitely generated). We will denote by \(\ell^2(S)\) the complex Hilbert space of square-summable functions \(\psi: S \to \mathbb{C}\), and by \(B(\ell^2(S))\) the space of bounded operators on it. We canonically embed \(\ell^\infty(S)\) into \(B(\ell^2(S))\) as diagonal operators. The canonical orthonormal basis of \(\ell^2(S)\) will be denoted by \(\{\delta_x\}_{x \in S}\). The norm of an operator \(T \in B(\ell^2(S))\) is denoted by \(\|T\|\). Given two sets \(X_1, X_2\) we denote their disjoint union by \(X_1 \sqcup X_2\) and the cardinality of \(X_1\) by \(|X_1|\).

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2. INVERSE SEMIGROUPS AND C*-ALGEBRAS

In this section we introduce the definition of inverse semigroup as well as some important structures and examples that will be needed later. Some standard textbooks for additional motivation and proofs are [19, 20, 22, 32]. Given an inverse semigroup $S$ there is a C*-algebra $\mathcal{R}_S$ which naturally generalizes the uniform Roe algebra of a discrete countable group (cf., [8, Proposition 5.1.6]). For results relating the amenability of $S$ and C*-properties of $\mathcal{R}_S$ we refer to [5].

**Definition 2.1.** An inverse semigroup is a non-empty set $S$ equipped with an associative binary operation, such that for all $s \in S$ there is a unique $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$. Given an inverse semigroup $S$ there are some standard textbooks for additional motivation and proofs are [19, 20, 22, 32].

In this article $S$ will always denote a discrete and countable inverse semigroup. We say that $S$ is unital if there is an element $1 \in S$ such that $1s = s = s1$ for all $s \in S$. An element $e \in S$ is a projection, or idempotent, if $e = e^2$. Observe that this implies that $e$ is self-adjoint, i.e., $e = e^*$. The set of projections is denoted by $E(S)$. Note that $s^*s$ is a projection for all $s \in S$, and thus $E(S)$ is never empty. Furthermore, $E(S)$ is a commutative inverse sub-semigroup of $S$ (see [40, Theorem 3]). Any group is trivially an inverse semigroup with $s^* = s^{-1}$. Conversely, it is easy to see that $S$ is a group if and only if $E(S)$ has exactly one element, namely the identity of the group.

We say that $s, t \in S$ are $\sigma$-equivalent if there is some projection $e \in E(S)$ such that $se = te$. Note this is equivalent to the existence of some projection $f \in E(S)$ such that $fs = ft$ (take $f = ses^* = tet^* \in E(S)$). It is well known that $\sigma$ is a congruence in $S$ and we denote by $G(S) := S/\sigma$ the corresponding quotient, which is a group called the maximal homomorphic image of $S$. For simplicity we will also denote by $\sigma$ the canonical projection $\sigma: S \to G(S)$.

Any inverse semigroup $S$ has a natural partial order: $s \leq t$ if there is some projection $e \in E(S)$ such that $s = te$. Observe that, again, this is equivalent to the existence of an idempotent $f \in E(S)$ satisfying $s = ft$ (just take $f := tet^*$). From the preceding definitions we have that if $s \leq t$ then $s \sigma t$, so that the partial order $\leq$ restricts to a partial order within the $\sigma$-classes.

Inverse semigroups have a canonical representation as partial bijections in $S$ and this dynamical picture will be useful in this article. In fact, given an element $s \in S$, we define the domain of $s$ by $D_s := \{ x \in S \mid ssx = x \}$. It can be shown that left multiplication by $s$ is a bijection between $D_s$ and $D_{ss^*}$, the so-called range of $s$. Moreover, if $s \leq t$ then $s^*s \leq t^*t$ and $D_{s^*s} \subset D_{t^*t}$.

**Example 2.2.** For any $n \in \mathbb{N} \cup \{ \infty \}$ the polycyclic monoid $P_n$ (see, e.g., [20, 22]) is the inverse monoid given by the presentation:

$$P_n := \{ a_1, \ldots, a_n \mid a_i^*a_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \}.$$  

If $n = 1$, for instance, it can be shown that $P_1 = \{ a^i(a^*)^j \mid i, j \in \mathbb{N} \} \cup \{0\}$, where $E(P_1) = \{ a^i(a^*)^j \mid i \in \mathbb{N} \} \cup \{0\}$. Moreover, one can see that $D_{a^i(a^*)^j} = \{ a^p(a^*)^q \mid p \geq i \} \cup \{0\}$.

Now we give the construction of the C*-algebra $\mathcal{R}_S$, which generalizes the uniform Roe algebra of a discrete group. Given an inverse semigroup $S$, consider its left regular representation:

$$V: S \to B(\ell^2(S)), \quad V_s(\delta_x) := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise} \end{cases}.$$  

$V$ is then a faithful representation of $S$ by partial isometries on $\ell^2(S)$ (see [40, Proposition 2.1.4]). Note the condition $x \in D_{s^*s}$ is necessary to guarantee that $V_s$ is bounded. Moreover, we consider $\ell^\infty(S)$ as multiplication (i.e., diagonal) operators in $\ell^2(S)$. The C*-algebra $\mathcal{R}_S$ is the norm completion of the *-algebra generated by the products $fV_s$, where $f \in \ell^\infty(S)$ and $s \in S$:

$$\mathcal{R}_S := C^*(\ell^\infty(S) \cdot \{ V_s \}_{s \in S}) \subset B(\ell^2(S)).$$  

Note that, in case $S$ is unital, the C*-algebra $\mathcal{R}_S$ will be generated by $\{ V_s \}_{s \in S}$ and $\ell^\infty(S)$. However, if $S$ is not unital neither will $\mathcal{R}_S$. Another C*-algebra of interest to us is the so-called reduced C*-algebra:

$$C^*_r(S) := C^*(\{ V_s \}_{s \in S}) \subset \mathcal{R}_S.$$
2.1. The reduced crossed product. Given a C*-algebra $\mathcal{A}$ and a (continuous) action of a group $G$ by *-automorphisms on $\mathcal{A}$ the crossed product of $\mathcal{A}$ by $G$ is a larger C*-algebra containing $\mathcal{A}$ and a copy of $G$ as unitaries that implement the action (see, e.g., [8, Section 4.1]). This construction was later generalized to the setting of inverse semigroups (see, for instance, [37, 12, 25]). Since in the subsequent sections we will only deal with the commutative case $\mathcal{A} = C_0(X)$, that is the only case we introduce.

**Definition 2.3.** Let $X$ be a locally compact Hausdorff space and $S$ be a countable inverse semigroup.

1. A partial automorphism of $C_0(X)$ is a triple $(\phi, E_1, E_2)$, where $E_1 \subset C_0(X)$ are closed two-sided ideals and $\phi: E_1 \to E_2$ is a *-isomorphism. The set of partial automorphisms of $C_0(X)$ is denoted by $\text{PAut}(X)$. We equip $\text{PAut}(X)$ with the binary operation given by composition of maps whenever defined.

2. An action of $S$ on $X$ is a homomorphism $\alpha: S \to \text{PAut}(X)$, where $s \mapsto (\alpha_s, E_{1,s}, E_{2,s})$.

Observe that, for the action defined in (2) above, the domain of the map $\alpha_s$ (namely $E_{1,s}$) only depends on $s^*$, and its range (namely $E_{2,s}$) only depends on $ss^*$. Indeed, this follows from the fact that $\alpha$ is a semigroup homomorphism and $s^*$ acts as the identity on $D_{ss^*}$. Therefore, and for the sake of simplicity, we will henceforth denote $E_{1,s}$ only by $E_{ss^*}$ and $E_{2,s}$ by $E_{ss^*}$. We will be particularly interested in the following action.

**Proposition 2.4.** Let $S$ be a countable and discrete inverse semigroup. Given $s \in S$ let

$$E_{ss^*} := \{ f \in \ell^\infty(S) \text{ such that supp}(f) \subset D_{ss^*} \}.$$ 

Then the map $\alpha: s \mapsto \alpha_s$ given by $(\alpha_s f)(x) = f(s^*x)$, where $f \in E_{ss^*}$, defines an action.

**Proof.** The sets $E_{ss^*}$ are clearly (two-sided) closed ideals in $\ell^\infty(S)$. Furthermore, since left multiplication by $s^*$ defines a bijection from $D_{ss^*}$ onto $D_{ss^*}$ it follows that $\alpha_s: E_{ss^*} \to E_{ss^*}$ is a *-isomorphism. Moreover it is routine to show that

$$\alpha_s^{-1}(E_{ss^*} \cap E_{t^*}) = E_{t^*ss^*}.$$ 

Finally, observe

$$\alpha_{st}f(x) = f(s^*t^*x) = (\alpha_s f)(t^*x) = \alpha_t(\alpha_s f)(x)$$

and, thus, $\alpha_{st}f = \alpha_s(\alpha_t f)$ for any $f \in E_{t^*ss^*}$. \qed

Since no other action will be considered for crossed products we will denote $\alpha_s f$ simply by $sf$. In order to construct the reduced crossed product of $\ell^\infty(S)$ by $S$ recall that the canonical representation of $\ell^\infty(S)$ as multiplication operators in $\ell^2(S)$ is faithful, and consider

$$\pi: \ell^\infty(S) \to B(\ell^2(S) \otimes \ell^2(S)), \quad (\pi(f))((\delta_x \otimes \delta_y) := \begin{cases} f(xy) \delta_x \otimes \delta_y & \text{if } x \in D_{y^*y}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$1 \otimes V: S \to B(\ell^2(S) \otimes \ell^2(S)), \quad (1 \otimes V_s)((\delta_x \otimes \delta_y) := \begin{cases} \delta_x \otimes \delta_{sy} & \text{if } y \in D_{s^*s}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{\delta_x\}_{x \in S}$ denotes the canonical orthogonal basis of $\ell^2(S)$. It follows from straightforward computations that $\pi$ and $1 \otimes V$ are faithful *-representations of $\ell^\infty(S)$ and $S$, respectively. Observe the representations intertwine the action in the following covariant way

$$(1 \otimes V_s) \pi(f)(1 \otimes V_s)^* = \pi(sf)$$

for all $s \in S$ and $f \in E_{ss^*}$. The reduced crossed product $\ell^\infty(S) \rtimes_r S$ is then the C*-algebra generated by the images of $\pi$ and $1 \otimes V$, that is:

$$\ell^\infty(S) \rtimes_r S := C^*\left( (\pi(\ell^\infty(S)) \cdot \{1 \otimes V_s \mid s \in S\} \right) \subset B(\ell^2(S) \otimes \ell^2(S)).$$

The following result relates this construction to the algebra $R_S$, and generalizes a standard result for groups (see [8, Proposition 5.1.3]).
Theorem 2.5. Let $S$ be a countable and discrete inverse semigroup and consider the action of $S$ on $\ell^\infty(S)$ defined in Proposition 2.4. Then the $C^*$-algebra $R_S$ and the reduced crossed product are isomorphic, i.e.,

$$ R_S \cong \ell^\infty(S) \rtimes_r S. $$

Proof. Consider the bounded linear operator $W: \ell^2(S) \otimes \ell^2(S) \to \ell^2(S) \otimes \ell^2(S)$ given by

$$ W(\delta_x \otimes \delta_y) = \begin{cases} \delta_x \otimes \delta_{yx} & \text{if } xx^* = y^*y \\ 0 & \text{otherwise.} \end{cases} $$

It can be checked that $W$ is a partial isometry whose adjoint is

$$ W^*(\delta_u \otimes \delta_v) = \begin{cases} \delta_u \otimes \delta_{vu^*} & \text{if } u^*u = v^*v \\ 0 & \text{otherwise.} \end{cases} $$

Moreover, its initial projection $W^*W$ is the orthogonal projection onto the closure of the subspace generated by $\delta_x \otimes \delta_y$ where $xx^* = y^*y$, and its final projection $WW^*$ projects onto the subspace generated by $\delta_u \otimes \delta_v$ where $u^*u = v^*v$. In addition, it is routine to check that

$$ W\pi(f)W^* = (1 \otimes f)WW^* = WW^*(1 \otimes f) \quad \text{and} \quad W(1 \otimes V_s) = (1 \otimes V_s)W. $$

It follows that the map $\text{Ad}(W)$ restricts to an $*-isomorphism between $\ell^\infty(S) \rtimes_r S$ and $(1 \otimes R_S) \subset 1 \otimes B(\ell^2(S))$. Indeed, from the commutation relations above we have that

$$ W(\ell^\infty(S) \rtimes_r S)W^* = \text{cl}_{\ell^2}(\{1 \otimes fv_s \mid f \in \ell^\infty(S) \text{ and } s \in S\}) \cdot WW^* = (1 \otimes R_S)WW^*, $$

which, in turn, is $*$-isomorphic to $1 \otimes R_S$. \hfill $\Box$

3. INVERSE SEMIGROUPS AND GRAPHS

In this section we will give a third characterization of $R_S$ as a uniform Roe algebra over a metric space naturally associated to a countable inverse semigroup $S$. Concretely, we will show in Theorem 3.20 when $R_S \cong C^*_u(\Lambda_S)$, where $\Lambda_S$ is an undirected graph (to be defined below) endowed with the path metric.

We begin recalling the construction and properties of the uniform Roe algebra $C^*_u(X,d)$ of an extended metric space $(X,d)$. We refer to [33, 8, 28, 38, 7] for proofs and additional motivation in the context of metric spaces. In [4] this class of algebras was generalized to so-called extended metric spaces $(X,d)$ where the metric $d:X \times X \to [0,\infty]$ is allowed to take the value $\infty$ (see [3, Section 2.1]). This generalization is crucial for the main result of this section as $\Lambda_S$ splits into a disjoint union of connected components that, necessarily, are pairwise at infinite distance (see also Proposition 3.1 below).

An extended metric space $(X,d)$ is of bounded geometry if for every radius $R > 0$ the number of points within the balls of radius $R$ is uniformly bounded, i.e., for every $R > 0$ we have $\sup_{x \in X} |B_R(x)| < \infty$. Given an extended metric space of bounded geometry $(X,d)$, the propagation of a bounded and linear operator $T \in B(\ell^2(S))$ is defined by

$$ p(T) := \sup \{d(x,y) \mid x,y \in X \text{ and } T_{y,x} = \langle \delta_y,T\delta_x \rangle \neq 0\} $$

and $T$ has bounded propagation if $p(T) < \infty$. The uniform Roe algebra $C^*_u(X,d)$ is the $C^*$-algebra generated by the $*$-algebra $C^*_u,alg(X,d)$ of operators with bounded propagation. The next proposition shows the need to consider extended metrics on $S$ if one wants to realize the equality $R_S \cong C^*_u(\Lambda_S)$.

Proposition 3.1. Let $S$ be a countable inverse semigroup. Consider the matrix units

$$ M_{x,y} : \ell^2(S) \to \ell^2(S), \quad \text{where } x,y \in S \text{ and } M_{x,y}(\delta_z) = \begin{cases} \delta_y & \text{if } z = x \\ 0 & \text{otherwise.} \end{cases} $$

Let $d : S \to [0,\infty]$ be an extended metric on $S$. For any $x,y \in S$ the following hold:

1. If $d(x,y) < \infty$ then $M_{x,y} \in C^*_u(S,d)$.
2. If $x^*x + y^*y$ then $M_{x,y} \notin R_S$. 


Proof. (1) follows from the fact that $M_{x,y} \in C_{n,\text{alg}}^+(X,d) \subset C_n^+(S,d)$. For (2), let $x,y \in S$ be such that $x^*x \neq y^*y$. Then the matrix unit $M_{x,y}$ is uniformly bounded away from any linear combination $\sum_{i=1}^n f_i V_{s_i} \in \mathcal{R}_{S,\text{alg}}$:

$$\left\| M_{x,y} - \sum_{i=1}^n f_i V_{s_i} \right\|^2 \geq \left\| M_{x,y}(\delta_x) - \left( \sum_{i=1}^n f_i V_{s_i} \right)(\delta_x) \right\|^2_{\ell^2(S)} = \left\| \delta_y - \sum_{x \in D_{s_i s_i}^*} f(s_i x) \delta_{s_i x} \right\|^2_{\ell^2(S)} = 1 + \left\| \sum_{x \in D_{s_i s_i}^*} f(s_i x) \delta_{s_i x} \right\|^2_{\ell^2(S)} \geq 1,$$

where the second equality follows from the fact that there can be no $s_i$ such that $x \in D_{s_i s_i}^*$ and $s_i x = y$, because otherwise $y^* y = x^* s_i^* s_i x = x^* x$, contradicting the hypothesis. \qed

### 3.1. Schützenberger graphs

Given an inverse semigroup $S$, the preceding proposition suggests that any pair $x,y \in S$ such that $x^* x \neq y^* y$ must be at infinite distance. We will construct in this section a graph $\Lambda_S$ which will be disconnected unless $S$ is a group, in which case it is its’ left Cayley graph. The connected components of $\Lambda_S$ are the so-called *left Schützenberger graphs* associated to each $\mathcal{L}$-class of $S$. We begin introducing these notions.

From now on we denote by $S$ a countable and discrete inverse semigroup with fixed symmetric generating set $K = K^+ \subset S$. Note that, in general, we do not assume $K$ to be finite. Recall some of the so-called Green’s equivalence relations (see, for instance, [14, 20, 22]). For $x,y \in S$, we write $xLy$ if $x^* x = y^* y$. Similarly, we say $xRy$ if $x^* x = yy^*$. Note that, by definition, in each $\mathcal{L}$-class there is exactly one projection, namely $x^* x$ for any $x \in L$. Thus we can use the set $E(S)$ to label each class:

$$S = \cup_{e \in E(S)} L_e \quad \text{where for each } \mathcal{L}\text{-class we have } L_e \cap E(S) = \{e\}.$$  

The next simple result will be used throughout the rest of the paper, without mentioning it explicitly.

**Lemma 3.2.** Let $S$ be an inverse semigroup and let $s \in S$. Then $x \in D_{s^* s}$ if and only if $xLs x$.

**Proof.** Assume $xLs x$. Then $x^* x = x^* s^* s x$ and, hence, multiplying by $x$ from the left we obtain $x = xx^* x = xx^* s^* s x = s^* s x$ which shows $x \in D_{s^* s}$. The reverse is done similarly. \qed

Let $L$ be an $\mathcal{L}$-class of $S$. The *left Schützenberger graph* of $L$ is the edge-labeled undirected graph $\Lambda(L,K)$, where $K$ is a fixed symmetric generating set of $S$. Its vertex set is $L$ and two vertices $x,y \in L$ are joined by an edge labeled by $k \in K$ if $kx = y$ (see, e.g., [35, 40, 13]). Observe that for inverse semigroups $\Lambda(L,K)$ is an undirected graph, in the sense that if $x,y \in L$ and $kx = y$ then $y^* y = x^* k^* k x$ and, thus, $k^* y = k^* k x = x$. Therefore there is a $k^*$-labeled edge going from $y$ to $x$. We will denote by $\Lambda(S,K)$ the disjoint union of the left Schützenberger graphs $\Lambda(L_e,K)$, where $e \in E(S)$. That is, the vertex set of $\Lambda(S,K)$ is $S$ and two vertices $x,y \in S$ are joined by an edge labeled by $k \in K$ if and only if $xLy$ and $kx = y$.

**Remark 3.3.** Note that $\Lambda(S,K)$ is not the usual (left) Cayley graph of a semigroup (see, e.g., [15]). Indeed, observe that the Cayley graph of $S$ is in general a directed graph while $\Lambda(S,K)$ is always undirected. For instance, if $S = \{0,1\}$ using the product as operation, then the Cayley graph of $S$ with $K = S$ has a directed edge going from 1 to 0, while in $\Lambda(S,K)$ the vertices 0 and 1 are in different connected components. In fact, it can be shown that $\Lambda(S,K)$ is the graph resulting from deleting the directed edges in the left Cayley graph of $S$.

Alternatively one could construct the undirected graph $\Sigma(S,K)$, whose connected components are the *right Schützenberger graphs* $\Sigma(R,K)$ of each $\mathcal{R}$-class $R$. For general semigroups the left and right version of these graphs need not even be coarsely equivalent, since an arbitrary semigroup could, for instance, have a distinct number of $\mathcal{L}$ and $\mathcal{R}$ classes (cf., [15, Example 1]). However, for inverse semigroups these graphs are isomorphic.

**Lemma 3.4.** The graphs $\Lambda(S,K)$ and $\Sigma(S,K)$ are isomorphic (as graphs).
Example 3.6. Consider the bicyclic semigroup
and the operation is given by
Then
is an inverse semigroup with
are of the form
Remark 3.5. Observe that, contrary to the group case, infinitely many edges might connect two vertices \(x, y \in S\). Indeed, let \(S := G \times \mathbb{N}\), where \(G\) is a discrete and countable group generated by \(K\) and the operation is given by
Then \(S\) is an inverse semigroup with \((g, n)^\ast := (g^{-1}, n)\), and any \((g, 1), (kg, 1) \in G \times \{1\}\) are connected by infinitely many edges of the form \((k, m) \in K \times \mathbb{N}\).

Example 3.6. Consider the bicyclic semigroup \(P_1 = \langle a, a^* \mid a^* a = 1 \rangle \sqcup \{0\}\) (see Example 2.2). Fix the canonical symmetric generating set \(K := \{a, a^*, 0\}\) and note that any non-zero elements of \(P_1\) are of the form \(a^i a^j\), for some \(i, j \in \mathbb{N}\), of which exactly those of the form \(a^i a^j\) are idempotents. Given \(i, j, p, q \in \mathbb{N}\) observe that \(a^i a^j L a^p a^q\) if and only if \(j = q\) and, thus, the map
is a bijection. The graph \(\Lambda_{P_1}\) is the disjoint union of copies of the usual Cayley graph of \(\mathbb{N}\) (see Figure 1), with an extra isolated component corresponding to the element \(0 \in P_1\). In general, the left Schützenberger graph of an \(L\)-class of \(P_n\) is the \(n\)-ary complete rooted tree, unless it is the \(L\)-class of 0, in which case the component is an isolated point.

![Figure 1. Left-Schützenberger graphs of the semigroup \(P_1 = \langle a, a^* \mid a^* a = 1 \rangle \sqcup \{0\}\). Any two lines are at infinite distance from each other.](image)

As usual, we consider the graph \(\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{Le}\), where \(\Lambda_{Le}\) is equipped with the path distance between the vertices. We complete this section stating some facts about the graph \(\Lambda_S\).

Proposition 3.7. Let \(S\) be a finitely generated inverse semigroup. Then \(\Lambda_S\) is of bounded geometry.

Proof. Denoting by \(K\) the finite and symmetric generating set we have the uniform estimate \(|B_R(x)| \leq |K|^R\). □
Our setting is that of semigroups which are countable but not necessarily finitely generated. We do, however, require $\Lambda_S$ to be of bounded geometry, which is an important condition to define the uniform Roe algebra $C^u(\Lambda_S)$ and study property A in the following sections. The next result shows that our setting reduces to the usual one in the case of groups. Recall that the Cayley graph of a group $G$ is bounded geometry if and only if only if $G$ is finitely generated.

**Proposition 3.8.** Let $S$ be an inverse semigroup. If $\Lambda_S$ is of bounded geometry, then so is the left Cayley graph of the maximal homomorphic image $G(S)$.

**Proof.** Given $R > 0$, since $\Lambda_S$ has bounded geometry, we have that

$$m := \sup_{x \in S} |B_R(x)| < \infty .$$

We claim that $m$ also uniformly bounds the cardinality of the $R$-balls in the Cayley graph of $G(S)$, thus proving that $G(S)$ has bounded geometry. Indeed, assume that this is not the case. Consider $\sigma : S \to G(S)$ and let $B_R(\sigma(x_1))$ be an $R$-neighborhood (with respect to the path distance) of some $\sigma(x_1) \in G(S)$ having at least $m + 1$ different points, i.e.,

$$\{ \sigma(x_1), \ldots, \sigma(x_{m+1}) \} \subset B_R(\sigma(x_1)) .$$

Any point $\sigma(x_i)$ is connected with $\sigma(x_1)$ by a geodesic path $\sigma(s_i) \in G(S)$ of length at most $R$, and in particular $\sigma(x_i) = \sigma(s_i x_1)$ for every $i = 2, \ldots, m + 1$. Consider then the idempotent generated by

$$e := x_1^* s_2^* s_2 x_1 \cdots x_{i-1}^* s_i x_i e = x_{i-1}^* s_i x_i e = x_{i-1}^* s_i x_i x_{i-1}^* x_i = e = y_1$$

and let $y_i := x_i e$ and $y_i := s_i y_1$, where $i = 2, \ldots, m + 1$. We claim that the points $\{y_1 \}^{m+1}$ are pairwise different and within distance $R$ of $y_1$, thus proving that $m$ is not a bound on the cardinality of the $R$-balls of $S$ and contradicting the hypothesis. Indeed, first note that, when $i \neq j$, it follows that $y_i \neq y_j$ since $\sigma(y_i) = \sigma(s_i x_1) = \sigma(\sigma(x_1)) = \sigma(x_1) = \sigma(x_j)$ and $\sigma(y_j)$. Secondly, observe

$$s_i^* s_i y_1 = s_i^* s_i x_1 e = x_1 x_i^* x_i e = x_1 = y_1$$

and hence $y_i \in D e s_i^*$ for every $i = 2, \ldots, m + 1$, proving that $y_1, \ldots, y_{m+1}$ are all $\mathcal{L}$-related (cf., Lemma 3.2). Finally, the points $y_1$ and $y_i$ are connected by a path of length less than $R$ by construction, since $s_i y_1 = y_i$, proving the claim. $\square$

**Remark 3.9.** The difficulty of the preceding proof is the fact that the elements $x_1, \ldots, x_{m+1} \in S$ need not be $\mathcal{L}$-related, that is, they might sit in different Schützenberger classes of $S$. Moreover, the edges connecting $\sigma(x_1), \ldots, \sigma(x_{m+1})$ in $G(S)$ need not be present in a certain $\mathcal{L}$-class, and thus we have to move the point $x_1 \in S$ via multiplication with a suitable projection $e$ in order to replicate those edges in a certain $\mathcal{L}$-class $L_e \subset S$. Therefore, what the proof above actually says is that the local structure of $G(S)$, i.e., a certain $R$-ball $B_R(\sigma(x_1)) \subset G(S)$ may be seen in a $\mathcal{L}$-class $L_e \subset S$, provided that $e \in E(S)$ is a sufficiently small idempotent. In particular, the left Cayley graph of $G(S)$ naturally is the inductive limit of the Schützenberger graphs of $S$ (see also the proof of Proposition 4.32).

It is well known that not every $K$-labeled graph of bounded geometry is the left Cayley graph of a group generated by the set $K$. The following proposition shows that a large class of graphs can be realized as Schützenberger graphs.

**Proposition 3.10.** Let $G = (V,E)$ be a non-empty, connected and undirected graph without multiple edges and suppose that every vertex is connected with itself via a loop. Then there is an inverse semigroup $S$ and an $\mathcal{L}$-class $L \subset S$ such that $\Lambda_L$ and $G$ are isomorphic as graphs.

**Proof.** We will only sketch the main ideas of the proof (see also [39, 24]). It is useful for the construction to think of the undirected edges of $G$ as a pair of edges with opposite orientations. We will denote these as $E \cup E^*$, where $(v_2, v_1)^* = (v_1, v_2)$. Fix an arbitrary vertex $v_0 \in V$ and consider the set of cycles in $G$ starting at $v_0$:

$$C(v_0) := \{ (v_0, v_p) \cdots (v_2, v_1)(v_1, v_0) \text{ such that } (v_{i+1}, v_i), (v_0, v_p) \in E \cup E^* \} .$$

\[\text{We would like to thank Nóra Szakács for pointing out the proof of the following proposition.}\]
Consider the inverse semigroup $S$ formally generated by $V \cup E \cup E^* \cup \{0\}$ and with relations given by:

- $v = v^2 = v^* = (v, v)$ for all $v \in V$.
- $v_2(v_2, v_1) = (v_2, v_1) = (v_2, v_1)v_1$ for all edges $(v_2, v_1) \in E$.
- $v_3(v_2, v_1) = 0$ if $v_2 \neq v_3$.
- $(v_2, v_1)v_3 = 0$ if $v_1 \neq v_3$.
- $\omega = v_0^*v_0$ for all $\omega \in C(v_0)$.

Then, taking $K := V \cup E \cup E^* \cup \{0\}$, the left Schützenberger graph of the $L$-class of $v_0$ can naturally be seen as oriented paths in $G$ starting at $v_0$. Indeed, note that non-zero elements in $S$ are formal expressions $p = (v_p, v_{p-1}) \cdots (v_2, v_1)$, where $(v_i, v_{i-1})$ are edges in $G$, and $pLx_0$ if and only if $p$ starts at $v_0$. Moreover, if $p, q$ are two paths in $G$ with the same initial vertex $v$ and final vertex $w$, it follows that $q^*p = v$ and $pq^* = w$. Therefore

$$p = pv = pq^*p \text{ and } q^* = q^*w = q^*pq^*,$$

which implies that $p = q$ whenever they share the initial and final vertices. Hence the map $p \mapsto r(p)$, sending each path $p$ to its’ final vertex, is a natural bijection between the $L$-class of $v_0$ in $S$ and the graph $G$. Observe, as well, that two elements $p, q \in S$ are joined by an edge in $\Lambda_S$ if one is a prefix of the other, and therefore the map above is a graph isomorphism.

**Remark 3.11.** The construction of $S$ in the proof of the preceding proposition can be also done in the more general case where the graph $G = (V, E)$ is $K$-labeled with $K$ a set, as long as the labeling of the graph is deterministic (see [21]). Moreover, in that case the graph isomorphism respects the $K$-labeling. Also, observe that the construction above is not the so-called *path inverse semigroup* (see [6]).

Note, as well, that we require the vertices of $G$ to be decorated with a loop. This condition is irrelevant from a large-scale geometry point of view and, therefore, any simple connected graph can be quasi-isometrically realized as a Schützenberger graph of an inverse semigroup.

In the final part of the section we will state some results concerning the graph $\Lambda_S$ when seen as a metric space with the path length metric. The next useful lemma relates different distances one may consider in inverse semigroups. In particular, if $S = \langle K \rangle$, we denote by $\ell(\cdot)$ the minimal length of a word in the alphabet $K$, by $d_\Sigma$ the path distance in $\Lambda(S, K)$ and by $d_{G(S)}$ the path distance in the left Cayley graph of $G(S)$ with respect to $\sigma(K)$.

**Lemma 3.12.** Let $S = \langle K \rangle$ be an inverse semigroup, and let $\sigma : S \to G(S)$ be the canonical projection onto the maximal homomorphic image.

1. For any $s, x \in S$ such that $x \in D_{ss^*}$, we have

$$d_{G(S)}(\sigma(x), \sigma(sx)) \leq d_\Sigma(x, sx) \leq d_\Sigma(ss^*, s, s) \leq \ell(s).$$

2. For any $s, x \in S$ such that $xx^* = ss^*$ (hence, in particular, $x \in D_{ss^*}$), we have

$$d_\Sigma(x, sx) = d_\Sigma(ss^*, s, s).$$

**Proof.** For (1), observe that if $\ell(s) = d$ and $s = k_d \ldots k_1 \in K^d$, then $ss^*$ and $s$ are joined by a path in the $L$-class $L_{ss^*}$ labeled by $k_d \ldots k_1$, and thus $\ell(s) \geq d_\Sigma(ss^*, s, s)$. Moreover, for $x \in D_{ss^*}$ a geodesic path joining $ss^*$ with $s$ will define by multiplication from the right with $x$ a path of the same length joining $x$ with $sx$ on $L_{xx^*}$ and, therefore, $d_\Sigma(x, sx) \leq d_\Sigma(ss^*, s, s)$. Similarly, any geodesic path joining $x$ with $sx$ will define via quotient map $\sigma$ a path joining $\sigma(x)$ with $\sigma(sx)$ in the Cayley graph of $G(S)$, proving the last inequality.

Part (2) follows from (1) since

$$d_\Sigma(x, sx) \leq d_\Sigma(ss^*, s, s) = d_\Sigma(xx^*, ssx^*) \leq d_\Sigma(x, sx),$$

where the last inequality is, again, due to the fact that a geodesic connecting $x$ and $sx$ in $\Lambda_S$ defines a path between $xx^*$ and $ssx^*$ when multiplied on the right by $x^*$. 

\[\square\]
Note that Lemma 3.12 (2) relates distances between pairs of points in different $\mathcal{L}$-classes, since $x$ and $xx^*$ need not be $\mathcal{L}$-related in general. Moreover, Lemma 3.12 (2) is exactly a right invariance condition on the path metric $d_S$, generalizing the usual right invariance of the path metric in groups.

The following proposition, which might be known to experts, proves that some of the Schützenberger graphs $\Lambda_L$ are isomorphic to each other.

**Proposition 3.13.** Let $S$ be a countable inverse semigroup. Given $s \in S$, let $\Lambda_{s^*s}$ and $\Lambda_{ss^*}$ be the Schützenberger graphs of the $\mathcal{L}$-classes of $s^*s$ and $ss^*$, respectively. Then

$$\rho: \Lambda_{s^*s} \to \Lambda_{ss^*}, \text{ where } \rho(x) := xs^*$$

defines a graph isomorphism between $\Lambda_{s^*s}$ and $\Lambda_{ss^*}$.

**Proof.** First note that $(x^s)^*xs^* = sx^*xs^* = ss^*ss^* = ss^*$ and thus $xs^*\mathcal{L}ss^*$. If $xs^* = ys^*$ then $x = xx^*x = xs^*s = ys^*s = yy^*y = y$ for any $x, y \in \Lambda_{s^*s}$, which proves that $\rho$ is injective. Given an arbitrary $t \in \Lambda_{s^*s}$, observe that $t^*t = ss^*$ and, therefore, $t = tss^* = \rho(ts)$ since $ts\mathcal{L}s^*s$, proving that $\rho$ is a bijection. Finally, since $\rho$ is defined by right multiplication it is clear that it preserves adjacency. Indeed, the points $x, kx \in \Lambda_{s^*s}$ are joined by an edge labeled by $k$ if and only if the points $xs^*, kxs^* \in \Lambda_{ss^*}$ are joined by an edge labeled by $k$. \qed

We conclude this section introducing quasi-isometries, which are an important special case of coarse equivalences between metric spaces. This concept is central when viewing an infinite discrete group as a (coarse) geometric object. Given two extended metric spaces $(X, d_X)$ and $(Y, d_Y)$, we say that they are quasi-isometric (see [33, 28, 15]) if there is a map $\phi: X \to Y$ such that

1. There are some constants $M > 0, C \geq 0$ such that, for any $x, x' \in X$

   $$\frac{1}{M}d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq M d_X(x, x') + C.$$

2. There exists $R > 0$ such that for any $y \in Y$ there is $x \in X$ with $d_Y(y, \phi(x)) \leq R$.

A function $\phi$ satisfying both conditions above is called a quasi-isometry, while an injective map $\phi$ satisfying only (1) is called a quasi-isometric embedding.

In the following proposition we generalize a well known result in the case of groups (see [28, Theorem 1.3.12]). See also [15, Proposition 4] for a similar statement for semigroups considered as semi-metric spaces.

**Proposition 3.14.** Let $S$ be an inverse semigroup and let $K_1, K_2 \subset S$ be two finite and symmetric generating sets. Then the graphs $\Lambda(S, K_1)$ and $\Lambda(S, K_2)$ are quasi-isometric.

**Proof.** The identity function $\text{id}: \Lambda(S, K_1) \to \Lambda(S, K_2)$ is a quasi-isometry. Indeed, since it is surjective it is enough to prove there is some constant $M > 0$ such that

$$\frac{1}{M}d_{K_1}(x, y) \leq d_{K_2}(x, y) \leq Md_{K_1}(x, y) \text{ for any } x, y \in S,$$

where $d_{K_i}$ denotes the path distance in the graph $\Lambda(S, K_i)$. Note that if $x$ and $y$ are not $\mathcal{L}$-related then $d_{K_1}(x, y) = d_{K_2}(x, y) = \infty$, so we may suppose that $x, y$ belong to the same $\mathcal{L}$-class. In this case the inequalities follow by choosing

$$M := \max \{d_{K_1}(k^*k, k), d_{K_2}(k^*k, k) \mid k \in K_1 \cup K_2\}.$$ \qed

### 3.2. The uniform Roe-algebra

In this section we will show that the graph $\Lambda_S := \bigsqcup_{e} L_e$ (seen as a metric space with the path length metric) allows to witness the algebra $\mathcal{R}_S$ as an honest uniform Roe algebra $C^0_u(\Lambda_S)$. We begin introducing an important notion for the semigroup $S$ (see in relation with Theorem 3.20).

**Definition 3.15.** Let $S = (K)$ be an inverse semigroup with countable and symmetric generating set $K$ and let $L \subset S$ be an $\mathcal{L}$-class.
The following classes of inverse semigroups are FL:

(1) We say that \( L \) is \textit{finitely labeleable}, or FL for short, if there are \( C > 0 \) and a finite \( K_1 \subset K \) such that for any \( x, y \in L \) with \( y \in Kx \), we have that \( y \in K_1^p x \) for some \( p \in \{1, \ldots, C\} \), where \( K_1^p \) denotes the words of length \( p \) in the alphabet \( K_1 \).

(2) We say that \( S \) is \textit{finitely labeleable} if every \( L \)-class of \( S \) is FL uniformly over the classes, that is, if there are \( C > 0 \) and a finite \( K_1 \subset K \) such that, for any \( x, y \in S \) with \( xLy \) and \( y \in Kx \), we have that \( y \in K_1^p x \) for some \( p \in \{1, \ldots, C\} \).

The preceding definition, although technical, is essential to various arguments in the article. It is, in particular, an algebraic characterization of the equality of C*-algebras \( \mathcal{R}_S = C^*_v(\Lambda_S) \) (see Theorem 3.20). The two following propositions aim to relate this notion with some other notions in inverse semigroups or geometric group theory. The first proposition gives sufficient conditions for an inverse semigroup to be FL (see Proposition 3.16), while the second gives necessary conditions (see Proposition 3.17). In particular, Proposition 3.16 shows there are important classes of examples which are FL. Recall that an inverse semigroup is called F-inverse if each \( \sigma \)-class has exactly one greatest element. This class semigroups are unital and contain all free inverse semigroups. Moreover, every inverse semigroup has an F-inverse cover (see [22, p. 230]).

**Proposition 3.16.** The following classes of inverse semigroups are FL:

1. The class of finitely generated inverse semigroups.
2. The class of F-inverse semigroups such that \( \Lambda_S \) is of bounded geometry.

**Proof.** It is clear that every finitely generated inverse semigroup is FL: take \( C := 1 \) and \( K_1 := K \), which is finite by assumption. For the second statement let \( S = (K) \) be an F-inverse semigroup such that \( \Lambda_S \) is of bounded geometry. Denote the maximal group homomorphic image \( G(S) \) simply by \( G \). Consider then the projection \( \sigma \) restricted to the following set of greatest elements:

\[
A_{\text{great}} := \{ s \in S \mid s \text{ is greatest in } \sigma(s) \text{ and } \exists x \in D_s^* s \text{ with } ds(x, sx) \leq 1 \},
\]

i.e., consider

\[
\sigma: A_{\text{great}} \to \{ g \in G \mid d_G(1_G, g) \leq 1 \}.
\]

Note that \( A_{\text{great}} \) is not empty because \( 1 \in A_{\text{great}} \) and since the greatest element in each class is unique the preceding map is injective. Moreover, the right hand side is a finite set since \( \Lambda_S \) is of bounded geometry (see Proposition 3.3) and, hence, \( A_{\text{great}} \) is finite too. Therefore there is a finite \( K_1 \subset K \) such that every element in \( A_{\text{great}} \) is a word in \( K_1 \) of length less than \( C := \max \{ \ell(s) \mid s \in A_{\text{great}} \} \). It then follows that \( S \) is FL for those \( K_1 \) and \( C \). In fact, let \( xLy \) with \( y = kx \) for some \( k \in K \) (i.e., \( ds(x, y) \leq 1 \)). Denote by \( s_0 \) the greatest element in \( \sigma(k) \). Since \( D_{k^*k} \subset D_{s_0^*s_0} \) we have \( sx = kx = y \) and, therefore, \( y \in K_1^p x \) for some \( p \in \{1, \ldots, C\} \), proving that \( S \) is FL. \hfill \Box

The following proposition gives necessary conditions for an inverse semigroup to be FL. Even though its proof is straightforward, it highlights some key ideas behind the definition. Note that the construction of \( \Lambda(S, K) \), carried out in Subsection 3.1 for a generating set \( K \), can be done similarly for any subset of labels \( K_1 \subset K \), and the corresponding graph will be denoted by \( \Lambda(S, K_1) \), which is then a subgraph of \( \Lambda(S, K) \).

**Proposition 3.17.** Let \( S = (K) \) be an FL inverse semigroup, and let \( K_1 \subset K \) be as in Definition 3.16. Then the following hold:

1. The identity map \( \Lambda(S, K) \to \Lambda(S, K_1) \) is a quasi-isometry.
2. Every \( s \in S \) can be written as a word \( s = k_d \ldots k_1 e \), where \( e \in E(S) \) and \( k_i \in K_1 \), that is, \( S \) is finitely generated modulo the idempotents.

**Proof.** (1) follows from the fact that for every \( x, y \in S \)

\[
d_K(x, y) \leq d_{K_1}(x, y) \leq C d_K(x, y)
\]

where \( d_K \) and \( d_{K_1} \) denote the path distances in \( \Lambda(S, K) \) and \( \Lambda(S, K_1) \), respectively, and \( C > 0 \) is as in Definition 3.16.

(2) holds since, by the hypothesis, any \( s \in S \) can be written as \( s = k_d \ldots k_1 \) where \( k_i \in K_1 \) that is, \( s = ss^* s = k_d \ldots k_1 s^* s \). \hfill \Box
Example 3.18. The polycyclic semigroup $\mathcal{P}_n$ is FL since it is finitely generated (cf., Example 3.12). A simple example of an inverse semigroup that is not FL is $S = (\mathbb{N}, \min)$, where $m := \min\{n, m\}$ and $n^* = n$. In this case, the generating set must necessarily be $K = S = E(S)$, and hence $\Lambda_S = \cup_{n \in \mathbb{N}}\{n\}$ (infinitely many isolated points pairwise at infinite distance, where any vertex $\{n\}$ has infinitely loops labeled by $m \geq n$). It is clear that for any finite $K_1 \subset K$ there is an $n \in \mathbb{N}$ such that $n \geq k$ for any $k \in K_1$. Letting $x = y := n$ in Definition 3.14 it then follows that $S$ is not FL, even though any $\mathcal{L}$-class consisting of a single point is trivially FL. Similarly, the graph $\Lambda_T$ associated to the semigroup $T := (\mathbb{N}, \max)$ consists of infinitely many isolated points pairwise at infinite distance, and any vertex $\{n\}$ has precisely $n$ loops labeled by $k \in \{1, \ldots, n\}$. In this case, $\Lambda_T$ is FL because $T$ is $F$-inverse.

Note also that the converse implications in the preceding proposition are false, that is, there are non-FL inverse semigroups $S$ satisfying both (1) and (2) in Proposition 3.17. Indeed, consider again $S = (\mathbb{N}, \min)$, which is not FL but satisfies that $\Lambda(S, K)$ and $\Lambda(S, K_1)$ are isometric for any $K_1 \subset \mathbb{N}$ as well as the second condition.

We turn to the proof of the main result, i.e., we want to compare the algebras $\mathcal{R}_S$ and $C^*_u(\Lambda_S)$. The following preliminary result shows that one inclusion always holds.

Proposition 3.19. Let $S$ be an inverse semigroup. Let $\Lambda_S$ be the disjoint union of the left-Schützenberger graphs of $S$. Then $\mathcal{R}_S \subseteq C^*_u(\Lambda_S)$.

Proof. Note first that any $f \in \ell^\infty(S)$ corresponds to a diagonal operator hence has propagation $0$. Moreover, the propagation of the generators $V_s, s \in S$ is given by

$$p(V_s) := \sup_{x \in D_{s^*s}} d(x, sx) = d(s^*s, s).$$

In fact, it is clear that $p(V_s) \geq d(s^*s, s)$ since $s^*s \in D_{s^*s}$. The reverse inequality follows from Lemma 3.12. Finally, since $s, s^*s \in L_{s^*s}$ it follows that $d(s^*s, s) < \infty$ and, therefore, all generators $V_s$ have bounded propagation too, showing the desired inclusion. \qed

Theorem 3.20. Let $S$ be a countable and discrete inverse semigroup such that $\Lambda_S$ is of bounded geometry. Then the following conditions are equivalent:

1. $S$ is finitely labelable (see Definition 3.15).
2. The $*$-algebras $\mathcal{R}_{S, \text{alg}}$ and $C^*_u(\Lambda_S)$ are equal, and hence $\mathcal{R}_S = C^*_u(\Lambda_S)$.

Proof. (1) $\Rightarrow$ (2). Let $C > 0$ and $K_1 \subset K$ be as in Definition 3.13. Given an operator $T \in C^*_u(\Lambda_S)$, say of propagation $R > 0$, let $K_{RC} := \cup_{j \geq 1} K_j$ be the set of words of length at most $RC$ in the alphabet $K_1$. Since $K_1$ is finite then so is $K_{RC}$. Moreover, observe that for every pair $x, y \in S$ such that $d(x, y) \leq R$ there is $t \in K_{RC}$ such that $tx = y$. Let $t_{x,y} \in K_{RC}$ be such a possible choice and consider the functions $\xi_s \in \ell^\infty(S), s \in S$, defined by

$$\xi_s(y) := \begin{cases} \langle dy, t_{d^*y} \rangle & \text{if } y L s^* y \text{ and } s = t_{s^*y,y}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\xi_s = 0$ if $s \notin K_{RC}$ and we claim that

$$T = \sum_{s \in S} \xi_s V_s = \sum_{s \in K_{RC}} \xi_s V_s \quad \text{(finite sum)}.$$

Observe that, on the one hand, if $x$ and $y$ are not $\mathcal{L}$-related then $\langle dy, t_{d^*x} \rangle = 0$ since $T$ is of bounded propagation. In addition, if $x \in D_{s^*s}$ then $x L s x$ and, therefore, $sx \neq y$, which implies that $\langle dy, (\sum_{s \in S} \xi_s V_s) dy \rangle = 0$. Finally, on the other hand, if $x L y$ then:

$$\delta_y \left( \sum_{s \in S} \xi_s V_s \right) \delta_x = \delta_y \left( \sum_{s \in K_{RC}} \xi_s (sx) \delta_{sx} \right) = \sum_{s \in K_{RC}} \xi_s (y) \delta_{dy},$$

since, by construction, there is exactly one $s \in K_{RC}$ such that $sx = y$ and $\xi_s(y) \neq 0$, namely $s = t_{x,y}$. \qed
\( \Rightarrow \) \( \Rightarrow \). Suppose \( S \) is not FL. Thus, by Proposition 3.18, \( S \) is not finitely generated. Let \( K = \{ k_1, k_2, \ldots \} \) be a enumeration of \( K \), and let \( K_n := \{ k_1, \ldots, k_n \} \) be the first \( n \) generators. Note that, since \( S \) is not FL, for every \( n \in \mathbb{N} \) there are points \( x_n, y_n \in S \) such that \( x_n, y_n \in K x_n \) and \( x_n \) and \( y_n \) are not joined by a path labeled by \( K_1 \) of length less than \( n \). Note, as well, that since \( \Lambda_S \) is of bounded geometry we may suppose that \( x_n \neq x_{n'} \) for any \( n \neq n' \). Consider the operator:

\[
T : \ell^2(S) \rightarrow \ell^2(S), \quad T \delta_{z} = \begin{cases} \delta_{y_n} & \text{if } z = x_n, \\ 0 & \text{otherwise.} \end{cases}
\]

The operator \( T \) has finite propagation since \( \sup_{n \in \mathbb{N}} d(x_n, y_n) \leq 1 \), and, thus, \( T \in C^*_{u,alg}(\Lambda_S) \). Moreover, we will show that \( T \) cannot be approximated by elements in \( R_{S,alg} \) and, therefore, \( T \notin R_S \).

Indeed, given any \( \sum_{i=1}^{m} f_i v_i \in R_{S,alg} \), let \( n \in \mathbb{N} \) be sufficiently large so that \( s_i \) is a word in \( K_n \) for every \( i = 1, \ldots, m \) and \( n \) is greater than the maximum length of the elements \( s_i \). In this case

\[
\left\| T - \sum_{i=1}^{m} f_i v_i \right\| \geq \left\| T(\delta_{x_n}) - \left( \sum_{i=1}^{m} f_i v_i (\delta_{x_n}) \right) \right\|_{\ell^2(S)} = \left\| \delta_{y_n} - \sum_{i=1}^{m} f_i (s_i x_n) \delta_{s_i x_n} \right\|_{\ell^2(S)} \geq 1,
\]

where the last inequality follows from the fact that, by construction, \( s_i x_n \neq y_n \) for all \( i = 1, \ldots, m \). \( \square \)

Note that, in general, the uniform Roe algebra decomposes as a direct sum over the algebras associated to the corresponding Schützenberger classes

\[
C^*_{u}(\Lambda_S) \cong \bigoplus_{e \in E(S)} C^*_{u}(\Lambda_{Le}).
\]

Based on the strategy of the proof of the preceding theorem one can prove a similar result relating the uniform Roe algebra of an \( \mathcal{L} \)-class in \( S \) and the corresponding corner of \( R_S \).

**Theorem 3.21.** Let \( S \) be a countable and discrete inverse semigroup and let \( L \subset S \) be an \( \mathcal{L} \)-class such that \( \Lambda_L \) is of bounded geometry. Denote by \( p_L \) be the orthogonal projection from \( \ell^2(S) \) onto \( \ell^2(L) \subset \ell^2(S) \). Then the following conditions are equivalent:

1. The graph \( \Lambda_L \) is finitely labelable (see Definition 3.15).
2. The \( C^* \)-algebras \( p_L R_{S,L} \) and \( C^*_{u}(\Lambda_L) \) are equal.

We apply next Theorem 3.20 to the classes of inverse semigroups satisfying condition FL (cf., Proposition 3.18).

**Corollary 3.22.** Let \( S \) be a finitely generated inverse semigroup or \( F \)-inverse semigroup such that \( \Lambda_S \) has of bounded geometry. Then

\[
R_S = C^*_{u}(\Lambda_S) = \ell^\infty(S) \rtimes S.
\]

**Example 3.23.** Recall from Example 3.18 that the semigroup \( S = (\mathbb{N}, \text{min}) \) is not FL. Thus, by Theorem 3.20 we have \( R_S \neq C^*_{u}(\Lambda_S) \). Indeed, observe that \( \Lambda_S = \bigcup_{n \in \mathbb{N}} \{ n \} \) and, therefore, \( C^*_{u}(\Lambda_S) = \ell^\infty(\mathbb{N}) \). On the other hand, \( R_{S,alg} = c_{\text{fin}}(\mathbb{N}) \) (sequences with finite support) and, hence, \( R_S = c_0(\mathbb{N}) \).

4. Quasi-isometric Invariants

In this final section we study some of the large scale properties of the graph \( \Lambda_S \) constructed in Section 3.1 (see also [13] and references therein) in relation to \( C^* \)-properties of the reduced semigroup \( C^* \)-algebra \( C^*_{u}(S) \) and the uniform Roe algebra \( C^*_{u}(\Lambda_S) \). Recall that a property \( \mathcal{P} \) is said to be a quasi-isometric invariant if, given two quasi-isometric extended metric spaces \( (X, d_X) \) and \( (Y, d_Y) \) such that \( X \) has \( \mathcal{P} \), then so does \( Y \). We will be particularly interested in two notions, namely amenability and property \( A \).

According to Day’s original definition in [9], a semigroup \( S \) is amenable if there exists a probability measure on \( S \) which is invariant under taking preimages (see also [10] [19]). In the context of inverse semigroups this condition splits into two conditions which we called domain measurability and domain localization (see [5] Section 4.1 for details). The former condition is the most important
one and captures the dynamical invariance of the measure: $S$ is \textit{domain measurable} if there exists a finitely additive (total) probability measure $\mu: \mathcal{P}(S) \to [0, 1]$ satisfying
\begin{equation}
\mu(sA) = \mu(A) \quad \text{for all } s \in S, \ A \subset D_{s^*s}.
\end{equation}

Amenability and domain measurability of inverse semigroups also allow a \textit{Følner type characterization} which we will need later in this section (see, e.g., Theorem 4.4.2). For proofs of the following result and additional motivation we refer to Theorems 4.23 and 4.27 in [5].

\textbf{Theorem 4.1.} Let $S$ be a countable and discrete inverse semigroup.

1. $S$ is domain measurable if and only if for every finite $F \subset S$ and $\varepsilon > 0$ there is some finite and non-empty $G \subset S$ such that $|s(F \cap D_{s^*s}) \setminus F| \leq \varepsilon|F|$ for all $s \in F$.
2. $S$ is amenable if and only if for every finite $F \subset S$ and $\varepsilon > 0$ there is some finite and non-empty $G \subset S$ such that $F \subset D_{s^*s}$ and $|sF \setminus F| \leq \varepsilon|F|$ for all $s \in F$.

It is clear that in the case of groups amenability and domain measurability coincide and both Følner type characterizations in (1) and (2) above are equivalent. However, these two notions are different in general. The stronger notion of amenability requires, in addition, that the Følner sets are localized \textit{within} the corresponding domains. For example, $S := \mathbb{F}_2 \cup \{1\}$, where $1 \in S$ denotes a unit, is a domain measurable semigroup (take $F := \{1\}$), but it is not amenable (since the free group $\mathbb{F}_2$ is not amenable).

It is well known that amenability is a quasi-isometric invariant for groups (see [28, Theorem 3.1.5]).

\textbf{Theorem 4.2.} Let $G$ and $H$ be quasi-isometric finitely generated groups. If $H$ is amenable then so is $G$.

\textit{Property A} is a metric property of a space that can be seen as a weak version of amenability (cf., [28, Example 4.1.2]). It was introduced by Yu in [43] and has been studied extensively since then (see, for example, [8, 28, 33]).

\textbf{Definition 4.3.} Let $(X, d)$ be a metric space, and let $\{(X_e, d_e)\}_{e \in E}$ be a family of metric spaces.

1. $(X, d)$ has property A if for every $\varepsilon > 0$ and $R > 0$ there is $C > 0$ and $\xi: X \to \ell^1(X)$ such that:
   (a) For every $x \in X$ the function $\xi_x$ is positive, has norm 1 and its support is contained in $B_C(x)$, the ball of center $x$ and radius $C$.
   (b) $\|\xi_x - \xi_y\|_1 \leq \varepsilon$ for every $x, y \in X$ such that $d(x, y) < R$.
2. The family $\{(X_e, d_e)\}_{e \in E}$ has uniform property A if every $(X_e, d_e)$ satisfies property A with constants $C_e > 0$ which are uniformly bounded, i.e., $C := \sup_{e \in E} C_e < \infty$.

Property A was introduced as a sufficient condition to ensure that $(X, d)$ coarsely embeds into a Hilbert space (see [8, 28, 33, 43] and the survey [11]. It is a particularly interesting notion in the context of the so-called \textit{Baum-Connes conjecture} (see [43]). Even though the class of property A groups is plentiful (containing all amenable groups, free groups and being preserved by various constructions, see [28, Chapter 4]) it turned out that not every group has property A. The only proofs known to the authors of the existence of non-property A groups were initially given by Gromov in [17] and then by Osajda in [29] (see also [41]). Even though these constructions are quite exotic, due to the rigid nature of groups, there are several constructions of non-property A metric spaces. For instance, the \textit{coarse disjoint union} of the Cayley graphs of $\{\mathbb{Z}/n\}_{n \in \mathbb{N}}$ is a metric space that does not have property A but is not of bounded geometry either (see [27] and [28, Theorem 4.5.3]).

\textbf{Example 4.4.} An interesting class of examples are the so-called \textit{box spaces} (see [28, Example 4.4.7] and [33, 21]). We consider the construction in the context of inverse semigroups (see [2, Example 4.11]) as well as [15, 22] for a similar construction for groupoids). Let $G$ be a residually finite group, and let $\{N_i\}_{i \in \mathbb{N}}$ be a descending chain of normal subgroups of finite index of $G$ with trivial intersection. Let $S = \bigcup_{i \in \mathbb{N}} G/N_i$, equipped with the operation
\begin{equation}
q_i(g) \cdot q_j(h) := q_{\min(i,j)}(gh),
\end{equation}
where $q_i: G \to G/N_i$ is the canonical quotient map. Observe in particular that the $\mathcal{L}$ and $\mathcal{R}$ relations are equal, i.e., $S$ is a bundle of groups.
This construction is of particular interest because of the following two known results needed later. For a proof of the first one see [28, Theorem 4.4.6]:

**Proposition 4.5.** Let $G$ be a residually finite group and $S$ be constructed as in Example 4.4. Then $\Lambda_S$ has property A if and only if $G$ is amenable.

For a proof of the second statement see [12, Theorem 1.2] (and [2, Example 4.11]) and recall that $C^*(S)$ denotes the maximal or full $C^*$-algebra of $S$.

**Proposition 4.6.** Let $\mathbb{F}_2$ be the free group on 2 generators and $S$ be constructed as in Example 4.4. If the sequence $\{N_i\}_{i \in \mathbb{N}}$ is as in Lemma 2.8 in [42], then $C^*(S) = C^*_r(S)$ (that is, $S$ has the weak containment property), and $C^*_r(S)$ is non-nuclear.

In particular, if $G = \mathbb{F}_2$ and $\{N_i\}_{i \in \mathbb{N}}$ is as in Lemma 2.8 in [12], then every $\mathcal{L}$-class of $S$ has property A, since all of them are either finite or quasi-isometric to a tree, but the graph $\Lambda_S$ does not have property A. Moreover, the $C^*$-algebra $C^*_r(S)$ is non-nuclear and non-exact (by Theorem 4.19 below). Finally, we also mention that the box space construction is also useful to give examples of bounded geometry metric spaces not having property A. The corresponding uniform Roe algebras are examples of non-nuclear $C^*$-algebras having amenable traces (see [3, Remark 4.14]).

To finish this brief introduction to property A we recall the following known characterization that motivates our analysis in this section. For a proof see, e.g., [8, Theorems 5.1.6 and 5.5.7].

**Theorem 4.7.** Let $G$ be a finitely generated group. The following are equivalent:

1. The left Cayley graph of $G$ has property A.
2. The uniform Roe algebra $\ell^\infty(G) \rtimes_r G$ is nuclear.
3. The reduced group $C^*$-algebra $C^*_r(G)$ is exact.

4.1. **Amenability and domain measurability.** In this subsection we show that domain measurability, defined via the invariance condition in Eq. (4.1), is a quasi-isometric invariant of the graph $\Lambda_S$. We begin showing that the Hilbert sets introduced in Theorem 4.1 may be localized within $\mathcal{L}$-classes.

**Lemma 4.8.** Let $S$ be domain measurable. Then for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$, there exists some $\mathcal{L}$-class $L \subset S$ and a finite non-empty $F \subset L$ such that $|s(F \cap D_{s^*s}) \cup F| \leq (1 + \varepsilon)|F|$ for all $s \in \mathcal{F}$.

**Proof.** The proof is essentially given in Lemma 5.2 of [5]. In fact, it is enough to note that by Lemma 3.2 the equivalence relation $\approx$ used in [5] is precisely the $\mathcal{L}$-relation.

The following two lemmas will allow us to reduce the proof of Theorem 4.12 to the case of a surjective quasi-isometry.

**Lemma 4.9.** Let $S, T$ be quasi-isometric semigroups of bounded geometry. Then there is a finite group $G$ and a surjective quasi-isometry $\varphi: S \times G \rightarrow T$.

**Proof.** Let $\phi: S \rightarrow T$ be a quasi-isometry. In particular, there is some $R > 0$ such that $d_T(t, \phi(s)) \leq R$ for every $t \in T$. Let $m > 0$ be an upper-bound of the cardinality of the $R$-balls in $T$, and let $G$ be any finite group of cardinality $m$. For any $t \in T$ let $\theta_t$ be an embedding of $B_R(t)$ into $G$ and consider the map

$$\varphi: S \times G \rightarrow T, \quad \text{where } \varphi(s, g) := \begin{cases} \theta_{\phi(s)}^{-1}(g) & \text{if } g \in \text{im}(\theta_{\phi(s)}) \\ \phi(s) & \text{otherwise.} \end{cases}$$

Then $\varphi$ is surjective and a local perturbation of $\phi$ and, thus, a quasi-isometry.

**Lemma 4.10.** Let $S$ be an inverse semigroup, and let $G$ be a finite group. If $S \times G$ is domain measurable (resp. amenable) then so is $S$.

**Proof.** Observe the product in $S \times G$ is defined coordinate-wise. In particular $(s, g)^*(s, g) = (s^*s, 1)$ and, therefore, $(x, g) \in D_{(s^*s, 1)^*}$ if and only if $x \in D_{s^*s}$.

Given $\varepsilon > 0$ and a finite $\mathcal{F} \subset S$ let $F_G \subset S \times G$ witness the $(\varepsilon/|G|, \mathcal{F} \times G)$-Følner condition of $S \times G$ (cf., Theorem 4.1). Consider the set

$$F := \{s \in S \mid (s, g) \in F_G \text{ for some } g \in G\}.$$
Then $F \subset S$ is clearly finite and non-empty. Furthermore $|F| \cdot |G| \geq |F \times G|$ and, for any $s \in \mathcal{F}$,
\[
\frac{|s(F \cap D_{s^{*}s}) \setminus F|}{|F|} \leq \frac{(s,1)(F \times G \cap D_{(s^{*},1)}) \setminus F \times G)}{|F \times G|} \cdot |G| \leq \varepsilon \cdot |G| \leq \varepsilon.
\]
Therefore $F$ witnesses the $(\varepsilon, \mathcal{F})$-Følner condition of $S$ (cf., Theorem 4.1).

If, in addition, $S \times G$ is amenable, then the set $F$ constructed before is contained in $D_{s^{*}s}$. □

The following proposition gives an alternative Følner type characterization of domain measurability in the case that $S$ is finitely labelable (FL) (cf., Theorem 4.1). Given $R > 0$ and $A \subset S$ we denote by $\mathcal{N}_{RA}$ the $R$-neighborhood of $A$, i.e., the set of points $x \in S$ such that $d(x, A) \leq R$. Note that, in particular, if $A \subset L$, where $L \subset S$ is an $\mathcal{L}$-class, then $A \subset \mathcal{N}_{RA} \subset L$.

**Proposition 4.11.** Let $S = \langle K \rangle$ be a FL inverse semigroup. Then $S$ is domain measurable if and only if for every $R > 0$ and $\varepsilon > 0$ there is a finite non-empty $F \subset L \subset S$, where $L$ is an $\mathcal{L}$-class, such that
\[
|\mathcal{N}_{R}F| \leq (1 + \varepsilon)|F|.
\]

**Proof.** It is standard to show that, if $S$ is domain measurable, the Følner condition in Theorem 4.1 implies the corresponding inequality of the $R$-neighborhood (cf., [3, Section 2]). To show the reverse implication observe that, since $S$ is FL, there is a finite $K_{1} \subset K$ and $C > 0$ such that every edge labeled in $K$ can be obtained as a path labeled in $K_{1}$ of length at most $C$. Given $\varepsilon > 0$ and $R > 0$ let $K_{RC} := \cup_{p \in [1]} K_{1}$. By Lemma 4.9 there is an $\mathcal{L}$-class $L$ and a finite $F \subset L$ such that for every $s \in K_{RC}$
\[
\frac{|s(F \cap D_{s^{*}s}) \cup F)}{|F|} \leq 1 + \frac{\varepsilon}{|K_{RC}|}.
\]
It follows that such an $F$ satisfies the Følner type condition required. □

We can now generalize Theorem 1.2 to the context of inverse semigroups.

**Theorem 4.12.** Let $S$ and $T$ be finitely labelable inverse semigroups and suppose $S$ and $T$ are quasi-isometric. If $T$ is domain measurable then so is $S$.

**Proof.** Let $\varphi: S \to T$ be a quasi-isometry. By Lemma 4.9 and Lemma 4.10 we may suppose that $\varphi$ is a surjective quasi-isometry. Let $M > 0$ and $C > 0$ be some constants such that for every $x, y \in S$
\[
\frac{1}{M} \cdot d_{S}(x, y) - C \leq d_{T}(\varphi(x), \varphi(y)) \leq M \cdot d_{S}(x, y) + C,
\]
where $d_{S}$ and $d_{T}$ denote the path distances in $\Lambda_{S}$ and $\Lambda_{T}$, respectively. By Proposition 4.11 and given $R > 0$ and $\varepsilon > 0$ there is some finite non-empty $F_{T} \subset T$ within some $\mathcal{L}$-class and with small $(MR + C)$-neighborhood, that is, $|\mathcal{N}_{MR+C}F_{T}| \leq (1 + \varepsilon)|F_{T}|$. We claim the set
\[
F_{S} := \varphi^{-1}(F_{T})
\]
has a small $R$-neighborhood in $S$ and, therefore, by Proposition 4.11 $S$ is domain measurable too. Observe that $F_{S}$ is non-empty since $\varphi$ is surjective. Furthermore, the distance between two any $\mathcal{L}$-classes (in either $\Lambda_{S}$ or $\Lambda_{T}$) is infinite, and thus, since $\varphi$ is a quasi-isometry, it takes $\mathcal{L}$-classes of $S$ onto $\mathcal{L}$-classes of $T$. Therefore $F_{S}$ is contained within some $\mathcal{L}$-class. Moreover, $F_{S}$ must be finite since $T$ is of bounded geometry and $\varphi$ is a quasi-isometry. Finally:
\[
\frac{|\mathcal{N}_{R}F_{S}|}{|F_{S}|} \leq \frac{|\mathcal{N}_{MR+C}F_{T}|}{|F_{T}|} \leq 1 + \varepsilon,
\]
which proves that $F_{S}$ has small $R$-neighborhood in $S$. □

**Remark 4.13.** Note that Theorem 4.12 only proves that domain measurability is a quasi-isometry invariant. However, it is not clear whether amenability of inverse semigroups has this property (see Theorem 4.1). The problem arises in the Følner type characterization of amenability and from the fact that a quasi-isometry $\varphi: \Lambda_{S} \to \Lambda_{T}$ needs not respect the partial order of the semigroups. Therefore, the quasi-isometry does not have to respect the domains $D_{s^{*}s}$ where the Følner sets have to be localized.
4.2. Property A. Given an inverse semigroup $S$ recall first the construction of the $K$-labeled undirected graph $\Lambda_S = \cup_{x \in E(S)} \Lambda_x$ (see Section 3.1), where the left-Schützenberger graphs $\Lambda_x$ correspond to the different connected components of $\Lambda_S$. We aim to characterize first when the left Schützenberger graph $\Lambda_L$ of an $L$-class $L \subset S$ has property A, generalizing the result for groups in Theorem 4.17.

To that end we first need to introduce the C*-algebraic notions of nuclearity and exactness. Both notions have distinct characterizations (see [8]) and have been studied extensively (see, for example, [22][23][29][29][1]). We will use the characterization in terms of contractive completely positive (ccp) matrix approximations. A map between C*-algebras $\theta: A \to B$ is nuclear if for every $\varepsilon > 0$ and every finite $F \subset A$ there exist ccp maps $\varphi: A \to M_n$ and $\psi: M_n \to B$ such that $\|\psi \circ \varphi(A) - \theta(A)\| \leq \varepsilon$ for every $A \in F$. A C*-algebra $A$ is nuclear if the identity map $\text{id}: A \to A$ is nuclear. A concretely represented C*-algebra $A \subset B(H)$ is called exact if the inclusion map $\iota: A \to B(H)$ is nuclear.

We recall next two useful results needed later.

**Proposition 4.14.** Let $q > 0$ and let $X$ be a locally compact Hausdorff space. Every $C^*$-subalgebra $A \subset C_0(X) \otimes M_q$ is nuclear.

**Proof.** Observe that every irreducible representation of $A$ is of dimension, at most, $q$ and, thus, $A$ is subhomogeneous (see [8, Definition 2.7.6]). Hence $A$ is nuclear by [8, Proposition 2.7.7].

The following result proves that for nuclearity it is enough to show that the identity map factors through nuclear algebras instead of matrices.

**Proposition 4.15.** A $C^*$-algebra $R$ is nuclear if and only if for every finite $F \subset R$ and $\varepsilon > 0$ there is a nuclear $C^*$-algebra $A$ and completely positive and contractive maps $\varphi: R \to A$ and $\psi: A \to R$ such that $\|\psi \circ \varphi(A) - A\| \leq \varepsilon$ for every $A \in F$.

**Proof.** The claim follows applying the nuclearity of $A$ and an $\varepsilon/2$-argument.

We prove now one of the main theorems of the section.

**Theorem 4.16.** Let $S$ be a countable and discrete inverse semigroup, and let $L \subset S$ be an $L$-class such that the left Schützenberger graph $\Lambda_L$ is of bounded geometry. Let $p_L$ be the orthogonal projection from $\ell^2(S)$ onto $\ell^2(L) \subset \ell^2(S)$. Consider the following statements:

1. The graph $\Lambda_L$ has property A.
2. The $C^*$-algebra $p_L(R_S)p_L$ is nuclear.
3. The $C^*$-algebra $p_L(C_0(S)p_L$ is exact.
4. The $C^*$-algebra $p_L C^0(S)p_L$ is exact.

Then $\{1\} \Rightarrow \{2\} \Rightarrow \{3\} \Leftrightarrow \{4\}$. Moreover, if $\Lambda_L$ is finitely labelable then $\{3\} \Rightarrow \{1\}$, and hence all conditions are equivalent.

**Proof.** For convenience, in this proof we denote by $p$ the projection $p_L$, and $R$ stands for $R_S$. Furthermore, note that $p V_S p = V_S$, since $p$ projects onto an $L$-class (see Lemma 3.4).

$\{1\} \Rightarrow \{2\}$. Let $F \subset pR p$ be finite and $\varepsilon > 0$. Observe that, without loss of generality, we may suppose that every element of $F$ is of the form $fV_S$ where the support of $f$ is contained in $L \cap D_{ss}^+$. Let $R > 0$ be larger than the length of $s$ for every $s$ such that $fV_S \in F$. Since $\Lambda_L$ has property A there are $C > 0$ and $\xi: \Lambda_L \to \ell^1(\Lambda_L)_{+}$ such that $\text{supp}(\xi_x) \subset B_C(x)$, $x \in L$, and

\[\|\xi_x - \xi_y\|_1 \leq \frac{\varepsilon}{M}\] for every $x, y \in L$ such that $d(x, y) \leq R$,

where $M := \max\{\|fV_S\| | fV_S \in F\}$. Consider then the map

$$\varphi: pR p \to \prod_{x \in L} M_{B_C(x)}, \text{ where } \varphi(a) := \left(p_{B_C(x)} a p_{B_C(x)} \right)_{x \in L}.$$ Recall that $M_{B_C(x)}$ denotes the full matrix algebra with rows and columns labeled by elements in $B_C(x)$ and $p_{B_C(x)} \in \ell^\infty(S)$ is the characteristic function of $B_C(x)$. Since $\Lambda_L$ is of bounded geometry there is some $q > 0$ such that $|B_R(x)| \leq q$ for every $x \in L$ and, therefore,

$$\text{im}(\varphi) \subset \prod_{x \in L} M_{B_C(x)} \subset \ell^\infty(L) \otimes M_q.$$
Let $\mathcal{A}$ be the closure in norm of the image of $\varphi$ and consider $\varphi : p\mathcal{R}p \to \mathcal{A}$. Moreover, let

$$\psi : \mathcal{A} \to p\mathcal{R}p,$$  

where $\psi((b_x)_{x \in L}) := \sum_{x \in L} \xi^*_x b_x \xi_x$

where we identify $\xi_x$ with the diagonal operator $\xi_x\delta_y := \xi_y(x)\delta_y$ (note the flip between the argument and the index of $\xi$). It is clear that both $\varphi$ and $\psi$ are ccp maps. Furthermore

$$\psi(\varphi(fV_s)) = \sum_{x \in L} f_x V_s \xi_x = fV_s \left( \sum_{x \in L} (s^* \xi_x) \xi_x \right) \in p\mathcal{R}p,$$

where $s^* \xi_x(\delta_y) := \xi_y(x)$ if $y \in D_{s,s^*}$ and $s^* \xi_x(\delta_y) = 0$ otherwise. From Eq. (4.12) we have for all $s \in S$ such that $fV_s \in \mathcal{F}$ the following estimate

$$\left\| V_{s^*s} - \sum_{x \in L} (s^* \xi_x) \xi_x \right\| = \sup_{y \in L} \left| V_{s^*s} \delta_y - \sum_{x \in L} ((s^* \xi_x) \xi_x) \delta_y \right| = \sup_{y \in L \cap D_{s,s^*}} \left| 1 - \sum_{x \in L} \xi_y(x) \xi_y(x) \right| \leq \frac{\varepsilon}{M}.$$

Therefore, we can estimate

$$\left\| fV_s - \psi(\varphi(fV_s)) \right\| = \left\| fV_s \left( \sum_{x \in L} (s^* \xi_x) \xi_x \right) \right\| \leq \left\| fV_s \right\| \left\| V_{s^*s} - \sum_{x \in L} (s^* \xi_x) \xi_x \right\| \leq \varepsilon.$$

Since, by Proposition 4.14, $A$ is nuclear it follows that $p\mathcal{R}p$ is nuclear as well by Proposition 4.15

(2) $\Rightarrow$ (3) follows since nuclear algebras are exact.

(3) $\Rightarrow$ (1) follows since exactness passes to subalgebras.

(1) $\Rightarrow$ (2). Given $\varepsilon > 0$ and a finite $\mathcal{F} \subseteq p\mathcal{R}p$, without loss of generality we may suppose again that every element in $\mathcal{F}$ is of the form $fV_s$, where the support of $f$ is contained in $L \cap D_{s,s^*}$. By the exactness of $p\mathcal{C}^*_\varphi(S)p$ there are ccp maps

$$\varphi : p\mathcal{R}p \to \ell^\infty(L) \otimes M_n$$

and

$$\psi : M_n \to B(\ell^2(L)),$$

such that $\|V_sp - \tilde{\psi}(V_sp)\| \leq \varepsilon/M$ for all $fV_s \in \mathcal{F}$, where $M := \max \left\{ \|fV_s\| : fV_s \in \mathcal{F} \right\}$. Consider the ccp maps

$$\varphi : p\mathcal{R}p \to \ell^\infty(L) \otimes M_n \quad \text{and} \quad \psi : \ell^\infty(L) \otimes M_n \to B(\ell^2(L)),$$

$$fV_s \to f \otimes \tilde{\varphi}(V_s) \quad \text{and} \quad f \otimes b \to f\tilde{\psi}(b).$$

Then

$$\left\| \psi(\varphi(fV_s)) - fV_s \right\| \leq \left\| fV_s \right\| \left\| V_{s^*s} - \sum_{x \in L} (s^* \xi_x) \xi_x \right\| \leq \varepsilon,$$

which proves, again using that $\ell^\infty(L) \otimes M_n$ is nuclear (see Proposition 4.14), that $p\mathcal{R}p$ is exact.

Finally, if the L-class $L$ is FL then by Theorem 3.24 the corner $p\mathcal{R}p$ is a uniform Roe algebra, i.e., $p\mathcal{R}p \cong C^*_\varphi(\Lambda_L)$. Using recent results by Sato (see [34, Theorem 1.1]) we have that $C^*_\varphi(\Lambda_L)$ is exact $\Leftrightarrow C^*_\varphi(\Lambda_L)$ is nuclear $\Leftrightarrow \Lambda_L$ has property A and, therefore, all the statements in the theorem are equivalent in this case.

**Remark 4.17.** Observe that implication of (1) $\Rightarrow$ (2) in Theorem 4.16 is independent of $\Lambda_L$ being FL or not. This is remarkable since, in general, nuclearity does not pass to subalgebras and if $\Lambda_L$ is not FL then the corner $p_L(\mathcal{R}_S)p_L$ is properly contained in the uniform Roe algebra $C^*_\varphi(\Lambda_L)$ (see Theorem 3.20 for a similar argument). In general, it is known that a metric space (say $\Lambda_L$) has property A if and only if its uniform Roe algebra (say $C^*_\varphi(\Lambda_L)$) is nuclear (see, for example, [8, Theorem 5.5.7]). Moreover, in Theorem 4.16 (1) the corner $p_LC^*_\varphi(S)p_L$ needs not be closed. Indeed, observe that, in general, $p_L$ is not contained in $C^*_\varphi(S)$.

Some examples of L-classes with property A are every finite graphs or every graph quasi-isometric to a tree (cf., [28, Example 4.1.5]). Nevertheless, contrary to the group case, an amenable inverse semigroup can have L-classes without property A. Indeed, let $G$ be a group without property A (see, for example, [11, 29]) and adjoin to it a zero element. Then $S := G \cup \{0\}$ is an amenable
inverse semigroup where $G$ forms an $L$-class without property A. We, however, would like to point out the following question, which is, in our opinion, more intriguing.

Q: Is there any class of inverse semigroups such that every/any of their Schützenberger graphs are non-property A?

We now extend Theorem 4.16 to the whole graph $Λ_S = \bigcup_{e \in E(S)} Λ_{L_e}$. In particular, this allows us to characterize when the graph $Λ_S$ has property A via the $C^*$-algebras $R_S$ and $C^*_r(S)$. For similar results relating the amenability of the maximal homomorphic image $G(S)$ and the weak containment of $S$ (see [31, Proposition 4.1] and [23]). For the relation between the nuclearity of $R_S$ and the exactness or the nuclearity of $C^*_r(S)$ see [11, 2]. We first note that $Λ_S$ has property A when all of its connected components have uniform property A (see Definition 4.3).

Lemma 4.18. $Λ_S$ has property A if and only if the family $\{Λ_{L_e}\}_{e \in E(S)}$ has uniform property A.

Proof. The proof is a direct consequence of the definitions involved. Use, for example, that for any $x \in S$ one has supp$x \subset B_C(x) \subset L_{e_x}$.

Observe that there are inverse semigroups, necessarily with infinitely many $L$-classes, without property A and such that every of its’ Schützenberger graphs do have property A (see Example 4.4).

We generalize next Theorem 4.16 to $Λ_S$.

Theorem 4.19. Let $S$ be a countable and discrete inverse semigroup such that $Λ_S$ is of bounded geometry. Consider the assertions:

1. The graph $Λ_S$ has property A.
2. The $C^*$-algebra $R_S$ is nuclear.
3. The $C^*$-algebra $R_S$ is exact.
4. The $C^*$-algebra $C^*_r(S)$ is exact.

Then $\{1\} \Rightarrow \{2\} \Rightarrow \{3\} \Leftrightarrow \{4\}$. Moreover, if $Λ_S$ is finitely labelable then $\{3\} \Rightarrow \{1\}$, and hence all conditions are equivalent.

Proof. By Lemma 4.18 condition $\{1\}$ is equivalent to the family $\{Λ_{L_e}\}_{e \in E(S)}$ having uniform property A. Observe that all the implications follow from similar arguments as those in the proof of Theorem 4.16. For instance, in this context the ccp maps are given for each $L$-class $L_e$ and one needs to use the uniform bound on the constants $C_e$, $e \in E$ (see Definition 4.3).

The rest of the section aims to give a relation between the property A of the graph $Λ_S$ and that of $G(S)$ (see Proposition 4.21). Before getting to the proof we need the next simple lemma (see [5, Lemma 4.26]).

Lemma 4.20. Every countable inverse semigroup $S$ has a decreasing sequence of projections $\{e_n\}_{n \in \mathbb{N}}$ that is eventually below every other projection, that is, $e_n \geq e_{n+1}$ and for every $f \in E(S)$ there is some $n_0 \in \mathbb{N}$ such that $f \geq e_{n_0}$.

Proof. Since $S$ is countable we can enumerate the set of projections $E(S) = \{f_1, f_2, \ldots\}$. The lemma follows putting $e_n := f_1 \cdots f_n$.

The proof of the following proposition is based on the facts that the left Cayley graph of $G(S)$ is the inductive limit of the Schützenberger graphs of $S$ and that property A is closed under inductive limits with injective connecting maps. We give an explicit proof because, in general, the natural connecting maps $L_e \to L_{e'f}$, where $xe \mapsto xef$, need not be injective for certain $L$-classes.

Proposition 4.21. Let $S$ be a countable and discrete inverse semigroup such that $Λ_S$ is of bounded geometry. If $Λ_S$ has property A, then so does the maximal homomorphic image $G(S)$.

Proof. Fix a free ultrafilter $ω \in β\mathbb{N} \setminus \mathbb{N}$, and let $\{e_n\}_{n \in \mathbb{N}} \subset E(S)$ be a decreasing sequence of projections as in Lemma 4.20. In particular, observe that for all $s \in S$ we have that $se_nL_{e_n}$ for all $n \geq n_0$ and $n_0 \in \mathbb{N}$ large enough. Let $L_n \subset S$ be the $L$-class of $e_n \in E(S)$. By Lemma 4.18 the family $\{Λ_{L_n}\}_{n \in \mathbb{N}}$
has uniform property A and, thus, given \( \varepsilon > 0 \) and \( R > 0 \), let \( \xi(n)_\Lambda \colon \Lambda_{L_n} \to \ell^1(\Lambda_{L_n}) \) witness the \((R, \varepsilon)\)-uniform property A of the family \( \{\Lambda_{L_n}\}_{n \in \mathbb{N}} \). Consider

\[
\xi \colon G(S) \to \ell^1(G(S)), \quad \text{where } \xi_{\sigma(s)}(\sigma(t)) := \lim_{n \to \infty} \xi(n)_{se_n}(tc_n).
\]

Note that, since \( se_n \mathcal{L} e_n \) for all \( n \geq n_0 \), the function \( \xi_{\sigma(s)} \) is well defined. Furthermore, \( \xi_{\sigma(s)} \) is positive and of norm 1, since

\[
\|\xi_{\sigma(s)}\|_1 = \sum_{\sigma(t) \in G(S)} \lim_{n \to \infty} \xi(n)_{se_n}(tc_n) = \lim_{n \to \infty} \sum_{\sigma(t) \in G(S)} \xi(n)_{se_n}(tc_n) = \lim_{n \to \infty} \|\xi(n)_{se_n}\|_1 = 1.
\]

In addition, for any \( \sigma(s), \sigma(t) \in G(S) \) such that \( \xi_{\sigma(s)}(\sigma(t)) \neq 0 \), by the limit construction it follows that there is some positive \( \delta > 0 \) with \( \xi(n)_{se_n}(tc_n) \geq \delta \) for all sufficiently large \( n \in \mathbb{N} \). Therefore, using the comparison of distances given in Lemma 4.3, we obtain

\[
d_{G(S)}(\sigma(s), \sigma(t)) \leq d_{L_n}(se_n, tc_n) \leq C < \infty
\]

where \( C > 0 \) is a constant bounding the diameter of the supports of \( \xi(n)_{se_n} \) (see Definition 4.3). It thus follows that \( \supp(\xi_{\sigma(s)}) \) is contained in a ball of radius \( C > 0 \) around \( \sigma(s) \). Finally, let \( \sigma(s), \sigma(t) \in G(S) \) such that \( d_{G(S)}(\sigma(s), \sigma(t)) \leq R \). Then

\[
\|\xi_{\sigma(s)} - \xi_{\sigma(t)}\|_1 = \lim_{n \to \infty} \|\xi_{se_n} - \xi_{tc_n}\|_1 \leq \varepsilon
\]

since \( \|\xi_{se_n} - \xi_{tc_n}\|_1 \leq \varepsilon \) for all sufficiently large \( n \in \mathbb{N} \).

\[\Box\]

4.3. Property A in the E-unitary case. We conclude the article applying Theorem 4.19 to the special case where \( S \) is E-unitary. An inverse semigroup is E-unitary if the relation generated by \( \sigma \) and \( \mathcal{L} \) is the equality, i.e., if \( s \mathcal{L} t \) and \( \sigma(s) = \sigma(t) \) then \( s = t \) for every \( s, t \in S \). Recall, that all F-inverse semigroups (see Section 5.2) are E-unitary and unital (cf., [22, Proposition 3, Chapter 7]). The following proposition is an improvement of Proposition 5.3.

**Proposition 4.22.** Let \( S \) be an E-unitary inverse semigroup, and let \( G(S) \) be its maximal homomorphic image. Then the following are equivalent:

1. \( \Lambda_S \) is of bounded geometry.
2. The left Cayley graph of \( G(S) \) is of bounded geometry.

**Proof.** The implication \( (1) \Rightarrow (2) \) is already proved in Proposition 4.8. To show the reverse implication let \( R > 0 \) and note that, by hypothesis, the \( R \)-balls of \( G(S) \) have uniformly bounded cardinality, i.e., \( \sup_{x \in S} |B_R(\sigma(x))| < \infty \). Since \( S \) is E-unitary the canonical projection \( \sigma \) gives an injective map from \( B_R(x) \subset S \) to \( B_R(\sigma(x)) \subset G(S) \) for every \( x \in S \) and, thus,

\[
\sup_{x \in S} |B_R(x)| \leq \sup_{x \in S} |B_R(\sigma(x))| < \infty,
\]

which proves that \( \Lambda_S \) is of bounded geometry.

\[\Box\]

Observe Proposition 4.22 (and Proposition 5.3) actually prove that any upper bound on the cardinality of the \( R \)-balls of \( S \) is also an upper bound on the cardinality of the \( R \)-balls of \( G(S) \). The reverse implication is also true if \( S \) is, in addition, E-unitary. Moreover, note that the E-unitary assumption on \( S \) is essential in Proposition 4.22. Indeed, as an example of a non-E-unitary semigroup that fails to have this property let \( \Gamma \) be any infinitely generated group and define \( S := \Gamma \cup \{0\} \), where 0 denotes a zero element. Then \( G(S) \) is trivial (hence of bounded geometry), but the left Schützenberger graph of the \( \mathcal{L} \)-class of \( 1_\Gamma \) is the left Cayley graph of \( \Gamma \) and, thus, not of bounded geometry.

Recall that it was proven in [1, Proposition 8.5] that an E-unitary inverse semigroup \( S \) is exact (in the sense that \( C^*_e(S) \) is an exact \( C^* \)-algebra) if and only if \( G(S) \) is an exact group. With the techniques presented previously we can give a new proof of this result relating exactness of the \( C^* \)-algebra to the property A of the graph \( \Lambda_S \).

**Theorem 4.23.** Let \( S \) be an E-unitary countable and discrete inverse semigroup. Suppose the graph \( \Lambda_S \) is finitely labelable. Then the following are equivalent:

1. \( \Lambda_S \) is of bounded geometry.
2. \( \Lambda_S \) is of uniform bounded geometry.
3. \( \Lambda_S \) is E-unitary.
4. \( \Lambda_S \) is exact.

**Proof.**
The graph $\Lambda_S$ has property A.
(2) The family of graphs $\{\Lambda_{L_e}\}_{e \in E(S)}$ has uniform property A.
(3) The $C^*$-algebra $C^*_r(S)$ is exact.
(4) The maximal homomorphic image $G(S)$ has property A.

Proof. The equivalences (1) $\iff$ (2) and (1) $\iff$ (3) were proven in Lemma 4.18 and Theorem 4.19 respectively. Moreover, the implication (1) $\Rightarrow$ (4) is proved in Proposition 4.21 and, thus, it suffices to show (1) $\Rightarrow$ (3). It is well known (see, e.g., [32, Theorem 4.4.2]) that $C^*_r(S) = C_0(X) \rtimes_r G(S)$, where $X$ is a certain locally compact Hausdorff space (the spectrum of $S$). Thus, following similar reasonings as in the proof of the same implication of Theorems 4.10 and 4.19 it follows that if $G(S)$ is exact then so is $C^*_r(S)$.

We conclude the article with a remark and a question in relation to Theorem 4.23.

Remark 4.24. The proof of the preceding theorem shows that the property A of $\Lambda_S$ and that of $G(S)$ are strongly related in the E-unitary case. However, the proof is indirect (in the sense that it is based on $C^*$-properties). Moreover, it is well known (to experts) that, in general, the canonical projection $\sigma: S \to G(S)$ needs not coarsely embed any Schützenberger graphs of $S$ into $G(S)$, even in the finitely generated and E-unitary case. This suggests the following question.

$Q$: Is there a direct (coarse) geometric technique translating property A of the maximal homomorphic image $G(S)$ into property A of any Schützenberger graph of $S$ in the E-unitary case?

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