The number of rational points on a family of varieties over finite fields*

Shuangnian Hu and Shaofang Hong*

Mathematical College, Sichuan University, Chengdu 610064, P.R. China

E-mails: sfhong@scu.edu.cn, s-f.hong@tom.com, hongsf02@yahoo.com (S. Hong); hushuangnian@163.com (S. Hu)

Abstract. Let $F_q$ stand for the finite field of odd characteristic $p$ with $q$ elements ($q = p^n, n \in \mathbb{N}$) and $F_q^*$ denote the set of all the nonzero elements of $F_q$. Let $m$ and $t$ be positive integers. In this paper, by using the Smith normal form of the exponent matrix, we obtain a formula for the number of rational points on the variety defined by the following system of equations over $F_q$:

$$\sum_{j=0}^{t-1} \sum_{i=1}^{r_j} a_{k,r_j+i} x_1^{(k)} \ldots x_{n_j+1}^{(k)} = b_k, \ k = 1, \ldots, m.$$ 

where the integers $t > 0$, $r_0 = 0 < r_1 < r_2 < \ldots < r_t$, $1 \leq n_1 < n_2 < \ldots < n_t$, $0 \leq j \leq t-1$, $b_k \in F_q$, $a_{k,i} \in F_q^*$ ($k = 1, \ldots, m, i = 1, \ldots, r_t$), and the exponent of each variable is a positive integer. Furthermore, under some natural conditions, we arrive at an explicit formula for the number of the above variety. It extends the results obtained previously by Wolfmann, Sun, Wang, Song, Chen, Hong, Hu and Zhao et al. Our result also answers completely an open problem raised by Song and Chen.

Keywords and phrases: Finite field, hypersurface, rational point, exponent matrix, Smith normal form.

AMS Subject Classification: 11T06, 11T71

1 Introduction and statement of main result

Let $F_q$ be the finite field of $q$ elements with odd characteristic $p$ ($q = p^n, n \in \mathbb{N}$ (the set of positive integers)) and $F_q^*$ denote the nonzero elements of $F_q$. By $f_i(x_1, \ldots, x_n)$ ($i = 1, \ldots, m$) we denote some polynomials with $n$ variables over $F_q$ and $V$ stands for the following algebraic variety over $F_q$:

*Hong is the corresponding author and was supported partially by National Science Foundation of China Grant #11371260.
Let $N_q(V)$ denote the number of $F_q$-rational points on the algebraic variety $V$ in $F_q^n$. That is, $N_q(V) = \# \{(x_1, \ldots, x_n) \in F_q^n | f_i(x_1, \ldots, x_n) = 0, \ i = 1, \ldots, m \}$. Especially, we use $N_q(f)$ to denote $N_q(V)$ if $m = 1$.

Studying the exact value of $N_q(V)$ is one of the main topics in finite fields. The degrees $\deg(f_i)$ play an important role in the estimate of $N_q(V)$. Let $[x]$ denote the least integer $\geq x$ and $\ord_q$ denote the additive valuation such that $\ord_q q = 1$. In 1964, Ax [3] generalized the Chevalley-Warning theorem by showing that

$$\ord_q N_q(V) \geq \left\lfloor \frac{n - \sum_{i=1}^{m} \deg f_i}{\sum_{i=1}^{m} \deg f_i} \right\rfloor.$$

Later, further works were done by Katz [17], Adolphson-Sperber [11 -2], Moreno-Moreno [19] and Wan [25-27].

It is difficult to give an explicit formula for $N_q(V)$ in general. Finding explicit formula for $N_q(f)$ under certain conditions has attracted many authors for many years (see, for instance, [15] et al). It is well known that there exists an explicit formula for $N_q$ with $\deg(f) \leq 2$ in $F_q$ (see, for example, [10] and [18]). One first considered the diagonal hypersurface:

$$a_1x_1^{e_1} + \ldots + a_nx_n^{e_n} - b = 0, \ 1 \leq i \leq n, \ a_i \in F_q^*, \ b \in F_q, \ e_i > 0,$$

and much work has been done to seek for the number of rational points of the hypersurface (1.1), see, for instance, [23] and [30-31]. Carlitz [7], Cohen [10] and Hodges [11] counted the rational points on the following $k$-linear hypersurface

$$a_1x_{11} \ldots x_{1k} + a_2x_{21} \ldots x_{2k} + \ldots + a_nx_{n1} \ldots x_{nk} - b = 0,$$

with $a_i \in F_q^*, \ b \in F_q$. Cao [4], Cao and Sun [5] [6] studied the rational points on the following more general diagonal hypersurface

$$a_1x_1^{e_{11}} \ldots x_1^{e_{1n_1}} + a_2x_2^{e_{21}} \ldots x_2^{e_{2n_2}} + \ldots + a_rx_r^{e_{r1}} \ldots x_r^{e_{rn_r}} = 0,$$

with $1 \leq i \leq r, \ 1 \leq j \leq n_i, \ e_{ij} \in N, \ a_i \in F_q^*$. Clearly, this extends (1.1) and (1.2) when $b = 0$. On the other hand, Pan, Zhao and Cao [20] considered the following hypersurface $(a_1x_1^{m_1} + \ldots + a_nx_n^{m_n})^{\lambda} - bx_1^{k_1} \ldots x_n^{k_n} = 0$ which extended the results of Carlitz in [8] and [9].

If $f = a_1x_1^{e_{11}} \ldots x_1^{e_{1n}} + \ldots + a_rx_r^{e_{r1}} \ldots x_r^{e_{rn}} - b$ with $e_{ij} > 0$ and $a_i \in F_q^*$ for $1 \leq i \leq s$ and $1 \leq j \leq n, \ b \in F_q$, then a formula for $N_q(f)$ was given by Sun [24]. Moreover, if $s = n$ and $\gcd(\det(e_{ij}), q - 1) = 1$ (det($e_{ij}$)) represents the determinant of the $n \times n$ matrix $(e_{ij})$, then Sun [24] gave the explicit formula for the number of rational points as follows:
$$N(f) = \begin{cases} q^n - (q - 1)^n, & \text{if } b = 0, \\ \frac{(q-1)^n - (-1)^n(q-1)}{q}, & \text{if } b \neq 0. \end{cases}$$

Wang and Sun [28] gave the formula for the number of rational points of the following hypersurface

$$a_1x_1^{e_{11}} + a_2x_1^{e_{12}} + \ldots + a_nx_1^{e_{n1}}x_2^{e_{n2}} \ldots x_n^{e_{nn}} - b = 0,$$

with $e_{ij} \geq 0$, $a_i \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$. In 2005, Wang and Sun [29] extended the results of [24] and [28]. Recently, Hu, Hong and Zhao [13] generalized Wang and Sun’s results. In fact, they used the Smith normal form to present a formula for $N_q(f)$ with $f$ being given by:

$$f = \sum_{j=0}^{t-1} \sum_{i=1}^{r_j - r_{j+1}} a_{r_j + i}x_1^{e_{r_j+i,1}} \ldots x_{n_j+1}^{e_{r_j+i,n_j+1}} - b,$$

(1.3)

where the integers $t > 0$, $r_0 = 0 < r_1 < r_2 < \ldots < r_t$, $1 \leq n_1 < n_2 < \ldots < n_t$, $b \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^*$ ($1 \leq i \leq r_t$) and the exponents $e_{ij}$ of each variable are positive integers.

On the other hand, Yang [32] followed Sun’s method and gave a formula for the rational points $N_q(V)$ on the following variety $V$ over $\mathbb{F}_q$:

$$\left\{ \begin{array}{l}
a_{11}x_1^{e_{11}} \ldots x_n^{e_{1n}} + \ldots + a_{1s}x_1^{e_{1s}} \ldots x_n^{e_{1n}} - b_1 = 0, \\
\vdots \\
a_{m1}x_1^{e_{m1}} \ldots x_n^{e_{m1}} + \ldots + a_{ms}x_1^{e_{ms}} \ldots x_n^{e_{ms}} - b_m = 0.
\end{array} \right.$$

Very recently, Song and Chen [22] continued to make use of Sun’s method and obtained a formula for $N_q(V)$ with $V$ being the variety over $\mathbb{F}_q$ defined by:

$$\left\{ \begin{array}{l}
\sum_{j=1}^{s_1} a_{1j}x_1^{e_{1j}} \ldots x_{n_1}^{e_{1n_1}} + \sum_{j=s_1+1}^{s_2} a_{1j}x_1^{e_{1j}} \ldots x_{n_2}^{e_{1n_2}} - b_1 = 0, \\
\vdots \\
\sum_{j=1}^{s_1} a_{mj}x_1^{e_{mj}} \ldots x_{n_1}^{e_{mj}} + \sum_{j=s_1+1}^{s_2} a_{mj}x_1^{e_{mj}} \ldots x_{n_2}^{e_{mj}} - b_m = 0.
\end{array} \right.$$

Meanwhile, they proposed an open problem. To state this question, we need to introduce some notation. In what follows, we always let $t, m, r_1, \ldots, r_t, n_1, \ldots, n_t$ be positive integers such that $r_1 < \ldots < r_t$, $1 \leq n_1 < \ldots < n_t$ and $r_0 = 0$. For any integers $i, j$ and $k$ with $1 \leq i \leq r_t$, $0 \leq j < t - 1$ and $1 \leq k \leq m$, let $e_{ij}^{(k)} > 0$ be integers, $b_k \in \mathbb{F}_q$ and $a_{ki} \in \mathbb{F}_q^*$, and let $f_k(x) := f_k(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, \ldots, x_n]$ be defined by

$$f_k(x) := f_k(x_1, \ldots, x_n) = \sum_{i=1}^{r_t} a_{ki}x_i^{e_{ij}^{(k)}} - b_k,$$

(1.3)
For any integer \( k \), the exponent matrix of \( E^{(k)}_i \) defined as follows:

\[
\begin{align*}
E^{(k)}_1 &= (e^{(k)}_{11}, \ldots, e^{(k)}_{1 n_t}, 0, \ldots, 0), \quad x E^{(k)}_i = x_{e^{(k)}_{i,1}} \ldots x_{e^{(k)}_{i,n_t}}, \\
E^{(k)}_{r_1} &= (e^{(k)}_{r_1,1}, \ldots, e^{(k)}_{r_1, n_t}, 0, \ldots, 0), \quad x E^{(k)}_{r_1} = x_{e^{(k)}_{r_1,1}} \ldots x_{e^{(k)}_{r_1,n_t}}, \\
E^{(k)}_{r_1+1} &= (e^{(k)}_{r_1+1,1}, \ldots, e^{(k)}_{r_1+1, n_t}, 0, \ldots, 0), \quad x E^{(k)}_{r_1+1} = x_{e^{(k)}_{r_1+1,1}} \ldots x_{e^{(k)}_{r_1+1,n_t}}, \\
E^{(k)}_{r_2} &= (e^{(k)}_{r_2,1}, \ldots, e^{(k)}_{r_2, n_t}, 0, \ldots, 0), \quad x E^{(k)}_{r_2} = x_{e^{(k)}_{r_2,1}} \ldots x_{e^{(k)}_{r_2,n_t}}, \\
E^{(k)}_{r_{t-1}+1} &= (e^{(k)}_{r_{t-1}+1,1}, \ldots, e^{(k)}_{r_{t-1}+1, n_t}), \quad x E^{(k)}_{r_{t-1}+1} = x_{e^{(k)}_{r_{t-1}+1,1}} \ldots x_{e^{(k)}_{r_{t-1}+1,n_t}}, \\
E^{(k)}_{r_t} &= (e^{(k)}_{r_t,1}, \ldots, e^{(k)}_{r_t, n_t}), \quad x E^{(k)}_{r_t} = x_{e^{(k)}_{r_t,1}} \ldots x_{e^{(k)}_{r_t,n_t}}.
\end{align*}
\]

The following interesting question was raised in [22].

**Problem 1.1.** [22] Let \( f_1(x), \ldots, f_m(x) \) be given as in (1.3). What is the formula for the number of rational points on the variety defined by the following system of equations over \( \mathbb{F}_q \):

\[
\begin{align*}
f_1(x) &= 0, \\
&\quad \ldots \\
\text{for } i = 2, \ldots, m.
\end{align*}
\]

(1.4)

When \( m = 1 \), this question has been answered by Hu, Hong and Zhao [13]. When \( t = 2 \), this question was answered by Song and Chen [22]. However, if \( m \geq 2 \) and \( t \geq 3 \), then this problem has not been solved yet and is still kept open so far. Note that a more general question was proposed in [13] and a partial answer to this general question was given in [12].

In this paper, our main goal is to investigate Problem 1.1. We will follow and develop the method of [13] to study Problem 1.1. To state the main result, we first need to introduce some related concept and notation. For \( 1 \leq k \leq m \), the exponent matrix of \( f_k(x) \), denoted \( E_{f_k} \), is defined to be the \( r_t \times n_t \) matrix

\[
E_{f_k} := \begin{pmatrix}
E^{(k)}_1 \\
\vdots \\
E^{(k)}_{r_t}
\end{pmatrix}_{r_t \times n_t}.
\]

For any integer \( l \) with \( 1 \leq l \leq t \), let \( E_{f_k}^{(l)} \) be the \( r_l \times n_t \) submatrix of \( E_{f_k} \) consisting of the first \( r_l \) rows and the first \( n_t \) columns of \( E_{f_k} \). Furthermore, we define

With \( E_{f_k}^{(l)} \) being the vectors of non-negative integer components of dimension \( n_t \)
where such that α to the primitive element
\[ F \]

\[ \text{denoted by } \text{ind} \alpha \]

tees the existence of unimodular matrices \( U^{(l)} \) and \( V^{(l)} \) of order \( m r_l \) and \( n_l \) such that

\[ \begin{pmatrix} E^{(l)} \end{pmatrix}_{mr_l \times n_l} = \begin{pmatrix} E_{l, l}^{(l)} & \cdots & E_{l, m}^{(l)} \\ \vdots & \ddots & \vdots \\ E_{m, l}^{(l)} & \cdots & E_{m, m}^{(l)} \end{pmatrix} \]

Then the famous Smith normal form (see \[ \text{[21, 14]} \] or Section 2 below) guarantees the existence of unimodular matrices \( U^{(l)} \) and \( V^{(l)} \) of order \( m r_l \) and \( n_l \) such that

\[ U^{(l)} E^{(l)} V^{(l)} = \begin{pmatrix} D^{(l)} & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( D^{(l)} := \text{diag}(d_1^{(l)}, \ldots, d_{s_l}^{(l)}) \) with all the diagonal elements \( d_1^{(l)}, \ldots, d_{s_l}^{(l)} \) being positive integers such that \( d_1^{(l)} | \cdots | d_{s_l}^{(l)} \). Throughout pick \( \alpha \in \mathbb{F}_q^* \) to be a fixed primitive element of \( \mathbb{F}_q \). For any \( \beta \in \mathbb{F}_q^* \), there exists exactly an integer \( r \in [1, q-1] \) such that \( \beta = \alpha^r \). Such an integer \( r \) is called index of \( \beta \) with respect to the primitive element \( \alpha \) (or called the logarithm of \( \beta \) w.r.t. base \( \alpha \)), and is denoted by \( \text{ind}_\alpha(\beta) := r \).

Let \( k \) and \( l \) be integers with \( 1 \leq k \leq m \) and \( 1 \leq l \leq t \). For the variety defined by the following system of linear equations over \( \mathbb{F}_q^* \):

\[
\begin{align*}
\sum_{i=1}^{r_1} a_{1i} u_{1i} &= b_1, \\
\vdots & \quad \vdots \\
\sum_{i=1}^{r_t} a_{mi} u_{mi} &= b_m, 
\end{align*}
\]

(1.5)

where \( a_{k1}, \ldots, a_{kr_t}, b_k \in \mathbb{F}_q^* \) being given in (1.4), we use \( N_l \) to denote the number of rational points \( (u_{11}, \ldots, u_{1r_1}, \ldots, u_{m1}, \ldots, u_{mr_l}) \in (\mathbb{F}_q^*)^{m r_l} \) of (1.5) under the following extra conditions:

\[ \begin{align*}
\gcd(q-1, d_1^{(l)} | h'_i) & \quad \text{for } i = 1, \ldots, s_l, \\
(q-1) | h'_i & \quad \text{for } i = s_l + 1, \ldots, m r_l, 
\end{align*} \]

(1.6)

where

\[ (h'_1, \ldots, h'_{m r_l})^T := U^{(l)} \text{ind}_\alpha(u_{11}), \ldots, \text{ind}_\alpha(u_{1r_1}), \ldots, \text{ind}_\alpha(u_{m1}), \ldots, \text{ind}_\alpha(u_{mr_l})^T. \]

Note that by Lemma 2.5 below, one knows that \( N_l \) is independent of the choice of the primitive element \( \alpha \). In what follows, we let \( N^{(n_1, \ldots, n_t)} \) stand for the number of rational points on the variety defined by (1.4). Now let

\[ N_{r_0}^{(n_1, \ldots, n_t)} := q^{n_1-1} \prod_{j=1}^{s_t} \gcd(q-1, d_j^{(l)}). \]

(1.7)

\[ N_{r_t}^{(n_1, \ldots, n_t)} := N_t(q-1)^{n_1-s_t} \prod_{j=1}^{s_t} \gcd(q-1, d_j^{(l)}) \]

(1.8)

and for \( 1 \leq l \leq t-1 \), let
Then we are in a position to state the main result of this paper.

**Theorem 1.2.** We have

\[
N_{r_1,\ldots,r_t} := \sum_{i=0}^t N_{r_i}^{(n_1,\ldots,n_t)}(q^{-1})^{n_i-n_1}(q^{n_i+1-n_1}-(q-1)^{n_i+1-n_1}) \prod_{j=1}^{s_j} \gcd(q-1, d_j^{(i)}).
\]

(1.9)

Evidently, Theorem 1.2 extends the main results of [13] and [22], and answers completely Problem 1.1.

This paper is organized as follows. In Section 2, we recall some useful known lemmas which will be needed later. Subsequently, in Section 3, we make use of the results presented in Section 2 to show Theorem 1.2. Also we supply some interesting corollaries. Finally, in Section 4, we provide two examples to demonstrate the validity of Theorem 1.2.

Throughout this paper, we use gcd\((a, m)\) to denote the greatest common divisor of any positive integers \(a\) and \(m\).

## 2 Preliminary lemmas

In this section, we present some useful lemmas that are needed in section 3. We first recall two well known definitions.

**Definition 2.1.** [14] Let \(M\) be a square integer matrix. If the determinant of \(M\) is \(\pm 1\), then \(M\) is called a **unimodular matrix**.

**Definition 2.2.** [14] For given any positive integers \(m\) and \(n\), let \(P\) and \(Q\) be two \(m \times n\) integer matrices. Suppose that there are two modular matrices \(U\) of order \(m\) and \(V\) of order \(n\) such that \(P = UQV\). Then we say that \(P\) and \(Q\) are equivalent and we write \(P \sim Q\).

Clearly, the equivalence has the three properties of being reflexive, symmetric and transitive.

**Lemma 2.1.** [14] [21] Let \(P\) be a nonzero \(m \times n\) integer matrix. Then \(P\) is equivalent to a block matrix of the following form

\[
\begin{pmatrix}
D & 0 \\
0 & 0
\end{pmatrix},
\]

(2.1)
where $D = \text{diag}(d_1, \ldots, d_r)$ with all the diagonal elements $d_i$ being positive integers and satisfying that $d_i | d_{i+1}$ $(1 \leq i < r)$. In other words, there are unimodular matrices $U$ of order $m$ and $V$ of order $n$ such that

$$UPV = \begin{pmatrix} \text{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix}. $$

We call the diagonal matrix in (2.1) the Smith normal form of the matrix $P$. Usually, one writes the Smith normal form of $P$ as SNF($P$). The elements $d_i$ are unique up to multiplication by a unit and are called the elementary divisors, invariants, or invariant factors.

For any system of linear congruences

$$\begin{align*}
\sum_{j=1}^{n} h_{1j} y_j &\equiv b_1 \pmod{m}, \\
\ldots &\ldots \\
\sum_{j=1}^{n} h_{sj} y_j &\equiv b_s \pmod{m},
\end{align*}$$

(2.2)

let $Y = (y_1, \ldots, y_n)^T$ be the column of indeterminates $y_1, \ldots, y_n$, $B = (b_1, \ldots, b_s)^T$ and $H = (h_{ij})$ be the matrix of its coefficient. Then one can write (2.2) as

$$HY \equiv B \pmod{m}. $$

(2.3)

By Lemma 2.1, there are unimodular matrices $U$ of order $s$ and $V$ of order $n$ such that

$$UHV = \text{SNF}(H) = \begin{pmatrix} \text{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{pmatrix}. $$

Consequently, we have the following lemma.

**Lemma 2.2.** [13] Let $B' = (b'_1, \ldots, b'_s)^T = UB$. Then the system (2.3) of linear congruences is solvable if and only if $\text{gcd}(m, d_i) | b'_i$ for all integers $i$ with $1 \leq i \leq r$ and $m | b'_i$ for all integers $i$ with $r + 1 \leq i \leq s$. Further, the number of solutions of (2.3) is equal to $m^{n-r} \prod_{i=1}^{r} \text{gcd}(m, d_i)$.

The following result is due to Sun [24].

**Lemma 2.3.** [24] Let $c_1, \ldots, c_k \in \mathbb{F}_q^*$ and $c \in \mathbb{F}_q$, and let $N(c)$ denote the number of rational points $(x_1, \ldots, x_k) \in (\mathbb{F}_q^*)^k$ on the hypersurface $c_1 x_1 + \ldots + c_k x_k = c$. Then

$$N(c) = \begin{cases} 
\frac{(q-1)^k + (-1)^k(q-1)}{q}, & \text{if } c = 0, \\
\frac{q^k - (-1)^k k}{q}, & \text{otherwise.}
\end{cases}$$

**Lemma 2.4.** Let $c_{ij} \in \mathbb{F}_q^*$ for all integers $i$ and $j$ with $1 \leq i \leq m$ and $1 \leq j \leq k$ and $c_1, \ldots, c_m \in \mathbb{F}_q$. Let $N(c_1, \ldots, c_m)$ denote the number of rational
points \((x_{11}, ..., x_{1k}, ..., x_{m1}, ..., x_{mk}) \in (\mathbb{F}_q)^{mk}\) on the following variety

\[
\begin{align*}
c_{11}x_{11} + ... + c_{1k}x_{1k} &= c_1, \\
... & \quad \quad \\
c_{m1}x_{m1} + ... + c_{mk}x_{mk} &= c_m.
\end{align*}
\]

Then

\[
N(c_1, ..., c_m) = \frac{(q-1)^r}{q^m}((q-1)^{k-1} + (-1)^k)^r((q-1)^k - (-1)^k)^{m-r},
\]

where \(r := \#\{1 \leq i \leq m | c_i = 0\}\).

Proof. For \(1 \leq i \leq m\), let \(N(c_i)\) denote the number of rational points \((x_{i1}, ..., x_{ik}) \in (\mathbb{F}_q)^k\) on the hypersurface \(c_{i1}x_{i1} + ... + c_{ik}x_{ik} = c_i\). Since for any rational points \((x_{i1}, ..., x_{1k}, ..., x_{m1}, ..., x_{mk}) \in (\mathbb{F}_q)^{mk}\) on the variety (2.4), the involved variables are different from equation to equation, one has

\[
N(c_1, ..., c_m) = \prod_{i=1}^{m} N(c_i).
\]

So Lemma 2.3 applied to \(N(c_i)\) gives us the required result. Hence Lemma 2.4 is proved.

Definition 2.3. Let \(k\) be a positive integer. We say that the column vector \(\begin{pmatrix} c_1 \\
... \\
c_k \end{pmatrix} \in \mathbb{Z}^k\) of dimension \(k\) divides the column vector \(\begin{pmatrix} d_1 \\
... \\
d_k \end{pmatrix} \in \mathbb{Z}^k\) of dimension \(k\) if \(c_i\) divides \(d_i\) for all integers \(i\) with \(1 \leq i \leq k\). The divisibility between two row integer vectors can be defined similarly.

Lemma 2.5. Let \(\alpha, \beta\) be two primitive elements of \(\mathbb{F}_q^*\) and \(k\) be a positive integer. Let \(u_1, ..., u_k\) be nonzero elements of \(\mathbb{F}_q\) and each of \(d_1, ..., d_k\) divides \(q-1\). Let \(U\) be a unimodular matrix. Then \((d_1, ..., d_k)^T\) divides \(U(\ind_\alpha u_1, ..., \ind_\alpha u_k)^T\) if and only if \((d_1, ..., d_k)^T\) divides \(U(\ind_\beta u_1, ..., \ind_\beta u_k)^T\).

Proof. First of all, for any integer \(i\) with \(1 \leq i \leq k\), one has

\[
\ind_\alpha u_i \equiv \ind_\alpha \beta \cdot \ind_\beta u_i \pmod{q-1}.
\]

It then follows that

\[
U \begin{pmatrix} \ind_\alpha u_1 \\
... \\
\ind_\alpha u_k \end{pmatrix} \equiv \ind_\alpha \beta \cdot U \begin{pmatrix} \ind_\beta u_1 \\
... \\
\ind_\beta u_k \end{pmatrix} \pmod{q-1}.
\]

On the one hand, since all of \(d_1, ..., d_k\) divide \(q-1\), by (2.5) one knows that \((d_1, ..., d_k)^T\) divides \(U(\ind_\alpha u_1, ..., \ind_\alpha u_k)^T\) if and only if \((d_1, ..., d_k)^T\) divides \(\ind_\alpha \beta \cdot U(\ind_\beta u_1, ..., \ind_\beta u_k)^T\).
On the other hand, since $\alpha$ and $\beta$ are primitive elements, $\text{ind}_\alpha \beta$ is coprime to $q - 1$. Hence $\text{ind}_\alpha \beta$ is coprime to each of $d_1, ..., d_k$. Then one can derive that $(d_1, ..., d_k)^T$ divides $\text{ind}_\alpha \beta \cdot U(\text{ind}_\beta u_1, ..., \text{ind}_\beta u_k)^T$ if and only if $(d_1, ..., d_k)^T$ divides $U(\text{ind}_\beta u_1, ..., \text{ind}_\beta u_k)^T$.

Finally, the desired result follows immediately. So Lemma 2.5 is proved.

Remark 2.1. If $N_l(\alpha)$ stands for the number of rational points $(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in (\mathbb{F}_q^*)^{m_{r_1}}$ of (1.5) under the extra conditions (1.6) with respect to the primitive element $\alpha$, then by Lemma 2.5 we have that $N_l(\beta) = N_l(\gamma)$ for any primitive elements $\beta$ and $\gamma$. So we can use $N_l$ to denote the number of rational points $(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in (\mathbb{F}_q^*)^{m_{r_1}}$ of (1.5) under the extra conditions (1.6) with respect to any given primitive element $\alpha$.

3 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. First, we present some notation and two lemmas. For any given $(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in \mathbb{F}_q^{m_{r_1}}$, we use

$$N(x^{E_{(k)}} = u_{ki}, k = 1, ..., m, i = 1, ..., r_t)$$

to denote the number of rational points $(x_1, ..., x_{n_1}) \in \mathbb{F}_q^{n_1}$ on the following algebraic variety over $\mathbb{F}_q$:

$$x^{E_{(k)}} = u_{ki}, k = 1, ..., m, i = 1, ..., r_t, \quad (3.1)$$

Define $T$ to be the set of rational points $(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in \mathbb{F}_q^{m_{r_1}}$ on the variety $\sum_{i=1}^{r_1} a_{ki} u_{ki} = b_k, k = 1, ..., m$. Namely,

$$T := \{(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in \mathbb{F}_q^{m_{r_1}} : \sum_{i=1}^{r_1} a_{ki} u_{ki} = b_k, k = 1, ..., m\},$$

with $b_k$ and $a_{ki}$ ($1 \leq k \leq m, 1 \leq i \leq r_t$) being given as in (1.3).

Let $T(0)$ consist of zero vector of dimension $m_{r_1}$. For any integer $n$ with $1 \leq n \leq r_t$, let $T(n)$ denote the subset of $T$ in which the vector holds exactly $mn$ nonzero components. Define

$$M^{(n_1, ..., n_t)} := \sum_{(u_{11}, ..., u_{1r_1}, ..., u_{m_1}, ..., u_{m_{r_1}}) \in T(n)} N(x^{E_{(k)}} = u_{ki}, k = 1, ..., m, i = 1, ..., r_t). \quad (3.2)$$

Then we have the following lemma.

Lemma 3.1. One has

$$N^{(n_1, ..., n_t)} = \sum_{n=0}^{r_t} M^{(n_1, ..., n_t)}. \quad (3.2)$$
Proof. First of all, by (1.3) and (3.1) we obtain that
\[ N(n_1,\ldots,n_t) = \sum_{(u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T^{mr_t}} N(x^{(k)}_{ki} = u_{ki}, k = 1,\ldots,m, i = 1,\ldots,r_t). \]

It follows from the definition of \( T \) that
\[ N(n_1,\ldots,n_t) = \sum_{(u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T} N(x^{(k)}_{ki} = u_{ki}, k = 1,\ldots,m, i = 1,\ldots,r_t). \]

(3.3)

Now let \((u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T\) and consider the rational points \((x_1,\ldots,x_{n_t}) \in \mathbb{P}_{q_t}^{n_t}\) on the variety (3.1). For any integer \( k \) with \( 1 \leq k \leq m \), one can easily deduce that for any given integer \( i \) with \( 0 < i < r_t \), if \( u_{ki} = 0 \), then \( u_{k,i+1} = 0 \). Thus if \((u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T\) is a nonzero vector, then it must hold exactly \( mn \) nonzero components for some integer \( n \) with \( 1 \leq n \leq r_t \). For any integer \( n \) with \( 1 \leq n \leq r_t \), let \( T(n) \) denote the subset of \( T \) in which the vector holds exactly \( mn \) nonzero components. So if \((u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T(n)\), then we have that \( u_{k1} \neq 0,\ldots,u_{kn} \neq 0 \) and \( u_{k,n+1} = \ldots = u_{k,r_t} = 0 \) for all integers \( k \) with \( 1 \leq k \leq m \). We claim that if \((u_{11},\ldots,u_{1r_{1}},\ldots,u_{m1},\ldots,u_{mr_t}) \in T\) holds exactly \( mn \) nonzero components, then \( n \) equals one of \( r_1,\ldots,r_t \). In the following we will prove the claim. Let \( n \in \{1,\ldots,r_t\} \setminus \{r_1,\ldots,r_t\} \). Then one can find an integer \( l \) with \( 1 \leq l \leq t \) such that \( r_{l-1} < n < r_l \). Assume that \( T(n) \) is nonempty. On the one hand, by the definition of \( T(n) \), one has \( u_{k1} \neq 0,\ldots,u_{kn} \neq 0 \) and \( u_{k,n+1} = \ldots = u_{k,r_t} = 0 \) for all integers \( k \) with \( 1 \leq k \leq m \). On the other hand, from \( u_{k,n+1} = 0 \), and noting that \( u_{kn} = x_1^{(k)}_{n,1} \ldots x_{n_t}^{(k)}_{n,1} \) and \( u_{k,n+1} = x_1^{(k)}_{n+1,1} \ldots x_{n_t}^{(k)}_{n+1,1} \), we infer that \( u_{kn} = 0 \). This contradicts to \( u_{kn} \neq 0 \). So the assumption is not true, and the claim is proved.

Now let \( T(0) \) consist of zero vector of dimension \( mr_t \). Then by the claim we have
\[ T = \bigcup_{n=0}^{r_t} T(n). \]

Thus from (3.3) one derives that
\[ N(n_1,\ldots,n_t) = \sum_{n=0}^{r_t} M_n^{(n_1,\ldots,n_t)} \]
as required. This ends the proof of Lemma 3.1. \( \square \)

Consequently, we compute \( M_n^{(n_1,\ldots,n_t)} \) for integers \( l \) with \( 1 \leq l \leq t \).

**Lemma 3.2.** For all integers \( l \) with \( 1 \leq l \leq t \), we have
\[ M_l^{(n_1,\ldots,n_t)} = N_{r_l}^{(n_1,\ldots,n_t)}. \]
Proof. Evidently, one has
\[ M_{r_i}^{(n_1, \ldots, n_t)} = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in T(r_i)} N(x_{E_i}^{(k)} = u_{k_i}, k = 1, \ldots, m, \ i = 1, \ldots, r_i) \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr_i}} N(x_{E_i}^{(k)} = u_{k_i}, k = 1, \ldots, m, \ i = 1, \ldots, r_i). \quad (3.4) \]

Now we let \( l \) be a given integer with \( 1 \leq l \leq t - 1 \). Then
\[ M_{r_i}^{(n_1, \ldots, n_t)} \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in T(r_i)} N(x_{E_i}^{(k)} = u_{k_i}, k = 1, \ldots, m, \ i = 1, \ldots, r_i) \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr_i}} N(x_{E_i}^{(k)} = u_{k_i}, k = 1, \ldots, m, \ i = 1, \ldots, r_i) \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr_i}} N(x_{E_i}^{(k)} = u_{k_i} \text{ and } x_{E_i}^{(r_i+1)} = 0, \]
\[ \text{for } 1 \leq k \leq m, 1 \leq i \leq r_i \text{ and } r_i+1 \leq j \leq r_l \). \quad (3.5) \]

But the definition of \( x_{E_i}^{(k)} \) tells us that the fact that \( x_{E_i}^{(k)} = 0 \) for \( r_i + 1 \leq j \leq r_l \)

is reduced to saying that \( x_{E_i}^{(r_i+1)} = 0 \). It then follows from (3.5) that
\[ M_{r_i}^{(n_1, \ldots, n_t)} \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr_i}} N(x_{E_i}^{(k)} = u_{k_i} \text{ for } 1 \leq k \leq m, 1 \leq i \leq r_i \text{ and } x_{E_i}^{(r_i+1)} = 0). \quad (3.6) \]

It is easy to see that \( x_{E_i}^{(r_i+1)} = 0 \) is equivalent to \( x_1 \ldots x_{n_i} \ldots x_{n_i+1} = 0 \).

Since \( u_{k_i} \neq 0 \) and \( u_{k_i} = x_{E_i}^{(k)} \), one has \( x_1 \ldots x_{n_i} \neq 0 \). So \( x_{E_i}^{(r_i+1)} = 0 \) is equivalent to \( x_{n_i+1} \ldots x_{n_i+1} = 0 \). Then by (3.6), one gets that for \( 1 \leq l \leq t - 1 \),
\[ M_{r_i}^{(n_1, \ldots, n_t)} \]
\[ = \sum_{(u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr_i}} N(x_{E_i}^{(k)} = u_{k_i}, 1 \leq k \leq m, 1 \leq i \leq r_i \text{ and } x_{n_i+1} \ldots x_{n_i+1} = 0). \quad (3.7) \]

For any given \((u_1, \ldots, u_{r_i}, \ldots, u_m) \in (\mathbb{F}_q)^{mr} \) with \( \sum_{i=1}^{r_i} a_{k_i} u_{k_i} = b_k (k = 1, \ldots, m) \), one has
\[ N(x_{E_i}^{(k)} = u_{k_i}, 1 \leq k \leq m, 1 \leq i \leq r_i \text{ and } x_{n_i+1} \ldots x_{n_i+1} = 0) = \]
\[ \#\{(x_1, \ldots, x_{n_i}) \in (\mathbb{F}_q)^{mr} : x_{E_i}^{(k)} = u_{k_i}, 1 \leq k \leq m, 1 \leq i \leq r_i \text{ and } x_{n_i+1} \ldots x_{n_i+1} = 0\} \].
Since each of the components \( x_{n_{i+1}} \) can run over the whole finite field \( \mathbb{F}_q \) independently, it then follows that

\[
N(x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l \text{ and } x_{n_{i+1}} \cdot x_{n_{i+1}} = 0)
\]

\[
= q^{n_{i+1}-n_i} \times \# \{(x_1, ..., x_{n_{i+1}}) \in (\mathbb{F}_q)^{n_{i+1}} : x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l \text{ and } x_{n_{i+1}} \cdot x_{n_{i+1}} = 0\}. \tag{3.8}
\]

Notice that the choice of \((x_1, ..., x_{n_i}) \in (\mathbb{F}_q)^{n_i}\) satisfying that \(x^{E_i(k)} = u_{ki}\) \((k = 1, ..., m, i = 1, ..., r_l)\) is independent of the choice of \((x_{n_{i+1}}, ..., x_{n_{i+1}}) \in (\mathbb{F}_q)^{n_{i+1}-n_i}\) satisfying that \(x_{n_{i+1}} \cdot x_{n_{i+1}} = 0\). We then derive that

\[
\# \{(x_1, ..., x_{n_{i+1}}) \in (\mathbb{F}_q)^{n_{i+1}+n_i} : x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l \text{ and } x_{n_{i+1}} \cdot x_{n_{i+1}} = 0\}
\]

\[
= \# \{(x_1, ..., x_{n_i}) \in (\mathbb{F}_q)^{n_i} : x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l\} \times \# \{(x_{n_{i+1}}, ..., x_{n_{i+1}}) \in (\mathbb{F}_q)^{n_{i+1}-n_i} : x_{n_{i+1}} \cdot x_{n_{i+1}} = 0\}. \tag{3.9}
\]

On the other hand, we can easily compute that

\[
\sum_{i=1}^{n_i} \binom{n_{i+1}-n_i}{i} (q-1)^{n_{i+1}-n_i-i} = q^{n_{i+1}-n_i} - (q-1)^{n_{i+1}-n_i}. \tag{3.10}
\]

So by (3.8) to (3.10), one obtains that

\[
N(x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l \text{ and } x_{n_{i+1}} \cdot x_{n_{i+1}} = 0)
\]

\[
= q^{n_{i+1}-n_i}(q^{n_{i+1}-n_i} - (q-1)^{n_{i+1}-n_i}) \times \# \{(x_1, ..., x_{n_i}) \in (\mathbb{F}_q)^{n_i} : x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l\}
\]

\[
= q^{n_{i+1}-n_i}(q^{n_{i+1}-n_i} - (q-1)^{n_{i+1}-n_i})N(x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l). \tag{3.11}
\]

Then by (3.7) together with (3.11), we have

\[
M_{r_l}^{(n_1, ..., n_i)} = q^{n_{i+1}-n_i}(q^{n_{i+1}-n_i} - (q-1)^{n_{i+1}-n_i}) \times \sum_{(u_1, ..., u_{r_l}, ..., u_{m_{r_l}}) \in (\mathbb{F}_q)^{m_{r_l}}} \binom{n_i}{k_{r_1} + \ldots + k_{r_l}} u_{k_{r_1}} = b_{k_{r_1}, k_{r_2}, \ldots, k_{r_{r_l}} = b_k, k = 1, ..., m}. \tag{3.12}
\]

Now we treat with the sum

\[
\sum_{(u_1, ..., u_{r_l}, ..., u_{m_{r_l}}) \in (\mathbb{F}_q)^{m_{r_l}}} \binom{n_i}{k_{r_1} + \ldots + k_{r_l}} u_{k_{r_1}} = b_{k_{r_1}, k_{r_2}, \ldots, k_{r_{r_l}} = b_k, k = 1, ..., m}, \tag{3.13}
\]

where \(l = 1, ..., t\). First, for any given \((u_1, ..., u_{r_l}, ..., u_{m_{r_l}}) \in (\mathbb{F}_q)^{m_{r_l}}\) with \(\sum_{i=1}^{r_l} a_{k_{r_1}} u_{k_{r_1}} = b_k (k = 1, ..., m)\), \(N(x^{E_i(k)} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_l)\) equals the number of rational points \((x_1, ..., x_{n_i}) \in (\mathbb{F}_q)^{n_i}\) on the following algebraic variety:
Since $u_{k1} \ldots u_{kr1} \neq 0$ ($1 \leq k \leq m$), we infer that the number of the rational points $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^m$ of (3.13) is equal to the number of nonnegative integral solutions $(\text{ind}_{\alpha}(x_1), \ldots, \text{ind}_{\alpha}(x_n)) \in \mathbb{N}^n$ of the following system of congruences

$$
\begin{align*}
\sum_{i=1}^{n} c_{1i}^{(1)} \text{ind}_{\alpha}(x_i) &\equiv \text{ind}_{\alpha}(u_{11}) \pmod{q - 1}, \\
\cdots & \\
\sum_{i=1}^{n} c_{r1i}^{(1)} \text{ind}_{\alpha}(x_i) &\equiv \text{ind}_{\alpha}(u_{1r1}) \pmod{q - 1}, \\
\cdots & \\
\sum_{i=1}^{n} c_{1i}^{(m)} \text{ind}_{\alpha}(x_i) &\equiv \text{ind}_{\alpha}(u_{m1}) \pmod{q - 1}, \\
\cdots & \\
\sum_{i=1}^{n} c_{r1i}^{(m)} \text{ind}_{\alpha}(x_i) &\equiv \text{ind}_{\alpha}(u_{mr1}) \pmod{q - 1}.
\end{align*}
$$

But Lemma 2.2 tells us that (3.14) has solutions $(\text{ind}_{\alpha}(x_1), \ldots, \text{ind}_{\alpha}(x_n)) \in \mathbb{N}^n$ if and only if the extra conditions (1.6) hold. Further, Lemma 2.2 gives us the number of solutions $(\text{ind}_{\alpha}(x_1), \ldots, \text{ind}_{\alpha}(x_n)) \in \mathbb{N}^n$ of (3.14) which is equal to $(q - 1)^{n_1 - s_1} \prod_{i=1}^{s_1} \gcd(q - 1, d_i^{(l)})$. Hence

$$N\left(E_{\alpha}^{(k)}=u_{k1}, 1 \leq k \leq m, 1 \leq l \leq r_1\right) = (q - 1)^{n_1 - s_1} \prod_{i=1}^{s_1} \gcd(q - 1, d_i^{(l)}), \quad (3.15)$$

Notice that

$$N_l = \sum_{(u_{11}, \ldots, u_{1r1}; \ldots; u_{m1}, \ldots, u_{mr1}) \in (\mathbb{F}_q^n)^m} 1.$$
It then follows from (3.15) that
\[
\sum_{(u_{11}, ..., u_{1r}, ..., u_{m1}, ..., u_{mr}) \in (\mathbb{F}_q^*)^{mr}} N(x^{s_{i(k)}} = u_{ki}, 1 \leq k \leq m, 1 \leq i \leq r_i)
\]
\[
= \sum_{(u_{11}, ..., u_{1r}, ..., u_{m1}, ..., u_{mr}) \in (\mathbb{F}_q^*)^{mr}} (q - 1)^{n_i - s_i} \prod_{i=1}^{s_j} \gcd(q - 1, d_i^{(l)})
\]
\[
= (q - 1)^{n_i - s_i} \prod_{i=1}^{s_j} \gcd(q - 1, d_i^{(l)}) \sum_{(u_{11}, ..., u_{1r}, ..., u_{m1}, ..., u_{mr}) \in (\mathbb{F}_q^*)^{mr}} 1
\]
\[
= N_i(q - 1)^{n_i - s_i} \prod_{i=1}^{s_j} \gcd(q - 1, d_i^{(l)}).
\]
(3.16)

For $1 \leq l \leq r$, using (3.3), (3.12) and (3.16), we obtain the desired result
\[
M_{r_1}^{(n_1, ..., n_t)} = N_{r_1}^{(n_1, ..., n_t)}.
\]
This concludes the proof of Lemma 3.2. 

We can now turn our attention to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** First we show that the following is true:
\[
M_0^{(n_1, ..., n_t)} := \begin{cases} 
N_0^{(n_1, ..., n_t)}, & \text{if } b_1 = ... = b_m = 0, \\
0, & \text{otherwise.}
\end{cases}
\]
(3.17)

Let $b_i \neq 0$ for some integer $i$ with $1 \leq i \leq m$. Then the following variety
\[
\begin{align*}
a_{11}u_{11} + ... + a_{1,r_1}u_{1,r_1} &= b_1 \\
&\vdots \\
a_{m1}u_{m1} + ... + a_{m,r_m}u_{m,r_m} &= b_m
\end{align*}
\]
does not contain the original point $(0, ..., 0) \in \mathbb{P}^{mr}$. So $T(0)$ is empty. It then follows from (3.2) that $M_0^{(n_1, ..., n_t)} = 0$. That is, the second part of (3.17) is true.

Now let $b_1 = ... = b_m = 0$. Then $T(0)$ consists of zero vector of dimension $mr$. Thus by (3.2), we have
\[
M_0^{(n_1, ..., n_t)} = N(x^{E_{i(k)}} = 0, k = 1, ..., m, i = 1, 2, ..., r_i)
\]
\[
= N(x^{E_{i(1)}} = 0)
\]
\[
= N(x_1^{(1)}, ..., x_{r_1}^{(1)} = 0)
\]
\[
= q^{n_t - n_1} \sum_{j=1}^{n_1} \binom{n_1}{j} (q - 1)^{n_1 - j}
\]
\[
= q^{n_t - n_1}(q^{n_1} - (q - 1)^{n_1}) = N_0^{(n_1, ..., n_t)}
\]

14
as required. This completes the proof of (3.17).

Consequently, we show that for all integers $n \in \{1, \ldots, r_t\} \setminus \{r_1, \ldots, r_t\}$, one has
\[
M_k^{(n_1, \ldots, n_t)} = 0. \tag{3.18}
\]
To prove it, we choose an $n \in \{1, 2, \ldots, r_t\} \setminus \{r_1, \ldots, r_t\}$. Then there exists an integer $l$ with $1 \leq l \leq t$ such that $r_{l-1} < n < r_l$. Claim that $T(n)$ is empty which will be shown in what follows.

Suppose that $T(n)$ is nonempty. On the one hand, the definition of $T(n)$ gives us that for all integers $k$ with $1 \leq k \leq m$, we have $u_{k1} \neq 0, \ldots, u_{kn} \neq 0$ and $u_{kn+1} = \ldots = u_{k,r_t} = 0$. On the other hand, from $u_{k,n+1} = 0$, and noting that
\[
u_{kn} = x_1^{e_n(1)} \ldots x_{n_i}^{e_n(n_i)}, \quad u_{k,n+1} = x_1^{e_{n+1}(1)} \ldots x_{n_i}^{e_{n+1}(n_i)},
\]
we deduce that $u_{kn} = 0$. This contradicts with the fact $u_{kn} \neq 0$. So the assumption is not true. Hence $T(n) = \emptyset$. Thus by (3.2),
\[
M_k^{(n_1, \ldots, n_t)} = 0.
\]
This finishes the proof of (3.18).

Finally, using Lemmas 3.1 and 3.2, (3.17) and (3.18), the desired result follows immediately. So Theorem 1.2 is proved.

In concluding this section, we present an interesting corollary.

**Corollary 3.1.** If $\text{SNF}(E^{(l)}) = (D^{(l)} \ 0)$ and $\gcd(\det D^{(l)}, q - 1) = 1$ for all integers $l$ with $1 \leq l \leq t$, then the number $N^{(n_1, \ldots, n_t)}$ of rational points on the variety (1.4) is given by
\[
N^{(n_1, \ldots, n_t)} = \left\{ \begin{array}{ll}
\sum_{i=0}^{t} \hat{N}^{(n_1, \ldots, n_t)}_i, & \text{if } b_1 = \ldots = b_m = 0, \\
\sum_{i=1}^{t} \hat{N}^{(n_1, \ldots, n_t)}_i, & \text{otherwise},
\end{array} \right.
\]
where
\[
\hat{N}^{(n_1, \ldots, n_t)}_0 := q^{n_1-n_1_1}(q^{n_1_1} - (q - 1)^{n_1_1}), \quad \hat{N}^{(n_1, \ldots, n_t)}_i := \hat{N}_t(q - 1)^{n_i-s_i} \quad \text{and} \quad \hat{N}^{(n_1, \ldots, n_t)}_i := q^{n_i-n_i_1+1}(q^{n_i_1-1} - (q - 1)^{n_i_1-1})\hat{N}_l(q - 1)^{n_i-s_i} \quad (l = 1, \ldots, t - 1),
\]
and for all integers $k$ with $1 \leq k \leq t$, one has
\[
\hat{N}_k := \frac{(q - 1)^r}{q^m}((q - 1)^{r_{k-1}} + (-1)^r)((q - 1)^{r_{k}} - (-1)^{r_{k}})^{n_k-r},
\]
where $r := \#\{1 \leq i \leq m | b_i = 0\}$.

**Proof.** Since $\text{SNF}(E^{(l)}) = (D^{(l)} \ 0)$ and $\gcd(\det D^{(l)}, q - 1) = 1$ for all integers $l$ with $1 \leq l \leq t$, we derive that
\[
s_l = m r_l \quad \text{and} \quad \gcd(q - 1, d^{(l)}_j) = 1
\]
for all integers $j$ with $1 \leq j \leq s_t$. Thus for any solution $(u_{11}, \ldots, u_{1,r_t}, \ldots, u_{m,1}, \ldots, u_{m,r_t}) \in (\mathbb{F}_q)^m r_t$ of (1.5), the extra conditions (1.6) are always true. It then follows from Theorem 1.2 and Lemma 2.4 that the desired result follows immediately. This finishes the proof of Corollary 3.1.

\[\square\]
4 Two examples

In this section, we supply two examples to illustrate the validity of our main result.

Example 4.1. We use Corollary 3.1 to compute the number $N(n_1, n_2)$ of rational points on the following variety over $\mathbb{F}_{11}$:

$$\begin{align*}
x_1x_2^2x_3^2 + x_1^2x_2^2x_3^2 + x_1^2x_3^2 + x_4^2x_5^2 + x_6^2 &= 4, \\
x_1x_2^2x_3 + x_1x_2x_3^2 + x_4x_5 + x_6 &= 0.
\end{align*}$$

Clearly, we have $n_1 = 3$, $n_2 = 6$, $r_1 = 1$, $r_2 = 3$, $m = 2$,

$$E^{(1)} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 5 & 3 \end{pmatrix} \quad \text{and} \quad E^{(2)} = \begin{pmatrix} 1 & 3 & 2 & 0 & 0 & 0 \\ 5 & 7 & 5 & 1 & 2 \\ 5 & 4 & 3 & 2 & 6 & 3 \\ 1 & 5 & 3 & 0 & 0 & 0 \\ 3 & 5 & 6 & 5 & 4 & 7 \\ 3 & 1 & 5 & 7 & 3 & 7 \end{pmatrix}.$$ 

One can easily deduce that the Smith normal forms of $E^{(1)}$ and $E^{(2)}$ are given as follows:

$$\text{SNF}(E^{(1)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{SNF}(E^{(2)}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 291 \end{pmatrix}.$$ 

Thus $s_1 = 2$ and $s_2 = 6$. It follows from Corollary 3.1 that

$$N(n_1, n_2) = \hat{N}_{r_1}(n_1, n_2) + \hat{N}_{r_2}(n_1, n_2) = 0 + \frac{(10^3 + 1)(10^3 - 10)}{11^2} = 8190.$$ 

Example 4.2. We use Theorem 1.2 to compute the number $N(n_1, n_2, n_3)$ of rational points on the following variety over $\mathbb{F}_7$:

$$\begin{align*}
x_1x_2^2x_3^2 + x_1^2x_2^2x_3^2 + x_1^2x_3^2 + x_4^2x_5^2 + x_6^2 &= 1, \\
x_1x_2^2x_3^2 + x_1x_2x_3^2 + x_4x_5 + x_6 &= 3, \\
x_1x_2x_3^2 + x_1x_2^2x_3^2 + x_4x_5 + x_6 &= 5.
\end{align*}$$

Clearly, we have $n_1 = 3$, $n_2 = 5$, $n_3 = 7$, $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, $m = 3$, 

$$E^{(1)} = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 5 \\ 2 & 5 & 2 \end{pmatrix},$$
Using elementary transformations, we obtain that

\[ E^{(2)} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 5 & 3 \\ 3 & 2 & 5 & 0 \\ 1 & 3 & 5 & 1 \end{pmatrix} \quad \text{and} \quad E^{(3)} = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 5 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 2 & 4 & 1 & 1 \\ 3 & 2 & 5 & 0 & 0 & 0 & 0 \\ 1 & 3 & 5 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 4 & 3 & 2 & 3 \\ 2 & 5 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 & 3 & 2 & 1 \end{pmatrix}. \]

We first calculate \( N^{(n_1,n_2,n_3)} \). Using elementary transformations, we obtain two unimodular matrices

\[ U^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 11 & -1 & -4 \end{pmatrix} \quad \text{and} \quad V^{(1)} = \begin{pmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \]

such that

\[ U^{(1)} E^{(1)} V^{(1)} = \text{SNF}(E^{(1)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}. \]

Thus \( d_1^{(1)} = 1, d_2^{(1)} = 1, d_3^{(1)} = 9 \) and \( s_1 = 3 \). Clearly, the vectors \((u_{11}, u_{21}, u_{31}) \in (\mathbb{F}_7^*)^3\) such that

\[
\begin{align*}
& \quad u_{11} = 1 \\
& u_{21} = 3 \\
& u_{31} = 5
\end{align*}
\]

are

\((u_{11}, u_{21}, u_{31}) = (1, 3, 5)\).

Choose the primitive element 3 of \( \mathbb{F}_7^* \). Then we have

\[(h'_1, h'_2)^T = U^{(1)}(\text{ind}_31, \text{ind}_33, \text{ind}_35)^T = U^{(1)}(6, 1, 5)^T \equiv (0, 5, 3)^T \pmod{6}.
\]

We deduce that the conditions (1.6) that \( \gcd(6, 9)|3 \) hold. It follows that \( N_1 = 1 \). So

\[
N^{(n_1,n_2,n_3)} = N_1 q^{n_1-n_2}(q-1)^{n_1-s_1}(q^{n_2-n_1} - (q-1)^{n_2-n_1}) \prod_{j=1}^{s_1} \gcd(q - 1, d_j^{(1)})
\]

\[= 7^2 \times (7^2 - 6^2) \times 3 = 1911.\]

Consequently, we turn our attention to the computation of \( N^{(n_1,n_2,n_3)} \). Using the elementary transformations, one gets that

\[ U^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 & 0 & 0 \\ 8 & 6 & -6 & -4 & 0 & 1 \\ -1 & 5 & -2 & -1 & 2 & -3 \\ 7 & -9 & 1 & 0 & -5 & 9 \end{pmatrix} \quad \text{and} \quad V^{(2)} = \begin{pmatrix} 1 & -2 & 2 & 0 & 10 \\ 0 & 1 & -2 & -1 & 10 \\ 0 & 0 & 1 & 1 & -15 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 17 \end{pmatrix}. \]

17
such that

\[ U^{(2)} E^{(2)} V^{(2)} = \text{SNF}(E^{(2)}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} . \]

Thus \( d_1^{(2)} = d_2^{(2)} = d_3^{(2)} = d_4^{(2)} = 1, d_5^{(2)} = 5 \) and \( s_2 = 5 \). By Lemma 2.4, one has that the number of the vectors \((u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}) \in (\mathbb{F}_q)^6\) such that

\[
\begin{cases}
u_{11} + u_{12} = 1 \\
u_{21} + u_{22} = 3 \\
u_{31} + u_{32} = 5
\end{cases}
\]

is 125. Choose the primitive element 3 of \( \mathbb{F}_q^* \). The argument for calculating \( N_1 \) and using Matlab we compute that \( N_2 = 21 \). Hence

\[
N_{r_2}^{(n_1, n_2, n_3)} = N_2 q^{n_3-n_3} (q-1)^{n_2-n_2} (q^{n_3-n_2} - (q-1)^{n_3-n_2}) \prod_{j=1}^{s_2} \gcd(q-1, d_j^{(2)})
\]

\[= 21 \times (7^2 - 6^2) = 273.\]

Let us now calculate \( N_{r_3}^{(n_1, n_2, n_3)} \). Using the elementary transformations, we obtain that

\[ U^{(3)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-7 & -1 & -2 & 1 & 3 & 1 & 2 & 0 & 0 \\
-13 & -3 & -3 & 2 & 5 & 0 & 3 & 2 & 1 \\
-15 & 1 & 0 & 2 & 3 & 0 & 4 & -2 & 0 \\
16 & 0 & 4 & -2 & -6 & -1 & -5 & 2 & -1 \\
7 & -9 & 0 & 1 & 0 & 0 & -5 & 9 & 0
\end{pmatrix} \]

and

\[ V^{(3)} = \begin{pmatrix}
1 & -2 & 4 & 10 & 10 & 10 & -20 \\
0 & 1 & -3 & -7 & -7 & -7 & 15 \\
0 & 0 & 1 & 2 & 2 & 2 & -5 \\
0 & 0 & 0 & 1 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \]

such that
\[
U^{(3)}E^{(3)}V^{(3)} = \text{SNF}(E^{(3)}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Thus \(d^{(3)}_1 = d^{(3)}_2 = d^{(3)}_3 = d^{(3)}_4 = d^{(3)}_5 = 1\), \(d^{(3)}_6 = 5\) and \(s_3 = 7\). Lemma 2.4 gives that the number of the vectors \((u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}) \in (\mathbb{F}_7^*)^9\) such that

\[
\begin{align*}
&u_{11} + u_{12} + u_{13} = 1 \\
u_{21} + u_{22} + u_{23} = 3 \\
u_{31} + u_{32} + u_{33} = 5
\end{align*}
\]

is equal to 29791. Choose the primitive element 3 of \(\mathbb{F}_7^*\). By the argument for calculating \(N_1\) and using Matlab, we compute that \(N_3 = 823\). Thus one has

\[N_{r_3}^{(n_1, n_2, n_3)} = N_3(q - 1)^{n_3 - s_3} \prod_{j=1}^{s_3} \gcd(q - 1, d^{(3)}_j) = 823.\]

Finally, by Theorem 1.2, we have

\[N^{(n_1, n_2, n_3)} = \sum_{i=1}^{3} N_{r_i}^{(n_1, n_2, n_3)} = 1911 + 273 + 823 = 3007.\]

References

[1] A. Adolphson and S. Sperber, \(p\)-Adic estimates for exponential sums and the theorem of Chevalley-Warning, Ann. Sci. École Norm. Sup. 20 (1987), 545-556.

[2] A. Adolphson and S. Sperber, \(p\)-Adic estimates for exponential sums, \(p\)-Adic analysis (Trento, 1989), 11-22, Lecture Notes in Math., 1454, Springer, Berlin, 1990.

[3] J. Ax, Zeros of polynomials over finite fields, Amer. J. Math. 86 (1964), 255-261.

[4] W. Cao, A special degree reduction of polynomials over finite fields with applications, Int. J. Number Theory 7 (2011), 1093-1102.

[5] W. Cao and Q. Sun, A reduction for counting the number of zeros of general diagonal equation over finite fields, Finite Fields Appl. 12 (2006), 681-692.
[6] W. Cao and Q. Sun, On a class of equations with special degrees over finite fields, Acta Arith. 130 (2007), 195-202.

[7] L. Carlitz, Pairs of quadratic equations in a finite field, Amer. J. Math. 76 (1954), 137-154.

[8] L. Carlitz, The number of solutions of some equations in a finite field, Portug. Math. 13 (1954), 25-31.

[9] L. Carlitz, Certain special equations in a finite field, Monatsh. Math. 58 (1954), 5-12.

[10] E. Cohen, Congruence representations in algebraic number field, Trans. Amer. Math. Soc. 75 (1953), 444-470.

[11] J.H. Hodges, Representations by bilinear forms in a finite field, Duke Math. J. 22 (1955), 497-509.

[12] S. Hu and S. Hong, Counting rational points on an algebraic variety over finite fields, preprint.

[13] S. Hu, S. Hong and W. Zhao, The number of rational points of a family of hypersurfaces over finite fields, J. Number Theory 156 (2015), 135-153.

[14] L.-K. Hua, Introduction to number theory, Springer-Verlag, Berlin Heidelberg, 1982.

[15] L.-K. Hua and H.S. Vandiver, Characters over certain types of rings with applications to the theory of equations in a finite field, Proc. Nat. Acad. Sci. 35 (1949), 94-99.

[16] K. Ireland and M. Rosen, A classical introduction to modern number theory, 2nd ed., GTM 84, Springer-Verlag, New York, 1990.

[17] N.M. Katz, On a theorem of Ax, Amer. J. Math. 93 (1971), 485-499.

[18] R. Lidl and H. Niederreiter, Finite fields, second edition, Encyclopedia of Mathematics and its Applications, vol. 20, Cambridge University Press, Cambridge, 1997.

[19] O. Moreno and C.J. Moreno, Improvement of Chevalley-Warning and the Ax-Katz theorem, Amer. J. Math. 117 (1995), 241-244.

[20] X. Pan, X. Zhao and W. Cao, A problem of Carlitz and its generalizations, Arch. Math. (Basel) 102 (2014), 337-343.

[21] H.J.S. Smith, On systems of linear indeterminate equations and congruences, Philos. Trans. Royal Soc. London, 151 (1861), 293-326.

[22] J. Song and Y. Chen, The number of solutions of certain system of equations over a finite field (in Chinese), Sci. Sin. Math., to appear.
[23] Q. Sun, On diagonal equations over finite fields, Finite Fields Appl. 3 (1997), 175-179.

[24] Q. Sun, On the formula of the number of solutions of some equations over finite fields (in Chinese), Chin. Ann. Math. 18A (1997), 403-408.

[25] D. Wan, An elementary proof of a theorem of Katz, Amer. J. Math. 111 (1989), 1-8.

[26] D. Wan, A Chevalley-Warning proof of the Ax-Katz theorem and character sums, Proc. Amer. Math. Soc. 123 (1995), 1681-1686.

[27] D. Wan, Modular counting of rational points over finite fields, Found. Comput. Math. 8 (2008), 597-605.

[28] W. Wang and Q. Sun, The number of solutions of certain equations over a finite field, Finite Fields Appl. 11 (2005), 182-192.

[29] W. Wang and Q. Sun, An explicit formula of solution of some special equations over a finite field (in Chinese), Chin. Ann. Math. 26A (2005), 391-396.

[30] J. Wolfmann, The number of solutions of certain diagonal equations over finite fields, J. Number Theory 42 (1992), 247-257.

[31] J. Wolfmann, New results on diagonal equations over finite fields from cyclic codes, Finite fields: theory, applications, and algorithms (Las Vegas, NV, 1993), 387-395, Contemp. Math., 168, Amer. Math. Soc., Providence, RI, 1994.

[32] J. Yang, A class of systems of equations over a finite field (in Chinese), Academic Forum of Nan Du (Natural Sciences Edition) 20 (2000), 7-12.