A new bound on quantum Wielandt inequality

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July 19, 2018

Abstract

A new bound on quantum version of Wielandt inequality for positive (not necessarily completely positive) maps has been established. Also bounds for entanglement breaking and PPT channels are put forward which are better bound than the previous bounds known. We prove that a primitive positive map $E$ acting on $\mathcal{M}_d$ that satisfies the Schwarz inequality becomes strictly positive after at most $2(d-1)^2$ iterations. This is to say, that after $2(d-1)^2$ iterations, such a map sends every positive semidefinite matrix to a positive definite one. This finding does not depend on the number of Kraus operators as the map may not admit any Kraus decomposition. The motivation of this work is to provide an answer to a question raised in the article [17] by Sanz-García-Wolf and Cirac.

1 Introduction

A $d \times d$ stochastic matrix $W$ is called primitive if there exists a number $n \in \mathbb{N}$ such that $(W^n)_{i,j} > 0$ for all $(i,j)$, that is all the entries of $W^n$ is strictly positive. The minimum $n$ for which this occurs, denoted by $p(W)$, is called the (classical) index of primitivity of $W$. The Wielandt’s inequality (22) states that for a primitive matrix $W \in \mathcal{M}_d$, we have

$$p(W) \leq (d^2 - 2d + 2).$$

The interesting part is that the above inequality only cares about the dimension and does not depend on the matrix elements. Wielandt inequality has broad applications in graph theory and combinatorics, number theory and Markov chains ([8, 18]).

Sanz et al. (see [17]) extended this concept of the classical Wielandt inequality to quantum channels (trace preserving and completely positive maps) and derived an upper bound on the number of iterations of a channel required to ensure that all the output density matrices must be of full rank. Their result states that if $E : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is a primitive quantum channel with $n$ linearly
independent Kraus operators, then the quantum Wielandt inequality or the quantum primitive index, denoted by $\omega(\mathcal{E})$ satisfies the following inequality:

$$\omega(\mathcal{E}) \leq (d^2 - n + 1) d^2.$$  \hfill (1)

Our work is motivated by one of the questions raised in the Section VI in [17] which asks for optimal bounds of primitivity index of positive maps as opposed to that of quantum channels. Since positive maps do not admit Kraus decompositions, any bound on primitivity index, must therefore will be different from the bound given above in Equation 1. Indeed, we show that for a primitive trace preserving positive map defined on $\mathcal{M}_d$ which satisfies the Schwarz inequality, the primitivity index $\omega(\mathcal{E})$ satisfies the following inequality:

$$\omega(\mathcal{E}) \leq 2(d - 1)^2.$$ 

Note that in this case, only dimension of the matrix algebra plays a role and not the linear map itself which is very similar to the spirit of the bound given in (classical) Wielndt’s inequality.

2 Index of primitivity and quantum Wielandt bound

We begin with some definitions and analyze closely the work of Sanz et all. The following definitions are the key concepts of the so-called non-commutative Perron-Fobinius theory. We refer to the articles [3], [23], [4], [17] for some preliminary background on this topic.

**Definition 2.1.** A positive linear map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called irreducible if $\mathcal{E}(p) \leq \lambda p$, for any projection $p \in \mathcal{M}_d$ implies that $p = 0$ or $p = 1$.

**Definition 2.2.** A positive linear map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called primitive if it is irreducible and moreover, $\text{Spec}(\mathcal{E}) \cap \mathbb{T} = \{1\}$. This means that the only peripheral spectrum of $\mathcal{E}$ is the identity element.

**Definition 2.3.** A positive linear map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is called strictly positive if it sends every positive semidefinite element in $\mathcal{M}_d$ to a positive definite element.

Note that whether a completely positive map on $\mathcal{M}_d$ is strictly positive or not is an NP-hard problem [6].

**Definition 2.4.** [see [17]] For a primitive positive map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$, the **index of primitivity** (denoted by $\omega(\mathcal{E})$) is the least natural number $n$, such that $\mathcal{E}^n(a)$ is positive definite for every positive semidefinite $a \in \mathcal{M}_d$.

The index of primitivity as defined above is a generalization of classical primitivity index of a non-negative primitive matrix.
Theorem 2.5. [see [17]] Let $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ be a primitive quantum channel with $n$ linearly independent Kraus operators, that is, there are linearly independent elements $\{a_i : 1 \leq i \leq n\}$ such that

$$\mathcal{E}(x) = \sum_{i=1}^{n} a_i x a_i^*.$$

Then

$$\omega(\mathcal{E}) \leq (d^2 - n + 1)d^2.$$

The above bound is called the quantum version of the **Wielandt bound** which is a generalization of the classical Wielandt number of primitive matrices. The bound $(d^2 - n + 1)d^2$ was obtained by looking at the following quantity:

$$i(\mathcal{E}) = \min\{k \in \mathbb{N} : \text{Span}\{a_{i_1}, \ldots, a_{i_k}\} = \mathcal{M}_d\}.$$

Which can be viewed as the minimum number $k$ for which the the Choi matrix of $\mathcal{E}^k$ has full rank, that is, $\text{rank}(C_{\mathcal{E}^k}) = d^2$. It was proved in [17] that $\omega(\mathcal{E}) \leq i(\mathcal{E})$ and then it was proved that $i(\mathcal{E}) \leq (d^2 - n + 1)d^2$. For finding a bound of the primitive index for positive we can not follow the above procedure because our maps do not admit any Kraus decomposition and hence a very different approach must be taken to achieve this goal. We discuss this approach in the following section.

3 A new Wielandt bound for positive maps

We begin with generalizing the concept of irreducibility of a linear map. Note that two projections $p, q \in \mathcal{M}_d$ are said to be (Murray-von Neumann) equivalent (written as $p \sim q$) if there is an operator $v \in \mathcal{M}_d$ such that $vv^* = p$ and $v^*v = q$. It follows that $p \sim q$ if and only if $\text{Tr}(p) = \text{Tr}(q)$.

**Definition 3.1.** A positive linear map $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ is defined to be **fully irreducible** if $\mathcal{E}(p) \leq \lambda q$, for two projections $p, q$ with $p \sim q$ and $\lambda > 0$ implies $p, q \in \{0, 1\}$.

Note that the above definition arose in [9] and these maps were called fully indecomposable however we are avoiding this terminology because there is a concept of indecomposibility in the theory of positive maps. The above definition clearly generalizes the irreducibility (see Definition [2.7]) so we will call such maps **fully irreducible**. It is evident that fully irreducibility is stronger than irreducibility as in the later case, trivially one can put $q = p$ and vacuously $p \sim p$.

We begin with a lemma.

**Lemma 3.2.** For a positive element $a \in \mathcal{M}_d$, if for a projection $p$ we have $pap = 0$, then $(1 - p)ap = 0 = pa(1 - p)$.
Proof. If $\xi \in \text{Range}(p)$ and $\eta \in \text{Range}(p)$, then $p\xi = \xi$ and $p\eta = 0$. By positivity of $a$, we have for every $\lambda \in \mathbb{C}$,

$$\langle a(\lambda \xi + \eta), \lambda \xi + \eta \rangle \geq 0.$$ 

Now using $pap = 0$, we get from the above inequality $\langle a\eta, \eta \rangle + 2\text{Re}(\lambda \langle a \xi, \eta \rangle) \geq 0$. This implies $\langle a\xi, \eta \rangle = 0$. This yields

$$\langle (1 - p)ap(\xi + \eta), \xi + \eta \rangle = \langle ap(\xi + \eta), \xi + \eta \rangle = 0.$$ 

Similarly $pa(1 - p) = 0$. □

**Proposition 3.3.** For a unital positive map $\mathcal{E}$, if $\mathcal{E}(p) \leq \lambda q$, for some $\lambda > 0$, then we have $\mathcal{E}(p) \leq q$.

Proof. We first note that $\mathcal{E}(p) \leq \lambda q$, implies that

$$(1 - q)\mathcal{E}(p)(1 - q) = 0.$$ 

Now by the previous lemma we have

$$q\mathcal{E}(p)(1 - q) = 0 = (1 - q)\mathcal{E}(p)q.$$ 

These equations result in

$$q\mathcal{E}(p)q = q\mathcal{E}(p) = \mathcal{E}(p)q.$$ 

Now expanding the Equation 2 we get

$$0 = (1 - q)\mathcal{E}(p)(1 - q) = \mathcal{E}(p) - \mathcal{E}(p)q - q\mathcal{E}(p) + q\mathcal{E}(p)q = \mathcal{E}(p) - q\mathcal{E}(p)q.$$ 

Now using the unitality of $\mathcal{E}$ we know that $\|\mathcal{E}\| = 1$ and we obtain

$$\mathcal{E}(p) = q\mathcal{E}(p)q \leq q\|\mathcal{E}(p)\|1q \leq q.$$ 

□

**Proposition 3.4.** For a unital trace preserving map $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$, $\mathcal{E}$ is fully irreducible if and only if there are no projections $p, q$ with $p \sim q$ such that $\mathcal{E}(p) = q$.

Proof. If part is obvious as $\mathcal{E}(p) = q$ clearly violates the definition of fully irreducibility. Conversely, suppose $\mathcal{E}$ is not fully irreducible. Then there are projections $p, q$ and $p \sim q$ such that $\mathcal{E}(p) \leq \lambda q$ which by proposition 3.3 we have $\mathcal{E}(p) \leq q$. Now using the trace preservation of $\mathcal{E}$ and the faithfulness of trace, we get $\mathcal{E}(p) = q$. □

We note down an observation here that every unital and trace preserving positive map is rank increasing.
Proposition 3.5. Let $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ a unital and trace preserving positive map. Then if $a$ is a positive element in $\mathcal{M}_d$, then it follows that

$$\text{Rank}(\mathcal{E}(a)) \geq \text{Rank}(a).$$

Proof. It is a consequence of Uhlmann’s theorem (see Theorem 4.33 in [21]) that $b = \Phi(a)$, for a unital and trace preserving positive map $\Phi$ if and only if

$$\lambda(b) \prec \lambda(a),$$

that is the vector of eigenvalues of $a$ majorizes the vector of eigenvalues of $\Phi(a)$.

Now it is enough to prove the proposition for projections. If $p$ is a projection, then by Uhlmann’s theorem $\lambda(\mathcal{E}(p)) \prec \lambda(p)$. Then it follows that

$$\text{Rank}(\mathcal{E}(p)) \geq \text{Rank}(p).$$

A unital positive map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ satisfies the following inequality (see [20], Theorem 1.3.1)

$$\Phi(aa^*) \geq \Phi(a)\Phi(a^*),$$

for all elements $a$ satisfying $aa^* = a^*a$. Using the Schwarz inequality for positive maps on hermitian elements, we can derive a stronger result for fully irreducible maps. The following result first appeared as Proposition 1.25 in [9]. Here we give a different proof.

Theorem 3.6. Let $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ be a unital and trace preserving positive map. Then $\mathcal{E}$ is fully irreducible if and only if for all $a \in \mathcal{M}_d$, we have

$$\text{Rank} \mathcal{E}(a) > \text{Rank}(a),$$

that is, $\mathcal{E}$ is fully irreducible if and only if it is strictly rank increasing or equivalently it is strictly kernel reducing.

Proof. A unital positive linear map satisfies the Schwarz inequality on normal elements. So using this inequality on a positive element $a^{1/2}$ one gets

$$\mathcal{E}(a) = \mathcal{E}(a^{1/2}a^{1/2}) \geq \mathcal{E}(a^{1/2})\mathcal{E}(a^{1/2}).$$

As $x \mapsto \sqrt{x}$ is an operator monotone function we get $\mathcal{E}(a)^{1/2} \geq \mathcal{E}(a^{1/2})$. We can continue this process to get

$$\mathcal{E}(a)^{1/2^n} \geq \mathcal{E}(a^{1/2^n}), \forall n.$$

Now if $q$ is the projection onto the Range($\mathcal{E}(a)$) and $p$ is the projection onto Range($a$), by the spectral theorem for positive elements, taking limit as $n \to \infty$ in the above equation we obtain

$$q \geq \mathcal{E}(p).$$
Now if $a$ and $\mathcal{E}(a)$ have the same rank, then $p \sim q$ and this violates the fully irreducibility property.

Conversely, suppose there are projections $p, q$ with $p \sim q$ such that $\mathcal{E}(p) \leq \lambda q$, for $\lambda > 0$. Using Proposition 3.3 we have $\mathcal{E}(p) \leq q$. Then $q - \mathcal{E}(p) \geq 0$. Using the trace preservation property of $\mathcal{E}$ and faithfulness of trace we get $q = \mathcal{E}(p)$. Since

$$\text{rank}(p) = \text{Tr}(p) = \text{Tr}(\mathcal{E}(p)) = \text{Tr}(q) = \text{rank}(q),$$

it violates the (strictly)rank increasing property. \(\square\)

Now we introduce one more concept related to a linear map acting on $\mathcal{M}_d$.

**Definition 3.7.** The multiplicative domain $\mathcal{M}_\Phi$ of a linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is the following set:

$$\mathcal{M}_\Phi = \{ a \in \mathcal{M}_d : \Phi(ab) = \Phi(a)b, \Phi(ba) = \Phi(b)a \forall b \in \mathcal{M}_d \}.$$

**Definition 3.8.** We say a positive linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is a Schwarz map if it satisfies the Schwarz inequality $\Phi(aa^*) \geq \Phi(a)\Phi(a^*)$, for every element $a \in \mathcal{M}_d$.

It is a consequence of the Stinespring dilation theorem for completely positive maps that every unital completely positive map acting on a $C^*$ algebra satisfies the Schwarz inequality. However, following [2], the map $\Phi : \mathcal{M}_2 \to \mathcal{M}_2$ defined by

$$\Phi\left( \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} x_{11} + \frac{x_{11} + x_{22}}{2} & x_{21} + \frac{x_{11} + x_{22}}{2} \\ x_{12} & x_{22} \end{bmatrix},$$

is a Schwarz map but fails to be 2-positive. For a Schwarz map $\Phi$, the set $\mathcal{M}_\Phi$ is a $C^*$-subalgebra of $\mathcal{M}_d$ (see Corollary 2.2.6 in [20]). Following [15], given a Schwarz map $\Phi$ on $\mathcal{M}_d$, one obtains a decreasing chain of $C^*$-subalgebras

$$\mathcal{M}_\Phi \supseteq \mathcal{M}_{\Phi^2} \supseteq \cdots \supseteq \mathcal{M}_{\Phi^n} \supseteq \cdots.$$

For finite dimensionality, the above chain stabilizes to the subalgebra

$$\mathcal{M}_{\Phi^\infty} = \bigcap_{n \geq 1} \mathcal{M}_{\Phi^n}.$$

The minimum number $n$ required for the channel to reach to this subalgebra $\mathcal{M}_{\Phi^\infty}$ is called the multiplicative index and denoted by $\kappa(\Phi)$.

**Remark 3.9.** It should be noted here that the stabilized multiplicative domain ($\mathcal{M}_{\Phi^\infty}$) and the multiplicative index ($\kappa(\Phi)$) of a linear map $\Phi$ can be defined as long as the map is a Schwarz map. Indeed the map need not be a channel as these concepts originated ([15]) exploiting only the Schwarz inequality and trace preservation property of $\Phi$.

**Proposition 3.10.** A trace preserving Schwarz map $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ is fully irreducible if and only if it has trivial multiplicative domain, that is, $\mathcal{M}_{\mathcal{E}} = \mathbb{C}1$.  

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Proof. First, we note that a trace preserving Schwarz map is unital. Indeed, 
\( E(xx^*) \geq E(x)E(x^*) \) implies \( \|E(x)\|^2 = \|E(xx^*)\| \leq \|E\| \|x\|^2 \). Using the Russodye theorem (Corollary 2.9 in [13]) we get from the above inequality with \( x = 1 \), 
\[ \|E\|^2 \leq \|E\| \Rightarrow \|E\| \leq 1. \]

As \( E \) is a contraction, we get \( E(1) \leq 1 \) and hence by trace preservation we get \( E(1) = 1 \).
If \( M_E \) is not trivial, then there exists a projection \( p \in M_E \). Now by definition of multiplicative domain, \( E(p) \) is again a projection, call it \( q \). Using the trace preserving property we get \( p \sim q \). This is a contradiction following Proposition 3.4. The converse follows exactly in the similar way. \( \square \)

It should be noted here that being fully irreducible or equivalently having \( M_E = \mathbb{C}1 \), does not force the map to be strictly positive. The following example verifies this fact.

Example 3.11. Consider the map \( E : M_3 \to M_3 \) defined by 
\[ E(x) = \frac{1}{2}(\text{Tr}(x)1 - x^t). \]

It is easy to verify that \( E \) is unital and trace preserving Schwarz map. It follows that any rank one projection is mapped to a rank 2 element, so rank one projections can not be in the multiplicative domain. Since this domain is a unital C*-subalgebra, it follows that there is no rank 2 projection in the multiplicative domain as well. Hence \( M_E = \mathbb{C}1 \). Now it is easily seen that the image of the rank one matrix unit \( E_{11}, E(E_{11}) = \frac{1}{2}(E_{22} + E_{33}) \) which is of rank 2\((\neq 3)\).

We are ready to state and prove the main theorem of this article.

Theorem 3.12. Let \( E : M_d \to M_d \) be a trace preserving primitive Schwarz map with the multiplicative index \( \kappa \). Then \( E^{\kappa(d-1)} \) sends every positive semi definite matrix to a positive definite matrix. That is, 
\[ \omega(E) \leq \kappa(E)(d-1). \]

Proof. As \( E \) is a trace preserving Schwarz map, it is unital. Following the Corollary 3.5 in [15], \( E \) is primitive implies that \( M_{E^{\infty}} = \mathbb{C}1 \). First, note the following chain has length \( \kappa \): 
\[ M_E \supseteq M_{E^2} \supseteq \cdots \supseteq M_{E^{\infty}} = \mathbb{C}1. \]

First of all observe that \( \kappa(E) \leq \omega(E) \). This is because if we take a projection \( p \in M_E \) such that \( E(p) \in M_E \), then \( E(p) \) is again a projection. By the definition of \( M_{E^2} \) (see [15]), \( p \in M_{E^2} \). If \( M_{E^2} \) is still not \( \mathbb{C}1 \), we get \( E(p) \) is not positive definite. Repeating the argument for \( E^2, E^3, \cdots E^{(\kappa-1)} \) we see that if \( p \in M_{E^{\kappa-1}} \), then \( E^\kappa(p) \in \mathbb{C}1 \) which then makes it invertible and hence \( E^{(\kappa)} \) maps every projection in \( M_E \) to an invertible operator. Thus \( \kappa(E) \leq \omega(E) \).
Now we will show $\omega(\mathcal{E}) = \kappa(\mathcal{E})(d - 1)$. Strict positivity of any map $\Phi$ will be guaranteed if $\Phi(p)$ is invertible for any rank one projection $p$. Indeed, given any projection $q$, there exists a rank one projection $p$ such that $p \leq q$. Hence for any positive linear map $\Phi$, $\Phi(p) \leq \Phi(q)$. So if $\Phi(p)$ is invertible, then so is $\Phi(q)$. Now if the spectral decomposition of positive element $a \in \mathcal{M}_d$ be given by

$$a = \sum_{j=1}^{k} \lambda_j p_j,$$

where $p_j$’s are spectral projections onto the the eigenspace corresponding to the eigenvalue $\lambda_j$, then $\Phi(a) = \sum_{j=1}^{k} \lambda_j \Phi(p_j)$. Now if for every $\xi \in \mathbb{C}^d$, $\langle \Phi(p_j)\xi, \xi \rangle > 0$, then $\langle \Phi(a)\xi, \xi \rangle > 0$. Hence it is enough to show that for any rank one projection $p$, $\Phi(p)$ is invertible.

Take a rank one projection $p \in \mathcal{M}_d$. Since $\mathcal{E}^{\kappa(\mathcal{E})}$ has multiplicative domain $\mathcal{M}_{\mathcal{E}} \subseteq \mathbb{C}1$, we get $\mathcal{E}^{\kappa(\mathcal{E})}$ does not have any non-trivial multiplicative domain. Now for the unital map $\Phi (= \mathcal{E}^{\kappa(\mathcal{E})})$ with trivial multiplicative domain, by Proposition 3.10 and Theorem 3.6, it is strictly kernel reducing, that is,

$$\dim \ker(\Phi(a)) < \dim \ker(a), \ \forall \ a \in \mathcal{M}_{d^+}.$$

Here $\mathcal{M}_{d^+}$ denotes the set of all positive semidefinite elements of $\mathcal{M}_d$. Now taking a rank one projection $p$, we evaluate

$$\dim \ker \Phi^{(d-1)}(p) < \dim \ker \Phi^{(d-2)}(p) < \cdots < \dim \ker \Phi(p) < \dim \ker(p).$$

Since $p$ has rank 1, the kernel has dimension $d - 1$ and since the dimension is a non negative integer function, the above inequality yields

$$\dim \ker \Phi^{(d-1)}(p) = 0.$$

Hence $\Phi^{(d-1)}(p)$ is invertible. Since this holds for every rank one projection, $\Phi^{d-1}$ is strictly positive and hence $\mathcal{E}^{\kappa(\mathcal{E})(d-1)}$ is strictly positive.

Now it is important to find a suitable bound for $\kappa(\mathcal{E})$ for a trace preserving Schwarz map. In [10], Theorem 3.6 such a bound was put forward. The key point is that the bound was obtained by utilizing the C*-algebra structure of the subalgebras $\mathcal{M}_\mathcal{E}, \mathcal{M}_{\mathcal{E}^2}$ etc. As for a Schwarz map $\mathcal{E}$, these subalgebras are all C*-algebras, we can use this bound in our context.

**Corollary 3.13.** For a trace preserving primitive Schwarz map $\mathcal{E}$ acting on $\mathcal{M}_d$, we have

$$\omega(\mathcal{E}) \leq 2(d - 1)^2.$$

**Proof.** As was shown in [10], $\kappa(\mathcal{E})$ must be less than the maximum length of the chain of subalgebras

$$\mathcal{M}_\mathcal{E} \supseteq \mathcal{M}_{\mathcal{E}^2} \supseteq \cdots \supseteq \mathcal{M}_{\mathcal{E}^n} \supseteq \mathcal{M}_{\mathcal{E}^\infty}.$$
It follows that \( \kappa(\mathcal{E}) \leq 2(d-1) \). So it shows that the Weilandt number \( \omega(\mathcal{E}) \leq 2(d-1)(d-1) \).

**Remark 3.14.** It should be noted here that similar to the bound given in [17], we don’t know whether the inequality given in Corollary 3 is sharp or not. Even for a quantum channel, the optimal value for \( \kappa \) is still unknown and hence deciding whether a positive map attains the exact Wielandt bound is an avenue for future research.

We utilize these findings to quantum channels to get better bounds of some classes of channels.

**Corollary 3.15.** For PPT and entanglement breaking channels \( \mathcal{E} \) acting on \( \mathcal{M}_d \), we have

\[
\omega(\mathcal{E}) \leq d(d-1).
\]

**Proof.** We know the PPT channels and the entanglement breaking channels have abelian multiplicative domain (see [16]). It is not hard to see that the multiplicative index of these channels can be maximum \( d \) (see Proposition 3.2 in [10]). So

\[
\omega(\mathcal{E}) = \kappa(\mathcal{E})(d-1) \leq d(d-1).
\]

**Remark 3.16.** Since \( d(d-1) < d^2 \), we have a better bound of Weilandt inequality than that given in [17] for PPT and entanglement breaking channels.

**Proposition 3.17.** Let \( \mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d \) be a unital primitive map that \( \mathcal{E} \) and the adjoint map \( \mathcal{E}^* \) satisfies the Schwarz inequality. Then the Wielandt bounds for \( \mathcal{E} \) and \( \mathcal{E}^* \) are same.

**Proof.** From [10], \( \kappa(\mathcal{E}) = \kappa(\mathcal{E}^*) \). Also if \( \mathcal{E} \) is primitive, then so is \( \mathcal{E}^* \). Hence the assertion follows from the Theorem 3.12.

### 4 Wielandt bound for tensor product channels

Since the Wielandt bound as given in Theorem 3.12 involves the multiplicative index, we can get a handle of Wielandt inequality for tensor products of channels. This is possible because multiplicative domain (and hence multiplicative index) of tensor products of unital channels behave nicely. We state this result below:

**Theorem 4.1** (See [10]). If \( \Phi, \Psi \) are two unital channels on \( \mathcal{M}_d \), then the multiplicative domain of \( \Phi \otimes \Psi \) splits, that is,

\[
\mathcal{M}_{\Phi \otimes \Psi} = \mathcal{M}_\Phi \otimes \mathcal{M}_\Psi.
\]

Moreover,

\[
\kappa(\Phi \otimes \Psi) = \max\{\kappa(\Phi), \kappa(\Psi)\} = \max\{2(d_1-1), 2(d_2-1)\}.
\]
Using the Theorem 3.12 we immediately get

**Proposition 4.2.** For unital primitive channels

\[ \Phi : M_{d_1} \rightarrow M_{d_1} \text{ and } \Psi : M_{d_2} \rightarrow M_{d_2} \]

we have the Weilandt bound

\[ \omega(\Phi \otimes \Psi) = \max \{ \omega(\Phi), \omega(\Psi) \} \leq \max \{ 2(d_1 - 1)^2, 2(d_2 - 1)^2 \}. \]

**Proof.** First of all note that the tensor product of two primitive maps is primitive. Indeed, this fact was proved in [10], Theorem 2.10 utilizing the splitting property:

\[ \mathcal{M}_{(\Phi \otimes \Psi)\infty} = \mathcal{M}_{\Phi\infty} \otimes \mathcal{M}_{\Psi\infty}. \]

Hence the result follows immediately from Theorem 3.12.

\[ \square \]

5 **Dichotomy result for the zero-error quantum capacity**

Similar to one given in [17], we can establish a dichotomy result for unital channels with respect to the quantum capacity using the Wielandt bound.

**Definition 5.1.** The one shot zero-error classical capacity \( C_0(\Phi) \) of a channel \( \Phi \) is defined to be as \( \sup \log |S| \), where \( \mathcal{S} \) is the set of all families \( \{\rho_i\} \) such that \( \text{Tr}(\Phi(\rho_i)\Phi(\rho_j)) = 0 \) for \( i \neq j \).

Consider the following dichotomy theorem for one shot zero-error classical capacity of channels:

**Theorem 5.2.** (Sanz-García-Wolf-Ciraq, [17]). If \( \mathcal{E} \) is a quantum channel with a full fixed rank fixed point, then either \( C_0(\mathcal{E}^n) \geq 1 \) for all \( n \) or \( C_0(\mathcal{E}^{\omega}(\mathcal{E})) = 0 \). Here \( \omega(\mathcal{E}) \) is the Wielandt bound for \( \mathcal{E} \).

Now we will prove a dichotomy theorem for another capacity of quantum channel.

**Definition 5.3.** (see [17]) Let \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) be a channel where \( \dim(\mathcal{H}) < \infty \). Then the one shot zero-error quantum capacity \( Q_0(\Phi) \) of a channel \( \Phi \) is defined to be \( \sup \log \dim(\mathcal{K}) \), where \( \mathcal{K} \) is the collection of subspaces \( \mathcal{H}_0 \) of \( \mathcal{H} \) such that there exists a channel \( \Psi \), satisfying \( \Psi(\Phi(\rho)) = \rho \), for all \( \rho \) supported on \( \mathcal{H}_0 \).

Now we write down the dichotomy theorem for this capacity of channel:

**Theorem 5.4.** Let \( \mathcal{E} \) be a unital channel on \( M_d \). Then \( Q_0(\mathcal{E}^n) > 0 \) for all \( n \) or \( Q_0(\mathcal{E}^{\omega}(\mathcal{E})) = 0 \).
Proof. Let the Kraus representation of $\mathcal{E}$ be given by $\mathcal{E}(x) = \sum_j a_j x a_j^*$. We encounter two mutually exclusive situations: either $\mathcal{E}$ is primitive or not.

Case 1: Suppose $\mathcal{E}$ is primitive. So $\mathcal{E}^{\omega}(\mathcal{E})$ sends every density operators to density operators with full rank. Now if $Q_0(\mathcal{E}^{\omega}(\mathcal{E})) > 0$, there is a proper-subspace $\mathcal{H}_0$ and channel $\Psi$ such that $\Psi(\mathcal{E}^{\omega}(\mathcal{E})(\rho)) = \rho$ for $\rho$ supported on $\mathcal{H}_0$. Observe that the Kraus representation of $\mathcal{E}^{\omega}(\mathcal{E})$ is given by

$$\mathcal{E}^{\omega}(\mathcal{E})(x) = \sum_{i_1, \ldots, i_{\omega}(\mathcal{E}) = 1} a_{i_1} \cdots a_{i_{\omega}(\mathcal{E})}^* x a_{i_{\omega}(\mathcal{E})} \cdots a_{i_1}^*.$$  

Now note that by the Knill-Laflamme ([11]) condition of reversibility of the channel $\mathcal{E}^{\omega}(\mathcal{E})$ is equivalent to the condition that, $\forall \xi, \eta \in \mathcal{H}_0$ with $\langle \xi, \eta \rangle = 0$,

implies that

$$\langle \xi, (a_{j_1}^* \cdots a_{j_{\omega}(\mathcal{E})}^*) a_{i_1} \cdots a_{i_{\omega}(\mathcal{E})} \eta \rangle = 0.$$  

(3)

Now if $\mathcal{E}^{\omega}(\mathcal{E})$ is strictly positive, then so is $\mathcal{E}^{\omega}(\mathcal{E}) \circ \mathcal{E}^{\omega}(\mathcal{E})$. Indeed, for any unit vector $\psi(q = \psi \psi^*)$ and rank one projection $p = \phi \phi^*$, we have

$$\langle \mathcal{E}^{\omega}(\mathcal{E}) \circ \mathcal{E}^{\omega}(\mathcal{E})(p) \psi, \psi \rangle = \text{Tr}(\mathcal{E}^{\omega}(\mathcal{E}) \circ \mathcal{E}^{\omega}(\mathcal{E})(p)q) = \text{Tr}(\mathcal{E}^{\omega}(\mathcal{E})(p)\mathcal{E}^{\omega}(\mathcal{E})(q)) > 0.$$  

Hence

$$\text{Span}\{a_{j_1}^* \cdots a_{j_{\omega}(\mathcal{E})}^* a_{i_1} \cdots a_{i_{\omega}(\mathcal{E})}\} = \mathcal{M}_d.$$  

Clearly this violates the Equation 3.

Case 2: Suppose $\mathcal{E}$ is not primitive. So $\mathcal{M}_{\mathcal{E}^{\infty}}$ is non-trivial. Following Theorem 2.5 in [15] we get $\mathcal{M}_d = \mathcal{M}_{\mathcal{E}^{\infty}} \oplus \mathcal{M}_{\mathcal{E}^{\infty}}^\perp$ and $\mathcal{E}$ is an automorphism on $\mathcal{M}_{\mathcal{E}^{\infty}}$ with inverse being $\mathcal{E}^*$. It follows that $\mathcal{E}^n$ also is an automorphism on $\mathcal{M}_{\mathcal{E}^{\infty}}$ of every $n$ with the inverse $\mathcal{E}^{*n}$. Now as $\mathcal{M}_{\mathcal{E}^{\infty}}$ is non-trivial algebra, there exists a projection $p$ whose support is $\mathcal{H}_0$ say. Then for every $n \in \mathbb{N}$, we have $\mathcal{E}^{*n} \circ \mathcal{E}^n(p) = p$. So for every $n$, there is a recovery channel $\mathcal{E}^{*n}$ for $\mathcal{E}^n$ and hence $Q_0(\mathcal{E}^n) > 0$.  

Remark 5.5. The dichotomy result for classical capacity given in the Theorem 5.2 works for channel with full rank fixed point(example-unital channels). Although Theorem 5.4 deals with unital channels, it can be shown that any fully irreducible channel, when properly scaled, can be made into a unital channel (see [7]). So in relevant contexts one does not lose much by choosing to work with unital channels.

6 Wielandt bound and strictly contractive channels

Strictly contractive channels were first introduced in [14] where (strict)contractivity of channels with respect to the metric induced by the trace norm ($\| \cdot \|_1$) was
considered. Later, in [5] these contractions were studied with respect to the Bures metric from a more operator algebraic viewpoint. For convergence analysis and entropy production of bi-stochastic channels, these strictly contractive maps are key objects to look at ([1], [12]).

**Definition 6.1.** A channel $\Phi$ is said to be strictly contractive if

$$
\| \Phi(\rho) - \Phi(\sigma) \|_1 \leq c(\Phi) \| \rho - \sigma \|_1
$$

for all density matrices $\rho, \sigma$ in $\mathcal{M}_d$, with $\rho \neq \sigma$ and $0 \leq c < 1$.

The constant $c(\Phi)$ is called the contractive modulus of $\Phi$. The $\| \cdot \|_1$-norm is defined by

$$
\| x \|_1 = \text{Tr}[(xx^*)^{\frac{1}{2}}],
$$

for all $x \in \mathcal{M}_d$. It can be proved that the strictly contractive channels are primitive. The converse follows once the primitive map becomes strictly positive.

**Theorem 6.2.** Given a primitive channel $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$, $\mathcal{E}^{\omega(\mathcal{E})}$ is a strictly contractive map.

**Proof.** Let the Choi matrix of $\mathcal{E}$ be denoted by $C_{\mathcal{E}}$. It is easy to see that the Choi matrix of the depolarizing channel $\Omega(x) = \text{Tr}(x)\frac{1}{d}$ is the identity element $1 \otimes 1$ in $\mathcal{M}_d \otimes \mathcal{M}_d$. Now, since $\mathcal{E}^{\omega(\mathcal{E})}$ is strictly positive, it follows that the corresponding Choi matrix, $C_{\mathcal{E}^{\omega(\mathcal{E})}}$ is of full rank. Hence, there must exists a $\lambda > 0$, such that the operator

$$
\lambda C_{\mathcal{E}^{\omega(\mathcal{E})}} - 1 \otimes 1,
$$

is positive semidefinite. This implies, that there exists a $\delta > 0$, such that the linear map

$$
\Psi_{\delta} = (1 + \delta)\mathcal{E}^{\omega(\mathcal{E})} - \delta \Omega,
$$

is completely positive. This map is trace preserving, hence a channel. Now from the above equation, we get

$$
\mathcal{E}^{\omega(\mathcal{E})} = \frac{1}{1 + \delta} \Psi_{\delta} + \frac{\delta}{1 + \delta} \Omega.
$$

Now one calculates for $\rho \neq \sigma$,

$$
\| \mathcal{E}^{\omega(\mathcal{E})}(\rho - \sigma) \|_1 = \|((1 + \delta) \Psi_{\delta} + \frac{\delta}{1 + \delta} \Omega)(\rho - \sigma)) \|_1
$$

$$
= \frac{1}{1 + \delta} \| \Psi_{\delta}(\rho - \sigma) \|_1
$$

$$
\leq \frac{1}{1 + \delta} \| \rho - \sigma \|_1.
$$

Here the last inequality follows from the fact that any channel is a contraction with respect to the trace norm. This shows that $\mathcal{E}^{\omega(\mathcal{E})}$ is strictly contractive. □
7 Acknowledgements

MR is supported by a Postdoctoral fellowship at the Department of Pure Mathematics, University of Waterloo. The author would like to thank Professor Vern Paulsen and Sam Jaques for many insightful discussion.

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