Non-local biased random walks and fractional transport on directed networks

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In this paper, we study non-local random walk strategies generated with the fractional Laplacian matrix of directed networks. We present a general approach to analyzing these strategies by defining the dynamics as a discrete-time Markovian process with transition probabilities between nodes expressed in terms of powers of the Laplacian matrix. We analyze the elements of the transition matrices and their respective eigenvalues and eigenvectors, the mean first passage times and global times to characterize the random walk strategies. We apply this approach to the study of particular local and non-local random walks on different directed networks; we explore circulant networks, the biased transport on rings and the dynamics on random networks. We discuss the efficiency of a fractional random walker with bias on these structures.

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I. INTRODUCTION

The study and understanding of dynamical processes taking place on networks have a significant impact in science and engineering with important applications in physics, biology, social and computer systems among many others [1, 2]. In particular, the diffusion problem associated to the dynamics of a random walker that hops visiting the nodes of the network following different strategies is an important and challenging field of research due to connections with interdisciplinary topics like ranking and searching on the web [3–5], aging and accumulation of damage [6], the understanding of human mobility in urban settlements [7–11], epidemic spreading [12, 13], algorithms for extracting useful information from data [14], just to mention a few examples. Several types of random walk strategies on networks have been introduced in the last decades, some of them only require local information of each node and in this way, the walker moves from one node to one of its nearest neighbors [15–17], whereas in other cases, the total architecture of the network comes into play and non-local strategies can use all this information to define long-range hops between distant nodes [18, 19].

In addition to the non-local random walks mentioned before, we have the fractional diffusion on undirected networks [20–22], a process associated with a Lévy like dynamics where the transition probabilities between nodes are defined in terms of powers of the Laplacian matrix of the network [20, 27, 28]. This mechanism to generate non-locality combines the information of all possible paths connecting two nodes on the network [21] improving the capacity to visit the nodes, a result that offers a significant advantage in networks with large average distances between nodes like lattices and trees but that also is evident in small-world networks [20]. Beyond the study of non-local dynamical processes on networks; recently, the concept of fractional Laplacian of a network has been implemented in semi-supervised learning algorithms for the classification of data structures [30]. Other potential applications of non-local dynamics on networks require the extension of all this formalism to the case of directed weighted networks [31].

In this paper, we explore the dynamics of a random walker with transition probabilities defined in terms of the fractional Laplacian matrix in directed networks. In the first part, we discuss the general definitions and properties of the fractional transport on connected directed networks. The formalism introduced allows generalizing different results and techniques developed in the context of the fractional Laplacian of a graph [28]. Once discussed the general case, we analyze the characteristics of random walk strategies that emerge in networks defined by circulant matrices, the complex eigenvalues associated with the fractional transport, mean-first passage times and the capacity of different strategies to explore circulant networks as well as the relation of these strate-
gies with non-local random walks similar to Lévy flights. We also apply this approach to investigate the fractional transport on stochastic networks generated with an algorithm similar to the Erdős-Rényi model to produce random directed structures. Our findings show different characteristics of the eigenvalues of the transition matrix in the complex plane associated with non-locality showing how the efficiency of non-local strategies is affected by the asymmetry of the network. The general approach introduced reveals several cases where the combination of non-local displacements and the bias generated by the directions of lines produce a global effect that can reduce or improve the efficiency of a random walker to visit all the nodes or reach a particular target on the network.

II. FRACTIONAL LAPLACIAN OF DIRECTED NETWORKS

In this section, we introduce a generalization of the fractional Laplacian of undirected networks (see Refs. [20, 27, 28]) to a general class of directed weighted networks. In terms of this operator, we define transition probabilities of a Markovian random walker associated to the biased fractional transport on networks.

We consider directed weighted networks with $N$ nodes $i = 1, \ldots, N$. The topology of the network is described by an adjacency matrix $A$ with elements $A_{ij} = 1$ if there is an edge between the nodes $i$ and $j$ and $A_{ij} = 0$ otherwise. In addition to the network structure, we have a $N \times N$ matrix of weights $\Omega$ with elements $\Omega_{ij} \geq 0$. The matrix $\Omega$ can include information of the structure of the network or incorporate additional data describing the flow capacity of each link [4, 52, 53]. In the simplest case, $\Omega$ coincides with the adjacency matrix $A$.

Since the matrix of weights in general is not symmetric, we define two types of degrees associated to each node. First, we have the in-degree given by

$$k_i^{(\text{in})} = \sum_{l=1}^{N} \Omega_{il}. \quad (1)$$

This degree determines the total flow to reach the node $i$ from all the nodes. In a similar way, we have the out-degree

$$k_i^{(\text{out})} = \sum_{l=1}^{N} \Omega_{li}, \quad (2)$$

that quantifies the total flow from the node $i$ to all the nodes in the network. Without loss in the generality of the formalism, we assume also that $\Omega_{ii} = 0$ for $i = 1, 2, \ldots, N$. In the following, we consider connected directed networks for which $k_i^{(\text{out})} > 0$ for all the nodes.

In terms of the matrix of weights, we define the Laplacian matrix $L$ with elements $i, j$, given by

$$L_{ij} = k_i^{(\text{out})} \delta_{ij} - \Omega_{ij} \quad (3)$$

where $\delta_{ij}$ denotes the Kronecker’s delta. Equation (3) is a generalization of the Laplacian matrix for binary undirected networks [24, 27], to include the possibility of weights in the connections and asymmetry in the flow on some lines, for these particular connections $\Omega_{ij} \neq \Omega_{ji}$.

On the other hand, in the context of the fractional diffusion on networks is introduced the fractional Laplacian matrix $L^\gamma$, where $\gamma$ is a real number ($0 < \gamma < 1$). The resulting operator models the fractional dynamics on general networks [20, 27, 28]. Using Dirac’s notation for the eigenvectors, we have a set of right eigenvectors $\{|\Psi_j\rangle\}_{j=1}^{N}$ that satisfy the eigenvalue equation $L |\Psi_j\rangle = \mu_j |\Psi_j\rangle$ for $j = 1, \ldots, N$. With this information, we define the matrix $Q$ with elements $Q_{ij} = \langle i |\Psi_j\rangle$ and the diagonal matrix $\Lambda = \text{diag}(\mu_1, \mu_2, \ldots, \mu_N)$. These matrices satisfy $L = Q\Lambda Q^{-1}$, therefore

$$L = Q\Lambda Q^{-1}, \quad (4)$$

where $Q^{-1}$ is the inverse of $Q$. Using the matrix $Q^{-1}$, we define the set of left eigenvectors $\{|\tilde{\Psi}_m\rangle\}_{m=1}^{N}$ with components $\langle\tilde{\Psi}_m|j\rangle = (Q^{-1})_{mj}$. Therefore

$$L^\gamma = Q\Lambda^\gamma Q^{-1} = \sum_{m=1}^{N} \mu_m^\gamma |\Psi_m\rangle \langle\Psi_m|, \quad (5)$$

where $\Lambda^\gamma = \text{diag}(\mu_1^\gamma, \mu_2^\gamma, \ldots, \mu_N^\gamma)$ for $0 < \gamma \leq 1$.

In this case, due to the asymmetry of $\Omega$, the eigenvalues $\mu_i$ can take complex values. However, as a consequence of $\sum_{i=1}^{N} L_{ii} = 0$, the definition of the out-degree in Eq. (2) and the conditions $L_{ii} > 0$, $L_{ij} \leq 0$ for $i \neq j$, the fractional Laplacian $L^\gamma$ of a directed weighted network has real entries and the following properties for $0 < \gamma \leq 1$:

(i) For the fractional out-degree, we have

$$k_i^{(\gamma)} \equiv (L^\gamma)_{ii} = -\sum_{m \neq i} (L^\gamma)_{im}. \quad (6)$$

(ii) The diagonal elements of $L^\gamma$ are positive real values; in this way $k_i^{(\gamma)} > 0$ for $i = 1, 2, \ldots, N$.

(iii) The non-diagonal elements of $L^\gamma$ are real values satisfying $(L^\gamma)_{ij} \leq 0$ for $i \neq j$. See Refs. [20, 28, 51] for a detailed discussion on these properties for undirected and directed networks.

Considering the following integral representation of the fractional Laplacian matrix

$$L^\gamma = \int_{0}^{\infty} \frac{t^{\gamma-1}}{\Gamma(\gamma)} (1 - e^{-Lt}) \, dt, \quad 0 < \gamma < 1 \quad (7)$$

shows that the properties (i)-(iii) of $L$ are conserved in the interval of convergence $\gamma \in (0,1)$ of Eq. (7) (see Appendix VII for a brief demonstration). In this relation
\[ I = (\delta_{ij}) \text{ indicates the } N \times N \text{ identity matrix and } \Gamma(\gamma) \text{ stands for the Gamma-function.} \]

The characteristics of the fractional Laplacian matrix allow to define the fractional diffusion on directed weighted networks as a discrete time Markovian process determined by a transition matrix \( W^{(\gamma)} \) with elements \( w_{ij}^{(\gamma)} \) representing the probability to hop from \( i \) to \( j \) given by

\[ w_{ij}^{(\gamma)} = \delta_{ij} - \frac{(L_{ij})^{\gamma}}{k_{i}^{\gamma}} \quad 0 < \gamma \leq 1. \quad (8) \]

Above properties (i)-(iii) of the fractional Laplacian matrix indeed guarantee stochasticity of the fractional transition matrix \( \mathcal{S} \) in the interval \( 0 < \gamma \leq 1 \) (see Appendix VII). At each time \( t = 0, 1, 2, \ldots \), fractional random walkers move visiting nodes on the network in a process without memory. The probability \( P_{ij}(t; \gamma) \) to start at time \( t = 0 \) in \( i \) and reach the node \( j \) at time \( t \) satisfies the master equation

\[ P_{ij}(t+1; \gamma) = \sum_{m=1}^{N} P_{im}(t; \gamma) w_{m,j}^{(\gamma)}. \quad (9) \]

In the following part, we analyze the consequences of the fractional dynamics defined by Eqs. (8)-(10) on different directed weighted networks that include circulant directed networks, biased transport on rings and random networks. We explore the characteristics of the eigenvalues of the transition matrix, the probabilities of transition in Eq. (8), mean first passage times and global times that describe the efficiency of the random walker to reach any node on the network.

### III. FRACTIONAL RANDOM WALKS ON DIRECTED CIRCULANT NETWORKS

In this section, we explore the fractional transport with transition probabilities given by Eq. (8). We analyze directed networks defined by matrices of weights \( \Omega \) with a circulant matrix structure.

#### A. Circulant matrices

A circulant matrix \( C \) is an \( n \times n \) matrix with the form

\[ C = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{pmatrix}, \quad (10) \]

with elements \( C_{ij} \). Thus, each column has real elements \( c_0, c_1, \ldots, c_{n-1} \) ordered in such a way that \( c_0 \) describes the diagonal elements and \( C_{ij} = c_{(i-j) \mod n} \). In addition to \( C \), the elementary circularizing matrix \( E \) is defined, which has all its null elements except \( c_1 = 1 \). From \( E \), the integer powers \( E^l \) for \( l = 0, 1, 2, \ldots, n-1 \) are also circulant matrices with null elements except \( c_1 = 1 \). Therefore, Eq. (10) can be expressed as

\[ C = c_0 I + c_1 E + c_2 E^2 + \ldots + c_{n-1} E^{n-1} = \sum_{m=0}^{n-1} c_m E^m, \quad (11) \]

where \( I = E^0 \) is the \( n \times n \) identity matrix. Furthermore, the relation \( E^n = I \) requires that the eigenvalues \( \nu \) of \( E \) satisfy \( \nu^n = 1 \); therefore, those eigenvalues are given by

\[ \nu = e^{\frac{2\pi i (l-1)}{n}} \quad \text{for} \quad l = 1, \ldots, n, \quad (12) \]

with \( i = \sqrt{-1} \). The respective eigenvectors \( \{ | \Psi_m \rangle \}_{m=1}^n \) have the components \( \langle l | \Psi_m \rangle = \frac{1}{\sqrt{n}} e^{i \frac{2\pi l (m-1)}{n}} \) (see Ref. [38] for details). Now, using Eq. (11), the eigenvectors \( | \Psi_l \rangle \) satisfy \( C | \Psi_l \rangle = \eta_l | \Psi_l \rangle \), where the eigenvalues \( \eta_l \) are given by

\[ \eta_l = \sum_{m=0}^{n-1} c_m e^{\frac{2\pi i (l-1)m}{n}} \quad (13) \]

for \( l = 1, 2, \ldots, n \). This result defines the eigenvalues of \( C \) in terms of the coefficients \( c_0, c_1, \ldots, c_{n-1} \).
FIG. 2. (Color online) Eigenvalues of the transition matrix $W^{(\gamma)}$ for circulant directed networks with $N = 100$ represented in the complex plane. Each eigenvalue $\lambda^{(\gamma)}_i$, $i = 1, 2, \ldots, N$, is determined by Eq. (14) with eigenvalues $\mu_l$ given by Eq. (15). In (a)-(f) we specify the non-null coefficients $c_m = 1$ that define the adjacency matrix of the network. For each $\gamma$, we have $N = 100$ points and, the effect of the fractional parameter $\gamma$ modifies the set of eigenvalues represented with different colors codified in the colorbar in the interval $0 < \gamma \leq 1$. The studied networks have a similar topology to those presented in Fig. 1.

B. Directed circulant networks

The circulant matrix $C$ defined in Eq. (10) allows us to explore different directed structures with $N$ nodes and an adjacency matrix $A$ with specific values $c_1, c_2, \ldots, c_{N-1}$ equal to 0 or 1, the result is a network with a periodic structure. When non-null elements appear in pairs $c_i$ and $c_{N-i}$ ($i = 1, 2, \ldots, N-1$) in the adjacency matrix, the structure is an undirected network with symmetric $A$. Fractional dynamics on undirected circulant networks has been studied in detail for continuous-time random walks [21, 28], quantum transport [22] and diffusion on multilayer networks [29].

In the following, we explore cases when the matrix of weights $\Omega$ that defines the Laplacian in Eq. (3) is $\Omega = A$, this adjacency matrix is not symmetric with a net direction in some lines. For example, the particular network with non-null $c_1 = 1$ (or $c_{N-1} = 1$) produces a directed ring. In Fig. 1 we illustrate several directed circulant structures with $N = 10$ nodes. In Fig. 1(a) we have a directed ring, whereas in Figs. 1(b)-(f) other networks are generated by adding new sets of lines defined with non-null elements $c_i$ ($i = 1, 2, \ldots, N-1$). In some cases, two nodes are connected with links in both directions. We represent this particular type of connection with lines between nodes as shown in Figs. 1(d)-(f).

The advantage in the study of the fractional dynamics on circulant networks is that we know all the eigenvalues and eigenvectors of the Laplacian matrix $L$, since this is also a circulant matrix. In addition, circulant networks are regular with the same fractional out-degree $k^{(\gamma)} = \frac{1}{N} \sum_{m=1}^{N} \mu_m^{(\gamma)}$ for all the nodes. Therefore, the eigenvalues of the transition matrix $W^{(\gamma)} = I - \frac{L}{k^{(\gamma)}}$ in Eq. (8) for a fractional random walker are

$$\lambda^{(\gamma)}_i = 1 - \frac{\mu_i^{(\gamma)}}{k^{(\gamma)}}, \quad i = 1, 2, \ldots, N,$$

where, applying the result in Eq. (13) for a network defined by $A$ with a set of lines with a particular sequence of values 0 and 1 for the coefficients $\{c_m\}_{m=1}^{N-1}$, the eigenvalues of the Laplacian matrix $L$ are

$$\mu_l = c_0 - \sum_{m=1}^{N-1} c_m e^{i \frac{2\pi}{N}(l-1)m},$$
where the diagonal element $c_0 \equiv \sum_{m=1}^{N-1} c_m$ is the out-degree of each node.

The result in Eq. (15) shows that in circulant directed networks the eigenvalues of the Laplacian matrix are complex numbers and, as a consequence, the eigenvalues of $W^{(\gamma)}$ in Eq. (14) are also complex numbers. In Fig. 2 we illustrate the effect of the fractional parameter $\gamma$ on the eigenvalues $\lambda_i^{(\gamma)}$ in Eq. (14) for different structures with topologies similar to the networks in Fig. 1. We show the real and imaginary parts of $\lambda_i^{(\gamma)}$ for directed networks with $N = 100$ nodes and $0 < \gamma \leq 1$.

C. Long-range dynamics and Lévy flights

We have analyzed the spectral properties of circulant matrices associated to the Laplacian $L$ and fractional transition probabilities $W^{(\gamma)}$. In the following part, we calculate the probabilities $w^{(\gamma)}_{i\rightarrow j}$ and the relation with the distance $d_{ij}$ that gives the shortest-path length between nodes $i$ and $j$. Using Eq. (15) and the respective eigenvectors of circulant matrices, we have for the fractional Laplacian

$$
(L^\gamma)_{ij} = \frac{1}{N} \sum_{\ell=1}^{N} \mu_{\ell}^{\gamma} \langle i|\Psi_{\ell}\rangle \langle \Psi_{\ell}|j\rangle = \frac{1}{N} \sum_{\ell=1}^{N} \mu_{\ell}^{\gamma} e^{i \frac{\pi}{\delta} (\ell-1)(j-i)}. 
$$

With the spectral characteristics of the eigenvalues $\mu_{\ell}^{\gamma}$ and the respective $\lambda_i^{(\gamma)}$ illustrated in Fig. 2, the sums in Eq. (17) in the interval $0 < \gamma \leq 1$ produce well-defined probabilities of transition between nodes. In Fig. 3 we present the transition probabilities $w^{(\gamma)}_{i\rightarrow j}$ for circular networks with $N = 10^5$ nodes. Probabilities are presented as a function of the distance $d_{ij}$ connecting the nodes $i$ and $j$ ($d_{ij}$ is the length of the shortest path on the directed structure). In the case of the directed ring [Fig. 3(a)], we see the relation $w^{(\gamma)}_{i\rightarrow j} \propto d_{ij}^{-1-\gamma}$, a power-law decay also observed in the large-world network explored in Fig. 3(b) and exemplified in Fig. 1(b). This particular relation between transition probabilities and distances show an emergent dynamics generated through the fractional Laplacian related with a Lévy-like dynamics in directed structures. Lévy flights in undirected networks have been explored in a series of works [18-21, 24, 25, 28, 29, 39, 41], revealing that long-range displacements in undirected networks always improve the capacity to explore a network by inducing dynamically the small-world property [18, 20]. Similar long-range strategies have been identified in human mobility [9] in the movement of cyclists between stations in bike-sharing systems in Chicago and New York [10], in taxi trips in New York City [11] and in the infection spreading through the United States’ highly-connected air travel network [42].

![Figure 3](image1.png)

**FIG. 3.** (Color online) Transition probabilities $w^{(\gamma)}_{i\rightarrow j}$ between two nodes as a function of the distance $d_{ij}$ in directed circulant networks with $N = 10^5$ nodes. (a) A directed ring defined with $c_{N-1} = 1$, (b) a network with $c_{N-1} = c_{N-2} = 1$. The probabilities $w^{(\gamma)}_{i\rightarrow j}$ are deduced from the analytical relation in Eq. (17) defined in terms of the eigenvalues $\lambda_i^{(\gamma)}$ in Eq. (15) for $\gamma = 0.25, 0.5, 0.75$ and $\gamma = 0.9$. Dashed lines represent the inverse power-law relation $w^{(\gamma)}_{i\rightarrow j} \propto d_{ij}^{-1-\gamma}$. (54x-87)
D. Infinite directed ring

Now, we explore the fractional Laplacian matrix and transition probabilities in a directed ring. This network is illustrated in Fig. 11(a), in the limit $N \to \infty$. In the particular case of a directed ring with $N$ nodes, Eq. (18) takes the form (we use the value $c_{N-1} = 1$)

$$
(L^γ)_{ij} = \frac{1}{N} \sum_{l=1}^{N} (1 - e^{-i2\pi(l-1)})^γ e^{i2\pi(l-1)(j-i)}. \quad (18)
$$

However, in the limit of $N$ large, we can define a continuous variable $\varphi = \frac{2\pi}{N}(l-1)$ and $d\varphi = \frac{2\pi}{N}$. Therefore, the elements of the fractional Laplacian matrix for an infinite directed ring are given by

$$
(L^γ)_{ij} = \frac{1}{2\pi} \int_{0}^{2\pi} (1 - e^{-i\varphi}) e^{i\varphi(j-i-\gamma)} d\varphi
= (-1)^{1+\gamma} \csc(\pi\gamma) \sin[\pi(j-i-\gamma)] \Gamma(j-i+1) \Gamma(1-\gamma) \Gamma(\gamma+1)
= \frac{\Gamma(j-i-\gamma)}{\Gamma(j-i+1)}. \quad (19)
$$

In particular, $(L^γ)_{ij} = 0$ for $i > j$ and $(L^γ)_{jj} = 1$ for the diagonal elements. On the other hand, in the limit $j \gg i$, using the relation $\Gamma(n+\alpha) \approx \Gamma(n)n^\alpha$ for $n$ large, we have for Eq. (19)

$$
(L^γ)_{ij} \approx \frac{1}{\Gamma(\gamma)} \frac{1}{(j-i)^{1+\gamma}} \quad \text{for} \quad j \gg i. \quad (20)
$$

The asymptotic result in Eq. (20) shows analytically that the fractional dynamics in the infinite directed ring produces transition probabilities $w_{i\to j}^{(\gamma)} \propto d_{ij}^{-1-\gamma}$ for $j \gg i$ with distances $d_{ij} = j - i$. This relation is also valid for networks with $N$ large but not necessarily infinite (see Appendix V for a detailed discussion). The behavior observed differs from the directed ring, where a similar analysis reveals a Lévy-like dynamics with $w_{i\to j}^{(\gamma)} \propto d_{ij}^{-2-\gamma}$ (see Refs. [20, 21, 22]), a result that in the general case of undirected $n$-dimensional lattices is $w_{i\to j}^{(\gamma)} \propto d_{ij}^{-n-\gamma}$ [23, 24, 25].

E. Efficiency of the fractional transport in circulant structures

In connected circulant networks, each node has the same fractional degree $\kappa^{(\gamma)}$. In addition, if the structure is connected we have only one eigenvalue $\lambda_1^{(\gamma)} = 1$ and; in this particular case, the ergodic condition is fulfilled since, for sufficiently large time, the random walker can reach any node of the network independently of the initial node. Therefore, well-known results for the mean first passage time $(T_{ij})$, which gives the average number of steps of a discrete-time random walker to start in node $i$ and reach for the first time $j$, still apply for circulant directed structures [2].

In terms of the left and right eigenvectors ($\langle \phi_i \rangle$ and $|\phi_i\rangle$, respectively) of the transition matrix $W$ of a Markovian random walker and the associated eigenvalues $\lambda_i$ (we denote $\lambda_1 = 1$), we have for $i \neq j$ [18, 27, 28]

$$
(T_{ij}) = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l} \frac{\langle j | \phi_1 \rangle \langle \phi_1 | j \rangle}{\langle j | \phi_1 \rangle \langle \phi_1 | j \rangle}, \quad (21)
$$

and the mean first return time $(T_{ii}) = (\langle i | \phi_1 \rangle \langle \phi_1 | i \rangle)^{-1}$. In addition, in structures with the same fractional degree we have the global time [18, 27, 28]

$$
T = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l}. \quad (22)
$$

This is the Kemeny’s constant that quantifies the capacity of the process to explore the network in regular structures and only depends on the eigenvalues of the transition matrix $W$.

In the fractional transport on circulant networks, the eigenvectors of $L$, the fractional Laplacian $L^γ$ and the transition probability matrix $W^{(\gamma)}$ coincide, since all of them are circulant matrices. Hence, $\langle i | \phi_1 \rangle \langle \phi_1 | j \rangle = \langle i | \Psi_1 \rangle \langle \Psi_1 | j \rangle = \frac{1}{N} e^{i\frac{2\pi}{N}(l-1)(j-i)}$. Then, for the fractional transport in circulant networks, Eq. (21) takes the form ($i \neq j$)

$$
\langle T_{ij}(\gamma) \rangle = \sum_{l=2}^{N} \frac{1}{1 - \lambda_l^{(\gamma)}} \left[1 - e^{i\frac{2\pi}{N}(l-1)(j-i)}\right],
$$

and $\langle T_{ii}(\gamma) \rangle = N$, $\lambda_1^{(\gamma)}$ is given by Eq. (14). Therefore

$$
\langle T_{ij}(\gamma) \rangle = \left(\frac{1}{N} \sum_{m=2}^{N} \mu_m^{(\gamma)} \right) \sum_{l=2}^{N} \frac{1}{1 - e^{i\frac{2\pi}{N}(l-1)(j-i)}} \mu_l^{(\gamma)}, \quad (23)
$$

where we use the result $\mu_1 = 0$. In a similar way, we obtain for the Kemeny’s constant in Eq. (22)

$$
T(\gamma) = \left(\frac{1}{N} \sum_{m=2}^{N} \mu_m^{(\gamma)} \right) \sum_{l=2}^{N} \frac{1}{1 - \mu_l^{(\gamma)}}. \quad (24)
$$

Our findings in Eqs. (23)–(24) are valid for directed and undirected connected circulant networks and allow to calculate analytically the mean first passage time (MFPT) and the Kemeny’s constant through the specification of the coefficients $c_0, c_1, \ldots, c_{N-1}$ in Eq. (15).

In Fig. 3 we depict the MFPT for a directed ring and an undirected ring, both with $N = 10^4$ nodes. The results illustrate a completely different behavior in the fractional dynamics in directed and undirected rings. First of all, as we can see in Fig. 3(a), for the directed ring defined with an adjacency matrix $c_{N-1} = 1$, the results show that for $\gamma = 1$ and $i \neq j$, $(T_{ij}(\gamma = 1)) = d_{ij}$, where $d_{ij} = j - i$ for
FIG. 4. (Color online) Mean first passage time \( \langle T_{ij}(\gamma) \rangle \) as a function of the distance \( d_{ij} \) in circulant networks with \( N = 10^4 \) nodes. (a) A directed ring defined with \( c_{N-1} = 1 \), (b) an undirected ring with \( c_{N-1} = c_1 = 1 \). Numerical values are obtained from Eq. (23) defined in terms of the eigenvalues in Eq. (14). We codified in the colorbar the different values \( 0 < \gamma \leq 1 \).

FIG. 5. (Color online) Kemeny’s constant \( T(\gamma) \) as a function of \( \gamma \) for different directed networks with \( N = 10^4 \) nodes. The networks have a topology similar to those in Fig. 1 and are defined with the particular set of coefficients equal to 1: (a) \( \{c_{N-1}\} \), (b) \( \{c_{N-1}, c_{N-2}\} \), (c) \( \{c_{N-1}, c_2\} \), (d) \( \{c_{N-1}, c_1, c_{N-2}\} \), (e) \( \{c_{N-1}, c_{N-2}, c_2\} \), (f) \( \{c_{N-1}, c_1, c_{N-2}, c_2, c_{N-3}\} \). The results were obtained with Eq. (24) and the Laplacian eigenvalues in Eq. (15). The dashed horizontal line represents \( T_{\text{complete}} = (N-1)^2/N \) for a random walker in a complete (fully connected) network.

In the directed ring, the value \( \gamma = 1 \) produces a deterministic dynamics where at each step the walker visits a new adjacent node with a cover time \( N \). On the other hand, in the interval \( 0 < \gamma < 1 \), the temporal evolution is stochastic with a biased Lévy like dynamics increasing the MFPT but maintaining these times below or equal to the value \( N \). The results for the biased transport that emerge in the fractional dynamics on the directed ring agree with previous studies showing that Lévy flights do not always optimize the search problem in the presence of an external drift \[43\]. In Fig. 4(b), we present the results for times \( \langle T_{ij}(\gamma) \rangle \) in a symmetric ring. In this case, the fractional dynamics with Lévy flights reduce the MFPT found in the local limit \( \gamma = 1 \) for which the random walker at each step moves with probability 1/2 from a node to one of its two neighbors.

Finally, it is important to have a global time that characterizes the capacity of the random walker to explore the network. In structures such as circulant networks, the Kemeny’s constant \( T(\gamma) \) defined in Eq. (24) gives a global value quantifying the efficiency of the fractional random walker to reach all the nodes. In Fig. 5 we present the values of \( T(\gamma) \) as a function of \( \gamma \) \( (0 < \gamma \leq 1) \) in circulant networks with \( N = 10^4 \) nodes. We explore different directed structures with topologies and properties discussed in Figs. 1-2. In Fig. 5 we observe a similar behavior with an optimal value in the local-limit \( \gamma \to 1 \).

In directed structures with more lines like in curves (c)-(f) we see a different behavior where a particular value of \( \gamma^* < 1 \) produces a maximum in the Kemeny’s constant; however, with the reduction of \( \gamma \) in the limit \( \gamma \to 0 \) all the
cases approach to the value $T_{\text{complete}}$. The results show particular cases where the combination of non-local displacements and the bias generated by the directions of the lines produce a global effect that reduces the efficiency to visit all the nodes of the network.

**IV. BIASED TRANSPORT ON RINGS**

In the previous section, we studied random walks on circulant directed networks for which we considered the weights $\Omega = A$. We are now interested in the effects of the fractional transport when the matrix $\Omega$ describes some type of bias determined by weights in the links. We explore the transport on a ring with transition probabilities different than the directed and undirected rings studied before.

Let us now consider a probability $0 \leq p \leq 1$ and a ring with $N$ nodes which are connected only to their first neighbors. In addition, the coefficients $\Omega_{ij}$ are defined by a circulant matrix with non-null elements $c_1 = p$ and $c_{N-1} = 1 - p$. In this way, we have the probability $p$ to hop in one direction, and $1 - p$ to the opposite direction. Using Eq. (15) for the eigenvalues of the Laplacian matrix, and $c_0 = c_1 + c_{N-1} = 1$, we have

$$\mu_l = 1 - pe^{i\varphi_l} - (1 - p)e^{-i\varphi_l}$$

where $\varphi_l \equiv \frac{2\pi}{N}(l - 1)$ and $1 \leq l \leq N$.

In the general case, the eigenvalues $\lambda_{\ell}^{(\gamma)}$ of the fractional transition matrix $W^{(\gamma)}$ are complex values. By applying Eq. (14), we obtain

$$\lambda_{\ell}^{(\gamma)} = 1 - \frac{1}{k^{(\gamma)}}(1 - pe^{i\varphi_l} - (1 - p)e^{-i\varphi_l})^\gamma$$

with a fractional degree

$$k^{(\gamma)} = \frac{1}{N} \sum_{l=1}^{N} (1 - pe^{i\varphi_l} - (1 - p)e^{-i\varphi_l})^\gamma.$$  

In particular, for $p = 1/2$ and $\gamma = 1$, we recover the eigenvalues $\lambda_{\ell}^{(\gamma=1)} = \cos(\varphi_l)$ for the local random walk in a symmetric ring.

In Fig. 6 we show the eigenvalues $\lambda_{\ell}^{(\gamma)}$ of the transition matrix $W^{(\gamma)}$ for the biased fractional transport in a ring with $N = 100$ nodes. The particular limit $p = 1$ recovers the transport on the directed ring presented in Fig. 4(a). We see how the bias modeled with the parameter $0.6 \leq p \leq 0.9$ reduces the imaginary component of the eigenvalues when $p \to 1/2$. In the limit $p = 1/2$ all the eigenvalues are real. We obtain similar results for $0 \leq p \leq 0.5$ since, in this interval, the random walker has the same dynamics but with a change in all the directions of the walker.

Once deduced the eigenvalues of the Laplacian matrix, we can quantify globally the capacity of the fractional random walker to explore the network. Since the fractional degrees are the same for all the nodes, we use the relation for the Kemeny’s constant in Eq. (24). Therefore

$$T^{(\gamma)} = \frac{1}{N} \sum_{l=2}^{N} \sum_{m=2}^{N} \left( \frac{1 - pe^{i\varphi_l} - (1 - p)e^{-i\varphi_l}}{1 - pe^{i\varphi_m} - (1 - p)e^{-i\varphi_m}} \right)^\gamma$$

for $0 < p \leq 1$ and $0 < \gamma \leq 1$. In particular, in the limit $p = 1/2$ we recover the result for the symmetric ring

$$T_S^{(\gamma)} = k^{(\gamma)} \sum_{m=2}^{N} \left( \frac{1}{1 - \cos \varphi_m} \right)^\gamma.$$ 

with the fractional degree $k^{(\gamma)} = \frac{1}{N} \sum_{l=2}^{N} (1 - \cos \varphi_l)^\gamma$. The global time $T_S^{(\gamma)}$ is analyzed in detail in Ref. [28] to characterize the fractional transport on undirected rings.

In Fig. 7 we present the results obtained with Eq. (28) for the Kemeny’s constant describing the fractional transport with bias in rings for different values of $p$. We observe in the behaviour of the Kemeny’s constant that, for $p$ in the interval $0.53 \leq p \leq 0.96$, $T^{(\gamma)}$ presents a maximum for a particular value $\gamma$ denoted as $\gamma^*$. Calculating $\frac{d}{d\gamma} T^{(\gamma)}|_{\gamma=\gamma^*} = 0$, we obtain that the value $\gamma^*$

\[\text{FIG. 6. (Color online) Eigenvalues of the transition matrix } W^{(\gamma)} \text{ for biased fractional transport on a ring with } N = 100 \text{ nodes. For } \gamma = 1, \text{ the transition probabilities to the two nearest neighbors are } p \text{ to move in one direction and } 1 - p \text{ in the opposite direction. We explore the effect of } \gamma \text{ and } p \text{ in the eigenvalues of } W^{(\gamma)} \text{ for } p = 0.9, 0.8, 0.7, 0.6. \text{ Each color represents a different set of eigenvalues, obtained with a given } \gamma \text{ codified in the colorbar. The eigenvalues are calculated with Eq. (25). The limit } p = 1 \text{ represents a directed ring with eigenvalues analyzed in Fig. 4(a).}\]
A random walk in a uniform random network is similar to the Erdős-Rényi (ER) model [44]. For directed networks generated stochastically with an algorithm as described in [44], we decide to include the value 1 in the network, in comparison with the normal random walk where $p = 1/2$ recovers a symmetric random walker with Levy flights whereas the limit $p = 1$ describes the transport on a directed ring. These two limits are explored in Figs. 3-4.

In the inset we present the value $\gamma^*$, as a function of $p$, that maximizes $T(\gamma)$ and is obtained numerically with Eq. (30).

maximizing the Kemeny’s constant satisfies

$$\sum_{l,m=2}^{N} \log(z_l) \left( \frac{z_l}{z_m} \right)^{\gamma^*} = \sum_{l,m=2}^{N} \log(z_m) \left( \frac{z_l}{z_m} \right)^{\gamma^*}$$

with $z_l = 1 - pe^{i\varphi_l} - (1-p)e^{-i\varphi_l}$. In the inset in Fig. 7 we show the values $\gamma^*$ as a function of $p$, calculated numerically with Eq. (30). This result determines which Levy flight strategy is the least efficient to explore the network, i.e., for which value of $\gamma$ the Kemeny’s constant $T(\gamma)$ has a maximum in the interval $0 < \gamma < 1$.

V. DIRECTED RANDOM NETWORKS

In this section, we explore the fractional transport on directed networks generated stochastically with an algorithm similar to the Erdős-Rényi (ER) model [44]. For $N$ nodes we have an adjacency matrix $A$ and, in each non-diagonal entry $A_{ij}$, we decide to include the value 1 or 0 randomly with probabilities $p$ and $1-p$, respectively. However, the difference with the traditional ER model is that the choice of $A_{ij}$ is independent of $A_{ji}$, and as a result, $A$ is the non-symmetric matricial representation of a directed network where $pN(N-1)$ is the average number of links in this structure.

In Fig. 8 we explore three random networks with size $N = 50$ generated with the value $p = 0.04$, following the same approach presented for the analysis of regular topologies. We use connected networks for all the nodes to analyze the modifications in the eigenvalues introduced by the fractional transport. The eigenvalues of the transition probabilities and the respective networks are shown in Figs. 8(a)-(c). In this representation, for each value $\gamma$, we calculate numerically the fractional Laplacian matrix given by Eq. (23) and in this way, we have the elements that define the transition matrix $W(\gamma)$, for which we calculate the eigenvalues $\lambda_l^{(\gamma)}$. Due to the connectivity of the network, in the cases explored, the eigenvalue $\lambda^*_1 =1$ is unique for all $\gamma$. The remaining eigenvalues are represented as black dots in the complex plane for $\gamma = 1$, and with different colors we show how the reduction of $\gamma$ concentrates the eigenvalues around the origin with $\lambda_l^{(\gamma)} = -1/(N-1)$ for $l = 2, 3, \ldots, N$ in the limit $\gamma \rightarrow 0$.

In addition to the eigenvalues, it is useful to quantify the capacity of the fractional random walker to explore the whole network. In this case, we use the global MFPT $\langle T(\gamma) \rangle$ defined as

$$\langle T(\gamma) \rangle = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle T_{ij}(\gamma) \rangle$$

(31)

gives the average of the mean first passage time $\langle T_{ij}(\gamma) \rangle$ considering all the initial nodes $i$ and all the nodes target $j$. In Fig. 8(d), we show the results obtained with the numerical values of the eigenvalues and left and right eigenvectors of the transition matrix $W(\gamma)$ for $0 < \gamma \leq 1$. In addition, in the inset in this figure, we include the values of the Kemeny’s constant $T(\gamma) = \sum_{l=2}^{N} \left( 1 - \lambda_l^{(\gamma)} \right)^{-1}$ that only includes information of the eigenvalues of $W(\gamma)$.

The results in Fig. 8 illustrate the different cases that can appear in the fractional transport on directed structures for the same type of random network. In the information in Fig. 8(d), the global MFPT $\langle T(\gamma) \rangle$ shows that in some cases, like in the network (a), the effect of $\gamma$ can improve the efficiency of the random walker to reach a target, in comparison with the normal random walk $\gamma \rightarrow 1$. In networks (b) and (c) we see two cases where the long-range dynamics increases the values of $\langle T(\gamma) \rangle$ in comparison with the normal dynamics. In addition, the eigenvalues in (a)-(c) reveal different spectral properties with the change of $\gamma$. However, in contrast with the regular cases explored with circulant matrices, in this case the Kemeny’s constant is not a good descriptor of the global activity in the fractional transport, revealing that in the networks analyzed the eigenvectors of $W(\gamma)$ contained important information for a global characterization of the dynamical process.
VI. CONCLUSIONS

In this work, we presented a general approach to examining Markovian non-local random walks generated through the fractional Laplacian of directed networks. This formalism is explored for different types of networks defined in terms of circulant matrices and also for random directed networks. We analyzed the eigenvalues of transition matrices that define the random walker and the effect of the fractional dynamics in the spectrum, showing how eigenvalues are modified in the complex plane. In addition, for regular networks, we analyze the Kemeny’s constant, which gives a good description of the global dynamics. With this global time, defined only in terms of the eigenvalues, we analyzed the effect of biased non-local transport, showing that the exploration of the network can be more effective in some particular cases reducing Kemeny’s constant. We also identified different configurations for which the capacity of the fractional random walker to reach any node is reduced in comparison with a random walker with hops to neighbor nodes. This is a fundamental difference with the results observed in undirected networks where the fractional dynamics always improve the transport through long-range displacements. Finally, we explored the transport of random directed networks; in this case, the global activity is analyzed using the average of the mean first passage time to all the nodes considering all the initial conditions. This quantity depends on the eigenvalues and eigenvectors of the transition matrix that defines the process.

VII. APPENDIX A: PROPERTIES OF THE FRACTIONAL LAPLACIAN

In this appendix we demonstrate briefly that the essential good properties (i)-(iii) of the Laplacian in Eq. (3) are conserved by the fractional Laplacian matrix $L^{\gamma}$ in the interval $0 < \gamma \leq 1$ in order to guarantee that the fractional transition matrix $W^{(\gamma)}$ is stochastic, i.e. $0 \leq w_{ij}^{(\gamma)} \leq 1$ with $\sum_{l=1}^{N} w_{il}^{(\gamma)} = 1$. To this end let us introduce $\Lambda > \max(k_i^{\text{(out)}})$ thus the matrix $\Lambda I - L$ has uniquely positive entries $(\Lambda - k_i^{\text{(out)}})\delta_{ij} + \Omega_{ij} > 0$. Then it follows that $(\Lambda I - L)^n$ and hence $e^{t(\Lambda I - L)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\Lambda I - L)^n = e^{\Lambda t} e^{-tL}$ ($t > 0$) also conserve this property. Then since $e^{\Lambda t} > 0$ this remains also true for the matrix exponential $(e^{-tL})_{ij} > 0$. It follows then that the non-diagonal elements $(I - e^{-tL})_{ij} = -(e^{-tL})_{ij} < 0$
(i ≠ j) are uniquely negative. Since the zero eigenvalue to the constant eigenvector \((i|\Psi_1)\) of \(L\) is conserved by the matrix function \((1 - e^{-tL})|\Psi_1\) = 0, it follows from the negativeness of the non-diagonal-elements, the positiveness of the diagonal-elements, i.e. it holds

\[
(1 - e^{-tL})_{ii} = -\sum_{j \neq i}^N (1 - e^{-tL})_{ij} > 0. \tag{32}
\]

Applying (31) on both sides of this relation yields Eq. (30) and conserves the signs in (32), i.e. \(k^{(\gamma)}_i = (L^\gamma)_{ii} > 0\) together with \((L^\gamma)_{ij} < 0\) for \(i \neq j\). In this way we have demonstrated that the good properties (i)-(iii) of \(L\) are indeed conserved by the fractional Laplacian matrix \(L^\gamma\) in the interval of convergence 0 < \(\gamma < 1\) of the integral representation in Eq. (7). For a more detailed analysis we refer to Ref. \[28\].

VIII. APPENDIX B: FRACTIONAL LAPLACIAN FOR DIRECTED RINGS

In this Appendix, we analyze the elements of \(L^\gamma\) for a directed ring of finite size \(N\) where \(N\) is not necessarily large. Let us consider the Laplacian \(L\) for a directed ring defined with elements \(c_0 = c_{N-1} = 1\) and \(c_m = 0\) for \(m = 1, 2, N - 2\). Using Eq. (18) we have the eigenvalues

\[
\mu_\ell = \sum_{m=1}^N c_m (1 - e^{i\varphi \ell m}), \quad \varphi_\ell = \frac{2\pi}{N} (\ell - 1). \tag{33}
\]

Therefore, the Laplacian eigenvalue is given by

\[
\mu_\ell = 1 - e^{i\varphi \ell (N-1)} = 1 - e^{-i\varphi_\ell} \tag{34}
\]

In addition, the elements of the fractional Laplacian matrix given by Eq. (13) for 0 < \(\gamma \leq 1\)

\[
(L^\gamma)_{pq} = (L^\gamma)_{q-p} = \sum_{\ell=1}^N \mu_\ell \gamma (p|\Psi_\ell) \langle \Psi_\ell |q\rangle = \frac{1}{N} \sum_{\ell=1}^N (1 - e^{-i\varphi_\ell})^\gamma e^{i\varphi_\ell (q-p)}. \tag{35}
\]

Now we can expand the fractional Laplacian eigenvalue (this series is converging)

\[
(1 - e^{-i\varphi_\ell})^\gamma = \sum_{m=0}^{\infty} (-1)^m \frac{\gamma^m}{m!} e^{-i\varphi_\ell m} = \sum_{s=0}^{N-1} e^{-i\varphi_\ell s} A_s \tag{36}
\]

with

\[
A_s = (-1)^{s+Nt} \sum_{t=0}^{\infty} \gamma^s (s + Nt). \tag{37}
\]

In this result, we put \(m = s + Nt\) and apply the \(N\)-periodicity condition \(e^{i\varphi \ell m} = e^{-i\varphi \ell (s + Nt)} = e^{-i\varphi \ell s}\) (with \(e^{-i\varphi \ell Nt} = e^{-2\pi i (\ell - 1)} = 1\)). We see especially that Eq. (37) indeed holds for finite \(N \geq 2\).

Now, using the orthogonality property

\[
\frac{1}{N} \sum_{q,s} e^{i(q-p-s)\varphi_\ell} = \delta_{q-p,s}, \quad s = 0, \ldots, N - 1, \tag{38}
\]

and combining Eqs. (34)-(38), we obtain the decomposition of the elements of the circulant fractional Laplacian matrix \([33, 35]\), i.e. the fractional Laplacian matrix for the finite ring in Eq. (18), for \(N\) finite but not necessarily large

\[
(L^\gamma)_{pq} = \frac{1}{N} \sum_{\ell=1}^N (1 - e^{-i\varphi_\ell})^\gamma e^{i\varphi_\ell (q-p)} = A_{q-p} = (-1)^{q-p+Nt} \sum_{t=0}^{\infty} \gamma (q - p + Nt) \tag{39}
\]

for \(q - p = 0, \ldots, N - 1\). Hence

\[
(L^\gamma)_{pq} = (-1)^{q-p} \left( \gamma \frac{\Gamma(q - p + 1)}{\Gamma(1 - (q - p))} \right) \tag{40}
\]

We see in this result for the \textit{finite ring} that the first term in the series \((t = 0)\), namely

\[
(-1)^{q-p} \left( \gamma \frac{\Gamma(q - p + 1)}{\Gamma(1 - (q - p))} \right)
\]

is the matrix element of Eq. (19) for the \textit{infinite ring}. In Eq. (37) additionally to the infinite ring elements we have the image series \(\sum_{t=1}^{\infty} (\ldots)\) where this additional contribution of the image terms for \(N\) finite but large can be (roughly) estimated as

\[
\sum_{t=1}^{\infty} (\frac{Nt}{\Gamma(-\gamma)} \approx \int_N^{\infty} \frac{\tau^{-\gamma-1}}{\Gamma(-\gamma)} d\tau - \frac{N^{-\gamma}}{\Gamma(1 - \gamma)}. \tag{42}
\]

It follows that the infinite ring matrix elements (19) already are a good approximation for rings with \(N\) large but not necessarily infinite. In Ref. \[28\], in section 6.2.3. \textit{Fractional Laplacian of the finite ring}, we consider the fractional Laplacian of finite undirected rings and obtain analogue results (see there Eqs. (6.27)-(6.31)) where we employ the same periodicity argument as here.
