Explicit construction of nilpotent covariants in $N = 4$ SYM

P.S. Howe$^a$, C. Schubert$^b$, E. Sokatchev$^b$ and P.C. West$^a$

$^a$ Department of Mathematics, King’s College, London, UK

$^b$ Laboratoire d’Annecy-le-Vieux de Physique Théorique, LAPTH, Chemin de Bellevue, B.P. 110, F-74941 Annecy-le-Vieux, France

Abstract

Some aspects of correlation functions in $N = 4$ SYM are discussed. Using $N = 4$ harmonic superspace we study two and three-point correlation functions which are of contact type and argue that these contact terms will not affect the non-renormalisation theorem for such correlators at non-coincident points. We then present a perturbative calculation of a five-point function at two loops in $N = 2$ harmonic superspace and verify that it reproduces the derivative of the previously found four-point function with respect to the coupling. The calculation of this four-point function via the five-point function turns out to be significantly simpler than the original direct calculation. This calculation also provides an explicit construction of an $N = 2$ component of an $N = 4$ five-point nilpotent covariant that violates $U(1)_Y$ symmetry.
1 Introduction

The superconformal Ward identities play a central rôle in the study of superconformal field theories, and in particular, in $N = 4$ Yang-Mills theory. These Ward identities can be conveniently expressed in harmonic superspaces - either $N = 4$ on-shell harmonic superspace $[1]$ which has the advantage that the gauge-invariant operators in short multiplets are represented by single-component analytic superfields, or $N = 2$ off-shell harmonic superspace $[2]$ in which, although the $N = 4$ Yang-Mills multiplet decomposes into the $N = 2$ Yang-Mills multiplet plus a hypermultiplet, one can carry out perturbation theory calculations whilst maintaining manifest $N = 2$ supersymmetry.

Using the harmonic superspace approach to $N = 4$ SYM and motivated by earlier works $[3]$ which indicated that the Ward identities for correlation functions of constrained superfield operators in superconformal quantum field theories are stronger than one might naively expect some interesting results were found. In particular $[4, 5]$, the SYM field strength is a covariantly analytic superfield $W$ carrying no indices from which one can build a set of analytic gauge-invariant operators by tracing products of $W$. The members of this set are in one-to-one correspondence with the Kaulza-Klein multiplets of IIB supergravity on $AdS_5 \times S^5$ $[9]$ and the Ward identities for correlation functions of operators of this type are easy to solve for in terms of prefactors times functions of superconformal invariants, largely due to the fact that the fields carry no indices. Since there are no three-point superinvariants for the superspaces we are considering, it is possible to obtain the functional form of three- (and two-) point functions exactly $[4, 10]$, although the Ward identities do not determine the dependence of the coefficients on the coupling. For four and more points, the Ward identities, when combined with analyticity, do put constraints on the functions of superinvariants that can arise $[11, 12]$, although these constraints do not seem to be enough to determine completely the $N = 2$ correlators that contain four harmonic matter fields of charge two contrary to the conjecture made in $[4, 5]$. However, it is not ruled out that this line of argument cannot be used to show that other correlators can be found explicitly.

To make further progress, therefore, it seems that in general some additional input is required. A field-theoretic trick one can use is to derive a relation between $n$- and $(n + 1)$-point functions by differentiating the path integral representation of the $n$-point function with respect to the coupling. In the present context we shall refer to the resulting equation as the reduction formula; it was first applied to $N = 4$ SCFT in $[13]$. An important aspect of this formula is that the $(n + 1)$-point function includes an integration over the point of insertion of the Lagrangian.

One application of the reduction formula is to use the known explicit form of all three- and four-point superconformal invariants to prove the non-renormalisation theorem for two- and three-point functions $[14]$. We mentioned above that the Ward identities fix the form of these functions but not their dependence on the coupling although, for the Green’s function of three supercurrents, there is an argument $[10]$, based on the absence of counterterms beyond one loop in $N = 4$ conformal supergravity $[15]$ which implies that this Green’s function depends trivially on the coupling (see also $[14]$). The result of $[14]$ extends this non-renormalisation theorem to three-point functions of short multiplet operators with arbitrary charges. In a sense it is 2Indeed, in a paper in preparation $[6]$ we shall use harmonic superspace arguments of the above type to show that extremal correlators of analytic operators are free. This is in accordance with the recent results of $[7]$ in AdS and also with the results obtained in $[8]$ for the one-loop and instanton contributions to the corresponding correlators on the field theory side.
an expression of the $U(1)_Y$ “bonus” symmetry first proposed in \cite{17}, and advocated in \cite{13}. Although this is not a true symmetry of interacting $N = 4$ Yang-Mills theory, it nevertheless seems to be a symmetry of a large class of superinvariants and through them of $n$-point functions of short operators with $n \leq 4$ \cite{13, 14}.

These results on two- and three-point functions are in accord with the conjectured relation between $N = 4$ SYM and IIB supergravity on an $AdS_5 \times S^5$ background \cite{18}. However, it was emphasised in \cite{19} that contact terms can arise in the field theory and that such terms can in principle have an effect in the reduction formula because of the presence of an integrated insertion. Contact terms have been observed in two-point functions at two loops in the $N = 2$ harmonic superspace formalism in \cite{10} and studied in more detail in $N = 1$ superfields in \cite{20}. The authors of \cite{13} were particularly interested in the effect of such terms on anomalies, but the contact terms found at two loops are actually consistent with the Ward identities. Since the superconformal anomaly is related to the divergences of $N = 4$ conformal supergravity, and since, as we remarked above, there are no such divergences beyond one loop, it follows that the contact terms which could potentially arise in the reduction formula should be consistent with the superconformal Ward identities.

In this article we begin by discussing contact term solutions to the $N = 4$ superconformal Ward identities using $N = 4$ superfields. For two points we find that there are covariant contact terms for short multiplet operators with arbitrary charges. For three points we then find all possible contact terms some of which are nilpotent and some which are not. The latter are not so important because they cannot contribute in the reduction formula. For the former we find that there only exists a solution for the case of three supercurrents. A consequence of this is that, if the formula is to be valid for contact terms as well as at non-coincident points, it should be the case that no contact terms should occur for two-point functions with higher charges than the supercurrent. It has to be admitted that this is not easy to verify directly in perturbation theory due to difficulties that arise in defining the graphs for a small number of points. However, one can certainly say that the only three-point contact nilpotent covariant does not affect the two-point non-renormalisation theorem; inserted in the reduction formula it simply reproduces the two-point contact term for two supercurrents. Although we have not investigated the situation at four points in as much detail, we have identified the four-point nilpotent contact covariant which gives rise to the three-point one using the reduction formula, and we present an argument that suggests that there are no additional four-point contact covariants which could interfere with the proof of the three-point non-renormalisation theorem. It should be borne in mind, however, that analyticity can be violated by harmonic delta-function terms and the effects of this are difficult to analyse in the $N = 4$ formalism as it is on-shell. This caveat also applies to the $N = 4$ version of the reduction formula (see below) which cannot be derived directly in contrast to the $N = 2$ version. We are reassured by $N = 2$ perturbative calculations that these difficulties should not affect our results.

Another possible use of the reduction formula is to obtain information about four-point correlators by first trying to guess or compute the corresponding five-point ones. It should be emphasised that, starting with five points, one can have nilpotent superconformal covariants of the non-contact type. In the reduction formula, after the integration over the insertion point, they can become non-nilpotent. In order to reproduce the known four-point correlators in this way, one has to assume the existence of a five-point covariant violating $U(1)_Y$ invariance \cite{13}. In \cite{14} it was shown that such covariants can only be of the nilpotent type and their expression to lowest order in fermions was given up to a multiplicative non-nilpotent function.
In the second part of the paper we are able to construct explicitly an $N = 2$ component of such an $N = 4$ covariant by calculating a certain five-point function at two loops in $N = 2$ harmonic superspace. This confirms the existence of the nilpotent covariant and gives its form in detail. The five-point correlator is related by the reduction formula to the correlator of four $N = 2$ hypermultiplet bilinear (or charge two) composites. In [21] we derived expressions for these correlators in terms of three functions of the spacetime variables $A_1, A_2$ and $A_3$. The first two of these come out as functions of the two independent spacetime conformal cross-ratios. In fact, $A_1, A_2$ are expressed in terms of the one-loop scalar box integral. However, $A_3$ is given by a generic two-loop integral for which conformal invariance is far from obvious. Subsequently, in [11], we used the superconformal Ward identities combined with harmonic analyticity to find a relation between $A_3$ and the other two. This enabled us to show that $A_3$ is expressed in terms of the same one-loop scalar box integral. Thus we could verify the conformal invariance of the entire four-point amplitude. This result was then confirmed by a numerical study of the integral formula for $A_3$. The calculation we present here is a two-loop computation at five points with four charge two hypermultiplet composites and one Yang-Mills composite $\text{Tr} W^2$. This fifth operator introduces a chiral point which one is to integrate over in the Intriligator formula. As expected, we reproduce the two-loop four-point function in a way which makes the simplified form of $A_3$ immediately apparent. Furthermore the calculation confirms the existence of a five-point $N = 4$ nilpotent superconformal invariant which is not invariant under $U(1)_Y$. The calculation also gives direct support to the assertion that contact terms do not make any significant difference to the reduction formula, since none are required in this case.

2 Contact covariants

We briefly recall the analytic superspace formalism. $N = 4$ analytic superspace $\mathcal{M}$ has coordinates

$$X^{\alpha A} = \begin{pmatrix} x^{\alpha \dot{\alpha}} \\ \pi^{\dot{\alpha} \dot{\gamma}} \\ \eta^{\alpha a'} \end{pmatrix}$$

where each lower case index can take on 2 values. The even coordinates $x$ and $y$ are coordinates for complex spacetime and the internal space $S(U(2) \times U(2)) \backslash SU(4)$ respectively. The odd coordinates $\lambda$ and $\pi$ number 8 in all, half the number of odd coordinates of $N = 4$ super Minkowski space. An infinitesimal superconformal transformation takes the form

$$\delta X = VX = B + AX + XD + XCX$$

where each of the parameter matrices is a $(2|2) \times (2|2)$ supermatrix and where

$$\delta g = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix} \in \mathfrak{sl}(4|4).$$

One can show that the central elements in the superalgebra $\mathfrak{sl}(4|4)$ do not act on $\mathcal{M}$ so that one really has an action of the superalgebra $\mathfrak{psl}(4|4)$.

From (3) one can read off the vector fields for each of the parameters. They divide into translational ($B$), linear ($A, D$) and quadratic ($C$) types. The translations are ordinary spacetime translations, half of the $Q$-supersymmetry transformations and translations in the internal $y$ space, which is locally the same as spacetime. The corresponding vector fields are
The linearly realised symmetries are Lorentz transformations (\(SL(2) \times SL(2)\) in complex space-time) and dilations, a corresponding set of internal transformations, the other half of the \(Q\)-supersymmetries and half of the \(S\)-supersymmetries. The \(SL(2)\) transformations are handled in the usual way so that we do not need to write them down. The vector fields generating dilations \((D)\) and internal dilations \((D')\) are

\[
V(D) = x^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \frac{1}{2}(\lambda^{\alpha\alpha'} \partial_{\alpha\alpha'} + \pi^{a\dot{a}} \partial_{a\dot{a}}),
\]

\[
V(D') = y^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \frac{1}{2}(\lambda^{\alpha\alpha'} \partial_{\alpha\alpha'} + \pi^{a\dot{a}} \partial_{a\dot{a}}).
\]

The vector fields generating linearly realised \(Q\)-supersymmetry are

\[
V(Q)_{\alpha} = \pi^{a\dot{a}} \partial_{a\dot{a}} + y^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}},
\]

\[
V(Q)_{\dot{\alpha}} = \lambda^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} - y^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}}.
\]

while those generating linearly realised \(S\)-supersymmetry are

\[
V(S)_{\alpha} = x^{\alpha\dot{\alpha}} \partial_{a\dot{a}} + \lambda^{a\dot{a}} \partial_{a\dot{a}},
\]

\[
V(S)_{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \partial_{a\dot{a}} - \lambda^{a\dot{a}} \partial_{a\dot{a}}.
\]

The remaining supersymmetry transformations are the non-linearly realised \(S\)-supersymmetries generated by

\[
V(S)_{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \partial_{a\dot{a}} + \lambda^{a\dot{a}} \partial_{a\dot{a}} + \frac{1}{2}(\pi^{\dot{a}a} \partial_{\dot{a}a} + \pi^{a\dot{a}} \partial_{a\dot{a}}),
\]

\[
V(S)_{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \partial_{a\dot{a}} - \lambda^{a\dot{a}} \partial_{a\dot{a}} + \frac{1}{2}(\pi^{\dot{a}a} \partial_{\dot{a}a} + \pi^{a\dot{a}} \partial_{a\dot{a}}).
\]

Finally, we have conformal boosts \((K)\) and internal conformal boosts \((K')\) generated by

\[
V(K)_{\alpha} = x^{\beta\dot{\beta}} \pi^{\alpha\beta} \partial_{\beta\dot{\beta}} + x^{\beta\dot{\beta}} \lambda^{a\dot{b}} \partial_{a\dot{b}} - \pi^{b\dot{a}} \pi^{\alpha\beta} \partial_{\beta\dot{a}} - \pi^{b\dot{a}} \lambda^{a\dot{b}} \partial_{b\dot{b}},
\]

\[
V(K')_{\dot{\alpha}} = x^{\beta\dot{\beta}} \pi^{\alpha\beta} \partial_{\beta\dot{\beta}} + x^{\beta\dot{\beta}} \lambda^{b\dot{a}} \partial_{b\dot{a}} + \pi^{b\dot{a}} \pi^{\alpha\beta} \partial_{\beta\dot{a}} + \pi^{b\dot{a}} \lambda^{a\dot{b}} \partial_{b\dot{b}}.
\]

The gauge-invariant operators in short multiplets in \(N = 4\) SYM are \(A_q = \text{Tr} (W^q)\) where \(W\) is the \(N = 4\) SYM field strength tensor which takes its values in the Lie algebra \(\mathfrak{su}(N_c)\) of the gauge group. These operators transform as

\[
\delta A_q = V A_q + q \Delta A_q
\]
where $\Delta = \text{str}(A + XC)$. A correlation function of such operators

$$G(X_1, \ldots, X_n) = \langle A_{q_1}(X_1) \cdots A_{q_n}(X_n) \rangle$$

should satisfy the Ward identity

$$\sum_{i=1}^{n} (V_i + q_i \Delta_i) G = 0 .$$

We define a contact covariant to be a Green’s function which satisfies the Ward identities and which is local in the sense that it involves at least one spacetime delta-function. Translational symmetries imply immediately that any $n$-point Green’s function depends only on $n - 1$ coordinate differences, $X_{12}, X_{23} \ldots X_{(n-1)n}$ where $X_{12} = X_1 - X_2$. Furthermore, assuming that there are no delta-functions in the internal space (which would be inconsistent with analyticity), linear $Q$-supersymmetry can be used to eliminate a further set of odd coordinates. Thus, after imposing translational and linear $Q$-supersymmetry we find that a Green’s function may be taken to depend on $n - 2 \lambda$ and $\pi$ coordinates of the form

$$\lambda_{123} = \lambda_{12} y_{12}^{-1} - \lambda_{23} y_{23}^{-1} , \quad \pi_{123} = y_{12}^{-1} \pi_{12} - y_{23}^{-1} \pi_{23} .$$

as well as $n - 1 y$ differences and $n - 1$ modified $x$ differences, $\hat{x}_{12}, \ldots$, with

$$\hat{x}_{12} = x_{12} - \lambda_{12} y_{12}^{-1} \pi_{12} .$$

In the preceding three equations we have suppressed the indices as the quantities involved in each expression are arranged in such a way that matrix multiplication is natural provided that the indices on $y^{-1}$ are taken to be a pair of subscripts in the order $a' a$.

From the foregoing it follows that any two-point Green’s function, whether contact or not, cannot depend explicitly on the odd coordinates. For a contact two-point function spacetime dilations fix the dependence on $\hat{x}_{12}$ to be of the form of powers of the d’Alembertian acting on the delta-function. Internal dilations then give the dependence on $y_{12}$ and so we arrive at the candidate two-point functions

$$\langle A_{q}(1) A_{q}(2) \rangle \sim (y_{12})^{2q-2} \sqrt{\Delta} \delta(\hat{x}_{12}) .$$

It is now a straightforward exercise to check that the expression on the right-hand side does indeed satisfy the remaining Ward identities. We note that the case of $q = 2$, i.e. the two-point function of two supercurrents, is the example previously encountered in perturbation theory $[10, 20]$, although the complete expression for the entire multiplet has not been derived before to our knowledge.

We now turn to three-point functions. We are primarily interested in nilpotent three-point covariants as they can contribute to the reduction formula. We remind the reader that this reads, in $N = 4$ superspace,
\[
\frac{\partial}{\partial \tau} < A_{q_1} \ldots A_{q_n} > = \frac{1}{\tau^2} \int d\mu < TA_{q_1} \ldots A_{q_n} >
\]  
(22)

where the integral is over the point of the inserted supercurrent \( T(= A_2) \). The measure \( d\mu \) involves an integral over the internal coset and a fermionic integral over \( \lambda \), i.e. \( d\mu \sim d^4x du d^4\lambda \).

Using linear \( S \)-supersymmetry one can show that a three-point nilpotent Green’s function with non-coincident points cannot depend on \( \lambda_{123} \) or \( \pi_{123} \) and hence must vanish completely. However, the same is not true in the presence of delta-functions. Consider the following expression which satisfies the translational, \( Q \)-supersymmetry and dilational Ward identities for three supercurrents,

\[
<T(1)T(2)T(3)> \sim (\lambda_{123})^4(y_{12})^4(y_{23})^4\delta(\hat{x}_{12})\delta(\hat{x}_{23}) .
\]  
(23)

The variation of \( (\lambda_{123})^4 \) under the first linear \( S \)-supersymmetry \((\Pi)\) is zero, but under the second linear \( S \) transformation the variation depends linearly on \( \hat{x}_{12} \) and \( \hat{x}_{23} \). Explicitly,

\[
V(S)\frac{\partial}{\partial \lambda_{123}}(\lambda_{123})^4 = (\hat{x}_{12}^{\alpha\dot{\alpha}}(y_{12}^{-1})_{\dot{a}'a} - \hat{x}_{23}^{\alpha\dot{\alpha}}(y_{23}^{-1})_{\dot{a}'a})(\lambda_{123})_{a}^3 .
\]  
(24)

However, this variation vanishes for the entire right-hand side of (23) due to the presence of the delta-functions. To complete the proof that the proposed three-point function satisfies all the Ward identities, it is sufficient to check the quadratic \( S \)-supersymmetry transformations \((\Pi)\) since the supersymmetries generate the entire super Lie algebra. After some algebra one can verify straightforwardly that the function does transform in the right way under this symmetry.

We remark that integrating the right-hand side of (23) over point 1, one recovers the functional form of the two-point contact term given above with \( q = 2 \). In other words, the two- and three-point contact terms for supercurrent Green’s functions are related to each other by the reduction formula, and the contact three-point function therefore has no effect on the two-point function at non-coincident points. We note further that, although the three-point function given above does not seem to be symmetric under the interchange of any two points, in fact it is. This is partly due to the presence of the delta-functions and partly because the \( \lambda^4 \) term can be written in the form

\[
(\lambda_{123})^4 = \frac{(\lambda_{123})^4}{(y_{13})^4}(y_{13})^4 .
\]  
(25)

The \( (y_{13})^4 \) factor here combines with the other two \( y \)-factors to give a symmetrical expression while \( \frac{(\lambda_{123})^4}{(y_{13})^4} \) is symmetrical by itself.

At first sight it might seem that the above Green’s function could be generalised to higher charges by the inclusion of appropriate d’Alembertians. However, this is not the case. Consider a general, nilpotent contact three-point function, \( G \), of the type that can contribute in the reduction formula. It will have the form

\[
G = (\lambda_{123})^4F(\hat{x}_{12}, \hat{x}_{23}, y_{12}, y_{23}) .
\]  
(26)

Under the second linear \( S \)-supersymmetry the fermionic factor will contribute a term
\[ V(S)_{\alpha'}^\hat{\alpha} G \sim (\hat{x}_{12}^{\alpha'} (y_{12}^{-1})_{a'a'} - \hat{x}_{23}^{\alpha'} (y_{23}^{-1})_{a'a'}) (\lambda_{123}^3)^a F. \] (27)

However, it is easily seen that this term cannot be cancelled by terms arising from the variations of the \(y\)'s or the \(\hat{x}\)'s because the fermion structures are different. Therefore the above term must vanish identically. The unique permissible \(\hat{x}\)-dependence of \(F\) which ensures this is the product of two delta-functions, \(\delta(\hat{x}_{12})\delta(\hat{x}_{23})\). However, this \(\hat{x}\) structure demands the \(y\) structure appearing in (23) and so we conclude that this contact covariant is the only one of its type.

One can also consider four-point contact terms that only have a \(\lambda^4\) multiplied by a non-nilpotent factor. By suitable labelling we may choose this to be \(\lambda^{123}_{123}\). Using S-supersymmetry and repeating the argument leading to (28) we find that the non-nilpotent factor must contain \(\delta(\hat{x}_{12})\delta(\hat{x}_{23})\). Using the reduction formula we see that such a term cannot lead to a three-point function that contains no delta functions. As a result we conclude that contact terms cannot invalidate the non-renormalisation theorem for two- and three-point functions shown in [14].

It is straightforward to construct a sequence of nilpotent contact covariants for an arbitrary number of points which are related by the reduction formula. For four points the covariant is

\[ < T(1)T(2)T(3)T(4) > \sim (\lambda_{123})^4(y_{12}^4(y_{23})^4(y_{34})^6\delta(\hat{x}_{12})\delta(\hat{x}_{23})\delta(\hat{x}_{34}) \] (28)

The proof that this satisfies the appropriate Ward identity is straightforward; one simply observes that the right-hand side is almost a product of two three-point functions of the type of (23); in fact, morally it is the product of two such functions divided by a non-nilpotent two-point contact function of the type given in (22) with \(q = 2\). Using this, one can show in a few lines that the four-point Ward identity is indeed satisfied by (28).

This construction can be extended to an arbitrary number of points straightforwardly. The contact covariant for \(n\) \(T\)'s is simply:

\[ < T(1)\ldots T(n) > \sim \prod_{i=1}^{n-2} (\lambda_{i(i+1)(i+2)})^4 \prod_{i=1}^{n-1} ((y_{i(i+1)})^4 \delta(\hat{x}_{i(i+1)}) \] (29)

This sequence of terms is clearly related to each other by the reduction formula and thus finally to the non-nilpotent two-point function (22) with \(q = 2\). One can also show that all contact terms which are not nilpotent and have no derivatives acting on the delta functions can only have the form

\[ (y_{12}^2)\ldots(y_{nn-1}^2)\delta(\hat{x}_{12})\ldots\delta(\hat{x}_{nn-1}) \] (30)

This can be expressed as a product of two-point functions and so corresponds to a disconnected Greens function. It should be straightforward to extend this result to include derivatives on the delta functions.

We shall now argue that all contact terms which are not disconnected and do not have derivatives on delta functions are of the form of equation (29). Given any non-nilpotent contact term we can integrate over the variable associated with a given leg to produce a contact term with one less external leg. Repeating this process and assuming one does not get zero one will arrive at a non-nilpotent contact term that must be of the form of the above equation. If we further assume that Greens functions that begin as connected do not become disconnected then we would conclude that the only connected Greens contact Greens functions are those that lead by
repeated integration to the non-nilpotent two-point function and so are as given in equation (29). If we assume that this result also holds for contact terms with derivatives on the delta functions we can conclude that all the connected contact Green’s functions are fixed by a single coefficient. The correlation functions of the supercurrent can be generated from an effective supergravity action obtained by coupling \( N = 4 \) SYM to a background supergravity and integrating over the Yang-Mills fields. The contact terms must then arise from only one term in this effective action which is superconformally invariant in four dimensions. The unique superconformal function of the supergravity fields is the \( N = 4 \) conformal supergravity action. Differentiating this with respect to the fields of this multiplet and setting them equal to their flat space values should then give the contact covariants described above.

3 Two-loop calculation

In this section we shall carry out a perturbative \( N = 2 \) calculation at two loops which explicitly demonstrates how the reduction formula works. Our main aim will be to reproduce the four-point correlator of hypermultiplet bilinears from refs. \([21, 11]\) as the integral of a five-point one. The latter is obtained by inserting the \( N = 2 \) SYM Lagrangian into the four-point correlator. It provides a more explicit form of the nilpotent five-point superconformal covariant which was constructed to lowest order in \([14]\). This term violates the \( U(1)_Y \) symmetry of ref. \([13]\).

3.1 \( N = 4 \) SYM in terms of \( N = 2 \) harmonic superfields

The absence of an off-shell formulation of \( N = 4 \) SYM theory does not allow one to do perturbation theory calculations in a manifestly \( N = 4 \) covariant way. The best one can do is reformulate the theory in terms of off-shell \( N = 2 \) harmonic superfields and then apply the existing Feynman graph technique for such superfields.

The two \( N = 2 \) ingredients of the \( N = 4 \) SYM theory are the \( N = 2 \) SYM multiplet and the \( N = 2 \) matter (hyper)multiplet. Both of them can be described as superfields in the Grassmann (G-)analytic superspace \([2]\) with coordinates \( x^A_{\dot{\alpha}}, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^+_i \). Here \( u^+_i \) are the harmonic variables which form a matrix of \( SU(2) \) and parametrise the sphere \( S^2 \sim SU(2)/U(1) \). A harmonic function \( F^{(q)}(u^\pm) \) of \( U(1) \) charge \( q \) is a function of \( u^+_i \) invariant under the action of the group \( SU(2) \) (which rotates the index \( i \) of \( u^+_i \)) and homogeneous of degree \( q \) under the action of the group \( U(1) \) (which rotates the index \( \pm \) of \( u^+_i \)). Such functions have infinite harmonic expansions on \( S^2 \) whose coefficients are \( SU(2) \) tensors (multispinors). The superspace is called G-analytic since it only involves half of the Grassmann variables, the \( SU(2) \)-covariant harmonic projections \( \theta^{+\alpha} = u^+_i \theta^{i\alpha}, \bar{\theta}^{+\dot{\alpha}} = u^+_i \bar{\theta}_{i\dot{\alpha}} \).

In this framework the hypermultiplet is described by a G-analytic superfield of charge +1, \( q^+(x_A, \theta^+, \bar{\theta}^+, u) \) (and its conjugate \( \bar{q}^+(x_A, \theta^+, \bar{\theta}^+, u) \) where \( \sim \) is a special conjugation on \( S^2 \) preserving G-analyticity). Note that this \( N = 2 \) supermultiplet cannot exist off shell with a finite set of auxiliary fields \([27]\). This only becomes possible if an infinite number of auxiliary fields (coming from the harmonic expansion on \( S^2 \)) are present. On shell these auxiliary fields are eliminated by the harmonic (H-)analyticity condition (equation of motion)

\[
D^{++} q^+ = 0 \tag{31}
\]

Here \( D^{++} \) is the harmonic derivative on \( S^2 \) (the raising operator of the group \( SU(2) \) realised
on the \(U(1)\) charges, \(D^{++}u^+ = 0, \ D^{++}u^- = u^+\). In the G-analytic superspace it becomes a supercovariant operator involving spacetime derivatives:

\[
D^{++} = u^+ \frac{\partial}{\partial u^-} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}} \frac{\partial}{\partial x_{\alpha \dot{\alpha}}}.
\]  

The field equation \((31)\) can be derived from an action given by an integral over the G-analytic superspace:

\[
S_{HM} = - \int dud^4x_{A}d^2\theta d^2\bar{\theta} \bar{q}^+ D^{++}q^+.
\]  

(33)

This action is real (with respect to the \(\bar{\cdot}\) conjugation) which can be seen by integrating \(D^{++}\) by parts. In this sense the action \((33)\) resembles the Dirac action for fermions, although the superfield \(q^+\) is bosonic.

The SYM gauge potential is introduced by covariantising the action \((33)\) with respect to a Yang-Mills group with G-analytic parameters \(\lambda(x_A, \theta^+, \bar{\theta}^+, u)\). To this end one replaces the harmonic derivative in \((33)\) by the following covariant one:

\[
D^{++} \rightarrow \nabla^{++} = D^{++} + igV^{++}(x_A, \theta^+, \bar{\theta}^+, u)
\]  

(34)

where \(g\) is the gauge coupling constant. The gauge potential is described by a real \((\bar{\cdot})\) G-analytic superfield of charge +2 (equal to the charge of \(D^{++}\)). The matter and gauge superfields are subject to the usual gauge transformations:

\[
q^+ \rightarrow e^{ig\lambda}q^+, \quad V^{++} \rightarrow -\frac{i}{g}e^{ig\lambda}D^{++}e^{-ig\lambda} + e^{ig\lambda}V^{++}e^{-ig\lambda},
\]  

(35)

so that the covariantised action \((33)\)

\[
S_{HM/SYM} = - \int dud^4x_{A}d^2\theta d^2\bar{\theta} \bar{q}^+ \nabla^{++}q^+
\]  

(36)

is indeed gauge invariant.

The gauge invariant action for \(V^{++}\) can be written down in terms of the gauge field strength \(W(x_L, \theta^{\alpha})\). Unlike the G-analytic potential, \(W\) is a (left-handed) chiral superfield which is harmonic-independent, \(\Delta^{++}W = 0\). It can be expressed as a power series in \(V^{++}\) \[28\]:

\[
W = \frac{i}{4} u^+_1 u^+_2 \bar{D}_d^i D^{i\dot{a}} \sum_{n=1}^{\infty} \int du_1 \ldots du_n \frac{(-ig)^n v^{++}(u_1) \ldots v^{++}(u_n)}{(u^+_1 u^+_2) \ldots (u^+_n u^+_n)}
\]  

(37)

where \((u^+_m u^+_n) \equiv u_{m+n}^{++}\). The SYM action is then given by the chiral superspace integral \[7\]

\[
S_{N=2 \text{ SYM}} = \frac{1}{4g^2} \int d^4x_L d^4\theta \ Tr \ W^2.
\]  

(38)

The details of how to fix the gauge and introduce ghosts can be found in \[29\].

When the hypermultiplet matter is taken in the adjoint representation of the gauge group, the two actions \((36)\) and \((38)\) describe the \(N = 4\) SYM theory,

\[
S_{N=4 \text{ SYM}} = S_{N=2 \text{ SYM}} + S_{HM/SYM}.
\]  

(39)

As mentioned earlier, the main advantage of the \(N = 2\) formulation is the possibility to quantise the theory in a straightforward way \[29\].

\[4\]In fact, there exists an alternative form given by the right-handed chiral integral \(\int d^4x_R d^4\bar{\theta} \ Tr \ W^2\). In a topologically trivial background the two forms are equivalent (up to a total derivative).
3.2 The reduction formula in $N = 2$

The main aim of our perturbative calculation is to explicitly show how the general formula used by Intriligator in ref. [13] works. It relates the correlation function of $n$ composite operators to an $(n + 1)$-point one where the extra point is obtained by inserting the $N = 4$ SYM Lagrangian. Here we shall derive this formula in the context of the $N = 2$ harmonic superspace formulation of the $N = 4$ theory.

Consider a set of $n$ composite gauge invariant operators $O_a$, $a = 1, \ldots, n$, each made out of $r_a$ hypermultiplets $\tilde{q}^+$ and $s_a$ hypermultiplets $q^+$,

$$O_a = (\tilde{q}^+)^{r_a} (q^+)^{s_a}.$$  

Their correlator is given by the functional integral

$$G_n = Tr \langle O_1 \ldots O_n \rangle = \frac{1}{Z} \int DqDV \, e^{iS_{N=4 \text{ SYM}}} O_1 \ldots O_n.$$  

(40)

Now we want to differentiate equation (40) with respect to the coupling constant $g$. By inspecting the two ingredients (36) and (38) of the action (39), one sees that, after the change of variables

$$V^{++} \rightarrow \frac{1}{g} V^{++}$$  

(41)

in the functional integral, the only dependence on $g$ is given by the overall factor $g^{-2}$ in the $N = 2$ SYM part (38) of the action. Thus, we find

$$\frac{\partial G_n}{\partial g} = \frac{1}{Z} \int DqDV \, e^{iS_{N=4 \text{ SYM}}} \frac{\partial(iS_{N=2 \text{ SYM}})}{\partial g} O_1 \ldots O_n$$

$$= -\frac{2i}{g} \int_{n+1} \langle O_1 \ldots O_n \frac{1}{4g^2} Tr W^2_{n+1} \rangle$$

(42)

$$= -\frac{2i}{g} \int_{n+1} \langle O_1 \ldots O_n L_{N=2 \text{ SYM}}(n + 1) \rangle.$$  

Note that throughout the derivation we have used the gauge-invariant SYM Lagrangian instead of the gauge-fixed one. This is possible since, on the one hand, the composite operators $O$ are gauge invariant and on the other, the difference between the two forms of the gauge action amounts to a gauge (or BRST) transformation. So, this formula relates the $n$-point correlator of composite hypermultiplet operators to the $(n + 1)$-point one obtained by inserting the $N = 2$ SYM Lagrangian (without the matter part).

Intriligator’s proposal was to use the formula (42) to try to learn something about the $n$-point function by first predicting (or computing) the $(n + 1)$-point one. The first half of the present paper was devoted to the possibility of predicting such correlators based on their superconformal properties. Now we shall undertake a direct calculation of the right-hand side of eq. (42). We will deal with the correlation functions of four (two) bilinear composite operators made out of hypermultiplets and a fifth (third) bilinear representing the insertion of the SYM Lagrangian into the four (two)-point correlator. After integrating over the insertion point, we will recover the known results for the four (two)-point correlators of hypermultiplet bilinears. The five-point correlator (before integration) is an example of a nilpotent superconformal invariant preserving harmonic analyticity, but violating the $U(1)_r$ invariance of ref. [13].

In its original version [13] the formula involves a complex coupling constant $\tau$. This corresponds to including the topological part of the SYM action with a separate parameter $\theta$. Here we only consider a background of trivial topology, so our $g$ is real.
3.3 Graphs and Feynman rules

We will be interested in the five-point correlator

\[
\langle (\tilde{q}^+ (1))^2 (q^+ (2))^2 (\tilde{q}^+ (3))^2 (q^+ (4))^2 \frac{1}{4g^2} (W(5))^2 \rangle .
\]

We want to perform the computation at the lowest non-trivial level of perturbation theory, i.e., at two loops. For this reason we do not need to consider non-Abelian vertices and can restrict ourselves to the minimal coupling SYM/HM from eq. (36). The non-trivial graph topologies relevant to the computation are shown in Figure 1:

They have been obtained from the corresponding four-point graphs (see [21] for details about the four-point calculation) by inserting the $N=2$ SYM linearised Lagrangian $W^2$ into each of the gluon lines. This amounts to replacing the gluon propagator

\[
1' \begin{array}{c} \text{\includegraphics{fig1a.png}} \\ \text{a} \end{array} 2' \quad \langle V^{++}(1) V^{++}(2) \rangle
\]

by the product of modified propagators

\[
3' \begin{array}{c} \text{\includegraphics{fig1b.png}} \\ \text{b} \end{array} 2' \quad \langle V^{++}(1) W(3) \rangle \frac{1}{16g^2} \langle W(3) V^{++}(2) \rangle .
\]

The modified SYM propagator $\langle W(1)V^{++}(2) \rangle$ has one chiral end (the field strength $W(x_{1L}, \theta_1)$) and one G-analytic end (the SYM potential $V^{++}(x_{2A}, \theta_1^+, \bar{\theta}_2^+, u_2)$). One way to construct it is to take the standard SYM propagator $\langle V^{++}(1)V^{++}(2) \rangle$ in the Feynman gauge and convert the G-analytic end 1 to a chiral one using the linearised version of the expression (37) of the field strength $W$ in terms of the potential $V^{++}$. However, it is easier to guess the form of this mixed chiral-G-analytic object based only on its dimension and supersymmetry properties. We recall that the natural $N=2$ superspace for describing (left-handed) chiral objects is the chiral one with coordinates $x^\alpha_L, \theta^\alpha$, and that for G-analytic objects is the Grassmann-analytic harmonic superspace with coordinates $x^\alpha_A, \theta^{+\alpha}, \bar{\theta}^{+\alpha}, u^\pm_i$.

\(^5\)The subscripts of $x_L$ and $x_A$ refer to the appropriate bases in superspace where chirality or G-analyticity become manifest.
under \( N = 2 \) supersymmetry are:

\[
\text{Chiral superspace:} \quad \text{G-analytic superspace:}
\]
\[
\begin{align*}
\delta x_{12}^{\alpha \dot{\alpha}} &= -4i\theta^{(\alpha i} i_{i}^{\dot{\alpha})} \\
\delta \theta^{\alpha i} &= i^{\alpha} \\
\delta u_{i}^{+} &= 0.
\end{align*}
\]  

(44)

Then, given a chiral point 1 and a G-analytic point 2, we can form the following coordinate differences with simple transformation laws:

\[
\begin{align*}
\delta x_{12}^{\alpha \dot{\alpha}} &= -4i\theta^{(\alpha i} i_{i}^{\dot{\alpha})} \\
\delta \theta_{12}^{\alpha} &= u_{2i}^{+}(\theta_{1}^{\alpha} - \theta_{2}^{\alpha}) \\
\delta u_{12}^{+} &= 0.
\end{align*}
\]  

(45)

Now, combining these two differences, one can easily construct a supersymmetric invariant with all the required properties of the propagator \( \langle W(1)V^{++}(2) \rangle \) (a Lorentz scalar of dimension +1, chiral at point 1, G-analytic with \( U(1) \) charge +2 at point 2):

\[
\frac{1}{2g} \langle W_{a}(1)V_{b}^{++}(2) \rangle = \frac{\delta_{ab}}{4i\pi^{2}} (\theta_{12})^{2} \hat{x}_{12}^{-2}.
\]

Here \( a, b \) are indices of the adjoint representation of the YM group. The coefficient has been fixed by finding the complex scalar of the SYM multiplet in \( W(x, \theta, u) = \ldots - i\sqrt{2}(\theta^{+})^{2}\bar{\phi}(x) + \ldots \) and in \( V^{++}(x, \theta, u) = \ldots - i\sqrt{2}(\theta^{+})^{2}\phi(x) + \ldots \) and thus relating \( \langle W(1)V^{++}(2) \rangle \) to the standard scalar propagator \( \langle \phi(1)\phi(2) \rangle = 1/4i\pi^{2} x_{12}^{-2} \).

Similarly, the hypermultiplet propagator \( \langle \bar{q}^{+}(1)q^{+}(2) \rangle \) can be built out of the coordinate difference

\[
\hat{x}_{12} = x_{12}^{A} - x_{2A}^{A} + \frac{4i}{(12)} [(1-2)\theta_{1}^{+} \theta_{1}^{-} + (2-1)\theta_{2}^{+} \theta_{2}^{-} + \theta_{1}^{+} \theta_{2}^{-} + \theta_{2}^{+} \theta_{1}^{-}]
\]

(46)

where \( (12), (1-2), \ldots \) is a shorthand for contractions of harmonics, e.g., \( (12) = u_{1}^{+} u_{2i}^{+} \), \( (1-2) = u_{1}^{+} u_{2i}^{+} \). Unlike the mixed chiral-analytic one \( \hat{x}_{12} \), this purely G-analytic difference is invariant under supersymmetry, \( \delta \hat{x}_{12} = 0 \). Thus, the hypermultiplet propagator (a Lorentz scalar of dimension 2, G-analytic with \( U(1) \) charges +1 at both points 1 and 2) can be written down as follows:

\[
\frac{1}{2g} \langle \bar{q}_{a}^{+}(1)q_{b}^{+}(2) \rangle = \frac{\delta_{ab}}{4i\pi^{2}} (12) \hat{x}_{12}^{-2}.
\]

Once again, the coefficient has been fixed by examining the isodoublet scalar of the hypermultiplet, \( q^{+}(x, \theta, u) = f^{i}(x) u_{1i}^{+} \ldots \).

\[\text{This is the } N = 2 \text{ analog of the difference (44).}\]
3.4 Building blocks

Let us now return to the graphs in Figure 1. It is clear that the two topologies in Figure 1a,b can be reduced to products of hypermultiplet propagators and the following three-point building block:

```
✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★✠ ★升学

Figure 2

The interaction point 4 is G-analytic, so one has to integrate over the G-analytic superspace \( \int du_4^4 d^2\theta_4^+ d^2\bar{\theta}_4^+ \). The resulting expression is

\[
I = \frac{igf_{abc}}{(2\pi)^6} \int du_4^4 d^2\theta_4^+ d^2\bar{\theta}_4^+ \frac{(14) \theta_3^+ \theta_4^+}{x_{14}^2 x_{42}^2 x_{34}^2} \frac{(34)^2}{(13)}.
\]

(47)

It is clear that the nilpotent factor \((\theta_3^+)\) serves as a Grassmann delta-function which identifies the left-handed G-analytic variable \(\theta_4^{+\alpha} \) with the harmonic projection \(\theta_3^{+\alpha} \equiv u_4^{+\alpha} \). This allows us to immediately do the left-handed half of the Grassmann integral at point 4. The easiest way to do the remaining right-handed integration is to make use of the supersymmetry of the expression in (47). The idea is to shift away the two external G-analytic variables \(\theta_1^{+\alpha}, \theta_2^{+\alpha} \) by means of a finite supersymmetry transformation:

\[
(\theta_1^{+\alpha}, \theta_2^{+\alpha})' = \theta_1^{+\alpha}, \theta_2^{+\alpha} + u_1^{+\alpha}, u_2^{+\alpha} \epsilon^{i\alpha, \dot{\alpha}} = 0
\]

(48)

whose parameter is

\[
\epsilon^{i\alpha, \dot{\alpha}} = \frac{u_2^{+i}}{(12)} \theta_1^{+\alpha}, \theta_2^{+\alpha} - \frac{u_1^{+i}}{(12)} \theta_1^{+\alpha}, \theta_2^{+\alpha}.
\]

(49)

After this, the integral (47) becomes

\[
I = \frac{gf_{abc}}{(2\pi)^6} \int du_4^4 d^2\theta_4^+ (14) (42) [x_{34} - 4i\theta_3^{-\alpha} \theta_4^+]^{-2} \times \]

\[
[x_{14} + 4i \theta_3^{-\alpha} \theta_4^+]^{-2} [x_{42} + 4i \theta_3^{-\alpha} \theta_4^+]^{-2}.
\]

(50)

where \(\theta_3^{-\alpha} \equiv u_1^{+\alpha} \theta_3^{+\alpha} \) and the differences \(x_{14}, x_{42}, x_{34} \) involve just \(x_{1,2,4_A} \) and \(x_{3_L} \). The next step is to perform a shift of the integration variable \(x_4 \rightarrow x_4 - 4i\theta_3^{-\alpha} \theta_4^+ \) and to use the harmonic cyclic identity, e.g., \((4 - 1)\theta_3^{+\alpha} + (14)\theta_3^{-\alpha} = \theta_3^{+\alpha} \), which leads to the following simplification of the integrand:

\[
I = \frac{gf_{abc}}{(2\pi)^6} \int du_4^4 d^2\theta_4^+ (14) (42) [x_{34}]^{-2} \times \]

\[
[x_{14} - \frac{4i}{(14)} \theta_3^{+\alpha} \theta_4^+ ]^{-2} [x_{42} - \frac{4i}{(42)} \theta_3^{+\alpha} \theta_4^+]^{-2}.
\]

(51)
In this form one realises that the entire dependence of the integrand on \( \hat{\theta}_4^\pm \) can be represented as a shift of the external points \( x_1 \) and \( x_2 \):

\[
I = \frac{g f_{abc}}{(2\pi)^6} \int du_4 d^2\hat{\theta}_4^+ \quad (14)(42) \times \\
\exp \left\{ \frac{2i}{(14)} \theta_3^+ \partial_1 \theta_4^+ - \frac{2i}{(24)} \theta_3^+ \partial_2 \theta_4^+ \right\} \int \frac{d^4x_4}{x_1^2 x_2^2 x_3^2 x_4^2} .
\]

Expanding the exponent in (52) and doing the integral \( \int d^2\theta_4^+ \) is now straightforward and the result is

\[
I = \frac{g f_{abc}}{(2\pi)^6} \int du_4 \left[ \frac{(24)}{(14)}(\theta_3^+)^2 \square_1 + \frac{(14)}{(24)}(\theta_3^+)^2 \square_2 + \theta_3^+ \theta_3^+ \theta_3^+ \theta_3^+ \right] \int \frac{d^4x_4}{x_1^2 x_2^2 x_3^2 x_4^2} 
\]

where \( (\sigma_{\mu \nu})_{\alpha \beta} = \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)_{\alpha \beta} \). The harmonic integration in the last two terms is trivial (\( \int du_4 = 1 \)) and in the first two is done as follows, e.g.,

\[
\int du_4 \frac{(24)}{(14)} = \int du_4 \frac{D_4^{++}(24^-)}{(14)} = \int du_4 (24^-)\delta(1,4) = (21^-) .
\]

Here we have used the property \( D_4^{++}(14)^{-1} = -\delta(1,4) \) of the singular harmonic distribution \( 1/(14) \) (see [29] for details). Finally, using the properties of the one-loop spacetime integral (see [30] for a discussion of such integrals), one easily finds

\[
\square_1 \int \frac{d^4x_4}{x_1^2 x_2^2 x_3^2 x_4^2} = \frac{4i\pi^2}{x_1^2 x_2^2 x_3^2} , \quad \partial'_1 \partial'_2 \int \frac{d^4x_4}{x_1^2 x_2^2 x_3^2 x_4^2} = -\frac{4i\pi^2}{x_1^2 x_2^2 x_3^2} .
\]

Putting all of this together, we obtain

\[
I = \frac{ig f_{abc}}{(2\pi)^6} \left[ \frac{\theta_3^+}{x_1^2 x_2^2} - \frac{(12)\theta_3^+/x_1^2 x_2^2}{x_3^2 x_4^2} + \frac{(12)\theta_3^+/x_1^2 x_2^2}{x_3^2 x_4^2} + 2i\theta_3^+ \theta_3^+ \theta_3^+ \theta_3^+ \right] .
\]

Now, we have to recall that the above computation has been done in the special frame where \( \theta_1^+ = \theta_2^+ = 0 \). The way to obtain the result in the original frame is to perform a supersymmetry transformation with the parameter \( \epsilon \) [14]. In the process the harmonic projections \( \theta_3^+/1, \theta_3^+/2 \) give rise to the supersymmetric invariants \( \theta_{31,32} \) (see [13]) and the analytic-analytic difference \( x_{12} \) becomes the invariant \( \bar{x}_{12} \) [40]. Further, the harmonic projections \( \theta_3^+/1, \theta_3^+/2 \) are converted into \( \theta_{31,32} \) [43], e.g.,

\[
\theta_{3/1}^+ \rightarrow \frac{1}{(12)} \theta_{31} - \frac{(21^-)}{(12)} \theta_{31} ,
\]

whereas the mixed chiral-analytic differences \( x_{31,32} \) become supersymmetric invariants with the help of both G-analytic \( \theta_{1,2}^+ \) , e.g.,

\[
\bar{x}_{31,2} = \bar{x}_{31} + \frac{4i}{(12)} \theta_{31}(\theta_2^+ - (21^-)) \theta_1^+ \Rightarrow \delta\bar{x}_{31,2} = 0 .
\]

Note that despite the presence of various nilpotent terms, the invariants \( \bar{x} \) and \( \hat{x} \) still satisfy the usual cyclic identity

\[
\bar{x}_{32,1} = \bar{x}_{31,2} = \hat{x}_{12} .
\]
So, the building block from Figure 2 needed for the Feynman graphs in Figure 1a,b has the following expression:

\[
I = \frac{g f_{abc}}{(2\pi)^4} \left\{ \theta_{31} \theta_{32} \left[ \frac{1}{x_{312}^2 x_{321}^2} - \frac{1}{x_{123}^2 x_{312}^2} - \frac{1}{x_{123}^2 x_{312}^2} \right] + \frac{(21^-)(\theta_{31})^2 + (12^-)(\theta_{32})^2}{x_{123}^2 x_{312}^2} \right\}.
\]

The graph in Figure 1c is made out of the building block shown in Figure 3:

![Figure 3](image)

The computation is similar to that of the block in Figure 2. Firstly, one uses the nilpotent factors from the two SYM propagators 3 → 4 and 3 → 5 to do the integrals \( \int d^2\theta_{4,5} \). Secondly, by means of a supersymmetry transformation one eliminates \( \theta_{4,5}^+ \). Thirdly, after shifts of the integration variables \( x_{4,5} \) one frees all the propagators in the loop from any \( \theta \) dependence. The result is:

\[
J = \frac{ig^2 f_{ac} f_{bd}}{(2\pi)^{10}} \int du_{4,5} d^4 x_{4,5} d^2 \theta_4^+ d^2 \theta_5^+ \left[ \frac{1}{x_{14} x_{25}} \theta_{31} \theta_{32} \right]^{-2} d^2 \theta_4^+ d^2 \theta_5^+ \left[ \frac{18}{x_{14} x_{25}} \theta_{31} \theta_{32} \right]^{-2}.
\]

This time the expansion and integration with respect to \( \theta_{4,5}^+ \) is very easy, giving rise to spacetime delta-functions \( \delta(x_{14}) \) and \( \delta(x_{25}) \). The harmonic integral is then reduced to

\[
\int du_{4,5} \frac{18}{(14)(52)} = -(1-2^-)
\]

(see (54)). Finally, after restoring the supersymmetry one finds:

\[
J = \frac{ig^2 f_{ac} f_{bd}}{(2\pi)^6} \left[ (1-2^-)(\theta_{31})^2(\theta_{32})^2 \right] \frac{1}{x_{123}^2 x_{312}^2 x_{321}^2}.
\]

### 3.5 Results

At this stage what remains to do is to multiply the above building blocks together with the relevant hypermultiplet propagators and obtain the complete expressions for the five-point graphs in Figure 1. This involves a lot of elementary algebra, therefore we shall only do it in full detail in the simpler case of the two-loop three-point correlator

\[
\langle (\tilde{q}^+(1))^2 (q^+(2))^2 \frac{1}{4g^2} (W(3))^2 \rangle.
\]

The corresponding graphs are shown in Figure 4:
One sees that they are made out of the same building blocks as before. However, the multiplication in the case of the graph in Figure 4a is considerably simplified by the identification of the end-point $\theta$'s since the two Lorentz structures in (58) (the scalar and the antisymmetric tensor) become orthogonal. Thus, this graph produces the expression (up to an overall factor)

$$\frac{(12)(1-2^-)(\theta_{31})^2(\theta_{32})^2}{x_{12}^2 x_{31}^2 x_{32}^2 x_{31;2}^2 x_{32;1}^2}.$$  \hspace{1cm} (63)

Clearly, one finds exactly the same result when completing the building block \( \boxed{[a]} \) from Figure 3 to the graph in Figure 4b by multiplying it by a hypermultiplet propagator \((12)/x_{12}^2\). The careful computation of the group and combinatorial factors shows that the two contributions cancel,

$$\langle (\tilde{q}^+ (1))^2 (q^+ (2))^2 \frac{1}{4g^2} (W (3))^2 \rangle = 0.$$  \hspace{1cm} (64)

This confirms the absence of quantum corrections to three-point correlators at two loops (see Section 2), other than possible contact terms. Concerning the latter, note one subtle point. The above multiplication of singular distributions of the type \(1/x_{12}^2 \times 1/x_{12}^2 = 1/x_{12}^4\) should be done with care, using a suitable regularisation scheme. The complete result may then contain contact terms which are lost in the formal manipulations presented here.

Finally, because of the (purely algebraic) complexity of the calculation of the five-point correlator in Figure 1 in full generality, we shall do it by setting the external $\theta$'s at the four hypermultiplet ends to zero, $\theta_{1,2,3,4}^+ = 0$. The only surviving Grassmann variable will be the chiral one $\theta_5^{\alpha i}$ at the point of insertion of the SYM Lagrangian. In other words, we will only be interested in the leading component corresponding to the correlator of four bilinears made out of the hypermultiplet scalars with a fifth bilinear composed of the SYM scalars. Our aim will be to show that after integrating over the insertion point, one correctly reproduces the known result for the four-point correlator of \([21, 11]\). So, multiplying two building blocks of the type \( \boxed{[b]} \) and doing all the necessary permutations, one finds the following surprisingly simple result:

$$\langle (\tilde{q}^+ (1))^2 (q^+ (2))^2 (q^+ (3))^2 (q^+ (4))^2 \frac{1}{4g^2} (W (5))^2 \rangle_{\theta_{1,2,3,4}^+ = 0} = -g^2 \frac{f_{abc}f_{abc}}{(2\pi)^4} (\theta_5)^4 \left[ \frac{(12)^2 (34)^2}{x_{12}^2 x_{34}^4} + \frac{(14)^2 (23)^2}{x_{14}^2 x_{23}^4} + \frac{(12)(23)(34)(41)}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \left( \frac{x_{13}^2 x_{34}^2}{x_{12}^2 x_{34}^2} - \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} - 1 \right) \right].$$  \hspace{1cm} (65)
One recognises the chiral Grassmann delta-function $(\theta_5)^4$ which gives the correlator the required $R$ weight of $W^2$. The dependence on the point $x_5$ is concentrated in a simple rational factor. The integration over the point of insertion $\int d^4\theta_5 d^4x_5$ removes $(\theta_5)^4$ and produces the well-known one-loop scalar box integral \[ \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = -\frac{i\pi^2}{x_{12}^2 x_{34}^2} \Phi^{(1)}(s, t) \] where

$$s = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}, \quad t = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}$$

are the two conformal cross-ratios. Thus, the end result for the four-point correlator (or, rather, its derivative with respect to the coupling constant, in accordance with eq. (42)) is

$$\langle (q^+(1))^2(q^+(2))^2(q^+(3))^2(q^+(4))^2 \rangle_{\theta_1,2,3,4=0} \sim \Phi^{(1)}(s, t) \left[ \frac{(12)^2 (34)^2}{x_{12}^2 x_{34}^2} + \frac{(14)^2 (23)^2}{x_{14}^2 x_{23}^2} s + \frac{(12)(23)(34)(41)}{x_{12}^2 x_{34}^2 x_{23}^2 x_{41}^2} (t - s - 1) \right].$$

This is in complete agreement with the results of refs. [21, 1].

4 Conclusions

The subject of this work has been the explicit construction of nilpotent superconformal covariants in $N = 4$ SYM theory. In particular, we have used the superconformal Ward identities to find all two- and three-point contact terms. We have also investigated the existence of such contact terms for more than three points and we have argued that such terms cannot affect the proof of the non-renormalisation of two- and three-point function given in [14]. We have also argued, subject to some assumptions, that all contact terms arise from the addition of a finite local counterterm to the effective action, namely the superconformal action.

The explicit contact terms which were found in reference [19] followed from non-contact contributions as a result of the Ward identities. Such contact terms are automatically encoded in the approach advocated in the works by the authors of this paper. Neither in our attempts at constructing them nor in our application of that formula to an explicit two-loop calculation have we found evidence for the existence of contact terms of the more malignant type that would invalidate the non-renormalisation theorem for two and three point functions given in [14].

We have also carried out a two-loop calculation in $N = 2$ harmonic superspace and as a result have been able to prove explicitly the existence of a five-point nilpotent $N = 2$ superconformal covariant which in turn strongly suggests the existence of a corresponding $N = 4$ covariant. If the multiplication of the building blocks in this calculation is done in full detail (i.e., without setting $\theta_{1,2,3,4}^+ = 0$) one arrives at an explicit expression for this five-point superconformal invariant which is of the type discussed in [14]. As first suggested in [14, 33] such new invariants must exist if the $N = 4$ harmonic superspace is to be consistent with the known facts about the Green’s functions in $N = 4$ Yang-Mills theory. These new invariants do however, have the other general properties postulated in [14]. These properties are harmonic analyticity $D^{++}G = 0$, which is evident from (65), and superconformal invariance (which follows from the $N = 4$ SYM context of the calculation).
As a byproduct of this investigation, we have achieved a further significant simplification in the calculation of the two-loop four-point correlators of gauge invariant operators first considered in [21, 32]. In the direct calculation of the four-point correlator carried out in ref. [21], the fact that all the three harmonic structures in (67) have the same non-trivial dependence on the conformal cross ratios was not obvious at all. An indirect argument based on conformal supersymmetry and harmonic analyticity allowed us to establish this relationship in [11]. The present calculation reproduces it directly, due to the remarkably simple structure of the five-point correlator. Moreover, in the variant of this calculation presented here the appearance of any two-loop integrals was completely avoided. This was made possible by the fact that for the two basic building blocks of our two-loop diagrams, depicted in figs. 2 and 3, the integration over the internal point can be performed trivially using supersymmetry. This fact is of independent interest, and may possibly lead to simplifications in other contexts.

Acknowledgements: We are indebted to M. Bianchi, F. Delduc, E. D’Hoker, E. Ivanov, R. Stora and S. Theisen for many useful discussions. This work was supported in part by the British-French scientific programme Alliance (project 98074) and by the EU network on Integrability, non-perturbative effects, and symmetry in quantum field theory (FMRX-CT96-0012).

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