An Explicit Upper Bound for Modulus of Divided Difference on A Jordan Arc in the Complex Plane

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April 6, 2016

Abstract

An explicit upper bound is derived for the modulus of divided difference for a smooth (not necessarily analytic) function defined on a smooth Jordan arc (or a smooth Jordan curve) in the complex plane. As an immediate application, an error estimate for complex polynomial interpolation on a Jordan arc (or a Jordan curve) is given, which extends the well-known error estimate for polynomial interpolation on the unit interval. Moreover, this upper bound is independent of the parametrization of the curve.

1 Introduction

Suppose \( f \) is a smooth function on \([0, 1]\). The problem of interpolating \( f \) at \( n + 1 \) distinct nodes \( z_1, \ldots, z_{n+1} \in [0, 1] \) using a polynomial of degree \( n \) has a satisfying answer, for which we have the following well-known error estimate (cf. [1] p.314)

\[
|f(z) - p_n(z)| \leq \frac{\sup_{0 < \xi < 1} |f^{(n+1)}(\xi)|}{(n + 1)!} \prod_{k=1}^{n+1} |z - z_k|.
\] (1)

However, if we consider polynomial interpolation with a complex variable for a smooth non-analytic function \( f \) defined on a smooth Jordan arc or a smooth Jordan curve in the complex plane, no result that resembles (1) is available.

Even though efforts have been made over the years in complex polynomial interpolation, for example, with monographs on this topic by J.L.Walsh ([16]), D.Gaier ([10]), etc., and numerous papers such as [11], [8], [6], [2], [4], [3], [13], [9], etc., the number of literatures investigating a possible extension of (1) to the complex plane is quite limited. Moreover, most of the results in existing literatures on complex polynomial interpolation require \( f \) be analytic in certain domain of interest (cf. [11], [10], [6], [13], [9]), or the curve be analytic (cf. [8]), and all of them focus on interpolation on a boundary curve (instead of a piece of arc) due to various needs, for example, in conformal mapping (cf. [13]) and in solving Dirichlet problems (cf. [5]), etc.

In terms of extending (1) to the complex plane, we note that since (1) can be deduced by estimating divided difference for \( f \), it then boils down to estimate the divided difference for the general case where \( f \) is not necessarily analytic and the Jordan arc (or Jordan curve) is not analytic, either. The only paper
we can find that deals with this issue is [3], in which the author showed the uniform boundedness of the divided difference for \( f \) on a Jordan curve in the complex plane. Though no explicit bound was given in [3], by following the proof in [3] to bound the divided difference, where the problem for a general Jordan curve was transformed into the problem for a unit circle by a change of variable (cf. [3, Lemma 3.1, Lemma 3.3]), leading to an estimate highly sensitive to the parametrization of the curve. Therefore, it is the aim of this paper to employ a direct approach to provide an upper bound independent of the parametrization of the curve and hopefully in a similar form as in the real case. As a straightforward application, an error estimate for polynomial interpolation with a complex variable on a Jordan arc or a Jordan curve in the complex plane will be obtained, which can be viewed as an extension of (1).

Since estimates for divided differences on Jordan curves as in [3] can all be derived by transforming the problem into estimating divided differences on Jordan arcs, it suffices for us to focus only on divided differences on Jordan arcs, and the case for Jordan curves can be immediately obtained as a byproduct.

We define divided difference as follows. For \( n + 1 \) distinct points \( z_1, \ldots, z_{n+1} \) on the complex plane, and a function \( f \) defined on a set containing those points, the divided difference for \( f \) of order \( k (k = 0, \ldots, n) \) with respect to those points is defined recursively by

\[
\begin{align*}
  d_0 &= d_0(f|z_1) = f(z_1) \\
  d_k &= d_k(f|z_1, \ldots, z_{k+1}) \\
  &= \frac{d_{k-1}(f|z_1, z_2, \ldots, z_k) - d_{k-1}(f|z_k+1, z_2, \ldots, z_k)}{z_1 - z_{k+1}}, \quad 1 \leq k \leq n. 
\end{align*}
\]

It will be shown later how to define divided difference properly at those points when \( z_i = z_j \) for some \( i \neq j \).

There are several definitions or representations for divided difference. Recall the Newton divided difference interpolation formula (cf. [12], [1])

\[
p_n(z) = d_0(f|z_1) + d_1(f|z_1, z_2)(z - z_1) + d_2(f|z_1, z_2, z_3)(z - z_1)(z - z_2) + \cdots + d_n(f|z_1, \ldots, z_{n+1})(z - z_1)(z - z_2)\cdots(z - z_n), \tag{3}
\]

or in the form (cf. [12])

\[
f(z) = p_n(z) + d_{n+1}(f|z_1, \ldots, z_{n+1}, z)(z - z_1)(z - z_2)\cdots(z - z_{n+1}), \tag{4}
\]

with \( p_n(z) \) given in [3], and the Lagrange interpolation formula (cf. [12], [1])

\[
p_n(z) = \sum_{k=1}^{n+1} f(z_k) \frac{w_k(z)}{w_k(z_k)}, \tag{5}
\]

where

\[
w_k(z) := \prod_{i \neq k}(z - z_i).
\]
Since the interpolating polynomial of degree at most \( n \) is unique, by comparing the coefficient of \( z^n \) in (3) and (5), we see that
\[
d_n(f|z_1, \ldots, z_{n+1}) = \sum_{k=1}^{n+1} \frac{f(z_k)}{w_k(z_k)},
\]
(6)

In addition to (2), Eq. (6) above provides another definition for divided difference, from which it can be seen that the divided difference \( d_n(f|z_1, \ldots, z_{n+1}) \) is invariant under any permutation of interpolation nodes \( z_1, \ldots, z_{n+1} \).

Besides, suppose either \( f \) is analytic in a neighborhood of the convex hull of \( S = \{z_1, \ldots, z_{n+1}\} \subset \mathbb{C} \), or \( f \) is smooth in an open interval containing \( S \) if \( S \subset \mathbb{R} \), an integral representation known as Hermite formula (or Genocchi-Hermite formula, cf. [7]) can be derived (cf. [11], [12]):
\[
d_n(f|z_1, \ldots, z_{n+1}) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} f^{(n)} \left((1-t_1)z_1 + (t_1-t_2)z_2 + \ldots + (t_{n-1} - t_n)z_n + t_n z_{n+1}\right) dt_n.
\]
(7)

From this representation, it is straightforward to obtain an upper bound of \(|d_n|\), namely,
\[
|d_n(f|z_1, \ldots, z_{n+1})| \leq \frac{\|f^{(n)}\|_{\text{sup}}}{n!},
\]
(8)
where the \( \text{sup} \) of \(|f^{(n)}|\) is taken inside the convex hull of \( S \).

However, if the assumptions above on \( f \) and \( S \) do not hold, namely, if \( S \not\subset \mathbb{R} \) and \( f \) is not analytic, no result on the upper bound of \(|d_n|\) can be found other than the one in [3]. As mentioned before, the result in [3] is too pessimistic and is dependent on the parametrization of the curve. We aim to find an upper bound for \(|d_n|\) that is independent of the parametrization of the curve, and is in a similar form as the one in (8). Namely, we are looking for an upper bound \( C_f \alpha(n)/n! \), where \( C_f \) is a positive constant depending on certain derivatives (in the sense of (9) described later) of \( f \) and \( \alpha(n) \) is a positive function of \( n \) that grows at most exponentially in \( n \), i.e., \( \alpha(n) \leq ab^n \) for some constant \( a,b > 0 \) (hence \( \lim_{n \to \infty} \alpha(n)/n! = 0 \)), and \( \alpha(n) \) is independent of the parametrization of the curve.

The rest of this paper is organized as follows. In Section 2 we present some technical tools developed in [3] that will be used in our proof. In Section 3 we derive an upper bound for modulus of divided difference of a smooth function defined on a Jordan arc (or a Jordan curve) in the complex plane, and thus establish an explicit error estimate for complex polynomial interpolation on a Jordan arc (or a Jordan curve).

2 Properties of divided difference as a function of one variable

By a Jordan arc, we mean the image of the closed unit interval \([0, 1]\) under a homeomorphism into the complex plane, while a Jordan curve on the complex plane is the homeomorphic image of the unit circle. We say a Jordan arc (or
Lemma 2.1. Let \( f \) be a function defined on an admissible Jordan arc \( \gamma \) with parametrization \( \phi \) such that \( f^{(n-1)} \) exists everywhere on \( \gamma \) and \( f^{(n-1)} \circ \phi(t) \) is absolutely continuous on \([0,1]\). Then we have

\[
d_{n}^{k} f|z_{1}, \ldots , z_{n+1} = \frac{\partial^{k} d_{n}(f|z_{1}, \ldots , z_{n+1})}{\partial z_{1}^{k}}.
\]

The following lemma is same to Lemma 3.2 in [3] by simply replacing "Jordan curve" in [3] with "Jordan arc" in our setting, and the proof is almost the same.

**Lemma 2.1.** Let \( f \) be a function defined on an admissible Jordan arc \( \gamma \) with parametrization \( \phi \) such that \( f^{(n-1)} \) exists everywhere on \( \gamma \) and \( f^{(n-1)} \circ \phi(t) \) is absolutely continuous on \([0,1]\). Then we have

\[
d_{1}^{k-1}(f|z_{1}, z_{2}) = \frac{\int_{z_{2}}^{z_{1}} (z - z_{2})^{k-1} f^{(k)}(z)dz}{(z_{1} - z_{2})^{k}}, \quad k = 1, 2, \ldots , n,
\]

where \( z_{1}, z_{2} \in \gamma \), and \( z_{1} \neq z_{2} \).
In order to derive the smoothness of divided difference for a smooth function, we need the following boundedness result, whose proof can be found in [3].

**Lemma 2.2.** Let \( f \) be a function defined on an admissible Jordan arc \( \gamma \) such that \( |f \circ \phi| \) is uniformly bounded on \([0, 1]\) except on a set of measure zero. We define

\[
J_k(z_1, z_2) = \int_{z_2}^{z_1} (z - z_2)^k f(z)dz, \quad z_1, z_2 \in \gamma,
\]

to be an integration on \( \gamma \). Then for each nonnegative integer \( k \), there exists a constant \( M_k \), depending only on \( g \) and \( \gamma \), such that

\[
|J_k(z_1, z_2)| \leq M_k (z_1 - z_2)^{k+1}
\]

for all \( z_1, z_2 \in \gamma \) with \( z_1 \neq z_2 \).

Next we show that if proper value is defined at \( z_1 = z_2 \), then \( d_1(f|z_1, z_2) \), as a function of \( z_1 \), inherits the smoothness of \( f' \), which resolves our concern in the definition of divided difference given in [2] when \( z_1 = z_j \) for some \( i \neq j \), assuming \( f \) is smooth enough. This property was mentioned in [3] where the proof was omitted. For completeness, we give a rigorous proof below.

**Lemma 2.3.** Let \( f, \gamma, \phi \) be given as in Lemma 2.2 with \( n \geq 2 \), and we further assume that \( f^{(n-1)} \circ \phi \) is Lipschitz continuous. Then for any fixed \( z_2 = \phi(t_2) \in \gamma \), \( |d_1^{n-1}(f|z_1, z_2)| \) is uniformly bounded in \( z_1 \in \gamma \setminus \{z_2\} \), and for each integer \( k \) with \( 0 \leq k \leq n-2 \), we can assign a proper value at \( t_1 = t_2 \) (i.e., \( z_1 = \phi(t_1) = z_2 \)) such that \( d_1^k(f|\phi(t_1), z_2) \) is absolutely continuous as a function of \( t_1 \in [0, 1] \).

**Proof.** For the following proof, we assume \( k \in \{1, \ldots, n-1\} \).

Since \( f^{(n-1)} \circ \phi \) is Lipschitz continuous, its derivative exists almost everywhere and is uniformly bounded. Lemma 2.2 and Lemma 2.3 then imply that \( |d_1^{n-1}(f|z_1, z_2)| \) is uniformly bounded in \( z_1 \) on \( \gamma \setminus \{z_2\} \). In particular, \( |d_1^{n-1}(f|z_1, z_2)| \) is uniformly bounded.

With the uniform boundedness of \( |d_1^{n-1}| \), we are able to define \( d_1^{k-1}(f|z_1, z_2) \) at \( z_1 = z_2 \) such that \( d_1^{k-1}(f|\phi(t_1), z_2) \) is absolutely continuous in \( t_1 \in [0, 1] \). To do this, we first observe that the total variation of \( d_1^{k-1}(f|\phi(t_1), z_2) \) is uniformly bounded on any subinterval of \([0, 1]\) \( \setminus \{t_2\} \). This implies that (cf. [15], p.371, Ex.6), as \( t_1 \) approaches \( t_2 \) from either side, the limit of \( d_1^{k-1}(f|\phi(t_1), z_2) \) exists, and assuming two limits coincide, if we define \( d_1^{k-1}(f|\phi(t_2), z_2) \) to be equal to the limit, \( d_1^{k-1}(f|\phi(t_1), z_2) \) is absolutely continuous as a function of \( t_1 \) on \([0, 1]\). Thus if \( t_2 \) is an endpoint in \([0, 1]\), we can assign to \( d_1^{k-1}(f|z_2, z_2) \) the unique limit as \( t_1 \rightarrow t_2 \). If \( t_2 \) is an interior point in \([0, 1]\), we shall prove that the two limits coincide as \( t_1 \) approaches \( t_2 \) from either side. In fact, from (12) and the continuity of \( f^{(k)} \circ \phi(t) \) in the assumption, we deduce by using L'Hospital's rule that

\[
\lim_{t_1 \rightarrow t_2} d_1^{k-1}(f|\phi(t_1), \phi(t_2)) = \lim_{t_1 \rightarrow t_2} \frac{f(t_1)(\phi(t_1) - \phi(t_2))^{k-1} f^{(k)} \circ \phi(t) \phi'(t)dt}{(\phi(t_1) - \phi(t_2))^{k}} \cdot
g = \lim_{t_1 \rightarrow t_2} \frac{(\phi(t_1) - \phi(t_2))^{k-1} f^{(k)} \circ \phi(t) \phi'(t)}{k(\phi(t_1) - \phi(t_2))^{k-1} \phi'(t_1)}
\]

(13)
Hence by setting $d_k^{n-1}(f|z_2, z_3)$ to be equal to the limit above, we conclude that $d_k^{n-1}(f|\phi(t_1), z_2)$ is absolutely continuous in $t_1 \in [0, 1]$.  

The proof above implies that, with $d_k^{n-1}(f|z_2, z_3)(k = 1, \ldots, n - 1)$ properly defined, $d_k^{n-1}(f|\phi(t_1), z_2)$ will be absolutely continuous in $t_1 \in [0, 1]$, and $|d_k^{n-1}(f|z_1, z_2)|$ will be uniformly bounded in $z_1 \in \gamma \{z_2\}$, as long as the following two conditions are all satisfied:

1. $f^{(n-1)} \circ \phi$ is absolutely continuous (consequently the representation formula (12) in Lemma 2.4 holds for $d_k^{n-1}(f|z_1, z_2)$ ($k = 1, \ldots, n$));

2. $|f^{(n)} \circ \phi(t)|$ is uniformly bounded in $[0, 1]$ except on a set of measure zero (hence Lemma 2.4 can be applied to $f^{(n)}$).

We next show that higher order divided differences can also be made continuous by recursively verifying the two conditions above.

We set $g(z) = d_1(f|z, z_2)$. Lemma 2.3 shows that $g^{(n-2)} \circ \phi(t)$ is absolutely continuous in $[0, 1]$ and $|g^{(n-1)}(z)|$ uniformly bounded on $\gamma \setminus \{z_2\}$. Hence condition 1 and condition 2 are both satisfied, and the absolute continuity of $d_k^{n-1}(g(\phi(t_1), z_3)(k = 1, \ldots, n - 2)$ in $t_1 \in [0, 1]$ follows, with $d_k^{n-1}(g|z_3, z_3)$ properly defined.

Note that

$$d_k^{n-1}(f|z_1, z_2, z_3) = \frac{\partial^{k-1}}{\partial z_1^{k-1}} d_1(f|z_1, z_2, z_3) = \frac{\partial^{k-1}}{\partial z_1^{k-1}} d_1(d_1(f|z, z_2)|z_1, z_3) = d_k^{n-1}(g|z_1, z_3).$$

Therefore, we have established the absolute continuity of $d_k^{n-1}(f|z_1, z_2, z_3)(k = 1, \ldots, n - 2)$, as a function of $t_1 = \phi^{-1}(z_1) \in [0, 1]$, as well as the uniform boundedness of $|d_k^{n-2}(f|z_1, z_2, z_3)|$.

Similarly, we can then set $h(z) = d_2(f|z, z_2, z_3)$ and use Lemma 2.3 to deduce the absolute continuity of $d_k^{n-1}(k = 1, \ldots, n - 3)$ and uniform boundedness of $|d_k^{n-3}|$.

Therefore, by iteratively using Lemma 2.3 to verify the two conditions mentioned above, we arrive at the following theorem.

**Theorem 2.1.** Let $f$ be a function defined on an admissible Jordan arc $\gamma$ with parametrization $\phi$ such that $f^{(n-1)}(n \geq 2)$ exists everywhere on $\gamma$ and $f^{(n-1)} \circ \phi(t)$ is Lipschitz continuous on $[0, 1]$. Then for any integer $k$ with $1 \leq k \leq n$, $|d_k^{n-k}(f|\phi(t_1), z_2, \ldots, z_{k+1})|$ is uniformly bounded almost everywhere on $[0, 1]$ as a function of $t_1$, and for $m = 0, 1, \ldots, n - k - 1$, $d_k^m(f|\phi(t_1), z_2, \ldots, z_{k+1})$ is absolutely continuous in $t_1$ when proper value is defined at $z_1 = \phi(t_1) = z_{k+1}$. Moreover, the following equation holds as a generalization of Equation (12),

$$d_k^m = \frac{\int_{z_{k+1}}^{z_{k+1}} (z - z_{k+1})^m d_k^{m+1}(f|z, z_2, \ldots, z_k)dz}{(z_1 - z_{k+1})^{m+1}}, \quad k = 1, \ldots, n, \quad m = 0, \ldots, n - k,$$

where $z_1 \neq z_{k+1}$, $d_0^n = d_n$, and $d_0^0 = f^{(n)}$.

Equation (14) will be the main tool that we use to derive the desired estimate in the next section.
3 An upper bound for modulus of divided difference on a Jordan arc

In [3], the boundedness of \(|d_n|\) was obtained by a change of variable to convert the problem on a general Jordan curve to the problem on the unit circle. However, this indirect approach makes the bound (which can be computed by following the proof in [3]) too pessimistic if the shape of the curve is not close to a circle. In this section, as opposed to [3], we employ a direct approach to derive an explicit upper bound that does not depend on the parametrization of \(\gamma\) and that resembles the estimate in (8).

To start with, we first compute upper bounds for \(|d_k^n|\) \((k = 1, 2, \ldots, n - 1)\).

**Lemma 3.1.** Let \(f\) be defined on an admissible Jordan arc \(\gamma\) such that \(f^{(n+1)}\) exists and is continuous on \(\gamma\). Then

\[
|d_k^n(f|z_1, z_2)| \leq \frac{C_{\gamma,k}}{k+1}, \quad \forall z_1 \neq z_2 \in \gamma, \quad k = 1, 2, \ldots, n - 1,
\]

where \(C_{\gamma,k}\) is a nonnegative constant only depending on \(f, \gamma, k\).

**Proof.** The main idea of this proof is to use the representation in (12) and then apply integration by parts. Indeed, suppose \(z_1 \neq z_2\), we deduce from (12) that, for \(1 \leq k \leq n - 1\),

\[
d_k^n(f|z_1, z_2) = \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+1)}(z) dz}{(z_1 - z_2)^{k+1}}
= \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+1)}(z) dz}{(z_1 - z_2)^{k+1}} - \left. \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+2)}(z) dz}{(z_1 - z_2)^{k+1}} \right|_{z=z_2}^{z_1}
= \frac{f^{(k+1)}(z_1)}{k+1} - \frac{1}{k+1} \int_{z_2}^{z_1} (z - z_2)^k f^{(k+2)}(z) dz.
\]

By using L’Hospital’s Rule as in (13), we find that

\[
\lim_{z_1 \to z_2} \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+2)}(z) dz}{(z_1 - z_2)^{k+1}} = \lim_{z_1 \to z_2} \frac{(z_1 - z_2)^{k+1} f^{(k+2)}(z_1)}{(k+1)(z_1 - z_2)^k}
= \lim_{z_1 \to z_2} \frac{f^{(k+2)}(z_1)}{k+1} = 0,
\]

since \(f^{(k+2)}\) is continuous on \(\gamma\). Thus we have

\[
M_{\gamma,k} := \sup_{z_1, z_2 \in \gamma, z_1 \neq z_2} \left| \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+2)}(z) dz}{(z_1 - z_2)^{k+1}} \right| < \infty.
\]

Hence it follows from (15) that

\[
|d_k^n(f|z_1, z_2)| \leq \frac{\sup_{z \in \gamma} |f^{(k+1)}(z)| + M_{\gamma,k}}{k+1} = \frac{C_{\gamma,k}}{k+1},
\]

where

\[
C_{\gamma,k} = \sup_{z \in \gamma} |f^{(k+1)}(z)| + M_{\gamma,k}
= \sup_{z \in \gamma} |f^{(k+1)}(z)| + \sup_{z_1, z_2 \in \gamma, z_1 \neq z_2} \left| \frac{f_{z_2}^{z_1}(z - z_2)^k f^{(k+2)}(z) dz}{(z_1 - z_2)^{k+1}} \right| \quad \text{(16)}
\]
only depends on \( f, \gamma, k \), and \( C_{\gamma,k} \to \sup_{z \in \gamma} |f^{(k+1)}(z)| \) as \( \text{diam}(\gamma) \to 0. \)

In order to estimate \( |d_n| \) using (14), we need the following elementary result.

**Lemma 3.2.** Let \( \{I_{k,n}\}_{k,n=0}^{\infty} \) be a double sequence of nonnegative numbers satisfying

\[
I_{1,n} \leq \frac{C}{n+1}
\]

\[
I_{k,n} \leq \frac{1}{n+1}(I_{k-1,n+1} + L_k I_{k-1,n+2}), \quad k = 2, \ldots, n = 0, 1, \ldots,
\]

where for each \( k \), \( L_k \) is a nonnegative constant. Then

\[
I_{n,0} \leq \frac{C \prod_{k=2}^{n}(1 + L_k)}{n!}, \quad \forall n \geq 2.
\]  

(17)

**Proof.** We first define an upper bound \( \hat{I}_{k,n} \) of \( I_{k,n} \) as follows. Let the double sequence \( \{\hat{I}_{k,n}\}_{k,n=0}^{\infty} \) be given by

\[
\hat{I}_{1,n} = \frac{C}{n+1}
\]

\[
\hat{I}_{k,n} = \frac{1}{n+1}(\hat{I}_{k-1,n+1} + L_k \hat{I}_{k-1,n+2}), \quad k = 2, \ldots, n = 0, 1, \ldots.
\]

It is easy to verify by induction on \( k \) that \( I_{k,n} \leq \hat{I}_{k,n} \). Thus it suffices to bound \( \hat{I}_{n,0} \).

We observe that \( \hat{I}_{k,n+1} \leq \hat{I}_{k,n} \). Indeed, for \( k = 1 \), \( \hat{I}_{1,n+1} = \frac{C}{n+2} \leq \frac{C}{n+1} = \hat{I}_{1,n} \) by definition. Assume the inequality holds with first index \( k-1 \). Then we see from the definition of \( \hat{I}_{k,n} \) and the hypothesis for \( k-1 \) that

\[
\hat{I}_{k,n+1} = \frac{1}{n+2}(\hat{I}_{k-1,n+2} + L_k \hat{I}_{k-1,n+3}) \leq \frac{1}{n+1}(\hat{I}_{k-1,n+1} + L_k \hat{I}_{k-1,n+2}) = \hat{I}_{k,n}.
\]

Hence the above induction implies that \( \hat{I}_{k,n+1} \leq \hat{I}_{k,n} \) for all indices. Consequently, we have

\[
\hat{I}_{k,n} = \frac{1}{n+1}(\hat{I}_{k-1,n+1} + L_k \hat{I}_{k-1,n+2})
\]

\[
\leq \frac{1}{n+1}(\hat{I}_{k-1,n+1} + L_k \hat{I}_{k-1,n+1}) = \frac{1 + L_k}{n+1} \hat{I}_{k-1,n+1}.
\]

Now we are able to estimate \( \hat{I}_{n,0} \) by induction below.

\[
\hat{I}_{n,0} \leq \frac{(1 + L_n)}{1} \hat{I}_{n-1,1}
\]

\[
\leq \frac{(1 + L_n)(1 + L_{n-1})}{2!} \hat{I}_{n-2,2}
\]

\[
\leq \cdots \leq \prod_{k=2}^{n}(1 + L_k) \hat{I}_{1,n-1} = \frac{C \prod_{k=2}^{n}(1 + L_k)}{(n-1)!}, \quad \forall n \geq 2.
\]

The inequality (17) then follows since \( I_{n,0} \leq \hat{I}_{n,0} \).
We are now in a position to state the main result.

**Theorem 3.1.** Let \( f \) be a function defined on an admissible Jordan arc \( \gamma \) such that \( f^{(n+1)} \) exists and is continuous on \( \gamma \). Then

\[
|d_n(f[z_1, \ldots, z_{n+1}])| \leq \frac{C_n \prod_{k=2}^{n} (1 + L_k)}{n!} \leq \frac{C_n (1 + \text{diam}(\gamma))^{n-1}}{n!}, \quad \forall n \geq 2,
\]

where \( z_1, \ldots, z_{n+1} \in \gamma \) are distinct, \( L_k = |z_1 - z_{k+1}|, \text{diam}(\gamma) := \max_{u,v \in \gamma} |u-v| \) and

\[
C_n = \max_{1 \leq k \leq n-1} C_{\gamma,k}
\]

with \( C_{\gamma,k} \) defined in (16) independent of \( z_1, \ldots, z_{n+1} \) and the parametrization of \( \gamma \). Furthermore, if \( \gamma \) is an admissible Jordan curve then (18) still holds.

**Proof.** For a fixed set of points \( z_1, \ldots, z_{n+1} \in \gamma \), we define

\[
I_{k,m} = |d^n_{m}(f[z_1, \ldots, z_{n+1}])|, \quad k = 1, 2, \ldots, n, \quad m = 0, 1, \ldots, n-k.
\]

Let \( C_{\gamma} = \max_{1 \leq k \leq n-1} C_{\gamma,k} \) with \( C_{\gamma,k} \) given in (16). From Lemma 3.1 we know that

\[
I_{1,m} = |d^n_{m}| \leq \frac{C_{\gamma,m}}{m+1} \leq \frac{C_{\gamma}}{m+1}, \quad m = 1, 2, \ldots, n-1. \tag{19}
\]

More generally, since \( f^{(n+1)} \) is continuous on \( \gamma \) (hence \( f^{(n)} \) is Lipschitz continuous), the discussion in previous section shows that \( d^n_{m}(f[\phi(t), z_2, \ldots, z_k]) \) is absolutely continuous in \( t \in [0,1] \) as long as \( k + m \leq n \), from which integration by parts is justified. Therefore, based on (14), integration by parts as in (15) yields that

\[
I_{k,m} = \left| \frac{\int_{z_{k+1}}^{z_1} (z - z_{k+1})^m d_{k-1}^{m+1}(f[z, z_2, \ldots, z_k])dz}{(z_1 - z_{k+1})^{m+1}} \right|
\]

\[
= \frac{|d_{k-1}^{(m+1)}(f[z_1, \ldots, z_k])|}{m+1} - \frac{z_1 - z_{k+1}}{m+1} \frac{\int_{z_{k+1}}^{z_1} (z - z_{k+1})^{m+1} d_{k-1}^{(m+2)}(f[z, z_2, \ldots, z_k])dz}{(z_1 - z_{k+1})^{m+2}}
\]

\[
= \frac{|d_{k-1}^{(m+1)}(f[z_1, \ldots, z_k])|}{m+1} - \frac{z_1 - z_{k+1}}{m+1} \frac{d_{k-1}^{(m+2)}(f[z_1, \ldots, z_k])}{m+1}
\]

\[
\leq \frac{1}{m+1} (I_{k-1,m+1} + L_k I_{k-1,m+2}), \quad k = 1, 2, \ldots, n, \quad m = 0, 1, \ldots, n-k, \tag{20}
\]

where \( L_k := |z_1 - z_{k+1}| \). It follows from (19), (20) that the assumptions in Lemma 3.2 are satisfied by \( I_{k,m} \). With the help of (17), we have

\[
|d_n(f[z_1, \ldots, z_{n+1}])| = |d^n_{0}| = I_{n,0} \leq \frac{C_n \prod_{k=2}^{n} (1 + L_k)}{n!} \leq \frac{C_n (1 + \text{diam}(\gamma))^{n-1}}{n!}, \quad \forall n \geq 2,
\]

which establishes (18).

Since both the integral given in (11) and the derivative defined in (9) are independent of the parametrization of \( \gamma \), we see that \( C_{\gamma,k} \) in (16) is independent of the parametrization of \( \gamma \).
Suppose now $\gamma$ is a Jordan curve satisfying the hypothesis in the claim and assume that we fix the orientation of the curve. Let $\gamma_0$ be an Jordan arc on $\gamma$ passing through the nodes $z_1, \ldots, z_{n+1}$. Then it is easily seen that the bound in (21) still holds if we replace $\gamma$ in (21) by $\gamma_0$. Furthermore, it follows immediately from (16) that $C_{\gamma_0} \leq C_\gamma$ if $\gamma_0 \subset \gamma$, where $C_{\gamma_0} = \max_{1 \leq k \leq n-1} C_{\gamma_0,k}$. Hence (18) follows, which completes the proof.

Note that divided difference is invariant under any permutation of nodes while the bound involving $L_k = |z_1 - z_{k+1}|$ in (18) depends on the ordering of $z_1, \ldots, z_{n+1}$. We can then find a sharper bound by permuting the nodes to minimize the corresponding quantity in (18).

**Corollary 3.1.** Let $f, \gamma$ be given as in Theorem 3.1. For distinct nodes $z_1, \ldots, z_{n+1} \in \gamma$, we have

$$|d_n(f|z_1, \ldots, z_{n+1})| \leq C_{\gamma} \min_{\sigma \in S_{n+1}} \prod_{k=2}^{n+1} \left(1 + \frac{|z_{\sigma(k)} - z_{\sigma(k+1)}|}{n!}\right), \quad \forall n \geq 2,$$

where $S_{n+1}$ denotes the symmetric group of degree $n + 1$ and $C_\gamma$ is given in Theorem 3.1.

With an estimate of divided difference above, an extension of (1) for polynomial interpolation error in complex plane readily follows.

**Theorem 3.2.** Let $f$ be a function defined on an admissible Jordan arc $\gamma$ such that $f^{(n+1)}$ $(n \geq 2)$ exists and is continuous on $\gamma$. If $p_n(z)$ interpolates $f$ at the $n+1$ nodes $z_1, \ldots, z_{n+1} \in \gamma$, then we have the error estimate

$$|f(z) - p_n(z)| \leq C_{\gamma} \min_{\sigma \in S_{n+2}} \prod_{k=2}^{n+2} \left(1 + \frac{|z_{\sigma(k)} - z_{\sigma(k+1)}|}{(n+1)!}\right) \prod_{k=1}^{n+2} |z - z_k|,$$

where we have set $z_{n+2} = z$, $C_{\gamma} := \max_{1 \leq k \leq n-1} C_{\gamma,k}$ with $C_{\gamma,k}$ defined in (16) independent of $z_1, \ldots, z_{n+1}$, and $S_{n+2}$ denotes the symmetric group of degree $n + 2$. Furthermore, if $\gamma$ is an admissible Jordan curve then (23) still holds.

**Proof.** This is an immediate result of Newton interpolation formula (4) and an estimate for $|d_n(f|z_1, \ldots, z_{n+1}, z_{n+2})|$ using (22).

**References**

[1] U.M. Ascher and C. Greif. *A First Course on Numerical Methods*. Computational Science and Engineering. Society for Industrial and Applied Mathematics, 2011.

[2] J. H. Curtiss. Necessary conditions in the theory of interpolation in the complex domain. *Annals of Mathematics*, 42(3):634–646, 1941.

[3] J. H. Curtiss. Limits and bounds for divided differences on a jordan curve in the complex domain. *Pacific J. Math.*, 12(4):1217–1233, 1962.

[4] J. H. Curtiss. Polynomial interpolation in points equidistributed on the unit circle. *Pacific J. Math.*, 12(3):863–877, 1962.
[5] J. H. Curtiss. Solution of the dirichlet problem by interpolating harmonic polynomials. *Bull. Amer. Math. Soc.*, 68(4):333–337, 07 1962.

[6] J. H. Curtiss. Convergence of complex lagrange interpolation polynomials on the locus of the interpolation points. *Duke Math. J.*, 32(2):187–204, 06 1965.

[7] C. de Boor. Divided differences. *Surveys in Approximation Theory (SAT)*, 1:46–69, 2005.

[8] L. Fejér. Interpolation und konforme abbildung. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:319–331, 1918.

[9] B. Fischer and L. Reichel. Newton interpolation in fejér and chebyshev points. *Mathematics of Computation*, 53(187):265–278, 1989.

[10] D. Gaier. *Vorlesungen über Approximation im Komplexen*. Birkhäuser, 1980.

[11] Ch. Hermite. Sur la formule d’interpolation de lagrange. *J. Reine Angew. Math.*, 84:70–79, 1878.

[12] N. E. Nørlund. *Vorlesungen über Differenzenrechnung*. Springer, 1924.

[13] L. Reichel. On polynomial approximation in the complex plane with application to conformal mapping. *Mathematics of Computation*, 44(170):425–433, 1985.

[14] E. M. Stein and R. Shakarchi. *Complex analysis*. Princeton Lectures in Analysis, II. Princeton University Press, Princeton, NJ, 2003.

[15] E.C. Titchmarsh. *The Theory of Functions*. Oxford science publications. Oxford University Press, 1939.

[16] J.L. Walsh. *Interpolation and Approximation by Rational Functions in the Complex Domain*. American Mathematical Society: Colloquium publications. American Mathematical Society, 1965.