HOPFIAN AND BASSIAN ALGEBRAS

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Abstract. A ring $A$ is Hopfian if $A$ cannot be isomorphic to a proper homomorphic image $A/J$. $A$ is Bassian, if there cannot be an injection of $A$ into a proper homomorphic image $A/J$. We consider classes of Hopfian and Bassian rings, and tie representability of algebras and chain conditions on ideals to these properties. In particular, any semiprime algebra satisfying the ACC on semiprime ideals is Hopfian, and any semiprime affine PI-algebra over a field is Bassian. We also characterize the ACC on semiprime ideals.

1. Introduction

By “affine” we mean finitely generated as an algebra over a Noetherian commutative base ring, often a field. In this paper we are interested in structural properties of affine algebras, especially those satisfying a polynomial identity (PI). Although there is an extensive literature on affine PI-algebras, several properties involving endomorphisms have not yet been fully investigated, so that is the subject of this note.

The main objective in this paper is to study the Hopfian property, that an algebra $A$ cannot be isomorphic to a proper homomorphic image $A/J$ for any $0 \neq J \triangleleft A$. Our specific motivation is to gain information about group algebras and enveloping algebras. We collect, unify, and extend some known facts and verify the Hopfian property in one more case:

**Theorem 2.12.** Every weakly representable affine algebra over a commutative Noetherian ring is Hopfian.

It is easy to see that any semiprime Hopfian ring satisfies the ACC on semiprime ideals, leading us to characterize this condition. (One direction is standard, but we do not have a reference for the other direction.)

**Proposition 2.11.** A ring $A$ satisfies the ACC on semiprime ideals, iff $A$ satisfies the ACC on prime ideals, with each semiprime ideal being the intersection of finitely many prime ideals.

We then turn to an even stronger property, the Bassian property, that there cannot be an injection of $A$ into a proper homomorphic image of $A$.

**Theorem 4.12.** Any semiprime affine PI-algebra $A$ over a field $F$ is Bassian.
To compare algebras and their isomorphisms, we need a nicely behaved dimension with which to work, and for this purpose recall the **Gelfand-Kirillov dimension** of an affine algebra \( A = F\{a_1, \ldots, a_\ell\} \), which is \( \text{GKdim}(A) := \lim_{n \to \infty} \log n d_n \), where \( A_n = \sum_n F a_i \cdots a_i \) and \( d_n = \dim_F A_n \).

The standard reference on Gelfand-Kirillov dimension is [20], from which we recall some well-known facts:

- \( \text{GKdim}(A) \) is independent of the choice of generating set \( \{a_1, \ldots, a_\ell\} \) of \( A \).
- If \( A \subset A' \) then \( \text{GKdim}(A) \leq \text{GKdim}(A') \).
- \( \text{GKdim}(A) \geq \text{GKdim}(A/I) \) for every ideal \( I \) of \( A \).
- \( \text{GKdim}(A) > \text{GKdim}(A/I) \) for every ideal \( I \) of \( A \) containing a regular element of \( A \).
- If \( A \) is a finite subdirect product of algebras \( A_{\mathcal{I}_1}, \ldots, A_{\mathcal{I}_t} \), then
  \[ \text{GKdim}(A) = \max\{\text{GKdim}(A/I_j) : 1 \leq j \leq t\} \].
- The GK dimension of any algebra finite over \( A \) equals \( \text{GKdim}(A) \).

## 2. Hopfian rings

**Definition 2.1.** A ring is **residually finite** if it is a subdirect product of finite rings.  
An algebra over a field is **residually finite dimensional** if it is a subdirect product of finite dimensional algebras.

**Remarks 2.2.**

(i) Any finite ring is Hopfian, since any onto map must be 1:1 by the pigeonhole principle.

(ii) [29] Any commutative affine algebra over a field is Hopfian.

(iii) [30] Any commutative affine algebra over a commutative ring \( C \) is Hopfian with respect to \( C \)-homomorphisms.

(iv) [5] There is an affine PI-algebra over a field, that is not Hopfian.

(v) [27] There is a group which is not Hopfian in the group-theoretic sense, so its group algebra is not Hopfian.

(vi) [1] Furthermore, there is a finitely presented solvable group which is not Hopfian in the group-theoretic sense.

(vii) Mal’tsev proved that the free algebra is Hopfian, a fact used in [14] in conjunction with the Jacobian conjecture. Short proofs are given in [12, 30], and [8], the latter using GK dimension.

(viii) [21, Theorem 3] Every residually finite ring is Hopfian, proved by Lewin via the fact that any subring of finite index in a finitely generated ring is also finitely generated.

(ix) Mal’tsev proved that residually finite dimensional (called approximable in [9, 10]) affine algebras over a field are all Hopfian.

(x) Positively graded affine algebras are residually finite dimensional, and thus Hopfian by (x).

(xi) Mal’tsev also proved that every representable affine algebra over a field is residually finite dimensional and thus Hopfian, cf. [11, Theorems 5 and 6].

(xii) The enveloping algebra of the positive part of the Witt algebra is positively graded and thus Hopfian. But it is not known whether the enveloping algebra of the Witt algebra itself is Hopfian.
(xiii) On the other hand, Irving [18] produced a graded affine PI-algebra over a field, that is not representable but is positively graded and thus Hopfian.
(xiv) [24, 25] If the enveloping algebra of a Lie algebra \( L \) is residually finite dimensional, then \( L \) is residually finite dimensional as a Lie algebra.

For a given property designated \( P \), we say that an algebra \( A \) satisfies ACC\((P)\) if any ascending chain of ideals \( J_1 \subseteq J_2 \subseteq \ldots \) for which each \( A/J_i \) has property \( P \), must stabilize. The following obvious observation is crucial.

**Lemma 2.3.** For any given property \( P \), if every proper homomorphic image of \( A \) satisfies ACC\((P)\), then \( A \) is Hopfian.

**Proof.** Otherwise take \( J_1 \) such that \( \varphi : A \cong A/J_1 \), and \( J_2 \supset J_1 \) such that \( A \cong A/J_1 \cong (A/J_1)/(J_2/J_1) \cong A/J_2 \) and so forth, but \( J_1 \subset J_2 \subset \ldots \), a contradiction. \( \square \)

In particular, we have the following instant consequences:

**Corollary 2.4.** Any algebra satisfying the ACC on two-sided ideals is Hopfian.

**Corollary 2.5.** Any prime ring \( A \) satisfying ACC\((\text{prime ideals})\) is Hopfian.

We say that an ideal \( P \) is **completely prime** if \( A/P \) is a domain.

**Corollary 2.6.** Any domain \( A \) satisfying ACC\((\text{completely prime ideals})\) is Hopfian.

**Proof.** Otherwise taking \( \varphi : A \cong A/J \subseteq A \), we see that \( J \) is a completely prime ideal, so we use the previous argument of Lemma 2.3. \( \square \)

The reason we became interested in Hopfian rings is in the converse, that any non-Hopfian domain cannot satisfy ACC\((\text{completely prime ideals})\), so if certain enveloping algebras are non-Hopfian, they cannot satisfy ACC\((\text{ideals})\), by Remark 2.2. But the opposite is the case.

**Remark 2.7.** Iyudu and Sierra [19, Theorem 1.2, Proposition 6.4] have verified the ACC on completely prime ideals for the most important of these enveloping algebras the positive Witt algebra \( U(W^+) \), the Witt algebra \( U(W) \), and the Virasoro algebra \( U(\text{Vir}) \). Their method is to show that the first two algebras and central factors of \( U(\text{Vir}) \) have just infinite GK-dimension; this instantly implies the ACC\((\text{completely prime ideals})\) by an easy argument (since modding out by the first completely prime ideal in an ascending chain would yield a domain of finite GK dimension). Their argument for \( U(\text{Vir}) \) is somewhat more intricate. One concludes that these algebras are all Hopfian, by Corollary 2.6.

**Remark 2.8.** Along the same lines, Passman and Small [31, Lemma 1.1] proved that for any epimorphism of finitely presented algebras, the kernel is a finitely generated two-sided ideal. Hence if a finitely presented algebra satisfies the ACC on finitely generated two-sided ideals, it must be Hopfian.

We turn to semiprime rings. Suppose \( J \triangleleft A. \sqrt{J} \), called the **prime radical** of the ideal \( J \), denotes the intersection of the prime ideals containing \( J \). The ideal \( J \) is **semiprime** if \( \sqrt{J} = J \). A ring \( A \) is **semiprime** if \( \sqrt{0} = 0 \).

**Corollary 2.9.** Any semiprime ring \( A \) satisfying the ACC on semiprime ideals is Hopfian.
Non-Hopfian now says $J$ is a nonzero ideal such that $A \cong A/J$. Then clearly the ideal $J$ is semiprime, so Lemma 2.3 is applicable.

This applies in particular to affine PI-algebras (although this result is strengthened in Theorem 4.12), and many enveloping algebras of Lie algebras. Note that Remark 2.2 gives a counterexample for non-semiprime rings.

Corollary 2.9 motivates us to study the ACC on semiprime ideals. There is a natural criterion in terms of the prime spectrum.

We say that a decomposition $J = P_1 \cap \ldots P_t$ as an intersection of prime ideals is irredundant if $P_i \not\subseteq P_j$ for each $i, j$. The following observation is from Herstein [17].

Lemma 2.10. Any irredundant decomposition $J = P_1 \cap \ldots P_t$ of $J$ is unique (up to permutation), and thus consists of minimal prime ideals.

Proposition 2.11. A ring $A$ satisfies the ACC on semiprime ideals, iff $A$ satisfies the ACC on prime ideals, with each semiprime ideal being the intersection of finitely many prime ideals.

Proof. In either direction, we may assume that $A$ is semiprime by modding out some ideal in the chain. ($\Rightarrow$) is a standard Noetherian-type induction argument, cf. [32 Proposition 2.3]. ($\Leftarrow$) We define the depth of a semiprime ideal $J$ to be the size of an irredundant decomposition, i.e., the minimal number of prime ideals whose intersection is $J$, and for any infinite ascending chain of semiprime ideals $0 = J_1 \subseteq J_2 \subseteq \ldots$, we induct on the depth vector $d = (d_1, d_2, \ldots)$ (lexicographic order) where $d_i$ is the depth of $J_i$. Namely, we assume that there are no infinite chains of depth $< d$, and want to prove that there are no infinite chains of depth $d$. (The assumption is vacuous for $d_1 = 1$)

We take a semiprime ideal $P$ of depth $d$ maximal with respect to the assertion failing in $A/P$; passing to $A/P$ we may assume that the assertion holds in every proper homomorphic image of $A$. Write $J_1 = P_1 \cap \cdots \cap P_t$ and $J_2 = Q_1 \cap \cdots \cap Q_r$ for prime ideals $P_j$ and $Q_j$. As in the proof of Lemma 2.10 for each $Q_j$ we have some $P_{i(j)} \subseteq Q_j$. Clearly $\cap_j P_{i(j)} \subseteq \cap_j Q_j = J_2$, so we could discard any $i$ not appearing as an $i(j)$ and conclude by induction on depth. Hence all the $i$ appear as $i(j)$ for suitable $j$, and we could build an infinite ascending sequence of prime ideals, contrary to assumption.

The ACC on semiprime ideals also comes up in [22 Proposition 14]. But we can push these ideas further. Following Malt’sev and Amitsur [3], we define property $\text{Emb}_n$ to be that $A/J^n$ can be embedded in matrix rings $M_n(K)$ over commutative ring $K$, with $n$ bounded.

Amitsur [3 Theorem 1] proved that any affine commutative Noetherian algebra satisfies ACC($\text{Emb}_n$). An algebra $A$ is called weakly representable if $A$ can be embedded into $M_n(K)$ for some commutative ring $K$. Consequently, one has:

Theorem 2.12. Every weakly representable affine algebra over a commutative Noetherian ring is Hopfian.

Proof. Apply Lemma 2.3 to Amitsur’s theorem.

We can get information about the kernel of an epimorphism from the following observation.
**Lemma 2.13.** \( \varphi(J) \subseteq J \) for any epimorphism \( \varphi \) of \( A \), and thus \( \varphi \) induces an epimorphism \( \varphi' \) of \( A/J \), under any of the following conditions:

- \( J \) is the prime radical of \( A \).
- \( J \) is the upper nilradical of \( A \).
- \( J \) is the Jacobson radical of \( A \).

**Proof.** \( \varphi(J) \) is an ideal since \( \varphi \) is onto, and it is built up inductively from nilpotent (resp. nil, resp. quasi-invertible) ideals, so is contained in \( J \). But this means \( \varphi \) induces a homomorphism of \( A/J \) to itself, which clearly is onto. \( \square \)

**Proposition 2.14.** Under the hypotheses and notation of Lemma 2.13, if \( \varphi \) is any epimorphism of \( A \) with \( A/J \) Hopfian, then \( \ker \varphi \subseteq J \).

**Proof.** \( \ker \varphi/J = 0 \). \( \square \)

**Corollary 2.15.** If \( A \) satisfies ACC(semiprime ideals) and \( \varphi \) is an epimorphism of \( A \), then \( \ker \varphi \) is contained in the prime radical of \( J \).

### 2.1. Hopfian modules.

Ironically, we often get more with a stronger definition, by turning to modules.

**Definition 2.16.** An \( A \)-module \( M \) is **Hopfian** if every epimorphism \( M \to M \) is an isomorphism.

Note that this generalizes the definition for rings, since we can always take \( A \) to be an \( A^{\text{op}} \otimes A \) bimodule.

As above, for a given property designated \( P \), we say that a module \( M \) satisfies ACC(\( P \)) if any ascending chain of submodules \( N_1 \subseteq N_2 \subseteq \ldots \) for which each \( M/N_i \) has property \( P \), must stabilize. We have the following analog of Lemma 2.3.

**Lemma 2.17.** For any given property \( P \), if \( M \) has property \( P \) and every proper homomorphic image of \( M \) satisfies ACC(\( P \)), then \( M \) is Hopfian.

**Proof.** Otherwise take \( N_1 \) such that \( \varphi : M \cong M/N_1 \), and \( N_2 = \varphi^{-1}(N_1) \) (so that \( M \cong M/N_1 \cong M/N_2 \) and so forth), but \( N_1 \subset N_2 \subset \ldots \), a contradiction. \( \square \)

**Corollary 2.18.** Any Noetherian module is Hopfian.

**Proof.** Property \( P \) is vacuous. \( \square \)

In fact, any finitely generated module over a commutative ring is Hopfian, by a theorem of Vasconcelos [36], with a short proof given in [30].

### 3. Counterexamples

Many counterexamples are Hopf algebras, when they are built as group algebras or enveloping algebras of Lie algebras. This provides a method of showing that a group algebra or enveloping algebra does not satisfy the ACC on ideals.

#### 3.1. Hopfian and non-Hopfian groups.

Analogously to our earlier definitions, we have:

**Definition 3.1.** A group \( G \) is **Hopfian** if every group epimorphism \( G \to G \) is an isomorphism.
The free group is Hopfian, by [23, Theorem 2.13]. A broader class of examples is given in [23, Corollary 2.13.2]; and [23, p. 111] provides other examples dating back to B.H. Neumann [27]. An example of a finitely presented non-Hopfian group $G$ is given in [23, p. 260]: $G = \langle b, t; t^{-1}b^2t = b^3 \rangle$. (Indeed, defining $c = b^{-1}t^{-1}bt$ one mods out by the relation $b = c^2$ to get the image $H = \langle c, t; t^{-1}c^2t = c^3 \rangle$.)

Ollivier and Wise [28] embed any countable group into a non-Hopfian group which is the outer automorphism group of a group satisfying Kazhdan’s property (T), and a subgroup of a hyperbolic group. De Cornulier produces an example of a finitely presentable, non-Hopfian group with property (T) and infinite outer automorphism group (but not a subgroup of a hyperbolic group), by means of the following lemma:

**Lemma 3.2 ([13, Lemma 2.3]).** Let $G$ be a group and $Z$ a central subgroup. Any automorphism $\varphi$ of $G$ that induces a surjective, endomorphism of $G/Z$, whose kernel is $\varphi^{-1}(Z)/Z$.

**Example 3.3** (Cornulier [13, Theorem 3.1]). $I_m$ denotes the identity $m \times m$ matrix, each $A_{jk} \in M_{j,k}(R)$ and $B_{ij} \in SL_j(R)$.

(i) Let $G$ be the set of matrices over $R = \mathbb{Z}[\frac{1}{p}]$ of the form $\begin{pmatrix} I_{n_1} & A_{12} & A_{13} \\ 0 & B_{22} & A_{23} \\ 0 & 0 & I_{n_3} \end{pmatrix}$. The diagonal matrix $(pI_{n_1}, I_{n_2+n_3})$ acts by conjugation on $G$, and sends the center $Z = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}$ to the proper subset $\begin{pmatrix} I_{n_1} & 0 & pA_{13} \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}$, thereby yielding a non-injective surjection (whose kernel is finite). Hence $G/Z$ is non-Hopfian, and is seen in [13, Proposition 2.7] to be finitely generated.

(ii) For finite presentation De Cornulier needs to add a component. Let $G$ be the set of matrices over $R = \mathbb{Z}[\frac{1}{p}]$ of the form $\begin{pmatrix} I_{n_1} & A_{12} & A_{13} & A_{14} \\ 0 & B_{22} & A_{23} & A_{24} \\ 0 & 0 & B_{33} & A_{44} \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix}$. The diagonal matrix $(pI_{n_1}, I_{n_2+n_3+n_4})$ acts by conjugation on $G$, and sends the center $Z$ to a proper subset of $Z$, thereby yielding a non-injective surjection (whose kernel is finite). Hence $G/Z$ is non-Hopfian, and is seen in [13, Proposition 2.7] to be finitely generated. Finite presentation requires an argument using dominant weights, given in [13, Lemma 3.2].

3.1.1. **Algebra counterexamples arising from group theory.**

For any ring $R$, any group homomorphism $\varphi_N : G \to G/N$ extends naturally to a ring homomorphism $\tilde{\varphi} : R[G] \to R[G/N]$. Applying the reverse direction, we see that $\tilde{\varphi}$ is an isomorphism if and only if $\varphi_N$ is an isomorphism.

**Lemma 3.4.** If a group $G$ is non-Hopfian, then its group algebra $R[G]$ is non-Hopfian, for any ring $R$.

**Proof.** For the group homomorphism $\varphi : G \to G/N$ with $N \neq \{1\}$, the kernel of $\tilde{\varphi}$ is easily seen to contain all elements of the form $1 - g$ for $g \in N$ (and in fact is generated by them), so is nontrivial. $\square$
Thus Example 3.3 yields non-Hopfian group algebras. Passman and Small [31, Theorem 2.3] complete a result of Baumslag, showing that a group $G$ is finitely presented, iff the group algebra $R[G]$ is finitely presented, so the finitely presented (resp. non-finitely presented) group counterexamples give finitely presented (resp. non-finitely presented) non-Hopfian group algebras.

On the other hand, one way of obtaining Hopfian groups is to apply the group version of Lemma 2.3. Zelmanov [38, Theorems 2.3.2, 2.3.3, 2.3.5] shows that various ascending chain conditions terminate, and thus the corresponding kinds of groups are Hopfian.

### 3.2. Hopfian and non-Hopfian Lie algebras.

**Definition 3.5.** A Lie algebra $L$ is **Hopfian** if every Lie epimorphism $L \to L$ is an isomorphism.

#### 3.2.1. Counterexamples arising from Lie theory.

**Lemma 3.6.** If a Lie algebra $L$ is non-Hopfian, then its enveloping algebra $C[L]$ is non-Hopfian, for any integral domain $C$, and consequently $C[L]$ does not satisfy the ACC on prime ideals.

**Proof.** As for Lemma 3.4, since the kernel of a Lie homomorphism extends to the kernel of the induced homomorphism of enveloping algebras. □

For restricted Lie algebras, one can just take the Lie version ("i.e., logarithm") of Example 3.3 to get a non-Hopfian restricted enveloping algebra.

**Example 3.7.** $I_m$ denotes the identity $m \times m$ matrix, each $A_{jk} \in \mathfrak{gl}_{j,k}(R)$ and $B_{jj} \in \mathfrak{sli}_j(R)$.

(i) Let $L$ be the set of matrices over $R = \mathbb{Z}[\frac{1}{p}]$ of the form

\[
\begin{pmatrix}
0 & A_{12} & A_{13} \\
0 & B_{22} & A_{23} \\
0 & 0 & 0
\end{pmatrix}
\]

The diagonal matrix $(a^p, 0, 0, 0)$ acts by the adjoint action on $L$, and sends the center $Z = \begin{pmatrix} 0 & 0 & A_{13} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$ to a proper subset, thereby yielding a non-injective surjection (whose kernel is finite). Hence $L/Z$ is non-Hopfian.

(ii) For finite presentation one could follow Example 3.3(ii), but this would require implementing the theory of Lie weights as in [2, Theorem 6.4.3], which we have not yet verified.

Passman and Small [31, Lemma 3.2] give a semi-direct product modification $\mathcal{L}$ of Example 3.7(i), for which $\mathcal{L}/Z(\mathcal{L})$ is non-Hopfian (not finitely presented) for the analogous reason, taking the Lie version of Lemma 3.2. (The suitable adjoint map is an isomorphism with a nontrivial kernel on the center.)

#### 3.2.2. Hopf counterexamples.

Since group algebras and enveloping algebras are both examples of Hopf algebras, one might ask if there is a common Hopf algebra generalization, and indeed Passman and Small [31, § 5] present such a candidate which they call $\mathcal{R}$. Note that here $Z(\mathcal{R})$ is the component in the 1,3 position, which is an ideal.
Conjecture 3.8.

- The analog to Example 3.3(i) yields a semi-direct product \( R \) and an algebra \( R/Z(R) \) that is non-Hopfian in the sense of this paper.
- The analog \( H \) of Example 3.3(ii) yields a finitely presented Hopf algebra \( H/Z(H) \) that is non-Hopfian in the sense of this paper.

4. Bassian rings

Bass [6] strengthened the Hopfian property.

Definition 4.1. A ring \( A \) is **Bassian** if for any injection \( f : A \rightarrow A/J \) for \( J \triangleleft A \), we must have \( J = 0 \).

Lemma 4.2. Any Bassian ring is Hopfian.

Proof. Given an epimorphism \( \varphi : A \rightarrow A \) which is not injective, we take \( J = \ker \varphi \) and have an isomorphism \( A/J \rightarrow A \), implying \( J = 0 \). \(\square\)

In this way, Bassian is a natural generalization of Hopfian.

Lemma 4.3. Any Bassian ring satisfies the following property: If \( \varphi : A \rightarrow A \) is a homomorphism and \( J \triangleleft A \) with \( J \cap \ker \varphi = 0 \), then \( J = 0 \).

Proof. The composite of \( \varphi \) with the canonical map \( A \rightarrow A/J \) is an injection, so \( J = 0 \). \(\square\)

One gets instances of Bassian rings by means of growth.

Lemma 4.4. Suppose \( A \) is an affine algebra in which the growth of \( A/I \) is less than the growth of \( A \), for each ideal \( I \) of \( A \). Then \( A \) is Bassian.

In particular, if \( \text{GKdim}(A/I) < \text{GKdim}(A) \) for all ideals \( I \) of \( A \), then \( A \) is Bassian.

Proof. \( A \cong \varphi(A) \cong A/\ker \varphi \), but \( \varphi(A) \) and \( A \) have the same growth rates, implying \( \ker \varphi = 0 \). \(\square\)

The hypothesis of Lemma 4.4 holds for prime affine PI-algebras over a field, cf. [7, Theorem 11.2.12], so we have:

Corollary 4.5. Any prime affine PI-algebra over a field is Bassian.

Likewise, we have:

Corollary 4.6. Any algebra with just infinite GK-dimension is Bassian.

Proof. Again, Lemma 4.4 applies. \(\square\)

Corollary 4.7. Any prime affine domain of finite GK-dimension is Bassian.

Proof. It is Ore, by a result of Jategaonkar [35, Proposition 6.2.15], so any ideal \( I \) contains a regular element and \( \text{GKdim}(A/I) < \text{GKdim}(A) \) by [35, Proposition 6.2.24]. \(\square\)

Recall from [26] that an algebra is called an **almost PI-algebra** if \( A \) is not PI (polynomial identity) but \( A/I \) is PI for each nonzero ideal \( I \) of \( A \).

Likewise, an algebra over a field is **nearly finite dimensional** if \( A \) is not finite dimensional but \( A/I \) is finite dimensional for each nonzero ideal \( I \) of \( A \), [10].
Proposition 4.8. Any almost PI-algebra (resp. nearly finite dimensional algebra) is Bassian.

Proof. A is not PI but \( A/I \) is PI, so cannot be isomorphic to \( A \). Likewise for nearly finite dimensional algebras. \( \square \)

But here are examples of a different flavor altogether.

Lemma 4.9. If a group \( G \) is non-Bassian, then its group algebra \( R[G] \) is non-Bassian, for any ring \( R \).

If a Lie algebra is non-Bassian, then its enveloping algebra is non-Bassian.

Proof. The same as for Lemma 4.3. \( \square \)

Example 4.10. (i) Although Hopfian, the free algebra \( F_m := F \{ x_1, \ldots, x_m \} \) \((m \leq 3)\) is far from Bassian. Indeed, it is well-known that \( F_2 \) contains the free algebra as generated by \( \{ \hat{x}_k = x_1 x_k^2 x_1 : k \geq 1 \} \), so for \( J = (x_k : k \geq 3) \), the map \( \phi : F_k \to F_k/J \) given by \( x_k \mapsto \hat{x}_k \) is injective.

(ii) The Lie algebra \( W \) from the proof of [31] is non-Bassian.

Lemma 4.11. If \( A \) satisfies the ACC on semiprime ideals, then there is an ideal \( J \triangleleft A \) maximal for which we have an embedding \( f : A \to A/J \), and that \( J \) is a semiprime ideal.

Proof. Take a semiprime ideal \( J_0 \triangleleft A \) maximal for semiprime ideals for which we have an embedding \( f : A \to A/J_0 \). We claim that \( J = J_0 \). Otherwise there is some \( J_1 \triangleleft A \) properly containing \( J_0 \), maximal for which we have an embedding \( f : A \to A/J_1 \). The composite \( \phi : A \xrightarrow{f} A/J_1 \to A/\sqrt{J_1} \) is injective. Indeed, \( \sqrt{J_1}/J_1 \), if not 0, contains a nonzero nilpotent ideal of \( A/J_1 \), which thus intersects \( f(A) \) at a nilpotent ideal of \( f(A) \cong A \) which is thus 0. But then

\[
A \xrightarrow{f} A/J_1 \to A/\sqrt{J_1}
\]

is injective, implying that we could replace \( J_1 \) by \( \sqrt{J_1} \), which is semiprime, so by maximality of \( J_0 \), we see that \( \sqrt{J_1} = J_0 \), a contradiction. \( \square \)

Bass [6, Theorem 1.1] proved that if \( A \) is commutative reduced and finitely presented, then it is Bassian. The main result of this section is a noncommutative generalization.

Theorem 4.12. Any semiprime affine PI-algebra \( A \) over a field \( K \) is Bassian.

Proof. We recall some well-known basic facts about a semiprime affine PI-algebra \( A \) over a field, taken from [31].

- \( A \) satisfies the ACC on semiprime ideals, and thus, \( A \) has a finite number of minimal prime ideals \( P_1 \cap \cdots \cap P_\ell \), whose intersection is 0.
- \( A \) is Goldie, and its ring of fractions \( Q(A) \) is obtained by inverting regular central elements, and

\[
Q(A) \cong A_1 \times \cdots \times A_\ell
\]

where \( A_i = Q(A)/M_i = Q(A/P_i) \) are the simple components, with \( M_i \) a maximal ideal of \( Q(A) \).
• If $J$ is an annihilator left ideal of $A$, then $Q(A)J$ is an annihilator left ideal of $Q(A)$, and taking right annihilators yields a 1:1 correspondence between left and right annihilators.

We follow Bass’ proof, using the PI-structure theory where appropriate. Given an injection $A \to A/J$, we want to show that $J = 0$. By Lemma \[\text{4.11}\] we may assume that $J$ is semiprime, since $A$ satisfies the ACC on semiprime ideals. Hence, for some $t$,

$$J = \sqrt{J} = P_1^t \cap \cdots \cap P_t^t,$$

an irredundant intersection of prime ideals of $A$, which thus are minimal prime over $J$. But furthermore $J$ cannot contain any regular elements since $\text{GKdim } A/J \geq \text{GKdim } A$. Hence $\sqrt{J} := JZ(Q(A))$ is a proper ideal of $Q(A)$ and thus has some nonzero left and right annihilator ideal $\hat{J}$ in $Q(A)$, which is precisely the sum of those simple components of $Q(A)$ that miss $J$. Let $\hat{J} = \text{Ann}_{Q(A)} \hat{J} = \text{Ann}_{Q(A)} J$. (This is the same if we take the annihilator from the left or right.) Then $J' := A \cap \hat{J}$ is $\text{Ann}_A J$, so is also a semiprime ideal and $J = \text{Ann}_A J'$. Also note that $\hat{J}$ annihilates $J'$, so $A \cap \hat{J} = J$.

Reordering the $P_i$, we may assume that $\sqrt{J} = \bigcap_{i=1}^t P_i^t$, and $\sqrt{J'} = \bigcap_{i=j+1}^t P_i^t$, for suitable $j$. Then $\bigcap_{i=1}^t P_i^t$ is radical over $A \cap \hat{J} = J$.

Let $P_i = A \cap P_i^t$, $1 \leq i \leq t$, which are prime ideals of $A$. Then $A \cap \hat{J} = \bigcap_{i=1}^t P_i = J$. It follows that $A \hookrightarrow A/P_1 \times \cdots \times A/P_t$.

We consider the function $\ell(R)$ for the length of a maximal chain of left annihilator ideals of a ring $R$. It is well-known that $\ell(W) \leq \ell(R)$ for any subring $W$ of $R$, since any chain $\text{Ann}_R(S_1) \subset \text{Ann}_R(S_1) \subset \cdots$ lifts to a chain $\text{Ann}_W(S_1) \subset \text{Ann}_W(S_1) \subset \cdots$ (which will intersect back to the original chain). In particular, noting that

$$A \hookrightarrow f(A) \subseteq A/J \hookrightarrow Q(A/J) \cong A/P_1 \times \cdots \times A/P_j,$$

implying $\ell(A) \leq \sum_{i=1}^j \ell(A/P_i)$.

On the other hand, $\bigcap_{i=1}^t P_i$ is nilpotent modulo $J \cap J' = 0$, implying $\bigcap_{i=1}^t P_i = 0$, so

$$\ell(A) = \ell(A/P_1 \times \cdots \times A/P_t) = \sum_{i=1}^t \ell(A/P_i).$$

Together we get $\sum_{i=1}^t \ell(A/P_i) \leq \sum_{i=1}^j \ell(A/P_i)$, implying $j = t$. Hence $J = 0$, as desired.

Reviewing the proof, we can make do with the following hypotheses. We leave the details to the reader.

• $A$ satisfies ACC(semiprime ideals).
• $A$ is Goldie.
• There is some dimension function such that $\dim A \leq \dim A'$ whenever $A \subseteq A'$, and any semiprime ideal of $A$ lifts up to a proper ideal of $Q(A)$ (i.e., does not contain regular elements).

The last condition is rather technical. In order to guarantee it, Bass used transcendence degree, and we used GK dimension in Theorem \[\text{4.12}\] but we are not familiar with other dimensions satisfying these properties, so we settle for the following result:
Theorem 4.13. Any semiprime affine Goldie algebra satisfying ACC(semiprime ideals) and of finite GK dimension is Bassian.

Proof. (sketch) $A$ has a finite number of minimal prime ideals $P_1 \cap \cdots \cap P_m$, whose intersection is 0. The ring of fractions $Q(A)$ satisfies

$$Q(A) \cong A_1 \times \cdots \times A_m$$

where $A_i = Q(A)/M_i = Q(A/P_i)$ are the simple components, with $M_i$ a maximal ideal of $Q(A)$.

Assume that $A$ is embedded in $A/J$. $J$ cannot contain any regular elements since $\text{GKdim } A/J \geq \text{GKdim } A$. Hence $J$ is contained in a proper ideal of $Q(A)$ which, being semisimple, satisfies the ACC on annihilator ideals. A Faith-Utumi type argument \[15\] shows that $\text{Ann}_{Q(A)} J$ is the sum of those components of $Q(A)$ disjoint from $J$, and they intersect to $\text{Ann}_A J$.

Now we follow the previous proof. □

Unfortunately, the most familiar examples satisfying these hypotheses fit into the previous known results: Affine semiprime PI-algebras, and enveloping algebras of finite dimensional Lie algebras (since they are Noetherian).

5. Open questions

The following questions are related to the Hopfian and Bassian properties.

- What condition on an algebra $A$ satisfying ACC(prime ideals) makes it Hopfian? (Being affine PI is not enough, in view of Remark \[22\].) Is it enough that $A$ is finitely presented?
- Does the enveloping algebra of the centerless Virasoro algebra satisfy ACC(prime ideals)\[1\]? ACC(semiprime ideals)? ACC(ideals)?
- Is the enveloping algebra of the centerless Virasoro algebra Bassian? Hopfian? \[2\]

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\[1\] Now proved for completely prime ideals by Iyudu, N.K. and Sierra \[19\] Proposition 6.4
\[2\] Now proved along with the positive Witt algebra, and also the Virasoro algebra being Hopfian, by Iyudu, N.K. and Sierra \[19\] Proposition 6.5
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