RIGIDITY OF ENTIRE CONVEX SELF-SHRINKING SOLUTIONS TO HESSIAN QUOTIENT FLOWS

WENLONG WANG

Abstract. We prove that all entire smooth strictly convex self-shrinking solutions on $\mathbb{R}^n$ to the Hessian quotient flows must be quadratic. This generalizes the rigidity theorem for entire self-shrinking solutions to the Lagrangian mean curvature flow in pseudo-Euclidean space due to Ding-Xin [5]. Moreover, we show that our argument works for a larger class of equations. In particular, we obtain rigidity results for entire self-shrinking solutions on $\mathbb{C}^n$ to the Kähler-Ricci flow under certain conditions.

1. Introduction

For a $n$-dimensional symmetric matrix $B$, let $\lambda = (\lambda_1, ..., \lambda_n)$ denote the eigenvalues of $B$. Let $\sigma_l(B)$ be the $l$-th elementary symmetric polynomial of $\lambda$ given by

$$\sigma_l (B) = \sum_{i_1 < ... < i_l} \lambda_{i_1} \cdots \lambda_{i_l} \quad (1 \leq l \leq n);$$

$$\sigma_0 (B) = 1.$$ 

We say $B$ is $k$-positive if $\sigma_l(B) > 0$ for all $0 \leq l \leq k$. Let $0 \leq n_2 < n_1 \leq n$, for any $n_1$-positive matrix $B$, we define the quotient $q_{n_1,n_2}(B)$ by

$$q_{n_1,n_2}(B) = \frac{\sigma_{n_1}(B)}{\sigma_{n_2}(B)}.$$ 

In the present paper, we prove the following main theorem.

Theorem 1.1. Let $u$ be an entire smooth strictly convex solution on $\mathbb{R}^n$ to the Hessian quotient equation

$$\ln q_{n_1,n_2}(D^2u(x)) = \frac{1}{2} x \cdot Du(x) - u(x).$$

Then $u$ is quadratic.

Any solution to (1.1) leads to an entire self-shrinking solution

$$v(x,t) = -tu \left( \frac{x}{\sqrt{-t}} \right).$$

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to a parabolic Hessian quotient equation
\[(1.2) \quad v_t = \ln q_{n_1,n_2} \left(D^2v\right)\]
on \(\mathbb{R}^n \times (-\infty, 0)\). In [12], Trudinger and Wang used this flow under the fixed boundary condition to study a Poincaré type inequality for Hessian integrals (see [13] for the Monge-Ampère integral). In fact, (1.2) is the negative logarithmic gradient flow of the following functional (cf. [13, 15])
\[I_{n_1,n_2}(u) = \frac{1}{n_1 + 1} \int (-u) \cdot \sigma_{n_1}(D^2u) - \frac{1}{n_2 + 1} \int (-u) \cdot \sigma_{n_2}(D^2u).\]
When \(n_1 = n, n_2 = 0\), (1.1) becomes the Monge-Ampère equation
\[(1.3) \quad \ln \det D^2u(x) = \frac{1}{2} x \cdot Du(x) - u(x).\]
Any solution to (1.3) leads to an entire self-shrinking solution
\[v(x,t) = -tu \left(\frac{x}{\sqrt{-t}}\right)\]
to a parabolic Monge-Ampère equation
\[(1.2) \quad v_t = \ln \det D^2v\]
on \(\mathbb{R}^n \times (-\infty, 0)\) and the family of embeddings \(F(x,t) = (x, Dv(x,t))\) from \(\mathbb{R}^n\) into \(\mathbb{R}^{2n}\) solves the mean curvature flow with respect to the pseudo-Euclidean background metric \(ds^2 = \sum_{i=1}^{n} dx^i dy^i\) on \(\mathbb{R}^{2n}\) (cf. [4, 8, 10, 11]).
Rigidity of entire smooth convex solutions to (1.3) has been studied in [3, 5, 8, 9]. In [3] and [9], the authors proved that any smooth convex solution to (1.3) must be quadratic under the condition that the Hessian is bounded below inversely quadratically. Later in [5], Ding-Xin gave a complete improvement by dropping additional assumptions.

The common part of the arguments in [3, 5] and here is proving the constancy of a natural quantity, the phase \(\phi = \ln \det D^2u\) (\(\phi = \ln q_{n_1,n_2}(D^2u)\) in the Hessian quotient case). Then the homogeneity of the self-similar term on the right-hand side of the equation leads to the quadratic conclusion. The phase satisfies an elliptic equation without zeroth order term (shown below in (2.21)). In [3], using the inversely quadratic decay assumption, Chau-Chen-Yuan constructed a specific barrier function to force the supremum of the phase in \(\mathbb{R}^n\) to be attained at some point. Then the strong maximum principle implies the constancy of the phase. In [5], Ding-Xin first obtained the properness of \(u\), then proved the constancy of the phase via the integral method.

Our approach is to construct a barrier function to force the supremum of the phase to be attained at some point. However, we cannot construct a specific barrier function as in [3], which requires the specific decay rate of the Hessian. We turn to estimate the growths of the solution \(u\) and \(|Du|\), then construct a non-concrete barrier function. To begin with, we establish a second order ordinary differential inequality for the spherical mean of \(u\), a univariate function depending on the radius of the sphere. Then using
some ODE techniques, we prove that the spherical mean of $u$ has at most a quadratic growth and the ball mean of $\Delta u$ is bounded. Combining these with the convexity of $u$, we obtain that $u$ has at most a quadratic growth, $|Du|$ has at most a linear growth and the negative part of $u$ has a sublinear growth. Having these estimates, we finally construct a suitable barrier function based on $u$ and $\phi$.

In fact, our argument for Theorem 1.1 does not depend on the particular structure of (1.1). This enables us to generalize the rigidity result to a larger class of equations.

Let $\mathcal{S}_n^+$ be the cone of $n$-dimensional positive-definite matrices. Let $F$ be a $C^1$ function defined on $\mathcal{S}_n^+$. For any $B = (b_{ij}) \in \mathcal{S}_n^+$, define the coefficient matrix $DF$ by

$$(DF)^{ij}(B) = \frac{\partial F}{\partial b_{ij}}(B).$$

**Theorem 1.2.** Assume for any $B \in \mathcal{S}_n^+$, $F$ satisfies the following conditions:

(i) $DF(B)$ is positive-definite;

(ii) $\exp F(B) \leq C \left[ (\text{tr } B)^{k_1} + 1 \right]$ for certain positive constants $k_1$ and $C$.

(iii) $\|DF(B) \cdot B\| \leq k_2$ for a certain positive constant $k_2$.

Let $u$ be an entire smooth strictly convex solution on $\mathbb{R}^n$ to the equation

$$(1.4) \quad F(D^2 u(x)) = \frac{1}{2}x \cdot Du(x) - u(x).$$

Then $u$ is quadratic.

Condition (i) guarantees the ellipticity of (1.4). Conditions (ii) and (iii) say that (1.4) has exponential or super-exponential nonlinearity for the quadratic self-similar term on the right-hand side in a sense. We are about to show that some common operators satisfy above conditions.

Let us first verify that $\ln q_{n_1,n_2}$ satisfies these conditions. For condition (i), $DF(B)$ is positive-definite when $B$ is $n_1$-positive. Namely, equation (1.1) is elliptic when $u$ is $n_1$-admissible (cf. [11, 12, 15]). Since $u$ is strictly convex, it is $n_1$-admissible. We can also check condition (i) directly by diagonalizing $B$ and using Newton’s inequality (cf. [7]).

For condition (ii), also by Newton’s inequality we have

$$(1.5) \quad q_{n_1,n_2}(B) \leq C(n,n_1,n_2) (\text{tr } B)^{n_1-n_2}.$$ 

For condition (iii), since $q_{n_1,n_2}(B)$ is a homogeneous order $n_1-n_2$ function of $B$, by Euler’s homogeneous function theorem we have

$$(1.6) \quad \text{tr } (D \ln q_{n_1,n_2}(B) \cdot B) = n_1 - n_2.$$ 

Because $\ln q_{n_1,n_2}$ is invariant under orthogonal transformations, $D \ln q_{n_1,n_2}(B)$ and $B$ can be diagonalized simultaneously. Thus $D \ln q_{n_1,n_2}(B)$ commutes with $B$. Then $D \ln q_{n_1,n_2}(B) \cdot B$ is positive-definite. Consequently,

$$(1.7) \quad \|D \ln q_{n_1,n_2}(B) \cdot B\| < \text{tr } (D \ln q_{n_1,n_2}(B) \cdot B) = n_1 - n_2.$$
We can verify that the operator \( \text{tr} (\arctan B) \) also satisfies above three conditions. The corresponding equation

\[
\sum_{i=1}^{n} \arctan \lambda_i (x) = \frac{1}{2} x \cdot Du (x) - u (x)
\]

describes the potential of the self-shrinking solution \((x, Du(x))\) to the Lagrangian mean curvature flow in \(\mathbb{R}^{2n}\) (cf. [2, 3, 4, 5, 8, 9, 10, 11]). In [3], Chau-Chen-Yuan first proved that any entire smooth solution to (1.8) on \(\mathbb{R}^{n}\) must be quadratic.

The Hermitian counterpart of (1.3) is the following complex Monge-Ampère equation

\[
\ln \det \partial \bar{\partial} u (x) = \frac{1}{2} x \cdot Du (x) - u (x).
\]

Any solution to (1.9) leads to an entire self-shrinking solution

\[
v (x, t) = -tu \left( \frac{x}{\sqrt{-t}} \right)
\]
to a parabolic complex Monge-Ampère equation

\[v_t = \ln \det \partial \bar{\partial} v\]
on \(\mathbb{C}^{n} \times (-\infty, 0)\). Note that the above equation of \(v\) is the potential equation of the Kähler-Ricci flow \(\partial_t g_{\alpha \bar{\beta}} = -R_{\alpha \bar{\beta}}\). In fact, the corresponding metric \((u_{\alpha \bar{\beta}})\) is a shrinking Kähler-Ricci (non-gradient) soliton (cf. [3]).

Rigidity of entire solutions to (1.9) has been studied in [3, 5, 6, 14]. In [6], Drugan-Lu-Yuan proved that any complete (with respect to the corresponding Kähler metric \(\partial \bar{\partial} u\)) solution has to be quadratic. In [14], completeness assumption is removed for complex one dimensional case. Using our argument, we can obtain two new rigidity theorems which are described now.

We know

\[
\det \partial \bar{\partial} u = 4^{-n} \sqrt{\det (D^2 u + J^T \cdot D^2 u \cdot J)},
\]

where \(J\) denotes the standard complex structure of \(\mathbb{R}^{2n}\) and \(J^T\) is the transpose of \(J\) with \(J^T = -J\). Accordingly, the “complex determinant” operator \(\det_{J}\) for \(B \in \mathcal{S}_+^{2n}\) is defined by

\[
\det_{J} B = 4^{-n} \sqrt{\det (B - JBJ)}.
\]

Let us verify that \(\ln \det_{J}\) satisfies conditions (i) and (ii). For condition (i), we have

\[
D (\ln \det_{J} B) = (B - JBJ)^{-1}.
\]

Since \(B > 0\), we have \(B - JBJ > 0\). Then \(D (\ln \det_{J} B)\) is positive-definite. Actually, \(D (\ln \det_{J} \partial \bar{\partial} u)\) is a quarter of the real representation of \((\partial \bar{\partial} u)^{-1}\). Equation (1.9) is elliptic if and only if \(u\) is pluri-subharmonic. Since \(u\) is
strictly convex, it is pluri-subharmonic. For condition (ii), by the arithmetic mean-geometric mean inequality we have

\[
\det J B \leq 4^{-n} \left( \frac{1}{2n} \operatorname{tr} (B - J \cdot B \cdot J) \right)^n = \frac{1}{(4n)^n} (\operatorname{tr} B)^n.
\]

Because \( \det J B \) is a homogeneous order \( n \) function of \( B \), by Euler’s homogeneous function theorem we have \( \operatorname{tr} (D \ln \det J B \cdot B) = n \). However, \( (B - J B)^{-1} \) and \( B \) do not commute in general. So \( \ln \det J \) does not satisfy condition (iii), our method is not suitable to a general convex function \( u \). But if \( u \) satisfies one of the following conditions, the rigidity theorem still holds.

**Definition 1.** For a pluri-subharmonic function \( u \) on \( \mathbb{C}^n \), we say the eigenvalues of \( \partial \bar{\partial} u \) are comparable, if there is a constant \( \Lambda \geq 1 \) such that

\[
\mu_{\max}(x) \leq \Lambda \mu_{\min}(x) \quad \text{for any } x \in \mathbb{C}^n,
\]

where \( \mu_{\max}(x) \) and \( \mu_{\min}(x) \) are the largest and the smallest eigenvalues of \( \partial \bar{\partial} u(x) \) respectively.

**Definition 2.** A function \( u \) on \( \mathbb{C}^n \) is called toric if

\[
u(z^1, ..., z^n) = u(e^{\sqrt{-1} t^1} z^1, ..., e^{\sqrt{-1} t^n} z^n) \quad \text{for any } (t^1, ..., t^n) \in \mathbb{R}^n.
\]

**Theorem 1.3.** Let \( u \) be an entire smooth strictly convex solution on \( \mathbb{C}^n \) to (1.9). Assume the eigenvalues of \( \partial \bar{\partial} u \) are comparable. Then \( u \) is quadratic.

**Theorem 1.4.** Let \( u \) be an entire smooth convex solution on \( \mathbb{C}^n \) to (1.9). Assume \( u \) is toric. Then \( u \) is quadratic.

Equation (1.1) has a relationship with Legendre transformation (cf. [9]). Suppose that \( u \) is a strictly convex solution to (1.1), then the Legendre transform of \( u \) denoted by \( u^* \) satisfies

\[
\ln q_{n-n_2,n-n_1} (D^2 u^*) = \frac{1}{2} x \cdot Du^* - u^*.
\]

In particular, when \( n_1 + n_2 = n \), (1.1) is invariant under Legendre transformation. Taking advantage of this relation, we have the following theorem.

**Theorem 1.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain. Assume \( u \) is a smooth strictly convex solution to (1.1) in \( \Omega \). Then \( u \) is improper.

**Proof.** We proceed by contradiction. Assume \( u \) is proper, then \( u^* \) is an entire smooth strictly convex solution on \( \mathbb{R}^n \) to (1.12). According to Theorem 1.1, \( u^* \) is quadratic. By the property of Legendre transform, \( u \) is also quadratic. Since a quadratic function cannot be proper in a bounded domain, the assumption is not true. Therefore \( u \) is improper. \( \square \)

Although Theorem 1.1 is a special case of Theorem 1.2, its proof is more original and explicit. And readers can get the proof of Theorem 1.2 from the proof of Theorem 1.1 easily with only change of symbols and constants.
So we only prove Theorem 1.1 in the following. For the proof of Theorem 1.3 and Theorem 1.4, we skip the common part with Theorem 1.1 and only talk about the difference.

2. Proof of Theorem 1.1

To get appropriate estimates for the solution \( u \), we establish four lemmas. In these lemmas, we prove that if \( u \) satisfies the conditions of Theorem 1.2, then \( u \) has at most a quadratic growth, \( |Du| \) has at most a linear growth and \( u^- \) has a sublinear growth.

In the first two lemmas, we derive a second order ordinary differential inequality for the spherical mean of \( \tilde{u} \), where \( \tilde{u} \) is related to \( u \) by a simple linear transform (shown below in (2.10)). Then we prove that the spherical mean of \( \tilde{u} \) has at most a quadratic growth and this property is passed on to \( u \).

**Definition 3.** For a \( C^2 \) function \( h(x) \) on \( \mathbb{R}^n \), define

(i) the spherical mean of \( h \) by

\[
S_h(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} h(x) \, dS_x,
\]

and

(ii) the ball mean of \( \Delta h \) by

\[
\Phi_h(r) = \frac{n}{\omega_n r^n} \int_{B_r(0)} \Delta h(x) \, dx,
\]

where \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

**Lemma 2.1.** Let \( h(x) \) be a \( C^2 \) function on \( \mathbb{R}^n \) satisfying

\[
\Delta h(x) \geq \exp \left[ x \cdot Dh(x) - 2h(x) \right].
\]

Then \( S_h(r) \) has at most a quadratic growth and \( \Phi_h(r) \) is bounded.

**Proof.** First of all, we derive a differential inequality for \( S_h \). By definition,

\[
S_h(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} h(r \omega) \, d\omega.
\]

Taking one derivative, we have

\[
S'_h(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} h_r(r \omega) \, d\omega.
\]

Multiplying \( r^{n-1} \) on both sides of (2.2) and using Stokes’s formula,

\[
r^{n-1} S'_h(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} h_r(r \omega) r^{n-1} \, d\omega
\]

\[
= \frac{1}{\omega_n} \int_{\partial B_r(0)} \frac{\partial h}{\partial \nu}(x) \, dS_x
\]

\[
= \frac{1}{\omega_n} \int_{B_r(0)} \Delta h(x) \, dx.
\]
A differentiation of (2.3) yields
\[
\left[r^{n-1}S'_h(r)\right]' = \frac{1}{\omega_n} \int_{\partial B_r(0)} \Delta h(x) \, dS_x.
\]

Dividing both sides of above equation by \(r^{n-1}\) and using (2.1), we get
\[
S''_h(r) + \frac{n-1}{r} S'_h(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} \Delta h(x) \, dS_x
\geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} \exp \left[ x \cdot Dh(x) - 2h(x) \right] \, dS_x.
\]

By Jensen’s inequality we obtain
\[
\frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} \exp \left[ x \cdot Dh(x) - 2h(x) \right] \, dS_x
\geq \exp \left\{ \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} \left[ x \cdot Dh(x) - 2h(x) \right] \, dS_x \right\}
= \exp \left[ rS'_h(r) - 2S_h(r) \right].
\]

Thus \(S'_h\) satisfies the following second order ordinary differential inequality
\[
(2.4) \quad S''_h(r) + \frac{n-1}{r} S'_h(r) \geq \exp \left[ rS'_h(r) - 2S_h(r) \right].
\]

Then we analyze above ordinary differential inequality. From (2.3) we see \(S'_h(r) > 0\) for \(r > 0\). Define an auxiliary function \(p(r)\) by
\[
p(r) = rS'_h(r) - 3S_h(r).
\]

We claim \(p(r) < 0\) when \(r \geq 4n\). Otherwise, there exists \(r_1 \geq 4n\) such that \(p(r_1) \geq 0\). Define \(r_2 = \sup\{t \mid p(r) \geq 0, r \in [r_1, t]\}\). If \(r_2 < +\infty\), then
\[
p'(r_2) = r_2 S''_h(r_2) - 2S'_h(r_2)
\geq r_2 \left\{ \exp \left[ r_2 S'_h(r_2) - 2S_h(r_2) \right] - \frac{n-1}{r_2} S'_h(r_2) \right\} - 2S'_h(r_2)
= r_2 \exp \left[ \frac{1}{3} r_2 S'_h(r_2) \right] - (n+1)S'_h(r_2)
\geq \left( \frac{r_2^2}{3} - n - 1 \right) S'_h(r_2) > 0.
\]
This contradicts the definition of $r_2$. Therefore $p(r) \geq 0$ holds on $[r_1, +\infty)$. Thus
\[
S_h''(r) \geq \exp \left[ r S_h'(r) - 2S_h(r) \right] - \frac{n-1}{r} S_h'(r)
\geq \exp \left[ \frac{1}{3} r S_h'(r) \right] - \frac{n-1}{r} S_h'(r)
> \frac{1}{2} \exp S_h'(r).
\]
By Osgood’s criterion, $S_h'(r)$ blows up in finite time, which contradicts the assumption that $h$ is entire. So the claim is true. For $r \geq 4n$ we have
\[
(2.5) \quad \frac{S_h'(r)}{S_h(r)} < \frac{3}{r}.
\]
Integrating (2.5), we get
\[
S_h(r) < \frac{S_h(4n)}{64 n^3} r^3 \quad \text{for } r \geq 4n.
\]
Substituting above inequality into (2.5), we obtain
\[
(2.6) \quad S_h'(r) < c_1 r^2 \quad \text{for } r \geq 4n,
\]
where $c_1 = S_h'(4n)$.

Now we have proved $S_h$ has at most a cubic growth. To get a finer estimate, we introduce another auxiliary function $q(r)$ given by
\[
q(r) = r S_h'(r) - 2S_h(r) - r.
\]
We claim $q(r) < 0$ when $r \geq n(c_1 + 4)$. The proof is similar. If the claim is not true, then there exists $r_3 \geq n(c_1 + 4)$ such that $q(r_3) \geq 0$. Define
\[
r_4 = \sup \{ t | q(r) \geq 0, r \in [r_3, t] \}.
\]
If $r_4 < +\infty$, then
\[
q'(r_4) = r_4 S_h''(r_4) - S_h'(r_4) - 1 \geq r_4 \left\{ \exp \left[ r_4 S_h'(r_4) - 2S_h(r_4) \right] - \frac{n-1}{r_4} S_h'(r_4) \right\} - S_h'(r_4) - 1
= r_4 \exp r_4 - n S_h'(r_4) - 1 \geq r_4 \exp r_4 - nc_1 r_4 - 1 > 0.
\]
This contradicts the definition of $r_4$. Hence $r S_h'(r) - 2S_h(r) \geq r$ holds on $[r_3, +\infty)$. It follows that
\[
S_h''(r) \geq \exp \left[ r S_h'(r) - 2S_h(r) \right] - \frac{n-1}{r} S_h'(r)
> \exp r - nc_1 r.
\]
Thus $S_h'(r)$ has an exponential growth as $r \to +\infty$, which contradicts (2.6).

Consequently, we have $r S_h'(r) - 2S_h(r) < r$ for $r \geq n(c_1 + 4)$. Or equivalently,
\[
\left[ \frac{S_h(r)}{r^2} \right]' < \frac{1}{r^2}.
\]
Integrating above inequality, we see when $r \geq 1$,

\[
\frac{S_h(r)}{r^2} < S_h(c_1 n + 4n) + 1.
\]

Clearly $S_h$ has at most a quadratic growth. According to (2.3),

\[
\Phi_h(r) = \frac{nS_h'(r)}{r}.
\]

Combining (2.5), (2.7) and (2.8), we conclude that $\Phi_h(r)$ is bounded. \[\square\]

**Lemma 2.2.** Let $u$ be as stated in Theorem 1.2. Then $S_u(r)$ has at most a quadratic growth, and $\Phi_u(r)$ is bounded.

**Proof.** According to condition (ii) and (1.4), we have

\[
C \left[ (\Delta u)^{k_1} + 1 \right] \geq \exp \left( \frac{1}{2} x \cdot Du - u \right).
\]

Since $u$ is strictly convex, $\Delta u > 0$. If $k_1 \geq 1$, we have

\[
(\Delta u + 1)^{k_1} \geq (\Delta u)^{k_1} + 1 \geq \exp \left( \frac{1}{2} x \cdot D u - u - \ln C \right).
\]

Set

\[
\hat{u}(x) = \frac{1}{2k_1} \left[ u(x) + \frac{1}{2n} |x|^2 + k_1 \ln 2k_1 + \ln C \right].
\]

Then it follows that

\[
\Delta \hat{u}(x) = \frac{1}{2k_1} [\Delta u(x) + 1]
\]

\[
\geq \frac{1}{2k_1} \exp \frac{1}{2k_1} \left( x \cdot Du - 2u - 2 \ln C \right)
\]

\[
= \exp \left( x \cdot D \hat{u} - 2\hat{u} \right).
\]

According to Lemma 2.1, $S_{\hat{u}}(r)$ has at most a quadratic growth, and $\Phi_{\hat{u}}(r)$ is bounded. Since we have the following relations

\[
S_u(r) = 2k_1 S_{\hat{u}}(r) - \frac{1}{2n} r^2 - k_1 \ln 2k_1 - \ln C
\]

and

\[
\Phi_u(r) = 2k_1 \Phi_{\hat{u}}(r) - 1,
\]

we conclude that $S_u(r)$ has at most a quadratic growth and $\Phi_u(r)$ is bounded. For the case $k_1 < 1$, we have

\[
2C (\Delta u + 1) > C \left[ (\Delta u)^{k_1} + 1 \right] \geq \exp \left( \frac{1}{2} x \cdot Du - u \right).
\]

In a very similar manner, we also draw the conclusion. \[\square\]

For a convex function, once we know the growth of its spherical mean, we know the growth of itself as well as its gradient.
Lemma 2.3. Let \( h(x) \) be a \( C^1 \) convex function on \( \mathbb{R}^n \). Assume that \( S_h(r) \) has at most a quadratic growth. Then \( h(x) \) has at most a quadratic growth, and \( |Dh(x)| \) has at most a linear growth.

Proof. By the assumption, there exist positive constants \( A \) and \( B \) such that

\[
S_h(r) \leq Ar^2 + B \quad \text{for all } r \geq 0.
\]

Since \( h \) is convex, there exist positive constants \( A' \) and \( B' \) such that

\[
h(x) + A'|x|^2 + B' \geq 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

As \( h(x) + A'|x|^2 + B' \) is subharmonic, it satisfies mean value inequality. Then it follows that

\[
h(x) + A'|x|^2 + B' \leq \frac{n}{\omega_n|x|^n} \int_{B_2(x)} \left[ h(y) + A'|y|^2 + B' \right] \, dy
\]

\[
\leq \frac{n}{\omega_n|x|^n} \int_{B_2(0)} \left[ h(y) + A'|y|^2 + B' \right] \, dy
\]

\[
= \frac{n}{\omega_n|x|^n} \int_0^{2|x|} \int_{\partial B_t(0)} \left[ h(z) + A'|z|^2 + B' \right] \, dS_z \, dt
\]

\[
\leq \frac{n}{\omega_n|x|^n} \int_0^{2|x|} \omega_n [(A + A')t^2 + (B + B')] t^{n-1} \, dt
\]

\[
\leq 2^n (A + A')|x|^2 + 2^n (B + B').
\]

The first inequality of (2.14) holds by the mean value inequality. The second one holds because of (2.13). And the third one holds due to (2.12).

Hence \( h^+(x) = \max\{h(x), 0\} \) has at most a quadratic growth. Since \( h(x) \) is convex, \( h^-(x) = \max\{-h(x), 0\} \) has at most a linear growth. In conclusion, \( h \) has at most a quadratic growth.

Attributable to the convexity of \( h \), for an arbitrary unit vector \( \xi \in \mathbb{R}^n \) we have

\[
\xi \cdot Dh(x) \leq \frac{h(x + |x|\xi) - h(x)}{|x|}.
\]

This implies that \( |Dh(x)| \) has at most a linear growth. \( \square \)

Suppose that \( u \) satisfies the conditions of Theorem 1.2. According to Lemma 2.2, \( S_u(r) \) has at most a quadratic growth and \( \Phi_u(r) \) is bounded. Then by Lemma 2.3 \( u(x) \) has at most a quadratic growth and \( |Du(x)| \) has at most a linear growth. The next lemma states that \( u^-(x) \) grows sublinearly.

Lemma 2.4. Let \( h(x) \) be a \( C^1 \) convex function on \( \mathbb{R}^n \). Suppose that \( S_h(r) \) has at most a quadratic growth. And assume that for a certain positive constant \( \alpha \), the ball mean of \( \exp \alpha(x \cdot Dh - 2h) \) is bounded. Then

\[
\lim_{|x| \to +\infty} \frac{h^-(x)}{\sqrt{|x|}} = 0.
\]
Proof. We proceed by contradiction. If the proposition is not true, then there exist a sequence \( \{x_i\}_{i=1}^{\infty} \subset \mathbb{R}^n \) and a positive constant \( c_2 \) such that

\[
h^-(x_i) \geq 3c_2|x_i|^\frac{3}{2}, \quad \text{and} \quad \lim_{i \to +\infty} |x_i| = \infty.
\]

According to Lemma 2.3, \( |Dh| \) has at most a linear growth. Namely, there is a positive constant \( c_3 \) such that

\[
(2.15) \quad |Dh(x)| \leq c_3 |x| \quad \text{for} \quad |x| \geq 1.
\]

Set \( r_i = \frac{c_2}{c_3} |x_i|^{-\frac{3}{2}} \). Choose large enough \( i \) for which \( |x_i| \geq (\frac{c_2}{c_3})^2 + 2 \). Then \( r_i < 1 \). By (2.15) we have

\[
(2.16) \quad h^-(x) \geq c_2 |x_i|^\frac{3}{2} \quad \text{for} \quad x \in B_{r_i}(x_i).
\]

It follows that

\[
\int_{B_{|x_i|+1}(0)} \exp \left[ x \cdot Dh(x) - 2h(x) \right] dx \\
\geq \int_{B_{|x_i|+1}(0)} \exp \left[ -h(x) - h(0) \right] dx \\
\geq \exp \left[ -ah(0) \right] \int_{B_{r_i}(x_i)} \exp \left[ ah^-(x) \right] dx \\
\geq \frac{\omega_n}{n} \left( \frac{c_2}{c_3} \right)^n |x_i|^{-\frac{n}{2}} \exp \left( c_2 \alpha |x_i|^\frac{1}{2} \right).
\]

The first inequality of (2.17) holds due to the convexity of \( h \). The second one holds because \( B_{r_i}(x_i) \subset B_{|x_i|+1}(0) \). And the third one holds because of (2.16).

It follows that

\[
\lim_{i \to \infty} \frac{n}{\omega_n (|x_i| + 1)^n} \int_{B_{|x_i|+1}(0)} \exp \left[ x \cdot Dh(x) - 2h(x) \right] dx \\
\geq C(n, \alpha, h(0), c_2, c_3) \lim_{i \to \infty} |x_i|^{-\frac{3n}{2}} \exp \left( c_2 \alpha |x_i|^\frac{1}{2} \right) = +\infty.
\]

This contradicts the assumption that the ball mean of \( \exp \alpha (x \cdot Dh - 2h) \) is bounded. So the proposition is true. \( \square \)

Because \( \Phi_u(r) \) is bounded, from (2.9) and (2.11) we see that the ball mean of \( \exp \alpha (x \cdot Du - 2u) \) is bounded for \( \alpha = \min\{1/2, 1/2k_1\} \). Since \( u \) is also convex, by Lemma 2.3 we have

\[
\lim_{|x| \to +\infty} \frac{u^-(x)}{\sqrt{|x|}} = 0.
\]

Having such estimates, we are in a position to construct a barrier function to prove the constancy of \( \phi \).
Proof of Theorem 1.1. Define the phase \( \phi = \ln q_{n_1,n_2} (D^2 u) \). By (1.1), we have

\[
\phi(x) = \frac{1}{2} x \cdot Du(x) - u(x).
\]

Taking two derivatives of (2.18), we obtain

\[
\phi_{ij} = \frac{1}{2} x^s u_{ij}.
\]

Define the coefficients \( a_{ij} (D^2 u) \) by

\[
a_{ij} (D^2 u) = \frac{\partial \ln q_{n_1,n_2} (D^2 u)}{\partial u_{ij}}.
\]

As shown above, \( (a_{ij}) \) is positive-definite. A differentiation of (1.1) with respect to \( x^s \) yields

\[
a_{ij} u_{ij s} = \phi_s.
\]

Combining (2.19) and (2.20), we get

\[
a_{ij} \phi_{ij} - \frac{1}{2} x \cdot D \phi = 0.
\]

Thus \( \phi \) satisfies an elliptic equation without zeroth order term (cf. [3, 8]).

Define the corresponding elliptic operator by

\[
\mathcal{L} := a_{ij} \partial^2_{ij} - \frac{1}{2} x \cdot D.
\]

By (1.6), we have

\[
a_{ij} u_{ij} = n_1 - n_2.
\]

For simplicity, denote \( n_1 - n_2 \) by \( N \). It follows that

\[
\mathcal{L} u = N - \frac{1}{2} x \cdot Du.
\]

Define \( \hat{u}(x) = u(x) - Du(0) \cdot x \). Since \( u \) is strictly convex, \( \hat{u} \) is proper. And we have

\[
\mathcal{L} \hat{u} = N - \frac{1}{2} x \cdot D \hat{u}.
\]

Set \( M = |u(0)| + 1 \). Define \( l(x) \) by

\[
l(x) = x \cdot Du(x) - u(x) + M.
\]

Note that \( l(x) \geq 1 \), and

\[
\mathcal{L} l = 2 \mathcal{L} \phi + \mathcal{L} u = N - \frac{1}{2} x \cdot Du.
\]

Define \( g(x) \) by

\[
g(x) = \ln [\hat{u}(x) + M] .
\]

Note that \( g(x) \geq 0 \), and

\[
\mathcal{L} g = \frac{\mathcal{L} \hat{u}}{\hat{u} + M} - \frac{a_{ij} \hat{u}_i \hat{u}_j}{(\hat{u} + M)^2} \leq \frac{1}{\hat{u} + M} \left( N - \frac{1}{2} x \cdot D \hat{u} \right) .
\]
Then there holds
\[ (2.23) \]
\[ \mathcal{L}(lg) = \mathcal{L}g + 2a^{ij}g_{lj} + g\mathcal{L}l \]
\[ \leq \frac{l}{\hat{u} + M} \left( N - \frac{1}{2} x \cdot D\hat{u} \right) + \frac{2\hat{u}a^{ij}u_{js}x^s}{\hat{u} + M} + \left( N - \frac{1}{2} x \cdot Du \right) \cdot \ln (\hat{u} + M). \]

Denote the three terms on the right-hand side of (2.23) by \( I_1, I_2 \) and \( I_3 \) respectively.

As talked above, \( u \) has at most a quadratic growth and \( |Du| \) has at most a linear growth. As well, \( \hat{u} \) has at most a quadratic growth and \( |D\hat{u}| \) has at most a linear growth. More precisely, there exists a positive constant \( K_1 \) such that
\[ (2.24) \]
\[ \hat{u}(x) + M \leq K_1|x|^2 \quad \text{for } |x| \geq 1, \]
\[ (2.25) \]
\[ |D\hat{u}(x)| \leq K_1|x| \quad \text{for } |x| \geq 1. \]

Attributable to the convexity and properness of \( \hat{u} \), there is a positive constant \( K_2 \) such that
\[ (2.26) \]
\[ \hat{u}(x) + M \geq K_2|x| \quad \text{for } |x| \geq 1, \]
\[ (2.27) \]
\[ x \cdot D\hat{u}(x) \geq K_2|x| \quad \text{for } |x| \geq 1. \]

As shown in Lemma (2.4), \( u^-(x) = o(|x|^{\frac{3}{2}}) \) as \( x \to \infty \). Namely there exists a positive constant \( K_3 \) such that
\[ (2.28) \]
\[ u(0) - u(x) \leq K_3|x|^\frac{3}{2} \quad \text{for } |x| \geq 1. \]

By convexity, \(-x \cdot Du(x) \leq u(0) - u(x)\). Thus we get
\[ (2.29) \]
\[ -x \cdot Du(x) \leq K_3|x|^\frac{3}{2} \quad \text{for } |x| \geq 1. \]

Since \( l(x) \geq 1 \), from (2.27) we see when \( |x| \) is large enough,
\[ (2.30) \]
\[ I_1 = \frac{l}{\hat{u} + M} \left( N - \frac{1}{2} x \cdot D\hat{u} \right) \leq 0. \]

Define \( E(x) = \hat{u}_i(x) a^{ij}(x) u_{js}(x) x^s \). By (1.7) we have
\[ (2.31) \]
\[ E(x) \leq \text{tr} \left( a^{ij}u_{js} \right) \cdot |x| \cdot |D\hat{u}| = N|x| |D\hat{u}(x)|. \]

Then it follows from (2.31), (2.25) and (2.26) that for \( |x| \geq 1 \),
\[ (2.32) \]
\[ I_2 = \frac{2E(x)}{\hat{u} + M} \leq 2NK_1K_2^{-1}|x|. \]

According to (2.24) and (2.29), when \( |x| \geq 1 \) we have
\[ (2.33) \]
\[ I_3 = \left( N - \frac{1}{2} x \cdot Du \right) \cdot \ln (\hat{u} + M) \leq \left( \frac{K_3}{2} |x|^{\frac{3}{2}} + N \right) \left( 2 \ln |x| + \ln K_1 \right). \]
Substituting (2.30), (2.32) and (2.33) into (2.23), for large enough \( |x| \), we have
\[
\mathcal{L} (lg) \leq 2 N K_1 K_2^{-1} |x| + K_3 |x|^{\frac{3}{2}} (\ln |x| + \ln K_1) + 2 N (\ln |x| + \ln K_1).
\]
Equations (2.22) and (2.27) then imply there exist \( R_0 \geq 1 \) and a large enough positive constant \( K_4 \) such that
\[
\mathcal{L} (lg + K_4 \hat{u}) \leq 0 \quad \text{when} \quad |x| \geq R_0.
\]
For any \( \varepsilon > 0 \), we take a barrier function \( w(x) \) defined by
\[
w(x) = \varepsilon \{ l(x) g(x) + K_4 [\hat{u}(x) + M] \} + \max_{\partial B_{R_0}} \phi.
\]
Clearly we have
\[
\mathcal{L} w \leq 0 = \mathcal{L} \phi \quad \text{for} \quad |x| \geq R_0,
\]
and
\[
w(x) \geq \phi(x) \quad \text{on} \quad \partial B_{R_0}.
\]
The last thing to check is
\[
w(x) > \phi(x) \quad \text{as} \quad |x| \to +\infty.
\]
We claim that above inequality holds when
\[
|x| \geq \frac{1}{K_2} \exp \frac{1}{\varepsilon} + \left( \frac{2 K_3}{K_2 K_4 \varepsilon} \right)^{\frac{2}{3}} + \frac{2}{K_2 K_4 \varepsilon} \max_{\partial B_{R_0}} \phi + R_0.
\]
By (2.26) we have
\[
\varepsilon K_4 \left( \hat{u}(x) + M \right) > \max_{\partial B_{R_0}} \phi,
\]
and
\[
\varepsilon g(x) > 1.
\]
Simple calculation yields
\[
\varepsilon K_2 K_4 |x| > K_3 |x|^{\frac{3}{2}}.
\]
Next we discuss the following two cases.

Case 1. \( \phi(x) < \frac{\varepsilon}{2} K_2 K_4 |x| \). Directly from (2.26) and (2.34) we see
\[
\phi(x) < \frac{\varepsilon K_4}{2} [\hat{u}(x) + M] \leq w(x).
\]

Case 2. \( \phi(x) \geq \frac{\varepsilon}{2} K_2 K_4 |x| \). By convexity, (2.28) and (2.36) we get
\[
\frac{1}{2} x \cdot Du \geq \frac{\varepsilon}{2} K_2 K_4 |x| + u(x)
\]
\[
\geq \frac{\varepsilon}{2} K_2 K_4 |x| - K_3 |x|^{\frac{3}{2}} + u(0)
\]
\[
\geq u(0).
\]
Thus
\[
l(x) - \phi(x) = \frac{1}{2} x \cdot Du + M \geq u(0) + M > 0.
\]
Combing (2.34), (2.35) and (2.37), we also have \( w(x) > \phi(x) \).

The weak maximum principle then implies

\[
\varepsilon \{ l(x) g(x) + K u(x) + M \} + \max_{\partial B_{R_0}} \phi \geq \phi(x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_{R_0}.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
\max_{\partial B_{R_0}} \phi \geq \phi(x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_{R_0}.
\]

So \( \phi \) attains its global maximum in the closure of \( B_{R_0} \). Hence \( \phi \) is a constant by the strong maximum principle. Using \( \phi = \frac{1}{2} x \cdot Du - u \), we have

\[
\frac{1}{2} x \cdot D[u(x) + \phi(0)] = u(x) + \frac{1}{2} \phi(0).
\]

Finally, it follows from Euler’s homogeneous function theorem that smooth \( u(x) + \phi(0)/2 \) is a homogeneous order 2 polynomial.

\( \square \)

3. Proof of Theorem 1.3 and Theorem 1.4

The whole proof of Theorem 1.1 can be copied here except inequality (2.31). Actually, we only need to prove a (2.31)-type inequality under the new conditions. For convenience and clarity, for the corresponding objects we use the same notations as in the proof of Theorem 1.1.

Proof. When the eigenvalues of \( \partial \bar{\partial} u \) are comparable, namely inequality (1.11) holds, for any \( i, s \) we have

\[
\sum_{i=1}^{2n} a^{ij}(x) u_{js}(x) < \frac{\Delta u(x)}{4 \mu_{\min}(x)} \leq \frac{n \mu_{\max}(x)}{\mu_{\min}(x)} \leq n \Lambda.
\]

So \( E(x) \leq n \Lambda |x| |D \hat{u}(x)| \), where \( \hat{u}(x) = u(x) - D u(0) \cdot x \).

Now we talk about the toric case. Since \( u \) is invariant under \( \mathbb{T}^n \)-actions, we have \( D u(0) = 0 \), \( \hat{u}(x) = u(x) \). And \( u(x) \) can be reduced to a function \( f(r^1, \ldots, r^n) \) depending only on each polar radius \( r^i = |x^i + \sqrt{-1} x^{n+i}| \). Simple calculation gives:

\[
\begin{align*}
    u_i &= f_i \cdot \frac{x^i}{r^i}, \quad u_{n+i} = f_i \cdot \frac{x^{n+i}}{r^i} \\
    u_{ij} &= f_{ij} \cdot \frac{x^i x^j}{r^i r^j} + f_i \cdot \delta_{ij} \cdot \frac{(x^{n+i})^2}{(r^i)^3}, \\
    u_{i,n+j} &= f_{ij} \cdot \frac{x^i x^{n+j}}{r^i r^j} - \delta_{ij} \cdot f_i \cdot \frac{x^i x^{n+j}}{(r^i)^3}, \\
    u_{n+i,n+j} &= f_{ij} \cdot \frac{x^{n+i} x^{n+j}}{r^i r^j} + f_i \cdot \delta_{ij} \cdot \frac{(x^i)^2}{(r^i)^3}.
\end{align*}
\]
for $1 \leq i, j \leq n$. So at $x = (r, 0)$ where $r = (r^1, \ldots, r^n)$, we have $Du = (Df(r), 0)$ and
\[
D^2u = \begin{pmatrix} D^2f(r) & 0 \\ 0 & \Omega(r) \end{pmatrix},
\]
where $\Omega(r) = \text{diag} \left( \frac{r^1}{r}, \ldots, \frac{r^n}{r} \right)$.

As noted in the introduction, $(a^{ij}) = (D^2u - J \cdot D^2u \cdot J)^{-1}$. $E(x)$ can be viewed as the matrix product $E(x) = (Du)^T \cdot (D^2u - J \cdot D^2u \cdot J)^{-1} \cdot (D^2u \cdot x)$,

where $(Du)^T$ is the transpose of $Du$.

Since $u$ is $\mathbb{T}^n$-invariant and $J$ is an infinitesimal generator of $\mathbb{T}^n$-actions, $E(x)$ is $\mathbb{T}^n$-invariant. So $E(x) = E(r, 0)$. Then it follows that
\[
E(r, 0) = (Df)^T \cdot (D^2f + \Omega)^{-1} \cdot D^2f \cdot r
\]
\[
= (Df)^T \cdot r - (Df)^T \cdot (D^2f + \Omega)^{-1} \cdot \Omega \cdot r
\]
\[
= (Df)^T \cdot r - (Df)^T \cdot (D^2f + \Omega)^{-1} \cdot Df
\]
\[
\leq (Df)^T \cdot r.
\]
Consequently, we have $E(x) \leq x \cdot Du(x)$. $\square$

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SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, HAIDIAN, BEIJING, 100871, P.R. CHINA

E-mail address: wwlpkumath@yahoo.com