AN $l_1$-ORACLE INEQUALITY FOR THE LASSO IN MIXTURE-OF-EXPERTS REGRESSION MODELS\textsuperscript{*,*\textsuperscript{\textdagger}}

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Abstract. Mixture-of-experts (MoE) models are a popular framework for modeling heterogeneity in data, for both regression and classification problems in statistics and machine learning, due to their flexibility and the abundance of available statistical estimation and model choice tools. Such flexibility comes from allowing the mixture weights (or gating functions) in the MoE model to depend on the explanatory variables, along with the experts (or component densities). This permits the modeling of data arising from more complex data generating processes when compared to the classical finite mixtures and finite mixtures of regression models, whose mixing parameters are independent of the covariates. The use of MoE models in a high-dimensional setting, when the number of explanatory variables can be much larger than the sample size, is challenging from a computational point of view, and in particular from a theoretical point of view, where the literature is still lacking results for dealing with the curse of dimensionality, for both the statistical estimation and feature selection problems. We consider the finite MoE model with soft-max gating functions and Gaussian experts for high-dimensional regression on heterogeneous data, and its $l_1$-regularized estimation via the Lasso. We focus on the Lasso estimation properties rather than its feature selection properties. We provide a lower bound on the regularization parameter of the Lasso function that ensures an $l_1$-oracle inequality satisfied by the Lasso estimator according to the Kullback–Leibler loss.

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1. INTRODUCTION

Mixture-of-experts (MoE) models, a flexible generalization of classical finite mixture models, were introduced by [22] in a problem decomposition context, and are widely used in statistics and machine learning, thanks to their flexibility and the abundance of available statistical estimation and model choice tools. The main idea
of MoE modeling is a divide-and-conquer principle that proposes dividing a complex problem into a set of simpler subproblems and then one or more specialized problem-solving tools, or experts, are assigned to each of the subproblems. The flexibility of MoE models comes from allowing the mixture weights (or the gating functions) to depend on the explanatory variables, along with the experts (or the component densities). This permits the modeling of data arising from more complex data generating processes than those that can be analyzed using the classical finite mixtures and finite mixtures of regression models, whose mixing parameters are independent of the covariates. A reason for the popularity of finite mixture-type models is because of their universal approximation properties, which have been extensively studied in [15, 17, 18, 45, 46, 49, 52]. Analogously, the approximation properties of MoE models for conditional density estimation and mean function estimation have been established in [19, 24, 41, 42, 43, 50].

Statistically, the MoE models are used to estimate the conditional distribution of a random variable $Y \in \mathbb{R}^q$, given certain features from $n$ observations $\{x_i\}_{i \in [n]} = \{(x_{i1}, \ldots, x_{ip})\}_{i \in [n]} \in (\mathbb{R}^p)^n$, where $q, p, n \in \mathbb{N}^*$, $[n] := \{1, \ldots, n\}$, $\mathbb{N}^*$ denotes the positive integer numbers, and $\mathbb{R}^p$ means the $p$-dimensional real number. In the context of regression, finite MoE models with Gaussian experts and soft-max gating functions are a standard choice and a powerful tool for modeling more complex non-linear relationships between response and predictors, arising from different subpopulations, compared to the finite mixture of Gaussian regression models. The reader is referred to [31, 40, 44, 61] for recent reviews on the topic.

The use of MoE models in the high-dimensional regression setting, when the number of explanatory variables can be much larger than the sample size, remains a challenge, particularly from a theoretical point of view, where there is still a lack of results in the literature regarding both statistical estimation and model selection. In such settings, we are required to reduce the dimension of the problem by seeking the most relevant relationships, to avoid numerical identifiability problems.

We focus on the use of an $l_1$-penalized maximum likelihood estimator (MLE), as originally proposed as the Lasso by [56], which tends to produce sparse solutions and can be viewed as a convex surrogate for the non-convex $l_0$-penalization problem. These methods have attractive computational and theoretical properties (cf. Fan and Li [14]). First introduced in [56] for the linear regression model, the Lasso estimator has since been extended to many statistical problems, including for high-dimensional regression of non-homogeneous data by using finite mixture regression models as considered by [27], [55], and [28]. In [55], it is assumed that, for $i \in [n], n \in \mathbb{N}^*$, the random response variable $Y_i$, conditionally on $X_i = x_i$, come from a conditional density $s_{\psi_0}(y_i|x_i)$, which is a finite mixture of $K \in \mathbb{N}^*$ Gaussian conditional densities with mixing proportions $(\pi_0,1,\ldots,\pi_0,K)$, where

$$Y_i|X_i = x_i \sim s_{\psi_0}(y_i|x_i) = \sum_{k=1}^{K} \pi_{0,k} \phi(y_i; \beta_{0,k}^T x_i, \sigma_{0,k}^2). \tag{1.1}$$

Here

$$\phi(\cdot; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(\cdot - \mu)^2}{2\sigma^2} \right)$$

is the univariate Gaussian probability density function (PDF), with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}^+$, and $\psi_0 = (\pi_{0,k}, \beta_{0,k}, \sigma_{0,k})_{k \in [K]}$ is the vector of model parameters.

We consider a model $S$, defined in (1.1). To estimate the true generative model $s_{\psi_0}$, [55] proposed a Lasso-

regularization based estimator, which consists of a minimiser of the penalized negative conditional log-likelihood that is defined by

$$\hat{s}^L(\lambda) = \arg\min_{s_\psi \in S} \left\{ \frac{1}{n} \sum_{i=1}^{n} \ln (s_\psi (y_i|X_i)) + \text{pen}_\lambda (\psi) \right\},$$

$$\text{pen}_\lambda (\psi) = \lambda \sum_{k=1}^{K} \sigma_{k}^{-1} \sum_{j=1}^{p} \left| \beta_{k,j} \right|, \lambda > 0, \psi = (\pi, \beta_k, \sigma_k)_{k \in [K]}. \tag{1.2}$$
For this estimator, the authors provided an $l_0$-oracle inequality, satisfied by $\hat{s}^L(\lambda)$, conditional on the restricted eigenvalue condition, namely the Fisher information matrix is positive definite. Furthermore, they have to introduce some margin conditions so as to link the Kullback–Leibler loss function to the $l_2$-norm of the parameters. More precisely, they firstly define a so-called excess risk as

$$\mathcal{E}(\psi|\psi_0) = -\int \log \left[ \frac{s_\psi}{s_{\psi_0}} \right] s_{\psi_0} dy.$$  

Then, they assume several regularity and identifiability conditions on score function and the Fisher information such that the behavior of the excess risk $\mathcal{E}(\psi|\psi_0)$ near $\psi_0$ satisfies a so-called margin condition. Lemma 1 from [55] implies that the margin $c_0$ satisfies

$$\inf_{x \in X} \frac{\mathcal{E}(\psi|\psi_0(x))}{\|\psi - \psi_0(x)\|_2^2} \geq \frac{1}{c_0^2}.$$  

Another direction of study regarding $\hat{s}^L(\lambda)$ is to look at its $l_1$-regularization properties; see, for example, [33, 38], and [10]. As indicated by [10], contrary to results for the $l_0$ penalty, some results for the $l_1$ penalty are valid with no assumptions, neither on the Gram matrix nor on the margin. However, such results can only produce bounds that converge at a rate of $1/n$, rather than $1/\sqrt{n}$.

In the framework of finite mixtures of Gaussian regression models, [38] considered the case for a univariate response, and [10] extended these results to the case of a multivariate responses, i.e., the Gaussian conditional pdf in (1.1) is replaced by a multivariate Gaussian PDF of the form $\phi(\cdot; \mu, \Sigma)$ with mean vector $\mu$ and a covariance matrix $\Sigma$. In particular, [10] considered an extension of the Lasso-estimator (1.2), with a regularization term defined by $pen_\lambda(\psi) = \lambda \sum_{k=1}^{K} \sum_{j=1}^{p} \sum_{z=1}^{q} |\beta_{kj}|_2$.

In this article, we shall extend such result for the finite mixture of Gaussian regressions models, which is considered as a special case of the MoE models to the more general mixture of Gaussian experts regression models with soft-max gating functions, as defined in (2.1). Since each mixing proportion is modeled by a soft-max function of the covariates, the dependence on each feature appears both in the experts pdfs and in the mixing proportion functions (gating functions), which allows us to capture more complex non-linear relationships between the response and predictors arising from different subpopulations, compared to the finite mixture of Gaussian regression models. This is demonstrated via numerical experiments in several articles such as [6], [7] [39], and [40].

In the context of studying the statistical properties of the penalized maximum likelihood approach for MoE models with soft-max gating functions, we may consider the prior works of [26] and [39]. In [26], for feature selection, two extra penalty terms are applied to the $l_2$-penalized conditional log-likelihood function. Their penalized conditional log-likelihood estimator is given by

$$\hat{s}^{\text{PL}}(\lambda) = \arg\min_{s_\psi \in S} \left\{ -\frac{1}{n} \sum_{i=1}^{n} \ln (s_\psi(y_i|x_i)) + \text{pen}_\lambda(\psi) \right\}, \quad (1.3)$$

$$s_\psi(y|x) = \sum_{k=1}^{K} g_k(x; \gamma) \phi \left( y; \beta_{k0} + \beta_k^T x, \sigma_k^2 \right), \quad \psi = (\gamma_k, \beta_k, \sigma_k)_{k \in [K]}, \quad (1.4)$$

$$\text{pen}_\lambda(\psi) = \sum_{k=1}^{K} \lambda_k^{[1]} \sum_{j=1}^{p} |\gamma_{kj}| + \sum_{k=1}^{K} \lambda_k^{[2]} \sum_{j=1}^{p} |\beta_{kj}| + \sum_{k=1}^{K} \frac{\lambda_k^{[3]}}{2} \|\gamma_k\|_2^2, \quad (1.5)$$
where $\lambda = (\lambda^{[1]}_1, \ldots, \lambda^{[1]}_K, \lambda^{[2]}_1, \ldots, \lambda^{[2]}_K, \frac{\lambda^{[3]}_0}{2})$ is a vector of non-negative regularization parameters, $S$ contains all functions of form (1.4), $\|x\|_2^2$ is the Euclidean norm in $\mathbb{R}^p$, and

$$g_k(x; \gamma) = \frac{\exp (\gamma_k x_0 + \gamma_k^T x)}{\sum_{k=1}^K \exp (\gamma_k x_0 + \gamma_k^T x)}$$

is a soft-max gating function. Note that the first two terms from (1.5) are the normal Lasso functions ($l_1$ penalty function), while the $l_2$ penalty function for the gating network is added to excessively wildly large estimates of the regression coefficients corresponding to the mixing proportions. This behavior can be observed in logistic/multinomial regression when the number of potential features is large and highly correlated (see e.g., [51] and [5]). However, this also affects the sparsity of the regularization model, which is confirmed via the numerical experiments of [6] and [7].

By extending the theoretical developments for mixture of linear regression models in [27], standard asymptotic theorems for MoE models are established in Khalili [26]. More precisely, under several strict regularity conditions on the true joint density function $s_{\psi_0}(y, x)$ and the choice of tuning parameter $\lambda$, the estimator of the true parameter vector $\hat{\psi}^{PL}_n(\lambda)$, defined via $s^{PL}(\lambda)$ from (1.3) but using the Scad penalty function from [14], instead of Lasso, is proved to be both consistent in feature selection and maintains root-$n$ consistency. Differing from Scad, for Lasso, the estimator $\hat{\psi}^{PL}_n(\lambda)$ cannot achieve both properties, simultaneously. In other words, Lasso is consistent in feature selection but introduces bias to the estimators of the true nonzero coefficients. Therefore, our non-asymptotic result in Theorem 3.2 can be considered as a complement to such asymptotics for MoE regression models with soft-max gating functions. To obtain our oracle inequality, Theorem 3.2, we shall restrict our study to the Lasso estimator without the $l_2$-norm.

Other related results to our work are the weak oracle inequalities for several MoE models [11, 12, 39, 47, 48]. In particular, regarding the MoE model with soft-max gating functions, Montuelle et al. [39, Theorem 1] focused on the model selection procedure instead of investigating the $l_1$-regularization properties of the Lasso estimator. A detailed comparison between our work and these related results can be found in Section 4.

While studying the oracle inequality within the context of the $(l_1 + l_2)$-norm may also be interesting. It has been demonstrated, in [21], that the regularized maximum-likelihood estimation of MoE models for generalized linear models, better encourages sparsity under the $l_1$-norm, compared to when using the $(l_1 + l_2)$-norm, which may affect sparsity. We shall not discuss such approaches, further.

To the best of our knowledge, we are the first to study the $l_1$-regularization properties of the MoE regression models. In the current paper, we focus on a simplified but standard setting in which the means of the experts are linear models, better encourages sparsity under the $l_1$-norm, with respect to explanatory variables. Although simplified, this model captures the core of the MoE regression problem, which is the interactions among the different mixture components. We believe that the general techniques that we develop here can be extended to more general experts, such as Gaussian experts with polynomial means (e.g., 37) or hierarchical MoE for exponential family regression models, as in [24]. We leave such nontrivial developments for future work.

The main contribution of our paper is a theoretical result: an oracle inequality, that provides the lower bound on the regularization parameter of the Lasso penalty that ensures non-asymptotic theoretical control of the Kullback–Leibler loss of the Lasso estimator for the mixtures of Gaussian experts regression models with soft-max gating functions. That is, we assume that we have $n$ observations, $((x_i, y_i))_{i \in [n]} \in ([0, 1]^p \times \mathbb{R}^q)$, coming from the unknown conditional mixture of Gaussian experts regression models $s_0 := s_{\psi_0} \in S$, cf. (2.6), and a Lasso estimator $\hat{s}^L(\lambda)$, defined as in (2.8), where $\lambda \geq 0$ is a regularization parameter to be tuned. Then, if $\lambda \geq \frac{\lambda_1}{\sqrt{n}}$, for some absolute constant $\lambda \geq 148$, the estimator $\hat{s}^L(\lambda)$ satisfies the following $l_1$-oracle inequality:

$$\mathbb{E} \left[ \text{KL}_n \left(s_0, \hat{s}^L(\lambda)\right) \right] \leq (1 + \kappa^{-1}) \inf_{s_{\psi} \in S} \left( \text{KL}_n \left(s_0, s_{\psi}\right) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right) + \lambda + \sqrt{\frac{K}{n}} \epsilon_2 \ln n,$$
where we suppress the dependence of positive constants $c_1, c_2$ on $p, q, K$, and boundedness constants defining our conditional class $S$, for ease of interpretation. For the explicit dependences, we refer the reader to Theorem 3.2. Note that this result is non-asymptotic; i.e., the number of observations $n$ is fixed, while the number of predictors $p$ and the dimension of the response $q$ can grow, with respect to $n$, and can be much larger than $n$. Good discussions regarding non-asymptotic statistics are provided in [32] and [59].

The goal of this paper is to provide a treatment regarding penalizations that guarantee an $l_1$-oracle inequality for finite MoE models in particular for high-dimensional non-linear regression. As such, the remainder of the article progresses as follows. In Section 2, we discuss the construction and framework of finite mixture of Gaussian experts regression models with soft-max gating functions. In Section 3, we state the main result of the article, which is an $l_1$-oracle inequality satisfied by the Lasso estimator in the finite mixture of Gaussian experts regression models. Some discussion and comparison of this work in the literature is provided in Section 4. Section 5 is devoted to the proof of these main results. The proof of technical lemmas can be founded in Section 6. Some conclusions are provided in Section 7, and additional technical results are relegated to Appendix A.

## 2. Notation and framework

### 2.1. Mixtures of Gaussian experts regression models with soft-max gating functions

We consider the statistical framework in which we model a sample of high-dimensional regression data generated from a heterogeneous population via the mixtures of Gaussian experts regression models with soft-max gating functions, named soft-max-gated MoE (SGaME) regression models. We observe $n$ independent couples $((x_i, y_i))_{i \in [n]} \in (\mathcal{X} \times \mathcal{Y})^n \subset (\mathbb{R}^p \times \mathbb{R}^q)^n$ ($p, q, n \in \mathbb{N}^*$), where typically $p \gg n$. $x_i$ is fixed and $y_i$ is a realization of the random variable $Y_i$, for all $i \in [n]$. We assume that $\mathcal{X}$ is a compact subset of $\mathbb{R}^p$. We also assume that the response variable $Y_i$ depends on the set of explanatory variables (covariates) through a regression-type model. The conditional probability density function (PDF) of the model is approximated by mixture of Gaussian experts regression models with soft-max gating functions.

More precisely, we assume that, conditionally to the $\{x_i\}_{i \in [n]}$, $\{Y_i\}_{i \in [n]}$ are independent and identically distributed with conditional density $s_0(\cdot|x_i)$, which is approximated by a MoE model. Our goal is to estimate this conditional density function $s_0$ from the observations.

For any $K \in \mathbb{N}^*$, the $K$-component MoE model can be defined as

$$\text{MoE}(y|x; \theta) = \sum_{k=1}^{K} g_k(x; \gamma) f_k(y|x; \eta),$$

where $g_k(x; \gamma) > 0$ and $\sum_{k=1}^{K} g_k(x; \gamma) = 1$, and $f_k(y|x; \eta)$ is a conditional PDF (cf. [40]). In our proposal, we consider the MoE model of [25], which extended the original MoE from [22], for a regression model. More precisely, we utilize the following mixtures of Gaussian experts regression models with soft-max gating functions:

$$s_\psi(y|x) = \sum_{k=1}^{K} g_k(x; \gamma) \phi(y; v_k(x), \Sigma_k),$$

(2.1)

to estimate $s_0$, where given any $k \in [K]$, $\phi(\cdot; v_k, \Sigma_k)$ is the multivariate Gaussian density with mean $v_k$, which is a function of $x$ that specifies the mean of the $k$th component, and with covariance matrix $\Sigma_k$. Here, $(v, \Sigma) := ((v_1, \ldots, v_K), (\Sigma_1, \ldots, \Sigma_K)) \in (\mathcal{Y} \times V)$, where $\mathcal{Y}$ is a set of $K$-tuples of mean functions from $\mathcal{X}$ to $\mathbb{R}^q$ and $V$ is a sets of $K$-tuples of symmetric positive definite matrices on $\mathbb{R}^q$, and the soft-max gating function $g_k(x; \gamma)$ is defined as in (2.2):

$$g_k(x; \gamma) = \frac{\exp(w_k(x))}{\sum_{l=1}^{K} \exp(w_l(x))},$$

$$w_k(x) = \gamma_{k0} + \gamma_k^\top x, \gamma = (\gamma_{k0}, \gamma_k)_{k \in [K]} \in \Gamma = \mathbb{R}^{(p+1)K}.$$  

(2.2)
We shall define the parameter vector $\psi$ in the sequel.

2.1.1. Fixed predictors and number of components with linear mean functions

Inspired by the framework in [38] and [10], the explanatory variables $x_i$ and the number of components $K \in \mathbb{N}^*$ are both fixed. We assume that the observed $x_i, i \in [n]$, are finite. Without loss of generality, we choose to rescale $x$, so that $\|x\|_{\infty} \leq 1$. Therefore, we can assume that the explanatory variables $x_i \in \mathcal{X} = [0, 1]^p$, for all $i \in [n]$. Note that such a restriction is also used in [10]. Under only the assumption of bounded parameters, we provide a lower bound on the Lasso regularization parameter $\lambda$, which guarantees an oracle inequality. Note that in this non-random explanatory variables setting, we focus on the Lasso for its regularization properties rather than as a model selection procedure, as in the case of random explanatory variables and unknown $K$, as in [39].

For simplicity, we consider the case where the means of Gaussian experts are linear functions of the explanatory variables; i.e.,

$$\mathbf{Y} = \left\{ v : \mathcal{X} \mapsto v_\beta(x) := (\beta_{k0} + \beta_k x)_{k \in [K]} \in (\mathbb{R}^q)^K \mid \beta = (\beta_{k0}, \beta_k)_{k \in [K]} \in \mathcal{B} = \left(\mathbb{R}^{q \times (p+1)}\right)^K \right\},$$

where $\beta_{k0}$ and $\beta_k$ are respectively the $q \times 1$ vector of bias and the $q \times p$ regression coefficients matrix for the $k$th expert.

In summary, we wish to estimate $s_0$ via conditional densities belonging to the class:

$$\left\{ (x, y) \mapsto s_\psi(y|x) \mid \psi = (\gamma, \beta, \Sigma) \in \Psi \right\},$$

where $\Psi = \Gamma \times \Xi$, and $\Xi = B \times V$.

From hereon in, for a vector $x \in \mathbb{R}^p$, we assume that $x = (x_1, \ldots, x_p)$ is in the column form. Similarly, the parameter of the entire model, $\psi = (\gamma, \beta, \Sigma)$, is also a column vector, where we consider any matrix as a vector produced using $\text{vec}(\cdot)$: the vectorization operator that stacks the columns of a matrix into a vector.

2.1.2. Boundedness assumption on the soft-max gating and Gaussian parameters

For a matrix $A$, let $m(A)$ be the modulus of the smallest eigenvalue, and $M(A)$ the modulus of the largest eigenvalue. We shall restrict our study to estimate $s_0$ by conditional PDFs belonging to the model class $S$, which has boundedness assumptions on the softmax gating and Gaussian expert parameters. Specifically, we assume that there exists deterministic constants $A_\gamma, A_\beta, a_\Sigma, A_\Sigma > 0$, such that $\psi \in \tilde{\Psi}$, where

$$\tilde{\Psi} = \Gamma \times \Xi, \tilde{\Gamma} = \left\{ \gamma \in \Gamma \mid \forall k \in [K], \sup_{x \in \mathcal{X}} (|\gamma_{k0}| + |\gamma_k x|) \leq A_\gamma \right\},$$

$$\tilde{\Xi} = \left\{ \xi \in \Xi \mid \forall k \in [K], \max_{z \in [q]} \sup_{x \in \mathcal{X}} (|\beta_{k0} z| + |\beta_k x|) \leq A_\beta, a_\Sigma \leq m \left(\Sigma_k^{-1}\right) \leq M \left(\Sigma_k^{-1}\right) \leq A_\Sigma \right\}.$$  (2.4)

Since

$$a_G := \frac{\exp(-A_\gamma)}{\sum_{k=1}^{K} \exp(A_\gamma)} \leq \sup_{x \in \mathcal{X}, \gamma \in \tilde{\Gamma}} \frac{\exp(\gamma_{k0} + \gamma_k x)}{\sum_{l=1}^{K} \exp(\gamma_{l0} + \gamma_l x)} \leq \frac{\exp(A_\gamma)}{\sum_{l=1}^{K} \exp(-A_\gamma)} =: A_G,$$

there exists deterministic positive constants $a_G, A_G$, such that

$$a_G \leq \sup_{x \in \mathcal{X}, \gamma \in \tilde{\Gamma}} g_k(x; \gamma) \leq A_G.$$  (2.5)

We wish to use the model class $S$ of conditional PDFs to estimate $s_0$, where

$$S = \left\{ (x, y) \mapsto s_\psi(y|x) \mid \psi = (\gamma, \beta, \Sigma) \in \Psi \right\}.$$  (2.6)
To simplify the proofs, we shall assume that the true density \( s_0 \) belongs to \( S \). That is to say, there exists \( \psi_0 = (\gamma_0, \beta_0, \Sigma_0) \in \hat{\Psi} \), such that \( s_0 = s_{\psi_0} \).

### 2.2. Losses and the penalized maximum likelihood estimator

In maximum likelihood estimation, we consider the Kullback–Leibler information as the loss function, which is defined for densities \( s \) and \( t \) by

\[
\text{KL}(s, t) = \begin{cases} \int_{\mathbb{R}^q} \ln \left( \frac{s(y)}{t(y)} \right) s(y) dy & \text{if } sdy \text{ is absolutely continuous with respect to } tdy, \\ +\infty & \text{otherwise.} \end{cases}
\]

Since we are working with conditional PDFs and not with classical densities, we define the following adapted Kullback–Leibler information that takes into account the structure of conditional PDFs. For fixed explanatory variables \((x_i)_{1 \leq i \leq n}\), we consider the average loss function

\[
\text{KL}_n(s, t) = \frac{1}{n} \sum_{i=1}^{n} \text{KL}(s(\cdot | x_i), t(\cdot | x_i)) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^q} \ln \left( \frac{s(y|x_i)}{t(y|x_i)} \right) s(y|x_i) dy.
\]

The maximum likelihood estimation approach suggests to estimate \( s_0 \) by the conditional PDF \( s_{\psi} \) that maximizes the log likelihood, conditioned on \((x_i)_{1 \leq i \leq n}\), defined as

\[
\ln \left( \prod_{i=1}^{n} s_{\psi}(y_i|x_i) \right) = \sum_{i=1}^{n} \ln \left( s_{\psi}(y_i|x_i) \right).
\]

Or equivalently, that minimizes the empirical contrast:

\[-\frac{1}{n} \sum_{i=1}^{n} \ln \left( s_{\psi}(y_i|x_i) \right).\]

However, since we want to handle high-dimensional data, we have to regularize the maximum likelihood estimator (MLE) in order to obtain reasonable estimates. Here, we shall consider \( l_1 \)-regularization and the associated so-called Lasso estimator, which is the \( l_1 \)-norm penalized MLE and is defined as in (1.2) but with a modified penalty function as follows:

\[
\bar{s}^k(\lambda) := \underset{s_{\psi} \in S}{\text{argmin}} \left\{ -\frac{1}{n} \sum_{i=1}^{n} \ln \left( s_{\psi}(y_i|x_i) \right) + \text{pen}_\lambda(\psi) \right\}, \tag{2.8}
\]

where \( \lambda \geq 0 \) is a regularization parameter to be tuned, \( \psi = (\gamma, \beta, \Sigma) \) and

\[
\text{pen}_\lambda(\psi) = \lambda \left\| \psi^{[1,2]} \right\|_1 := \lambda \left( \left\| \psi^{[1]} \right\|_1 + \left\| \psi^{[2]} \right\|_1 \right), \tag{2.9}
\]

\[
\left\| \psi^{[1]} \right\|_1 = \|\gamma\|_1 = \sum_{k=1}^{K} \sum_{j=1}^{p} |\gamma_{kj}|, \quad \left\| \psi^{[2]} \right\|_1 = \|\text{vec}(\beta)\|_1 = \sum_{k=1}^{K} \sum_{j=1}^{p} \sum_{z=1}^{q} |[\beta_k]_{z,j}|. \tag{2.10}
\]

From now on, we denote \( \|\beta\|_p \ (p \in \{1, 2, \infty\}) \) by the induced \( p \)-norm of a matrix; see Definition A.3, which differs from \( \|\text{vec}(\beta)\|_p \).

Note that \( \text{pen}_\lambda(\psi) \) is a Lasso regularization term encouraging sparsity for both the gating and expert parameters. Recall that this penalty is also studied in [26], [6], and [7], in which the authors studied the univariate
1.5

Indeed, by considering \( \lambda = \min \left\{ \lambda_1^{(1)}, \ldots, \lambda_1^{(l)}, \lambda_2^{(1)}, \ldots, \lambda_2^{(l)} \right\} \), the condition for a regularization parameter’s lower bound, (3.2) from Theorem 3.2, can also be applied to model (1.3), which leads to an \( l_1 \)-oracle inequality.

3. An \( l_1 \)-oracle inequality for the Lasso estimator

In this section, we state Theorem 3.2, which is proved in Section 5.3. This result provides an \( l_1 \)-oracle inequality for the Lasso estimator for mixtures of Gaussian experts regression models with soft-max gating functions. It is the primary contribution of this article and is motivated by the problem studied in [38] and [10].

Firstly, we aim to prove that the negative of differential entropy (see its definition, e.g., from Mansuripur [30, Chapter 9]) of the true unknown conditional density \( s_0 \in S \), defined in (2.1), is finite, see more in Lemma 3.1, which is proved in Section 6.3.

**Lemma 3.1** (Differential entropy of SGaME regression model with boundedness assumptions on parameter spaces). There exists a nonnegative constant

\[
H_{s_0} = \max \{0, \ln C_{s_0} \}, \text{ where } C_{s_0} = (4\pi)^{-q/2} A_{s_0}^{q/2};
\]

such that

\[
\max \left\{ 0, \sup_{x \in \mathcal{X}} \int_{S} \ln (s_0(y|x)) s_0(y|x) \, dy \right\} \leq H_{s_0} < \infty.
\] (3.1)

Then, we are able to state our main result via Theorem 3.2, which is proved in Section 5.3.

**Theorem 3.2** (\( l_1 \)-oracle inequality). We observe \( (x_i, y_i)_{i \in [n]} \in ([0,1]^p \times \mathbb{R}^q) \), coming from the unknown conditional mixture of Gaussian experts regression models \( s_0 := s_{\psi_0} \in S \), cf. (2.6). We define the Lasso estimator \( \hat{s}^l(\lambda) \), by (2.8), where \( \lambda \geq 0 \) is a regularization parameter to be tuned. Then, if

\[
\lambda \geq \kappa \frac{KB'}{\sqrt{n}} \left( q \ln n + \ln(2p + 1) + 1 \right), B'_n = \max (A_{\Sigma}, 1 + KA_G) \left( 1 + 2q A_{\Sigma} (5A_\beta^2 + 4A_{\Sigma} \ln n) \right),
\] (3.2)

for some absolute constants \( \kappa \geq 148 \), the estimator \( \hat{s}^l(\lambda) \) satisfies the following \( l_1 \)-oracle inequality:

\[
\mathbb{E} \left[ \text{KL}_n \left( s_0, \hat{s}^l(\lambda) \right) \right] \leq \left( 1 + \kappa^{-1} \right) \inf_{s_\psi \in S} \left( \text{KL}_n \left( s_0, s_\psi \right) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right) + \lambda \sqrt{\frac{K}{n}} \left( \frac{e^{q/2 - 1/2} \pi^{q/2}}{A_{\Sigma}^{q/2}} + H_{s_0} \right) \sqrt{2q A_{\gamma}}
\]

\[
+ 302q \sqrt{\frac{K}{n}} \max (A_{\Sigma}, 1 + KA_G) \left( 1 + 2q A_{\Sigma} (5A_\beta^2 + 4A_{\Sigma} \ln n) \right)
\]

\[
\times K \left( 1 + A_{\gamma} + qA_{\beta} + \frac{q \sqrt{q} \sqrt{A_{\Sigma}}}{\sqrt{\Sigma}} \right)^2.
\] (3.3)

Next, we state Theorem 3.3, which is proved in Section 5.4 and is an \( l_1 \)-ball MoE regression model selection theorem for \( l_1 \)-penalized maximum conditional likelihood estimation in the Gaussian mixture framework. Note that Theorem 3.2 is an immediate consequence of Theorem 3.3.

**Theorem 3.3.** Assume that we observe \( (x_i, y_i)_{i \in [n]} \) with unknown conditional Gaussian mixture PDF \( s_0 \). For all \( m \in \mathbb{N}^* \), consider the \( l_1 \)-ball

\[
S_m = \left\{ s_\psi \in S, \left\| \psi^{[1,2]} \right\|_1 \leq m \right\},
\] (3.4)
and let \( \hat{s}_m \) be a \( \eta_m \)-in-likelihood estimator in \( S_m \) for some \( \eta_m \geq 0 \):

\[
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m (y_i | x_i)) \leq \inf_{s_m \in S_m} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (s_m (y_i | x_i)) \right) + \eta_m.
\]  

(3.5)

Assume that, for all \( m \in \mathbb{N}^* \), the penalty function satisfies \( \text{pen}(m) = \lambda m \), where \( \lambda \) is defined later. Then, we define the penalized likelihood estimator \( \hat{s}_m \), where \( \hat{m} \) is defined via the satisfaction of the inequality

\[
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_{\hat{m}} (y_i | x_i)) + \text{pen}(\hat{m}) \leq \inf_{m \in \mathbb{N}^*} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m (y_i | x_i)) + \text{pen}(m) \right) + \eta,
\]  

(3.6)

for some \( \eta \geq 0 \). Then, if

\[
\lambda \geq \kappa \frac{KB'}{\sqrt{n}} (q \ln n \sqrt{\ln(2p + 1)} + 1), B'_n = \max (A_{\Sigma}, 1 + K A_C) \left( 1 + 2q \sqrt{q A_{\Sigma}} (5A_B^2 + 4A_{\Sigma} \ln n) \right),
\]  

(3.7)

for some absolute constants \( \kappa \geq 148 \), then

\[
\mathbb{E} [KL_n(s_0, \hat{s}_m)] \leq \left( 1 + \kappa^{-1} \right) \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + \text{pen}(m) + \eta \right) + \eta + \sqrt{\frac{K}{n}} \left( \frac{e^{q/2-1} q^{q/2}}{A_{\Sigma}^{q/2}} + H_{s_0} \right) \sqrt{2q A_{\gamma}}
\]  

\[
+ 302q \sqrt{\frac{K}{n}} \max (A_{\Sigma}, 1 + K A_C) \left( 1 + 2q \sqrt{q A_{\Sigma}} (5A_B^2 + 4A_{\Sigma} \ln n) \right) K \left( 1 + \left( A_{\gamma} + q A_{\beta} + \frac{q \sqrt{q}}{a_{\Sigma}} \right)^2 \right).
\]  

(3.8)

4. DISCUSSION AND COMPARISONS

Theorem 3.2 provides information about the performance of the Lasso as an \( l_1 \) regularization estimator for mixtures of Gaussian experts regression models. If the regularization parameter \( \lambda \) is properly chosen, the Lasso estimator, which is the solution of the \( l_1 \)-penalized empirical risk minimization problem, behaves in a comparable manner to the deterministic Lasso (or so-called oracle), which is the solution of the \( l_1 \)-penalized true risk minimization problem, up to an error term of order \( \lambda \). Note that the best model defined via

\[
\inf_{s_\psi \in S} \left( KL_n(s_0, s_\psi) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right)
\]  

(4.1)

is the one with the smallest \( l_1 \)-penalized true risk. However, since we do not have access to the true density \( s_0 \), we can not select that best model, which we call the oracle. The oracle is by definition the model belonging to the collection that minimizes the \( l_1 \)-penalized risk (4.1), which is generally assumed unknown.

Our result is non-asymptotic: the number \( n \) of observations is fixed while the number of covariates \( p \) can grow with respect to \( n \), and in fact can be much larger than \( n \). Note that, as in [26], the true order \( K \) of the MoE model (the true number of experts in our model) is assumed to be known. From a pragmatic perspective, one may estimate it via the AIC of [1], the BIC of [54], or slope heuristic of [4]. Our result directly follows the lineage of research of [38] and [10]. In fact, our theorem combined Vapnik’s structural risk minimization paradigm (e.g., [58]) and theory of model selection for conditional density estimation (e.g., [8]), which is an extended version of the density estimation results from [32].

A great amount of attention has been paid to obtaining a lower bound for \( \lambda \), cf. (3.2), in the oracle inequality, with optimal dependence on \( p \) and \( q \), which are the only parameters not to be fixed and which can grow the possibility that \( p \gg n \). In fact, the condition that \( \kappa \geq 148 \) is implied by Theorem 3.3, namely, the following
condition:
\[
\lambda \geq \kappa \frac{KB_n'}{\sqrt{n}} \left( q \ln n \sqrt{\ln(2p+1) + 1} \right), \quad B_n' = \max \left( A_{\Sigma} + 1 + KA_G, 1 + 2q\sqrt{q}A_{\Sigma} (5A_{\Sigma}^2 + 4A_{\Sigma} \ln n) \right),
\]
for some absolute constants \( \kappa \geq 148 \). Such conditions are obtained from the proof of Theorem 3.3, which requires the results of Proposition 5.1. In the proof of Proposition 5.1, the original condition for \( \kappa \) is as follows: let \( \kappa \geq 1 \) and assume that \( \text{pen}(m) = \lambda m \), for all \( m \in \mathbb{N}^* \) with
\[
\lambda \geq \kappa \frac{4KB_n}{\sqrt{n}} \left( 37q \ln n \sqrt{\ln(2p+1) + 1} \right).
\]

Then, to simplify the constant on the lower bound for \( \lambda \), we replaced \( 4 \times 37\kappa, \kappa \geq 1 \), by \( \kappa' = 4 \times 37\kappa \geq 148 \).

Note that, we recover the same dependence of form \( \sqrt{\ln(2p+1)} \) as for the homogeneous linear regression in [55] and of form \( \sqrt{\ln(2p+1) \ln n} \) for the mixture Gaussian regression models in [38]. On the contrary, the dependence on \( q \) for the mixture of multivariate Gaussian regression models in [10] was of form \( q^2 + q \), while we obtain the form \( q^2 / q \), here. The main reason is that we need to control the larger class, \( S \), of the finite mixture-of-experts models with soft-max gating functions and Gaussian experts, and we use a different technique to evaluate the upper bound on the uniform norm of the gradient for each element in \( S \). Furthermore, the dependence on \( n \) for the homogeneous linear regression in [55] was of order \( 1/\sqrt{n} \), while we have an extra \( (\ln n)^2 \) factor, here. In fact, the same situation can be found in the \( l_1 \)-oracle inequalities of [38], and [10]. As explained in [38], using a non-linear Kullback–Leibler information leads to a scenario where the linearity arguments developed in [55] with the quadratic loss function can not be exploited. Instead, we need to use the entropy arguments to handle our model, which leads to an extra \( (\ln n)^2 \) factor. Motivated by the frameworks from [38], [10], we have paid attention to giving an explicit dependence not only on \( n, p \) and \( q \), but also on the number of mixture components \( K \) as well as on the regressors and \( A_{\beta}, A_{\Sigma}, A_G \), all the quantities bounding the parameters of the model. However, we should be aware of the fact that these dependences may not be optimal. In our lower bound, we obtain the factor \( K^2 \) dependence instead of \( K \), as in [38], [10]. This can be explained via the fact that we used another technique to handle the more complex model when dealing with the upper bound on the uniform norm of the gradient of \( \ln s_{\psi} \) for \( \psi \in S \), in Lemma 5.6. We refer to Meynet [38, Remark 5.8] for some data sets for which the dependence on \( K \) might be reduced to an order of \( \sqrt{K} \) for the mixture Gaussian regression models. Establishing the optimal rates for such problems is still open.

We further note that our theorem ensures that there exists a \( \lambda \) sufficiently large for which the estimate has good properties, but does not give an explicit value for \( \lambda \). However, in Theorem 3.2, we provide at least the lower bound for the value of \( \lambda \) via the bound \( \lambda \geq \kappa C(p,q,n,K) \), where \( \kappa \geq 148 \), even though this value is obviously overly pessimistic. Possible solutions for calibrating penalties from the data are the AIC and BIC approaches, motivated by asymptotic arguments. Another method is the slope heuristic introduced first by [4] and further discussed in [3], which is a non-asymptotic criterion for selecting a model among a collection of models. Such strategy, however, will not be further considered here as the implementations will result in an overextension of the length and scope of the manuscript. Furthermore, the technical developments required for such methods is non-trivial and constitutes a significant research direction that we hope to pursue in the future. We refer the reader to the numerical experiments from [39] (using the slope heuristic), and [6], and [7] (using BIC), for the practical implementation of penalization methods in the framework of MoE regression models.

As in [10], we suppose that the regressors belong to \( \mathcal{X} = [0,1]^p \), for simplicity. However, the arguments in our proof are valid for covariates of any scale.

To the best of our knowledge, we are the first to prove a non-asymptotic \( l_1 \)-oracle inequality, of the form in Theorem 3.2, for the mixture of Gaussian experts regression models with \( l_1 \)-regularization. Note that by extending the theoretical developments for mixture of linear regression models in [27], a standard asymptotic theory for MoE models is established in Khalili [26]. Therefore, our non-asymptotic result in Theorem 3.2 can be considered as complementary to such asymptotic results for MoE models with soft-max gating functions.
Note that Theorem 3.2 is also complementary to Theorem 1 of [39], where [39] also considered mixture of Gaussian experts regression models with soft-max gating functions. Notice that they focused on model selection and obtained a weak oracle inequality for the penalized MLE, while we aimed to study the $l_1$-regularization properties of the Lasso estimators. However, we can compare their procedure with Theorem 3.3.

The main reason explaining their result being considered a weak oracle inequality is that we can see that Theorem 1 of [39] uses a difference divergence on the left (the JKL$^\otimes_\rho$, tensorized Jensen-Kullback-Leibler divergence), to that on the right (the KL$^\otimes_\rho$, tensorized Kullback-Leibler divergence). However, under a strong assumption, the two divergences are equivalent for the conditional PDFs considered. This strong assumption is nevertheless satisfied if we assume that $\mathcal{X}$ is compact, as is the case of $\mathcal{X} = [0, 1]^p$ in Theorem 3.3, $s_0$ is compactly supported, and the regression functions are uniformly bounded, and there is a uniform lower bound on the eigenvalues of the covariance matrices.

To illustrate the strictness of the compactness assumption for $s_0$, we only need to consider $s_0$ as a univariate Gaussian PDF, which obviously does not satisfy such a hypothesis. Therefore, in such case, Theorem 1 in [39] is actually weaker than Theorem 3.3, with respect to the compact support assumption on the true conditional PDF $s_0$. On the contrary, the only assumption used to establish Theorem 3.3 is the boundedness of the parameters of the mixtures, which is also assumed in Montuelle et al. [39, Theorem 1].

Furthermore, these boundedness assumptions also appeared in [55], [38], and [10], and are quite natural when working with maximum likelihood estimation [2, 34], at least when considering the problem of the unboundedness of the likelihood on the boundary of the parameter space [36, 53], and to prevent the likelihood from diverging. Nevertheless, by using the smaller divergence: JKL$^\otimes_\rho$ (or more strict assumptions on $s_0$ and $s_m$, so that the same divergence KL$^\otimes_\rho$ appears on both side of the oracle inequality in Theorem 3.3), Montuelle et al. [39, Theorem 1] obtained the faster rate of convergence of order $1/n$, while in Theorem 3.3, we only seek a rate of convergence of order $1/\sqrt{n}$. Therefore, in cases where there are no guarantees on the strict conditions such as the compactness of the support of $s_0$ and the uniform boundedness of the regression functions, Theorem 3.3 provides a theoretical foundation for the Lasso estimators with the order of convergence of $1/\sqrt{n}$ with only a boundedness assumption on the parameter space.

Note that the constant $1 + \kappa^{-1}$ from the upper bound in Theorem 3.3 and $C_1$ from Montuelle et al. [39, Theorem 1] can not be taken to be equal to 1. This fact is consequential when $s_0$ does not belong to the approximation class, i.e., when the model is misspecified. This problem also occurred in the $l_1$-oracle inequalities from [38] and [10]. Deriving an oracle inequality such that $1 + \kappa^{-1}$ can be replaced by 1, for the Kullback-Leibler loss, is still an open problem. However, one way to handle this difficulty is to use the approximation capacity of the class MoE regression models (see more in [42, 43, 45, 46]). Indeed, if we take large enough number of mixture components, $K$, we could approximate well a wide class of densities, then the term on the first right-hand side, $(1 + \kappa^{-1}) \inf_{s_0\in S} (\text{KL}_n(s_0, s_\psi) + \lambda \|\psi^{[1:2]}\|_1)$, in which $S$ depends on $K$, is small for $K$ well-chosen.

To the best of our knowledge, Theorem 2.8 from [35] is the only result in the literature which studies the minimax estimator for Gaussian mixture models. We believe that the establishment of such an extension to MoE regression models is not trivial. We hope to overcome such a challenge in the future.

5. Proof of the oracle inequality

We shall firstly provide a sketch of the proof. Motivated by ideas from [38] and [10], we study the Lasso as the solution of a penalized maximum likelihood model selection procedure over countable collections of models in an $l_1$-ball. Therefore, the main Theorem 3.2 is an immediate consequence of Theorem 3.3, which is an $l_1$-ball MoE regression model selection theorem for $l_1$-penalized maximum conditional likelihood estimation in the Gaussian mixture framework. The proof of Theorem 3.3 can be deduced from Proposition 5.1 and Proposition 5.2, which address the cases for large and small values of $Y$. Proposition 5.1 constitutes our main technical contribution. Its proof follows the arguments developed in the proof of a more general model selection theorem for maximum likelihood estimators: Massart [32, Theorem 7.11]. More precisely, the proof of Proposition 5.1 is in the spirit of Vapnik’s method of structural risk minimization, which is established initially in [58] and briefly summarized.
in Section 8.2 in [32]. In particularly, to obtain an upper bound of the empirical process in expectation, we shall use concentration inequalities combined with symmetrization arguments.

5.1. Main propositions used in this proof

Theorem 3.33 can be deduced from the two following propositions, which address the cases for large and small values of $Y$.

Proposition 5.1. Assume that we observe $((x_i, y_i))_{i \in [n]}$, with unknown conditional PDF $s_0$. Let $M_n > 0$ and consider the event

$$\mathcal{T} = \left\{ \max_{i \in [n]} \| Y_i \|_\infty = \max_{i \in [n]} \max_{z \in [q]} |Y_i[z]| \leq M_n \right\}.$$ 

For all $m \in \mathbb{N}^*$, consider the $l_1$-ball

$$S_m = \left\{ s_\psi \in S, \left\| \psi^{[1,2]} \right\|_1 \leq m \right\}$$

and let $\hat{s}_m$ be a $\eta_m$-ln-likelihood estimator in $S_m$, for some $\eta_m \geq 0$:

$$-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) \leq \inf_{s_m \in S_m} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (s_m(y_i|x_i)) \right) + \eta_m.$$ 

Assume that for all $m \in \mathbb{N}^*$, the penalty function satisfies $\text{pen}(m) = \lambda m$, where $\lambda$ is defined later. Next, we define the penalized likelihood estimator $\hat{s}_\tilde{m}$ with $\tilde{m}$ defined via the inequality

$$-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_\tilde{m}(y_i|x_i)) + \text{pen}(\tilde{m}) \leq \inf_{m \in \mathbb{N}^*} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(m) \right) + \eta, \quad (5.1)$$

for some $\eta > 0$. Then, if

$$\lambda \geq \frac{K B_n}{\sqrt{n}} \left( q \ln n \sqrt{\ln(2p + 1)} + 1 \right), \quad B_n = \max \left( A_\Sigma, 1 + K A_G \right) \left( 1 + q \sqrt{q} (M_n + A_\beta)^2 A_\Sigma \right),$$

for some absolute constants $\lambda \geq 148$, it holds that

$$\mathbb{E} [\text{KL}_n (s_0, \hat{s}_\tilde{m}) \mathbb{1}_\mathcal{T}] \leq \left( 1 + \frac{1}{\lambda} \right)^{-1} \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} \text{KL}_n (s_0, s_m) + \text{pen}(m) + \eta_m \right)$$

$$+ \frac{302 K^{3/2} q B_n}{\sqrt{n}} \left( 1 + \left( A_\gamma + q A_\beta + \frac{q \sqrt{q}}{A_\Sigma} \right)^2 \right) + \eta. \quad (5.2)$$

Proposition 5.2. Consider $s_0$, $\mathcal{T}$, and $\hat{s}_m$ as defined in Proposition 5.1. Denote by $\mathcal{T}^C$ the complement of $\mathcal{T}$, i.e.,

$$\mathcal{T}^C = \left\{ \max_{i \in [n]} \| Y_i \|_\infty = \max_{i \in [n]} \max_{z \in [q]} |Y_i[z]| > M_n \right\}.$$ 

Then,

$$\mathbb{E} [\text{KL}_n (s_0, \hat{s}_\tilde{m}) \mathbb{1}_{\mathcal{T}^C}] \leq \left( \frac{e^{q/2-1} q^{q/2}}{A_\Sigma^{q/2}} \right) + H_{s_0} \sqrt{2 Knq A_\gamma} e^{-\frac{M_n^2 - 2M_n A_\beta}{4 A_\Sigma}},$$
where $H_{s_0}$ is as defined in Lemma 3.1.

Theorem 3.3, and Propositions 5.1 and 5.2 are proved in the Sections 5.4, 5.5 and 5.6, respectively.

5.2. Additional notation

We first introduce some definitions and notations that we shall use in the proofs. For any measurable function $f : \mathbb{R}^q \to \mathbb{R}$, consider its empirical norm

$$\|f\| := \sqrt{\frac{1}{n} \sum_{i=1}^{n} f^2(Y_i|x_i)},$$

and its conditional expectation

$$E_{Y|X=x}[f] := \mathbb{E}[f(Y|X)|X=x] = \int_{\mathbb{R}^q} f(y|x)s_0(y|x)dy.$$

Furthermore, we also define its empirical process

$$P_n(f) := \frac{1}{n} \sum_{i=1}^{n} f(Y_i|x_i),$$

with expectation

$$P(f) = \frac{1}{n} \sum_{i=1}^{n} E_{Y_i|X_i=x_i}[f(Y_i|X_i)|X_i=x_i] = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^q} f(y|x_i)s_0(y|x_i)dy,$$

and the recentered process

$$\nu_n(f) := P_n(f) - P(f) = \frac{1}{n} \sum_{i=1}^{n} \left[ f(Y_i|x_i) - \int_{\mathbb{R}^q} f(y|x_i)s_0(y|x_i)dy \right].$$

For all $m \in \mathbb{N}^*$, recall that we consider the model

$$S_m = \left\{ s_\psi \in S, \|\psi^{[1,2]}\|_1 \leq m \right\},$$

and define

$$F_m = \left\{ f_m = -\ln \left( \frac{s_m}{s_0} \right) = \ln(s_0) - \ln(s_m), s_m \in S_m \right\}.$$ 

By using the basic properties of the infimum: for every $\epsilon > 0$, there exists $x_\epsilon \in A$, such that $x_\epsilon < \inf A + \epsilon$. Then let $\delta_{KL} > 0$ for all $m \in \mathbb{N}^*$, and let $\eta_m \geq 0$. It holds that there exist two functions $\hat{s}_m$ and $\bar{s}_m$ in $S_m$, such that

$$P_n(-\ln \hat{s}_m) \leq \inf_{s_m \in S_m} P_n(-\ln s_m) + \eta_m,$$

and

$$\text{KL}_n(s_0, \bar{s}_m) \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \delta_{KL}.$$ 

(5.7)

(5.8)
Define
\[
\hat{f}_m := -\ln \left( \frac{\hat{s}_m}{s_0} \right), \quad \text{and} \quad \hat{\eta}_m := -\ln \left( \frac{\eta_m}{s_0} \right).
\] (5.9)

Let \( \eta \geq 0 \) and fix \( m \in \mathbb{N}^* \). Further, define
\[
\hat{M}(m) = \{ m' \in \mathbb{N}^* | P_n (-\ln \hat{s}_{m'}) + \text{pen}(m') \leq P_n (-\ln \hat{s}_m) + \text{pen}(m) + \eta \}.
\] (5.10)

5.3. Proof of Theorem 3.2

Let \( \lambda > 0 \) and define \( \hat{m} \) to be the smallest integer such that \( \hat{s}^L(\lambda) \) belongs to \( S_{\hat{m}} \), i.e.,
\[
\hat{m} := \left\lfloor \left\| \psi^{[1,2]} \right\|_1 \right\rfloor + 1.
\]
Then using the definition of \( \hat{m} \), (2.8), (3.4), and \( S = \bigcup_{m \in \mathbb{N}^*} S_m \), we get

\[
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m^L(y_i|x_i)) + \lambda \hat{m} \leq -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m^L(y_i|x_i)) + \lambda \left( \left\| \psi^{[1,2]} \right\|_1 + 1 \right) = \inf_{s_\psi \in S} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (s_\psi(y_i|x_i)) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right) + \lambda = \inf_{m \in \mathbb{N}^*} \left( \inf_{s_\psi \in S_m} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (s_\psi(y_i|x_i)) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right) \right) + \lambda \leq \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (s_m(y_i|x_i)) + \lambda m \right) \right) + \lambda,
\]

which implies
\[
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m^L(y_i|x_i)) + \text{pen}(\hat{m}) \leq \inf_{m \in \mathbb{N}^*} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(m) \right) + \eta
\]

with \( \text{pen}(m) = \lambda m, \eta = \lambda, \) and \( \hat{s}_m \) is a \( \eta_m \)-in-likelihood minimizer in \( S_m \), with \( \eta_m \geq 0 \) defined by (3.5). Thus, \( \hat{s}_m^L(\lambda) \) satisfies (3.6) with \( \hat{s}_m^L(\lambda) \equiv \hat{s}_m \), i.e.,
\[
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(\hat{m}) \leq \inf_{m \in \mathbb{N}^*} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(m) \right) + \eta.
\] (5.11)

Then, Theorem 3.3 implies that if
\[
\lambda \geq \frac{K B_n'}{\sqrt{n}} + B_n' = \max (A_\Sigma, 1 + KA_G) \left( 1 + 2q \sqrt{\ln(A_\Sigma \ln n)} \right),
\]

where
\[
\lambda \geq \kappa \frac{K B_n'}{\sqrt{n}} \left( q \ln n \sqrt{\ln(2p + 1)} + 1 \right).
\]
for some absolute constants $\kappa \geq 148$, it holds that

$$
\mathbb{E} \left[ KL_n \left( s_0, \hat{s}^k(\lambda) \right) \right] \leq (1 + \kappa^{-1}) \inf_{s_\psi \in S} \left( KL_n \left( s_0, s_\psi \right) + \lambda \left\| \psi^{[1,2]} \right\|_1 \right) + \lambda \\
+ \sqrt{\frac{K}{n}} \left( \frac{e^{a/2 - 1} \pi^{q/2}}{A^q/2} + H_{a_\gamma} \right) \sqrt{2qA_\gamma} \\
+ 302q \sqrt{\frac{K}{n}} \max (A_{\Sigma}, 1 + KA_G) \left( 1 + 2\sqrt{qA_{\Sigma}} \left( 5A_{\beta}^2 + 4A_{\Sigma} \ln n \right) \right) \\
\times K \left( 1 + \left( A_\gamma + qA_{\beta} + \frac{q\sqrt{q}}{a_{\Sigma}} \right)^2 \right),
$$

as required.

5.4. Proof of Theorem 3.3

Let $M_n > 0$ and $\kappa \geq 148$. Assume that, for all $m \in \mathbb{N}^*$, the penalty function satisfies $\text{pen}(m) = \lambda_m$, with

$$
\lambda \geq \kappa \frac{KB_n}{\sqrt{n}} \left( q \ln n \sqrt{\ln(2p + 1)} + 1 \right).
$$

We derive, from Propositions 5.1 and 5.2, that any penalized likelihood estimator $\hat{s}_m$ with $\hat{m}$, satisfying

$$
-\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(\hat{m}) \leq \inf_{m \in \mathbb{N}^*} \left( -\frac{1}{n} \sum_{i=1}^{n} \ln (\hat{s}_m(y_i|x_i)) + \text{pen}(m) \right) + \eta,
$$

for some $\eta \geq 0$, yields

$$
\mathbb{E} \left[ KL_n \left( s_0, \hat{s}_m \right) \right] = \mathbb{E} \left[ KL_n \left( s_0, \hat{s}_m \right) 1_T \right] + \mathbb{E} \left[ KL_n \left( s_0, \hat{s}_m \right) 1_{\bar{T}} \right] \\
\leq (1 + \kappa^{-1}) \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n \left( s_0, s_m \right) + \text{pen}(m) + \eta_m \right) \\
+ \frac{302K^{3/2}qB_n}{\sqrt{n}} \left( 1 + \left( A_\gamma + qA_{\beta} + \frac{q\sqrt{q}}{a_{\Sigma}} \right)^2 \right) + \eta \\
+ \left( \frac{e^{q/2 - 1} \pi^{q/2}}{A^q/2} + H_{a_\gamma} \right) \sqrt{2KqA_\gamma} e^{-\frac{M_n^2 - 2M_n A_\beta}{4A_{\Sigma}}}.
$$

To obtain inequality (3.8), it only remains to optimize the inequality (5.13), with respect to $M_n$. Since the two terms depending on $M_n$, in (5.13), have opposite monotonicity with respect to $M_n$, we are looking for a value of $M_n$ such that these two terms are the same order with respect to $n$. Consider the positive solution

$$
M_n = A_\beta + \sqrt{A_{\beta}^2 + 4A_{\Sigma} \ln n} \text{ of the equation } X(X - 2A_\beta) - \ln n = 0. 
$$

Then, on the one hand,

$$
e^{-\frac{M_n^2 - 2M_n A_\beta}{4A_{\Sigma}}} \sqrt{n} = e^{-\ln n \sqrt{n}} = \frac{1}{\sqrt{n}}.$$
On the other hand, using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\), we have
\[
B_n = \max(A_\Sigma, 1 + K A_G) \left(1 + q \sqrt{q} (M_n + A_\beta)^2 A_\Sigma\right)
\]
\[
= \max(A_\Sigma, 1 + K A_G) \left(1 + q \sqrt{q} A_\Sigma \left(2 A_\beta + \sqrt{A_\beta^2 + 4 A_\Sigma \ln n}\right)^2\right)
\]
\[
\leq \max(A_\Sigma, 1 + K A_G) \left(1 + 2q \sqrt{q} A_\Sigma \left(5 A_\beta^2 + 4 A_\Sigma \ln n\right)\right),
\]
hence (5.13) implies (3.8).

5.5. **Proof of Proposition 5.1**

For every \(m' \in \mathcal{M}(m)\), from (5.10), (5.9), and (5.7), we obtain
\[
P_n \left(\hat{f}_{m'}\right) + \text{pen}(m') = P_n \left(\ln(s_0) - \ln(\hat{s}_{m'})\right) + \text{pen}(m') \text{ (using (5.9))}
\]
\[
\leq P_n \left(\ln(s_0) - \ln(s_\hat{m})\right) + \text{pen}(m) + \eta \text{ (using (5.10))}
\]
\[
\leq P_n \left(\ln(s_0) - \ln(\bar{s}_{m})\right) + \eta_m + \text{pen}(m) + \eta \text{ (using (5.7))}
\]
\[
= P_n \left(\bar{F}_m\right) + \text{pen}(m) + \eta_m + \eta \text{ (using (5.9))}
\]

By the definition of recentered process, \(\nu_n(\cdot)\), in (5.5), it holds that
\[
P \left(\hat{f}_{m'}\right) + \text{pen}(m') \leq P \left(\bar{F}_m\right) + \text{pen}(m) + \nu_n(\bar{F}_m) - \nu_n(\hat{f}_{m'}) + \eta + \eta_m.
\]

Taking into account (2.7) and (5.3), we obtain
\[
\text{KL}_n(s_0, \hat{s}_{m'}) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^q} \ln \left(\frac{s_0(y|x_i)}{\hat{s}_{m'}(y|x_i)}\right) s_0(y|x_i) dy = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^q} \hat{f}_{m'}(y|x_i) s_0(y|x_i) dy \text{ (using (5.9))}
\]
\[
= P \left(\hat{f}_{m'}\right) \text{ (using (5.4))}.
\]

Similarly, we also obtain \(\text{KL}_n(s_0, \bar{s}_{m}) = P \left(\bar{F}_m\right)\). Hence, (5.8) implies that
\[
\text{KL}_n(s_0, \hat{s}_{m'}) + \text{pen}(m') \leq \text{KL}_n(s_0, \bar{s}_{m}) + \text{pen}(m) + \nu_n(\bar{F}_m) - \nu_n(\hat{f}_{m'}) + \eta + \eta_m
\]
\[
\leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \nu_n(\bar{F}_m) - \nu_n(\hat{f}_{m'}) + \eta_m + \delta_{\text{KL}} + \eta.
\]

All that remains is to control the deviation of \(-\nu_n(\hat{f}_{m'}) = \nu_n(-\hat{f}_{m'})\). To handle the randomness of \(\hat{f}_{m'}\), we shall control the deviation of \(\sup_{f_{m'} \in F_{m'}} \nu_n(\cdot - f_{m'})\), since \(\hat{f}_{m'} \in F_{m'}\). Such control is provided by Lemma 5.3.

**Lemma 5.3 (Control of deviation).** Let \(M_n > 0\). Consider the event
\[
\mathcal{T} = \left\{\max_{i \in [n]} \|Y_i\|_\infty = \max_{i \in [n]} \max_{z \in [q]} |Y_{i,z}| \leq M_n\right\},
\]
and set
\[
B_n = \max(A_\Sigma, 1 + K A_G) \left(1 + q \sqrt{q} (M_n + A_\beta)^2 A_\Sigma\right), \text{ and}
\]
\[
\Delta_{m'} = m' \sqrt{\ln(2p + 1) \ln n + 2\sqrt{K} \left(A_\gamma + q A_\beta + \frac{q \sqrt{q}}{A_\Sigma}\right)}. 
\]
Then, for all \(m' \in \mathbb{N}^*\), and for all \(t > 0\), with probability greater than \(1 - e^{-t}\),

\[
\sup_{f_{m'} \in F_{m'}} |\nu_n(-f_{m'})| \leq \frac{4KB_n}{\sqrt{n}} \left[ 37q\Delta_{m'} + \sqrt{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) \sqrt{t} \right].
\]  

(5.17)

**Proof.** The proof appears in Section 6.1. \(\square\)

From (5.14) and (5.17), we derive that on the event \(\mathcal{T}\), which means that we multiply the considered term by the indicator function \(1_{\mathcal{T}}\), for all \(m \in \mathbb{N}^*\), and \(t > 0\), with probability larger than \(1 - e^{-t}\),

\[
\text{KL}_n(s_0, \hat{s}_{m'}) + \text{pen}(m') \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \nu_n(\hat{f}_m) - \nu_n(\hat{f}_{m'}) + \eta_m + \delta_{KL} + \eta.
\]

\[
\leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \nu_n(\hat{f}_m) + \eta_m + \delta_{KL} + \eta
\]

\[
+ \frac{4KB_n}{\sqrt{n}} \left[ 37q\Delta_{m'} + \sqrt{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) \sqrt{t} \right]
\]

\[
\leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \nu_n(\hat{f}_m) + \eta_m + \delta_{KL} + \eta
\]

\[
+ \frac{4KB_n}{\sqrt{n}} \left[ 37q\Delta_{m'} + \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + (z + m + m') \right],
\]  

if \(m' \in \hat{\mathcal{M}}(m)\).  

(5.18)

Here, we use the fact that \(\hat{f}_{m'} \in F_{m'}\) and get the last inequality using the fact that \(2ab \leq a^2 + b^2\) for \(b = \sqrt{t}\), and \(a = \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) / \sqrt{2}\).

It remains to sum up the tail bounds (5.18) over all possible values of \(m \in \mathbb{N}^*\) and \(m' \in \hat{\mathcal{M}}(m)\). To get an inequality valid on a set of high probability, we need to adequately choose the value of the parameter \(t\), depending on \(m \in \mathbb{N}^*\) and \(m' \in \hat{\mathcal{M}}(m)\). Let \(z > 0\), for all \(m \in \mathbb{N}^*\) and \(m' \in \hat{\mathcal{M}}(m)\), and apply (5.18) to obtain \(t = z + m + m'\). Then, on the event \(\mathcal{T}\), for all \(m \in \mathbb{N}^*\), with probability larger than \(1 - e^{-(z+m+m')}\),

\[
\text{KL}_n(s_0, \hat{s}_{m'}) + \text{pen}(m') \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \nu_n(\hat{f}_m) + \eta_m + \delta_{KL} + \eta
\]

\[
+ \frac{4KB_n}{\sqrt{n}} \left[ 37q\Delta_{m'} + \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + (z + m + m') \right],
\]  

if \(m' \in \hat{\mathcal{M}}(m)\).  

(5.19)

Here, (5.19) is equivalent to

\[
\text{KL}_n(s_0, \hat{s}_{m'}) - \nu_n(\hat{f}_m) \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \text{pen}(m) + \frac{4KB_n}{\sqrt{n}} m + \eta_m + \delta_{KL} + \eta
\]

\[
+ \left[ \frac{4KB_n}{\sqrt{n}} (37q\Delta_{m'} + m') - \text{pen}(m') \right]
\]

\[
+ \frac{4KB_n}{\sqrt{n}} \left[ \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + z \right].
\]  

(5.20)

Note that with probability larger than \(1 - e^{-z}\), (5.19) holds simultaneously for all \(m \in \mathbb{N}^*\) and \(m' \in \hat{\mathcal{M}}(m)\). Indeed, by defining the event

\[
\cap_{(m, m') \in \mathbb{N}^* \times \hat{\mathcal{M}}(m)} \Omega_{m, m'} = \{ w : w \in \Omega \text{ such that the event in (5.19) holds} \},
\]
it holds that, on the event \( \mathcal{T} \),

\[
\mathbb{P} \left( \bigcap_{(m,m') \in \mathbb{N}^* \times \tilde{\mathcal{M}}(m)} \Omega_{m,m'} \right) = 1 - \mathbb{P} \left( \bigcup_{(m,m') \in \mathbb{N}^* \times \tilde{\mathcal{M}}(m)} \Omega_{m,m'}^C \right) \\
\geq 1 - \sum_{(m,m') \in \mathbb{N}^* \times \tilde{\mathcal{M}}(m)} \mathbb{P} \left( \Omega_{m,m'}^C \right) \\
\geq 1 - \sum_{(m,m') \in \mathbb{N}^* \times \tilde{\mathcal{M}}(m)} e^{-z + m + m'} \\
\geq 1 - \sum_{(m,m') \in \mathbb{N}^* \times \mathbb{N}^*} e^{-z + m + m'} \\
= 1 - e^{-z} \left( \sum_{m \in \mathbb{N}^*} e^{-m} \right)^2 \\
\geq 1 - e^{-z},
\]

where we get the last inequality by using the geometric series

\[
\sum_{m=1}^{\infty} (e^{-1})^m = \sum_{m=0}^{\infty} (e^{-1})^m - 1 = \frac{1}{1 - e^{-1}} - 1 = \frac{e}{e - 1} - 1 = \frac{1}{e - 1} < 1.
\]

Taking into account (5.16), we get

\[
\text{KL}_n (s_0, \hat{s}_{m'}) - \nu_n (\mathcal{F}_m) \leq \inf_{s_m \in S_m} \text{KL}_n (s_0, s_m) + \left[ \text{pen}(m) + \frac{4KB_n}{\sqrt{n}} m \right] + \eta_m + \delta_{KL} + \eta \\
+ \left[ \frac{4KB_n}{\sqrt{n}} \left( 37q \ln n \sqrt{\ln(2p+1) + 1} \right) m' - \text{pen}(m') \right] \\
+ \frac{4KB_n}{\sqrt{n}} \left[ \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + 74qK \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) + z \right].
\]

(5.21)

Now, let \( \kappa \geq 1 \) and assume that \( \text{pen}(m) = \lambda m \), for all \( m \in \mathbb{N}^* \) with

\[
\lambda \geq \kappa \frac{4KB_n}{\sqrt{n}} \left( 37q \ln n \sqrt{\ln(2p+1) + 1} \right),
\]

(5.22)
Then, (5.21) implies

\[
\begin{align*}
&\text{KL}_n(s_0, \tilde{s}_{m'}) - \nu_n(\tilde{f}_m) \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \left[ \lambda m + \frac{4KB_n}{\sqrt{n}}m \right] + \eta_m + \delta_{KL} + \eta \\
&\quad + \left[ \frac{4KB_n}{\sqrt{n}} \left( 37q \ln n \sqrt{\ln(2p + 1) + 1} \right) m' - \lambda m' \right] \\
&\quad + \left[ \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + 74q\sqrt{K} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) + z \right] \\
&\quad \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + \left[ \text{pen}(m) + \frac{4KB_n}{\sqrt{n}}m \right] + \eta_m + \delta_{KL} + \eta \\
&\quad + \left[ \lambda \kappa^{-1}m' - \lambda m' \right] \\
&\quad + \left[ \frac{1}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + 74q\sqrt{K} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) + z \right].
\end{align*}
\]

Next, using the inequality \(2ab \leq \beta^{-1}a^2 + \beta^{-1}b^2\) for \(a = \sqrt{K}, b = K \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)\), and \(\beta = \sqrt{K}\), and the fact that \(K \leq K^{3/2}\), for all \(K \in \mathbb{N}^*\), it follows that

\[
\begin{align*}
&\text{KL}_n(s_0, \tilde{s}_{m'}) - \nu_n(\tilde{f}_m) \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m + \delta_{KL} + \eta \\
&\quad + 4B_n \left[ \frac{qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + 74q\sqrt{K} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) + Kz \right] \\
&\quad \leq \inf_{s_m \in S_m} \text{KL}_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m + \delta_{KL} + \eta \\
&\quad + 4B_n \left[ 37qK^{1/2} + \frac{75qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + Kz \right].
\end{align*}
\] (5.23)
By (5.1) and (5.10), $$\hat{m}$$ belongs to $$\hat{M}(m)$$, for all $$m \in \mathbb{N}^*$$, so we deduce from (5.23) that on the event $$\mathcal{T}$$, for all $$z > 0$$, with probability greater than 1 $$- e^{-z}$$,

$$KL_n(s_0, \hat{s}_m) - \nu_n(\mathcal{T}_m) \leq \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m \right) + \eta + \delta_{KL} + 4B_n \sqrt{n} \left[ 37qK^{1/2} + \frac{75qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + Kz \right].$$ \hspace{1cm} (5.24)

Note that for any non-negative random variable $$Z$$ and any $$a > 0$$, $$\mathbb{E}[Z] = a \int_{z \geq 0} \mathbb{P}(Z > az)dz$$. Indeed, if we let $$t = az$$, then $$dz = adt$$ and

$$a \int_{z \geq 0} \mathbb{P}(Z > az)dz = a \int_{az}^{\infty} \mathbb{P}(Z > u)du = a \int_{az}^{\infty} \mathbb{P}(Z > uz)du = \int_{z \geq 0} \mathbb{P}(Z > az)dz = a \int_{z \geq 0} e^{-z}dz = a. \hspace{1cm} (5.25)$$

Then, we define the following random variable w.r.t. the random response $$Y_{[n]} := (Y_i)_{i \in [n]}$$:

$$Z := KL_n(s_0, \hat{s}_m) - \nu_n(\mathcal{T}_m) - \left[ \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m \right) + \eta + \delta_{KL} \right] - 4B_n \sqrt{n} \left[ 37qK^{1/2} + \frac{75qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 \right].$$

Then by (5.24), on the even $$\mathcal{T}$$, it holds that $$P(Z \leq az) \geq 1 - e^{-z}$$ and if $$Z \leq 0$$ then $$P(Z < az) = 1$$, for all $$z > 0$$, where $$a = \frac{4B_nK}{\sqrt{n}} > 0$$. Therefore, it is sufficient to consider $$Z \geq 0$$. In this case, by (5.26), it holds that

$$\mathbb{E}_{Y_{[n]}}[Z \mathbb{1}_{\mathcal{T}}] \leq \mathbb{E}_{Y_{[n]}}[Z] = a \int_{z \geq 0} \mathbb{P}(Z > az)dz \leq a \int_{z \geq 0} e^{-z}dz = a. \hspace{1cm} (5.27)$$

Then, by integrating (5.24) over $$z > 0$$ using (5.27), the fact that $$\mathbb{E}_{Y_{[n]}}[\nu_n(\mathcal{T}_m)] = \mathbb{E}_{Y_{[n]}}[P_n(\mathcal{T}_m) \mathbb{1}_{\mathcal{T}}] = \mathbb{E}_{Y_{[n]}}[P(\mathcal{T}_m) \mathbb{1}_{\mathcal{T}}] = 0$$, that $$\delta_{KL} > 0$$ can be chosen arbitrary small, and $$\mathbb{E}_{Y_{[n]}}[\mathbb{1}_{\mathcal{T}}] = \mathbb{P}(\mathcal{T}) \leq 1$$, we obtain that

$$\mathbb{E}[KL_n(s_0, \hat{s}_m) \mathbb{1}_{\mathcal{T}}] \leq \left[ \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m \right) + \eta \right] \mathbb{E}_{Y_{[n]}}[\mathbb{1}_{\mathcal{T}}] + \frac{4B_n}{\sqrt{n}} \left[ 37qK^{1/2} + \frac{75qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + K \right] \mathbb{E}_{Y_{[n]}}[\mathbb{1}_{\mathcal{T}}]$$

$$\leq \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m \right) + \eta + \frac{4B_n}{\sqrt{n}} \left[ 37qK^{3/2} + \frac{75qK^{3/2}}{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 + qK^{3/2} \right]$$

$$\leq \inf_{m \in \mathbb{N}^*} \left( \inf_{s_m \in S_m} KL_n(s_0, s_m) + (1 + \kappa^{-1}) \text{pen}(m) + \eta_m \right) + \eta + \frac{302K^{3/2}qB_n}{\sqrt{n}} \left( 1 + \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)^2 \right). \hspace{1cm} (5.28)$$
5.6. Proof of Proposition 5.2

By the Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ \text{KL} \left( s_0, \hat{s}_m \right) \mathbb{I}_{T^c} \right] \leq \sqrt{\mathbb{E} \left[ \text{KL}^2 \left( s_0, \hat{s}_m \right) \right] \mathbb{E} \left[ T^c \right]} = \sqrt{\mathbb{E} \left[ \text{KL}^2 \left( s_0, \hat{s}_m \right) \right] \mathbb{P} \left( T^c \right)}. \tag{5.29}
\]

We seek to bound the two terms of the right-hand side of (5.29).

For the first term, let us bound \( \text{KL} \left( s_0 \left( \cdot | x \right), s_{\psi} \left( \cdot | x \right) \right) \), for all \( s_\psi \in S \) and \( x \in \mathcal{X} \). Let \( s_\psi \in S \) and \( x \in \mathcal{X} \). Then, we obtain

\[
\text{KL} \left( s_0 \left( \cdot | x \right), s_{\psi} \left( \cdot | x \right) \right) = \int_{\mathbb{R}^q} \ln \left( \frac{s_0(y|x)}{s_{\psi}(y|x)} \right) s_0(y|x)dy \\
= \int_{\mathbb{R}^q} \ln \left( s_0(y|x) \right) s_0(y|x)dy - \int_{\mathbb{R}^q} \ln \left( s_{\psi}(y|x) \right) s_0(y|x)dy \\
\leq - \int_{\mathbb{R}^q} \ln \left( s_{\psi}(y|x) \right) s_0(y|x)dy + H_{s_0}, \forall x \in \mathcal{X} \text{ (using (3.1))}. \tag{5.30}
\]

Thus, for all \( y \in \mathbb{R}^q \),

\[
\ln \left( s_{\psi}(y|x) \right) s_0(y|x) \geq \ln \left[ \sum_{k=1}^{K} \frac{a_G \det\left( \Sigma_k^{-1} \right)^{1/2}}{(2\pi)^{q/2}} \exp \left( - \left( y^T \Sigma_k^{-1} y + (\beta_{k0} + \beta_k x)^T \Sigma_k^{-1} (\beta_{k0} + \beta_k x) \right) \right) \right] \\
\times \sum_{k=1}^{K} \frac{a_G \det\left( \Sigma_{0,k}^{-1} \right)^{1/2}}{(2\pi)^{q/2}} \exp \left( - \left( y^T \Sigma_{0,k}^{-1} y + (\beta_{0,k0} + \beta_{0,k} x)^T \Sigma_{0,k}^{-1} (\beta_{0,k0} + \beta_{0,k} x) \right) \right) \] (using (2.5) and \( -(a - b)^T A (a - b)/2 \geq -(a^T A a + b^T A b), \text{ e.g., } a = y, b = \beta_{k0} + \beta_k x, A = \Sigma_k \) \]

\[
\geq \ln \left[ \sum_{k=1}^{K} \frac{a_G q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^T \Sigma_k^{-1} y + (\beta_{k0} + \beta_k x)^T \Sigma_k^{-1} (\beta_{k0} + \beta_k x) \right) \right) \right] \\
\times \sum_{k=1}^{K} \frac{a_G q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^T \Sigma_{0,k}^{-1} y + (\beta_{0,k0} + \beta_{0,k} x)^T \Sigma_{0,k}^{-1} (\beta_{0,k0} + \beta_{0,k} x) \right) \right) \] (using (2.4))

\[
\geq \ln \left[ K \frac{a_G q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^T y + q A_\beta^2 \right) A_\Sigma \right) \right] \times K \frac{a_G q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^T y + q A_\beta^2 \right) A_\Sigma \right) \] (using (2.4)), \tag{5.31}
\]

where, in the last inequality, we use the fact that for all \( u \in \mathbb{R}^q \). By using the eigenvalue decomposition of \( \Sigma_1 = P^T DP \),

\[
| u^T \Sigma_1 u | = | u^T P^T D Pu | \leq \| Pu \|_2 \leq M(D) \| Pu \|_2 \leq A_\Sigma \| u \|_2 \leq A_\Sigma q \| u \|_\infty^2 ,
\]
where in the last inequality, we used the fact that (A.4). Therefore, setting \( u = \sqrt{2A_\Sigma}y \) and \( h(t) = t \ln t \), for all \( t \in \mathbb{R} \), and noticing that \( h(t) \geq h(e^{-1}) = -e^{-1} \), for all \( t \in \mathbb{R} \), and from (5.30) and (5.31), we get that

\[
\text{KL}(s_0 (\cdot | x), s_\psi (\cdot | x)) - H_{s_0} \leq - \int_{\mathbb{R}^q} \ln \left[ K \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) \right] \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) dy
\]

\[
= - \int_{\mathbb{R}^q} \ln \left[ K \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) \right] \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) dy
\]

\[
= - \int_{\mathbb{R}^q} \ln \left[ K \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) \right] \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) dy
\]

\[
= - \int_{\mathbb{R}^q} \ln \left[ K \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) \right] \frac{a_{\gamma/2}q/2}{(2\pi)^{q/2}} \exp \left( - \left( y^\top y + qA_\Sigma^2 \right) A_\Sigma \right) dy
\]

\[
\leq - \sum_{i=1}^n \text{KL}(s_0 (\cdot | x_i), s_\psi (\cdot | x_i)) \leq \frac{e^{q/2-1} \pi^{q/2}}{A_\Sigma^{q/2}} + H_{s_0},
\]

and note that \( \tilde{s}_m \in S \), thus

\[
\sqrt{\mathbb{E} \left[ \text{KL}^2_n (s_0, \tilde{s}_m) \right]} \leq \frac{e^{q/2-1} \pi^{q/2}}{A_\Sigma^{q/2}} + H_{s_0}.
\]

We now provide an upper bound for \( \mathbb{P} (\mathcal{T}^C) \):

\[
\mathbb{P} (\mathcal{T}^C) \leq \sum_{i=1}^n \mathbb{P} (\|Y_i\|_\infty > M_n).
\]

For all \( i \in [n] \),

\[
Y_i | x_i \sim \sum_{k=1}^K g_k (x_i; \gamma) \mathcal{N}_q (\beta_k + \beta_k x_i, \Sigma_k),
\]

so we see from (5.34) that we need to provide an upper bound on \( \mathbb{P} (\|Y_x\| > M_n) \), with

\[
Y_x \sim \sum_{k=1}^K g_k (x; \gamma) \mathcal{N}_q (\beta_k + \beta_k x, \Sigma_k), x \in \mathcal{X}.
\]
First, using Chernoff’s inequality for a centered Gaussian variable (see Lemma A.8), and the fact that \( \psi \) belongs to the bounded space \( \Psi \) (defined by (2.4)), and that \( \sum_{k=1}^{K} g_k (x; \gamma) = 1 \), we get

\[
\Pr (\|Y_x\|_\infty > M_n) = \sum_{k=1}^{K} g_k (x; \gamma) \frac{g_k (x; \gamma) \Pr (\|Y_x,k\|_\infty > M_n)}{\sum_{k=1}^{K} g_k (x; \gamma) \sum_{z=1}^{q} \Pr (|Y_{x,k}|_z > M_n)} \leq \sum_{k=1}^{K} g_k (x; \gamma) \sum_{z=1}^{q} \left( \Pr (U > M_n - [\beta_k + \beta_k x]_z) + \Pr \left( U < -M_n + [\beta_k + \beta_k x]_z \right) \right)
\]

\[
\leq 2 \sum_{k=1}^{K} g_k (x; \gamma) \sum_{z=1}^{q} \exp \left( -\frac{1}{2} \frac{(M_n - [\beta_k + \beta_k x])^2}{\|\Sigma_k\|_{z,z}^2} + \frac{1}{2} \frac{M_n + [\beta_k + \beta_k x]_z^2}{\|\Sigma_k\|_{z,z}^2} \right) \leq 2KA_\gamma q e^{-\frac{M_n^2 - 2M_n A_\beta}{4A_\Sigma}},
\]

where

\[
Y_{x,k} \sim \mathcal{N} (\beta_k + \beta_k x, \Sigma_k), |Y_{x,k}|_z \sim \mathcal{N} \left( [\beta_k + \beta_k x]_z, [\Sigma_k]_{z,z} \right), \text{ and } U = \frac{|Y_{x,k}|_z - [\beta x]_z}{\|\Sigma_k\|_{z,z}^2} \sim \mathcal{N}(0, 1),
\]

and using the facts that \( e^{-\frac{1}{2} \frac{[\beta_0 + \beta_k]_z^2}{\|\Sigma_k\|_{z,z}^2}} \leq 1 \) and \( \max_{1 \leq z \leq q} \|\Sigma_k\|_{z,z} \| \leq \|\Sigma_k\|_2 = M (\Sigma_k) = m (\Sigma_k^{-1}) \leq A_\Sigma \). We derive from (5.34) and (5.35) that

\[
\Pr (T^c) \leq 2KnqA_\gamma e^{-\frac{M_n^2 - 2M_n A_\beta}{4A_\Sigma}},
\]

and finally from (5.29), (5.33), and (5.36), we obtain

\[
\mathbb{E} [\text{KL}_{\gamma} (s_0, \hat{s}_m) \mathbb{1}_{T^c}] \leq \left( \frac{e^{q/2 - 1} e^{q/2}}{A_\Sigma^{q/2}} + H_{s_0} \right) \sqrt{2KnqA_\gamma e^{-\frac{M_n^2 - 2M_n A_\beta}{4A_\Sigma}}},
\]
6. Proofs of technical lemmas

6.1. Proof of Lemma 5.3

Let \( m \in \mathbb{N}^* \), to control the deviation

\[
\sup_{f_m \in F_m} \left| \nu_n (-f_m) \right| \mathbb{I}_T = \sup_{f_m \in F_m} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ f_m (Y_i | x_i) - \mathbb{E} [ f_m (Y_i | x_i) ] \right\} \mathbb{I}_T \right|, \tag{6.1}
\]

we shall use concentration and symmetrization arguments. We shall first use the following concentration inequality, Lemma 6.1, which is an adaption of Wainwright [59, Theorem 4.10].

**Lemma 6.1** (Theorem 4.10 from [59]). Let \( Z_1, \ldots, Z_n \) be independent random variables with values in some space \( \mathcal{Z} \) and let \( \mathcal{F} \) be a class of integrable real-valued functions with domain on \( \mathcal{Z} \). Assume that

\[
\sup_{f \in \mathcal{F}} \| f \|_\infty \leq R_n \quad \text{for some non-random constants } R_n < \infty. \tag{6.2}
\]

Then, for all \( t > 0 \),

\[
\mathbb{P} \left( \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} [ f(Z_i) - \mathbb{E} [ f(Z_i) ] ] \right| > \mathbb{E} \left( \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} [ f(Z_i) - \mathbb{E} [ f(Z_i) ] ] \right| \right) + 2\sqrt{2R_n \frac{T}{n}} \right) \leq e^{-t}. \tag{6.3}
\]

That is, with probability greater than \( 1 - e^t \),

\[
\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} [ f(Z_i) - \mathbb{E} [ f(Z_i) ] ] \right| \leq \mathbb{E} \left( \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} [ f(Z_i) - \mathbb{E} [ f(Z_i) ] ] \right| \right) + 2\sqrt{2R_n \frac{T}{n}} \tag{6.4}
\]

Then, we propose to bound \( \mathbb{E} \left( \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} [ f(Z_i) - \mathbb{E} [ f(Z_i) ] ] \right| \right) \) due to the following symmetrization argument. The proof of this result can be found in [57].

**Lemma 6.2** (See Lemma 2.3.6 in [57]). Let \( Z_1, \ldots, Z_n \) be independent random variables with values in some space \( \mathcal{Z} \) and let \( \mathcal{F} \) be a class of real-valued functions on \( \mathcal{Z} \). Let \( (\epsilon_1, \ldots, \epsilon_n) \) be a Rademacher sequence independent of \( (Z_1, \ldots, Z_n) \). Then,

\[
\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(Z_i) - \mathbb{E} [ f(Z_i) ] \right| \right] \leq 2\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(Z_i) \right| \right]. \tag{6.5}
\]

From (6.5), the problem is to provide an upper bound on

\[
\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(Z_i) \right| \right].
\]

To do so, we shall apply the following lemma, which is adapted from Lemma 6.1 in [32].

**Lemma 6.3** (See Lemma 6.1 in [32]). Let \( Z_1, \ldots, Z_n \) be independent random variables with values in some space \( \mathcal{Z} \) and let \( \mathcal{F} \) be a class of real-valued functions on \( \mathcal{Z} \). Let \( (\epsilon_1, \ldots, \epsilon_n) \) be a Rademacher sequence, independent of \( (Z_1, \ldots, Z_n) \). Define \( R_n \), a non-random constant, such that

\[
\sup_{f \in \mathcal{F}} \| f \|_n \leq R_n. \tag{6.6}
\]
Then, for all $S \in \mathbb{N}^*$,

$$
E \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(Z_i) \right| \right] \leq R_n \left( \frac{6}{\sqrt{n}} \sum_{s=1}^{S} 2^{-s} \sqrt{\ln \left[ 1 + M \left( 2^{-s} R_n, \| \cdot \|_n \right) \right]} + 2^{-S} \right), \quad (6.7)
$$

where $M(\delta, \mathcal{F}, \| \cdot \|_n)$ stands for the $\delta$-packing number (see Definition A.5) of the set of functions $\mathcal{F}$, equipped with the metric induced by the norm $\| \cdot \|_n$.

We are now able to prove Lemma 5.3. Indeed, given any fixed values $x_1, \ldots, x_n \in \mathcal{X}$, in order to control $\sup_{f_m \in F_m} |\nu_n (-f_m)| \mathbb{1}_T$ from the (6.1), we would like to apply Lemmas 6.1–6.3. On the one hand, we see from (6.6) that we need an upper bound of $\sup_{f_m \in F_m} \| f_m \|_{\infty} \mathbb{1}_T$. On the other hand, we see from (6.7) that on the event $T$, we need to bound the entropy of the set of functions $F_m$, equipped with the metric induced by the norm $\| \cdot \|_n$. Such bounds are provided by the two following lemmas.

Given $M_n > 0$ and the event

$$
T = \left\{ \max_{i \in [n]} \| Y_i \|_{\infty} = \max \max_{i \in [n]} \| Y_i \| \leq M_n \right\},
$$

let $B_n = \max (A, 1 + KA) \left( 1 + q\sqrt{q}(M_n + A)^2 A \right)$.

**Lemma 6.4.** For all $m \in \mathbb{N}^*$,

$$
\sup_{f_m \in F_m} \| f_m \|_{\infty} \mathbb{1}_T \leq 2KB_n \left( A + qA\beta + \frac{q\sqrt{q}}{A\beta} \right) =: R_n. \quad (6.8)
$$

**Proof.** See Section 6.2.1. \qed

**Lemma 6.5.** Let $\delta > 0$ and $m \in \mathbb{N}^*$. On the event $T$, we have the following upper bound of the $\delta$-packing number of the set of functions $F_m$, equipped with the metric induced by the norm $\| \cdot \|_n$:

$$
M(\delta, F_m, \| \cdot \|_n) \leq \left( 2 + 1 \right)^{\frac{(2p + 1)2^{2(p+1)^2}K^2m^2}{\delta}} \left( 1 + \frac{18B_n K \beta}{\delta} \right)^K \left( 1 + \frac{18B_n K A \beta}{\gamma \delta} \right)^K \left( 1 + \frac{18B_n K q \sqrt{q}}{\alpha \beta \gamma} \right)^K.
$$

**Proof.** See Section 6.2.2. \qed

**Lemma 6.6** (Lemma 5.9 from [38]). Let $\delta > 0$ and $(x_{ij})_{i \in [n]; j = 1, \ldots, p} \in \mathbb{R}^{np}$. There exists a family $\mathcal{B}$ of $(2p + 1)\| x \|_{\max, n}/\delta^2$ vectors in $\mathbb{R}^p$, such that for all $\beta \in \mathbb{R}^p$, with $\| \beta \|_1 \leq 1$, where $\| x \|_{\max, n} = \frac{1}{p} \sum_{i=1}^{n} \max_{j \in \{1, \ldots, p\}} x_{ij}^2$, there exists $\beta' \in \mathcal{B}$, such that

$$
\frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{p} (\beta_j - \beta'_j) x_{ij} \right)^2 \leq \delta^2.
$$

**Proof.** See in the proof of Lemma 5.9 [38]. \qed

Via the upper bounds provided in Lemmas 6.4 and 6.5, we can apply Lemma 6.3 to get an upper bound on

$$
E \left[ \sup_{f_m \in F_m} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f_m(Y_i) x_i \right| \right] \mathbb{1}_T.
$$

Lemmas 6.1 and 6.3 can be utilized via defining a suitable class of integrable real-valued functions as follows:

$$
\mathcal{F} := \left\{ f := f_m \mathbb{1}_{|f_m| \leq R_n} : f_m \in F_m, \right\}, \quad Z_i := Y_i x_i, \forall i \in [n]. \quad (6.9)
$$
Indeed, by definition, it holds that
\[
\sup_{f \in \mathcal{F}} \|f\|_n \leq \sup_{f \in \mathcal{F}} \|f\|_\infty = \sup_{f \in \mathcal{F}, z \in \mathcal{Z}} |f(z)| = \sup_{f \in \mathcal{F}, z \in \mathcal{Z}} |f_m(z)|_{\|f_m(z)\| \leq R_n} \leq R_n.
\] (6.10)

We thus obtain the following results.

**Lemma 6.7.** Let \( m \in \mathbb{N}^* \), consider \( (\epsilon_1, \ldots, \epsilon_n) \), a Rademacher sequence independent of \( (Y_1, \ldots, Y_n) \). Then, it holds that
\[
\mathbb{E} \left[ \sup_{f_m \in \mathcal{F}_m} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f_m(Y_i|x_i) \right| \right] \leq \frac{74KB_nq}{\sqrt{n}} \Delta_m, \quad \text{where}
\]
\[
\Delta_m := m \sqrt{\ln(2p + 1) \ln n + 2\sqrt{K} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right)}.
\] (6.11)

**Proof.** See Section 6.2.3. \( \square \)

Finally, for all \( m \in \mathbb{N}^* \) and \( t > 0 \), with probability greater than \( 1 - e^{-t} \), we obtain
\[
\sup_{f_m \in \mathcal{F}_m} |\nu_n(-f_m)| \leq \sup_{f_m \in \mathcal{F}_m} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ f_m(Y_i|x_i) - \mathbb{E} [f_m(Y_i|x_i)] \right\} \right| \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ f(Z_i) - \mathbb{E} [f(Z_i)] \right\} \right| \right] \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ f(Z_i) - \mathbb{E} [f(Z_i)] \right\} \right| \right] \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(Z_i) \right| \right] \leq \frac{148KB_nq}{\sqrt{n}} \Delta_m + 4\sqrt{2KB_n} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) \sqrt{\frac{t}{n}}
\] (using Lemma 6.7 and \( R_n = 2KB_n \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) \))
\[
\leq \frac{4KB_n}{\sqrt{n}} \left[ 37q \Delta_m + \sqrt{2} \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_\Sigma} \right) \sqrt{t} \right].
\] (6.12)

This leads to the end of the proof of Lemma 5.3.
6.2. Proofs of Lemmas 6.4–6.7

The proofs of Lemmas 6.4–6.5 require an upper bound on the uniform norm of the gradient of \( \ln s_\psi \), for \( s_\psi \in S \). We begin by providing such an upper bound.

**Lemma 6.8.** Given \( s_\psi \), as described in (2.6), it holds that

\[
\sup_{x \in X} \sup_{\psi \in \Psi} \left\| \frac{\partial \ln (s_\psi (|x|))}{\partial \psi} \right\|_\infty \leq G(\cdot),
\]

where

\[
G : \mathbb{R}^d \ni y \mapsto G(y) = \max \left( A_\Sigma, 1 + KA_G \right) \left( 1 + q \sqrt{q} (\|y\|_\infty + A_\beta)^2 \right). \tag{6.13}
\]

**Proof.** Let \( s_\psi \in S \), with \( \psi = (\gamma, \beta, \Sigma) \). From now on, we consider any \( x \in X \), any \( y \in \mathbb{R}^d \), and any \( k \in [K] \). We can write

\[
\ln (s_\psi (y|x)) = \ln \left( \sum_{k=1}^K g_k (x; \gamma) \phi (y; \beta_{k0} + \beta_k x, \Sigma_k) \right) = \ln \left( \sum_{k=1}^K f_k (x, y) \right),
\]

\[
g_k (x; \gamma) = \frac{\exp (w_k(x))}{\sum_{l=1}^K \exp (w_l(x))}, \quad w_k(x) = \gamma_{k0} + \gamma_k^T x,
\]

\[
\phi (y; \beta_{k0} + \beta_k x, \Sigma_k) = \frac{1}{(2\pi)^{d/2} \det (\Sigma_k)^{1/2}} \exp \left( -\frac{(y - (\beta_{k0} + \beta_k x))^\top \Sigma_k^{-1} (y - (\beta_{k0} + \beta_k x))}{2} \right),
\]

\[
f_k (x, y) = g_k (x; \gamma) \phi (y; \beta_{k0} + \beta_k x, \Sigma_k) = \frac{g_k (x; \gamma)}{(2\pi)^{d/2} \det (\Sigma_k)^{1/2}} \exp \left[ -\frac{1}{2} (y - (\beta_{k0} + \beta_k x))^\top \Sigma_k^{-1} (y - (\beta_{k0} + \beta_k x)) \right].
\]

By using the chain rule, for all \( l \in [K] \),

\[
\frac{\partial \ln (s_\psi (y|x))}{\partial \gamma_{l0}} = \sum_{k=1}^K \frac{f_k (x, y)}{g_k (x; \gamma)} \frac{\partial g_k (x; \gamma)}{\partial w_l(x)} \frac{\partial w_l(x)}{\partial \gamma_{l0}}, \text{ and}
\]

\[
\frac{\partial \ln (s_\psi (y|x))}{\partial \gamma^l} = \sum_{k=1}^K \frac{f_k (x, y)}{g_k (x; \gamma)} \frac{\partial g_k (x; \gamma)}{\partial w_l(x)} \frac{\partial w_l(x)}{\partial \gamma^l}.
\]

Furthermore,

\[
\frac{\partial g_k (x; \gamma)}{\partial w_l(x)} = \frac{\partial}{\partial w_l(x)} \left( \frac{\exp (w_k(x))}{\sum_{i=1}^K \exp (w_i(x))} \right) = \frac{\partial}{\partial w_l(x)} \left( \frac{\exp (w_k(x))}{\sum_{i=1}^K \exp (w_i(x))} \right) - \frac{\exp (w_k(x))}{\sum_{i=1}^K \exp (w_i(x))} \frac{\partial}{\partial w_l(x)} \sum_{i=1}^K \exp (w_i(x))
\]

\[
\left( \text{using } \frac{\partial}{\partial x} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \right)
\]

\[
= \frac{\delta_{lk} \exp (w_k(x))}{\sum_{l=1}^K \exp (w_l(x))} - \frac{\exp (w_l(x)) \exp (w_k(x))}{\sum_{l=1}^K \exp (w_l(x))} \frac{\exp (w_l(x))}{\sum_{l=1}^K \exp (w_l(x))} = g_k (x; \gamma) (\delta_{lk} - g_l (x; \gamma)),
\]

where \( \delta_{lk} = \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \neq k. \end{cases} \)
Therefore, we obtain

\[
\frac{\partial \ln (s_\psi(y|x))}{\partial (\gamma_0 x)} = \frac{\partial \ln (s_\psi(y|x))}{\partial \gamma_0} = \sum_{k=1}^{K} \frac{f_k(x,y)}{g_k(x;\gamma)} \sum_{k=1}^{K} f_k(x,y) g_k(x;\gamma) (\delta_{ik} - g_i(x;\gamma)) = \sum_{k=1}^{K} \frac{f_k(x,y)}{\sum_{k=1}^{K} f_k(x,y)} (\delta_{ik} - g_i(x;\gamma)) \left( \text{since } \frac{f_k(x,y)}{\sum_{k=1}^{K} f_k(x,y)} \leq 1 \right)
\]

\[
\leq \sum_{k=1}^{K} (\delta_{ik} - g_i(x;\gamma)) \left( \text{since } \frac{f_k(x,y)}{\sum_{k=1}^{K} f_k(x,y)} \leq 1 \right)
\]

\[
= 1 - \sum_{k=1}^{K} (\delta_{ik} - g_i(x;\gamma)) \leq 1 - Kg_i(x;\gamma) \leq 1 + Kg_i(x;\gamma) \leq 1 + KA_G \left( \text{using (2.5)} \right).
\]

Similarly, by using the fact that \( \psi \) belongs to the bounded space \( \tilde{\Psi} \), \( f_t(x,y)/\sum_{k=1}^{K} f_k(x,y) \leq 1 \),

\[
\left\| \frac{\partial \ln (s_\psi(y|x))}{\partial \beta_{i0}} \right\|_\infty \leq \left\| \frac{\partial \ln (s_\psi(y|x))}{\partial (\beta_{i0})} \right\|_\infty \leq \left\| \frac{f_t(x,y)}{\sum_{k=1}^{K} f_k(x,y)} \frac{\partial}{\partial (\beta_{i0} + \beta_{i1})} \left[ -\frac{1}{2} (y - (\beta_{i0} + \beta_{i1}))^\top \Sigma_{i}^{-1} (y - (\beta_{i0} + \beta_{i1})) \right] \right\|_\infty \leq \left\| \Sigma_{i}^{-1} (y - (\beta_{i0} + \beta_{i1})) \right\|_\infty \leq \left\| \Sigma_{i}^{-1} \right\|_\infty \left\| (y - (\beta_{i0} + \beta_{i1})) \right\|_\infty \leq \sqrt{q} \left\| \Sigma_{i}^{-1} \right\|_2 \left( \left\| y \right\|_\infty + \left\| \beta_{i0} + \beta_{i1} \right\|_\infty \right) \leq \sqrt{q} \left( \left\| y \right\|_\infty + \left\| \beta_{i0} + \beta_{i1} \right\|_\infty \right) \leq \sqrt{q} A_1 \left( \left\| y \right\|_\infty + A_2 \right) \left( \text{using (2.4)} \right).
\]

Now, we need to calculate the gradient w.r.t. to the covariance matrices of the Gaussian experts. To do this, we need the following result: given any \( l \in [K] \), \( v_l = \beta_{i0} + \beta_{i1} \), it holds that

\[
\frac{\partial}{\partial \Sigma_l} \phi(x;v_l,\Sigma_l) = \frac{\partial}{\partial \Sigma_l} \left[ \left( 2\pi \right)^{-p/2} \det(\Sigma_l)^{-1/2} \exp \left( -\frac{(x - v_l)^\top \Sigma_l^{-1} (x - v_l)}{2} \right) \right]
\]

\[
= \phi(x;v_l,\Sigma_l) \left[ \left. \frac{1}{2} \frac{\partial}{\partial \Sigma_l} \left( (x - v_l)^\top \Sigma_l^{-1} (x - v_l) \right) + \det(\Sigma_l)^{1/2} \frac{\partial}{\partial \Sigma_l} \left( \det(\Sigma_l)^{-1/2} \right) \right|_{\Sigma_l = \Sigma_l} \right]
\]

\[
= \phi(x;v_l,\Sigma_l) \left[ \left. \frac{1}{2} \Sigma_l^{-1} (x - v_l) (x - v_l)^\top \Sigma_l^{-1} \right|_{\Sigma_l = \Sigma_l} - \frac{1}{2} \det(\Sigma_l)^{-1} \frac{\partial}{\partial \Sigma_l} \left( \det(\Sigma_l) \right) \left( \Sigma_l^{-1} \right)^\top \right]
\]

\[
= \phi(x;v_l,\Sigma_l) \left[ \left. \frac{1}{2} \Sigma_l^{-1} (x - v_l) (x - v_l)^\top \Sigma_l^{-1} \right|_{\Sigma_l = \Sigma_l} - \left( \Sigma_l^{-1} \right)^\top \right] \tag{6.14}
\]

noting that

\[
\frac{\partial}{\partial \Sigma_l} \left( (x - v_l)^\top \Sigma_l^{-1} (x - v_l) \right) = -\Sigma_l^{-1} (x - v_l) (x - v_l)^\top \Sigma_l^{-1} \tag{using Lemma A.1} ,
\]

\[
\frac{\partial}{\partial \Sigma_l} \left( \det(\Sigma_l) \right) = \det(\Sigma_l) \left( \Sigma_l^{-1} \right)^\top \tag{using Jacobi formula, Lemma A.2} .
\]
For any $l \in [K]$,
\[
\left| \partial \ln (s_\psi(y|x)) \right| \leq \left\| \frac{\partial \ln (s_\psi(y|x))}{\partial \Sigma} \right\|_2 \tag{using (A.9)}
\]
\[
= \left| \frac{\sum_{k=1}^K f_k(x, y)}{\sum_{k=1}^K f_k(x, y)} \right| \left\| \frac{\partial}{\partial \Sigma} \left[ -\frac{1}{2} \left( (\beta_{10} + \beta_1 x)^T \Sigma_i^{-1} (y - (\beta_{10} + \beta_1 x)) \right) \right] \right\|_2 \leq \left| \frac{\partial}{\partial \Sigma} \left[ -\frac{1}{2} \left( (\beta_{10} + \beta_1 x)^T \Sigma_i^{-1} (y - (\beta_{10} + \beta_1 x)) \right) \right] \right\|_2
\]
\[
= \frac{1}{2} \left\| \Sigma_i^{-1} (y - (\beta_{10} + \beta_1 x)) (y - (\beta_{10} + \beta_1 x))^T \Sigma_i^{-1} - (\Sigma_i^{-1})^T \right\|_2 \tag{using (6.14)}
\]
\[
\leq \frac{1}{2} \left[ A_\Sigma + q \sqrt{q} \left( (y - (\beta_{10} + \beta_1 x)) (y - (\beta_{10} + \beta_1 x))^T \right) \right] \| A_\Sigma^2 \|_{\infty} \tag{using (A.10)}
\]
\[
\leq \frac{1}{2} \left[ A_\Sigma + q \sqrt{q} (\| y \|_\infty + A_\beta)^2 A_\Sigma^2 \right] \tag{using (2.4)}
\]
where, in the last inequality given $a = y - (\beta_{10} + \beta_1 x)$, we use the fact that
\[
\| \sum_{i=1}^q a_i a_j \|_\infty = \max_{1 \leq i, j \leq q} \sum_{i=1}^q |a_i a_j| = \max_{1 \leq i \leq q} \sum_{j=1}^q |a_i| \leq q \| a \|_\infty^2.
\]
Thus,
\[
\sup_{x \in \mathcal{X}_\psi} \sup_{y \in \mathcal{Y}} \left| \partial \ln (s_\psi(y|x)) \right| \|_\infty \leq \max \left[ 1 + KA_\Sigma, \sqrt{q} (\| y \|_\infty + A_\beta) A_\Sigma, \frac{1}{2} \left( A_\Sigma + q \sqrt{q} (\| y \|_\infty + A_\beta)^2 A_\Sigma^2 \right) \right]
\]
\[
\leq \max \left[ 1 + KA_\Sigma, \max (A_\Sigma, 1) \left( 1 + q \sqrt{q} (\| y \|_\infty + A_\beta)^2 A_\Sigma \right) \right]
\]
\[
\leq \max (A_\Sigma, 1 + KA_\Sigma) \left( 1 + q \sqrt{q} (\| y \|_\infty + A_\beta)^2 A_\Sigma \right) =: G(y),
\]
where we use the fact that
\[
\sqrt{q} (\| y \|_\infty + A_\beta) A_\Sigma =: \theta \leq 1 + \theta^2 = 1 + q (\| y \|_\infty + A_\beta)^2 A_\Sigma^2 \leq \max (A_\Sigma, 1) \left( 1 + q \sqrt{q} (\| y \|_\infty + A_\beta)^2 A_\Sigma \right).
\]

6.2.1. Proof of Lemma 6.4

Let $m \in \mathbb{N}^*$ and $f_m \in F_m$. By (5.6), there exists $s_m \in S_m$, such that $f_m = -\ln (s_m/s_0)$. For all $x \in \mathcal{X}$, let $\psi(x) = (\gamma_{k_0}, \gamma_k^T x, \beta_{k_0}, \beta_k x, \Sigma_k)_{k \in [K]}$ be the parameters of $s_m (\cdot |x)$. In our case, we approximate $f(\psi) = \ln (s_\psi (y_i|x_i))$ around $\psi_0 (x_i)$ by the $n = 0^{th}$ degree Taylor polynomial of $f(\psi)$. That is,
\[
\ln (s_m (y_i|x_i)) - \ln (s_0 (y_i|x_i)) =: |f(\psi) - f(\psi_0)| = |R_0 (\psi)| \text{ (defined in Lemma A.9)}
\]
\[
\leq \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \left| \frac{\partial \ln (s_\psi(y|x))}{\partial \psi} \right| \| \psi (x_i) - \psi_0 (x_i) \|_1.
\]
First applying Taylor’s inequality and then Lemma 6.8 on the event $\mathcal{T}$. For all $i \in [n]$, it holds that

$$|f_m(y_i|x_i)| \mathbb{1}_\mathcal{T} = |\ln (s_m(y_i|x_i)) - \ln (s_0(y_i|x_i))| \mathbb{1}_\mathcal{T} \leq \sup_{x \in X} \sup_{\psi \in \Psi} \left\| \frac{\partial \ln (s_\psi(y_i|x))}{\partial \psi} \right\|_\infty \| \psi(x_i) - \psi_0(x_i) \|_1 \mathbb{1}_\mathcal{T}$$

$$\leq \max (A_{\Sigma}, 1 + KA_G) \left(1 + q\sqrt{q}(M_n + A_\beta)^2 A_{\Sigma}\right) \| \psi(x_i) - \psi_0(x_i) \|_1 \text{(using Lemma 6.8)}$$

$$\leq B_n \sum_{k=1}^{K} \left( |\gamma_{k0} - \gamma_{0,k0}| + |\gamma^T_{k}x_i - \gamma^T_{0,k}x_i| + \|\beta_{k} - \beta_{0,k}\|_1 + \|\beta_{k}x_i - \beta_{0,k}x_i\|_1 + \|\text{vec} (\Sigma_k - \Sigma_{0,k})\|_1 \right)$$

$$\leq 2B_n \sum_{k=1}^{K} \left( |\gamma_{k0} - \gamma_{0,k0}| + |\gamma^T_{k}x_i| + \|\beta_{k}\|_1 + \||\beta_{k}x_i\|_1 + q \|\Sigma_k\|_1 \right) \text{(using (A.7))}$$

$$\leq 2KB_n \left( A_\gamma + q \|\beta_0\|_{\infty} + q \|\beta_kx_i\|_{\infty} + q\sqrt{q}\|\Sigma_k\|_{2} \right) \text{(using (2.4), (A.2), (A.3), (A.11))}$$

$$\leq 2KB_n \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_{\Sigma}} \right) \text{(using (2.4)).}$$

Therefore,

$$\sup_{f_m \in F_m} \| f_m \|_{\infty} \mathbb{1}_\mathcal{T} \leq 2KB_n \left( A_\gamma + qA_\beta + \frac{q\sqrt{q}}{a_{\Sigma}} \right) =: R_n.$$ 

6.2.2. Proof of Lemma 6.5

Let $m \in \mathbb{N}^*$, $f^1_m \in F_m$, and $x \in [0,1]^p$. By (5.6), there exists $s^1_m \in S_m$, such that $f^1_m = -\ln \left(s^1_m/s_0\right)$. Introduce the notation $s^{[2]}_m \in S$ and $f^{[2]}_m = -\ln \left(s^{[2]}_m/s_0\right)$. Let

$$\psi^{[1]}(x) = \left( \gamma^{[1]}_{k0}, \gamma^{[1]}_k x, \beta^{[1]}_k x, \Sigma_k \right)_{k \in [K]} \text{, and } \psi^{[2]}(x) = \left( \gamma^{[2]}_{k0}, \gamma^{[2]}_k x, \beta^{[2]}_k x, \Sigma_k \right)_{k \in [K]},$$

be the parameters of the PDFs $s^1_m (\cdot|x)$ and $s^2_m (\cdot|x)$, respectively. By applying Taylor’s inequality and then Lemma 6.8 on the event $\mathcal{T}$, for all $i \in [n]$, it holds that

$$\left| f^1_m(y_i|x_i) - f^{[2]}_m(y_i|x_i) \right| \mathbb{1}_\mathcal{T} = \left| \ln \left(s^1_m(y_i|x_i)\right) - \ln \left(s^{[2]}_m(y_i|x_i)\right) \right| \mathbb{1}_\mathcal{T} \leq \sup_{x \in X} \sup_{\psi \in \Psi} \left\| \frac{\partial \ln (s_\psi(y_i|x))}{\partial \psi} \right\|_\infty \| \psi^{[1]}(x_i) - \psi^{[2]}(x_i) \|_1 \mathbb{1}_\mathcal{T} \text{(using Taylor’s inequality in Lemma A.9)}$$

$$\leq \max (A_{\Sigma}, C(p,K)) \left(1 + q\sqrt{q}(M_n + A_\beta)^2 A_{\Sigma}\right) \| \psi^{[1]}(x_i) - \psi^{[2]}(x_i) \|_1 \text{(using Lemma 6.8)}$$

$$\leq B_n \sum_{k=1}^{K} \left( |\gamma^{[1]}_{k0} - \gamma^{[2]}_{k0}| + |\gamma^{[1]}_k x_i - \gamma^{[2]}_k x_i| \right.$$

$$\left. + \|\beta^{[1]}_k - \beta^{[2]}_k\|_1 + \|\beta^{[1]}_k x_i - \beta^{[2]}_k x_i\|_1 + \|\text{vec} (\Sigma^1_k - \Sigma^2_k)\|_1 \right).$$
By the Cauchy-Schwarz inequality, \((\sum_{i=1}^{m} a_i)^2 \leq m \sum_{i=1}^{m} a_i^2 \) \((m \in \mathbb{N}^*)\), we get
\[
\left| f^{[1]}_m(y|x_i) - f^{[2]}_m(y|x_i) \right|^2 \leq 3B^2_n \left[ \left( \sum_{k=1}^{K} \left| \gamma_k^1 x_i - \gamma_k^2 x_i \right|^2 \right)^{1/2} + \left( \sum_{k=1}^{K} \sum_{z=1}^{q} \left| \beta_k^1 x_i^z - \beta_k^2 x_i^z \right|^2 \right)^{1/2} \right] 
\]

**Lemma 6.6**

To bound \(a \) and \(b \), we write
\[
 a = Km^2 \sum_{k=1}^{K} \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{p} \frac{\gamma_j^1}{m} x_{ij} - \sum_{j=1}^{p} \frac{\gamma_j^2}{m} x_{ij} \right)^2, \quad \text{and} \quad b = Km^2 \sum_{k=1}^{K} \sum_{z=1}^{q} \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{p} \frac{\beta_j^1}{m} x_{ij}^z - \sum_{j=1}^{p} \frac{\beta_j^2}{m} x_{ij}^z \right)^2.
\]

Then, we apply Lemma 6.6 to obtain
\[
\gamma_k^1 = \left( \frac{\gamma_k^1}{m} \right)_{j \in [q]} \quad \text{and} \quad \beta_k^1 = \left( \frac{\beta_k^1}{m} \right)_{j \in [q]}, \quad \text{for all } k \in [K], z \in [q].
\]

Since \(s_m^1 \in S_m \), and using (3.4), we have
\[
\gamma_k^1 \leq m \quad \text{and} \quad \left\| \beta_k^1 \right\|_1 \leq m, \quad \text{which leads to} \quad \sum_{k=1}^{K} \frac{\gamma_k^1}{m} \leq 1
\]
and
\[
\sum_{z=1}^{q} \sum_{j=1}^{p} \frac{\beta_j^1}{m} \leq 1, \quad \text{respectively. Furthermore, given } x \in \mathcal{X} = [0,1]^p, \quad \text{we have } \left\| x \right\|^2_{\max, n} = 1. \quad \text{Thus,}
\]
there exist families \( \mathcal{A} \) of \((2p+1)^{36B^2_nK^2m^2/\delta^2} \) vectors and \( \mathcal{B} \) of \((2p+1)^{16B^2_nq^2K^2m^2/\delta^2} \) vectors of \( \mathbb{R}^p \), such that
for all $k \in [K]$, $z \in [q]$, $\gamma_k^{[1]}$, and $[\beta_k^{[1]}]_{_{z^*}}$, there exist $\gamma_k^{[1]} \in \mathcal{A}$ and $[\beta_k^{[2]}]_{_{z^*}} \in \mathcal{B}$, such that

$$\frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\gamma_{kj}^{[1]}}{m} x_{ij} - \sum_{j=1}^{n} \frac{\gamma_{kj}^{[2]}}{m} x_{ij} \right)^2 \leq \frac{\delta^2}{36B_n^2 K^2 m^2},$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{[\beta_k^{[1]}]_{_{z^*}}}{m} x_{ij} - \sum_{j=1}^{n} \frac{[\beta_k^{[2]}]_{_{z^*}}}{m} x_{ij} \right)^2 \leq \frac{\delta^2}{36B_n^2 q^2 K^2 m^2},$$

which leads to $a \leq \delta^2/36B_n^2$ and $b \leq \delta^2/36B_n^2$. Moreover, (2.4) leads to

$$\|\beta_0^{[1]}\|_1 = \sum_{k=1}^{K} \|\beta_0^{[1]}\|_1 \leq K q \|\beta_0^{[1]}\|_\infty \leq K q A_\gamma \text{ (using (A.2))}, \quad \|\gamma_0^{[1]}\|_1 \leq K A_\gamma, \quad \|\text{vec}(\Sigma^{[1]})\|_1 \leq \frac{K q \sqrt{q}}{a \Sigma}.$$

Therefore, on the event $\mathcal{T}$,

$$M(\delta, F_m, \|\cdot\|_n) \leq N(\delta/2, F_m, \|\cdot\|_n) \text{ (using Lemma A.7)}$$

$$\leq \text{card}(\mathcal{A}) \text{ card}(\mathcal{B}) N \left( \frac{\delta}{18B_n}, B_n^K (K q A_\beta), \|\cdot\|_1 \right)$$

$$\times N \left( \frac{\delta}{18B_n}, B_n^K (K A_\gamma), \|\cdot\|_1 \right) \times \left( \frac{\delta}{18B_n}, B_n^K \left( \frac{K q \sqrt{q}}{a \Sigma} \right), \|\cdot\|_1 \right)$$

$$\leq (2p + 1) \left( \frac{27B_n^2 K^2 m^2}{\delta^2} \right)^{K} \left( 1 + \frac{18B_n K q A_\beta}{\delta} \right)^{K} \left( 1 + \frac{18B_n K A_\gamma}{\delta} \right)^{K} \left( 1 + \frac{18B_n K q \sqrt{q}}{a \Sigma \delta} \right)^{K}.$$

6.2.3. Proof of Lemma 6.7

Let $m \in \mathbb{N}^*$. From Lemma 6.4,

$$\sup_{f_m \in F_m} \|f_m\|_n \mid \mathcal{T} \leq 2KB_n \left( A_\gamma + q A_\beta + \frac{q \sqrt{q}}{a \Sigma} \right) =: R_n. \quad (6.17)$$

From Lemma 6.5, on the event $\mathcal{T}$ for all $S \in \mathbb{N}^*$, with $\delta = 2^{-s} R_n$,

$$\sum_{s=1}^{S} 2^{-s} \sqrt{\ln \left[ 1 + M(2^{-s} R_n, F_m, \|\cdot\|_n) \right]} \leq \sum_{s=1}^{S} 2^{-s} \sqrt{\ln \left[ 2M(\delta, F_m, \|\cdot\|_n) \right]}$$

$$\leq \sum_{s=1}^{S} 2^{-s} \left[ \sqrt{\ln 2 + 6 \sqrt{q} B_n q K m \frac{q}{\delta}} \sqrt{\ln (2p + 1)} + \sqrt{K \ln \left( \left( 1 + \frac{18B_n K q A_\beta}{\delta} \right) \left( 1 + \frac{18B_n K A_\gamma}{\delta} \right) \left( 1 + \frac{18B_n K q \sqrt{q}}{a \Sigma \delta} \right) \right)} \right]$$

$$\leq \sum_{s=1}^{S} 2^{-s} \left[ \sqrt{\ln 2 + 6 \sqrt{q} B_n q K m \frac{R_n}{\delta}} \sqrt{\ln (2p + 1)} + \sqrt{K \ln \left( \left( 1 + \frac{2qB_n K q A_\beta}{R_n} \right) \left( 1 + \frac{2qB_n K A_\gamma}{R_n} \right) \left( 1 + \frac{2qB_n K q \sqrt{q}}{a \Sigma R_n} \right) \right)} \right] \cdot \left( 6.18 \right)$$

Notice from (6.17), that $R_n \geq 2KB_n \max \left( A_\gamma, q A_\beta, \frac{2 \sqrt{q}}{a \Sigma} \right)$. Moreover, it holds that $1 \leq 2^{s+3}$, and $\sum_{s=1}^{S} 2^{-s} = 1 - 2^{-S} \leq 1, \sum_{s=1}^{S} \left( \sqrt{\pi}/2 \right)^s \leq \sqrt{\pi}/(2 - \sqrt{\pi})$, and since for all $s \in \mathbb{N}^*$, $e^s \geq s$, and thus $2^{-s} \sqrt{s} \leq (\sqrt{\pi}/2)^s.$
Therefore, from (6.18):

\[
\sum_{s=1}^{S} 2^{-s} \sqrt{\ln |1 + M(2^{-s} R_n, F_m, \|\cdot\|_n)|} \\
\leq \sum_{s=1}^{S} 2^{-s} \left[ \sqrt{\ln 2} + \frac{2^s \sqrt{2} B_n q K m}{R_n} \sqrt{\ln (2p + 1)} + \sqrt{K \ln [(2^{s+1} 3^2)(2^{s+1} 3^2)(2^{s+1} 3^2)]} \right] \\
= \sum_{s=1}^{S} 2^{-s} \left[ \sqrt{\ln 2} + \frac{2^s \sqrt{2} B_n q K m}{R_n} \sqrt{\ln (2p + 1)} + \sqrt{K \sqrt{3} ((s + 1) \ln 2 + 2 \ln 3)} \right] \\
\leq \frac{6 \sqrt{2} B_n K m q}{R_n} \sqrt{\ln (2p + 1) S + \sqrt{K} \sqrt{3} \ln 2} \sum_{s=1}^{S} 2^{-s} \sqrt{s} + \sqrt{\ln 2} \left( 1 + \sqrt{3K} \right) + 6 \ln 3K \\
\leq \frac{6 \sqrt{2} B_n K m q}{R_n} \sqrt{\ln (2p + 1) S + \sqrt{K} \sqrt{3} \ln 2} \sum_{s=1}^{S} \left( \frac{\ln 2}{2} \right)^s + \sqrt{\ln 2} \left( 1 + \sqrt{3K} \right) + 6 \ln 3K \\
\leq \frac{6 \sqrt{2} B_n q K m}{R_n} \sqrt{\ln (2p + 1) S + \sqrt{K} \ln 2} \left( \frac{\ln 2}{2 - \sqrt{e}} + 1 + \sqrt{3} + \sqrt{6 \ln 3} \ln 2 \right). 
\tag{6.19}
\]

Then, for all \( S \in \mathbb{N}^* \):

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f_m(Z_i) \right] \mathbb{I}_\tau = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f_m(Z_i) \mathbb{I}_{f_m(Z_i) \leq R_n} \right] \mathbb{I}_\tau \quad (\text{using Lemma 6.4}) \\
= \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(Z_i) \right] \mathbb{I}_\tau \\
\leq R_n \left( \frac{6}{\sqrt{n}} \sum_{s=1}^{S} 2^{-s} \sqrt{\ln |1 + M(2^{-s} R_n, F, \|\cdot\|_n)|} + 2^{-S} \right) \mathbb{I}_\tau \quad (\text{using Lemma 6.3}) \\
\leq R_n \left[ \frac{6}{\sqrt{n}} \left( \frac{6 \sqrt{2} B_n K m q}{R_n} \sqrt{\ln (2p + 1) S + \sqrt{K} \ln 2 C_1} \right) + 2^{-S} \right] \quad (\text{using Lemma 6.19}). 
\tag{6.20}
\]

We choose \( S = \ln n / \ln 2 \) so that the two terms depending on \( S \) in (6.20) are of the same order. In particular, for this value of \( S \), \( 2^{-S} \leq 1/n \), and we deduce from (6.20) and (6.17) that

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f_m(Z_i) \right] \mathbb{I}_\tau \leq \frac{36 \sqrt{2} B_n K m q}{\sqrt{n}} \sqrt{\ln (2p + 1) \ln n / \ln 2} + 2 K B_n \left( A_\gamma + q A_\beta + \frac{q \sqrt{q}}{a_\Sigma} \right) \left( 6 \sqrt{2} C_1 \frac{\sqrt{K}}{\sqrt{n}} + \frac{1}{n} \right) \\
\leq \frac{B_n K m q}{\sqrt{n}} \sqrt{\ln (2p + 1) \ln n} + \frac{36 \sqrt{2}}{\ln 2} + \frac{K \sqrt{K}}{\sqrt{n}} \approx 73.45 \\
\leq \frac{74 K B_n}{\sqrt{n}} \left[ m q \sqrt{\ln (2p + 1) \ln n + 2 \sqrt{K}} \left( A_\gamma + q A_\beta + \frac{q \sqrt{q}}{a_\Sigma} \right) \right].
\]
6.3. Proof of Lemma 3.1

Since \( \ln(z) \) is concave in \( z \), Jensen’s inequality (see e.g. [9, 23]) implies that \( \ln \left( \mathbb{E}_Z [Z] \right) \geq \mathbb{E}_Z [\ln (Z)] \), where \( Z \) is a random variable. Thus, for all \( x \in \mathcal{X} \), Jensen’s inequality and Lemma 6.9 lead us the following upper bound

\[
\int_{\mathbb{R}^d} \ln \left( s_0 \left( y \mid x \right) \right) s_0 \left( y \mid x \right) dy \\
= \int_{\mathbb{R}^d} \ln \left[ s_0 \left( y \mid x \right) \right] \sum_{k=1}^{K} \left[ g_k \left( x; \gamma_0 \right) \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) \right] dy \\
= \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) \int_{\mathbb{R}^d} \ln \left[ s_0 \left( y \mid x \right) \right] \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) dy \\
\leq \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) \ln \left[ \int_{\mathbb{R}^d} s_0 \left( y \mid x \right) \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) dy \right] \\
= \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) \ln \left[ \int_{\mathbb{R}^d} \sum_{l=1}^{K} \left[ g_l \left( x; \gamma_0 \right) \phi \left( y; v_{0l} (x), \Sigma_{0l} \right) \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) \right] dy \right] \\
= \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) \ln \left[ \sum_{l=1}^{K} g_l \left( x; \gamma_0 \right) \int_{\mathbb{R}^d} \phi \left( y; v_{0l} (x), \Sigma_{0l} \right) \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) dy \right] \\
\leq \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) \ln \left[ \sum_{k=1}^{K} g_k \left( x; \gamma_0 \right) C_{s_0} \right], C_{s_0} = \left( 4\pi \right)^{-q/2} A_{\Sigma}^{q/2}, (\text{using Lemma 6.9}) \\
= \ln C_{s_0} < \infty. \tag{6.21}
\]

Therefore, we obtain

\[
\max \left\{ 0, \sup_{x \in \mathcal{X}} \int_{\mathbb{R}^d} \ln \left( s_0 \left( y \mid x \right) \right) s_0 \left( y \mid x \right) dy \right\} \leq \max \left\{ 0, \ln C_{s_0} \right\} =: H_{s_0} < \infty.
\]

Next, we state the following important Lemma 6.9, which is used in the proof of Lemma 3.1.

**Lemma 6.9.** There exists a positive constant \( C_{s_0} := \left( 4\pi \right)^{-q/2} A_{\Sigma}^{q/2} \), \( 0 < C_{s_0} < \infty \), such that for all \( k \in [K], l \in [L], \)

\[
\int_{\mathbb{R}^d} \phi \left( y; v_{0l} (x), \Sigma_{0l} \right) \phi \left( y; v_{0k} (x), \Sigma_{0k} \right) dy < C_{s_0}, \quad \forall x \in \mathcal{X}. \tag{6.22}
\]

**Proof of Lemma 6.9.** Firstly, for all \( k \in [K], l \in [L], \) given

\[
c_{lk} (x) = C_{lk} \left[ \Sigma_{0l}^{-1} v_{0l} (x) + \Sigma_{0k}^{-1} v_{0k} (x) \right], \quad C_{lk} = \left( \Sigma_{0l}^{-1} + \Sigma_{0k}^{-1} \right)^{-1},
\]
Lemma A.10 leads to
\[
\int_{\mathbb{R}^q} \left[ \phi (y; v_{0l}(x), \Sigma_{0l}) \phi (y; v_{ok}(x), \Sigma_{ok}) \right] dy \\
= Z_{lk}^{-1} \int_{\mathbb{R}^q} \phi (y; c_{lk}(x), C_{lk}) dy, \text{ where} \\
= (2\pi)^{-q/2} \det (\Sigma_{0l} + \Sigma_{ok})^{-1/2} \exp \left( -\frac{1}{2} (v_{0l}(x) - v_{ok}(x))^\top (\Sigma_{0l} + \Sigma_{ok})^{-1} (v_{0l}(x) - v_{ok}(x)) \right) 
\]

(6.23)

Next, since the determinant is the product of the eigenvalues, counted with multiplicity, and Weyl’s inequality, see e.g., Lemma A.11, for all \( k \in [K], l \in [L] \), we have
\[
\det (\Sigma_{0l} + \Sigma_{ok}) \geq [m (\Sigma_{0l} + \Sigma_{ok})]^q \\
\geq [m (\Sigma_{0l}) + m (\Sigma_{ok})]^q \quad \text{(using (A.18) from Lemma A.11)} \\
= \left[ M (\Sigma_{0l}^{-1})^{-1} + M (\Sigma_{ok}^{-1})^{-1} \right]^q \\
\geq (2A_{\Sigma}^{-1})^q \quad \text{(using boundedness assumptions in (2.4))}. 
\]

Therefore, for all \( k \in [K], l \in [L] \), it holds that
\[
\det (\Sigma_{0l} + \Sigma_{ok})^{-1/2} \leq 2^{-q/2} (A_{\Sigma}^{-1})^{q/2} \quad \text{(using boundedness assumptions in (2.4))}. 
\]

(6.24)

Since \((\Sigma_{0l} + \Sigma_{ok})^{-1}\) is a positive definite matrix, it holds that
\[
(v_{0l}(x) - v_{ok}(x))^\top (\Sigma_{0l} + \Sigma_{ok})^{-1} (v_{0l}(x) - v_{ok}(x)) \geq 0, \quad \forall x \in \mathcal{X}, l \in [L], k \in [K]. 
\]

Then, since the exponential function is increasing, \( \forall x \in \mathcal{X}, l \in [L], k \in [K] \), we have
\[
\exp \left( -\frac{1}{2} (v_{0l}(x) - v_{ok}(x))^\top (\Sigma_{0l} + \Sigma_{ok})^{-1} (v_{0l}(x) - v_{ok}(x)) \right) \leq \exp(0) = 1. 
\]

(6.25)

Finally, from (6.23), (6.24) and (6.25), we obtain
\[
\int_{\mathbb{R}^q} \left[ \phi (y; v_{0l}(x), \Sigma_{0l}) \phi (y; v_{ok}(x), \Sigma_{ok}) \right] dy \leq (2\pi)^{-q/2} 2^{-q/2} A_{\Sigma}^{q/2} = (4\pi)^{-q/2} A_{\Sigma}^{q/2} =: C_{\alpha_0} < \infty. 
\]

\( \square \)

7. Conclusions

We have studied an \( l_1 \)-regularization estimator for finite mixtures of Gaussian experts regression models with soft-max gating functions. Our main contribution is the proof of the \( l_1 \)-oracle inequality that provides the lower bound on the regularization of the Lasso that ensures non-asymptotic theoretical control on the Kullback–Leibler loss of the estimator. Other than some remaining questions regarding the tightness of the bounds and the form of penalization functions, we believe that our contribution helps to further popularize mixtures of Gaussian experts regression models by providing a theoretical foundation for their application in high-dimensional problems.
APPENDIX A. TECHNICAL RESULTS

We denote the vector space of all \( q \times q \) real matrices by \( \mathbb{R}^{q \times q} \) \( (q \in \mathbb{N}^*) \):

\[
A \in \mathbb{R}^{q \times q} \iff A = \begin{bmatrix}
a_{1,1} & \cdots & a_{1,q} \\
\vdots & \ddots & \vdots \\
a_{q,1} & \cdots & a_{q,q}
\end{bmatrix}, \; a_{i,j} \in \mathbb{R}.
\]

If a capital letter is used to denote a matrix (e.g., \( A, B \)), then the corresponding lower-case letter with subscript \( i, j \) refers to the \((i, j)\)th entry (e.g., \( a_{i,j}, b_{i,j} \)). When required, we also designate the elements of a matrix with the notation \([A]_{i,j}\) or \( A(i, j)\). Denote the \( q \times q \) identity and zero matrices by \( I_q \) and \( 0_q \), respectively.

**Lemma A.1** (Derivative of quadratic form, see [29]). Assume that \( X \) and \( a \) are non-singular matrix in \( \mathbb{R}^{q \times q} \) and vector in \( \mathbb{R}^{q \times 1} \), respectively. Then

\[
\frac{\partial a^\top X^{-1} a}{\partial X} = -X^{-1} a a^\top X^{-1}.
\]

**Lemma A.2** (Jacobi’s formula, Theorem 8.1 from [29]). If \( X \) is a differentiable map from the real numbers to \( q \times q \) matrices, 

\[
\frac{d}{dt} \det (X(t)) = \text{tr} \left( \text{Adj} (X(t)) \frac{dX(t)}{dt} \right), \; \frac{\partial \det (X)}{\partial X} = (\text{Adj} (X))^\top = \det (X) \left( X^{-1} \right)^\top.
\]

**Lemma A.3** (Operator (induced) \( p \)-norm). We recall an operator (induced) \( p \)-norms of a matrix \( A \in \mathbb{R}^{q \times q} \) \( (q \in \mathbb{N}^*, \; p \in \{1, 2, \infty\}) \),

\[
\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{x \neq 0} \left\| A \left( \frac{x}{\|x\|_p} \right) \right\|_p = \max_{\|x\|_p = 1} \|Ax\|_p,
\]

where for all \( x \in \mathbb{R}^q \),

\[
\|x\|_\infty \leq \|x\|_1 = \sum_{i=1}^q |x_i| \leq q \|x\|_\infty,
\]

\[
\|x\|_2 = \left( \sum_{i=1}^q |x_i|^2 \right)^{\frac{1}{2}} = (x^\top x)^{\frac{1}{2}} \leq \|x\|_1 \leq \sqrt{q} \|x\|_2, \text{ and}
\]

\[
\|x\|_\infty = \max_{1 \leq i \leq q} |x_i| \leq \|x\|_2 \leq \sqrt{q} \|x\|_\infty.
\]

**Lemma A.4** (Some matrix \( p \)-norm properties, [16]). By definition, we always have the important property that for every \( A \in \mathbb{R}^{q \times q} \) and \( x \in \mathbb{R}^q \),

\[
\|Ax\|_p \leq \|A\|_p \|x\|_p,
\]

and every induced \( p \)-norm is submultiplicative, i.e., for every \( A \in \mathbb{R}^{q \times q} \) and \( B \in \mathbb{R}^{q \times q} \),

\[
\|AB\|_p \leq \|A\|_p \|B\|_p.
\]
In particular, it holds that
\[
\|A\|_1 = \max_{1 \leq j \leq q} \sum_{i=1}^{q} |a_{ij}| \leq \sum_{j=1}^{q} \sum_{i=1}^{q} |a_{ij}| := \|\text{vec}(A)\|_1 \leq q\|A\|_1, \tag{A.7}
\]
\[
\|\text{vec}(A)\|_\infty := \max_{1 \leq i \leq q, 1 \leq j \leq q} |a_{ij}| \leq \|A\|_\infty = \max_{1 \leq j \leq q} \sum_{i=1}^{q} |a_{ij}| \leq q\|\text{vec}(A)\|_\infty, \tag{A.8}
\]
\[
\|\text{vec}(A)\|_\infty \leq \|A\|_2 = \lambda_{\text{max}}(A) \leq q\|\text{vec}(A)\|_\infty, \tag{A.9}
\]
where $\lambda_{\text{max}}$ is the largest eigenvalue of a positive definite symmetric matrix $A$. The $p$-norms, when $p \in \{1, 2, \infty\}$, satisfy
\[
\frac{1}{\sqrt{q}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{q}\|A\|_\infty, \tag{A.10}
\]
\[
\frac{1}{\sqrt{q}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{q}\|A\|_1. \tag{A.11}
\]

Given $\delta > 0$, we need to define the $\delta$-packing number and $\delta$-covering number.

**Definition A.5** ($\delta$-packing number, e.g., Definition 5.4 from [59]). Let $(\mathcal{F}, \|\cdot\|)$ be a normed space and let $\mathcal{G} \subset \mathcal{F}$. With $(g_i)_{i=1, \ldots, m} \in \mathcal{G}$, \{g_1, \ldots, g_m\} is an $\delta$-packing of $\mathcal{G}$ of size $m \in \mathbb{N}^*$, if $\|g_i - g_j\| > \delta, \forall i \neq j, i, j \in \{1, \ldots, m\}$, or equivalently, $\cap_{i=1}^{m} B(g_i, \delta/2) = \emptyset$. Upon defining $\delta$-packing, we can measure the maximal number of disjoint closed balls with radius $\delta/2$ that can be “packed” into $\mathcal{G}$. This number is called the $\delta$-packing number and is defined as
\[
M(\delta, \mathcal{G}, \|\cdot\|) := \max \{m \in \mathbb{N}^* : \exists \delta\text{-packing of } \mathcal{G} \text{ of size } m\}. \tag{A.12}
\]

**Definition A.6** ($\delta$-covering number, Definition 5.1 from [59]). Let $(\mathcal{F}, \|\cdot\|)$ be a normed space and let $\mathcal{G} \subset \mathcal{F}$. With $(g_i)_{i \in [n]} \in \mathcal{G}$, \{g_1, \ldots, g_n\} is an $\delta$-covering of $\mathcal{G}$ of size $n$ if $\mathcal{G} \subset \cup_{i=1}^{n} B(g_i, \delta)$, or equivalently, $\forall g \in \mathcal{G}, \exists i$ such that $\|g - g_i\| \leq \delta$. Upon defining the $\delta$-covering, we can measure the minimal number of closed balls with radius $\delta$, which is necessary to cover $\mathcal{G}$. This number is called the $\delta$-covering number and is defined as
\[
N(\delta, \mathcal{G}, \|\cdot\|) := \min \{n \in \mathbb{N}^* : \exists \delta\text{-covering of } \mathcal{G} \text{ of size } n\}. \tag{A.13}
\]

The covering entropy (metric entropy) is defined as follows $H_{\|\cdot\|}(\delta, \mathcal{G}) = \ln(N(\delta, \mathcal{G}, \|\cdot\|))$.

The relation between the packing number and the covering number is described in the following lemma.

**Lemma A.7** (Lemma 5.5 from [59]). Let $(\mathcal{F}, \|\cdot\|)$ be a normed space and let $\mathcal{G} \subset \mathcal{F}$. Then
\[
M(2\delta, \mathcal{G}, \|\cdot\|) \leq N(\delta, \mathcal{G}, \|\cdot\|) \leq M(\delta, \mathcal{G}, \|\cdot\|). \tag{A.14}
\]

**Lemma A.8** (Chernoff’s inequality, e.g., Chapter 2 in [59]). Assume that the random variable has a moment generating function in a neighborhood of zero, meaning that there is some constant $b > 0$ such that the function $\varphi(\lambda) = \mathbb{E}[e^{\lambda(U - \mu)}]$ exists for all $\lambda \leq |b|$. In such a case, we may apply Markov’s inequality to the random variable $Y = e^{\lambda(U - \mu)}$, thereby obtaining the upper bound
\[
P(U - \mu \geq a) = P\left( e^{\lambda(U - \mu)} \geq e^{\lambda a} \right) \leq \frac{\mathbb{E}[e^{\lambda(U - \mu)}]}{e^{\lambda a}}. \tag{A.15}
\]
Optimizing our choice of $\lambda$ so as to obtain the tightest result yields the Chernoff bound
\[
\ln (\mathbb{P} (U - \mu \geq a)) \leq \sup_{\lambda \in [0,b]} \left\{ \lambda t - \ln \left( \mathbb{E} \left[ e^{\lambda(U-\mu)} \right] \right) \right\}.
\] (A.14)

In particular, if $U \sim \mathcal{N}(\mu, \sigma)$ is a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. By a straightforward calculation, we find that $U$ has the moment generating function
\[
\mathbb{E} \left[ e^{\lambda U} \right] = e^{\mu \lambda + \frac{\sigma^2 \lambda^2}{2}}, \text{ valid for all } \lambda \in \mathbb{R}.
\]

Substituting this expression into the optimization problem defining the optimized Chernoff bound (A.14), we obtain
\[
\sup_{\lambda \geq 0} \left\{ \lambda t - \ln \left( \mathbb{E} \left[ e^{\lambda(U-\mu)} \right] \right) \right\} = \sup_{\lambda \geq 0} \left\{ \lambda t - \frac{\sigma^2 \lambda^2}{2} \right\} = -\frac{t^2}{2\sigma^2},
\]
where we have taken derivatives in order to find the optimum of this quadratic function. So, (A.14) leads to
\[
\mathbb{P} (X \geq \mu + t) \leq e^{-\frac{t^2}{2\sigma^2}}, \text{ for all } t \geq 0.
\] (A.15)

Recall that a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p), \alpha_i \in \mathbb{N}^*, \forall i \in \{1, \ldots, p\}$ is an $p$-tuple of non-negative integers. Let
\[
|\alpha| = \sum_{i=1}^{p} \alpha_i, \quad \alpha! = \prod_{i=1}^{p} \alpha_i!, \quad x^\alpha = \prod_{i=1}^{p} x_i^{\alpha_i}, \quad x \in \mathbb{R}^p, \quad \partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_p^{\alpha_p} \frac{\partial^{\alpha}|f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_p^{\alpha_p}}.
\]
The number $|\alpha|$ is called the order or degree of $\alpha$. Thus, the order of $\alpha$ is the same as the order of $x^\alpha$ as a monomial or the order of $\partial^\alpha$ as a partial derivative.

**Lemma A.9** (Taylor’s Theorem in Several Variables from [13]). Suppose $f : \mathbb{R}^p \to \mathbb{R}$ is in the class $C^{k+1}$, of continuously differentiable functions, on an open convex set $S$. If $a \in S$ and $a + h \in S$, then
\[
f(a + h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h),
\]
where the remainder is given in Lagrange’s form by
\[
R_{a,k}(h) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a + ch)}{\alpha!} h^\alpha \text{ for some } c \in (0,1),
\]
or in integral form by
\[
R_{a,k}(h) = (k+1) \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a + th)}{\alpha!} \int_0^1 (1-t)^k \partial^\alpha f(a + th) dt.
\]
In particular, we can estimate the remainder term if $|\partial^\alpha f(x)| \leq M$ for $x \in S$ and $|\alpha| = k + 1$, then
\[
|R_{a,k}(h)| \leq \frac{M}{(k+1)!} \|h\|_{k+1}^1, \quad \|h\|_1 = \sum_{i=1}^{p} |h_i|.
\]
Recall that the multivariate Gaussian (or Normal) distribution has a joint density given by
\[
\phi(y; \mu, \Sigma) = (2\pi)^{-q/2} \det(\Sigma)^{-1/2} \exp \left( -\frac{1}{2} (y - \mu)^\top \Sigma^{-1} (y - \mu) \right),
\] (A.16)
where \(\mu\) is the mean vector (of length \(q\)) and \(\Sigma\) is the symmetric, positive definite covariance matrix (of size \(q \times q\)). Then, we have the following well-known Gaussian identity, see more in Lemma A.10.

Lemma A.10 (Product of two Gaussians, see e.g., Equation (A.7) in [60]). The product of two Gaussians gives another (un-normalized) Gaussian
\[
\phi(y; a, A) \phi(y; b, B) = Z^{-1} \phi(y; c, C),
\] (A.17)
where,
\[
c = C (A^{-1} a + B^{-1} b) \quad \text{and} \quad C = (A^{-1} + B^{-1})^{-1},
\]
\[
Z^{-1} = (2\pi)^{-q/2} \det(A + B)^{-1/2} \exp \left( -\frac{1}{2} (a - b)^\top (A + B)^{-1} (a - b) \right).
\]

We recall the following inequality of Hermann Weyl, see e.g., Horn and Johnson [20, Theorem 4.3.1]

Lemma A.11 (Weyl’s inequality, e.g., Theorem 4.3.1 from [20]). Let \(A, B \in \mathbb{R}^{q \times q}\) be Hermitian and let the respective eigenvalues of \(A, B,\) and \(A + B\) be \(\{\lambda_i(A)\}_{i=1}^q, \{\lambda_i(B)\}_{i=1}^q,\) and \(\{\lambda_i(A + B)\}_{i=1}^q,\) each algebraically nondecreasing order as follows:
\[
m(A) = \lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_q(A) = M(A).
\]

Then, for each \(i \in [q],\)
\[
\lambda_i(A + B) \leq \lambda_{i+j}(A) + \lambda_{q-j}(B), \quad j \in \{0\} \cup [q - i],
\]
\[
\lambda_{i+j+1}(A) + \lambda_{j}(B) \leq \lambda_i(A + B), \quad j \in [i].
\]
In particular, we have
\[
M(A + B) \leq M(A) + M(B),
\]
\[
m(A + B) \geq m(A) + m(B).
\] (A.18)

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