INTERSECTION COHOMOLOGY OF THE UHLENBECK
COMPACTIFICATION OF THE CALOGERO–MOSER SPACE

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To Joseph Bernstein on his 70th birthday, with gratitude and admiration

Abstract. We study the natural Gieseker and Uhlenbeck compactifications of the rational Calogero–Moser phase space. The Gieseker compactification is smooth and provides a small resolution of the Uhlenbeck compactification. We use the resolution to compute the stalks of the IC-sheaf of the Uhlenbeck compactification.

I’d say that if one can compute the Poincaré polynomial for intersection cohomology without a computer then, probably, there is a small resolution which gives it.

(J. Bernstein)

1. Introduction

1.1. The Calogero–Moser space. The Calogero–Moser space $M^n$ [12] is the quotient modulo a free action of $\text{PGL}_n$ of the space of pairs of complex $n \times n$-matrices $(X,Y)$ such that $[X,Y] – \text{Id}$ has rank 1. The Calogero–Moser space is a smooth connected affine algebraic variety of dimension $2n$ [21].

1.2. The Gieseker and Uhlenbeck compactifications. More generally, for a parameter $\tau \in \mathbb{C}^\times$, we consider a graded algebra $A^\tau$ with generators $x,y,z$, of degree 1, and the following commutation relations

\[ [x,z] = [y,z] = 0, \quad [x,y] = \tau z^2. \]  

(1.2.1)

This algebra is a very special case of the Sklyanin algebras studied in [18], specifically, it corresponds to the case of a degenerate plane cubic curve equal to a triple line. We set $\mathbb{P}_\tau^2 = \text{Proj}(A^\tau)$, a non-commutative $\text{Proj}$ in the sense of [1], see also [10], and write $\text{coh}(\mathbb{P}_\tau^2) = \text{agr}(A^\tau)$ for the corresponding abelian category $\text{coh}(\mathbb{P}_\tau^2)$ of “coherent sheaves”. Associated with an object $E \in \text{coh}(\mathbb{P}_\tau^2)$ there is a well defined triple $(r = \text{rk} E, d = \deg E, n = c_2(E))$, of nonnegative integers, the rank, the degree, and the second Chern class of $E$, respectively.

Given a triple $(r,d,n)$, where $r$ and $d$ are coprime, we introduce two different moduli spaces, $G^r_M(r,d,n)$ and $U^r_M(r,d,n)$, of coherent sheaves on $\mathbb{P}_\tau^2$. These moduli spaces are defined by stability conditions. The moduli space $G^r_M(r,d,n)$, the Gieseker moduli space, is defined using Gieseker stability. The moduli space $U^r_M(r,d,n)$, the Uhlenbeck moduli space, is defined using Mumford stability. These moduli spaces are projective varieties which provide two different compactifications of the moduli space of locally free sheaves. The variety $G^r_M(r,d,n)$ is a particular case of moduli spaces studied in [18] (cf. also [10]) in greater generality. The variety $U^r_M(r,d,n)$ is more mysterious; it does not fit into the
framework of [18] and it has not been considered there. In fact, even in the commutative case, a satisfactory construction of the Uhlenbeck compactification of the moduli space of locally free sheaves on an arbitrary smooth surface is not known so far, cf. [3]. In the case we are interested in, i.e., in the case of the noncommutative surface $\mathbb{P}^2$, the variety $\mathcal{M}_r(r, d, n)$ will be studied in section 2. In particular, using an interpretation of our moduli spaces in terms of certain moduli spaces of quiver representations, we construct a projective morphism $\gamma_r : \mathcal{Q}_r(r, d, n) \to \mathcal{U}_r(r, d, n)$. This morphism turns out to be a resolution of singularities, provided $r$ and $d$ are coprime.

In this paper, we will mostly be interested in the case where $r = 1$, $d = 0$, and $\tau \neq 0$. The moduli space of locally free sheaves $E$ on $\mathbb{P}^2$ such that $\text{rk} E = 1$, $\text{deg} E = 0$, and $c_2(E) = n$ has an ADHM type description. Specifically, according to [17] and [10], this moduli space has an ADHM type description. Specifically, according to [17] and [10], this moduli space is isomorphic to the variety $\mathcal{M}_n$ defined as a quotient of the space of pairs $(X, Y)$, of $n \times n$-matrices such that $\text{rk}([X, Y] - \tau \text{Id}) = 1$, by the (free) action of the group $\text{PGL}_n$ by conjugation. Note that the rescaling map $(X, Y) \mapsto (\frac{1}{\tau}X, Y)$ gives a canonical isomorphism of $M_n$ with the Calogero–Mosser space $M^n$. Therefore, the varieties $\mathcal{Q}_n = \mathcal{Q}_r(1, 0, n)$ and $\mathcal{U}_n = \mathcal{U}_r(1, 0, n)$ provide two different compactifications of the Calogero–Mosser space. Since 1 and 0 are coprime, the corresponding morphism $\gamma_r : \mathcal{Q}_n \to \mathcal{U}_n$ is a resolution of singularities. Moreover, we show that this morphism is small in the sense of Goresky–MacPherson.

One can allow the parameter $\tau$ to vary in $\mathbb{A}^1$. Similarly to the above, one constructs the family of Gieseker, resp. Uhlenbeck, compactifications $\mathcal{Q}M^n$, resp. $\mathcal{U}M^n$, equipped with maps to $\mathbb{A}^1$ such that the fibers over the point $\tau \in \mathbb{A}^1 \setminus \{0\}$ are $\mathcal{Q}M^n$, resp. $\mathcal{U}M^n$. Furthermore, we construct a small resolution of singularities $\gamma : \mathcal{Q}M^n \to \mathcal{U}M^n$. In fact, over $\mathbb{A}^1 \setminus \{0\}$ the maps $\gamma_r : \mathcal{Q}M^n \to \mathcal{U}M^n$ are identified with the maps discussed above, while the fiber over $\tau = 0$ is the Hilbert–Chow morphism $\gamma_0 : \text{Hilb}^n \mathbb{P}^2 \to S^n \mathbb{P}^2 = (\mathbb{P}^2)^n / \mathbb{S}_n$.

It is well known that $\gamma_0$ is only semismall. The reason of this difference between $\gamma_0$ and $\gamma_r$, $\tau \neq 0$, is due to the difference between the stratifications of the commutative and noncommutative Uhlenbeck compactifications, respectively. Namely, we have a distinguished (classical, commutative) projective line subscheme $\mathbb{P}^1 \subset \mathbb{P}^2$, and similarly, we have $\mathbb{P}^1 \subset \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus \mathbb{P}^1 = \mathbb{A}^2$. There is a stratification

$$\mathcal{U}_n = \bigsqcup_{0 \leq m \leq n} \mathcal{M}_n \times S^{n-m} \mathbb{P}^1,$$

and $\gamma_r$ is an isomorphism over the open part $\mathcal{M}_n$. Similarly, we have a stratification

$$\mathcal{U}_0 = S^n \mathbb{P}^2 = \bigsqcup_{0 \leq m \leq n} S^m \mathbb{A}^2 \times S^{n-m} \mathbb{P}^1,$$

but $\gamma_0$ is not an isomorphism over $S^n \mathbb{A}^2$, only a semismall resolution of singularities.

Let us remark that the readers experienced with the classical Uhlenbeck compactifications might expect another stratification with strata of $\mathcal{U}_n$ being $\mathcal{M}_n \times S^{n-m} \mathbb{P}_2$ (the reason of semismallness of the classical Gieseker resolution); however $\mathbb{P}^2$ is not a classical scheme, so only its “classical part” $\mathbb{P}^1$ survives in the classical moduli scheme $\mathcal{U}_n$, yielding the stratification of the previous paragraph.
Note also that the Gieseker moduli spaces of [18] carry a natural Poisson structure. We expect it to descend to the Uhlenbeck compactification. However, even normality of $U^\tau$ seems to be a hard question and it is out of the scope of this paper.

1.3. The main Theorem. Let $\mathfrak{P}(n)$ denote the set of partitions of an integer $n \geq 0$ and for an algebraic variety $T$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ put

$$S_\lambda T = \{ \sum \lambda_i P_i \mid P_1 \neq P_2 \neq \cdots \neq P_l \in T \} \subset S^n T = T^n / \mathfrak{S}_n.$$ 

so that $S^n T = \bigsqcup_{\lambda \in \mathfrak{P}(n)} S_\lambda T$ is a stratification, which we call the diagonal stratification. Let $IC(U^\tau)$ be the IC sheaf (see [3]) of the Uhlenbeck compactification. Our main result is the computation of the stalks of the IC sheaf.

**Theorem 1.3.1.** The IC sheaf of the Uhlenbeck compactification is smooth along the stratifications

$$U^\tau = \bigsqcup_{0 \leq m \leq n} (M^m \times S_\lambda \mathbb{P}^1).$$

For $0 \leq m \leq n$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathfrak{P}(n - m)$, the stalk of the sheaf $IC(U^\tau)$ at a point of a stratum $M^m \times S_\lambda \mathbb{P}^1$ is isomorphic to

$$\bigotimes_{i=1}^k \left( \bigoplus_{\mu \in \mathfrak{P}(\lambda_i)} \mathbb{C}[2\ell(\mu)] \right) [2m]$$

as a graded vector space.

The proof employs the small resolution of the family $\gamma : \mathcal{R}^\tau \to U^\tau$ and reduces the study of the fibers for $\tau \neq 0$ to the well known properties of the fibers of the Hilbert–Chow morphism for $\tau = 0$.

**Remark 1.3.3.** Given a complex semisimple simply connected group $G$ one can consider the moduli space of $G$-bundles on $\mathbb{P}^2$ equipped with a trivialization at the infinite line $\mathbb{P}^1 \subset \mathbb{P}^2$. There is also an Uhlenbeck space $\mathcal{U}_G(\mathbb{A}^2)$ that contains the above moduli space as a Zariski open subset (the variety $\mathcal{U}_G(\mathbb{A}^2)$ is not proper in this setting). Assume that the group $G$ is almost simple, and let $G_{aff}$ be the affinization of $G$, a Kac-Moody group such that $g_{aff} = Lie(G_{aff})$ is an affine Lie algebra. Then, the Uhlenbeck space $\mathcal{U}_G(\mathbb{A}^2)$ may be viewed as a slice in the affine Grassmannian for the group $G_{aff}$, that is, in the double affine Grassmannian for $G$. The IC stalks of $\mathcal{U}_G(\mathbb{A}^2)$ may be identified, in accordance with the predictions based on the geometric Satake correspondence, with certain graded versions of the weight spaces of the basic integrable representation of $g_{aff}$, the Langlands dual of the Lie algebra $g_{aff}$. In the simply-laced case, the Dynkin diagram of the Lie algebra $g_{aff}$ is an affine Dynkin diagram of types $A$, $D$, $E$, and we have $g_{aff}^\vee = g_{aff}$.

It is often useful to view the graph with one vertex and one edge-loop at that vertex as a Dynkin diagram of type $\tilde{A}_0$. It is known that the Kac-Moody Lie algebra associated with $\tilde{A}_0$ is the Heisenberg Lie algebra $\mathfrak{h}$. By definition, we have $\mathfrak{h} := \mathfrak{c}^0 \times \overline{\mathbb{C}(t)}$, where $\overline{\mathbb{C}(t)}$ is a central extension of the abelian Lie algebra $\mathbb{C}(t)$ and $\delta := t \frac{d}{dt}$, a derivation. The Fock representation of $\mathfrak{h}$ plays the role of the basic integrable representation of an affine
Lie algebra. The tensor factors of the graded vector space in (1.3.2) may be identified in a natural way with certain weight spaces of the Fock space (the action of the derivation $\delta$ gives a grading on the Fock space). This suggests, in view of the above, that our variety $U_M^\tau$ might play the role of a slice in some kind of an affine Grassmannian for the Heisenberg group and Theorem 1.3.1 is a manifestation of (a certain analogue of) the geometric Satake correspondence in the case of Dynkin diagram of type $\tilde{A}_0$.

1.4. **Organization of the paper.** In Section 2 we study coherent sheaves on a noncommutative projective plane and the corresponding representations of a Kronecker-type quiver. We introduce Gieseker and Mumford stabilities of sheaves and interpret them as stabilities of quiver representations. We construct the Gieseker and the Uhlenbeck moduli spaces of sheaves as GIT moduli spaces of quiver representations and a map $\gamma$ between the moduli spaces as the map coming from a variation of GIT quotients. In Section 3 we discuss the special case of sheaves of rank 1 and degree 0. In this case the Gieseker and the Uhlenbeck moduli spaces are compactifications of the Calogero–Moser space. We investigate in detail the map $\gamma$ between the compactifications and compute the stalks of the IC sheaf on the Uhlenbeck compactification. In the Appendix we provide proofs of some of the results of Section 2.

**Notation.** Given a vector space $V$, we write $V^\vee$ for the dual vector space and $S^\bullet V = \bigoplus_{i \geq 0} S^i V$ for the Symmetric algebra of $V$.

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2. **Sheaves on the noncommutative plane $\mathbb{P}^2_\tau$ and quiver representations**

To construct the Gieseker and the Uhlenbeck compactifications of the Calogero-Moser space we use an interpretation of the latter as moduli spaces of coherent sheaves on a noncommutative projective plane.

2.1. **Sheaves on the noncommutative projective plane.** We start with a slightly more invariant definition of the Calogero–Moser space. We consider a symplectic vector space $H$ of dimension 2 with a symplectic form $\omega \in \Lambda^2 H^\vee$, a vector space $V$ of dimension $n$, a nonzero complex number $\tau \in \mathbb{C}^\times$, and consider the subvariety $\tilde{M}_\tau(V) \subset \text{Hom}(V,V \otimes H)$ defined by

$$\tilde{M}_\tau(V) = \{a \in \text{Hom}(V,V \otimes H) | \text{rank}(\omega(a \circ a) - \tau \text{id}_V) = 1\},$$

where $\omega(a \circ a)$ is defined as the composition $V \xrightarrow{a} V \otimes H \xrightarrow{a \otimes \text{id}_H} V \otimes H \otimes H \xrightarrow{\text{id}_V \otimes \omega} V$, consider the action of $\text{PGL}(V)$ on $\tilde{M}_\tau(V)$ by conjugation, and define

$$M_\tau(V) = \tilde{M}_\tau(V)/\text{PGL}(V).$$
A choice of a symplectic basis in $H$ allows to rewrite $\alpha$ as a pair of operators $(X, Y)$, then $\omega(\alpha \circ \alpha)$ becomes as $[X, Y]$, and so this definition agrees with the standard one.

Denote $\tilde{H} := H \oplus \mathbb{C}$. We define a twisted symmetric algebra of $\tilde{H}$ by

$$A^r = S^*\tilde{H} = \mathbb{C}(H \oplus \mathbb{C}z)/([H, z] = 0, \ [h_1, h_2] = \tau \omega(h_1, h_2)z^2).$$

Choosing a symplectic basis $x, y$ in $H$ the defining relations in $A^r$ take the form (1.2.1).

The algebra $A^r$ is a graded noetherian algebra and we let

$$\mathbb{P}^2_\tau := \text{Proj}(A^r)$$

be the non-commutative “projective spectrum” of $A^r$ in the sense of [1]. The category of “coherent sheaves” on the non-commutative scheme $\mathbb{P}^2_\tau$ is defined as $\text{coh}(\mathbb{P}^2_\tau) := \text{qgr}(A^r)$, a quotient of the abelian category of finitely generated graded $A^r$-modules by the Serre subcategory of finite-dimensional modules. Note that the group $\text{SL}(H)$ acts on the algebra $A^r$ by automorphisms. The action on $A^r$ induces an $\text{SL}(H)$-action on the category of coherent sheaves $\text{coh}(\mathbb{P}^2_\tau)$.

As it was shown in [1] [10] [2] [17] and other papers, coherent sheaves on such a non-commutative projective plane behave very similarly to those on the usual (commutative) plane $\mathbb{P}^2$. For instance, one can define the cohomology spaces of sheaves, local $\text{Ext}$ sheaves, the notions of torsion free and locally free sheaves, one has the sequence $\{\mathcal{O}(i)\}_{i \in \mathbb{Z}}$ of “line bundles”, one can prove Serre duality and construct the Beilinson spectral sequence.

The main differences from the commutative case are

- in general there is no tensor product of sheaves (due to the noncommutativity); however, one can tensor with $\mathcal{O}(i)$ and thus define the twist functors $F \mapsto F(i)$ since sheaves $\mathcal{O}(i)$ correspond to graded $A^r$-modules having a natural bimodule structure (alternatively, the twist functor can be thought of as the twist of the grading functor in the category of graded $A^r$-modules);
- the dual of a sheaf on $\mathbb{P}^2_\tau$ is a sheaf on $\text{Proj}((A^r)^{\text{opp}})$, the “opposite” noncommutative projective plane; in fact, one has $\text{Proj}((A^r)^{\text{opp}}) = \mathbb{P}^2_{-\tau}$ since $(A^r)^{\text{opp}} \cong A^{-\tau}$.
- the noncommutative projective plane $\mathbb{P}^2_\tau$ has less points than the usual plane $\mathbb{P}^2$, and as a consequence the category $\text{coh}(\mathbb{P}^2_\tau)$ has more locally free sheaves than $\text{coh}(\mathbb{P}^2)$.

Below, we summarize the results of [1] [2] [10] [17] that we are going to use later in the paper.

By [1] Theorem 8.1(3)], the cohomology groups of the sheaves $\mathcal{O}(i)$ are given by the following formulas (similar to those in the commutative case):

$$H^p(\mathbb{P}^2_\tau, \mathcal{O}(i)) = \begin{cases} A^r_i = S^i\tilde{H}, & \text{if } p = 0 \text{ and } i \geq 0 \\ (A^r_{-i-3})^\vee = S^{-i-3}\tilde{H}^\vee, & \text{if } p = 2 \text{ and } i \leq -3 \\ 0 & \text{otherwise} \end{cases}$$

One has a functorial Serre duality isomorphism

$$\text{Ext}^i(E, F) \cong \text{Ext}^{2-i}(F, E(-3))^\vee.$$
sequence involves the sheaves $Q_0, Q_1$ and $Q_2$ on $\mathbb{P}^2_\tau$ defined by

$$Q_0 = \mathcal{O}, \quad 0 \to \mathcal{O}(x,y,z) \to \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \to Q_1 \to 0, \quad Q_2 = \mathcal{O}(3). \quad (2.1.1)$$

Sometimes another resolution for $Q_1$ is more convenient

$$0 \to Q_1 \to \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathcal{O}(3) \to 0 \quad (2.1.2)$$

We remark that each of the two sequences above is a truncation of the Koszul complex.

The Beilinson spectral sequence has the form

$$E_1^{-p,q} = \text{Ext}^q(Q_p(-p), \mathcal{E}) \otimes \mathcal{O}(-p) \implies E_\infty^i = \begin{cases} E, & \text{for } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

where $p = 0, 1, 2$. Using the Beilinson spectral sequence one shows, cf. [2, §7.2], that any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^2_\tau$ admits a resolution of the form

$$0 \to V' \otimes \mathcal{O}(k-2) \to V \otimes \mathcal{O}(k-1) \to V'' \otimes \mathcal{O}(k) \to E \to 0 \quad (2.1.3)$$

for some $k \in \mathbb{Z}$ and vector spaces $V', V, V''$.

The dual $E^* := \text{Hom}(\mathcal{E}, \mathcal{O})$, of any sheaf $\mathcal{E}$ is a sheaf on the opposite plane $\mathbb{P}^2_\tau$. The sheaf $\mathcal{E}$ is called \emph{locally free} if $\text{Ext}^i(\mathcal{E}, \mathcal{O}) = 0$ for $i > 0$.

The following statements are proved in [2, Proposition 2.0.4]. For any sheaf $\mathcal{E}$, the sheaf $\mathcal{E}^*$ is locally free, furthermore, $\mathcal{E}$ is locally free if and only if its canonical map $\mathcal{E} \to \mathcal{E}^* \to \mathcal{E}^{**}$ is an isomorphism. The kernel of a morphism of locally free sheaves is always locally free.

Let $\mathcal{E}$ be a locally free sheaf. Writing (2.1.3) for $\mathcal{E}^*$ and dualizing, one deduces that any locally free sheaf $\mathcal{E}$ has a resolution of the form

$$0 \to \mathcal{E} \to U' \otimes \mathcal{O}(-k) \to U \otimes \mathcal{O}(1-k) \to U'' \otimes \mathcal{O}(2-k) \to 0. \quad (2.1.4)$$

A sheaf $\mathcal{E}$ is called \emph{torsion free} if it can be embedded in a locally free sheaf. This can be shown, e.g. using [2, Proposition 2.0.6], to be equivalent to the injectivity of the canonical map $\mathcal{E} \to \mathcal{E}^{**}$.

For a coherent sheaf $\mathcal{E}$ its \emph{Hilbert polynomial} is defined by the usual formula

$$h_\mathcal{E}(t) = \sum_{i=0}^{2} (-1)^i \dim H^i(\mathbb{P}^2_\tau, \mathcal{E}(t)).$$

For sheaves $\mathcal{O}(i)$ it is the same as in the commutative case $h_{\mathcal{O}(i)}(t) = (t+i+1)(t+i+2)/2$. So, using (2.1.3) one sees that the Hilbert polynomial of any sheaf can be written as

$$h_\mathcal{E}(t) = r(\mathcal{E}) \frac{(t+1)(t+2)}{2} + \deg(\mathcal{E}) \frac{2t+3}{2} + \frac{\deg(\mathcal{E})^2}{2} - c_2(\mathcal{E}) \quad (2.1.5)$$

for some integers $r(\mathcal{E}), \deg(\mathcal{E})$ and $c_2(\mathcal{E})$ defined by this equality and called the rank, degree and second Chern class of $\mathcal{E}$ respectively. It is clear from the definition that the Hilbert polynomial as well as the rank and the degree are additive in exact sequences. Further, one can check that they behave naturally with respect to dualization

- for any sheaf $\mathcal{E}$ one has $r(\mathcal{E}^*) = r(\mathcal{E})$;
- for a torsion free sheaf $\mathcal{E}$ one also has $\deg(\mathcal{E}^*) = -\deg(\mathcal{E})$;
- for a locally free sheaf $\mathcal{E}$ one also has $c_2(\mathcal{E}^*) = c_2(\mathcal{E})$. 

...
Sometimes, instead of the second Chern class \(c_2(E)\) it is more convenient to use
\[
\text{ch}_2(E) := \deg(E)^2/2 - c_2(E)
\]
(this can be thought of as the second coefficient of the Chern character). Its obvious advantage is additivity in exact sequences.

For any sheaf \(E\) the rank \(r(E)\) is nonnegative. If \(E\) is torsion free and nonzero then \(r(E) > 0\), moreover, if \(r(E) = 0\) then the degree \(\deg(E)\) is nonnegative. The sheaf \(F\) is called Artin sheaf of length \(n = h_F = \text{ch}_2(F)\) if both the rank and degree of \(F\) are equal to zero, equivalently, the Hilbert polynomial of \(F\) is constant. In this case, the integer \(n := h_F = \text{ch}_2(F)\) is nonnegative and it is called the length of \(F\).

A special feature of the noncommutative plane \(\mathbb{P}_x^2\) is that it has less points than the commutative \(\mathbb{P}^2\): all points of \(\mathbb{P}_x^2\) are contained, in a sense, in the projective line \(\mathbb{P}^1\) ‘at infinity’. In more detail, note we have \(\text{Proj}(S^1) = \mathbb{P}(H^1) \cong \mathbb{P}(H) = \mathbb{P}^1\), where we identify \(H^1 = H\) via \(\omega\). Heuristically, one may view the graded algebra morphism
\[
A^r \to A^r/(z) \cong S^*(H) \cong \mathbb{C}[x,y].
\]
as being induced by a ‘closed imbedding’ \(\mathbb{P}^1 \hookrightarrow \mathbb{P}_x^2\), of the projective line ‘at infinity’. Specifically, there is a pair of adjoint functors \(i_* : \text{coh}(\mathbb{P}(H)) \to \text{coh}(\mathbb{P}_x^2)\) and \(i^* : \text{coh}(\mathbb{P}_x^2) \to \text{coh}(\mathbb{P}(H))\). The pushforward functor \(i_*\) extends a graded \(S^*(H)\)-module structure to a graded \(A^r\)-module structure by setting the action of \(z\) to be zero. The pullback functor \(i^*\) takes a graded \(A^r\)-module \(M\) to \(M/Mz\). The projection \(A^r \to S^*(H)\) is clearly \(\text{SL}(H)\)-equivariant, hence so are the functors \(i_*\) and \(i^*\). The functor \(i_*\) is exact. The functor \(i^*\) is right exact, and it has a sequence of left derived functors \(L_p i^*, p > 0\). In fact
- for any sheaf \(E\) one has \(L_{>1} i^* E = 0\);
- for a torsion free sheaf \(E\) one also has \(L_1 i^* E = 0\);
- for a locally free sheaf \(E\) the sheaf \(i^* E\) is also locally free.

We will use the following result.

**Proposition 2.1.6.** ([2] Proposition 3.4.14) For any \(\tau \neq 0\) one has
(1) If \(i^* E = 0\) then \(E = 0\).
(2) If \(\phi \in \text{Hom}(E,F)\) and \(i^* \phi\) is an epimorphism, then \(\phi\) is an epimorphism.
(3) If \(\phi \in \text{Hom}(E,F)\) and both \(i^* \phi\) and \(L_1 i^* \phi\) are isomorphisms then \(\phi\) is an isomorphism.
(4) If \(\phi \in \text{Hom}(E,F)\), \(i^* \phi\) is a monomorphism and \(L_1 i^* F = 0\) then \(\phi\) is a monomorphism.
(5) A sheaf \(E\) is locally free iff \(L_{\geq 0} i^* E = 0\) and \(i^* E\) is locally free.

We deduce the following properties of Artin sheaves:

**Proposition 2.1.7.** Let \(F\) be an Artin sheaf and \(h_F(t) = n\).
(1) For sufficiently general \(h \in H \subset H = H^0(\mathbb{P}_x^2, \mathcal{O}(1))\), the map \(h : F(-1) \to F\), of right multiplication by \(h\), is an isomorphism.
(2) For any locally free sheaf \(\mathcal{E}\) we have
\[
\dim \text{Ext}^m(\mathcal{E}, F) =\begin{cases} 0, & \text{if } m > 0 \\ nr(\mathcal{E}), & \text{if } m = 0. \end{cases}
\]
(3) The sheaf \(F\) has a filtration \(0 = F_0 \subset F_1 \subset \ldots \subset F_n = F\) with \(F_k/F_{k-1} = i_* \mathcal{O}_P\) for some points \(P_1, \ldots, P_n \in \mathbb{P}(H)\) on the line at infinity. In particular, if \(h_F(t) = 1\) then \(F \cong i_* \mathcal{O}_P\) for some point \(P \in \mathbb{P}(H)\).
Proof. (1) Since both \( i^* F \) and \( L_1 i^* F \) are torsion sheaves on \( \mathbb{P}(H) \) the maps \( i^* h \) and \( L_1 i^* h \) are isomorphisms for generic \( h \). Hence \( h : F(-1) \to F \) is also an isomorphism for generic \( h \) by Proposition 2.1.6(3).

(2) By (2.1.4) it is enough to consider the case \( E = \mathcal{O}(p) \) for some \( p \in \mathbb{Z} \). In this case for \( p \ll 0 \) the result is clear and for arbitrary \( p \) it follows from (1).

(3) The map \( F \to i_* i^* F \) is an epimorphism. On the other hand, \( i^* F \) is a nontrivial sheaf on \( \mathbb{P}(H) \), hence there is an epimorphism \( i^* F \to \mathcal{O}_P \) for some \( P \in \mathbb{P}(H) \). The composition gives an epimorphism \( F \to i_* \mathcal{O}_P \). Its kernel is an Artin sheaf on \( \mathbb{P}^2_\tau \) of length \( n - 1 \) and we can apply induction in \( n \).

\[ \Box \]

Let \( F \) be an Artin sheaf and take an arbitrary \( p \in \mathbb{Z} \). Consider the canonical map
\[
H^0(\mathbb{P}^2_\tau, F(p)) \otimes H \to H^0(\mathbb{P}^2_\tau, F(p + 1))
\]
induced by the embedding \( H \subset \mathbb{H} = H^0(\mathbb{P}^2_\tau , \mathcal{O}(1)) \). Let \( n = h_F \) be the length of \( F \), so that both cohomology spaces above are \( n \)-dimensional. A component of the \( n \)-th wedge power of the above map is a map
\[
\det(H^0(\mathbb{P}^2_\tau, F(p))) \otimes S^n H \to \det(H^0(\mathbb{P}^2_\tau, F(p + 1))).
\]
Its partial dualization gives a map
\[
\det(H^0(\mathbb{P}^2_\tau, F(p))) \otimes \det(H^0(\mathbb{P}^2_\tau, F(p + 1)))^\vee \to S^n H^\vee.
\] (2.1.8)
We consider the projectivization of the right hand side as the space of degree \( n \) divisors on \( \mathbb{P}(H) \) and denote by
\[
\text{supp}(F) \in \mathbb{P}(S^n H^\vee) = S^n \mathbb{P}(H^\vee)
\]
the image of the map. The next Lemma shows it is well defined.

Lemma 2.1.9. For any Artin sheaf \( F \) of length \( n \) the map \( (2.1.8) \) is injective, \( \text{supp}(F) \) is well defined and is independent of the choice of \( p \). If \( 0 \to F_1 \to F_2 \to F_3 \to 0 \) is an exact sequence of Artin sheaves then
\[
\text{supp}(F_2) = \text{supp}(F_1) + \text{supp}(F_3).
\]
If \( 0 \neq h \in H \) then \( h : F(-1) \to F \) is an isomorphism if and only if \( h \notin \text{supp}(F) \).

Proof. Clearly, evaluation of the image of \( (2.1.8) \) on \( h \in H \) is the determinant of the map
\[
H^0(\mathbb{P}^2_\tau, F(p)) \to H^0(\mathbb{P}^2_\tau, F(p + 1))
\]
induced by \( h \). We know that for generic \( h \) the map is an isomorphism, hence its determinant is nonzero. This means that the image of \( (2.1.8) \) is not identically zero and proves the first claim of the Lemma. It also proves the “only if” part of the last claim. Moreover, the “if” part also follows for Artin sheaves of length 1. The additivity of the support under extensions is evident (the determinant of a block upper triangular matrix is the product of the determinants of blocks). It follows that if \( F_\bullet \) is a filtration on \( F \) with \( F_k/F_{k-1} \cong \mathcal{O}_{P_k} \) then \( \text{supp}(F) = \sum P_k \). In particular, \( \text{supp}(F) \) does not depend on the choice of the integer \( p \). Finally, this observation and the additivity of the support also proves the “if” part of the last claim in general.

\[ \Box \]

To finish this introductory section let us mention that one can consider families of sheaves on a noncommutative plane \( \mathbb{P}^2_\tau \). More precisely, for each affine scheme \( S \) one can define a notion of a coherent sheaf on \( S \times \mathbb{P}^2_\tau \) (see [18]) which is the standard way to think about
S-families of sheaves on $\mathbb{P}^2$. This allows to define moduli spaces of sheaves on $\mathbb{P}^2$ with appropriate stability conditions, which is the goal of this section.

In fact, a significant part of the results of this section are proved in a more general setting (i.e., for an arbitrary Artin–Schelter algebra instead of $A_\tau$) in [18], in particular, the Gieseker moduli space we construct coincides with the moduli space of Nevins–Stafford. However, the case we consider is significantly simpler than the general case, this is the reason for us to present most of the constructions here, while suppressing some proofs. The really new content of the section is the definition, construction, and investigation of the Uhlenbeck moduli space. To make its relation to the Gieseker moduli space more clear, we use a GIT construction of the latter moduli space which is different from that of [18].

2.2. Coherent sheaves and quiver representations. Let $A_\tau^1$ be the quadratic dual algebra of $A_\tau$. From the quadratic relations for $A_\tau$ we deduce that $A_\tau^1$ is isomorphic to a twisted exterior algebra $\Lambda^\bullet(\bar{H}^\vee)$ of the vector space $\bar{H}^\vee = H^\vee \oplus \mathbb{C}\zeta$. Specifically, writing \{-, -\} for the anticommutator, we have

$$A_\tau^1 \cong \Lambda^\bullet(\bar{H}^\vee) = \mathbb{C}\langle H^\vee \oplus \mathbb{C}\zeta \rangle/\{H^\vee, H^\vee\} = \{H^\vee, \zeta\} = \zeta^2 + \tau \omega = 0\}.$$ 

The group $SL(H)$ acts on $A_\tau^1$ by algebra automorphisms.

Choosing a symplectic basis $\xi, \eta$ in $H^\vee$ we can rewrite the above as follows

$$A_\tau^1 = \mathbb{C}\langle \xi, \eta, \zeta \rangle/\langle \zeta^2 = \eta^2 = \eta \xi + \xi \eta = \zeta \xi + \xi \zeta = \eta \zeta + \zeta \eta = \zeta^2 + \tau(\xi \eta - \eta \xi) = 0\rangle.$$ 

Here, the grading on $A_\tau^1$ corresponds to the grading $\text{deg} \xi = \text{deg} \eta = \text{deg} \zeta = 1$. Let $Q_\tau$ be the following quiver:

\[
\begin{array}{c}
1 \\
(A_\tau^1)_1 \\
2 \\
(A_\tau^1)_2 \\
3 \\
(A_\tau^1)_3
\end{array}
\]

with the spaces of arrows given by the components $(A_\tau^1)_1$ and $(A_\tau^1)_2$ of the dual algebra and the composition of arrows given by the multiplication $(A_\tau^1)_1 \otimes (A_\tau^1)_1 \to (A_\tau^1)_2$ in $A_\tau^1$. The $SL(H)$ action on $A_\tau^1$ induces an action on the quiver $Q_\tau$, on the category of its representations $\text{Rep}(Q_\tau)$, and on its derived category $D(\text{Rep}(Q_\tau))$.

**Proposition 2.2.1.** The functors between the bounded derived categories

$$D(\text{coh}(\mathbb{P}^2_{\tau})) \to D(\text{Rep}(Q_\tau)), \quad E \mapsto (\text{Ext}^\bullet(Q_2(-1), E), \text{Ext}^\bullet(Q_1, E), \text{Ext}^\bullet(Q_0(1), E)),$$

$$D(\text{Rep}(Q_\tau)) \to D(\text{coh}(\mathbb{P}^2_{\tau})), \quad R_\bullet \mapsto \{R_1 \otimes O(-1) \to R_2 \otimes O \to R_3 \otimes O(1)\}$$

are mutually inverse $SL(H)$-equivariant equivalences.

**Proof.** Follows from the fact that $(O(-1), O, O(1))$ is a strong exceptional collection in $D(\text{coh}(\mathbb{P}^2_{\tau}))$, and $(Q_2(-1), Q_1, Q_0(1))$ is its dual collection. The quiver $Q_\tau$ is in fact the quiver of morphisms of the latter sequence. \qed

We consider the restrictions of these functors to the abelian categories. Given a representation $R_\bullet = (R_1, R_2, R_3)$ of $Q_\tau$ one constructs a complex of sheaves

$$\mathcal{E}(R_\bullet) := \{R_1 \otimes O(-1) \to R_2 \otimes O \to R_3 \otimes O(1)\}.$$  

(2.2.2)
Denote by \( D(R_\bullet) \), \( i = 1, 2, 3 \), its cohomology sheaves. Recall that a three-term complex is a monad if its cohomology at the first and last terms vanish.

Analogously, given a sheaf \( E \) we consider a representation of \( Q_\tau \)

\[
V_\bullet(E) = (\Ext^1(Q_2(-1), E), \Ext^1(Q_1, E), \Ext^1(Q_0(1), E)). \tag{2.2.3}
\]

This is equivalent to applying the functor of Proposition 2.2.1 and then taking the first cohomology in the derived category of quiver representations.

**Lemma 2.2.4.** If \( \mathcal{C}(R_\bullet) \) is a monad and \( E = D^2(\mathcal{C}(R_\bullet)) \) is its middle cohomology sheaf then \( V_\bullet(E) \cong R_\bullet \) and \( \Ext^i(Q_p(1-p), E) = 0 \) for \( i \neq 1 \) and \( p = 0, 1, 2 \).

*Proof.* If \( \mathcal{C}(R_\bullet) \) is a monad then \( E = D^2(\mathcal{C}(R_\bullet)) \) is isomorphic to the complex \( \mathcal{C}(R_\bullet) \) in the derived category \( D(\coh(P^2_\tau)) \), hence the complex can be used to compute \( \Ext^i(Q_p(1-p), E) \).

The computation gives the required result.

Vice versa, under the conditions of the Lemma we have \( V_\bullet(E) \) is the image of \( E \) in \( D(\Rep(Q_\tau)) \) under the equivalence of Proposition 2.2.1 hence \( \mathcal{C}(V_\bullet(E)) \cong E \) in \( D(\coh(P^2_\tau)) \). This means that the complex is a monad and its middle cohomology is \( E \).

\[ \square \]

### 2.3. Stability of sheaves and quiver representations.

The notions of Gieseker and Mumford (semi)stability of coherent sheaves are standard in the commutative context. We refer to [9] for more details and for proofs of standard facts. These notions have generalizations for sheaves on \( \mathbb{P}^2 \).

Given a sheaf \( E \) on \( \mathbb{P}^2_\tau \) with \( r(E) > 0 \), we define its Mumford and Gieseker slopes as

\[
\begin{align*}
\mu_M(E) &= \frac{\deg(E)}{r(E)} \in \mathbb{Q}, \\
\mu_G(E) &= \frac{h_E(t)}{r(E)} = \frac{(t+1)(t+2)}{2} + \mu_M(E) \frac{2t+3}{2} + \frac{\deg(E)^2 - 2c_2(E)}{2r(E)} \in \mathbb{Q}[t].
\end{align*}
\]

Let \( p(t) \) and \( q(t) \) be polynomials. We say that \( p < q \) (resp. \( p \leq q \)) if for all \( t \gg 0 \) we have \( p(t) < q(t) \) (resp. \( p(t) \leq q(t) \)).

**Definition 2.3.1.** A sheaf \( E \) is Gieseker stable (resp. Gieseker semistable) if \( E \) is torsion free and for any subsheaf \( 0 \subsetneq F \subsetneq E \) we have \( \mu_G(F) < \mu_G(E) \) (resp. \( \mu_G(F) \leq \mu_G(E) \)). Sheaves \( E \) and \( F \) are called Gieseker \( S \)-equivalent if both of them are Gieseker semistable and have isomorphic composition factors in the category of Gieseker semistable sheaves.

Similarly, a sheaf \( E \) is Mumford stable (resp. Mumford semistable) if any torsion subsheaf in \( E \) is Artin and for any \( F \subset E \) such that \( 0 < r(F) < r(E) \) we have \( \mu_M(F) < \mu_M(E) \) (resp. \( \mu_M(F) \leq \mu_M(E) \)). A pair of sheaves \( E \) and \( F \) are called Mumford \( S \)-equivalent if both of them are Mumford semistable and have isomorphic composition factors in the category of Mumford semistable sheaves.

Both Gieseker and Mumford stabilities of sheaves on \( \mathbb{P}^2 \) behave analogously to those on the commutative projective plane \( \mathbb{P}^2 \). For example, by [20] each sheaf \( F \) has a Harder–Narasimhan filtration, i.e. a filtration

\[ 0 = F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 = F \]
such that $F_{i-1}/F_i$ are stable and $\mu(F_{n-1}/F_n) > \mu(F_{n-2}/F_{n-1}) > \cdots > \mu(F_0/F_1)$.

To check Gieseker stability (semistability) it is enough to consider only subsheaves $F \subset E$ such that $E/F$ is torsion free (in particular, $r(F) < r(E)$). So, the following is clear.

**Lemma 2.3.2.** Any torsion free sheaf of rank 1 is Gieseker stable and Mumford stable.

Note also that $\mu_M(E) > \mu_M(F)$ implies $\mu_G(E) > \mu_G(F)$ and $\mu_G(E) \geq \mu_G(F)$ implies $\mu_M(E) \geq \mu_M(F)$. It follows that Gieseker semistability implies Mumford semistability, while Mumford stability for torsion free sheaves implies Gieseker stability. Moreover, if the rank and the degree of a torsion free sheaf are coprime then semistability implies stability.

The following Lemma is standard

**Lemma 2.3.3.** (1) If $E, F$ are Mumford semistable sheaves with $F$ torsion free and $\mu_M(E) > \mu_M(F)$ then $\text{Hom}(E,F) = 0$.

(2) If $E, F$ are Mumford stable sheaves, $E$ is locally free and $\mu_M(E) \geq \mu_M(F)$ then any nontrivial homomorphism $E \to F$ is an isomorphism.

The notion of stability for a representation of a quiver depends on a choice of a polarization, see [13]. A **polarization** in case of the quiver $Q_r$ amounts to a triple $\theta = (\theta_1, \theta_2, \theta_3)$ of real numbers. The **$\theta$-slope** of a representation $R_\bullet = (R_1, R_2, R_3)$ of $Q_r$ is defined as

$$\mu_\theta(R_\bullet) = \langle \theta, \dim R_\bullet \rangle := \theta_1 \dim R_1 + \theta_2 \dim R_2 + \theta_3 \dim R_3.$$ 

**Definition 2.3.4.** A representation $R_\bullet$ is **$\theta$-stable** (resp. **$\theta$-semistable**) if $\mu_\theta(R_\bullet) = 0$ and for any subrepresentation $R'_\bullet \subset R_\bullet$ such that $0 \neq R'_\bullet \neq R_\bullet$ we have $\mu_\theta(R'_\bullet) > 0$ (resp. $\mu_\theta(R'_\theta) \geq 0$). Representations $R_\bullet$ and $R'_\bullet$ are called $S$-equivalent with respect to $\theta$ if both of them are $\theta$-semistable and have isomorphic composition factors in the category of $\theta$-semistable representations.

Let $\theta, \theta'$ be a pair of polarizations. It is well known (e.g. [8]) that, for all sufficiently small and positive $\varepsilon \in \mathbb{R}$, stability, semistability and $S$-equivalence with respect to $\theta + \varepsilon\theta'$ does not depend on $\varepsilon$.

**Definition 2.3.5.** A representation $R_\bullet$ is $(\theta, \theta')$-stable (resp. $(\theta, \theta')$-semistable) if $R_\bullet$ is $(\theta + \varepsilon\theta')$-stable (resp. semistable) for sufficiently small positive $\varepsilon$.

There is an analogue of Lemma 2.3.3 for representations of the quiver $Q_r$.

### 2.4. From sheaves to quiver representations.

Let $\mathcal{F}_{r,d}$ be the The following result is essentially a combination of Lemma 6.4 and Theorem 5.6 from [18]. The only new statement is the exactness claim. We provide a proof for the reader’s convenience.

**Theorem 2.4.1.** Let $-r \leq d < r$. Then, the assignment $E \mapsto V_\bullet(E)$ gives an exact functor from the category of Mumford semistable torsion free sheaves $E$ on $\overline{\mathbb{P}_r^1}$ such that $r(E) = r$ and $\text{deg}(E) = d$ to the category of representations of the quiver $Q_r$. For such a sheaf $E$, the representation $V_\bullet(E)$ gives a monad

$$V_1(E) \otimes \mathcal{O}(-1) \to V_2(E) \otimes \mathcal{O} \to V_3(E) \otimes \mathcal{O}(1)$$

such that its cohomology at the middle term is isomorphic to $E$. Furthermore, we have $\dim V_\bullet(E) = (n - d(d - 1)/2, 2n - d^2 + r, n - d(d + 1)/2)$, where $n = c_2(E)$; in particular, $c_2(E) \geq d(d + 1)/2$. 

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Proof. First we note that all $Q_p$ are Mumford stable of slopes equal to 0, 3/2 and 3 respectively. Indeed, for $p = 0$ and $p = 2$ this follows from Lemma 2.3.3 and the definitions of $Q_p$. So let $p = 1$. The sheaf $Q_1$ is locally free because by (2.1.2) it is the kernel of a morphism of locally free sheaves, and moreover $r(Q_1) = 2$, deg($Q_1$) = 3. So, it is enough to check that if $F \subset Q_1$ is a subsheaf of rank 1 with $Q_1/F$ torsion free then deg($F$) \leq 1. Assume deg($F$) \geq 2. As $Q_1/F$ is torsion free, $F$ is locally free. Since $Q_1$ is a subsheaf in $\mathcal{O}(2)^{\oplus 3}$ there is a nontrivial homomorphism from $F$ to $\mathcal{O}(2)$. On the other hand both $F$ and $\mathcal{O}(2)$ are Mumford stable by (1), $F$ is locally free, and $\mu_M(F) \geq 2 = \mu_M(\mathcal{O}(2))$. Hence $F \cong \mathcal{O}(2)$ by Lemma 2.3.3(2). But applying the functor $\text{Hom}(\mathcal{O}(2), -)$ to (2.1.2) we see that $\text{Hom}(\mathcal{O}(2), Q_1) = 0$. 

The proved stability implies that

$$\text{Hom}(Q_0(1), E) = \text{Hom}(Q_1, E) = \text{Hom}(Q_2(-1), E) = 0.$$ 

Indeed, the slopes of the first arguments are 1, 3/2, and 2 respectively, while the slope of the second argument is $d/r < 1$, so Lemma 2.3.3(1) applies. Analogously,

$$\text{Hom}(E(3), Q_0(1)) = \text{Hom}(E(3), Q_1) = \text{Hom}(E(3), Q_2(-1)) = 0$$

since the slope of the first argument is $d/r + 3 > 2$. By Serre duality we then have

$$\text{Ext}^2(Q_0(1), E) = \text{Ext}^2(Q_1, E) = \text{Ext}^2(Q_2(-1), E) = 0.$$ 

Therefore, Lemma 2.2.4 applies to $E$ and shows that (2.4.2) is a monad and $E$ is its cohomology. The dimensions of the spaces $V_p(E)$ are computed directly by using the formula (2.1.5) for the Hilbert polynomial of a sheaf. The exactness of the functor $V_\bullet$ is clear from its definition and vanishing of Hom and Ext spaces. □

**Proposition 2.4.3.** The functor $F \mapsto \mathcal{C}(R_\bullet(F))$ yields, for an Artin sheaf $F$, a canonical exact sequence

$$0 \to W_1(F) \otimes \mathcal{O}(-1) \to W_2(F) \otimes \mathcal{O} \to W_3(F) \otimes \mathcal{O}(1) \to F \to 0$$

The resulting functor $W_\bullet$ from the category of Artin sheaves on $\mathbb{P}_{\tau}^2$ to the category of representations of the quiver $Q_{\tau}$ is exact and we have $\dim W_\bullet(F) = (n, 2n, n)$, where $n$ is the length of $F$.

**Proof.** The proof is analogous to the proof of Lemma 2.2.4. We apply the equivalence of Proposition 2.2.1 to the sheaf $F$. By Proposition 2.1.7(2) applying the functor of Proposition 2.2.1 to $F$ we obtain nothing but representation

$$W_\bullet(F) = (\text{Hom}(Q_2(-1), F), \text{Hom}(Q_1, F), \text{Hom}(Q_0(1), F))$$

and its dimension vector is $(n, 2n, n)$. Since the functor is an equivalence, it follows that the complex $\mathcal{C}(W_\bullet(F))$ is left exact and $\mathcal{H}^3(\mathcal{C}(W_\bullet(F))) \cong F$, which amounts to the above exact sequence. Exactness of the functor $W_\bullet$ follows from the vanishing of $\text{Ext}^1(Q_p(1 - p), F)$ by Proposition 2.1.7(2). □

If $0 \neq h \in H$, $P \in \mathbb{P}(H)$ is the corresponding point and $F = \mathcal{O}_P$, then

$$W_\bullet(\mathcal{O}_P) = \{\mathbb{C} \xrightarrow{(h)} \mathbb{C}^2 \xrightarrow{(-\zeta, h)} \mathbb{C}\}.$$ 

(2.4.4)
2.5. **From sheaf stability to quiver stability.** In this section we show that Gieseker and Mumford semistability correspond to semistability of quiver representations.

From now on we fix a triple \((r,d,n)\) such that

\[
0 \leq d < r \quad \text{and} \quad n \geq d(d+1)/2. \tag{2.5.1}
\]

Put

\[
\alpha(r,d,n) = \left( n - d(d-1)/2, 2n - d^2 + r, n - d(d+1)/2 \right). \tag{2.5.2}
\]

According to Theorem 2.4.1 if \(E\) is a Mumford semistable sheaf such that \(r(E) = r, \deg(E) = d\) and \(c_2(E) = n\) then \(\dim V_\bullet(E) = \alpha(r,d,n)\).

We choose the following pair of polarizations

\[
\theta^0 = (-r - d, d - r - d), \\
\theta^1 = (2n - d^2 + r, d^2 - 2n, 2n - d^2 + r). \tag{2.5.3}
\]

Note that \(\theta^0\) does not depend on \(n\). Note also that

\[
\langle \theta^0, \alpha(r,d,n) \rangle = \langle \theta^1, \alpha(r,d,n) \rangle = 0.
\]

In what follows we frequently consider \(\theta^0\)-stability and \((\theta^0, \theta^1)\)-stability of representations. In fact the notion of \((\theta^0, \theta^1)\)-(semi)stability of a quiver representation is equivalent to the notion of (semi)stability of a Kronecker complex considered in \[18\].

**Lemma 2.5.4.** Let \(V_\bullet\) be an \(\alpha(r,d,n)\)-dimensional representation of the quiver \(Q_\tau\) and let \(\mathcal{C}(V_\bullet)\) be the associated complex \((2.2.2)\). Then \(V_\bullet\) is \((\theta^0, \theta^1)\)-(semi)stable if and only if \(\mathcal{C}(V_\bullet)\) is (semi)stable in the sense of \[18\] Def. 6.8.

**Proof.** Just note that a subcomplex in \(\mathcal{C}(V_\bullet)\) always corresponds to a subrepresentation \(U_\bullet \subset V_\bullet\), and the expression (1) from \[18\] Definition 6.8 for \(\mathcal{C}(U_\bullet)\) equals \(\mu_{\theta^0}(U_\bullet)\), while the expression (2) equals \(\mu_{\theta^1}(U_\bullet)\), \(\square\)

The following crucial observation relating stability for sheaves and for quiver representations, respectively, is due to Sergey Kuleshov. Parts (2) and (3) (as well as a part of Lemma 2.6.3 below) are also proved in Proposition 6.20 of \[18\].

**Lemma 2.5.5.** Let \(E\) be a torsion free sheaf with \(r(E) = r, \deg(E) = d, c_2(E) = n\) and let \(V_\bullet(E)\) be the corresponding representation of the quiver \(Q_\tau\). Then

(1) if \(E\) is Mumford semistable then \(V_\bullet(E)\) is \(\theta^0\)-semistable;
(2) if \(E\) is Gieseker semistable then \(V_\bullet(E)\) is \((\theta^0, \theta^1)\)-semistable;
(3) if \(E\) is Gieseker stable then \(V_\bullet(E)\) is \((\theta^0, \theta^1)\)-stable.

**Proof.** (1) Assume that \(E\) is Mumford semistable. Let \(U_\bullet\) be a subrepresentation in \(V_\bullet(E)\) and \(W_\bullet = V_\bullet(E)/U_\bullet\) be the quotient representation, and put \(u_i = \dim U_i\). Then we have a
short exact sequence of complexes

\[ \begin{array} {c} \vline \end{array} \begin{array} {ccc} 0 & \longrightarrow & U_1 \otimes \mathcal{O}(-1) \longrightarrow U_2 \otimes \mathcal{O} \longrightarrow U_3 \otimes \mathcal{O}(1) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & V_1(E) \otimes \mathcal{O}(-1) \longrightarrow V_2(E) \otimes \mathcal{O} \longrightarrow V_3(E) \otimes \mathcal{O}(1) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_1 \otimes \mathcal{O}(-1) \longrightarrow W_2 \otimes \mathcal{O} \longrightarrow W_3 \otimes \mathcal{O}(1) \longrightarrow 0 \end{array} \ \text{(2.5.6)} \]

Considering it as an exact triple (with respect to the vertical maps) of 3-term complexes and applying the Snake Lemma we obtain a long exact sequence of cohomology sheaves

\[ 0 \rightarrow \mathcal{H}^1(U_\bullet) \rightarrow 0 \rightarrow \mathcal{H}^1(W_\bullet) \rightarrow \mathcal{H}^2(U_\bullet) \rightarrow E \rightarrow \mathcal{H}^2(W_\bullet) \rightarrow \mathcal{H}^3(U_\bullet) \rightarrow 0 \rightarrow \mathcal{H}^3(W_\bullet) \rightarrow 0 \]

of these complexes. In particular, we have \( \mathcal{H}^1(U_\bullet) = \mathcal{H}^3(W_\bullet) = 0 \). Denote

\[ r^U_i = r(\mathcal{H}^i(U_\bullet)), \quad d^U_i = \deg(\mathcal{H}^i(U_\bullet)), \quad r^W_i = r(\mathcal{H}^i(W_\bullet)), \quad d^W_i = \deg(\mathcal{H}^i(W_\bullet)). \]

Let \( I \) be image of the morphism \( \mathcal{H}^2(U_\bullet) \rightarrow E \) in the above sequence and let

\[ r_I = r(I), \quad d_I = \deg(I). \]

Then using additivity of rank and degree we can rewrite the slope of \( U_\bullet \) as

\[ \mu_{\phi}(U_\bullet) = r(u_3 - u_1) + d(u_2 - u_1 - u_3) = r(d^U_3 - d^U_1) + d(r^U_2 - r^U_3) = r(d^U_3 - d^W_3 - d_I) + d(r^W_1 + r_I - r^U_3) = (rd^U_3 - r^U_3 d) + (r^W_1 d - rd^W_1) + (r_I d - rd_I). \]

Now we will show that all three summands in the right-hand-side are nonnegative.

First, note that \( \mathcal{H}^3(U_\bullet) \) is a quotient of \( U_3 \otimes \mathcal{O}(1) \). The latter sheaf is semistable by Lemma 2.3.2(1) and we have \( \mu_M(U_3 \otimes \mathcal{O}(1)) = 1 \). Therefore, \( d^U_3 \geq r^U_3 \) and hence

\[ rd^U_3 - r^U_3 d \geq r^U_3 (r - d) \geq 0. \]

Note that the inequality \( rd^U_3 - r^U_3 d \geq 0 \) is strict unless \( r^U_3 = d^U_3 = 0 \), that is unless \( \mathcal{H}^3(U_\bullet) \) is an Artin sheaf.

Further, note that \( \mathcal{H}^1(W_\bullet) \) is a subsheaf of \( W_1 \otimes \mathcal{O}(-1) \). The latter sheaf is semistable and one has \( \mu_M(W_1 \otimes \mathcal{O}(-1)) = -1 \). Therefore, \( d^W_1 \leq -r^W_1 \leq 0 \) and hence

\[ r^W_1 d - rd^W_1 \geq dr^W_1 \geq 0. \]

Note that this inequality is strict unless \( r^W_1 = d^W_1 = 0 \), that is unless \( \mathcal{H}^1(W_\bullet) \) is an Artin sheaf. But since it is a subsheaf in \( W_1 \otimes \mathcal{O}(-1) \) it is torsion free, hence this is equivalent to \( \mathcal{H}^1(W_\bullet) = 0 \).

Finally, \( I \) is a subsheaf of the Mumford semistable torsion free sheaf \( E \). Hence either \( I = 0 \), or else \( r_I > 0 \) and \( \mu_M(I) \leq \mu_M(E) \). In both cases we have

\[ r_I d - d_I r \geq 0. \]

Note that this inequality is strict unless \( I = 0 \) or \( \mu_M(I) = \mu_M(E) \).

Combining all these inequalities we see that any subrepresentation in \( V_\bullet(E) \) has a non-negative \( \theta^0 \)-slope. Thus \( V_\bullet(E) \) is \( \theta^0 \)-semistable.
(2) Assume that the sheaf $E$ is Gieseker semistable but $V_\bullet(E)$ is not $(\theta^0, \theta^1)$-semistable. Let $U_\bullet \subset V_\bullet(E)$ be a destablizing subrepresentation. Since $E$ is automatically Mumford semistable and hence $V_\bullet(E)$ is $\theta^0$-semistable by (1), we conclude that $\mu_{\theta^0}(U_\bullet) = 0$. Then as we observed in the above argument $\mathcal{K}^3(U_\bullet)$ is an Artin sheaf, $\mathcal{K}^1(W_\bullet) = 0$ and hence $\mathcal{K}^2(U_\bullet) = I$ is a subsheaf in $E$.

Let $c = ch_2(E) = d^2/2 - n$ and $c_2' = ch_2(\mathcal{K}^i(U_\bullet))$. Then

$$\mu_{\theta^i}(U_\bullet) = (r - 2c)u_1 + 2cu_2 + (r - 2c)u_3 = r(u_1 + u_3) + 2c(u_2 - u_1 - u_3)$$

$$= 2r(c_3' - c'_2) + 2c(r_2' - r_3'c) = 2(rc_3' - r_3'c) + 2(r_2'c - rc_2').$$

Since $\mathcal{K}^3(U_\bullet)$ is an Artin sheaf we have $r_3'c = rc_2' = 0$ and $c_3' \geq 0$, hence

$$rc_3' - r_3'c = rc_3' \geq 0.$$  

On the other hand, $\mathcal{K}^2(U_\bullet) = I$ and hence either $\mathcal{K}^2(U_\bullet) = 0$, or $\mu_{\theta^0}(\mathcal{K}^2(U_\bullet)) = \mu_{\theta^0}(E)$. In both cases

$$r_2'c - rc_2' = r_2'c = 0,$$

because $E$ is Gieseker semistable. Thus $\mu_{\theta^i}(U_\bullet) \geq 0$ which contradicts to the assumption that $U_\bullet$ destablizes $V_\bullet(E)$.

(3) In the notation of (2) assume that $\mu_{\theta^0}(U_\bullet) = \mu_{\theta^1}(U_\bullet) = 0$. Then $c_2' = 0$, hence $\mathcal{K}^3(U_\bullet) = 0$. Moreover, either $\mathcal{K}^2(U_\bullet) = 0$, or $\mu_{\theta^0}(\mathcal{K}^2(U_\bullet)) = \mu_{\theta^0}(E)$, and so $\mathcal{K}^2(U_\bullet) = E$ since $E$ is Gieseker stable. In the first case the first line of (2.5.6) is exact, hence $U_\bullet = 0$ by Proposition 2.2.1. In the second case the first line of (2.5.6) is a resolution of $E$, hence $U_\bullet = V_\bullet(E)$. \hfill \hfill \square

Lemma 2.5.7. Let $F$ be an Artin sheaf. Then $W_\bullet(F)$ is $\theta^0$-semistable. If the length of $F$ is equal to 1 then $W_\bullet(F)$ is $\theta^0$-stable.

Proof. Since by Proposition 2.1.7(3) any Artin sheaf is an extension of the structure sheaves of points and the functor $W_\bullet$ is exact, it is enough to verify the $\theta^0$-stability of $W_\bullet(O_P)$. The latter is clear from the explicit form of the monad — it is easy to see that the only nontrivial subrepresentations of $W_\bullet(O_P)$ have dimension $(0, 0, 1), (0, 1, 1)$ and $(0, 2, 1)$, and their $\theta^0$-slope is clearly positive with our assumptions on $d$ and $r$. \hfill \square

2.6. From quiver stability to sheaf stability. In this section we show that stable representations of the quiver, in turn, give rise to stable sheaves.

Definition 2.6.1. A representation $V_\bullet$ is called:

- Artin, if $\mathcal{K}^1(V_\bullet) = \mathcal{K}^2(V_\bullet) = 0$ and $\mathcal{K}^3(V_\bullet)$ is an Artin sheaf;
- monadic, if $\mathcal{K}^1(V_\bullet) = \mathcal{K}^3(V_\bullet) = 0$;
- supermonadic, if both $V_\bullet$ and $V_\bullet^\vee$ are monadic.

Lemma 2.6.2. A monadic representation $V_\bullet$ is supermonadic iff $\mathcal{K}^2(V_\bullet)$ is locally free.

Proof. Since $V_\bullet$ is monadic, the complex $\mathcal{C}(V_\bullet)$ is isomorphic to $\mathcal{K}^2(V_\bullet)$ in the derived category $\mathbf{D}(\text{coh}(\mathbb{P}^2))$. Therefore the complex $\mathcal{C}(V_\bullet^\vee) = \mathcal{C}(V_\bullet)^\vee$ is isomorphic to the derived dual of $\mathcal{K}^2(V_\bullet)$. In other words, $\mathcal{K}^1(\mathcal{C}(V_\bullet^\vee)) \cong \text{Ext}^{-2}(\mathcal{K}^2(V_\bullet), \mathcal{O})$. So, $V_\bullet$ is supermonadic iff $\text{Ext}^i(\mathcal{K}^2(V_\bullet), \mathcal{O}) = 0$ for $i \neq 0$, i.e., iff $\mathcal{K}^2(V_\bullet)$ is locally free. \hfill \square
Lemma 2.6.3. Let \( V_\bullet \) be a monadic representation of \( Q_f \) and let \( E = \mathcal{H}^2(V_\bullet) \). If \( V_\bullet \) is \( \theta^0 \)-semistable then \( E \) is Mumford semistable and if \( V_\bullet \) is \( (\theta^0, \theta^1) \)-semistable then \( E \) is Gieseker semistable. In both cases \( V_\bullet = V_\bullet(E) \).

Proof. Assume \( E \) is not Mumford semistable and consider its Harder–Narasimhan filtration. Breaking it up at slope \( \mu_M(E) \) we can represent \( E \) as an extension

\[
0 \to E' \to E \to E'' \to 0,
\]

such that the slopes of all quotients in the Harder–Narasimhan filtration of \( E' \) (resp. \( E'' \)) are greater than (resp. less than or equal to) \( \mu_M(E) \). Let \( (r', d', n') \) be the rank, the degree and the second Chern class of \( E' \). Note that both \( E' \) and \( E'' \) are the cohomology sheaves of the monads \( V_\bullet(E') \) and \( V_\bullet(E'') \) respectively. Indeed, for \( E' \) the argument of Theorem 2.4.1 shows that \( \text{Ext}^2(Q_p(1 - p), E') = 0 \). On the other hand, since \( E' \) is a subsheaf in \( E \) we have \( \text{Hom}(Q_p(1 - p), E') \subset \text{Hom}(Q_p(1 - p), E) = 0 \). Analogously, for \( E'' \) the argument of Theorem 2.4.1 gives the vanishing of \( \text{Hom}'s \), while the surjectivity of the map from \( E \) gives the vanishing of \( \text{Ext}^2 \). It follows that we have an exact sequence of monads

\[
0 \to V_\bullet(E') \to V_\bullet \to V_\bullet(E'') \to 0.
\]

Finally, note that

\[
\langle \theta^0, \alpha(r', d', n') \rangle = r'd - r d' = rr' \left( \frac{d}{r} - \frac{d'}{r'} \right) = rr' (\mu_M(E) - \mu_M(E')) < 0.
\]

Hence the subrepresentation \( V_\bullet(E') \subset V_\bullet \) violates the \( \theta^0 \)-semistability of \( V_\bullet \). This proves the first part.

If \( E \) is Mumford semistable but not Gieseker semistable, we take again \( E' \) to be the part of the Harder–Narasimhan filtration of \( E \) with the slopes greater than \( \mu_G(E) \). Then \( \langle \theta^0, \alpha(r', d', n') \rangle = 0 \) but

\[
\langle \theta^1, \alpha(r', d', n') \rangle = r'(d^2 - 2n) - r(d^2 - 2n') = 2rr' \left( \frac{d^2 - 2n}{2r} - \frac{d^2 - 2n'}{2r'} \right) < 0
\]

Hence the subrepresentation \( V_\bullet(E') \subset V_\bullet \) violates the \( (\theta^0, \theta^1) \)-semistability of \( V_\bullet \). This proves the second part.

Finally, we have \( V_\bullet = V_\bullet(E) \) by Lemma 2.2.4. \( \square \)

Proposition 2.6.4. Let \( V_\bullet \) be a \( \theta^0 \)-semistable representation of \( Q_f \) of dimension \( \alpha(r, d, n) \). Then \( V_\bullet \) is \( S \)-equivalent to a direct sum \( U_\bullet \oplus W_\bullet \), where \( U_\bullet \) is supermonadic of dimension \( \alpha(r, d, n - k) \) and \( W_\bullet \) is Artin of dimension \( (k, 2k, k) \) for some \( 0 \leq k \leq n \).

Proof. Assume that \( \mathcal{H}^3(V_\bullet) \neq 0 \). Then \( i^*(\mathcal{H}^3(V_\bullet)) \neq 0 \) by Proposition 2.2.4(1), hence there is a surjective morphism \( \mathcal{H}^3(V_\bullet) \to i_* \mathcal{O}_P \) for some point \( P \in \mathbb{P}(H) \). Since \( \mathcal{H}^3(V_\bullet) \) is the top cohomology of the complex \( \mathcal{C}(V_\bullet) \), there is a canonical morphism \( \mathcal{C}(V_\bullet) \to \mathcal{H}^3(V_\bullet) \). Composing these morphisms we get a nontrivial morphism \( \mathcal{C}(V_\bullet) \to \mathcal{O}_P \). By Proposition 2.2.4 this corresponds to a nontrivial morphism \( V_\bullet \to W_\bullet(\mathcal{O}_P) \). Since \( V_\bullet \) is \( \theta^0 \)-semistable and \( W_\bullet(\mathcal{O}_P) \) is \( \theta^0 \)-stable, the morphism is surjective. Taking \( V'_\bullet \) to be the kernel of the morphism, we see that \( V'_\bullet \) is \( \theta^0 \)-semistable and \( V_\bullet \) is \( S \)-equivalent to \( V'_\bullet \oplus W_\bullet(\mathcal{O}_P) \). The dimension of \( V'_\bullet \) is strictly less than that of \( V_\bullet \), so iterating the construction we reduce to the case when \( \mathcal{H}^3(V_\bullet) = 0 \).
Assume now that $\mathcal{H}^3(V\bullet) = 0$, but $\mathcal{H}^3(V\bullet') \neq 0$. Then applying the same argument to $V\bullet'$ we obtain an injection $W\bullet(\mathcal{O}_P) \to V\bullet$. Taking $V\bullet'$ to be the cokernel of this morphism, we see that $V\bullet'$ is $\theta^0$-semistable and $V\bullet$ is $S$-equivalent to $V\bullet' \oplus W\bullet(\mathcal{O}_P)$. Iterating the construction we reduce to the case when $\mathcal{H}^3(V\bullet) = \mathcal{H}^3(V\bullet') = 0$.

Finally, assume that $\mathcal{H}^3(V\bullet) = 0$ and $\mathcal{H}^3(V\bullet') \neq 0$. Let us show that $V\bullet$ is supermonadic. Indeed, if $\mathcal{F} := \mathcal{H}^1(V\bullet) \neq 0$ then $\mathcal{F}$ is a locally free sheaf and we have a left exact sequence $0 \to \mathcal{F} \to V_1 \otimes \mathcal{O}(-1) \to V_2 \otimes \mathcal{O}$. After dualization we get a complex

$$V_2' \otimes \mathcal{O} \to V_3' \otimes \mathcal{O}(1) \to \mathcal{F}^*$$

in which the second arrow is nontrivial (since after second dualization it gives back the embedding $\mathcal{F} \to V_1 \otimes \mathcal{O}(-1)$). This means that $\mathcal{H}^1(V\bullet') \neq 0$, thus contradicting the assumption. Therefore $\mathcal{H}^3(V\bullet) = 0$. Analogous argument with $V\bullet$ replaced by $V\bullet'$ shows that $\mathcal{H}^1(V\bullet') = 0$, so $V\bullet$ is indeed supermonadic.

As at each step of the above procedure the dimension of the representation has decreased by $(1, 2, 1)$, the dimension of the supermonadic part we end up with is equal to

$$\alpha(r, d, n) - (k, 2k, k) = \alpha(r, d, n - k).$$

Since all the components of a dimension vector are nonnegative, we have $n - k \geq d(d + 1)/2$, in particular $k \leq n$.

Note that we have

$$(n, 2n, n) = \alpha(0, 0, n).$$

**Corollary 2.6.5.** Let $W\bullet$ be a $\theta^0$-semistable representation of $Q_\tau$ of dimension $(n, 2n, n)$. Then $W\bullet$ is an Artin representation.

**Proof.** Applying Proposition 2.6.4 we see that $W\bullet$ is $S$-equivalent to a sum of an Artin representation and a supermonadic representation $U_\bullet$ of dimension $\alpha(0, 0, n - k)$ for some $k$.

By definition and Lemma 2.6.2 the corresponding complex $C(U\bullet)$ is a monad and its middle cohomology is a locally free sheaf of rank 0. Thus the cohomology is zero and the complex $C(U\bullet)$ is acyclic. By Proposition 2.2.1 this means that $U\bullet = 0$, so we have no supermonadic part. It follows that $W\bullet$ is $S$-equivalent to an Artin representation. It follows immediately that $\mathcal{H}^1(W\bullet) = \mathcal{H}^2(W\bullet) = 0$, and $\mathcal{H}^3(W\bullet)$ is an iterated extension of Artin sheaves. Hence it is an Artin sheaf itself, and so $W\bullet$ is also an Artin representation. \[ \square \]

The first part of the following result can be found in Lemma 6.14 of [18].

**Proposition 2.6.6.** Let $V\bullet$ be a $(\theta^0, \theta^1)$-semistable representation of $Q_\tau$ of dimension $\alpha(r, d, n)$ with $0 \leq d < r$. Then $V\bullet$ fits into a short exact sequence

$$0 \to W\bullet \to V\bullet \to U\bullet \to 0,$$

where $U\bullet$ is supermonadic and $W\bullet$ is Artin. Moreover, $V\bullet = V\bullet(E)$, where $E$ is a Gieseker semistable sheaf of rank $r$, degree $d$ and $c_2 = n$, $U\bullet = V\bullet(E^{**})$, and $W\bullet = W\bullet(E^{**}/E)$.

**Proof.** The argument of Proposition 2.6.4 proves that there is a filtration on $V\bullet$ in which there are several factors which are Artin representations of dimension $(1, 2, 1)$ and one supermonadic factor of dimension $\alpha(r, d, n - k)$ for some $0 \leq k \leq n$. But

$$\langle \theta^1, (1, 2, 1) \rangle = 2r > 0,$$
hence Artin factors can appear only before the supermonadic factor. This proves that the filtration gives the required exact sequence.

Applying the functor $\mathcal{C}$ to the exact sequence and taking into account $\mathcal{H}^2(W\bullet) = \mathcal{H}^3(U\bullet) = 0$, we get the long exact sequence of cohomology sheaves

$$0 \to \mathcal{H}^2(V\bullet) \to \mathcal{H}^2(U\bullet) \to \mathcal{H}^3(W\bullet) \to \mathcal{H}^3(V\bullet) \to 0.$$  

If $\mathcal{H}^3(V\bullet) \neq 0$ then the argument of the proof of Proposition 2.6.4 shows that there is a surjection $V\bullet \to W\bullet(\mathcal{O}_P)$ which, as we observed, contradicts to $(\theta^0, \theta^1)$-semistability of $V\bullet$. Thus $V\bullet$ is monadic. Denote $E = \mathcal{H}^2(V\bullet)$, $\mathcal{E} = \mathcal{H}^2(U\bullet)$ and $F = \mathcal{H}^3(W\bullet)$. Then the above sequence can be rewritten as

$$0 \to E \to \mathcal{E} \to F \to 0.$$  

Note that $\mathcal{E}$ is locally free by Lemma 2.6.2 and $F$ is Artin since $W\bullet$ is. Dualizing the sequence and taking into account that $\operatorname{Hom}(F, \mathcal{O}) = \operatorname{Ext}^1(F, \mathcal{O}) = 0$ since $F$ is Artin, we deduce that $E^* = \mathcal{E}^*$. Therefore $E^{**} = \mathcal{E}^{**} = \mathcal{E}$ since $\mathcal{E}$ is locally free and the map $E \to \mathcal{E} = E^{**}$ is the canonical embedding. Thus $E$ is torsion free and $F \cong E^{**}/E$. Moreover, by Proposition 2.2.1 it follows that $V\bullet = V\bullet(E)$ and $U\bullet = V\bullet(E^{**})$, while $W\bullet = W\bullet(E^{**}/E)$.

We finish by noting that $E$ is Gieseker semistable by Lemma 2.6.3.  

Corollary 2.6.7. If $r$ and $d$ are coprime then a $\theta^0$-semistable representation is $(\theta^0, \theta^1)$-semistable if and only if it has no Artin quotients, i.e., if it is monadic.

Proof. First let us show that if $r$ and $d$ are coprime and $V\bullet$ is a supermonadic $\theta^0$-semistable representation then it is $\theta^0$-stable. Indeed, $V\bullet = V\bullet(E)$ for a Mumford semistable sheaf $E$ by Lemma 2.6.3 and moreover, the sheaf $E$ is locally free by Lemma 2.6.2. So, if $0 \to V\bullet' \to V\bullet \to V\bullet'' \to 0$ is an exact sequence of representations with both $V\bullet'$ and $V\bullet''$ nonzero and $\mu_{\theta^0}(V\bullet') = \mu_{\theta^0}(V\bullet'') = 0$ then by Lemma 2.5.5 we have an exact sequence

$$0 \to \mathcal{H}^2(V\bullet') \to E \to \mathcal{H}^2(V\bullet'') \to \mathcal{H}^3(V\bullet') \to 0,$$

and moreover, $V\bullet''$ is monadic, $\mu_M(\mathcal{H}^2(V\bullet'')) = \mu_M(E) = d/r$, and $\mathcal{H}^3(V\bullet')$ is Artin. But since $r$ and $d$ are coprime either the rank and the degree of $\mathcal{H}^2(V\bullet'')$ are zero, or equal to $r$ and $d$ respectively. The first is impossible since then $F := \mathcal{H}^2(V\bullet'')$ is an Artin sheaf and $V\bullet'' = \operatorname{Ext}^1(Qp(1 - p), F) = 0$, so $V\bullet'' = 0$. The second is impossible since then the rank of $\mathcal{H}^2(V\bullet')$ is zero, and as $\mathcal{H}^2(V\bullet')$ being a subsheaf in $E$ is torsion free, it should be zero. Thus $V\bullet'$ is a nonzero Artin subrepresentation in $V\bullet$ which means that $V\bullet'$ has a nonzero Artin quotient representation and hence cannot be monadic.

Now let $V\bullet$ be an arbitrary $\theta^0$-semistable but not $(\theta^0, \theta^1)$-semistable representation. Consider all composition factors of $V\bullet$ in the category of $\theta^0$-semistable representations. By Proposition 2.6.4 and the above argument these are Artin representations and the supermonadic part of $V\bullet$. Among those Artin representations have positive $\theta^1$-slope, hence the supermonadic part is the only composition factor with negative $\theta^1$-slope. So, the only way how $V\bullet$ can be not $(\theta^0, \theta^1)$-semistable is when $V\bullet$ has an Artin quotient.  

2.7. Moduli spaces. Let $\mathcal{M}^\theta_{r_1, r_2, r_3}$ denote the moduli space of $\theta$-semistable $(r_1, r_2, r_3)$-dimensional representations of the quiver $\mathcal{Q}_r$, as defined by King [13]. It is a coarse moduli space for families of $\theta$-semistable representations of the quiver $\mathcal{Q}_r$ of dimension $(r_1, r_2, r_3)$.
In particular, its closed points are in a bijection with S-equivalence classes of \( \theta \)-semistable representations.

For rational \( \theta \) there is an explicit GIT construction of the moduli space. One starts with the representation space of \( Q_\tau \):

\[
R_\tau(r_1, r_2, r_3) \subset \text{Hom}(C^{r_1} \otimes (A_1^r)^1_1, C^{r_2}) \times \text{Hom}(C^{r_2} \otimes (A_1^r)^1_1, C^{r_3}),
\]

consisting of those pairs of maps \( f : C^{r_1} \otimes (A_1^r)^1_1 \to C^{r_2}, g : C^{r_2} \otimes (A_1^r)^1_1 \to C^{r_3} \) such that the composition \( g \circ (f \otimes \text{id}) : C^{r_1} \otimes (A_1^r)^1_1 \otimes (A_1^r)^1_1 \to C^{r_3} \) factors through the multiplication map \( C^{r_1} \otimes (A_1^r)^1_1 \otimes (A_1^r)^1_1 \to C^{r_1} \otimes (A_1^r)^2_1 \). Clearly, (2.7.1) is a Zariski closed subset in an affine space. The group

\[
\text{GL}(r_1, r_2, r_3) = \text{GL}(r_1) \times \text{GL}(r_2) \times \text{GL}(r_3)
\]

acts naturally on \( R_\tau(r_1, r_2, r_3) \). Given a rational polarization \( \theta \), in the trivial bundle, let \( \mathbb{C} [R_\tau(r_1, r_2, r_3)]^{\text{GL}(r_1, r_2, r_3), p\theta} \) be the vector space of polynomial \( \text{GL}(r_1, r_2, r_3) \)-semiinvariants of weight \( p\theta \) (this space is declared to be zero unless \( p\theta \) is an integral weight). One defines an associated GIT quotient by

\[
R_\tau(r_1, r_2, r_3) // \theta \text{ GL}(r_1, r_2, r_3) := \text{Proj} \left( \bigoplus_{p=0}^{\infty} \mathbb{C} [R_\tau(r_1, r_2, r_3)]^{\text{GL}(r_1, r_2, r_3), p\theta} \right).
\]

Then, according to [13], one has \( M_\tau^\theta(r_1, r_2, r_3) \cong R_\tau(r_1, r_2, r_3) // \theta \text{ GL}(r_1, r_2, r_3) \). Further, it turns out that the space of all polarizations \( \theta \) has a chamber structure and the moduli space \( M_\tau^\theta(r_1, r_2, r_3) \) depends only on the chamber in which \( \theta \) sits. This allows to define \( M_\tau^\theta \) for arbitrary (real) polarization \( \theta \) by taking rational \( \theta' \) in the same chamber as \( \theta \) and setting \( M_\tau^\theta(r_1, r_2, r_3) := M_\tau^{\theta'}(r_1, r_2, r_3) \).

Analogously one constructs a coarse moduli space \( M_\tau^{(\theta, \theta')}(r_1, r_2, r_3) \) for a pair of polarizations \( (\theta, \theta') \) by taking an arbitrary polarization in the chamber containing \( \theta + \varepsilon \theta' \) for all sufficiently small and positive \( \varepsilon \).

It has been shown in [13] that the moduli space of semistable representations of any quiver that has no oriented cycles is a projective variety. It follows, since the quiver \( Q_\tau \) has no oriented cycles, that each of the above moduli spaces \( M_\tau^\theta(r_1, r_2, r_3) \) is a projective variety. This variety comes equipped with a natural \( \text{SL}(H) \)-action. Finally, we remark that if the dimension vector \( (r_1, r_2, r_3) \) is primitive, i.e., indivisible, then \( M_\tau^\theta(r_1, r_2, r_3) \) is a fine moduli space.

Below we discuss moduli spaces of several classes of representations of the quiver \( Q_\tau \). First, recall that by Corollary 2.6.5 any \( \theta^0 \)-semistable representation of dimension \( (n, 2n, n) \) is an Artin representation. So, we refer to the corresponding moduli space as to the moduli space of Artin representations and denote it by \( A M_\tau(n, 2n, n) \). Thus we have

\[
A M_\tau(n, 2n, n) := M_\tau^{\theta^0}(n, 2n, n) = R_\tau(n, 2n, n) // _{\theta^0} \text{ GL}(n, 2n, n).
\]

The moduli space of Artin representations is highly non-reduced. In what follows, however, we will only need a description of the underlying reduced scheme which we denote by \( A M_\tau(n, 2n, n)_{\text{red}} \). The proof of the following Proposition can be found in the Appendix.
Proposition 2.7.2. The map \( W_\bullet \mapsto \text{supp}(\mathcal{H}^3(W_\bullet))) \) gives an \( \text{SL}(H) \)-equivariant isomorphism
\[
A^\text{M}_r(n,2n,n)_{\text{red}} \cong S^n(\mathbb{P}(H)).
\]
Dimension vector \((n,2n,n)\) considered here is a special case of the vector \(\alpha(r,d,n)\) for \(r = d = 0\). We also consider a general moduli space of \(\theta^0\)-semistable \(\alpha(r,d,n)\)-dimensional representations of \(Q_\tau\) which we call the Uhlenbeck moduli space (the reasons behind this choice of a name will become clear later) of sheaves on \(\mathbb{P}^2_{\tau}\). We denote it
\[
^U\text{M}_r(r,d,n) := \text{M}^{(\theta^0)}_r(\alpha(r,d,n)) \cong R_r(\alpha(r,d,n))//\theta^0 \text{GL}(\alpha(r,d,n)).
\]
The last moduli space we consider is the moduli space of \((\theta^0,\theta^1)\)-semistable \(\alpha(r,d,n)\)-dimensional representations
\[
^G\text{M}_r(r,d,n) := \text{M}^{(\theta^0,\theta^1)}_r(\alpha(r,d,n)) \cong R_r(\alpha(r,d,n))//\theta^0,\theta^1 \text{GL}(\alpha(r,d,n)).
\]
We call this moduli space the Gieseker moduli space of sheaves on \(\mathbb{P}^2_{\tau}\). The reason for this is the following

Proposition 2.7.3. The Gieseker moduli space \(^G\text{M}_r(r,d,n)\) is isomorphic to the moduli space of Gieseker semistable sheaves on \(\mathbb{P}^2_{\tau}\) constructed in [18]. Moreover, the open subset of \(^G\text{M}_r(r,d,n)\) of \((\theta^0,\theta^1)\)-stable representations corresponds, via the isomorphism, to the open set of Gieseker stable sheaves on \(\mathbb{P}^2_{\tau}\).

Proof. This follows immediately from Lemma 2.5.4, as the functor of \((\theta^0,\theta^1)\)-(semi)stable representations of the quiver \(Q_\tau\) is isomorphic to the functor of (semi)stable Kronecker complexes considered in [18].

Corollary 2.7.4. If \(\text{gcd}(r,d,n) = 1\), then \(^G\text{M}_r(r,d,n)\) is a fine moduli space, moreover, this moduli space is smooth.

Proof. By [18] Prop. 7.15] the moduli space of semistable Kronecker complexes is fine. As the functor of semistable Kronecker complexes is isomorphic to the functor of \((\theta^0,\theta^1)\)-semistable representations of \(Q_\tau\), we conclude that \(^G\text{M}_r(r,d,n)\) is also a fine moduli space. Moreover, from \(\text{gcd}(r,d,n) = 1\) it follows that all \((\theta^0,\theta^1)\)-semistable representations of the quiver are \((\theta^0,\theta^1)\)-stable, hence all Gieseker semistable sheaves are stable, and so the smoothness of the moduli space is proved in [18, Thm. 8.1].

2.8. Stratifications. Recall that by Proposition 2.6.6 any \((\theta^0,\theta^1)\)-semistable representation \(V_\bullet\) can be written as \(V_\bullet(E)\) for a Gieseker semistable sheaf \(E\). This gives a decomposition of the moduli space \(^G\text{M}_r(r,d,n)\) into pieces by the length of \(E^{**}/E\). It will be shown in the Appendix at the end of the paper that this decomposition is, in fact, an algebraic stratification.

The proofs of other results of this subsection stated below are also deferred to section 4.

Lemma 2.8.1. The Gieseker moduli space \(^G\text{M}_r(r,d,n)\) is naturally stratified by locally closed \(\text{SL}(H)\)-invariant subsets
\[
^G\text{M}_r(r,d,n) = \bigsqcup_{0 \leq k \leq n} ^G\text{M}_r^k(r,d,n),
\]
where the stratum $\mathcal{G}M_k^\tau(r,d,n)$ corresponds to the locus of Gieseker semistable sheaves $E$ on $\mathbb{P}^2$ with $c_2(E^{**}/E) = k$.

In particular, the open stratum $\mathcal{G}M^0_\tau(r,d,n) \subset \mathcal{G}M_\tau(r,d,n)$ parameterizes locally free Gieseker semistable sheaves.

There is also an analogous stratification of the Uhlenbeck moduli space.

**Lemma 2.8.2.** The Uhlenbeck moduli space $\mathcal{U}M_\tau(r,d,n)$ is naturally stratified by locally closed $\text{SL}(H)$-invariant subsets

$$\mathcal{U}M_\tau(r,d,n) = \bigsqcup_{0 \leq k \leq n} \mathcal{U}M^k_\tau(r,d,n),$$

where the stratum $\mathcal{U}M^k_\tau(r,d,n)$ corresponds to the locus of Mumford semistable sheaves $E$ on $\mathbb{P}^2$ with $c_2(E^{**}/E) = k$.

The natural stratifications of the Gieseker and the Uhlenbeck moduli spaces have highly nonreduced strata. The reason for that is the nonreducedness of the moduli space of Artin sheaves, or going one step deeper, the nonreducedness of the scheme of “commutative points” of $\mathbb{P}^2$. However, as we have only topological applications in mind, the nilpotents in the structure sheaves of the strata are irrelevant for our purposes, so by this reason we replace each stratum of both moduli spaces by the reduced scheme underlying the corresponding natural stratum. Thus, from now on we will abuse the notation and write $\mathcal{G}M^k_\tau(r,d,n)$, resp. $\mathcal{U}M^k_\tau(r,d,n)$, for the stratum of the corresponding stratification equipped with reduced scheme structure.

It follows from standard results of geometric invariant theory (see [8]) that there is a canonical $\text{SL}(H)$-equivariant projective morphism

$$\gamma_\tau : \mathcal{G}M_\tau(r,d,n) \to \mathcal{U}M_\tau(r,d,n),$$

resulting from a specialization of $(\theta^0, \theta^1)$-semistability to $\theta^0$-semistability.

**Remark 2.8.3.** We do not know how to define the morphism $\gamma_\tau$ in terms of coherent sheaves on $\mathbb{P}^2$, without using identifications of moduli spaces of coherent sheaves with the corresponding moduli spaces of quiver representations.

The main result of this section establishes a compatibility between the constructed stratifications of the Gieseker and Uhlenbeck moduli spaces and describes the relation between the strata.

**Theorem 2.8.4.** (1) The map $\gamma_\tau : \mathcal{G}M_\tau(r,d,n) \to \mathcal{U}M_\tau(r,d,n)$ is compatible with the stratifications, i.e., $\gamma_\tau(\mathcal{G}M^k_\tau(r,d,n)) \subset \mathcal{U}M^k_\tau(r,d,n)$.

In the case where $\tau \neq 0$ and the integers $r$ and $d$ are coprime, the Gieseker compactification is smooth and the following holds:

(2) The open set $\mathcal{G}M^0_\tau(r,d,n)$ is the locus of Gieseker stable supermonadic representations; furthermore, this open set corresponds, via the isomorphism of Proposition 2.7.3, to the locus of locally free Gieseker stable sheaves on $\mathbb{P}^2$. Moreover, the map $\gamma_\tau$ yields an isomorphism $\mathcal{G}M^0_\tau(r,d,n) \isom \mathcal{U}M^0_\tau(r,d,n)$.
(3) For any $k > 0$ one has an $\text{SL}(H)$-equivariant isomorphism
\[ \overline{U}_\tau^k(r, d, n) \cong \overline{G}_\tau^0(r, d, n - k) \times A\tau_\tau(k, 2k, k)_{\text{red}} \cong \overline{G}_\tau^0(r, d, n - k) \times S^k \mathbb{P}(H). \]
Using this isomorphism, for $E \in \overline{G}_\tau^k(r, d, n)$ we have
\[ \gamma_\tau(E) = (E^{**}, \text{supp}(E^{**}/E)) \]
In particular, the fiber of $\gamma_\tau$ over a point $(E, D) \in \overline{G}_\tau^0(r, d, n - k) \times S^k \mathbb{P}^1 \subset \overline{U}_\tau^r(r, d, n)$ is the underlying reduced scheme for the moduli space of subsheaves $E \subset E$ with $\text{supp}(E/E) = D$.

Remark 2.8.5. The relation between the moduli spaces $\overline{G}_\tau(r, d, n)$ and $\overline{U}_\tau(r, d, n)$ is completely analogous to that of the Gieseker and Uhlenbeck compactifications of the moduli spaces of vector bundles on commutative algebraic surfaces (this justifies the use of these names in our situation). An important difference between commutative and noncommutative cases is that in the dimensions of the strata: in the commutative case the second factors in the product expression for Uhlenbeck strata are symmetric powers of the surface, while in the noncommutative case we only have a symmetric power of a curve. This fact will play a crucial role in subsequent sections.

3. Rank 1 sheaves and the Calogero–Moser space

In this section we study the Gieseker and the Uhlenbeck moduli spaces of rank 1 and degree 0 torsion free sheaves on $\mathbb{P}^2$.

3.1. The compactifications. To unburden the notation we write
\[ \overline{G}_\tau^n = \overline{G}_\tau(1, 0, n), \quad \overline{U}_\tau^n = \overline{U}_\tau(1, 0, n), \quad \overline{G}_\tau^{n,k} = \overline{G}_\tau(1, 0, n), \quad \overline{U}_\tau^{n,k} = \overline{U}_\tau^{k}(1, 0, n). \]
It is well known, cf. [18, Prop. 8.13], that the open strata $\overline{G}_\tau^{n,0} \subset \overline{G}_\tau^n$ and $\overline{U}_\tau^{n,0} \subset \overline{U}_\tau^n$ can be identified with the Calogero–Moser space. Thus, the varieties $\overline{G}_\tau^n$ and $\overline{U}_\tau^n$ provide two different compactifications of the Calogero–Moser space, to be called the Gieseker and the Uhlenbeck compactifications, respectively. Furthermore, the variety $\overline{G}_\tau^n$ being smooth, the morphism $\gamma_\tau : \overline{G}_\tau^n \to \overline{U}_\tau^n$ is a resolution of singularities. Later on, we will use this resolution to compute the stalks of the IC sheaf of the Uhlenbeck compactification.

Theorem 3.1.1. For $\tau \neq 0$ we have $\text{SL}(H)$-equivariant isomorphisms
\[ \overline{G}_\tau^{n,0} = \overline{U}_\tau^{n,0} \cong \overline{M}_\tau^n. \]
Proof. The first isomorphism follows from Proposition 2.7.3 and [18, Prop. 8.13]. The second is a consequence of the first and Theorem 2.8.4(2).

In Section 2.7 we have introduced a contraction map $\gamma_\tau : \overline{G}_\tau^n \to \overline{U}_\tau^n$. By Theorem 2.8.4, it sends a torsion free sheaf $E$ to $(E^{**}, \text{supp}(E^{**}/E))$, and for any $0 \leq k \leq n$, we have
\[ \gamma_\tau(\overline{G}_\tau^{n,k}) = \overline{U}_\tau^{n,k} = \overline{M}_\tau^{n-k} \times S^k \mathbb{P}(H). \]
Below, we are going to describe the fibers of the map $\gamma_\tau$. Choose $0 \neq h \in H$ and let
\[ \mathbb{A}_h^1 := \mathbb{P}(H) \setminus \{h\}. \]
Thus, $A^1_\mathbb{A}$ is an affine line and, for any $n \geq 0$, the set $S^n A^1_\mathbb{A}$ is Zariski open and dense in $S^n \mathbb{P}(H)$. It is clear that these sets for all $h \in H$ form an open covering of $S^n \mathbb{P}(H)$.

Consider the zeroth Calogero–Moser space $M^0_t$. Clearly, this is just a point, and under the isomorphism of Theorem 2.8.4(3) it corresponds to the trivial line bundle $O_{\mathbb{P}^2}$. For each nonzero vector $h \in H$ consider the open subset $\{0\} \times S^n A^1_{\mathbb{A}} = M^0_t \times S^n A^1_{\mathbb{A}} \subset M^0_t \times S^n \mathbb{P}(H) = U M^m_{\mathbb{A}}$ and its preimage under the map $\gamma_t : GM^m_{\mathbb{A}} \rightarrow U M^m_{\mathbb{A}}$:

$$B^k_h = \gamma_t^{-1}(\{0\} \times S^n A^1_h). \quad (3.1.2)$$

Analogously, we can take arbitrary locally free sheaf $E$ of rank 1 and degree 0, consider the locally closed subset $\{E\} \times S^k A^1_h \subset M^m_{\mathbb{A}} \times S^k A^1_h \subset M^m_{\mathbb{A}} \times S^k \mathbb{P}(H) = U M^m_{\mathbb{A}}$, and its preimage under the map $\gamma_t : GM^m_{\mathbb{A}} \rightarrow U M^m_{\mathbb{A}}$:

**Proposition 3.1.3.** For any locally free sheaf $E$ of rank 1, $E \in M^m_{\mathbb{A}}$, there is an isomorphism

$$\gamma_t^{-1}(\{E\} \times S^k A^1_h) \cong B^k_h.$$  

**Proof.** There is an integer $p \in \mathbb{Z}$ and two maps

$$\phi : O(-p) \rightarrow E \quad \text{and} \quad \phi' : E \rightarrow O(p)$$

such that $i^*(\phi) = h^p$ and $i^*(\phi') = h^{-p}$. Indeed, take $p$ sufficiently large to have

$$\text{Ext}^1(E, O(p-1)) = \text{Ext}^1(O(-p), E(-1)) = 0$$

and define $\phi$ and $\phi'$ as lifts of the compositions in the next two diagrams

$$\begin{array}{ccc}
\mathbb{O}(-p) & \xrightarrow{i_* i^*} & \mathbb{O}(-p) \\
\phi \downarrow & \downarrow & \downarrow \phi' \\
0 \rightarrow E(-1) & \xrightarrow{\mathbb{I}} & E & \xrightarrow{i_* i^*} & 0 \\
\end{array}$$

$$\begin{array}{ccc}
\mathbb{E}' & \xrightarrow{i_* i^*} & \mathbb{E}' \\
\phi \downarrow & \downarrow & \downarrow \phi' \\
0 \rightarrow O(p-1) & \xrightarrow{\mathbb{I}} & O(p) & \xrightarrow{i_* i^*} & 0 \\
\end{array}$$

The maps $\phi$ and $\phi'$ give morphisms between the moduli spaces of surjections $E \rightarrow F$ and $\mathbb{O} \rightarrow F$ of length $k$ with $\text{supp}(F) \subset A^1_h$:

$$(f : E \rightarrow F) \mapsto (f \circ \phi(p) : O \rightarrow F) \quad \text{and} \quad (g : \mathbb{O} \rightarrow F) \mapsto (g(p) \circ \phi' : E \rightarrow F).$$

where in both cases we identify $F$ with $F(p)$ via $h^p$. It is straightforward to check that these maps are mutually inverse. As the preimages $\gamma_t^{-1}(E \times S^k A^1_h)$ and $B^k_h = \gamma_t^{-1}(E \times S^k A^1_h)$ are identified by Theorem 2.8.4(3) with the reduced schemes underlying these moduli spaces, the constructed maps provide an isomorphism between them as well. \hfill \Box

The spaces $B^k_h$ come with a natural map $\gamma : B^k_h \rightarrow S^n A^1_h$. In fact they enjoy the following factorization property. Define the open subset $(S^{k_1} A^1 \times S^{k_2} A^1)_{\text{disj}} \subset S^{k_1} A^1 \times S^{k_2} A^1$ as

$$(S^{k_1} A^1 \times S^{k_2} A^1)_{\text{disj}} = \{(D_1, D_2) \in S^{k_1} A^1 \times S^{k_2} A^1 \mid \supp(D_1) \cap \supp(D_2) = \emptyset\} \quad (3.1.4)$$

and

$$(B^k_{h_1} \times B^k_{h_2})_{\text{disj}} := (\gamma_{h_1} \times \gamma_{h_2})^{-1}((S^{k_1} A^1 \times S^{k_2} A^1)_{\text{disj}}) \subset B^k_{h_1} \times B^k_{h_2}.$$  

**Proposition 3.1.5.** The collection of spaces $B^k_h$ has a factorization property, i.e. there is a collection of maps

$$\psi_{k_1, k_2} : (B^k_{h_1} \times B^k_{h_2})_{\text{disj}} \rightarrow B^{k_1+k_2}_h$$

for all positive integers $k_1, k_2$ which has the following properties:
• (associativity) \( \psi_{k_1+k_2,k_3} \circ (\psi_{k_1,k_2} \times \text{id}) = \psi_{k_1,k_2+k_3} \circ (\text{id} \times \psi_{k_1,k_2}) \) for all \( k_1, k_2, k_3 \);

• (commutativity) the maps \( \psi_{k,k} : (B^k_h \times B^k_h)_{\text{disj}} \to B^k_h \) commute with the transposition of the factors on the source for all \( k \);

• (compatibility with the addition) the following diagram is Cartesian

\[
\begin{array}{ccc}
(B^k_h \times B^k_h)_{\text{disj}} & \xrightarrow{\psi_{k_1,k_2}} & B^{k_1+k_2}_h \\
\gamma \times \gamma & & \gamma \\
(S^k \mathbb{A}^1 \times S^k \mathbb{A}^1)_{\text{disj}} & \xrightarrow{a_{k_1,k_2}} & S^{k_1+k_2} \mathbb{A}^1
\end{array}
\]

where the bottom arrow is the addition morphism: \((D_1, D_2) \mapsto D_1 + D_2\).

**Proof.** A point of \((B^k_h \times B^k_h)_{\text{disj}}\) can be represented by a pair of Artin sheaves \( F_1, F_2 \) of length \( k_1 \) and \( k_2 \) respectively with epimorphisms \( \mathcal{O} \to F_1 \) and \( \mathcal{O} \to F_2 \). Consider the sum \( F = F_1 \oplus F_2 \) and the map \( \mathcal{O} \to F \) given by the sum of the two above maps. Let us show it is surjective. By Proposition 2.1.6(2) it is enough to check that the map \( \mathcal{O}_{P(H)} \to i^*F = i^*F_1 \oplus i^*F_2 \) is surjective. But as the supports of the sheaves \( i^*F_1 \) and \( i^*F_2 \) are disjoint, this is equivalent to the surjectivity of each of the maps \( \mathcal{O}_{P(H)} \to i^*F_1 \) and \( \mathcal{O}_{P(H)} \to i^*F_2 \) which we have again by Proposition 2.1.6(2). This means that the sheaf \( F \) with the constructed epimorphism \( \mathcal{O} \to F \) give a point of \( B^k_{h+1} \), and thus a morphism

\[ \psi^R_{k_1,k_2} : (B^k_h \times B^k_h)_{\text{disj}} \to B^{k_1+k_2}_h \]

is defined. Let us show it is a factorization. Indeed, the associativity and the commutativity properties are evident, so it remains to check the compatibility with the addition, i.e. that the corresponding diagram is Cartesian. The commutativity of the diagram follows from Lemma 2.1.9 so it remains to note that if \( F \) is an Artin sheaf of length \( k_1 + k_2 \) such that \( \text{supp}(F) = D_1 + D_2 \) with disjoint divisors \( D_1 \in S^k_1 \mathbb{A}^1 \) and \( D_2 \in S^k_2 \mathbb{A}^1 \), then \( F \) has a unique representation as a direct sum \( F = F_1 \oplus F_2 \) with \( \text{supp}(F_1) = D_1 \) and \( \text{supp}(F_2) = D_2 \) (this follows easily from Proposition 2.1.7).

The variety \( B^k_h \) has a nice linear algebra description. Fix a vector space \( V \) of dimension \( k \).

Let

\[ \tilde{B}^k_h = \{ (Y, Z, v) \in \text{End} (V) \times \text{End} (V) \times V \mid [Y, Z] = \tau Z \} \text{ and } v \text{ is cyclic} \}. \]

(3.1.6)

Here we say that a vector \( v \) is cyclic for a pair of matrices \((Y, Z)\) if there is no proper vector subspace \( V' \subset V \) that contains \( v \) and is both \( Y \)-stable and \( Z \)-stable. One has a natural \( \text{GL}(V) \)-action on \( \tilde{B}^k_h \) given by \( g : (Y, Z, v) \mapsto (gYg^{-1}, gZg^{-1}, gv) \).

**Theorem 3.1.7.** The action of \( \text{GL}(V) \) on \( \tilde{B}^k_h \) is free and

\[ B^k_h \cong (\tilde{B}^k_h / \text{GL}(V))_{\text{red}}. \]

Under this isomorphism the map \( \gamma : B^k_h \to S^k \mathbb{A}^1_h \) is induced by the map \( \tilde{B}^k_h \to S^k \mathbb{A}^1_h \) which takes \((Y, Z, v)\) to \( \text{Spec}(Y) \).

**Proof.** Assume that \( g \in \text{GL}(V) \) acts trivially on a triple \((Y, Z, v)\). Let \( V^g \subset V \) be the space of invariants of \( g \). Then \( v \in V^g \) and \( Y(V^g) \subset V^g \), \( Z(V^g) \subset V^g \), hence \( V^g = V \) as \( v \) is cyclic, and so \( g = 1 \).
Now consider the moduli space of surjections $\mathcal{O} \to F$ with $F$ an Artin sheaf of length $k$ with $\text{supp}(F) \subset \mathbb{A}_k^1$. Let us show it is isomorphic to the quotient $\tilde{B}_h^k/\text{GL}(V)$. Then passing to the underlying reduced schemes will prove the Theorem.

Choose symplectic coordinates $x,y$ in $H$ such that the point $h \in \mathbb{P}(H)$ is given by the equation $x = 0$. Let $(Y,Z,v)$ be a point of $\tilde{B}_h^k$. Consider a graded vector space $V[x] := V \otimes \mathbb{C}[x]$ with deg $x = 1$, with $x$ acting by multiplications, and with the action of $y$ and $z$ defined by

$$y = xY - \tau x^2 Z^2 \partial_x, \quad z = xZ.$$  

The commutation $[x,z] = 0$ is clear. Moreover, we have

$$[y,z] = [xY - \tau x^2 Z^2 \partial_x, xZ] = x^2 [Y,Z] - \tau x^2 Z^2 [\partial_x, x]Z = \tau x^2 Z^3 - \tau x^2 Z^3 = 0$$

and

$$[x,y] = [x, xY - \tau x^2 Z^2 \partial_x] = -\tau x^2 Z^2 [x, \partial_x] = \tau x^2 Z^2 = \tau z^2.$$

This shows that $V[x]$ is a graded $A^*$-module. Let $F$ be the corresponding coherent sheaf on $\mathbb{P}_\tau^2$. By definition the map $x : V[x] \to V[x]$ is injective with finite dimensional cokernel, hence the map $x : F \to F(1)$ is an isomorphism. In particular, the Hilbert polynomial $h_F(t)$ is constant, hence $F$ is an Artin sheaf with supp($F$) $\subset \mathbb{A}_h^1$ by Lemma 2.1.9. Moreover, the length of $F$ is equal to dim $V = k$ and the vector $v \in V \subset V[x]$ gives a morphism of $A^*$-modules $A^* \to V[x]$. By cyclicity assumption the map is surjective in all components of sufficiently large degree, hence the corresponding morphism of sheaves $\mathcal{O} \to F$ is surjective. Note that the construction is $\text{GL}(V)$-invariant.

Vice versa, let $\mathcal{O} \to F$ be a surjection with $F$ an Artin sheaf of length $k$ and supp($F$) $\subset \mathbb{A}_h^1$. Choose an isomorphism $V \cong H^0(\mathbb{P}_\tau^2, F)$. Note that by Lemma 2.1.9 the map $x : F \to F(1)$ is an isomorphism, hence it also induces an isomorphism on the spaces of global sections $x : H^0(\mathbb{P}_\tau^2, F) \xrightarrow{\sim} H^0(\mathbb{P}_\tau^2, F(1))$. We denote by $x^{-1}$ the inverse isomorphism. We put $Y = x^{-1}y$, $Z = x^{-1}z$ considered as endomorphisms of $V = H^0(\mathbb{P}_\tau^2, F)$. Finally, we take $v$ to be the image of $1 \in H^0(\mathbb{P}_\tau^2, \mathcal{O})$ in $V$ under the map $\mathcal{O} \to F$.

Let us show that (3.1.3) holds. First, we have $y = xY$, $z = xZ$ which gives relations

$$x^2 Z = xZx, \quad xyxz = xZxy, \quad \text{and} \quad x^2 Y - xyx = \tau xz xz.$$

It follows that

$$x^2(YZ - ZY) = x^2 YZ - xZ^2 Y = xyxz + \tau xz xz^2 - xZxy = \tau x^2 Z^3.$$

In the second equality we used the third relation, and in the third equality we used the first two relations. As $x$ is an isomorphism, we deduce $[Y,Z] = \tau Z^3$. So, it remains to show that $v$ is cyclic. For this take an arbitrary subspace $V' \subset V$ containing $v$ and closed under the action of $Y$ and $Z$. The triple $(Y,Z,v)$ in the vector space $V'$ then gives an Artin sheaf $F'$ and a surjection $\mathcal{O} \to F'$. The embedding $V' \subset V$ gives an embedding of sheaves $F' \hookrightarrow F$ and it is clear that the original map $\mathcal{O} \to F$ factors as $\mathcal{O} \to F' \hookrightarrow F$. It follows that $F' = F$ and hence $V' = H^0(\mathbb{P}_\tau^2, F') = H^0(\mathbb{P}_\tau^2, F) = V$.

The two constructions are clearly mutually inverse and thus prove an isomorphism of moduli spaces and hence the first part of the Theorem. For the second part it remains to show that supp($F$) = Spec($Y$). But this is clear since the action of the coordinate $y$ from $H^0(\mathbb{P}_\tau^2, F) = V$ to $H^0(\mathbb{P}_\tau^2, F(1)) = V$ is given by the operator $Y$. $\square$
We use the identification \( B^k_h \cong (\tilde{B}^k_h/\text{GL}(V))_{\text{red}} \) to investigate the properties of \( B^k_h \).

**Lemma 3.1.8.** If a pair \((Y, Z)\) satisfies (3.1.6) then \( Z \) is nilpotent.

*Proof.* Note that \([Y, Z] = p \tau Z^{p+2}\) for \( p \geq 3 \). Since \( \tau \neq 0 \), it follows that \( \text{Tr} Z^{p+2} = 0 \) for any \( p \geq 3 \). Hence \( Z \) is nilpotent. \( \square \)

**Lemma 3.1.9.** For any nilpotent \( Z \) there exist \( Y \) and \( v \) such that (3.1.6) holds.

*Proof.* First, for any \( u \in \mathbb{C} \) take \( V = \mathbb{C}[t]/t^k \), \( Y = u + \tau t^3 \partial_t \), \( Z = t, \ v = 1 \).

Clearly, (3.1.6) holds, so we have an example in case when \( Z \) is just one Jordan block. Note that \( \text{Spec}(Y) = ku \in S^k \mathbb{A}^1_h \).

For arbitrary nilpotent \( Z \) the Jordan decomposition of \( Z \) is a direct sum decomposition \( Z = Z_1 \oplus \cdots \oplus Z_m \) with blocks of size \( k_1, \ldots, k_m \). Choosing \( m \) distinct complex numbers \( u_1, \ldots, u_m \) we construct triples \((Y_i, Z_i, v_i)\) on \( V_i = \mathbb{C}[t]/t^{k_i} \) such that \( \text{supp}(Y_i, Z_i, v_i) = k_i u_i \).

Factorization property of Proposition 3.1.5 then shows that the direct sum \( \oplus Y_i, \oplus Z_i, \oplus v_i)\) is a point of \( \tilde{B}^{k_1+\cdots+k_m}_h \). \( \square \)

We will use a natural one-to-one correspondence \( \lambda \mapsto O_\lambda \), between partitions of \( k \) and the nilpotent conjugacy classes in \( \text{End}(V) \), provided by Jordan normal form. Let \( B^\lambda_h \subset B^k_h \) denote the set of all triples \((Y, Z, v)\) satisfying (3.1.6) with \( Z \in O_\lambda \).

**Theorem 3.1.10.** We have a decomposition into a union of connected components

\[
B^k_h = \bigsqcup_{\lambda \in \mathfrak{P}(k)} B^\lambda_h.
\]

The component \( B^\lambda_h \) is smooth, connected and \( k \)-dimensional.

*Proof.* Let \( O = O_\lambda \) be a nilpotent orbit of the group \( \text{GL}(V) \). Let

\[
N_O^* \text{End}(V) = \{ (Y, Z) \mid [Y, Z] = 0 \} \subset \text{End}(V) \times \text{End}(V)
\]

be the conormal bundle of \( O \). Let

\[
\tau N_O^* \text{End}(V) = \{ (Y, Z) \mid Z \in O, \ [Y, Z] = \tau Z^3 \}.
\]

Then according to Lemma 3.1.9 the space \((\tau N_O^* \text{End}(V))_{\text{red}}\) is a \( \text{GL}(V) \)-equivariant \( N_O^* \text{End}(V) \)-torsor over \( O \). In particular \((\tau N_O^* \text{End}(V))_{\text{red}}\) is smooth and \( k^2 \)-dimensional.

It is clear that for any pair \((Y, Z)\) the set of cyclic \( v \in V \) is open. Therefore,

\[
\tilde{B}^k_h \subset \bigsqcup_{\lambda \in \mathfrak{P}(k)} (\tau N_O^* \text{End}(V) \times V)
\]

is an open subset. Moreover, by Lemma 3.1.9 it has a nonempty intersection with every component above. The theorem now follows from Theorem 3.1.7. \( \square \)

**Corollary 3.1.11.** Consider the map \( \gamma_{\tau} : B^k_h \to S^k \mathbb{A}^1_h \). The set of points \( D \in S^k \mathbb{A}^1_h \) such that \( \dim \gamma_{\tau}^{-1}(D) \geq m \) has codimension at least \( m \) in \( S^k \mathbb{A}^1_h \), and is empty for \( m \geq k \).
Proof. As $B_h^k$ is equidimensional of dimension $k$ by Theorem 3.1.10 it is enough to show that no component of $B_h^k$ is contained in the fiber of $\gamma : B_h^k \to S^kA^1_h$. For this note that the map $\gamma$ is equivariant with respect to the action of the group $G_a \subset SL(H)$, the unipotent radical of the parabolic which fixes $h \in H$, and that its action on $A^1_h$ is free. □

Theorem 3.1.12. The map $\gamma : qM^n_r \to UM^n_r$ is small.

Proof. Let $(UM^n_r)_m \subset UM^n_r$ be the set of points over which the fiber of $\gamma$ has dimension $m$. Take any $0 \neq h \in H$. By Proposition 3.1.3 for any $(E_D) \in M^{n-k}_r \times S^kA^1_h$ the fiber $\gamma(E_D)$ is isomorphic to the fiber of the map $\gamma : B^k_h \to S^kA^1_h$ over $D$. In particular, by Corollary 3.1.11 the codimension of the set $(UM^n_r)_m \cap (M^{n-k}_r \times S^kA^1_h)$ in $M^{n-k}_r \times S^kA^1_h$ is at least $m$, and moreover $k > m$. Therefore
\[
\dim((UM^n_r)_m \cap (M^{n-k}_r \times S^kA^1_h)) \leq \dim(M^{n-k}_r \times S^kA^1_h) - m = 2(n-k) + k - m = 2n - k - m < 2n - 2m.
\]
Since the sets $M^{n-k}_r \times S^kA^1_h$ form an open covering of the stratum $M^{n-k}_r \times S^kA^1 = UM^{n-k,k}_r$ of a stratification of $UM^n_r$, the result follows. □

3.2. Deformation of $qM^n_r$ and $UM^n_r$. The goal of this section is to show that the Gieseker and the Uhlenbeck compactifications form a family over $A^1$ (with coordinate $\tau$) and check that the former is smooth. To be more precise, consider the following graded algebra:
\[
A = \langle x, y, z, t \rangle / \langle [x, z] = [y, z] = [t, x] = [t, y] = [t, z] = 0, [x, y] = t z^2 \rangle,
\]
\[
\deg x = \deg y = \deg z = 1, \quad \deg t = 0.
\]

As $t$ is central of degree 0, this is an algebra over $C[t]$. In particular, we can specialize $t$ to any complex number $\tau$, which gives back the algebra $A^\tau$ we considered before.

Analogously, we consider the Koszul dual of $A$ over $C[t]$:
\[
A^! = C[\xi, \eta, \zeta, t] / (t \eta^2 = t\eta \xi + \xi \eta = \zeta \xi + \xi \zeta = \eta \zeta + \zeta \eta = \zeta^2 + t(\xi \eta - \eta \xi) = 0),
\]
\[
\deg \xi = \deg \eta = \deg \zeta = 1, \quad \deg t = 0.
\]

This is a graded $C[t]$-algebra. Note that each of its graded components $A^!_0$, $A^!_1$, $A^!_2$, $A^!_3$ is a free $C[t]$-module of finite rank (equal to 1, 3, 3, and 1 respectively).

Further, we consider the quiver $Q$ over $C[t]$ defined as
\[
\begin{array}{c}
\begin{array}{ccc}
A^!_2 & \longrightarrow & A^!_3 \\
\downarrow & & \\
A^!_1 & \longrightarrow & A^!_2 \\
\end{array}
\end{array}
\]
(analogously to the quiver $Q_\tau$), and its representations in the category of $C[t]$-modules. By definition such a representation is the data of three $C[t]$-modules $(V_1, V_2, V_3)$ and two morphisms of $C[t]$-modules $V_1 \otimes C[t] A^!_1 \to V_2$ and $V_2 \otimes C[t] A^!_2 \to V_3$ such that the composition $V_1 \otimes C[t] A^!_1 \otimes C[t] A^!_1 \to V_2 \otimes C[t] A^!_1 \to V_3$ factors through $V_1 \otimes C[t] A^!_2 \to V_3$.

Assuming each of $V_i$ is a free $C[t]$-module of finite rank, the space
\[
\text{Hom}_{C[t]}(V_1 \otimes C[t] A^!_1, V_2) \oplus \text{Hom}_{C[t]}(V_2 \otimes C[t] A^!_1, V_3)
\]
is also a free \( \mathbb{C}[t] \)-module of finite rank. We consider the associated vector bundle over \( \text{Spec}(\mathbb{C}[t]) = \mathbb{A}^1 \) and its total space \( \text{Tot}(\text{Hom}_{\mathbb{C}[t]}(\mathbf{V}_1 \otimes_{\mathbb{C}[t]} \mathbf{A}_1^1, \mathbf{V}_2) \oplus \text{Hom}_{\mathbb{C}[t]}(\mathbf{V}_2 \otimes_{\mathbb{C}[t]} \mathbf{A}_1^1, \mathbf{V}_3)) \) which is fibered over \( \mathbb{A}^1 \) with fiber an affine space. The above factorization condition defines a Zarisky closed subspace

\[
\text{Rep}_Q(\mathbf{V}_\bullet) \subset \text{Tot}(\text{Hom}_{\mathbb{C}[t]}(\mathbf{V}_1 \otimes_{\mathbb{C}[t]} \mathbf{A}_1^1, \mathbf{V}_2) \oplus \text{Hom}_{\mathbb{C}[t]}(\mathbf{V}_2 \otimes_{\mathbb{C}[t]} \mathbf{A}_1^1, \mathbf{V}_3))
\]

parameterizing all representations of the quiver \( Q \) in \( \mathbf{V}_\bullet \). By definition this is an affine variety over \( \mathbb{A}^1 \).

Now we take a relatively prime triple \( (r, d, n) \) such that \( (2.5.1) \) hold, consider a triple of free \( \mathbb{C}[t] \)-modules \( (\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \) of ranks given by the dimension vector \( \alpha(r, d, n) \) of \( (2.5.2) \), and put

\[
\text{Rep}_Q(\alpha(r, d, n)) = \text{Rep}_Q(\mathbf{V}_\bullet).
\]

The group \( \text{GL}(\alpha(r, d, n)) \) acts naturally on the space \( \text{Rep}_Q(\alpha(r, d, n)) \) along the fibers of the projection \( \text{Rep}_Q(\alpha(r, d, n)) \rightarrow \mathbb{A}^1 \). Any rational polarization \( \theta \) (in the sense Section 2.3) linearizes this action and thus gives rise to the GIT quotient

\[
\mathcal{M}_Q^\theta(\alpha(r, d, n)) = \text{Rep}_Q(\alpha(r, d, n))//_\theta \text{GL}(\alpha(r, d, n)).
\]

By construction it comes with a map \( \mathcal{M}_Q^\theta(\alpha(r, d, n)) \rightarrow \mathbb{A}^1 \), and clearly its fiber over a point \( \tau \in \mathbb{A}^1 \) identifies with the moduli space \( \mathcal{M}_Q^\theta(\alpha(r, d, n)) \).

Using this construction for \( \theta = \theta^0 \) and for \( \theta = \theta^0 + \varepsilon \theta^1 \) we construct the relative versions of the Gieseker and the Uhlenbeck compactifications

\[
\mathcal{G}M(r, d, n) = \mathcal{M}_Q^{(\theta^0, \theta^1)}(\alpha(r, d, n)) \quad \text{and} \quad \mathcal{U}M(r, d, n) = \mathcal{M}_Q^{\theta^0}(\alpha(r, d, n)).
\]

By standard GIT we have a contraction \( \gamma : \mathcal{G}M(r, d, n) \rightarrow \mathcal{U}M(r, d, n) \) commuting with the morphisms to \( \mathbb{A}^1 \).

**Proposition 3.2.1.** If \( \text{gcd}(r, d, n) = 1 \) then the map \( \mathcal{G}M(r, d, n) \rightarrow \mathbb{A}^1 \) is smooth and projective. In particular, \( \mathcal{G}M(r, d, n) \) is a smooth variety.

**Proof.** By Lemma 2.5.4 the moduli space \( \mathcal{G}M(r, d, n) \) coincides with the moduli space of Gieseker semistable sheaves of rank \( r \), degree \( d \) and second Chern class \( n \) for the family \( A \) of Artin–Schelter algebras over \( \mathbb{C}[t] \) constructed in [17]. The smoothness and the projectivity of the latter is proved in Theorem 8.1 of loc. cit. \( \square \)

When considering the case \( r = 1, d = 0 \) we abbreviate \( \mathcal{G}M(1, 0, n) \) to just \( \mathcal{G}M^n \) and \( \mathcal{U}M(1, 0, n) \) to \( \mathcal{U}M^n \).

### 3.3. Fixed points.

We choose a torus \( T \subset \text{SL}(H) \) and consider its action on the Calogero–Moser space and its Gieseker and Uhlenbeck compactifications. The stratifications and the map \( \gamma \) are \( \text{SL}(H) \)-equivariant and hence \( T \)-equivariant as well. We aim at a description of the set of \( T \)-fixed points on \( \mathcal{U}M^n \). Recall first what is known about the \( T \)-fixed locus of \( M^n \).

**Lemma 3.3.1.** ([21 Proposition 6.11]) For \( \tau \neq 0 \) the set of \( T \)-fixed points in \( M^n_\tau \) is in a natural bijection with the set \( \mathcal{P}(n) \) of partitions of \( n \).
We denote by \( c^n_\lambda \in M^n_\tau \) the T-fixed point corresponding to a partition \( \lambda \in \mathcal{P}(n) \). In particular, \( c^0 \in M_0^\tau \) is the unique point (it is automatically T-fixed). Let also \( P_0, P_{\infty} \in \mathbb{P}(H) \) be the T-fixed points on the line \( \mathbb{P}(H) \).

**Lemma 3.3.2.** For any \( \tau \neq 0 \) the set of T-fixed points in \( U^n_\tau \) is finite. Moreover,

\[
(U^n_\tau)^T = \{ (c^m_\lambda, k_0 P_0 + k_{\infty} P_{\infty}) \in M^m_\tau \times S^{n-m} \mathbb{P}^1 | \lambda \in \mathcal{P}(m), k_0 + k_{\infty} = n - m \}.
\]

**Proof.** It is enough to describe \( T \)-fixed points on each of the strata \( M^m_\tau \times S^{n-m} \mathbb{P}(H) \) of the stratification of \( U^n_\tau \). As the product decomposition is \( \text{SL}(H) \)-invariant, it is enough to describe fixed points on each factor. On first factor we use Lemma 3.3.1 and on \( S^k \mathbb{P}(H) \) a description of fixed points is evident. \( \square \)

Recall that a \( T \)-fixed point \( P \) is called attracting if all the weights of the \( T \)-action on the tangent space at point \( P \) are positive.

**Lemma 3.3.3.** The Uhlenbeck compactification \( U^n_\tau \) of the Calogero–Moser space has a unique attracting \( T \)-fixed point \( (c^0, nP_0) \in M^0_\tau \times S^n \mathbb{P}(H) \subset U^n_\tau \).

**Proof.** Since \( U^n_\tau \) is a projective variety, the \( T \)-action on it should have at least one attracting point. On the other hand, \( M^n_\tau \) is a sympletic manifold, and the \( T \)-action preserves the sympletic structure \([12]\), hence for \( m > 0 \) the weights of \( T \) on the tangent spaces at points \( c^m_\lambda \) are pairwise opposite, and thus for \( m > 0 \) the \( T \)-fixed points \( (c^m_\lambda, k_0 P_0 + k_{\infty} P_{\infty}) \) are not attracting. Therefore, each attracting point of the \( T \)-action on \( U^n_\tau \) lies on \( M^0_\tau \times S^n \mathbb{P}(H) = S^n \mathbb{P}(H) \). As it also should be an attracting point for the \( T \)-action on \( S^n \mathbb{P}(H) \), it should coincide with \( (c^0, nP_0) \). \( \square \)

### 3.4. The IC sheaf of the Uhlenbeck compactification

In this section we will prove Theorem 1.3.1. The statement of the Theorem and the arguments we use are purely topological. We refer to [3] for the notion of IC sheaf and the general machinery.

We start with computing the stalks of the IC-sheaf at the deepest stratum of the Uhlenbeck stratification. Since for \( n = 0 \) the Calogero–Moser space \( M^0_\tau \) is just a point, by Theorem 2.8.4 we have \( S^n \mathbb{P}(H) = S^n \mathbb{P}(H) = M^0_\tau \times S^n \mathbb{P}(H) \subset U^n_\tau \). Recall also the diagonal stratification (1.3) of \( S^n \mathbb{P}(H) \) and its deepest stratum \( S_{(n)} \mathbb{P}(H) \subset S^n \mathbb{P}(H) \).

**Proposition 3.4.1.** For any \( P \in \mathbb{P}(H) \) the stalk of the sheaf \( \text{IC}(U^n_\tau) \) at the point \( (\emptyset, nP) \) of the stratum \( M^n_\tau \times S_{(n)} \mathbb{P}(H) \subset U^n_\tau \) is isomorphic to

\[
\text{IC}(U^n_\tau)_{(\emptyset, nP)} = \bigoplus_{\mu \in \mathcal{P}(n)} \mathbb{C}[2l(\mu)].
\]  

(3.4.2)
Proof. Let $T \subset \text{SL}(H)$ be a torus such that $P = P_0$ is the attracting point for the action of $T$ on $\mathbb{P}(H)$. The computation is based on the following “deformation diagram”:

$\xymatrix{ \mathcal{C} \ar[r]^{\xi} \ar[d]^{\gamma_0} & \mathcal{C} \ar[r]^{\gamma} \ar[d]^{\gamma_\eta} & \mathcal{C} \ar[d]^{\gamma_\eta} \\ \mathcal{U} \ar[r]^{\sigma} \ar[d]^{\sigma} & \mathbb{A}^1 \ar[d]^{\sigma} \\ \{0\} \ar[r] & \mathbb{A}^1 \setminus \{0\} }$

Here the middle column is the deformation family over $\mathbb{A}^1$ of Proposition 3.2.1 with $p$ being the structure map. The left column is the fiber over the point 0, while the right column is the base change to $\mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$. Finally, the map $\sigma : \mathbb{A}^1 \to \mathcal{U}$ is defined as follows.

For any $\tau \neq 0$ we put $\sigma(\tau) = n \cdot P \in \mathcal{U}^{\mathbb{A}^1} \subset \mathcal{U}$. Clearly, this is a regular map $\mathbb{A}^1 \setminus \{0\} \to \mathcal{U}$. By properness of $\mathcal{U}$ over $\mathbb{A}^1$ it extends to a map $\sigma : \mathbb{A}^1 \to \mathcal{U}$. Note that $\sigma$ is a section of the map $\mathcal{U}$. Indeed, this is clear over $\mathbb{A}^1 \setminus \{0\}$ by definition of $\sigma$, and over 0 this is true by continuity.

Let $\mathbb{C}^x_{\leq 1} \subset \mathbb{C}^x = T$ be the sub-semigroup of formed by the complex numbers with absolute value $\leq 1$, and let $F = \sigma(\mathbb{A}^1) \subset \mathcal{U}^{\mathbb{A}^1}$ be the image of the section $\sigma$. Further, let $\mathcal{U} \subset \mathcal{U}^{\mathbb{A}^1}$ be a small open neighborhood (in the analytic topology) of the point $\sigma(0)$. Without loss of generality, we may choose the set $\mathcal{U}$ to be $\mathbb{C}^x_{\leq 1}$-stable. Note that $F$ is the attracting connected component of $(\mathcal{U}^{\mathbb{A}^1})^T$, by Lemma 3.3.3. Therefore, shrinking $\mathcal{U}$ further, if necessary, one may assume in addition that we have $\mathcal{U}^T = F \cap \mathcal{U}$. The action of $\mathbb{C}^x_{\leq 1}$ preserves the fibers of $p : \mathcal{U} \to \mathbb{A}^1$, and contracts $\mathcal{U}$ to the section $F = \sigma(\mathbb{A}^1)$. According to [5, Lemma 6], for any $\mathbb{C}^x$-equivariant complex $\mathcal{F}$ of constructible sheaves on $\mathcal{U}^{\mathbb{A}^1}$, the natural morphism $\sigma^* \mathcal{F} \to p_! (\mathcal{F}|_\mathcal{U})$ is an isomorphism. In other words, for any $\tau \in \mathbb{A}^1$, there is a natural isomorphism

$$H^\bullet(\mathcal{U} \cup \mathcal{U}^{\mathbb{A}^1}, \mathcal{F}) \cong \mathcal{F}|_{\sigma(\tau)}. \quad (3.4.3)$$

Next, let $\psi_p$, resp. $\psi_{p \circ \gamma}$, denote the nearby cycles functor [11, 8.6] with respect to the function $p$, resp. $p \circ \gamma$. Note that the morphism $p \circ \gamma$ being smooth, we have $\psi_{p \circ \gamma}(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}}) = \mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}}$. Therefore, using the proper base change for nearby cycles (see e.g. [11, Exercise VIII.15]) we obtain

$$(\gamma_0)_*(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}}) = (\gamma_0)_*(\psi_{p \circ \gamma}(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}})) = \psi_p((\gamma_0)_*(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}})).$$

The map $\gamma_\eta : \mathcal{C} \to \mathcal{U}$ is a small and proper morphism. Hence, we have an isomorphism $\text{IC}(\mathcal{U}) \cong (\gamma_\eta)_*(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}})[2n]$. Combining the above isomorphisms and taking stalks at the point $\sigma(0)$ yields

$$(\gamma_0)_*(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}})|_{\sigma(0)} \cong (\psi_p((\gamma_\eta)_*(\mathcal{C}^\mathbb{A}^1_{\mathcal{M}_\mathcal{U}}))|_{\sigma(0)} \cong (\psi_p(\text{IC}(\mathcal{U}))|_{\sigma(0)}[-2n]. \quad (3.4.4)$$
Further, by definition of the functor \( \psi_p \), for a sufficiently small open set \( \mathcal{U} \) as above and for any \( \tau \neq 0 \) with a sufficiently small absolute value one has

\[
(\psi_p(\text{IC}(t^1M^n_\eta)))|_{\sigma(0)} \cong H^\bullet(\mathcal{U} \cap p^{-1}(\tau), \text{IC}(t^1M^n_\eta)) \cong H^\bullet(\mathcal{U} \cap M^n_\tau, \text{IC}(t^1M^n_\tau)).
\]

Thus, comparing the LHS and the RHS in (3.4.4), we obtain

\[
H^\bullet(\gamma^{-1}_0(\sigma(0)))[2n] \cong \left((\gamma_0), \bigoplus \text{IC}(\mathcal{M}^n_\eta)\right)|_{\sigma(0)}[2n] \cong \left(\psi_p(\text{IC}(t^1M^n_\eta))\right)|_{\sigma(0)} \tag{3.4.5}
\]

\[
\cong H^\bullet(\mathcal{U} \cap M^n_\tau, \text{IC}(t^1M^n_\tau)) \cong \text{IC}(t^1M^n_\tau)|_{\sigma(\tau)},
\]

where the last isomorphism is a special case of (3.4.3) for \( \mathcal{F} = \text{IC}(t^1M^n_\tau) \).

To complete the proof, we observe that the fiber \( \gamma^{-1}_0(\sigma(0)) \) is the “central fiber” of the Hilbert–Chow morphism \( \text{Hilb}^n(\mathbb{P}^2) \rightarrow S^n\mathbb{P}^2 \). In other words, the variety \( \gamma^{-1}_0(\sigma(0)) \) is nothing but \( \text{Hilb}^n_0(\mathbb{A}^2) \), the punctual Hilbert scheme of \( n \) infinitesimally close points in \( \mathbb{A}^2 \). The Betti numbers of the punctual Hilbert scheme are well known, cf. e.g. [15]. Specifically, all odd Betti numbers vanish and one has the formula

\[
\dim H^{2k-2}(\text{Hilb}^n_0(\mathbb{A}^2)) = \# \{ \mu \in \mathfrak{P}(n) \mid l(\mu) = k \}.
\]

It follows from (3.4.5) that, for \( \tau \) sufficiently small, the dimensions of the cohomology groups of the stalk \( \text{IC}(t^1M^n_\tau)|_{\sigma(\tau)} \) are given by the same formula. This is equivalent to the statement of the proposition. \( \square \)

Now Theorem 1.3.1 follows from Proposition 3.4.1 and the factorization property of Proposition 3.1.1. In effect, due to \( \text{SL}(H) \)-equivariance, it suffices to find the stalks of \( \text{IC}(t^1M^n_\tau) \) at \( S^1A^1_\lambda \subset S^1\mathbb{P}(H) \). Given a point \( \mathcal{E} \in M^n \) and \( D \in S^1A^1_\lambda \), due to the smallness of \( \gamma_\tau \),

\[
\text{IC}(t^1M^n_\tau)|_{\mathcal{E}, D} = H^\bullet(\gamma_\tau^{-1}(\mathcal{E} \times D), \mathbb{C}).
\]

Further, by Proposition 3.1.3 we have

\[
H^\bullet(\gamma_\tau^{-1}(\mathcal{E} \times D), \mathbb{C}) \cong H^\bullet(\gamma_\tau^{-1}(\emptyset \times D), \mathbb{C}).
\]

Now if \( D = \sum k_iP_i \) with pairwise distinct points \( P_i \in A^1_\lambda \), then according to Proposition 3.1.3 \( \gamma_\tau^{-1}(\emptyset \times D) \cong \bigoplus \gamma_i^{-1}(\emptyset \times k_iP_i) \), and hence

\[
H^\bullet(\gamma_\tau^{-1}(\emptyset \times D), \mathbb{C}) \cong \bigotimes_i H^\bullet(\gamma_i^{-1}(\emptyset \times k_iP_i), \mathbb{C}).
\]

Due to the smallness of \( \gamma_\tau \), \( H^\bullet(\gamma_i^{-1}(\emptyset \times k_iP_i), \mathbb{C}) = \text{IC}(t^1M^n_\tau)|_{\emptyset, k_iP_i} \), and the latter stalk is known from Proposition 3.4.1. This completes the proof of Theorem 1.3.1. \( \square \)

4. Appendix

In this Appendix we collect the proofs of some results from section Section 2. Throughout, we assume that \( \tau \neq 0 \).
4.1. Moduli spaces of Artin sheaves. Denote by $\mathbb{P}^3_3$ the third infinitesimal neighborhood of the line at infinity $\mathbb{P}(H)$ in $\mathbb{P}^2_3$, i.e. the projective spectrum of a commutative graded algebra $\mathbb{C}[x, y, z]/z^3$.

**Lemma 4.1.1.** The moduli space $\mathcal{M}_{0}(1, 2, 1)$ is a fine moduli space. It is isomorphic to the third infinitesimal neighborhood of a line on a plane: $A\mathcal{M}_{0}(1, 2, 1) \cong \mathbb{P}^3_3$.

**Proof.** The data of a $(1, 2, 1)$-dimensional representation of $\mathbb{Q}_r$ amounts to two maps

$$
\mathbb{C} \xrightarrow{f} \mathbb{C}^2 \otimes A_1^r \quad \text{and} \quad \mathbb{C}^2 \xrightarrow{g} \mathbb{C} \otimes A_1^r
$$

with the condition saying that the composition

$$
\mathbb{C} \xrightarrow{f} \mathbb{C}^2 \otimes A_1^r \xrightarrow{g \otimes 1} \mathbb{C} \otimes A_1^r \otimes A_1^r \to \mathbb{C} \otimes A_2^r
$$

is zero. In other words, it can be rewritten as saying that

$$
\sigma := (g \otimes 1)(f(1)) \in K := \text{Ker}(A_1^r \otimes A_1^r \to A_2^r).
$$

The $\theta^0$-semistability is equivalent to the injectivity of the maps $f^T, g: \mathbb{C}^2 \to \mathbb{C} \otimes A_1^r$. This means that the element $\sigma$ considered as an element of $A_1^r \otimes A_1^r = \text{Hom}((A_1^r)^*, A_1^r)$ has rank 2 (then $\mathbb{C}^2$-component of the representation is just the image of $\sigma$). Thus the moduli space is nothing but the degeneration scheme of the morphism

$$
(A_1^r)^* \otimes \mathcal{O}_{\mathbb{P}(K)}(-1) \to A_1^r \otimes \mathcal{O}_{\mathbb{P}(K)}
$$

on $\mathbb{P}(K)$. As $K \subset A_1^r \otimes A_1^r$ can be written as

$$
K = \{u(y \otimes z - z \otimes y) + v(x \otimes z - z \otimes x) + w(x \otimes y - y \otimes x - \tau z \otimes z) \mid u, v, w \in \mathbb{C}\},
$$

the above morphism is given by the matrix

$$
\begin{pmatrix}
0 & w & v \\
-w & 0 & u \\
-v & -u & -\tau w
\end{pmatrix}
$$

(4.1.2)

and the degeneration condition is given by its determinant which is equal to

$$
\det\begin{pmatrix}
0 & w & v \\
-w & 0 & u \\
-v & -u & -\tau w
\end{pmatrix} = -\tau w^3.
$$

This means that the moduli space is the subscheme of $\mathbb{P}(K)$ given by the equation $w^3 = 0$, i.e. the third infinitesimal neighborhood $\mathbb{P}^3_3$ of the line $\mathbb{P}^1 = \{w = 0\}$ in the plane $\mathbb{P}^2$.

To show that the moduli space is fine we should construct a universal family. For this we restrict the map $(A_1^r)^* \otimes \mathcal{O}_{\mathbb{P}(K)}(-1) \to A_1^r \otimes \mathcal{O}_{\mathbb{P}(K)}$ to $M := \mathbb{P}^3_3$. This is a morphism of constant rank 2 (the rank does not drop to 1 since among the 2-by-2 minors of the matrix (4.1.2) one easily finds $w^2$, $v^2$, and $wv$), hence its image is a rank 2 vector bundle $\mathcal{V}_2$.

It comes equipped with a surjective map $(A_1^r)^* \otimes \mathcal{O}_M(-1) \to \mathcal{V}_2$ and an injective map $\mathcal{V}_2 \to A_1^r \otimes \mathcal{O}_M$. Clearly these two maps provide $(\mathcal{O}_M(-1), \mathcal{V}_2, \mathcal{O}_M)$ with a structure of a family of representations of the quiver $\mathbb{Q}_r$. The above arguments show it is a universal family. \hfill \Box

Now we give a description of the reduced structure of the space $\mathcal{A}\mathcal{M}_r(k, 2k, k)$ for $k > 1$. \hfill 32
Proof of Proposition 2.7.2. Consider the subset $A^1R^0_\tau(k, 2k, k) \subset A^1R_\tau(k, 2k, k)$ of all $\theta^0$-semistable $(k, 2k, k)$-dimensional representations of $Q_\tau$ and let $W_\bullet$ be the universal representation of the quiver over $P^2_\tau$. Let $S$ be the universal sheaf on the product $A^1R^0_\tau(k, 2k, k) \times P^2_\tau$, i.e., the sheaf defined by exact sequence

$$0 \to W_1 \boxtimes \mathcal{O}(-1) \to W_0 \boxtimes \mathcal{O} \to W_1 \boxtimes \mathcal{O}(1) \to J \to 0.$$ 

Then the support map defined in Lemma 2.1.9 gives a map $\text{supp} : A^1R^0_\tau(k, 2k, k) \to S^kP^1$. The map is clearly $GL(k, 2k, k)$-equivariant, hence descends to a map from the moduli space

$$\text{supp} : A^1M_\tau(k, 2k, k) \to S^kP^1.$$ 

On the other hand, we clearly have an embedding which takes a $k$-tuple of $(1, 2, 1)$-dimensional Artin representations $W^{1}_\bullet, W^{2}_\bullet, \ldots, W^{k}_\bullet$ to their direct sum

$$(A^1R_\tau(1, 2, 1))^k \to A^1R_\tau(k, 2k, k), \quad (W^{1}_\bullet, W^{2}_\bullet, \ldots, W^{k}_\bullet) \mapsto W^{1}_\bullet \oplus W^{2}_\bullet \oplus \cdots \oplus W^{k}_\bullet.$$ 

This map is equivariant with respect to the action of the group $GL(1, 2, 1)^k \times \mathcal{S}_k$ on the source, such that the $i$-th factor $GL(1, 2, 1)$ acts naturally on the $i$-th factor of $(A^1R_\tau(1, 2, 1))^k$ and $\mathcal{S}_k$ permutes the factors, and the action on the target is given by a natural embedding $GL(1, 2, 1)^k \times \mathcal{S}_k \subset GL(k, 2k, k)$. The Proj construction of the GIT quotient implies that the map induces a morphism of the GIT quotients

$$(A^1R_\tau(1, 2, 1))^k / \theta^0(\text{GL}(1, 2, 1)^k \times \mathcal{S}_k) \to A^1R_\tau(k, 2k, k) / \theta^0 \text{GL}(k, 2k, k).$$ 

The quotient on the right is just the moduli space $A^1M_\tau(k, 2k, k)$. The quotient on the left can be identified with $(A^1M_\tau(1, 2, 1))^k / \mathcal{S}_k$, so it is isomorphic to $S^k(P^1_3)$ by Lemma 4.1.1. Restricting to the reduced subscheme we obtain a map

$$\Sigma : S^kP^1 = S^k(P^1_3)_{\text{red}} \to A^1M_\tau(k, 2k, k).$$ 

We are going to show that the constructed maps $\text{supp}$ and $\Sigma$ induce isomorphisms between $S^kP^1$ and the reduced moduli space $A^1M_\tau(k, 2k, k)_{\text{red}}$.

For this we note that the maps give bijections between the sets of closed points of $S^kP^1$ and $A^1M_\tau(k, 2k, k)$, since by Proposition 2.1.7(3) any Artin sheaf is $S$-equivalent to a direct sum of structure sheaves for a unique collection of points (which are given back by the support map). Note also that both $S^kP^1$ and $A^1M_\tau(k, 2k, k)$ are projective varieties, hence the map $\Sigma$ is proper. Finally, $S^kP^1 \cong P^k$ is normal.

So, it is enough to show that any proper regular map from a reduced normal scheme to a reduced scheme inducing a bijection on the sets of closed points is an isomorphism. Locally, we just have an integral (due to properness) extension of rings with the bottom ring being integrally closed (by normality), hence it is an isomorphism. 

4.2. Stratifications. Here we construct the required stratifications of the Gieseker and Uhlenbeck moduli spaces.

Proof of Lemma 2.8.1. Let $G^R := G^R_{\alpha(r, d, n)}(\theta^0, \theta^1) \subset G^R_{\alpha(r, d, n)}(\alpha(r, d, n))$ be the open subset of $(\theta^0, \theta^1)$-semistable $\alpha(r, d, n)$-dimensional representations of $Q_\tau$. Let $V_\bullet$ be the universal family of representations over $G^R_{\tau}$. Consider the universal monad

$$V_1 \boxtimes \mathcal{O}(-1) \to V_2 \boxtimes \mathcal{O} \to V_3 \boxtimes \mathcal{O}(1)$$
on $G\mathcal{R}_\tau \times \mathbb{P}^2$ and denote its cohomology sheaf by $E$. For each point $s \in G\mathcal{R}_\tau$ we denote by $E_s$ the restriction of $E$ to $\{s\} \times \mathbb{P}^2$. Note that this is just the cohomology sheaf of the monad $V_{1s} \otimes \mathcal{O}(-1) \to V_{2s} \otimes \mathcal{O} \to V_{3s} \otimes \mathcal{O}(1)$. In particular, the sheaf $E$ is flat over $G\mathcal{R}_\tau$.

Consider also the dual monad on $S \times \mathbb{P}^2$

$$V^\vee_s \otimes \mathcal{O}(-1) \to V^\vee_2 \otimes \mathcal{O} \to V^\vee_1 \otimes \mathcal{O}(1)$$

and let $\mathcal{F}$ be the cokernel of the last map

$$\mathcal{F} := \text{Coker}(V^\vee_2 \otimes \mathcal{O} \to V^\vee_1 \otimes \mathcal{O}(1)).$$

For each point $s \in S$ we have

$$\mathcal{F}_s \cong \text{Coker}(V^\vee_{1s} \otimes \mathcal{O} \to V^\vee_{1s} \otimes \mathcal{O}(1)) \cong \text{Ext}^1(E_s, \mathcal{O}) \cong \text{Ext}^2(E^*_s/E_s, \mathcal{O}).$$

Thus it is an Artin sheaf, but its length may vary from point to point. Consider the flattening stratification of $S$ for $\mathcal{F}$:

$$G\mathcal{R}_\tau = G\mathcal{R}^{\geq 0}_\tau \supset G\mathcal{R}^{\geq 1}_\tau \supset G\mathcal{R}^{\geq 2}_\tau \supset \cdots \supset G\mathcal{R}^{\geq n}_\tau \supset G\mathcal{R}^{\geq n+1}_\tau = \emptyset,$$

where $G\mathcal{R}^{\geq k}_\tau$ is the subscheme of points $s \in G\mathcal{R}_\tau$ where the length of $\mathcal{F}_s$ is at least $k$.

This stratification is GL$(\alpha(r,d,n))$-invariant, so it gives a stratification of the GIT quotient $G\mathcal{R}_\tau/(\theta^0,\theta^1)\text{GL}(\alpha(r,d,n))$, i.e., of the Gieseker moduli space $\mathcal{M}_\tau(r,d,n)$. Finally, we replace each stratum by its underlying reduced subscheme.

Below we will need the following result on universal families.

**Proposition 4.2.1.** Let $\mathcal{V}_\bullet$ be the universal family of $(\theta^0,\theta^1)$-semistable $\alpha(r,d,n)$-dimensional representations of $Q_\tau$ over $G\mathcal{R}^k_\tau := G\mathcal{R}^{\geq k}_\tau \setminus G\mathcal{R}^{\geq k+1}_\tau$. Then there is a natural exact sequence

$$0 \to \mathcal{W}_\bullet \to \mathcal{V}_\bullet \to \mathcal{U}_\bullet \to 0$$

of families of representations over $G\mathcal{R}^k_\tau$ where $\mathcal{W}_\bullet$ is a family of Artin representations of dimension $(k,2k,k)$ and $\mathcal{U}_\bullet$ is a family of supermonadic $(\theta^0,\theta^1)$-semistable representations.

**Proof.** We use freely the notation introduced in the proof of Lemma 2.8.1. By assumption $\mathcal{F}$ is a flat (over $G\mathcal{R}^k_\tau$) family of Artin sheaves of length $k$. Consider its Beilinson resolution

$$\mathcal{W}^\prime_1 \otimes \mathcal{O}(-1) \to \mathcal{W}^\prime_2 \otimes \mathcal{O} \to \mathcal{W}^\prime_3 \otimes \mathcal{O}(1),$$

By flatness of $\mathcal{F}$ we know that $\mathcal{W}^\prime_1$, $\mathcal{W}^\prime_2$, $\mathcal{W}^\prime_3$ are vector bundles of ranks $k$, $2k$, and $k$ respectively. By functoriality of the Beilinson resolution there is a morphism of resolutions

$$\mathcal{V}^\vee_3 \otimes \mathcal{O}(-1) \to \mathcal{V}^\vee_2 \otimes \mathcal{O} \to \mathcal{V}^\vee_1 \otimes \mathcal{O}(1)$$

$$\mathcal{W}^\prime_1 \otimes \mathcal{O}(-1) \to \mathcal{W}^\prime_2 \otimes \mathcal{O} \to \mathcal{W}^\prime_3 \otimes \mathcal{O}(1)$$

Note that the induced morphisms $\mathcal{V}^\vee_i \to \mathcal{W}^\prime_{i+1}$ of vector bundles on $G\mathcal{R}^k_\tau$ are surjective. Indeed, this can be verified pointwise, i.e. just for one representation instead of a family. In this case note that both $\mathcal{V}^\vee_i$ and $\mathcal{W}^\prime_i$ are $\theta^0$-semistable, hence so is the image of the map. But any $\theta^0$-semistable subrepresentation of an Artin representation is also Artin. If $\mathcal{F}' \subset \mathcal{F}$ is the corresponding Artin sheaf then it follows that the map $\mathcal{V}^\vee_1 \to \mathcal{F}$ factors through $\mathcal{F}'$ which by definition of $\mathcal{F}$ implies $\mathcal{F}' = \mathcal{F}$.
Let $\mathcal{W}_i = (\mathcal{W}_{i-1}^r)^\vee$ and $U_i = \text{Ker}(\mathcal{V}_i^r \to \mathcal{W}_{i-1}^r)^\vee$, so that we have an exact sequence of monads

$\xymatrix{ \mathcal{W}_1 \otimes \mathcal{O}(-1) \ar[d] & \mathcal{W}_2 \otimes \mathcal{O} \ar[r] \ar[d] & \mathcal{W}_3 \otimes \mathcal{O}(1) \ar[d] \\ \mathcal{V}_1 \otimes \mathcal{O}(-1) \ar[d] & \mathcal{V}_2 \otimes \mathcal{O} \ar[r] \ar[d] & \mathcal{V}_3 \otimes \mathcal{O}(1) \ar[d] \\ U_1 \otimes \mathcal{O}(-1) \ar[d] & U_2 \otimes \mathcal{O} \ar[r] \ar[d] & U_3 \otimes \mathcal{O}(1) \ar[d] }

By construction, the family $U_\bullet$ is $(\theta^0, \theta^1)$-semistable and supermonadic, so this exact sequence is the one we need. $\square$

And now we are ready to construct the stratification of the Uhlenbeck moduli space. The situation here is a bit more complicated than in the Gieseker case, since Artin representations can appear both as subrepresentations and as quotient representations of a $\theta^0$-semistable representation. So, we perform a two-step construction, first dealing with the latter, and then with the former.

**Proof of Lemma 2.8.2.** Let $U R_\tau := \{ \mathcal{V}^\theta_\mathcal{V} (\alpha(r, d, n)) \subset U R_\tau (\alpha(r, d, n)) \}$ be the open subset of $\theta^0$-semistable $\alpha(r, d, n)$-dimensional representations of $Q_\tau$. Let $\mathcal{V}_\bullet$ be the universal family of representations over $U R_\tau$. Consider the family of sheaves $\mathcal{F} := \text{Coker}(\mathcal{V}_2 \otimes \mathcal{O} \to \mathcal{V}_3 \otimes \mathcal{O}(1))$ over $U R_\tau \times \mathbb{P}_T^2$. Note that these are Artin sheaves of length at most $n$. Let

$$U R_\tau = U R_{\tau}^{0, \bullet} \supset U R_{\tau}^{1, \bullet} \supset U R_{\tau}^{2, \bullet} \supset \cdots \supset U R_{\tau}^{n, \bullet} \supset U R_{\tau}^{n+1, \bullet} = \emptyset,$$

be the flattening stratification for the sheaf $\mathcal{F}$. Restricting the family $\mathcal{V}_\bullet$ to each stratum $U R_\tau^{k, \bullet} = U R_{\tau}^{k, \bullet} \setminus U R_{\tau}^{k+1, \bullet}$ and repeating the arguments of Proposition 4.2.1 (without dualization) we construct on $U R_\tau^{k}$ a natural exact sequence

$$0 \to \mathcal{V}_\bullet \to \mathcal{V}_\bullet \to \mathcal{W}_\bullet \to 0$$

of representations with $\mathcal{W}_\bullet$ being a $(k, 2k, k)$-dimensional family of Artin representations, and $\mathcal{V}_\bullet$ being a $(\theta^0, \theta^1)$-semistable family of $\alpha(r, d, n - k)$-dimensional representations of the quiver $Q_\tau$. Applying to the latter family the arguments of the proof of Lemma 2.8.1 we obtain a natural stratification

$$U R_{\tau}^{k, \bullet} = U R_{\tau}^{k, \geq 0} \supset U R_{\tau}^{k, \geq 1} \supset U R_{\tau}^{k, \geq 2} \supset \cdots \supset U R_{\tau}^{k, \geq n-k} \supset U R_{\tau}^{k, \geq n+1-k} = \emptyset,$$

by the length of the Artin sheaf $\text{Coker}(\mathcal{V}_2^\mathcal{V} \otimes \mathcal{O} \to (\mathcal{V}_1^\mathcal{V}) \otimes \mathcal{O}(1))$, and moreover, on the stratum $U R_{\tau}^{k,l} := U R_{\tau}^{k, \geq l} \setminus U R_{\tau}^{k, \geq l+1}$ an exact sequence of representations

$$0 \to \mathcal{W}_\bullet \to \mathcal{V}_\bullet \to \mathcal{U}_\bullet \to 0$$

with $\mathcal{W}_\bullet$ being a $(l, 2l, l)$-dimensional family of Artin representations, and $\mathcal{U}_\bullet$ being a $(\theta^0, \theta^1)$-semistable family of $\alpha(r, d, n - k - l)$-dimensional supermonadic representations.

It is clear that the subsets

$$U R_{\tau}^{\geq m} := \bigcup_{k+l \geq m} U R_{\tau}^{k,l} \subset U R_{\tau}$$
are \( \text{GL}(\alpha(r, d, n)) \)-invariant closed subsets, hence they induce a stratification of the GIT quotient \( U R_\tau /_{/\vartheta_0} \text{GL}(\alpha(r, d, n)) \), i.e., of the Uhlenbeck moduli space \( U M_\tau(r, d, n) \). Again, we finish by replacing each stratum with its reduced underlying scheme. \( \square \)

**Remark 4.2.2.** We could start with splitting of Artin subrepresentations first (and define in this way closed subsets \( U R_\tau^{\geq l} \subset U R_\tau \)) and then continue with splitting Artin quotient representations. Note that this will give different two-index stratification of the space \( U R_\tau \), but the resulting total stratification will be the same.

**Remark 4.2.3.** We always have an embedding \( G R_\tau^k \subset U R_\tau^{0,k} \). Moreover, if \( r \) and \( d \) are coprime then this inclusion becomes an equality

\[
G R_\tau^k = U R_\tau^{0,k}.
\]

Indeed, the union of \( U R_\tau^{0,k} \) over all \( k \) parameterizes all monadic \( \theta^0 \)-semistable representations, and by Corollary 2.6.7 these are precisely all \( (\theta^0, \theta^1) \)-semistable representations.

Before we go to the proof of the main Theorem we need one more result. We will use the notation of the proof of Lemma 2.8.2 Consider the stratum \( U R_\tau^{0,0} \) of the space \( U R_\tau \). We checked in the proof of Lemma 2.8.2 that the universal representation over it fits into an exact sequence

\[
0 \to \mathcal{U}_* \to \mathcal{V}_* \to \mathcal{W}_* \to 0 \quad (4.2.4)
\]

with \( \mathcal{U}_* \) being supermonadic and \( \mathcal{W}_* \) being Artin.

**Lemma 4.2.5.** The subset \( U R_\tau^{k,\text{split}} \subset U R_\tau^k \) of all points over which the exact sequence \( (4.2.4) \) splits is closed.

**Proof.** Indeed, \( U R_\tau^{k,\text{split}} = U R_\tau^k \cap U R_\tau^{\geq k} \cap U R_\tau^{\geq k} \) and both \( U R_\tau^{\geq k} \) and \( U R_\tau^{\geq k} \) are closed subsets in \( U R_\tau \) by their construction. \( \square \)

**4.3. Proof of Theorem 2.8.4.** (1) Follows immediately from the inclusion \( G R_\tau^k \subset U R_\tau^{0,k} \).

(2) Equality \( U M_\tau^0(r, d, n) = \overline{O M}_\tau^0(r, d, n) \) for \( r \) and \( d \) coprime follows immediately from the equality \( G R_\tau^0 = U R_\tau^{0,0} \) which is the only component of the stratum of \( U R_\tau \) giving \( U M_\tau^0(r, d, n) \).

Further, consider the subset \( U R_\tau^{k,\text{split}} \subset U R_\tau^k \). By Lemma 4.2.5 it is closed. It is also \( \text{GL}(\alpha(r, d, n)) \)-invariant. Finally, each point in \( U R_\tau^k \) is \( S \)-equivalent to a point in \( U R_\tau^{k,\text{split}} \). Therefore

\[
U M_\tau^k(r, d, n) = U R_\tau^k /_{/\vartheta_0} \text{GL}(\alpha(r, d, n)) = U R_\tau^{k,\text{split}} /_{/\vartheta_0} \text{GL}(\alpha(r, d, n)).
\]

So, it is enough to find a direct product decomposition for the right hand side of the equality. For this recall that the universal family of representations restricted to \( U R_\tau^{k,\text{split}} \) splits canonically as a direct sum

\[
\mathcal{V}_* = \mathcal{U}_* \oplus \mathcal{W}_*
\]

of a supermonadic \( (\theta^0, \theta^1) \)-semistable \( \alpha(r, d, n - k) \)-dimensional representation and of an Artin representation of dimension \( (k, 2k, k) \). This decomposition gives a \( \text{GL}(\alpha(r, d, n)) \)-equivariant morphism to the homogeneous space

\[
U R_\tau^{k,\text{split}} \to \text{GL}(\alpha(r, d, n)) / (\text{GL}(\alpha(r, d, n - k) \times \text{GL}(k, 2k, k))
\]
such that the fiber over a point is the product $U^R_\tau(r, d, n - k) \times A^R_\tau(k, 2k, k)$. It follows that

\[
U^R_k \text{split} \sslash \theta^0 \text{GL}(\alpha(r, d, n))

\cong (U^R_\tau(r, d, n - k) \times A^R_\tau(k, 2k, k)) \sslash \theta^0 \text{GL}(\alpha(r, d, n - k) \times \text{GL}(k, 2k, k))

\cong (U^R_\tau(r, d, n - k)) / \theta^0 \text{GL}(\alpha(r, d, n - k)) \times (A^R_\tau(k, 2k, k)) / \theta^0 \text{GL}(k, 2k, k)

\cong U^M_\tau(r, d, n - k) \times A^M_\tau(k, 2k, k).
\]

This together with Proposition 2.7.2 proves part (2) of the Theorem.

(3) The split representation in a closure of the $\text{GL}(\alpha(r, d, n))$-orbit of a $(\theta^0, \theta^1)$-semistable representation $V_\bullet(E)$ is just the direct sum of the supermonadic quotient and the Artin subrepresentation of $V_\bullet$. By Proposition 2.6.6 these correspond to the sheaves $E^{**}$ and $E^{**} / E$ respectively, hence the claim. \[\Box\]

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