Reference Point Methods and Approximation in Multicriteria Optimization

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Abstract

Operations research applications often pose multicriteria problems. Mathematical research on multicriteria problems predominantly revolves around the set of Pareto optimal solutions, while in practice, methods that output a single solution are more widespread. In real-world multicriteria optimization, reference point methods are widely used and successful examples of such methods. A reference point solution is the solution closest to a given reference point in the objective space.

We study the approximation of reference point solutions. In particular, we establish that approximating reference point solutions is polynomially equivalent to approximating the Pareto set. Complementing these results, we show for a number of general algorithmic techniques in single criteria optimization how they can be lifted to reference point optimization. In particular, we lift the link between dynamic programming and FPTAS, as well as oblivious LP-rounding techniques. The latter applies, e.g., to Set Cover and several machine scheduling problems.

1 Introduction

In many applications of combinatorial optimization, trade-offs between conflicting objectives play a crucial role. For example, route guidance systems are a classical application of the shortest path problem. Yet, a good route guidance should allow the driver to make an informed choice to balance travel time and fuel consumption.

It is well-known that even for this basic example, the bicriteria shortest path problem, the number of Pareto optimal (i.e., non-dominated) solutions can grow exponentially with the size of the network. Decision makers may have different preferences how much extra fuel to spend on less travel time. Thus, a central task of multicriteria optimization is to either find a single solution based on a priori expressed trade-off preferences of the decision maker, or to identify a set of solutions that is of manageable (in mathematical terms: polynomial) size but still reflects all possible trade-off options at least approximately.

A straightforward way to a single solution is the weighted-sum method: The trade-off preferences are specified by two non-negative weights for time and fuel consumption. The navigation system then chooses a route minimizing the weighted sum of the two objectives. Unfortunately, this method deprives the decision maker of essential solutions: Consider an instance with three possible routes with corresponding objective value vectors (10, 1), (6, 6), and (1, 10), respectively. The route with fuel consumption 6 and travel time 6 will never be the optimum for any choice of weights, despite being a balanced and thus attractive alternative for many drivers.

Formally, this shortcoming of the weighted-sum approach means that it cannot reach every point of the Pareto set. This motivates the concept of compromise solutions and reference point solutions as defined by Yu [25], which returns a solution closest to a given reference point, where the distance is measured by some norm in the objective space. Compromise solutions use the
component-wise optimum over all solutions as a reference point. The trade-off preferences are reflected by the choice of the norm in the objective space. Every point in the Pareto set is a reference point solution for some norm. Reference point methods are widely used in practice, serving as a core concept of MCDM\textsuperscript{1} tools (cf. Caballero et al. \cite{Caballero1999} and Opricovic and Tzeng \cite{Opricovic2004} for particular examples and Ehrgott et al. \cite{Ehrgott2003} for an overview). Still, they did not attract a lot of theoretical interest so far.

We show that approximating reference point solutions is polynomially equivalent to approximating the Pareto set as proposed by Papadimitriou and Yannakakis \cite{Papadimitriou2000}. Further, we provide general techniques for approximation algorithms, by means of which reference point solutions can often be approximated with the same factor as the single-criterion problem, most notably for the case of LP-rounding. A byproduct of our results are approximation algorithms for the Pareto sets of many hard combinatorial optimization problems.

Related work. Multicriteria optimization has a long tradition. The central notion of Pareto optimality goes back to works by Vilfredo Pareto in the late 19th and early 20th century. Ever since then, solution concepts in multicriteria optimization have been studied. The notion of compromise solutions was introduced in 1973 by Yu \cite{Yu1973} and further studied and extended in the following years by Freimer and Yu \cite{Freimer1976}, Gearhardt \cite{Gearhardt1983}, Choo and Steuer \cite{Choo1985} and many others. The concept was later extended to more general reference points and is incorporated in many MCDM tools (cf. Caballero et al. \cite{Caballero1999}, Opricovic and Tzeng \cite{Opricovic2004}, Ehrgott et al. \cite{Ehrgott2003}). Recently, Voorneveld et al. \cite{Voorneveld2008} gave an axiomatization of compromise solutions, in particular those w.r.t. the Euclidean norm.

Also the approximation of Pareto sets has been studied for several decades now. It was initiated by Hansen in 1979 \cite{Hansen1979}, followed by several publications on specific problems such as shortest paths (Warburton \cite{Warburton1983}) and scheduling (Cheng et al. \cite{Cheng1993}). More general results on the existence and computability of approximate Pareto sets were presented by Safer in his PhD thesis \cite{Safer1992} in 1992, and in 2000 by Papadimitriou and Yannakakis \cite{Papadimitriou2000}. Some of our results are based on the latter.

The results by Papadimitriou and Yannakakis \cite{Papadimitriou2000} were extended by Vassilvitskii and Yannakakis \cite{Vassilvitskii2008} and, under stronger assumptions on the problems, further improved by Diakonikolas and Yannakakis \cite{Diakonikolas2009, Diakonikolas2012}. The latter publication is particularly related to our results on the equivalence between the approximability of the weighted sum problem and the approximability of the Pareto set (Corollary 4.2), as the authors show a similar statement for convex approximate Pareto sets.

Also several other works have studied the relationship between approximate Pareto sets and aggregations of the objectives into one single objective, and are thus related to reference point methods. Ackermann et al. \cite{Ackermann2012} use approximate Pareto sets to optimize an aggregation that is assumed to be (partially) differentiable. Their results are restricted to bi-objective problems, however. Recently, Mittal and Schulz \cite{Mittal2015, Mittal2016} have used approximate Pareto sets to approximately optimize low-rank functions over polytopes and discrete sets. While one of our results can be seen as a special case of their framework, the remainder of our work also implies the reverse direction of their results: If one can approximately optimize a certain class of low-rank functions, one can also compute an approximate Pareto set.

Multicriteria optimization and in particular compromise solutions are also closely related to robust optimization, in particular to min-max regret robustness. This connection has also been noted and exploited by others, e.g. Aissi et al. \cite{Aissi2012, Aissi2013}. We extend some of their results to reference point methods.

Our contribution. Our research mainly focuses on minimization problems, and we will restrict ourselves to this setting throughout most parts of this paper. We note that this is not

\textsuperscript{1}Multicriteria Decision Making
without loss of generality, and some of the results do not hold in the context of maximization. We discuss the differences in Section 5.

In Section 3, we establish an algorithmic link between reference point solutions and approximation of the Pareto set. As a main result, we show that approximating reference point solutions, approximating compromise solutions, and approximating the Pareto set are polynomially equivalent. An overview over the reductions that are proven in this paper is given in Figure 1. We also show that any point in the Pareto set can be obtained as reference point solution for two classes of popular norms with polynomially sized norm parameters, extending a result by Gearhardt [11].

Combining these results with an easy constant factor approximation for reference points, through optimization of the weighted sum, yields the following interesting corollary: For any discrete minimization problem with a fixed number of linear criteria, there is a constant factor approximation for the Pareto set if and only if there is a constant factor approximation for the single-criterion version of the problem. The approximation guarantee of the thus obtained set is increased by a factor of $k$ (the number of criteria), but it remains constant.

In Section 4, we show how to solve the reference point problem approximately for many combinatorial optimization problems. As a main result in this section, we show that single-objective approximations obtained by oblivious LP-rounding directly can be transferred to approximation algorithms for reference point methods. Along the way, we also prove that reference point solutions for linear objectives on convex sets can be found efficiently. From this we get a short alternative proof to Papadimitriou and Yannakakis [20] for the existence of an FPTAS for the Pareto set of such problems. Finally, we extend a technique by Aissi et al. [2] from robust optimization to multicriteria optimization, allowing us to construct an FPTAS for reference point problems from pseudopolynomial algorithms.

In Section 5 we analyze maximization problems and present both positive and negative answers to the question which of the results from Section 3 carry over to maximization.

2 Preliminaries

Throughout the paper, we let $P$ denote a multicriteria discrete optimization problem with $k$ objectives. As usual in multicriteria optimization, we assume the number of objectives to be fixed. With the exception of Section 5, we consider only minimization objectives. As we want to study approximation, we also restrict to non-negative objective values. An instance $I$ of $P$ is thus given by the set of feasible solutions $X$ and the vector of objective functions $c : X \rightarrow \mathbb{Z}_{\geq 0}^k$.

The objective vector set of the instance is defined by $Y := c(X) \subseteq \mathbb{Z}_{\geq 0}^k$. A solution $y \in Y$ is Pareto optimal if there is no $y' \in Y \setminus \{y\}$ with $y' \leq y$, where $y' \leq y$ is defined as $y'_i \leq y_i \forall i \in [k]$. By $[k]$ here and throughout the paper we denote the set $\{1, 2, \ldots, k\}$. The Pareto set $Y_P$ is the set of all Pareto optimal solutions.

Similar to Papadimitriou and Yannakakis [20], we will assume throughout this paper that for any instance $I$, we can compute an exponential bound on the objective values of all solutions, i.e., a number $M > 0$ such that $Y \subseteq [0, M]^k$ and such that there is a polynomial $\pi$ with $M \leq 2^{\pi(|I|)}$, where $|I|$ is the encoding length of the instance. This is not a major restriction in usual discrete optimization problems.

Reference point methods. To model the decision maker’s preferences, reference point methods take two types of additional input: a reference point $y^{rp} \in \mathbb{Z}_{\geq 0}^k$ and a weight vector $\lambda \in \mathbb{Q}_{\geq 0}^k$ on the objectives. The reference point is a—usually unattainable—vector of aspired values for each criterion. The weights are used to adjust a fixed norm $\|\cdot\|$ on $\mathbb{R}^k$ by letting $\|\cdot\|^{\lambda}$ be the norm defined by $\|y\|^{\lambda} := \|\lambda_1 y_1, \ldots, \lambda_k y_k\|$.

The goal is to find a solution that is as close as possible to the reference point w.r.t. $\|\cdot\|^{\lambda}$. Conceive of this distance as the price to pay to attain a compromise among the criteria. The
objective value of an optimal reference point solution is the value of the reference point, degraded by the price of compromise. For minimization, the reference point objective function thus reads:

\[ r_{y^{rp},\lambda}(y) = \|y^{rp}\|^{\lambda} + \|y - y^{rp}\|^{\lambda}. \]

Of particular interest in this context is the ideal point \( y^{id} \in \mathbb{Z}_0^k \), which is defined as the point in the objective space obtained by optimizing each objective individually, i.e., \( y^{id}_i := \min_{y \in \mathcal{Y}} y_i \). Throughout this paper, we will restrict ourselves to reference points \( y^{rp} \) with \( y^{rp} \leq y^{id} \). We call these points feasible reference points.

Formally, we define the problem of reference point solutions, \( \text{RP}((\mathcal{P}, \|\cdot\|)) \) for short, as follows: Given an instance of \( \mathcal{P} \), a feasible reference point \( y^{rp} \in \mathbb{Z}_0^k \), and a weight vector \( \lambda \in \mathbb{Q}_{\geq 0}^k \) as input, find a solution \( x \in \mathcal{X} \) that minimizes \( r_{y^{rp},\lambda}(c(x)) \). Given the particular interest of the ideal point, we will also consider the problem \( \text{CP}((\mathcal{P}, \|\cdot\|)) \), which is known as compromise programming: Given an instance of \( \mathcal{P} \) and \( \lambda \in \mathbb{Q}_{\geq 0}^k \), find a solution \( x \in \mathcal{X} \) that minimizes \( r_{y^{id},\lambda}(c(x)) \).

The constant in the objective. As \( \|y^{rp}\| \) is a constant, exact minimization of \( r(y) \) boils down to minimizing the distance \( \|y - y^{rp}\| \), as the level sets of this function are identical to that of the reference point objective function. Still, for judging the quality of an approximation, this short-cut is not permissible, as the following trivial example shows. Consider a multicriteria problem defined by \( k \) unrelated copies of a single criteria optimization problem, for which we have a tight approximation algorithm with factor \( \alpha \). Let the distance be measured in any norm, and choose the ideal point as a reference point. As the single criteria problems are unrelated, one expects that solving each problem separately by the approximation algorithm gives an \( O(\alpha) \)-approximation for the reference point solution. This is indeed true for the reference point objective function. However, for minimizing the distance, the ratio to the optimum is infinite, because the optimum attains the ideal point for the unrelated problems.

Conversely, any approximation algorithm for the distance \( \|y - y^{rp}\| \) could be turned into an algorithm that solves the single-criterion problem exactly (as the minimal distance to the ideal point is 0 when focusing on a single criterion). Thus, we can not hope for approximating the distance \( \|y - y^{rp}\| \) for any problem that is NP-hard in the single criterion version. In contrast to that, for the objective \( r(y) \) we do get positive approximation results also for NP-hard problems.

Caveat on complexity. Note that although the concept of reference point solutions is a generalization of compromise solutions, in terms of complexity \( \text{CP} \) is not a special case of \( \text{RP} \). In the former problem, the ideal point is not given, while in the latter case the reference point is given in the input. This leads to different consequences if the underlying single-criterion problem can not be solved in polynomial time. In this case, the objective function of \( \text{CP} \) is hard to evaluate. However, in the context of approximability this is only a minor issue, as Corollary 3.5 shows. For \( \text{RP} \), on the other hand, it becomes hard to verify feasibility of the input (i.e., checking whether \( y^{rp} \leq y^{id} \)). The best we can expect from an algorithm is to approximately distinguish between feasible and infeasible instances, i.e., an \( \alpha \)-approximation algorithm needs to accept all feasible inputs and reject all instances where \( y^{rp}_i > \alpha \cdot y^{id}_i \) for some \( i \in [k] \), but it might also accept instances with slightly infeasible reference points, as long as \( y^{rp} \leq \alpha y^{id} \).

Norms. Throughout this paper, we will restrict to norms fulfilling the following two properties. A norm \( \|\cdot\| \) is called monotone, if \( y' \leq y'' \) implies \( \|y'\| \leq \|y''\| \) for any \( y', y'' \in \mathbb{R}_0^k \). It is called polynomially decidable, if we can decide whether \( \|y'\| \leq \|y''\| \) in time polynomial in the encoding length of \( y' \) and \( y'' \).

We will mainly use the following families of norms: the infinity-norm \( \|y\|_{\infty} := \max_i |y_i| \) (which we will sometimes also denote by \( \langle y \rangle_{\infty} \) for convenience), the standard \( l^p \)-norm \( \|y\|_p := \)
(\sum_i |y_i|^p)_{\frac{1}{p}}$, and the cornered $p$-norm \( \langle \langle y \rangle \rangle_p \) := \max_i y_i + \frac{1}{p} \sum_i |y_i| \) (both for $p \geq 1$). The cornered norm has been considered in the context of compromise programming before, e.g. by Gearhart \cite{gearhart1984}. Our motivation to use this norm is twofold. Firstly, for general values of $p$, it will be hard to minimize a distance measured in the $\ell^p$-norm because of the exponents. The cornered $p$-norms are simpler, but still have properties similar to the $\ell^p$-norms: Their unit spheres are nested within each other, and for increasing values of $p$ they approach the axis parallel square. This allows to control the degree of balancing of the criteria in the reference point solution. Secondly, the infinity-norm (often referred to as Chebyshev-norm in this context) is very popular in MCDM-tools. Often it is augmented by a small linear term to avoid weakly Pareto optimal solutions (cf. Choo and Steuer \cite{choo1993}), similar to the addition of the term \( \frac{1}{p} \|y\|_p \).

Note that all $\ell^p$- and cornered $p$-norms are monotone and polynomially decidable.

Approximation of the Pareto set. We extend the well-known concept of approximation algorithms for the single-objective case to approximability of the Pareto set in a similar way as done in Papadimitriou and Yannakakis \cite{papadimitriou1993}, with the slight difference of including constant factor approximations. For $\alpha > 1$, an $\alpha$-approximate Pareto set is a set $Y_\alpha \subseteq Y$ such that for all $y \in Y_P$ there is $y' \in Y_\alpha$ with $y' \leq \alpha y$. An $\alpha$-approximation algorithm for the Pareto set is an algorithm that constructs an $\alpha$-approximate Pareto set in time polynomial in the encoding length of the instance of $P$. An FPTAS for the Pareto set is a family of algorithms that, for all $\varepsilon > 0$, contains a $(1 + \varepsilon)$-approximation algorithm for the Pareto set with running time polynomial in $\frac{1}{\varepsilon}$ and the encoding length of the instance.

3 Equivalence of Approximation

In this section, we investigate the relation between approximation of the Pareto set, reference point methods, and compromise programming. Our main theorem states these three notions of approximability are essentially equivalent: A constant approximation factor for one of these problems implies constant (although possibly different) approximation factors for the others, and the same is true for approximation schemes.

**Theorem 3.1.** Let $P$ be a multicriteria discrete minimization problem. The following statements are equivalent.

- There is a constant factor approximation (FPTAS, respectively) for the Pareto set of $P$.
- There is a constant factor approximation (FPTAS, respectively) for $\text{RP}(P, \|\cdot\|)$ for every monotone and polynomially decidable norm $\|\cdot\|$.
- There is a constant factor approximation (FPTAS, respectively) for $\text{RP}(P, \|\cdot\|_\infty)$.
- There is a family of algorithms that, for each $p \geq 1$, contains a constant factor approximation (FPTAS, respectively) for $\text{RP}(P, \|\cdot\|_p)$ or $\text{RP}(P, \langle \langle \cdot \rangle \rangle_p)$, and the running time of all algorithms is bounded by a polynomial in the input size and $\log(p)$.
- There is a constant factor approximation (FPTAS, respectively) for $\text{CP}(P, \|\cdot\|)$ for every monotone and polynomially decidable norm $\|\cdot\|$.
- There is a constant factor approximation (FPTAS, respectively) for $\text{CP}(P, \|\cdot\|_\infty)$.
- There is a family of algorithms that, for each $p \geq 1$, contains a constant factor approximation (FPTAS, respectively) for $\text{CP}(P, \|\cdot\|_p)$ or $\text{CP}(P, \langle \langle \cdot \rangle \rangle_p)$, and the running time of all algorithms is bounded by a polynomial in the input size and $\log(p)$.
Reference point solutions and the Pareto set. Gearhardt [11] showed that for both the \( \ell^p \)-norm and the cornered \( p \)-norm, if \( p \) tends to infinity, the distance between the Pareto set and the set of compromise solutions with respect to all non-negative normalized weight vectors tends to zero. This means that for discrete optimization problems with a finite set of feasible solutions, there is a finite value \( p_0 \) for which the two sets coincide. We show that, under the assumption that there is an exponential bound on the objectives, also \( p \) can be chosen in such a way that it is polynomially encodable.

**Theorem 3.2.** Let \( y^{rp} \in \mathbb{Z}^k_{\geq 0} \) be a feasible reference point. If the objective vector set \( Y \) is contained in \([0, M]^k\), then the following statements hold true.

1. If \( p > \frac{\log k}{\log(1 + \frac{1}{M})} \), then for any Pareto optimal solution \( y \in Y \) there is a weight vector \( \lambda \in \mathbb{Q}^k_{\geq 0} \) such that \( y \) minimizes \( \|y - y^{rp}\|^\lambda_p \).

2. If \( p > kM \), then for any Pareto optimal solution \( y \in Y \) there is a weight vector \( \lambda \in \mathbb{Q}^k_{\geq 0} \) such that \( y \) minimizes \( \langle \langle y - y^{rp} \rangle \rangle^\lambda_p \).

**Proof.** We first consider the cornered norm \( \langle \langle \cdot \rangle \rangle^\lambda_p = \max_{i \in [k]} \{ \lambda_i y_i \} + \frac{1}{p} \sum_{i \in [k]} \lambda_i y_i \). Therefore, let \( p > kM \). Further let \( y \in Y \) be a Pareto optimal cost vector, and let \( I := \{ i \in [k] : y_i = y_i^{rp} \} \).

We set the weight vector \( \lambda \) as follows:

\[
\lambda_i = \begin{cases} 
1 + k & \text{if } i \in I, \\
\frac{1}{y_i - y_i^{rp}} & \text{otherwise}.
\end{cases}
\]

The weighted distance of \( y \) to the reference point is

\[
\langle \langle y - y^{rp} \rangle \rangle^\lambda_p = \max_{i \notin I} \{ \lambda_i(y_i - y_i^{rp}) \} + \frac{1}{p} \sum_{i \notin I} \lambda_i(y_i - y_i^{rp}) = 1 + \frac{1}{p} (k - |I|) \leq 1 + \frac{k}{p}.
\]
Consider any \( y' \in \mathcal{Y} \setminus \{ y \} \). If there is an index \( j \in I \) with \( y'_j > y_j = y^P_j \), then, since \( \mathcal{Y} \subseteq \mathbb{Z}^k \), we know that \( y'_j - y^P_j \geq 1 \), and therefore
\[
\langle y' - y^P \rangle_p^\lambda = \frac{\lambda_j(y'_j - y^P_j)}{y_j - y^P_j} = 1 + \frac{1}{p} \lambda_j > 1 + k > 1 + \frac{k}{p},
\]
so in this case \( y \) is closer to \( y^P \) than \( y' \).

Otherwise, since \( y \) is Pareto optimal, there is some \( j \in [k] \) such that \( y'_j < y_j \), or with integrality, \( y'_j - y_j \geq 1 \). On the other hand, we know that \( y_j - y^P_j \leq M \). Therefore for this index \( j \),
\[
\lambda_j(y'_j - y^P_j) = \frac{y'_j - y^P_j}{y_j - y^P_j} = 1 + \frac{1}{M},
\]
and as a consequence
\[
\langle y' - y^P \rangle_p^\lambda \geq \max_{i \in [k]} \{ \lambda_i(y'_i - y^P_i) \} \geq 1 + \frac{1}{M} > 1 + \frac{k}{p},
\]
so again \( y \) is closer to \( y^P \) than \( y' \).

For the \( \ell^p \)-norm we let \( p > \frac{\log k}{\log(1 + \frac{1}{k})} \), and set the weight vector \( \lambda \) as before. We get
\[
\left( \| y - y^P \|_p^\lambda \right)^p = \sum_{i \in I} \left( \frac{y_i - y^P_i}{y^P_i} \right)^p = k - |I| \leq k.
\]

If there is a \( j \in I \) with \( y'_j > y_j = y^P_j \), then
\[
\left( \| y' - y^P \|_p^\lambda \right)^p \geq \left( \lambda_j(y'_j - y^P_j) \right)^p \geq \lambda_j^p > k.
\]

Otherwise, with the same choice of \( j \in [k] \) as above,
\[
\left( \| y' - y^P \|_p^\lambda \right)^p \geq \left( \lambda_j(y'_j - y^P_j) \right)^p \geq \left( 1 + \frac{1}{M} \right)^p > k,
\]
where the last inequality holds by the choice of \( p \). Thus again in both cases \( y \) is closer to \( y^P \) than \( y' \), completing the proof. \( \square \)

From approximate Pareto sets to approximating reference point solutions. We start the proof of Theorem 3.1 by showing that from an \( \alpha \)-approximate Pareto set we can always choose an \( \alpha \)-approximate solution to \( \mathcal{P} \).

Lemma 3.3. Let \( y^P \) be a feasible reference point, and let \( \mathcal{Y}_\alpha \) be an \( \alpha \)-approximate Pareto set of \( \mathcal{P} \). Then for any monotone norm \( \| \cdot \| \), \( \min_{y \in \mathcal{Y}_\alpha} r(y) \leq \alpha \cdot \min_{y \in \mathcal{Y}} r(y) \), where \( r(y) = \| y^P \| + \| y - y^P \| \).

Proof. Let \( y^* \in \mathcal{Y} \) be an optimal solution to \( \min_{y \in \mathcal{Y}} r(y) \). By monotonicity, we can w.l.o.g. assume \( y^* \) to be Pareto optimal. Thus, there is \( y' \in \mathcal{Y}_\alpha \) such that \( y' \leq \alpha y^* \). Using monotonicity and triangle inequality, we get
\[
\| y' - y^P \| \leq \| \alpha(y^* - y^P) + (\alpha - 1)y^P \| \leq \alpha \| y^* - y^P \| + (\alpha - 1)\| y^P \|. \]

Reformulation yields
\[
\min_{y \in \mathcal{Y}_\alpha} r(y) \leq r(y') = \| y^P \| + \| y' - y^P \| \leq \alpha(\| y^P \| + \| y^* - y^P \|) = \alpha r(y^*). \] \( \square \)
Corollary 3.4. If there is an \( \alpha \)-approximation algorithm for the Pareto set of \( \mathcal{P} \), then there is an \( \alpha \)-approximation for \( \text{RP}(\mathcal{P}, \| \cdot \|) \), for every monotone and polynomially decidable norm \( \| \cdot \| \).

Remark 1. Corollary 3.4 can also be proved using a result by Mittal and Schulz [16], showing how to use an \( \alpha \)-approximate Pareto set in order to obtain an \( \alpha' \)-approximation algorithm for any monotone low-rank function \( h : \mathcal{Y} \to \mathbb{R}_{\geq 0} \) fulfilling \( h(\mu y) \leq \mu^c h(y) \) for some constant \( c > 0 \), all \( y \in \mathcal{Y} \), and all \( \mu > 1 \). Indeed, it can be shown that the reference point objective function \( r \) fulfills this requirement with \( c = 1 \) for every monotone norm. Since showing this property is not less effort than proving the result directly, and for reasons of self-containment, we have included the direct proof here.

In fact, we can also approximate the compromise solution without knowing the exact ideal point: As \( \mathcal{Y}_\alpha \) contains an \( \alpha \)-approximate optimal solution for each objective, we can obtain a reference point \( y^{\text{RP}} \) with \( \frac{1}{\alpha} y^{\text{id}} \leq y^{\text{RP}} \leq y^{\text{id}} \). By choosing the point closest to \( y^{\text{RP}} \) from \( \mathcal{Y}_\alpha \) we get an \( \alpha^2 \)-approximation to the compromise solution.

Corollary 3.5. If there is an \( \alpha \)-approximation algorithm for the Pareto set of \( \mathcal{P} \), then there is an \( \alpha^2 \)-approximation algorithm for \( \text{CP}(\mathcal{P}, \| \cdot \|) \), for every monotone and polynomially decidable norm \( \| \cdot \| \).

Proof. Let \( \mathcal{Y}_\alpha \) be an \( \alpha \)-approximation to the Pareto set. Observe that

\[
y_i^{\text{RP}} := \left\lfloor \frac{1}{\alpha} \min_{y \in \mathcal{Y}_\alpha} y_i \right\rfloor
\]

yields a feasible reference point with \( y^{\text{RP}} \leq y^{\text{id}} \leq \alpha y^{\text{RP}} \).

Now let \( y^* := \arg\min_{y \in \mathcal{Y}} \| y^{\text{RP}} \| + \| y - y^{\text{RP}} \| \), which by Corollary 3.4 is an \( \alpha \)-approximation to the reference point solution for \( y^{\text{RP}} \). Thus, for the compromise solution \( y^* \), we get

\[
\| y^* - y^{\text{id}} \| \leq \| y^* - y^{\text{RP}} \| \leq \alpha \| y^* - y^{\text{RP}} \| + (\alpha - 1) \| y^{\text{RP}} \|
\]

where the first inequality follows from monotonicity. Observe that \( y^{\text{id}} - y^{\text{RP}} \leq (\alpha - 1) y^{\text{RP}} \) and thus, again by monotonicity,

\[
\| y^* - y^{\text{RP}} \| \leq \| y^* - y^{\text{id}} \| + (\alpha - 1) \| y^{\text{RP}} \|
\]

This finally yields

\[
\| y^{\text{id}} \| + \| y^* - y^{\text{id}} \| \leq \| y^{\text{id}} \| + \alpha \| y^* - y^{\text{id}} \| + (\alpha - 1) \| y^{\text{RP}} \| + (\alpha - 1) \| y^{\text{RP}} \|
\]

\[
\leq \alpha^2 \| y^{\text{id}} \| + \alpha \| y^* - y^{\text{id}} \| \quad \Box
\]

From approximating reference point solutions to an approximate Pareto set. In order to show the converse of the result proven above, we use a characterization from Papadimitriou and Yannakakis [20], stating that approximability of the Pareto set is equivalent to tractability of the so-called GAP problem.

Definition 3.6 (GAP Problem). Given an instance of \( \mathcal{P} \) and a vector \( y \in \mathbb{Q}_+^k \) as input, the GAP problem for approximation factor \( \alpha > 1 \), denoted by \( \text{GAP}(\mathcal{P}, \alpha) \), is to find a solution \( y' \in \mathcal{Y} \) with \( y' \leq y \), or to guarantee that there is no solution \( y'' \in \mathcal{Y} \) with \( y'' \leq \frac{1}{\alpha} y \).

Theorem 3.7 (Papadimitriou & Yannakakis, 2000). Let \( \mathcal{P} \) be a multicriteria discrete minimization problem, and let \( \alpha > 1 \). If there is an \( \alpha \)-approximation algorithm for the Pareto set, then \( \text{GAP}(\mathcal{P}, \alpha) \) is solvable in polynomial time. If \( \text{GAP}(\mathcal{P}, \alpha) \) is solvable in polynomial time, then there is an \( \alpha^2 \)-approximation algorithm for the Pareto set.
We now show how to use an approximation algorithm for RP to solve the GAP problem with a slight increase in the approximation factor. In fact, our result does not even require the algorithm to solve RP for arbitrary reference points. It suffices to find a particular reference point, on an instance-by-instance basis, that can be approximated. We formalize this by introducing two algorithms, the first acting as an oracle computing a suitable reference point, which then can be approximated by the second algorithm.\footnote{Note that it is not sufficient for the first algorithm to simply return a trivial feasible reference point such as 0, as it has to ensure that the second algorithm can provide an approximation for this point. E.g., in the proof of Corollary 3.9, it needs to return a point close to the ideal point.}

**Lemma 3.8.** Let $\alpha > 1$ and set $\beta := \frac{\alpha^2}{2(\alpha - 1)}$. There is a polynomial time algorithm for GAP($\mathcal{P}, \alpha$), if there are two polynomial time algorithms $A_1, A_2$ such that,

- given an instance of $\mathcal{P}$, algorithm $A_1$ computes a feasible reference point $y^{\text{ref}} \in \mathbb{Z}^k_{\geq 0}$ for that instance, and,
- additionally given $y^{\text{ref}}$ and $\lambda \in \mathbb{Q}_{\geq 0}$, algorithm $A_2$ computes in polynomial time a solution $y' \in \mathcal{Y}$ with $r(y') \leq \beta \min_{z \in \mathcal{Y}} r(z)$, for $r(z) := \|y^{\text{ref}}\|_{\infty}^\lambda + \|z - y^{\text{ref}}\|_{\infty}^\lambda$.

**Proof.** Let $y \in \mathbb{Q}^k_{\geq 0}$ be the input to the GAP problem. W.l.o.g., we can assume that $y \geq \alpha y^{\text{ref}}$ for the reference point $y^{\text{ref}}$ computed by $A_1$, as otherwise there is no $y' \leq \frac{1}{\alpha} y$ and GAP can be answered negatively.

We will solve the GAP problem with a single call of the $\beta$-approximation algorithm for RP($\mathcal{P}, \|\cdot\|_{\infty}$). For $i \in [k]$, let $\lambda_i := \frac{1}{y_i - y_i^{\text{ref}}}$ if $y_i > y_i^{\text{ref}}$, and $\lambda_i := 2$ if $y_i = y_i^{\text{ref}} = 0$. Let $y'$ be a $\beta$-approximation to $\min_{z \in \mathcal{Y}} r(z)$.

If $r(y') \leq r(y)$, we return $y'$ as answer to the GAP problem:

$$
\lambda_i(y_i - y_i^{\text{ref}}) \leq \|y' - y^{\text{ref}}\|_{\infty}^\lambda \leq \|y - y^{\text{ref}}\|_{\infty}^\lambda \leq 1
$$

for all $i \in [k]$ by choice of the weights. Dividing by $\lambda_i$ yields $y_i' \leq y_i$ if $y_i > 0$, or $y_i' \leq \frac{1}{2}$ if $y_i = 0$. In the latter case, integrality of $y_i'$ implies $y_i' = 0$.

If $r(y') > r(y)$, we answer GAP negatively: Let $y'' \in \mathcal{Y}$. We show there is an $i \in [k]$ with $y_i'' > \frac{1}{\alpha} y_i$. First observe that $\beta r(y'') \geq r(y') > r(y)$, which implies

$$
\beta \|y'' - y^{\text{ref}}\|_{\infty}^\lambda > \|y - y^{\text{ref}}\|_{\infty}^\lambda - (\beta - 1)\|y^{\text{ref}}\|_{\infty}^\lambda.
$$

Substituting the weights and using $y \geq \alpha y^{\text{ref}}$ yields

$$
\beta \frac{y_i'' - y_i^{\text{ref}}}{y_i - y_i^{\text{ref}}} > \frac{y_j - y_j^{\text{ref}}}{y_j - y_j^{\text{ref}}} - (\beta - 1) \frac{y_j^{\text{ref}}}{y_j' - y_j^{\text{ref}}} \geq 1 - \frac{\beta - 1}{\alpha - 1},
$$

with $i, j, j'$ being the indices of those components attaining the maxima in the norms. (If either of the denominators is 0, then $y_i'' > \frac{1}{\alpha} y_i$ follows directly.) Using the fact that $1 < \beta \leq \alpha$, we get

$$
\beta y_i'' > \frac{1}{\alpha} (1 - \frac{\beta - 1}{\alpha - 1})(y_i - y_i^{\text{ref}}) + \beta y_i^{\text{ref}} \geq (1 - \frac{\beta - 1}{\alpha - 1})y_i.
$$

It is easy to verify that $\beta = \frac{\alpha^2}{2(\alpha - 1)}$ now implies $y_i'' > \frac{1}{\alpha} y_i$ and the negative answer to the GAP problem is correct. $\square$

As a particular application of Lemma 3.8, we can show now that also an approximation to CP suffices to approximate the Pareto set:

**Corollary 3.9.** Let $\alpha > 1$ and set $\beta := \sqrt{\frac{\alpha^2}{2(\alpha - 1)}}$. There is a polynomial time algorithm for GAP($\mathcal{P}, \alpha$), if there is a $\beta$-approximation algorithm for CP($\mathcal{P}, \|\cdot\|_{\infty}$).
Proof. We show that algorithm $A_1$ and $A_2$ exist, as required by Lemma 3.8.

Algorithm $A_1$: For every $i \in [k]$, let $y^{(i)}$ be a $\beta$-approximation to $\min_{z \in \mathcal{Y}} r_{y^{(i)},\lambda}(z)$ for weights $\bar{\lambda}_i = 1$ and $\bar{\lambda}_j = 0$ for $j \in [k] \setminus \{i\}$. Then $y_{i}^{\text{FP}} := \left\lfloor \frac{1}{\beta} y^{(i)} \right\rfloor$ defines a feasible reference point with $y_{i}^{\text{FP}} \leq y_{i}^{\text{id}} \leq \beta y_{i}^{\text{FP}}$.

Algorithm $A_2$: Let $\lambda \in \mathbb{Q}^k$. Let $y'$ be a $\beta$-approximation to $\min_{z \in \mathcal{Y}} r_{y',\lambda}(z)$, and let $y^* = \arg\min_{z \in \mathcal{Y}} r_{y^*,\lambda}(z)$ be an optimal solution to RP. We show that $r_{y',\lambda}(y') \leq \beta^2 r_{y^*,\lambda}(y^*)$, which concludes the proof.

\[
\|y^\text{FP}\|_\infty + \|y' - y^\text{FP}\|_\infty \leq \|y^\text{FP}\|_\infty + \|y' - y^\text{id}\|_\infty + \|y^\text{id} - y^\text{FP}\|_\infty \\
\leq \beta\|y^\text{FP}\|_\infty + \|y' - y^\text{id}\|_\infty \\
\leq \beta\|y^\text{FP}\|_\infty + \beta\|y^* - y^\text{id}\|_\infty + (\beta - 1)\|y^\text{id}\|_\infty \\
\leq \beta^2\|y^\text{FP}\|_\infty + \beta\|y^* - y^\text{id}\|_\infty .
\]

Corresponding versions of Lemma 3.8 and Corollary 3.9 with the same approximation factors can be shown for the $\|\cdot\|_p$ and $\langle\cdot\rangle_p$-norms. These results have been moved to the appendix.

Remark 2. In order to show the result for approximation schemes, let $\alpha = 1 + \varepsilon$ and $\beta = 1 + \delta$. In both cases it suffices to choose $\delta$ such that $1/\delta \in \mathcal{O}(1/\varepsilon^2)$, maintaining polynomiality in $1/\delta$.

## 4 Approximating Reference Point Solutions

In this section, we discuss several general techniques for obtaining approximation algorithms for RP$(\mathcal{P}, \|\cdot\|)$). We start with a simple constant factor approximation based on the weighted sum method, then turn our attention to convex optimization and LP rounding, and close with approximation schemes arising from pseudopolynomial algorithms.

**Approximation by weighted sum.** Although not all Pareto optimal solutions can be reached by minimizing a weighted sum, this method still provides an easy way to transfer approximability results from the single-criterion world to reference point methods.

In [8], Diakonikolas and Yannakakis show that an approximate convex Pareto set can be computed, if the weighted sum can be optimized. Our results show that via reference point solutions, also the Pareto set can be approximated. The approximation factor, however, increases by a factor of $k$.

**Theorem 4.1.** If there is an $\alpha$-approximation for $\min_{y \in \mathcal{Y}} \lambda^T y$, then there is a $k\alpha$-approximation for RP$(\mathcal{P}, \|\cdot\|_\infty)$.

**Proof.** Let $y^\text{FP}$ be a feasible reference point and $\lambda \in \mathbb{Q}_+^k$. Let $y^* = \arg\min_{y \in \mathcal{Y}} r_{y^*,\lambda}(y)$ and let $y' \in \mathcal{Y}$ be an $\alpha$-approximation to $\min_{y \in \mathcal{Y}} \lambda^T y$. Then

\[
\|y^\text{FP}\|_\infty^\lambda + \|y' - y^\text{FP}\|_\infty^\lambda \leq \|y^\text{FP}\|_\infty^\lambda + \lambda^T (y' - y^\text{FP}) \\
\leq \|y^\text{FP}\|_\infty^\lambda + \alpha \lambda^T y^* - \lambda^T y^\text{FP} \\
\leq \|y^\text{FP}\|_\infty^\lambda + \alpha \lambda^T (y^* - y^\text{FP}) + (\alpha - 1)\lambda^T y^\text{FP} \\
\leq \|y^\text{FP}\|_\infty^\lambda + \alpha k\|y^* - y^\text{FP}\|_\infty^\lambda + (\alpha - 1)\|y^\text{FP}\|_\infty^\lambda \\
\leq k\alpha(\|y^\text{FP}\|_\infty^\lambda + \|y^* - y^\text{FP}\|_\infty^\lambda) .
\]

In combination with Theorem 3.1, this implies the following result.

**Corollary 4.2.** For any multicriteria combinatorial minimization problem $\mathcal{P}$ with a constant number of linear objectives, there is a constant factor approximation for the Pareto set of $\mathcal{P}$, if and only if there is a constant factor approximation for the single-criterion version of $\mathcal{P}$.
Convex optimization with linear objectives. For optimization problems where the solution space is convex and the objectives are linear (e.g. linear programming), we can compute reference point solutions w.r.t. the cornered norm exactly:

**Theorem 4.3** (Reference point solutions for convex optimization). For a multicriteria minimization problem \( \min_{x \in X} Cx \), with a convex solution set \( X \subseteq \mathbb{R}^n \) for which a polynomial separation algorithm exists, and a cost matrix \( C \in \mathbb{Q}^{k \times n} \), the problem \( \min_{x \in X} r(Cx) \) with \( r(y) = \langle y^p \rangle_p + \langle y - y^p \rangle_p \), for any feasible reference point \( y^p \) and any \( p \in [1, \infty] \), is again a convex optimization problem with linear objectives and thus solvable in polynomial time.

**Proof.** The problem can be formulated as follows:

\[
\min_{x \in X} r(Cx) = \|y^p\|_\infty + \left\{ \begin{array}{ll}
\min & \Delta + \frac{1}{p} \cdot 1^t Cx \\
\text{s.t.} & Cx - y^p \leq \Delta \cdot 1 \\
& x \in X \\
& \Delta \in \mathbb{R}.
\end{array} \right.
\]

Here, \( 1 \) denotes the vector of ones of corresponding dimension. In the optimum, \( \Delta = \max_i \{ c_i x - y^p_i \} \), and therefore the two programs are equivalent. The objective is clearly linear, and the solution space is \( X \times \mathbb{R} \), intersected with the halfspaces defined by the inequalities \( c_i x - y^p_i \leq \Delta, \) \( i \in [k] \), and thus convex.

Since we can solve the separation problem for the original set \( X \), we can also solve it for the set with the added inequalities. By the equivalence of separation and optimization (Grötschel et al. [12]) we can solve \( \min_{x \in X} r(Cx) \) in polynomial time. \( \square \)

**Remark 3.** A special case of convex optimization problems are linear programs (LPs). From our result it follows that we can exactly compute reference point solutions for multicriteria LPs. It also yields a nice alternative proof of the existence of an FPTAS for the Pareto set, which has first been proven in Papadimitriou and Yannakakis [20] using an involved geometric argument.

A different argument for the approximability of Pareto sets of linear programs has independently been noted by Mittal and Schulz [17].

**Corollary 4.4.** Let \( \mathcal{P} \) be a multicriteria minimization problem with convex feasible set and linear objective functions. Assume that there is a positive polynomial \( \pi \) such that \( \mathcal{Y} \subseteq \{ y \in \mathbb{Q}^k : y_i \geq \frac{1}{\pi(|I|)} \forall i \in [k] \} \), where \( |I| \) is the encoding length of the instance. If there is a polynomial time algorithm for the separation problem of \( \mathcal{P} \), then there is an FPTAS for the Pareto set.

**Remark 4** (Convex sets and the integrality assumption). Note that our general integrality assumption \( \mathcal{Y} \subseteq \mathbb{Z}_0^k \) for discrete optimization problems, introduced in Section 2, does not hold for the case of convex optimization problems in Theorem 4.3 and Corollary 4.4. However, by assuming \( y_i \geq \frac{1}{\pi(|I|)} \) for all occurring objective values in Corollary 4.4, we ensure that all prerequisites stated in Papadimitriou and Yannakakis [20] for Theorem 3.7 are still fulfilled. Furthermore observe that, while our proof of Lemma 3.8 also assumed integral objectives, we used this integrality assumption only for showing that if the solution \( y' \) computed by algorithm \( A_2 \) fulfills \( r(y') \leq r(y) \), then \( y_i = 0 \) implies \( y'_i = 0 \). However, we can ignore this case, as by our assumption all objectives are strictly positive and thus \( y_i = 0 \) already implies that the answer to GAP is negative. Thus, both Lemma 3.8 and Theorem 3.7 are still valid for convex optimization problems fulfilling the condition of Corollary 4.4.

**Proof of Corollary 4.4.** By Theorem 4.3, we can compute an optimal solution to \( \text{RP}(\mathcal{P}, \|\cdot\|_\infty) \) for any reference point in polynomial time. Thus, by Lemma 3.8, we can solve \( \text{Gap}(\mathcal{P}, 1 + \varepsilon) \) in polynomial time for any \( \varepsilon > 0 \) (with running time independent of \( \varepsilon \)), which by Theorem 3.7 gives an FPTAS for the Pareto set. \( \square \)
Approximation through LP rounding. One of the most successful techniques for the design of approximation algorithms for integer problems is LP rounding: The problem is formulated as a linear integer program (IP), then the integrality constraints are relaxed and the resulting LP is solved, and finally the optimal fractional solution is rounded to a feasible integral solution, losing only a certain factor in the objective.

Many important LP rounding algorithms are oblivious in the sense that the rounding procedure is independent of the cost function. We show that these algorithms can be adapted such that they also solve the reference point version of the problem, with the same approximation factor.

**Theorem 4.5.** Consider a multicriteria minimization problem \( \min_{x \in X} Cx \) with a solution set \( X \subseteq \mathbb{Z}_{\geq 0}^n \) and a cost matrix \( C \in \mathbb{Q}^{k \times n} \). If there exist

- a convex relaxation \( X' \) for which the separation problem can be solved in polynomial time,

- and a polynomial time rounding procedure \( R : X' \to X \) such that \( c^TR(x') \leq \alpha c^T x' \) for all \( c \in \mathbb{Q}_{\geq 0}^n \) and all \( x' \in X' \),

then for any feasible reference point \( y^{\text{RP}} \) and any \( p \in [1, \infty] \) there is an \( \alpha \)-approximation algorithm for \( \min_{x \in X} r(Cx) \), with \( r(y) = \langle y^{\text{RP}} \rangle_p + \langle y - y^{\text{RP}} \rangle_p \).

**Proof.** From Theorem 4.3 it follows that we can compute in polynomial time a fractional solution \( x' \in X' \) minimizing \( r(Cx) \). Let \( x = R(x') \). Then

\[
  r(Cx) = \max_{i \in [k]} \{y^{\text{RP}}_i\} + \max_{i \in [k]} \{(Cx)_i - y^{\text{RP}}_i\} + \frac{1}{p} \sum_{i \in [k]} (Cx)_i
  \leq \max_{i \in [k]} \{y^{\text{RP}}_i\} + \max_{i \in [k]} \{\alpha(Cx')_i - y^{\text{RP}}_i\} + \alpha \cdot \frac{1}{p} \cdot \sum_{i \in [k]} (Cx')_i
  = \max_{i \in [k]} \{y^{\text{RP}}_i\} + \max_{i \in [k]} \{\alpha((Cx')_i - y^{\text{RP}}_i) + (\alpha - 1)y^{\text{RP}}_i\} + \alpha \cdot \frac{1}{p} \cdot \sum_{i \in [k]} (Cx')_i
  \leq \alpha \cdot \max_{i \in [k]} \{y^{\text{RP}}_i\} + \alpha \cdot \max_{i \in [k]} \{(Cx')_i - y^{\text{RP}}_i\} + \alpha \cdot \frac{1}{p} \cdot \sum_{i \in [k]} (Cx')_i
  = \alpha \cdot r(Cx') .
\]

Theorem 4.5 immediately results in the approximability, with a factor independent of \( k \), of reference point solutions and the Pareto set for several classical combinatorial optimization problems. We give two examples here.

For **Set Cover**, in 1982 Hochbaum [15] presented an LP-based \( \kappa \)-approximation algorithm, where \( \kappa \) is the maximum cardinality of a set. Thus, there is a \( \kappa \)-approximation algorithm for the corresponding reference point version, and a \( O(\kappa^2) \)-approximation algorithm for the Pareto set. A notable special case is **Vertex Cover**, where \( \kappa = 2 \).

For the scheduling problem of minimizing the weighted sum of completion times on a single machine with release dates \((1|r_j\sum w_jC_j), \) Hall et al. [13] gave a 3-approximation algorithm based on an LP-relaxation, resulting in a 3-approximation for compromise solutions, which gives a constant factor approximation for the Pareto set as well. Möhring et al. [18] extended this to stochastic scheduling with random processing times \((P|p_j \sim \text{stoch}, r_j|E[\sum w_jC_j]), \) for which we consequently also get constant factor approximations for the multicriteria problems.

**Remark 5.** While we usually restrict ourselves to the case of a constant number of criteria, the results on convex optimization and LP-rounding also hold for a polynomial number of criteria. This is due to the fact that we can still solve the linear program if we add a polynomial number of constraints.
From pseudopolynomial algorithms to approximation schemes. Multicriteria optimization, and in particular the concept of compromise solutions, is closely related to robust optimization. If each criterion is considered as one scenario in the robust setting, then a compromise solution w.r.t. $\|\cdot\|_\infty$ is exactly the same as a min-max regret robust solution.

Assi et al. [2] consider this robust setting for binary optimization problems and show that if we can compute upper and lower bounds on the optimum which only differ by a polynomial factor, and if there is a pseudopolynomial algorithm whose running time depends on the encoding length of the instance and the upper bound, then there is an FPTAS for the min-max regret robust problem. We show that this result can be extended to reference point solutions.

**Theorem 4.6.** Consider a multicriteria minimization problem $\min_{x \in X} C x$ with a set of feasible solutions $X \subseteq \{0,1\}^n$ and cost matrix $C \in \mathbb{Z}^{k \times n}_{\geq 0}$. For any $p \in [1,\infty]$, if

1. for any instance $I = (X,C)$, and any feasible reference point $y^{IP}$, a lower and an upper bound $L$ and $U$ on $\min_{x \in X} r(C x)$ can be computed in time $\pi_1(|I|)$, such that $U \leq \pi_2(|I|) L$, where $\pi_1$ and $\pi_2$ are non-decreasing polynomials,

2. and there exists an algorithm that solves $\min_{x \in X} r(C x)$ for any instance $I = (X,C)$ in time $\pi_3(|I|,U)$, where $\pi_3$ is a non-decreasing polynomial,

then there is an FPTAS for $\min_{x \in X} r(C x)$, where $r(y) = \langle y^{IP} \rangle_p + \langle y - y^{IP} \rangle_p$.

By $|I|$ we denote the encoding length of the instance $I$.

**Proof.** To compute a $(1+\varepsilon)$-approximation to the reference point solution, we set $\varepsilon' = \varepsilon \cdot (1 + \frac{k}{p})^{-1}$ and apply the pseudopolynomial algorithm to a modified instance $\tilde{T}$ with cost coefficients $\tilde{c}_{ij} := \left\lfloor \frac{3n}{\varepsilon' L} c_{ij} \right\rfloor$. Observe that

$$\frac{\varepsilon' L}{3n} \tilde{c}_{ij} \leq c_{ij} < \frac{\varepsilon' L}{3n} (\tilde{c}_{ij} + 1).$$

The reference point for the modified instance is defined by $y^{IP}_i := \left\lfloor \frac{3n}{\varepsilon' L} y^{IP}_i \right\rfloor$. This reference point is feasible for the modified instance, and it holds that

$$\frac{\varepsilon' L}{3n} y^{IP}_i \leq y^{IP}_i < \frac{\varepsilon' L}{3n} y^{IP}_i + \frac{\varepsilon' L}{3n} < \frac{\varepsilon' L}{3n} y^{IP}_i + \frac{\varepsilon' L}{3}.$$

Let $x^*$ and $\overline{x}^*$ be reference point solutions for $I$ and $\tilde{T}$, respectively. We now bound the value of $\overline{x}^*$ w.r.t. the original costs $c$. Let $r$ and $\overline{r}$ denote the reference point objective function for the original and the modified costs, respectively. We get

$$r(C \overline{x}^*) = \langle y^{IP} \rangle_p + \langle C x^* - y^{IP} \rangle_p$$

$$\leq \frac{\varepsilon' L}{3n} \langle y^{IP} \rangle_p + \frac{\varepsilon' L}{3} \left( 1 + \frac{k}{p} \right)$$

$$+ \max_{i \in [k]} \left\{ \frac{\varepsilon' L}{3n} (\tilde{c}_{i} \overline{x}^* - y^{IP}_i) \right\} + \frac{\varepsilon' L}{3n} \sum_{i \in [k]} \left( \frac{\varepsilon' L}{3n} (\tilde{c}_{i} x^* - y^{IP}_i) + \frac{\varepsilon' L}{3} \right)$$

$$= \frac{\varepsilon' L}{3n} \langle y^{IP} \rangle_p + \frac{\varepsilon' L}{3n} \langle C \overline{x}^* - y^{IP}_i \rangle_p + \frac{2 \varepsilon' L}{3} \left( 1 + \frac{k}{p} \right)$$

$$\leq \frac{\varepsilon' L}{3n} \langle y^{IP} \rangle_p + \frac{\varepsilon' L}{3n} \langle C x^* - y^{IP}_i \rangle_p + \frac{2 \varepsilon' L}{3} \left( 1 + \frac{k}{p} \right)$$

$$\leq \langle y^{IP} \rangle_p + \frac{\varepsilon' L}{3n} \left( 3n \varepsilon' L \langle C x^* - y^{IP} \rangle_p + n \left( 1 + \frac{k}{p} \right) \right) + \frac{2 \varepsilon' L}{3} \left( 1 + \frac{k}{p} \right)$$

$$= \langle y^{IP} \rangle_p + \langle C x^* - y^{IP} \rangle_p + \varepsilon L \left( 1 + \frac{k}{p} \right)$$

$$\leq (1 + \varepsilon) r(C x^*).$$
It remains to be shown that \( \mathbf{x}^* \) can be computed in time polynomial in \( |I| \) and \( \frac{1}{\varepsilon} \). For this, denote by \( \mathcal{T} \) and \( \mathcal{U} \) the lower and upper bounds on the optimal value \( \text{OPT} \) of the modified instance \( \mathcal{I} \). According to the prerequisites of the theorem, we can compute \( L \) and then \( \mathbf{x}^* \) in time

\[
\pi_1(|I|) + \pi_3(|\mathcal{I}|, \mathcal{T}) \leq \pi_1(|I|) + \pi_3(|\mathcal{I}|, \pi_2(|\mathcal{I}|)\mathcal{L}) \\
\leq \pi_1(|I|) + \pi_3(|\mathcal{I}|, \pi_2(|\mathcal{I}|)\text{OPT}) \\
\leq \pi_1(|I|) + \pi_3 \left( |\mathcal{I}|, \pi_2(|\mathcal{I}|) \left( \frac{3}{2} \pi_2(|I|) + n\left(1 + \frac{k}{p}\right) \right) \right)
\]

where the last inequality holds because

\[
\overline{\text{OPT}} \leq \frac{3n}{\varepsilon L} \langle (y^p) \rangle_p + \frac{3n}{\varepsilon L} \langle (C\mathbf{x}^* - y^p) \rangle_p + n \left(1 + \frac{k}{p}\right)
\]

\[
\leq \frac{3n}{\varepsilon L} \cdot U + n \left(1 + \frac{k}{p}\right)
\]

\[
\leq \frac{3n}{\varepsilon L} \cdot \pi_2(|I|) + n \left(1 + \frac{k}{p}\right)
\].

Finally note that \( |\mathcal{I}| \leq \pi_4(|I|, \log \frac{1}{\varepsilon}, \log \frac{k}{p}) \) for some polynomial \( \pi_4 \). Thus the above calculations prove that the running time is indeed polynomial. \( \square \)

**Remark 6.** Theorem 4.6 also holds for \( \text{CP} (\mathcal{P}, \langle \cdot \rangle_p) \).

**Proof.** For compromise solutions, we can not choose the reference point of the modified instance as we see fit. However, also for the ideal point, Equation (1) still holds. To see this, denote the respective ideal points by \( y^{id} \) and \( \bar{y}^{id} \), and let \( x^{(i)}, \bar{x}^{(i)} \) for \( i \in \{k\} \) be optimal solutions of \( \min_{\mathbf{x} \in X} \mathbf{c}_i \mathbf{x} \) and \( \min_{\mathbf{x} \in X} \bar{\mathbf{c}}_i \mathbf{x} \).

It holds that

\[
y^{id}_i = c_i \bar{x}^{(j)} \geq c_i x^{(j)} \geq \frac{\varepsilon' L}{3n} c_i \bar{x}^{(j)} = \frac{\varepsilon' L}{3n} y^{id}_i,
\]

\[
y^{id}_i = c_i \bar{x}^{(j)} \leq \frac{\varepsilon' L}{3n} (\bar{\mathbf{c}}_i + \mathbf{1}^T) x^{(j)} \leq \frac{\varepsilon' L}{3n} \bar{\mathbf{c}}_i x^{(j)} + \frac{\varepsilon' L}{3n} \bar{\mathbf{c}}_i x^{(j)} + \frac{\varepsilon' L}{3n} \bar{\mathbf{c}}_i x^{(j)} + \frac{\varepsilon' L}{3n} \bar{\mathbf{c}}_i x^{(j)},
\]

so Equation (1) also holds for the ideal points. \( \square \)

**Remark 7.** For the running time, it is essential that \( p \) is fixed, or at least bounded from below by a positive constant (e.g. \( p \geq 1 \)), as the running time is only polynomial in \( \frac{1}{p} \). Since for \( p \to 0 \) compromise programming becomes equivalent to the weighted sum problem, this is only a minor restriction.

Similarly to Proposition 1 in Aissi et al. [2], we can show that the necessary bounds \( U \) and \( L \) can be computed, if the single-objective problem is tractable. This is a direct implication of the weighted sum approximation described in Theorem 4.1.

**Corollary 4.7.** If there is an \( \alpha \)-approximation for the single-criterion version of \( \mathcal{P} \), then for all instances of \( \text{RP}(\mathcal{P}, \| \cdot \|) \), we can compute \( L \) such that \( L \leq \min_{y \in \mathcal{Y}} r(y) \leq \alpha k L \).

The pseudopolynomial algorithms for the shortest path problem (SP) and the minimum spanning tree problem (MST) presented in Aissi et al. [2] can be used to compute reference point solutions as well, as they both compute all (non-dominated) regret vectors (that obey the upper bound \( U \)), and the reference point solution always has a non-dominated regret vector.

**Corollary 4.8.** There is an FPTAS for \( \text{RP} (\text{SP}, \langle \cdot \rangle_p) \) and \( \text{RP} (\text{MST}, \langle \cdot \rangle_p) \) for any \( p \in [1, \infty] \).
5 Maximization

We now investigate which of the results from Section 3 hold for maximization problems. Note that for the problem \( \max_{y \in \mathcal{Y}} y \), the ideal point \( y^{\text{id}} \) is defined by \( y^{\text{id}} = \max_{y \in \mathcal{Y}} y_i \), and a reference point \( y^{\text{RP}} \in \mathbb{Z}^k_{\geq 0} \) is called feasible if \( y^{\text{RP}} \geq y^{\text{id}} \). A solution \( y \in \mathcal{Y} \) is Pareto optimal if there is no \( y' \in \mathcal{Y} \setminus \{ y \} \) with \( y' \geq y \). An \( \alpha \)-approximate Pareto set has to contain, for all \( y \in \mathcal{P} \), a solution \( y' \) with \( y' \geq \frac{1}{\alpha} y \). Accordingly, feasible answers to \( \text{Gap}(P, \alpha) \) for an input vector \( y \) are either a vector \( y' \in \mathcal{Y} \) with \( y' \geq y \), or the guarantee that there is no vector \( y'' \in \mathcal{Y} \) with \( y'' \geq \alpha y \).

Recall that the objective function for compromise and reference point solutions is the value of the reference point, degraded by the price of compromise. For maximization, we have to subtract the price of compromise, i.e., \( r(y) = \|y^{\text{RP}}\| - \|y^{\text{RP}} - y\| \). This objective function is then aimed to be maximized. To simplify the presentation, in this section we restrict to statements about monotone norms and about the infinity-norm.

We begin our considerations with an observation.

**Observation 5.1.** It is not true that whenever there is a constant factor approximation algorithm for the weighted sum problem \( \max_{y \in \mathcal{Y}} \lambda^T y \), then there also is an approximation algorithm for the Pareto set.

**Proof.** Suppose there is an \( \alpha \)-approximation algorithm for \( \max_{y \in \mathcal{Y}} \lambda^T y \). Consider an instance with \( k = 2 \) and \( \mathcal{Y} = \{(1, 1), (3, 0), (0, 3)\} \). For any \( \lambda \in \mathbb{Q}^k_{\geq 0} \), the approximation algorithm could either return \((3, 0)\) or \((0, 3)\). Hence an algorithm that only relies on the existence of a weighted sum approximation can not tell whether the element \((1, 1)\) exists or not. Any approximate Pareto set, however, has to contain the point \((1, 1)\), if it exists.

This shows that at least one of the implications of approximabilty depicted in Figure 1 no longer holds for maximization problems. Some of them, however, continue to hold. We get analogues of Lemma 3.3 and Corollaries 3.4 and 3.5, implying the approximability of compromise respectively reference point solutions, in case the Pareto set is approximable.

**Lemma 5.2.** Let \( y^{\text{RP}} \) be a feasible reference point for \( \max_{y \in \mathcal{Y}} y \), and let \( \mathcal{Y}_\alpha \) be an \( \alpha \)-approximate Pareto set. Then for any monotone norm \( \|\cdot\| \), \( \max_{y \in \mathcal{Y}_\alpha} r(y) \geq \frac{1}{\alpha} \cdot \max_{y \in \mathcal{Y}} r(y) \), where \( r(y) = \|y^{\text{RP}}\| - \|y^{\text{RP}} - y\| \).

**Proof.** Let \( y^* \in \mathcal{Y} \) be an optimal solution to \( \min_{y \in \mathcal{Y}} r(y) \), and let \( y' \in \mathcal{Y}_\alpha \) with \( y' \geq \frac{1}{\alpha} y^* \). Then
\[
\|y^{\text{RP}} - y'\| \leq \frac{1}{\alpha} \|y^{\text{RP}} - y^*\| + \left(1 - \frac{1}{\alpha}\right) \|y^{\text{RP}}\|
\]
and hence
\[
\max_{y \in \mathcal{Y}_\alpha} r(y) \geq \|y^{\text{RP}}\| - \|y^{\text{RP}} - y'\| \geq \frac{1}{\alpha}((\|y^{\text{RP}}\| - \|y^* - y^{\text{RP}}\|)) .
\]

**Corollary 5.3.** If there is an \( \alpha \)-approximation algorithm for the Pareto set of \( \max_{y \in \mathcal{Y}} y \), then there is an \( \alpha \)-approximation for \( \max_{y \in \mathcal{Y}} \|y^{\text{RP}}\| - \|y^{\text{RP}} - y\| \), for every monotone and polynomially decidable norm \( \|\cdot\| \).

**Corollary 5.4.** If there is an \( \alpha \)-approximation algorithm for the Pareto set of \( \max_{y \in \mathcal{Y}} y \), then there is an \( \alpha^2 \)-approximation for \( \max_{y \in \mathcal{Y}} \|y^{\text{id}}\| - \|y^{\text{id}} - y\| \), for every monotone and polynomially decidable norm \( \|\cdot\| \).

Interestingly, in the reverse direction, compromise programming and reference point methods are suddenly of different complexities: For compromise solutions, there is no analogue of Corollary 3.9:
Observation 5.5. For any 0 < \varepsilon < 1 and sufficiently large M, there are instances \( \mathcal{Y}(M, \varepsilon) \) and \( \mathcal{Y}'(M, \varepsilon) \), with encoding length \( O(\log M) \), that have different \((1 + \varepsilon)\)-approximate Pareto sets, but a \((1 + \delta)\)-approximation algorithm for \( \max_{y \in \mathcal{Y}} ||y||_\infty^\lambda - ||y^\text{id} - y||_\infty^\lambda \) can only distinguish between the two instances for \( \delta \in O(1/M) \).

Proof. Let \( r_\lambda(y) := ||y^\text{id}||_\infty^\lambda + ||y^\text{id} - y||_\infty^\lambda \). Since approximation is invariant under scaling of \( \lambda \), we can assume w.l.o.g. that \( ||\lambda||_\infty = 1 \). Consider the two sets

\[
\mathcal{Y}(M, \varepsilon) = \left\{ y = \left( \frac{1}{M + 1} \right), y' = \left( \frac{(1 + \varepsilon)^{-2}}{M + (1 + \varepsilon)^{-2}} \right), y'' = \left( \frac{1}{M^2 + 1} \right), \left( \frac{0}{2M + 1} \right) \right\},
\]

\[
\mathcal{Y}'(M, \varepsilon) = \mathcal{Y}(M, \varepsilon) \setminus \{ y \}.
\]

For sufficiently large values of \( M \), a \((1 + \varepsilon)\)-approximate Pareto set of \( \mathcal{Y} \) has to contain \( y \). However, for any \( \lambda \in \mathbb{Q}_{\geq 0} \), either \( y' \) (for \( \lambda_2 \geq 2/3 \)) or \( y'' \) (for \( \lambda_2 \leq 2/3 \)) is a \((1 + \delta)\)-approximation to \( r_\lambda(y) \), unless \( \delta \in O(1/M) \).

If the reference point can be chosen freely, however, we do get an analogue of Lemma 3.8:

Lemma 5.6. Let \( \mathcal{P} \) be a multicriteria maximization problem, and let \( \alpha > 1 \). There is a polynomial time algorithm for \( \text{Gap}(\mathcal{P}, \alpha) \), if for any feasible reference point \( y^\text{ref} \) and any \( \lambda \in \mathbb{Q}_{\geq 0} \) there is an \( \alpha \)-approximation algorithm for \( \max_{z \in \mathcal{Y}} r_\lambda(z) \), where \( r_\lambda(z) = ||y^\text{ref}||_\infty^\lambda - ||y^\text{ref} - z||_\infty^\lambda \).

Proof. Let \( y \in \mathbb{Q}^k \) be the input to the Gap problem, w.l.o.g. \( y \neq 0 \). Let \( I = \{ i \in [k] : y_i \neq 0 \} \), and let \( M \) be an upper bound on the objective values. Further, let \( c = \max_{i \in I} M y_i \), and set \( y^\text{ref} = c \cdot y \). Note that this is a feasible reference point. We now set the weight vector to

\[
\lambda_i = \begin{cases} 1/y_i & \text{for } i \in I, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( y' \) be an \( \alpha \)-approximate solution to \( \max_{z \in \mathcal{Y}} r_\lambda(z) \).

If \( r_\lambda(y') \geq r_\lambda(y) \), then \( y'_i \geq y_i \) for all \( i \in I \), and \( y'_i \geq 0 = y_i \) for all other \( i \), so \( y' \) is a positive answer to the Gap problem. Otherwise, i.e. if \( r_\lambda(y') < r_\lambda(y) \), for any \( y'' \in \mathcal{Y} \), we know \( r_\lambda(y'') \leq \alpha r_\lambda(y') < \alpha r_\lambda(y) \). Let \( j = \arg \max_{i \in I} \frac{y''_i - y^\text{ref}_i}{y_i} \). Then,

\[
r_\lambda(y) = c||y||_\infty^\lambda - (c - 1)||y||_\infty^\lambda = ||y||_\infty^\lambda = 1
\]

\[
\Rightarrow \alpha = \alpha r_\lambda(y) > r_\lambda(y'') = c \cdot ||y''||_\infty^\lambda - ||y^\text{ref} - y''||_\infty^\lambda = c - \frac{c y_j - y''_j}{y_j}
\]

\[
\Rightarrow y''_j < \alpha y_j - c y_j + c y_j = \alpha y_j.
\]

We can therefore conclude that there is no \( y'' \in \mathcal{Y} \) with \( y'' \geq \alpha y \), and answer the Gap problem negatively.

All approximability reductions for compromise and reference point solutions for maximization problems are depicted in Figure 2 below.

6 Conclusion

A multicriteria optimization problem lacks a single, unifying objective function. A priori, there is no metric justifying a preference on the set of Pareto optimal solutions. Still, the Pareto solutions are not equivalent like, e.g., the set of optima for a single criterion. Decision makers can have preferences among the Pareto solutions. Reference point methods model such preferences. These methods are widespread in practice and are a more powerful model than a simple weighing
of the objectives. They are the most powerful model in the sense that every Pareto solution can become the unique optimum, for some choice of the additional input. To the best of our knowledge, this paper provides the first extensive theoretical study of these methods in the context of approximation.

Our main results establishes computational equivalence between the approximation of the Pareto set and the approximation of reference point solutions, thus linking the rich body of mathematical research on Pareto sets to the practically widespread reference point methods. Moreover, this work lifts a number of important and general algorithmic techniques known for single criteria optimization to the setting of reference point solutions.

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Appendix

Lemma 6.1 (Lemma 3.8 revisited, for $\langle \cdot \rangle_p$ and $\| \cdot \|_p$). Let $\alpha > 1$ and set $\beta := \frac{\sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 - 1} - 1}$. There is a polynomial time algorithm for $\text{GAP}(\mathcal{P}, \alpha)$. Let $\alpha > 1$ and set $\beta := \frac{\sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 - 1} - 1}$. There is a polynomial time algorithm for $\text{GAP}(\mathcal{P}, \alpha)$, if there are two polynomial time algorithms $A_1, A_2$ such that,

- given an instance of $\mathcal{P}$, algorithm $A_1$ computes in polynomial time a feasible reference point $y^p \in \mathbb{Z}_{\geq 0}^k$ for that instance, and,

- additionally given $y^p$ and $\lambda \in \mathbb{Q}_{\geq 0}^k$ and $p \geq 1$, algorithm $A_2$ computes in polynomial time a solution $y' \in \mathcal{Y}$ with $r(y') \leq \beta \min_{z \in \mathcal{Y}} r(z)$, for $r(z) = \langle y^p \rangle_p^\lambda + \langle y^p - z \rangle_p^\lambda$ or $r(z) = \|y^p\|_p^\lambda + \|y^p - z\|_p^\lambda$, respectively.

Proof. Let $y \in \mathbb{Q}_{\geq 0}^k$ be the input to the GAP problem. W.l.o.g., we can assume that $\alpha \geq \alpha y^p$ for the reference point $y^p$ computed by $A_1$, as otherwise there is no $y' \leq \frac{1}{\alpha} y$ and GAP can be answered negatively.

We will solve the GAP problem with a single call of the $\beta$-approximation algorithm for $\text{RP}(\mathcal{P}, \langle \cdot \rangle_p)$ (or $\text{RP}(\mathcal{P}, \| \cdot \|_p)$, respectively) with

$$p := \max \left\{ \frac{\log k}{\log(1 + \frac{1}{2p})}, 2kMq \right\},$$

where $q$ is the largest denominator of all the components in $y$, and $M$ is an upper bound on the objectives in $\mathcal{Y}$.

Let $I := \{ i \in [k] : y_i = y^p = 0 \}$. For $i \in [k], \lambda_i = \left\{ \begin{array}{ll} \frac{1}{y_i - y^p} & \text{if } i \in I, \\ 1 & \text{otherwise.} \end{array} \right.$

Let $y'$ be a $\beta$-approximation to $\min_{z \in \mathcal{Y}} r(z)$. If $r(y') \leq r(y)$, we return $y'$ as a positive answer to the GAP problem. Observe that

$$\lambda_i(y'_i - y^p) \leq \langle y' - y^p \rangle_p^\lambda \leq \langle y - y^p \rangle_p^\lambda \leq 1 + \frac{k}{p}.$$ 

If $i \in I$, we have $y'_i \leq \frac{1}{\alpha} (1 + \frac{1}{2kp}) < 1$. If $i \notin I$, then $y'_i \leq (1 + \frac{k}{p})y_i < y_i + 1$. In both cases, integrality of $y'$ implies $y'_i \leq y_i$. The same holds for the $\| \cdot \|_p$-norm with $1 + \frac{k}{p}$ replaced by $\sqrt{k}$, in which case, the choice of $p$ guarantees $\sqrt{k} \cdot y'_i < y'_i + 1$.

If $r(y') > r(y)$, we answer GAP negatively: Let $y'' \in \mathcal{Y}$. We show that there is an $i \in [k]$ with $y''_i > \frac{1}{\alpha} y_i$. This is true if $y''_i > 0 = y_i$ for any $i \in I$. Thus, we can restrict to the projection of $\mathbb{Q}^k$ to the components in $[k] \setminus I$, and w.l.o.g. assume $I = \emptyset$. First observe that $\beta r(y''_i) \geq r(y') > r(y)$, which implies

$$\beta \langle y'' - y^p \rangle_p^\lambda > \langle y - y^p \rangle_p^\lambda - (\beta - 1) \langle y^p \rangle_p^\lambda.$$ 

It is easy to verify that $\langle z \rangle_p^\lambda \leq (1 + \frac{k}{p}) \|z\|_\infty^\lambda$ for all $z \in \mathbb{Q}^k$, and furthermore $\langle y - y^p \rangle_p^\lambda = 1 + \frac{k}{p}$, as $I = \emptyset$. This yields

$$(1 + \frac{k}{p})\beta \|y'' - y^p\|_\infty^\lambda > (1 + \frac{k}{p})\|y - y^p\|_\infty^\lambda - (1 + \frac{k}{p})(\beta - 1)\|y^p\|_\infty^\lambda,$$

which brings us back to the case of the $\| \cdot \|_\infty$-norm. The same holds true for the $\| \cdot \|_p$-norm, with the factor $1 + \frac{k}{p}$ replaced by $\sqrt{k}$.

$\square$

Corollary 6.2 (Corollary 3.9 revisited, for $\langle \cdot \rangle_p$ and $\| \cdot \|_p$). Let $\alpha > 1$ and set $\beta := \sqrt{\frac{\sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 - 1} - 1} - 1}$. There is a polynomial time algorithm for $\text{GAP}(\mathcal{P}, \alpha)$, if there is a $\beta$-approximation algorithm for $\text{CP}(\mathcal{P}, \langle \cdot \rangle_p)$ (or $\text{CP}(\mathcal{P}, \| \cdot \|_p)$, respectively) for every $p \geq 1$ and the running time of all algorithms is bounded by a polynomial in the instance size and $\log(p)$.

Proof. The proof is identical to that of Corollary 3.9 given in the paper. In fact, the second part of this proof only uses properties of monotone norms. $\square$