Thermodynamics of squashed Kaluza–Klein black holes and black strings: a comparison of reference backgrounds

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Abstract
We investigate thermodynamics constructed on different background reference spacetimes for squashed Kaluza–Klein (SqKK) black hole and electrically charged black string in the five-dimensional Einstein–Maxwell system. Two spacetimes are possible to be reference spacetimes giving finite gravitational classical actions: one is four-dimensional Minkowski times a circle and the other is the KK monopole. The boundary of the SqKK black hole cannot be matched perfectly to that of the former reference spacetime because of the difference in topology. However, the resultant classical action coincides with that calculated by the counterterm subtraction scheme. The boundary of the KK monopole has the same topology as that of the SqKK black hole and can be matched to the boundary of the black hole perfectly. The resultant action takes a different value from the result given by using the former reference spacetime. After a brief review of thermodynamic quantities of the black hole solutions, we calculate thermodynamic potentials relevant for several thermodynamic environments. The most stable state is different for each environment: for example, the KK monopole is the most stable state in an isothermal environment with fixed gravitational tension. On the other hand, when the size of the extra dimension is fixed, the Minkowski times a circle is the most stable. It is shown that these two spacetimes can be reference spacetimes of the five-dimensional black string.

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1. Introduction
The gravitational path integral in the saddle-point approximation has a long history in the context of black hole thermodynamics \cite{1}. In general, there is a problem that must be faced
with this approach, that is, the classical action diverges. Thus, in order to obtain finite action, an appropriate regularization is needed. There are two well-known regularization schemes: ‘background subtraction’ which is a traditional approach and ‘counterterm subtraction’.

In the background subtraction scheme, we consider the values of the action relative to a background reference spacetime, where the choice of the reference spacetime is ambiguous in some cases. An example is the Euclidean Taub-bolt solution; both Minkowski spacetime [2] and the self-dual Taub-NUT [3] are possible to be reference spacetimes. In the case of Minkowski spacetime, the asymptotic boundary cannot be matched perfectly to that of the Taub-bolt because of the difference in topology, whereas the boundary of the self-dual Taub-NUT solution has the same topology and can be matched perfectly. The resulting finite actions evaluated on these two reference spacetimes are different.

Motivated by the AdS/CFT correspondence [4–6], the counterterm subtraction method was developed by Balasubramanian and Kraus [7], who proposed adding a counterterm to gravitational action at the asymptotic boundary. One remarkable thing is that this method does not require to choose any reference spacetime and there is no ambiguity mentioned above. The counterterm method was found originally for asymptotically AdS or asymptotically locally AdS spacetime [7], and then extended for a class of \((d + 1)\)-dimensional asymptotically flat spacetime with boundary \(S^\mu \times R^{d-\mu}\) [9, 10, 39] and five-dimensional asymptotically locally flat spacetime with a fibered boundary topology \(R^2 \hookrightarrow S^2\) in [11]. This progress makes it possible to investigate the gravitational mass of an asymptotically locally flat spacetime like the Kaluza–Klein (KK) monopole [12, 13] as was done in [11].

By applying the counterterm method to Taub-bolt-AdS and taking zero cosmological constant limit after the evaluation of the gravitational action, in [14], Emparan, Johnson and Myers compared the action with that computed by the background subtraction method. They showed that the resultant action of the Taub-bolt instanton by the counterterm method agrees precisely with the one by the background subtraction method with the different boundary topology given in [2]. Thus, the generating functional in the CFT side seems to prefer the gravitational action evaluated using the spacetime boundary with different topology. This might imply that, in the background subtraction scheme, topology of the asymptotic boundary is not so important.

Recently, for the purpose of searching the end state of the Gregory–Laflamme instability [15, 16], much effort has been devoted to the investigation of the so-called caged Kaluza–Klein (KK) black holes, which are defined to be solutions with a spherical event horizon that asymptotes to Minkowski space times a circle (see, e.g., [17, 18] and the references therein). The thermodynamic first law of the caged KK black holes was obtained in [19–21], showing that the caged KK black holes can be characterized by mass and gravitational tension.

In the five-dimensional Einstein–Maxwell theory, there are analytic solutions representing electrically charged black holes with squashed horizons [22] as a generalization of the solutions given in [23, 24]. Intriguingly, the spacetime far from the black hole is locally a product of the four-dimensional Minkowski spacetime and a circle. In this sense, the black hole resides in KK spacetime and is worth being named a squashed Kaluza–Klein (SqKK) black hole. The dynamical stability of the SqKK black hole was investigated in [25]. The thermodynamic properties of the SqKK black hole were investigated by use of the counterterm method in [26] and by the background subtraction scheme in [27]. The thermodynamics of the rotating uncharged SqKK black hole was also investigated in [28].

When one tries to evaluate the classical action of the SqKK black hole by the background subtraction scheme, there is an ambiguity in the choice of reference spacetime. As far as we

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3 The circle is fibered over the four-dimensional spacetime with non-trivial twisting.

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are aware, there are two different reference spacetimes; one is Minkowski spacetime times $S^1$ and the other is the KK monopole. In this paper, we call the former reference spacetime the flat background, in short. The asymptotic boundary of the flat background is not the same as that of the SqKK black hole in topology, but the boundary of the KK monopole has the same topology and can be matched to that of the SqKK black hole perfectly. Therefore, in the counterterm method, the counterterm relevant for the KK monopole is applicable also to the SqKK black hole spacetime.

In this paper, we compare these two reference spacetimes from the viewpoint of thermodynamics. In the background subtraction scheme, free energy can be interpreted as a value relative to the reference background which one chooses. We calculate free energy, or equivalently, thermodynamic potentials in various thermodynamic environments characterized by thermodynamic variables obtained in [27]. It will be shown that the most stable state among these solutions depends on the choice of thermodynamic environment. For example, the KK monopole can be the most stable state in isothermal environments with fixed gravitational tension, but the flat background is the most stable in isothermal environments with fixed size of the extra dimension at spatial infinity.

We will also show that these two reference spacetimes can be reference spacetimes of five-dimensional electrically charged black string. We calculate its free energy and thermodynamic potentials in some thermodynamic environments.

The organization of the present paper is as follows. In the following section, we review the SqKK black hole and give the metrics of the two reference spacetimes. In section 3, calculations of the classical actions are given in detail by two methods: the background subtraction method with respect to the two reference spacetimes and the counterterm method. In section 4, we give a brief review of the thermodynamic quantities of the SqKK black hole and expressions of the first law obtained in [27]. In section 5, thermodynamic potentials relevant for thermodynamic environments are calculated and the most stable state in each environment is clarified. In section 6, using these two reference spacetimes, we apply the background subtraction method to the five-dimensional charged black string. In section 7, we summarize the results and discuss a remaining problem.

2. The solution

Let us review the SqKK black hole [22], which is a solution of the five-dimensional Einstein–Maxwell theory. The action is given by

$$I = \frac{1}{16\pi G} \int_M d^5x \sqrt{-g} \left[ R - F_{\mu\nu} F^{\mu\nu} \right] + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{-h} d^4x,$$

(2.1)

where $G$ is the five-dimensional Newton constant, $g_{\mu\nu}$ is the metric, $R$ is scalar curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the five-dimensional $U(1)$ gauge field $A_\mu$. The second term is the so-called Gibbons–Hawking term, in which $h$ is the determinant of the induced metric and $K$ is the trace of the extrinsic curvature of the boundary $\partial M$, respectively. It is required so that upon variation with metric fixed at the boundary, the action yields the Einstein equations [1].

The metric of the SqKK black hole considered in this paper is given by

$$ds^2 = -V(\rho) dt^2 + \frac{B(\rho)}{V(\rho)} d\rho^2 + \rho^2 B(\rho) d\Omega^2 + \frac{r_\infty^2}{4B(\rho)} (d\psi + \cos \theta \, d\phi)^2,$$

(2.2)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$ is the metric of the unit two-dimensional sphere and

$$V(\rho) = \left( 1 - \frac{\rho_+}{\rho} \right) \left( 1 - \frac{\rho_-}{\rho} \right), \quad B(\rho) = 1 + \frac{\rho_0}{\rho}, \quad r_\infty = 2\sqrt{(\rho_+ + \rho_0)(\rho_- + \rho_0)}.$$

(2.3)
Here, the coordinate ranges are $0 \leq \theta < 2\pi$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 4\pi$. The gauge potential is given by
\[
A = \mp \frac{\sqrt{3}}{2} \frac{\rho_\ast \rho_-}{\rho} \, d\tau.
\tag{2.4}
\]
When $\rho_0 > 0$, we consider parameter region $\rho_\ast \geq \rho_- \geq 0$, which does not lose any generality. We note that even when $\rho_0$ is negative, the metric (2.2) describes a black hole geometry if the parameters satisfy $-\rho_0 = |\rho_0| < \rho_- \leq \rho_\ast$ [29].

It is easy to see the apparent singularity at $\rho_\ast$ corresponds to the outer horizon of the SqKK black hole. The inner horizon at $\rho_-$ is analogous to that of the Reissner–Nordström black holes. It should be noted that the shape of the horizon is a squashed $S^3$ as was discussed in [22]. The spatial infinity corresponds to a limit $\rho \to \infty$. In the limit, the metric asymptotes to
\[
ds^2 = -d\tau^2 + d\rho^2 + \rho^2 \frac{d\Omega_1^2}{\rho_\ast^2} + r_\infty^4 \, d\psi^2,
\tag{2.5}
\]
which is a metric of a ‘twisted’ $S^1$ fiber bundle over four-dimensional Minkowski spacetime. From the metric (2.5), it is seen that the $S^1$ circle parametrized by a coordinate $\psi$ has finite size even at spatial infinity. The non-trivial twisting of the $S^1$ circle fibered over the $S^2$ base space leads a four-dimensional U(1) gauge field by KK reduction. Actually, in no horizon limit $\rho_\pm \to 0$ with $\rho_0 > 0$, the SqKK black hole spacetime becomes the KK monopole spacetime [12, 13]. The metric of the KK monopole is
\[
ds^2 = -d\tau^2 + \left(1 + \frac{\rho_0}{\rho}\right) d\rho^2 + \rho(\rho + \rho_0) \, \Omega_2^2 + \frac{\rho_0^2 \rho}{(\rho + \rho_0)} (d\psi + \cos \theta \, d\phi)^2,
\tag{2.6}
\]
which is a vacuum solution of the five-dimensional Einstein gravity. In the limit $r_\infty \to \infty$, the KK monopole becomes five-dimensional Minkowski spacetime and the SqKK black hole reduces to the five-dimensional Reissner–Nordström black hole (for details, see [22]). Thus, this limit is the five-dimensional spherically symmetric limit, i.e., the spacetime admits $SO(4)$ isometry group.

The metric (2.5) has a curvature singularity at $\rho = 0$ because of the non-trivial twisting. Therefore, the action and Hamiltonian diverge and the spacetime is not useful as a reference spacetime to regularize the action and Hamiltonian of the SqKK black hole. If the $S^1$ fiber is untwisted, the spacetime becomes a product $S^1$ bundle over four-dimensional Minkowski, with the metric
\[
ds^2 = -d\tau^2 + d\rho^2 + \rho^2 \, \Omega_2^2 + \frac{r_\infty^2}{4} \, d\psi^2.
\tag{2.7}
\]
This is a completely flat solution of the five-dimensional vacuum Einstein equations. Thus, the flat spacetime is useful as a reference spacetime in evaluation of the action and the Hamiltonian. This is the flat background mentioned in section 1.

Given the metric (2.2), we can calculate various physical quantities. The surface gravity is calculated as
\[
\kappa_\ast = \frac{\rho_\ast - \rho_-}{2 \rho_\ast \sqrt{\rho_\ast (\rho_\ast + \rho_0)}},
\tag{2.8}
\]
which gives the Hawking temperature of the SqKK black hole as $T = \kappa_\ast / 2\pi$ [30]. The graybody factor of the SqKK black hole was investigated in [31, 32]. We assume that the entropy of the SqKK black hole is given by the Bekenstein–Hawking formula [33, 34]
\[
S = \frac{A_\ast}{4G} = \frac{4\pi^2}{G} \rho_\ast (\rho_\ast + \rho_0) \sqrt{\rho_\ast (\rho_- + \rho_0)},
\tag{2.9}
\]

4 In this paper, we consider the definition of the Hamiltonian given in [42].
where $A_+$ is the area of the outer horizon, which is consistent with Wald's entropy formula [35, 36]. The electric charge and electrostatic potential of the SqKK black hole are also calculated as [22]

$$Q = \frac{1}{8\pi G} \int_\infty dS_{\mu\nu} F^{\mu\nu} = \pm \frac{\sqrt{3\pi}}{G} r_\infty \sqrt{\rho_+ \rho_-}, \quad (2.10)$$

$$\Phi = A_\uparrow |_{\rho \to \infty} - A_\uparrow |_{\rho = \rho_+} = \pm \frac{\sqrt{3}}{2} \sqrt{\frac{\rho_-}{\rho_+}}. \quad (2.11)$$

The Komar mass is a meaningful mass of black holes which possess a timelike Killing vector. Using the timelike Killing vector $\partial_\tau$, we can calculate the Komar mass of the SqKK black hole as

$$M_K = -\frac{3}{32\pi G} \int dS_{\mu\nu} D^\mu \xi^\nu = \frac{3\pi}{4G}(\rho_+ + \rho_-) r_\infty, \quad (2.12)$$

where the integral is taken over the squashed three-dimensional sphere at spatial infinity. The Smarr-type formula was shown generally in [37] as

$$M_K - Q\Phi = \frac{1}{2} TS, \quad (2.13)$$

which is sometimes called the integrated expression for the first law.

The Abbott–Deser mass (AD mass) [38] is also a meaningful mass of the SqKK black hole as was discussed in [26]. It is a definition of mass for spacetimes with arbitrary asymptotic behavior\footnote{As for the definition of the AD mass for the SqKK black hole, see [26], for example.}. Using the flat background\footnote{In the calculation of the AD mass in [26], the locally flat spacetime with the metric (2.5) was chosen as the reference spacetime. However, as mentioned before, the locally flat background is not suitable for a reference spacetime in the evaluation of the classical action and the Hamiltonian. The flat background, instead, is a suitable reference spacetime for all of them. To maintain consistency in the choice of reference spacetime, we consider the flat background in this paper.}, the AD mass is calculated as

$$M_{AD} = \frac{\pi}{2G} (2\rho_+ + 2\rho_- + \rho_0) r_\infty. \quad (2.14)$$

For asymptotically flat stationary spacetimes, the Komar mass equals the ADM mass as far as we are aware. However, interestingly, for the SqKK black hole we consider here, which is not asymptotically flat, the AD mass has a different value from the Komar mass.

### 3. The classical action and free energy of the SqKK black hole

The Euclidean action is

$$I_E = \frac{1}{16\pi G} \int_M (R_E - F_{\mu\nu} F^{\mu\nu}) \sqrt{g_E} d^5x + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{h} d^4x, \quad (3.1)$$

where $R_E$ is the Euclidean metric and $R_E$ is the scalar curvature. As mentioned earlier, in order to evaluate a finite classical action, we need some regularization scheme\footnote{Such regularization is almost always needed for non-compact spacetime. As far as we know, the only exceptional case is lower-dimensional braneworld black hole spacetime (for details, see Kudoh and Kurita [41]).}.

Firstly, let us consider the background subtraction method, which requires a background spacetime as a reference. We set the boundary of the SqKK black hole spacetime, $\partial M$, at $\rho = \rho_B = \text{const}$, and calculate $I_E(M, \partial M)$. Similarly, for the reference spacetime $M_R$ with $\partial M_R$, we obtain $I_E(M_R, \partial M_R)$. After the subtraction $I_E(M, \partial M) - I_E(M_R, \partial M_R)$, we will take a limit $\rho_B \to \infty$. In order to obtain the finite action, the reference spacetime is
required to have the same proper length along the imaginary time, the same size of the $S^1$ fiber and the same circumference radius of the $S^2$ base space as those of the SqKK black hole at the boundary. The period of the imaginary time, $\beta$, should be related to the inverse of the temperature as

$$\beta = T^{-1} = \frac{4\pi \rho_+ \sqrt{\rho_+ (\rho_+ + \rho_0)}}{\rho_+ - \rho_-}. \tag{3.2}$$

Thus, to match the period of imaginary time $\tau_E$ means that the two spacetimes have the same temperature.

Now, we consider the flat background with the following metric as a reference spacetime,

$$ds^2 = V(\rho_B) d\tau^2_E + \rho^2 d\Omega_2^2 + \frac{r^2_{\infty R}}{4} d\psi^2, \tag{3.3}$$

where the parameter $r_{\infty R}$ should be determined by $r^2_{\infty R} = r^2_{\infty B}(\rho_B)^{-1}$, and the imaginary time has a period $\tau_E \sim \tau_E + \beta$. In order to match the circumference radius of the $S^2$ base space at the boundary, the boundary of the reference spacetime is set at the hypersurface $\rho = \rho_{BR} := \rho_B \sqrt{B(\rho_B)}$. Therefore, the classical action of the SqKK black hole relative to the reference (3.3) is calculated as

$$I_E = \beta \pi \frac{\rho_+ + \rho_0}{2G} r_{\infty}, \tag{3.4}$$

which leads to a free energy

$$F = \frac{I_E}{\beta} = \frac{\pi}{2G} (\rho_+ + \rho_0) r_{\infty}. \tag{3.5}$$

This quantity can be interpreted as the free energy of the SqKK black hole relative to the flat background. In the KK monopole limit $\rho_\pm \to 0$, the parameter $\rho_0$ becomes $r_{\infty}/2$ and (3.5) becomes

$$F \to \hat{F} = \frac{\pi}{4G} r_{\infty}^2, \tag{3.6}$$

which is the free energy of the KK monopole relative to the flat background. This free energy equals that calculated by the counterterm method obtained in [11].

We should note that the $S^1$ fiber parametrized by $\psi$ in the flat background is not twisted. Therefore, the boundary of the flat background is topologically different from that of the SqKK black hole spacetime, where the $S^1$ fiber is twisted. In spite of this difference in topology, we can obtain the finite result.

Since the boundary of the KK monopole background is the same as that of the SqKK black hole, the KK monopole would be a natural reference spacetime for the SqKK black hole. We consider the metric of the KK monopole

$$ds^2 = V(\rho_B) d\tau^2_E + B_R(\rho) d\rho^2 + \rho^2 B_R(\rho) d\Omega^2 + \frac{r^2_{\infty R}}{B_R(\rho)} d\chi^2, \tag{3.7}$$

where

$$B_R(\rho) = 1 + \frac{\rho_{BR}}{\rho}. \tag{3.8}$$

The boundary in the SqKK black hole spacetime $\rho = \rho_B$ is matched to the surface $\rho = \rho_{BR}$ in the KK monopole background. The parameters $\rho_{BR}$ and $\rho_{BR}$ should satisfy

$$\rho^2_{BR} B(\rho_B) = \rho^2_{BR} B(\rho_{BR}), \quad \frac{r^2_{\infty R}}{4B(\rho_B)} = \frac{\rho^2_{BR}}{B(\rho_{BR})}. \tag{3.9}$$

In this paper, $\hat{A}$ means quantity $A$ of the KK monopole evaluated on the flat background.
Then, we have
\[ \rho_{0R} = \frac{r_\infty}{2} \left[ \frac{1}{2} \rho_B + \rho_0\rho_B - \frac{1}{2} \rho_B r_\infty \right]^{1/2}, \quad \rho_{BR} = \left[ \frac{1}{2} \rho_B^2 + \rho_0\rho_B - \frac{1}{2} \rho_B r_\infty \right]^{1/2}. \] (3.10)

With these parameters, the circumference radii of \( S^2 \) and the \( \psi \) circle at the boundary are matched. Again, the imaginary time has the period \( \beta \). Then, the classical action of the SqKK black hole relative to the KK monopole background is calculated as
\[ \tilde{I}_E = \frac{\pi}{2G} \beta r_\infty \left( \rho_+ + \rho_0 - \frac{r_\infty}{2} \right). \] (3.11)

which gives the free energy of the SqKK black hole evaluated on the KK monopole background as
\[ \tilde{F} = \frac{\tilde{I}_E}{\beta} = \frac{\pi}{2G} r_\infty \left( \rho_+ + \rho_0 - \frac{r_\infty}{2} \right). \] (3.12)

Of course, the above free energy vanishes in the KK monopole limit \( \rho_+ \to 0 \) with \( \rho_0 > 0 \).

It is easily seen that the difference between \( F \) and \( \tilde{F} \) is the free energy of the KK monopole relative to the flat background, or equivalently,
\[ F = \tilde{F} + \tilde{F}, \] (3.13)
as expected. Therefore, it is checked that these free energies are well defined as relative quantities between any two spacetimes.

Secondly, the action of the SqKK black hole can also be calculated by the counterterm method. The counterterm for five-dimensional asymptotically locally flat spacetime with a fibered boundary topology \( R^2 \hookrightarrow S^2 \) was proposed in [10] as
\[ I_{ct} = \frac{1}{8\pi G} \int_{\partial M} \sqrt{2\overline{R}} \sqrt{-h} \, d^4x, \] (3.14)
where \( \overline{R} \) is the Ricci scalar of the induced metric on the boundary. The result is
\[ \tilde{I}_E + I_{ct} = \beta \frac{\pi}{2G} (\rho_+ + \rho_0) r_\infty. \] (3.15)

Then, the free energy is obtained as
\[ F_{ct} = \frac{I_E + I_{ct}}{\beta} = \frac{\pi}{2G} (\rho_+ + \rho_0) r_\infty, \] (3.16)
which is equal to the free energy relative to the flat background (3.5)\(^9\).

Furthermore, the AD mass of the SqKK black hole evaluated on the flat background (2.14) is the same as the counterterm mass obtained in [26]\(^10\). Therefore, for the SqKK black hole, the counterterm method is equivalent to the background subtraction method using the flat background as a reference spacetime.

### 4. Thermodynamical relation between Hamiltonian, Abbott–Deser mass and Komar mass

In this section, we review thermodynamic quantities of the SqKK black hole shown in [27] and give more detailed description of their definitions and calculations. Firstly, we review the definitions of the Hamiltonian and the gravitational tension.

\(^9\) The same finite action (3.15) and free energy can be obtained by use of the counterterm proposed in [8].

\(^10\) The counter-term mass can be calculated by use of the stress–energy tensor obtained in [40].
The Hamiltonian for the Einstein–Maxwell system is \[ H := -\frac{1}{8\pi G} \int_{B_\tau} \sqrt{\sigma} \left[ N_\tau k + u_i (\Theta^{ij} - \Theta h^{ij}) N_j + 2 A_\tau F^\tau_i u_i N_L \right], \quad (4.1) \]
where \( B_\tau \) is the boundary at infinity of a \( \tau \) constant hypersurface \( \Sigma_\tau \), \( N_\tau \) is the lapse function, \( \sqrt{\sigma} \) is the area element of \( B_\tau \), \( k \) is the trace of the extrinsic curvature of \( B_\tau \) embedded in \( \Sigma_\tau \), \( u \) is the outward pointing unit normal to \( B_\tau \), \( \Theta^{ij} \) is the extrinsic curvature of \( \Sigma_\tau \) embedded in the spacetime manifold, \( \Theta \) is the trace of the extrinsic curvature.

The gravitational tension is a thermodynamic quantity of the SqKK black hole. It was originally introduced for the black \( p \)-branes or black strings and contributes to their thermodynamical first law as an intensive quantity conjugate to the size of the compact dimension \([19–21, 43]\). In \([44]\), the gravitational tension for a non-asymptotically flat spacetime relative to a reference spacetime was defined by using the Hamiltonian formalism to a foliation of the spacetime along asymptotically translationally-invariant spatial direction. The gravitational tension of the SqKK black hole along the \( \psi \) direction is given by \[ T := -\frac{1}{8\pi G \beta} \int_{B_\psi} \sqrt{\bar{\sigma}} \left[ \bar{N}_\psi \bar{k} + \bar{u}_i (\bar{\Theta}^{ij} - \bar{\Theta} \bar{h}^{ij}) \bar{N}_j + 2 A_\psi F^{\psi}_i \bar{u}_i \bar{N}_L \right], \quad (4.2) \]
where \( \beta \) is the inverse of the temperature \((3.2)\), \( B_\psi \) is the boundary at infinity of a \( \psi \) constant hypersurface \( \Sigma_\psi \), \( \bar{N}_\psi \) is the lapse function, \( \sqrt{\bar{\sigma}} \) is the area element of \( B_\psi \), \( \bar{k} \) is the trace of the extrinsic curvature of \( B_\psi \) embedded in \( \Sigma_\psi \), \( \bar{u} \) is the outward pointing unit normal to \( B_\psi \), \( \bar{\Theta}^{ij} \) is the extrinsic curvature of \( \Sigma_\psi \) embedded in the spacetime manifold and \( \bar{\Theta} \) is the trace of the extrinsic curvature.

Here, we note that \( B_\tau \) in \((4.1)\) or \( B_\psi \) in \((4.2)\) is the boundary at infinity of a \( \tau \) or \( \psi \) constant hypersurface and does not include the Misner strings. In the evaluation, we consider the Euclidean section outside the outer horizon and there is no fixed point of the isometry generated by \( \partial_\psi \). However, there are the Misner strings along \( \theta = 0 \) and \( \theta = \pi \). If one wishes to reconstruct the classical action \((3.4)\) or \((3.11)\) in this Hamiltonian formalism along the direction \( \partial_\psi \), the surface terms on the Misner string should be considered \([45]\).

### 4.1. Using the flat background

Now, we consider the boundary adjusted flat background \((3.3)\). Then, the Hamiltonian is calculated as \[ H = -\frac{\pi}{2G r_\infty} (2\rho_+ - \rho_- + \rho_0), \quad (4.3) \]
which is also interpreted as a relative quantity to the flat background. The gravitational tension of the SqKK black hole along \( \partial_\psi \) is calculated as \[ T = \frac{1}{4G} (\rho_+ + \rho_- + 2\rho_0). \quad (4.4) \]
With this form of the tension, we can write expressions for the first law as \[ dF = -S dT - Q d\Phi + T dL, \quad (4.5) \]
\[ dH = T dS - Q d\Phi + T dL, \quad (4.6) \]
\[ dM_{AD} = T dS + \Phi dQ + T dL, \quad (4.7) \]
where \( L := 2\pi r_\infty \) is the size of the \( S^1 \) fiber at infinity. The quantities \( F, H \) and \( M_{AD} \) are related to each other by the Legendre transformations as \[ F = H - TS = M_{AD} - Q \Phi - TS. \quad (4.8) \]
Therefore, $F$, $H$ and $M_{AD}$ are thermodynamic potentials having $(T, \Phi, L)$, $(S, \Phi, L)$ and $(S, Q, L)$ as natural variables, respectively. The transformations (4.8) are summarized as follows:

$$
F \xrightarrow{T \rightarrow S} H \xrightarrow{\Phi \rightarrow Q} M_{AD},
$$

where the long arrow means Legendre transformation between thermodynamic potentials and the small arrow on the long arrow represents conversion of variables. The first law (4.5) is quite natural because in the evaluation of the free energy $F$, we fixed the temperature, the electrostatic potential and the size of the extra dimension at the boundary, and hence, the free energy should be associated with thermodynamic environment with fixed $(T, \Phi, L)$.

The Legendre transformations of $H$ or $M_{AD}$ with respect to $T L$ do not give the Komar mass. It implies that the gravitational tension is not a thermodynamic variable related to the Komar mass. Instead, $\epsilon$ and $\Sigma$ defined as

$$
\epsilon = L^2, \quad \Sigma = \frac{T}{2L} = \frac{1}{16\pi Gr_{\infty}}(\rho_+ + \rho_- + 2\rho_0)
$$

are a pair of thermodynamic variables related to the Komar mass. Actually, these quantities satisfy

$$
dM_K = T dS + \Phi dQ - \epsilon d\Sigma
$$

and

$$
F = M_K - TS - Q\Phi + \epsilon \Sigma.
$$

Thus, the Komar mass can be interpreted as a thermodynamic potential with natural variables $(S, Q, \Sigma)$. Furthermore, from the Legendre transformations (4.8) and (4.12), $\epsilon$ can be a thermodynamic variable of $F$, $H$ and $M_{AD}$. That is,

$$
dF = - S dT - Q d\Phi + \Sigma d\epsilon, \quad dH = T dS - Q d\Phi + \Sigma d\epsilon, \quad dM_{AD} = T dS + \Phi dQ + \Sigma d\epsilon.
$$

These expressions are consistent with the interpretation that $F$, $H$ or $M_{AD}$ is the thermodynamic potential with natural variables $(T, \Phi, L)$, $(S, \Phi, L)$ or $(S, Q, L)$, because the system with fixed $L$ is equivalent to that with fixed $\epsilon$ due to the relation $\epsilon \propto L^2$.

Using the quantities $(\epsilon, \Sigma)$, the Legendre transform of the Hamiltonian gives

$$
H - \epsilon \Sigma = \frac{3}{2} TS.
$$

The Legendre transformations (4.8), (4.12) and (4.16) are summarized as follows:

$$
F \xrightarrow{T \rightarrow S} H \xrightarrow{\Phi \rightarrow Q} M_{AD} \xrightarrow{\epsilon \rightarrow \Sigma} W \xrightarrow{\Phi \rightarrow Q} M_K,
$$

where we set $W = \frac{3}{2} TS$. Note that the Legendre transformation between $M_K$ and $W$ is nothing but the Smarr-type formula (2.13). The quantity $W$ can also be interpreted as a thermodynamic potential with natural variables $(S, \Phi, \Sigma)$. 

4.2. Using the KK monopole background

Next, we evaluate the thermodynamic potentials and the gravitational tension by use of the boundary matched KK monopole background. The Hamiltonian of the SqKK black hole relative to the KK monopole background is calculated as

\[ \tilde{H} = \frac{\pi^2}{2G} r_\infty \left( 2\rho_+ - \rho_- + \rho_0 - \frac{r_\infty}{2} \right), \]  
(4.17)

which is related to the free energy on the KK monopole background as

\[ \tilde{F} = \tilde{H} - TS. \]  
(4.18)

This relation is the same as in the previous case using the flat background.

The AD mass (2.14) is calculated using the flat background. One might think that the KK monopole is useful as a reference spacetime for the AD mass, but the AD mass calculated by use of the boundary matched KK monopole spacetime diverges. Indeed, in the evaluation of the finite AD mass obtained in [26], the boundary of the KK monopole was not matched. However, its physical meaning is not clear because the disagreement of the period of the imaginary time between the SqKK black hole and the reference spacetime implies the disagreement in temperature, and thus the system is not in thermodynamic equilibrium. Actually, as discussed in [26], the AD mass evaluated on the boundary ‘unmatched’ KK monopole does not seem to be any sensible thermodynamic quantity because it does not satisfy any consistent expression for the first law. Therefore, we do not consider the AD mass in the thermodynamic formulation with the KK monopole background.

The gravitational tension using the boundary matched KK monopole background is

\[ \tilde{T} = \frac{1}{4G} (\rho_+ + \rho_- + 2\rho_0 - r_\infty), \]  
(4.19)

which contributes to the following expressions for the first law:

\[ d\tilde{F} = -SdT - Qd\Phi + \tilde{T}dL, \]  
(4.20)

\[ d\tilde{H} = TS - Qd\Phi + \tilde{T}dL. \]  
(4.21)

As in the case of the flat background, the Legendre transformation with respect to TL does not give the Komar mass, and \( T \) and \( L \) are not related to the first law including the Komar mass. Instead,

\[ \epsilon = L^2, \quad \Sigma = \frac{\tilde{T}}{2L} = \frac{1}{16\pi Gr_\infty} (\rho_+ + \rho_- + 2\rho_0 - r_\infty) \]  
(4.22)

are a couple of variables related to the Komar mass, because the following differential relation is satisfied:

\[ dM_K = TdS + \Phi dQ - \epsilon d\Sigma. \]  
(4.23)

Furthermore, the Legendre transformation between the free energy and the Komar mass is given as

\[ \tilde{F} = M_K - TS - Q\Phi + \epsilon \Sigma. \]  
(4.24)

One might think that the expression (4.23) conflicts with (4.11) because the Komar mass does not depend on the choice of the reference spacetime. But, (4.23) and (4.11) are consistent, because the difference between \( \Sigma \) and \( \tilde{\Sigma} \) is just a constant, i.e., \( d\Sigma = d\tilde{\Sigma} \). Again, \( (\epsilon, \Sigma) \) contribute to the differential relation satisfied by the free energy and the Hamiltonian as

\[ d\tilde{F} = -SdT - Qd\Phi + \tilde{\Sigma}d\epsilon, \]  
(4.25)
\[ d\tilde{H} = T \, dS - Q \, d\Phi + \Sigma \, d\epsilon. \]  \hfill (4.26)

Furthermore, the Legendre transform of the Hamiltonian with respect to \( \epsilon \Sigma \) gives \( W \). If we introduce a new quantity \( \tilde{M} \) as the Legendre transform of \( \tilde{H} \) with respect to \( \Phi \),

\[ \tilde{M} = \tilde{H} + Q\Phi, \]  \hfill (4.27)

it satisfies

\[ d\tilde{M} = T \, dS + \Phi \, dQ + \Sigma \, d\epsilon. \]  \hfill (4.28)

Thus, \( \tilde{M} \) can be considered as a thermodynamic potential having \( (S, Q, \epsilon) \) as natural variables.

Therefore, \( \tilde{M} \) corresponds to \( M_{AD} \) of the flat background.

The Legendre transformations between thermodynamic potentials evaluated on the KK monopole background are summarized as

\[ \begin{array}{ccc}
F & \rightarrow & S \\
H & \rightarrow & M \\
\epsilon & \rightarrow & \Sigma \\
W & \rightarrow & M_X.
\end{array} \]

In the formulation using the KK monopole background, we can take the \( r_\infty \rightarrow \infty \) limit, under which the \( \tau \) constant hypersurface becomes asymptotically Euclid space. As is expected, in this limit, \( \epsilon \Sigma \) and \( TL \) become zero, and \( M_K, F \) and \( H \) become those of a five-dimensional Reissner–Nordström black hole evaluated on the five-dimensional Minkowski reference spacetime. Thus, this formulation for the SqKK black hole includes the usual thermodynamic formulation for the five-dimensional Reissner–Nordström black hole as a limit. In this sense, it is a generalized formulation of thermodynamics for five-dimensional electrically charged static black holes.

5. Thermodynamic potentials in various thermodynamic environments

In thermodynamics, fixing a set of variables corresponds to specifying a thermodynamic environment characterized by those variables. In the previous section, we have obtained various thermodynamic quantities. Each mass and free energy has a set of natural variables or control parameters and can be interpreted as a thermodynamic potential for the environment characterized by the variables.

In this section, we obtain some free energies for different isothermal environments by use of the Legendre transformations, and discuss the state having the lowest value of the free energy, i.e., the so-called globally stable state in the sense of thermodynamics\(^{11}\). Generally, thermodynamical stability requires not only global stability but also local stability. However, we do not discuss the local stability of the SqKK black hole in this paper. This point will be mentioned in section 7.

As was discussed in [27], the thermodynamic environments with fixed \( L \) and those with fixed \( \epsilon \) are the same, but the thermodynamic environments with fixed \( T \) are different from that with fixed \( \Sigma \). Therefore, we treat \( L, T \) and \( \Sigma \) as independent thermodynamic variables related to the compact extra dimension.

As for the thermodynamic variable with respect to the electric field, we consider only \( \Phi \) in this section, because the quantity \( Q\Phi \) is positive definite and the Legendre transformation by use of \( Q\Phi \) does not change the relative order between the thermodynamic potentials. Therefore, the globally stable state in each environment with fixed \( Q \) is the same as that in the environment with fixed \( \Phi \).

\(^{11}\) Of course, it is not proved that the known solutions are all solutions in the five-dimensional Einstein–Maxwell system. Therefore, the meaning of the global stability discussed here is restricted to the solutions treated in this paper.
5.1. Environment with fixed \( (T, \Phi, L) \)

As discussed in section 3, equations (4.5) and (4.20) show that the free energy \( F \) or \( \tilde{F} \) determines the suitable thermal state for the thermodynamic environment with fixed \( (T, \Phi, L) \). Note that \( F \), which is the free energy of the SqKK black hole relative to the flat background, is positive definite. Hence the free energy of the flat background is the lowest, which implies that it is globally stable in this environment.

5.2. Environment with fixed \( (T, \Phi, T) \)

The thermodynamic potential available for the environment with fixed \( (T, \Phi, T) \) is given by

\[
\Omega_{T\Phi T} = F - TL = -\frac{\pi}{2G}(\rho_+ + \rho_0) r_\infty.
\]

(5.1)

This potential satisfies

\[
d\Omega_{T\Phi T} = -SdT - Qd\Phi - LdT.
\]

(5.2)

Thus, \( \Omega_{T\Phi T} \) is the thermodynamic potential having \( (T, \Phi, T) \) as natural variables. We note that \( \Omega_{T\Phi T} \) is always negative, which implies that the SqKK black hole may be more preferable than the flat background in this environment.

The thermodynamic potential of the KK monopole relative to the flat background can be obtained by taking the KK monopole limit \( \rho_\pm \to 0 \) with \( \rho_0 > 0 \) as

\[
\tilde{\Omega}_{T\Phi T} = -\frac{\pi}{4G} r_\infty^2.
\]

(5.3)

This is also negative definite. Thus, the KK monopole is more preferable than the flat background. The potential of the SqKK black hole relative to the KK monopole is

\[
\tilde{\Omega}_{T\Phi T} = \tilde{F} - TL = \frac{\pi}{2G} r_\infty \left(-\rho_- - \rho_0 + \frac{r_\infty}{2}\right).
\]

(5.4)

which satisfies

\[
d\tilde{\Omega}_{T\Phi T} = -SdT - Qd\Phi - Ld\tilde{T}.
\]

(5.5)

From (5.1), (5.3) and (5.4), it is checked that the thermodynamic potentials are well-defined relative quantities, i.e.,

\[
\Omega_{T\Phi T} = \tilde{\Omega}_{T\Phi T} + \tilde{\Omega}_{T\Phi T}.
\]

(5.6)

Therefore, as in the previous case, we can determine the most stable state among these solutions as the one that has the lowest value of the potential. Now, \( \Omega_{T\Phi T} \) is positive definite, because

\[
\tilde{\Omega}_{T\Phi T} = \frac{\pi}{2G} r_\infty \sqrt{\rho_- + \rho_0} \left(\sqrt{\rho_+ + \rho_0} - \sqrt{\rho_- + \rho_0}\right) > 0.
\]

(5.7)

Therefore, the KK monopole is the most stable among these solutions.

5.3. Environment with fixed \( (T, \Phi, \Sigma) \)

Now, we consider the thermodynamic environment controlled by \( \Sigma \). The thermodynamic potential relevant for this environment is

\[
\Omega_{T\Phi \Sigma} = F - \epsilon \Sigma = \frac{\pi}{4G} r_\infty (\rho_+ - \rho_-).
\]

(5.8)

which is positive definite and satisfies

\[
d\Omega_{T\Phi \Sigma} = -SdT - Qd\Phi - \epsilon d\Sigma.
\]

(5.9)
The potential of the SqKK black hole relative to the KK monopole is
\[ \tilde{\Omega}_{T\Phi} = \hat{F} - \epsilon \hat{\Sigma} = \frac{\pi}{4G}\infty(\rho_+ - \rho_-). \] (5.10)

It is seen that \( \tilde{\Omega}_{T\Phi} = \Omega_{T\Phi}, \) which implies that the potential of the KK monopole relative to the flat background equals zero. Actually, by taking the KK monopole limit in (5.8), we obtain
\[ \hat{\Omega}_{T\Phi} = 0. \] (5.11)

Therefore, in this environment, the flat background and the KK monopole can always be in phase equilibrium. From the viewpoint of statistical mechanics, it means that the contributions of these solutions to the partition function are the same. Both the flat background and the KK monopole are always the most stable.

Furthermore, for the extremal SqKK black hole, i.e., \( \rho_+ = \rho_- \), where horizons are degenerate and temperature becomes zero, it is clear that \( \Omega_{T\Phi} = \hat{\Omega}_{T\Phi} = 0 \). Therefore, if the extremal SqKK black hole is locally stable, the flat background, the KK monopole and the extremal SqKK black hole can be in phase equilibrium in the environment with \( T = 0 \).

6. Application to the black string

Now, we investigate the thermodynamical formulation for the charged black string using the two reference spacetimes: the flat background and the KK monopole. The metric of the black string is known as [46]
\[ ds^2 = -V(\rho) \, d\tau^2 + \left( 1 - \frac{\rho_+}{\rho} \right)^{-1} d\rho^2 + \rho^2 \left( 1 - \frac{\rho_-}{\rho} \right) d\Omega^2 + \left( 1 - \frac{\rho_-}{\rho} \right)^{-1} dz^2, \] (6.1)

where \( V(\rho) = (1 - \frac{\rho_+}{\rho})(1 - \frac{\rho_-}{\rho}) \) and \( z \) has a range \( 0 \leq z \leq L \). Then, \( L \) is the size of the extra dimension at infinity. The gauge field is given by (2.4). The black string is characterized by three parameters \( \rho_+ \), \( \rho_- \) and \( L \).

Firstly, we obtain thermodynamic quantities of the black string and investigate the first law, and then calculate the thermodynamic potential of the black string relative to the most stable state in each environment.

The thermodynamical quantities: the ADM mass, inverse of the temperature, entropy, electric charge and electrostatic potential of the black string are given by
\[ M_{\text{ADM}} = \frac{L}{4G}(2\rho_+ + \rho_-), \] (6.2)
\[ \beta = T^{-1} = 4\pi\rho_+ \sqrt{\rho_+/\rho_-}, \] (6.3)
\[ S = \frac{A_H}{4G} = \frac{\pi L}{G} \rho_+^2 \sqrt{1 - \rho_-/\rho_+}, \] (6.4)
\[ Q = \pm \sqrt{3} L \sqrt{\rho_+\rho_-}, \] (6.5)
\[ \Phi = \pm \sqrt{3} \sqrt{\rho_-/\rho_+}. \] (6.6)

Unlike the SqKK black hole, the boundary topology of the flat background is the same as that of the black string. If the boundary of the black string is set at the surface \( \rho = \rho_B \), the metric
of the boundary matched flat background is
\[ ds^2 = V(\rho_B) d\tau^2 + d\rho^2 + \rho^2 d\Omega^2 + \left(1 - \frac{\rho_\pm}{\rho_B}\right)^{-1} d\zeta^2. \] (6.7)

Here, the boundary of the flat background is set at
\[ \rho_{BR} = \rho_B \left(1 - \frac{\rho_\pm}{\rho_B}\right)^{1/2}, \] (6.8)
so as to match the circumference radius of the $S^2$. Then, the classical action relative to the flat background is
\[ I_E = \frac{L}{4G} \beta(\rho_+ - \rho_-). \] (6.9)

The free energy is obtained as
\[ F = \frac{I_E}{\beta} = \frac{L}{4G} (\rho_+ - \rho_-). \] (6.10)

With this reference spacetime, the Hamiltonian and gravitational tension of the black string can be calculated as
\[ H = \frac{L}{2G} (\rho_+ - \rho_-), \quad T = \frac{1}{4G} (\rho_+ - \rho_-). \] (6.11)

As is well known, the black string has the gravitational tension as a thermodynamic quantity and the following relations are satisfied:
\[ dF = -SdT - Qd\Phi + TdL, \] (6.12)
\[ dH =TdS - Qd\Phi + TdL, \] (6.13)
\[ dM_{ADM} = TdS + \Phi dQ + TdL. \] (6.14)

These potentials are related to each other via the Legendre transformations as
\[ F = H - TS = M_{ADM} - Q\Phi - TS. \] (6.15)

The five-dimensional Komar mass of the black string is calculated as
\[ M_K = -\frac{3}{32\pi G} \int dS_{\mu\nu} \nabla^\mu \xi^\nu = \frac{3}{8G} L(\rho_+ + \rho_-). \] (6.16)

As is the case for the SqKK black hole, the quantities
\[ \epsilon = L^2 \quad \text{and} \quad \Sigma = \frac{T}{2L} = \frac{1}{8GL}(\rho_+ - \rho_-) \] (6.17)
are relevant for the Komar mass, in the sense that the Komar mass satisfies
\[ dM_K = TdS + \Phi dQ - \epsilon d\Sigma. \] (6.18)

Furthermore, the relation $M_{ADM} = M_K + \epsilon \Sigma$ gives another expression for the first law for the ADM mass
\[ dM_{ADM} = TdS + \Phi dQ + \Sigma d\epsilon. \] (6.19)

Then, the Legendre transformations (6.15) lead to
\[ dF = -SdT - Qd\Phi + \Sigma d\epsilon, \] (6.20)
\[ dH = TdS - Qd\Phi + \Sigma d\epsilon. \] (6.21)
Again, the Smarr-type relation is obtained as
\[ M_K - Q\Phi = \frac{1}{2} TS =: W. \] (6.22)

The relations between thermodynamic potentials are summarized as
\[ F \rightarrow S \rightarrow H \rightarrow M_{ADM} \]
\[ \epsilon \rightarrow \Sigma \rightarrow W \rightarrow M_K. \]

This is the same as the case of the SqKK black hole, where the ADM mass of the string corresponds to the AD mass of the SqKK black hole.

The KK monopole background can be useful as a reference spacetime of the black string, though the boundary topology of the black string is not the same as that of the KK monopole. We set the boundary of the black string at the surface \( \rho = \rho_B \). The metric of the KK monopole can be rewritten as
\[ ds_E^2 = V(\rho_B) d\tau_E^2 + B_R(\rho) d\rho^2 + \rho^2 B_R(\rho) d\Omega_1^2 + \rho^2 (d\psi + \cos \theta d\phi)^2. \] (6.23)

where \( B_R(\rho) = 1 + \frac{\rho_0}{\rho} \). The boundary of the KK monopole is set at surface \( \rho = \rho_{BR} \). In order to match the period of the imaginary time and the size of \( S^1 \) at the boundary, we set parameters \( \rho_{BR} \) and \( \rho_{0R} \) as
\[ \rho_{BR} = \rho_B \left(1 - \frac{\rho_B - \rho}{4\pi \rho_B}\right)^{1/2}, \] (6.24)
\[ \rho_{0R} = \frac{\rho_B}{4\pi \rho_{BR}}. \] (6.25)

Then, the finite classical action is obtained as
\[ I_E = \frac{L}{4G} \beta \left(\rho_+ - \rho_- - \frac{L}{4\pi}\right). \] (6.26)

The free energy of the black string relative to the KK monopole is
\[ \tilde{F} = \frac{L}{4G} \left(\rho_+ - \rho_- - \frac{L}{4\pi}\right). \] (6.27)

The Hamiltonian and the gravitational tension can also be calculated by use of the KK monopole (6.23) as reference spacetime as
\[ \tilde{H} = \frac{L}{2G} \left(\rho_+ - \rho_- - \frac{L}{8\pi}\right), \quad \tilde{T} = \frac{1}{4G} \left(\rho_+ - \rho_- - \frac{L}{2\pi}\right). \] (6.28)

Using these quantities, the free energy and Hamiltonian satisfy
\[ d\tilde{F} = -S dT - Q d\Phi + \tilde{T} dL, \] (6.29)
\[ d\tilde{H} = T dS - Q d\Phi + \tilde{T} dL. \] (6.30)

The expression of the first law by the Komar mass can be written as
\[ dM_K = T dS + \Phi dQ - \epsilon d\Sigma, \] (6.31)

where
\[ \epsilon := L^2, \quad \Sigma := \frac{\tilde{T}}{2L} = \frac{1}{8GL} \left(\rho_+ - \rho_- - \frac{L}{2\pi}\right). \] (6.32)
The thermodynamic potentials are related by the following Legendre transformations:

\[ M_K - Q \Phi = W = \tilde{H} - \epsilon \tilde{\Sigma} = \tilde{F} + T S - \epsilon \tilde{\Sigma}, \] (6.33)

and thus, \( \tilde{\Sigma} \) can contribute to the expression of the first law for \( \tilde{F} \) and \( \tilde{H} \) as

\[
\begin{align*}
\tilde{F} &= -S dT - Q d\Phi + \tilde{\Sigma} d\epsilon, \\
\tilde{H} &= T dS - Q d\Phi + \tilde{\Sigma} d\epsilon.
\end{align*}
\] (6.34) (6.35)

If we introduce a new quantity \( \tilde{M} \) as the Legendre transform of \( \tilde{H} \) with respect to \( \Phi \),

\[ \tilde{M} = \tilde{H} + Q \Phi = \frac{L}{4G} \left( 2\rho_+ + \rho_- - \frac{L}{4\pi} \right), \] (6.36)

it satisfies

\[
\begin{align*}
d\tilde{M} &= T dS + \Phi dQ + \tilde{\Sigma} d\epsilon.
\end{align*}
\] (6.37)

Thus, \( \tilde{M} \) can be considered as a thermodynamic potential having \( (S, Q, \epsilon) \) as natural variables, and is a counterpart of the AD mass with respect to the KK monopole background.

Therefore, we can write the following map for the black string with the KK monopole background:

\[
\begin{array}{c}
\tilde{F} \xrightarrow{T \rightarrow S} \tilde{H} \xrightarrow{\Phi \rightarrow Q} \tilde{M} \\
\epsilon \rightarrow \tilde{\Sigma} \\
W \xrightarrow{\Phi \rightarrow Q} \tilde{M}_K.
\end{array}
\]

Now, it is found that we can formulate the thermodynamics of the black string in quite a similar way to that of the SqKK black hole, where the thermodynamic environments are characterized by the same sets of variables as those in the case of the SqKK black hole. In the remaining part of this section, we calculate the thermodynamic potential of the black string in two thermodynamic environments with fixed \((T, \Phi, \Sigma)\) and \((T, \Phi, \Sigma)\), taking the most stable state in each environment as a reference background.

When the set \((T, \Phi, \Sigma)\) is fixed, the most stable state is the KK monopole. The potential of the black string relative to the KK monopole is given as

\[ \Omega_{T \Phi \Sigma} = \tilde{F} - T L = \frac{L^2}{16\pi G}, \] (6.38)

This is positive definite. It is found that the potential (6.38) depends on neither \( \rho_+ \) nor \( \rho_- \). It implies that the potential relative to the flat background equals zero. Actually, it is calculated as

\[ \Omega_{T \Phi \Sigma} := F - T L = 0, \] (6.39)

which always vanishes. Thus, if the black string is locally stable in the sense of the thermodynamics, then the flat spacetime and the black string are in phase equilibrium.

This result holds generally for solutions without Misner string and bolt singularity with respect to isometry related to the gravitational tension. The reason is as follows: the gravitational tension can be obtained as a Hamiltonian along isometry in a compact spatial direction. In such a Hamiltonian formulation, the classical Euclidean action can be reconstructed from the gravitational tension (Hamiltonian) and contribution from Misner strings and bolt singularities as [45]

\[ I_E = \beta (LT + H_{MS}) - \frac{1}{4G} (A_{bolts} + A_{MS}), \] (6.40)
where $H_{MS}$ is the Hamiltonian surface term on the Misner strings, $A_{bolts}$ and $A_{MS}$ are, respectively, the total areas of the bolts and the Misner strings in the spacetime. In the present case, there is no Misner string nor bolt singularity in the spacetime as well as in the reference spacetime. Therefore,

$$I_E = \beta LT,$$  \hspace{1cm} (6.41)

which means that the Legendre transform of $F (= I_E / \beta)$ with respect to $L$, which is the thermodynamic potential for isothermal environment with fixed gravitational tension, vanishes.

When $\Sigma$ is a variable of the system, the KK monopole background is preferable as the flat background and either reference spacetime gives the same result. Here, we consider the flat background. For example, the potential relevant for the environment with fixed $(T_0, \Phi, \Sigma)$ is

$$\Omega_{T\Phi\Sigma} = F - \epsilon \Sigma = \frac{L}{8G}(\rho_+ - \rho_-),$$  \hspace{1cm} (6.42)

which is positive definite. As in the case of the SqKK black hole, for an extremal black string, i.e., $\rho_+ = \rho_-$, the potential vanishes.

As shown in this section, the KK monopole can be a background spacetime for the black string as well as the flat background. Especially, for thermodynamic environment with fixed $(T, \Phi, \Sigma)$, the KK monopole should be considered as a reference spacetime because the potential relative to the flat background is always negative.

7. Summary and discussion

The formulation of thermodynamics of the charged squashed Kaluza–Klein (SqKK) black hole and the charged black string has been investigated by the background subtraction scheme with two reference spacetimes. The natural reference spacetime for the SqKK black hole is the KK monopole because the boundaries of these spacetimes have the same topology. With this reference spacetime, we can take the spherically symmetric limit and reproduce the usual thermodynamic formulation for the five-dimensional Reissner–Nordström black hole in the limit. It has been shown for the SqKK black hole that the counterterm method is equivalent to the background subtraction method with the flat background, in which the boundary topology is different from that of the SqKK black hole. As in the case of the Taub-bolt instanton, the counterterm method prefers the gravitational action without perfect matching.

We have seen that the free energy, the Hamiltonian, the AD mass and the Komar mass are related by the Legendre transformation as thermodynamic potentials. By calculating thermodynamic potentials for a given thermodynamic environment, we can determine the most stable state by these quantities. In isothermal environments with fixed size of the extra dimension $L$, the flat background is the most stable. On the other hand, in those with fixed gravitational tension $T$, the KK monopole is the most stable. Furthermore, when we fix $\Sigma$ which is conjugate to $\epsilon = L^2$, the flat background and the KK monopole can be in phase equilibrium and both are the most stable.

We have constructed a thermodynamic formulation for the five-dimensional charged black string concretely, which shows that the thermodynamic variables are the same as those of the SqKK black hole. It has been shown that the KK monopole is useful as a reference spacetime for the charged black string. In particular, the KK monopole should be considered as a reference spacetime in isothermal environments with fixed gravitational tension, because the KK monopole is the most stable.

We are considering several future works related to the present paper. The thermodynamic formulation will be able to be generalized for the rotating charged SqKK black hole, which was recently found in [48]. This would be a combination of the work in [28] and the present
work. Further, it is interesting to investigate thermodynamic stability of the SqKK black hole and the black string using the thermodynamic formulation given in this paper, because there seems to be phase transition between the black string and the SqKK black hole and between the SqKK black hole and the flat background in some environments. However, when we discuss phase transition, the property of thermodynamical local stability must be clarified. If the entropy of a given system is additive, then the parameter region of local stability can be shown by examining the Hessian matrix of the entropy, but it is not trivial that black hole entropy is additive or not (see, for example, [49]). Therefore, other methods to investigate the local stability should be applied to this problem. We will report thermodynamic stability of these solutions in a future publication.

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