An averaging scheme for the efficient approximation of time-periodic flow problems

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We study periodic solutions to the Navier-Stokes equations. The transition phase of a dynamic Navier-Stokes solution to the periodic-in-time state can be excessively long and it depends on parameters like the domain size and the viscosity. Several methods for an accelerated identification of the correct initial data that will yield the periodic state exist. They are mostly based on space-time frameworks for directly computing the periodic state or on optimization schemes or shooting methods for quickly finding the correct initial data that yields the periodic solution. They all have a large computational overhead in common. Here we describe and analyze a simple averaging scheme that comes at negligible additional cost. We numerically demonstrate the efficiency and robustness of the scheme for several test-cases and we will theoretically show convergence for the linear Stokes problem.

1 Introduction

We study periodic solutions to the Navier-Stokes equations. Such solutions appear in the laminar regime for the flow around an obstacle but they can also be induced by a periodic forcing. Here we investigate this second case which is easier as the frequency of oscillation is known from the problem data. In the following we will consider the incompressible Navier-Stokes equations on a domain $\Omega \subset \mathbb{R}^d \ (d = 2, 3)$

$$\text{div } v = 0, \quad \partial_t v + (v \cdot \nabla)v - \nu \Delta v + \nabla p = f \text{ in } \mathbb{R}^d_+ \times \Omega, \quad v = v_0 \text{ on } \{0\} \times \Omega, \quad v = 0 \text{ on } \mathbb{R}^d_+ \times \partial \Omega, \quad (1)$$

where $v$ is the velocity, $p$ the pressure, $\nu > 0$ the viscosity and $f$ the right hand side, which we assume to be $P$-periodic in time

$$f(t + P) = f(t), \quad (2)$$

where $P > 0$ is a fixed period. We are looking for periodic solutions to (1) that satisfy $v(t + P) = v(t)$ and $p(t + P) = p(t)$. In general, two strategies for identifying such periodic solutions exist: starting with arbitrary initial values $v_0$ one lets the system run into the periodic state for $t \to \infty$, or, one tries to identify the correct initial value $v_0$ that will directly give the periodic state.

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The necessity to compute such cyclic states of the Navier-Stokes equations arises e.g. in the context of
temporal multi-scale schemes, where oscillatory periodic short-scale solutions guide efficient time-stepping
schemes that govern the long-scale dynamics [6, 26, 8, 7]. Another application is found in optimization
and steering processes like in simulated moving bed processes in chemical engineering [18, 15, 29].

The transition phase of a dynamic Navier-Stokes solution to the periodic state can be long and it
depends, usually exponentially, on the domain size, on problem parameters like the viscosity or on dis-
cretization parameters like temporal and spatial resolutions.

Several methods for an acceleration exist. One approach is based on a space-time framework for
directly computing the periodic state [18, 27]. This transforms the problem into a higher-dimensional
one with substantial numerical overhead. Another possibility is to find efficient ways for identifying
the correct initial data \( v_0 \) for the velocity \( v(0) = v_0 \) that gives a periodic temporal solution of period
\( P > 0 \) with \( v(t + P) = v(t) \) for all \( t \geq 0 \). The classical approach is the shooting method and it has
been demonstrated for systems of ordinary differential equations [25, 12]. Without special adaptions as
discussed by these authors, the shooting method requires the solution of a high-dimensional problem
which comes at significant costs if partial differential equations are considered. Yet another approach
casts the task of finding a periodic solution into the identification of the initial value \( v_0 \) that solves
the nonlinear problem \( F(v_0) = 0 \) where \( F(v_0) := v_{v_0}(P) - v_0 \), where \( v_{v_0}(t) \) is the dynamic solution
starting with \( v(0) = v_0 \). Applying Newton’s method to this problem yields a similar structure as the
shooting method. The authors of [19, 10] derived special preconditioners to avoid the large effort of
high-dimensional problems within the Newton scheme. Nonlinear parabolic problems are analyzed in [17]
and monotone iterative schemes are presented for converging to periodic solutions. Finding the zero of
\( F(v_0) = 0 \) can also be tackled as an optimization scheme for minimizing \( \| F(v_0) \|^2 \to 0 \). The authors
of [1, 24] apply different optimization schemes to accelerate this problem. This approach requires the
solution of backward in time adjoint problems. Finally, the authors of [15] propose an acceleration tool
for the forward simulation based on a cascadic multilevel method. While this approach requires the
forward simulation only, it might still suffer from a long transient phase.

Here, an accelerated scheme for the forward simulation is presented that is based on updating the initial
values using the solution of a stationary auxiliary problem. For the Stokes equations we give a proof of
the robust convergence of the scheme with a rate that does not depend on problem or discretization
parameters. We numerically demonstrate the efficiency of this scheme for the nonlinear Navier-Stokes
case.

In the following section we shortly introduce the Navier-Stokes equations and the notation required for
specifying cyclic states. Section 3 presents the averaging algorithm for accelerating convergence to such
periodic states. We give a complete analysis for the linear case of the Stokes equations and some hints
on treating the nonlinear Navier-Stokes equations. In Section 5 we present different numerical test cases
describing the robustness and efficiency of the suggested scheme. We conclude in Section 6 with a short
outlook to open problems.

2 Periodic-in-time flow problems

For the following, let \( \Omega \subset \mathbb{R}^d \) be a domain \( (d = 2 \) or \( d = 3 \)). By \( H^s(\Omega) \), for \( s \in \mathbb{N} \) with \( s \geq 1 \) we denote
the Sobolev space of \( L^2 \) functions on \( \Omega \) with \( s \)-th weak derivative in \( L^2 \). By \( H^1_0(\Omega) \) we denote the space
of \( H^1 \)-functions with trace zero on the boundary \( \partial \Omega \). By \( \| \cdot \| \) we denote the spatial \( L^2 \)-norm. Further
norms are specified by a corresponding index. We start by discussing the linear Stokes equations
Problem 1 (Stokes). Let $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$ be a domain with a boundary that is either smooth ($C^2$-parametrization) or polygonal with convex corners. Let $P > 0$ be the fixed period and

$$f \in L^\infty(\mathbb{R}; H^{-1}(\Omega)^d)$$

with $f(t, \cdot) = f(t + P, \cdot)$. For $v_0 \in L^2(\Omega)$ let

$$\text{div } v = 0, \quad \partial_t v - \nu \Delta v + \nabla p = f \text{ in } \mathbb{R}_+ \times \Omega, \quad v = v_0 \text{ on } \{0\} \times \Omega, \quad v = 0 \text{ on } \mathbb{R}_+ \times \partial \Omega \quad (3)$$

be the solution to the Stokes equations for $\nu > 0$.

Standard existence and regularity results [28] give a unique solution to Problem 1 that satisfies

$$\|v(t)\|^2 + \int_0^t \nu \|\nabla v(s)\|^2 \, ds \leq \|v_0\|^2 + \int_0^t \nu^{-1} \|f(s)\|_{H^{-1}}^2 \, ds \quad \forall t \in \mathbb{R}_+. \quad (4)$$

By Poincaré’s inequality $\|v\| \leq c_p \|\nabla v\|$ we obtain an integral equation for $\|v(t)\|^2$

$$\|v(t)\|^2 + \int_0^t \frac{\nu}{c_p^2} \|v(s)\|^2 \, ds \leq \|v_0\|^2 + \int_0^t \frac{1}{\nu} \|f(s)\|_{H^{-1}}^2 \, ds, \quad (5)$$

which shows that the solution is bounded for all $t \geq 0$

$$\|v(t)\|^2 \leq \|v_0\|^2 \exp \left(-\frac{\nu}{c_p^2} t\right) + \frac{c_p^2}{\nu^2} \sup_{s \in [0,t]} \|f\|_{H^{-1}}^2. \quad (6)$$

Next, let $w(t) := v(t + P) - v(t)$. This function satisfies the homogeneous version of equations (4) and (5) such that it holds

$$\|w(t)\| \leq \|v(P) - v(0)\| \exp \left(-\frac{\nu}{c_p^2} t\right) \xrightarrow{t \to \infty} 0. \quad (6)$$

Hence, the solution $v(t)$ runs into a unique periodic-in-time state for $t \to \infty$. Further (6) shows the potentially slow decay to the periodic solution depending on the viscosity $\nu$ and the Poincaré constant $c_p$.

These simply results do not carry over to the Navier-Stokes equations. This is mainly due to the small-data assumption that is required for the existence of global solutions and the uniqueness of stationary solutions, which is also of relevance for identifying periodic-in-time states.

Problem 2 (Navier-Stokes). Let $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$ be a domain with a boundary that is either smooth ($C^2$-parametrization) or polygonal with convex corners. Let $P > 0$ be the fixed period and

$$f \in L^\infty(\mathbb{R}; H^{-1}(\Omega)^d)$$

with $f(t, \cdot) = f(t + P, \cdot)$. For $v_0 \in L^2(\Omega)$ let

$$\text{div } v = 0, \quad \partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p = f \text{ in } \mathbb{R}_+ \times \Omega, \quad v = v_0 \text{ on } \{0\} \times \Omega, \quad v = 0 \text{ on } \mathbb{R}_+ \times \partial \Omega \quad (7)$$

be the solution to the Navier-Stokes equations for $\nu > 0$. 

3
It has been shown by Kyed and Galdi [14, 9] that the solution runs into a periodic-in-time state \((v^\pi, p^\pi)\) if the data is sufficiently small, i.e. if the right hand side \(f\), the initial value \(v_0\), the non-homogenous Dirichlet data \(v^D\) as well as the period length \(P\) are small and the viscosity \(\nu\) is sufficiently large.

If the proper initial \(v_0^\pi\) is known, one cycle of the Stokes or Navier-Stokes equations on \([0, P]\) will directly give the periodic solution. If however the exact initial value is not given it might take a tremendous number of cycles (of length \(P\)) to sufficiently reduce the periodicity error \(\|v^\pi(t) - v(t)\|\). Here we describe an acceleration approach that is based on projecting the approximated solution to one that already satisfies a correct temporal average condition. While the analysis is rigorous for the linear Stokes equations it remains an heuristic computational acceleration scheme in the nonlinear case. In contrast to the various approaches presented in the previous section, our averaging scheme only requires the solution of one linear stationary problem in each cycle as computational overhead.

### 3 Averaging scheme for identifying periodic solutions

A very slow decay to the periodic solution must be expected in the general case. Convergence rates

\[
\|v(t) - v^\pi(t)\| \leq C \exp \left(-\frac{\nu}{c_P t}\right) \|v_0 - v_0^\pi\|
\]

(8)

are also numerically observed, both for the Stokes and the Navier-Stokes equations, see Section 5 for a numerical illustration.

In this section we will derive an averaging scheme for accelerating the convergence to the periodic solution \(v^\pi(t)\) for arbitrary initial values \(v_0\). This averaging scheme is based on a splitting of the solution into its average and the fluctuations. On the interval \(I = [0, P]\) we introduce

\[
v(t) := \bar{v} + \tilde{v}(t), \quad \bar{v} := \frac{1}{P} \int_0^P v(s) \, ds, \quad \tilde{v}(t) := v(t) - \bar{v}.
\]

To speedup the process of finding the periodic solution we try to quickly adapt the average \(\bar{v}\). Averaging the Navier-Stokes equation, Problem 2, gives

\[
\begin{align*}
\frac{v(P) - v(0)}{P} + (\bar{v} \cdot \nabla)\bar{v} + \frac{1}{P} \int_0^P (\tilde{v}(s) \cdot \nabla)\tilde{v}(s) \, ds - \nu \Delta \bar{v} + \nabla \bar{p} &= f, \\
\text{div } \tilde{v}(t) &= 0
\end{align*}
\]

(9)

and reveals that average and fluctuation are coupled and cannot be computed separately. Assuming that \(v\) is the periodic solution \(v^\pi\) with \(v^\pi(t) = v^\pi(t + P)\) it holds

\[
(\tilde{v}^\pi \cdot \nabla)\tilde{v}^\pi + \frac{1}{P} \int_0^P (\tilde{v}^\pi(s) \cdot \nabla)\tilde{v}^\pi(s) \, ds - \nu \Delta \tilde{v}^\pi + \nabla \tilde{p}^\pi = \bar{f}, \quad \text{div } \tilde{v}^\pi = 0
\]

\[
\partial_t \tilde{v}^\pi(t) + (\tilde{v}^\pi(t) \cdot \nabla)\tilde{v}^\pi + (\tilde{v}^\pi \cdot \nabla)\tilde{v}^\pi(t) - \nu \Delta \tilde{v}^\pi(t) + \nabla \tilde{p}^\pi(t) = \bar{f}(t), \quad \text{div } \tilde{v}^\pi(t) = 0,
\]

In the averaging scheme we aim at correcting the initial value such that the correct average \(\tilde{v}^\pi\) is well approximated. For this, we introduce a problem for the correction \((\bar{w}, \tilde{q}) := (\tilde{v}^\pi - \bar{v}, \tilde{p}^\pi - \bar{p})\), i.e. for the difference between periodic solution and current approximation

\[
(\bar{w} \cdot \nabla)\bar{w} + (\bar{w} \cdot \nabla)\bar{v} + (\bar{v} \cdot \nabla)\bar{w} - \nu \Delta \bar{w} + \nabla \tilde{q}
\]

\[
= \frac{v(P) - v(0)}{P} + \frac{1}{P} \int_0^P \left( (\tilde{v}(s) \cdot \nabla)\tilde{v}(s) - (\tilde{v}^\pi(s) \cdot \nabla)\tilde{v}^\pi(s) \right) \, ds, \quad \text{div } \bar{w} = 0.
\]

(10)
Naturally, the fluctuation term on the right hand side cannot be computed without knowledge of the periodic fluctuations \( \tilde{v}^\pi(s) \). However, since \( \tilde{v} \) shall eventually converge to \( \tilde{v}^\pi \), and since adequate initial values will be required anyway, we simplify the equation by dropping this term. Likewise, we will drop the quadratic term \( (\bar{w} \cdot \nabla)\bar{w} \) since this is second order in \( \bar{w} \), which will converge to zero.

With these preparations we can formulate the approximated averaging scheme for finding periodic-in-time solutions to the Navier-Stokes equations

**Algorithm 3** (Averaging Scheme for the Navier-Stokes equations). Let \( v_0^{(0)} \in L^2(\Omega)^d \) be an initial value. For \( l = 1, 2, \ldots \) iterate

1. Solve the Navier-Stokes equation on \([0, P]\)
   \[ \partial_t v^{(l)}(t) - (v^{(l)} \cdot \nabla)v^{(l)} - \nu \Delta v^{(l)} + \nabla p^{(l)} = f, \quad \text{div } v^{(l)} = 0, \quad v^{(l)}(0) = v_0^{(l-1)}. \]

2. Compute the average in time
   \[ \bar{v}^{(l)} := \frac{1}{P} \int_0^P v^{(l)}(s) \, ds. \]

3. Compute the approximated stationary update problem
   \[ (\bar{w}^{(l)} \cdot \nabla)\bar{v}^{(l)} + (\bar{v}^{(l)} \cdot \nabla)\bar{w}^{(l)} - \nu \Delta \bar{w}^{(l)} + \nabla q^{(l)} = \frac{v^{(l)}(P) - v^{(l)}(0)}{P}, \quad \text{div } \bar{w}^{(l)} = 0. \]

4. Update the initial value
   \[ v_0^{(l)} := v^{(l)}(P) + \bar{w}^{(l)}. \]

The main effort of this scheme is still in the computation of the dynamic problems in Step 1. We will however observe an significant reduction of cycles required to approximate the periodic solution.

### 3.1 Analysis for the Stokes equations

The application of the averaging scheme to the Stokes equations significantly simplified the setting. First, this is due to a separation of average and fluctuations in equation (9). This allows to exactly compute the update problem (10) without further simplifications. Step 3 of Algorithm 3 can be replaced by solving

\[ -\nu \Delta \bar{w}^{(l)} + \nabla q^{(l)} = \frac{v^{(l)}(P) - v^{(l)}(0)}{P}, \quad \text{div } \bar{w}^{(l)} = 0. \] (11)

Second, the symmetric Stokes operator allows for a very simple analysis based on diagonalization. To be precise, under the assumptions of Problems 1 and Problem 2 there exists an orthonormal basis of weakly divergence free eigenfunctions \( \omega_i \in H^1_0(\Omega) \) and corresponding eigenvalues \( \lambda_i \in \mathbb{R} \) for \( i = 1, \ldots, \infty \) such that (see [28])

\[
(\nu \nabla \omega_i, \nabla \phi) - (\zeta_i, \text{div } \phi) + (\text{div } \omega_i, \xi) = \lambda_i(\omega_i, \phi) \quad \forall \phi \in H^1_0(\Omega), \xi \in L^2(\Omega) \setminus \mathbb{R}
\]

with \( (\omega_i, \omega_j)_\Omega = \delta_{ij} \) and \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) (12)

for all \( i, j \in \mathbb{N} \). Given \( v(t) \), we denote the expansion in the eigenfunctions as

\[ v(t) = \sum_{i \geq 1} v_i(t) \omega_i, \quad v_i(t) = (v(t), \omega_i)_\Omega. \]
A corresponding notation is used for further functions like the right hand side $f(t)$, the periodic solution $\mathbf{v}_i^\pi(t)$, the update $\mathbf{w}(t)$ or the initial data $\mathbf{v}_0$. Hereby the Stokes equation can be diagonalized and a simple decoupled system of ode’s results
\begin{equation}
\partial_t v_i(t) - \nu \lambda_i v_i(t) = f_i(t), \quad i = 1, 2, \ldots
\end{equation}
Let value $v_i(0) = v_{i,0} \in \mathbb{R}$ be the coefficients of the initial values, the solution to (13) is given by
\begin{equation}
v_i(t) = \exp\left(-\lambda_i \nu t\right)v_i(0) + \int_0^t f_i(s) \exp\left(-\lambda_i \nu (t-s)\right) \, ds.
\end{equation}

**Lemma 4** (Averaging Scheme for the Stokes equation). Let $\Omega \in \mathbb{R}^d$ for $d = 2$ or $d = 3$ be a domain with either convex polygonal or smooth ($C^2$-parametrization) boundary, $f \in L^\infty([0, P], H^{-1}(\Omega)^d)$ and $\mathbf{v}_0 \in L^2(\Omega)$. The averaging scheme applied to the Stokes equations converges like
\begin{equation}
\|\mathbf{v}_0^{(l)} - \mathbf{v}_0^\pi\| \leq 0.3 \cdot \|\mathbf{v}_0^{(l-1)} - \mathbf{v}_0^\pi\|,
\end{equation}
where $\mathbf{v}_0^\pi$ is the initial value that yields the periodic-in-time solution.

**Proof.** Let $v_{i,0}$ be the coefficients of the initial value $\mathbf{v}_0$, by $v_i(t)$ we denote the coefficient functions of the dynamic solution on $I = [0, P]$, by $v_i^\pi(t)$ the coefficient functions of the unknown periodic-in-time solution. We analyze one iteration of Algorithm 3 applied to the Stokes equations. Hence, $l = 1$ and we will drop this index. For the error $w_i(t) := v_i^\pi(t) - v_i(t)$ equation (14) gives
\begin{equation}
v_i^\pi(t) - v_i(t) = \exp\left(-\lambda_i \nu t\right)(v_i^\pi(0) - v_i(0))
\end{equation}
Step 2 can be omitted since the average does not enter the update equation in the linear case, compare (11). The solution to the stationary update equation of Step 3 is given by
\begin{equation}
\lambda_i \nu \bar{w}_i = \frac{v_i(P) - v_i(0)}{P} \quad \Rightarrow \quad \bar{w}_i = \frac{(v_i^\pi(0) - v_i(0)) - (v_i^\pi(P) - v_i(P))}{\lambda_i \nu P},
\end{equation}
where we used that $v_i^\pi(0) = v_i^\pi(P)$. With (15) this gives
\begin{equation}
\bar{w}_i = \frac{1}{\lambda_i \nu P} \left(1 - \exp\left(-\lambda_i \nu P\right)\right)(v_i^\pi(0) - v_i(0))
\end{equation}
Then, the newly computed initial value from Step 4 in the algorithm satisfies
\begin{equation}
v_i^\pi - (v_i(P) + \bar{w}_i) = (v_i^\pi(P) - v_i(P)) - \bar{w}_i = \left(\exp\left(-\lambda_i \nu P\right) \left(1 + \frac{1}{\lambda_i \nu P}\right) - \frac{1}{\lambda_i \nu P}\right)(v_i^\pi(0) - v_i(0))
\end{equation}
where we used (17) and (15). Let $s = \lambda_i \nu P > 0$. We study the function
\begin{equation}
\rho(s) = \exp(-s) \left(1 + \frac{1}{s}\right) - \frac{1}{s} = \frac{\exp(-s)}{s} \left(1 + s - \exp(s)\right),
\end{equation}
which is negative for $s \in \mathbb{R}_+$ since $\exp(s) \geq 1 + s$. Further it holds $\rho(0) = 0$ and $\rho(s) \to 0$ for $s \to \infty$. $\rho(s)$ has only one extreme point in $\mathbb{R}_+$ that is easily found numerically and gives the bound $|\rho(x)| < 0.299$ for $x \in \mathbb{R}_+$.

The convergence rate of this averaging scheme is always at least 0.3 and does in particular not depend on the stiffness rate $\lambda_i \nu > 0$ or the period length $P > 0$. 

6
4 Discrete setting

For discretization of the dynamic Navier-Stokes and Stokes equations we employ standard techniques which we summarize briefly. Details are found in [21, Sections 4.1 and 4.2]. All implementations are done in Gascoigne 3D [4]. In time we use the \( \theta \)-time-stepping scheme that can be considered as a variant of the Crank-Nicolson scheme [16, 11] with better smoothing properties capable of giving robust long time solutions. Let \( k = P/N \) be the time step size and \( t_n = nk \) for \( n = 0,1,\ldots \) be the uniform partitioning of \([0,P]\). By \( v_n \approx v(t_n) \) and \( p_n \approx p(t_n) \) we denote the approximation to the solution at time \( t_n \). Likewise \( f^n := f(t_n) \) is the right hand side evaluated at time \( t_n \).

Let \( V_h \times Q_h \subset H^1_0(\Omega; \Gamma^D)^d \times L^2(\Omega) \) be a suitable finite element pair. For simplicity assume that \( V_h \times Q_h \) is an inf-sup stable pressure-velocity pair like the Taylor-Hood element [5]. Alternatively, stabilized equal order finite elements, e.g. based on the local projection stabilization scheme [3], can be used. This is indeed the standard setting of our implementation in Gascoigne 3D [4]. The complete space-time discrete formulation of the incompressible Navier-Stokes equation is given by

\[
v^n \in V_h, \quad p^n \in V_h, \quad v^0 := v_0, \quad \text{for } n = 1,2,\ldots
\]

\[
\left( v^n - v^{n-1}, \phi \right) + k \left( (1-\theta)(v^{n-1} \cdot \nabla)v^{n-1} + \theta(v^n \cdot \nabla)v^n, \phi \right) + \nu k \left( \nabla((1-\theta)v^{n-1} + \theta v^n) \cdot \nabla \phi \right) - k \left( p^n, \nabla \cdot \phi \right) + k \left( \nabla \cdot v^n, \xi \right) = k \left( (1-\theta)f^{n-1} + \theta f^n, \phi \right) \quad \forall (\phi, \xi) \in V_h \times Q_h. \tag{20}
\]

The parameter \( \theta \) is chosen in \([1/2,1]\). For \( \theta = 1 \) this scheme corresponds to the backward Euler method, for \( \theta = 1/2 \) to the Crank-Nicolson scheme and for \( \theta > 1/2 \) to the shifted Crank-Nicolson scheme which has better smoothing properties. For \( \theta = 1/2 + O(\epsilon) \) we get global stability and still have second order convergence, see [16, 20, 23]. In addition to (20) we also consider the discretization of the Stokes equation, which is realized by skipping the convective terms.

To transfer the averaging scheme to the discrete setting we start by indicating a discrete counterpart of the averaging operator that is conforming with the discretization based on the \( \theta \)-scheme. We sum (20) over all time steps using the same pair of test functions \((\phi, \xi)\) for all steps and divide by the period \( P \)

\[
\frac{1}{P} \left( v^N - v^0, \phi \right) + \left( (v \cdot \nabla)v^k, \phi \right) + \nu \left( \nabla v^k, \nabla \phi \right) - \left( \bar{p}^{k,0}, \nabla \cdot \phi \right) + \left( \nabla \cdot v^k, \xi \right) = \left( \bar{f}^k, \phi \right) \tag{21}
\]

with the discrete averaging operators

\[
v^k := \frac{k}{P} \sum_{n=1}^{N} (1-\theta)v_{n-1} + \theta v_n, \quad \bar{p}^{k,0} := \frac{k}{P} \sum_{n=1}^{N} p_n. \tag{22}
\]

Directly summing the divergence term in (20) yields the average \((\nabla \cdot \bar{v}^{k,0}, \xi) = 0\), this however is equivalent to \((\nabla \cdot \bar{v}^{k,0}, \xi) = (\nabla \cdot \bar{v}^k, \xi)\). The discrete periodic solution \((v^\pi, p^\pi)\) satisfies the equation

\[
\left( (v^\pi \cdot \nabla)v^{\pi,k}, \phi \right) + \nu \left( \nabla v^{\pi,k}, \nabla \phi \right) - \left( \bar{p}^{\pi,k,0}, \nabla \cdot \phi \right) + \left( \nabla \cdot v^{\pi,k}, \xi \right) = \left( \bar{f}^k, \phi \right), \tag{23}
\]

such that the approximated discrete update equation, i.e. the equation for approximating the difference \((\bar{w}^k, \bar{q}^{k,0}) \approx (v^{\pi,k} - v^k, \bar{p}^{\pi,k,0} - \bar{p}^k)\) is given by

\[
\left( (\bar{w}^k \cdot \nabla)v^k + (\bar{v}^k \cdot \nabla)\bar{w}^k, \phi \right) + \nu \left( \nabla v^k, \nabla \phi \right) - \left( \bar{q}^{k,0}, \nabla \cdot \phi \right) + \left( \nabla \cdot \bar{w}^k, \xi \right) = \frac{1}{P} \left( v_N - v^0, \phi \right), \tag{24}
\]
where the same approximations are applied as in the continuous case: we neglect the averaging error \((w \cdot \nabla)w^k - (w^k \cdot \nabla)w^k\) as well as the quadratic nonlinearity \((w^k \cdot \nabla)w^k\).

**Algorithm 5** (Discrete averaging Scheme for the Navier-Stokes equations). Let \(v^{(0)}_0 \in V_h\) be an initial value. For \(l = 1, 2, \ldots\) iterate

1. Solve the Navier-Stokes (20) equation on \([0, P]\) for \((v_{n}^{(l)}, p_{n}^{(l)})\), \(n = 1, \ldots, N\) with \(v_0^{(l)} = v_0^{(l-1)}\).
2. Compute \(\bar{v}^{(l)}_k\), the discrete average in time, by (23).
3. Compute the approximated (stationary) update problem (24) for \(w^{(l)}_k, q^{(l)}_{k,0}\).
4. Update the initial value
   \[ v_0^{(l)} := v_N^{(l)} + \bar{w}^{(l)}_k. \]

Since we consider conforming finite element discretizations the proof for the linear case can be transferred to the discrete setting. Some care is required to obtain sufficient stability of the time stepping scheme.

**Lemma 6** (Discrete averaging Scheme for the Stokes equation). Let \(\Omega \subset \mathbb{R}^d\) for \(d = 2\) or \(d = 3\) be a domain with either convex polygonal or smooth \((C^2\text{-parametrization})\) boundary, \(f \in L^\infty([0, P], H^{-1}(\Omega)^d)\) and \(v_0 \in L^2(\Omega)\). Let \(V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2(\Omega)\) be an inf-sup stable finite element space. Let \(I = [0, P]\) be discretized in \(N \geq 4\) time steps of size \(k = P/N\) with time-scheme parameter
   \[ \theta_N = \frac{1}{2} + \frac{1}{2N}. \]

Then, the discrete averaging Algorithm 5 applied to the Stokes equations converges like
   \[ \|v_0^{(l)} - v_0^\pi\| \leq 0.42 \cdot \|v_0^{(l-1)} - v_0^\pi\|. \]

**Proof.** Let \((\omega_i, \mu_i, \lambda_i) \in V_h \times Q_h \times \mathbb{R}\) for \(i = 1, \ldots, N_h := \text{div}(V_h)\) be a system of \(L^2\)-orthonormal, discretely divergence free eigenfunctions and eigenvalues of the discrete Stokes operator. By \(v_{i,0}, f_i \in \mathbb{R}\) and \(v_i(t) \in \mathbb{R}\) we denote the coefficient (function) of an initial value \(v_0\) of the right hand side and of the solution \(v(t)\) with respect to this basis. Comparable to the continuous case we analyze one single step of the discrete averaging scheme given in Algorithm 5.

\((i)\) The discretized Stokes equation can be decoupled into a system of \(N_h\) difference equations
   \[ v_i^n - v_i^{n-1} + \nu k \lambda_i ((1 - \theta)v_i^{n-1} + \theta v_i^n) = \nu k ((1 - \theta)f_i^{n-1} + \theta f_i^n), \quad i = 1, \ldots, N_h. \]

We measure the difference \(v_i^{\pi,n} - v_i^n\) between the solution to the correct initial value \(v_{0,i}\) and to an arbitrary starting value \(v_{0,i}\)
   \[ v_i^{\pi,n} - v_i^n = 1 - \frac{\nu k (1 - \theta) \lambda_i}{1 + \nu k \theta \lambda_i} w_i^{n-1} = q_i (v_{0,i} - v_{0,i}). \]

The coefficients of the stationary update equation in Step 3 of Algorithm 5 are determined by
   \[ \bar{w}_i^k = \frac{v_i^n - v_i^0}{\nu \lambda_i P}. \]
and with \( v_i^{\pi,N} = v_i^{\pi,0} = v_{0,i}^{\pi} \) the new initial computed in Step 4 of Algorithm 5 is carrying the error

\[
v_{0,i}^{\pi} - (v_i^N + w_i^k) = v_i^{\pi,N} - v_i^N + \frac{v_i^{\pi,N} - v_i^{\pi,0} - (v_{0,i}^{\pi} - v_{0,i}^{0})}{\nu \lambda_i P},
\]

which, together with (25), is estimated as

\[
v_{0,i}^{\pi} - (v_i^N + w_i^k) = (q_i^N (1 + \frac{1}{\nu \lambda_i P}) - \frac{1}{\nu \lambda_i P}) (v_{0,i}^{\pi} - v_{0,i}^{0}).
\]

With \( s_i := \nu \lambda_i P \in [s_1, s_{|V_h|}] \subset (0, \infty) \) we identify the reduction factor

\[
\rho^N(s) := \left(1 - \frac{s}{N + \theta s}\right)^N \left(1 + \frac{1}{s}\right) - \frac{1}{s},
\]  

which for \( N \to \infty \) converges to the reduction rate \( \rho(s) \) of the continuous case, see (19).

(ii) For estimating (26) we consider two cases. First, let \( 0 \leq s \leq N \). Then, it holds

\[
0 \leq 1 - \frac{s}{N} \leq 1 - \frac{s}{N + \theta s} \leq 1 - \frac{s}{N(1 + \theta)}
\]  

and hereby, we can estimate \( \rho^N(s) \) to both sides by

\[
\left(1 - \frac{s}{N}\right)^N \left(1 + \frac{1}{s}\right) - \frac{1}{s} \leq \rho^N(s) \leq \left(1 - \frac{s}{N(1 + \theta)}\right)^N \left(1 + \frac{1}{s}\right) - \frac{1}{s}
\]

\[
\Leftrightarrow \exp(-s) \left(1 + \frac{1}{s}\right) - \frac{1}{s} \leq \rho^N(s) \leq \exp\left(-\frac{s}{1 + \theta}\right) \left(1 + \frac{1}{s}\right) - \frac{1}{s}.
\]

The lower bound is \( \rho(s) \), see Eq. (19), with \(-0.29 < \rho(s) \) for all \( s \geq 0 \). The upper bound takes its maximum at \( s \to 0 \)

\[
\exp\left(-\frac{s}{1 + \theta}\right) \left(1 + \frac{1}{s}\right) - \frac{1}{s} \xrightarrow{s \to 0} \frac{\theta}{1 + \theta}
\]

such that it holds

\[
-0.3 \leq \rho^N(s) \leq \frac{\theta}{1 + \theta}, \quad \forall 0 \leq s \leq N.
\]

For \( \theta = \theta_N \) and \( N \geq 4 \) this gives the bound

\[
|\rho^N|_{\theta=\theta_N} \leq 0.39 \quad \forall 0 \leq s \leq N.
\]

Next, let \( s \geq N \). The function

\[
q(s) := 1 - \frac{s}{N + \theta s}
\]

is monotonically decreasing, such that for \( s \geq N \) it holds

\[
1 - \frac{1}{\theta} \leq q(s) = 1 - \frac{s}{N + \theta s} \leq 1 - \frac{1}{1 + \theta} \quad \Rightarrow \quad |q(s)| \leq \frac{1}{\theta} - 1.
\]

Choosing \( \theta = \theta_N \) and \( N \geq 2 \) gives

\[
|q(s)^N| \leq \left|\frac{N - 1}{N + 1}\right|^N \leq \left|1 - \frac{2}{N + 1}\right|^N \leq \exp(-2).
\]
Altogether we estimate for \( s \geq N \) with \( N \geq 4 \) we get the bound
\[
|\rho_N(s)|_{\theta=\theta_N} \leq \exp(-2) \left( 1 + \frac{1}{N} \right) + \frac{1}{N} \leq 1.25 \exp(-2) + 0.25 \leq 0.42.
\]

\[\square\]

**Remark 7.** Numerical results show that this estimate is not sharp, convergence rates close to 0.3 are observed (which is the reduction rate in the continuous setting). The specific choice of
\[
\theta_N = \frac{1}{2} + \frac{1}{2N}
\]
corresponds to a slightly shifted version of the Crank-Nicolson scheme. It is of second order and has improved stability properties. The choice of \( \theta_N \) corresponds to \( \theta = \frac{1}{2} + \frac{2}{T}k \) and it can be generalized to \( \theta = \frac{1}{2} + \alpha k \) for \( \alpha > 0 \). In the numerical test cases we observe no problems with the standard Crank-Nicolson scheme \( \theta = 1/2 \).

## 5 Numerical test cases

We discuss different numerical test cases to highlight the efficiency and robustness of the averaging scheme for the computation of cyclic states. We directly consider the Navier-Stokes equations but include problems at very low Reynolds numbers. The linear Stokes problem gives comparable results. Here, perfect robustness of the averaging scheme for arbitrary variations of the viscosity, the domain size, the velocity, etc. if found.

Before presenting the specific test cases we shortly describe the computational setting implemented in the finite element software library *Gascoigne 3D* [4]: Discretization (in 2d) is based on quadrilateral meshes. To cope with the saddle-point structure we utilize stabilized quadratic equal-order elements. Stabilization is based on the local projection scheme [3]. As the appearing Reynolds numbers are very moderate we do not require any stabilization of convective terms. Nonlinear problems are approximated with a Newton scheme using analytic Jacobians. The equal-order setup allows us to use an efficient geometric multigrid solver for all linear problems, see [2] for the general setup and [13] for the efficient implementation. Although the analysis shows superior robustness for a shifted version of the Crank-Nicolson scheme with \( \theta > 1/2 \) we do not observe any difficulties with the choice \( \theta = 1/2 \) which will be used throughout this section. In [22] an application of the averaging scheme to the efficient simulation of temporal multiscale problems is given. Here we also include a study on a three dimensional test case. These results are in perfect agreement to the following two dimensional cases.

### 5.1 Robustness of the averaging scheme

For \( L \in \mathbb{R}_+ \) let \( \Omega = (-L,L)^2 \) and \( I = [0,P] \) for a given \( P \in \mathbb{R}_+ \). We solve

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0, & & \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{f} \quad \text{in } I \times \Omega, \\
\mathbf{v} &= \mathbf{v}_0 \quad \text{on } \{0\} \times \Omega, & & \mathbf{v} = 0 \quad \text{on } I \times \partial \Omega,
\end{align*}
\]

and try to identify a time-periodic solution \( \mathbf{v}(P) = \mathbf{v}(0) \). The forcing \( \mathbf{f} \) is \( P \)-periodic and given by
\[
f(x,y,t) = \frac{\tanh(y)}{LP} \sin \left( \frac{2\pi t}{P} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
We start by analyzing the dependency of the two approaches “forward simulation” (F) and “averaging scheme” (A) on the domain size $L$, which enters the Poincaré constant and the smallest eigenvalue, the viscosity $\nu$ that directly influences the reduction rate (19), further the period length $P$. The results are shown in Fig. 1. We also tested the robustness versus changes in the temporal step size $k = P/N$ and the mesh size $h$. There was however no influence of these parameters on the iteration counts, neither in the case of the forward simulation nor for the averaging scheme.

Enlarging the domain has a dramatic effect on the forward simulation (F) as it is shown in the upper/left plot of Figure 1. The rate of convergence strongly changes. For $L = 4$ or $L = 8$ the required tolerance of $10^{-8}$ could no be reached within the allowed 50 cycles. In contrast, the averaging scheme (A) is very robust. We see a slight dependency of its convergence rate on the parameter $P$.

In Table 1 we indicate the convergence rates of the forward simulation and the averaging scheme in dependency of the parameters, i.e. the experimental reduction rate $\sigma$ defined by

$$
\sigma^{(l)} := \frac{\|v_0^{(l)} - v_0^{(l-1)}\|}{\|v_0^{(l-1)} - v_0^{(l-2)}\|}.
$$
Table 1: Convergence rate $\sigma = \|v_0^{(l)} - v_0^{(l-1)}\|/\|v_0^{(l-1)} - v_0^{(l-2)}\|$. Left: forward simulation. Right: averaging scheme. Variation of the domain size $L$, the period $P$ and the viscosity $\nu$.

The rate of the averaging scheme is very robust. All test cases show $\sigma < 0.29$, which is the limit shown in the continuous case of Lemma 4. In contrast, the simple forward iteration shows deteriorating convergence rates for smaller viscosities and period lengths and larger domains and behaves like

$$\sigma = O\left(\exp\left(-\frac{\nu P}{L^2}\right)\right)$$

which is expected from the theoretical analysis, see (6).

The results for the averaging scheme suggest that the analysis in the discrete case in Lemma 6 is not sharp. The bound 0.29 for the convergence rate identified in the continuous case in Lemma 4 also appears to be valid in the discrete setting.

### 5.2 Robustness with regard to the Reynolds number

As second test case we study the Navier-Stokes flow in an annulus with outer radius $R = 5$ and inner radius $r = 0.5$, see Fig. 2. On both rings we drive the flow by a Dirichlet condition. While the outer ring is rotating, an inflow/outflow condition with an oscillating direction is prescribed on the inner ring. The maximum velocity on the boundary reaches $|v| = 0.5$. The period is chosen as $P = 1$ and we choose $N = 20$ time steps for each cycle. Biquadratic finite elements on an isoparametric mesh with a total of $N_{dofs} = 49,920$ degrees of freedom are used. Like in the previous test case we could not identify any dependency of the convergence rates on the temporal step size $k = 1/N$ or the number of spatial unknowns $N_{dofs}$.

In Figure 3 we show the required number of steps for different viscosities $\nu$ in order to investigate the influence of the nonlinearity. With $|v| = 1/2$, $diam(\Omega) = 10$ we compute the Reynolds number as

$$Re = \frac{|v|R}{\nu} = \frac{5}{\nu}.$$ 

Variation of the viscosity from $\nu = 1$ to $\nu \approx 0.0078$ corresponds to a range of Reynolds numbers from $Re = 5$ to $Re = 640$. For smaller viscosities we could not identify any periodic solution. In the case of very large viscosity parameters $\nu = 1$ there is no benefit of the averaging scheme (A).

Increasing the Reynolds number causes a significant increase of required iterations in the case of the forward simulation (F) while the necessary iterations for the averaging scheme (A) stays constant until we leave the regime of periodic solutions. In between the savings are substantial.
\[ v_r = \frac{1}{2} \begin{pmatrix} \cos(2\pi t/P) \\ \sin(2\pi t/P) \end{pmatrix} \]

\[ v_R = \frac{\sin(2\pi t/P)}{2R} \begin{pmatrix} -y \\ x \end{pmatrix} \]

\[ r = \frac{1}{2}, \quad R = 5, \quad P = 1 \]

---

**Figure 2:** Configuration of the second Navier-Stokes test case. The flow is driven by periodic Dirichlet conditions on both boundaries, the inner circle \( \Gamma_r \) with radius \( r = 0.5 \) and the outer circle \( \Gamma_R \) with radius \( R = 5 \).

---

**Figure 3:** Required number of cycles for the averaging scheme and the forward iteration to reach the tolerance error \( 10^{-8} \) depending on the Reynolds number \( Re \).

---

### 6 Conclusion

We have presented an acceleration scheme for the computation of periodic solutions to the Stokes and Navier-Stokes equations. In the linear Stokes case we could show robust convergence of the scheme with a rate that does not depend on the problem parameters or the eigenvalues of the Stokes operator. Applied to the Navier-Stokes equations numerical tests predict the same robustness and efficiency of the algorithm. The only numerical overhead of the proposed algorithm is the computation of one stationary averaged problem in every cycle of the dynamic process. Depending on the problem data, which strongly effects the decay rate of direct forward simulations, we get a significant speed-up.

The proof of convergence is based on the linearity and symmetry of the Stokes operator. The extension to the nonlinear Navier-Stokes equations will call for a different approach.

An interesting but open extension of the averaging scheme will be the application to problems with unknown periodicity. A possible application is the laminar vortex shedding of the flow around an obstacle. Here, predictions of the frequency are available with the Strouhal number, the exact value however is
depending on the specific configuration, in particular on the geometry. To tackle such problems we aim at the combination of the averaging scheme for obtaining initial values $v_0$ with an optimization approach to identify the period length $P$. The difficulty of such settings is the higher Reynolds number that is usually involved. There is only a small regime, where the solution is nonstationary with a clear periodic-in-time solution.

Finally, the proposed scheme allows to accelerate several problems where the computation of cyclic states is an algorithmic sub-task such as temporal multi-scale problems [8, 7, 22].

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