MONOPOLES, CURVES AND RAMANUJAN

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Abstract. We develop the Ercolani-Sinha construction of SU(2) monopoles and make this effective for (a five parameter family of centred) charge 3 monopoles. In particular we show how to solve the transcendental constraints arising on the spectral curve. For a class of symmetric curves the transcendental constraints become a number theoretic problem and a recently proven identity of Ramanujan provides a solution.

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1. Introduction

This article deals with the construction of magnetic monopoles, algebraic curves subject to certain transcendental constraints and number theory. Magnetic monopoles, or the topological soliton solutions of Yang-Mills-Higgs gauge theories in three space dimensions, have been objects of fascination for over a quarter of a century. BPS monopoles in particular have been the focus of much research (see [MS04] for a recent review). These monopoles arise as a limit in which the Higgs potential is removed and satisfy a first order Bogomolny equation

$$B_i = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} F^{jk} = D_i \Phi$$

(together with certain boundary conditions, the remnant of the Higgs potential). Here $F_{ij}$ is the field strength associated to a gauge field $A$, and $\Phi$ is the Higgs field. These equations may be viewed as a dimensional reduction of the four dimensional self-dual equations upon setting all functions independent of $x_4$ and identifying $\Phi = A_4$. Just as Ward’s twistor transform relates instanton solutions in $\mathbb{R}^4$ to certain holomorphic vector bundles over the twistor space $\mathbb{C}\mathbb{P}^3$, Hitchin showed [Hit82] that the dimensional reduction leading to BPS monopoles could be made at the twistor level as well. Mini-twistor space is a two dimensional complex manifold isomorphic to $T\mathbb{P}^1$, and BPS monopoles may be identified with certain bundles over this space. In particular a curve $\mathcal{C} \subset T\mathbb{P}^1$, the spectral curve, arises in this construction and, subject to certain nonsingularity conditions, Hitchin was able to prove all monopoles could be obtained by this approach [Hit83]. Nahm also gave a transform of the ADHM instanton construction to produce BPS monopoles [Nah82]. The resulting Nahm’s equations have Lax form and the corresponding spectral curve is again $\mathcal{C}$.

These curves $\mathcal{C}$ are the curves of the title. Hitchin’s construction (as we shall soon recall) places various transcendental constraints upon them and the outstanding and difficult problem is to construct curves satisfying these constraints. Now, given an appropriate curve, Ercolani and Sinha showed in their seminal paper [ES89] how one could solve (a gauge transform of) the Nahm equations in terms of a Baker-Akhiezer function for the curve $\mathcal{C}$. The authors have extended this work [BE06] and, given a curve, one can solve both the Nahm data and reconstruct the desired monopole in terms of standard integrable systems constructions. Our purpose here is not to focus on this integrable system side of the story but on the curves $\mathcal{C}$ and we shall only cite various formulae as needed to illustrate the type of detail needed of a curve in order to implement the construction. The number theory (and in particular ‘Ramanujan’ of the title) appear when we try to implement the transcendental constraints on the curve and construct the required quantities associated with the curve. This is the story we now recount.

An outline of our article is as follows. In section 2 we recall the relevant aspects of the Hitchin and Ercolani-Sinha constructions. This recounts the constraints of Hitchin and we highlight the ingredients needed to make effective the integrable system construction and show how these reduce to evaluating quantities intrinsic to the curve. The first hurdle in implementing the construction is to analytically determine the period matrix for $\mathcal{C}$ and then understand the theta divisor. In section 3 we will introduce a class of (genus 4) curves for which we can do this. They are of the form

$$\eta^3 + \chi(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)(\zeta - \lambda_4)(\zeta - \lambda_5)(\zeta - \lambda_6) = 0,$$

where $\lambda_i$, $i = 1, \ldots, 6$ are distinct complex numbers. (For appropriate $\lambda_i$ this yields a charge 3 monopole.) This class of curves was studied by Wellsstein over one hundred years...
ago [Wel99] and more recently by Matsumoto [Mat01]. Here we will introduce our homology basis and define branch points in terms of $\theta$-constants following [Mat01].

Corresponding to (some of) Hitchin’s nonsingularity conditions Ercolani and Sinha obtain restrictions on the allowed period matrices for the spectral curve. Equivalent formulations of these conditions were given in [HMR00]. The Ercolani-Sinha conditions are transcendental constraints and to solve these is the next (perhaps the) major hurdle to overcome in the construction. In section 4 we do this for our curves. At this stage we have replaced the constraints by relations between various hypergeometric integrals. To simplify matters for the present paper we next demand more symmetry and consider in section 5 the genus 4 curves

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$$

where $b$ is a certain real parameter. This restriction has the effect of reducing the number of hypergeometric integrals to be calculated to two. Interestingly the relations we demand of these integrals are assertions of Ramanujan only recently proven. We will denote curves of the form (1.2) as symmetric monopole curves (though in fact they may not satisfy all of Hitchin’s nonsingularity conditions). The tetrahedrally symmetric charge 3 monopole is of this form.

The curve (1.2) covers a hyperelliptic curve of genus two and two elliptic curves. We discuss these coverings. Using Weierstrass-Poincaré reduction theory we are able to express the theta function behaviour of these symmetric monopoles in terms of elliptic functions and fairly comprehensive results may be obtained. Finally, in section 6, we shall consider the curve (1.2) associated with tetrahedrally symmetric 3-monopole when the above parameter $b = 2\sqrt{5}$. This genus 4 curve covers 4 elliptic curves and all entries to the period matrices are expressible in terms of elliptic moduli. The analytical means which we are using for our analysis involve Thomae-type formulae, Weierstrass-Poincaré reduction theory, multivariable hypergeometric function and higher hypergeometric equalities of Goursat. Our conclusions in section 7 will highlight various of our results.

This article is based upon the second part of the preprint [BE06]. New theoretical results and the unwieldy length of that paper have led us to separate general constructions from the exploration of the transcendental constraints upon which we now report.

2. The Monopole Spectral Curve

In this section we shall recount the transcendental constraints placed upon the spectral curve coming from Hitchin’s construction. We shall then describe those quantities related to the curve needed in the Ercolani-Sinha construction and its extensions.

2.1. Hitchin Data. Using twistor methods Hitchin [Hit83] has shown that each static $SU(2)$ Yang-Mills-Higgs monopole in the BPS limit with magnetic charge $n$ is equivalent to a spectral curve of a restricted form. If $\zeta$ is the inhomogeneous coordinate on the Riemann sphere, and $(\zeta, \eta)$ are the standard local coordinates on $\mathbb{P}^1$ (defined by $(\zeta, \eta) \rightarrow \eta \frac{d}{d\zeta}$), the spectral curve is an algebraic curve $C \subset \mathbb{P}^1$ which has the form

$$P(\eta, \zeta) = \eta^n + \eta^{n-1}a_1(\zeta) + \ldots + \eta^r a_{n-r}(\zeta) + \ldots + \eta a_{n-1}(\zeta) + a_n(\zeta) = 0.$$  

Here $a_r(\zeta)$ (for $1 \leq r \leq n$) is a polynomial in $\zeta$ of maximum degree $2r$.

The Hitchin data constrains the curve $C$ explicitly in terms of the polynomial $P(\eta, \zeta)$ and implicitly in terms of the behaviour of various line bundles on $C$. If the homogeneous coordinates of $\mathbb{P}^1$ are $[\zeta_0 : \zeta_1]$ we consider the standard covering of this by the open sets $U_0 = \{[\zeta_0 : \zeta_1] | \zeta_0 \neq 0 \}$ and $U_1 = \{[\zeta_0 : \zeta_1] | \zeta_1 \neq 0 \}$, with $\zeta = \zeta_1/\zeta_0$ the usual coordinate
on $U_0$. We will denote by $\hat{U}_{0,1}$ the pre-images of these sets under the projection map $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$. Let $L^\lambda$ denote the holomorphic line bundle on $T\mathbb{P}^1$ defined by the transition function $g_{01} = \exp(-\lambda \eta/\zeta)$ on $\hat{U}_0 \cap \hat{U}_1$, and let $L^\lambda(m) \equiv L^\lambda \otimes \pi^*\mathcal{O}(m)$ be similarly defined in terms of the transition function $g_{01} = \zeta^m \exp(-\lambda \eta/\zeta)$. A holomorphic section of such line bundles is given in terms of holomorphic functions $f_\alpha$ on $\hat{U}_\alpha$ satisfying $f_\alpha = g_{\alpha \beta} f_\beta$. We denote line bundles on $C$ in the same way, where now we have holomorphic functions $f_\alpha$ defined on $C \cap \hat{U}_\alpha$.

The Hitchin data constrains the curve to satisfy:

**H1.** Reality conditions

\begin{equation}
(2.2) \quad a_r(\zeta) = (-1)^r \zeta^{2r} a_r(-1/\zeta).
\end{equation}

**H2.** $L^2$ is trivial on $C$ and $L(n-1)$ is real.

**H3.** $H^0(C, L^\lambda(n-2)) = 0$ for $\lambda \in (0, 2)$.

The reality conditions express the requirement that $C$ is real with respect to the standard real structure on $T\mathbb{P}^1$

\begin{equation}
(2.3) \quad \tau : (\zeta, \eta) \mapsto (-\frac{1}{\zeta}, -\frac{\bar{\eta}}{\zeta^2}).
\end{equation}

This is the anti-holomorphic involution defined by reversing the orientation of the lines in $\mathbb{R}^3$. A consequence of the reality condition is that we may parameterise $a_r(\zeta)$ as follows,

\begin{equation}
(2.4) \quad a_r(\zeta) = \sum_{k=0}^{2r} a_{rk} \zeta^k = \chi_r \left[ \prod_{l=1}^{r} \left( \frac{\eta}{\alpha_l} \right) \right]^{1/2} \prod_{k=1}^{r} (\zeta - \alpha_k)(\zeta + \frac{1}{\alpha_k}), \quad \alpha_r \in \mathbb{C}, \: \chi_r \in \mathbb{R}.
\end{equation}

Thus each $a_r(\zeta)$ contributes $2r+1$ (real) parameters. Certainly this constraint may be readily implemented. The remaining two constraints are however transcendental. The triviality of $L^2$ on $C$ means that there exists a nowhere-vanishing holomorphic section. In terms of our open sets $\hat{U}_{0,1}$ we then have two, nowhere-vanishing holomorphic functions, $f_0$ on $\hat{U}_0 \cap C$ and $f_1$ on $\hat{U}_1 \cap C$, such that on $\hat{U}_0 \cap \hat{U}_1 \cap C$

\begin{equation}
(2.5) \quad f_0(\eta, \zeta) = \exp \left\{ -\frac{2\eta}{\zeta} \right\} f_1(\eta, \zeta).
\end{equation}

Ercolani and Sinha utilize this to give an alternate characterisation we will present in due course.

For a generic $n$-monopole the spectral curve is irreducible and has genus $g_C = (n-1)^2$. This may be calculated as follows. For fixed $\zeta$ the $n$ roots of $P(\eta, \zeta) = 0$ yield an $n$-fold covering of the Riemann sphere. The branch points of this covering are given by

\[ 0 = \text{Resultant}_\eta(P(\eta, \zeta), \partial_\eta P(\eta, \zeta)) = \prod_{i=1}^{n} \partial_\eta P(\eta_i, \zeta), \quad \text{where} \quad P(\eta_i, \zeta) = 0. \]

This expression is of degree $n \times \deg a_{n-1} = n(2n-2)$ in $\zeta$ and so by the Riemann-Hurwitz theorem we have that

\[ 2g_C - 2 = 2n(g_{P_1} - 1) + n(2n-2) = 2(n-1)^2 - 2, \]

whence the genus as stated.
The \( n = 1 \) monopole spectral curve is given by
\[
\eta = (x_1 + i x_2) - 2 x_3 \zeta - (x_1 - i x_2) \zeta^2,
\]
where \( x = (x_1, x_2, x_3) \) is any point in \( \mathbb{R}^3 \). In general the three independent real coefficients of \( a_1(\zeta) \) may be interpreted as the centre of the monopole in \( \mathbb{R}^3 \). Strongly centred monopoles have the origin as center and hence \( a_1(\zeta) = 0 \). The group \( SO(3) \) of rotations of \( \mathbb{R}^3 \) induces an action on \( TP^1 \) via the corresponding \( PSU(2) \) transformations. If
\[
\begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \in PSU(2), \quad |p|^2 + |q|^2 = 1,
\]
the transformation on \( TP^1 \) given by
\[
\zeta \mapsto \frac{p \zeta - \bar{q}}{q \zeta + \bar{p}}, \quad \eta \mapsto \frac{\eta}{(q \zeta + \bar{p})^2}
\]
corresponds to a rotation by \( \theta \) around \( n \in S^2 \), where \( n_1 \sin(\theta/2) = \text{Im} q, n_2 \sin(\theta/2) = -\text{Re} q, n_3 \sin(\theta/2) = -\text{Im} q, \) and \( \cos(\theta/2) = \text{Re} p \). (Here the \( \eta \) transformation is given by the derivative of the \( \zeta \) transformation.) The \( SO(3) \) action commutes with the real structure \( \tau \). Although a general Möbius transformation does not change the period matrix of a curve \( \mathcal{C} \) only the subgroup \( PSU(2) < PSL(2, \mathbb{C}) \) preserves the desired reality properties. We have that
\[
\alpha_k \rightarrow \tilde{\alpha}_k = \frac{p \alpha_k + q}{\bar{p} - \alpha_k q}, \quad \chi_r \rightarrow \tilde\chi_r \equiv \chi_r \prod_{k=1}^r \left[ \frac{(p - \alpha_k q)(\bar{\alpha}_k \bar{p} + q)(\alpha_k p + q)}{\alpha_k \bar{\alpha}_k} \right]^{1/2},
\]
and
\[
a_r \rightarrow \frac{\tilde{\alpha}_r}{(q \zeta + \bar{p})^{2r}} = \frac{\tilde{\chi}_r}{(q \zeta + \bar{p})^{2r}} \prod_{l=1}^r \left( \frac{\tilde{\alpha}_l}{\tilde{\alpha}_l} \right)^{1/2} \prod_{k=1}^r (\zeta - \tilde{\alpha}_k)(\zeta + \frac{1}{\tilde{\alpha}_k}).
\]
In particular the form of the curve does not change: that is, if \( a_r = 0 \) then so also \( \tilde{a}_r = 0 \). It is perhaps worth emphasising that the reality conditions are an extrinsic feature of the curve (encoding the space-time aspect of the problem) whereas the intrinsic properties of the curve are invariant under birational transformations or the full Möbius group. Such extrinsic aspects are not a part of the usual integrable system story.

2.2. Quantities associated to the curve needed for reconstruction. In order to implement the reconstruction of the Nahm data and monopole associated to a curve \( \mathcal{C} \) various other quantities are also required. We shall now describe these and in so doing express Ercolani and Sinhas’s alternate characterisation of one of Hitchin’s transcendental constraints.

The spectral curve (2.1) is an \( n \)-sheeted cover of \( P^1 \). By a rotation if necessary it is always possible to choose \( n \) distinct preimages \( \{ \infty_j \} \) of \( \zeta = \infty \) and consequently the roots of \( P(\eta, \zeta)/\zeta^{2n} \) near \( \zeta = \infty \) behave as
\[
P(\eta, \zeta)/\zeta^{2n} \sim \prod_{j=1}^n \left( \frac{\eta}{\zeta^2} - \rho_j \right),
\]
where the \( \rho_j \) may be assumed distinct. As a consequence we see that at \( \infty_j \) we have
\[
\frac{\eta}{\zeta} \sim \rho_j \zeta, \quad d\left( \frac{\eta}{\zeta} \right) \sim \rho_j d\zeta = \left( -\frac{\rho_j}{\zeta^3} + O(1) \right) dt,
\]
where \( t = 1/\zeta \) is a local coordinate. Also from (2.4) we have that at \( \zeta = 0 \)

\[
P(\eta, 0) = \prod_{j=1}^{n} (\eta + \rho_j).
\]

(2.8)

In view of (2.7) there exists a differential \( \gamma_\infty \), a sum of differentials of the second kind, such that

\[
\gamma_\infty(P) = \left( \frac{\rho_l}{t^2} + O(1) \right) dt, \quad \text{as} \quad P \to \infty_l,
\]

(2.9)

\[
\int_{a_k} \gamma_\infty(P) = 0, \quad \forall k = 1, \ldots, g.
\]

(2.10)

Here \( \{a_k, b_k\} \) are a canonical homology basis for \( C \). On the integrable systems side of the story this differential yields the flow on the Jacobian corresponding to the time evolution of the system. This direction is given by

\[
U_k = \frac{1}{2\pi i} \oint_{b_k} \gamma_\infty(P).
\]

(2.11)

In practice one does not need \( \gamma_\infty(P) \) explicitly but (2.11) and

\[
\nu_j = \lim_{P \to -\infty_j} \left( \int_{0}^{P} \gamma_\infty(P) + \frac{\eta}{\zeta} \right).
\]

(2.12)

At this stage we have not imposed the Hitchin constraints H2, H3 on our curve \( C \). Ercolani and Sinha now make the following connection between (2.11) and H2. Consider the logarithmic derivative of (2.5) representing the triviality of \( L^2 \) on \( C \),

\[
d\log f_0 = d\left( -\frac{2\eta}{\zeta} \right) + d\log f_1.
\]

(2.13)

(Hurtubise considered a similar construction in the \( n = 2 \) case [Hur83].) Now in order to avoid essential singularities in \( f_{0,1} \) we have from (2.6, 2.8) that

\[
d\log f_1(P) = \left( \frac{2\eta_j(0)}{\zeta^2} + O(1) \right) d\zeta = \left( \frac{2\rho_j(0)}{\zeta^2} + O(1) \right) d\zeta, \quad \text{at} \quad P \to 0_j,
\]

(2.14)

\[
d\log f_0(P) = \left( \frac{2\rho_j}{t^2} + O(1) \right) dt, \quad \text{at} \quad P \to \infty_j.
\]

(2.15)

Because \( f_0 \) is a function on \( U_0 = C \setminus \{P_j\}_{j=1}^{n} \), then

\[
\exp \oint_{\lambda} d\log f_0 = 1,
\]

(2.16)

for all cycles \( \lambda \) from \( H_1(\mathbb{Z}, C) \). A similar result follows for \( f_1 \) and upon noting (2.13) we may define

\[
m_j = -\frac{1}{2\pi i} \oint_{a_j} d\log f_0 = -\frac{1}{2\pi i} \oint_{a_j} d\log f_1,
\]

(2.17)

\[
n_j = \frac{1}{2\pi i} \oint_{b_j} d\log f_0 = \frac{1}{2\pi i} \oint_{b_j} d\log f_1.
\]

(2.18)
Further, in view of (2.9) and (2.15), we may write

\[(2.19) \quad \gamma_\infty(P) = \frac{1}{2} d\log f_0(P) + i\pi \sum_{j=1}^{g} m_j v_j(P),\]

where \(v_j\) are canonically \(a\)-normalized holomorphic differentials. Integrating \(\gamma_\infty\) around \(b\)-cycles leads to the Ercolani-Sinha constraints

\[\oint_{b_k} \gamma_\infty = i\pi n_k + i\pi \sum_{l=1}^{g} \tau_{kl} m_l,\]

which are necessary and sufficient conditions for \(L^2\) to be trivial when restricted to \(\mathcal{C}\). Thus

\[(2.20) \quad U = \frac{1}{2} n + \frac{1}{2} \tau m.\]

Therefore the vector \(2U \in \Lambda\), the period lattice for the curve \(\mathcal{C}\), and so the “winding-vector” vector is a half-period. Note that \(U \neq 0\) or otherwise \(\gamma_\infty\) would be holomorphic contrary to our choice.

Using the bilinear relations we have that, for any holomorphic differential \(\Omega\),

\[\sum_{i=1}^{g} U_i \oint_{a_i} \Omega = \frac{1}{2\pi i} \sum_{i=1}^{g} \left( \oint_{a_i} \Omega \oint_{b_i} \gamma_\infty(P) - \oint_{b_i} \Omega \oint_{a_i} \gamma_\infty(P) \right) = \sum_{i=1}^{n} \text{Res}_{P \to \infty, \gamma_\infty(P)} \int_{P_0}^{P} \Omega.\]

Houghton, Manton and Ramão utilise this expression to express a dual form of the Ercolani-Sinha constraints (2.20). Define the 1-cycle

\[(2.21) \quad \mathbf{c} = \sum_{i=1}^{g} (n_i a_i + m_i b_i).\]

Then (upon recalling that \(\tau_{ik} \oint_{a_k} \Omega = \oint_{b_l} \Omega\), where \(\tau\) is the period matrix) we have the equivalent constraint:

\[(2.22) \quad \oint_{\mathbf{c}} \Omega = 2 \sum_{i=1}^{n} \text{Res}_{P \to \infty, \gamma_\infty(P)} \int_{P_0}^{P} \Omega.\]

The right-hand side of this equation is readily evaluated. We may express an arbitrary holomorphic differential \(\Omega\) as,

\[(2.23) \quad \Omega = \frac{\beta_0 \eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\sum_{i=1}^{n} \prod_{j \neq i} (\eta/\zeta^2 - \mu_j(1/\zeta))} d\zeta,

\[= \frac{\beta_0(\eta/\zeta^2)^{n-2} + \hat{\beta}_1(1/\zeta)(\eta/\zeta^2)^{n-3} + \ldots + \hat{\beta}_{n-2}(1/\zeta)}{\sum_{i=1}^{n} \prod_{j \neq i} (\eta/\zeta^2 - \mu_j(1/\zeta))} d\zeta/\zeta^2,\]

where \(\beta_j(\zeta) \equiv \zeta^{2j} \hat{\beta}_j(1/\zeta)\) is a polynomial of degree at most 2\(j\) in \(\zeta\). Thus using (2.6) we obtain

\[\sum_{i=1}^{n} \text{Res}_{P \to \infty, \gamma_\infty(P)} \int_{P_0}^{P} \Omega = -\sum_{i=1}^{n} \frac{\beta_0 \rho_i^{n-1} + \hat{\beta}_1(0)\rho_i^{n-2} + \ldots + \hat{\beta}_{n-2}(0)\rho_i}{\prod_{j \neq i} (\rho_i - \rho_j)} = -\beta_0,\]

upon using Lagrange interpolation. At this stage we have from the condition H2,

**Lemma 2.1 (Ercolani-Sinha Constraints).** The following are equivalent:

1. \(L^2\) is trivial on \(\mathcal{C}\).
There exists a 1-cycle \( \mathbf{c} = n \cdot \mathbf{a} + m \cdot \mathbf{b} \) such that for every holomorphic differential \( \Omega \):

\[
\oint_{\mathbf{c}} \Omega = -2 \beta_0,
\]

(3) \( 2U \in \Lambda \iff \)

\[
U = \frac{1}{2\pi i} \left( \oint_{b_1} \gamma_\infty, \ldots, \oint_{b_g} \gamma_\infty \right)^T = \frac{1}{2} n + \frac{1}{2} \tau m.
\]

Here (2) is the dual form of the Ercolani-Sinha constraints given by Houghton, Manton and Ramão. Their 1-cycle generalises a similar constraint arising in the work of Corrigan and Goddard [CG81]. The only difference between (3) and that of Ercolani-Sinha Theorem II.2 is in the form of \( U \) in which we disagree. We also know that \( U \neq 0 \).

The Ercolani-Sinha constraints impose \( g \) conditions on the period matrix of our curve. We have seen that the coefficients \( a_r(\zeta) \) each give \( 2r + 1 \) (real) parameters, thus the moduli space of charge \( n \) centred \( SU(2) \) monopoles is

\[
\sum_{r=2}^{n} (2r + 1) - g = (n + 3)(n - 1) - (n - 1)^2 = 4(n - 1)
\]

(real) dimensional.

The 1-cycle appearing in the work of Houghton, Manton and Ramão further satisfies

**Corollary 2.2** (Houghton, Manton and Ramão, 2000). \( \tau, \mathbf{c} = -\mathbf{c} \).

This result is the dual of Hitchin’s remark [Hit83, p164] that the triviality of \( L^2 \) together with the antiholomorphic isomorphism \( L \cong L^* \) yields an imaginary lattice point with respect to \( H^1(\mathcal{C}, \mathbb{Z}) \subset H^1(\mathcal{C}, \mathcal{O}) \).

The final constraint of Hitchin and various quantities needed for our purposes are encoded in the expressions

\[
\Psi_j(z, P) = g_j(P) \frac{\theta_{\mp, \frac{\tau}{2}} \left( \phi(P) - \phi(\infty_j) + z \mathbf{U} - \mathbf{K} \right)}{\theta_{\mp, \frac{\tau}{2}} \left( \phi(P) - \phi(\infty_j) - \mathbf{K} \right)} \frac{\theta_{\mp, \frac{\tau}{2}} \left( z \mathbf{U} - \mathbf{K} \right)}{e^{\int_0^P \gamma_\infty \cdot z \nu_j}},
\]

which we shall now unpack. Here \( \theta_{\mp, \frac{\tau}{2}} \) are theta functions with characteristics, \( \phi \) is the Abel map, \( z \in (-1, 1) \), and \( P \in \mathcal{C} \). (Our conventions for theta functions are given in the Appendix.) The vector \( \mathbf{K} \) plays a special role in the monopole construction. It is defined by

\[
\mathbf{K} = \mathbf{K} + \phi \left( (n - 2) \sum_{k=1}^{n} \mathbf{\infty}_k \right),
\]

where \( \mathbf{K} \) is the vector of Riemann constants (our conventions regarding this are also given in the Appendix). We have that

1. \( \mathbf{K} \) is independent of the choice of base point of the Abel map;
2. \( \theta(\mathbf{K}) = 0 \);
3. \( 2\mathbf{K} \in \Lambda \);
4. for \( n \geq 3 \) we have \( \mathbf{K} \in \Theta_{\text{singular}} \).
The point $\tilde{K}$ is the distinguished point Hitchin uses to identify degree $g - 1$ line bundles with $\text{Jac}(C)$. The proof of these properties together with the following lemma further constraining the Ercolani-Sinha vector may be found in [BE06]:

**Lemma 2.3.** $U \pm \tilde{K}$ is a non-singular even theta characteristic.

The expressions (2.20) are the components of a Baker-Akhiezer function; they are sections of a line bundle $L^{z+1}(n-1)$. The $g_j(P)$ which we have left unspecified form a basis of the holomorphic sections of $L(n-1)$. (They can be explicitly described in terms of theta functions and the quantities introduced already, but we do not need this for our present purposes.) Now the full condition $H_3$ is that $L^{z+1}(n-2) \in \text{Jac}^{g-1}(C) \setminus \Theta$ for $z \in (-1,1)$. This constraint must be checked using knowledge of the $\Theta$ divisor. The exact sequence $\mathcal{O}(L^s) \hookrightarrow \mathcal{O}(L^s(n-2))$ given by multiplication by a section of $\pi^*\mathcal{O}(n-2)|_C$ does however give us the necessary condition

\begin{equation}
H^0(C, \mathcal{O}(L^s(n-2))) = 0 \implies H^0(C, \mathcal{O}(L^s)) = 0, \quad s \in (0,2).
\end{equation}

If $L^s$ were trivial we would have a section, contradicting this vanishing result. The same treatment given to the triviality of $L^2$ shows that if $L^s$ were trivial then $sU \in \Lambda$. Therefore (2.27) shows that $sU \not\in \Lambda$ for $s \in (0,2)$. Thus $2U$ is a primitive vector in $\Lambda$ and we obtain the final part of the Ercolani-Sinha constraints,

\begin{equation}
2U \quad \text{is a primitive vector in } \Lambda \iff \epsilon \quad \text{is primitive in } H_1(C, \mathbb{Z}).
\end{equation}

The Ercolani-Sinha constraints (2.23) or (2.24) place $g$ transcendental constraints on the spectral curve $C$ and a major difficulty in implementing this construction has been in solving these, even in simple examples. The case of $n = 2$ has been treated by several authors and going beyond these results we consider the case of $n = 3$ in the remaining portion of this work. Before doing this let us record the remaining expressions needed for the integrable system reconstruction which are to be found in an object $Q_0(z)$ used by Ercolani-Sinha to reconstruct the Nahm data. The work [BE06] gives

**Theorem 2.4.** The matrix $Q_0(z)$ (which has poles of first order at $z = \pm 1$) may be written

\begin{equation}
Q_0(z)_{ij} = \epsilon_{ij} \frac{\rho_j - \rho_i}{E(\infty, \infty)} e^{2\pi i \tilde{q}(\operatorname{ph}(\infty))} \frac{\theta(\operatorname{ph}(\infty) - \operatorname{ph}(\infty_j)) + [z + 1]U - \tilde{K}}{\theta([z + 1]U - \tilde{K})} \epsilon^{(n_i - n_j)}.
\end{equation}

Here $E(P, Q)/(\sqrt{dx(P)dx(Q)})$ is the Schottky-Klein prime form, $U - \tilde{K} = \frac{1}{2}\hat{p} + \frac{1}{2}\tau \hat{q}$ \ ($\hat{p}, \hat{q} \in \mathbb{Z}^g$) is a non-singular even theta characteristic, and $\epsilon_{ij} = \epsilon_{ij} = \pm 1$ is determined (for $j < 1$) by $\epsilon_{ij} = \epsilon_{ij+1} \epsilon_{ij+1+j} \cdots \epsilon_{i-1}$. The $n - 1$ signs $\epsilon_{ij+1} = \pm 1$ are arbitrary.

In passing we note that a formula with similar features was obtained by Dubrovin [Dub77] when giving a theta-functional solution to the Euler equation describing motion of the $n$-dimensional rigid body. (Dubrovin’s curve was of course different and was free of Hitchin’s constraints.) This theorem encodes the data needed to reconstruct the Nahm data and monopoles given a suitable spectral curve. The whole construction is predicated on the theta functions built from the spectral curve. Thus we need

1. To construct the period matrix $\tau$ associated to $C$.
2. To determine the half-period $\tilde{K}$.
3. To determine the Ercolani-Sinha vector $U$.
4. For normalised holomorphic differentials $\nu$ to calculate $\int_{\infty_i}^{\infty_j} \nu = \frac{\theta(\infty_j) - \theta(\infty_i)}{\theta(\infty_j - \infty_i)}$.
5. To determine $E(\infty_j, \infty_i)$. 

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(6) To determine $\gamma_\infty(P)$ and $\nu_i = \lim_{P \to \infty} \left( \int_{P_0}^P \gamma_\infty(P') + \frac{2}{\xi}(P) \right)$.

3. The trigonal curve

We shall now introduce the class of curves that will be the focus of our attention. These are

\begin{equation}
\eta^3 + \hat{\chi}((\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)(\zeta - \lambda_4)(\zeta - \lambda_5)(\zeta - \lambda_6) = 0.
\end{equation}

For suitable $\lambda_i$ they correspond to centred charge three monopoles restricted by $a_2(\zeta) = 0$. Thus the eight dimensional moduli space of centred monopoles has been reduced to three dimensions. The asymptotic behaviour of the curve gives us

\begin{equation}
\rho_k = -\frac{1}{3} e^{2\pi k/3}.
\end{equation}

For notational convenience we will study \((3.1)\) in the form \((w = -\hat{\chi} - \frac{1}{4} \eta, z = \zeta)\)

\begin{equation}
w^3 = \prod_{i=1}^6 (z - \lambda_i).
\end{equation}

The moduli space of such curves with an homology marking can be regarded as the configuration space of six distinct points on $\mathbb{P}^1$. This class of curves has been studied by Picard \cite{Pic83}, Wellstein \cite{Wel99}, Shiga \cite{Shi88} and more recently by Matsumoto \cite{Mat01}; we shall recall some of their results. To make concrete the $\theta$-functions arising in the Ercolani-Sinha construction we need to have the period matrix for the curve, the vector of Riemann constants, and to understand the special divisors. We shall now make these things explicit, beginning first with our choice of homology basis.

3.1. The curve and homologies. Let $\mathcal{C}$ denote the curve \((3.3)\) of genus four where the six points $\lambda_i \in \mathbb{C}$ are assumed distinct and ordered according to the rule $\arg(\lambda_1) < \arg(\lambda_2) < \ldots < \arg(\lambda_6)$. Let $\mathcal{R}$ be the automorphism of $\mathcal{C}$ defined by

\begin{equation}
\mathcal{R} : (z, w) \mapsto (z, \rho w), \quad \rho = \exp\{2\pi i/3\}.
\end{equation}

The bilinear transformation $(z, w) \leftrightarrow (Z, W)$

\begin{equation}
Z = \frac{(\lambda_2 - \lambda_1)(z - \lambda_4)}{(\lambda_2 - \lambda_4)(z - \lambda_1)},
\end{equation}

\begin{equation}
W = -\frac{w}{(z - \lambda_1)^{\frac{1}{3}}} \left( \prod_{k=2}^6 (\lambda_1 - \lambda_k) \right)^{\frac{1}{3}} \left( \frac{(\lambda_1 - \lambda_4)(\lambda_1 - \lambda_2)}{\lambda_2 - \lambda_4} \right)^{\frac{1}{6}}
\end{equation}

and its inverse

\begin{equation}
z = \frac{Z\lambda_1(\lambda_2 - \lambda_4) + \lambda_4(\lambda_1 - \lambda_2)}{Z(\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_1)}
\end{equation}

\begin{equation}
w = -\frac{W}{(Z(\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_1))^2} \left( \prod_{k=2}^6 (\lambda_1 - \lambda_k) \right)^{\frac{1}{3}} \left( \lambda_1 - \lambda_2 \right)^{\frac{1}{6}}(\lambda_1 - \lambda_4)^{\frac{1}{6}}(\lambda_2 - \lambda_4)^{\frac{1}{6}}
\end{equation}

leads to the following normalization of the curve \((3.14)\)

\begin{equation}
W^3 = Z(Z - 1)(Z - \Lambda_1)(Z - \Lambda_2)(Z - \Lambda_3),
\end{equation}

\begin{footnote}{Here $\{\lambda_i\}_{i=1}^6 = \{\alpha_j, -1/3\}_{j=1}^3$ and $\hat{\chi} = \chi_3 \left[ \prod_{i=1}^3 \left( \frac{\pi}{\alpha_i} \right)^{1/2} \right]$.}
\end{footnote}
The following lexicographical ordering of independent canonical holomorphic differentials of $\mathcal{C}$,

$$
du_1 = \frac{dz}{w}, \quad du_2 = \frac{dz}{w^2}, \quad du_3 = \frac{zdz}{w^2}, \quad du_4 = \frac{z^2dz}{w^2}. $$

To construct the symplectic basis $(a_1, \ldots, a_4; b_1, \ldots, b_4)$ of $H_1(\mathcal{C}, \mathbb{Z})$ we introduce oriented paths $\gamma_i(z_i, z_j)$ going from $P_i = (z_i, w_i)$ to $P_j = (z_j, w_j)$ in the $k$-th sheet. Define 1-cycles $a_i, b_i$ on $\mathcal{C}$ as follows:

$$
a_1 = \gamma_1(\lambda_1, \lambda_2) + \gamma_2(\lambda_2, \lambda_1), \quad b_1 = \gamma_1(\lambda_2, \lambda_1) + \gamma_3(\lambda_1, \lambda_2),
$$

$$
a_2 = \gamma_1(\lambda_3, \lambda_4) + \gamma_2(\lambda_4, \lambda_3), \quad b_2 = \gamma_1(\lambda_4, \lambda_3) + \gamma_3(\lambda_3, \lambda_4),
$$

$$
a_3 = \gamma_1(\lambda_5, \lambda_6) + \gamma_2(\lambda_6, \lambda_5), \quad b_3 = \gamma_1(\lambda_6, \lambda_5) + \gamma_3(\lambda_5, \lambda_6),
$$

$$
a_4 = \gamma_3(\lambda_1, \lambda_2) + \gamma_1(\lambda_2, \lambda_6) + \gamma_3(\lambda_6, \lambda_5) + \gamma_2(\lambda_5, \lambda_1), \quad b_4 = \gamma_2(\lambda_2, \lambda_1) + \gamma_3(\lambda_6, \lambda_2) + \gamma_2(\lambda_5, \lambda_6) + \gamma_1(\lambda_1, \lambda_5).$$

The $a$-cycles of the homology basis are given in Figure 1, with the $b$-cycles shifted by one sheet. We have the pairings $a_k \circ a_i = b_k \circ b_i = 0$, $a_k \circ b_i = -b_k \circ a_i = \delta_{ik}$, and therefore $(a_1, \ldots, a_4; b_1, \ldots, b_4)$ is a symplectic basis of $H_1(\mathcal{C}, \mathbb{Z})$. In the homology basis introduced we have

$$
\mathcal{R}(b_i) = a_i, \quad i = 1, 2, 3, \quad \mathcal{R}(b_4) = -a_4.
$$

As $(1 + \mathcal{R} + \mathcal{R}^2) \mathbf{c} = 0$ for any cycle $\mathbf{c}$ we have, for example, that $\mathcal{R}(a_i) = -a_i - \mathcal{R}^2(a_i) = -a_i - b_i$ for $i = 1, 2, 3$ and $\mathcal{R}(a_4) = -a_4 + b_4$, so completing the $\mathcal{R}$ action on the homology basis.

\[ \text{Figure 1. Homology basis: } a\text{-cycles} \]

\[ \text{Figure 2. Cycles } a_1 \text{ and } b_1 \]
3.2. The Riemann period matrix. Denote vectors

\[
\mathbf{x} = (x_1, x_2, x_3, x_4)^T = \left( \oint_{a_1} du_1, \ldots, \oint_{a_4} du_1 \right)^T,
\]

\[
\mathbf{b} = (b_1, b_2, b_3, b_4)^T = \left( \oint_{a_1} du_2, \ldots, \oint_{a_4} du_2 \right)^T,
\]

\[
\mathbf{c} = (c_1, c_2, c_3, c_4)^T = \left( \oint_{a_1} du_3, \ldots, \oint_{a_4} du_3 \right)^T,
\]

\[
\mathbf{d} = (d_1, d_2, d_3, d_4)^T = \left( \oint_{a_1} du_4, \ldots, \oint_{a_4} du_4 \right)^T.
\]

Crucial for us is the fact that the symmetry (3.4) allows us to relate the matrices of \(a\) and \(b\)-periods. For any contour \(\Gamma\) and one form \(\omega\) we have that \(\oint_{\Gamma} R(\Gamma)\omega = \oint_{\Gamma} R^*\omega\). If \((\tilde{z}, \tilde{w}) = (z, \rho w) = R(z, w)\) then, for example,

\[
R^* \left( \oint_{b_2} du_2 \right) = R^* \left( \oint_{b_2} \frac{dz}{\rho^2 w^2} \right) = \int_{b_2} \frac{dz}{\rho^2 w^2} = \rho \int_{b_2} du_2.
\]

leading to

\[
\oint_{a_1} \oint_{b_2} du_2 = \oint_{b_1} \oint_{b_2} du_2 = \oint_{b_1} R^* (du_2) = \oint_{b_1} \frac{dz}{\rho^2 w^2} = \rho \oint_{b_1} du_2.
\]

We find that

\[
\mathbf{A} = (A_{ki}) = \left( \int_{b_1} du_i \right)_{i,k=1,\ldots,4} = (\mathbf{x}, \mathbf{b}, \mathbf{c}, \mathbf{d})
\]

(3.12)

\[
\mathbf{B} = (B_{ki}) = \left( \int_{b_1} du_i \right)_{i,k=1,\ldots,4} = (\rho H \mathbf{x}, \rho^2 H \mathbf{b}, \rho^2 H \mathbf{c}, \rho^2 H \mathbf{d}) = H \Lambda \mathbf{A},
\]

where \(H = \text{diag}(1, 1, 1, -1)\) and \(\Lambda = \text{diag}(\rho, \rho^2, \rho^2, \rho^2)\). This relationship between the \(a\) and \(b\)-periods leads to various simplifications of the Riemann identities,

\[
\sum_i \left( \oint_{a_i} \oint_{b_i} du_i - \oint_{a_i} du_i \oint_{b_i} du_i \right) = 0.
\]

For \(k = 1\) and \(l = 2, 3, 4\) we obtain (respectively) that

\[
\mathbf{x}^T H \mathbf{b} = \mathbf{x}^T H \mathbf{c} = \mathbf{x}^T H \mathbf{d} = 0,
\]

relations we shall employ throughout the paper.

Given \(\mathbf{A}\) and \(\mathbf{B}\) we now construct the Riemann period matrix which belongs to the Siegel upper half-space \(S^4\) of degree 4. If one works with canonically \(a\)-normalized differentials the period matrix (in our conventions) is \(\tau_a = \mathbf{B} \mathbf{A}^{-1}\) while for canonically \(b\)-normalized differentials it is \(\tau_b = \mathbf{A} \mathbf{B}^{-1}\). Clearly \(\tau_b = \tau_a^{-1}\) and we shall simply denote the period matrix by \(\tau\) if neither normalization is necessary.
Proposition 3.1 (Wellstein, 1899; Matsumoto, 2000). Let \( \mathbb{C} \) be the triple covering of \( \mathbb{P}^1 \) with six distinct point \( \lambda_1, \ldots, \lambda_6 \),

\[
(3.14) \quad w^3 = \prod_{i=1}^{6} (z - \lambda_i).
\]

Then the Riemann period matrix is of the form

\[
(3.15) \quad \tau_b = \rho \left( H - (1 - \rho) \frac{x x^T}{x^T H x} \right),
\]

where \( H = \text{diag}(1, 1, 1, -1) \). Then \( \tau_b \) is positive definite if and only if

\[
(3.16) \quad \bar{x}^T H x < 0.
\]

Both Wellstein and Matsumoto give broadly similar proofs of (3.15) and we shall present another variant as we need to use an identity established in the proof later in the text.

Proof. From (3.13) we see that we have

\[
A^T H x = (\Delta, 0, 0, 0)^T, \quad \Delta := x^T H x.
\]

We know that \( A \) is nonsingular and consequently \( x \neq 0 \) and \( \Delta \neq 0 \). Now \( H x = A^{-1}(\Delta, 0, 0, 0)^T \) which gives

\[
(3.17) \quad (H x)_\mu = A^{-1}_\mu \Delta.
\]

Now from (3.12) we see that

\[
B A^{-1} = \rho^2 H + (\rho - \rho^2) H(x, 0, 0, 0) A^{-1}.
\]

From (3.17) we obtain

\[
H(x, 0, 0, 0) A^{-1} = \frac{1}{\Delta} H x x^T H
\]

and therefore

\[
B A^{-1} = \rho^2 H + \frac{(\rho - \rho^2)}{\Delta} H x x^T H.
\]

Finally one sees that

\[
\begin{bmatrix}
\rho^2 H + \frac{(\rho - \rho^2)}{\Delta} H x x^T H
\end{bmatrix}
\begin{bmatrix}
\rho H - \frac{(\rho - \rho^2)}{\Delta} x x^T
\end{bmatrix}
= 1,
\]

whence the result (3.15) follows for \( \tau_b = A B^{-1} \). The remaining constraint arises by requiring \( \text{Im} \tau \) to be positive definite. We note that (3.16) ensures that both \( x \neq 0 \) and \( \Delta \neq 0 \).

The branch points can be expressed in terms of \( \theta \)-constants. Following Matsumoto [Mat01] we introduce the set of characteristics

\[
(3.18) \quad (a, b), \quad b = -a H, \quad a_i \in \left\{ \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \right\}
\]

and denote \( \theta_{a_i-H^b}(\tau) = \theta_{6a}(\tau) \) (see Appendix A for out theta function conventions). The characteristics (3.18) are classified in [Mat01] by the representations of the braid group. Further, the period matrix determines the branch points as follows.

Proposition 3.2 (Diez 1991, Matsumoto 2000). Let \( \tau_b \) be the period matrix of (3.7) given in Proposition 3.1. Then

\[
(3.19) \quad \Lambda_1 = \left( \frac{\theta\{3, 3, 3\}}{\theta\{1, 1, 3\}} \right)^3, \quad \Lambda_2 = -\left( \frac{\theta\{1, 5, 3\}}{\theta\{1, 1, 5\}} \right)^3, \quad \Lambda_3 = -\left( \frac{\theta\{1, 1, 3\}}{\theta\{5, 1, 1\}} \right)^3.
\]
These results have the following significance for our construction of monopoles. First we observe that the period matrix is invariant under $x \rightarrow \lambda x$. Thus to our surface we may associate a point $[x_1 : x_2 : x_3 : x_4] \in \mathbb{B}^3 = \{x \in \mathbb{P}^3 | x^T H x < 0 \} \subset \mathbb{P}^3$ and from this point we may obtain the normalized curve (3.7). It is known that a dense open subset of $\mathbb{B}^3$ arises in this way from curves with distinct roots with the complement corresponding to curves with multiple roots. Correspondingly, if we choose a point $[x_1 : x_2 : x_3 : x_4] \in \mathbb{B}^3$ we may construct a period matrix and corresponding normalized curve.

We note that with $d\bar{u} = (du_1, \ldots, du_4)$ then

$$\tau^*(d\bar{u}) = \bar{u} \cdot T,$$

and so we obtain

$$\oint_{\tau \epsilon} d\bar{u} = \oint_{\epsilon} \tau^*(d\bar{u}) = \oint_{\epsilon} \bar{u} \cdot T = \left(\oint_{\epsilon} d\bar{u}\right) \cdot T = \left(6\chi_1^+ (1 0 0 0)\right) \cdot T = -6\chi_1^+ (1 0 0 0) = \oint_{-\epsilon} d\bar{u},$$

and as a consequence corollary (2.2) of Houghton, Manton and Romão, $\tau_\epsilon \epsilon = -\epsilon$. More generally, let us write for an arbitrary cycle

$$\gamma = p \cdot a + q \cdot b = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \tau(\gamma) = \begin{pmatrix} p & q \end{pmatrix} \mathcal{M} \begin{pmatrix} a \\ b \end{pmatrix}.$$ Then the equality

$$\oint_{\tau(\gamma)} d\bar{u} = \oint_{\gamma} \tau^*(d\bar{u}) = \oint_{\gamma} \bar{u} \cdot T = \left(\oint_{\gamma} d\bar{u}\right) \cdot T$$

leads to the equation

$$\begin{pmatrix} p & q \end{pmatrix} \mathcal{M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \cdot T.$$

We have then that the matrix $\mathcal{M}$ representing the involution $\tau$ on homology and Ercolani-Sinha vector satisfy

$$\mathcal{M}^2 = \text{Id}, \quad \mathcal{M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \cdot T, \quad \mathcal{U} \mathcal{M} = \begin{pmatrix} n & m \end{pmatrix} \mathcal{M} = -\begin{pmatrix} n & m \end{pmatrix}.$$

A calculation employing the algorithm of Tretkoff and Tretkoff [TT84] to describe the homology basis generators and relations, together with some analytic continuation of the paths associated to our chosen homology cycles (with the sheet conventions described later in the text), yields that for our curve

$$\mathcal{M} = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\
0 & -1 & 1 & 2 & -1 & 0 & 1 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\
1 & -1 & 0 & 2 & 0 & -1 & 1 & -2
\end{pmatrix}.$$
The matrix $\mathcal{M}$ is not symplectic but satisfies
\[\mathcal{M}J\mathcal{M}^T = -J,\]
where $J$ is the standard symplectic form. (The minus sign appears here because of the reversal of orientation under the antiholomorphic involution.)

3.3. The vector $\tilde{K}$ and $\int_{\infty_i}^{\infty_j} v$. We shall now describe the vector $\tilde{K}$ and various related results, including the quantity $\phi(\infty_i) - \phi(\infty_j)$.

First let us record some elementary facts about our curve. For ease in defining various divisors of the curve (3.3) let $\infty_1, \infty_2, \infty_3$ be the three points over infinity and $Q_i = (\lambda_i, 0)$ ($i = 1, \ldots, 6$) be the branch points. Then
\[
\text{Div}(z - \lambda_i) = \frac{Q_i^3}{\infty_1\infty_2\infty_3}, \quad \text{Div}(w) = \frac{\prod_{i=1}^6 Q_i}{(\infty_1\infty_2\infty_3)^2}, \quad \text{Div}(dz) = \frac{(\prod_{i=1}^6 Q_i)^2}{(\infty_1\infty_2\infty_3)^2},
\]
\[
\text{Div} \left( \frac{dz}{w} \right) = \prod_{i=1}^6 Q_i, \quad \text{Div} \left( \frac{dz}{w^2} \right) = (\infty_1\infty_2\infty_3)^2, \quad \text{Div} \left( \frac{(z - \lambda_i)dz}{w^2} \right) = Q_i^3\infty_1\infty_2\infty_3,
\]
\[
\text{Div} \left( \frac{(z - \lambda_i)^2dz}{w^2} \right) = Q_i^6.
\]

Consideration of the function $(z - \lambda_i)/(z - \lambda_j)$ shows that $3 \int_{Q_i}^Q v \in \Lambda$. The order of vanishing of the differentials $d(z - \lambda_i)/w^2$, $d(z - \lambda_i)/w$, $(z - \lambda_i)d(z - \lambda_i)/w^2$ and $(z - \lambda_i)d(z - \lambda_i)/w^2$ at the point $Q_i$ are found to be 0, 1, 3 and 6 respectively, which means that the gap sequence at $Q_i$ is 1, 2, 4 and 7. From this we deduce that the index of speciality of the divisor $Q_i^3$ is $i(Q_i^3) = 2$. Because the genus four curve $C$ has the function $w$ of degree 3 then $C$ is not hyperelliptic. The function $1/(z - \lambda_i)$ has divisor $U/D$, with $U = \infty_1\infty_2\infty_3$ and $D = Q_i^3$ such that $D^2$ is canonical. This means that any other function of degree 3 on $C$ is a fractional linear transformation of $w$ and that $\Theta_{\text{sing}}$ consists of precisely one point which is of order 2 in $\text{Jac}(C)$ [FK80, III.8.7, VII.1.6]. The vector of Riemann constants $K_{Q_i}$ is a point of order 2 in $\text{Jac}(C)$ because $Q_i^6$ is canonical [FK80, VI.3.6]. Let us fix $Q_1$ to be our base point. Then as $K_{Q_1}, \phi_Q(Q_1^3) + K_{Q_1}$ we have that $K_{Q_1} \in \Theta$. Because $i(Q_1^3) = 2$ we may identify $K_{Q_1}$ as the unique point in $\Theta_{\text{sing}}$. We may further identify $K_{Q_1}$ as the unique even theta characteristic belonging to $\Theta$.

With $Q_1$ as our base point $\phi(\sum_k \infty_k)$ corresponds to the image under the Abel map of the divisor of the function $1/(z - \lambda_1)$, and so vanishes (modulo the period lattice). Thus for our curve $\tilde{K} = K_{Q_1} + \phi(\sum_k \infty_k) = K_{Q_1} = \Theta_{\text{sing}}$ is the unique even theta characteristic. The point $K_{Q_1}$ may be constructed several ways: directly, using the formula (A.6) of the Appendix (the evaluation of the integrals of normalised holomorphic differentials between branch points is described in Appendix B); by enumeration we may find which of the 136 even theta characteristics $\left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right]$ leads to the vanishing of $\theta \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right](z; \tau)$; using a monodromy argument of Matsumoto [Mat01]. One finds that the relevant half period is $\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$.

The analysis of the previous paragraph, together with Lemma 2.3, tells us that $U$ must also be an even theta characteristic.

Again using that $\sum_k \infty_k \sim 0$ we have that $\infty_i - \infty_j \sim 2\infty_i + \infty_k$ (with $i$, $j$, $k$ distinct) and so $\theta(\phi(\infty_j) - \phi(\infty_i) - \tilde{K}) = \theta(\phi(2\infty_i + \infty_k) + K) = 0$. One sees from the above
divisors (in particular Div\((dz/w^2)\)) that \(\text{Dim} H^0(\mathcal{C}, L_{2\infty_i + \infty_k}) = i(2\infty_i + \infty_k) = 1\). Thus 
\(\theta(w + \phi(\infty_j) - \phi(\infty_i) - \overline{K})\) and \(\theta(w - \overline{K})\) have order of vanishing differing by one for (generic) \(w \to 0\).

3.4. Calculating \(\nu_i - \nu_j\). From the results of the previous section we see that 
\[
\text{Div}\left(\frac{z^4 dz}{w^2}\right) = \left(0, 0, 0, 0, 0\right),
\]
This has precisely the same divisor of poles as \(\gamma_{\infty}\) and we will use this to represent \(\gamma_{\infty}\). It is convenient to introduce the (meromorphic) differential 
\[
dr_1(P) = \frac{z^4 dz}{3w^2},
\]
the factor of three here being introduced to give the pairing 
\[
\sum_{s=1}^{3} \text{Res}_{P=\infty_s} \dr_1(P) \int_{P_0}^{P} du_1(P') = 1.
\]
We may therefore write
\[
(3.21) \quad \gamma_{\infty}(P) = -3dr_1(P) + \sum_{i=1}^{4} c_i v_i(P).
\]
The constants \(c_i\) are found from the condition of normalisation 
\[
\oint_{a_k} \gamma_{\infty}(P) = 0 \iff c_k = 3 \oint_{a_k} dr_1(P) \equiv 3y_k, \quad k = 1, \ldots, 4,
\]
where we have defined the vector of \(a\)-periods \(y^T = \left(\oint_{a_1} dr_1(P), \ldots, \oint_{a_4} dr_1(P)\right)\). The vector of \(b\)-periods of \(dr_1\) is found to be \(\rho^2 H y\). The pairing with \(du_1\) then yields the Legendre relation
\[
(3.22) \quad y \cdot H x = -\frac{2\pi}{\sqrt{3}}.
\]
Now the \(b\)-periods of the differential \(\gamma_{\infty}\) give the Ercolani-Sinha vector. Using (3.21) we then obtain the equality 
\[
(3.23) \quad -3(\rho^2 H - \tau_a)y = \pi n + \pi i \tau m.
\]
Finally, using (3.22), we may write 
\[
(3.24) \quad \nu_i - \nu_j = 3y \cdot \int_{\infty_i}^{\infty_j} v + \int_{\infty_j}^{\infty_i} \left[ d\left(\frac{w}{z}\right) - 3dr_1\right].
\]

4. Solving the Ercolani-Sinha constraints

Although we have now described how to evaluate most of the quantities needed for the reconstruction of a monopole, at least up to the evaluation of the integrals \(x\), we have not described how to solve the Ercolani-Sinha constraints for the spectral curve (3.1) which encode one of Hitchin’s transcendental constraints. To this we now turn. As we shall see, this reduces to constraints just on the four periods \(x\). Later we shall restrict attention to the curves (1.2), which has the effect of reducing the number of integrals to be evaluated to two and consequently simplifies our present analysis.
We shall work with the Ercolani-Sinha constraints in the form (2.24). Let the holomorphic differentials be ordered as in (3.9). Then there exist two integer 4-vectors \( n, m \in \mathbb{Z}^4 \) and values of the parameters \( \lambda_1, \ldots, \lambda_6 \) and \( \chi \) such that

\[ n^T A + m^T B = \nu (1, 0, 0, 0). \]

Here \( \nu \) depends on normalizations. For us this will be

\[ \nu = 6 \chi. \]

To see this observe that (2.24) requires that

\[ -2 \delta_{1k} = \oint_{n \cdot a + m \cdot b} \Omega^{(k)} \]

for the differentials \( \Omega^{(1)} = \frac{\eta^{n-2} d\zeta}{\partial \eta}, \Omega^{(2)} = \frac{\eta^{n-3} d\zeta}{\partial \eta}, \ldots \)

In the parameterisation (3.3) we are using we have that

\[ x_i = \oint_{a_i} dz = \oint_{a_i} \frac{d\zeta}{\zeta^\nu} = -3 \frac{\chi^\nu}{\zeta^\nu} \Omega^{(1)}. \]

We wish

\[ -2 = \oint_{n \cdot a + m \cdot b} \Omega^{(1)} = \frac{-1}{3 \chi^\nu} (n \cdot x + \rho m \cdot H \cdot x) \]

and so

\[ n \cdot x + \rho m \cdot H \cdot x = \nu, \]

with the value of \( \nu \) stated. Consideration of the other differentials then yields (4.1), transcendental constraints on the curve \( \mathcal{C} \). These constraints may be solved using the following result.

**Proposition 4.1.** The Ercolani-Sinha constraints (4.1) are satisfied for the curve (3.1) if and only if

\[ x = \xi (H n + \rho^2 m), \]

where

\[ \xi = \frac{\nu}{|n \cdot H n - m \cdot n + m \cdot H m|} = \frac{6 \chi^\nu}{|n \cdot H n - m \cdot n + m \cdot H m|}. \]

**Proof.** Rewriting (4.1) we have that

\[ n^T + m^T B A^{-1} = \nu (1, 0, 0, 0) A^{-1} = \nu A_{14}. \]

Upon using (3.17) we obtain

\[ n^T + m^T B A^{-1} = \frac{\nu}{\Delta} x^T H. \]

Therefore

\[ x = \frac{\Delta}{\nu} (H n + H (B A^{-1})^T m) \]

\[ = \frac{\Delta}{\nu} (H n + \rho^2 m + \left( \frac{D - \rho^2}{\Delta} \right) x x^T H m) \]
upon using that the period matrix is symmetric and our earlier expression for $B {A}^{-1}$. Rearranging now gives us that

$$\begin{align*}
(1 + \frac{\rho^2 - \rho}{\nu} x^T H m)x &= \frac{\Delta}{\nu} (H n + \rho^2 m) \\
\text{and so we have established (4.3) where}
\end{align*}$$

$$\xi = \frac{\Delta}{\nu} (1 + \frac{\rho^2 - \rho}{\nu} x.H m)^{-1}.$$  

There are several constraints. First, the Ercolani-Sinha condition (4.2) is that

1. The Ercolani-Sinha condition

$$\nu \geq 0 \quad \text{and} \quad \nu \neq \rho.$$

Thus establishing (4.4). We remark that if $\tilde{\chi}$ is real, then $\tilde{\chi}^T$ may be chosen real and hence $\xi$ is real. We observe that (4.4) and (4.6) are consistent with

$$\nu = 6 \tilde{\chi}^3,$$

thus establishing (4.4). We remark that if $\tilde{\chi}$ is real, then $\tilde{\chi}^T$ may be chosen real and hence $\xi$ is real. We observe that (4.4) and (4.6) are consistent with

$$\Delta = x^T H x = \xi^2 (n^T H + \rho^2 m^T) H (n + \rho^2 m) = \xi^2 [n.H n + 2\rho^2 m.n + \rho m.H m].$$

A further consistency check is given by (3.23). Using the form of the period matrix, the Legendre relation (3.22) and the proposition (with $\nu = -6$) we obtain (3.23). 

At this stage we have reduced the Ercolani-Sinha constraints to one of imposing the four constraints (4.3) on the periods $x_k$. In particular this means we must solve

$$\frac{x_1}{n_1 + \rho^2 m_1} = \frac{x_2}{n_2 + \rho^2 m_2} = \frac{x_3}{n_3 + \rho^2 m_3} = \frac{x_4}{-n_4 + \rho^2 m_4} = \xi,$$

which means $x_i/x_j \in \mathbb{Q} [\rho]$. Further we have from the conditions (3.16) that

$$\frac{x^T H x}{|\xi|^2} = [n.H n - m.n + m.H m] = \sum_{i=1}^{3} (n_i^2 - n_i m_i + m_i^2) - n_4^2 - m_4^2 - m_4 n_4 < 0.$$

Our result admits another interpretation. Thus far we have assumed we have been given an appropriate curve and sought to satisfy the Ercolani-Sinha constraints. Alternatively we may start with a curve satisfying (most of) the Ercolani-Sinha constraints and seek one satisfying the reality constraints (and any remaining Ercolani-Sinha constraints). How does this progress? First note that the period matrix (3.15) for a curve satisfying (4.8) is

$$\text{ranging now gives us that} \\
\text{we may start with a curve satisfying (most of) the Ercolani-Sinha constraints and seek} \\
\text{an appropriate curve and sought to satisfy the Ercolani-Sinha constraints. Alternatively} \\
\text{this question may be answered, with the roots $\alpha_i$ being determined up to an overall rotation.} \\
\text{At this stage we have (using the rotational freedom) a curve of the form} \\
W^3 = Z(Z - a)(Z + \frac{1}{a})(Z - w)(Z + \frac{1}{w}), \quad a \in \mathbb{R}, \ w \in \mathbb{C}.$$
period. Then using (4.8) and (4.4) we determine $\hat{\chi}$. This is a constraint. For a consistent monopole curve we require

$$\arg(\xi) = \arg \left[ \frac{w}{w^{1/6}} \right].$$

Of course, to complete the construction we need to check there are no roots of the theta function in $[-1, 1]$. Although the procedure outlined involves several transcendental calculations it is numerically feasible and gives a means of constructing putative monopole curves.

To conclude we state when there exists a Möbius transformation of the set $S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}$ to one of the form $H = \{\alpha_j, -1/\alpha_j\}_{j=1}^3$. For simplicity we give the case of distinct roots:

**Theorem 4.2.** The roots $S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}$ are Möbius equivalent to $H = \{\alpha_j, -1/\alpha_j\}_{j=1}^3$ if and only if

1. If just one of the roots, say $\Lambda_1$, is real and
   - $\Lambda_1 < 0$ then $\Lambda_2 = \Lambda_1$,
   - $0 < \Lambda_1 < 1$ then $\frac{\Lambda_2 - \Lambda_3}{\Lambda_2 - 1 - \Lambda_3} = \frac{\Lambda_1}{1 - \Lambda_1}$,
   - $1 < \Lambda_1$ then $(1 - \Lambda_2)(1 - \Lambda_3) = 1 - \Lambda_1$.

   If all three roots are real then, up to relabelling, one of the above must hold.

2. All three roots are complex and, up to relabelling,

   $$0 < \Lambda_1, \Lambda_2 \in \mathbb{R}, \quad 1 < \frac{\Lambda_1}{\Lambda_2}, \quad \Lambda_3 = \Lambda_2 \frac{1 - \Lambda_1}{1 - \Lambda_2}.$$  

5. **Symmetric 3-monopoles**

In this section we shall consider the curve $C$ specialised to the form

$$(5.1) \quad \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0,$$

where $b$ is a real parameter. In this case branch points are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\alpha, \rho^2 \beta, \rho \alpha, \beta, \rho^2 \alpha, \rho \beta),$$

where $\alpha$ and $\beta$ are real,

$$\alpha = \sqrt[3]{-b + \sqrt{b^2 + 4}} > 0, \quad \beta = \sqrt[3]{-b - \sqrt{b^2 + 4}} < 0, \quad \alpha^3 \beta^3 = -1.$$  

Here $\chi = \hat{\chi}$ is real and we choose our branches so that $\hat{\chi}^{1/3}$ is also real.

The effect of choosing such a symmetric curve will be to reduce the four period integrals $x_i$ to two independent integrals. The tetrahedrally symmetric monopole is in the class (5.1). We note that a general rotation will alter the form of $a_3(\zeta)$. Thus the dimension of the moduli space is reduced from three by the 3 degrees of freedom of the rotations yielding a discrete space of solutions. We are seeking then a discrete family of spectral curves.

We shall begin by calculating the period integrals, and then imposing the Ercolani-Sinha constraints. We shall also consider the geometry of the curves (5.1).

5.1. **The period integrals.** In terms of our Wellstein parameterisation we are working with

$$w^3 = z^6 + bz^3 - 1 = (z^3 - \alpha^3)(z^3 + \frac{1}{\alpha^3})$$

$(1/\alpha^3 = -\beta^3 = (b + \sqrt{b^2 + 4})/2)$. We choose the first sheet so that $w = \sqrt[3]{(z^3 - \alpha^3)(z^3 + 1/\alpha^3)}$ is negative and real on the real $z$-axis between the branch points $(-1/\alpha, \alpha)$. 

Introduce integrals computed on the first sheet

\[ I_1(\alpha) = \int_0^\alpha \frac{dz}{w} = -\frac{2\pi\sqrt{3}\alpha}{9} _2F_1 \left( \frac{1}{3}, \frac{1}{3}; 1; -\alpha^2 \right), \]

\[ J_1(\alpha) = \int_0^\beta \frac{dz}{w} = \frac{2\pi\sqrt{3}}{9\alpha} _2F_1 \left( \frac{1}{3}, \frac{1}{3}; 1; -\alpha^{-2} \right). \]

Here \( _2F_1(a, b; c; z) \) is the standard Gauss hypergeometric function and we have, for example, evaluated the first integral using the substitution \( z = \alpha^{1/3} \) and our specification of the first sheet. We also have that

\[ \int_0^\beta \frac{dz}{w} = \rho^k I_1(\alpha), \quad \int_0^\beta \frac{dz}{w} = \rho^k J_1(\alpha), \quad k = 1, 2. \]

Our aim is to express the periods for our homology basis \( (3.10) \) in terms of the integrals \( I_1(\alpha) \) and \( J_1(\alpha) \). Consider for example

\[ x_1 = \int_{a_1}^\gamma \frac{du}{a_1} = \int_{\gamma_1(\lambda_1, \lambda_2)} \frac{dz}{w} + \int_{\gamma_2(\lambda_2, \lambda_1)} \frac{dz}{w} = \int_{\lambda_1}^{\lambda_2} \frac{dz}{w} - \rho^2 \int_{\lambda_1}^{\lambda_2} \frac{dz}{w} = (1 - \rho^2) \int_0^\beta \frac{dz}{w} = (1 - \rho^2) \left[ -I_1(\alpha) + \rho^2 J_1(\alpha) \right] \]

and our specification of the first sheet. The Ercolani-Sinha constraints.

\[ (5.3) \quad x_1 = -(2J_1 + I_1)\rho - 2I_1 - J_1, \quad x_2 = (J_1 - I_1)\rho + I_1 + 2J_1, \]

\[ x_3 = (J_1 + 2I_1)\rho - J_1 + I_1, \quad x_4 = 3(J_1 - I_1)\rho + 3J_1. \]

Note that

\[ (5.4) \quad x_2 = \rho x_1, \quad x_3 = \rho^2 x_1. \]

5.2. The Ercolani-Sinha constraints. We next reduce the Ercolani-Sinha constraints to a number theoretic one. Using \( (4.3) \) and \( (5.3) \) we may rewrite the constraints as

\[ (5.5) \quad x_i = \xi(e_i n_i + \rho^2 m_i) = (\alpha_i I_1 + \beta_i J_1) + (\gamma_i I_1 + \delta_i J_1) \rho. \]

We may solve for the various \( n_i, m_i \) in terms of \( n_1, m_1 \) as follows. Set

\[ C_i = \begin{pmatrix} \epsilon_i & -1 \\ 0 & -1 \end{pmatrix}, \quad D_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \quad \tilde{I} = I_1/\xi, \quad \tilde{J} = J_1/\xi. \]

Then \( (5.5) \) may be rewritten as

\[ C_i \begin{pmatrix} n_i \\ m_i \end{pmatrix} = D_i \begin{pmatrix} \tilde{I} \\ \tilde{J} \end{pmatrix} \]

giving

\[ \begin{pmatrix} n_i \\ m_i \end{pmatrix} = C_i^{-1} D_i \begin{pmatrix} \tilde{I} \\ \tilde{J} \end{pmatrix} = C_i^{-1} D_i D_1^{-1} C_1 \begin{pmatrix} n_1 \\ m_1 \end{pmatrix}. \]
Now given (5.6) we find that

\[ n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} n_1 \\ m_1 - n_1 \\ -m_1 \\ 2n_1 - m_1 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} m_1 \\ -n_1 \\ n_1 - m_1 \\ -3n_1 \end{pmatrix}. \]

One may verify that for vectors of this form then \((\mathbf{n}, \mathbf{m})M = -\langle \mathbf{n}, \mathbf{m} \rangle\) as required by (3.20). Recall further that \((\mathbf{n}, \mathbf{m})\) is to be a primitive vector: that is one for which the greatest common divisor of the components is 1, and hence a generator of \(\mathbb{Z}^8\). We see that \((\mathbf{n}, \mathbf{m})\) is primitive if and only if

\[ \langle n_1, m_1 \rangle = 1. \]

From

\[ \frac{\tilde{x}}{\tilde{J}} = D_i^{-1} C_i \begin{pmatrix} n_1 \\ m_1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ m_1 \end{pmatrix}, \]

we obtain

\[ \frac{\tilde{x}}{\tilde{J}} = \frac{I}{\tilde{J}} = \frac{m_1 - 2n_1}{m_1 + n_1}, \]

\[ I_1 = m_1 - 2n_1 \quad \text{and} \quad I_1 = \frac{1}{3} \frac{\pi}{3} \alpha \quad 2 \text{F}_1 \left( 1, \frac{1}{3}, 1, -\alpha^6 \right), \]

\[ J_1 = \frac{m_1 + n_1}{\xi} \quad \text{and} \quad J_1 = \frac{1}{3} \frac{\pi}{3} \alpha \quad 2 \text{F}_1 \left( 1, \frac{1}{3}, 1, -\alpha^6 \right). \]

Now given (5.6) we find that

\[ \mathbf{n} \cdot \mathbf{H} \mathbf{n} - \mathbf{m} \cdot \mathbf{n} + \mathbf{m} \cdot \mathbf{H} \mathbf{m} = 2(m_1 + n_1)(m_1 - 2n_1) \]

and so the constraint (3.10) is satisfied if

\[ \bar{x}^T H x = \xi^2 [\mathbf{n} \cdot \mathbf{H} \mathbf{n} - \mathbf{m} \cdot \mathbf{n} + \mathbf{m} \cdot \mathbf{H} \mathbf{m}] = 2\xi^2(m_1 + n_1)(m_1 - 2n_1) < 0. \]

This requires

\[ (m_1 + n_1)(m_1 - 2n_1) < 0. \]

In particular we have from (3.7) that

\[ \xi = \frac{3\chi^\frac{1}{2}}{(m_1 + n_1)(m_1 - 2n_1)}. \]

Thus we have to solve

\[ I_1 = \frac{\chi^\frac{1}{2}}{n_1 + m_1} = \frac{2\pi}{3\sqrt{3}} \alpha \quad 2 \text{F}_1 \left( \frac{1}{3}, \frac{1}{3}, 1, -\alpha^6 \right), \]

\[ J_1 = \frac{\chi^\frac{1}{2}}{m_1 - 2n_1} = \frac{2\pi}{3\sqrt{3}} \alpha \quad 2 \text{F}_1 \left( \frac{1}{3}, \frac{1}{3}, 1, -\alpha^6 \right). \]

Using the identity

\[ 2 \text{F}_1 \left( \frac{1}{3}, \frac{1}{3}, 1, x \right) = (1 - x)^{-1/3} \quad 2 \text{F}_1 \left( \frac{1}{3}, \frac{2}{3}, 1, \frac{x}{x - 1} \right) \]

we then seek solutions of

\[ \frac{I_1}{J_1} = \frac{m_1 - 2n_1}{m_1 + n_1} = \frac{-2 \text{F}_1 \left( \frac{1}{2}, \frac{1}{3}, 1, t \right)}{2 \text{F}_1 \left( \frac{1}{3}, \frac{1}{3}, 1, 1 - t \right)}, \quad t = \frac{\alpha^6}{1 + \alpha^6} = \frac{b + \sqrt{b^2 + 4}}{2\sqrt{b^2 + 4}}. \]
Figure 3. The function \( f(t) = \frac{\binom{2}{1} \binom{2}{1;1-t}}{\binom{2}{1} \binom{2}{1;1-t}} \).

From (5.8) the ratio of \( I_1/J_1 \) is negative. Consideration of the function

\[
f(t) = \frac{\binom{2}{1} \binom{2}{1;1-t}}{\binom{2}{1} \binom{2}{1;1-t}}.
\]

(see Figure 2 for its plot) shows that there exists unique root \( t \in (0,1) \) for each value \( f(t) \in (0,\infty) \) and correspondingly a unique real positive \( \alpha = \sqrt{t/(1-t)} \).

Bringing these results together we have established:

**Proposition 5.1.** To each pair of relatively prime integers \((n_1, m_1) = 1\) for which

\[(m_1 + n_1)(m_1 - 2n_1) < 0\]

we obtain a solution to the Ercolani-Sinha constraints for a curve of the form (5.1) as follows. First we solve for \( t \), where

\[
\frac{2n_1 - m_1}{m_1 + n_1} = \frac{\binom{2}{1} \binom{2}{1;1-t}}{\binom{2}{1} \binom{2}{1;1-t}}.
\]

Then

\[
b = \frac{1 - 2t}{\sqrt{t(1-t)}}, \quad t = \frac{-b + \sqrt{b^2 + 4}}{2\sqrt{b^2 + 4}},
\]
and we obtain $\chi$ from

$$\chi^{\frac{1}{3}} = -(n_1 + m_1) \frac{2\pi}{3\sqrt{3}} \frac{\alpha}{(1 + \alpha^6)^{\frac{1}{3}}} 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right)$$

with $\alpha^6 = t/(1 - t)$.

5.3. Ramanujan. Thus far we have reduced the problem of finding an appropriate monopole curve within the class (5.1) to that of solving the transcendental equation (5.10) for which a unique solution exists. Can this ever be solved apart from numerically? Here we shall recount how a (recently proved) result of Ramanujan enables us to find solutions.

Let $n$ be a natural number. A modular equation of degree $n$ and signature $r$ ($r = 2, 3, 4, 6$) is a relation between $\alpha, \beta$ of the form

$$n \frac{2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1 - \alpha\right)}{2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1 - \beta\right)}{2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}.$$

When $r = 2$ we have the complete elliptic integral $K(k) = \frac{\pi}{2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ and (5.13) yields the usual modular relations. By interchanging $\alpha \leftrightarrow \beta$ we may interchange $n \leftrightarrow 1/n$. This, together with iteration of these modular equations, means we may obtain relations with $n$ being an arbitrary rational number. Our equation (5.10) is precisely of this form for signature $r = 3$ and starting with say $\alpha = 1/2$.

Ramanujan in his second notebook presents results pertaining to these generalised modular equations and various theta function identities. For example, if $n = 2$ in signature $r = 3$ then $\alpha$ and $\beta$ are related by

$$(\alpha \beta)^{\frac{1}{3}} + ((1 - \alpha)(1 - \beta))^{\frac{1}{3}} = 1.$$

He also states that (for $0 \leq p < 1$)

$$\left(1 + p + p^2\right) 2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1, \frac{3p^2(2 + p)}{1 + 2p}\right) = \sqrt{1 + 2p} \frac{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, \frac{27p^2(1 + p)^2}{4(1 + p + p^2)^2}\right)}{2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, \frac{27p^2(1 + p)^2}{4(1 + p + p^2)^2}\right)}.$$

Ramanujan’s results were derived in [BBG95] (see also [Cha98]), though some related to expansions of $1/\pi$ had been obtained earlier by J.M. and P.B. Borwein [BB87]. An account of the history and the associated theory of these equations may be found in the last volume dedicated to Ramanujan’s notebooks [Ber98]. The associated theory of these modular equations presented in the accounts just cited is largely based on direct verification that appropriate expressions of hypergeometric functions satisfy the same differential equations and initial conditions and so are equal: we shall present a more geometric picture in due course.

Analogous expressions to (5.14) are known for $n = 3, 5, 7$ and $11$ [Ber98 7.13, 7.17, 7.24, 2.28 respectively]. Thus by iteration we may solve (5.10) for rational numbers whose numerator and denominator have these as their only factors. We include some examples of these in the table below. Thus to get the value $2$ for the ratio $(2n_1 - m_1)/(m_1 + n_1)$ we set $\alpha = \frac{1}{2}$ in (5.14) and solve for

$$t^{\frac{1}{3}} + (1 - t)^{\frac{1}{3}} = 2^{\frac{1}{3}},$$
taking the larger value \( t = \frac{1}{2} + \frac{5\sqrt{3}}{18} \) (the smaller value yielding the ratio \( \frac{1}{2} \)).

\[
\begin{array}{|c|c|c|c|}
\hline
m_1 & m_1 & (2n_1 - m_1)/(m_1 + n_1) & t & b \\
\hline
2 & 1 & 1 & 1 & 0 \\
1 & 0 & 2 & \frac{1}{2} + \frac{5\sqrt{3}}{18} & 5\sqrt{2} \\
1 & 1 & \frac{1}{2} & \frac{1}{2} - \frac{5\sqrt{3}}{18} & 5\sqrt{2} \\
4 & -1 & 3 & (63 + 171\sqrt{2} - 18\sqrt{4})/250 & (44 + 38\sqrt{2} + 26\sqrt{4})/3 \\
5 & -2 & 4 & \frac{1}{2} + \frac{153\sqrt{3} - 99\sqrt{2}}{250} & 9\sqrt{458} + 187\sqrt{6} \\
\hline
\end{array}
\]

A theory exists then for solving (5.13) and this has been worked out for various low primes. These results enable us to reduce the Ercolani-Sinha conditions (5.10) to solving an algebraic equation.

5.4. Covers of the sextic. We shall now describe some geometry underlying our curves (5.1) which will lead to an understanding of the results of the last section. We shall first present a more computational approach, useful in actual calculations, and then follow this with a more invariant discussion. We begin with the observation that our curves each cover four elliptic curves.

Lemma 5.2. The curve \( C := \{(x, y)|y^3 + x^6 + bx^3 - 1 = 0\} \) with arbitrary value of the parameter \( b \) is a simultaneous covering of the four elliptic curves \( E_{\pm}, E_{1,2} \) as indicated in the diagram, where \( C^* \) is an intermediate genus two curve:

\[
\begin{align*}
C &= (x, y) \\
C^* & \rightarrow \pi^* \\
\pi_+ & \rightarrow \pi_- \\
\pi_1 & \rightarrow \pi_2 \\
E_+ &= (z_+, w_+) \\
E_- &= (z_-, w_-) \\
E_1 &= (z_1, w_1) \\
E_2 &= (z_2, w_2)
\end{align*}
\]

The equations of the elliptic curves are

\[
\begin{align*}
(5.16) & \quad E_{\pm} = \{(z_\pm, w_\pm)|w_\pm^2 = z_\pm(1 - z_\pm)(1 - k_\pm^2 z_\pm)\}, \\
(5.17) & \quad E_1 = \{(z_1, w_1)|z_1^3 + w_1^3 + 3z_1 + b = 0\}, \\
(5.18) & \quad E_2 = \{(z_2, w_2)|w_2^3 + z_2^2 + bz_2 - 1 = 0\},
\end{align*}
\]

where the Jacobi moduli, \( k_{\pm} \) are given by

\[
(5.19) \quad k_\pm^2 = -\frac{\rho(\rho M \pm 1)(\rho M \mp 1)^3}{(M \pm 1)(M \mp 1)^3}
\]

with

\[
(5.20) \quad M = \frac{K}{L}, \quad K = (2t - b)^\sharp, \quad L = (b^2 + 4)^\sharp.
\]
The covers $\pi_\pm, \pi_{1,2}$ are given by

$$
\begin{align*}
\pi_\pm :& \quad z_\pm = \frac{K^2 - L^2}{K^2 - \rho L^2} \frac{Kx - y}{\rho Kx - y} \frac{L^2x - Ky}{L^2x - K\rho y}, \\
w_\pm :& = i\sqrt{2 + \rho} \sqrt{\frac{L \pm K}{L \mp K}} \frac{L^2 - \rho K^2}{L \pm K} \frac{(Lx \mp y)(x^6 + 1)}{\rho L^2 - K^2 (\rho Kx - y)^2(L^2x - K\rho y)^2}
\end{align*}
$$

and

$$
\begin{align*}
\pi_1 :& \quad z_1 = x - \frac{1}{x}, \quad w_1 = \frac{y}{x}, \\
\pi_2 :& \quad z_2 = x^3, \quad w_2 = y.
\end{align*}
$$

The elliptic curves $E_{1,2}$ are equianharmonic ($g_2 = 0$) and consequently have vanishing $j$-invariant, $j(E_{1,2}) = 0$.

**Proof.** The derivation of the covers $\pi_{1,2}$ and the underlying curves is straightforward. The pullbacks $\pi_{1,2}^{-1}$ of these covers are

$$
\begin{align*}
\pi_1^{-1} :& = \begin{cases} 
\mu = y, \\
\nu = -x^3 - \frac{1}{x^3}
\end{cases}, \\
\pi_2^{-1} :& = \begin{cases} 
\mu = \rho \sqrt{z_2}, \\
\nu = w_2
\end{cases}
\end{align*}
$$

showing that the degrees of the cover are 2 and 3 respectively. A direct calculation putting these elliptic curves into Weierstrass form shows $g_2 = 0$ and hence the elliptic curves $E_{1,2}$ are equianharmonic. Their $j$-invariants are therefore vanishing and $E_{1,2}$ are birationally equivalent.

To derive the covers $\pi_\pm$ we first note that the curve $C$ is a covering of the hyperelliptic curve $C^*$ of genus two,

$$
\begin{align*}
C^* = \{ (\mu, \nu) | \nu^2 = (\mu^3 + b^2 + 4) \}.
\end{align*}
$$

The cover of this curve is given by the formulae

$$
\begin{align*}
\pi^* :& \quad \mu = \frac{y}{x}, \quad \nu = -x^3 - \frac{1}{x^3}. \\
\end{align*}
$$

The curve $C^*$ covers two-sheetedly the two elliptic curves $E_\pm$ given in (5.16)

$$
\begin{align*}
\pi^* :& = \begin{cases} 
\mu = \frac{y}{x}, \\
\nu = -x^3 - \frac{1}{x^3}
\end{cases}, \\
\end{align*}
$$

$$
\begin{align*}
\pi_{1,2} :& = \begin{cases} 
\mu = \rho \sqrt{z_2}, \\
\nu = w_2
\end{cases}
\end{align*}
$$

Composition of (5.23) and (5.24) leads to (5.21). □

Using these formulae direct calculation then yields

**Corollary 5.3.** The holomorphic differentials of $C$ are mapped to holomorphic differentials of $E_{\pm, 1,2}$ as follows

$$
\begin{align*}
\frac{dz_\pm}{w_\pm} = & \sqrt{1 + 2\rho} \frac{L}{K} \sqrt{(L \pm K)(L \mp K)^3} \frac{Lx \pm y}{y^2} \frac{dy}{\nu}, \\
= & \sqrt{1 + 2\rho} \frac{L}{K} \sqrt{(L \mp K)(L \pm K)^3} \frac{(L \pm K)^3}{(L \pm K)^3} \frac{dy}{\nu}, \\
\frac{dz_1}{w_1} = & \frac{x^2 + 1}{y^2} \frac{dx}{\nu},
\end{align*}
$$

(5.25)
\[ \frac{dz_2}{w_2} = \frac{3x^2}{y^2} dx, \]

where \( L, K \) are given in (5.20).

The absolute invariants \( j_\pm \) of the curves \( E_\pm \) are

\[ j_\pm = 108 \frac{L^3 (5L^3 \pm 4b)^3}{(L^3 \pm b)^3} \]

Evidently \( j_\pm \neq 0 \) in general, as well \( j_+ \neq j_- \); therefore these elliptic curves are not birationally equivalent to that one appearing in Hitchin’s theory of the tetrahedral monopole which is equianharmonic [HMM95]. We observe that the substitution

\[ M = 1 + 2\rho \]

leads to the parameterisation of Jacobi moduli being

\[ k_+^2 = \frac{(p + 1)^3(3 - p)}{16p}, \quad k_-^2 = \frac{(p + 1)(3 - p)^3}{16p^3} \]

which Ramanujan used in his hypergeometric relations of signature 3, see e.g. [BBG95].

The \( \theta \)-functional representation of the moduli \( k_\pm \) and parameter \( p \) can be found in [Law89, Section 9.7],

\[ k_+ = \frac{\vartheta_2^2(0|\tau)}{\vartheta_3^2(0|\tau)}, \quad k_- = \frac{\vartheta_2^2(0|3\tau)}{\vartheta_3^2(0|3\tau)}, \quad p = \frac{3\vartheta_3^2(0|3\tau)}{\vartheta_3^2(0|\tau)}. \]

We shall now describe the geometry of the covers we have just presented explicitly. Our curve has several explicit symmetries which lie behind the covers described. We will first describe these symmetries acting on the field of functions \( \mathcal{C} \) of our curve as this field does not depend on whether we have a singular or nonsingular model of the curve; we will subsequently give a projective model for these, typically working in weighted projective spaces where the curves will be nonsingular.

Viewing \( \tilde{y} = y/x \) and \( x \) as functions on \( \mathcal{C} \) we see that

\[ \tilde{y}^3 = x^3 + b - \frac{1}{x^3} \]

has symmetries \( (\rho = e^{2\pi i/3}) \)

\[ \begin{align*}
  a : x &\rightarrow x, \quad \tilde{y} \rightarrow \rho \tilde{y}, \\
  b : x &\rightarrow \rho x, \quad \tilde{y} \rightarrow \tilde{y}, \\
  c : x &\rightarrow -1/x, \quad \tilde{y} \rightarrow \tilde{y}.
\end{align*} \]

Together these yield the group \( G = C_3 \times S_4 \), with \( C_3 = \langle a^3 = 1 \rangle \) and \( S_3 = \langle b, c | b^3 = 1, c^2 = 1, cbc = b^2 \rangle \). When \( b = 5\sqrt{2} \), the dihedral symmetry \( S_3 \) is enlarged to tetrahedral symmetry by

\[ t : x \rightarrow \frac{\sqrt{2} - x}{1 + \sqrt{2}x}, \quad \tilde{y} \rightarrow \frac{3x\tilde{y}}{(1 + \sqrt{2}x)(x - \sqrt{2})}, \quad t^2 = 1, \]

with \( A_4 \) being generated by \( b \) and \( t \). Now to each subgroup \( H \leq G \) we have the fixed field \( \mathcal{C}^H \) associated to the quotient curve \( \mathcal{C}/H \).

The canonical curve of a non-hyperelliptic curve of genus 4 is given by the intersection of an irreducible quadric and cubic surface in \( \mathbb{P}^3 \). In our case the quadric is in fact a cone and
we may represent our curve \( C \) as the nonsingular curve in the weighted projective space \( \mathbb{P}^{1,1,2} = \{ [z, t, w] \mid [z, t, w] \sim [\lambda z, \lambda t, \lambda^2 w] \} \) given by the vanishing of
\[
f(z, t, w) = z^6 + b z^3 t^3 - t^6 - w^3.
\]
The group \( G \) acts on this as \((x = z/t, \bar{y} = w/(zt))\)
\[
a : [z, t, w] \rightarrow [z, t, \rho w] \sim [\rho z, pt, w],
\]
\[
b : [z, t, w] \rightarrow [\rho z, t, \rho w] \sim [\rho^2 z, pt, w],
\]
\[
c : [z, t, w] \rightarrow [t, -z, -w] \sim [t, -iz, w].
\]
The fixed points of these actions on \( C \) and quotient curves are as follows:

a: There are 6 fixed points, \([1, \rho^k \alpha_\pm, 0]\), where \(\alpha_\pm\) are the two roots of \(a^2 - ba - 1 = 0\). For other points we have a \(3 : 1\) map \( C \rightarrow C/\langle a \rangle \). An application of the Riemann-Hurwitz theorem shows the genus of \( C/\langle a \rangle \) to be \(g_{C/\langle a \rangle} = 0\).

b: This has no fixed points and an application of the Riemann-Hurwitz theorem shows the genus of \( C/\langle b \rangle \) to be \(g_{C/\langle b \rangle} = 2\).

c: There are 6 fixed points, \([1, \pm i, \rho^k \beta_\pm]\), where \(\beta_\pm\) is a root of \(\beta_\pm^2 = 2 \pm ib\). Here the Riemann-Hurwitz theorem shows the genus of \( C/\langle c \rangle \) to be \(g_{C/\langle c \rangle} = 1\).

By using the invariants of \( H \) we may obtain nonsingular projective models of \( t^H \). Take for example \( H = \langle c \rangle \) with invariants \( u = zt, v = z^2 - t^2 \) and \( w \) (in degree 2). Then we obtain the quotient curve \( w^3 = v^3 + 3u^2v + bu^3 \) in \( \mathbb{P}^{2,2,2} \sim \mathbb{P}^{1,1,1} = \{ [u, v, w] \} \). The genus of the quotient is seen to be 1. We recognise this as the curve \( E_1 \). One verifies that
\[
c^* \left( \frac{x^2 + 1}{y^2} \right) = \frac{x^2 + 1}{y^2} dx
\]
giving us the invariant differential \( \{5, 26\} \). Similarly, by taking \( H = \langle bc \rangle \) and \( H = \langle b^2c \rangle \), we also obtain equianharmonic elliptic curves. The invariants of the involution \( b^2c \) are again all in degree 2 and now are \( u = zt, v = \rho^{1/2} z^2 - \rho^{-1/2} t^2 \) and \( w \).

By taking \( H = \langle a^2b \rangle \) we may identify \( E_2 \). The invariant of \( \langle a^2b \rangle : [z, t, w] \rightarrow [\rho z, t, w] \) is \( u = z^3 \) and the curve \( w^3 = u^2 - bu^3 + t^6 \) in \( \mathbb{P}^{3,1,2} = \{ [u, t, w] \} \). Using the formula for the genus of a smooth curve of degree \( d \) in \( \mathbb{P}^{a_0, a_1, a_2} \),
\[
g = \frac{1}{2} \left( \frac{d^2}{a_0a_1a_2} - d \sum_{i<j} \frac{\gcd(a_i, a_j)}{a_ia_j} + \sum_{i=0}^{2} \frac{\gcd(a_i, d)}{a_i} - 1 \right),
\]
the genus is seen to be 1. Now \( \{5, 27\} \) is the invariant differential for this action. If we had taken \( H = \langle a \rangle \) with invariants \( u = z^3, v = t^3 \) and \( w \) we obtain the curve \( w^3 = u^2 + buv - v^2 \) in \( \mathbb{P}^{3,3,2} \) (which is equivalent to \( W = u^2 + buv - v^2 \) in \( \mathbb{P}^{1,1,2} \)). The genus of this quotient is seen to be 0.

We obtain the genus 2 curve \( C^* \) as follows. The invariants of \( H = \langle b \rangle \) are \( U = zt, V = z^3, T = t^3 \) and \( w \), subject to the relation \( U^3 = VT \). The curve \( C \) may be written \( T^2 = -u^3 + bU^3 + V^2 \), and hence \( U^6 = V^2T^2 = V^2(-u^3 + bU^3 + V^2) \). This curve has genus 2 in \( \mathbb{P}^{2,3,2} = \{ [U, V, w] \} \) and may be identified with \( C^* \). By setting \( \nu = 2V^2 - (u^3 - bU^3) \) this curve takes the form
\[
\nu^2 = (w^3 - bU^3)^2 + 4U^6
\]

\footnote{Had we represented \( C \subset \mathbb{P}^2 \) as the plane curve given by the vanishing of \( z^6 + b z^3 t^3 - t^6 - w^3 \) the curve is singular. When \( b \) is real the point \( [z, t, w] = [0, 0, 1] \) is the only singular point of \( C \) with delta invariant 6 and multiplicity 3 yielding \( g_C = 4 \).}
in \( \mathbb{P}^{1,3,1} = \{[U, \nu, w]\} \) and the identification with \( C^* \) in the affine chart of earlier is given by \( \mu = -w, U = 1 \). In this latter form we find that the action of \( c \) is given by \([U, \nu, w] \rightarrow [-U, \nu, -w] \sim [U, -\nu, w]\) which is the hyperelliptic involution; further quotienting yields a genus 0 curve.

The remaining genus 1 curves \( \mathcal{E}_{\pm} \) are identified with the quotients of \( C^* \) by \( U \rightarrow \pm w/\sqrt[6]{4+b^2} \), \( w \rightarrow \pm \sqrt[6]{4+b^2}U, \nu \rightarrow \nu \). This action has invariants \( A = Uw \) (in degree 2), \( B = w \pm \sqrt[6]{4+b^2}U \) (in degree 1), and \( \nu \) (in degree 3). The resulting degree 6 curve is

\[
\nu^2 = B^6 + 6LAB^4 + 9L^2A^2B^2 + 2L^3A^3 - 2bA^3,
\]

where, as previously, \( L = \sqrt[6]{4+b^2} \). These curves have genus 1 in \( \mathbb{P}^{2,1,3} = \{[A, B, \nu]\} \). To complete the identification with \( \mathcal{E}_{\pm} \) we compute the \( j \)-invariants of these curves. In the affine patch with \( B \neq 0 \) which looks like \( C^2 \) (the other affine patches have orbifold singularities and hence this choice) the curve takes the form

\[
Y^2 = 1 \mp 6LX + 9L^2X^2 - 2(b \pm L^3)X^3.
\]

The \( j \)-invariants of these curves agree with \([5.28]\) and hence the identifications as stated.

Both the differentials \( dx/y \) and \( x \, dx/y^2 \) are invariant under \( b \). These may be obtained by linear combinations of \( dz/\pm w \). The latter differentials are those invariant under the symmetry of \([5.22]\)

\[
\mu \rightarrow L^2 \mu, \quad \nu \rightarrow \pm L^3 \nu,
\]

which yield the quotients \( \mathcal{E}_{\pm} \). A birational transformation makes this symmetry more manifest\(^4\)

\[
T = \frac{L + \mu}{L - \mu}, \quad S = \frac{8\nu}{(L - \mu)^3}, \quad \mu = L \frac{T - 1}{T + 1}, \quad \nu = \frac{L^3S}{(T + 1)^3}.
\]

Then \([5.22]\) transforms to

\[
S^2 = (T - 1)^6 + 2 \frac{b}{L^3}(T^2 - 1)^3 + (T + 1)^6
\]

which is manifestly invariant under \( T \rightarrow -T, S \rightarrow \mp S \). The substitution \( W = T^2 \) reduces the canonical differentials \( dt/S \) and \( t \, dt/S^2 \) to the canonical differentials the elliptic curves

\[
\mathcal{E}_+ : \quad S^2 = 2(1 + \frac{b}{L^3})W^4 + 6(5 - \frac{b}{L^3})W^2 + 6(5 + \frac{b}{L^3})W + 2(1 - \frac{b}{L^3}),
\]

\[
\mathcal{E}_- : \quad S^2 = 2(1 - \frac{b}{L^3})W^4 + 6(5 - \frac{b}{L^3})W^3 + 6(5 + \frac{b}{L^3})W^2 + 2(1 - \frac{b}{L^3})W,
\]

which correspond to our earlier parameterisations.

5.5. **Role of the higher Goursat hypergeometric identities.** We have seen that complete Abelian integrals of the curve \( C \) \([1.2]\) are given by hypergeometric functions. The same is true for the various curves given in lemma \([5.2]\) covered by \( C \). Relating the periods of \( C \) and the curves it covers leads to various relations between hypergeometric functions, and this underlies the higher hypergeometric identities of Goursat \([Gou81]\). Goursat gave detailed tables of transformations of hypergeometric functions up to order four that will be enough for our purposes.

The simplest example of this is the cover \( \pi : C \rightarrow C^* \) for which \( \pi^*(\mu \, d\mu/\nu) = dx/y \) and \( \pi^*(d\mu/\nu) = x \, dx/y^2 \). One then finds for example that

\[
\int_0^\infty \frac{dx}{y} = \int_0^\infty \frac{\mu \, d\mu}{\nu},
\]

\[^4\text{We thank Chris Eilbeck for this observation.}\]
where both \( y \) and \( \nu \) are evaluated on the first sheet. A change of variable shows that
\[
\int_0^\infty \frac{\mu \, d\mu}{\nu} = \frac{2\pi}{3\sqrt{3}} (b - 2t)^{-1/3} \text{}_2F_1 \left( \frac{1}{2}, \frac{1}{3}; \frac{1}{2}; \frac{4t}{b - 2t} \right).
\]
Now the left-hand side of equation (5.30) is \(-I_1\) (the minus sign arising when we go to Weilstein variables \( y \to -w \)) and this has been evaluated in (5.2). Comparison of these two representations yields the hypergeometric equality
\[
\text{F} \left( \frac{1}{2}, \frac{1}{3}; \frac{1}{2}; \frac{4t}{b - 2t} \right) = \frac{2(b - 2t)}{b + \sqrt{b^2 + 4}} \text{F} \left( \frac{1}{3}, \frac{1}{3}; 1; \frac{b - \sqrt{b^2 + 4}}{b + \sqrt{b^2 + 4}} \right),
\]
which is one of Goursat’s quadratic equalities [Gou81]; see also [BE55, Sect. 2.11, Eq. (31)]. Further identities ensue from the coverings \( C \to E \) \( \pm \) and we shall describe these as needed below.

We remark that the curve (5.22) already appeared in Hutchinson’s study [Hut02] of automorphic functions associated with singular, genus two, trigonal curves in which he developed earlier investigations of Burkhardt [Bur93]. These results were employed by Grava and one of the authors [EG04] to solve the Riemann-Hilbert problem and associated Schlesinger system for certain class of curves with \( Z_N \)-symmetry.

5.6. Weierstrass reduction. It is possible for the theta functions associated to a period matrix \( \tau \) to simplify (or admit reduction) and be expressible in terms of lower dimensional theta functions. Such happens when the curve covers a curve of lower genus, but it may also occur without there being a covering. Reduction may be described purely in terms of the Riemann matrix of periods (see [Mar92b]; for more recent expositions and applications see [BE01, BE02]). A \( 2g \times g \) Riemann matrix \( \Pi = \begin{pmatrix} A \\ B \end{pmatrix} \) is said to admit reduction if there exists a \( g \times g_1 \) matrix of complex numbers \( \lambda \) of maximal rank, a \( 2g_1 \times g_1 \) matrix of complex numbers \( \Pi_1 \) and a \( 2g \times 2g_1 \) matrix of integers \( M \) also of maximal rank such that
\[
(5.31) \quad \Pi \lambda = M \Pi_1,
\]
where \( 1 \leq g_1 < g \). When a Riemann matrix admits reduction the corresponding period matrix may be put in the form
\[
(5.32) \quad \tau = \begin{pmatrix} \tau_1 & Q \\ Q^T & \tau^\# \end{pmatrix},
\]
where \( Q \) is a \( g_1 \times (g - g_1) \) matrix with rational entries and the matrices \( \tau_1 \) and \( \tau^\# \) have the properties of period matrices. Because \( Q \) here has rational entries there exists a diagonal \( (g - g_1) \times (g - g_1) \) matrix \( D = \text{Diag}(d_1, \ldots, d_{g-g_1}) \) with positive integer entries for which \((QD)_{jk} \in \mathbb{Z}\). With \((z, w) = (z_1, \ldots, z_{g_1}, w_1, \ldots, w_{g-g_1})\) the theta function associated with \( \tau \) may then be expressed in terms of lower dimensional theta functions as
\[
\theta((z, w); \tau) = \sum_{m = (m_1, \ldots, m_{g-g_1})} \theta(z + Qm; \tau_1) \theta \left[ \begin{array}{c} D^{-1}m \\ 0 \end{array} \right] \theta(Dw; D\tau^\# D).
\]

Our curve admits many reductions. Of itself this just means that the theta functions may be reduced to theta functions of fewer variables. It is only when the Ercolani-Sinha vector correspondingly reduces that we obtain real simplification. In the remainder of this section we shall describe these reductions and later see how dramatic simplifications occur.
First let us describe the Riemann matrix of periods. We may evaluate the remaining period integrals as follows. Let

\[ \int_0^\alpha du_i = I_i(\alpha), \quad \int_0^\beta du_i = J_i(\alpha), \quad i = 1, \ldots, 4. \]

Then for \( k = 1, 2 \) we have that

\[
\begin{align*}
\int_0^{\rho^k\alpha} du_{1,2} &= \rho^k I_{1,2}(\alpha), \\
\int_0^{\rho^k\beta} du_{1,2} &= \rho^k J_{1,2}(\alpha), \\
\int_0^{\rho^k\alpha} du_3 &= \rho^k I_3(\alpha), \\
\int_0^{\rho^k\beta} du_3 &= \rho^k J_3(\alpha), \\
\int_0^{\rho^k\alpha} du_4 &= I_4(\alpha), \\
\int_0^{\rho^k\beta} du_4 &= J_4(\alpha),
\end{align*}
\]

where it is again supposed that the integrals \( I_\ast \) and \( J_\ast \) are computed on the first sheet. We have already computed \( I_1(\alpha) \) and \( J_1(\alpha) \). The integrals \( I_\ast \) and \( J_\ast \) are found to be

\[
\begin{align*}
I_1(\alpha) &= -\frac{2\pi\alpha}{3\sqrt{3}} 2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{1}{1} - \alpha^6\right) = -\frac{2\pi\alpha}{3\sqrt{3}} \frac{\alpha}{(1 + \alpha^6)^{3/2}} 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right), \\
J_1(\alpha) &= \frac{2\pi}{3\sqrt{3}\alpha} 2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{1}{1} - \frac{1}{\alpha^6}\right) = \frac{2\pi}{3\sqrt{3}} \frac{\alpha}{(1 + \alpha^6)^{3/2}} 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, 1 - t\right), \\
I_2(\alpha) &= \frac{4\pi^2}{9\Gamma\left(\frac{1}{4}\right)^3} \frac{\alpha}{(1 + \alpha^6)^{3/2}}, \\
J_2(\alpha) &= -\frac{4\pi^2}{9\Gamma\left(\frac{1}{4}\right)^3} \frac{\alpha}{(1 + \alpha^6)^{3/2}}, \\
I_3(\alpha) &= \frac{2\pi\alpha^2}{3\sqrt{3}} 2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{1}{1} - \alpha^6\right) = \frac{2\pi}{3\sqrt{3}} \frac{\alpha^2}{(1 + \alpha^6)^{3/2}} 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right), \\
J_3(\alpha) &= \frac{2\pi}{3\sqrt{3}\alpha^2} 2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{1}{1} - \frac{1}{\alpha^6}\right) = \frac{2\pi}{3\sqrt{3}} \frac{\alpha^2}{(1 + \alpha^6)^{3/2}} 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, 1 - t\right), \\
I_4(\alpha) &= \frac{\alpha^3}{4} 2F_1\left(\frac{2}{3}, 1; \frac{4}{3} - \alpha^6\right), \\
J_4(\alpha) &= -\frac{\alpha}{4} 2F_1\left(\frac{2}{3}, 1; \frac{4}{3} - \frac{1}{\alpha^6}\right),
\end{align*}
\]

with \( t = \alpha^6/(1 + \alpha^6) \).

We observe that the relations

\[
(5.34) \quad \mathcal{R} \equiv \frac{I_1(\alpha)}{J_1(\alpha)} = -\frac{I_3(\alpha)}{J_3(\alpha)}, \quad I_2(\alpha) + J_2(\alpha) = 0, \quad I_4(\alpha) - J_4(\alpha) = I_2(\alpha),
\]

follow from the above formulae.
The vectors \( x, \ldots, d \) are

\[
\begin{align*}
x &= \begin{pmatrix} -2J_1 + J_3 - J_1 \\ (J_1 - J_3) + J_3 + 2J_1 \\ (J_1 + 2J_3) + J_1 - J_1 \\ 3(J_1 - J_3) + 3J_1 \end{pmatrix}, \quad b &= I_2 \begin{pmatrix} 1 + 2\rho \\ -2 - \rho \\ 1 - \rho \\ 0 \end{pmatrix}, \\
c &= \begin{pmatrix} (I_3 + 2J_3) - J_3 - J_3 \\ (I_3 - J_3) + J_1 + 2J_3 \\ -2J_3 - J_3 - 2J_3 - J_3 \\ 3(I_3 - J_3) + 3J_3 \end{pmatrix}, \quad d &= (\rho - 1)I_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.
\end{align*}
\]

One may easily check that

\[
x^T H b = x^T H c = x^T H d = 0.
\]

We then have that

\[
(5.36) \quad A = \begin{pmatrix} -1 - 2\rho - (2 + \rho)R & 1 + 2\rho & 1 + 2\rho + (1 - \rho)R & -1 + \rho \\ 2 + \rho + (1 - \rho)R & -2 - \rho & 1 - \rho - (2 + \rho)R & -1 + \rho \\ -1 + \rho + (1 + 2\rho)R & 1 - \rho & -2 - \rho + (1 + 2\rho)R & -1 + \rho \\ 3 + 3\rho - 3\rho R & 0 & -3\rho - 3(1 + \rho)R & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 + \rho & 1 - \rho & -2 + \rho & 2 + \rho \\ 2 + \rho + (1 - \rho)R & 1 - \rho & 1 - \rho & 2 + \rho \\ -1 + \rho + (1 + 2\rho)R & 1 + 2\rho & -2 + \rho & 2 + \rho \\ -1 - 2\rho - (2 + \rho)R & -2 - \rho & 1 + 2\rho + (1 - \rho)R & 2 + \rho \\ 3 + 3(1 + \rho)R & 0 & 3 + 3\rho R & 0 \end{pmatrix}.
\]

The Ercolini-Sinha conditions, \( n^T A + m^T B = 6 \frac{\chi^4}{\pi}(1, 0, 0, 0) \) written for the vectors

\[
(5.37) \quad n = \begin{pmatrix} n_1 \\ m_1 - n_1 \\ -m_1 \\ 2n_1 - m_1 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 \\ -n_1 \\ n_1 - m_1 \\ -3n_1 \end{pmatrix}
\]

lead to the equations

\[
(5.38) \quad \mathcal{R} = \frac{2n_1 - m_1}{m_1 + n_1}, \quad \mathcal{J}_1 = \frac{\chi^4}{m_1 - 2n_1},
\]

which were obtained earlier. A calculation also shows that the relation

\[
\mathcal{M} \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} A \\ B \end{array} \right) \cdot T
\]

yielding a nontrivial check of our procedure.

The integrals between infinities may be reduced to our standard integrals by writing

\[
\int_{-\infty}^{\infty} \mu = \int_{\tau(0, r)}^{\tau(0, j)} \mu = \int_{0(0, r)}^{0(0, j)} \tau^*(d\mu) = \int_{0(0, r)}^{0(0, j)} d\mu \cdot T = \left( \int_{0(0, r)}^{0(0, j)} d\mu - \int_{0(0, r)}^{0(0, j)} d\mu \right) \cdot T,
\]
where we write \( \tau(\infty_i) = 0_{\tau(i)} \) and \( \lambda_\ast \) is any of the branch points. These are then calculated to be

\[
\int_{\infty_1}^{\infty_2} du = \left( \begin{array}{c}
\rho^2 - \rho J_1 \\
-\rho \rho - 1 J_4 \\
(\rho^2 - 1) J_3 \\
-\rho \rho - 1 J_2
\end{array} \right). 
\]

Our Riemann matrix admits a reduction with respect to any of its columns. We will exemplify this with the first column, a result we will use next; similar considerations apply to the other columns. Now from the above and (5.39) it follows that

\[
\Pi \lambda = \left( \begin{array}{c} A \\ B \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} j_{a_1} du_1 \\ j_{b_1} du_1 \end{array} \right) = \left( \begin{array}{c} \xi(n + \rho^2 m) \\ \xi(\rho n + H m) \end{array} \right) = \xi M \left( \begin{array}{c} 1 \\ \rho \end{array} \right),
\]

where \( M \) is the \( 2g \times 2 \) integral matrix

\[
M^T = \left( \begin{array}{cccc}
n_1 - m_1 & n_2 - m_2 & n_3 - m_3 & -n_4 - m_4 \\
-m_1 & -m_2 & -m_3 & -m_4 \\
n_1 & n_2 & n_3 & -n_4
\end{array} \right).
\]

Then to every two Ercolani-Sinha vectors \( n, m \) we have that

\[
M^T J M = d \left( \begin{array}{ccc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad d = n.H n - m.n + m.H m = \sum_{j=1}^{4} (\varepsilon_j n_1^2 - n_j m_j + \varepsilon m_j^2).
\]

The number \( d \) here is often called the Hopf number. In particular for \( d \neq 0 \) then \( M \) is of maximal rank and consequently our Riemann matrix admits reduction.

Let us now focus on the consequences of reduction for symmetric monopoles.

**Theorem 5.4.** For the symmetric monopole we may reduce by the first column using the vector (5.41) whose elements are related by (7.0), with \( (n_1, m_1) = 1 \). Then

\[
d = 2(n_1 + m_1)(m_1 - 2n_1)
\]

and for \( d \neq 0 \) there exists an element \( \sigma \) of the symplectic group \( Sp_{2g}(\mathbb{Z}) \) such that

\[
\tau^\prime_b = \sigma \circ \tau_b = \left( \begin{array}{cccc}
(\rho + 2)/d & \alpha/d & 0 & \ldots & 0 \\
\alpha/d & \rho^2/d & 0 & \ldots & 0 \\
0 & 0 & \tau^\# & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \tau^\#
\end{array} \right).
\]

Letting \( pm_1 + q n_1 = 1 \) then

\[
\alpha = \gcd(m_1 + 4n_1 - q [m_1 - 2n_1], n_1 - 2m_1 - p [m_1 - 2n_1]).
\]

When \( \alpha = 1 \) a further symplectic transformation allows the simplification \( \tau^\prime_{11} = \rho/d \).

Under \( \sigma \) the Ercolani-Sinha vector transforms as

\[
\sigma \circ U = \sigma \circ (m^T + n^T \tau_b) = (1/2, 0, 0, 0).
\]

The proof of the theorem is constructive using work of Krazer, Weierstrass and Kowalewski. Martens [Mar92a, Mar92b] has given an algorithm for constructing \( \sigma \) which we have implemented using Maple. Because \( \sigma \) depends on number theoretic properties of \( n_1 \) and \( m_1 \) the
The form is rather unilluminating and we simply record the result (though an explicit example will be given in the following section). What is remarkable however is the simple universal form the Ercolani-Sinha vector takes under this transformation. This has great significance for us as we next describe.

Using \((5.33), (5.43)\) and say \(D = \text{Diag}(d, 1, 1)\) we have that
\[
\theta((z, w); \tau_b) = \sum_{m=0}^{d-1} \theta(z + \frac{ma}{d}, \frac{\rho}{d} + 2) \theta\left[ \begin{array}{c} \frac{m}{d} \\ 0 \\ 0 \\ 0 \end{array} \right] (Dw; D\tau^#D) = \sum_{m=0}^{d-1} \theta\left[ \begin{array}{c} \frac{m}{d} \\ 0 \end{array} \right] (dz; d(\rho + 2)) \theta((w_1 + \frac{ma}{d}, w_2, w_3); \tau^#),
\]
where we have genus one and three theta functions on the right hand-side here. Comparison of \((2.26)\) and \((5.45)\) then reveals that the theta function dependence of the Baker-Akhiezer is given wholly by the genus one theta functions. Further simplifications ensue from the identity
\[
\theta\left[ \begin{array}{c} \frac{\xi}{e} \\ \tau \end{array} \right] (dz; d\tau) = \mu(\tau) \prod_{l=0}^{d-1} \theta\left[ \begin{array}{c} \frac{\xi}{\tau} + \frac{d-1}{d} \frac{(1+2l)}{d} \\ \tau \end{array} \right] (z; \tau),
\]
where \(\mu(\tau)\) is a constant. We then have

**Theorem 5.5.** For symmetric monopoles the theta function \(z\)-dependence of \((2.26)\) is expressible in terms of elliptic functions.

Thus far we have not discussed the final Hitchin constraint for symmetric monopoles. This theorem reduces the problem to one of the zeros of elliptic functions. The graph in Figure 4 shows the real and imaginary parts of the theta function denominator of \(Q_0(z)\) for the \(n_1 = 2, m_1 = 1\) symmetric monopole, the \(b = 0\) Ramanujan case. These vanish at

\[5\]When \(\gcd(a, d) \neq 1\) a smaller multiple than \(d_1 = d\) would suffice here with correspondingly fewer terms in the sums \(0 \leq m \leq d_1 - 1\).
The Legendre relation (3.22) gives a non trivial consistency check of our calculations. This may be written in the form of the following hypergeometric equality
\[ n \]
Thus the \( Q \) is no corresponding vanishing and consequently \( Q_0(z) \) yields unwanted poles in \( z \in (0, 2) \). Thus the \( n_1 = 2, m_1 = 1 \) curve does not yield a monopole.

A similar evaluation of the relevant \( n_1 = 4, m_1 = -1 \) and \( n_1 = 5, m_1 = -2 \) theta functions also reveals unwanted zeros and of the those cases from our table of symmetric 3-monopoles only the tetrahedrally symmetric case has the required vanishing. Indeed extensive numerical calculations suggests:

**Conjecture 5.6.** For a symmetric monopole the denominator of \( Q_0(z) \) has \( 2(|n_1| - 1) \) zeros and consequently the tetrahedrally symmetric monopole is the only monopole in this class.

Although other techniques exist for the study of monopoles which might allow one to prove that the tetrahedrally symmetric monopole is the only monopole in the class of symmetric monopoles, we don’t know of a suitable theory that counts the number of times a real line (interval) intersects the theta divisor. We would welcome such a theory. Our results do however allow us to say more about the curve of the tetrahedrally symmetric monopole. Before turning to a more detailed examination of this case in our next section we first describe how to calculate the remaining quantities appearing in the formula for \( Q_0(z) \).

### 5.7. Calculating \( \nu_i - \nu_j \)

Here we follow section §5.4. We calculate the \( \text{a}\)-periods of the differential \( dr_1 \) in a manner similar to the period integrals already calculated. Introduce integrals on the first sheet

\[ K_1(\alpha) = \int_0^\alpha \frac{z^4dz}{3w^2}, \quad L_1(\beta) = \int_0^\beta \frac{z^4dz}{3w^2}, \quad \beta = -\frac{1}{\alpha}. \]

Evidently \( K_1(\rho^k\alpha) = \rho^{2k}K(\alpha) \) and \( L_1(\rho^k\beta) = \rho^{2k}L(\beta) \) and one finds that

\[ K_1 = -\frac{4\sqrt{3}\pi}{27} \alpha^5 2F_1 \left(\frac{2}{3}, \frac{5}{3}; 2; -\frac{1}{\alpha^6}\right), \quad L_1 = \frac{4\sqrt{3}\pi}{27} \frac{1}{\alpha^5} 2F_1 \left(\frac{2}{3}, \frac{5}{3}; 2; -\frac{1}{\alpha^6}\right). \]

We find, as before in the case of holomorphic differentials, that

\[ y_1 = (K_1 + 2L_1)\rho - K_1 + L_1, \quad y_2 = (K_1 - L_1)\rho + 2K_1 + L_1 \]

\[ y_3 = -(2K_1 + L_1)\rho - K_1 - 2L_1, \quad y_4 = 3(K_1 - L_1)\rho + 3K_1. \]

The Legendre relation (3.22) gives a non trivial consistency check of our calculations. This may be written in the form of the following hypergeometric equality

\[ \frac{27}{4\sqrt{3}\pi} = \alpha^4 2F_1 \left(\frac{1}{3}, \frac{1}{3}, 1; -\frac{1}{\alpha^6}\right) 2F_1 \left(\frac{2}{3}, \frac{5}{3}; 2; -\alpha^6\right) + \frac{1}{\alpha^4} 2F_1 \left(\frac{1}{3}, \frac{1}{3}, 1; -\alpha^6\right) 2F_1 \left(\frac{2}{3}, \frac{5}{3}; 2; -\frac{1}{\alpha^6}\right) \]

and this may be established by standard means.

To calculate \( \nu_i - \nu_j \) using (3.24) introduce the differential of the second kind,

\[ s = d \left(\frac{w}{z}\right) \left(\frac{z}{w}\right) (P) - 3dr_1(P) = \frac{dz}{z^2w^2}, \]

with second order pole at 0 on all sheets,

\[ \left. \frac{dz}{z^2w^2} \right|_{P=0_k} = \left\{ \frac{1}{w(0_k)^2} \frac{1}{\xi^2} + \frac{2b}{3} \xi + \ldots \right\} d\xi = \left\{ -\frac{w(0_k)^2}{\xi^2} + \frac{2b}{3} \xi + \ldots \right\} d\xi. \]

Here we took into account \( w(0_k)^3 = -1 \) for \( k = 1, 2, 3 \). Then

\[ \nu_i - \nu_j = 3y_i \int_{\infty_j}^{\infty_j} v + \int_{\infty_j}^{\infty_j} \frac{dz}{z^2w^2}. \]
The last integral in (5.49) may also be expressed in terms of hypergeometric functions as follows. First we remark that
\[
\int_{\infty}^{\infty} \frac{dz}{z^2 w^2} = (\rho_i - \rho_j) \int_{\alpha}^{\infty} \frac{dz}{z^2 w^2},
\]
where \(\rho_i = \rho^{i-1}\). Next, for the integrals on the first sheet we have
\[
\int_{\alpha}^{\infty} \frac{dz}{z^2 w^2} = \frac{4\sqrt{3}\pi}{27} \alpha^5 F \left(\frac{2}{3}, \frac{5}{3}; 2; -\frac{1}{\alpha^6}\right) = \mathcal{L}_1,
\]
\[
\int_{-1}^{\infty} \frac{dz}{z^2 w^2} = -\frac{4\sqrt{3}\pi}{27} \alpha^5 F \left(\frac{2}{3}, \frac{5}{3}; 2; -\alpha^6\right) = \mathcal{K}_1.
\]

6. The tetrahedral 3-monopole

The curve of the tetrahedrally symmetric monopole is of the form
\[
(6.1) \quad \eta^3 + \chi(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0.
\]

In this case we may take
\[
t = \frac{1}{2} - \frac{5\sqrt{3}}{18}, \quad \alpha = \frac{\sqrt{3} - 1}{\sqrt{2}}, \quad J_1(\alpha) = -2I_1(\alpha).
\]

For these values we may explicitly evaluate the various hypergeometric functions. Using Ramanujan’s identity (5.15) together with the standard quadratic transformation of the hypergeometric function
\[
2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1, z\right) = (1 + \sqrt{z})^{-1} 2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1, \frac{4\sqrt{z}}{(1 + \sqrt{z})^2}\right),
\]
(valid for \(|z| < 1, \arg z < \pi\)) we find that
\[
2F_1 \left(\frac{1}{3}, \frac{2}{3}; 1, t\right) = \frac{3\sqrt{3}}{4} 2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1, \frac{2 - \sqrt{3}}{4}\right).
\]

(In verifying this we note that \(p = 4 + 3\sqrt{3} - 2\sqrt{6} - 3\sqrt{2}\) is the relevant value leading to our \(t\) in (5.15).) Now this last hypergeometric function is related to an elliptic integral we may evaluate [Law89, p 86],
\[
K \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) = \frac{\pi}{2} 2F_1 \left(\frac{1}{2}, \frac{1}{2}; 4, \frac{2 - \sqrt{3}}{4}\right) = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{3\pi^{\frac{3}{2}}}.
\]

Bringing these results together we finally obtain
\[
(6.2) \quad 2F_1 \left(\frac{1}{3}, \frac{2}{3}; 1, t\right) = \frac{3\sqrt{3}}{8\pi^{\frac{3}{2}}} \Gamma(\frac{1}{6})\Gamma(\frac{1}{3}).
\]

Then from (5.12) we obtain that
\[
(6.3) \quad \chi^{\frac{1}{2}} = -2 \frac{2\pi}{3\sqrt{3}} \frac{\alpha}{(1 - \alpha^6)^{\frac{1}{2}}} 2F_1 \left(\frac{1}{3}, \frac{2}{3}; 1, t\right) = -\frac{1}{2\sqrt{3}} \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{2\sqrt{3}^{\frac{3}{2}}}.
\]

This agrees with the result of [HMR00]. We also note that upon using Goursat’s identity [Gou81 (39)]
\[
2F_1 \left(\frac{1}{3}, \frac{2}{3}; 1, \frac{2}{x}\right) = (1 - 2x)^{-\frac{1}{2}} 2F_1 \left(\frac{1}{6}, \frac{2}{3}; 1, \frac{4x(x - 1)}{(2x - 1)^2}\right),
\]
we may establish the result of [HMR00] based on numerical evaluation, that
\[ 2F_1 \left( \frac{1}{6}, \frac{2}{3}, 1, -\frac{2}{25} \right) = \frac{5 + 3}{8} \frac{\Gamma \left( \frac{1}{5} \right) \Gamma \left( \frac{1}{3} \right)}{\sqrt{3} \pi^3}. \]

Using these results and those of the previous section we have,

**Theorem 6.1.** The tetrahedral 3-monopole for which \( b = 5\sqrt{2} \) admits the \( \tau \)-matrix of the form

\[
\tau = \frac{1}{98} \begin{pmatrix}
-73 + 51i\sqrt{3} & 9 - 13i\sqrt{3} & 15 + 11i\sqrt{3} & 42 - 28i\sqrt{3} \\
9 - 13i\sqrt{3} & -34 + 60i\sqrt{3} & 2i\sqrt{3} - 24 & 21 + 35i\sqrt{3} \\
15 + 11i\sqrt{3} & 2i\sqrt{3} - 24 & -40 + 36i\sqrt{3} & -63 - 7i\sqrt{3} \\
42 - 28i\sqrt{3} & 21 + 35i\sqrt{3} & -63 - 7i\sqrt{3} & 49 + 49i\sqrt{3}
\end{pmatrix}
\]

We have already seen that the symmetric monopole curve \( C \) covers two equianharmonic tori \( E_{1,2} \). For the value of the parameter \( b = 5\sqrt{2} \) the curve covers three further equianharmonic elliptic curves. These may be described as follows. For \( i = 3, 4, 5 \) let \( \pi_i : C \to E_i \) be defined by the formulae

\[
\mu_3 = -\frac{i}{27} \frac{(1 + z\alpha)^4 + (z - \alpha)^4}{\alpha^2 w^2}, \quad \nu_3 = \frac{1 + i(1 + \alpha^2)(z^2 + 1)}{\sqrt{2}(z - \alpha)(z\alpha + 1)},
\]

\[
\mu_4 = -\frac{2}{27} \frac{(1 + z\alpha)^4 + (z - \alpha)^4}{\alpha w(1 + z\alpha)(z - \alpha)}, \quad \nu_4 = \frac{2}{27} \frac{(1 + z\alpha)^4 - (z - \alpha)^4}{\alpha^3(z\alpha + 1)^3},
\]

\[
\mu_5 = -\sqrt{3i} \frac{(z^2 + 1)(z^2 - 2\sqrt{2}z - 1)}{(z^2 + \sqrt{2}z - 1)^2}, \quad \nu_5 = -4\sqrt{6i} \frac{w(z^4 - \sqrt{2}z^3 + 3z^2 + \sqrt{2}z + 1)}{(z^2 + \sqrt{2}z - 1)^3}.
\]

Then

\[
E_3 : \{ (\nu_3, \mu_3) : |\nu_3|^2 - \mu_3^2 - 2\mu_3 = 0 \},
\]

\[
E_4 : \{ (\nu_4, \mu_4) : |\nu_4|^2 - \mu_4(\mu_4^3 + 4) = 0 \},
\]

\[
E_5 : \{ (\nu_5, \mu_5) : |\nu_5|^3 + 24\sqrt{6i}(\mu_5^2 - 1)^2 = 0 \},
\]

and we have the following relations between holomorphic differentials

\[
du_2 = z \frac{dz}{w^2} = \frac{1}{2\sqrt{3}} \left( (1 - 1)\pi_3^2 \left( \frac{d\mu_3}{\nu_3} \right) + \pi_4^2 \left( \frac{d\mu_4}{\nu_4} \right) \right)
\]

\[
du_1 = \frac{dz}{w} = \pi_5^2 \left( \frac{d\mu_5}{\nu_5} \right).
\]

The final of these rational maps was introduced by [HMR00] and has the following significance.

**Proposition 6.2.** Let \( x \) and \( y \) be the \( a \) and \( b \)-periods of the differential \( du_1 \) and denote by \( X, Y \) the \( a \) and \( b \)-periods of the elliptic differential \( d\mu_5/\nu_5 \). Then

\[
\begin{pmatrix} x \\ y \end{pmatrix} = M_5 \begin{pmatrix} X \\ Y \end{pmatrix},
\]
where $M_5$ is the matrix

\begin{equation}
M_5^T = \begin{pmatrix}
-1 & 1 & 0 & 3 & 1 & 0 & -1 & 1 \\
0 & 1 & -1 & 2 & 1 & -1 & 0 & 3 \\
\end{pmatrix}
\end{equation}

satisfying the condition

\begin{equation}
M_5^T \begin{pmatrix}
0_4 & 1_4 \\
-1_4 & 0_4 \\
\end{pmatrix} M_5 = 4 \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix}.
\end{equation}

Proof. Introduce the homology basis for the elliptic curve as shown in Figure 6 and set

\[ K(\alpha) = \int_{-1}^{1} d\mu_5. \]

Then

\begin{equation}
X = (2 + \rho)K(\alpha), \quad Y = -(2\rho + 1)K(\alpha).
\end{equation}

From the reduction formula (6.8) we next conclude that

\begin{equation}
\int_{\alpha\rho}^{\alpha\rho^2} \frac{dz}{w} = K(\alpha)
\end{equation}

and therefore have that

\begin{equation}
2I_1(\alpha) + \rho I_1(\alpha) = \rho K(\alpha), \\
-2\rho I_1(\alpha) - I_1(\alpha) = K(\alpha).
\end{equation}

Equations (6.12) and (6.14) permit us to express

\begin{equation}
I_1(\alpha) = -\frac{Y}{3}, \quad \rho I_1(\alpha) = -\frac{X}{3}
\end{equation}

and comparison with (5.36) yields the given $M_5$. The condition (6.11) is checked directly. The number 4 appearing in (6.11) means that the cover $\pi_5$ given in (6.5) be of degree 4.

We remark that the matrix $M_5$ of the proposition is obtained from the $M$ of (5.41) by

\[ M_5 = -\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} M, \]

which simply reflects our choice of homology basis. Thus we are
discussing the reduction of the previous section. Indeed with
\[
\sigma = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & -1 & 1 & 1 & -1 & 0 & -3 \\
5 & -1 & 0 & 3 & 0 & -1 & 1 & -2 \\
-6 & 0 & 0 & -3 & 0 & 0 & -1 & 2 \\
7 & 0 & 0 & 3 & 0 & 0 & 0 & -2
\end{bmatrix}
\equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

we find that the \( \tau \) matrix \([6.4]\) transforms to
\[
\tau' = \sigma \circ \tau = (a\tau + b)(c\tau + d)^{-1} = \begin{bmatrix}
\rho/4 & 1/4 & 0 & 0 \\
1/4 & 5\rho/4 & \rho & 0 \\
0 & \rho & 2\rho & \rho \\
0 & 0 & \rho & 2/7 + 6\rho/7
\end{bmatrix}.
\]

Combined with Theorem \([5.4]\) we may reduce our expression for the tetrahedral monopoles \(Q_0(z)\) to one built out of Jacobi elliptic theta functions.

### 7. Conclusions

Although monopoles have been studied now for many years and from various perspectives, relatively few analytic solutions are known. The outstanding problem is constructing curves satisfying the transcendental constraints of Hitchin placed upon it. Using the connection with integrable systems this article has sought to make effective such solutions. It is nevertheless only early steps upon this road. Here we have seen how one class of transcendental constraints may be replaced by a number theoretic problem.

In applying our techniques beyond the known case of charge two we considered the restricted class of charge three monopoles \([1.1]\) which includes the tetrahedrally symmetric monopole. This family of curves has many arithmetic properties that facilitates analytic integration. In particular the period matrix may be explicitly expressed in terms of just four integrals. Using this we were able to explicitly solve the Ercolani-Sinha constraints that are equivalent to Hitchin’s transcendental condition \((H2)\) of the triviality of a certain line bundle over the spectral curve (Proposition \([4.1]\)). Our approach reduces the problem to that of determining certain rationality properties of the \(n\) relevant periods. (Our result also admits another approach to seeking monopole curves: we may solve the Ercolani-Sinha constraints and then seek to impose Hitchin’s reality conditions on the resulting curves.) To proceed further in this rather uncharted territory we further restricted our attention to what we have referred to as “symmetric 3-monopoles” whose spectral curve has the form \([1.2]\). This reduced the required independent integrals from four to two, each of which were hypergeometric in form, and the rationality requirement is now for the ratio of these \([Proposition \([5.1]\)]\). Extensions of work by Ramanujan mean this latter question may be replaced by number theory and of seeking solutions of various algebraic equations (depending on the primes involved in the rational ratio). Examples of such solutions were given (again including the tetrahedral case). We further examined the symmetries and coverings of these symmetric curves and their relation to higher Goursat hypergeometric identities. Having at hand
now many putative spectral curves we proceeded to evaluate the remaining integrals needed in our construction. Remarkably we discovered that application of Weierstrass reduction theory showed that the Ercolani-Sinha vector transformed to a universal form and that all of the theta function \( z \)-dependence for symmetric 3-monopoles was expressible in terms of elliptic functions (Theorems 5.4, 5.5). The final selection of permissible spectral curves at last reduced to the question of zeros of these elliptic functions. We are hampered by not knowing a suitable theory for the calculation of the number of times a real line interval intersects the theta divisor. Numerical calculation of these zeros led to a universal conjecture (5.6) suggesting that of the symmetric 3-monopoles only the tetrahedral monopole has the required zeros.

Our final section then was devoted to the charge three tetrahedrally symmetric monopole. Here we were able to substantially simplify known expressions for the period matrix of the spectral curve as well as prove a conjectured identity of earlier workers. Again an explicit map was given and we have been able to reduce entirely to elliptic functions. The final comparison with the Nahm data of [HMM95] will be left for a subsequent work.

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Appendix A. Theta Functions

For \( r \in \mathbb{N} \) the canonical Riemann \( \theta \)-function is given by

\[
\theta(z; \tau) = \sum_{n \in \mathbb{Z}^r} \exp(i \pi n^T \tau n + 2i \pi z^T n).
\]

The \( \theta \)-function is holomorphic on \( \mathbb{C}^r \times \mathbb{S}^r \) and satisfies

\[
\theta(z + p; \tau) = \theta(z; \tau), \quad \theta(z + pr; \tau) = \exp\{-i \pi (p^T \tau p + 2z^T p)\} \theta(z; \tau),
\]

where \( p \in \mathbb{Z}^r \).

The Riemann \( \theta \)-function \( \theta_{a,b}(z; \tau) \) with characteristics \( a, b \in \mathbb{Q} \) is defined by

\[
\theta_{a,b}(z; \tau) = \exp \left\{ i \pi (a^T \tau a + 2a^T (z + b)) \right\} \theta(z + \tau a + b; \tau)
\]

\[
= \sum_{n \in \mathbb{Z}^r} \exp \left\{ i \pi (n + a)^T \tau (n + a) + 2i \pi (n + a)^T (z + b) \right\}.
\]
where \( a, b \in \mathbb{Q}^r \). This is also written as
\[
\theta_{a,b}(z; \tau) = \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z; \tau).
\]

For arbitrary \( a, b \in \mathbb{Q}^r \) and \( a', b' \in \mathbb{Q}^r \) the following formula is valid
\[
(A.3) \quad \theta_{a,b}(z + a' \tau + b'; \tau) = \exp \left\{ -i\pi a'^T \tau a' - 2i\pi a'^T z - 2i\pi (b + b')^T a' \right\} \times \theta_{a+a', b+b'}(z; \tau).
\]

The function \( \theta_{a,b}(\tau) = \theta_{a,b}(0; \tau) \) is called the \( \theta \)-constant with characteristic \( a, b \). We have
\[
\theta_{-a,-b}(z; \tau) = \theta_{a,b}(-z; \tau)
\]
\[
\theta_{a+p,b+q}(z; \tau) = \exp(2\pi i a^T q) \theta_{a,b}(z; \tau)
\]

The following transformation formula is given in [Igu72, p85, p176].

**Proposition A.1.** For any \( g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(2g, \mathbb{Z}) \) and \( (a, b) \in \mathbb{Q}^{2g} \) we put
\[
g \cdot (a, b) = (a, b) g^{-1} + \frac{1}{2} (\text{diag}(CD^T), \text{diag}(AB^T))
\]
\[
\phi_{a,b}(g) = -\frac{1}{2} (aD^T Ba^T - 2aB^T Cb^T + bC^T Ab^T) + \frac{1}{2} (aD^T - bC^T)T \text{diag}(AB^T),
\]
where \( \text{diag}(A) \) is the row vector consisting of the diagonal components of \( A \). Then for every \( g \in \text{Sp}(2g, \mathbb{Z}) \) we have
\[
(A.4) \quad \theta_{g \cdot (a,b)}(0; (A\tau_b + B)(C\tau_b + D)^{-1}) = \kappa(g) \exp(2\pi i \phi_{a,b}(g)) \det(C\tau_b + D)^{\frac{1}{2}} \theta_{(a,b)}(0; \tau_b)
\]
in which \( \kappa(g)^2 \) is a 4-th root of unity depending only on \( g \) while
\[
(A.5) \quad \theta_{g \cdot (a,b)}(z(C\tau_b + D)^{-1}; (A\tau_b + B)(C\tau_b + D)^{-1}) = \mu \exp \left( i\pi z(C\tau_b + D)^{-1} Cz^T \right) \det(C\tau_b + D)^{\frac{1}{4}} \times \theta_{(a,b)}(z; \tau_b)
\]
and \( \mu \) is a complex number independent of \( \tau \) and \( z \) such that \( |\mu| = 1 \).

**The Vector of Riemann Constants** The convention we adopt for our vector of Riemann constants is
\[
\theta \left( \phi(P) - \phi \left( \sum_{i=1}^{g} Q_i \right) - K \right) = 0
\]
in the Jacobi inversion. This is the convention used by Farkas and Kra and the negative of that of Mumford; the choice of signs appears in the actual construction of \( K \), such as (2.4.1) of Farkas and Kra. Then
\[
(K_Q)_j = \frac{1}{2} \tau_{jj} - \sum_k \oint_{P} \omega_k(P) \int_{Q} \omega_j,
\]
\[
(A.6) \quad = \frac{1}{2} (\tau_{jj} + 1) - \sum_{k \neq j} \oint_{P} \omega_k(P) \int_{Q} \omega_j.
\]
The vector of Riemann constants depends on the homology basis and base point \( Q \). If we change base points of the Abel map \( \phi_Q \to \phi_{Q'} \) then \( K_Q = K_{Q'} + \phi_{Q'}(Q^{g-1}) \). With this convention

\[
(A.7) \quad \phi_Q(\text{Div}(K_C)) = -2K_Q.
\]

**Theta Characteristics** The set \( \Sigma \) of divisor classes \( D \) such that \( 2D = K_C \), the canonical class, is called the set of *theta characteristics* of \( C \). The set \( \Sigma \) is a principal homogeneous space for the group \( J_2 \), the group of 2-torsion points of the group \( \text{Pic}^0(C) \) of degree zero line bundles on \( C \). Equivalently this may be viewed as the 2-torsion points of the Jacobian, \( J_2 = \frac{1}{2} \Lambda/\Lambda \). Geometrically if \( \xi \) is a holomorphic line bundle on \( C \) such that \( \xi^2 \) is holomorphically equivalent to \( K_C \), then the divisor of \( \xi \) is a theta characteristic. If \( L \) is a holomorphic line bundle of order 2, that is \( L^2 \) is holomorphically trivial, then the divisor of \( \xi \otimes L \) is also a theta characteristic. Thus there are \(|J_2| = 2^{2g} \) theta characteristics.

We may view \( J_2 = \{ v \in \text{Pic}^0(C)|2v = 0 \} \) as a vector space of dimension \( 2g \) over \( \mathbb{F}_2 \). This vector space has a nondegenerate symplectic (and hence symmetric as the field is \( \mathbb{F}_2 \)) form defined by the Weil pairing. If \( D \) and \( E \) are divisors with disjoint support in the classes of \( u \) and \( v \) respectively, and \( 2D = \text{div}(f) \), \( 2E = \text{div}(g) \) then the Weil Pairing is

\[
\lambda_2 : J_2 \times J_2 \to \mathbb{F}_2, \quad \lambda_2(u, v) = \frac{g(D)}{f(E)},
\]

where if \( D = \sum_j n_j x_j \) then \( g(D) = \prod_j g(x_j)^{n_j} \). Mumford identifies \( \mathbb{F}_2 \) with \( \pm 1 \) by sending 0 to 1 and 1 to \(-1\). (In general we may consider \( J_r \), the \( r \)-torsion points of \( \text{Pic}^0(C) \), and the Weil pairing gives us a nondegenerate antisymmetric map \( \lambda_2 : J_r \times J_r \to \mu_r \), where \( \mu_r \) are \( r \)-th roots of unity.) The \( \mathbb{F}_2 \) vector space \( J_2 \) may be identified with \( H^1(C, \mathbb{F}_2) \) and with this identification \( \lambda_2 \) is simply the cup product.

Define \( \omega_\xi : J_2 \to \mathbb{F}_2 \) by

\[
(A.8) \quad \omega_\xi(u) = \text{Dim} H^0(C, \xi \otimes u) - \text{Dim} H^0(C, \xi) \quad \text{( mod 2)},
\]

where \( u = L_D \) is the line bundle with divisor \( D \). Then

\[
\lambda_2(u, v) = \omega_\xi(u \otimes v) - \omega_\xi(u) - \omega_\xi(v).
\]

Any function \( \omega_\xi \) satisfying this identity is known as an Arf function, and any Arf function is given by \( \omega_\xi \) for some theta characteristic with corresponding line bundle \( \xi \). Thus the space of theta characteristics may be identified with the space of quadratic forms \([A,0] \).

**APPENDIX B. INTEGRALS BETWEEN BRANCH POINTS**

We shall now describe how to integrate holomorphic differentials between branch points. We use the fact that for non-invariant holomorphic differentials (as we have)

\[
\sum_{i=1}^3 \int_{\gamma_1(\lambda_A, \lambda_B)} \omega = \int_{\gamma_A} (\omega + R_2 \omega + R_3 \omega) = 0.
\]

Indeed, if \( \omega \) is any holomorphic differential on a compact Riemann surface which is an \( N \)-fold branched cover of \( \mathbb{C}P^1 \) then \( \sum_{j=1}^N \omega(P^{(j)}) = 0 \), where \( P^{(j)} \) are the preimages of \( P \in \mathbb{C}P^1 \). Then

\[
\oint_{a_1 - b_1} \omega = 3 \int_{\gamma_1(\lambda_1, \lambda_2)} \omega, \quad \oint_{a_2 - b_2} \omega = 3 \int_{\gamma_1(\lambda_3, \lambda_4)} \omega, \quad \oint_{a_3 - b_3} \omega = 3 \int_{\gamma_1(\lambda_5, \lambda_6)} \omega,
\]
and consequently

\[
\begin{align*}
\int_{\gamma_1(\lambda_1, \lambda_2)} \omega &= \frac{1}{3} \oint_{a_1-b_1} \omega, \\
\int_{\gamma_2(\lambda_1, \lambda_2)} \omega &= \int_{\gamma_1(\lambda_1, \lambda_2)-a_1} \omega = \frac{1}{3} \oint_{-2a_1-b_1} \omega, \\
\int_{\gamma_3(\lambda_1, \lambda_2)} \omega &= \frac{1}{3} \oint_{2b_1+a_1} \omega,
\end{align*}
\]

with similar expressions obtained for \(\gamma_i(\lambda_3, \lambda_4)\) and \(\gamma_i(\lambda_5, \lambda_6)\).

Further utilising \(\gamma_1(\lambda_2, \lambda_6) = \gamma_1(\lambda_2, \lambda_1) + \gamma_1(\lambda_1, \lambda_6)\) and \(\gamma_2(\lambda_5, \lambda_1) = \gamma_2(\lambda_5, \lambda_6) + \gamma_2(\lambda_6, \lambda_1)\) we may write

\[
\begin{align*}
a_4 &= b_1 - b_3 - a_3 + \gamma_1(\lambda_1, \lambda_6) + \gamma_2(\lambda_6, \lambda_1), \\
b_4 &= a_1 + b_1 - a_3 + \gamma_1(\lambda_1, \lambda_6) + \gamma_3(\lambda_6, \lambda_1).
\end{align*}
\]

Appropriate linear combinations of these yield \(\int_{\gamma_i(\lambda_1, \lambda_6)} \omega\) for \(i = 1, 2, 3\). For example

\[
\int_{\gamma_1(\lambda_1, \lambda_6)} \omega = \frac{1}{3} \oint_{2a_3-2b_1-a_1+b_3+a_4+b_4} \omega.
\]

In order to be able to integrate a holomorphic differential between any branch point we must show how we may integrate such between \(\lambda_4\) and \(\lambda_5\) on any branch. Now we use that there exist meromorphic functions \(f = w/(z - \lambda_1)^2\) and \(g = (z - \lambda_i)/(z - \lambda_j)\) (for each \(i, j\)) with (respective) divisors

\[
(f) = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 5\lambda_1, \quad (g) = 3(\lambda_i - \lambda_j).
\]

Thus for any normalized holomorphic differential \(\mathbf{v}\)

\[
\Lambda \ni \int_{\lambda_1}^{\lambda_2} \mathbf{v} + \int_{\lambda_2}^{\lambda_3} \mathbf{v} + \int_{\lambda_1}^{\lambda_4} \mathbf{v} + \int_{\lambda_1}^{\lambda_5} \mathbf{v} + \int_{\lambda_1}^{\lambda_6} \mathbf{v} = 4 \int_{\lambda_1}^{\lambda_2} \mathbf{v} + 2 \int_{\lambda_1}^{\lambda_4} \mathbf{v} + \int_{\lambda_1}^{\lambda_5} \mathbf{v} + \int_{\lambda_1}^{\lambda_6} \mathbf{v},
\]

and \(3 \int_{\lambda_1}^{\lambda_6} \mathbf{v} \in \Lambda\), where \(\Lambda\) is the period lattice. These equalities hold (modulo a lattice vector) for a path of integration on any branch and so, for example,

\[
\int_{\gamma_4(\lambda_4, \lambda_5)} \mathbf{v} = \int_{\gamma_1(\lambda_3, \lambda_4)} \mathbf{v} - \int_{\gamma_1(\lambda_1, \lambda_2)} \mathbf{v} - \int_{\gamma_1(\lambda_1, \lambda_6)} \mathbf{v} \quad \text{mod} \ \Lambda.
\]

APPENDIX C. MöBIUS TRANSFORMATIONS

We wish to determine when there is a Möbius transformation between the sets \(H = \{\alpha_1, -1/\alpha_1, \alpha_2, -1/\alpha_2, \alpha_3, -1/\alpha_3\}\) and \(S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}\). The former corresponds to reality constraints on our data arising from \((H1)\) while the latter may be constructed from the period matrix of the curve in terms of various theta constants. If we have a period matrix satisfying \((H2)\) then we must satisfy \((H1)\).

At the outset we note that the Möbius transformation \(M\) sending \(a \to 0, b \to 1, c \to \infty\) and its inverse \(M^{-1}\)

\[
\begin{align*}
M(a) &= 0 & M^{-1}(0) &= a, \\
M(b) &= 1 & M^{-1}(1) &= b, \\
M(c) &= \infty & M^{-1}(\infty) &= c.
\end{align*}
\]
are given by

\[(C.1)\quad M(z) = \frac{b - c}{b - a} \frac{z - a}{z - c} \quad M^{-1}(z) = \frac{z c(b - a) - a(b - c)}{z(b - a) - (b - c)}.\]

The transformation

\[M(z) = \lambda \cdot \frac{z - a}{z - c} = \frac{\alpha z + \beta}{\gamma z + \delta}\]

may be represented by the \(SL(2, \mathbb{C})\) matrix

\[(C.2)\quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{i \sqrt{c}}{\sqrt{a}+c} & -\frac{i \sqrt{c}}{\sqrt{a}+c} \\ -\frac{i \sqrt{a}}{\sqrt{a}+c} & -\frac{i \sqrt{a}}{\sqrt{a}+c} \end{pmatrix}\]

and upon setting \(\lambda = (b - c)/(b - a)\) we may determine a \(SL(2, \mathbb{C})\) representation of \((C.1)\).

A Möbius transformation is conjugate to a rotation if and only if it is of the form \(M(z) = (\alpha z + \beta)/(−\beta Z + \gamma)\). In terms of \((C.2)\) this means

\[\alpha c = -1 \quad \text{and} \quad \lambda \overline{\lambda} = \frac{1}{\alpha^2} \]

Then \(M(0)M(\infty) = -1.\)

The rotation \(\begin{pmatrix} \overline{\alpha_1} & 1 \\ \sqrt{1 + |\alpha_1|^2} & \sqrt{1 + |\alpha_1|^2} \end{pmatrix}\) transforms the set \(H\) to one of the form \(\{0, \infty, \tilde{\alpha}_2, -1/\tilde{\alpha}_2, \tilde{\alpha}_3, -1/\tilde{\alpha}_3\}\) where \(\tilde{\alpha}_r = M(\alpha_r) = (1 + \overline{\alpha_1} \alpha_r)/(\alpha_1 - \alpha_r)\) \((r = 2, 3)\). Upon setting \(\tilde{\alpha}_2 = a e^{i\theta},\ a = |\tilde{\alpha}_2|\) the rotation \(\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\) will transform the latter set to one of the form \(\{0, \infty, a, -1/a, w, -1/w\}\). Finally the scaling \(z \to z/a\) given by \(\begin{pmatrix} 1/\sqrt{a} & 0 \\ 0 & \sqrt{a} \end{pmatrix}\) transforms \(H\) to \(H_s = \{0, 1, \infty, -1/a^2, w/a, -1/(a\overline{w})\}\). Such a set is of the desired form \(S\) and is characterised by 3 \((\text{real})\) parameters. With \(\lambda_1 = -1/a^2,\ \lambda_2 = w/a, \lambda_3 = -1/a\overline{w}\) we see we have \(\lambda_1 \in \mathbb{R}, \lambda_1 < 0, \lambda_2 \lambda_3 = \lambda_1\). From a set \(H_s\) and a choice of \(\theta\) and \(\alpha_1\) \((\text{equivalently, a rotation})\) we may reconstruct \(H\).

More generally, let us consider images \(M(H)\) under Möbius transformations. Up to a relabelling of roots we have four possibilities of those roots we map to \(\{0, 1, \infty\}\):

a. \(\alpha_1 \to 0, \quad \alpha_2 \to 1, \quad -1/\overline{\alpha_1} \to \infty,\)

b. \(\alpha_1 \to 0, \quad -1/\overline{\alpha_1} \to 1, \quad \alpha_2 \to \infty,\)

c. \(\alpha_1 \to 0, \quad -1/\overline{\alpha_2} \to 1, \quad \alpha_2 \to \infty,\)

d. \(\alpha_1 \to 0, \quad \alpha_3 \to 1, \quad \alpha_2 \to \infty.\)

We have already considered \((a)\) in the previous paragraph. For completeness let us give \(\lambda_1, \lambda_2, \lambda_3\) for the various cases and the various restrictions arising from

\(a.\)

\[\lambda_1 = M(\alpha_2) = -\frac{(1 + \overline{\alpha}_1 \alpha_2)(1 + \overline{\alpha}_2 \alpha_1)}{(\alpha_1 - \alpha_2)(\overline{\alpha}_1 - \overline{\alpha}_2)} < 0,\]

\[\lambda_2 = M(\alpha_3) = \frac{\alpha_1 - \alpha_3 + \overline{\alpha}_1 \alpha_2}{\alpha_1 - \alpha_2 + \overline{\alpha}_1 \overline{\alpha}_2},\]

\[\lambda_3 = M(\alpha_3) = -\frac{1 + \alpha_1 \overline{\alpha}_3 + \alpha_2 \overline{\alpha}_1}{\alpha_1 - \overline{\alpha}_3 + \alpha_1 - \alpha_2}.\]
\[ \Lambda_2 \overline{\Lambda}_4 = \Lambda_1; \]

b.

\[
\begin{align*}
\Lambda_1 &= M(-\frac{1}{\alpha_2}) = -\frac{1 + \overline{\alpha}_1 \alpha_2}{1 + \alpha_1 \overline{\alpha}_2}, \\
\Lambda_2 &= M(\alpha_3) = \frac{1 + \overline{\alpha}_1 \alpha_2 \alpha_3 - \alpha_1}{1 + \alpha_1 \overline{\alpha}_2 \alpha_3 - \alpha_2}, \\
\Lambda_3 &= M(-\frac{1}{\alpha_3}) = \frac{1 + \overline{\alpha}_1 \alpha_2}{1 + \alpha_1 \overline{\alpha}_2}. \\
\end{align*}
\]

\[ \frac{\Lambda_2 - 1}{\Lambda_2 - 1} \left(\frac{\Lambda_3}{\Lambda_3 - 1}\right) = \frac{\Lambda_1}{\Lambda_1 - 1}; \]

c.

\[
\begin{align*}
\Lambda_1 &= M(-\frac{1}{\alpha_1}) = -\frac{1 + \alpha_2 \overline{\alpha}_2}{1 + \alpha_1 \overline{\alpha}_2}, \\
\Lambda_2 &= M(\alpha_3) = \frac{1 + \alpha_2 \overline{\alpha}_2 \alpha_3 - \alpha_1}{1 + \alpha_1 \overline{\alpha}_2 \alpha_3 - \alpha_2}, \\
\Lambda_3 &= M(-\frac{1}{\alpha_3}) = \frac{1 + \alpha_2 \overline{\alpha}_2}{1 + \alpha_1 \overline{\alpha}_2}. \\
\end{align*}
\]

\[ (1 - \Lambda_2)(1 - \Lambda_3) = 1 - \Lambda_1; \]

d.

\[
\Lambda_r = M(-\frac{1}{\alpha_r}) = \frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_1}, \quad r = 1, 2, 3, \\
\]

\[ 0 < \alpha_1 \overline{\alpha}_2 \in \mathbb{R}, \quad 1 < \frac{\Lambda_1}{\Lambda_2} \in \mathbb{R}, \quad \Lambda_3 = \Lambda_2 \frac{(1 - \overline{\Lambda}_1)}{1 - \overline{\Lambda}_2}. \]

The constraints \((C.4)\) for case \((b)\) may be obtained as follows. Further composing the Möbius transformation leading to \((b)\) with that giving \(0 \to 0, 1 \to \infty, \infty \to 1\) gives us case \((a)\) for which we know the constraint. This second Möbius transformation is given by \(M(z) = M^{-1}(z) = z/(z - 1)\) and we may transfer the constraint of \((a)\) to \((b)\). Similarly composing \((c)\) with \(M(z) = 1 - z\) yields case \((a)\) up to a relabelling of roots. Geometrically cases \((a), (b), (c)\) consist of the following. A circle passes through \(\{\alpha_1, -1/\overline{\alpha}_1, \alpha_2, -1/\overline{\alpha}_2\}\) and \(\alpha_3\). Under a Möbius transformation to the set \(\{0, 1, \infty, \mu\}\) the circle becomes the real axis and so \(\mu \in \mathbb{R}\). This is the real parameter appearing in each of these cases. A similar argument composing \((d)\) with \(M(z) = z/(z - \Lambda_1)\) will give the constraints \((C.6)\).

In each case, given \(\alpha\), and a choice of \(\theta\) (a rotation) we can construct \(S\) from \(H\).

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