EQUATIONS AND CHARACTER SUMS WITH MATRIX POWERS, KLOOSTERMAN SUMS OVER SMALL SUBGROUPS AND QUANTUM ERGODICITY

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Abstract. We obtain a nontrivial bound on the number of solutions to the equation

\[ A^{x_1} + \ldots + A^{x_{n\nu}} = A^{x_1+1} + \ldots + A^{x_{n\nu}}, \quad 1 \leq x_1, \ldots, x_{n\nu} \leq \tau, \]

with a fixed \( n \times n \) matrix \( A \) over a finite field \( \mathbb{F}_q \) of \( q \) elements of multiplicative order \( \tau \). We give applications of our result to obtaining a new bound of additive character sums with a matrix exponential function, which is nontrivial beyond the square-root threshold. For \( n = 2 \) this equation has been considered by Kurlberg and Rudnick (2001) (for \( \nu = 2 \)) and Bourgain (2005) (for large \( \nu \)) in their study of quantum ergodicity for linear maps over residue rings. Here we use a new approach to improve their results. We also obtain a bound on Kloosterman sums over small subgroups, of size below the square-root threshold.

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1. Introduction

1.1. Set-up and motivation. For a positive integer $n$ and a prime power $q$ we use $\text{GL}(n, q)$ and $\text{SL}(n, q)$ to denote the general and special linear groups of $n \times n$ matrices over the finite field $\mathbb{F}_q$ of $q$ elements, respectively.

For a matrix $A \in \text{GL}(n, q)$ and a positive integer $\nu$, we denote by $Q_{\nu,n,q}(A)$ the number of solutions to the matrix equation

$$A^{x_1} + \ldots + A^{x_\nu} = A^{x_{\nu+1}} + \ldots + A^{x_{2\nu}}, \quad 1 \leq x_1 \ldots, x_{2\nu} \leq \tau,$$

where $\tau$ is the multiplicative order of $A$, that is, the smallest $t \geq 1$ such that $A^t$ is the identity matrix. We also set

$$E_{n,q}(A) = Q_{2,n,q}(A) \quad \text{and} \quad F_{n,q}(A) = Q_{3,n,q}(A).$$

In particular, the quantity $E_{n,q}(A)$ is called the additive energy of the multiplicative subgroup $\langle A \rangle$ generated by $A$ in the ring of $n \times n$ matrices, see [47] for a background and exposition of the role of additive energy.

We first recall that for $n = 1$, that is, in the scalar case, a variety of bounds on the additive energy of multiplicative subgroups of $\mathbb{F}_p^*$ can
be found in \[17, 31, 39, 41\], the case of arbitrary finite fields is more involved \[29, 49\].

Furthermore, for \( n = 2 \), a prime \( q = p \) and a matrix \( A \in \text{SL}(2, p) \), Kurlberg and Rudnick \[26\] have shown links between such results and the problem of equidistribution of eigenfunctions of the “quantised cat map”, which is a toy model of quantum chaos, we refer to \[16, 19, 23–27, 33\] for further references and concrete results. Bourgain \[4\] has given a stronger version of \[26\]. Here we obtain a further improvement of \[4, 26\] and using a different approach, give an explicit version of the bound of Bourgain \[4, \text{Theorem 3}\].

This motivates to study the case of arbitrary \( n \) and also of more general fields and matrices. It is also well-known that such estimates lead to new bounds of exponential sums and thus in turn apply to some additive problems. We present such applications as well.

Furthermore, we apply our results to obtain an explicit bound on Kloosterman sums over small subgroups of \( \mathbb{F}_p^* \) for a prime \( p \). While general results of Bourgain \[3\] apply to very small subgroups, they are not explicit and making them explicit appears to be very nontrivial. Thus, till now, such explicit bounds have been known only in the monomial case.

1.2. Previous results. For \( n = 2 \) and also a prime \( q = p \) and a matrix \( A \in \text{SL}(2, p) \), Kurlberg and Rudnick \[26\] have essentially shown that

\[
E_{2,q}(A) \leq 3\tau^2.
\]

Here we use some ideas which stem from \[9\] to obtain a nontrivial bound, that is, better than \( \tau^3 \), for any dimension and arbitrary finite field, and we also relax the condition \( A \in \text{SL}(n, p) \).

In a higher dimension, also for \( A \in \text{SL}(n, p) \), Bourgain \[4\] used bounds of exponential sums over small subgroups to obtain an asymptotic formula for \( Q_{n,n,q}(A) \).

Here we obtain new bounds on this quantity, which are based on new estimates of \( F_{n,q}(A) \) given by \(1.2\). It is easy to see that \( F_{n,q}(A) \leq \tau^2 E_{n,q}(A) \), so we are interested in obtaining stronger bounds. Although we believe our approach can deliver such better estimates for any \( n \), here, to exhibit the main ideas, we concentrate on the case of \( n = 2 \), a prime \( q = p \), and \( A \in \text{SL}(2, p) \), which corresponds to the settings of Kurlberg and Rudnick \[26\]. In turn, this result leads to a new bound on certain operators considered by Kurlberg and Rudnick \[26\] and to an explicit form of a result of Bourgain \[4, \text{Proposition 1}\], see Theorem 3.3.

Our approach uses the results of \[46\] which allows us to obtain nontrivial bounds on the number of rational points on curves of very high
degree over finite fields, in the regime where the Weil bound (see, for example, [28, Section X.5, Equation (5.2)]) becomes trivial. We believe this approach is of independent interest and may have several other applications.

1.3. **Notation.** We denote by $\overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$. For $\lambda \in \overline{\mathbb{F}}_q^*$, we denote by $\text{ord}_\lambda$ the multiplicative order of $\lambda$.

We note that hereafter in a matrix-vector multiplication we always assume that the dimensions are properly matched, that is, the vectors on the left of a matrix are always rows, while the vectors on the right of a matrix are always columns. In particular $u v$ is the scalar product of the vectors $u$ and $v$.

We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which throughout this work, all implied constants may depend only on $n$.

2. **Main results**

2.1. **Bounds on the number of solutions to matrix equations.**

We start with the following general bound on $E_{n,q}(A)$ given by (1.2).

**Theorem 2.1.** Let $A \in \text{GL}(n, q)$ be diagonalisable. Then we have

$$E_{n,q}(A) \ll \tau^3 \min \left\{ t \tau^{-1/n^2}, t^{n/(n-1)} \tau^{-1/n(n-1)} \right\}$$

for any $A$ and

$$E_{n,q}(A) \ll t \tau^{-3-1/n}$$

if the characteristic polynomial of $A$ is irreducible over $\mathbb{F}_q$, where $\tau$ is the multiplicative order of $A$ and $t$ is the multiplicative order of $\det A$.

In particular, for $A \in \text{SL}(n, q)$ the bounds of Theorem 2.1 become

$$E_{n,q}(A) \ll \tau^{3-1/n(n-1)}$$

and

$$E_{n,q}(A) \ll \tau^{-3-1/n}$$

for any $A$ and $A$ with an irreducible over $\mathbb{F}_q$ characteristic polynomial, respectively.

Next we estimate $F_{2,p}(A)$ given by (1.2) in the split case.

**Theorem 2.2.** Let $p$ be prime and let $A \in \text{SL}(2, p)$ be diagonalisable with both eigenvalues in $\mathbb{F}_p$. Then we have

$$F_{2,p}(A) \ll \tau^{11/3},$$

where $\tau$ is the multiplicative order of $A$.

In the irreducible case we have a slightly weaker bound.
Theorem 2.3. Let $p$ be prime and let $A \in \text{SL}(2,p)$ be diagonalisable with both eigenvalues in $\mathbb{F}_p^* \setminus \mathbb{F}_p$. Then we have

$$F_{2,p}(A) \ll \tau^{19/5} + \tau^5 p^{-1},$$

where $\tau$ is the multiplicative order of $A$.

We note that both Theorems 2.2 and 2.3 are nontrivial for any $\tau$. Moreover, we note that similarly to (1.3), we have

$$F_{2,q}(A) \ll \tau^4.$$

Thus, Theorem 2.2 improves this bound for any $\tau$, while Theorem 2.3 improves it when $\tau < p^{1-\varepsilon}$ for some fixed $\varepsilon > 0$.

2.2. Bounds on exponential sums with matrices. We now use Theorem 2.1 to obtain a new bound on exponential sums with a matrix exponential function, which is non-trivial below the square-root threshold, that is, for $\tau < q^{n/2}$. We recall that for $\tau \geq q^{n/2}$ a non-trivial bound, see (2.2) below, can be achieved via well-known methods in the case of arbitrary finite fields, which stem from the work of Postnikov [32, Chapter I, Section 4, Lemma 1], which in turn is a slight variation of a classical result of Korobov [22]. We note that both Korobov [22] and Postnikov [32] formulate their bounds only for prime fields, but the proofs extend to arbitrary finite fields without any changes (at the cost of essentially only typographical changes). On the other hand, in the case of prime fields or finite fields of large characteristic, several better bounds of this kind are known [2,3,5,6,8,12,40] but their underlying methods, based on additive combinatorics, do not extend to arbitrary finite fields (and sometimes apply only to special cases).

We fix a nontrivial additive character $\psi$ of $\mathbb{F}_q$ and consider the complete character sum

$$S_{n,q}(a,b;A) = \sum_{x=1}^\tau \psi(aA^x b)$$

with $a, b \in \mathbb{F}_q^n$, where $\tau$ is the multiplicative order of $A \in \text{GL}(n,q)$.

We note that it is easy to see that the sequence $aA^x b$ is a linear recurrence sequence, and the other way around, any linear recurrence sequence over $\mathbb{F}_q$ can be represented in this for some vectors $a, b$ and a matrix $A$, see [14, Section 1.1.12].

We now define

$$\kappa_n = \left(4n \left| n - 1 \left| \frac{n-1}{2} \right| \right| \right)^{-1}.$$
Thus \( \kappa_n \sim (3n^2)^{-1} \), when \( n \to \infty \).

**Theorem 2.4.** Let \( \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n \) and let \( A \in \text{GL}(n,q) \) be diagonalisable. Assume that each of the groups of the following \( n \) vectors
\[
\mathbf{A}^i, \quad i = 0, \ldots, n-1,
\]
and
\[
A^i \mathbf{b}, \quad i = 0, \ldots, n-1,
\]
are linearly independent over \( \mathbb{F}_q \). Then, we have
\[
S_{n,q}(\mathbf{a}, \mathbf{b}; A) \ll \begin{cases} 
 t^{1/4} \tau^{1/2-\kappa_n} q^{n/4}, & \text{if } \tau > q^{n/2}, \\
 t^{1/4} \tau^{3/4-\kappa_n} q^{n/8}, & \text{if } \tau \leq q^{n/2},
\end{cases}
\]
where \( \tau \) is the multiplicative order of \( A \) and \( t \) is the multiplicative order of \( \det A \).

Using the idea of Korobov [22], one can easily obtain the bound
\[
(2.2) \quad |S_{n,q}(\mathbf{a}, \mathbf{b}; A)| \leq q^{n/2},
\]
which is nontrivial if \( \tau \geq q^{n/2+\varepsilon} \) for some fixed \( \varepsilon > 0 \). The main interest of Theorem 2.4 is that it remains nontrivial below the square-root threshold, see, for example, (3.1), which is a notoriously difficult range for problems of this flavour.

We have a stronger bound for matrices with irreducible over \( \mathbb{F}_q \) characteristic polynomial, which also makes redundant the linear independence conditions for both families of vectors \( \mathbf{A}^i \) and \( A^i \mathbf{b} \), \( i = 0, \ldots, n-1 \).

**Theorem 2.5.** Let \( \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n \) be non-zero vectors and let \( A \in \text{GL}(n,q) \) be such that the characteristic polynomial of \( A \) is irreducible over \( \mathbb{F}_q \). Then we have
\[
S_{n,q}(\mathbf{a}, \mathbf{b}; A) \ll \begin{cases} 
 t^{1/4} \tau^{1/2-1+1/4n} q^{n/4}, & \text{if } \tau > q^{n/2}, \\
 t^{1/4} \tau^{3/4-1+1/4n} q^{n/8}, & \text{if } \tau \leq q^{n/2},
\end{cases}
\]
where \( \tau \) is the multiplicative order of \( A \) and \( t \) is the multiplicative order of \( \det A \).

**Remark 2.6.** One can easily check that our method extends, at the cost of only marginal typographical changes, to the twisted sums
\[
\sum_{x=1}^{\tau} \psi(\mathbf{a}A^ix\mathbf{b}) \exp(2\pi i \alpha x), \quad \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n, \quad \alpha \in \mathbb{R}.
\]

Thus using the standard completing technique, see [18, Section 12.2], one can extend the bounds of Theorems 2.4 and 2.5 to incomplete sums (with just an additional factor \( \log q \)).
Now we obtain stronger results in the case $n = 2$.
First we estimate $S_{2,p}(a, b; A)$ in the split case.

**Theorem 2.7.** Let $p$ be prime. Let $a, b \in \mathbb{F}_p^2$ and let $A \in \text{SL}(2, p)$ be diagonalisable with eigenvalues in $\mathbb{F}_p^*$. Assume that in each pair

$$(a, aA) \quad \text{and} \quad (b, Ab)$$

the vectors are linearly independent over $\mathbb{F}_p$. Then, we have

$$S_{2,p}(a, b; A) \ll \min\{\tau^{23/36}p^{1/6}, \tau^{20/27}p^{1/9}\},$$

where $\tau$ is the multiplicative order of $A$.

In the case when the characteristic polynomial of $A$ is irreducible over $\mathbb{F}_p$ we have a weaker result. In fact we formulate it in the setting of matrices $A \in \text{SL}(2, p)$ but with eigenvalues avoiding $\mathbb{F}_p$.

**Theorem 2.8.** Let $p$ be prime. Let $a, b \in \mathbb{F}_p^2$ and let $A \in \text{SL}(2, p)$ be diagonalisable with eigenvalues in $\mathbb{F}_p^* \setminus \mathbb{F}_p$. Assume that in each pair

$$(a, aA) \quad \text{and} \quad (b, Ab)$$

the vectors are linearly independent over $\mathbb{F}_p^2$. Then, we have

$$S_{2,p}(a, b; A) \ll \min\{\tau^{1/2}p^{1/4}, \tau^{13/20}p^{1/6}, \tau^{34/45}p^{1/9}\},$$

where $\tau$ is the multiplicative order of $A$.

Given a multiplicative subgroup $G \subseteq \mathbb{F}_q^*$ and $a, b \in \mathbb{F}_q$ we consider Kloosterman sums over a subgroup,

$$K_q(G; a, b) = \sum_{u \in G} \psi(au + bu^{-1}).$$

It is easy to see that the Weil bound of exponential sums with rational functions (see, for example, [30]) implies that

$$K_q(G; a, b) \ll q^{1/2},$$

unless $a = b = 0$, which becomes trivial for subgroups $G$ of order $\tau < q^{1/2}$.

If $g$ is a generator of $G \subseteq \mathbb{F}_p^*$ then, applying Theorems 2.7 and 2.8 to

$$a = (a, 1), \quad b = (1, b)^t, \quad A = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix},$$

(where $u^t$ means the transpose of a vector $u$) we obtain the following result.

**Corollary 2.9.** Let $p$ be prime and let $G$ be a multiplicative subgroup of $\mathbb{F}_p^*$ of order $\tau$. Then

$$K_p(G; a, b) \ll \min\{\tau^{23/36}p^{1/6}, \tau^{20/27}p^{1/9}\}.$$
We now denote by $N_q$ the norm subgroup of $\mathbb{F}_q^*$, that is, the subgroup, formed by elements $z \in \mathbb{F}_q$ of norm $Nm(z) = 1$.

Let $\mathbb{F}_q$ be of characteristic $p$. We recall that, given an additive character $\psi$ of $\mathbb{F}_q$, there exists an element $\alpha \in \mathbb{F}_q$ such that

$$\psi(z) = e_p(\text{Tr}(\alpha z)),$$

where

$$e_p(u) = \exp(2\pi i u/p)$$

and $\text{Tr}(z)$ is the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$.

We also observe that for $\lambda \in N_{p^2}$ we have $\lambda^{-1} = \lambda^p$. Hence

$$\text{Tr}(\alpha \lambda^{-1}) = \text{Tr}(\alpha \lambda^p) = \text{Tr}(\alpha^p \lambda).$$

Furthermore, it is easy to see that for any $a \in \mathbb{F}_p^*$ and $\lambda \in N_{p^2}$ there are $a, b \in \mathbb{F}_p^2$ and $A \in \text{SL}(2, p)$ such that

$$(2.3) \quad aA^x b = \text{Tr}(a \lambda^x), \quad x = 1, 2, \ldots.$$ 

Indeed, if $f(X) = X^2 - uX + 1 \in \mathbb{F}_p[X]$ is the minimal polynomial of $\lambda$, that is, $u = \text{Tr}(\lambda) = \lambda + \lambda^{-1}$, then, for

$$A = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix},$$

we see that $f$ is the characteristic polynomial of $A$. Thus, if we define

$$a = (\text{Tr}(a), \text{Tr}(a \lambda)), \quad b = (1, 0)^t,$$

both sequences

$$aA^x b \quad \text{and} \quad \text{Tr}(a \lambda^x), \quad x = 0, 1, \ldots,$$

satisfy the same binary linear recurrence, and one also verifies that they have the same initial values

$$a I_2 b = \text{Tr}(a) \quad \text{and} \quad aA b = \text{Tr}(a \lambda),$$

where $I_2 = A^0$ is the $2 \times 2$ identity matrix, which implies (2.3). Hence, we see that Theorem 2.8 allows us to estimate the Gauss sums over a subgroup:

$$G_q(\mathcal{G}; a) = \sum_{u \in \mathcal{G}} \psi(au).$$

**Corollary 2.10.** Let $p$ be prime, $a \in \mathbb{F}_p^*$ and let $\mathcal{G}$ be a multiplicative subgroup of $N_{p^2}$ of order $\tau$. Then

$$G_{p^2}(\mathcal{G}; a) \ll \min \left\{ \tau^{1/2} p^{1/4}, \tau^{13/20} p^{1/6}, \tau^{34/45} p^{1/9} \right\}.$$
Remark 2.11. Clearly Theorem 2.7 is nontrivial whenever
\[ \tau > p^{3/7+\epsilon} \]
for any fixed \( \epsilon > 0 \), while Theorem 2.8 is nontrivial for
\[ \tau > p^{5/11+\epsilon}, \]
and similarly for Corollaries 2.9 and 2.10. On the other hand, Bourgain [3] gives a nontrivial bound of the form \( S_{2,n}^2(p, A) \ll \tau^{-\delta} \) provided \( \tau > p^\delta \) with some \( \delta > 0 \) depending only on \( \epsilon \), however this dependence is not explicit and it is not obvious how to get such an explicit result. For applications, see Section 3.2, it is essential to have nontrivial bounds with a power saving for any \( \tau > p^{1/2} \).

Remark 2.12. Corollary 2.9 is the first known bound of this type which is nontrivial below the square-root threshold. We note that for Gauss sums
\[ G_p(g; a) = \frac{1}{s} \sum_{x \in \mathbb{F}_p^s} e_p(ax^s) \]
where \( s = (p-1)/\tau \), the first bound of this type has been obtained in [37] and then improved and extended in various directions, see [3, 5–8, 12, 17, 20, 21, 29, 38, 40, 49] and references therein. However the explicit bound of Corollary 2.10 is new.

3. Applications

3.1. Additive properties of matrix orbits. In the special case of \( A \in \text{SL}(n, q) \) with an irreducible characteristic polynomial, the bound in Theorem 2.5 is nontrivial for
\[ \tau \geq q^{n/2-n/(2n+2)+\epsilon} \]
for any fixed \( \epsilon > 0 \) and applies to all nonzero vectors \( a, b \in \mathbb{F}_q^n \). This allows us some applications to additive properties of orbits of cyclic matrix groups
\[ a\langle A \rangle = \{aA^x : x = 1, \ldots, \tau\}. \]

Our next result is motivated by results of Schoen and Shkredov [34, Corollary 49] and Shkredov and Vyugin [42, Corollary 5.6] on additive properties of small multiplicative subgroups in finite fields.

Similarly to [34, 42] studying small subgroups, we are interested in results for matrices of small order. Furthermore, for a set \( S \subseteq \mathbb{F}_q^n \) and an integer \( k \geq 1 \), we denote
\[ kS = \{s_1 + \ldots + s_k : s_1, \ldots, s_k \in S\}. \]
To exhibit the main ideas we only consider that case of matrices $A \in \text{SL}(n,q)$ with an irreducible characteristic polynomial.

**Theorem 3.1.** Let $\varepsilon > 0$ be fixed and let $q$ be sufficiently large. Assume that the characteristic polynomial of $A \in \text{SL}(n,q)$ is irreducible and the multiplicative order $\tau$ of $A$ satisfies (3.1) for some fixed $\varepsilon > 0$. Then, for any nonzero vector $a \in \mathbb{F}_q^n$ and integer $k > \max \left\{ \frac{2n}{n+1} \varepsilon^{-1}, 3 \right\}$, we have

$$k (a \langle A \rangle) = \mathbb{F}_q^n.$$

For $n = 2$ and prime $q = p$, we have a stronger result. We recall that, by Remark 2.11, Theorem 2.8 is nontrivial for $\tau > p^{5/11+\varepsilon}$.

**Theorem 3.2.** Let $\varepsilon > 0$ be fixed and let $p$ be a sufficiently large prime. Assume that the characteristic polynomial of $A \in \text{SL}(2,p)$ is irreducible and the multiplicative order $\tau$ of $A$ satisfies

$$\tau \geq p^{5/11+\varepsilon}$$

for some fixed $\varepsilon > 0$. Then, for any nonzero vector $a \in \mathbb{F}_p^2$ and integer $k > \max \left\{ \frac{45}{11} \varepsilon^{-1} - 3, 6 \right\}$, we have

$$k (a \langle A \rangle) = \mathbb{F}_p^2.$$

### 3.2. Quantum ergodicity of linear maps on a torus

The goal of [26] has been to show that for almost all $N$, all the eigenfunctions of the “quantum cat map” become uniformly distributed in a suitable sense. Bourgain [4] has given a quantitative improvement of [26] with some unspecified power saving in the bounds on the non-uniformity of distribution.

We now present our improvement of the results of [4, 26] and make the bound of Bourgain [4, Theorem 3] explicit. However, before doing this we have to introduce some notation, we refer to [4, 26, 27] for detailed description.

First, given an integer $N \geq 1$, we introduce the Hilbert space $L^2(\mathbb{Z}_N)$ of functions $\varphi : \mathbb{Z}_N \rightarrow \mathbb{C}$, acting on the residue ring $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ modulo $N$, and equipped with the scalar product

$$\langle \varphi, \psi \rangle = \frac{1}{N} \sum_{u \in \mathbb{Z}_N} \varphi(u) \overline{\psi(u)}, \quad \varphi, \psi \in L^2(\mathbb{Z}_N).$$
Next, we define the family of operators $T_N(a)$ on $L^2(\mathbb{Z}_N)$ with $a = (a_1, a_2) \in \mathbb{Z}^2$, which act as follows: For a function $\psi \in L^2(\mathbb{Z}_N)$, we have

$$(T_N(a)\psi)(u) = \exp(\pi i a_1 a_2/N) \exp(2\pi i a_2 u/N) \psi(u + a_1).$$

Given a Fourier expansion

$$f(z) = \sum_{a \in \mathbb{Z}^2} \hat{f}(a) \exp(2\pi az)$$

of an infinitely differentiable function $f \in C^\infty(T_2)$ defined on a two-dimensional unit torus $T_2 = (\mathbb{R}/\mathbb{Z})^2$, we define the quantised operator

$$\text{Op}_N(f) = \sum_{a \in \mathbb{Z}^2} \hat{f}(a) T_N(a).$$

Furthermore, given a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

with distinct real eigenvalues (that is, with the trace satisfying $|\text{tr}(A)| > 2$) and such that

$$a_{11}a_{12} \equiv a_{21}a_{22} \equiv 0 \pmod{2},$$

once can associate with $A$ a unitary operator $U_N(A)$ called quantised cat map which satisfies

$$U_N(A)^* \text{Op}_N(f) U_N(A) = \text{Op}_N(f),$$

where $U_N(A)^* = \overline{U_N(A)}^t$ is the Hermitian transpose operator.

Finally, for a function $f \in C^\infty(T_2)$ we set

$$\Delta(N, f) = \sup_{\psi \in \Psi(A)} \left| \langle \text{Op}_N(f)\psi, \psi \rangle - \int_{T_2} f(v) dv \right|,$$

where $\Psi(A)$ is the set of all $L^2$-normalised $\langle \psi, \psi \rangle = 1$ eigenfunctions $\psi$ of $U_N(A)$.

As usual, we say that some property holds for almost all integers $N \geq 1$ if when $X \to \infty$, the number of $N \in [1, X]$ for which it fails is $o(X)$.

Kurlberg and Rudnick [26, Theorem 1] have proved the uniformity of distribution property of quantised operators and established that for all functions $f \in C^\infty(T_2)$ we have $\Delta(N, f) \to 0$ as $N \to \infty$ over a certain sequence consisting of almost all integers. Bourgain [4, Theorem 3] gives a quantitative improvement of this result and shows that there is an absolute constant $\delta > 0$ such that for almost all integers $N \geq 1$, for all functions $f \in C^\infty(T_2)$ we have

$$\Delta(N, f) \leq N^{-\delta}.$$
We now obtain an explicit form of (3.2).

**Theorem 3.3.** For almost all integers \( N \geq 1 \), for all functions \( f \in C^\infty(T_2) \) we have

\[
\Delta(N, f) \leq N^{-1/60+o(1)}.
\]

4. Preliminaries

4.1. **Multiplicative orders of matrices and eigenvalues.** The following result is perhaps well-known. We supply a short proof for the sake of completeness.

**Lemma 4.1.** Let \( A \in \text{GL}(n, q) \) be diagonalisable, and let \( \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q^* \) be the eigenvalues of \( A \in \text{GL}(n, q) \) (not necessarily distinct). Then

\[
\tau = \text{lcm}[\text{ord} \lambda_1, \ldots, \text{ord} \lambda_n],
\]

where \( \tau \) is the multiplicative order of \( A \).

**Proof.** Since \( A \in \text{GL}(n, q) \) is diagonalisable, there exist \( V \in \text{GL}(n, q) \) such that

\[
A = VDV^{-1}, \quad D = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]

Since \( \tau \) is the multiplicative order of \( A \), we have

\[
D^\tau = V^{-1}A^\tau V = I,
\]

which implies that \( \lambda_i^\tau = 1 \), \( i = 1, \ldots, n \).

Moreover, if \( D^r = I \) for some \( r < \tau \), then from the diagonalisation of \( A \) we obtain \( A^r = I \), which contradicts the definition of \( \tau \). \( \square \)

4.2. **Equations with products of exponential functions.** Here we estimate the number of solutions to the equation

\[
(4.1) \quad \prod_{j=1}^n (\xi_j - \lambda_j^x) = \xi_0, \quad x = 1, \ldots, \tau,
\]

with some \( \xi_0, \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q^* \) and \( \xi_1, \ldots, \xi_n \in \mathbb{F}_q \), where

\[
\tau = \text{lcm}[\text{ord} \lambda_1, \ldots, \text{ord} \lambda_n].
\]

Clearly, one can now directly use various bounds on the number of solutions to various equations and congruences with linear recurrence sequences, see, for example, [1, Lemma 6], [9, Lemma 7], [10, Lemma 9], [11, Proposition A.1], [15, Lemma 6], [35, Lemma 2], [36, Theorem 1] (some, but not all, of these works are also summarised in [14, Section 5.4]).

However, we obtain a better bound via an application of the method of [9], adjusted to the special shape of the equation (4.1). In particular, our next result is a multiplicative analogues of [9, Lemma 7].
Lemma 4.2. For $n \geq 1$, given $\xi_1, \ldots, \xi_n \in \mathbb{F}_q$ not all zeros and $\xi_0, \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q^*$, we denote by $N$ the number of solutions to the equation (4.1). Suppose that $L = \text{ord} \lambda_1 \geq \ldots \geq \text{ord} \lambda_n \geq 1$ and $\xi_1 \not= 0$.

Then

$$N \ll \tau L^{-1/n},$$

where

$$\tau = \text{lcm}[\text{ord} \lambda_1, \ldots, \text{ord} \lambda_n].$$

Proof. Extending $\mathbb{F}_q$, without loss of generality we can assume that $\xi_0, \xi_1, \ldots, \xi_n, \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q$.

Let $\vartheta$ be a primitive root of $\mathbb{F}_q$. Putting $\lambda_i = \vartheta^{r_i}$, we see that

$$(4.2) \quad N = \frac{\tau}{q - 1} R,$$

where $R$ is the number of solutions of the equation

$$\prod_{i=1}^{n} (\xi_i - \vartheta^{r_i y}) = \xi_0, \quad 0 \leq y \leq q - 2.$$

Let

$$D = \min_{1 \leq j \leq n} \gcd(r_j, q - 1) = \gcd(r_1, q - 1) \quad \text{and} \quad M = \left\lfloor (q - 1)L^{-1/n} \right\rfloor.$$ 

In particular,

$$L = (q - 1)/D.$$

As in the proof of [9, Lemma 7] (however with $n$ instead of $n - 1$) we see that by the pigeonhole principle there exists $\ell$ with $1 \leq \ell \leq L - 1$ such that the remainders $s_i = r_i \ell \mod q - 1$, taken in the interval $-(q - 1)/2 \leq s_i \leq q/2$, satisfy the inequality

$$|s_i| \leq M, \quad i = 1, \ldots, n.$$

Let $d = \gcd(\ell, q - 1)$. Clearly for any $y$, $0 \leq y \leq q - 2$, there is a unique representation of the form

$$y = dw + \nu, \quad 0 \leq w \leq (q - 1)/d - 1, \quad 0 \leq \nu \leq d - 1.$$

We now define $z$ by the congruence

$$(\ell/d)z \equiv w \mod (q - 1)/d, \quad 0 \leq z \leq (q - 1)/d - 1.$$

Therefore we have a unique representation of the form

$$y \equiv \ell z + \nu \mod q - 1, \quad 0 \leq z \leq (q - 1)/d - 1, \quad 0 \leq \nu \leq d - 1.$$
Then

\begin{equation}
R \leq \sum_{\nu=0}^{d-1} R_{\nu},
\end{equation}

where $R_{\nu}$, $\nu = 0, \ldots, d-1$, is the number of solutions of the equation

\[ \prod_{i=1}^{n} \left( \xi_i - \vartheta^{r_i(z+\nu)} \right) = \xi_0, \quad 0 \leq z \leq (q-1)/d-1. \]

It is obvious that

\begin{equation}
R_{\nu} = \frac{1}{d} Q_{\nu}, \quad \nu = 0, \ldots, d-1,
\end{equation}

where $Q_{\nu}$ is the number of solutions of the exponential equation

\[ \prod_{i=1}^{n} \left( \xi_i - \vartheta^{r_i \vartheta^{s_i z}} \right) = \xi_0, \quad 0 \leq z \leq q-2, \]

which does not exceed the number of zeros of the rational function

\[ F(U) = \prod_{i=1}^{n} \left( \xi_i - \vartheta^{r_i U^{s_i}} \right) - \xi_0 \in \mathbb{F}_q(U). \]

Since $L = \text{ord} \lambda_1$, using the inequality $dD \leq (L-1)D < q-1$, one easily verifies that

\[ s_1 \equiv r_1 \ell \not\equiv 0 \pmod{q-1} \]

and thus

\[ s_1 \neq 0. \]

Let $\zeta$ be any root of the equation $\xi_1 - \vartheta^{r_1 \vartheta} U^{s_1} = 0$. Since by our assumption $\xi_1 \neq 0$ we see that $\zeta \neq 0$. Hence, $\zeta$ is not a pole of $F(U)$ and thus $F(\zeta) = -\xi_0 \neq 0$ has a solution and thus the rational function $F(U)$ is not identical to zero.

Therefore, $Q_{\nu}$ does not exceed the number of zeros of a non-zero polynomial of degree at most

\[ Q_{\nu} \leq nM = O \left( qL^{-1/n} \right). \]

Substituting this in (4.4) and recalling (4.3) we obtain

\[ R = O \left( qL^{-1/n} \right), \]

which after substitution in (4.2) implies the result. \qed
4.3. **Rational points on absolutely irreducible curves.** It is well-known that by the Weil bound we have

\[(4.5) \quad \{(x, y) \in \mathbb{F}_q^2 : F(x, y) = 0\} = q + O\left(d^2q^{1/2}\right)\]

for any absolutely irreducible polynomial \(F(X, Y) \in \mathbb{F}_q[X, Y]\) of degree \(d\) (see, for example, [28, Section X.5, Equation (5.2)]). One can see that (4.5) is a genuine asymptotic formula only for \(d \leq O(q^{1/4})\) and is in fact weaker than the trivial bound

\[\{(x, y) \in \mathbb{F}_q^2 : F(x, y) = 0\} = O(dq)\]

for \(d \geq q^{1/2}\), which is exactly the range of our interest. To obtain nontrivial bounds for such large values of \(d\) we use some idea and results from [48].

We start with establishing absolutely irreducibility of polynomials relevant to our applications.

**Lemma 4.3.** For a positive integer \(s\) with \(\gcd(s, q) = 1\) and \(a, b \in \mathbb{F}_q\) with \(ab(ab-1) \neq 0\), the polynomial

\[F(X, Y) = (X^s + Y^s + a)(X^s + Y^s + bX^sY^s) - X^sY^s \in \mathbb{F}_q[X, Y]\]

is absolutely irreducible.

**Proof.** Let us begin by considering the case \(s = 1\), so

\[F(X, Y) = (X + Y + a)(X + Y + bXY) - XY.\]

Since \(b \neq 0\), this polynomial is a cubic and its homogeneous term of degree 3 is \(bXY(X + Y)\). The curve in the projective plane defined by the homogeneisation of \(F\) has the points

\[(0 : 1 : 0), \ (1 : 0 : 0), \ (1 : -1 : 0)\]

at infinity. If the curve is not absolutely irreducible, it has a factor of degree one that passes through one of these points. That means that \(F\) has to have a factor of the form \(X - c, Y - c\) or \(X + Y - c\) for some constant \(c\). We look at each in turn.

First,

\[F(c, Y) = (bc + 1)Y^2 + (2c + bc^2 + a + abc - c)Y + c(a + c)\]

and, for this to be identically zero, we must have \(c = -1/b \neq 0\) and \(c(a + c) = 0\), so \(c = -a\). Hence \(-a = -1/b\), so \(ab = 1\), contradicting the hypothesis. So \(X - c\) is not a factor of \(F\) and, by symmetry, neither is \(Y - c\).

Now,

\[F(X, c - X) = c(a + c) + (b(a + c) - 1)X(c - X)\]
and, for this to be identically zero, we must have \( c(a + c) = 0 \). If \( c = 0 \), we get \( ab = 1 \), looking at the coefficient of \( X^2 \), which is a contradiction. Otherwise, \( c = -a \) and the coefficient of \( X^2 \) in \( F(X, c - X) \) is 1 so the polynomial is not identically zero and \( X + Y - c \) is not a factor of \( F \).

We have shown that, for \( s = 1 \), the polynomial \( F \) is absolutely irreducible. We consider the algebraic curve \( C \) which is a non-singular projective model of \( F = 0 \) (still with \( s = 1 \)). Recall that we assume that \( a \neq 0 \). The point \( P = (0, -a) \) is a simple point on the curve \( F = 0 \) with
\[
\frac{\partial F}{\partial X}(0, -a) = 0 \quad \text{and} \quad \frac{\partial F}{\partial Y}(0, -a) = -a \neq 0.
\]
So \( P \) corresponds to a point on \( C \). We consider the functions \( x, y \) on \( C \) that satisfy the equation \( F(x, y) = 0 \). The function \( x \) has a simple zero at \( P \), hence is not a power of another function on \( C \). It follows from [45, Proposition 3.7.3], that the equation \( Z^s = x \) is irreducible over the function field of \( C \) and defines a cover \( D \) of \( C \). Now, consider any point \( Q \) on \( D \) above the point \( (-a, 0) \) on \( C \). Since \( x \) is not zero at \( (-a, 0) \) (since \( a \neq 0 \)), the curve \( D \) is locally isomorphic to \( C \) near \( Q \) and we conclude, as above, that the function \( y \) on \( D \) has a simple zero at \( Q \) and, in particular, is not a power of another function on \( D \). Again, we conclude that the equation \( W^s = y \) is irreducible over the function field of \( D \) and defines a cover \( E \) of \( D \). In other words, \( F(Z^s, W^s) = 0 \) is an absolutely irreducible equation defining the curve \( E \), which concludes the proof.

We first recall the following result [48, Theorem (i)] on the number of points on curves over \( \mathbb{F}_p \).

**Lemma 4.4.** Let \( p \) be prime and let \( F(X, Y) \in \mathbb{F}_p[X, Y] \) be an absolutely irreducible polynomial of degree \( d \) with \( d < p \). Then
\[
\#\{(x, y) \in \mathbb{F}_p^2 : F(x, y) = 0\} \leq 4d^{1/3}p^{2/3}.
\]

We note that in Lemma 4.4 we dropped the condition \( d > p^{1/4} \) of [48, Theorem (i)] since otherwise the Weil bound (4.5) is stronger.

Unfortunately, Lemma 4.4 applies only to prime fields which is restrictive to our applications. However, in the special case of polynomials of our interest, we can obtain a version of Lemma 4.4, which is suitable for such applications.

**Lemma 4.5.** Let \( p \) be prime and let \( s = k(p - 1) \), where \( k \) is a positive integer with \( \gcd(k, p) = 1 \). Then for \( a, b \in \mathbb{F}_{p^2} \) with \( ab(ab - 1) \neq 0 \), for the polynomial
\[
F(X, Y) = (X^s + Y^s + a)(X^s + Y^s + bX^sY^s) - X^sY^s \in \mathbb{F}_{p^2}[X, Y]
\]
we have

$$\#\{(x, y) \in \mathbb{F}_p^2 : F(x, y) = 0\} \ll s^{6/5} p^{8/5} + p^3.$$  

Proof. Clearly we can assume that

$$k < 3^{-5/4} p$$

as otherwise the result is trivial.

From Lemma 4.3, we know that the equation $F(X, Y) = 0$ defines an absolutely irreducible curve, of degree $d = 3s$, denoted by $E$ there. Let $\alpha \in \mathbb{F}_p^*$ satisfy $\alpha^8 = -a$. The point $P = (0, \alpha)$ defines a point on $E$ and the line $Y = \alpha$ meets $E$ at $P$ with multiplicity $2s$, since $F(X, \alpha) = X^{2s}(1 - ab)$. We denote by $x, y$ the functions on $E$ satisfying $F(x, y) = 0$.

We want to bound the number $R$ of solutions of $F = 0$ in $\mathbb{F}_p^2$. We follow the proof of Lemma 4.4 given in [48, Theorem (i)]. It proceeds by considering, for some integer $m$, the embedding of $E$ in $\mathbb{P}^n$, $n = (m + 2)(m + 1)/2$, given by the monomials in $X, Y$ of degree at most $m$. If this embedding is Frobenius classical in the sense of [46], then by [46, Theorem 2.13]

$$R \leq (n - 1)d(d - 3)/2 + md(p^2 + n)/n.$$  

If

$$m < \min\{p/2, d - 1\} \quad \text{and} \quad p \nmid \prod_{i=1}^{m} \prod_{j=-m}^{m-i} (2si + j)$$

then we claim that the above embedding is classical. Indeed, the order sequence of the embedding at the point $P$ defined above consists of the integers $2ni + j, i, j \geq 0, i + j \leq m$ as follows by considering the order of vanishing at $P$ of the functions $x^j(y - \alpha)^i, i, j \geq 0, i + j \leq m$. The claim now follows from [46, Corollary 1.7].

If the embedding is Frobenius classical, we get the inequality (4.7) as mentioned above. If the embedding is classical but Frobenius non-classical then, by [46, Corollary 2.16], every rational point of $E$ is a Weierstrass point for the embedding and, as the embedding is classical, the number of Weierstrass points satisfies

$$R \leq n(n + 1)d(d - 3)/2 + md(n + 1).$$

We now choose

$$m = \min\{\lceil (p/k)^{1/5} \rceil, 2k - 1\}.$$  

If $|i|, |j| \leq m$, then for the choice of $m$ as in (4.10) we have

$$0 < | - 2ki + j| \leq 3km \leq 3k^{4/5}p^{1/5} < p.$$
provided that (4.6) holds.

Note also that
\[ 2s_i + j \equiv -2ki + j \pmod{p}, \]
as \( s = k(p-1) \). Hence,
\[ \prod_{i=1}^{m} \prod_{j=-m}^{m-i} (2s_i + j) \equiv \prod_{i=1}^{m} \prod_{j=-m}^{m-i} (-2ki + j) \pmod{p}. \]

Thus from the definition of \( m \) in (4.10) and the inequalities (4.11) we see that the conditions (4.8) are satisfied. We note that (4.7) and (4.9) can be simplified and combined as
\[ R \ll \max\{m^2s^2 + sp^2/m, m^4s^2\} \ll sp^2/m + m^4s^2. \]

Since \( m \ll (p/k)^{1/5} \ll (p^2/s)^{1/5} \), we have \( m^4s^2 \ll sp^2/m \) and thus we obtain
\[ R \ll sp^2/m. \]

Recalling the choice of \( m \) in (4.10), we obtain the desired result. \( \Box \)

We note that for \( k \gg p^{1/6} \) the bound of Lemma 4.5 is \( O((s^{6/5}p^{8/5}) \).

5. Proofs of bounds on the number of solutions to matrix equations

5.1. Proof of Theorem 2.1. Assume that \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( A \) in the algebraic closure \( \overline{F}_q \) of \( F_q \) (which are not necessary distinct). Multiplying the equation
\[ A^{x_1} + A^{x_2} = A^{x_3} + A^{x_4}, \quad 1 \leq x_1, x_2, x_3, x_4 \leq \tau, \]
by an eigenvector \( \mathbf{v}_i \) for each eigenvalue \( \lambda_i, \ i = 1, \ldots, n \), we see that \( E_{n,q}(A) \) is the number of solutions to the system of equations
\[ \lambda_i^{x_1} + \lambda_i^{x_2} = \lambda_i^{x_3} + \lambda_i^{x_4}, \quad i = 1, \ldots, n, \]
\[ 1 \leq x_1, x_2, x_3, x_4 \leq \tau. \]

Now, suppose that
\[ L = \text{ord} \lambda_1 \geq \ldots \geq \text{ord} \lambda_n \geq 1. \]

Hence, the contribution \( T_0 \) to the number of solutions to (5.1) from quadruples \( (x_1, x_2, x_3, x_4) \) with
\[ \lambda_i^{x_1} + \lambda_i^{x_2} = \lambda_i^{x_3} + \lambda_i^{x_4} = 0 \]
satisfies
\[ T_0 \ll \tau^3/L. \]

Indeed, fixing \( x_3 \) we see that \( x_4 \) is uniquely defined modulo \( \text{ord} \lambda_1 = L \).

Now, when \( x_3 \) and \( x_4 \) are fixed (in one of \( O(\tau^2/L) \) possible ways), we see from Lemma 4.1 that for each \( x_2 \) there is a unique value of \( x_1 \) satisfying (5.1) and thus we obtain (5.2).
Therefore, fixing $x_3$ and $x_4$, we now see that it is enough to estimate the number of solutions to at most $\tau^2$ systems of equations of the form
\begin{equation}
\lambda_j^{x_1} + \lambda_j^{x_2} = \xi_j, \quad j = 1, \ldots, n, \quad 1 \leq x_1, x_2 \leq \tau,
\end{equation}
with some $\xi_1, \ldots, \xi_n \in \mathbb{F}_q$, where $\xi_1 \neq 0$.

We now derive from (5.3) that
\begin{equation}
\prod_{j=1}^n (\xi_j - \lambda_j^{x_1}) = \prod_{j=1}^n \lambda_j^{x_2} = (\det A)^{x_2}.
\end{equation}

If the multiplicative order of $\det A$ is $t$, the equation (5.4) is reduced to $t$ equations of the form
\begin{equation}
\prod_{j=1}^n (\xi_j - \lambda_j^{x_1}) = \prod_{j=1}^n \lambda_j^{x_2} = \xi_0
\end{equation}
for $t$ distinct values $\xi_0 \in \mathbb{F}_q^*$. Recalling the bound (5.2) and Lemma 4.2, we obtain
\begin{equation}
E_{n,q}(A) \ll T_0 + t\tau^3L^{-1/n} \ll t\tau^3L^{-1/n}.
\end{equation}

It remains to obtain a lower bound on $L$.

First, we observe that
\begin{equation}
\text{lcm}[\text{ord} \lambda_1, \ldots, \text{ord} \lambda_n] \leq \prod_{i=1}^n \text{ord} \lambda_i \leq L^n,
\end{equation}
and thus by Lemma 4.1 we have
\begin{equation}
L \geq \tau^{1/n}.
\end{equation}

Second, we have
\begin{equation}
\left( \prod_{i=1}^n \lambda_i \right)^t = \det A^t = 1.
\end{equation}

Denoting $\tau_0 = \text{lcm}[\text{ord} \lambda_1, \ldots, \text{ord} \lambda_{n-1}]$, we see that
\begin{equation}
\lambda_n^{\tau_0} = \left( \prod_{i=1}^{n-1} \lambda_i^{\tau_0} \right)^{-t} = 1.
\end{equation}

Hence
\[ \text{ord} \lambda_n \mid t\tau_0, \]
which implies $\tau \mid t\tau_0$. Since obviously $\tau_0 \leq L^{n-1}$, we now obtain
\begin{equation}
L \geq (\tau/t)^{1/(n-1)}.
\end{equation}
Substituting the bound (5.7) and (5.8) in (5.5), we conclude the first bound of Theorem 2.1.

If the characteristic polynomial of \( A \) is irreducible over \( \mathbb{F}_q \), then all eigenvalues \( \lambda_i, i = 1, \ldots, n \), have same multiplicative order, which is equal to \( \tau \), the order of the matrix \( A \). Thus, in the above we have \( L = \tau \) and substituting this in (5.5) we conclude the proof.

5.2. Proof of Theorem 2.2. Let \( \lambda, \lambda^{-1} \in \mathbb{F}_p^* \) be the eigenvalues of \( A \). In particular, \( \tau \) is the multiplicative order of \( \lambda \) in \( \mathbb{F}_p^* \).

Arguing as in the proof of Theorem 2.1 we see that \( F_{2,p}(A) \) is the number of solutions to the system of equations
\[
\begin{align*}
\lambda^{x_1} + \lambda^{x_2} + \lambda^{x_3} &= \lambda^{x_4} + \lambda^{x_5} + \lambda^{x_6} \\
\lambda^{-x_1} + \lambda^{-x_2} + \lambda^{-x_3} &= \lambda^{-x_4} + \lambda^{-x_5} + \lambda^{-x_6} \\
1 &\leq x_1, \ldots, x_6 \leq \tau.
\end{align*}
\]

We fix \( x_3, x_4, x_5 \) and denote
\[
a = \lambda^{x_3} - \lambda^{x_1} - \lambda^{x_5} \quad \text{and} \quad b = \lambda^{-x_3} - \lambda^{-x_4} - \lambda^{-x_5}.
\]

In this notation we can rewrite (5.9) as
\[
\begin{align*}
\lambda^{x_1} + \lambda^{x_2} + a &= \lambda^{x_6} \\
\lambda^{-x_1} + \lambda^{-x_2} + b &= \lambda^{-x_6} \\
1 &\leq x_1, x_2, x_6 \leq \tau.
\end{align*}
\]

Multiplying the equations in (5.10) we obtain
\[
(\lambda^{x_1} + \lambda^{x_2} + a) (\lambda^{-x_1} + \lambda^{-x_2} + b) = 1, \quad 1 \leq x_1, x_2 \leq \tau.
\]

We first consider the case when \( ab(ab-1) = 0 \). Obviously there are at most \( \tau^2 \) such choices of \( (x_3, x_4, x_5) \), for each of them the equation (5.11) has at most \( \tau \) solutions in \( (x_1, x_2) \) after which \( x_6 \) is uniquely defined. Hence the total contribution from such solutions is \( O(\tau^3) \) which is admissible.

Therefore, from now on we investigate the number \( T_{a,b} \) of solutions to (5.11) for at most \( \tau^3 \) choices of \( (x_3, x_4, x_5) \), with
\[
ab(ab-1) \neq 0.
\]

Let \( s = (p-1)/\tau \). Clearly \( T_{a,b} = s^{-2} R_{a,b} \), where \( R_{a,b} \) is the number of solutions to the equation
\[
(x^s + y^s + a) (x^{-s} + y^{-s} + b) = 1, \quad x, y \in \mathbb{F}_p^*.
\]
or, equivalently, to the polynomial equation
\[
(x^s + y^s + a) (x^s + y^s + bx^s y^s) - x^s y^s = 0, \quad x, y \in \mathbb{F}_p^*.
\]
Applying Lemmas 4.3 and 4.4, we obtain
\begin{equation}
T_{a,b} = s^{-2}R_{a,b} \ll s^{-2/3}p^{2/3} \ll \tau^{2/3}.
\end{equation}
After this $x_6$ is uniquely defined. Hence the total contribution from such solutions is $O(\tau^{2/3})$ which concludes the proof.

5.3. Proof of Theorem 2.3. We proceed as in the proof of Theorem 2.2 however with the eigenvalues $\lambda, \lambda^{-1} \in \mathbb{F}^*_{p^2} \setminus \mathbb{F}_p$. In particular, we arrive to the same equation (5.12), however with the variables $x, y \in \mathbb{F}_{p^2}$ and with $s = (p^2 - 1)/\tau$. Clearly we can assume that $\lambda$ and $\lambda^{-1}$ are distinct as otherwise $\tau \leq 2$ and the bound is trivial. In this case $\lambda^{-1} = \lambda^p$ or $\lambda^{p+1} = 1$. Hence $\tau | p + 1$ and we see that the conditions of Lemma 4.5 are satisfied with $s = k(p - 1)$ and $k = (p + 1)/\tau$. Since for similarly defined quantities $T_{a,b}$ and $R_{a,b}$, applying Lemma 4.5, instead of (5.13) we obtain
\begin{equation}
T_{a,b} = s^{-2}R_{a,b} \ll s^{-4/5}p^{8/5} + p^3s^{-2} \ll \tau^{4/5} + \tau^2p^{-1},
\end{equation}
and the result follows.

6. Proofs of BOUNDS ON EXPONENTIAL Sums WITH MATRICES

6.1. Proof of Theorem 2.4. For any integer $k \geq 1$ we have
\begin{align*}
S_{n,q}(a, b; A)^k &= \sum_{x_1, \ldots, x_k = 1}^\tau \psi(a (A^{x_1} + \ldots + A^{x_k}) b) \\
&= \sum_{\mathbf{u} \in \mathbb{F}_q^n} \nu_k(\mathbf{u}) \psi(\mathbf{ub}),
\end{align*}
where $\nu_k(\mathbf{u})$ is the number of solutions to the equation
\begin{align*}
a (A^{x_1} + \ldots + A^{x_k}) = \mathbf{u}, \quad 1 \leq x_1, \ldots, x_k \leq \tau.
\end{align*}
Clearly $\nu_k(\mathbf{u}) = \nu_k(\mathbf{u}A^u)$ for any integer $u$. As $\mathbf{u}A^u$ runs through all vectors of $\mathbb{F}_q^n$ when $\mathbf{u}$ does. Hence
\begin{align*}
S_{n,q}(a, b; A)^k &= \frac{1}{\tau} \sum_{u = 1}^\tau \sum_{\mathbf{u} \in \mathbb{F}_q^n} \nu_k(\mathbf{u}A^u) \psi(\mathbf{u}A^u\mathbf{b}) \\
&= \frac{1}{\tau} \sum_{\mathbf{u} \in \mathbb{F}_q^n} \nu_k(\mathbf{u}) \sum_{u = 1}^\tau \psi(\mathbf{u}A^u\mathbf{b}) \\
&= \frac{1}{\tau} \sum_{\mathbf{u} \in \mathbb{F}_q^n} \nu_k(\mathbf{u}) S_{n,q}(\mathbf{u}, b; A).
\end{align*}
Writing
\begin{equation}
\nu_k(\mathbf{u}) = \nu_k(\mathbf{u})^{1-1/\ell}(\nu_k(\mathbf{u})^2)^{1/2\ell},
\end{equation}
by the Hölder inequality, for any integer $\ell$ we have

$$|S_{n,q}(a, b; A)|^{2k\ell} \leq \frac{1}{\tau^{2\ell}} \left( \sum_{u \in \mathbb{F}_q^n} \nu_k(u) \right)^{2\ell - 2} \sum_{u \in \mathbb{F}_q^n} |S_{n,q}(u, b; A)|^{2\ell}.$$  

(6.1)

Obviously

$$\sum_{u \in \mathbb{F}_q^n} \nu_k(u) = \tau^k$$  

(6.2)

and

$$\sum_{u \in \mathbb{F}_q^n} \nu_k(u)^2 = J_k,$$  

(6.3)

where $J_k$ is the number of solutions to the equation

$$a (A^{x_1} + \ldots + A^{x_k} - A^{x_{k+1}} - \ldots - A^{x_{2\ell}}) = 0,$$  

$$1 \leq x_1, \ldots, x_{2\ell} \leq \tau.$$  

(6.4)

Furthermore, by the orthogonality of exponential functions we also have

$$\sum_{u \in \mathbb{F}_q^n} |S_{n,q}(u, b; A)|^{2\ell} = q^n K_{\ell},$$  

(6.5)

where $K_{\ell}$ is the number of solutions to the equation

$$(A^{x_1} + \ldots + A^{x_{\ell}} - A^{x_{\ell+1}} - \ldots - A^{x_{2\ell}}) b = 0,$$  

$$1 \leq x_1, \ldots, x_{2\ell} \leq \tau.$$  

(6.6)

Substituting (6.2), (6.3) and (6.5) in (6.1), we derive

$$|S_{n,q}(a, b; A)|^{2k\ell} \leq q^n \tau^{2k\ell - 2k - 2\ell} J_k K_{\ell}.$$  

Now, if we denote $B = A^{x_1} + \ldots + A^{x_k} - A^{x_{k+1}} - \ldots - A^{x_{2k}}$, for a solution $(x_1, \ldots, x_{2k})$ to (6.4), we have $aB = 0$. Multiplying by powers of $A$, we also obtain the equations

$$(aA^i)B = 0, \quad i = 0, \ldots, n - 1,$$

and thus,

$$\begin{pmatrix} a \\ aA \\ \vdots \\ aA^{n-1} \end{pmatrix} B = O_n,$$
where $O_n$ is the $n \times n$ zero matrix. Since the vectors $a_j A^j$, $j = 0, \ldots, n-1$, are linearly independent, we obtain that $B = 0$. Therefore, $J_k$ is the number of solution to the equation
\[ A^{x_1} + \ldots + A^{x_k} - A^{x_{k+1}} - \ldots - A^{x_{2k}} = O_n, \quad 1 \leq x_1, \ldots, x_{2k} \leq \tau, \]
and similarly for $K_\ell$.

We now observe that
\[ J_2 = K_2 = E_n, \]
while we trivially have
\[ J_1 = K_1 = \tau. \]

Using (6.6) with $k = 2$ and $\ell = 1$ we see from (6.7) and (6.8) that
\[ |S_{n,q}(a, b; A)| \leq q^{n/4} \tau^{-1/4} E_n(A)^{1/4}. \]
Similarly, taking $k = \ell = 2$ we see from (6.7) that (6.6) implies
\[ |S_{n,q}(a, b; A)| \leq q^{n/8} E_n(A)^{1/4}. \]

At this stage we can already apply Theorem 2.1 to obtain a nontrivial estimate on $S_{n,q}(a, b; A)$. However this gives our bound with $\kappa_n$ about $(2n)^{-2}$, which we now improve using the argument below. Thus, as we have mentioned, this argument gives $\kappa_n \sim (3n^2)^{-1}$, when $n \to \infty$.

Let $P_A \in \mathbb{F}_q[X]$ be the characteristic polynomial of $A$. We first note that the bound of Theorem 2.4 is trivial if $\tau^{1/4 + \kappa_n} \leq q^{n/8}$. Hence we now assume that
\[ \tau > q^{n/(2(1+4\kappa_n))}. \]

There are some integers $1 \leq d_1 < \ldots < d_r$, such that we can factor $P_A(X) = g_1(X) \ldots g_r(X)$, where $g_i$ is a product of irreducible over $\mathbb{F}_q$ polynomials of the same degree $d_i$, $i = 1, \ldots, r$. Let $\deg g_i = m_i d_i$, $i = 1, \ldots, r$. Thus
\[ \sum_{i=1}^r m_i d_i = n \]
and we also see that $P_A$ has
\[ m = \sum_{i=1}^r m_i \]
irreducible factors. For each of this polynomials we fix one of its roots and denote them $\mu_1, \ldots, \mu_m$. 
Since all roots of an irreducible over \( \mathbb{F}_q \) polynomial have the same multiplicative order, instead of (5.6) in the proof of Theorem 2.1 we obtain
\[
\text{lcm}[\text{ord } \lambda_1, \ldots, \text{ord } \lambda_n] \leq \prod_{j=1}^{m} \text{ord } \mu_j \leq L^m,
\]
and thus (5.7) becomes
\[
(6.14) \quad L \geq \tau^{1/m}.
\]
Hence, it remains to estimate \( m \).

First we remark that
\[
\tau \leq \text{lcm} \left[ q^{d_1} - 1, \ldots, q^{d_r} - 1 \right] \leq q^{d_1 + \ldots + d_r},
\]
and together with (6.11) we conclude
\[
(6.15) \quad d_1 + \ldots + d_r \geq \left\lfloor \frac{n}{2 \left( 1 + 4\kappa_n \right)} \right\rfloor + 1.
\]

We consider two cases.

First, assume that \( d_1 = 1 \). Then, recalling (6.13) and using (6.12), we obtain
\[
(6.16) \quad m = \sum_{i=1}^{r} m_i \leq m_1 + \frac{1}{2} \sum_{i=2}^{r} m_i d_i = n/2 + m_1/2.
\]
On the other hand, from (6.15) we derive
\[
m_1 = n - \sum_{i=2}^{r} m_i d_i \leq n - \sum_{i=2}^{r} d_i = n + 1 - \sum_{i=1}^{r} d_i \leq n - \left\lfloor \frac{n}{2 \left( 1 + 4\kappa_n \right)} \right\rfloor,
\]
which together with (6.16) implies
\[
(6.17) \quad m \leq \left\lfloor n - \frac{1}{2} \left\lfloor \frac{n}{2 \left( 1 + 4\kappa_n \right)} \right\rfloor \right\rfloor.
\]

Second, if \( d_1 \geq 2 \), we obviously have a slightly better bound
\[
m = \sum_{i=1}^{r} m_i \leq \frac{1}{2} \sum_{i=1}^{r} m_i d_i = n/2.
\]

From the definition of \( \kappa_n \) given by (2.1), we see that \( \kappa_n \leq (4n)^{-1} \) and thus
\[
\frac{n}{2} > \frac{n}{2(1 + 4\kappa_n)} = \frac{n}{2} - \frac{2n\kappa_n}{1 + 4\kappa_n} > \frac{n - 1}{2}.
\]
Therefore
\[ \left\lfloor \frac{n}{2(1 + 4\kappa_n)} \right\rfloor = \left\lfloor \frac{n - 1}{2} \right\rfloor. \]
Now, using (2.1) again, we derive the identity
\[ \left\lfloor n - \frac{1}{2} \left( \frac{n}{2(1 + 4\kappa_n)} \right) \right\rfloor = \left\lfloor n - \frac{1}{2} \left( \frac{n - 1}{2} \right) \right\rfloor = \frac{1}{4n\kappa_n}. \]
Hence, the bound (6.17) implies
\[ m \leq \frac{1}{4n\kappa_n}. \]
Therefore we see from (6.14) that
\[ L \geq \tau^{4n\kappa_n}, \]
which after substituting in (5.5) in the proof of Theorem 2.1 implies
\[ E_{n,q}(A) \ll t\tau^{3-4\kappa_n}. \]
and together with (6.9) and (6.10) concludes the proof.

6.2. Proof of Theorem 2.5. It is enough to show that if the characteristic polynomial \( P_A \) of \( A \) is irreducible over \( \mathbb{F}_q \) then the conditions of Theorem 2.4 are satisfied for any nonzero vectors \( a, b \in \mathbb{F}_q^n \). In this case, exactly as in the proof of Theorem 2.4 one concludes from (6.9) and (6.10) and Theorem 2.1 that the desired bounds hold.

To show that the conditions of Theorem 2.4 on \( a, b \in \mathbb{F}_q^n \) are now redundant we note that if \( P_A \) is irreducible over \( \mathbb{F}_q \) then all its roots \( \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q \) are distinct, and thus the matrix \( A \) is diagonalisable.

If the vectors \( aA^i, i = 0, \ldots, n-1 \), are linearly dependent over \( \mathbb{F}_q \) for some nonzero \( a \in \mathbb{F}_q^n \), then there is a linear relation
\[ \sum_{i=0}^{n-1} c_i aA^i = 0 \]
for some \( c_i \in \mathbb{F}_q \) not all zero. Let \( v_\ell \) be an eigenvector corresponding to the eigenvalue \( \lambda_\ell, \ell = 1, \ldots, n \), and choose \( \ell \) such that \( a \) is not orthogonal to \( v_\ell \). This is indeed possible since all the vectors \( v_1, \ldots, v_n \) are linearly independent (because they correspond to distinct eigenvalues) and thus their span is the whole space \( \mathbb{F}_q^n \).

Multiplying now (6.18) on the right with \( v_\ell \) (which is a column vector), we obtain
\[ \sum_{i=0}^{n-1} c_i aA^i v_\ell = \sum_{i=0}^{n-1} c_i a\lambda_\ell^i v_\ell = \sum_{i=0}^{n-1} d_\ell,i \lambda_\ell^i = 0, \]
where $d_{\ell,i} = c_i a \mathbf{v}_\ell \in \mathbb{F}_q$. Since $a \mathbf{v}_\ell \neq 0$ by our choice of $\ell$, and since not all coefficients $c_i$ are zero, we note that $d_{\ell,i}$, $i = 0, \ldots, n - 1$, are not all zero. Thus, the polynomial

$$f(X) = \sum_{i=0}^{n-1} d_{\ell,i} X^i$$

is nonzero. Therefore $\lambda_\ell$ is a root of $f$ which is of degree $n - 1$, and thus impossible by the irreducibility of $P_A$.

Similar argument also applies to the vectors $A_i b_i$, $i = 0, \ldots, n - 1$, which concludes the proof.

### 6.3. Proof of Theorem 2.7

We proceed as in the proof of Theorem 2.4. First, as we have noted in (1.3), for $n = 2$ and $t = 1$, the argument of Kurlberg and Rudnick [26] gives

$$J_2 = K_2 \ll \tau^2.$$ 

Furthermore, we also have,

$$J_3 = K_3 = F_{2,p}(A).$$ 

Invoking Theorem 2.2 and using (6.6) with $(k, \ell) = (2, 3)$, we obtain

$$|S_{2,p}(a, b; A)|^{12} \ll p^2 \tau^{12 - 4 - 6 + 2 + 11/3} = p^2 \tau^{23/3},$$

while the choice $(k, \ell) = (3, 3)$, gives

$$|S_{2,p}(a, b; A)|^{18} \ll p^2 \tau^{18 - 6 - 6 + 22/3} = p^2 \tau^{40/3},$$

and the desired bound follows.

### 6.4. Proof of Theorem 2.8

First, we proceed as in the proof of Theorem 2.4, thus with $(k, \ell) = (2, 2)$, using (1.3), we obtain the bound

$$S_{2,q}(a, b; A) \ll \tau^{1/2} q^{1/4}.$$ 

Note that these bounds hold for any $q$ and also for any characteristic polynomial of $A$. It is also easy to check that for $\tau > p^{5/6}$ we have

$$\tau^{1/2} p^{1/4} < \min \{\tau^{13/20} p^{1/6}, \tau^{34/45} p^{1/9}\}.$$ 

Hence we can now assume that

$$\tau \leq p^{5/6}. \quad (6.19)$$

Next we proceed as in the proof of Theorem 2.7 however in an appropriate place we use Theorem 2.3 instead of Theorem 2.2. We also remark that $\tau^{19/5} > \tau^{5} p^{-1}$ under the assumption (6.19).

Thus, with the choice $(k, \ell) = (2, 3)$, we obtain

$$|S_{2,p}(a, b; A)|^{12} \ll p^2 \tau^{12 - 4 - 6 + 2 + 19/5} = p^2 \tau^{39/5},$$
while the choice \((k, \ell) = (3, 3)\), gives
\[
|S_{2, p}(a, b; A)|^{18} \ll p^{2} \tau^{18 - 6 - 38/5} = p^{2} \tau^{68/5},
\]
and the desired bound follows.

7. Proofs of additive properties of matrix orbits

7.1. Proof of Theorem 3.1. We have to show that for \(k\), which satisfies the inequality of Theorem 3.1 and sufficiently large \(q\), for any \(u \in \mathbb{F}_{q}^n\), the equation
\[
(aA^{x_1} + \ldots + aA^{x_k} = u, 1 \leq x_1, \ldots, x_k \leq \tau,
\]
has a solution.

Using the orthogonality of additive characters, we can express the number \(N_k(u)\) of solutions to (7.1) as
\[
N_k(u) = \sum_{x_1, \ldots, x_k=1}^{\tau} \frac{1}{q^n} \sum_{b \in \mathbb{F}_q^n} \psi((aA^{x_1} + \ldots + aA^{x_k} - u)b)
\]
\[
= \frac{1}{q^n} \sum_{b \in \mathbb{F}_q^n} S_{n,q}(a, b; A)^k \psi(-ub).
\]

We recall that \(aA^x\) and \(u\) are row vectors while \(b\) is column vector, thus the multiplications above are well-defined.

Separating the contribution \(\tau^k/q^n\) from the zero vector \(b = 0\) we obtain
\[
|N_k(u) - \tau^k| \leq R,
\]
where
\[
R = \frac{1}{q^n} \sum_{b \in \mathbb{F}_q^n \setminus \{0\}} |S_{n,q}(a, b; A)|^k.
\]

Clearly we can assume that \(k \geq 4\).

By Theorem 2.5 we now have
\[
S_{n,q}(a, b; A) \ll \tau^{3/4 - 1/4n} q^{n/8},
\]
(we note this bound also holds for \(\tau > q^{n/2}\) as in this case \(\tau^{1/2 - 1/4n} q^{n/4} < \tau^{3/4 - 1/4n} q^{n/8}\)). Hence, we derive
\[
R \leq \frac{1}{q^n} \left(\tau^{3/4 - 1/4n} q^{n/8}\right)^{k-4} \sum_{b \in \mathbb{F}_q^n \setminus \{0\}} |S_{n,q}(a, b; A)|^4.
\]
As in the proof of Theorem 2.4, see (6.5), we have
\[ \frac{1}{q^n} \sum_{b \in F_q^n \setminus \{0\}} |S_{n,q}(a, b; A)|^4 \leq \frac{1}{q^n} \sum_{b \in F_q^n \setminus \{0\}} |S_{n,q}(a, b; A)|^4 = E_{n,q}(A), \]
(however as in the proof of Theorem 2.5 we do not need to impose the linear independence of \(a, A^i, i = 0, \ldots, n-1\)). Therefore, by Theorem 2.1 we obtain
\[ R \ll \left( \tau^{3/4-1/4n}q^{-n/8} \right)^{k-4} \tau^{3-1/n} = \tau^{3k/4-k/4n}q^{kn/8-n/2}. \]
Hence, we see from (7.2) that
\[ (7.4) \quad N_k(u) = \frac{\tau^k}{q^n} \left( 1 + O \left( \tau^{-k/4-k/4n}q^{kn/8+n/2} \right) \right). \]
Recalling (3.1), which we write in an equivalent form \(\tau \geq q^{n^2/(2n+2)+\varepsilon}\), we see that
\[ \tau^{-k/4-k/4n}q^{kn/8+n/2} \leq q^{-(n^2/(2n+2)+\varepsilon)(k/4+k/4n)+kn/8+n/2}. \]
Thus, by (7.4) it is enough to ensure that
\[ 0 > -(n^2/(2n+2)+\varepsilon)(k/4+k/4n)+kn/8+n/2 \]
\[ = -\varepsilon(k/4+k/4n)+n/2. \]
This is equivalent to
\[ k > 2n \frac{\varepsilon}{n+1}, \]
which concludes the proof.

7.2. Proof of Theorem 3.2. We proceed as in the proof of Theorem 3.1 however now we can assume that \(k \geq 7\).
First we consider that case when \(\tau < p^{5/6}\). Then the last bound of Theorem 2.8 simplifies as
\[ S_{2,p}(a, b; A) \ll \tau^{34/45}p^{1/9} + \tau^{8/9} \ll \tau^{34/45}p^{1/9}. \]
Hence, instead of (7.3) we derive
\[ R \leq \frac{1}{p^2} \left( \tau^{34/45}p^{1/9} \right)^{k-6} \sum_{b \in F_p^k \setminus \{0\}} |S_{2,p}(a, b; A)|^6. \]
As in the proof of Theorem 2.4, see (6.5), we have
\[ \frac{1}{p^2} \sum_{b \in F_p^k \setminus \{0\}} |S_{2,p}(a, b; A)|^6 \leq \frac{1}{p^2} \sum_{b \in F_p^k} |S_{2,p}(a, b; A)|^6 = F_{2,p}(A), \]
and by Theorem 2.3 we obtain
\[ R \ll \left( \tau^{34/45}p^{1/9} \right)^{k-6} \left( \tau^{19/5} + \tau^5p^{-1} \right) \ll \left( \tau^{34/45}p^{1/9} \right)^{k-6} \tau^{19/5} \]
since $\tau^{19/5} > \tau^5 p^{-1}$ under our assumption $\tau < p^{5/6}$. Hence, in the notation of the proof of Theorem 3.1, we see from (7.2) that

$$N_k(u) = \frac{\tau^k}{p^2} \left(1 + O \left(\tau^{-11k/45 - 11/15} p^{k/9 + 4/3}\right)\right).$$

Recalling the condition on $\tau$, we see that by (7.5) it is enough to ensure that

$$0 > -(5/11 + \varepsilon)(11k/45 + 11/15) + k/9 + 4/3$$
$$= -11k\varepsilon/45 - 11\varepsilon/15 + 1.$$

This is equivalent to

$$k > \frac{45}{11}\varepsilon^{-1} - 3,$$

which concludes the argument for $\tau < p^{5/6}$.

For $\tau \geq p^{5/6}$ we use the bound

$$S_{2,p}(a, b; A) \ll \tau^{1/4} p^{1/2}$$

of Theorem 2.8 and also the bound (1.3) getting

$$R \ll \left(\tau^{1/4} p^{1/2}\right)^{k-4} \tau^2 \ll \tau^{k/4 + 1} p^{k/2 - 2}.$$ 

Hence instead of (7.5), and recalling that $\tau \geq p^{5/6}$, we now obtain

$$N_k(u) = \frac{\tau^k}{p^2} \left(1 + O \left(\tau^{-3k/4 + 1} p^{k/2}\right)\right) = \frac{\tau^k}{p^2} \left(1 + O \left(p^{-5k/8 + 5/6 + k/2}\right)\right)$$
$$= \frac{\tau^k}{p^2} \left(1 + O \left(p^{-k/8 + 5/6}\right)\right) = \frac{\tau^k}{p^2} \left(1 + O \left(p^{-1/24}\right)\right)$$

for $k \geq 7$, which concludes the proof.

8. Proof of uniformity of distribution of quantised operators

8.1. Preliminary bounds. We start with deriving an explicit form of [4, Proposition 1], which could be of independent interest.

Lemma 8.1. Under the condition of Theorem 3.3, for a prime $N = p$ such that the multiplicative order $\tau$ of $A$ modulo $p$ satisfies $\tau \geq p^{1/2 + o(1)}$, uniformly over $a \in \mathbb{Z}^2$ such that $a$ and $aA$ are linearly independent modulo $p$, we have

$$\sup_{\psi \in \Psi(A)} |\langle T_p(a)\psi, \psi\rangle| \leq p^{-1/60 + o(1)}.$$
Proof. We note that it is easy to see that in the argument of Bourgain \cite[Equations (2.6), (2.9) and (2.11)]{4} one can replace $2\ell$ with an arbitrary integer $\nu \geq 1$, not necessary even. Hence together with \cite[Equation (2.4)]{4}, we obtain that
\begin{equation}
\sup_{\psi \in \Psi(A)} |\langle T_p(\mathbf{a})\psi, \psi \rangle|^{2\nu} \leq \tau^{-2\nu} pQ_{\nu,2,p}(A),
\end{equation}
where $Q_{\nu,n,p}(A)$ is defined by \eqref{1.1}. We now consider two cases.

If $\tau \geq p^{5/6}$ we choose $\nu = 2$ and use the bound \eqref{1.3}, getting from \eqref{8.1} that
\begin{equation}
\sup_{\psi \in \Psi(A)} |\langle T_p(\mathbf{a})\psi, \psi \rangle| \leq (\tau^{-4} p E_{2,p}(A))^{1/4} \ll (\tau^{-2} p)^{1/4} \leq p^{-1/6}.
\end{equation}

For $p^{5/6} > \tau \geq p^{1/2+o(1)}$ we choose $\nu = 3$ and use the bound of Theorem 2.3 (since the bound of Theorem 2.2 is stronger), and so we derive
\begin{equation}
\sup_{\psi \in \Psi(A)} |\langle T_p(\mathbf{a})\psi, \psi \rangle| \leq (\tau^{-6} p F_{2,p}(A))^{1/6} \ll (\tau^{-11/5} p)^{1/6} \leq p^{-1/60+o(1)}.
\end{equation}
Combining the bounds \eqref{8.2} and \eqref{8.3} we conclude the proof. \qed

We remark that in applications, such as the proof of Theorem 3.3 below, the quantity $o(1)$ in the condition $\tau \geq p^{1/2+o(1)}$ of Lemma 8.1, is negative. In fact, using the argument of Erdős and Murty \cite[Theorem 3]{13}, one can avoid the quantity $o(1)$ and use instead a version of Lemma 8.1, which under the condition $\tau \geq p^{1/2}$ gives the bound
\begin{equation*}
\sup_{\psi \in \Psi(A)} |\langle T_p(\mathbf{a})\psi, \psi \rangle| \ll p^{-1/60}.
\end{equation*}

8.2. Proof of Theorem 3.3. The proof follows exactly the same way as the proof of \cite[Theorem 3]{4} however it is using the explicit bound of Lemma 8.1 instead of the bound of \cite[Proposition 1]{4}.

Namely, the starting point is the expansion
\begin{equation*}
\langle Op_N(f)\psi, \psi \rangle = - \int_{T_2} f(\mathbf{v})d\mathbf{v} = \sum_{\mathbf{a} \in \mathbb{Z}^2 \setminus \{0\}} \hat{f}(\mathbf{a}) \langle T_N(\mathbf{a})\psi, \psi \rangle,
\end{equation*}
where $\mathbf{0} = (0,0)$. Thus, taking into account the rapid decay of the coefficients $\hat{f}(\mathbf{a})$ for $f \in C^\infty(T_2)$, we see that it is enough to obtain a bound of the form
\begin{equation}
\sup_{\psi \in \Psi(A)} |\langle T_N(\mathbf{a})\psi, \psi \rangle| \leq \|\mathbf{a}\|^{O(1)} N^{-1/60+o(1)},
\end{equation}
where $\|\mathbf{a}\|$ is the Euclidean norm of $\mathbf{a}$.
Next, we use the inequality
\[
|\langle T_p(a)\psi, \psi \rangle|^{2^\nu} \leq \tau_N^{-2^\nu} NR_{\nu,N}(a, A),
\]
where \(\tau_N\) is the multiplicative order of \(A\) modulo \(N\) and \(R_{\nu,N}(a, A)\) is
the number of solutions to the following matrix congruence
\[
a(A^{x_1} + \ldots + A^{x_{2^\nu}} - A^{x_{2^\nu+1}} - \ldots A^{x_{2^{2^\nu}}}) \equiv 0^t \pmod{N},
\]
\[1 \leq x_1, \ldots, x_{2^\nu} \leq \tau_N,
\]
see [4, Equations (2.4), (2.6), (2.11) and (3.5)].

The reduction from estimating \(R_{\nu,N}(a, A)\) to estimating \(Q_{\nu,2^\nu}(A)\) as
defined by (1.1) is based on the elementary fact that for almost all \(N\),
the largest square divisor of \(N\) is small and on the following two much
more involved facts established in [26]:

(i) for almost all integers \(N \geq 1\) the following two quantities
\[
lcm[\tau_p : p | N] \quad \text{and} \quad \prod_{p|N} \tau_p,
\]

taken over all prime divisors \(p | N\), are of a the same order of
magnitude, see [26, Proposition 11];

(ii) for almost all primes \(p\), the multiplicative order \(\tau_p\) of \(A\) satisfies
the inequality \(\tau_p > p^{1/2 + o(1)}\), see [26, Lemma 15].

Thus, using (i) and the essentially square-freeness of \(N\), allows us to
reduce bounding \(R_{\nu,N}(a, A)\) to bounding of \(Q_{\nu,n,p}(A)\) for \(p | N\). Then
taking \(\nu = 3\), using (ii) and arguing as in the proof of Lemma 8.1, we
obtain (8.4) and the desired result follows.

9. Comments

We note that Theorems 2.2 and 2.3 imply upper bounds on the 6-th
moments of Kloosterman and Gauss sums over small subgroups. More
precisely, we have
\[
\sum_{a,b \in \mathbb{F}_p} |K_p(\mathcal{G}; a, b)|^6 \ll \tau^{11/3}
\]
and
\[
\sum_{a \in \mathbb{F}_{p^2}} |G_{p^2}(\mathcal{G}; a)|^6 \ll \tau^{19/5} + \tau^5 p^{-1}.
\]

Clearly Corollary 2.9 gives bounds for exponential sums with Lau-
rent binomials \(aX^s + bX^{-s}\). It seems plausible than one can obtain
versions of Corollary 2.9 for exponential sums with more general binom-
ials \(aX^s + bX^t\), we refer to [43,44] for the currently known results in
this direction.
We also note that the idea behind the proofs of Theorems 2.2 and 2.3, can in principle be used for matrices of higher dimension. However investigating irreducible factors of the corresponding bivariate polynomials, similar to those in Lemma 4.3, however with more terms, becomes rather difficult.

Finally, we note that Kelmer [19] has studied quantum ergodicity of linear maps on a $2d$-dimensional torus, associated with $2d$-dimensional symplectic matrices, in particular, see [19, Propositions 3.5 and 3.6]. We believe our ideas and results can be used in this setting as well.

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