Equivalence between different classical treatments of the $O(N)$ non-linear sigma model and their functional Schrödinger equations.

A. A. Deriglazov*, W. Oliveira† and G. Oliveira-Neto‡

Departamento de Física, Instituto de Ciencias Exatas, Universidade Federal de Juiz de Fora, CEP 36036-330, Juiz de Fora, Minas Gerais, Brazil.

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Abstract

In this work we derive the Hamiltonian formalism of the $O(N)$ non-linear sigma model in its original version as a second-class constrained field theory and then as a first-class constrained field theory. We treat the model as a second-class constrained field theory by two different methods: the unconstrained and the Dirac second-class formalisms. We show that the Hamiltonians for all these versions of the model are equivalent. Then, for a particular factor-ordering choice, we write the functional Schrödinger equation for each derived Hamiltonian. We show that they are all identical which justifies our factor-ordering choice and opens the way for a future quantization of the model via the functional Schrödinger representation.

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*alexei@fisica.ufjf.br
†wilson@fisica.ufjf.br
‡gilneto@fisica.ufjf.br
I. INTRODUCTION

The Hamiltonian formulation of a classical constrained system with second-class constraints can be obtained, usually, in two different ways [1]. In the first one, the unconstrained formalism, one starts solving the classical constraints and substituting the result in the action. Then, one proceeds with the derivation of the Hamiltonian in the usual way because the theory is written, now, in terms of the physical degrees of freedom only. In the second one, the Dirac second-class, one writes the Hamiltonian formalism of the theory with the Dirac brackets which take in account the constraints explicitly. After that, the constraints can be solved and part of the variables can be omitted from consideration. In general, the remaining physical variables are not canonically conjugated to their momenta through the Dirac bracket. Then, one finds transformations to new variables which are canonically conjugated to their momenta through the Dirac bracket. Finally, one writes the reduced Hamiltonian, obtained after the use of the relations derived from the constraints in the original Hamiltonian.

For some time now, other methods of treating a second-class constrained system have been developed [2–5]. One converts the original theory in a theory with first-class constraints and derives its Hamiltonian. The main motivation for this conversion are the symmetries that the first-class systems possess. Through the symmetries, it is possible to determine many physical properties of the system in a more general way. Therefore, one expects to use those symmetries to study the properties of a second-class system after the conversion to a first-class one.

Although one expects that the different ways to treat a second-class constrained system leads to the same Hamiltonian theory, it is by no means trivial to show explicitly. In the present work we would like to show this equivalence for the $O(N)$ non-linear sigma model, which is a well-know second-class constrained field theory [6,7].

We shall consider the $O(N)$ non-linear sigma model described in a 1 + 1-dimensional Minkowski space-time. Therefore, it cannot directly describe physical phenomena in the
real world. On the other hand, since it was shown to have many properties similar to physically relevant, 3 + 1-dimensional, non-Abelian field theories [6–8], we believe that our result will be easily extended to more physically relevant theories. Over the years, many works have been dedicated to the quantization of the $O(N)$ non-linear sigma model using different techniques [7,9–13]. As we shall see below, some of the works dealing with the canonical quantization of the model have few points of contact with our work.

Besides the purely classical treatment, we shall also write down the functional Schrödinger equation [14,15] for each derived Hamiltonian. As we shall see, for a particular factor-ordering choice, the functional Schrödinger equations are identical. This result, along with the fact that in the study of the first class constrained version of the model the ordering is consistent with the operatorial version of the classical constraint algebra (see Sec. III), justifies our factor-ordering choice and opens the way for a future quantization of the model via the functional Schrödinger representation. It is important to notice that, we shall not demonstrate that our factor-ordering choice is the only one to satisfy the above mentioned properties. It means that, there may be other choices that also satisfy those properties.

The functional Schrödinger representation has recently been systematically used in order to quantize different field theories, including gravity [14–17]. Many theoretical as well as some physical predictions have been derived, for different theories, from the wave-functionals obtained so far. One example of an important theoretical feature of gauge theories established in the context of the functional Schrödinger representation, without any ‘instanton’ approximation, is the so-called vacuum angle [15]. On the other hand, from the wave-functional of the quantum Schwarzschild-de Sitter black hole one is able to predict how it depends on the mass and cosmological constants [17].

In the next section, Sec. II, we shall treat the $O(N)$ non-linear sigma model as a second-class constrained field theory. We shall use the unconstrained and the Dirac second-class formalisms, both described above. As we shall see, the resulting Hamiltonians coming from both formalisms, written in terms of the initial fields, will be the same. Therefore, they are classically equivalent. It also means that, the two formalisms will lead to the same
functional Schrödinger equation, if we apply the same factor-ordering choice for each of them. Here, besides the main result, further novelties in relation to previous works in this area will appear. In the treatment of the model using the unconstrained formalism, we shall not use the standard field transformation in order to eliminate one of the fields [10,11,13]. Rather, we shall express one of the fields in terms of the others through the constraint. Therefore, our Lagrangian present in the action eq. (5) and Hamiltonian eq. (12) will be different from the ones in [10,11,13]. Since, in Ref. [13] they also work in the functional Schrödinger representation, the difference in the Hamiltonians implies that the functional Schrödinger equations will not be the same. Nevertheless, due to the fact that the difference comes from the use of different field basis in order to describe the model we should obtain the same results from our Hamiltonian as the ones found in Ref. [13]. In a recent work on the analogous quantum mechanical problem of a particle moving on a sphere the authors used the unconstrained formalism in order to write the Hamiltonian of that system [18]. Then, if we keep in mind the differences between the two systems our Lagrangian and Hamiltonian will be similar to theirs. In the treatment of the model with the Dirac second-class formalism, we shall explicitly write transformations eq. (28) from the original fields and conjugated momenta to new ones, such that, the Dirac brackets between the new fields and their conjugated momenta have the canonical form. These transformations eq. (28) and consequently the new fields and their conjugated momenta are different from the ones introduced in previous works [9,11].

In Sec. III, we shall treat the $O(N)$ non-linear sigma model as a first-class constrained field theory. The Hamiltonian that we shall manipulate was first derived in [5]. We shall show that this Hamiltonian leads to the same classical theory than the other two Hamiltonians obtained in Sec. II. Then, we shall derive the functional Schrödinger equation using the Dirac first-class quantization technique [19]. There, one writes the operatorial versions of the constraints and forces them to annihilate the wave-functional. Then, this wave-functional that satisfies the operatorial version of the constraints, must be a solution to the functional Schrödinger equation. In the present case, as we shall see, after we use the
information coming from the annihilation of the wave-functional by the quantum constraints, the functional Schrödinger equation reduces to the one obtained in Sec. II.

Finally, in Sec. IV we summarize the main points and results of the paper.

II. THE $O(N)$ NON-LINEAR SIGMA MODEL AS A SECOND-CLASS CONSTRAINED FIELD THEORY.

The $O(N)$ nonlinear sigma model is described by the action,

$$S = \int d^2x \left( \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a \right),$$  \hspace{1cm} (1)

where it is implied the kinematic constraint,

$$T_1 = |\phi|^2 - 1.$$  \hspace{1cm} (2)

Here, $\mu = 0, 1, a$ is an index related to the $O(N)$ symmetry group, the metric has signature $(+, -)$, we are using the convention of sum over repeated index and $|\phi|^2 \equiv \phi^a \phi^a$.

In the present section we shall treat the $O(N)$ non-linear sigma model as a second-class constrained field theory. We shall obtain its Hamiltonian formulation in two different ways [1]. In the first one, Subsec. II A we shall start expressing one of the fields in terms of the others, through the constraint, and substituting the result in the Lagrangian. Next, we shall find the Hamiltonian. The theory will be written, then, in terms of the physical degrees of freedom only. Finally, using the Hamiltonian and a particular factor-ordering choice we shall write the functional Schrödinger equation. As explained above, the Lagrangian present in the action eq. (5) and the Hamiltonian eq. (12) will be different from the ones in [10, 11, 13] and similar to the ones in [18], if we keep in mind the differences between our model and the one in [18]. In the second way, Subsec. II B we shall write the Hamiltonian formalism of the theory with the Dirac brackets which take in account the constraints explicitly, following [9, 11]. As we shall see, the initial variables are not canonically conjugated to their momenta through the Dirac brackets. Here, a novelty with respect to the treatments of Refs. [9, 11]
will appear: we shall explicitly introduce transformations eq. (28) to new variables which are canonically conjugated to their momenta through the Dirac brackets. Then, we shall impose the constraints and write the reduced Hamiltonian, obtained after the use of the relations derived from the constraints in the original Hamiltonian. Finally, we re-write the reduced Hamiltonian in terms of the new variables. We shall call it physical Hamiltonian \( H_{\text{phys}} \). As we shall see this physical Hamiltonian is identical to the one obtained in Subsec. II A. Naturally, if one uses the same factor-ordering choice of Subsec. II A, the functional Schrödinger equation derived from that physical Hamiltonian must be the same as the one computed in Subsec. II A.

### A. Unconstrained formalism.

We start by strongly imposing the constraint \( T_1 \) eq. (2). Then, we write one of the fields, say \( \phi^N \), in terms of the other \( N - 1 \) fields,

\[
\phi^N = \sqrt{1 - \phi^i \phi^i}, \tag{3}
\]

where \( i = 1, 2, ..., N - 1 \). From eq. (3) it is straightforward to compute \( \partial_\mu \phi^N \) as,

\[
\partial_\mu \phi^N = \frac{\phi^i \partial_\mu \phi^i}{\sqrt{1 - \phi^i \phi^i}}. \tag{4}
\]

Introducing both results eqs. (3) and (4) in the action eq. (1), we obtain the theory described in terms of the \( N - 1 \) physical fields.

\[
S_{\text{phys}} = \int d^2 x \left( \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial_\mu \phi^j \right), \tag{5}
\]

where \( g_{ij} \) is given by,

\[
g_{ij} = \delta_{ij} + \frac{\phi^i \phi^j}{1 - \phi^i \phi^i}. \tag{6}
\]

Now, we would like to construct the Hamiltonian of the model for posterior quantization. The initial step is the derivation of the momenta, through the usual definition,
\[ \pi_i = \frac{\partial L}{\partial (\partial_0 \phi^i)}, \]  

(7)

where \( L \) is the density of Lagrangian which can be read directly from eq. (5) and \( \partial_0 \) means partial derivative with respect to the time coordinate. So, we may compute the momenta to obtain,

\[ \pi_i = g_{ij} \partial_0 \phi^j. \]  

(8)

In order to re-write the theory in its Hamiltonian form we must know how to invert eq. (8), so that, we may write the velocities in terms of the momenta. It is accomplished by the computation of the inverse of \( g_{ij} \) which is,

\[ \tilde{g}^{ij} = \delta^{ij} - \phi^i \phi^j. \]  

(9)

Therefore,

\[ \partial_0 \phi^i = \tilde{g}^{ij} \pi_j. \]  

(10)

The Hamiltonian of the theory, which general expression is,

\[ H = \int dx (\pi_i \partial_0 \phi^i - L), \]  

(11)

takes the particular form,

\[ H = \int dx \left( \frac{1}{2} \tilde{g}^{ij} \pi_i \pi_j + \frac{1}{2} g_{ij} \partial_x \phi^i \partial_x \phi^j \right). \]  

(12)

Where \( \partial_x \) means partial derivative with respect to the spatial coordinate.

By definition the \((\phi^i, \pi_i)\) form canonically conjugated pairs which have the usual Poisson brackets,

\[ \{\phi^i(x_0, x), \pi_j(x_0, x')\} = \delta^{i}_j \delta^{N-1}(x - x'). \]  

(13)

As we have mentioned above, the unconstrained formalism was recently used in the study of the analogous quantum mechanical problem of a particle moving on a sphere [18].
Therefore, if we keep in mind the differences between the two systems, some of the above equations (3-12) are similar to theirs.

Now, we would like to write the functional Schrödinger equation of the model [14,15]. Therefore, we start introducing the wave-functional $\Psi[\phi^i, t]$. Then, we consider the $\phi^i$’s and the $\pi_i$’s as quantum operators, it means that in the fields representation the momenta are replaced by the following functional derivatives,

$$
\pi_i(x) \rightarrow -i \frac{\delta}{\delta \phi^i(x)},
$$

where we have set $\hbar$ equal to one.

The wave-functional $\Psi$ satisfies the functional Schrödinger equation,

$$
i \frac{\partial}{\partial t} \Psi[\phi^i, t] = \hat{H}[\phi^i, t] \Psi[\phi^i, t],
$$

where $\hat{H}$ is the operatorial version of $H$ eq. (12).

It is important to notice that since $\tilde{g}^{ij}$ depends on the fields, the kinetic term in the Hamiltonian eq. (12) will develop factor-ordering ambiguities upon quantization. Here, we shall solve this problem by choosing a particular factor-ordering. We shall write all field functions to the left of the momenta operators. We justify this choice by two different facts. Firstly, its application in the Hamiltonians obtained in the present paper leads to the same functional Schrödinger equation. Secondly, in the study of the first class constrained version of the model the ordering is consistent with the operatorial version of the classical constraint algebra (see Sec. III). The situation is similar to the one with the Wheeler-DeWitt equation [20]. As we have mentioned above, we shall not demonstrate that our factor-ordering choice is the only one to satisfy the above mentioned properties. It means that, there may be other choices that also satisfy those properties.

Taking in account the explicit expression of $H$ eq. (12) and the particular factor-ordering choice mentioned above, the functional Schrödinger equation for the O(N) non-linear sigma model is given by,

$$
\int dx \left( \frac{1}{2} \tilde{g}^{ij} \frac{\delta^2 \Psi}{\delta \phi^i \delta \phi^j} + \frac{1}{2} g_{ij} \partial_x \phi^i \partial_x \phi^j \Psi \right) = i \frac{\partial}{\partial t} \Psi.
$$

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Since the Hamiltonian eq. (12) does not explicitly depend on time, we may separate out the time dependence of the wave-functional and write,

$$\Psi[\phi^i, t] = e^{-iEt}\Psi[\phi^i].$$  \hspace{1cm} (17)

From eq. (16), $\Psi[\phi^i]$ satisfies the time-independent functional Schrödinger equation,

$$\int dx \left( \frac{1}{2} \delta^{ij} \frac{\delta^2 \Psi}{\delta \phi^i \delta \phi^j} + \frac{1}{2} g_{ij} \partial_x \phi^i \partial_x \phi^j \Psi \right) = E \Psi.$$  \hspace{1cm} (18)

It is clear from the above equation (18) that the energies $E$, the eigenvalues of the Hamiltonian, will be determined for the present model when we solve this equation.

**B. Dirac second-class formalism.**

In the present formulation, it is more appropriated to write the action of the model in the following way,

$$S = \int d^2x \left[ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \lambda (|\phi|^2 - 1) \right],$$  \hspace{1cm} (19)

where $\lambda$ is a Lagrange’s multiplier and the geometrical constraint was introduced in the action. It is not difficult to see that the Lagrange's equations for both actions eqs. (1) and (19) are the same.

In order to write the Hamiltonian for the action eq. (19), we compute the canonically conjugated momenta. They are given by eq. (7) which for the present case reduce to,

$$\pi_a = \partial_0 \phi^a, \quad \pi_\lambda = 0.$$  \hspace{1cm} (20)

Using the values of the momenta eq. (20) and the Lagrangian present in the action (19), the Hamiltonian eq. (11) becomes,

$$H = \int dx \left[ \frac{1}{2} \pi_a \pi_a + \frac{1}{2} \partial_x \phi^a \partial_x \phi^a - \lambda (|\phi|^2 - 1) + v_\lambda \pi_\lambda \right],$$  \hspace{1cm} (21)

where $v_\lambda$ is a new Lagrange’s multiplier associated to the constraint $\pi_\lambda = 0.$
We may now, derive all second-class constraints of the model by computing the time evolution of the known constraints. Starting with the known constraint \( \pi_\lambda = 0 \) we obtain the complete set,

\[
\pi_\lambda = 0, \quad |\phi|^2 \lambda + \frac{1}{2} \pi_a \pi_a + \partial \phi^a \partial \phi^a = 0, \tag{22}
\]

\[
G_1 = |\phi|^2 - 1 = 0 \quad G_2 = \phi^a \pi_a = 0. \tag{23}
\]

Following Dirac’s procedure \([19]\), we take into account the above constraints eqs. (22) and (23) by constructing the Dirac bracket. After that, we shall be able to use explicitly the constraints in the theory.

The sector of the Dirac bracket involving the constraints eq. (22) is trivial. Therefore, it will get contributions just from \( G_1 \) and \( G_2 \) eq. (23),

\[
\{A, B\}_D = \{A, B\} + \frac{1}{2|\phi|^2}\{A, |\phi|^2 - 1\}\{\phi^a \pi_a, B\} - \frac{1}{2|\phi|^2}\{A, \phi^a \pi_a\}\{|\phi|^2 - 1, B\}, \tag{24}
\]

where \( A \) and \( B \) are functions of the canonical variables and all brackets in the right hand side of eq. (24) are Poisson brackets.

Computing the Dirac brackets of the fields and their conjugated momenta we obtain the below values,

\[
\{\phi^a(x), \phi^b(x')\}_D = 0, \quad \{\pi_a(x), \pi_b(x')\}_D = \left( \frac{\pi^a \phi^b - \pi^b \phi^a}{|\phi|^2} \right) \delta(x - x'),
\]

\[
\{\phi^a(x), \pi_b(x')\}_D = \left( \delta^a_b - \frac{\phi^a \phi^b}{|\phi|^2} \right) \delta(x - x'). \tag{25}
\]

At this stage we may use explicitly the results coming from the constraints. From the constraints eq. (22) we learn that \( \pi_\lambda = 0 \) and the value of \( \lambda \) in terms of the other variables.

From \( G_1 \) eq. (23) we obtain eq. (3) and from \( G_2 \) eq. (23) we obtain,

\[
\pi_N = -\frac{\phi^i \pi_i}{\sqrt{1 - \phi^i \phi^i}}. \tag{26}
\]

Now, the model can be written in terms of \( N - 1 \) independent fields and their conjugated momenta. Using the results coming from the constraints, the Dirac brackets for the \( N - 1 \) independent fields and their conjugated momenta become,
\[
\{ \phi^i(x), \phi^j(x') \}_D = 0, \quad \{ \pi_i(x), \pi_j(x') \}_D = \left( \pi_i \phi^j - \pi_j \phi^i \right) \delta(x - x'),
\]
\[
\{ \phi^i(x), \pi_j(x') \}_D = \left( \delta^i_j - \phi^i \phi^j \right) \delta(x - x').
\] (27)

Since \( \phi^i \)'s and \( \pi_i \)'s do not form canonically conjugated sets, the next step \([19,21]\) is to find new variables which form canonical pairs with relation to the Dirac brackets eq. (27). It allows one to apply the standard quantization methods, in particular, to write the functional Schrödinger equation for the theory.

The variables are given by the following transformations,

\[
\tilde{\phi}^i = \phi^i, \quad \tilde{\pi}_i = \pi_i + \left( \frac{\pi_j \phi^j}{1 - \phi^i \phi^i} \right) \phi^i.
\] (28)

They are easily inverted resulting in,

\[
\phi^i = \tilde{\phi}^i, \quad \pi_i = \tilde{\pi}_i - \left( \frac{\tilde{\pi}_j \tilde{\phi}^j}{1 + \tilde{\phi}^i \tilde{\phi}^i} \right) \tilde{\phi}^i.
\] (29)

Finally, we are in position to write the physical Hamiltonian \((H_{\text{phys}})\) starting from \(H\) eq. (21). For this we start substituting the constraints eqs. (22) and (23) in \(H\) eq. (21). Then, we eliminate the non-physical variable \(\phi^N\) and its conjugated momentum \(\pi_N\) through the eqs. (3) and (26). After that, we re-write the whole expression in terms of the canonically conjugated pairs \((\tilde{\phi}^i, \tilde{\pi}_i)\) eq. (28). We obtain,

\[
H_{\text{phys}} = \int dx \left( \frac{1}{2} \tilde{g}^{ij} \tilde{\pi}_i \tilde{\pi}_j + \frac{1}{2} \tilde{g}_{ij} \partial_x \tilde{\phi}^i \partial_x \tilde{\phi}^j \right).
\] (30)

Comparing \(H_{\text{phys}}\) eq. (30) with \(H\) eq. (12) we can see that they are the same\(^1\). Naturally, if one uses the same factor ordering choice of Subsec. II A, the functional Schrödinger equations for both must be identical since they are written in terms of pairs of fields and momenta that are canonically conjugated.

\(^1\)Note that, eq. (28) means that the variables \((\phi^i, \pi_i)\) of the theory eq. (19) and the corresponding variables of the unconstrained formulation eq. (12) are related by the noncanonical transformations eq. (28). We are grateful to the referee for pointing this fact.
III. THE $O(N)$ NON-LINEAR SIGMA MODEL AS A FIRST-CLASS
CONSTRAINED FIELD THEORY.

The Hamiltonian for the $O(N)$ non-linear sigma model written as a first-class constrained field theory that we shall use is [5],

$$H = \int dx \left[ \frac{1}{2} \bar{g}^{ab} \pi_a \pi_b + \frac{1}{2} \partial_x \phi^a \partial_x \phi^a - \lambda(|\phi|^2 - 1) + v_\lambda \pi_\lambda \right],$$

where,

$$\bar{g}^{ab} = \delta^{ab} - \frac{\phi^a \phi^b}{|\phi|^2}.$$ (32)

The first-class constraints are $\pi_\lambda = 0$ eq. (22) and $G_1 = 0$ eq. (23). Note that they are in involution with the Hamiltonian eq. (31). The formulation eq. (31) is classically equivalent to the initial one eq. (12). This means that, in the appropriate gauge, the equations of motion for the physical variables in eq. (31) are the same as for eq. (12). One may demonstrate that in the following way. Firstly, choose the remaining constraints in eqs. (22) and (23),

$$\pi_\lambda = 0, \quad |\phi|^2 \lambda + \frac{1}{2} \pi_a \pi_a + \partial_x \phi^a \partial_x \phi^a = 0,$$ (33)
as the gauge fixing conditions for the first class constraints $\pi_\lambda = 0$ and $G_1 = 0$. Then, supposing that the corresponding Dirac bracket eq. (24) is constructed, one can impose the constraints upon the Hamiltonian eq. (31). The resulting expression will be identical to the Hamiltonian eq. (21). Finally, one follows all the steps presented in subsection II B which showed that the model described by the Hamiltonian eq. (21) is classically equivalent to the one described by the Hamiltonian eq. (12). Therefore, $H$ eq. (31) represents correctly the model at the classical level.

Let us point out, also, that the formulation eq. (31) can be used to represent the sigma-model dynamics in a simple form. Namely, instead of (33), one can now choose the following gauge: $\lambda = 0$ and $\phi^a \pi_a = 0$, for the constraints $\pi_\lambda = 0$ and $G_1 = 0$. It leads to the free equations of motion: $\partial_0 \phi^a = p_a, \partial_0 p_a = \partial_i \partial_0 \phi^a$ or $\Box \phi^a = 0$, for the configuration space
variables. Contrary to the previous section, they can be immediately solved in terms of the creation and annihilation operators which obey the brackets following from eq. (25). It can lead to the possible quantization of the model in the Fock-space representation [22].

We would like to write the functional Schrödinger equation for \( H \) eq. (31). For this, we shall use the Dirac’s prescription to canonically quantize first-class constrained systems [19]. As we shall see the functional Schrödinger equation will be the same as the ones derived in Subsecs. II A and II B.

We start by noting that the functional Schrödinger method described in Subsec. II A will have a single modification in order to comply with the Dirac’s prescription to treat first-class constrained systems. The wave-functional will have to be annihilated by the operatorial version of the constraints besides satisfying the functional Schrödinger equation [16,17].

Observing the constraint \( G_1 = 0 \) eq. (23), we notice that the condition that its operatorial version (\( \hat{G}_1 \)) annihilates the wave-functional (\( \Psi \)) does not result in any condition upon \( \Psi \). It is in fact a condition upon the fields, since in the fields representation all the operators in \( \hat{G}_1 \) have a multiplicative application upon \( \Psi \). One way to obtain a restriction upon \( \Psi \), from \( \hat{G}_1 \) is by considering the pair \( (\phi^N, \pi_N) \) as the corresponding non-physical variables. Then, without affecting the physical sector variables, one can make the canonical transformation,

\[
\phi^N \rightarrow -\pi_N, \quad \pi_N \rightarrow \phi^N.
\]

This transformation changes \( G_1 = 0 \) eq. (23) and \( H \) eq. (31) to,

\[
\tilde{G}_1 = \pi_N \pi_N + \phi^i \phi^i - 1 = 0,
\]

\[
\tilde{H} = \int dx \left\{ \frac{1}{2} \left[ \pi_i \pi_i - \left( \frac{\phi^i \phi^j}{\pi_N \pi_N + \phi^k \phi^k} \right) \pi_i \pi_j + \partial_x \phi^i \partial_x \phi^i + \partial_x \pi_N \partial_x \pi_N \right] + \frac{1}{2} \left[ \phi^N \phi^N + 2 \left( \frac{\phi^i \phi^N}{\pi_N \pi_N + \phi^i \phi^i} \right) \pi_i \pi_N - \left( \frac{\phi^N \phi^N}{\pi_N \pi_N + \phi^i \phi^i} \right) \pi_N \pi_N \right] \right\}.
\]

Now, we may write the equations for the wave-functional \( \Psi[\phi^N, \phi^i, \lambda] \). The first two will be obtained by demanding that the operatorial version of the constraints \( \pi_\lambda = 0 \) eq. (22)
and $\tilde{G}_1$ eq. (35) annihilate $\Psi$. The last one is the functional Schrödinger equation and will be derived from the operatorial version of the Hamiltonian eq. (36) ($\tilde{H}$). The operatorial version of all the above mentioned quantities will be obtained, in the fields representation, by the substitution of the momenta by $-i$ times the functional derivatives with respect to canonical conjugated fields eq. (14). They are,

$$\frac{\delta \Psi}{\delta \lambda} = 0,$$

$$-\frac{\delta^2 \Psi}{\delta (\phi^N)^2} + (\phi^i \phi^i - 1) \Psi = 0,$$

where we have explicitly used eq. (38) in order to substitute the result of the operation of the denominator present in $\tilde{H}$ upon $\Psi$.

Note that the particular factor-ordering chosen in (37-39), the same one introduced in Subsec. II A, preserves the classical constraint algebra and involution of the constraints with the Hamiltonian.

Now, we may proceed to solve eqs. (37) and (38) in order to learn what restrictions they will impose upon $\Psi$. They are not difficult to solve and result, respectively, in,

$$\Psi[\phi^i, \phi^N, \lambda, t] = \Psi[\phi^i, \phi^N, t],$$

$$\Psi[\phi^i, \phi^N, t] = \exp\left[ \int dy \phi^N \sqrt{\phi^i \phi^i - 1} \right] \Psi_{phys}[\phi^i, t].$$

Finally, we must introduce $\Psi$ eq. (41) in the functional Schrödinger equation (39) to obtain,

$$\int dx \left( \frac{1}{2} \tilde{g}^{ij} \frac{\delta^2 \Psi_{phys}}{\delta \phi^i \delta \phi^j} + \frac{1}{2} g_{ij} \partial_x \phi^i \partial_x \phi^j \Psi_{phys} \right) = E \Psi_{phys},$$
where $g_{ij}$ and $\tilde{g}^{ij}$ are given, respectively, by eqs. (6) and (9) and we have supposed the time dependence of $\Psi_{phys}$, given in eq. (17). It is important to mention that few terms proportional to the Dirac delta function of the point zero ($\delta(0)$) appear in the derivation of eq. (42). They contribute an infinity amount of energy for the system that can be removed by the usual regularization techniques [23,24].

Comparing eq. (42) with eq. (18) we notice that they are the same.

IV. CONCLUSIONS.

In this work we derived the Hamiltonian formalism of the $O(N)$ non-linear sigma model in its original version as a second-class constrained field theory and then as a first-class constrained field theory. We treated the model as a second-class constrained field theory, by two different methods: the unconstrained and the Dirac second-class formalisms. We showed that the Hamiltonians for all these versions of the model are equivalents. Then, for a particular factor-ordering choice, we wrote the functional Schrödinger equation for each derived Hamiltonian. We showed that they are all identical which justifies our factor-ordering choice and opens the way for a future quantization of the model via the functional Schrödinger representation.

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REFERENCES

[1] E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (John Wiley & Sons, New York, 1974).

[2] I. A. Batalin and E. S. Fradkin, Phys. Lett. B 180, 157 (1986); Nucl. Phys. B 279, 514 (1987); I. A. Batalin, E. S. Fradkin, and T. E. Fradkina, *ibid.* 314, 158 (1989); 323, 734 (1989).

[3] I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. A 6, 3255 (1991).

[4] C. Wotzasek, Int. J. Mod. Phys. A 5, 1123 (1990).

[5] J. Ananias Neto, A. C. R. Mendes, W. Oliveira, C. Neves and D. C. Rodrigues, *Embedding Second Class Systems via Symplectic Gauge-Invariant Formalism*, hep-th 0109089.

[6] For a review see: V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rep. 116, 103 (1984).

[7] For a review see: E. Abdalla, M. C. B. Abdalla and K. D. Rothe, *Non-perturbative methods in 2-dimensional quantum field theory*, (World Scientific, Singapore, 1991).

[8] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov and M. A. Shifman, Sov. J. Part. Nucl. 17, 204 (1986).

[9] J. Maharana, Phys. Lett. B 128, 411 (1983).

[10] W. A. Bardeen, B. W. Lee and R. E. Shrock, Phys. Rev. D 14, 985 (1976).

[11] T. R. Gaztelurrutia and A. C. Davis, Nucl. Phys. B 347, 319 (1990).

[12] N. Banerjee, S. Ghosh and R. Banerjee, Nucl. Phys. B 417, 257 (1994).

[13] D. K. Kim and C. K. Kim, J. Phys. A 31, 6029 (1998).

[14] For a review see: B. Hatfield, *Quantum field theory of point particles and strings* (Addison-Wesley, New York, 1992), pp. 199-210.
[15] For a review see: R. Jackiw, *Diverse topics in theoretical and mathematical physics* (World Scientific, Singapore, 1995).

[16] J. Goldstone and R. Jackiw, Phys. Lett. B 74, 81 (1978); M. Henneaux, Phys. Rev. Lett. 54, 959 (1985); R. Floreanini, C. T. Hill and R. Jackiw, Ann. of Phys. 175, 345 (1987); D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B 321, 193 (1994); D. Cangemi, R. Jackiw and B. Zwiebach, Ann. of Phys. 245, 408 (1996); J. Hallin and P. Liljenberg, Phys. Rev. D 54, 1723 (1996); E. Benedict, R. Jackiw and H. J. Lee, ibid., 6213; S. Cassemero F. F. and V. O. Rivelles, Phys. Lett. B 452, 234 (1999).

[17] G. Oliveira-Neto, Phys. Rev. D 58, 24010 (1998).

[18] E. Abdalla and R. Banerjee, Braz. J. of Phys. 31, 80 (2001).

[19] P. A. M. Dirac, Can. J. Math. 2, 129 (1950); *Lectures on quantum mechanics* (Yeshiva University, New York, 1964).

[20] C. J. Isham, *Canonical quantum gravity and the problem of time*, gr-qc/9210011.

[21] D. M. Gitman and I. V. Tyutin, *Quantization of fields with constraints*, (Springer-Verlag, Berlin, 1990), p. 32.

[22] A. Deriglazov, Phys. Lett. B 530, 235 (2002).

[23] L. H. Ryder, *Quantum field theory*, (Cambridge University Press, Cambridge, 1985).

[24] C. Itzykson and J. B. Zuber, *Quantum field theory*, (McGraw-Hill, Singapore, 1987).