Dynamics of Universe in Problems

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Preface

We do not know books of problems on cosmology including its recent achievements. However we believe that such a book would be extremely useful for the youth poured in the last decade into this one of the most actively developing field of science. The only way to rise over "popular" level in any science is to master its alphabet, i.e. to learn to solve the problems, even the simplest on the first stage. A.Sommerfeld wrote: "Just solve problems carefully. Then you make clear what you have already understood and what — not yet."

Of course most of modern wonderful courses include problems. However a reader tired by high theory does not have force and time to solve them. May be it is worth to change the tactics and immediately throw into water those who wish to learn to swim?

We suggest to cosmological community to create a joint collaboration "Dynamics of Universe in Problems". In first release we propose 500 problems as our contribution into that beginning. In current release we add about 170 new problems and reorganize the old in more accurate way. If our project will make an interest then it is necessary to think how to organize technically the international collaboration in that direction. In any case we plan to update our "database" monthly, including 30-50 new problems every time.

Of course each problem has an author. However many problems already became a scientific folklore and it is difficult to point out correctly the real author. We justify ourselves by the fact that most of books of problems do not cite the authors and it does not offend anybody. We believe that only answers should be copyrighted, not the questions themselves, although well posed question normally worth half of the answer.

So welcome to the rage water of cosmology problems and good luck!
Chapter 1

Cosmo-warm-up

1. How the Earth radius could be determined in 350 B.C.?

2. From what distance will one astronomical unit length have visible size of one angular second?

3. What is angular dimension of our Galaxy for an observer situated in the Andromeda galaxy, if the distance to it is about 700\(kpc\)? Compare it with the Sun angular size viewed from the Earth.

4. A supernova outburst in the Andromeda nebula have been observed on the Earth. Estimate time passed after the star explosion.

5. A galaxy situated at distance \(R\) from us at the moment of observation recedes with velocity \(V\). At what distance was it situated in the moment of emission of the observed light?

6. A glance on the night sky makes impression of invariability of the Universe. Why the stars seem to us practically static?

7. What is maximum sum of angles in a triangle on a sphere?

8. Consider the surface of a two-dimensional sphere of radius \(R\). Circle is drawn on the sphere which has radius \(r\) as measured on the surface on the sphere. Find circumference of such a circle as function of \(r\).

9. Suppose that galaxies are distributed evenly in a two-dimensional sphere of radius \(R\) with a number density \(n\) per unit area. Determine the total number \(N\) of galaxies inside a radius \(r\). Do you see more or fewer galaxies out to the same radius, if the geometry is spherical rather than flat?

10. An object of size \(A\) is situated at distance \(B\). Determine the angle at which the object is viewed in flat space and in spaces of constant (positive and negative) curvature.

11. Every second about 1400\(J\) of solar energy falls onto square meter of the Earth. Estimate absolute emittance of the Sun.
12. Assuming that the constant emittance stage for the Sun is of order of $10^{10}$ years, find the part of solar mass lost due to radiation.

13. Why the connection between the emittance of variable stars (Cepheids) and the period of their brightness variation was discovered from observation of stars in Great Magellan cloud rather than in our Galaxy?

14. A supernova in its brightness maximum reaches absolute stellar magnitude $M = -21$. How often the supernova outbursts will be registered if the observation is provided on the whole sky up to limiting magnitude $m = 14$? Assume that in a typical galaxy a supernova bursts in average once per 100 years, and galaxies are distributed with spatial density of one galaxy per $10^3 Mpc$.

15. From the fundamental constants $c$, $G$, $\hbar$ construct dimensions of length, mass, temperature, density and find their values (the corresponding magnitudes are called Planck’s units).

16. Estimate the perception delay time for an object situated on 1 meter distance from a plain mirror and compare it with the Planck’s time.

17. Show that Compton wave length for a particle of Planck’s mass coincides with its gravitational radius $r_g = 2Gm/c^2$ (the gravitational radius in general relativity coincides with the radius of a spherically symmetric body for which the classical second orbital velocity on the surface would be equal to the light speed).

18. Show that in the $c = \hbar = 1$ units

$$1 GeV \approx 1.8 \cdot 10^{-24} g; \ 1 GeV^{-1} \approx 0.7 \cdot 10^{-24} sec \approx 2 \cdot 10^{-14} cm.$$  

19. Express $M_{Pl}$ in $K$, $cm^{-1}$, $sec^{-1}$.

20. Show that the fine structure constant $\alpha = e^2/\hbar c$ is dimensionless only in the space of dimension $D = 3$.

21. From the constants $c$, $\hbar$, $e$, $G$ construct a dimensionless combination in the space of arbitrary dimension.

22. To study charge distribution inside a nucleus one commonly uses high-energy electrons (not protons) as they do not take part in strong interactions and therefore they are sensitive only to the charge distribution in the nucleus. What is minimum energy of electrons needed for that purpose?

23. Estimate radius and mass of a white dwarf.

24. Estimate radius and mass of a neutron star.

25. Why neutron stars must have strong magnetic fields?
26. What is average density of a white dwarf with the mass equal to solar one, brightness — one thousand times less than the solar one, and the surface temperature — twice greater than the solar one?

27. Suppose that we have concentrated whole cosmic history (14 billion years) in one day. Display main events in the Universe history using logarithmic time scale. Start from the Planck’s time to avoid degeneracies.

28. Effect of variation of observed frequency of oscillations at relative motion of the source and the receiver was predicted by Doppler for light and sound waves in 1842 year. Doppler believed that the effect could explain the distinction in the stars colors: a star moving towards the Earth looks ”bluer”, and outwards the Earth — ”reddens”. Explain why the Doppler effect cannot change the star ”coloring” significantly.

29. Estimate mass of Milky Way and number of stars in it. Our Galaxy has diameter about 100 000 light years and the Sun is situated approximately two thirds of the way from the Galaxy center. Period of the Sun rotation around the galaxy center is of order 250 millions years.

30. Estimate density of luminous matter in the Universe assuming that the Milky Way containing $\sim 10^{11}$ stars of solar type is a typical galaxy, and average intergalactic distance is of order $L = 1\text{Mpc}$.

*Age of Universe is one of the most discussed question in cosmology. Unfortunately no chronometer was started in the moment of the Big Bang and we have no direct methods to determine the cosmic time. However common sense allows to construct many ways to estimate the lower bound of that magnitude supposing that the Universe cannot be younger than any of its component, or, in other words, the Universe age must be greater than the age of the oldest object discovered in it. Different methods for determination of lower bound for the Universe age use the following objects as the ”cosmic long-livers”: long-living isotopes, globular clusters, white dwarfs. In particular, nuclear cosmochronology allows to estimate the age of Universe comparing current relative abundance of some long-living radioactive isotopes with the initial one, which is determined in frames of certain cosmological models.*

31. According to the Big Bang model the initial ratio of the uranium isotopes abundances was $U^{235}/U^{238} \approx 1.65$, and the presently observed one is $U^{235}/U^{238} \approx 0.0072$. Accounting that the half-value periods of the isotopes equal $t_{1/2}(U^{235}) = 1.03 \cdot 10^9$ years and $t_{1/2}(U^{238}) = 6.67 \cdot 10^9$ years, determine the age of Universe.

32. In frames of nuclear cosmochronology propose a method to determine age of the Earth (i.e. Solar system).

33. Show that in the hydrogen atom ratio of electrical forces to gravitational ones is close to ratio of the Universe size to the size of electron (first noted by P.Dirac).

34. Compare constants of strong, weak, electromagnetic and gravitational interactions.
35. Estimate order of magnitude of the Great Unification temperature (the temperature when intensity of gravitation comes up with intensities of three other interactions).

36. Express Bohr radius through fine structure constant and Compton wavelength.

37. Estimate free path of hydrogen atom in the intergalactic space.

38. Protons accelerated on LHC ($E = 7 TeV$) and photons are participants of cosmic race on Earth-Sun distance. How much will the protons remain behind in time and distance?

39. Determine Fermi energy of cosmic neutrino background (gas).

40. Assume that the space is infinite and in average uniformly filled with matter. Estimate the distance from our observable part of Universe where the same (with identical distribution of galaxies and the same Earth) part of Universe is situated.

41. If the Universe would be infinitely old and infinitely extended, and stars could eternally shine, then whatever direction you look the ray of your vision should get on surface of a star, and therefore all sky should be as bright as the Sun surface. This notion is known under name of Olbers' paradox. Formulate the Olbers' paradox quantitatively.

42. There are $n$ trees per hectare in a forest. Diameter of each one is $D$. Starting from what distance "the trees will hide the forest"? How this question is connected with the Olbers' paradox?

43. In a more romantic formulation that problem looks as the following. Suppose that in Sherwood Forest, the average radius of a tree is $30 cm$ and average number of trees per unit area is $n = 0.005 m^{-2}$. If Robin Hood shoots an arrow in a random direction, how far, on average, will it travel before it strikes a tree?

44. The same problem in the cosmological formulation looks this way. Suppose you are in a infinitely large, infinitely old Universe in which the average density of stars is $n_{st} = 10^{11} Mpc^{-3}$ and the average stellar radius is equal to the Sun’s radius: $R_{st} = R_{\odot} = 7 \cdot 10^8 m$. How far, on average, could you see in any direction before your vision line struck a star?

45. Accretion process is the gravitational capture of matter and its subsequent fall down on a cosmic body (a star for example) due to gravitational forces. In such process the kinetic energy of the falling mass $m$ with some efficiency transforms into radiation energy which leads to additional contribution to brightness of the accreting system. Determine the limiting brightness due to the accretion (the Eddington limit).

46. Give considerations in favor of electrical neutrality of the Universe on large scales.

47. What is the difference (quantitative and qualitative) between the gravitational waves and the electromagnetic ones?
48. Estimate total amount of energy collected by optical telescopes during the past XX century and compare it with the energy needed to turn over a page of a book.

49. Find probability that transition between two atomic states occurs due to gravitation rather than electromagnetic forces.

50. What cosmological process releases maximum amount of energy for one time after the Big Bang?

51. One of possible formulations of the Fermi paradox, or the ”great silence” paradox, is the following: if there is an intelligent life in the Universe, so why it emits no signal into the space and generally manifest itself in no way? This paradox is connected with name of Fermi, because once having listened arguments of his colleagues stating that there exist a great many of highly developed technological civilizations, he asked after some pause: ”Well, and where are they so far?” Give arguments to support or disprove the paradox.

52. Evidently role of gravitation growth with the mass of a body. Show that gravitation dominates if the number of atoms in the body exceeds some critical value

\[ N_{cr} \simeq \left( \frac{\alpha}{\alpha_G} \right)^{3/2} \simeq 10^{54}, \]

where \( \alpha = e^2/(\hbar c) \) is the fine structure constant and \( \alpha_G \equiv G m_p^2/(\hbar c) \) is the ”fine structure constant” for gravitation, \( m_p \) is the proton mass.

53. In frames of Newtonian mechanics calculate the collapse time for homogeneous spherical mass with density \( \rho_0 \).

54. Black holes—one of the most mysterious astrophysical objects—can be small (of several solar masses) and super-massive (millions times heavier than the Sun). Probably super-massive black holes exist in centers of many galaxies, and in center of our galaxy too. Such black holes can be formed when the central part of galaxy collapses. Black holes can be detected by indirect signs. In particular, detection of stars rapidly rotating around an invisible gigantic mass gives serious confirmation of a super-massive black hole presence. Recently (1995-2008) such stars rotating on elliptic orbits around a common center were discovered in the center of our galaxy. Estimate the invisible mass (the black hole mass), if the velocities of the stars are of order \( V \sim 10^4 \text{ km/s} \), and their motion is observed on distance of 100 a.u. from the center.

55. A gas cloud of mass \( M \) consisting of molecules of mass \( \mu \) is unstable if the gravitational energy exceeds the kinetic energy of thermal motion. Derive the stability condition for the spherically symmetric homogeneous cloud of radius \( R \) (the Jeans criterion).

56. Estimate the critical density for hydrogen cloud of solar mass at temperature \( T = 1000 \text{ K} \).
57. Compare gravitational pressure in the centers of the Sun \(\rho = 1.4 \text{ g/cm}^3\) and the Earth \(\rho = 5.5 \text{ g/cm}^3\).

58. Hydrogen burning (hydrogen to helium transformation) in stars is realized in the so-called \(p-p\) cycle, which starts from the reaction of deuterium formation \(p + p \rightarrow d + e^+ + \nu\). To support such a reaction the colliding protons have to overcome Coulomb barrier (in order to enter the region where nuclear forces act: \(r_{nf} \approx 10^{-13} \text{ cm}\).) It requires energy as high as \(E_c = e^2/r_{nf} \approx 1.2 \text{ MeV}\). Typical solar temperature is only \(T_{\odot} = 10^7 K \approx 0.9 \text{ keV}\). The Coulomb barrier can be overcome due to the quantum tunneling effect (classical probability to overcome the barrier for the protons in the tail of Maxwell distribution is too low). Estimate the probability of tunneling through the Coulomb barrier for protons on the Sun.
Chapter 2

Dynamics of Expanding Universe

1. The main assumption lying in the basis of overwhelming majority of cosmological models is spatial homogeneity and isotropy of Universe. Give examples showing that this two notions do not imply each other automatically.

2. Show that if some distribution is everywhere isotropic then it will be also homogeneous. Is the opposite true?

3. What three-dimensional objects are both homogeneous and isotropic?

4. Why the term "Big Bang" should not be literally understood concerning to the first stage of Universe evolution?

5. Show that Hubble law is invariant under Galilean transformations.

6. Show that Hubble law is the only one expansion law compatible with homogeneity and isotropy of Universe.

7. Show that if the Universe expansion follows Hubble law then initial homogeneous distribution will remain homogeneous in all further moments of time.

8. Bondi, Gold and Hoil in forties of last century proposed a stationary model of Universe based on generalized cosmological principle implying absence of privileged position not only in space, but in time as well. In a Universe described by such a model all global properties and characteristics (density, Hubble parameter and others) remain constant in time. Estimate rate of matter creation in such a model.

9. Typical peculiar velocities of galaxies are $V_p \approx 100 km/s$. How far from us must a galaxy be in order to have small peculiar velocity compared to the velocity of Hubble expansion $V_H = H_0 R$?

10. Using the observed value of Hubble constant, estimate age of Universe (the Hubble time $t_H$).
11. Introduce notion of Hubble radius and clarify its physical sense.

12. Show that the expanding Universe model allows to resolve Olbers paradox.

13. Can cosmological objects recede with velocities exceeding the light speed?

14. Can we see galaxies receding with velocities exceeding the light speed?

15. How will distance $r_{AB}$ between two points $A$ and $B$ on a two-dimensional sphere and their relative velocity $v_{AB} = dr_{AB}/dt$ change if the sphere radius $a(t)$ depends on time?

16. Having integrated the Hubble law, introduce notions of scale factor and comoving distance.

17. Obtain equation of motion for free particle in a space-time with metric $g_{\mu\nu}$.

18. Determine distance between two infinitesimally close points on the 3-sphere surface.

19. Show that the Friedman-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

can be rewritten in the form

$$ds^2 = dt^2 - a^2(t) \left[ d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

where

$$\Sigma^2(\chi) = \begin{cases} 
\sin^2 \chi & k = +1 \\
\chi^2 & k = 0 \\
\sinh^2 \chi & k = -1.
\end{cases}$$

20. Show that metric of a homogeneous and isotropic three-dimensional space which may expand (or contract) as a function of time is necessarily the FRW metric.

21. Find volume, surface area and equator length for closed Universe.

22. Give arguments in favor of the fact that in a closed Universe the total electrical charge equals to zero.

23. Derive the Hubble law using the FRW metrics.

24. Obtain Einstein equations from variational principle.

25. Show that Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$$

can be presented in the form

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \right) : T \equiv SpT_{\mu\nu} = g^{\mu\nu}T_{\mu\nu}.$$
26. Calculate the connectivity components (Christoffel symbols) in FRW metric.

27. Calculate Ricci tensor components in FRW metric.

28. Calculate scalar curvature $R$ in FRW metric.

29. Find components of the energy-momentum tensor satisfying the cosmology principle. Calculate its trace.

30. Obtain zero geodesics for a particle with initial condition $r(0) = 0$ and boundary condition $r(T) = R$, at radial motion $\theta = \theta_1$, $\varphi = \varphi_1$, in the de Sitter space-time

$$ds^2 = dt^2 - e^{2Ht} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)],$$

where $H = \text{const}$.

31. Obtain Friedman equations from Einstein equations.

32. Are solutions of Friedman equations Lorenz-invariant?

33. Using the first Friedman equation, make a classification of the solutions describing the Universe dynamics in frames of the Newtonian mechanics.

34. Determine value of the critical density — the density corresponding to spatially flat Universe.

35. What is "physical" reason for appearance of coefficient 3 in the second Friedman equation:

$$\frac{\ddot{a}}{a} = - \frac{4\pi G}{3}(\rho + 3p)?$$

36. Show that the first Friedman equation can be written in the form

$$\sum_i \Omega_i = 1,$$

where $\Omega_i$ are relative densities of components:

$$\Omega_i \equiv \frac{\rho_i}{\rho_{cr}}.$$

The summation includes contribution of spatial curvature

$$\rho_{\text{curv}} = - \frac{3}{8\pi G a^2} k.$$

37. Express the curvature radius of Universe through the Hubble radius and total relative density.

38. Show that the relative density of curvature in some region can be interpreted as measure of difference between mean potential energy and mean kinetic energy in the region.
39. Show that in a closed Universe $\dot{H} = -4\pi G(\rho + p)$.

40. Obtain the Raychadhuri equation:

$$H^2 + \dot{H} = -\frac{4\pi G}{3}(\rho + 3p).$$

41. From Friedman equations obtain the conservation equation for expanding Universe

$$\dot{\rho} + 3H(\rho + p) = 0.$$

42. Obtain the conservation equation for expanding Universe using invariance property for energy-momentum tensor.

43. Show that the conservation equation can be presented in the form

$$\frac{d\ln \rho}{d\ln a} + 3(1 + w) = 0.$$

44. Show that the first Friedman equation is the first integral of the second one.

45. Show that the second Friedman equation can be presented in the form:

$$HH' = -4\pi G(\rho + p),$$

where

$$H' = \frac{dH}{d\ln a}.$$

46. Show that in homogeneous and isotropic Universe filled by ideal liquid the energy-momentum tensor trace is:

$$T_\lambda^\lambda = \frac{3}{4\pi G} \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right].$$

47. In flat Universe express pressure through Hubble parameter and its time derivative.

48. Express state parameter $w = p/\rho$ through Hubble parameter and its time derivative.

49. Find upper limit for the state parameter $w = p/\rho$.

50. Show that among the three equations (two Friedman equations + conservation equation) only two are independent, i.e. from any pair of those equations the third can be obtained.

51. Rewrite the conservation equation in terms of $e$-convolutions number for scale factor $N(t) = \ln(a(t)/a_0)$ and find its solution.

52. Obtain the first Friedman equation using only Newtonian mechanics.

53. Obtain an analogue of the second Friedman equation in Newtonian mechanics.
54. Obtain the conservation equation for expanding Universe using only thermodynamic considerations.

55. Obtain the conservation equation for non-relativistic matter from the continuity equation for ideal liquid.

56. Show that for non-relativistic particles the parameter $w$ in the state equation is much less than unity.

57. Show that the only source of gravitation in the General Relativity is the quantity $p + 3\rho$.

58. What effect makes the pressure magnitude on the expansion rate?

59. Show that the equations of Newtonian hydrodynamics and gravity theory do not allow existence of homogeneous, isotropic, static cosmological model, i.e. time-invariable Universe, uniformly filled by ideal liquid.

60. Why scale factor normalization is arbitrary in the case of Euclidean geometry of Universe?

61. Show that only sign of curvature makes physical sense: rescaling of the scale factor leads to rescaling of the curvature.

62. Propose a scheme for experimental determination of curvature in a static Universe.

63. Find connection between the scale factor and the redshift.

64. Consider a set of the following four parameters:

\[ H(t) \equiv \frac{1}{a} \frac{da}{dt}, \]

\[ q(t) \equiv -\frac{1}{a} \frac{d^2a}{dt^2} \left( \frac{1}{a} \frac{da}{dt} \right)^{-2}, \]

\[ j(t) \equiv \frac{1}{a} \frac{d^3a}{dt^3} \left( \frac{1}{a} \frac{da}{dt} \right)^{-3}, \]

\[ s(t) \equiv \frac{1}{a} \frac{d^4a}{dt^4} \left( \frac{1}{a} \frac{da}{dt} \right)^{-4}. \]

Using the above parameters, decompose the scale factor in Taylor series in time.

65. Expand the redshift in Taylor series in time.

66. For "close" galaxies ($z \ll 1$) formulate Hubble law in terms of the redshift.
67. Show that the standard definition of the redshift can be used only for cosmological objects situated within the Hubble sphere.

In order to simplify the description of Universe dynamics it is often convenient to replace the physical time \( t \) by the conformal one \( \eta \), defined as

\[
\eta = \int \frac{dt}{a(t)},
\]

so that \( dt = a(\eta)d\eta \). The conformal time can be interpreted as the time showed by the clock which decelerate as the Universe expands. Therefore a light signal will be able to pass increased physical distance during the same conformal time interval.

68. Construct the FRW metric in the conformal time.

69. For an arbitrary function of time \( f(t) \) express \( \dot{f} \) and \( \ddot{f} \) through derivatives over the conformal time.

70. Find trajectory of a photon in the conformal time for isotropic spatially-flat Universe.

71. Obtain the Friedman equations for the conformal time.

72. Show that logarithmic derivative of the scale factor over the conformal time determines evolution of the scale factor in physical time.

73. Obtain the conservation equation in terms of the conformal time.

74. Calculate derivatives of the state parameter \( w \) with respect to cosmological time \( t \) and conformal time \( \eta \).

75. Find connection between the scale factor and the redshift, using properties of the conformal time interval.

76. Find connections between the physical time and the redshift, and between the conformal time and the redshift.

77. Show that in the comoving coordinates and conformal time the FRW metrics is conformally-flat, i.e. it differs from the flat one only by global stretching.

78. Can an open Universe evolve into a closed one?
Chapter 3

Dynamics of Universe in the Big Bang model

1. A spider sits on a wall. In the place where it sits a 1 meter long elastic rubber cord is attached to the wall, and a man holds in hand the second end of it. The man starts moving off the wall with velocity 1 m/s. At the same moment the spider starts to run along the cord with velocity 1 cm/s. Will the spider come up with the man?

2. What main observations is the Big Bang model based on?

3. Show that in early Universe one can neglect the curvature-proportional term in Friedman equation.

4. Estimate upper limit of contribution to the first Friedman equation by the term with curvature in electro-weak and nuclear-synthesis epochs.

5. Obtain the dependencies $\rho(t)$ and $a(t)$ for flat Universe, composed of
   a) radiation,
   b) matter.

6. Let us consider two ”Universes”. One is filled by radiation, another — by non-relativistic matter. Energy density in both cases is initially identical. Compare energy densities after period of time $t$.

   If space expands, does it mean that everything stretches in it? And Galaxies? And atoms too? A superficial answer is that ”bounded” systems do not participate the expansion. But if the space stretches, how these systems can avoid at least minimal stretching? Must the bounded systems stretch less intensively? And must less bounded systems stretch stronger? In the following set of problems we try to clarify the situation with help of a simple model—a classical atom, composed of a negatively charged electron with negligibly small mass, rotating around positively charged massive nucleus. Let us place the classical atom in the homogeneous Universe with expansion described by the scale factor $a(t)$. In order to describe the atom we will
use two sets of spatial coordinates. Both of them are spherical coordinates with origin in the nucleus. The first set consists of physical coordinates $R, \theta, \varphi$, where $R$ is the distance from nucleus to the electron at a given moment of time. The second set $r, \theta, \varphi$—the comoving coordinates—represent fixed points participating the cosmological expansion. The two sets are linked by the following relation

$$R = a(t)r.$$  

The angular coordinates are the same for both sets as we assume that the cosmological expansion is radial.

7. How to answer, in terms of physical and comoving coordinates, the question whether an atom participate the cosmological (Hubble) expansion?

8. Obtain equation of motion for electron in the atom of the considered model, taking into account the cosmological expansion.

9. For the case of exponential expansion $a(t) = e^{\beta t}$ construct effective potential for the electron in atom and with the help of it analyze dynamics of the electron for the case $L^2 = C$, where $C$ is the electrostatic interaction constant.

10. If all the Universe expand, why the Solar system does not? Give quantitative arguments.

11. In flat Universe composed from matter and radiation find time dependence for scale factor and densities of all components in case of domination of one of them. Present the results graphically.

12. In flat Universe, composed from matter and radiation only, obtain exact solutions for Friedmann equations. Plot time dependencies for the scale factor and both densities.

13. Present the first Friedman equation in terms of the redshift and analyze contributions of separate terms in different cosmological epochs.

14. Obtain the dependence $a(t)$ for flat Universe composed only of matter with the state equation $p = wp$ assuming that during all the period of the evolution the parameter $w$ does not depend of time.

15. Using the first Friedman equation, construct the effective potential $V(a)$, governing the one-dimensional motion of fictive particle with coordinate $a$.

16. Construct the effective one-dimensional potential $V(a)$ (using notions of the previous problem) in the Universe composed of non-relativistic matter and radiation. Show that in such a potential only decelerated expansion is possible.

17. Obtain time dependence for Hubble parameter in flat Universe, where either matter or radiation dominates.
18. Find time dependence for Hubble parameter for flat Universe, filled by a substance with state equation \( p = w \rho \) assuming that the parameter \( w \) is time-independent during all the evolution.

19. Find critical density of Universe expansion from the expression for second orbital velocity \( v = \sqrt{2gR} \).

20. At what moment after birth of Universe the density of matter exceeded that of radiation for the first time?

21. Find age of Universe in the case of domination of
   a) matter,
   b) radiation.

22. Show that the deceleration parameter \( q \) is connected with Hubble parameter by the relation

\[
q(t) = \frac{d}{dt} \left( \frac{1}{H} \right) - 1.
\]

23. Let \( t_a \) be the moment in the Universe history when the decelerated expansion turned into the accelerated one, i.e. the deceleration parameter \( q(t_a) = 0 \). Show that if \( t_1 \) and \( t_2 \) are two moments of time lying in the vicinity of \( t_a \) but on different sides of it, then

\[
\Delta t \equiv t_1 - t_2 = \frac{1}{H_1} - \frac{1}{H_2}.
\]

24. Show that surface of Hubble sphere recedes with velocity \( V = c(1 + q) \), where \( q \) is the deceleration parameter. Explain this result.

25. Show that in the accelerated expanding Universe the comoving Hubble radius decreases with time.

26. Show that dependence of the deceleration parameter on the red shift can be presented in the form

\[
q(z) = \frac{1 + z}{H} \frac{dH}{dz} - 1.
\]

27. Show that for a Universe composed of only one component with state equation \( p = w \rho \), the deceleration parameter is

\[
q = \frac{1}{2}(1 + 3w).
\]

28. Show that for a Universe composed of several components with state equations \( p_i = w_i \rho_i \), the deceleration parameter equals

\[
q = \frac{\Omega}{2} + \frac{3}{2} \sum_i w_i \Omega_i,
\]

where \( \Omega \) is total relative density.
29. Show that the Hubble parameter is connected with the deceleration parameter by the integral relation

\[ H = H_0 \exp \left[ \int_0^z \left[ q(z') + 1 \right] d \ln (1 + z') \right] . \]

30. Show that \( k = \text{sign}(\Omega - 1) \).

31. Closed \( (k = 1) \) Friedmannian Universe has current Hubble constant \( H_0 \) and deceleration parameter \( q_0 \). Find lifetime of such a Universe.

32. Closed Friedmannian Universe has current Hubble constant \( H_0 \) and deceleration parameter \( q_0 \). Find maximum size of such a Universe.

33. Consider closed Friedmannian Universe, where radiation is predominant only during negligibly small rate of total lifetime of the Universe. How many times a photon can run round this Universe for the time period from its ”birth” to the ”death”?

34. For a Universe containing only matter \( (p = 0, \ \rho = \rho_0 / a^3) \), show that the solutions of Friedmann equations can be presented in the form

\[ a(\theta) = \frac{4\pi G \rho_0}{3k} (1 - \cos \theta); \ t(\theta) = \frac{4\pi G \rho_0}{3k^{3/2}} (\theta - \sin \theta); \ \ 0 < \theta < 2\pi. \]

35. For a closed Universe filled by matter, find relation between the total life time and the maximum size.

36. Closed Friedmannian Universe has Hubble constant \( H_0 \) and deceleration parameter \( q_0 \). Suppose that in that Universe matter always dominated.

a) What is total proper volume of the Universe in the present epoch?

b) What is total proper volume of space that we embrace by view looking on the sky?

c) What is total proper volume of space presently occupied by matter, that we can see looking at the firmament?

37. For an open Universe \( (k = -1) \), composed only of matter, find dependence \( \rho(t) \) for the period of evolution, when the curvature term dominates in Friedmann equation.

38. In a Universe composed of matter and radiation only obtain exact solutions for Friedmann equations with arbitrary curvature.

39. Let density of some component depends on the scale factor as \( \rho(t) \sim a(t)^{-n} \). How much time is needed for the density of that component to change from value \( \rho_1 \) to value \( \rho_2 \)?

40. Find time dependence for the quantity \( |\Omega - 1| \) for Universe dominated by
a) radiation;

b) matter.

41. Using time dependence of Hubble parameter calculate the deceleration parameter in the cases with domination of

a) radiation;

b) matter.

42. Find solutions of Friedman equations for Miln Universe: open Universe \( k = -1 \) in the limit \( \rho \rightarrow 0 \).

43. For Miln Universe find dependence of density on distance to the horizon, and also total number of galaxies in Euler and Lagrange coordinate systems.

44. Calculate the particle horizon in Universe dominated by

a) radiation;

b) matter.

45. Show that the comoving particle horizon equals the age of Universe in conformal time.

46. Show that if the Universe density is provided by ultra-relativistic matter only, then the particle horizon coincides with the Hubble radius.

47. For flat Universe composed of a single component with the state equation \( p = w \rho \) find dependence of the comoving Hubble radius \( R_H/a \) on the scale factor.

48. Express the comoving particle horizon \( l_p/a \) through comoving Hubble radius \( R_H/a \).

49. As it is known the Universe became transparent at \( z \sim 1100 \) (the moment of neutral hydrogen creation — the recombination process), i.e. when it was about 1100 times smaller in size than today. Therefore in practice our ability to observe the Universe optically is limited by the so-called optical horizon: the maximum distance passed by the light after the recombination moment. Find ratio of the particle horizon to the optical one in the matter-dominated Universe.

50. Show that in an open Universe, filled only by matter, the number of observed galaxies grows with time.

51. Estimate ratio of the Hubble sphere volume to the total volume of Universe, if it is a 3-sphere of radius

\[
R_U = \frac{1}{H_0 \sqrt{\Omega_{\text{curv}}}}.
\]

52. Find dependence of particle horizon on the red shift for a Universe composed of matter and radiation with relative densities \( \Omega_{m0} \) and \( \Omega_{r0} \) respectively.
53. Show that signals, emitted from the cosmological horizon, come to the observer with infinite redshift.

54. Show that even in early Universe the particle horizon scale was much less than the curvature radius. Therefore inside the horizon the curvature plays insignificant role and on that scale Newtonian mechanics is applicable.

55. Show that momentum of a particle decreases with expansion of Universe as \( p(t) \propto a(t)^{-1} \).

56. Does the inertia law hold in the expanding Universe?

57. Let Universe was initially filled by gas of non-relativistic particles with mass density \( \rho_m \) and ration \( c_p/c_v = \gamma \). Construct the state equation for such system.

58. In General Relativity an important role is played by the so-called energy conditions, which impose restrictions on the energy-momentum tensor components basing on physical considerations. For an Universe filled by ideal liquid with density \( \rho \) and pressure \( p \) those conditions take the following form:

- **Zero energy condition:** \( \rho + p \geq 0 \)
- **Weak energy condition:** \( \rho \geq 0, \rho + p \geq 0 \)
- **Strong energy condition:** \( \rho + 3p \geq 0, \rho + p \geq 0 \)
- **Dominant energy condition:** \( \rho \geq 0, -\rho \geq p \geq \rho \)

Express energy conditions in terms of scale factor and its derivatives.

59. Express energy conditions in terms of Hubble parameter and the redshift.

60. Find dependence of conformal time on the scale factor for a Universe dominated by radiation and non-relativistic matter.

61. Find connection between the redshift and time; consider case of the matter-dominated Universe.

62. Find dependence of the scale factor on the conformal time for the radiation-dominated Universe.

63. Express physical time through the conformal one for the radiation-dominated Universe.

64. Find dependence of the scale factor on the conformal time for the matter-dominated Universe.

65. For a flat Universe, filled by mixture of radiation and matter, find scale factor dependence on the conformal time.

66. For the case, when the state equation parameter \( w_i = p_i/\rho_i \) for some component \( i \) depends on time, find dependence of density of that component on the redshift.
67. In an Universe composed of only non-relativistic matter find dependence of Hubble parameter on the redshift.

68. In an Universe composed of only non-relativistic matter find dependence of relative density on the redshift.

69. Show that the deceleration parameter \( q \) can be presented in the form

\[
q(x) = \frac{H'(x)}{H(x)} x - 1, \quad x = 1 + z.
\]

70. Express derivatives \( dH/dz \) and \( d^2H/dz^2 \) through the parameters \( q \) and

\[
r \equiv \frac{\ddot{a}}{aH^3}.
\]

71. Redshift of any object slowly changes due to the acceleration (or deceleration) of the Universe expansion. Find rate of the red shift variation in the non-relativistic matter-dominated Universe.

72. Show that in the flat matter-dominated Universe the visible angular size as a function of distance to the object of linear size \( d \) has minimum at \( z = 1.25 \).

73. Show that if one has a combination of radiation and matter components, then the sound speed is

\[
c_s^2 = \frac{1}{3} \left( \frac{3 \rho_m}{4 \rho_r} + 1 \right)^{-1}.
\]
CHAPTER 3. DYNAMICS OF UNIVERSE IN THE BIG BANG MODEL
Chapter 4

Observational cosmology

'The great tragedy of Science — the slaying of a beautiful hypothesis by an ugly fact.'
T. H. Huxley

1. Calculate the ratio of the Earth illuminance caused by the Sun to illuminance caused by all other light sources in the Universe (the quantitative resolution of the Olbers' paradox).

2. Certain luminous object has visible magnitude $m = 20$, and absolute magnitude $M = 15$. Calculate the distance to this object.

3. Due to accidental coincidence, Balmer series spectral lines of singly ionized helium from distant star overlap with Balmer lines of hydrogen from the Sun. How fast that star is receding from us?

4. The angular size of the elliptic galaxy is $d = 3'$ and certain hydrogen absorption line in its spectrum has wavelength $\lambda = 4866\,\text{Å}$ and width $\sigma = 3\,\text{Å}$. Estimate the galaxy’s mass, supposing that the laboratory wavelength of this line is $\lambda = 4861\,\text{Å}$.

5. Gas cloud is rotating around supermassive black hole $M = 3.6 \times 10^6M_\odot$ (one of the possible interpretations of the recent observations). Supposing that distance between these objects is around 60 light years, estimate the amount of expected Doppler shift.

6. In expanding flat Universe determine the physical distance to the object, that have emitted the light with redshift $z$.

7. Calculate the comoving distance to observable at present time galaxy as a function of the redshift.

8. Solve the previous problem for the flat Universe dominated by the nonrelativistic matter.

9. In expanding flat Universe determine the velocity of the object, that have emitted light with the redshift $z$, which is caused by the expansion.
10. In the Universe, consisted only from nonrelativistic matter, obtain the dependence of the Hubble parameter $H$ on the redshift.

11. In the Universe, consisted only from nonrelativistic matter, determine the dependence of the relative density on the redshift.

12. Construct the method of determination of the scale factor acceleration sign, basing on measuring the characteristics of the explosions of supernovae.

13. Show, that the rate of the redshift for the light, that was emitted at the time $t$ and is registered at present time $t_0$, is determined by the expression

$$
\dot{z} \equiv \frac{dz}{dt_0} = H(t_0)(1 + z) - H(t).
$$

14. In the flat one-component Universe with equation of state $p = w\rho$ the light with redshift $z$ is registered at time $t_0$. For what values of the parameter $w$ the derivative $dz/dt_0 > 0$? Explain the physical meaning of this result.

15. Express photometric distance in the terms of the redshift of the registered radiation.

16. Show that for $z \ll 1$ in the first order by $z$ one obtains $d_L \sim zH_0^{-1}$.

17. Calculate the second order term of the expansion of $d_L$ by $z$ for $z \ll 1$.

18. Redshift for certain radiation source is $z = 0.05$. Estimate the distance to this source.

19. Express the Hubble parameter $H(z)$ in terms of $d_L(z)$.

20. Express angular diameter size $d_A(z)$ in terms of $H(z)$.

21. Express the age of the Universe in terms of $H(z)$.

22. Compare the dependence of refraction angle $\Theta$ on aiming parameter $p$ for optical lens and gravitational lens, using general physical principles and find the differences between them.

23. Obtain the formula $\Theta = r_g/p$ for refraction of light ray using Newtonian theory. $r_g = 2GM/c^2$ is the gravitational radius.

24. Calculate the angle of refraction of light in the gravitational field of the Sun.

25. Propagation of light in gravitational field could be considered as propagation in the ”medium”. Calculate the effective refractive index for such medium.

26. Determine the dependence of ray shifting from the axes of symmetry after the refraction on nontransparent lens. Find the region of shadow and estimate its size, considering the Sun as a lens.
27. What scales of angles and distances determine the position of the images of the light source after the passage through the gravitational lens (GL)? Consider two cases: 1) source and lens are at cosmological distances from observer; 2) distance from observer to lens is much smaller than distance to source.

28. Show that when the gravitational lens is placed between source and observer the two images of the source would be observed in general. How are the images placed relative to the lens and observer?

29. How should source, gravitational lens and observer be placed relative to each other to observe the Einstein ring? Calculate the radius of the ring.

30. How would Einstein ring change, if we take into account the finite size of the source? Estimate the space characteristics of the observed image, assuming, that radius of the lens is much smaller than the radius of the lens.

31. Qualitatively consider the general situation, when source of finite size, lens and observer are not lie on a one line. Estimate the angular sizes of the observed images.

32. Calculate the angular shift of the Einstein ring from the circle of the GL. Estimate it, considering Sun as a lens. Is its value observable?

33. Recently the exceptional phenomenon was observed using the Hubble space telescope: the double Einstein ring, formed by the influence of gravitational field of the galaxy on the light from two other more distant galaxies. What conditions are necessary for the observation of this phenomenon?

34. Determine the dependence of the registered radiation on the distance from observer to nontransparent lens and distance from the systems’ axis of symmetry, assuming that the source is at infinite distance from lens (parallel input beam). What peculiarities does this dependence have? Compare with optical lens.

35. The light from the Earth is billiard times weaker than light from Sun. Not only Earth, but huge planets near other stars (the so-called exoplanets), could not be seen in telescope, even the cosmic, as Hubble telescope—they are completely immersed to the light of the star. Propose the methods of the detections of such planets.

36. Express the probability of finding two galaxies in the infinitely small volumes $dV_1$ and $dV_2$ in the terms of the correlation function $\xi(\vec{r}_1, \vec{r}_2)$, if the mean density of the galaxies in the considered volume is $\bar{n}$ and total number of galaxies is $N$. Specify the main features of the correlation function.

37. Determine the relation between the space $\xi(r)$ and angular $w(\theta)$ correlation function.

38. Show that power law of decrease of space correlations lead to power law for decrease of angular correlations.
39. Calculate the two-point correlation function for $N$ galaxies, that are distributed on the line with mean density $\bar{n}$ on non-overlapping clusters with length $a$. Density of the galaxies inside the cluster is constant and equals $n_c$. Clusters are distributed randomly.

40. Estimate the deceleration of the clock at the surface of the pulsar with parameters $R = 10km$, $M = 10^{30}kg$.

41. White dwarf with mass $M = 0.9M_\odot$ and radius $R = 6000km$ is moving towards the Sun with velocity $v = 60km/s$. In what part of the spectrum are its spectral lines shifted: red or blue?

42. Light emitted from the Sun is observed on the Earth. Determine the shift of the observed frequency comparing to the corresponding frequency of light, emitted by Earth atoms.

43. Some radioactive nucleus emits photons with energy $e = h\nu$. Detector is placed at value $L$ below the emitter. What frequency will be registered by the detector? Estimate the relative shift of the frequency $\Delta\nu/\nu$ in the gravitational field of the Earth with $L = 22.6m$.

44. Determine the gravitational redshift of the light, arrived from the artificial satellite, when observing at the point exactly under the perihelion of the satellite’s orbit.

45. Show that Newtonian mechanics could be formally adopted for the description of the gravitational redshift.

46. The bullet is shot to the Universe with FRW metrics with velocity $v_0$ relative to the cosmological observers. Later, when Universe expands in $(1 + z)^{-1}$ times, bullet has another velocity $v$, relative to the cosmological observers. Express $v$ in terms of $v_0$ and $z$ and show that in the limit $v \to c$ this dependence leads to the formula for redshift of the photons.

47. Calculate the visible bolometric emittance $E$ of the object with given distance $r$, redshift $z$ and bolometric emittance $L$.

48. Find the relation between bolometric star value and redshift for the object with bolometric emittance $L$ in the closed Universe with $\Omega > 1$.

49. Let’s suppose that for the observer, which is in rest relative to the distant motionless stars, the distribution of these stars is isotropic. Another observer is moving with velocity $\beta$ ($c = 1$) in direction $e_x$. What distribution will the second observer see? Does this formulation collide with cosmological principle?

50. If the age of the Universe is $t_0$, how could we see objects that are much more distant than $ct_0$?
51. Optical telescope with effective diameter $D$ is used for research of some light source. The number of photons, that are registered by the unit surface of the telescope in unit time in unit interval of frequencies is $n(\nu)$. The source is observed for time $\Delta t$ in the interval of frequencies $\Delta \nu$. Show that the time necessary for obtaining of certain signal-noise ratio is scaled as $D^{-2}$ if the main source of noise are fluctuations of number of counted photons and $\Delta t \sim \Omega D^{-2}$ ($\Omega$—angle of the observation of considered part of the sky) if the main source of noise is CMB.
Chapter 5

Cosmic Microwave Background (CMB)

1. In 1953 year the article ”Extended Universe and creation of Galactics” by G.A.Gamov was published in ”Det Kongelige Danske Videnskabernes Selskab”. In that paper Gamov took two numbers — age of the world and average density of matter in Universe — and basing on them he determined the third number — the relict radiation temperature. Try to repeat the scientific feat of Gamov.

2. Show that in expanding Universe the quantity $aT$ is an approximate invariant.

3. Find the CMB temperature 1 second after the Big Bang.

4. Why CMB cannot help to heat food like in the microwave oven?

5. Estimate moment of time when the CMB energy density was comparable to that in the microwave oven.

6. Estimate moment of time when the CMB wavelength will be comparable to that in the microwave oven, which is $\lambda = 12.6 \text{ cm}$.

7. The binding energy of electron in hydrogen atom equals 13.6 $eV$. What is temperature of Planck distribution, whose photons have such average energy?

8. Show that creation of the relict radiation (the photon decoupling) took place in the matter-dominated epoch.

9. Let a black hole absorb cosmic background radiation. Express the rate of the black hole mass growth through the black hole mass, cosmic background radiation temperature and fundamental constants.

10. Two stars with presumably equal absolute brightness (standard candles) are observed. Maximum of the observed Planck spectrum for one of them lies at 700$nm$, and for the other — at 350$nm$. What is ratio of their radii?

11. Calculate the presently observed density of photons for the cosmic microwave background and express it in Planck units.
12. Find ratio of CMB photons energy density to that of neutrino background.

13. Determine average energy of the CMB photons in the present time.

14. Why, calculating energy density of electromagnetic radiation in the Universe, we can limit ourself to the CMB photons?

15. Can hydrogen burning in the thermonuclear reactions provide the observed energy density of the relict radiation?

16. Find ratio of relict radiation energy density in the last scattering epoch to the present one.

17. Find ratio of average number densities of photons and baryons in the Universe.

18. Explain qualitatively why temperature of photons on the last scattering surface (0.3 eV) is considerably less than the ionization energy in hydrogen atom (13.6 eV)?

19. Magnitude of dipole component of anisotropy generated by motion of the Solar system relative to the relict radiation equals $\Delta T_d \simeq 3.35 mK$. Determine velocity of the Solar system relative to the relict radiation.

20. Estimate magnitude of annual variations of CMB anisotropy produced by rotation of the Earth around the Sun.

21. Estimate the starting moment of time for the recombination — the transition from ionized plasma to gas of neutral atoms.

22. Determine age of Universe at the moment of photon decoupling (free path for photons is greater than the current observable size of Universe).

23. How will the results of problems 21 and 22 change if we take into account possibility of neutral hydrogen creation in excited states?

   Ionization rate of atomic hydrogen in thermal equilibrium can be described by the Saha equation
   \[
   \frac{1 - X}{X^2} = n\lambda_{Te}^2 e^{I/kT},
   \]
   where $X = n_e/n$ is the ionization rate, $n_e$ and $n$ are concentrations of electrons and atoms (both neutral and ionized) respectively,
   \[
   \lambda_{Te}^2 = \frac{2\pi\hbar^2}{m_e kT},
   \]
   is the electron’s thermal de Broglie wave length and $I = 13.6eV$ is the ionization energy for hydrogen. It is often used in astrophysics for description of stellar dynamics.

24. Using the Saha equation, determine the hydrogen ionization rate

   a) 100 seconds after the Big Bang;
b) at the recombination epoch;
c) at present time.

Assume \( \Omega = 1 \) for simplicity.

25. Determine size of the sound horizon on the last scattering surface.

26. Determine radius of the last scattering surface.

27. Estimate thickness of the last scattering layer.

28. Show that the angular resolution \( \Delta \theta \) is connected with maximum harmonic \( l_{\text{max}} \) (in the spherical harmonics decomposition) by the relation

\[
\Delta \theta = \frac{180^\circ}{l_{\text{max}}}
\]

29. Determine position of the first acoustic peak in the CMB power spectrum, produced by baryon oscillations on the last scattering surface in the Einstein-de Sitter Universe model.

30. Using the fact that the relict radiation has absolutely black body spectrum, show that \( d_L/d_A = (1 + z)^2 \).

31. Estimate probability of the fact that a photon observed on the Earth have already experienced Thomson scattering after the moment when it left the last scattering surface.

32. Show that in the comoving coordinates and conformal time the action for electromagnetic field coincides with that in the flat space-time.

33. Show that the electromagnetic radiation frequency decreases with Universe expansion as \( \omega(t) \propto a(t)^{-1} \).

34. Construct differential equation for the photons distribution function \( \varphi(\omega, t) \) in homogeneous and isotropic Universe.

35. Show that if a radiation spectrum was equilibrium at initial moment of time then it will remain equilibrium during all Universe expansion.

36. Why in the present time the neutrino relict radiation temperature is lower than the one for CMB?

37. Find entropy density for photon gas.

38. Find adiabatic curve equation for CMB.

39. Does the measurement of velocity relative to CMB mean violation of the relativity principle and an attempt to introduce an absolute reference frame?
40. Find the force acting on electron moving through the CMB with velocity $v \ll c$.

41. Estimate characteristic time of energy loss by high-energy electrons with energy of order $100\, GeV$ passing through CMB.

42. Find energy limit upper which the $\gamma$-rays interacting with the CMB should not be observed.

43. The "tired light" models assume graduate energy loss by photons while they travel in Cosmos, resulting in dependence of the red shift on the distance. According to those models there is no Universe expansion. Give arguments refuting models of such type.

44. Show that spontaneous photon decay does not explain the redshift.

45. Relation $\rho_\gamma \propto a^{-4}$ presumes conservation of photon number. Strictly speaking such presumption is inaccurate. The Sun for example emits of order $10^{45}$ photons per second. Estimate accuracy of the presumption about the photon number conservation.

46. Propose a scheme how to use acoustic oscillations of CMB for determination of the Universe geometry.

47. When passing through hot intergalactic gas the relict radiation is scattered on electrons. Estimate variation of the relict radiation temperature due to that process (Zeldovich-Sunyaev effect).
Chapter 6

Thermodynamics of Universe

In the mid-forties G.A.Gamov proposed an idea of ”hot” origin of the World. Therefore thermodynamics was introduced in cosmology, and nuclear physics too. Before him the science about the evolution of Universe contained only dynamics and geometry of the World.

A.D.Chernin.

1. Find energy and number densities for bosons and fermions.

2. Find number densities and energy densities, normalized on the photon energy density at the same temperature, for neutrinos, electrons, positrons, pions and nucleons in the relativistic limit.

3. Find the average energy per particle in relativistic and in non-relativistic limits.

4. Find number of internal degrees of freedom for quark.

5. Find entropy density for bosons and fermions with zero chemical potential.

6. Estimate current entropy density of the Universe.

7. Estimate entropy of the observable part of Universe.

8. Find effective number of internal degrees of freedom for mixture of relativistic bosons and fermions.

9. Generalize results of the previous problem for the case when some $i$-components have temperature $T_i$ different from the CMB temperature $T$.

10. Find effective number of internal degrees of freedom for Standard Model particles at temperature $T > 1 TeV$.

11. Find variation of internal degrees of freedom in the quark hadronization process.

12. Find relation between the energy density and temperature at $10^{10} \, K$. 
13. Find ratio of the energy density at temperature $10^{10} \text{ K}$ to that at temperature $10^8 \text{ K}$.

14. Why the Universe expansion described by Friedman equations is adiabatic?

15. Show that entropy conserves during the Universe expansion described by Friedman equations.

16. Show that in the expanding Universe the entropy density behaves as $s \propto a^{-3}$.

17. Using only thermodynamical considerations, show that if in an expanding Universe the energy density of some component $\rho = const$ than the state equation for that component reads $p = -\rho$.

18. Show that in expanding Universe where relativistic particles dominate the product $aT$ is an approximate invariant.

19. In early flat Universe find temperature dependence for Hubble parameter.

20. Find time dependence of the early Universe temperature by direct integration of the first Friedman equation.

21. Prove that results of the problems 19 and 20 are equivalent.

22. What was the law of time dependence for temperature on early stages of Universe evolution?

23. Determine energy density of Universe at Planck time.

24. Show that in Planck time the energy density in Universe corresponded to $10^{77}$ proton masses in one proton volume.

25. What was the temperature of radiation-dominated Universe at Planck time?

26. Determine age of Universe when its temperature was equal $1 \text{ MeV}$.

27. In the first cyclic accelerator — the cyclotron (1931) — particles were accelerated up to energies of order $1 \text{ MeV}$. In the next generation accelerators — the bevatrons — energy was risen to $1 \text{ GeV}$. In the last generation accelerator — the LHC — protons are accelerated to energy of $1 \text{ TeV}$. What times in the Universe history do those energies allow to investigate?

28. Show, that in epoch when energy density in Universe is determined by ultra-relativistic matter and effective number of internal degrees of freedom does not change, $\dot{T}/T \propto -T^2$.

29. Find dependence of radiation temperature on the red shift in the expanding Universe.

30. Find dependence of free non-relativistic gas temperature on the red shift in the expanding Universe.
31. Estimate the starting moment of time for the recombination — the transition from ionized plasma to gas of neutral atoms. The recombination temperature is \( T_{\text{rec}} \approx 0.3 \, \text{eV} \).

32. Find ratio of thermal capacities of matter in form of monatomic gas and radiation.

33. Expansion of Universe tends to violate thermal equilibrium between the radiation \((T \propto a^{-1})\) and non-relativistic particles gas \((T \propto a^{-2})\). Which of those two components determines the Universe temperature?

34. Show that in the non-relativistic limit \((kT \ll mc^2)\) \( p \ll \rho \).

35. Find entropy density of photon gas.

36. Use the result of previous problem to alternative proof of the fact that \( aT = \text{const} \) in adiabatically expanding Universe.

37. Find the adiabat exponent for photon gas in the expanding Universe.

38. Show that the ratio of CMB entropy density to baryon number density \( s_\gamma/n_b \) remains constant in the process of Universe expansion.

39. Assuming that the ionization rate at last scattering was 10\%, determine the decoupling temperature using the Saha equation.

40. How many iterations in the Saha equation are needed in order to obtain the decoupling temperature with 1K accuracy? Write down analytically the approximate result.

41. Estimate the duration of the recombination epoch: how long did it take for hydrogen ionization rate to change from 90\% to 10\% according to the Saha equation?

42. Using the Saha equation, determine the hydrogen ionization rate in the center of the Sun \((\rho = 100 \, g/cm^3, \, T = 1.5 \cdot 10^7 \, K)\).

43. Estimate the baryon-antibaryon asymmetry

\[
A \equiv \frac{n_b - n_{\bar{b}}}{n_{\bar{b}}}
\]

in early Universe.

44. Define monopoles number density and their contribution to the Universe energy density at the great Unification temperature. Compare the latter with photons energy density at the same temperature.

45. At what temperature and time the contribution of monopoles into the Universe energy density becomes comparable to the contribution of photons?

46. Find ratio of neutrons number density to protons number density in the case of thermal equilibrium between them.
47. Up to what temperature the reaction $n\nu_e \leftrightarrow p e^-$ can support thermal equilibrium between protons and neutrons in the expanding Universe?

48. Determine ratio $n_n/n_p$ at freezing temperature.

49. Determine the age of Universe when it reached the freezing temperature.

50. Determine the time period during which the light elements synthesis took place.

51. At what temperature and at what time efficient deuterium synthesis starts?

52. Determine ratio of neutrons number density to protons number density on temperature interval from freezing temperature to deuterium creation temperature.

53. Determine relative abundance of $^4He$ in Universe.

54. How many helium atoms account for one hydrogen atom?

55. What changes in relative $^4He$ abundance would cause
   a) decreasing of average neutron lifetime $\tau_n$;
   b) decreasing or increasing of the freezing temperature $T_f$?

56. In our Universe the neutron half-value period (the life-time) approximately equals 600 seconds. What would the relative helium abundance be if the neutron life-time decreases to 100 seconds?

57. At what temperature in Universe the synthesis reactions stop?

58. If the Universe is electrically neutral then how many electrons fall on a baryon?

59. What nuclear reactions provided $^4He$ synthesis in early Universe?

60. Why synthesis of elements heavier than $^7Li$ is restricted in early Universe?

61. Explain why thermonuclear processes in first stars considerably influenced the evolution of Universe as a whole?

62. Why the present relative abundance of chemical elements is approximately the same as right after creation of the Solar system?
Perturbation Theory

Perturbation theory in the expanding Universe has a number of specific features. Strictly speaking the theory should be built in frame of General Relativity. However if the inhomogeneities are small, i.e. much less than characteristic sizes in Universe, then we can neglect the effects connected with the curvature and finiteness of the interaction propagation speed, and we can work in frame of Newtonian dynamics.

For description of the fluctuations under interest we will need the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

and Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{\nabla p}{\rho} + \nabla \Phi = 0.$$

Newtonian gravitational potential satisfies the Laplace equation

$$\Delta \Phi = 4\pi G \rho.$$

Non-relativistic theory of small perturbations

1. Express the velocity deflection from the Hubble expansion law in terms of physical and comoving coordinates.

2. Assume that the unperturbed state represents resting gas uniformly distributed in space. Find equations for perturbations in linear approximation.

3. Show that in the case of stationary unperturbed solution the perturbations depend on time exponentially.

4. Consider time-dependent adiabatic perturbations and find characteristic scale of the instability (Jeanse instability).

5. Using results of the previous problem, consider the cases of:

   a) short-wave $\lambda < \lambda_J$;
b) long-wave $\lambda > \lambda_J$; perturbations. Consider the limiting case of short waves $\lambda \ll \lambda_J$.

6. Neglecting entropy perturbations in frames of Newtonian approximation, compose the equation for small relative fluctuations of density

$$\delta = \frac{\delta \rho}{\rho}.$$ 

7. Represent the equation obtained in the previous problem in form of Fourier transform, having excluded Lagrange coordinates. Find order of magnitude for ”physical” Jeans wavelength for matter dominated Universe.

8. Find time dependence for density fluctuations for flat Universe in the case of domination of

a) radiation;

b) matter.

9. Assume that particular solution of the differential equation obtained in the problem has the form $\delta_1(t) \sim H(t)$. Construct the general solution for $\delta(t)$, consider case of flat Universe filled by a medium with the state equation $p = w \rho$.

10. Show that transversal or rotating mode in the expanding Universe tends to decrease.

11. Investigate the quasi-static limit ($a = \text{const}$, $\dot{a} = 0$) of equation for small density fluctuations obtained in the problem.

12. Show that the quasi-static approximation cannot be used to describe evolution of small perturbations of density in the expanding Universe.

13. Find dependence of density fluctuations on the scale factor for flat Universe in the case of domination of

a) radiation;

b) matter.

14. Find time dependence of density fluctuations for closed ($k = 1$) Universe.

15. Find time dependence of density fluctuations for open ($k = -1$) Universe.

16. Analyze stability in the linear approximation of the static solution for Universe composed from matter ($p = 0$) and a substance with the state equation $p = w \rho$.

17. Build correlation function for Fourier components of relative density fluctuations satisfying the cosmological principle.

18. Express the relative density fluctuations correlation function through the power spectrum of those fluctuations.
Relativistic theory of small perturbations

19. Inhomogeneity in matter distribution in Universe induces metric perturbations of different type. In linear approximation the different perturbation types do not interact one with another (evolve independently). Make classification of the perturbations.

20. Point out the number of independent functions needed to describe the perturbations from the previous problem.

21. In the first order in small perturbation $h_{ik}$ calculate components of energy-momentum tensor for the Universe filled by ideal liquid with the state equation $p = w\rho$ and described by metric

$$ds^2 = dt^2 - a^2(t)(\delta_{ik} - h_{ik})dx^i dx^k \quad (i = 1, 2, 3).$$

22. In the linear approximation find transformation of the metric tensor $g_{\mu\nu}$ generated by the coordinate transformation of the form $x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha$, where $\xi^\alpha$ is infinitesimal scalar function.

23. Using results of the previous problem, find the metric perturbations generated by transformation of the form $x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha$, where 4-vector $\xi^\alpha = (\xi^0, \xi^i)$ satisfies the condition $\xi^i = \xi^i_\perp + \zeta^i$, where $\xi^i_\perp$ is a 3-vector with zero divergence $\xi^i_\perp, i = 0$ and $\zeta^i$ is a scalar function.

24. Find the Christoffel symbols for conformally Newtonian reference system with the metrics defined as

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 + 2\Psi)\delta_{ij}dx^i dx^j],$$

where $\Phi$ and $\Psi$ are scalars.

25. Obtain equations resulting from the condition $T^3_{\alpha\beta} = 0$ for a Universe filled by ideal liquid with state equation $p = w\rho$ and described by metrics from the preceding problem.

26. Obtain equation of motion for photon in linear approximation for $\Phi$ in the metrics

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j].$$

27. For a Universe dominated by a substance with the state equation $p = w\rho$, relate CMB fluctuations with the gravity potential in the first approximation.

28. Obtain equation of motion for scalar field $\varphi$ from variational principle in the metrics

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j].$$

29. Obtain the equation for scalar field fluctuations $\tilde{\varphi}(\vec{r}, t) = \varphi(t) + \delta\varphi(\vec{r}, t)$, where $\varphi(t)$ satisfies the Klein-Gordon equation.
Chapter 8

Inflation

...some researchers to question whether inflationary cosmology is a branch of science at all'.
J.Barrow

"Inflation hasn't won the race,
But so far it's the only horse.
A.Linde

Although inflation is remarkably successful as a phenomenological model for the dynamics of the very early Universe, a detailed understanding of the physical origin of the inflationary expansion has remained elusive.
D. Baumann, L. McAllister

Problems of the Hot Universe Model

"Sad the week without a paradox,
a difficulty, an apparent contradiction!
For how can one then make progress?"
John Wheeler

1. Determine the Universe size at Planck temperature (The problem of the universe size).

2. If in the present time deviation of density from the critical one is \( \Delta \) then what was it at \( t \sim t_{P1} \) (The problem of the Universe flatness).

3. Show that both in the radiation-dominated and matter-dominated times the combination \( a^2 H^2 \) is a decreasing function of time. Relate this result with the Universe flatness problem.

4. Show that both in the radiation-dominated and matter-dominated cases \( x = 0 \) is unstable fixed point for the quantity

\[
x \equiv \frac{\Omega - 1}{\Omega}
\]
5. Determine number of causally disconnected regions at the red shift \( z \), represented in our causal volume today.

6. What is expected angular scale of CMB isotropy (The horizon problem)?

7. Show, that in Big Bang model any mechanism of initial inhomogeneities generation violates causality principle.

8. How should early Universe evolve to provide the more fast decrease of initial perturbations \( \lambda_p \) than decrease of Hubble radius \( l_H \), when we move back in time?

9. Show that in the radiation-dominated Universe there are causally disconnected regions at any time moment in the past.

10. Suppose that in some initial moment of time the homogeneity scale in our Universe was greater than the causality scale. Show that in gravitation-dominated Universe such scale relation will preserve in all subsequent moments of time.

11. If the presently observed CMB would be strictly homogeneous, then in what number of causally independent regions should constant temperature be supported in Planck time?

12. Formulate the horizon problem in terms of entropy of the Universe.

13. Let initial homogeneous matter distribution in the Universe is given. The initial velocities must obey Hubble law (otherwise the initially homogeneous matter distribution will be quickly destroyed). What must the accuracy of the initial velocity field determination be in order to preserve the homogeneous matter distribution up to the present time?

14. Estimate the present density of relict monopoles in frame of the Hot Universe model.

15. Show that standard model of Big Bang must include the huge dimensionless parameter—initial entropy of the Universe—as an initial condition.

16. Cyclic model of the Universe is interesting because it avoids the intrinsic problem of the Big Bang model—the "initial singularity" problem. However, as it happens often, avoiding old problems, model produce new. Try to determine the main problems of the cyclic model of Universe.

17. In Big Bang model universe is homogeneous and isotropic. In this model momentum of the particle decreases as \( p(t) \propto a(t)^{-1} \) as Universe expands. At first sight it seems that due to homogeneity of the Universe the translational invariance must ensure the conservation of the momentum. Explain this apparent contradiction.
Inflation theory

"Inflation hasn’t won the race,
But so far it’s the only horse"

Andrei Linde

18. A scalar field $\varphi(r, t)$ in a potential $V(\varphi)$ is described by Lagrangian

$$L = \frac{1}{2} (\dot{\varphi}^2 - \nabla \varphi \cdot \nabla \varphi) - V(\varphi)$$

From the least action principle obtain the equation of motion (evolution) for that field.

19. Using the action for free scalar field connected with gravitation in minimal way

$$S_\varphi = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right)$$

obtain action for that field in the FRW metric.

20. Using the action obtained in the previous problem, obtain evolution equation for scalar field in the expanding Universe.

21. Calculate density and pressure of homogeneous scalar field $\varphi(t)$ in the potential $V(\varphi)$ in the FRW metric.

22. Starting from the scalar field action of the form

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \frac{\partial \varphi}{\partial x_\mu} \frac{\partial \varphi}{\partial x_\nu} - V(\varphi) \right)$$

obtain equation of motion for that field in the case of FRW metric.

23. Construct Lagrange function describing dynamics of Universe filled by scalar field in potential $V(\varphi)$. Using the obtained Lagrangian, obtain the Friedman equations and Klein-Gordon equation.

24. Obtain equation of motion for homogeneous scalar field $\varphi(t)$ in potential $V(\varphi)$ starting from the conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a} (\rho + p) = 0.$$ 

25. Show, that Klein-Gordon equation could be expressed in dimensionless form

$$\varphi'' + (2 - q) \varphi' = \chi; \quad \chi \equiv -\frac{1}{H^2} \frac{dV}{d\varphi},$$

where prime denotes the derivative on $\ln a$, and $q = -\frac{\ddot{a}}{a^2}$ is deceleration parameter.
26. In the equation for scalar field the term $3H\dot{\varphi}$ formally acts as friction term that damps the inflation evolution. However this term does not lead to dissipative energy production, since it origins from the coupling of the scalar field with the background FRW metric. Show that.

27. In the conformal time obtain the system of equations describing the scalar field dynamics in the expanding Universe containing radiation and matter.

28. Calculate pressure of homogeneous scalar field in the potential $V(\varphi)$ using the above obtained energy density of the field and its equation of motion.

29. What condition should the homogeneous scalar field $\varphi(t)$ in potential $V(\varphi)$ satisfy in order to provide accelerated expansion of Universe?

30. What considerations led A.Guth to name the theory describing early Universe dynamics as ”inflation theory”?

31. Obtain equation of motion for homogeneous scalar field $\varphi(t)$ in potential $V(\varphi)$ using analogy with Newtonian dynamics.

32. Show that Friedman equations for scalar field $\varphi(t)$ in potential $V(\varphi)$ can be presented in the form

$$H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right),$$

$$\dot{H} = -4\pi G \dot{\varphi}^2$$

33. Provided the scalar field $\varphi(t)$ is a single-valued function of time, transform the second order equation for the scalar field into a system of first order equations.

34. Express equations for scalar field in terms of the conformal time.

35. Show that the condition $\dot{H} > 0$ can be realized for the scalar field with positively defined kinetic energy.

36. The inflation is defined as any epoch for which the scale factor of Universe has accelerated growth, i.e. $\ddot{a} > 0$. Show that the condition is equivalent to requirement of decreasing of the comoving Hubble radius with time.

37. What conditions should the scalar field satisfy in order to provide close to exponential Universe expansion?

38. Show that in the inflation process the curvature term in Friedman equation becomes unessential. Even if that condition was not initially satisfied, the inflation quickly realizes it.

39. Show that the scalar field equations in flat Universe can be presented in the form

$$3H^2 = V + 2H'^2$$

$$\dot{\varphi} = -2H'$$
where the dot denotes derivative with respect to cosmological time, and prime - with respect to $\varphi$.

40. Obtain the evolution equations for scalar field in expanding universe in the inflationary slow-roll limit.

41. Find the time dependence of scale factor in slow-roll inflation regime for the potential $V(\varphi) = \frac{m^2 \varphi^2}{2}$.

42. Find the dependence of scale factor on the scalar field in the slow-roll regime.

43. Find the particle horizon in the inflationary regime, assuming $H \approx \text{const}$.

44. Show that the conditions for realization of the slow-roll inflation regime can be presented in the form:

$$
\varepsilon(\varphi) \equiv \frac{M_{Pl}^2}{2} \left( \frac{V'}{V} \right)^2 \ll 1; \quad |\eta(\varphi)| \equiv \left| \frac{M_{Pl}^2 V''}{V} \right| \ll 1; \quad M_{Pl}^2 \equiv (8\pi G)^{-1/2}.
$$

45. Show that the conditions $\varepsilon \ll 1$ for realization of the slow-roll inflation regime obtained in the previous question is the sufficient condition for the inflation.

46. Find the slow-roll condition for power law potentials.

47. Show, that condition $\varepsilon \ll \eta$ is satisfied in the vicinity of inflection point of the potential $V(\varphi)$.

48. Show that the inflation parameter $\varepsilon$ can be expressed through the parameter $w$ present in the scalar field state equation.

49. Show that the second Friedmann equation

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)
$$

can be presented in the form

$$
\frac{\ddot{a}}{a} = H^2(1 - \varepsilon).
$$

50. Show that in the slow-roll regime $\varepsilon_H \to \varepsilon$, and $\eta_H \to \eta - \varepsilon$.

51. Show that the inflation parameters $\varepsilon_H$, $\eta_H$ can be presented in the following symmetric form

$$
\varepsilon_H = -\frac{d \ln H}{d \ln a}; \quad \eta_H = -\frac{d \ln H'}{d \ln a}.
$$

52. Prove that the definition of the inflation as the regime for which $\ddot{a} > 0$ is equivalent to the condition $\varepsilon_H < 1$.

53. Show that the inflation appears every time when the scalar field value exceeds the Planck mass.
54. In the model $V(\varphi) = \lambda \varphi^4$ ($\lambda \ll 1$) estimate the interval of scalar field values corresponding to the inflation period.

55. Show that the classical analysis of the Universe evolution is applicable for the scalar field value $\varphi \gg M_{Pl}$, allowing inflation to start.

56. Show that in the inflation period the relative density $\Omega$ exponentially tends to unity.

57. Estimate the Universe temperature in the end of inflation.

58. Estimate the Universe size in the end of inflation.

59. Find the number $N_e$ of $e$-multiple increasing of the scale factor in the inflation period.

60. Find the number $N_e$ of $e$-multiple increasing of the scale factor in the model $V(\varphi) = \lambda \varphi^4$ ($\lambda = 10^{-10}$).

61. Show that inflation transforms the unstable fixed point $x = 0$ for the magnitude $x \equiv \frac{\Omega - 1}{\Omega}$ into the stable one, therefore solving the Universe flatness problem.

62. Find solution of the horizon problem in frames of the inflation theory.

63. Find exact solution of the monopole problem in frame of the inflation theory.

64. Find exact particular solution of the system of equations for the scalar field in the potential $V(\varphi) = r \exp(-\lambda \varphi)$.

65. Compare the solution obtained in previous problem with the solution of evolution equations for scalar field in the expanding Universe in the inflation limit.

66. Find time dependence for the scale factor in the inflation regime for the potential $(1/n)\lambda \varphi^n$, assuming $\varphi \gg M_{Pl}$.

67. The inflation conditions deliberately break near minimum of the inflation potential and Universe leaves the inflation regime. The scalar field starts to oscillate near the minimum. Assuming that the oscillations period is much less than the cosmological time scales, determine effective state equation near the inflation potential minimum.

68. Show, that effective state equation for scalar field, obtained in previous problem, in potential $V \propto \varphi^n$ for $n = 2$ the equation corresponds to non-relativistic matter and for $n = 4$ — to the ultra-relativistic component (radiation).

69. Find energy momentum tensor for homogeneous scalar field in the slow-roll regime.

70. Find the time dependence of scalar field near minimum of the potential.
71. Find energy-momentum tensor of homogeneous scalar field in the regime of fast oscillations near the potential minimum.

72. Show that the dependence \( H(\varphi) = \varphi^{-\beta} \) leads to the so-called intermediate inflation (The Universe expansion goes faster than any power law and slower than the exponential one), at which

\[
a(t) \propto \exp(At^f), \quad 0 < f < 1, \quad A > 0, \quad f = (1 + \beta/2)^{-1}.
\]

73. Show that the dependence

\[
H(\varphi) \propto \exp\left(-\sqrt{\frac{1}{2p} \frac{\varphi}{M_{Pl}}}\right)
\]

leads to the power law inflation \( a(t) = a_0 t^p \).

74. Determine the dynamics of scalar field in potential \( V = 3\sigma H^4 \).

75. Did entropy change during the inflation period? If yes, then estimate what its change was.

76. Does inflation theory explain the modern value of entropy?

77. Formulate the differences between the models of cold and warm inflation.

78. What is the difference between the chaotic inflation model by Linde and its original version by Starobinsky-Guth?

79. The Standard Model of particles allows instability of baryons, which can decay to leptons in frames of the model. But as the rate of baryon number conservation violating processes is very low (for example, the proton lifetime \( \tau_p \geq 10^{30} \text{years} \)), it is extremely difficult to find experimental proofs of the proton instability. Try to formulate the ”inflation proof” of the proton instability.

80. In what way is the inflation theory connected with solution of the problem: why mathematics is so efficient in description of our world and prediction of its evolution?
Chapter 9

Dark Energy

To paraphrase Winston Churchill speaking about democracy, it may be that the cosmological constant is the worst form of accelerating physics, except for all those other forms that have been tried from time to time.

E. Linder

The observed accelerated expansion of Universe requires either modification of General Relativity or existence in frame of the latter of smooth energy component with negative pressure, called the ”dark energy”. That component is usually described with help of the state equation \( p = w \rho \). As it follows from Friedman equation,

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p).
\]

For cosmological acceleration it is needed that \( w < -1/3 \). Current observations give

\[-1.48 < w(t_0) < -0.72\]

with confidence level of 95%. Any physical field with positive energy and negative pressure can play role of dark energy. The allowable region for \( w \) values can be split into three intervals. The first interval \(-1 < w < -1/3\) includes the scalar fields named the quintessence. The substance with the state equation \( p = -\rho \) \( (w = -1) \) was named the cosmological constant, because \( \rho = \text{const} \) in that case. At last the scalar fields with \( w < -1 \) were called the fantom fields.

1. Show that the \( \Lambda \)-term introduced by Einstein (1917)

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

is a constant (the cosmological constant): \( \partial_\mu \Lambda = 0 \).

2. Obtain the Einstein equations in presence of cosmological constant by variation of the gravitational field action with the additional term

\[
S_\Lambda = -\frac{\Lambda}{8\pi G} \int \sqrt{-g} d^4x,
\]

introduction of which is not forbidden by the relativistic invariance requirements.
3. Obtain Friedman equations in presence of cosmological constant.

4. Find natural scales for length and time connected with the introduction of cosmological constant in General Relativity.

5. Show that the relativity principle results in the state equation \( p = -\rho \) for dark energy in form of cosmological constant if it is treated as the vacuum energy.

6. Show that the cosmological constant state equation \( p = -\rho \) ensures Lorentz-invariance of the vacuum energy-momentum tensor.

7. Show that the state equation \( p = -\rho \) is Lorenz-invariant.

8. Does the energy conservation law hold in presence of dark energy in form of cosmological constant?

9. Show that assigning energy to vacuum we do not revive the notion of "ether", i.e. we do not violate the relativity principle or in other words we do not introduce notion of absolute rest and motion relative to vacuum.

10. Obtain analog of the conservation equation \( \dot{\rho} + 3H(\rho + p) = 0 \) for the case when the gravitation constant \( G \) and the cosmological "constant" depend on time.

11. Estimate the upper limit for the cosmological constant. What physical condition determines its lower limit?

12. Suppose that density of the dark energy as cosmological constant is equal to the present critical density, \( \rho_\Lambda = \rho_{cr} \). What is the total amount of dark energy inside the Solar System then? Compare this number with \( M_\odot c^2 \).

13. Knowing age of the oldest objects in Universe determine the lower physical limit of the physical vacuum density.

14. Find the redshift in the cosmological constant dominated flat Universe at which a source of linear size \( d \) has minimum visible angular dimension.

15. Find static solution of Friedman equations with cosmological constant (static Einstein’s Universe).

16. Show that the static Einstein’s Universe must be closed. Find volume and mass of such Universe.

17. Estimate radius of the static Einstein’s Universe if zero-point energy of radiation field is cut off at the classical electron radius.

18. Analyze the static Einstein’s model of Universe and show its instability.

19. Construct the effective one-dimensional potential \( V(a) \) for a flat Universe filled by non-relativistic matter and dark energy in form of cosmological constant.
20. Show that the static Einstein’s Universe may be realized only in the maximum of the effective potential \( V(a) \) (see the previous problem).

21. Derive time dependence of the scale factor for flat Universe with the cosmological constant.

22. Derive time dependence of the scale factor for flat Universe with the \( \Lambda \)-dynamical constant \( \Lambda = \Lambda_0 (1 + \alpha t) \).

23. Show that in the case when the cosmological constant depends on time the energy density related to the latter can be converted into matter.

24. Build dynamics of Universe in the cosmological model where \( \Lambda = \sigma H \), \( \sigma > 0 \).

25. For flat Universe composed of matter and dark energy in form of the cosmological constant with \( \Lambda < 0 \), show that the Universe collapses into point in the time period

\[
t_{\text{col}} = \frac{2\pi}{\sqrt{3|\Lambda|}}
\]

26. Find dependence of dark energy density on the red shift for the state equation \( p_{DE} = w(z)\rho_{DE} \).

27. Find dependence of Hubble parameter on the red shift in a flat Universe filled by non-relativistic matter with current relative density \( \Omega_{m0} \) and dark energy with the state equation \( p_{DE} = w(z)\rho_{DE} \).

28. Show that in a flat Universe filled by non-relativistic matter and arbitrary component with the state equation \( p = w(z)\rho \) the first Friedman equation can be presented in the form:

\[
w(z) = -1 + \frac{1}{3} \frac{d \ln(\delta H^2/H_0^2)}{d \ln(1+z)},
\]

where

\[
\delta H^2 = H^2 - \frac{8\pi G}{3} \rho_m
\]

describes contribution into the Universe expansion rate of any component other than matter.

29. Consider a Universe filled by dark energy with the state equation depending on the Hubble parameter and its derivatives,

\[
p = w\rho + g(H, \dot{H}, \ddot{H}, \ldots; t).
\]

What equation does the Hubble parameter satisfy in that case?

30. Show that taking the function \( g \) (see the previous problem) in the form

\[
g(H, \dot{H}, \ddot{H}) = -\frac{2}{\kappa^2} \left( \dot{H} + \ddot{H} + \omega_0^2 H + \frac{3}{2}(1 + w)H^2 - H_0 \right), \quad \kappa^2 = 8\pi G
\]

leads to the equation for Hubble parameter identical to the harmonic oscillator one, and find its solution.
31. Find time dependence of the Hubble parameter in the case of function $g$ (see the problem 29) taken in the form

$$
g(H; t) = -\frac{2\dot{f}(t)}{\kappa^2 f(t)} H, \ \kappa^2 = 8\pi G$$

where $f(t)$ is arbitrary function of time.

32. Find dependencies $a(\rho)$ and $t(\rho)$ for flat Universe filled by a substance with the state equation $p = -\rho - f(\rho)$.

33. Find dependencies $a(\rho)$ and $t(\rho)$ for flat Universe filled by a substance with the state equation considered in the previous problem in the case $f(\rho) = A\rho^\alpha, \ \alpha = \text{const.}$

34. Express time derivative through the derivative over the red shift.

35. Show that dark energy density with the state equation $p = w(a)\rho$ can be presented as a function of the scale factor in the form

$$\rho = \rho_0 a^{-3(1+\bar{w}(a))},$$

where $\bar{w}(a)$ is the parameter $w$ averaged in the logarithmic scale

$$\bar{w}(a) \equiv \frac{\int w(a)d\ln a}{\int d\ln a}.$$
42. Estimate magnitude of the scalar field variation $\Delta \varphi$ during the Universe evolution time $\Delta t$.

43. Show that in the radiation-dominated or matter-dominated epoch the variation of the scalar field is small, and its smallness measure is given by relative density of the scalar field.

44. Barotropic liquid is the substance for which pressure is single-valued function of density. Is the quintessence generally a barotropic liquid?

45. Show that a scalar field oscillating near a potential minimum is not barotropic substance.

46. Calculate the sound speed in the quintessence field $\varphi(t)$ with potential $V(\varphi)$.

47. Show that in hyperbolic Universe the scalar field potential $V[\varphi(\eta)]$ depends on the conformal time monotonously.

48. Consider spatially flat Universe where non-relativistic matter and spatially homogeneous complex scalar field $\Phi$ dominate. Obtain the equations describing dynamics of such a Universe.

49. Reconstruct the dependence of the scalar field potential $V(a)$ on the scale factor given the dependencies for the field energy density $\rho_{\varphi}(a)$ and the state equation parameter $w(a)$.

50. Relate $\varphi, \rho_{\varphi}, H, V(\varphi)$ with the red shift dependent parameter of the state equation $w(z)$ for Universe composed of non-relativistic matter and quintessence.

51. Reconstruct the state equation parameters for the quintessence basing on the supernova bursts observations.

52. Show that in the quintessence ($w > -1$) dominated Universe the condition $\dot{H} < 0$ always holds.

53. Express the quintessence time derivative $\dot{\varphi}$ through its density $\rho_{\varphi}$ and the state equation parameter $w_{\varphi}$.

54. Often used parametrization for the quintessence equation of state $p = w(a)\rho$ has the following form: $w(a) = w_0 + w_1(1 - a)$. Show that in such parametrization energy density and pressure of the scalar field take the form:

$$\rho(a) \propto a^{-3(1+w_{\text{eff}}(a))}, \quad p(a) \propto (1 + w_{\text{eff}}(a))\rho(a),$$

where

$$w_{\text{eff}}(a) = (w_0 + w_1) + (1 - a)w_1/\ln a.$$

In the models including the dark energy in different forms it is useful to introduce a pair of cosmological parameters $\{r, s\}$, which is called the statefinder:

$$r \equiv \frac{\ddot{a}}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)}.$$
The dimensionless parameters are constructed from the scale factor and its derivatives. The parameter $r$ is the next member in the sequence of the kinematic characteristics describing the Universe expansion (after the Hubble parameter $H$ and the deceleration parameter $q$). The parameter $s$ is the combination of $q$ and $r$ chosen in such way that it is independent on the dark energy density. Values of the parameters can be reconstructed with high precision basing on the available cosmological data. After that the statefinder can be successfully used to identify different dark energy models.

55. Explain the advantages for description of the current Universe dynamics brought by introduction of the characteristics like the statefinder.

56. Express the statefinder \{r, s\} through total density, pressure and their time derivatives for spatially flat Universe.

57. Show that for a flat Universe filled by two-component liquid composed from non-relativistic matter (dark matter + baryons) and dark energy with relative density $\Omega_X = \rho_X/\rho_{cr}$ the statefinder takes the form:

$$r = 1 + \frac{9}{2} \Omega_X w(1 + w) - \frac{3}{2} \Omega_X \frac{\dot{w}}{H};$$

$$s = 1 + w - \frac{1}{3} \frac{\ddot{w}}{wH}; \quad w \equiv \frac{p_X}{\rho_X}.$$

58. Express the statefinder in terms of Hubble parameter $H(z)$ and its derivatives.

59. Find the statefinders
   a) for dark energy in form of cosmological constant;
   b) for the case of time-independent state equation parameter $w$;
   c) for dark energy in form of the quintessence.

60. Express the photometric distance $d_L(z)$ through current values of the parameters $q$ and $s$.

61. Explain the difference between the tracker solutions and attractors in dynamical systems.

62. Show that the initial value of the tracker field must satisfy the condition $\varphi_0 \ll M_{Pl}$.

63. Show that the kinetic and potential energy densities of the scalar field $\varphi$ in the exponential potential $V(\varphi) = M^4 \exp(-\alpha \varphi/M)$, ($M = M_{Pl}^2/16\pi$) are proportional to density of the comoving component (matter or radiation), realizing the tracker solutions.
64. Let a scalar field potential equals

\[ V(\varphi) = \frac{A}{n} \varphi^{-n}, \]

where \( A \) is a dimensional parameter and \( n > 2 \). Show that if \( a(t) \propto t^m \), where \( m = 1/2 \) or \( 2/3 \) (either radiation or non-relativistic matter dominates), then the solution \( \varphi^*(t) \propto t^{2/(n+2)} \) is a tracker one.

65. Show that energy density of the scalar field corresponding to the tracker solution in the potential of problem [64] decreases slower than the energy density of radiation or that of non-relativistic matter.

66. For the scalar field with the potential of problem [64] find the parameter \( w_\varphi \) in the state equation \( p_\varphi = w_\varphi \rho_\varphi \).

67. Using explicit form for tracker field for the potential of problem [64] check the parameter value \( w_\varphi \) obtained in the previous problem.

68. Solving the scalar field equation we assumed that the scale factor time dependence needed to calculate the Hubble parameter in that equation is determined by the dominant component. As the scalar field energy density falls down slower than the energy density of radiation or matter, than, starting from some moment of time the approximated solution obtained this way is no more valid. What is the scalar field value for the potential of problem [64] at that time moment?

69. Consider an oscillating scalar field as a model of dark energy.

70. Consider a quintessence model representing the scalar field in vicinity of the potential minimum.

71. Show that for the case \( w < -1/3 \) (dark energy) the horizon does not exist because the corresponding integral diverge.

72. Show that for \( -1 \leq w < -1/3 \) number of observed galaxies decreases with time.

73. Find the scale factor dependence for density of the substance with the state equation \( p = -A/\rho \), \( A > 0 \) (the Chaplin gas).

74. Show that in early Universe the Chaplin gas behaves as matter with zero pressure, and in later times — as the cosmological constant.

75. Find the range of the energy density \( \rho_{ch} \) corresponding to accelerated expansion of Universe filled by dark energy in form of Chaplin gas with the pressure \( p = -A/\rho_{ch} \) and non-relativistic matter with density \( \rho_m \).

76. Show that the sound speed in the Chaplin gas behaves as \( c_s \propto t^2 \) in the matter-dominated period.
77. Find dependence of density on the scale factor in the generalized Chapligin gas model with the state equation

\[ p = -\frac{A}{\rho^\alpha}, \quad (A > 0, \alpha > 0). \]

78. In the generalized Chapligin gas model (see the previous problem) find the state equation parameter \( w \).

79. Define the sound speed in the generalized Chapligin gas model (see problem 77). Can it exceed the light speed?

80. Let us present the energy density in the generalized Chapligin gas model of problem 77 in form of the sum \( \rho_{\text{ch}}^{(\text{gen})} = \rho_{DM} + \rho_{DE} \), where \( \rho_{DM} \) is the component with properties of non-relativistic matter \( (p_{DM} = 0) \) and \( \rho_{DE} \) is the component with properties of dark energy described by the state equation \( p_{DE} = w_{DE} \rho_{DE} \). Find restriction on the model parameter \( w_{DE} \).

81. Find dependence of density on the scale factor for Chapligin gas model with the state equation

\[ p = (\gamma - 1) \rho - \frac{A}{\rho^\alpha}, \quad 0 \leq \alpha \leq 1 \]

(the so-called modified Chapligin gas.)

82. Construct effective potential for the Chapligin gas considering it as a scalar field. Do the same for the generalized Chapligin gas (see problem 77) and modified Chapligin gas from previous problem.

*The fantom fields \( w < -1 \) on finite times in future lead to divergences of the scale factor \( a \), Hubble parameter \( H \), its derivative \( \dot{H} \) and scalar curvature. This singularity was initially called the Big Smash, and then the Big Rip. Although mechanisms to avoid the Big Rip are presently known, the possible variants of the events development (the density of Universe) make an interest.*

83. Show that the scalar field action minimally connected to gravitation

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right] \]

under the condition \( \varphi^2/2 < V(\varphi) \) leads to the condition \( w_\varphi < -1 \), i.e. the field is fantom.

84. Obtain equation of motion for the fantom scalar field described by action from the previous problem.

85. Show that the fantom energy density grows with time. Find dependence \( \rho(a) \) for \( w = -4/3 \).

86. Make the divergencies classification for the substance described by the state equation \( p = -\rho - A\rho^\alpha \).
87. Show that in the fantom scalar field \((w < -1)\) dominated Universe the condition \(\dot{H} > 0\) always holds.

88. Show that in a Universe filled by substance with the state equation
\[
p = w \rho + \frac{2}{\kappa^2} H \frac{\dot{f}(t)}{f(t)}; \quad \kappa^2 = 8\pi G, \quad f(t) = H_1 + H_2 \sin \omega_0 t, \quad H_1 > H_2 > 0,
\]
the non-fantom \((\dot{H} < 0)\) and fantom \((\dot{H} > 0)\) epochs multiply change each other.

89. Show that for flat Universe filled by non-relativistic matter with density \(\rho_m\) and fantom field \(\varphi\) with density \(-\varphi^2/2 + V(\varphi)\) the Hubble parameter satisfies the equation
\[
H^2 = \frac{8\pi G \rho_m + V(\varphi)}{3 (1 + \frac{8\pi G}{6} \varphi'^2)},
\]
where
\[
\varphi' = \frac{d\varphi}{d \ln a}.
\]

90. For the flat Universe composed of matter \((\Omega_m \approx 0.3)\) and fantom energy \((w = -1.5)\) find the time period separating us from the Big Rip.

91. Show that in the fantom component of dark energy the sound speed exceeds the light speed.

**Immediate consequence of approaching the Big Rip is the dissociation of bound systems due to negative pressure inside them. In order to study the Universe expansion effect on the bound systems it is necessary (in the simplest case) to build the test particle dynamics in vicinity of the point mass \(M\) put into the expanding cosmological background. Adequate metric for that problem is the one representing the interpolation between the static Schwartzschild metric in vicinity of \(M\) and FRW metric on large distances.**

92. Show that in Universe filled by non-relativistic matter a hydrogen atom will remain a bounded system forever.

93. Show that the condition of finite motion on an orbit of radius \(R\) around an object of mass \(M\) takes the following form
\[
-\frac{4\pi}{3} (\rho + 3p)R^3 \leq M.
\]

94. Show that for \(w \geq -1\) a system gravitationally bound in some moment of time (Milky Way for example) remains bound forever.

95. Show that in the fantom energy dominated Universe any gravitationally bound system will dissociate with time.

96. Build the metric representing the approximation between the Schwartzschild metric on small distances (in vicinity of mass \(M\)) and FRW metric on large distances.
97. Find the geodesic equation for the metric build in the previous problem.

98. Show that the geodesic equation found in the previous problem is equivalent to Newton equation with time-dependent potential.

99. Let the state equation for the dark energy is $p_{DE} = w(z)\rho_{DE}$. Neglecting the radiation in the flat Universe case express $w(z)$ through the current value of relative density of matter $\Omega_{m0}$ and Hubble parameter $H(z)$.

100. Build the fantom energy model with negative kinetic term in the potential satisfying the slow-roll conditions

$$\frac{1}{V} \frac{dV}{d\varphi} \ll 1$$

and

$$\frac{1}{V} \frac{d^2V}{d\varphi^2} \ll 1.$$  

101. Let parameter $w$ in the state equation for dark energy varies in the interval $-1.2 < w < -0.8$. Plot $\rho_{DE}(1+z)(GeV^4)$ in double logarithmic scale.

102. Consider possibility to observe the dark energy in laboratory experiments at researching of quantum noises in dissipative systems.
Chapter 10
Dark Matter

Since the original suggestion of the existence of dark matter, the evidence has become overwhelming. The question has changed from "Does dark matter exist?" to "What is this most common of substances?"
G. Jungman, M. Kamionkowski, K. Griest

In the beginning of thirties of the last century a Swiss cosmologist F. Zwicky applied the virial theorem (in the gravitational field $2\langle E_{\text{kin}} \rangle + \langle E_{\text{pot}} \rangle = 0$) in order to estimate the Coma cluster (Berenice’s Hair). He was surprised to discover that in order to support finite motion of the galaxies belonging to the cluster its mass must be at least two orders of magnitude greater than the observed mass (in form of the luminous galaxies). He was the first to introduce the term "dark matter" which strongly entered into the modern cosmology vocabulary. In the present time the term is understood as non-baryon matter component which neither emits nor absorb electromagnetic waves in any range.

1. Analyze the problem of small satellite galaxy moving in gravitational field of a large galaxy (The problem of non-decreasing behavior of the rotational curves.)

2. What radial dependence of spherically symmetric density of galactic halo corresponds to constant velocity of satellite galaxies?

3. Using the virial theorem, express mass of a galaxy cluster through observed quantities — average velocity of galaxies in the cluster and its size. Estimate mass of Coma cluster (Berenice’s hair) for $R \approx 10^{23} m$, $\langle v^2 \rangle^{1/2} \approx 2 \cdot 10^6 m/s$.

4. Show that age of matter dominated Universe contradicts the observations.

5. A galaxy has visible mass of $10^{11} M_\odot$ and horizontal rotation curve up to distance of $30 kpc$ at velocity $250 km/s$. What is the dark to visible mass ratio in the galaxy?

6. Show that conservation of thermal equilibrium for certain component in the expanding Universe is possible only under the condition $\Gamma \ll H$, where $\Gamma$ is rate of the reaction needed to support the equilibrium.
7. Find lower limit for mass of the dark matter particles in the case when they are:
   a) bosons;
   b) fermions.

8. Find lower limit for mass of the fermion particles composing a compact spherical object of dark matter with radius $R$ and mass $M$.

9. Show that particles composing galaxy of mass $M = 10^{10} M_\odot$ and radius $R = 3kpc$ must be non-relativistic.

10. In assumption that neutrinos have mass $m_\nu$ and decoupling temperature $T_D \simeq 1MeV$, define their contribution to the presently observed energy density.

   If WIMPs represent dark matter they will not just generate the background density of Universe, but they will also cluster together with usual baryon matter in galactic halo. In particular they will be presented in our own galaxy — the Milky Way. It gives hope to detect the relict WIMPs immediately on the Earth. What those hopes are connected with? By definition WIMPs do not interact with photons. However WIMPs could annihilate into usual matter (quarks and leptons) in early Universe. Otherwise they would have too high relative abundance today. Due to the crossover symmetry the temperature of annihilation of e.g. WIMPs into quarks is connected with the amplitude of elastic scattering of WIMPs on quarks. Therefore WIMPs must have rather weak but non-zero interaction with usual matter. Due to that interaction WIMPs can elastically scatter on target nuclei and lead to recoil of detector nuclei partially transferring energy to them. Therefore search of events of elastic scattering of WIMPs on detector nuclei is one of prospective ways of galactic dark matter research.

11. Show that at high temperatures $T \gg m_\chi$ ($m_\chi$ is the WIMPs mass) the ratio of equilibrium densities of WIMPs and photons is constant: $n_\chi^{eq}/n_\gamma^{eq} = const$.

12. Show that Boltzmann equation describing evolution of WIMPs number density $n_\chi$ in particle number preserving interactions leads to the usual relation for non-relativistic matter $n_\chi \propto a^{-3}$.

13. Show that in early Universe for $T \geq m_\chi$ the WIMPs number density follows its equilibrium value at temperature decreasing.

14. What temperature does neutrino density ”freezing” take place at?

15. Considering the WIMPs as the thermal relicts of early Universe, estimate their current density.

16. Estimate local density of dark halo in vicinity of the Earth, assuming that the halo density decreases as $\rho_g = C/r^2$.

17. Build a model of the spherically symmetric dark halo density corresponding to the observed galactic rotation curves.
18. In frames of the halo model considered in the previous problem determine local dark matter density $\rho_0$ basing on the given rotation velocities of satellite galaxy on the outer border of the halo $v_\infty \equiv v(r \to \infty)$ and in some point $r_0$.

19. For the halo model considered in the problem 17 obtain dependencies $\rho(r)$ and $v(r)$ in terms of $\rho_0$ and $v_\infty$. Plot the dependencies $\rho(r)$ and $v(r)$ for that model.

20. Many clusters are sources of X-ray radiation. It is emitted by hot intergalactic gas filling the cluster volume. Assuming that the hot gas ($kT \approx 10keV$) is in equilibrium state in the cluster with linear size $R = 2.5\,Mpc$ and core radius $r_c = 0.25\,Mpc$, estimate mass of the cluster.

21. Estimate the WIMPs flow on the Earth’s surface.

22. What are main processes thanks to which the WIMPs can be detected?

23. Show that WIMPs of mass $\sim 100GeV$ being elastically scattered on Xenon nuclei with mass $\sim 130GeV$ lead to energy recoil $\leq 40keV$.

24. Show that the WIMPs elastic scattering experiments will be the most efficient if the target nuclei mass is compared to the WIMPs mass.

25. Determine the minimum velocity of WIMPs that can transmit energy $Q$ to a nucleus with mass $m_N$.

26. Estimate the counting rate for the detector fixing elastic events of WIMPs.

27. Obtain expression for the counting rate of the detector fixing elastic events of WIMPs.

28. Reconstruct one-dimensional WIMPs distribution over velocities $f_1(v)$ basing on the given counting rate of the elastic events.

29. Build a model-independent scheme for WIMPs mass determination using the results of WIMPs elastic scattering events detection for two and more sets of experimental data with detectors of different composition.

30. Show that if WIMP has mass of order $100\,GeV$ and velocity of order $300\,km/s$ then it coherently interacts with nucleons of detector nuclei.

31. Build the WIMP-nucleus total cross-section determining the elastic events counting rate.

32. How will number of counts for the elastic events detector be affected by transition to heavier target nuclei at fixed detector mass?

33. Show that if the WIMPs mass is of order $100\,GeV$ then the elastic spin-independent cross-section for WIMPs on a nucleus with $A \sim 100$ is eight orders of magnitude larger then the corresponding cross-section on a nucleon.
34. What consequences follow from the $R$-parity conservation?

35. Estimate annual modulations amplitude of the WIMPs elastic scattering cross-section.

36. Estimate diurnal modulations amplitude of the WIMPs elastic scattering cross-section.

37. Find velocity change for a spaceship rotating around the Earth with period $T$ as the result of dark matter particles scattering on nuclei of particles composing the spaceship.

38. Assume that due to interaction with dark matter particles the spaceship velocity changed on $\Delta v$ for one turn. Given its mass is $m$ and it moves around the Earth on a circular orbit, estimate dark matter density in the Earth neighborhood.

39. Estimate how much will the period of rotation of the Earth around the Sun change in one year due to gravitational capture of dark matter particles.

40. Estimate rate of energy outcome due to the WIMP-annihilation process using the parameter values $m_{WIMP} = 100\text{GeV}$ and $\langle \sigma v \rangle_{\text{ann}} = 3 \cdot 10^{-26}\text{cm}^3/\text{sec}$ for the annihilation cross section.

41. It is theorized that the dark matter particles annihilation processes could be a competitive energy source in the first stars. Why those processes played an important role only in early Universe and they are not important nowadays?

   Dark energy is the main component of Universe energy budget, thus it is necessary to consider possibility of its interaction with other Universe components, in particular with the component second in importance — the dark matter. Additional interest to that possibility is connected with the fact that in its frame it is possible to solve the so-called "coincidence problem": the coincidence (in the present time) of dark energy and dark matter densities by order of magnitude (0.7 and 0.3 respectively). As the nature of those two components is yet unknown we cannot describe interaction between them starting from the first principles and we are forced to turn to the phenomenology. In the base of the phenomenology one can put the conservation equation

   $$\dot{\rho_i} + 3H(\rho_i + p_i) = 0.$$  

   In the case of interaction between the components it is necessary to introduce the interaction (a source) into the right-hand side of the equation. It is naturally to assume that the interaction is proportional to the energy density multiplied by a constant of inverse time dimension. For that constant it is naturally to choose the Hubble constant.

42. Construct a model of Universe containing only interacting dark energy and dark matter, while their total energy density is conserved.

43. In frame of model of the problem 42 find effective state parameters $w_{\text{eff}}^{(\varphi)}$ and $w_{\text{eff}}^{(m)}$ allowing to treat the components as non-interacting.
44. Show that in frames of the model of Universe described in the problem 42 the Klein-Gordon equation for the scalar field takes the form:
\[ \ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = -\Gamma; \quad \Gamma \equiv \frac{Q}{\dot{\varphi}} \]
Here \( Q \) is the constant of interaction between dark energy and dark matter.

45. Study the model from the problem 42 for the exponential potential
\[ V(\varphi) = V_0 \exp(-k\lambda\varphi), \]
where \( k^2 \equiv 8\pi G, \lambda \) is a dimensionless constant and \( V_0 > 0 \).

46. In the model of problem 42 find scale factor dependence for the dark matter density assuming that the constant of interaction between the dark matter and the dark energy equals \( Q = \delta(a)H\rho_{DM} \).

47. In the model of problem 42 find scale factor dependence for the dark matter and the dark energy density assuming that \( Q = \delta H\rho_{DE} \) (\( \delta = const \)).

48. Show that in the model of interacting dark matter and dark energy in form of quintessence the relation \( R \equiv \rho_m/\rho_\varphi \) satisfies the equation
\[ \dot{R} = 3Hw_\varphi + \frac{Q}{\rho_\varphi}(1 + R). \]

49. Let the dark energy state equation be \( p_{DE} = w\rho_{DE} \), where \( w = const \). In frame of problem 42 find dependence of the dark energy density on the scale factor, assuming that \( \rho_{DM} = \rho_{DM0}a^{-3+\delta} \), where \( \delta \) characterizes deviation of the dark matter density evolution from the standard one (in absence of the interaction).

50. Let in model of interacting dark energy and dark matter the densities ratio has the form
\[ \frac{\rho_{DM}}{\rho_{DE}} \propto Aa^{-\xi}. \]
Determine the interaction \( Q \) between the components.

51. Determine the statefinder \( \{r, s\} \) (see Chapter 9) in the model of interacting dark energy and dark matter with the interaction intensity \( Q = -3\alpha H \).

52. Consider a flat Universe filled by dark energy in form of the Chapligin gas (\( p_{ch} = -A/\rho_{ch} \)) and dark matter. Let the components interact with each other with intensity \( Q = 3\Gamma H\rho_{ch} \) (\( \Gamma > 0 \)). Show that for large \( a \) (\( a \to \infty \)) the parameter \( w_{ch} \equiv p_{ch}/\rho_{ch} < -1 \), i.e. in such a model the Chapligin gas behaves as the fantom energy.

53. Interaction between the dark matter and the dark energy leads to non-conservation of matter, or equivalently to scale dependence for mass of particles composing the dark matter. Show that in frame of the model considered in the problem 42 the relative change of particles mass per Hubble time equals the interaction constant.
54. Consider a model of flat homogeneous and isotropic Universe filled by matter (baryon and dark), radiation and negative pressure component (dark energy in form of quintessence). Assuming that the baryon matter and radiation conserve separately and the dark components interact with each other, describe dynamics of such system.

55. Assume that the dark matter particles mass $m_{DM}$ depends on scalar field $\phi$. Construct the model of interacting dark energy and dark matter in that case.

56. Find equation of motion for the scalar field interacting with the dark matter whose particles mass depends on the scalar field.
Chapter 11

Standard Cosmological Model

"The history of cosmology shows that in every age devout people believe that they have at last discovered the true nature of the Universe”

E.R. Harrison

On the frontier between XX and XXI centuries the Standard Cosmological Model (SCM) became the dominant model of Universe. It is based on two the most important observational results:
1. Accelerated expansion of Universe
2. Euclidean character of space geometry.

The theoretical basis of the SCM is the General Relativity theory. Besides that it is assumed that early Universe is adequately described by the inflation theory (see Chapter 8). SCM fixes a row of Universe parameters and in particular its energy composition. According to SCM in the present Universe the two components dominate — the dark energy (in form of cosmological constant $\Lambda$) and the cold dark matter ($CDM$). Therefore the model was named $\Lambda CDM$.

1. Calculate the dark energy density in the SCM. Find the cosmological constant value in the model.

2. Using the SCM parameters find dependence of relative density of dark energy $\Omega_\Lambda$ on the red shift. Plot $\Omega_\Lambda(z)$.

3. Estimate number of stars in the Universe described by SCM.

4. Find ratio of dark energy density in SCM to energy density of electric field of intensity $1V/m$. Compare the dark energy density with gravitational field energy density on the Earth surface.

5. Estimate distance between two neutral hydrogen atoms at which the gravitational force of their attraction is balanced by the repulsion force generated by dark energy in form of the cosmological constant. Make the same estimates for the Sun-Earth system.

6. Can an open Universe recollapse or a closed Universe expand forever?
7. Current observations show that the Universe is flat with high precision: $|\Omega_{\text{curv}}| < 0.02$. Consider a hypothetic situation: all dark energy in the Universe instantly disappears, but the curvature remains as small as it is, and $k = +1$ ($\Omega_{\text{curv}} < 0$) (closed Universe). What ratio of scale factors $a(t)/a_0$ corresponds to the moment when expansion change to contraction in such a Universe?

8. Calculate physical acceleration magnitude in the SCM.

9. Using the SCM parameters, determine the red shift value corresponding to equality of radiation and matter densities.

10. Rewrite the first Friedmann equation in the SCM in terms of red shift and analyze contributions of separate components on different stages of Universe evolution.

11. Construct effective one-dimensional potential (see problem 15 of Chapter 3) corresponding to the SCM parameters.

12. In the SCM described Universe photons with $z = 0.1, 1, 100, 1000$ are registered. What was the Universe age $t_U$ in the moment of their emission? What period of time $t_{\text{ph}}$ were the photons on the way? Plot $t_U(z)$ and $t_{\text{ph}}(z)$.

13. In the flat expanding Universe described by SCM determine the physical distance to the object emitted light with red shift $z$.

14. Find the time dependence for the scale factor in the SCM and analyze asymptotes of the dependence. Plot $a(t)$.

15. Find age of Universe corresponding to the SCM parameters.

16. Give a qualitative explanation why the age of Universe in SCM is considerably greater than the age of matter dominated Universe (Einstein-de Sitter model).

17. Let light is emitted at time moment $t$ and is registered at time moment $t_0$ with red shift $z$. Find and plot the dependence of emission time on the red shift $t(z)$.

18. What will relative density of dark energy be a billion years later?

19. Find variation rate for relative density of dark energy in the SCM. What are asymptotic values of the quantity? Plot it.

20. Estimate size of the cosmological horizon in the SCM.

21. Find time dependence of Hubble parameter in the SCM. Plot the dependence.

22. Find asymptotic (in time) value of the Hubble parameter in SCM.

23. In the present time age of Universe $t_0 \simeq 13.7$ billion years is close to the Hubble time $t_H = H_0^{-1} \simeq 14$ billion years. Is it true for any moment in evolution of SCM described Universe that its age $t^*$ satisfies the relation $t^* \simeq t_H(t^*) = H^{-1}(t^*)$?
24. What value of state parameter is needed for the dark energy in order to make its current value of relative density to provide accelerated expansion of Universe in the present time?

25. Find current value of the deceleration parameter in the SCM.

26. For the SCM described Universe find the redshift dependence of the deceleration parameter. Analyze the limiting cases.

27. Find and plot time dependence of the deceleration parameter in the SCM.

28. For the SCM described Universe find the time moment when dark energy starts dominate over dark matter. What red shift does it correspond to?

29. For the Universe described by the SCM determine the time moment and red shift corresponding to the transition from decelerated Universe expansion to the accelerated one took place.

30. Solve the previous problem using the derivative \( d\eta/d\ln a \).

31. Is dark energy domination necessary for transition to accelerated expansion of Universe?

32. For flat Universe composed of matter and dark energy in form of cosmological constant find connection between the red shift corresponding to equality of densities for both the components \( \rho_m(z_{eq}) = \rho_\Lambda(z_{eq}) \) and the red shift value corresponding to start of the accelerated Universe expansion.

33. Imagine that in the SCM described Universe the dark energy was instantly switched off. Analyze further dynamics of the Universe in that case.

34. Estimate the density of dark energy in form of cosmological constant using the Hubble diagram (see fig.11.1) for vicinity of the Local group.

35. Estimate the Local group mass by methods used in the previous problem.

36. Find "weak" points in the argumentation of the two preceding problems.

37. For the SCM described Universe estimate time when the "cosmology end" sets in, i.e. when it will be impossible to see any galaxy on the sky.

38. Product of the Universe age and current Hubble parameter (the Hubble’s constant) is a very important test (Sandage consistency test) of internal consistency for any model of Universe. Analyze the parameters on which the product \( H_0 t_0 \) depends in the Big Bang model and in the SCM.

39. Show that for a fixed source of radiation the luminosity distance for high red shift values for flat Universe dominated by dark energy will be greater compared with the matter dominated flat Universe.
Figure 11.1: To problem 34 Hubble diagram for vicinity of the Local group.
40. In the experiments discovered the accelerated expansion of Universe the researches in particular detected two 1a type supernovae: 1992P, $z = 0.026$, $m = 16.08$ and 1997ap, $z = 0.83$, $m = 24.32$. Show that those observables accord with the SCM parameters.

41. Compare the observed value of the dark energy density with the one expected from the dimensionality considerations (the cosmological constant problem).

42. Determine density of vacuum energy using Planck scale as the cutoff parameter.

43. Identifying the vacuum fluctuations density with the observable dark energy value in SCM, find the frequency cutoff magnitude in the fluctuation spectrum.

44. Determine the inflation period duration in frames of SCM.

45. Plot the dependence of luminosity distance $d_L$ (in units of $H_0^{-1}$) on the red shift $z$ for the two-component flat Universe with non-relativistic liquid ($w = 0$) and cosmological constant ($w = -1$). Consider the following cases:

   a) $\Omega_0^\Lambda = 0$;
   b) $\Omega_0^\Lambda = 0.3$;
   c) $\Omega_0^\Lambda = 0.7$;
   d) $\Omega_0^\Lambda = 1$.

46. Determine position of the first acoustic peak in the CMB power spectrum, produced by baryon oscillations on the last scattering surface in the SCM.

47. Compare asymptotes of time dependence of the scale factor $a(t)$ for the SCM and de Sitter models. Explain physical reasons of their distinction.

48. Redshift for any object slowly changes due to the acceleration (or deceleration) of the Universe expansion. Estimate change of velocity in one year period in the SCM described Universe.

49. If the Universe is described by SCM then what is lower limit of ratio of total Universe volume to the observed one?

50. What is difference between the inflationary expansion in early Universe and the present accelerated expansion?

51. Compare the values of Hubble parameter in the beginning of the inflation period and in the beginning of the present accelerated expansion of Universe.

52. Consider the quantity

$$O(x) = \frac{h^2(x) - 1}{x^3 - 1},$$

where $x = 1 + z$, $h(x) = H(x)/H_0(x)$. Show that if the state parameter $w = const$ then $O(x) = \Omega_{0m}$ for cosmological constant, $O(x) > \Omega_{0m}$ for quintessence and $O(x) < \Omega_{0m}$ for phantom energy, and therefore $O(x)$ can be used to probe the dark energy state equation.
53. Plot the dependencies \( h(x) = H(x)/H_0(x) \), \( x = 1 + z \) in the SCM, for quintessence and for phantom energy.