CHARACTERIZATION OF $n$-DIMENSIONAL NORMAL AFFINE $\text{SL}_n$-VARIETIES

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ABSTRACT. We show that any normal irreducible affine $n$-dimensional $\text{SL}_n$-variety $X$ is determined by its automorphism group seen as an ind-group in the category of normal irreducible affine varieties. In other words, if $Y$ is an irreducible affine normal algebraic variety such that $\text{Aut}(Y) \simeq \text{Aut}(X)$ as an ind-group, then $Y \simeq X$ as a variety. If we drop the condition of normality on $Y$, then this statement fails. In case $n \geq 3$, the result above holds true if we replace $\text{Aut}(X)$ by $U(X)$, where $U(X)$ is the subgroup of $\text{Aut}(X)$ generated by all one-dimensional unipotent subgroups. In dimension 2 we have some interesting exceptions.

1. Introduction and Main Results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine variety $X$ the automorphism group $\text{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions and results in Section 2. The classical example is $\text{Aut}(\mathbb{A}^n)$, $n > 1$, the group of automorphisms of the affine $n$-space $\mathbb{A}^n$. Recently, HANSPEETER KRAFT proved the following result which shows that the affine $n$-space is determined by its automorphism group (see [Kr15]).

**Theorem 0.** Let $Y$ be a connected affine variety. If $\text{Aut}(Y) \simeq \text{Aut}(\mathbb{A}^n)$ as ind-groups, then $Y \simeq \mathbb{A}^n$ as varieties.

Note that this result was generalised in [CRX19] (see also [KrS19] and in a similar spirit, see [LRU20, Theorem 1]) where the authors proved Theorem 0 under a weaker condition, namely, that groups $\text{Aut}(Y)$ and $\text{Aut}(\mathbb{A}^n)$ are isomorphic only as abstract groups. Moreover, recently Theorem 0 was generalized in [LRU18, Theorem 1.4] (see also [RvS21, Main Theorem 1]) where it was proved that an affine toric variety different from the algebraic torus is determined by its automorphism group seen as an ind-group in the category of normal affine irreducible varieties. If we drop the normality condition in [LRU18, Theorem 1.4], the situation changes. In this paper we show that for “most” $n$-dimensional affine normal varieties $X$ endowed with a non-trivial regular $\text{SL}_n = \text{SL}_n(\mathbb{C})$-action, there are infinitely many affine varieties $Y$ such that $\text{Aut}(Y) \simeq \text{Aut}(X)$ as an ind-group and we classify all such $Y$.

Let $d > 1$. Consider the action of $\mu_d = \{\xi \in \mathbb{C}^* | \xi^d = 1\}$ on $\mathbb{A}^n$ by scalar multiplication and denote by $A_{d,n}$ the quotient of $\mathbb{A}^n$ by $\mu_d$. Note that $A_{d,n}$ is normal. Denote also by $\pi: \mathbb{A}^n \to A_{d,n}$ the quotient map. This means that $A_{d,n}$ is

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an affine variety with coordinate ring
\[ \mathcal{O}(A_{d,n}) = \mathbb{C}[x_1, \ldots, x_n]^{\mu_d} = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk}, \]
the algebra of invariants, where \( \mathbb{C}[x_1, \ldots, x_n]_{dk} \) denotes the homogeneous polynomials of degree \( dk \). Note that \( A_{d,n} \) is indeed an orbit space, because \( \mu_d \) is finite. For \( d > 1 \), \( A_{d,n} \) has an isolated singularity in \( \pi(0) \) and \( \pi \) induces an étale covering \( \mathbb{A}^n \setminus \{0\} \to A_{d,n} \setminus \{p(0)\} \) with Galois group \( \mu_d \).

**Remark 1.** We will see in Lemma 6 and Proposition 4 that any affine normal variety endowed with a regular non-trivial \( SL_n \)-action is isomorphic to either \( SL_2/T \), \( SL_2/N \) or to \( A_{d,n} \) for some \( d \in \mathbb{N} \), where \( T \subset SL_2 \) is the standard subtorus and \( N \subset SL_2 \) is the normalizer of \( T \). This implies that Theorem 1 and Theorem 2 below indeed provide the characterization of \( n \)-dimensional normal affine \( SL_n \)-varieties.

Consider the affine variety \( A^n_{d,n} \) with coordinate ring
\[ \mathcal{O}(A^n_{d,n}) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk} \subset \mathcal{O}(A_{d,n}), \quad s \geq 1. \]
Then the induced morphism \( \eta: A_{d,n} \to A^n_{d,n} \) is the normalization and has the property that the induced map \( \eta': A_{d,n} \setminus \{\ast\} \xrightarrow{\sim} A^n_{d,n} \setminus \{\ast\} \) is an isomorphism, where \( \ast \) denotes the points corresponding to the homogeneous maximal ideals. In fact, \( \eta \) is \( SL_n \)-equivariant, and \( A_{d,n} \setminus \{\ast\} \) is an \( SL_n \)-orbit. We prove the following result.

**Theorem 1.** Let \( X \) be an irreducible affine variety. Then \( Aut(X) \) and \( Aut(A_{d,n}) \) are isomorphic as ind-groups if and only if \( X \simeq A^n_{d,n} \) as a variety for some \( s \in \mathbb{N} \).

Theorem 1 and the following result shows that \( SL_2/T \) and \( SL_2/N \) are the only affine \( n \)-dimensional \( SL_n \)-varieties (except \( \mathbb{A}^n \)) that are determined by their automorphism groups in the category of affine irreducible varieties.

**Theorem 2.** Let \( X \) be an irreducible variety such that \( Aut(X) \simeq Aut(SL_2/T) \) respectively \( Aut(X) \simeq Aut(SL_2/N) \) as ind-groups. Then \( X \simeq SL_2/T \) respectively \( X \simeq SL_2/N \) as varieties.

For an affine variety \( X \) we denote by \( U(X) \subset Aut(X) \) the subgroup generated by the one-dimensional unipotent subgroups. We do not know whether \( U(X) \) has the structure of an ind-subgroup (i.e., whether \( U(X) \subset Aut(X) \) is closed). That is why we introduce the definition of an algebraic isomorphism. This is an isomorphism \( \phi: U(X) \xrightarrow{\sim} U(Y) \) such that for any subgroup \( U \subset U(X) \), where \( U \) is a closed one-dimensional unipotent subgroup of \( Aut(X) \), the image \( \phi(U) \subset Aut(Y) \) is a closed one-dimensional unipotent subgroup and \( \phi|_U: U \xrightarrow{\sim} \phi(U) \) is an isomorphism of algebraic groups.

**Theorem 3.** Let \( X \) be \( A_{d,n} \), \( SL_2/T \) or \( SL_2/N \) and \( Y \) be an irreducible affine variety. Assume that there is an algebraic isomorphism \( U(X) \xrightarrow{\sim} U(Y) \). Then
(a) if \( X \simeq A_{2,s} \), then \( Y \simeq SL_2/T \) or \( Y \simeq A^n_{2,s} \) for some \( s \in \mathbb{N} \),
(b) if \( X \simeq SL_2/T \), then \( Y \simeq SL_2/T \) or \( Y \simeq A^n_{2,s} \) for some \( s \in \mathbb{N} \),
(c) if \( X \simeq A_{4,s} \), then \( Y \simeq SL_2/N \) or \( Y \simeq A^n_{4,s} \) for some \( s \in \mathbb{N} \),
(d) if \( X \simeq SL_2/N \), then \( Y \simeq SL_2/N \) or \( Y \simeq A^n_{4,s} \) for some \( s \in \mathbb{N} \),
(e) if \( X = A_{d,n} \), where \( (d,n) \notin \{(2,2),(2,4)\} \), then \( Y \simeq A^n_{d,n} \) for some \( s \geq 1 \).
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2. Preliminaries

2.1. Ind-groups. The notion of an ind-group goes back to Shafarevich who called such objects *infinite dimensional groups* (see [Sh66]). We refer to [Kum02] and [FK18] for basic notions in this context.

**Definition 1.** By an *ind-variety* we mean a set $V$ together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ such that the following holds:

1. $V = \bigcup_{k \in \mathbb{N}} V_k$;
2. each $V_k$ has the structure of an affine algebraic variety;
3. for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism from an ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ to an ind-variety $W = \bigcup_{m \in \mathbb{N}} W_m$ is a map $\phi: V \to W$ such that for any $k$ there is an $m$ such that $\phi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $V = \bigcup_{k \in \mathbb{N}} V_k$ and $V' = \bigcup_{k \in \mathbb{N}} V'_k$ are called *equivalent* if for every $k$ there is an $m$ such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$.

An ind-variety $V$ has a natural topology: a subset $S \subset V$ is open (resp. closed), if $S_k = S \cap V_k \subset V_k$ is open (resp. closed), for all $k$. A locally closed subset $S \subset V$ has the induced structure of an ind-variety. It is called an *ind-subvariety*. A subset $S \subset V$ that is a closed subset of some $V_k$ is called an *algebraic subset*.

The product of two ind-varieties is defined in the usual way. This allows to give the following definition.

**Definition 2.** An ind-variety $G$ is said to be an *ind-group* if the underlying set $G$ is a group such that the map $G \times G \to G$, $(g,h) \mapsto gh^{-1}$, is a morphism.

An ind-group $G$ is called *connected* if for every $g \in G$ there is an irreducible curve $C$ and a morphism $C \to G$ whose image contains the neutral element $e$ and $g$.

A closed subgroup $H$ of $G$ (i.e., $H$ is a subgroup of $G$ and is a closed subset) is again an ind-group under the closed ind-subvariety structure on $G$. A closed subgroup $H$ of an ind-group $G$ is called an *algebraic subgroup* if and only if $H$ is an algebraic subset of $G$.

**Proposition 1** ([FK18, Theorem 0.3.1]). Let $X$ be an affine variety. Then $\text{Aut}(X)$ has the structure of an ind-group such that for any algebraic group $G$, there is a correspondence between regular $G$-actions on $X$ and ind-group homomorphisms $G \to \text{Aut}(X)$.

If $G$ is an algebraic group acting regularly and faithfully on $X$, then, by Proposition 1, we can consider $G$ as an algebraic subgroup of $\text{Aut}(X)$. We will often switch between these two points of view.
2.2. Locally nilpotent derivations and $\mathbb{G}_a$-actions. Additive group actions on affine varieties can be described by a certain kind of derivations. We recall some of the basics here (see [Fre06] for details). Let $\lambda: \mathbb{G}_a \to \text{Aut}(X)$ be a $\mathbb{G}_a$-action on an affine variety $X$. Such an action induces a derivation on the level of regular functions $\mathcal{O}(X)$ by

$$\delta_\lambda: \mathcal{O}(X) \to \mathcal{O}(X), \quad f \mapsto \left[ \frac{d}{ds} \lambda(s)^*(f) \right]_{s=0},$$

where $\mathbb{G}_a = \text{Spec}(\mathbb{C}[s])$. We have that for every $f \in \mathcal{O}(X)$ there exists an $k \in \mathbb{N}$ with $\delta_\lambda^k(f) = 0$. Derivations that have such a property are called locally nilpotent. Moreover, every $\mathbb{G}_a$-action on $X$ arises from a certain locally nilpotent derivation $\delta$ and the $\mathbb{G}_a$-action $\lambda_d: \mathbb{G}_a \times X \to X$ is obtained from $\delta$ via

$$(\alpha_s(\lambda))^*: \mathcal{O}(X) \to \mathcal{O}(X)[s], \quad f \mapsto \exp(s\delta)(f) := \sum_{i=0}^{\infty} \frac{s^i \delta^i(f)}{i!}.$$

3. Automorphisms

Proposition 2. Let $\pi: \mathbb{A}^n \to A_{d,n}$. Then every automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{A}^n$ which commutes with each element of $\mu_d$.

Proof. The quotient map $\pi: \mathbb{A}^n \to A_{d,n}$ induces a natural embedding of $\mathcal{O}(A_{d,n})$ into $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, \ldots, x_n]$. So, we assume that $\mathcal{O}(A_{d,n})$ is a subring of $\mathbb{C}[x_1, \ldots, x_n]$ and equals $\bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{dk}$, where $\mathbb{C}[x_1, \ldots, x_n]_{dk}$ denotes the homogeneous polynomials of degree $dk$. Let $\phi \in \text{Aut}(A_{d,n})$. First we claim that $p_i = \phi^*(x_i^d)$ and $p_j = \phi^*(x_j^d)$ are coprime in $\mathbb{C}[x_1, \ldots, x_n]$, where $i \neq j$ and $\phi^*$ is the pull-back of $\phi$.

Let $p$ be a common factor of $p_i$ and $p_j$. Then $\tilde{p} = \prod_{q \in \mu_d} qp$ divides $p_i^d$ and $p_j^d$. By construction it is clear that $\tilde{p} \in \mathcal{O}(A_{d,n})$. Hence, $(\phi^*)^{-1}(\tilde{p})$ is a common factor of $(\phi^*)^{-1}(p_i^d) = x_i^d$ and $(\phi^*)^{-1}(p_j^d) = x_j^d$. Therefore, $\tilde{p} \in \mathbb{C}$ and then, $p \in \mathbb{C}$.

We have

$$\phi^*(x_i^d)\phi^*((x_j^d)^{d-1}) = \phi^*(x_i^d x_j^{d(d-1)}) = \phi^*(x_i x_j^{d-1})^d$$

which means that $p_i p_j^{d-1} = q^d$ for some $q \in \mathcal{O}(A_{d,n})$. Because $p_i$ is coprime with $p_j$, it follows that $p_i = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \ldots, x_n]$.

Since for an automorphism $\phi: A_{d,n} \to A_{d,n}$ we have that $\phi^*(x_i^d) = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \ldots, x_n]$, we define the morphism

$$\hat{\phi} = (q_1, \ldots, q_n): \mathbb{A}^n \to \mathbb{A}^n$$

given by the map

$$(x_1, \ldots, x_n) \mapsto (q_1, \ldots, q_n).$$

If $\phi$ is the identity automorphism, then the restriction of $\hat{\phi}^*$ to $\mathcal{O}(A_{d,n})$ is the identity and we have that $q_i^d = x_i^d$ which implies that $q_i = w_i x_i$ for some $w_i \in \mathbb{C}^*$, $w_i^d = 1$. In this case $\hat{\phi}$ is an automorphism, i.e., the trivial automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{A}^n$ which we denote by $\Delta(w_1, \ldots, w_n)$.

Let now $\theta: A_{d,n} \to A_{d,n}$ be the inverse automorphism of $\phi \in \text{Aut}(A_{d,n})$. Since $\phi \circ \theta$ is the trivial automorphism of $A_{d,n}$, it lifts to an automorphism $\Delta(w_1, \ldots, w_n)$ of $\mathbb{A}^n$ and so $\hat{\phi}$ is an automorphism with the inverse $\hat{\theta} \circ \Delta(w_1, \ldots, w_n)^{-1}$. To finish the proof we need to show that $\hat{\phi}$ commutes with $\mu_d$. Indeed, since $\hat{\phi}^*$ preserves
\( \mathbb{C}[x_1, \ldots, x_n]^{\mu_d} \), we have that \( q^d_i(\xi_1, \ldots, \xi_n) = q^d_i(x_1, \ldots, x_n) \), where \( \xi \in \mu_d \).

Hence,

\[
q_i(\xi_1, \ldots, \xi_n) = \xi_1^l q_i(x_1, \ldots, x_n)
\]

for some \( l = 1, \ldots, d-1 \). Since polynomial \( q_i \) has a linear summand it follows that \( l = 1 \). The proof follows. \( \square \)

Let \( X \) be an affine variety, \( H \) be a finite group that acts faithfully on \( X \) and let \( \pi: X \to X/H \) be the quotient morphism. Since \( H \) acts faithfully, \( H \) naturally embeds into \( \text{Aut}(X) \) and we identify \( H \) with its image in \( \text{Aut}(X) \). Denote by \( \text{Aut}^H(X) \subset \text{Aut}(X) \) the subgroup of all automorphisms of \( X \) which normalize \( H \), i.e., the subgroup of those automorphisms \( \phi \) such that \( \phi^{-1} \circ H \circ \phi = H \).

**Lemma 1.** (a) There is a canonical homomorphism of groups \( \phi: \text{Aut}^H(X) \to \text{Aut}(X/H) \).

(b) If \( X \) is normal and contains only finitely many fixed points of \( H \) then every \( \mathbb{C}^+ \)-action on \( X/H \) lifts to a \( \mathbb{C}^+ \)-action on \( X \).

**Proof.** (a) Let \( h \in H \), \( f \in \mathcal{O}(X)^H \) and \( \phi \in \text{Aut}^H(X) \). Then \( \phi^*: \mathcal{O}(X) \to \mathcal{O}(X) \) is an isomorphism and

\[
h(\phi^*(f)) = \phi^*((\phi^*)^{-1} \circ h \circ \phi^*)(f) = (\phi^* \circ h')(f) = \phi^*(f)
\]

for some \( h' \in H \). Therefore \( \phi^*(f) \in \mathcal{O}(X)^H \), which means that \( \phi \) induces an automorphism of \( X/H \).

(b) follows from [MM09, Theorem 1.3]. \( \square \)

Let us recall that a closed algebraic subgroup \( U \) of \( \text{Aut}(X) \) is a 1-dimensional unipotent subgroup if \( U \cong \mathbb{C}^+ \).

**Proposition 3.** The homomorphism \( \phi_d: \text{Aut}^{nd}(\mathbb{A}^n) \to \text{Aut}(A_{d,n}) \) is surjective with kernel \( \mu_d \). Moreover, every 1-dimensional unipotent subgroup of \( \text{Aut}(A_{d,n}) \) is the image of some 1-dimensional unipotent subgroup of \( \text{Aut}^{nd}(\mathbb{A}^n) \).

**Proof.** The surjectivity of \( \phi_d \) follows from Proposition 2. The last claim of the statement follows from Lemma 1 (b). What remains is to compute the kernel of \( \phi_d \).

It is clear that

\[
\text{Aut}^{nd}(\mathbb{A}^n) = \{ f = (f_1, \ldots, f_n) \in \text{Aut}(\mathbb{A}^n) \mid f_i \in \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{kd+1}, i = 1, \ldots, n \}.
\]

Now let \( f = (f_1, \ldots, f_n) \in \text{Aut}^{nd}(\mathbb{A}^n) \) be such that the map \( f' \) induced by \( f \) on \( \mathbb{A}^n/\mu_d \) is the identity. This means that \( f' \) acts trivially on

\[
\mathcal{O}(\mathbb{A}^n/\mu_d) = \mathbb{C} \oplus \bigoplus_{k \geq 1} \mathbb{C}[x_1, \ldots, x_n]_{kd}.
\]

Hence, \( f'(x^d_i) = x^d_i \) for any \( i \) which implies that \( f = (\xi_1 x_1, \ldots, \xi_n x_n) \), where \( \xi^d_i = 1 \) for \( i = 1, \ldots, n \). In particular, \( f'(x^d_{i-1} x_j) = x^{d-1} x_j \) which implies that \( \xi^d_{i-1} \xi_j = 1 \) for any \( i, j \). Because \( \xi^d_{i-1} \xi_i = 1 \) we conclude that \( \xi_i = \xi_j \). The claim follows. \( \square \)
4. Root subgroups

Let $G$ be an ind-group, and let $T \subset G$ be a closed torus.

**Definition 3.** A closed subgroup $U \subset G$ isomorphic to $\mathbb{C}^+$ and normalized by $T$ is called a **root subgroup** with respect to $T$. The **character** of $T$ on $\text{Lie} U \simeq \mathbb{C}$ i.e., the algebraic action of $T$ on $\text{Lie} U$ is called the **weight character** of $U$.

Let $X$ be an affine variety and consider a nontrivial algebraic action of $\mathbb{C}^+$ on $X$, given by $\lambda : \mathbb{C}^+ \to \text{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{C}^+$-invariant, then the **modification** $f \cdot \lambda$ of $\lambda$ is defined in the following way:

$$(f \cdot \lambda)(s) x = \lambda(f(x)) s x$$

for $s \in \mathbb{C}$ and $x \in X$. It is easy to see that this is again a $\mathbb{C}^+$-action. In fact, the corresponding locally nilpotent derivation to $f \cdot \lambda$ is $f \delta_{\lambda}$, where $\delta_{\lambda}$ is the locally nilpotent derivation which correspond to $\lambda$ (see Section 2.2 for details). It is clear that if $X$ is irreducible and $f \neq 0$, then $f \cdot \lambda$ and $\lambda$ have the same invariants. If $U \subset \text{Aut}(X)$ is a closed subgroup isomorphic to $\mathbb{C}^+$ and if $f \in \mathcal{O}(X)^U$ is a $U$-invariant, then in a similar way we define the modification $f \cdot U$ of $U$. Choose an isomorphism $\lambda : \mathbb{C}^+ \to U$ and set

$$f \cdot U = \{(f \cdot \lambda)(s) \mid s \in \mathbb{C}^+\}.$$  

Note that $\text{Lie}(f \cdot U) = f \text{Lie} U \subset \text{Vec}(X)$, where $\text{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on $X$, i.e., $\text{Vec}(X) = \text{Der}(\mathcal{O}(X))$, the Lie algebra of derivations of $\mathcal{O}(X)$.

If a torus $T$ acts linearly and rationally on a vector space $V$, then we call $V$ **multiplicity-free** if the weight spaces $V_\alpha$ are all of dimension less than or equal to 1.

**Lemma 2** (Lemma 6.2, [Kr15]). Let $X$ be an irreducible affine variety and let $T \subset \text{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subset \text{Aut}(X)$ with respect to $T$ such that the $T$-module $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.

The next result is going to be of use in the sequel and can be found in [Lie11, Theorem 1]. We denote by $\text{SAut}(\mathbb{A}^n)$ the subgroup of $\text{Aut}(\mathbb{A}^n)$ of the following form

$$\{f = (f_1, \ldots, f_n) \in \text{Aut}(\mathbb{A}^n) \mid \text{jac}(f) = \det \left[ \frac{\partial f_i}{\partial x_j} \right]_{i,j} = 1\}$$

and by $T'_n$ a maximal subtorus of $\text{SAut}(\mathbb{A}^n)$ of the form

$$\{(t_1 x_1, \ldots, t_n x_n) \mid t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1\}.$$  

**Lemma 3.** Let $U \subset \text{SAut}(\mathbb{A}^n)$ be a one-dimensional unipotent subgroup. Then $U$ is a root subgroup with respect to $T'_n$ if and only if $U = U_\lambda = \{(x_1, \ldots, x_i + cm_i, \ldots, x_n) \mid c \in \mathbb{C}\}$, where $m_i = x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n}$. The character $\xi_\lambda$ corresponding to the root subgroup $U$ is the following: $\xi_\lambda : T'_n \to \mathbb{C}^*, t = (t_1, \ldots, t_n) \mapsto t_1 t_2^{-\lambda_1} \cdots t_i^{-\lambda_1} \cdots t_n^{-\lambda_n}$.

**Remark 2.** The last lemma can also be expressed in the following way (see [KS13, Remark 2]): there is a bijective correspondence between the $T'_n$-stable one-dimensional unipotent subgroups $U \subset \text{Aut}(\mathbb{A}^n)$ and the characters of $T'_n$ of the
form $\lambda = \sum_j \lambda_j \epsilon_j$ where one $\lambda_i$ equals 1 and the others are $\leq 0$. We will denote this set of characters by $X_u(T'_n)$:

$$X_u(T'_n) = \{ \lambda = \sum \lambda_j \epsilon_j \mid \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i \}.$$ 

If $\lambda \in X_u(T'_n)$, then $U_\lambda$ denotes the corresponding one-dimensional unipotent subgroup normalized by $T'_n$.

5. A Special Subgroup of $\text{Aut}(X)$

For any affine variety $X$ consider the normal subgroup $U(X)$ of $\text{Aut}(X)$ generated by closed one-dimensional unipotent subgroups. The group $U(X)$ was introduced and studied in [AFK13], where the authors called it the group of special automorphisms of $X$. Following [Kr15], we introduce the notion of an algebraic homomorphism between these groups.

**Definition 4.** A homomorphism $\phi: U(X) \to U(Y)$ is algebraic if for any subgroup $U \subset U(X)$ such that $U \subset \text{Aut}(X)$ is closed, $U \cong \mathbb{C}^*$, the image $\phi(U) \subset \text{Aut}(Y)$ is closed and $\phi|_{U}: U \to \phi(U)$ is a homomorphism of algebraic groups. We say that $\phi$ is an algebraic isomorphism if $\phi$ is an isomorphism of groups and $\phi|_{U}: U \xrightarrow{\sim} \phi(U)$ is an isomorphism of algebraic groups.

A subgroup $G \subset U(X)$ is called algebraic if $G \subset \text{Aut}(X)$ is a closed algebraic subgroup. The next lemma can be found in [Kr15, Lemma 4.2].

**Lemma 4.** Let $\phi: U(X) \to U(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subset U(X)$ generated by one-dimensional unipotent subgroups of $\text{Aut}(X)$, the image $\phi(G)$ is an algebraic subgroup of $U(Y)$ and $\phi|_{G}: G \to \phi(G)$ is a homomorphism of algebraic groups.

Let $X$ be an affine variety and let $\eta: \tilde{X} \to X$ be a normalization map. It is well-known that any automorphism of $X$ lifts uniquely to the automorphism of $\tilde{X}$. Indeed, for a given automorphism $\phi: X \to X$, the composition $\phi \circ \eta: \tilde{X} \to X$ is a morphism, which by the universal property of normalization factors through a morphism $\tilde{\phi}: \tilde{X} \to \tilde{X}$ such that $\phi \circ \eta = \tilde{\eta} \circ \tilde{\phi}$. It remains to argue that $\tilde{\phi}$ is an automorphism. But for the same reason, the inverse $\phi^{-1}$ lifts to an automorphism $\psi: \tilde{X} \to \tilde{X}$. Since $\eta: \tilde{X} \to X$ is birational, the compositions $\psi \circ \tilde{\phi}$ and $\tilde{\phi} \circ \psi$ are equal to the identity on a dense open subset of an irreducible variety, hence everywhere. This shows that $\eta$ induces a well-defined injective homomorphism $\tilde{\eta}: \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X})$. Moreover, in [FK18, Proposition 12.1.1] it is proved that $\tilde{\eta}$ is a closed immersion of ind-groups, i.e., $\tilde{\eta}(\text{Aut}(X)) \subset \text{Aut}(\tilde{X})$ is a closed subgroup and $\tilde{\eta}$ induces the isomorphism of ind-groups $\text{Aut}(X) \xrightarrow{\sim} \tilde{\eta}(\text{Aut}(X))$. Hence, we have the following statement.

**Lemma 5.** Let $X$ be an irreducible affine variety, and let $\eta: \tilde{X} \to X$ be its normalization. Then every automorphism of $X$ lifts uniquely to an automorphism of $\tilde{X}$ and the induced map $\tilde{\eta}: \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X})$ is a closed immersion of ind-groups.

**Proposition 4.** Let $n \geq 3$ and let $X$ be an $n$-dimensional irreducible affine variety endowed with a non-trivial $\text{SL}_n$-action. Then

$$\mathcal{O}(X) = \sum_{i=1, \ldots, l} \bigoplus_{k \geq 0} \mathbb{C}[x_1, \ldots, x_n][t^i].$$
for some \( d_1, \ldots, d_l \in \mathbb{N} \), where \((d_1, \ldots, d_l) = d\) and the normalization of \( X \) is isomorphic to \( A_{d,n} \). The same holds when \( n = 2 \) and the normalization of \( X \) is \( A_{d,2} \) for some \( d \in \mathbb{N} \).

**Proof.** First, let \( n \geq 3 \). If \( X \) is normal, then from [KRZ20, Theorem 1.6 and Proposition 4.4(2), see also Example 4.5] it follows that \( X \simeq A_{d,n} \) for some \( d \in \mathbb{N} \). It is well known that 

\[
O(A_{d,n}) = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{kd}
\]

is a direct sum of irreducible pairwise non-isomorphic \( SL_n \)-modules \( \mathbb{C}[x_1, \ldots, x_n]_{kd} \).

Now, consider any \( n \)-dimensional irreducible affine variety \( X \) endowed with a non-trivial \( SL_n \)-action and a normalization morphism \( \eta: A_{d,n} \to X \). Since any \( SL_n \)-action on \( O(X) \) lifts to an \( SL_n \)-action on \( O(A_{d,n}) \), it follows that \( O(X) \) is a \( SL_n \)-submodule of \( O(A_{d,n}) \) and therefore 

\[
O(X) = \bigoplus_{k \in \Omega} \mathbb{C}[x_1, \ldots, x_n]_{kd},
\]

where \( \Omega \) is a submonoid of \( \mathbb{N} \) under addition. Since \( O(X) \) is finitely generated, \( \Omega \subset \mathbb{N} \) is a finitely generated submonoid i.e., there exist \( d_1, \ldots, d_l \in \mathbb{N} \) such that

\[
\Omega = d_1 \mathbb{N} + \cdots + d_l \mathbb{N}.
\]

The claim follows. \( \Box \)

### 6. 2-DIMENSIONAL CASE

#### 6.1. Two dimensional normal affine \( SL_2 \)-surfaces.

The next result can be found in [Pop73], §3 (see also [Giz71] and [Kr84], §4).

**Lemma 6.** Let \( X \) be an affine normal irreducible variety of dimension two endowed with a non-trivial \( SL_2 \)-action. Then \( X \) is \( SL_2 \)-equivariantly isomorphic to one of the following varieties:

- (a) \( A_{d,2} \) for some \( d \in \mathbb{N} \), where \( SL_2 \)-action on \( A_{d,2} \) is induced by the standard \( SL_2 \)-action on \( \mathbb{A}^2 \),
- (b) \( SL_2 / T \), where \( T \) is the standard subtorus of \( SL_2 \) and \( SL_2 \) acts on \( SL_2 / T \) by left multiplication,
- (c) \( SL_2 / N \), where \( N \) is the normalizer of \( T \) and \( SL_2 \) acts on \( SL_2 / N \) by left multiplication.

The \( SL_2 \)-action on \( SL_2 / T \) and on \( SL_2 / N \) from Lemma above is transitive. The \( SL_2 \)-variety \( A_{d,2} \) is the union of a fixed point and the orbit \((\mathbb{A}^2 \setminus \{0\})/ \mu_d \simeq SL_2 / U_d \), where \( \mu_d \) acts by scalar multiplication on \( \mathbb{A}^2 \setminus \{0\} \) and 

\[
U_d = \left\{ \begin{bmatrix} \xi & t \\ 0 & \xi^{-1} \end{bmatrix} \right| t \in \mathbb{C}, \xi \in \mathbb{C}^*, \xi^d = 1 \right\}.
\]

Moreover, any closed subgroup of \( SL_2 \) of codimension less than or equal to 2 is conjugate to either \( T \), or \( N \), or \( U_d \) for some \( d \geq 1 \), or \( B = \left\{ \begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} \right| t \in \mathbb{C}, a \in \mathbb{C}^* \}
\]

(see for example [Pop73, page 803]).

The next result can be found in [Kr84, III.2.5, Folgerung 3].

**Proposition 5.** If a reductive group \( G \) acts on an affine variety \( X \) and if the stabilizer of a point \( x \in X \) contains a maximal torus, then the orbit \( Gx \) is closed.
Proposition 6. Let $X$ be a two-dimensional $\text{SL}_2$-variety and let $O = \text{SL}_2 x$ be the orbit of $x \in X$. Then we are in one of the following cases:

(a) $x$ is a fixed point;
(b) the orbit $O$ is closed and $\text{SL}_2$-isomorphic to $\text{SL}_2 / T$ or $\text{SL}_2 / N$;
(c) $\overline{O} = O \cup \{x_0\}$, where $\overline{O}$ is the closure of the orbit $O$ and $x_0$ is a fixed point. Moreover, either $\overline{O} \simeq \mathbb{A}^2$ or $x_0$ is an isolated singular point.

Proof. If the stabilizer of $x$ contains a maximal torus then we are in case (a) or (b) by Proposition 5 and Lemma 6. Otherwise, from the classification of closed subgroups of $\text{SL}_2$ it follows that the stabilizer of $x$ coincides with $U_d$ for some $d \geq 1$ and $\overline{O}$ does not contain orbits of dimension one. Hence, $\overline{O} = O \cup \{x_0\}$. It is clear that if $\overline{O}$ is singular, then $x_0$ is an isolated singular point. If $\overline{O}$ is smooth, then from Lemma 6 it follows that $\overline{O}$ is isomorphic to $\mathbb{A}^2$. □

Remark 3. Note that $\text{SL}_2 / T \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where $\Delta$ is the diagonal, and $\text{SL}_2 / N \simeq \mathbb{P}^2 \setminus C$, where $C$ is a smooth conic (see [Pop73, Lemma 2]).

6.2. The structure of $\text{Aut}(\text{SL}_2 / T)$. The variety $\text{SL}_2 / T$ is isomorphic to the following so-called Danielewski surface, i.e., the smooth 2-dimensional affine quadric $V(xz - y^2 + y) \subset \mathbb{A}^3$ (see [DP09]) and the quotient map $\pi: \text{SL}_2 \to \text{SL}_2 / T$ is given by $\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \mapsto (ab, ad, cd)$. It is not difficult to see that $X = V(xz + y^2 - 1) \simeq V(xz - y^2 + y) \subset \mathbb{A}^3$. From now on and until end of Section 6 we identify $\text{SL}_2 / T$ with $X = V(xz + y^2 - 1)$.

Consider the orthogonal group $O_3 = \text{O}(3, \mathbb{C})$ associated with the quadratic form $y^2 + xz$ generated by $\tau: \mathbb{A}^3 \to \mathbb{A}^3$ given by the following map: $(x, y, z) \mapsto (-x, -y, -z)$ and by the group $\text{SO}_3 = \text{SO}(3, \mathbb{C})$ that is composed of the matrices

$$\frac{1}{ad - bc} \begin{pmatrix} a^2 & 2ab & -b^2 \\ ac & ad + bc & -bd \\ -c^2 & 2cd & d^2 \end{pmatrix} \text{ with } \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \text{PSL}_2.$$

Following [Lam05, Theorem 6] (see also [MM90]), $\text{Aut}(X)$ is the amalgamated product of the orthogonal group $O_3 = \text{SO}_3 \times \langle \tau \rangle$ and $J \rtimes \langle \tau \rangle$ along their intersection $C$, where $J$ is the subgroup of $\text{Aut}(X)$ of the automorphisms of the form

$$(x, y, z) \mapsto (ax + 2\alpha yP(z) - \alpha zP^2(z), y - zP(z), \frac{1}{\alpha}z); \quad \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z].$$

Note that $\text{SO}_3$ is generated by $\mathbb{C}^+$-actions. Define $\hat{J}$ to be the subgroup of $J$ generated by $\mathbb{C}^+$-actions. The subgroup of $\text{Aut}(X)$ generated by $\text{SO}_3$ and $\hat{J}$ coincides with the subgroup of $\text{Aut}(X)$ generated by $\text{SO}_3$ and $J$ and is a subgroup of $U(X)$. Moreover, because $\tau$ normalizes $(J, \text{SO}_3)$ we have that $\text{Aut}(X) = \langle J, \text{SO}_3 \rangle \rtimes \langle \tau \rangle$. This implies that $\text{Aut}(X)$ is not connected and $\tau \not\in \text{Aut}(X)$. Since the closure of $U(X) \subset \text{Aut}(X)$ is connected as $U(X)$ is generated by connected subgroups and since $(J, \text{SO}_3) \subset U(X)$ we have that $U(X) \subset \text{Aut}(X)$ coincides with $\text{Aut}(X)$ and hence is closed. Moreover, $\text{Aut}(X) = U(X) \rtimes \langle \tau \rangle$.

The following proposition is going to be of use later (see [Neu48, Corollary 8.11]).

Proposition 7. In the amalgamated product $G = A \ast_C B$ with the unified subgroup $C = A \cap B$, consider two subgroups $\hat{A} \subset A$ and $\hat{B} \subset B$, and let $\hat{G} = \langle \hat{A}, \hat{B} \rangle$. Assume that $\hat{A} \cap C = \hat{C} = \hat{B} \cap C$. Then $\hat{G} = \hat{A} \ast_{\hat{C}} \hat{B}$. 

Lemma 7. The group \( U(X) \) is the amalgamated product of \( \text{SO}_3 \) and \( J \) along their intersection.

Proof. We know that \( \text{Aut}(X) \) is the amalgamated product of \( \text{O}_3 = \text{SO}_3 \times \langle \tau \rangle \) and \( J \rtimes \langle \tau \rangle \) along their intersection \( C \). Moreover, since \( \text{SO}_3 \cap C = J \cap \text{SO}_3 = J \cap C \) we have by Proposition 7 that \( \langle J, \text{SO}_3 \rangle \subset \text{Aut}(X) \) is the amalgamated product of \( J \) and \( \text{SO}_3 \) along their intersection. As \( U(X) = \langle J, \text{SO}_3 \rangle \) the claim follows. \( \square \)

Lemma 8. The subgroup \( \text{Aut}_{\text{SO}_3}(X) \subset \text{Aut}(X) \) of those automorphisms that commute with \( \text{SO}_3 \) is \( \langle \tau \rangle \). Moreover, \( X/\langle \tau \rangle \simeq \text{SL}_2/N \).

Proof. Let \( \varphi \in \text{Aut}_{\text{SO}_3}(X) \). Since \( \text{Aut}(X) \) is the amalgamated product of \( \text{O}_3 \) and \( J \) along their intersection, we can write \( \varphi \) as the product \( a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \), where \( a_i \in \text{O}_3 \) and \( b_i \in J \). Since \( a = \varphi \circ a \circ \varphi^{-1} \) for any \( a \in \text{SO}_3 \), we have that \( b_k \in \text{O}_3 \cap J \) and \( \varphi \) can be written as \( \tilde{a}_1 \circ b_1 \circ \cdots \circ \tilde{a}_k \) for some \( \tilde{a}_i \in \text{O}_3 \) and \( b_i \in J \). Assume first that \( k > 1 \). Hence, the expression

\[
\tilde{a}_1 \circ b_1 \circ \cdots \circ \tilde{a}_k \circ a \circ (\tilde{a}_1 \circ b_1 \circ \cdots \circ \tilde{a}_k)^{-1} = a
\]

implies that \( \tilde{a}_k \circ a \circ \tilde{a}_k^{-1} \in \text{O}_3 \cap J \). Since this should hold for any \( a \in \text{SO}_3 \) we get a contradiction. Therefore, \( k = 1 \) and hence, \( \varphi \in \text{O}_3 \). This implies that \( \text{Aut}_{\text{SO}_3}(X) = \langle \tau \rangle \) as the centralizer of \( \text{SO}_3 \) in \( \text{O}_3 \) is \( \langle \tau \rangle \).

To finish the proof we need to argue that \( X/\langle \tau \rangle \simeq \text{SL}_2/N \). We first note that \( X/\langle \tau \rangle \) is normal and since \( \text{SL}_2 \)-action on \( X \) is transitive, it follows that the induced action of \( \text{SL}_2 \) on \( X/\langle \tau \rangle \) is transitive too. Hence, from Lemma 6 it follows that \( X/\langle \tau \rangle \simeq \text{SL}_2/N \) as \( X \neq X/\langle \tau \rangle \). \( \square \)

Note that the subgroup \( U(X) = \text{Aut}(X) \) closed (see [Kr15, Lemma 6.3]), where \( \text{Aut}(X) \) is the neutral component of \( \text{Aut}(X) \). Hence, \( U(X) \) is an ind-group.

Proposition 8. We have the following properties.

(a) All closed subgroups \( S \subset \text{Aut}(X) \) isomorphic to \( \text{PSL}_2 \) are conjugate.

(b) The root subgroups with respect to a maximal torus \( \tilde{T} \) of any \( S \simeq \text{PSL}_2 \) are multiplicity-free with weights \( 1, 2, 3, \ldots \) up to an automorphism of \( \tilde{T} \).

Proof. (a) Since \( \text{Aut}(X) \) is the amalgamated product of \( \text{O}_3 \) and \( J \) over their intersection we have that by [Sr80] any closed subgroup \( S \subset \text{Aut}(X) \) isomorphic to \( S \) is conjugate to one of the factors \( \text{O}_3 \) or \( J \). Since all unipotent subgroups of \( J \) commute, \( S \) can not be embedded into \( J \) and hence \( S \) is conjugate to a subgroup of \( \text{O}_3 \), i.e., to \( \text{SO}_3 \). The claim follows.

Now we are going to prove (b). Let \( U \subset \text{Aut}(X) \) be a root subgroup with respect to \( \tilde{T} \). This means that \( \tilde{T} \rtimes U \) is an algebraic subgroup of \( \text{Aut}(X) \) and by [Sr80], \( \tilde{T} \rtimes U \) is conjugate to a subgroup of either \( \text{O}_3 \) or \( J \). If \( \tilde{T} \rtimes U \) is conjugate to a subgroup of \( \text{O}_3 \), then the weight of \( U \) with respect to \( \tilde{T} \) is either 1 or \(-1\), i.e., up to an automorphism of \( \tilde{T} \) we can assume that the weight is 1. If \( \tilde{T} \rtimes U \) is conjugate to a subgroup of \( J \), then without loss of generality we can assume that \( \tilde{T} \rtimes U \) is an algebraic subgroup of \( J_{\leq k} \) generated by elements of the form

\[
(x, y, z) \mapsto (\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - z P(z)), \frac{1}{\alpha} z); \quad \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z]_{\leq k}.
\]
for some natural $k$, where $\mathbb{C}[z]_{\leq k}$ denotes the polynomials of degree less or equal than $k$. Moreover, since all tori in $J_{\leq k}$ are conjugate we can assume that

$$\tilde{T} = \{(tx, y, t^{-1}z) \mid t \in \mathbb{C}^*\}.$$ 

By the following computation

$$(tx, y, t^{-1}z) \circ (x + 2yP(z) - zP^2(z), (y - zP(z)), z) \circ (t^{-1}x, y, tz) = (x + 2ytP(tz) - zt^2P^2(tz), (y - ztP(tz)), z),$$

it is easy to see that a root subgroup $U_i \subset J_{\leq k}$ should have the form

$$U_i = \{(x + 2cyP_i(z) - c^2zP_i^2(z), (y - czP_i(z)), z) \mid c \in \mathbb{C}, P_i(z) = z^i\}$$

for some natural $i \leq k$. Note that the root subgroup $U_i$ with respect to $\tilde{T}$ has the weight $i + 1$. The claim follows. \qed

6.3. The structure of $\text{Aut}(\text{SL}_2/N)$. By Lemma 8, there is an automorphism $\tau \in \text{Aut}_{SO}(X)$ and the quotient $Y = X/\langle \tau \rangle$ is isomorphic to $\text{SL}_2/N$. In particular, $O(Y) = O(X)\langle \tau \rangle$. An automorphism $\phi$ of $X$ descends to an automorphism on $Y$ if and only if $\phi$ sends $\langle \tau \rangle$-orbits to $\langle \tau \rangle$-orbits. In fact, such an automorphism induces the automorphism of $O(X)$ that sends $\langle \tau \rangle$-invariant functions of $O(X)$ to $\langle \tau \rangle$-invariant functions of $O(X)$. This condition for $\phi$ is equivalent to the condition that $\phi$ normalizes $\langle \tau \rangle$. Moreover, since $\tau$ has order two, $\phi$ commutes with $\tau$. Recall that by $\text{Aut}^{(\tau)}(X)$ we denote the subgroup of those elements of $\text{Aut}(X)$ that normalize $\langle \tau \rangle$, but in this particular case $\text{Aut}^{(\tau)}(X)$ is even the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with $\tau$.

As we have mentioned above, $\phi \in \text{Aut}(X)$ induces an automorphism of $Y \simeq \text{SL}_2/N$ if and only if $\phi \in \text{Aut}^{(\tau)}(X)$. On the other hand, since $X \simeq \text{SL}_2/T$ is simply connected and the quotient map $\pi: X \to X/\langle \tau \rangle = Y$ is an étale covering, every automorphism $\varphi$ of $Y$ can be lifted to a continuous analytical automorphism of $X$ and hence by [Sr58, Proposition 20], $\varphi$ can be lifted to an automorphism $\tilde{\varphi}$ of $X$, i.e., $\tilde{\varphi} \in \text{Aut}^{(\tau)}(X)$. Hence, we have the surjective homomorphism $\text{Aut}^{(\tau)}(X) \to \text{Aut}(Y)$ with the kernel $\langle \tau \rangle$ and so

$$\text{Aut}(Y) \simeq \text{Aut}^{(\tau)}(X)/\langle \tau \rangle.$$ 

Observe that $SO_3 \times \langle \tau \rangle$ is the subgroup of $\text{Aut}^{(\tau)}(X)$. Define the subgroup $J^{(\tau)} \subset \text{Aut}(X)$ of those automorphisms from $J$ which normalize $\langle \tau \rangle$. It is not difficult to see that $J^{(\tau)}$ is comprised of the following automorphisms:

$$\{(x, y, z) \mapsto (ax + 2ayP(z) - azP^2(z), y - zP(z), \frac{1}{a}z); \ a \in \mathbb{C}^*, P \in \bigoplus_{l=0}^{\infty} \mathbb{C}z^l\}.$$ 

We have the following statement.

**Lemma 9.** The subgroup $\text{Aut}^{(\tau)}(X) \subset \text{Aut}(X)$ is the direct product of $\langle \tau \rangle$ and the amalgamated product of $SO_3$ and $J^{(\tau)}$ along their intersection.

**Proof.** As we have mentioned above, $\text{Aut}^{(\tau)}(X)$ is the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with $\tau$. Assume $\phi \in \text{Aut}(X)$ commutes with $\tau$. Since $\text{Aut}(X)$ is the amalgamated product of $SO_3 \times \langle \tau \rangle$ and $J \rtimes \langle \tau \rangle$ one can write $\phi$
as a product \( a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \), where \( a_i \in \text{SO}_3 \times \langle \tau \rangle \) and \( b_i \in \text{J} \times \langle \tau \rangle \). Further, because \( \phi \) commutes with \( \tau \), \( \tau \circ \phi \tau = \phi \) or equivalently,

\[
\tau \circ a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ \tau = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k.
\]

Since \( \tau \) commutes with \( \text{SO}_3 \) one can rewrite this equation as follows:

\[
a_1 \circ (\tau \circ b_1 \circ \tau) \circ \cdots \circ a_k \circ (\tau \circ b_k \circ \tau) = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k.
\]

From the amalgamated product structure of \( \text{Aut}(X) \) it follows that \( \tau \circ b_1 \circ \tau = c_i \circ b_i \)
for some \( c_i \in (\text{SO}_3 \times \langle \tau \rangle) \cap (\text{J} \times \langle \tau \rangle) \). This can happen only if \( b_i \) commutes with \( \tau \), i.e., \( b_i \in \text{J}^{(\tau)} \times \langle \tau \rangle \). Therefore, \( \text{Aut}^{(\tau)}(X) \) is generated by \( \text{SO}_3 \times \langle \tau \rangle \) and \( \text{J}^{(\tau)} \times \langle \tau \rangle \).

Moreover, \( \text{Aut}^{(\tau)}(X) = \langle \text{SO}_3, \text{J}^{(\tau)} \rangle \times \langle \tau \rangle \) and by Proposition 7, \( \langle \text{SO}_3, \text{J}^{(\tau)} \rangle \) is the amalgamated product of \( \text{SO}_3 \) and \( \text{J}^{(\tau)} \) over their intersection. \( \square \)

From Lemma 9 and (1) we have the following statement.

**Lemma 10.** The automorphism group \( \text{Aut}(Y) \) is isomorphic to the amalgamated product of \( \text{SO}_3 \) and \( \text{J}^{(\tau)} \). In particular, \( \text{Aut}(Y) = \text{U}(Y) \).

**Remark 4.** Lemma 10 can also be retrieved from [KPZ17, Remark 3.9] and Remark 3 (see also [DG77, (2.4.3)]).

**Corollary 1.** We have the following properties.

(a) All closed subgroups \( S \subset \text{Aut}(Y) \) isomorphic to \( \text{PSL}_2 \) are conjugate.

(b) The root subgroups of \( \text{Aut}(Y) \) with respect to a maximal torus \( \hat{T} \) of any \( S \simeq \text{PSL}_2 \) are multiplicity-free with weights 1, 3, 5, \ldots up to an automorphism of \( \hat{T} \). In particular, \( \text{U}(\text{SL}_2/N) \) and \( \text{U}(\text{SL}_2/T) \) are not algebraically isomorphic.

**Proof.** (a) By Lemma 10, \( \text{Aut}(Y) \) is the amalgamated product of \( \text{SO}_3 \) and \( \text{J}^{(\tau)} \) and by [Sr80] any algebraic subgroup of the amalgamated product is conjugate to one of the factors. Since, \( \text{J}^{(\tau)} \) does not contain a copy of \( \text{PSL}_2 \) it follows that \( S \) is conjugate to a subgroup of \( \text{SO}_3 \), i.e., to \( \text{SO}_3 \) itself.

(b) Without loss of generality we can assume that \( S \) equals \( \text{SO}_3 \) and \( \hat{T} \subset \text{SO}_3 \) is the subtorus of the form

\[
\{(tx, y, t^{-1}z) \mid t \in C^*\}.
\]

Any root subgroup of \( \text{Aut}(Y) \) with respect to \( \hat{T} \) lifts to a root subgroup \( U \) of \( \text{Aut}^{(\tau)}(X) \) (see Lemma 1) with respect to the subtorus \( p^{-1}(\hat{T})^\circ \subset \text{Aut}^{(\tau)}(X) \). As it follows from the proof of Proposition 8(b), \( U \) coincides with

\[
U_{2i} = \{(x + 2yP(z) - zP^2(z), (y - zP(z)), z) \mid P(z) = z^{2i}\}
\]

for some \( i \in \mathbb{N} \cup \{0\} \). The weight of the root subgroup \( U_{2i} \subset \text{Aut}^{(\tau)}(X) \) with respect to \( p^{-1}(\hat{T})^\circ \) is \( 2i + 1 \). Since the kernel of \( p^{-1}(\hat{T}) \to \hat{T} \) is trivial we have that the set of weights of root subgroups of \( \text{Aut}(Y) \) with respect to \( \hat{T} \) is \( \{2i + 1 \mid i \in \mathbb{N}\} \). This proves the first part of the statement. The second part follows because if there is an algebraic isomorphism \( \varphi: \text{U}(X) \to \text{U}(Y) \), then \( \varphi \) maps root subgroups of \( \text{U}(X) \) with respect to a subtorus \( \hat{T} \subset \text{U}(X) \) to root subgroups of \( \text{U}(Y) \) with respect to \( \varphi(\hat{T}) \) that have the same weights. But as follows from Proposition 8 and the first part of this proof it is not the case. \( \square \)

**Remark 5.** Analogously as in the proof of Lemma 8, using amalgamated product structure of \( \text{Aut}(Y) \) described in Lemma 10 we can show that the subgroup \( \text{Aut}_{\text{SO}_3}(X) \subset \text{Aut}(X) \) of those automorphisms that commute with \( \text{SO}_3 \) is trivial.
6.4. On the automorphism group of $A_{d,2}$. Recall that by Proposition 3, there is a surjective homomorphism $\phi_d : \text{Aut}^{zd}(\mathbb{A}^n) \to \text{Aut}(A_{d,n})$ of groups. Consider now the maximal subtorus

$$T_n = \{(t_1 x_1, \ldots, t_n x_n) | t_i \in \mathbb{C}^* \} \subset \text{Aut}(\mathbb{A}^n)$$

and recall that by $T_n$ we denote the subtorus of the form

$$\{ (t_1 x_1, \ldots, t_n x_n) | t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1 \} \subset U(\mathbb{A}^n)$$

that has dimension $n - 1$. Then $T_{d,n}' = \phi_d(T_n')$ is a maximal subtorus of $U(A_{d,n}) \subset \text{Aut}(A_{d,n})$.

**Lemma 11.** Let $U \subset \text{Aut}(A_{d,n})$ be a root subgroup with respect to $T_{d,n}'$ which has a character $\chi$. Then $U$ lifts to a root subgroup $\tilde{U} = (\phi_d^{-1}(U))^o \subset \text{Aut}^{zd}(\mathbb{A}^n)$ with respect to $T_n' = (\phi_d^{-1}(T_{d,n}')^o$ with character $\tilde{\chi} = \psi^*(\chi)$ such that the following diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & \mu_{(n,d)} & \longrightarrow & T_n' & \stackrel{\psi}{\longrightarrow} & T_{d,n}' & \longrightarrow & 1 \\
& & \downarrow \tilde{\chi} & & \downarrow \chi & & \downarrow & & \\
& & \mathbb{C}^* & \overset{=} \longrightarrow & \mathbb{C}^* \\
\end{array}
$$

commutes, where $\psi = \phi_d|T_{d,n}'$ and $\psi^*(\chi)$ is a pull-back of $\chi$.

**Proof.** From Proposition 2 it follows that any root subgroup $U$ of $\text{Aut}(A_{d,n})$ with respect to $T_{d,n}'$ lifts to a unipotent subgroup $\tilde{U} = (\phi_d^{-1}(U))^o \subset \text{Aut}^{zd}(\mathbb{A}^n)$. Moreover, $\tilde{U}$ is normalized by $(\phi_d^{-1}(T_{d,n}'))^o = T_n'$. Now, let $\tilde{u} \in \tilde{U}$ be a non-trivial element and $u = \phi_d(\tilde{u}) \in U$. We have group isomorphisms

$$\mathbb{C}^+ \overset{\sim} \longrightarrow \tilde{U}, \ s \mapsto \tilde{u}(s)$$

and

$$\mathbb{C}^+ \overset{\sim} \longrightarrow U, \ s \mapsto u(s).$$

Now the proof follows from the formula

$$\phi_d(\tilde{u}(\chi \circ \psi(t)s)) = u(\chi(t)s).$$

\[\square\]

Observe that the homomorphism $\phi_d : \text{Aut}^{zd}(\mathbb{A}^n) \to \text{Aut}(A_{d,n})$ induces the homomorphism $\tilde{\phi}_d : U^{zd}(\mathbb{A}^n) \to U(A_{d,n})$ which has the kernel $\mu_{(n,d)}$, where $U^{zd}(\mathbb{A}^n) \subset \text{Aut}^{zd}(\mathbb{A}^n)$ is a subgroup generated by $\mathbb{C}^+$-actions.

In [BH03] it is proved that any faithful action of an $(n - 1)$-dimensional torus on an $n$-dimensional toric $T_Z$-variety $Z$ is conjugate to a subtorus of the big torus $T_Z$. This result is used in order to prove the following lemma.

**Lemma 12.** Let $T$ be an algebraic subtorus of $U(A_{d,n})$ of dimension $(n - 1)$. Then there exists an algebraic isomorphism $F : U(A_{d,n}) \to U(A_{d,n})$ such that $F(T) = T_{d,n}'$.

**Proof.** Since $T \subset U(A_{d,n}) \subset \text{Aut}(A_{d,n})$ is an algebraic subtorus of dimension $n - 1$ and since $A_{d,n}$ is toric, by [BH03, Theorem p. 2] there exists $\varphi \in \text{Aut}(A_{d,n})$ such that $\varphi \circ T \circ \varphi^{-1} \subset T_{d,n}'$. Moreover, since $U(A_{d,n})$ is a normal subgroup of $\text{Aut}(A_{d,n})$, $\varphi \circ T \circ \varphi^{-1} \subset T_{d,n}'$ and hence since $\text{dim} T_{d,n}' = n - 1$, $\varphi \circ T \circ \varphi^{-1} = T_{d,n}'$. This proves that an algebraic isomorphism $F : U(A_{d,n}) \to U(A_{d,n})$, $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$ maps $T$ to $T_{d,n}'$.\[\square\]
Let $Z$ be an irreducible affine variety of dimension $n \geq 2$ and $\psi: U(Z) \to U(A_{d,n})$ be an algebraic isomorphism. Let $T$ be an $(n-1)$-dimensional algebraic subtorus of $U(Z)$. Then, after composing $\psi$ with a suitable algebraic isomorphism $F: U(A_{d,n}) \to U(A_{d,n})$ (see Lemma 12), we can assume that $\psi(T) = T'_{d,n}$.

**Lemma 13.** Root subgroups $U$ and $\psi(U)$ have the same weight characters with respect to $T$ and $\psi(T) = T'_{d,n}$ respectively, i.e., if $\chi: T'_{d,n} \to \mathbb{C}^*$ is the weight of $\psi(U)$, then the weight of $U$ is $\chi \circ \psi$.

**Proof.** Let $U$ be a root subgroup of $U(Z)$ with respect to $T$ and $\text{Lie} U = \mathbb{C} \nu$, where $\nu$ is a generator. Then $\psi(U)$ is the root subgroup of $U(A_{d,n})$ with respect to $T'_{d,n}$. The algebraic isomorphism $\psi$ induces an isomorphism $\psi_U: \text{Lie} U \to \text{Lie} \psi(U)$. Note that the action of $T$ on $U$ induces the action of $T$ on $\text{Lie} U$. Then

$$\psi(t) \circ d\psi^U_{\nu}(\nu) \circ \psi(t^{-1}) = \chi(\psi(t))d\psi^U_{\nu}(\nu) = d\psi^U_{\nu}(\chi(\psi(t))\nu) = d\psi^U_{\nu}(\chi \circ \psi(t)\nu),$$

where $t \in T$. On the other hand,

$$\psi(t) \circ d\psi^U_{\nu}(\nu) \circ \psi(t^{-1}) = d\psi^U_{\nu}(t \circ \nu \circ t^{-1}).$$

The claim follows. \[\square\]

**Lemma 14.** Let $d$ be even. Then the set of weights of root subgroups of $\text{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ is $\{kd+2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_{d,2}$.

**Proof.** By Proposition 3, any root subgroup of $\text{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ lifts to a root subgroup of $\text{Aut}^{d,i}(A^2)$ with respect to $\phi_d^{-1}(T'_{d,2})^o = T_2'$. By Lemma 3, any root subgroup of $\text{Aut}(A^2)$ with respect to $T_2'$ is equal either to

$$U_s = \{(x + cy^s, y) \mid c \in \mathbb{C}\}$$

or to

$$U_l = \{(x, y + cx^l) \mid c \in \mathbb{C}\}$$

for some $s, l \in \mathbb{N} \cup \{0\}$. Root subgroups $U_s$ and $U_l$ belong to $\text{Aut}^{d,i}(A^2)$ if and only if $s, l \in d\mathbb{N} + 1$. The weight of the action of $T_2' = \{(cx, c^{-1}y) \mid c \in \mathbb{C}^*\}$ on $U_s$ by $t \circ u \circ t^{-1}$, $t \in T_2'$ and $u \in U_s$, equals $s + 1$. Analogously, the weight of $T_2'$-action on $U_l$ is $-l - 1$. Therefore, the set of weights of root subgroups of $\text{Aut}^{d,i}(A^2)$ with respect to $T_2'$ is $\{kd+2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T_2'$. Moreover, since the kernel of the map $\phi_d: T_2' \to T_{d,2}$ is $\mu_2$ as $d$ is even, the statement follows from Lemma 11. \[\square\]

By the Jung-Van der Kulk Theorem (see [Ju42] and [Ku53]) $\text{Aut}(A^2) = \text{Aff}_2 \ast C J$, where $\text{Aff}_2$ is the group of affine transformations of $\mathbb{A}^2$,

$$J = \{(ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\}$$

and $C = \text{Aff}_2 \cap J$. Now, the subgroup $\text{Aut}^{d,i}(A^2) \subset \text{Aut}(A^2)$ contains the standard $\text{GL}_2 \subset \text{Aut}(A^2)$ and

$$J_d = \{(ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C} x^{ld+1}\}.$$

By [AZ13, Theorem 4.2], $\text{Aut}(A_{d,2}) \simeq \text{Aut}^{d,i}(A^2)/\mu_d$ is the amalgamated product of $\text{GL}_2/\mu_d$ and $J_d/\mu_d$ along their intersection. Moreover, as we will see in Lemma 15.
below, such an amalgamated product structure induces the amalgamated product structure of $U(A_d)$. Denote by $\tilde{J}_d$ the subgroup of $J_d$ of the following form:
\[
\{(ax + c, a^{-1}y + f(x)) \mid a \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C}x^{ld+1}\}
\]
and by $\tilde{T}_{d,2}$ the one-dimensional subtorus of $\text{Aut}(A_{d,2})$ induced by the $\mathbb{C}^*$-action on $\mathbb{A}^2$ given by the maps $\{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\}$. We have the following statement.

**Lemma 15.** The group $\text{Aut}(A_{d,2})$ is the semidirect product $U(A_{d,2}) \ltimes \tilde{T}_{d,2}$. Moreover, $U(A_{d,2})$ is the amalgamated product of $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ and $\tilde{J}_d/(\mu_d \cap \tilde{J}_d)$ along their intersection.

**Proof.** As $\text{Aut}(A_{d,2}) \cong \text{Aut}^{d}(\mathbb{A}^2)/\mu_d$, it is clear that $\text{Aut}(A_{d,2})$ is generated by $U(A_{d,2})$ and $\tilde{T}_{d,2}$. Further, the subgroup $U(A_{d,2}) \subset \text{Aut}(A_{d,2})$ is normal and the subgroups $U(A_{d,2})$ and $\tilde{T}_{d,2}$ do not intersect. Indeed, if the subgroups $\tilde{T}_{d,2}$ and $U(A_{d,2})$ have a non-trivial intersection, then as $\text{Aut}(A_{d,2}) \cong \text{Aut}^{d}(\mathbb{A}^2)/\mu_d$, the subgroups $\{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\}$ of $\text{Aut}(\mathbb{A}^2)$ also have a non-trivial intersection which is not the case. Hence, we conclude that $\text{Aut}(A_{d,2}) = U(A_{d,2}) \ltimes \tilde{T}_{d,2}$. 

Recall that $\text{Aut}(A_{d,2})$ is the amalgamated product of $\text{GL}_2/\mu_d$ and $\tilde{J}_d/\mu_d$ along their intersection and $\text{GL}_2/\mu_d = \text{SL}_2/(\mu_d \cap \text{SL}_2) \cong \tilde{T}_{d,2}$ and $\tilde{J}_d/\mu_d = \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \cong \tilde{T}_{d,2}$. Since $\tilde{T}_{d,2}$ is contained in the intersection $\text{GL}_2/\mu_d \cap \tilde{J}_d/\mu_d$, it follows that
\[
\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \ltimes (\text{SL}_2/(\mu_d \cap \text{SL}_2) \ast C \tilde{J}_d/(\mu_d \cap \tilde{J}_d)),
\]
where $C$ is the intersection of $A = \text{SL}_2/(\mu_d \cap \text{SL}_2)$ and $B = \tilde{J}_d/(\mu_d \cap \tilde{J}_d)$. As both $A$ and $B$ are generated by unipotent subgroups it follows that $A \ast C B \subset U(A_{d,2})$. The other inclusion follows from (2). The claim follows.

**Remark 6.** Define the homomorphism of abstract groups
\[
\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \ltimes U(A_{d,2}) \to \tilde{T}_{d,2}
\]
by projection onto the first factor. Such a homomorphism is a morphism of ind-groups which implies that $U(A_{d,2}) \subset \text{Aut}(A_{d,2})$ is a closed subgroup.

**Remark 7.** By Lemma 15, $U(A_{d,2})$ is the amalgamated product of $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ and $\tilde{J}_d/(\tilde{J}_d \cap \mu_d)$ along their intersection. Note that if $d$ is even, then $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ is isomorphic to $\text{PSL}_2$. If $d$ is odd, then $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ is isomorphic to $\text{SL}_2$.

The following result was pointed out to me by Hanspeter Kraft.

**Proposition 9.** Let $Z$ be an irreducible affine normal variety of dimension 2.
(a) The groups $U(\text{SL}_2/T)$ and $U(Z)$ are algebraically isomorphic if and only if $Z \cong \text{SL}_2/T$ or $Z \cong A_{2,2}$.
(b) The groups $U(\text{SL}_2/N)$ and $U(Z)$ are algebraically isomorphic if and only if $Z \cong \text{SL}_2/N$ or $Z \cong A_{1,2}$.

**Proof.** Let $X$ be isomorphic either to $\text{SL}_2/T$ or to $\text{SL}_2/N$. Then $U(X)$ contains a copy of $\text{PSL}_2$ (see Lemma 7 and Lemma 10 respectively). Hence, by Lemma 6, $Z$ is isomorphic either to $\text{SL}_2/T$, to $\text{SL}_2/N$, or to $A_{d,2}$ for some $d \in \mathbb{N}$. We claim that $Z$ can be isomorphic to $A_{2,2}$ only if $d$ is even. Indeed, assume that $\text{Aut}(A_{d,2})$ contains an algebraic subgroup $S$ isomorphic to $\text{PSL}_2$. Since $S \subset U(A_{d,2})$ it follows from Lemma 15 that $S$ is conjugate either to a subgroup of $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ or
we conclude that and Lemma 13 we have \( U(SL_2) \approx \frac{\tilde{J}_d}{\mu_d \cap \tilde{J}_d} \) (see [Sr80]). Moreover, since \( \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \) does not contain a copy of \( PSL_2 \), \( \tilde{S} \) should be conjugate to a subgroup of \( SL_2/(\mu_d \cap SL_2) \). Hence, by Remark 7 we conclude that \( d \) is even.

By Corollary 1 we have \( U(SL_2/T) \neq U(SL_2/N) \). Hence, to prove (a) we first need to show that an algebraic isomorphism \( \phi: U(A_{d,2}) \sim \sim U(SL_2/T) \) implies that \( d = 2 \). By Lemma 14, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T_{d,2} \) is \( \{ \frac{k\pi}{2} \mid k \in \mathbb{N} \cup \{0\} \} \) up to an automorphism of \( T_{d,2} \). Since \( T_{d,2} \) is a subgroup of some \( S \subset U(A_{d,2}) \) isomorphic to \( PSL_2 \) we have by Proposition 8 that the set of weights of root subgroups of \( U(X \simeq SL_2/T) \) with respect to \( \phi(T_{d,2}) \) is \{1, 2, 3, \ldots\} up to an automorphism of \( \phi(T_{d,2}) \). By Lemma 13, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T_{d,2} \) and of \( U(SL_2/T) \) with respect to \( \phi(T_{d,2}) \) are equal. Therefore, \( d \) indeed equals 2. To finish the proof of (a) we need to show that \( U(A_{2,2}) \) and \( U(X \simeq SL_2/T) \) are algebraically isomorphic. To do so we first note that by Lemma 15 and Lemma 7, the first factors \( SL_2/\mu_2 \) and \( SO_3 \) from the amalgamated product structure of \( U(A_{2,2}) \) and of \( U(X) \) respectively are isomorphic to \( PSL_2 \). Moreover, \( \tilde{J}_2 \) and \( J \) are algebraically isomorphic, as both \( \tilde{J}_2 \) and \( J \) are direct limits of isomorphic algebraic groups. Finally, the intersections \( SL_2/\mu_2 \cap \tilde{J}_2 \subset \text{Aut}(A_{2,2}) \) and \( SO_3 \cap J \subset \text{Aut}(X) \) are also isomorphic as algebraic groups as they are both isomorphic to a Borel subgroup of \( PSL_2 \).

Define a homomorphism \( \varphi: U(A_{d,2}) \rightarrow U(X) \) that sends isomorphically the first factor \( SL_2/\mu_2 \) of the amalgamated product of \( U(A_{d,2}) \) to the first factor \( SO_3 \) of the amalgamated product of \( U(X) \) in a way that \( \varphi(SL_2/\mu_2 \cap \tilde{J}_2) = SO_3 \cap J \subset \text{Aut}(X) \) and the second factor \( \tilde{J}_2 \) of the amalgamated product of \( U(A_{d,2}) \) to the second factor \( J \) of the amalgamated product of \( U(X) \). Such a map is well-defined and is an isomorphism as follows from the amalgamated product structure of \( U(A_{2,2}) \) and \( U(X \simeq SL_2/T) \). The proof of (a) follows.

To prove (b) we first need to show that an algebraic isomorphism \( \phi: U(A_{d,2}) \sim \sim U(SL_2/N) \) implies that \( d = 4 \). As we have already mentioned above in this proof, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T_{d,2} \) is \( \{ \frac{k\pi}{2} \mid k \in \mathbb{N} \cup \{0\} \} \) up to an automorphism of \( T_{d,2} \) (see Lemma 14). Further, analogously as in the first part of the proof we have that the set of weights of root subgroups of \( U(Y \simeq SL_2/N) \) with respect to \( \phi(T_{d,2}') \) is \{1, 3, 5, \ldots\} up to an automorphism of \( \phi(T_{d,2}') \) (see Corollary 1). By Lemma 13, the set of weights of root subgroups of \( U(A_{d,2}) \) with respect to \( T_{d,2} \) and of \( U(Y) \) with respect to \( \phi(T_{d,2}) \) coincide which implies that \( d = 4 \). To finish the proof of (b) we need to show that groups \( U(A_{d,2}) \) and \( U(Y \simeq SL_2/N) \) are algebraically isomorphic. This follows analogously as in the previous paragraph in the case of groups \( U(A_{2,2}) \) and \( U(X \simeq SL_2/T) \). \( \square \)

7. Higher-dimensional case

Consider the action of \( SL_n \) on \( A_{d,n} \) induced by the standard \( SL_n \)-action on \( \mathbb{A}^n \). Denote by \( S_{d,n} \subset \text{Aut}(A_{d,n}) \) the image of \( SL_n \) under the natural homomorphism \( SL_n \rightarrow \text{Aut}(A_{d,n}) \).

**Lemma 16.** We have an isomorphism \( S_{d,n} \simeq SL_n/\mu_{(d,n)} \), where \( (d,n) \) denotes the greatest common divisor of \( d \) and \( n \). Moreover, \( S_{d,n} \subset U(A_{d,n}) \).

**Proof.** By Proposition 3, there is a surjective homomorphism \( \phi_d: \text{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \text{Aut}(A_{d,n}) \) of groups with \( \ker \phi_d = \mu_d \). Hence, \( \text{Aut}(A_{d,n}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^n)/\mu_d \) which
shows that $S_{d,n} \simeq \text{SL}_n / (\mu_d \cap \text{SL}_n) \simeq \text{SL}_n / \mu_{(d,n)}$. The second claim is clear since $S_{d,n}$ is generated by unipotent subgroups. \hfill \Box

**Lemma 17.** If there is an injective algebraic homomorphism

$$\varphi : S_{d,n} = \text{SL}_n / \mu_{(n,d)} \hookrightarrow U(A_{l,n}),$$

then $(n,d) = (n,l)$. In particular, if $U(A_{d,n})$ and $U(A_{l,n})$ are algebraically isomorphic, then $(d,n) = (l,n)$. 

**Proof.** Applying Lemma 12 we can assume $\varphi(T_{d,n}) = T_{l,n}$. Hence, intersection $\varphi(S_{d,n}) \cap S_{l,n}$ contains $T_{l,n}$. We claim that $\varphi(S_{d,n}) = S_{l,n}$. To show this we first note that the subgroup $T_{l,n} \subset \varphi(S_{d,n})$ lifts to $T_{k,n}$ and by Proposition 3, each root subgroup of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ lifts to a one-dimensional unipotent subgroup of $\text{Aut}(\mathbb{A}^n)$. Moreover, the subgroup $G$ of $\text{Aut}(\mathbb{A}^n)$ generated by all one-dimensional unipotent subgroups $U_i$ lifted from root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ is algebraic subgroup of $\text{Aut}(\mathbb{A}^n)$. Indeed, if $G$ is not algebraic, then $G$ can not be written as a finite product of $U_i$. In contrast, $\varphi(S_{d,n})$ can be written as a finite product of root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$. Moreover, $\phi_d$ induces a homomorphism of groups $G \rightarrow \varphi(S_{d,n})$ with a kernel $\mu_{(l,n)}$.

Hence, a homomorphism of groups $G \rightarrow \varphi(S_{d,n})$ is a homomorphism of algebraic groups with the kernel $\mu_{(l,n)}$ and $G$ is isomorphic to $\text{SL}_n$ that contains $T'_{n}$ as a maximal subtorus. It follows from [KRZ20, Theorem 1.1] that all subgroups of $\text{Aut}(\mathbb{A}^n)$ isomorphic to $\text{SL}_n$ are conjugate. Therefore, $G$ is conjugate to the standard $\text{SL}_n$ in $\text{Aut}(\mathbb{A}^n)$, i.e., there exists $\psi \in \text{Aut}(\mathbb{A}^n)$ such that $\psi^{-1} \circ G \circ \psi = \text{SL}_n$. Since $T_{n} \subset G$ we have that $\psi^{-1} \circ T'_{n} \circ \psi$ is a subtorus in $\text{SL}_n$ which implies that $\psi$ is a linear map that moreover belongs to $\text{GL}_n$. Now it is easy to see that $G$ coincides with $\text{SL}_n$. Therefore, $\varphi(S_{d,n}) = S_{l,n}$.

Therefore, $S_{d,n}$ is isomorphic to $S_{l,n}$ as an algebraic group. Hence, from Lemma 16 it follows that $(d,n) = (l,n)$. The second part of the statement follows from the first one directly since $U(A_{d,n})$ contains a copy of $S_{d,n}$. \hfill \Box

**Proposition 10.** Let $X$ be $A_{d,n}$, $\text{SL}_2 / T$ or $\text{SL}_2 / N$ and $Y$ be an irreducible affine variety. Assume that there is an algebraic isomorphism $U(X) \xrightarrow{\sim} U(Y)$. Then $\dim Y \leq \dim X$. Moreover, if additionally $Y$ is normal, then

(a) if $X \simeq \text{SL}_2 / T$, then $Y \simeq A_{2,2}$ or $Y \simeq \text{SL}_2 / T$,

(b) if $X \simeq A_{2,2}$, then $Y \simeq A_{2,2}$ or $Y \simeq \text{SL}_2 / T$,

(c) if $X \simeq \text{SL}_2 / N$, then $Y \simeq A_{2,2}$ or $Y \simeq \text{SL}_2 / N$,

(d) if $X \simeq A_{4,2}$, then $Y \simeq A_{4,2}$ or $Y \simeq \text{SL}_2 / N$,

(e) if $X = A_{d,n},$ where $(d,n) \notin \{(2,2), (2,4)\}, Y \simeq X$.

**Proof.** Fix an algebraic isomorphism $\psi : U(X) \xrightarrow{\sim} U(Y)$ and denote by $T'$ the image of $T'_{n}$ if $X = A_{d,n}$ or the image of a maximal subtorus $T$ of $U(X)$ if $X = \text{SL}_2 / T$ or $\text{SL}_2 / N$. By Lemma 11, Proposition 8 and Corollary 1, all root subgroups $U \subset U(Y)$ with respect to $T'$ have different weights. In particular, the root subgroups $O(Y)^U : U \subset U(Y)$ have different weights, which implies that $O(Y)^U$ is multiplicity-free, because the map $O(Y)^U \to O(Y)^U \cdot U$ is injective. Hence, by Lemma 2, we have that

$$\dim Y \leq \dim T' + 1 = n,$$

which proves the first part of the proposition.

Now (a), (b), (c) and (d) follow from Proposition 9.
To prove (c), we note that $U(A_{d,n})$ contains a copy of $\text{SL}_n / \mu_{(n,d)}$ which implies that $\text{SL}_n$ acts non-trivially on $Y$ and thus, by Proposition 4, Lemma 6 and Proposition 10 (a)-(d), $Y \simeq A_{l,n}$ for some $l \in \mathbb{N}$. Hence, $\psi: U(A_{d,n}) \overset{\sim}{\twoheadrightarrow} U(A_{l,n})$. By Lemma 12, there exists an algebraic isomorphism $F: U(A_{l,n}) \overset{\sim}{\twoheadrightarrow} U(A_{l,n})$ such that $F(\psi(T_{d,n}')) = T_{l,n}'$. Therefore, we can assume that $\psi(T_{d,n}') = T_{l,n}'$.

Consider the $\mathbb{C}^+$-action

$$\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}$$

on $\mathbb{A}^n$. It induces the $\mathbb{C}^+$-action $U$ on $A_{d,n}$ which is normalized by $T_{d,n}'$. Hence, $\psi(U) \subset U(A_{l,n})$ is a root subgroup with respect to $\psi(T_{d,n}') = T_{l,n}'$. By Proposition 3, $\psi(U)$ lifts to a $\mathbb{C}^+$-action on $\mathbb{A}^n$ normalized by $T_{l,n}'$. Since $(n, d) = (n, l)$ by Lemma 17 and because $U$ and $\psi(U)$ have the same weight characters with respect to $T_{d,n}'$ and $T_{l,n}'$ respectively (see Lemma 11), Lemma 11 implies that $\psi(U)$ lifts to a root subgroup of $\text{Aut}(\mathbb{A}^n)$ with respect to $T_{l,n}'$. Therefore, $l \leq d$. Analogously, $d \leq l$, i.e., $d = l$. The proof follows.

**Proof of Theorem 3.** Let $\psi: U(X) \overset{\sim}{\twoheadrightarrow} U(Y)$ be an algebraic isomorphism. Proposition 10 implies that $\dim Y \leq \dim X$. Since $\text{SL}_n$ acts regularly and non-trivially on $X$, $\text{SL}_n$ also acts non-trivially and regularly on $Y$.

First, let $X$ be isomorphic to $A_{d,n}$. Then by Lemma 6 and by Proposition 4, the normalization of $Y$, which we denote by $\tilde{Y}$, is isomorphic to $\text{SL}_2 / T$, $\text{SL}_2 / N$ or $A_{l,n}$ for some $l \geq 1$. First, assume that $\tilde{Y} \simeq A_{l,n}$. Hence, Proposition 4 implies that

$$\mathcal{O}(Y) = \sum_{i=1, \ldots, l} \bigoplus \mathbb{C}[x_1, \ldots, x_n]_{kd_i}$$

for some $d_1, \ldots, d_l \in \mathbb{N}$, where $(d_1, \ldots, d_l) = l$.

Let $\eta: A_{l,n} \to Y$ be the normalization morphism, which by Lemma 5 induces the algebraic homomorphism $\eta: U(Y) \hookrightarrow U(A_{l,n})$. Note that $\text{SL}_n / \mu_{(n,d)}$ acts faithfully on $X$. Then $\text{SL}_n / \mu_{(n,d)}$ also acts faithfully on $Y$ and therefore on $A_{l,n}$. Hence, by Lemma 17 we have that $(n, d) = (n, l)$.

Consider the $\mathbb{C}^+$-action

$$\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}$$

on $\mathbb{A}^n$. It induces the $\mathbb{C}^+$-action $U$ on $A_{d,n}$ which is normalized by $T_{d,n}'$. Hence, $\psi(U) \subset U(Y)$ is a root subgroup with respect to $\psi(T_{d,n}')$. By Lemma 12 there is an algebraic isomorphism $U(A_{d,n}) \overset{\sim}{\twoheadrightarrow} U(A_{d,n})$ that maps $T_{d,n}'$ to $\psi^{-1}(\tilde{\eta}^{-1}(T_{l,n}'))$ and so there is an isomorphism $U(Y) \overset{\sim}{\twoheadrightarrow} U(Y)$ that maps $\psi(T_{d,n}')$ to $\tilde{\eta}^{-1}(T_{l,n}')$. Hence, we can assume that $\psi(U)$ is a root subgroup with respect to $\tilde{\eta}^{-1}(T_{l,n}')$. By Lemma 5, $\psi(U)$ lifts to a $\mathbb{C}^+$-action on $A_{l,n}$ which is normalized by $T_{l,n}'$ and then by Lemma 1(c), $\psi(U)$ lifts to a $\mathbb{C}^+$-action on $\mathbb{A}^n$ normalized by $T_{d,n}'$. Since $(n, d) = (n, l)$ and because $U$ and $\psi(U)$ have the same weight characters with respect to $\tilde{\eta}^{-1}(T_{l,n}')$ and $T_{d,n}'$ respectively (i.e., if $\chi: T_{d,n}' \to \mathbb{C}^*$ is the weight of $\psi(U)$, then the weight of $U$ is $\chi \circ \psi$), Lemma 11 implies that $\psi(U)$ lifts to a root subgroup

$$\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}$$
of Aut(\(\mathbb{A}^n\)) with respect to \(T'_n\). Hence,
\[
\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}
\]
induces an action on \(O(\mathbb{A}^n)^\mu\)
which implies that \(l \mid d\). Moreover, \(\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}\)
induces an action on \(O(Y)\). This implies that
\[
d + l_i \in \mathbb{N}l_1 + \cdots + \mathbb{N}l_s
\]
for any \(i\).

The \(\mathbb{C}^+\) action on \(\mathbb{A}^n\) of the form \(\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}\) induces an action on \(O(Y)\). Since \((d, n) = (l, n)\) it follows that
\[
\{(x_1 + cx_2^{d+1}, x_2, \ldots, x_n) \mid c \in \mathbb{C}\}\)
induces an action on \(O(A^n)^\mu d\).

Hence, \(d \mid l_i\) for any \(i\) and then \(d \mid (l_1, \ldots, l_s) = l\). Therefore, \(d = l\). Now, because \(d \mid l_i\) for any \(i\), \(d + l_i \in \mathbb{N}l_1 + \cdots + \mathbb{N}l_s\) implies that
\[
\mathbb{N}l_1 + \cdots + \mathbb{N}l_s = \mathbb{N}_{\geq k} \{m \in \mathbb{N} \mid m \geq k\}.
\]

Now assume that \(\bar{Y}\) is isomorphic to \(SL_2/T\) or to \(SL_2/N\), then by Proposition 6, \(Y = \bar{Y}\). Then (e) follows from Proposition 9.

Let now \(X \simeq SL_2/T\). Then by Lemma 6, \(\bar{Y}\) can only be isomorphic to \(SL_2/T\), \(SL_2/N\) or \(A_{2,2}\). By Proposition 9, \(\bar{Y}\) is isomorphic to \(SL_2/T\) or to \(A_{2,2}\). If \(\bar{Y} \simeq SL_2/T\), from Proposition 6, it follows that \(Y = \bar{Y}\). Hence, (b) follows from the first part of the proof. Analogously follows (d).

**Proof of Theorem 1.** The isomorphism \(\text{Aut}(X) \xrightarrow{\sim} \text{Aut}(A_{d,n})\) induces an algebraic isomorphism \(U(X) \xrightarrow{\sim} U(A_{d,n})\). Note that \(X\) admits a torus action of dimension \(n\). From Theorem 3 it follows that \(X\) can only be isomorphic to \(A_{d,n}^s\). On the other hand, since normalization of \(A_{d,n}^s\) is equal to \(A_{d,n}\), it follows from Lemma 5 that there is a closed embedding \(\text{Aut}(A_{d,n}^s) \hookrightarrow \text{Aut}(A_{d,n})\) of ind-groups. Now the proof follows from [RvS21, Proposition 9.1(3)].

**Proof of Theorem 2.** Let \(Z\) be isomorphic either to \(SL_2/T\) or to \(SL_2/N\). Then an isomorphism \(\text{Aut}(X) \xrightarrow{\sim} \text{Aut}(Z)\) induces an algebraic isomorphism \(U(X) \xrightarrow{\sim} U(Z)\).

By Theorem 3, \(X\) is isomorphic either to \(Z\) or to \(A_{2k,2}^s\) for some \(s \in \mathbb{N}\) and \(k \in \{1, 2\}\). To finish the proof we need to show that \(\text{Aut}(Z)\) can not be isomorphic to \(\text{Aut}(A_{2k,2}^s)\). But this is clear as all \(A_{2k,2}^s\) admit an action of a two-dimensional torus and varieties \(SL_2/T\) and \(SL_2/N\) do not admit such an action.

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