Entanglement Percolation in Quantum Networks

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Quantum networks are composed of nodes which can send and receive quantum states by exchanging photons [1]. Their goal is to facilitate quantum communication between any nodes, something which can be used to send secret messages in a secure way [2, 3], and to communicate more efficiently than in classical networks [4]. These goals can be achieved, for instance, via teleportation [5]. Here we show that the design of efficient quantum communication protocols in quantum networks involves intriguing quantum phenomena, depending both on the way the nodes are displayed, and the entanglement between them. These phenomena can be employed to design protocols which overcome the exponential decrease of signals with the number of nodes. We relate the problem of establishing maximally entangled states between nodes to classical percolation in statistical mechanics [6], and demonstrate that quantum phase transitions [7] can be used to optimize the operation of quantum networks.

Introduction. The future of quantum communication will be based on quantum networks (cf. [1, 8–12, 13, 14]), where different nodes are entangled, leading to quantum correlations which can be exploited by performing local measurements in each node. For instance, a set of quantum repeaters [10] can be considered as a simple quantum network where the goal is to establish quantum communication over long distances. In order to optimize the operation of such a network, it is required to establish efficient protocols of measurements in such a way that the probability of success in obtaining maximally entangled states between different nodes is maximized. This probability may behave very differently as a function of the number of nodes if we use different protocols: in some cases it may decay exponentially, something which makes the repeaters useless, whereas for some protocols it may decay only polynomially, something which would make them very efficient.

A general network may be characterized by a quantum state, \( \rho \), shared by the different nodes. The goal is then: given two nodes, \( A \) and \( B \), find the measurements to be performed in the nodes, assisted with classical communication, such that \( A \) and \( B \) share a maximally entangled state, or singlet, with maximal probability. We call this probability the singlet conversion probability (SCP). This, or other related quantities like the localizable entanglement [13, 10], can be used as a figure of merit to characterize the state \( \rho \) and therefore the performance of the quantum network. Here, we focus on the SCP because of its operational meaning. These quantities cannot be determined in general, given that they require the optimization over all possible measurements in the different nodes, which is a formidable task even for small networks.

In this work we concentrate in some particular quantum networks which, despite its apparent simplicity, contain a very rich and intriguing behavior. The simplification comes from two facts (see Fig. 1): first, the nodes are spatially distributed in a regular way according to some geometry. Second, each pair of nodes are connected by a pure state \( |\varphi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \). Up to local change of bases, any of these states can be written as

\[
|\varphi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |ii\rangle,
\]

where \( \lambda_i \) are the (real) Schmidt coefficients such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0 \). This configuration reminds of the states underlying the so-called projected pair entangled states [18], and thus we call these networks pair-entangled pure networks (PEPN). For these geometries, we first introduce a series of protocols which are closely related to classical percolation [6], a concept that appears in statistical mechanics. We then determine the optimal protocols for several 1-dimensional (1D) configurations, where some counterintuitive phenomena occur. We use these phenomena to introduce various protocols in more complex 2-dimensional (2D) configurations.

We show that these new protocols provide a dramatic improvement over those based on classical percolation, in the sense that one can obtain perfect quantum communication even though the percolation protocols give rise to an exponential decay of the success probability with the number of nodes. In fact, we will argue that there exists a quantum phase transition in the quantum networks which may be exploited to obtain very efficient protocols. Thus, this work opens a new set of problems in quantum information theory which are related to statistical physics, but pose completely new challenges in these fields. As opposed to most of the recent work on entanglement theory, which has been devoted to using some of the tools developed so far in quantum informa-
tion theory to analyze problems in statistical mechanics [18, 19, 29, 22], the present work makes a step in the converse direction.

**Classical entanglement percolation.** A first natural measurement consists of all the pairs of nodes nodes locally transforming their states into singlets with optimal probability, \( p^{ok} \). Recall that the SCP for a state [1] is known to be equal to \( p^{ok} = \min(1, 2(1 - \lambda_1)) \) [22]. Then, a perfect quantum channel between the nodes is established with probability \( p^{ok} \), otherwise no entanglement is left. This problem is equivalent to a standard bond percolation situation [6], where one distributes connections among the nodes of a lattice in a probabilistic way: with probability \( p \) an edge connecting a pair of nodes is established, otherwise the nodes are kept unconnected. We call this measurement strategy classical entanglement percolation (CEP). In bond percolation, for each lattice geometry there exists a percolation threshold probability, \( p_{th} \), such that an infinite connected cluster can be established if and only if \( p > p_{th} \) (see also Table 1). The probability \( \theta(p) \) that a given node belongs to an infinite cluster, or percolation probability, is strictly positive for \( p > p_{th} \), and zero otherwise (in the limit of an infinite number of nodes). Then, the probability that two given distant nodes can be connected by a path is distance independent and given, correspondingly, by \( \theta^2(p) \) for \( p > p_{th} \); for \( p < p_{th} \) this probability decays exponentially with the number of nodes, \( N \).

The threshold probabilities define a minimal amount of entanglement for the initial state such that CEP is possible. In the case of 1D chains, see Fig. 2 percolation is possible if, and only if, \( p = 1 \). Therefore, the SCP decays exponentially with \( N \) unless the states are more entangled than the singlet, in the sense that \( p^{ok} = 1 \). In a square 2D lattice, the entanglement threshold derived from percolation arguments, see Table 1 is \( p^{ok} = 2(1 - \lambda_1) = 1/2 \).

It is natural to wonder whether CEP is optimal for any geometry and number of nodes. And if not, to see if, at least, it predicts the correct decay of entanglement in the asymptotic limit. Next, we show that, for 1D chains, although CEP is not optimal for some finite \( N \), it gives the right asymptotic behavior. Moving to 2D networks, we prove that CEP is not optimal even in the asymptotic case. Thus, the problem of entanglement distribution through quantum networks defines a new type of critical phenomenon, that we call entanglement percolation.

**1D Chains** The scenario of a 1D chain configuration, see Fig. 2 consists of two end nodes connected by several repeaters [10]. As said, all the bonds are equal to \( |\varphi\rangle \).

We start out with the case of qubits, \( d = 2 \). A surprising result already appears in the first non-trivial situation consisting of one repeater. An upper bound to the SCP in the one-repeater scenario is obtained by putting nodes \( A \) and \( R_1 \) together, which implies that the SCP cannot be larger than \( p^{ok} \). This bound can indeed be achieved by means of a rather simple protocol involving entanglement swapping [23] at the repeater. However, if CEP is applied, the obtained SCP is simply \( (p^{ok})^2 \). This proves that CEP is not optimal already for the one-repeater configuration. We find quite counter-intuitive that the intermediate repeater does not decreases the optimal SCP. This behavior, however, does not survive in the asymptotic limit. In this limit, the so-called concurrence [24], another measure of entanglement, decreases exponentially with the number of repeaters, unless the connecting states are maximally entangled (see Methods). The exponential decay of the SCP automatically follows.

Most of these results can be generalized to higher dimensional systems, \( d > 2 \). For the one-repeater configuration, the SCP is again equal to \( p^{ok} \). It suffices to map the initial state into a two-qubit state, without changing the SCP, and then apply the previous protocol. Moving to the asymptotic limit, an exponential decay of the SCP with \( N \) can be proven in the scenario where the measurement strategies only involve one-way communication: first, a measurement is performed at the first repeater. The result is communicated to the second repeater, where a second measurement is applied. The results of the two measurements are communicated to the third, and so on until the last repeater, where the final measurement depends on all the previous results.

Putting all these results together, a unified picture emerges for the distribution of entanglement in 1D chains: despite some remarkable effects for finite \( N \), the SCP decreases exponentially with the number of repeaters whenever the connecting bonds have less entanglement than a singlet. The CEP strategy fails for some finite configuration, but predicts the correct behavior in the asymptotic limit.

**2D Lattices.** The situation becomes much richer for 2D geometries. First, we consider finite 2D lattices. The non-optimality of CEP can be shown already for the simplest \( 2 \times 2 \) square lattice and qubits. Consider the two non-neighboring sites in the main diagonal of the square. The SCP obtained by CEP is \( 1 - (1 - (p^{ok})^2)^2 \). By concatenating the optimal measurement strategy for the one-repeater configuration, the SCP is \( 1 - (1 - p^{ok})^2 \). However none of these strategies exploits the richness of the 2D configuration. Indeed, one can design strategies such that a singlet can be established with probability one whenever \( |\varphi\rangle \) satisfies \( 1/2 \leq \lambda_1 \lesssim 0.6498 \). Thus, there are 2D network geometries where, although the connections are not maximally entangled, the entanglement is still sufficient to establish a perfect quantum channel.

Let us now see whether the thresholds defined by standard percolation theory are optimal for asymptotically large networks. In the next lines, we construct an example that goes beyond the classical percolation picture, proving that the CEP strategy is not optimal. The key ingredient for this construction is the measurement derived above for the one-repeater configuration, which gave raise to a SCP equal to \( p^{ok} \). Our example considers a honeycomb lattice where each node is connected by two copies of the same two-qubit state \( |\varphi\rangle = |\varphi_2\rangle^{\otimes 2} \), see

...
If, as above, the Schmidt coefficients of the two-qubit state are \( \lambda_1 \geq \lambda_2 \), the SCP of \( |\varphi\rangle \) is given by \( p^{ok} = \frac{1}{2}(1 - \lambda_2^2) \). We choose this conversion probability smaller than the percolation threshold for the honeycomb lattice, which gives
\[
\lambda_1 = \sqrt{\frac{1}{2} + \sin\left(\frac{\pi}{18}\right)} \approx 0.82. \tag{2}
\]

So, CEP is useless. Now, half of the nodes perform the optimal strategy for the one-repeater configuration, mapping the honeycomb lattice into a triangular lattice, as shown in Fig. 3a. The SCP for the new bonds is exactly the same as for the state \( |\varphi_2\rangle \), that is \( 2\lambda_2 \). This probability is larger than the percolation threshold for the triangular lattice, since
\[
2\lambda_2 = 2 \left(1 - \sqrt{\frac{1}{2} + \sin\left(\frac{\pi}{18}\right)}\right) \approx 0.358 > 2 \sin\left(\frac{\pi}{18}\right). \tag{3}
\]

The nodes can now apply CEP to the new lattice and succeed. Thus, this strategy, which combines entanglement swapping and CEP, allows to establish a perfect quantum channel in a network where CEP fails.

**Conclusions.** We have shown that the distribution of entanglement through quantum networks defines a framework where statistical methods and concepts, such as classical percolation theory and beyond, naturally apply. It leads to a novel type of quantum phase transitions, that we call entanglement percolation, where the critical parameter is the minimal amount of entanglement necessary to establish a perfect quantum channel with significant (non-exponentially decaying) probability. Further understanding of optimal entanglement percolation strategies is necessary for the future development and prosperity of quantum networks.

**METHODS**

**1D chains**

We start by showing that the concurrence decays exponentially with the number of nodes in a 1D chain of qubits when the connecting states are not maximally entangled. Recall that, given a two-qubit pure state \( |\varphi\rangle = \sum_{i,j} t_{ij} |ij\rangle \), its concurrence reads \( C(\varphi) = 2|\det(T)| \), where \( T \) is the \( 2 \times 2 \) matrix such that \( (T)_{ij} = t_{ij} \).

When considering the repeater configuration, the maximization of the averaged concurrence turns out to be equal to
\[
C_N = \sup_{\mathcal{M}} \sum_r 2|\det(\varphi_1 M_r \varphi_2 \ldots M_r \varphi_{N+1})|. \tag{4}
\]

Here \( \mathcal{M} \) briefly denotes the choice of measurements, while \( \varphi_k \) represent the \( 2 \times 2 \) diagonal matrices given by the Schmidt coefficients of the states \( |\varphi_k\rangle \). \( M_r \) are also \( 2 \times 2 \) matrices, corresponding to the pure state \( |\mu_k\rangle \) associated to the measurement result \( r_k \) of the \( k \)-th repeater, that is \( |r_k\rangle = \sum_{i,j} (M_{r_k})_{ij} |ij\rangle \). Note that the computational basis \( i \) and \( j \) in the previous expressions are the Schmidt bases for the states \( |\varphi_k\rangle \) and \( |\varphi_{k+1}\rangle \) entering the repeater \( k \). Using the fact that \( \det(AB) = \det(A)\det(B) \), the previous maximization gives
\[
C_N = \prod_{k=1}^{N} 2|\det(\varphi_k)|. \tag{5}
\]

Note that \( |\det(\varphi_k)| = 1 \) if and only if \( |\varphi_k\rangle \) is maximally entangled, which proves the announced result.

Most of the results derived in the qubit case can be generalized to arbitrary dimension. Let us first consider the one-repeater configuration. Given a state \( |\varphi\rangle \), it is always possible to transform in a deterministic way this state into a two-qubit state of Schmidt coefficients \( (\lambda_1, 1 - \lambda_1) \) by local operations and classical communication (LOCC). This follows from the application of majorization theory to the study of LOCC transformations between entangled states [23]. Note that the SCP for the two states is the same, \( p^{ok} = \min(2(1 - \lambda_1), 1) \).

In the case of arbitrary \( N \), an exponential decay for the qubit concurrence can be shown for protocols with one-way communication. Given an arbitrary chain, we consider the almost identical chain where the first state is replaced by a two-qubit entangled state. It is relatively easy to prove that the SCP decays exponentially in the first chain if, and only if, it does it in the second one. We start with the simplest one-repeater configuration. The quantity to be optimized reads
\[
C_1 = 2|\det(\varphi_1)| \sup_{\mathcal{M}} \sum_r 2|\det(M_r \varphi_2)|, \tag{6}
\]

where, as above, \( \varphi_1 \) is the \( 2 \times 2 \) (\( d \times d \)) matrix corresponding to \( |\varphi_1\rangle \) \( (|\varphi_2\rangle) \), while \( M_r \) is a \( 2 \times d \) matrix associated to the measurement outcome \( r \) at the repeater. Thus, we recognize in the r.h.s. of Eq. (6) the optimal average concurrence we can obtain out of \( |\varphi_2\rangle \) by measurements on one particle which correspond to operators of rank 2. We denote this quantity by \( \bar{C} \), by \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \) the Schmidt coefficients corresponding to \( \varphi_2 \) and by \( p^{ok} \) its SCP, as above. For the outcome \( r \), which occurs with probability \( p_r \), \( \mu_1^r \geq \mu_2^r \) denote the Schmidt coefficients corresponding to the resulting two-qubit state \( |\varphi_r\rangle \). With this notation, we have
\[
\bar{C} = 2 \sum_r p_r \sqrt{\mu_1^r \mu_2^r} \leq 2\sqrt{x\sqrt{1 - x}}, \tag{7}
\]

where
\[
x = \sum_r p_r \mu_2^r \leq 1 - \lambda_1, \tag{8}
\]

where the last inequality follows from the majorization criterion [24]. The optimal value is obtained for \( x = p^{ok}/2 \), which is achieved when \( p_1 = 1 \). Thus, we obtain
\[
C_1 = 2|\det(\varphi_1)| \sqrt{p^{ok}(2 - p^{ok})}. \tag{9}
\]
Percolation Threshold Probability

| Lattice   | Percolation Threshold Probability |
|-----------|----------------------------------|
| Square    | $\frac{1}{2}$                    |
| Triangular| $2 \sin \left( \frac{\pi}{8} \right) \approx 0.3473$ |
| Honeycomb | $1 - 2 \sin \left( \frac{\pi}{16} \right) \approx 0.6527$ |

TABLE I: Bond Percolation Threshold Probabilities for some examples of 2D lattices.

Note that $\sqrt{p^{ok}(2 - p^{ok})} \leq 1$, with equality if and only if $p^{ok} = 1$. Note also, and this is important for what follows, that the optimal strategy only depends on $|\varphi_2\rangle$, and not on the first two-qubit state, $|\varphi_1\rangle$.

This strategy can be generalized to the case of $N$ repeaters when the measurements proceed from left to right. We show this generalization for the case $N = 2$, the case of arbitrary $N$ will immediately follow. Consider the measurement step in the second repeater. After receiving the information about the measurement result in the first repeater $r_1$, $R_2$ has to measure his particles. For each value of $r_1$, and since $A$ is a qubit, $A$ and $R_2$ share a two-qubit pure state, $|\varphi_1\rangle$. Therefore, for each measurement result, $R_2$ is back at the previous one-repeater situation. The optimal measurement strategy in this case was independent of the entanglement of the first two-qubit state. Thus, up to local unitary transformations, the measurement to be applied in the second repeater is independent of $r_1$, and

$$C_2 = C_1|\sqrt{p^{ok}(2 - p^{ok})}|,$$

where $p^{ok}$ is defined as above for the state $|\varphi_2\rangle$. It is straightforward that this reasoning generalizes to an arbitrary number of repeaters, so

$$C_N = 2|\det(\varphi_1)| \prod_{j=2}^{N+1} \sqrt{p_j^{ok}(2 - p_j^{ok})}.$$  \hspace{1cm} (11)

Therefore, the average concurrence decreases exponentially with the number of repeaters unless the connecting pure states have $p^{ok} = 1$. A non-exponential decay of the SCP when $p^{ok} < 1$ would contradict this result.

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(a) ![Diagram](image1)

(b) ![Diagram](image2)

FIG. 1: Quantum networks. A general quantum communication network consists of an arbitrary number of nodes in a given geometry sharing some quantum correlations, given by a global state $\rho$, as in (a). Here we consider a simplified network where the nodes are disposed according to a well-defined geometry, e.g. a 2D square lattice in (b), and where each pair of nodes is connected by the same pure state $|\varphi\rangle$.

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FIG. 2: Quantum repeaters in the 1D chain. The upper figure shows the simplest one-repeater configuration, that is generalized below. The first step of the optimal strategy for the one-repeater configuration and qubits, where $|\varphi_1\rangle = |\varphi_2\rangle = \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle$, consists of entanglement swapping at the repeater. The resulting states between nodes $A$ and $B$ are $(\lambda_1|00\rangle \pm \lambda_2|11\rangle)/\sqrt{\lambda_1 + \lambda_2}$ with probability $p = \lambda_1/\lambda_2$. Collecting all these terms, the average SCP between $A$ and $B$ is equal to $2(\lambda_1 + \lambda_2) = 2\lambda_2 = p^{\text{opt}}$, which is known to be optimal.

FIG. 3: Example of quantum network where entanglement percolation and CEP are not equivalent. Each node is connected by a state consisting of two copies of the same two-qubit state, $|\varphi\rangle = |\varphi_2\rangle^\otimes 2$. The nodes marked in (a) perform the optimal measurement for the one-repeater configuration on pairs of qubits belonging to different connections, as shown in the inset. A triangular lattice is then obtained where the SCP for each connection is the same as for the two-qubit state $|\varphi_2\rangle$. The remaining nodes perform CEP on the new lattice.