Two enriched poset polytopes

Soichi Okada∗ and Akiyoshi Tsuchiya†

Abstract

Stanley introduced and studied two lattice polytopes, the order polytope and chain polytope, associated to a finite poset. Recently Ohsugi and Tsuchiya introduce an enriched version of them, called the enriched order polytope and enriched chain polytope. In this paper, we give a piecewise-linear bijection between these enriched poset polytopes, which is an enriched analogue of Stanley’s transfer map and bijectively proves that they have the same Ehrhart polynomials. Also we construct explicitly unimodular triangulations of two enriched poset polytopes, which are the order complexes of graded posets.

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1 Introduction

We assume that readers are familiar with the definition of a poset presented in [10, Chapter 3]. Let $P$ be a finite poset with $d$ elements. We denote by $\mathbb{R}^P$ the vector space of all real-valued functions on $P$, and identify $\mathbb{R}^P$ with the Euclidean space $\mathbb{R}^d$. The order polytope $O(P)$ of $P$ is the subset of $\mathbb{R}^P$ consisting of all functions $f : P \rightarrow \mathbb{R}$ satisfying the following two conditions:

(i) $0 \leq f(v) \leq 1$ for all $v \in P$;
(ii) If $x < y$ in $P$, then we have $f(x) \leq f(y)$.

And the chain polytope $C(P)$ of $P$ is the subset of $\mathbb{R}^P$ consisting of all functions $g : P \rightarrow \mathbb{R}$ satisfying the following two conditions:

(i) $g(v) \geq 0$ for all $v \in P$;
(ii) If $v_1 > \cdots > v_r$ is a chain in $P$, then we have $g(v_1) + \cdots + g(v_r) \leq 1$.

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Then it is known (see [9 Corollary 1.3 and Theorem 2.2]) that $O(P)$ and $C(P)$ are convex polytopes whose vertex sets are given by
\[
O(P) = \text{conv } \mathcal{F}(P), \quad C(P) = \text{conv } \mathcal{A}(P),
\]
respectively, where $\chi_S$ is the characteristic function of a subset $S \subset P$ defined by $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. Here an order filter of $P$ is a subset $F \subset P$ such that if $v \in F$ and $v < w$ then $w \in F$. In particular, we have
\[
O(P) = \text{conv } \mathcal{F}(P), \quad C(P) = \text{conv } \mathcal{A}(P),
\]
where conv $S$ denotes the convex hull of $S$. These poset polytopes are related via the transfer map.

**Theorem 1.1.** (Stanley [9 Theorem 3.2]) We define a piecewise-linear map $\Phi : \mathbb{R}^P \to \mathbb{R}^P$, called the transfer map, by
\[
(\Phi f)(v) = \begin{cases} f(v) & \text{if } v \text{ is minimal in } P, \\ f(v) - \max\{f(w) : v \text{ covers } w \text{ in } P\} & \text{if } v \text{ is not minimal in } P \end{cases}
\]
for $f \in \mathbb{R}^P$ and $v \in P$. Then $\Phi$ induces a continuous bijection from $O(P)$ to $C(P)$. In particular, $\Phi$ provides a bijection between $mO(P) \cap \mathbb{Z}^P$ and $mC(P) \cap \mathbb{Z}^P$ for any nonnegative integer $m$, where $m\mathcal{P} = \{mf : f \in \mathcal{P}\}$ is the $m$th dilation of a polytope $\mathcal{P}$ and $\mathbb{Z}^P$ is the set of all integer-valued functions on $P$.

The transfer map enables us to compare certain properties of $O(P)$ and $C(P)$. For example, the two polytopes $O(P)$ and $C(P)$ have the same Ehrhart polynomials, i.e.,
\[
\#(mO(P) \cap \mathbb{Z}^P) = \#(mC(P) \cap \mathbb{Z}^P). \tag{2}
\]
We note that the polynomial $\#(mO(P) \cap \mathbb{Z}^P)$ in $m$ is the order polynomial (with a shifted argument) of the poset $P$, which counts the number of $P$-partitions. A map $h : P \to \mathbb{Z}_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers, is called a $P$-partition if $v \leq w$ implies $h(v) \leq h(w)$. Then $mO(P) \cap \mathbb{Z}^P$ is the set of all $P$-partitions $h : P \to \mathbb{Z}_{\geq 0}$ such that $h(v) \leq m$ for all $v \in P$. Hence the transfer map $\Phi$ gives a bijection between such $P$-partitions and lattice points in the $m$th dilation of the chain polytope $C(P)$. In a very recent work, Higashitani [4] proves that $O(P)$ and $C(P)$ are combinatorially mutation-equivalent by using the transfer map $\Phi$. The notion of combinatorial mutation was introduced from viewpoints of mirror symmetry for Fano manifolds. Also we can transfer a canonical triangulation of $O(P)$, which is the order complex of a graded poset as simplicial complexes, to $C(P)$ via the transfer map $\Phi$.

**Theorem 1.2.** (Stanley [9 Section 5]) For a chain $C = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k\}$ of order filters of $P$, we put
\[
S_C = \text{conv}\{\chi_{F_1}, \ldots, \chi_{F_k}\}, \quad T_C = \text{conv}\{\Phi(\chi_{F_1}), \ldots, \Phi(\chi_{F_k})\}. \tag{3}
\]
Then we have
Theorem 1.3. We define a piecewise-linear map \( \Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P \), which we call the \textit{enriched transfer map}, inductively on the ordering of \( P \) such that the value of \( \Phi^{(e)}(f) \) at \( v \) is equal to

\[
\Phi^{(e)}(f) = \begin{cases} 
1 & \text{if } f(v) = 0, \\
0 & \text{if } f(v) = 1, \\
-1 & \text{if } f(v) = -1,
\end{cases}
\]

for any nonnegative integer \( m \). It is a natural problem to find a bijective proof of this equality \((6)\).
canonical triangulations

S
induced subposet of
F
sets of defining inequalities of facets of the triangulation
S
and give an explicit bijection between left enriched
P
triangulations with Ohsugi–Tsuchiya’s triangulations algebraically obtained in [6, 7].

Then \( \Phi^{(e)} \) induces a continuous bijection from \( \mathcal{O}^{(e)}(P) \) to \( \mathcal{C}^{(e)}(P) \). In particular, \( \Phi^{(e)} \) provides a bijection between \( m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P \) and \( m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P \) for any nonnegative integer \( m \).

Moreover, by composing with \( \Phi^{(e)} \), we also obtain an explicit bijection between \( m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P \) and \( \mathcal{E}_m(P) \) for any nonnegative integer \( m \) (Proposition 2.9).

It can be shown (see Proposition 2.7) that the restriction of \( \Phi^{(e)} \) to \( \mathcal{O}(P) \) gives a continuous piecewise-linear bijection between \( \mathcal{O}(P) \) and \( \mathcal{C}(P) \), which coincides with the restriction of Stanley’s transfer map \( \Phi \) in Theorem 1.1. Also, by the same technique of [4], we can show that \( \mathcal{O}^{(e)}(P) \) and \( \mathcal{C}^{(e)}(P) \) are combinatorially mutation-equivalent by using the enriched transfer map \( \Phi^{(e)} \) (see [4] Section 5).

Ohsugi–Tsuchiya [6, 7] constructed triangulations of enriched order and chain polytopes by using the algebraic technique of Gröbner bases. Also, Kohl–Olsen–Sanyal [5] constructed triangulations of enriched chain polytopes from a viewpoint of convex geometry. Another main result of this paper is an explicit combinatorial description of triangulations of two enriched poset polytopes, which are the order complexes of graded posets as simplicial complexes and are transferred by the enriched transfer map \( \Phi^{(e)} \). Our result is analogous to Stanley’s canonical triangulations of two poset polytopes (see Theorem 1.2).

**Theorem 1.4.** We equip \( \mathcal{F}^{(e)}(P) \) with a poset structure by the partial ordering given in Definition 3.1. For a chain \( K \) in \( \mathcal{F}^{(e)}(P) \), we define

\[
S_K^{(e)} = \text{conv } K, \quad T_K^{(e)} = \text{conv } \Phi^{(e)}(K).
\]

Then we have

(a) The set \( S_P^{(e)} = \{ S_K^{(e)} : K \text{ is a chain in } \mathcal{F}^{(e)}(P) \} \) is a unimodular triangulation of \( \mathcal{O}^{(e)}(P) \).

(b) The set \( T_P^{(e)} = \{ T_K^{(e)} : K \text{ is a chain in } \mathcal{F}^{(e)}(P) \} \) is a unimodular triangulation of \( \mathcal{C}^{(e)}(P) \).

Remark that the partial ordering on \( \mathcal{F}^{(e)}(P) \) given in Definition 3.1 is an extension of the inclusion ordering on the set of order filters of \( P \), so the poset \( \mathcal{F}(P) \) is the induced subposet of \( \mathcal{F}^{(e)}(P) \). Stanley gave the defining inequalities of facets of the canonical triangulations \( S_P \) and \( T_P \) of \( \mathcal{O}(P) \) and \( \mathcal{C}(P) \) (Proposition 3.12). We also give sets of defining inequalities of facets of the triangulation \( S_P^{(e)} \) and \( T_P^{(e)} \) of \( \mathcal{O}^{(e)}(P) \) and \( \mathcal{C}^{(e)}(P) \) (Corollary 3.11 and Proposition 3.13). On the other hand, we identify these triangulations with Ohsugi–Tsuchiya’s triangulations algebraically obtained in [6, 7] (Propositions 3.2 and 3.3).

The rest of this paper is organized as follows. In section 2 we prove Theorem 1.3 and give an explicit bijection between left enriched \( P \)-partitions and lattice points of
the dilated enriched order polytope. Section 3 is devoted to the proof of Theorem 1.4. We also give sets of defining inequalities for the maximal faces. In Section 4, we prove that the triangulations described in Theorem 1.4 coincide with the Ohsugi–Tsuchiya’s triangulations.

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2 Enriched transfer map

In this section, we give a proof of Theorem 1.3, and we use the enriched transfer map to describe a bijection between left enriched \( P \)-partitions and lattice points of the dilated enriched order polytope.

2.1 Notations

In what follows, we use the following notations and terminologies. Let \( P \) be a finite poset. For \( v, w \in P \), we say that \( v \) covers \( w \), written \( v \gtrdot w \), if \( v > w \) and there is no element \( u \) such that \( v > u > w \). Given an antichain \( A \), we denote by \( \langle A \rangle \) the smallest order filter containing \( A \). Given an element \( v \in P \), we put

\[
P_{\leq v} = \{ w \in P : w \leq v \}, \quad P_{< v} = \{ w \in P : w < v \}.
\]

For a subposet \( Q \) of \( P \), we denote by \( \max Q \) and \( \min Q \) the set of maximal and minimal elements of \( Q \) respectively. For a chain \( C = \{ v_1 > v_2 > \cdots > v_r \} \) of \( Q \), we say that

- \( C \) is saturated if \( v_i \gtrdot v_{i+1} \) for \( i = 1, \ldots, r-1 \);
- \( C \) is maximal if it is saturated and \( v_1 \in \max Q \) and \( v_r \in \min Q \).

Let \( C(Q), SC(Q) \) and \( MC(Q) \) be the sets of all chains, all saturated chains and all maximal chains respectively. We denote by \( \top C \) the maximum element of a chain \( C \). For \( f \in \mathbb{R}^P \) and a chain \( C = \{ v_1 > \cdots > v_r \} \), we define

\[
S(f; C) = |f(v_1)| + \cdots + |f(v_r)|,
\]

\[
T^+(f; C) = -f(v_1) - 2f(v_2) - \cdots - 2^{r-2}f(v_{r-1}) + 2^{r-1}f(v_r),
\]

\[
T^-(f; C) = -f(v_1) - 2f(v_2) - \cdots - 2^{r-2}f(v_{r-1}) - 2^{r-1}f(v_r).
\]

Note that, if \( C \) is a one-element chain \( \{ v \} \), then \( T^+(f; \{ v \}) = f(v) \) and \( T^-(f; \{ v \}) = -f(v) \).
2.2 Defining inequalities for enriched poset polytopes

Our proof of Theorem 1.3 is based on the defining inequalities of \( O^{(e)}(P) \) and \( C^{(e)}(P) \) given by [7].

**Proposition 2.1.** ([6] Lemma 1.1, [7] Proposition 6.1 and Theorem 6.2) We have
\[
O^{(e)}(P) = \left\{ f \in \mathbb{R}^P : T^+(f; C) \leq 1 \text{ for all } C \in SC(P) \text{ with top } C \in \text{max}(P) \right\},
\]
and
\[
C^{(e)}(P) = \left\{ g \in \mathbb{R}^P : S(g; C) \leq 1 \text{ for all } C \in \text{MC}(P) \right\}.
\]

**Example 2.2.** Let \( \Lambda \) be the three-element poset on \( \{u, v, w\} \) with covering relations \( u \lessdot w \) and \( v \lessdot w \). If we identify \( \mathbb{R}^\Lambda \) with \( \mathbb{R}^3 \) by the correspondence \( f \leftrightarrow (f(u), f(v), f(w)) \), we have
\[
\mathcal{F}^{(e)}(\Lambda) = \left\{ (0, 0, 0), (0, 0, 1), (0, 0, -1), (1, 0, 1), (-1, 0, 1), (0, 1, 1), (0, -1, 1) \right\},
\]
\[
\mathcal{A}^{(e)}(\Lambda) = \left\{ (1, 0, 0), (1, 1, 0), (1, -1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1) \right\},
\]
and
\[
O^{(e)}(\Lambda) = \left\{ f \in \mathbb{R}^\Lambda : \frac{f(w)}{f(w)} \leq 1, \quad \frac{f(u) + 2f(w)}{f(u) + 2f(w)} \leq 1, \quad \frac{-f(u) + 2f(w)}{f(u) - 2f(w)} \leq 1, \quad \frac{-f(u) - 2f(w)}{f(u) - 2f(w)} \leq 1 \right\},
\]
\[
C^{(e)}(\Lambda) = \left\{ g \in \mathbb{R}^\Lambda : |g(u)| + |g(w)| \leq 1, \quad |g(v)| + |g(w)| \leq 1 \right\}.
\]

2.3 Proof of Theorem 1.3

In this subsection we prove Theorem 1.3 The inductive definition [7] of \( \Phi^{(e)} \) can be written as
\[
\left( \Phi^{(e)}(f) \right)(v)
= \begin{cases} f(v) & \text{if } v \text{ is minimal in } P, \\ f(v) - \max\{S(\Phi^{(e)}(f); C) : C \in C(P_{<v})\} & \text{if } v \text{ is not minimal in } P. \end{cases}
\]
(11)

It is easy to see that the map \( \Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P \) is a bijection.

**Lemma 2.3.** The map \( \Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P \) is a bijection with inverse map \( \Psi^{(e)} \) given by
\[
\left( \Psi^{(e)}(g) \right)(v)
= \begin{cases} g(v) & \text{if } v \text{ is minimal in } P, \\ g(v) + \max\{S(g; C) : C \in C(P_{<v})\} & \text{if } v \text{ is not minimal in } P. \end{cases}
\]
(12)
Here we note that
\[
\max\{S(g; C) : C \in C(P_{\leq v})\} = \max\{S(g; C) : C \in \text{MC}(P_{\leq v})\},
\]
hence we may replace \(C(P_{\leq v})\) with \(\text{MC}(P_{\leq v})\) in \([11]\) and \([12]\). The following proposition follows from the definitions of \(\Phi^{(e)}\) and \(\Psi^{(e)}\).

**Proposition 2.4.** (a) For \(f \in \mathcal{F}^{(e)}(P)\), we have
\[
(\Phi^{(e)}(f))(v) = \begin{cases} 
  f(v) & \text{if } v \text{ is minimal in } \text{supp}(f), \\
  0 & \text{otherwise.}
\end{cases}
\]
In particular, \(\Phi^{(e)}(f) \in \mathcal{A}^{(e)}(P)\) and \(\text{supp} \Phi^{(e)}(f) = \text{min}(\text{supp}(f))\).

(b) For \(g \in \mathcal{A}^{(e)}(P)\), we have
\[
(\Psi^{(e)}(g))(v) = \begin{cases} 
  1 & \text{if } v \in \langle \text{supp}(g) \rangle \setminus \text{min}(\text{supp}(g)), \\
  g(v) & \text{if } v \in \text{min}(\text{supp}(g)), \\
  0 & \text{otherwise.}
\end{cases}
\]
In particular, \(\Psi^{(e)}(g) \in \mathcal{F}^{(e)}(P)\) and \(\text{supp} \Psi^{(e)}(g) = \langle \text{supp}(g) \rangle\).

(c) The map \(\Phi^{(e)}\) induces a bijection between \(\mathcal{F}^{(e)}(P)\) and \(\mathcal{A}^{(e)}(P)\).

In order to prove Theorem \([L.3]\) we need to prepare two lemmas. We put
\[
M(g; P_{\leq v}) = \max\{S(g; C) : C \in \text{MC}(P_{\leq v})\},
\]
\[
M(g; P_{< v}) = \max\{S(g; C) : C \in \text{MC}(P_{< v})\}.
\]

**Lemma 2.5.** Let \(f \in \mathbb{R}^P\) and \(v \in P\). We put
\[
\mathcal{T}(f; v) = \{T^+(f; C) : C \in \text{SC}(P_{\leq v}) \text{ with } \text{top} C = v\} \cup \{T^-(f; C) : C \in \text{MC}(P_{\leq v})\}.
\]
Then, for any \(C \in \text{MC}(P_{\leq v})\), there exists an element \(T \in \mathcal{T}(f; v)\) such that \(S(\Phi^{(e)}(f); C) \leq T\).

**Proof.** We write \(g = \Phi^{(e)}(f)\). We proceed by induction on the ordering of \(P\). If \(v\) is a minimal element, then \(C\) is a one-element chain \(\{v\}\) and
\[
S(g; C) = |g(v)| = |f(v)| = \begin{cases} 
  f(v) = T^+(f; C) & \text{if } f(v) \geq 0, \\
  -f(v) = T^-(f; C) & \text{if } f(v) \leq 0.
\end{cases}
\]
If \(v\) is not a minimal element, then by definition
\[
g(v) = f(v) - M(g; P_{< v}).
\]
Let \(C = \{v = v_1 \succ v_2 \succ \cdots \succ v_r\}\). Since \(C \setminus \{v\} = \{v_2 \succ \cdots \succ v_r\} \in \text{MC}(P_{< v})\), we have
\[
S(g; C \setminus \{v\}) \leq M(g; P_{< v}).
\]
If \( g(v) = f(v) - M(v; P_{\leq v}) \geq 0 \), then we have
\[
S(g; C) = f(v) - M(g; P_{\leq v}) + S(g; C \setminus \{v\}) \\
\leq f(v) = T^+(f; \{v\}).
\]

If \( g(v) \leq 0 \), then we have
\[
S(g; C) = -f(v) + M(g; P_{\leq v}) + S(g; C \setminus \{v\}) \\
\leq -f(v) + 2M(g; P_{\leq v}).
\]

Let \( C' \in \text{MC}(P_{\leq v}) \) be a chain which attains the maximum \( M(g; P_{\leq v}) \). Then, by applying the induction hypothesis to \( C' \) and \( w = \text{top} C' \), there exists a chain \( C'' \) satisfying one of the following conditions:

(i) \( C'' \in \text{SC}(P_{\leq w}) \) with \( \text{top} C'' = w \) and \( S(g; C') \leq T^+(g; C'') \);

(ii) \( C'' \in \text{MC}(P_{\leq w}) \) and \( S(g; C') \leq T^-(g; C'') \).

In the case (i), we have
\[
S(g; C) \leq -f(v) + 2S(g; C') \leq -f(v) + 2T^+(g; C'') = T^+(g; \{v\} \cup C''),
\]
and in the case (ii), we have
\[
S(g; C) \leq -f(v) + 2S(g; C') \leq -f(v) + 2T^-(g; C'') = T^-(g; \{v\} \cup C'').
\]
Since \( v \gg w \), we can complete the proof.

\[
\square
\]

**Lemma 2.6.** Let \( g \in \mathbb{R}^P \) and \( v \in P \). For a chain \( C = \{v_1 \gg v_2 \gg \cdots \gg v_r\} \in \text{SC}(P_{\leq v_1}) \), we have
\[
2^{r-1}(|g(v_r)| + M(g; P_{\leq v_r})) + \sum_{i=1}^{r-1} 2^{r-i-1} (|g(v_{r-i})| - M(g; P_{\leq v_{r-i}})) \leq M(g; P_{\leq v_1}).
\]

**Proof.** We proceed by induction on \( r \). If \( r = 1 \), then
\[
|g(v_1)| + M(g; P_{\leq v_1}) = |g(v_1)| + \max\{S(g; C') : C' \in \text{MC}(P_{\leq v_1})\} \\
= \max\{|g(v_1)| + S(g; C') : C' \in \text{MC}(P_{\leq v_1})\} \\
= \max\{S(g; C) : C \in \text{MC}(P_{\leq v_1})\} \\
= M(g; P_{\leq v_1}).
\]

Let \( r \geq 2 \). Since \( \{v_r\} \cup C' \in \text{MC}(P_{\leq v_{r-1}}) \) for any \( C' \in \text{MC}(P_{\leq v_r}) \), we have
\[
|g(v_r)| + M(g; P_{\leq v_r}) = |g(v_r)| + \max\{S(g; C') : C' \in \text{MC}(P_{\leq v_r})\} \\
\leq \max\{S(g; C'') : C'' \in \text{MC}(P_{\leq v_{r-1}})\} = M(g; P_{\leq v_{r-1}}).
\]
Hence we have
\[
2^{r-1} (|g(v_r)| + M(g; P_{\leq v_r})) + 2^{r-2} (|g(v_{r-1})| - M(g; P_{\leq v_{r-1}}))
\]

\[ \leq 2^{r-1}M(g; P_{<v_r}) + 2^{r-2} \left( |g(v_{r-1})| - M(g; P_{<v_{r-1}}) \right) \]
\[ = 2^{r-2} \left( |g(v_{r-1})| + M(g; P_{<v_{r-1}}) \right). \]

Therefore, by using the induction hypothesis, we see that
\[ 2^{r-1} \left( |g(v_r)| + M(g; P_{<v_{r}}) \right) + \sum_{i=1}^{r-1} 2^{r-i-1} \left( |g(v_{r-i})| - M(g; P_{<v_{r-i}}) \right) \]
\[ \leq 2^{r-2} \left( |g(v_{r-1})| + M(g; P_{<v_{r-1}}) \right) + \sum_{i=2}^{r-1} 2^{r-i-1} \left( |g(v_{r-i})| - M(g; P_{<v_{r-i}}) \right) \]
\[ \leq M(g; P_{\leq v_1}). \]

This completes the proof. \(\square\)

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** First we shall prove that \( f \in \mathcal{O}(e)(P) \) implies \( \Phi(e)(f) \in \mathcal{C}(e)(P) \). Let \( f \in \mathcal{O}(e)(P) \) and put \( g = \Phi(e)(f) \). We show that \( S(g; C) \leq 1 \) for all maximal chains \( C = \{v_1 \gg v_2 \gg \cdots \gg v_r\} \) \( \in \text{MC}(P) \). By Lemma 2.5 there exists a chain \( C' \) satisfying one of the following conditions:

(i) \( C' \in \text{SC}(P_{\leq v_1}) \) with \( \text{top } C' = v_1 \) and \( S(g; C) \leq T^+(f; C') \);

(ii) \( C' \in \text{MC}(P_{\leq v_1}) \) and \( S(g; C) \leq T^-(f; C') \).

Then it follows from (9) in Proposition 2.1 that \( S(g; C) \leq 1 \). Hence, by using (10), we conclude that \( g \in \mathcal{C}(e)(P) \).

Conversely, we show that \( g \in \mathcal{C}(e)(P) \) implies \( \Psi(e)(g) \in \mathcal{O}(e)(P) \). Let \( g \in \mathcal{C}(e)(P) \) and put \( f = \Psi(e)(g) \). We need to prove that \( T^+(f; C) \leq 1 \) for all \( C \in \text{SC}(P) \) with \( \text{top } C \in \text{max}(P) \) and that \( T^-(f; C) \leq 1 \) for all \( C \in \text{MC}(P) \).

Suppose \( C = \{v_1 \gg v_2 \gg \cdots \gg v_r\} \in \text{SC}(P) \) with \( v_1 \in \text{max}(P) \). Then by definition
\[ T^+(f; C) = 2^{r-1} f(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} f(v_{r-i}) \]
\[ = 2^{r-1} \left( g(v_r) + M(g; P_{<v_1}) \right) - \sum_{i=1}^{r-1} 2^{r-i-1} \left( g(v_{r-i}) + M(g; P_{<v_{r-i}}) \right). \]

By using \( x \leq |x| \) and \( -x \leq |x| \), we see that
\[ T^+(f; C) \leq 2^{r-1} \left( |g(v_r)| + M(g; P_{<v_1}) \right) + \sum_{i=1}^{r-1} 2^{r-i-1} \left( |g(v_{r-i})| - M(g; P_{<v_{r-i}}) \right). \]

Then by using Lemma 2.6 we obtain
\[ T^+(f; C) \leq M(g; P_{\leq v_1}) = \max \{ S(g; C') : C' \in \text{MC}(P_{\leq v_1}) \}. \]

Since \( S(g; C') \leq 1 \) for all \( C' \in \text{MC}(P_{\leq v_1}) \) by (10), we have \( T^+(f; C) \leq 1 \).
Suppose \( C = \{ v_1 \succ v_2 \succ \cdots \succ v_r \} \in \text{MC}(P) \). Then \( v_1 \in \max(P) \) and \( v_r \in \min(P) \).

It follows from the definition that

\[
T^-(f; C) = -2^{r-1} f(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} f(v_{r-i}) \\
= -2^{r-1} g(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} (g(v_{r-i}) + M(g; P_{<v_{r-i}})) .
\]

By using \( x \leq |x| \) and \(-x \leq |x|\), we see that

\[
T^-(f; C) \leq 2^{r-1} |g(v_r)| + \sum_{i=1}^{r-1} 2^{r-i-1} (|g(v_{r-i})| - M(g; P_{<v_{r-i}})) .
\]

Since \( \{v_r\} \in \text{MC}(P_{<v_{r-1}}) \), we have \( |g(v_r)| \leq M(g; P_{<v_{r-1}}) \). Hence we have

\[
T^-(f; C) \\
\leq 2^{r-1} M(g; P_{<v_{r-1}}) + 2^{r-2} (|g(v_{r-1})| - M(g; P_{<v_{r-1}})) \\
+ \sum_{i=2}^{r-1} 2^{r-i-1} (|g(v_{r-i})| - M(g; P_{<v_{r-i}})) \\
= 2^{r-2} (|g(v_{r-1})| + M(g; P_{<v_{r-1}})) + \sum_{i=2}^{r-1} 2^{r-i-1} (|g(v_{r-i})| - M(g; P_{<v_{r-i}})) .
\]

Now we can use Lemma 2.6 and (9) to obtain \( T^-(f; C) \leq M(g; P_{v_1}) \leq 1 \).

Therefore we conclude that \( f \in \text{O}(P) \).

Here we show that the bijection \( \Phi^{(e)} : \text{O}^{(e)}(P) \to \text{C}^{(e)}(P) \) restricts to the bijection \( \Phi : \text{O}(P) \to \text{C}(P) \).

**Proposition 2.7.** The restriction of the enriched transfer map \( \Phi^{(e)} \) to \( \text{O}(P) \) coincides with the restriction of the transfer map \( \Phi \) to \( \text{O}(P) \).

**Proof.** Let \( f \in \text{O}(P) \) and put \( g = \Phi(f), \tilde{g} = \Phi^{(e)}(f) \). By using the induction on the ordering of \( P \), we prove

\[
\max\{ f(w) : w \preceq v \} = \max\{ g(v_1) + \cdots + g(v_r) : \{ v_1 \succ \cdots \succ v_r \} \in \text{MC}(P_{<v}) \} , \quad (13)
\]

\[
\tilde{g}(v) = g(v) \geq 0 . \quad (14)
\]

If \( v \) is minimal in \( P \), then \( f(v) = g(v) = \tilde{g}(v) \). If \( v \) is not minimal in \( P \) and \( \{w \in P : w \preceq v\} = \{w_1, \ldots, w_k\} \), then it follows from the induction hypothesis for (13) that

\[
\max\{ g(v_1) + \cdots + g(v_r) : \{ v_1 \succ \cdots \succ v_r \} \in \text{MC}(P_{<v}) \} \\
= \max_{1 \leq i \leq k} \{ g(w_i) + \max\{ g(v_2) + \cdots + g(v_r) : \{ v_2 \succ \cdots \succ v_r \} \in \text{MC}(P_{<w_i}) \} \} .
\]
By using the induction hypothesis for (13) and (1), we obtain

\[
\max \{|\tilde{g}(v_1)| + \cdots + |\tilde{g}(v_r)| : \{v_1 \gg \cdots \gg v_r\} \in \text{MC}(P_{<v})\} \\
= \max_{1 \leq i \leq k} \{ g(w_i) + \max \{ f(u_i) : u_i \ll w_i \} \} \\
= \max_{1 \leq i \leq k} f(w_i) = \max \{ f(w) : w \ll v \}.
\]

Hence, comparing (11) with (1), we obtain (13) and (14). □

2.4 Left enriched \(P\)-partitions

In this subsection, we use the enriched transfer map to find a bijection from left enriched \(P\)-partitions to lattice points of the dilated enriched order polytope.

Recall the definition of left enriched \(P\)-partition introduced by Petersen [8]. A map \(h : P \rightarrow \mathbb{Z}\) is called a left enriched \(P\)-partition if it satisfies the following two conditions:

(i) If \(v \leq w\), then \(|h(v)| \leq |h(w)|\);

(ii) If \(v \leq w\) and \(|h(v)| = |h(w)|\), then \(h(w) \geq 0\).

We denote by \(\mathcal{E}_m(P)\) the set of left enriched \(P\)-partitions \(h : P \rightarrow \mathbb{Z}\) such that \(|h(v)| \leq m\) for all \(v \in P\). Note that \(F^{(e)}(P) = \mathcal{E}_1(P)\). Ohsugi–Tsuchiya [6] gave an explicit bijection between \(\mathcal{E}_m(P)\) and \(mC^{(e)}(P) \cap \mathbb{Z}^P\).

**Proposition 2.8.** ([6] Theorem 0.2 and its proof) Let \(\Pi : \mathcal{E}_m(P) \rightarrow \mathbb{R}^P\) be the map defined by

\[
(\Pi(h))(v) = \begin{cases} 
    h(v) & \text{if } v \text{ is minimal in } P, \\
    h(v) - \max\{|h(w)| : w \ll v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) \geq 0, \\
    h(v) + \max\{|h(w)| : w \ll v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) < 0.
\end{cases} \tag{15}
\]

Then \(\Pi\) gives a bijection from \(\mathcal{E}_m(P)\) to \(mC^{(e)}(P) \cap \mathbb{Z}^P\).

By composing this bijection \(\Pi\) with the inverse enriched transfer map \(\Psi^{(e)} : mC^{(e)}(P) \cap \mathbb{Z}^P \rightarrow mO^{(e)}(P) \cap \mathbb{Z}^P\), we obtain an explicit bijection from \(\mathcal{E}_m(P)\) to \(mO^{(e)}(P) \cap \mathbb{Z}^P\).

**Proposition 2.9.** Let \(\Theta : \mathcal{E}_m(P) \rightarrow \mathbb{R}^P\) be the map defined by

\[
(\Theta(h))(v) = \begin{cases} 
    h(v) & \text{if } v \text{ is minimal in } P \text{ or } h(v) \geq 0, \\
    h(v) + 2 \max\{|h(w)| : w \ll v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) < 0.
\end{cases} \tag{16}
\]

Then \(\Theta\) gives a bijection from \(\mathcal{E}_m(P)\) to \(mO^{(e)}(P) \cap \mathbb{Z}^P\).
Proof. We show that Θ = Ψ(e) ◦ Π. Let h ∈ 𝒟(P) and put g = Π(h). By comparing (11) with (15) and (16), it is enough to show
\[ \max\{S(g; C) : C ∈ MC(P_{<v})\} = \max\{|h(w)| : w < v\}. \]  
(17)

We proceed by induction on the ordering of P. If v is minimal in P, there is nothing to prove. Suppose that v is not minimal in P. Since h ∈ 𝒟(P), we have |h(v)| ≥ \max\{|h(w)| : w < v\}. Then it follows from (15) that
\[ |g(v)| = |h(v)| - \max\{|h(w)| : w < v\}. \]  
(18)

If \{w ∈ P : w < v\} = \{w_1, \ldots, w_k\}, then we have
\[ \max\{S(g; C) : C ∈ MC(P_{<v})\} = \max\ \{\max\{|g(w_i)| + \max\{S(g; C') : C' ∈ MC(P_{<w_i})\}\} : 1 ≤ i ≤ k\}. \]

By using (18), we have
\[ \max\{S(g; C) : C ∈ MC(P_{<v})\} = \max\ \{\max\{|g(w_i)| + \max\{|h(u_i)| : u_i < w_i\}| : 1 ≤ i ≤ k\}\} = \max\ \{\max\{|h(u_i)|\} : 1 ≤ i ≤ k\}, \]
from which (17) follows. □

2.5 Vertices of enriched poset polytopes

In this subsection, we determine the vertex sets of the enriched order polytope \( \mathcal{O}^{(e)}(P) \) and the enriched chain polytope \( \mathcal{C}^{(e)}(P) \).

In order to state the result, we need a partial ordering ≤ on \( \mathcal{F}^{(e)}(P) \) or \( \mathcal{A}^{(e)}(P) \). For \( f, f' ∈ \mathcal{F}^{(e)}(P) \) (or \( \mathcal{A}^{(e)}(P) \)), we write \( f ≤ f' \) if \( \text{supp}(f) ⊆ \text{supp}(f') \) and \( f|_{\text{supp}(f)} = f'|_{\text{supp}(f)} \).

Example 2.10. If \( Λ = \{u, v, w\} \) is the three-element chain with covering relations \( u < w \) and \( v < w \), then the Hasse diagrams of \( \mathcal{F}^{(e)}(P) \) and \( \mathcal{A}^{(e)}(P) \) with respect to ≤ are shown in Figures 1 and 2 respectively. The enriched order polytope \( \mathcal{O}^{(e)}(Λ) \) is the pyramid with five vertices \((1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1) \) and \((0, 0, -1)\), while the enriched chain polytope \( \mathcal{C}^{(e)}(Λ) \) is the bipyramid with six vertices \((1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1), (0, 0, 1) \) and \((0, 0, -1)\).

Proposition 2.11. (a) A point \( f ∈ \mathcal{F}^{(e)}(P) \) is a vertex of \( \mathcal{O}^{(e)}(P) \) if and only if \( f \) is maximal with respect to the ordering ≤.

(b) A point \( f ∈ \mathcal{A}^{(e)}(P) \) is a vertex of \( \mathcal{C}^{(e)}(P) \) if and only if \( f \) is maximal with respect to the ordering ≤.

Note that \( f ∈ \mathcal{A}^{(e)}(P) \) is maximal with respect to ≤ if and only if \( \text{supp}(f) \) is a maximal antichain.
Proof. (a) Let $f$ be a maximal element of $\mathcal{F}^{(e)}(\Lambda)$ with respect to $\preceq$. Assume to the contrary that $f$ is not a vertex of $\mathcal{O}^{(e)}(P)$. Then there exist elements $g_1, \ldots, g_r \in \mathcal{F}^{(e)}(P)$ and positive real numbers $\lambda_1, \ldots, \lambda_r$ such that $g_i \neq f$ and

$$f = \sum_{i=1}^{r} \lambda_i g_i, \quad \sum_{i=1}^{r} \lambda_i = 1.$$ 

Considering the value at $v \in P$, we have

$$\sum_{i=1}^{r} \lambda_i g_i(v) = f(v) = \sum_{i=1}^{r} \lambda_i f(v).$$

If $f(v) = 1$, then we see that $\sum_{i=1}^{r} \lambda_i (1 - g_i(v)) = 0$. Since $\lambda_i > 0$ and $1 - g_i(v) \geq 0$, we obtain $g_i(v) = 1$ for all $i$. By a similar reasoning, we see that, if $f(v) = -1$, then we have $g_i(v) = -1$ for all $i$. Hence we have $\text{supp}(f) \subset \text{supp}(g_i)$ and $f|_{\text{supp}(f)} = g_i|_{\text{supp}(f)}$. Since $f$ is maximal with respect to $\preceq$, we have $f = g_i$, which contradicts to the assumption $g_i \neq f$. Therefore $f$ is a vertex of $\mathcal{O}^{(e)}(P)$.

Conversely, suppose that $f$ is not maximal with respect to $\preceq$. Then there exists $g \in \mathcal{F}^{(e)}(P)$ such that $\text{supp}(f) \subset \text{supp}(g)$ and $f|_{\text{supp}(f)} = g|_{\text{supp}(f)}$. We take a
maximal element $u$ of $\text{supp}(g) \setminus \text{supp}(f)$ and define $f', f'': P \to \mathbb{R}$ by

$$f'(v) = \begin{cases} f(v) = g(v) & \text{if } v \in \text{supp}(f), \\ g(u) & \text{if } v = u, \\ 0 & \text{otherwise}, \end{cases} \quad f''(v) = \begin{cases} f(v) = g(v) & \text{if } v \in \text{supp}(f), \\ -g(u) & \text{if } v = u, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\text{supp}(f') = \text{supp}(f'') = \text{supp}(f) \cup \{u\}$ is an order filter of $P$ and $u$ is an minimal element of $\text{supp}(f') = \text{supp}(f'')$. Hence $f' \in \mathcal{F}(e)(P)$. Since $f = (f' + f'')/2$, we see that $f$ is not a vertex of $\mathcal{O}(e)(P)$.

(b) Similar to (a).

Remark 2.12. (1) A characterization of the vertex set of the enriched chain polytope $\mathcal{C}(e)(P)$ is also given in [5, Section 7].

(2) In general, the image $\Phi(e)(f)$ of a vertex $f$ of $\mathcal{O}(e)(P)$ under the enriched transfer map $\Phi(e)$ is not a vertex of $\mathcal{C}(e)(P)$, and the number of vertices of $\mathcal{O}(e)(P)$ is different from that of $\mathcal{C}(e)(P)$ (see Example 2.10).

3 Triangulations

In this section we prove Theorem 1.4, which describes triangulations of enriched order and chain polytopes.

3.1 Poset structure on $\mathcal{F}(e)(P)$

We introduce a partial ordering $\geq$ on $\mathcal{F}(e)(P)$, which is an extension of the inclusion ordering on the set of order filters of $P$. Note that this ordering $\geq$ is different from the ordering $\succeq$ used in Section 2.5.

Definition 3.1. For $f, g \in \mathcal{F}(e)(P)$, we write $f \geq g$ if the following three conditions hold:

(i) $\text{supp}(f) \supseteq \text{supp}(g)$;
(ii) $f(v) \geq g(v)$ for any $v \in \text{supp}(g)$;
(iii) If $v \in \text{supp}(g)$ and $v$ is minimal in $\text{supp}(f)$, then $f(v) = g(v)$.

Also we write $f \geq g$ if $f = g$ or $f \geq g$.

The following lemma is obvious, but will be used in several places.

Lemma 3.2. If $F \supset G$ are order filters of $P$ and $v \in G$ is minimal in $F$, then $v$ is minimal in $G$.

By using this lemma, we can prove that $\mathcal{F}(e)(P)$ is equipped with a poset structure with respect to the binary relation $\geq$.

Lemma 3.3. The binary relation $\geq$ given in Definition 3.1 is a partial ordering on $\mathcal{F}(e)(P)$.
Proof. It is enough to show the transitivity. Let \( f, g, h \in F(e)(\Lambda) \) satisfy \( f > g \) and \( g > h \). Then it is clear that \( \text{supp}(f) \supseteq \text{supp}(h) \) and \( f(v) \geq h(v) \) for any \( v \in \text{supp}(h) \). Since \( \text{supp}(f) \supseteq \text{supp}(g) \supseteq \text{supp}(h) \), it follows from Lemma 3.2 that, if \( v \in \text{supp}(h) \) is minimal in \( \text{supp}(f) \), then we have \( f(v) = g(v) = h(v) \).

Example 3.4. Let \( \Lambda \) be the three-element poset on \( \{u, v, w\} \) with covering relations \( u \preceq w \) and \( v \preceq w \). Figure 3 shows the Hasse diagram of \( (F(e)(\Lambda), \geq) \).

We collect several properties of this partial ordering on \( F(e)(P) \).

Proposition 3.5. The resulting poset \( F(e)(P) \) has the following properties.

(a) For order filters \( F \) and \( G \), we have \( F \supset G \) if and only if \( \chi_F \geq \chi_G \) in \( F(e)(P) \), where \( \chi_S \) is the characteristic function of \( S \).

(b) The zero map \( 0 \) is the unique minimal element of \( F(e)(P) \).

(c) If \( f \) covers \( g \) in \( F(e)(P) \), then \( \# \text{supp}(f) = \# \text{supp}(g) + 1 \).

(d) If \( f \) is a maximal element in \( F(e)(P) \), then \( \text{supp}(f) = P \).

(e) All maximal chains of \( F(e)(P) \) have the same length \( d = \#P \).

Proof. (a) and (b) are obvious.

(c) It is enough to show that, if \( f > g \), then there exists \( h \in F(e)(P) \) such that \( f \geq h > g \) and \( \# \text{supp}(h) = \# \text{supp}(g) + 1 \).

Since \( \text{supp}(f) \supseteq \text{supp}(g) \) and they are order filters of \( P \), there exists \( u \in \text{supp}(f) \) such that \( \text{supp}(g) \cup \{u\} \) is an order filter of \( P \). Then we define \( h : P \to \{1, 0, -1\} \) by putting

\[
h(v) = \begin{cases} 
  f(v) & \text{if } v \in \text{supp}(g) \cup \{u\}, \\
  0 & \text{otherwise.}
\end{cases}
\]

We see that \( h \in F(e)(P) \), \( \text{supp}(h) = \text{supp}(g) \cup \{u\} \), and \( f \geq h > g \).

(d) Suppose that \( \text{supp}(g) \neq P \). Since \( \text{supp}(g) \) is a proper order filter of \( P \), there exists \( u \notin \text{supp}(g) \) such that \( \text{supp}(g) \cup \{u\} \) is an order filter. Define \( f : P \to \{1, 0, -1\} \)
Proof. It follows from Definition 3.6 that $(\text{supp}(v))$.

We may assume $C, \phi$ are in bijection with pairs $(F, \phi)$.

Proof. Then we have $f \in \mathcal{F}(P)$, $\text{supp}(f) = \text{supp}(g) \cup \{u\}$ and $f \succ g$.

(e) follows from (b), (c) and (d).

Next we consider chains in the poset $\mathcal{F}(P)$.

**Definition 3.6.** Given a chain $K = \{f_1 > f_2 > \cdots > f_k\}$ of $\mathcal{F}(P)$, we define its *support* $\text{supp}(K)$ and *signature* $\text{sgn}(K)$ as follows. The support $\text{supp}(K)$ is the chain \{supp($f_1$) $\supseteq$ supp($f_2$) $\supseteq$ $\cdots$ $\supseteq$ supp($f_k$)\} of order filters. The signature $\text{sgn}(K)$ is the map $\phi : P \to \{1, 0, -1\}$ given by

(i) If $v$ is not minimal in supp($f_i$) for any $i$, then $\phi(v) = 0$;

(ii) If $v$ is minimal in supp($f_i$) for some $i$, then $\phi(v) = f_i(v)$.

The following lemma guarantees that the definition of $\phi(v)$ in the case (ii) is independent of the choice of $i$.

**Lemma 3.7.** Let $K = \{f_1 > f_2 > \cdots > f_k\}$ be a chain of $\mathcal{F}(P)$. If $v$ is minimal in both supp($f_i$) and supp($f_j$), then we have $f_i(v) = f_j(v)$.

**Proof.** We may assume $i < j$. Then $f_i > f_j$ and supp($f_i$) $\supseteq$ supp($f_j$). Since $v \in$ supp($f_j$) and minimal in supp($f_i$), we have $f_i(v) = f_j(v)$ by the condition (iii) in Definition 3.6.

A key property of support and signature is the following.

**Proposition 3.8.** Let $X(P)$ be the set of all chains of $\mathcal{F}(P)$ (including the empty chain), and $Y(P)$ the set of all pairs $(C, \phi)$ of chains $C = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k\}$ of order filters of $P$ and maps $\phi : P \to \{1, 0, -1\}$ satisfying

$$\text{supp}(\phi) = \bigcup_{i=1}^{k} \text{min } F_i,$$

where $\text{min } F_i$ is the set of minimal elements of $F_i$. Then the map $X(P) \ni K \mapsto (\text{supp}(K), \text{sgn}(K)) \in Y(P)$ is a bijection. In particular, maximal chains in $\mathcal{F}(P)$ are in bijection with pairs $(C, \phi)$ of maximal chains $C$ of order filters and maps $\phi : P \to \{1, -1\}$.

It follows that the number of maximal chains in $\mathcal{F}(P)$ is equal to $2^{d\varepsilon(P)}$, where $d = \#P$ and $\varepsilon(P)$ is the number of linear extensions of $P$.

**Proof.** It follows from Definition 3.6 that $(\text{supp}(K), \text{sgn}(K)) \in Y(P)$ for $K \in X(P)$.

Given a chain $C = \{F_1 \supseteq \cdots \supseteq F_k\}$ of order filters and a map $\phi : P \to \{1, 0, -1\}$ satisfying (19), we define $f_1, \cdots, f_k \in \mathbb{R}^P$ by

$$f_i(v) = \begin{cases} 1 & \text{if } v \in F_i \text{ and } v \text{ is not minimal in } F_i, \\ \phi(v) & \text{if } v \in F_i \text{ and } v \text{ is minimal in } F_i, \\ 0 & \text{if } v \notin F_i. \end{cases}$$
Then we see that \( f_i \in \mathcal{F}(e)(P) \) and \( \text{supp}(f_i) = F_i \).

We show that \( f_i > f_{i+1} \) for \( 1 \leq i \leq k - 1 \). Firstly one has \( \text{supp}(f_i) = F_i \supseteq F_{i+1} = \text{supp}(f_{i+1}) \). Secondly we check that \( f_i(v) \geq f_{i+1}(v) \) for \( v \in \text{supp}(f_{i+1}) \). Since \( v \in \text{supp}(f_{i+1}) \subset \text{supp}(f_i) \), we have \( f_i(v), f_{i+1}(v) \in \{1, -1\} \), and there is nothing to prove in the case \( f_i(v) = 1 \). If \( f_i(v) = -1 \), then \( v \) is minimal in \( \text{supp}(f_i) \), so \( v \) is minimal in \( \text{supp}(f_{i+1}) \) by Lemma 3.2. Then we have \( \varphi(v) = -1 \) and \( f_{i+1}(v) = -1 = f_i(v) \). Lastly, if \( v \in \text{supp}(f_{i+1}) \) and \( v \) is minimal in \( \text{supp}(f_i) \), then \( v \) is minimal in \( \text{supp}(f_{i+1}) \) by Lemma 3.2 and \( f_i(v) = \varphi(v) = f_{i+1}(v) \).

Therefore \( K = \{f_1 > f_2 > \cdots > f_k\} \) is a chain in \( \mathcal{F}(e)(P) \), and \( \text{supp}(K) = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k\} \), \( \text{sgn}(K) = \varphi \). \( \square \)

### 3.2 Triangulation of \( C(e)(P) \)

In this subsection, we use the triangulation of \( C(P) \) given in Theorem 1.2 to construct a unimodular triangulation of \( C(e)(P) \). We transfer this triangulation of \( C(e)(P) \) to \( O(e)(P) \) via the inverse enriched transfer map \( \Psi(e) \) in the next subsection.

A (lattice) triangulation of a lattice polytope \( P \subset \mathbb{R}^d \) of dimension \( d \) is a finite collection \( \Delta \) of (lattice) simplices such that

(i) every face of a member of \( \Delta \) is in \( \Delta \),

(ii) the union of the simplices in \( \Delta \) is \( P \), and

(iii) any two elements of \( \Delta \) intersect in a common (possibly empty) face.

We say that a triangulation \( \Delta \) is unimodular if all maximal faces of \( \Delta \) are unimodular, i.e., have the Euclidean volume \( 1/d! \).

Recall that the simplices \( S_C = \text{conv}\{\chi_F : F \in C\} \) and \( T_C = \text{conv}\{\Phi(\chi_F) : F \in C\} \) of the triangulation given in Theorem 1.2 are described as follows.

**Proposition 3.9.** (Stanley [9, Section 5]) If \( C = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k\} \) is a chain of order filters of \( P \), then we have

\[
S_C = \left\{ f \in \mathbb{R}^P : \begin{array}{l}
(i) \text{ \text{\textit{f}}} \text{ is constant on the subsets} P \setminus F_1, F_1 \setminus F_2, \ldots, F_{k-1} \setminus F_k, F_k, \\
(ii) 0 = f(P \setminus F_1) \leq f(F_1 \setminus F_2) \leq \cdots \leq f(F_{k-1} \setminus F_k) \leq f(F_k) = 1.
\end{array} \right\}
\]

and

\[
T_C = \Phi(S_C). \tag{20}
\]

If \( C = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k\} \) is a chain of order filters of \( P \), then \( \chi_C = \{\chi_{F_1} > \chi_{F_2} > \cdots > \chi_{F_k}\} \) is a chain in \( \mathcal{F}(e)(P) \) by Proposition 3.5 (a), and

\[
S_{\chi_C} = S_C, \quad T_{\chi_C}^{(e)} = T_C. \tag{21}
\]

First we show that any \( T_K^{(e)} = \text{conv} \Phi(e)(K) \) is obtained from \( T_C \) by a composition of reflections. For \( \varphi : P \to \{1, 0, -1\} \), we define a linear map \( R_{\varphi} : \mathbb{R}^P \to \mathbb{R}^P \) by

\[
(R_{\varphi}g)(v) = \begin{cases} 
g(v) & \text{if } \varphi(v) = 1 \text{ or } 0, \\
-g(v) & \text{if } \varphi(v) = -1.
\end{cases}
\]

The linear map \( R_{\varphi} \) is a composition of reflections along coordinate hyperplanes.
Proposition 3.10. For a chain $K$ in $\mathcal{F}_P^{(e)}$, we obtain
\[ T_K^{(e)} = R_{\text{sgn}(K)}(T_{\text{supp}(K)}). \] (22)

Proof. Let $K = \{f_1 > \cdots > f_k\}$ and put $C = \text{supp}(K) = \{F_1 \supseteq \cdots \supseteq F_k\}$ ($F_i = \text{supp}(f_i)$) and $\varphi = \text{sgn}(K)$. Since $T_C = \text{conv} \Phi(\chi_C)$, we have
\[ R_{\varphi}T_C = R_{\varphi}(\text{conv} \Phi^{(e)}(\chi_C)) = \text{conv}(R_{\varphi}(\Phi^{(e)}(\chi_C))). \]

Hence it is enough to show that $R_{\varphi}(\Phi^{(e)}(\chi_{F_i})) = \Phi^{(e)}(f_i)$ for each $i$.

By the definition of the enriched transfer map, we have
\[ \Phi^{(e)}(\chi_{F_i})(v) = \begin{cases} 1 & \text{if } v \text{ is minimal in } F_i, \\
0 & \text{otherwise}, \end{cases} \]
\[ \Phi^{(e)}(f_i)(v) = \begin{cases} f_i(v) & \text{if } v \text{ is minimal in } F_i, \\
0 & \text{otherwise}. \end{cases} \]

On the other hand, it follows from the definition of $\varphi = \text{sgn}(K)$ that
\[ \varphi(v) = \begin{cases} f_i(v) & \text{if } v \text{ is minimal in some } \text{supp}(f_i), \\
0 & \text{otherwise}. \end{cases} \]

Hence we obtain $R_{\varphi}(\Phi^{(e)}(\chi_{F_i})) = \Phi^{(e)}(f_i)$. $\square$

In order to prove Theorem 1.4 (b), we prepare several lemmas. Given $\varphi \in \{1, 0, -1\}^P$, we put
\[ V_{\varphi} = \left\{ g \in \mathbb{R}^P : \begin{array}{l}
(\text{i}) \text{ if } \varphi(v) = 1, \text{ then } g(v) \geq 0, \\
(\text{ii}) \text{ if } \varphi(v) = 0, \text{ then } g(v) = 0 \\
(\text{iii}) \text{ if } \varphi(v) = -1, \text{ then } g(v) \leq 0 \end{array} \right\}. \]

For $\varepsilon \in \{1, -1\}^P$, we put
\[ C_\varepsilon^{(e)}(P) = C^{(e)}(P) \cap V_\varepsilon, \quad A_\varepsilon^{(e)}(P) = A^{(e)}(P) \cap V_\varepsilon. \]

Since $\mathbb{R}^P = \bigcup_{\varepsilon \in \{1, -1\}^P} V_\varepsilon$, we have
\[ C^{(e)}(P) = \bigcup_{\varepsilon \in \{1, -1\}^P} C_\varepsilon^{(e)}(P), \quad A^{(e)}(P) = \bigcup_{\varepsilon \in \{1, -1\}^P} A_\varepsilon^{(e)}(P). \]

Lemma 3.11. ([6] lemma 1.1]) For $\varepsilon \in \{1, -1\}^P$, we have
\[ C_\varepsilon^{(e)}(P) = \text{conv} \left( A_\varepsilon^{(e)}(P) \right) = R_\varepsilon(C(P)). \]

Proof. The first equality is proved in [6] Lemma 1.1. We prove the second equality. Let $\varepsilon_0$ be the map given by $\varepsilon_0(v) = 1$ for all $v \in P$. Then $A_\varepsilon^{(e)}(P) = A(P)$ and $C_\varepsilon^{(e)}(P) = \text{conv} \left( A(P) \right) = C(P)$. Since $A_\varepsilon^{(e)}(P) = R_\varepsilon(A_\varepsilon^{(e)}(P)) = R_\varepsilon(A(P))$, we have
\[ C_\varepsilon^{(e)}(P) = \text{conv} \left( R_\varepsilon(A(P)) \right) = R_\varepsilon(\text{conv} \left( A(P) \right)) = R_\varepsilon(C(P)). \] $\square$
Lemma 3.12. Suppose that $\varphi \in \{1, 0, -1\}^P$ and $\varepsilon \in \{1, -1\}^P$ satisfy $\varphi|_{\text{supp}(\varphi)} = \varepsilon|_{\text{supp}(\varphi)}$. Then we have

(a) $V_{\varphi} \subset V_{\varepsilon}$.
(b) $R_{\varphi}|_{\text{supp}(\varphi)} = R_{\varepsilon}|_{\text{supp}(\varphi)}$, where $|\varphi|$ is defined by $|\varphi|(v) = |\varphi(v)|$.

**Proof.** (a) Let $g \in V_{\varphi}$. If $\varepsilon(v) = 1$, then $\varphi(v) = 1$ or $0$ and $g(v) \geq 0$. If $\varepsilon(v) = -1$, then $\varphi(v) = -1$ or $0$ and $g(v) \leq 0$. Hence $g \in V_{\varepsilon}$.
(b) Let $g \in V_{\varphi}$. If $v \in \text{supp}(\varphi)$, then we have $\varphi(v) = \varepsilon(v)$ and $(R_{\varphi}g)(v) = \varphi(v)g(v) = \varepsilon(v)g(v) = (R_{\varepsilon}g)(v)$. If $v \notin \text{supp}(\varphi)$, then we have $\varphi(v) = g(v) = 0$, thus $(R_{\varphi}g)(v) = g(v) = 0$ and $(R_{\varepsilon}g)(v) = \varepsilon(v)g(v) = 0$. \hfill \Box

Lemma 3.13. (a) Let $C = \{F_1 \supseteq \cdots \supseteq F_k\}$ be a chain of order filters of $P$. If $\varphi \in \{1, 0, -1\}^P$ satisfies $\text{supp}(\varphi) = \bigcup_{i=1}^k \text{min} F_i$, then $T_C \subset V_{\text{sgn}(K)}$.
(b) If $K$ is a chain in $\mathcal{F}(e)(P)$, then we have $T^{(e)}_K \subset V_{\text{sgn}(K)}$.
(c) If $K$ is a chain in $\mathcal{F}(e)(P)$ and $\varepsilon \in \{1, -1\}^P$ satisfies $\text{sgn}(K)|_{\text{supp}(\text{sgn}(K))} = \varepsilon|_{\text{supp}(\text{sgn}(K))}$, then we have $T^{(e)}_K = R_{\varepsilon}T_{\text{sgn}(K)}$.

**Proof.** (a) Let $g \in T_C$. It is enough to show that $\varphi(v) = 0$ implies $g(v) = 0$. By Theorem 1.2 (a), there exists $f \in S_C$ such that $g = \Phi(f)$. Let $i$ be the largest index such that $v \in F_i$, where we use the convention $F_0 = P$. If $i = 0$, then $f(v) = 0$ and $g(v) = 0$. Suppose that $i \geq 1$ and $\varphi(v) = 0$. Then $v \in F_i \setminus F_{i+1}$ by the maximality of $i$. Since $v$ is not minimal in $F_i$, there exists $w \in F_1$ such that $w \prec v$. If $w \in F_{i+1}$, then $v \in F_{i+1}$ (since $F_{i+1}$ is an order filter), which contradicts to the maximality of $i$. Hence we have $w \in F_i \setminus F_{i+1}$. Then by (20), we have $f(v) = f(w)$. Therefore $g(v) = (\Phi(f))(v) = f(v) - \max\{f(u) : u \prec v\} = f(v) - f(w) = 0$.
(b) Let $C = \text{supp}(K)$ and $\varphi = \text{sgn}(K)$. By (a), we have $T_C \subset V_{\text{sgn}(K)}$. Since $R_{\varphi}V_{\text{sgn}(K)} = V_{\varphi}$, we obtain $T_K = R_{\varphi}(T_C) \subset V_{\varphi}$.
(c) follows from Proposition 3.10 (a) and Lemma 3.12 (b). \hfill \Box

Note that $\Phi^{(e)}$ gives a bijection between $\mathcal{F}(e)(P)$ and $\mathcal{A}(e)(P)$ (Proposition 2.4 (c)), and that $R_{\varphi}$ preserves $\mathcal{A}(e)(P)$ for any $\varphi \in \{1, 0, -1\}^P$.

Lemma 3.14. Given $f_1, f_2 \in \mathcal{F}(e)(P)$ and $\varphi \in \{1, 0, -1\}^P$, we define $f'_1, f'_2 \in \mathcal{F}(e)(P)$ by the condition

$$R_{\varphi}\Phi^{(e)}(f_1) = \Phi^{(e)}(f'_1), \quad R_{\varphi}\Phi^{(e)}(f_2) = \Phi^{(e)}(f'_2).$$

Then $f_1 > f_2$ implies $f'_1 > f'_2$.

**Proof.** We may assume that there exists a unique $u \in P$ such that $\varphi(u) = -1$, i.e.,

$$(R_{\varphi}g)(v) = \begin{cases} g(v) & \text{if } v \neq u, \\ -g(v) & \text{if } v = u. \end{cases}$$
Then it follows from Proposition 2.4 that, if \( R_\varphi(\Phi^e(f)) = \Phi^e(f') \), then
\[
f'(v) = \begin{cases} 
-f(v) & \text{if } u \in \text{min}(\text{supp}(f)) \text{ and } v = u, \\
f(v) & \text{otherwise},
\end{cases}
\]
and \( \text{supp}(f') = \text{supp}(f) \).

Now we assume that \( f_1 > f_2 \). Then it is enough to prove the following two claims:

1. If \( u \in \text{supp}(f_2') \), then \( f'_1(u) \geq f_2'(u) \).
2. If \( u \in \text{supp}(f_2') \) and \( u \) is minimal in \( \text{supp}(f'_1) \), then \( f'_1(u) = f_2'(u) \).

First we prove (1) by dividing into four cases. If \( u \in \text{min}(\text{supp}(f_1)) \) and \( u \in \text{min}(\text{supp}(f_2)) \), then we have \( f_1(u) = f_2(u) \), thus \( f'_1(u) = -f_1(u) = -f_2(u) = f_2'(u) \). If \( u \in \text{min}(\text{supp}(f_1)) \) and \( u \not\in \text{min}(\text{supp}(f_2)) \), then it follows from Lemma 3.2 that \( u \not\in \text{supp}(f_2) \), which contradicts to the assumption \( u \in \text{supp}(f_2') = \text{supp}(f_2) \). If \( u \not\in \text{min}(\text{supp}(f_1)) \) and \( u \in \text{min}(\text{supp}(f_2)) \), then we have \( f'_1(u) = f_1(u) = 1 \), thus \( f'_1(u) \geq f_2'(u) \). If \( u \not\in \text{min}(\text{supp}(f_1)) \) and \( u \not\in \text{min}(\text{supp}(f_2)) \), then we have \( f'_1 = f_2 \) and \( f'_2 = f_2 \), thus \( f'_1(u) = f_2'(u) \).

Next we prove (2). If \( u \in \text{supp}(f_2') \) and \( u \) is minimal in \( \text{supp}(f'_1) \), then it follows from Lemma 3.2 that \( u \) is minimal in \( \text{supp}(f'_2) \), hence we see that \( f'_1(u) = -f_1(u) = -f_2(u) = f_2'(u) \). This completes the proof.

Now we are in position to prove Theorem 1.4 (b). 

**Proof of Theorem 1.4 (b).** We need to show the following four claims:

1. If \( K \) is a chain in \( \mathcal{F}^e(P) \), then \( T_K^e \) is a unimodular simplex.
2. If \( K \) is a chain in \( \mathcal{F}^e(P) \), then \( T_K^e \subset C^e(P) \).
3. \( \bigcup_K T_K^e = C^e(P) \), where \( K \) runs over all chains in \( \mathcal{F}^e(P) \).
4. If \( K \) and \( L \) are chains in \( \mathcal{F}^e(P) \), then \( T_K^e \cap T_L^e = T_{K \cap L}^e \).

Recall that \( T_K^e = \text{conv} \Phi^e(K) \) and \( R_\varphi : \mathbb{R}^P \rightarrow \mathbb{R}^P \) is a linear map given by
\[
(R_\varphi g)(v) = \begin{cases} 
g(v) & \text{if } \varphi(v) = 1 \text{ or } 0, \\
-g(v) & \text{if } \varphi(v) = -1,
\end{cases}
\]
for \( \varphi : P \rightarrow \{1, 0, -1\} \).

1. If we put \( C = \text{supp}(K) \) and \( \varphi = \text{sgn}(K) \), then \( T_K^e = R_\varphi(T_C) \) by Proposition 3.10. Since \( T_C \) is a unimodular simplex (Theorem 1.2 (b)) and \( R_\varphi \) is a composition of reflections, we see that \( T_K^e \) is a unimodular simplex.

2. We put \( C = \text{supp}(K) \) and \( \varphi = \text{sgn}(K) \), and take \( \varepsilon \in \{1, -1\}^P \) such that \( \varphi|_{\text{supp}(\varphi)} = \varepsilon|_{\text{supp}(\varphi)} \). Then, by using Lemma 3.13 (c) and Lemma 3.11 we have
\[
T_K^e = R_\varepsilon(T_C) \subset R_\varepsilon(C(P)) = C^e_\varepsilon(P) \subset C^e(P).
\]
follows from Lemma 3.14 that there exists a chain $C$. Similarly there exists a chain $D$. Therefore we have $R\left(\Phi\right) = \bigcup_{\varepsilon \in \{1,-1\}^P} R\left(\mathcal{C}(P)\right) = \bigcup_{\varepsilon \in \{1,-1\}^P} \bigcup_{C} R\left(T_C\right)$, where $C$ runs over all chain of order filters of $P$. Given a chain $C$ of order filters of $P$ and $\varepsilon \in \{1,-1\}^P$, we define $\varphi : P \to \{1,0,-1\}$ by putting

$$\varphi(v) = \begin{cases} 
\varepsilon(v) & \text{if } v \text{ is minimal in some } F_i, \\
0 & \text{otherwise}.
\end{cases}$$

Then it follows from Lemma 3.13 (c) that $R\varepsilon T_C = T_K^{(e)}$, where $K$ is the chain in $\mathcal{F}(P)$ corresponding to $(C,\varphi)$ under the bijection of Proposition 3.8.

(4) We put $C = \text{supp}(K)$, $\varphi = \text{sgn}(K)$, $D = \text{supp}(L)$ and $\psi = \text{sgn}(L)$. Then we have $T_K^{(e)} \subset V_\varphi$ and $T_L^{(e)} \subset V_\psi$ by Lemma 3.13 (b). If we define $\eta : P \to \{1,0,-1\}$ by putting

$$\eta(v) = \begin{cases} 
1 & \text{if } \varphi(v) = \psi(v) = 1, \\
-1 & \text{if } \varphi(v) = \psi(v) = -1, \\
0 & \text{otherwise},
\end{cases}$$

then we have $V_\varphi \cap V_\psi = V_\eta$. Hence we have

$$T_K^{(e)} \cap T_L^{(e)} = T_K^{(e)} \cap T_L^{(e)} \cap V_\eta.$$ 

Since $T_K^{(e)} = \text{conv}(\Phi^{(e)}(K))$ by definition, and $V_\eta$ is a “boundary” of $V_\varphi$, we see that

$$T_K^{(e)} \cap V_\eta = \text{conv}(\Phi^{(e)}(K)) \cap V_\eta = \text{conv}(\Phi^{(e)}(K) \cap V_\eta).$$

We take $\varepsilon \in \{1,-1\}^P$ satisfying $\eta|_{\text{supp}(\eta)} = \varepsilon|_{\text{supp}(\eta)}$. Then we have $\Phi^{(e)}(K) \cap V_\eta$, $\Phi^{(e)}(L) \cap V_\eta \subset A^{(e)}(P)$. Since $R\varepsilon$ gives a bijection between $A^{(e)}(P)$ and $A(P)$, it follows from Lemma 3.13 that there exists a chain $C'$ of order filters of $P$ such that $R\varepsilon(\Phi^{(e)}(\chi_{C'})) = \Phi^{(e)}(K) \cap V_\eta$. Hence we have

$$T_K^{(e)} \cap V_\eta = \text{conv}(R\varepsilon(\Phi^{(e)}(\chi_{C'}))) = R\varepsilon \text{conv}(\Phi^{(e)}(\chi_{C'})).$$

Similarly there exists a chain $D'$ of order filters of $P$ such that

$$T_L^{(e)} \cap V_\eta = \text{conv}(R\varepsilon(\Phi^{(e)}(\chi_{D'}))) = R\varepsilon \text{conv}(\Phi^{(e)}(\chi_{D'})).$$

Therefore we have

$$T_K^{(e)} \cap T_L^{(e)} = (T_K^{(e)} \cap V_\eta) \cap (T_L^{(e)} \cap V_\eta) = R\varepsilon \text{conv}(\Phi^{(e)}(\chi_{C'})) \cap R\varepsilon \text{conv}(\Phi^{(e)}(\chi_{D'})) = R\varepsilon \left(\text{conv}(\Phi^{(e)}(\chi_{C'})) \cap \text{conv}(\Phi^{(e)}(\chi_{D'}))\right) = R\varepsilon(T_{C'} \cap T_{D'}).$$

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By Theorem 1.2 (b), we see that \( T_{C'} \cap T_{D'} = \text{conv}(\Phi(e)(\chi_{C' \cap D'})) \). Hence we have

\[
T^{(e)}_K \cap T^{(e)}_L = R_{\varepsilon} \left( \text{conv}(\Phi^{(e)}(\chi_{C' \cap D'})) \right) \\
= R_{\varepsilon} \left( \text{conv}(\Phi^{(e)}(\chi_{C'}) \cap \Phi^{(e)}(\chi_{D'})) \right) \\
= \text{conv} \left( R_{\varepsilon}(\Phi^{(e)}(\chi_{C'})) \cap R_{\varepsilon}(\Phi^{(e)}(\chi_{D'})) \right) \\
= \text{conv} \left( (\Phi^{(e)}(K) \cap V_\eta) \cap (\Phi^{(e)}(L) \cap V_\eta) \right) \\
= \text{conv} \left( \Phi^{(e)}(K) \cap \Phi^{(e)}(L) \cap V_\eta \right) \\
= \text{conv} \left( \Phi^{(e)}(K \cap L) \right) \\
= T^{(e)}_{K \cap L}.
\]

This completes the proof of Theorem 1.4 (b). \( \square \)

We conclude this subsection with giving a set of defining inequalities of a facet \( T^{(e)}_K \), where \( K \) is a maximal chain in \( \mathcal{F}^{(e)}(P) \). Recall the result of Stanley [9] on the defining inequalities of facets of the triangulations of \( \mathcal{O}(P) \) and \( \mathcal{C}(P) \). To a maximal chain \( C = \{F_0 \supseteq F_1 \supseteq \cdots \supseteq F_d\} \) of order filters of \( P \), we associate a linear extension \((v_1, \ldots, v_d)\) and chains \( C_1, \ldots, C_d \) of \( P \) as follows. The linear extension \((v_1, \ldots, v_d)\) is defined by

\[ F_i = F_{i-1} \cup \{v_i\} \quad (i = 1, \ldots, d). \]

The chain \( C_i \) is given inductively by

(i) If \( v_i \) is minimal, then we put \( C_i = \{v_i\} \);

(ii) If \( v_i \) is not minimal and \( j \) is the largest index satisfying \( v_j \preceq v_i \), then we put \( C_i = \{v_i\} \cup C_j \).

**Proposition 3.15.** (Stanley [9] Section 5) Let \( C \) be a maximal chain of order filters of \( P \). Let \((v_1, \ldots, v_d)\) be the associated linear extension of \( P \) and \( C_1, \ldots, C_d \) the associated chains of \( P \). Then we have

(a) The facet \( S_C \) of the triangulation \( S_P \) of \( \mathcal{O}(P) \) is given by

\[ S_C = \{f \in \mathbb{R}^P : 0 \leq f(v_1) \leq f(v_2) \leq \cdots \leq f(v_d) \leq 1\}. \]

(b) If \( f \in S_C \), then we have

\[ (\Phi(f))(v_i) = f(v_i) - f(v_j), \]

where \( j \) is the largest index satisfying \( v_j \preceq v_i \).
(c) If we define
\[ L_i^C(g) = \sum_{v \in C_i} g(v), \]
then the facet \( T_C \) of the triangulation \( T_P \) of \( \mathcal{C}(P) \) is given by
\[ T_C = \{ g \in \mathbb{R}^P : 0 \leq L_1^C(g) \leq L_2^C(g) \leq \cdots \leq L_d^C(g) \leq 1 \}. \]

(d) If \( g \in T_C \), then we have
\[ (\Psi(g))(v_i) = \sum_{v \in C_i} g(v), \]
where \( \Psi : \mathcal{C}(P) \to \mathcal{O}(P) \) is the inverse transfer map.

Corollary 3.16. Let \( K \) be a maximal chain in \( \mathcal{F}^{(e)}(P) \) and put \( C = \text{supp}(K) \), \( \varepsilon = \text{sgn}(K) \). Let \( C_1, \ldots, C_d \) be the chains of \( P \) associated to \( C \), and define
\[ \bar{L}_i^K(g) = \sum_{v \in C_i} \varepsilon(v)g(v) \quad (g \in \mathbb{R}^P). \]

Then the face \( T_K^{(e)} \) of the triangulation \( T_P^{(e)} \) of \( \mathcal{C}^{(e)}(P) \) is given by
\[ T_K^{(e)} = \{ g \in \mathbb{R}^P : 0 \leq L_1^K(g) \leq L_2^K(g) \leq \cdots \leq L_d^K(g) \leq 1 \}. \quad (23) \]

**Proof.** It follows from Proposition 3.10 and Proposition 3.15(c).

### 3.3 Triangulation of \( \mathcal{O}^{(e)}(P) \).

In this subsection, we transfer the triangulation of \( \mathcal{C}^{(e)}(P) \) to \( \mathcal{O}^{(e)}(P) \) via the inverse map \( \Psi^{(e)} \) of the enriched transfer map \( \Phi^{(e)} \). In order to prove Theorem 1.4(a), it is enough to show that \( S_K^{(e)} = \text{conv} K = \Psi^{(e)}(T_K^{(e)}) \) and it is a unimodular simplex.

**Lemma 3.17.** Let \( K \) be a maximal chain in \( \mathcal{F}^{(e)}(P) \) and put \( C = \text{supp}(K) \), \( \varepsilon = \text{sgn}(K) \). Let \( (v_1, \ldots, v_d) \) be the linear extension and \( C_1, \ldots, C_d \) the chains of \( P \) associated to \( C \). For \( g \in T_K^{(e)} \), we have
\[ (\Phi^{(e)}(g))(v_i) = g(v_i) + \sum_{v \in C_i \setminus \{v_i\}} \varepsilon(v)g(v). \]

**Proof.** Since \( T_K^{(e)} \subset V_\varepsilon \) by Lemma 3.13(b), we have \( |g(v)| = \varepsilon(v)g(v) \) for \( g \in T_K^{(e)} \) and \( v \in P \), thus \( |g| \in T_C \). By Proposition 3.15 we see that
\[ \max \{ S(|g|; B) : B \in \text{MC}(P_{\leq v_j}) \} = \sum_{v \in C_j} |g(v)| = \sum_{v \in C_j} \varepsilon(v)g(v), \]
where we recall \( S(f; C) = \sum_{v \in C} |f(v)| \) for \( f \in \mathbb{R}^P \) and a chain \( C \) of \( P \), and \( \text{MC}(P_{\leq v_j}) \) is the set of maximal chains of the subposet \( P_{\leq v_j} \). Hence we obtain the desired identity. \( \square \)
Proof of Theorem 1.4 (a). If \( K \) is a maximal chain in \( \mathcal{F}^{(e)}(P) \), then it follows from Lemma 3.17 that \( \Psi^{(e)} \) is a unimodular linear map on \( T^{(e)}_K \). Hence, if \( L \) is a chain in \( \mathcal{F}^{(e)}(P) \) contained in \( K \), then we see that \( S^{(e)}_L = \Psi^{(e)}(T^{(e)}_L) \) is a unimodular simplex because \( T^{(e)}_L \) is a unimodular simplex (Theorem 1.4 (b)). □

We can use Lemma 3.17 to give a set of defining inequalities of a facet \( S^{(e)}_K \), where \( K \) is a maximal chain in \( \mathcal{F}^{(e)}(P) \).

Proposition 3.18. Let \( K \) be a maximal chain in \( \mathcal{F}^{(e)}(P) \) and put \( C = \text{supp}(K) \), \( \varepsilon = \text{sgn}(K) \). Let \((v_1, \ldots, v_d)\) be the linear extension and \( C_1, \ldots, C_d \) the chains of \( P \) associated to \( C \). If we put

\[
\widetilde{M}^K_i(f) = \sum_{l=1}^{r} \varepsilon(u_l) \prod_{j=l+1}^{r} (1 - \varepsilon(u_j)) f(u_l) \quad (f \in \mathbb{R}^P),
\]

where \( C_i = \{ u_1 \prec u_2 \prec \cdots \prec u_r = v_i \} \), then the face \( S^{(e)}_K \) of the triangulation \( S^{(e)}_P \) of \( \mathcal{O}^{(e)}(P) \) is given by

\[
S^{(e)}_K = \{ f \in \mathbb{R}^P : 0 \leq \widetilde{M}^K_1(f) \leq \cdots \leq \widetilde{M}^K_d(f) \leq 1 \}. \quad (24)
\]

Proof. It is easy to prove by induction on \( k \) that

\[
f(u_k) = g(u_k) + \sum_{i=1}^{k} \varepsilon(u_i) g(u_i) \quad (k = 1, \ldots, r)
\]

if and only if

\[
g(u_k) = f(u_k) - \sum_{i=1}^{k-1} \varepsilon(u_i) \prod_{j=i+1}^{k-1} (1 - \varepsilon(u_j)) f(u_i) \quad (k = 1, \ldots, r).
\]

Hence we have

\[
\widetilde{M}^K_i(f) = \bar{M}^K_i(\Phi^{(e)}(f)).
\]

On the other hand, by Theorem 1.4 (b) and Corollary 3.16 we see that \( f \in S^{(e)}_K \) if and only if

\[
0 \leq \bar{L}^K_1(\Phi^{(e)}(f)) \leq \cdots \leq \bar{L}^K_d(\Phi^{(e)}(f)) \leq 1.
\]

Hence we obtain (24). □

4 Identification with Ohsugi–Tsuchiya’s triangulations

Ohsugi–Tsuchiya [6, 7] computed the squarefree initial ideals of the toric ideals of \( \mathcal{O}^{(e)}(P) \) and \( \mathcal{C}^{(e)}(P) \) with respect to certain monomial orderings. This gives regular unimodular triangulations of \( \mathcal{O}^{(e)}(P) \) and \( \mathcal{C}^{(e)}(P) \). In this section, we show that these triangulations coincide with the triangulations given in Theorem 1.4.
Let \( \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables \( x_1, \ldots, x_n \) over a field \( \mathbb{K} \) and \( \Delta \) a simplicial complex on \([n] := \{1, 2, \ldots, n\}\). To a subset \( F \subset [n] \), we associate a monomial
\[
x_F = \prod_{i \in F} x_i.
\]
The Stanley–Reisner ideal of \( \Delta \) is the ideal \( I_\Delta \) of \( \mathbb{K}[x] \) which is generated by those squarefree monomials \( x_F \) with \( F \not\in \Delta \). On the other hand, given an arbitrary squarefree monomial ideal \( I \) of \( \mathbb{K}[x] \), there is a unique simplicial complex \( \Delta(I) \) such that \( I = I_\Delta(I) \).

### 4.1 Triangulation of enriched order polytope

In this subsection, we prove that the triangulation \( \mathcal{S}_P^{(e)} \) of \( \mathcal{O}^{(e)}(P) \) given in Theorem [1.3] (a) coincides with the one algebraically defined in [7].

Let \( P \) be a finite poset with \( d \) elements and let \( R[\mathcal{O}^{(e)}] = \mathbb{K}[\{x_f : f \in \mathcal{F}^{(e)}(P)\}] \) be the polynomial ring in variables \( x_f \ (f \in \mathcal{F}^{(e)}(P)) \).

**Proposition 4.1.** ([7, Theorem 5.2]) Let \( I_{\mathcal{O}^{(e)}(P)} \) be the ideal of \( R[\mathcal{O}^{(e)}] \) generated by all squarefree monomials \( x_f x_g \) satisfying either of the following conditions:

(i) there exists \( v \in \min(\text{supp}(f)) \cap \min(\text{supp}(g)) \) such that \( f(v) \neq g(v) \);

(ii) \( \text{supp}(f) \not\subset \text{supp}(g) \) and \( f(v) = g(v) \) for each \( v \in \min(\text{supp}(f)) \cap \min(\text{supp}(g)) \),

where the symbol \( A \not\subset B \) means that \( A \nsubseteq B \) and \( A \not\supseteq B \). Then \( \Delta(I_{\mathcal{O}^{(e)}(P)}) \) is a regular unimodular triangulation of \( \mathcal{O}^{(e)}(P) \).

Now we can show that this triangulation \( \Delta(I_{\mathcal{O}^{(e)}(P)}) \) coincides with the triangulation given in Theorem [1.3] (a).

**Proposition 4.2.** With the notations above, we have \( \Delta(I_{\mathcal{O}^{(e)}(P)}) = \mathcal{S}_P^{(e)} \).

**Proof.** Since both of \( \Delta(I_{\mathcal{O}^{(e)}(P)}) \) and \( \mathcal{S}_P^{(e)} \) are unimodular triangulations of \( \mathcal{O}^{(e)}(P) \), the number of maximal simplices are same. Hence it is enough to show that \( x_{f_0} \cdots x_{f_d} \not\in I_{\mathcal{O}^{(e)}(P)} \) for any maximal chain \( K = \{f_0 \triangleright \cdots \triangleright f_d\} \) of \( \mathcal{F}^{(e)}(P) \).

Let \( K = \{f_0 \triangleright \cdots \triangleright f_d\} \) be a maximal chain of \( \mathcal{F}^{(e)}(P) \), and assume to the contrary that \( x_{f_0} \cdots x_{f_d} \in I_{\mathcal{O}^{(e)}(P)} \). Then there exists a pair of indices \( i < j \) such that \( f_i \) and \( f_j \) satisfy the condition (i) or (ii) in Proposition 4.1. Since \( f_i > f_j \) in \( \mathcal{F}^{(e)}(P) \), one has \( \text{supp}(f_i) \supseteq \text{supp}(f_j) \) and \( f_i \) and \( f_j \) do not satisfy the condition (ii). Hence there exists \( v \in \min(\text{supp}(f_i)) \cap \min(\text{supp}(f_j)) \) with \( f_i(v) \neq f_j(v) \). However, since \( f_i > f_j \) in \( \mathcal{F}^{(e)}(P) \) and \( v \in \text{supp}(f_j) \) and \( v \) is minimal in \( \text{supp}(f_i) \), we obtain \( f_i(v) = f_j(v) \), which is a contradiction. Thus it follows that \( x_{f_0} \cdots x_{f_d} \not\in I_{\mathcal{O}^{(e)}(P)} \). \( \square \)

### 4.2 Triangulation of enriched chain polytope

In this subsection, we prove that the triangulation \( \mathcal{T}_P^{(e)} \) of \( \mathcal{C}^{(e)}(P) \) given in Theorem [1.4] (b) coincides with the one algebraically defined in [6]. Let \( R[\mathcal{C}^{(e)}] \) be the polynomial ring in variables \( y_g \ (g \in \mathcal{A}^{(e)}(P)) \).
Proposition 4.3. ([6, Theorem 1.4]) Let \( I_{C^{(e)}(P)} \) be the ideal of \( R[O^{(e)}] \) generated by all squarefree monomials \( y_g y_h \) satisfying either of the following conditions:

(i) there exists \( v \in \text{supp}(g) \cap \text{supp}(h) \) such that \( g(v) \neq h(v) \);

(ii) \( \langle \text{supp}(g) \rangle \not\sim \langle \text{supp}(h) \rangle \) and \( g(v) = h(v) \) for any \( v \in \text{supp}(g) \cap \text{supp}(h) \).

Then \( \Delta(I_{C^{(e)}(P)}) \) is a regular unimodular triangulation of \( C^{(e)}(P) \).

Finally, we show that this triangulation \( \Delta(I_{C^{(e)}(P)}) \) coincides with the triangulation given in Theorem 1.4 (b).

Proposition 4.4. With the same notation as above, we have \( \Delta(I_{C^{(e)}(P)}) = T^{(e)}_P \).

Proof. This follows from the fact that the map \( x_f \mapsto y_{\Phi^{(e)}(f)} \) induces the ring isomorphism

\[
\frac{R[O^{(e)}(P)]}{I_{O^{(e)}(P)}} \cong \frac{R[C^{(e)}(P)]}{I_{C^{(e)}(P)}}.
\]

Declarations
Conflict of Interest
On behalf of all authors, the corresponding author states that there is no conflict of interest.

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