ON ANTISYMMETRIC INFINITESIMAL CONFORMAL BIALGEBRAS

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ABSTRACT. In this paper, we construct a bialgebra theory for associative conformal algebras, namely antisymmetric infinitesimal conformal bialgebras. On the one hand, it is an attempt to give conformal structures for antisymmetric infinitesimal bialgebras. On the other hand, under certain conditions, such structures are equivalent to double constructions of Frobenius conformal algebras, which are associative conformal algebras that are decomposed into the direct sum of another associative conformal algebra and its conformal dual as $\mathbb{C}[\partial]$-modules such that both of them are subalgebras and the natural conformal bilinear form is invariant. The coboundary case leads to the introduction of associative conformal Yang-Baxter equation whose antisymmetric solutions give antisymmetric infinitesimal conformal bialgebras. Moreover, the construction of antisymmetric solutions of associative conformal Yang-Baxter equation is given from $\mathcal{O}$-operators of associative conformal algebras as well as dendriform conformal algebras.

1. Introduction

The theory of Lie conformal algebras appeared as a formal language describing the algebraic properties of the operator product expansion in two-dimensional conformal field theory ([16]). In particular, Lie conformal algebras turn out to be valuable tools in studying vertex algebras and Hamiltonian formalism in the theory of nonlinear evolution equations ([3]). Moreover, Lie conformal algebras have close connections to infinite-dimensional Lie algebras satisfying the locality property ([17]). The conformal analogues of associative algebras, namely, associative conformal algebras naturally appeared in the representation theory of Lie conformal algebras ([9]). They were studied widely on the structure theory ([1] [5] [6] [8] [10] [11] [18] [20] [22] [26] [27] [28], [29] [30] [31] [32]) as well as representation theory ([7] [21] [23]). We would like to point that there are the “conformal analogues” for certain algebras besides Lie and associative algebras or the “conformal structures” of these algebras such as left-symmetric conformal algebras ([13]) and Jordan conformal algebras ([19]).

It is natural to extend such structures to the bialgebras, that is, consider the conformal analogues of bialgebras. In the case of Lie bialgebras, Liberati in [24] developed a theory of Lie conformal bialgebras including the introduction of the notions of conformal classical Yang-Baxter equation, conformal Manin triples and conformal Drinfeld’s doubles. Similarly, a theory of left-symmetric conformal bialgebras was developed in [14], which are equivalent to a class of special Lie conformal algebras named parakähler Lie conformal algebras and the notion of conformal $S$-equation was introduced in the coboundary case. Moreover, the operator forms of the conformal classical Yang-Baxter equation and the conformal $S$-equation were studied in [12], which shows that the antisymmetric solutions of the conformal classical Yang-Baxter equation and the symmetric solutions of the conformal $S$-equation can be interpreted in terms of a kind of operators called $\mathcal{O}$-operators in the conformal sense.

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But as far as we know, there is not a conformal analogue of “associative bialgebras” yet. In fact, there are two kinds of “associative bialgebras”. One is the usual bialgebras in the theory of Hopf algebras, which the coproducts are homomorphisms of the products. Another is the infinitesimal bialgebras, which the coproducts are “derivations” of the products in certain sense, introduced by Joni and Rota in [15] in order to provide an algebraic framework for the calculus of divided difference. In particular, for the latter, in the case of antisymmetric infinitesimal (ASI) bialgebras which are called “associative D-algebras” in [33] or “balanced infinitesimal bialgebras” in the sense of the opposite algebras in [1], there is a systematic study in terms of their equivalences with double constructions of Frobenius algebras as well as their relationships with associative Yang-Baxter equation ([2]).

In this paper, we develop a “conformal” theory for antisymmetric infinitesimal bialgebras, namely antisymmetric infinitesimal (ASI) conformal bialgebras. It is also a bialgebra theory for associative conformal algebras. That is, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{ASI bialgebras} & \xrightarrow{\text{conformal structures}} & \text{ASI conformal bialgebras} \\
\downarrow & & \downarrow \\
\text{bialgebra structures} & & \text{bialgebra structures} \\
\text{associative algebras} & \xrightarrow{\text{conformal structures}} & \text{associative conformal algebras}
\end{array}
\]

We would like to point out that such an approach might not be available for considering the conformal analogues of the usual (associative) bialgebras and even Hopf algebras, but can help to shed light on further studies on the latter.

The main idea is to extend the study of ASI bialgebras given in [2] to the “conformal case”. Explicitly, we first introduce the notion of double constructions of Frobenius conformal algebras as a conformal analogue of double constructions of Frobenius algebras, which are associative conformal algebras that are decomposed into the direct sum of another associative conformal algebra and its conformal dual as a \(\mathbb{C}[\partial]\)-module such that both of them are subalgebras and the natural conformal bilinear form is invariant. Such structures are interpreted equivalently in terms of matched pairs of associative conformal algebras which were introduced in [11]. Finally the notion of antisymmetric infinitesimal (ASI) conformal bialgebras is introduced as equivalent structures of the aforementioned matched pairs of associative conformal algebras as well as the double constructions of Frobenius conformal algebras. Note that the notion of ASI conformal bialgebras is available for any associative conformal algebra, whereas the equivalence with double constructions of Frobenius conformal algebras is available for the associative conformal algebras which are finitely generated and free as \(\mathbb{C}[\partial]\)-modules.

As in the case of ASI bialgebras, the definition of coboundary ASI conformal bialgebra is introduced and its study is also meaningful. It leads to the introduction of associative conformal Yang-Baxter equation as a conformal analogue of the associative Yang-Baxter equation. In particular, its antisymmetric solutions give ASI conformal bialgebras. The associative conformal Yang-Baxter equation is interpreted in terms of its operator forms by introducing the notion of \(\mathcal{O}\)-operators of associative conformal algebras, especially an antisymmetric solution of the associative conformal Yang-Baxter equation corresponds to the skew-symmetric part of a conformal linear map \(T\), where \(T_0 = T_\lambda \big|_{\lambda=0}\) is an \(\mathcal{O}\)-operator in the conformal sense and
moreover, an $\mathcal{O}$-operator of an associative conformal algebra gives an antisymmetric solution of associative conformal Yang-Baxter equation in a semi-direct product associative conformal algebra. Furthermore, we introduce the notion of dendriform conformal algebras and show that for a dendriform conformal algebra which is finite and free as a $\mathbb{C}[\partial]$-module, there is a natural $\mathcal{O}$-operator on the associated conformal associative algebra. Therefore there are constructions of antisymmetric solutions of associative conformal Yang-Baxter equation and hence ASI conformal bialgebras from $\mathcal{O}$-operators of associative conformal algebras as well as dendriform conformal algebras.

This paper is organized as follows. In Section 2, the notions of an associative conformal algebra, its bimodule and a matched pair of associative conformal algebras are recalled. In Section 3, we introduce the notion of double constructions of Frobenius conformal algebras and study their relationship with matched pairs of associative conformal algebras. In Section 4, the notion of ASI conformal bialgebras is introduced as (under certain conditions) equivalent structures of the aforementioned matched pair of associative conformal algebras as well as the double constructions of Frobenius conformal algebras. Section 5 is devoted to studying the coboundary case of ASI conformal bialgebras and the associative conformal Yang-Baxter equation is introduced. In Section 6, we introduce the notions of $\mathcal{O}$-operators of associative conformal algebras and dendriform conformal algebras to construct (antisymmetric) solutions of associative conformal Yang-Baxter equation and hence give ASI conformal bialgebras.

Throughout this paper, we denote by $\mathbb{C}$ the field of complex numbers. All tensors over $\mathbb{C}$ are denoted by $\otimes$. We denote the identity map by $I$. Moreover, if $A$ is a vector space, then the space of polynomials of $\lambda$ with coefficients in $A$ is denoted by $A[\lambda]$.

2. Preliminaries on associative conformal algebras

We recall the notions of an associative conformal algebra, its bimodule and a matched pair of associative conformal algebras. The interested readers may consult [16] and [11] for more details.

**Definition 2.1.** A conformal algebra $R$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-bilinear map $\cdot_{\lambda} : R \times R \to R[\lambda]$, $(a, b) \mapsto a\lambda b$ satisfying

$$(\partial a)_{\lambda} b = -\lambda a_{\lambda} b, \quad a_{\lambda}(\partial b) = (\partial + \lambda)a_{\lambda} b, \quad \forall \ a, b \in R. \quad \text{(conformal sesquilinearity)} \quad (1)$$

An associative conformal algebra $R$ is a conformal algebra with the $\mathbb{C}$-bilinear map $\cdot_{\lambda} : R \times R \to R[\lambda]$ satisfying

$$(a_{\lambda} b)_{\lambda + \mu} c = a_{\lambda}(b_{\mu} c), \quad \forall \ a, b, c \in R. \quad (2)$$

A conformal algebra is called finite if it is finitely generated as a $\mathbb{C}[\partial]$-module. The rank of a conformal algebra $R$ is its rank as a $\mathbb{C}[\partial]$-module. The notions of a homomorphism, an ideal and a subalgebra of an associative conformal algebra are defined as usual.

**Example 2.2.** Let $(A, \cdot)$ be an associative algebra. Then $\text{Cur}(A) = \mathbb{C}[\partial] \otimes A$ is an associative conformal algebra with the following $\lambda$-product:

$$(p(\partial) a)_{\lambda} (q(\partial) b) = p(-\lambda) q(\lambda + \partial)(a \cdot b), \quad \forall \ p(\partial), q(\partial) \in \mathbb{C}[\partial], \ a, b \in A.$$
Proposition 2.5. Let \( a \rightarrow _\lambda v = -\lambda a \rightarrow _\lambda v, \) \( a \rightarrow _\lambda (\partial v) = (\partial + \lambda)(a \rightarrow _\lambda v), \)
\( (\partial a) \rightarrow _\lambda v = -\lambda a \rightarrow _\lambda v, \)
\( (\partial a) \rightarrow _\lambda (\partial v) = (\partial + \lambda)(a \rightarrow _\lambda v). \)
We denote it by \((M, \rightarrow _\lambda)\).

A right module \( M \) over an associative conformal algebra \( A \) is a \( \mathbb{C}[\partial] \)-module endowed with a \( \mathbb{C} \)-bilinear map \( M \times A \rightarrow M[\lambda], \) \((v, a) \mapsto v \leftarrow _\lambda a, \) satisfying the following axioms:
\( (\forall a, b \in A, v \in M): \)
\( \text{(RM1)} (\partial v) \leftarrow _\lambda a = -\lambda v \leftarrow _\lambda a, v \leftarrow _\lambda (\partial a) = (\partial + \lambda)(v \leftarrow _\lambda a), \)
\( \text{(RM2)} v \leftarrow _\lambda a \leftarrow _\lambda + \mu b = v \leftarrow _\lambda (a \mu b). \)
We denote it by \((M, \leftarrow _\lambda)\).

An \( A \)-bimodule is a triple \((M, \rightarrow, \leftarrow)\) such that \((M, \rightarrow _\lambda)\) is a left \( A \)-module, \((M, \leftarrow _\lambda)\) is a right \( A \)-module, and they satisfy the following condition:
\[ (a \rightarrow _\lambda v) \leftarrow _\lambda + \mu b = a \rightarrow _\lambda (v \leftarrow _\lambda b), \quad \forall a, b \in A, v \in M. \] (3)

**Definition 2.4.** Let \( U \) and \( V \) be two \( \mathbb{C}[\partial] \)-modules. A conformal linear map from \( U \) to \( V \) is a \( \mathbb{C} \)-linear map \( a : U \rightarrow V[\lambda], \) denoted by \( a_\lambda : U \rightarrow V, \) such that \([\partial, a_\lambda] = -\lambda a_\lambda \). Denote the vector space of all such maps by \( \text{Chom}(U, V) \). It has a canonical structure of a \( \mathbb{C}[\partial] \)-module:
\[ (\partial a)_\lambda = -\lambda a_\lambda. \]

Define the conformal dual of a \( \mathbb{C}[\partial] \)-module \( U \) as \( U^{*c} = \text{Chom}(U, \mathbb{C}) \) where \( \mathbb{C} \) is viewed as the trivial \( \mathbb{C}[\partial] \)-module, that is,
\[ U^{*c} = \{ a : U \rightarrow \mathbb{C}[\lambda] \mid \text{\( \mathbb{C} \)-linear and} \ a_\lambda(\partial b) = \lambda a_\lambda b, \forall b \in U \}. \]

In the special case \( U = V \), set \( \text{Cend}(V) = \text{Chom}(V, V) \). If \( V \) is a finite \( \mathbb{C}[\partial] \)-module, then the \( \mathbb{C}[\partial] \)-module \( \text{Cend}(V) \) has a canonical structure of an associative conformal algebra defined by
\[ (a_\lambda b)_\mu v = a_\lambda (b_{\mu - \lambda} v), \quad \forall a, b \in \text{Cend}(V), v \in V. \]

Set \( a \rightarrow _\lambda v = l_A(a)_\lambda v \) and \( v \leftarrow _\lambda a = r_A(a)_\lambda - \partial v. \) Then a structure of a bimodule \( M \) over an associative conformal algebra \( A \) is the same as two \( \mathbb{C}[\partial] \)-module homomorphisms \( l_A \) and \( r_A \) from \( A \) to \( \text{Cend}(M) \) such that the following conditions hold:
\[ l_A(a_\lambda b)_\lambda + \mu v = l_A(a)_\lambda (l_A(b)_\mu v), \] (4)
\[ r_A(b)_\lambda - \lambda - \mu - \partial (r_A(a)_\lambda - \partial v) = r_A(a_\mu b)_\lambda - \partial v, \] (5)
\[ l_A(a)_\lambda (r_A(b)_\lambda - \partial v) = r_A(b)_\lambda - \lambda - \mu - \partial (l_A(a)_\lambda v), \] (6)
for all \( a, b \in A \) and \( v \in M. \) We denote this bimodule by \((M, l_A, r_A)\).

**Proposition 2.5.** Let \((M, l_A, r_A)\) be a finite bimodule of an associative conformal algebra \( A \). Let \( l_A^* \) and \( r_A^* \) be two \( \mathbb{C}[\partial] \)-module homomorphisms from \( A \) to \( \text{Cend}(M^{*c}) \) defined by
\[ (l_A^*(a)f)_\mu u = f_{\mu - \lambda}(l_A(a)_\lambda u), \quad (r_A^*(a)f)_\mu u = f_{\mu - \lambda}(r_A(a)_\lambda u), \quad \forall a \in A, f \in M^{*c}, u \in M. \] (7)
Then \((M^{*c}, r_A^*, l_A^*)\) is a bimodule of \( A. \)
Proof. Let \(a, b \in A\), \(f \in M^{*c}\) and \(u \in M\). Since

\[
(r_A^*(a)\lambda(r_A(b)_\mu f))_\nu u = (r_A^*(a)_\mu f)_\nu (r_A^*(b)_\lambda u) = f_{\nu - \lambda - \mu}(r_A(b)_\mu (r_A(a)_\lambda u))
\]

\[
= f_{\nu - \lambda - \mu}(r_A(a)_\lambda b + \mu u) = (r_A^*(a)_\lambda b)_\lambda u;
\]

\[
(l_A^*(a)_\lambda (l_A(b)_\mu f))_\nu u = (l_A^*(a)_\lambda (l_A(b)_\mu f))_\lambda u = f_{\lambda}(l_A(a)_\mu (l_A(b)_\nu - \lambda - \mu u))
\]

\[
= f_{\lambda}(l_A(a)_\mu (l_A(b)_\nu - \lambda - \mu u)) = (l_A^*(a)_\lambda b)_\lambda u;
\]

\[
(r_A^*(a)_\lambda (l_A^*(b)_\mu f))_\nu u = (r_A^*(a)_\lambda (l_A^*(b)_\mu f))_\nu u = (r_A^*(a)_\lambda f)_\lambda \mu (l_A(b)_\nu - \lambda - \mu u)
\]

\[
= (l_A^*(b)_\lambda (l_A^*(a)_\lambda f))_\nu u,
\]

we have

\[
r_A^*(a)_\lambda (r_A^*(b)_\mu f) = r_A^*(a)_\lambda b + \mu f, \quad l_A^*(a)_\lambda (l_A^*(b)_\mu f) = l_A^*(a)_\mu b - \lambda - \nu f.
\]

Hence \((M^{*c}, r_A^*, l_A^*)\) is a bimodule of \(A\). \(\square\)

Example 2.6. Let \(A\) be a finite associative conformal algebra. Define two \(\mathbb{C}[\partial]\)-module homomorphisms \(L_A\) and \(R_A\) from \(A\) to \(\text{Cend}(A)\) by \(L_A(a)_\lambda b = a_\lambda b\) and \(R_A(a)_\lambda b = b - \lambda - \partial a\) for all \(a, b \in A\). Then \((A, L_A, R_A)\) is a bimodule of \(A\). Moreover, \((A^{*c}, R_A^*, L_A^*)\) is a bimodule of \(A\).

Proposition 2.7. (\cite{proposition 4.4}) Let \(A\) and \(B\) be two associative conformal algebras. Suppose that there are \(\mathbb{C}[\partial]\)-module homomorphisms \(l_A, r_A : A \rightarrow \text{Cend}(B)\) and \(l_B, r_B : B \rightarrow \text{Cend}(A)\) such that \((B, l_A, r_A)\) is a bimodule of \(A\) and \((A, l_B, r_B)\) is a bimodule of \(B\) and they satisfy the following relations:

\[
l_A(a)_\lambda (x_\mu y) = (l_A(a)_\lambda x)_\lambda u + l_A(r_B(x)_\mu - \lambda - \partial a)_\lambda u,
\]

\[
r_B(x)_\mu - \lambda - \partial (a_\lambda b) = a_\lambda (r_B(x)_\mu - \lambda - \partial b) + r_B(l_A(b)_\mu (x))_\mu - \lambda - \partial a,
\]

\[
l_B(x)_\lambda a + b = l_B(r_A(a)_\mu - \lambda - \partial x)_\lambda + \mu b + l_B(r_A(a)_\mu - \lambda - \partial x)_\lambda + \mu b,
\]

\[
r_A(l_B(y)_\mu a)_\lambda - \mu - \partial x + x_\lambda (r_A(a)_\mu - \lambda - \partial y) = r_A(a)_\lambda - \mu - \partial (x_\lambda y),
\]

\[
r_A(r_B(y)_\mu - \lambda - \partial a)_\lambda - \mu - \partial x + x_\lambda (l_A(a)_\mu _\lambda + y + (r_A(a)_\mu - \lambda - \partial x)_\lambda + \mu y,
\]

\[
a_\lambda (l_B(x)_\mu b) + r_B(r_A(b)_\mu - \lambda - \partial x)_\lambda - \partial a = (r_B(x)_\mu - \lambda - \partial a)_\lambda + \mu b + l_B(l_A(a)_\lambda x)_\lambda + \mu b,
\]

for all \(a, b \in A\) and \(x, y \in B\). Then there is an associative conformal algebra structure on the \(\mathbb{C}[\partial]\)-module \(A \oplus B\) given by

\[
(a + x)_\lambda (b + y) = (a_\lambda b + l_B(x)_\lambda b + r_B(y)_\lambda - \partial a) + (x_\lambda y + l_A(a)_\lambda y + r_A(b)_\lambda - \partial x),
\]

for all \(a, b \in A\) and \(x, y \in B\). We denote this associative conformal algebra by \(A \bowtie B\). \((A, B, l_A, r_A, l_B, r_B)\) satisfying the above relations is called a matched pair of associative conformal algebras. Moreover, any associative conformal algebra \(E = A \oplus B\) where the sum is the direct sum of \(\mathbb{C}[\partial]\)-modules and \(A, B\) are two associative conformal subalgebras of \(E\), is \(A \bowtie B\) associated to some matched pair of associative conformal algebras.

Remark 2.8. If \(l_B, r_B\) and the \(\lambda\)-product on \(B\) are trivial in Proposition 2.7, that is, \(B\) is exactly a bimodule of \(A\), then \(A \bowtie B\) is the semi-direct product of \(A\) and its bimodule \(B\), which is denoted by \(A \bowtie_{l_A, r_A} B\).
3. Double constructions of Frobenius conformal algebras

We introduce the notion of double constructions of Frobenius conformal algebras. The relationship between double constructions of Frobenius conformal algebras and matched pairs of associative conformal algebras is investigated.

**Definition 3.1.** Let $V$ be a $\mathbb{C}[\partial]$-module. A **conformal bilinear form** on $V$ is a $\mathbb{C}$-bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}[\lambda]$ satisfying

$$\langle \partial a, b \rangle_\lambda = -\lambda \langle a, b \rangle_\lambda, \quad \langle a, \partial b \rangle_\lambda = \lambda \langle a, b \rangle_\lambda, \quad \forall a, b \in V. \quad (15)$$

A conformal bilinear form is called **symmetric** if $\langle a, b \rangle_\lambda = \langle b, a \rangle_{-\lambda}$ for any $a, b \in V$.

Suppose that there is a conformal bilinear form on a $\mathbb{C}[\partial]$-module $V$. Then we have a $\mathbb{C}[\partial]$-module homomorphism $\varphi : V \to V^{*c}$, $v \mapsto \varphi_v$ defined by

$$\langle \varphi_v w, a \rangle_\lambda = \langle v, w \rangle_\lambda, \quad \forall v, w \in V.$$ 

A conformal bilinear form is called **non-degenerate** if $\varphi$ gives an isomorphism of $\mathbb{C}[\partial]$-modules between $V$ and $V^{*c}$.

**Definition 3.2.** An associative conformal algebra $A$ is called a **Frobenius conformal algebra** if there is a non-degenerate conformal bilinear form on $A$ such that

$$\langle a_\lambda b, c \rangle_\mu = \langle a, b_{\mu - \lambda} c \rangle_\lambda, \quad \forall a, b, c \in A. \quad (16)$$

A Frobenius conformal algebra $A$ is called **symmetric** if the conformal bilinear form on $A$ is symmetric.

**Example 3.3.** Let $A = \mathbb{C}[\partial]a \oplus \mathbb{C}[\partial]b$. Suppose that $A$ is an associative conformal algebra with the following $\lambda$-product:

$$a_\lambda a = (\partial^2 + \lambda \partial + \lambda^2)b, \quad a_\lambda b = b_\lambda a = b_\lambda b = 0.$$ 

Then $A$ is a Frobenius conformal algebra with the following conformal bilinear form:

$$\langle a_\lambda b, b \rangle_\lambda = 0, \quad \langle a_\lambda b, a \rangle_\lambda = 1.$$ 

**Example 3.4.** Let $(A, \langle \cdot, \cdot \rangle)$ be a Frobenius algebra, that is, $A$ is an associative algebra with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle ab, c \rangle = \langle a, bc \rangle, \quad \forall a, b \in A.$$ 

Then $(\text{Cur}(A), \langle \cdot, \cdot \rangle_\lambda)$ is a Frobenius conformal algebra with $\langle \cdot, \cdot \rangle_\lambda$ defined by

$$\langle p(\partial) a, q(\partial) b \rangle_\lambda = p(-\lambda) q(\lambda) \langle a, b \rangle, \quad \forall p(\partial), \quad q(\partial) \in \mathbb{C}[\partial], \quad a, b \in A.$$ 

**Definition 3.5.** If a Frobenius conformal algebra $A$ satisfies the following conditions:

1. $A = B \oplus B^{*c}$ where the sum is the direct sum of $\mathbb{C}[\partial]$-modules;
2. $B$ and $B^{*c}$ are two associative conformal subalgebras of $A$;
3. the conformal bilinear form on $A$ is naturally given by

$$\langle a + f, b + g \rangle_\lambda = f_\lambda(b) + g_{-\lambda}(a), \quad \forall a, b \in B, \quad f, g \in B^{*c}, \quad (17)$$

then $A$ is called a **double construction of Frobenius conformal algebra** associated to $B$ and $B^{*c}$.
Theorem 3.6. Let $A$ be a finite associative conformal algebra which is free as a $\mathbb{C}[\partial]$-module. Suppose that there is an associative conformal algebra structure on $A^{ac}$. Then there is a double construction of Frobenius conformal algebra associated to $A$ and $A^{ac}$ if and only if $(A, A^{ac}, R_A^*, L_A^*, R_{A^{ac}}^*, L_{A^{ac}}^*)$ is a matched pair of associative conformal algebras.

Proof. Suppose that $(A, A^{ac}, R_A^*, L_A^*, R_{A^{ac}}^*, L_{A^{ac}}^*)$ is a matched pair of associative conformal algebras. Then $A \bowtie A^{ac}$ is endowed with an associative conformal algebra structure as follows.

\[(a + f)\lambda(b + g) = (a\lambda b + R_{A^{ac}}^*(f)\lambda b + L_{A^{ac}}^*(g)\lambda - \partial a) + (f\lambda g + R_A^*(a)\lambda g + L_A^*(b)\lambda - \partial f), \quad (18)\]

for all $a, b \in A$ and $f, g \in A^{ac}$. By this $\lambda$-product, $A$ and $A^{ac}$ are two subalgebras of $A \bowtie A^{ac}$.

Obviously, the conformal bilinear form on $A \bowtie A^{ac}$ given by $(17)$ is symmetric and non-degenerate. For all $a, b, c \in A$, and $f, g, h \in A^{ac}$, we have

\[
\langle (a + f), (b + g) \rangle_{\lambda}(c + h) \mu = \langle a\lambda b + R_{A^{ac}}^*(f)\lambda b + L_{A^{ac}}^*(g)\lambda - \partial a + f\lambda g + R_A^*(a)\lambda g + L_A^*(b)\lambda - \partial f, (c + h) \mu \rangle
\]

\[
= (f\lambda g + R_A^*(a)\lambda g + L_A^*(b)\lambda - \partial f)\mu(c) + h - \mu(a\lambda b + R_{A^{ac}}^*(f)\lambda b + L_{A^{ac}}^*(g)\lambda - \partial a)
\]

\[
= (f\lambda g)\mu(c) + g_{\mu - \lambda}(R_A(a)c) + f\lambda(L_A(b)\mu - \lambda c) + h - \mu(a\lambda b)
\]

\[
+ (R_{A^{ac}}(f)\lambda h)\mu - \lambda b + (L_{A^{ac}}(g)\mu - \lambda h)\lambda a
\]

\[
= (f\lambda g)\mu(c) + g_{\mu - \lambda}(c - \lambda \partial a) + f\lambda(b - \mu c) + h - \mu(a\lambda b) + (h - \lambda f)\lambda - \mu b + (g_{\mu - \lambda} h)\lambda a
\]

\[
= (f\lambda g)\mu(c) + g_{\mu - \lambda}(c - \lambda \partial a) + f\lambda(b - \mu c) + h - \mu(a\lambda b) + (h - \lambda f)\lambda - \mu b + (g_{\mu - \lambda} h)\lambda a,
\]

and

\[
\langle (a + f), (b + g) \rangle_{\mu - \partial (c + h)} \lambda
\]

\[
= \langle a + f, b - \partial c + R_{A^{ac}}(g) - \partial c + L_A^*(h) - \partial b + g_{\mu - \partial h} + R_A^*(b) - \partial h + L_A^*(c) - \mu g \rangle \lambda
\]

\[
= f\lambda(b - \partial c) + f\lambda(R_{A^{ac}}^*(g) - \partial c) + f\lambda(L_{A^{ac}}^*(h) - \partial b)
\]

\[
+ (g_{\mu - \partial h}) - \lambda a + (R_A^*(b) - \partial h - \lambda a + (L_A^*(c) - \mu g) - \lambda a
\]

\[
= f\lambda(b - \lambda c) + (R_{A^{ac}}^*(g) - \lambda f) - \partial c + (L_{A^{ac}}^*(h) - \partial f) - \lambda b
\]

\[
+ (g_{\mu - \lambda} h) - \lambda a + h - \mu(R_A(b) - \lambda c - \mu a) + g_{\mu - \lambda}(L_A(c) - \mu a)
\]

\[
= f\lambda(b - \lambda c) + (f\lambda g) - \partial c + (h - \lambda f) - \mu b + (g_{\mu - \lambda} h) - \lambda a + h - \mu(a\lambda b) + g_{\mu - \lambda}(c - \mu a).
\]

Hence this conformal bilinear form on $A \bowtie A^{ac}$ is invariant. Therefore $A \bowtie A^{ac}$ is a double construction of Frobenius conformal algebra associated to $A$ and $A^{ac}$.

Conversely, suppose that there is a double construction of Frobenius conformal algebra associated to $A$ and $A^{ac}$. Therefore there is an associative conformal algebra structure on $A \oplus A^{ac}$ associated to a matched pair $(A, A^{ac}, l_A, r_A, l_{A^{ac}}, r_{A^{ac}})$. Note that in $A \oplus A^{ac}$,

\[a\lambda f = r_{A^{ac}}(f) - \lambda - \partial a + l_A(a) f, \quad f\lambda a = l_{A^{ac}}(f) - \lambda a + r_A(a) - \lambda - \partial f, \quad \forall a \in A, f \in A^{ac}.
\]

For all $a, b \in A$, and $f, g \in A^{ac}$, we have

\[
\langle (l_A(a) f), (b) \rangle_{\mu} = \langle a, f\mu - \partial b \rangle = \langle f\mu - \lambda b, a \rangle - \lambda = \langle f, b - \mu a \rangle - \lambda
\]

\[
= f\mu - \lambda(R_A(a) - \partial b) = (R_A^*(a) - \lambda f)\mu b = (r_A^*(a) f, b)_{\mu}.
\]

Since $\langle \cdot, \cdot \rangle_{\mu}$ is non-degenerate, $l_A(a) f = R_A^*(a) \lambda f$ for all $a \in A$ and $f \in A^{ac}$ and hence $l_A = R_A^*$. Similarly, we have

\[
r_A = L_A^*, \quad l_{A^{ac}} = R_{A^{ac}}^*, \quad r_{A^{ac}} = L_{A^{ac}}^*.
\]

Thus $(A, A^{ac}, R_A^*, L_A^*, R_{A^{ac}}^*, L_{A^{ac}}^*)$ is a matched pair of associative conformal algebras. \qed
Theorem 3.7. Let $A$ be a finite associative conformal algebra which is free as a $\mathbb{C}[\partial]$-module. Assume that there is an associative conformal algebra structure on the $\mathbb{C}[\partial]$-module $A^{sc}$. Then $(A, A^{sc}, R^*_A, L^*_A, R^*_{A^{sc}}, L^*_{A^{sc}})$ is a matched pair of associative conformal algebras if and only if

$$R^*_A(a)(f_\lambda g) = (R^*_A(a)f_\lambda)\lambda + \mu g + R^*_A(L^*_{A^{sc}}(f_\lambda - \delta_\partial a)\lambda + \mu g, \quad (19)\]$$

$$R^*_A(R^*_{A^{sc}}(f_\lambda a)\lambda + \mu g + (L^*_A(a_\lambda - \delta_\partial f_\lambda)\lambda + \mu g = L^*_A(R^*_A(g) - \delta a) - \delta_\partial f + f_\lambda(R^*_A(a)\mu g), \quad (20)$$

for all $a \in A$ and $f, g \in A^{sc}$.

Proof. Obviously, (19) is exactly (8) and (20) is exactly (12) when

$$l_A = R^*_A, \quad r_A = L^*_A, \quad l_B = R^*_{A^{sc}}, \quad r_B = L^*_{A^{sc}},$$

in Proposition 2.7. Then the conclusion follows if we prove that (19), (9), (10) and (11) are mutually equivalent, (20) and (13) are equivalent. As an example, we give an explicit proof that (20) holds if and only if (13) holds. The other cases can be proved similarly.

Let $\{e_1, \cdots, e_n\}$ be a $\mathbb{C}[\partial]$-basis of $A$ and $\{e^*_1, \cdots, e^*_n\}$ be a dual $\mathbb{C}[\partial]$-basis of $A^{sc}$ in the sense that $(e^*_\lambda)\lambda e_i = \delta_i$. Set

$$e_i \lambda e_j = \sum_{k=1}^n P^{ij}_k(\lambda, \partial) e_k, \quad e^*_i \lambda e^*_j = \sum_{k=1}^n R^{ij}_k(\lambda, \partial) e^*_k,$$

where $P^{ij}_k(\lambda, \partial)$ and $R^{ij}_k(\lambda, \partial) \in \mathbb{C}[\lambda, \partial]$. Since

$$(L^*_A(e_i)\lambda e^*_j)_\mu e_k = e^*_j e_i \lambda e^*_j = e^*_j e_i e_k = (\sum_{j=1}^n P^{ik}_j(\lambda, \partial) e_j) = P^{ik}_j(\lambda, \mu - \lambda),$$

we have

$$L^*_A(e_i)\lambda e^*_j = \sum_{k=1}^n P^{ik}_j(\lambda, -\lambda - \partial)e^*_k.$$  

Similarly, we have

$$L^*_{A^{sc}}(e^*_i)\lambda e_j = \sum_{k=1}^n P^{ik}_j(\lambda, -\lambda - \partial)e_k, \quad R^*_A(e_i)\lambda e^*_j = \sum_{k=1}^n P^{ki}_j(\partial, -\lambda - \partial)e^*_k,$$

$$R^*_A(e^*_i)\lambda e_j = \sum_{k=1}^n R^{ki}_j(\partial, -\lambda - \partial)e_k.$$

Therefore (20) holds for any $a \in A$, $f, g \in A^{sc}$ if and only if

$$(R^*_A(R^*_{A^{sc}}(e^*_i)\lambda e_i)\lambda + \mu e_k + (L^*_A(e_i) - \lambda - \delta e^*_i)\lambda + \mu e_k =$$

$$-L^*_A(L^*_{A^{sc}}(e^*_k)\lambda - \delta e^*_i) - \lambda - \delta e^*_i + e^*_j(\lambda R^*_A(e_i)\mu e^*_k))e_s = 0, \quad \forall \ i, j, k, s,$$

if and only if the following equation holds:

$$\sum_{t=1}^n (R^{ij}_t(\lambda - \lambda, \mu) P^{st}_k(\lambda - \lambda - \delta, \nu \partial) e^*_s) + R^{ik}_j(\lambda + \mu, -\nu) P^{ji}_t(\mu, \lambda)$$

$$-R^{ik}(\lambda - \lambda - \mu, \mu) P^{is}_j(\lambda - \lambda - \nu) - R^{ij}_s(\lambda + \mu, -\nu) P^{is}_j(\lambda - \lambda - \nu) = 0, \quad \forall \ i, j, k, s. \quad (21)$$

On the other hand, (13) holds for any $a, b \in A$, $x \in A^{sc}$ if and only if

$$e^*_j(\lambda R^*_A(R^*_{A^{sc}}(e^*_k)\mu e_s) + L^*_A(L^*_{A^{sc}}(e_i) - \lambda - \delta e^*_i) - \lambda - \delta e^*_i$$

$$-L^*_{A^{sc}}(e^*_k)\lambda - \delta e^*_i) + \mu e_s - R^*_A(e_i)\lambda e^*_k)\lambda + \mu e_s = 0, \quad \forall \ i, j, k, s,$$
if and only if the following equation holds:
\[
\sum_{i=1}^{n} (R^{ij}_i (-\lambda - \nu, \lambda) P^{st}_k (-\lambda - \mu - \nu, \mu) + R^{ik}_i (-\lambda + \mu, \lambda) P^{tt}_k (-\lambda - \mu - \nu, \mu)) \]
\[
- R^{kt}_j (-\lambda + \mu, \lambda) P^{ts}_j (-\lambda - \mu - \nu, \mu) - R^{it}_j (-\lambda - \mu - \nu, \mu) P^{st}_k (-\lambda + \mu, \lambda)) = 0, \forall i,j,k,s. \tag{22}
\]
Note that (22) is exactly (21) when we replace \( \lambda \) by \( \nu \) by \( \lambda \), and \( \mu \) by \( \nu - \lambda - \mu \) in (22).
Therefore (20) holds if and only if (13) holds. \( \Box \)

4. **Antisymmetric infinitesimal conformal bialgebras**

We introduce the notion of antisymmetric infinitesimal conformal bialgebras as (under certain conditions) the equivalent structures of the matched pairs of associative conformal algebras as well as double constructions of Frobenius conformal algebras given in the previous section.

**Definition 4.1.** An associative conformal coalgebra is a \( \mathbb{C}[\partial] \)-module \( A \) endowed with a \( \mathbb{C}[\partial] \)-module homomorphism \( \Delta : A \rightarrow A \otimes A \) such that
\[
(I \otimes \Delta) \Delta(a) = (\Delta \otimes I) \Delta(a), \quad \forall a \in A, \tag{23}
\]
where the module action of \( \mathbb{C}[\partial] \) on \( A \otimes A \) is defined as \( \partial(a \otimes b) = (\partial a) \otimes b + a \otimes (\partial b) \) for any \( a, b \in A \).

**Proposition 4.2.** Let \((A, \Delta)\) be a finite associative conformal coalgebra. Then \( A^{ac} = \text{Chom}(A, \mathbb{C}) \) is endowed with an associative algebra structure with the following \( \lambda \)-product:
\[
(f \lambda g)_{\mu}(a) = \sum f_{\lambda}(a_{(1)}) g_{\mu-\lambda}(a_{(2)}) = (f \otimes g)_{\lambda-\lambda}(\Delta(a)), \quad \forall f, g \in A^{ac}, \tag{24}
\]
where \( \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \) for all \( a \in A \).

**Proof.** By (21), the conformal sesquilinearity of the \( \lambda \)-product is naturally satisfied. For all \( a \in A, f, g \) and \( h \in A^{ac} \), we have
\[
((f \lambda g)_{\lambda+\mu} h - f \lambda (g_{\lambda} h))_{\nu}(a) = \sum (f \lambda g)_{\lambda+\mu}(a_{(1)}) h_{\nu-\lambda-\mu}(a_{(2)}) - \sum f_{\lambda}(a_{(1)}) (g_{\lambda} h)_{\nu-\lambda}(a_{(2)})
\]
\[
= \sum f_{\lambda}(a_{(1)}) g_{\mu}(a_{(2)}) h_{\nu-\lambda-\mu}(a_{(2)}) - \sum f_{\lambda}(a_{(1)}) g_{\mu-\lambda}(a_{(2)}) h_{\nu-\lambda-\mu}(a_{(2)})
\]
\[
= (f \otimes g \otimes h)_{\lambda, \mu, \nu-\lambda-\mu}((\Delta \otimes I) \Delta - (I \otimes \Delta) \Delta)(a) = 0.
\]
Hence the conclusion holds. \( \Box \)

**Proposition 4.3.** Let \( A \) be a finite associative conformal algebra which is free as a \( \mathbb{C}[\partial] \)-module, that is, \( A = \sum_{i=1}^{n} \mathbb{C}[\partial] e_i \), where \( \{e_1, \cdots, e_n\} \) is a \( \mathbb{C}[\partial] \)-basis of \( A \). Then \( A^{ac} = \text{Chom}(A, \mathbb{C}) = \sum_{i=1}^{n} \mathbb{C}[\partial] e_i^{*} \) is an associative conformal algebra with the following coproduct:
\[
\Delta(f) = \sum_{i,j} f_{\mu}(e_i \otimes e_j)(e_i^{*} \otimes e_j^{*})|_{\lambda=\partial, \mu=-\partial-1-1}, \tag{25}
\]
where \( \{e_1^{*}, \cdots, e_n^{*}\} \) is a dual \( \mathbb{C}[\partial] \)-basis of \( A^{ac} \). More precisely, if \( e_i \lambda e_j = \sum_k P^{ij}_{k} (\lambda, \partial) e_k \), then
\[
\Delta(e_k^{*}) = \sum_{i,j} Q^{ij}_{k} (\partial \otimes 1, 1 \otimes \partial)(e_i^{*} \otimes e_j^{*}),
\]
where \( Q^{ij}_{k}(x,y) = P^{ij}_{k}(x,-x-y) \).
Proof. By \((25)\), \(\Delta\) is a \(\mathbb{C}[\partial]\)-module homomorphism. By the definition of \(\Delta\), we have
\[
(I \otimes \Delta)\Delta(e_\mu^k) = (\Delta \otimes I)\Delta(e_\mu^k)
\]
\[
= \sum_{i,j,l,r} Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial)
\]
\[
\times Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial) (e_i \otimes e_l \otimes e_r) - \sum_{i,j,l,r} Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1, 1 \otimes 1 \otimes \partial)
\]
\[
\times Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1) (e_i \otimes e_l \otimes e_r).
\]
On the other hand, since \(e_{i\lambda}(e_{l\mu}e_r) = (e_{i\lambda}e_l)_{\lambda+\mu}e_r\), we have
\[
\sum_j P_{ij}^l (\mu, \lambda + \partial) P_{ij}^l (\lambda, \partial) = \sum_j P_{ij}^l (\lambda, -\lambda - \mu) P_{ij}^l (\lambda + \mu, \partial).
\]
Since \(Q_{ij}^l(x,y) = P_{ij}^l(x,-x-y)\), we have
\[
\sum_j Q_{ij}^l (\mu, -\lambda - \mu - \partial) Q_{ij}^l (\lambda, -\lambda - \partial) = \sum_j Q_{ij}^l (\lambda, \mu) Q_{ij}^l (\lambda + \mu, -\lambda - \mu - \partial).
\]
Set
\[
\lambda = \partial \otimes 1 \otimes 1, \quad \mu = 1 \otimes \partial \otimes 1, \quad \partial = -\partial \otimes 1 \otimes 1 - 1 \otimes \partial \otimes 1 - 1 \otimes 1 \otimes \partial.
\]
Then by \((26)\), we have
\[
\sum_j Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial) Q_{ij}^l (\partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1, 1 \otimes 1 \otimes \partial)
\]
\[
= \sum_j Q_{ij}^l (\partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1, 1 \otimes 1 \otimes \partial) Q_{ij}^l (\partial \otimes 1 \otimes 1, 1 \otimes \partial \otimes 1).
\]
Therefore for all \(k \in \{1, \ldots, n\}\), we have
\[
(I \otimes \Delta)\Delta(e_\mu^k) = (\Delta \otimes I)\Delta(e_\mu^k).
\]
Hence the conclusion holds. \(\square\)

Corollary 4.4. Let \((A = \mathbb{C}[\partial]a, \Delta)\) be an associative conformal coalgebra which is free of rank 1 as a \(\mathbb{C}[\partial]\)-module. Then \(\Delta(a) = ka \otimes a\) for some \(k \in \mathbb{C}\).

Proof. If \(A = \mathbb{C}[\partial]a\) is an associative conformal algebra, then \(a_\lambda a = ka\) for some \(k \in \mathbb{C}\). Therefore this conclusion follows from Proposition 4.3. \(\square\)

In the sequel, denote \(\partial \otimes 1 + 1 \otimes \partial\) by \(\partial^{\otimes 2}\) and \(\partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial\) by \(\partial^{\otimes 3}\). Moreover, for any vector space \(V\), let \(\tau : V \otimes V \rightarrow V \otimes V\) be the flip map, that is,
\[
\tau(x \otimes y) = y \otimes x, \quad \forall x, y \in V.
\]

Theorem 4.5. Let \(A\) be a finite associative conformal algebra which is free as a \(\mathbb{C}[\partial]\)-module. Suppose there is another associative conformal algebra structure on the \(\mathbb{C}[\partial]\)-module \(A^{\text{ac}}\) obtained from a \(\mathbb{C}[\partial]\)-module homomorphism \(\Delta : A \rightarrow A \otimes A\). Then \((A, A^{\text{ac}}, R_A^*, L_A^*, R_{A^{\text{ac}}}^*, L_{A^{\text{ac}}}^*)\) is a matched pair of associative conformal algebras if and only if \(\Delta\) satisfies
\[
\Delta(a \lambda b) = (I \otimes L_A(a))\Delta(b) + (R_A(b)_{-\lambda - \partial^{\otimes 2}} \otimes I)\Delta(a),
\]
\[
(L_A(b)_{-\lambda - \partial^{\otimes 2}} \otimes I - I \otimes R_A(b)_{-\lambda - \partial^{\otimes 2}})\Delta(a) + \tau(L_A(a)_{\lambda} \otimes I - I \otimes R_A(a)_{\lambda})\Delta(b) = 0,
\]
for all \(a, b \in A\).
Proof. With the same assumption as that in the proof of Theorem 3.7 and by Proposition 4.3, we have
\[ \Delta(e_k) = \sum_{i,j} Q^{ij}_k (\partial \otimes 1, 1 \otimes \partial) e_i \otimes e_j, \text{ where } Q^{ij}_k (x, y) = R^{ij}_k (x, -x - y). \]

Considering the coefficient of \( e_j \otimes e_k \) in
\[ \Delta(e_{s \lambda} e_i) = (I \otimes L_A(e_s)_\lambda) \Delta(e_i) + (R_A(e_i)_-\lambda-\partial \otimes 1 \otimes \partial \otimes I) \Delta(e_s), \]
we have
\[
\sum_{i=1}^{n} P_s^{s i} (\lambda, \partial \otimes 1 + 1 \otimes \partial) R^{jk}_i (\partial \otimes 1, -1 \otimes \partial - \partial \otimes 1) \\
= \sum_{i=1}^{n} R^{j i}_i (\partial \otimes 1, -\lambda - \partial \otimes 1 - 1 \otimes \partial) P^{s t}_k (\lambda, 1 \otimes \partial) \\
+ \sum_{i=1}^{n} R^{j k}_i (-\lambda - 1 \otimes \partial, \partial) P^{j i}_j (\lambda + 1 \otimes \partial, \partial \otimes 1). \tag{29}
\]

On the other hand, (19) holds for any \( a \in A, f, g \in A^c \) if and only if
\[ (R^*_A(e_i)(e_j^* e_k^*))_{\nu} e_s = ((R_A(e_i)\lambda e_j^*)_{\lambda+\mu} e_k^* + R_A(L^*_A(e_j^*)_{-\lambda-\partial} e_i)_{\lambda+\mu} e_k^*)_{\nu} e_s, \forall i, j, k, s, \]
if and only if the following equation holds:
\[
\sum_{i=1}^{n} P_s^{s i} (-\nu, -\lambda - \mu) R^{j k}_i (\mu, \lambda - \nu) = \sum_{i=1}^{n} R^{j i}_i (\mu, \lambda) P^{s t}_k (-\nu, -\lambda - \mu) \\
+ \sum_{i=1}^{n} R^{j k}_i (\mu + \nu, -\mu) P^{j i}_j (-\lambda - \mu, \mu), \forall j, k, s, t. \tag{30}
\]

Obviously, (29) is exactly (30) by replacing \( \lambda \) by \( -\nu, 1 \otimes \partial \) by \( -\nu - \lambda - \mu \), and \( \partial \otimes 1 \) by \( \mu \) in (29). Then (27) holds if and only if (19) holds. With a similar discussion, (28) holds if and only if (20) holds. Then this conclusion follows from Theorem 3.7. □

Definition 4.6. Let \((A, \cdot, \cdot)\) be an associative conformal algebra and \((A, \Delta)\) be an associative conformal bialgebra. If \( \Delta \) satisfies (27), then \((A, \cdot, \cdot, \Delta)\) is called an infinitesimal conformal bialgebra. Moreover, if \( \Delta \) also satisfies (28), then \((A, \cdot, \cdot, \Delta)\) is called an antisymmetric infinitesimal (ASI) conformal bialgebra.

Remark 4.7. By the definition of infinitesimal bialgebra given in [1], the corresponding definition of conformal version of infinitesimal bialgebra should be given as follows. \((A, \circ, \lambda)\) is an associative conformal algebra and \((A, \Delta)\) is an associative conformal coalgebra satisfying
\[ \Delta(a \circ \lambda b) = (L_A(a) \lambda \otimes I) \Delta(b) + (I \otimes R_A(b)_{-\lambda-\partial \otimes 2}) \Delta(a), \forall a, b \in A. \tag{31} \]

We would like to point out that the definition of infinitesimal conformal bialgebra given in Definition 4.6 is equivalent to this definition if we replace \((A, \circ, \lambda)\) by the opposite algebra of \((A, \cdot, \cdot)\) in the sense that \( a \circ \lambda b = b_{-\lambda-\partial} a \) for all \( a, b \in A \).

Combining Theorems 3.6 and 4.5 together, we have the following conclusion.

Corollary 4.8. Let \( A \) be a finite associative conformal algebra which is free as a \( \mathbb{C}[\partial] \)-module. Suppose there is another associative conformal algebra structure on the \( \mathbb{C}[\partial] \)-module \( A^{c} \) obtained from a \( \mathbb{C}[\partial] \)-module homomorphism \( \Delta : A \rightarrow A \otimes A \). Then the following conditions are equivalent:
Proposition 5.2. Let $A$, then 

Definition 5.1.\( \text{Conformal bialgebras.} \)

By a straightforward computation, the map defined by (32) gives an associative conformal coalgebra is an odd polynomial, i.e., $p$ is a Frobenius conformal algebra associated

Example 4.11. Let $p(\lambda) \in \mathbb{C}[\lambda]$ and $A = \mathbb{C}[\partial]a \otimes \mathbb{C}[\partial]b$ be a rank two associative conformal algebra with the following $\lambda$-product:

$\lambda a = p(\lambda + \partial) b, \quad \lambda b = b \lambda a = b \lambda b = 0.$

Define a $\mathbb{C}[\partial]$-module homomorphism $\Delta : A \rightarrow A \otimes A$ by

$\Delta(a) = a \otimes b, \quad \Delta(b) = b \otimes b.$

By a straightforward computation, $(A, \cdot \lambda, \Delta)$ is an ASI conformal bialgebra if and only if $p(\lambda)$ is an odd polynomial, i.e., $p(\lambda) = -p(-\lambda)$.

5. Coboundary antisymmetric infinitesimal conformal bialgebras

We consider a special class of ASI conformal bialgebras which are called coboundary ASI conformal bialgebras.

Definition 5.1. For an ASI conformal bialgebra $(A, \cdot \lambda, \Delta)$, if there exists $r \in A \otimes A$ such that

$$\Delta(a) = (I \otimes L_A(a)_\lambda - R_A(a)_\lambda \otimes I)r|_{\lambda=-\partial^2}, \quad \forall \ a \in A,$$

then $(A, \cdot \lambda, \Delta)$ is called coboundary.

For $r = \sum_i r_i \otimes l_i \in A \otimes A$, define

$$r \cdot r = \sum_{i,j} r_i \otimes r_j \otimes l_i \mu_{\lambda=\partial^2} - r_i \otimes r_j \mu_{\lambda=\partial^2} l_i \otimes l_j |_{\mu=1 \otimes \partial^2} l_i \otimes l_j |_{\mu=1 \otimes \partial^2}.$$

Proposition 5.2. Let $A$ be an associative conformal algebra and $r = \sum_i r_i \otimes l_i \in A \otimes A$. Then the map defined by (32) gives an associative conformal coalgebra $(A, \Delta)$ if and only if

$$(I \otimes I \otimes L_A(a)_{\lambda=\partial^3} - R_A(a)_{-\partial^3} \otimes I \otimes I)(r \cdot r) = 0, \quad \forall \ a \in A.$$
Proof. By the definition of $\Delta$, we have
\[
(I \otimes \Delta)\Delta(a) = (I \otimes \Delta)(\sum_i r_i \otimes a_{\lambda} l_i - \sum_i r_i \lambda - \theta \otimes a \otimes l_i)|_{\lambda = \theta^2}
\]
\[
= (I \otimes \Delta)(\sum_i r_i \otimes a_{\lambda} l_i - \sum_i r_{11 \otimes \theta} a \otimes l_i)
\]
\[
= \sum_i r_i \otimes \Delta(a_{\lambda} l_i) - \sum_i r_{11 \otimes \theta} a \otimes \Delta(l_i)
\]
\[
= \sum (r_i \otimes r_j \otimes (a_{\theta^2} l_i))_\mu l_j - r_i \otimes r_{1 \otimes \theta} (a_{\theta^2} l_i) \otimes l_j - (r_{11 \otimes \theta} a \otimes r_j \otimes l_i) l_j - r_{11 \otimes \theta} a \otimes r_{1 \otimes \theta} (l_i \otimes l_j)\mu = -1 \otimes \theta^2
\]
\[
= \sum r_i \otimes r_j \otimes a_{\theta^3} (l_i \otimes l_j) - r_i \otimes r_{1 \otimes \theta} (a_{\theta^2} l_i) \otimes l_j
\]
\[
= (1 \otimes 1 \otimes I_A(a)_{\theta^3}) \sum (r_i \otimes r_j \otimes l_i \otimes l_j) - r_i \otimes r_{1 \otimes \theta} (a_{\theta^2} l_i) \otimes l_j
\]
\[
= (R_A(a)_{\theta^3} \otimes 1 \otimes 1) \sum r_i \otimes r_j \otimes l_i l_j - r_i \otimes r_{1 \otimes \theta} (l_i \otimes l_j).
\]
Similarly, we have
\[
(\Delta \otimes I)\Delta(a) = (\Delta \otimes I)(\sum r_i \otimes a_{\lambda} l_i - r_{11 \otimes \theta} a \otimes l_i)
\]
\[
= \sum_i (\Delta(r_i) \otimes a_{\lambda} l_i - \Delta(r_{11 \otimes \theta} a) \otimes l_i)
\]
\[
= \sum (r_j \otimes r_{i \otimes \theta} l_j \otimes a_{\lambda} l_i - r_{j \otimes 1 \otimes \theta} r_i \otimes l_j \otimes a_{\theta^3} l_i)
\]
\[
- \sum (r_i \otimes (r_{1 \otimes \theta} a)_{\theta^2} l_i \otimes l_j - r_{j \otimes 1 \otimes \theta} (r_{1 \otimes \theta} a) \otimes l_j \otimes l_i)
\]
\[
= (I \otimes I \otimes L_A(a)_{\theta^3}) \sum (r_i \otimes r_{j \otimes \theta^2} l_i \otimes l_j - r_{1 \otimes \theta} r_j \otimes l_i \otimes l_j)
\]
\[
- \sum r_i \otimes r_{j \otimes \theta} (a_{\theta^2} l_i) \otimes l_j
\]
\[
+ (R_A(a)_{\theta^3} \otimes 1 \otimes 1) \sum r_{1 \otimes \theta} r_j \otimes l_i \otimes l_j.
\]
Therefore $\Delta$ satisfies $\text{(23)}$ if and only if $\text{(31)}$ holds. \hfill \QED

**Theorem 5.3.** Let $(A, \lambda)$ be an associative conformal algebra and $r = \sum_i r_i \otimes l_i \in A \otimes A$. Then the map defined by $\text{(32)}$ gives an associative conformal coalgebra $(A, \Delta)$ such that $(A, \cdot, \lambda, \Delta)$ is an ASI conformal bialgebra if and only if $\text{(33)}$ and the following equation hold:
\[
(L_A(b)_{\lambda - \theta^2} \otimes I - I \otimes R_A(b)_{\lambda - \theta^2})(I \otimes L_A(a)_{\theta^2} - R_A(a)_{\theta^2} \otimes I)(r + \tau r) = 0,
\]
for all $a, b \in A$.\]
Proof. Let \( a, b \in A \). We first check that \((27)\) holds automatically. In fact, we have
\[
(I \otimes L_A(a))_\lambda \Delta(b) + (R_A(b)_{-\lambda\otimes^2} \otimes I) \Delta(a)
\]
\[
= (I \otimes L_A(a))_\lambda \sum_i (r_i \otimes b_{-\lambda\otimes^2} l_i - r_{i1\otimes^2} b_i \otimes l_i)
+ (R_A(b)_{-\lambda\otimes^2} \otimes I) \sum_i (r_i \otimes a_{-\lambda\otimes^2} l_i - r_{i1\otimes^2} a_i \otimes l_i)
\]
\[
= \sum_i (r_i \otimes a_\lambda(b_{-\lambda\otimes^2} l_i) - r_{i\lambda+1\otimes^2} b_i \otimes a_\lambda l_i)
+ r_{i\lambda+1\otimes^2} b_i \otimes a_\lambda l_i - (r_{i1\otimes^2} a_i)_{\lambda+1\otimes^2} b_i \otimes l_i
\]
\[
= \sum_i (r_i \otimes a_\lambda(b_{-\lambda\otimes^2} l_i) - (r_{i1\otimes^2} a_i)_{\lambda+1\otimes^2} b_i \otimes l_i)
\]
\[
= \sum_i (r_i \otimes (a_\lambda b)_{-\lambda\otimes^2} l_i - r_{i1\otimes^2} (a_\lambda b) \otimes l_i)
= \Delta(a_\lambda b).
\]

Therefore \((27)\) holds.

Obviously, we have
\[
(L_A(b)_{-\lambda\otimes^2} \otimes I - I \otimes R_A(b)_{-\lambda\otimes^2}) \Delta(a)
\]
\[
= (L_A(b)_{-\lambda\otimes^2} \otimes I - I \otimes R_A(b)_{-\lambda\otimes^2})(I \otimes L_A(a)_{-\lambda\otimes^2} - R_A(a)_{-\lambda\otimes^2} \otimes I)\tau r.
\]

Moreover, we have
\[
\tau(L_A(a))_\lambda \otimes I - I \otimes R_A(a)_\lambda \Delta(b)
\]
\[
= \tau(L_A(a))_\lambda \otimes I - I \otimes R_A(a)_\lambda \sum_i (r_i \otimes b_{-\lambda\otimes^2} l_i - r_{i1\otimes^2} b_i \otimes l_i)
\]
\[
= \sum_i (a_\lambda r_i \otimes b_{-\lambda\otimes^2} l_i - a_\lambda (r_{i1\otimes^2} b_i) \otimes l_i - r_i \otimes (b_{-\lambda\otimes^2} l_i)_{-\lambda\otimes^2} a_i + r_{i\lambda+1\otimes^2} b_i \otimes l_{i-\lambda\otimes^2} a_i)
\]
\[
= \sum_i (b_{-\lambda\otimes^2} l_i \otimes a_\lambda r_i - l_i \otimes a_\lambda (r_{i\lambda+1\otimes^2} b_i) - (b_{-\lambda\otimes^2} l_i)_{-\lambda\otimes^2} a_i \otimes r_i + l_{i-\lambda\otimes^2} a_i \otimes r_{i\lambda+1\otimes^2} b_i)
\]
\[
= \sum_i (b_{-\lambda\otimes^2} l_i \otimes a_\lambda r_i - l_i \otimes (a_\lambda r_i)_{\lambda+1\otimes^2} b_i - b_{-\lambda\otimes^2} (l_{i1\otimes^2} a_i) + l_{i-\lambda\otimes^2} a_i \otimes r_{i\lambda+1\otimes^2} b_i)
\]
\[
= (L_A(b)_{-\lambda\otimes^2} \otimes I - I \otimes R_A(b)_{-\lambda\otimes^2})(I \otimes L_A(a)_{-\lambda\otimes^2} - R_A(a)_{-\lambda\otimes^2} \otimes I)(\tau r).
\]

Therefore \((28)\) holds if and only if \((35)\) holds. Hence by Proposition \(5.2\), the conclusion holds. \(\square\)

**Corollary 5.4.** Let \( A \) be an associative conformal algebra and \( r \in A \otimes A \). Suppose that \( r \) is antisymmetric. Then the map defined by \((72)\) gives an associative conformal coalgebra \((A, \Delta)\) such that \((A, \cdot, \Delta)\) is an ASI conformal bialgebra if
\[
r \cdot r \equiv 0 \mod (\otimes^3).
\]

**Proof.** If \( r \cdot r \equiv 0 \mod (\otimes^3) \), \((34)\) naturally holds by conformal sesquilinearity. Then this conclusion directly follows from Theorem \(5.3\). \(\square\)

**Definition 5.5.** Let \( A \) be an associative conformal algebra and \( r \in A \otimes A \). \((36)\) is called associative conformal Yang-Baxter equation in \( A \).
Remark 5.6. In fact, the associative conformal Yang-Baxter equation (36) is regarded as a conformal analogue of the associative Yang-Baxter equation in an associative algebra (2).

Definition 5.7. Let \((A, \cdot, \Delta_A)\) and \((B, \cdot, \Delta_B)\) be two ASI conformal bialgebras. If \(\varphi : A \to B\) is a homomorphism of associative conformal algebras satisfying
\[
(\varphi \otimes \varphi)\Delta_A(a) = \Delta_B(\varphi(a)), \quad \forall a \in A,
\]
then \(\varphi\) is called a homomorphism of ASI conformal bialgebras.

Theorem 5.8. Let \((A, \cdot, \Delta_A)\) be a finite ASI conformal bialgebra where \(A\) is free as a \(\mathbb{C}[\partial]\)-module. Then there is a canonical ASI conformal bialgebra structure on the \(\mathbb{C}[\partial]\)-module \(A \oplus A^\ast\) such that the inclusions \(i_1 : A \to A \oplus A^\ast\) and \(i_2 : A^\ast \to A \oplus A^\ast\) are homomorphisms of ASI conformal bialgebras. Here the ASI conformal bialgebra structure on \(A^\ast\) is \((A^\ast, \cdot, -\Delta_{A^\ast})\), where \(\cdot\) and \(\Delta_{A^\ast}\) on \(A^\ast\) are defined by (24) and (25) respectively.

Proof. Let \(\{e_1, \cdots, e_n\}\) be a \(\mathbb{C}[\partial]\)-basis of \(A\) and \(\{e_1^\ast, \cdots, e_n^\ast\}\) be the dual basis in \(A^\ast\). Since \((A, \cdot, \Delta_A)\) is an ASI conformal bialgebra, \((A, A^\ast, R_A^1, L_A^1, R_A^2, L_A^2)\) is a matched pair of associative conformal algebras by Corollary 4.8 where the associative conformal algebra structure on \(A^\ast\) is obtained from \(\Delta_A\). Then by Proposition 2.7 there is an associative conformal algebra structure on the \(\mathbb{C}[\partial]\)-module \(A \oplus A^\ast\) associated with such a matched pair. Set
\[
e_{i\lambda}e_j = \sum_k P^{ij}_k(\lambda, \partial)e_k, \quad \Delta_A(e_j) = \sum_{i,k} R^{ik}_j(\partial \otimes 1, -\partial \otimes 1 - 1 \otimes \partial)e_i \otimes e_k,
\]
where \(P^{ij}_k(\lambda, \partial), R^{ik}_j(\lambda, \partial) \in \mathbb{C}[\lambda, \partial]\). Then by Proposition 2.7 again, we have
\[
e^\ast_{i\lambda}e^\ast_j = \sum_k R^i_k(\lambda, \partial)e^\ast_k,
\]
\[
e_{i\lambda}e^\ast_j = R_A^1(e_i)\lambda e_j^\ast + L_A^1(e_j^\ast)\lambda - \partial e_i = e_{i\lambda}e^\ast_j + \sum_k P^{ki}_j(\partial, -\partial - \lambda) + \sum_k R^{ik}_j(-\partial - \lambda) e_k,
\]
\[
e^\ast_{i\lambda}e_j = R_A^2(e_i^\ast)\lambda e_j^\ast + L_A^2(e_j^\ast)\lambda - \partial e_i^\ast = e^\ast_{i\lambda}e_j^\ast - \sum_k R^{ki}_j(\partial, -\partial) + \sum_k P^{ik}_j(-\partial - \lambda) e_k^\ast.
\]
Let \(r = \sum_{i=1}^n e_i \otimes e_i^\ast \in (A \oplus A^\ast) \otimes (A \oplus A^\ast)\) and define
\[
\Delta_{A \oplus A^\ast}(a) = (I \otimes L_A(a) \lambda - R_A(a) \lambda \otimes I)r |_{\lambda = -\partial \otimes 2}, \quad \forall a \in A \oplus A^\ast.
\]
Note that
\[
r \cdot r = \sum_{i,j} (e_i \otimes e_j \otimes e_i^\ast \otimes e_j^\ast |_{\mu = \partial \otimes 1})
\]
\[
- e_i \otimes e_j \otimes e_i^\ast \otimes e_j^\ast |_{\mu = -\partial^2 \otimes 1} + e_{i\mu}e_j \otimes e_i^\ast \otimes e_j^\ast |_{\mu = 1 \otimes \partial \otimes 1}
\]
\[
= \sum_{i,j,k} (R^{ij}_k(\partial \otimes 1 \otimes 1, 1 \otimes 1 \otimes \partial)e_i \otimes e_j \otimes e_k^\ast - P^{kj}_i(1 \otimes \partial \otimes 1, \partial \otimes 1 \otimes 1)e_i \otimes e_k^\ast \otimes e_j^\ast
\]
\[
- R^{ik}_j(\partial \otimes 1 \otimes 1, -\partial^2 \otimes 1)e_i \otimes e_k \otimes e_j^\ast + P^{ij}_k(1 \otimes \partial \otimes 1, \partial \otimes 1 \otimes 1)e_k \otimes e_i^\ast \otimes e_j^\ast)
\]
\[
= \sum_{i,j,k} (R^{ij}_k(\partial \otimes 1 \otimes 1, 1 \otimes 1 \otimes \partial) - R^{ij}_k(\partial \otimes 1 \otimes 1, -\partial^2 \otimes 1))e_i \otimes e_i \otimes e_k^\ast
\]
\[
\equiv 0 \mod \partial ^2 \otimes 1.
\]
Then \( 34 \) holds by conformal sesquilinearity. Moreover, for all \( e_i \in A \), we have

\[
(I \otimes L_A(e_i) - \vartheta \otimes 2 - R_A(e_i) - \vartheta \otimes 2 \otimes I)(r + \tau r)
\]

\[
= \sum_j (I \otimes L_A(e_i) - \vartheta \otimes 2 - R_A(e_i) - \vartheta \otimes 2 \otimes I)(e_j \otimes e_j^* + e_j^* \otimes e_j)
\]

\[
= \sum_j (e_j \otimes e_i - \vartheta \otimes 2 e_j^* + e_j^* \otimes e_i - e_j 1 \otimes \vartheta e_i \otimes e_j^* - e_j^* 1 \otimes \vartheta e_i \otimes e_j)
\]

\[
= \sum_{j,k}(P^{k_i}_j(1 \otimes \vartheta, \vartheta \otimes 1) e_j \otimes e_k + R^{i k}_j(\vartheta \otimes 1, -\vartheta \otimes 2) e_j \otimes e_k + P^{i j}_k(-\vartheta \otimes 2, 1 \otimes \vartheta) e_j^* \otimes e_k
\]

\[
- P^{i j}_k(1 \otimes \vartheta, \vartheta \otimes 1) e_k \otimes e_j^* - R^{k j}_i(\vartheta \otimes 1, -\vartheta \otimes 2) e_k \otimes e_j - P^{i k}_j(-\vartheta \otimes 2, 1 \otimes \vartheta) e_k^* \otimes e_j^* = 0.
\]

Similarly, for all \( e_i^* \in A^{sc} \), we have

\[
(I \otimes L_A(e_i^*) - \vartheta \otimes 2 - R_A(e_i^*) - \vartheta \otimes 2 \otimes I)(r + \tau r) = 0.
\]

Therefore \( 35 \) holds. Hence by Theorem \( 53 \), \( \Delta_{A \oplus A^{sc}} \) gives an ASI conformal bialgebra structure on \( A \oplus A^{sc} \).

Furthermore, for all \( e_i \in A \), we have

\[
\Delta_{A \oplus A^{sc}}(e_i) = \sum_{j=1}^n (I \otimes L_{A \oplus A^{sc}}(e_i) \lambda - R_{A \oplus A^{sc}}(e_i) \lambda \otimes I)(e_j \otimes e_j^*)|_{\lambda = -\vartheta \otimes 2}
\]

\[
= \sum_{j=1}^n (e_j \otimes e_i - \vartheta \otimes 2 e_j^* - e_j 1 \otimes \vartheta e_i \otimes e_j^*)
\]

\[
= \sum_{j=1}^n (e_j \otimes (\sum_k P^{k i}_j(1 \otimes \vartheta, \vartheta \otimes 1) e_k^* + \sum_k R^{k j}_i(\vartheta \otimes 1, -\vartheta \otimes 2) e_k)
\]

\[
- \sum_k P^{i j}_k(1 \otimes \vartheta, \vartheta \otimes 1) e_k \otimes e_j^*)
\]

\[
= \sum_{j=1}^n \sum_k R^{k j}_i(\vartheta \otimes 1, -\vartheta \otimes 2) e_j \otimes e_k = \Delta_A(e_i).
\]

Similarly, for all \( e_i^* \in A^{sc} \), we have

\[
\Delta_{A \oplus A^{sc}}(e_i^*) = -\Delta_{A^{sc}}(e_i).
\]

Therefore \( i_1 : A \rightarrow A \oplus A^{sc} \) and \( i_2 : A^{sc} \rightarrow A \oplus A^{sc} \) are homomorphisms of ASI conformal bialgebras. Hence the proof is finished. \( \square \)

6. \( \mathcal{O} \)-OPERATORS OF ASSOCIATIVE CONFORMAL ALGEBRAS AND DENDRIFORM CONFORMAL ALGEBRAS

We introduce the notion of \( \mathcal{O} \)-operators of associative conformal algebras to interpret the associative conformal Yang-Baxter equation. In particular, an \( \mathcal{O} \)-operator of an associative conformal algebra \( A \) associated to a bimodule gives an antisymmetric solution of associative conformal Yang-Baxter equation in a semi-direct product associative conformal algebra. We also introduce the notion of dendriform conformal algebras to construct \( \mathcal{O} \)-operators of their associated associative conformal algebras and hence give (antisymmetric) solutions of associative conformal Yang-Baxter equation.
Let $A$ be a finite associative conformal algebra which is free as a $\mathbb{C}[\partial]-$module. Define a linear map $\varphi : A \otimes A \to Chom(A^c, A)$ as

$$\varphi(u \otimes v)_{\lambda}(g) = g_{-\lambda - \partial A}(u)v, \ \forall \ u, v \in A, g \in A^c.$$ (39)

Here $\partial A$ represents the action of $\partial$ on $A$. Obviously, $\varphi$ is a $\mathbb{C}[\partial]-$module homomorphism. Similar to Proposition 6.1 in [7], we show that $\varphi$ is a $\mathbb{C}[\partial]-$module isomorphism.

Set $r = \sum r_i \otimes l_i \in A \otimes A$. By $\varphi$, we associate a conformal linear map $T^r \in Chom(A^c, A)$ given by

$$T^r_\lambda(f) = \sum_i f_{-\lambda - \partial A}(r_i)l_i, \ \forall \ f \in A^c.$$ (40)

For all $f \in A^c$ and $a \in A$, we define $\langle a, f \rangle_\lambda = \{f, a\}_- = f_\lambda(a)$. Obviously,

$$\langle \partial a, f \rangle_\lambda = -\lambda \langle a, f \rangle_\lambda.$$ (41)

We also define

$$\langle a \otimes b \otimes c, f \otimes g \otimes h \rangle_{(\lambda, \mu, \theta)} = \langle a, f \rangle_\lambda \langle b, g \rangle_{\mu} \langle c, h \rangle_{\theta}, \ \forall \ a, b, c \in A, f, g, h \in A^c.$$ (42)

By Proposition 2.5, we have

$$\langle a \lambda b, f \rangle_\mu = \langle b, L^*_A(a)_\lambda f\rangle_{\mu - \lambda}, \ \langle b_{\mu - \lambda}a, f \rangle_\mu = \langle b, R^*_A(a)_\lambda f\rangle_{\mu - \lambda}, \ \forall a, b \in A, f \in A^c.$$ (43)

**Theorem 6.1.** Let $A$ be a finite associative conformal algebra which is free as a $\mathbb{C}[\partial]-$module and $r \in A \otimes A$ be antisymmetric. Then $r$ is a solution of associative conformal Yang-Baxter equation if and only if $T^r \in Chom(A^c, A)$ satisfies

$$T^r_0(f)_\lambda T^r_0(g) - T^r_0(R^*_A(T^r_0(f))_{\lambda}g) - T^r_0(L^*_A(T^r_0(g))\lambda - \partial f) = 0, \ \forall \ f, g \in A^c.$$ (44)

**Proof.** Since $r$ is antisymmetric, we have

$$T^r_\lambda(f) = \sum_i \langle r_i, f \rangle_{\lambda + \partial A}l_i = -\sum_i \langle l_i, f \rangle_{\lambda + \partial A}r_i, \ \forall \ f \in A^c.$$ (45)

Obviously, the fact that $r \bullet r \mod (\partial^{\otimes 3}) = 0$ holds if and only if the following equation holds:

$$\langle r \bullet r \mod (\partial^{\otimes 3}), f \otimes g \otimes h \rangle_{(\lambda, \eta, \nu)} = 0, \ \forall \ f, g, h \in A^c,$$

if and only if the following equation holds:

$$\langle r \bullet r, f \otimes g \otimes h \rangle_{(\lambda, \eta, \nu)} = 0 \text{ mod } (\lambda + \eta + \nu), \ \forall \ f, g, h \in A^c.$$ (46)

Let $f, g, h \in A^c$. Then we have

$$\langle \sum_{i,j} r_i \otimes r_j \otimes l_i l_j |_{\mu = \partial \otimes 1 \otimes 1}, f \otimes g \otimes h \rangle_{(\lambda, \eta, \nu)}$$

$$= \sum_{i,j} \langle r_i, f \rangle_{\lambda} \langle r_j, g \rangle_{\eta} \langle l_i - \lambda l_j, h \rangle_{\nu} = \langle (\sum_i \langle r_i, f \rangle_{\lambda} l_i) - \lambda (\sum_j \langle r_j, g \rangle_{\eta} l_j), h \rangle_{\nu}$$

$$= \langle T^r_{\lambda - \partial}(f) - \lambda T^r_{\eta - \partial}(g), h \rangle_{\nu} = (T^r_0(f) - \lambda T^r_{\lambda + \eta + \nu}(g), h \rangle_{\nu},$$ (47)
and

\[
\langle \sum_{i,j} r_i \otimes r_j \mu | l_i \otimes l_j \rangle_{\mu = -\partial \otimes 1, f \otimes g \otimes h} = \langle r_{ij} f, \lambda \rangle_{\lambda+\eta \langle l_j, h \rangle_{\eta \nu}} = \sum_{i,j} \langle r_{ij} f, \lambda \rangle_{\lambda+\eta \langle l_j, h \rangle_{\eta \nu}} = \sum_{i,j} \langle r_{ij} f, \lambda \rangle_{\lambda+\eta \langle l_j, h \rangle_{\eta \nu}} = \langle (\sum_{j} r_j, R^*_A(T^r_0(f))-\lambda g)_{\lambda+\eta \langle l_j, h \rangle_{\eta \nu}} = \langle (T^r_0(R^*_A(T^r_0(f))-\lambda g), h, l_j)_{\eta \nu} = \langle T^r_0(R^*_A(T^r_0(f))-\lambda g), h, l_j \rangle_{\eta \nu}.
\]

Therefore (43) holds if and only if

\[
\langle T^r_0(f)-\lambda f, T^r_0(g)-T^r_0(R^*_A(T^r_0(f))-\lambda g), h, l_j \rangle_{\eta \nu} = \langle T^r_0(L^*_A(T^r_0(g))-\eta f), h, l_j \rangle_{\eta \nu} = 0 \mod (\lambda + \eta + \nu), \forall f, g, h \in A^{*c}.
\]

It is straightforward that the above equality holds if and only if

\[
T^r_0(f)-\lambda T^r_0(g)-T^r_0(R^*_A(T^r_0(f))-\lambda g) - L^*_A(T^r_0(g))_{\lambda-\partial f} = 0, \forall f, g \in A^{*c}. \tag{44}
\]

Therefore the conclusion follows by replacing \( \lambda \) by \(-\lambda \) in (44). \( \square \)

**Corollary 6.2.** Let \( A \) be a finite associative conformal algebra which is free as a \( \mathbb{C}[\partial] \)-module, \( r \in A \otimes A \) be antisymmetric and \( \Delta_A \) be the map defined by (32) through \( r \). Suppose the associative conformal algebra structure on \( A^{*c} \) is obtained from \( \Delta_A \). Let \( T^c \in \text{Chom}(A^{*c}, A) \) be the element corresponding to \( r \) through the isomorphism \( A \otimes A \cong \text{Chom}(A^{*c}, A) \). Then \( T^c : A^{*c} \rightarrow A \) is a homomorphism of associative conformal algebras.

**Proof.** Let \( \{e_1, \cdots, e_n\} \) be a \( \mathbb{C}[\partial] \)-basis of \( A \) and \( \{e_1^*, \cdots, e_n^*\} \) be the dual \( \mathbb{C}[\partial] \)-basis. Set

\[
e_i \lambda e_j = \sum_k P_{ij}^k e_k, \quad \tau = \sum_{i,j} a_{ij}(\partial \otimes 1, 1 \otimes \partial) e_i \otimes e_j,
\]

where \( a_{ij}(\lambda, \mu) \in \mathbb{C}[\lambda, \mu] \). Then

\[
\Delta_A(e_k) = (I \otimes L_A(e_k) - R_A(a)_{\lambda \otimes 1}) r |_{\lambda = -\partial \otimes 2} = \sum_{i,j} a_{ij}(\partial \otimes 1, \lambda + 1 \otimes \partial) e_i \otimes e_k \lambda e_j - \sum_{i,j} a_{ij}(\lambda + \partial \otimes 1, 1 \otimes \partial) e_i \lambda - \partial e_k \otimes e_j = \sum_{i,j,s} (a_{ij}(\partial \otimes 1, -\partial \otimes 1) P_{ij}^k (-\partial \otimes 1, 1 \otimes \partial) - a_{js}(-1 \otimes \partial, 1 \otimes \partial) P_{jk}^i (1 \otimes \partial, \partial \otimes 1)) e_i \otimes e_s.
\]
Note that $T^*_0(e_t^*) = \sum_j a_{kj}(-\partial, \partial)e_j$. Then by Proposition 4.2,

$$e_t^*e_t^* = \sum_{j,k} (a_{ij}(\lambda, -\lambda)P^*_k(\partial, -\lambda - \partial) - a_{ij}(\lambda + \partial, -\lambda - \partial)P^*_j(\lambda - \partial, \lambda))e_k^*$$

$$= \sum_{j,k} (a_{ij}(\lambda, -\lambda)e_t^*_{-\lambda-\partial A^{sc}}(e_k\partial\partial A^{sc}e_j) - a_{ij}(\lambda + \partial, -\lambda - \partial)e_t^*_{\lambda}(e_j - \lambda - \partial A^{sc}e_k))e_k^*$$

$$= \sum_{j,k} (a_{ij}(\lambda, -\lambda)e_t^*_{-\lambda-\partial A^{sc}}(e_k\partial\partial A^{sc}e_j) + a_{ij}(-\lambda - \partial, \lambda + \partial)e_t^*_{\lambda}(e_j - \lambda - \partial A^{sc}e_k))e_k^*$$

$$= \sum_k (e_t^*_{-\lambda-\partial A^{sc}}(e_k\partial\partial A^{sc}T^*_0(e_t))) + e_t^*_{\lambda}(T^*_0(e_t) - \lambda - \partial A^{sc}e_k))e_k^*$$

$$= R^*_A(T^*_0(e_t))_\lambda e_t^* + L^*_A(T^*_0(e_t))_{-\lambda - \partial e_t^*}.$$

Thus by Theorem 6.1, we have

$$T^*_0(e_t^*) = T^*_0(e_t^*), \quad \forall l, t \in \{1, \ldots, n\}.$$

Therefore $T^*_0 : A^{sc} \to A$ is a homomorphism of associative conformal algebras. \hfill \square

Let $A$ be a Frobenius conformal algebra with a non-degenerate invariant conformal bilinear form $\langle \cdot, \cdot \rangle_\lambda$, which is finitely generated and free as a $\mathbb{C}[\partial]$-module. Define

$$\langle a \otimes b, c \otimes d \rangle_{(\lambda, \mu)} = \langle a, c \rangle_\lambda \langle b, d \rangle_\mu, \quad \forall a, b, c, d \in A. \quad (45)$$

Let $r = \sum_i r_i \otimes l_i \in A \otimes A$. Define a linear map $P^r : A \to A[\lambda]$ by

$$\langle r, u \otimes v \rangle_{(\lambda, \mu)} = \langle P^r_{\lambda-\partial}(u), v \rangle_\mu, \quad \forall u, v \in A.$$

Obviously, $P^r \in \text{Cend}(A)$.

**Corollary 6.3.** Let $A$ be a symmetric Frobenius conformal algebra which is finitely generated and free as a $\mathbb{C}[\partial]$-module and $r \in A \otimes A$ be antisymmetric. Then $r$ is a solution of associative conformal Yang-Baxter equation in $A$ if and only if $P^r \in \text{Cend}(A)$ satisfies

$$P^r_0(a)_\lambda P^r_0(b) = P^r_0(P^r_0(a)_\lambda b) + P^r_0(a_\lambda P^r_0(b)), \quad \forall a, b \in A. \quad (46)$$

**Proof.** Since $A$ has a non-degenerate symmetric invariant conformal bilinear form, $\varphi : A \to A^{sc}$, $a \mapsto \varphi_a$ defined by

$$(\varphi_a)_\lambda b = \langle a, b \rangle_\lambda, \quad \forall a, b \in A,$$

is an isomorphism of $\mathbb{C}[\partial]$-modules. By the definitions of $T^r$ in Theorem 6.1 and $P^r$, we get $P^r = T^r \circ \varphi$. Therefore $P^r_0 = T^r_0 \circ \varphi$. Since $\varphi$ is a $\mathbb{C}[\partial]$-module isomorphism, for any $f, g \in A^{sc}$, we assume that $\varphi(a) = f$ and $\varphi(b) = g$. Then (42) becomes

$$T^r_0(\varphi(a))_\lambda T^r_0(\varphi(b)) - T^r_0(R^*_A(T^r_0(\varphi(a)))_\lambda \varphi(b)) - T^r_0(L^*_A(T^r_0(\varphi(b)) - \lambda - \partial \varphi(a)) = 0. \quad (47)$$

Thus

$$P^r_0(a)_\lambda P^r_0(b) - T^r_0(R^*_A(P^r_0(a))_\lambda \varphi(b)) - T^r_0(L^*_A(P^r_0(b)) - \lambda - \partial \varphi(a)) = 0, \quad \forall a, b \in A. \quad (48)$$

For all $a, b, c \in A$, we have

$$(R^*_A(P^r_0(a))_\lambda \varphi(b))_\mu c = \varphi(b)_\mu - \lambda(R_A(P^r_0(a))_\lambda c) = \langle b, R_A(P^r_0(a))_\lambda c \rangle_\mu - \lambda$$

$$= \langle b, c_\lambda - \partial P^r_0(a) \rangle_\mu - \lambda = \langle b_\mu - \lambda c, P^r_0(a) \rangle - \lambda$$

$$= \langle P^r_0(a), b_\mu - \lambda c \rangle_\lambda = \langle P^r_0(a)_\lambda b, c \rangle_\mu.$$
Therefore we have
\[ R'_\lambda(P^0_0(a))\lambda\varphi(b) = \varphi(P^0_0(a)b), \quad \forall \ a, b \in A. \]
Similarly, we have
\[ L'_\lambda(P^0_0(b))_{-\lambda-\partial}\varphi(a) = \varphi(a\lambda P^0_0(b)), \quad \forall \ a, b \in A. \]
Hence (46) follows from (48).

Similar to the above correspondence. Then by Theorem 6.1, the conclusion holds.

By the fact that the \( T^*_0 \) in Theorems 6.1 and the \( P^*_0 \) in Corollary 6.3 are \( \mathbb{C}[\partial] \)-module homomorphisms, we present the following notions.

**Definition 6.4.** Let \( A \) be an associative conformal algebra and \((M, l_A, r_A)\) be a bimodule of \( A \). A \( \mathbb{C}[\partial] \)-module homomorphism \( T : M \to A \) is called an \( \mathcal{O} \)-operator associated with \((M, l_A, r_A)\) if \( T \) satisfies
\[ T(u)\lambda T(v) = T(l_A(T(u))\lambda v) + T(r_A(T(v))_{-\lambda-\partial}u), \quad \forall u, v \in M. \tag{49} \]
In particular, an \( \mathcal{O} \)-operator \( T : A \to A \) associated with the bimodule \((A, L_A, R_A)\) is called a Rota-Baxter operator (of weight zero) on \( A \), that is, \( T \) is a \( \mathbb{C}[\partial] \)-module homomorphism satisfying
\[ T(a)\lambda T(b) = T(T(a)\lambda b) + T(a\lambda T(b)), \quad \forall \ a, b \in A. \tag{50} \]

**Example 6.5.** Let \( A \) be a finite associative conformal algebra which is free as a \( \mathbb{C}[\partial] \)-module and \( r \in A \otimes A \) be antisymmetric. Then \( r \) is a solution of associative conformal Yang-Baxter equation if and only if \( T^*_0 \) is an \( \mathcal{O} \)-operator associated with the bimodule \((A^c, R^*_A, L^*_A)\). If in addition, \( A \) is a symmetric Frobenius conformal algebra, that is, \( A \) has a non-degenerate symmetric invariant conformal bilinear form, then \( r \) is a solution of associative conformal Yang-Baxter equation if and only if \( P^*_0 \) is a Rota-Baxter operator (of weight zero) on \( A \).

**Example 6.6.** Let \( A \) be an associative conformal algebra. The identity map \( I \) is an \( \mathcal{O} \)-operator associated with the bimodule \((A, L_A, 0)\) or \((A, 0, R_A)\).

Let \((M, l_A, r_A)\) be a bimodule of an associative conformal algebra \( A \). Then \((M^c, r_A^*, l_A^*)\) is a bimodule of \( A \) by Proposition 2.5. Suppose that \( M \) is a \( \mathbb{C}[\partial] \)-module of finite rank. By Proposition 6.1 in [7], \( M^c \otimes A \cong \text{Chom}(M, A) \) as \( \mathbb{C}[\partial] \)-modules through the isomorphism \( \varphi \) defined as
\[ \varphi(f \otimes a)\lambda v = f\lambda_{\partial}a(v), \quad \forall \ a \in A, v \in M, f \in M^c. \]
By the \( \mathbb{C}[\partial] \)-module actions on \( M^c \otimes A \), we also get \( M^c \otimes A \cong A \otimes M^c \) as \( \mathbb{C}[\partial] \)-modules. Therefore as \( \mathbb{C}[\partial] \)-modules, \( \text{Chom}(M, A) \cong A \otimes M^c \). Consequently, for any \( T \in \text{Chom}(M, A) \), we associate an \( r_T \in A \otimes M^c \subset (A \otimes r_A^* l_A^*) M^c \otimes (A \otimes r_A^* l_A^*) M^c \).

**Theorem 6.7.** Let \( A \) be a finite associative conformal algebra and \((M, l_A, r_A)\) be a finite bimodule of \( A \). Suppose that \( A \) and \( M \) are free as \( \mathbb{C}[\partial] \)-modules. Let \( T \in \text{Chom}(M, A) \) and \( r_T \in A \otimes M^c \subset (A \otimes r_A^* l_A^*) M^c \otimes (A \otimes r_A^* l_A^*) M^c \) be the element corresponding to \( T \) under the above correspondence. Then \( r = r_T - \tau_T \) is an antisymmetric solution of the associative conformal Yang-Baxter equation in \( A \otimes r_A^* l_A^*) M^c \) if and only if \( T_0 = T_{\lambda=0} \) is an \( \mathcal{O} \)-operator associated with the bimodule \((M, l_A, r_A)\).
Proof. Let \( \{e_1, \cdots, e_n\} \) be a \( \mathbb{C}[\partial]\)-basis of \( A \), \( \{v_1, \cdots, v_m\} \) be a \( \mathbb{C}[\partial]\)-basis of \( M \) and \( \{v_1^*, \cdots, v_m^*\} \) be the dual \( \mathbb{C}[\partial]\)-basis of \( M^{**} \). Assume that

\[
T_\lambda(v_i) = \sum_{j=1}^n g_{ij}(\lambda, \partial)e_j, \quad \forall \ i = 1, \cdots, m,
\]

where \( g_{ij}(\lambda, \partial) \in \mathbb{C}[\lambda, \partial] \). Then we have

\[
r_T = \sum_{j=1}^n \sum_{i=1}^m g_{ij}(-\partial^2, \partial \otimes 1)e_j \otimes v_i^*.
\]

Therefore we have

\[
r = \sum_{i,j} (g_{ij}(-\partial^2, \partial \otimes 1)e_j \otimes v_i^* - g_{ij}(-\partial^2, 1 \otimes \partial)v_i^* \otimes e_j).
\]

Moreover, by the definition of \( (M^{**}, r^*_A, l^*_A) \), we have

\[
l^*_A(e_i)\lambda v_j^* = \sum_k v_j^* - \lambda - \partial (l_A(e_i)\lambda v_k)v_k^*, \quad r^*_A(e_i)\lambda v_j^* = \sum_k v_j^* - \lambda - \partial (r_A(e_i)\lambda v_k)v_k^*.
\]

Then we get

\[
\begin{align*}
r \cdot r & \equiv \sum_{i,j,k,l} (-g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, -1 \otimes \partial \otimes 1)e_j \otimes v_k^* \otimes e_l \mu e_i|_{\mu = -\partial^3 \otimes 1} \\
& - g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)v_i^* \otimes e_l \otimes e_j \mu v_k^|_{\mu = -\partial^3 \otimes 1} \\
& + g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)v_i^* \otimes e_l \otimes e_j \mu v_k^|_{\mu = -\partial^3 \otimes 1} \\
& - g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)e_j \otimes e_l \mu v_i^* \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& + g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)e_j \otimes e_l \mu v_i^* \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& - g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)v_i^* \otimes e_l \mu e_j \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& + g_{ij}(0, -\partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)v_i^* \otimes e_l \mu e_j \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& - g_{ij}(0, \partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)e_j \otimes e_l \mu v_i^* \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& - g_{ij}(0, \partial \otimes 1 \otimes 1)g_{kl}(0, 1 \otimes \partial \otimes 1)e_j \otimes e_l \mu v_i^* \otimes v_k^|_{\mu = -\partial^3 \otimes 1} \\
& \equiv \sum_{i,k}((-T_0(v_i) \otimes v_k^* \otimes v_i^* - v_i^* \otimes T_0(v_k) \otimes v_i^*) - v_i^* \otimes T_0(v_k) \otimes T_0(v_i)\mu v_k^* \\
& + v_i^* \otimes v_k^* T_0(v_i)\mu T_0(v_k)|_{\mu = -\partial \otimes 1 \otimes 1} \\
& + (-T_0(v_i) \otimes T_0(v_k)\mu v_i^* \otimes v_k^* + v_i^* \otimes T_0(v_k)\mu T_0(v_i) \otimes v_k^* \\
& - v_i^* \otimes v_k^* T_0(v_i) \otimes T_0(v_k)|_{\mu = -\partial \otimes 1 \otimes 1} \\
& + (T_0(v_i)\mu T_0(v_k) \otimes v_i^* \otimes v_k^* - T_0(v_i)\mu v_i^* \otimes v_k^* \otimes T_0(v_k) \\
& - v_i^* \mu T_0(v_k) \otimes T_0(v_i) \otimes v_k^*)|_{\mu = 1 \otimes \partial \otimes 1} \ mod \ (\partial^3).
\end{align*}
\]
Since $T_0$ is a $\mathbb{C}[\partial]$-module homomorphism, we have
\[
\sum_{i,k} T_0(v_i) \otimes v_k^* \otimes v_i^* \mu T_0(v_k)|_{\mu = \partial \otimes 1 \otimes 1} = \sum_{i,k} T_0(v_i) \otimes v_k^* \otimes l_A^\partial(T_0(v_k)) - \mu \otimes v_i^* |_{\mu = \partial \otimes 1 \otimes 1} = \sum_{i,k} T_0(v_i) \otimes v_k^* \otimes l_A^\partial(T_0(v_k))_{1 \otimes \partial \otimes 1} v_i^* \ mod (\partial^{\otimes 3}) \\
\Rightarrow \sum_{i,j,k} T_0(v_i) \otimes v_k^* \otimes v_i^* \partial \otimes 1 \otimes 1 (i, k) (l_A(T_0(v_k))_{1 \otimes \partial \otimes 1} v_j) v_j^* \ mod (\partial^{\otimes 3}) \\
\Rightarrow \sum_{i,j,k} T_0(v_i) \otimes v_k^* \otimes (l_A(T_0(v_k))_{1 \otimes \partial \otimes 1} v_j) v_j^* \ mod (\partial^{\otimes 3}) \\
\Rightarrow \sum_{i,j,k} T_0(l_A((T_0(v_k))_\mu v_j)) \otimes v_k^* \otimes v_j^* |_{\mu = 1 \otimes \partial \otimes 1} \ mod (\partial^{\otimes 3}).
\]

Similarly, we have
\[
\forall r \in \mathbb{C}[\partial], \ r \cdot r \ mod (\partial^{\otimes 3}) \Rightarrow \sum_{i,k} ((T_0(v_k)), T_0(v_i) - T_0(l_A(T(v_k)))_\mu v_i) - T_0(r_A(T_0(v_i)) - \lambda \otimes \partial \otimes 1 \otimes 1) \otimes v_k^* \otimes v_i^* |_{\mu = 1 \otimes \partial \otimes 1} \\
+ (v_k^* \otimes (T_0(v_k)_\mu T_0(v_i) - T_0(l_A(T(v_k)))_\mu v_i) - T_0(r_A(T_0(v_i)) - \lambda \otimes \partial \otimes 1 \otimes 1) \otimes v_k^* |_{\mu = 1 \otimes \partial \otimes 1} \\
+ (v_k^* \otimes v_i^* \otimes (T_0(v_k)_\mu T_0(v_i) - T_0(l_A(T(v_k)))_\mu v_i) - T_0(r_A(T_0(v_i)) - \lambda \otimes \partial \otimes 1 \otimes 1) \otimes v_k^* |_{\mu = 1 \otimes \partial \otimes 1}).
\]

Therefore $r$ is a solution of associative conformal Yang-Baxter equation in the associative conformal algebra $A \ltimes v_i^* i_A M$ if and only if
\[
T_0(v_k)_\mu T_0(v_i) = T_0(l_A(T(v_k)))_\mu v_i + T_0(r_A(T_0(v_i)))_\mu v_i, \ \forall i, k \in \{1, \cdots, m\}.
\]

Thus this conclusion holds. \qed

At the end of this paper, we introduce a class of conformal algebras, namely, dendriform conformal algebras, which are used to construct Q-operators naturally and hence give solutions of associative conformal Yang-Baxter equation.

**Definition 6.8.** Let $A$ be a $\mathbb{C}[\partial]$-module with two bilinear products $\prec_\lambda$ and $\succ_\lambda$: $A \times A \to A[\lambda]$. If for all $a, b, c \in A$,
\[
(\partial a) \succ_\lambda b = -\lambda a \succ_\lambda b, \ a \succ_\lambda (\partial b) = (\partial + \lambda)(a \succ_\lambda b), \ (51) \\
(\partial a) \prec_\lambda b = -\lambda a \prec_\lambda b, \ a \prec_\lambda (\partial b) = (\partial + \lambda)(a \prec_\lambda b), \ (52) \\
(a \prec_\lambda b) \prec_{\lambda + \mu} c = a \prec_{\lambda} (b \star_{\mu} c), \ (53) \\
(a \prec_{\lambda} b) \prec_{\lambda + \mu} c = a \succ_{\lambda} (b \prec_{\mu} c), \ (54) \\
(a \succ_\lambda (b \succ_{\mu} c) = (a \prec_{\lambda} b) \succ_{\lambda + \mu} c, \ (55)
\]

where $a \prec_{\lambda} b = a \prec_{\lambda} b + a \succ_{\lambda} b$, then $(A, \prec_{\lambda}, \succ_{\lambda})$ is called a **dendriform conformal algebra**.

**Remark 6.9.** It is obvious that $(A, \prec_{\lambda}, \succ_{\lambda})$ with $\succ_{\lambda}$ being trivial (or $\prec_{\lambda}$ being trivial) is a dendriform conformal algebra if and only if $(A, \prec_{\lambda})$ (or $(A, \succ_{\lambda})$) is an associative conformal algebra.
Example 6.10. Let $\langle A, \langle, \rangle \rangle$ be a dendriform algebra [25]. Then $\text{Cur}(A) = \mathbb{C}[\partial] \otimes A$ is endowed a natural dendriform conformal algebra as follows.
\[ a \triangleleft_\lambda b = a \langle b, \ a \triangleright_\lambda b = a \rangle b, \ \forall a, b \in A. \tag{56} \]
Moreover, $(\text{Cur}(A), \langle, \triangleright_\lambda \rangle)$ is called a current dendriform conformal algebra. It is straightforward that any dendriform conformal algebra which is free and of rank one as a $\mathbb{C}[\partial]$-module is current.

Proposition 6.11. Let $\langle A, \langle, \rangle \rangle$ be a dendriform conformal algebra. Define
\[ a \ast_\lambda b = a \langle b + a \rangle b, \ \forall a, b \in A. \tag{57} \]
Then $(A, \ast_\lambda)$ is an associative conformal algebra. We call $(A, \ast_\lambda)$ the associated associative conformal algebra of $(A, \langle, \rangle)$ and $(A, \langle, \rangle)$ is called a compatible dendriform conformal algebra structure on the associative conformal algebra $(A, \ast_\lambda)$.

Proof. It is straightforward. \hfill \Box

Let $(A, \langle, \rangle)$ be a dendriform conformal algebra. Set
\[ L_\triangleright(a)\lambda(b) = a \triangleright_\lambda b, \ L_\langle(a)\lambda(b) = a \langle b, \ R_\triangleright(a)\lambda(b) = b \triangleright_{\lambda - \partial} a, \ R_\langle(a)\lambda(b) = b \langle_{\lambda - \partial} a, \ \forall a, b \in A. \]

Proposition 6.12. Let $\langle A, \langle, \rangle \rangle$ be a dendriform conformal algebra. Then $(A, L_\triangleright, R_\langle)$ is a bimodule of the associated associative conformal algebra $(A, \ast_\lambda)$. Hence the identity $I$ is an $\partial$-operator of the associative conformal algebra $(A, \ast_\lambda)$ associated with $(A, L_\triangleright, R_\langle)$.

Proof. It is straightforward. \hfill \Box

Proposition 6.13. Let $A$ be an associative conformal algebra and $(M, l_A, r_A)$ be a bimodule of $A$. Suppose $T : M \to A$ is an $\partial$-operator associated with $(M, l_A, r_A)$. Then the following $\lambda$-product
\[ u \triangleright_\lambda v = l_A(T(u))\lambda v, \ u \langle_\lambda v = r_A(T(v)) -_\lambda - \partial u, \ u, v \in M, \tag{58} \]
endows a dendriform conformal algebra structure on $M$. Therefore there is an associated conformal algebra structure on $M$ and $T : M \to A$ is a homomorphism of associative conformal algebras. Moreover, $T(M) \subset A$ is an associative conformal subalgebra of $A$ and there is also a dendriform conformal algebra structure on $T(M)$ defined by
\[ T(u) \triangleright_\lambda T(v) = T(u \triangleright_\lambda v), \ T(u) \langle_\lambda T(v) = T(u \langle_\lambda v), \ u, v \in M. \tag{59} \]
Furthermore, the associated associative conformal algebra on $T(M)$ is a subalgebra of $A$ and $T : M \to A$ is a homomorphism of dendriform conformal algebras.

Proof. For all $u, v, w \in M$, we have
\[ (u \langle_\lambda v) \langle_\lambda \mu w - u \langle_\lambda (v \langle_\mu w + v \rangle_\mu w) \]
\[ = r_A(T(w)) -_\lambda - \mu - \partial(r_A(T(v)) -_\lambda - \partial u) - r_A(r_A(T(w)) -_\mu - \partial v) -_\lambda - \partial u - r_A(l_A(T(v)) -_\mu \partial w -_\lambda - \partial u \]
\[ = r_A(T(v))_\mu T(w) - r_A(T(v)) -_\mu - \partial v - l_A(T(v))_\mu w -_\lambda - \partial u \]
Similarity, we have
\[ (u \triangleright_\lambda v) \triangleright_\lambda \mu w = (u \triangleright_\lambda v + u \langle_\lambda v) \triangleright_\lambda \mu w, \ \forall u, v, w \in M. \]
Moreover, since \((M, l_A, r_A)\) is a bimodule of \(A\), we have
\[
(u \succ_\lambda v) \prec_{\lambda + \mu} w = u \succ_\lambda (v \prec_\mu w), \quad \forall \ u, v, w \in M.
\]
Hence \((M, \prec_\lambda, \succ_\lambda)\) is a dendriform conformal algebra. Moreover, the other conclusions follow straightforwardly. Therefore the conclusion holds. \(\square\)

**Corollary 6.14.** Let \((A, *_\lambda)\) be an associative conformal algebra. There is a compatible dendriform conformal algebra structure on \(A\) if and only if there exists a bijective \(O\)-operator \(T : M \rightarrow A\) associated with some bimodule \((M, l_A, r_A)\) of \(A\).

**Proof.** Suppose that there is a compatible dendriform conformal algebra structure \((A, \succ_\lambda, \prec_\lambda)\) on \(A\). Then by Proposition 6.12, the identity map \(I : A \rightarrow A\) is a bijective \(O\)-operator of \(A\) associated with \((A, L_\succ, R_\prec)\).

Conversely, suppose that there exists a bijective \(O\)-operator \(T : M \rightarrow A\) of \((A, *_\lambda)\) associated with a bimodule \((M, l_A, r_A)\). Then by Proposition 6.13 with a straightforward checking, we have
\[
a \succ_\lambda b = T(l_A(a)T^{-1}(b)), \quad a \prec_\lambda b = T(r_A(b) - \lambda - \partial T^{-1}(a)), \quad \forall \ a, b \in A,
\]
defines a compatible dendriform conformal algebra structure on \(A\). \(\square\)

Finally, there is a construction of (antisymmetric) solutions of associative conformal Yang-Baxter equation from dendriform conformal algebras.

**Theorem 6.15.** Let \((A, \succ_\lambda, \prec_\lambda)\) be a finite dendriform conformal algebra which is free as a \(\mathbb{C}[\partial]\)-module. Then
\[
r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)
\]
is a solution of associative conformal Yang-Baxter equation in the associative conformal algebra \(A \ltimes_{L_A^*, R_A^*} A^c\), where \(\{e_1, \cdots, e_n\}\) is a \(\mathbb{C}[\partial]\)-basis of \(A\) and \(\{e_1^*, \cdots, e_n^*\}\) is the dual \(\mathbb{C}[\partial]\)-basis of \(A^c\).

**Proof.** By Proposition 6.12 \(T = I : A \rightarrow A\) is an \(O\)-operator associated with \((A, L_\succ, R_\prec)\). Then by Theorem 6.7, the conclusion holds. \(\square\)

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