A NEW CLASS OF THE ENTIRE FUNCTION OF ORDER ONE: A CASE STUDY

XIAO-JUN YANG\textsuperscript{1,2}

Abstract. In this article, a new class of the entire function of order one, expressed by the series and product representations with the real positive coefficients and complex zeros, is investigated for the first time. The entire function on the critical line deduces an even entire function of order one. It is proved that the real part of the complex zeros is equal to the critical line. An equivalent representation theorem is obtained to set up the sufficient conditions for the critical line for the entire function. As a typical example, the critical line for the special hyperbolic cosine function obtained by the present theorem agrees with the result of Euler. We also discover the new products of the hyperbolic cosine and sinc functions.

Contents

1. Introduction
2. New results
   2.1. The functional equation for $\mathcal{H}(s)$
   2.2. The different products for $\mathcal{H}(s)$
   2.3. The symmetric lines for $\mathcal{H}(s)$
3. The proof of Theorem 1
   3.1. A class of $\mathcal{H}(s)$
   3.2. A class of $\psi(\vartheta)$
   3.3. A class of $\overline{\psi}(\vartheta)$
   3.4. Two products and convergence of $\psi(\vartheta)$
   3.5. A detailed proof of Theorem 1
4. An equivalent representation of Theorem 1
5. A typical application associated with the work of Euler
6. Conclusion and further remarks
References

2020 Mathematics Subject Classification. Primary: 30D10; Secondary: 30D15, 30D99.

Key words and phrases. entire function, series representation, product representation, complex zeros, critical line.
1. Introduction

The distribution of the zeros of the entire functions is one of important topics in the study of the analytic functions of complex variable [1, 2]. There exists a class of the entire function with their series and product representations. This idea for this class of the entire functions was proposed by Laguerre [3] and Pólya [4]. Let \( \mathbb{C}, \mathbb{R}, \) and \( \mathbb{A} \) denote the sets of the complex, real and integer numbers and let \( \Xi = \mathbb{C} \setminus \{0\}, \mathbb{H} = \mathbb{R} \setminus \{0\} \) and \( \Pi = \mathbb{A} \cup \{0\} \). Suppose that \( m \in \Pi \) and \( k \in \mathbb{A} \). This is so-called Laguerre-Pólya (L − P) class of the real entire function \( G(t) \), which have the form (see, for instance, [5, 6, 7, 8] and the references therein)

\[
G(t) = \sum_{m=0}^{\infty} \frac{\gamma_m}{m!} t^m = \alpha t^n \exp \left( -bt^2 + at \right) \prod_{k=1}^{\infty} \left( 1 + \frac{t}{t_k} \right) \exp \left( -\frac{t}{t_k} \right),
\]

where \( \alpha, a, t_k \in \mathbb{R}, t_k \neq 0, b \geq 0, n \in \Pi, 0 \leq \omega \leq \infty, \) and \( \sum_{k=1}^{\infty} t_k^{-2} \) is convergent.

A family of the entire functions in the L − P class has an important relation with the well-known conjectures [8, 9], for instance, the Riemann hypothesis [10], the conjecture of Pólya [11, 12], the de Bruijn-Newman constant [13, 14], and the conjecture of Karlin [15]. The zeros of the successive derivatives of the even L − P functions were considered by Shen in [16]. A generalized version of the L − P class of the entire functions was proposed by Suárez in [17]. The Laguerre polynomials were developed by Dimitrov and Cheikh as the Jensen polynomials in sense of the L − P entire functions [18]. The partial theta function was investigated by Bohdanov and Vishnyakova in the L − P class [19].

In the working paper, we now consider a new class of the entire function as follows:

**Definition 1.** Let \( \ell \in \mathbb{H} \) and \( s \in \mathbb{C} \). A entire function \( \mathcal{H}(s) \) of order \( \nu = 1 \), given by the series

\[
\mathcal{H}(s) = \sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m},
\]

is said to be in the class \( \mathcal{J} \), written \( \mathcal{H} \in \mathcal{J} \), if \( \mathcal{H}(s) \) can be expressed in the product

\[
\mathcal{H}(s) = \mathcal{H}(0) \prod_{\rho_k} \left( 1 - s/\rho_k \right),
\]

where \( \rho_k \in \Xi \) take over the zeros of \( \mathcal{H}(s) \), \( \Omega_m > 0 \) are the real coefficients for \( \mathcal{H}(s) \),

\[
\mathcal{h} = \sum_{k=1}^{\infty} |\rho_k|^{-2}
\]

is convergent and

\[
\mathcal{H}(0) \neq 0.
\]

Here, we allow that \( \alpha_m \) can be replaced by the Taylor series of the entire function \( \mathcal{H}(s) \). It is clear that \( \mathcal{H} \in \mathcal{J} \) has the infinity of the complex zeros, \( \mathcal{H}(0) = \sum_{m=0}^{\infty} \Omega_m \ell^{2m} > 0 \) and
\( H(\ell) = \Omega_0 > 0 \). In fact, two special cases of the class \( \mathcal{J} \) have been considered in [20, 21]. The behaviors for the complex zeros of the class of the entire function considered in the paper have not been investigated. By the motivation of the idea, the main aim of the present paper is to prove the following:

**Theorem 1.** If \( H \in \mathcal{J} \), then all zeros of \( H(s) \) lie on the critical line \( \text{Re}(s) = \ell \).

Let \( \text{Re}(s) \) and \( \text{Im}(s) \) be the real and imaginary parts of a complex variable \( s \in \mathbb{C} \). To prove Theorem 1, we establish a class of the function \( H(s) \), given as

\[
H(s) = \sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \left( \frac{s - \ell}{\rho_k - \ell} \right)^2 \right].
\]

With the aid of (6), we may derive that

\[
\psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right],
\]

where \( \vartheta \in \mathbb{C} \). Let \( \overline{\rho_k} \) be the complex conjugate of the complex zeros \( \rho_k \) and let \( \overline{\vartheta} \in \mathbb{C} \) be the complex conjugate of a complex variable \( \vartheta \in \mathbb{C} \). Adopting (7) to find the complex conjugate \( \overline{\psi(\vartheta)} \) of the function \( \psi(\vartheta) \), we present

\[
\psi(\overline{\vartheta}) = \sum_{m=0}^{\infty} \Omega_m \overline{\vartheta}^{2m} = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\overline{\vartheta}^2}{(\rho_k - \ell)^2} \right] = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\overline{\vartheta}^2}{(\rho_k - \ell)^2} \right] = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\overline{\vartheta}^2}{(\rho_k - \ell)^2} \right].
\]

to obtain the convergent series

\[
\sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2} = \sum_{k=1}^{\infty} \frac{1}{(\overline{\rho_k} - \ell)^2}.
\]

Making use of (9), we obtain

\[
(\rho_k - \ell)^2 = (\overline{\rho_k} - \ell)^2
\]

to reduce to Theorem 1.

The structure of the paper is designed as follows. In Section 2 we address the functional equation, products and symmetric lines for the entire function \( H(s) \). In Section 2 we give the detailed proof of Theorem 1. In Section 3 we propose an equivalent representation of Theorem 1. In Section 4 we investigate the critical line of the special hyperbolic cosine function \( F(s) = \cosh(s - 6) \). Finally, we draw the conclusion in Section 5.

2. **New results**

2.1. **The functional equation for** \( H(s) \). To be begin with we present the functional equation of \( H(s) \).

**Lemma 1.** Assume that \( s \in \mathbb{C} \) and \( H \in \mathcal{J} \). Then we have

\[
H(s) = H(2\ell - s).
\]
Proof. By using \((2)\), we consider
\[
\mathcal{H}(2\ell - s) = \sum_{m=0}^{\infty} \Omega_m [(2\ell - s) - \ell]^{2m} = \sum_{m=0}^{\infty} \Omega_m (\ell - s)^{2m} = \sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m},
\]
which reduce to \((11)\).

We thus complete the proof of Lemma 1. \(\square\)

2.2. The different products for \(\mathcal{H}(s)\).

Lemma 2. Let \(s \in \mathbb{C}\) and \(\mathcal{H} \in \mathcal{J}\). Then there exist any constant \(\beta \in \mathbb{C}\) with \(\beta \neq \rho_k\) such that
\[
\mathcal{H}(s) = \mathcal{H}(\beta) \prod_{\rho_k} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right).
\]

Proof. Since \(\mathcal{H} \in \mathcal{J}\), we have
\[
\mathcal{H}(s) = \mathcal{H}(0) \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) = \mathcal{H}(0) \prod_{\rho_k} \frac{\rho_k - s}{\rho_k}
\]
\[
= \mathcal{H}(0) \prod_{\rho_k} \left(\frac{\rho_k - \beta}{\rho_k - \beta} \cdot \frac{\rho_k - s}{\rho_k - \beta}\right)
\]
\[
= \mathcal{H}(0) \prod_{\rho_k} \frac{\rho_k - \beta}{\rho_k} \cdot \prod_{\rho_k} \frac{\rho_k - s}{\rho_k - \beta}
\]
\[
= \mathcal{H}(0) \prod_{\rho_k} \frac{\rho_k - \beta}{\rho_k} \cdot \prod_{\rho_k} \frac{\rho_k - s - (s - \beta)}{\rho_k - \beta}
\]
\[
= \mathcal{H}(0) \prod_{\rho_k} \left(1 - \frac{\beta}{\rho_k}\right) \cdot \prod_{\rho_k} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right).
\]

Taking \(s = \beta\) in \((2)\), we may get
\[
\mathcal{H}(\beta) = \mathcal{H}(0) \prod_{\rho_k} \left(1 - \frac{\beta}{\rho_k}\right).
\]

By substituting \((15)\) into the last term of \((14)\), we carry out
\[
\mathcal{H}(s) = \mathcal{H}(\beta) \prod_{\rho_k} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right).
\]

Thus, we finish the proof of Lemma 2. \(\square\)
Corollary 1. Assume $s \in \mathcal{C}$, $\mathcal{H} \in \mathcal{J}$ and $\ell \in \mathcal{H}$. Then we have

\begin{equation}
\mathcal{H} (s) = \mathcal{H} (\ell) \prod_{\rho_k} \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right).\end{equation}

Proof. Following the definition of $\mathcal{H} \in \mathcal{J}$, we have $\mathcal{H} (\ell) \neq 0$ such that (17) is valid if we make $\beta = \ell$ in Lemma 2. Thus, we finish the proof. □

Corollary 2. Suppose that $s \in \mathcal{C}$, $\mathcal{H} \in \mathcal{J}$ and $\ell \in \mathcal{H}$. Then we have

\begin{equation}
\mathcal{H} (s) = \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \left( \frac{s - \ell}{\rho_k - \ell} \right)^2 \right].\end{equation}

Proof. By using Lemma 1, we have

\begin{equation}
\mathcal{H} (\rho_k) = \mathcal{H} (2\ell - \rho_k),
\end{equation}

which implies that

\begin{equation}
\begin{aligned}
\mathcal{H} (s) &= \mathcal{H} (\ell) \prod_{\rho_k} \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right) \\
&= \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right) \left[ 1 - \frac{s - \ell}{(2\ell - \rho_k) - \ell} \right] \\
&= \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left\{ \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right) \left[ 1 - \frac{s - \ell}{\rho_k - \ell} \right] \right\} \\
&= \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left\{ \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right) \left( 1 + \frac{s - \ell}{\rho_k - \ell} \right) \right\} \\
&= \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \left( \frac{s - \ell}{\rho_k - \ell} \right)^2 \right].
\end{aligned}
\end{equation}

□

Corollary 3. If $s \in \mathcal{C}$, $\mathcal{H} \in \mathcal{J}$ and $\ell \in \mathcal{H}$, then there exists any $\beta \in \mathcal{C}$ with $\beta \neq \rho_k$ such that

\begin{equation}
\mathcal{H} (s) = \mathcal{H} (\beta) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{(s - \ell)^2 - (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2} \right].
\end{equation}
Proof. By Lemmas 1 and 2, we obtain

\[ H(s) = H(\beta) \prod_{\rho_k} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right) = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right) \left(1 - \frac{s - \beta}{2\ell - \rho_k - \beta}\right) \]

\[ H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{(s - \ell) - (\beta - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \left[1 - \frac{(s - \ell) + (\ell - \beta)}{(\ell - \rho_k) + (\ell - \beta)}\right] \]

\[ = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) - (s - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \left[\frac{(\ell - \rho_k) - (s - \ell)}{(\ell - \rho_k) + (\ell - \beta)}\right], \]

where

\[ \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{(s - \ell) - (\beta - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] = \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) - (s - \ell) - (\beta - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \]

\[ = \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) - (s - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \]

and

\[ \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{(s - \ell) + (\ell - \beta)}{(\ell - \rho_k) + (\ell - \beta)}\right] = \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\ell - \rho_k) + (\ell - \beta) - (s - \ell) - (\ell - \beta)}{(\ell - \rho_k) + (\ell - \beta)}\right] \]

\[ = \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) + (s - \ell)}{(\rho_k - \ell) + (\beta - \ell)}\right]. \]

To simplify (14), we obtain

\[ H(s) = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) - (s - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) + (s - \ell)}{(\rho_k - \ell) + (\beta - \ell)}\right] \]

\[ = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell) - (s - \ell)}{(\rho_k - \ell) - (\beta - \ell)}\right] \frac{(\rho_k - \ell) + (s - \ell)}{(\rho_k - \ell) + (\beta - \ell)} \]

\[ = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell)^2 - (s - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2}\right] \]

\[ = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[\frac{(\rho_k - \ell)^2 - (\beta - \ell)^2 - (s - \ell)^2 + (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2}\right] \]

\[ = H(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{(s - \ell)^2 - (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2}\right]. \]

We then complete the proof. \qed
2.3. The symmetric lines for $\mathcal{H}(s)$. The symmetric function $\Lambda(s)$ is defined as

$$\Lambda(s, \beta) := \frac{\mathcal{H}(s)}{\mathcal{H}(\beta)} = \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{(s - \ell)^2 - (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2} \right].$$

**Corollary 4.** There exist the first symmetric line $s = \ell$ and the second symmetric line $\beta = \ell$ for the symmetric function $\Lambda(s)$ for $s \in \mathbb{C}$.

**Proof.** By using the definition of the symmetric function (26), we reduce to the required result. $\square$

**Corollary 5.** There exist the first symmetric line $s = \ell$ and the second symmetric line $\beta = \ell$ for the entire function $\mathcal{H}(s)$ for $s \in \mathbb{C}$.

**Proof.** Making use of (21) in Corollary 3, we obtain the result. $\square$

**Remark.** By (26), we have

$$\Lambda(s, \ell) = \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \left( \frac{s - \ell}{\rho_k - \ell} \right)^2 \right],$$

and

$$\Lambda(s, 2\ell) = \Lambda(s, 0) = \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{(s - \ell)^2 - \ell^2}{(\rho_k - \ell)^2 - \ell^2} \right],$$

such that

$$\mathcal{H}(s) = \mathcal{H}(0) \Lambda(s, 0),$$

(30) $$\mathcal{H}(s) = \mathcal{H}(\ell) \Lambda(s, \ell),$$

and

$$\mathcal{H}(s) = \mathcal{H}(2\ell) \Lambda(s, 2\ell).$$

Removing the effect of the second symmetric line $\beta = \ell$ and taking $\beta = \ell$ into (13) and (21), we obtain (17) and (18), which can be applied to find the critical line of the function $\mathcal{H}(s)$.

3. The proof of Theorem 1

3.1. A class of $\mathcal{H}(s)$. Since $\mathcal{H} \in \mathcal{J}$, $\mathcal{H}(s)$ can be expressed as

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(0) \prod_{\rho_k} \left( 1 - \frac{s}{\rho_k} \right).$$

If we combine (32) and Lemma 2, then (32) is equivalent to

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\beta) \prod_{\rho_k} \left( 1 - \frac{s - \beta}{\rho_k - \beta} \right).$$
Taking $\beta = \ell$ in (33) implies that $H(s)$ can be rewritten as

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = H(\ell) \prod_{\rho_k} \left(1 - \frac{s - \ell}{\rho_k - \ell}\right).$$

(34)

Here, (34) is obtained by (33) when we remove the second symmetric line $\beta = \ell$ in (33).

By Corollary 2 and (34), we have

$$H(\ell) \prod_{\rho_k} \left(1 - \frac{s - \ell}{\rho_k - \ell}\right) = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \left(\frac{s - \ell}{\rho_k - \ell}\right)^2\right]$$

(35)

such that

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \left(\frac{s - \ell}{\rho_k - \ell}\right)^2\right].$$

(36)

Remark. Clearly, the identity (36) is also derived from Corollary 3 when we remove the second symmetric line $\beta = \ell$ in (21) and is in agreement with (6).

3.2. A class of $\psi(\vartheta)$.

Taking $\vartheta = s - \ell$ in (36) for $\vartheta \in \mathbb{C}$, we get

$$\psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m}$$

(37)

and

$$\psi(\vartheta) = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{\vartheta^2}{(\rho_k - \ell)^2}\right].$$

(38)

Combining (37) and (38), we have the class of $\psi(\vartheta)$, that is,

$$\sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = H(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{\vartheta^2}{(\rho_k - \ell)^2}\right].$$

(39)

Obviously, $\psi(\vartheta)$ and $H(s)$ are entire functions of order $\nu = 1$.

3.3. A class of $\overline{\psi(\vartheta)}$.

It follows from (37) that the complex conjugate $\overline{\psi(\vartheta)}$ of the function $\psi(\vartheta)$ reads

$$\overline{\psi(\vartheta)} = \sum_{m=0}^{\infty} \Omega_m \overline{\vartheta}^{2m} = \sum_{m=0}^{\infty} \Omega_m \overline{\vartheta}^{2m}$$

(40)

due to the fact

$$\Omega_m > 0.$$  

(41)

From (40) we show that

$$\overline{\psi(\vartheta)} = \psi(\overline{\vartheta}).$$

(42)
By using (39) and (42), we have

\[ \psi(\vartheta) = \psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] \]

because of \( \mathcal{H}(\ell) = \Omega_0 > 0 \). Finding \( \psi(\vartheta) \) in (38) implies that

\[ \psi(\vartheta) = \left\{ \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] \right\} \]

(44)

\[ \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right]. \]

Combining (43) and (44), we get

\[ \psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] = \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] \]

where \( \vartheta \in \mathcal{C} \).

3.4. **Two products and convergence of** \( \psi(\vartheta) \). By replacing \( \vartheta \in \mathcal{C} \) by \( \vartheta \in \mathcal{C} \) in (45), we obtain

\[ \psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] = \mathcal{H}(\ell) \prod_{I_{m(\rho_k)} > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] \]

(46)

Considering the fact that \( \psi(\vartheta) \) is the even entire function of order \( \nu = 1 \), there exists any \( r > 0 \) such that

\[ \sum_{\rho_k} |\rho_k - \ell|^{-(1+r)} \]

(47)

and

\[ \sum_{\rho_k} |\rho_k - \ell|^{-(1+r)} \]

(48)

are convergent.

Taking \( r = 1 \) in (47) and (48) implies that

\[ \sum_{\rho_k} |\rho_k - \ell|^{-2} \]

(49)

and

\[ \sum_{\rho_k} |\rho_k - \ell|^{-2} \]

(50)
are absolutely convergent. By (49) and (50), both

\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2}
\end{equation}

and

\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{(\overline{\rho_k} - \ell)^2}
\end{equation}

are convergent.

It follows from (46), (51) and (52) that

\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2} = \sum_{k=1}^{\infty} \frac{1}{(\overline{\rho_k} - \ell)^2}.
\end{equation}

3.5. A detailed proof of Theorem 1. From (51) we have

\begin{equation}
(\rho_k - \ell)^2 = (\overline{\rho_k} - \ell)^2,
\end{equation}

which leads to

\begin{equation}
[(\rho_k - \ell) + (\overline{\rho_k} - \ell)] [\rho_k - \ell - (\overline{\rho_k} - \ell)] = 0.
\end{equation}

Since

\begin{equation}
(\rho_k - \ell) - (\overline{\rho_k} - \ell) = \rho_k - \overline{\rho_k} = 2i \text{Im} \,(\rho_k),
\end{equation}

we have from (55) that

\begin{equation}
(\rho_k - \ell) + (\overline{\rho_k} - \ell) = 0.
\end{equation}

By (57), we get

\begin{equation}
2 \text{Re} \,(\rho_k) - 2\ell = 0
\end{equation}

or, alternatively,

\begin{equation}
\text{Re} \,(\rho_k) = \ell.
\end{equation}

We hence finish the proof of Theorem 1.

4. An equivalent representation of Theorem 1

Taking

\begin{equation}
\text{Im} \,(\rho_k) = \chi_k > 0
\end{equation}
and using (45) and (59), we arrive at

\[
\psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m} = \mathcal{H}(\ell) \prod_{I m(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] = \mathcal{H}(\ell) \prod_{\chi_k} \left( 1 + \frac{\vartheta^2}{\chi_k^2} \right)
\]

(61)

Clearly, all nontrivial zeros of \( \psi(\vartheta) \) read \( \pm i\chi_k \), where \( i = \sqrt{-1} \).

By the combination of (21), (32), (33), (35) and (59), we have

\[
(62)
\mathcal{H}(s) = \sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(0) \prod_{\rho_k} \left( 1 - \frac{s}{\rho_k} \right) = \mathcal{H}(\beta) \prod_{\rho_k} \left( 1 - \frac{s - \beta}{\rho_k - \beta} \right)
\]

\[
= \mathcal{H}(\ell) \prod_{I m(\rho_k) > 0} \left[ 1 - \frac{(s - \ell)^2 - (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2} \right]
\]

(62)

\[
= \mathcal{H}(\ell) \prod_{\rho_k} \left( 1 - \frac{s - \ell}{\rho_k - \ell} \right)
\]

\[
= \mathcal{H}(\ell) \prod_{I m(\rho_k) > 0} \left[ 1 - \left( \frac{s - \ell}{\rho_k - \ell} \right)^2 \right]
\]

\[
= \mathcal{H}(\ell) \prod_{k=1}^{\infty} \left( 1 + \frac{(s - \ell)^2}{\chi_k^2} \right).
\]

Obviously, all nontrivial zeros of \( \mathcal{H}(s) \) are written as

\[\rho_k = \ell \pm i\chi_k.\]

By using (63) and

\[
(64)
\mathcal{H}(\rho_k) = \mathcal{H}(2\ell - \rho_k),
\]

we obtain

\[
\mathcal{H}(s) = \mathcal{H}(0) \prod_{\rho_k} \left( 1 - \frac{s}{\rho_k} \right) = \mathcal{H}(0) \prod_{I m(\rho_k) > 0} \left( 1 - \frac{s}{\rho_k} \right) \left( 1 - \frac{s}{2\ell - \rho_k} \right)
\]

\[
= \mathcal{H}(0) \prod_{I m(\rho_k) > 0} \left\{ \left( 1 - \frac{s}{\rho_k} \right) \left( 1 - \frac{s}{\ell - i\chi_k} \right) \right\}
\]

(65)

\[
= \mathcal{H}(0) \prod_{I m(\rho_k) > 0} \left\{ \left( 1 - \frac{s}{\rho_k} \right) \left( 1 - \frac{s}{\ell - i\chi_k} \right) \right\}
\]

\[
= \mathcal{H}(0) \prod_{I m(\rho_k) > 0} \left\{ \left( 1 - \frac{s}{\rho_k} \right) \left( 1 - \frac{s}{\rho_k} \right) \right\}.
\]
It is obvious that $\rho_k$, $\overline{\rho}_k$, $2\ell - \rho_k$ and $2\ell - \overline{\rho}_k$ are the nontrivial zeros of $\mathcal{H}(s)$.

As a direct result of (62), we have the following:

**Corollary 6.** Suppose $s \in \mathcal{C}$, $\mathcal{H} \in \mathcal{J}$ and $\ell \in \mathcal{H}$. Then there exist the following equivalent representations:

(L1) All zeros of $\mathcal{H}(s)$ lie on the critical line $\text{Re}(\rho_k) = \ell$.

(L2) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(0) \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right).$$

(L3) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\ell) \prod_{\rho_k} \left(1 - \frac{s - \ell}{\rho_k - \ell}\right).$$

(L4) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\ell) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \left(\frac{s - \ell}{\rho_k - \ell}\right)^2\right].$$

(L5) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\ell) \prod_{k=1}^{\infty} \left(1 + \frac{(s - \ell)^2}{\chi_k^2}\right).$$

(L6) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\beta) \prod_{\rho_k} \left(1 - \frac{s - \beta}{\rho_k - \beta}\right).$$

(L7) There exists

$$\sum_{m=0}^{\infty} \Omega_m (s - \ell)^{2m} = \mathcal{H}(\beta) \prod_{\text{Im}(\rho_k) > 0} \left[1 - \frac{(s - \ell)^2 - (\beta - \ell)^2}{(\rho_k - \ell)^2 - (\beta - \ell)^2}\right].$$

**Remark.** As a matter of fact, the equivalent relationship between (L1) and (L2) was adopted in [20] if $\mathcal{H}(s)$ is considered as the Riemann xi function. The result considered in [21] is (L4) at $\ell = 0$. In Corollary 6, we adopt (L7) to reduce to (L4) because this process removes the influence of the second symmetric line $\beta = \ell$ for the entire function $\mathcal{H}(s)$. There always exists an entire function $\psi(\vartheta)$ of order $\nu = 1$, i.e.,

$$\psi(\vartheta) = \sum_{m=0}^{\infty} \Omega_m \vartheta^{2m}.$$
such that
\begin{equation}
\psi (\vartheta) = \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] = \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right].
\end{equation}

Since \( \psi (\vartheta) \) is an even entire function of order \( \nu = 1 \), this implies that
\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2} = \sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2}
\end{equation}
or, alternatively,
\begin{equation}
\Re (\rho_k) = \ell.
\end{equation}

If we allow to take the value \( \ell = 0 \) in (73), (74) and (75), this is the key work presented in [21]. Thus, Corollary 6 is an equivalent representation theorem for the critical line for the entire function \( \mathcal{H} (s) \) considered in the class \( \mathcal{J} \).

5. A TYPICAL APPLICATION ASSOCIATED WITH THE WORK OF EULER

Suppose the special hyperbolic cosine function is represented in the form
\begin{equation}
F (s) = \cosh (s - 6),
\end{equation}
where \( s \in \mathbb{C} \).

From (76) we obtain the series representation as follows:
\begin{equation}
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!}.
\end{equation}

It follows from (77) that
\begin{equation}
F (s) = F (12 - s).
\end{equation}

Let us recall that
\begin{equation}
\cos (\vartheta) = \sum_{m=0}^{\infty} \frac{(-1)^m \vartheta^{2m}}{(2m)!}
\end{equation}
and
\begin{equation}
\cosh (\vartheta) = \sum_{m=0}^{\infty} \frac{\vartheta^{2m}}{(2m)!}.
\end{equation}

Since \( \cos (\vartheta) \) is an even function of order \( \nu = 1 \), there exist
\begin{equation}
\cosh (\vartheta) = \prod_{\lambda_k} \left( 1 - \frac{s}{\lambda_k} \right)
\end{equation}
and
\begin{equation}
\cos (\vartheta) = \prod_{\lambda_k} \left( 1 + \frac{s}{\lambda_k} \right),
\end{equation}

such that
\begin{equation}
\psi (\vartheta) = \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right] = \mathcal{H} (\ell) \prod_{\text{Im}(\rho_k) > 0} \left[ 1 - \frac{\vartheta^2}{(\rho_k - \ell)^2} \right].
\end{equation}

Since \( \psi (\vartheta) \) is an even entire function of order \( \nu = 1 \), this implies that
\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2} = \sum_{k=1}^{\infty} \frac{1}{(\rho_k - \ell)^2}
\end{equation}
or, alternatively,
\begin{equation}
\Re (\rho_k) = \ell.
\end{equation}

If we allow to take the value \( \ell = 0 \) in (73), (74) and (75), this is the key work presented in [21]. Thus, Corollary 6 is an equivalent representation theorem for the critical line for the entire function \( \mathcal{H} (s) \) considered in the class \( \mathcal{J} \).

5. A TYPICAL APPLICATION ASSOCIATED WITH THE WORK OF EULER

Suppose the special hyperbolic cosine function is represented in the form
\begin{equation}
F (s) = \cosh (s - 6),
\end{equation}
where \( s \in \mathbb{C} \).

From (76) we obtain the series representation as follows:
\begin{equation}
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!}.
\end{equation}

It follows from (77) that
\begin{equation}
F (s) = F (12 - s).
\end{equation}

Let us recall that
\begin{equation}
\cos (\vartheta) = \sum_{m=0}^{\infty} \frac{(-1)^m \vartheta^{2m}}{(2m)!}
\end{equation}
and
\begin{equation}
\cosh (\vartheta) = \sum_{m=0}^{\infty} \frac{\vartheta^{2m}}{(2m)!}.
\end{equation}

Since \( \cos (\vartheta) \) is an even function of order \( \nu = 1 \), there exist
\begin{equation}
\cosh (\vartheta) = \prod_{\lambda_k} \left( 1 - \frac{s}{\lambda_k} \right)
\end{equation}
and
\begin{equation}
\cos (\vartheta) = \prod_{\lambda_k} \left( 1 + \frac{s}{\lambda_k} \right),
\end{equation}
where $\lambda_k \in \mathbb{C}$ run over the zeros of $\cosh(\vartheta)$.

By combining (77), (80) and (82), we suggest

$$F(s) = \cosh(s - 6) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = \prod_{\lambda_k} \left( 1 - \frac{s - 6}{\lambda_k} \right).$$

From (83) we reduce to

$$\phi_k = \lambda_k + 6.$$ 

By using (84), we write (83) as

$$F(s) = \cosh(s - 6) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = \prod_{\phi_k} \left( 1 - \frac{s - 6}{\phi_k - 6} \right),$$

which leads to

$$F(6) = 1.$$ 

Combining (85) and (86), $F(s) = \cosh(s - 6)$ can be represented as

$$\sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = F(6) \prod_{\phi_k} \left( 1 - \frac{s - 6}{\phi_k - 6} \right).$$

Let us recall that $\cos(\vartheta)$ is an even function of order $\nu = 1$. Then, by (81), there exists any $\hbar > 0$ such that $\sum_{\lambda_k} |\lambda_k|^{-(1+h)}$ is convergent. This implies that we have from (84) that

$$\sum_{\lambda_k} |\lambda_k|^{-(1+h)} = \sum_{\phi_k} |\phi_k - 6|^{-(1+h)}$$

is convergent.

This implies that $F \in \mathcal{F}$ and that by (A3) in Corollary 6 we have

$$\text{Re}(\phi_k) = 6.$$ 

From (84) and (89) we deduce that

$$\text{Re}(\lambda_k) = 0.$$
Let \( \text{Im} (\lambda_k) = \mu_k > 0 \). Then we have from (82) and (90) that

\[
\cos (\vartheta) = \prod_{\lambda_k} \left( 1 + \frac{\vartheta}{i \lambda_k} \right) = \prod_{\text{Im}(\lambda_k) > 0} \left( 1 + \frac{\vartheta}{i \lambda_k} \right) \left( 1 - \frac{\vartheta}{i \lambda_k} \right) \\
= \prod_{k=1}^{\infty} \left[ \left( 1 + \frac{\vartheta}{i \mu_k} \right) \left( 1 - \frac{\vartheta}{i \mu_k} \right) \right] \\
= \prod_{k=1}^{\infty} \left( 1 - \frac{\vartheta^2}{\mu_k^2} \right).
\]

(91)

Adopting (91), we obtain \( \mu_k^2 > 0 \), which agrees with the result of Euler, i.e.,

\[ (2k - 1)^2 \pi^2/4 > 0 \]

since there exists (see [22]; also see [23], p.114)

\[
\cos (\vartheta) = \sum_{m=0}^{\infty} \frac{(-1)^m \vartheta^{2m}}{(2m)!} = \prod_{k=1}^{\infty} \left[ 1 - \frac{4\vartheta^2}{(2k - 1)^2 \pi^2} \right].
\]

(93)

From (84) and (87) we conclude that

\[ \phi_k = 6 \pm i \mu_k. \]

(94)

Considering \( \text{Im} (\lambda_k) = \mu_k > 0 \) and using (94), the identity (87) yields that

\[
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} \\
= F (6) \prod_{\phi_k} \left( 1 - \frac{s - 6}{\phi_k - 6} \right) \\
= F (6) \prod_{\text{Im}(\phi_k) > 0} \left( 1 - \frac{s - 6}{\phi_k - 6} \right) \left( 1 - \frac{s - 6}{6 - \phi_k} \right) \\
= F (6) \prod_{\text{Im}(\phi_k) > 0} \left( 1 - \frac{s - 6}{\phi_k - 6} \right) \left( 1 + \frac{s - 6}{\phi_k - 6} \right) \\
= F (6) \prod_{\text{Im}(\phi_k) > 0} \left[ 1 - \left( \frac{s - 6}{\phi_k - 6} \right)^2 \right] \\
= F (6) \prod_{k=1}^{\infty} \left[ 1 - \left( \frac{s - 6}{i \mu_k} \right)^2 \right].
\]

(95)
To simplify (95), we obtain

\begin{equation}
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = F (6) \prod_{k=1}^{\infty} \left[ 1 + \frac{(s - 6)^2}{\mu_k^2} \right].
\end{equation}

Comparing between (93) and (96), we present

\begin{equation}
\mu_k = \frac{(2k - 1) \pi}{2}.
\end{equation}

Thus,

\begin{equation}
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = F (6) \prod_{k=1}^{\infty} \left[ 1 + \frac{4(s - 6)^2}{(2k - 1)^2 \pi^2} \right]
\end{equation}

and

\begin{equation}
F (s) = \sum_{m=0}^{\infty} \frac{(s - 6)^{2m}}{(2m)!} = F (6) \prod_{k=1}^{\infty} \left[ 1 + \frac{4(s - 6)^2}{(2k - 1)^2 \pi^2} \right]
\end{equation}

With (96) and (97), we may get

\begin{equation}
\sum_{k=1}^{\infty} \mu_k^{-2} = \sum_{k=1}^{\infty} \left[ \frac{(2k - 1) \pi}{2} \right]^{-2} = 2\pi^{-2} \sum_{k=1}^{\infty} \left( k - \frac{1}{2} \right)^{-2}.
\end{equation}

Define the Hurwitz zeta function \( \zeta (\eta, s) \) by (see [24], p.607)

\begin{equation}
\zeta (\eta, s) = \sum_{m=0}^{\infty} (m + \eta)^{-s}.
\end{equation}

In view of (100) and (101), we have

\begin{equation}
\sum_{k=1}^{\infty} \left( k - \frac{1}{2} \right)^{-2} = \zeta \left( -\frac{3}{2}, 2 \right) - \frac{4}{9}
\end{equation}

such that

\begin{equation}
\sum_{k=1}^{\infty} \mu_k^{-2} = 2\pi^{-2} \left[ \zeta \left( -\frac{3}{2}, 2 \right) - \frac{4}{9} \right].
\end{equation}
By using (94) and (L7) in Corollary 6, we may find that

\[ F(s) = \sum_{m=0}^{\infty} \frac{(s-6)^{2m}}{(2m)!} \]

\[ = \Phi(\beta) \prod_{m(\phi_k) > 0} \left[ 1 - \frac{(s-6)^2 - (\beta-6)^2}{(\phi_k - 6)^2 - (\beta-6)^2} \right] \]

\[ = \Phi(\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{(s-6)^2 - (\beta-6)^2}{\mu_k^2 + (\beta-6)^2} \right] \]

\[ = \Phi(\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{(s-6)^2 - (\beta-6)^2}{(2k-1)^2 \pi^2 + (\beta-6)^2} \right]. \tag{104} \]

It follows from (104) that \( F(s) \) has the first symmetric line \( s = 6 \) and the second symmetric line \( \beta = 6 \).

As a direct result of Corollary 3, we obtain the followings:

**Corollary 7.** If \( s \in \mathbb{C} \) and \( \ell \in \mathbb{H} \), then there exists any \( \beta \in \mathbb{C} \) with \( \beta \neq \ell + i(2k-1)\pi/2 \) such that

\[ \cosh (s - \ell) = \cosh (\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{(s-\ell)^2 - (\beta-\ell)^2}{(2k-1)^2 \pi^2 + (\beta-\ell)^2} \right]. \tag{105} \]

**Corollary 8.** If \( s \in \mathbb{C} \) and \( \ell \in \mathbb{H} \), then there exists any \( \beta \in \mathbb{C} \) with \( \beta \neq \ell + ik\pi \) such that

\[ \sinh (s - \ell) = \sinh (\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{(s-\ell)^2 - (\beta-\ell)^2}{k^2 \pi^2 + (\beta-\ell)^2} \right], \]

where

\[ \sinh (s) = \sum_{m=0}^{\infty} \frac{s^{2n}}{(2n+1)!}. \]

**Corollary 9.** If \( s \in \mathbb{C} \) and \( \ell \in \mathbb{H} \), then there exists any \( \beta \in \mathbb{C} \) with \( \beta \neq \ell + (2k-1)\pi/2 \) such that

\[ \cos (s - \ell) = \cos (\beta) \prod_{k=1}^{\infty} \left[ 1 - \frac{(s-\ell)^2 - (\beta-\ell)^2}{(2k-1)^2 \pi^2 - (\beta-\ell)^2} \right]. \tag{107} \]

**Corollary 10.** If \( s \in \mathbb{C} \) and \( \ell \in \mathbb{H} \), then there exists any \( \beta \in \mathbb{C} \) with \( \beta \neq \ell + k\pi \) such that

\[ \text{sinc} (s - \ell) = \text{sinc} (\beta) \prod_{k=1}^{\infty} \left[ 1 - \frac{(s-\ell)^2 - (\beta-\ell)^2}{k^2 \pi^2 - (\beta-\ell)^2} \right], \tag{108} \]

where \( \text{sinc} (s) \) is the sinc function, denoted by \([25]\)

\[ \text{sinc} (s) = \sum_{m=0}^{\infty} \frac{(-1)^n s^{2n}}{(2n+1)!}. \]
Remark. The identity (104) is a special case of (105) at the point $\ell = 6$. Taking $\ell = 0$ into (105), (106), (107) and (108), we may have the followings:

\begin{align}
(109) \quad \cosh (s) &= \cosh (\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{s^2 - \beta^2}{(2k-1)^2\pi^2 + \beta^2} \right], \\
(110) \quad \sinh (s) &= \sinh (\beta) \prod_{k=1}^{\infty} \left[ 1 + \frac{s^2 - \beta^2}{k^2\pi^2 + \beta^2} \right], \\
(111) \quad \cos (s) &= \cos (\beta) \prod_{k=1}^{\infty} \left[ 1 - \frac{s^2 - \beta^2}{(2k-1)^2\pi^2 - \beta^2} \right] \\
and \\
(112) \quad \text{sinc} (s) &= \text{sinc} (\beta) \prod_{k=1}^{\infty} \left[ 1 - \frac{s^2 - \beta^2}{k^2\pi^2 - \beta^2} \right].
\end{align}

Putting $\beta = 0$ into (109), (110), (111) and (112), we can obtain

\begin{align}
(113) \quad \cosh (s) &= \cosh (0) \prod_{k=1}^{\infty} \left[ 1 + \frac{s^2}{(2k-1)^2\pi^2} \right] = \prod_{k=1}^{\infty} \left[ 1 + \frac{s^2}{(2k-1)^2\pi^2} \right], \\
(114) \quad \sinh (s) &= \sinh (0) \prod_{k=1}^{\infty} \left( 1 + \frac{s^2}{k^2\pi^2} \right) = \prod_{k=1}^{\infty} \left( 1 + \frac{s^2}{k^2\pi^2} \right), \\
(115) \quad \cos (s) &= \cos (0) \prod_{k=1}^{\infty} \left[ 1 - \frac{s^2}{(2k-1)^2\pi^2} \right] = \prod_{k=1}^{\infty} \left[ 1 - \frac{s^2}{(2k-1)^2\pi^2} \right] \\
and \\
(116) \quad \text{sinc} (s) &= \text{sinc} (0) \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{k^2\pi^2} \right) = \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{k^2\pi^2} \right).
\end{align}

Obviously, (113) and (115) are the results of Euler (see [22]; also see [24], p.126 and p.118), and (116) is the result reported in [25].

By using (110), (112), (113) and (115), we may carry out

\begin{align}
(117) \quad \sinh (s) &= s \times \sinh (s) = \sinh (\beta) s \prod_{k=1}^{\infty} \left[ 1 + \frac{s^2 - \beta^2}{k^2\pi^2 + \beta^2} \right], \\
(118) \quad \sin (s) &= s \times \text{sinc} (s) = \text{sinc} (\beta) s \prod_{k=1}^{\infty} \left[ 1 - \frac{s^2 - \beta^2}{k^2\pi^2 - \beta^2} \right], \\
(119) \quad \sinh (s) &= s \prod_{k=1}^{\infty} \left( 1 + \frac{s^2}{k\pi^2} \right)
\end{align}
and
\begin{equation}
\sin (s) = s \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{\pi^2 k^2} \right).
\end{equation}

Clearly, (119) and (120) are the results of Euler (see [22]; also see [24], p.126 and p.118).

As a similar way of (100), we have
\begin{equation}
\sum_{k=1}^{\infty} (k\pi)^{-2} = \pi^{-2} \zeta(2),
\end{equation}
where \( \zeta(s) \) is the Riemann zeta function, denoted as (see [23], p.151)
\[ \zeta(s) = \sum_{k=1}^{\infty} k^{-s}. \]

It is obvious that (121) holds because \( \sinch(s) \) is an even function of order one.

As a direct result, from Corollaries 7 and 8 we obtain the following:

**Corollary 11.** Assume the notations of Corollaries 7 and 8. Then \( \cosh(s-\ell) \) and \( \sinch(s-\ell) \) belong to the class \( \mathcal{J} \).

6. **Conclusion and further remarks**

In the present work we proposed a new class of the entire function of order one with the real positive coefficients and infinity of complex zeros. We suggested a sufficient condition for the same critical line for its complex zeros. We presented the equivalent representation theorem for it. We also gave a typical example for the special hyperbolic cosine function, whose result is in accord with the result of Euler. The obtained result is proposed as a new mathematical tool to obtain the critical line of the class of the entire function of order one.

**References**

[1] B. Y. Levin, Distribution of zeros of entire functions, Vol. 150, American Mathematical Society, 1980.
[2] I. Markushevich, Entire functions, Elsevier, 2014.
[3] E. Laguerre, Sur les fonctions du genre zéro et du genre un, Comptes rendus de l'Acad’emie des Sciences Paris, 95 (1882), 828-831.
[4] G. Pólya, Über Annäherung durch Polynome mit lauter reellen Wurzeln, Rendiconti del Circolo Matematico di Palermo, 36 (1913) (2), 279-295.
[5] I. Wagner, On a new class of Laguerre-Pólya type functions with applications in number theory, Pacific Journal of Mathematics, 320 (2022) (2), 177-192.
[6] Á. Baricz and S. Singh, Zeros of some special entire functions, Proceedings of the American Mathematical Society, 146 (2018) (6), 2207-2216.
[7] G. Csordas and A. Vishnyakova, The generalized Laguerre inequalities and functions in the Laguerre-Pólya class, Central European Journal of Mathematics, 11 (2013) (10), 1643-1650.
[8] G. Csordas and D. K. Dimitrov, Conjectures and theorems in the theory of entire functions, Numerical Algorithms, 25 (2000) (2), 109-122.
[9] Y. O. Kim, Critical points of real entire functions and a conjecture of Pólya, Proceedings of the American Mathematical Society, 124 (1996) (4), 819-830.
[10] G. Csordas, R. S. Varga and I. Vincze, Jensen polynomials with applications to the Riemann $\xi$-function, Journal of Mathematical Analysis and Applications, 153 (1990) (2), 112-135.

[11] G. Pólya, Sur une question concernant les fonctions entieres, Comptes Rendus de l'Académie des Sciences, Paris, 158 (1914), 330-333.

[12] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, Transactions of the American Mathematical Society, 227 (1977), 227-249.

[13] N. G. De Bruijn, The roots of trigonometric integrals, Duke Mathematical Journal, 17 (1950) (4), 197-226.

[14] C. M. Newman, Fourier transforms with only real zeros, Proceedings of the American Mathematical Society, 61 (1976) (3), 245-251.

[15] S. Karlin, Total positivity, Vol. 1, Stanford University Press, 1968.

[16] L. C. Shen, On the zeros of successive derivatives of even Laguerre-Pólya functions, Transactions of the American Mathematical Society, 298 (1986) (3), 643-652.

[17] D. Suárez, A generalization of the Laguerre-Pólya class of entire functions, Journal of approximation theory, 101 (1999) (2), 37-48.

[18] D. K. Dimitrov and Y. B. Cheikh, Laguerre polynomials as Jensen polynomials of Laguerre-Pólya entire functions, Journal of Computational and Applied Mathematics, 233 (2009) (4), 703-707.

[19] A. Bohdanov and A. Vishnyakova, On the conditions for entire functions related to the partial theta function to belong to the Laguerre-Pólya class, Journal of Mathematical Analysis and Applications, 434 (2016) (3), 1740-1752.

[20] X. J. Yang, All nontrivial zeros for the Riemann zeta function are on the critical line $\Re(s) = 1/2$, arXiv: 1811.02418v15.

[21] X. J. Yang, On all real zeros for a class of the even entire function, arXiv: 2107.04005v2.

[22] L. Euler, Introductio in analysin infinitorvm, Apud Marcum-Michaelem Bousquet, 1748.

[23] E. C. Titchmarsh, The theory of functions, Oxford University Press, 1939.

[24] F. W. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, (Eds.), NIST handbook of mathematical functions, Cambridge university press, 2010.

[25] W. B. Gearhart and H. S. Shultz, The Function $\sin x/x$, College Mathematics Journal, 21 (1990) (2), 90-99.

Email address: dyangxiaojun@163.com; xjyang@cumt.edu.cn

1 School of Mathematics, and State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China

2 Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia