On Ramsey \((P_3, C_7)\)-minimal graphs

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Abstract. Let \(G, H\) be graphs. Notation \(F \rightarrow (G, H)\) means that there is any two-coloring, say red and blue, of all edges of \(F\) which contains red subgraph isomorphic to \(G\) or blue subgraph isomorphic to \(H\). The graph \(F\) is Ramsey \((G, H)\)-minimal if \(F \rightarrow (G, H)\) but \(F - e \nrightarrow (G, H)\) for any \(e \in E(F)\). The class of all Ramsey \((G, H)\)-minimal graphs will be denoted by \(\mathcal{R}(G, H)\). According to the previous study, we know that graphs in \(\mathcal{R}(P_3, C_7)\) have at least 13 edges. In this paper, we find graphs in \(\mathcal{R}(P_3, C_7)\) having seven vertices and 13 and 14 edges, respectively. Further, we give some necessary conditions for Ramsey \((P_3, C_7)\)-minimal graphs with seven vertices.

1. Introduction

Ramsey theory was first applied in graph theory [1]. In their study, they found a new concept related to Ramsey theory called Ramsey number. In 1978, Erdos et al. [2] introduced size Ramsey numbers. Then a new concept called Ramsey minimal graph was introduced by Burr et al. [3]. Here, we follow the notations in [4]. Let \(G, H\) be graphs. Notation \(F \rightarrow (G, H)\) means that there is any two-colorings, say red and blue, of all edges of \(F\) which contains red subgraph isomorphic to \(G\) or blue subgraph isomorphic to \(H\). Let a graph \(F\) satisfy \(F \rightarrow (G, H)\). The minimum number of vertices and edges of \(F\) is called Ramsey number and size Ramsey number, respectively. We write a graph \(F\) without any fixed edge \(e \in E(F)\) by \(F - e\). Graph \(F\) is a Ramsey \((G, H)\)-minimal graph if \(F \rightarrow (G, H)\) but \(F - e \nrightarrow (G, H)\), \(\forall e \in (F)\). The class of all Ramsey \((G, H)\)-minimal graphs will be denoted by \(\mathcal{R}(G, H)\).

Many problems arise in determining the class of Ramsey \((G, H)\)-minimal graphs. In 2005, Borowiecki et al. [4] found all graphs in \(\mathcal{R}(K_{1,2}, K_3)\). Then in 2008, Buskoro et al. [5] gave a family of graphs with diameter 2 that belongs to \(\mathcal{R}(K_{1,2}, C_4)\). Veterik et al. [6] found an infinite family of Ramsey \((K_{1,2}, C_4)\)-minimal graphs with diameter at least 4. Then Cyman and Dzido [7] found the restricted size Ramsey number for \(P_3\) versus cycle. Nisa et al. [8] studied \(\mathcal{R}(P_3, C_6)\). In this paper, we prove that there are some graphs that have seven vertices and 13 and 14 edges, respectively, in \(\mathcal{R}(P_3, C_7)\). Further, we give some necessary conditions for graphs with seven vertices in \(\mathcal{R}(P_3, C_7)\).

2. Graph

Some basic notations and terminologies, we follow of that [9]. Graph \(G\) is a set of an ordered pair \((V(G), E(G))\), where \(V(G)\) is a nonempty set of vertices and \(E(G)\) is a set of edges (it can be empty).

Two graphs \(G\) and \(H\) are called isomorphic if there is a bijection \(\theta : V(G) \rightarrow V(H)\) and \(\varphi : E(G) \rightarrow E(H)\) such that \(\varphi_G(e) = uv\) if and only if \(\varphi_H(\varphi(e)) = \theta(u)\theta(v)\). Graph \(H\) is called subgraph of \(G\), denoted by \(H \subseteq G\), if \(V(H) \subseteq V(G)\) and \(E(H) \subseteq E(G)\). If \(H\) is a subgraph of \(G\), where \(H \neq G\), then...
is called a proper subgraph of \( G \). Let \( e \) and \( f \) be edges of graph \( G \). If \( e \neq f \) and these edges are incident with different vertices, then \( e \) and \( f \) are called independent edges. A set of all independent edges is called a matching.

![Figure 1. Some types of graphs.](image1)

A cycle \( C_n \) is a connected graph with \( n \) vertices, where each vertex has degree two. If the number of vertices is even, then it is called even cycle, otherwise, it is called odd cycle. A path \( P_n \) is a connected graph with \( n \) vertices and \( n - 1 \) edges, where its end vertices have one degree and the others have degree one. Consequently, the degree of each vertex of \( K_n \) is \( n - 1 \). Star graph \( K_{1,n} \) is a connected graph with \( n + 1 \) vertices where one vertex has degree \( n \) and each other vertex has degree one. A vertex that has one degree is called pendant. Examples of \( C_6 \), \( P_5 \), \( K_6 \) and \( K_{1,5} \) are as in Figure 1.

3. Ramsey number and restricted size Ramsey number

Let \( F, G \) and \( H \) be graphs. The Ramsey number \( R(G, H) \) is the minimum number of vertices of graph \( F \) such that any red-blue coloring of the edges of \( F \) contains a red subgraph \( G \) or a blue subgraph \( H \), it is also defined as \( \min \{|V(F)| \colon F \rightarrow (G, H)\} \). The size Ramsey number \( r(G, H) \) is the minimum number of edges of graph \( F \) such that any red-blue coloring of the edges of \( F \) contains a red subgraph \( G \) or a blue subgraph \( H \). Additionally, if the order of \( F \) in the size Ramsey number equals \( R(G, H) \), then it is called the restricted size Ramsey number which is denoted by \( r^*(G, H) \) and also defined as \( \min \{|E(F)| \colon F \rightarrow (G, H), |V(F)| = R(G, H)\} \).

4. Main results

The results in the previous research (see e.g [5], [6], [8]) explain that there are graphs that satisfy \( R(P_n, C_n) \) for \( n \leq 6 \). We are interested to investigate graphs in \( R(P_n, C_7) \). There are results from the previous studies that we will use in our main result as follow.

**Theorem 1** [10] \( R(C_m, P_n) = \max\{m + \left\lceil \frac{n}{2} \right\rceil - 1, 2n - 1\} \) for \( 2 \leq n \leq m, m \text{ odd} \).

**Theorem 2** [7] \( r(P_3, C_7) = 13 \).

![Figure 2. Graph \( F_1 \) and graph \( F_2 \).](image2)
Next, we will give our main results. Firstly, we find graphs in $\mathcal{R}(P_3, C_7)$ having seven vertices and 13 and 14 edges, respectively. We find all graphs in this section analytically. In addition, we give some necessary conditions for Ramsey $(P_3, C_7)$-minimal graphs with seven vertices.

We use graph $F_1$ and $F_2$ each with seven vertices and 13 edges as in Figure 2 above in our results.

**Theorem 3** Graph $F_1$ is in $\mathcal{R}(P_3, C_7)$.

**Proof.** According to Theorem 1 and Theorem 2, we know that $R(P_3, C_7) = 7$ and $r(P_3, C_7) = 13$. Now, we will prove that $F_1$ is in $\mathcal{R}(P_3, C_7)$.

First, we show that $F_1 \to (P_3, C_7)$. Consider any red blue coloring of the edges in $F_1$. Suppose that there is no red copy of $P_3$ in coloring. Let us consider maximum red coloring of the edges of graph $F_1$ is $v_1v_2, v_3v_4$, and $v_5v_6$, then we have blue cycle $v_1v_3v_2v_7v_5v_4v_6$. If we give another maximum red coloring of the edges of graph $F_1$, say $v_1v_7, v_2v_3, v_4v_5$, we have blue cycle $C_7$ that is $v_1v_2v_7v_5v_6v_4v_3$.

For other red coloring of the edges of graphs $F_1$, we will find blue cycle $C_7$.

Next we will prove that $F_1 - e \not\to (P_3, C_7), \forall e \in (F_1)$. We will explain this proof by several cases:

Case 1. If $e = v_6, v_7$ or $v_1v_7$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 2. If $e = v_1v_2$ or $v_6v_5$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 3. If $e = v_2v_3$ or $v_4v_5$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 4. If $e = v_1v_3$ or $v_6v_4$, then give red color for $v_2v_3, v_4v_5$ and $v_6v_7$.

Case 5. If $e = v_2v_7$ or $v_5v_7$, then give red color for $v_2v_3, v_4v_5$ and $v_6v_7$.

Case 6. If $e = v_1v_6$, then give red color for $v_2v_5, v_3v_4$ and $v_6v_7$.

Case 7. If $e = v_2v_5$, then give red color for $v_1v_6, v_2v_7$ and $v_3v_4$.

Case 8. If $e = v_3v_4$, then give red color for $v_1v_6, v_2v_5$ and $v_2v_5$.

For each case, color remaining edges by blue. By this coloring we obtain that $F_1 - e \not\to (P_3, C_7), \forall e \in E(F_1)$. $\blacksquare$

**Theorem 4** Graph $F_2$ is in $\mathcal{R}(P_3, C_7)$.

**Proof.** First, we show that $F_2 \to (P_3, C_7)$. Consider any red blue coloring of the edges in $F_2$. Suppose that there is no red copy of $P_3$ in coloring. Let us consider maximum red coloring of the edges of graph $F_1$ is $v_1v_2, v_3v_4$, and $v_5v_6$, then we have blue cycle $v_1v_3v_2v_7v_5v_4v_6$. If we give any other maximum red coloring of the edges of graph $F_2$, say $v_1v_7, v_2v_3, v_4v_5$, we have blue cycle $C_7$ that is $v_1v_2v_7v_5v_6v_4v_3$.

Next we will prove $F_1 - e \not\to (P_3, C_7), \forall e \in (F_2)$. We will explain this proof by several cases:

Case 1. If $e = v_6, v_7$ or $v_1v_7$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 2. If $e = v_1v_2$ or $v_6v_5$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 3. If $e = v_2v_3$ or $v_4v_5$, then give red color for $v_1v_6, v_2v_5$ and $v_3v_4$.

Case 4. If $e = v_1v_3$ or $v_6v_4$, then give red color for $v_2v_3, v_4v_5$ and $v_6v_7$.

Case 5. If $e = v_2v_7$ or $v_5v_7$, then give red color for $v_2v_3, v_4v_5$ and $v_6v_7$.

Case 6. If $e = v_1v_6$, then give red color for $v_2v_3, v_4v_5$ and $v_6v_7$.

Case 7. If $e = v_2v_5$, then give red color for $v_1v_6, v_2v_7$ and $v_3v_4$.

Case 8. If $e = v_3v_4$, then give red color for $v_1v_6, v_2v_5$ and $v_4v_5$.

For each case above, color remaining edges by blue. By this coloring we obtain that $F_2 - e \not\to (P_3, C_7), \forall e \in E(F_2)$. $\blacksquare$
Next we give two graphs with seven vertices and 14 edges as in Figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.pdf}
\caption{Graph $F_3$, graph $F_4$ and graph $F_5$.}
\end{figure}

**Theorem 5** Graph $F_3$ is in $\mathcal{R}(P_3, C_7)$.

**Proof.** First, we show that $F_3 \rightarrow (P_3, C_7)$. Consider any red-blue coloring of the edges in $F_3$. Suppose that there is no red copy of $P_3$ in coloring. Let us consider maximum red coloring of the edges of graph $F_3$ is $v_1v_2, v_3v_4,$ and $v_5v_6$, then we have blue cycle $v_1v_3v_2v_7v_4v_5v_6$. Then we give another maximum red coloring of the edges of graph $F_3v_1v_7, v_2v_3, v_4v_5$ we have blue cycle $C_7v_1v_3v_4v_6v_5v_7v_2$. For other red coloring of the edges of graphs $F_3$ we will find blue cycle $C_7$.

Next we will prove $F_3 - e \rightarrow (P_3, C_7), \forall e \in (F_3)$ we will explain this proof by several cases:

- **Case 1.** If $e = v_6, v_7$ or $v_1, v_7$, then give red color for $v_1v_5, v_3v_4$ and $v_2v_6$.
- **Case 2.** If $e = v_1v_2$ or $v_6v_5$, then give red color for $v_1v_5, v_3v_4$ and $v_2v_7$.
- **Case 3.** If $e = v_2v_3$, then give red color for $v_1v_5, v_2v_6$ and $v_3v_4$.
- **Case 4.** If $e = v_1v_2$ or $v_2v_6$, then give red color for $v_1v_2, v_3v_6$ and $v_4v_5$.
- **Case 5.** If $e = v_2v_7$ or $v_5v_7$, then give red color for $v_1v_7, v_2v_6$ and $v_3v_4$.
- **Case 6.** If $e = v_1v_6$, then give red color for $v_1v_2, v_7v_5$ and $v_3v_4$.
- **Case 7.** If $e = v_4v_7$, then give red color for $v_1v_5, v_2v_6$ and $v_3v_4$.
- **Case 8.** If $e = v_3v_4$, then give red color for $v_2v_3, v_4v_5$ and $v_1v_6$.
- **Case 9.** If $e = v_4v_5$, then give red color for $v_3v_4, v_3v_2$ and $v_6v_7$.

For every case, color remaining edges by blue and by this coloring we obtain that $F_3 - e \rightarrow (P_3, C_7), \forall e \in E(F_3)$. 

**Theorem 6** Graph $F_4$ is in $\mathcal{R}(P_3, C_7)$.

**Proof.** First, we show that $F_4 \rightarrow (P_3, C_7)$. Consider any red-blue coloring of the edges in $F_4$. Suppose that there is no red copy of $P_3$ in coloring. Let us consider maximum red coloring of the edges of graph $F_4$ is $v_1v_2, v_3v_4,$ and $v_5v_6$, then we have blue cycle $v_1v_3v_2v_7v_4v_5v_6$. Then we give another maximum red coloring of the edges of graph $F_4v_1v_7, v_2v_3, v_4v_5$ we have blue cycle $C_7v_1v_3v_4v_6v_5v_7v_2$. For other red coloring of the edges of graphs $F_4$, we will find blue cycle $C_7$.

Next we will prove $F_4 - e \rightarrow (P_3, C_7), \forall e \in (F_4)$ we will explain this proof by several cases:

- **Case 1.** If $e = v_6, v_7$ or $v_1, v_7$, then give red color for $v_2v_6, v_3v_4$ and $v_5v_7$.
- **Case 2.** If $e = v_1v_2$ or $v_6v_5$, then give red color for $v_1v_6, v_4v_5$ and $v_2v_7$.
- **Case 3.** If $e = v_2v_3$ or $v_4v_5$, then give red color for $v_1v_7, v_2v_6$ and $v_3v_4$.
- **Case 4.** If $e = v_1v_5$ or $v_2v_6$, then give red color for $v_2v_6, v_3v_4$ and $v_5v_7$. 
Case 5. If \( e = v_2v_7 \) or \( v_5v_7 \), then give red color for \( v_1v_5, v_2v_6 \) and \( v_3v_4 \).
Case 6. If \( e = v_1v_6 \), then give red color for \( v_1v_5, v_2v_6 \) and \( v_3v_4 \).
Case 7. If \( e = v_1v_3 \) or \( v_4v_6 \), then give red color for \( v_1v_5, v_2v_6 \) and \( v_3v_4 \).
Case 8. If \( e = v_3v_4 \), then give red color for \( v_2v_3, v_4v_5 \) and \( v_1v_6 \).

For every case, color remaining edges by blue and by this coloring we obtain that \( F_4 - e \leftrightarrow (P_3, C_7) \), \( \forall e \in E(F_4) \).

**Theorem 7** Graph \( F_5 \) is in \( \mathcal{R}(P_3, C_7) \).

**Proof.** First, we show that \( F_5 \rightarrow (P_3, C_7) \). Consider any red blue coloring of the edges in \( F_5 \). Suppose that there is no red copy of \( P_3 \) in coloring. Let us consider maximum red coloring of the edges of graph \( F_5 \) is \( v_1v_2, v_2v_4, v_5v_6 \), then we have blue cycle \( v_1v_2v_3v_4v_5v_6v_3 \). Then we give another maximum red coloring of the edges of graph \( F_5 \) we have blue cycle \( C_7 \) as graph \( v_1v_2v_3v_4v_5v_6v_3 \). For other red coloring of the edges of graphs \( F_5 \) we will find blue cycle \( C_7 \).

Next we will prove \( F_5 - e \leftrightarrow (P_3, C_7) \), \( \forall e \in E(F_5) \) we will explain this proof by several cases:

Case 1. If \( e = v_6, v_7 \) or \( v_1v_7 \), then give red color for \( v_2v_6, v_3v_4 \) and \( v_5v_7 \).
Case 2. If \( e = v_1v_2 \) or \( v_6v_7 \), then give red color for \( v_2v_7, v_3v_4 \) and \( v_5v_6 \).
Case 3. If \( e = v_2v_3 \) or \( v_4v_5 \), then give red color for \( v_1v_6, v_2v_5 \) and \( v_3v_4 \).
Case 4. If \( e = v_1v_5 \) or \( v_2v_4 \), then give red color for \( v_2v_3, v_4v_5 \) and \( v_5v_7 \).
Case 5. If \( e = v_2v_7 \) or \( v_5v_7 \), then give red color for \( v_1v_6, v_2v_5 \) and \( v_3v_4 \).
Case 6. If \( e = v_1v_6 \), then give red color for \( v_4v_7, v_2v_6 \) and \( v_3v_4 \).
Case 7. If \( e = v_1v_4 \) or \( v_3v_6 \), then give red color for \( v_1v_6, v_2v_6 \) and \( v_3v_4 \).
Case 8. If \( e = v_3v_4 \), then give red color for \( v_2v_3, v_4v_5 \) and \( v_1v_6 \).

For every case, color remaining edges by blue and by this coloring we obtain that \( F_5 - e \leftrightarrow (P_3, C_7) \), \( \forall e \in E(F_5) \).

Next, we give necessary conditions for graphs with seven vertices in \( \mathcal{R}(P_3, C_7) \) as in the following theorem.

**Theorem 8** Let \( G \) be any connected graph with seven vertices in \( \mathcal{R}(P_3, C_7) \) then:

a. The minimum degree of \( G \) is three.

b. If \( G \) has exact two vertices of degree three, then the vertices have no common neighbour.

**Proof.** (a) Assume that there is a vertex have degree two \( (v_1v_2v_3) \), suppose that any red coloring \( v_1v_2 \) then the edge \( v_2v_3 \) must be blue, to make any \( C_7 \) we need at least 2 degree. So the vertices must be have degree 3 to make \( C_7 \). Contradiction. (b) Assume a pair of vertices have joining vertices said \( v_1v_2v_3v_4v_5 \), and \( v_2v_4 \) adjacent. Suppose red coloring on \( v_1v_2 \) and \( v_4v_5 \) then the coloring blue \( v_2v_3v_4 \) is \( C_3 \). To make another graph cycle, we need another edge. Contradiction.

**5. Conclusion**

All graphs in \( \mathcal{R}(K_{1,2}, K_3) \) had been characterized by Borowiecki, et al. Error! Reference source not found. Some graphs in \( \mathcal{R}(P_3, C_4), \mathcal{R}(P_3, C_5), \) and \( \mathcal{R}(P_3, C_6) \) has been studied. Recently, Nisa et al. studied about \( \mathcal{R}(P_3, C_6) \). However, the characterization of all graphs in \( \mathcal{R}(P_3, C_4), \mathcal{R}(P_3, C_5), \) and \( \mathcal{R}(P_3, C_6) \) are still open. In this paper, we show that the graphs with seven vertices as in Figure 4, namely \( F_1, F_2, F_3, F_4, F_5 \), are in \( \mathcal{R}(P_3, C_7) \).
We also give two necessary conditions for graphs with seven vertices in $\mathcal{R}(P_3, C_7)$, as in Theorem 8. However, the characterization of all graphs in $\mathcal{R}(P_3, C_7)$ are still open. It is interesting to find an algorithm to construct all graphs in $\mathcal{R}(P_3, C_7)$. So the graphs in $\mathcal{R}(P_3, C_7)$ will be determined computationally.

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Figure 4. Some graphs with seven vertices in $\mathcal{R}(P_3, C_7)$. 