GLOBAL STRONG SOLUTIONS TO THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES SYSTEM IN THE EXTERIOR OF A CYLINDER

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Abstract. In this paper, the global strong axisymmetric solutions for the inhomogeneous incompressible Navier-Stokes system are established in the exterior of a cylinder subject to the Dirichlet boundary conditions. Moreover, the vacuum is allowed in these solutions. One of the key ingredients of the analysis is to obtain the $L^2(s, T; L^\infty(\Omega))$ bound for the velocity field, where the axisymmetry of the solutions plays an important role.

1. Introduction and main results

The mixture of incompressible and non-reactant flows, flows with complex structure fluids containing a melted substance, etc (37), can be described by the following inhomogeneous incompressible Navier-Stokes system

\begin{align}
\begin{cases}
(\rho u)_t + \div (\rho u \otimes u) - \mu \Delta u + \nabla P = 0, & \text{in } \Omega \times [0, T), \\
\rho_t + \div (\rho u) = 0 & \text{in } \Omega \times [0, T) \\
\div u = 0, & \text{in } \Omega \times [0, T),
\end{cases}
\end{align}

where $\rho, u, P,$ and $\mu$ are the density, velocity field, pressure, and viscosity coefficient of fluid, respectively. In this paper, the viscosity coefficient is assumed to be a constant. Without loss of generality, one assumes $\mu = 1$. Furthermore, in the domain $\Omega$ where the fluid occupies, the system (1.1) is usually supplemented with the following initial conditions and no slip boundary conditions

\begin{align}
(\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0) & \text{ in } \Omega; \quad u = 0 & \text{ on } \partial\Omega \times (0, T).
\end{align}

Since Leray’s pioneering work 35 on the global existence of weak solutions to the homogeneous incompressible Navier-Stokes system (corresponding to the case $\rho \equiv 1$), there have been many important progresses on the homogeneous incompressible Navier-Stokes system. When the initial density is away from vacuum, there is a counterpart theory of inhomogeneous Navier-Stokes system to Leray’s results. The global existence of weak solutions and

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local existence of strong solutions for inhomogeneous Navier-Stokes system were established in [1,5,31]. Furthermore, the strong solution exists globally in two dimensional case [4]. Recently, there are many studies on the well-posedness for the inhomogeneous Navier-Stokes system in various critical spaces, see [3,15,16,39] and references therein.

When the vacuum is allowed, the local and global existence of weak solutions to system (1.1) was established in [29,42]. However, the uniqueness and smoothness of weak solutions to the inhomogeneous Navier-Stokes system, even for the two dimensional case, are still open problems. This is very different from the two dimensional homogeneous Navier-Stokes system ( [33]). A local strong solution under some compatibility conditions on the initial data was established in [8]. More precisely, given $(\rho_0, u_0)$ satisfying

\begin{align}
0 \leq \rho_0 &\in L^2(\Omega) \cap H^2(\Omega), \\
u_0 &\in H^1_0(\Omega) \cap H^2(\Omega),
\end{align}

and the compatibility conditions

\begin{align}
-\mu \Delta u_0 + \nabla P_0 &= \frac{4}{3} \rho_0 g, \quad \text{and} \quad \text{div} \ u_0 = 0 \quad \text{in} \ \Omega,
\end{align}

with some $(P_0, g)$ belonging to $D^{1,2}(\Omega) \times L^2(\Omega)$, there exists a unique local strong solution $(\rho, u)$ to the initial boundary value problem (1.1)–(1.2). For the further studies on local well-posedness of strong solutions for inhomogeneous Navier-Stokes system, see [13,14,23,30,45] and references therein. A natural question is whether the general local strong solutions away from the vacuum can be prolonged globally in time. Suppose the local strong solution blows up in finite time $T^*$, a Serrin type blow-up criterion was established in [28],

\begin{align}
\int_0^{T^*} \|u(t)\|_{L^r_w}^s \, dt = \infty, \quad \text{for any} \ (r, s) \ \text{with} \ \frac{2}{s} + \frac{n}{r} = 1, \ n < r \leq \infty,
\end{align}

where $n$ is the dimension of space, and $L^r_w$ is the weak $L^r$ space. With the aid of this blow up criterion, for the initial data even with the vacuum, the global strong solutions for the inhomogeneous Navier-Stokes system in two dimensional case were established in [19,21,22].

Global existence of strong solutions for three dimensional Navier-Stokes system even in the homogeneous case is a long standing challenging problem. However, it was proved in [32,44] that for the axisymmetric solutions without swirls, the global Leray-Hopf weak solution for homogeneous Navier-Stokes system is regular for all time $t > 0$. The proof in [32,44] was based on important facts that the vorticity $\omega = \nabla \times u$ satisfies the maximum principle and the global a priori estimate

\begin{align}
\left\| \frac{\omega}{r} \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \frac{\omega_0}{r} \right\|_{L^2(\mathbb{R}^3)}
\end{align}

holds. However, when the swirl velocity is present, the global well-posedness for the axisymmetric Navier-Stokes system becomes much more difficult. There are many important
progresses on this problem, see [7, 9–12, 20, 25, 27, 34] and references therein. On the other hand, the significant partial regularity results in [6] (see also [17, 36, 40, 43]) assert that the one-dimensional Hausdorff measure of the set for singular points is zero. This implies that the singularity of axisymmetric solutions can only happen at the axis. When the domain is the exterior of a cylinder, the global existence of unique axisymmetric strong solution was proved in [33] and [1] when the no slip and Navier boundary conditions were supplemented, respectively. The crucial points for the analysis in [33] and [1] are an interpolation inequality and the maximum principle (1.6), respectively.

The axisymmetric solutions for inhomogeneous Navier-Stokes system without swirls were studied in [2] and references therein. For the inhomogeneous Navier-Stokes system in an exterior domain, the local existence of weak solutions was proved in [38] when the initial density is positive almost everywhere. The main goal of this paper is to study the global existence of axisymmetric strong solutions for the inhomogeneous Navier-Stokes system in the exterior of a cylinder subject to the no slip boundary conditions, where (1.6) may not be true and it seems difficult to apply the interpolation inequality used in [33]. Without loss of generality, in this paper, one assumes that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r^2 = x_1^2 + x_2^2 > 1, x_3 \in \mathbb{R}\}.$$ 

The key idea in this paper is to get some bound for $$\|u\|_{L^2(s,T;L^\infty(\Omega))}$$ from the energy inequality, which corresponds to a regularity criterion of Serrin type [41].

Before stating the main results in the paper, the following notations are introduced. For $$1 \leq q \leq \infty$$, let $$L^q(\Omega)$$ denote the usual scalar-valued and vector-valued $$L^q$$-space over $$\Omega$$. Let

$$W^{m,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), |\alpha| \leq m, m \in \mathbb{N}\}.$$ 

When $$q = 2$$, one abbreviates $$H^m(\Omega) = W^{m,2}(\Omega)$$. Denote the closure of $$C^\infty_0(\Omega)$$ in $$H^1(\Omega)$$ by $$H^1_0(\Omega)$$. Let

$$C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) : \text{div } u = 0, \text{ in } \Omega\}.$$ 

Denote the closure of $$C^\infty_{0,\sigma}(\Omega)$$ in $$H^1(\Omega)$$ by $$H^1_{0,\sigma}(\Omega)$$.

Our main result can be stated as follows.

**Theorem 1.1.** Let $$(\rho_0, u_0)$$ be axisymmetric initial data and satisfy compatibility condition (1.4) and the following regularity conditions

(1.7) \[ \rho_0 \geq 0, \quad \rho_0 - \bar{\rho} \in L^2(\Omega) \cap H^2(\Omega), \quad u_0 \in H^2(\Omega) \cap H^1_0(\Omega), \]

with $$\bar{\rho} > 0$$ is a constant. Then for every $$T > 0$$, there exists a unique axisymmetric strong solution $$(\rho, u)$$ to the problem (1.1)–(1.2) with

$$\rho - \bar{\rho} \in C([0, T]; L^2(\Omega) \cap H^2(\Omega)), \quad u \in C([0, T]; H^1_{0,\sigma}(\Omega)) \cap L^\infty(0, T; H^2(\Omega)),$$
and

$$\nabla u_t \in L^2([0, T]; \mathbb{L}^2(\Omega)), \ (\rho_t, \sqrt{\rho}u_t) \in \mathbb{L}^\infty([0, T]; \mathbb{L}^2(\Omega)).$$

There are a few remarks in order.

Remark 1.1. Together with the analysis in [21], one can also show that this result holds for inhomogeneous MHD equations.

Remark 1.2. Together with the method in [30] for the proof of the local existence of solutions for the inhomogeneous Navier-Stokes system in bounded domains even when the initial data violate the compatibility conditions (1.4), the compatibility conditions (1.4) should also be removed in Theorem 1.1. The major aim of this paper is to highlight the a priori estimate to get global strong axisymmetric solutions for the inhomogeneous Navier-Stokes system so that we try to avoid including a more complicated local existence result in this paper.

Remark 1.3. The method in this paper can also be used to prove the global existence of axisymmetric strong solutions to Navier-Stokes system in the exterior of a cylinder subject to Navier boundary condition.

Remark 1.4. The analysis in this paper should be also helpful for the study on the helically symmetric flows.

The rest of this paper is organized as follows. Some elementary results on axisymmetric functions, the critical Sobolev inequalities, and the estimates regarding Stokes equations are collected in Section 2 which are important for the analysis in the whole paper. The proof of Theorem 1.1 is presented in Section 3 after one assumes the local well-posedness of the problem. In Section 4 the local existence and uniqueness of strong solutions are sketched.

2. Preliminaries

For \((x_1, x_2, x_3) \in \mathbb{R}^3\), introduce the cylindrical coordinate

$$r = \sqrt{(x_1)^2 + (x_2)^2}, \ \theta = \arctan \frac{x_2}{x_1}, \ z = x_3,$$

and denote \(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\) the standard basis vectors in the cylindrical coordinate:

$$\mathbf{e}_r(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \ \mathbf{e}_\theta(\theta) = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \ \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A function \(f\) or a vector-valued function \(\mathbf{u} = (u^r, u^\theta, u^z)\) is said to be axisymmetric if \(f, u^r, u^\theta\) and \(u^z\) do not depend on \(\theta\):

$$\mathbf{u}(x_1, x_2, x_3) = u^r(r, z)\mathbf{e}_r + u^\theta(r, z)\mathbf{e}_\theta + u^z(r, z)\mathbf{e}_z.$$
The following lemma shows that for axisymmetric initial data the local strong solution to (1.1) is also axisymmetric.

**Lemma 2.1.** Assume that the initial data \((\rho_0, u_0)\) is axisymmetric. Then the local strong solution \((\rho, u)\) to (1.1)–(1.2) is also axisymmetric.

**Proof.** For every \(\eta \in [0, 2\pi)\), define the rotation matrix

\[ R(\eta) = (e_\rho(\eta), e_\theta(\eta), e_z). \]

Let

\[ \varrho(x_1, x_2, x_3, t) = \rho(R(x_1, x_2, x_3), t) \quad \text{and} \quad v(x_1, x_2, x_3, t) = R^t u(R(x_1, x_2, x_3), t) \]

where \( "R^t" \) is the transpose of the matrix \( R \). Since the inhomogeneous Navier-Stokes system (1.1) is rotation invariant and \( \rho_0 \) and \( u_0 \) are axisymmetric, it is easy to check that \((\varrho, v)\) is also a solution to (1.1) with the initial data \((\rho_0, u_0)\). Due to the uniqueness of strong solutions to (1.1)–(1.2), one has \( \varrho = \rho \) and \( v = u \). Hence the solution \((\rho, u)\) is axisymmetric. \( \square \)

The following critical Sobolev inequality of Logarithmic type plays an important role to obtain the bound of \( \|u\|_{L^2(s,T;L^\infty(\Omega))} \).

**Lemma 2.2.** Suppose \( D \) is a domain in \( \mathbb{R}^2 \), for every function \( f \in L^2(s,T;H^1_0(D)) \cap L^2(s,t;W^{1,q}(D)) \) with some \( q > 2 \), and \( s < t \), it holds that

\[
\|f\|_{L^2(s,T;L^\infty(D))} \leq C \|\nabla f\|_{L^2(s,T;L^2(D))} \left[ \ln(\varepsilon + \|f\|_{L^2(s,T;W^{1,q}(D))}) \right]^{\frac{1}{2}} + C,
\]

where \( C \) is independent of the function \( s, t, \) and the domain \( D \).

**Proof.** The inequality (2.1) has been proved in [21] for \( D = \mathbb{R}^2 \). If \( D \) is a domain in \( \mathbb{R}^2 \), it can be proved by zero extension, so we omit the details here. \( \square \)

The next lemma gives the uniform regularity estimates for solutions to the Stokes equations with Dirichlet boundary condition.

**Lemma 2.3.** Let \( \mathcal{D} \) be a domain of \( \mathbb{R}^3 \), whose boundary is uniformly of class \( C^3 \). Assume \( u \in H^1_{0,\sigma}(\mathcal{D}) \) is a weak solution to the following Stokes equations

\[
\begin{cases}
- \Delta u + \nabla P = f, & \text{in } \mathcal{D}, \\
\text{div } u = 0, & \text{in } \mathcal{D}, \\
u = 0, & \text{on } \partial \mathcal{D}.
\end{cases}
\]

Then for \( f \in L^q(\mathcal{D}) \), \( 1 < q < \infty \), it holds that

\[
\|u\|_{W^{2,q}(\mathcal{D})} \leq C\|f\|_{L^q(\mathcal{D})} + C\|u\|_{W^{1,q}(\mathcal{D})},
\]
where the constant $C$ depends only on $q$ and the $C^3$-regularity of $\partial \mathcal{D}$ (not on the size of $\partial \mathcal{D}$ or $\mathcal{D}$).

**Proof.** The proof of (2.3) for the particular case $q = 2$ can be found in [24] Lemma 2.2. With the aid of “local” estimate up to the boundary for the Stokes problem (2.2) (\cite{18,33}), the estimate (2.3) for the general $q > 1$ can be proved in the same spirit of [24]. For readers’ convenience, we give a sketch of the proof for the case with general $q > 1$.

In a neighbourhood of a given point $\xi \in \partial \mathcal{D}$, let the boundary $\partial \mathcal{D}$ be represented by $y_3 = F(y_1, y_2)$ in local Cartesian coordinates $(y_1, y_2, y_3)$ chosen so that the positive $y_3$-axis coincides with the inward normal to $\partial \mathcal{D}$ at $\xi$. Suppose that $d$ is a sufficiently small number determined by $C^3$-regularity of $\partial \mathcal{D}$ near $\xi$. Denote $Q = \{x \in \mathcal{D} : |y_1|, |y_2| < d \text{ and } F(y_1, y_2) < y_3 < F(y_1, y_2) + 2d\}$ and $Q' = \{x \in \mathcal{D} : |y_1|, |y_2| < d/2 \text{ and } F(y_1, y_2) < y_3 < F(y_1, y_2) + d\}$. It follows from \cite{18,33} that

\begin{equation}
\|D^2 u\|_{L^q(Q')} \leq C_{\partial,d}\|u\|_{W^{1,q}(Q)} + C_{\partial,d}\|f\|_{L^q(Q)}. \tag{2.4}
\end{equation}

Moreover, let $G' = \{x \in \mathcal{D} : \text{dist}(x, \partial \mathcal{D}) > d/2\}$ be an open bounded subset of $\mathcal{D}$, with $\bar{G}' \subset \mathcal{D}$. Choose a function $\zeta \in C^2_0(\mathcal{D})$ satisfying that $\zeta \equiv 1$ in $G'$ and $0 \leq \zeta \leq 1$ elsewhere in $\mathcal{D}$, such that $|\nabla \zeta|$ and $|\Delta \zeta|$ are bounded by some constant $C_{\partial,d}$ which depends on $d$ and the $C^2$-regularity of $\partial \mathcal{D}$. Then one has

\begin{equation}
\|D^2 u\|_{L^q(\mathcal{D} \setminus G')} \leq C_{d}\|\nabla u\|_{L^q(\mathcal{D})} + C_{d}\|f\|_{L^q(\mathcal{D})} + C_{d}\|D^2 \zeta u\|_{L^q(\mathcal{D})} + C_{d}\|u\|_{W^{1,q}(\mathcal{D})}, \tag{2.5}
\end{equation}

\begin{equation*}
\leq C_{\partial,d}\|u\|_{W^{1,q}(\mathcal{D})} + C_{d}\|f\|_{L^q(\mathcal{D})}.
\end{equation*}

On the other hand, since the boundary is uniformly of class $C^3$, $d$ is sufficiently small, and the mean curvature of the surface near the given point is bounded, then the boundary strip $\mathcal{D} - G'$ can be covered with a collection of “cubes” $Q'_i$, of the type described in (2.4) in such a way that no point of $\mathcal{D}$ belongs to more than ten of the associated larger “cubes” $Q_i$. Thus it follows from (2.4) that

\begin{equation*}
\|D^2 u\|_{L^q(\mathcal{D})} \leq \sum_i \|D^2 u\|_{L^q(Q'_i)} \leq \sum_i \left(C_{\partial,d}\|u\|_{W^{1,q}(Q'_i)} + C_{\partial,d}\|f\|_{L^q(Q'_i)}\right) \leq 10 \left(C_{\partial,d}\|u\|_{W^{1,q}(\mathcal{D})} + C_{\partial,d}\|f\|_{L^q(\mathcal{D})}\right).
\end{equation*}

This, together with (2.5), implies the desired estimate

\begin{equation*}
\|D^2 u\|_{L^q(\mathcal{D})} \leq C_{\partial} \left(\|u\|_{W^{1,q}(\mathcal{D})} + \|f\|_{L^q(\mathcal{D})}\right),
\end{equation*}

since $d$ is determined by the $C^3$-regularity of $\partial \mathcal{D}$. Hence the proof of the lemma is completed. \hfill \square
3. A priori estimate and the proof of the main result

This section devotes to the proof of Theorem 1.1. Given initial data \((\rho_0, u_0)\) satisfying (1.7) and the compatibility condition (1.4), Theorem 4.6 asserts that there exists a unique local strong solution \((\rho, u)\). According to Lemma 2.1, the solution is axisymmetric. Define the quantity \(\Phi(T)\) as follows:

\[
\Phi(T) = \sup_{0 \leq t \leq T} \left( \|\rho - \bar{\rho}\|_{L^3(\Omega)} + \|\rho - \bar{\rho}\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)}^2 + \|\sqrt{\rho} u_t\|_{L^\infty(0, T; L^2(\Omega))}^2 \right).
\]

Suppose this local strong solution blows up at some \(T^* < \infty\), the key issue is to prove that in fact there exists a constant \(\bar{M} < \infty\) depending only on the initial data and \(T^*\) such that

\[
\sup_{0 \leq T < T^*} \Phi(T) \leq \bar{M}.
\]

This, together with Theorem 4.6, implies that the local strong solution can be extended beyond \(T^*\), and thus gives a contradiction. Therefore, the local strong solution does not blow up in finite time.

**Proof Theorem 1.1:** First, it is easy to see that for the strong solutions, the system (1.1) is equivalent to

\[
\begin{align*}
\rho u_t + (\rho u \cdot \nabla) u - \Delta u + \nabla P &= 0, \quad \text{in } \Omega \times [0, T), \\
\rho_t + u \cdot \nabla \rho &= 0, \quad \text{in } \Omega \times [0, T), \\
\text{div } u &= 0, \quad \text{in } \Omega \times [0, T).
\end{align*}
\]

The proof is divided into 5 steps.

**Step 1.** \(L^\infty\) bound for \(\rho\). The second equation in (3.2) is in fact a transport equation, due to the divergence free property of \(u\). Hence, for every \(0 \leq t < T^*\), it holds that

\[
\|\rho(\cdot, t)\|_{L^\infty(\Omega)} = \|\rho_0\|_{L^\infty(\Omega)}
\]

and

\[
\|\rho(\cdot, t) - \bar{\rho}\|_{L^2(\Omega)} = \|\rho_0 - \bar{\rho}\|_{L^2(\Omega)}.
\]

**Step 2.** Basic energy estimate. The energy estimate can be stated as the following proposition.

**Proposition 3.1.** There exists some constant \(M_1\), which depends only on \(\|\sqrt{\rho_0 u_0}\|_{L^2(\Omega)}^2, \|\rho_0 - \bar{\rho}\|_{L^2(\Omega)}, \|\rho_0\|_{L^\infty(\Omega)}, \bar{\rho}^{-1}\), such that

\[
\sup_{0 \leq T < T^*} \left\{ \|\sqrt{\rho} u\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|u\|_{L^2(0, T; H^1(\Omega))}^2 \right\} \leq M_1.
\]
Proof. Multiplying the first equation of (3.2) by $u$ and integrating by parts over $\Omega$ yield that for every $0 < T < T^*$,
\[
\frac{d}{dt} \int_{\Omega} \rho |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = 0.
\]
Hence,
\[
\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2(\Omega))}^2 + 2 \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 \, dt \leq \int_{\Omega} \rho_0 |u_0|^2 \, dx.
\]
Moreover, note that
\[
\|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \leq C_2 \|\nabla u\|_{L^2(s,T;L^2(\Omega))} \ln \Psi(T) + C_2.
\]

Step 3. Estimates for $\|\sqrt{\rho} u_t\|_{L^2(0,T;L^2(\Omega))}$ and $\|\nabla u\|_{L^\infty(0,T;L^2(\Omega))}$. This is the key step of the whole proof. Higher order estimates of the density and the velocity can be done in a standard way provided that $\|u(\cdot,t)\|_{H^1}$ is uniformly bounded with respect to time. Let
\[
\Psi(t) = e + \sup_{0 \leq \tau \leq t} \|\nabla u(\cdot,\tau)\|_{L^2(\Omega)}^2 + \int_0^t \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 \, d\tau, \quad 0 \leq t < T^*.
\]
Let \( \mathbf{\nabla} = (\partial_r, \partial_\theta) \) be the two-dimensional gradient operator, and \( \tilde{W}^{1,6}(D_2) \) be the Sobolev space defined on \( D_2 \). By Lemma 2.2,

\[
\int_s^T \| \mathbf{u} \|^2_{L^2(\Omega)} \, d\tau \\
\leq C \int_s^T \left( \| u^r \|^2_{L^2(D_2)} + \| u^\theta \|^2_{L^2(D_2)} + \| u^z \|^2_{L^2(D_2)} \right)^2 \, d\tau \\
\leq C \| \nabla (u^r, u^\theta, u^z) \|^2_{L^2(s,T;L^2(D_2))} \ln \left( e + \| (u^r, u^\theta, u^z) \|_{L^2(s,T;\tilde{W}^{1,6}(D_2))} \right) + C.
\]

Note that

\[
\partial_t u^r = \partial_1 u^1 \cos^2 \theta + \partial_2 u^1 \sin \theta \cos \theta + \partial_1 u^2 \cos \theta \sin \theta + \partial_2 u^2 \sin^2 \theta,
\]

\[
\partial_t u^\theta = -\partial_1 u^1 \cos \theta \sin \theta - \partial_2 u^1 \sin^2 \theta + \partial_1 u^2 \cos^2 \theta + \partial_2 u^2 \sin \theta \cos \theta,
\]

and

\[
\partial_t u^z = \partial_1 u^1 \cos \theta + \partial_2 u^1 \sin \theta.
\]

Hence one has

\[
\| \nabla (u^r, u^\theta, u^z) \|_{L^2(s,T;L^2(D_2))} \leq C \| \mathbf{u} \|_{L^2(s,T;L^2(\Omega))}
\]

and

\[
\| (u^r, u^\theta, u^z) \|_{L^2(s,T;\tilde{W}^{1,6}(D_2))} \leq C \| \mathbf{u} \|_{L^2(s,T;\tilde{W}^{1,6}(\Omega))}.
\]

Combining (3.10) and (3.11)-(3.15) together gives

\[
\int_s^T \| \mathbf{u} \|^2_{L^2(\Omega)} \, d\tau \leq C \| \nabla \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} \ln \left( e + \| \mathbf{u} \|_{L^2(s,T;\tilde{W}^{1,6}(\Omega))} \right) + C.
\]

It follows from Sobolev embedding inequality and Lemma 2.3 that one has

\[
\| \mathbf{u} \|^2_{L^2(s,T;W^{1,6}(\Omega))} \leq C \left( \| \nabla \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} + \| \nabla^2 \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} \right) \\
\leq C \left( \| \mathbf{u} \|^2_{L^2(s,T;H^1(\Omega))} + \| \sqrt{\rho} \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} + \| (\rho \mathbf{u} \cdot \nabla) \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} \right) \\
\leq C \left[ \| \mathbf{u} \|^2_{L^2(s,T;H^1(\Omega))} + \| \sqrt{\rho} \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} + \| \mathbf{u} \|^2_{L^2(s,T;L^\infty(\Omega))} \| \nabla \mathbf{u} \|^2_{L^\infty(s,T;L^2(\Omega))} \right] \\
\leq C \left[ \| \mathbf{u} \|^2_{L^2(s,T;H^1(\Omega))} + \Psi(T) + \| \mathbf{u} \|^2_{L^2(s,T;L^\infty(\Omega))} \Psi(T) \right].
\]

This, together with the estimate (3.10), implies that

\[
\| \mathbf{u} \|^2_{L^2(s,T;L^\infty(\Omega))} \leq C \| \nabla \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} \ln \left[ \| \mathbf{u} \|^2_{L^2(s,T;H^1(\Omega))} + \Psi(T) + \| \mathbf{u} \|^2_{L^2(s,T;L^\infty(\Omega))} \Psi(T) \right] + C \\
\leq C \| \nabla \mathbf{u} \|^2_{L^2(s,T;L^2(\Omega))} \ln \Psi(T) + C_1 \ln \left( 1 + \| \mathbf{u} \|^2_{L^2(s,T;L^\infty(\Omega))} \right) + C.
\]
Choose $N_1$ sufficiently large such that

$$C_1 \ln(1 + \gamma) \leq \frac{1}{2} \gamma, \quad \text{for } \gamma \geq N_1.$$  

Then one has

$$C_1 \ln \left(1 + \|u\|_{L^2(s,T; L^\infty(\Omega))}^2\right) \leq \frac{1}{2} \|u\|_{L^2(s,T; L^\infty(\Omega))}^2 + \frac{1}{2} N_1. \quad (3.19)$$

Combining (3.18) and (3.19) yields

$$\int_s^T \|u\|_{L^\infty(\Omega)}^2 \, d\tau \leq C_2 \|\nabla u\|_{L^2(s,T; L^2(\Omega))}^2 \ln \Psi(T) + C_2. \quad (3.20)$$

This finishes the proof of the proposition. \( \Box \)

With the estimate (3.9) at hand, one can prove the following estimate.

**Proposition 3.2.** It holds that

$$\sup_{0 < T < T^*} \left\{ \|\nabla u\|^2_{L^2(\Omega)} + \int_0^T \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 \, dt + \int_0^T \|\nabla u\|^2_{H^1(\Omega)} \, dt \right\} < +\infty. \quad (3.21)$$

**Proof.** Multiplying the first equation of (3.2) by $\partial_t u$ and integrating over $\Omega$ lead to

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \rho |u_t|^2 \, dx = - \int_{\Omega} (\rho u \cdot \nabla) u \cdot u_t \, dx. \quad (3.22)$$

By Hölder inequality and Young’s inequality,

$$\left| \int_{\Omega} (\rho u \cdot \nabla) u \cdot u_t \, dx \right| \leq C \|\sqrt{\rho} u_t\|_{L^2(\Omega)} \|u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{2} \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 + C \|u\|_{L^\infty(\Omega)}^2 \|\nabla u\|_{L^2(\Omega)}^2. \quad (3.23)$$

Substituting (3.23) into (3.22) gives

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \rho |u_t|^2 \, dx \leq C \|u\|^2_{L^\infty(\Omega)} \|\nabla u\|^2_{L^2(\Omega)}. \quad (3.24)$$

Hence, for every $0 \leq s < T < T^*$,

$$\|\nabla u(T)\|_{L^2(\Omega)}^2 + \int_s^T \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 \, d\tau \leq \|\nabla u(s)\|_{L^2(\Omega)}^2 \exp \left\{ C \int_s^T \|u\|^2_{L^\infty(\Omega)} \, d\tau \right\}. \quad (3.25)$$

Consequently,

$$\Psi(T) \leq \Psi(s) \exp \left\{ C \int_s^T \|u\|^2_{L^\infty(\Omega)} \, d\tau \right\} + \int_0^s \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 \, d\tau + \Psi(s) \quad (3.26)$$

$$\leq 3 \Psi(s) \exp \left\{ C \int_s^T \|u\|^2_{L^\infty} \, d\tau \right\}. \quad (3.26)$$
This, together with Proposition 3.1 gives
\[ \Psi(T) \leq 3\Psi(s) \exp \left\{ C_3 \| \nabla u \|^2_{L^2(s,T;L^2(\Omega))} \ln \Psi(T) + C_3 \right\} \]
\[ \leq C\Psi(s)\Psi(T)^{C_3\| \nabla u \|^2_{L^2(s,T;L^2(\Omega))}}. \]
Recalling the basic energy inequality, one can choose some \( s_0 \) close enough to \( T^* \), such that
\[ C_3\| \nabla u \|^2_{L^2(s_0,T;L^2(\Omega))} \leq \frac{1}{2}. \]
Therefore, for every \( s_0 < T < T^* \), one has
\[ \Psi(T) \leq C\Psi(s_0)^2 < +\infty. \]
Combining the estimates in Proposition 3.1 and Proposition 3.2 yields
\[ \sup_{0 < T < T^*} \int_0^T \| u \|^2_{L^\infty(\Omega)} \, dt < +\infty. \]
For every \( T \in (0, T^*) \), it follows from the inequality (3.17) that
\[ \int_0^T \| \nabla^2 u \|^2_{L^2(\Omega)} \, dt \leq C \left( \| u \|^2_{L^2(0,T;H^1(\Omega))} + \Psi(T) + \| u \|^2_{L^2(s,T;L^\infty(\Omega))} \Psi(T) \right) < +\infty. \]
This finishes the proof of the proposition. \( \square \)

Step 4. Estimates for \( \| \sqrt{\rho} u_t \|^2_{L^\infty(0,T;L^2(\Omega))} \) and \( \| \nabla u_t \|^2_{L^2(0,T;L^2(\Omega))} \). These estimates can be stated as the following proposition.

**Proposition 3.3.** Suppose that \((\rho, u)\) is a local strong solution to the problem (1.1)–(1.2), it holds that
\[ \sup_{0 < T < T^*} \left\{ \| \sqrt{\rho} u_t \|^2_{L^2(\Omega)} + \| u_t \|^2_{H^1(\Omega)} + \int_0^T \| \nabla u_t \|^2_{L^2(\Omega)} \, dt \right\} < +\infty. \]

**Proof.** Taking the derivative of the first equation in (3.2) with respect to \( t \), gives
\[ \rho u_t + (\rho u \cdot \nabla) u_t - \Delta u_t + \nabla P_t = -\rho_t u_t - (\rho_t u \cdot \nabla) u - (\rho u_t \cdot \nabla) u. \]
Multiplying (3.32) by \( u_t \) and integrating over \( \Omega \) yield
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} |\nabla u_t|^2 \, dx \]
\[ = -\int_{\Omega} \rho_t |u_t|^2 \, dx - \int_{\Omega} (\rho_t u \cdot \nabla) u \cdot u_t \, dx - \int_{\Omega} (\rho u_t \cdot \nabla) u \cdot u_t \, dx. \]
One can estimate the three terms on the right-hand side of (3.32) one by one. Taking the second equation of (3.2) into account, and using Gagliardo-Nirenberg inequality yield

\[
- \int_\Omega \rho_t |u_t|^2 \, dx = \int_\Omega \text{div} (\rho u) |u_t|^2 \, dx
\]

\[
= -2 \int_\Omega \rho u \cdot \nabla u_t \cdot u_t \, dx
\]

(3.33)

\[
\leq C \|u\|_{L^6(\Omega)} \|\nabla u_t\|_{L^2(\Omega)} \|\rho u_t\|_{L^3(\Omega)}
\]

\[
\leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla u_t\|_{L^2(\Omega)} \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2
\]

\[
\leq \frac{1}{16} \|\nabla u_t\|_{L^2(\Omega)}^2 + C \|\sqrt{\rho} u_t\|_{L^2(\Omega)}^2 \|\nabla u\|_{L^2(\Omega)}^4.
\]

For the second term of (3.32), one has

\[
- \int_\Omega (\rho \cdot \nabla) u \cdot u_t \, dx = \int_\Omega \text{div} (\rho u) (u \cdot \nabla) u \cdot u_t \, dx
\]

\[
= - \int_\Omega (\rho u) \cdot \nabla [(u \cdot \nabla) u \cdot u_t] \, dx
\]

\[
\leq \int_\Omega |\rho u_t| |u| \|\nabla u\|^2 \, dx + \int_\Omega |\rho u_t| |u|^2 |\nabla^2 u| \, dx
\]

\[
+ \int_\Omega \rho |u|^2 |\nabla u| \|\nabla u_t| \, dx.
\]

By Sobolev inequality, one has

\[
\int_\Omega |\rho u_t| |u| \|\nabla u\|^2 \, dx \leq \|\rho\|_{L^\infty(\Omega)} \|u_t\|_{L^6(\Omega)} \|u\|_{L^6(\Omega)} \|\nabla u\|_{L^3(\Omega)}^2
\]

(3.34)

\[
\leq C \|\nabla u\|_{L^2(\Omega)}^2 \|\nabla u_t\|_{L^2(\Omega)} \|\nabla u\|_{H^1(\Omega)}
\]

\[
\leq \frac{1}{16} \|\nabla u_t\|_{L^2(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)}^4 \|\nabla u\|_{H^1(\Omega)}^2.
\]

Using Hölder and Young’s inequalities gives

\[
\int_\Omega |\rho u_t| |u|^2 |\nabla^2 u| \, dx \leq C \|\rho\|_{L^\infty(\Omega)} \|u_t\|_{L^6(\Omega)} \|u\|_{L^6(\Omega)} \|\nabla^2 u\|_{L^2(\Omega)}
\]

\[
\leq C \|\nabla u_t\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla^2 u\|_{H^1(\Omega)}
\]

(3.35)

\[
\leq \frac{1}{16} \|\nabla u_t\|_{L^2(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)}^4 \|\nabla u\|_{H^1(\Omega)}^2.
\]

Similarly, one has

\[
\int_\Omega \rho |u|^2 |\nabla u| \|\nabla u_t| \, dx \leq \frac{1}{16} \|\nabla u_t\|_{L^2(\Omega)}^2 + C \|\nabla u\|_{L^2(\Omega)}^4 \|\nabla u\|_{H^1(\Omega)}^2.
\]

(3.36)
For the third term on the right-hand side of (3.32), similar to the estimate in (3.33), one has

\[
(3.37) \quad - \int_{\Omega} (\rho \mathbf{u}_t \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_t \, dx \leq \frac{1}{16} \| \nabla \mathbf{u}_t \|_{L^2(\Omega)}^2 + C \| \sqrt{\rho} \mathbf{u}_t \|_{L^2(\Omega)}^2 \| \nabla \mathbf{u} \|_{L^2(\Omega)}^4.
\]

Therefore, collecting all the estimates (3.32)-(3.37) and taking (3.21) into account yield

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\sqrt{\rho} \mathbf{u}_t|^2 \, dx + \frac{1}{4} \int_{\Omega} |\nabla \mathbf{u}_t|^2 \, dx \leq C \| \nabla \mathbf{u}_t \|_{L^2(\Omega)}^2 \| \sqrt{\rho} \mathbf{u}_t \|_{L^2(\Omega)}^2 + C \| \nabla \mathbf{u} \|_{H^1(\Omega)}^2 \| \nabla \mathbf{u} \|_{L^2(\Omega)}^4.
\]

This, together with Gronwall’s inequality, shows

\[
(3.38) \quad \sup_{0 \leq t < T^*} \left( \int_{\Omega} |\sqrt{\rho} \mathbf{u}_t|^2 \, dx + \int_{0}^{T} \int_{\Omega} |\nabla \mathbf{u}_t|^2 \, dx \, dt \right) < +\infty.
\]

Furthermore, the second equation of (3.2), together with Lemma 2.3, gives

\[
\| \mathbf{u} \|_{H^2(\Omega)} \leq C \| \mathbf{u} \|_{H^1(\Omega)} + C \| \rho \mathbf{u}_t \|_{L^2(\Omega)} + C \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{L^2(\Omega)}
\]

\[
\leq C \| \mathbf{u} \|_{H^1(\Omega)} + C \| \sqrt{\rho} \mathbf{u}_t \|_{L^2(\Omega)} + C \| \rho \|_{L^\infty(\Omega)} \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{L^2(\Omega)}
\]

\[
\leq C \| \mathbf{u} \|_{H^1(\Omega)} + C \| \sqrt{\rho} \mathbf{u}_t \|_{L^2(\Omega)} + C \| \mathbf{u} \|_{L^5(\Omega)} \| \nabla \mathbf{u} \|_{L^2(\Omega)}
\]

\[
\leq C \| \mathbf{u} \|_{H^1(\Omega)} + C \| \sqrt{\rho} \mathbf{u}_t \|_{L^2(\Omega)} + C \| \nabla \mathbf{u} \|_{L^2(\Omega)}^{\frac{3}{2}} \| \mathbf{u} \|_{H^1(\Omega)}^{\frac{1}{2}}.
\]

It follows from Young’s inequality and the bounds for \( \| \mathbf{u} \|_{L^\infty(0,T;H^1(\Omega))} \) and \( \| \sqrt{\rho} \mathbf{u}_t \|_{L^\infty(0,T;L^2(\Omega))} \) that

\[
\sup_{0 \leq t < T^*} \| \mathbf{u}(\cdot, T) \|_{H^2(\Omega)} < +\infty.
\]

Hence, the proof of Proposition 3.3 is completed. \(\square\)

**Step 5.** Estimates for \( \| \nabla \rho \|_{L^\infty(0,T;H^1(\Omega))} \) and \( \| \rho_t \|_{L^\infty(0,T;H^1(\Omega))} \). These estimates can be summarized as follows.

**Proposition 3.4.** It holds that

\[
(3.39) \quad \sup_{0 < t < T^*} \left( \| \nabla \rho \|_{L^\infty(0,T;H^1(\Omega))} + \| \rho_t \|_{L^\infty(0,T;H^1(\Omega))} \right) < +\infty.
\]

**Proof.** Differentiating the second equation of (3.2) with respect to \( x_j \ (j = 1, 2, 3) \) yields

\[
(\rho_{x_j})_t + \mathbf{u} \cdot \nabla \rho_{x_j} = -\mathbf{u}_{x_j} \cdot \nabla \rho.
\]

Multiplying the resulting equation by \( \rho_{x_j} \), integrating over \( \Omega \), and summing up gives

\[
\frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 \, dx \leq C \int_{\Omega} |\nabla \mathbf{u}| |\nabla \rho|^2 \, dx \leq C \| \nabla \mathbf{u} \|_{L^\infty(\Omega)} \| \nabla \rho \|_{L^2(\Omega)}^2.
\]
A similar argument shows that
\[
\frac{d}{dt} \int_\Omega |\nabla^2 \rho|^2 \, dx \leq C \int_\Omega |\nabla u| |\nabla^2 \rho|^2 \, dx + \int_\Omega |\nabla^2 u| |\nabla \rho| |\nabla^2 \rho|\, dx
\]
\[
\leq C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla^2 \rho\|_{L^2(\Omega)}^2 + \|\nabla^2 u\|_{L^6(\Omega)} \|\nabla \rho\|_{L^3(\Omega)} \|\nabla^2 \rho\|_{L^2(\Omega)}.
\]
It follows from Sobolev embedding inequality and Gronwall’s inequality that
\[
\|\nabla \rho\|_{H^1(\Omega)}^2 \leq C \|\nabla \rho_0\|_{H^1(\Omega)}^2 \exp \left(C \int_0^T \|\nabla u\|_{W^{1,6}(\Omega)} \, dt\right).
\]
Herein, by Lemma 2.3,
\[
\|\nabla u\|_{W^{1,6}(\Omega)} \leq C \|u\|_{W^{1,6}(\Omega)} + C \|\rho_0\|_{L^6(\Omega)} + C \|\rho_0 \cdot \nabla u\|_{L^6(\Omega)}
\]
\[
\leq C \|u\|_{H^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega)} + C \|u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^6(\Omega)}
\]
\[
\leq \|u\|_{H^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega)} + C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla u\|_{H^1(\Omega)}.
\]
By Proposition 3.3 one has
\[
\sup_{0 \leq t < T^*} \|\nabla \rho\|_{H^1(\Omega)} < +\infty.
\]
Moreover, according to the second equation of (3.2) and Proposition 3.3 it holds that
\[
\sup_{0 \leq t < T^*} \|\rho_t\|_{H^1(\Omega)} \leq \sup_{0 \leq t < T^*} \|u \cdot \nabla \rho\|_{H^1(\Omega)} < +\infty.
\]
This finishes the proof of Proposition 3.4.

Combining all the estimates in (3.21), (3.31) and (3.39) yields (3.1). This, together with Theorem 4.6 shows that the local strong solution \((\rho, u)\) does not blow up at \(T^*\). Hence, \((\rho, u)\) is in fact a global solution so that the proof of Theorem 1.1 is completed.

4. Local well-posedness of the strong solutions in the exterior domains

The local existence and uniqueness of strong solutions for inhomogeneous Navier-Stokes system in bounded domains has been proved in [8] by Galerkin approximation. In this section, the local existence of strong solutions in the exterior domain is established as a limit of local strong solutions in a sequence of bounded domains constructed in [8]. To prove the convergence of approximate solutions, one of the key observations is that the lifespan of each local strong solution depends only on the \(C^3\)-regularity of the domain, \(\|\rho_0\|_{L^\infty(\Omega)}\), \(\|\rho_0 - \bar{\rho}\|_{L^2(\Omega)}\), \(\bar{\rho}^{-1}\), \(\|\mathbf{u}_0\|_{H^1(\Omega)}\), but is independent of the size of \(\partial \Omega\) and \(\Omega\). This observation relies heavily on the uniform estimates for the Stokes problem (cf. Lemma 2.3).
In this section, assume that $\bar{\Omega}$ is a bounded domain of $\mathbb{R}^3$, the boundary of which is uniformly of class $C^3$. And for simplicity of notations, denote $L^r = L^r(\bar{\Omega})$, $W^{k,p} = W^{k,p}(\bar{\Omega})$, $H^k = H^k(\bar{\Omega})$, and
\[
\int f \, dx = \int_{\bar{\Omega}} f \, dx.
\]

Given initial data $(\rho_0, u_0) \in L^\infty \times (H^2 \cap H^3_{0,\sigma})$, suppose that $(\rho_0, u_0)$ satisfy
\[
\rho_0 \geq 0, \quad \rho_0 - \bar{\rho} \in H^2 \cap L^\infty, \quad -\mu \Delta u_0 + \nabla P_0 = \rho_0^{-\frac{1}{2}} g, \quad \text{in } \bar{\Omega},
\]
with some $(P_0, g)$ belonging to $D^{1,2} \times L^2$. The following is the local existence result proved in [8].

**Lemma 4.1.** Assume that the initial data $(\rho_0, u_0)$ satisfies $\rho_0 \in L^\infty$, $u_0 \in H^3_{0,\sigma} \cap H^2$ and (4.1). There exists a positive time $T_0$ and a unique strong solution $(\rho, u)$ to the initial boundary value problem (1.1) - (1.2) such that
\[
\rho - \bar{\rho} \in C([0, T_0); L^\infty \cap H^2), \quad u \in C([0, T_0); H^3_{0,\sigma}) \cap L^\infty(0, T_0; H^2), \quad \rho_t \in L^\infty(0, T_0; H^1), \quad \sqrt{\rho} u_t \in L^\infty(0, T_0; L^2), \quad \nabla u_t \in L^2(0, T_0; L^2).
\]

Next, one can prove that $T_0$ has a uniform lower bound, which depends only on $\|\rho_0 - \bar{\rho}\|_{L^\frac{3}{2}}$, $\|\rho_0\|_{L^\infty}$, $\bar{\rho}^{-1}$, $||u_0||_{H^1}$ and the $C^3$-regularity of $\partial \bar{\Omega}$. Note that the lower bound does not depend on the size of $\partial \bar{\Omega}$ or $\bar{\Omega}$. And also some uniform estimates for solutions independent of the size of $\partial \bar{\Omega}$ and $\bar{\Omega}$, can be established.

**Lemma 4.2.** For every $0 \leq t < T_0$, it holds that
\[
\|\rho(\cdot, t) - \bar{\rho}\|_{L^\frac{3}{2}} = \|\rho_0 - \bar{\rho}\|_{L^\frac{3}{2}}, \quad \text{and} \quad \|\rho(\cdot, t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}.
\]

**Lemma 4.3.** There exists some constant $C$, which depends on $\|\rho_0 - \bar{\rho}\|_{L^\frac{3}{2}}$, $\|\sqrt{\rho_0} u_0\|_{L^2}$, $\bar{\rho}^{-1}$, such that for every $0 < T < T_0$,
\[
\|\sqrt{\rho} u\|_{L^\infty(0, T; L^2)} + \|u\|_{L^2(0, T; H^1)} \leq C.
\]

The proofs of Lemmas 4.2, 4.3 are the same as those for (3.3) - (3.5).

**Lemma 4.4.** There exist some positive constants $C$ and $T_0^*$, which depend on $\|\rho_0\|_{L^\infty}$, $\|\rho_0 - \bar{\rho}\|_{L^\frac{3}{2}}$, $\|u_0\|_{H^1}$, $\bar{\rho}^{-1}$ and $C^3$-regularity of $\partial \bar{\Omega}$, such that for every $0 < T < T_0^*$,
\[
\sup_{0 \leq t < T} \|u\|_{H^1}^2 + \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt + \int_0^T \|u\|_{H^2}^2 \, dt \leq C.
\]
Proof. Multiplying the first equation of (3.2) by $\partial_t u$ and integrating yield
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 \, dx + \int \rho |u_t|^2 \, dx = - \int (\rho u \cdot \nabla) u \cdot u_t \, dx.
\]
By Sobolev embedding inequality and Young’s inequality,
\[
\left| \int (\rho u \cdot \nabla) u \cdot u_t \, dx \right| \leq \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho} u_t \|_{L^2} \| u \|_{L^\infty} \| \nabla u \|_{L^2}
\]
(4.5)
\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} \| u \|_{H^\frac{3}{2}} \| u \|_{H^\frac{3}{2}}
\]
\[
\leq \frac{1}{2} \| \sqrt{\rho} u_t \|_{L^2}^2 + C \| u \|_{H^1}^3 \| u \|_{H^2}.
\]
Herein, using Lemma 2.3 gives
\[
\| u \|_{H^2} \leq C \| \rho u_t \|_{L^2} + C \left( \| \rho u \cdot \nabla \|_{L^2} + C \| u \|_{H^1} \right)
\]
(4.6)
\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u \|_{L^2} + C \| u \|_{H^1}
\]
\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| u \|_{H^1}^\frac{3}{2} \| u \|_{H^1}^\frac{3}{2} + C \| u \|_{H^1}.
\]
It follows from Young’s inequality and (3.7) that
\[
\| u \|_{H^2} \leq C \left( \| \sqrt{\rho} u_t \|_{L^2} + \| u \|_{H^1}^\frac{3}{2} + \| u \|_{H^1} \right)
\]
(4.7)
\[
\leq C \left( \| \sqrt{\rho} u_t \|_{L^2} + \left[ \rho^{-1} \int \rho |u|^2 \, dx + \rho - \bar{\rho} \right] \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right)^\frac{3}{2} + 1
\]
\[
\leq C \left( \| \sqrt{\rho} u_t \|_{L^2} + \| \nabla u \|_{L^2}^2 + 1 \right).
\]
Hence, substituting (4.7) and (3.7) into (4.5) gives
\[
\frac{d}{dt} \int |\nabla u|^2 \, dx + \int \rho |u_t|^2 \, dx \leq C_3 \left(1 + \| \nabla u \|_{L^2}^2 \right)^3.
\]
(4.8)
Let $T_0^* = \frac{1}{8C_3 \left(1 + \| \nabla u_0 \|_{L^2}^2 \right)}$. It follows from (4.8) that
\[
\sup_{0 \leq t \leq \min \{ T_0, T_0^* \}} \left( \| \nabla u \|_{L^2}^2 + 1 \right) + \int_0^{\min \{ T_0, T_0^* \}} \int \rho |u_t|^2 \, dx \, dt \leq 2 \left( \| \nabla u_0 \|_{L^2}^2 + 1 \right).
\]
(4.9)
According to the blow up criterion obtained in [28], the estimate (4.9) implies that the local strong solution does not blow up before the time $T_0^*$, i.e., $T_0 \geq T_0^*$.
Moreover, combining (4.9) and (4.7) together gives that
\[
\sup_{0 < T < T_0} \int_0^T \| u \|_{H^2}^2 \, dt \leq C.
\]
(4.10)
Thus the proof of the lemma is completed. \qed
Lemma 4.5. There exists a constant $C$, which depends on the $C^3$-regularity of $\partial\tilde{\Omega}$, $\|\rho_0\|_{L^\infty}$, $\|\rho_0 - \bar{\rho}\|_{L^2}$, $\|\rho_0 - \bar{\rho}\|_{H^2}$, $\|u_0\|_{H^2}$, $\bar{\rho}^{-1}$, $T_0^*$, such that for every $0 < T < T_0^*$,

$$
\sup_{0 \leq T \leq T_0^*} \left\{ \|\sqrt{\rho} u_t\|_{L^2} + \|u\|_{H^2} + \|\nabla\rho\|_{H^1} + \int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \right\} \leq C.
$$

The proof for Lemma 4.5 follows exactly the same as that for Propositions 3.3–3.4.

Now we are in position to prove the local existence of strong solutions for the inhomogeneous Navier-Stokes system in an exterior domain.

Theorem 4.6. Assume that $(\rho_0, u_0)$ satisfies conditions (1.7) and (1.4), then there exist a positive time $T_0^*$ and a unique strong solution $(\rho, u)$ to the initial boundary value problem (1.1)–(1.2) satisfying

$$
\rho - \bar{\rho} \in C([0, T_0^*); H^2(\Omega)), \quad u \in C([0, T_0^*); H^1(\Omega) \cap L^\infty(0, T_0^*); H^2(\Omega)),
$$

$$
\rho_t \in L^\infty(0, T_0^*; H^1(\Omega)), \quad \sqrt{\rho} u_t \in L^\infty(0, T_0^*; L^2(\Omega)), \quad \nabla u_t \in L^2(0, T_0^*; L^2(\Omega)).
$$

Proof. Given $k \in \mathbb{N}$, let $\Omega_k := \Omega \cap \{|x| < k\}$. In each domain $\Omega_k$, choose the initial density and velocity $(\rho_{0,k}, u_{0,k})$, which satisfy that

$$
\rho_{0,k} = \rho_0 + \epsilon_k, \quad \text{with} \quad \lim_{k \to \infty} \|\epsilon_k\|_{H^2(\Omega_k)} = 0, \quad \inf_{\Omega_k} \epsilon_k > 0,
$$

$$
(4.11) \quad u_{0,k} \in H^2(\Omega_k) \cap H^1_{0,\sigma}(\Omega_k), \quad \text{with} \quad \|u_{0,k}\|_{H^2(\Omega_k)} \leq 2 \|u_0\|_{H^2(\Omega)}.
$$

and $u_{0,k}$ converges to $u_0$ in $H^2(\Omega')$, for each compact subdomain $\Omega'$.

By Lemma 4.1, for each $k \in \mathbb{N}$, there exists a unique strong solution $(\rho_k, u_k)$ to the equations (1.1) with the initial data $(\rho_{0,k}, u_{0,k})$ over some time interval $[0, T_k)$. As proved above, there exists a positive time $T_0^*$, which depends only on the $C^3$-regularity of $\partial\Omega_k$, $\|\rho_{0,k} - \bar{\rho}\|_{L^2(\Omega_k)}$, $\bar{\rho}^{-1}$, $\|\rho_{0,k}\|_{L^\infty(\Omega_k)}$, $\|u_{0,k}\|_{H^2(\Omega_k)}$, such that

$$
T_k \geq T_0^*.
$$

It means that the lifespans of the approximate solutions $(\rho_k, u_k)$ have a uniform lower bound $T_0^*$. Moreover, as proved above, there exists some constant $C$ which does not depend on the size of $\Omega$ or $\partial\Omega$, such that

$$
(4.12) \quad \sup_{0 \leq T \leq T_0^*} \left( \|\rho_k - \bar{\rho}\|_{L^2(\Omega_k)} + \|\rho_k - \bar{\rho}\|_{H^2(\Omega_k)} + \|\partial_t \rho_k\|_{H^1(\Omega_k)} + \|u_k\|_{H^2(\Omega_k)} \right) \leq C,
$$

and

$$
(4.13) \quad \int_0^{T_0^*} \|\nabla \partial_t u_k\|_{L^2(\Omega_k)}^2 \, dt \leq C.
$$
Hence, there exists a subsequence of \((\rho_k, u_k)\) (which is still labelled by \((\rho_k, u_k)\)), and the limit function \((\rho, u)\), such that for every compact subdomain \(\Omega'\),

\[
\rho_k - \bar{\rho} \rightharpoonup^{*} \rho - \bar{\rho} \text{ in } L^\infty(0, T^*_0; H^2(\Omega')), \quad u_k \rightharpoonup^{*} u \text{ in } L^\infty(0, T^*_0; H^2(\Omega')),
\]

\[
\partial_t \rho_k \rightharpoonup^{*} \partial_t \rho \text{ in } L^\infty(0, T^*_0; H^1(\Omega')),
\]

\[
\nabla \partial_t u_k \rightharpoonup^{*} \nabla \partial_t u \text{ in } L^2(0, T^*_0; L^2(\Omega')).
\]

and

\[
\sup_{0 \leq T \leq T^*_0} \left( \| \rho - \bar{\rho} \|_{L^2(\Omega)} + \| \rho - \bar{\rho} \|_{H^2(\Omega)} + \| u \|_{H^2(\Omega)} + \| \rho_t \|_{H^1(\Omega)} \right) \leq C,
\]

\[
\int_0^{T^*_0} \| \nabla u_t \|_{L^2(\Omega)}^2 \, dt \leq C.
\]

By Aubin-Lions Lemma,

\[
\rho_k - \bar{\rho} \rightarrow \rho - \bar{\rho} \text{ in } C([0, T^*_0]; H^1(\Omega')), \quad u_k \rightarrow u \text{ in } C([0, T^*_0]; H^1(\Omega')).
\]

Hence, \((\rho, u)\) is a strong solution to the initial value problem (1.1)–(1.2). Furthermore, it follows from the equations that \(\rho - \bar{\rho} \in C([0, T^*_0]; H^2(\Omega))\). The proof of uniqueness is now standard and one can refer to [8] for details. Hence, the proof of Theorem 4.6 is completed. \(\square\)

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