Optimal quantum repeaters with doubly entangled state purification

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Recently, Briegel et al. present a quantum repeaters protocol using nest entanglement purification for long distance quantum communication (Physical Review Letters \textbf{81},5932). In this paper we present a modified scheme for constructing an optimal quantum repeater using doubly entangled photon pairs that overcomes the limitations of quantum communication via noisy channels. Based on the imperfect quantum operations and classical communications, the polarization entangled quantum channels can be built with a better efficiency.

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I. INTRODUCTION

The idea of quantum key distribution (QKD) was first proposed by Bennett and Brassard in the year 1984 \[1\]. Later, it is proved to be unconditional secure \[2, 3\]. Two fundamental principles, the uncertain principle and non-cloning theorem, protect the quantum signal not to be eavesdropped without been discovered by the authorized parties. QKD also provides such a secure way for creating private keys with which Alice and Bob can exchange the secret message securely using the proven secure cryptosystem and it has progressed quickly in the following decades \[4, 5, 6, 7, 8, 9, 10\]. The main obstacle of long distance quantum communication is the loss of quantum signals in noisy channels between the communication parties. After long time research of experimental quantum communication, the length of quantum communication using single photons is bounded at about 150km both in free space \[11\] and in optical fibers \[12\]. The number of trials of quantum signals is exponentially enlarged as the length of the channel increased. Another problem of long distance quantum communication is the increasing of bit error rate in accordance with the enlarging of the channel distance. Thus the fidelity of the quantum state is limited in quantum communication.

In 1998, Briegel et al. proposed the idea of quantum repeaters as the role of imperfect local operations in quantum communication which overcomes the error problems and the communication distance problems in quantum communication via noisy channels \[13, 14\]. The protocol utilizes entangled purification protocol that was proposed by Bennett in which the Werner states can be purified. They introduce the imperfect local operations to build long distance quantum channel by connecting the distant nodes. An entangled photon channel is established by such nested purification protocols. Then the long distance quantum channel can be built with a high fidelity.

Later, Duan et al. proposed the theoretical protocol of long distance QKD using atomic ensembles as quantum repeaters \[16\]. Recently, remarkable progress has been reported on quantum repeaters \[17, 18, 19\]. Meanwhile, Collins et al. proposed the quantum relay protocols to overcome the problems in long distance quantum communications \[20\] and the related topics also progressed quickly \[21, 22\].

In this study, we first introduce the idea of polarization and frequency doubly entangled photon states (DEPs) to realize the photon transmission and entanglement purification process and then present an optimal quantum repeater protocol that allows the creation of quantum channels using DEPs over arbitrary long distance. The paper is organized as follows: in section \(\text{III}\) we introduce the DEPs purification protocol to construct the optimal quantum repeater protocol by imperfect local operations. In section \(\text{IV}\) we analyze the efficiency and the main features of the optimal quantum repeaters protocol utilize the generic error model. The last section is our conclusion and acknowledgement.

II. QUANTUM REPEATERS WITH DEPS PURIFICATION PROTOCOL

Entanglement purification protocol based on DEPs and its modified version were proposed by Xiao and Wang et al \[23, 24\]. This protocol consists of two steps: first
step is the bit-flip error correction process and second step is the phase-flip error correction process.

The doubly entangled photon state was experimentally realized by Ravaro et al. [25] and it can be written in the formula:

$$|\Phi^+_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|H, \omega_i\rangle + |V, \omega_s\rangle|V, \omega_i\rangle),$$  \hspace{1cm} (1)

here $|H\rangle$ and $|V\rangle$ represent the horizontal and the vertical polarizations of the photons respectively, $\omega_s(i')$ and $\omega_i(i')$ correspond to the frequencies of entangled photons. The transmission of DEPs through noisy channels in quantum communication will cause the polarizations of each photon effected by the channel noise. Then the initial state will change to a Werner state which is represented as

$$\rho = F|\Phi^+_{ab}\rangle\langle \Phi^+_{ab}| + \frac{1-F}{2}|\Phi_{ab}\rangle\langle \Phi_{ab}| + \frac{1-F}{2}|\Psi^\pm_{ab}\rangle\langle \Psi^\pm_{ab}|$$
$$+ \frac{1-F}{2}|\Gamma^\pm_{ab}\rangle\langle \Gamma^\pm_{ab}| + \frac{1-F}{2}|\Upsilon^\pm_{ab}\rangle\langle \Upsilon^\pm_{ab}|,$$ \hspace{1cm} (2)

here the coefficient $F = \langle \Phi^+_{ab}|\rho|\Phi^+_{ab}\rangle$ is the fidelity of the state $\rho$ relative to $|\Phi^+_{ab}\rangle$ and the eight states in Equ.(2) are

$$|\Phi^+_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|H, \omega_i\rangle \pm |V, \omega_s\rangle|V, \omega_i\rangle);$$ \hspace{1cm} (3)

$$|\Psi^\pm_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|V, \omega_i\rangle \pm |V, \omega_s\rangle|H, \omega_i\rangle);$$ \hspace{1cm} (4)

$$|\Gamma^\pm_{ab}\rangle = \frac{1}{\sqrt{2}}(|V, \omega_s\rangle|H, \omega_i\rangle \pm |H, \omega_s\rangle|V, \omega_i\rangle);$$ \hspace{1cm} (5)

$$|\Upsilon^\pm_{ab}\rangle = \frac{1}{\sqrt{2}}(|V, \omega_s\rangle|V, \omega_i\rangle \pm |H, \omega_s\rangle|H, \omega_i\rangle).$$ \hspace{1cm} (6)

In the entanglement purification process, the devices shown in Fig.1 is used. In this step, two photons are sent to the left and right WDMs respectively. The state has no bit-flip errors if the two photons come out of port 1 and port 3, respectively. When a photon comes out of the lower spatial mode (port 2 or port 4), a bit-flip error takes place. Then the two HWPs on port 2 and port 4 is used to correct the bit-flip errors and the DEPs degenerate to Bell state. The DEPs with phase-flip errors can be further distilled in the same way as in the existing schemes, as Ref. 24, 27.

After the first step, the fidelity $F$ of the state $|\Phi^+\rangle$ increased to $(4F + 3)/7$. When the two steps operations finish, the state becomes to

$$\rho''' = F'\rho'_{ab} + (1-F')|\Psi^+_{ab}\rangle\langle \Psi^+_{ab}|,$$ \hspace{1cm} (7)

here the fidelity

$$F' = \frac{(4F + 3)^2}{32F^2 - 8F + 25}.$$ \hspace{1cm} (8)

If the initial coefficient $(4F + 3)/7$ is larger than $1/2$, which means that the initial fidelity $F > 1/8$, the fidelity $F'$ after purification is larger than $(4F + 3)/7$ and the entanglement purification succeeds. So we can perform these purification operations iteratively to increase the fidelity of the ensemble.

Long distance quantum communication can be realized by quantum repeaters between distant locations in the communication network. We have learned the nested purification protocol proposed by Briegel et al. and improved using our DEPs purification protocols. The scheme of nest purification quantum repeater is shown in Fig.2. The two remote parties Alice(A) and Bob(B) need to build an entangled quantum channel. There are many nodes at distant locations between them and each node performs an entanglement purification protocol and a Bell state measurement on their two particles. Then Alice and Bob could build their quantum channel with a better fidelity.

In our protocol, each neighboring two nodes share a pair of DEPs in the state $|\Phi^+_{ab}\rangle$. The procedure of entanglement distribution is realized by transferring one par-
ticle of DEPs through the channels. By applying DEPs entanglement purification protocol on the state between the nodes, the polarization and frequency doubly entangled photon state degenerate to Bell state. After that, Alice and $C_1$, $C_1$ and $C_2$, ..., $C_{n-1}$ and Bob are connected by DEPs. Then the channel can be connected by the method of entanglement swapping. For example, at the first level, $C_1$ apply Bell state measurement on his two particles and announce the results, as a result, Alice and $C_2$ are sharing a quantum channel by performing a local unitary operation. The same operations are performed by $C_i$ here $i = 1, 3, 5, ...$. At the second level, the $C_{2i}$ nodes perform the same procedures. The process performs repeatedly by $C_{4i}$, $C_{8i}$ at each level. Finally, only the two photons at Alice and Bob’s side are kept and the quantum channel is established.

III. EFFICIENCY OF LONG DISTANCE QUANTUM COMMUNICATION WITH DEPS QUANTUM REPEATERS

In ideal case, Alice and Bob could establish their quantum channel with high fidelity by perfect operations and measurements. However, in realistic conditions, Alice and Bob’s imperfect operations and measurements will reduce the fidelity of the states in the entanglement purification and Bell state measurement process.

Here in this section we will discuss the efficiency of the quantum repeaters protocol improved by our DEPs purification protocol. The imperfect measurement on the single qubit is described as

$$P_0 = \eta|0\rangle\langle 0| + (1 - \eta)|1\rangle\langle 1|.$$  
$$P_1 = \eta|1\rangle\langle 1| + (1 - \eta)|0\rangle\langle 0|.$$  

These $|0\rangle$ and $|1\rangle$ represent the quantum state of single photon respectively and $\eta$ is the parameter of the projection quality.

In our purification procedures, single qubit operation and state measurement are needed. The imperfect single qubit operation on the density matrix of the state is described by the map

$$\rho \rightarrow O_1\rho p_1 O_1^{\text{ideal}} \rho + \frac{1 - p_1}{2}tr_1\rho \otimes I_1.$$  

Here $O_1^{\text{ideal}}$ is the ideal operations. $I_1$ is the subspace where the ideal operation acts.

We analyze the two procedures of nest DEPs purification protocol in the error models shown above. The quantum repeater protocol consists two element: entanglement purification and the connection of photon pairs. The protocol takes place where the entanglement channels between the nodes are in the Werner state:

$$\rho = F|\Phi_{ab}^+\rangle\langle \Phi_{ab}^+| + \frac{1 - F}{7}|\Phi_{ab}^-\rangle\langle \Phi_{ab}^-| + \frac{1 - F}{7}|\Psi_{ab}^+\rangle\langle \Psi_{ab}^+| + \frac{1 - F}{7}|\Psi_{ab}^-\rangle\langle \Psi_{ab}^-| + \frac{1 - F}{7}|\Gamma_{ab}^+\rangle\langle \Gamma_{ab}^+| + \frac{1 - F}{7}|\Gamma_{ab}^-\rangle\langle \Gamma_{ab}^-|,$$  

The fidelity of the state is $F = \langle \psi^+|\rho|\psi^+\rangle$.

Considering the imperfect operations, the nodes between Alice and Bob will perform Bell state measurement on their two particles. Then Alice’s and Bob’s particles become entangled with a certain fidelity. Briegel et al. illustrate that the fidelity decrease exponentially with $N$. By connecting the $N$ neighboring pairs.

Note that in the optimal quantum purification protocol, the quantum repeater protocol can be described as follows. In the first step purification process, two half wave plates are operated on the port 2 and 4 to correct the bit-flip errors. In this process, the density matrix $\rho$ of the ensemble changes to $\rho'$.

$$\rho' = F|\Phi_{ab}^+\rangle\langle \Phi_{ab}^+| + \frac{1 - F}{7}|\Phi_{ab}^-\rangle\langle \Phi_{ab}^-| + p_1 \frac{1 - F}{7}|\Phi_{ab}^\pm\rangle\langle \Phi_{ab}^\pm| + p_1 \frac{1 - F}{7}|\Phi_{ab}^\pm\rangle\langle \Phi_{ab}^\pm| + p_1 \frac{1 - F}{7}|\Psi_{ab}^\pm\rangle\langle \Psi_{ab}^\pm| + C(p_1)$$  

Here $C(p_1)$ denotes the errors introduced by the imperfect single qubit operations. Also the density matrix can be written as

$$\rho' = (F + 2p_1 \frac{1 - F}{7} + p_1 \frac{1 - F}{7})(|\Phi_{ab}^\pm\rangle\langle \Phi_{ab}^\pm| + (1 - F) + 2p_1 \frac{1 - F}{7} + p_1 \frac{1 - F}{7})(|\Phi_{ab}^\pm\rangle\langle \Phi_{ab}^\pm| + C(p_1)$$  

Then we perform the second step purification and correct the phase-flip errors. Another pair of entangled photon pairs selected from the ensemble is needed. Then four half-wave plates operate on each of the photons in this process. The density matrix becomes to

$$\rho'' = p_1 F'|\Phi_{ab}^+\rangle\langle \Phi_{ab}^+| + p_1 (1 - F')|\Psi_{ab}^+\rangle\langle \Psi_{ab}^+| + C(p_1)$$

where

$$F' = \frac{(F + 2p_1 \frac{1 - F}{7} + p_1 \frac{1 - F}{7})^2 + (1 - F + 2p_1 \frac{1 - F}{7} + p_1 \frac{1 - F}{7})^2}{(p_1 F'|\eta^2 + (1 - \eta)^2| + p_1 (1 - F')\eta(1 - \eta) + (1 - p_1)^2/8(17)}$$

Using DEPs entanglement purification protocol, $F''$ is solved as the final fidelity of the new pair after the entanglement purification process.
FIG. 3: The efficiency of entanglement purification protocols. Thick line is the DEPs purification efficiency and the dashed line is Bennett’s method for purification at one time in quantum repeaters protocol. Parameters are \( p_1 = p_2 = 0.99, \eta = 1 \).

The efficiency of the purification scheme in quantum repeaters is shown in Fig. 3. We can see from the curve that DEPs purification is better at each time purification. So the efficiency of this protocol is higher than the traditional quantum repeater protocols.

Here we introduce the modified DEPs entanglement purification into the nest purification quantum repeater protocol. Comparing with the traditional protocols, our scheme avoids using the two qubit quantum operations which is difficult to realize in experiment. When Alice and Bob start the long distance quantum communication, they need to build a quantum channel. Using the DEPS nest purification protocols, they first distribute the DEPs between the nodes. Then the nodes \( C_i \) perform the entanglement purifications on the photon pairs at each level and finally they share the state with fidelity \( F'' \). The last step is to perform the Bell state measurement and quantum teleportation on the two particles at the nodes. These steps iterate at the nodes \( C_{2i}, C_{4i}, C_{8i} \) and so on. At last, a high quantum channel is established.

IV. CONCLUSION

In this study, we have proposed an optimal quantum repeaters protocol using DEPs entanglement purification protocol and show that creating DEPs channels via a noisy channel is possible. The additional degree of freedom of the DEPs increases the efficiency of purification, so it makes the two elements in the quantum repeater improved. Compared with the Bennett purification protocols, the efficiency of the quantum repeaters using DEPs purification with imperfect quantum operations are discussed.

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