Sharp estimates for the gradient of the generalized Poisson integral for a half-space

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Dedicated to Vakhtang Kokilashvili on the occasion of his 80th birthday

Abstract. A representation of the sharp coefficient in a pointwise estimate for the gradient of the generalized Poisson integral of a function $f$ on $\mathbb{R}^n$ is obtained under the assumption that $f$ belongs to $L^p$. The explicit value of the coefficient is found for the cases $p = 1$ and $p = 2$.

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1 Introduction

In the paper [3] (see also [6]) a representation for the sharp coefficient $K_p(x)$ in the inequality

$$|\nabla u(x)| \leq K_p(x)\|u\|_p$$

was found, where $u$ is harmonic function in the half-space $\mathbb{R}_+^{n+1} = \{x = (x', x_{n+1}) : x' \in \mathbb{R}^n, x_{n+1} > 0\}$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^n)$, $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $x \in \mathbb{R}_+^{n+1}$. It was shown that

$$K_p(x) = \frac{K_p}{x_{n+1}^{(n+p)/p}}$$

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and explicit formulas for $K_1$ and $K_2$ were given. Namely,
\[ K_1 = \frac{2n}{\omega_{n+1}}, \quad K_2 = \sqrt{\frac{n(n+1)}{2^{n+1}\omega_{n+1}}}, \]
where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$.

In [3] it was shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of a harmonic function in the half-space coincide for the case $p = 1$ as well as for the case $p = 2$.

Similar results for the gradient and the radial derivative of a harmonic function in the multidimensional ball with boundary values from $L^p$ for $p = 1, 2$ in [4] were obtained.

Thus, the $L^1, L^2$-analogues of Khavinson’s problem [1] were solved in [3, 4] for harmonic functions in the multidimensional half-space and the ball.

We note that explicit sharp coefficients in the inequality for the first derivative of analytic function in the half-plane and the disk with boundary values of the real-part from $L^p$ in [2, 5, 7] were found.

In this paper we treat a generalization of the problem considered in our work [3]. Here we consider the generalized Poisson integral
\[ u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^{\alpha}}{|y-x|^{n+\alpha}} f(y')dy' \]
with $f \in L^p(\mathbb{R}^n)$, $\alpha > -(n/p)$, $1 \leq p \leq \infty$, where $x \in \mathbb{R}^{n+1}$, $y = (y',0)$, $y' \in \mathbb{R}^n$, and $k_{n,\alpha}$ is a normalization constant. In the case $\alpha = 1$ the last integral coincides with the Poisson integral for a half-space.

In Section 2 we obtain a representation for the sharp coefficient $C_p(x)$ in the inequality
\[ \left| \nabla u_f(x) \right| \leq C_p(x) \| f \|_p, \]
where
\[ C_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}} \]
and the constant $C_p$ is characterized in terms of an extremal problem on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$.

In Section 3 we reduce this extremal problem to that of finding of the supremum of a certain double integral, depending on a scalar parameter and show that
\[ C_1 = k_{n,\alpha} n \]
if $-n < \alpha \leq n$, and
\[ C_2 = \sqrt{\omega_{n-1}k_{n,\alpha}} \left\{ \frac{\sqrt{\pi(n+\alpha)n(n+2)}\Gamma\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{n}{2} + \alpha\right)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2} \]
if $-(n/2) < \alpha \leq n(n+1)/2$.

It is shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of the generalized Poisson integral for a half-space coincide in the case $p = 1$ as well as in the case $p = 2$. 

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2 Representation for the sharp constant in inequality for the gradient in terms of an extremal problem on the unit sphere

We introduce some notation used henceforth. Let \( \mathbb{R}^{n+1}_+ = \{ x = (x', x_{n+1}) : x' = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_{n+1} > 0 \} \), \( \mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \), \( \mathbb{S}^n_+ = \{ x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} > 0 \} \) and \( \mathbb{S}^n_- = \{ x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} < 0 \} \). Let \( e_\sigma \) stand for the \( n+1 \)-dimensional unit vector joining the origin to a point \( \sigma \) on the sphere \( \mathbb{S}^n \).

By \( || \cdot ||_p \) we denote the norm in the space \( L^p(\mathbb{R}^n) \), that is

\[
||f||_p = \left\{ \int_{\mathbb{R}^n} |f(x')|^p \, dx' \right\}^{1/p},
\]

if \( 1 \leq p < \infty \), and \( ||f||_\infty = \text{ess sup}\{||f(x')|| : x' \in \mathbb{R}^n\} \).

Let the function in \( \mathbb{R}^n_+ \) be represented as the generalized Poisson integral

\[
u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^{\alpha}}{|y - x|^{n+\alpha}} f(y') \, dy', \tag{2.1}
\]

with \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), where \( y = (y',0), y' \in \mathbb{R}^n \),

\[
k_{n,\alpha} = \left\{ \int_{\mathbb{R}^n} \frac{x_{n+1}^{\alpha}}{|y - x|^{n+\alpha}} \, dy \right\}^{-1} = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)}, \tag{2.2}
\]

and

\[
\alpha > -\frac{n}{p}. \tag{2.3}
\]

Now, we find a representation for the best coefficient \( C_p(x; z) \) in the inequality for the absolute value of derivative of \( \nu_f(x) \) in an arbitrary direction \( z \in \mathbb{S}^n \), \( x \in \mathbb{R}^{n+1}_+ \). In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

**Proposition 1.** Let \( x \) be an arbitrary point in \( \mathbb{R}^{n+1}_+ \) and let \( z \in \mathbb{S}^n \). The sharp coefficient \( C_p(x;z) \) in the inequality

\[
| (\nabla \nu_f(x), z) | \leq C_p(x; z) \| f \|_p
\]

is given by

\[
C_p(x; z) = \frac{C_p(z)}{|x|^{(n+p)/p}}, \tag{2.4}
\]

where

\[
C_1(z) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)|(e_\sigma, e_{n+1})^{n+\alpha}, \tag{2.5}
\]

\[
C_p(z) = k_{n,\alpha} \left\{ \int_{\mathbb{S}^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)|^p (e_\sigma, e_{n+1})^{(n-1)p(n+1)} \, d\sigma \right\}^{\frac{1}{p}}, \tag{2.6}
\]

for \( 1 < p < \infty \), and

\[
C_\infty(z) = k_{n,\alpha} \int_{\mathbb{S}^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)|(e_\sigma, e_{n+1})^{\alpha-1} \, d\sigma. \tag{2.7}
\]
In particular, the sharp coefficient \( C_p(x) \) in the inequality
\[
|\nabla u_f(x)| \leq C_p(x) \| f \|_p
\]
is given by
\[
C_p(x) = \frac{C_p}{x^{(n+p)/p}},
\]
where
\[
C_p = \sup_{|z|=1} C_p(z).
\]

**Proof.** Let \( x = (x', x_{n+1}) \) be a fixed point in \( \mathbb{R}^{n+1}_+ \). The representation (2.11) implies
\[
\frac{\partial u_f}{\partial x_i} = k_{n, \alpha} \int_{\mathbb{R}^n} \left[ \frac{\alpha e_{n+1}}{|y - x|^{n+\alpha}} + \frac{(n + \alpha)x_{n+1}(y_i - x_i)}{|y - x|^{n+2+\alpha}} \right] f(y') dy',
\]
that is
\[
\nabla u_f(x) = k_{n, \alpha}x_{n+1}^{-1} \int_{\mathbb{R}^n} \left[ \frac{\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z}{|y - x|^{n+\alpha}} \right] f(y') dy',
\]
where \( e_{xy} = (y - x)|y - x|^{-1} \). For any \( z \in S^n \),
\[
(\nabla u_f(x), z) = k_{n, \alpha}x_{n+1}^{-1} \int_{\mathbb{R}^n} \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z)}{|y - x|^{n+\alpha}} f(y') dy'.
\]
Hence,
\[
C_1(x; z) = k_{n, \alpha}x_{n+1}^{\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z)}{|y - x|^{n+\alpha}} \right|,
\]
and
\[
C_p(x; z) = k_{n, \alpha}x_{n+1}^{\alpha-1} \left\{ \int_{\mathbb{R}^n} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z)}{|y - x|^{(n+\alpha)q}} \right|^q dy \right\}^{1/q}
\]
for \( 1 < p \leq \infty \), where \( p^{-1} + q^{-1} = 1 \).

Taking into account the equality
\[
\frac{x_{n+1}}{|y - x|} = (e_{xy}, -e_{n+1}),
\]
by (2.11) we obtain
\[
C_1(x; z) = k_{n, \alpha}x_{n+1}^{\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z)}{|y - x|^{n+\alpha}} \right| \left( \frac{x_{n+1}}{|y - x|} \right)^{n+\alpha}
\]
\[
= k_{n, \alpha} \sup_{x_{n+1}} \left| \frac{(\alpha e_{n+1} - (n + \alpha)(e_{xy}, e_{n+1})e_{xy}, z)}{|e_{xy} - e_{n+1}|^{n+\alpha}} \right|.
\]
Replacing here $\mathbf{e}_\sigma$ by $-\mathbf{e}_\sigma$, we arrive at (2.4) for $p = 1$ with the sharp constant (2.5).

Let $1 < p \leq \infty$. Using (2.13) and the equality

\[
\frac{1}{|y - x|^{(\alpha + \alpha)q}} = \frac{1}{x_{n+1}^{(\alpha + \alpha)p - n}} \left( \frac{x_{n+1}}{|y - x|} \right)^{(n+\alpha)q - n - 1} \frac{x_{n+1}}{|y - x|^{n+1}},
\]

and replacing $q$ by $p/(p - 1)$ in (2.12), we conclude that (2.4) holds with the sharp constant

\[C_p(z) = k_{n,\alpha} \left\{ \int_{\mathbb{S}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right|^{\frac{p}{p-1}} \left( \mathbf{e}_{\sigma}, -\mathbf{e}_{n+1} \right)^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}},
\]

where $\mathbb{S}^n = \{ \sigma \in \mathbb{S}^n : (\mathbf{e}_\sigma, \mathbf{e}_{n+1}) < 0 \}$. Replacing here $\mathbf{e}_\sigma$ by $-\mathbf{e}_\sigma$, we arrive at (2.6) for $1 < p < \infty$ and at (2.7) for $p = \infty$.

By (2.10) we have

\[
|\nabla u_1(x)| = k_{n,\alpha} x_{n+1}^{(\alpha-1)} \sup_{|z|=1} \int_{\mathbb{R}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right| \left| y - x \right|^{(\alpha + \alpha)q} f(y') dy'.
\]

Hence, by the permutation of suprema, (2.12), (2.11) and (2.14),

\[
C_p(x) = k_{n,\alpha} x_{n+1}^{(\alpha-1)} \sup_{|z|=1} \left\{ \int_{\mathbb{R}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right|^{\frac{q}{p-1}} \left| y - x \right|^{(\alpha + \alpha)q} dy' \right\}^{\frac{1}{q}}
\]

\[
\sup_{|z|=1} C_p(x; z) = C_p(z) x_{n+1}^{-(n+1)}
\]

(2.14)

for $1 < p \leq \infty$, and

\[
C_1(x) = k_{n,\alpha} x_{n+1}^{(\alpha-1)} \sup_{|z|=1} \sup_{y \in \partial \mathbb{R}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right| \left| y - x \right|^{(\alpha + \alpha)q}
\]

\[
\sup_{|z|=1} C_1(x; z) = C_1(z) x_{n+1}^{-(n+1)}.
\]

(2.15)

Using the notation (2.9) in (2.14) and (2.15), we arrive at (2.8).

Remark. Formula (2.6) for the coefficient $C_p(z)$, $1 < p < \infty$, can be written with the integral over the whole sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$,

\[
C_p(z) = k_{n,\alpha} \left\{ \int_{\mathbb{S}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right|^{\frac{p}{p-1}} \left( \mathbf{e}_{\sigma}, \mathbf{e}_{n+1} \right)^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}}
\]

A similar remark relates (2.7):

\[
C_\infty(z) = k_{n,\alpha} \int_{\mathbb{S}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right| \left| \left( \mathbf{e}_{\sigma}, \mathbf{e}_{n+1} \right) \right|^{\alpha-1} d\sigma,
\]

(2.16)

as well as formula (2.3):

\[
C_1(z) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n} \left| (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_{x}, \mathbf{e}_{n+1})(\mathbf{e}_{xy}, z) \right| \left( \mathbf{e}_{\sigma}, \mathbf{e}_{n+1} \right)^{n+\alpha}.
\]
3 Reduction of the extremal problem to finding of the supremum by parameter of a double integral. The cases \( p = 1 \) and \( p = 2 \)

The next assertion is based on the representation for \( C_p \), obtained in Proposition 1.

**Proposition 2.** Let \( f \in L^p(\mathbb{R}^n) \), and let \( x \) be an arbitrary point in \( \mathbb{R}^{n+1} \). The sharp coefficient \( C_p(x) \) in the inequality

\[
|\nabla u_f(x)| \leq C_p(x) \| f \|_p
\]

is given by

\[
C_p(x) = \frac{C_p}{x_{(n+p)/p}},
\]

where

\[
C_p = (\omega_{n-1})^{(p-1)/p} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1 + \gamma^2}} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} F_{n,p}(\varphi, \vartheta; \gamma) d\vartheta \right\}^{\frac{p-1}{p}},
\]

if \( 1 < p < \infty \). Here

\[
F_{n,p}(\varphi, \vartheta; \gamma) = \left| G_n(\varphi, \vartheta; \gamma) \right|^{p/(p-1)} \cos^{((\alpha-1)\varrho + n+1)/(p-1)} \vartheta \sin^{n-1} \vartheta \sin^{n-2} \vartheta.
\]

with

\[
G_n(\varphi, \vartheta; \gamma) = \left( (n + \alpha) \cos^2 \vartheta - \alpha \right) + \gamma(n + \alpha) \cos \vartheta \sin \vartheta \cos \varphi.
\]

In addition,

\[
C_1 = k_{n,\alpha} n
\]

if \(-n < \alpha \leq n\).

In particular,

\[
C_2 = \sqrt{\omega_{n-1} k_{n,\alpha}} \left\{ \frac{\sqrt{\pi(n + \alpha)n(n + 2)\Gamma \left( \frac{n}{2} - 1 \right) \Gamma \left( \frac{n}{2} + \alpha \right)}}{8(n + 1 + \alpha)\Gamma(n + \alpha)} \right\}^{1/2}
\]

for \(-(n/2) < \alpha \leq n(n + 1)/2\).

For \( p = 1 \) and \( p = 2 \) the coefficient \( C_p(x) \) is sharp in conditions of the Proposition also in the weaker inequality obtained from (3.1) by replacing \( \nabla u_f \) by \( \partial u_f/\partial x_{n+1} \).

**Proof.** The equality (3.2) was proved in Proposition 1.

(i) Let \( p = 1 \). Using (2.5), (2.9) and the permutability of two suprema, we find

\[
C_1 = k_{n,\alpha} \sup_{|z|=1} \sup_{\sigma \in \mathbb{S}^n_+} |(\alpha e_{n+1} - (n + \alpha)(e_{\sigma}, e_{n+1})e_{\sigma}, z)| (e_{\sigma}, e_{n+1})^{n+\alpha}
\]

\[
= k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n_+} |\alpha e_{n+1} - (n + \alpha)(e_{\sigma}, e_{n+1})e_{\sigma}| (e_{\sigma}, e_{n+1})^{n+\alpha}.
\]

(3.7)
Taking into account the equality

\[ |\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma| \]

\[ = \left( \alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, \alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma \right)^{1/2} \]

\[ = \left( \alpha^2 + ((n + \alpha)^2 - 2\alpha(n + \alpha))(e_\sigma, e_{n+1})^2 \right)^{1/2}, \]

and using (2.3), (3.7), we arrive at the sharp constant (3.6) for \(-n < \alpha \leq n\).

Furthermore, by (2.5),

\[ C_1(e_{n+1}) = k_{n,\alpha} \sup_{\sigma \in S_n^+} |\alpha - (n + \alpha)(e_\sigma, e_{n+1})^2|(e_\sigma, e_{n+1})^{n+\alpha} \geq k_{n,\alpha} n. \]

Hence, by \( C_1 \geq C_1(e_{n+1}) \) and by (3.6) we obtain \( C_1 = C_1(e_{n+1}) \), which completes the proof in the case \( p = 1 \).

(ii) Let \( 1 < p < \infty \). Since the integrand in (2.6) does not change when \( z \in S^n \) is replaced by \(-z\), we may assume that \( z_{n+1} = (e_{n+1}, z) > 0 \) in (2.9).

Let \( \sigma' = z - z_{n+1}e_{n+1} \). Then \((\sigma', e_{n+1}) = 0\) and hence \( z_{n+1}^2 + |\sigma'|^2 = 1 \). Analogously, with \( \sigma = (\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \in S_n^+ \), we associate the vector \( \sigma' = e_\sigma - \sigma_{n+1}e_{n+1} \).

Using the equalities \((\sigma', e_{n+1}) = 0\), \( \sigma_{n+1} = \sqrt{1 - |\sigma'|^2} \) and \((z', e_{n+1}) = 0\), we find an expression for \((\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z)\) as a function of \( \sigma' \):

\[ (\alpha e_{n+1} - (n + \alpha)(e_\sigma, e_{n+1})e_\sigma, z) = \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}(e_\sigma, z) \]

\[ = \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}(\sigma' + \sigma_{n+1}e_{n+1}, z' + z_{n+1}e_{n+1}) \]

\[ = \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}[(\sigma', z') + z_{n+1}\sigma_{n+1}] \]

\[ = -[(n + \alpha)(1 - |\sigma'|^2) - \alpha] z_{n+1} - (n + \alpha) \sqrt{1 - |\sigma'|^2} (\sigma', z'). \tag{3.8} \]

Let \( B^n = \{ x' = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x'| < 1 \} \). By (2.6) and (3.8), taking into account that \( d\sigma = d\sigma'/\sqrt{1 - |\sigma'|^2} \), we may write (2.9) as

\[ C_p = k_{n,\alpha} \sup_{z \in S_n^+} \left\{ \int_{B^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) \left( 1 - |\sigma'|^2 \right)^{(\alpha p + n + 1)/2(p-1)} d\sigma' \right\}^{\frac{p-1}{p}} \]

\[ = k_{n,\alpha} \sup_{z \in S_n^+} \left\{ \int_{B^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) \left( 1 - |\sigma'|^2 \right)^{(\alpha - 2)p + n + 2)/2(p-1)} d\sigma' \right\}^{\frac{p-1}{p}}, \tag{3.9} \]

where

\[ \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) = \left[ (n + \alpha)(1 - |\sigma'|^2) - \alpha \right] z_{n+1} + (n + \alpha) \sqrt{1 - |\sigma'|^2} (\sigma', z') \right]^{p/(p-1)}. \tag{3.10} \]

Using the well known formula (see e.g. [8], 3.3.2(3)),

\[ \int_{B^n} g(|x|, (a, x)) dx = \omega_{n-1} \int_0^1 r^{n-1} dr \int_0^{\pi} g(r, |a| r \cos \varphi) \sin^{n-2} \varphi \, d\varphi, \]

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we obtain
\[\int_{\mathbb{R}^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) (1 - |\sigma'|^2)^{(\alpha - 2)p + n + 2}/(p - 1) \, d\sigma' = \omega_{n-1} \int_0^1 r^{n-1} (1 - r^2)^{(\alpha - 2)p + n + 2}/(p - 1) \, dr \int_0^\pi \mathcal{H}_{n,p}(r, r|z'| \cos \varphi) \sin^{n-2} \varphi d\varphi.\]

Making the change of variable \( r = \sin \vartheta \) in the right-hand side of the last equality, we find
\[
\int_{\mathbb{R}^n} \mathcal{H}_{n,p}(|\sigma'|, (\sigma', z')) (1 - |\sigma'|^2)^{(\alpha - 2)p + n + 2}/(p - 1) \, d\sigma' = \omega_{n-1} \int_0^\pi \sin^{n-2} \varphi d\varphi \int_0^{\pi/2} \mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) \sin^{-1} \vartheta \cos \left(\frac{(\alpha - 1)p + n + 1}{p - 1}\right) \vartheta d\vartheta, \tag{3.11}
\]
where, by \((3.10)\),
\[
\mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) = \left| ((n + \alpha) \cos^2 \vartheta - \alpha) z_{n+1} + (n + \alpha) |z'| \cos \vartheta \sin \vartheta \cos \varphi \right|^{p/(p - 1)}. \tag{3.12}
\]

Introducing here the parameter \( \gamma = |z'|/z_{n+1} \) and using the equality \(|z'|^2 + z_{n+1}^2 = 1\), we obtain
\[
\mathcal{H}_{n,p}(\sin \vartheta, |z'| \sin \vartheta \cos \varphi) = (1 + \gamma^2)^{-p/2(p - 1)} |\mathcal{G}_n(\varphi, \vartheta; \gamma)|^{p/(p - 1)}, \tag{3.12}
\]
where \(\mathcal{G}_n(\varphi, \vartheta; \gamma)\) is given by \((3.5)\).

By \((3.9)\), taking into account \((3.11)\) and \((3.12)\), we arrive at \((3.3)\).

(iii) Let \( p = 2 \). By \((3.3)\), \((3.4)\) and \((3.5)\),
\[
C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\gamma^2} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} \mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) \, d\vartheta \right\}^{1/2}, \tag{3.13}
\]
where
\[\mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) = \left[ ((n + \alpha) \cos^2 \vartheta - \alpha) + \gamma (n + \alpha) \cos \vartheta \sin \vartheta \cos \varphi \right]^2 \cos^{n-1+2\alpha} \vartheta \sin^{n-1} \vartheta \sin^{n-2} \varphi.\]

The last equality and \((3.13)\) imply
\[
C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\gamma^2} \left\{ \mathcal{I}_1 + \gamma^2 \mathcal{I}_2 \right\}^{1/2}, \tag{3.14}
\]
where
\[
\mathcal{I}_1 = \int_0^\pi \sin^{n-2} \varphi \, d\varphi \int_0^{\pi/2} ((n + \alpha) \cos^2 \vartheta - \alpha)^2 \sin^{n-1} \vartheta \cos^{n-1+2\alpha} \vartheta \, d\vartheta
\]
\[
= \frac{\sqrt{\pi} n(n + 2)(n + \alpha) \Gamma \left(\frac{n + 1}{2}\right) \Gamma \left(\frac{n + 2 + \alpha}{2}\right)}{4(n + 2\alpha)(n + 1 + \alpha) \Gamma(n + \alpha)} \tag{3.15}
\]
\[ \mathcal{I}_2 = (n + \alpha)^2 \int_0^\pi \sin^{n-2} \varphi \cos^2 \varphi \, d\varphi \int_0^{\pi/2} \sin^{n+1} \varphi \cos^{n+1+2\alpha} \varphi \, d\varphi \]

\[ = \frac{\sqrt{\pi} (n + \alpha) \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n+2+2\alpha}{2} \right)}{4(n + 1 + \alpha) \Gamma(n + \alpha)} . \tag{3.16} \]

By (3.14) we have

\[ C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \max \{ I_1^{1/2}, I_2^{1/2} \} . \tag{3.17} \]

Further, by (3.15) and (3.16),

\[ \frac{I_1}{I_2} = \frac{n(n + 2)}{n + 2\alpha} . \]

Therefore,

\[ \frac{I_1}{I_2} - 1 = \frac{n^2 + n - 2\alpha}{n + 2\alpha} . \]

Taking into account (3.17) and that \( n + 2\alpha > 0 \) for \( p = 2 \) by (2.3), we see that inequality

\[ \frac{I_1}{I_2} \geq 1 \]

holds for \( \alpha \leq (n+1)/2 \). So, we arrive at the representation for \( C_2 \) with \(- (n/2) < \alpha \leq (n+1)/2 \) given in formulation of the Proposition.

Since \( z \in S^n \) and the supremum in \( \gamma = |z'|/z_{n+1} \) in (3.13) is attained for \( \gamma = 0 \), we have \( C_2 = C_2(e_{n+1}) \) under requirements of the Proposition. \( \square \)

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