CRITICAL BEHAVIOUR OF A NON-LOCAL $\phi^4$ FIELD THEORY AND ASYMPTOTIC FREEDOM

R. TRINCHERO

Abstract. The critical behaviour of a non-local scalar field theory is studied. This theory has a non-local kinetic term which involves a real power $1 - 2\alpha$ of the Laplacian. The interaction term is the usual local $\phi^4$ interaction. The lowest order Feynman diagrams corresponding to coupling constant renormalization, mass renormalization and field renormalization are computed. Particular features appearing in the renormalization of these non-local theory that differ from the case of local theories are studied. The previous calculations lead to the perturbative computation of the coupling constant beta function and critical exponents $\nu$ and $\eta$. In four dimensions for $\alpha < 0$ this beta function presents asymptotic freedom in the UV. This is remarkable since no non-abelian vector fields are included. However this comes at the expense of loosing reflection positivity.

1. Introduction

The computation of critical exponents for the 3-dimensional Ising model using the $\epsilon$-expansion provides a concrete example of the relevance of the renormalization group ideas [1][2]. This is done by considering a self-interacting $\phi^4$ theory in $d = 4 - \epsilon$ dimensions, where $\epsilon$ is allowed to take real values. This procedure led to a qualitative understanding of the 3-dimensional Ising model physics and to predictions for critical exponents in reasonable agreement with the exact values. The results for the theory in $d$-dimensions are obtained by computing the theory in an integer number $n$ of dimensions and then replacing $n$ by $d$.

The renormalization group consists in the study of the evolution of a system under scale transformations. This system involves all possible interactions of any range for all kinds of dynamical variables. Different physical systems correspond to the study of particular fixed points in this huge space of couplings. This paper studies a particular example of system described near the corresponding fix point by a non-local field theory. The use of non-local field theories in the description of critical phenomena is not new [3],[4],[5],[6]. Such models appear in statistical systems with long range interactions. In this paper the critical behaviour of a non-local field theory is studied. This non-local theory is motivated by an alternative approach to non-integer dimensional spaces (NIDS) [7]. Free scalar theories on these spaces has been studied in this last reference. This theory has been employed to compute loop corrections and compare the results with dimensional regularization [8][9], showing that the structure of singularities is the same as in dimensional regularization. In addition, the fullfilment or not of the requirement of reflection positivity for the corresponding Euclidean field theory has been considered [10]. There, it is shown that for negative values of the non-integer power mentioned above the theory fullfills reflection positivity. This means that the corresponding theory in Minkowski space is unitary for those values of the non-integer power. The aim in this work is to add a $\phi^4$ interaction and study
the renormalization and critical properties of the resulting non-local theory\footnote{This theory can also be obtained as the analytic regularized\cite{11} version of the usual $\phi^4$ local field theory.} This study shows the relevance of this model in describing non-trivial fixed points. The features and results of this work are summarized as follows,

- The theory to be considered is the free scalar theory studied in \cite{7} with the addition of a $\phi^4$ interaction term.
- The contribution of the lowest order Feynman diagrams corresponding to coupling constant renormalization, mass renormalization and field renormalization are computed. This computation exemplifies general issues about the renormalization of non-local field theories. The procedure employed involves features that do not appear in the local case.
- The previous calculation allows to compute the fixed point value for the coupling constant and the critical exponents $\nu$ and $\eta$, respectively. The corresponding results describe a theory which shows asymptotic freedom in the UV and a non-trivial infrared fixed point at finite coupling. The corresponding theory does not fulfill the condition of reflection positivity.
- In addition assuming the usual $n$-dimensional conformal algebra to be a symmetry of the theory, the unitarity bounds are studied. They agree with the ones obtained by requiring the condition of reflection positivity.

2. The action

The free part of the action to be considered is essentially the same as in \cite{7} for $M = 0$. The interaction part is $\phi^4$\footnote{No infrared regulator is required for the following computations.}. In terms of the scalar product of form fields mentioned above and described in \cite{7}, the action is given by,

\begin{equation}
S = S_0 + S_I, \quad S_0 = \frac{1}{2} < d\phi, d\phi >, \quad S_I = \frac{\lambda_0}{4!} < \phi^2, \phi^2 >
\end{equation}

evaluating the scalar products appearing in the last equation leads to the following expression in terms of an integral over the integer $n$-dimensional space,

\begin{equation}
S_0 = \int d^n x \frac{1}{2} \phi(-\Box + m_0^2)(-\Box)^{-2\alpha} \phi, \quad S_I = \frac{\lambda_0}{4!} \int d^n x \phi^4
\end{equation}

where, anticipating renormalization effects, a explicit mass term has been included\footnote{This way of introducing a mass term is motivated by the calculation of perturbative corrections appearing below.}. In what follows bare mass and coupling will be indicated by $m_0$ and $\lambda_0$, the corresponding renormalized quantities will be $m$ and $\lambda$. The Fourier transform of the free two point function is therefore given by,

\begin{equation}
< \phi\phi > (p) = \frac{1}{(p^2 + m_0^2)(p^2)^{-2\alpha}} = \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)} \int_0^1 da (1-a)^{-2\alpha} \frac{1}{(p^2 + m_0^2 a)^{1-2\alpha}}
\end{equation}

the second equality in the last equation is obtained using Feynman parametrization. This last expression will be employed in the computations below.
The integral to be computed is,

$$a_S(p, \alpha) = \left( \frac{\Gamma(1 - 2\alpha)}{\Gamma(-2\alpha)} \right)^3 \int_0^1 \left( \prod_{i=1}^3 da_i (1 - a_i)^{-1-2\alpha} \right) I_S(p, \alpha, a_1, a_2, a_3)$$

$$I_S(p, \alpha, a_1, a_2, a_3) = \lambda_0^2 \int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \frac{1}{(q_1^2 + a_1 m_0^2)^{1-2\alpha}(q_2^2 + a_2 m_0^2)^{1-2\alpha}[(p + q_1 + q_2)^2 + a_3 m_0^2]^{1-2\alpha}}$$



For the purposes of this work it is convenient to expand the integrand as a power series in $$m^2$$. This leads to the following expression,

$$I_S(p, \alpha, a_1, a_2, a_3) = I_0(p, \alpha) + m_0^2(a_1 + a_2 + a_3)I_2(p, \alpha) + \mathcal{O}(m_0^4)$$

where,

$$I_0(p, \alpha) = \lambda_0^2 \int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \frac{1}{(q_1^2 + a_1 m_0^2)^{1-2\alpha}(q_2^2 + a_2 m_0^2)^{1-2\alpha}[(p + q_1 + q_2)^2 + a_3 m_0^2]^{1-2\alpha}}$$

$$I_2(p, \alpha) = 3\lambda_0^2 \int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \frac{1}{(q_1^2 + a_1 m_0^2)^{1-2\alpha}(q_2^2 + a_2 m_0^2)^{1-2\alpha}[(p + q_1 + q_2)^2 + a_3 m_0^2]^{1-2\alpha}}$$

, introducing Feynman parametrizations to rewrite the integrands, performing the momentum integrals and the integrals on the Feynman parameters and taking $$n = 4$$ leads to,

$$I_0(p, \alpha) = p^{2(6\alpha + 1)} \frac{4^{-2\alpha - 3}\lambda_0^2 \csc(4\pi \alpha)\Gamma(-6\alpha - 1)\Gamma\left(\frac{3}{2} - 2\alpha\right) B_1(4\alpha + 2, 2\alpha + 1)}{\pi^{7/2}\Gamma(3 - 8\alpha)\Gamma(1 - 2\alpha)^2\Gamma(4\alpha)}$$

$$I_2(p, \alpha) = (p^2)^{6\alpha} \frac{\lambda_0^2 \Gamma(-6\alpha)\Gamma(2\alpha)\Gamma(2\alpha + 1)^2}{256\pi^4\Gamma(1 - 2\alpha)^2\Gamma(2 - 2\alpha)\Gamma(6\alpha + 2)}$$

The coupling $$\lambda_0$$ has dimension $$4 - n - 8\alpha$$ in momentum units, therefore for $$n = 4$$, it can be written as follows in terms of an adimensional coupling $$g_0$$ as follows,

$$\lambda_0 = g_0 \mu^{-8\alpha} \Rightarrow g = \lambda \mu^{8\alpha}$$

noting that,

$$\left( \frac{\Gamma(1 - 2\alpha)}{\Gamma(-2\alpha)} \right)^3 \int_0^1 \left( \prod_{i=1}^3 da_i (1 - a_i)^{-1-2\alpha} \right) (a_1 + a_2 + a_3) = \frac{3}{1 - 2\alpha}$$

$$\left( \frac{\Gamma(1 - 2\alpha)}{\Gamma(-2\alpha)} \right)^3 \int_0^1 \left( \prod_{i=1}^3 da_i (1 - a_i)^{-1-2\alpha} \right) = 1$$
leads to,

$$a_S(p, \alpha) = \frac{g_0^2}{(4\pi)^4} \mu^{-16\alpha} (p^2)^{6\alpha} \left[ \frac{p^2}{12\alpha} + m_0^2 \left( -\frac{1}{4\alpha^2} + \frac{1}{2\alpha} \right) + \mathcal{O}(\alpha^0) \right]$$

$$= \frac{g_0^2}{(4\pi)^4} \mu^{-16\alpha} \frac{(p^2)^{6\alpha}}{12\alpha} \left[ p^2 + m_0^2 \left( -\frac{3}{\alpha} + 6 \right) + \mathcal{O}(\alpha) \right]$$

This result shows that $I_0(p, \alpha)$ is the relevant integral for the field renormalization and that $I_2(p, \alpha)$ contributes to mass renormalization. At this stage a recurrent situation in the renormalization of these non-local theories shows up. Similar to what happens in dimensional regularization the correction provided by a given diagram, in this case the sunrise diagram, is proportional to a power of the momentum which is not in general the same as the one that originally appears in the Lagrangian. The integral $I_0(p, \alpha)$ gives a contribution proportional to $p^{2(1+6\alpha)}$, while the original Lagrangian has the power $p^{2(1-2\alpha)}$. The choice of the power of $p^2$ that appears in the kinetic term of the renormalized Lagrangian fixes the finite contribution of this diagram. In other words if a different power of $p^2$ is choosen than the finite contribution of the diagram will also be different. The following way of rewriting $a_S(p, \alpha)$ illustrates this point,

$$a_S(p, \alpha) = \frac{g_0^2}{(4\pi)^4} \mu^{-16\alpha} (p^2)^{2\alpha} \left[ p^2 + m_0^2 \left( -\frac{3}{\alpha} + 6 \right) + \mathcal{O}(\alpha) \right]$$

$$= \frac{g_0^2}{(4\pi)^4} \frac{(p^2)^{2\alpha}}{12\alpha} \left[ p^2 + m_0^2 \left( -\frac{3}{\alpha} + 6 \right) + \mathcal{O}(\alpha) \right] \left( 1 + 8\alpha \log \left( \frac{p^2}{\mu^2} \right) + \mathcal{O}(\alpha^2) \right)$$

$$= \frac{g_0^2}{(4\pi)^4} \frac{(p^2)^{2\alpha}}{12\alpha} \left[ p^2 + m_0^2 \left( -\frac{3}{\alpha} + 6 \right) + \mathcal{O}(\alpha) \right]$$

(3.2)

For $\alpha \rightarrow 0$ the pole term of the last expression multiplied by the symmetry factor $\frac{1}{6}$ is the one to be substracted. It is given by,

$$\left( \frac{1}{6} a_S(p, \alpha) \right)_{\text{pole}} = \frac{g_0^2}{(4\pi)^4} \frac{1}{72\alpha} p^{2(1-2\alpha)}$$

which leads to the renormalization constant,

$$Z_\phi = 1 + \frac{g_0^2}{(4\pi)^4} \frac{1}{72\alpha}$$

the function $\gamma$ is defined and given by,

$$\gamma(g) = \mu \frac{\partial}{\partial \mu} \log Z_\phi^\gamma \bigg|_{\lambda_{\text{fixed}}} = \frac{1}{2} \frac{\partial}{\partial \mu} \log \left( 1 + \frac{\lambda_0^2 \mu^{16\alpha}}{(4\pi)^4} \frac{1}{72\alpha} \right)$$

$$= \frac{1}{2} Z_\phi \left( \frac{\lambda_0^2 \mu^{16\alpha}}{(4\pi)^4 72\alpha} \mu^{16\alpha} \right) = \frac{g_0^2}{9(4\pi)^4} + \mathcal{O}(g^4)$$

(3.3)

3.2. The fixed point and coupling constant renormalization. The diagram to be considered is the one corresponding to the one loop correction to the quartic coupling, i.e.,

$^{4}$In dimensional regularization of the usual local $\phi^4$ theory , the correction provided by the sunrise diagram is proportional to $p^{2(d-3)} = p^{2(1-\epsilon)}$. 
The integral to be computed is,

\[ a_F(p, \alpha) = \left( \frac{\Gamma(1 - 2\alpha)}{\Gamma(-2\alpha)} \right)^2 \int_0^1 \left( \prod_{i=1}^2 da_i (1 - a_i)^{-1-2\alpha} \right) I_F(p, \alpha, a, b) \]

\[ I_F(p, \alpha, a, b) = \frac{3}{2} \lambda_0 \int_0^1 d^n q \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + m_0^2 a)^{1-2\alpha} ((p + q)^2 + m_0^2 b)^{1-2\alpha}} \]

the factor \( \frac{3}{2} \) coming from the \( \frac{1}{2} \) of the second order term of the exponential and the contributions of 3 diagrams which give the same contribution. Introducing the Feynman parametrization and integrating over the n-moment \( q \), leads to,

\[ I_F(p, \alpha, a, b) = \frac{3}{2} \lambda_0 \int_0^1 d^2 x \frac{(m^2[a x + b(1 - x)] + p^2(1 - x) x)^{\frac{n}{2}-2+4\alpha}}{(1 - x) x^{2\alpha}} \]

\[ = \frac{3}{2} \lambda_0 \mu^{8\alpha} \frac{\Gamma(-4\alpha)}{\Gamma(1 - 2\alpha)^2} \int_0^1 dx \left[ \frac{(m_0^2 + p^2(1 - x) x)^{2\alpha}}{\mu^4(1 - x) x} \right] \]

\[ = \frac{3}{4(4\pi)^2} \left[ -\frac{1}{2\alpha} + \int_0^1 dx \log \left( \frac{\mu^4(1 - x) x}{(m_0^2 - p^2(1 - x) x)^2} \right) \right] \]

where in the second equality a parameter \( \mu \) with dimensions of mass has been introduced in order to make adimensional the argument of the logarithm. In addition in the last equality only terms up to \( \mathcal{O}(\alpha^0) \) has been kept. In the minimal substraction scheme only the first term in the square bracket of the last expression will be relevant in defining the renormalized coupling \( \lambda_R \). This term is independent of \( a \) and \( b \), therefore,

\[ a_F(p, \alpha) = \left( \frac{\Gamma(1 - 2\alpha)}{\Gamma(-2\alpha)} \right)^2 \int_0^1 da db (1 - a)^{-1-2\alpha} (1 - b)^{-1-2\alpha} \frac{3(\lambda_0 \mu^{4\alpha})^2}{4(4\pi)^2} \left( -\frac{1}{2\alpha} \right) \]

Taking into account the computation in the last subsection, this leads to the following renormalized coupling,

\[ \lambda = \frac{\lambda_0 Z_g^2}{Z_g}, \quad Z_g = 1 + \frac{3}{4(4\pi)^2} \frac{\lambda_0}{(-2\alpha)} + \mathcal{O}(\lambda_0^2) \]

\[ Z_\phi = 1 + \frac{g_R^2}{(4\pi)^2} \frac{1}{72\alpha} \]

the beta function corresponding to the renormalized adimensional coupling \( g \) fulfills,

\[ \beta(g_R) = \mu d \mu g_R = \mu d \mu \left( \frac{\lambda Z_\phi^2}{Z_g} \right)^{8\alpha} = 8\alpha g_R + 2g_R Z_\phi^{-1} \mu d \mu Z_\phi + g_R Z_\phi^{-1} \mu d \mu Z_\phi = 8\alpha g_R + 4g_R \gamma + g_R \frac{3}{8\alpha} \frac{1}{(4\pi)^2} \beta(\lambda_R) \]

which implies,

\[ \beta(g_R) = 8\alpha g_R + \frac{6g_R^2}{2(4\pi)^2} + 4 \frac{g_R^3}{9(4\pi)^4} \]

Neglecting negative values of \( g \), which make the theory unstable, the figure below shows a plot of this function for \( \alpha = \pm 0.01 \).
This figure shows that for $\alpha > 0$ the theory has asymptotic freedom in the infrared. However for $\alpha < 0$ this 4-dimensional theory presents asymptotic freedom (AF) in the ultraviolet (UV). This a remarkable result since it is usually believed that non-abelian gauge bosons are required in order to get AS in the UV. However as the analysis in [10] shows, the theory for $\alpha < 0$ does not satisfy the requirement of reflection positivity (RP). This means that the Wick rotated theory in Minkowski space does not provide a unitary representation of the Poincaré group, which implies that no unitary evolution can be defined in this space. Alternatively, as will be shown in the next section, the unitarity bounds are violated for $\alpha < 0$. This does not mean that the Euclidean theory is useless, indeed many useful statistical mechanical models fail to satisfy RP.

The fixed point $g^*$ is defined by $\beta(g^*) = 0$. Writing the solution of this last equation as a power series in $\alpha$,

$$g^* = g_0 + g_1 \alpha + g_2 \alpha^2 + \cdots$$

leads to two solutions, the Gaussian fixed point $g^* = 0$ and,

$$g^* = -\frac{8}{3} (4\pi)^2 \alpha - \frac{256}{243} (4\pi)^2 \alpha^2$$

3.3. Mass renormalization. The one loop correction to the two point function is given by the following diagram,

The integral to be computed is,

\begin{equation}
(3.4) \quad a_T(\alpha) = \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)} \int_0^1 da (1-a)^{-1-2\alpha} I_T(\alpha, a)
\end{equation}

\begin{equation}
I_T(\alpha, a) = -\lambda_0 \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + m_0^2 a)^{1-2\alpha}}
\end{equation}

leading to,

\begin{equation}
I_T(\alpha, a) = -\frac{\lambda_0}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(1-\frac{n}{2}-2\alpha)}{\Gamma(1-2\alpha)} (m_0^2 a)^{\frac{2}{2}-1+2\alpha}
\end{equation}
replacing in (3.4) leads to,
\[
a_T(\alpha) = \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)} \int_0^1 \frac{da}{(1-a)^{1-2\alpha}} \left( -\frac{\lambda_0}{(4\pi)^2} \frac{\Gamma(1-\frac{\alpha}{2} - 2\alpha)}{\Gamma(1-2\alpha)} (m_0^2 a)^{\frac{\alpha}{2} - 1 + 2\alpha} \right)
\]
\[
\approx \frac{-g_0 \mu^{-2\alpha}}{16\pi \sin(2\pi\alpha)} (m_0^2)^{1+2\alpha} = \frac{-g_0}{16\pi \sin(2\pi\alpha)} (m_0^2)^{-2\alpha} \left( \frac{m_0^2}{\mu^2} \right)^{4\alpha}
\]
\[
= (m_0^2)^{1-2\alpha} \left( \frac{-g_0}{(4\pi)^2 2\alpha} + \mathcal{O}(\alpha^0) \right)
\]
where in the second equality the dimensional coupling \( \lambda \) has been expressed in terms of the adimensional coupling \( g \) by means of (3.1). As was mentioned for the case of the sunrise diagram, in this case also the power of \( p^2 \) appearing in the correction is different from the one appearing in the Lagrangian. In a similar way as for the sunrise, the choice of the power to appear in the renormalized Lagrangian fixes the finite contribution of this diagram. This last point is illustrated by the following computation of the correction to the proper two point function\[5\]
\[
\Gamma_2(p) = (p^2 + m_0^2)(p^2)^{-2\alpha} - \frac{1}{2} a_T(\alpha)
\]
\[
= (p^2 + m_0^2)(p^2)^{-2\alpha} + (m_0^2)^{1-2\alpha} \frac{g_0}{(4\pi)^2 4\alpha} + C
\]
\[
= (p^2 + m_0^2)(p^2)^{-2\alpha} + m_0^2(p^2)^{-2\alpha} \left( \frac{m_0^2}{p^2} \right)^{-2\alpha} \frac{g_0}{(4\pi)^2 4\alpha} + C
\]
\[
= (p^2 + m_0^2 \left( 1 + \frac{g_0}{(4\pi)^2 4\alpha} \right))(p^2)^{-2\alpha} + C'
\]
where \( C \) and \( C' \), denote terms that converge for \( \alpha \to 0 \). Therefore, taking into account the computation in subsection 3.1, the renormalized mass \( m \) in the minimal substraction scheme is given by,
\[
m^2 = m_0^2 \frac{Z_\phi}{Z_{m^2}}
\]
where, up to \( \mathcal{O}(g^2) \),
\[
Z_{m^2} = 1 - \frac{g_0}{(4\pi)^2} \frac{1}{4\alpha}
\]
the beta function \( \gamma_m \) for the mass is given by,
\[
\gamma_m(g) = \frac{\mu}{m} \frac{\partial m}{\partial \mu} = \frac{1}{2} \left( \frac{\mu}{Z_\phi} \frac{\partial \log Z_\phi}{\partial \mu} - \mu \frac{\partial Z_{m^2}}{\partial \mu} \right) = \gamma + \frac{1}{2} \beta(g) \frac{\beta(g)}{2(4\pi)^2 4\alpha}
\]
\[
= \frac{1}{2} \beta(g) \left( \frac{1}{(4\pi)^2 4\alpha} + \frac{2g_0}{(4\pi)^4 72\alpha} \right) = \frac{g}{(4\pi)^2} + \frac{g^2}{9(4\pi)^4} + \mathcal{O}(g^3)
\]
\[5\]It is worth noting that if the mass were included with a kinetic term of the form, \( \mathcal{L}_0 = \phi(-\Box + m^2)^{1-2\alpha} \phi \) then the singular contribution of this diagram when \( \alpha \to 0 \) could not be absorbed by mass renormalization, in other words the counterterm required to cancel the divergence when \( \alpha \to 0 \) would not be of the form \( \mathcal{L}_0 \).
3.4. The critical exponents $\nu$ and $\eta$. These critical exponents are related to the fixed point values $\gamma^*$ and $\gamma_m^*$ of the functions $\gamma$ and $\gamma_m$. They are given by,

$$\nu = \frac{1}{2 - 2\gamma_m^*}, \eta = 2\gamma^*$$

The non-trivial fixed point is given by,

$$g^* = -\frac{8}{3}(4\pi)^2\alpha - \frac{256}{243}(4\pi)^2\alpha^2$$

the fixed point values $\gamma_m^*$ and $\gamma^*$ are therefore given by,

$$\gamma_m^* = \gamma_m(g^*) = \frac{-8\alpha}{3} + \frac{256}{243}(12\pi^2 - 1)\alpha^2 + \frac{65536\pi^2\alpha^3}{6561} + O(\alpha^3)$$

$$\gamma^* = \gamma(g^*) = \frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561}$$

which imply,

$$\nu = \frac{1}{2} - \frac{4\alpha}{3} + \frac{32}{243}(23 + 48\pi^2)\alpha^2 - \frac{256(171 + 36\pi^2)\alpha^3}{6561}$$

$$\eta = 2\left(\frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561}\right)$$

(3.5)

It is worth noting that the value of $\alpha$ is related to the dimension of space. A free propagator at the Gaussian fixed point in 4-dimensions, should behave as $\frac{1}{|x|^2}$, this corresponds to small values of $\alpha$, as the ones employed in the last figure. The critical exponents for the non-Gaussian fixed point for $\alpha = -0.01$ are,

$$\nu^{a=-0.01} = 0.52, \eta^{a=-0.01} = 0.0001$$

Following the same reasoning in 3 dimensions, the $\frac{1}{|x|}$ behaviour of the free propagator gives $\alpha = -\frac{1}{4}$. This is the value of $\alpha$ which corresponds to the $\epsilon = 1$ in the $\epsilon$-expansion. In the same spirit as in the case of the $\epsilon$-expansion, the critical exponents for the non-Gaussian fixed point can be computed for this last value of $\alpha$. Replacing $\alpha = -\frac{1}{4}$ in (3.5) leads to the following values for the critical exponents,

$$\nu^{a=-\frac{1}{4}} = 4.92, \eta^{a=-\frac{1}{4}} = 0.079$$

which, in comparison with the values obtained with the $\epsilon$-expansion, significantly differs from the 3d-Ising model critical exponents. This shows that this fixed point does not describe the 3d-Ising model critical point.

4. Relation with $\epsilon$ expansion

For each diagram there is a way to obtain its divergent contribution (when $\alpha \to 0$) from the corresponding one in the $\epsilon$ expansion. In order to show this let us consider the superficial degree of divergence (SDD) for both theories, the one considered in this paper described by the action (2.1), from now on the $\alpha$-theory and the usual $\phi^4$ theory dimensionaly regularized to a dimension $d = 4 - \epsilon$, from now on the $\epsilon$-theory. The SDD for a proper graph $G$ in the $\epsilon$-theory is given by,

$$\omega_\epsilon(G) = 4 - \epsilon(1 + V) + \left(\frac{\epsilon}{2} - 1\right)E$$
where $V$ denotes the number of vertices and $E$ the number of external legs. For the case of the $\alpha$-theory the SDD can be computed to give,

$$\omega_\alpha(G) = 4 + 8\alpha V - (1 + 2\alpha)E$$

which of course coincide for $\alpha = \epsilon = 0$. Note that there is no replacement of $\epsilon$ as a function of $\alpha$ such that for any $V$ and $E$ the following equality holds$^6$:

$$\omega_\epsilon(\alpha)(G) = \omega_\alpha(G)$$

However for each given $V$ and $E$ there is a replacement. This is shown in the table below, which compares the SDD and the renormalization constants for the diagrams considered in the previous section,

$$Z(\epsilon) \rightarrow Z(\alpha)$$

| Diagram | $\omega_\epsilon$ | $\omega_\alpha$ | $Z(\epsilon)$ | $Z(\alpha)$ | $\omega_\epsilon \rightarrow \omega_\alpha$ | $Z(\epsilon) \rightarrow Z(\alpha)$ |
|---------|------------------|----------------|----------------|----------------|---------------------------------|---------------------------------|
| $\circ$ | $2 - \epsilon$   | $2 + 4\alpha$ | $Z_{\mu^2}(\epsilon) = 1 + \frac{g}{(4\pi^2)^2\epsilon}$ | $Z_{\mu^2}(\alpha) = 1 - \frac{g}{(4\pi^2)^24\alpha}$ | $\epsilon \rightarrow -4\alpha$ | $\epsilon \rightarrow -4\alpha$ |
| $\times$ | $-\epsilon$      | $8\alpha$     | $Z_{g}(\epsilon) = 1 + \frac{3g}{(4\pi^2)^2\epsilon}$ | $Z_{g}(\alpha) = 1 - \frac{3g}{(4\pi^2)^28\alpha}$ | $\epsilon \rightarrow -8\alpha$ | $\epsilon \rightarrow -8\alpha$ |
| $\circ$ | $2 - 2\epsilon$  | $2 + 12\alpha$| $Z_{\phi}(\epsilon) = 1 - \frac{g^2}{(4\pi^2)^212\epsilon}$ | $Z_{\phi}(\alpha) = 1 + \frac{g^2}{(4\pi^2)^272\alpha}$ | $\epsilon \rightarrow -6\alpha$ | $\epsilon \rightarrow -6\alpha$ |

This table shows that knowing the SDD of a given diagram in both theories allows to obtain the renormalization constant in one theory knowing the renormalization constant in the other. In other words for a given diagram $G$ the same replacement that sends $\omega_\epsilon(G)$ to $\omega_\alpha(G)$, sends $Z_G(\epsilon)$ to $Z_G(\alpha)$. This fact shows that an expansion in powers of $\alpha$ and the $\epsilon$ expansion, are not same and describe different critical theories, this is so because of the non-trivial dependence of this replacement on the diagram considered.

5. **Unitarity bounds**

5.1. **The conformal algebra in $n$-dimensions.** The action (2.1) is invariant under conformal transformations$^7$. It is assumed that there exists conserved charges implementing these transformations at the level of the field. The conformal algebra for dimensions $n \geq 3$ is given by,

$$[D, P_\mu] = iP_\mu$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)$$
$$[D, K_\mu] = -iK_\mu$$
$$[K_\rho, P_\mu] = 2i(\eta_{\rho\mu}D - L_{\mu\nu})$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\sigma}L_{\mu\rho} + \eta_{\rho\sigma}L_{\nu\mu} - \eta_{\rho\mu}L_{\nu\sigma} - \eta_{\rho\sigma}L_{\mu\nu})$$

$^6$If such a replacement where posible then an expansion in powers of $\alpha$ would be the same as the $\epsilon$ expansion.

$^7$Given that this theory can be thought as a theory depending on derivatives of the field of any order, then there should be an infinite number of conserved charges. This assertion is not analysed in this paper.
where $P_\mu$ are the generators of translations, $L_{\mu\nu}$ the generators of rotations in the $\mu-\nu$ plane, $D$ the generator of dilatations and $K_\mu$ the generators of special conformal transformations. In cylindrical coordinates the hermiticity properties of operators are such that $P_\mu^\dagger = K_\mu$.

5.2. **Positive definite inner products and bounds for $\alpha$.** For a spinless primary state $|\Delta>$ the commutation relation between $P_\mu$ and $K_\nu$ can be used to show that,

$$|P_\mu|\Delta>|^2 > 0 \Rightarrow \Delta > 0$$

(5.1)

$$|P_\mu P_\nu|\Delta>|^2 > 0 \Rightarrow \Delta > \frac{n-2}{2}$$

for a space of dimension $n$. For the free theory $\lambda = 0$, the dimension of the field $\phi$ is,

$$[\phi] = \frac{n-2+4\alpha}{2}$$

thus the unitarity bound (5.1) implies,

$$\alpha > 0$$

For the interacting theory,

$$[\phi] = \frac{n-2+4\alpha}{2} + \frac{\eta}{2}$$

thus the unitarity bound implies,

$$\alpha > -\frac{\eta}{4}$$

therefore, using (3.5), this implies that,

$$\alpha + \frac{1}{2} \left( \frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561} \right) > 0$$

the polynomial on the l.h.s. of the last inequality has only one real root at $\alpha = 0$, and the last inequality is equivalent to $\alpha > 0$. Showing that the free theory unitarity bound is stable under the corrections computed in this work.

6. **Concluding remarks**

Conclusions and further research motivated by this work are summarized in the series of remarks given below,

- It was shown that introducing a non-local kinetic term for a scalar field with interaction $\phi^4$, it is possible to get asymptotic freedom in the UV without including non-abelian vector fields. For $\alpha < 0$, the resulting theory can not be Wick rotated to obtain a field theory over Minkowski space realizing a unitary representation of the Poincaré group. This is so because the condition of reflection positivity is not fulfilled for these values of $\alpha$.

- In order to obtain that result the renormalization of the theory was considered. This was done for the first corrections to the two and four point functions. The concrete renormalization procedure shows features different from the case of local theories, which all the same make sense. In this respect although the usual renormalization program in field theory is formulated for local interactions, nevertheless the properties of a system involving non-local terms under rescaling of distances is in the same footing as a system involving only local ones from the point of view of the Wilsonian renormalization group.
CRITICAL BEHAVIOUR OF A NON-LOCAL $\phi^4$ FIELD THEORY AND ASYMPTOTIC FREEDOM

The fact that reflection positivity does not hold for $\alpha < 0$, is confirmed by the violation for $\alpha < 0$ of the unitarity bounds obtained assuming that the theory provides a representation of the conformal algebra. This rise up the question of the symmetries for the non-local action (2.1), which is an interesting subject to be considered.

Summarizing, it is believed that the study of non-local field theories can enlarge our knowledge about the fixed points and renormalization group flows in the space of all possible couplings mentioned in the introduction.

Acknowledgements.

I am deeply indebted to G. Torroba for sharing his expertise on the renormalization group and for many enlightening discussions.

References

[1] Kenneth G. Wilson and Michael E. Fisher. Critical exponents in 3.99 dimensions. Phys. Rev. Lett., 28:240–243, 1972.
[2] K. G. Wilson and John B. Kogut. The Renormalization group and the epsilon expansion. Phys. Rept., 12:75–200, 1974.
[3] V. A. Fateev and A. B. Zamolodchikov. Parafermionic Currents in the Two-Dimensional Conformal Quantum Field Theory and Selfdual Critical Points in Z(n) Invariant Statistical Systems. Sov. Phys. JETP, 62:215–225, 1985. [Zh. Eksp. Teor. Fiz.89,380(1985)].
[4] A. S. Reddy. Nonlocal Field Theories at Finite Temperature and Density. PhD thesis, Univ. of Minnesota, 2014.
[5] Miguel F. Paulos, Slava Rychkov, Balt C. van Rees, and Bernardo Zan. Conformal Invariance in the Long-Range Ising Model. Nucl. Phys., B902:246–291, 2016.
[6] Peter W. Egolf and Kolumban Hutter. The mean field theories of magnetism and turbulence. Entropy, 19:589, 2017.
[7] R. Trinchero. Scalar field on non-integer dimensional spaces. Int. J. Geom. Meth. Mod. Phys., 09:1250070, 2012.
[8] CG Bollini and JJ Giambiagi. Dimensional renorinalization: The number of dimensions as a regularizing parameter. Il Nuovo Cimento B (1971-1996), 12(1):20–26, 1972.
[9] Gerard ’t Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. Nucl. Phys., B44:189–213, 1972.
[10] Roberto Trinchero. Examples of reflection positive field theories. Int. J. Geom. Meth. Mod. Phys., 15(02):1850022, 2017.
[11] C. G. Bollini, J. J. Giambiagi, and A. Gonzáles Domínguez. Analytic regularization and the divergences of quantum field theories. Il Nuovo Cimento (1955-1965), 31(3):550–561, Feb 1964.
[12] H. Kleinert and V. Schulte-Frohlinde. Critical properties of phi**4-theories. 2001.
[13] Joshua D. Qualls. Lectures on Conformal Field Theory. 2015.