The Length of the Shortest Closed Geodesics on a Positively Curved Manifold

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Abstract

We give a metric characterization of the Euclidean sphere in terms of the lower bound of the sectional curvature and the length of the shortest closed geodesics.

1 Introduction

Let $M$ be a complete connected Riemannian manifold of dimension $d$ and class $C^\infty$. The study of global structure of closed geodesics on $M$ *vis a vis* certain quantitative restrictions on the sectional curvature $K$ of $M$ has attracted considerable interest among researchers. Henceforth, we assume $k$ to be a positive constant. It follows straightforwardly from Morse-Schoenberg index comparison that if $K \geq k^2$ on all tangent 2-planes of $M$, then there must exist on $M$ a closed geodesic whose length is $\leq 2\pi/k$ representing the lowest dimensional homology class of $M$. The purpose of the present paper is to describe a rigidity phenomenon observed when this length is extremal on $M$. More precisely, we prove

**Main Theorem.** If $M$ satisfies $K \geq k^2$ and if the shortest closed geodesics on $M$ have length $= 2\pi/k$, then $M$ is isometric to $S^d_k$, the Euclidean sphere of radius $1/k$ in $\mathbb{R}^{d+1}$.

Note that we make no assumption about the geodesics’ having no self-intersections. There exists an example of a 2-dimensional smooth surface all

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of whose shortest closed geodesics have self-intersections. These examples have some regions where the curvature is negative. Calabi and Cao [CC] has proved that on a positively curved surface, at least one of the shortest closed geodesics is always without self-intersections.

We now mention some related rigidity phenomena. Previously, Sugimoto [Su], improving on an earlier work of Tsukamoto [Ts], proved

**Theorem A.** Suppose that $M$ satisfies $4k^2 \geq K \geq k^2$. If $d$ is odd, assume that $M$ is simply connected. Then, if $M$ has a closed geodesic of length $2\pi/k$, it is isometric to $S^d_k$.

Recall that under the curvature assumption of Theorem A, if $M$ is simply connected, the celebrated Injectivity radius theorem, which is primarily due to Klingenberg (see [CE (§§5.9,10)], [GKM, §§7.5,7] and also [CG], [K] and [Sa2]) states that all closed geodesics on $M$ have length $\geq \pi/k$.

However, we point out that, in general, an assumption on the length of the shortest closed geodesic is a nontrivially weaker condition than an upper bound on the sectional curvature. In fact, it is possible to construct, for any given $k$ and $\delta$, a Riemannian metric on $S^2$ with $K \geq k^2$ and the length of the shortest closed geodesic $\delta$-close to $2\pi/k$ but whose curvature grows arbitrarily large ($S^2$ like surface with highly curved “equator”). This construction means that, from the viewpoint of rigidity theorems in Riemannian geometry, imposing an upper bound on the curvature is not natural in characterizing a Euclidean sphere among complete Riemannian manifolds with $K \geq k^2$ having a shortest closed geodesic of length just $2\pi/k$. For more informations on the curvature bounds and the lengths of closed geodesics, see, for instance, [Sa2].

In the spacial case of dimension 2, we have

**Theorem B (Toponogov [T]).** Suppose that $M$ is an abstract surface with Gauss curvature $K \geq k^2$. If there exists on $M$ a closed geodesic without self-intersections whose length $= 2\pi/k$, then $M$ is isometric to $S^2_k$.

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However, in higher dimensions, there are lens spaces of constant sectional curvature $k^2$ whose geodesics are all closed with the prime ones have no self-intersections, and they are either
(a) homotopic to 0 and have length $= 2\pi/k$, or
(b) homotopically nontrivial and can be arbitrarily short.
See [Sa1]

Of course, it follows from our Main Theorem that

**Corollary.** If $K \geq k^2$ and the shortest closed geodesics that are homotopic to 0 in $M$ have the length $2\pi/k$, then the universal covering of $M$ must be isometric to $S^d$.

Note also that Theorem B is false without the assumption that the closed geodesics have no self-intersections. In fact, for any $k$, one can construct an ellipsoid in $\mathbb{R}^3$ which possesses a prime closed geodesic of length $= 2\pi/k$ and whose curvature is $> k^2$.

Finally, we mention a previous related result of the first author which gives another rigidity solution for the nonsimply connected case.

**Theorem C (Itokawa [I1,2]).** If the Ricci curvature of $M$ is $\geq (d - 1)k^2$ and if the shortest closed geodesics on $M$ have the length $\geq \pi/k$, then either $M$ is simply connected or else $M$ is isometric with the real projective space all of whose prime closed geodesics have length $= \pi/k$.

It is not yet known if our Main Theorem remains true if the assumption on the sectional curvature is weakened to that on the Ricci curvature. However, we point out that examples were shown in [I1,2] so that for the Ricci curvature assumption, the shortest closed geodesics may have length arbitrarily close to $2\pi/k$ without the manifold’s even being homeomorphic to $S^d$. This indicates how delicate the Ricci curvature assumption could be.

## 2 Preliminaries

The purpose of this section is to collect together all the well-known facts and results which will be used in proving the Main Theorem and also to
set straight our notational conventions and normalizations. In this paper, we agree that by the term curve we mean an absolutely continuous mapping \( c : \mathbb{R} \to M \) whose derivative is defined almost everywhere and is \( L^2 \) on each closed interval. We refer to the restriction of a curve to any closed interval as an arc. If \( c \) is a curve and \( a < b \) are reals, we write \( c_{a,b} \) to denote the arc \( c \mid_{[a,b]} \). If \( c \) happens to be differentiable, the normal bundle; respectively, the unit normal bundle of \( c \), which are in fact bundles over \( \mathbb{R} \), are denoted \( \bot c \); respectively, \( U \bot c \). We shall call a curve \( c \) closed if \( c(s+1) = c(s) \) for all \( s \). We denote the set of all absolutely continuous closed curves with \( L^2 \) derivative in \( M \) by \( \Omega \).

For fixed \( a, b \), let \( A_{a,b} \) denote the set of all arcs \( [a,b] \to M \). It is known that \( A_{a,b} \) has the structure of a Riemannian Hilbert manifold where the inner product is given by the natural \( L^2 \) inner product of variation vector fields along a curve. The restriction map \( c \mapsto c_{0,1} \) embeds \( \Omega \) in \( A_{0,1} \) as a closed submanifold so that the Riemannian structure pulls back on \( \Omega \). We refer to the paper [GM2], the book [K], and references cited therein for details.

For \( \gamma \in A_{a,b} \), we define the space \( V'_\gamma \) of all square integrable vector fields \( v \in T_\gamma A_{a,b} \) along \( \gamma \) such that \( v(a) = 0 \) and \( v(b) = 0 \) and such that \( v(s) \in \bot_s \gamma \) for all \( s \), wherever \( \gamma \) is differentiable. If \( c \in \Omega \), we also define the space \( V_c \) of all \( v \in T_c \Omega \) with \( v(s) \in \bot_s c \) almost everywhere. Then, \( V'_{c,0,1} \) is canonically embedded in \( V_c \).

We normalize the energy of \( \gamma \in A_{a,b} \) by

\[
E(\gamma) := \int_a^b |\gamma'(s)|^2 \, ds .
\]

Also, we denote by \( L(\gamma) \) the length of \( \gamma \) in the usual sense. Thus, in our convention, \( L(\gamma)^2 \leq (b-a)E(\gamma) \) with equality if and only if \( \gamma \) is parametrized proportional to arclength. The term geodesic is always understood to mean a nonconstant geodesic. For \( u \in UTM \), the unit tangent bundle, we denote by \( c_u \) the geodesic \( s \mapsto \exp su \). Recall that the critical points of \( E \) on \( \Omega \) are closed geodesics and the constant curves. See either [CE], [GKM], or [Mi].

Let \( c \) be a geodesic and \( a < b \in \mathbb{R} \). The Hessian of \( E \) at \( c_{a,b} \), here regarded as a symmetric bilinear form on \( T A_{a,b} \), is denoted \( H^b_a \). We remind the reader
that if $v \in V'_{c,a,b}$ and is differentiable outside of finitely many points, or if $c \in \Omega$, $a = 0$, $b = 1$, and $v \in V_c$ is differentiable outside of finitely many points in $(a, b)$, then $H^b_a(v, v)$ is given by the index integral

$$-2 \left\{ \int_a^b (\langle v''(s), v(s) \rangle + |v(s)|^2 |c'(s)|^2 K_{v(s)}|c'(s)|ds + \sum_s \langle v(s), \Delta_s v'(s) \rangle \right\}$$

where

$$\Delta_s v'(s) = v'(s_+) - v'(s_-)$$

denotes the jump in $v'(s)$ at one of its finitely many points of discontinuity in the open interval $(a, b)$. See, for example, [Mi], [Bo], or [BTZ]. We write $\iota'(c_{a,b})$ to denote the index of $H^b_a | V'_{c,a,b}$. If $c$ is closed, we put $H^1_0 := H^1_0 | V_c$ and $\iota(c)$ its index. We recall the basic inequality

$$\iota(c) \geq \iota'(c) = \sum_{0 < s < 1} \nu'(c_{0,s})$$

where $\nu'(c_{0,s})$ is the dimension of the space of Jacobi fields in $V'_{c_{0,s}}$. In this notation, we state the following well-known theorem, which is primarily due to Fet [F].

**Theorem D.** Assume that $M$ satisfies $K \geq k^2$. Then there exists a closed geodesic $c$ on $M$ such that $L(c) \leq 2\pi/k$ and $\iota(c) \leq d - 1$.

For each $r \in \mathbb{R}$, we denote by $\Omega^r$ (respectively, $\Omega^{=r}$ and $\Omega^{<r}$) the subspaces \(\{c \in \Omega : E(c) \leq r \text{ (respectively, } = r \text{ and } < r)\}\). However, $\Omega^0 = \Omega^{=0}$ is identified with $M$ itself and so denoted also by $M$. It is well-known that the energy functional $E$ satisfies the famous Condition C of Palais and Smale. See, for example [GM2]. The significance of this for us is that, as far as global variational-theoretic properties are concerned, we can treat $E$ as if it were a proper function defined on a locally compact manifold.

Alternatively, we can work on the finite-dimensional approximation of $\Omega^r$ à la [Mi (§16)] or [Bo]. While this has the advantage that it simplifies the analytical aspect of the argument, we prefer to use the infinite-dimensional argument in §3 because of the ease by which we can write the variational vector fields explicitly. Not that we could write the corresponding fields explicitly in the finite-dimensional approximation, but the actual expressions would be unpleasantly complicated.
We must later consider a more general functional $F$ on $\Omega$ or $\Omega'$ (in §4). Let $c \in \Omega$ be a critical point of $F$. Then $T_c \Omega$ decomposes into a direct sum

$$T_c \Omega = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{Z}$$

where $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{Z}$ are the spaces on which the Hessian $H_F$ of $F$ at $c$ is positive definite, negative definite and zero respectively. We write $\| \cdot \|$ for the norm in $T_c \Omega$. Then, we can state the following important fact due to Gromoll and Meyer [GM1] (See especially the note on p.362).

**Theorem E** (Generalized Morse Lemma). In the setting described above, there exists a neighborhood $U$ of $c$, a coordinate chart

$$\xi_c : U \longrightarrow T_c \Omega,$$

with respect to which $F$ takes the form

$$F \circ \xi_c^{-1}(v) = \|x\|^2 - \|y\|^2 + f(z) + F(c)$$

where $x$, $y$ and $z$ are the orthogonal projections of $v \in \xi_c^{-1}(U)$ on $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{Z}$ respectively, and $f$ is a function whose Taylor series expansion at $z = 0$ starts with the term of degree at least 3 in $z$ or equivalently with vanishing Hessian. For this decomposition, $c$ need not be an isolated critical point of $F$, but if $F$ has other critical points in $U$, their images in $T_c \Omega$ are all contained in $\mathcal{Z}$.

The chart $\xi_c$ is often called the *Gromoll-Meyer-Morse chart* at $c$ with respect to $F$.

We put $U_c^{-} := \xi_c^{-1}(\mathcal{N})$ and $U_c^{0} := \xi_c^{-1}(\mathcal{N} \oplus \mathcal{Z})$ and call them the *strong unstable submanifold* and the *weak unstable submanifold* of $F$ at $c$ respectively, even though we make no assumption that $\dim \mathcal{Z}$ is finite.

Suppose that $a \in \mathbb{R}$. We set $\Omega^a_{F} := \{c \in \Omega : F(c) \leq a\}$. Let $I$ be the interval $[-1, 1]$. Suppose $c$ is a critical point of $F$ with $a := F(c)$ and $\iota := \text{index } H_{F|c} = \dim \mathcal{N}$. Let $U$ be a neighborhood of $c$ as defined in Theorem E.

We call a differentiable embedding $\sigma : (I^\iota, \partial I^\iota) \longrightarrow (\Omega, \Omega^a_{F} - U)$ a *weak unstable simplex* (resp. *strong unstable simplex*), if there exists a smaller neighborhood $W$ of $c$, $c \in W \subset U$, so that $\sigma(I^\iota) \cap W$ coincides with $\xi^{-1}(\mathcal{N} \oplus$
\[ \mathcal{Z} \text{ (resp. } \xi^{-1}(N) \cap W). \] If \( \sigma \) is a weak unstable simplex of \( F \) at \( c \) in this sense, it is clear that \( \sigma \cap W \) must be contained in the topological cone

\[ \{ \gamma \in W ; H_F(\xi(\gamma), \xi(\gamma)) \leq 0 \} \]

containing \( \xi^{-1}(N) \). Distinguishing from weak or strong unstable simplex just defined above, we mean, by \textit{unstable simplex} of \( F \) at \( c \), a differentiable embedding \( \sigma : (I^t, \partial I^t) \rightarrow (\Omega, \Omega^k - U) \) such that \( \sigma(0) = c \) and \( F|\sigma \leq a \).

We shall say that a critical point \( c \) of \( F \) is \textit{nondegenerate} if \( \mathcal{Z} = \{0\} \). Note that this agreement is different from the often-used convention of calling a closed geodesic nondegenerate if \( \mathcal{Z} \) is the \( S^1 \) orbit of the geodesic. With our convention, a closed geodesic is never a nondegenerate critical point for \( E \) because of the \( S^1 \)-action. We put \( a := F(c) \) and write \( \Omega^r := \{ \gamma \in \Omega ; F(\gamma) \leq r \} \). If \( c \) is a nondegenerate critical point of \( F \), then of course \( c \) is an isolated critical point and, for some \( \varepsilon > 0 \), the strong unstable simplexes at \( c \) represent a nontrivial class in the relative homotopy group \( \pi_\imath(X^{a+\varepsilon}, X^{a-\varepsilon}) \).

### 3 Proof of the Main Theorem

It is clear that, in order to prove Theorem 1, we need to consider only one specific \( k \). So, hereafter we assume that \( M \) satisfies \( K \geq k^2 \) where \( k := 2\pi \). In the present section, we further assume that \( M \) contains no closed geodesic of length < 1, or equivalently that there are no critical points of \( E \) in \( \Omega^{<1} - M \). It now remains for us to prove that then \( M \) is isometric to \( S^d_{2\pi} \).

We set

\[ \mathcal{C} := \{ c \in \Omega ; c \text{ is a closed geodesic of length } 1 \text{ and } \imath(c) = d - 1 \} \]

and

\[ \mathcal{C}^* := \{ c \in \mathcal{C} ; \text{ an unstable simplex of } E \text{ at } c \text{ represents a nontrivial element in } \pi_{d-1}(\Omega, M) \} . \]
Theorem D and the Morse-Schoenberg index comparison assert that $C \neq \emptyset$. If we can assume that each $c \in C$ has an isolated critical $S^1$-orbit, the technique of Gromoll and Meyer [GM2] fairly readily shows that $C^*$ too is non-empty. In our case, however, it will be precisely one of our points that no $c \in C^*$ has an isolated critical orbit. Under the stronger hypothesis of $4k^2 \geq K \geq k^2$, Ballman [Ba] showed that all closed geodesics have nontrivial unstable simplexes. However, he makes essential use of the upper bound for $K$ which is not available to us. Nonetheless, we shall still prove in the next section,

**Lemma 1.** Under the assumptions of this section, $C^*$ is nonempty and is a closed set in $\Omega$.

In this section, we accept Lemma 1 for the time being, and prove

**Lemma 2.** For each $c \in C^*$, there is a neighborhood $U$ of $c'(0)$ in $UT_{c(0)}M$ such that whenever $u \in U$ and $\tau$ is any tangent 2-plane containing $c_u'(s)$ for some $s \in \mathbb{R}$, then $K(\tau) = k^2$ ($c_u$ being the geodesic determined by the initial condition $c_u'(0) = u$).

We prove Lemma 2 by proving a sequence of other Lemmas (from 3 to 8). The idea for proving Lemma 2 is to construct for every $c \in C$ a specific unstable simplex $\tau$ which is homotopic to the strong unstable simplex and a deformation of such a $\tau$ so that, unless the conclusion of the lemma is met, $\tau$ is deformed into $M \subset \Omega$, which is a contradiction if $c \in C^*$. First, we show

**Lemma 3.** If $c \in C$, then for any $s \in \mathbb{R}$ and any $v \in \mathbb{R}$, $K(c'(s) \wedge v) \equiv k^2$.

*Proof.* Assume that, for some $s_1 \in \mathbb{R}$ and $v_1 \in \mathbb{R}$, $K(c'(s_1) \wedge v_1) > k^2$. By virtue of the natural $S^1$-action on $\Omega$, it is no loss of generality to assume that $0 < s_1 < 1/2$. Now, we define a real number $\delta$ as follows. If there is a point in $(0, s_1]$ which is conjugate to 0 along $c$, we choose any $\delta$ so that $s_1 < \frac{1}{2} - \delta < \frac{1}{2}$. If, on the other hand, there is no conjugate point in $(0, s_1]$, there is a unique Jacobi field $Y$ along $c$ with $Y(0) = 0$ and $Y(s_1) = v_1$, and by a consequence of the original Rauch comparison theorem [CE (§1.10, Remark, p.35)], there is an $s_2$, $s_1 < s_2 < 1/2$ so that $Y(s_2) = 0$. In this case,
we choose $\delta$ so that $s_2 < \frac{1}{2} - \delta < \frac{1}{2}$. In either case, we have $\iota'(c_{0, \frac{1}{2} - \delta}) \geq 1$.

On the other hand, by the Morse-Schoenberg index comparison with $S^d_k$, we have $\iota'(c_{\frac{1}{2} - \delta, 1}) \geq d - 1$, since $L(c_{\frac{1}{2} - \delta, 1}) \geq \frac{1}{2}$. Therefore, we have

$$\iota(c) \geq \iota'(c) \geq \iota'(c_{0, \frac{1}{2} - \delta}) + \iota'(c_{\frac{1}{2} - \delta, 1}) \geq 1 + d - 1 = d,$$

which is a contradiction. $\square$

As a consequence, we have

**Lemma 4.** Jacobi fields in the space $\mathcal{V}'_{c_0,1}$ are constant multiples of the fields $\sin(ks)V(s); 0 \leq s \leq 1$ while the negative eigenfields of $H_E$ are constant multiples of the fields $\sin(\frac{ks}{2})V(s)$, where $V$ is any parallel section in the bundle $U(\perp c|_{[0,1]})$ ($\mathcal{V}'_{c_0,1}$ being the space of all square integrable normal vector fields along the arc $c_{0,1} := c|_{[0,1]}$ with Dirichlet boundary condition, as is defined in §2).

**Remark.** A priori, the holonomy along the loop $c$ might be non-trivial. So, a parallel vector field $V$ of elements in $U(\perp c|_{[0,1]})$ might not close up at $s = 1$. Later we will show that the holonomy along $c$ is trivial.

**Construction of the Araki Simplex.** Let $V_1, \ldots, V_{d-1}$ be parallel vector fields of orthonormal elements in $U(\perp c|_{[0,1]})$. By compactness argument, there exists an $\eta > 0$ so that each orthogonally trajectories geodesics $t \mapsto \exp tx$ where $x \in U \perp c$ has no point focal to $c$ in $t < \arctan \eta$. Define $2(d-1)$ vector fields $X_i(s)$ and $Y_i(s)$ ($0 \leq s \leq 1$) along $c$ as follows. These vector fields are not continuous at $s = 0$ and $s = \frac{1}{2}$.

$$X_i(s) = \begin{cases} V_i(s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < s < 1 \end{cases}$$

and

$$Y_i(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \frac{1}{2} \\ V_i(s) & \text{if } \frac{1}{2} < s < 1 \end{cases}.$$

Let $x = (x_1, \ldots, x_{d-1}) \in I \subset \mathbb{R}^{d-1}$ and $y = (y_1, \ldots, y_{d-1}) \in I \subset \mathbb{R}^{d-1}$, where $I$ is a small interval in $\mathbb{R}^{d-1}$ centered at the origin. We define a $2(d-1)$-dimensional differentiable simplex $\tilde{\sigma}$ in $\Omega$ (here we regard $\Omega$ as a Riemannian
Hilbert manifold consisting of absolutely continuous closed curves with $L^2$ inner product) as follows

$$\tilde{\sigma}(x, y)(s) = \exp_{c(s)} \arctan\{\eta \sin(2\pi s) (\sum_{i=1}^{d-1} (x_i X_i(s) + y_i Y_i(s)))\}.$$ 

Here, by “arctan” of a vector, we will mean for a vector $x \in U \perp c$ the resized vector $(\arctan \|x\|) \frac{x}{\|x\|}$. W. Ballman pointed out to us that Araki [A] constructed a simplex in the same way, i.e., varying Jacobi fields independently outside the zero set, when $M$ is a symmetric space. So, such a simplex may be called Araki simplex. The Araki simplex $\tilde{\sigma}$ consists of curves all passing through $c(0)$ and $c(\frac{1}{2})$.

**Deformation of the Araki Simplex.** We deform the Araki Simplex just constructed in the following way. If $x = y$ we make no change on the corresponding loop. If $x \neq y$, then we make suitable short cuts at the non-trivial angle created by the discrepancy $x \neq y$ at $s = \frac{1}{2}$. For instance, we fix a small positive number $\delta$ and make a short cut between points corresponding to $s = \frac{1}{2} - \delta$ and $s = \frac{1}{2} + \delta$ by a small geodesic arc. After performing this modification and reparametrizing the corresponding loops by arc length, we get a $2(d - 1)$-dimensional simplex $\sigma$ (we call this the “short cut modification”). We note that

(i) the intersection $\tilde{\sigma} \cap \sigma$ consists of those closed curves that are generated by $x = y$ where $x(= y)$ satisfies the condition that the parallel vector field $\sum_{i=1}^{d-1} x_i V_i$ along $c$ closes up at $s = 1$, i.e., variations which integrate global Jacobi fields on $c |_{[0,1]}$;

and

(ii) the vector fields $\sin(2\pi s)X_i(s)$ and $\sin(2\pi s)Y_i(s)$ are naturally regarded as Jacobi fields along $c |_{[0,\frac{1}{2}]}$; respectively, $c |_{[\frac{1}{2},1]}$ which vanish at end points.

If $x \neq y$, then, after performing the above modification, we see that $\sigma(x, y)$ is strictly under the level set $\Omega = 1$ of $E = 1$. We see this, by applying
Rauch type comparison theorem of Berger (Rauch’s second comparison; see [CE (§1.10)]) to variations

\[ \exp_{c(s)} \arctan \{ \sin(2\pi s) \sum_{i=1}^{d-1} x_i X_i(s) \} , \quad s \in [0, \frac{1}{2}] , \quad t \in [0, \eta] \]

of \( c \mid_{[0, \frac{1}{2}]} \), and

\[ \exp_{c(s)} \arctan \{ \sin(2\pi s) \sum_{i=1}^{d-1} x_i Y_i(s) \} , \quad s \in [\frac{1}{2}, 1] , \quad t \in [0, \eta] \]

of \( c \mid_{[\frac{1}{2}, 1]} \) in \( \tilde{\sigma} \). Note that the corresponding comparison variations in \( S_k \) generate great semicircles fixed at the north and south poles.

Summing up, we have

**Lemma 5.** There exists a neighborhood \( W \) of \( c \in C \) in \( \Omega \) so that the \( 2(d-1) \)-dimensional simplex \( \sigma \cap W \) is contained in \( \Omega^1 \).

We now exhibit the unstable simplex \( \tau \) mentioned just after the statement of Lemma 2. In fact, \( \tau \) is the \( (d-1) \)-dimensional subsimplex of the modified Araki simplex \( \sigma \) corresponding to the parameters \( x = -y \). The simplex \( \tau \) is not itself the strong unstable simplex. However, because \( \tau \) and the strong unstable simplex \( \tau' \) constructed by exponentiating the negative eigenspace of \( H_E \) are, downstairs in \( M \), both contained in a tubular neighborhood of the geodesic \( c \), we see

**Lemma 6.** There exists a neighborhood \( U \) of \( c \) in \( \Omega \) such that, for \( \varepsilon > 0 \) sufficiently small and a subneighborhood \( W \subset U \), \( \tau \) constructed above represents the same homotopy class as \( \tau' \) in \( \pi_{d-1}(W, W \cap \Omega^{1-\varepsilon}) \).

**Proof.** Both strong unstable simplex \( \tau' \) and the simplex \( \tau \) are constructed by exponentiating certain variation vector field in a negative cone in \( T_{\tilde{c}} \Omega \) of the \( H_E \). Moreover, choosing a neighborhood \( U \) of \( c \) in \( \Omega \) sufficiently small, we may assume that the set of critical points of \( E \) in \( U \) coincides with the connected critical submanifold, say, \( C \), containing \( c \) which, at each point \( \tilde{c} \in C \), is tangent to \( Z \), where \( Z \) is the zero eigenspace in \( T_{\tilde{c}} \Omega \) of \( H_E \) (ref. Theorem
Therefore, if we choose sufficiently small subneighborhood $W \subset U$ of $c$ (and therefore $\varepsilon > 0$ sufficiently small), we see that there exists a homotopy connecting $\tau$ and $\tau'$ in the space $\pi_{d-1}(W, W \cap \Omega^{1-\varepsilon})$. □

Lemma 6 implies:

Lemma 7. If $c \in C^*$, then there is a neighborhood $U$ of $c$ in $\Omega$ so that, for $\varepsilon > 0$ sufficiently small and a subneighborhood $W \subset U$, $\tau$ constructed inside the modified Araki simplex represents a nontrivial element in $\pi_{d-1}(W, W \cap \Omega^{1-\varepsilon})$.

The reason why we have chosen the “short cut modification” is explained in the following way. If we define a $(d-1)$-dimensional simplex $\tau$ by

$$\tau(x) := \exp_{c(s)} \arctan\{\eta \sin(\pi s) \sum_{i=1}^{d-1} x_i V_i(s)\},$$

then direct calculation of the Hessian implies that $\tau$ also defines an unstable simplex at $c$ and belongs to the same class as $\tau$ in the relative homotopy group. This unstable simplex corresponds to the eigenvector of the index form with negative eigenvalue. In this sense, $\tau$ is more natural than $\tau$. Now define a $(2d-2)$-simplex $\sigma$, which also contains the $(d-1)$-simplex (the one defined by $x = y$ in our simplex $\sigma$) corresponding to the global Jacobi field on $c$, by

$$\sigma(x, y)(s) := \exp_{c(s)} \arctan\{\eta \sum_{i=1}^{d-1} (x_i \sin(\pi s) + y_i \sin(ks)) V_i(s)\} \quad (k = 2\pi).$$

Although this construction is natural, it turns out that it is not clear whether there exists an interval $I$ containing 0 such that $\sigma(I \times I)$ is contained in $\Omega^1$. This is the reason why our construction of $(2d-2)$-simplex $\sigma$ is based on the short cut argument of broken geodesics in the model space, although the unstable simplex $\tau$ does not directly integrate the negative eigenspace of the Hessian of the energy functional $E$ at $c$.

We return to our “short cut modification” and consider the holonomy problem mentioned just after Lemma 4. One of the following two cases is possible. Namely, either
(A) For at least one choice of $x_0 \in I$, there is some $\varepsilon > 0$ such that
\[
\exp_{c(s)} \arctan \left\{ t \sin(2\pi s) (\sum_{i=1}^{d-1} x_{0,i} X_i(s) + x_{0,i} Y_i(s)) \right\}
= \exp_{c(s)} \arctan \left\{ t \sin(2\pi s) (\sum_{i=1}^{d-1} x_{0,i} V_i(s)) \right\},
\]
is contained in $\Omega^{1-2\varepsilon}$ for all $t \in (\eta/2, \eta]$. In the picture of this situation, we find two variation vector fields $V$ and $Y$ along $c$ of the form
\[
Y = \sin(2\pi s) \sum_{i=1}^{d-1} x_i V_i(s) \quad \text{(Jacobi fields)},
\]
\[
V = \sin(2\pi s) \sum_{i=1}^{d-1} (x_i X_i(s) - x_i Y_i(s)) \quad \text{(tangent to $\tau$)},
\]
outside a small neighborhood of $s = \frac{1}{2}$. In the picture of the simplex $\sigma$, we find a $(d-1)$-dimensional simplex $\tau \cap W$ which lies in $\Omega^{<1}$ except at $c$ and moreover we have extra one direction represented by $x_0$ which also behaves exactly like a strong unstable simplex,

or else,

(B) There exists an $\alpha; 0 < \alpha < 1$ so that whenever $|x_1|, \ldots, |x_{d-1}| \leq \alpha$,
\[
\exp_{c(s)} \arctan \left\{ \sin(2\pi s) (\sum_{i=1}^{d-1} x_i V_i(s)) \right\}
\]
lies in $\Omega^{=1}$. (In particular, each parallel vector field $\sum_{i=1}^{d-1} x_i V_i(s)$ closes up at $s = 1$.)

If Case (A) prevails, $\tau \cap W$ rides on a $d$-dimensional submanifold of $\sigma \cap W$ which lies in $\Omega^{<1}$ except at $c$ and hence $\tau \cap W$ can be deformed into $W \cap \Omega^{1-\varepsilon}$. This contradicts the conclusion of Lemma 7. Hence $c \not\in C^*$. If, on the other hand, we start out with a $c \in C^*$, then Case (B) must really be the case. In particular, the holonomy along $c \in C^*$ must be trivial. We thus get a $(d-1)$-dimensional local submanifold $S$ of $\Omega^{-1}$ which is tangent to the $0$ eigenspace of the Hessian of $E$ defined on $V_{c_0,1}$ through $c$.

**Lemma 8.** In the present situation, each parallel vector field $\sum_{i=1}^{d-1} x_i V_i(s)$ closes up at $s = 1$, i.e., the holonomy along $c$ is trivial, and each member $\tilde{c}$ of $S$ is a (smooth) closed geodesic in $C$. 

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Proof. We need to prove the second assertion. If \( \tilde{c} \) is not a critical point of \( E \), there exists at least one \( x_0 \in I \subset \mathbb{R}^{d-1} \) such that the \( (d-1) \)-dimensional simplex defined by the \( (d-1) \)-dimensional affine subspace through \( x_0 \) orthogonal to the linear subspace defined by \( x = y \) contains no critical point. Then, by following the trajectory of \( -\text{grad} \ E \), \( \tau \) is deformed into \( W \cap \Omega^{1-\varepsilon} \), which contradicts the assumption that we started with \( c \in C^* \). Hence all \( \tilde{c} \in S \) are closed geodesics. If some \( \tilde{c} \not\in C \), then, it follows from Lemma 3 and its proof that \( \iota(\tilde{c}) > d-1 \). So \( \tau \) is again deformed into \( W \cap \Omega^{1-\varepsilon} \) via the unstable simplex of \( \tilde{c} \). Hence, either way, we get a contradiction. \( \Box \)

By construction, we also see that for any \( \tilde{c} \in S \), \( \tilde{c}(0) = c(0) \). Translated into \( M \), this means that there is an open tube \( B \) (cone-like at \( s = 0 \) and \( \frac{1}{2} \)) around the set \( c(0, \frac{1}{2}) \cup c(\frac{1}{2}, 1) \) such that for each \( q \in B \), a geodesic joining \( c(0) \) to \( q \) extends to a closed geodesic in \( C \) whose image lies in \( B \) except at \( s \in \frac{1}{2}\mathbb{Z} \). Applying Lemma 3 to each geodesics proves Lemma 2. \( \Box \)

Even more is true.

Lemma 9. Let \( c \in C^* \) and let \( U \subset T_{c(0)} M \) be the set in Lemma 2. Then, there exists an open set \( U^* : c'(0) \in U^* \subset U \), so that, for all \( u \in U^* \), \( c_u \in C^* \).

Proof. Since \( c \in C^* \) is a closed geodesic in the compact Riemannian manifold \( M \), there exists an \( \varepsilon > 0 \) such that there are no critical values for \( E \) on \( \Omega \) in the intervals \((1-\varepsilon, 1)\) and \((1, 1+\varepsilon)\). Lemma 8 implies that there exists an open set \( U^* \) in \( T_{c(0)} M \) containing \( c'(0) \) satisfying the condition that \( u \in U^* \) implies \( c_u \in C \). The strong unstable simplex constructed by exponentiating the negative eigenspace of \( H_E \) defines a differentiable simplex \( \tau : (I, \partial I) \rightarrow (\Omega^{1+\varepsilon}, \Omega^{1-\varepsilon}) \) \( (I \subset \mathbb{R}^{d-1}) \). We introduce the he compact-open topology to the set \( \Sigma \) of all absolutely continuous maps \( (I, \partial I) \rightarrow (\Omega^{1+\varepsilon}, \omega^{1-\varepsilon}) \), by which we can argue the closeness of maps in \( \Sigma \). Then if \( U^* \) is a sufficiently small open set containing \( c'(0) \) in \( T_{c(0)} M \), then, for \( \forall u \in U^* \), we can construct, by exponentiating \( (d-1) \)-dimensional negative eigenspace of \( H_E \) at \( c_u \), a \( (d-1) \)-dimensional strong unstable simplex which is homotopic in \( \Sigma \) to \( \tau \) constructed above. \( \Box \)

That is to say, the set
\[ U^* = \{ u \in UT_{c(0)} M ; c_u \in C^* \} \]

is an open set in \( UT_{c(0)} M \). On the other hand, by Lemma 1 and the continuous dependence of geodesics on their initial values, the set \( U^* \) is also a closed set. Since \( UT_{c(0)} M \) is connected, \( U^* \) must in fact be all of \( UT_{c(0)} M \). Together with Lemma 2, we summarize our result as

**Lemma 10.** Let \( M \) be assumed in this section. Then, there exists a point \( p \in M \) such that for all \( u \in T_p M \), \( c_u \) is a closed geodesic of prime length 1 and \( K(\tau) = k^2 \) for all 2-planes \( \tau \) tangent to the radial direction from \( p \).

**Proof.** Take a \( c \in C^* \) and let \( p := c(0) \). \( \square \)

Now it is a standard technique to construct an explicit isometry from \( M \) onto \( S^d_k \) exactly as in Toponogov’s Maximum diameter theorem (see, for instance, [CE (§6.5)] or [GKM (§7.3)]). Thus, the Main Theorem is proven as soon as Lemma 1 is established.

## 4 Proof of Lemma 1

In this section we work on the finite-dimensional approximation (because we need some analytic argument which is described simpler in the finite-dimensional approximation). Recall that \( \Omega \) is the space of all absolutely continuous closed curves with \( L^2 \) derivative and in particular contains piecewise differentiable curves. It is well-known that for each \( r > 0 \), \( \Omega^r \) (it is defined just after Theorem D in §2) contains a submanifold \( '\Omega_r \) which is diffeomorphic to an open set in some finite product \( M \times \cdots \times M \) and homotopy equivalent to \( \Omega^r \). \( '\Omega = ' \Omega_r \) consists of broken geodesics. The functional \( E \) becomes a proper function on \( '\Omega_r \). The space \( '\Omega_r \) contains all the critical points in \( \Omega^r \) and the Hessian of \( E \mid_{\Omega_r} \) retains the same index as \( E \) at each critical point. Moreover, Theorem E in §2 remains true in this finite-dimensional setting. For details, see [Mi (§16)] and [Bo]. If \( a < r \), we put \( '\Omega_r^a := '\Omega_r \cap \Omega^a \).

In this section, we continue to assume \( K \geq 4\pi^2 \). The following proposition is essentially contained in some earlier works of M. Berger and is easy to prove by Morse-Schoenberg index comparison with \( S^d_k \) and the tautological isomorphism \( \pi_i(\Omega) \cong \pi_{i+1}(M) \).
Proposition 1. If $M$ contains no closed geodesic of length $\leq 1/2$, then $M$ has the homotopy type of a sphere. In particular, we have

$$\pi_i(\Omega, M) \cong \begin{cases} 
\mathbb{Z} & \text{if } i = d - 1 \\
0 & \text{for } 0 \leq i \leq d - 2
\end{cases}$$

for the relative homotopy groups $\pi_i(\Omega, M)$ up to $i \leq d - 1$.

We now return to the assumption that the length of the shortest closed geodesics on $M$ is 1. Let $C$ and $C^*$ be as defined in §3. We wish to prove that a strong unstable simplex at at least one $c \in C$ represents a nontrivial class of $\pi_{d-1}(\Omega, M)$. Our technique will be to approximate $E$ with other functionals that are guaranteed to have nontrivial unstable simplexes. Although all our arguments carry through in all of $\Omega$ in an $S^1$-invariant fashion, essentially because the functional $E$ satisfies the Condition (C) of Palais and Smale and because an $S^1$-invariant formulation of Theorem E is available [GM2], we find it a little easier to work in a finite dimensional space.

More precisely, choose $r$ sufficiently large, say $r > 2$. Then, all closed geodesics not in $\Omega^r$ will have index $> 2(d - 1)$. Let $'\Omega := '\Omega_r$. Then

$$\pi_i('\Omega, M) \cong \pi_i(\Omega, M)$$

for all $i; 0 \leq i \leq 2d - 3$, and $d - 1 \leq 2d - 3$ if $d \geq 2$. Using Theorem E and a partition of unity on $'\Omega$, we can approximate $E$ with a sequence $\{E_n\}_{n=1}^\infty$ of functionals on $'\Omega$ with the following properties.

(i) $\lim_{n \to \infty} E_n = E$ in the $C^2$ topology.

(ii) For some $\varepsilon > 0$, all critical points of $E_n$ in the closure of the set $L := '\Omega_{1+\varepsilon} - '\Omega_{1-\varepsilon}$ either belong to $\Omega_{=1}$ or have index $\geq 2d - 2$, and outside $L$, each $E_n$ agrees with $E$.

(iii) Each $E_n$ has only nondegenerate critical points in the set $L$, all of which have index $\geq d - 1$.

Let $C$ be the set of all closed geodesics in $'\Omega=1$ and let $C_n$ be the set of all critical points of $E_n$ that lie in $L$. 

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Lemma 11. For each \( n \), there exists in \( C_n \), at least one critical point of \( E_n \) that possesses a strong unstable simplex that represents a nontrivial element in \( \pi_{d-1}(\Omega, M) \).

Proof. From the topology described in Proposition 1, there must exist a nontrivial element \( \rho \) of \( \pi_{d-1}(\Omega, M) \). We first deform \( \rho \) so that the only points of \( C_n - (M \cap C_n) \) that lies on the image of \( \rho \) are the relative maxima of \( E_n \circ \rho \). In fact, since there are no critical points of index \( < d - 1 \) except in \( M \), at every critical point of \( E_n \) lying on \( \rho \), say \( c \), other than relative maxima, the unstable dimension of \( E_n \) in \( \Omega \) is strictly greater than the unstable dimension of \( E_n \circ \rho \) in the image of \( \rho \). Therefore, in some neighborhood of \( c \) in which a chart of the form described in Theorem E is valid, we can deform \( \rho \) in a direction transversal to itself and which decreases \( E_n \). Since the critical points of \( E_n \) are isolated and \( \rho \) is contained in a compact region, by repeating this deformation a finite number of times and by deforming \( \rho \) along the trajectory of \( -\text{grad } E_n \), we can deform \( \rho \) until it is expressed as a sum of disjoint simplexes, each summand of which is a simplex in \( (\Omega, M) \), hanging from a single critical point of index \( = d - 1 \). Such critical points must be in \( C_n \), and at least one summand must be nontrivial itself. We deform this simplex by a differentiable homotopy, if necessary, into a strong unstable simplex (in the sense defined just after Theorem E) in \( \pi_{d-1}(\Omega, M) \).

Of course, it is not necessarily true that a sequence of critical points \( \{c_n\} \) of \( C_n \) converges to a closed geodesic. However, that \( \lim_{n \to \infty} C_n \subset C \) in the following weaker sense is clear.

Lemma 12. Given any open neighborhood \( U \) of \( C \) in \( \Omega \), whenever \( n \) is large enough, \( C_n \subset U \).

In fact, since the convergence is specified in the \( C^2 \) topology, we can state the even stronger

Lemma 13. Let \( \{U_c^- \subset U_c^{0-}\}_{c \in C} \) be a family of pairs of open sets in \( \Omega \) so that, for each \( c \in C \), \( U_c^- \) is a neighborhood of the strong unstable submanifold \( U_c^- \) of \( E \) at \( c \) and \( U_c^{0-} \) is a neighborhood of the unstable submanifold \( U_c^{-0} \). Then, for \( n \) sufficiently large, for each \( c_n \in C_n \), there exists some \( c \in C \), so that \( U_{c_n}^- \), the strong unstable manifold of \( E_n \) at \( c_n \) is contained in \( U_c^{0-} \).
Moreover, for such $c_n$ and $c$, a strong unstable simplex $\tau_n$ of $c_n$ contains a subsimplex $\tau'_n$ with $\dim \tau'_n = \dim U_c^- = \iota(c)$ which is actually contained in $U_c^-$. 

To see the above, we can take a local coordinate expression around each $c \in C$ as described in Theorem E and look at the partial derivatives. By taking $n$ large, if $c_n \in C_n$ is close to $c \in C$, the corresponding second derivatives respectively of $E_n$ at $c_n$, $E$ at $c_n$ and $E$ at $c$ can all be made arbitrarily close to each other by the property (i). But, in $U$, the strong unstable submanifolds and unstable submanifolds are determined by the second partial derivatives.

Now, for each $n$, let $c_n$ be the critical point in Lemma 11 which has a strong unstable simplex $\tau_n$ that is nontrivial in $\pi_{d-1}(\Omega, M)$. For such a $c_n$, $\tau_n \cap U$ must itself be contained in a neighborhood $U_c^-$ of the strong unstable submanifold at some $c \in C$ by index comparison and the dimensional consideration. From the construction of $\tau_n$, this $c$ must be $\in C$. Let $\tau$ be a strong unstable simplex at $c$ with $\tau(\partial I^{d-1}) \subset M$. By repeating the standard Morse theoretic arguments as in Lemmas 6-7, we see that $\tau$ represents a nontrivial element in $\pi_{d-1}(\Omega, M)$. Hence $c \in C^*$. Then, that $C^*$ is closed follows from the topological arguments in the proof of Lemma 9. This completes the proof of Lemma 1 and thus of Main Theorem. □

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