Topology Data Analysis of 9×9 Range Image Patches

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Abstract. We use the NEB method to deal with data with the property of high dimension from a range image database. From the Brown database we select a large random sample of log-valued, high contrast, normalized, 9×9 range image patches. We create a Morse function via a density estimator. We make 1-dimensional cell complexes from the image data. We detect some topological properties of 9×9 range image patches, find that there exist two types of subsets of 9×9 range image patches modelled as a ring.

1. Introduction
In data analysis Computational topology turns into an efficient method to analyze large sets of multidimensional data for all branches of science and engineering [1], [2], [3], [4]. To investigate multidimensional data, we create a series of complexes from the range image patch data to produce a simple bulletin. The constructed complexes usually compose of lots of simplexes, often they are too large to deal with. Adams [5] used the NEB to make cell complexes by density functions of the given data, they made models for nonlinear data sets with cell complexes, effectually detected the homology. It reveals that cell complex models are the ways for analyzing multidimensional data with high efficiency.

We find a ring model for 9 × 9 range image patches through the method of [5], and give some topological characteristics of 9 × 9 range image patches. With different parameters of σ and subsets of X, we compute and get the 0-cells and 1-cells with certain densities. All the 1-cells make various circles. This method can be used to deal with bigger size range image patches than 9 × 9.

2. Background
We briefly introduce some topics: CW complexes, Morse theory.

2.1. CW complexes.
A CW complex is a type of cell complex [5]. For k a nonnegative integer, a k-cell is the closed ball \[ \{ y \in R^k : |y| \leq 1 \} \] of dimension k. So a 0-cell is a point, a 1-cell is a line segment, a 2-cell is a disk, etc. We form the k-skeleton \( W^{(k)} \) by gluing the boundaries of k-cells to the (k-1)-skeleton \( W^{(k-1)} \). If \( W \) is a finite-dimensional CW complex, then this process terminates and we have \( W = W^{(k)} \) for some k.

2.2. Morse theory.
The following introduction to Morse theory is informal; see [6] for thorough treatment. Suppose \( M \) is a compact manifold of dimension d and Morse function \( f: M \rightarrow R \) is smooth with non-degenerate critical points \( m_1, \ldots, m_k \in M \) satisfying

\[ a_0 < f(m_2) < a_1 < f(m_2) < \cdots < a_{k-1} < f(m_k) < a_k \]
The index $\lambda_i$ of critical point $m_i$ is the number of linearly independent directions around $m_i$ in which $f$ decreases. So a minimum has index 0, a maximum has index $d$, and a saddle point has index between 1 and $(d-1)$. Let $M_a = f^{-1}\left((-\infty, a]\right)$ be the sublevel set corresponding to $a \in \mathbb{R}$. Morse theory tells us that each $M_a$ is homotopy equivalent to a CW complex with one $\lambda_i$-cell for each critical point $m_i$. In particular, $M_a$ is homotopy equivalent to $M_{a-1}$ with a single $\lambda_i$-cell attached. For instance, $M_{a_1}$ is homotopy equivalent to a point and is obtained from $M_{a_0}$, the empty set, by attaching a single 0-cell. Figure 1 contains an example.

![Figure 1](Morse theory example. Manifold M is a torus. There are four critical points $m_1, \ldots, m_4$ with indices 0, 1, 1, and 2 respectively. Let $a_0 < f(m_1) < a_1 < \cdots < f(m_4) < a_4$. Moving from left to right, we attach cells of dimension 0, 1, 1, and 2 in order to obtain $M_{a_1}$ from $M_{a_{i-1}}$.)

3. Data set of range image patches
The Brown range image database by Lee and Huang is a set of 197 range images from indoor and outdoor scenes, mostly 444 × 1440 pixels. The operational range for the Brown scanner is typically 2–200 meters, and the distance values for the pixels are stored in units of 0.008 meters. The database can be found at the following webpage: http://www.dam.brown.edu/ptg/brid/index.html. From the Brown database we obtain a space of range image patches through the following steps [5]:

1. We randomly select about $5 \cdot 10^5$ size $9 \times 9$ patches from the images in the database. Each patch is represented by a vector $x \in \mathbb{R}^{81}$ with logarithm values.

2. We compute the contrast norm $\|x\|_2 = \sqrt{\sum_{i,j}(x_i - x_j)^2}$ of each patch and select the patches with contrast norm in the top 20% of the entire sample.

3. We subtract from each patch the average of its coordinates and divide by the contrast norm.

4. We change to the DCT basis $\{e_1, \ldots, e_{80}\}$ for $9 \times 9$ patches, normalized to have contrast norm one. This maps the patches to an 80-dimensional sphere.

Let $R$ be the resulting set of high-contrast, normalized, $9 \times 9$ range patches. Our data set is a random subset $X \subset R$ of size 5,000. See Figure 2 for an example of range image patches from the Brown database.

![Figure 2](Forest scene -- Sample range image from the Brown database.)
4. Computing method
In this part, the steps of calculation procedure are given. refer to the paper [5] for more details of the method. For \( X \subset \mathbb{R}^n \) from probability density function \( f: \mathbb{R}^n \to (0, \infty) \). We get superlevel sets \( X^\alpha = f^{-1}([\alpha, \infty)) = \{ x \in \mathbb{R}^n | f(x) \geq \alpha \} \), important topological information is given by the high dense regions. We may make CW complex models \( Z^\alpha \) to estimate \( X^\alpha \).

Only the 1-dimensional skeleton of the cell complex is given. We give a differentiable function to estimate the unknown density function, then we can get the partial maximum of the density estimate in order to make 0-cells, we make original bands randomly, then discover the convergent bands through NEB [8], so we can get the 1-cells.

5. Experimental results
Now we analyze subsets of \( X_9 \) with two types subsets of \( X_n \): (1) \( \overline{X}_9(50000) \) of \( X_n \) with size 50000 randomly; (2) core subsets \( \overline{X}_9(k, p) \), with \( k=200,300 \) and \( p=30 \). Table 1 gives the detailed parameters used in the experiments [9], [10].

| Table 1. Data set information |
|-----------------------------|
|                           | \( \overline{X}_9(50000) \) | \( \overline{X}_9(200,30) \) | \( \overline{X}_9(300,30) \) |
| Size of data set           | 50000                        | 15000                        | 15000                        |
| Dimension n                | 9×9                          | 9×9                          | 9×9                          |
| Standard deviation \( \sigma \) | 0.37                         | 0.31                         | 0.34                         |

We deal with the set \( \overline{X}_9(50000) \) with \( \sigma=0.37 \), finding four 0-cells, whose densities are 960.6,1359,912.7,882.6, and four 1-cells with densities of 918.7,796.6,888.0,773.7, these cells make a ring. (Figure 3).

![Figure 3](image)

**Figure 3** \( \overline{X}_9(50000) \) and the ring \( Z^{773.7} \), projected to a plane.

For \( \overline{X}_9(200,30) \), with \( \sigma=0.31 \), there are four 0-cells with densities in 3713,1761,2379,2040 and four 1-cells with densities in 1903,1698,1277,1156, and they form a ring (Figure 4).
For $\mathcal{X}_g(300,30)$, by $\sigma = 0.34$, there are four 0-cells with densities in 1770, 2055, 2402, 3763 and four 1-cells with densities in 1157, 1284, 1717, 1929, and they form a circle too (Figure 5).

6. Conclusions
We utilize the NEB method to analyze the data of 9×9 Range image patches, the experimental results indicate that the spaces of 9×9 range image patches having different subsets modeled as a ring. There are four quarter-circular 1-cells forming a loop. We test the way on range image patch data sets and find compact complexes showing important nonlinear topology structures. The results got with this method are native properties of Range image patches, and they do not change with different methods and data sets. But there are still lots of problems to make higher dimensional cells with this method [5].

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