Inverse source problem with a posteriori boundary measurement for fractional diffusion equations

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Communicated by: F. Colombo
Funding information
Agence Nationale de la Recherche, Grant/Award Number: ANR-17-CE40-0029; Estonian Research Council, Grant/Award Number: PRG832

In this article, we study inverse source problems for time-fractional diffusion equations from a posteriori boundary measurement. Using the memory effect of these class of equations, we solve these inverse problems for several class of space- or time-dependent source terms. We prove also the unique determination of a general class of space–time-dependent separated variables source terms from such measurement. Our approach is based on the study of singularities of the Laplace transform in time of boundary traces of solutions of time-fractional diffusion equations.

KEYWORDS
fractional diffusion equations, inverse problems, inverse source problems, memory effect
MSC CLASSIFICATION
35R30, 35R11

1 INTRODUCTION

1.1 Statement

Let \( \Omega \subset \mathbb{R}^d \) \((d \geq 2)\) be an open bounded and connected subset with a \( C^2 \) boundary \( \partial \Omega \). We define an elliptic operator \( \mathcal{A} \) on the domain \( \Omega \) by

\[
\mathcal{A}u(x) := -\sum_{i,j=1}^{d} \partial_{x_i} \left( a_{i,j}(x) \partial_{x_j} u(x) \right) + q(x)u(x), \quad x \in \Omega,
\]

where the potential \( q \in L^\infty(\Omega) \) is nonnegative, and the diffusion coefficient matrix \( a := (a_{i,j})_{1 \leq i,j \leq d} \in C^1(\bar{\Omega}; \mathbb{R}^{d \times d}) \) is symmetric, that is, \( a_{i,j}(x) = a_{j,i}(x) \), for any \( x \in \bar{\Omega} \), \( i, j = 1, \ldots, d \), and fulfills the following ellipticity condition:

\[
\exists c > 0 : \sum_{i,j=1}^{d} a_{i,j}(x) \xi_i \xi_j \geq c |\xi|^2, \quad x \in \bar{\Omega}, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.
\]

Let the weight function \( \rho \in L^\infty(\Omega) \) satisfy the condition

\[
0 < c_0 \leq \rho(x) \leq C_0 < +\infty \text{ in } \Omega,
\]

with \( c_0, C_0 \) two positive constants. From now on and in all the remaining parts of this article, we set \( \mathbb{R}_+ := (0, +\infty) \) and \( \mathbb{N} := \{1, 2, \ldots\} \). We introduce the Riemann–Liouville integral operator \( I^\beta \) and the Riemann–Liouville fractional derivative \( D_t^\beta \), of order \( \beta \in (0, 1) \), as follows:
\[ I^\beta h(\cdot, t) := \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(\cdot, \tau)}{(t - \tau)^{1-\beta}} \, d\tau, \quad D_t^\beta := \partial_t I^{1-\beta}. \]

We define also the Caputo fractional derivative \( \partial_t^\beta \), of order \( \beta \in (0, 1) \), by

\[ \partial_t^\beta h = D_t^\beta (h - h(\cdot, 0)), \quad h \in C([0, +\infty); L^2(\Omega)). \]

We recall that, for all \( \beta \in (0, 1) \), we have

\[ \partial_t^\beta h = I^{1-\beta} \partial_t h, \quad h \in W_{loc}^{1,1}(\mathbb{R}^+_t; L^2(\Omega)). \]

Fixing \( F \in L^1(\mathbb{R}^+_t; L^2(\Omega)) \) and \( \alpha \in (0, 1) \), we consider weak solutions (in the sense of Definition 2.1) \( u \) of the following initial boundary value problem:

\[
\begin{cases}
\rho \partial_t^\alpha u + Au = F(x, t), & \text{in } \Omega \times \mathbb{R}_+, \\
u = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\
u = 0, & \text{in } \Omega \times \{0\}.
\end{cases}
\tag{1.4}
\]

With reference to Kian and Yamamoto [1, 2], one can check that (1.4) admits a unique weak solution lying in \( L^1_{loc}(\mathbb{R}^+_t; H^s(\Omega)) \), \( 1 \leq s < 2 \). Fixing \( T \in \mathbb{R}_+ \), in the present article, we assume that the source term \( F \) satisfies the following condition:

\[ F(x, t) = \sigma(t) f(x), \quad t \in (0, T), \quad x \in \Omega, \tag{1.5} \]

with \( \sigma \in L^1(0, T) \) and \( f \in L^2(\Omega) \). We denote by \( \partial_{\nu} \) the conormal derivative associated with the coefficient \( \alpha \) of \( \partial\Omega \) defined by

\[ \partial_{\nu} v(x) = \sum_{j=1}^d a_{ij}(x) \partial_\nu v(x) v_i(x), \quad x \in \partial\Omega, \]

where \( \nu = (v_1, \ldots, v_d) \) denotes the outward unit normal vector of \( \partial\Omega \). We assume that there exists \( \delta \in (0, T) \) such that the time-dependent part \( \sigma \) of the source term in (1.4) satisfies the following condition:

\[ \sigma(t) = 0, \quad t \in (T - \delta, T). \tag{1.6} \]

Then, we consider three inverse problems:

(IP1) Assuming that \( \sigma \) is known and \( \sigma \neq 0 \), determine \( f \) from knowledge of \( \partial_{\nu} u(x, t), \quad (x, t) \in \Gamma \times (T - \epsilon, T) \) with \( \Gamma \) an arbitrary open subset of \( \partial\Omega \) and with \( \epsilon \in (0, T) \) arbitrary small.

(IP2) Determine the space–time-dependent function \( \sigma(t) f(x), \quad t \in (0, T) \) and \( x \in \Omega \), from knowledge of \( \partial_{\nu} u(x, t), \quad (x, t) \in \Gamma \times (T - \epsilon, T) \) with \( \Gamma \) an arbitrary open subset of \( \partial\Omega \) and with \( \epsilon \in (0, T) \) arbitrary small.

(IP3) Assuming that \( f \) is known and \( f \neq 0 \), determine \( \sigma \) from the knowledge of \( \partial_{\nu} u(x_0, t), \quad t \in (T - \epsilon, T) \) for some \( x_0 \in \partial\Omega \) and for \( \epsilon \in (0, T) \) arbitrary small.

Note that the condition (1.6) will be fulfilled if for instance the time-dependent part \( \sigma \) of the source take the form

\[ \sigma(t) = g(t) \chi(t), \quad t \in (0, T), \]

with \( g \in L^1(0, T) \) and \( \chi \) any function supported on \([0, T - \delta]\).

### 1.2 Motivations

We recall that systems of the form (1.4) model anomalous diffusion phenomena appearing in applied sciences. This includes models of geophysics, environmental science, and biology [3, 4]. For such models, sub-diffusive processes are described by (1.4) and the kinetic equation (1.4) describes the corresponding macroscopic model to microscopic diffusion phenomena driven by continuous time random walk [5]. The equation in (1.4) is derived from models of continuous in time random walk associated with the probability density function describing the probability of a walker to be at time \( t \) to a position \( x \) [6–8]. In these models, the chosen waiting time is proportional to \( t^{-1-\alpha} \) which is justified by the experiments.
In (1.4), we consider general class of time-fractional diffusion equations generated by the source \( F \) with variable coefficients of diffusion \( \rho \) and \( a \) associated to the density of the medium. Such models of time-fractional diffusion equations and the corresponding fractional derivatives and integrals have been the subject of study of many scientific contributions [9–13].

In this context, the goal of the inverse problems (IP1)–(IP3) is to determine a source of anomalous diffusion from a posteriori measurement. This class of problems may appear in several practical situations such as environmental accident where the data are often available only after the occurrence of such accident. We study this problem by exploiting the memory effect of solutions of (1.4).

### 1.3 Known results

We recall that among the different formulations of inverse problems, inverse source problems for time-fractional diffusion equations have received a lot of attention last decades. The interested reader can refer to Jin and Rundell [3] and Liu et al. [14] for an overview of this class of problems. Most of the results for this class of inverse problems are uniqueness equations have received a lot of attention last decades. The interested reader can refer to Jin and Rundell [3] and Liu et al. [14] for an overview of this class of problems. Most of the results for this class of inverse problems are uniqueness results stated for source terms with separated variables of the form \( F(x, t) = \sigma(t)f(x) \), \( x \in \Omega \), \( t \in (0, T) \). For this class of source terms, one can refer to previous studies [15–18] for the determination of the time-dependent component \( \sigma(t) \) from measurement at one point and to Jiang et al. [19] and Kian et al. [20] for the determination of the space-dependent component \( f(x) \) from internal data. We refer also to Kilbas et al. [21] for the stable recovery of the space-dependent component \( f(x) \) in this class of problems. We can also mention the work of Jin et al. [22] and Kian and Yamamoto [23] dealing with the determination of a source term independent of one space direction in a cylindrical domain and the work of Kinash and Janno [24] who have considered the determination of a general space–time source term from the full knowledge of the solution close to the final time.

Most of the above-mentioned results are stated with measurement throughout the full interval of time \((0, T)\). We are only aware of the three results [24–26] considering measurement during an interval of time of the form \((T - \varepsilon, T)\) with \( \varepsilon \in (0, T) \) arbitrary small. In all these three works, the authors use the memory effect of time-fractional diffusion equations exhibited by Kinash and Janno [24, Theorem 1] which cannot be applied in the context of the inverse problems (IP1)–(IP3).

### 1.4 Main results

Our main results will be devoted to the study of each of the inverse problems (IP1)–(IP3).

For (IP1), we obtain the following result.

**Theorem 1.1.** Let \( \sigma \in L^1(0, T) \), \( f \in L^2(\Omega) \) with \( \sigma \not\equiv 0 \) satisfying (1.6). Consider also \( F \in L^1(\mathbb{R}_+; L^2(\Omega)) \) satisfying (1.5) and let \( \Gamma \) be an open and not empty subset of \( \partial \Omega \). Then, for any \( \varepsilon \in (0, T) \), the following implication

\[
\partial_x u|_{\Gamma \times (T-\varepsilon, T)} \equiv 0 \implies f \equiv 0
\]

holds true.

As a consequence of Theorem 1.1, we obtain the following result for (IP2).

**Theorem 1.2.** Let \( \alpha \) be irrational and let \( \sigma_j \in L^1(0, T) \), \( f_j \in L^2(\Omega) \), \( j = 1, 2 \). Assume also that condition (1.6) is fulfilled with \( \sigma = \sigma_j \), \( j = 1, 2 \). For \( j = 1, 2 \), consider \( F \in L^1(\mathbb{R}_+; L^2(\Omega)) \). For \( j = 1, 2 \), satisfying (1.5) with \( \sigma = \sigma_j \) and \( f = f_j \), and let \( u_j \) be the weak solution of (1.4) corresponding to the source term \( F = F_j \). Let also \( \Gamma \) be an open and not empty subset of \( \partial \Omega \). Then, for any \( \varepsilon \in (0, T) \), the following implication

\[
\partial_x u_1|_{\Gamma \times (T-\varepsilon, T)} \equiv \partial_x u_2|_{\Gamma \times (T-\varepsilon, T)} \implies F_1|_{\Omega \times (0, T)} \equiv F_2|_{\Omega \times (0, T)}
\]

holds true.

Let us introduce the operator \( A = \rho^{-1}A \) acting on \( L^2(\Omega; \rho dx) \) with domain \( D(A) := \{ h \in H^1_0(\Omega); \rho^{-1}Ah \in L^2(\Omega; \rho dx) \} \). Here, we denote by \( L^2(\Omega; \rho dx) \) the space of measurable functions \( v \) satisfying \( \int_{\Omega} |v|^2 \rho dx < \infty \) endowed with the inner product \( \langle u, v \rangle = \int_{\Omega} uv \rho dx \). Note that under condition (1.3), we have \( L^2(\Omega; \rho dx) = L^2(\Omega) \) with equivalent norm, and thus, we distinguish only the inner products but not the spaces. With reference to Section 2, we consider, for all \( r > 0 \), \( D(A^r) \) and we denote by \( \rho D(A^r) \) the space of functions \( h \) such that \( \rho^{-1}h \in D(A^r) \). Assuming that \( \Omega \) has a \( C^{\frac{1}{2}+[\frac{1}{4}]} \) boundary, \( a \in C^{1+[\frac{1}{4}]}(\bar{\Omega}; \mathbb{R}^d) \), \( \rho \in C^{1+[\frac{1}{4}]}(\bar{\Omega}) \), and applying the elliptic regularity of the operator \( \rho^{-1}A \), one can check that \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). In addition, from Grisvard [28, Theorem 2.5.1.1], we deduce that, for all \( \ell = 1, \ldots, \lceil \frac{d}{4} \rceil + 1 \), we have
By interpolation, for all \( r > 0, \) \( D(A^r) \) is embedded continuously into \( H^{2r}(\Omega) \). Combining this with the fact that \( 0 \) is not in the spectrum of \( A \) and applying the Sobolev embedding theorem, we deduce that for all \( f \in \rho D(A^s), \) with \( s > \frac{d-2}{2} \), we have \( A^{-k}f \in C^r(\tilde{\Omega}), k \in \mathbb{N}, \) and we can define the set

\[
G(f) := \{ x \in \partial \Omega : \text{there exists } k \in \mathbb{N} \cup \{ 0 \} \text{ such that } \partial_{\nu}A^{-k}f(x) \neq 0 \text{ and } a(k+1) \notin \mathbb{N} \}. \tag{1.10}
\]

For (IP3), our result is a uniqueness result which can be stated as follows.

**Theorem 1.3.** We assume that \( \Omega \) has a \( C^{2,1} \) boundary, \( \alpha \in C^{1,1}([0, \tilde{T}); \mathbb{D}^d) \), \( \rho \in C^{1,1}([0, \tilde{T}); \mathbb{D}^d) \). Let \( \sigma \in L^1(0, T) \) satisfy the condition (1.6), \( f \in \rho D(A^s), \) with \( s > \frac{d-2}{4} \), be a non-identically vanishing function and let \( F \in L^1(\mathbb{R}_+; L^2(\Omega)) \) satisfy (1.5). Then, the weak solution \( u \) of (1.4) is lying in \( L^1(0, T; C^1(\tilde{\Omega})) \). Moreover, the set \( G(f) \), defined by (1.10), is dense on \( \partial \Omega \) and for all \( x_0 \in G(f) \) and all \( \varepsilon \in (0, T) \) the implication

\[
(\forall t \in (T - \varepsilon, T), \quad \partial_{\nu}u(x_0, t) = 0) \Rightarrow \sigma \equiv 0
\]

holds true. In addition, assuming that there exists \( k_1 \in \mathbb{N} \cup \{ 0 \} \) and \( g \in D(A^{k_1+s}) \) of constant sign such that \( f = \rho A^{k_1}g, \) we have \( G(f) = \partial \Omega \) and the implication (1.11) holds true for any \( x_0 \in \partial \Omega \).

Let us observe that the results of Theorems 1.1, 1.2, and 1.3 are all stated with a posteriori boundary measurement restricted to an arbitrary small interval of time of the form \( (T - \varepsilon, T) \) where \( T \) denotes the final time. We are only aware of the two articles [25, 26] studying this class of inverse source problems with such data. While the results of Kian et al. [26] are stated with internal data, in the results of Kian [25], the measurements are given by \( \partial_{\nu}u(x, t) \) and \( \partial_{\nu}^{d-1}\partial_{\nu}u(x, t) \), \( (x, t) \in \Gamma \times (T - \varepsilon, T) \) with \( \Gamma \) an arbitrary open subset of \( \partial \Omega \) and with \( \varepsilon \in (0, T) \) arbitrary small. In both of these results, the authors apply the memory effect exhibited in Kinash and Janno [24, Theorem 1] in order to determine from these class of data restricted to in interval of time of the form \( (T - \varepsilon, T) \) similar type of data on the full interval \( (0, T) \). When the boundary measurement are only restricted to \( \partial_{\nu}u(x, t) \), \( (x, t) \in \Gamma \times (T - \varepsilon, T) \), this approach does not work and a different strategy should be considered for the resolution of this problem. In the present article, we introduce a new strategy based on the study of the singularities of the Laplace transform in time of solutions of (1.4), with \( F = 0 \) on \( \Omega \times (T, +\infty) \), in order to analyze the inverse problems (IP1)–(IP3). Our results show that the memory effect of (1.4) can also be applied to the boundary measurement under consideration in inverse problems (IP1)–(IP3).

Note that in all the results of this article, the source terms \( F \) of (1.4) are unknown on \( \Omega \times (T, +\infty) \), and we do not try to determine such values. This can be equivalently seen as the statement of our inverse problems on the set \( \Omega \times (0, T) \) instead of \( \Omega \times \mathbb{R}_+ \).

During the preparation of this article, we realized that a result similar to Theorem 1.1 has been obtained, simultaneously and independently of this article, in Yamamoto [29, Theorem 2] as a consequence of the main results of the same article. Our proof of Theorem 1.1 is completely different from the one under consideration in Yamamoto [29] where the author uses asymptotic properties of solutions while our analysis is mostly based on the study of singularities of Laplace transform in time of solution of (2.2) which coincides with the solution of (1.4) with \( F = 0 \) on \( \Omega \times (T, +\infty) \).

Let us mention that, to the best of our knowledge, in Theorem 1.2, we obtain the first result of full determination of a general space–time-dependent source term of the form \( \sigma(t)f(x) \) from boundary measurement. Indeed, in all the results that we are aware of (see, e.g., previous studies [19, 20, 26]), some a priori knowledge of the time-dependent component \( \sigma(t) \) or the space-dependent component \( f(x) \) of such source terms are required for the determination of this type of source terms. We recall also that there is a natural obstruction for the determination of general source terms of the form \( \sigma(t)f(x) \) for classical diffusion equations corresponding to (1.4) with \( \alpha = 1 \) (see, e.g., Kian et al. [20, Theorem 7.1]). In that sense, Theorem 1.2 emphasizes the particularity of time-fractional diffusion equations where in contrast to classical diffusion equations such inverse problems can be solved.

Notice that the result of Theorem 1.3 is stated with measurement at one point \( x_0 \) located on the explicit subset \( G(f) \) of \( \partial \Omega \) defined by (1.10). This formulation allows to state the result of Theorem 1.3 for general non-uniformly vanishing space-dependent part \( f(x) \) of the source term \( \sigma(t)f(x) \) while other similar results are often stated with some extra condition imposed to \( f \) (see, e.g., previous studies [15, 17, 18]). In addition, we prove in Theorem 1.3 that the set \( G(f) \) is dense in \( \partial \Omega \) which means that the result of Theorem 1.3 holds true for at least one point in any arbitrary chosen open set of \( \partial \Omega \). We exhibit also in Theorem 1.3 a general condition guarantying that \( G(f) = \partial \Omega \).
All the results of this article can be easily extended to (1.4) with homogeneous Neumann boundary condition instead of the homogeneous Dirichlet boundary condition. Moreover, with some minor modifications, our results can also be applied to super-diffusive models given by (1.4) with \( \alpha \in (1, 2) \). Nevertheless, Theorem 1.2 is not true for classical diffusion equations of the form (1.4) with \( \alpha = 1 \), and it is not clear whether Theorems 1.1 and 1.3 hold true for classical diffusion equations.

1.5 | Outline

This article is organized as follows. In Section 2, we recall some properties of solutions of (1.4) and we collect some tools for our inverse problems. Then, in Section 3, we prove Theorem 1.1 while Sections 4 and 5 will be devoted to the proof of Theorems 1.2 and 1.3.

2 | PRELIMINARY PROPERTIES

In this section, we introduce several preliminary properties related to the definition of solutions, the well-posedness, and regularity properties for the initial boundary value problem (1.4). We start by introducing the definition of weak solutions for problem (1.4) as follows.

**Definition 2.1.** We say that \( u \in L^1_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \) is a weak solution to (1.4) if it satisfies the following conditions.

1. The identity \( \rho(x)D_t^\alpha u(x, t) + Au(x, t) = F(x, t), (x, t) \in \Omega \times \mathbb{R}_+ \) holds true in the sense of distributions in \( \Omega \times \mathbb{R}_+ \).
2. \( L^{1-\alpha}u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+; H^{-2}(\Omega)) \) and \( L^{1-\alpha}u(0, x) = 0, \ x \in \Omega \).
3. \( \rho_0 = \inf \{ \rho > 0 : e^{-\rho t}u \in L^1(\mathbb{R}_+; L^2(\Omega)) \} < \infty \) and there exists \( r_1 \geq r_0 \) such that for all \( p \in \mathbb{C} \) satisfying \( \Re p > r_1 \) we have

\[
\hat{u}(\cdot, p) := \int_0^\infty e^{-\rho t}u(\cdot, t) dt \in H_0^1(\Omega).
\]

According to Kian and Yamamoto [1, 2], assuming that \( F \in L^1(\mathbb{R}_+; L^2(\Omega)) \), one can check that problem (1.4) admits a unique weak solution in the sense of Definition 2.1.

Recall that the spectrum of the operator \( A \) consists of an increasing sequence of strictly positive eigenvalues \( (\lambda_n)_{n \geq 1} \). In the Hilbert space \( L^2(\Omega; \rho dx) \), for each eigenvalue \( \lambda_n \), we fix also \( m_n \in \mathbb{N} \) the algebraic multiplicity of \( \lambda_n \) and the family \( \{ \phi_{n,k} \}_{k=1}^{m_n} \) of eigenfunctions of \( A \), which forms an orthonormal basis in \( L^2(\Omega; \rho dx) \) of the algebraic eigenspace of \( A \) associated with \( \lambda_n \). For all \( s \geq 0 \), we denote by \( A^s \) the operator defined by

\[
A^s g = \sum_{n=1}^{+\infty} \sum_{k=1}^{m_n} \langle g, \phi_{n,k} \rangle \lambda_n^s \phi_{n,k}, \quad g \in D(A^s) = \left\{ h \in L^2(\Omega) : \sum_{n=1}^{+\infty} \sum_{k=1}^{m_n} |\langle h, \phi_{n,k} \rangle|^2 \lambda_n^{2s} < \infty \right\},
\]

and in \( D(A^s) \), we introduce the norm

\[
\|g\|_{D(A^s)} = \left( \sum_{n=1}^{+\infty} \sum_{k=1}^{m_n} |\langle g, \phi_{n,k} \rangle|^2 \lambda_n^{2s} \right)^{\frac{1}{2}}, \quad g \in D(A^s).
\]

With reference to Podlubny [11], we introduce the Mittag-Leffler function \( E_{\alpha,\beta}(\tau) \) defined by

\[
E_{\alpha,\beta}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(n\alpha + \beta)}, \quad \tau \in \mathbb{C},
\]

and we define the operator

\[
S(t)h = \sum_{n=1}^{+\infty} \sum_{k=1}^{m_n} t^{\alpha-1} E_{\alpha,\beta}(-\lambda_n t^\alpha) \langle h, \phi_{n,k} \rangle \phi_{n,k}, \quad h \in L^2(\Omega), \ t \in \mathbb{R}_+.
\]

Let us consider the following initial boundary value problem:

\[
\begin{cases}
\rho \partial_t^\alpha v + Av = \sigma(t)f(x), & \text{in } \Omega \times \mathbb{R}_+, \\
v = 0, & \text{on } \partial \Omega \times \mathbb{R}_+, \\
v = 0, & \text{in } \Omega \times \{0\}.
\end{cases}
\]
where \( \sigma \) denotes the extension of the function \( \sigma \in L^1(0, T) \) by 0 into an element of \( L^1(\mathbb{R}_+) \).

It is well known that the weak solution of problem (2.2) is given by

\[
v(\cdot, t) = \int_0^t \sigma(s)S(t-s)(\rho^{-1}f)ds, \quad t \in \mathbb{R}_+
\]

and that \( v = u \) on \( \Omega \times (0, T) \), where \( u \) denotes the solution of (1.4) with \( F \in L^1(\mathbb{R}^+; L^2(\Omega)) \) satisfying (1.5). In addition, following Jin and Kian [30, Proposition 2.2], Jin and Kian [31, Lemma 2.2], and Kian et al. [20, Proposition 2.1.], one can check the following results.

**Lemma 2.2.** Let \( f \in L^2(\Omega) \) and \( \sigma \in L^1(0, T) \) satisfy (1.6). Then, the problem (2.2) admits a unique weak solution \( v \in L^1_{loc}(\mathbb{R}_+; H^1(\Omega)) \) satisfying

\[
\partial_v v(\cdot, t)|_{\Gamma} = \int_0^t \sigma(s)\partial_v [S(t-s)\rho^{-1}f]|_{\Gamma}ds, \quad t \in \mathbb{R}_+.
\]

Moreover, the map \( t \mapsto v(\cdot, t) \) is analytic with respect to \( t \in (T - \delta, +\infty) \) as a map taking values in \( H^1(\Omega) \). Finally, fixing \( w = \partial_v v|_{\Gamma \times \mathbb{R}_+} \), we have

\[
\inf\{p > 0 : e^{-pt}w(\cdot, t) \in L^1(\mathbb{R}_+; L^2(\Gamma))\} = 0,
\]

and for all \( p \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\} \), we have

\[
\partial_v \hat{v}(\cdot, p)|_{\Gamma} = \hat{w}(\cdot, p) = \delta(p) \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \left\{p^{-1}f, \phi_{n,k}\right\}_{\lambda_n + p^a} \partial_v \phi_{n,k} |_{\Gamma}.
\]

**Lemma 2.3.** Assume that \( \Omega \) has a \( C^{2[\frac{d}{2}] + 1} \) boundary, \( a \in C^{1[\frac{d}{2}] + 1}(\overline{\Omega} \cup \mathbb{R}^d) \), \( \rho \in C^{1[\frac{d}{2}] + 1}(\overline{\Omega}) \). Let \( f \in \rho \mathcal{D}(A^s), \) with \( s > \frac{d-2}{4} \) and let \( \sigma \in L^1(0, T) \) satisfy (1.6). Then, the problem (2.2) admits a unique weak solution \( v \in L^1_{loc}(\mathbb{R}_+; C^1(\overline{\Omega})) \) satisfying

\[
\partial_v v(x, t) = \int_0^t \sigma(s)\partial_v [S(t-s)\rho^{-1}f](x)ds, \quad t \in (0, +\infty), \ x \in \partial \Omega.
\]

Moreover, the map \( t \mapsto v(\cdot, t) \) is analytic with respect to \( t \in (T - \delta, +\infty) \) as a map taking values in \( C^1(\overline{\Omega}) \). Finally, for any \( x_1 \in \partial \Omega \) fixing \( w_1 := \mathbb{R}_+ \ni t \mapsto \partial_v v(x_1, t) \), we have

\[
\inf\{p > 0 : e^{-pt}w_1(t) \in L^1(\mathbb{R}_+)\} = 0,
\]

and for all \( p \in \mathbb{C}_+ \), we have

\[
\hat{w}_1(p) = \partial_v (A + p^a)^{-1}p^{-1}f(x_1) = \delta(p) \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \left\{p^{-1}f, \phi_{n,k}\right\}_{\lambda_n + p^a} \partial_v \phi_{n,k}(x_1).
\]

Armed with these results, we are now in position to complete the proof of Theorems 1.1, 1.2, and 1.3.

### 3 | PROOF OF THEOREM 1.1

Let \( \epsilon \in (0, T) \) and let \( \Gamma \) be an open and not empty subset of \( \partial \Omega \). Assume that the solution \( u \) of (1.4) satisfies \( \partial_v u|_{(\Gamma \times (T-\epsilon, T))} \equiv 0 \). Then, recalling that the solution \( v \) of (2.2) satisfies \( v = u \) on \( \Omega \times (0, T) \), we deduce that \( \partial_v v|_{(\Gamma \times (T-\epsilon, T))} \equiv 0 \).

Without loss of generality, we assume that \( \epsilon = \delta \) with \( \delta > 0 \) appearing in (1.6). Applying Lemma 2.2, we deduce that \( (T - \delta, +\infty) \ni t \mapsto \partial_v v(\cdot, t)|_{\Gamma} \) is an analytic function taking values in \( L^1(\Gamma) \). Therefore, fixing \( w = \partial_v v|_{\Gamma \times \mathbb{R}_+} \) and applying unique continuation for holomorphic functions, we find

\[
w(\cdot, t) = \partial_v v(\cdot, t)|_{\Gamma} \equiv 0, \quad t \in (T - \epsilon, +\infty).
\]
Thus, the Laplace transform in time \( \hat{w}(-, p) \) of \( w \) is holomorphic with respect to \( p \in \mathbb{C} \) as a function taking values in \( L^2(\Gamma) \). On the other hand, applying (2.4), we deduce that the following identity

\[
\hat{w}(-, p) = \hat{\sigma}(p) \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + p^n} \partial_v \phi_{n,k} \Gamma
\]  

(3.2)

holds true for all \( p \in \mathbb{C}_+ \). In the same way, one can easily check that the map

\[
\mathcal{K} := p \mapsto \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + p^n} \partial_v \phi_{n,k} \Gamma
\]

admits an holomorphic extension to \( \mathbb{C} \setminus (-\infty, 0] \) as a map taking values in \( L^2(\Gamma) \). Here and in all the remaining parts of the article, for any \( p \in \mathbb{C} \setminus (-\infty, 0] \), we set \( p^a = e^{a \log(p)} \) with log the complex logarithm defined on \( \mathbb{C} \setminus (-\infty, 0] \). Therefore, the identity (3.2) holds true for all \( p \in \mathbb{C} \setminus (-\infty, 0] \). Now, let us fix \( R > 0 \) and consider the identity (3.2) for \( p = \text{Re}^{i\theta} \) with \( \theta \in (-\pi, \pi) \). Sending \( \theta \to \pm \pi \) and using the fact that \( \hat{w}(-, p) \) and \( \hat{\sigma}(p) \) are holomorphic with respect to \( p \in \mathbb{C} \), we obtain

\[
\hat{w}(-, R) = \hat{w}(-, \text{Re}^{i\pi}) = \hat{\sigma} \left( \text{Re}^{i\pi} \right) \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\pi}} \partial_v \phi_{n,k} \Gamma \]

Taking the difference of these two expressions, we obtain

\[
\hat{\sigma}(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\pi}} \partial_v \phi_{n,k} \Gamma \right) \left( \frac{1}{\lambda_n + \text{Re}^{i\pi}} - \frac{1}{\lambda_n + \text{Re}^{i\pi}} \right) = \hat{w}(-, -R) - \hat{w}(-, -R) \equiv 0.
\]

It follows that

\[
0 \equiv \hat{\sigma}(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\pi}} \partial_v \phi_{n,k} \Gamma \right) \left( \frac{\text{Re}^{i\alpha \pi} - e^{i\alpha \pi}}{(\lambda_n + \text{Re}^{i\alpha \pi})(\lambda_n + \text{Re}^{i-\alpha \pi})} \right)
\]

\[
= \hat{\sigma}(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\pi}} \partial_v \phi_{n,k} \Gamma \right) \left( \frac{-2i\text{Re}^{i\alpha \pi} \sin(\alpha \pi)}{(\lambda_n + \text{Re}^{i\alpha \pi})(\lambda_n + \text{Re}^{-2i\alpha \pi} \text{Re}^{\alpha \pi})} \right).
\]

Recalling that \( \alpha \in (0, 1) \), we find \( \sin(\alpha \pi) \neq 0 \) and we deduce that

\[
\hat{\sigma}(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \frac{\langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\alpha \pi}} \partial_v \phi_{n,k} \Gamma \right) \equiv 0.
\]

In the same way, using the fact that \( \hat{\sigma} \) is holomorphic and \( \sigma \neq 0 \), we can find \( R_1, R_2 \in \mathbb{R}_+ \) with \( R_1 < R_2 \) such that \( |\hat{\sigma}(-R)| > 0 \) for \( R \in (R_1, R_2) \). Then, it follows that

\[
\sum_{n=1}^{\infty} \left( \frac{\sum_{k=1}^{m_n} \langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + \text{Re}^{i\pi}} \partial_v \phi_{n,k} \Gamma \right) \equiv 0, \quad r \in (R_1, R_2).
\]

(3.3)

Now, let us consider the set \( \mathcal{O} := \mathbb{C} \setminus \{ -\lambda_n, -e^{-2i\alpha \pi} \lambda_n : n \in \mathbb{N} \} \) and the map

\[
\mathcal{H} : \mathcal{O} \ni z \mapsto \sum_{n=1}^{\infty} \left( \frac{\sum_{k=1}^{m_n} \langle \rho^{-1} f, \phi_{n,k} \rangle}{\lambda_n + z} \partial_v \phi_{n,k} \Gamma \right).
\]

Since \( f \in L^2(\Omega) \), one can easily check that the map \( \mathcal{H} \) is an holomorphic function on \( \mathcal{O} \) as a map taking values in \( L^2(\Gamma) \). Combining this with (3.3) and applying the unique continuation for holomorphic functions, we obtain the following identity:
\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \left( \frac{\rho^{-1} f, \phi_{n,k}}{(\lambda_n + z)(\lambda_n + e^{-2ix\pi}} \right) \right) \equiv 0, \quad z \in \mathcal{O}. \tag{3.4}
\]

Fixing \( n \in \mathbb{N} \) and multiplying (3.4) by \((z + \lambda_n)\) and sending \( z \to -\lambda_n \), we find
\[
\sum_{k=1}^{m_n} \left( \rho^{-1} f, \phi_{n,k} \right) \partial_z \phi_{n,k} \| \Gamma \equiv 0, \quad n \in \mathbb{N}. \tag{3.5}
\]

On the other hand, it is well known that the maps \( \partial_z \phi_{n,k} \| \Gamma, k = 1, \ldots, m_n \) are linearly independent (see, e.g., Canuto and Kavian [32, Lemma 2.1] or Kian et al. [20, Step 4 in the proof of Theorem 1.1] for the proof of an equivalent property), and applying (3.5), we find
\[
\left( \rho^{-1} f, \phi_{n,k} \right) = 0, \quad n \in \mathbb{N}, \quad k = 1, \ldots, m_n.
\]

From this last identity, we deduce that \( \rho^{-1} f \equiv 0 \) and (1.3) implies that \( f \equiv 0 \). This completes the proof of the theorem.

### 4 | PROOF OF THEOREM 1.2

Let us assume that
\[
\partial_u u_{11} \| \Gamma \times (-c, T) \equiv \partial_u u_{21} \| \Gamma \times (-c, T). \tag{4.1}
\]

Let \( v_j, j = 1, 2, \) be the solutions of (2.2) that correspond to the pairs \((f, \sigma) = (f_j, \sigma_j), j = 1, 2, \) respectively, and notice that (4.1) and Lemma 2.2 imply that
\[
\partial_u v_1 \| \Gamma \times (-c, +\infty) \equiv \partial_u v_2 \| \Gamma \times (-c, +\infty). \tag{4.2}
\]

Further, let us define \( w = \partial_u v_{11}(., t) \| \Gamma - \partial_u v_{21}(., t) \| \Gamma, t > 0 \). In view of (4.2) and condition (1.6) with \( \sigma = \sigma_j, j = 1, 2, \) the Laplace transforms of \( w, \sigma_1 \) and \( \sigma_2 \) are holomorphic in \( \mathbb{C} \) and the formula
\[
\hat{\psi}(\cdot, p) = \hat{\sigma}_1(p) \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{m_n} \left( \rho^{-1} f_1, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma}{\lambda_n + p^n} - \hat{\sigma}_2(p) \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{m_n} \left( \rho^{-1} f_2, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma}{\lambda_n + p^n} \tag{4.3}
\]
is valid for \( p \in \mathbb{C} \setminus (-\infty, 0) \). We fix \( R > 0 \) and we set \( p = Re^{ix} \) with \( \theta \in (-\pi, \pi) \) in (4.3). Sending \( \theta \to \pm \pi \) and taking the difference of expressions obtained, we get
\[
\hat{\sigma}_1(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \left( \rho^{-1} f_1, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma \right) \frac{1}{(\lambda_n + R e^{i\alpha x})(\lambda_n + R e^{-i\alpha x})} - \hat{\sigma}_2(-R) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \left( \rho^{-1} f_2, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma \right) \frac{1}{(\lambda_n + R e^{i\alpha x})(\lambda_n + R e^{-i\alpha x})} \equiv 0,
\]
where \( R > 0 \). Replacing \( R^n \) by \( r > 0 \), for all \( r \in \mathbb{R}_+ \), we have
\[
\hat{\sigma}_1\left(-\frac{1}{r} \right) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \left( \rho^{-1} f_1, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma \right) \frac{1}{(\lambda_n + re^{i\alpha x})(\lambda_n + re^{-i\alpha x})} - \hat{\sigma}_2\left(-\frac{1}{r} \right) \sum_{n=1}^{\infty} \left( \sum_{k=1}^{m_n} \left( \rho^{-1} f_2, \phi_{n,k} \right) \partial_u \phi_{n,k} \| \Gamma \right) \frac{1}{(\lambda_n + re^{i\alpha x})(\lambda_n + re^{-i\alpha x})} \equiv 0.
\]

Let us define the set \( \mathcal{O}_1 := \mathbb{C} \setminus \{ e^{iz} \lambda_n : n \in \mathbb{N} \} \cup (-\infty, 0) \) and the map
\[
H_1 : \mathcal{O}_1 \ni z \mapsto \hat{\sigma}_1\left(-\frac{1}{r} \right) \psi_1(\cdot, z) + \hat{\sigma}_2\left(-\frac{1}{r} \right) \psi_2(\cdot, z),
\]
with
\[
\Psi_1(\cdot, z) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{\rho^{-1} f_1(\phi_{n,k})}{(\lambda_n + z e^{iax})(\lambda_n + z e^{-iax})} \right).
\]
\[
\Psi_2(\cdot, z) = -\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{\rho^{-1} f_2(\phi_{n,k})}{(\lambda_n + z e^{iax})(\lambda_n + z e^{-iax})} \right).
\]

Then, for all \( r > 0 \), we have \( H_1(r) = 0 \). Moreover, in a similar way to Theorem 1.1, one can check that the map \( H_1 \) is holomorphic in \( \mathcal{O}_1 \). Therefore, by unique continuation of holomorphic functions, \( H_1 \) vanishes in \( \mathcal{O}_1 \). Now, we set \( z = \varrho e^{i\theta} \) in the relation \( H_1(z) \equiv 0 \) and send \( \theta \) to \( \pm \pi \). We obtain the following system of equations:

\[
\begin{align*}
\hat{\sigma}_1 \left( -\varrho^2 e^{iax} \right) \Psi_1(\cdot, -\varrho) + \hat{\sigma}_2 \left( -\varrho^2 e^{-iax} \right) \Psi_2(\cdot, -\varrho) &\equiv 0, \\
\hat{\sigma}_1 \left( -\varrho e^{iax} \right) \Psi_1(\cdot, -\varrho) + \hat{\sigma}_2 \left( -\varrho e^{-iax} \right) \Psi_2(\cdot, -\varrho) &\equiv 0,
\end{align*}
\]

where \( \varrho > 0 \). Since the maps \( \Psi_j \) are holomorphic in the set \( \mathcal{O}_2 = \mathbb{C} \setminus \{ -e^{\pm ia} \lambda_n : n \in \mathbb{N} \} \), either

(1) \( \Psi_1 = \Psi_2 \equiv 0, \quad j = 1, 2 \) or
(2) there exist \( j_0 \in \{ 1, 2 \} \) and \( r_0 > 0 \) such that \( \| \Psi_{j_0}(\cdot, \varrho) \|_{L^1(\Gamma)} > 0 \) for \( 0 < |\varrho| < r_0 \).

In case (1), using the method presented in the proof of Theorem 1.1, we deduce that \( f_1 \equiv 0 \) and \( f_2 \equiv 0 \) that imply \( F_1|_{\mathcal{O}_0(0,T)} = F_2|_{\mathcal{O}_0(0,T)} \equiv 0 \). In case (2), the determinant of the system (4.4) and (4.5) is zero for \( 0 < \varrho < r_0 \), that is,

\[
\det \begin{pmatrix}
\hat{\sigma}_1 \left( -\varrho^2 e^{iax} \right) & \hat{\sigma}_2 \left( -\varrho^2 e^{-iax} \right) \\
\hat{\sigma}_1 \left( -\varrho e^{iax} \right) & \hat{\sigma}_2 \left( -\varrho e^{-iax} \right)
\end{pmatrix} = 0, \quad 0 < \varrho < r_0.
\]

If \( \hat{\sigma}_1 \equiv 0 \), combining (4.4) with the arguments used at the end of the proof of Theorem 1.1 , we deduce that either \( \sigma_2 \equiv 0 \) or \( f_2 \equiv 0 \) and it follows that \( F_1|_{\mathcal{O}_0(0,T)} = F_2|_{\mathcal{O}_0(0,T)} \equiv 0 \). In the same way, assuming that \( \hat{\sigma}_2 \equiv 0 \), we deduce that \( F_1|_{\mathcal{O}_0(0,T)} = F_2|_{\mathcal{O}_0(0,T)} \equiv 0 \). Therefore, we need to consider the situation where (2) holds true and \( \hat{\sigma}_j \not\equiv 0, \quad j = 1, 2 \). Since \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) are entire and not uniformly vanishing, we can find \( r_1 \in (0, r_0) \) such that

\[
\min(|\hat{\sigma}_1(z)|, |\hat{\sigma}_2(z)|) > 0, \quad 0 < |z| < r_1.
\]

In that situation, either \( \hat{\sigma}_1(z)/\hat{\sigma}_2(z) \) or \( \hat{\sigma}_2(z)/\hat{\sigma}_1(z) \) will have a finite limit as \( z \to 0 \) and then can be extended to an holomorphic function with respect to \( z \in D_{r_1} := \{ \eta \in \mathbb{C} : |\eta| < r_1 \} \). From now on, without loss of generality, we assume that \( \hat{\sigma}_2(z)/\hat{\sigma}_1(z) \) admits an holomorphic extension with respect to \( z \in D_{r_1} \). Then, there exists a function \( C \) holomorphic in \( D_{r_1} \) such that

\[
\hat{\sigma}_2(z) = C(z)\hat{\sigma}_1(z), \quad z \in D_{r_1}.
\]

Using this notation, we can transform Equation (4.6) into

\[
\hat{\sigma}_1 \left( -\varrho^2 e^{iax} \right) \hat{\sigma}_1 \left( -\varrho e^{iax} \right) \left( C \left( -\varrho^2 e^{iax} \right) - C \left( -\varrho e^{iax} \right) \right) = 0, \quad 0 < \varrho < r_1.
\]

This implies that \( C \left( -\varrho^2 e^{iax} \right) \equiv C \left( -\varrho e^{iax} \right) \) for \( 0 < \varrho < r_1 \).

Due to the holomorphy of \( C \), we deduce that

\[
C(z) = C(ze^{iax}), \quad |z| < \frac{1}{\varrho}.
\]

Differentiating this relation at \( z = 0 \), we have

\[
C^{(n)}(0) = C^{(n)}(0) e^{-\frac{2\pi i}{\varrho}} , \quad n \in \mathbb{N}.
\]
Since $a$ is irrational, we have $e^{-\frac{2na}{\pi}} \neq 1$, $n \in \mathbb{N}$. Consequently, $C^{(0)}(0) = 0$, $n \in \mathbb{N}$. Hence, the function $C$ is constant and we have $\sigma_2 \equiv C \sigma_1$. Using the fact that $\delta_j \neq 0$, $j = 1, 2$, we deduce that $C \neq 0$. Then, we can rewrite $\sigma_2(t)f_2(x)$ in the form

$$\sigma_2(t)f_2(x) = \sigma_1(t)\tilde{f}_2(x), \quad (x, t) \in \Omega \times (0, T),$$

where $\tilde{f}_2 \equiv C f_2$. Then, the function $u = v_1 - v_2$ solves (1.4) with the source function $F(x, t) = (f_1(x) - \tilde{f}_2(x))\sigma_1(t)$ and satisfies the condition $\partial_uu_{|F \times (\Omega \times T)} \equiv 0$. Thus, Theorem 1.1 implies $f_1 - \tilde{f}_2 \equiv 0$. Consequently, we have

$$\sigma_2(t)f_2(x) = \sigma_1(t)C f_2(x) = \sigma_1(t)f_1(x), \quad (x, t) \in \Omega \times (0, T),$$

which implies that $F_1|_{\Omega \times (0, T)} \equiv F_2|_{\Omega \times (0, T)}$. This completes the proof of Theorem 1.2.

## 5 | PROOF OF THEOREM 1.3

In this section, we use the notation of Section 2. Recall that $\rho^{-1}f \in D(A^s)$, with $s > \frac{d-2}{4}$, and we deduce that $(A + re^{iax})^{-1}(A + re^{-iax})^{-1}\rho^{-1}f \in H^{2s+2}(\Omega)$, $r > 0$, and the Sobolev embedding theorem implies that $(A + re^{iax})^{-1}(A + re^{-iax})^{-1}f \in C^1(\Omega)$. Therefore, we can define the set $J(f)$ of points $x_0 \in \partial\Omega$ such that for all $\epsilon > 0$, there exists $r_0 \in (0, \epsilon)$ such that the condition

$$\partial_u(A + r_0e^{iax})^{-1}(A + r_0e^{-iax})^{-1}\rho^{-1}f(x_0) \neq 0 \quad (5.1)$$

is fulfilled. The proof of Theorem 1.3 will be divided into four steps. We will start by proving that for $G(f)$, defined by (1.10), we have $G(f) = J(f)$. Then, using this identity, we will prove that the set $G(f)$ is dense in $\partial\Omega$. Then, we prove that, for all $x_0 \in G(f)$, the implication (1.11) holds true. Finally, we will show that when $f = \rho A^{k}$, with $g \in D(A^{k + \varepsilon})$ of constant sign, we have $G(f) = \partial\Omega$.

**Step 1.** In this step, we will show that $J(f) = G(f)$. For this purpose, it would be enough to prove that $\partial\Omega \setminus G(f) = \partial\Omega \setminus J(f)$. Let us fix $\delta_1 = \left( 2\left\| A^{-1} \right\|_{D(A^{k + \varepsilon})} \right)^{-1}$. Since $\rho^{-1}f \in D(A^s)$, we deduce that the map $r \mapsto (A + re^{iax})^{-1}(A + re^{-iax})^{-1}\rho^{-1}f$ is analytic with respect to $r \in (0, +\infty)$ as a function taking values in $D(A^{s+1})$. Combining this with the continuous embedding $D(A^{s+1}) \subset C^1(\Omega)$, one can check that the map $r \mapsto (A + re^{iax})^{-1}(A + re^{-iax})^{-1}\rho^{-1}f$ is analytic with respect to $r \in (0, +\infty)$ as a function taking values in $C^1(\Omega)$. From this last property and the unique continuation for analytic functions, we deduce that $x \in \partial\Omega \setminus J(f)$ if and only if $x \in \partial\Omega$ satisfies the following condition:

$$\exists \epsilon_1 \in (0, \delta_1), \quad \partial_u(A + re^{iax})^{-1}(A + re^{-iax})^{-1}\rho^{-1}f(x) = 0, \quad r \in (0, \epsilon_1). \quad (5.2)$$

On the other hand, for $z_1, z_2 \in D_{\delta_1} := \{ \eta \in \mathbb{C} : |\eta| < \delta_1 \}$, we have

$$(A + z_1e^{iax})^{-1}(A + z_2e^{-iax})^{-1}\rho^{-1}f = (Id + z_1e^{iax}A^{-1})^{-1}(Id + z_2e^{-iax}A^{-1})^{-1}A^{-2}\rho^{-1}f$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-e^{iax})^j z_1^k (-e^{-iax})^j z_2^{-1 - k - 2j} A^{-j - k - 2}\rho^{-1}f.$$

Using the fact $\rho^{-1}f \in D(A^s)$ and applying the Sobolev embedding theorem, we deduce that, for all $z_1, z_2 \in D_{\delta_1}$, we have
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\| \left( -e^{iax} \right)^k c_k \left( -e^{-iax} \right)^j z_j A^{-j-k-2} \rho^{-1} f \right\|_{C^1(\Omega)} \\
\leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\| \left( -e^{iax} \right)^k c_k \left( -e^{-iax} \right)^j z_j A^{-j-k-2} \rho^{-1} f \right\|_{H^{n+2}(\Omega)} \\
\leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\| \left( -e^{iax} \right)^k c_k \left( -e^{-iax} \right)^j z_j A^{-j-k-2} \rho^{-1} f \right\|_{D^{(A+1)}} \\
\leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta_j \left\| A^{-1} \right\|_{B_{(D(A+1))}}^{j+1} \delta_k \left\| A^{-1} \right\|_{B_{(D(A+1))}}^{k} \left\| A^{-1} \right\|_{B(D(A+1),D(A^j))} \left\| \rho^{-1} f \right\|_{D(A^j)} \\
\leq C \left\| A^{-1} \right\|_{B(D(A+1),D(A^j))} \left\| \rho^{-1} f \right\|_{D(A)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-k-2-j} < \infty.
\]

Therefore, the sequence
\[
\sum_{N_1, N_2}^{N_1, N_2} \left( -e^{iax} \right)^k c_k \left( -e^{-iax} \right)^j z_j A^{-j-k-2} \rho^{-1} f, \quad N_1, N_2 \in \mathbb{N}
\]
converges uniformly with respect to \( z_1, z_2 \in D_{\delta_i} \) in the sense of \( C^1(\Omega) \) valued function and, for any \( x \in \partial \Omega \), we have
\[
\partial_{\nu} (A + z_1 e^{iax})^{-1} (A + z_2 e^{-iax})^{-1} \rho^{-1} f(x) \\
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( -e^{iax} \right)^k c_k \left( -e^{-iax} \right)^j z_j A^{-j-k-2} \rho^{-1} f(x), \quad z_1, z_2 \in D_{\delta_i}.
\]

In the above expression, fixing \( z_1 = z_2 = z \in D_{\delta_i} \), we obtain
\[
\partial_{\nu} (A + z e^{iax})^{-1} (A + z e^{-iax})^{-1} \rho^{-1} f(x) \\
= \sum_{n=0}^{\infty} \left[ \sum_{k+j=n} \left( -e^{iax} \right)^k \left( -e^{-iax} \right)^j \right] \partial_{\nu} A^{-n-2} \rho^{-1} f(x) z^n \\
= \sum_{n=0}^{\infty} \left[ \sum_{k+j=n} e^{iax(k-j)} \right] (-1)^n \partial_{\nu} A^{-n-2} \rho^{-1} f(x) z^n \\
= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} e^{iax(2k-n)} \right] (-1)^n \partial_{\nu} A^{-n-2} \rho^{-1} f(x) z^n \\
= \sum_{n=0}^{\infty} \left[ \frac{1 - e^{2iax(n+1)}}{1 - e^{2iax}} \right] e^{-iax n} (-1)^n \partial_{\nu} A^{-n-2} \rho^{-1} f(x) z^n.
\]

Recalling that \( x \in \partial \Omega \setminus G(f) \) if and only if \( x \in \partial \Omega \) satisfies the following condition:
\[
\partial_{\nu} A^{-n-2} \rho^{-1} f(x) = 0, \quad n \in \mathbb{N} \cup \{ 0 \}, \quad \alpha(n+1) \not\in \mathbb{N},
\]
we deduce that
\[
\left[ \frac{1 - e^{2iax(n+1)}}{1 - e^{2iax}} \right] e^{-iax n} (-1)^n \partial_{\nu} A^{-n-2} \rho^{-1} f(x) = 0, \quad x \in \partial \Omega \setminus G(f), n \in \mathbb{N} \cup \{ 0 \}.
\]

Therefore, we have
\[
\partial_{\nu} (A + z e^{iax})^{-1} (A + z e^{-iax})^{-1} \rho^{-1} f(x) = 0, \quad z \in D_{\delta_i}, \quad x \in \partial \Omega \setminus G(f),
\]
which implies that for all \( x \in \partial \Omega \setminus G(f) \), condition (5.2) is fulfilled. This implies that \( \partial \Omega \setminus G(f) \subset \partial \Omega \setminus J(f) \) and we deduce \( J(f) \subset G(f) \). In the same way, from the above argumentation, we deduce that for all \( x \in \partial \Omega \), the map
is holomorphic with respect to $z \in D_{\delta_i}$ and we have

$$\partial_{\nu}(A + ze^{ix})^{-1}(A + ze^{-ix})^{-1} \rho^{-1} f(x) = \sum_{n=0}^{\infty} \left[ \frac{1 - e^{i2ax(n+1)}}{1 - e^{2ix}} \right] e^{-ixn}(-1)^n \partial_{\nu}A^{-n-2} \rho^{-1} f(x)z^n.$$  

Thus, fixing $x \in \partial \Omega \setminus J(f)$, condition (5.2) implies

$$\partial_{\nu}(A + ze^{ix})^{-1}(A + ze^{-ix})^{-1} \rho^{-1} f(x) = 0, \ z \in \Omega.$$  

Then, it follows that

$$n! \left[ \frac{1 - e^{i2ax(n+1)}}{1 - e^{2ix}} \right] e^{-ixn}(-1)^n \partial_{\nu}A^{-n-2} \rho^{-1} f(x) = \partial_x^n \left[ \partial_{\nu}(A + ze^{ix})^{-1}(A + ze^{-ix})^{-1} \rho^{-1} f(x) \right]_{z=0} = 0, \ n \in \mathbb{N} \cup \{0\}.$$  

Therefore, we have

$$\left[ \frac{1 - e^{i2ax(n+1)}}{1 - e^{2ix}} \right] \partial_{\nu}A^{-n-2} \rho^{-1} f(x) = 0, \ x \in \partial \Omega \setminus J(f), \ n \in \mathbb{N} \cup \{0\}, \ (n + 1)\alpha \notin \mathbb{N}.$$  

From this identity, one can easily check that for any $x \in \partial \Omega \setminus J(f)$, we have $x \in \partial \Omega \setminus G(f)$, and combining this with the fact that $\partial \Omega \setminus G(f) \subset \partial \Omega \setminus J(f)$, we deduce that $\partial \Omega \setminus G(f) = \partial \Omega \setminus J(f)$. It follows that $G(f) = J(f)$. 

**Step 2.** In this step, we will prove that $G(f)$ is dense in $\partial \Omega$. In view of Step 1, it would be enough to show that $J(f)$ is dense in $\partial \Omega$. We will show this result by contradiction. Assuming that $J(f)$ is not dense in $\partial \Omega$, we deduce that there exists an open not empty set $\Gamma$ of $\partial \Omega$ such that $\Gamma \subset \partial \Omega \setminus J(f)$. Then, for all $x \in \Gamma$, there exists $e_x > 0$ such that

$$\partial_{\nu}(A + re^{ix})^{-1}(A + re^{-ix})^{-1} \rho^{-1} f(x) = 0, \ r \in (0, e_x).$$  

On the other hand, using the fact that $\rho^{-1} f \in D(A^*)$, we deduce that for all $x \in \partial \Omega$, we have

$$\partial_{\nu}(A + z)^{-1}(A + e^{-2iax}z)^{-1} \rho^{-1} f(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \left( \rho^{-1} f, \phi_{n,k} \right) \partial_{\nu} \phi_{n,k}(x), \ z \in \mathcal{O},$$  

and the map $z \mapsto \partial_{\nu}(A + z)^{-1}(A + e^{-2iax}z)^{-1} \rho^{-1} f(x)$ is holomorphic with respect to $z \in \mathcal{O}$. Thus, the condition (5.4) implies that

$$\partial_{\nu}(A + z)^{-1}(A + e^{-2iax}z)^{-1} \rho^{-1} f(x) = 0, \ z \in \mathcal{O}, x \in \Gamma.$$  

and we deduce that (3.4) holds true. Therefore, repeating the arguments used at the end of the proof of Theorem 1.1, we deduce that (3.4) implies that $f \equiv 0$ which contradicts the fact that $f \neq 0$. This proves the density of the set $G(f) = J(f)$ in $\partial \Omega$. 

**Step 3.** In this step, we will show that for any $x_0 \in G(f)$, the implication (1.11) holds true. For this purpose, we fix $x_0 \in G(f)$, and from Step 1, we deduce that $x_0 \in G(f) = J(f)$. Therefore, for all $e' > 0$, there exists $r_0 \in (0, e')$ such that (5.1) is fulfilled. Let $u$ be the weak solution of problem (1.4) satisfying

$$\partial_{\nu} u(x_0, t) = 0, \ t \in (T - \epsilon, T).$$
Step 4. In this step, we prove the last statement of Theorem 1.1. For this purpose, we fix

and fixing \( h := \mathbb{R}_+ \) \( \ni t \mapsto \partial_n v(x_0, t) \), we deduce that \( \text{supp}(h) \subset [0, T - \epsilon] \). Therefore, the Laplace transform \( \hat{h} \) of \( h \) admits an holomorphic extension to \( \mathbb{C} \). Moreover, (2.5) implies that

\[
\hat{h}(p) = \hat{\sigma}(p)\partial_n(A + p^n)^{-1}\rho^{-1}f(x_0), \quad p \in \mathbb{C}_+.
\]

and it follows that

\[
\hat{\sigma}(-R)\partial_n(A + R^n)e^{i\alpha x} - (A + R^\alpha e^{-i\alpha x})^{-1}\rho^{-1}f(x_0) = 0, \quad R \in \mathbb{R}_+.
\]

Since \( x_0 \in \mathcal{G}(f) \), condition (5.1) implies that, for all \( n \in \mathbb{N} \), there exists \( r_n \in (0, 2^{-n}) \) such that

\[
\partial_n(A + r_n^\alpha e^{i\alpha x})^{-1}(A + r_n^\alpha e^{-i\alpha x})^{-1}\rho^{-1}f(x_0) \neq 0.
\]

Then, (5.7) implies

\[
\hat{\sigma}(-r_n) = 0, \quad n \in \mathbb{N}.
\]

It is clear that the sequence \( (r_n)_{n \in \mathbb{N}} \) admits an accumulation at 0 and using the fact that \( \hat{\sigma} \) is holomorphic on \( \mathbb{C} \), we deduce that \( \hat{\sigma} \equiv 0 \). Therefore, we have \( \sigma \equiv 0 \). This proves that the implication (3.1) holds true.

Step 4. In this step, we prove the last statement of Theorem 1.1. For this purpose, we fix \( k_1 \in \mathbb{N} \cup \{0\} \) and \( g \in \mathbb{D}(A^{k_1}) \) of constant sign. Then, we fix \( f = \rho A^{k_1} g \). By eventually replacing \( f \) by \( -f \), we may assume without loss of generality that \( g \leq 0 \). We fix \( k_2 \in \mathbb{N} \) such that \( k_2 \geq k_1 \) and \( a(k_2 + 1) \notin \mathbb{N} \). According to the above discussion, the proof will be completed if we prove that for any \( x_0 \in \partial\Omega \), we have \( \partial_n A^{-2-k_2} \rho^{-1}f(x_0) \neq 0 \).

Set \( x_0 \in \partial\Omega \) and \( w = A^{-2-k_2} \rho^{-1}f = A^{-2-k_2}g \). Using the fact that \( A A^{-1}g = \rho g \leq 0 \) and \( A^{-1}g|_{\partial\Omega} = 0 \), the maximum principle implies that \( A^{-1}g \leq 0 \). In the same way by iteration, we can prove that \( A^{-1-k_2} g \leq 0 \). Therefore, since \( A w = \rho A^{-1-k_2}g \leq 0 \) and \( w|_{\partial\Omega} = 0 \), the strong maximum principle (see, e.g., Gilbarg and Trudinger [32, Corollary 3.5]) implies that \( w(x) < 0 = A^{-2-k_2}g(x_0), x \in \Omega \). Thus, the Hopf lemma (see Gilbarg and Trudinger [32, Lemma 3.4]) implies that

\[
\partial_n A^{-2-k_2} \rho^{-1}f(x_0) = \partial_n w(x_0) > 0.
\]

On the other hand, recalling that \( w|_{\partial\Omega} = 0 \), we deduce that \( \nabla w(x_0) = (\partial_n w(x_0))v(x_0) \). Then, fixing the matrix \( C = (a_{ij}(x_0))_{1 \leq i, j \leq d} \), we get

\[
\partial_n w(x_0) = [\nabla w(x_0)] \cdot [C v(x_0)] = [(\partial_n w(x_0))v(x_0)] \cdot [C v(x_0)] = (\partial_n w(x_0))v(x_0)^T C v(x_0).
\]
Moreover, according to (1.2), we have
\[ \nu(x_0)^T C \nu(x_0) = \sum_{i,j=1}^{d} a_{ij}(x_0) \nu_i(x_0) \nu_j(x_0) \geq c|\nu(x_0)|^2 = c > 0. \]

Combining this with (5.9), we deduce that \( \partial_{x_0} A^{-2-k} \rho^{-1} f(x_0) = \partial_{x_0} w(x_0) > 0. \) Therefore, we have \( x_0 \in \mathcal{G}(f), \) and it follows that \( \mathcal{G}(f) = \partial \Omega. \) Moreover, according to Step 3 of the proof of this theorem, the implication (1.11) holds true for any \( x_0 \in \partial \Omega. \) This completes the proof of the last statement of the theorem and by the same way the proof of the theorem.

6  |  CONCLUSION

In this article, we have proved how the memory effect of fractional diffusion equations can be exploited for the determination of general class of separated variables time-dependent source of anomalous diffusion from a posteriori boundary measurement restricted to an arbitrary small interval of time of the form \( (T - \epsilon, T). \) We have applied our results to several class of time-dependent source term with separated variables including the general example of Theorem 1.2, whose result is not true for classical diffusion equations. Our results can be applied to several important practical situations, such as environmental accident, where the data are often available only after the occurrence of such accident. Moreover, the different properties of our main results exhibit several particularities of time-fractional diffusion equations and the corresponding anomalous diffusion phenomena compared with classical diffusion process.

ACKNOWLEDGEMENTS

The work of the first author is supported by the Grant PRG832 of the Estonian Research Council. The work of the second author is partially supported by the Agence Nationale de la Recherche (project MultiOnde) under grant ANR-17-CE40-0029.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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**How to cite this article**: J. Janno and Y. Kian, *Inverse source problem with a posteriori boundary measurement for fractional diffusion equations*, Math. Meth. Appl. Sci. 46 (2023), 15868–15882. DOI 10.1002/mma.9432