A CUT FINITE ELEMENT METHOD FOR THE DARCY PROBLEM ∗, **

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Abstract. We present and analyze a cut finite element method for the weak imposition of the Neumann boundary conditions of the Darcy problem. The Raviart-Thomas mixed element on both triangular and quadrilateral meshes is considered. Our method is based on the Nitsche formulation studied in [12] and can be considered as a first attempt at extension in the unfitted case. The key feature is to add two ghost penalty operators to stabilize both the velocity and pressure fields. We rigorously prove our stabilized formulation to be well-posed and derive a priori error estimates for the velocity and pressure fields. We show that an upper bound for the condition number of the stiffness matrix holds as well. Numerical examples corroborating the theory are included.

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INTRODUCTION

Mesh generation is one of the major bottlenecks for the classical finite element method for the numerical solution of partial differential equations (PDEs). It is a very costly process because it necessitates not only of important computational power, but also of ad hoc human intervention. Mathematical problems for which mesh generation becomes a very demanding task are, in general, problems set in complicated geometries or such that the domain changes during the simulation. The last few decades have seen the flourishing of an abundance of numerical methods trying to overcome this issue. The most popular approaches belong to the family of the so-called fictitious domain methods (from the pioneering work [27]) where the possibly complicate domain Ω is immersed in a much simpler geometry ΩT, for which the generation of the mesh is a simple task. In the last few years, the Cut Finite Element Method (CutFEM) [7], a particular fictitious domain method, gained a lot of attention and showed its potential in different applications in science and engineering [10, 20, 25]. Unlike other methods, the CutFEM relies on solid theoretical foundations, and its key feature is to add weakly consistent stabilization operators [6] to the variational formulation of the discrete problem to transfer the stability and approximation properties from the finite element scheme constructed on the background mesh to its cut finite element counterpart.

In this contribution we study the weak imposition of the Neumann boundary conditions for the Darcy problem when the mesh does not fit the boundary of the domain. Let us recall that the Darcy problem combines a constitutive equation describing the flow of a fluid in porous media with a conservation of mass equation, and it is coupled with suitable boundary conditions. This first-order system of PDEs can also be derived as a

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dual formulation of the Poisson problem. An important difference between the variational formulations of the Poisson and the Darcy problems is that the Dirichlet boundary conditions that are enforced by modifying the trial space in the Poisson case are now natural, i.e., they appear as an integral in the right hand side, and the Neumann boundary conditions that are before natural, have to be enforced as essential boundary conditions in the Darcy case.

The weak imposition of Dirichlet boundary conditions for the Poisson problem is a quite well-understood matter; let us refer, for instance, to [13,16,32]. On the other hand, the problem of weakly imposing the Neumann boundary conditions for the Darcy problem has not been sufficiently explored in the literature, especially in the context of a fictitious domain method. We base our analysis on the classical $H(\text{div})$-conforming Raviart-Thomas discretization in simplicial and quadrilateral meshes and adopt the Nitsche-type method introduced in [12] for the weak imposition of the essential boundary conditions. The discrete formulation is very ill-posed because of the mismatch between the computational mesh and the physical domain where the PDEs live. We show that this affects not only the accuracy of the approximation scheme, but also the conditioning of the arising linear system. Our strategy to recover the well-posedness of the discrete formulation is in line with [6,8,9] and consists of adding to the variational formulation at the discrete level two weakly consistent ghost penalty operators acting separately on the velocity and on the pressure fields. The discrete functional setting is unusual since it is based on mesh-dependent norms scaling as $H^1 \times H^1$, instead of the standard $H(\text{div}) \times L^2$. Hence, we derive a priori error estimates for the velocity and pressure fields which are optimal for the chosen topology, but not for the usual ones. A further drawback of our method is lost of the divergence-free property of the Raviart-Thomas element.

An outline of the remainder of the paper follows. We set the notation and introduce the model problem in Section 1. In Section 2, we introduce the model problem and its Raviart-Thomas discretization for both triangular and quadrilateral meshes. In Section 3 we explain how we interpolate regular functions when the mesh does not fit the boundary of the physical domain. Section 4 contains the discrete stabilized formulation and its numerical analysis: we rigorously derive the estimates guaranteeing the stability of our formulation and prove the a priori error estimates. Section 5 is devoted to the study of the condition number of the stiffness matrix. We prove that the ghost penalty stabilization restores the usual conditioning of the boundary-fitted case. In Section 6 we explain how to deal and what changes in the case of pure Dirichlet boundary conditions. Finally, in Section 7 we present some numerical experiments illustrating the theory.

1. Notation and model problem

We briefly introduce some useful notations and definitions for the forthcoming analysis. Let $d \in \{2,3\}$ and $D$ be a Lipschitz-regular domain (subset, open, bounded, connected) of $\mathbb{R}^d$, with boundary $\partial D$ and unit outer normal $n$. Let $L^2(D)$ denote the space of square integrable functions on $D$, equipped with the usual norm $\|\cdot\|_{L^2(D)}$. Let $L^0_0(D)$ be the subspace of $L^2(D)$ of functions with zero average, where the average of $v \in L^2(D)$ is $\mathfrak{v} := \frac{1}{|D|} \int_D v$. For a given $\varphi : D \to \mathbb{R}$ sufficiently regular and $\alpha$ multi-index with $|\alpha| := \sum_{i=1}^d \alpha_i$, we define $D^\alpha \varphi := \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$ and $\partial_\varphi \mathfrak{n} := \sum_{|\alpha| = j} D^\alpha \varphi \mathfrak{n}^\alpha$, where $\mathfrak{n}^\alpha := n_1^{\alpha_1} \cdots n_d^{\alpha_d}$, when $\varphi : \Omega \to \mathbb{R}$ is smooth enough. We indicate by $H^k(D)$, for $k \in \mathbb{N}$, the standard Sobolev space of functions in $L^2(D)$ whose $k$-th order weak derivatives belong to $L^2(D)$, equipped with the norm $\|\varphi\|_{H^k(D)}^2 := \sum_{|\alpha| \leq k} \|D^\alpha \varphi\|_{L^2(D)}^2$. Sobolev spaces of fractional order $H^r(D)$, $r \in \mathbb{R}$, can be defined by interpolation techniques, see [1]. The space $H^1_{0,\Sigma}(D)$ consists of functions in $H^1(D)$ with vanishing trace on $\Sigma$. We write $H^1_{1,\partial D}(D)$ instead of $H^1_{0,\partial D}(D)$. Let us denote $L^2_0(D) := (L^2(D))^d$ and $H^k(D) := (H^k(D))^d$. We define the Hilbert space $H(\text{div}; D)$ of vector fields in $L^2(D)$ with divergence in $L^2(D)$, endowed with the graph norm, denotes as $\|\cdot\|_{H(\text{div}; D)}$. Moreover, we set $H_{0,\Sigma}(\text{div}; D) := \{v \in H(\text{div}; D) : v \cdot n = 0 \text{ on } \Sigma\}$ and $H_{0}(\text{div}; D) := H_{0,\partial D}(\text{div}; D)$. Let $H^{1}(\partial D)$ be the range of the trace operator of functions in $H^{1}(D)$ and we define its restriction to $\Sigma$ as $H^{1/2}(\Sigma)$. Both $H^{1/2}(\partial D)$ and $H^{1/2}(\Sigma)$ can be endowed with an intrinsic norm, see [33]. The dual space of $H^{1/2}(\Sigma)$ is denoted $H^{-1/2}(\Sigma)$. 
The duality pairing between $H^\frac{1}{2}(\Sigma)$ and $H^{-\frac{1}{2}}(\Sigma)$ will be denoted with a formal integral notation. Finally, let $H^\frac{1}{2}(\partial D) := \left( H^\frac{1}{2}(\partial D) \right)^d$, $H^\frac{1}{2}(\Sigma) := \left( H^\frac{1}{2}(\Sigma) \right)^d$ and $H^{-\frac{1}{2}}(\Sigma) := \left( H^{-\frac{1}{2}}(\Sigma) \right)^d$. We will denote as $\mathcal{Q}_{r,s,t}(D)$ the vector space of polynomials of degree at most $r$ in the first variable, at most $s$ in the second and at most $t$ in the third one (analogously for th case $d = 2$) in $D$, $\mathcal{P}_u(D)$ the vector space of polynomials of degree at most $u$ in $D$, $\mathcal{P}_f(D)$ the vector space of homogeneous polynomials of degree $f$ in $D$. We may write $\mathcal{Q}_k(D)$ instead of $\mathcal{Q}_{k,k,k}(D)$ or $\mathcal{Q}_{k,k,k}(D)$. In the same way, we can define vector spaces of polynomials defined in hypersurfaces of $\mathbb{R}^d$. With a slight abuse of notation, we will use the same symbol $|\cdot|$ to denote both the $d$-dimensional Lebesgue measure and the $(d - 1)$-dimensional Hausdorff measure. Given $D \subset \mathbb{R}^d$ and $\Sigma$ a hypersurface of $\mathbb{R}^d$ or a subset of it, $|D|$ and $|\Sigma|$ denote the $d$-dimensional Lebesgue measure of $D$ and the $(d - 1)$-dimensional Hausdorff measure of $\Sigma$, respectively. $E^o$ and $\text{int} E$ denotes the interior of $E \subset \mathbb{R}^d$. Given $x,y \in \mathbb{R}$, we will write $x \lesssim y$ if there exists $c > 0$, independent of $x,y$, such that $x \leq cy$ and $x \sim y$ if $x \lesssim y$ and $y \lesssim x$. $O$ will denote the classical Landau symbol. $\mathcal{C}$ will denote generic positive constants that may change with each occurrence throughout the document but are always independent of the local mesh size and the mutual position of mesh and physical domain unless otherwise specified.

Let $\Omega$ be a Lipschitz-regular domain of $\mathbb{R}^d$ with boundary $\Gamma$ such that $\Gamma = \Gamma_N \cup \Gamma_D$, where $\Gamma_N$, $\Gamma_D$ are non-empty, open, and disjoint. The Darcy problem is a linear system of partial differential equations modeling the flow of groundwater through a porous medium, here represented by $\Omega$, with permeability $\kappa$. Given $f \in L^2(\Omega)$, $g \in L^2(\Omega)$, $u_N \in H^{-\frac{1}{2}}(\Gamma_N)$, $p_D \in H^\frac{1}{2}(\Gamma_D)$, we look for $u : \Omega \rightarrow \mathbb{R}^d$ and $p : \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
\kappa^{-1} u - \nabla p &= f, & \text{in } \Omega, \\
\text{div } u &= g, & \text{in } \Omega, \\
u \cdot n &= u_N, & \text{on } \Gamma_N, \\
p &= p_D, & \text{on } \Gamma_D.
\end{align*}
$$

The unknowns $u$ and $p$ represent, respectively, the seepage velocity and the pressure of the fluid. The first equation of (1) is called Darcy law relating the velocity and the pressure gradient of the fluid, the second one expresses mass conservation (when $g \equiv 0$), the third and the fourth equations are, respectively, a Neumann boundary condition for the velocity field and a Dirichlet boundary condition for the pressure. Moreover, $\kappa \in \mathbb{R}^{d \times d}$ is symmetric positive definite with eigenvalues $\lambda_i$ such that $0 < \lambda_{\min} \leq \lambda_i \leq \lambda_{\max} < +\infty$, for every $i = 1, \ldots, d$.

In the subsequent analysis we are going to consider, for the sake of simplicity, $\kappa = I$ the identity matrix.

### 2. The finite element discretization

Let us introduce $(\mathcal{T}_h)_{h>0}$, a family of admissible triangular or quadrilateral meshes (see, for instance, Chapter 3 of [29]) such that, for every $h > 0$, $\overline{\Omega} \subseteq \Omega_T$, $\Omega_T$ being the fictitious domain, i.e., $\Omega_T := \text{int} \cup_{K \in \mathcal{T}_h} K$. Let us denote the collection of all the facets (edges if $d = 2$ and faces if $d = 3$) as $\mathcal{F}_h$ and partition it into two disjoint sets: the faces lying on the boundary of $\Omega_T$, denoted as $\mathcal{F}^o_h$, and $\mathcal{F}^b_h$, the internal ones. For every cut element $K \in \mathcal{T}_h$, let us denote its intersection with the boundary as $\Gamma_K$. It will be clear from the context if with $\Gamma_K$ we mean the intersection with the whole boundary, i.e., $\Gamma_K := \Gamma \cap K$ or with just one of its disjoint components $\Gamma_N$, i.e., $\Gamma_K := \Gamma_N \cap K$, or $\Gamma_D$, i.e., $\Gamma_K := \Gamma_D \cap K$. It will also be useful to define the collection of the cut-elements, namely $\mathcal{G}_h := \{ K \in \mathcal{T}_h : |\Gamma_K| \neq 0 \}$, its two sub-collections $\mathcal{G}_h(\Gamma_N) := \{ K \in \mathcal{T}_h : |\Gamma_N \cap K| \neq 0 \}$ and $\mathcal{G}_h(\Gamma_D) := \{ K \in \mathcal{T}_h : |\Gamma_D \cap K| \neq 0 \}$, that we assume to be disjoint, and the interior part $\Omega_{I,h} := \Omega \setminus \cup_{K \in \mathcal{G}_h} K$. Let $\mathcal{T}_h(\Omega_{I,h}) := \{ K \in \mathcal{T}_h : K \subset \overline{\Omega}_{I,h} \}$. The collections of the internal and boundary faces entirely contained in the bulk of the domain are respectively denoted as $\mathcal{F}^o_h(\Omega_{I,h})$ and $\mathcal{F}^b_h(\Omega_{I,h})$. The collection of facets in the boundary region are denoted as $\mathcal{F}^b_h := \{ f \in \mathcal{F}^b_h : \exists K \in \mathcal{G}_h \text{ such that } f \subset \partial K \}$.

Let us assign to each element $K \in \mathcal{T}_h$ its diameter $h_K$ and denote $h := \max_{K \in \mathcal{T}_h} h_K$. We assume the background mesh to be shape-regular, i.e., there exists $\sigma > 0$, independent of $h$, such that $\min_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq \sigma$, $\rho_K$ being the diameter of the largest ball inscribed in $K$. Moreover, $\mathcal{T}_h$ is supposed to be quasi-uniform in the
sense that there exists \( \tau > 0 \), independent of \( h \), such that \( \min_{K \in T_h} h_K \geq \tau h \). We fix an orientation for the internal faces, i.e., given \( f \in F_h^i \) such that \( f = \partial T_1 \cap \partial T_2 \), we assume that the unit normal on \( f \) points from \( K_1 \) toward \( K_2 \). Let \( \varphi : \Omega \to \mathbb{R} \) be smooth enough. Then, for all \( f \in F_h^i \) and a.e. \( x \in f \), we define the jump of \( \varphi \) as

\[
[\varphi](x) := \varphi|_{K_1}(x) - \varphi|_{K_2}(x),
\]

where \( f = \partial K_1 \cap \partial K_2 \). We may remove the subscript \( f \) when it is clear from the context to which facet we refer to.

The following mild assumptions on how the mesh may be intersected by the boundary \( \Gamma \) will be helpful. First, let us require that the number of facets to be crossed to move from a cut element \( K \) to an uncut element \( K' \) is uniformly bounded with respect to \( h \).

**Assumption 2.1.** There exists \( N > 0 \) such that, for every \( h > 0 \) and \( K \in \mathcal{G}_h \), there exist \( K' \in T_h(\Omega_f,h) \) and at most \( N \) elements \( (K_i)_{i=1}^N \subset T_h \) such that \( K_1 = K, K_N = K' \) and \( K_i \cap K_{i+1} \) is a cut facet, for every \( 1 \leq i \leq N - 1 \).

Then, we assume that it is possible to subdivide the boundary region into a fixed (i.e., independent of \( h \)) number of patches such that every cut element belongs to one patch, every patch contains an uncut element, the diameter of every patch is \( O(h) \), and all the patches are diffeomorphic to a reference patch.

**Assumption 2.2.** The boundary zone \( \bigcup_{K \in \mathcal{G}_h} K \) can be decomposed into \( N_P \) patches \( (P_\ell)_{\ell=1}^{N_P} \), \( \bigcup_{K \in P_\ell} K, 1 \leq \ell \leq N_P \), satisfying:

(i) for every \( K \in \mathcal{G}_h \) there exists \( 1 \leq \ell \leq N_P \) such that \( K \subset P_\ell \);

(ii) for every \( 1 \leq \ell \leq N_P \) there exists \( K' \in T_h(\Omega_f,h) \) and \( P_\ell \);

(iii) there exists \( C_h > 0 \) such that, for every \( 1 \leq \ell \leq N_P \) and \( K \in P_\ell \), it holds \( h_K \geq C_h h_{P_\ell} \), where \( h_{P_\ell} := \text{diam}(P_\ell) \);

(iv) for every \( 1 \leq \ell \leq N_P \) there exist an affine map \( F_\ell \) and a reference patch \( \tilde{P} \) with \( \text{diam}(\tilde{P}) = 1 \), such that \( F_\ell : \tilde{P} \to P_\ell \) is invertible.

For every \( 1 \leq \ell \leq N_P \), we denote as \( F_\ell := \{ f \in F_h^i : f \subset P_\ell, f \subset \partial P_\ell \} \).

Let \( V_h \subset H(\text{div}; \Omega_T) \) and \( Q_h \subset L^2(\Omega_T) \) be the Raviart-Thomas finite element space defined in the whole fictitious domain \( \Omega_T \). For the sake of completeness, we recall its construction (for the boundary-fitted case the interested reader is referred, for instance, to \([3, 12, 26, 29]\)).

In the case of triangles and tetrahedra, we consider as reference element \( \hat{K} \) the unit \( d \)-simplex, i.e., the triangle of vertices \((0,0), (1,0), (0,1)\) if \( d = 2 \) and the tetrahedron of vertices \((0,0,0), (1,0,0), (0,1,0), (0,0,1)\) when \( d = 3 \). For quadrilaterals, the reference element \( \hat{K} \) is the unit \( d \)-cube \([0,1]^d\).

Let us construct the spaces of polynomials on the reference element. For the triangular meshes (see \([4]\)),

\[
\mathbb{RT}_k(\hat{K}) := \left( \mathbb{P}_k(\hat{K}) \right)^d \oplus \mathbb{X}\mathbb{P}_k(\hat{K}), \quad \mathbb{M}_k(\hat{K}) := \mathbb{P}_k(\hat{K}),
\]

while, in the case of quadrilaterals (see \([3, 4]\)), it reads as follows

\[
\mathbb{RT}_k(\hat{K}) := \begin{cases} 
\mathbb{Q}_{k+1,k}(\hat{K}) \times \mathbb{Q}_{k,k+1}(\hat{K}), & \text{if } d = 2, \\
\mathbb{Q}_{k+1,k,k}(\hat{K}) \times \mathbb{Q}_{k,k+1,k}(\hat{K}) \times \mathbb{Q}_{k,k,k+1}(\hat{K}), & \text{if } d = 3
\end{cases}, \quad \mathbb{M}_k(\hat{K}) := \mathbb{Q}_k(\hat{K}).
\]

Let \( \mathbf{F}_K : \hat{K} \to K \) be a diffeomorphism mapping the reference element to a general \( K \in T_h \). For triangles we consider an affine bijection \( \mathbf{F}_K(\hat{x}) := B_K \hat{x} + b_K \), where \( B_K \in \mathbb{R}^{d \times d} \) is non-singular and invertible, and \( b_K \in \mathbb{R}^d \). For quadrilateral meshes, an affine mapping would constrain us to parallelograms, hence we let \( \mathbf{F}_K \).
being a bijection bilinear for each component, so that we can map the reference element to arbitrary convex quadrilaterals. The diffeomorphism $\mathbf{F}_K$ induces the pull-back operators

$$
\mathbf{F}_K^p : L^2(K) \to L^2(\hat{K}), \quad \mathbf{F}_K^v(q) := q \circ \mathbf{F}_K, \\
\mathbf{F}_K^n : \mathbf{H}(\text{div}; K) \to \mathbf{H}(\text{div}; \hat{K}), \quad \mathbf{F}_K^n(v) := |\text{det}(D\mathbf{F}_K)| D\mathbf{F}_K^{-1}v.
$$

Let us observe that $\mathbf{F}_K^p$ and $\mathbf{F}_K^n$ are isometric isomorphisms (see, for instance, [31]). For the construction of our discrete spaces, we will use $\mathbf{F}_K^p$ and $\mathbf{F}_K^n$. The inverse of $\mathbf{F}_K^p$ is commonly known as the contravariant Piola transform or, more simply, Piola transform, and we denote it as

$$
\mathcal{P}_K : \mathbf{H}(\text{div}; \hat{K}) \to \mathbf{H}(\text{div}; K), \quad \mathcal{P}_K(\mathbf{v}) := |\text{det}(D\mathbf{F}_K)|^{-1} D\mathbf{F}_K \mathbf{v}.
$$

We define the following finite-dimensional vector spaces

$$
V_h := \{ \mathbf{v}_h \in \mathbf{H}(\text{div}; \Omega) : \mathbf{F}_K^n(\mathbf{v}_h|_K) \in \mathbb{R}\mathcal{T}_k(\hat{K}), \ \forall \ K \in \mathcal{T}_h \} = \{ \mathbf{v}_h \in \mathbf{H}(\text{div}; \Omega) : \mathbf{v}_h|_K \in \mathbb{R}\mathcal{T}_k(K), \ \forall \ K \in \mathcal{T}_h \},
$$

$$
Q_h := \{ q_h \in L^2(\Omega) : \mathbf{F}_K^n(q_h|_K) \in M_k(\hat{K}), \ \forall \ K \in \mathcal{T}_h \} = \{ q_h \in L^2(\Omega) : q_h|_K \in M_k(K), \ \forall \ K \in \mathcal{T}_h \},
$$

where $\mathbb{R}\mathcal{T}_k(K) := \{ \mathcal{P}_K \mathbf{w}_h : \mathbf{w}_h \in \mathbb{R}\mathcal{T}_k(\hat{K}) \}$. Remember that in the pure Neumann case, i.e., $\Gamma = \Gamma_N$, we have to filter out the constant functions from $Q_h$ by imposing the zero average constraint.

### 3. Interpolation strategy

Let us construct the interpolation operators by using the degrees of freedom of $V_h$ and $Q_h$. For every $\mathbf{v} \in \mathbf{H}^s(\hat{K}), s > \frac{1}{2}, r_{\hat{K}}$ is uniquely defined by:

$$
\int_{\hat{K}} r_{\hat{K}} \mathbf{v} \cdot \mathbf{n}_{\hat{h}} = \int_{\hat{K}} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{h}}, \quad \forall \mathbf{n}_{\hat{h}} \in \Psi_k(\hat{f}),
$$

$$
\int_{\hat{K}} r_{\hat{K}} \mathbf{v} \cdot \mathbf{w}_h = \int_{\hat{K}} \hat{\mathbf{v}} \cdot \hat{\mathbf{w}}_h, \quad \forall \hat{\mathbf{w}}_h \in \Psi_k(\hat{K}), \quad \text{if } k > 0,
$$

where, for triangles,

$$
\Psi_k(\hat{K}) := \left( \mathbf{P}_{k-1}(\hat{K}) \right)^d, \quad \Psi_k(\hat{f}) := \mathbf{P}_k(\hat{f}),
$$

and, for quadrilaterals,

$$
\Psi_k(\hat{K}) := \begin{cases} 
Q_{k-1}(\hat{K}) \times Q_{k-1}(\hat{K}), & \text{if } d = 2, \\
Q_{k-1}(\hat{K}) \times Q_{k-1}(\hat{K}) \times Q_{k-1}(\hat{K}) \times Q_{k-1}(\hat{K}), & \text{if } d = 3,
\end{cases}
$$

$$
\Psi_k(\hat{f}) := \begin{cases} 
\mathbf{P}_k(\hat{f}), & \text{if } d = 2, \\
\mathbf{P}_k(\hat{f}), & \text{if } d = 3,
\end{cases}
$$

for all facets (edges if $d = 2$, faces if $d = 3$) $\hat{f}$ of $\hat{K}$. We define $r_K : \mathbf{H}^s(\hat{K}) \to \mathbb{R}\mathcal{T}_k(K)$, so that $r_K = (\mathbf{F}_K)^{-1} r_{\hat{K}} \circ \mathbf{F}_K = \mathcal{P}_K \circ r_{\hat{K}} \circ \mathbf{F}_K$. The global interpolant $r_h^T : \mathbf{H}(\text{div}; \Omega_T) \cap \prod_{K \in \mathcal{T}_h} \mathbf{H}^s(K) \to V_h$ is readily defined by gluing together the local interpolation operators, that is $r_h^T|_K := r_K, K \in \mathcal{T}_h$. Let us move to
the pressure case. We start from the reference element \( \hat{K} \), and define \( \Pi_{\hat{K}} : L^2(\hat{K}) \to M_k(\hat{K}) \) which acts on \( \xi \in L^2(\hat{K}) \) as

\[
\int_K \Pi_{\hat{K}} \xi q_h = \int_{\hat{K}} \xi q_h, \quad \forall \ q_h \in M_k(\hat{K}).
\]

Then, for a general \( K \in T_h \), we define \( \Pi_K : L^2(K) \to M_k(K) \) such that \( \Pi_K = (F_K^p)^{-1} \circ \pi_{\hat{K}} \circ F_K^p \). Finally, let \( \Pi_{\hat{K}}^T : L^2(\Omega_T) \to Q_h \) such that for every \( K \in T_h \), \( \Pi_{\hat{K}}^T|_K := \Pi_K \).

**Remark 3.1.** Given \( v \in H(\text{div}; \Omega_T) \cap \bigcap_{K \in T_h} H^s(K) \), the degrees of freedom for \( r^T_{\hat{K}}(v|_{\hat{K}}) \) given by (4) are invariant under \( F_K^p \), for every \( K \in T_h \). Similarly, given \( q \in L^2(\Omega_T) \), the degrees of freedom for \( \Pi_{\hat{K}}(q|_{\hat{K}}) \) given by (5) are invariant under \( F_K^p \), for every \( K \in T_h \).

The key property of \( r^T_{\hat{K}} : H(\text{div}; \Omega_T) \cap \bigcap_{K \in T_h} H^s(K) \to V_h, \ s > \frac{1}{2} \), and \( \Pi_{\hat{K}}^T : L^2(\Omega_T) \to Q_h \) is that the following diagram commutes and, in particular, \( \text{div} V_h = Q_h \).

\[
\begin{array}{ccc}
H(\text{div}; \Omega_T) \cap \bigcap_{K \in T_h} H^s(K) & \xrightarrow{\text{div}} & L^2(\Omega_T) \\
\downarrow r^T_{\hat{K}} & & \downarrow \Pi_{\hat{K}}^T \\
V_h & \xrightarrow{\text{div}} & Q_h
\end{array}
\]

From [21], there exist \( E : H^t(\Omega) \to H^t(\mathbb{R}^d), t \geq 1 \), and \( E : H^r(\Omega) \to H^r(\mathbb{R}^d), r \geq 1 \), universal (degree-independent) Sobolev-Stein extensions such that \( \text{div} \circ E = E \circ \text{div} \). We define, for \( t \geq 1 \) and \( r \geq 1 \),

\[
\begin{align*}
r_h : H^t(\Omega) & \to V_h, \quad v \mapsto r^T_{\hat{K}}(E(v)|_{\Omega_T}), \\
\Pi_h : H^r(\Omega) & \to Q_h, \quad q \mapsto \Pi_{\hat{K}}^T(E(q)|_{\Omega_T}).
\end{align*}
\]

**Remark 3.2.** By construction, the commutativity of diagram (6) is preserved when restricting to the physical domain \( \Omega \), namely when employing \( V_h|_{\Omega}, \ H(\text{div}; \Omega), \ r_h, \), and \( Q_h|_{\Omega}, \ L^2(\Omega), \ \Pi_h \) instead of \( V_h, \ H(\text{div}; \Omega_T), \ r^T_{\hat{K}}, \) and \( Q_h, \ L^2(\Omega_T), \ \Pi^T_{\hat{K}} \). See Section 7.3.

### 4. The Stabilized Formulation

Given \( k \in \mathbb{N} \), the order of the Raviart-Thomas element employed for the discretization, we introduce two ghost penalty jumps-based operators to enhance the stability of our discrete formulation and, in particular, to recover stability estimates independent of the mesh-boundary intersection (see [6,9,17,24]). We define

\[
\begin{align*}
j_h(w_h,v_h) & := \sum_{f \in F^+_h} \sum_{j=0}^k h^{2j+1} \int_f [\partial_n^j w_h][\partial_n^j v_h], \quad w_h, v_h \in V_h, \\
j_h(r_h,q_h) & := \sum_{f \in F^+_h} \sum_{j=0}^k h^{2j-1} \int_f [\partial_n^j r_h][\partial_n^j q_h], \quad r_h, q_h \in Q_h.
\end{align*}
\]
Remark 4.1. A wide zoo of ghost penalty operators has been proposed in the literature. For instance, it is possible to show that \( j_h(\cdot, \cdot) \) and \( j_h(\cdot, \cdot) \) are equivalent to the following operators.

\[
\begin{align*}
    s_h(w_h, v_h) &:= \sum_{\ell=1}^{N_p} \int_{P_\ell} (w_h - \pi_h w_h) v_h, \quad w_h, v_h \in V_h, \\
g_h(w_h, v_h) &:= \sum_{\ell=1}^{N_p} \int_{P_\ell} (w_h - E_h w_h) v_h, \quad w_h, v_h \in V_h, \\
s_h(r_h, q_h) &:= \sum_{\ell=1}^{N_p} h^{-2} \int_{P_\ell} (r_h - \pi_h r_h) q_h, \quad r_h, q_h \in Q_h, \\
g_h(r_h, q_h) &:= \sum_{\ell=1}^{N_p} h^{-2} \int_{P_\ell} (r_h - E_h r_h) q_h, \quad r_h, q_h \in Q_h,
\end{align*}
\]

namely it holds, respectively,

\[
\begin{align*}
    j_h(v_h, v_h) &\leq s_h(v_h, v_h) \leq j_h(v_h, v_h), \\
    j_h(v_h, v_h) &\leq g_h(v_h, v_h) \leq j_h(v_h, v_h), \quad \forall \ v_h \in V_h, \\
    j_h(q_h, q_h) &\leq s_h(q_h, q_h) \leq j_h(q_h, q_h), \\
    j_h(q_h, q_h) &\leq g_h(q_h, q_h) \leq j_h(q_h, q_h), \quad \forall \ q_h \in Q_h,
\end{align*}
\]

Here,

\[
\pi_\ell : L^2(P_\ell) \to \mathbb{RT}_k(P_\ell), \quad \pi_\ell : L^2(P_\ell) \to \mathbb{M}_k(P_\ell),
\]

denote the \( L^2 \)-orthogonal projections onto the respective finite-dimensional spaces, and

\[
\mathcal{E}_\ell : \mathbb{RT}_k(K') \to \mathbb{RT}_k(P_\ell), \quad \mathcal{E}_\ell : \mathbb{P}_k(K') \to \mathbb{M}_k(P_\ell),
\]

are the canonical extensions of the respective polynomials, see [23,28], where \( K' \) is the uncut element of the \( \ell \)-th patch (see Assumption 2.2). From the implementation point of view, the operators \( s_h(\cdot, \cdot), \ s_h(\cdot, \cdot) \) and \( g_h(\cdot, \cdot), \ g_h(\cdot, \cdot) \) turn out to be a more convenient choice of \( j_h(\cdot, \cdot), \ j_h(\cdot, \cdot) \), respectively, when a higher-order discretization is employed because of the evaluation of the high order derivatives. In the numerical experiments of Section 7 we use the projection-based operators \( s_h(\cdot, \cdot) \) and \( s_h(\cdot, \cdot) \).

We are now ready to introduce our stabilized discrete formulation. The idea is to employ the Nitsche formulation for the Darcy flow, which has been proposed and analyzed in [12], stabilizing it with the ghost penalty operators introduced above.

Find \((u_h, p_h) \in V_h \times Q_h\) such that

\[
a_h(u_h, v_h) + j_h(u_h, v_h) + b_1(v_h, p_h) = \int_\Omega f \cdot v_h + \int_{\Gamma_D} p_D v_h \cdot n + h^{-1} \int_{\Gamma_N} u_N v_h \cdot n, \quad \forall \ v_h \in V_h, \\
b_1(u_h, q_h) - j_h(p_h, q_h) = \int_\Omega q_h - \int_{\Gamma_N} u_N q_h, \quad \forall \ q_h \in Q_h,
\]

where

\[
\begin{align*}
    a_h(w_h, v_h) &:= \int_\Omega w_h \cdot v_h + h^{-1} \int_{\Gamma_N} w_h \cdot n v_h \cdot n, \quad w_h, v_h \in V_h, \\
b_1(v_h, q_h) &:= \int_\Omega q_h \ \text{div} \ v_h - \int_{\Gamma_N} q_h v_h \cdot n, \quad v_h \in V_h, \ q_h \in Q_h.
\end{align*}
\]
It will be convenient to rewrite (8) in the following more compact form.

Find \((u_h, p_h) \in V_h \times Q_h\) such that

\[
\mathcal{A}_h ((u_h, p_h); (v_h, q_h)) = \mathcal{F}_h (v_h, q_h), \quad \forall (v_h, q_h) \in V_h \times Q_h,
\]

where, for \((w_h, r_h), (v_h, q_h) \in V_h \times Q_h,\)

\[
\mathcal{A}_h ((w_h, r_h); (v_h, q_h)) := a_h (w_h, v_h) + b_1 (w_h, v_h) + b_1 (v_h, r_h) - j_h (r_h, q_h),
\]

\[
\mathcal{F}_h (v_h, q_h) := \int_\Omega f \cdot v_h + \int_{\Gamma_D} p_D v_h \cdot n + h^{-1} \int_{\Gamma_N} u_N v_h \cdot n + \int_\Omega g q_h - \int_{\Gamma_N} u_N q_h.
\]

**Proposition 4.2** (Weak Galerkin Orthogonality). Let \((u, p) \in H(\text{div}; \Omega) \times L^2 (\Omega)\) be the solution of (1) and \((u_h, p_h)\) the one of (9). Then,

\[
\mathcal{A}_h ((u - u_h, p - p_h); (v_h, q_h)) = j_h (u, v_h) - j_h (p, q_h), \quad \forall (v_h, q_h) \in V_h \times Q_h.
\]

**Proof.** The proof is trivial, hence we skip it. \(\square\)

**Remark 4.3.** We note that the formulation (9) no longer fits into the framework of saddle-point problems. Hence, in order to study its stability, we will need to resort to the more general Banach-Nečas-Babuška Theorem [15]. The case of pure Dirichlet boundary conditions will be covered separately in Section 6. Moreover, note that all the dimensionless parameters have been set for simplicity to 1, unlike for the standard Nitsche method for the Poisson problem [16], where the dimensionless parameter needs to be taken large enough.

We endow \(V_h\) and \(Q_h\) with the following mesh-dependent norms.

\[
\|v_h\|_{0_h,\Omega_T}^2 := \|v_h\|_{L^2 (\Omega_T)}^2 + \sum_{K \in G_h (\Omega)} h^{-1} \|v_h \cdot n\|_{L^2 (\Gamma_K)}^2, \quad v_h \in V_h,
\]

\[
\|q_h\|_{1_h,\Omega_T}^2 := \sum_{K \in G_h (\Omega)} \|\nabla q_h\|_{L^2 (K)}^2 + \sum_{f \in F_h (\Omega)} h^{-1} \|q_h\|_{L^2 (f)}^2, \quad q_h \in Q_h,
\]

\[
\|q_h\|_{1_h,\Omega_T}^2 := \sum_{K \in T_h} \|\nabla q_h\|_{L^2 (K)}^2 + \sum_{f \in F_h} h^{-1} \|q_h\|_{L^2 (f \cap \Omega)}^2 + \sum_{K \in G_h (\Gamma_D)} h^{-1} \|q_h\|_{L^2 (\Gamma_K)}^2, \quad q_h \in Q_h.
\]

The space \(V_h \times Q_h\) is equipped with the product norm

\[
\|(v_h, q_h)\|^2 := \|v_h\|_{0_h,\Omega_T}^2 + \|q_h\|_{1_h,\Omega_T}^2, \quad (v_h, q_h) \in V_h \times Q_h.
\]

Let us illustrate the salient properties of the ghost penalty operators that are needed to study the well-posedness of formulation (8).

**Lemma 4.4.** The bilinear forms \(j_h (\cdot, \cdot)\) and \(j_h (\cdot, \cdot)\) induce semi-inner products on \(V_h\) and \(Q_h\), respectively. In particular,

\[
j_h (w_h, v_h) \leq j_h (w_h, v_h)^{\frac{1}{2}} j_h (v_h, v_h)^{\frac{1}{2}}, \quad \forall w_h, v_h \in V_h,
\]

\[
j_h (r_h, q_h) \leq j_h (r_h, r_h)^{\frac{1}{2}} j_h (q_h, q_h)^{\frac{1}{2}}, \quad \forall r_h, q_h \in Q_h.
\]
Proof. It suffices to apply the Cauchy-Schwarz inequality, first in the $L^2$-setting and then in the $\ell^2$-setting. Let us show, for instance, the bound for $j_h(\cdot, \cdot)$. Given $r_h, q_h \in Q_h$, it holds
\[
j_h(r_h, q_h) = \sum_{f \in F_h^i} \sum_{j=0}^k h^{2j-1} \int_f [\partial^n r_h] [\partial^n q_h] \leq \sum_{f \in F_h^i} \sum_{j=0}^k h^{2j-1} \| [\partial^n r_h] \|_{L^2(f)} h^{2j-1} \| [\partial^n q_h] \|_{L^2(f)} \]
\[
\leq \left( \sum_{f \in F_h^i} \sum_{j=0}^k h^{2j-1} \| [\partial^n r_h] \|_{L^2(f)}^2 \right)^{\frac{1}{2}} \left( \sum_{f \in F_h^i} \sum_{j=0}^k h^{2j-1} \| [\partial^n q_h] \|_{L^2(f)}^2 \right)^{\frac{1}{2}}
\]
\[
= j_h(r_h, r_h)^{\frac{1}{2}} j_h(q_h, q_h)^{\frac{1}{2}}. \]

The inequality for $j_h(\cdot, \cdot)$ follows in a similar fashion. \hfill \Box

Lemma 4.5. Let $K_1, K_2 \in T_h$ with $f = \partial K_1 \cap \partial K_2$. Let $\varphi_h$ be a piecewise polynomial such that $\varphi_1 := \varphi_h|_{K_1} \in M_{K_1}(K_1)$ and $\varphi_2 := \varphi_h|_{K_2} \in M_{K_2}(K_2)$, and let $k := \max\{k_1, k_2\}$. There exist $C_1, C_2 > 0$, independent of $h > 0$, but dependent on the shape-regularity constant and on $k$, such that
\[
\| \varphi_1 \|_{L^2(K_1)}^2 \leq C_1 \left( \| \varphi_2 \|_{L^2(K_2)}^2 + \sum_{j=0}^k h^{2j+1} \| [\partial^n \varphi_h] \|_{L^2(f)}^2 \right),
\]
\[
\| \partial \varphi_1 / \partial x_j \|_{L^2(K_1)}^2 \leq C_2 \left( \| \partial \varphi_2 / \partial x_j \|_{L^2(K_2)}^2 + \sum_{j=0}^k h^{2j+1} \| [\partial^n \varphi_h] \|_{L^2(f)}^2 \right), \quad \forall 1 \leq j \leq d. \tag{11} \]

Proof. The first inequality in (11) has been proven in Lemma 5.1 in [24]. For the second inequality, see Lemma 5.2 of [23]. \hfill \Box

The following results enable us to control the norms in the whole $\Omega_T$ in terms of the norms in the domain $\Omega_{I,h}$ through the ghost penalty operators.

Theorem 4.6. The following inequalities hold.
\[
\| v_h \|_{L^2(\Omega_T)}^2 \lesssim \| v_h \|_{L^2(\Omega_{I,h})}^2 + j_h(v_h, v_h), \quad \forall v_h \in V_h, \tag{12}
\]
\[
\sum_{K \in T_h} \| \nabla q_h \|_{L^2(K)}^2 \lesssim \sum_{K \in T_h(\Omega_{I,h})} \| \nabla q_h \|_{L^2(K)}^2 + j_h(q_h, q_h), \quad \forall q_h \in Q_h, \tag{13}
\]
\[
\sum_{f \in F_h^i} h^{-1} \| q_h \|_{L^2(f)}^2 \lesssim \sum_{f \in F_h^i(\Omega_{I,h})} h^{-1} \| q_h \|_{L^2(f)}^2 + j_h(q_h, q_h), \quad \forall q_h \in Q_h. \tag{14}
\]

Proof. Let us start with the proof of (12). Since we can decompose $\Omega_T = \Omega_{I,h} \cup \bigcup_{K \in \mathcal{G}_h} K$, it is sufficient to show
\[
\sum_{K \in \mathcal{G}_h} \| v_h \|_{L^2(K)}^2 \lesssim \| v_h \|_{L^2(\Omega_{I,h})}^2 + j_h(v_h, v_h), \quad \forall v_h \in V_h,
\]
which holds by Assumption 2.1, the shape-regularity of $T_h$, and Lemma 4.5. See also Lemma 2 of [17] and Proposition 5.1 of [24]. Inequality (13) for the pressures follows in a similar fashion. Let us move to (14). Note that in view of Remark 4.1, we may replace $j_h(\cdot, \cdot)$ with $s_h(\cdot, \cdot)$. Without loss of generality, let $f \in F_h^i \setminus F_h^i(\Omega_{I,h})$. Assumption 2.2 guarantees the existence of a patch $P_t$ such that $f \in F_t$ and of an internal face $f' \in F_t \setminus F_h^i(\Omega_{I,h})$. Let us take $q_h \in Q_h$ and map $P_t$ to the reference patch $\hat{P}$. We denote $\hat{f} := F_t^{-1}(f)$, $\hat{f}' := F_t^{-1}(f')$, $\hat{q}_h := q_h \circ F_t$, and...
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and \( \hat{\pi} : L^2(\hat{P}) \to M_k(\hat{P}) \) the \( L^2 \)-orthogonal projection. Moreover, let \( \tilde{\mathcal{P}} := \{ f \in \mathcal{F}_h^1 : f \subseteq \hat{P}, f \not\subseteq \partial P \} \). We also write

\[
\hat{s}(\hat{q}_h, \hat{q}_h) := \int_{\hat{P}} (\hat{q}_h - \hat{\pi}\hat{q}_h) \hat{q}_h.
\]

It is sufficient to show that

\[
\|\hat{q}_h\|^2_{L^2(\hat{f})} \lesssim \|\hat{s}(\hat{q}_h, \hat{q}_h)\|^2_{L^2(\hat{P})}.
\]

We decompose \( \hat{q}_h = \hat{q}_1 + \hat{q}_2 \), where \( \hat{q}_1 \in \ker \hat{s} \) and \( \hat{q}_2 \in (\ker \hat{s})^\perp \), where the orthogonal complement is taken with respect to the \( L^2 \)-scalar product on \( \hat{P} \). Note that \( \ker \hat{s} = M_k(\hat{P}) \). We have, of course,

\[
\|\hat{q}_1 + \hat{q}_2\|^2_{L^2(\hat{f})} \leq \|\hat{q}_1\|^2_{L^2(\hat{f})} + \|\hat{q}_2\|^2_{L^2(\hat{f})}.
\]

From norms equivalence on discrete spaces, it holds

\[
\sum_{f \in \tilde{\mathcal{P}}} \|\hat{q}_2\|^2_{L^2(f)} \lesssim \|\hat{q}_2\|^2_{L^2(\hat{P})}.
\]

Indeed, it is easy to check that both terms in (16) are norms on \( (\ker \hat{s})^\perp \). In particular, (16) entails

\[
\|\hat{q}_2\|^2_{L^2(f)} \lesssim \|\hat{q}_2\|^2_{L^2(\hat{P})}, \quad \forall f \in \tilde{\mathcal{P}}.
\]

By combining (15) and (17), we have

\[
\|\hat{q}_1 + \hat{q}_2\|^2_{L^2(f)} \leq \|\hat{q}_1\|^2_{L^2(f)} + \|\hat{q}_2\|^2_{L^2(f)} + \hat{s}(\hat{q}_1 + \hat{q}_2, \hat{q}_1 + \hat{q}_2) \lesssim \|\hat{q}_2\|^2_{L^2(\hat{P})}.
\]

The reader can easily check \( \hat{s}(\hat{q}_1 + \hat{q}_2, \hat{q}_1 + \hat{q}_2) = \|\hat{q}_2\|^2_{L^2(\hat{P})} \). Hence,

\[
\|\hat{q}_1 + \hat{q}_2\|^2_{L^2(f)} \leq \|\hat{q}_1\|^2_{L^2(f)} + \|\hat{q}_2\|^2_{L^2(f)} + \hat{s}(\hat{q}_1 + \hat{q}_2, \hat{q}_1 + \hat{q}_2).
\]

The claim follows by scaling back to the physical patch \( P_t \), summing over all the patches, using Assumption 2.2, and the shape-regularity of the mesh.

4.1. Stability estimates

Let us prove the main ingredients that allow us to show the well-posedness of formulation (9).

**Proposition 4.7.** The bilinear forms appearing in the weak formulation (9) are continuous, namely, there exist \( M_a, M_b, M_j, M_j > 0 \), such that

\[
|a_h(\mathbf{w}_h, \mathbf{v}_h)| \leq M_a \|\mathbf{w}_h\|_{0,h,\Omega_T} \|\mathbf{v}_h\|_{0,h,\Omega_T}, \quad \forall \mathbf{w}_h, \mathbf{v}_h \in V_h,
\]

\[
|b_1(\mathbf{v}_h, q_h)| \leq M_b \|\mathbf{v}_h\|_{0,h,\Omega_T} \|q_h\|_{1,h,\Omega_T}, \quad \forall \mathbf{v}_h \in V_h, \forall q_h \in Q_h,
\]

\[
|f_j(\mathbf{w}_h, \mathbf{v}_h)| \leq M_J \|\mathbf{w}_h\|_{0,h,\Omega_T} \|\mathbf{v}_h\|_{0,h,\Omega_T}, \quad \forall \mathbf{w}_h, \mathbf{v}_h \in V_h,
\]

\[
|j_h(r_h, q_h)| \leq M_J \|r_h\|_{1,h,\Omega_T} \|q_h\|_{1,h,\Omega_T}, \quad \forall r_h, q_h \in Q_h.
\]

**Proof.** Let us fix any \( \mathbf{w}_h, \mathbf{v}_h \in V_h \) and \( r_h, q_h \in Q_h \). By Cauchy-Schwartz’s inequality

\[
|a_h(\mathbf{w}_h, \mathbf{v}_h)| \leq \|\mathbf{w}_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^2(\Omega)} + h^{-\frac{1}{2}} \|\mathbf{w}_h \cdot \mathbf{n}\|_{L^2(\Gamma_N)} h^{-\frac{1}{2}} \|\mathbf{v}_h \cdot \mathbf{n}\|_{L^2(\Gamma_N)}
\]

\[
\leq \|\mathbf{w}_h\|_{0,h,\Omega_T} \|\mathbf{v}_h\|_{0,h,\Omega_T},
\]
hence $M_n = 1$. By integration by parts, we get
\[
b_1(v_h, q_h) = \int_{\Omega} q_h \text{div} v_h - \int_{\Gamma_N} q_h v_h \cdot n = \sum_{K \in T_h} \int_{K \cap \Omega} q_h \text{div} v_h - \sum_{K \in \mathcal{G}(\Gamma_D)} \int_{\Gamma_K} q_h v_h \cdot n
\]
\[= - \sum_{K \in T_h} \int_{K \cap \Omega} \nabla q_h \cdot v_h + \sum_{f \in F_h^D} \int_{f \cap \Omega} [q_h] v_h \cdot n + \sum_{K \in \mathcal{G}(\Gamma_D)} \int_{\Gamma_K} q_h v_h \cdot n.
\]

From Cauchy-Schwarz's inequality, Lemma A.2, and a standard inverse estimate (Proposition 6.3.2 of [29]), we obtain
\[
\sum_{f \in F_h^D} \int_{f \cap \Omega} [q_h] v_h \cdot n \leq C \left( \sum_{f \in F_h^D} h^{-\frac{1}{2}} \| [q_h] \|_{L^2(f \cap \Omega)} \right)^\frac{1}{2} \left( \sum_{K \in T_h} \| v_h \|_{L^2(K)}^2 \right)^\frac{1}{2}
\]
\[\leq C \| q_h \|_{1,h,\Omega_T} \| v_h \|_{0,h,\Omega_T},
\]
and, analogously,
\[
\sum_{K \in \mathcal{G}(\Gamma_D)} \int_{\Gamma_K} q_h v_h \cdot n \leq C \left( \sum_{K \in \mathcal{G}(\Gamma_D)} h^{-\frac{1}{2}} \| q_h \|_{L^2(K \cap \Omega)} \right)^\frac{1}{2} \left( \sum_{K \in T_h} \| v_h \|_{L^2(K)}^2 \right)^\frac{1}{2}
\]
\[\leq C \| q_h \|_{1,h,\Omega_T} \| v_h \|_{0,h,\Omega_T}.
\]

Thus,
\[
|b_1(v_h, q_h)| \leq \sum_{K \in T_h} \| \nabla q_h \|_{L^2(K \cap \Omega)} \| v_h \|_{L^2(K \cap \Omega)} + C \| q_h \|_{1,h,\Omega_T} \| v_h \|_{0,h,\Omega_T}
\]
\[\leq C \| q_h \|_{1,h,\Omega_T} \| v_h \|_{0,h,\Omega_T}.
\]

The bounds for $j_h(\cdot, \cdot)$ and $j_h(\cdot, \cdot)$ follow as well straightforward. \hfill \Box

**Proposition 4.8.** There exists $\alpha > 0$, such that,
\[
a_h(v_h, v_h) + j_h(v_h, v_h) \geq \alpha \| v_h \|_{0,h,\Omega_T}^2, \quad \forall v_h \in V_h.
\]

**Proof.** This is just a consequence of Theorem 4.6. \hfill \Box

**Proposition 4.9.** There exists $\beta_1 > 0$ such that
\[
\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b_1(v_h, q_h)}{\| v_h \|_{0,h,\Omega_T} \| q_h \|_{1,h,\Omega_T}} \geq \beta_1 \| q_h \|_{1,h,\Omega_T}.
\]
Proof. Let us fix \( q_h \in Q_h \) and construct \( v_h \in V_h \) using just the internal degrees of freedom of \( V_h \), namely

\[
\int_f v_h \cdot n \varphi_h = h^{-1} \int_f [q_h] \varphi_h, \quad \forall \ f \in \mathcal{F}_h^1(\Omega_I), \forall \ \varphi_h \in \Psi_k(f),
\]

\[
\int_f v_h \cdot n \varphi_h = 0, \quad \forall \ f \in \mathcal{F}_h^0(\Omega_I), \forall \ \varphi_h \in \Psi_k(f),
\]

\[
\int_K v_h \cdot \psi_h = - \int_K \nabla q_h \cdot \psi_h, \quad \forall K \in \mathcal{T}_h(\Omega_I), \forall \ \psi_h \in \Psi_k(K), \quad \text{if } k > 0.
\]

We refer the reader to Section 2 for the definitions of \( \Psi_k(K) \) and \( \Psi_k(f) \). Let us extend \( v_h \) to zero outside \( \Omega_{I,h} \). It holds

\[
b_k(v_h, q_h) = - \sum_{K \in \mathcal{T}_h(\Omega_{I,h})} \int_K \nabla q_h \cdot v_h + \sum_{f \in \mathcal{F}_h^1(\Omega_{I,h})} \int_f [q_h] v_h \cdot n
\]

\[
= \sum_{K \in \mathcal{T}_h(\Omega_{I,h})} m_K \| \nabla q_h \|_{L^2(K)}^2 + \sum_{f \in \mathcal{F}_h^1(\Omega_{I,h})} h^{-1} \| [q_h] \|_{L^2(f)}^2 = \| q_h \|_{1,h,\Omega_{I,h}}^2.
\]

Now, we prove that \( \| v_h \|_{0,h,\Omega_I} \lesssim \| q_h \|_{1,h,\Omega_{I,h}} \). By construction of \( v_h \), it is sufficient to show

\[
\sum_{K \in \mathcal{T}_h(\Omega_{I,h})} \| v_h \|_{L^2(K)}^2 \lesssim \sum_{K \in \mathcal{T}_h(\Omega_{I,h})} \| \nabla q_h \|_{L^2(K)}^2 + \sum_{f \in \mathcal{F}_h^1(\Omega_{I,h})} h^{-1} \| [q_h] \|_{L^2(f)}^2.
\]

We mimic the proof of Proposition 2.1 in [11]. Let \( K \in \mathcal{T}_h(\Omega_{I,h}) \) and \( \tilde{K} \) be the reference element. Let \( f \) be a face of \( K \) and \( \tilde{f} \) be its preimage through \( F_K \) (see Section 2 for the definitions of \( \tilde{K} \) and \( F_K \)). Finite dimensionality implies

\[
\| \tilde{v}_h \|_{L^2(\tilde{K})}^2 \lesssim \| \pi_{\tilde{K},K-1} \tilde{v}_h \|_{L^2(\tilde{K})}^2 + \| \tilde{v}_h \cdot n \|_{L^2(\tilde{f})}^2,
\]

where \( \pi_{\tilde{K},K-1} \) is the \( L^2 \)-projection onto \( \Psi_k(\tilde{K}) \). By pushing forward to the element \( K \), a scaling argument implies

\[
\| v_h \|_{L^2(K)}^2 \lesssim \| \pi_{K,K-1} v_h \|_{L^2(K)}^2 + h \| v_h \cdot n \|_{L^2(f)}^2.
\]

This time \( \pi_{K,K-1} \) is the \( L^2 \)-projection onto \( \Psi_k(K) \). By construction of \( v_h \),

\[
\| v_h \|_{L^2(K)}^2 \lesssim \| \pi_{K,K-1} \nabla q_h \|_{L^2(K)}^2 + h^{-1} \| \pi_{f,K} [q_h] \|_{L^2(f)}^2 = \| \nabla q_h \|_{L^2(K)}^2 + h^{-1} \| [q_h] \|_{L^2(f)}^2,
\]

where \( \pi_{f,K} \) denotes the \( L^2 \)-projection onto \( \Psi_k(f) \). \( \square \)

We are left with the proof of the well-posedness of formulation (9). In order to do that, we verify that the bilinear form \( A_h(\cdot,\cdot) \) satisfies the hypotheses of the Banach-Necas-Babuška Theorem.

**Theorem 4.10.** There exists \( \eta > 0 \) such that

\[
\inf_{(v_h,q_h) \in V_h \times Q_h} \sup_{(w_h,r_h) \in V_h \times Q_h} \frac{A_h ((v_h,q_h);(w_h,r_h))}{\| (v_h,q_h) \| \| (w_h,r_h) \|} \geq \eta.
\]
For the volumetric term, let us apply the standard Deny-Lions argument (Chapter 3 of [29]) to the interpolant

\[ -b_1(w_h, q_h) = \|q_h\|_{1, h, \Omega_{t, h}}^2, \quad \|w_h\|_{0, h, \Omega_T} \lesssim \|q_h\|_{1, h, \Omega_{t, h}}. \]

We have

\[ A_h ((v_h, q_h); (-w_h, 0)) = a_h (v_h, w_h) - j_h (v_h, w_h) - b_1 (w_h, q_h) + b_1 (v_h, q_h) - j_h (q_h, 0) \]

\[ \geq -2 \|v_h\|_{0, h, \Omega_T} \|w_h\|_{0, h, \Omega_T} + \|q_h\|_{1, h, \Omega_{t, h}}^2 + 0 + 0 \]

\[ \geq -2 \|v_h\|_{0, h, \Omega_T} \|q_h\|_{1, h, \Omega_{t, h}} + \|q_h\|_{1, h, \Omega_{t, h}}^2 \]

\[ \geq -\frac{1}{\varepsilon} \|v_h\|_{0, h, \Omega_T} + (1 - \varepsilon) \|q_h\|_{1, h, \Omega_{t, h}}^2, \]

where \( \varepsilon > 0 \) arises from the Young inequality. On the other hand, it holds

\[ A_h ((v_h, q_h); (v_h - \delta w_h, -q_h)) = a_h (v_h, v_h) + j_h (v_h, v_h) + b_1 (v_h, q_h) + b_1 (v_h, q_h) + j_h (q_h, q_h) \]

\[ \geq \|v_h\|_{0, h, \Omega_T}^2 + j_h (q_h, q_h). \]

By choosing \((v_h - \delta w_h, -q_h) \in V_h \times Q_h\), for \( \delta > 0 \) to be set later on, we get

\[ A_h ((v_h, q_h); (v_h - \delta w_h, -q_h)) \lesssim \left(1 - \frac{\delta}{\varepsilon}\right) \|v_h\|_{0, h, \Omega_T}^2 + \delta (1 - \varepsilon) \|q_h\|_{1, h, \Omega_{t, h}}^2 + j_h (q_h, q_h). \]

Let us take, for instance, \( \varepsilon = \frac{1}{2} \) and any \( 0 < \delta < \frac{1}{2} \), so that

\[ A_h ((v_h, q_h); (v_h - \delta w_h, -q_h)) \lesssim \|v_h\|_{0, h, \Omega_T}^2 + \|q_h\|_{1, h, \Omega_{t, h}}^2 + j_h (q_h, q_h) \]

\[ \lesssim \|v_h\|_{0, h, \Omega_T}^2 + \|q_h\|_{1, h, \Omega_{t, h}}^2, \]

where in the last inequality we used \( \|q_h\|_{1, h, \Omega_{t, h}}^2 \lesssim \|q_h\|_{1, h, \Omega_{t, h}}^2 + j_h (q_h, q_h) \), which follows from Theorem 4.6. We are left with proving that \( \|(v_h - \delta w_h, q_h)\| \lesssim \|(v_h, q_h)\| \), which is a consequence of (18). \( \square \)

4.2. A priori error estimates

**Theorem 4.11.** There exists \( C > 0 \) such that, for every \((v, q) \in H^1(\Omega) \times H^r(\Omega), t \geq 1, r \geq 1\), it holds

\[ \|E(v) - r_h v\|_{0, h, \Omega_T} + \|E(q) - \Pi_h q\|_{1, h, \Omega_T} \leq Ch^s \left(\|v\|_{H^r(\Omega)} + \|q\|_{H^r(\Omega)}\right), \]

where \( s := \min\{t - 1, r - 1, k\} \), \( E \) and \( E' \) have been introduced in Section 3.

**Proof.** Let us start with the velocity. We have

\[ \|E(v) - r_h v\|_{0, h, \Omega_T}^2 = \|E(v) - r_h v\|_{L^2(\Omega_T)}^2 + h^{-1} \sum_{K \in \mathcal{T}_h(\Gamma_N)} \|E(v) - r_h v\|_{L^2(\Gamma_K)}^2. \]

For the volumetric term, let us apply the standard Deny-Lions argument (Chapter 3 of [29]) to the interpolant \( r_h \) and get

\[ \|E(v) - r_h v\|_{L^2(\Omega_T)}^2 = \sum_{K \in \mathcal{T}_h} \|E(v) - r_h v\|_{L^2(\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_h} h^{2t} \|E(v)\|_{H^r(K)}^2 \leq Ch^{2t} \|v\|_{H^r(\Omega)}^2. \]
For the boundary part, let us first use the multiplicative trace inequality of Lemma A.1 (componentwise) and then, again, the Deny-Lions argument:

\[
\begin{align*}
    h^{-1} \sum_{K \in \mathcal{G}_h(\Gamma_N)} \| (E(v) - r_h v) \cdot n \|_{L^2(\Gamma_K)} & \leq h^{-1} \sum_{K \in \mathcal{G}_h(\Gamma_N)} \| E(v) - r_h v \|_{L^2(\Gamma_K)} \\
    & \leq \mathcal{C} h^{-1} \sum_{K \in \mathcal{G}_h(\Gamma_N)} \| E(v) - r_h v \|_{H^1(K)} \| E(v) - r_h v \|_{L^2(K)} \\
    & \leq \mathcal{C} h^{-1} \sum_{K \in \mathcal{F}_h} h^{t-1} \| E(v) \|_{H^1(K)} h^t \| E(v) \|_{H^r(K)} \\
    & \leq \mathcal{C} h^{2(\ell-1)} \| v \|_{H^r(\Omega)}^2.
\end{align*}
\]

We move to the pressure case.

\[
\begin{align*}
    \| E(q) - \Pi_h q \|_{1,\Omega}^2 = & \sum_{K \in \mathcal{F}_h} \| \nabla (E(q) - \Pi_h q) \|_{L^2(K)}^2 + \sum_{f \in \mathcal{F}_h^1} h^{-1} \| (E(q) - \Pi_h q) \|_{L^2(f \cap \Omega)}^2 \\
    & + \sum_{K \in \mathcal{G}_h(\Gamma_D)} h^{-1} \| E(q) - \Pi_h q \|_{L^2(\Gamma_K)}^2.
\end{align*}
\]

For the volumetric term one we may proceed as in the case of the velocity to easily obtain

\[
\sum_{K \in \mathcal{F}_h} \| \nabla (E(q) - \Pi_h q) \|_{L^2(K)}^2 \leq \mathcal{C} h^{2(\ell-1)} \| q \|_{H^r(\Omega)}^2.
\]

Let us focus on the jump part of (19) and take \( f \in \mathcal{F}_h^1 \) such that \( f = K_1 \cap K_2 \). Then, by Lemma A.1 and the Deny-Lions Lemma, we have

\[
\begin{align*}
    h^{-1} \| [E(q) - \Pi_h q] \|_{L^2(f \cap \Omega)}^2
    = & \left( \| E(q) - \Pi_h q \|_{K_1} - \| E(q) - \Pi_h q \|_{K_2} \right)_{L^2(f)}^2 \\
    \leq & \mathcal{C} h^{-1} \| E(q) - \Pi_h q \|_{L^2(K_1)} \| E(q) - \Pi_h q \|_{H^1(K_1)} \\
    & + \mathcal{C} h^{-1} \| E(q) - \Pi_h q \|_{L^2(K_2)} \| E(q) - \Pi_h q \|_{H^1(K_2)} \\
    \leq & \mathcal{C} h^{2(\ell-1)} \left( \| E(q) \|_{H^r(K_1)}^2 + \| E(q) \|_{H^r(K_2)}^2 \right).
\end{align*}
\]

Hence,

\[
\sum_{f \in \mathcal{F}_h^1} h^{-1} \| [E(q) - \Pi_h q] \|_{L^2(f \cap \Omega)}^2 \leq \mathcal{C} h^{2(\ell-1)} \| q \|_{H^r(\Omega)}^2.
\]

The bound for the boundary part of (19) follows in a similar fashion. \( \square \)

**Lemma 4.12.** For every \( v \in H^\ell(\Omega) \) and \( q \in H^r(\Omega) \), \( t \geq 1 \), \( r \geq 1 \),

\[
\left( j_h(r_h v_r, r_h v_r) \right)^\frac{1}{2} \leq h^m \| v \|_{H^\ell(\Omega)}, \quad \left( j_h(\Pi_h q, \Pi_h q) \right)^\frac{1}{2} \leq h^\ell \| q \|_{H^r(\Omega)},
\]

where \( m := \min\{k + 1, t\} \) and \( \ell := \min\{k, r - 1\} \).

**Proof.** Note that

\[
j_h(\pi_h q - E(q), \pi_h q - E(q)) = j_h(\Pi_h q, \Pi_h q) - j_h(\Pi_h q, E(q)) - j_h(E(q), \Pi_h q) + j_h(E(q), E(q)). \tag{21}
\]
Proof.
Firstly, we observe that it suffices to bound
By Theorem 4.10, for \((u, p)\), Theorem 4.11 implies, for
the solution \((u, p)\)

Now, by using Lemma A.1 and a standard approximation argument, we obtain

Given \(0 \leq k \leq r - 1\), for every \(0 \leq j \leq k\), \(\alpha\) multi-index such that \(|\alpha| = j\), \(D^\alpha E(q) \in H^{r-j}(\Omega_T) \subset H^1(\Omega_T)\), hence \([\partial^j E(q)]_f\) vanishes across every \(f \in F_h\). Thus, (21) implies that

\[
j_h(\Pi_h q, \Pi_h q) = j_h(\Pi_h q - E(q), \Pi_h q - E(q)).
\]

Now, by using Lemma A.1 and a standard approximation argument, we obtain

\[
j_h(\Pi_h q - E(q), \Pi_h q - E(q)) = \sum_{f \in F_h} \sum_{i=0}^{k} h^{2i-1} \left\| \partial^i (\Pi_h q - E(q)) \right\|^2_{L^2(f)} \leq C \sum_{K \in T_h} \sum_{i=0}^{k} h^{2i-1} \left\| D^i (\Pi_h q - E(q)) \right\|^2_{L^2(K)} \leq C h^{2k} \left\| E(q) \right\|^2_{H^r(\Omega)} \leq C h^{2k} \left\| q \right\|^2_{H^r(\Omega)},
\]

where \(\ell := \min\{k, r-1\}\) and the last inequality follows from the boundedness of \(E\). The bound for \(j_h(\cdot, \cdot)\) follows in a completely similar fashion.

**Theorem 4.13.** Let \((u, p) \in H^t(\Omega) \times H^r(\Omega), t \geq 1, r \geq 1\), be the solution of problem (1). Then, the finite element solution \((u_h, p_h) \in V_h \times Q_h\) of (8) satisfies

\[
\|u - u_h\|_{L^2(\Omega)} + \left( \sum_{K \in T_h} \|\nabla (p - p_h)\|^2_{L^2(K \cap \Omega)} \right)^{\frac{1}{2}} \leq Ch^s \left( \|u\|_{H^r(\Omega)} + \|p\|_{H^r(\Omega)} \right),
\]

where \(s := \min\{t-1, r-1, k\}\).

**Proof.** Firstly, we observe that

\[
\|u - u_h\|_{L^2(\Omega)} + \left( \sum_{K \in T_h} \|\nabla (p - p_h)\|^2_{L^2(K \cap \Omega)} \right)^{\frac{1}{2}} \leq \|E(u) - u_h\|_{0, h, \Omega_T} + \|E(p) - p_h\|_{1, h, \Omega_T} \leq \sqrt{d} \|(E(u) - u_h, E(p) - p_h)\|,
\]

it suffices to bound \(\|(E(u) - u_h, E(p) - p_h)\|\). Let us proceed by triangular inequality:

\[
\|(E(u) - u_h, E(p) - p_h)\| \leq \sum_{i} \left( \|(E(u) - r_h u, E(p) - \Pi_h p)\| + \|(r_h u - u_h, \Pi_h p - p_h)\| \right).
\]

Theorem 4.11 implies, for \(s = \min\{t-1, r-1, k\}\),

\[
I \leq Ch^s \left( \|u\|_{H^r(\Omega)} + \|p\|_{H^r(\Omega)} \right).
\]

By Theorem 4.10, for \((u_h - r_h u, p_h - \pi_h p)\) there exists \((v_h, q_h) \in V_h \times Q_h\) such that

\[
II = \|(u_h - r_h u, p_h - \Pi_h p)\| \lesssim \frac{A_h (\|(u_h - r_h u, p_h - \Pi_h p)\|, (v_h, q_h))}{\|(v_h, q_h)\|}.
\]
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From Propositions 4.2, 4.7, and the definition of $\|\cdot\|$, it holds

$$A_h((u_h - r_h u, p_h - \Pi_h p); (v_h, q_h)) = A_h((u_h - u, p_h - p); (v_h, q_h))$$

$$+ A_h((u - r_h u, p - \Pi_h p); (v_h, q_h)) \leq \|j_h(r_h u, v_h)\| + \|j_h(\Pi_h p, q_h)\| + \|\Pi_h p\| \|(v_h, q_h)\|.$$  \hspace{1cm} (25)

Lemmas 4.4, 4.12 entail, for $s := \min\{t - 1, r - 1, k\}$,

$$|j_h(r_h u, v_h)| + |j_h(\Pi_h p, q_h)| \leq j_h(r_h u, r_h u)^{\frac{1}{2}} j_h(v_h, v_h)^{\frac{1}{2}} + j_h(\Pi_h p, \Pi_h p)^{\frac{1}{2}} j_h(q_h, q_h)^{\frac{1}{2}}$$

$$\lesssim h^s \|u\|_{H^s(\Omega)} \|v_h\|_{0, h, \Omega_T} + h^s \|p\|_{H^s(\Omega)} \|q_h\|_{1, h, \Omega_T}$$

$$\lesssim h^s \left(\|u\|_{H^s(\Omega)} + \|p\|_{H^s(\Omega)}\right) \|(v_h, q_h)\|,$$  \hspace{1cm} (26)

where $s := \min\{t - 1, r - 1, k\}$. By plugging (26) back into (25) and using Theorem 4.11, we obtain

$$A_h((u_h - r_h u, p_h - \Pi_h p); (v_h, q_h)) \lesssim h^s \left(\|u\|_{H^s(\Omega)} + \|p\|_{H^s(\Omega)}\right) \|(v_h, q_h)\|.$$  \hspace{1cm} (27)

We combine (22), (23), (24) and (27), getting

$$\|(E(u) - u_h, E(p) - p_h)\| \lesssim h^s \left(\|u\|_{H^s(\Omega)} + \|p\|_{H^s(\Omega)}\right),$$

where $s := \min\{t - 1, r - 1, k\}$. \hspace{1cm} \square

**Remark 4.14.** The convergence rates given by Theorem 4.11 are optimal for the chosen discrete norms. On the other hand, the scaling of the energy norm $\|\cdot\|$ does not allow us to obtain optimal convergence rates for the velocities with respect to the $L^2$-norm by simply applying Theorem 4.11. This is due to the term $h^{-1} \int_{\Gamma} \nabla u_h \cdot \nabla v_h : \nabla n$; the natural weight, mimicking the $H^{-\frac{1}{2}}$-scalar product, would be $h$ instead of $h^{-1}$. However, as already discussed in [12] such weight does not lead to an optimally converging scheme.

### 5. The Condition Number

Following [14] and [24], we want to show that the Euclidean condition number of the matrix arising from the discretization (9) is uniformly bounded by $C h^{-2}$, where $C > 0$ is independent of how the underlying mesh cuts the boundary. Let us observe that the usual scaling of the condition number for a finite element discretization of the Darcy problem is $O(h^{-1})$. We pay with a factor $h^{-1}$ because of the choice of the discrete norms.

Denoting $N = \dim V_h \times Q_h$, we can expand an arbitrary element $(v_h, q_h) \in V_h \times Q_h$ as $(v_h, q_h) = \sum_{i=1}^N \psi_i \varphi_i$, where $(\varphi_i)_{i=1}^N$ is the finite element basis for the product space $V_h \times Q_h$ and $\mathbf{v} \in \mathbb{R}^N$ is its coordinate vector. The previous expansion of the elements of $V_h \times Q_h$ uniquely defines a canonical isometric isomorphism between $V_h \times Q_h$ and $\mathbb{R}^N$, namely

$$\mathcal{C} : V_h \times Q_h \to \mathbb{R}^N, \quad (v_h, q_h) \mapsto \mathbf{v}.$$  

Here, $\mathbb{R}^N$ is equipped with the standard Euclidean scalar product, denoted as $(\cdot, \cdot)_{\mathbb{R}^N}$, and the induced norm $|\cdot|_{\mathbb{R}^N}$. Given $M \in \mathbb{R}^{N \times N}$, we denote as $\|M\|_{2}$ the matrix norm of $M$ induced by $|\cdot|_{\mathbb{R}^N}$. Let $A \in \mathbb{R}^{N \times N}$ denote the matrix associated to the discrete formulation (9), namely

$$(A\mathcal{C}(v_h, q_h), \mathcal{C}(v_h, q_h)) = A_h((v_h, q_h); (w_h, r_h)), \quad \forall \mathcal{C}(v_h, q_h), \mathcal{C}(w_h, r_h) \in V_h \times Q_h.$$
Remark 5.1. In the pure Neumann case $\Gamma = \Gamma_N$, then the solution to (9) for the pressure is determined up to a constant, hence $A$ is singular and $\ker A = \text{span}\{C(0,1)\}$. We shall consider, instead of $A$, its bijective restriction $A_{|\widehat{\Gamma}_N} : \widehat{\Gamma}_N \to \mathbb{R}^N$, where $\widehat{\Gamma}_N := \mathbb{R}^N / \ker A$ and $\mathbb{R}^N := \text{Im}(A)$.

As already said, the goal is to analyze the Euclidean condition number $\kappa_2(A) := \|A\|_2 \|A^{-1}\|_2$. From [29], we know that the main ingredients for the conditioning analysis are the following:

1. the stability of the discrete formulation with respect to a given norm $\|\cdot\|_a$;
2. the $\ell^2$-stability of the basis with respect to a given norm $\|\cdot\|_b$;
3. the equivalence between the norms $\|\cdot\|_a$ and $\|\cdot\|_b$.

In the subsequent exposition, the product norm (10) plays the role of $\|\cdot\|_b$, while $\|\cdot\|_{L^2(\Omega_T)}$, defined as $\|(v_h, q_h)\|_{L^2(\Omega_T)}^2 := \|v_h\|_{L^2(\Omega_T)}^2 + \|q_h\|_{L^2(\Omega_T)}^2$, corresponds to $\|\cdot\|_a$.

Lemma 5.2. There exist $C_1, C_2 > 0$ such that, for every $(v_h, q_h) \in V_h \times Q_h$,

$$C_1 h^\frac{d}{2} |V|_{L^2} \leq \|(v_h, q_h)\|_{L^2(\Omega_T)} \leq C_2 h^\frac{d}{2} |V|_{L^2},$$

where $V = C(v_h, q_h)$.

Proof. The result holds because the background mesh is shape-regular and quasi-uniform. We refer the interested reader to Lemma A.1 in [15].

Lemma 5.3. There exist $C_1, C_2 > 0$ such that, for every $(v_h, q_h) \in V_h \times Q_h$,

$$C_1 h^{-1}\|(v_h, q_h)\|_{L^2(\Omega_T)} \leq \|(v_h, q_h)\| \leq C_2 h^{-1}\|(v_h, q_h)\|_{L^2(\Omega_T)}.$$

Proof. The first bound follows because of a Poincaré-Friedrichs inequality for piecewise $H^1$-functions (Section 10.6 of [5]). For the second inequality it is sufficient to apply standard inverse estimates for boundary-fitted finite elements.

Theorem 5.4. There exists $C > 0$ such that

$$\kappa_2(A) \leq Ch^{-2}.$$

Proof. Let us start by bounding $\|A\|_{L^2}$. Given $(v_h, q_h), (w_h, r_h) \in V_h \times Q_h$ such that $C(v_h, q_h) = V, C(w_h, r_h) = W$, we have

$$(AV, W)_{L^2} = A_h(\{v_h, q_h\}; \{w_h, r_h\}) \lesssim \|(v_h, q_h)\|\|(w_h, r_h)\| h^{-2} \|(v_h, q_h)\|_{L^2(\Omega_T)} \|(w_h, r_h)\|_{L^2(\Omega_T)} \lesssim h^{d-2} |V| |W|_{L^2}.$$ 

In the previous inequalities we used, respectively, the continuity of $A_h(\cdot; \cdot)$, Lemma 5.3 and Lemma 5.2. Hence, $\|A\|_{L^2} \lesssim h^{d-2}$. We need to bound $\|A^{-1}\|_{L^2}$. Since (the restriction of) $A^{-1}$ is invertible, we can write

$$\|A^{-1}\|_{L^2} = \sup_{\mathcal{V} \in \mathbb{R}^N \backslash \{0\}} \frac{\|A^{-1}\mathcal{V}\|}{\|\mathcal{V}\|_{L^2}} = \sup_{A\mathcal{V} \in \mathbb{R}^N \backslash \{0\}} \frac{|\mathcal{V}|}{\|A\|} = \sup_{A\mathcal{V} \in \mathbb{R}^N \backslash \{0\}} \sup_{\mathcal{W} \in \mathbb{R}^N \backslash \{0\}} \frac{|\mathcal{V}|_{L^2}}{|A\mathcal{V}, \mathcal{W}|_{L^2}}. \quad (28)$$

From Theorem 4.10, we know that, for every $\mathcal{V} \in \mathbb{R}^N \backslash \{0\}$, there exists $\mathcal{W}$ such that

$$(AV, W)_{L^2} = A_h(\{v_h, q_h\}; \{w_h, r_h\}) \gtrsim \|(v_h, q_h)\|\|\mathcal{W}\|_{L^2(\Omega_T)} \|(w_h, r_h)\|_{L^2(\Omega_T)} \gtrsim h^{d-2} |V| |W|_{L^2}. \quad (29)$$

Moreover in the last two inequalities we used, respectively, Lemma 5.3 and Lemma 5.2. By combining (28) and (29) we get $\|A^{-1}\|_{L^2} \lesssim h^{-d}$. Hence, we are done.
6. The purely Dirichlet case

The goal of this section is to sketch the main steps required to analyze formulation (8) in the purely Dirichlet case. If $\Gamma = \Gamma_D$, then we consider the following Raviart-Thomas finite element discretization of problem (1).

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$
\int_{\Omega} u_h \cdot v_h + b_0(v_h, p_h) = \int_{\Omega} f \cdot v_h + \int_{\Gamma} p_D v_h \cdot n, \quad \forall v_h \in V_h,
$$

$$
b_0(u_h, q_h) = \int_{\Omega} g q_h, \quad \forall q_h \in Q_h.
$$

(30)

It is natural to equip the discrete spaces $V_h$ and $Q_h$ with $\|\cdot\|_{H(div;\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$, respectively. It is readily seen that formulation (30) satisfies the standard stability estimates for saddle point problems (see the hypotheses of Theorem 5.2.5 of [4]). Hence a priori error estimates can be obtained by standard techniques (again, we refer the reader to Section 5.2 of [4]). This time, the convergence rates are optimal because of the choice of the norms. However, the conditioning of the arising linear system will still strongly depend on the way the boundary cuts the mesh. As for the general case with mixed boundary conditions, we propose to cure this issue with a ghost penalty-based stabilization.

$$
\tilde{j}_h(w_h, v_h) := \sum_{f \in F_h^k} \sum_{j=0}^{k} h^{2j+1} \int_f [\partial_n^j w_h][\partial_n^j v_h], \quad w_h, v_h \in V_h,
$$

$$
\tilde{j}_h(r_h, q_h) := \sum_{f \in F_h^k} \sum_{j=0}^{k} h^{2j+1} \int_f [\partial_n^j r_h][\partial_n^j q_h], \quad r_h, q_h \in Q_h.
$$

Let us observe that the ghost penalty operators scale differently than in (7) because of the different choices of the norms to the mixed case. The stabilized formulation reads as follows.

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$
\int_{\Omega} u_h \cdot v_h + \tilde{j}_h(u_h, v_h) + b_0(v_h, p_h) = \int_{\Omega} f \cdot v_h + \int_{\Gamma} p_D v_h \cdot n, \quad \forall v_h \in V_h,
$$

$$
b_0(u_h, q_h) + \tilde{j}_h(p_h, q_h) = \int_{\Omega} g q_h, \quad \forall q_h \in Q_h.
$$

(31)

By mimicking the same lines of Section 5, it is possible to show that the condition number of (31) goes as $O(h^{-1})$.

7. Numerical examples

In the following, we use the ghost penalty projection-based operators $s_h(\cdot, \cdot)$ and $s_h(\cdot, \cdot)$ defined in Remark 4.1. As already observed, this choice is convenient from the implementation point of view since it spares us to calculate the jumps of possibly high-order normal derivatives through the facets in the vicinity with the cut boundary. We limit the scope of our numerical investigations to the case of Cartesian quadrilateral meshes in 2D. To integrate in the cut elements, we employ the strategy depicted in [2]: the cut elements are reparametrized using polynomials with the same approximation order of the Raviart-Thomas space employed for the space discretization.
7.1. Convergence rates

7.1.1. Cut pentagon

Let \( \Omega_0 = (0, 1)^2 \), \( \Omega_1 \) be the triangle with vertices \((0, 0.25 + \varepsilon) - (0, 1) - (0.75 - \varepsilon, 1) \) and \( \Omega = \Omega_0 \setminus \overline{\Omega}_1 \), with \( \varepsilon = 10^{-9} \), see Figure 1c. The reference solutions are

\[
\mathbf{u}_{\text{ex}} = \left( \begin{array}{c} y \sin(x) \cos(y) \\ -x \sin(y) \cos(x) \end{array} \right), \quad p_{\text{ex}} = x^3 y.
\]

Neumann boundary conditions are imposed on the whole boundary, weakly just on the sides that do not fit the underlying mesh. We compute the approximation errors of the velocity and pressure fields for different degrees \( k \in \{0, 1, 2\} \), see Figure 2. We have optimal convergence, despite the sub-optimal result of Theorem 4.13.

7.1.2. Cut circle

Let us consider \( \Omega = B_r(x_0) \), with \( x_0 = (0.5, 0.5) \) and \( r = 0.45 \), see Figure 1d. The manufactured solution for the pressure is

\[
p_{\text{ex}} = \sin(2\pi x) \cos(2\pi),
\]

and the velocity field is computed from Darcy’s law (1) when \( f \) is taken to be zero. We weakly prescribe Neumann boundary conditions on the whole boundary, which does not fit the underlying mesh. The \( L^2 \)-errors for the velocity and pressure fields are plotted in Figure 3. We can see optimal orders of convergence and better accuracy in the stabilized case.


**7.2. Condition number**

7.2.1. Cut rectangle

Let us consider as physical domain the cut rectangle \( \Omega = (0,1) \times (0, 0.75 + \varepsilon) \) where \( \varepsilon = 10^{-7} \), see Figure 1a. We impose Neumann boundary conditions weakly on the whole boundary. In Figure 4 we compare the conditioning of the stabilized and non-stabilized formulations. Similarly, in Figure 5 we compare the conditioning of the stabilized and non-stabilized formulations when Dirichlet boundary conditions are imposed. The results are in agreement with the theory developed in Sections 5, 6. In particular, we observe that without stabilization, the condition number is negatively affected by the presence of cut elements and seems to grow without control,
while in the stabilized case, the expected scaling of the conditioning is restored: $O(h^{-2})$ for the Neumann case and $O(h^{-1})$ for the purely Dirichlet case.

### 7.3. On mass conservation

Mass conservation is an important feature for finite element discretizations of incompressible flows, whose violation is not tolerable in many applications [22]. As observed in Remark 3.2, the Raviart-Thomas finite element satisfies $\text{div} V_h = Q_h$ in the unfitted configuration as well. The formulation (8) as it stands is bound to fail to satisfy the incompressibility constraint in a weak sense, which is why, to exploit this property when the right-hand side $g$ vanishes, we consider the following non-symmetric variant of formulation (8).

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$
\begin{align*}
    a_h(u_h, v_h) + b_1(v_h, p_h) + j_h(u_h, v_h) &= \int_{\Omega} \mathbf{f} \cdot v_h + \int_{\Gamma_D} p_D v_h \cdot n + h^{-1} \int_{\Gamma_N} u_N v_h \cdot n, & \forall v_h \in V_h, \\
    b_0(u_h, q_h) + j_h(p_h, q_h) &= 0, & \forall q_h \in Q_h,
\end{align*}
$$

(32)
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\begin{align*}
  k &= 0 \\
  k &= 1 \\
  k &= 2 \\
  k &= 3
\end{align*}

\begin{align*}
  k &= 0 \text{ with GP} \\
  k &= 1 \text{ with GP} \\
  k &= 2 \text{ with GP} \\
  k &= 3 \text{ with GP}
\end{align*}

\begin{align*}
  \kappa_2(A) &= 10^{11} \\
  \kappa_2(A) &= 10^{21} \\
  \kappa_2(A) &= 10^{31} \\
  \kappa_2(A) &= 10^{41}
\end{align*}

\begin{align*}
  h^{-1} &= 2^{-6} \\
  h^{-1} &= 2^{-5} \\
  h^{-1} &= 2^{-4} \\
  h^{-1} &= 2^{-3} \\
  h^{-1} &= 2^{-2} \\
  h^{-1} &= 2^{-1}
\end{align*}

Figure 5. Condition number for the cut rectangle with Dirichlet boundary conditions.

where

\begin{align*}
  b_0(w_h, q_h) &= \int_\Omega q_h \nabla w_h, \quad w_h \in V_h, q_h \in Q_h.
\end{align*}

Let us test formulation (32) in the stabilized and non-stabilized cases. We take as reference solutions

\begin{align*}
  u_{ex} &= \begin{pmatrix} \cos(x) \sinh(y) \\ \sin(x) \cosh(y) \end{pmatrix}, \\
  p_{ex} &= -\sin(x) \sinh(y) - (\cos(1) - 1)(\cosh(1) - 1).
\end{align*}

Note that \( \nabla u_{ex} = 0 \). We impose Dirichlet boundary conditions on \( \{(x, y) : x = 2, 0 \leq y \leq 2\} \) and on \( \{(x, y) : 0 \leq x \leq 2, y = 2\} \), and weak Neumann boundary conditions on the rest of the boundary. The computed divergence of the discrete solution for the velocity is shown in Figure 6. We observe that the ghost penalty stabilization pollutes the divergence of the velocity, hence also the non-symmetric formulation (32) fails to the mass conservation at the discrete level.
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Figure 6. \( \text{div} \mathbf{u}_h \) in the square with circular cut obtained with (32).

APPENDIX A. AUXILIARY RESULTS

Lemma A.1. There exists \( C > 0 \) depending on \( \Gamma \), but not on the way it cuts the mesh, such that for every \( K \in \mathcal{G}_h \):

\[
\|v\|_{L^2(K \cap \Gamma)}^2 \leq C \|v\|_{L^2(K)} \|v\|_{H^1(K)}, \quad \forall \ v \in H^1(K).
\]

Proof. See, for instance, Lemma 3 in [18], Lemma 3 in [19], or Lemma 4.1 of [30].

Lemma A.2. There exists \( C > 0 \) depending on \( \Gamma \), but not on the way it cuts the mesh, such that for every \( K \in \mathcal{G}_h \):

\[
h^{\frac{1}{2}} \|\mathbf{v}_h \cdot \mathbf{n}\|_{L^2(\Gamma_K)} \leq C \|\mathbf{v}_h\|_{L^2(K)}, \quad \forall \ \mathbf{v}_h \in \mathcal{V}_h.
\]

Proof. It follows by Lemma A.1, finite dimensionality and a scaling argument.

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