ASYMPTOTICALLY CONSERVED QUANTITIES FOR NONLINEAR ODES AND CONNECTION FORMULAE

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Abstract. We introduce a new rigorous method, based on Borel summability and asymptotic constants of motion generalizing [9] and [12], to analyze singular behavior of nonlinear ODEs in a neighborhood of infinity and provide global information about their solutions. In equations with the Painlevé-Kowalevski (P-K) property (stating that movable singularities are not branched) it allows for solving connection problems. The analysis in carried in detail for \( P_1 \), \( y'' = 6y^2 + z \), for which we find the Stokes multipliers in closed form and global asymptotics for solutions having power-like behavior in some direction in \( \mathbb{C} \), in particular for the tritronquées.

Calculating the Stokes multipliers solely relies on the P-K property and does not use linearization techniques such as Riemann-Hilbert or isomonodromic reformulations. We discuss how the approach would work to calculate connection constants for a larger class of P-K integrable equations.

We develop methods for finding asymptotic expansions in sectors where solutions have infinitely many singularities. These techniques do not rely on integrability and apply to more general second order ODEs which, after normalization, are asymptotically close to autonomous Hamiltonian systems.

1. Introduction

1.1. Overview of the paper and motivation. We provide a new method for studying solutions of nonlinear second order equations in singular regions, containing singularities which may accumulate towards infinity. The method relies on obtaining and analyzing asymptotically conserved quantities; these may not exist globally, but rather on regions bordered by antistokes lines, and they can be matched to each other and to the asymptotic expansions valid in regular regions, which are bordered by antistokes lines and where solutions are analytic towards infinity. The asymptotically conserved quantities determine solutions and their behavior, and their use as dependent variables desingularizes the problem.

In regions where solutions are regular where the independent variable is large enough, solutions have asymptotic expansions which are Borel summable \([7, 8, 9]\); these expansions are shown to match to the asymptotically conserved quantities of the singular regions, providing global information about solutions. The approach does not use linearization such as a Riemann-Hilbert reformulation. For equations with the Painlevé-Kowalevski property (P-K) – stating that all solutions are single-valued on a common Riemann surface—the asymptotically conserved quantities provide explicit connection formulas using the P-K property alone. The P-K property entails a consistency condition, that the solution returns to the same asymptotic representation after a \( 2\pi \) rotation in a neighborhood of infinity, and this is a nontrivial equation for the Stokes multiplier, see \([2]\).

In the present paper we carry out this program for the Painlevé equation \( P_1 \)

\[
y'' = 6y^2 + z
\]

(1)

We discuss in \([7, 4]\) how the approach can be applied to the Painlevé equation \( P_2 \) and more general equations.

For \( P_1 \) we obtain the asymptotic behavior of tronqué solutions with exponential accuracy in the pole-free sectors, with \( O(z^{-25}) \) relative errors in pole regions where the associated asymptotic elliptic functions become trigonometric ones, and \( O(z^{-\frac{45}{8}}) \) where the elliptic functions are nondegenerate, see Theorem\([1], [13]\) and \([13.3]\) below. This precision exceeds the one needed to keep track of the Stokes multipliers and then to determine them based on the single-valued consistency mentioned above.
Until now the Stokes multiplier has been calculated using linearization methods; one interest in developing an alternative approach is that while there is no known method of generating an associated Riemann-Hilbert problem from the P-K property, establishing the P-K property appears a more tractable problem.

1.2. A brief overview of integrability, linearization, the R-H problem and connection formulas. Eq. (1) is the first of the six Painlevé equations. These, together with equations reducible to equations of classical functions, constitute the complete set of differential equations of degree at most two, in a quite general class [15], that are P-K integrable [1], modulo equivalences.

In the realm of linear differential equations, the classical special functions such as the Airy, Bessel or hypergeometric ones, play an important role due the existence of integral representations, allowing in particular for a global description: an integral formula allows for explicitly linking the behavior of one solution at various critical points, and along different directions at infinity. These links are connection problems and their solutions are connection formulae. Until the late 1970s integral formulas were essentially the only tools in solving connection problems.

The general solution of (1) is highly transcendental; in particular there are no integral representations in terms of simpler functions. This is a deep result with a long history starting with a partial proof by Painlevé himself, and a complete argument due to Umemura, [25], [26].

All six Painlevé equations turned out to have explicit connection formulas. These have been obtained over a span of about two decades starting in the late seventies, via linearization methods. The first of these was achieved via a determinantal representation, followed by the isomonodromic reformulation, cf. the fundamental papers by Ablowitz and Segur [1], McCoy, Wu and Tracy [21], and Jimbo, Miwa, Mori and Sato [16]; for a good survey of the vast literature see [4] and [13].

Linearization techniques fall in some sense under the Riemann-Hilbert (RH) reformulation umbrella, [13]. The classical RH problem is to find an analytic function in a cut plane with a given jump across the cut and prescribed behavior at infinity. The solution is provided by Plemelj’s formulas. More generally, one seeks a matrix-valued analytic function with cuts along a number of rays with prescribed matrix jumps. The existence of such a function requires a compatibility condition to be satisfied. Roughly speaking, following ideas going back to Birkhoff, a RH reformulation of an equation is finding a RH problem for which the equation is equivalent to this compatibility condition. Since a RH problem is linear, this effectively linearizes the equation; see [24] §32.4(i) for a short description of the isomonodromic linearization. The six Painlevé equations were shown in the late last century to be RH compatibility equations, and the P-K property follows from this presentation.

For the Painlevé transcendent P_1 the Stokes multiplier was obtained via linearization in 1988 by Kapaev [18], corrected in a 1993 paper by Kitaev-Kapaev [19]. The Painlevé transcendents are now as important in nonlinear mathematical physics as the classical special functions play in linear physics [13].

Conversely, knowing the jumps across the cuts determines the associated RH problem. For a differential equation the jumps follow from its connection formulas. The circle is closed once the P-K property is shown to determine the jump conditions in closed form.

1.3. Definitions, setting and general properties of P_1.

1.3.1. Normalization. It is convenient to normalize (1) as described in [9]. The change of variables

\[ z = 24^{-1} 30^{4/5} x^{4/5} e^{-\pi i/5}; \quad y(z) = i \sqrt{z/6} (1 - \frac{4}{z} e^{-2} + h(x)) \] (2)
where the branch of the square root is the usual one, which is positive for \( z > 0 \), brings (1) to the Boutroux-like form

\[
h'' + \frac{h'}{x} - h - \frac{h^2}{2} - \frac{392}{625} \frac{1}{x^4} = 0
\]  

(3)

1.3.2. Symmetries. Eq. (1) has a five-fold symmetry: if \( y(z) \) solves (1), then so does \( \rho^2 y(\rho z) \) if \( \rho^5 = 1 \). Relatedly, (3) is invariant under the transformations \( h(x) \mapsto h(xe^{\pm i\pi}) \) and note also the symmetry \( h(x) \mapsto h(x) \).

1.3.3. Regularity. There are five special directions of (1) for solutions having asymptotic power series in some sectors (see Notes 45, 34 and \$7.1\). Bordered by these directions, we have the symmetry sectors

\[
S_k = \left\{ z \in \mathbb{C} \mid \frac{2k - 1}{5}\pi < \arg z < \frac{2k + 1}{5}\pi \right\}, \; k \in \mathbb{Z}_5
\]  

(4)

Tronquées and tritronquées solutions. Generic solutions have poles accumulating at \( \infty \) in all \( S_k \).

Any solution has poles in at least one \( S_k \). For any two adjacent sectors \( S_k \) there is a one-parameter family of solutions, called tronquées solutions, with the behavior \( y = i\sqrt{\frac{z}{6}}(1 + o(1)) \) as \( z \to \infty \) in these sectors (so they do not have poles for large \( z \) in these sectors). In particular, for any set of four adjacent sectors there is exactly one solution with this behavior, see e.g., [20]; these particular tronquées solutions which are maximally regular solutions are called tritronquées. The five tritronquées are obtained from each other via the five-fold symmetry. (Note that although in any \( S_k \) there is also a solution with behavior \( \sim -i\sqrt{z/6}(1 + o(1)) \) as \( z \to \infty \), it turns out that this is just some tritronquée.)

We will study the tritronquée \( y_t \) with

\[
y_t(z) = i\sqrt{\frac{z}{6}}(1 + o(1)) \text{ as } z \to \infty \text{ with } \arg z \in \left[ -\frac{3\pi}{5}, \pi \right]
\]  

(5)

In the normalization (3), a sector \( S_k \) in \( z \) corresponds to a quadrant in \( x \) and the sector \(-\pi < \arg z \leq \pi \) corresponds to the sector \(-\pi < \arg x \leq 3\pi/2 \). The solution \( h_t \) corresponding to \( y_t \) satisfies

\[
h_t(x) = o(1) \text{ as } x \to \infty \text{ with } \arg x \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right]
\]  

(6)

The solution \( h_t \) is analytic for large \( x \) in the sector (6) and has arrays of poles beyond its edges, see [22,24]. General results about the solutions of (3) which decay in some direction at infinity, which correspond to tronquée of (1), are overviewed in the Appendix, \$8.1\.

2. Main results in sectors of analyticity

2.1. The Stokes constants for tronquée solutions. Theorem\[\Box\] gives the value of the Stokes multiplier \( \mu \) for any tronquée solution. While it is formulated for solutions analytic in the sector \( S_0 \), it can be easily adapted to solutions analytic in any of the sectors in (4).

**Theorem 1.** Let \( h \) be a solution of (3) satisfying

\[
\forall \alpha \in [0, \frac{1}{2}\pi], \quad h(x) = Ce^{-x}x^{-\frac{1}{2}} + o(x^{-1/2}) \text{ as } x \to e^{i\alpha} \infty
\]  

(7)

Then

\[
\forall \alpha \in [-\frac{1}{2}\pi, 0], \quad h(x) = (C + \mu)x^{-\frac{1}{2}}e^{-x} + o(x^{-\frac{1}{2}}) \text{ as } x \to e^{i\alpha} \infty
\]  

(8)
The existence of a Stokes multiplier such that (7) and (8) hold is known in a wide class of differential equations, see in [7] formula following (1.15) and also see in [8], (182). A complete Borel summed expansion of the solutions \( h \) satisfying (7), (8) is given in the Appendix, §8, where \( C_+ = C, C_- = C + \mu \).

On the other hand, the existence of an explicit expression for \( \mu \) is expected only in special cases such as integrable equations. The value (9) was calculated before using Riemann-Hilbert associated problems, as mentioned in §1.2. In the present paper, (9) is a byproduct of more general asymptotic formulas we obtain by matching Borel summed expansions valid in the regular sector to asymptotic constants of motion explained in §2.2, which are shown to give suitable representations in the sectors with singularities.

In particular, for the tritronquée \( h_t \) obtained from \( y_t \) via (2), Theorem 1 gives

**Proposition 1.** The tritronquée \( h_t \) defined by (6) satisfies

\[
h_t(x) = O(x^{-4}) \quad \text{as} \quad x \to +i\infty
\]

(impling \( C = 0 \) in (7)) and

\[
h_t(x) = \mu x^{-\frac{1}{2}} e^{-x}(1 + o(1)) \quad \text{as} \quad x \to -i\infty
\]

where \( \mu \) is given by (9).

It is clear from Proposition 1 and Theorem 1 that it suffices to obtain \( \mu \) for \( h_t \). The proof of Proposition 1 is given in Section 7; the equation that \( \mu \) solves is (202).

### 2.2. Sectors with singularities. Setting and heuristics.

#### 2.2.1. Arrays of poles near regular sectors of tronquée solutions.

Solutions \( h \) satisfying (7) are analytic for large \( x \) in the right half plane. Beyond the edges of the sector \(-\pi/2 \leq \arg x \leq \pi/2\), \( h \) develops arrays of poles (unless \( h \) is tritronquée). These facts are proved, and the location of the first few arrays of singularities is given in [9] and are overviewed below.

Given \( h \) as in Theorem 1 there is a unique constant \( C_+ \) with the following properties. Denoting \( \xi(x) = C_+ x^{-1/2} e^{-x} \) the leading behavior of \( h \) for large \( |x| \) with \( \arg x \) close to \( \pi/2 \) is

\[
h \sim H_0(\xi) + \frac{H_1(\xi)}{x} + \frac{H_2(\xi)}{x^2} + \cdots \quad (x \to i\infty \text{ with } |\xi - 12| > \epsilon, \ |\xi| < M) \tag{10}
\]

(if \( \xi \) is small, the terms may need to be reordered) where

\[
H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2}, \quad H_1(\xi) = -\frac{1}{60} e^{4+3\xi^3+210\xi^2+216\xi} (12-\xi)^2, \quad \ldots, \quad H_n(\xi) = \frac{P_n(\xi)}{\xi^n(\xi-12)^{n+2}} \tag{11}
\]

with \( P_n \) polynomials of degree \( 3n + 2 \).

The first array of poles beyond \( i\mathbb{R}^+ \) is located at points \( x = p_n \) near the solutions \( \tilde{p}_n \) of the equation \( \xi(x) = 12 \), namely

\[
p_n = \tilde{p}_n + o(1) = 2n\pi i - \frac{1}{2} \ln(2n\pi i) + \ln C_+ - \ln 12 + o(1), \quad (n \to \infty) \tag{12}
\]

Rotating \( x \) further into the second quadrant, \( h \) develops successive arrays of poles separated by distances \( O(\ln x) \) of each other as long as \( \arg(x) = \pi/2 + o(1) \) [9].
we first rewrite equation (3) as a system

\[ c \text{ certain the case in the region where (10) holds. It is then natural to take} \]

angle-like independent solutions to be asymptotically periodic, and (3) is close to the autonomous Hamiltonian system

\[ - \text{the sector} \]

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In this section we give the heuristics of the approach in the present paper to studying solutions in sectors with poles; rigorous proofs are given in [5]. As it is often the case, rigorous arguments are more involved and sometimes depart from the heuristic ideas. It is natural to describe the intuition first.

Our method of analysis resembles an adiabatic invariants one. Note that for large \( x \) is close to the autonomous Hamiltonian system

\[ h'' - h - h^2/2 = 0 \quad \text{with Hamiltonian} \quad s/2 \]

where

\[ s = h^2 - h^2 - h^3/3 \]

The solutions of (14) are elliptic functions, doubly periodic in \( \mathbb{C} \). For (3) we expect solutions to be asymptotically periodic, and \( s \) to be a slow varying quantity; this is certainly the case in the region where (10) holds. It is then natural to take \( h = u \) as an independent angle-like variable and treat \( s \) and \( x \) as dependent variables. With \( w = u' \) we first rewrite equation (3) as a system

\[ u' = w \]

\[ w' = u + \frac{u^2}{2} - \frac{w}{x} + \frac{392}{625} \frac{1}{x^4} \]

and then, with

\[ R(u, s) = \sqrt{u^3/3 + u^2 + s} \]

we transform (10), (17) into a system for \( s(u) \) and \( x(u) \):

\[ \frac{ds}{du} = -\frac{2w}{x} + \frac{784}{625} \frac{1}{x^4} = -\frac{2R(u, s)}{x} + \frac{784}{625} \frac{1}{x^4} \]

\[ \frac{dx}{du} = \frac{1}{w} = \frac{1}{R(u, s)} \]

Note 2. The array of poles developed near the other edge of the sector of analyticity, for \( \arg(x) = -\pi/2 + o(1) \), is obtained by the conjugation symmetry in (13.2) in (10) and (12) \( i \) is replaced by \( -i \) and \( C_+ \) by a different constant, still unique, \( C_- \). The tritronquée \( h_t \) has the sector of analyticity as in (6) and \( C_- = \mu; h_t \) has an array of poles for \( \arg(x) = -\pi/2 + o(1) \). By the rotation symmetry, near \( \arg(x) = 3\pi/2 + o(1) \), the other edge of the sector of analyticity of \( h_t \), (10) and (11) hold with \( e^{-x} \) replaced by \( e^x \) and with \( C_3 = \mu \) instead of \( C_+ \) (see also (245) below).

2.2.2. Sectors with poles. Setting and heuristics. As mentioned, the general solution of \( P_1 \) has poles in any sector in \( \mathbb{C} \), and any solution has at least a sector of width \( 2\pi/5 \) with singularities. In particular, any truncated solution \( h \) as in Theorem 1 has poles outside the sector \( -\pi/2 \leq \arg x \leq \pi/2 \) and \( h_t \) has poles outside the sector \( -\pi/2 \leq \arg x \leq 3\pi/2 \), in particular \( h_t \) has poles for \( x \) in the sector

\[ \Sigma = \{ x \mid -\pi < \arg x < -\pi/2 \} \]

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we transform (10), (17) into a system for \( s(u) \) and \( x(u) \):

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\[ \frac{dx}{du} = \frac{1}{w} = \frac{1}{R(u, s)} \]

Note 3. Given an initial condition \( u_I, s(u_I), x(u_I) \) such that the right side of (19), (20) is analytic, the system \( \{ 19, 20 \} \) admits a locally analytic solution \( x(u), s(u) \). The function \( x(u) \) is analytically invertible by the inverse function theorem since \( 1/R \neq 0 \).

Using (19) this determines an analytic \( s(x) \). From \( s \) and \( u \), we define an analytic branch of \( w = u' \). The systems \( \{ 19, 17 \} \) and \( \{ 19, 20 \} \) are then equivalent in any domain in which \( u, u', s(u), x(u) \) are analytic.

Let \( h \) be a tritronquée solution (or one with \( s \) very small) of (3) having poles for large \( x \) in the sector \( \Sigma \). It turns out that there are closed curves \( C \), which will be used hereon, see Fig. 1, similar to the classical cycles [17], such that \( R(u, s(u)) \) does not vanish on \( C \).
and \( x(u) \) traverses \( \Sigma \) from edge to edge as \( u \) travels along \( C \) a number \( N_m \) times. More precisely, starting with \( u_0 \in C \) and writing \( u_n \) instead of \( u_0 \) to denote that \( u \) has traveled \( n \) times along \( C \), \( s_n = s(u_n) \) and \( x_n = x(u_n) \), the following hold: (i) \( x_0 = x(u_0) \) is close to the first array of poles explained in \ref{2.2.1} arg\((x_0) = -\pi / 2(1 + o(1))\), and \( s(u_0) \) is given by \ref{16} (recall that \( u = h \)) (ii) for some \( N = N_m(x_0) \), \( x_N \) is close to the last array of poles, \( \arg(x_N) = -\pi (1 + o(1)) \). The size of \( |x_n| \) is of the order \( |x_0| \) for all \( n \leq N \). Two roots of \( R(u, s_n) \), \( n = 0, 1, \ldots, N_0 \) are in the interior of \( C \) and a third one is in its exterior. Written in integral form, \ref{19} and \ref{20} become

\[
s(u) = s_n - 2 \int_{u_n}^u \left( \frac{R(v, s(v))}{x(v)} - \frac{392}{625} \frac{1}{x(v)^4} \right) dv \tag{21}
\]

\[
x(u) = x_n + \int_{u_n}^u \frac{1}{R(v, s(v))} dv \tag{22}
\]

where the integrals are along \( C \).

2.2.3. The Poincaré map. A first important ingredient in our analysis is the study of the Poincaré map for \ref{21}, \ref{22}, namely the study of \((s_{n+1}, x_{n+1})\) as a function of \((x_n, s_n)\). With the adiabatic invariants analogy in mind, the Poincaré map is used to eliminate the fast evolution. The asymptotic expansions of \( s(u) \) and \( x(u) \) when \( u \) is between \( u_n \) and \( u_{n+1} \) are straightforward local expansions of \ref{21} and \ref{22}. We denote

\[
J(s) = \oint_C R(v, s) \, dv; \quad L(s) = \oint_C \frac{dv}{R(v, s)} \tag{23}
\]

It is easily checked that

\[
J'' + \frac{1}{4} \rho(s) J = 0; \quad \text{where} \quad \rho(s) = \frac{5}{3s(3s+4)} \tag{24}
\]

and, since \( J' = L/2 \) we get

\[
L'' - \frac{\rho'(s)}{\rho(s)} L' + \frac{1}{4} \rho(s) L = 0 \tag{25}
\]

The points \( s = 0 \) and \( s = -4/3 \) are regular singular points of \ref{21} (and of \ref{25}) and correspond to the values of \( s \) for which the polynomial \( u^3/3 + u^2 + s \) has repeated roots. Simple asymptotic analysis of \ref{21} and \ref{22} shows that the Poincaré map satisfies

\[
s_{n+1} = s_n - \frac{2J_n}{x_n} (1 + o(1)) \quad \text{with} \quad J_n = J(s_n) \tag{26}
\]

\[
x_{n+1} = x_n + L_n (1 + o(1)) \quad \text{with} \quad L_n = L(s_n) \tag{27}
\]

Here, and in the following heuristic outline, \( o(1) \) stands for terms which are small for large \( x_n \) and large \( n \). The rigorous justification of these estimates is the subject of \ref{33}.

2.2.4. Solving \ref{26} and \ref{27}: asymptotically conserved quantities. We see from \ref{26} that \( s_{n+1} - s_n \ll s_n \) and \( x_{n+1} - x_n \ll x_n \). It is natural to take a “continuum limit” and approximate \( s_{n+1} - s_n \) by \( ds/dn \) and \( x_{n+1} - x_n \) by \( dx/dn \). We get

\[
\frac{ds}{dx} = \frac{ds/dn}{dx/dn} = \frac{-2J(s)}{xL(s)} (1 + o(1)) = -\frac{J(s)}{xJ'(s)} (1 + o(1)) \tag{28}
\]

which implies, by separation of variables and integration,

\[
Q(x, s) := xJ(s) = x_0 J(s_0) (1 + o(1)) \tag{29}
\]

\footnote{Later on we will choose \( u_0 = -4 \in C \).}
That is, \( Q \) is asymptotically a constant of motion. A second (nonautonomous) one is obtained using (26) and (29) as follows. We write

\[
\frac{1}{J^2(s)} \frac{ds}{dn} = -\frac{2}{x_0 J(s_0)} (1 + o(1))
\]  

Let \( \hat{J} \) be an independent solution of (24) with \( \hat{J}(0) = 0 \) and denote

\[
\mathcal{K}(s) := \kappa_0 \int_0^s \frac{ds}{J(s)} = \hat{J}(s) J(s) \quad (31)
\]

where \( \kappa_0 \) is the Wronskian of \( J \) and \( \hat{J} \). Integrating both sides of (30) from 0 to \( n \) we get

\[
\mathcal{K}(s) - \mathcal{K}(s_0) = -2 \frac{n}{\kappa_0 x_0 J(s_0)} (1 + o(1)) \Rightarrow \mathcal{K}(s) + 2 \frac{n}{\kappa_0 x_0 J(s_0)} = \mathcal{K}(s_0) + o(1) \quad (32)
\]

for \( n = O(x_0) \). \( \mathcal{K} \) is in fact a Schwarzian triangle function whose properties are studied in \[4.4\].

**Note 4.** The fact that the system (28), (30), can be integrated in closed form to leading order is not surprising: this system is an asymptotic reduction of (11) and is expected to be integrable too, with solutions expressible in terms of classical functions (here, hypergeometric functions).

**Note 5.** Near the edge \( \arg x = -\pi/2 - o(1) \) of \( \Sigma \) (or near the other edge, for \( \arg x = -\pi + o(1) \)) special care is needed for the truncated solutions since \( s = o(1/x) \) (respectively, \( s = -\frac{4}{3} + o(1/x) \)), so \( s \) is near singularities.

**Note 6.** One gets higher orders in the asymptotic expansions of \( (x_n, s_n) \) by formal Picard iterations, using (29) and (32) in (22) and (21); these lead, after inversion of (22) and (21), to an asymptotic expansion of \( h(x) \) and \( h'(x) \). This is quite straightforward and fairly short, but of course (28) has uncontrolled errors, and a good part of the technical sections of the paper deals with rigorizing the analysis.

### 2.3. Calculating \( \mu \).

#### 2.3.1. Is integrability needed to find \( \mu \)?

The P-K property, implying that the analytic continuation of \( h_t \) through \( \Sigma \) coincides with the one obtained using Borel summed expansions in \( \mathbb{C} \setminus \Sigma \), see \[8.1\], is crucial in this argument, to identify \( \mu \) as the Stokes multiplier. If we drop the term \( -\frac{392}{625x^4} \) in (3), the equation becomes Painlevé nonintegrable, in the sense that the movable singularities are branch points. Nonetheless, our asymptotic expansion of \( (s_n, x_n) \) does not change to the order used. The analysis in \[7\] using transseries does not rely on integrability either. However, no matching would be possible this time around, since, because of the ramification in the solutions, the two analytic continuations will now be done on different Riemann sheets, rendering the matching impossible.

#### 2.3.2. Generalizations.

Application of these methods to \( P_2 \) and more general equations are discussed in \[7.4\].
3. Main results about the singular sectors

3.1. Asymptotic expansions of solutions. We start with some fixed \( u_0, s_0, x_0 \) and as \( u \) travels along \( C \) back to \( u_0 \) we obtain \( s_1, x_1 \), and repeating this procedure we get \( s_2, x_2 \), etc. and in this way (21) and (22) provide a recurrence relation describing the evolution of \( s \) and \( x \) as \( u \) goes along \( C \).

Theorem\(^2\) provides asymptotic expansions of the asymptotically conserved quantities (29), (32) of the system (21), (22), up to \( O(x^{-5/4}) \) or better.

It turns out (Lemma 26) that for \( h = h_t \) the \( s_m \) are in the upper half plane with modulus proved to be less than 5 (it is in fact less than 2), \( s \in \mathbb{D}^+ \) cf. (15). Since \( J \) has singularities (square root branch points) at \( s = 0 \) and at \( s = -4/3 \) only, both \( J \) and \( L \) are single-valued for \( s \in \mathbb{D}^+_0 \).

3.1.1. Choosing initial data \( s_0, x_0, u_0 \). We obtain asymptotic expansions for large \( x \), therefore \( x_0 \) will be chosen large enough. Also, \( x_0 \) will be chosen near \( i\mathbb{R}_- \), the lower edge of the sector of analyticity of the tritronquée solution \( h_t \). Choosing \( s_0 \) near 0 (a singularity), and iterating the Poincaré map we will obtain \( x_0, x_1, \ldots, x_n, \ldots \) which go through the sector with singularities up to the other edge, as it will be proved that \( \arg x_n \) close to \(-\pi \). It will turn out that \( s_n \) is close to the other singularity, \(-4/3 \).

The iteration can also be done for other values for \( s_0 \), not necessarily close to singularities, in which case the estimates are simpler, and they can be used to obtain asymptotic conserved quantities in sectors with poles for any solution of \( P_I \). In this paper however, we are interested in the connection problem, and then we do need \( s_0 \) close to singularities; see also Note 7.

More precisely: let \( m > 0 \) be a large enough number (its estimate can be traced through the calculations of this paper).

Assumption. We assume that

\[
|x_0| > m, \quad \text{Im } x_0 < 0, \quad \text{Re } x_0 < \ln |x_0|, \quad \text{Im } s_0 > 0, \quad 1 < |s_0 x_0| < 10, \quad u_0 = -4 \tag{33}
\]

The choice of \( u_0 = -4 \) makes some calculations simpler, cf. Note 32 though we will use this precise value only later.

Note that (33) implies that \( x_0 = -i|x_0|e^{i\theta_x} \) where \( \theta_x \) is \( o(1) \), so indeed, \( x_0 \) is close to \( i\mathbb{R}_- \) and that \( s_0 = O(x_0^{-1}) \), close to 0 indeed.

3.1.2. The solutions \( J, \dot{J} \) used and other notations. In the rest of the paper \( J \) is the unique solution of (24) satisfying

\[
J(s) = 2A \left[ 1 - \left( \frac{5}{96} + \frac{5}{48} \ln s \right) s \right] + 2sB + O(s^2) \tag{34}
\]

with

\[
A = -\frac{12}{5}; \quad B = -\frac{3}{8} - \frac{1}{2} \ln(24) + \frac{\pi i}{4} \tag{35}
\]

and \( \dot{J}(s) \) is the unique solutions of (24) with

\[
\dot{J}(0) = 0, \quad \dot{J}'(0) = \pi i \tag{36}
\]

Denote

\[
Q(u, s) = \frac{2}{3} \frac{6s + 6su + 18u + 3u^2 - 4u^3 - u^4}{s(3s + 4) \sqrt{9s + 9u^2 + 3u^3}} = \rho(s) \frac{P(u, s)}{R(u, s)} \tag{37}
\]

where \( R \) is given by (18), \( \rho \) by (24) and the polynomial \( P \) equals

\[
P(u, s) = \frac{2}{15} \left[ u(2 - u)(3 + u)^2 + 6s(1 + u) \right] \tag{38}
\]
Denote
\[ N_0 = \lfloor |x_0|^{\frac{2}{3}} \rfloor, \quad c_0 = \frac{e^{-\frac{\pi i}{3}}}{\sqrt{3}}, \quad \zeta = \frac{5s_0 x_0}{48}, \quad \tilde{\zeta} = \frac{5i s_N x_N}{48}, \quad n' = n + \zeta, \quad \tilde{n}' = n + \tilde{\zeta} \] (39)

and define
\[
B_{n-1} = \frac{5}{24} \left( n' \ln \frac{n' e}{e} - \ln \Gamma(c_0 + n') + \ln \Gamma(\zeta + c_0) - \zeta \ln \frac{e}{c_0} \right)
\]
\[
\tilde{B}_{n-1} = -\frac{5}{24} \left( \tilde{n}' \ln \frac{\tilde{n}' e}{e} - \ln \Gamma \left( \frac{1}{2} + \tilde{n}' \right) + \ln \Gamma \left( \tilde{\zeta} + \frac{1}{2} \right) - \tilde{\zeta} \ln \frac{e}{\tilde{c}_0} \right)
\] (40)

As \( n \to \infty \) we have
\[
B_{n-1} = \frac{24n}{5} \left[ \frac{\sqrt{3} \ln n}{5} + g_a + O(n^{-1}) \right], \quad \text{where} \quad g_a := -\zeta \ln \frac{e}{c_0} + \ln \Gamma(\zeta + c_0) - \frac{1}{2} \ln(2\pi)
\]
\[
\tilde{B}_{n-1} = \frac{24\tilde{n}}{5} + o(1), \quad \text{where} \quad g_b := \tilde{\zeta} \ln \frac{e}{\tilde{c}_0} - \ln \Gamma \left( \tilde{\zeta} + \frac{1}{2} \right) + \frac{1}{2} \ln(2\pi)
\] (41)

**Theorem 2.** Under the assumption (39), there exists a curve \( \mathcal{C} \) such that the following hold as \( |x_0| \to 0 \).

(i) The system (21), (22) has a unique solution \((s, x)\) for \( u \) traveling \( n \) times on \( \mathcal{C} \) with \( 0 \leq n < N_m \), where \( N_m \in \mathbb{N} \) is the unique number such that \( 0 < \ln s N_m < 11/|x_0| \) and \( \Re s N_m + 4/3 < 2|x_0|^{-1/2} \).

(ii) We have (see (31) for notations)
\[
\mathcal{K}_n = \mathcal{K}(s_0) + \frac{48\pi in}{J_0 x_0} + \frac{2\pi i \phi_n}{x_0 J_0} + O(x_0^{-5/4} \ln x_0)
\] (42)

where
\[
\phi_n = \frac{n}{x_0} \left( g_a + \frac{4\sqrt{3}}{5} \ln \frac{5s_0 x_0}{48} \right) + \frac{1}{4\pi i} \int_{s_0}^{s_n} Q(u_0, s) \left( -\frac{2\pi i n}{x_0} J(s) - \hat{J}(s) \right) ds
\]

(iii) For \( n \in (N_0, N_m - N_0] \), \( \mathcal{G}_n := Q_n/x_0 \) (for notations see (29), (39)) we have
\[
\mathcal{G}_n = \mathcal{G}_{N_0} - \frac{1}{2} \frac{1}{x_0} \int_{s_{N_0}}^{s_n} Q(u_0, s) J(s) ds + O(x_0^{-3/2})
\] (43)

while for \( n \in [0, N_0] \) (resp. \( n \in (N_0, N_m - N_0] \)) where \( s_n \) is small, \( \mathcal{G}_n \) is given by
\[
\mathcal{G}_n = \mathcal{G}_0 + x_0^{-1} B_{n-1} + O(x_0^{-\frac{3}{4}} \ln x_0), \quad \text{resp.} \quad \mathcal{G}_{N_m} - x_0^{-1} \tilde{B}_{n-1} + O(x_0^{-\frac{5}{4}} \ln x_0)
\] (44)

Part (i) follows from Proposition (63), (ii) is proved in (63) and (iii) is shown in (62) with (10), (11) shown in Lemma (51). Higher order corrections can be obtained in the usual asymptotic way, iteratively order-by-order.

**Note 7.** In fact, (11) applies to more general conditions \( s_0 \in \mathbb{H} \), if \( s_n \in \mathbb{H} \) for \( n = 0, \ldots, N_m - N_0 \).

Based on Theorem (2) (ii) Theorem (2) (iii) more orders can be obtained for \( s_n \) and \( x_n \):

**Proposition 8.** (i) For \( N_m - N_0 \leq n \leq N_m \) we have
\[
s_n = -\frac{4}{3} - s_0 + \frac{24i}{5\pi} + \frac{1152n}{25J_0 x_0} - \frac{2\phi_n}{x_0^2} + O(x_0^{-2} \ln^2 x_0) + O \left( (s_n)^{-3/2} \right)
\] (45)

and thus \( N_m = \frac{|x_0|}{2\pi} + O(\ln x_0) \).

(ii) We have
\[
\Re s_{N_m} = -\frac{4}{3} - \Re s_0 + \frac{\Re (x_0 J_0)}{\pi |x_0|} + O(x_0^{-1})
\] (46)
\[ x_{N_m} = \frac{x_0 J_0}{J_{N_m}} + \frac{\sqrt{3}}{6} \ln x_0 + O(1) \]  

In particular

\[ x_{N_m} = -ix_0 + O(\ln x_0) \]

and

\[ \text{Im } x_{N_m} = \text{Im } \frac{x_0 J_0}{J_{N_m}} + O(1) \]

The proof is given in \( \text{[6.4]} \).

4. Proofs. I. General properties of the functions used in the proofs

4.1. The zeroes of \( R(u, s) \). We denote

\[ \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}, \quad \mathbb{D}_\rho^+ = \{ s \in \mathbb{H} \mid |s| < \rho \}, \quad \mathbb{D}_\rho^- = \{ s \in -\mathbb{H} \mid |s| < \rho \} \]  

Denote by \( P_s(u) \) the following polynomial in \( u \), with parameter \( s \in \mathbb{H} \)

\[ P_s(u) = u^3/3 + u^2 + s \]  

and note the symmetry \( P_s(u) = -P_{-s-4/3}(-u-2) \), or

\[ P_s(u) = -P_{-s^-}(-u^-) \quad \text{where } s^- = s + \frac{4}{3}, \quad u^- = u + 2 \]

which entails that results for \( s \) close to 0 can be translated into results for \( s \) close to \(-\frac{4}{3}\).

The only values of \( s \) for which two roots of \( P_s(u) \) coalesce are \( s = 0 \) and \( s = -4/3 \). Therefore the roots \( r_{1,2,3}(s) \) of \( P_s(u) \) are distinct and analytic for \( s \in \mathbb{H} \) (see, e.g. \([3]\)). Lemma 9 gives bounds for these roots and for distances between them, see Fig. 4.

**Lemma 9.** (0) For \( s \in \mathbb{H} \) we have

\[ r_1 \in -\mathbb{H}, \quad \text{Re } r_1 < -2 \]
\[ r_2 \in \mathbb{H}, \quad \arg r_2 > \text{arctan}(3/2), \quad \arg (r_2 + 2) < \pi - \text{arctan}(3/2) \]
\[ r_3 \in -\mathbb{H}, \quad \text{Re } r_3 > 0 \]

(i) For \( |s| < \mathbb{D}_1^{(+\sqrt{3})/3} \) (for notation see \([48]\)) with the choice \( \sqrt{s} > 0 \) if \( s > 0 \) (and a choice of labeling of the roots) we have

\[ |r_1 + 3 + s/3| < |s^2|, \quad |r_2 - i\sqrt{s}| < |s|, \quad |r_3 + i\sqrt{s}| < |s| \]  

(ii) For \( s^- \in \mathbb{D}_1^{(+\sqrt{3})/3} \) we have

\[ |2 + r_1 + \sqrt{s^-}| < |s^-|, \quad |2 + r_2 - \sqrt{s^-}| < |s^-|, \quad |r_3 - 1 + s^-/3| < |s^2| \]  

(iii) Let \( r_j(s; t) \) be the roots of \( tu^3/3 + u^2 + s \), labeled with the convention \( r_j(s; 1) = r_j(s) \). If \( |s| \in \mathbb{D}_1^{(1/10)} \) then \( r_{2,3}(s; t) \) are real analytic in \( t \in (0, 1) \). If \( |s^-| \in \mathbb{D}_1^{(1/10)} \) then \( r_{1,2}(s; t) \) are real analytic in \( t \in (0, 1) \).

(iv) If \( s \in \mathbb{D}_1^{(1/4)} \) then \( |r_{1,2,3}(s)| < 399/100 \).

The proof, elementary, but rather laborious, is given in the Appendix, \([48,3]\).

**Note 10.** Choosing the closed path. Let \( C \) consist of the polygonal line connecting \( 6, \frac{1+i}{3}, -(1+i), -2 + \frac{-1+i}{3}, -3 + \frac{i}{3}, u_0, -6 \) and a semicircle of radius 6 centered at the origin in the lower half plane, see Figure 4. For us \( u_0 = -4 \), cf. \([47,12]\).
ASYMPTOTICALLY CONSERVED QUANTITIES AND CONNECTION FORMULAE

We will also need the incomplete integrals $J$ and $L$:

$$ J(u, s) = \int_{u_0}^{u} R(v, s(v)) \, dv; \quad L(u, s) = \int_{u_0}^{u} \frac{dv}{R(v, s(v))} $$  \hspace{1cm} (54)

where the integration is along $C$ (specified above), which will be shown to surround two of the three roots of $u^3/3 + u^2 + s_0$. The contour is traveled upon multiple times, and we will use an index $n$ to specify the winding number.

Corollary 11. (i) We have $r_3(s), r_1(s) \in \mathbb{D}_5^-$ while $r_2(s) \in \mathbb{D}_5^+$ for all $s \in \mathbb{D}_{21/4}^+$.  

(ii) Consider the polygon $\hat{C}_0$ with vertices $-1, -1 - 6i, 6 - 6i, 6i, -6 + 6i, -6$, oriented anticlockwise. Then $r_2(s), r_3(s) \in \text{int}(\hat{C}_0)$ while $r_1(s) \in \text{ext}(\hat{C}_0)$ if $s \in \mathbb{D}_{21/4}^+$.  

(iii) For all $s \in \mathbb{D}_{21/4}^+$ the path $C$ defined in Note 10 encloses $r_1(s), r_3(s)$, leaving $r_2(s)$ outside. Moreover

$$ \alpha := \sup_{u \in C, s \in \mathbb{D}_{21/4}^+} |J(u, s)| < \infty $$  \hspace{1cm} (55)

where $J$ is as defined in (54).

Proof. By Lemma 9 we have $|r_j(s)| < 5$, $j = 1, 2, 3$, $\text{Im} \, r_{3,1}(s) < 0$ and $\text{Im} \, r_2(s) > 0$ implying (i). Continuity of $J$ is manifest, and $J(0)$ is an elementary integral.  

(ii) By Lemma 9 we have $r_1(s) \in \{ z : \text{Im} \, z < 0, \text{Re} \, z < 2 \}$ implying $r_1(s) \in \text{ext}(\hat{C}_0)$, $r_2(s) \in \{ z : \text{Im} \, z > 0, |z| < 5 \}$, and $r_3(s) \in \{ z : \text{Im} \, z < 0, \text{Re} \, z > 0, |z| < 5 \}$, which implies $r_2(s)$ and $r_3(s)$ are in $\text{int}(\hat{C}_0)$. Continuity of $J$ at zero is manifest, and it implies $J(0) = 0$; this together with the fact that $\hat{J}(s)$ satisfies (54) implies, by Frobenius theory, that it is analytic at zero (see also §4.3 below); the value of $\hat{J}'(0)$ is simply obtained by the residue theorem.  

(iii) This is obvious by Lemma 9 and continuity of $\hat{J}$.  \hfill \Box

4.2. Link between $J$ and $L$. We use the notations (18), (23) where $C$ can be any closed curve (piece-wise smooth). (37), (38).

Proposition 12. We have

$$ \frac{\partial Q}{\partial u} = \frac{1}{R(u, s)^3} - \rho(s)R(u, s) $$  \hspace{1cm} (56)
In particular, using (24) and (53) we have

$$\frac{dJ(s)}{ds} = \frac{1}{2}L(s); \quad \frac{dL(s)}{ds} = -\int_C \frac{du}{2R^3(v,s)} = -\frac{1}{2}p(s)J(s)$$

(57)

The proof of Proposition 12 is by direct verification. □

4.3. Integral representations of $J$, $\hat{J}$, $L$ and $\hat{L}$. We defined $J$ and $\hat{J}$ as solutions of (24) satisfying the initial conditions (34)-(36); we now derive some integral representations useful in the sequel.

Denote by $\kappa$ the elliptic modulus

$$\kappa = \frac{r_1 - r_3}{r_2 - r_3}$$

(58)

Lemma 13. The points $r_1, r_2, r_3$ are collinear only if $s \in (-4/3, 0)$. The roots $r_j$ are analytic in $s$ except for $s \in \{-4/3, 0\}$. Furthermore, the triangle $\Delta[r_1, r_2, r_3]$ preserves its orientation when $s$ traverses any curve $\gamma \subset \mathbb{C}$ which does not cross the real line.

As a consequence, for $s \neq -4/3, 0$ the roots satisfy

$r_3$ does not belong to the segment $[r_1, r_2] \subset \mathbb{C}$

(59)

Proof. By Vieta’s formulas, $r_2 + r_3 + r_1 = -3$, $r_2r_3 + r_2r_1 + r_3r_1 = 0$, $r_2r_3r_1 = -3s$, and a straightforward calculation gives

$$r_2 = -1 + \frac{\kappa - 2}{\sqrt{\kappa^2 - \kappa + 1}}, \quad r_3 = -1 + \frac{1 + \kappa}{\sqrt{\kappa^2 - \kappa + 1}}, \quad r_1 = -1 + \frac{1 - 2\kappa}{\sqrt{\kappa^2 - \kappa + 1}}$$

If $r_{1,2,3}$ are collinear then $\kappa \in \mathbb{R}$ which in turn implies $r_{1,2,3} \in \mathbb{R}$, hence $s \in (-4/3, 0)$, in which case $r_1 < r_2 < r_3$.

Analyticity is standard: the roots satisfy $F(r, s) = 0, r = r_j$, which, by the implicit function theorem defines analytic functions $r_j(s)$ in a neighborhood of any point where $F_r \neq 0$. But $F_r(r_j) = 0$ clearly means that the polynomial has a double root. If we take a curve $\gamma$ not intersecting $S_0$, then $r_1$ and $r_2$ are always distinct, and we can orient the line through $r_1$ and $r_2$ by choosing the direction from $r_1$ to $r_2$ as being positive. If $r_3$ is, for some $s$, to the left of the line (in the usual meaning) it stays to the left by continuity, since the distance between $s_0$ and the line is never zero. □

Proposition 14. (i) We have $L = 2L_{3,1}$, $J = 2J_{3,1}$, $\hat{J} = 2J_{3,2}$ and $\hat{L} = 2L_{3,2}$, where

$$L_{i;j} = \int_{r_1}^{r_3} \frac{1}{R(u,s)}du; \quad J_{i;j} = \int_{r_1}^{r_3} R(u,s)du$$

(60)

with branched of $R(u,s) = \sqrt{P_s(u)} = 3^{-1/2}(u-r_1)(u-r_2)(u-r_3)$ defined so that arg $P_s(u) = 0$ for $u \to +\infty$, namely they are analytic continuations for $s$ in the upper half planes of the branches, which for $s \in \mathbb{R}$ have arg $P_s(u) = 0$ for $u > r_3$, arg $P_s(u) = \pi$ for $r_2 < u < r_3$, arg $P_s(u) = 2\pi$ for $r_1 < u < r_2$, arg $P_s(u) = 3\pi$ for $u < r_1$.

(ii) We have, with the notation (58),

$$J_{3,1} = -3^{-1/2}(r_1 - r_3)^{5/2} \int_0^1 \sqrt{t(1-t)} \left(\frac{1}{\kappa - t}\right) dt$$

(61)

$$J_{3,2} = -3^{-1/2}(r_2 - r_3)^{5/2} \int_0^1 \sqrt{t(1-t)(\kappa - t)} dt$$

(62)

(iii) As $s^- = s + \frac{1}{4} \to 0$ we have

$$J(s) = -\frac{24i}{5} - \left(\frac{1}{2} \pi - i \ln 24 - \frac{1}{2} i\right) s^- - \frac{1}{2}i s^- \ln(-s^-) + O(s^2 \ln s^-)$$

(63)
Comparing with (72) we get (35) and thus

$$J(s) = \pi is + O(s^2 \ln s); \quad \hat{J}(s) = J(s) - 2J_{2,1} = J(s) + \pi s^- + O(s^2 \ln s^-)$$

In particular we have (34) and (36).

For the proof, we note that $J_{i;j}$ are solutions of (24). To identify them, we simply have to determine their behavior at $s = 0$. We note that $\sqrt{P_0(u)}$ has a square root singularity at $u = -3$ thus

$$\int_{-3-3s}^{-1} 1/R(u,s) = \int_{-3}^{-1} 1/R(u,0) + o(1).$$

Thus

$$L_{3,1} = \int_{-3}^{-1} \frac{1}{R(u,s)} du + o(1) \quad (s \to 0)$$

We re-express $R(u,s)$ as its approximation where we discard $u^3$ plus the corresponding difference:

$$L_{3,1} = \int_{-3}^{-1} \frac{1}{\sqrt{u^3/3 + u^2 + s}} = \frac{1}{\sqrt{u^2 + s}} + Q_1(u,s)$$

where

$$Q_1(u,s) := -\frac{3u^3}{\sqrt{3u^3 + 9u^2 + 9s \sqrt{u^2 + s} (3\sqrt{u^2 + s} + \sqrt{3u^3 + 9u^2 + s})}}$$

We note that $Q_1$ is continuous at $s = 0$, and, for $u < 0$,

$$Q_1(u,0) = \frac{\sqrt{3}}{\sqrt{u + 3} (3 + \sqrt{3\sqrt{u} + 3})}$$

and thus

$$L_{3,1} = \int_{-3}^{-1} \frac{1}{\sqrt{u^2 + s}} du + \int_{-3}^{-1} Q_1(u,0) du + o(1)$$

With the change of variable $u = -i\sqrt{s} \sinh v$ we get

$$\int_{-3}^{-1} \frac{1}{\sqrt{u^2 + s}} du = -\int_{-\frac{i\pi}{2}}^{\frac{i\pi}{2}} e^{\sin^{-1}(3/\sqrt{s})} dv = \frac{1}{2}i\pi - \sinh^{-1}(3/\sqrt{s}) = \frac{\pi i}{2} - \ln 6 + \frac{1}{2}\ln s$$

The second integral in (70) is $-2\ln 2$ and thus

$$L_{3,1} = \frac{1}{2}\ln s - \ln(24) + \frac{\pi i}{2} + o(1)$$

Applying Frobenius theory to (24) we get

$$J_{3,1} = A \left[1 - \left(\frac{5}{96} + \frac{5}{48} \ln s\right) s + \cdots\right] + B \left(s - \frac{5}{96} s^2 + \cdots\right)$$

We have $L_{3,1} = 2J_{3,1}'$, and thus

$$L_{3,1} = 2A \left(-\frac{5}{32} - \frac{5}{48} \ln s\right) + 2B + o(1)$$

Comparing with (72) we get (35) and thus $J_{3,1}$ has the asymptotic expansion (73) with $A, B$ given by (72). It follows that $J(s) = 2J_{3,1}$. Similarly one can show that $J_{3,2}(0) = 0$ and $L_{3,2}(0) = \pi i$ implying $\hat{J} = J_{2,3}$.  

\[ L(s) = -i \ln s^- - \pi + 2i \ln 24 + o(1) \]
Lemma 15. There is some $0 < \eta_1 < 1/100$ such that $|s| < 2\eta_1$ implies
\[
|J(s) + \frac{24}{5} + (\ln 24 + \frac{1}{2} - \frac{1}{2}\pi i) s - \frac{1}{2}s \ln s| < |s|^{3/2}
\]
and
\[
|\dot{J}(s) - \pi is| < |s|^{3/2}
\]
whereas $|s^-| = |s + 4/3| < 2\eta_1$ implies
\[
|J(s) + \frac{24i}{5} + \left(\frac{1}{2}\pi - \ln 24 - \frac{1}{2}i\right) s - \frac{1}{2}is^{-} \ln(-s^-)| < |s^-|^{3/2},
\]
and
\[
|J(s) - \dot{J}(s) + \pi s^-| < |s^-|^{3/2}
\]
and
\[
\text{Im } L(s) > \max(|4| \ln s/5, 2|\text{Re } L(s)|) > 4.
\]
In particular $|J(s) + 24/5| < \sqrt{|s|}$ for $|s| < 2\eta_1$ and $|J(s) + 24i/5| < \sqrt{|s^-|}$ for $|s^-| < 2\eta_1$.

Proof. This follows directly from \eqref{eq:58}, Proposition \ref{prop:14} \eqref{eq:63}, \eqref{eq:64} and \eqref{eq:62}. \qed

Finally,

Lemma 16. We have $\beta := \inf_{s \in \mathbb{D}_{21/4}^+} |J(s)| > 0$ and $\beta_1 := \inf_{s \in \mathbb{D}_{21/4}^+ \setminus \{|s+4/3| > \epsilon\}} |J_{2;1}(s)| > 0$ for any $\epsilon > 0$.

Proof. By the second line of \eqref{eq:51} we have $|r_1 - r_3| > 2$ for all $s \in \mathbb{H}$. By \eqref{eq:58} we have $\inf_{s \in \mathbb{D}_{21/4}^+} |\kappa| > 0$. Now the integral in \eqref{eq:61} does not vanish for $s \in \mathbb{D}_{21/4}^+$, since the integrand is in the open fourth quadrant for all $t \in (0, 1)$; therefore $J \neq 0$. The conclusion then follows from continuity of $J$ in $s \in \mathbb{D}_{21/4}^+$.

The proof for $J_{2;1}$ is similar except that there is a factor $(r_2 - r_1)^{5/2}$ which can vanish when $s = -4/3$, therefore we need the additional condition $|s + 4/3| > \epsilon$ in this case. \qed

4.4. Conformal mapping of the upper half plane $\mathbb{H}$ by $\mathcal{K} := \hat{J}/J$. After the substitution $s = -4t/3$ equation \eqref{eq:24} becomes a standard hypergeometric equation
\[
t(1-t)\frac{d^2 J}{dt^2} - \frac{5}{36}J = 0
\]
where the associated hypergeometric function is degenerate, with $c = 0$. Since the conformal map of ratios of solutions of \eqref{eq:24} does not appear to follow immediately from standard references such as \cite{22} and \cite{2}, we provide for completeness an independent analysis.

Proposition 17. (i) $\mathcal{K}(s) := \hat{J}(s)/J(s)$ is a conformal map of the upper half plane into the interior of $C_2$ where $C_2$ consists of a semicircle in the upper half plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, an arccircle $C_3$ tangent to the imaginary line at 0 passing through $e^{-\pi i/3}$ and the reflection of $C_3$ about $x = \frac{1}{2}$. In particular $C_2 \cap \mathbb{R} = \{0, 1\}$; see Fig. 4.

(ii) If $|s| > 5$ then $\text{Im } \mathcal{K}(s) < -\frac{5}{2}$. Furthermore, with $\eta_1$ as in Lemma \ref{lem:15} if $\eta_2 > 0$ is small enough, then
\[
\sup_{|s| < \eta_2; |s^-| < \eta_1; 0 < \text{Im } s < \eta_2} \text{Im } \mathcal{K}(s) > \eta_2 > 0
\]
Lemma 18. Let

\[ f_a(s) = \frac{2 F_1\left(-\frac{1}{3}; \frac{5}{6}; \frac{5}{6}; -\frac{4}{3}\right)}{s^{2/3}} \]

where \( F \) is the hypergeometric function. Then \( f_a \) maps the upper half plane conformally into the interior of \( C_1 \) where \( C_1 \) consists of the segment \( I_1 = [0, a] \), where the number \( a > 0 \) is given in [54] below, followed by an arcircle tangent at \( 1 \) to it and at \( e^{\pi i/3} \) to \( I_2 = e^{\pi i/3} I_1 \), and then followed by \( I_2 \). Furthermore, \( |f_a(s)| < \frac{1}{4} \) if \( |s| > 5 \).

Proof. (a) Since \( t = \infty \) is a regular singularity of (76), from the indicial equation we see that there is a fundamental set of solutions of the form

\[ f_2(s) = s^{5/6} A(1/s); \quad f_1(s) = s^{1/6} B(1/s) \]

where \( A \) and \( B \) are analytic and \( A(0) = B(0) = 1 \). We choose the natural branch of the roots, where \( s^{1/6} > 0 \) if \( s > 0 \). Since (76) has real coefficients, we can check that \( A \) and \( B \) have real-valued Taylor coefficients.

(b) The differential equation for \( g(z) = s^{5/6} J(-3/(4s)) \); \( z = -3/(4s) \) is

\[ z(1-z)g'' + \frac{1}{3} (1-z)g' + \frac{5}{36}g = 0 \]

a hypergeometric equation in standard form. The solution analytic at \( z = 0 \) is \( 2 F_1\left(-\frac{5}{6}; \frac{1}{6}; \frac{1}{3}; -\frac{4}{3s}\right) \), [2] 15.5.1. Thus,

\[ f_2(s) = s^{5/6} 2 F_1\left(-\frac{5}{6}; \frac{1}{6}; \frac{1}{3}; -\frac{4}{3s}\right) \]

is the solution of (76) with the properties in [54]. By [2] 15.3.1,

\[ f_2(s) = \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{6}\right) s^{5/6} \int_0^1 t^{-5/6} (1-t)^{-5/6} \left(1 + \frac{4t}{3s}\right)^{5/6} dt \]

Note that when \( s \) is in the upper half plane \( \text{Im}(1+4t/(3s))^{5/6} < 0 \). Thus we have \( s \in \mathbb{H} \Rightarrow f_2(s) \neq 0 \) and it follows that \( f_a \) is analytic in \( \mathbb{H} \).

(c) The function \( f_1 \) is given by

\[ f_1 = s^{1/6} 2 F_1\left(-\frac{1}{6}; \frac{5}{6}; \frac{5}{3}; -\frac{4}{3s}\right) = \frac{2}{3} s^{1/6} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{3}{3}\right)} \int_0^1 t^{-\frac{1}{6}} (1-t)^{-\frac{1}{6}} (1 + \frac{4t}{3s})^{\frac{1}{6}} dt \]
Indeed, (83) solves (76) and has the required behavior for large $s$.

(d) We note that $f_a = f_1/f_2$. Note also that for $s > 0$ or $s < -4/3$ the integrand is positive and $f_1(s)$ does not vanish. The integrals in (83) and (82) become elementary in the limit $s \to 0$ and we find that

$$a := f_a(0) = \frac{1}{10} \frac{\Gamma \left( \frac{1}{3} \right)^3}{2^{2/3} 3^{1/3} \pi^{3/2}}$$

(84)

Similarly, $f_a(-4/3)$ is also elementary and

$$f_a(-4/3) = -e^{\frac{4\pi}{3}} a; \quad \text{and also} \quad \lim_{s \to \infty} f_a(s) = 0$$

(85)

and clearly

(e) In fact, it follows just from Frobenius theory that neither $f_1$ nor $f_2$ are zero at 0 or $-3/4$, thus they are not analytic there. Indeed, the indicial equation at $-3/4$ and 0 has roots 0 and 1; the generic solution has a logarithmic singularity. If say, $f_1$ were analytic at zero, it is then analytic at $-3/4$ as well, else it would be single-valued. But if $f_1$ is analytic at 0, the log singularity at $-3/4$ is also incompatible with the $5/6$ branching at $\infty$. So $f_1$ is singular at both 1 and 0. Thus it is nonzero at those points (as the solution that vanishes say at 0 corresponds to the solution 1 of the indicial equation which is analytic by Frobenius theory).

(f) Note that $(f_1/f_2)' = -W/f_2^2 \neq 0$ where $W$ is the Wronskian of $f_1$ and $f_2$. Thus, $f_1/f_2$ maps $(0, \infty)$ one-to-one to the segment $(0, a)$ where $a = f_1(0)/f_2(0)$, see (54), and $\infty$ is mapped to 0.

(g) Similarly, by (79), $f_1/f_2$ maps $(-\infty, 0)$ one-to-one to the segment $e^{-4\pi i/3}(0, a)$, for the same $a$ by the symmetry implied by (79), and, as before $-\infty$ mapped to zero.

(h) By Frobenius theory and (81) above, both $f_1$ and $f_2$ have a singularity which, to leading order, is of the form $t \ln t$ at $-\frac{4}{3}$ (0, resp.), where $t = s + \frac{4}{3}$ (s, resp.). Thus, looking at the local mapping of a segment by $\ln s$ near $s = 0$, we see that at 0 the angle change is $\pi$ and the orientation of the arc is preserved. Thus, the segment $[-\frac{4}{3}, 0]$ is mapped into a curve which is tangent to both $I_1$ and $I_2$, and the angle change at $-\frac{4}{3}$ and 0 is, in absolute value, $\pi$.

(i) We now determine this curve. Between $-\frac{4}{3}$ and 0 we take, once more relying on the real-valuedness of the coefficients at a different pair of independent solutions $f_3$ and $f_4$ which are real-valued and analytic on $[-\frac{4}{3}, 0]$. Then $f_3/f_4$ maps $[-\frac{4}{3}, 0]$ onto a segment on the real line. But since $f_1, ..., f_4$ solve (66), $f_3$ and $f_4$ are linear combinations with constant coefficients of $f_1$ and $f_2$. Thus $f_3/f_4$ is a Möbius transformation of $f_1/f_2$, and the segment $[-\frac{4}{3}, 0]$ is mapped by $f_1/f_2$ onto a segment or an arccircle. Because the tangency showed in item (d), it must be an arccircle.

As in (d) above, $(f_3/f_4)' \neq 0$ and thus $f_3/f_4$ is one-to-one on $[-\frac{4}{3}, 0]$ and, since $f_a$ is a Möbius transformation of $f_3/f_4$, $f_a$ is one-to-one as well on $[-\frac{4}{3}, 0]$.

Combining with (d) and (e) above, the image of $\mathbb{H}$ is int$(C_1)$ and $f_a$ is one-to-one between $\mathbb{H}$ and int$(C_1)$ (see also [11] p. 227).

(j) By (d) and the argument principle, $f_a$ is one-to-one between $\mathbb{H}$ and int$(C_1)$ (see also [11] p. 227).

(k) If $|s| > 5$ then, using the bound $|1 + \frac{1}{15}|/56 - 1| < 4/15$ in (82) it follows that $|f_2 - 1| < \frac{4}{15}$. Thus $|f_2(s)| > 7|s|^{5/6}/10$. Similarly, for $|s| > 5$ we use (83) to obtain $|f_1(s)| < 4|s|^{1/6}/5$. Therefore $f_a = f_1/f_2$ satisfies $|f_a(s)| < 8|s|^{-1}/7 < \frac{4}{15}$ if $|s| > 5$. \(\Box\)
**Proof of Proposition** [17] Since $f_a = f_1/f_2$ where $f_{1,2}$ solve (24), as do $J$ and $\hat{J}$, $K$ is a linear fractional transformation of $f_a$, i.e.

$$K(s) = a_1 + \frac{a_2 f_a}{f_a + a_3} \quad (86)$$

for some constants $a_{1,2,3}$ that we now determine. It follows from (65) that $K(0) = 0$ and $K(-\frac{4}{3}) = 1$. For $\Re s = 0$ and $\Im s \to \infty$ the roots of $u^3/3 + u^2 + s$ approach the roots of $u^3/3 + s$, thus by Lemma 9 we see that $r_2 \sim i|3s|^{-1/3}$, $r_3 \sim e^{-\pi i/6}|3s|^{-1/3}$, and $r_1 \sim e^{-5\pi i/6}|3s|^{-1/3}$. It then follows from (61) and (62) that $K(s) \to e^{-\pi i/3}$. The corresponding values for $f_a$ can be obtained directly using its definition since the hypergeometric functions can be calculated explicitly for $s = 0, -\frac{4}{3}, \infty i$ (see also (84)). We have $f_a(0) = a$, $f_a(-\frac{4}{3}) = e^{4\pi i/3}a$, and $f_a(\infty i) = 0$, which allows us to solve for $a_{1,2,3}$ and obtain

$$K(s) = M(f_a(s)); \quad M(z) := e^{-\pi i/3} + i\sqrt{3}z \left(z - e^{2\pi i/3}a\right)^{-1}$$

A simple calculation shows that the Möbius transformation $M$ has the properties: $M(I_1)$ is the arc circle through 0 and $e^{-\pi i/3}$, tangent to $i\mathbb{R}$, $M(I_2)$ is the arc circle through $e^{-\pi i/3}$ and 1, and it maps the arc circle through $a$ tangent at $z = e^{4\pi i/3}a$ to $\{z : \arg z = 4\pi i/3\}$ into the arc circle through 0 tangent to $i\mathbb{R}$ and to 1 + $i\mathbb{R}$. It then follows from Lemma 15 that $K$ maps the upper half plane into the interior of $C_2$.

(ii) If $|s| > 5$, then by Lemma 15 $|f_a| < \frac{1}{3}$ implying $\left|i\sqrt{3}f_a \left(f_a - e^{2\pi i/3}a\right)^{-1}\right| < \frac{2}{5}$ and thus using (86) we get $\Im K(s) < -\frac{\sqrt{3}}{5} + \frac{2}{5} < -\frac{2}{5}$.

Consider $\mathcal{I}$, the image under $K$ of region $\mathbb{H} \cap \{s : |s| > \eta_1\} \cap \{s : |s| > \eta_2\}$ and the compact set $\mathcal{I}_I = \{z : \text{dist}(z, I_1) \leq \epsilon\}$. If $\epsilon$ is small enough then $\mathcal{I}_I \subset K(\mathbb{H})$ and thus $\text{dist}(K^{-1}(\mathcal{I}_I), \mathbb{R})$ is positive and increasing in $\epsilon$ implying (77).

(iii) This follows directly from Lemma 15. 

\[ \Box \]

5. Recurrence relations and Constants of Motion. Proof of Theorem 21

We will use the integral equations (21) and (22) to derive an asymptotic constant of motion formula for $x$ in the third quadrant.

5.1. **Notations.** (i) Denote $R_I(v) = R(v, s_I)$, $J_I(v) = J(v, s_I)$, etc., $R_n(v) = R(v, s_n)$, $J_n(v) = J(v, s_n)$, $L_n(v) = L(v, s_n)$, $J_n = J(s_n)$, $L_n = L(s_n)$ etc., and $s_I = s_I + \frac{1}{3}$.

(ii) Denote, consistent with the notations of Theorem 21

$$Q = xJ, \quad K = \hat{J}/J, \quad Q_n = x_n J_n, \quad G_n = Q_n/x_0, \quad K_n = \hat{J}_n/J_n \quad (87)$$

Note: since $Q_n$ is a large quantity (for large $x_0$) it is preferable to work with the normalized quantity $G_n$ which is $O(1)$.

(iii) We use $c_0, c_1, c_2 \ldots$ to denote constants independent of $n, s_0, x_0, s_I, x_I$ etc., and $c$ denotes a “generic” such constant.

(iv) Consider the segment $\ell$ and its symmetric about the line $u = -1$, $\ell^-$, contained in $\mathcal{C}$:

$$\ell = \{t(1 + i)/3 : t \in [-1, 1]\} \quad \text{and} \quad \ell^- = \{-2 + t(-1 + i)/3 : t \in [-1, 1]\} \quad (88)$$

($\ell, \ell^-$ are sub-segments of $[1+i/3, -(1+i)]$, respectively $[-(1+i), -2 + \frac{-1+i}{3}]$, see Note 10).

Note that $u \in \ell$ if and only if $-2 - \overline{u} \in \ell^-$, symmetry which will be used in the following, in conjunction with (50) and with $| -1 - \overline{u}| = |u^-|$.
5.2. Calculating the Poincaré map (the first return map). The values of \( s_n, x_n \) are obtained by iterating the first return map. To establish its properties consider (21), (22) with initial conditions \((s_I, x_I)\):

\[
\begin{align*}
    s(u) &= s_I - 2 \int_{u_0}^{u} \left( \frac{R(v, s(v))}{x(v)} - \frac{392}{625} \frac{1}{x(v)^3} \right) dv \\
    x(u) &= x_I + \int_{u_0}^{u} \frac{1}{R(v, s(v))} dv
\end{align*}
\]  

where \( u_0 \) is given by Lemma 15 and Corollary 11 (iii).

Proposition 19 shows that the system (89), (90) has a unique solution for \( u \) traveling once along \( C \) and for initial values \((s_I, x_I) \in \mathcal{R}\), a suitable region with the property that the final values \((s_f, x_f)\) of \((s(u), x(u))\) when \( u \) returns to \( u_0 \) are close to \((s_I, x_I)\), cf. (93), (105).

Once this fact is proved, then \((s_n, x_n)\) are found by iterating the first return map

\[
\begin{align*}
    s_f &= \Phi(s_I, x_I) = s_I - 2 \oint_C \left( \frac{R(v, s(v))}{x(v)} - \frac{392}{625} \frac{1}{x(v)^3} \right) dv \\
    x_f &= \Psi(s_I, x_I) = x_I + \oint_C \frac{1}{R(v, s(v))} dv
\end{align*}
\]

and \((s_n, x_n) = (\Phi, \Psi)^{on}(s_0, x_0)\) for all \( n = 1, 2, \ldots, N_m \) for which \((s_n, x_n)\) remain in \( \mathcal{R} \). We calculate asymptotically the Poincaré map (91), (92) in Lemma 23, information needed to determine \( N_m \) in Proposition 24.

5.2.1. The region \( \mathcal{R} \) of initial conditions \((s_I, x_I)\). As explained in §3.1.1, \( s_0 \) starts near 0, and \( s_{N_m} \) ends near \(-4/3\). The estimates must be worked out differently in these two regions in \( s \), though they are linked by the symmetry (50). Therefore we define \( \mathcal{R} \) as a union \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \), where \( \mathcal{R}_1 \) contains values of \( s \) close to 0 (but not very close) and far from \(-4/3\), while \( \mathcal{R}_2 \) has \( s \) close to \(-4/3\) (but not very close) and far from 0; both contain intermediate values of \( s \). They are defined as follows.

Let \( m > 0 \) be large enough (independent of any other parameter), \( \eta_1 \) given by Lemma 15 and \( \eta_2 \) small so that (77) holds. Recall that here \( u_0 = -4 \) (though other values can also be used).

**Region** \( \mathcal{R}_1 \) is the set of all \((s, x)\) with:

(i) \(|x| > m, |w_0^3/3 + \eta_0 + s| > \eta_2/2, s \in \mathbb{D}_5^+, |s^-| > \eta_1/2, |s x| > 1 \) and

(ii) for all \( w \in \ell \) we have \( s - 2J(w, s)/x \in \mathbb{D}_5^+ \) and \(|x - 2J(w, s)/x| > |x/8|\).

**Region** \( \mathcal{R}_2 \) is defined as \( \mathcal{R}_1 \), only with \( s \) interchanged with \( s^- \) and \( \ell \) replaced by \( \ell^- \).

Note that \((s, x) \in \mathcal{R}_1 \) if and only if \((s^-, x) \in \mathcal{R}_2 \).

5.2.2. Existence of the Poincaré map and estimates.

**Proposition 19.** For \((s_I, x_I) \in \mathcal{R} \), the system (89), (90) has a unique solution \((s(u), x(u))\) for \( u \) going once along \( C \). Furthermore, with \( \alpha \) defined in (55) the solution satisfies

\[
|s(u) - s_I + 2J(u)/x_I| \leq 7\alpha \ln |x_I|/|x_I|^2
\]

The proof, given in §5.2.4, needs the estimates of §5.2.3.

5.2.3. Estimates of sums of square roots. Let \( \eta_1 \) be given by Lemma 15 and \( r_j \) as in §4.1. By Lemma 9 and Corollary 11 (iii) we have

\[
\inf_{u \in \mathcal{C}, s \in \mathbb{D}_{21/4}^+} |u - r_j(s)| > 0
\]
Lemma 21. (i) If $\Phi$ (ii) Let $\tilde{\Phi}$ anticlockwise. This, of course, may result in a discontinuity at $R$. Note 20. Upon analytic continuation in $s$ from $s \in (-4/3,0)$ to $s > 0$ through $\mathbb{H}$ the branches specified in Proposition 1(i) give that $P_s(0) = s$ which for $s > 0$ has zero argument, hence in $R(0,s)$ we choose the usual branch of the square root (which is positive when the argument is in $\mathbb{R}^+$).

In the following, $f$ is either $\tilde{f}_1$ or $\tilde{f}_2$. Since $f + s \in \mathbb{H}$, by Lemma 20 none of the square roots vanishes on $C$. We analytically continue $\psi$ on $C$ from 0 to $u_0$ clockwise and anticlockwise. This, of course, may result in a discontinuity at $u_0$.

Lemma 21. (i) If $\sigma_1 = \sigma_2 = \sigma_3 = 1$, then

$$\inf_{u \in \mathbb{C}, |s| > \eta_1, |s^-| > \eta_1} |\psi(u)| > 0$$

(ii) If $|s| < \eta_1$ then, for all choices of $\sigma_i$ we have

$$\inf_{u \in \mathbb{C}} |\psi(u)(|u|^2 + |s|)^{-1/2}| > 0$$

(iii) Let $\Phi_{10}$ and $\Phi_{20}$ the expressions defined in (55) with $\sigma_1 = \sigma_2 = 0$ and $\tilde{f}_i(0)$ replaced by $\tilde{f}_i(0)$. If in addition $\tilde{f}_k$ satisfy $|\tilde{f}_k(v) - \tilde{f}_k(0)| \lesssim |v| + |s|^{-3/2}$, then

$$\sup_{u \in \mathbb{C}, |s| < \eta_1} \left| \frac{1}{\psi(u)} - \frac{1}{\Phi_{10} + \Phi_{20}} \right| \lesssim 1$$

(iv) Let $u^{-}, s^{-}$ as in (50). Similar statements hold for $\psi^{-} := \sqrt{\sigma_1(u^{-})^3/3 - (u^-)^2 + s^- + \tilde{f}_1(u^-) + \sigma_3\sqrt{\sigma_2(u^-)^3/3 - (u^-)^2 + s^- + \tilde{f}_2(u^-)}}$ where $\tilde{f}_{1,2}$ satisfies (95) with $\ell$ replaced by $\ell^-$, and $s$ by $s^-$. To be precise we have

(ii) If $|s^-| < \eta_1$ then, for all choices of $\sigma_i$ we have

$$\inf_{u \in \mathbb{C}} |\psi^{-}(u)(|u^-|^2 + |s^-|)^{-1/2}| > 0$$

(iii) Let $\Phi_{10}^-$ and $\Phi_{20}^-$ the expressions defined in (95) with $\sigma_1 = \sigma_2 = 0$ and $\tilde{f}_i(u)$ replaced by $\tilde{f}_i(-2)$. If in addition $\tilde{f}_k$ satisfy $|\tilde{f}_k(u) - \tilde{f}_k(-2)| \lesssim |u^- s^-| + |s^-|^{-3/2}$, then

$$\sup_{u \in \mathbb{C}, |s^-| < \eta_1} \left| \frac{1}{\psi^{-}(u)} - \frac{1}{\Phi_{10}^- + \Phi_{20}^-} \right| \lesssim 1$$

\footnote{Recall that we use here $u_0 = -4$, but the results are more general.}
Proof. (i) Note that \( \Phi_1^2/\Phi_2^2 = 1 + \lambda(u) \) where \( \lambda(u) = (\hat{f}_1 - \hat{f}_2)/\Phi_2^2 \). By (94) and (96) we have \( \sup_{u \in C} |\lambda| = a_1 < 1 \). By the choice of branches, see Note 20 we have \( \Phi_1 + \Phi_2 = \Phi_2(1 + \lambda)^{\frac{1}{2}} \) and thus \( \Phi_1 + \Phi_2 \) can only vanish if \( \Phi_2 \) does, and this is ruled out by (94).

(ii) For \( u \in \ell \), by the choice of branch, \( \Phi_{1,2} \) are in the first quadrant. Then, \( |\Phi_1 + \Phi_2| \geq \min \{ |\Phi_1|, |\Phi_2| \} \), so we can reduce the analysis to the case \( \sigma_3 = 0 \). If \( \sigma_1 = \sigma_3 = 0 \), then the estimate follows from the fact that \( |u^2| + |s + f| \leq 2|u^2| + 2|s| \). If \( \sigma_1 = 1 \) and \( \sigma_3 = 0 \) the proof is similar on \( \ell \), where \( |u^3/3 + u^2| = |u^2|^2 |1 + u/3| \) and \( |u/3| < \frac{1}{3} \). On the rest of \( C \) we have, using Lemma 9 and Note 20, \( \sqrt{\frac{2}{3}u^3 + u^2 + s + f} \) and \( \sqrt{u^2 + s + f} \) are the analytic continuations of \( u \sqrt{1 + \frac{2}{3} \sqrt{1 + g_1(u,s)}} \) and \( u \sqrt{1 + g_2(u,s)} \) respectively (these are the branches when \( u \in \ell \) and large relative to \( s \), and here \( g_1 = -1 + (1 - \frac{3 + r_i}{u}) (1 - \frac{R_i}{u}) (1 - \frac{R_i}{u}) \), where \( r_i \) are the roots of \( \frac{1}{3}u^3 + u^2 + s + f \), \( g_2 = (1 + \frac{s + f}{u^2}) - 1 \), \( |g_{1,2}| < 10\eta_1 \) and \( |1 + \frac{2}{3}x| > \frac{1}{3} \) and the estimate is immediate.

(iii) This follows by straightforward estimates using (ii):

\[
\left| \frac{1}{\psi} \Phi_1 + \Phi_2 \right| = \left| \frac{1}{\psi(\Phi_1 + \Phi_2)} \left( \frac{-\sigma_1 v^3/3 - (\hat{f}_1(v) - \hat{f}_1(0))}{\Phi_1 + \Phi_2} + \sigma_3 \frac{-\sigma_2 v^3/3 - (\hat{f}_2(v) - \hat{f}_2(0))}{\Phi_2 + \Phi_2} \right) \right| \lesssim \frac{|v|^3 + |s_nv|}{\sqrt{(|v|^2 + |s_n|^2}} + \frac{|v|^3 + |s_n|^3/2}{\sqrt{(|v|^2 + |s_n|^3}} \lesssim 1 \quad (98)
\]

(iv) The proof is very similar: in fact, it follows by replacing \((u,s)\) by \((u^-,s^-)\) and using the symmetry (50) which is essentially inherited by the whole problem. \( \square \)

5.2.4. Proof of Proposition \( \square \). Due to the symmetry (50), the calculations with \( s_I \) and \( s_I^- \) are similar, see the proof of Lemma 21 (iv), so we only give the proof for \( R_1 \).

Proof. Inserting (90) in (89) and using the identity \( \sqrt{a} - \sqrt{b} = (a - b)/(\sqrt{a} + \sqrt{b}) \) we get

\[
s(u) = s_I - \frac{2J_I(u)}{x_I} - \frac{2}{x_I} \int_{u_I}^u s(v) - s_I R_1(v) + R(v, s(v)) \, dv + \frac{2}{x_I} \int_{u_I}^u R(v, s(v))L(v, s) \, dv + \int_{u_I}^u \frac{784}{625(x_I + L(v, s))^4} \, dv \quad (99)
\]

Denoting \( s(u) - s_I + 2J_I(u)/x_I = \delta(u) \), \( R_I(v) = R(s_I - 2J_I(v)/x_I + \delta(v)) \), \( L_I^-(v) = L(s_I - 2J_I(v)/x_I + \delta(v)) \) becomes

\[
\delta(u) = \frac{2}{x_I^2} \int_{u_I}^u 2J_I(v)dv \frac{R_I(v) + R_I^-(v)}{R_I(v) + R_I^-(v)} - \frac{2}{x_I} \int_{u_I}^u \delta(v)dv \frac{R_I(v) + R_I^-(v)}{R_I(v) + R_I^-(v)} + \frac{2}{x_I} \int_{u_I}^u R_I^-(v)L_I^-(v) \, dv + \frac{784}{625} \int_{u_I}^u \frac{dv}{(x_I + L_I^-(v))^4} =: \mathcal{N}(\delta)(u) \quad (100)
\]

We show that \( \mathcal{N} \) is contractive in the ball \( B = \{ y : \sup |y(u)| \leq 7\alpha \ln |x_I|/|x_I^2| \} \) in the Banach space of continuous functions on one loop of the lifting of \( C \) on its universal covering, from \( u_0 \) to the lifting of \( u_0 \).

**Note.** Once existence has been established, solutions turn out to be analytic in \( u, s_I, x_I \) by standard theory of analytic ODEs.

We first check that the assumptions of Lemma 21 hold if \( \hat{f}_1(v) = -2J_I(v)/x_I + \delta(v) \), namely that \( s_I + \hat{f}_1 \in D_{21/4}^+ \) and the inequalities in (96) are satisfied if \( |x_I| \) is sufficiently large. By assumption \( s_I - 2J_I(v)/x_I \in D_5^+ \), we have \( s_I - 2J_I(v)/x_I + \delta(v) \in D_{21/4}^+ \) if
\[ |\delta(v)| < \eta_2/2. \] Since \(|s_I - 2J_I(u)/x_I| > |s_I|/8\), we have \(|s_I + \tilde{J}_I(v)| > |s_I|/8 - |\delta(v)| > \eta_3/10\) if \(|\delta(v)| < 1/\eta_3|x_I|\). Finally \(-2J_I(u)/x_I + \delta(v)| < 2\alpha/|x_I| + |\delta(v)| < \eta_3/2\) if \(|x_I|\) is sufficiently large. Thus Lemma \[21\] applies.

It follows that, for large \(m\) we have
\[
\delta \in B \Rightarrow |\mathcal{N}(\delta)(u)| \leq \frac{60\ln |x_I|}{|x_I|^2} + \frac{c\ln^2 x_I}{x_I^2} \quad \forall u \in \mathcal{C} \Rightarrow \mathcal{N}(\delta) \in B
\]

For \(\delta_{1,2} \in B\), denoting \(R_j^-(v) = R(v, s_I - 2J_I(v)/x_I + \delta_j(v))\) and \(L_j^-(v) = L(v, s_I - 2J_I(v)/x_I + \delta_j(v))\), \(j = 1, 2\), we have by Lemma \[21\]
\[
|R_j^-(v) - R_2^-(v)| = \left| \frac{\delta_1 - \delta_2}{R_1^-(v) + R_2^-(v)} \right| \leq \frac{c}{(\sqrt{|v|^2 + |s_I|})^{3/2}} (101)
\]

and
\[
\left| \frac{1}{R_I(v) + R_I^-(v)} - \frac{1}{R_I(v) + R_2^-(v)} \right| = \left| \frac{\delta_1(v) - \delta_2(v)}{(R_1(v) + R_2^-(v))(R_I(v) + R_1(v))(R_I(v) + R_2^-(v))} \right| \leq \frac{c}{\sqrt{|v|^2 + |s_I|}} (102)
\]

Thus
\[
|L_I^-(v) - L_2^-(v)| = \left| \int_{u_0}^u \frac{\delta_2(t) - \delta_1(t)}{R_1(t)R_2(t)(R_I(t) + R_2^-(t))} dt \right| \leq \frac{c}{\sqrt{|s_I|}} \sup |\delta_2 - \delta_1| (103)
\]

Using \[(101), \ (102)\ and \ (103)\ applied to \ (100)\ it follows that, for large \(m\) we have
\[
\left| \mathcal{N}(\delta_1)(u) - \mathcal{N}(\delta_2)(u) \right| \leq c|x_I|^{-1}\ln |x_I| \sup \left| \delta_2 - \delta_1 \right| \leq \frac{\epsilon}{2}.
\]

For initial data in \(\mathcal{R}\), it follows from Proposition \[19\] and Lemma \[21\] that
\[
x(u) = x_I + \int_{u_I}^u \frac{1}{R_I(v)} dv - \int_{u_I}^u \frac{s(v) - s_I}{R_I(v)R(v, s(v))(R(v, s(v)) + R_I(v))} dv
\]

implying
\[
|x(u) - x_I - L_I(u)| \leq \frac{\epsilon}{|x_I s_I|} (105)
\]

and \(|x(u) - x_I| \leq \epsilon(\ln |s_I| + 1)|, since \(L_I(u) \leq \epsilon(\ln |s_I| + 1)\) by Lemma \[21\] \(\Box\)

### 5.3. Calculating higher orders in the expansions of \(x\) and \(s\).

We can bootstrap Proposition \[19\] in \[21\] and \[22\] to obtain, in principle, any number of terms in the expansion of \(s\) and \(x\) for large \(x_I\). Proposition \[22\] and Lemma \[23\] give higher order terms in the asymptotic behavior of \(s_f, x_f\) and of \(Q, K\). These will be used in \[5.4\] to establish the number \(N_m\) of times we can iterate the Poincaré map while preserving the asymptotic formulas.

**Proposition 22.** I. For \((s_I, x_I) \in \mathcal{R}_1\) the following estimates hold.

**(i)** The values of \(s(u)\) for \(u\) going once along \(\mathcal{C}\) satisfy
\[
s(u) = s_I - \frac{2J_I(u)}{x_I} + 2\int_{u_I}^u \frac{J_I(v)}{R_I(v)} dv + \frac{2}{x_I^2} \int_{u_I}^u R(v, s(v)) L_I(v) dv + O(s_I^{-1}x_I^{-3})
\]
\[
= s_I - \frac{2J_I(u)}{x_I} + \frac{2J_I(u)L_I(u)}{x_I^2} + O(x_I^{-3}s_I^{-1}) (106)
\]

**(ii)** The final value \(x_f\) satisfies
\[
x_f = x_I + L_I - 2G_I + O(x_I^{-1}s_I^{-1/2}) (107)
\]
(iii) \( Q = xJ \) is approximately constant (in the sense that \( Q = x_0 J_0 (1 + o(1)) \)) and the first correction is

\[
Q_f - Q_I = x_f J_f - x_I J_I = -2J_f G_I - 2J_I F_I + O(\epsilon^{-1} s_I^{-1/2})
\]

where the notation \( F_I = F(s_I, x_I, J_I), G_I = G(s_I, x_I) \) where

\[
F(s, x, J) = \left( \frac{sx}{4J} - \frac{i}{2} \right) \left( \ln(s) - \ln(s - 2J/x) \right) - \frac{1}{2}
\]

\[
G(s, x) = \frac{1}{2} \left( \ln(s) - \ln(s - 2J_00/x) \right)
\]

where \( J_00 = J(0, 0) \) cf. (54), is an elementary integral which evaluates to:

\[
J_00 = - \frac{12}{5} + \frac{2(-2 + u_0)(3 + u_0)^{3/2}}{5\sqrt{3}} = - \frac{12 + 4\sqrt{3}i}{5} \quad \text{(since } u_0 = -4 \text{)}
\]

(iv) The functions \( F_I \) and \( G_I \) are \( O(\epsilon^{-1} s_I^{-1}) \).

II. For \( (s_I, x_I) \in \mathcal{R}_2 \) similar statements hold after replacing \( u, s, s_I \) by \( u^-, s^-, s_I^- \) (where

\[
s_I^- = s_I + \frac{i}{2}
\]

namely

\[
Q_f - Q_I = -2J_f G_I - 2J_I F_I + O(\epsilon^{-1} s_I^-)^{-1/2})
\]

where \( F_I = \hat{F}(s_I, x_I, J_I), G_I = \hat{G}(s_I, x_I) \) and

\[
\hat{F}(s, x, J) = \left( \frac{is^- x}{4J} - \frac{i}{2} \right) \left( \ln(s^-) - \ln(s^- - 2J/x) \right) - \frac{i}{2}
\]

and

\[
\hat{G} = \hat{G}(s, x) = \frac{i}{2} \left[ \ln s^- - \ln (s^- - 2J(-2, -\frac{4}{3}))/x \right]
\]

where cf. (54)

\[
J(-2, -\frac{4}{3}) = - \frac{12i}{5}
\]

The functions \( \hat{F}_I \) and \( \hat{G}_I \) are \( O(\epsilon^{-1} s_I^-)^{-1}) \).

Proof. It suffices to prove I., then II. follows due to the symmetry (50), cf. the proof of Lemma 21 (iv).

(i) We have, with the notation \( \epsilon_I = |v| + \sqrt{|s_I|} \),

\[
\frac{1}{R(v, s(v)) + R_I(v)} - \frac{1}{2R_I(v)} = \frac{s_I - s(v)}{2R_I(v)(R(v, s(v)) + R_I(v))^2} = O(\epsilon_I^{-3} x_I^{-1}) \quad (113)
\]

where we used Lemma 21 in the last equality. Thus Proposition 19 and (113) imply

\[
R(v, s(v)) = R_I(v) + \frac{s(v) - s_I}{R(v, s(v)) + R_I(v)}
\]

\[
= R_I(v) - \frac{J_I(v)}{x_I R_I(v)} + O(x_I^{-2} \epsilon_I^{-1} (\epsilon_I^2 + |\ln x_I|)) \quad (114)
\]

Using (113) and (114) we can rewrite (99) as (106).

(ii) To improve the estimate for \( x(u) \), we denote \( W_I = \sqrt{R_I^2 - \frac{2J_I}{x_I}} \) and note that (106) implies

\[
R(v, s(v)) - W_I(v) = O \left( \frac{|\ln s_I|}{\epsilon_I^2 x_I^2} \right)
\]
and thus
\[
\frac{s(v) - s_I}{R_I(v)(R(v, s(v))(R(v, s(v)) + R_I(v))} = \frac{-2J_I(v)}{x_I R_I(v) W_I(v)(W_I(v) + R_I(v))} + O\left(\frac{\ln s_I}{\epsilon^2 x_I^2}\right)
\]
(115)

Let \(\rho_I = \sqrt{v^2 + s_I}\) and \(\tilde{\rho}_I = \sqrt{v^2 + s_I - 2J_{00}/x_I}\). To simplify the estimate in (115), we use the fact that \(J(v, s_I) - J(v, 0) = O(s_I \ln s_I)\) and \(J(v, 0) - J_{00} = O(v)\) which together with Lemma 21 implies
\[
\frac{s(v) - s_I}{R_I(v)(R(v, s(v))(R(v, s(v)) + R_I(v))} = \frac{-2J_{00}}{x_I \rho_I (\rho_I + \tilde{\rho}_I)} + O\left(\epsilon^{-2} x_I^{-1}\right)
\]
(116)

The function on the right side of (116) can be integrated explicitly:
\[
\int_{u_I}^0 \frac{-2J_{00}(\rho_I + \tilde{\rho}_I)}{x_I \rho_I (\rho_I + \tilde{\rho}_I)} \, dv = -G_I(s) + O(x_I^{-1}) \quad \text{where} \quad 2G_I := \ln(s_I) - \ln(s_I - 2J_{00}/x_I)
\]
(117)

We apply (116) and (117) to (104) and obtain the recurrence relation (107).

(iii) To obtain the change in \(J\) we use the definition of \(J_I\) as well as (106) to get
\[
J_f - J_I = \int \frac{s_f - s_I}{R_I + R_f} \, dv = \frac{-2J_I}{x_I} \int \frac{1}{(R_I + R_f)} \, dv + O(x_I^{-2} \ln s_I)
\]
\[
= \frac{-2J_I}{x_I} \left(\frac{L_I}{2} - \int \frac{s_f - s_I}{2R_I(R_I + R_f)^2} \, dv\right) + O(x_I^{-2} \ln s_I)
\]
\[
= \frac{-2J_I}{x_I} \left(\frac{L_I}{2} + \frac{J_I}{x_I} \int \frac{1}{R_I(R_I + R_f)^2} \, dv\right) + O(x_I^{-2} \ln s_I)
\]
(118)

Using (106) and Lemma 21 with \(\tilde{f}_I(v) = s_f\) we get
\[
\frac{1}{R_I(v)(R_I(v) + R_f(v))^2} = \frac{1}{\rho_I (\rho_I + \sqrt{v^2 + s_I - 2J_I/x_I}^2) + O\left(\epsilon^{-2}\right)}
\]
(119)

Since the function on the right hand side of (119) can be integrated explicitly, we have
\[
\int \frac{dv}{\rho_I (\rho_I + \sqrt{v^2 + s_I - 2J_I/x_I}^2)} = \frac{x_IF_I(s_I, x_I, J_I)}{J_I} + O\left(s_I^{-1/2}\right)
\]
(120)

Thus
\[
\frac{J_I}{x_I} \int \frac{1}{R_I(R_I + R_f)^2} \, dv = F_I + O\left(x_I^{-1} s_I^{-1/2}\right)
\]
(121)

Applying (121) to (118) we get
\[
J_f = J_I - \frac{L_I J_I}{x_I} - \frac{2J_I F_I}{x_I} + O\left(x_I^{-2} s_I^{-1/2}\right)
\]
(122)

The conclusion then follows from a straightforward calculation using (107) and (122).

(iv) When \(|s_f| > 4 \max(|J_I|, |J_{00}|)/|x_I|\) the \(\ln\) in the expressions of \(F_I\) and \(G_I\) can be Taylor-expanded while if \(1/|x_I| < |s_I| \leq 4 \max(|J_I|, |J_{00}|)/|x_I|\), then \(F_I, G_I = O(1)\) by straightforward estimates using the definitions of \(F_I\) and \(G_I\).

\[\square\]

Lemma 23. If \((s_I, x_I) \in \mathcal{R}\) there exists a constant \(c_1 > 1\) so that
\[
|s_f - s_I + 2J_I/x_I| < c_1 |x_I^{-2}|(|\ln s_I| + |\ln s_I^-|)
\]
(123)
\[
|x_f - x_I - L_I| < c_1 |x_I^{-1}| \left(|s_I|^{-1} + |s_I^-|^{-1}\right)
\]
(124)
\[
|J_f - J_I + L_I J_I/x_I| < c_1 |x_I^{-2}| \left(|s_I|^{-1} + |s_I^-|^{-1}\right)
\]
(125)
\[ |Q_f - Q_I| = |x_f J_f - x_I J_I| < c_1 |x_I^{-1}| \left( |s_I|^{-1} + |s_I^-|^{-1} \right) \]  

(126)

\[ |K_f - K_I - \epsilon_0 \frac{Q_0}{Q_I^2}| < c_1 |x_I|^{-2} \left( |\ln s_I| + |\ln s_I^-| \right) \]  

(127)

where \( \epsilon_0 = -2\pi i / x_0 \).

**Proof.** The first four estimates follow directly from Lemma 21 and Propositions 28, 26, 29, and 15.

To show (127) we note that

\[ K' = \left( \frac{j}{J} \right)' = \frac{J L - J L'}{2 J^2} = \frac{c_0}{2 J^2}, \quad K'' = -\frac{c_0 L}{2 J^3}, \quad c_0 = -48\pi i / 5 \]  

(128)

where we used the fact that (24) has no first derivative term, and the value of \( c_0 \) follows from (33) and Proposition 14.

By (33) and Lemma 16 we have \( \beta \leq |J(s_I)| \leq \alpha \).

Frobenius theory applied to (25) shows that (24) has no first derivative term, and the value of \( c_0 \) follows from (33) and Proposition 14.

Thus Taylor theorem, (128), and (123) imply

\[ K(s_f) - K(s_I) = \frac{48\pi i}{5Q_I} + O \left( x_I^{-2} \left( |\ln s_I| + |\ln s_I^-| \right) \right) \]

which together with Lemma 15 leads to (127) (with \( K(s_f) = K_f \) etc.) \( \square \)

### 5.4. The solution of \( (21), (22) \) exists for \( n \) large enough so that \( x_0, x_1, \ldots, x_{N_m} \) traverse the sector from edge to edge.

The main result in this section are Proposition 24 and Corollary 29 which are proved under Assumption (33), which implies \( (s_0, x_0) \in R_1 \).

It will turn out that the iteration ends in \( R_2 \).

Denote

\[ \epsilon_0 = -2\pi i / x_0, \quad \epsilon_+ = |\epsilon_0| \]  

(129)

We note that

\[ |\epsilon_0 / \epsilon_+ - 1| \leq 2 \epsilon_+ |\ln \epsilon_+| \]  

(130)

by (33), for large enough \( m \). Let

\[ N_s = \left[ (\epsilon_+)^{-1} - (\epsilon_+)^{-1/2} \right], \quad j_+ = (\epsilon_+)^{-1} - j \quad \text{for} \quad 0 \leq j \leq N_s \]  

(131)

**Proposition 24.** Consider \( s_0, x_0 \) satisfying (33). Then there exists \( N_m > N_s \) so that the solution of the integral equations (21), (22) where we take \( n = 0 \) exists along \( C \) for \( N_m \) loops and so that we have

\[ 0 < \ln s_{N_m} < 11 |x_0|^{-1} \quad \text{and} \quad |\Re s_{N_m} + \frac{4\pi}{3}| < 2 |x_0|^{-1/2} \]

The proof of Proposition 24 is given in 5.4.1 for going along \( C \) the first \( n \leq N_s \) loops, followed by 5.4.2 for a number \( N_s < n < N_m \) loops.

#### 5.4.1. Iteration of the Poincaré map a number of \( n \leq N_s \) times.

While \( Q_n \) and \( K_n \) change from \( n \) to \( n + 1 \) by a term much smaller than their order, when expressed in terms of \( Q_0 \) and \( K_0 \) the corrections add up to a significant term; in this section we show that a better expression of discrete asymptotic conserved quantities are

\[ \tilde{Q}_j = Q(s_0) + a_j \ln(j + 1); \quad \tilde{K}_j = K(s_0) + j \epsilon_+ + b_j c_2 j \ln[(j + 1) j_+], \]

(132)

where \( a_j, b_j \) may depend on \( s_0, x_0 \), but are bounded by a constant \( c_2 \) independent of \( s_0, x_0 \):

\[ |a_j|, |b_j| \leq c_2 \quad \text{for all} \quad j = 0, 1, \ldots, N_s \quad \text{for some} \quad c_2 \quad \text{large enough} \]

(133)

proved in Proposition 29 with the help of Lemma 26 by complete induction on \( n \).

Then

\[ \tilde{s}_j = K^{-1}(\tilde{K}_j), \quad \tilde{x}_j = \tilde{Q}_j / J(\tilde{s}_j) \]  

(134)
are expected to be the leading order of \( s_j, x_j \), as expected form (170), (18).

**Note 25.** As before denote \( s^- = s + 4/3, \tilde{s}_j^- = \tilde{s}_j + 4/3 \) and so on. We only prove the results in this section for \( \tilde{s}_j \), and the proofs for \( \tilde{s}_j + \frac{4}{3} \) are similar due to the symmetry (50), see the proof of Lemma 9 (iii).

We write \( o(1) \) for quantities that vanish as \( x_0 \to \infty \) (therefore as \( \epsilon_0 \to 0 \)).

**Lemma 26.** Let \( n \leq N_s \). Assume (132), (133) hold for all \( j = 0, 1, \ldots, n \) and \( c_2 \) is large enough. Then there exist two constants \( c_3, c_4 > 0 \), independent of \( c_2, s_0, x_0 \), such that for all \( 1 \leq j \leq n \) and large \( x_0 \) we have

\[
\begin{align*}
(1) & \quad \tilde{s}_j \in \mathbb{H} \text{ and } |\tilde{s}_j| < 5 \\
(2) & \quad c_3 j \epsilon_+ \leq |\tilde{s}_j| \leq c_4 j \epsilon_+ \\
(3) & \quad \frac{1}{3} \leq |\tilde{x}_j/x| \leq \frac{2}{3} \text{ where } \alpha, \beta \text{ are given by Lemma 10 and 15.}
\end{align*}
\]

**Proof.** (132) implies \(|K(s_j) - j \epsilon_+| = O(j e_+^{2} \ln \epsilon_+^{-1})\) for large \( x_0 \) and all \( j \leq N_s \). Thus \( K_j \) traverses \([0, 1]\) up to small corrections.

(i) By the above, \( \text{Im} K(s_j) = O(j e_+ \ln \epsilon_+^{-1}) \). Now Proposition 17 (ii) implies \( \tilde{s}_j \in \mathbb{H} \) and \( |\tilde{s}_j| < 5 \). Lemma 10 and 15 now give

\[
\beta \leq |J(\tilde{s}_j)| \leq \alpha \quad (135)
\]

(ii) For small \( w \), Proposition 17 (iii) implies

\[
|K^{-1}(t) - \frac{2iw}{\alpha t}| \leq |t|^{3/2} \quad |K^{-1}(1 - t) + \frac{4}{3} - \frac{2iw}{\beta t}| \leq |t|^{3/2}
\]

and thus \( \tilde{s}_j/(j \epsilon_+) \) is bounded above and below when \( j \epsilon_+ \to 0, (j > 0) \) is small. The rest is immediate.

(iii) This follows by straightforward estimates from (132) and (135).

\[\square\]

**Proposition 27.** Let \( n \) be such that the assumptions of Lemma 26 hold. If \( s_0, x_0 \) satisfy (133) with \( m \) large enough, then \( (\tilde{s}_j, \tilde{x}_j) \) (defined in (134)) belong to \( \mathcal{R} \) (defined in (5.2.1)) for all \( j = 0, 1, \ldots, n \).

**Proof.** By Note 25 it suffices to look at those \( j \) for which \( |\tilde{s}_j^-| > \eta_1 \). Lemma 26 implies \( |\tilde{x}_j| \geq \frac{6}{27} x_0 \) and \( \tilde{s}_j \in \mathbb{D}_5^+ \). With \( u_0 = -4 \) we have \( |u_0^3/3 + u_0^2 + \tilde{s}_j| > \frac{16}{3} - 5 \) by Lemma 26 (i).

The property \( |\tilde{s}_j \tilde{x}_j| > 1 \) only needs to be checked when \( \tilde{s}_j = o(1) \), by Lemma 26 (ii) and (iii). That is, by Proposition 17 we look at those \( j \) for which \( j \epsilon_+ = o(1) \). In this case, by (132) we have

\[
J(\tilde{s}) = J(0)(1 + o(1)) \Rightarrow \tilde{x}_j = x_0(1 + o(1)) \quad (137)
\]

and the rest follows from the definition of \( \tilde{s}_j, \tilde{x}_j \), (132) and (136).

Using the definition of \( J \) we have \( J(u, s) - J(u, 0) \to 0 \) as \( s \to 0 \), and \( J(u, 0) \) is given by an elementary integral. Let \( J_T = -\frac{12}{5} u^2 + \frac{4\pi \sqrt{2}}{5} \) be the two term-Taylor expansion of \( J(u, 0) \); the Taylor remainder \( |J(u, 0) - J_T(u)| \) is bounded by \( 1/10 \) for \( u \in \ell \). Using this bound and calculating \( \text{Re} J_T, \text{Im} J_T \) for \( u \in \ell \) we get

\[
-16/5 < \text{Re} J(u, 0) < -2 \quad \text{and} \quad 2/3 < \text{Im} J(u, 0) < 11/5 \quad (138)
\]

For small \( s \) (138) and (137) imply

\[
\text{Re} \tilde{s}_j + \frac{3}{|x_j|} < \text{Re} \left( \tilde{s}_j - 2J_j(u, \tilde{s}_j) \right) < \text{Re} \tilde{s}_j + \frac{5}{|x_j|} \quad \text{and} \quad \text{Im} \left( \tilde{s}_j - 2J_j(u) \right) > \text{Im} \tilde{s}_j + \frac{3}{|x_j|}
\]

Since \( \tilde{s}_j \in \mathbb{H} \), using these inequalities, we see that \( \tilde{s}_j - 2J_j(u) \tilde{x}_j \in \mathbb{D}_5^+ \) and

\[
|\tilde{s}_j | \geq |\tilde{x}_j| \geq \max \{ \text{Re} \tilde{s}_j - 5/|x_j|, \text{Im} \tilde{s}_j + 3/|x_j| \} > |\tilde{s}_j|/8
\]
for $u \in \ell$. For $n$ close to $N_\ast$ see Note 25.

**Proposition 28** (The evolution “preserves” (132)).

Let $n \leq N_\ast$. Assume (132), (133) are true for $j = 1, \ldots, n - 1$.

Consider the initial conditions $(s_1, x_1) = (\bar{s}_{n-1}, \bar{x}_{n-1})$. By Propositions 27 and 29, the solution exists for one more loop and $(s_f, x_f)$ are well defined, if $m$ is large enough.

Then with $\tilde{Q}_n = x_f J_f$ and $\tilde{K}_n = K(s_f)$, $Q_j, \tilde{Q}_j$ satisfy (132), (133) for $j = 1, \ldots, n$ for some $c_2 > 0$.

**Proof of Proposition 28** Using Lemma 26 to estimate $1/x_f, 1/s_f$, we see that for large $x_0$ there is a constant $c_5$ (independent of $n$, $c_2, s_0, x_0$) so that

$$|x_f^{-1}|(|s_f|^{-1} + |s_f|^{-1}) \leq c_5(1/n + 1/N_\ast)$$

(139)

Now, (126) and (139) imply

$$|\tilde{Q}_n - \tilde{Q}_{n-1}| \leq c_1c_5 \left( \frac{1}{n-1} + \frac{1}{(n-1)^{-}} \right)$$

(140)

implying that (132), (133) hold for $\tilde{Q}_j$ for all $j = 1, \ldots, n$ if $c_2 > 2c_1c_5$. This fact, and (130), and Lemma 26 used to estimate $\bar{x}_j$ and $\bar{s}_j$ in (127), show that

$$|\mathcal{K}(\bar{s}_n) - \mathcal{K}(\bar{s}_{n-1}) - \mathcal{K}(\bar{s}_n - \mathcal{K}(\bar{s}_{n-1}) - \mathcal{K}(\bar{s}_n - \mathcal{K}(\bar{s}_{n-1}) - 1/2) < 1/2 |2c_2 \epsilon^2 \ln((j+1)j_) + 2c_1^2 \ln \epsilon_{+1} |$$

$$+ \frac{4c_1 \beta^2}{\alpha^2 |x_0|^2} (|\ln((j+1)j_+ | 2 |\ln(c_3 \epsilon_{+1}|))$$

(141)

Adding the errors in (141), and using the fact that $\epsilon_{+1} < 2(j+1)j_+$ it follows that $\tilde{K}_j$ satisfy (132), (133) for all $j = 1, \ldots, n$. □

We can now obtain estimates for $Q_n$ and $K_n$:

**Corollary 29** (Inductive construction of the solution of (21), (22)). The solution of the integral equations (21), (22) exists along $\mathcal{C}$ for $N_\ast$ loops. Furthermore,

$$|s_{N_\ast} - 4i|x_0|^{-1/2} \leq |x_0|^{-1/2}; \left| \frac{x_{N_\ast}}{x_0} + i \right| < (c_2 + 1) \ln |x_0|/|x_0|$$

(142)

In particular $(s_{N_\ast}, x_{N_\ast}) \in \mathcal{R}_2$.

**Proof.** With $s_I = s_0, x_I = x_0$, we get by Proposition 19 $x_1 = x_f, s_1 = s_f$, and (132) follow from (126) and (127). Thus, by Proposition 27 Proposition 19 applies, to yield $Q_2$ and $K_2$ which by Proposition 28 satisfy (132) and, inductively $x_j, s_j$ yield $Q_j, K_j$ for all $j \leq N_\ast$.

The estimate for $s_{N_\ast}$ follows from (136), and the estimate for $x_{N_\ast}$ follows from Lemma 15 and (132). □

5.4.2. **Proof of Proposition 24** for $n > N_\ast$ up to $n = N_m$. We prove by complete induction that Proposition 19 applies to $s_n, x_n$ with $n \geq N_\ast - 1$ as long as $\text{Im} s_n \geq 11/|x_0|$.

First note that Proposition 19 applies to $s_{N_\ast-1}, x_{N_\ast-1}$ by Corollary 29. Suppose for some $N_\ast \leq n < N_\ast + |x_0|^{1/2}$ we have that $(s_k, x_k) \in \mathcal{R}_2$, $|s_k^-| < \eta_1$, and $\text{Im} s_k \geq 11/|x_0|$ for all $k$ with $N_\ast - 1 < k < n$. We only need to verify the following conditions defining $\mathcal{R}_3$: $|x_n| > m, s_n \in \mathbb{D}_0^+, |x_n s_n^-| > 1$ and that for all $w \in \ell^-$ we have $s_n - 2J_n(w)/x_n \in \mathbb{D}_0^+$ and $|s_n^- - 2J_n(w)/x_n| > |s_n^-|/8$, since the other conditions are obvious.

By (124), (123), Lemma 15 we have $|x_k - x_{k-1}| < c_0 \ln |x_0|$ for some $c_0$, $|s_k^-| < 8(n-k)/|x_0|$, and

$$|s_k - s_{k-1} + \frac{48}{5} \frac{1}{x_0} | < \frac{1}{|x_0|}$$

(143)
for $N_s - 1 \leq k \leq n$. Thus by (142) we have $|s_n^-| < |s_{N_n}^-| + 11|x_0|^{-1/2} < 16|x_0|^{-1/2}$ and

$$|x_n/x_0 + i| < |x_{N_n}/x_0 + i| + c_6|x_0|^{-1/2} \ln |x_0| < (c_6 + 1)|x_0|^{-1/2} \ln |x_0|$$

which implies

$$|s_n^-| < \eta_1, \quad \text{and} \quad |x_n| = |x_0|(1 + o(1)) \quad (144)$$

and by (143)

$$\text{Im } s_n > \text{Im } \left( s_{n-1} + \frac{48}{5x_0} \right) - \frac{1}{|x_0|} > 0$$

Thus $|x_n| > 9|x_0|/10 > m$, $s_n \in \mathbb{D}_o^s$, and $|x_ns_n^-| > 9|x_0|\text{Im } s_n|/10 > 1$.

A calculation similar to that used in the proof of Proposition 22 shows that

$$|\text{Re } J(w, s_n)| < 1 \quad \text{and} \quad -13/4 < \text{Im } J(w, s_n) < -8/5 \quad \text{for } w \in \ell^-$$

Thus $\text{Im } (s_n^- - 2J_n(u)/x_n) > \text{Im } s_n - 7/|x_n| > 0$ and

$$|s_n^- - 2J_n(w)/x_n| \geq \max(|\text{Re } s_n^-| - 3/|x_n|, \text{Im } s_n - 7/|x_n|) > |s_n^-|/8$$

Thus $s_n$, $n$, are in Region 2, and Proposition 13 applies again.

Since $\text{Im } s_{N_n} < 5|x_0|^{-1/2}$ by (142) and $\text{Im } (s_k - s_{k-1}) < -8/|x_0|$ by (143), there must exist some $N_m < N_s + |x_0|^{1/2}$ such that $0 < \text{Im } s_{N_m} < 11/|x_0|$. By (143) we have

$$|\text{Re } (s_k - s_{k-1})| < |\text{Im } (s_k - s_{k-1})|/8. \quad \text{Thus by } (142) \text{ we have}$$

$$|\text{Re } s_{N_m^-}| < |\text{Re } s_{N_n^-}| + |\text{Im } s_{N_m}|/8 < 2|x_0|^{-1/2} \quad \square$$

**Corollary 30.** The solution of the integral equations (21), (22), with initial condition $(s_{N_s}, x_{N_s})$, exists along $C$ for $N_m - N_s$ loops. Furthermore, we have

$$0 < \text{Im } s_{N_m} < 11/|x_0|, \quad |\text{Re } s_{N_m} + 4M/3| < 2|x_0|^{-1/2}, \quad \left| \frac{x_n}{x_0} + i \right| < (c_7 + 1)|x_0|^{-1/2} \ln |x_0|$$

for all $N_s < n \leq N_m$ for some constant $c_7$.

6. Asymptotics of the discrete constants. Proof of Theorem 2 (ii)

6.1. Asymptotics of the discrete constants of motion. We derive two more orders of these formulas, needed in the calculation of $\mu$, cf. Proposition 1. For this we need more properties of functions $F$ and $G$ in Proposition 22.

Denote

$$F_n = F(s_n, x_n, J_n), \quad G_n = G(s_n, x_n), \quad \tilde{F}_n = \tilde{F}(s_n, x_n, J_n), \quad \tilde{G}_n = \tilde{G}(s_n, x_n) \quad (145)$$

$$F_{n,a} = F(s_0 + \frac{48n}{5x_0}, x_0, -\frac{2a}{3}) \quad \text{and} \quad G_{n,a} = G(s_0 + \frac{48n}{5x_0}, x_0) \quad (146)$$

$$\tilde{F}_{n,a} = \tilde{F}(s_{N_n} - \frac{48n}{5x_{N_n}}, x_{N_n}, -\frac{2a}{3}), \quad \tilde{G}_{n,a} = \tilde{G}\left(s_{N_n} - \frac{48n}{5x_{N_n}}, x_{N_n}\right) \quad (147)$$

Let by convention $B_0 = \tilde{B}_0 = 0$.

**Lemma 31.** $B_n$ and $\tilde{B}_n$ defined in (40) and (41) for $n \geq 1$ satisfy

$$B_n = \frac{48}{3} \sum_{k=0}^{n-1} (F_{k,a} + G_{k,a}), \quad \tilde{B}_n = \frac{48i}{3} \sum_{k=0}^{n-1} (\tilde{F}_{k,a} + \tilde{G}_{k,a})$$

**Proof.** The sums above are, up to elementary sums, telescopic; the calculations are straightforward. \(\square\)

**Note 32.** For generic $u_0$, $\tilde{B}_n$ would contain a term of order $\ln(n + 1)$, but the term vanishes for the special choice $u_0 = -4$, which makes the calculation simpler.

In the following $O(\cdot)$ denotes $n$–independent error terms.

With $N_0$ defined in (35) we study the regions $0 \leq n \leq N_0$ and $N_m - N_0 \leq n \leq N_m$. The following estimate is needed.
Lemma 33. For \( n \leq 2N_0 \) we have
\[
|F_n - F_{n:a}| + |G_n - G_{n:a}| + |\tilde{F}_n - \tilde{F}_{n:a}| + |\tilde{G}_n - \tilde{G}_{n:a}| = O(x_0^{-1} \ln x_0)
\] (148)

Proof. Define \( \delta_{J,n} \) and \( \delta_{z:n} \) by \( J_n = -48/5(1 + \delta_{J,n}) \) and \( s_n = (s_0 + 48n/5x_0)(1 + \delta_{z,n}) \). It follows from Lemma 23 that
\[
\delta_{J,n} = O \left( \frac{n + 1}{x_0} \right); \quad x_{n+1} - x_n = O(\ln x_0);
\]
\[
s_{n+1} - s_n = \frac{-2J_0}{x_n} + O \left( \frac{(n + 1) \ln x_0}{x_0^2} \right)
\]
(149)

implying
\[
s_n = s_0 + \frac{48n}{5x_0} + O(x_0^{-2} \ln x_0)
\]
and
\[
s_n x_n - s_{n-1} x_{n-1} = \frac{48}{5} + O \left( \frac{(n + 1) \ln x_0}{x_0} \right) \Rightarrow \delta_{z:n} = O((n + 1)x_0^{-1} \ln x_0)
\]
(150)

The estimates for \( F_n - F_{n:a} \) and \( G_n - G_{n:a} \) follow by Taylor expansion, using (149) and (150) and the fact that \( \frac{N_0}{x_0} = o(1) \). The proof for \( \tilde{F} \) and \( \tilde{G} \) is analogous. \( \square \)

6.2. Proof of Theorem 2 (ii). Case I. Consider \( n \leq N_0 \). By (149) we have
\[
(J_n + 24/5)(G_n + F_n) = O(\ln x_0/x_0)
\]
(151)

We then have by (108), Lemma 33 and (151)
\[
Q_n - Q_0 = B_n + O(x_0^{-1/4} \ln x_0)
\]
(152)

Case II. Consider \( n \) with \( N_0 < n \leq N_m - N_0 \). We first show that \( |s_n| > \frac{1}{2} |x_n|^{-\frac{1}{2}} \) and \( |s_{n-1}| > \frac{1}{2} |x_{n-1}|^{-\frac{1}{2}} \) for \( N_0/2 < n \leq N_m - N_0/2 \). We need to distinguish two subcases.

II.a For \( n \leq \eta_1 |x_0|/8 := N_2 \) (note that \( N_2 > N_0 \)), Lemma 15 and Lemma 23 imply
\[
\left| s_n - s_0 - \frac{48ni}{5|x_0|} \right| < 1/|x_0|; \quad |x_n/x_0 - 1| < 1/20
\]
(153)

and similarly
\[
\left| s_n - s_{N_m} - \frac{48(N_m - n)i}{5|x_0|} \right| < 1/|x_0|; \quad |x_n/x_0 + i| < 1/20
\]
(154)

for \( N_m - N_2 < n \leq N_m \).

II.b For \( N_2 < n \leq N_m - N_2 \) we have \( |s_n| \geq c_1 \eta_1/2 \) and \( |s_n^-| \geq c_1 \eta_1/2 \) by Lemma 26. A straightforward calculation using (153) and (154) shows that \( |s_n| > \frac{1}{2} |x_n|^{-\frac{1}{2}} \) and \( |s_n^-| > \frac{1}{2} |x_n|^{-\frac{1}{2}} \) for \( N_0/2 < n \leq N_m - N_0/2 \).

It follows from Lemma 21, (106) and Lemma 26 that
\[
\frac{1}{R(v,s(v))} - \frac{1}{R_n(v)} = \frac{s_n - s(v)}{R_n(v)R(v,s(v))(R_n(v) + R(v,s(v)))} = O \left( x_0^{-1}s_n^{-3/2} \right) + O \left( x_0^{-1}(s_n^-)^{-3/2} \right)
\]
(155)
This equation together with (113), (106), and (107) implies that
\[ x_{n+1} = x_n + L_n - \int \frac{s(v) - s_n}{2R_n(v)^3} dv + O \left( (x_0 s_n)^{-2} \right) + O \left( (x_0 s_n^{-3})^{-2} \right) \]
\[ = x_n + L_n + \int \frac{J_n(v)}{x_n R_n(v)^3} dv + O \left( x_0^{-3/2} \right) \]
\[ = x_n + L_n + \frac{1}{x_n} \int \tilde{J}_n(v) \frac{\partial Q(v, s_n)}{\partial v} dv + \frac{\rho(s_n) J_n^2}{2 x_n} + O \left( x_0^{-3/2} \right) \]
\[
(156)
\]
In (156) we used (56) to integrate by parts:
\[ \int \frac{\tilde{J}_n(v)}{R_n(v)^3} dv = \int \tilde{J}_n(v) \frac{\partial Q(v, s_n)}{\partial v} dv + \frac{\rho(s_n) J_n^2}{2} = J_n Q(u_n, s_n) + \frac{\rho(s_n) J_n^2}{2} \]
since \( R_n(v)Q(v, s_n) \) is analytic and its loop integral is 0. Therefore
\[ x_{n+1} = x_n + L_n + \frac{J_n(u_n, s_n)}{x_n} + \frac{\rho(s_n) J_n^2}{2 x_n} + O \left( x_0^{-3/2} \right) \]
\[
(157)
\]
We rewrite (118) using (106), (113) and (56) as
\[ J_{n+1} - J_n = \int \frac{s_{n+1} - s_n}{R_n(v) + R_{n+1}(v)} dv \]
\[ = - \frac{2J_n}{x_n} \int \frac{1}{(R_n(v) + R_{n+1}(v))} dv + \frac{J_n L_n^2}{x_n} + O \left( x_0^{-3} s_n^{-1} \right) + O \left( x_0^{-3} s_n^{-1} \right) \]
\[ = - \frac{2J_n}{x_n} \left( \frac{L_n}{2} - \int \frac{-J_n(v)}{x_n R_n(v)(R_n(v) + R_{n+1}(v))^2} dv \right) + \frac{J_n L_n^2}{x_n} + O \left( \frac{\ln s_n}{x_0^3 s_n} \right) + O \left( \frac{\ln s_n}{x_0^1 s_n} \right) \]
\[
(158)
\]
Now
\[ \int \frac{1}{R_n(v)(R_n(v) + R_{n+1}(v))^2} dv = \int \frac{1}{4 R_n^3(v)} dv + O \left( x_n^{-1} (s_n s_n^{-2}) \right) \]
\[ = \frac{\rho(s_n) J_n}{4} + O \left( x_n^{-1} (s_n s_n^{-2}) \right) \]
\[
(159)
\]
Using (158) and (159) we get
\[ J_{n+1} - J_n = - \frac{J_n L_n}{x_n} + \frac{J_n^2 L_n^2}{x_n^2} - \frac{\rho(s_n) J_n^3}{2 x_n^2} + O \left( x_0^{-5/2} \right) \]
\[
(160)
\]
which, combined with (157) implies
\[ x_{n+1} J_{n+1} - x_n J_n = x_n^{-1} Q(u_0, s_n) J_n^2 + O(x_0^{-3/2}) = -\frac{1}{2} Q(u_0, s_n) J_n (s_{n+1} - s_n) + O(x_0^{-3/2}) \]
\[
(161)
\]
On the other hand,
\[ \frac{dQ(u_n, s)}{ds} = O \left( (s_n s_n^{-2}) \right) = O(x_0^{1/2}) \]

implying
\[ Q(u_0, s_0) J(s_n) - Q(u_0, s) J(s) = O(x_0^{-1/2}) \]
\[
(162)
\]
for \( s \) between \( s_n \) and \( s_{n+1} \), and thus integrating (162) we get
\[ (s_{n+1} - s_n) Q(u_0, s_n) J_n = \int_{s_n}^{s_{n+1}} Q(u_0, s) J(s) ds + O(x_0^{-3/2}) \]
\[
(163)
\]
It follows from \((161)\), \((163)\) and Lemma \(23\) that
\[
x_{n+1}J_{n+1} - x_nJ_n = -\frac{1}{2} \int_{s_n}^{s_{n+1}} Q(u_0, s)J(s)ds + O(x_0^{-3/2})
\]

Summing in \(n\) we get
\[
Q_n = Q_{N_0} - \frac{1}{2} \int_{s_{N_0}}^{s_{n}} Q(u_0, s)J(s)ds + O(x_0^{-1/2}) \quad (164)
\]

Now by \(I\), we have \(Q_{N_0} - Q_0 = \frac{4\sqrt{2\pi}}{\pi} \ln(N_0) + g_0 + O(x_0^{-1/4}\ln x_0)\). Since \(Q(u_0, s)J(s) = -\frac{8\sqrt{2\pi}}{5s} + O(x_0\ln s)\) by definition and \(s_{N_0} = \frac{48N_0}{\pi x_0} + O(x_0^{-1})\) by \((153)\), we have
\[
\int_{s_0}^{s_{N_0}} Q(u_0, s)J(s)ds = -\frac{8\sqrt{3i}}{5} \ln \frac{48N_0}{5s_0x_0} + O(x_0^{-1/4}\ln x_0)
\]

Thus by \((164)\) we have
\[
Q_n = Q_0 + g_0 + \frac{4\sqrt{3i}}{5} \ln \frac{5s_0x_0}{48} - \frac{1}{2} \int_{s_0}^{s_n} Q(u_0, s)J(s)ds + O(x_0^{-1/4}\ln x_0)
\]

**Case III.** The remaining case \(N_m - N_0 < n \leq N_m\) is similar to \(I\). by symmetry and we omit the details. We get
\[
Q_{N_m} - Q_n = \tilde{B}_{N_m-n} + O(x_0^{-1/4}\ln x_0) \quad (165)
\]

### 6.3. Proof of Theorem 2 (iii)

It follows from \((166)\) that
\[
\frac{s_{n+1} - s_n}{J_n^2} = -\frac{2}{Q_n} + \frac{2L_n}{x_n^3J_n} + O(x_0^{-3}\ln^2 x_0)
\]

Now using the definitions of \(J\) and \(L\) we have for \(s(u)\) with \(u\) in the \(n\)th loop on \(C\) (between \(u_n\) and \(u_{n+1}\))
\[
\frac{1}{J_n^2} - \frac{1}{J_{n+1}^2} = \frac{(J_n + J(s))L_n(s - s_n)}{2J_n^2J_{n+1}^2} + O\left(\frac{(s - s_n)^2}{s_n}\right)
\]
\[
= \frac{L_n(s - s_n)}{J_{n+1}^2} + O\left(\frac{(s - s_n)^2}{s_n}\right) + O\left(\frac{(s - s_n)^2}{s_{n-1}}\right) + O\left(\frac{\ln x_0(s - s_n)}{x_0}\right) \quad (167)
\]

Integrating both sides gives
\[
\frac{s_{n+1} - s_n}{J_n^2} - \int_{s_n}^{s_{n+1}} \frac{1}{J^2(s)}ds = \frac{L_n(s_{n+1} - s_n)^2}{2J_n^2} + O\left(\frac{1}{s_n^3x_0^3}\right) + O\left(\frac{1}{s_n^{-3}x_0^3}\right) + O\left(\frac{\ln x_0}{x_0^3}\right)
\]

This together with \((166)\) implies
\[
\int_{s_n}^{s_{n+1}} \frac{1}{J^2(s)}ds = -\frac{2}{x_0J_0} + \frac{2(Q_n - Q_0)}{x_0^2J_0^2} + O\left(x_0^{-3}\ln^2 x_0\right) \quad (169)
\]

Using \((128)\) to integrate \(1/J^2\) we get
\[
\int_{s_n}^{s_{n+1}} \frac{1}{J^2(s)}ds = -\frac{5}{24\pi i}(K(s_{n+1}) - K(s_n))
\]
This together with (169) implies
\[ \mathcal{K}(s_{n+1}) - \mathcal{K}(s_n) = \frac{48\pi i}{5Q_0} + \frac{2\pi i (Q_n - Q_0)}{x_0^2J_0} + O(x_0^{-3}\ln x_0) \quad \text{for } 0 \leq n < N_m \]

Summing in \( n \) we get
\[ \mathcal{K}(s_n) = \mathcal{K}(s_0) + \frac{48\pi in}{5Q_0} + \frac{2\pi i \sum_{j=0}^{n-1}(Q_j - Q_0)}{x_0^2J_0} + O(x_0^{-2}\ln^2 x_0) \quad \text{(170)} \]

Now by (41) and Theorem 2 (ii) we have
\[ |B_k| \leq c\ln x_0. \]

Thus for \( 0 \leq n \leq N_0 \) we have
\[ \sum_{j=0}^{n}(Q_j - Q_0) = \sum_{j=0}^{N_0} B_j = O(x_0^{3/4}\ln x_0) \quad \text{(171)} \]

while for \( N_0 < n \leq N_m - N_0 \) we have
\[ \sum_{j=0}^{n}(Q_j - Q_0) = \sum_{j=0}^{N_0} B_j + \sum_{j=N_0+1}^{n}(Q_j - Q_{N_0}) + (n - N_0)B_{N_0} \]
\[ \quad = nB_{N_0} - \frac{1}{2} \sum_{j=N_0}^{n} \int_{s_{N_0}}^{s_j} Q(u_0, s)J(s)ds + O(x_0^{3/4}\ln x_0) \quad \text{(172)} \]

Now by Lemma 23 and Theorem 2 (ii) we have
\[ \frac{(s_{j+1} - s_j)}{2J_j^2} = -\frac{1}{x_0J_0} + O(x_0^{-2}\ln x_0) \]

Thus
\[ -\frac{1}{2} \sum_{j=N_0}^{n} \int_{s_{N_0}}^{s_j} Q(u_0, s)J(s)ds = x_0J_0 \sum_{j=N_0}^{n} \frac{(s_{j+1} - s_j)}{-2J_j^2} \left( -\frac{1}{2} \int_{s_{N_0}}^{s_j} Q(u_0, s)J(s)ds \right) + O(\ln x_0) \]
\[ = x_0J_0 \int_{s_{N_0}}^{s_n} \frac{1}{4J^2(s)} \int_{s_{N_0}}^{s} Q(u_0, s)J(s)ds + O(\ln x_0) \quad \text{(173)} \]

where we noted that the middle term is a Riemann sum, that we replaced by an integral plus the usual error bound in terms of the derivative. Using (128) to write \( 1/J^2 \) in terms of \( (\mathcal{K} - 1)' \) and integrating by parts we get
\[ \int_{s_{N_0}}^{s_n} \frac{1}{4J^2(s)} \int_{s_{N_0}}^{s} Q(u_0, t)J(t)dt \]
\[ = -\frac{5}{96\pi i} \int_{s_{N_0}}^{s_n} \left( \int_{s_{N_0}}^{s} Q(u_0, t)J(t)dt \right) (\mathcal{K}(s) - 1)'ds \]
\[ = -\frac{5}{96\pi i} (\mathcal{K}_n - 1) \int_{s_{N_0}}^{s_n} Q(u_0, s)J(s)ds - \frac{5}{96\pi i} \int_{s_{N_0}}^{s_n} Q(u_0, s)(J(s) - \hat{J}(s))ds + O(x_0^{-1/4}) \]
\[ = -\frac{5}{96\pi i} \int_{s_{N_0}}^{s_n} Q(u_0, s)(\mathcal{K}_nJ(s) - \hat{J}(s))ds + O(x_0^{-1/4}) \quad \text{(174)} \]

Combining (172), (173) and (174) we have
\[ \sum_{j=0}^{n}(Q_j - Q_0) = nB_{N_0} - \frac{5Q_0}{96\pi i} \int_{s_{N_0}}^{s_n} Q(u_0, s)(\mathcal{K}_nJ(s) - \hat{J}(s))ds + O(x_0^{3/4}\ln x_0) \quad \text{(175)} \]

Since Theorem 2 (ii) implies \( Q_n - \mathcal{Q}_{N_n} = O(1) \) for \( N_m - N_0 \leq n \leq N_m \), we see that (175) is also valid for \( N_m - N_0 \leq n \leq N_m \).
Since $K_n$ is bounded, (170) and (175) imply that

$$K_n = -\frac{2\pi i n}{x_0} + O(\ln x_0 / x_0)$$

(176)

Note also $Q_0 = -24x_0/5 + O(\ln x_0)$ since $J_0 = -24/5 + O(x_0^{-1} \ln x_0)$. This together with (175) and (170) imply that

$$\sum_{j=0}^{n} (Q_j - Q_0) = nB_{N_0} + \frac{1}{4\pi i} \int_{s_{N_0}}^{s} Q(u_0, s) (-2\pi i n J(s) - x_0 J(s)) ds + O(x_0^{5/4} \ln x_0)$$

(177)

Now, by definition, $Q(u_0, s) (-2\pi i n J(s) - x_0 J(s)) = -\frac{16\sqrt{3} \pi}{48} n + O(x_0 \ln s)$ and by (153) we have $s_{N_0} = \frac{4N_0}{48} x_0 + O(x_0^{-1})$. Thus

$$\int_{s_{N_0}}^{s} Q(u_0, s) (-2\pi i n J(s) - x_0 J(s)) ds =$$

$$\int_{s_0}^{s_{N_0}} Q(u_0, s) (-2\pi i n J(s) - x_0 J(s)) ds + \frac{16\sqrt{3} \pi}{48} n \ln \frac{48N_0}{5s_0 x_0} + O(x_0^{3/4} \ln x_0)$$

(178)

This together with (111) and (177) implies

$$\sum_{j=0}^{n} (Q_j - Q_0) = ng_{a} + \frac{1}{4\pi i} \int_{s_0}^{s_{N_0}} Q(u_0, s) (-2\pi i n J(s) - x_0 J(s)) ds + \frac{4\sqrt{3} \pi}{48} n \ln \frac{5s_0 x_0}{48} + O(x_0^{3/4} \ln x_0)$$

(179)

Comparing this with (171) we see that (179) is in fact valid for $0 \leq n \leq N_m$. The conclusion then follows from (170) and (179).

6.4. Proof of Proposition 8. (i) It follows from Lemma 15 and (170) that

$$1 + \frac{\pi (s_n + \frac{4}{3})}{J(-\frac{1}{3})} = \frac{\pi i s_0}{J(0)} - \frac{2\pi i J(0)n}{J_0 x_0} + \frac{2\pi i \phi_n}{x_0^2 J(0)} + O(x_0^{-5/4} \ln x_0) + O\left((s_n^{-})^{3/2}\right)$$

This implies (45). A calculation using (111) and Theorem 2 (iii) shows that $Re \frac{2\phi_n}{x_0^2} = O(x_0^{\alpha})$ and $Im \frac{2\phi_n}{x_0^2} = O(x_0^{-1} \ln x_0)$. Since $0 < Im s_{N_m} < 11/|x_0|$, (45) implies $N_m = \frac{|x_0|}{2\pi} + O(\ln x_0)$.

(ii) This follows directly from (i).

(iii) (47) follows from Theorem 2 (ii). By (i) and (ii) we have $s_{N_m} = O(x_0^{-1} \ln x_0)$, and thus by Lemma 15 we have $J_{N_m} = iJ_0 + O(x_0^{-1} (\ln x_0)^2)$. The rest follows from (47). □

7. APPLICATION: FINDING THE STOKES MULTIPLIER

As an application of the discrete constants of motion, in this section we find the Stokes multiplier $\mu$ by analyzing the tritronquée solution $y_t(z)$ of $P_1$ specified by the sector of analyticity [5].

7.1. Overview of the approach. The solution $y_t$ is meromorphic; this was known since Painlevé, and proving meromorphicity does not require a Riemann-Hilbert reformulation, see e.g. [10, 14] for direct proofs and references. Starting with a large $z \in \mathbb{R}^+$ we analytically continue $y_t$ (i) anticlockwise on an arccircle until arg $z = \pi$ and (ii) clockwise on an arccircle until arg $z = -\pi$. The continuation (ii) traverses the pole sector, arg $z \in (-\pi, -3\pi/5)$. Because of the above-mentioned meromorphicity, we must have

$$y_t(|z|e^{i\pi}) = y_t(|z|e^{-i\pi})$$

(180)
After the normalization (2) this tritonquée $y_t(z)$ becomes $h_t(x)$, solution of (4) specified by (4). The analytic continuation corresponds in the new variables to the following: We start with large $x$ with $\arg x = \pi/4$ and (i') analytically continue $h_t(x)$ anticlockwise, until $\arg x = 3\pi/2$, and (ii') analytically continue $h_t(x)$ clockwise, until $\arg x = -\pi$. The single-valuedness equation (180) implies

$$h_t(|x|e^{3\pi i/2}) = -h_t(|x|e^{-\pi i}) - 2 + \frac{8}{25|x|^2}$$

(181)

Recall that a Stokes line is a direction at which the constant $C$ in the transseries of solutions changes: the Stokes phenomenon, and in fact $C = C(\arg x)$ is piecewise constant, see §3.1; orthogonal to them are the antistokes lines, directions along which some exponential in the transseries solutions is purely oscillatory. By Theorem 2 (iii) of (7) the value of $C$ jumps by $\mu$, cf. also §3.2.

By Theorem 2 of (7), $\mathbb{R}^+$ and $\mathbb{R}^-$ are the (only) Stokes lines of (4) (the Stokes lines coincide with directions along which some exponential in the transseries has maxima decay) and the antistokes lines are $i\mathbb{R}^+$ and $i\mathbb{R}^-$. The tritonquée $h_t$, with zero constant in its transseries in the first quadrant, $C(\arg x) = C(0) = 0$ for $\arg x \in (0, \frac{\pi}{2})$, is analytically continued (i') traversing the antistokes line $\arg(x) = \frac{x}{2}$ ($C$ does not change) and reaches the Stokes line $\arg x = \pi$, where $C(\frac{\pi}{2} +) = \mu$; $h_t$ continues to have a transseries with the same $C$ until the next antistokes line $\arg x = \frac{3\pi}{2}$ beyond which it enters a pole region; upon analytic continuation (ii') $h_t$ traverses the Stokes line $\arg x = 0$ gives $C(0) = -\mu$, then crosses the antistokes line $\arg(x) = -\frac{x}{2}$ entering the pole sector.

For $y_t(z)$ continuation (i) means that $z$ traverses the antistokes line $\arg(z) = \pi/5$ and reaches the Stokes line $\arg z = 3\pi/5$, while (ii') traverses the Stokes line $\arg z = -\pi/5$, the antistokes line $\arg(z) = -3\pi/5$, entering the pole sector.

In variable $z$, and using the five-fold symmetry, we see that

**Note 34.** Their position in the original $z$ plane are $\arg z \in \{-\pi/5, 3\pi/5, 7\pi/5\}$ (Stokes) and $\arg z \in \{-3\pi/5, \pi/5, \pi\}$ (antistokes). The lines bordering the sectors of symmetry (4) are antistokes lines for some tritonquée.

Going back to the normalized for, $h_t(x)$ Along $\mathbb{R}^+$ the change is given by (see also (25))

$$C(0^-) = C(0^+) - \mu \quad \text{where} \quad S_\beta =: \frac{\mu}{2\Gamma(1 - \beta) \sin \pi \beta} \quad \beta_1 = 1/2$$

(182)

(7) 4 we give a short proof in the Appendix, §8.2. For the tritonquée $C(0) = 0$. Along $\mathbb{R}^-$, we have $C(\pi + 0) = C(\pi - 0) - \mu = C(0^+) - \mu = -\mu$ for the same $\mu$ as in (182) because of (149) and since the direction of continuation in (i) is opposite to that in (ii). In (ii), the third quadrant, a sector with poles in $x$ is traversed. In this region $h_t$ is described by constants of motion (cf. Theorem 2 (ii) and (151)), which are valid until $x$ reaches $\mathbb{R}^-$ when it is again described by a transseries; the asymptotic expansions of the constants of motion that we obtain depend on $C$. The transseries representation of $h_t$ also depends on $C$ in a way visible in the first few terms when $\arg x = -\pi$ or $3\pi/2$. Eq. (181) is a nontrivial equation for $\mu$ which determines it uniquely. The fact that $\mu$ is uniquely determined is not surprising given that there is only one solution, the tritonquée, with algebraic behavior in the region (5), cf. (9), Proposition 15.

**7.2. The transseries regions.** Our goal is to find the value of the Stokes multiplier $\mu$ using (181). By (9) $h$ has the asymptotic expansion (10) in the region $\text{Im} x < 0$, $\text{Re} x \in [-\frac{4}{3} \ln |x|, 0]$. Similarly, since $y(z)$ is continuous in $z$, by (2) we have when $h_t(x) \sim -2$.
when \( x \) is in the region \( \text{Re} x < 0, \text{Im} (x) \in [-\frac{4}{3} \ln |x|, 0] \). A calculation similar to (10) (cf. [9]) gives the asymptotic expansion
\[
h_t(x) \sim -2 - h_0(\tilde{\xi}) - \frac{1}{x} h_1(\tilde{\xi}) - \frac{1}{x^2} h_2(\tilde{\xi}) + \cdots
\]
where \( \tilde{\xi} = \mu e^{ix} \) and \( \mu = -\mu \) (see the discussion below (182)).

**Remark 35.** The fact that \( \tilde{\mu} = \mu \) is in fact not used for our purpose of calculating \( \mu \).

**Note 36.** There are infinitely many points \( x_0 \) so that \( h_t(x_0) = -4 \), and among them there are sequences with modulus going to \( \infty \).

**Proposition 37.** One can choose \( x_0 \) satisfying Assumption (83) with \( |x_0| \) is sufficiently large, such that the tritronquee solution with \( u_0 = h_t(x_0) = -4 \) satisfies
\[
s_0 = \frac{8(3 + \sqrt{3}i)}{5x_0} + O(x_0^{-3/2})
\]
and
\[
\frac{5}{2} x_0 J_0 = 24i\pi k_0 - 2i\sqrt{3} \ln k_0 + 12 \ln \left[ (1 + i) \left( \sqrt{3} + i \right)^{-1} \right] + 2 \left( \frac{3 + \sqrt{3}}{2} - i \right) (5\pi + 3i) - \sqrt{3}i (6 \ln 2 + 3 \ln 3 + 2 \ln 5) - 3 \ln \frac{100}{3} - 2\sqrt{3}i \ln \pi + O \left( \frac{\ln k_0}{k_0} \right)
\]

**Proof.** Since \( u_0 = -4 \), (10) implies that \( \xi(\xi/12 - 1)^{-2} = -4 \) for \( x \) near \( -i\mathbb{R} \). This equation has solutions \( \xi = 6(-1 \pm \sqrt{3}i) \). For convenience we choose \( \xi = 6(-1 + \sqrt{3}i) \). Let \( x_0 \) be a value of \( x \) corresponding to \( \xi \). A straightforward calculation using (10) shows (184).

We write \( x_0 = -2k_0\pi i + \tilde{x}_0 \) where \( k_0 \in \mathbb{N} \) is large, and \( \tilde{x}_0 = O(\ln k_0) \). By definition of \( \xi \) we see that \( \tilde{x}_0 \) solves the equation
\[
\frac{\mu e^{-\tilde{x}_0 + \pi i/4}}{\sqrt{2k_0\pi + i\tilde{x}_0}} = 6(-1 + \sqrt{3}i)
\]
Expanding the square root at \( \tilde{x}_0 = 0 \) and inverting the exponential we obtain
\[
\tilde{x}_0 = -\ln \left( 6(-1 + \sqrt{3}i)\mu^{-1} \sqrt{-2k_0\pi i} \right) + O(\frac{1}{k_0})
\]
Combining (186) and Proposition 14 we obtain (185).

Now we have

**Proposition 38.** Let \( x_0 \) as in Proposition 37 large enough so that \((s_0, x_0) \in \mathcal{R}_1\), so that \((s_n, x_n)\) exist for \( 0 \leq n \leq N_m \). Furthermore, \( x_{N_m} \) is in the transseries region \( \{ x \in \mathbb{C} : \text{Re} x < 0, \text{Im} x \in (-\frac{1}{3} \log |x|, 0) \} \) and (83) implies
\[
s_{N_m} = \frac{24}{5x_0} + O(x_0^{-2} \ln x_0)
\]

**Proof.** With \( u_0 = -4 \) and \( u_0, x_0, s_0 \) given by Proposition 37, the conditions of Proposition 24 are satisfied. By Lemma 15, Proposition 8 (iii) and (185) we have
\[
\text{Im} x_{N_m} = \text{Im} \frac{x_0 J_0}{J_{N_m}} + O(1) = \text{Im} \frac{5 |x_0| s_{N_m}^{-} \ln s_{N_m}^{-}}{48} + O(1)
\]
Since \( 0 < \text{Im} s_{N_m}^{-} < 11/|x_0| \) by Proposition 24 and \( \text{Re} s_{N_m}^{-} = O(1/|x_0|) \) by (46) and (185), we see that \( \text{Im} x_{N_m} > -\frac{55}{48} \ln |x_0| + O(1) \). Since the second quadrant is a transseries region (cf. (3) and (8.1)) and \( u_0 = -4 \), by (83) we must have \( \text{Im} x_{N_m} < 0 \).
It follows from (183) that and \(-\tilde{\xi}(\tilde{\xi}/12 - 1)^{-2} = -2\) for \(x\) near \(-\mathbb{R}\), with solutions \(\tilde{\xi} = 12(-4 \pm \sqrt{15})\), which implies (187) by (183). Note that \(x_{N_m} = -ix_0 + O(\ln x_0)\) by Proposition 8 (iii).

\[\Box\]

7.3. Calculating the Stokes multiplier. We now find the exact value of the Stokes multiplier using Proposition 8 (i) and (187).

Note 39. Eq. (45) gives a formula for \(s_{N_m}\) based on the constants of motion given in Theorem 2 (ii) and (iii), whereas (187) gives the value of \(s_{N_m}\) according to the asymptotic expansion (183) for the tritronquée. Thus by setting them equal to each other we establish an equation for \(\mu\), see (202) below.

We need to prove some estimates first.

Lemma 40. Let \(x_0, s_0\) as in Proposition 37. For \(N_0/2 < N < 2N_0\) we have

\[B_N = \frac{2}{5} \left(-i + \frac{1}{\sqrt{3}}\right) \pi + 6 + 2i\sqrt{3} + i\sqrt{3} \ln 3 + \ln 27 - 6 \ln(2\pi) + \frac{4\sqrt{3}i}{5} \ln N + O(x_0^{-1/4} \ln x_0) \tag{188}\]

where \(B_n\) is as defined in (31). Similarly for \(N_0/2 < m < 2N_0\) we have

\[\tilde{B}_m = \frac{12}{5} (1 - \ln \pi) + O\left(\frac{1}{m} + 1\right) + O(x_0^{-1/4} \ln x_0) \tag{189}\]

Equivalently, \(g_a = \frac{2}{5} \left(-i + \frac{1}{\sqrt{3}}\right) \pi + 6 + 2i\sqrt{3} + i\sqrt{3} \ln 3 + \ln 27 - 6 \ln(2\pi)\) and \(g_b = \frac{12}{5} (1 - \ln \pi)\) (\(g_a, g_b\) are as defined in (41)).

Proof. Since \(\xi_0 = -4\) and \(\xi = 6(-1 + \sqrt{3}i)\), we have \(s_0 = \frac{8}{3\xi_0}(3 + \sqrt{3}i) + O(\ln x_0/x_0^2)\) by direct calculation. Also recall that \(J_{00} = -\frac{12}{5} + \frac{4\sqrt{3}i}{5}\) by (109).

With this choice we have the following explicit formulas by direct calculation using the definitions of \(F\) and \(G\) (cf. Proposition 22):

\[F_{n,a} = l_n + O\left(\frac{\ln x_0}{x_0}\right)\]

where

\[l_n = -\frac{3}{4} - \frac{3 - i\sqrt{3} - 6n}{12} \ln \left(3 + i\sqrt{3} + 6n\right) + \frac{9 + i\sqrt{3} + 6n}{12} \ln \left(9 + i\sqrt{3} + 6n\right) \tag{190}\]

and

\[G_{n,a} = g_n + O\left(\frac{\ln x_0}{x_0}\right)\]

where

\[g_n = \frac{1}{2} \ln \left(\frac{6n + 3 + i\sqrt{3}}{6n + 6}\right)\]

Thus, by Theorem 2 (ii) we have for \(N < 2N_0\)

\[B_N = \frac{48}{5} \sum_{k=0}^{N-1} (l_n + g_n) + O(x_0^{-1/4} \ln x_0)\]
The sum is a telescopic sum plus an explicit sum, and we get
\[
\sum_{k=0}^{N-1} (l_n + g_n) = \frac{1}{12} \left( 3 + i\sqrt{3} + 6N \right) \ln \left( 3 + i\sqrt{3} + 6N \right) - 6N - 6\ln 6N - 6\ln(N!) \tag{191}
\]

Using Stirling’s formula \(\ln(n!) = (-1 + \ln n)n + \frac{1}{2} \left( -\ln \left( \frac{1}{n} \right) + \ln(2\pi) \right) + O(1/n)\) in \(\text{191}\) we get
\[
\sum_{k=0}^{N-1} (l_n + g_n) = \frac{1}{24} \left( 6 + 2i\sqrt{3} + i\sqrt{3} \ln 3 + \ln 27 - 6\ln(2\pi) \right)
+ \frac{(-3i + \sqrt{3})\pi}{72} + \frac{i}{4\sqrt{3}} \ln N + O(1/N) \tag{192}
\]

This shows \(\text{188}\).

The proof for \(\text{189}\) is similar. Straightforward calculations using \(\text{187}\) show that
\[
\tilde{B}_m = \frac{48i}{5} \sum_{n=1}^{m-1} (\tilde{l}_n + \tilde{g}_n) + O(x_0^{-1/4}\ln x_0)
\]

where
\[
\tilde{l}_n = -\frac{i}{4} \left( (2n+1)\ln(2n-1) - (2n+1)\ln(2n+1) + 2 \right); \quad \tilde{g}_n = \frac{i}{2} \left( \ln \left( \frac{i}{2} - in \right) - \ln n + \frac{i\pi}{2} \right)
\]

These can be summed in \(n\) explicitly implying \(\text{189}\). \(\square\)

**Lemma 41.** For \(t \neq 0\) we have
\[
\int_{t}^{\frac{4}{3}} Q(-4, s)J(s)ds = \frac{-8}{5} \left( 6\sqrt{3}i \ln 2 + \pi \left( \sqrt{3} - 4i \right) + 6\ln \left( 4 - \sqrt{15} \right) + 2\sqrt{3}i \ln 3 \right) + \frac{8\sqrt{3}i}{5} \ln t + O(t\ln(|t| + 1)) \tag{193}
\]

and
\[
\int_{0}^{\frac{4}{3}} Q(-4, s)\tilde{J}(s)ds = -\frac{16\sqrt{3}i\pi}{5} + \frac{16}{15} \ln \left( 4 - \sqrt{15} \right) + \frac{32}{3} \ln \left( 4 + \sqrt{15} \right) \tag{194}
\]

**Proof.** The proofs of \(\text{193}\) and \(\text{194}\) are very similar. We have by definition and Lemma \(\text{15}\)
\[
\int_{t}^{\frac{4}{3}} Q(-4, s)J(s)ds = \int_{t}^{\frac{4}{3}} \frac{-4J(s)}{s\sqrt{-48 + 9s}} ds
= \int_{0}^{\frac{4}{3}} \left( \frac{-4J(s)}{s\sqrt{-48 + 9s}} - \frac{iJ(0)\ln s}{\sqrt{3}s} \right) ds + \frac{iJ(0)\ln s}{\sqrt{3}s} \bigg|_{t}^{-\frac{4}{3}} + O(t\ln(|t| + 1))
= \lim_{e \to 0} \int_{C_0}^{\frac{4}{3}} \int_{t}^{-\frac{4}{3}} \left( \frac{-4\sqrt{U(e)} + s}{s\sqrt{-48 + 9s}} - \frac{i\sqrt{U(e)}}{\sqrt{3}s} \right) ds du + \frac{iJ(0)\ln s}{\sqrt{3}s} \bigg|_{t}^{-\frac{4}{3}} + O(t\ln(|t| + 1)) \tag{195}
\]

where \(U(e) = u^3 + u^2 + ei\), and \(C_0\) is as in Corollary \(\text{11}\). In particular it surrounds \(-3 - ei/3, -2 - \sqrt{e}i, \sqrt{-e}i, 1 - ei/3\) but neither \(-\sqrt{-e}i\) nor \(-2 + \sqrt{e}i\).
Similarly
\[
\int_0^{-\frac{4}{3}} Q(-4, s)\hat{J}(s) ds = \lim_{\epsilon \to 0} \int_{\check{C}_0} \int_0^{-\frac{4}{3}} \left( \frac{-4\sqrt{U(\epsilon)} + s}{s\sqrt{-48 + 9s}} - \frac{i\sqrt{U(\epsilon)}}{\sqrt{3}s} \right) ds \, du \quad (196)
\]
where \(\check{C}_0\) is as in Corollary 11. In particular it surrounds \(-2 + \sqrt{\epsilon}i, \pm \sqrt{-\epsilon}i,\) and 1, but neither \(-3\) nor \(-2 - \sqrt{\epsilon}i\).

Elementary integration gives
\[
\int_0^{-\frac{4}{3}} \left( \frac{-4\sqrt{U} + s}{s\sqrt{-48 + 9s}} - \frac{i\sqrt{U}}{\sqrt{3}s} \right) ds = \frac{1}{3} \left( 3\sqrt{U} + 4\ln \left( 16 - 8\sqrt{3}\sqrt{U} - 3U \right) \right.
\]
\[
- 4\ln \left( 24 - 3U - 4\sqrt{5}\sqrt{4 - 3U} \right) + i\sqrt{3}\sqrt{U} \left( \ln 48 + \ln U \right.
\]
\[
- 2\ln(16 + 3U) + \ln \left( 16 - 27U + 4\sqrt{15}\sqrt{U(-4 + 3U)} \right) ) \right) \quad (197)
\]
This function can be integrated in \(u\) explicitly as well; the calculation is tedious but straightforward and we omit the details. The branches of \(\ln\) and square roots are chosen according to analytic continuations along the contour \(C_0\) or \(\check{C}_0\) where the initial branch is consistent with \(J\) or \(\hat{J}\). Integrating (197) along \(C_0\) we obtain
\[
\int_0^{-\frac{4}{3}} \left( \frac{-4J(s)}{s\sqrt{-48 + 9s}} - \frac{iJ(0)}{\sqrt{3}s} \right) ds
\]
\[
= -\frac{8}{5} \left( 4i\sqrt{3}\ln 2 + 2 \left( -2i + \sqrt{3} \right) \pi + 6\ln \left( 4 - \sqrt{15} \right) + 3i\sqrt{3}\ln 3 \right) \quad (198)
\]
which together with (195) implies (193). Similarly integrating (197) along \(\check{C}_0\) and using (196) we obtain (194). \(\square\)

Now we calculate the Stokes multiplier \(\mu\) rigorously using Proposition 8 (i).

**Proof of Proposition 7**. We apply Proposition 8 (i) by first noting that
\[
\phi_{N_m} = \frac{g_a}{2\pi} + \frac{1}{4\pi i} \int_{s_0}^{-\frac{4}{3}} Q(u_0, s)(J(s) - \hat{J}(s)) ds + O(x_0^{3/4} \ln x_0) - \frac{2\sqrt{3}}{5\pi} \ln \frac{3 + \sqrt{3}i}{6} \quad (199)
\]
since \(N_m = \frac{|s_0|}{2\pi} + O(\ln x_0)\) by Proposition 8 and \(s_0 x_0 = \frac{8(3 + \sqrt{3}i)}{5} + O(x_0^{-1/2})\) by (184).

Now it follows from Lemma 11
\[
\frac{1}{4\pi i} \int_{s_0}^{-\frac{4}{3}} Q(u_0, s)(J(s) - \hat{J}(s)) ds =
\]
\[
\frac{1}{4\pi i} \int_{s_0}^{-\frac{4}{3}} Q(u_0, s)J(s) ds - \frac{1}{4\pi i} \int_{s_0}^{-\frac{4}{3}} Q(u_0, s)\hat{J}(s) ds + O(x_0^{-1} \ln x_0)
\]
\[
= \frac{2}{5} \left( \frac{\sqrt{3}\log (s_0/576)}{\pi} - i\sqrt{3} + 4 \right) + O(x_0^{-1} \ln x_0)
\]
\[
= -\frac{2\sqrt{3}\log |x_0|}{5\pi} + \frac{8}{5} - \frac{2i}{5\sqrt{3}} - \frac{\sqrt{3}\log(10800)}{5\pi} + O(x_0^{-1}) \quad (200)
\]
Applying (188) and (200) to (199) we obtain
\[
\phi_{N_m} = -\frac{i}{5\pi} \left( \pi \left( \sqrt{3} + 8i \right) - 2i\sqrt{3} - 6 - 6i\sqrt{3}\ln 2 - 4i\sqrt{3}\ln 3 - 2i\sqrt{3}\ln 5 \right. \\
- 2\ln 27 + 2\sqrt{3}i\ln \left( 3 + i\sqrt{3} \right) + 6\ln \left( 3 + i\sqrt{3} \right) + 6\ln(\pi) - 2i\sqrt{3}\ln |x_0| \bigg) + O(x_0^{3/4}\ln x_0)
\]
\tag{201}
\]

By (15), (185), (201) and Proposition 38, the matching equation (181) implies
\[
- 4 \frac{(\sqrt{3} - 6i)}{5\pi} = \frac{24i}{5\pi} (k_1 - N_m) + \frac{1}{5\pi^2} \left( 12\ln \left( \frac{(6 + 6i)(\sqrt{3} + i)}{\mu} \right) ight) + i\pi - 4\sqrt{3}\pi - 24\ln 2
\]
\[
- \ln 729 - 6\ln(5\pi) \bigg) + O(x_0^{-1/4}\ln x_0) \Rightarrow \mu = 6^{(k_1 - N_m)\pi} \sqrt{\frac{6}{5\pi}} i = \sqrt{\frac{6}{5\pi}} i \tag{202}
\]

since \(k_1 - N_m \in \mathbb{Z}\). \qed

7.4. The Painlevé equation \( P_2 \). The normal form of \( P_2 \) is (9)
\[
h'' + \frac{h'}{t} - \left( 1 + \frac{24\alpha^2 + 1}{9t^2} \right) h - \frac{8\alpha}{9} h^3 + \frac{8\alpha}{3t^2} h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0 
\]
\tag{203}

The associated asymptotic Hamiltonian equation, with Hamiltonian \( s \) is \( s'' - s - \frac{8}{9}s^3 = 0 \).

With \( R = \sqrt{9u^2 + 4u^4 + 18s(u)} \), the analog of the system (19), (20) is
\[
\frac{ds}{du} = -\frac{8\alpha u^2 + R}{3x} + \frac{u(1 + 24\alpha^2)}{9x^2} + \frac{8(\alpha^3 - \alpha)}{9x^3} 
\tag{204}
\]
\[
\frac{dx}{du} = \frac{3}{R} 
\tag{205}
\]

We integrating in \( u \) along cycle \( C \) surrounding two or three singularities, and use the notation (54). Since \( u^2 \) is single-valued we get
\[
s_{n+1} - s_n = -\frac{J_n}{3x_n} + O(x_n^{-2}); \quad x_{n+1} = x_n + 3L_n
\tag{206}
\]

the same as (26), (27) except for the fact that \( R^2 \) is now quartic. The leading order constants of motion are of the same form as those for \( P_1 \). We leave this analysis for a different paper.

7.4.1. Calculating the Stokes multiplier for more general integrable equations. Consider for simplicity second order differential equations for which the general solution is single-valued (the P-K property allows for finitely many branch points common to all solutions; this case can be accommodated too by an appropriate uniformization) and consider solutions for which the transseries is a pure power series along an antistokes line \( A_0 \). By [7] these solutions are unique and are regular up to the next antistokes \( A_1 \) and \( A_{-1} \), lines encountered by rotating the independent variable in both directions. Still by [7] and [9], there is only one solution regular in an open sector \( S \) bordered by \( A_1 \) and \( A_{-1} \) which contains a third antistokes line, \( A_0 \), in-between. These solution play the role of the tritronquées in being maximally regular; that is, the sector \( S \) connecting three antistokes lines is maximal. Assume that for large \( x \) in the complement of \( S^c \) of \( S \), the solution has poles. This is generic. At either edge of \( S^c \), the solution is of the form \( F_0(\xi) + o(1) \) [9] where \( \xi = \mu x^b e^{-\lambda x} \) where \( \mu \) is the Stokes multiplier while \( b \) and \( \lambda \) depend only on the equation (for \( P_1 \), \( \lambda = 1 \) and \( b = -1/2 \)). The solution is single valued, and we know that it needs to match a transseries at the other edge of \( S^c \). Since \( S \) is maximal, \( S^c \) is minimal. The condition that \( S^c \) is minimal means that the asymptotics in the pole region, which
contains $\mu$ in the leading orders, uniquely determines $\mu$ through the condition that the angular width of the pole region is minimal, equal to that of $S^c$. The equations determining the asymptotics in the pole regions are expected to be solvable in closed form, see Note [4].

8. Appendix

8.1. Asymptotic expansions, some known results and transseries representations. We write (23) as a system, as usual,

$$
\begin{pmatrix} h(t) \\ h'(t) \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ \frac{329}{829} t^{-4} & 1 \end{pmatrix} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h(t) \\ h'(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} h^2 \end{pmatrix}
$$

(207)

Simple algebra shows that the transformation

$$
\begin{pmatrix} h \\ h' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 - \frac{1}{4} t^{-1} \\ 1 - \frac{1}{4} t^{-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
$$

(208)

brings (207) to the normal form (in [7] eq. (1.1))

$$
y' + \left( \hat{\Lambda} + \frac{1}{x} \hat{B} \right) y = g(x^{-1}, y); \quad \hat{\Lambda} = \text{diag}(\lambda_i), \quad \hat{B} = \text{diag}(\beta_i); \quad \lambda_{1,2} = \mp 1; \quad \beta_{1,2} = \frac{1}{2}
$$

(209)

where $g = O(x^{-2}) + O(|y|^2)$.

In the following $L_\phi f(x) = \int_0^\infty e^{i\phi} e^{-xp} f(p) dp$ is the Laplace transform of $f$ in the direction $e^{i\phi}$ (by convention, $\phi = -\arg x$).

Note 42 (Results from [7] and [9]). (i) By Theorem (ii) in [7] if $y$ is a solution of the system (209) with $y = o(x^{-3})$ for $x \to \infty$, $x \in e^{-i\phi} \mathbb{R}$ (for some $\phi$) then $y$ has a unique Borel summed transseries: for some $C$

$$
y(x; C) = \sum_{k=0}^\infty C^k e^{-kx}(L_\phi Y_k)(x) \quad \text{for} \quad x \in e^{-i\phi} \mathbb{R}, \quad |x| \text{ large}
$$

(210)

where $Y_0 = p^3 A_0(p)$, $Y_k(p) = p^{k/2-1} A_k(p)$, with $A_k(p)$ independent of $C$ and analytic in $C \setminus \{\pm 1, \pm 2, \ldots\}$. All $Y_k(|p| e^{i\phi})$ are left and right continuous in $\phi$ at $\phi = 0$ and $\phi = \pi$.

There exist $\nu$ and $M$ independent of $k$ such that $\sup_{p \in C \setminus \mathbb{R}} |Y_k(p) e^{-|p|\nu}| \leq M k$.

We note that along Stokes directions, when $\phi \in \{0, \pi\}$, the Laplace transform does not exist as a usual integral, and must be considered in a generalized sense [8].

(ii) The constant $C$ in (210) depends on the direction $\phi$: $C = C(\arg x)$ is piecewise constant; it can only change at the Stokes rays.

(iii) We have

$$
L_\phi Y_k := y_k \sim c_k x^{-\frac{1}{2}}; \quad k \geq 1; \quad y_0 = O(x^{-4}) \quad \text{for} \quad x \to \infty, \quad x \in e^{-i\phi} \mathbb{R}
$$

(211)

where $c_1$ is taken to be 1 by convention, thus fixing $C$ (if $C \neq 0$).

(iv) For any $\delta > 0$ there is $b > 0$ so that for all $k \geq 0$ and $\phi$ in $(-\pi, 0) \cup (0, \pi)$ we have

$$
\int_0^\infty |Y_k(p e^{-i\phi})| e^{-by} dp < \delta^k \quad \text{(Proposition 20 in [7])}
$$

Note 43. (i) Algebraically, the equation is simpler in variables $(h, h')$ than in $y$, and it is more convenient to work directly with the second order equation (4); the results in [7], [8] and [9] translate easily through the linear substitution (208) into results about $h$ and $H := L^{-1} h$. In particular, (212) below holds for solutions $h = o(x^{-3})$, where $H_k$ satisfy all the analyticity properties and, up to constants, bounds satisfied by $Y_k$. 


Lemma 44. (i) Assume \( h \) solves (8) and satisfies (a) or (b) below:
(a) \( h(x) = o(1) \) as \( x \to \infty \) with \( \arg(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \);
(b) \( h(x) = o(x^{-3}) \) as \( x \to \infty \) with \( \arg(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \).
Then \( h \) also satisfies
(c) \( h(x) \sim C_{+} x^{-1/2} e^{-x} \) as \( x \to +i\infty \) for some \( C_{+} \).
Furthermore, for such \( h \) there is \( C_{-} \) such that \( h \sim C_{-} x^{-1/2} e^{-x} \) as \( x \to -i\infty \), and there exists a unique sequence \( \{c_k\}_k \) such that \( h \sim \frac{392}{625 \pi^2} + \sum_{k=5}^{\infty} c_k x^{-k} \) as \( x \to +\infty \) where the asymptotic expansion is differentiable. We denote \( C_{-} - C_{+} = \mu \).
(ii) A solution \( h \) as above has the Borel summed transseries representations:
\[
h(x) = \sum_{k=0}^{\infty} C_k e^{-kx} (L_{0} H_k)(x) \quad \text{for } \pm \phi \in (0, \pi/2]
\]
where \( H_k = p^{\frac{\phi}{2}} A_k(p) \) and \( A_k \) are analytic in \( C_r := \mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\} \). The functions \( H_k \) satisfy bounds of the type in Notes 44 and 42(iv).

Proof. (i) follows from formula (113) in [9].
Equation (13) can be rewritten in integral form as
\[
h(x) = C_1 x^{-\frac{1}{2}} e^{-x} + C_2 x^{-\frac{3}{2}} e^{x} + \mathcal{N}(f) \quad \text{where} \quad (212)
\]
which is well defined if \( h \) satisfies (a),(b), or (c), and furthermore, in these cases \( C_2 = 0 \) and we denote \( C_1 = C_{+} \). Equation (213) is contractive if \( x_0 > 0 \) is large enough in the ball of radius one, in the norm \( \|h\| = \sup_{x > x_0} |x^{1/2} \phi(x)| \) and its solution is of course unique. The same equation is also contractive in the norm \( \|h\| = \sup_{x > x_0} |x^{3} \phi(x)| \) and thus \( h = O(x^{-3}) \). Equation (213) now implies that \( h(x) = \frac{392}{625 \pi^2} + h_1(x) \) where \( h_1(x) = o(x^{-4}) \). Inductively, we obtain that \( h = \sum_{k=4}^{N} c_k x^{-k} + h_N(x) \) where \( c_k \) are unique and \( h_N = o(x^{-N}) \). Thus \( h \sim \tilde{h} = \sum_{k=4}^{\infty} c_k x^{-k} \), the form...
Their position in the original $z$ plane are $\arg z \in \{-\pi/5, 3\pi/5, 7\pi/5\}$ (Stokes) and $\arg z \in \{-3\pi/5, \pi/5, \pi\}$ (antistokes).

8.2. Derivation of (182). We rely on the summary of results provided in Note 42. The fact that $|\mathcal{L}Y_k| < \delta^k$ with $\delta$ independent of $\arg(x)$, shows that the change in $C$ can only occur due to $Y_0$ and possibly $Y_1$. Since $p^{-1/2}Y_1$ is analytic in a neighborhood of $|p| < 1$ and has continuous limits say $Y_\pm$ on the sides of the cut $[1, \infty]$, that satisfy $\sup_{p \in \mathbb{R}_+} |Y_\pm(p)e^{-i|p|}| \leq K$, Watson's Lemma implies that $|\mathcal{L}Y_\pm - \mathcal{L}Y_\pm(x)| = o(e^{-(2-\epsilon)x})$ for any $\epsilon > 0$ as $x \to +\infty$. So the change in $C$ can only occur due to $Y_0$ and we only need to evaluate $\delta Y_0 := (\mathcal{L}_0 - \mathcal{L}_0p)Y_0$. As in Note 42 $Y_0$ is analytic in $\mathbb{C} \setminus (\mathbb{Z} \setminus \{0\})$; assume $Y_0(p) = S\beta(1 - p)^{-\beta}(1 + o(1))$, $\beta > 1$. Then, with the choice of branch $(1 - p)^{-\beta} = 1 + o(1)$ for small $p$, we have $(1 - p)^{-\beta} = e^{\pm i\beta\pi}|1 - p|^{-\beta}$ for $|p| > 1, \arg(p) = \pm 0$. Using Watson's lemma and analyticity at zero, the result follows from

$$\delta Y_0 \sim -2i\sin \pi\beta S\beta \int_1^\infty |1 - p|e^{-px} dp = -2i\sin \pi\beta S\beta e^{-x\beta - 1} \Gamma(1 - \beta).$$

8.3. Proof of Lemma 9.

Proof. (i) Analyticity of the roots of a polynomial in $\mathbb{C} \setminus S_1$ where $S_1$ is the finite set of points where the roots coalesce is standard [3]; here $S_1 = \{0, -4/3\}$. As for the behavior near $S_1$, because of the symmetry [50], it suffices to analyze the roots near 0.

We write $e^z = -s$ and rewrite the equation as $v\sqrt{1 + v/3} = \sigma e, \sigma = \pm 1$. By symmetry, it is enough to analyze the case $\sigma = 1, v\sqrt{1 + v/3} = \epsilon$. We choose the branch of the square root with the cut $(-\infty, -3]$. The implicit function theorem (IFT) applies at $(v, \epsilon) = (0, 0)$ and gives a root, $r_2(\epsilon)$ which is analytic on the universal covering of $\mathbb{C} \setminus S_1$.

Consider the domain $S_2 := \{\epsilon : |\epsilon| < 2/3\}$. If $|r_2| = 1$ have $|r_2^3||1 + r_2/3| > 1 - 1/3 = 2/3$ and, by analyticity and the fact that $r_2(0) = 0$ we see that $|r_2| < 1$ for $\epsilon \in S_2$. By our choice of branch, we thus have throughout $S_2$,

$$|r_2| < 1; \quad \text{and Re} \sqrt{1 + r_2/3} > 0 \quad (216)$$

Using (216), we see that

$$|r_2 - \epsilon| = \left|\frac{\epsilon r_2}{1 + \frac{r_2^2}{3} + \sqrt{1 + \frac{r_2^2}{3}}}\right| \leq \frac{|\epsilon|}{2/3 + \sqrt{2/3}} \leq \frac{\epsilon}{4} \quad (217)$$

Estimating the right side of the equality in (217), now relying on $|r_2 - \epsilon| \leq \frac{\epsilon}{4}$ we get that $|r_2 - \epsilon| \leq |\epsilon|^2/3$. The result about $r_1$ follows similarly.

Using the symmetry [50], for $|s^-| < \sqrt{2/3}$, in some labeling, the three roots $\tilde{r}_i$ of $P$ satisfy

$$|\rho_1 - \sqrt{s^-}| < |s^-|; |\rho_3 - 3 + s^-/3| < |s^-|; |\rho_2 - \sqrt{s^-}| < |s^-|; \quad (218)$$

where $\rho_i = 2 + \tilde{r}_i, s^- = 4/3 + s$.

(ii) We let $s$ traverse a region in $\mathbb{H}$.

We note that the roots do not cross $\mathbb{R}$, otherwise we would have $s = -r_2^3 - r_3^3/3 \in \mathbb{R}$. As a consequence of this and by analyticity $\text{Im} r_j, j = 1, 2, 3$ do not change sign. For small $s, r_2 \in \mathbb{H},$ thus $r_2 \in \mathbb{H}$ for all $s \in \mathbb{H}$.

In (218), $\tilde{r}_2$ is the only root in $\mathbb{H}$, thus $\tilde{r}_2 = r_2$. Similarly, since $r_{3,1} \in -\mathbb{H}$ for small $s, r_{3,1} \in -\mathbb{H}$ for all $s$.

Letting $u = -ti$ where $t \in \mathbb{R}_+$ we see that $u^3/3 + u^2 + s = s + \frac{4u^3}{3} - t^2 \neq 0$ since its imaginary part is positive. Similarly letting $u = -2 - ti$ we have $u^3/3 + u^2 + s = s + \frac{4u^3}{3} + t^2 + \frac{4}{3} \neq 0$. Thus neither $r_3(s)$ nor $r_1(s)$ crosses the line $\text{Re} z = 0$ or $\text{Re} z = -2$. This together with [52] shows that $\text{Re} r_3(s) > 0$ and $\text{Re} r_1(s) < -2$ for all $s \in \mathbb{H}$. 

ASYMPTOTICALLY CONSERVED QUANTITIES AND CONNECTION FORMULAE 41
Comparing (39) we see that \( \tilde{r}_3 = r_3 \) and \( \tilde{r}_1 = r_1 \). Finally, for small \( s \in \mathbb{H} \), by (39), \( r_2 \) is between \( l_1 = \{ t(1 + i) : t \geq 0 \} \) and \( l_2 = \{ -2 + t(-1 + i) : t \geq 0 \} \). On the other hand, for \( s \in \mathbb{H} \), \( \text{Im} P_s(t(2 + 3i)) = \text{Im} \left[ s - \left( \frac{46}{3} - 3i \right) t^3 - (5 - 12i)t^2 \right] > 0 \) and similarly, \( \text{Im} P_s(-2 + t(-2 + 3i)) > 0 \). Thus \( r_2 \) stays in between the two rays \( l_1, l_2 \) for all \( s \in \mathbb{H} \).

(iii) Real analyticity in \( t \) follows again from the IFT. Near \( t = 0 \), the IFT applied to the equations \( v \pm (-tv^3/3 - s)^{1/2} = 0 \) at \( r = \mp\sqrt{-s}, t = 0 \) implies the existence of two roots of \( tv^3/3 + v^2 + s \) analytic in \( t \). When \( s \) is close to \(-4/3 \) the result follows from the symmetry \( T \).

(iv) This simply follows from the fact that for \( |u| \geq 399/100 \) we have \( |u^3/3 + u^2| \geq |(399/100)^3/3 - (399/100)^2| > 21/4 \). \( \square \)

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