The continuation method and the real analyticity of the accessory parameters: the general elliptic case

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Abstract
We apply the Le Roy–Poincaré continuation method to prove the real analytic dependence of the accessory parameters on the position of the sources in the Liouville theory in presence of any number of elliptic sources. The treatment is easily extended to the case of the torus with any number of elliptic singularities. A discussion is given of the extension of the method to parabolic singularities and higher genus surfaces.

Keywords: Liouville theory, accessory parameters, conformal field theory

1. Introduction

The accessory parameters first appeared in the Riemann–Hilbert problem asking for an ordinary differential equation whose solutions transform according to a given monodromy group [1]. They reappear in Liouville theory in the quest for an auxiliary differential equation in which all elements of the monodromy group belong to SU(1, 1). Such a request is the necessary and sufficient condition for having a single valued conformal Liouville field. Their determination also play a crucial role in 2 + 1 dimensional gravity [2] in presence of matter. This is also connected to the Polyakov relation which relates such accessory parameters to the variation of the on-shell action of Liouville theory under the change in the position of the sources [2–6]. They appear again in the classical limit of the conformal blocks of the quantum conformal theory [7–14].

In several developments it is important to establish the nature of the dependence of such accessory parameters on the source positions and on the moduli of the theory. To this end we have the result of Kra [15] which in the case of the sphere topology in presence of only parabolic and finite order elliptic singularities proved that such a dependence is real analytic.
(not analytic). The technique used to reach such a result was that of the Fuchsian mapping, a method which cannot be applied to the case of general elliptic singularities.

On the other hand in the usual applications, general elliptic, not finite order elliptic singularities appear. Finite order singularities are those for which the source strength is given by $\eta_k = (1 - 1/n)/2, n \in \mathbb{Z}^+$ (see section 2).

In the case when only one independent accessory parameter is present, like the sphere topology with four sources or the torus with one source, it was proven that such accessory parameters are real analytic functions of the source position or moduli, almost everywhere (i.e. everywhere except for a zero measure set) in the source position or moduli space [10, 14, 16, 17]. The qualification almost everywhere implies e.g. that we could not exclude the presence of a number of cusps in the dependence of the accessory parameters on the source positions, a phenomenon which may be expected in the solution of a system of implicit equations.

This result was obtained by applying complex variety techniques to the conditions which impose the SU(1, 1) nature of all the monodromies. In [16] an extension of such a technique was attempted to the case of two independent accessory parameters, like the sphere with five sources and the torus with two sources but results where obtained only under an irreducibility assumption.

The usual approach to the solution of the Liouville equation is the variational approach. Such an approach was suggested by Poincaré in [18] but not pursued by him due to some difficulties in proving the existence of the minimum of a certain functional. The variational approach was developed with success by Lichtenstein [19] and in a different context by Troyanov [20] by writing the conformal field as the sum of a proper background and a remainder. With such a splitting the problem is reduced to the search of the minimum of a given functional. One proves that such a minimum exists and solves the original problem [19–21]. Poincaré in [18] pursued and solved the same problem by means of a completely different procedure which became known as the Le Roy–Poincaré continuation method [22, 23].

The idea is to write the solution of the Liouville equation as a power series expansion in certain properly chosen parameters. Such a series turns out to be uniformly convergent over all the complex plane or Riemann surface.

This cannot be achieved in a single step. Once one has solved the equation with one of such parameter in a certain region one uses the obtained solution as the starting point of an other series in another parameter and thus at the end one has the solution as a series of series, each uniformly convergent.

The procedure is more lengthy than the variational approach but has the advantage that one can follow the dependence of each series on the input, the input being the Lichtenstein background field.

Such a field, to be called $\beta$ is a real positive function smooth everywhere except at the source positions, the singularity being characterized by the nature and the strength of the sources; apart from these requirements the choice of $\beta$ is free. Thus except at the singularities $\beta$ is a smooth, say $C^\infty$ function. The uniqueness theorem [19, 21] tells us that the final result does not depend on the specific choice of $\beta$.

Simple smoothness would not be a good starting point for proving the real analytic dependence of the result; on the other hand as we shall see, it is possible to provide a background field $\beta$ satisfying all the Lichtenstein requirements and real analytic in the moduli except obviously at the sources.

Starting from such a $\beta$ one sees that the zero order approximation in the Poincaré procedure gives rise to a conformal field which is real analytic in the position of the sources $z_k$ and in the argument $\zeta$ except at the source positions $z_k$. The problem is to show that such real analyticity properties are inherited in all the power expansion procedures and finally by
the conformal factor itself. This is what is proved in this paper in presence of any number of elliptic singularities. The final outcome is that the conformal factor depends in real analytic way both on the argument $z$ of the field and on the source positions. Once this result is established is not difficult to express the accessory parameters in terms of the conformal field and prove the real analytic dependence of the accessory parameters themselves on the source positions.

The procedure of [18] of writing the solution to the Liouville equation in terms of uniformly convergent series is quite powerful. The result on the real analyticity of the conformal factor is more general that the analyticity of the accessory parameters, being the last fact a simple consequence of the previous. The method may be applied to other problems both in deriving qualitative properties and quantitative bounds.

The paper is structured as follows. In section 2 we describe the Lichtenstein decomposition and provide a background field $\beta$ which is real analytic everywhere except at the sources. In section 3 we give the Poincaré procedure for the solution of the Liouville equation and in the following section 4 we give the method of solution for a class of linear inhomogeneous equation which appear in section 3. In section 5 we prove how the real analytic properties of the background field $\beta$ are inherited in all the iteration process and finally by the solution i.e. the Liouville field. In section 6, using the obtained result we prove the real analytic dependence of the accessory parameters on the source positions for the sphere topology for any number of general elliptic singularities. In section 7 we give the extension of the result to the torus with any number of sources.

Finally in section 8 we discuss the perspectives for the extension of the method to the parabolic singularities and higher genus. To make the paper more readable we have relegated in an appendix the proof of some technical lemmas which are employed in the text.

2. The Lichtenstein decomposition

The Liouville equation is

$$\Delta \phi = e^\phi$$

(1)

with the boundary conditions at the elliptic singularities

$$\phi + 2\eta_k \log |z - z_k|^2 = \text{bounded}, \quad \eta_k < \frac{1}{2}$$

(2)

and at infinity

$$\phi + 2 \log |z|^2 = \text{bounded}.$$  

(3)

The procedure starts by constructing a positive function $\beta$ everywhere smooth except at the sources where it obeys the inequalities

$$0 < \lambda_m < \beta |z - z_k|^{4\eta_k} < \lambda_M$$

(4)

and for $|z| > \Omega$, $\Omega$ being the radius of a disk which include all singularities

$$0 < \lambda_m < \beta |z|^4 < \lambda_M.$$  

(5)

Note that $\int \beta(z) d^2z < \infty$. In addition $\beta$ will be normalized as to have

$$- \sum_k 2\eta_k + \frac{1}{4\pi} \int \beta(z) d^2z = -2.$$  

(6)
for the sphere topology.

Apart from these requirements $\beta$ is free and due to the uniqueness theorem the final result for the field $\phi$ does not depend on the specific choice of $\beta$. On the other hand, as discussed in the introduction, it will be useful to start from a $\beta$ which is real analytic both in $z$ and in the source positions $z_k$, except at the sources. One choice is

$$\beta = c \prod_k \left[ \frac{(z - z_k)(\bar{z} - \bar{z}_k)}{1 + z z_k} \right]^{-2 \eta_k}, \quad \sigma = \sum_k \eta_k, \quad (7)$$

where the positive constant $c$ has to be chosen as to comply with the sum rule (6). Picard inequalities require the presence of at least three singularities, the case of three singularities being soluble in terms of hypergeometric functions. As is well known, by performing a projective transformation we can set $z_1 = 0, z_2 = 1, z_3 = i$. We shall be interested in the dependence of the accessory parameters on a given $z_k$ keeping the other fixed; we shall call such a source position $z_4$. Obviously (7) is not the only choice but it is particularly simple. Varying the position $z_4$ around a given initial position we shall need to vary the $c$ in order to keep (6) satisfied. It is easily seen that such $c$ depends on $z_4$ in real analytic way (see appendix, lemma 1). Given $\beta$ one constructs the function [19, 21]

$$\nu(z) = \phi_1(z) + \frac{1}{4\pi} \int \log |z - z'|^2 \beta(z') d^2z' \equiv \phi_1(z) + I(z) \quad (8)$$

with

$$\phi_1 = \sum_k (-2\eta_k) \log |z - z_k|^2 \quad (9)$$

and we define $u$ by

$$\phi = \nu + u. \quad (10)$$

With such a definition the Liouville equation becomes

$$\Delta u = e^\nu e^u - \beta \equiv \theta e^u - \beta. \quad (11)$$

The real analyticity of $\beta$ and $\theta$ need a little discussion. We recall that a real analytic function can be defined as the value assumed by an analytic function of two variables $f(z, z')$ when $z'$ assumes the value $z$. Equivalently it can be defined as a function of two real variables $x$ and $y$ which locally can be expanded in a convergent power series

$$f(x + \delta x, y + \delta y) - f(x, y) = \sum_{m,n} a_{m,n} \delta x^m \delta y^n. \quad (12)$$

In equation (7) we can write

$$[(z - z_k)(\bar{z} - \bar{z}_k)]^{-2 \eta_k} = [(x - x_k)^2 + (y - y_k)^2]^{-2 \eta_k} \quad (13)$$

which around a point $x, y$ with $x \neq x_k$ and/or $y \neq y_k$ can be expanded in a power series, obviously with bounded convergence radius. The function $\nu$ and consequently the function $\theta$ contain $\beta$ in the form

$$e^\nu = e^{\phi_1 + \frac{1}{4\pi} \int \log |z - z'|^2 \beta(z') d^2z'} = \prod_k [(z - z_k)(\bar{z} - \bar{z}_k)]^{-2 \eta_k} e^{I(z)}. \quad (14)$$
As we shall keep all \( z_k \) fixed except \( z_4 \) we shall write
\[
I(z, z_4) = \frac{1}{4\pi} \int \log|z - z'|^2 \beta(z', z_4) d^2z'
\] (15)
and \( \nu(z, z_4) \) for \( \nu \).

The analytic properties of \( I(z, z_4) \) both in \( z \) and \( z_4 \) are worked out in lemma 2 of the appendix.

3. The Poincaré procedure

After performing the decomposition of the field \( \phi \) as \( \phi = u + \nu \) the Liouville equation becomes
\[
\Delta u = \theta e^u - \beta \quad \text{with} \quad \theta = e^\nu \equiv r\beta
\] (16)
and as a consequence of the inequalities (4, 5) we have
\[
0 < r_1 < r < r_2
\] (17)
for certain \( r_1, r_2 \).

Let \( \alpha \) be the minimum
\[
\alpha = \min \left( \frac{\beta}{\theta} \right) = \frac{1}{\max r}
\] (18)
which due to (17) is a positive number. Then we can rewrite the equation as
\[
\Delta u = \theta e^u - \alpha \theta - \beta (1 - \alpha r).
\] (19)
As a consequence of the choice for \( \alpha \) we have \( \psi \equiv \beta (1 - \alpha r) \geq 0 \).

Convert the previous equation to
\[
\Delta u = \theta e^u - \alpha \theta - \lambda \psi
\] (20)
and write
\[
u = u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots
\] (21)
We have to solve the system
\[
\Delta u_0 = \theta (e^{u_0} - \alpha)
\]
\[
\Delta u_1 = \theta e^{u_0} u_1 - \psi
\]
\[
\Delta u_2 = \theta e^{u_0} (u_2 + w_2)
\]
\[
\Delta u_3 = \theta e^{u_0} (u_3 + w_3)
\]
\[
\ldots,
\] (22)
where
\[
w_2 = \frac{u_1^2}{2}, \quad w_3 = \frac{u_1^3}{6} + u_1 u_2, \quad w_4 = \frac{u_1^4}{24} + \frac{u_1^2 u_2 + u_3^2}{2} + u_1 u_3, \ldots
\] (23)
are all polynomials with positive coefficients. We see that in the \( n \)th equation the \( w_n \) is given in terms of \( u_k \) with \( k < n \) and thus each of the equation (22) is a linear equation.
Thus the previous is a system of linear inhomogeneous differential equation for the $u_k$. The first equation is solved by $u_0 = \log \alpha$. We shall see in the next section that each of the following equations in (22) can be solved by iterated power series expansion and that all the $u_k$ are bounded. From the properties of the Laplacian $\Delta$ and equation (22) we have

$$|u_1| \leq \max \left( \frac{\psi}{\epsilon^{u_0 \theta}} \right)$$
$$|u_2| \leq \max |w_2|$$
$$\ldots$$
$$|u_k| \leq \max |w_k|$$
$$\ldots$$

(24)

If at $z = z_{\text{max}}$, where $z_{\text{max}}$ is the point where $|u_k|$ reaches its maximum, $\Delta u_k$ is finite the above inequalities follow from the well known properties of the Laplacian. At the singular points it may happen that the Laplacian diverges but the inequalities (24) still hold. In fact if the maximum of $|u_k|$ is reached at the singular point $z_l$, with $u_k(z_l) > 0$ and the rhs in equation (22) is definite positive in a neighborhood of $z_l$, then the circular average

$$\frac{1}{2\pi} \int u_k(z_l + \rho e^{i\phi}) d\phi \equiv \bar{u}(\rho)$$

(25)

has a positive definite source. Thus it is increasing with $\rho$ which contradicts the fact that $u_k(z_l)$ is the maximum. The same reasoning works if at $z_l$ we have $u_k(z_l) < 0$.

Using the above inequalities one proves [18] (appendix, lemma 4) that the series (21) converges for

$$|\lambda| < \frac{\alpha(\log 4 - 1)}{\max |\psi|}$$

(26)

and such convergence is uniform.

It is not difficult to show [18] using the results of lemma 3 of the appendix, that the convergent series satisfies the differential equation (20) i.e. that one can exchange in (20) the Laplacian with the summation operation.

Thus we are able to solve the equation

$$\Delta u = \theta e^u - \alpha \theta - \lambda_0 \psi$$

(27)

for

$$0 < \lambda_0 < \frac{\alpha(\log 4 - 1)}{\max |\psi|}.$$  

(28)

If $\lambda_0$ can be taken equal to 1 the problem is solved. Otherwise one can extend the region of solubility of our equation by solving the equation

$$\Delta u = \theta e^u - \theta \alpha - \lambda_0 \psi - \lambda \psi \equiv \theta e^u - \varphi - \lambda \psi.$$  

(29)

Expanding as before in $\lambda$ one obtains \[6\]
\[ \Delta u_0 = \theta e^{u_0} - \varphi \]
\[ \Delta u_1 = \theta e^{u_0} u_1 - \psi \]
\[ \Delta u_2 = \theta e^{u_0} (u_2 + w_2) \]
\[ \Delta u_3 = \theta e^{u_0} (u_3 + w_3) \]
\[ \ldots \] \hspace{1cm} (30)

From the first equation using \( \varphi > 0 \) we have
\[ \min e^{u_0} > \min \left( \frac{\varphi}{\theta} \right) \] \hspace{1cm} (31)
and thus from the second
\[ \max |u_1| \leq \max \left| \frac{\psi}{\theta e^{u_0}} \right| \leq \max \left| \frac{\psi}{\theta} \right| \min \left( \frac{1}{\psi} \right) = \max \left| \frac{\psi}{\theta} \right| \min \left( \frac{\alpha + \lambda_0 \psi}{\alpha} \right) < \max \left| \frac{\psi}{\alpha} \right| . \] \hspace{1cm} (32)

Then following the procedure of the previous step we have convergence for
\[ |\lambda| < \frac{\alpha(\log 4 - 1)}{\max |\frac{\psi}{\theta}|} . \] \hspace{1cm} (33)
This is the same bound as (26) and thus repeating such extension procedure, in a finite number of steps we reach the solution of the original equation (16). We shall call these steps extension steps.

4. The equation \( \Delta u = \eta u - \varphi \)

In the previous section we met the problem of solving linear equations in \( u \) of the type
\[ \Delta u = \theta e^{U} u - \varphi, \] \hspace{1cm} (34)
where \( U \) is provided by the solution of a previous equation. Here we give the procedure for obtaining the solution of the more general equation
\[ \Delta u = \eta u - \varphi, \] \hspace{1cm} (35)
where \( \eta \) is positive and has the same singularities as \( \theta \) in the sense that \( 0 < c_1 < \frac{\eta}{\theta} < c_2 \) [18].

We start noticing that due to the positivity of \( \eta \), if \( u \) and \( v \) are two solutions of (35) then we have
\[ \int (u - v) \Delta (u - v) d^2 z = - \int \nabla (u - v) \cdot \nabla (u - v) d^2 z = \int \eta (u - v)^2 d^2 z = 0 \] \hspace{1cm} (36)
i.e. \( u = v \). To construct the solution one considers the equation
\[ \Delta u = \lambda \eta u - \varphi_0 - \lambda \psi \] \hspace{1cm} (37)
with \( \int \varphi_0 d^2 z = 0 \) and writes the \( u \) as
\[ u = (u_0 + c_0) + \lambda (u_1 + c_1) + \lambda^2 (u_2 + c_2) + \ldots \] \hspace{1cm} (38)
and then we have
\[
\begin{align*}
\Delta u_0 &= -\varphi_0 \\
\Delta u_1 &= \eta(u_0 + c_0) - \psi \\
\Delta u_2 &= \eta(u_1 + c_1) \\
\Delta u_3 &= \eta(u_2 + c_2)
\end{align*}
\]
\ldots
(39)
where the \( u_k \) are simply given by
\[
u_k = \frac{1}{4\pi} \int \log |z - z'|^2 s_k(z') d^2 z'
\]
being \( s_k \) the sources in equation (39). Due to the compactness of the domain, i.e. the Riemann sphere, equations of the type \( \Delta u = s \) are soluble only if \( \int s d^2 z = 0 \). The solutions of the \( \Delta u = s \) are determined up to a constant, a fact which has been explicitly taken into account in (38).

Then the \( c_k \) are chosen as to have the integral of the rhs of the equations in (39) equal to zero.
\[
\begin{align*}
c_0 \int \eta d^2 z &= \int \psi d^2 z - \int \eta u_0 d^2 z \\
c_1 \int \eta d^2 z &= - \int \eta u_1 d^2 z \\
c_2 \int \eta d^2 z &= - \int \eta u_2 d^2 z \\
\ldots
\end{align*}
\]
(41)
Thus we have \( |c_k| < \max |u_k| \) for \( k \geq 1 \). On the other hand we have from the inequality proven in lemma 2 of the appendix
\[
\max |u_2| \leq B \max |u_1 + c_1|
\]
from which
\[
\max |u_2 + c_2| \leq 2 \max |u_2| < 2B \max |u_1 + c_1|
\]
and similarly for any \( k \). Thus the series converges uniformly for \( |\lambda| < \frac{1}{2B} \). Again one can easily prove [18] using the results of lemma 3 of the appendix, that one can exchange the summation operation with the Laplacian and thus the series satisfies the differential equation (37). It is important to notice that the convergence radius does not depend on \( \varphi \). Then chosen any \( \lambda_1, \)

\[
0 < \lambda_1 < \frac{1}{2B}
\]
we can solve for any \( \varphi \)
\[
\begin{align*}
\Delta u &= \lambda_1 \eta u - \varphi_0 \equiv \lambda_1 \eta u - \varphi_0 - \psi
\end{align*}
\]
(44)
as the power expansion in \( \lambda \) of
\[
\begin{align*}
\Delta u &= \lambda \eta u - \varphi_0 - \frac{\lambda}{\lambda_1} \psi
\end{align*}
\]
(45)
converges for \( \lambda = \lambda_1 \).
Thus if $\frac{1}{2B} > 1$ the problem is solved. Otherwise one can extend the region of convergence in the following way.

Chosen $0 < \lambda_1 = \frac{1}{2B} - \varepsilon$ we consider the equation

$$\Delta u = \lambda_1 \eta u + \lambda \eta u - \varphi .$$

(46)

We are already able to solve

$$\Delta u = \lambda_1 \eta u - \varphi$$

(47)

and thus we shall expand in $\lambda$

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots$$

(48)

with

$$\Delta u_0 = \lambda_1 \eta u_0 - \varphi$$

$$\Delta u_1 = \lambda_1 \eta u_1 + \eta u_0$$

$$\Delta u_2 = \lambda_1 \eta u_2 + \eta u_1$$

$$\Delta u_3 = \lambda_1 \eta u_3 + \eta u_2$$

$$\ldots$$

(49)

all of which are of the form (44) and thus we are able to solve.

To establish the convergence radius in $\lambda$ we use the fact that in the solution of (49) we have

$$|u_{k+1}| < \frac{1}{\lambda_1} \max |u_k|, \quad k \geq 1$$

(50)

and thus we have uniform convergence of the series in $\lambda$ for $|\lambda| < \lambda_1$. We repeat now the procedure starting from the equation

$$\Delta u = \lambda_1 \eta u + \lambda_2 \eta u + \lambda \eta u - \varphi$$

(51)

with $0 < \lambda_2 < \lambda_1$ which is solved again by expanding in $\lambda$. From the same argument as before the convergence radius in $\lambda$ is

$$\lambda_1 + \lambda_2$$

(52)

which is even larger than the convergence radius in $\lambda_2$ and thus in a finite number of extension steps we are able to solve

$$\Delta u = (\lambda_1 + \lambda_2 + \ldots + \lambda_n) \eta u - \varphi$$

(53)

with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ which is our original equation (35).

5. The inheritance of real analyticity

Not to overburden the notation we shall write $f(z, z_4)$ for $f_{z_4}(z, z, z_4, \bar{z}_4)$ at $z' = \bar{z}$ and $z_4' = \bar{z}_4$ with $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$ and $\frac{\partial}{\partial z_4} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$. 

We need the detailed structure of the most important function which appears in the iteration procedure i.e. of $\theta = \beta r$. We are interested in the problem when $z_4$ varies in a domain $D_1$ around a $z_4^0$, say $|z_4 - z_4^0| < R_1$ which excludes all others singularities. We choose $R_4$ equal to $1/4$ the minimal distance of $z_4^0$ from the singularities $z_k, k \neq 4$. We know that $0 < r_1 < r < r_2$ where the bounds $r_1$ and $r_2$ can be taken independent of $z_4$ for $z_4 \in D_4$. For the function $\beta(z, z_4)$ we point out the uniform bound for $|z| > 2 \max |z_4|$ and $z_4 \in D_4$

$$\beta = \frac{c(z_4)}{(1 + \bar{z_4}z)^2} \left( 1 + \frac{1}{2z} \right)^{-2\frac{\sigma}{\gamma}} \left( 1 + \frac{1}{z} \right)^{-\frac{2\sigma}{\gamma}} \int \left( 1 + \frac{1}{2z} \right)^2 \int \left( 1 + \frac{1}{z} \right)^{-\frac{2\sigma}{\gamma}}$$

$$\int \frac{\text{const}}{(1 + z)^2} \left( 1 + \frac{1}{z} \right)^{-\frac{2\sigma}{\gamma}}$$

(54)

with $+$ or $-$ according to $\eta_k < 0$ or $\eta_k > 0$.

The function $\theta(z, z_4)$ is explicitly given by

$$\theta(z, z_4) = \prod_k (1 - z_i(z - z_4))(z - z_4)$$

(55)

where

$$I(z, z_4) = \frac{1}{4\pi} \int \log |z - z'|^2 \beta(z', z_4)|^2$$

(56)

In dealing with integrals of the type (56) to avoid the appearance of non integrable functions in performing the derivative w.r.t. $z_4$ it is instrumental to isolate a disk $R_1$ around $z_4$ of radius $R_1$ that for $z_4 \in D_4$ contains only the singularity $z_4$ and not the others $z_k$.

For the function $u(z)$ and $z_4$ it is useful to write for $|z - z_4| < R_1$, $u(z, z_4) = u(z, z_4)$ with $\zeta = z - z_4$ and thus also $\theta(z, z_4) = \theta(z, z_4)$ for $|\zeta| < R_1$. Thus for $|z - z_4| < R_1$ we shall have denoting with $R_{1\epsilon}$ the complement of $R_1$

$$I(z, z_4) = \frac{1}{4\pi} \int \log |\zeta - \zeta'|^2 \beta(z', z_4)|^2$$

$$+ \frac{1}{4\pi} \int \log |\zeta + z_4 - \zeta'|^2 \beta(z', z_4)|^2$$

(57)

We shall also consider an other disk centered in $z_4$ with radius $R_2 < R_1$ and write for

$$|z - z_4| > R_2$$

$$I(z, z_4) = \frac{1}{4\pi} \int \log |z - z_4 - \zeta|^2 \beta(z', z_4)|^2$$

$$+ \frac{1}{4\pi} \int \log |z - z_4|^2 \beta(z', z_4)|^2$$

(58)

In lemma 2 of the appendix it is proven that $I(z, z_4)$ equation (56), is continuous and it is real analytic in $z$ for $z \neq z_4$ and that $I(\zeta, z_4)$ equation (57) is real analytic in $z_4$ for $z_4 \in D_4$ and $I(z, z_4)$ equation (58) real analytic in $z_4 \in D_4$.

The typical transformation we where confronted with in the previous sections was

$$u(z, z_4) = \frac{1}{4\pi} \int \log |z - \zeta|^2 \beta(z', z_4)|^2$$

(59)

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For $|z - z_4| < R_1$ we have

$$\hat{u}(\zeta, z_4) = \frac{1}{4\pi} \int_{R_1} \log |\zeta' - \zeta|^2 \hat{\theta}(\zeta', z_4) \hat{s}(\zeta', z_4) d^2 \zeta'$$
$$+ \frac{1}{4\pi} \int_{R_{1c}} \log |\zeta + z_4 - z'|^2 \hat{\theta}(\zeta', z_4) s(\zeta', z_4) d^2 \zeta'$$

(60)

and for $|z - z_4| > R_2$ we have

$$u(z, z_4) = \frac{1}{4\pi} \int_{R_2} \log |z - z'|^2 \hat{\theta}(\zeta', z_4) \hat{s}(\zeta', z_4) d^2 \zeta'$$
$$+ \frac{1}{4\pi} \int_{R_{2c}} \log |z - \zeta|^2 \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta'.$$

(61)

We recall that we work under the condition

$$\int \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta' = 0$$

(62)

which can also be written as

$$\int \hat{\theta}(\zeta', z_4) \hat{s}(\zeta', z_4) d^2 \zeta' + \int \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta' = 0.$$  

(63)

A consequence of relation (62) is that we can work also with

$$\hat{u}(\zeta, z_4) = \frac{1}{4\pi} \int_{R_1} \log \left| 1 - \frac{\zeta'}{\zeta} \right|^2 \hat{\theta}(\zeta', z_4) \hat{s}(\zeta', z_4) d^2 \zeta'$$
$$+ \frac{1}{4\pi} \int_{R_{1c}} \log \left| 1 + \frac{z_4 - z'}{\zeta} \right|^2 \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta',$$

(64)

$$u(z, z_4) = \frac{1}{4\pi} \int_{R_2} \log \left| 1 - \frac{\zeta'}{z} \right|^2 \hat{\theta}(\zeta', z_4) \hat{s}(\zeta', z_4) d^2 \zeta'$$
$$+ \frac{1}{4\pi} \int_{R_{2c}} \log \left| 1 - \frac{z'}{z} \right|^2 \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta'.$$

(65)

This last form is useful in investigating the behavior of $u(z, z_4)$ at $z = \infty$.

We shall now show that some boundedness and real analyticity properties of the source $s(z, z_4)$ are inherited by $u(z, z_4)$ through the transformation (59). We shall always work with $z_4 \in D_4$ where $D_4$ was described at the beginning of the present section and does not contain any other singularity $z_k$. The real analyticity is proven by showing the existence of the complex derivatives w.r.t. $z$ and $\bar{z}$ or w.r.t. $z_4$ and $\bar{z}_4$. Due to the symmetry of the problem it is sufficient to prove analyticity w.r.t. $z$ and $z_4$.

We come now to the main theorem of this section

**Theorem 5.1.** Given the transformation

$$u(z, z_4) = \frac{1}{4\pi} \int \log |z - z'|^2 \theta(\zeta', z_4) s(\zeta', z_4) d^2 \zeta'.$$

(66)
with
\[
\int \theta(\zeta', z_4) s(\zeta', z_4) d\zeta' = 0 \tag{67}
\]
if \(s(z, z_4)\) satisfies the properties

- **P1** \(s\) is bounded and continuous in \(z, z_4, z_4 \in D_4\)
- **P2** \(\hat{s}(\zeta, z_4) \equiv s(\zeta + z_4, z_4)\) is analytic in \(\zeta\) for \(|\zeta| < R_1, \zeta \neq 0\).
- **P3** \(\hat{s}(\zeta, z_4)\) is analytic in \(z_4\) with \(\frac{\partial \hat{s}(\zeta, z_4)}{\partial z_4}\) bounded for \(z_4 \in D_4, |\zeta| < R_1\).
- **P4** \(s(z, z_4)\) is analytic in \(z\) for \(|z - z_4| > R_2, z = \infty\) included, except at \(z = z_4\).
- **P5** \(s(z, z_4)\) is analytic in \(z_4\) with \(\frac{\partial s(z, z_4)}{\partial z_4}\) bounded for \(z_4 \in D_4, |z - z_4| > R_2\).

then \(u(z, z_4)\) satisfies the same properties P1–P5.

**Proof.** The inheritance of property P1 is a consequence of the inequality proven in lemma 3 of the appendix.

The inheritance of properties P2 and P4 is proved by computing the derivative w.r.t. \(z\) using the method employed in lemma 3 of the appendix when dealing with the derivative of \(I(\zeta, z_4)\) and using the analyticity and boundedness of \(s(z, z_4)\).

As for P3 we shall use the expression (60) for \(\hat{u}, \hat{\theta}\) has the following structure
\[
\hat{\theta}(\zeta, z_4) = (\zeta'' + z_4)^{-2n} \prod_{k \neq 4} (|\zeta + z_4 - z_k|^2)^{-2n} \phi^2(\zeta' - z_4), \quad |\zeta| < R_1. \tag{68}
\]

With respect to the first term in (60), in taking the derivative w.r.t. \(z_4\) one easily sees that the conditions are satisfied for taking the derivative under the integral sign, for all \(\zeta\), provided \(s\) and \(\frac{\partial s}{\partial z_4}\) be bounded in \(R_1 \times D_4\), i.e. properties P1 and P3.

In fact the derivative of the product in equation (68) w.r.t. \(z_4\) is regular and the derivative of the exponential boils down to the derivative of \(I\) which we have shown in lemma 2 of the appendix to be analytic in \(z_4\) for all \(\zeta\) in \(R_1\).

Then we have to differentiate \(\hat{s}(\zeta, z_4)\) w.r.t. \(z_4\). As the \(\frac{\partial s}{\partial z_4}\) is uniformly bounded in \(R_1\), property P3, such differentiation under the integral sign is legal.

In taking the derivative w.r.t. \(z_4\) of the second term in (60) we must take into account the fact that the integration region \(R_{1z}\) moves as \(z_4\) varies. Then the derivative of the second integral appearing in (60) is
\[
\frac{1}{4\pi i} \int_{\partial R_{1z}} \frac{\partial}{\partial z_4} \left[ \log |\zeta + z_4 - \zeta'|^2 \theta(\zeta', z_4)s(\zeta', z_4) \right] d\zeta'
- \frac{1}{8\pi i} \int_{\partial R_{1z}} \log |\zeta + z_4 - \zeta'|^2 \theta(\zeta', z_4)s(\zeta', z_4)dz'. \tag{69}
\]

The logarithms in the above equation are not singular for \(\zeta \in R_1\) which makes the differentiation under the integral sign legal.

We come now to the \(u(z, z_4)\) with \(|z - z_4| > R_2\) where we use expression (61). In taking the derivative w.r.t. \(z_4\) the first integral does not present any problem as \(z \in R_2\) and \(\zeta \in R_2\) and thus the logarithm is non singular. The second integral gives two contributions, being the first provided by the derivative of \(\theta s\) which gives rise to an integrand bounded by an absolutely integrable function, independent of \(z_4\) for \(z_4 \in D_4\) and a contour integral due to the motion of \(R_{2z}\) as \(z_4\) varies.
We are left to examine the neighborhood of \( z = \infty \) which, with the behavior (5) for the \( \beta \) at infinity and the consequent behavior of the \( \theta \), is a regular point. We have with \( \tilde{u}(x, z_4) = u(1/x, z_4) \) and \( \tilde{\theta}(y, z_4) = \theta(1/y, z_4) \) and using (65)

\[
\tilde{u}(x, z_4) = \frac{1}{4\pi} \int \log |x - y|^2 \frac{\tilde{\theta}(y, z_4)}{(yy)^2} d^2y - \frac{1}{4\pi} \int \log |y|^2 \frac{\tilde{\theta}(y, z_4)}{(yy)^2} d^2y. \tag{70}
\]

Exploiting the analyticity of \( \tilde{\theta}(y, z_4)/(yy)^2 \) for \( |y| < 1/\Omega \) we have that (70) has complex derivative w.r.t. \( x \) for \( |x| < 1/\Omega \) thus proving the analyticity in \( x \) of \( \tilde{u} \) around \( x = 0 \).

From the previous equation (70) we see that \( \tilde{u}(x, z_4) \) is analytic in the polydisk \( |x| < 1/\Omega \) and \( z_4 \in D_4 \). This assures not only that \( u \) at infinity is bounded, a result that we knew already from the treatment of sections 3, 4 and lemma 3 of the appendix but that \( \frac{\partial u}{\partial z_4} \) is uniformly bounded for all \( z \) with \( |z - z_4| > R_2 \). Thus we have reproduced for \( u \) the properties P1–P5 completing the proof of the theorem.

We have now to extend the properties P1–P5 to all the \( u_k \) of sections 3 and 4 and to their sum.

**Theorem 5.2.**  The properties P1–P5 hold for all \( u_k \) of sections 3 and 4 and for their sums.

**Proof.**  First of all we notice that in solving equation (37) the constants \( c_k \) intervene. These are given by equation (41) i.e. by the ratio of two integrals where the one which appears at the denominator never vanishes. The real analytic dependence on \( z_4 \) of the denominator \( \int \eta d^2 \) is established by the method provided in lemma 1 of the appendix while the derivative of the numerator is again computed by splitting the integration region as \( R \cup R_c \) and using the fact that \( u_k \) and \( \frac{\partial u_k}{\partial z_4} \) are bounded.

To establish the real analyticity of the sum of the series we shall exploit the well known result that given a sequence of analytic functions \( f_n \) defined in a domain \( \Omega \) which converge to \( f \) uniformly on every compact subset of \( \Omega \) then their sum is analytic in \( \Omega \) and the series of the derivatives \( f'_n \) converge uniformly to \( f' \) on every compact subset of \( \Omega \).

We saw in sections 3 and 4 that in general more than one extension step is required to reach the complete solutions of equations (16) and (35) but these steps are always finite in number. Let us consider first the case in which a single step is sufficient

Then we have explicitly

\[
\Delta u = \theta e^u - \beta \tag{71}
\]

\[
u = \sum_{k=0}^{\infty} \lambda_k^1 u_k \tag{72}
\]

\[
u_0 = \log \alpha \tag{73}
\]

\[
\Delta u_1 = \alpha \theta u_1 - \psi \tag{74}
\]

\[
\Delta u_k = \alpha \theta(u_k + w_k), \quad k \geq 2 \tag{75}
\]

\[
u_k = \sum_{h=0}^{\infty} \lambda_k^h u_k,h. \tag{76}
\]
Now we climb back the above sequence. Starting from
\[
\Delta u_{1,0} = -\varphi_0 \\
\Delta u_{1,1} = \eta(u_{1,0} + c_{1,0}) - \psi \\
\Delta u_{1,2} = \eta(u_{1,1} + c_{1,1}) \\
\Delta u_{1,3} = \eta(u_{1,2} + c_{1,2}) \\
\ldots
\]
and applying the inheritance result proven above in theorem 5.1 we have that \( u_{1,k} \) are bounded in \( D_4 \) together with \( \partial_{u_{1,0}} \B_{i3} \), \( \partial_{u_{1,1}} \B_{i3} \). The convergence is uniform and thus we have that the sum i.e. \( u_1 = \sum_{h=0}^{\infty} \lambda_i^4 u_{1,h} \) is real analytic and bounded in \( D_4 \) and that \( \partial_{u_{1,0}} \B_{i3}, \partial_{u_{1,1}} \B_{i3} \) are bounded in \( |z_4 - z_4^0| \leq R_4 - \frac{\varepsilon}{2} \). The function \( u_2 \) is obtained by solving (75) where we recall that the source \( w_2 \) depends only on \( u_1 \) and \( w_k \) depends only on \( u_r \) with \( r < k \), in polynomial and thus analytic way.

Repeating the previous reasoning for \( u_2 \) we have analyticity and boundedness of \( u_2 \) and its derivative in \( |z_4 - z_4^0| \leq R_4 - \frac{\varepsilon}{2} \) and thus boundedness of \( u = \sum_{k=0}^{\infty} \lambda_i^k u_k \) in \( |z_4 - z_4^0| < R_4 - \frac{\varepsilon}{2} \) with its derivative bounded in \( |z_4 - z_4^0| < R_4 - \varepsilon \). Similarly one extends the analyticity of \( u_k \) in \( \zeta \) for \( \zeta \neq 0 \) and of \( u_k \) in \( z \), \( |z - z_4| > R_2 \) for \( z \neq z_4 \) to the sum of the series.

In the case the solution of equation (71) requires more that one extension step one repeats the same procedure for each extension step and the same for equation (75) keeping in mind that such extension steps are always finite in number. Suppose e.g. that the solution of equation (71) requires three extension steps. Then we allocate for each step \( \varepsilon/3 \) instead of \( \varepsilon \) and proceed as before. The same is done if the intermediate linear equations require more than one extension step. Here we employ the general result that given the equation \( \Delta u = \lambda \eta u - \phi_0 - \lambda \psi \) if the sources \( \phi_0 \) and \( \psi \) are of the form \( \eta s \) with \( s \) having the properties P1–P5 for \( z_4 \in D_4 \), then the solution \( u \) has the properties P1–P5 for \( |z_4 - z_4^0| < R_4 - \varepsilon \) for any \( \varepsilon > 0 \). Such a result is proven using exactly the treatment of equation (77) given above.

Remark. We recall now that the conformal field \( \phi(z, z_4) \) is given in terms of \( u \) by
\[
\phi(z, z_4) = u(z, z_4) + \nu(z, z_4) = u(z, z_4) - 2 \sum_k \eta_k \log |z - z_k|^2 + I(z, z_4),
\]
where the analytic properties of \( I(z, z_4) \) have already been given in the appendix, lemma 2. Thus we conclude that \( \phi(z, z_4) \) is real analytic in \( z_4 \) and in \( z \) for \( |z_4 - z_4^0| < R_4 - \varepsilon \) and for \( z \neq z_4 \).

6. The real analyticity of the accessory parameters

The real analyticity of the accessory parameters is a simple consequence of the previous theorem 5.2. In fact we have

**Theorem 6.1.** The accessory parameters are real analytic functions of the source positions.

**Proof.** Consider a singularity \( z_k \) with \( k \neq 4 \) and a circle \( C \) around it of radius such that no other singularity is contained in it. Given the conformal factor \( \phi = u + \nu \) we have that both \( u \) and \( \nu \) are real analytic functions of \( z \) and \( z_4 \) for \( z \) in an annulus containing \( C \) and \( z_4 \in D_4 \). The
accessory parameter $b_k$ can be expressed in terms of $\phi$ as

$$b_k = \frac{1}{i\pi} \oint_{\gamma_k} Q \, dz,$$  \hspace{1cm} (79)

where $Q$ is given by (see e.g. [17])

$$Q = -e^{2\phi} \frac{\partial^2}{\partial z^2} e^{-\phi} = \sum_k \frac{\eta_k (1 - \eta_k)}{(z - z_k)^2} + \sum_k \frac{b_k}{2(z - z_k)}.$$  \hspace{1cm} (80)

Due to the analyticity of $\phi$ we can associate to any point of $C$ a polydisk $D_z \times D_{\bar{z}}$ where the $\phi$ is real analytic. Due to the compactness of $C$ we can extract a finite covering provided by such polydisks.

It follows then that the integral (79) is a real analytic function of $z_4$ for $z_4 \in D_z$. Thus we have that all the accessory parameters $b_k$ with $k \neq 4$ are real analytic functions of $z_4$. With regard to the accessory parameter $b_4$ we recall that due to the Fuchs relations [17] it is given in terms of the other $b_k$ and thus also $b_4$ is real analytic in $z_4$. The reasoning obviously holds for the dependence on the position of any singularity keeping the others fixed, thus concluding the proof of the real analyticity on all source positions. $\square$

Remark. We considered explicitly the case of the sphere topology with an arbitrary number of elliptic singularities. This treatment extends the results of [10, 16, 17] where it was found that in the case of the sphere with four sources we had real analyticity almost everywhere. With the almost everywhere attribute we could not exclude the occurrence of a number of cusps in the dependence of the accessory parameters on the position of the sources. Here we proved real analyticity everywhere and for any number of sources and thus the occurrence of cusps is excluded. Obviously the whole reasoning holds when the positions of the singularities are all distinct. What happens when two singularities meet has been studied only in special cases in [15, 24, 25].

7. Higher genus

For the case of the torus i.e. genus 1 we can follow the treatment of [21]. In this case we know the explicit form of the Green function

$$G(z, z' | \tau) = \frac{1}{4\pi} \log[\theta_1(z - z', e^{i\pi\tau}) \times \theta_1(\bar{z} - z', e^{-i\pi\tau})] + \frac{i}{4(\tau - \bar{\tau})} (z - z' - \bar{z} + \bar{z}')^2,$$  \hspace{1cm} (81)

where $\theta_1$ is the elliptic theta function [26]. $G(z, z' | \tau)$ is a real analytic function in $z$, for $z \neq z'$, and in $\tau$. It satisfies

$$\Delta G(z, z' | \tau) = \delta^2(z - z') - \frac{2i}{\tau - \bar{\tau}}.$$  \hspace{1cm} (82)

As for the $\beta$ we can construct it using the Weierstrass $\wp$ function.

$$\beta(z | \tau) = c \prod_k \left[ (\wp(z - z_k | \tau) - \wp(\bar{z} - \bar{z}_k | \bar{\tau}))^{\eta_k} \right].$$  \hspace{1cm} (83)

Using the freedom of $c$ we normalize the $\beta$ as to have

$$\int \beta(z | \tau) d^2 z = 4\pi \sum_k 2\eta_k.$$  \hspace{1cm} (84)

15
consistent with the topological restriction \( \sum_k 2 \eta_k > 2(1 - g) = 0 \). The \( \nu \) is given by

\[
\nu = 4\pi \sum_k 2 \eta_k G(z, z_k | \tau) + \int G(z, z' | \tau) \beta(z' | \tau) d^2 z'.
\]  

(85)

with \( \phi = u + \nu \).

We proceed now as in sections 5 and 6 to obtain the real analytic dependence of \( \phi \) both on the position of the sources and on the modulus.

**Theorem 7.1.** The conformal factor \( \varphi \) for the torus with any number of elliptic sources depends in real analytic way both on the position of the sources and on the modulus \( \tau \).

**Proof.** The \( \phi(z) \) is translated to the two sheet representation of the torus \( \varphi(v, w) \) using \( v = \varphi(z | \tau) \) and

\[
\varphi(v, w) = \phi(z) + \log \left( \frac{dz}{dv} \right).
\]  

(86)

where

\[
w = \frac{\partial v}{\partial z} = \varphi'(z | \tau) = \sqrt{4(v - v_1)(v - v_2)(v - v_3)}
\]  

(87)

and thus

\[
\varphi(v, w) = \phi(z) - \frac{1}{2} \log \left[ 16(v - v_1)(\bar{v} - \bar{v}_1)(v - v_2)(\bar{v} - \bar{v}_2)(v - v_3)(\bar{v} - \bar{v}_3) \right].
\]  

(88)

Now we proceed as in section 6 where now [17] in the auxiliary equation we have

\[
Q = \frac{3}{16} \left( \frac{1}{(v - v_1)^2} + \frac{1}{(v - v_2)^2} + \frac{1}{(v - v_3)^2} \right) + \frac{b_1}{2(v - v_1)} + \frac{b_2}{2(v - v_2)} + \frac{b_3}{2(v - v_3)}
\]

\[
+ \sum_{k>3} \eta_k (1 - \eta_k) \frac{(w + w_k)^2}{4(v - v_k)^2 w^2} + \frac{b_1 (w + w_k)}{4(v - v_k) w}.
\]  

(89)

In equation (89) \( w = \sqrt{4(v - v_1)(v - v_2)(v - v_3)} \) takes opposite values on the two sheets and the factors \( \frac{w + w_k}{2} \) project the singularities on the sheet to which they belong. We recall that the accessory parameters \( b_k \) are related by the Fuchs relations which in the case of the torus are three in number [17] and thus the independent ones are as many as the sources. Then proceeding as in the previous section we can extract by means of a contour integral the real analytic dependence of the accessory parameters on the source positions and on the modulus. □
Remark. For higher genus we do not possess the explicit form of the Green function and we have a representation of the analogue of the Weierstrass $\wp$ function only for genus 2 [27]. Thus one should employ more general arguments for the analyticity of the Green function and for the expression of the $\beta$ function.

8. Discussion and conclusions

In the present paper we proved that on the sphere topology with any number of elliptic singularities the accessory parameters are real analytic functions of the source positions and the result has been extended to the torus topology with any number of elliptic singularities. This complements the result of Kra [15] where the real analytic dependence was proven for parabolic and elliptic singularities of finite order. Here the elliptic singularities are completely general. The extension of the present treatment to the case when one or more singularities are parabolic should be in principle feasible even though more complicated. Poincaré [18] in fact applied with success the continuation method also in presence of parabolic singularities but the treatment is far lengthier. The reason is that integrals of the type

$$\int \log |z - z'|^2 f(z')d^2z'$$

with $f(z')$ behaving like

$$\frac{1}{|z' - z_k|^2 \log^2 |z' - z_k|^2}$$

for $z'$ near $z_k$, diverge for $z \to z_k$, contrary to what happens in the elliptic case. On the other hand it is proven in [18] that the solution of equation (16) is finite even at the parabolic singularities; in different words even if each term of the series diverges for $z \to z_k$ their sum converges to a function which is finite for $z \to z_k$, a procedure which requires an higher number of iteration steps. Some of these intermediate steps employ $C^\infty$ but non analytic regularization. This does not mean that the continuation method does not work for parabolic singularities but simply that one should revisit the procedure of [18] keeping analyticity in the forefront.

The real analytic dependence of the accessory parameter on the sphere with four sources elliptic and/or parabolic and of the torus with one source was proven already in [10, 16, 17] almost everywhere in the moduli space using analytic variety techniques. Almost everywhere meant e.g. that we could not exclude the presence of a number of cusps in the dependence of the parameter on the source position or moduli. The results of the present paper remove such a possibility.

In section 7 we extended the procedure to the torus topology in a rather straightforward way and thus in presence of $n$ elliptic sources on the torus we have that the independent accessory parameters, which are $n$ in number, depend in real analytic way on the source positions and on the modulus.

For higher genus i.e. $g > 1$ the best approach appears to be the use of the representation of the Riemann surface using the Fuchsian domains in the upper half-plane. For carrying through the program, in absence of explicit forms of the Green function, one should establish its analytic dependence on the moduli and also one should provide an analytic $\beta$ satisfying the correct boundary conditions.
Data availability statement

No new data were created or analysed in this study.

Appendix

In the text the problem arises to establish the real analyticity of certain integrals. The problem can be dealt with in two equivalent ways. The integral in question is a function of two real variables $x, y$. To prove real analyticity we must show that around a real point $x^0, y^0$ the function for real values of $x, y$ is identical to the values taken by a holomorphic function of two complex variables, call them $x', y'$.

Alternatively one can use the complex variable $z$ and its complex conjugate $\bar{z}$. Then proving the real analyticity of $f(z, \bar{z})$ around $z_0, \bar{z}_0$ is equivalent to prove the analyticity of $f(a, b)$ in $a$ and $b$ taken as independent variables. We shall use this complex variable notation as it is simpler.

In proving the main theorem of section 5 we need a number of Lemmas which are stated and proven below.

In all these Lemmas the variable $z_4$ belongs to the domain $D_4$ introduced in section 5 whose definition we recall here: given a point $z_4 \in \mathbb{C}$, $(z_4 \neq z_k$ for $k \neq 4$) $D_4$ is given by

$$|z_4 - z_0| < R_4$$

where $R_4 = 1/4$ the minimal distance of $z_4$ from the other singularities $z_k, k \neq 4$.

Lemma 1. Let

$$f(z, z_4) = \prod_{k \neq 4} [(z - z_k)(\bar{z} - \bar{z}_k)]^{-2\eta_k/}[1 + \bar{z}z]^{-2\sigma + 2}[(z - z_4)(\bar{z} - \bar{z}_4)]^{-2\eta_4} \equiv g(z, \bar{z})[(z - z_4)(\bar{z} - \bar{z}_4)]^{-2\eta_4}$$

with $\sigma = \sum \eta_k$. The integral

$$A = \int \prod_{k \neq 4} [(z - z_k)(\bar{z} - \bar{z}_k)]^{-2\eta_k/}[1 + \bar{z}z]^{-2\sigma + 2}[(z - z_4)(\bar{z} - \bar{z}_4)]^{-2\eta_4} \frac{i}{2}dz \wedge d\bar{z}$$

is real analytic in $z_4$ for $z_4 \in D_4$.

Proof. In computing the derivative w.r.t. $z_4$ in order to avoid the occurrence of non integrable singularities it is expedient to apply the technique of writing

$$A = A_1 + A_2$$

being $A_1$ the integral extended inside a disk of center $z_4$ and radius $R$ excluding all other singularities and $A_2$ the integral outside.

Then we have

$$A_1 = \int_R (\zeta \bar{\zeta})^{-2\eta_4} g(\zeta + z_4, \bar{\zeta} + \bar{z}_4) \frac{i}{2}d\zeta \wedge d\bar{\zeta}$$

which has derivative w.r.t. $z_4$

$$\frac{\partial A_1}{\partial z_4} = \int_R (\zeta \bar{\zeta})^{-2\eta_4} \frac{\partial g(\zeta + z_4, \bar{\zeta} + \bar{z}_4)}{\partial z_4} \frac{i}{2}d\zeta \wedge d\bar{\zeta}.$$
It is justified to take the derivative operation inside the integral sign as due to the real analyticity of \( g \) in \( R \) the integrand in (96) can be bounded by a function \( h(\zeta, \bar{\zeta}) \) independent of \( z_4 \) whose integral over \( R \) is absolutely convergent, exploiting \(-2\eta + 1 > 0\).

For \( A_2 \) we have

\[
A_2 = \int_{R_c} f(z, z_4) \frac{dz \wedge d\bar{z}}{2},
\]

(97)

where \( R_c \) is the complement of \( R \), whose derivative is given by

\[
\frac{\partial A_2}{\partial z_4} = \int_{R_c} \frac{\partial f(z, z_4)}{\partial z_4} \frac{dz \wedge d\bar{z}}{2} + \oint_{\partial R_c} f(z, z_4) \frac{idz}{2}.
\]

(98)

Again it is legal to take the derivative operation inside the integral sign in the first term of (98) as we are working outside \( R \) and the contour integral arises from the fact that the domain \( R_c \) moves with \( z_4 \).

Remark. We have also the complex conjugate equation which give rise to the complex derivative w.r.t. \( x, y \) and as a consequence the \( c(z_4) = C(x_4, y_4) \) of section 2 is real analytic.

Lemma 2. The integral

\[
I(z, z_4) = \frac{1}{4\pi} \log |z - \bar{z}|^2 \beta(z', z_4) d^2 z'
\]

(100)

is everywhere finite continuous in \( z, z_4 \) analytic in \( z \) for \( z \neq z_4 \). \( I(z, z_4) \equiv I(\zeta, \bar{\zeta}, z_4) \) is analytic in \( z_4 \) for \( z_4 \in D_4, |\zeta| < R_1 \), while \( I(z, z_4) \) is analytic in \( z_4 \) for \( z_4 \in D_4, |z - z_4| > R_2 \). At infinity \( I(z, z_4) \) behaves like \( (\sum_k 2\eta_k - 2) \log |z|^2 \) and we have \( \left| \frac{\partial I(z, z_4)}{\partial z_4} \right| < \text{const} \log |z|^2 \).

Proof. We have

\[
I(z, z_4) = \frac{1}{4\pi} \log(z\bar{z}) \int \beta(z', z_4) d^2 z' + \frac{1}{4\pi} \int \log \left| 1 - \frac{z'}{z} \right|^2 \beta(z', z_4) d^2 z'.
\]

(101)

Let \( \Omega \) be the radius of a disk which encloses all singularities \( z_k \); outside such a disk we have

\[
\beta(z, z_4) < \frac{c}{(\Omega z)^2}
\]

(102)

with \( c \) independent of \( z_4 \) for \( z_4 \in D_4 \). Moreover we choose \( \Omega > 1 \). We have

\[
\int \log \left| 1 - \frac{z'}{z} \right|^2 \beta(z', z_4) d^2 z' =
\]

(103)

\[
|z|^2 \int_{|y| < \frac{1}{2}} \log |1 - y|^2 \beta(zy, z_4) d^2 y + |z|^2 \int_{|y| > \frac{1}{2}} \log |1 - y|^2 \beta(zy, z_4) d^2 y.
\]

(104)

First we examine the region \( |z| > 2\Omega \). The first integral is less than

\[
2 \log 2|z|^2 \int_{|y| < \frac{1}{2}} \beta(zy, z_4) d^2 y \leq 2 \log 2 \int \beta(z', z_4) d^2 z'.
\]

(105)
and the second, due to $\beta(z, z_4) < c/(z \bar{z})^2$ is less than

$$
\left| \frac{c}{z \bar{z}} \int_{|y| > \frac{1}{2}} \log |1 - y|^2 \frac{d^2 y}{(y \bar{y})^2} \right|.
$$

(106)

Thus for $|z| > 2\Omega$ we have

$$
|I(z, z_4)| \leq \frac{1}{4\pi} \log(z \bar{z}) \int \beta(z', z_4) d^2 z' + \frac{2}{4\pi} \log 2 \int \beta(z', z_4) d^2 z' + \frac{c}{4\pi z \bar{z}} \int_{|y| > \frac{1}{2}} \log |1 - y|^2 \frac{d^2 y}{(y \bar{y})^2}.
$$

(107)

(108)

For $|z| < 2\Omega$ we isolate the singularities $z_k$ of $\beta$ by non overlapping discs of radius $a < \frac{1}{2}$. In the complement $\beta(z, z_4)$ is majorized by $c/(1 + z \bar{z})^2$ with $c$ independent of $z_4$ for $z_4 \in D_a$. We bound $|I|$ by the sum of two terms being the first

$$
\frac{c}{4\pi} \int_{|\zeta| > 1} \log \zeta \bar{\zeta} \frac{1}{(1 + (z + \zeta)(z + \bar{\zeta}))^2} d^2 \zeta \leq.
$$

(109)

and the second is the contribution of $|\zeta| < 1$, where $\log \zeta \bar{\zeta}$ is negative

$$
\frac{c}{4\pi} \int_{|\zeta| < 1} \log \zeta \bar{\zeta} \frac{1}{(1 + (z + \zeta)(z + \bar{\zeta}))^2} d^2 \zeta \leq \frac{c}{4\pi} \int_{|\zeta| < 1} \log \zeta \bar{\zeta} d^2 \zeta.
$$

(110)

(111)

The singularity at $z' = z_k$ is dealt with $z' = z - z_k$. The contribution of the disk of radius $a$ is

$$
4\pi I_a = \int_a \log |\zeta - \zeta'|^2 \tilde{\beta}(\zeta', z_4) d^2 \zeta',
$$

(112)

where $\tilde{\beta}(\zeta', z_4) = \beta(\zeta + z_k, z_4)$. We have

$$
4\pi I_a = \log |\zeta|^2 \int_a \tilde{\beta}(\zeta', z_4) d\zeta' + \int_a \log \left| 1 - \frac{\zeta'}{\zeta} \right|^2 \tilde{\beta}(\zeta', z_4) d^2 \zeta'.
$$

(113)

and thus for $|\zeta| > 2a$ we have

$$
|4\pi I_a| \leq (\log |\zeta|^2) + 2 \log 2 \left( \int_a \tilde{\beta}(\zeta', z_4) d\zeta' \right)
$$

(114)

and as we are working for $|z| < 2\Omega$ the $|\log |\zeta|^2|$ is bounded. For $a < |\zeta| < 2a$ as $\log |\zeta - \zeta'|^2$ is always negative we have

$$
|4\pi I_a| \leq -M \int_a \log |\zeta - \zeta'|^2 (\zeta' \bar{\zeta'})^{2a} d^2 \zeta' = -\pi M \log(\zeta \bar{\zeta}) \frac{(a^2)^{1-2a}}{1-2a},
$$

(115)
where $M$ is such that $\bar{\beta}(\zeta', z_4) < M(\zeta', \bar{\zeta})^{-2\eta_k}$ for $\zeta$ in the disk of radius $a$ and $z_4 \in D_4$.

Finally for $|\zeta| < a$

$$|4\pi I_0| \leq \frac{\pi M}{1 - 2\eta_k} \left[ -(a^2)^{1-2\eta_k} \log a^2 + \frac{(a^2)^{1-2\eta_k} - (\zeta, \bar{\zeta})^{1-2\eta_k}}{1 - 2\eta_k} \right].$$

(116)

We conclude that $I(z, z_4)$ for any $z$ and $z_4 \in D_4$ is always finite and bounded by

$$|I(z, z_4)| \leq \frac{1}{4\pi} \log(z\bar{z} + 1) \int \beta(z, z_4)d^2z + c_1$$

(117)

with $c_1$ independent of $z_4$ for $z_4 \in D_4$.

We prove now that $I(z, z_4)$ is analytic in $z$ for $z \neq z_4$. Given a $z_0 \neq z_4$ let us consider a disk $D$ of center $z_0$ and radius $r$ such that such disk does not contain any singularity of $\beta(z, z_4)$. By standard arguments one shows that the derivative w.r.t. $z$ of the contribution of such disk to the integral (100) is

$$\frac{1}{4\pi} \int_D \frac{\beta(\zeta', z_4)}{z - \zeta} d^2\zeta.$$ 

(118)

The contribution of the complement $Dc$ of $D$ to the derivative is

$$\frac{1}{4\pi} \int_{Dc} \frac{\beta(\zeta', z_4)}{z - \zeta} d^2\zeta$$

(119)

as for $|z - z_0| < \frac{r}{2}$ we have that the integrand is bounded by

$$\frac{\beta(\zeta', z_4)}{|z' - z_4|} = \frac{1}{r}$$

(120)

which is absolutely convergent and independent of $z$. Thus $I(z, z_4)$ is analytic in $z$ for $z \neq z_4$ and its derivative is given by the sum of (118) and (119) i.e. by (118) with $D$ replaced by the whole $z$ plane.

In working out the derivative of $I(z, z_4)$ w.r.t. $z_4$ in order to avoid non integrable singularities we must isolate a disk $R_1$ of fixed radius $R_1$ with center $z_4$ and excluding all other $z_3$.

Thus as given in section 5 we write for $|\zeta| < R_1$, $\zeta = z - z_4$,

$$\hat{I}(\zeta, z_4) = \frac{1}{4\pi} \int_{R_1} \log(|\zeta - \zeta'|^2 \hat{\beta}(\zeta', z_4)) d^2\zeta'$$

$$+ \frac{1}{4\pi} \int_{R_1^c} \log(|\zeta + z_4 - \zeta'|^2 \beta(\zeta, z_4)) d^2\zeta'$$

(121)

where $R_1^c$ is the complement of $R_1$.

We notice that $\hat{I}(\zeta, z_4)$ does not depend on the specific choice of the radius $R_1$ of the domain used in (121) and in (122) below to compute the derivative w.r.t. $z_4$, provided that $R_1$ does not contain any other singularity except $z_4$.

Its derivative w.r.t. $z_4$ is

$$\frac{\partial \hat{I}(\zeta, z_4)}{\partial z_4} = \frac{1}{4\pi} \int_{R_1} \log(|\zeta - \zeta'|^2 \frac{\partial \hat{\beta}(\zeta', z_4)}{\partial z_4}) d^2\zeta'$$

$$+ \frac{1}{4\pi} \int_{R_1^c} \frac{1}{\zeta + z_4 - \zeta'} \beta(\zeta', z_4)) d^2\zeta'$$

(121)
\[ + \frac{1}{4\pi} \int_{R_{1\epsilon}} \log |z + z_4 - z'|^2 \frac{\partial \beta(z', z_4)}{\partial z_4} d^2 z' \]

\[ + \frac{i}{8\pi} \oint_{\partial R_{1\epsilon}} \log |z + z_4 - z'|^2 \beta(z', z_4) d^2 z'. \] 

(122)

The contour integral is the contribution of the dependence of the domain \(R_{1\epsilon}\) on \(z_4\). In the first term of (122) taking the derivative under the integral sign is legal because the integrand is of the form \((z_4 \zeta)^{-2\eta_4} \frac{\partial \beta(z_4)}{\partial z_4}\) and this expression can be majorized for \(|\zeta| < R_1\) by a function independent of \(z_4\) for \(z_4 \in D_4\) whose integral is absolutely convergent due to \(-2\eta_4 + 1 > 0\). In the second term the denominator of \(\zeta + z_4 - z'\) never vanishes and we can apply the same majorization. In the third term the log is not singular, \(\frac{\partial \beta(z_4)}{\partial z_4}\) has the singularity \(|z_4 - z_4|^2 (z' - z_4)\) which is non integrable for \(-4\eta_4 - 1 < 0\) but such a singularity lies outside the integration region \(R_{1\epsilon}\). As for the contour integral, on it both the logarithm and the \(\beta\) are regular.

We consider then another disk \(R_2\) centered again in \(z_4\) and of radius \(R_2 < R_1\). For \(|z - z_4| > R_2\) we use the expression

\[ I(z, z_4) = \frac{1}{4\pi} \int_{R_2} \log |z - z' - z_4|^2 \beta(z', z_4) d^2 z' \]

\[ + \frac{1}{4\pi} \int_{R_2} \log |z - z'| |\beta(z', z_4)| d^2 z'. \] 

(123)

Its derivative w.r.t. \(z_4\) is given by

\[- \frac{1}{4\pi} \int_{R_2} \frac{1}{z - z' - z_4} \beta(z', z_4) d^2 z' \]

\[ + \frac{1}{4\pi} \int_{R_2} \log |z - z' - z_4|^2 \frac{\partial \beta(z', z_4)}{\partial z_4} d^2 z' \]

\[ + \frac{1}{4\pi} \oint_{\partial R_2} \log |z - z'| |\beta(z', z_4)| d^2 z' \]

\[ + \frac{i}{8\pi} \oint_{\partial R_2} \log |z - z'|^2 \beta(z', z_4) d^2 z'. \] 

(124)

Regarding the first two terms in (124) as \(z - z' - z_4\) never vanishes in \(R_2\) it is legal to take the derivative under the integral sign. In the third term the integrand can be majorized by a \(z_4\) independent integrable function. The last term is the contribution of the moving integration region. Thus we have analyticity of \(I(z, z_4)\) for \(|z - z_4| > R_2\), \(z_4 \in D_4\).

We conclude that \(I(z, z_4)\) is everywhere finite, bounded by (117), continuous in \(z, z_4\). \(I(z, z_4)\) is analytic for \(z \neq z_4\). \(I(z, z_4)\) is analytic in \(z_4\) for \(z_4 \in D_4\), \(|\zeta| < R_1\) while \(I(z, z_4)\) is analytic in \(z_4\) for \(z_4 \in D_4\), \(|z - z_4| > R_2\). From (124) we have for large \(z |\frac{\partial \beta(z_4)}{\partial z_4}| < \text{const log}|z|\). 

**Lemma 3.** Given

\[ \theta(z, z_4) = \prod_k [(z - z_k)(z - \bar{z}_k)]^{-2\eta_k} e^{\beta(z, z_4)} = \beta(z, z_4)^{r(z, z_k)} \]

(125)

with \(0 < r_1 < r < r_2\) and \(s(z, z_4)\) with

\[ \int \theta(z, z_4)s(z, z_4) d^2 z = 0. \]

(126)
the integral
\[
\frac{1}{4\pi} \int \log |z - z'|^2 \theta(z', z_4) s(z', z_4) d^2 z' \tag{127}
\]
is bounded in absolute value by \( B \max |s(z, z_4)| \) with \( B \) independent of \( z_4 \) for \( z_4 \in D_4 \).

**Proof.** We recall that the function \( \theta(z, z_4) \) is positive with elliptic singularities and bounded at infinity by \( \text{const} |z|^{-2} \). As \( r_1 < r < r_2 \) most of the techniques for proving such a bound have been already worked out in the preceding lemma 2.

We have due to the condition (126) and the bound (54)
\[
\int \log |z - z'|^2 \theta(z', z_4) s(z', z_4) d^2 z' = \int \log \left| 1 - \frac{z'}{z} \right|^2 \theta(z', z_4) s(z', z_4) d^2 z'. \tag{128}
\]
For \( |z| > 2\Omega \) we use the second form in equation (128) replacing \( s(z', z_4) \) with \( \max |s(z', z_4)| \) and the logarithm by its absolute value and then proceeding as in lemma 2. It is important to notice that now due to the condition (126) the term \( \log z \) which diverges at infinity in equation (117) is absent and thus we have boundedness also at infinity. The region \( |z| < 2\Omega \) is treated exactly as in the previous lemma 2. The result is that
\[
\left| \frac{1}{4\pi} \int \log |z - z'|^2 \theta(z', z_4) s(z', z_4) d^2 z' \right| < B \max |s(z, z_4)| \tag{129}
\]
with \( B \) independent of \( z_4 \) for \( z_4 \in D_4 \). \( \square \)

**Lemma 4.** The series (21) of section 3 converges for \( |\lambda| < (\log 4 - 1)/\gamma_1 \), where \( \gamma_1 \equiv \max |u_1| \).

**Proof.** From the text we have for \( k \geq 2 \)
\[
\max |u_k| \leq \max |u_2|. \tag{130}
\]
Given \( \max |u_1| \equiv \gamma_1 \) one considers the series
\[
\nu = \lambda \gamma_1 + \lambda^2 \gamma_2 + \lambda^3 \gamma_3 + \ldots, \tag{131}
\]
where (see section 3)
\[
\gamma_2 = \frac{\gamma_1^2}{2}, \quad \gamma_3 = \frac{\gamma_1^3}{6} + \gamma_1 \gamma_2, \quad \gamma_4 = \frac{\gamma_1^4}{24} + \frac{\gamma_1^2 \gamma_2 + \gamma_1^3}{2} + \gamma_1 \gamma_3, \ldots \tag{132}
\]

Obviously such a series of positive terms majorizes term by term the series \( \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3 + \ldots \). We want to find the convergence radius of \( \nu \). For the function \( \nu \) we have
\[
e^{\nu} = 1 + \lambda \gamma_1 + \lambda^2 \gamma_2 + \lambda^3 \gamma_3 + \ldots
\]
\[
+ \lambda^2 \gamma_2 + \lambda^3 \gamma_3 + \ldots
\]
\[
= 1 + 2\nu - \lambda \gamma_1. \tag{133}
\]
Let us consider the implicit function defined by
\[
1 + 2\nu - e^{\nu} = \lambda \gamma_1. \tag{134}
\]
As for $\nu = 0$ we have $\lambda \gamma_1 = 0$ and
\[
\frac{\partial (1 + 2\nu - \epsilon^\nu)}{\partial \nu} = 2 - \epsilon^\nu
\] (135)
which equals 1 at $\nu = 0$ the analytic implicit function theorem assures us that the $\nu$ is analytic in $\lambda$ in a finite disk around $\lambda = 0$ and thus with a non zero radius of convergence in the power expansion. This suffices for the developments of the present paper.

Actually the radius of convergence $r_0$ of (131) is given by the vanishing of (135) i.e. $r_0 = (\log 4 - 1)/\gamma_1$.

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