INJECTIVITY FOR ALGEBRAS AND CATEGORIES WITH QUANTUM SYMMETRY

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Abstract. We study completely positive maps and injectivity for Yetter–Drinfeld algebras over compact quantum groups, and module categories over rigid C*-tensor categories. This gives a generalization of Hamana’s theory of injective envelopes to the framework of dynamical systems over quantum groups. As a byproduct we also establish a duality between the Yetter–Drinfeld algebras and certain bimodule categories with central generators.

1. Introduction

In this paper, we study completely positive maps and injectivity for Yetter–Drinfeld algebras over compact quantum groups, and module categories over rigid C*-tensor categories.

The study of injectivity of operator systems and operator spaces led to surprisingly rich applications to the structure theory of operator algebras old and new. While the foundation of the theory goes back to the 1950s, one early breakthrough in this context is Arveson’s result on the injectivity of $B(H)$ [Arv69]. This, together with subsequent works by Choi and Effros [Cho75, CE77] among others, proved the field to be a fruitful framework at the intersection of abstract functional analysis and more “applied” fields such as quantum information.

Following this direction, Hamana took on systematic study of injectivity [Ham78, Ham79b, Ham79, Ham85, Ham11], defining and proving the existence and uniqueness of injective envelopes of operator systems, C*-algebras, and C*-dynamical systems with dynamics given by a discrete group, and more generally by a Hopf–von Neumann algebra. Roughly speaking, his construction builds on two parts: first is to embed a given object into a bigger injective object, then next is to construct a ‘smallest’ injective one containing the original object as the image of minimal idempotent in a convex semigroup of completely positive maps acting on the injective one from the first step.

In another direction, Furstenberg and his school studied the probabilistic notion of boundary actions of semisimple Lie groups [Fur63, Moo73], and more generally of locally compact groups [Gla75]. A central object in their theory is the notion of a universal (initial) boundary action of $G$, now known as the Furstenberg boundary $\partial_F(G)$. In the case of a semisimple Lie group $G$, an important application of boundary theory is to the symmetric space $G/K$ corresponding to $G$, which appears through measure theoretic considerations on $\partial_F(G)$. Furstenberg also considered another kind of boundary, the Poisson boundary $B(G)$ of $G$, which shows up through the study of harmonic functions, and studied how $\partial_F(G)$ relates to $B(G)$.

While these two theories look completely separate at the outset, it turns out that the C*-algebra of continuous functions on the Furstenberg boundary of a discrete group $G$ agrees with the $G$-injective envelope of the trivial $G$-C*-algebra $\mathbb{C}$ [Ham78, KK17]. In recent years, this connection led to striking implications on the structure of group C*-algebras: this includes the equivalence between amenability of a discrete group and triviality of its Furstenberg–Hamana boundary, results on rigidity and C*-simplicity [BKKO17, Ken20].

Probabilistic boundary theory has been brought to the framework of operator algebraic quantum groups in various flavors in the last 30 years [Bia91, Izu02, Izu04, INT06, NT04, VV07, NY17].

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More recently, the analogue of Furstenberg–Hamana boundaries in this framework is also achieved by the work of Kalantar, Kasprzak, Skalski, and Vergnioux [KKSV22]. They define and prove both the existence and the uniqueness of Furstenberg–Hamana boundaries for discrete quantum groups, and relate them to several C*-algebraic concepts such as simplicity, exactness, and existence of KMS-states.

This was further brought to the case of Drinfeld doubles $D(G)$ of compact quantum groups $G$ in a joint work of the first named author with Habbestad and Neshveyev [HHN22]. When the compact quantum group is the $q$-deformation of a compact simple group, this Drinfeld double construction can be seen as a quantization of the corresponding complex simple Lie group, hence coming back to a setting close to Furstenberg’s original work. This also provides a conceptual explanation for the equality between the Furstenberg–Hamana boundary and the Poisson boundary of such quantum groups.

The boundary theory developed in [HHN22] works in the framework of braided commutative Yetter–Drinfeld $G$-C*-algebras for a compact quantum group $G$, and more generally C*-tensor categories under a rigid tensor category. Such correspondence follows from a Tannaka–Krein type duality [NY14], but more general types of duality for module categories and comodule algebras are known to hold [DCY13, Nes14], which can be traced back to Ostrik’s result in the purely algebraic setting [Ost03].

In this work, we look at a more general setting of module categories, and also the Yetter–Drinfeld algebras without braided commutativity. Let us explain the structure of our paper.

In Section 2 we fix our conventions and recall some basic results. Then in Section 3, we quickly prove our first main result, as follows.

**Theorem A** (Theorem 3.6). For a compact quantum group $G$, every unital Yetter–Drinfeld $G$-C*-algebra $A$ admits an injective envelope.

This result, showing that the injective envelopes of continuous $D(G)$-actions are still continuous, improves a result of Hamana.

In Section 4, we look at the categorical dual of quantum group actions. We begin with the study of injective envelope of a pointed module category, i.e., a module category with a fixed object. Here we make extensive use of the correspondence between algebra objects in a tensor category and cyclic pointed module categories, and the concept of multipliers between pointed module categories introduced in [JP17]. This allows us to bring ideas about completely positive maps between C*-algebras to the categorical setting. In fact, special cases of such maps have already been considered in [PV15, GJ16, NY16] to study representation theory and approximation properties, like the Haagerup property and property (T), for rigid C*-tensor categories, subfactors, and $\lambda$-lattices.

Given a Hilbert space object $H \in \text{Hilb}(\mathcal{C})$ (equivalently, an object in the ind-completion of $\mathcal{C}$ in the sense of [NY16]), there is a C*-algebra object $\mathcal{B}(H)$ in $\text{Vec}(\mathcal{C})$ playing the role of algebra of bounded operators in $H$. Our first main result in this setting is the following analogue of Arveson’s theorem.

**Theorem B** (Theorem 4.10). For any $H \in \text{Hilb}(\mathcal{C})$, the $C^{*}$-$\mathcal{C}$-module category $\mathcal{M}_{\mathcal{B}(H)}$ corresponding to $\mathcal{B}(H)$ is injective.

By a standard argument, this allows us to obtain an analogue of Hamana’s theorem for module categories (Theorem 4.13). We also show that the injectivity of a pointed module category $(\mathcal{M}, m)$ is actually a property of the module subcategory of $\mathcal{M}$ generated by $m$ (Proposition 4.9) through a generalization of the Choi matrix construction. In particular, the injectivity of $G$-C*-algebra is invariant under Morita equivalence with respect to finitely generated projective Hilbert modules.

Next in Section 5 we introduce a special type of module categories, which we call centrally pointed bimodule categories. These are bimodule categories endowed with an analogue of unitary
half-braiding on the distinguished object. Such bimodule categories and bimodule functors compatible with the half-braiding on a distinguished object again form a category $\mathcal{CB}(\mathcal{C})$. In particular, we are interested in its subcategory $\mathcal{CB}^c(\mathcal{C})$ consisting of the ones in which the distinguished objects are generating.

In Section 6, we look at the Tannaka–Krein type duality between $C^*$-module categories over $\text{Rep}(G)$ and $G$-$C^*$-algebras for compact quantum groups $G$.

**Theorem C** (Theorem 6.1). The category of Yetter–Drinfeld $G$-$C^*$-algebras is equivalent to $\mathcal{CB}^c(\text{Rep}(G))$. Under this equivalence, two morphisms $F, F': \mathcal{M} \to \mathcal{M}'$ in $\mathcal{CB}^c(\text{Rep}(G))$ induce the same homomorphism $B_M \to B_{M'}$ if and only if they are naturally isomorphic as centrally pointed bimodule functors.

This allows us to drop the braided commutativity assumption in [HHN22], and work with categorical analogues of general Yetter–Drinfeld $G$-$C^*$-algebras.

Finally, in Section 7 we investigate injectivity and boundary theory of centrally pointed bimodule categories based on the schemes of previous sections. Here our main result is the following analogue of Theorem A.

**Theorem D** (Theorem 7.8). Every centrally pointed module category has an injective envelope in $\mathcal{CB}(\mathcal{C})$.

In the boundary theory for $D(G)$ developed in [HHN22], one sees that every boundary action embeds into the Furstenberg–Hamana boundary, and as a corollary one sees that every boundary action is braided-commutative. Following a similar idea, we obtain the following generalization in the categorical framework, where a boundary of $\mathcal{C}$ is a centrally pointed module category over $\mathcal{C}$ such that any ucp multiplier to another category is completely isometric.

**Theorem E** (Theorem 7.10). Every boundary category $\mathcal{M}$ of $\mathcal{C}$ is a centrally pointed subcategory of $\partial_{\text{FH}}(\mathcal{C})$.

In Appendix A we give an intrinsic characterization of $C^*$-algebra objects in terms of the associated Hilbert modules over the $C^*$-algebra at the tensor unit, which avoids talking directly about the associated module category as was done in [JP17].

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2. Preliminaries

When $H$ is a Hilbert space, we denote the algebra of bounded operators on $H$ by $B(H)$, while the algebra of compact operators is denoted by $K(H)$. The conjugate Hilbert space is denoted by $\overline{H}$, and we write $\xi \mapsto \overline{\xi}$ for the anti-linear isomorphism $H \to \overline{H}$.

The multiplier algebra of a $C^*$-algebra $A$ is denoted by $\mathcal{M}(A)$. We freely identify $\mathcal{M}(K(H))$ with $B(H)$.

As tensor product of $C^*$-algebras we always take the minimal tensor product, that we denote by $A \otimes B$. For von Neumann algebras, their von Neumann algebraic tensor product is denoted by $M \otimes N$.

2.1. Operator systems. An operator system is a unital and selfadjoint closed subspace of a unital $C^*$-algebra, which we write as $1 \in S \subset A$. Such objects form a category with unital completely positive maps as morphisms. An operator system is injective if it is injective in this category.

When $S$ is an operator system, its injective envelope is given by an injective operator system $I(S)$ and a complete isometry $\phi: S \to I(S)$ such that $\phi$ is essential in the sense that for any ucp map $\psi: I(S) \to S'$ to another operator system, $\psi\phi$ is completely isometric if and only if $\psi$ is.
The injective envelope of an operator system always exists and is unique up to complete order isomorphisms [Ham79b], with various generalizations imposing additional structures on operator systems. A key technical step in the construction of injective envelopes is the following proposition.

**Proposition 2.1 ([IHNN22 Proposition 2.1]).** Assume $X$ is a subspace of a dual Banach space $Y^*$, and $S$ is a convex semigroup of contractive linear maps $X \to X$ such that, if we consider $S$ as a set of maps $X \to Y^*$, then $S$ is closed in the topology of pointwise weak* convergence. Then there is an idempotent $\phi_0 \in S$ such that

$$\phi_0 \psi \phi_0 = \phi_0 \quad (\psi \in S).$$

Moreover, $\phi_0$ is minimal with respect to the preorder relation $\preceq$ on $S$ defined as

$$\phi \preceq \psi \iff \forall x \in X : \|\phi(x)\| \leq \|\psi(x)\|.$$  

2.2. C*-tensor categories. Here we mostly follow [NT13,NY14].

Given a C*-category $\mathcal{C}$, we denote the spaces of morphisms from an object $X$ to another $Y$ as $\mathcal{C}(X,Y)$. The involution $\mathcal{C}(X,Y) \to \mathcal{C}(Y,X)$ is denoted by $T \mapsto T^*$, and the norm is by $\|T\|$, so that we have the C*-identity $\|T^*T\| = \|T\|^2$. We also assume that the direct sum $X \oplus Y$ makes sense with structure morphisms given by isometries $X \to X \oplus Y \to Y$. Under these assumptions, $\mathcal{C}(X,Y)$ is naturally a right Hilbert module over the unital C*-algebra $\mathcal{C}(X) = \mathcal{C}(X,X)$. We tacitly assume that $\mathcal{C}$ is closed under taking subobjects, i.e., any projection in $\mathcal{C}(X)$ corresponds to a direct summand of $X$.

A C*-tensor category is a C*-category endowed with monoidal structure given by: a *-bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $1_\mathcal{C}$, and unitary natural isomorphisms

$$1_\mathcal{C} \otimes U \to U \leftarrow U \otimes 1_\mathcal{C}, \quad \Phi : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

for $U,V,W \in \mathcal{C}$, satisfying a standard set of axioms. Without losing generality, we may and do assume that $\mathcal{C}$ is strict so that the above morphisms are identity, and that $1_\mathcal{C}$ is simple, unless explicitly stated otherwise.

A rigid C*-tensor category is a C*-tensor category where any object $U$ has a dual given by: an object $\bar{U}$ and morphisms

$$R : 1_\mathcal{C} \to \bar{U} \otimes U, \quad \bar{R} : 1_\mathcal{C} \to U \otimes \bar{U}$$

satisfying the conjugate equations for $U$.

When $\mathcal{C}$ is a C*-tensor category, a right C*-C-module category is given by a C*-category $\mathcal{M}$, together with a *-bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and unitary natural isomorphisms

$$X \otimes 1_\mathcal{C} \to X, \quad \Psi : (X \otimes U) \otimes V \to X \otimes (U \otimes V)$$

for $X \in \mathcal{M}$ and $U,V \in \mathcal{C}$, satisfying standard set of axioms. Again we may and do assume that module category structures are strict so that the above morphisms are identities.

A functor of right C- module categories is given by a functor $F : \mathcal{M} \to \mathcal{M}'$ of the underlying linear categories, together with natural isomorphisms

$$F_2 = F_{2,m,U} : F(m) \otimes U \to F(m \otimes U) \quad (m \in \mathcal{M}, U \in \mathcal{C})$$

satisfying the standard compatibility conditions with structure morphisms of $\mathcal{M}$ and $\mathcal{M}'$. If $\mathcal{M}$ and $\mathcal{M}'$ are C*-C-module categories, a functor $(F,F_2)$ as above is said to be a functor of right C*-C-module categories if it is a *-functor and the natural isomorphism $F_2$ is unitary.

2.3. Quantum groups. Again we mostly follow [NT13,NY14]. A compact quantum group $G$ is given by a unital C*-algebra $C(G)$ and a *-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ satisfying the coassociativity and cancellativity. We denote its invariant state (Haar state) by $h$, and always assume that $h$ is faithful on $C(G)$. The GNS Hilbert space of $(C(G), h)$ is denoted by $L^2(G)$. 
A finite dimensional unitary representation of \( G \) is given by a finite dimensional Hilbert space \( H \) and a unitary element \( U \in B(H) \otimes C(G) \) satisfying
\[
U_{12}U_{13} = (\text{id} \otimes \Delta)(U). \tag{2.1}
\]
We often write \( H = H_{U} \). These objects form a rigid C*-tensor category that we denote by \( \text{Rep}(G) \), with its tensor unit represented by the multiplicative unit \( 1 \in C(G) \simeq B(\mathbb{C}) \otimes C(G) \).

Our convention of duality is given by \( H_{U} = H^{\infty} \), with structure morphisms \( R_{U} : 1 \to U \otimes U \) and \( R_{\bar{U}} : 1 \to U \otimes \bar{U} \) given by
\[
R_{U} = \sum_{i} \xi_{i} \otimes \rho_{U, i}^{-1/2} \xi_{i}, \quad R_{\bar{U}} = \sum_{i} \rho_{U, i}^{-1/2} \xi_{i} \otimes \bar{\xi}_{i} \tag{2.2}
\]
for any choice of orthonormal basis \( (\xi_{i})_{i} \subset H_{U} \). Here \( \rho_{U} \) is the positive operator on \( H_{U} \) defined by
\[
\rho_{U} = (\text{id} \otimes f_{1})(U),
\]
where \( f_{1} \) is a special value of the Woronowicz characters \( \{ f_{x} \}_{x \in \mathbb{C}} \).

We denote the set of irreducible classes in \( \text{Rep}(G) \) by \( \text{Irr}(G) \), and write \( (U_{i})_{i \in \text{Irr}(G)} \) by a choice of representatives of finite dimensional irreducible unitary representations, with underlying Hilbert spaces \( (H_{i})_{i} \).

As the model of \( c_{0} \)-sequences on \( \hat{G} \), the discrete dual of \( G \), we take the C*-algebra
\[
C_{c}^{*}(G) = c_{0}(\hat{G}) = c_{0} \bigoplus_{i \in \text{Irr}(G)} B(H_{i}),
\]
where \( c_{0} \bigoplus \) denotes the C*-algebraic direct sum.

As usual, without finite dimensionality assumption on \( H \), an unitary representation of \( G \) is given by an unitary element \( U \in \mathcal{M}(\mathcal{K}(H) \otimes C(G)) \) satisfying (2.1), where we extend \( \text{id} \otimes \Delta \) to a unital \(*\)-homomorphism \( \mathcal{M}(\mathcal{K}(H) \otimes C(G)) \to \mathcal{M}(\mathcal{K}(H) \otimes C(G) \otimes C(G)) \). In particular, there is a distinguished representation (the regular representation) of \( G \) on \( L^{2}(G) \), given by the multiplicative unitary \( W_{G} \in \mathcal{M}(c_{0}(\hat{G}) \otimes C(G)) \subset \mathcal{M}(\mathcal{K}(L^{2}(G)) \otimes C(G)) \).

Purely algebraic models of functions on these quantum groups are given as follows. The regular subalgebra of \( C(G) \), or the algebra of matrix coefficients, is denoted by \( \mathcal{O}(G) \), which is spanned by the elements \( (\omega \otimes \text{id})(U) \) for \( U \in \text{Rep}(G) \) and \( \omega \in B(H_{U}) \). On the dual side, we take
\[
c_{c}(\hat{G}) = \bigoplus_{i \in \text{Irr}(G)} B(H_{i})
\]
which is the usual algebraic direct sum. There is a nondegenerate linear duality pairing between \( \mathcal{O}(G) \) and \( c_{c}(\hat{G}) \), so that the algebra structures and the coalgebra structures on the spaces are related by
\[
(\phi_{(1)}, f_{1})(\phi_{(2)}, f_{2}) = (\phi, f_{1}f_{2}), \quad (\phi_{1}f_{2}, f) = (\phi_{1}, f_{1})(\phi_{2}, f_{2}). \tag{2.3}
\]

The Drinfeld double of \( G \) is represented by \( \mathcal{O}_{c}(D(G)) \), which is modeled on the tensor product coalgebra \( c_{c}(\hat{G}) \otimes \mathcal{O}(G) \) endowed with the coproduct \( \Delta(\omega) = \omega_{(1)}a_{(1)} \otimes \omega_{(2)}a_{(2)} \), and the \(*\)-algebra structure induced by those of \( \mathcal{O}(G) \) and \( c_{c}(\hat{G}) \) together with the exchange rule
\[
\omega(a_{(1)})a_{(2)} = a_{(1)}\omega(a_{(2)}^{*})
\]
for \( \omega \in c_{c}(\hat{G}) \) and \( a \in \mathcal{O}(G) \). To be precise, this is to be interpreted as an algebraic model of convolution algebra of functions on the Drinfeld double quantum group \( D(G) \). Similarly, the function algebra on \( D(G) \) is given by the tensor product \(*\)-algebra \( \mathcal{O}_{c}(D(G)) = \mathcal{O}(G) \otimes c_{c}(\hat{G}) \) together with the twisted coproduct induced by the adjoint action of \( W_{G} \), see [NY10]. These algebras have completion as C*-algebraic models for locally compact quantum groups, but we refrain from working with them.
2.4. Quantum group actions. A continuous action of $G$ on a $C^*$-algebra $A$ is given by a nondegenerate and injective $*$-homomorphism $\alpha: A \to C(G) \otimes A$ satisfying $\Delta \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\alpha$. We say that $A$ is a $G$-$C^*$-algebra.

Given a $G$-$C^*$-algebra $(\mathcal{A}, \alpha)$, its regular subalgebra $\mathcal{A}$ is defined as the set of elements $a \in \mathcal{A}$ such that $\alpha(a)$ belongs to the algebraic tensor product $\mathcal{O}(G) \otimes \mathcal{A}$. This is a left $\mathcal{O}(G)$-comodule algebra.

A measurable group, $G$-von Neumann algebra, is given by a von Neumann algebra $M$ and a unital injective normal $*$-homomorphism $\alpha: M \to L^\infty(G) \otimes M$ satisfying the coassociativity condition $\Delta(\Delta \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\alpha$. The regular subalgebra $\mathcal{M} \subset \mathcal{M}$ makes sense as above, and we denote its $C^*$-algebraic closure by $\mathcal{R}(\mathcal{M})$. Then $\mathcal{R}(\mathcal{M})$ admits a continuous action of $G$.

Example 2.2. Let $(H, U)$ be a finite dimensional unitary representation of $G$. Under our convention, $H$ is a right $\mathcal{O}(G)$-comodule, but it can be considered as a left comodule by the coaction map $\xi \mapsto U_{21}^* (1 \otimes \xi)$. This becomes an equivariant right Hilbert $C$-module by the inner product

$$(\xi, \eta)_C = (U_1 \eta, U_1 \xi)_H h(U_2^* U_2) = (\hat{\rho}_U^{-1} \eta, \xi)_H$$

where $(\cdot, \cdot)_H$ denotes the original inner product on $H$, see [NY18]. Then $\mathcal{B}(H)$ admits the induced $G$-$C^*$-algebra structure, concretely given by the coaction $T \mapsto U_{21}^* T_2 U_{21}$. Analogously, when $(H, U)$ is an infinite dimensional unitary representation, the same formula makes $\mathcal{B}(H)$ a $G$-von Neumann algebra and $\mathcal{R}(\mathcal{B}(H))$ a $G$-$C^*$-algebra.

A Yetter–Drinfeld $G$-$C^*$-algebra is given by a $G$-$C^*$-algebra $(\mathcal{A}, \alpha)$ together with a left action $\mathcal{O}(G) \otimes \mathcal{A} \to \mathcal{A}$, $f \otimes a \mapsto f \triangleright a$

such that

$$\alpha(f \triangleright a) = f_{(1)} a_{(1)} S(f_{(3)}) \otimes (f_{(2)} \triangleright a_{(2)}).$$

Equivalently, it can be interpreted as a continuous action of the locally compact quantum group $D(G)$. Because of this, we also say $D(G)$-equivariance instead of Yetter–Drinfeld $G$-equivariance.

Let $\mathcal{A}$ be a left $\mathcal{O}(G)$-module algebra. By duality, we have a homomorphism

$$\beta_{\mathcal{A}}: \mathcal{A} \to \prod_{i \in \text{Irr}(G)} A \otimes \mathcal{B}(H_i), \quad a \mapsto \left( \sum_{k,l} (u_{ki} \triangleright a) \otimes m_{ki}^{(2)} \right)_i,$$

which can be regarded as a right comodule algebra over the multiplier Hopf algebra representing $\hat{G}$ with the convention of [23].

This construction makes sense as a unitary coaction of $\ell^\infty(\hat{G})$ when $\mathcal{A}$ is a Yetter–Drinfeld $G$-$C^*$-algebra. Moreover, when we consider an action of $\hat{G}$ on a von Neumann algebra $\mathcal{N}$, the target of coaction map becomes

$$\ell^\infty \prod_{i \in \text{Irr}(G)} N \otimes \mathcal{B}(H_i) \simeq N \otimes \ell^\infty(\hat{G}).$$

By an analogue of the Tannaka–Krein–Woronowicz duality, the category of unital $G$-$C^*$-algebras is equivalent to the category of right $C^*$-$(\text{Rep}(G))$-module categories [DCY13, Nes14].

Concretely, given a unital $G$-$C^*$-algebra $B$, one takes the category $\mathcal{D}_B$ of finitely generated projective $G$-equivariant right Hilbert modules over $B$. Thus, an object of $\mathcal{D}_B$ is a right Hilbert module $E_B$ with a left coaction $\delta$ of $C(G)$, such that the action of $B$ is equivariant. Given an object $U$ of $\text{Rep}(G)$, its right action $E_B \otimes U$ is represented by the equivariant right Hilbert module $H_U \otimes E_B$, where the underlying left comodule is given as the tensor product of $E_B$ and the left comodule $H_U$ as explained in Example 2.2. Explicitly, the new coaction on $E_B \otimes U$ is given by

$$\xi \otimes a \mapsto (U_{21}^* (1 \otimes \xi \otimes 1)) \delta(a)_{13}.$$
Conversely, given a right $C^\ast$-(Rep($G$))-module category $\mathcal{D}$ and an object $X \in \mathcal{D}$, we take the left $\mathcal{O}(G)$-comodule

$$B_{\mathcal{D},X} = \bigoplus_{i \in \text{Irr}(\mathcal{G})} \mathcal{H}_i \otimes \mathcal{D}(X, X \otimes U_i),$$

which admits an associative product from irreducible decomposition of monoidal products. Together with the involution coming from duality of representations, we obtain a pre $C^\ast$-algebra which admits a canonical completion supporting a coaction of $C(G)$.

**Remark 2.3.** The formula above explains why left $C(G)$-comodule structures give rise to right Rep($G$)-module categories. Indeed, given two finite dimensional unitary representations $(H_U, U)$ and $(H_V, V)$ of $G$, and given $\xi \in H_U$, $\eta \in H_V$ and $a \in E_B$, the $C(G)$-coaction on $\xi \otimes \eta \otimes a \in (H_V \otimes E_B) \otimes U$ is given by

$$\xi \otimes \eta \otimes a \mapsto U_{21} V_{31}^\ast (a(1) \otimes \xi \otimes \eta \otimes a(2)),$$

where $a(1) \otimes a(2) = \delta(a)$. Flipping the second and third legs, we obtain

$$U_{21} V_{31}^\ast (a(1) \otimes \eta \otimes a(2)) \mapsto (V \otimes U)_2 (a(1) \otimes (\eta \otimes \xi) \otimes a(2)).$$

This computation shows that the flip map $H_U \otimes H_V \to H_V \otimes H_U$ induces an equivariant isomorphism $(H_V \otimes E_B) \otimes U \simeq H_V \otimes E_B$.

The generalization of module categories to the nonunital setting is given by multiplier module categories [AV20].

### 2.5. Vector spaces and algebras over $C$

Let us review the correspondence between module categories and algebra objects for linear tensor categories. Further details on the following construction can be found in [JP17].

Let $\mathcal{C}$ be a rigid tensor category. We denote by Vec($\mathcal{C}$) the category whose objects are the contravariant linear functors $\mathcal{C} \to \text{Vec}$ and morphisms are natural transformations. For $V \in \text{Vec}(\mathcal{C})$, we call its values $V(X)$ on $X \in \mathcal{C}$ the fibers of $V$.

Due to the semisimplicity of $\mathcal{C}$, it is possible to give a concrete description of Vec($\mathcal{C}$): the morphism space between objects $V$ and $W$ of Vec($\mathcal{C}$) can be decomposed into the algebraic direct product as

$$\text{Vec}(\mathcal{C})(V, W) \cong \prod_{i \in \text{Irr}(\mathcal{C})} \mathcal{L}(V(U_i), W(U_i)),$$

while the fibers can be written as

$$V(Y) \cong \bigoplus_{i \in \text{Irr}(\mathcal{C})} V(U_i) \otimes \mathcal{C}(Y, U_i).$$

Under this presentation of $V$, its value on a morphism $\psi \in \mathcal{C}(X, Y) = \mathcal{C}^\text{op}(Y^\text{op}, X^\text{op})$ would then be

$$V(\psi) = \bigoplus_{i \in \text{Irr}(\mathcal{C})} \text{id}_{V(U_i)} \otimes \psi^\#$$

with $\psi^\#$ denoting pre-composition with $\psi$.

The category Vec($\mathcal{C}$) becomes a monoidal category by introducing the monoidal product

$$(V \otimes W)(X) = \bigoplus_{i,j \in \text{Irr}(\mathcal{C})} V(U_i) \otimes \mathcal{C}(X, U_i \otimes U_j) \otimes W(U_j) \quad (2.4)$$

together with naturally defined structure morphisms (see [JP17]). Then the Yoneda embedding $X \to \mathcal{C}(-, X)$ is a fully faithful monoidal functor from $\mathcal{C}$ to Vec($\mathcal{C}$). We regard $\mathcal{C}$ as a monoidal subcategory of Vec($\mathcal{C}$) by means of this embedding.

An algebra object in Vec($\mathcal{C}$) is given by $A \in \text{Vec}(\mathcal{C})$ together with natural transformations $m: A \otimes A \to A$ and $i: 1_\mathcal{C} \to A$ that make the algebra diagrams commute. We denote by Alg($\mathcal{C}$) the category whose object are algebras in Vec($\mathcal{C}$) and the morphisms are algebra natural
transformations. In other words, the algebra objects in Vec(\mathcal{C}) are exactly the lax tensor functors \mathcal{C}^{op} \to \text{Vec}.

**Definition 2.4.** A pointed right \mathcal{C}-module category is a pair \((\mathcal{M}, m)\), where \mathcal{M} is a right \mathcal{C}-module category and \(m\) is an object of \(\mathcal{M}\). A \mathcal{C}-module category is said to be cyclic if it has a generating object \(m\), in the sense that any object \(X \in \mathcal{M}\) is a direct summand of \(m \otimes U\) for some \(U \in \mathcal{C}\). By a pointed cyclic \mathcal{C}-module category we mean a pointed category \((\mathcal{M}, m)\) such that \(m\) is a generator of \(\mathcal{M}\).

When the choice of \(m\) is implicit, we just write \(\mathcal{M}\) instead of \((\mathcal{M}, m)\), and \(\mathcal{M}_\ast = m\).

**Definition 2.5.** A functor of pointed right \mathcal{C}-module categories is given by a pair consisting of a \mathcal{C}-module functor \(F: \mathcal{M} \to \mathcal{M}'\) and an isomorphism \(F_0: m_{\mathcal{M}'} \to F(m_{\mathcal{M}})\). We denote by \(\text{Mod}_\ast^\mathcal{C}(\mathcal{C})\) the category of pointed cyclic right \mathcal{C}-module categories, with these functors as morphisms.

Let us denote the category of cyclic pointed right \mathcal{C}-module categories and functors of pointed \mathcal{C}-module categories by \(\text{Mod}_\ast^\mathcal{C}(\mathcal{C})\).

**Theorem 2.6** ([Ost03, JP17]). There is an equivalence of categories \(\text{Alg}(\mathcal{C}) \simeq \text{Mod}_\ast^\mathcal{C}(\mathcal{C})\).

Let us sketch this correspondence. Given \(A \in \text{Alg}(\mathcal{C})\), we get a right \mathcal{C}-module category by taking the category \(\mathcal{M}_A\) of left \(A\)-module objects in \(\mathcal{C}\). Concretely, we can start with objects \(X_A \in \mathcal{M}_A\) for \(X \in \mathcal{C}\) (corresponding to the left \(A\)-modules \(A \otimes X\)) and morphism sets

\[\mathcal{M}_A(X_A, Y_A) = A(X \otimes Y)\]

and then take the idempotent completion. In this presentation the \(\mathcal{C}\)-module structure is induced by \(X_A \otimes U = (X \otimes U)_A\).

Conversely, given \(\mathcal{M} \in \text{Mod}_\ast^\mathcal{C}(\mathcal{C})\), we define \(A_M \in \text{Vec}(\mathcal{C})\) by setting its fibers as

\[A_M(X) = \mathcal{M}(m_{\mathcal{M}} \otimes X, m_{\mathcal{M}}),\]

and its action on \(\psi \in \mathcal{C}(X, Y)\) as

\[A_M(\psi): \mathcal{M}(m_{\mathcal{M}} \otimes Y, m_{\mathcal{M}}) \to \mathcal{M}(m_{\mathcal{M}} \otimes X, m_{\mathcal{M}}), \quad f \mapsto f(id \otimes \psi)\]

Now, let us assume that \(\mathcal{C}\) is a rigid \(\mathcal{C}^\ast\)-tensor category.

**Definition 2.7** ([JP17]). An algebra \(A \in \text{Alg}(\mathcal{C})\) is said to be a \(\mathcal{C}^\ast\)-algebra object in \(\mathcal{C}\) when the corresponding pointed cyclic \(\mathcal{C}\)-module category \(\mathcal{M}_A\) admits a compatible structure of \(\mathcal{C}^\ast\)-category, and a \(W^\ast\)-category when \(\mathcal{M}_A\) admits a compatible structure of \(W^\ast\)-category.

See Appendix [A] for an intrinsic characterization of \(\mathcal{C}^\ast\)-algebra objects. When \(A\) is a \(\mathcal{C}^\ast\)-algebra object in the above sense, we frequently regard \(A(U \otimes U)\) as a \(\mathcal{C}^\ast\)-algebra up to the isomorphism \(A(U \otimes U) \cong \mathcal{M}_A(U_A, U_A)\).

**Definition 2.8.** The category \(\text{Hilb}(\mathcal{C})\) is the subcategory of \(\text{Vec}(\mathcal{C})\) consisting of contravariant \(\ast\)-functors \(\mathcal{C} \to \text{Hilb}\), and having uniformly bounded natural transformations as morphisms: for \(H, K \in \text{Hilb}(\mathcal{C})\),

\[\text{Hilb}(\mathcal{C})(H, K) \cong \ell^\infty \prod_{i \in \text{Irr}(\mathcal{C})} B(H(U_i), K(U_i)).\]

\(\text{Hilb}(\mathcal{C})\) becomes monoidal subcategory of \(\text{Vec}(\mathcal{C})\) by giving the following inner product to the fibers of \(H \otimes K\). On the space \(H(U_i) \otimes \mathcal{C}(X, U_i \otimes U_j) \otimes K(U_j)\), consider the Hermitian inner product characterized by

\[\langle \xi_2 \otimes \alpha_2 \otimes \eta_2, \xi_1 \otimes \alpha_1 \otimes \eta_1 \rangle_1 X = \frac{1}{d_i d_j} \langle \xi_2, \xi_1 \rangle \langle \eta_2, \eta_1 \rangle \alpha_1^\ast \alpha_2.\]

Then \((H \otimes K)(X)\) has an inner product such that [2.4] gives an orthogonal decomposition.
Remark 2.9. An object $H \in \text{Hilb}(C)$ can be interpreted as the infinite direct sum $\bigoplus H(U_i) \otimes U_i$. This way $\text{Hilb}(C)$ can be identified with the ind-category $\text{ind}(C)$ of $C$ as defined in [NY16 Section 2].

We will work with the following $W^*$-algebra object $B(H)$ for $H \in \text{Hilb}(C)$ [JP17 Example 10]. The fiber is given by $B(H)(U) = \text{Hilb}(C)(H \otimes U, H)$, with a natural algebra structure such that the associated module category $M_{B(H)}$ is generated by objects of the form $U_{B(H)}$ for $U \in C$ and the morphism spaces are given by

$$M_{B(H)}(U_{B(H)}, V_{B(H)}) = \text{Hilb}(C)(H \otimes U, H \otimes V).$$

Later, we will use the following analogue of the Gelfand–Naimark Theorem to construct injective envelopes of $C^*$-module categories.

**Theorem 2.10 ([JP17 Theorem 4]).** Every $C^*$-algebra object in $\text{Vec}(C)$ admits a faithful representation into $B(H)$, for some $H \in \text{Hilb}(C)$.

Here, a **faithful representation** means a $^*$-algebra morphism $A \to B(H)$ where the associated functor of module categories $M_A \to M_{B(H)}$ is faithful, or equivalently, the corresponding natural transformation $A(U) \to B(H)(U)$ is given by injective maps.

**Remark 2.11.** When $C$ is $\text{Rep}(G)$ for a compact quantum group $G$, the category $\text{Hilb}(\text{Rep}(G))$ can be identified with the category of all unitary representations of $G$. Then the above theorem reduces to the fact that any $G$-$C^*$-algebra can be equivariantly embedded into one of the form $R(B(H))$ for some infinite dimensional unitary representation $(H, U)$ of $G$.

Recall that if $A$ and $B$ are $C^*$-algebra objects in $\text{Vec}(C)$, then the spaces $A(X \bar{X})$ and $B(X \bar{X})$ are $C^*$-algebras for all $X \in C$.

**Definition 2.12 ([JP17]).** Let $A$ and $B$ be $C^*$-algebra objects in $\text{Vec}(C)$. A **completely positive map** (or a $\text{cp}$ map) from $A$ to $B$ is a natural transformation of contravariant $C^*$-functors $\theta: A \to B$ for which the induced maps $\theta_X: A(X \bar{X}) \to B(X \bar{X})$, with $X \in C$, are positive. If they are also unital, $\theta$ is called a unital completely positive map, or a $\text{ucp}$ map.

**Definition 2.13 ([JP17]).** Let $(M, m)$ and $(M', m')$ be pointed $C^*$-C-module categories. A **multiplier $\Theta$: $(M, m) \to (M', m')$** is a collection of linear maps

$$\Theta_{X,Y}: M(m \otimes X, m \otimes Y) \to M'(m' \otimes X, m' \otimes Y)$$

for $X,Y \in C$ satisfying

$$\Theta_{X \otimes U, Z \otimes U}(((\text{id}_m \otimes \phi)f(\text{id}_m \otimes \psi)) \otimes \text{id}_U) = ((\text{id}_{m'} \otimes \phi)\Theta_{Y,W}(f)(\text{id}_{m'} \otimes \psi)) \otimes \text{id}_U$$

for all $U,X,Y,W,Z \in C$, $\psi \in C(X,Y)$, $\phi \in C(W,Z)$, and $f \in M(m \otimes Y, m \otimes W)$. A multiplier $\Theta$ for which $\Theta_{X,X}$ is positive for all $X$ is called a $\text{cp}$ multiplier. It is called $\text{ucp}$ multiplier if $\Theta_{X,X}$ is $\text{ucp}$ for $X$.

**Proposition 2.14 ([JP17 Proposition 7 and Corollary 5]).** Given two $C^*$-algebra objects $A, B$ in $\text{Vec}(C)$, the space of natural transformations between the contravariant functors $A$ and $B$ is in bijection with the space of multipliers from $M_A$ to $M_B$. Under this bijection, the $\text{cp}$ maps correspond to the $\text{cp}$ multipliers, as the $\text{ucp}$ maps correspond to the $\text{ucp}$ multipliers.

Concretely, given a natural transformation $\theta: A \to B$, the corresponding multiplier is given by

$$\Theta_{V,W}(T) = \sum_{U,\alpha} ((\theta_U(R_W(T \otimes \text{id}_W)v_\alpha)v_\alpha^*) \otimes \text{id}_W)(\text{id}_V \otimes R_W)$$

where $U$ runs over $\text{Irr}(C)$ and $(v_\alpha)_\alpha$ is an orthonormal basis of $C(U,V \otimes W)$. 
3. Injective envelopes for $D(G)$-$C^*$-algebras

In this section we are going to establish the existence of injective envelopes for unital Yetter–Drinfeld $G$-$C^*$-algebras, closely following [HHN22, Section 2].

Take a unitary representation $(H, U)$ of $G$ on $H$, and as before consider the coaction

$$\alpha_H(T) = U_{21}(1 \otimes T)U_{21}$$

on $B(H)$. Then $B(H) \otimes \ell^\infty(\hat{G})$ becomes a $G$-von Neumann algebra with coaction

$$\beta: B(H) \otimes \ell^\infty(\hat{G}) \to L^\infty(G) \otimes B(H) \otimes \ell^\infty(\hat{G}), \quad x \mapsto W_{31}(\alpha_H \otimes \text{id})(x)W_{31}, \quad (3.1)$$

where $W = W_G$ is the multiplicative unitary. Together with the $G$-von Neumann algebra structure given by $\text{id} \otimes \Delta$, we get a Yetter–Drinfeld von Neumann $G$-algebra structure on $B(H) \otimes \ell^\infty(\hat{G})$.

**Proposition 3.1** ([HHN22], Section 2). Under the above setting, the $G$-$C^*$-algebra $\mathcal{R}(B(H) \otimes \ell^\infty(\hat{G}))$ admits a structure of Yetter–Drinfeld $G$-$C^*$-algebra.

Now let us define injective envelopes for unital Yetter–Drinfeld $G$-$C^*$-algebras, in the quite standard way. In this section, by equivariance we always mean equivariance with respect to Yetter–Drinfeld structures.

**Definition 3.2.** A unital Yetter–Drinfeld $G$-$C^*$-algebra $A$ is said to be **injective** if for any equivariant completely isometric map $\phi: B \to C$ of unital Yetter–Drinfeld $G$-$C^*$-algebras and any equivariant ucp map $\psi: B \to A$, there is an equivariant ucp map $\hat{\psi}: C \to A$ that extends $\psi$ along $\phi$, i.e., $\hat{\psi} \circ \phi = \psi$.

**Definition 3.3.** An **injective envelope** of a unital Yetter–Drinfeld $G$-$C^*$-algebra $A$ is given by an injective Yetter–Drinfeld $G$-$C^*$-algebra $I$ and an equivariant complete isometry $\phi: A \to I$ which is **essential**, i.e., for any equivariant ucp map $\psi: I \to B$, $\psi$ is completely isometric if and only if $\psi \phi$ is.

The first step in proving the existence of injective envelopes is to show the existence of enough injective objects. This is accomplished by the following proposition.

**Proposition 3.4.** Let $(H, U)$ be a unitary representation of $G$. The Yetter–Drinfeld $G$-$C^*$-algebra $\mathcal{R}(B(H) \otimes \ell^\infty(\hat{G}))$ is injective.

**Proof.** The proof is essentially the same as [HHN22, Corollary 2.6]. By [HHN22, Proposition 2.5], for any Yetter–Drinfeld $G$-$C^*$-algebra $A$ there is a bijective correspondence between the $G$-equivariant completely bounded maps $\phi: A \to B(H)$ and $(D(G)$-equivariant cb maps $P: A \to B(H) \otimes \ell^\infty(\hat{G})$. Moreover, the image of such map $P$ would be in $\mathcal{R}(B(H) \otimes \ell^\infty(\hat{G}))$.

By the averaging argument (see [HHN22, Lemma 2.10]), the injectivity of $B(H)$ as an operator system implies its injectivity as a $G$-operator system. This implies the injectivity of $\mathcal{R}(B(H) \otimes \ell^\infty(\hat{G}))$ as a Yetter–Drinfeld $G$-$C^*$-algebra. \hfill $\square$

Let $A$ be a Yetter–Drinfeld $G$-$C^*$ algebra, and let us take a faithful representation $\pi: A \to B(H)$ together with a covariant unitary representation $U$ of $G$ on $H$. For example, we may start from a faithful nondegenerate representation $\pi_0: A \to B(H_0)$, and take

$$H = L^2(G) \otimes H_0, \quad \pi = (\lambda \otimes \pi_0)\alpha, \quad U = W_{13} \in \mathcal{M}(\mathcal{K}(L^2(G) \otimes H_0) \otimes C(G)) .$$

**Definition 3.5.** Under the above setting, the **Poisson integral** of $\pi$ is defined as the $D(G)$-equivariant embedding map

$$P_\pi: A \to B(H) \otimes \ell^\infty(\hat{G}), \quad a \mapsto (\pi \otimes \text{id})\beta_A(a).$$

**Theorem 3.6.** Every unital Yetter–Drinfeld $G$-$C^*$-algebra $A$ admits an injective envelope.
Proposition 4.1. Let $H \in \text{Hilb}(\mathcal{C})$ be finite dimensional, corresponding to an object $U \in \mathcal{C}$. Then the pointed cyclic $\mathcal{C}$-module category $\mathcal{M}_{\mathcal{B}(H)}$ is isomorphic to $\mathcal{C}_U$.

Proof. Let us write $\tilde{V}$ for the image of Yoneda embedding of $V \in \mathcal{C}$, so that we have $H \simeq \tilde{U}$. Since $V \mapsto \tilde{V}$ is a monoidal functor, we have

$$(H \otimes \tilde{V})(Z) \simeq \mathcal{C}(Z, U \otimes V).$$
Now, recall that we have
\[
\mathcal{M}_{B(H)}(V, W) = \mathcal{B}(H)(V \otimes \hat{W}) \simeq \ell_\infty \prod_{Z \in \text{Irr}(\mathcal{C})} \mathcal{B}\left((H \otimes \hat{V})(Z), (H \otimes \hat{W})(Z)\right).
\]
We thus have
\[
\mathcal{M}_{B(H)}(V, W) \simeq \ell_\infty \prod_{Z \in \text{Irr}(\mathcal{C})} \mathcal{B}(\mathcal{C}(Z, U \otimes V), \mathcal{C}(Z, U \otimes W)).
\]
From this we see that an element \( \theta \in \mathcal{M}_{B(H)}(V, W) \) is the same thing as a bounded family of linear operators
\[
\theta_Z : \mathcal{C}(Z, U \otimes V) \to \mathcal{C}(Z, U \otimes W) \quad (Z \in \text{Irr}(\mathcal{C})).
\]
This is precisely a natural transformation between the Yoneda embeddings of \( U \otimes V \) and \( U \otimes W \), hence given by a morphism \( U \otimes V \to U \otimes W \). We thus obtained an isomorphism \( \mathcal{M}_{B(H)}(V, W) \simeq \mathcal{C}_U(V, W) \).

**Definition 4.2.** A pointed \( C^* \)-\( \mathcal{C} \)-module category \((\mathcal{M}, m)\) is said to be injective when, for any ucp-multiplier \( \Phi : (\mathcal{N}, n) \to (\mathcal{M}, m) \) and another ucp-multiplier \( \Psi : (\mathcal{N}, n) \to (\mathcal{N}', n') \) such that the maps
\[
\Psi_{V,W} : \mathcal{N}(n \otimes V, n \otimes W) \to \mathcal{N}'(n' \otimes V, n' \otimes W)
\]
are completely isometric for \( V, W \in \mathcal{C} \), there exists a ucp-multiplier \( \tilde{\Phi} : (\mathcal{N}', n') \to (\mathcal{M}, m) \) such that \( \tilde{\Phi}\Psi = \Phi \).

When the generator \( m \) is understood from the context we also say \( \mathcal{M} \) is injective. We will later see that this definition is independent of the choice of \( m \) for cyclic module categories.

In the following lemmas, \((\mathcal{M}, m)\) is a fixed pointed \( C^* \)-\( \mathcal{C} \)-module category.

**Lemma 4.3.** Let \((\mathcal{M}', m')\) be another pointed \( C^* \)-\( \mathcal{C} \)-module category. Then there is a bijective correspondence between the cp-multipliers \( P : (\mathcal{M}, m) \to (\mathcal{M}', m' \otimes U) \) and the cp-multipliers \( Q : (\mathcal{M}, m \otimes \tilde{U}) \to (\mathcal{M}', m') \). Under this correspondence, a ucp-multiplier \( P \) corresponds to a ucp-multiplier \( Q \) satisfying
\[
Q_{U \otimes V, U \otimes V}(\text{id}_m \otimes (R_U R_U^* \otimes \text{id}_V)) = d_U^{-1}\text{id}_{m' \otimes U \otimes V} \tag{4.1}
\]
for all \( V \in \mathcal{C} \).

**Proof.** Let \( P : (\mathcal{M}, m) \to (\mathcal{M}', m' \otimes U) \) be a cp-multiplier, given by the maps
\[
P_{V,W} : \mathcal{M}(m \otimes V, m \otimes W) \to \mathcal{M}'(m' \otimes U \otimes V, m' \otimes U \otimes W).
\]
Then we get a multiplier \( Q : (\mathcal{M}, m \otimes \tilde{U}) \to (\mathcal{M}', m') \) by setting
\[
Q_{V,W} : \mathcal{M}(m \otimes \tilde{U} \otimes V, m \otimes \tilde{U} \otimes W) \to \mathcal{M}'(m' \otimes V, m' \otimes W),
\]
\[
T \mapsto d_U^{-1}(\text{id}_{m'} \otimes \tilde{R}_{U}^* \otimes \text{id}_V)P_{U \otimes V, U \otimes W}(T)(\text{id}_{m'} \otimes \tilde{R}_U \otimes \text{id}_V).
\]
If we have \( V = W \) and \( T \geq 0 \), then \( P_{U \otimes V, U \otimes W}(T) \) is positive by assumption, and since \( d_U^{-1/2} \tilde{R}_U \) is an isometry, we conclude that \( Q_{V,V}(T) \) is positive. This shows that \( Q \) is completely positive. Moreover, when \( P \) is unital, \( Q \) is unital by \( \tilde{R}_U \tilde{R}_U = d_U \), and (4.1) follows from \( \mathcal{C} \)-modularity.

In the other direction, given a cp-multiplier \( Q : (\mathcal{M}, m \otimes \tilde{U}) \to (\mathcal{M}', m') \), we get a multiplier \( P : (\mathcal{M}, m) \to (\mathcal{M}', m' \otimes U) \) by
\[
P_{V,W}(T) = d_U^{-1}Q_{U \otimes V, U \otimes W}(T)(\text{id}_{m} \otimes R_{U} \otimes \text{id}_V).
\]
This is again completely positive by the complete positivity of \( Q \). Moreover, when \( Q \) satisfies (4.1), the unitality of \( P \) is a direct consequence of the above definition.

It remains to check that these constructions are inverse to each other. Let us start from a multiplier \( P \) as above, and let \( Q \) be the corresponding one. Let \( P' : (\mathcal{M}, m) \to (\mathcal{M}', m' \otimes U) \)
denote the multiplier we obtain from $Q$. We then need to check that $P'$ is equal to $P$. Expanding
the definitions, we have
\[
P'(T) = (\text{id}_{m'} \odot \hat{R}_U \otimes \text{id}_U \otimes \text{id}_W) P_{U \otimes U \otimes V; U \otimes U \otimes W}((\text{id}_m \otimes R_U \otimes \text{id}_V) T((\text{id}_m \otimes R_U^* \otimes \text{id}_V)) (\text{id}_m' \otimes \hat{R}_U \otimes \text{id}_U \otimes \text{id}_V).
\]
Using
\[
P_{U \otimes U \otimes V; U \otimes U \otimes W}((\text{id}_m \otimes R_U \otimes \text{id}_V) T((\text{id}_m \otimes R_U^* \otimes \text{id}_V)) = (\text{id}_{m'} \otimes \text{id}_U \otimes R_{U} \otimes \text{id}_W) P_{V; W}(T)((\text{id}_m' \otimes \text{id}_U \otimes R_{U}^* \otimes \text{id}_V)
\]
and the conjugate equations for $U$, we see that $P'(T) = P(T)$ indeed holds. This can also be seen from the graphical calculus for multipliers established in [JP17].

\[\Box\]

**Remark 4.4.** In fact, the condition \([\mathbf{11}]\) for all $V \in C$ implies that $Q$ is a unital multiplier. To see this, given $W \in C$, take $V = U \otimes W$ and consider the operation
\[
T \mapsto (\text{id}_m' \otimes \hat{R}_U \otimes \text{id}_W) T((\text{id}_m' \otimes \hat{R}_U \otimes \text{id}_W)
\]
on both sides of \([\mathbf{11}]\). The left hand side gives $Q_{W; W}(\text{id}_{m \otimes W})$ by the conjugate equations, while the right hand side gives $S_{W; W}$ by $\hat{R}_U \hat{R}_U = d_U$.

**Lemma 4.5** (cf. [HHN22 Proposition 4.11]). There is a bijective correspondence between the ucp-multipliers $(M, m) \to (C, 1_C)$ and the states on the $C^*$-algebra $M(m)$.

**Proof.** Suppose we are given a ucp-multiplier $P$: $(M, m) \to (C, 1_C)$.

Given $V \in C$, define a state $\omega_V$ on $M(m \otimes V)$ by $\omega_V(T) = \text{tr}_V(P_{V; V}(T))$. By the multiplier property, we have
\[
\omega_V(T) = d_V^{-1} P_{1,1}((\text{id}_m \otimes \hat{R}_U^*)(T \otimes \text{id}_V)(\text{id}_m \otimes \hat{R}_U)). \tag{4.2}
\]
We claim that the state $\omega_1 = P_{1,1}$ determines both $\omega_V$ and $P_{V; V}$, which implies that the correspondence from ucp-multipliers to states is one-to-one.

First, \([\mathbf{12}]\) shows that $\omega_V$ can be written in terms of $\omega_1$.

Observe that $C(V)$ embeds into $M(m \otimes V)$ by $T \mapsto \text{id}_m \otimes T$, and the restriction of $\omega_V$ to $C(V)$ agrees with $\text{tr}_V$. Using the spherical structure, we see that $C(V)$ is in the centerizer of $\omega_V$, implying the existence of a unique state-preserving conditional expectation
\[
E_V: M(m \otimes V) \to C(V), \quad \text{tr}_V \circ E_V = \omega_V.
\]
By the uniqueness, we see that $P_{V; V} = E_V$, which shows that $E_V$ is indeed determined by $\omega_V$, hence by $\omega_1$.

Conversely, let $\omega: M(m) \to C$ be a state. Composing the conditional expectation
\[
M(m \otimes V) \to M(m), \quad T \mapsto d_V^{-1} (\text{id}_m \otimes \hat{R}_U^*)(T \otimes \text{id}_V)(\text{id}_m \otimes \hat{R}_U)
\]
with $\omega$, we obtain a state $\omega_V$ on $M(m \otimes V)$.

Again $C(V)$ sits in the centerizer of $\omega$. Setting $P_{V; V}: M(m \otimes V) \to C(V)$ to be the conditional expectation for $\omega_V$, we obtain a ucp-multiplier satisfying $\omega = P_{1,1}$.

\[\Box\]

**Lemma 4.6.** Let $\omega$ be a state on $M(m \otimes U)$, and $Q: (M, m \otimes U) \to (C, 1)$ be the ucp-multiplier corresponding to $\omega$ given by Lemma \(\mathbf{3.4}\). Then $Q$ satisfies \([\mathbf{11}]\) if and only if $\omega$ restricts to the normalized categorical trace on $C(U)$.

**Proof.** Let $\omega$ be a state on $M(m \otimes U)$. Recall that from $\omega$ we can construct state $\omega_V$ on the $C^*$-algebra $M(m \otimes (U \otimes V))$. Let $Q$ be the corresponding ucp-multiplier. Then, to check \([\mathbf{11}]\), it is enough to check
\[
\omega_{U \otimes V}((\text{id}_m \otimes R_U R_{U}^* \otimes \text{id}_V)(\text{id}_{m \otimes U} \otimes S)) = d_U^{-1} \text{tr}_{U \otimes V}(S) \quad (S \in C(U \otimes V))
\]
as $Q_{U⊗V, U⊗V}$ is the unique conditional expectation $\mathcal{M}(m⊗(U⊗V)) → \mathcal{C}(U⊗V)$ such that $tr_{U⊗V} ⊗ Q_{U⊗V, U⊗V} = ω_{U⊗V, U⊗V}$. By the definition of $ω_{U⊗V}$, the left hand side of the above is equal to

$$d_U^{-1}ω((id_m ⊗ (R_U^* ⊗ id^\mathcal{C}))(id_D ⊗ S' ⊗ id^\mathcal{C}))((id_D ⊗ R_U))$$

for $S' = (id ⊗ tr_V)(S)$. If $ω$ restricts to $tr_D$, by a standard sphericity argument we see that this is indeed equal to $d_U^{-1}tr_{U⊗V}(S)$.

Conversely, suppose that we know (4.1). We then have

$$d_UQ_{U,V}(id_m ⊗ (R_U R_U^*(id_D ⊗ S))) = S$$

for any $S ∈ \mathcal{C}(U)$. From the form of state $ω_U$, we obtain

$$ω((id_m ⊗ ((R_U^* ⊗ id_D)(id_D ⊗ ST ⊗ id_D)(id_D ⊗ R_U))) = tr_U(ST) \quad (T ∈ \mathcal{C}(U)).$$

Again by standard sphericity argument we obtain $ω(S') = tr_U(S')$ for

$$S' = (R_U^* ⊗ id_D)(id_D ⊗ S ⊗ id_D)(id_D ⊗ R_U).$$

As such $S'$ exhaust $\mathcal{C}(U)$, we obtain the claim.

We now move towards proving that the category $\text{Mod}^*_c(\mathcal{C})$ has enough injective objects. First we show that objects of a particular type are injective.

**Lemma 4.7.** The object $(\mathcal{C}, 1_\mathcal{C})$ is injective in $\text{Mod}^*_c(\mathcal{C})$.

**Proof.** For any $(\mathcal{M}, m) ∈ \text{Mod}^*_c(\mathcal{C})$, the ucp-multipliers $(\mathcal{M}, m) → (\mathcal{C}, 1_\mathcal{C})$ are completely determined by the induced states on $\mathcal{M}(m)$. Moreover, a completely isometric multiplier $(\mathcal{M}, m) → (\mathcal{M}', m')$ induces a (complete) isometric map of unital $C^*$-algebras $\mathcal{M}(m) → \mathcal{M}(m')$. Then the claim follows from the Hahn–Banach theorem. \(\square\)

**Proposition 4.8.** Let $(\mathcal{M}, m)$ be an injective pointed $\mathcal{C}$-module category. For any $U ∈ \mathcal{C}$ and a direct summand $m'$ of $m ⊗ U$, the pointed $\mathcal{C}$-module category $(\mathcal{M}, m')$ is also injective.

**Proof.** Lemma 4.3 shows that $(\mathcal{M}, m ⊗ U)$ is injective. Thus, it is enough to prove the assertion when $U = 1$. Let us take the projection $p ∈ \mathcal{M}(m)$ corresponding to $m'$, and let $m''$ be the summand corresponding to $1 − p$.

Suppose that $Φ: (\mathcal{N}, n) → (\mathcal{M}, m')$ is a ucp-multiplier, and $Ψ: (\mathcal{N}, n) → (\mathcal{N}', n')$ is a complete isometric multiplier as in Definition 4.2. Take a state $ω: \mathcal{N}(n) → \mathcal{C}$, and let $P: \mathcal{N} → \mathcal{C}$ be the corresponding ucp-multiplier. Composing this with the ucp-multiplier $Q: \mathcal{C} → (\mathcal{M}, m'')$ given by

$$\mathcal{C}(V,W) → \mathcal{M}(m'' ⊗ V, m'' ⊗ W), \quad T ↦ (1 − p) ⊗ T,$$

we obtain a ucp-multiplier $Φ': (\mathcal{N}, n) → (\mathcal{M}, m'')$. By taking the direct sum $Φ ⊕ Φ'$, we get a ucp multiplier $(\mathcal{N}, n) → (\mathcal{M}, m)$. Then the injectivity of $(\mathcal{M}, m)$ gives an ucp extension $Ψ': (\mathcal{N}', n') → (\mathcal{M}, m)$. Then the maps $Ψ_{V,W}(T) = (p ⊗ id_W)Ψ'_{V,W}(T)(p ⊗ id_V)$ give a desired ucp extension $(\mathcal{N}', n') → (\mathcal{M}, m')$. \(\square\)

**Proposition 4.9.** Let $H ∈ \text{Hilb}(\mathcal{C})$ be a finite dimensional object. Then $\mathcal{M}_{B(H)}$ is an injective pointed $\mathcal{C}$-module category.

**Proof.** Let $H$ be a finite dimensional Hilbert space object. By Proposition 4.1 there is $U ∈ \mathcal{C}$ such that $\mathcal{M}_{B(H)} ≃ \mathcal{C}U ≃ (\mathcal{C}, U)$. Since $(\mathcal{C}, 1_\mathcal{C})$ is injective by Lemma 4.7, $(\mathcal{C}, U)$ is injective by Proposition 4.8. \(\square\)

Now, following Arveson’s proof for the injectivity of $B(H)$ for Hilbert spaces, we obtain the following.

**Theorem 4.10.** For any $H ∈ \text{Hilb}(\mathcal{C})$, the $C^*$-$\mathcal{C}$-module category $\mathcal{M}_{B(H)}$ is injective.
\textbf{Proof.} Let $A$ and $B$ be $\mathcal{C}$*-algebra objects, and suppose that $\iota: A \to B$ is a completely isometric multiplier and $\psi: A \to \mathcal{B}(H)$ is a ucp multiplier. We then need to construct an extension $\hat{\psi}: B \to \mathcal{B}(H)$.

Take an increasing net $(H_\lambda)_{\lambda \in \Lambda}$ of finite dimensional Hilbert space objects such that $H_\lambda(X) \subset H(X)$ and $H(X) = \lim_{\lambda \in \Lambda} H_\lambda(X)$ for all $X \in \mathcal{C}$. Let us denote the orthogonal projections $H(X) \to H_\lambda(X)$ by $p_{\lambda,X}$. Then we get a morphism $p_\lambda: H \to H_\lambda$ by

$$
(p_{\lambda,X})_{X \in \text{Irr}(\mathcal{C})} \in \ell^\infty \prod_{X \in \text{Irr}(\mathcal{C})} \mathcal{B}(H(X), H_\lambda(X)) \simeq \text{Hilb}(\mathcal{C})(H, H_\lambda).
$$

The maps $\text{Ad}(p_\lambda): \mathcal{B}(H_\lambda) \to \mathcal{B}(H)$ and $\text{Ad}(p_\lambda^*): \mathcal{B}(H) \to \mathcal{B}(H_\lambda)$ are cp and ucp multipliers, respectively [JP17, Lemma 4.27].

For fixed $\lambda$, since $\text{Ad}(p_\lambda^*) \circ \psi$ is a ucp map from $A$ to $\mathcal{B}(H_\lambda)$, we obtain a ucp extension $\hat{\psi}_\lambda: B \to \mathcal{B}(H_\lambda)$ by Proposition 4.9. Then $\hat{\psi}_\lambda = \text{Ad}(p_\lambda) \circ \psi_\lambda$ is a completely contractive positive map from $B$ to $\mathcal{B}(H)$. We claim that a limit of the family $(\psi_\lambda)_\lambda$ is a desired ucp extension of $\psi$.

For each $V \in \mathcal{C}$, the map

$$
\hat{\psi}_{\lambda,V}: B(V) \to \mathcal{B}(H)(V) \simeq \ell^\infty \prod_{X \in \text{Irr}(\mathcal{C})} \mathcal{B}((H \otimes V)(X), H(X))
$$

is completely contractive. By passing to a subnet, we may assume that $\psi_V = \lim_\lambda \psi_{\lambda,V}$ exists as a complete contraction for each irreducible $V$, where we consider the topology of pointwise convergence with respect to the weak operator topology on $\mathcal{B}(H)(V)$ up to the above identification.

Now, collecting $\hat{\psi}_V$ as above for irreducible $V$, we get a natural transformation $\hat{\psi}: B \to \mathcal{B}(H)$, and by Proposition 2.14 a multiplier $\hat{\Psi}: \mathcal{M}_B \to \mathcal{M}_{\mathcal{B}(H)}$. Let $(\hat{\Psi}_{\lambda,V,W})_{\lambda,V,W}$ be the cp multiplier corresponding to $\psi_\lambda$. We then claim that $\hat{\Psi}_{V,W}(T) = \lim_\lambda \hat{\Psi}_{\lambda,V,W}(T)$ for the weak operator topology. This follows from the correspondence (2.5) and $\hat{\psi}_V(S) \otimes \text{id}_W = \lim_\lambda \psi_{\lambda,V}(S) \otimes \text{id}_W$ for $S \in B(U)$.

Then, setting $V = W$, we obtain the complete positivity of $\hat{\Psi}$. Moreover, $\text{id}_V \in \mathcal{M}_B(V_B, V_B)$ is the image of $\text{id}_{V_\lambda} \in \mathcal{M}_A(V_\lambda, V_\lambda) = \mathcal{M}_A(V_\lambda, V_\lambda)$, so we have

$$
\hat{\Psi}_{V,V}(\text{id}_V) = \lim_\lambda (p_\lambda^* p_\lambda)_V = \text{id}_{H(V)}.
$$

This proves that $\hat{\Psi}$ is unital, hence $\hat{\psi}$ is a ucp extension of $\psi$.

\begin{definition}
A functor $F: \mathcal{M} \to \mathcal{N}$ between $\mathcal{C}$*-C-module is said to be an embedding if it is faithful and norm-closed at the level of morphism spaces.
\end{definition}

\begin{definition}
An injective envelope of a $\mathcal{C}$*-C-module category $(\mathcal{M}, m)$ is an injective $\mathcal{C}$-module category $(\mathcal{T}, m')$ endowed with a faithful module functor $\iota: (\mathcal{M}, m) \to (\mathcal{T}, m')$ such that $\text{Id}_T$ is the only ucp extension of $\iota$.
\end{definition}

\begin{theorem}
Every cyclic $\mathcal{C}$*-C-module category $(\mathcal{M}, m)$ has an injective envelope.
\end{theorem}

\textbf{Proof.} As we only deal with cyclic module categories in this proof, for simplicity we write $\mathcal{M}$ instead for $(\mathcal{M}, m)$, and similarly for other cyclic module categories.

Theorems 2.10 and 4.10 imply that $\mathcal{M}$ embeds into an injective $\mathcal{C}$-module $W^*$-category $\mathcal{N}$, namely $\mathcal{N} = \mathcal{M}_{\mathcal{B}(H)}$ for some $H \in \text{Hilb}(\mathcal{C})$. Define $i: \mathcal{M} \to \mathcal{N}$ to be this embedding. Consider the semigroup of ucp multipliers

\[ S = \{ \Phi: \mathcal{N} \to \mathcal{N} | \Phi \circ i = i \}. \]
This can be identified with a weak*-closed convex set in the dual Banach space

\[ \ell^\infty \prod_{U,V \in \text{Irr}(C)} \mathcal{N}(U,V) = \left[ \ell^1 \bigoplus_{U,V \in \text{Irr}(C)} \mathcal{N}(U,V)_* \right]^\wedge. \]

By Proposition 2.1, \( \mathcal{C} \) has a minimal idempotent \( \Psi \).

Let us make sense of the image of \( \Psi \) as a \( \mathcal{C}^* \)-module category. For each \( U \) and \( V \) in \( \mathcal{C} \), set

\[ \Psi(\mathcal{N})(U,V) = \Psi(\mathcal{N}(U,V)). \]

When we have \( U = V \), we get a structure of \( \mathcal{C}^* \)-algebra on this space by the Choi–Effros product \( S \cdot T = \Psi(ST) \). Generally, by considering \( X = U \otimes V \otimes W \) and \( \Psi(\mathcal{N})(X,X) \), we get the composition maps

\[ \Psi(\mathcal{N})(V,W) \times \Psi(\mathcal{N})(U,V) \to \Psi(U,W) \]

that defines a \( \mathcal{C}^* \)-category \( \Psi(\mathcal{N}) \). Moreover, using the fact that \( \Psi \) is a multiplier, we obtain maps

\[ \Psi(\mathcal{N})(V,W) \otimes \mathcal{C}(V',W') \to \Psi(\mathcal{N})(V \otimes V', W \otimes W') \]

that defines a structure of \( \mathcal{C} \)-module category on \( \Psi(\mathcal{N}) \).

We next claim that \( \Psi(\mathcal{N}) \) is an injective envelope for \( \mathcal{M} \). Again by construction \( i \) induces maps

\[ \mathcal{M}(m \otimes V, m \otimes W) \to \Psi(\mathcal{N})(V,W), \]

and by the multiplicative domain argument this is a functor of \( \mathcal{C}^* \)-categories. It is straightforward to check the compatibility with \( \mathcal{C} \)-module structures. Moreover, the uniqueness of \( \text{Id}_{\Psi(\mathcal{N})} \) as a ucp multiplier stabilizing \( \mathcal{M} \) is obvious from the characterization of \( \Psi \). \( \square \)

Let \( (\mathcal{M},m) \) be a cyclic \( \mathcal{C}^* \)-module category, and let \( \Psi : (\mathcal{M},m) \to (\mathcal{M},m) \) be a ucp multiplier. The proof of Theorem 4.13 shows that the image of \( \Psi \) has a natural composition rule which makes it a cyclic \( \mathcal{C}^* \)-module category: this result does not depend on minimality of \( \Psi \), and it will be used later again to prove existence of injective envelopes for a class of bimodule categories, which can be equivalently understood as module categories over \( \mathcal{C} \otimes \mathcal{C}^{\text{op}} \).

5. Centrally pointed bimodule categories

Fix a rigid \( \mathcal{C}^* \)-tensor category \( \mathcal{C} \). Then a \( \mathcal{C} \)-bimodule category \( \mathcal{M} \) is defined in an analogous way to right \( \mathcal{C} \)-module categories with three sets of module associativity morphisms

\[ \Psi' : (V \otimes W) \otimes X \to V \otimes (W \otimes X), \quad \Psi : (X \otimes V) \otimes W \to X \otimes (V \otimes W) \]

\[ \Psi'' : (V \otimes X) \otimes W \to V \otimes (X \otimes W) \]

for \( X \in \mathcal{M} \) and \( V,W \in \mathcal{C} \), satisfying the pentagon-type equations with the associativity morphism of \( \mathcal{C} \). A \( \mathcal{C} \)-bimodule functor \( F : \mathcal{M} \to \mathcal{M}' \) is given by a linear functor of underlying categories, together with

\[ 2F_{V,X} : V \otimes F(X) \to F(V \otimes X), \quad F_{2,X,V} : F(X) \otimes V \to F(X \otimes V), \]

again compatible with structure morphisms of \( \mathcal{M} \) and \( \mathcal{M}' \).

Remark 5.1. An equivalent, more succinct way of defining \( \mathcal{C}^* \)-bimodule categories, is by means of the max tensor product \( \mathcal{C}^{\text{max}} \otimes \mathcal{C}^{\text{op}} \) defined in \cite{AV20}. Indeed, given a \( \mathcal{C}^* \)-category \( \mathcal{M} \), there is an isomorphism between the space of \( \mathcal{C}^* \)-functors from \( \mathcal{C}^{\text{max}} \otimes \mathcal{C}^{\text{op}} \) to \( \text{End}(\mathcal{M}) \) and that of bilinear \( * \)-functors from \( \mathcal{C} \times \mathcal{C}^{\text{op}} \) to \( \text{End}(\mathcal{M}) \) by \cite{AV20}, Proposition 3.3. This can be promoted to a correspondence between the \( \mathcal{C}^* \)-\( \mathcal{C} \)-\( \mathcal{C}^{\text{op}} \)-module category structures on \( \mathcal{M} \) and the \( \mathcal{C}^* \)-\( \mathcal{C} \)-bimodule category structures on \( \mathcal{M} \).

To simplify our presentation we further assume that \( \mathcal{C} \) and \( \mathcal{M} \) are strict.
Definition 5.2. Let $\mathcal{M}$ be a $C^*-C$-bimodule category, and $m \in \mathcal{M}$. We say that $m$ is central if there is a family of unitary morphisms

$$\sigma_U: U \otimes m \to m \otimes U$$

that is natural in $U$ and satisfy the braid relations

$$\sigma_{U \otimes V} = (\sigma_U \otimes \text{id}_V)(\text{id}_U \otimes \sigma_V),$$

for all $U, V \in \mathcal{C}$. We call $(\mathcal{M}, m)$ a centrally pointed bimodule category. We call it a cyclic centrally pointed bimodule category when furthermore $m$ is a generator of $\mathcal{M}$ under the action of $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$.

Remark 5.3. In the above definition, because of centrality, $m$ is a generator of $\mathcal{M}$ as a bimodule category if and only if it generates $\mathcal{M}$ under either the right or left action of $\mathcal{C}$ on $\mathcal{M}$.

Definition 5.4. Let $(\mathcal{M}, m)$ and $(\mathcal{M}', m')$ be centrally pointed bimodule categories. A central functor between them is a bimodule functor $F: \mathcal{M} \to \mathcal{M}'$ endowed with an isomorphism $F_0: m' \to F(m)$ which are compatible with central generators and structure morphisms, in the sense that the diagram

$$
\begin{array}{ccc}
U \otimes m' & \xrightarrow{\sigma_U'} & m' \otimes U \\
\downarrow & & \downarrow \\
U \otimes F(m) & \xrightarrow{F_2} & F(m) \otimes U \\
\downarrow & & \downarrow \\
F(U \otimes m) & \xrightarrow{F(\sigma_U)} & F(m \otimes U)
\end{array}
$$

is commutative.

Definition 5.5. Let $F$ and $F'$ be central functors from $(\mathcal{M}, m)$ to $(\mathcal{M}', m')$. A natural transformation of bimodule functors $\alpha: F \to F'$ is given by a natural transformation of the underlying linear functors $\alpha_X: F(X) \to F'(X)$ for $X \in \mathcal{M}$ such that the diagrams

$$
\begin{array}{ccc}
U \otimes F(X) & \xrightarrow{\alpha_U \otimes X} & F(U) \otimes X \\
\downarrow & & \downarrow \\
U \otimes F'(X) & \xrightarrow{F'\alpha} & F'(U) \otimes X \\
\downarrow & & \downarrow \\
F(U \otimes m) & \xrightarrow{F(\alpha_U)} & F'(m \otimes U)
\end{array}
$$

and

$$
\begin{array}{ccc}
F(X) \otimes U & \xrightarrow{\alpha_X \otimes U} & F(X) \otimes U \\
\downarrow & & \downarrow \\
F(X) \otimes U & \xrightarrow{\alpha_X \otimes U} & F(X) \otimes U \\
\downarrow & & \downarrow \\
F(m) & \xrightarrow{F_0} & F(m)
\end{array}
$$

and

$$
\begin{array}{ccc}
F(X) \otimes U & \xrightarrow{\alpha_X \otimes U} & F(X) \otimes U \\
\downarrow & & \downarrow \\
F(X) \otimes U & \xrightarrow{\alpha_X \otimes U} & F(X) \otimes U \\
\downarrow & & \downarrow \\
F'(X) \otimes U & \xrightarrow{F'(\alpha_X)} & F'(X) \otimes U \\
\downarrow & & \downarrow \\
F'(m) & \xrightarrow{F'\alpha} & F'(m)
\end{array}
$$

commute.

We denote by $\mathcal{CB}(\mathcal{C})$ be the category of centrally pointed $C^*-C$-bimodule categories and central bimodule functors, and by $\mathcal{CB}^c(\mathcal{C})$ the subcategory of cyclic centrally pointed $C^*-C$-bimodule categories.

Given a central object as above, we often work with the induced maps

$$\Sigma_{U,V,W}: \mathcal{M}(m \otimes V, m \otimes W) \to \mathcal{M}(m \otimes U \otimes V, m \otimes U \otimes W)$$

$$T \mapsto (\sigma_U \otimes \text{id}_W)(\text{id}_U \otimes T)(\sigma_U \otimes \text{id}_V).$$

(5.2)

Proposition 5.6. Let $(\mathcal{M}, m)$ a centrally pointed bimodule category. There is an embedding of centrally pointed bimodule categories $F: (\mathcal{C}, 1_C) \to (\mathcal{M}, m)$ which sends $U$ to $m \otimes U$.

Proof. For simplicity we assume that the right action of $\mathcal{C}$ on $\mathcal{M}$ is strict. Then we get a right $\mathcal{C}$-module functor $F(U) = m \otimes U$ with $F_2 = \text{id}_{m \otimes U}$. We extend it to a bimodule functor by setting

$$\sigma_U \otimes \text{id}_V: (U \otimes m) \otimes V \to m \otimes U \otimes V = m \otimes (U \otimes V).$$

Then consistency conditions of $\sigma_U$ follow from those of the half-braiding. □
Definition 5.7. Let $\mathcal{M}$ be a right $C^*-\mathcal{C}$-module category, and $m \in \mathcal{M}$. We denote by $\tilde{\mathcal{M}}_m$ the idempotent completion of $\mathcal{C}$ with the enlarged morphism sets

$$\tilde{\mathcal{M}}_m(V,W) = \text{Nat}_V(m \otimes \iota \otimes V, m \otimes \iota \otimes W) \simeq \ell^\infty \prod_{X \in \text{Irr}(\mathcal{C})} \mathcal{M}(m \otimes X \otimes V, m \otimes X \otimes W). \tag{5.3}$$

This is a bimodule category, with $1$ (which corresponds to $m \in \mathcal{M}$) being a central generator. The bimodule structure of $\tilde{\mathcal{M}}_m$ is as follows. At the level of objects, for $T \in \tilde{\mathcal{M}}_m(V,W)$ and $X \in \mathcal{C}$, we define the monoidal products of $T$ and $\text{id}_X$ by

$$(T \otimes \text{id}_X)_V = T_Y \otimes \text{id}_X, \quad (\text{id}_X \otimes T)_V = T_{Y \otimes X}.$$ 

Then we see that $1 \in \tilde{\mathcal{M}}_m$ is indeed a central object, with $\sigma_U$ given by $\text{id}_U$.

This is motivated by the ‘dual category’ $\check{\mathcal{C}}$ introduced in [NY17], which corresponds to case of $\mathcal{M} = \mathcal{C}$ and $m = 1_C$. In this case there is a natural $C^*$-tensor structure on $\check{\mathcal{C}}$ (with nonsimple unit) such that $U \mapsto U$ is a $C^*$-tensor functor from $\mathcal{C}$ to $\check{\mathcal{C}}$. If we take $m = U$ instead, because of the centrality, the resulting category can be identified with $\check{\mathcal{C}}_U$, with morphism sets

$$\check{\mathcal{C}}_U(V,W) = \check{\mathcal{C}}(U \otimes V, U \otimes W) = \ell^\infty \prod_{i \in \text{Irr}(\mathcal{C})} \mathcal{C}(U \otimes U \otimes V, U \otimes U \otimes W)$$

for $V,W \in \mathcal{C}$. The evaluation at the unit $1_C$ gives a multiplier $\check{\mathcal{C}}_U \rightarrow \mathcal{C}_U$ that is a conditional expectation (completely idempotent onto $\mathcal{C}_U$).

Suppose $\mathcal{C} = \text{Rep}(G)$ for some compact quantum group $G$. Up to the Tannaka–Krein type duality for actions, the $C^*-\mathcal{C}$-module category $\check{\mathcal{C}}$ corresponds to the discrete dual quantum group with adjoint action, see [NY14 Section 4.1]. When $(H,U)$ is a finite dimensional unitary representation of $G$, the module category corresponding to $\mathcal{R}(\mathcal{B}(H) \otimes \ell^\infty(\hat{G}))$ is $\check{\mathcal{C}}_U$, cf. Proposition 4.1. Our constructions leading up to Theorem 3.6 shows that, with a suitable generalization to infinite dimensional $H$, the categorical dual of any Yetter–Drinfeld $C^*$-algebra can be embedded in a category analogous to $\check{\mathcal{C}}_U$, in a way compatible with extra bimodule structures as we will see in Section 6.

5.1. Trivialization of central structure. Let us consider a construction that we call trivialization of the central structure. Let $(\mathcal{M}, m, \sigma)$ be a centrally pointed $C^*-\mathcal{C}$-bimodule category. Then, as usual, we consider the idempotent completion of $\mathcal{C}$ with morphism sets $\text{Mor}(V,W) = \mathcal{M}(m \otimes V, m \otimes W)$. In other words, our objects are projections $p$ in the $C^*$-algebras $\mathcal{M}(m \otimes V)$. Given such $p$, and another $U \in \mathcal{C}$, consider the projection

$$\Sigma_{U,V}(p) = (\sigma_U \otimes \text{id}_V)(\text{id}_U \otimes p)(\sigma_U^{-1} \otimes \text{id}_V) \in \mathcal{M}(m \otimes U \otimes V),$$

which defines a subobject of $U \otimes V$ in our category. Of course, this is isomorphic to the left action of $U$ on the subobject of $m \otimes V$ defined by $p$, i.e., isomorphic to the subobject of $U \otimes m \otimes V$ defined by the projection $\text{id}_U \otimes p$. This way (assuming $\mathcal{C}$ is strict) we get a strict bimodule category. Moreover, $1$ is a central object, with the structure morphisms $\sigma_U' = \text{id}_U$.

Proposition 5.8. The centrally pointed bimodule category $\tilde{\mathcal{M}}$ obtained by trivialization of the central structure is isomorphic to the original one.

Proof. Define $F_\sigma : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ as (the canonical extension of) the functor $V \mapsto m \otimes V$ at the level of objects, and by the tautological map at the level of morphisms. This is a right module functor. The compatibility with left module structures,

$$2(F_\sigma) : V \otimes m \otimes W = V \otimes F_\sigma(W) \rightarrow m \otimes V \otimes W = F_\sigma(V \otimes W),$$

is given by $\sigma_V \otimes \text{id}_W$. Then the commutativity of (5.1) holds by construction.

In the other direction, to define $F : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$, we choose $V_X \in \mathcal{C}$ and an isometry $j_X : X \mapsto m \otimes V_X$ for each $X \in \mathcal{M}$. At the level of $C^*$-functor, for each $X \in \mathcal{M}$ the object $F(X)$ is
represented by the projection \( j_X j_X^* \in \tilde{M}(V_X) \), while for each \( T \in \mathcal{M}(X,Y) \) the morphism \( F'(T) \) is represented by \( j_Y T j_X^* \in \mathcal{M}(V_X,V_Y) \).

Now, we can define the natural isomorphisms giving compatibility with bimodule structure as follows. For the right module structure, we take

\[
F_2 = j_X \otimes U (j_X^* \otimes \text{id}_U) : F'(X) \otimes U \to F(X \otimes U)
\]

while for the left module structure we take

\[
2F' = j_U \otimes X (\text{id}_U \otimes j_X^*)(\sigma_V \otimes \text{id}_{V_X}) : \mathcal{U} \otimes F(X) \to F(U \otimes X)
\]

Then the composition \( F_\sigma F : \mathcal{M} \to \mathcal{M} \) is naturally isomorphic to the identity functor \( \text{Id}_\mathcal{M} \). Indeed, the assumption on cyclicity says that \( \mathcal{M} \ni X \mapsto (m \otimes V_X, j_X j_X^*) \) extends to an equivalence of right \( \mathcal{C}^\ast\mathcal{C} \)-module categories. Under this equivalence, \( F_\sigma F \) corresponds to the identity functor on \( \mathcal{M} \), and this correspondence is by construction compatible with the module structures. A similar argument shows that \( FF_\sigma \) is naturally isomorphic to the identity functor on \( \mathcal{M} \).

**Proposition 5.9.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be \( \mathcal{C} \)-bimodule categories with central generators \( m_1 \) and \( m_2 \), respectively. Let \( F : \mathcal{M}_1 \to \mathcal{M}_2 \) be a functor of right \( \mathcal{C} \)-module categories such that \( F(m_1) = m_2 \). Suppose that the maps \( f_{5.2} \) satisfy

\[
F_{2,m_1, U \otimes W}^{-1} F(S_{U,V,W}(T)) F_{2,m_1, U \otimes V} = \sum_{U,V,W} F_{2,m_1, W}(T) F_{2,m_1, V} (T \in \mathcal{M}_1(m_1 \otimes V, m_1 \otimes W)). \tag{5.4}
\]

Then we have a strict bimodule functor \( \tilde{F} : \tilde{\mathcal{M}}_1 \to \tilde{\mathcal{M}}_2 \) characterized by

\[
\tilde{F}(T) = F_{2,m_1, W}^{-1} F(T) F_{2,m_1, V}
\]

for morphisms \( T \in \tilde{\mathcal{M}}_1(V,W) = \mathcal{M}_1(m_1 \otimes V, m_1 \otimes W) \).

**Proof.** By the above formula, an object of \( \tilde{\mathcal{M}}_1 \) represented by a projection \( p \in \mathcal{M}_1(m_1 \otimes V) \) is mapped to the object of \( \tilde{\mathcal{M}}_2 \) represented by the projection \( F_{2,m_1, V}^{-1} F(p) F_{2,m_1, V} \).

Then the above claim says that \( \tilde{F} \) is a strict functor of left \( \text{Rep}(G) \)-module categories. The compatibility of \( F_2 \) and the (trivial) associativity morphisms imply that \( \tilde{F} \) is also a strict functor of right module categories.

The above functor \( \tilde{F} \) satisfies \( \tilde{F}(\tilde{\sigma}_U) = \tilde{\sigma}_U \) for the half-braidings of \( \tilde{\mathcal{M}}_1 \) and \( \tilde{\mathcal{M}}_2 \), since \( \tilde{\sigma}_U \) is given by \( \text{id}_{m_1 \otimes U} \in \mathcal{M}(m_1 \otimes U) \). Combined with Proposition 5.8, we obtain a central functor \( \mathcal{M}_1 \to \mathcal{M}_2 \) from a right module functor satisfying 5.4. Consequently, \( \mathcal{CB}(\mathcal{C}) \) is equivalent to the category with the same objects but right module category functors satisfying 5.4 as morphisms.

### 6. Duality for Yetter–Drinfeld \( \mathcal{C}^\ast \)-algebras

In this section, we establish the Tannaka–Krein duality for the unital Yetter–Drinfeld \( \mathcal{G} \)-\( \mathcal{C}^\ast \)-algebras. In our presentation, their categorical dual are cyclic bimodules over \( \text{Rep}(G) \) from Section 5. More precisely, let \( \mathcal{YD}(\mathcal{G}) \) denote the category of Yetter–Drinfeld \( \mathcal{G} \)-\( \mathcal{C}^\ast \)-algebras and equivariant unital *-homomorphisms. We then have the following correspondence.

**Theorem 6.1.** The categories \( \mathcal{YD}(\mathcal{G}) \) and \( \mathcal{CB}(\text{Rep}(\mathcal{G})) \) are equivalent. Under this equivalence, two morphisms \( F, F' : \mathcal{M} \to \mathcal{M}' \) in \( \mathcal{CB}(\text{Rep}(\mathcal{G})) \) induce the same homomorphism \( B_\mathcal{M} \to B_{\mathcal{M}'} \) if and only if they are naturally isomorphic as central bimodule functors.

The proof of this theorem will occupy the rest of the section.

**Remark 6.2.** An alternative presentation of duality as above was recently given in [FTW23] in terms of weak monoidal functors [PoissonNesi].
6.1. **From Yetter–Drinfeld algebras to centrally pointed bimodules.** Let $B$ be a Yetter–Drinfeld C*-algebra with $G$-action $\alpha$ and $G$-action $\beta$. Let us show that the category $\mathcal{D}_B$ admits a structure of centrally pointed bimodule category.

Given a unitary finite dimensional representation $U$ of $G$, we fix a choice $\{m_{ij}^U\}_{i,j}$ of matrix units for $B(H_U)$ and write

$$U = \sum_{i,j} m_{ij}^U \otimes u_{ij}.$$  

As we did in Section 2.4 on the right Hilbert $B$-module $H_U \otimes B$ we consider the coaction of $C(G)$ given by

$$\delta_U(\xi \otimes b) = U_{21}^*(b_1 \otimes \xi \otimes b_2),$$

where $\alpha(b) = b_1 \otimes b_2$ denotes the coaction of $G$ on $B$.

**Proposition 6.3.** For a finite dimensional unitary representation $U$ of $G$, consider the homomorphism $\pi_U: B \rightarrow \text{End}_B(H_U \otimes B)$ given by

$$\pi_U(a) = \sum_{i,j} m_{ij}^U \otimes (u_{ij} \triangleright a).$$

Then we have the equivariance condition

$$\delta_U(\pi_U(a)(\xi \otimes b)) = ((\text{id} \otimes \pi_U)\alpha(a)) (\delta_U(\xi \otimes b)) \quad (a, b \in B, \xi \in H_U).$$

**Proof.** Let $\{\xi_i\}_i$ be the basis of $H_U$ corresponding to the matrix units $\{m_{ij}^U\}_{ij}$. Let us check the claim for $\xi = \xi_i$.

The left hand side is

$$\delta_U(\pi_U(a)(\xi_i \otimes b)) = \delta_U \left( \sum_{jk} m_{jk}^U(\xi_i) \otimes (u_{jk} \triangleright a)b \right)$$

$$= \sum_{j,r,s} u_{rs}^*(u_{ji} \triangleright a)(1) b_1 \otimes m_{rs}^U(\xi_j) \otimes (u_{ji} \triangleright a)(2) b_1$$

$$= \sum_{j,s,k,l} u_{js}^* u_{jk} a_1 u_{kl}^* b_1 \otimes \xi_s \otimes (u_{kl} \triangleright a_2) b_2,$$

since

$$\alpha(u_{ij} \triangleright a) = (u_{ij} \triangleright a)(1) \otimes (u_{ij} \triangleright a)(2) = \sum_{k,l} u_{jr} a_1 u_{kl}^* \otimes (u_{kl} \triangleright a_2)$$

holds by the Yetter–Drinfeld property. As $\sum_j u_{js}^* u_{jk} = \delta_{sk}$

$$\delta_U(\pi_U(a)(\xi_i \otimes b)) = \sum_{kl} a_1 u_{kl}^* b_1 \otimes \xi_k \otimes (u_{kl} \triangleright a_2) b_2$$

(6.1)

For the right hand side, we have

$$((\text{id} \otimes \pi_U)\alpha(a)) \delta_U(\xi_i \otimes b) = (a_1 \otimes \pi_U(a_2)) \left( \sum_{j,k} u_{jk}^* b_1 \otimes m_{kj}^U(\xi_i) \otimes b_2 \right)$$

$$= \sum_k a_1 u_{ik}^* b_1 \otimes \pi_U(a_2)(\xi_k \otimes b_2)$$

$$= \sum_{k,r,s} a_1 u_{rk}^* b_1 \otimes m_{rs}^U(\xi_k) \otimes (u_{rs} \triangleright a_2) b_2$$

$$= \sum_{k,l} a_1 u_{kl}^* b_1 \otimes \xi_l \otimes (u_{lk} \triangleright a_2) b_2,$$

which proves the claim. □
Thus, $H_U \otimes B$ becomes an equivariant bimodule over $B$. We use this structure to define balanced tensor products

$$(H_U \otimes B) \otimes_B (H_U \otimes B),$$

which is again a equivariant Hilbert $B$-module. Note that the canonical isomorphism $B \otimes_B (H_U \otimes B) \simeq H_U \otimes B$ is $G$-equivariant. Given $U, V \in \text{Rep}(G)$, the map

$S_{U,V} : (H_V \otimes B) \otimes_B (H_U \otimes B) \to H_{U \otimes V} \otimes B,$

$(\zeta \otimes b) \otimes_B (\xi \otimes a) \mapsto \sum_{i,j} m^V_{ij}(\zeta) \otimes \zeta \otimes (u_{ij} \triangleright b)a$

is an equivariant isomorphism of right Hilbert $B$-modules.

**Proposition 6.4.** The map $S_{U,V}$ is a $B$-bimodule isomorphism.

**Proof.** We need to show that the left $B$-actions, which is equivalent to the equality

$$\pi_V(b) \otimes \text{id} = S_{U,V}^{-1}\pi_{U \otimes V}(b)S_{U,V}$$

for $b \in B$.

Let $a, b, c \in B$ and $\xi_{i_0} \in H_U, \zeta_{k_0} \in H_V$ be basis elements corresponding to the respective matrix units of $U$ and $V$. We then have

$$S_{U,V}(\pi_V(c)(\zeta_{k_0} \otimes b) \otimes_B (\xi_{i_0} \otimes a)) = S_{U,V}\left(\sum_k (\zeta_k \otimes (v_{k_0} \triangleright c)b) \otimes_B (\xi_{i_0} \otimes a)\right)$$

$$= \sum_{i,k} \xi_i \otimes \zeta_k \otimes (u_{i_0i} \triangleright c)(u_{ji} \triangleright b)a.$$

On the other hand, we also have

$$S_{U,V}(\pi_V(c)(\zeta_{k_0} \otimes b) \otimes_B (\xi_{i_0} \otimes a)) = S_{U,V}\left(\sum_k (\zeta_k \otimes (v_{k_0} \triangleright c)b) \otimes_B (\xi_{i_0} \otimes a)\right)$$

$$= \sum_{i,k} \xi_i \otimes \zeta_k \otimes (u_{i_0i} \triangleright ((v_{k_0} \triangleright c)b))a$$

$$= \sum_{i,j,k} \xi_i \otimes \zeta_k \otimes (u_{ij}v_{k_0} \triangleright c)(u_{ji} \triangleright b)a$$

$$= \pi_{U \otimes V}S_{U,V}((\zeta_{k_0} \otimes b) \otimes_B (\xi_{i_0} \otimes a)),
$$

which proves the claim. \qed

**Proposition 6.5.** We have

$$S_{U \otimes V,W}(\text{id}_{H_{U \otimes B} \otimes S_{U,V}}) = S_{U \otimes V,W}S_{V,W} \otimes \text{id}_{H_{U \otimes B}}.$$

**Proof.** Let us compare the action of both sides on vectors of the form $(\zeta_{k_0} \otimes c) \otimes (\eta_{j_0} \otimes b) \otimes (\xi_{i_0} \otimes a)$. The left hand side gives

$$S_{U \otimes V,W}\left(\sum_{i_1} (\zeta_{k_0} \otimes c) \otimes (\xi_{i_1} \otimes \eta_{j_0} \otimes (u_{i_1i} \triangleright b)a)\right)$$

$$= \sum_{i_1,i_2,j} \xi_{i_1} \otimes \eta_j \otimes \zeta_{k_0} \otimes (u_{i_1i_2}v_{j_0} \triangleright c)(u_{i_2i_0} \triangleright b)a = \sum_{i,j} \xi_i \otimes \eta_j \otimes \zeta_{k_0} \otimes (u_{i_0i} \triangleright (v_{j_0} \triangleright c)b))a.$$

The right hand side gives

$$S_{U \otimes V,W}\left(\sum_j (\eta_j \otimes \zeta_{k_0}) \otimes (v_{j_0} \triangleright c)b) \otimes (\xi_{i_0} \otimes a)\right) = \sum_{i,j} \xi_i \otimes \eta_j \otimes \zeta_{k_0} \otimes (u_{i_0i} \triangleright (v_{j_0} \triangleright c)b))a,$$

hence we obtain the claim. \qed
Given \( W \in \text{Rep}(G) \), let us denote by \( \beta_W \) the composition of the \( \hat{G} \)-action \( \beta: B \to M(B \otimes c_0(\hat{G})) \) with the projection \( \text{id} \otimes \pi_W \), where \( \pi_W: \ell^\infty(\hat{G}) \to B(\mathcal{H}_W) \) is the representation homomorphism.

**Definition 6.6.** Define a left action of \( \text{Rep}(G) \) on the category \( \mathcal{D}_B \) by
\[
V \otimes X = X \otimes_B (H_V \otimes B),
\]
and for \( T \in \mathcal{D}_B(X, Y) \) and \( W \in \text{Rep}(G) \),
\[
\text{id}_W \otimes T = T \otimes \text{id}_{H_W \otimes_B}.
\]

**Proposition 6.7.** Together with the left \( \text{Rep}(G) \) action given by Definition 6.6, \( C_B \) is a \( \text{Rep}(G) \)-bimodule \( C^* \)-category.

**Proof.** Proposition 6.5 gives natural isomorphisms
\[
(U \otimes B) \otimes V = H_V \otimes (B \otimes_B B_U) \simeq B_V \otimes_B B_U = U \otimes (B \otimes V).
\]

We need to show that for \( T \in \text{Mor}_{\text{Rep}(G)}(U, V) \), \( T \otimes \text{id}_B \in C_B(B_U, B_V) \) is a left \( B \)-module map. For \( a, b \in B \) and \( \xi \in H_U \),
\[
(T \otimes \text{id}_B)(\pi_U(b)(\xi \otimes a)) = (T \otimes \text{id}_B)(m_{ij}(\xi) \otimes (u_{ij} \triangleright b)a) = (T \otimes \text{id}_B)(m_{ij}(\xi) \otimes ((\text{id} \otimes u_{ij})\beta(b))a).
\]
By the intertwiner condition for \( T \), we see that this is equal to
\[
m_{kl}(T\xi) \otimes ((\text{id} \otimes v_{kl})\beta(b))a = \pi_V(b)(T\xi \otimes a),
\]
hence we obtain the claim. \( \square \)

We conclude that the \( \text{Rep}(G) \)-bimodule category \( \mathcal{D}_B \) has \( B \) as a central generator, with the half-braiding \( \sigma_U: U \otimes B \to B \otimes U \) given by the identity map up to the standard identification
\[
U \otimes B = B \otimes_B (H_U \otimes B) \simeq H_U \otimes B.
\]
Thus, a Yetter–Drinfeld \( G \)-\( C^* \)-algebra gives rise to a \( \text{Rep}(G) \)-bimodule category with a central generator.

### 6.2. From centrally pointed bimodules to Yetter–Drinfeld algebras.

In the other direction, let \( \mathcal{M} \) be a \( \text{Rep}(G) \)-bimodule \( C^* \)-category with a central generator \( m \), with central structure \( \sigma_U: U \otimes m \to m \otimes U \) for \( U \in \text{Rep}(G) \). Being a cyclic right \( \text{Rep}(G) \)-module \( C^* \)-category, to \( \mathcal{M} \) there is associated a unital \( G \)-\( C^* \)-algebra \( B_\mathcal{M} \), which is a completion of the vector space
\[
B_\mathcal{M} = \bigoplus_{U \in \text{Irr}(G)} \tilde{H}_U \otimes \mathcal{M}(m, m \otimes U),
\]
which we call regular part of \( B_\mathcal{M} \), see Section 2.4.

Let us briefly recall the \( \ast \)-algebra structure on this space. For further reference, see [Nes14, NY14]. It is built using the auxiliary universal algebra
\[
\tilde{B}_\mathcal{M} = \bigoplus_{U \in \mathcal{C}} \tilde{H}_U \otimes \mathcal{M}(m, m \otimes U),
\]
called the universal cover of \( B_\mathcal{M} \). Its product is defined by
\[
(\tilde{\xi} \otimes T) \bullet (\tilde{\eta} \otimes S) = (\xi \otimes \eta) \otimes (T \otimes \text{id}_V)S
\]
for \( \tilde{\xi} \otimes T \in \tilde{H}_U \otimes \mathcal{M}(m, m \otimes U) \) and \( \tilde{\eta} \otimes S \in \tilde{H}_V \otimes \mathcal{M}(m, m \otimes V) \).

There is a surjective idempotent linear map \( \pi: \tilde{B}_\mathcal{M} \to B_\mathcal{M} \) defined by the decompositions of arbitrary elements of \( \text{Rep}(G) \) into irreducible objects, and for \( x, y \in B_\mathcal{M} \) the product \( xy \) is
\( \pi(x \cdot y) \). The specific form of \( \pi \) will be given in the proof of Lemma 6.8. The involution \( x \mapsto x^\dagger \) on \( \mathcal{B}_\mathcal{M} \) is given by

\[
\tilde{H}_U \otimes \mathcal{M}(m, m \otimes U) \to H_U \otimes \mathcal{M}(m, m \otimes \hat{U}), \quad \tilde{\xi} \otimes T \mapsto \rho_u^{-1/2} \xi \otimes (T^* \otimes \text{id}_U)\tilde{R}_U,
\]
and we obtain the involution on \( \mathcal{B}_\mathcal{M} \) by \( \pi(x)^* = \pi(x^\dagger) \).

The construction above can be applied to the Yetter-Drinfeld algebra \( C(G) \). The regular part of it is \( \mathcal{O}(G) \), the algebra of polynomial functions on \( G \), and its universal cover is \( \tilde{\mathcal{O}}(G) \).

We claim that the *-algebra \( \mathcal{B}_\mathcal{M} \) has a Yetter-Drinfeld structure. Consider the action

\[
\triangleright : \tilde{\mathcal{O}}(G) \otimes \mathcal{B}_\mathcal{M} \to \mathcal{B}_\mathcal{M}
\]
defined by

\[
(\xi \otimes \zeta) \triangleright (\eta \otimes T) = (\xi \otimes \eta \otimes \rho^{-1/2}\zeta) \otimes \left((\sigma_U \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes T \otimes \text{id}_G)(\sigma_U^{-1} \otimes \text{id}_G)\tilde{R}_U\right)
\]
(6.2)

for \( T \in \mathcal{M}(m, m \otimes V) \), \( \tilde{\eta} \in \tilde{H}_V \), and \( \xi, \zeta \in H_U \). For \( x \in \mathcal{O}(G) \) and \( a \in \mathcal{B}_\mathcal{M} \), we put \( x \triangleright a = \pi(x \triangleright a) \).

From now on, \( U, V \) and \( W \) will always stand for finite dimensional unitary representations of \( G \), and to avoid cumbersome formulas we will often write \( UV \) instead of \( U \otimes V \).

**Lemma 6.8.** For all \( x \in \tilde{\mathcal{O}}(G) \) and every \( a \in \mathcal{B}_\mathcal{M} \),

\[
\pi_G(x) \triangleright \pi(a) = \pi(x \triangleright a).
\]

**Proof.** Take \( x = \tilde{\xi} \otimes \zeta \in \tilde{H}_U \otimes H_U \) and \( a = \tilde{\eta} \otimes T \in \tilde{H}_V \otimes \mathcal{M}(m, m \otimes V) \). Take partial isometries \( u_i : H_{si} \to H_U \) and \( v_j : H_{sj} \to H_V \) defining decompositions of \( U \) and \( V \) into irreducible finite dimensional unitary representations. Then we have

\[
\pi_G(x) \triangleright \pi(a) = \pi\left(\sum_{i,j} (u_i^* \tilde{\xi} \otimes u_j^* \tilde{\zeta}) \triangleright (v_j^* \tilde{\eta} \otimes v_j^* T)\right),
\]
which can be expanded as

\[
\pi\left(\sum_{i,j} (u_i^* \tilde{\xi} \otimes u_j^* \tilde{\eta} \otimes \rho_u^{-1/2}(u_i^* \tilde{\zeta})) \otimes (\sigma_U \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes v_j^* T \otimes \text{id}_G)(\sigma_U^{-1} \otimes \text{id}_G)\tilde{R}_U\right).
\]

On the other hand, we also have

\[
\pi(x \triangleright a) = \pi\left(\tilde{\xi} \otimes \tilde{\eta} \otimes \rho_u^{-1/2} \tilde{\zeta} \otimes (\sigma_U \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes T \otimes \text{id}_G)(\sigma_U^{-1} \otimes \text{id}_G)\tilde{R}_U\right)
\]
which can be expanded as

\[
\pi\left(\sum_{i,j,k} (u_i^* \tilde{\xi} \otimes u_j^* \tilde{\eta} \otimes \rho_k^{-1/2} \tilde{\zeta}) \otimes (\sigma_U \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes v_j^* T \otimes \text{id}_G)(\sigma_U^{-1} \otimes \text{id}_G)(u_i^* \otimes u_k^*)\tilde{R}_U\right).
\]

We have \( u_i^* \rho_U = \rho_{U_i} u_i^* \) and \( \tilde{R}_U = \sum_i (u_i \otimes \tilde{u}_i)\tilde{R}_{si} \). Moreover, the partial isometries \( u_i \) have mutually orthogonal ranges. We conclude that

\[
\pi_G(x) \triangleright \pi(a) = \pi(x \triangleright a),
\]
establishing the claim. \( \square \)

**Lemma 6.9.** The map \( \triangleright \) makes \( \mathcal{B}_\mathcal{M} \) into a left \( \mathcal{O}(G) \)-module.
Proof. Taking \( x = \bar{\xi} \otimes \zeta \in \bar{H}_U \otimes H_U, y = \bar{\mu} \otimes \nu \in \bar{H}_W \otimes H_W \) and \( \alpha = \bar{\eta} \otimes T \in \bar{H}_V \otimes M(m, m \otimes V) \), we must show that
\[
\pi(x \triangleright (y \triangleright a)) = \pi((xy) \triangleright a).
\]

Expanding the left hand side, we obtain
\[
\pi \left( (\xi \otimes \mu \otimes \eta \otimes \rho_W^{-1/2} \nu \otimes \rho_U^{-1/2} \zeta) \right) \otimes \left( (\sigma_U \otimes \text{id}_{VW}) \right) \left( \text{id}_{UW} \otimes T \otimes \text{id}_{WU} \right) \left( \sigma_V^* \otimes \text{id}_{UV} \right) \left( \text{id}_U \otimes \bar{R}_W \otimes \text{id}_O \right) \bar{R}_U.
\]

The right hand side is,
\[
\pi \left( (\xi \otimes \mu \otimes \rho_U^{-1/2} \zeta \otimes \rho_W^{-1/2} \nu) \otimes \left( (\sigma_U \otimes \text{id}_{VW}) \right) \left( \text{id}_{UW} \otimes T \otimes \text{id}_{WU} \right) \left( \sigma_V^* \otimes \text{id}_{UV} \right) \bar{R}_{UV} \right).
\]

The equality between the two expressions comes from the fact that \( \gamma: \bar{H}_W \otimes \bar{H}_U \rightarrow \bar{H}_U \otimes \bar{H}_W \), \( \gamma(\alpha \otimes \beta) = \beta \otimes \alpha \)
is an equivalence of representations, and that we can take
\[
\bar{R}_{UV} = (\text{id}_{UV} \otimes \gamma) (\text{id}_U \otimes \bar{R}_W \otimes \text{id}_O) \bar{R}_U,
\]
hence the claim.

\[\square\]

**Lemma 6.10.** The map \( \triangleright \) is an algebra-module map, that is,
\[
x \triangleright (ab) = (x(1) \triangleright a)(x(2) \triangleright b)
\]
holds for any \( x \in \mathcal{O}(G) \) and \( a, b \in \mathcal{B}_M \).

**Proof.** Take \( a = \bar{\eta} \otimes T \in \bar{H}_V \otimes M(m, m \otimes V) \), \( b = \bar{\xi} \otimes \zeta \in \bar{H}_W \otimes M(m, m \otimes W) \), and take \( x \) to be \( u_{ij} = \pi_G(\xi_i \otimes \xi_j) \), where \( \{\xi_j\}_j \) is an orthonormal basis of \( H_U \), so that \( \{u_{ij}\} \) are the matrix coefficients of \( U \) with respect to it.

By \( \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \), our claim is equivalent to
\[
\pi \left( (\xi_i \otimes \xi_j) \triangleright (ab) \right) = \sum_k \pi \left( ((\xi_i \otimes \xi_k) \triangleright a)((\xi_k \otimes \xi_j) \triangleright b) \right).
\]

The left hand side is
\[
\pi \left( (\xi_i \otimes \eta \otimes \zeta \otimes \rho_U^{-1/2} \xi_j) \otimes \left( (\sigma_U \otimes \text{id}_{VW}) \right) \left( \text{id}_U \otimes (T \otimes \text{id}_W)S \otimes \text{id}_O \right) \left( \sigma_V^* \otimes \text{id}_O \right) \bar{R}_U \right),
\]

while the right hand side is
\[
\sum_k \pi \left( \left( (\xi_i \otimes \eta \otimes \rho_U^{-1/2} \xi_k \otimes \xi_k \otimes \zeta \otimes \rho_U^{-1/2} \xi_j) \otimes X \right) \right),
\]
where
\[
X = (\sigma_U \otimes \text{id}_{UVWU}) (\text{id}_U \otimes T \otimes \text{id}_{UWU}) \left( \sigma_V^* \otimes \text{id}_{UVW} \right) \left( \bar{R}_U \otimes \text{id}_{WU} \right) \left( \text{id}_U \otimes S \otimes \text{id}_O \right) \left( \sigma_U^* \otimes \text{id}_O \right) \bar{R}_U.
\]

We have \( \sum_k \rho_U^{-1/2} \xi_k \otimes \xi_k = R_U(1) \), and the operator \( d_U^{-1/2} R_U \) is an isometric embedding of the trivial representation 1 into \( U \otimes U \). Therefore
\[
\sum_k \pi \left( \left( (\xi_i \otimes \eta \otimes \rho_U^{-1/2} \xi_k \otimes \xi_k \otimes \zeta \otimes \rho_U^{-1/2} \xi_j) \otimes X \right) \right)
\]
is equal to
\[
\pi \left( (\xi_i \otimes \eta \otimes \zeta \otimes \rho_U^{-1/2} \xi_j) \otimes (\text{id}_{UV} \otimes R_U^* \otimes \text{id}_W) X \right),
\]
which is equal to \( \pi \left( (\bar{\xi}_i \otimes \bar{\xi}_j) \triangleright (ab) \right) \) since \( (R_U^* \otimes \text{id}_U)(\text{id}_U \otimes \bar{R}_U) = \text{id}_U \). \(\square\)
Lemma 6.11. The $\mathcal{O}(G)$-action on $\mathcal{B}_M$ is compatible with the $*$-structure:

$$x \triangleright a^* = (S(x)^* \triangleright a)^*.$$  

Proof. For $x = \xi \otimes \zeta \in \bar{H}_U \otimes H_V$, denote $\zeta \otimes \xi$ by $x^\dagger$. We then have $S(\pi_G(x))^* = \pi_G(x^\dagger)$, see [NY14]. Thus, the claim is equivalent to

$$\pi(x \triangleright a) = \pi((x^\dagger \triangleright a)^\dagger).$$

The left hand side is

$$\pi\left(\xi \otimes \rho_U^{-1/2} \eta \otimes \rho_U^{-1/2} \zeta \otimes (\sigma_U \otimes \text{id}_{V_U})(\text{id}_U \otimes T^* \otimes \text{id}_{V_U})(\text{id}_U \otimes \bar{R}_V \otimes \text{id}_U)(\sigma_U^* \otimes \text{id}_U)\bar{R}_U\right).$$

The right hand side is

$$\pi\left((\rho_U^{-1/2} \xi \otimes \rho_U^{-1/2} \eta \otimes \xi) \otimes (\bar{R}_U \otimes \text{id}_{V_U})(\sigma_U \otimes \text{id}_{UUV}) (\text{id}_U \otimes T^* \otimes \text{id}_{UUV}) (\sigma_U^* \otimes \text{id}_{V_UUV}) \bar{R}_{UVU}\right).$$

The unitary $K: H_U \otimes H_V \otimes H_U \rightarrow H_{UVU}$, $\xi \otimes \eta \otimes \xi \mapsto \zeta \otimes \eta \otimes \xi$ is an equivalence of representations, and

$$\bar{R}_{UVU} = (\text{id}_{UUV} \otimes K)(\text{id}_{UV} \otimes \bar{R}_U \otimes \text{id}_{U\bar{U}})(\text{id}_U \otimes \bar{R}_V \otimes \text{id}_U)\bar{R}_U.$$

With this and the identity $(\bar{R}_U^* \otimes \text{id}_U)(\text{id}_U \otimes \bar{R}_U) = \text{id}_U$, we see that the claim holds. \hfill \Box

Lemma 6.12. The Yetter–Drinfeld condition for $\triangleright$ is satisfied: if $\alpha$ is the action of $G$ on $\mathcal{B}_M$, then

$$\alpha(x \triangleright a) = x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)})$$

for all $x \in \mathcal{O}(G)$ and for all $a \in \mathcal{B}_M$.

Proof. Fix orthonormal basis $\{\xi_i\}$ and $\{\eta_k\}_{k}$ of $H_U$ and $H_V$, respectively, consisting of eigenvectors of the corresponding operators $\rho_U$ and $\rho_V$. Let $\rho_i$ be the associated eigenvalues of $\rho_U$, so that we have $\rho_U \xi_i = \rho_i \xi_i$. Take $x = u_{i_0,j_0} = \pi_G(\xi_{i_0} \otimes \eta_{j_0}) \in \mathcal{O}(G)$ and $a = \pi(\eta_{k_0} \otimes T) \in \pi(\bar{H}_V \otimes M(m, m \otimes V))$. Recall that the $G$-action on $\mathcal{B}_M$ is defined by

$$\alpha(\pi(\eta_{k_0} \otimes T)) = \sum_k v_{k_0 k} \otimes \pi(\eta_k \otimes T),$$

with $v_{k_1,k_2}$ denoting the matrix coefficients of $V$ with respect to the basis $\{\eta_k\}_k$. From the definitions, we get that $x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)})$ equals to

$$\sum_{i,j,k} \rho_j^{-1/2} u_{i_0 i} v_{k_0 k} S(u_{j_0 j}) \otimes \pi\left((\xi_i \otimes \eta_k \otimes \xi_j) \otimes (\sigma_U \otimes \text{id}_{V_U})(\text{id}_U \otimes T \otimes \text{id}_U) (\sigma_U^* \otimes \text{id}_U) \bar{R}_U\right).$$

The contragredient representation of $U$ has $\{u_{i,j}\}_{i,j}$ as matrix coefficients. As for the dual representation $\bar{U}$, the matrix coefficients are $\{\bar{u}_{i,j}\}_{i,j} = \rho_i^{1/2} \rho_j^{-1/2} u_{i,j}^* = \rho_i^{1/2} \rho_j^{-1/2} S(u_{j,i})_{i,j}$. Therefore we have

$$\rho_j^{-1/2} S(u_{j_0,j}) = \rho_j^{-1/2} \bar{u}_{j_0,j},$$

and the expression above is the same as

$$\rho_j^{-1/2} \sum_{i,j,k} u_{i_0 i} v_{k_0 k} \bar{u}_{j_0 j} \otimes \pi\left((\xi_i \otimes \eta_k \otimes \xi_j) \otimes (\sigma_U \otimes \text{id}_{V_U})(\text{id}_U \otimes T \otimes \text{id}_U) (\sigma_U^* \otimes \text{id}_U) \bar{R}_U\right).$$

This is the right hand side of the equation in the Lemma; let us compute the left hand side. As $u_{i_0,j_0} \triangleright (\eta_{k_0} \otimes T) \in \pi(\bar{H}_{UVU} \otimes M(m, m \otimes (U \otimes V \otimes \bar{U})))$,

$$\alpha(u_{i_0,j_0} \triangleright (\eta_{k_0} \otimes T)) = (U \otimes V \otimes \bar{U})_{i_0,j_0}^*(1 \otimes (u_{i_0,j_0} \triangleright (\eta_{k_0} \otimes T))).$$

Expanding on the definitions, we see that it is the same as $x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)})$. \hfill \Box
Lemmas 6.8, 6.12 say that the closure \( B_M \) of \( B_M \) becomes then a Yetter–Drinfeld \( G \)-\( C^* \)-algebra.

6.3. Duality. Our next goal is to prove that, up to isomorphisms, the above constructions are inverse to each other. Given a (strict) centrally pointed bimodule category \( \mathcal{M} \), let us write \( \mathcal{D}_M = \mathcal{D}_{B_M} \). Then by the duality for \( G \)-\( C^* \)-algebras, there is an equivalence of right \( \text{Rep}(G) \)-module \( C^* \)-categories, \( \Psi : \mathcal{M} \to \mathcal{D}_M \) ([Nes14]).

Proposition 6.13. The right module category equivalence functor \( \Psi \) extends to a central functor.

Proof. By Proposition 5.8 it is enough to construct a corresponding extension from \( \hat{\mathcal{M}} \) to \( \hat{\mathcal{D}}_M \).

First we note that \( \hat{\mathcal{D}}_M \) is the category \( \mathcal{C}_M = \mathcal{C}_{B_M} \) in [NY14], i.e., the idempotent completion of \( \mathcal{C} \) with respect to the morphism sets

\[
\mathcal{C}_{B_M}(U, V) = \text{Hom}_{\mathcal{D}_M}(H_U \otimes B_M, H_V \otimes B_M),
\]

the right-hand-side meaning \( G \)-equivariant Hilbert \( B_M \)-module maps. Then, the right module category equivalence \( \Psi : \mathcal{M} \to \mathcal{C}_{B_M} \) is characterized by \( m \otimes U \mapsto U \) at the level of objects, and at the level of morphisms by

\[
\Psi(T) = \sum_{i,j} \theta_{\xi_i,\xi_2} \otimes \pi \left( (\xi_i \otimes \rho^{-1/2} \xi_j) \otimes (T \otimes \text{id}_U)(\text{id}_m \otimes \bar{R}_U) \right) \tag{6.3}
\]

for \( T \in \mathcal{M}(m \otimes U, m \otimes V) \), where \( (\xi_i)_i \) is an orthonormal basis of \( H_U \), \( (\xi_j)_j \) is an orthonormal basis of \( H_V \), and \( \theta_{\xi_i,\xi_j} \) is the linear map \( \eta \mapsto (\eta, \xi_i) \xi_j \), see [NY14].

Now let us denote the central structure on \( \mathcal{M} = (\mathcal{M}, m) \) by \( \sigma \), and the one on \( \mathcal{C}_M \) by \( S \). Let us fix \( T \in \mathcal{M}(m_1 \otimes V, m_2 \otimes W) \), and take orthonormal bases \( (\xi_i)_i \) on \( H_U \), \( (\xi_j)_j \) on \( H_V \), and \( (\eta_k)_k \) on \( H_W \).

On the one hand, the morphism \( \Psi((\sigma \otimes \text{id}_W)(\text{id}_U \otimes T)(\sigma^{-1} \otimes \text{id}_V)) \) can be expanded as

\[
\sum_{i,j,k,l} \theta_{\xi_i \otimes \eta_k,\xi_2 \otimes \eta_l} \otimes \pi \left( (\xi_i \otimes \eta_k \otimes \rho^{-1/2} (\xi_j \otimes \eta_l)) \otimes ((\sigma \otimes \text{id}_W)(\text{id}_U \otimes T)(\sigma^{-1} \otimes \text{id}_V) \otimes \text{id}_{TT}) \bar{R}_{UV} \right),
\]

which is equal to

\[
\sum_{i,j,k,l} m_{ij}^{kl} \otimes \theta_{\eta_k,\eta_l} \otimes \left( u_{ij} \triangleright \pi \left( (\eta_k \otimes \rho^{-1/2} \xi_j) \otimes (T \otimes \text{id}_V) \bar{R}_V \right) \right).
\]

On the other, writing \( \Psi(T) = \sum_i \theta_i \otimes b_i \) with \( \theta_i \in B(H_U, H_W) \) and \( b_i \in B_M \), we have

\[
(S \otimes \text{id}_W) \left( \text{id}_U \otimes \sum_i T_i \otimes b_i \right) (S^{-1} \otimes \text{id}_V) = \sum_{i,j} m_{ij}^{kl} \otimes T_i \otimes (u_{ij} \triangleright b_i).
\]

Comparing these we get

\[
\Psi((\sigma \otimes \text{id}_W)(\text{id}_U \otimes T)(\sigma^{-1} \otimes \text{id}_V)) = (S \otimes \text{id}_W)(\text{id}_U \otimes \Psi(T))(S^{-1} \otimes \text{id}_V),
\]

which gives the desired extension by Proposition 5.9.

Going the other way, let \( B \) be a Yetter–Drinfeld \( G \)-\( C^* \)-algebra, and consider the new algebra \( B_{\mathcal{D}_B} \) we get by the above procedure. Let us denote their regular subalgebras by \( B \) and \( B_{\mathcal{D}_B} \). Then the isomorphism of \( G \)-algebras \( \lambda : B_{\mathcal{D}_B} \to B \) is given by

\[
\lambda(\bar{\eta} \otimes T) = (\eta^* \otimes \text{id}_B)(T)
\]

for \( \bar{\eta} \in \hat{R}_V \) and \( T \in (H_V \otimes B)^G \simeq \mathcal{D}_B(B, B \otimes V) \), where \( \eta^* \) is the functional \( \xi \mapsto (\xi, \eta) \) on \( H_V \).

Proposition 6.14. \( \lambda \) is \( \hat{G} \)-equivariant.
Proof. Fix an orthonormal basis $(\zeta_k)$ in $H_V$ and take $T \in \mathcal{D}_B(B, B \otimes V)$, represented by $\sum_k \zeta_k \otimes b_k \in H_V \otimes B$ up to the above correspondence.

The central structure $S$ on $\mathcal{D}_B$ satisfies

$$(S \otimes \text{id}_V)(\text{id}_U \otimes T \otimes \text{id}_V)(S^{-1} \otimes \text{id}_V) = \sum_{i,j,k} m_{ij}^U \otimes \zeta_k \otimes 1 \otimes (u_{ij} \triangleright b_k)$$

in $\mathcal{D}_B(B \otimes U \otimes \hat{U}, B \otimes U \otimes V \otimes \hat{U})$.

From our definition of the action $\triangleright$, we see that $u_{i0j0} \triangleright \pi(\tilde{\zeta}_{k0} \otimes T)$ is equal to

$$\pi\left((\tilde{\zeta}_{i0} \otimes \xi_{j0}) \triangleright (\tilde{\zeta}_{k0} \otimes T)\right) = \pi\left((\xi_{i0} \otimes \zeta_{k0} \otimes \rho_U^{-1/2} \xi_{j0}) \otimes (S \otimes \text{id}_V)(\text{id}_U \otimes T \otimes \text{id}_V)(S^{-1} \otimes \text{id}_V)\bar{R}_U\right)$$

$$= \pi\left((\xi_{i0} \otimes \zeta_{k0} \otimes \rho_U^{-1/2} \xi_{j0}) \otimes \left(\sum_{i,j,k} m_{ij}^U \otimes \zeta_k \otimes 1 \otimes (u_{ij} \triangleright b_k)\right)\bar{R}_U\right).$$

From (2.2), this is equal to

$$\sum_{i,j,k} (\xi_{i0} \otimes \zeta_{k0} \otimes \rho_U^{-1/2} \xi_{j0}) \otimes (\xi_i \otimes \zeta_k \otimes \rho_U^{1/2} \xi_j) \otimes (u_{ij} \triangleright b_k).$$

We thus obtain

$$\lambda\left(u_{i0j0} \triangleright \pi(\tilde{\zeta}_{k0} \otimes T)\right) = u_{i0j0} \triangleright b_{k0} = u_{i0j0} \triangleright \lambda(\tilde{\zeta}_{k0} \otimes T),$$

establishing the claim. \hfill \Box

6.4. Moduli of module functors. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two centrally pointed $\text{Rep}(G)$-bimodule categories, with respective generators $m_1$ and $m_2$ and half-braiding $\sigma$ and $\sigma'$, and denote their corresponding Yetter–Drinfeld $G$-$\text{C}^*$-algebras by $B_1$ and $B_2$, respectively. Consider a right $\text{Rep}(G)$-module functor $F: \mathcal{M}_1 \to \mathcal{M}_2$ such that $F(m_1) = m_2$. Let us work out the condition on $F$ such that the induced $G$-equivariant homomorphism $f: B_1 \to B_2$ becomes a $D(G)$-equivariant homomorphism.

Restricting to the regular subalgebras, $f$ is given by the collection of

$$\bar{R}_U \otimes \mathcal{M}_1(m_1, m_1 \otimes U) \to \bar{R}_U \otimes \mathcal{M}_2(m_2, m_2 \otimes U), \quad T \mapsto \text{id}_{\bar{R}_U} \otimes F_2^{-1}F(T)$$

for the irreducible representations $U$. In view of (6.2), $f$ will be $\hat{G}$-equivariant if and only if

$$\left(F_{2, m_1 U, V}^{-1}F((\sigma_U \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes T \otimes \text{id}_G)(\sigma_U^{-1} \otimes \text{id}_G)(\text{id}_{m_1} \otimes \bar{R}_U))\right)$$

$$= \left((\sigma_U' \otimes \text{id}_V \otimes \text{id}_G)(\text{id}_U \otimes F_{2, m_1 U, V}^{-1}F(T) \otimes \text{id}_G)((\sigma_U^{-1} \otimes \text{id}_G)(\text{id}_{m_2} \otimes \bar{R}_U))\right).$$

As before, consider the braiding maps $\Sigma_{U:V}: \mathcal{M}_1(m_1, m_1 \otimes U, m_1 \otimes U \otimes V)$ given by $\Sigma_{U:1,V}$ in (6.3). We will write the analogous braiding maps on $\mathcal{M}_2$ as $\Sigma'$. Using $\Sigma_{U,V}(T) = (\text{id}_{m_1} \otimes \sigma_U \otimes \text{id}_V \otimes \sigma_U')(\text{id}_U \otimes T \otimes \text{id}_V)(\sigma_U^{-1} \otimes \text{id}_U)(\text{id}_{m_2} \otimes \bar{R}_U \otimes \text{id}_U)$, we see that $f$ will be $\hat{G}$-equivariant if and only if

$$F_{2, m_1 U, V}^{-1}F(\Sigma_{U:V}(T))F_2 = \Sigma'_{U:V}(F_{2, m_1 U, V}^{-1}F(T))$$

(6.4)

for all $U, V \in \text{Rep}(G)$ and $T \in \mathcal{M}_1(m_1, m_1 \otimes V)$.

Proposition 6.15. Let $F: \mathcal{M}_1 \to \mathcal{M}_2$ be a right $C^*$-$\text{Rep}(G)$-module functor between centrally pointed bimodule categories. If $F$ satisfies (6.4), then there is a strict bimodule functor $\hat{F}: \mathcal{M}_1 \to \mathcal{M}_2$ between the categories obtained by trivialization of the central structures, that is naturally isomorphic to $F$ up to the equivalence of Proposition 6.3.
Proof. From the assumption we actually have
\[ F_{m_1,UW}^{-1} F(\Sigma_U;V(T)) F_{m_1,UV} = \Sigma_U;V(F_{m_1,V}^{-1} F(T)) \quad (T \in \mathcal{M}_1(m_1 \otimes V, m_1 \otimes W). \]
Indeed, given \( T \in \mathcal{M}_1(m_1 \otimes V, m_1 \otimes W) \), we can take \( S = (T \otimes \text{id}_W)(\text{id}_{m_1} \otimes \tilde{R}_W) \) and combine with the conjugate equation to get this claim. Then the claim follows from Proposition \ref{5.9}.

Finally, let us compare different choices of bimodule functors leading to the same homomorphisms. Suppose that \( F, F' : \mathcal{M} \to \mathcal{M}' \) are central functors that are naturally isomorphic. Then the induced homomorphisms \( B_M \to B_{M'} \) agree. Indeed, we just need to look at the induced maps
\[ \mathcal{M}(m, m \otimes U) \to \mathcal{M}'(m', m' \otimes U) \]
which are given by \( T \mapsto (F_0^{-1} \otimes \text{id}_U)F_2^{-1}F(T)F_0 \) and a similar formula for \( F' \). Using the commutativity of diagrams in Definition \ref{5.5} we see that these maps indeed agree.

Proposition 6.16. Suppose that the induced homomorphisms \( B_M \to B_{M'} \) agree. Then \( F \) and \( F' \) are naturally isomorphic as bimodule functors.

Proof. Let \( f_F : B_M \to B_{M'} \) denote the homomorphism induced by \( F \). By Proposition \ref{6.13} the assertion follows once we can check that the bimodule functor \( f_F : \mathcal{D}_M \to \mathcal{D}_{M'} \) induced by \( f_F \) is naturally isomorphic to \( F \) up to the bimodule equivalence \( \mathcal{M} \to \mathcal{D}_M \). Looking at the corresponding functor \( \mathcal{M} \to \mathcal{M}' \) induced by \( F \), the claim is tautological. Then we get the claim by Proposition \ref{5.8}.

7. Operator system theory for centrally pointed bimodule categories

7.1. Multipliers and injectivity.

Definition 7.1. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be centrally pointed \( C \)-bimodule categories, with central generators \( m_1 \) and \( m_2 \). A central ucp \( C \)-linear multiplier \( F : \mathcal{M}_1 \to \mathcal{M}_2 \) is a ucp right \( C \)-linear multiplier \( F = \{ F_{U,V} : \mathcal{M}_1(m_1 \otimes U, m_1 \otimes V) \to \mathcal{M}_2(m_2 \otimes U, m_2 \otimes V) \} \) such that
\[ F_{UV,UW} \Sigma_{U;V,W} = \Sigma_{U;V,W} F_{V,W} \quad (U,V,W \in \mathcal{C}). \quad (7.1) \]

We denote by \( \text{CBOS}(\mathcal{C}) \) the category of centrally pointed \( C \)-bimodule categories and central ucp \( C \)-linear multipliers.

Remark 7.2. In Section \ref{5} we showed that the category of centrally pointed bimodule categories is equivalent, through trivialization of the central structures, to a category of pointed cyclic bimodules \( (\mathcal{M}, m) \) on which the left bimodule structure is an extension of \( U \otimes (m \otimes V) = m \otimes U \otimes V \). At the level of morphisms, the left action on such a trivialized category was given by
\[ \text{id}_U \otimes T = \Sigma_{U;V,W}(T) \quad (T \in \mathcal{M}(m \otimes V, m \otimes W)). \]
Equation (7.1) is saying, therefore, that \( F \) is a ucp-multiplier of the corresponding bimodule categories with trivialized central structures.

Recall that a pointed right \( C \)-module category \( \mathcal{M}' \) gives rise to a centrally pointed \( C \)-module category \( \mathcal{M}' \), see Definition \ref{5.7}.

Proposition 7.3. Given \( \mathcal{M} \in \text{CBOS}(\mathcal{C}) \) and \( \mathcal{M}' \in \text{Mod}_c^r(\mathcal{C}) \), there is a bijective correspondence between the morphisms \( \mathcal{M} \to \mathcal{M}' \) in \( \text{CBOS}(\mathcal{C}) \) and the ucp-multipliers \( \mathcal{M} \to \mathcal{M}' \) of pointed right \( C \)-modules.

Proof. Let \( F \) be a ucp-multiplier \( \mathcal{M} \to \mathcal{M}' \). Given \( T \in \mathcal{M}(m \otimes V, m \otimes W) \) and \( U \in \mathcal{C} \), we define \( \hat{F}_{V,W}(T)_U \in \mathcal{M}'(m' \otimes U \otimes V, m' \otimes U \otimes W) \) by
\[ \hat{F}_{V,W}(T)_U = F_{V,UW}((\sigma_U \otimes \text{id}_W)(\text{id}_U \otimes T)(\sigma_U^{-1} \otimes \text{id}_V)) = F_{V,UW}(\Sigma_{U;V,W}(T)). \]
This is natural in $U$ by construction, hence we get a multiplier $\hat{F}: \mathcal{M} \to \hat{\mathcal{M}}'$. This is completely positive because $\sigma_U$ is unitary.

Moreover, $\Sigma_{U,V,W}$ on $\mathcal{M}'$ is the left module structure map $S \mapsto \text{id}_U \otimes S$. Thus, \((7.1)\) for $\hat{F}$ becomes

$$\hat{F}_{UV,UW}(\Sigma_{U,V,W}(T)) = \text{id}_U \otimes \hat{F}_{V,W}(T) \quad (T \in \mathcal{M}(m \otimes V, m \otimes W)).$$

This follows from the multiplicativity of $\sigma_U$ in $U$.

In the other direction, given a ucp-multiplier $F: \mathcal{M} \to \hat{\mathcal{M}}'$ (satisfying \((7.1)\)), we get a ucp-multiplier $\hat{F}: \mathcal{M} \to \mathcal{M}'$ by $\hat{F}_{V,W}(T) = F_{V,W}(T)_1$ for $T \in \mathcal{M}(m \otimes V, m \otimes W)$.

These correspondences are inverse to each other: since $\sigma_1 = \text{id}_m$, if $F: \mathcal{M} \to \mathcal{M}'$ is a right-module ucp-multiplier, $G = \hat{F}$ satisfies

$$\hat{G}_{V,W} = [\hat{F}_{V,W}]_1 = F_{V,W}.$$ 

Conversely, suppose $F: \mathcal{M} \to \hat{\mathcal{M}}'$ is a central ucp-multiplier, and write $G = \hat{F}$. Then we have

$$[\hat{G}_{V,W}(T)]_U = \hat{F}_{UV,UW}(\Sigma_{UV,UW}(T)) = F_{UV,UW}(\Sigma_{UV,UW}(T))_1 = \Sigma_{U,V,W}'(F_{V,W}(T))_1 = [\text{id}_U \otimes F_{V,W}(T)]_1 = [F_{V,W}(T)]_U$$

for $T \in \mathcal{M}(m \otimes V, m \otimes W)$. \hfill \Box

Now, consider the notions of injectivity and injective envelopes for centrally pointed bimodule categories analogously to Definitions \[4.2\] and \[4.12\] by restricting to central ucp-multipliers. In other words, we consider only central ucp-extensions of central ucp-multipliers.

**Theorem 7.4.** The categories $\mathcal{M}_{\mathcal{B}(H)}$ are injective in $\text{CBOS}(\mathcal{C})$ for $H \in \text{Hilb}(\mathcal{C})$.

**Proof.** By Proposition \[7.3\], the central ucp-multipliers $\mathcal{M}' \to \hat{\mathcal{M}}_{\mathcal{B}(H)}$ bijectively correspond to the right module ucp-multipliers $\mathcal{M}' \to \mathcal{M}_{\mathcal{B}(H)}$. We then get the claim by Theorem \[4.10\]. \hfill \Box

As a consequence, any object in $\text{CBOS}(\mathcal{C})$ embeds into a injective object.

**Remark 7.5.** As the above proof shows, $\hat{\mathcal{M}}$ is injective as a centrally pointed bimodule category whenever $\mathcal{M}$ is an injective pointed right $\mathcal{C}$-module category.

**Remark 7.6.** Consider a right $\mathcal{C}$-module $\mathcal{C}$-category $\mathcal{M}$ and an embedding of $\mathcal{M}$ into $\mathcal{M}_{\mathcal{B}(H)}$, for some $H \in \text{Hilb}(\mathcal{C})$. The composition

$$\mathcal{M} \to \mathcal{M}_{\mathcal{B}(H)} \to \hat{\mathcal{M}}_{\mathcal{B}(H)}$$

is an analogue of the Poisson transform considered in \[KKSV22, HHN22\], which we call a categorical Poisson transform.

**Definition 7.7.** A central bimodule functor $F: \mathcal{M} \to \mathcal{N}$ is said to be rigid when $\text{Id}_\mathcal{N}$ is the only central ucp multiplier $G: \mathcal{N} \to \mathcal{N}$ satisfying $GF = F$.

**Theorem 7.8.** Every object in $\text{CBOS}(\mathcal{C})$ has an injective envelope $\mathcal{I}_{\text{CBOS}}(\mathcal{M})$. Moreover the embedding $i: \mathcal{M} \to \mathcal{I}_{\text{CBOS}}(\mathcal{M})$ is rigid.

**Proof.** The first claim can be proved with a slight modification of the proof of Theorem \[4.13\]. Start with a centrally pointed bimodule category $(\mathcal{M}, m)$, and embed it into $\hat{\mathcal{M}}_{\mathcal{B}(H)}$ for some $H \in \text{Hilb}(\mathcal{C})$.

Then consider the semigroup $S$ of central ucp multipliers from $\hat{\mathcal{M}}_{\mathcal{B}(H)}$ to itself that are identity on the image of $\mathcal{M}$. This is still a closed convex set for the weak* topology with respect to the presentation \[5.3\], ensuring then the existence of a minimal idempotent $\Phi \in S$. Consider the image of $\Phi$, endowed with the Choi–Effros product. Since $\Phi$ is central, its image inherits the centrally pointed bimodule structure of $\mathcal{M}_{\mathcal{B}(H)}$. This is a model of $\mathcal{I} = \mathcal{I}_{\text{CBOS}}(\mathcal{M})$. 
The rest of the proof is also a close analogue of the usual one for injective envelope of C*-algebras. Let \( \Psi: \mathcal{I} \to \mathcal{I} \) be a central ucp multiplier with \( \Psi i = i \), and put \( \Phi' = \Psi \Phi \). Then by the complete contractivity of \( \Psi \), we have
\[
\|\Phi'(T)\| \leq \|\Phi(T)\|
\]
for any \( U, V \in \mathcal{C} \) and any \( T \in \hat{\mathcal{M}}_{B(H)}(U, V) \). By the minimality of \( \Phi \), we obtain \( \Phi' = \Phi \), which means \( \Psi = \text{Id} \).

7.2. Boundary theory for centrally pointed bimodule categories. Let us quickly recall the relevant concepts from [HHL22]: A \( \mathcal{C} \)-tensor category is a C*-tensor category \( \mathcal{D} \) endowed with a dominant faithful C*-tensor functor \( F: \mathcal{C} \to \mathcal{D} \). (For simplicity we assume that \( \mathcal{C} \) is a subcategory of \( \mathcal{D} \), and that any object of \( \mathcal{D} \) is a subobject of some \( U \in \mathcal{C} \).) A \( \mathcal{C} \)-linear transformation \( \Theta: \mathcal{D}_1 \to \mathcal{D}_2 \) between \( \mathcal{C} \)-tensor categories is given by a family of linear maps
\[
\Theta_{U,V}: \mathcal{D}_1(U,V) \to \mathcal{D}_2(U,V) \quad (U,V \in \mathcal{C})
\]
satisfying
\[
\Theta_{U_2,V_1}(S_1T_{S_2}) = S_1\Theta_{V_2,U_1}(T)S_2 \quad (T \in \mathcal{D}_1(V_2,U_1), S_1 \in \mathcal{C}(U_1,V_1)),
\]
\[
\Theta_{Z \otimes U \otimes V, Z \otimes U \otimes V}(id_Z \otimes T \otimes id_Y) = id_Z \otimes \Theta_{U,V}(T) \otimes id_Y \quad (T \in \mathcal{D}_1(U,V)).
\]
A \( \mathcal{C} \)-linear transformation \( \Theta \) is ucp if \( \Theta_{U,U} \) is ucp for all \( U \in \mathcal{C} \). The \( \mathcal{C} \)-injectivity for \( \mathcal{C} \)-tensor categories is defined using this class of maps. Finally, the Furstenberg–Hamana boundary \( \partial_{FH}(\mathcal{C}) \) of \( \mathcal{C} \) is an injective \( \mathcal{C} \)-tensor category such that any \( \mathcal{C} \)-linear transformation \( \partial_{FH}(\mathcal{C}) \to \mathcal{D} \) is completely isometric. Concretely, \( \partial_{FH}(\mathcal{C}) \) can be realized inside \( \hat{\mathcal{C}} \) using Proposition 2.1 applied to the semigroup of \( \mathcal{C} \)-linear ucp transformations from \( \hat{\mathcal{C}} \) to itself.

Now, observe that any \( \mathcal{C} \)-tensor category \( \mathcal{D} \) has a canonical structure of centrally pointed \( \mathcal{C} \)-bimodule category: as the generating object we take the tensor unit, and the central structure \( \sigma_U: U \otimes 1_\mathcal{C} \to 1_\mathcal{C} \otimes U \) is given by the composition of structure morphisms of the monoidal unit. Under this correspondence, a ucp \( \mathcal{C} \)-linear transformations between \( \mathcal{C} \)-tensor categories is exactly a ucp central \( \mathcal{C} \)-linear transformation. This motivates the following definition of boundary bimodules.

Definition 7.9. A centrally pointed \( \mathcal{C} \)-bimodule category \( \mathcal{M} \) is called a boundary category when any central ucp multiplier \( \mathcal{M} \to \mathcal{N} \) to any centrally pointed bimodule category \( \mathcal{N} \) is completely isometric.

Theorem 7.10. Every boundary \( \mathcal{M} \) is a centrally pointed subcategory of \( \partial_{FH}(\mathcal{C}) \).

Proof. By Proposition 5.6, we have central embedding of \( \mathcal{C} \) into \( \mathcal{M} \). Since \( \mathcal{M} \) is a boundary and \( \partial_{FH}(\mathcal{C}) \) is injective, we get a completely isometric central ucp multiplier \( i: \mathcal{M} \to \partial_{FH}(\mathcal{C}) \).

Take an embedding \( \pi \) of \( \mathcal{M} \) to \( \hat{\mathcal{M}}_{B(H)} \) for some \( H \in \text{Hilb}(\mathcal{C}) \), as a centrally pointed bimodule category. By the injectivity of \( \hat{\mathcal{M}}_{B(H)} \), the multiplier \( i \) extends to a ucp central multiplier \( \Phi: \partial_{FH}(\mathcal{C}) \to \hat{\mathcal{M}}_{B(H)} \). Conversely, by the injectivity of \( \partial_{FH}(\mathcal{C}) \), we also get a ucp central map \( \Psi: \hat{\mathcal{M}}_{B(H)} \to \partial_{FH}(\mathcal{C}) \).

The composition \( \Psi \Phi \) is a \( \mathcal{C} \)-linear ucp transformation on \( \partial_{FH}(\mathcal{C}) \). Thus, \( \Phi \) must be completely isometric. Moreover, by the above construction of \( \partial_{FH}(\mathcal{C}) \) via Proposition 2.1, \( \Psi \Phi \) must be given by the identity functor. This implies that \( E = \Phi \Psi \) is a conditional expectation on \( \hat{\mathcal{M}}_{B(H)} \), with image \( \Phi(\partial_{FH}(\mathcal{C})) \). As before, the Choi–Effros product
\[
\Phi(g) \cdot \Phi(f) = E(\Phi(g)\Phi(f))
\]
turns this image into a category. Since we worked with central ucp central multipliers throughout, this category is again centrally pointed.
By construction \(\pi(M)\) is contained in \(\Phi(\partial_{FH}(C))\). Moreover, the Choi–Effros product coincides with the original product of \(\pi(M)\), by
\[
\pi(g) \cdot \pi(f) = E(\pi(g)\pi(f)) = \Phi(\Psi(\pi(g)f)) = \Phi(i(g)f) = \pi(g)f = \pi(g)\pi(f).
\]
Thus \(\pi\) defines a functor \(M \rightarrow \Phi(\partial_{FH}(C))\), which can be upgraded to a central functor of centrally pointed bimodules. By the boundary property of \(M\), this must be an embedding. □

**Corollary 7.11.** The boundary objects in \(\text{CBOS}(C)\) are exactly the centrally pointed \(C\)-bimodule subcategories of \(\partial_{FH}(C)\).

**Appendix A. Intrinsic characterization of \(C^*\)-algebra objects**

Definition [2.7] does not give an intrinsic condition on \(C^*\)-algebra objects. Here we give an intrinsic characterization of such structures, without directly referring to the corresponding module category.

When \(A\) is an algebra object in \(\text{Vec}(C)\), let us denote the product of \(a \in A(U)\) and \(b \in A(V)\) by \(a \cdot b \in A(V \otimes U)\). We also write \(A_0 = A(1_C)\).

Now, let \(A\) be a \(*\)-algebra object in \(\text{Vec}(C)\) [JP17]. This means that there is a family of conjugate linear maps
\[
A(U) \rightarrow A(\bar{U}), \quad a \mapsto a^\natural
\]
(denoted by \(j_U\) in [JP17]) satisfying the following conditions:

- the naturality
  \[
  A(T)(a^\natural) = A(T^{\ast \natural})(a^\natural) \quad (T \in \mathcal{C}(U,V), a \in A(V)), \quad (A.1)
  \]
- the involutivity \((a^\natural)^\natural = a\) up to the canonical isomorphism \(\bar{U} \simeq U\),
- the unitality \(1^\natural_{A_0} = 1_{A_0}\) in \(A_0\) up to the canonical choice \(1_C = 1_C\), and
- the antimultiplicativity \(a^\natural \cdot b^\natural = (b \cdot a)^\natural\) up to the natural isomorphism \(\bar{U} \otimes \bar{V} \simeq \bar{V} \otimes \bar{U}\).

Here, we recall that \(T^{\ast \natural} \in \mathcal{C}(\bar{U}, \bar{V})\) is the morphism characterized by
\[
(id_U \otimes T^{\ast \natural})\bar{R}_U = (T^{\ast} \otimes id_{\bar{V}})\bar{R}_V.
\]

Such a structure induces a dagger category structure on the associated module category \(\mathcal{M}_A\). In particular, \(A(U \otimes \bar{U}) \simeq \mathcal{M}_A(U_A, U_A)\) is a \(*\)-algebra for every \(U \in \mathcal{C}\). First note that we have \(*\)-algebra embeddings
\[
j_{U,V} : A(U \otimes \bar{U}) \rightarrow A(U \otimes V \otimes \bar{V} \otimes \bar{U}), \quad a \mapsto A(id_U \otimes R_{\bar{U}}^V \otimes id_{\bar{V}})(a),
\]
corresponding to
\[
\mathcal{M}_A(U_A, U_A) \rightarrow \mathcal{M}_A(U_A \otimes V, U_A \otimes V), \quad T \mapsto T \otimes id_V.
\]
This has a left inverse \(a \mapsto \sigma_V^{-1}A(id_U \otimes R_{\bar{U}}^V \otimes id_{\bar{V}})(a)\). Moreover,
\[
A_0(a, b) = A(R_{\bar{U}})(a \cdot b^\natural) \quad (A.2)
\]
defines an \(A_0\)-valued inner product on the left \(A_0\)-module \(A(U)\).

**Definition A.1.** A pre-\(C^*\)-algebra object in \(\text{Vec}(\mathcal{C})\) is a \(*\)-algebra object \(A\) such that \(A_0\) is a \(C^*\)-algebra and the inner product \((A.2)\) is Hermitian, i.e., \(A_0(a, a)\) is a positive element of \(A_0\) for any \(a \in A(U)\).

Let us now assume that \(A\) is a pre-\(C^*\)-algebra object. By taking fiberwise completion, we may assume that \(A(U)\) is a left Hilbert module over \(A_0\). Our goal is to see that this is a \(C^*\)-algebra object.

*There seems to be a typo in [JP17].*
Proposition A.2. Let $A$ be a pre-$C^*$-algebra object in $\text{Vec}(\mathcal{C})$. Then we have
\[\|A(T)(a)\| \leq \|T\| \|a\| \quad (T \in \mathcal{C}(V, U), a \in A(U))\]
for the norm of left Hilbert $A_0$-modules. Similarly, we have
\[\|a \cdot b\| \leq \|a\| \|b\| \quad (a \in A(U), b \in A(V)).\]

Proof. We have
\[A_0(A(T)(a), A(T)(a)) = A_0(A(T^*)(a), a)\]
by (A.1) and the way we define inner product, (A.2). From $TT^* \leq \|T\|^2$, there is $S$ such that $\|T\|^2 - TT^* = SS^*$. Then we get
\[\|T\|^2 A_0(a, a) - A_0(A(T)(a), A(T)(a)) = A_0(A(S)(a), A(S)(a)) \geq 0,\]
hence the first claim.

As for the second claim, we can start from $A_0(a, a) \leq \|a\|^2$ and use a similar argument. \qed

Theorem A.3. Let $A$ be a pre-$C^*$-algebra object in $\text{Vec}(\mathcal{C})$. Then the fiberwise completion of $A$ with respect to the Hilbert module structure over $A_0$ is a $C^*$-algebra object.

Proof. The natural action of $A(U \otimes \overline{U})$ on $A(U)$ is by bounded and adjointable $A_0$-homomorphisms by Proposition A.2. We can thus assume that each $A(U)$ is already a Hilbert $A_0$-module.

From the associativity of product · and the antimultiplicativity of involution \(\overline{\cdot}\), we see that the natural action of $A(U \otimes \overline{U} \otimes U \otimes \overline{U})$ on $A(U \otimes \overline{U})$ is by adjointable homomorphisms. Thus, the norm
\[\|a\| = \|j_{U, U}(a)\|_{B_{A_0}(A(U \otimes \overline{U}))}\]
on $A(U \otimes \overline{U})$ is a pre-$C^*$-norm. We are going to show that this norm is equivalent to the Hilbert module norm on $A(U \otimes \overline{U})$. Then the fiberwise completion of $A$ gives a $C^*$-module category, hence the completion will be a $C^*$-algebra object in the sense of Definition 2.7.

On one hand, the action of \(j(a) = j_{U, U}(a)\) on $\eta_{U} = A(R^*_{\overline{U}})(1_{A_0})$ (the unit of $A(U \otimes \overline{U})$) is $a$, hence we have
\[\|a\| \geq \|a\|_{A(U \otimes \overline{U})} \|\eta_{U}\|^{-1}_{A(U \otimes \overline{U})} .\]
On the other, Proposition A.2 implies
\[\|R_{U}\| \|a\|_{A(U \otimes \overline{U})} \geq \|a\|',\]
and we obtain the claim. \qed

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