The leaking soft stadium

J. S. Espinoza Ortiz\textsuperscript{1}, R. E. Lagos-Monaco\textsuperscript{2}

\textsuperscript{1}Instituto de Física, Universidade Federal de Catalão, Catalão, GO 75704020, Brasil
\textsuperscript{2}Departamento de Física, IGCE, UNESP, Rio Claro, SP 13500970, Brasil

Abstract. We soften the non zero $y$-boundary on a Bunimovich like quarter-stadium. The smoothing procedure is performed via an exponent monomial potential, the system becomes partially reflective, preserving the particle’s translation and rotational motion. By increasing the exponent value, the stadium’s boundaries become rigid and the system’s dynamics reaches a chaotic regime. We set a leaking soft stadium family by opening a limited region located at some place of its basis’s boundary, throughout which the particles can leak out. This work is an extension of our recently reported paper on this matter. We chase the particle’s trajectory and focus on the stadium transient behavior by means of the statistical analysis of the survival probability on the marginal orbits that never leave the system, the so called bouncing ball orbits. We compare these family orbits with the billiard’s transient chaos orbits.

1. Introduction

Billiards are simple systems with a complex dynamics, so different cavity shapes have been used as theoretical models to understand experiments measuring the conductivity of quantum dots [1], and in many experimental research on micro wave cavities, optical and acoustical resonances [2]. In leaking cavities (open billiards) a tiny region in phase space can leak classical trajectories, with some saddle points ruling the motion of the long-lasting transient behavior [3, 4]. Different issues have been addressed using these constructions, ranging from geometrical acoustics, ergodicity, controlling chaos, cosmology, and conductance fluctuations in heterostructure devices [5]. Here we consider the soft potential given in [6], and extend the preliminary calculation reported by us in [7]. We begin by briefly reviewing the formalism presented in [7] and in section 3 we present the novel part.

A generalization of the hard wall stadium described by the Hamiltonian governed by a potential with real positive parameters $\alpha$ and $a$ [6] is,

$$H = \frac{p^2_x}{2} + \frac{p^2_y}{2} + V(x, y); \quad V(x, y) = \begin{cases} \frac{y^{2\alpha}}{2} & y \geq 0, x \in (-a, 0) \text{ (region1)}; \\ (x^2 + y^2)^{\alpha}/2 & x, y \geq 0 \text{ (region2)}; \\ \infty & (x = -a; \forall y \geq 0), (\forall x \geq -a; y = 0). \end{cases}$$

The soft stadium’s equipotential lines match with those of the corresponding billiards’ boundaries. By changing the exponent $\alpha$, the system covers all the dynamical possibilities, ranging from fully integrable motion when $\alpha = 1$ to the fully chaotic motion when $\alpha \to \infty$ (the quarter stadium billiard). We construct a leaking soft stadium by piercing a hole in its lower hard boundary (at $y = 0$), and probe the response of the marginal unstable bouncing ball periodic orbits to the survival trajectories probability. We use our section method strategy to chase the particle’s classical trajectories, thus improving the computation of the orbit’s surviving
probability by the Sprinkler method, therefore reaching longer evolution times. In section 2 we review the cavities’ leaking theory and the transient chaos theory scheme. In section 3 we describe the Sprinkler method, and rewrite our algorithm to compute the particle’s trajectory evolution. The novel contribution is to consider turning points analysis. In section 4 the numerical results are presented, and finally in section 5 we present our concluding remarks.

2. General theory
We consider a 2-Dim cavity with perimeter $\mathcal{L}$ and area $\mathcal{A}$, which is filled with $N$ particles. There is a tiny hole of length $\triangle L$ set on the cavity’s boundary and our objective is to probe how the cavity is emptied out [7, 8]. Via a kinetic theory scheme, we define a phase space density of particles $g(v, t)$ as function of the velocity $v$ and time $t$, then $\int g(v, t) d^2v = N(t)/\mathcal{A}$. The rate of particles escaping from the cavity is $dN(t)/dt = -\triangle L \int v \cdot n g(v, t) d^2v$, where $n$ is the external normal vector to the leaking boundary. For a small hole and for a time scale shorter that the average lifetime, we have a quasi-equilibrium particle distribution sharing the same closed system’s properties. If we assume the phase space isotropic and homogeneous, then $d^2v = 2\pi v dv$ and $v \cdot n = v \cos \phi$ with $\phi \in (-\pi/2, \pi/2)$, leading to $N(t)/\mathcal{A} = 2\pi \int g(v, t) v dv$. The probability density for the velocity modulus is defined as $\omega(v) dv = 2\pi g(v, t) v dv \mathcal{A}/N(t)$, it follows $dN(t)/dt = -N(t) \triangle L \langle v \rangle /\pi \mathcal{A}$ and the mean velocity is given by $\langle v \rangle = \int \omega(v) v dv$. By defining the average lifetime as $\langle \tau \rangle = 1/\kappa = \pi \mathcal{A}/\triangle L \langle v \rangle$, we find the survival probability $P(t) = N(t)/N(0) = \exp(-\kappa t)$, and the escape time probability density becomes $p(t) dt = -dP(t)$. Here, we define the collision time, a parameter measured experimentally, as $\langle t_{\text{coll}} \rangle = \mathcal{A}/\triangle L \langle v \rangle$ and we write $\kappa = \mu(\mathcal{I})/\langle t_{\text{coll}} \rangle$, where $\mu(\mathcal{I}) = \triangle L/\mathcal{A}$ is a Lebesgue measure.

Usually, neither the phase space is kept unchanged due to the small size of the leak nor the correlation time decays exponentially due to the particles’ unstable dynamics; however the exponential behavior of the survival probability would remain valid for some finite time while the system leaks. Therefore, it is strongly necessary to improve the escape rate estimate. We utilize a conditional invariant measure, as in [7], $\mu_c : \mu(\mathcal{I}) \rightarrow \mu_c(\mathcal{I})$, taking into account long living particles, so $\langle t_{\text{coll}} \rangle \rightarrow \langle t_{\text{coll}} \rangle_c = \int t_{\text{coll}}(x) d\mu_c$, with $x$ the phase space coordinates along the cavity boundary.

Transient chaos theory arises from a set of non-attracting chaotic saddle in phase space. In both weak and hard chaotic leaky cavities, the saddle possess a stable and an unstable manifold causing an exponential decay of the orbit survival probability. Regions with regular motion are composed by stable periodic orbits and quasi-periodic KAM tori [9]. It is well known that in closed Hamiltonian systems, these orbits can coexist together with chaotic motion regions. For instance, any trajectory into the chaotic region approximating some KAM surface, must in closed Hamiltonian systems, these orbits can coexist together with chaotic motion regions. It is well known that in closed Hamiltonian systems, these orbits can coexist together with chaotic motion regions. In turn, stickiness and weak chaos are driven by trajectories of measure zero. Therefore, the existence of sticky regions must modify the asymptotic decay of the survival probability, i.e. initially behaving as an exponential ($e^{-\kappa t}$), then followed by a power law behavior ($t^{-z}$) [8]. As the escape rate $\kappa$ is a property of the saddle, the exponent $z$ is related to the chaotic region’s properties close to the regular regions of measure zero. Thus, we can expect a crossover between both different temporal behaviors marked by a critical time $t_c$, characterizing the first trajectory approaching the sticky region.

3. The sprinkler method and the trajectories’ evolution
In the framework of the transient theory [7, 10], fast calculations are performed by applying the Sprinkler method. We consider a set of trajectories uniformly distributed in phase space $N_0 \gg 1$, and then select an appropriate time $t^* \gg 1/\kappa$. The set of trajectories evolves towards $t^*$, and we keep only the subgroup of no escaping trajectories ($\approx N_0 e^{-\kappa t^*}$). Notice that, the subset of trajectories with long lifetime comes very close to the neighborhood of the saddle, while following
their dynamical evolution pathway. Therefore, the escaping subgroup of orbits are going to be distributed along the unstable manifold, exhibiting a conditional invariant measure [10]. This stationary distribution is known as a conditionally invariant measure.

Next, for the soft stadium’s classical trajectories [6], in the rectangular region, the initial conditions are set as: $(0, y')$ for the position and $(-P_{xt}, \pm p_{yt})$ for the momentum. The rectangular height is $y_m = (2 - p_{x'}^2)^{1/2}\alpha$ and the vertical momentum $p_{yt} = p(y')$, where $p(y) = \sqrt{2 - p_{x'}^2 - y'^2}$. The zigzagging particle travels a distance of $2a$ units along the x-direction until it reaches the point $(0, \tilde{y})$, satisfying $\tau(\tilde{y}) = \text{sign}(p_{\tilde{y}}) \left( \text{sign}(p_{yt}) \tau(y') + 2 \{\Lambda_T\} \tau(y_m) \right)$; $\tau(y) = \int_0^y dy/p(y)$, where the fractional-part function is $\{\Lambda_T\} = \frac{a/|p_{x'}|}{\tau(y_m) - \frac{a/|p_{x'}|}{\tau(y_m)}}$ and the momentum’s modulus $|p_{\tilde{y}}| = p(\tilde{y})$. Inside the quarter circle, now the initial conditions becomes $(0, \tilde{y})$ and $(+\nu, \pm p(\tilde{y}))$, with angular momentum $\nu = \tilde{y}p_{x'}$. Observe that, the radial momentum $p(r) = \sqrt{2 - \nu^2/r^2 - r'^2}$ possesses as least two real turning points $p(r = 0) = r_1 < r_2$, $r_2$ corresponding to the circle radius. If the particle rotates an angle $\pi$ in order to reach its final position $(0, y'\nu)$, we have $\theta(y'\nu) = \text{sign}(p_{y'\nu}) \left[ \text{sign}(p_{\tilde{y}}) \theta(\tilde{y}) + 2 \{\Lambda_R\} \theta(r_2) \right]$; $\theta(y) = \int_{r_1}^y \nu dr/r^2 p(r)$ and $\{\Lambda_R\} = \frac{\pi/2}{\nu(r_2)} - \left[ \frac{\pi/2}{\nu(r_2)} \right]$; with $p_{y'\nu} = p(y'\nu)$ and $P_{xt} = -\frac{\nu}{\tau(\tilde{y})} p_{xt}$. With these dynamical information, we are able to chase the particle’s trajectory time-evolution by finding its position and momentum at successive contact points on the lower billiard boundary at $y = 0$, as it was com-

**Figure 1.** The spreading of orbits evolution in phase space (at $y = 0$) for the soft stadium with $\alpha = 1.35$, 24.25 and the billiard case (from left to right).

**Figure 2.** The number of bounces $N_b$ collisions on the stadium boundaries as a function of time, in a log-log plot. From left to right, $\alpha = 4.35$, 8.25 and 24.25.
Figure 3. The orbits survival probability $P$ as time evolves. From left to right upper-row for $\alpha = 1.35$ and $2.35$, and in the lower-row for $\alpha = 17.25$ and the billiard case.

puted in [7].

4. Numerical Results

We set up the soft stadium with parameters $\alpha \geq 1$ and $a = 1$, so the minimal stadium’s perimeter when $\alpha \to \infty$ reads $L = 2a (2 + \pi)$. In order to facilitate analogies with the hard billiard, the hole is centered on the stadium’s lower rigid wall at $y = 0$ $x = -\sqrt{2}/2$ and the width defined as $\Delta L = 10^{-4}L$. In order to facilitate analogies with the hard billiard, the hole is centered on the stadium’s lower rigid wall at $y = 0$ $x = -\sqrt{2}/2$ and the width defined as $\Delta L = 10^{-4}L$. The set of trajectories traveling through the stadium depart from the position $x_{1i} = 0$, $y_{1i} \in (y_{1i} - \Delta y/2, y_{1i} + \Delta y/2)$ with momentum $p_{x1i} = p \sin \theta_{1i}$, $p_{y1i} = p \cos \theta_{1i}$; $\theta_{1i} \in (\theta_{1i} - \Delta \theta/2, \theta_{1i} + \Delta \theta/2)$, for $i \in (1, 2, 3, \ldots , N)$. The particular parameters are chosen so the set of trajectories departs very close to a bouncing ball trajectory. Then we choose $y_{1i} = 0.175$, $\theta_{1i} = 1.147 \times (\pi/2)$ and the momentum modulus $p = (\sqrt{2} - y_{1i}^{2m} + \sqrt{2} - y_{1i}^{2p})/2$, with $y_{1i}^{2m} = (y_{1i} \pm \Delta y/2)$, $\Delta y = 10^{-5}$, $\Delta \theta = 10^{-5}$ and the number of evolved orbits $N = 7500$.

In Figure 1 it is shown how the orbits spreads in phase space, with $\alpha = 1.35$, $24.25$ and the billiard case ($\alpha \to \infty$), respectively. We observe, the bigger the $\alpha$-values the more the orbits spread in phase space, signalizing that the soft stadium system’s family reaches the hard chaotic regime. Notice that in the billiard’s phase space, there is an unreachable region in the neighborhood of $x \leq 0$ with a small momentum $p_x$. This region is almost filled up in the soft stadium case. Thus, for the former, some family of bouncing ball trajectories require even longer time evolution in order to be observed. Figure 2 shows the number of bounces the particle’s surviving orbit has done as a function of time, for different values of $\alpha = 4.35$, $8.25$, $24.25$, in a log-log plot. We are able to examine the orbits’ surviving-probability as a function of time up to $t = 10^5$, and observe a transition from an exponential to a power law decay behavior. In Figure 3 this transition is displayed for the soft stadium with exponent $\alpha = 1.35$, $2.35$, $17.25$ and the billiard case. The first trajectory approximation to the sticky region is found for a critical time $t_c \approx 4 \times 10^4$ in the hard case and for the soft stadium is found to occur earlier at $t_c \approx 2 \times 10^4$. The escape rate $\kappa$ and the $z$-exponent were computed for several values of $\alpha$. For $\alpha = 1.35 : \kappa = 2.24 \times 10^{-4}$, $z = 0.97$; for $\alpha = 2.35 : \kappa = 3.00 \times 10^{-4}$, $z = 0.99$; and for $\alpha = 17.25 : \kappa = 3.77 \times 10^{-4}$, $z = 2.10$. Finally, in Figure 4 we plot the values of $\kappa$ and $z$ computed as a function of $\alpha$. We also plot the billiard case.
5. Concluding remarks
We applied the Sprinkler method [10] to chase a set of orbits’ trajectories evolving in a stadium shaped cavity, and analyze their long temporal evolution; until they leak through a given hole. The set of orbits departs with initial conditions closer to the marginal bouncing ball orbits, i.e. with a small horizontal momentum, so that the orbits travel for a long time ($\approx 10^5$), allowing us to observe the orbits survival probability transient behavior and the transition regime at the critical time $t_c$ [8, 10]. For time values smaller than $t_c$, by varying $\alpha$ the typical $\kappa$-values slowly increase from $2.4 \times 10^{-4}$ to $3.6 \times 10^{-4}$, as its shown in the left side of Figure 4. For the billiard case $\kappa$ is close to $10^{-4}$ as shown in the same figure. Notice that, the stadium-cavity’s trajectories involve turning points (i.e. acceleration) with a shorter time evolution than the billiard case.

For time values bigger than $t_c$, the behavior of $z$ as a function of $\alpha$ is shown in the right-side of Figure 4. By varying $\alpha$ from 1.25 to 12.25, the parameter $z$ increases from a value close to one until to reach its maximum value equal to 2.8. When increasing $\alpha$ further up to $\alpha = 17.25$ the corresponding $z$-values start decreasing down to the value 2.1. For the billiard case we find $z = 2, 3$.

We have significantly improved our results for the billiard case reported in [7], and extended our analysis of the soft stadium shaped cavity. We would like to emphasize that our results are in agreement with those reported for the billiard case, by numerical computation in [11] and by asymptotic analysis in [12]. By varying $\alpha$ the leaking cavity shows a complex dynamical behavior ranging from soft to hard chaos and it also presents sticky regions on phase space modifying the survival probability exponential behavior to a power law behavior. Work is in progress in order to improve the computed parameter values of the leaking soft stadium system, by using the Lebesgue measure and by exploring even longer trajectories’ evolution times.

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