ON THE STABILIZATION OF A HYPERBOLIC STOKES
SYSTEM UNDER GEOMETRIC CONTROL CONDITION

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ABSTRACT. In this paper, we study the stabilization problem for a hyperbolic
type Stokes system posed on a bounded domain. We show that when the
damping effects are restricted to a subdomain satisfying the geometrical control
condition the system decays exponentially. The result is a consequence of a
new quasi-mode estimate for the Stokes system.

1. Introduction and Main Results

Let Ω ⊂ Rd (d ≥ 2) be a bounded connected open set whose boundary ∂Ω is
regular enough, ω be a small subset of Ω and let T > 0.

In this paper, we are interested in the stabilization problem for the following
hyperbolic Stokes system:

\[
\begin{align*}
\partial_t^2 u - \Delta u + \nabla p + a(x) \partial_t u &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
u &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
(u(0, x), \partial_t u(0, x)) &= (u_0, v_0) \in V \times H,
\end{align*}
\]

(1.1)

where V and H are the usual spaces in the context of fluid mechanics, i.e.,

\[V = \{u \in H^1_0(\Omega)^d : \text{div } u = 0\}\]

and

\[H = \{u \in L^2(\Omega)^d : \text{div } u = 0, u \cdot \nu|_{\partial \Omega} = 0\},\]

and \(\nu(x)\) is the outward normal to Ω at the point \(x \in \partial \Omega\). In (1.1), the damping
term \(a \in L^\infty(\Omega)\) and satisfies \(a(x) \geq 0\), for all \(x \in \Omega\).

If \(u = u(x, t)\) is a (sufficiently smooth) solution of the system, we define its
energy as

\[E[u](t) = \frac{1}{2} \int_\Omega (|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2) dx, \quad \forall t \in \mathbb{R},\]

and when there is no damping, namely \(a \equiv 0\), the energy is conserved, while in
general we only have that \(E[u](t)\) is non-increasing:

\[\frac{dE[u]}{dt} = -\int_\Omega a(x)|\partial_t u(t, x)|^2 dx \leq 0.\]

As for other hyperbolic systems, the stabilization problem for (1.1) concerns
about the decay rate in time of the energy \(E[u](t)\) under appropriate assumptions
on the damping term.

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It is well-known that stabilization problems are closely related to observability and exact controllability problems in abstract settings. In fact, if we consider the undamped system

$$\begin{cases}
\partial_t^2 u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R} \times \Omega, \\
\text{div} \, u = 0 & \text{in } \mathbb{R} \times \Omega, \\
u = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
(u(0, x), \partial_t u(0, x)) = (u_0, v_0) & \in V \times H,
\end{cases}$$

(1.2)

we say that (1.2) is observable at time $T$ with observation in $\omega$ if there exists $C > 0$ such that

$$\|u_0\|^2_V + \|v_0\|^2_H \leq C \int_0^T \int_\omega |\partial_t u(t, x)|^2 \, dx \, dt,$$

(1.3)

for every $(u_0, v_0) \in V \times H$.

When (1.3) holds, one can show that for any $(u_0, v_0) \in V \times H$ there exists $f \in L^2((0, T) \times \omega)^d$ such that the solution of

$$\begin{cases}
\partial_t^2 u - \Delta u + \nabla p = f \mathbf{1}_\omega & \text{in } \mathbb{R} \times \Omega, \\
\text{div} \, u = 0 & \text{in } \mathbb{R} \times \Omega, \\
u = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
(u(0, x), \partial_t u(0, x)) = (u_0, v_0) & \in V \times H,
\end{cases}$$

(1.4)

satisfies

$$u(T, x) = 0, \quad \partial_t u(T, x) = 0,$$

that is to say, system (1.4) is exact controllable at $T$ with control localized in $\omega$. Nevertheless, it is important to mention that a complete characterization of the sets $\omega$ for which (1.3) is true remains open. A partial answer to this question was given by the first author in [16].

The motivation for studying the stabilization of system (1.1) is two folded. First, system (1.2) is the hyperbolic counterpart of Stokes system, which is the linearized version of the well-known Navier-Stokes equation in fluid mechanics. In fact, if we know that system (1.2) is exact controllable at some time $T > 0$, with control applied to some control region $\omega$, then the so-called Control Transmutation Method can be applied to obtain the null controllability at any time and the optimal cost of controllability (in time) for the Stokes system (for more details, see [16]). On the other hand, system (1.2) comes from simple models of dynamical elasticity for incompressible materials. More precisely, it can be derived as a limit model of Lamé system in linear elastic theory when one parameter tends to infinity (12). For the sake of completeness, in the Appendix we give a derivation of system (1.2) from Lamé system. It is important to remark that the stabilization problem for the Lamé system has been already studied in [5].

To state our main results, let us introduce several concepts. Some terminologies and notation will be clear in the next section.

**Definition 1.1.** We say that the support of a non-negative function $a \in C(\overline{\Omega})$ satisfies the geometric control condition (GCC in short) if there exists $T > 0$, such that each generalized bicharacteristic ray $\gamma(t)$ with speed 1 issued from a point $\rho \in b \cdot T^* \overline{\Omega}$ enters the set $\{x \in \overline{\Omega} : a(x) > 0\}$ in a time $t < T$.

We recall that an open set $\Omega$ has no infinite order of contact, if in the decomposition

$$T^* \partial \Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G},$$
we have
\[ \mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j. \]

Here, the sets \( \mathcal{E}, \mathcal{H}, \mathcal{G} \) are called elliptic zone, hyperbolic zone and glancing zone, respectively, and \( \mathcal{G}^j \) are the sets of points with \( j \)-th order of contact. The precise definition of this sets will be given in the next section.

Our first main result is as follows.

**Theorem 1.2.** Suppose \( \Omega \subset \mathbb{R}^d \) is a bounded open set with no infinite order of contact and \( a \in C(\overline{\Omega}) \) is a non-negative function whose support satisfies the geometric control condition. Then, there exist positive constants \( C_0 \) and \( \alpha \) such that for any \( (u_0, v_0) \in V \times H \), the corresponding solution \( u(t) \) to (1.1) has the exponential decay:
\[ E[u](t) \leq C_0 E[u](0) e^{-\alpha t}, \quad \forall t \geq 0. \] (1.5)

In what follows, we say that the stabilization of (1.1) holds if (1.5) holds true.

**Remark 1.3.** As a byproduct of the proof of Theorem 1.2, we obtain the null (exact) controllability at some time \( T \) of system (1.4). Namely, there exists \( T > 0 \) and a control \( f \in L^2([0, T] \times \omega) \) such that the corresponding solution \( u \) to (1.4) satisfies \( (u(T), \partial_t u(T)) = (0, 0) \). However, we do not know the control time \( T \) explicitly, since we prove the observability inequality (1.3) by reducing it to a quasi-mode estimate.

Let us mention that if \( a \) is supported in a neighborhood of boundary \( \partial \Omega \), the same result is true by adapting the strategy in [10], where the author has proved the exact controllability of the system (1.4) with \( \omega \) be a neighborhood of \( \partial \Omega \). Our result is somewhat generalization of the result in [10].

The pioneering work of J.Rauch and M.Taylor [19] related the exponential decay of damped wave equation to geometric control condition (GCC) of damped region on compact Riemannian manifold without boundary.

Until the celebrated work of C. Bardos, G. Lebeau, and J. Rauch [2], the presence of the boundary has been understood and the exactly controllability for wave equation as well as the exponential stabilization are obtained under (GCC). The proof mainly relies on the propagation of singularity under Melrose-Sjöstrand flow. Later on, the tool of micro-local defect measure, introduced by P.Gérard and L.Tartar independently, has been used to simplify the proof of these results and adapt to many other problems, see for example [5] for Lamé systems and [6] for a coupled wave system. The key ingredient of the measure-based proof is the propagation formula, which can be viewed as a transport equation for defect measure. As a consequence, the propagation of singularity can be derived as a special case of measure invariance under bicharacteristic flow.

For the present system (1.1), the presence of the pressure term \( \nabla p \) introduces nontrivial difficulties if we want to adapt the strategy in [5] directly, due to the rough regularity of time-dependent harmonic function \( p(t, x) \). However, follow the semi-classical reduction in [4], it turns out that the exponentially stabilization of (1.1) can be reduced to the following semi-classical version observability estimate:

**Proposition 1.4.** Assume that \( a \in L^\infty(\Omega) \cap C^0(\overline{\Omega}) \) and \( \int_{\Omega} \, dx > 0 \). Suppose the following statement holds true:
\[
\exists h_0 > 0, C > 0 \text{ such that } \forall 0 < h < h_0, \forall (u, q, f) \in H^2(\Omega) \cap V \times H^1(\Omega) \times H
\]
solves the equation

\[- h^2 \Delta u - u + h\nabla q = f, \quad (1.6)\]

implies

\[
\|u\|_{L^2(\Omega)} \leq C \left( \| a^{1/2} u\|_{L^2(\Omega)} + \frac{1}{h} \|f\|_{L^2(\Omega)} \right). \quad (1.7)
\]

Then we have the stabilization of \((1.1)\).

Note that the system \((1.7)\) is just a quasi-mode equation of stationary Stokes system, and in particular, if \(f = 0\), the solution \(u(h)\) is an eigenfunction of Stokes operator corresponding to eigenvalues \(h^{-2}\).

The proof of \((1.7)\) is based on the propagation of semi-classical measure \(\mu\) in the recent work \([20]\) of the second author. We give a brief recall here. The sequence of pressure \(q\) are harmonic, and their impact on the solution only occurs at the boundary. It has been shown that the measure is propagated along bi-characteristic rays which is invariant under the flow. When a ray touches the boundary, more careful analysis between the wave-like propagation phenomenon and the impact of the pressure yield the propagation of the support of the measure \(\mu\) along generalized bi-characteristic ray defined in \([14]\).

We organize this paper as follows. In section 2, we give some notations, definitions and classical results. In section 3, we follow the strategy in \([4]\) to reduce the stabilization to semi-classical observability \((1.7)\). In section 4, we prove the semi-classical observability by adapting the propagation result. Finally in Appendix, we give the derivation from Lamé system to system \((1.1)\).

2. Preliminary

2.1. Notations. For a manifold \(M\), we let \(TM\) be its tangent bundle and \(T^*M\) be the cotangent bundle with canonical projection

\[\pi: TM \ (or \ T^*M) \to M.\]

In the turbulence neighborhood of boundary, we can identify the \(\Omega\) locally as \([0, \epsilon_0) \times X, X = \{x' \in \mathbb{R}^{d-1} : |x'| < 1\}\). For \(x \in \Omega\), we note \(x = (y, x')\), where \(y \in [0, \epsilon_0), x' \in X\), and \(x \in \partial \Omega\) if and only if \(x = (0, x')\). In this coordinate system, the Euclidean metric \(dx^2\) can be written as matrices

\[g = \begin{pmatrix} 1 & 0 \\ 0 & M(y, x') \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(y, x') \end{pmatrix},\]

with \(|\xi'|_{\alpha(y, x')}^2 = \langle \xi', \alpha(y, x')\xi\rangle_{C^{d-1}}\) be the induced metric on \(T^*\partial \Omega\), parametrized by \(y\). Note that \(|\xi'|_{\alpha(0, x')}^2 = \langle \xi', \alpha(0, x')\xi\rangle_{C^{d-1}}\) is the natural norm on \(T^*\partial \Omega\), dual of the norm on \(T\partial \Omega\), induced by the canonical metric on \(\Omega\). Write \((x, \xi) = (y, x', \eta, \xi')\) and denote by \(|\xi|\) the Euclidean norm on \(T^*\mathbb{R}^d\).

We define the \(L^2\) norms and inner product on \([0, \epsilon_0) \times X\) via

\[
\|u\|_{L^2_{y,\cdot,x'}}^2 := \int_0^{\epsilon_0} \int_X |u|^2 d g(y, \cdot) x' dy, \\
(u|v)_{L^2_{y,\cdot,x'}} := \int_0^{\epsilon_0} \int_X u \cdot v d g(y, \cdot) x' dy, \\
\|u(y, \cdot)\|_{L^2_{\cdot,x'}}^2 := \int_X |u(y, \cdot)|^2 d g(y, \cdot) x',
\]

where \(d g(y, \cdot) = \sqrt{\det(g)} d\lambda\) with \(\lambda\) the Lebesgue measure and \(\det(g) = \det(g_{ij})\).
where the measure $d_{g(y,\cdot)}x'$ is the induced measure on $X$, parametrized by $y \in [0, \epsilon_0)$ such that $d_{g(y,\cdot)}x'\,dy = \mathcal{L}(dx)$, the Lebesgue measure on $\mathbb{R}^d$. Note that the measure $d_{g(0,\cdot)}x'$ is nothing but the surface measure on $\partial\Omega$. In certain situations we perform using global notation for inner product:

$$(u|v)_{\Omega} := \int_{\Omega} u \cdot \nabla dx,$$

$$(f|g)_{\partial\Omega} := \int_{\partial\Omega} f \cdot \nabla \sigma(x)$$

In the turbulence neighborhood, we can write a vector field $X = (X_\parallel, X_\perp)$, where $X_\parallel$ stands for the components parallel to the boundary while $X_\perp$ stands for the normal component with the following convention: $(0, a) = -a \nu$.

As in [17], we will write down system (1.1) in the turbulence neighborhood. For $u = (u_\parallel, u_\perp)$, equation (1.1) can be rewritten:

$$
\begin{aligned}
&(-h^2 \Delta_{\parallel} - 1)u_\parallel + h \nabla_x q = f_\parallel, \\
&(-h^2 \Delta_{\perp} - 1)u_\perp + h \partial_y q = f_\perp, \\
h \text{ div } \parallel u_\parallel + \frac{h}{\sqrt{\det g}} \partial_{\lambda}(\sqrt{\det g u_\perp}) = 0
\end{aligned}
$$

(2.1)

where

$$
\begin{aligned}
h^2 \Delta_{\parallel} &= h^2 \partial_y^2 - \Lambda^2(y, x', hD_x) + hM_{\parallel}(y, x', hD_x') + hM_1(y, x') h \partial_y, \\
h^2 \Delta_{\perp} &= h^2 \partial_y^2 - \Lambda^2(y, x', hD_x) + hM_{\perp}(y, x', hD_x') + hN_1(y, x') h \partial_y,
\end{aligned}
$$

$h^2 \Lambda^2(y, x', hD_x')$ has the symbol $\lambda^2 = |\xi'|_g^2(y, \cdot)$, and $M_{\parallel, \perp}$ are both first-order matrix-valued semi-classical differential operators.

2.2. **Geometric Preliminaries.** Denote by $bT^*\overline{\Omega}$ the vector bundle whose sections are the vector fields $X(p)$ on $\overline{\Omega}$ with $X(p) \in T_p\partial\Omega$ if $p \in \partial\Omega$. Moreover, denote by $bT^*\overline{\Omega}$ the Melrose’s compressed cotangent bundle which is the dual bundle of $bT^*\overline{\Omega}$.

Let

$$
j : T^*\overline{\Omega} \to bT^*\overline{\Omega}
$$

be the canonical map. In our geodesic coordinate system near $\partial\Omega$, $bT^*\overline{\Omega}$ is generated by the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, y \frac{\partial}{\partial y}$ and thus $j$ is defined by

$$
j(y, x'; \eta, \xi') = (y, x'; v = y\eta, \xi').
$$

The principal symbol of operator $P_h = -(h^2 \Delta + 1)$ is

$$
p(y, x', \eta, \xi') = \eta^2 + |\xi'|_{\alpha(y, x')}^2 - 1.
$$

By Car($P$) we denote the characteristic variety of $p$:

$$
\text{Car}(P) := \{ (x, \xi) \in T^*\mathbb{R}^d_{|\overline{\Omega}} : p(x, \xi) = 0 \}, Z := j(\text{Car}(P)).
$$

By writing in another way

$$
p = \eta^2 - r(y, x', \xi'), r(y, x', \xi') = 1 - |\xi'|_{\alpha}^2,
$$
we have the decomposition
\[ T^*\partial \Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G}, \]
according to the value of \( r_0 := r|_{y=0} \) where
\[ \mathcal{E} = \{ r_0 < 0 \}, \mathcal{H} = \{ r_0 > 0 \}, \mathcal{G} = \{ r_0 = 0 \}. \]
The sets \( \mathcal{E}, \mathcal{H}, \mathcal{G} \) are called elliptic, hyperbolic and glancing, with respectively.

For a symplectic manifold \( S \) with local coordinate \((z, \xi)\), a Hamiltonian vector field associated with a real function \( f \) is given by
\[ H_f = \frac{\partial f}{\partial \xi} \frac{\partial}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial}{\partial \xi}. \]
Now for \((x, \xi) \in \Omega \) far away from the boundary, the Hamiltonian vector field associated to the characteristic function \( p \) is given by
\[ H_p = 2\xi \frac{\partial}{\partial x}. \]
We call the trajectory of the flow
\[ \phi_s : (x, \xi) \mapsto (x + s\xi, \xi) \]
bicharacteristic or simply ray, provided that the point \( x + s\xi \) is still in the interior.

To classify different scenarios as a ray approaching the boundary, we need more accurate decomposition of the glancing set \( \mathcal{G} \). Let \( r_1 = \partial_y r|_{y=0} \) and define
\[ \mathcal{G}^{k+3} = \{(x', \xi') : r_0(x', \xi') = 0, H^{(j)}_p(r_1) = 0, \forall j \leq k; H^{k+1}_p(r_1) \neq 0\}, k \geq 0 \]
\[ \mathcal{G}^{2 \pm} := \{(x', \xi') : r_0(x', \xi') = 0, \pm r_1(x', \xi') > 0\}, \mathcal{G}^2 := \mathcal{G}^{2+} \cup \mathcal{G}^{2-}. \]
No infinite order of contact means that we can decompose \( \mathcal{G} \) into
\[ \mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j. \]

Given a ray \( \gamma(s) \) with \( \pi(\gamma(0)) \in \Omega \) and \( \pi(\gamma(s_0)) \in \partial \Omega \) be the first point who attaches the boundary. If \( \gamma(s_0) \in \mathcal{H} \), then \( \eta_{\pm}(\gamma(s_0)) = \pm \sqrt{r_0(\gamma(s_0))} \) be the two different roots of \( \eta^2 = r_0 \) at this point. Notice that the ray starting with direction \( \eta_- \) will leave \( \Omega \), while the ray with direction \( \eta_+ \) will enter the interior of \( \Omega \). This motivates the following definition of broken bicharacteristic:

**Definition 2.1** ([10]). *A broken bicharacteristic arc of \( p \) is a map:*
\[ s \in I \setminus B \mapsto \gamma(s) \in T^*\Omega \setminus \{0\}, \]
*where \( I \) is an interval on \( \mathbb{R} \) and \( B \) is a discrete subset, such that*

1. *If \( I \) is an interval contained in \( I \setminus B \), then \( s \in I \mapsto \gamma(s) \) is a bicharacteristic of \( p \) over \( \Omega \).*
2. *If \( s \in B \), then the limits \( \gamma(s^+) \) and \( \gamma(s^-) \) exist and belongs to \( T^*_x\Omega \setminus \{0\} \) for some \( x \in \partial \Omega \), and the projections in \( T^*_x\partial \Omega \setminus \{0\} \) are the same hyperbolic point.*

When a ray \( \gamma(s) \) arrives at some point \( \rho_0 \in \mathcal{G} \), there are several situations. If \( \rho_0 \in \mathcal{G}^{2+} \), then the ray passes transversally over \( \rho_0 \) and enters \( T^*\Omega \) immediately. If \( \rho_0 \in \mathcal{G}^{2-} \) or \( \rho_0 \in \mathcal{G}^k \) for some \( k \geq 3 \), then we can continue it inside \( T^*\partial \Omega \) as long as it can not leave the boundary along the trajectory of the Hamiltonian flow of \( H_{-\rho_0} \). We now give the precise definition.
Definition 2.2 ([10]). A generalized bicharacteristic ray of $p$ is a map:

$$s \in I \setminus B \mapsto \gamma(s) \in (T^*\Sigma \setminus T^*\partial \Omega) \cup \mathcal{G}$$

where $I$ is an interval on $\mathbb{R}$ and $B$ is a discrete set of $I$ such that $p \circ \gamma = 0$ and the following:

1. $\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\Sigma \setminus T^*\partial \Omega$ or $\gamma(s) \in \mathcal{G}^{2,+}$.
2. Every $s \in B$ is isolated, $\gamma(s) \in T^*\Sigma \setminus T^*\partial \Omega$ if $s \neq t$ and $|s - t|$ is small enough, the limits $\gamma(s^{\pm})$ exist and are different points in the same fibre of $T^*\partial \Omega$.
3. $\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_{-r_0}(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \setminus \mathcal{G}^{2,+}$.

Remark 2.3. The definition above does not depend on the choice local coordinate, and in the geodesic coordinate system, the map

$$s \mapsto (y(s), \eta^2(s), x'(s), \xi'(s))$$

is always continuous and

$$s \mapsto (x'(s), \xi'(s))$$

is always differentiable and satisfies the ordinary differential equations

$$\frac{dx'}{dt} = \frac{\partial r}{\partial \xi'(s)} \frac{d\xi'}{dt} = \frac{\partial r}{\partial x'(s)}$$

the map $s \mapsto y(s)$ is left and right differentiable with derivative $2\eta(s^{\pm})$ for any $s \in B$ (hyperbolic point).

Moreover, there is also the continuous dependence with the initial data, namely the map

$$(s, \rho) \mapsto (y(s, \rho), \eta^2(s, \rho), x'(s, \rho), \xi'(s, \rho))$$

is continuous. We denote the flow map by $\gamma(s, \rho)$.

Remark 2.4. Under the map $j : T^*\Sigma \rightarrow T^*\Sigma$, one could regard $\gamma(s)$ as a continuous flow on the compressed cotangent bundle $bT^*\Sigma$, and it is called the Melrose-Sjöstrand flow. We will also call each trajectory generalized bicharacteristic or simply ray in the sequel.

It is well-known that if there is no infinite contact in $\mathcal{G}$, a generalized bicharacteristic is uniquely determined by any one of its points. In other words, the Melrose-Sjöstrand flow is globally well-defined. See [10] for more discussion.

3. Review of Semi classical propagation of singularity

3.1. Definition of defect measure. We follow closely as in [3] and the one can find in [7] for a little different but comprehensive introduction.

Define the partial symbol class $S^m_{\xi'}$ and the class of boundary $h$-pseudo-differential operators $\mathcal{A}^m_h$ as follows

$$S^m_{\xi'} := \{a(y, x', \xi') : \sup_{\alpha, \beta, y \in [0, r_0]} |\partial^\alpha_y \partial^\beta_{\xi} a(y, x', \xi')| \leq C_{m, \alpha, \beta}(1 + |\xi'|)^{m-\beta}\}.$$  

$$\mathcal{A}^m_h := \text{Op}_h^\text{comp}(S^m) + \text{Op}_h(S^m_{\xi'}) := \mathcal{A}^m_{h,1} + \mathcal{A}^m_{h,\partial}.$$  

Denote by $U$ a turbulence neighborhood of $\partial \Omega$. Consider functions of the form $a = a_1 + a_\partial$ with $a_1 \in C^\infty_c(\Omega \times \mathbb{R}^d)$ which can be viewed as a symbol in $S^0$, and
\( a_\beta \in C_0^\infty(U \times \mathbb{R}^{d-1}) \) can be viewed as a symbol in \( S^0_{\xi} \). We quantize \( a \) via the formula (in local coordinate)

\[
\text{Op}_h(a)f(y, x') = \frac{1}{(2\pi \hbar)^d} \int_{\mathbb{R}^{2d}} e^{i(x'-x)\xi} a_i(x, \xi)f(z)dz d\xi + \frac{1}{(2\pi \hbar)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{i(x'-x')\xi'} a_\beta(y, x', \xi')f(y, z')dz'd\xi'.
\]

Notice that the acting of tangential operator \( \text{Op}_h(a_\beta) \) can be viewed as pseudodifferential operator on the manifold \( \partial \Omega \), parametrized by the parameter \( y \in [0, \epsilon_0) \). No doubt that the definition of the operator \( \text{Op}_h(a_\beta) \) depends on the choice of local coordinate of \( \partial \Omega \). However, the bounded family of operators \( \mathcal{A}^m_{h, \beta} \) is defined uniquely up to a family of operators with norms uniformly dominated by \( \text{Ch} \), as \( h \to 0 \). See \[9\] for more details. Moreover, for any family \( (A_h) \), such that

\[
\|A_h - \text{Op}_h(a_\beta)\|_{L^2 \to L^2} = O(h),
\]

the principal symbol \( \sigma(A) \) is determined uniquely as a function on \( T^* \partial \Omega \), smoothly depending on \( y \), i.e. \( \sigma(A) \in C^\infty([0, \epsilon_0) \times T^* \partial \Omega) \).

When we deal with vector-valued functions, we could require the symbol \( a \) to be matrix-valued. Now for any sequence of vector-valued function \( w_k \), uniformly bounded in \( L^2(\Omega) \), there exists a subsequence (still use \( w_k \) for simplicity), and a nonnegative definite Hermitian matrix-valued measure \( \mu_i \) on \( T^* \Omega \) such that

\[
\lim_{k \to 0} (\text{Op}_h(a_i)w_k|w_k)_{L^2} = \langle \mu_i, a \rangle := \int_{T^*\Omega} \text{tr} (ad\mu_i).
\]

For a proof, see for example \[3\], and the micro-local version was appeared in \[8\].

From now on we will only deal with scalar-valued operator, even though we will encounter vector-valued functions in the analysis. Suppose \( u_k \) be a sequence of solutions to (5.1), under the assumptions below:

\[
\|u_k\|_{L^2(\Omega)} = O(1), f_k \in H \quad \text{and} \quad \|f_k\|_{L^2(\Omega)} = o(h_k),
\]

\[
\|h\nabla q_k\|_{L^2(\Omega)} = O(1), \int_{\Omega} q_k dx = 0, \tag{3.1}
\]

The following result shows that the interior measure \( \mu_i \) is supported on the \( \text{Car}(P) \).

**Proposition 3.1.** Let \( a_i \in C_0^\infty(\Omega \times \mathbb{R}^d) \) be equal to 0 near \( \text{Car}(P) \), then we have

\[
\lim_{k \to \infty} (\text{Op}_h(a_i)u_k|u_k)_{L^2} = 0.
\]

**Proof.** Note that the symbol \( b(x, \xi) = \frac{a_i(x, \xi)}{\xi_i} \in S^0 \) is well-defined from the assumption on \( a_i \). From symbolic calculus, we have

\[
\text{Op}_h(a_i) = B_h(-h_k^2\Delta - 1) + O_{L^2 \to L^2}(h_k).
\]

Therefore

\[
(B_h(-h_k^2\Delta - 1)u_k|u_k)_{L^2} = (B_h f_k|u_k)_{L^2} - (B_h h_k\nabla q_k|u_k)_{L^2}
\]

\[
= o(1) + [(h_k \nabla, B_h q_k)|u_k)_{L^2} - (h_k \nabla B_h q_k|u_k)_{L^2}
\]

\[
= o(1), \text{ as } k \to \infty,
\]

where in the last line we have used the symbolic calculus, integrating by part, and Lemma [5.3].
Now we denote by $Z = J(\text{Car}(P))$. Proposition 3.1 indicates that the interior defect measure $\mu_i$ is supported on $Z$. To define the defect measure up to the boundary, we have to check that if $a_0 \in C^\infty_c(U \times \mathbb{R}^{d-1})$ vanishing near $Z$ (i.e. $a_0$ is supported in the elliptic region for all $y$ small) then
\[
\lim_{k \to \infty} (\text{Op}_{h_k}(a_0)u_k|u_k)_{L^2} = 0.
\]
Indeed, this can be ensured by the analysis of boundary value problem in the elliptic region, and the reader can consult section 6. Now for any family of operator $A_h \in \mathcal{A}_h^0$, let $a = \sigma(A_h)$ be the principal symbol of $A_h$ and we define $\kappa(a) \in C^0(Z)$ via $\kappa(a)(\rho) := a(j^{-1}(\rho))$. Note that $Z$ is a locally compact metric space and the set
\[
\{\kappa(a) : a = \sigma(A_h), A_h \in \mathcal{A}_h^0\}
\]
is a locally dense subset of $C^0(Z)$. We then have the following proposition, which guarantees the existence of a Radon measure on $Z$:

**Proposition 3.2.** There exists a subsequence of $u_k, h_k$ and a nonnegative definite Hermitian matrix-valued Radon measure $\mu$, such that
\[
\lim_{k \to \infty} (A_{h_k}u_k|u_k)_{L^2} = (\mu, \kappa(a)), a = \sigma(A_h), \forall A_h \in \mathcal{A}_h^0.
\]

The proof of this result can be found in [3], see also [5] and [8] for its micro-local counterpart. Notice that if we write $a = a_i + a_\theta$, then
\[
(A_ku_k) \to \int_T \int_{\Omega} \text{tr} (a_i(\rho)d\mu_i(\rho)) + \int_{Z} \text{tr} (a_\theta(\rho)d\mu(\rho)).
\]

The following result shows that information of frequencies higher than the scale $h_k^{-1}$ does not lost, and the measure $\mu$ contains the relevant information of the sequence $(u_k)$.

**Proposition 3.3** ([20]). The sequence of solution $(u_k)$ is $h_k$-oscillating in the following sense:
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{|\xi| \leq Rh_k^{-1}} \left| \hat{\psi}u_k(\xi) \right|^2 d\xi = 0, \forall \psi \in C^\infty_c(\Omega),
\]
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{|\xi'| \leq Rh_k^{-1}} \left| \hat{\psi}u_k(y, \xi') \right|^2 d\xi' = 0, \forall \psi \in C^\infty_c(\Omega),
\]
where in the second formula, the Fourier transform involved is only the $x'$ direction.

A direct consequence is the following:

**Corollary 3.4.** Suppose $a^{1/2}u_k \to 0$ in $L^2(\Omega)$, and $\mu$ is the defect measure associated with $(u_k, h_k)$, then
\[
\langle \mu, a \rangle = 0.
\]

3.2. **Recall of propagation theorem.** Now let us recall the several results proved in [20].

In the interior, the full transport property of defect measure is proved.

**Proposition 3.5** ([20]). For any real-valued scalar function $a \in C^\infty_c(\Omega \times \mathbb{R}^d)$ vanishing near $\xi = 0$, we have
\[
\frac{d}{ds} \langle \mu, a \circ \gamma(s, \cdot) \rangle = 0.
\]
The following proposition illustrates that near an elliptic point on the boundary, there is no accumulate of singularity.

**Proposition 3.6** (20). \( \mu 1_E = 0 \). If we let \( \nu \) be the semi-classical defect measure of the sequence \((h_k \partial_t u_k |_{\partial \Omega}, h_k)\), then \( \nu 1_E = 0 \).

When a ray travels near a hyperbolic point or point in the glancing surface, the knowledge of the singularity is much less. Nevertheless, we have

**Theorem 3.7** (20). Assume that \( \Omega \) is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose \((u_k)\) is a sequence of solutions to the quasi-mode problem (1.1) with semi-classical parameters \( h = h_k \). Assume that \( f_k \in H \), \( \| f_k \|_{L^2(\Omega)} = o(h_k) \) and \( u_k \) converges weakly to 0 in \( L^2(\Omega) \). Assume that \( \mu \) is any semi-classical measure associated to some subsequence of \((u_k, h_k)\), then \( \text{supp} \mu \) is invariant under Melrose-Sjöstrand flow.

4. Reduction to Semi-classical observability

This section is devoted to the proof of Proposition 1.4. In fact, it is classical from [9] that stabilization or observability of a self-adjoint evolution system is equivalent to resolvent estimates. See also [4], [23].

Recall that the damped system is given by

\[
\begin{aligned}
\partial_t^2 u - \Delta u + a(x) \partial_t u + \nabla p &= 0, \quad (t, x) \in \mathbb{R} \times \Omega \\
\text{div} u &= 0, \text{in } \Omega \\
u(t, .)|_{\partial \Omega} &= 0 \\
(u(0), \partial_t u(0)) &= (u_0, v_0) \in V \times H \\
\end{aligned}
\]

(4.1)

We always assume that \( \Omega \subset \mathbb{R}^d \) is a bounded domain (open, connected set). We use \( \nu \) to denote the outward normal vector on \( \partial \Omega \) and the damping term \( a \in L^\infty(\Omega) \) with \( a(x) \geq 0 \).

We also consider the undamped system

\[
\begin{aligned}
\partial_t^2 u - \Delta u + \nabla p &= 0, \quad (t, x) \in \mathbb{R} \times \Omega \\
\text{div} u &= 0, \text{in } \Omega \\
u(t, .)|_{\partial \Omega} &= 0 \\
(u(0), \partial_t u(0)) &= (u_0, v_0) \in V \times H \\
\end{aligned}
\]

(4.2)

4.1. *Some functional analysis preliminaries.* We work with a Hilbert space \( \mathcal{H} := V \times H \), equipped with the norm

\[
\| (f,g) \|_{\mathcal{H}}^2 := \| \nabla f \|_{L^2(\Omega)}^2 + \| g \|_{L^2(\Omega)}^2.
\]

and denote \( \Pi : L^2(\Omega)^N \to H \) be the orthogonal projector (Leray-projector) and \( A = \Pi \Delta \) be the Stokes operator. We consider the operator:

\[
A = \begin{pmatrix}
0 & \text{Id} \\
A & -\Pi a
\end{pmatrix}
\]

(4.3)

with domain

\[
D(A) = (V \cap H^2(\Omega)) \times V.
\]
In order to use semi-group theory, we first show that for some $\lambda > 0$, the operator $(A - \lambda)$ is invertible: Take $(f, g) \in V \times H$, and consider the system
\[
\begin{cases}
v - \lambda u = f \\
Au - (\Pi a + \lambda)v = g
\end{cases}
\]  
(4.4)

We consider the bilinear form
\[
B(u_1, u_2) = \int_\Omega \nabla u_1 \cdot \nabla u_2 dx + \int_\Omega (\lambda^2 + \lambda a(x))u_1 \cdot u_2 dx,
\]
defined on $V \times V$. We then conclude from Lax-Milgram that for $\lambda > 0$, there exists $u \in V$ such that for any $w \in V$, we have
\[
B(u, w) = -\int_\Omega (g \cdot w + (a(x) + \lambda)f \cdot w) dx.
\]
Set $v = \lambda u + f$, we have solved the system (4.4) in weak sense. Standard regularity argument gives that for $\lambda > 0$, 
\[
(A - \lambda)^{-1} : H \to D(A),
\]
is a bounded, and $(A - \lambda)^{-1} : H \to H$ is compact. Moreover, if $\lambda \in \text{Spec}(A)$, we must have $\text{Re} \lambda < 0$. This will be clear in the proof of Proposition 4.6.

However, since the operator $A$ is not maximal dissipative, the Hille-Yoshida theorem is not applicable. A slightly general modification ensures the existence of semi-group $e^{tA}$ which evolves the initial data in $D(A)$ and solves the equation (4.1) with more regular data.

For solution $u, \partial_t u$ to (4.1), we consider the energy functional
\[
E[u](t) := \frac{1}{2} \int_\Omega (|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2) dx,
\]
and we calculate
\[
\frac{d}{dt} E[u](t) = \int_\Omega \partial_t u \cdot (\partial_t^2 u - \Delta u) dx
= -\int_\Omega \partial_t u \cdot \nabla p dx - \int_\Omega a(x)|\partial_t u|^2 dx
\leq 0,
\]
thus
\[
E[u](t) \leq E[u](s), \forall s \leq t.
\]  
(4.5)
By density argument, we can solve (4.1) with initial data in $H$ such that the energy dissipation (4.5) still holds.

4.2. Observability and Stabilization. In this section, we will prove the stabilization for damped system is equivalent to observability for undamped system. For this part, we follow closely in the appendix of [4] in which the authors have sketched the standard argument for damped wave equation.

We first introduce the quantity
\[
D[u](T) = \int_0^T \int_\Omega a(x)|\partial_t u(t, x)|^2 dx dt,
\]
and it quantifies the dissipation of the energy:
\[
E[u](T) = E[u](0) - D[u](T).
\]
Proposition 4.1. The following assertions are equivalent:

(1) Stabilization: There exists $C_0, \alpha > 0$, such that for every solution $u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the damped system (4.1), we have
$$E[u](t) \leq C_0 E[u](0) e^{-\alpha t}, \forall t \geq 0.$$ 

(2) Observability: There exists $C > 0$ and $T > 0$, such that, for every solution $v \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the undamped system (4.2), the observability inequality holds:
$$E[v](0) \leq C D[v](T).$$ 

Proof. We first claim that the stabilization of damped system is equivalent to the observability of damped system.

It is clear that
$$E[u](0) = E[u](t) - D[u](t).$$

Let us first assume the stabilization of damped system. Argue by contradiction, suppose the observability of damped system does not hold. We first choose $T_0 > 0$ large enough such that $C_0 e^{-\alpha T_0} < \frac{1}{2}$. We can select a sequence of solutions $(u_k)$ such that
$$E[u_k](0) = 1, D[u_k](T_0) \to 0, \text{ as } k \to \infty.$$ 

We thus have
$$\frac{1}{2} > C_0 e^{-\alpha T_0} \geq E[u_k](T_0) = E[u_k](0) - D[u_k](T_0) = 1 + o(1), \text{ as } k \to \infty,$$

which leads to a contradiction.

Let us now assume the observability for damped system, i.e.
$$E[u](0) \leq C D[u](T),$$

We may assume that $C > 1$, from the energy dissipation and observability, we have
$$E[u](2T) = E[u](0) - D[u](2T) \leq \left( 1 - \frac{1}{C} \right) E[u](0).$$

For any $t > 0$, we write $m = \left\lfloor \frac{t}{2T} \right\rfloor$, therefore we have
$$E[u](t) \leq E[u](m) \leq \left( 1 - \frac{1}{C} \right)^m E[u](0),$$

after choosing $C_0, \alpha$ appropriately, we have the stabilization of damped system.

Our second step is to justify the equivalence between observability of damped system (4.1) and undamped system (4.2). To do this, we denote $u$ and $v$ be solutions of the damped and of the undamped system, respectively, with the same initial data at $t = 0$. Let $w = u - v$, and simple calculations yield
$$\partial_t^2 w - \Delta w = -a \partial_t u - \nabla q,$$
$$\partial_t^2 w - \Delta w + a \partial_t w = -a \partial_t v - \nabla q,$$

with some pressure function $q$.

We calculate
$$\frac{d}{dt} E[w](t) = - \int_{\Omega} a(x) |\partial_t u|^2 dx + \int_{\Omega} a(x) \partial_t u \cdot \partial_t v dx - \int_{\Omega} \partial_t w \cdot \nabla q dx,$$
and the last term of left hand side vanishes, thanks to \( \partial_t w \in C(\mathbb{R}; V) \). Thus we can write
\[
\frac{d}{dt} E[w](t) = - \int_{\Omega} a(x) |\partial_t u|^2 dx + \int_{\Omega} a(x) \partial_t u \cdot \partial_t v dx
\]
or equivalently
\[
\frac{d}{dt} E[w](t) = - \int_{\Omega} a(x) \partial_t u \cdot \partial_t w dx.
\]

Integrating the two expressions above and using the inequality of the type
\[
|ab| \leq |a|^2 + C(\epsilon)|b|^2, \forall \epsilon > 0,
\]
one easily get
\[
E[w](T) \leq B \min (D[u](T), D[v](T)), \forall T > 0, \tag{4.6}
\]
where \( B \) is another absolute constant.

Now suppose we have observability for the damped system \((4.1)\), if \( D[u](T) \leq D[v](T) \), the observability of undamped system \((4.2)\) is trivial. Now assume that \( D[u](T) > D[v](T) \), we deduce from \((4.6)\) that
\[
E[u](0) = E[u](0) \leq CD[u](T) \leq CD[W](T) + CD[v](T)
\]
\[
\leq C(E[u](T) + D[v](T)) \leq CD[v](T).
\]
The derivation of observability from undamped system to the damped follows in the same way, and we omit the details.

\[\blacksquare\]

Remark 4.2. Since the domain \( D(A) \) is dense in \( \mathcal{H} \) and the observability and energy decay only involves the \( L^2 \) norm of \( \nabla u \) and \( \partial_t u \), thus the same result of proposition 4.1 holds if we replace \( u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V) \) to \( u \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; H) \).

4.3. Resolvent estimates and stabilization. Recall that from the previous sections, the study of damped system \((4.1)\) is equivalent to the project system
\[
\frac{d}{dt} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & \text{Id} \\ A & -\Pi a \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \tag{4.7}
\]
We will use the notation \( U = (u, \partial_t u)^t \) in the sequel.

In this part, we follows almost the same way as in the appendix of [4], only to pay attention to the changing of working spaces (appearance of the pressure term and divergence free structure). Moreover, we add some technical details which may seems standard to experts in analysis but not disposable for many applied people.

From last section, we know that the observability of undamped system \((4.2)\) is equivalent to the stabilization of damped system \((4.1)\), therefore we will concentrate ourselves to the study of stabilization of \((4.1)\). The following result is standard in semigroup theory:

Proposition 4.3. Consider a semi-group \( e^{tL} \) on a Hilbert space \( \mathcal{X} \), with infinitesimal generator \( L \) defined on \( D(L) \). Then if there exists \( C > 0, \delta > 0 \) such that the resolvent of \( L \), \( (L - \lambda)^{-1} \), exists for \( \text{Re} \lambda \geq -\delta \) and satisfies
\[
\forall \lambda \in \mathbb{C}^\delta := \{ z \in \mathbb{C} : \text{Re} z > -\delta \}, \|(L - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C.
\]
Then there exists $M > 0$ such that for any $t > 0$,

$$\|e^{tL}\|_{\mathcal{L}(X)} \leq Me^{-\frac{a}{2}t}.$$ 

We need a lemma from complex analysis. We temporarily use the convention of Fourier transform

$$\hat{u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} u(t) dt.$$ 

**Lemma 4.4.** Let $u, v$ be two continuous functions with support in $\mathbb{R}^+ = (0, \infty)$. Assume moreover that $u, v \in L^2(\mathbb{R}^+)$ and $v$ has compact support. From Wiener-Paley theory, we know that the Fourier transform $\hat{v}$ admits a holomorphic extension to $\mathbb{C}$ and of exponential type. Given $a_0 > 0$, suppose that the Fourier transform $\hat{u}$ is also holomorphic in $S_{a_0} = \{ z \in \mathbb{C} : \text{Im} z < a_0 \}$ and satisfies

$$|\hat{u}(z)| \leq C|\hat{v}(z)|, \forall z \in S_{a_0}.$$ 

Then for any $a < a_0$, $e^{at}u(t) \in L^2(\mathbb{R}^+)$ and

$$\int_0^\infty e^{2at} |v(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{v}(\tau + ia)|^2 d\tau.$$ 

**Proof.** We first claim that

$$\int_0^\infty e^{2at} |v(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{v}(\tau + ia)|^2 d\tau, \forall a \in \mathbb{R}. \quad (4.8)$$

Indeed, since $v$ is compactly supported,

$$\hat{v}(\tau + ia) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iat} e^{-it\tau} v(t) dt$$

which is analytic in $a$ and rapidly decreasing in $\tau$ for each fixed $a \in \mathbb{R}$. Thus one easily deduce from the Plancherel (or calculate the integral directly) that (4.8) is true.

As a consequence, $\hat{u}(\tau + ia) \in L^2(\mathbb{R})$ for each $a < a_0$. Notice also that $u \in L^2(\mathbb{R}^+)$, thus for each $a$ with $\text{Re} a < 0$, the formula

$$\int_0^\infty e^{2at} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(\tau + ia)|^2 d\tau \quad (4.9)$$

holds true and analytic with respect to $a$. In particular, $|\hat{u}(z)| \leq C|\hat{v}(z)|, z \in S_{a_0}$ implies that $\hat{u}(\tau + ia)$ is rapidly decreasing in $\tau$ for each fixed $a < a_0$. For $z = \tau + ia$ with $a < a_0$, consider the integral

$$F(a, t) = \frac{e^{at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz} \hat{u}(z) d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\tau} \hat{u}(\tau) d\tau \in L^2(\mathbb{R}).$$

From Cauchy integral theorem, we have that

$$F(a, t) = \frac{e^{at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\tau) e^{it\tau} d\tau = e^{at}u(t) \in L^2(\mathbb{R}).$$

From this, we conclude that (4.9) follows for all $a < a_0$. 

**Remark 4.5.** In the previous lemma, the same results hold true if we replace $u, v$ to be Hilbert-space valued functions.
proof of proposition 4.3. The basic tool to prove this proposition is the Fourier-
Laplace transform in time variable. From the property of strongly continuous semi-
group, we know that there exists \( \omega_0 > 0 \) such that (see [21])
\[
\|e^{tL}\|_{\mathcal{L}(X)} \leq e^{\omega_0 t}, \forall t \geq 1.
\]
Take \( u_0 \in D(L) \), and pick a nonnegative cut-off \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi \equiv 0, \forall t \leq 1 \) and \( \chi \equiv 1, \forall t > 2 \). We define \( u(t) := \chi(t)e^{tL-\omega t}u_0 \) for some \( \omega > \omega_0 \) and thus \( u \in L^\infty(\mathbb{R}; \mathcal{X}) \). Moreover, we have the equation
\[
(\partial_t + \omega - L)u = \chi(t)e^{tL-\omega t}u_0 =: v(t).
\]
By taking Fourier transform we get
\[
(i\tau + \omega - L)\widehat{u} = \widehat{v}(\tau).
\]
Since \( v \) is compactly supported in positive axis in time variable, the \( \widehat{v}(\tau) \) has a holomorphic and bounded extension in any domain of the form
\[
S_\alpha = \{ \tau \in \mathbb{C} : \text{Im} \tau < \alpha \}, \alpha > 0.
\]
From the assumption on the resolvent, we deduce that \( (i\tau + \omega - L) \) is invertible if \( \tau \in S_{\delta+\omega} \) and thus \( \widehat{u}(\tau) \) admits a bounded holomorphic extension to \( S_{\delta+\omega} \) which satisfies
\[
\|\widehat{u}(\tau)\|_X \leq C\|\widehat{v}(\tau)\|_X.
\]
Apply Lemma 4.4, we deduce that
\[
\int_{-\infty}^{\infty} \|e^{(\omega_0+\delta)t}u\|_X^2 dt = \int_{-\infty}^{\infty} \|\widehat{u}(\xi + i(\omega_0 + \delta))\|_X^2 d\xi
\leq C \int_{-\infty}^{\infty} \|\widehat{v}(\xi + i(\omega_0 + \delta))\|_X^2 d\xi
\leq C \int_{-\infty}^{\infty} \|e^{(\omega_0+\delta)t}v\|_X^2 dt \leq C\|u_0\|_X^2.
\]
We remark that one need use various types of Winer-Paley theorems to justify the
above calculations, thanks to the fact that \( u(t), v(t) \) is supported on \([1, \infty)\) and furthermore \( v(t) \) has compact support. Take \( \omega < \omega_0 + \frac{\delta}{2} \) in the definition of \( u \), we have that
\[
\|e^{\frac{\delta}{2}}e^{tL}u_0\|_{L^2(\mathbb{R}_+; \mathcal{X})} \leq C_1\|u_0\|_X.
\]
Thanks to the semi-group structure and uniform bound principal, we have that there exists \( M_0 > 0 \), such that for any interval \( I \subset (0, +\infty) \) of length 1,
\[
\sup_{t \in I, s > 0, t + s \in I} \frac{|f(t + s)|}{|f(t)|} \leq M_0.
\]
with \( f(t) = \|e^{tL}u_0\|_X \). Therefore, for any \( T > 0 \),
\[
\int_T^{T+1} e^{\delta t}|f(t)|^2 dt \geq e^{\delta T} \min_{t \in [T, T+1]} |f(t)|^2.
\]
Therefore,
\[
|f(T + 1)|^2 \leq M_0^2 \min_{t \in [T, T+1]} |f(t)|^2 \leq e^{-\delta T} \int_T^{T+1} e^{\delta u} |f(t)|^2 dt.
\]
This implies the exponential decay
\[
\|e^{tL}u_0\|_X \leq Me^{-\frac{\delta}{2}t}\|u_0\|_X.
\]
Now we can introduce the semi-classical observability

**Proposition 4.6.** Assume that \( a \in L^\infty(\Omega) \cap C^0(\overline{\Omega}) \) and \( \int_\Omega adx > 0 \). Then the stabilization of system (4.1) is implied by the following statement:

\[
\exists h_0 > 0, C > 0 \text{ such that } \forall 0 < h < h_0, \forall (u, q, f) \in H^2(\Omega) \cap V \times H^1(\Omega) \times H
\]
solves the equation

\[
-h^2 \Delta u - u + h\nabla q = f,
\]
we have

\[
\|u\|_{L^2(\Omega)} \leq C \left( \|a^{1/2}u\|_{L^2(\Omega)} + \frac{1}{h} \|f\|_{L^2(\Omega)} \right).
\] (4.10)

For the proof, we need two lemmas.

**Lemma 4.7.** Let \( L \) be a linear operator on Hilbert space \( \mathcal{X} \) with a compact resolvent \((L - \text{Id})^{-1}\). Suppose the spectrum \( \text{Spec}(L) \subset \{z : \text{Re}z < 0\} \) and satisfies that for any \( \sigma \in \mathbb{R} \), \( L - i\sigma \) is invertible and satisfies the uniform bound

\[
\sup_{\sigma \in \mathbb{R}} \| (L - i\sigma)^{-1} \| < \infty.
\]

Then there exists \( \delta > 0 \), such that

\[
\sup_{\lambda \in \mathbb{C}^\sigma} \| (L - \lambda)^{-1} \| < \infty,
\]
where \( \mathbb{C}^\sigma := \{z \in \mathbb{C} : \text{Re}z > -\sigma\} \) for any \( \sigma \in \mathbb{R} \).

**Proof.** Write

\[
\sup_{\sigma \in \mathbb{R}} \| (L - i\sigma)^{-1} \| = C
\]

We denote \( R(z) = (L - z)^{-1} \) for \( z \in \rho(L) := \{z : z \in \mathbb{C} \setminus \text{Spec}(L)\} \). Take \( z_0 \in \rho(L) \), we write

\[ L - z = (L - z_0)(\text{Id} + (L - z_0)^{-1}(z_0 - z)), \]

and for \( |z - z_0| < \frac{1}{\|(L - z_0)^{-1}\|} \), we have

\[
\|R(z)\| \leq \|R(z_0)\| \sum_{n=0}^\infty |z - z_0|^n \|(L - z_0)^{-1}\|^n \leq R(z_0).
\]

Therefore, for \( \lambda \) with \( |\text{Re}\lambda| \leq \delta \), where \( 0 < \delta < \frac{1}{2C} \), we have \( \|R(\lambda)\| \leq C \). To conclude, we only need show that there exists \( C_1 > 0 \), such that

\[
\sup_{\text{Re}z > \delta} \|(L - z)^{-1}\| \leq C_1.
\]

Consider the holomorphic equivalence \( \varphi : \mathbb{C}^0 \to \mathbb{D}, \psi = \varphi^{-1} \).

\[
\varphi(z) = \frac{z - 1}{z + 1}, \psi(\zeta) = \frac{1 + \zeta}{1 - \zeta},
\]
where \( \mathbb{D} := \{\zeta : |\zeta| < 1\} \) be the unit disk. One easily verifies that the operator-valued function

\[
\Phi(\zeta) = R(\psi(\zeta)) : \mathbb{D} \to \mathcal{L}(\mathcal{X})
\]
is analytic and satisfies the Cauchy integral formula

\[
\Phi(\zeta_0) = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{\Phi(\zeta)}{\zeta - \zeta_0} d\zeta, \forall \zeta_0 \in \mathbb{D}.
\]
STABILIZATION OF A HYPERBOLIC STOKES TYPE SYSTEM

Since\(\text{dist}\left(\partial D, \varphi(C^{-\delta})\right) \geq \epsilon_0 > 0\) for some \(\epsilon_0\) depends only on \(\delta\), we deduce that for any \(z \in C^{-\delta}\),

\[
\|R(z)\| \leq \left\| \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\Phi(\zeta)}{\zeta - \varphi(z)} d\zeta \right\| \leq \frac{C}{\epsilon_0}.
\]

Lemma 4.8. [Unique Continuation of Stoke Operator] Let \(\sigma > 0\) and \(u \in V\) satisfies that

\[Au = \sigma^2 u.\]

Then if \(u|_\omega \equiv 0\), we must have \(u \equiv 0\).

Proof. It is equivalent to write

\[-\Delta u + \nabla p = \sigma^2 u, \ \text{div} \ u = 0, \ u \in V, \int_\Omega p dx = 0.\]

Take divergence of the equation, we have \(\Delta p = 0\). The vanishing of \(u\) in \(\omega\) implies that \(p\) equals to a constant in a component of \(\omega\). Now since \(\Omega\) is connected, the maximum principal implies that \(p \equiv 0\) in \(\Omega\). Therefore we have reduced to unique continuation of eigenfunction of Laplace operator, and this implies that \(u \equiv 0\) in \(\Omega\).

proof of proposition 1.4. We need show that the semi-classical observability implies the stabilization.

Note that the operator \((A - \lambda)\) is invertible for any \(\lambda > 0\). One write

\[A - z = (\text{Id} + (1 - z)(A - \text{Id})^{-1})(A - 1), \forall z \in \mathbb{C}.\]

Since \(\text{Id} + (1 - z)(A - \text{Id})^{-1}\) is Fredholm with index 0, we infer that \(A - z\) is invertible iff it is injective. In light of the previous lemmas and the Proposition 4.3, we only have to prove the fact that

\[\exists C > 0, \text{ such that } \forall \sigma \in \mathbb{R}, U \in D(A), F \in V \times H, (A - i\sigma)U = F \implies \|U\|_H \leq C\|F\|_H.\]

We argue by contradiction. If it is not true then we can find sequences \((\sigma_n), (U_n)\), and \((F_n)\) such that

\[\|\sigma_n\| \to \sigma, \text{ and we write} \]

\[U_n = (u_n, v_n)^t, F_n = (f_n, g_n)^t.\]

We have several cases to analyse, according to the limit value \(\sigma\).

1. \(\sigma = 0\): In this case, we have \(Au_n = o(1)_H\), which is equivalent to

\[v_n = o(1)_H, Au_n - \Pi e_n = o(1)_L,\]

thus \(Au_n = o(1)_L\). Taking inner product with \(u_n\) and integrating by part we have

\[\int_\Omega |\nabla u_n|^2 dx = o(1).\]

This contradicts to \(\|U_n\|_H = 1\).
\(0 < |\sigma| < \infty\): In this case we have \(AU_n - i\sigma U_n = o(1)_H\), or equivalently,
\[
v_n - i\sigma u_n = o(1)_{H^1_0}, Au_n - (i\sigma + \Pi a)v_n = o(1)_{L^2}.
\]
Thanks to Poincaré inequality, we deduce that
\[
Au_n - i\sigma(i\sigma + \Pi a)u_n = o(1)_{L^2}.
\]
Applying Rellich compact embedding theorem followed by extracting to suitable sub-sequences, we may assume that
\[
u_n \to u, \text{ in } L^2(\Omega), u_n \to u, \text{ in } V.
\]
Taking inner product with \(u_n\), we have
\[
- \int_{\Omega} |\nabla u_n|^2 dx = -\sigma \int_{\Omega} |u_n|^2 dx + i\sigma \int_{\Omega} a(\xi)|u_n|^2 dx + o(1),
\]
which implies that \(au \equiv 0\) in \(\Omega\). Thus we can conclude that \(u\) is an eigenfunction of Stokes operator \(A\) and vanishes in a non trivial open subset of \(\Omega\). The unique continuation property for the system
\[
-\Delta u + \nabla p = \sigma^2 u, \text{ div } u = 0
\]
implies that \(u \equiv 0\). As a consequence, we have that \(u_n = o(1)_{H^1_0}, v_n = o(1)_{L^2}\). This contradicts to the original assumptions.

(3) \(|\sigma| = \infty\): We only study the case \(\sigma_n \to +\infty\) (the other one is obtained by considering \(U_n\)).

Let \(h_n = \sigma_n^{-1}\), and we deduce from the system \(AU_n = \sigma_n U_n = o(1)_H\):
\[
h_n^2 Au_n + u_n - i h_n \Pi a u_n = h_n^2 \Pi a f_n + i h_n f_n + h_n^2 g_n = o_{L^2}(h_n)
\]
\[
h_n v_n - i v_n = h_n f_n = o(h_n)_{H^1_0},
\]
\[
h_n^2 Av_n + v_n - i h_n \Pi a v_n = i h_n g_n - h_n^2 A f_n = o_{L^2}(h_n) + o_{H^{-1}}(h_n).
\]
Define the operator \(P_h = h^2 A + \text{Id} - ih\Pi a\) on \(H\) with domain \(H^2(\Omega) \cap V\), we have (dropping the subindex \(n\) for the moment)
\[
(P_h u)_{L^2} = \|u\|_{L^2(\Omega)}^2 - \|h\nabla u\|_{L^2(\Omega)}^2 - i\|a^{1/2} u\|_{L^2(\Omega)}^2.
\]
Taking imaginary part, we have
\[
\|a^{1/2} u\|_{L^2(\Omega)}^2 \leq C \frac{\|P_h u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}}{h}.
\]
Applying the semi-classical observability to the equation
\[
h^2 Au + u = ih\Pi a + \tilde{f}
\]
with \(\tilde{f} = o_{L^2}(h)\), we have
\[
\|u\|_{L^2(\Omega)}^2 \leq C \left( \|a^{1/2} u\|_{L^2(\Omega)}^2 + \frac{1}{h^2} (\|f\|_{L^2(\Omega)}^2 + h^2 \|a^{1/2} u\|_{L^2(\Omega)}^2) \right)
\]
\[
\leq \frac{C}{h^2} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \frac{C}{h^2} \|f\|_{L^2(\Omega)}^2.
\]
This implies that
\[
\|u_n\|_{L^2(\Omega)} \leq \frac{C}{h_n} \|f_n\|_{L^2(\Omega)} = o(1).
\]
To conclude, observe that $v_n$ satisfies
\[ h_n^2 Av_n + v_n = o_H(h_n) + o_{H^{-1}}(h_n^2), \]
and we claim that if $(h^2 A + 1)v = f_1 + f_2$, then
\[
\|v\|_{L^2(\Omega)} + \|h\nabla v\|_{L^2(\Omega)} \\
\leq C \left( \|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2} \right). \tag{4.12}
\]
Assume the claim for the moment, we thus have $\|h_n \nabla v_n\|_{L^2(\Omega)} = o(1)$, and $\|\nabla u_n\|_{L^2(\Omega)} = o(1)$, thanks to $u_n + ih_n v_n = ih_n f_n$. This contradicts to the original assumption.

Now we turn to the proof of the claim. By density, (4.10) still valid when $v \in V$. Taking inner product of $v$ with $P_h v$, we have
\[
(P_h v|v)_{L^2} = \|v\|_{L^2(\Omega)}^2 - \|h\nabla v\|_{L^2(\Omega)}^2 - ih\|a^{1/2}v\|_{L^2(\Omega)}^2.
\]
Therefore, by taking real part and injecting (4.10), we have
\[
\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C \left( \|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|P_h v\|_{L^2(\Omega)}}{h} \right). \tag{4.13}
\]
By taking real and imaginary part of $(P_h v|v)_{L^2}$, we have
\[
\|a^{1/2}v\|_{L^2(\Omega)}^2 \leq \frac{\|P_h v\|_{L^2(\Omega)}^2}{h} ||v||_{L^2(\Omega)}. \tag{4.14}
\]
Substituting (4.14) into (4.13), we obtain that
\[
\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C \left( \frac{\|P_h v\|_{L^2(\Omega)}^2}{h} \right),
\]
and this implies that
\[
\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C \frac{||P_h v||_{L^2(\Omega)}}{h}.
\]
Thus $P_h$ is bijective from $H^2(\Omega) \cap V$ to $H$ and hence invertible. From the fact that
\[
P_h = (1 + (2 - ih\Pi a) (h^2 A - 1)^{-1})(h^2 A - 1),
\]
$P_h$ can be written as composition of a positive operator and a Fredholm operator of index 0. From the estimate above, we conclude that
\[
\|P_h^{-1}\|_{L^2 \rightarrow L^2} \leq \frac{C}{h}, \|P_h^{-1}\|_{L^2 \rightarrow H^1} \leq \frac{C}{h^2}.
\]
Now come back to the equation $(h^2 A + 1)v = f_1 + f_2$. Taking $g \in H$, and letting $w = P_h^{-1} g$, we have
\[
(v|g)_{L^2} = (h^2 A + 1)v|w)_{L^2} + ih(v|\Pi a w)_{L^2}
\]
\[
= (f_1 + f_2|w)_{L^2} + ih(av|w)_{L^2}
\]
\[
\leq \|f_1\|_{L^2(\Omega)} \|P_h^{-1} g\|_{L^2(\Omega)}
\]
\[
+ \|f_2\|_{H^{-1}(\Omega)} \|P_h^{-1} g\|_{L^2(\Omega)} + h \|av\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2} \right) \|g\|_{L^2(\Omega)}. \tag{4.15}
\]
This completes the proof.

5. Apriori Estimates for the Quasi-mode System

Now we consider the quasi-modes of Stokes system

\[
\begin{align*}
- h_k^2 \Delta u_k - u_k + h_k \nabla q_k &= f_k, (u_k, f_k) \in (H^2(\Omega) \cap V) \times H, \\
h_k \text{div} u_k &= 0, \text{in } \Omega 
\end{align*}
\]

(5.1)

To simplify the notation, we drop the sub index \( k \) and just keep the semi-classical parameter \( h \) everywhere. Note that the functions \( u, v \), etc. should be understood as \( u(h), v(h) \), etc. We fix the geometric assumption on the domain \( \Omega \subset \mathbb{R}^d \) is smooth and connected and \( \partial \Omega = \bigcup_{j=1}^{N_j} \Gamma_j \) with \( \Gamma_j \cap \Gamma_k = \emptyset, i \neq k \) and each \( \Gamma_j \) is smooth and connected.

Now assume that

\[
\| u \|_{L^2(\Omega)} = O(1), \| f \|_{L^2(\Omega)} = o(h). 
\]

Taking inner product with \( u \) and integrate by part, we have

\[
\| h \nabla u \|_{L^2(\Omega)} = O(1). 
\]

One can always assume that \( \int_{\Omega} q dx = 0 \), since \( q \in L^2(\Omega)/\mathbb{R} \). From the regularity theory of steady Stokes system, (see [22], page 33), and Poincaré inequality, we have

\[
\| h^2 \nabla^2 u \|_{L^2(\Omega)} = O(1), \| q \|_{L^2(\Omega)} = O(h^{-1}), \| h \nabla q \|_{L^2(\Omega)} = O(1). 
\]

We now give some estimates on the trace. Write \( q_0 = q|_{\partial \Omega} \).

**Lemma 5.1.** \( \| q \|_{L^2(\Omega)} = O(h^{-1}), \| q_0 \|_{H^{s/2}(\partial \Omega)} = O(h^{-1}), \| q_0 \|_{L^2(\partial \Omega)} = O(h^{-1}) \).

**Proof.** Since \( q \) is harmonic function, then one can apply trace theorem \( H^s(\Omega) \to H^{s-1/2}(\partial \Omega) \) for any \( s \in \mathbb{R} \). Hence the conclusions follows from these and interpolations. ■

**Lemma 5.2.** \( h \partial_x u|_{\partial \Omega} = (h \partial_x u|, 0) \), and \( \| h \partial_x u|_{\partial \Omega} \|_{L^2(\partial \Omega)} = O(1) \).

**Proof.** The first assertion follows from \( h \text{div} u = 0 \) and Dirichlet boundary condition, while we apply a multiplier method to prove the second. From the geometric assumption on \( \Omega \), we can find a vector field \( L \in C^1(\Omega) \) such that \( L|_{\partial \Omega} = \nu \) (see [13], page 36). In global coordinate system, we write \( L = L_j(x) \partial_{x_j} \). By using the equation, we have

\[
\begin{align*}
\int_{\Omega} Lu \cdot f dx &= \int_{\Omega} Lu \cdot (h^2 \Delta u - u + h \nabla q) dx \\
- \int_{\Omega} Lu \cdot u dx &= - \int_{\Omega} L_j(x) \partial_{x_j} u^i u^i dx \\
&= - \int_{\Omega} \partial_{x_j} (L_j(x) u^i) u^i dx + \int_{\Omega} \text{div } L(x)|u|^2 dx \\
&= \int_{\Omega} L_j(x) u^i(x) \partial_{x_j} u^i dx + \int_{\Omega} \text{div } L(x)|u|^2 dx \\
&= \int_{\Omega} Lu \cdot u dx + \int_{\Omega} \text{div } L(x)|u|^2 dx,
\end{align*}
\]
thus
\[
h \int_{\Omega} Lu \cdot \nabla qdx = -h \int_{\Omega} u^i \partial_{x_i} (L_j \partial_{x_j} q) dx
\]
\[
= -h \int_{\Omega} u \cdot L(\nabla q) dx - h \int_{\Omega} (\text{div } L(x)) u \cdot \nabla qdx
\]
\[
= -h \int_{\Omega} u \cdot [L, \nabla]qdx - h \int_{\Omega} \text{div } L(x)u \cdot \nabla qdx
\]
\[
= O(1),
\]
and \[\int_{\Omega} Lu \cdot udx = - \frac{1}{2} \int_{\Omega} \text{div } L(x)|u|^2 dx = O(1),\]
\[-h^2 \int_{\Omega} Lu^i \Delta u^i dx = -h^2 \int_{\partial \Omega} |\partial_{\nu} u^i|^2 d\sigma + h^2 \int_{\Omega} \nabla L(\nabla u^i, \nabla u^i) dx
\]
\[
+ h^2 \int_{\Omega} L_j(x) \partial_{x_j}^2 u^i \partial_{x_k} u^i
\]
\[
= -h^2 \int_{\partial \Omega} |\partial_{\nu} u^i|^2 d\sigma + h^2 \int_{\Omega} \nabla L(x)(\nabla u^i, \nabla u^i) dx
\]
\[
+ h^2 \int_{\Omega} \partial_{x_j} (L_j \partial_{x_k} u^i) \partial_{x_k} u^i dx - h^2 \int_{\Omega} \text{div } L(x)\nabla u^i \cdot \nabla u^i(x) dx,
\]
\[
h^2 \int_{\Omega} \partial_{x_j} (L_j \partial_{x_k} u^i) \partial_{x_k} u^i dx = h^2 \int_{\Omega} L \cdot \nu |\partial_{\nu} u^i|^2 d\sigma - h^2 \int_{\Omega} L_j(x) \partial_{x_j} u^i \partial_{x_j}^2 u^i dx,
\]
\[-h^2 \int_{\Omega} Lu^i \Delta u^i dx = - \frac{h^2}{2} \int_{\partial \Omega} |\partial_{\nu} u^i|^2 d\sigma + \int_{\Omega} \nabla L(x)(h \nabla u^i, h \nabla u^i) dx - \frac{h^2}{2} \int_{\Omega} \text{div } L(x) |\nabla u^i|^2 dx.
\]
Observing that \[\int_{\Omega} Lu \cdot f dx = o(1),\] we have
\[
\int_{\partial \Omega} |h \partial_{\nu} u^i|^2 d\sigma = O(1).
\]

**Lemma 5.3.** Under additional assumption that
\[
||a^{1/2}u_k||_{L^2(\Omega)} = o(1),
\]
after extracting to subsequences, we have \(h_k \nabla q_k \to 0 \) \(L^2(\Omega)\) and \(u_k \to 0 \) weakly in \(L^2(\Omega)\). Therefore from Rellich theorem, we have \(h q \to 0\), strongly in \(L^2(\Omega)\).

**Proof.** We may assume that \(a \nabla q \to r\), weakly in \(L^2(\Omega)\), and Rellich theorem implies that \(h q \to P\), strongly in \(L^2(\Omega)\), and thus \(\nabla P = r\), with the property \(\int_{\Omega} P = 0\).

Now we claim that \(\Delta P = 0\) in \(\Omega\).

Indeed, take any \(\varphi \in C_0^\infty(\Omega)\),
\[
\int_{\Omega} \nabla P \cdot \nabla \varphi = \lim_{h \to 0} \int_{\Omega} h \nabla q \cdot \nabla \varphi = 0.
\]
Now suppose \(u_k \to U\), weakly in \(L^2(\Omega)\), \(w_k = h_k^2 u_k \to W\), weakly in \(H^2(\Omega)\), by taking the weak limit in the equation, we have
\[
-\Delta W - U + \nabla P = 0, \text{ in } L^2(\Omega).
\]
Notice that \(a^{1/2}u_k \to 0, a^{1/2}w_k \to 0\), strongly in \(L^2(\Omega)\), and this implies that \(U|_\omega = W|_\omega = 0\). Therefore, in a connected component \(\omega'\) of \(\omega\), we have \(\nabla P \equiv 0\). However, \(P\) is a harmonic function, then \(P \equiv \text{const.}\), thanks to the fact that \(\Omega\) is
connected. Note that \( \int_{\Omega} P = 0 \), hence \( P = 0 \). Moreover, from Rellich theorem that \( w_k \to W \) strongly in \( L^2(\Omega) \), and on the other hand \( \|h_k^2u_k\|_{L^2(\Omega)} = o(1) \) we must have \( W = 0 \). Therefore \( U = 0 \). ■

6. Proof of the Observability Estimates

In this part, we will prove the Proposition 1.4 under the assumption in Theorem 1.2 on \( \Omega \) and \( \omega \).

We argue by contradiction, suppose (1.7) is not true, we can then choose a sequence \( (u_n, h_n, q_n, f_n) \in H^2(\Omega) \cap V \times \mathbb{R}^+ \times H^1(\Omega) \times H \) satisfies equation

\[-h_n^2 \Delta u_n - u_n + h_n \nabla q_n = f_n \quad (6.1)\]

with the following properties:

\[\|u_n\|_{L^2(\Omega)} = 1, \|f_n\|_{L^2(\Omega)} = o(h_n), \|a^{1/2}u_n\|_{L^2(\Omega)} = o(1), n \to \infty.\]

Up to extracting to subsequence, we can associate \((u_n, h_n)\) with a semi-classical defect measure \( \mu \). Therefore we have \( \omega \cap \pi(\text{supp}(\mu)) = \emptyset \) from Corollary 3.4, where we denote \( \pi : T^*\Omega \to \Omega \) be the canonical projection.

Denote \( \phi(s, \rho) \) be the globally defined generalized bicharacteristic flow, thanks to the geometric assumption that \( \Omega \) has no infinite contact. Pick any point \( \rho_0 \) with \( \pi(\phi(s, \rho_0)) \in \omega \). For any time segment \([0, s_0]\), there are several situations:

Either \( \phi([0, s_0], \rho_0) \subset \Omega \), or there exist \( \pi(\phi([0, s_0], \rho_0)) \cap \partial\Omega \neq \emptyset \), then from the assumption on \( \Omega \), all points \( \phi(s, \rho_0) \) with \( s \in [0, s_0] \) and \( \pi(\phi(s, \rho_0)) \in \partial\Omega \) must lie in \( \mathcal{H} \cup G^{2, \pm} \cup G^{2, -} \cup \bigcup_{k \geq 3} G^k \). Now Theorem 3.7 implies that

\[\text{supp}(\phi(\cdot, \cdot), \mu) \subset \text{supp}(\mu).\]

Therefore, we have

\[\phi([0, s_0], \rho_0) \cap \text{supp}(\mu) = \emptyset.\]

We now invoke the geometric control condition to deduce that

\[\overline{\Omega} \subset \pi \left( \bigcup_{\rho_0 \in \omega} \phi([0, T_0], \rho_0) \right)\]

for some \( T_0 > 0 \) and thus \( \mu = 0 \). This contradicts to the assumption that

\[\int_{\Omega} |u_n(x)|^2 dx = 1.\]

7. Appendix

We will derive the hyperbolic Stokes system (1.2) from certain limit procedure of Lamé system from elastic theory:

\[
\begin{aligned}
\partial_t^2 w - \mu \Delta w - (\lambda + \mu) \nabla \text{div} w &= 0, (t, x) \in [0, T] \times \Omega \\
w(t, \cdot) |_{\partial \Omega} &= 0 \quad (w(0), \partial_t w(0)) = (w_0, z_0) \in (H^1_0(\Omega) \times L^2(\Omega))^d
\end{aligned}
\]

(7.1)

where the solution \( w(t, x) \) is vector-valued.

Define \( u(t, x) := w(t/\sqrt{\mu}, x) \), then we find that

\[\partial_t^2 u - \Delta u - \frac{\lambda + \mu}{\mu} \nabla \text{div} u = 0.\]
We let $\epsilon = \frac{\mu}{\mu + \lambda} \ll 1$, in the case that $\lambda \gg \mu > 0$. Thus we obtain a family of equations

$$\begin{cases}
\partial_t^2 u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = 0, (t, x) \in [0, T] \times \Omega \\
u_\epsilon(t, .)|_{\partial \Omega} = 0 \\
(u_\epsilon(0), \partial_t u_\epsilon(0)) = (u_{0, \epsilon}, v_{0, \epsilon}) \in (H^1_0(\Omega) \times L^2(\Omega))^d
\end{cases}$$

(7.2)

where $p_\epsilon = -\frac{1}{\epsilon} \text{div} u_\epsilon$ and satisfies $\int_{\Omega} p_\epsilon \, dx = 0$.

We make further assumption on the family of initial data $(u_{0, \epsilon}, v_{0, \epsilon})$ so that

$$\| (u_{0, \epsilon}, v_{0, \epsilon}) - (u_0, v_0) \|_{H^1 \times L^2} \leq C \epsilon$$

for some divergence free data $(u_0, v_0) \in V \times H$. In particular, we have

$$\| \text{div} u_{0, \epsilon} \|_{L^2(\Omega)} \leq C \epsilon.$$

From the well-posedness of Lame system, we have that $u_\epsilon \in C([0, T]; H^1_0(\Omega))$, $\partial_t u_\epsilon \in C([0, T]; L^2(\Omega))$, and $p_\epsilon \in C([0, T]; L^2(\Omega))$. Moreover, we have the conservation of energy

$$E[u_\epsilon] = \frac{1}{2} \int_{\Omega} \left( |\partial_t u_\epsilon|^2 + |\nabla u_\epsilon|^2 + \epsilon |p_\epsilon|^2 \right) \, dx$$

and therefore

$$E[u_\epsilon] = \frac{1}{2} \int_{\Omega} \left( |u_{0, \epsilon}|^2 + |v_{0, \epsilon}|^2 + \frac{1}{\epsilon} |\text{div} u_{0, \epsilon}|^2 \right) \, dx.$$

From this, we have, up to some subsequence of $(u_\epsilon, \partial_t u_\epsilon)$

$$\text{div} u_\epsilon \to 0, \text{ in } L^\infty([0, T]; L^2(\Omega)),$$

$$u_\epsilon \rightharpoonup u, \text{ weakly in } L^\infty([0, T]; H^1_0(\Omega)),$$

$$\partial_t u_\epsilon \rightharpoonup \partial_t u, \text{ weakly in } L^\infty([0, T]; L^2(\Omega)).$$

From the uniform bound of $\| \partial_t u_\epsilon \|_{L^\infty([0, T]; L^2(\Omega))}$, apply Ascoli theorem, we have that (up to some subsequence)

$$u_\epsilon \to u, \text{ in } C([0, T]; L^2(\Omega)).$$

Using the equation, we conclude that $\| \nabla p_\epsilon \|_{L^\infty([0, T]; H^{-1}(\Omega))}$ is uniformly bounded. Combine with the fact $\int_{\Omega} p_\epsilon = 0$, we have that $\| p_\epsilon \|_{L^\infty([0, T]; L^2(\Omega))}$ is uniformly bounded, thus up to some subsequence, we may assume that

$$p_\epsilon \rightharpoonup p, \text{ weakly in } L^\infty([0, T]; L^2(\Omega)).$$

Now it is not difficult to verify that $(u, p)$ is a weak solution to (1.1). Moreover, $p$ satisfies the zero mean condition

$$\int_{\Omega} p \, dx = 0.$$

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