The Futaki Invariant of Kähler Blowups with Isolated Zeros via Localization

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Abstract

We present an analytic proof of the relationship between the Calabi-Futaki invariant for a Kähler manifold relative to a holomorphic vector field with a nondegenerate zero and the corresponding invariant of its blowup at that zero, restricting to the case that zeros on the exceptional divisor are isolated. This extends the results of Li and Shi [15] for Kähler surfaces. We also clarify a hypothesis regarding the normal form of the vector field near its zero. An algebro-geometric proof was given by Székelyhidi [17] by reducing the situation to the case of projective manifolds for rational data and using Donaldson-Futaki invariants. Our proof will be an application of degenerate localization.

1 Introduction

Let $M$ be a compact Kähler manifold with Kähler metric $\omega$. A fundamental question in Kähler geometry asks whether the class $[\omega]$ contains a canonical Kähler metric. When the first Chern class $c_1(M)$ is zero, celebrated work of Yau [20] established the existence of a unique Ricci-flat Kähler metric in every Kähler class, while in the case of negative first Chern class, Yau [20] and Aubin [2] independently proved existence and uniqueness of a Kähler-Einstein metric in $c_1(M)$. The existence of Kähler-Einstein metrics when $c_1(M) > 0$ has recently been addressed by Chen-Donaldson-Sun [6].

More generally, one could ask whether a Kähler class $\Omega \in H^{1,1}(M, \mathbb{R})$ contains a constant scalar curvature Kähler (cscK) metric. The question is quite subtle and conjecturally related to the algebro-geometric stability of $M$; see [16] for a survey and references. One obstruction to the existence of a cscK metric is a generalization due to Calabi [5] of Futaki’s famous obstruction to Kähler-Einstein metrics first defined in [11]. For any Kähler class $\Omega$, this Calabi-Futaki invariant is a certain character on the Lie algebra of holomorphic vector fields $\mathfrak{h}$

$$\text{Fut}(\Omega, \cdot) : \mathfrak{h} \to \mathbb{C}.$$
whose vanishing on $\mathfrak{h}$ is necessary for $\Omega$ to support a cscK metric.

One approach to calculating the Calabi-Futaki invariant, at least when $M$ is algebraic, is via the algebraic Donaldson-Futaki invariant [9]. Another method is localization, which will be our approach in this paper. When the zero locus of $X$ is nondegenerate, Tian [18] gave a complete formula reducing $\text{Fut}(\Omega, X)$ to a calculation on $\text{Zero}(X)$. When the zero locus of $X$ is degenerate, localization calculations are quite difficult. See section 2 for details.

In this paper we study the following situation where degenerate localization calculations naturally arise: With $M$ as above, suppose that $p \in M$ is a zero of $X \in \mathfrak{h}$. The blowup $\pi : \tilde{M} \to M$ at $p$ then admits a holomorphic lift $\tilde{X}$ of $X$ as well as a natural Kähler class

$$\tilde{\Omega} = \pi^* \Omega - \epsilon [E],$$

where $E$ is the exceptional divisor and $\epsilon$ is sufficiently small [14]. A natural question is, what is the relationship between $\text{Fut}(\Omega, X)$ and $\text{Fut}(\tilde{\Omega}, \tilde{X})$?

We will limit ourselves to the following assumption on $X$ at $p$:

\begin{itemize}
  \item[(\star)] The Jordan canonical form of the linearization $DX$ at $p$ does not contain multiple Jordan blocks for the same eigenvalue.
\end{itemize}

Geometrically, (\star) means $\tilde{X}$ does not contain a positive dimensional zero locus.

Our main theorem is

**Theorem 1.1** Let $\pi : \tilde{M} \to M$ be the blow-up of $n$-dimensional Kähler manifold $M$ at isolated non-degenerate zeros $\{p_1, \ldots, p_k\}$ of a holomorphic vector field $X$ with zero-average holomorphy potential $\theta_X$. When (\star) holds, the Futaki invariant for $\tilde{M}$ with respect to the class $\tilde{\Omega} = \pi^* \Omega - \sum \epsilon_i [E_i]$ and the natural holomorphic extension $\tilde{X}$ of $X$ to $\tilde{M}$ satisfies

$$\text{Fut}_{\tilde{M}}(\tilde{\Omega}, \tilde{X}) = \text{Fut}_M(\Omega, X) - \sum_{i=1}^{k} n(n-1) \theta_X(p) \epsilon_i^{n-1} + O(\epsilon^n)$$

Li and Shi established the result for Kähler surfaces [15]. In fact, they addressed when $p$ is not isolated or when the zero locus of $\tilde{X}$ is $E \cong \mathbb{C}P^1$, which are necessarily nondegenerate situations for surfaces but would be quite formidable more generally.

An algebro-geometric proof was given by Székelyhidi [17] by reducing the situation to the case of projective manifolds for rational data and using Donaldson-Futaki invariants, with related results recently appearing in the work of Dervan-Ross [8] and Dyrefelt [10]. We give an analytic proof via localization, for which the $n = 2$ case of Li and Shi is a special case.

One application of Theorem 1.1 is to show the nonexistence of cscK metrics on certain blow-ups:
Corollary 1.2 If a Kähler manifold $M$ admits a constant scalar curvature metric $\omega \in \Omega$ and there exists a holomorphic vector field $X$ on $M$ vanishing at $p$ and such that $\theta_X(p) \neq 0$, then the blowup $\tilde{M}$ at $p$ does not admit cscK metrics in any class $\pi^*\Omega - \epsilon[E]$ for small $\epsilon$.

The organization of this paper is as follows. Section 2 provides background concerning the Futaki invariant, summarizes Tian’s application of Bott localization to its calculation, and sets up the blowup calculation. Section 3 gives a proof of Theorem 1.1 in the special case that the linearization of $X$ at $p$ has a single Jordan block. It relies on Lemma 3.1, which generalizes the main calculation of [15]. Section 4 addresses the general case of multiple Jordan blocks with distinct eigenvalues. Section 5 discusses the normal forms of a holomorphic vector field $X$ about a singular point and establishes that for the purposes of Theorem 1.1, the simplifying assumption that $X$ is locally biholomorphic to its linearization as in Section 4 is sufficient. Section 6 is an appendix that contains a proof of Lemma 4.1 used to sum localization calculations.

2 Background

A reference for much of this section is [18]; to align conventions, we too define the Kähler form $\omega$ and Ricci form $\text{Ric}$ without the usual $\sqrt{-1}$, and let the Kähler class be

$$\Omega = \left[ \frac{\sqrt{-1}}{2\pi} \omega \right] \in H^{1,1}(M, \mathbb{R})$$

We first recall the definition of the Calabi-Futaki invariant. Let $(M, \omega)$ be a compact Kähler manifold and $F$ the smooth function uniquely determined by

$$S(\omega) - \bar{S} = \Delta F \quad \int_M (e^F - 1)\omega^n = 0$$

where $S(\omega)$ is scalar curvature, $\Delta$ is the Laplacian with respect to $\omega$, and

$$\bar{S} = \frac{\int S(\omega)\Omega^n}{\int \Omega^n} = \frac{n \int c_1(M) \cup \Omega^{n-1}}{\int \Omega^n}$$

is the average scalar curvature.

The *Calabi-Futaki invariant* is defined for each Kähler class $\Omega$ to be a functional on the Lie algebra of holomorphic vector fields $\mathfrak{h}$

$$\text{Fut}(\Omega, \cdot) : \mathfrak{h} \to \mathbb{C}$$

given by
Futaki [12] and Calabi [5] showed $\text{Fut}(\Omega, \cdot)$ is a Lie algebra character and that
the definition is in fact independent of the choice of metric $\omega$ in its Kähler class $\Omega$, justifying
the notation and making its vanishing for all $X \in \mathfrak{h}$ necessary for $\Omega$ to contain a cscK metric.

Following Tian [18], we now explain the localization of $\text{Fut}(\Omega, X)$. For every $X \in \mathfrak{h}$, Hodge theory provides a harmonic $(0, 1)$-form $\alpha$ and a smooth function $\theta_X$, unique up to addition of a constant, such that
\[ i_X \omega = \alpha - \bar{\partial} \theta_X. \tag{1} \]

Equivalently, $\theta_X$ is holomorphy potential for $X^1$. By applying $\bar{\partial}^*$ to both sides of (1) and using integration by parts,
\[
\text{Fut}(\Omega, X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (\Delta \theta_X) F \omega^n 
= \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M \theta_X \Delta F \omega^n 
= \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M \theta_X S \omega^n - \bar{S} \int_M \theta_X \omega^n 
= \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M n \theta_X \text{Ric} \wedge \omega^{n-1} - \frac{S}{n+1} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (\theta_X + \omega)^{n+1} \tag{2}
\]

which expresses the Calabi-Futaki invariant without explicit reference to $F$. This expression (2) also shows we may assume $\alpha = 0$ in (1) for our purposes.

Let $A = (-\Delta \theta_X + \text{Ric})$ and $B = (\theta_X + \omega)$. By using the identity
\[
\sum_{j=0}^{l} (-1)^j \binom{l}{j} (l-2j)^k = \begin{cases} 0 & \text{if } k < l \text{ or } k = l + 1 \\ 2^l & \text{if } k = l \end{cases}
\tag{3}
\]

one checks that

\footnote{Or rather, recalling that $\omega$ is defined without a $\sqrt{-1}$ on it, $\sqrt{-1} \theta$ is holomorphy potential in the sense that $f : M \to \mathbb{C}$ is holomorphy potential for holomorphic vector field $X = g^{ij} (\partial_j f) \partial_i$, i.e. $X$ is the $(1, 0)$ part of the Riemannian gradient of $f$ (up to a factor of 2).}
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_M (A + (n - 2j)B)^{n+1} - (-A + (n - 2j)B)^{n+1}
\]
\[
= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_M \sum_{k=0}^{n+1} \binom{n+1}{k} \left[ A^{n+1-k}(n - 2j)^kB^k - (-A)^{n+1-k}(n - 2j)^kB^k \right]
\]
\[
= 2^n n! \int_M (n + 1)2AB^n \quad \text{(only } k = n \text{ is non-zero)}
\]
\[
= 2^{n+1}(n + 1)! \int_M \left[ n\theta_X \text{Ric} \wedge \omega^{n-1} - \Delta \theta_X \omega^n \right]
\]
\[
= 2^{n+1}(n + 1)! \int_M n\theta_X \text{Ric} \wedge \omega^{n-1}
\]

Dividing this expression by \( n! \) and substituting into the previous calculation (2) yields

\[
2^{n+1}(n + 1)\text{Fut}(X, \Omega) = \sum_{j=0}^{n} \frac{(-1)^j}{j!(n - j)!} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M \left[ (-\Delta \theta_X + \text{Ric} + (n - 2j)(\theta_X + \omega))^{n+1} - (\Delta \theta_X - \text{Ric} + (n - 2j)(\theta_X + \omega))^{n+1} \right]
\]
\[
- \bar{\partial} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j}(n + 1 - 2j)^{n+1} \frac{\sqrt{-1}}{2\pi} \int_M (\theta_X + \omega)^{n+1}
\]

(4)

The point of expressing \( \text{Fut}(X, \Omega) \) in this manner is that, as we will see, each integral (for each fixed \( j \)) may be realized as a Futaki-Morita type invariant, to which holomorphic localization applies.

### 2.1 Holomorphic Localization

We turn for a moment to a general description of Bott’s holomorphic localization and its application to Futaki-Morita integrals. Let \((M, g)\) be a Hermitian manifold and \(E\) a holomorphic vector bundle on \(M\) with Hermitian metric \(h\) and curvature \(R(h) \in \Omega^{1,1}(\text{End}(E))\) of its Chern connection. Suppose that there exists smooth \(\theta(h) \in \Gamma(\text{End}(E))\) satisfying

\[
i_X R(h) = -\bar{\partial} \theta_X(h)
\]

Given an invariant polynomial

\[
\phi : \mathfrak{gl}(\text{rank}(E), \mathbb{C}) \to \mathbb{C}
\]
of degree \(n + k\), the Futaki-Morita integral is defined as

\[
f_{\phi}(X) := \int_M \phi \left( \theta_X(h) + \frac{\sqrt{-1}}{2\pi} R(h) \right),
\]

which turns out to be independent of the chosen metrics [13].

Futaki-Morita integrals may generally be computed via holomorphic localization: Define a \((1, 0)\) form \(\eta\) on \(M/\text{Zero}(X)\) by

\[
\eta(\cdot) = \frac{g(\cdot, \bar{X})}{\|X\|^2}
\]

Bott’s transgression argument [4] [14] shows

\[
f_{\phi}(X) = -\sum_{\lambda} \lim_{r \to 0^+} \int_{\partial B_r(Z_\lambda)} \phi(\theta_X(h) + R(h)) \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial}\eta)^k
\]

where \(\{Z_\lambda\}\) is the zero locus of \(X\) and \(B_r(Z_\lambda)\) is any small neighborhood of \(Z_\lambda\).

We say that \(\text{Zero}(X)\) is nondegenerate when the endomorphism \(L_\lambda(X)\) of the normal bundle \(N_\lambda\) to \(Z_\lambda\) induced by \(X\) is invertible\(^2\). The work of Bott [3] essentially showed that when \(\text{Zero}(X)\) is nondegenerate,

\[
f_{\phi}(X) = \sum_{\lambda} \int_{Z_\lambda} \frac{\phi(\theta_X(h) + \frac{\sqrt{-1}}{2\pi} R(h))}{\det \left( L_\lambda(X) + \frac{\sqrt{-1}}{2\pi} K_\lambda \right)}
\]

where \(K_\lambda\) is the curvature form of the connection induced on \(N_\lambda\).

On the other hand, when \(\text{Zero}(X)\) consists of isolated degenerate points:

**Theorem 2.1 (Cherveny [7])** If the zero locus of \(X \in \mathfrak{h}\) is an isolated degenerate zero \(p\) such that in local coordinates centered at \(p\)

\[
z_i^\alpha_{i+1} = \sum b_{ij} X_j
\]

for some matrix \(B = (b_{ij})\) of holomorphic functions, then

\[
f_{\phi}(X) = \frac{1}{\prod \\alpha_i!} \left. \frac{\partial^{\alpha_1} (\phi(\theta_X(h)) \det B)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right|_{z=0}
\]

If \(\text{Zero}(X)\) consists of multiple isolated degenerate zeros, then \(f_{\phi}(X)\) is the sum of such contributions.

\(^2\)One may verify this endomorphism is given by \(L_\lambda(X)(Y) = (\nabla_Y X)^\perp\)
A special case of both (6) and (7) is when \( p \) is an isolated nondegenerate zero: As \( DX \) is invertible near \( p \), take \( B = (DX)^{-1} \) and \( \alpha_i = 0 \), giving

\[
f_\phi(X) = \phi(DX_p) \det B = \frac{\phi(DX_p)}{\det DX_p}
\]  

(8)

Localization involving a positive dimensional degenerate zero locus is quite complicated and not understood in the general Kähler setting. The calculations in this paper may be viewed as a step in this direction.

2.2 Localization of \( \text{Fut}(X, \Omega) \)

Returning to the localization of \( \text{Fut}(\Omega, X) \), suppose without loss of generality that \( \Omega = c_1(L) \) where \( L \) is a positive line bundle. Applying the above Futaki-Morita framework to the bundle \( E = K_M^+ \otimes L^{n-2j} \), standard computations yield

\[
R_E = \pm \text{Ric} + (n - 2j) \omega
\]

\[
i_X R_E = \pm i_X \text{Ric} + (n - 2j) i_X \omega
\]

\[
-\bar{\partial} \theta_E = \pm i_X \text{Ric} - (n - 2j) \bar{\partial} \theta_X
\]

\[
\theta_E = \mp \Delta \theta_X + (n - 2j) \theta_X
\]

Take \( \phi \) to be the invariant polynomial \( \phi(A) = \text{Tr}(A^{n+1}) \). The first integral in (4) is then recognized to be

\[
f_{\phi, E}(X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (-\Delta \theta_X + \text{Ric} + (n - 2j)(\theta_X + \omega))^{n+1}
\]

for \( E = K_M^+ \otimes L^{n-2j} \); the second is

\[
f_{\phi, E}(X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (\Delta \theta_X - \text{Ric} + (n - 2j)(\theta_X + \omega))^{n+1}
\]

for \( E = K_M^- \otimes L^{n-2j} \); and the third

\[
f_{\phi, E}(X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n (n + 1 - 2j)^{n+1} \int_M (\theta_X + \omega)^{n+1}
\]

for \( E = L^{n+1-2j} \). The Calabi-Futaki invariant is thus fully expressible in terms of Futaki-Morita integral invariants.

Applying localization (5) to each Futaki-Morita invariant in this expression and using the combinatorial identity (3) again to resolve summations yields

\[
\text{Fut}(\Omega, X) = \lim_{r \to 0} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \sum_{\lambda} \int_{\partial B_r(Z)} \left[ (-\Delta \theta_X + \text{Ric})(\theta_X + \omega)^n + \frac{\tilde{S}(\theta_X + \omega)^{n+1}}{n+1} \right] \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k
\]
When Zero($X$) is nondegenerate, this expression was evaluated by Tian using (6), with explicit cohomological simplifications in the specific cases of nondegenerate isolated zeros, or nondegenerate zero loci on a Kähler surface, or the Fano case $\Omega = c_1(M)$. See Theorem 6.3 in [18]; also p. 31 in [19].

2.3 Blowup Situation

Our interest will be the blowup scenario where degenerate contributions to localization naturally arise. To align notation with Li-Shi [15], let $\mu = \frac{\hat{\mu}}{n}$ and define

$$I_{Z\lambda} := \sum_{\lambda} \lim_{r \to 0^+} \left( \sqrt{-1} \over 2\pi \right)^n \int_{\partial B_r(Z\lambda)} (\Delta \theta_X - \text{Ric})(\theta_X + \omega)^n \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k$$

$$J_{Z\lambda} := - \sum_{\lambda} \lim_{r \to 0^+} \left( \sqrt{-1} \over 2\pi \right)^n \int_{\partial B_r(Z\lambda)} (\theta_X + \omega)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k$$

$$\text{Fut}_{Z\lambda}(X, \Omega) := I_{Z\lambda} - \frac{n\mu}{n+1} J_{Z\lambda}$$

so that

$$\text{Fut}(\Omega, X) = \sum_{\lambda} \text{Fut}_{Z\lambda}(X, \Omega) = \sum_{\lambda} \left( I_{Z\lambda} - \frac{n\mu}{n+1} J_{Z\lambda} \right). \quad (9)$$

Also define the summed contribution

$$J_M(\Omega, X) := \sum_{\lambda} J_{Z\lambda}(\Omega, X) = \left( \sqrt{-1} \over 2\pi \right)^n \int_M (\theta_X + \omega)^{n+1}$$

We close by expressing a relationship between local contributions to $\text{Fut}(X, \Omega)$ on $M$ and those for $\text{Fut}(\tilde{X}, \tilde{\Omega})$ on $\tilde{M}$, where notation is as in the introduction.

Lemma 2.2 (Li-Shi [15], Lemma 3.1) Let $X \in h$ vanish at $p$, $\tilde{M}$ be the blowup of $M$ at $p$, $\tilde{\Omega}$ be the Kähler class $\pi^*\Omega - \epsilon c_1(E)$, and $\tilde{X}$ be the extension of $X$ to $\tilde{M}$. Define $\delta = \tilde{\mu} - \mu$. Then

$$\text{Fut}_{\tilde{M}}(\tilde{\Omega}, \tilde{X}) = \text{Fut}_M(\Omega, X) - \frac{n\delta}{n+1} J_M(\Omega, X) + \sum_{E} \text{Fut}_{\tilde{Z}\lambda}(\tilde{\Omega}, \tilde{X}) + \frac{n\delta}{n+1} J_p(\Omega, X) - \text{Fut}_p(\Omega, X).$$

The lemma is a consequence of the localization formula for the Futaki invariant (9) and the above definitions after separating the fixed components of $\tilde{X}$ on $\tilde{M}$ into those contained in the exceptional divisor and those not. The latter type is in one-to-one correspondence with the fixed components of $X$ on $M$ apart from $p$, and moreover these local Futaki invariants agree after an adjustment to $J_{Z\lambda}$ by $\delta$. 
Recall that $\theta_X$ is defined up to addition of a constant. Without loss of generality, we may prove our main theorem under the simplifying assumption that this constant is chosen so $\theta_X$ has average value zero, and consequently $J_M(\Omega, X) = 0$. As $p$ is nondegenerate, the term $\text{Fut}_p(\Omega, X)$ is immediate using (8):

$$\text{Fut}_p(\Omega, X) = I_p - \frac{n\mu}{n+1} J_p = \frac{\text{Tr}(A)}{\det A} \theta^n_p - \frac{n\mu}{n+1} \frac{\theta^{n+1}_p}{\det A}$$

(10)

where we have simplified the notation using $A := DX_p$ and $\theta_p := \theta_X(p)$.

With this choice of $\theta_X$, and in light of Lemma 2.2 and (10),

**Lemma 2.3** To prove Theorem 1.1, it is sufficient to show

$$\frac{\text{Tr}(A)\theta^n_p}{\det A} - \frac{n(\mu + \delta)}{n+1} \frac{\theta^{n+1}_p}{\det A} - \sum_{j=1}^m \text{Fut}_{q_j}(\tilde{\Omega}, \tilde{X}) = n(n-1)\theta_p\epsilon^{n-1} + O(\epsilon^n)$$

where $\{q_1, \ldots, q_m\}$ are the isolated, possibly degenerate zeros of $\tilde{X}$ in $E$.

### 3 Case I: Maximally Degenerate Zero

In this section we will prove Theorem 1.1 when $\tilde{X}$ has a single isolated zero in the exceptional locus, necessarily of maximal degeneracy. The case of multiple zeros builds on these computations and will be given in the next section.

Let $p \in \text{Zero}(X)$ be the zero at which we will blow up $M$. Choosing coordinates about $p$ such that $DX_p$ is in Jordan form, a maximal degenerate zero on the blowup corresponds to $DX_p$ having a single Jordan block.

To be precise, let $A$ denote the $n \times n$ Jordan matrix

$$A = (A_{ij}) = \begin{bmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & a \end{bmatrix}$$

(11)

By Poincaré’s Theorem 5.1, if $DX_p = A$ then $X$ is biholomorphically equivalent to its linearization on some neighborhood $U$ of $p$ provided that $a \neq 0$ (the zero is nondegenerate). Therefore, without loss of generality, we may assume $X = \sum X_i \frac{\partial}{\partial z_i}$ is given on $U$ by

$$X = \sum_{i=1}^{n-1} (az_i + z_{i+1}) \frac{\partial}{\partial z_i} + az_n \frac{\partial}{\partial z_n}$$

(12)
Following [15], we now describe $X$’s natural extension $\tilde{X}$ to the blowup in local coordinates. Let $\tilde{U} = \pi^{-1}(U)$ be the neighborhood of the exceptional divisor on $\tilde{M} = \text{Bl}_p M$ given by

$$\tilde{U} = \{ ((z_1, \ldots, z_n), [\eta_1, \ldots, \eta_n]) | z_i \eta_j = z_j \eta_i \} \subseteq \mathbb{C}^n \times \mathbb{C}P^{n-1}.$$ 

Cover $\tilde{U}$ with charts $\tilde{U}_i = \{ \eta_i \neq 0 \}$ having local coordinates

$$(u_1, \ldots, u_n) = \left( \frac{\eta_1}{\eta_i}, \ldots, z_i, \ldots, \frac{\eta_n}{\eta_i} \right)$$

Note that the slice of $\tilde{U}_i$ with $z_i = 0$ is just the standard cover for $E \cong \mathbb{C}P^{n-1}$.

In these coordinates the holomorphic extension $\tilde{X}$ of any $X \in h$ vanishing at $p$ to the blow-up is given by

$$\tilde{X} \big|_{\tilde{U}_i} = X_i \frac{\partial}{\partial u_i} + \sum_{j \neq i} \frac{1}{u_i} (X_j - u_j X_i) \frac{\partial}{\partial u_j} \tag{13}$$

In particular, $\tilde{X}$ over $\tilde{U}_1$ for the one block case (12) presently being considered is

$$\tilde{X} \big|_{\tilde{U}_1} = u_1 (a + u_2) \frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1} \left[ (u_{i+1} - u_i u_2) \frac{\partial}{\partial u_i} \right] + (-u_2 u_n) \frac{\partial}{\partial u_n} \tag{14}$$

One may verify that the isolated zero at the origin in this chart, which we denote $q$, is indeed the only zero of $\tilde{X}$ in the exceptional divisor. It is “maximally degenerate” in the sense that zero is an eigenvalue of $D\tilde{X}_q$ with algebraic multiplicity $n - 1$. This is all geometrically obvious when one considers the action induced by $X$ on lines through $p$, where the zero locus of $\tilde{X}$ in the exceptional divisor corresponds to eigenspaces for $DX_p$.

We now choose a convenient metric in the class $\tilde{\Omega} = \pi^* \Omega - \epsilon c_1([E])$ following [14] (p. 185). Denote by $B_r \subseteq U$ the ball of radius $r$ centered at $p$. Suppose for simplicity $B_1 \subseteq U$, and let $\tilde{B}_1 = \pi^{-1}(B_1)$. The fiber of $[E]$ over $\tilde{B}_1$ at a point is simply

$$[E]_{(z, \eta)} = \{ \lambda(\eta_1, \ldots, \eta_n), \lambda \in \mathbb{C} \}.$$ 

Denote by $h_1$ the metric on $[E]|_{\tilde{B}_1}$ given by $\| (\eta_1, \ldots, \eta_n) \|^2$. Also let $\sigma \in H^0(\tilde{M}, \mathcal{O}([E]))$ be the holomorphic section whose zero divisor is $E$ and denote by $h_2$ the metric on $[E]|_{\tilde{M}/E}$ such that $\| \sigma \|_{h_2} \equiv 1$. Finally choose a partition of unity $\{ \rho_1, \rho_2 \}$ subordinate to the cover $\{ \tilde{B}_1, \tilde{M}/\tilde{B}_{1/2} \}$. The metric

$$h = \rho_1 h_1 + \rho_2 h_2$$

has nonzero curvature only on $\tilde{B}_1$. Our Kähler form will be

$$\tilde{\omega} = \pi^* \omega + \epsilon \partial \bar{\partial} \log h.$$
A short calculation shows the holomorphy potential $\tilde{\theta}$ for $\tilde{X}$ relative to this $\tilde{\omega}$ on $\tilde{U}$ is

$$\tilde{\theta} = \pi^* \theta_X - \epsilon \left[ a + \frac{u_2 + \bar{u}_2 u_3 + \bar{u}_3 u_4 + \cdots + \bar{u}_{n-1} u_n}{1 + |u_2|^2 + \cdots + |u_n|^2} \right]$$  \hspace{1cm} (15)

For brevity, denote $\theta_X(p)$ by $\theta_p$ and likewise for $\tilde{\theta}_q := \theta_{\tilde{X}}(q)$, so that

$$\tilde{\theta}_q = \theta_p - a\epsilon. \hspace{1cm} (16)$$

All derivatives of all orders of $\tilde{\theta}$ vanish at $q$ with the exception of

$$\frac{\partial \theta_{\tilde{X}}}{\partial u_2}(q) = -\epsilon. \hspace{1cm} (17)$$

Another short calculation shows that

$$\Delta_{\omega} \tilde{\theta} = a - (n - 1)u_2 \hspace{1cm} (18)$$

By Theorem 2.1, the local contribution to any Futaki-Morita integral in the present situation (and in particular the Futaki invariant) is given by the residue formula

$$\text{Res}_q \phi = \frac{1}{\prod (\alpha_j - 1)!} \frac{\partial^{|\alpha|}(\phi \det B)}{\partial u_1^{\alpha_1-1}\partial u_2^{\alpha_2-1}\cdots \partial u_n^{\alpha_n-1}} \bigg|_q$$  \hspace{1cm} (19)

where $\alpha_j$ are natural numbers and $B = (b_{ij})$ is an $n \times n$ matrix such that

$$u_j^{\alpha_j} = \sum b_{ij} \tilde{X}_i. \hspace{1cm} (20)$$

We now construct $B$ in order to calculate (19). Choose $k$ such that $2^k < n \leq 2^{k+1}$. It is straightforward to verify for (14) that

$$u_1 = \left[ \frac{1}{a} - \sum_{i=2}^{n} \left( \frac{-1}{a} \right)^i u_i \right] \tilde{X}_1 + \sum_{i=2}^{n} \left[ -u_1 \left( \frac{-1}{a} \right)^i \right] \tilde{X}_i$$

$$u_2^n = \sum_{i=2}^{n} \left[ -u_2^{n-i} \right] \tilde{X}_i$$

For $j = 3, \ldots, n$, one calculates

$$u_j^{2^{k+1}} = \sum_{l=2}^{j-1} \left[ -u_2^{(j-1)2^{k+1}-l} + u_2^{(j-l-1)2^{k+1}} \prod_{i=0}^{k} (u_i^{2i} + u_2^{2i} u_1^{2i}) \right] \tilde{X}_l + \sum_{l=j}^{n} -u_2^{(j-l-1)2^{k+1}-l} \tilde{X}_l$$
The idea is to repeatedly factor \( u_j^{2k+1} - (u_2 u_{j-1})^{2k+1} \) into binomials, one of which is eventually \( \tilde{X}_j \), and insert the above expression for \( u_j^2 \) in terms of the \( \tilde{X}_i \). These relations contain the information necessary to construct \( B \) for (20) with parameters \( \alpha_1 = 1, \alpha_2 = n, \alpha_j = 2^{k+1} \) for \( j = 3, \ldots, n \) (these are certainly not minimal \( \alpha_i \) for every \( n \), but are convenient for a general setup).

With these choices, the determinant of \( B \) is found by row reduction to be

\[
\det B = (-1)^{n-1} \left[ \frac{1}{a} - \sum_{i=2}^{n} \left( \frac{-1}{a} \right)^i u_i \right] \prod_{j=3}^{n} \prod_{i=0}^{k} (u_j^{2i} + u_2^{2i} u_{j-1})
\]

(21)

Applying (19), the residue of interest is

\[
\text{Res}_q \phi = \frac{1}{(n-1)![(2k)!]^{n-2}} \cdot \frac{\partial^{[a]}(\phi \det B)}{(\partial u_2)^{n-1}(\partial u_3)^{2^{k}} \cdots (\partial u_n)^{2^{k}}} \bigg|_q
\]

Since \( \phi \) is a function of \( \theta_{\tilde{X}} \), which depends only on \( u_2 \) and its derivatives in our case, all other derivatives must be applied to \( \det B \). Doing so, the coefficient of \( u_2^{2k} \cdots u_n^{2k} \) in \( \det B \) is found to be

\[
(-1)^{n-1} \sum_{i=0}^{n-1} \frac{(-u_2)^i}{a^{i+1}}
\]

(22)

so that the residue after taking appropriate derivatives and evaluating at \( u_3 = \ldots = u_n = 0 \) is

\[
\text{Res}_q \phi = \frac{1}{(n-1)!} \frac{\partial}{(\partial u_2)^{n-1}} \left( \phi \sum_{i=0}^{n-1} (-1)^{n-1} \frac{(-u_2)^i}{a^{i+1}} \right) \bigg|_q
\]

\[
= \frac{1}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} \frac{(-1)^{n-1-j}(n-1-j)!}{a^{n-1-j+1}} \cdot \frac{\partial^j \phi}{(\partial u_2)^j}(0)
\]

\[
= \sum_{j=0}^{n-1} \frac{(-1)^j j! a^{n-j}}{j! a^{n-j}} \frac{\partial^j \phi}{(\partial u_2)^j}(0)
\]

We have shown

**Lemma 3.1** If \( \phi \) is an invariant polynomial whose value depends only on \( u_2 \) in the above situation, and \( DX_p \) is a single Jordan block with eigenvalue \( a \), then the residue contribution to the Futaki-Morita of the blowup at \( p \) at the unique isolated zero \( q \) is

\[
\text{Res}_q \phi = \sum_{i=0}^{n-1} \frac{(-1)^i}{i! a^{n-i}} \frac{\partial^i \phi}{(\partial u_2)^i}(0)
\]
When \( n = 2 \) we recover Lemma 3.6 of [15], which is the main calculation of the paper and obtained by brute force calculus.

We can now give a direct proof of Theorem 1.1 in the case \( DX_p = A \) by verifying the identity in Lemma 2.3. The term to calculate is \( \text{Fut}_q(\tilde{\Omega}, \tilde{X}) = I_q - \frac{n}{n+1}(\delta + \mu)J_q \).

For \( J_q \), apply Lemma 3.1 with \( \phi = \theta^{n+1}_X \). By (16) and (17),

\[
J_q = \sum_{i=0}^{n-1} \frac{(n+1)!}{i!} \frac{(\theta_p - ae)^{n+1-i} \epsilon^i}{a_{n-i}(n+1-i)!}
\]

Binomial expansion and interchanging summations yields

\[
J_q = (n+1)! \sum_{j=2}^{n+1} \sum_{i=0}^{n+1-j} \frac{(-1)^{n+1-i-j}}{i! j! (n+1-i-j)!} a_{j} \epsilon^{n+1-j} + O(\epsilon^n).
\]

The coefficient of \( \epsilon^{n+1-j} \) is proportional to \( \sum_{i=0}^{n+1-j} (-1)^i \binom{n+1-j}{i} \), which vanishes by symmetry of binomial coefficients unless \( j = n+1 \). It follows that

\[
J_q = \frac{\theta_p^{n+1}}{a^n} + O(\epsilon^n).
\]

On the other hand, to calculate \( I_q \) use \( \phi = (-\Delta \theta_X) \theta^n_X \) in Lemma 3.1. Again by (16), (17), and (18),

\[
I_q = \sum_{i=0}^{n-1} \frac{n! \epsilon^{i-1}}{i! a_{n-i}(n-i)!} \left[ \frac{(n-1)\epsilon a_p - a\epsilon}{n-i+1} + a(\theta_p - ae)^{n-i} \right]
\]

By a similar expansion, the summed second term simplifies to

\[
\frac{\theta^n_p}{a^{n-1}} + O(\epsilon^n)
\]

while the first simplifies to

\[
\frac{(n-1)\theta^n_p}{a^{n-1}} - n(n-1)\theta_p \epsilon^{n-1} + O(\epsilon^n)
\]

yielding

\[
I_q = \frac{n\theta^n_p}{a^{n-1}} - n(n-1)\theta_p \epsilon^{n-1} + O(\epsilon^n)
\]

Putting everything together in Lemma 2.3,

\[
\text{Fut}_p(X, \Omega) - \text{Fut}_q(\tilde{X}, \tilde{\Omega}) - \frac{n\delta}{n+1} J_p
\]

\[
= \left[ \frac{an\theta^n_p}{a^n} - \frac{n(\mu + \delta) \theta^{n+1}_p}{a^n} \right] - \left[ \frac{n\theta^n_p}{a^{n-1}} - n(n-1)\theta_p \epsilon^{n-1} - \frac{n(\mu + \delta) \theta^{n+1}_p}{a^n} + O(\epsilon^n) \right]
\]

\[
= n(n-1)\theta_p \epsilon^{n-1} + O(\epsilon^n)
\]
which completes our verification of Theorem 1.1 in this special case.

4 Case II: Multiple Degenerate Zeros

In this section we complete the proof of Theorem 1.1. Suppose the linearization of $X$ at $p$ now has multiple Jordan blocks. Hypothesis ($\star$) means that each Jordan block corresponds to an isolated degenerate zero in the exceptional divisor, and by a change of coordinate we may assume a particular degenerate zero corresponds to the first block. We extend the computations from Section 3 to calculate the contribution to the Futaki invariant from the first block under the influence of other blocks. The net contribution from all degenerate zeros is then a sum given by symmetrizing that formula, which we evaluate using Lemma 4.1 (proved via integration of meromorphic differentials in the appendix).

Suppose coordinates centered at $p \in \text{Zero}(X)$ have been chosen such that the linearization $DX$ of $X$ is in Jordan form at $p$:

$$DX_p = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & : \\ : & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_m \end{bmatrix}$$

Each Jordan block $A_i$ is of the form $(11)$ with diagonal entries $a_i$ and dimension $n_i$. By Lemma 5.2, we may assume $X$ is biholomorphic to its linearization near $p$.

Let $s_j = \sum_{k=1}^j n_k$, so that $s_m = n$. In the coordinates introduced in Section 3,

$$\tilde{X}|_{\tilde{U}_1} = u_1(a_1 + u_2) \frac{\partial}{\partial u_1} + \sum_{j=1}^m \left[ \sum_{i=s_j-1}^{s_j-1} [u_{i+1} - u_i(u_2 + a_1 - a_j)] \frac{\partial}{\partial u_i} - u_{s_j}(u_2 + a_1 - a_j) \frac{\partial}{\partial u_{s_j}} \right]$$

(23)

(which of course reduces to the one block formula (14) when $m = 1$). The zero at the origin in $\tilde{U}_1$ will be denoted $q_1$.

Our first task is to construct the appropriate $B$ to apply Theorem 2.1. Let $k$ be the natural such that $2^k < n_1 \leq 2^{k+1}$. The work of Section 3 constructs a matrix $B_1$ expressing powers of $u_1, \ldots, u_{s_1}$ in terms of $\tilde{X}_{s_1}, \ldots, \tilde{X}_{s_1}$. For $1 < j \leq m$, one checks

$$u_{s_j} = \tilde{X}_{s_j} \prod_{i=0}^{k-1} \left[ (a_j - a_1)^{2^i} + u_2^{2^i} \right] + u_2^{2^{k+1}} u_{s_j}$$

(24)

The idea here is to factor $u_{s_j}(u_2^{2^{k+1}} - (a_{s_j} - a_1)^{2^{k+1}})$. We are then done since $u_2^{2^{k+1}}$ is a known linear combination of $\tilde{X}_{s_1}, \ldots, \tilde{X}_{s_1}$ from Section 3. For $u_{s_{j-1}} < u_i < u_{s_j}$,
\[
\frac{u_i}{(a_j - a_1)^{2k+1}} = \prod_{i=0}^{k} \frac{(a_j - a_1)^{2i} + u_2^{2i}}{(a_j - a_1)^{2k+1}} \bar{X}_i + \sum_{k=0}^{\infty} \frac{u_2^{k+1}}{(a_j - a_1)^{2k+1}} u_{i+1} \tag{25}
\]

so that, in light of (24), we may recursively solve to obtain

\[
\frac{u_i}{(a_j - a_1)^{2k+1}} = \prod_{i=0}^{k} \frac{(a_j - a_1)^{2i} + u_2^{2i}}{(a_j - a_1)^{2k+1}} \bar{X}_i + \text{linear combination of } \{\bar{X}_1, \ldots, \bar{X}_{n_1}, \bar{X}_{i+1}, \ldots, X_j\}. \tag{26}
\]

It follows that \(B\) has the form

\[
B = \begin{bmatrix}
B_1 & 0 & 0 & 0 \\
* & B_2 & 0 & 0 \\
* & 0 & \ddots & 0 \\
* & 0 & 0 & B_m
\end{bmatrix} \tag{27}
\]

where \(B_1\) was constructed in the previous section, and each \(B_j\) for \(j > 1\) is upper triangular with the entries recoverable from (24) and (25).

We have calculated \(\det B_1\) in (21), while for \(j > 1\),

\[
\det B_j = \left( \prod_{i=0}^{k} \frac{(a_j - a_1)^{2i} + u_2^{2i}}{(a_j - a_1)^{2k+1}} \right)^{n_j}
\]

Clearly the only derivative that may be applied to \(\det B_j\) for \(j > 1\) is the \(u_2\)-derivative. The \(i\)-th \(u_2\)-derivative of \(\det B_j\) evaluated at \(u_2 = 0\) is

\[
\frac{\partial}{(\partial u_2)^i} \det B_j = \frac{(n_j + i - 1)!}{(n_j - 1)! (a_j - a_1)^{n_j+i}}
\]

Putting everything into (19), the residue of interest

\[
\text{Res}_{q_1} \phi = \frac{1}{(n_1 - 1)! (\partial u_2)^{n_1-1}} \left( \phi \prod_{j=1}^{m} \frac{\partial \det B_j}{(\partial u_2)^{\mu_j}} \right) \left. \frac{\partial \phi}{(\partial u_2)^i} \right|_{u=0}
\]

\[
= \frac{1}{(n_1 - 1)!} \sum \left( \begin{array}{c}
\frac{n_1 - 1}{i} \\
i, \mu_1, \ldots, \mu_m
\end{array} \right) \prod_{j=1}^{m} \frac{\partial \det B_j}{(\partial u_2)^{\mu_j}} \left. \frac{\partial \phi}{(\partial u_2)^i} \right|_{u=0}
\]

\[
= \sum_{i=0}^{n_1-1} \sum_{\mu} \frac{1}{i!} \left( \prod_{j=1}^{m} \frac{1}{\mu_j!} \frac{\partial \det B_j}{(\partial u_2)^{\mu_j}} \right) \left. \frac{\partial \phi}{(\partial u_2)^i} \right|_{u=0}
\]

\[
= \sum_{i=0}^{n_1-1} \sum_{\mu} \frac{(-1)^{n_1+i+1}}{i! a_1^{n_1+i+1}} \left( \prod_{j=2}^{m} \frac{(n_j+\mu_j-1)}{a_j-a_1^{n_j+\mu_j}} \right) \left. \frac{\partial \phi}{(\partial u_2)^i} \right|_{u=0} \tag{28}
\]
where \( \mu = (\mu_1, \ldots, \mu_m) \) runs over all partitions of \( n_1 - i - 1 \) of length \( m \). In the last line we have used (22). This generalizes Lemma 3.1 (\( m = 1 \) is the lemma).

We now complete the proof of Theorem 1.1 by verifying the identity in Lemma 2.3. The holomorphy potential of \( \tilde{X} \) as in (23) is

\[
\theta_{\tilde{X}} = \pi^* \theta_X - \epsilon \left[ a_1 + \frac{u_2 + \bar{u}_2u_3 + \cdots + \bar{u}_{n-1}u_n + \sum_{j=1}^{\mu} \sum_{k=\mu_j+1}^{\mu_j+\mu_j-1} (a_j - a_1)|u_k|^2}{1 + |u_2|^2 + \cdots + |u_n|^2} \right],
\]

generalizing (15). As in (16) and (17), \( \theta_{\tilde{X}} \) satisfies

\[
\tilde{\theta}_{q_1} = \theta_p - a_1 \epsilon \quad \frac{\partial \tilde{\theta}_{\tilde{X}}}{\partial u_2}(q_1) = -\epsilon
\]

while all other derivative of \( \theta_{\tilde{X}} \) vanish at \( q_1 \), and the Laplacian generalizing (18) is

\[
\Delta_{\tilde{X}} \theta_{\tilde{X}} = \text{Tr}(A) - (n-1)(u_2 + a_1).
\]

With (29) in mind, \( J_{q_1} \) is calculated by applying (28) to \( \phi = \theta_{\tilde{X}}^{n+1} \):

\[
J_{q_1} = \sum_{i=0}^{n_1-1} \sum_{\mu} \frac{(-1)^{n_1+\mu_1-1}}{i!a_1^{\mu_1+1}} \left( \prod_{j=2}^{m} \frac{(n_j+\mu_j-1)}{(a_j - a_1)^{n_j+\mu_j}} \right) \frac{(n_1+1)\epsilon_i^j(\theta_p - a_1 \epsilon)^{n+1-i}}{(n_1+1-i)!}
\]

where \( \mu \) is still runs over partitions of \( n_j - i - 1 \) of length \( m \).

Interchanging \( 1 \leftrightarrow j \) gives the sum over all zeros \( \{q_1, \ldots, q_m\} \) in the exceptional divisor to be

\[
\sum_j J_{q_j} = \sum_{j=1}^{m} \sum_{i=0}^{n_j-1} \frac{(n_1+1)\epsilon_i^j(\theta_p - a_j \epsilon)^{n+1-i}}{i!(n_1-i)!} \sum_{\mu} \frac{(-1)^{n_j+\mu_j-1}}{a_j^{\mu_j+1}} \left( \prod_{l \neq j}^{m} \frac{(n_l+\mu_l-1)}{(a_l - a_j)^{n_l+\mu_l}} \right)
\]

\[
= \sum_{j=1}^{m} \sum_{i=0}^{n_j-1} \sum_{k=0}^{n_j-1-i} \frac{(-1)^{n-j}(n_1+1)!}{j!k!(n_1-i-k)!} \theta_p^k \epsilon^{n+1-k} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+i+k+n}} \left( \prod_{l \neq j}^{m} \frac{(n_l+\mu_l-1)}{(a_l - a_j)^{n_l+\mu_l}} \right)
\]

\[
= \sum_{k=0}^{n+1} (-1)^{n-k} \frac{(n_1+1)!}{k!} \theta_p^k \epsilon^{n+1-k} G_k
\]

where

\[
G_k = \sum_{j=1}^{m} \sum_{i=0}^{n_j+1-k} \frac{(n_1+1-k)}{i} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+i+k+n}} \left( \prod_{l \neq j}^{m} \frac{(n_l+\mu_l-1)}{(a_l - a_j)^{n_l+\mu_l}} \right).
\]
We have simplified the notation by allowing $i$ to run into values that make the partition $\mu$ of $n_j - i - 1$ a partition of a negative number. The term is understood to be zero when this happens.

**Lemma 4.1**

$$G_k = \begin{cases} \frac{-1}{\det A} & k = n + 1 \\ 0 & 1 < k < n + 1 \\ (-1)^n & k = 1 \end{cases}$$

See the appendix in Section 6 for a proof via integration of meromorphic differentials. It follows from Lemma 4.1 that

$$\sum_{j} J_{q_j} = \frac{\theta_{p}^{n+1}}{\det A} + O(\epsilon^n)$$

Likewise, to compute the contribution $I_{q_j}$ from each fixed point $q_j$, use $\phi = (-\Delta \theta_X)^\theta_X$ in (28) along with (29) and (30). The contribution is

$$\sum_{j} I_{q_j} = \sum_{j=1}^{m} \sum_{i=0}^{n_j-1} \sum_{\mu} \frac{(-1)^{n_j+\mu_j-1}}{i! a_j^{\mu_j+1}} \left( \prod_{l \neq j} \frac{(n_j+\mu_j-1)}{(a_j-a_l)^{n_j+\mu_j}} \right) \left[ \frac{-n!(\text{Tr}(A) - (n-1)a_j)(-\epsilon)^i(\theta_p - a_j\epsilon)^{n-i}}{(n-i)!} + \frac{i(n-1)n!(-\epsilon)^{i-1}(\theta_p - a_j\epsilon)^{n-i+1}}{(n-i+1)!} \right]$$

This summation is of the form

$$\sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} \theta_{p}^{k} e^{n-k} [G_k' + G_k'' + G_k'''] + O(\epsilon^n)$$

where

$$G_k' = \text{Tr}(A) \sum_{j=1}^{m} \sum_{i=0}^{n_j-k} \binom{n-k}{i} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+i+k-n}} \left( \prod_{l \neq j} \frac{(n_j+\mu_j-1)}{(a_l-a_j)^{n_j+\mu_j}} \right)$$

$$G_k'' = -(n-1) \sum_{j=1}^{m} \sum_{i=0}^{n_j-k} \binom{n-k}{i} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+i+k-n}} \left( \prod_{l \neq j} \frac{(n_j+\mu_j-1)}{(a_l-a_j)^{n_j+\mu_j}} \right)$$

$$G_k''' = -(n-1) \sum_{j=1}^{m} \sum_{i=1}^{n_j-k-1} \binom{n-k}{i-1} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+i+k-n}} \left( \prod_{l \neq j} \frac{(n_j+\mu_j-1)}{(a_l-a_j)^{n_j+\mu_j}} \right)$$

and, as with $G_k$ above, any index $i$ producing a partition of a negative number contributes zero.
Clearly $G_k' = \text{Tr}(A)G_{k+1}$, while for each $1 \leq k \leq n$, combinatorial manipulation shows

$$G_k'' + G_k''' = -(n - 1)G_k.$$ 

By Lemma 4.1, (32) evaluates to

$$\sum_j I_{q_j} = \frac{\text{Tr}(A)}{\det A} \theta_p^n - n(n + 1)\theta_p e^{n-1} + O(\epsilon^n).$$

Putting these results into Lemma 2.3,

$$\text{Fut}_p(\Omega, X) - \sum_{j=1}^m \left( I_{q_j} - \frac{n\tilde{\mu}}{n + 1} J_{q_j} \right) - \frac{n\delta}{n + 1} J_p(\Omega, X)$$

$$= \left( \frac{\text{Tr}(A)\theta_p^n}{\det A} - \frac{n(\mu + \delta)}{n + 1} \theta_p^{n} \right) - \left( \frac{\text{Tr}(A)\theta_p^n}{\det A} - n(n + 1)\theta_p e^{n-1} - \frac{n\tilde{\mu}}{n + 1} \theta_p^{n+1} + O(\epsilon^n) \right)$$

$$= n(n - 1)\theta_X(p)e^{n-1} + O(\epsilon^n)$$

which concludes the verification of Theorem 1.1.

5 Normal Forms at Singularities

Sections 3 and 4 rely on the assumption that $X$ is holomorphically equivalent to its linearization on a neighborhood of $p$, which is in general not true (not even smoothly). Our main result of this section is that for the purposes of proving Theorem 1.1 it is sufficient to assume such a normal form.

We recall a well-known condition originally due to Poincaré under which this assumption is true. A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is called resonant if there exists a relation

$$\lambda_k = \sum m_i \lambda_i$$

where $m_i$ are nonnegative integers and $\sum m_i \geq 2$. The vector $\lambda$ is said to belong to the Poincaré domain if the convex hull of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ does not contain the origin.

**Theorem 5.1 (Poincaré [1])** If the eigenvalues of the linear part of a holomorphic vector field at a zero belong to the Poincaré domain and are nonresonant, then the vector field is biholomorphically equivalent to its linearization on a neighborhood of that zero.

This condition applies to the one non-trivial Jordan block needed in Li-Shi [15] and is key to simplifying their calculation. It remains valid more generally for the case of a single Jordan block in Section 3, but does not apply in Section 4 in general.
We now show that it is sufficient to prove Theorem 1.1 under the assumption $X$ is locally biholomorphic to its linearization. To be precise, let $Y$ be a holomorphic vector field on $M$ with an isolated nondegenerate zero at $p$ satisfying $(\star)$, so that in coordinates centered at $p$,

$$Y = \sum_j (A_{ij} z_i + O(z_k z_l)) \frac{\partial}{\partial z_j} \quad (33)$$

Let $X$ denote the linearization of $Y$, defined on a neighborhood of $p$:

$$X = \sum_j (A_{ij} z_i) \frac{\partial}{\partial z_j}$$

The extension $\tilde{X}$ of $X$ to $\tilde{M} = Bl_p M$ is defined on a neighborhood of the exceptional divisor (see (13) for the explicit local formula).

**Lemma 5.2** Let $X, Y, \tilde{X}, \tilde{Y}$ be as above. The zero loci of $\tilde{X}$ and $\tilde{Y}$ on a neighborhood of the exceptional divisor $E$ agree, and for each isolated zero $q \in E$,

$$\text{Fut}_q(\tilde{X}, \tilde{\Omega}) = \text{Fut}_q(\tilde{Y}, \tilde{\Omega})$$

**Proof** In coordinates centered at $p$, $Y$ has the form in (33). As $p$ is an isolated and nondegenerate zero of $Y$, it is also isolated and nondegenerate for $X$. From expression (13) it is clear that any zeros of $\tilde{Y}$ in a neighborhood of $E$ must be on $E$ (one must have $Y_1 = \cdots = Y_n = 0$, and this can only happen on $\pi^{-1}(p)$). Likewise for the zeros of $\tilde{X}$. But the zeros on the exceptional divisor correspond with the eigenspaces of $DX_p = DY_p$, and so the zero loci agree as claimed. In particular, the zeros of $\tilde{Y}$ are isolated iff the zeros of $\tilde{X}$ are isolated.

The local Futaki invariant for $X$ was calculated in Section 4 using (28). We will show that adding in higher order terms to form $Y$ changes $\theta_{\tilde{X}}, \Delta \theta_{\tilde{X}}$, and $\det B$ each by $O(u_1)$, so that

$$-(\Delta \theta_{\tilde{Y}}) \theta^n_{\tilde{Y}} \det C = -(\Delta \theta_{\tilde{X}}) \theta^n_{\tilde{X}} \det B + O(u_1)$$

$$\theta^{n+1}_{\tilde{X}} \det C = \theta^{n+1}_{\tilde{Y}} \det B + O(u_1) \quad (34)$$

and moreover still no $u_1$ derivatives will be involved. The lemma then follows, as $I_q$ and $J_q$ are calculated via (7) with no $u_1$-derivatives applied to $-(\Delta \theta_{\tilde{Y}}) \theta^n_{\tilde{Y}} \det C$ and $\theta^{n+1}_{\tilde{Y}} \det C$, respectively, followed by evaluation involving $u_1 = 0$.

By assumption,

$$Y_i = X_i + O(z_k z_l)$$

for each $i = 1, \ldots, n$. Using the coordinates in Section 3 in which the zero of $\tilde{Y}$ is at the origin in $\tilde{U}_1$, by (13)

$$\tilde{Y}_1 = \tilde{X}_1 + O(u_1^2)$$
while for $2 \leq i \leq n$,

$$
\tilde{Y}_i = \frac{1}{u_1} [A_{ij}z_j + O(u_1^2) - u_i(A_{ij}z_j + O(u_1^2))]
= \tilde{X}_i + O(u_1)
$$

Straightforward calculation then gives

$$
\Delta\omega_\theta \tilde{Y} = \sum_i \frac{\partial \tilde{Y}_i}{\partial u_i}
= \frac{\partial}{\partial u_1}[\tilde{X}_1 + O(u_1^2)] + \sum_{i>1} \frac{\partial}{\partial u_i}[\tilde{X}_i + O(u_1)]
= \Delta\omega_\theta \tilde{X} + O(u_1)
$$

(35)

Next we consider the holomorphy potential itself. On $U$,

$$
\bar{\partial}\theta_\tilde{X} = \bar{\partial}\theta_\tilde{Y} + O(z_iz_k)
$$

so that

$$
\bar{\partial}\theta_\tilde{X} = \pi^* \bar{\partial}\theta_\tilde{X} - \bar{\partial}(\tilde{X} \log h)
\bar{\partial}\theta_\tilde{X} = \pi^* \bar{\partial}(\theta_\tilde{X} + O(z_kz_l)) - \bar{\partial}(\tilde{Y} \log h) + \bar{\partial}(O(u_1))
\theta_\tilde{X} = \theta_\tilde{Y} + O(u_1)
$$

(36)

Lastly, let $B = (b_{ij})$ be the matrix of holomorphic functions near $q$ constructed in Sections 3 and 4 satisfying $u_{\alpha j}^j = \sum_j b_{ij}X_j$. We will show that there is similarly a matrix $C = (c_{ij})$ such that $u_{\alpha j}^{\alpha j}' = \sum_j c_{ij}Y_j$ for the same values of $\alpha_j$ (and in particular $\alpha_1 = 1$), such that

$$
\det C = \det B.
$$

(37)

Theorem 5.1 does adapt to the first Jordan block of $DY_p$, providing a holomorphic coordinate system in which $Y_1, \ldots, Y_{n_1}$ agree with $X_1, \ldots, X_{n_1}$, and consequently $\tilde{Y}_1, \ldots, \tilde{Y}_{n_1}$ agree with $\tilde{X}_1, \ldots, \tilde{X}_{n_1}$ by (13). As a result, the matrix $C$ sought begins with a block $C_1$ identical to block $B_1$ in (27).

For the $u_i$ where $i > n_1$, it is known from (26) that

$$
u_i = \prod_{k=0}^k \frac{[(a_j - a_1)^{2^{i+1}} + u_2^{2^{i+1}}]}{(a_j - a_1)^{2^{i+1}}} \tilde{X}_i + \text{linear combination of } \{\tilde{X}_1, \ldots, \tilde{X}_{n_1}, \tilde{X}_{i+1}, \ldots, X_s\}.
$$

Using that $\tilde{Y}_j = \tilde{X}_j + O(u_1)$ and that $u_1 = \sum_{j=1}^n c_{1j} \tilde{Y}_j$, we have

$$
u_i = \prod_{k=0}^k \frac{[(a_j - a_1)^{2^{i+1}} + u_2^{2^{i+1}}]}{(a_j - a_1)^{2^{i+1}}} \tilde{X}_i + \text{linear combination of } \{\tilde{X}_1, \ldots, \tilde{X}_{n_1}, \tilde{X}_{i+1}, \ldots, X_s\}.
$$
\[ u_i = \frac{\prod_{i=0}^{k}[(a_j - a_1)^{2i} + u_2^i]}{(a_j - a_1)^{2k+1}} \tilde{Y}_i + \text{linear combination of } \{ \tilde{Y}_1, \ldots, \tilde{Y}_{n_1}, \tilde{Y}_{i+1}, \ldots, Y_{s_j} \}. \]

It follows that the diagonal entries for \( C_j \) are the same as the diagonal entries for \( B_j \) for \( j > 1 \) and that \( C \) has the same form of \( B \) in (27), which is sufficient to conclude \( \det C = \det B \).

With (35), (36), and (37), we have shown (34) and the proof is complete.

6 Appendix: Proof of Lemma 4.1

We now prove

**Lemma 6.1** (Lemma 4.1) *Let*

\[ G_k = \sum_{j=1}^{m} \sum_{i=0}^{n+1-k} \left( n + 1 - k \right) \sum_{i} (-1)^{n_j+\mu_j} \frac{\prod_{l \neq j}^{m} \left( \frac{(n_l+\mu_l-1)}{(a_l - a_j)^{n_l+\mu_l}} \right)}{a_j^{\mu_j+\sum_{l \neq j}^{m} \mu_l} \cdot z^{-\sum_{l \neq j}^{m} \mu_l} (z - a_j)} \]

*Then*

\[ G_k = \begin{cases} 
\frac{-1}{\det A} & k = n + 1 \\
0 & 1 < k < n + 1 \\
(-1)^n & k = 1 
\end{cases} \]

The proof is by integration of meromorphic differentials over the Riemann sphere, where the residues will correspond to contributions to \( G_k \). If \( k = n + 1 \), consider

\[ \psi_j = \sum_{\mu \perp n_j-1} \frac{(-1)^{n_j+\mu_j}}{a_j^{\mu_j+\sum_{l \neq j}^{m} \mu_l} \cdot z^{-\sum_{l \neq j}^{m} \mu_l} (z - a_j)} \left( \prod_{l \neq j}^{m} \left( \frac{(n_l+\mu_l-1)}{(a_l - a_j)^{n_l+\mu_l}} \right) \right) dz. \]

There are always poles at \( z = a_1, \ldots, a_m \). There is also a pole at \( z = 0 \) when \( \sum_{l \neq j} \mu_l = 0 \), i.e. the partition is \( \mu_j = n_j - 1 \) and \( \mu_l = 0 \) for \( l \neq j \). There is never a pole at infinity. It is immediate that

\[ \sum_{j=1}^{m} \text{Res}_{a_j} \psi_j = G_{n+1} \]

and

\[ \text{Res}_0 \psi_j = \frac{1}{\prod_{l=1}^{m} a_l^{n_l}} = \frac{1}{\det A}, \]

while standard power series expansion and a consideration of partition recovery shows...
\[ \text{Res}_{a_j} \psi_l = \text{Res}_{a_j} \psi_j \]

for \( l \neq j \). As the sum of residues over a closed Riemann surface is zero,

\[
0 = \sum_{j=1}^{m} \left[ \text{Res}_0 \psi_j + \sum_{l=1}^{m} \text{Res}_{a_l} \psi_j \right] = \frac{m}{\det A} + m \sum_{j=1}^{m} \text{Res}_{a_j} \psi_j = \frac{m}{\det A} + m G_n
\]

and the \( k = n + 1 \) case follows.

For \( 1 < k < n + 1 \), instead use

\[
\psi_{j,k} = \sum_{i=0}^{n+1-k} \binom{n+1-k}{i} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{a_j^{n_j+i+k-n+\sum_{i \neq j} \mu_i} \cdot z^{-\sum_{i \neq j} \mu_i} (z - a_j)} \left( \prod_{l \neq j} \frac{(n_l+\mu_l-1)}{a_l - z}^{n_l+\mu_l} \right) \, dz
\]

The changed power in the denominator results in poles at \( a_1, \ldots, a_n \), but never at \( z = 0 \) or \( z = \infty \). It is again immediate that the contribution of interest is

\[
\sum_{j=1}^{m} \text{Res}_{a_j} \psi_{j,k} = G_k
\]

while power series expansion for this range of \( k \) shows \( \psi_{j,k} \) has the nice property

\[
\text{Res}_{a_l} \psi_{l,k} = \text{Res}_{a_j} \psi_{j,k}
\]

for all \( l \neq j \) (this fails when \( k = 1 \)). As the sum of residues is zero,

\[
0 = \sum_{j=1}^{m} \left[ \sum_{l=1}^{m} \text{Res}_{a_l} \psi_{j,k} \right] = m \sum_{j=1}^{m} \text{Res}_{a_j} \psi_{j,k} = m G_k
\]

and the second part of the lemma is established.

Finally for the \( k = 1 \) case, use

\[
\psi_j = \sum_{i=0}^{n} \binom{n+1-k}{i} \sum_{\mu} \frac{(-1)^{n_j+\mu_j}}{z^{1+i-n+\mu_j} (z - a_j)} \left( \prod_{l \neq j} \frac{(n_l+\mu_l-1)}{a_l - z}^{n_l+\mu_l} \right) \, dz.
\]

The term of interest is

\[
\sum_{j=1}^{m} \text{Res}_{a_j} \psi_j = G_1
\]

There is a pole at \( z = \infty \) with \( \text{Res}_\infty \psi_j = (-1)^{n+1} \), and again combinatorial manipulation shows
\[
\text{Res}_{a,j} \psi_j = \text{Res}_{a,l} \psi_l
\]
for all \( l \neq j \). We have

\[
0 = \sum_{j=1}^{m} \left[ \sum_{l=1}^{m} \text{Res}_{a,l} \psi_j + \text{Res}_\infty \psi_j \right] = m(G_1 + (-1)^{n+1})
\]

and the final \( k = 1 \) case follows.
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