Commuting Charges of the Quantum Korteweg-deVries and Boussinesq Theories from the Reduction of $W_\infty$ and $W_{1+\infty}$ Algebras

Andrew J. Bordner*
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan
(April 1, 2022)

Abstract

Integrability of the quantum Boussinesq equation for $c = -2$ is demonstrated by giving a recursive algorithm for generating explicit expressions for the infinite number of commuting charges based on a reduction of the $W_\infty$ algebra. These charges exist for all spins $s \geq 2$. Likewise, reductions of the $W_{\infty/2}$ and $W_{(1+\infty)/2}$ algebras yield the commuting quantum charges for the quantum KdV equation at $c = -2$ and $c = \frac{1}{2}$, respectively.

I. INTRODUCTION

Both the classical KdV and Boussinesq equations are integrable Hamiltonian systems, i.e., there are an infinite number of integrals of local polynomials in the fields and their derivatives, or charges, whose Poisson brackets with both the Hamiltonian and each other vanish. Of course, this implies an infinite hierarchy of equations, however we refer only to the lowest-dimensional member. The algebra of fields induced by the second Poisson bracket for the KdV and Boussinesq equations give, respectively, the Virasoro and classical $W_3$ algebra. We will study quantum versions of the KdV and Boussinesq theories, which share the classical theories’ integrability.

Sasaki and Yamanaka first considered the quantum KdV equation, in which the second Poisson brackets are replaced by the commutator as $i\{, \} \rightarrow [\, , ]$ and classical Hamiltonian $\int u^2(x)dx$ is replaced by $\frac{1}{2\pi i} \oint \mathcal{F}(TT)(z)dz$, where the parentheses denote a normal-ordering of the quantum fields, to be defined below. They found the existence of quantum charges $H_s$ of spin $s$, one for each odd spin $1 \leq s \leq 9$, and postulated that charges exist for each odd spin. They also noticed that these conserved charges took a particularly simple form when the central charge, $c = -2$.

The quantum Boussinesq theory has fields whose commutators are those of the $W_3$ extended conformal algebra and likewise has a integrable classical limit as the central charge

*e-mail address: bordner@yukawa.kyoto-u.ac.jp
$c \to \infty$ and the commutators are replaced with Poisson brackets. The equations of motion for the fields $T(z)$ and $W(z)$ in this theory are

$$\dot{T} = [T, P],$$
$$\dot{W} = [W, P],$$
$$P = \frac{1}{2\pi i} \oint W dz.$$  \hfill (1)\hfill (2)\hfill (3)

The dot indicates the derivative with respect the “time” variable, actually $\bar{z}$. (Note that the field theory is formulated using light-cone quantization and that the fields $T$ and $W$ depend on both $z$ and $\bar{z}$, however we drop the $\bar{z}$ dependence for notational simplicity.) The fields $T$ and $W$ are the currents of the $W_3$ extended conformal algebra with central charge $c$ and with operator product expansions (OPEs) \[3\]

$$T(y)T(z) = \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)} + O(1),$$
$$T(y)W(z) = \frac{3W(z)}{(y-z)^2} + \frac{\partial W(z)}{(y-z)} + O(1),$$
$$W(y)W(z) = \frac{c/3}{(y-z)^6} + \frac{2T(z)}{(y-z)^4} + \frac{\partial T(z)}{(y-z)^3} + \frac{1}{(y-z)^2} \left[ \frac{3}{10} \partial^2 T(z) + \frac{32}{22 + 5c} \Lambda(z) \right] + O(1)$$

with

$$\Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z).$$  \hfill (4)\hfill (5)\hfill (6)\hfill (7)

Parentheses around operators denote normal ordering defined by

$$(AB)(z) = \frac{1}{2\pi i} \oint \frac{dx}{x-z} A(x)B(z).$$  \hfill (8)

Assuming spatial periodicity, Fourier modes for the fields may be introduced, which then gives the usual commutation relations for $W_3$ modes.

Kupershmidt and Mathieu considered the quantum Boussinesq theory for generic central charge and found quantum charges for the first few lowest spins $s$ for $s \neq 0 \mod 3$ \[4\]. The classical Boussinesq equation also lacks charges for these values of the spin.

The quantum charges for the quantum KdV and Boussinesq theories may be related, through a quantum Miura transformation, to the charges of $A_1$ and $A_2$ Toda field theory, respectively. Quantum Toda field theories have been proven to be integrable, i.e., there exist an infinite number of mutually commuting charges which also commute with the Hamiltonian \[5\]. However, this proof is not constructive in the sense that explicit forms for the conserved charges are not given.

In this paper the commuting charges for the quantum Boussinesq theory for $c = -2$ are constructed using the reduction of the $W_\infty$ algebra to normal-ordered products of $W_3$ currents as described in Ref. \[6\]. We find similar reductions of the infinite-dimensional algebras $W_{\infty/2}$ and $W_{(1+\infty)/2}$ which allow a construction of the commuting charges for the quantum KdV theory at $c = -2$ and $c = 1/2$. Finally, the construction of the commuting conserved charges for the associated $A_1$ and $A_2$ Toda field theories are briefly described.
II. QUANTUM CONSERVED CHARGES FROM $W_\infty$

First the construction of $W_N$ from $W_\infty$ at $c = -2$ given in Ref. [3] is reviewed. The authors used a representation of the $W_\infty$ currents $V^{(i)}$, with spin $i + 2$, for $c = -2$ in terms of fermionic ghost fields $b(z)$ and $c(z)$ as

$$V^{(i)}(z) = \sum_{j=1}^{i+1} a_j(i) \partial^j c(z) \partial^{i+1-j} b(z)$$

(9)

with $a_j(i)$ the constants

$$a_j(i) = \binom{i+1}{j} \frac{(-1)^{i-j+1}(4q)^{j}(i+3-j)j_{j+1-j}}{(i+2)_{i+1}}$$

(10)

and $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ [7]. The only non-zero OPE of the fermion fields is $b(z)c(w) \sim 1/(z-w)$. $q$ is an arbitrary parameter which fixes the normalization of the currents; $q^2 = 1/24$ is chosen to agree with the normalization of Eqs. [3] with the definitions $T(z) \equiv V^{(0)}(z)$ and $W(z) \equiv V^{(1)}(z)$. By requiring the vanishing of quartic fermion terms resulting from the normal-ordered product of two $W_\infty$ currents in the fermion representation Eq. [3] it was shown that the $W_\infty$ current $V^{(i)}$ for $i \geq 2$ may be expressed in terms of only currents of lower spin as

$$V^{(i)} = \beta_i \sum_{p=0}^{i-2} b_p^{(i)} V^{(p)} V^{(i-2-p)} + \sum_{p=1}^{\lfloor \frac{i}{2} \rfloor} \gamma_p^{(i)} \partial^{2p} V^{(i-2p)},$$

(11)

where $[x]$ denotes the integer part of $x$. The coefficients $\beta_i$, $b_p^{(i)}$, and $\gamma_p^{(i)}$ are

$$\beta_i = \frac{c_i}{S_i(i)},$$

(12)

$$b_p^{(i)} = \frac{a_{i-1-p}(i-2)}{2a_1(p)a_{i-1-p}(i-2-p)},$$

(13)

$$\gamma_p^{(i)} = -\frac{\beta_i S_{i-p}(i)(2i+3-2p)!}{(2i+3-p)!c_{i-p}}.$$  

(14)

$c_i$ is defined by

$$c_i = \frac{(2q)^{2i}(i+2)!2i+2)!}{8(2i+1)!!c}$$

(15)

with the central charge $c = -2$ and

$$S_j(i) = \sum_{p=0}^{i-2} \sum_{k=1}^{i-1-p} \sum_{l=1}^{j+1} \sum_{m=1}^{j+1} b_p^{(i)} (-1)^{k+j} a_k(p)a_l(i-2-p)a_m(j)$$

$$\times \left( \frac{(i-l+1+m)!(j+1+l-m)!}{(i-p+k-l)} + (-1)^p \frac{(i-p-1-l+m)!(p+l+3+j-m)!}{(p-k+l+2)} \right).$$

(16)
For example, the first few lowest spin currents may be written as

\[ V^{(2)} = \frac{2}{3} \left[ (V^{(0)}V^{(0)}) - \frac{3}{10} \partial^2 V^{(0)} \right], \]

\[ V^{(3)} = \frac{2}{5} \left[ (V^{(0)}V^{(1)}) + (V^{(1)}V^{(0)}) - \frac{13}{14} \partial^2 V^{(1)} \right], \]

\[ V^{(4)} = \frac{4}{93} \left[ 5(V^{(0)}V^{(2)}) + 12(V^{(1)}V^{(1)}) + 5(V^{(2)}V^{(0)}) - \frac{2}{3} \partial^2 V^{(2)} - \frac{10}{21} \partial^4 V^{(0)} \right], \]

\[ V^{(5)} = \frac{5}{168} \left[ \frac{7}{2} (V^{(0)}V^{(3)}) + 15(V^{(1)}V^{(2)}) + (V^{(2)}V^{(1)}) + \frac{7}{2} (V^{(3)}V^{(0)}) - \frac{1203}{44} \partial^2 V^{(3)} \right] - \frac{265}{126} \partial^4 V^{(1)}, \]

\[ V^{(6)} = \frac{2}{907} \left[ 21(V^{(0)}V^{(4)}) + 140(V^{(1)}V^{(3)}) + 250(V^{(2)}V^{(2)}) + 140(V^{(3)}V^{(1)}) + 21(V^{(4)}V^{(0)}) - \frac{6454}{13} \partial^2 V^{(4)} - \frac{6005}{99} \partial^4 V^{(2)} - \frac{103}{81} \partial^6 V^{(0)} \right]. \]

If one defines the mode expansion for \( W_\infty \) currents as

\[ V^{(i)}(z) \equiv \sum_m V_m^{(i)} z^{-m-i-2} \]  

the commutation relations for the modes then have the standard form for Virasoro quasiprimary fields \( \mathbb{F} \)

\[ [V^{(i)}_m, V^{(j)}_n] = \sum_{k=0}^{i+j} \sum_{r,s=0}^{\Delta_i+\Delta_j-\Delta_k-1} C_{ij}^k p_{ijk}(m, n) V_{m+n}^{k} + \delta_{n,-m} D_{ij} \left( \frac{m+i+1}{2i+3} \right) \]  

where \( C_{ij}^k = 0 \) for \( i + j - k \) an odd number, \( p_{ijk}(m, n) \) are the universal polynomials

\[ p_{ijk}(m, n) = \sum_{r,s=0}^{\Delta_i+\Delta_j-\Delta_k-1} \delta_{r+s,\Delta_i+\Delta_j-\Delta_k-1} c_{r,s}^{ijk} \left( \frac{m+\Delta_i-1}{r} \right) \left( \frac{n+\Delta_j-1}{s} \right) \]  

with

\[ c_{r,s}^{ijk} = (-1)^s \frac{(2\Delta_k - 1)!}{(\Delta_i + \Delta_j + \Delta_k - 2)!} (-\Delta_i + \Delta_j + \Delta_k)_r (\Delta_i - \Delta_j + \Delta_k)_s. \]

\( \Delta_i = i + 2 \) is the conformal dimension of the field \( V^{(i)} \). The structure constants are \( \mathbb{F} \)

\[ C_{ij}^k = q_{i+j-k}(i + j - k + 4)! \sum_{r=0}^{(i+j-k)/2} \frac{(r-\frac{1}{2})!}{2(2k+3)!} \frac{(r-\frac{1}{2})!}{r!(-i - \frac{1}{2})_r(-j - \frac{1}{2})_r(k + \frac{1}{2})_r}. \]

for \( i + j - k \) even and 0 otherwise. The central term is \( D_{ij} = \delta_{ij} c_i \) with \( c_i \) defined in Eq. \( \mathbb{F} \).

The \( W_\infty \) algebra has an infinite-dimensional Abelian subalgebra consisting of the modes \( V^{(i)}_{-1} \). These are the commuting charges of the quantum Boussinesq equation. The charges \( V^{(i)}_{-1} \) in the \( c = -2 \) fermion representation, Eq. \( \mathbb{F} \), are those used in Ref. \( \mathbb{F} \) to construct charges of the quantum KdV equation. They are
\( H_{j+1} = \frac{1}{2\pi i} \oint \frac{V^{(j)}}{dz} = V^{(j)}_{j-1} \)  
(23)

\[
\sim \frac{1}{2\pi i} \oint (\partial^{j+1} c(z) b(z)) dz
\]

with the tilde denoting proportionality up to an (inessential) constant. In fact the modes \( V^{(i)}_{-i-1} \) of the dimension \( i + 2 \) currents of any Lie algebra of Virasoro quasi-primary fields form an Abelian subalgebra if the structure constants \( C^{i+j+1}_{ij} \) vanish.

The procedure to generate the quantum conserved charges for the Boussinesq theory is similar to the construction of the \( c = -2 \) \( W_3 \) currents in terms of \( W_\infty \) currents described in Ref. [6]. First the \( W_\infty \) currents \( T \equiv V^{(0)} \) and \( W \equiv V^{(1)} \) are regarded as fundamental and the currents with higher spin are replaced by their expressions quadratic in lower spin currents using Eq. \( 11 \). This is a recursive procedure which results in a representation of the \( W_\infty \) algebra by normal-ordered products of \( W_3 \) currents. The commuting charges are then the contour integral of these reduced \( W_\infty \) currents as given in Eq. \( 23 \). This procedure readily yields the charges for quantum Boussinesq theory as follows

\[
H_1 = \frac{1}{2\pi i} \oint T(z) dz,
\]
(24)

\[
H_2 = \frac{1}{2\pi i} \oint W(z) dz,
\]

\[
H_3 = \frac{1}{2\pi i} \oint (TT)(z) dz,
\]

\[
H_4 = \frac{1}{2\pi i} \oint (TW)(z) dz,
\]

\[
H_5 = \frac{1}{2\pi i} \oint \left[ (WW)(z) + \frac{5}{9}(T(TT))(z) + \frac{1}{6}(\partial T\partial T)(z) \right] dz,
\]

\[
H_6 = \frac{1}{2\pi i} \oint \left[ (\partial T\partial W)(z) + 2(W(TT))(z) \right] dz,
\]

\[
H_7 = \frac{1}{2\pi i} \oint \left[ (\partial W\partial W)(z) - \frac{119}{31}(T(WW))(z) - \frac{4295}{2232}(T(T(TT)))(z) + \frac{6139}{8928}(\partial^2 T\partial^2 T)(z) \right.
\]

\[
- \frac{4949}{2232}(\partial^2 T(TT))(z) \right] dz.
\]

These expressions have been multiplied by constants for convenience since this does not affect their commutativity. The following equation for the normal-ordered commutator

\[
([A, B])(z) = \sum_{n>0} \frac{(-1)^{n+1}}{n!} \partial^n \{ AB \}_n(z)
\]
(25)

in terms of the operators \( \{ AB \}_n(z) \) appearing in the OPE of \( A(y) \) and \( B(z) \)

\[
A(y)B(z) = \sum_{n=-\infty}^{M} \frac{\{ AB \}_n(z)}{(y-z)^n},
\]
(26)

with \( M \) is a positive integer, as well as the rearrangement relation

\[
(A(BC))(z) - (B(AC))(z) = ([A, B])C)(z)
\]
(27)
have been used to simplify the resulting expressions. Unlike for generic values of the central charge, one finds that charges exist for all integer spins \( s \geq 1 \) for \( c = -2 \).

\( \mathcal{W}_N \) minimal models, whose Verma modules contain an infinite number of singular vectors, exist for certain values of the central charge \( c \) specified by two relatively prime integers \( p \) and \( p' \) according to

\[
c = N - 1 - \frac{(p - p')^2}{pp'} N (N^2 - 1).
\] (28)

In the case of \( c = -2 \) \( \mathcal{W}_N \) minimal models exist for all \( N \left( p' = N - 1, \ p = N \right) \). Consider the conservation of the charge \( H_3 \). The commutator depends only on the simple pole term and gives

\[
\dot{H}_3 = \left[ \frac{1}{2\pi i} \oint dz (TT)(z), \frac{1}{2\pi i} \oint dy W(y) \right]
\]

\[
= \frac{1}{2\pi i} \oint dz \frac{1}{5} \left( -12\partial(TW)(z) - \partial^3 W(z) + \Xi(z) \right)
\] (29)

with \( \Xi(z) \equiv -8(T\partial W)(z) + 12(\partial TW)(z) + \partial^3 W(z) \). The total derivative terms as well as \( \Xi(z) \) vanish resulting in a conserved \( H_3 \). \( \Xi(z) \) vanishes since it is a null descendent of the Virasoro primary field \( W(z) \) at level 3. In other words, defining

\[
\hat{L}_{-n} \phi(w) = \frac{1}{2\pi i} \oint \frac{dz}{(z - w)^{n-1}} T(z) \phi(w)
\]

then \( \Xi(z) = (-8\hat{L}_{-2}\hat{L}_{-1} + 12\hat{L}_{-3} + \hat{L}_3^3)W(z) = 0 \). One may show that \( \Xi(z) \), and all other singular vectors, vanish identically in the fermion representation since all products of modes of fermion fields with equal number of modes of \( b(z) \) and \( c(z) \) may be expressed in terms of modes of \( T(z) \) and \( W(z) \). The singular vectors are then states in the nonsingular Fock space with zero inner product with all other states and thus are the unique state with zero norm in the fermion Fock space.

### III. REDUCTION OF \( \mathcal{W}_\infty/2 \)

The \( \mathcal{W}_\infty \) algebra contains an infinite-dimensional subalgebra \( \mathcal{W}_\infty/2 \) consisting of only the even spin currents of \( \mathcal{W}_\infty \). One may see from Eq. [17] that a different set of normal-ordered products of lower-dimensional currents is necessary for the reduction of \( \mathcal{W}_\infty/2 \), however the central charge is the same as in the \( \mathcal{W}_\infty \) reduction, namely \( c = -2 \). The general form of the reduction is

\[
V^{(2i)} = \beta_i \sum_{p=0}^{i-1} b_{p}^{(2i)} (T\partial^{2p} V^{(2i-2p-2)}) + \sum_{p=1}^{i} \gamma_{2p}^{(2i)} \partial^{2p} V^{(2i-2p)}.
\] (32)

Although there is no simple expression for the coefficients they may be found in the same manner as for the reduction of \( \mathcal{W}_\infty \). The coefficients \( b_{p}^{(2i)} \) are determined by the vanishing of quadratic fermion terms in the sum of normal-ordered products of currents. This leads to a linear system of equations.
\[ \sum_{p=0}^{i-1} b^{(2i)}_p \sum_{k=0}^{\text{min}(2m,l-1)} \binom{2m}{k} a_{l-k}(2i-2p-2) = 0, \quad l = 2, \ldots, i \]  

(33)

with \( a_j(i) \) given by Eq. [10]. This has a unique solution up to an overall multiplicative factor which may be absorbed in the coefficient \( \beta_i \). The remaining coefficients may then be found by considering the central term in the operator product expansion of two currents as explained in Ref. [3]. The reduction of \( W_{\infty/2} \) for the lowest spin currents is then

\[ V^{(2)} = \frac{2}{3} \left[ (TT) - \frac{3}{10} \partial^2 T \right], \]

(34)

\[ V^{(4)} = \frac{8}{9} \left[ (TV^{(2)}) + \frac{1}{5} (T^2 T) \right] - \frac{2}{9} \partial^2 V^{(2)} - \frac{32}{945} \partial^4 T, \]

\[ V^{(6)} = (TV^{(4)}) + \frac{10}{9} (T \partial^2 V^{(2)}) + \frac{4}{63} (T \partial^4 T) - \frac{9}{13} \partial^2 V^{(4)} - \frac{5}{33} \partial^4 V^{(2)} \]

- \frac{5}{567} \partial^6 T,

\[ V^{(8)} = \frac{16}{15} \left[ (TV^{(6)}) + \frac{35}{13} (T \partial^2 V^{(4)}) + \frac{70}{99} (T \partial^4 V^{(2)}) + \frac{2}{81} (T \partial^6 T) \right] \]

- \frac{148}{85} \partial^2 V^{(6)} - \frac{56}{117} \partial^4 V^{(4)} - \frac{4592}{57915} \partial^6 V^{(2)} - \frac{64}{22275} \partial^8 T,

\[ V^{(10)} = \frac{10}{9} \left[ (TV^{(8)}) + \frac{84}{17} (T \partial^2 V^{(6)}) + \frac{112}{39} (T \partial^4 V^{(4)}) + \frac{1568}{3861} (T \partial^6 V^{(2)}) \right] \]

+ \frac{16}{1485} (T \partial^8 T) - \frac{215}{63} \partial^2 V^{(8)} - \frac{4480}{2907} \partial^4 V^{(6)} - \frac{1960}{5967} \partial^6 V^{(4)}

- \frac{1360}{34749} \partial^8 V^{(2)} - \frac{112}{104247} \partial^{10} T.

Following the same procedure as in the preceding section, one then obtains the quantum commuting charges for the quantum KdV at \( c = -2 \). These are the charges found previously in Ref. [11], namely \( H_{2n-1} = \frac{1}{2m} \oint (\ldots ((TT)T) \ldots T)(z)dz \) with \( n \) factors of \( T \).

### IV. REDUCTION OF \( W_{(1+\infty)/2} \)

\( W_{1+\infty} \) is another infinite-dimensional Lie algebra generated by fields quasi-primary with respect to \( T \equiv V^{(0)} \) of spins \( i \geq 1 \) [11]. One finds a reduction of this algebra for \( c = \frac{1}{2} \). The commutation relations for the modes are given by Eq. [19] with structure constants

\[ C^{k}_{ij} = \sum_{r=0}^{(i+j-k+4)!} \frac{(\frac{1}{2})_r (\frac{1}{2})_r (-\frac{1}{2}) (\frac{1}{2})_r (\frac{1}{2})_r (-\frac{1}{2}) (\frac{1}{2})_r (\frac{1}{2})_r (\frac{1}{2})_r (\frac{1}{2})_r}{2(2k+3)!} \]

(35)

and central terms

\[ D_{ij} = \delta_{ij} \frac{(2q)^{2i}((i+1)!)^2(2i+2)!!}{4(2i+1)!!c}. \]

(36)

Since we would like to consider only currents of conformal dimension \( \Delta_i \geq 2 \) we restrict to the subalgebra \( W_{(1+\infty)/2} \) of even-dimensional currents. We find a representation of this subalgebra at central charge \( c = \frac{1}{2} \) by a real fermion \( \psi \) as
\[ V^{(2i)}(z) = \frac{2^{2i-1}(2i+1)!}{(4i+1)!!} q^{2i} \sum_{k=0}^{i} (-1)^k \binom{2i+1}{k} : \partial^{2i+1-k} \psi(z) \partial^k \psi(z) : \]  

(37)

with \( \psi(z) \psi(w) \sim 1/(z-w) \). Note that this is not the usual representation of \( W_{1+\infty} \) by complex fermions, which has central charge \( c = 1 \) \([12]\). Again, using the methods of Ref. \([6]\) and a reduction of the form in Eq. \([11]\) with a summation only over even-spin currents, we obtain the following reduction for the lowest spin currents:

\[ V^{(2)} = \frac{4}{7} \left[ (TT) - \frac{3}{10} \partial^2 T \right], \]

\[ V^{(4)} = \frac{400}{441} \left[ (TV^{(2)}) - \frac{1}{6} \partial^2 V^{(2)} \right], \]

\[ V^{(6)} = \frac{1323}{5743} \left[ (TV^{(4)}) + \frac{2500}{567} (V^{(2)2})(2) - \frac{2581}{702} \partial^2 V^{(4)} - \frac{2375}{8019} \partial^4 V^{(2)} \right. \]

\[ \left. - \frac{50}{15309} \partial^6 T \right], \]

\[ V^{(8)} = \frac{3744}{87545} \left[ (TV^{(6)}) + \frac{5145}{143} (V^{(2)2} V^{(4)}) - \frac{54237}{2431} \partial^2 V^{(6)} - \frac{117649}{66924} \partial^4 V^{(4)} \right. \]

\[ \left. - \frac{85750}{4969107} \partial^6 V^{(2)} \right]. \]

The quantum commuting charges for \( c = \frac{1}{2} \) quantum KdV may be found from this reduction using the same method as for the reduction of the \( W_\infty \) algebra. The corresponding charges are:

\[ H_3 = \frac{1}{2\pi i} \oint (TT)(z)dz \]

\[ H_5 = \frac{1}{2\pi i} \oint \left[ (T(TT))(z) + \frac{3}{10} (\partial T \partial T)(z) \right] dz \]

\[ H_7 = \frac{1}{2\pi i} \oint \left[ (T(T(TT)))(z) + \frac{77}{85} \partial^2 T (TT)(z) + \frac{809}{4420} (\partial^2 T \partial^2 T)(z) \right] dz \]

\[ H_9 = \frac{1}{2\pi i} \oint \left[ \left( \frac{1089043}{897915} \partial^3 T (TT)(z) - \frac{998002}{179583} (\partial T (TT))(z) \right) + \frac{22912073}{86199840} (\partial^4 T(TT))(z) + \frac{1414087427}{244232880} (\partial^2 T (\partial T \partial T))(z) + \frac{57504805453}{571504939200} (\partial^3 T \partial^3 T)(z) \right] dz. \]

V. COMMUTING CHARGES FOR A\(_1\) AND A\(_2\) TODA FIELD THEORIES

The quantum commuting charges found above may be used to construct charges for \( A_1 \) and \( A_2 \) Toda field theories via the quantum Miura transformation. However these charges only commute modulo singular vectors of the corresponding Virasoro or \( W_3 \) algebras, respectively. The holomorphic part of the imaginary coupling \( A_N \) Toda field theory interaction Hamiltonian, in light-cone coordinates, is

\[ V_0 \equiv \sum_{i=1}^{N} \frac{1}{2\pi i} \int dz : \exp (-i\beta \alpha_i \cdot \phi(z)) :. \]  

(40)
expressions for commuting quantum charges of the quantum Boussinesq and quantum KdV primary fields formed from modes of commutators for the unique spin 4 field, one finds that there is only one algebra, the conformal field theory with central charge $c = N \left[ 1 - (N + 1)(N + 2)(2\beta - 1/\beta)^2 \right]$ \[15\].

The algebra of the quantum charges for $A_N$ Toda field theory is the $W_{N+1}$ algebra whose currents are found from the quantum Miura transformation. This has the following generating expression:

$$- \sum_{k=0}^{N+1} U_k(z) (\gamma_0 \partial)^{N+1-k} = ((\gamma_0 \partial - \epsilon_1 \cdot i \partial \phi(z)) \cdots (\gamma_0 \partial - \epsilon_{N+1} \cdot i \partial \phi(z)))$$ \[41\]

where $(\epsilon_i, i = 1, \ldots, N + 1)$ are the weights of the vector representation of $A_N$ with $\epsilon_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, N, \epsilon_i \cdot \epsilon_j = \delta_{ij} - 1/(N+1)$, and $\sum \epsilon = 0$ \[14\]. $\gamma_0$ is a constant related to the central charge by $c = N(1 - (N + 1)(N + 2)\gamma_0^2)$. Then $T = U_2, W = U_3 - \frac{1}{2}(N-1)\gamma_0 \partial U_2$, and the higher spin currents of the $W_N$ algebra are found by projecting to Virasoro quasi-primary combinations of the $U_i$ \[13\]. The $U_i(z)$, and hence any products of modes of them, commute with each term in the Toda Hamiltonian Eq. \[40\]. These terms in the potential are generalization of the screening currents of the Feigin-Fuchs formulation of minimal conformal models, i.e., they have conformal dimension one.

Consider, more specifically, $A_2$ Toda field theory. Any sum of normal-ordered products of $T$ and $W$ and their derivatives commute with the Hamiltonian $V_0$. The difficulty is to find a commuting set. Charges which commute, modulo singular vectors of the $W_3$ representation, are provided by Eq. \[24\] for $c = -2$. Unlike in the fermion representation of Eq. \[8\] the singular vectors do not vanish identically in the boson representation. The boson operators corresponding to these singular vectors should be set equal to zero, which results in an irreducible representation of the $W_3$ algebra. Thus, one obtains an infinite set of charges which commute modulo singular vectors for $A_1$ Toda field theory with couplings $\beta_+ = (\sqrt{2}/8)(1 + \sqrt{17})$ or $\beta_+ = (\sqrt{3}/24)(1 + \sqrt{97})$ and for the $A_2$ theory with coupling $\beta_+ = \sqrt{2}/3$ using the preceding reductions. Because of a coupling duality, these are also charges for the corresponding Toda field theories with coupling $\beta_- = -1/(2\beta_+)$. 

VI. CONCLUSION

We have presented a single reduction of each of $W_\infty, W_\infty/2$, and $W_{(1+\infty)/2}$ for given values of the central charge. One may show that these are the only reductions of these algebras. This is demonstrated by considering the general form of the reduction of the spin 4 current as $V^{(2)} = b_1(TT) + \gamma_2 \partial^2 T$. Then by requiring that $\hat{L}_1 V^{(2)} = 0$ and that $D_{02}, D_{22}, C_{02}^0,$ and $C_{22}^2$ match those for the appropriate algebra, one finds only the single solution, stated in the previous sections, for each algebra.

Furthermore, one may also try to find other infinite-dimensional Lie algebras of quasi-primary fields formed from modes of $W_3$ fields $T$ and $W$. However, by examining the commutators for the unique spin 4 field, one finds that there is only one algebra, the $W_\infty$ algebra with $c = -2$, with exactly one quasi-primary field for each conformal dimension $\Delta_i \geq 2$ generated by modes of $T$ and $W$.

There are several interesting questions to investigate further. Since we have explicit expressions for commuting quantum charges of the quantum Boussinesq and quantum KdV
theories, one may be able to find the common eigenvalues and eigenstates of these charges. In addition, it is not known whether there are similar infinite-dimensional Lie algebras of Virasoro quasi-primary fields, which likewise give the commuting quantum charges, for theories whose fields are currents of other finite-dimensional $W$ algebras. Finally, since $W$ algebras for certain values of the central charge may be represented via the Kac-Moody algebra coset construction, one may find that the Abelian subalgebra of the quantum charges has a simpler form in that case [10].

ACKNOWLEDGEMENTS

We thank Ryu Sasaki for many helpful discussions. The author was supported by the National Science Foundation under grant no. 9703595 and the Japan Society for the Promotion of Science.
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