A Stochastic Game Approach to Masking Fault-Tolerance: Bisimulation and Quantification

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Abstract. We introduce a formal notion of masking fault-tolerance between probabilistic transition systems based on a variant of probabilistic bisimulation (named masking simulation). We also provide the corresponding probabilistic game characterization. Even though these games could be infinite, we propose a symbolic way of representing them, such that it can be decided in polynomial time if there is a masking simulation between two probabilistic transition systems. We use this notion of masking to quantify the level of masking fault-tolerance exhibited by almost-sure failing systems, i.e., those systems that eventually fail with probability 1. The level of masking fault-tolerance of almost-sure failing systems can be calculated by solving a collection of functional equations. We produce this metric in a setting in which the minimizing player behaves in a strong fair way (mimicking the idea of fair environments), and limit our study to memoryless strategies due to the infinite nature of the game. We implemented these ideas in a prototype tool, and performed an experimental evaluation.

1 Introduction

Fault-tolerance is one important characteristic of modern software. This is particularly true for critical software like banking software, automotive applications, communication protocols, avionics software, etc. However, in practice, it is hard to quantify the level of fault-tolerance provided by computing systems. In most cases fault-tolerant systems are built using ad-hoc techniques which are based on experience and, many times, lack a mathematical foundation. Furthermore, faults usually have a probabilistic nature. Therefore, concepts coming from probability theory become necessary when developing fault-tolerant software. In this paper we provide a framework aimed at analysing the fault-tolerance exhibited by concurrent probabilistic systems. This encompasses the probability of occurrence of faults as well as the use of randomized algorithms for improving the fault-tolerance of systems.

In practice, there are different types of fault-tolerance, \textit{masking fault-tolerance} (when both the safety and liveness properties are preserved under the occurrence
of faults), non-masking fault-tolerance (when only liveness properties are preserved) and failsafe fault-tolerance (when only safety properties are preserved). Among them, masking fault-tolerance is often acknowledged as the most desirable kind of fault-tolerance, because all the properties of the nominal (i.e., non-faulty) system are preserved under faulty behavior. However, in many settings, requiring full masking fault-tolerance is unrealistic. In particular, for those systems that are not designed to terminate and the degradation of hardware, or software, components will eventually lead to a failure (i.e., a behavior that deviates from the expected system’s behavior). One of the main applications of the framework described in the forthcoming sections is the quantification of the amount of masking fault-tolerance provided by systems before they enter into a failure. This measure provides a tool for selecting a fault tolerance mechanism, or for balancing multiple mechanisms (e.g., to which extent it is worth the cost of efficient hardware redundancy against time demanding software artifacts).

During the last decade, significant progress has been made towards defining suitable metrics or distances for diverse types of quantitative models including real-time systems [18], probabilistic models [23][15][30], and metrics for linear and branching systems [8][13][17][26][31]. Some authors have already pointed out that these metrics can be useful to reason about the robustness and correctness of a system, notions related to fault-tolerance. In [6], we presented a notion of masking fault-tolerance between systems built on a simulation relation and a corresponding game representation with quantitative objectives. In this paper we review these ideas in a probabilistic setting, and define a probabilistic version of this characterization of masking fault-tolerance.

More specifically, we start characterizing probabilistic masking fault-tolerance via a variant of probabilistic bisimulation. This masking simulation relates two probabilistic transition systems. The first one acts as a specification of the intended behavior (i.e., nominal model) and the second one as the fault-tolerant implementation (i.e., the extended model with faulty behavior). The existence of a masking simulation implies that the implementation masks the faults. This simulation relation can be captured as a stochastic game played by a Verifier and a Refuter. If the Verifier wins, there is a probabilistic masking simulation. If instead the Refuter wins, the implementation is not masking fault-tolerant. These games rely on the notion of couplings between probabilistic distributions and, as a consequence, the numbers of vertices of the game graphs is infinite. To tackle this problem, we introduce a symbolic representation of these games where the couplings are symbolically captured by means of equation systems. The size of this symbolic graph is polynomial on the size of the input systems. Moreover, the simulation games can be solved via their symbolic representation.

In practice masking fault-tolerance comes in a quantitative fashion, and thus we enrich the games with quantitative objectives. This makes it possible to quantify the amount of masking tolerance provided by the implementations. We focus on games that almost-surely fail when the Refuter plays fairly (i.e., those systems that eventually fail with probability 1). Due to the infinite nature of the games, we restrict this result to randomized memoryless strategies and show
that the game is determined under these conditions. We show that the problem of deciding if the game is almost surely failing under fairness is polynomial. Moreover, the value of the game can be computed by solving a collection of functional equations via a Value Iteration algorithm \cite{9111221}. We take such a value as the measure of fault-tolerance.

Summarizing our contribution, (1) we define a notion of probabilistic masking simulation, (2) provide a game characterization for it, and (3) show that it can be decided in polynomial time (Sec. 3). Moreover, in Sec 4 (4) we define an extension of the game with rewards and provide a payoff function that counts the number of “milestones” achieved by the implementation; (5) we show that the game is determined provided it is almost-surely failing under fairness and memoryless strategies, and (6) provide an algorithm to calculate it. In addition, (7) we provide a polynomial time algorithm to decide if a game is almost-surely failing under fairness. (8) We finally present an experimental evaluation on some well-known case studies (Sec. 5). Full details and proofs can be found in the Appendix.

2 Preliminaries

We first introduce some basic definitions that will be necessary across the paper.

A (discrete) probability distribution \( \mu \) over a denumerable set \( S \) is a function \( \mu : S \to [0,1] \) such that \( \mu(S) \triangleq \sum_{s \in S} \mu(s) = 1 \). Let \( \mathcal{D}(S) \) denote the set of all probability distributions on \( S \). \( \Delta_s \in \mathcal{D}(S) \) denotes the Dirac distribution for \( s \), i.e., \( \Delta_s(s) = 1 \) and \( \Delta_s(s') = 0 \) whenever \( s' \neq s \). The support set of \( \mu \) is defined by \( \text{Supp}(\mu) = \{ s | \mu(s) > 0 \} \).

A Probabilistic Transition System (PTS) is a structure \( A = \langle S, \Sigma, \rightarrow, s_0 \rangle \) where (i) \( S \) is a denumerable set of states containing the initial state \( s_0 \in S \), (ii) \( \Sigma \) is a set of actions, and (iii) \( \rightarrow \subseteq S \times \Sigma \times \mathcal{D}(S) \) is the (probabilistic) transition relation. We assume that there is always some transition leaving from every state.

A distribution \( w \in \mathcal{D}(S \times S') \) is a coupling for \( (\mu, \mu') \), with \( \mu \in \mathcal{D}(S) \) and \( \mu' \in \mathcal{D}(S') \), if \( w(s, \cdot) = \mu' \) and \( w(\cdot, S') = \mu \). \( \mathcal{C}(\mu, \mu') \) denotes the set of all couplings for \( (\mu, \mu') \). It is worth noting that this defines a polytope. \( \mathcal{V}(\mathcal{C}(\mu, \mu')) \) denotes the set of all vertices of the corresponding polytope. This set is finite if \( S \) and \( S' \) are finite. For \( R \subseteq S \times S' \), we say that a coupling \( w \) for \( (\mu, \mu') \) respects \( R \) if \( \text{Supp}(w) \subseteq R \) (i.e., \( w(s, s') > 0 \Rightarrow s R s' \)). We define \( R^\# \subseteq \mathcal{D}(S) \times \mathcal{D}(S') \) by \( \mu R^\# \mu' \) if and only if there is an \( R \)-respecting coupling for \( (\mu, \mu') \).

A stochastic game graph \cite{10} is a tuple \( \mathcal{G} = \langle V, E, V_1, V_2, V_R, v_0, \delta \rangle \), where \( V \) is a set of vertices with \( V_1, V_2, V_R \subseteq V \) being a partition of \( V \), \( v_0 \in V \) is the initial vertex, \( E \subseteq V \times V \), and \( \delta : V_R \to \mathcal{D}(V) \) is a probabilistic transition function such that, for all \( v \in V_R \) and \( v' \in V \): \( (v, v') \in E \) iff \( v' \in \text{Supp}(\delta(v)) \). If \( V_R = \emptyset \), then \( \mathcal{G} \) is called a 2-player game graph. Moreover, if \( V_1 = \emptyset \) or \( V_2 = \emptyset \), then \( \mathcal{G} \) is a Markov Decision Process (or MDP). Finally, in case that \( V_1 = \emptyset \) and \( V_2 = \emptyset \), \( \mathcal{G} \) is a Markov chain (or MC). For all states \( v \in V \) we define \( \text{Post}(v) = \{ v' \in V | (v, v') \in E \} \), the set of successors of \( v \). Similarly,
$Pre(v') = \{v \in V \mid (v, v') \in E\}$ as the set of predecessors of $v'$. We assume that $Post(s) \neq \emptyset$ for every $v \in V_1 \cup V_2$.

Given a game as defined above, a play is an infinite sequence $\omega = \omega_0, \omega_1, \ldots$ such that $(\omega_k, \omega_{k+1}) \in E$ for every $k \in \mathbb{N}$. The set of all plays is denoted by $\Omega$, and the set of plays starting at vertex $v$ is written $\Omega_v$. A strategy (or policy) for Player 1 is a function $\pi_1 : V^* \times V_1 \to D(V)$ that assigns a probabilistic distribution to each finite sequence of states such that $\text{Supp}(\pi_1(w \cdot v)) \subseteq Post(v)$ for all $w \in V^*$ and $v \in V_1$. The set of all the strategies for Player 1 is named $\Pi_1$. A strategy $\pi_1$ is said to be pure if, for every $w \in V^*$ and $v \in V_1$, $\pi_1(w \cdot v)$ is a Dirac distribution, and it is called memoryless if $\pi_1(w \cdot v) = \pi_1(v)$, for every $w \in V^*$ and $v \in V_1$. Given two strategies $\pi_1 \in \Pi_1$, $\pi_2 \in \Pi_2$ and a starting state $v$, the result of the game is a Markov chain, denoted by $G^p_{\pi_1, \pi_2}$. As any Markov chain, $G^p_{\pi_1, \pi_2}$ defines a probability measure $\text{Prob}_{\pi_1, \pi_2}$. An event $A$ is a measurable set in the Borel $\sigma$-algebra generated by the cones of $\Omega$. Thus, $\text{Prob}_{\pi_1, \pi_2}(A)$ is the probability that strategies $\pi_1$ and $\pi_2$ generate a play belonging to $A$ from state $v$. It would normally be convenient to use LTL notation to define events. For instance, $\Diamond v' = \{\omega = \omega_0, \omega_1, \ldots \in \Omega \mid \exists i : \omega_i \in V'\}$ defines the event in which some state in $V'$ is reached. The outcome of the game, denoted by $\text{outcome}(\pi_1, \pi_2)$, is the set of possible paths of $G^p_{\pi_1, \pi_2}$ starting at vertex $v$ (i.e., the possible plays when strategies $\pi_1$ and $\pi_2$ are used). When the initial state $v$ is fixed, we write $\text{outcome}(\pi_1, \pi_2)$ instead of $\text{outcome}(\pi_1, \pi_2)$. A Boolean objective for $G$ is a set $\Phi \subseteq \Omega$, we say that a play $\omega$ is winning for Player 1 at vertex $v$ if $\omega \in \Phi$, otherwise we say that it is winning for Player 2 (i.e., we consider zero-sum games). A strategy $\pi_1$ is a sure winning strategy for Player 1 from vertex $v$ if, for every strategy $\pi_2$ for Player 2, $\text{outcome}(\pi_1, \pi_2) \subseteq \Phi$. $\pi_1$ is said to be almost-sure winning if for every strategy $\pi_2$ for Player 2, we have $\text{Prob}_{\pi_1, \pi_2}(\Phi) = 1$. Sure and almost-sure winning strategies for Player 2 are defined in a similarly. Reachability games are games with Boolean objectives of the style: $\Diamond v'$, for some set $V' \subseteq V$. A standard result is that, if a reachability game has a sure winning strategy, then it has a pure memoryless sure winning strategy [10].

A quantitative objective is a measurable function $f : \Omega \to \mathbb{R}$. Given a measurable function we define $E_{G, v}^{\pi_1, \pi_2}[f]$ as the expectation of function $f$ under probability $\text{Prob}_{G, v}^{\pi_1, \pi_2}$. The goal of Player 1 is to maximize the value $f$ of the play, whereas the goal of Player 2 is to minimize it. Sometimes quantitative objective functions can be defined via rewards. These are assigned by a reward function $r : V \to \mathbb{R}$. A stochastic game with rewards is a structure $(V, E, V_1, V_2, V_0, \delta, r)$ composed of a stochastic game and a reward function. The value of the game for Player 1 under strategy $\pi_1$ at vertex $v$, denoted $\text{val}_1(\pi_1)(v)$, is defined as the infimum over all the values resulting from Player 2 strategies when the game starts at $v$, i.e., $\text{val}_1(\pi_1)(v) = \inf_{\pi_2 \in \Pi_2} E_{G, v}^{\pi_1, \pi_2}[f]$. The value of the game for Player 1 from vertex $v$ is defined as the supremum of the values of all Player 1 strategies, i.e., $\sup_{\pi_1 \in \Pi_1} \text{val}_1(\pi_1)(v)$. Analogously, the value of the game for a Player 2 strategy $\pi_2$ and the value of the game for Player 2 are defined as $\text{val}_2(\pi_2)(v) = \sup_{\pi_2 \in \Pi_2} E_{G, v}^{\pi_1, \pi_2}[f]$ and $\inf_{\pi_2 \in \Pi_2} \text{val}_2(\pi_2)(v)$, re-
spectively. We say that a game is determined if both values are equal, that is,
\[ \sup_{\pi_1 \in \Pi_1} \mathrm{val}(\pi_1)(v) = \inf_{\pi_2 \in \Pi_2} \mathrm{val}(\pi_2)(v), \] for every vertex \( v \).

3 Probabilistic Masking Simulation

In this section we introduce probabilistic masking simulation which is a proba-
ibilistic extension of the masking simulation relation introduced in [6]. We a lso
give a symbolic version of the stochastic game characterization of the relation,
and provide an algorithm to solve it.

3.1 The relation.

Roughly speaking, a probabilistic masking simulation is a relation between PTSs
that extends probabilistic bisimulation [19] in order to account for fault masking.
Intuitively, one of the PTSs acts as the nominal model (i.e., the specification),
while the other one models the implementation of the system under faults. The
nominal model describes the ideal behavior of the system (i.e., when no faults
are considered), while the implementation describes a fault-tolerant version of
the system, where the occurrence of faults are taken into account and a fault
tolerance mechanism is expected to act upon them. Probabilistic masking sim-
ulation allows one to analyze whether the implementation is able to mask the
faults while preserving the behavior of the nominal model. More specifically, for
non-faulty transitions, the relation behaves as probabilistic bisimulation, which
is captured by means of couplings and relations respecting these couplings (as
done for instance in [19]). The novel part is given by the occurrence of faults: if
the implementation performs a fault, the nominal model matches it by doing
nothing.

For a set of actions \( \Sigma \), and a (finite) set of
fault labels \( F \), with \( F \cap \Sigma = \emptyset \), we
define \( \Sigma_F = \Sigma \cup F \). Intuitively, the elements of \( F \) indicate the occurrence of a
fault in a faulty implementation. Furthermore, when useful we consider the set
\( \Sigma_i = \{ e^i | e \in \Sigma \} \), containing the elements of \( \Sigma \) indexed with superscript \( i \).

Definition 1. Let \( A = (S, \Sigma, \rightarrow, s_0) \) and \( A' = (S', \Sigma_F, \rightarrow', s'_0) \) be two PTSs rep-
resenting the nominal and the implementation model, respectively. \( A' \) is strong
probabilistic masking fault-tolerant with respect to \( A \) if there exists a relation
\( M \subseteq S \times S' \) such that: (a) \( s_0 M s'_0 \), and (b) for all \( s, s' \in S' \) with \( s M s' \) and all \( e \in \Sigma \) and \( F \in \mathcal{F} \) the following holds:

1. if \( s \xrightarrow{e} \mu \), then \( s' \xrightarrow{e} \mu' \) and \( \mu M^{\#} \mu' \) for some \( \mu' \);
2. if \( s' \xrightarrow{e} \mu' \), then \( s \xrightarrow{e} \mu \) and \( \mu M^{\#} \mu' \) for some \( \mu \);
3. if \( s' \xrightarrow{F} \mu' \), then \( \Delta_s M^{\#} \mu' \).

If such relation exists we say that \( A' \) is a strong probabilistic masking fault-
tolerant implementation of \( A \), denoted \( A \preceq_m A' \).
module NOMINAL
b : [0..1] init 0; // 0 = normal, 1 = refreshing
m : [0..1] init 0; // 0 = normal, 1 = refreshing

[w0] (m=0) -> (b'= 0);
[w1] (m=0) -> (b'= 1);
[r0] (m=0) & (b=0) -> true;
[r1] (m=0) & (b=1) -> true;
[tick] (m=0) -> p : (m'= 1) + (1-p) : true;
[rfsh] (m=1) -> (m'= 0);
endmodule

module FAULTY
v : [0..3] init 0;
s : [0..2] init 0; // 0 = normal, 1 = faulty, 2 = refreshing
f : [0..1] init 0; // fault limiting artifact

[w0] (s!=2) -> (v'= 0) & (s'= 0);
[w1] (s!=2) -> (v'= 3) & (s'= 0);
[r0] (s!=2) & (v<=1) -> true;
[r1] (s!=2) & (v>=2) -> true;
[tick] (s!=2) -> p : (s'= 2) + q : (s'= 1) + (1-p-q) : true;
[rfsh] (s=2) -> (s'=0)
& (v'= (v<=1) ? 0 : 3);
[fault] (s=1) & (f<1) -> (v'= (v<3) ? (v+1) : 2)
& (s'= 0) & (f'= f+1);
[fault] (s=1) & (f<1) -> (v'= (v>0) ? (v-1) : 1)
& (s'= 0) & (f'= f+1);
endmodule

Fig. 1. Nominal and fault-tolerant models for the memory cell.

Example 1. Consider a memory cell storing one bit of information that periodically refreshes its value. The memory supports writing and reading operations, whereas a refresh performs a read operation and overwrites the value with itself. Obviously, in this system, the result of a reading depends on the value stored in the cell. Thus, a property associated with the system is that the value read from the cell coincides with that of the last performed writing. This is captured by the nominal model given at the left of Figure 1 in PRISM notation [24]. Actions $r_i$ and $w_i$ (for $i = 0, 1$) represent the actions of reading or writing value $i$. The bit stored in the memory is saved in variable $b$. A tick action indicates the passing of one time unit and in doing so, with probability $p$, it enables the refresh action ($rfsh$). Variable $m$ indicates whether the system is in write/read mode, or producing a refresh.

A potential fault in this scenario occurs when a cell unexpectedly changes its value (e.g., as a consequence of some electromagnetic interference). In practice, the occurrence of such an error has a certain probability. A typical technique to deal with this situation is redundancy; for instance, using three memory bits instead of one. Then, writing operations are performed simultaneously on the three bits while reading returns the value read by majority (i.e., by voting). The right hand-side model of Figure 1 represents this implementation with the occurrence of the fault implicitly modeled (ignore, by the time being the red part). Variable $v$ counts the votes for the value 1. Thus writing 1 ($w_1$) sets $v$ in 3, and writing 0 ($w_0$) sets it in 0. The read actions would return 1 ($r_1$) if $v \geq 2$ and 0 ($r_0$) otherwise. In addition to enabling the refresh action, a tick may also enable the occurrence of a fault with probability $q$, with the restriction that $p + q \leq 1$. Variable $s$ indicates that the system is in normal mode ($s = 0$), in a state where a fault may occur ($s = 1$), or producing a refresh ($s = 2$). Notice that reading and writing are allowed as long as the system is not producing a refresh. The red coloured text of the figure is an artifact to limit the number of faults to 1. Under this condition, it is easy to check that the relation $M = \{((b,m),(v,s,f)) \mid 2b \leq v \leq 2b+1 \land (m = 1 \leftrightarrow s = 2)\}$ is a probabilistic
The 2-players stochastic masking game graph is defined as follows:

The game is similar to a bisimulation game [29], and it is played by two players, $G_1$ and $G_2$, to formalize this game. Thus, the probabilistic step of the fault can only be matched by a transition $s \xrightarrow{a} \mu$ from the nominal model. Therefore, the Refuter's choice $a$ from the opposite model, and the only possible transition $s' \xrightarrow{a} \mu'$ from the implementation. In addition, $V$ chooses a coupling $w$ for $(\mu, \mu')$.

If the play continues forever, then the Verifier wins; otherwise, the Refuter wins.

1) $R$ chooses either a transition $s \xrightarrow{a} \mu$ from the nominal model or a transition $s' \xrightarrow{a} \mu'$ from the implementation.
2.a) If $a \notin F$, $V$ chooses a transition matching action from the opposite model, i.e., a transition $s' \xrightarrow{a} \mu'$ if $R$'s choice was from the nominal, or a transition $s \xrightarrow{a} \mu$ otherwise. In addition, $V$ chooses a coupling $w$ for $(\mu, \mu')$.
2.b) If $a \in F$, $V$ can only select the Dirac distribution $\Delta_r$ and the only possible coupling $w$ for $(\Delta_r, \mu')$.
3) The successor pair of states $(t, t')$ is chosen probabilistically according to $w$.

If the play continues forever, then the Verifier wins; otherwise, the Refuter wins. Step 2.b is the only one that differs from the usual bisimulation game. It is necessary for the asymmetry produced by transitions that represent the occurrence of faults: if the Refuter chooses to play a fault in the implementation, then it needs to be masked, and therefore the Verifier cannot produce any move in the nominal model. Thus, the probabilistic step of the fault can only be matched by a Dirac distribution on the same state of the nominal model.

In the following, we define a stochastic masking game graph that allows us to formalize this game.

**Definition 2.** Let $A = (S, \Sigma, \rightarrow, s_0)$ and $A' = (S', \Sigma', \rightarrow', s'_0)$ be two PTMs. The 2-players stochastic masking game graph $G_{A,A'} = (V^G, E^G, V_R^G, V_V^G, V_P^G, v_0^G, \delta^G)$, is defined as follows:

$$V^G = V_R^G \cup V_V^G \cup V_P^G,$$

where:

$$V_R^G = \{(s,\epsilon, s',\epsilon, \epsilon, \epsilon, R) | s \in S \land s' \in S'\} \cup \{v_{err}\}$$

$$V_V^G = \{(s,\sigma^1, s',\epsilon, \mu, \epsilon, \epsilon, V) | s \in S \land s' \in S' \land \sigma \in \Sigma \land (s,\sigma,\epsilon, \mu, \epsilon) \rightarrow \} \cup$$

$$\{(s,\sigma^2, s',\epsilon, \epsilon, \mu', \epsilon, V) | s \in S \land s' \in S' \land \sigma \in \Sigma \land (s',\sigma,\mu') \rightarrow \}$$

$$V_P^G = \{(s,\epsilon, s',\mu, \mu', w, P) | s \in S \land s' \in S' \land \mu \in D(s) \land \mu' \in D(s') \land w \in C(\mu, \mu')\}$$

$$v_0^G = (s_0,\epsilon, s'_0,\epsilon, \epsilon, \epsilon, R) \quad \text{(the Refuter starts playing)}$$

$$\delta^G : V_P^G \rightarrow D(V_R^G), \text{ defined by } \delta^G((s,\epsilon, s',\mu, \mu', w, P))((t,\epsilon, t',\epsilon, \epsilon, \epsilon, R)) = w(t, t').$$
and \( E^G \) is the minimal set satisfying the following rules:
\[
\begin{align*}
  &s \xrightarrow{\sigma} \mu \Rightarrow \langle (s, -, s', -), (s, \sigma^1, s', \mu, -, V) \rangle \in E^G \\
  &s' \xrightarrow{\sigma'} \mu' \Rightarrow \langle (s, -, s', -), (s, \sigma^2, s', \mu', -, V) \rangle \in E^G \\
  &s' \xrightarrow{\sigma'} \mu' \land w \in C(\mu, \mu') \Rightarrow \langle (s, \sigma^1, s', \mu, -, V), (s, \sigma^2, s', \mu', w, P) \rangle \in E^G \\
  &\sigma \notin \mathcal{F} \land s \xrightarrow{\sigma} \mu \land w \in C(\mu, \mu') \Rightarrow \langle (s, \sigma^2, s', \mu', -, V), (s, \sigma^1, s', \mu, w, P) \rangle \in E^G \\
  &F \in \mathcal{F} \land w \in \mathcal{C}(\Delta_s, \mu') \Rightarrow \langle (s, F^2, s', \mu', -, V), (s, \sigma^1, s', \mu, w, P) \rangle \in E^G \\
  &(s, -, s', \mu, \mu', w, P) \in V^G \land (t, t') \in \text{Supp}(w) \Rightarrow \langle (s, -, s', \mu, \mu', w, P), (t, -, t', -, -) \rangle \in E^G \\
  &v \in (V^G \cup \{v_{\text{err}}\}) \land (\not\exists v' \neq v_{\text{err}} : \langle v, v' \rangle \in E^G) \Rightarrow \langle v, v_{\text{err}} \rangle \in E^G
\end{align*}
\]

The definition follows the idea of the game previously described. A round of the game starts in the Refuter’s state \( v_0^G \). Notice that, at this point, only the current states of the nominal and implementation models are relevant (all other information is not yet defined in this round and hence marked with “−”). Step 1 of the game is encoded in rules (1) and (2), where the Refuter chooses a transition, thus defining the action and distribution that need to be matched, and moving to a Verifier’s state. Thus, a Verifier’s state in \( V^G \) is a tuple that has also defined which action and distribution need to be matched and which model the Refuter has moved. Step 2.a of the game is given by rules (2.a) and (2.a.1) in which the Verifier chooses a matching move from the opposite model (hence defining the other distribution) and an appropriate coupling, and moving to a probabilistic state. Step 2.b of the game is encoded in rule (2.b). Here the Verifier has no choice since it is obliged to choose the Dirac distribution \( \Delta_s \) and the only available coupling in \( C(\Delta_s, \mu') \). A probabilistic state in \( V^G \) has everything defined to probabilistically resolve the next step through function \( \delta^G \) (rule (3)). Finally, if a player has no move, then it can only move to the error state \( v_{\text{err}} \) (rule (err)). This can only happen in a Verifier’s state or in \( v_{\text{err}} \).

The notion of probabilistic masking simulation can be captured by the corresponding stochastic masking game with the appropriate Boolean objective.

**Theorem 1.** Let \( A = (S, \Sigma, \rightarrow, s_0) \) and \( A' = (S', \Sigma', \rightarrow', s'_0) \) be two PTSs. We have \( A \preceq_m A' \) iff the Verifier has a sure winning strategy for the stochastic masking game graph \( G_{A,A'} \) with the Boolean objective \( \neg \Diamond v_{\text{err}} \).

Notice that the graph for a stochastic masking game could be infinite. Indeed, each probabilistic node of the graph includes a coupling between the two contending distributions, and there can be uncountably many of them. It nonetheless
induces an algorithm as follows. We define regions $W^i$ of the graph vertices. Intuitively, each $W^i$ represents a collection of vertices from which the Refuter has a strategy (in the infinite game) with probability greater than 0 of reaching the error state in at most $i$ steps (these sets can be thought of as a probabilistic version of attractors [20]).

**Definition 3.** Let $G_{A,A'} = (V^G, E^G, V^G_R, V^G_V, V^G_P, v_0^G, \delta^G)$ be a stochastic masking game graph for PTSs $A$ and $A'$. We define sets $W^i$ (for $i \geq 0$) as follows:

\[ W^0 = \{ v_{err} \}, \]

\[ W^{i+1} = \{ v' \mid v' \in V^G_R \land Post(v') \cap W^i \neq \emptyset \} \cup \]

\[ \{ v' \mid v' \in V^G_V \land \forall (Post(v')) \subseteq \bigcup_{j \leq i} W^j \land \forall (Post(v')) \cap W^i \neq \emptyset \} \cup \]

\[ \{ v' \mid v' \in V^G_P \land \sum_{v'' \in Post(v')} \delta^G(v')(v'') > 0 \} \]

where $\forall (V') = \{(s,-,s',\mu,\mu',w,P) \in V' \cap V^G_P \mid w \in \forall{(C(\mu,\mu'))}\}$. Finally, let $W = \bigcup_{i \geq 0} W^i$.

The sets $W^i$ can be used to solve the game $G_{A,A'}$. Notice, in particular, that we do not take into account all possible couplings but only those that are vertices of the polytope $C(v[3], v[4])$. ($(x_0, \ldots, x_n)[i]$ is the $(i+1)$-th projection, i.e., $x_i$.) This is sufficient to determine the winner of the game since every coupling in $C(v[3], v[4])$ can be expressed as a convex combination of its vertices. Thus, if there is a positive probability of reaching the error state with some coupling, there is also a positive probability of reaching it through a vertex coupling. By taking only the vertex couplings, only a finite number of graph vertices are collected in each $W_i$ and hence $W$ can be effectively computed with a fix point algorithm.

The following result is a straightforward adaptation of the results for reachability games over finite graphs [10].

**Theorem 2.** Let $G_{A,A'} = (V^G, E^G, V^G_R, V^G_V, V^G_P, v_0^G, \delta^G)$ be a stochastic masking game graph for PTSs $A$ and $A'$. Then, the Verifier has a sure winning strategy from vertex $v$ iff $v \notin W$.

It is worth noting that, for a probabilistic vertex $(s,-,s',\mu,\mu',w,P) \in V^G_P$, the two-way transportation polytope $C(\mu,\mu')$ has at least $\frac{(\max\{m,n\})!}{(\max\{m,n\} - \min\{m,n\} + 1)!}$ vertices (and at most $m^{n-1}n^{m-1}$ vertices) [22], where $m = \text{Supp}(\mu)$ and $n = \text{Supp}(\mu')$. Therefore, it could be computationally impractical to calculate such sets.

### 3.3 A Symbolic Game Graph.

In this section, we introduce a finite representation of stochastic masking games through a symbolic representation which enables a more efficient algorithm. We define the symbolic graph for a stochastic masking game in two parts. The first
part captures the non-stochastic behaviour of the game by removing the stochastic choice \( (\delta^p) \) of the game graph, as well as the couplings on the vertices. The second part adds an equation system to each probabilistic vertex whose solution space is the polytope defined by the set of all couplings for the contending distributions.

**Definition 4.** Let \( A = (S, \Sigma, \rightarrow, s_0) \) and \( A' = (S', \Sigma, \rightarrow', s'_0) \) be two PTSs. The symbolic game graph for the stochastic masking game \( G_{A,A'} \) is defined by the structure \( SG_{A,A'} = (V_{SG}, E_{SG}, V_{SG}^p, V_{SG}^V, v_{SG}^0) \), where:

\[
V_{SG} = V_{SG}^R \cup V_{SG}^V \cup V_{SG}^p,
\]

\[
V_{SG}^R = \{(s, s', r, -) \mid s \in \sup(S) \land s' \in \sup(S') \} \cup \{ \vtext{err} \},
\]

\[
V_{SG}^V = \{(s, \sigma, s', \mu, \nu, v) \mid s \in S \land s' \in S' \land \sigma \in \Sigma \land (s, \sigma, \mu) \rightarrow \} \cup
\]

\[
\{(s, \sigma, s', \mu, v) \mid s \in S \land s' \in S' \land \sigma \in \Sigma \land (s', \sigma, \mu') \rightarrow \}
\]

\[
V_{SG}^p = \{(s, s', \mu, \mu', P) \mid s \in S \land s' \in S' \land \mu \in \text{dom}(S) \land \mu' \in \text{dom}(S') \}
\]

\[
v_{SG}^0 = (s_0, s'_0, -, -, R),
\]

and \( E_{SG} \) is the minimal set satisfying the following rules:

\[
s \xrightarrow{\sigma} \mu \Rightarrow \{(s, s', r, -) \mid (s, \sigma, s', \mu, V) \in E_{SG}^V \}
\]

\[
s' \xrightarrow{\sigma'} \mu' \Rightarrow \{(s, s', r, -) \mid (s, \sigma, s', \mu, V) \in E_{SG}^V \}
\]

\[
s' \xrightarrow{\sigma} \mu' \Rightarrow \{(s, \sigma, s', \mu, V) \mid (s, s', \mu') \in E_{SG}^V \}
\]

\[
\sigma \notin F \land s \xrightarrow{\sigma} \mu \Rightarrow \{(s, \sigma, s', \mu, V) \mid (s, s', \mu') \in E_{SG}^V \}
\]

\[
F \in F \Rightarrow \{(s, F, s', \mu, V) \mid (s, \sigma, \mu, V) \in E_{SG}^V \}
\]

\[
(s, s', \mu, \mu', P) \in V_{SG}^p \land
\]

\[
(t, t') \in \text{Supp}(\mu) \times \text{Supp}(\mu') \Rightarrow \{(s, s', \mu, \mu', P) \mid (t, t', -, -) \in E_{SG}^V \}
\]

\[
v \in (V_{SG}^V \cup \{ \vtext{err} \}) \land (\exists v' : v \neq v' \in E_{SG}^V) \Rightarrow \langle v, v' \rangle \in E_{SG}^V
\]

In addition, for each \( v = (s, s', \mu, \mu', P) \in V_{SG}^p \), define the set of variables \( X(v) = \{ x_{s_k, s_j} \mid s_k \in \text{Supp}(\mu) \land s_j \in \text{Supp}(\mu') \} \), and the system of equations

\[
\text{Eq}(v) = \left\{ \sum_{s_j \in \text{Supp}(\mu')} x_{s_k, s_j} = \mu(s_k) \mid s_k \in \text{Supp}(\mu) \right\} \cup
\]

\[
\left\{ \sum_{s_j \in \text{Supp}(\mu')} x_{s_k, s_j} = \mu'(s_j) \mid s_j \in \text{Supp}(\mu') \right\} \cup
\]

\[
\{ x_{s_k, s_j} \geq 0 \mid s_k \in \text{Supp}(\mu) \land s_j \in \text{Supp}(\mu') \}
\]

Notice that \( \{ x_{s_k, s_j} \}_{s_k, s_j} \) is a solution of \( \text{Eq}(v) \) if and only if there is a coupling \( w \in \text{Cons}(\mu, \mu') \) such that \( w(s_k, s_j) = x_{s_k, s_j} \) for all \( s_k \in \text{Supp}(\mu) \) and \( s_j \in \text{Supp}(\mu') \).

In addition, given a set of game vertices \( V' \subseteq V_{SG}^V \), we define \( \text{Eq}(v, V') \) by extending \( \text{Eq}(v) \) with an equation limiting the couplings in such a way that vertices in \( V' \) are not reached. Formally, \( \text{Eq}(v, V') = \text{Eq}(v) \cup \{ \sum_{(s, s', r, -) \in V'} x_{s_k, s_j} = 0 \} \). By properly defining a family of sets \( V' \), we will show that probabilistic masking simulation can be checked in polynomial time through the symbolic game graph.
In the following we propose to use the symbolic game graph to solve the infinite game. By doing so, we obtain a polynomial time procedure. Similarly to Definition 3, we provide an inductive construction of the Refuter-winning nodes using equation systems in place of sets of polytope vertices, as follows.

**Definition 5.** Let $SG_{A,A'} = (V^{SG}, E^{SG}, V^{SG}_R, V^{SG}_V, V^{SG}_P, v_0^{SG})$ be a symbolic game graph for PTSs $A$ and $A'$. Sets $U_i$ (for $i \geq 0$) are defined as follows:

\[
U^0 = \{v_{err}\}, \\
U^{i+1} = \{v' \mid v' \in V^{SG}_R \land \text{Post}(v') \cap U^i \neq \emptyset\} \cup \\
\{v' \mid v' \in V^{SG}_V \land \text{Post}(v') \subseteq \bigcup_{j \leq i} U^j \land \text{Post}(v') \cap U^i \neq \emptyset\} \cup \\
\{v' \mid v' \in V^{SG}_P \land \text{Post}(v') \cap U^i \neq \emptyset \land \text{Eq}(v', \text{Post}(v') \cap U^i) \text{ has no solution}\}
\]

Furthermore, we define $U = \bigcup_{i \geq 0} U_i$.

The construction of each $U^{i+1}$ follows a similar idea as the construction of $W_i$, only varying significantly in the case of the probabilistic vertices. The first and second line correspond to the Refuter and Verifier players, respectively. The last one corresponds to the probabilistic player. Notice that, if $\text{Eq}(v', \text{Post}(v') \cap U^i)$ has no solution, it means that every possible coupling will inevitably lead with some probability to a “losing” state of a smaller level, since, in particular, equation $\sum_{(s,s',r,r',R) \in (\text{Post}(v') \cap U^i)} x_{s,s'} = 0$ cannot be satisfied.

There is a strong connection between sets $W_i$ and $U_i$: a vertex is in $W_i$ if and only if its abstract version is in $U^i$. This is formally stated in the next theorem.

**Theorem 3.** Let $G_{A,A'}$ be a stochastic masking game graph for PTSs $A$ and $A'$ and let $SG_{A,A'}$ be the corresponding symbolic game graph. For every $v \in V^G$, $u \in V^{SG}$ such that $v[i] = u[i]$ (for $0 \leq i \leq 4$) and $v[6] = u[5]$, and for all $k \geq 0$, $v \in U^k$ if and only if $u \in W^k$.

The following theorem is a direct consequence of Theorems 2 and 3.

**Theorem 4.** Let $G_{A,A'}$ be a stochastic game graph for PTSs $A$ and $A'$, and let $SG_{A,A'}$ be the corresponding symbolic game graph. Then, the Verifier has a sure winning strategy in $G_{A,A'}$ if and only if $v_0^{SG} \notin U$.

As a consequence of this last theorem and Theorem 1, it suffices to calculate the set $U$ over $SG_{A,A'}$ to decide whether there is a probabilistic masking simulation between $A$ and $A'$. This can be done in polynomial time, since $\text{Eq}(v, C)$ can be solved in polynomial time and the number of iterations to construct $U$ is bounded by $|V^{SG}|$. Since $V^{SG}$ linearly depends on the transitions of the involved PTSs, we have the following theorem.

**Theorem 5.** Let $A$ and $A'$ be PTSs. $A \preceq_m A'$ can be decided in time $O(\text{Poly}(m \cdot m'))$, where $m$ and $m'$ are the size of the transitions of $A$ and $A'$, respectively.
4 Quantifying Fault Tolerance in Almost-Sure Failing Systems

Probabilistic masking simulation determines whether a fault-tolerant implementation is able to completely mask faults. However, in practice, this kind of masking fault-tolerance is uncommon. Usually, fault-tolerant systems are able to mask a number of faults before suffering a failure. Our main goal in this section is to extend the game theory presented in the previous section to be able to measure the amount of masking tolerance exhibited by a system before it fails. To do this, we extend the probabilistic masking games with a quantitative objective function. This is done in such a way that the expected value of this function indicates the number of “milestones” that the fault-tolerant implementation is expected to cross before failing. A milestone is any interesting event that may occur during the execution of the system. For instance, a milestone may be the number of faults occurring during the execution of the system, and therefore this measure will reflect the number of faults that are tolerated by the system before crashing. Another milestone may be successful acknowledgments in a transmission protocol, in this case we measure the expected number of chunks that the protocol is able to transfer under the occurrence of faults before failing.

To do this type of measuring, we need stochastic games with quantitative objective functions. Intuitively, these objective functions count the number of “milestones” observed during a play. Therefore, we first extend stochastic masking games as follows.

**Definition 6.** Let $\mathcal{A} = (S, \Sigma, \rightarrow, s_0)$ and $\mathcal{A}' = (S', \Sigma_F, \rightarrow', s_0')$ be two PTSs. A stochastic masking game graph with milestones $\mathcal{M}$ is a tuple $\mathcal{MG}_{\mathcal{A}, \mathcal{A}'} = (V^G, E^G, V^G_R, V^G_P, V^G_V, v^G_0, \delta^G, M)$ where: (i) $(V^G, E^G, V^G_R, V^G_P, V^G_V, v^G_0, \delta^G)$ is a stochastic masking game graph, (ii) $\mathcal{M} \subseteq \Sigma^2_F$ is the set of milestones, and (iii) $r^G(v) = \chi_M(v[1])$ is a reward function.

In this definition, $\chi_B$ is the characteristic function over set $B$ defined as usual: $\chi_B(a) = 1$ whenever $(a \in B)$ and $\chi_B(a) = 0$ otherwise. If $B = \{b\}$ is a singleton set, we simply write $\chi_b$.

Given a stochastic masking game graph with milestones and reward function $r^G$, for any play $\rho = \rho_0, \rho_1, \ldots$, we define the **masking payoff function** by $f_m(\rho) = \lim_{n \to \infty} (\sum_{i=0}^{n} r^G(\rho_i))$.

Intuitively, the payoff function $f_m$ characterizes the number of milestones that a fault-tolerant implementation is able to achieve until the error state is reached. This type of payoff functions are usually called **total rewards** in the literature. One may think of this as a game played by the fault-tolerance built-in mechanism and a (malicious) player that chooses the way in which faults occur. In this game, the Verifier is the maximizer (she intends to obtain as much milestones as possible) and the Refuter is the minimizer (she intends to prevent the Verifier from achieving milestones).

For the game of expected total reward to be determined, we need that the stochastic game is almost-sure stopping, i.e., that the game reaches a sink vertex
A Stochastic Game Approach to Masking Fault-Tolerance

We manage to extend the determinacy property to games that are almost-sure stopping under the condition that the minimizing player is fair.

In our setting, this amounts to considering almost-sure failing masking games, that is, games in which the error state $v_{err}$ is reached with probability 1. Moreover, we require that the Refuter plays fair. This is necessary to avoid the Refuter stalling the game in an unproductive loop. Indeed, consider the scenario described in Example 1 and set the stochastic masking game between the nominal and faulty model of Figure 1 (omitting the red part). One would expect that the game leads to a failure with probability 1. However, the Refuter has strategies for which the probability of reaching $v_{err}$ is less than 1. For instance, the Refuter may always play the reading action, and hence the Verifier has to mimic this action forever, this yields a probability of 0 of reaching the error state. Observe that, in this scenario, the Refuter is behaving in a benevolent manner, playing with the aim to avoid the error state. Clearly, this is against the spirit of the behaviour of faults which one expects they happen if waiting long enough. Therefore the assumption that the Refuter is fair in the sense that, if some action or fault is infinitely often enabled for the Refuter, it will eventually play such action or fault.

The setting for the stochastic game with the masking payoff function as objective that we present in the rest of the section stands on [7], only that here special care is needed due to the infinite nature of the stochastic game graphs. For this reason we limit the results of the rest of the section to (randomized) memoryless strategies and postpone the general result for further work. Thus, we let $\Pi^M_V$ and $\Pi^M_R$ denote the sets of all (randomized) memoryless strategies for the Verifier and the Refuter, respectively, and similarly, we let $\Pi^{MD}_V$ and $\Pi^{MD}_R$ denote the sets of all pure (or deterministic) memoryless strategies.

A Refuter’s fair play is defined as a play in which the Refuter commits to follow a strong fair pattern, i.e., that includes infinitely often any transition that is enabled infinitely often. A fair strategy for the Refuter, is a strategy that always measures 1 on the set of all the Refuter’s fair plays, regardless of the strategy of the Verifier. The definition provided below follows the style in [5,4,7].

**Definition 7.** Given a masking game $G_{A,A'} = (V^G, E^G, V^G_R, V^G_P, v^G_0, \delta^G)$, the set of all Refuter’s fair plays is defined by $RFP = \{\omega \in \Omega | v \in \text{inf}(\omega) \cap V^G_R \Rightarrow Post(v) \subseteq \text{inf}(\omega)\}$. A Refuter strategy $\pi_R$ is said to be almost-sure fair iff, for every Verifier’s strategy $\pi_V$, $\text{Prob}_{G_{A,A'}}^{\pi_V,\pi_R}(RFP) = 1$.

Under this concept, the stochastic masking game is almost-sure failing under fairness if for every Verifier’s strategy and every Refuter’s fair strategy, the game leads to an error with probability 1. This is formally defined as follows.

**Definition 8.** Let $A$ and $A'$ be two PTSs. We say that the stochastic masking game $G_{A,A'}$ is almost-sure failing under fairness (and memoryless strategies) iff, for every memoryless strategy $\pi_V \in \Pi^{MD}_V$ and any fair memoryless strategy $\pi_R \in \Pi^M_R$, $\text{Prob}_{G_{A,A'}}^{\pi_V,\pi_R}(v_{err}) = 1$. 

Interestingly, under strong fairness assumptions the determinacy of games is preserved [7]. Furthermore, in finite stochastic games with fairness restrictions the value of the game can be computed by calculating the greatest fixed point of the following Bellman functional. We can adapt this result to our games by using the vertices of the polytopes when computing the values of the probabilistic states.

**Theorem 6.** Let $MG_{A,A'}$ be a stochastic game with milestones for some PTSs $A$ and $A'$ that is almost-sure failing for fair Refuter’s strategies. Then, we have:

$$
\inf_{w \in R_1} \sup_{v \in R_0} E_{MG_{A,A'}v}^r[v_m] = \inf_{v \in R_0} \inf_{w \in R_1} E_{MG_{A,A'}w}^r[v_m] < \infty.
$$

Moreover, the value of the game for memoryless strategies for the Verifier and fair memoryless Refuter’s strategies is the greatest fixed point of the following functional $L$:

$$
L(f)(v) = \begin{cases} 
\min\{u, \max_{w \in V(C(v[3],v[4]))} \{\chi_{M}(v[1]) + \sum_{v' \in Post(v)} w(v'[0], v'[2])f(v')\}\} & \text{if } v \in V_{SG}^d \\
\min\{u, \max_{v} \chi_{M}(v[1]) + f(v') \} & \text{if } v \in V_V^d, \\
\min\{u, \min_{v} \chi_{M}(v[1]) + f(v') \} & \text{if } v \in V_R^d, \\
0 & \text{if } v = v_{err}.
\end{cases}
$$

where $u$ is a number such that $u \geq \inf_{w \in R_1} \sup_{v \in R_0} E_{MG_{A,A'}v}^r[v_m]$.

Also, we can check whether a game is almost-sure failing under fairness by computing predecessor sets in the symbolic game graph. To do so, we define the symbolic version of the predecessor sets. Given a game $G_{A,A'}$ and its symbolic version $SG_{A,A'}$, let $\exists Pre_f^S(C)$ and $\forall Pre_f^S(C)$, for a given set $C$ of symbolic vertices, be defined as follows:

$$
\exists Pre_f^S(C) = \{ v \in V_{SG}^d \mid \exists v' \in C \cap V_{SG}^d : v'[0] \in Supp(v[3]) \land v'[2] \in Supp(v[4]) \} \\
\cup \{ v \in V_{V}^S \cup V_{R}^S \mid \exists v' \in C : (v, v') \in E_{A,A'}^S \}
$$

$$
\forall Pre_f^S(C) = \{ v \in V_{SG}^d \mid Eq(v, C) \text{ has no solution } \} \\
\cup \{ v \in V_{V}^S \mid \forall v' \in C : (v, v') \in E_{SG}^d \Rightarrow v' \in C \} \\
\cup \{ v \in V_{R}^S \mid \exists v' \in C : (v, v') \in E_{SG}^d \}
$$

In particular, the first set in the definition of $\exists Pre_f^S(C)$ collects all probabilistic vertices $v$ for which there is a coupling that leads to a Refuter vertex $v'$ in $C$. For this is sufficient to check that the states $v'[0]$ and $v'[2]$ that define $v'$ are on the respective support sets of the probabilities $v[3]$ and $v[4]$ that define $v$ (since it is always possible to define a coupling that assigns positive probability to a pair of states in the respective support sets). The first set in the definition of $\forall Pre_f^S(C)$ collects all probabilistic vertices $v$ for which there is no coupling “avoiding” $C$, that is, no coupling that leads with probability 0 to the set of all pair of states.
defining a vertex in $C$. A coupling avoiding $C$ will solve $Eq(v, C)$. By using $\exists Pre^S_f$ and $\forall Pre^S_f$ recursively, we can decide whether a game is almost-sure failing under fairness as follows.

**Theorem 7.** Given a masking game $G_{A,A'}$ and its symbolic version $SG_{A,A'}$, we have that $G_{A,A'}$ is almost-sure failing under fairness iff $v^S_0 \in V^{SG} \setminus \exists Pre^S_f(V^{SG} \setminus \forall Pre^S_f(v_{err}))$, where $v^S_0$ is the initial state of $SG_{A,A'}$ and $V^{SG}$ its sets of vertices.

As $Eq(v, C)$ can be computed in polynomial time, so do the predecessor sets $\exists Pre^S_f(C)$ and $\forall Pre^S_f(C)$. As a consequence, the problem of deciding whether a stochastic masking game is almost-sure failing under fairness is also polynomial.

## 5 Experimental Evaluation

We implemented the approach described in this paper in a prototype tool, available at [1]. Tables 1, 2 and 3 report the results obtained for three case studies: a Redundant Cell Memory (our running example); N-Modular Redundancy (NMR), a standard example of fault-tolerance [28]; and a NMR processor/memory architecture with N voters [23]. In the tables, $M_t$ and $M_r$ are used to indicate the measurement results for the tick and refresh actions, considered as milestones, respectively.

Some words are useful to interpret the results. For the memory cell example, either increasing the redundancy, or augmenting the frequency of refreshing, have positive effects in the measures. In practice, these values can be taken into account when designing a fault-tolerant component that provides an optimal balance between fault-tolerance and hardware costs. For example, assuming a fault probability of 0.05, one might prefer 3 bits and more frequent refreshing, over 5 bits with a less often refreshing, despite the software overhead. NMR consists of N modules that independently perform a task, and whose results are processed by a perfect voter to produce a single output. These modules may exhibit an unexpected behavior with a given probability, in which case they output an incorrect value. The results for this case study are similar to the memory cell example when there is 0 probability of refreshing. The last case study consists of N processors that output a value to a memory module through N voters. Both the voters and processors may output an incorrect value with certain probability. The experiment outputs the same results if the fault probability of voters and processors are exchanged. This suggests that, provided that the probability that a pair processor-voter fails remains the same, the system is more tolerant when faults occur equally distributed on voters and processors.

We have run our experiments on a MacBook Air with processor 1.3 GHz Intel Core i5 and a memory of 4 Gb. The tool and case studies are available in the tool repository [1].
Table 1. Experimental results on a Redundant Memory Cell.

| Bits Fault Prob. | Refresh Prob. | Modules Fault Prob. |
|------------------|---------------|---------------------|
| 0.5 1.0 1.0      | 4.44 0.5      | 0.5 4.2 0.21        |
| 0.5 0.5 1.0      | 25 0.5        | 0.5 30 0.5          |
| 0.5 0.1 1.0      | 250.02 0.5    | 0.5 80 0.5          |
| 0.5 0.05 1.0     | 60 0.5        | 0.5 60 0.5          |
| 0.5 0.0 1.0      | 14 0.5        | 0.5 14 0.5          |
| 0.5 0.0 0.5      | 4.39 0.5      | 0.5 70 0.7          |
| 0.05 0.0 1.0     | 47.5 0.5      | 0.05 140 0.5        |
| 0.05 0.0 0.05    | 3.5 0.5       | 0.05 30 0.5         |
| 0.05 0.0 0.0     | 2660.1 0.5    | 0.05 1000 0.5       |
| 0.05 0.0 0.0     | 4.39 0.5      | 0.05 1000 0.5       |
| 0.05 0.0 0.0     | 2660.1 0.5    | 0.05 1000 0.5       |
| 0.05 0.0 0.0     | 2660.1 0.5    | 0.05 1000 0.5       |

Table 2. Experimental results on a N-Modular Redundant System.

| N P Fault Prob. | V Fault Prob. | Modules Fault Prob. |
|-----------------|---------------|---------------------|
| 0.5 0.0 0.1     | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.08    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |
| 0.5 0.0 0.05    | 25 0.5        | 0.5 4.2 0.21        |

Table 3. Experimental results on a NMR processor/memory architecture with N voters.

6 Related Work

The games introduced in [2, 3, 14, 15] are based on probabilistic bisimulation, so they are symmetric. Furthermore, in [14, 15] the nodes of the game graph are modeled using subsets of states of the PTSs, in our formulation we do not use subsets of states. The games defined in [2, 3] use Kantorovich’s and Hausdorff’s liftings to deal with probabilistic distributions and non-determinism, respectively. In addition, the authors use the vertices of the transportation polytopes to model the probabilistic vertices. In contrast, we introduced a symbolic representation of games to avoid the state explosion caused by the vertices of the polytopes. Also note that the metrics introduced in [2, 3] measure the (probabilistic) bisimulation distance between two PTSs, which for almost-sure failing systems is always 1.

Another related framework is defined in [25]. Therein, the authors introduce a notion of weak simulation quasimetric tailored for reasoning about the evolution of gossip protocols. This makes it possible to compare network protocols that have similar behaviour up to a certain tolerance; being 0 and 1 the minimum and maximum distance, respectively. Note that using this quasimetric to compare a network protocol with an almost-sure failing implementation will always return 1, thus that approach cannot be used to quantify the masking fault-tolerance of almost-sure failing systems.

After the case studies of Section 5, Mean-Time To Failure (MTTF) [27] may come to mind. Though this metric (lifted to games) may be the result of a particular case study, we present a much more general framework. Indeed, on
the one hand, we do not necessarily have to count time units, and other events may be set as milestones. On the other hand, the computation of MTTF would normally require the identification of failures states in an ad hoc manner, while we do this at a higher level of abstraction: the failure situation appears in the game as a result of comparing the implementation model against the nominal model.

7 Conclusions and Future Work

We presented a relation of masking fault-tolerance between probabilistic transition systems, which is accompanied by a corresponding probabilistic game characterization. Even though the game could be infinite, we proposed an alternative finite symbolic representation by means of which the game can be solved in polynomial time. We extended the game with quantitative objectives based on counting “milestones” thus providing a way to quantify the amount of masking fault tolerance provided by a given implementation. As this game inherits the characteristic of total reward objectives, some stopping criterion is necessary and thus the game is required to be almost-sure failing under a fair Refuter. By restricting to (randomize) memoryless strategies, we could show that the resulting game is determined and can be computed by solving a collection of functional equations. We also provided a polynomial technique to decide whether a game is almost-sure failing under fairness.

There are many directions for future work. As an immediate one, we have pending to extend the result on quantitative objectives to non-memoryless strategies. Given the result in [7], we believe this is possible but special care is needed to deal with the infinite nature of the game. In a different direction, in this paper we introduced a strong version of probabilistic masking simulation. However, for analyzing non-trivial systems, a weak version of this kind of relation is needed since it could abstract away from internal transitions which are mostly associated with the fault-tolerant machinery of the implementation. Besides, we have so far only worked with masking fault-tolerance. Similar ideas to those presented in this paper could be extrapolated to other levels of fault-tolerance like fail-safe and non-masking. Finally, we have presented a prototype tool for measuring some well-known small case studies. Our goal is to develop an automated tool to support the use of these measurements in practice.
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A Proofs of Properties

Proof of Theorem 1 Let $A = (S, \Sigma, \rightarrow, s_0)$ and $A' = (S', \Sigma_{\mathcal{F}}, \rightarrow', s'_0)$ be two PTSs. We have $A \preceq_m A'$ iff the Verifier has a sure winning strategy for the stochastic masking game graph $G_{A,A'}$ with the Boolean objective $\Phi = \{\omega_0, \omega_1, \cdots \in \Omega \mid \forall i \geq 0 : \omega_i \neq v_{err}\}$.

Proof. “Only If”: Assume $A \preceq_m A'$, thus there is a probabilistic masking simulation $M \subseteq S \times S'$. Let us define a sure winning strategy $\pi_V$ for the Verifier as follows. Given a state $(s, \sigma^1, s', \mu, \omega, V)$ (resp. $(s, \sigma^2, s', \mu, \omega, V)$), if $s M s'$, $\pi_V$ selects a transition $((s, \sigma^1, s', \mu, \omega, V), (s, \sigma^2, s', \mu, \omega, P))$ (resp. $((s, \sigma^2, s', \mu, \omega, V), (s, \sigma^2, s', \mu, \mu', w, P))$) such that $w$ is a $M$-respecting coupling for $(\mu, \mu')$ (which is guaranteed to exist by Def. 1). Otherwise, $\pi_V$ selects an arbitrary vertex. Let us show that this strategy is sure winning for the Verifier in the initial state. We have to prove that, for any Refuter’s strategy $\pi_R$, we have $out(\pi_V, \pi_R) \subseteq \Omega \setminus \Phi$, where $out(\pi_V, \pi_R)$ denotes the set of paths generated when strategies $\pi_R$ and $\pi_V$ are used. Let $\pi_R$ be any strategy for the Refuter, and $\omega = \omega_0, \omega_1, \ldots$ the corresponding play in $out(\pi_V, \pi_R)$. We prove by induction that $\forall i \geq 0 : \omega_i \neq v_{err} \land (\omega[i] \notin F \Rightarrow \omega[i] = 0) M (\omega[i] \notin 2)$. For $i = 0$, the proof is straighforward. Assume that the property holds for $\omega_i$, if $\omega_i$ is a Verifier’s vertex and $\omega[1] = \sigma^1$ (resp. $\sigma^2$) with $\sigma \notin F$, then by definition of $\pi_V$ and Def. 1 $\omega_{i+1} = (s, \sigma^1, s', \mu, \mu', w, P)$ (resp. $\omega_{i+1} = (s, \sigma^2, s', \mu, \mu', w, P)$). Thus, we have by inductive hypothesis that $s M s'$ and also that $\omega_{i+1} \neq v_{err}$. If $\omega_i[1] \in F$, then the proof is similar, but taking into account that $\mu = \Delta$. If $\omega_i$ is a Refuter’s vertex, then $\omega_{i+1}$ is a Verifier’s vertex, and it cannot be $v_{err}$ because by construction only Verifier’s nodes are adjacent to the $v_{err}$. If $\omega_i[1] \in F$, then note that $\text{Supp}(\omega_i[5]) = \emptyset$ and therefore $v_{err} \neq \omega_{i+1}$. Thereby, $\omega_{i+1} = (s, \sigma^2, s', \mu, \mu', w, P)$.

“If”: Suppose that the Verifier has a sure winning strategy $\pi_V$ from the initial state. Then, we define a probabilistic masking simulation relation as follows: $M = \{ (s, s') \mid (s, s', \mu, \mu', w, P) \in \pi_V(V^D) \}$ for some sure winning strategy $\pi_V$. We know by our assumption that this set is not empty and it is direct to see that $(s_0, s'_0) \in M$. First, let us prove that for any $(s, s', \mu, \mu', w, P) \in \pi_V(V^D)$ we have $\mu M^\# \mu'$. Assume $(s, s', \mu, \mu', w, P) \in \pi_V(V^D)$ and it is not the case that $\mu M^\# \mu'$, or equivalently $\exists t, t' : w(t, t') > 0 \land \neg (t M t')$. Thus, we have a successor $(t, t', \omega, -\omega, R)$ of $(s, s', \mu, \mu', w, P)$ which can be chosen with probability greater than 0 and $(t, t') \notin M$. Furthermore, there exists a $t \xrightarrow{s} t_0$ (or $t' \xrightarrow{s'} t'_0$) such that $(t_0, \sigma, t', \mu, \omega, V) \in Post((t, t', \omega, -\omega, R))$ (resp. $(t, \sigma', t'_0, \mu', -\omega, V) \in Post((t, t', \omega, -\omega, R)))$. This state cannot be $v_{err}$ and there must be a winning strategy for the Verifier from it; otherwise, the Refuter would have a winning strategy from $(t, t', \omega, -\omega, R)$, and $(s, s', \mu, \mu', w, P) \notin Im(\pi)$ for some winning strategy $\pi_V$ for the Verifier. But then $(t, t') \notin M$, contradicting the assumption above. Thus, $\mu M^\# \mu'$.

Let us now prove that $M$ is a probabilistic masking simulation. Assume that $s M s'$ and $s \xrightarrow{a} \mu$, for any successor $(s, a, s', \mu, -\omega, V)$ of $(s, s', \omega, -\omega, R)$ we have a sure winning strategy $\pi_V$ for $V$, such that: $\pi((s, a, s', \mu, -\omega, V)) = (s, a, s', \mu, -\omega, V)$ and also by the property proven.
above, we have $\mu M^k \neq \mu'$. Similarly for the cases: $s' \xrightarrow{a} \mu$ and $s' \xrightarrow{E} \mu'$ for $F \in F$. Finally, since $V$ has a winning strategy from $(s_0, \cdot, s'_0, \cdot, \cdot, R)$ we have that $s_0 M s'_0$. Thus, all the requirements of Def. 1 holds, and so $M$ is a probabilistic masking relation.

Proof of Theorem 2. Let $G_{A,A'} = (V^G, E^G, V^G_R, V^G_P, v^G_0, \delta^G)$ be a stochastic masking game graph for some PTSs $A$ and $A'$, we have that the Verifier has a sure winning strategy from vertex $v$ iff $v \notin W$.

Proof. First, we can define a 2-player reachability game obtained from $G_{A,A'}$ by considering the probabilistic nodes as Refuter’s nodes, and ignoring the probabilistic distribution, let $H^G_{A,A'}$ be that game. It is clear that a Verifier’s strategy is sure winning in $G_{A,A'}$ iff this strategy is winning in $H^G_{A,A'}$. Then, proving the theorem reduces to show that the sets $W^i$ determines the winning strategies of the Verifier in $H^G_{A,A'}$ (recall that only the vertices of the polytopes are taken into account for defining the $W^i$’s). If the Verifier has a winning strategy from vertex $v$ let us prove that $v \notin W^k$ for every $k$ by induction. For $k=1$ it is direct. Now, assume that the property holds for $W^k$, let $v$ be an arbitrary vertex such that the Verifier has a winning strategy named $\pi_V$ from $v$. If $v$ is a Verifier’s node and $v \in W^{k+1}$, then $V(\text{Post}(v)) \subseteq W^k$. Thus, by inductive hypothesis $\pi_V(v) \notin V(\text{Post}(v))$, that is, $\pi_V(v)$ is a probabilistic vertex whose coupling is not a vertex. Furthermore, it is a Refuter’s node in $H^G_{A,A'}$. For this node we have $v' \in \text{Post}(\pi_V(v))$ iff $\pi_V(v)[5](v'[0], v'[2]) > 0$. Note that $\pi_V(v)[5]$ is a point of the polytope defined by $C(\pi_V(v)[3], \pi_V(v)[4])$, since polytopes do not contain lines, either $\pi_V(v)[5]$ is a vertex or there is a polytope’s vertex $v'$ such that $w'(\pi_V(v)[3], \pi_V(v)[4]) > 0$ iff $\pi_V(v)[5](\pi_V(v)[3], \pi_V(v)[4]) > 0$. Thereby, there is a $v'' \in V(\text{Post}(v))$ such that $\text{Post}(v'') = \text{Post}(\pi_V(v))$, that is, $v'' \in W^k$ implies that $\pi_V(v) \in W^k$ which is a contradiction and so $v \notin W^{k+1}$.

Let us define a strategy $\pi_V$ which is winning strategy in $H^G_{A,A'}$ for any Verifier’s node $v \notin W$. If $v \notin W$, then $\pi_V(v) = v'$ for some $v' \in \text{Post}(v) \cap (V^G \setminus W)$ (which is guaranteed to exist by assumption), moreover, if $v \notin W$, then $\pi_V(v) = v'$ for an arbitrary node $v'$. Let us prove that for any play generated by $\pi_V$: $v_0, v_1, \ldots$ we have $v_i \notin W$, the proof is by induction on $i$. For $i=0$ it is direct, assuming that $v_i \notin W$ let us prove that $v_{i+1} \notin W$. If $v_i$ is a Refuter’s node by Def. 3 $\text{Post}(v_i) \cap W = \emptyset$, and therefore, $v_{i+1} \notin W$. If $v_i$ is Verifier’s node, by definition of $\pi_V$: $v_{i+1} = \pi_V(v_i) \notin W$ and therefore the result follows.

Proof of Theorem 3. Given a stochastic masking game graph $G_{A,A'}$ for some PTSs $A$ and $A'$ and the corresponding symbolic game $S_G_{A,A'}$. For any states $v \in V^G$, $u \in V^SG$ such that $v[i] = u[i]$ (for $0 \leq i \leq 4$) and for any $k > 0$ we have that: $v \in U^k$ iff $u \in U^k$.

Proof. The proof is by induction on $k$. For $k=1$, we have that $W^1 = \{v_{\text{err}}\} = U^1$. For the inductive case, consider arbitrary nodes $v \in V^SG$ and $u \in V^G$, such that $v[i] = u[i]$ for $0 \leq i \leq 4$. Note that these nodes also coincide in their last components, that is, either both are Refuter’s nodes, Verifier’s nodes, or probabilistic nodes. Assume these are Refuter’s nodes, if $v \in U^k$ then $\text{Post}(v) \cap U^{k-1} \neq \emptyset$. 
Thus, there is some \( v' \in Post(v) \cap U^{k-1} \) which is a probabilistic node. By Def. 5 we have many \( u' \in Post(u) \) such that \( v'[i] = u'[i] \) (for \( 0 \leq i \leq 4 \)), and by induction we have that \( u' \in W^{k-1} \) and therefore \( u \in W^k \). Similarly, if \( u \in W^k \) we have that \( Post(u) \cap W^{k-1} \neq \emptyset \), and we proceed as before. If \( v \) and \( u \) are Verifier’s nodes the proof is similar. Now, assume that \( v \) and \( u \) are probabilistic nodes. If \( v \in U^k \), then \( Post(v) \cap U^{k-1} \neq \emptyset \). As proved above, we also have that \( Post(u) \cap W^{k-1} \neq \emptyset \). Moreover, if \( Eq(v, Post(v) \cap U^{k-1}) \) has no solutions, then we have at least a coupling (say \( u \)) for distributions \( v[3] \) and \( v[4] \) that satisfies the equations and also \( w(v'[0], v'[2]) > 0 \) for some \( v' \in Post(v) \cap U^{k-1} \). By Def. 2 we have a vertex \( u' \in Post(u) \) such that \( u'[5] = w \), and thus \( \sum_{u' \in Post(u) \cap W^{k-1}} \delta^v(u') > 0 \), which means that that \( u' \in W^k \). Similarly, if \( \sum_{u' \in Post(u) \cap W^{k-1}} \delta^v(u') > 0 \) then \( Eq(v, Post(v) \cap U^{k-1}) \) has no solutions, and then \( u \in W^k \) implies \( v \in U^k \).

**Proof of Theorem 6** Let \( MG_{A,A'} \) be a stochastic game with milestones for some PTSs \( A \) and \( A' \) that is almost-sure failing for fair Refuter’s strategies. Then:

\[
\inf_{\pi_r \in H_R^M} \sup_{\pi_V \in H_V^M} \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] = \inf_{\pi_V \in H_V^M} \sup_{\pi_r \in H_R^M} \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] < \infty
\]

Furthermore, the value of the game for memoryless strategies for the Verifier and fair Memoryless Refuter’s strategies is the greatest fixpoint of the following functional \( L \):

\[
L(f)(v) = \begin{cases} 
\min\{u_0, \max_{v' \in Post(v)} \sum_{v' \in Post(v)} w(v'[0], v'[2]) f(v')\} & \text{if } v \in V_0^F \\
\min\{u_0, \max_{v' \in Post(v)} \chi_M(v[1]) + f(v') \mid v' \in Post(v)\} & \text{if } v \in V_0^S \\
\min\{u_0, \min_{v' \in Post(v)} \chi_M(v'[1]) + f(v') \mid v' \in Post(v)\} & \text{if } v \in V_0^D \\
0 & \text{if } v = v_{err}
\end{cases}
\]

where \( u \) is a number such that \( u \geq \inf_{\pi_r \in H_R^M} \sup_{\pi_V \in H_V^M} \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] \), for every \( v \).

**Proof.** First we prove that we can safely restrict to deterministic strategies when computing the value of the game for memoryless strategies. To do so, we prove that for every memoryless strategies \( \pi_V \) and \( \pi_R \), there is a memoryless and deterministic strategy \( \pi_V' \) such that: \( \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] \geq \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] \). To do so, note that any memoryless strategy satisfies the following equation for every \( v \in V_V^F \):

\[
\mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] \leq r(v) + \sum_{v' \in Post(v)} \delta^v,\pi(v, v') \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v'}[f_m] \quad (1)
\]

\[
\mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v}[f_m] \leq r(v) + \max_{v' \in Post(v)} \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v'}[f_m] \quad (2)
\]

The first inequality follows from the definition of expected value, the second inequality follows since \( \sum_{v' \in Post(v)} \delta^v,\pi(v, v') \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v'}[f_m] \) is a convex combination. That is, defining \( \pi_V'(v) = \arg\max_{v' \in Post(v)} \mathbb{E}^\pi_V,\pi_R_{MG_{A,A'},v'}[f_m] \), for every
$v$, we obtain $E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m] \leq E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m]$. Similarly we can prove that for every memoryless strategies $\pi_R$ and $\pi_v$, there is a memoryless, deterministic and fair strategy $\pi_R$ such that $E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m] \leq E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m]$. These properties imply that:

$$\inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m] = \inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m]$$

and similarly:

$$\sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m] = \sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m].$$

Now, we prove the theorem. We define a restricted (finite) game just taking into account the vertices of the polytope defined by the couplings. Consider the subgame $\cal H_{A,A'}$ obtained from $\cal M \cal G_{A,A'}$ by restricting the successors of Verifier’s vertices to the following sets:

- $\{(s, s^*, s', s^*, \mu', \nu, V), (s, s^*, \mu, \nu, w, P)\}$ for any $\nu \in \Sigma : s \rightarrow \mu \wedge w \in \cal V(\cal C(\mu, \mu')) \} \subseteq E', \forall \sigma \notin F$
- $\{(s, s^*, s', \mu, \nu, w, P)\}$ for any $\nu \in \Sigma : s \rightarrow \mu, w \in \cal V(\cal C(\mu, \mu')) \} \subseteq E'$
- $\{(s, F^2, s^*, s', \mu', \nu, V), (s, s^*, \Delta, \mu', \nu, w, P)\}$ for any $\nu \in \Sigma : s \rightarrow \mu$, $w \in \cal V(\cal C(\mu, \mu')) \} \subseteq E'$

That is, we restrict the couplings to the vertices of the polytope $\cal C(\mu, \mu')$. Note that since the set of vertices is finite, the game $\cal H_{A,A'}$ is finite. We show now that:

$$\sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal H_{A,A'},v}^{\pi_v,\pi_R}[f_m] \leq \sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m],$$

and:

$$\inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal H_{A,A'},v}^{\pi_v,\pi_R}[f_m] \leq \inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m].$$

Note that, by the property proven above, these are equivalent to:

$$\sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal H_{A,A'},v}^{\pi_v,\pi_R}[f_m] \leq \sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m],$$

and:

$$\inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal H_{A,A'},v}^{\pi_v,\pi_R}[f_m] \leq \inf_{\pi_R \in \Pi_R} \sup_{\pi_v \in \Pi_v} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m].$$

\[\text{E1}\] holds since $Post^{\cal H_{A,A'}}(v) \subseteq Post^{\cal M \cal G_{A,A'}}(v)$ for $v \in \cal V_{\cal H_{A,A'}}$ and $Post^{\cal H_{A,A'}}(v) = Post^{\cal M \cal G_{A,A'}}(v)$ for $v \in \cal V_{\cal H_{A,A'}}$. For proving \[\text{E1}\] fix a fair strategy $\pi_R \in \Pi_R$, the optimal strategy for the Verifier in game $\cal G_{A,A'}$, is attained only in probabilistic vertices that are vertices of $\cal C(\mu, \mu')$, thus the probabilistic vertices of $\cal H_{A,A'}$, thus $\sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal H_{A,A'},v}^{\pi_v,\pi_R}[f_m] \leq \sup_{\pi_v \in \Pi_v} \inf_{\pi_R \in \Pi_R} E_{\cal M \cal G_{A,A',v}}^{\pi_v,\pi_R}[f_m]$ for any fair and memoryless $\pi_R$. \[\text{E1}\] follows.
Furthermore, the value of game $\mathcal{H}_{A,A'}$ is given by the greatest fixed point of the equations [7]:

$$L(f)(v) = \begin{cases} 
\min\{u, \chi_M(v[1]) + \sum_{v' \in \text{Post}(v)} \delta(v)(v')f(v')\} & \text{if } v \in V^\mathcal{H}_{A,A'}_P \\
\min\{u, \max\{\chi_M(v[1]) + f(v') | v' \in \text{Post}(v)\}\} & \text{if } v \in V^\mathcal{H}_{A,A'}_R \\
\min\{u, \min\{\chi_M(v[1]) + f(v') | v' \in \text{Post}(v)\}\} & \text{if } v \in V^\mathcal{H}_{A,A'}_L \\
0 & \text{if } v = v_{err}.
\end{cases}$$

for some $u \geq \inf_{\pi \in P_R} \sup_{v \in P_R} \mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{H}_{A,A'},v}[f_m]$. That is, we have:

$$\inf_{\pi \in P_R} \sup_{v \in P_R} \mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{H}_{A,A'},v}[f_m] = \sup_{\pi \in P_R} \inf_{v \in P_R} \mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{H}_{A,A'},v}[f_m] < \infty \quad (8)$$

Thus, because of $\mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{H}_{A,A'},v}[f_m]$ we have:

$$\inf_{\pi \in P_R} \sup_{v \in P_R} \mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{M}^G_{A,A'},v}[f_m] = \sup_{\pi \in P_R} \inf_{v \in P_R} \mathbb{E}^{\pi \rightarrow \pi_R}_{\mathcal{M}^G_{A,A'},v}[f_m] < \infty.$$

This proves a part of the theorem. Now, consider the following functional over the symbolic game:

$$L'(f)(v) = \begin{cases} 
\min\{u, \max_{w \in \mathbb{V}(C(v[3],v[4]))} \left\{ \sum_{v' \in \text{Post}(v)} w(v'[0],v'[2])f(v') \right\} & \text{if } v \in V^\mathcal{G}_{A,A'}^S \\
\min\{u, \max\{\chi_M(v[1]) + f(v') | v' \in \text{Post}(v)\}\} & \text{if } v \in V^\mathcal{G}_{A,A'}^S \\
\min\{u, \min\{\chi_M(v[1]) + f(v') | v' \in \text{Post}(v)\}\} & \text{if } v \in V^\mathcal{G}_{A,A'}^S \\
0 & \text{if } v = v_{err}.
\end{cases}$$

we will prove that this can be used to solve $L$. First, note that $L'$ is monotone, it is defined over a complete lattice $[0,u]$ and it is Scott-complete. Thus, it has a greatest fixpoint. Let $\nu L'$ the greatest fixpoint of $L'$, we prove that $\nu L(v) = \nu L'((v[0],v[1],v[2],v[3],v[4]))$, for every $v \in V^\mathcal{H}_{A,A'} \cup V^\mathcal{H}_{A,A'}$. For doing so, consider for each symbolic vertex the following mapping:

$$- \left\{ (s,\sigma,s',\mu,\mu',X) \right\} = (s,\sigma,s',\mu,\mu',-X), \text{ for } X \in \{R,V\}$$

$$- \left\{ (s,\sigma,s',\mu,\mu',P) \right\} = (s,\sigma,s',\mu,\mu',w,P), \text{ where } w = \text{argmax}_{w \in \mathbb{V}(C(\mu,\mu'))} \left\{ \sum_{v' \in \text{Post}(v)} w(v'[0],v'[2]) \nu L'(v') \right\}$$

Similarly, we can define a mapping from concrete vertices to symbolic ones as:

$$- \left\{ (s,\sigma,s',\mu,\mu',Y,X) \right\} = (s,\sigma,s',\mu,\mu',X), \text{ for } X \in \{R,V\} \text{ and } Y \in \{-\} \cup \mathbb{V}(C(\mu,\mu'))$$

Now, we prove that $\alpha(v) = \nu L'((\langle v \rangle))$ is a fixpoint of $L$. We proceed by cases:

If $v$ is a Refuter’s vertex, then:

$$L(\alpha)(v) = \min\{u, \min\{\chi_M(v[1]) + \alpha(v') | v' \in \text{Post}(v)\}\} \quad (9)$$
The first line is by definition of \( L \), the second line is obtained applying the definition of \( \alpha \), the third line is due to surjectivity of \( \nu \) and definition of \( L' \) and the fact that \( \nu L'(v) \) is a fixed point of \( L' \). If \( v \) is a Probabilistic vertex, the proof similar:

\[
L(\alpha)(v) = \min\{u, \max_{v' \in \text{Post}(v)} \{ \sum_{v'' \in \text{Post}(v)} w(v'[0], v''[2])\alpha(v'') \} \}
\]

\[
= \nu L'(\nu(v))
\]

\[
= \alpha(v)
\]

If \( v \) is a Verifier’s vertex, then:

\[
L(\alpha)(v) = \min\{u, \max_{v' \in \text{Post}(v)} \{ \sum_{v'' \in \text{Post}(v)} w(v'[0], v''[2])\nu L'(\nu(v'')) \} \}
\]

\[
= \nu L'(\nu(v))
\]

\[
= \alpha(v)
\]

Hence, \( \alpha \) is a fixpoint of \( L \). Furthermore, we prove that it is greatest one. Assume for the sake of contradiction that there is some \( \alpha' \) such that is a fixpoint of \( L \) and \( \alpha'(v) \geq \alpha(v) \) for every \( v \), and \( \alpha'(v') > \alpha(v') \) for some \( v' \). We can define \( \beta(v) = \alpha'(v) \), as above we can prove that it is a fixpoint of \( L' \) and, furthermore, for every symbolic vertex we have \( \beta(v) = \alpha'(v) \geq \alpha(v) \) and \( \beta(v') > \alpha(v') \), which is a contradiction since \( \nu L' \) is the greatest fixpoint of \( L' \).

**Proof of Theorem** \( \Box \) Given a masking game \( G_{A,A'} \) and its symbolic version \( SG_{A,A'} \), we have that \( G_{A,A'} \) is stopping under fairness iff \( v_0^G \in \text{Post}^{SG} \) \( \exists \text{Pre}^{SG} (V^{SG} \wedge \forall \text{Pre}^{SG} (v_{\text{err}})) \), where \( v_0^G \) is the initial state of \( SG_{A,A'} \) and \( V^{SG} \) its sets of vertices.

**Proof.** Consider the game \( H_{A,A'} \) as defined in the proof of Theorem \( \Theta \). First, we prove that the game \( G_{A,A'} \) is almost-sure failing for fair Refuter’s strategies iff \( H_{A,A'} \) is too almost-sure failing for fair Refuter’s strategies. This is equivalent to prove that \( \inf_{\pi} \text{Prob}_{G_{A,A'}^\nu, v_0^G} (\Diamond v_{\text{err}}) = 1 \) iff \( \inf_{\pi} \text{Prob}_{H_{A,A'}^\nu, v_0^H} (\Diamond v_{\text{err}}) = 1 \) for every strategy memoryless and fair \( \pi_R \). Now, note we have:

\[
\text{Prob}_{H_{A,A'}^\nu, v_0^H} (\Diamond v_{\text{err}}) = \min\{ \sum_{v' \in \text{Post}(v)} w(v'[0], v''[2]) \text{Prob}_{H_{A,A'}^\nu, v} (\Diamond v_{\text{err}}) \mid v \in C(v[3], v[4]) \}
\]
we proceed as follows. If $v \pi$ then, we have a deterministic and memoryless $u$ relating distributions $u$ similar. $u$ the unique successor of $u$ the same successors (up to removal of dummy notation). If $n$ for every $0 \leq n \leq 4$, we have: $v \in \forall \text{Post}(v)$ if $v' \in \forall \text{Post}(v)$, for every $n$. The proof is by induction on $n$. The base case is direct. The inductive cases for Refuter’s nodes and Probabilistic nodes are also direct (as the predecessors in both games are the same for those vertices module dummy notation). For Verifier’s nodes we proceed as follows. If $v \in \forall \text{Post}(v)$, then for all $t \in \forall \text{Post}(v)$ we have $\delta(t)(\forall \text{Post}^{n-2}(v)) > 0$, that is, $\forall \text{Eq}(t')(\exists \text{Post}^{n-2}(v))$ has no solutions (with $t'$ satisfying $t[i] = t'[i]$ for $0 \leq i \leq n$). Thus, $t' \in \forall \text{Post}^{n-1}(v)$, and since $t'$ is the unique successor of $v$ we have that $v' \in \forall \text{Post}^n(v)$, the other direction is similar.

Now, given sets $C \subseteq V^H$ and $C^* \subseteq V^S$ with $C^* = \{v | \exists v' \in C : \forall 0 \leq i \leq 4 : v[i] = v'[i]\}$. We prove that, for every $u \in V^H$ and $u' \in V^S$ such that $u[i] = u'[i]$ for every $0 \leq i \leq 4$, we have $u \in \exists \text{Post}^n(C)$ iff $u' \in \exists \text{Post}^n(C^*)$. The proof is by induction on $n$, for $n = 0$ it is direct. For $n > 0$, assume $w \in \exists \text{Post}^n(C)$, for Refuter’s (or Verifier’s) vertices the proof is direct, since in both games they have the same successors (up to removal of dummy notation). If $u$ is a Probabilistic vertex and $u \in \exists \text{Post}^n(C)$, then there is $t \in C$ such that $t \in \forall \text{Post}(u)$, that is, $u[5](t[3]) > 0$. But then $t[0] \in \forall \text{Post}(u)$ and $t[2] \in \forall \text{Post}(u)$ which implies that $u' \in \exists \text{Post}^n(C)$. The other direction is similar, but noting that if we have $t[0] \in \forall \text{Post}(u)$ and $t[2] \in \forall \text{Post}(u)$, then we can construct a coupling relating distributions $u[3]$ and $u[4]$. Thus, we have that $v_0^G \in V \forall \text{Post}^n(V \forall \text{Post}^n(v_0))$ if $v_0^G \in V^S \forall \text{Post}^n(V^S \forall \text{Post}^n(v_0))$, from there the theorem follows. \[\square\]