ASYMPTOTIC BEHAVIOR OF DIMENSIONS OF SYZYGIES

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Abstract. Let \( R \) be a commutative noetherian local ring and \( M \) be a finitely generated \( R \)-module of infinite projective dimension. It is well-known that the depths of the syzygy modules of \( M \) eventually stabilize to the depth of \( R \). In this paper, we investigate the conditions under which a similar statement can be made regarding dimension. In particular, we show that if \( R \) is equidimensional and the Betti numbers of \( M \) are eventually non-decreasing, then the dimension of any sufficiently high syzygy module of \( M \) coincides with the dimension of \( R \).

Introduction

Throughout this paper \( R \) will denote a commutative noetherian local ring with identity element, unique maximal ideal \( m := R/m \). Let \( M \) be a finitely generated \( R \)-module. The \( i \)th Betti number of \( M \) is given by \( \beta_i(M) := \dim_k(\text{Tor}_i^R(k,M)) \). A minimal free resolution of \( M \) then has the form

\[
\cdots \to R^{\beta_2(M)} \xrightarrow{\delta_2} R^{\beta_1(M)} \xrightarrow{\delta_1} R^{\beta_0(M)} \to 0.
\]

The \( i \)th syzygy module of \( M \) is \( \Omega_i(M) := \text{Coker}(\delta_{i+1}) \). We let \( \text{Min}(M) \) denote the set of minimal elements under inclusion of \( \text{Supp}(M) := \{ p \in \text{Spec}(R) | M_p \neq 0 \} \).

Our main result is the following, which is part of Theorem 8.

**Theorem 1.** Let \( R \) be a noetherian local ring and \( M \) a finitely generated \( R \)-module with eventually non-decreasing Betti numbers. Then for all \( i \gg 0 \) we have \( \text{Min}(\Omega_i(M)) \subseteq \text{Min}(R) \) and \( \text{Supp}(\Omega_i(M)) = \text{Supp}(\Omega_{i+2}(M)) \).

An important consequence of Theorem 8 is the following corollary, which follows immediately from Corollary 9.

**Corollary 2.** Let \( R \) be an equidimensional noetherian local ring and \( M \) a finitely generated \( R \)-module with eventually non-decreasing Betti numbers. Then the sequence \( (\dim(\Omega_i(M)))_{i=0}^{\infty} \) is constant for \( i \gg 0 \).
This raises the following open question.

**Question 3.** Let $R$ be a noetherian local ring and $M$ a finitely generated $R$-module. Is $(\dim(\Omega_i(M)))_{i=0}^\infty$ constant for all $i \gg 0$?

This question was also explored in the last section of [5]. In [5] Remark 5.2 (i) it is noted that if $R$ is unmixed and equidimensional, then $(\dim(\Omega_i(M)))_{i=0}^\infty$ is constant for $i \gg 0$. This is clear since the associated primes of any submodule of $R^\beta_i(M)$ are also associated primes of $R$ and are therefore primes of maximal dimension by assumption.

It is worth noting that the asymptotic behavior of the depths of syzygy modules is known. Given $M$ the sequence $(\depth(\Omega_i(M)))_{i=0}^\infty$ is constant for all $i \gg 0$. Let $\pd(M)$ denote the projective dimension of $M$. In particular if $\pd(M) = \infty$, then $\depth(\Omega_n(M)) \geq \depth(R)$ for $n \geq \max\{0, \depth(R) - \depth(M)\}$, with at most one strict inequality at either $n = 0$ or $n = \depth(R) - \depth(M) + 1$; see [9 Proposition 10] or [2 Proposition 1.2.8]. It follows therefore that if $\pd(M) = \infty$ and $R$ is Cohen-Macaulay, then $\dim(\Omega_n(M)) = \dim(R)$ for $n \gg 0$.

All of our results are for modules whose Betti numbers are eventually non-decreasing. Therefore finding a proof for the following conjecture of L. Avramov would improve our results.

**Conjecture 4 ([1]).** The Betti numbers of any finitely generated module over an arbitrary noetherian local ring are eventually non-decreasing.

There are a plethora of cases for which this conjecture is known to be true. J. Lescot [3 Corollaire 6.5] showed that over a Golod ring which is not a hypersurface any finitely generated module of infinite projective dimension will have eventually increasing Betti numbers. Also L.-C. Sun [11 Corollary] showed that over rings of codepth less than or equal to three and Gorenstein rings of codepth four all finitely generated modules have eventually non-decreasing Betti numbers. Several other interesting cases are also proven in [3, 4] and [12].

Whenever the Betti numbers of a module are eventually strictly increasing it is known that the dimension of a sufficiently high syzygy will have the dimension of the ring. This is clear from the next lemma, which is mentioned without proof in [5 Remark 5.2 (iii)].

**Results**

We denote the length of an $R$-module $M$ by $\lambda_R(M)$ or simply $\lambda(M)$ when the ring is unambiguous.

**Lemma 5.** Let $R$ be a noetherian local ring and $M$ a finitely generated $R$-module. If $\beta_i(M) > \beta_{i-1}(M)$ for some $i > 0$, then $\Supp(\Omega_{i+1}(M)) = \Spec(R)$; hence $\dim(\Omega_{i+1}(M)) = \dim(R)$.

**Proof.** Given $q \in \Spec(R)$ there exists $p \in \Min(R)$ such that $p \subseteq q$. Localizing the exact sequence

$$0 \rightarrow \Omega^R_{i+1}(M) \rightarrow R^\beta_i(M) \rightarrow R^\beta_{i-1}(M)$$

at $p$ we obtain the following inequalities:

$$\lambda_{R_p}(\Omega_{i+1}(M)_p) \geq \lambda_{R_p}(R^\beta_i(M)) - \lambda_{R_p}(R^\beta_{i-1}(M)) = \lambda_{R_p}(R_p)(\beta_i(M) - \beta_{i-1}(M)) > 0.$$

Thus $p \in \Supp(\Omega_{i+1}(M))$; hence $q \in \Supp(\Omega_{i+1}(M))$ and the result follows. \qed
The following lemma is used in the proof of our main result, Theorem 8.

Lemma 6. Let \( R \) be a noetherian local ring and \( M \) a finitely generated \( R \)-module. For a given \( n \in \mathbb{N} \) suppose that \( \beta_0(M) \leq \beta_1(M) \leq \ldots \leq \beta_{2n-1}(M) \) and that \( \text{Supp}(\Omega_{2n}(M)) \neq \text{Spec}(R) \). Then we have the following:

(a) \( \beta_{2i}(M) = \beta_{2i+1}(M) \) for \( i = 0, \ldots, n-1 \);
(b) \( \text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M)) \) for \( i = 0, \ldots, n-1 \); and
(c) \( \text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R) = \text{Supp}(M) \cap \text{Min}(R) \).

Proof. Choose \( p \in \text{Min}(R) \setminus \text{Supp}(\Omega_{2n}(M)) \). Localizing part of a minimal free resolution of \( M \) at \( p \), we get an exact sequence of finite-length \( R_p \)-modules of the following form:

\[
0 \to R^\beta_{2n-1}(M)_p \xrightarrow{\varphi_{2n-1}} R^\beta_{2n-2}(M)_p \to \cdots \to R^\beta_0(M)_p \xrightarrow{\varphi_0} M_p \to 0.
\]

Since \( \varphi_{2n-1} \) is an injection, \( \lambda(R^\beta_{2n-2}(M)_p) \geq \lambda(R^\beta_{2n-1}(M)_p) \); hence \( \beta_{2n-2}(M)_p \geq \beta_{2n-1}(M)_p \). It follows that \( \beta_{2n-2}(M) = \beta_{2n-1}(M) \). Since \( R_p \) has finite length it follows that \( \varphi_{2n-1} \) is an isomorphism and \( \varphi_{2n-2} \) is the zero map. By repeating this argument, one sees that \( \varphi_{2i+1} \) is an isomorphism, \( \varphi_{2i} \) is the zero map, and \( \beta_{2i}(M) = \beta_{2i+1}(M) \) for each \( i = 0, 1, \ldots, n-1 \). In particular, we have shown (a).

Since \( \varphi_0 \) is the zero map, \( M_p = 0 \); hence \( p \notin \text{Supp}(M) \). It follows that

\[
\text{Supp}(M) \cap \text{Min}(R) \subseteq \text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R).
\]

Let \( q \in \text{Spec}(R) \setminus \text{Supp}(\Omega_{2}(M)) \) for some \( i \) with \( 0 \leq i \leq n-1 \). Localizing part of a minimal free resolution of \( M \) at \( q \) we obtain an exact sequence of the following form:

\[
0 \to \Omega_{2i+2}(M)_q \to R^\beta_{2i+1}(M)_q \to R^\beta_{2i}(M)_q \to 0.
\]

Since \( \beta_{2i+1}(M) = \beta_{2i}(M) \) it follows that \( \Omega_{2i+2}(M)_q = 0 \) and \( q \notin \text{Supp}(\Omega_{2i+2}(M)) \). Thus \( \text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M)) \) for \( i = 0, \ldots, n-1 \), which proves (b). Consequently \( \text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R) \subseteq \text{Supp}(M) \cap \text{Min}(R) \). Since (1) provides the reverse containment, (c) is now immediate.

We make the following fact explicit in order to clarify some of our argumentation.

Fact 7. Let \((R, m)\) be a noetherian local ring. Let \( B \) be an \( n \times m \) matrix with entries in \( R \) defining a map from \( R^n \) to \( R^m \). Applying invertible row and column operations to \( B \), one can obtain a matrix \( B' = I_h \oplus A \), where \( I_h \) is the \( h \times h \) identity matrix for some \( h \geq 0 \) and \( A \) is an \((n-h) \times (m-h)\) matrix with entries in \( m \).

Theorem 8. Let \( R \) be a noetherian local ring and \( M \) a finitely generated \( R \)-module with eventually non-decreasing Betti numbers. Then for all \( n \geq 0 \) we have the following:

(a) \( \text{Min}(\Omega_n(M)) \subseteq \text{Min}(R) \);
(b) \( \text{Supp}(\Omega_n(M)) = \text{Supp}(\Omega_{n+2i}) \) for all \( i \geq 0 \); and
(c) if \( \text{Supp}(\Omega_n(M)) \neq \text{Spec}(R) \), then \( \beta_{n+2i}(M) = \beta_{n+2i+1}(M) \) for all \( i \geq 0 \).

Proof. We may assume that \( \text{pd}(M) = \infty \). By replacing \( M \) by a sufficiently high syzygy, one may assume that \( \beta_{i+1}(M) \geq \beta_i(M) \) for all \( i \geq 0 \). Assuming \( M \) was replaced by an even (odd) syzygy, if \( \text{Supp}(\Omega_{2i}(M)) = \text{Spec}(R) \) for \( i \geq 0 \), then all of the statements hold for even (odd) syzygies. Therefore we may suppose that there exist infinitely many \( i \in \mathbb{N} \) such that \( \text{Supp}(\Omega_{2i}(M)) \neq \text{Spec}(R) \).
Since $\operatorname{Min}(R)$ is a finite set we may choose $p \in \operatorname{Min}(R)$ so that there are infinitely many $i \in \mathbb{N}$ for which $p \notin \operatorname{Supp}(\Omega_{2i}(M))$. For each positive integer $c$ such that $p \notin \operatorname{Supp}(\Omega_{2c}(M))$, Lemma 5 implies that we have $\operatorname{Supp}(\Omega_{2i+1}(M)) \subseteq \operatorname{Supp}(\Omega_{2i}(M))$ and $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $0 \leq i < c$. Since $c$ can be chosen to be arbitrarily large we have $\operatorname{Supp}(\Omega_{2i+2}(M)) \subseteq \operatorname{Supp}(\Omega_{2i}(M))$ and $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $i \geq 0$. Since closed sets in the Zariski topology satisfy the descending chain condition, it follows that we may choose $m \gg 0$ such that $(\operatorname{Supp}(\Omega_{2m+2}(M)))_n^{\infty}$ is constant, proving (3). Therefore the assumption that there exist infinitely many $i \in \mathbb{N}$ such that $\operatorname{Supp}(\Omega_{2i}(M)) \neq \operatorname{Spec}(R)$ is equivalent to assuming that $\operatorname{Supp}(\Omega_{2m}(M)) \neq \operatorname{Spec}(R)$ and (3) follows.

Therefore it remains to show that $\operatorname{Min}(\Omega_{2i}(M)) \subseteq \operatorname{Min}(R)$ for $i \gg 0$. Choose $q \in \operatorname{Min}(\Omega_{2m}(M))$. Let $S := R_q$, $M_i := (\Omega_{2m+2i}(M))_q$ for $i \geq 0$ and $n := qR_q$. Note that $n$ is the maximal ideal for $S$. For all $i \geq 0$ we obtain a commutative diagram of the form

$$
\begin{array}{ccccccccc}
0 & \rightarrow & M_{i+1} & \rightarrow & S^{b_i} & \rightarrow & S^{b_i} & \rightarrow & M_i & \rightarrow & 0 \\
& & \phi_i & \downarrow & \psi_i & & \downarrow & & \\
& & N_i & & \\
\end{array}
$$

where the top row is exact and $N_i := \operatorname{Im}(\alpha_i)$. If the matrix $A_i$ defining the map $\alpha_i : S^{b_i} \rightarrow S^{b_i}$ has some entries which are units, then by Fact 7 we can reduce this sequence by taking away free summands; hence we may assume that $A_i$ has all of its entries in $n$.

Let $H^j_n(-)$ denote the $i$th local cohomology functor with respect to $n$. For background on local cohomology see [7]. Since $M_{i+1}$ has finite length, $H^0_n(M_{i+1}) \cong M_{i+1}$ and $H^j_n(M_{i+1}) = 0$ for all $j > 0$. From the long exact sequence of local cohomology modules associated to the short exact sequence

$$
0 \rightarrow M_{i+1} \rightarrow S^{b_i} \xrightarrow{\phi_i} N_i \rightarrow 0
$$

we get an exact sequence

$$
0 \rightarrow M_{i+1} \rightarrow H^0_n(S^{b_i}) \rightarrow H^0_n(N_i) \rightarrow 0
$$

and isomorphisms $H^j_n(\phi_i) : H^j_n(S^{b_i}) \rightarrow H^j_n(N_i)$ for all $j \geq 1$. Also the exact sequence

$$
0 \rightarrow N_i \xrightarrow{\psi_i} S^{b_i} \rightarrow M_i \rightarrow 0
$$

yields an exact sequence

$$
0 \rightarrow H^0_n(N_i) \rightarrow H^0_n(S^{b_i}) \rightarrow M_i \xrightarrow{\gamma_i} H^1_n(N_i) \rightarrow H^1_n(S^{b_i}) \rightarrow 0
$$

and isomorphisms $H^j_n(\psi_i) : H^j_n(N_i) \rightarrow H^j_n(S^{b_i})$ for all $j \geq 2$. Here we are defining $\gamma_i : M_i \rightarrow H^1_n(N_i)$ to be the map found in exact sequence (3). By the additivity of length we get the first and third steps in the next display from sequences (3) and (2) respectively:

\[
\lambda(M_i) = \lambda(H^0_n(S^{b_i})) - \lambda(H^0_n(N_i)) + \lambda(\operatorname{Im}(\gamma_i)) \\
\geq \lambda(H^0_n(S^{b_i})) - \lambda(H^0_n(N_i)) \\
= \lambda(M_{i+1}).
\]
Since the sequence \((λ(M_i))_{i=0}^{∞}\) is positive and non-increasing it is eventually constant. Choose \(ℓ \in \mathbb{N}\) such that \(λ(M_ℓ) = λ(M_{ℓ+1})\). Then \(λ(\Im(γ_ℓ)) = 0\). Therefore \(γ_ℓ\) is the zero map. From (34), it follows that \(H^1_n(ψ_ℓ) : H^1_n(N_ℓ) \rightarrow H^1_n(S^{b_ℓ})\) is an isomorphism. We have shown that \(H^1_n(ψ_ℓ)\) and \(H^1_n(φ_ℓ)\) are isomorphisms for all \(j \geq 1\). Using the commutativity of (2) it follows that

\[H^1_n(α_ℓ) = H^1_n(ψ_ℓ) \circ H^1_n(φ_ℓ) : H^1_n(S^{b_ℓ}) \rightarrow H^1_n(S^{b_ℓ})\]

is an isomorphism for all \(j \geq 1\). Since \(H^1_n(−)\) is an \(S\)-linear functor the map \(H^1_n(α_ℓ)\) is defined by matrix multiplication from the matrix \(A_ℓ\) applied to the components of \(H^1_n(S^{b_ℓ})\). Since \(A_ℓ\) has entries in \(n\) it must kill socle elements of \(H^1_n(S^{b_ℓ})\). Therefore \(H^1_n(S^{b_ℓ})\) has no socle elements. Since \(H^1_n(S^{b_ℓ})\) is \(n\)-torsion it follows that \(H^1_n(S^{b_ℓ}) = 0\) for all \(j \geq 1\). By [7, Theorem 9.3] we get the second equality in the next display:

\[\dim(R_q) = \dim(S) = \sup\{j | H^1_n(S) \neq 0\} = 0.\]

Thus \(q \in \Min(R)\); hence \(\Min(Ω_{2i}(M)) \subseteq \Min(R)\) for all \(i \gg 0\), and (23) follows. □

**Corollary 9.** Let \(R\) be a noetherian local ring and \(M\) a finitely generated \(R\)-module with eventually non-decreasing Betti numbers. Then \((\dim(Ω_{2i}(M)))_{i=0}^{∞}\) and \((\dim(Ω_{2i+1}(M)))_{i=0}^{∞}\) are constant for \(i \gg 0\). If \(pd(M) = ∞\), then one sequence stabilizes to \(\dim(R)\) and the other sequence stabilizes to \(\dim(R/p)\) for some \(p \in \Min(R)\).

**Proof.** By Theorem 5[13], both sequences are constant for \(i \gg 0\). If \(pd(M) = ∞\), then \(\Min(Ω_i(M)) \neq 0\) for all \(i\). Therefore by Theorem 8[14] one sequence will stabilize to \(\dim(R/p)\) and the other to \(\dim(R/q)\) for some \(p, q \in \Min(R)\). Since \(\Supp(Ω_{2i}(M)) \cup \Supp(Ω_{2i+1}(M)) = \Spec(R)\), it follows that \(\dim(Ω_{2i}(M)) = \dim(R)\) or \(\dim(Ω_{2i+1}(M)) = \dim(R)\). □

Corollary 2 follows immediately. Note that if \(R\) is a domain or if \(\dim(R) \leq 1\), then \(R\) is equidimensional; hence, one can apply Corollary 2.

**Remark 10.** It should be noted that [5, Remark 5.6] claims that using [5, Proposition 5.5] one can show that if \(R\) is equidimensional and Conjecture 3 is true, then \(\dim(Ω_n(M))\) is constant for \(n \gg 0\). However, [5, Proposition 5.5] requires the assumption that \(\dim(R) \geq 2\). Therefore although the conclusions of [5, Remark 5.6] are correct, the justification given for these conclusions is invalid. One should note that the justification uses a localization argument, so it is invalid in every positive dimension, not just dimension 1.

We now turn our attention to determining how quickly \((\Supp(Ω_{2i}(M)))_{i=0}^{∞}\) stabilizes once the Betti numbers of \(M\) become non-decreasing.

**Lemma 11.** Let \(R\) be a noetherian local ring and \(M\) a finitely generated \(R\)-module.

If \(β_i(M) = β_{i+1}(M)\) for some \(i > 0\), then \(\Supp(Ω_i(M)) = \Supp(Ω_{i+2}(M))\).

Suppose \(β_0(M) = β_1(M)\). Then we have the following:

(a) If \(\Supp(M) \setminus \Supp(Ω_2(M)) \neq ∅\), then \(M\) is not a first syzygy.

(b) If \(p \in \Min(M) \setminus \Supp(Ω_2(M))\), then \(\height(p) = 1\).

**Proof.** Suppose \(β_0(M) = β_1(M)\) and \(\Supp(M) \setminus \Supp(Ω_2(M)) \neq ∅\). Consider the exact sequence

\[0 \rightarrow Ω_2(M) \rightarrow R^{β_1(M)} \rightarrow R^{β_0(M)} \rightarrow M \rightarrow 0.\]
Choose $p \in \text{Min}(M) \setminus \text{Supp}(\Omega_2(M))$. Since $M_p$ has finite length as an $R_p$-module, the complex

$$0 \to R_p^{\beta_1(M)} \to R_p^{\beta_2(M)} \to 0$$

has non-zero finite length homology. By the New Intersection Theorem \textsuperscript{10} it follows that $\dim(R_p) \leq 1$.

Fact \textsuperscript{7} implies that there exists a minimal $R_p$-free resolution of $M_p$ of the form

$$0 \to R_p^n \to R_p^n \to M_p \to 0$$

for some $n > 0$. Therefore $1 \geq \dim(R_p) \geq \text{pd}_{R_p}(M_p) = 1$; hence height($p$) = dim($R_p$) = 1, depth($M_p$) = 0 and depth$_R(R_p) = \text{depth}(M_p) + \text{pd}(M_p) = 1$.

Assume that $M = \Omega_1(L)$ for some $R$-module $L$. We will obtain a contradiction. Since $M_p$ is finite length and dim($R_p$) = 1, it follows that $M_p$ has no $R_p$-free summands. Therefore $M_p = \Omega_1^{R_p}(L_p)$. Since $M_p$ is a syzygy,

$$0 = \text{depth}(M_p) \geq \min\{1, \text{depth}(R_p)\} = 1.$$

This is a contradiction; hence $M$ is not a first syzygy.

Now suppose that $\beta_i(M) = \beta_{i+1}(M)$ for some $i > 0$. Since $\Omega_1(M)$ is a first syzygy of $\Omega_{i-1}(M)$ it follows that $\text{Supp}(\Omega_1(M)) \subseteq \text{Supp}(\Omega_{i+2}(M))$. By Lemma \textsuperscript{6} we get the opposite inclusion and the result follows. \hfill $\square$

The following is an example where we have $\beta_1(M) = \beta_0(M)$ and $\text{Supp}(M) \not\subseteq \text{Supp}(\Omega_2(M))$.

**Example 12.** Let $S = k[x, y, z]$ and $m = (x, y, z)$. Let $R = S_m/yzS_m$ and let $M = R/xyR$. The complex

$$\cdots \to R \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{xy} R \to 0$$

is a minimal free resolution of $M$. We have $\Omega_2(M) \cong zR \cong R/(y)$. The prime ideal $p = (x, z)$ of height 1 is in $\text{Supp}(R)$, but it is not in $\text{Supp}(\Omega_2(M))$.

**Proposition 13.** Let $R$ be a noetherian local ring and $M$ a finitely generated $R$-module with non-decreasing Betti numbers. Then either $\text{Supp}(\Omega_{2i}(M))$ is constant for all $i \geq 1$ or there exists $n \geq 1$ such that $\text{Supp}(\Omega_{2i}(M)) = \text{Spec}(R)$ for all $i > n$ and $\text{Supp}(\Omega_{2j}(M))$ is constant for $1 \leq j \leq n$.

**Proof.** Suppose $\text{Supp}(\Omega_{2i}(M)) \neq \text{Spec}(R)$ for some $i \geq 2$. By Lemma \textsuperscript{6} it follows that $\beta_{2j}(M) = \beta_{2j+1}(M)$ for all $j$ with $0 \leq j < i$. From Lemma \textsuperscript{11} we get that $\text{Supp}(\Omega_{2j}(M))$ is constant for $1 \leq j \leq i$.

Now suppose there exists $n$ such that $\text{Supp}(\Omega_{2n+2}(M)) = \text{Spec}(R)$. Assume that $\text{Supp}(\Omega_{2n+2}(M)) \neq \text{Spec}(R)$ for some $i > 0$. Then by Lemma \textsuperscript{6} it follows that

$$\text{Supp}(\Omega_{2n+2}(M)) \cap \text{Min}(R) = \text{Supp}(\Omega_{2n+2}(M)) \cap \text{Min}(R) = \text{Min}(R).$$

Thus $\text{Supp}(\Omega_{2n+2}(M)) = \text{Spec}(R)$, a contradiction. Therefore $\text{Supp}(\Omega_{2n+2}(M)) = \text{Spec}(R)$ for all $i > 0$ and the result follows. \hfill $\square$

We conclude with a few key examples. The following example is due to Hamid Rahmati and can be found in [5].

**Example 14.** Let $R = k[[x, y]]/(x^2, xy)$ and $M = R/(y)$. A minimal free resolution of $M$ has the form

$$\cdots \to \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \to R^2 \xrightarrow{[x, y]} R \xrightarrow{x} R \xrightarrow{y} R \to 0.$$
Also \( \dim(M) = \dim(\Omega_2(M)) = 0 \) and \( \dim(\Omega_i(M)) = 1 = \dim(R) \) for \( i \neq 0, 2 \).

In the following example we construct a module such that the support of each of its odd syzygies is equal to \( \text{Spec}(R) \) while the support of each of its even syzygies is not.

**Example 15.** Let \( R = [a, b, c, d, e]/(ade - bce) \). Let \( M \) be the cokernel of the first map in the following matrix factorization:

\[
\ldots \rightarrow \begin{bmatrix} a \\ c \\ d \end{bmatrix} \rightarrow R^2 \rightarrow \begin{bmatrix} d \\ e \\ -be \\ ce \\ ae \end{bmatrix} \rightarrow R^2 \rightarrow \begin{bmatrix} a \\ c \\ d \end{bmatrix} \rightarrow \ldots
\]

Then \( \text{Supp}(\Omega_{2i+1}(M)) = \text{Spec}(R) \) and \( \text{Supp}(\Omega_{2i}(M)) = \text{Supp}(R/(ad - bc)) \neq \text{Spec}(R) \) for all \( i \geq 0 \).

The following example is due to Craig Huneke and can be found in [5].

**Example 16.** Let \( S = \mathbb{Q}[x, y, z, u, v] \) and let \( I \subseteq S \) be the ideal

\[
I = \langle x^2, xz, z^2, xu, zv, u^2, v^2, zv + xv + uv, yu, yv, yx - zu, yz - xv \rangle.
\]

Let \( R = S/I \), which is a 1-dimensional ring of depth 0. A computation using Macaulay2 [6] yields that \( y \) is a parameter, \( (0 : y) = (u, v, z^2) \) and \( (y) = (0 : R (0 : R y)) \). Let \( M \) be the cokernel of the rightmost map in the following exact complex:

\[
\ldots \rightarrow R^3 \rightarrow \begin{bmatrix} u \\ v \\ z^2 \end{bmatrix} \rightarrow R \rightarrow \begin{bmatrix} y \\ z^2 \end{bmatrix} \rightarrow R^3 \rightarrow \ldots
\]

Then the first and third syzygy modules of \( M \) are \( R/(y) \) and \( (0 : y) \) respectively. These are both modules of finite length since \( y \) is a parameter, but all other syzygies have dimension 1.

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