Generalized nonlocal Robin Laplacian on arbitrary domains

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Abstract. In this paper, we prove that it is always possible to define a realization of the Laplacian $\Delta_{\kappa,\theta}$ on $L^2(\Omega)$ subject to nonlocal Robin boundary conditions with general jump measures on arbitrary open subsets of $\mathbb{R}^N$. This is made possible by using a capacity approach to define an admissible pair of measures $(\kappa, \theta)$ that allows the associated form $E_{\kappa,\theta}$ to be closable. The nonlocal Robin Laplacian $\Delta_{\kappa,\theta}$ generates a sub-Markovian $C_0$-semigroup on $L^2(\Omega)$ which is not dominated by the Neumann Laplacian semigroup unless the jump measure $\theta$ vanishes.

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1. Introduction. The main goal of the present paper is to prove, by means of a capacity approach, the existence of the realization in $L^2(\Omega)$ of the Laplacian with nonlocal Robin boundary conditions involving general jump measures on the boundary. Given a Borel measure $\kappa$ on $\partial\Omega$ and a symmetric Radon measure $\theta$ on $\partial\Omega \times \partial\Omega \setminus d$, where $d$ indicates the diagonal of $\partial\Omega \times \partial\Omega$, we define the bilinear symmetric form $E_{\kappa,\theta}$ on $L^2(\Omega)$ by

$$E_{\kappa,\theta}(u,v) = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \kappa + \int_{\partial\Omega \times \partial\Omega \setminus d} (u(x) - u(y))(v(x) - v(y)) \, d\theta$$

whenever the right hand side is meaningful in $H^1(\Omega) \cap C_c(\bar{\Omega})$. The first question to answer in this paper is whether the symmetric positive form $E_{\kappa,\theta}$ is closable. This will be done by introducing the notion of an admissible pair of measures;
i.e., the measure $\kappa + \hat{\theta}$ charges no relative polar sets, where $\hat{\theta}(dx) := \theta(dx, \partial \Omega)$. This shows that the closability of the form $\mathcal{E}_{\kappa, \theta}$ is inherent to the measure $\hat{\theta}$ rather than to the measure $\theta$ itself. The associated self-adjoint operator $\Delta_{\kappa, \theta}$ generates a sub-Markovian semigroup which dominates the Dirichlet Laplacian semigroup but is not dominated by the Neumann Laplacian semigroup; i.e.,

$$0 \leq e^{t\Delta_D} \leq e^{t\Delta_{\kappa, \theta}} \not\preceq e^{t\Delta_N} \quad (t \geq 0)$$

holds in the positive operators sense. This is essentially due to the nonlocality term expressed with the jump measure $\theta$.

Many authors have investigated local and nonlocal Robin boundary conditions. For local boundary conditions, see e.g. [1–3,5,6,11,21], where the existence of the realization of the Robin Laplacian, regularity, domination results, and qualitative properties of the semigroup are the main discussed subjects. The linear and nonlinear nonlocal Robin boundary conditions have captured a particular attention in recent years, see [14,18–20]. All previous works about linear nonlocal Robin boundary conditions deal with jump measures of the form $\theta(dx, dy) = \frac{1}{|x-y|^{N+2}} \kappa(dx)\kappa(dy)$, $x, y \in \partial \Omega$, while here we consider general symmetric Radon measures $\theta$ which justifies the designation “generalized” nonlocal Robin boundary conditions.

The rest of this paper is structured as follows. In Section 2, we recall the concept of relative capacity and some results from Dirichlet forms theory. In Section 3, we give a characterization of the realization on $L^2(\Omega)$ of the Laplacian with general nonlocal Robin boundary conditions. In Section 4, we prove that the self-adjoint operator $\Delta_{\kappa, \theta}$ generates a sub-Markovian $C_0$-semigroup on $L^2(\Omega)$. The results about the domination of the semigroup $e^{t\Delta_{\kappa, \theta}}$ are detailed in Section 5.

2. Preliminaries. Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$. The following space is fundamental in the rest of this paper

$$\widetilde{H}^1(\Omega) = \overline{H^1(\Omega) \cap C_c(\overline{\Omega})}^{H^1(\Omega)},$$

where $C_c(\overline{\Omega})$ denotes the space of all continuous real-valued functions with compact support in $\overline{\Omega}$ and $H^1(\Omega)$ the usual Sobolev space endowed with its usual norm. Given an arbitrary subset of $\mathbb{R}^N$, the relative capacity $\text{Cap}_{\overline{\Omega}}(A)$ (i.e., with respect to the fixed open set $\Omega$) is defined for an arbitrary open set $A$ of $\overline{\Omega}$ by

$$\text{Cap}_{\overline{\Omega}}(A) = \inf \{ ||u||_{H^1(\Omega)} : u \in \widetilde{H}^1(\Omega), \exists O \subset \mathbb{R}^N \text{ open such that } A \subset O \text{ and } u(x) \geq 1 \text{ a.e. on } \Omega \cap O \}.$$  

We consider the topological space $X = \overline{\Omega}$, the $\sigma$-algebra $\mathcal{B}(X)$ of all Borel sets in $X$, and the measure $m$ on $\mathcal{B}(X)$ given by $m(A) = \lambda(A \cap \Omega)$ for all $A \in \mathcal{B}(X)$ with $\lambda$ the Lebesgue measure. Denoting by $L^2(\Omega)$ the usual $L^2$-space with respect to the Lebesgue measure, we then have $L^2(\Omega) = L^2(X, \mathcal{B}(X), m)$. The introduction of $m$ is needed to ensure this identity in the case where $\partial \Omega$ has positive Lebesgue measure. Moreover, we have that $\widetilde{H}^1(\Omega) \cap C_c(\overline{\Omega})$ is dense in $\widetilde{H}^1(\Omega)$. Thus the Dirichlet form $\langle \mathcal{E}, \mathcal{D} \rangle$ on $L^2(X, \mathcal{B}(X), m)$ is regular.
As introduced in [5], the relative capacity of a subset $A$ of $\bar{\Omega}$ is exactly the capacity of $A$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mathcal{B}(X), m)$ in the sense of [7, 8.1.1, p. 52].

**Definition 2.1.**

1. A subset $A$ of $\bar{\Omega}$ is called relatively polar if $\text{Cap}_{\mathcal{E}}(A) = 0$.
2. We say that a property holds on $\bar{\Omega}$ relatively quasi-everywhere (r.q.e.) if it holds for all $x \in \bar{\Omega} \setminus N$, where $N \subset \Omega$ is relatively polar.
3. A scalar function $u$ on $\bar{\Omega}$ is called relatively quasi-continuous if for each $\varepsilon > 0$, there exists an open set $G \subset \mathbb{R}^N$ such that $\text{Cap}_{\mathcal{E}}(\Omega \cap \bar{\Omega}) < \varepsilon$ and $u$ is continuous on $\bar{\Omega} \setminus G$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and define $H^1_0(\Omega) := \overline{D(\Omega)^H}^{H^1(\Omega)}$, where $D(\Omega)$ denotes the space of all infinitely differentiable functions with compact support. In [5], the space $H^1_0(\Omega)$ is characterized with the help of relative capacity as follows

$$H^1_0(\Omega) = \{ u \in \overline{H^1(\Omega)} : \bar{u}(x) = 0 \text{ r.q.e on } \partial \Omega \}.$$ 

Now, we recall some results from Dirichlet form theory. Let $H$ be a real Hilbert space. A positive form on $H$ is a bilinear mapping $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ such that $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ and $\mathcal{E}(u, u) \geq 0$ for all $u, v \in D(\mathcal{E})$ (the domain of the form is a dense subspace of $H$). The form is closed if $D(\mathcal{E})$ is complete for the norm $\|u\|_{\mathcal{E}} = (\mathcal{E}(u, u) + \|u\|_H^2)^{1/2}$. Then the operator $A$ on $H$ associated with $\mathcal{E}$ is defined by

$$A := \{ u \in D(\mathcal{E}) \mid \exists v \in H, \mathcal{E}(u, \varphi) = (v, \varphi)_H \forall \varphi \in D(\mathcal{E}) \},$$

$$D(A) := \{ u \in D(\mathcal{E}) \mid \exists v \in H, \mathcal{E}(u, \varphi) = (v, \varphi)_H \forall \varphi \in D(\mathcal{E}) \}.$$ 

The operator $A$ is self-adjoint and $-A$ generates a contraction semigroup $(e^{-tA})_{t \geq 0}$ of symmetric operators on $H$. Moreover, the form $\mathcal{E}$ is called closable if for each Cauchy sequence $(u_n)_{n \in \mathbb{N}}$ in $(D(\mathcal{E}), \| \cdot \|_{\mathcal{E}})$, $\lim_{n \to \infty} u_n = 0$ in $H$ implies $\lim_{n \to \infty} \mathcal{E}(u_n, u_n) = 0$. In that case, the closure $\hat{\mathcal{E}}$ is the unique positive closed form extending $\mathcal{E}$ such that $D(\hat{\mathcal{E}})$ is dense in $D(\mathcal{E})$.

Now assume that $H = L^2(\Omega)$ where $(\Omega, \Sigma, \lambda)$ is a $\sigma$-finite measure space. We let $L^2(\Omega)^+ = \{ f \in L^2(\Omega) : f \geq 0 \text{ a.e.} \}$ and $F_+ = L^2(\Omega)^+ \cap F$ if $F$ is a subspace of $L^2(\Omega)$.

**Theorem 2.2** (First Beurling-Deny criterion). Let $S$ be the semigroup associated with a closed, positive form $\mathcal{E}$ on $L^2(\Omega)$. Then $S$ is positive (i.e., $S(t)L^2(\Omega)^+ \subset L^2(\Omega)^+$ for all $t \geq 0$) if and only if $u \in D(\mathcal{E})$ implies $|u| \in D(\mathcal{E})$ and $\mathcal{E}(|u|) \leq \mathcal{E}(u)$.

**Theorem 2.3** (Second Beurling-Deny criterion). Let $S$ be the semigroup associated with a closed, positive form $\mathcal{E}$ on $L^2(\Omega)$. Assume that $S$ is positive. Then $S$ is $L^\infty$-contractive (i.e., if $f \in L^2(\Omega)$ satisfies $0 \leq f \leq 1$, then $0 \leq S(t)f \leq 1$ for all $t \geq 0$) if and only $0 \leq u \in D(\mathcal{E})$ implies $u \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u \wedge 1) \leq \mathcal{E}(u)$.

We say that the semigroup $S$ is sub-Markovian if it is positive and $L^\infty$-contractive. A Dirichlet form is a closed positive form satisfying the two
Beurling-Deny criteria. Now let $\mathcal{E}$ and $\mathcal{F}$ be two closed, positive forms on $L^2(\Omega)$ such that the associated semigroups $S$ and $T$ are positive. We say that $D(\mathcal{E})$ is an ideal of $D(\mathcal{F})$ if $u \in D(\mathcal{E})$ implies $|u| \in D(\mathcal{F})$ and $0 \leq u \leq v$, $v \in D(\mathcal{E})$, $u \in D(\mathcal{F})$ implies $u \in D(\mathcal{E})$. The Ouhabaz domination criterion [15] says that $0 \leq S(t) \leq T(t)$ ($t \geq 0$) if and only if $D(\mathcal{E})$ is an ideal of $D(\mathcal{F})$ and $\mathcal{E}(u, v) \geq \mathcal{F}(u, v)$ for all $u, v \in D(\mathcal{E})_+$.

3. Closability. Let $\Omega$ be an open bounded set of $\mathbb{R}^N$. Let $\kappa$ be a Borel measure on $\partial \Omega$ and $\theta$ a symmetric Radon measure on $\partial \Omega \times \partial \Omega \setminus d$, where $d$ indicates the diagonal of $\partial \Omega \times \partial \Omega$. Let

$$E = \{u \in H^1(\Omega) \cap C_c(\overline{\Omega}) : \int_{\partial \Omega} |u|^2 d\kappa + \int_{\partial \Omega \times \partial \Omega \setminus d} (u(x) - u(y))^2 d\theta < \infty\}.$$ 

We define the bilinear symmetric form $\mathcal{E}_{\kappa, \theta}$ with domain $E$ on $L^2(\Omega)$ by

$$\mathcal{E}_{\kappa, \theta}(u, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\partial \Omega} uv d\kappa + \int_{\partial \Omega \times \partial \Omega \setminus d} (u(x) - u(y))(v(x) - v(y))d\theta.$$ 

It is natural to ask whether $(a_{\kappa, \theta}, E)$ is closable in $L^2(\Omega)$ or not. The cases $\theta \equiv 0$ and $\theta(dx, dy) = \frac{1}{|x-y|^{N+2}} \kappa(dx)\kappa(dy)$ are treated in [5] and [10], respectively.

Example. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Fix $z, z' \in \partial \Omega$ and let $d\kappa = d\delta_z$ and $d\theta = \frac{1}{2} (d\delta_{z} \otimes \delta_{z'} + d\delta_{z'} \otimes \delta_z)$, where $\delta_n$ is the Dirac measure in $a$ and $d\delta_{z} \otimes \delta_{z'}(x, y) = d\delta_{z}(x) d\delta_{z'}(y)$. For such measures, we have

$$\mathcal{E}_{\kappa, \theta}(u) = \int_{\Omega} |\nabla u|^2 dx + u^2(z) + (u(z) - u(z'))^2.$$ 

Since $H^1(\Omega) \cap C_c(\overline{\Omega})$ is dense in $H^1(\Omega)$, there exists $u_n \in H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $u_n \to 0$ in $H^1(\Omega)$ as $n \to \infty$ with $u_n(z) = 1$ and $u_n(z') = 0$ for all $n \geq 1$. For such sequence, we have $\lim_{n,m \to \infty} \mathcal{E}_{\kappa, \theta}(u_n - u_m) = 0$ but $\lim_{n,m \to \infty} \mathcal{E}_{\kappa, \theta}(u_n) = 2$.

The examples show that the form $(\mathcal{E}_{\kappa, \theta}, E)$ is not closable in general, but as it is a positive form, then by means of the Reed-Simon theorem [16, Theorem S15, p. 373], we know that there exists a closable positive form $(\mathcal{E}_{\kappa, \theta}, r) \leq \mathcal{E}_{\kappa, \theta}$ such that $\mathcal{F} \leq (\mathcal{E}_{\kappa, \theta}, r)$ whenever $\mathcal{F}$ is a closable form such that $\mathcal{F} \leq \mathcal{E}_{\kappa, \theta}$. Thus $(\mathcal{E}_{\kappa, \theta}, r)$ is the largest closable form smaller or equal than $\mathcal{E}_{\kappa, \theta}$. Clearly, $\mathcal{E}_{\kappa, \theta}$ is closable if $\kappa \neq 0$.

Define the Borel measure $\hat{\theta}(dx) := \theta(dx \times \partial \Omega \setminus d)$. We want to determine $(\mathcal{E}_{\kappa, \theta}, r)$ and, in particular, characterize when $\mathcal{E}_{\kappa, \theta}$ is closable. One can see that the closability of the form $\mathcal{E}_{\kappa, \theta}$ is inherent to the measure $\hat{\theta}$ rather than to the measure $\theta$. We say that a Borel measure $\nu$ on $\partial \Omega$ is locally infinite everywhere if for all $z \in \partial \Omega$ and $r > 0$, $\nu(\partial \Omega \cap B(z, r)) = \infty$. If we assume that $\kappa \neq 0$ or $\nu$ is not.
are locally infinite everywhere on $\partial \Omega$, then the form $\mathcal{E}_{\kappa, \theta}$ is closable and its closure, which we denote by $\mathcal{E}_\infty$, is given by

$$\mathcal{E}_\infty(u, v) = \int_\Omega \nabla u \nabla v \, dx,$$

with domain $H^1_0(\Omega)$. This can be proved by remarking that

$$\int_{\partial \Omega \times \partial \Omega \setminus d} (u(x) - u(y))^2 \, d\theta = 2 \int_{\partial \Omega} u^2(x) \hat{\theta}(dx) - 2 \int_{\partial \Omega \setminus \partial \Omega \setminus d} u(x)u(y) \, d\theta,$$

and by following the proof of [21, Proposition 3.2.1]. Now let $\Gamma_{\kappa, \theta} := \{ z \in \partial \Omega : \exists r > 0 \text{ such that } \kappa(\partial \Omega \cap B(z, r)) + \hat{\theta}(\partial \Omega \cap B(z, r)) < \infty \}$ be the part of $\partial \Omega$ on which $\kappa$ and $\hat{\theta}$ are locally finite. It is easy to verify that $\tilde{u} = 0$ on $\partial \Omega \setminus \Gamma_{\kappa, \theta}$ for all $u \in E$. Since $\Gamma_{\kappa, \theta}$ is a locally compact metric space, it follows from [17, Theorem 2.18, p.48] that $\kappa$ and $\hat{\theta}$ are regular Borel measures on $\Gamma_{\kappa, \theta}$. Therefore $\kappa$ and $\hat{\theta}$ are Radon measures on $\Gamma_{\kappa, \theta}$. We introduce the notion of an admissible pair of measures. We say that the pair of measures $(\kappa, \theta)$ is admissible if for each Borel set $A \subset \Gamma_{\kappa, \theta}$, one has

$$\text{Cap}_\infty(A) = 0 \Rightarrow (\kappa + \hat{\theta})(A) = 0,$$

where $\text{Cap}_\infty(A)$ is the relative capacity of the subset $A$.

The following lemma is an adaptation of [8, Lemma 4.1.1], and will be used to prove Theorem 3.2.

**Lemma 3.1.** Let $B$ be a Borel set of $\partial \Omega$ such that $\hat{\theta}(B) > 0$. Then there exist disjoint compact sets $K$ and $C$ such that $K \subset B$ and $\theta(K \times C) > 0$.

The proof of the following theorem is based on the techniques in the proofs of [21, Theorem 3.1.1] and [8, Theorem 4.1.2].

**Theorem 3.2.** The form $\mathcal{E}_{\kappa, \theta}$ is closable if and only if the pair $(\kappa, \theta)$ is admissible.

**Proof.** ($\Leftarrow$) Let $u_k \in E$ be such that $u_k \rightarrow 0$ in $L^2(\Omega)$ and $\lim_{n, k \rightarrow \infty} \mathcal{E}_{\kappa, \theta}(u_n - u_k, u_n - u_k) = 0$. It is clear that $u_k \rightarrow 0$ in $H^1(\Omega)$. By [21, Theorem 2.1.3] applied to the relative capacity, the sequence $(u_k)$ contains a subsequence which converges to zero r.q.e. on $\Omega$. Since $\kappa + \hat{\theta}$ charges no set of zero relative capacity, it follows that $u_k|_{\partial \Omega} \rightarrow 0$ $\kappa$ a.e. and $\hat{\theta}$ a.e. Since $u_k$ is a Cauchy sequence in $L^2(\partial \Omega, \kappa)$, it follows that $u_k \rightarrow 0$ in $L^2(\partial \Omega, \kappa)$. Since $u_k|_{\partial \Omega} \rightarrow 0$ $\hat{\theta}$ a.e., thus $u_k(x) - u_k(y) \rightarrow 0$ $\theta$ a.e. Since $u_k(x) - u_k(y)$ is a Cauchy sequence in $L^2(\partial \Omega \times \partial \Omega \setminus d, \theta)$, it follows that $u_k(x) - u_k(y) \rightarrow 0$ in $L^2(\partial \Omega \times \partial \Omega \setminus d, \theta)$ and thus $\lim_{k \rightarrow \infty} \mathcal{E}_{\kappa, \theta}(u_k, u_k) = 0$, which means that the form $(\mathcal{E}_{\kappa, \theta}, E)$ is closable.

($\Rightarrow$) Suppose that there exists a Borel set $B$ such that $\text{Cap}_\infty(B) = 0$ and $\hat{\theta}(B) > 0$ or $\kappa(B) > 0$. We show that $\mathcal{E}_{\kappa, \theta}$ is not closable. We just consider the case where $\hat{\theta}(B) > 0$. The case where $\kappa(B) > 0$ can be proved simultaneously.

By Lemma 3.1, we can choose disjoint compact sets $K$ and $C$ such that $\theta(C \times K) > 0$ and $K \subset B$ (and therefore $\text{Cap}_\infty(K) = 0$).
Since $\text{Cap}_{\mathbb{P}}(K) = 0$, by [21, Theorem 2.2.4], there exists a sequence $u_k \in H^1(\Omega) \cap C_c(\overline{\Omega})$ such that

$$0 \leq u_k \leq 1, \quad u_k = 1 \text{ on } K, \quad \text{and } \|u_k\|_{H^1(\Omega)} \to 0.$$ 

Let $(A_i)$ be a sequence of relatively open sets with compact closure satisfying

$$K \subset \overline{A_{i+1}} \subset A_i \subset \partial \Omega \text{ and } \bigcap_i \overline{A_i} = K.$$ 

There exists then a sequence $v_i \in \mathcal{D}(\mathbb{R}^N)$ such that $\text{supp}[v_i] \subset A_i$, $v_i = 1$ on $K$, and $0 \leq v_i \leq 1$. Clearly, $v_i \iota_\Omega \in H^1(\Omega) \cap C_c(\overline{\Omega})$ and $\|u_k v_i\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. For all $i \geq 1$, we have $u_k v_i \in H^1(\Omega) \cap C_c(\overline{\Omega})$. Moreover, for all $i, k$, $0 \leq u_k v_i \leq 1$ and $u_k v_i = 1$ on $K$. For all $i \geq 1$, we choose $k_i \in \mathbb{N}$ such that $\|u_{k_i} v_i\|_{H^1(\Omega)} \leq \frac{1}{2^i}$. Let $w_i = u_{k_i} v_i$. Then $w_i \to 0$ in $H^1(\Omega)$, $0 \leq w_i \leq 1$, and $w_i = 1$ on $K$. Moreover, $w_i \to \chi_K$ pointwise since $\text{supp}[w_i] \subset A_i$.

One has $\sup_i \|w_i\|_{\kappa, \theta}^2 < \infty$. In fact

$$\sup_i \int (w_i(x) - w_i(y))^2 d\theta \leq 2 \sup_i \int w_i^2(x) d\hat{\theta} \leq 2 \hat{\theta}(K) < \infty,$$

and

$$\sup_i \int w_i^2(x) d\kappa \leq \kappa(K) < \infty.$$

Since $K$ and $C$ are disjoint compact sets of $\partial \Omega$ and $H^1(\Omega) \cap C_c(\overline{\Omega})$ is dense in $(C_c(\overline{\Omega}), \|\cdot\|_\infty)$, we can choose $f \in H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $f(x) \geq 1$ for all $x \in K$ and $|f(x)| \leq \frac{1}{4}$ for all $x \in C$. Since $\mathcal{E}_{\kappa, \theta}$ is sub-Markovian, its closure is a Dirichlet form. Thus, by [7, Theorem 1.4.2, p. 14],

$$\|uv\|_{\kappa, \theta} \leq \|u\|_\infty \|v\|_{\kappa, \theta} + \|v\|_\infty \|u\|_{\kappa, \theta} \quad \forall u, v \in D(\mathcal{E}_{\kappa, \theta}).$$

We deduce that $\sup_i \|fw_i\|_{\kappa, \theta} < \infty$. Define $h_n = \frac{1}{n} \sum_{i=1}^{n} fw_i$. Thus, selecting a subsequence if necessary, we may assume that $(h_n)_{n \geq 0}$ is a Cauchy sequence in $(\mathcal{E}_{\kappa, \theta}, D(\mathcal{E}_{\kappa, \theta}))$, and therefore convergent in $L^2(\Omega)$. Since $h_n \to 0$ a.e., we have $h_n \to 0$ in $L^2(\Omega)$. Since $\mathcal{E}_{\kappa, \theta}$ is closable this implies that $\mathcal{E}_{\kappa, \theta}(h_n, h_n) \to 0$.

On the other hand and by the choice of $f$, we can choose $n_0$ large enough such that, for any $n \geq n_0$, we have $h_n(x) \geq \frac{3}{4} \forall x \in K$ and $h_n(y) \leq \frac{1}{2} \forall y \in C$. Thus

$$\limsup_{n \to \infty} \int (h_n(x) - h_n(y))^2 d\theta \geq \frac{1}{16} \theta(K \times C) > 0,$$
and therefore $\limsup_{n \to \infty} E_{\kappa, \theta}(h_n, h_n) > 0$. The existence of such a sequence contradicts the closability of $E_{\kappa, \theta}$. \hfill \Box

The proof of the following theorem is much the same as the similar result in [21] or in [5].

**Theorem 3.3.** Let $\kappa$ (resp. $\theta$) be a Radon measure on $\partial \Omega$ (resp. $\partial \Omega \times \partial \Omega \setminus d$). Suppose that the pair $(\kappa, \theta)$ is admissible. Then the closure of $(E_{\kappa, \theta}, E)$ is given by

$$D = \{ u \in \tilde{H}^1(\Omega) : \int_{\partial \Omega} |\tilde{u}|^2 d\kappa + \frac{1}{2} \int_{\partial \Omega \times \partial \Omega \setminus d} (\tilde{u}(x) - \tilde{u}(y))^2 d\theta < \infty \},$$

$$E_{\kappa, \theta}(u, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\partial \Omega} \tilde{u} \tilde{v} d\kappa + \frac{1}{2} \int_{\partial \Omega \times \partial \Omega \setminus d} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y)) d\theta,$$

where $\tilde{u}$ is a relative quasi-continuous representation of $u$.

**4. Markov property.** We denote by $-\Delta_{\kappa, \theta}$ the self-adjoint operator associated with the closure of the form $E_{\kappa, \theta}$; i.e.,

$$\begin{align*}
D(\Delta_{\kappa, \theta}) := & \{ u \in \mathcal{D} : \exists v \in L^2(\Omega) : E(u, \varphi) = (v, \varphi) \forall \varphi \in \mathcal{D} \}, \\
\Delta_{\kappa, \theta} := & -v.
\end{align*}$$

If we choose $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\langle -\Delta u, \varphi \rangle = \langle v, \varphi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{D}(\Omega)'$ and $\mathcal{D}(\Omega)$. Since $\varphi \in \mathcal{D}(\Omega)$ is arbitrary, it follows that

$$-\Delta u = v \quad \text{in } \mathcal{D}(\Omega)'.$$

Thus $\Delta_{\kappa, \theta}$ is a realization of the Laplacian in $L^2(\Omega)$.

**Proposition 4.1.** The operator $\Delta_{\kappa, \theta}$ generates a symmetric sub-Markovian semigroup on $L^2(\Omega)$.

**Proof.** a) Let $u \in D(E)$. We have to show that $|u| \in D(E)$ and $E(|u|, |u|) \leq E(u, u)$. Indeed, using the reverse triangle inequality $||u|(x) - |u|(y)| \leq |u(x) - u(y)|$, we get that $E(|u|, |u|) \leq E(u, u)$. It follows from the first Beurling-Deny criterion that $e^{t\Delta_{\kappa, \theta}} \geq 0$ for all $t \geq 0$.

b) Let $0 \leq u \in D(E)$. We have

$$\begin{align*}
\int_{\partial \Omega \times \partial \Omega \setminus d} ((u \wedge 1)(x) - (u \wedge 1)(y))^2 d\theta = & \int_{\{u \leq 1\} \times \{u \leq 1\} \setminus d} (u(x) - u(y))^2 d\theta \\
& + \int_{\{u \leq 1\} \times \{u > 1\} \setminus d} (u(x) - 1)^2 d\theta + \int_{\{u > 1\} \times \{u \leq 1\} \setminus d} (1 - u(y))^2 d\theta.
\end{align*}$$
On \( \{u \leq 1\} \times \{u > 1\} \setminus d \), we have \( 0 \leq u(x) \leq 1 < u(y) \). Thus \( 0 \leq 1 - u(x) < u(y) - u(x) \). Similarly, on \( \{u > 1\} \times \{u \leq 1\} \setminus d \), we have \( 0 \leq u(y) \leq 1 < u(x) \).

Thus \( 0 \leq 1 - u(y) < u(x) - u(y) \). It follows that

\[
\int_{\partial \Omega \times \partial \Omega \setminus d} \left( (u \wedge 1)(x) - (u \wedge 1)(y) \right)^2 d\theta = \int_{\{u \leq 1\} \times \{u \leq 1\} \setminus d} (u(x) - u(y))^2 d\theta
\]

\[
+ \int_{\{u \leq 1\} \times \{u > 1\} \setminus d} (u(y) - u(x))^2 d\theta + \int_{\{u > 1\} \times \{u \leq 1\} \setminus d} (u(x) - u(y))^2 d\theta
\]

\[
= \int_{\partial \Omega \times \partial \Omega \setminus d} (u(x) - u(y))^2 d\theta.
\]

It follows that \( u \wedge 1 \in D(\mathcal{E}_{\kappa, \theta}) \) and \( \mathcal{E}_{\kappa, \theta}(u \wedge 1, u \wedge 1) \leq \mathcal{E}_{\kappa, \theta}(u, u) \). By the second Beurling-Deny criterion, the semigroup \( e^{t \Delta_{\kappa, \theta}} \) is \( L^\infty \)-contractive. \( \square \)

5. Domination. Now, we want to verify if the semigroup generated by \( \Delta_{\kappa, \theta} \) is or is not sandwiched between the semigroup of the Dirichlet Laplacian and the semigroup of the Neumann Laplacian. The left hand side of the inequality is assured by the following theorem

**Theorem 5.1.** Let \((\kappa, \theta)\) be an admissible pair of measures. Then

\[ e^{t \Delta_D} \leq e^{t \Delta_{\kappa, \theta}} \quad (t \geq 0) \]

in the positive operators sense.

**Proof.** By the Ouhabaz domination criterion, it suffices to prove that \( H^1_0(\Omega) \) is an ideal of \( D(\mathcal{E}_{\kappa, \theta}) \) and \( \mathcal{E}_{\kappa, \theta}(u, v) \leq \int_{\Omega} \nabla u \nabla v dx \) for all \( u, v \in H^1_0(\Omega)_+ \). We may assume that functions in \( \tilde{H}^1(\Omega) \) are r.q.c.

(1) Let \( u \in H^1_0(\Omega) \) and \( v \in D(\mathcal{E}_{\kappa, \theta}) \) be such that \( 0 \leq v \leq u \). Since \( \overline{\Omega} \) is relatively open, it follows from \([13, Lemma 2.1.4]\) that \( 0 \leq v \leq u \) r.q.c. on \( \overline{\Omega} \). We have \( u = 0 \) r.q.e. on \( \partial \Omega \), then \( v = 0 \) r.q.e. on \( \partial \Omega \). Therefore \( v \in H^1_0(\Omega) \).

(2) Let \( u, v \in H^1_0(\Omega)_+ \). Then \( u = v = 0 \) r.q.e. on \( \partial \Omega \). Since the pair \((\kappa, \theta)\) is admissible, it follows that \( u = v = 0 \) \((\kappa + \hat{\theta})\)-a.e. on \( \Gamma_{\kappa, \theta} \). We finally obtain that

\[
\mathcal{E}_{\kappa, \theta}(u, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\partial \Omega} uv d(\kappa + 2\hat{\theta}) - 2 \int_{\partial \Omega \times \partial \Omega \setminus d} u(x)v(y)d\theta
\]

\[
= \int_{\Omega} \nabla u \nabla v dx - 2 \int_{\partial \Omega \times \partial \Omega \setminus d} u(x)v(y)d\theta
\]

\[
\leq \int_{\Omega} \nabla u \nabla v dx,
\]

and the proof is complete. \( \square \)
The right hand side of the sandwiched inequality is not valid. We have then the following theorem.

**Theorem 5.2.** Let \((\kappa, \theta)\) be an admissible pair of measures. Then

\[ e^{t\Delta_{\kappa,\theta}} \leq e^{t\Delta_N} \quad \text{if and only if} \quad \theta \equiv 0. \]

**Proof.** Similarly to the previous result, we use the Ouhabaz domination criterion and we may assume that functions in \(\tilde{H}^1(\Omega)\) are r.q.c.

(1) Suppose that we have \(e^{t\Delta_{\kappa,\theta}} \leq e^{t\Delta_N}\). By the Ouhabaz domination criterion, we know that \(\int_{\Omega} \nabla u \nabla v dx \leq \mathcal{E}_{\kappa,\theta}(u, v)\) for all \(u, v \in D(\mathcal{E}_{\kappa,\theta})_+\). Which means that for all \(u, v \in D(\mathcal{E}_{\kappa,\theta})_+\), we have

\[ \int_{\Omega} \nabla u \nabla v dx \leq \int_{\Omega} \nabla u \nabla v dx + \int_{\partial\Omega} u v (\kappa + \hat{\theta}) - \int_{\partial\Omega \times \partial\Omega \setminus d} u(x) v(y) d\theta. \]

Thus for all \(u, v \in D(\mathcal{E}_{\kappa,\theta})_+\),

\[ \int_{\partial\Omega \times \partial\Omega \setminus d} u(x) v(y) d\theta \leq \int_{\partial\Omega} u v (\kappa + \hat{\theta}). \]

We choose \(u, v \in D(\mathcal{E}_{\kappa,\theta})_+\) such that \(\text{supp}[u] \cap \text{supp}[v] = \emptyset\), then we get

\[ \int_{\partial\Omega \times \partial\Omega \setminus d} u(x) v(y) d\theta \leq 0, \]

which means that \(\int_{\partial\Omega \times \partial\Omega \setminus d} u(x) v(y) d\theta = 0\) for all \(u, v \in D(\mathcal{E}_{\kappa,\theta})_+\) such that \(\text{supp}[u] \cap \text{supp}[v] = \emptyset\). Thus \(\theta \equiv 0\) because \(\text{supp}(\theta) \subset \partial\Omega \times \partial\Omega \setminus d\).

(2) Now if \(\theta \equiv 0\), the assertion follows from the sandwiched property proved in [6] or [1]. \(\square\)

In the next theorem, we prove that the nonlocal Robin semigroup can be dominated by a particular local semigroup.

**Theorem 5.3.** Let \((\kappa, \theta)\) be an admissible pair of measures. Then

\[ e^{t\Delta_{\kappa+2\hat{\theta}}} \leq e^{t\Delta_{\kappa,\theta}} \quad (t \geq 0) \]

in the positive operators sense, where \(\Delta_{\kappa+2\hat{\theta}} := \Delta_{\kappa+2\hat{\theta},0}\).
Proof. (1) Let $u \in D(\mathcal{E}_{\kappa+2\hat{\theta}})$ and $v \in D(\mathcal{E}_{\kappa,\theta})$ be such that $0 \leq v \leq u$. We have
\begin{align*}
\mathcal{E}_{\kappa+2\hat{\theta}}(v) &= \int_{\Omega} |\nabla v|^2 \, dx + \int_{\partial\Omega} |v|^2 \, d\kappa + \int_{\partial\Omega \times \partial\Omega} (v(x) - v(y))^2 \theta(dx, dy) \\
&\quad + 2 \int_{\partial\Omega \times \partial\Omega} v(x)v(y)\theta(dx, dy) \\
&= \mathcal{E}_{\kappa,\theta}(v) + 2 \int_{\partial\Omega \times \partial\Omega} v(x)v(y)\theta(dx, dy) \\
&\leq \mathcal{E}_{\kappa,\theta}(v) + 2 \int_{\partial\Omega} u^2(x)\hat{\theta}(dx) \\
&\leq \mathcal{E}_{\kappa,\theta}(v) + \mathcal{E}_{\kappa+2\hat{\theta}}(u).
\end{align*}
This means that $v \in D(\mathcal{E}_{\kappa+2\hat{\theta}})$.

(2) Let $u, v \in D(\mathcal{E}_{\kappa+2\hat{\theta}})_+$. We have
\begin{align*}
\mathcal{E}_{\kappa,\theta}(u, v) &= \mathcal{E}_{\kappa+2\hat{\theta}}(u, v) - 2 \int_{\partial\Omega \times \partial\Omega \setminus \partial\Omega} u(x)v(y) \theta(dx, dy) \\
&\leq \mathcal{E}_{\kappa+2\hat{\theta}}(u, v).
\end{align*}
which completes the proof. \hfill \Box

Corollary 5.4. For any admissible measure $\nu$ on $\partial\Omega$ such that $\nu \geq \kappa + 2\hat{\theta}$, we have
\begin{align*}
0 \leq e^{t\Delta\nu} \leq e^{\Delta_{\kappa,\theta}} \quad (t \geq 0)
\end{align*}
in the positive operators sense.

Proof. This follows immediately from Theorem 5.3 and [6, Theorem 3.2]. \hfill \Box

Open question. For which admissible measure $\nu$ is the semigroup $e^{t\Delta\nu}$ the largest semigroup associated with local Robin boundary conditions, which is dominated by the nonlocal Robin semigroup. By [6, Theorem 3.2], we know that $\nu \leq \kappa + 2\hat{\theta}$. We conjecture that $\nu = \mu + 2\hat{\theta}$ satisfies this property.

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