Novel complete non-compact symmetries for the Wheeler–DeWitt equation in a wormhole scalar model and axion–dilaton string cosmology

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Received 8 March 2011, in final form 19 July 2011
Published 25 August 2011
Online at stacks.iop.org/CQG/28/185002

Abstract

We find the full symmetries of the Wheeler–DeWitt equation for the Hawking and Page wormhole model and an axion–dilaton string cosmology. We show that the Wheeler–DeWitt Hamiltonian admits a $U(1,1)$ hidden symmetry for the Hawking and Page model and $U(2,1)$ for the axion–dilaton string cosmology. If we consider the existence of matter–energy renormalization, for each of these models we find that the Wheeler–DeWitt Hamiltonian accepts an additional $SL(2,R)$ dynamical symmetry. In this case, we show that the $SL(2,R)$ dynamical symmetry generators transform the states from one energy Hilbert eigensubspace to another. Some new wormhole-type solutions for both models are found.

PACS numbers: 98.80.Qc, 11.30.Pb, 11.30.–j

1. Introduction

The study of the early universe has become one of the more intense research areas in physics. General relativity describes the universe at scales larger than the Planck scale and it is expected that quantum mechanics has to be taken into account at least at these small scales.

In the quantum cosmology framework the whole universe is represented by means of a wavefunction. The quantum cosmology formalism, including the definition of the wavefunction of the universe, its configuration space and its evolution according to the Wheeler–DeWitt equation, was set up in the late 1960s [1–5].

The development of quantum cosmology started at the beginning of 1980 when it was proposed that the universe could be spontaneously nucleated out of nothing [6], where nothing
means the absence of space and time. After nucleation the universe enters to a phase of inflationary expansion and continues its evolution to the present. However, there are several important questions that remain to be solved like the appropriate boundary conditions for the Wheeler–DeWitt equation. In the case of quantum mechanics there is an external setup and the boundary conditions can be imposed safely, but in four-dimensional quantum cosmology there is nothing external to the universe and the correct boundary condition remains unsolved. A number of proposals for such boundary conditions came out [7–11]. Once we have chosen the boundary conditions, different physical settings emerge, e.g. if the wavefunction is regular when the three-geometry collapses to zero and it is exponentially damped for large three-geometries we have wormholes [12]. Hawking and Page [12] considered a minimally coupled massless scalar field and found its wormhole solution.

On the other hand, it is believed that string theory could play an important role in describing the evolution of the early universe and gives some insights on the mechanism of inflation. The pre-Big Bang scenario [13] is very interesting since it uses stringy symmetries in order to give a novel mechanism for inflation. Besides, it is possible that string theory could provide resolution of the initial singularity problem in cosmology.

In string cosmology, the usual approach is to study time-dependent solutions to the lowest order string equations of motion. This standard approach applies on scales above those energies where the string symmetries are broken but on scales below the string scales [14]. The low-energy four-dimensional effective field theory action of string theory contains two massless fields [15]. One of these scalar fields is called the axion $\chi$ and it comes from the third rank field strength corresponding to the Kalb–Ramond field; the other one is called the dilaton $\phi$. The physical consequences of the axion in a curved spacetime has been investigated with the aim of finding possible indirect evidence of low-energy string theory [16–18]. The dilaton is very important in string theory since it defines the string coupling constant $g_s$ as $e^{\phi/2}$ and it determines the Newton constant, the gauge coupling constants and Yukawa couplings.

It is a well-known fact that symmetries are very important to understand several properties of diverse theories. In particular, it is very interesting to investigate the underlying symmetries of the Hawking and Page wormhole model and axion–dilaton string cosmology. For the first model, a $U(1)$ symmetry generated by the ‘angular momentum’ is present. For the second model, Maharana [19, 20] showed that the ‘angular part’ of the Wheeler–DeWitt equation is invariant under the $SO(2, 1)$ group of transformations. For both models the angular symmetries were employed to reduce the Wheeler–DeWitt equation to a one-dimensional radial equation. However, in this work we show that these systems have larger symmetry groups.

In this paper, we consider the axion–dilaton string cosmology studied by Maharana and find the $U(2, 1)$ symmetry for the complete Wheeler–DeWitt Hamiltonian. For the Hawking and Page wormhole model we find that the Wheeler–DeWitt Hamiltonian admits a $U(1, 1)$ symmetry. Also, for each of these models we show that the Wheeler–DeWitt Hamiltonian accepts an additional $SL(2, R)$ dynamical symmetry when matter–energy renormalization is allowed. In this case, we prove that the $SL(2, R)$ dynamical symmetry transform states from an energy Hilbert eigensubspace to another energy eigensubspace. The paper is organized as follows. In section 2, we find the $U(1, 1)$ and $SL(2, R)$ symmetries for the wormhole scalar model. In section 3, by choosing a factor ordering different from the used in [19, 20] we show that the groups $U(2, 1)$ and $SL(2, R)$ are symmetries for the axion–dilaton string cosmology. In section 4, for both models, we find some new solutions for the Wheeler–DeWitt equation, including wave packets. By imposing the Hartle–Hawking boundary conditions, wormhole-type solutions are found. Finally, in section 5, we give our concluding remarks.
2. Symmetries for the Hawking and Page wormhole scalar model

Hawking and Page considered the Wheeler–DeWitt equation for the massless scalar field $\phi$ in a Friedmann–Robertson–Walker (FRW) spacetime

$$\mathcal{H}(a, \phi) = \frac{1}{2} \left( \frac{\partial}{\partial a} a^2 \frac{\partial}{\partial a} a^2 - \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} \right) \psi(a, \phi) = 0,$$

whose independent solutions are [12]

$$\psi(a, \phi) = J_{\pm i m}^{\pm}(a^2/2) e^{i m \phi}.$$  

Note that there exists a solution for each integer $m$, corresponding to the angular momentum eigenvalue:

$$L \psi(a, \phi) \equiv -i \frac{\partial \psi(a, \phi)}{\partial \phi} = m \psi(a, \phi).$$

Thus, the Wheeler–DeWitt equation has an infinite number of eigenstates.

We define the creation and annihilation operators

$$a_0 = \frac{1}{\sqrt{2}} \left( -\sinh \phi \left( a + \frac{\partial}{\partial a} \right) + \cosh \phi \frac{\partial}{a \partial \phi} \right),$$

$$\bar{a}_0 = \frac{1}{\sqrt{2}} \left( -\sinh \phi \left( a - \frac{\partial}{\partial a} \right) - \cosh \phi \frac{\partial}{a \partial \phi} \right),$$

$$a_1 = \frac{1}{\sqrt{2}} \left( \cosh \phi \left( a + \frac{\partial}{\partial a} \right) - \sinh \phi \frac{\partial}{a \partial \phi} \right),$$

$$\bar{a}_1 = \frac{1}{\sqrt{2}} \left( \cosh \phi \left( a - \frac{\partial}{\partial a} \right) + \sinh \phi \frac{\partial}{a \partial \phi} \right),$$

which satisfy the commutation relations $[a_\mu, \bar{a}_\nu] = G^\mu_\nu = \text{diag}(-1, 1), \mu, \nu = 0, 1$. By means of the coordinate transformation $x = a \sinh \phi$ and $y = a \cosh \phi$, these operators become

$$a_0 = \frac{1}{\sqrt{2}} \left( -x + \frac{\partial}{\partial x} \right), \quad \bar{a}_0 = -\frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right),$$

$$a_1 = \frac{1}{\sqrt{2}} \left( y + \frac{\partial}{\partial y} \right), \quad \bar{a}_1 = \frac{1}{\sqrt{2}} \left( y - \frac{\partial}{\partial y} \right).$$

The angular momentum operator is

$$L = -i \frac{\partial}{\partial \phi} = -i \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)$$

$$= -i(a_0 \bar{a}_1 - a_1 \bar{a}_0),$$

and the Hamiltonian (1) can be written as

$$\mathcal{H} = \frac{1}{2} \left[ \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - (y^2 - x^2) \right]$$

$$= \bar{a}_0 a_0 - \bar{a}_1 a_1 - 1.$$  

The creation and annihilation operators defined above allow us to find the $U(1, 1)$ hidden symmetry generators $\bar{a}_0 a_0, \bar{a}_0 a_1, \bar{a}_1 a_0$ and $\bar{a}_1 a_1$, which commute with the Hamiltonian operator $\mathcal{H}$. Since the group $U(1, 1)$ is equal to $SU(1, 1) \times U(1)$ [26], we find that the $U(1)$ generator is

$$J_0 = -\bar{a}_0 a_0 + \bar{a}_1 a_1,$$
and the non-trivial $SU(1, 1)$ traceless generators are

$$J_{00} = \bar{a}_0 a_0 + \frac{1}{2} J_0, \quad J_{01} = \bar{a}_0 a_1,$$

$$J_{10} = \bar{a}_1 a_0, \quad J_{11} = \bar{a}_1 a_1 - \frac{1}{2} J_0. \quad (17)$$

These symmetry operators are such that $J_{00} = J_{11}$, $[J_0, H] = 0$, $[J_0, J_{\mu\nu}] = 0$ and $[J_{\mu\nu}, \mathcal{H}] = 0$.

If we consider the possibility of a matter–energy renormalization by introducing an arbitrary constant [7], equation (1) can be rewritten in the following form:

$$\frac{1}{2} \left( \frac{1}{a} \frac{\partial a}{\partial a} \frac{\partial}{\partial a} - \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} - a^2 - 2E \right) \psi_{Em}(a, \phi) = 0. \quad (18)$$

Note that this equation enforces us to introduce the energy $E$ to label the wavefunction. Since the operators $J_{\mu\nu}$ commute with the Hamiltonian, then they do not change the energy $E$ but the angular momentum quantum number $m$. Thus, these generators transform the degenerate states corresponding to a given energy between themselves, in particular those for the zero energy $E = 0$.

Also, we can define the set of operators

$$K_+ \equiv \frac{1}{2} \left( - \bar{a}_0^2 + \bar{a}_1^2 \right), \quad (19)$$

$$K_- \equiv \frac{1}{2} \left( - a_0^2 + a_1^2 \right), \quad (20)$$

$$K_0 \equiv \frac{1}{2} \left( - \bar{a}_0 a_0 + \bar{a}_1 a_1 + 1 \right) = - \frac{\mathcal{H}}{2}, \quad (21)$$

which satisfy the commutation relations

$$[K_+, K_-] = -2K_0, \quad [K_0, K_{\pm}] = \pm K_{\pm}. \quad (22)$$

This means that the operators $K_0, K_+$ and $K_-$ close the $SL(2, \mathbb{R})$ dynamical Lie algebra. A direct calculation shows that the Casimir operator $\tilde{K} \equiv K_0(K_0 - 1) - K_+ K_-$ is related to the angular momentum $L$ as $\tilde{K}^2 = -L^2 - \frac{1}{2}$. From this result, the common eigenfunctions for the Hamiltonian $K_0$ and the Casimir $K^2$ operators of the $SL(2, \mathbb{R})$ algebra can be chosen as those of the Hamiltonian and the angular momentum operators. Thus, from equation (21) we obtain $K_0|E m\rangle = -\frac{L}{2}|E m\rangle$, and from the second commutation relation we show that $K_0 K_{\pm}|E m\rangle = - \left( \frac{L^2 + 2}{2} \right) K_{\pm}|E m\rangle$. These results imply that $K_{\pm}|E m\rangle \propto |E \mp 2 m\rangle$. Hence, the operators (19) and (20) acting on the states $|E m\rangle$ change the energy and leave the angular momentum quantum number fixed. If we restrict the solutions to those of the Wheeler–DeWitt equation without matter–energy renormalization, we must consider the states with $E = 0$, and the above $SL(2, \mathbb{R})$ dynamical symmetry is not relevant.

### 3. Symmetries of axion–dilaton string cosmology

We begin summarizing some important points of the Maharana papers [19, 20] which are relevant to our work. In these references, the $SO(2, 1)$ symmetry of axion–dilaton string cosmology derived from the action in the Einstein frame has been found,

$$S = \int d^4 x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} e^{2\phi} \partial_{\mu} \chi \partial^{\mu} \chi \right). \quad (23)$$
where \( R \) is the scalar curvature, \( \sqrt{-g} \) is the determinant of the metric \( g_{\mu\nu} \), and \( \phi \) and \( \chi \) are the dilaton and axion fields, respectively. The homogeneous and isotropic FRW metric for closed universes \( (k = 1) \),

\[
ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right),
\]

was assumed, where \( a(t) \) is the scalar factor and \( t \) is the cosmic time. The corresponding Wheeler–DeWitt equation is

\[
H\Psi := \frac{1}{2} \left( \frac{\partial^2}{\partial a^2} + \frac{P}{a} \frac{\partial}{\partial a} - a^2 + \frac{1}{a^2} \hat{C} \right) \Psi = 0,
\]

where in order to solve the ordering ambiguity between \( a \) and \( \partial/\partial a \), the prescription \( p = 1 \) was adopted. Since the action \( S \) is invariant under the \( SO(2, 1) \) transformations (S-duality), also \( H \) is invariant under these transformations. \( \hat{C} \) is the \( SO(2, 1) \) Casimir operator, which expressed in the pseudospherical coordinate system

\[
x = a \sinh \alpha \cos \beta, \quad y = a \sinh \alpha \sin \beta, \quad z = a \cosh \alpha
\]

is just the Laplace–Beltrami operator given by

\[
\hat{C} = -\frac{1}{\sinh \alpha} \frac{\partial}{\partial \alpha} \left( \sinh \alpha \frac{\partial}{\partial \alpha} \right) - \frac{1}{\sinh^2 \alpha} \frac{\partial^2}{\partial \beta^2}.
\]

The axion and dilaton fields can be written in terms of the pseudospherical coordinates (26) as

\[
\chi = \frac{\sinh \alpha \cos \beta}{\cosh \alpha + \sinh \alpha \sin \beta}, \quad e^{-\phi} = \frac{1}{\cosh \alpha + \sinh \alpha \sin \beta}.
\]

The explicit solutions for the Wheeler–DeWitt constraint (25) on the pseudosphere were obtained from the \( SO(2, 1) \) group theory by identifying that the correct series involved in quantum cosmology is the continuous one \[20\]. These are

\[
\Psi(a, \alpha, \beta) = J_{\pm i\frac{\nu}{2}}(i\alpha^2/2) Y_{\pm \frac{\nu}{2} + i\lambda}^{m}(\cosh \alpha, \beta),
\]

where \( Y_{\pm \frac{\nu}{2} + i\lambda}^{m}(\cosh \alpha, \beta) \) = \( e^{i\nu \beta} P_{\pm \frac{\nu}{2} + i\lambda}^{m}(\cosh \alpha) \), and \( \nu^2 = (\lambda^2 + \frac{1}{4}) \) are the eigenfunctions for the non-compact operator \( \hat{C} \) and the compact generator \(-i\partial_\beta \), with \( P_{\pm \frac{\nu}{2} + i\lambda}^{m}(\cosh \alpha) \) the associated Legendre polynomials (also called toroidal functions). Note that for this case, by varying \( \lambda \) and \( m \) there exists an infinite degeneracy.

One of the main results of this paper is to find the full symmetries for the Wheeler–DeWitt equation (25). This is based on recognizing that equation (25) in coordinates (26) with factor ordering \( p = 2 \) can be written as

\[
H\Psi = \frac{1}{2} \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - (z^2 - y^2 - x^2) \right) \Psi = 0.
\]

Some aspects about the factor ordering operator in the context of string cosmology are important to remark. When one considers the Wheeler–DeWitt equation in the string frame the operator ordering is usually fixed by T-duality invariance of the Hamiltonian. This selection of the factor ordering is important in the graceful exit problem of quantum string cosmology of the pre-Big Bang scenario. In the case of homogeneous and isotropic cosmology without the axion, the T-duality is just the scale factor duality \( a \rightarrow \frac{1}{a} \) and this requirement constrains the choice of factor ordering to \( p = 1 \) \[21, 22\]. However, the scale factor duality is not adequate in the graceful exit in pre-Big Bang string cosmology when quantum loop corrections are taken into account \[23\]. Besides, the presence of a homogeneous axion field or spatial curvature is compatible with S-duality but breaks T-duality (O(d,d) symmetry) \[24, 25\]. In our case, the
use of the Einstein frame helps to show S-duality of the theory but it is not useful to fix the factor ordering because \( a \to \bar{a} \) under S-duality [19].

If we want to preserve reparametrization invariance of the Hamiltonian, following the arguments presented in [21], we need \( p = 2 \) because in our case the minisuperspace is three dimensional unlike that found in [12], where the minisuperspace is bidimensional and therefore the adequate factor ordering results to be \( p = 1 \).

For this model we propose the set of creation and annihilation operators

\[
a_0 = \frac{1}{\sqrt{2}} \left( -z + \frac{\partial}{\partial z} \right), \quad \bar{a}_0 = -\frac{1}{\sqrt{2}} \left( z + \frac{\partial}{\partial z} \right),
\]

\[
a_1 = \frac{1}{\sqrt{2}} \left( y + \frac{\partial}{\partial y} \right), \quad \bar{a}_1 = \frac{1}{\sqrt{2}} \left( y - \frac{\partial}{\partial y} \right),
\]

\[
a_2 = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad \bar{a}_2 = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right).
\]

These operators satisfy the commutation relations \([a_\mu, \bar{a}_\nu] = G_{\mu\nu} = \operatorname{diag}(-1, 1, 1), \mu, \nu = 0, 1, 2\), and factorize the Hamiltonian as follows:

\[
H = -\bar{a}_0 a_0 + \bar{a}_1 a_1 + \bar{a}_2 a_2 + \frac{3}{2}.
\]

Operators (31)\textendash;(33) allow us to define the angular operators

\[
J_z = -i(\bar{a}_1 a_2 - \bar{a}_2 a_1) = -i \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right),
\]

\[
J_y = -i(\bar{a}_0 a_1 - \bar{a}_1 a_0) = i \left( z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \right),
\]

\[
J_x = -i(\bar{a}_2 a_0 - \bar{a}_0 a_2) = -i \left( x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x} \right).
\]

These operators satisfy the \( SO(2, 1) \) commutation relations

\[
[J_x, J_y] = -iJ_z, \quad [J_z, J_x] = iJ_y, \quad [J_y, J_z] = iJ_x
\]

and reproduce the Casimir operator \( \hat{C} = J_x^2 + J_y^2 + J_z^2 \). This result reflects a non-compact symmetry (S-duality) on the angular part of the Hamiltonians (25) or (30).

The creation and annihilation operators (31)\textendash;(33) allow us to define the set of second-order operators

\[
J_{00} = \bar{a}_0 a_0 + \frac{1}{2} J_0, \quad J_{11} = \bar{a}_1 a_1 - \frac{1}{2} J_0,
\]

\[
J_{22} = \bar{a}_2 a_2 - \frac{1}{2} J_0, \quad J_{01} = \bar{a}_0 a_1,
\]

\[
J_{10} = \bar{a}_1 a_0, \quad J_{02} = \bar{a}_0 a_2,
\]

\[
J_{20} = \bar{a}_2 a_0, \quad J_{12} = \bar{a}_1 a_2,
\]

\[
J_{21} = \bar{a}_2 a_1.
\]

\[
J_0 = -\bar{a}_0 a_0 + \bar{a}_1 a_1 + \bar{a}_2 a_2,
\]

\[
= -a_0 \bar{a}_0 + a_1 \bar{a}_1 + a_2 \bar{a}_2 - 3.
\]
Note that the operator $J_0$ and the Wheeler–DeWitt Hamiltonian (30) are related by $J_0 = H - \frac{3}{2}$.

We prove that the operators $J_0$ and $J_{\mu\nu}$ satisfy the commutation relations

$$[J_0, J_{\mu\nu}] = 0.$$  (45)

This means that the operators $J_{\mu\nu}$ are the symmetries of the Hamiltonian $H$. In fact, the operators $J_{\mu\nu}$ are the non-trivial generators of $SU(2, 1)$, whereas $J_0$ is the $U(1)$ generator [26].

In a similar way to the wormhole model, we can introduce a matter–energy renormalization. Hence, the Wheeler–DeWitt equation (25) takes the form

$$\frac{1}{2} \left( \frac{\partial^2}{\partial a^2} + \frac{2}{a} \frac{\partial}{\partial a} - a^2 + \frac{1}{a^2} \hat{C} - 2E \right) \Phi_{E\lambda m} = 0.$$  (46)

By using the spherical functions $Y_{-\frac{1}{2}+i\lambda}^m (\cosh \alpha, \beta) = e^{i m \beta} P_{-\frac{1}{2}+i\lambda}^m (\cosh \alpha)$ as the correct wavefunctions for the Casimir operator $\hat{C}$, we propose $\Psi_{E\lambda m}$ to have the form

$$\langle a\alpha\beta | E\lambda m \rangle = e^{i m \beta} P_{-\frac{1}{2}+i\lambda}^m (\cosh \alpha) W_{E\lambda} (a).$$  (47)

This allows us to find the solution to the Wheeler–DeWitt equation with matter–energy renormalization for the scale factor $W_{E\lambda} (a)$. It is given by

$$W_{E\lambda} (a) = c_1 a^{-\frac{3}{2}} M_{-\frac{1}{2}, \lambda} (a^2) + c_2 a^{-\frac{3}{2}} W_{-\frac{1}{2}, \lambda} (a^2),$$  (48)

where $M$ and $W$ are the Whittaker functions, and $\Lambda = \sqrt{2 + 4 \lambda^2}$. Note that to set the correct coefficients, we need to impose on the functions $W_{E\lambda} (a)$ one of the well-known proposals for the boundary conditions [7–11].

The toroidal functions $P_{-\frac{1}{2}+i\lambda}^m (x)$ [27] can be expressed in terms of the hypergeometric functions [20]

$$P_j^m (x) = \frac{1}{\Gamma(1-m)} \left( \frac{x-1}{x+1} \right)^{-j} \binom{n}{j} (-j, j+1; 1-m; 1-\frac{x}{2+\frac{x}{2}}),$$  (49)

with $j = -\frac{1}{2} + i\lambda$. They satisfy the orthogonality relations [28]

$$\int_1^\infty P_{-\frac{1}{2}+i\lambda}^m (x) P_{-\frac{1}{2}-i\lambda}^m (x) dx = \delta (\lambda - \lambda') \left| \frac{\Gamma(i\lambda)}{\Gamma \left( \frac{1}{2} + i\lambda - m \right)} \right|^2,$$  (50)

and the completeness relation

$$\int_0^\infty P_{-\frac{1}{2}+i\lambda}^m (x) P_{-\frac{1}{2}-i\lambda}^m (x') dx = \delta (x - x') \left| \frac{\Gamma(i\lambda)}{\Gamma \left( \frac{1}{2} + i\lambda - m \right)} \right|^2.$$  (51)

Since operators $J_{\mu\nu}$ commute with the Hamiltonian $H$, it is immediate to show that the functions $J_{\mu\nu} | E\lambda m \rangle$ are also eigenfunctions of the Wheeler–DeWitt equation (46). Taking into account that $\langle a\alpha\beta | E\lambda m \rangle$ are a complete set of functions [28], we can use them to expand $J_{\mu\nu} | E\lambda m \rangle$. Thus,

$$J_{\mu\nu} | E\lambda m \rangle = \sum_{m=-\infty}^{\infty} \int_0^\infty C_{m,m} (\lambda, \lambda') | E\lambda' m' \rangle d\lambda'.$$  (52)

However, the analytical expression for the functions $C_{m,m} (\lambda, \lambda')$ is very difficult to obtain because the integrand involves both toroidal and Whittaker functions.

We define the new set of operators

$$\mathcal{K}_+ = \frac{1}{2} \left( -\partial_{\hat{a}^3}^2 + \partial_{\hat{a}^1}^2 + \partial_{\hat{a}^2}^2 \right).$$  (53)
\[ K_+ = \frac{1}{2} \left(-a_0^2 + a_1^2 + a_2^2\right), \]  
\[ K_0 = \frac{1}{2} \left(-\bar{a}_0 a_0 + \bar{a}_1 a_1 + \bar{a}_2 a_2 + \frac{3}{2}\right) = \frac{H}{2}, \]  
which satisfy the \( SL(2, R) \) commutation relations
\[ [K_+, K_-] = -2K_0, \]  
\[ [K_0, K_{\pm}] = \pm K_{\pm}. \]  
The Casimir operator \( \hat{K}_0 \) for this \( SL(2, R) \) algebra and the Casimir operator \( \hat{C} \) for the \( SO(2, 1) \) angular momentum are related by \( \hat{K}_0 = -\frac{1}{4} \hat{C} - \frac{3}{16} \). These results allow us to find the action of the operators \( K_- \) and \( K_+ \) on the non-zero energy eigenstates of the Wheeler–DeWitt equation,
\[ K_{\mp}|E \lambda m\rangle \propto |E_{\mp}^2 \lambda m\rangle. \]  
Thus, if we restrict to the solutions of the Wheeler–DeWitt equation (30) with \( E = 0 \), the generators \( K_+ \), \( K_- \) and \( K_0 \) of the \( SL(2, R) \) algebra do not play a relevant role as a symmetry group for the Wheeler–DeWitt equation. However, the \( SU(2, 1) \) symmetry generators are relevant for any energy \( E \) and describe the degeneracy in the quantum numbers \( \lambda \) and \( m \).

4. Wave packets solutions for the scalar and axion–dilaton string cosmology models

In a similar way to that followed to calculate wave packets for the Kantowski–Sachs spacetime [29], we can construct wave packet solutions by integrating over the quantum number \( m \):
\[ \psi_{WDW} = \int_{-\infty}^{\infty} e^{i(m\phi + \gamma)} J_{\pm i \frac{1}{2}} \left(ia^2/2\right) dm. \]  
This allows us to obtain
\[ \psi_{WP}^{\text{HP}} = A_{1,2} e^{\pm \frac{1}{4}a^2 \cosh(2\phi + \gamma_{1,2})}, \]  
where \( A \) and \( \gamma \) are constants. These functions are solutions for the Wheeler–DeWitt equation (1) for the Hawking–Page scalar model. The solution with the minus sign is the only which is a Gaussian wave packet and satisfies the Hartle–Hawking boundary condition (‘no-boundary proposal’) [7], i.e. the wavefunction of the universe is regular at \( a \to 0 \) and it is exponentially damped for a large scale factor \( a \to \infty \). Thus, the Gaussian wave packet represents a wormhole solution for the Wheeler–DeWitt equation [12].

For the string cosmology model, we find the wave packets
\[ \Psi_{SCP}^{\psi} = A_{\pm} e^{\pm \frac{1}{2}a^2 \cosh(2\phi + \gamma_{1,2})}, \]  
which are solutions for the Wheeler–DeWitt equations \( H \pm \frac{1}{2})\Psi = 0 \). These are completely analogous to those for the Hawking–Page scalar model (60). Therefore, the solution with the minus sign corresponds to a wormhole-type solutions for the axion–dilaton string cosmology model.

Also, we can show that the functions
\[ \Psi_{SC} = e^{\pm \frac{1}{2}(x^2+y^2)} \sqrt{(C_1 I_{\frac{1}{2}}(x^2/2) + C_2 K_{\frac{1}{2}}(x^2/2))} \]  
are solutions for the Wheeler–DeWitt equation \( H \Psi = 0 \), \( I_{\frac{1}{2}}(x^2/2) \) and \( K_{\frac{1}{2}}(x^2/2) \) being the modified Bessel functions and \( C_1 \) and \( C_2 \) constants. In these solutions if we interchange the \( x \) and \( y \) coordinates, the resulting functions are also solutions for the Wheeler–DeWitt equation. However, only the solution with the minus sign and \( C_2 = 0 \) is regular at the origin.
(this is because the only modified Bessel function which leads to a regular wavefunction as \( a \to 0 \) is \( I_1(x^2/2) \)). Thus, the functions

\[
\Psi_1 = Ce^{-\frac{1}{4}(z^2+y^2)} \sqrt{xI_1(\frac{x^2}{2})}
\]

(63)

and

\[
\Psi_2 = Ce^{-\frac{1}{4}(z^2+y^2)} \sqrt{yI_1(\frac{y^2}{2})}
\]

(64)

are wormhole-type solutions for the axion–dilaton string cosmology model. To our knowledge the solutions above in this section do not have been reported in the literature.

From the point of view of the group theory, it is more important the \( U(1) \) generator than the dynamical equation (in this case, the Wheeler–Dewitt equation). For the Hawking–Page and the axion–dilaton string cosmology models, we find the following solutions:

\[
\psi_{U(1)} = e^{\pm \frac{1}{2}a^2}, \quad \Psi_{U(1)} = e^{\pm \frac{1}{2}a^2}
\]

(65)

which satisfy the Wheeler–DeWitt equations \((\mathcal{H} \mp 1)\psi = 0\) and \((\mathcal{H} \mp \frac{1}{2})\Psi = 0\), respectively. In fact, these solutions are annihilated by either of the two forms for the \( U(1) \) generators \( J_0 \) and \( J_0 \), respectively. These are manifestly Lorentz invariant under rotations around the \( z \)-axis and under the \( SO(2, 1) \) group, respectively. However, from the quantum cosmology point of view, these solutions do not represent any interesting scenario because they do not involve the scalar field \( \phi \) or the axion–dilaton \( (\chi - \phi) \) fields.

5. Concluding remarks

In this paper, we have found the symmetries related to the Wheeler–DeWitt Hamiltonian for the Hawking and Page wormhole and the axion–dilaton quantum cosmology. We have shown that the Wheeler–DeWitt Hamiltonian for the wormhole model has the \( U(1, 1) \) non-compact symmetry which describes the degeneracy of the states with or without energy–matter renormalization. Also, we have found that the Wheeler–DeWitt Hamiltonian accepts an additional \( SL(2, R) \) dynamical symmetry when energy–matter renormalization is considered. In this case, we showed that the \( SL(2, R) \) dynamical symmetry generators transform the states from an energy Hilbert eigensubspace to another.

The factor ordering is frequently chosen by convenience [30–33]. For the axion–dilaton string cosmology we have set the factor ordering \( p = 2 \), which is necessary to preserve reparameterization invariance of the Hamiltonian. Indeed, this choice was crucial in order to find the closed Lie algebras representing the hidden symmetries of the axion–dilaton string cosmology. A similar setting has been taken by Pioline, et al [34] in order to fix the conformal symmetry \( SO(2, 1) \) for the one-dimensional Wheeler–DeWitt equation. For the axion–dilaton string cosmology Hamiltonian, the non-compact hidden symmetries \( U(2, 1) \) and \( SL(2, R) \) are permissible. The \( U(2, 1) \) symmetry is valid whenever energy–matter renormalization is or not present. Also, in this case, the \( SL(2, R) \) hidden symmetry transforms the states from an energy Hilbert eigensubspace to another.

The zero-energy eigenfunctions (2) and (29) are particular cases of the huge degeneracy on the states \( \psi_{Em} \) or \( \Psi_{Em} \) when matter–energy renormalization is considered. The huge degeneracy of the wavefunctions on the secondary quantum numbers \( m \) or \( \lambda m \) for the systems studied in this paper are fully described by the non-compact symmetries \( U(1, 1) \) or \( U(2, 1) \), respectively. On the other hand, the \( SL(2, R) \) symmetries are suitable to relate the states with different principal quantum numbers but maintaining the secondary quantum numbers fixed.

Other symmetries have been found for the Wheeler–DeWitt equation in different physical settings. For example, the \( SO(2, 1) \) conformal group has been found for the one-dimensional
radial Wheeler–DeWitt equation with cosmological constant [34]. We emphasize that the symmetries for the Wheeler–DeWitt equation found in this work are for the complete Hamiltonian. The wormhole-type solution for the Hawking–Page model, equation (60), and for the axion–dilaton string cosmology, equations (61)–(64), were found. To our knowledge these solutions have not been reported in previous works.

Finally, our procedure can be applied to find the symmetries of the Wheeler–DeWitt equation for other systems like multidimensional quantum wormholes [35, 36], or the Kantowski–Sachs quantum cosmological model [37, 38], which is work in progress.

Acknowledgments

This work was partially supported by SNI-México, CONACYT grant no J1-60621-I, COFAA-IPN, EDI-IPN, SIP-IPN projects nos 20100897, 20110127 and 20100684.

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