1. The physical meaning of the reconstructed quantity in phase-shifting holography when speckle grains are under-sampled.

Phase-shifting holography has been widely used to measure the phase of scattered light. However, in previous experiments, the speckle grains were well-sampled by camera pixels. When speckle grains are under-sampled, it is still unclear what quantity is measured by phase-shifting holography and whether optical focusing through optical phase conjugation can still be achieved. We investigate this problem here. In one implementation of phase-shifting holography [1-3], a planar reference beam and a sample beam with a distorted wavefront beat at a frequency of $f_b$, and a digital camera records their interference pattern at a frame rate of $4f_b$. Four successive frames are used to reconstruct the phase and amplitude of the sample beam. When speckle grains are under-sampled, the light power on a single pixel of the camera in each frame can be written as

$$R_k = A E_R^2 + A \sum_{j=1}^{F} E_S^2(\vec{r}_j) + A \sum_{j=1}^{F} 2 E_R E_S(\vec{r}_j) \cos[\phi_r(\vec{r}_j) + (k-1)\pi/2],$$ (S1)

where $A$ is the area of one pixel; $F$ is the number of speckle grains within one pixel, which corresponds to the under-sampling factor in the main text; $E_S$ is the amplitude of the reference beam, and we assume that the phase of the reference beam is 0 for simplicity; and $E_r(\vec{r}_j)$ and $\phi_r(\vec{r}_j)$ are the amplitude and phase of the electric field (along the polarization direction of the reference beam) of the $j$-th speckle grain located at $\vec{r}_j$ within one pixel. Each speckle grain is assumed to have the same size. Since the camera samples the beat at $4 \times$ the beat frequency, the interference term in each successive frame has a $\pi/2$ phase shift, which is reflected in the cosine term. Based on Eq. (S1), the reconstructed quantity in phase-shifting holography can be found by

$$E = (P_1 - P_3) + i(P_2 - P_3) = 4E_R A \sum_{j=1}^{F} E_S(\vec{r}_j) e^{i\phi_r(\vec{r}_j)} = C \sum_{j=1}^{F} E_S(\vec{r}_j) e^{i\phi_r(\vec{r}_j)},$$ (S2)

where $C$ is a constant prefactor. From Eq. (S2), we conclude that when speckle grains are under-sampled, the reconstructed quantity on each pixel in phase-shifting holography is proportional to the summation of the electric fields of all the speckle grains within that pixel.
2. Derivation of peak-to-background ratios of optical time reversal under different speckle sampling conditions

Here, we provide an analytical analysis of theoretical peak-to-background ratios (PBRs) of time-reversal-based wavefront shaping when speckle grains are well-sampled and under-sampled. For simplicity without losing generality, the incident field $E_{\text{in}}$ is set to be 1 for the first element and 0 for the rest of the elements. The total number of elements in the incident field is $N_i$. To model the scattering process, a transmission matrix $T$ that connects the incident field $E_{\text{in}}$ and the scattered field $E_S$ is introduced. The matrix elements $t_y$ satisfy a circular Gaussian distribution. That is to say, the real part ($x$) and the imaginary part ($y$) of each element are independent variables, and they satisfy the Gaussian distribution $f_x(\mu = 0, \sigma) = e^{-x^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$, $f_y(\mu = 0, \sigma) = e^{-y^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$, with a mean of zero and a standard deviation of $\sigma$. The dimensions of the matrix are $(S_{11})_{N_i \times \tilde{N}_S}$, $(S_{12})_{N_i \times \tilde{N}_S}$, $(S_{21})_{N_i \times \tilde{N}_S}$, $(S_{22})_{N_i \times \tilde{N}_S}$, where $\tilde{N}_S$ is the number of speckle grains on the PCM. The scattered field $E_S$ intercepted by the phase conjugation mirror (PCM) can be computed as follows:

$$E_S = TE_{\text{in}} = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1N_i} \\ t_{21} & t_{22} & \ldots & t_{2N_i} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N_i1} & t_{N_i2} & \ldots & t_{N_iN_i} \end{pmatrix}_{N_i \times \tilde{N}_S} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{1 \times \tilde{N}_S} = \begin{pmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{N_i1} \end{pmatrix}_{\tilde{N}_S \times 1}. \quad (S3)$$

We note that each element in $E_S$ describes the electric field of an independent speckle grain, and the total number of speckle grains on the PCM is $N_S$.

2.1 When speckle grains are well-sampled

We start by describing the wavefront measurement and wavefront reconstruction in the well-sampling situation, because similar mathematics will be adopted in the description of the under-sampling situation. When speckle grains are well-sampled, so that one speckle grain occupies multiple pixels, the experimentally measurable scattered field $E_{\text{S,well-sampled}}$ is identical to $E_S$:

$$E_{\text{S,well-sampled}} = \begin{pmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{N_i1} \end{pmatrix}_{\tilde{N}_S \times 1}. \quad (S4)$$

By multiplying the backward transmission matrix $T^T$ (the upper case $T$ stands for matrix transpose) by the conjugated scattered field $E_{\text{S,well-sampled}}^*$, the optical phase conjugated field $E_{\text{OPC}}$ exiting the scattering medium can be computed as

$$E_{\text{OPC}} = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1N_i} \\ t_{21} & t_{22} & \ldots & t_{2N_i} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N_i1} & t_{N_i2} & \ldots & t_{N_iN_i} \end{pmatrix}_{N_i \times \tilde{N}_S} \begin{pmatrix} t_{11}^* \\ t_{21}^* \\ \vdots \\ t_{N_i1}^* \end{pmatrix}_{N_i \times 1}. \quad (S5)$$

The first element of $E_{\text{OPC}}$ corresponds to the electric field of the peak of the focus, and is calculated by

$$E_{\text{peak}} = \sum_{m=1}^{N_S} t_{m1} t_{m1}^*, \quad (S6)$$

By taking the absolute square of the electric field and then performing an ensemble average $<\cdot>$, the peak intensity of the focus is

$$I_{\text{peak}} = \left< |E_{\text{peak}}|^2 \right>. \quad (S7)$$

To proceed, we define

$$V_m = t_{m1} t_{m1}^*, \quad n \neq 1,$$

$$U = \sum_{m=1}^{N_S} V_m^*, \quad (S10)$$

where both $V_m$ and $U$ have zero mean. Then, Eq. (S9) can be rewritten as

$$I_{\text{background}} = \left< |U|^2 \right> = \text{Var}(U), \quad (S11)$$

where $\text{Var}(\cdot) = \text{Var}_{\text{real}}(\cdot) + \text{Var}_{\text{imag}}(\cdot)$ expresses the summation of the variance for both the real part and the imaginary part of a complex random variable. From the central limit theorem, the variance of $U$ is $N_S$ times the variance of $V_m$, so we have

$$I_{\text{background}} = \text{Var}(U) = N_S \text{Var}(V_m) = N_S \text{Var}(t_{m1} t_{m1}^*). \quad (S12)$$

Since $t_{m1}$ and $t_{m1}^*$ are independent complex random variables with mean values of zero, it can be shown that the variance of their product equals the product of their variances. Thus, we have
\[ I_{\text{background}} = N_S \, \text{Var}(i_{\text{in}}, j_{\text{in}}), \]
\[ = N_S \, \text{Var}(i_{\text{in}}) \, \text{Var}(j_{\text{in}}), \]
\[ = N_S \times 2\sigma^2 \times 2\sigma^2 = 4\sigma^4 N_S. \]  
\[ \text{(S13)} \]

From the results in Eq. (S7) and Eq. (S13), the theoretical peak-to-background ratio (PBR) is
\[ \text{PBR} = \frac{I_{\text{peak}}}{I_{\text{background}}} = \frac{N_P}{N_S}. \]  
\[ \text{(S14)} \]

The same result when speckle grains are well-sampled has also been reported \[4,5\]. Eq. (S14) shows that when speckle grains are well-sampled, the theoretical PBR of the focus is determined by the number of speckle grains intercepted by the PCM. Since 3x3 pixels to 5x5 pixels were typically used to sample one speckle grain in previous experiments, \( N_S \) is usually \( 9 - 25 \) times smaller than the number of pixels \( (N_P) \) of a spatial light modulator (SLM).

### 2.2 When speckle grains are under-sampled

Now, we investigate the situation when speckle grains are under-sampled with under-sampling factor \( F = N_S / N_P \). In this case, multiple speckle grains are within one PCM pixel. With the knowledge that phase-shifting holography reconstructs the summation of the electric fields of all the speckle grains within one pixel, the experimentally measurable scattered field \( E_{\text{S,undersampled}} \) has the following form:

\[ E_{\text{S,undersampled}} = \left( \begin{array}{c}
  t_{11} + t_{21} + \cdots + t_{F1} \\
  t_{12} + t_{22} + \cdots + t_{F1} \\
  \vdots \\
  t_{1N_S} + t_{2N_S} + \cdots + t_{F1} \\
  t_{N_S-F+1,1} + t_{N_S-F+2,1} + \cdots + t_{N_S,1} \\
  t_{N_S-F+1,2} + t_{N_S-F+2,2} + \cdots + t_{N_S,2} \\
  \vdots \\
  t_{N_S-F+1,N_S} + t_{N_S-F+2,N_S} + \cdots + t_{N_S,N_S} \\
  \end{array} \right) \]
\[ \times \left( \begin{array}{c}
  F \text{ rows} \\
  F \text{ rows} \\
 \end{array} \right) \]  
\[ \text{(S15)} \]

By multiplying the backward transmission matrix \( T^T \) by the conjugated scattered field \( E_{\text{S,undersampled}}^* \), the optical phase conjugated field \( E_{\text{OPC}} \) exiting the scattering medium is computed as
\[ E_{\text{OPC}} = T^T E_{\text{S,undersampled}}^* = \left( \begin{array}{c}
  t_{11}^* + t_{21}^* + \cdots + t_{F1}^* \\
  t_{12}^* + t_{22}^* + \cdots + t_{F1}^* \\
 \vdots \\
  t_{1N_S}^* + t_{2N_S}^* + \cdots + t_{F1}^* \\
  t_{N_S-F+1,1}^* + t_{N_S-F+2,1}^* + \cdots + t_{N_S,1}^* \\
  t_{N_S-F+1,2}^* + t_{N_S-F+2,2}^* + \cdots + t_{N_S,2}^* \\
 \vdots \\
  t_{N_S-F+1,N_S}^* + t_{N_S-F+2,N_S}^* + \cdots + t_{N_S,N_S}^* \\
  \end{array} \right) \times \left( \begin{array}{c}
  F \text{ rows} \\
  F \text{ rows} \\
 \end{array} \right) \]  
\[ \text{(S16)} \]

To simplify the mathematical calculation, we introduce a new set of variables \( b_{ij} \) \((i = 1, 2, \ldots, N_P; j = 1, 2, \ldots, N_S)\):
\[ b_{ij} = t_{ij} + t_{ifi} + \cdots + t_{f,ij}. \]  
\[ b_{j} = t_{j} + t_{j} + \cdots + t_{j}. \]  
\[ \vdots \]
\[ b_{N_P} = t_{N_P} + t_{N_P} + \cdots + t_{N_P}. \]  
\[ \text{(S17)} \]

Since \( t_{ij} \) automatically satisfies another circular Gaussian distribution, \( b_{ij} \) usually 9 – 25 times smaller than the number of pixels \( (N_P) \) of a spatial light modulator (SLM).

We note that Eq. (S18) shares the same mathematical form as Eq. (S5), except that \( N_S \) becomes \( N_P \). Hence, following the same procedures described for Eqs. (S6) – (S14), the theoretical PBR when speckle grains are under-sampled can be found by
\[ \text{PBR} = N_P. \]  
\[ \text{(S19)} \]

Eq. (S19) shows that when speckle grains are under-sampled, the theoretical PBR equals the number of SLM pixels, which is usually \( 9 - 25 \) times higher than the theoretical PBR achieved when speckle grains are well-sampled.

### 3. Derivation of peak-to-background ratios of optical time reversal using different wavefront modulation schemes when speckle grains are under-sampled

To achieve a large pixel count or a fast modulation speed, many types of SLMs do not support full-field (amplitude plus phase) electric field modulation. For example, nematic liquid crystal based SLMs provide phase-only modulation, ferroelectric liquid crystal based SLMs provide binary-phase modulation, and digital micromirror devices provide binary-amplitude modulation. Thus, it is important to investigate the performance of time-reversal–based wavefront shaping under various modulation schemes, especially when speckle grains are under-sampled. Here, we start by describing various modulation schemes in the well-sampled condition. We note that when speckle grains are well-sampled, the PBRs under different wavefront modulation schemes were reported in the literature \[4-7\].

#### 3.1 When speckle grains are well-sampled

##### 3.1.1 Phase-only modulation

Based on Eq. (S4), when only the phase of the electric field is modulated, the conjugated scattered field is
\[ E_{\text{S,well-sampled, phase-only}}^* = \left( \begin{array}{c}
  t_{11}^* I_{F1} \\
  t_{12}^* I_{F2} \\
 \vdots \\
  t_{F1}^* I_{NF1} \\
 \end{array} \right) \]  
\[ \text{(S20)} \]

Thus, the optical phase conjugated field \( E_{\text{OPC}} \) exiting the scattering medium can be obtained by multiplying the backward transmission matrix \( T^T \) by \( E_{\text{S,well-sampled, phase-only}}^* \).
Specifically, the first element, which corresponds to the electric field of the peak of the focus, is computed as

\[ E_{\text{peak}} = \sum_{m=1}^{N_x} f_{m1}^* f_{m1} = \sum_{m=1}^{N_x} |f_{m1}|^2 \]  \hspace{1cm} \text{(S21)}

Thus, the peak intensity of the focus is computed by

\[ I_{\text{peak}} = \left\{ |E_{\text{peak}}|^2 \right\} \]

\[ = \left[ N_x \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dydx \cdot \left| f_x(0, \sigma) f_z(0, \sigma) \right|^2 \right] \]

\[ = \left[ N_x \int_{0}^{2\pi} \int_{0}^{\infty} d\sigma \rho^2 \cdot \left( \frac{1}{2\pi} \right) \right] \]

\[ = N_x \sqrt{2} \pi \left( \frac{3}{2} \right)^2 \]

Thus, the peak intensity can be calculated as

\[ I_{\text{peak}} = \left\{ |E_{\text{peak}}|^2 \right\} \]

\[ = N_x \int_{0}^{2\pi} \int_{0}^{\infty} d\sigma \rho^2 \cdot \left( \frac{1}{2\pi} \right) \]

\[ = \frac{N_x \sqrt{2} \pi \left( \frac{3}{2} \right)^2}{2} \]

\[ = \frac{N_x \sqrt{2} \pi \left( \frac{3}{2} \right)^2}{2} \]

\[ \text{where } \theta(\cdot) \text{ is the Heaviside step function. The remaining elements of } \mathbf{E}_{\text{OPC}} \text{ correspond to the fields of the background, and can be calculated as} \]

\[ E_{\text{background}} = \sum_{m=1}^{N_x} f_{mne} \mathbf{BP}(t_{m1}), \quad n \neq 1 \]

\[ \text{(S23)} \]

Thus, the background intensity is expressed as

\[ I_{\text{background}} = \left\{ \sum_{m=1}^{N_x} f_{\text{mne}} \mathbf{BP}(t_{m1}) \right\}^2, \quad n \neq 1 \]

\[ \text{which can be computed by following the same procedures as for Eqs. (S9) – (S12)}: \]

\[ I_{\text{background}} = N_x \text{Var}(f_{\text{mne}}) \text{Var}(\mathbf{BP}(t_{m1})) \]

\[ = N_x \times 2\sigma^2 \times 1 = 2\sigma^2 N_x \]

\[ \text{From the results in Eq. (S22) and Eq. (S25), the theoretical PBR for phase-only modulation is} \]

\[ \text{PBR} = \frac{I_{\text{peak}}}{I_{\text{background}}} = \frac{\pi}{4} N_x \text{ (Phase-only modulation)} \]

\[ \text{S46} \]

\[ \text{3.1.2 Binary-phase modulation} \]

\[ \text{When an SLM provides binary-phase modulation, the conjugated scattered field is} \]

\[ \mathbf{E}_{\text{S,well-sampled, binary phase}}^* = \begin{pmatrix} \mathbf{BP}(t_{11}) \\ \mathbf{BP}(t_{12}) \\ \vdots \\ \mathbf{BP}(t_{N_x}) \end{pmatrix}_{N_x \times 1} \]

\[ \text{In the above equation, the binary-phase operator } \mathbf{BP}(z = x + iy) \text{ is defined as} \]

\[ \mathbf{BP}(z) = \begin{cases} 1, & \text{if } y > 0, \\ -1, & \text{if } y < 0. \end{cases} \]

\[ \text{By multiplying the backward transmission matrix by } \mathbf{E}_{\text{S,well-sampled, binary phase}}^*, \text{ the optical phase conjugated field } \mathbf{E}_{\text{OPC}} \text{ exiting the scattering medium can be obtained. Again, the first element of } \mathbf{E}_{\text{OPC}} \text{ corresponds to the electric field of the peak of the focus, and it can be calculated as} \]

\[ E_{\text{peak}} = \sum_{m=1}^{N_x} f_{m1} \mathbf{BP}(t_{m1}) \]

\[ \text{Thus, the peak intensity can be calculated as} \]

\[ I_{\text{peak}} = \left\{ |E_{\text{peak}}|^2 \right\} \]

\[ = N_x \int_{0}^{2\pi} \int_{0}^{\infty} d\sigma \rho^2 \cdot \left( \frac{1}{2\pi} \right) \]

\[ = 2N_x \int_{0}^{2\pi} \int_{0}^{\infty} d\sigma \rho^2 \cdot \left( \frac{1}{2\pi} \right) \]

\[ = \frac{2N_x \sqrt{2} \pi \left( \frac{3}{2} \right)^2}{2} \]

\[ \text{where } \theta(\cdot) \text{ is the Heaviside step function. The remaining elements of } \mathbf{E}_{\text{OPC}} \text{ correspond to the fields of the background, and can be calculated as} \]

\[ E_{\text{background}} = \sum_{m=1}^{N_x} f_{mne} \mathbf{BP}(t_{m1}), \quad n \neq 1 \]

\[ \text{(S31)} \]

Thus, the background intensity is expressed as

\[ I_{\text{background}} = \left\{ \sum_{m=1}^{N_x} f_{\text{mne}} \mathbf{BP}(t_{m1}) \right\}^2, \quad n \neq 1 \]

\[ \text{which can be computed by following the same procedures as for Eqs. (S9) – (S12):} \]

\[ I_{\text{background}} = N_x \text{Var}(f_{\text{mne}}) \text{Var}(\mathbf{BP}(t_{m1})) \]

\[ = N_x \times 2\sigma^2 \times 1 = 2\sigma^2 N_x \]

\[ \text{From the results in Eq. (S30) and Eq. (S33), the theoretical PBR for binary-phase modulation is} \]

\[ \text{PBR} = \frac{I_{\text{peak}}}{I_{\text{background}}} = \frac{\pi}{4} N_x \text{ (Binary-phase modulation)} \]

\[ \text{S44} \]

\[ \text{3.1.3 Binary-amplitude modulation} \]

\[ \text{When an SLM provides binary-amplitude modulation, the conjugated scattered field is} \]

\[ \mathbf{E}_{\text{S,well-sampled, binary amplitude}}^* = \begin{pmatrix} \mathbf{BA}(t_{11}) \\ \mathbf{BA}(t_{12}) \\ \vdots \\ \mathbf{BA}(t_{N_x}) \end{pmatrix}_{N_x \times 1} \]

\[ \text{where the binary-amplitude operator } \mathbf{BA}(z = x + iy) \text{ is defined as} \]

\[ \mathbf{BA}(z) = \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y < 0. \end{cases} \]

\[ \text{By multiplying the backward scattering matrix } \mathbf{T}' \text{ by } \mathbf{E}_{\text{S,well-sampled, binary amplitude}}^*, \text{ the optical phase conjugated field } \mathbf{E}_{\text{OPC}} \text{ exiting the scattering medium can be obtained. The electric field of the peak of the focus corresponds to the first element of } \mathbf{E}_{\text{OPC}}, \text{ and can be calculated as} \]

\[ E_{\text{peak}} = \sum_{m=1}^{N_x} f_{m1} \mathbf{BA}(t_{m1}) \]

\[ \text{Thus, the peak intensity can be calculated as} \]
\[ I_{\text{peak}} = \left|F_{\text{peak}} \right| = \sqrt{N_s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy (x+iy) \theta(y) f_1(0, \sigma) f_1(0, \sigma)' \]
\[ = N_s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy (x+iy) \theta(y) f_1(0, \sigma) f_1(0, \sigma) \]
\[ = \left| N_s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy (x+iy) \theta(y) f_1(0, \sigma) f_1(0, \sigma)' \right| \]
\[ = \left| N_s \int_{-\infty}^{\infty} dy y \theta(y) f_1(0, \sigma) \right| \]
\[ = \left| N_s \sigma \sqrt{2 \pi} \right| = N_s^2 \sigma^2 \frac{\Gamma(1)}{2 \pi} \]

The remaining elements of \( E_{\text{spec}} \) correspond to the fields of the background, and can be calculated as
\[ E_{\text{background}} = \sum_{n=1}^{N_s} I_{\text{sw}} B A(t_{m n})_n \neq 1. \]

Thus, the background intensity is expressed as
\[ I_{\text{background}} = \left| \sum_{n=1}^{N_s} I_{\text{sw}} B A(t_{m n})_n \right|^2, n \neq 1, \]

which can be computed by following the same procedures as for Eqs. (S9) – (S12):
\[ I_{\text{background}} = N_s \text{Var}(t_{sw} B A(t_{m}))_n \]
\[ = N_s \frac{1}{2} \text{Var}(t_{sw})_n \]
\[ = N_s \times 2 \sigma^2 \times \frac{1}{2} = \sigma^2 N_s. \]

From the results in Eq. (S38) and Eq. (S41), the theoretical PBR for binary-amplitude modulation is
\[ \text{PBR} = \frac{I_{\text{peak}}}{I_{\text{background}}} = \frac{1}{2\pi N_s}. \text{(Binary-amplitude modulation)} \]

We note that when speckle grains are well-sampled, same results for the PBRs under different wavefront modulation schemes were also reported [4-7].

### 3.2 When speckle grains are under-sampled

Using the transformations in Eq. (S17) and following the same procedures for Eqs. (S20) – (S42), when multiple speckle grains are within one pixel, the theoretical PBRs can be derived as
\[ \text{PBR}_{\text{under-sampled}} = \frac{\pi}{4} N_s, \text{(phase-only modulation)} \]
\[ \text{PBR}_{\text{under-sampled}} = \frac{1}{\pi} N_s, \text{(binary-phase modulation)} \]
\[ \text{PBR}_{\text{under-sampled}} = \frac{1}{2\pi} N_s, \text{(binary-amplitude modulation)} \]

### 4. Numerical simulations of peak-to-background ratios of optical time reversal using various wavefront modulation schemes when speckle grains are under-sampled

To verify the analytical results in Eqs. (S19) and (S43), we performed numerical simulation based on random matrix theory. In the simulation, the pixel count of the PCM was fixed to be 100, while the under-sampling factor \( F \), which is the number of speckle grains within one pixel, was varied from 1 to 100 (\( F = N_s / N_p \)). Figure S1 shows the simulated PBR/\( N_p \) as a function of \( F \) for different wavefront modulation schemes. We observe that PBR/\( N_p \) remains close to a constant value independent of \( F \), and the constant values are close to their theoretical values of 1, \( \pi/4, 1/\pi \), and \( 1/(2\pi) \), for full-field, phase-only, binary-phase, and binary-amplitude modulation, respectively.

Fig. S1 Numerical simulation of peak-to-background ratios (PBRs) when speckle grains are under-sampled with different factors \( F \) (1 ≤ \( F \) ≤ 100). The black circles, blue squares, red diamonds, and green triangles denote PBR/\( N_p \) for full-field, phase-only, binary-phase, and binary-amplitude modulation, respectively. Each data point was obtained by averaging 200 independent simulation results. Error bars are not plotted due to their indiscernible lengths in the figure. The black solid line, blue dashed line, red dotted line, and green dash-dot line denote theoretical values obtained from the analytical theory for full-field, phase-only, binary-phase, and binary-amplitude modulation, respectively.

### 5. Under-sampling speckle grains improves the PBR of the focus when focusing light inside a scattering medium

Fig. S2 Focusing light inside a scattering medium comprising two diffusers using focused-ultrasound-guided digital optical phase conjugation. a. Schematic of the set-up for observing the focus. A beamsplitter creates a copy of the focus so we can measure it with a camera outside the water tank. b. The observed focus when speckle grains were well-
sampled (2.8 × 2.8 pixels per speckle grain on average) during wavefront measurement. c. The observed focus when speckle grains were under-sampled (F = 5) during wavefront measurement. The PBR of the focus was improved by 4 times compared with that in b. Because a 2.9-ms-long burst of ultrasound was employed, the focus was elongated along the x direction (the acoustic axis direction).

BS, beamsplitter; R, reference beam; SLM, spatial light modulator; US, ultrasound (50 MHz, numerical aperture = 0.4). Scale bar, 200 µm.

6. Signal-to-noise ratio (SNR) of wavefront measurement when speckle grains are under-sampled

To investigate the SNR of wavefront measurement when speckle grains are under-sampled, Eq. (1) is converted into a representation of the number of photoelectrons:

\[
N_k = n_R + \sum_{j=1}^{F} n_R(\tilde{r}_j) + \sum_{j=1}^{F} 2\frac{n_R}{F} n_S(\tilde{r}_j) \cos \left( \phi_k(\tilde{r}_j) + \frac{(k-1)v}{2} \right),
\]

\( k = 1, 2, 3, 4 \).

Here, \( N_k \) is the number of photoelectrons per pixel in each successive frame, \( n_R = \eta AE^2 t/\hbar v \) is the number of photoelectrons induced by the reference beam per pixel, \( n_S(\tilde{r}_j) = \eta (A/F) E^2(\tilde{r}_j) t/\hbar v \) is the number of photoelectrons induced by the \( j \)-th speckle grain, \( \eta \) is the quantum efficiency, \( t \) is the exposure time, \( h \) is Planck’s constant, and \( v \) is the frequency of a photon. Considering that \( \phi_k(\tilde{r}_j) \) is uniformly distributed between 0 and \( 2\pi \), the interference component \( \sum_{j=1}^{F} 2\frac{n_R}{F} n_S(\tilde{r}_j) \cos \left[ \phi_k(\tilde{r}_j) \right] \) is a random summation of \( F \) terms. Thus, the signal amplitude = max \( \left\{ \sum_{j=1}^{F} 2\frac{n_R}{F} n_S(\tilde{r}_j) \cos \left[ \phi_k(\tilde{r}_j) \right] \right\} \)

\(~\sqrt{F} \times 2\frac{n_R}{F} \times \text{NPS} = 2\sqrt{n_R \times \text{NPS}},\) (S45)

where NPS is the average number of photoelectrons induced by the light exiting the sample per speckle grain. Since the major noises during wavefront measurement are the shot-noise of the reference beam and the camera readout noise, the SNR can be estimated as

\[
\text{SNR}_{\text{under-sampled}} = \frac{2\sqrt{n_R \times \text{NPS}}}{\sqrt{\left(n_R^2 + n_{\text{camera}}^2\right)}},\) (S46)

where \( n_{\text{camera}} \) is the readout noise of the camera sensor. When detection is shot-noise limited (\( n_R \gg n_{\text{camera}} \)), the SNR can be further simplified as

\[
\text{SNR}_{\text{under-sampled}} = 2\sqrt{\text{NPS}}.\) (S47)

This result indicates that when speckle grains are under-sampled, the SNR is determined only by the average photoelectron number induced by speckle grains, and is not directly related to the under-sampling factor \( F \).

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