SYMMETRY PRESERVING SELF-ADJOINT EXTENSIONS OF
SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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Abstract. We develop a general technique for finding self-adjoint extensions
of a symmetric operator that respect a given set of its symmetries. Problems of this
type naturally arise when considering two- and three-dimensional
Schrödinger operators with singular potentials. The approach is based on con-
structing a unitary transformation diagonalizing the symmetries and reducing
the initial operator to the direct integral of a suitable family of partial opera-
tors. We prove that symmetry preserving self-adjoint extensions of the initial
operator are in a one-to-one correspondence with measurable families of self-
adjoint extensions of partial operators obtained by reduction. The general
construction is applied to the three-dimensional Aharonov-Bohm Hamiltonian
describing the electron in the magnetic field of an infinitely thin solenoid.

1. Introduction

It is well known that strong singularities in the potential may lead to the lack of
self-adjointness of the corresponding Schrödinger operator on its natural domain.
As a result, the quantum model is no longer fixed uniquely by the potential and
different quantum dynamics described by various self-adjoint extensions of the ini-
tial Schrödinger operator are possible. Without additional physical information,
it is generally impossible to choose a single extension giving the “true” dynamics.
However, the arbitrariness can be reduced if there are symmetries of the initial
Schrödinger operator: in this case, it is natural to require the extensions to also
respect these symmetries. In this paper, we propose a general technique for find-
ning all such symmetry preserving extensions and apply it to the analysis of the
Aharonov-Bohm Hamiltonian describing a charged particle in the magnetic field of
an infinitely thin solenoid.

Most generally, the problem of finding symmetry preserving self-adjoint exten-
sions can be posed as follows. Suppose \( H \) is a symmetric (not necessarily closed)
operator in a separable Hilbert space \( \mathcal{H} \) and \( \mathcal{X} \) is a set of symmetries of \( \mathcal{H} \), i.e.,
bounded everywhere defined operators in \( \mathcal{H} \) commuting\(^1\) with \( H \). Then our aim is
to find all self-adjoint extensions \( \tilde{H} \) of \( H \) that commute with all elements of \( \mathcal{X} \).

In this paper, we assume that the symmetries are normal pairwise commuting
operators. The procedure of finding symmetry preserving self-adjoint extensions of
\( H \) falls into three major steps:

- Diagonalization of symmetries.

\(^1\) The commutation of \( T \in \mathcal{X} \) with \( H \) means that \( T\Psi \in D_H \) and \( TH\Psi = HT\Psi \) for any \( \Psi \)
belonging to the domain \( D_H \) of \( H \) (see the beginning of Sec. 1).

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• Reduction of \( H \).
• Finding self-adjoint extensions of the partial operators obtained via reduction of \( H \).

By a diagonalization of \( \mathcal{X} \), we mean a unitary operator \( V : \mathcal{H} \to \int_{\mathcal{S}}^{\oplus} \mathcal{S}(s) \, d\nu(s) \) such that \( VTV^{-1} \) is the operator \( T_f \) of multiplication by some \( \nu \)-measurable complex function \( f \) in \( \int_{\mathcal{S}}^{\oplus} \mathcal{S}(s) \, d\nu(s) \) for any \( T \in \mathcal{X} \) (here, \( \nu \) is a measure on a measurable space \( S \) and \( \mathcal{S} \) is a \( \nu \)-measurable family of Hilbert spaces on \( S \); we shall briefly recall the notions related to direct integrals of Hilbert spaces in Sec. 5). We shall be mainly interested in a special class of diagonalizations, called exact, that satisfy the following condition:

(E) For any bounded everywhere defined operator \( R \) in \( \mathcal{H} \) that commutes with all elements of \( \mathcal{X} \), the operator \( VTV^{-1} \) commutes with \( T_f \) for any \( \nu \)-measurable bounded \( f \) on \( S \).

This condition allows us to apply the von Neumann’s reduction theory [11] (or, more precisely, its generalization due to Nussbaum [9] for the case of unbounded operators) and conclude that \( VTV^{-1} \) can be decomposed into a direct integral of closed operators for any closed \( R \) commuting with symmetries. For applications, it is important to have a criterion for deciding whether a given diagonalization is exact or not. To this end, we introduce the notion of a \( \nu \)-separating family of functions on \( S \) (see Definition 4.1) and prove, under very mild assumptions on \( \nu \), that a diagonalization is exact if and only if there is a \( \nu \)-separating family \( \{ f_\iota \}_{\iota \in I} \) of \( \nu \)-measurable complex functions on \( S \) such that \( VTV^{-1} \) is an extension of \( DV \) for any \( \iota \in I \) (see Theorem 5.3). The latter condition is usually easily checked for concrete examples.

By a reduction of \( H \) with respect to a given diagonalization for \( \mathcal{X} \), we mean a \( \nu \)-measurable family of operators \( a(s) \) acting in \( \mathcal{S}(s) \) such that \( \int_{\mathcal{S}}^{\oplus} a(s) \, d\nu(s) \) is an extension of \( VTV^{-1} \) and the image \( V(D_H) \) of \( D_H \) under \( V \) has a suitable density with respect to the domains of \( a(s) \) (see Definition 6.3 and Definition 6.4 for details).

In this paper, we do not give any general recipe for constructing diagonalizations and reductions: this has to be done separately for each concrete case. At the same time, we prove that exact diagonalizations and reductions always exist for any set \( \mathcal{X} \) of normal bounded pairwise commuting operators in \( \mathcal{H} \) and any densely defined closable operator \( H \) commuting with all elements of \( \mathcal{X} \) (Theorem 5.5 and Lemma 6.6).

Given an exact diagonalization for \( \mathcal{X} \) and a reduction of \( H \), we can describe all symmetry preserving extensions of \( H \). Namely, we prove (Theorem 6.5) that the operator

(1) \[ \hat{H} = V^{-1} \int_{\mathcal{S}}^{\oplus} \tilde{a}(s) \, d\nu(s) V \]

is a self-adjoint extension of \( H \) commuting with symmetries for any \( \nu \)-measurable family \( \tilde{a}(s) \) of self-adjoint extensions of \( a(s) \). Conversely, for any self-adjoint extension \( \hat{H} \) of \( H \) commuting with symmetries, there is a unique (up to \( \nu \)-equivalence) \( \nu \)-measurable family \( \tilde{a}(s) \) of self-adjoint extensions of \( a(s) \) such that (1) holds.

We illustrate the general construction described above by applying it to the three-dimensional model of an electron in the magnetic field of an infinitely thin solenoid. In this case, the Hamiltonian is formally given by the differential expression

(2) \[ \frac{\hbar^2}{2m_e} \left( i \nabla + \frac{e}{\hbar c} A \right)^2, \]
where $e$ and $m_e$ are the electron charge and mass respectively, $c$ is the velocity of light, and the vector potential $A = (A^1, A^2, A^3)$ has the form

$$
A^1(x, y, z) = -\frac{\Phi y}{2\pi(x^2 + y^2)}, \quad A^2(x, y, z) = \frac{\Phi x}{2\pi(x^2 + y^2)}, \quad A^3(x, y, z) = 0
$$

($\Phi$ is the flux of the magnetic field through the solenoid). Expression (2) is singular on the $z$-axis. For this reason, (2) naturally determines an operator $H$ in $L^2(\mathbb{R}^3)$ with the domain consisting of smooth functions with compact support separated from the $z$-axis. As the set $\mathcal{X}$ of symmetries, it is natural to choose the set of all operators in $L^2(\mathbb{R}^3)$ induced by translations along the $z$-axis and rotations around the $z$-axis (it is straightforward to check that $H$ commutes with all such operators).

We describe all self-adjoint extensions of $H$ commuting with the elements of $\mathcal{X}$ (Theorem 8.3).

The paper is organized as follows. In Sec. 2, we fix the measure-theoretic notation and recall some basic facts concerning the integration with respect to spectral measures. In Sec. 3 we show how the commutation properties of (unbounded) operators in a Hilbert space can be described in terms of von Neumann algebras, which provide a convenient setting for the study of diagonalizations and their exactness. In Sec. 4 we give the definition of $\nu$-separating families of functions and use it to describe the systems of generators of von Neumann algebras associated with spectral measures. In Sec. 5 we reformulate the definition of exact diagonalization in terms of von Neumann algebras, establish the existence of exact diagonalizations, and use the results of Secs. 3 and 4 to characterize them in terms of $\nu$-separating families. In Sec. 6 we prove the existence of reductions for any symmetric operator with respect to exact diagonalizations of symmetries and obtain the description of its symmetry preserving self-adjoint extensions. Secs. 7 and 8 are devoted to application of the abstract construction to Schrödinger operators. In Sec. 7 we derive a condition for the measurability of families of one-dimensional Schrödinger operators and their self-adjoint extensions. Combining this condition with the general results of Sec. 6 we find all symmetry preserving self-adjoint extensions of the Aharonov-Bohm Hamiltonian determined by (2).

2. Preliminaries on measures and spectral measures

Recall that a set $S$ is called a measurable space if it is equipped with a $\sigma$-algebra $\Sigma_S$ of subsets of $S$. Given a Borel space $S$, the elements of $\Sigma_S$ are called measurable subsets of $S$. Every subset $A$ of a measurable space $S$ has a natural structure of a measurable space: the $\sigma$-algebra $\Sigma_A$ consists of all sets of the form $A \cap B$, where $B \in \Sigma_S$. A map $f$ from a measurable space $S$ to a measurable space $S'$ is called measurable if $f^{-1}(A) \in \Sigma_S$ for any measurable subset $A$ of $S'$. A measurable map $f : S \to S'$ is called a measurable isomorphism if it is bijective and $f^{-1}$ is a measurable map from $S'$ to $S$.

If $S$ is a topological space, then it can be naturally made a measurable space by putting $\Sigma_S$ equal to the Borel $\sigma$-algebra of $S$ (i.e., the smallest $\sigma$-algebra on $S$ containing all open subsets of $S$). We shall assume, unless otherwise specified, that all considered topological spaces (in particular, $\mathbb{R}$ and $\mathbb{C}$) carry a measurable structure defined in this way.

A measure on a measurable space $S$ means a countably additive function $\nu$ from $\Sigma_S$ to the extended positive semi-axis $[0, \infty]$. A subset $N$ of $S$ is called a $\nu$-null set if $N \subset N'$, where $N'$ is measurable and $\nu(N') = 0$. A map $f$ is said to be
defined $\nu$-almost everywhere ($\nu$-a.e.) on $S$ if there is a $\nu$-null set $N$ such that $S \setminus N \subset D_f$, where $D_f$ is the domain of $f$. Given a set $S'$, a map $f$ is said to be an $\nu$-a.e. defined map from $S$ to $S'$ if there is a $\nu$-null set $N$ such that $S \setminus N \subset D_f$ and $f(s) \in S'$ for any $s \in S \setminus N$. Two $\nu$-a.e. defined maps $f$ and $g$ are called equal $\nu$-a.e. if there is a $\nu$-null set $N$ such that $S \setminus N \subset D_f \cap D_g$ and $f$ and $g$ coincide on $S \setminus N$. The $\nu$-essential supremum of a $\nu$-a.e. defined real function $f$ on $S$ (notation $\nu$-ess sup$_{s \in S} f(s)$) is the greatest lower bound of $C \in \mathbb{R}$ such that $f(s) \leq C$ for $\nu$-almost every ($\nu$-a.e.) $s \in S$. A complex $\nu$-a.e. defined function $f$ is said to be $\nu$-essentially bounded on $S$ if $\nu$-ess sup$_{s \in S} |f(s)| < \infty$. A map $f$ is called an $\nu$-measurable map from $S$ to a measurable space $S'$ if $f$ is defined $\nu$-a.e. on $S$ and there is a measurable map from $S$ to $S'$ that is $\nu$-a.e. equal to $f$.

All maps defined $\nu$-a.e. on $S$ fall into equivalence classes of maps that are equal $\nu$-a.e. Given a $\nu$-a.e. defined map $f$ on $S$, we denote its equivalence class by $[f]_\nu$. For any set $S'$, we denote by $\mathcal{F}(S, S', \nu)$ the set of all equivalence classes $[f]_\nu$ such that $\xi(s) \in S'$ for $\nu$-a.e. $s \in S$. If $S'$ is a vector space, then $\mathcal{F}(S, S', \nu)$ obviously has a natural structure of a vector space. Given a Hilbert space $\mathcal{H}$, we denote by $L^2(S, \mathcal{H}, \nu)$ the subspace of $\mathcal{F}(S, S', \nu)$ consisting of all $[f]_\nu$ such that $f$ is a $\nu$-measurable map from $S$ to $\mathcal{H}$ and $\int ||f(s)||^2 \, d\nu(s) < \infty$. For $\mathcal{H} = \mathbb{C}$, the space $L^2(S, \mathbb{C}, \nu)$ will be denoted by $L^2(S, \nu)$. If $\nu$ is the Lebesgue measure, the space $L^2(S, \nu)$ will be denoted by $L^2(S)$.

A measure $\nu$ on $S$ is called $\sigma$-finite if there is a sequence of measurable sets $A_1, A_2, \ldots$ such that $S = \bigcup_{j=1}^{\infty} A_j$ and $\nu(A_j) < \infty$ for all $j$. Throughout the paper, all measures will be assumed $\sigma$-finite.

Let $S$ be a measurable space, $\mathcal{H}$ be a Hilbert space, and $\mathcal{P}(\mathcal{H})$ be the set of orthogonal projections on $\mathcal{H}$. A map $E: \Sigma_S \to \mathcal{P}(\mathcal{H})$ is called a spectral measure for $(S, \mathcal{H})$ if it is countably additive with respect to the strong operator topology on $\mathcal{P}(\mathcal{H})$ and $E(S)$ is the identity operator in $\mathcal{H}$. If $E$ is a spectral measure, then $E(A_1 \cap A_2) = E(A_1)E(A_2)$ for any measurable $A_1, A_2 \subset S$ (see [2], Sec. 5.1, Theorem 1). For any $\Psi \in \mathcal{H}$, the finite positive measure $E_\Phi$ on $S$ is defined by setting $E_\Phi(A) = \langle E(A)\Psi, \Psi \rangle$ for any measurable $A$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathcal{H}$. A subset $N$ of $S$ is called an $E$-null set if $N \subset N'$, where $N'$ is measurable and $E(N') = 0$. The concepts of an $E$-a.e. defined function, an $E$-essentially bounded function, and an $E$-measurable function are defined in the same way as in the case of a positive measure with $\nu$-null sets replaced with $E$-null sets.

Let $E$ be a spectral measure on a measurable space $S$. Given an $E$-measurable complex function $f$ on $S$, the integral $J^E_f$ of $f$ with respect to $E$ is defined as the unique linear operator in $\mathcal{H}$ such that

\begin{equation}
D_{J^E_f} = \left\{ \Psi \in \mathcal{H} : \int |f(s)|^2 \, dE_\Phi(s) < \infty \right\}
\end{equation}

\begin{equation}
\langle \Psi, J^E_f \Phi \rangle = \int f(s) \, dE_\Phi(s), \quad \Psi \in D_{J^E_f}.
\end{equation}
For any \( E \)-measurable complex function \( f \) on \( S \), the operator \( J_f^E \) is normal\(^2\), and we have \( J_f^E = J_f^* \), where \( \bar{f} \) is the complex conjugate function of \( f \). For any \( E \)-measurable \( f \) and \( g \) on \( S \), we have
\[
J_{fg}^E = J_f^E J_g^E, \quad J_{f+g}^E = J_f^E + J_g^E,
\]
where the bar means closure. The operator \( J_{\bar{f}}^E \) is everywhere defined and bounded if and only if \( f \) is \( E \)-essentially bounded. In this case, we have
\[
\|J_f^E\| = \nu\text{-ess sup}_{s \in S} |f(s)|.
\]

For every normal operator \( T \), there is a unique spectral measure \( \mathcal{E}_T \) on \( \mathbb{C} \) such that \( J_{\text{id}_A}^E = T \), where \( \text{id}_A \) is the identity function on \( \mathbb{C} \). The operators \( \mathcal{E}_T(A) \), where \( A \) is a Borel subset of \( \mathbb{C} \), are called the spectral projections of \( T \). If \( f \) is an \( \mathcal{E}_T \)-measurable complex function on \( S \), then the operator \( J_f^E \) is also denoted as \( f(T) \).

Let \( S \) and \( S' \) be measurable spaces, \( \mathfrak{H} \) be a Hilbert space, \( E \) be a spectral measure for \( (S, \mathfrak{H}) \), and \( \varphi: S \to S' \) be an \( E \)-measurable map. We denote by \( \varphi, E \) the push-forward of \( E \) under \( \varphi \). By definition, this means that \( \varphi_* E \) is the spectral measure for \( (S', \mathfrak{H}) \) such that
\[
\varphi_* E(A) = E(\varphi^{-1}(A))
\]
for any measurable \( A \subset S' \). If \( f \) is an \( \varphi_* E \)-measurable complex function on \( S' \), then \( f \circ \varphi \) is an \( E \)-measurable function on \( S \), and we have
\[
J_{f \circ \varphi}^E = J_f^{E \circ \varphi}.
\]

Let \( \varphi \) be an \( E \)-measurable complex function on \( S \). Formula \((6)\) with \( f = \text{id}_A \) yields \( J_{\text{id}_A}^E = J_A^E \). In view of the uniqueness of \( \mathcal{E}_{J_A^E} \), this means that
\[
\mathcal{E}_{J_A^E} = \varphi_* E.
\]
In view of \((5)\), it follows that \( f \circ \varphi \) is \( E \)-measurable and
\[
f(J_{\varphi_* E}^E) = J_{f \circ \varphi}^E = J_{f \circ \varphi}^E
\]
for any \( \mathcal{E}_{J_{\varphi_* E}} \)-measurable complex function \( f \) on \( \mathbb{C} \).

3. Commutation of Operators and Von Neumann Algebras

Given a Hilbert space \( \mathfrak{H} \), we denote by \( L(\mathfrak{H}) \) the algebra of all bounded everywhere defined linear operators in \( \mathfrak{H} \). We say that operators \( T_1 \) and \( T_2 \) in \( \mathfrak{H} \) with the respective domains \( D_{T_1} \) and \( D_{T_2} \) commute if one of the following conditions is satisfied:

(A) \( T_1 \in L(\mathfrak{H}) \) and we have \( T_1 \Psi \in D_{T_2} \) and \( T_1 T_2 \Psi = T_2 T_1 \Psi \) for any \( \Psi \in D_{T_2} \).
(B) \( T_2 \in L(\mathfrak{H}) \) and we have \( T_2 \Psi \in D_{T_1} \) and \( T_1 T_2 \Psi = T_2 T_1 \Psi \) for any \( \Psi \in D_{T_1} \).
(C) \( T_1 \) and \( T_2 \) are both normal and their spectral projections commute.

Clearly, \( T_1 \) commutes with \( T_2 \) if and only if \( T_2 \) commutes with \( T_1 \). The next lemma shows that, wherever applicable, conditions (A), (B), and (C) are equivalent.

Lemma 3.1. Let \( T_1 \) and \( T_2 \) be commuting operators in \( \mathfrak{H} \). Then we have

1. If \( T_1 \in L(\mathfrak{H}) \), then (A) is satisfied.

\(^2\)Recall that a closed densely defined linear operator \( T \) in a Hilbert space \( \mathfrak{H} \) is called normal if the operators \( TT^* \) and \( T^* T \) have the same domain of definition and coincide thereon (as usual, \( T^* \) denotes the adjoint of \( T \)).
2. If \( T_2 \in L(H) \), then (B) is satisfied.
3. If \( T_1 \) and \( T_2 \) are both normal, then (C) is satisfied.

Proof. 1. Let \( T_1 \in L(H) \). If (B) is satisfied, then \( T_2 \in L(H) \) and \( T_1 \) and \( T_2 \) commute in the usual sense. Hence, (A) holds. If (C) is satisfied, then (A) is ensured by Theorem 1 of Sec. 5.4 in [2] applied to the spectral measure of \( L \) in the usual sense. Hence, (A) holds. If (C) is satisfied, then \( T_1 \) commutes with all spectral projections of \( T_2 \) by Theorem 1 in [5]. Applying the latter theorem again, we get (C). If (B) holds, then \( T_2 \) and \( T_1 \) satisfy (A), which again implies (C). The lemma is proved.

**Lemma 3.2.** Let \( R \in L(H) \) and \( T \) be a densely defined operator in \( H \) commuting with \( R \). Then \( R^* \) commutes with \( T^* \). If \( T \) is closable, then the closure \( \bar{T} \) of \( T \) commutes with \( R \).

Proof. By statement 1 of Lemma 3.1, the operators \( R \) and \( T \) satisfy (A). Let \( \Psi \in D_{T^*} \) and \( \Phi = T^*\Psi \). Then we have \( \langle T\Psi', \Psi \rangle = \langle \Psi', \Phi \rangle \) for any \( \Psi' \in D_T \) and, in view of (A), we obtain
\[
\langle T\Psi', R^*\Psi \rangle = \langle TR\Psi', \Psi \rangle = \langle R\Psi', \Phi \rangle = \langle \Psi', R^*\Phi \rangle, \quad \Psi' \in D_T.
\]
This means that \( R^*\Psi \in D_{T^*} \) and \( T^*R^*\Psi = R^*T^*\Psi \), i.e., \( R^* \) and \( T^* \) satisfy (A). If \( T \) is closable, then \( R \) commutes with \( \bar{T} \) because \( R = (R^*)^* \) and \( \bar{T} = (T^*)^* \). The lemma is proved.

Given a set \( \mathcal{X} \) of operators in \( H \), let \( \mathcal{X}' \) denote its commutant, i.e., the subalgebra of \( L(H) \) consisting of all operators commuting with every element of \( \mathcal{X} \). If all operators in \( \mathcal{X} \) are densely defined, we denote by \( \mathcal{X}^{**} \) the set consisting of the adjoints of the elements of \( \mathcal{X} \). The set \( \mathcal{X} \) is called involutive if \( \mathcal{X}^{**} = \mathcal{X} \). Lemma 3.2 implies that
\[
(\mathcal{X}')^* = (\mathcal{X}^{**})'
\]
whenever all elements of \( \mathcal{X} \) are closed and densely defined. Recall [4] that a subalgebra \( \mathcal{A} \) of \( L(H) \) is called a von Neumann algebra if it is involutive and coincides with its bicommutant \( \mathcal{A}'' \). By the well-known von Neumann’s theorem (see, e. g., [4], Sec. I.3.4, Corollaire 2), an involutive subalgebra \( \mathcal{A} \) of \( L(H) \) is a von Neumann algebra if and only if it contains the identity operator and is closed in the strong operator topology.

**Lemma 3.3.** If \( \mathcal{X} \) is an involutive set of densely defined closed operators in \( H \), then \( \mathcal{X}' \) is a von Neumann algebra.

Proof. By [9], the algebra \( \mathcal{X}' \) is involutive, and it suffices to show that \( \mathcal{X}' \) is closed in the strong operator topology. Given an operator \( T \) in \( H \), let \( C_T \) denote the set of all elements of \( L(H) \) commuting with \( T \) (in other words, \( C_T \) is the commutant of the one-point set \( \{T\} \)). Since \( \mathcal{X}' = \bigcap_{T \in \mathcal{X}} C_T \), it suffices to prove that \( C_T \) is strongly closed for any closed \( T \). Let \( R \) belong to the strong closure of \( C_T \). For every \( \Psi_1, \Psi_2 \in H \) and \( n = 1, 2, \ldots \), the set
\[
W_{\Psi_1, \Psi_2, n} = \{ \tilde{R} \in L(H) : \| (\tilde{R} - R)\Psi_i \| < 1/n, \ i = 1, 2 \}
\]
3The same is true for the weak operator topology because every involutive strongly closed subalgebra of \( L(H) \) is weakly closed (see [4], Sec. I.3.4, Théorème 2).
is a strong neighborhood of $R$ and, hence, has a nonempty intersection with $C_T$. Fix $\Psi \in D_T$ and choose $R_n \in C_T \cap W_{\Psi, T\Psi, n}$ for each $n$. Then $R_n \Psi \to R\Psi$ and $R_n T\Psi \to RT\Psi$ in $\mathfrak{H}$. As $R_n$ commute with $T$, we have $R_n \Psi \in D_T$ and $R_n T\Psi = TR_n\Psi$ for all $n$. In view of the closedness of $T$, it follows that $R\Psi \in D_T$ and $TR\Psi = RT\Psi$, i.e., $R \in C_T$. The lemma is proved.

Let $\mathcal{X}$ be a set of closed densely defined operators in $\mathfrak{H}$. Then the set $\mathcal{X} \cup \mathcal{X}^*$ is involutive. We set $A(\mathcal{X}) = (\mathcal{X} \cup \mathcal{X}^*)''$ and call $A(\mathcal{X})$ the von Neumann algebra generated by $\mathcal{X}$. If $\mathcal{X} \subset L(\mathfrak{H})$, then $A(\mathcal{X})$ is the smallest von Neumann algebra containing $\mathcal{X}$. If $\mathcal{X}$ consists of normal operators, then $(\mathcal{X} \cup \mathcal{X}^*)' = \mathcal{X}'$ (by Theorem 1 in [5], if $R \in L(\mathfrak{H})$ commutes with a normal operator $T$, then it commutes with $T^*$) and, therefore, we have $A(\mathcal{X}) = \mathcal{X}'$. If $T$ is a closed densely defined operator, then we shall write $A(T)$ instead of $A(\{T\})$, where $\{T\}$ is the one-point set containing $T$.

Given a spectral measure $E$ on a measurable space $S$, we denote by $\mathcal{P}_E$ the set of all operators $E(A)$, where $A$ is a measurable subset of $S$. Theorem 3 of Sec. 6.6 in [2] implies that $\{T, T^*\}' = \mathcal{P}_E'$ for any normal operator $T$ and, hence,

\[(10) \quad A(T) = A(\mathcal{P}_E').\]

**Lemma 3.4.** Let $T_1$ and $T_2$ be normal operators in $\mathfrak{H}$. Then the following statements are equivalent:

1. $T_1$ commutes with $T_2$.
2. $T_1$ commutes with every element of $A(T_2)$.
3. Every element of $A(T_1)$ commutes with every element of $A(T_2)$.

**Proof.** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the sets of all spectral projections of $T_1$ and $T_2$ respectively. By (10), we have

\[(11) \quad A(T_1) = A(\mathcal{P}_1), \quad A(T_2) = A(\mathcal{P}_2).\]

By Lemma 3.1, $T_1$ and $T_2$ commute if and only if they satisfy (C), i.e., if and only if $\mathcal{P}_1 \subset \mathcal{P}_2$. Since $\mathcal{P}_2 = A(\mathcal{P}_2)'$ and $A(\mathcal{P}_1)$ is the smallest von Neumann algebra containing $\mathcal{P}_1$, the latter inclusion is equivalent to the relation $A(\mathcal{P}_1) \subset A(\mathcal{P}_2)'$, which means, in view of (11), that all elements of $A(T_1)$ commute with all elements of $A(T_2)$. Thus, statements 1 and 3 are equivalent. By Lemma 3.2, $T_1$ commutes with every element of $A(T_2)$ if and only if $A(T_2) \subset \{T_1, T_1^*\}'$, i.e., if and only if $A(T_1) \subset A(T_2)'$. Hence, statements 2 and 3 are equivalent and the lemma is proved.

**Lemma 3.5.** A closed densely defined operator $T$ in $\mathfrak{H}$ is normal if and only if the algebra $A(T)$ is Abelian.

**Proof.** If $T$ is normal, then it commutes with itself and Lemma 3.4 shows that $A(T)$ is Abelian. Let $T$ be a closed densely defined operator such that $A(T)$ is Abelian. Let $|T| = (T^*T)^{1/2}$. Then $D_{|T|} = D_T$, the range $\text{Ran} |T|$ of $|T|$ coincides with $\text{Ran} T^*$, and we have the polar decompositions

\[(12) \quad T = U|T|, \quad T^* = |T|U^*\]

where $U$ is a partially isometric operator $U$ in $\mathfrak{H}$ with the initial space $\overline{\text{Ran} |T|}$ (see, e.g., Sec. 8.1 in [2] for details on polar decomposition). By Theorem 4 of Sec. 8.1 in [2], we have

\[(13) \quad TT^* = UT^*TU^*.\]
We now show that

(14) \[ U \in \mathcal{A}(T), \quad \mathcal{A}(|T|) \subset \mathcal{A}(T). \]

Let \( R \in \{T, T^*\} \). Then \( R \) commutes with \( T^*T \), and it follows from Theorem 2 of Sec. 6.3 and Theorem 8 of Sec. 5.4 in [2] that \( R \) commutes with \(|T|\). This implies the second relation in (14). Let \( \Psi \in \text{Ran} |T| \) and \( \Phi \in \mathcal{D}T \) be such that \( \Psi = |T|\Phi \). Then \( RU\Phi = DT \), and we have

\[ RU\Psi = RT\Phi = TR\Phi = U|T|R\Phi = URU\Psi. \]

If \( \Psi \in \text{Ran}|T|^{-1} \), then \( R\Psi \in \text{Ran}|T|^{-1} \) because \( R \) and \(|T|\) commute, and we have \( URU\Psi = URU\Phi = 0 \). We thus see that \( UR = UR \) everywhere on \( \mathfrak{H} \) and, therefore, the first relation in (14) holds. Hence, \( U \) is normal, and it follows from (14) and Lemma 3.4 that \( U \) commutes with \(|T|\). Now equalities (12) imply that \( U \) commutes with both \( T \) and \( T^* \), and Lemma 3.2 ensures that \( U^* \) also commutes with both \( T \) and \( T^* \). Hence, \( U^* \) commutes with \( UT^*T \), and it follows from (13) that \( TT^* \) is an extension of the operator \( U^*UT^*T \). But \( U^*U \) is the orthogonal projection onto the initial space \( \text{Ran}|T| = \text{Ran}T^* \) and, therefore, \( U^*UT^*T = T^*T \). Thus, \( TT^* \) is an extension of \( T^*T \). Since both operators are self-adjoint, this implies \( TT^* = T^*T \). The lemma is proved.

If all elements of an involutive set \( \mathcal{X} \subset L(\mathfrak{H}) \) pairwise commute, then the algebra \( \mathcal{A}(\mathcal{X}) \) is Abelian. Indeed, we have \( \mathcal{X} \subset \mathcal{X}' \) and, therefore, \( \mathcal{A}(\mathcal{X}) \subset \mathcal{X}' \), whence the statement follows because \( \mathcal{X}' = \mathcal{A}(\mathcal{X})' \).

**Lemma 3.6.** Let \( S \) be a measurable space, \( \mathfrak{H} \) be a separable Hilbert space, and \( E \) be a spectral measure for \((\mathfrak{H}, S)\). Then \( \mathcal{A}(\mathcal{P}_E) \) coincides with the set of all \( J^E \), where \( f \) is an \( E \)-measurable \( E \)-essentially bounded complex function on \( S \). A closed densely defined operator \( T \) in \( \mathfrak{H} \) is equal to \( J^E \) for an \( E \)-measurable complex function \( f \) on \( S \) if and only if

(15) \[ \mathcal{A}(T) \subset \mathcal{A}(\mathcal{P}_E) \]

**Proof.** By Theorem 8 of Sec. 5.4 in [2], we have \( \mathcal{A}(J^E) \subset \mathcal{A}(\mathcal{P}_E) \) for any \( E \)-measurable complex function \( f \) on \( S \). This implies that \( J^E \in \mathcal{A}(\mathcal{P}_E) \) for \( E \)-essentially bounded \( f \) because \( J^E \) belongs to \( L(\mathfrak{H}) \) for such \( f \) and, hence, is contained in \( \mathcal{A}(J^E) \). Conversely, Theorem 5 of Sec. 7.4 in [2] shows that any element of \( \mathcal{A}(\mathcal{P}_E) \) is equal to \( J^E \) for some \( E \)-measurable \( E \)-essentially bounded complex function \( f \) on \( S \). It remains to prove that any closed densely defined operator \( T \) such that (15) holds \( J^E \) for some \( E \)-measurable complex function \( f \) on \( S \). Since the elements of \( \mathcal{P}_E \) pairwise commute, the algebra \( \mathcal{A}(\mathcal{P}_E) \) is Abelian, and Lemma 3.5 implies that \( T \) is normal. Let \( \chi \) be a complex function on \( \mathbb{C} \) defined by the relation

\[ \chi(z) = \frac{z}{|z| + 1}. \]

It is easy to see that the function \( \chi \) is one-to-one and maps \( \mathbb{C} \) onto the open unit disc \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \). Its inverse function \( \chi^{-1} \) is given by

\[ \chi^{-1}(z) = \frac{2}{1 - |z|}, \quad |z| < 1. \]

Since \( \chi \) is bounded and measurable on \( \mathbb{C} \), we have \( \chi(T) \in \mathcal{A}(\mathcal{P}_E) \). By (15), it follows that \( \chi(T) \in \mathcal{A}(T) \). In view of (15), this implies that \( \chi(T) \in \mathcal{A}(\mathcal{P}_E) \).
and, therefore, \( \chi(T) = J_g^F \) for some \( E \)-measurable function \( g \) on \( S \). As \( \mathbb{C} \setminus \emptyset \) is a \( \chi_0 \mathcal{E}_T \)-null set and \( \chi^{-1} \) is a measurable function from \( \emptyset \) to \( \mathbb{C} \), the function \( \chi^{-1} \) is \( \chi_0 \mathcal{E}_T \)-measurable on \( \mathbb{C} \). By \( \text{(4)} \), we have \( \mathcal{E}(\chi(T)) = \mathcal{E}_T \) and, hence, \( \chi^{-1} \) is \( \mathcal{E}(\chi(T)) \)-measurable. It therefore follows from \( \text{(8)} \) that
\[
T = \chi^{-1}(\chi(T)) = \chi^{-1}(J_g^F) = J_f^F
\]
for \( f = \chi^{-1} \circ g \). The lemma is proved.

We say that two sets \( X \) and \( Y \) of closed densely defined operators in \( \mathfrak{H} \) are equivalent if \( A(X) = A(Y) \). We say that \( X \) is equivalent to a closed densely defined operator \( T \) if \( X \) is equivalent to the one-point set \( \{ T \} \). Two closed densely defined operators \( T_1 \) and \( T_2 \) are called equivalent if \( \{ T_1 \} \) and \( \{ T_2 \} \) are equivalent.

**Lemma 3.7.** Let \( \{ X_i \}_{i \in I} \) and \( \{ Y_i \}_{i \in I} \) be families of sets of closed densely defined operators in \( \mathfrak{H} \) and let \( X = \bigcup_{i \in I} X_i \) and \( Y = \bigcup_{i \in I} Y_i \). If \( X_i \) and \( Y_i \) are equivalent for every \( i \in I \), then \( X \) and \( Y \) are equivalent.

**Proof.** Set \( M_i = (X_i \cup X_i^*)' \) and \( M = (X \cup X^*)' \). Then we have \( M = \bigcap_{i \in I} M_i \). Hence \( M' = A(X) \) coincides with the von Neumann algebra generated by \( \bigcup_{i \in I} M_i' = \bigcup_{i \in I} A(X_i) \) (see \( \text{(4)} \), Sec. I.1.1, Proposition 1). Analogously, \( A(Y) \) is the von Neumann algebra generated by \( \bigcup_{i \in I} A(Y_i) \). Since \( A(X_i) = A(Y_i) \) for all \( i \), it follows that \( A(X) = A(Y) \). The lemma is proved.

**Lemma 3.8.** Let \( \mathcal{X} \) be a set of closed densely defined operators in a Hilbert space \( \mathfrak{H} \) and let \( T \) be a closed densely defined operator such that both \( T \) and \( T^* \) commute with all elements of \( \mathcal{X} \). Then \( T \) commutes with all elements of \( A(\mathcal{X}) \).

**Proof.** We first show that \( T \) commutes with all elements of \( A(R) \) for any \( R \in \mathcal{X} \). If \( T \in L(\mathfrak{H}) \), then Lemma 6.6 implies that \( R^* \) commutes with \( T \). This means that \( T \in \{ R, R^* \} \) and, therefore, \( T \) commutes with all elements of \( A(R) = \{ R, R^* \} \).

If \( R \in L(\mathfrak{H}) \), then \( R \in \{ T, T^* \} \). By Lemma 6.3 \( \{ T, T^* \} \) is a von Neumann algebra. Since \( A(R) \) is the smallest von Neumann algebra containing \( R \), we have \( A(R) \subset \{ T, T^* \} \) and, hence, \( T \) commutes with all elements of \( A(R) \). If neither \( T \) nor \( R \) belongs to \( L(\mathfrak{H}) \), then \( T \) and \( R \) are normal and the statement follows from Lemma 3.3. Interchanging the roles of \( T \) and \( T^* \), we conclude that \( T^* \) also commutes with all elements of \( A(R) \) for any \( R \in \mathcal{X} \). Let \( \mathcal{Y} = \bigcup_{R \in \mathcal{X}} A(R) \). Clearly, we have \( \mathcal{Y} \subset \{ T, T^* \} \), and Lemma 3.7 implies that \( A(\mathcal{Y}) = A(\mathcal{X}) \). Since \( A(\mathcal{X}) \) is the smallest von Neumann algebra containing \( \mathcal{Y} \), we have \( A(\mathcal{X}) \subset \{ T, T^* \} \) and, hence, \( T \) commutes with all elements of \( A(\mathcal{X}) \). The lemma is proved.

**Lemma 3.9.** Let \( \mathfrak{H} \) be a separable Hilbert space and \( \mathcal{X} \) be a set of closed densely defined operators in \( \mathfrak{H} \). Then there is a countable subset \( \mathcal{X}_0 \) of \( \mathcal{X} \) which is equivalent to \( \mathcal{X} \).

**Proof.** We first note that every subset of \( L(\mathfrak{H}) \) is separable in the strong topology. Indeed, for any \( M \subset L(\mathfrak{H}) \), we have \( M = \bigcup_{n=1}^{\infty} M \cap B_n \), where \( B_n = \{ T \in L(\mathfrak{H}) : \| T \| \leq n \} \) is the ball of radius \( n \) in \( L(\mathfrak{H}) \). Since \( \mathfrak{H} \) is separable, \( B_n \) endowed with the strong topology is a separable metrizable space for any \( n \) (see, e.g., \( \text{(4)} \), Sec. I.3.1). This implies that \( M \cap B_n \) is separable for any \( n \) and, hence, \( M \) is separable in the strong topology.

Let \( \mathfrak{A} = \bigcup_{\mathcal{Y} \subset \mathcal{X}} A(\mathcal{Y}) \), where \( \mathcal{Y} \) runs through all finite subsets of \( \mathcal{X} \). Obviously, \( A(T) \) is equivalent to \( T \) for any closed densely defined operator \( T \) and, therefore,
Lemma 3.7 implies that $\mathcal{X}$ is equivalent to $\bigcup_{T \in \mathcal{X}} \mathcal{A}(T)$. Since the latter set is contained in $\mathfrak{A}$ and $\mathfrak{A} \subset \mathcal{A}(\mathcal{X})$, we conclude that $\mathfrak{A}$ is equivalent to $\mathcal{X}$. We now note that $\mathfrak{A}$ is an involutive subalgebra of $L(\mathcal{S})$ containing the identity operator and, therefore, is strongly dense in $\mathfrak{A}' = \mathcal{A}(\mathcal{X})$ (\cite{4}, Sec. I.3.4, Lemma 6). Let $\mathfrak{R}$ be a strongly dense countable subset of $\mathfrak{A}$. For any $R \in \mathfrak{R}$, we choose a finite set $\mathcal{Y}_R \subset \mathcal{X}$ such that $R \in \mathcal{A}(\mathcal{Y}_R)$ and put $\mathcal{X}_0 = \bigcup_{R \in \mathfrak{R}} \mathcal{Y}_R$. Clearly, $\mathcal{X}_0$ is a countable set. The algebra $\mathcal{A}(\mathcal{X}_0)$ is strongly dense in $\mathcal{A}(\mathcal{X})$ because it contains $\mathfrak{R}$. On the other hand, $\mathcal{A}(\mathcal{X}_0)$ is a von Neumann algebra and, therefore, is strongly closed. We hence have $\mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{X})$, i.e., $\mathcal{X}_0$ is equivalent to $\mathcal{X}$. The lemma is proved. \[\square\]

**Lemma 3.10.** Let $\mathcal{Y}$ be a Hilbert space and $\mathcal{X}$ be a set of closed densely defined operators in $\mathcal{Y}$. The algebra $\mathcal{A}(\mathcal{X})$ is Abelian if and only if all elements of $\mathcal{X}$ are normal and pairwise commute.

**Proof.** Let $\mathcal{Y} = \bigcup_{T \in \mathcal{X}} \mathcal{A}(T)$. By Lemma 3.7, the sets $\mathcal{X}$ and $\mathcal{Y}$ are equivalent. If all elements of $\mathcal{X}$ are normal and pairwise commute, then Lemma 3.4 implies that all elements of $\mathcal{Y}$ pairwise commute. Since $\mathcal{Y} \subset L(\mathcal{S})$, this means that $\mathcal{A}(\mathcal{Y}) = \mathcal{A}(\mathcal{X})$ is Abelian. Conversely, if $\mathcal{A}(\mathcal{X})$ is Abelian, then all elements of $\mathcal{Y}$ pairwise commute. Hence, Lemma 3.6 implies that all elements of $\mathcal{X}$ are normal and Lemma 3.3 implies that all elements of $\mathcal{X}$ pairwise commute. The lemma is proved. \[\square\]

4. **Generators of von Neumann algebras associated with spectral measures**

Recall that a topological space $S$ is called a Polish space if its topology can be induced by a metric that makes $S$ a separable complete space. A measurable space $S$ is called a standard Borel space if its measurable structure can be induced by a Polish topology on $S$. A measure $\nu$ on a measurable space $S$ is called standard if there is a measurable set $S' \subset S$ such that $\nu(S \setminus S') = 0$ and $S'$, considered as a measurable subspace of $S$, is a standard Borel space. Standard spectral measures are defined in the same way.

A family of maps $\{f_i\}_{i \in I}$ is said to separate points of a set $S$ if for any two distinct elements $s_1$ and $s_2$ of $S$, there is $i \in I$ such that $f_i(s_1) \neq f_i(s_2)$.

**Definition 4.1.** Let $S$ be a measurable space and $\nu$ be a positive measure on $S$. A family $\{f_i\}_{i \in I}$ of maps is said to be $\nu$-separating on $S$ if $I$ is countable and $\{f_i\}_{i \in I}$ separates points of $S \setminus N$ for some $\nu$-null set $N$. The notion of an $E$-separating family for a spectral measure $E$ is defined analogously.

Given a spectral measure $E$ on $\mathcal{S}$, we denote by $\mathcal{P}_E$ the set of all operators $E(A)$, where $A$ is a measurable set. The main result of this section is the next theorem that gives a complete description of systems of generators for $\mathcal{A}(\mathcal{P}_E)$.

**Theorem 4.2.** Let $S$ be a measurable space, $\mathcal{S}$ be a separable Hilbert space, $E$ be a standard spectral measure for $(S, \mathcal{S})$, and $\mathcal{X}$ be a set of closed densely defined operators in $\mathcal{S}$. Then $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{P}_E)$ if and only if the following conditions hold

1. For every $T \in \mathcal{X}$, there is an $E$-measurable complex function $f$ on $S$ such that $T = J^E_f$.
2. There is an $E$-separating family $\{f_i\}_{i \in I}$ of $E$-measurable complex functions on $S$ such that $J^E_{f_i} \in \mathcal{X}$ for all $i \in I$. 
The rest of this section is devoted to the proof of Theorem 1.2. Given a topological space $S$, we denote by $C(S)$ the space of all continuous complex functions on $S$.

**Lemma 4.3.** Let $S$ be a Polish space, $\mathcal{H}$ be a Hilbert space, and $E$ be a spectral measure for $(S, \mathcal{H})$. Let $\mathcal{C}$ be a subset of $C(S)$ that separates the points of $S$ and $\mathcal{X}$ be the set of all operators $J_f^E$ with $f \in \mathcal{C}$. Then $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{P}_E)$.

In the proof below, all spectral integrals are taken with respect to $E$, and we write for brevity $J_f$ instead of $J_f^E$.

**Proof.** Since $\mathcal{A}(\mathcal{X}) \subset \mathcal{A}(\mathcal{P}_E)$ by Lemma 3.6, we have to show that $\mathcal{A}(\mathcal{P}_E) \subset \mathcal{A}(\mathcal{X})$.

Let $\mathcal{C}$ denote the set of functions, complex conjugate to the elements of $\mathcal{C}$, and let $\mathfrak{A}$ be the subalgebra of $C(S)$ generated by $\mathcal{C} \cup \bar{\mathcal{C}}$ and the constant functions. Fix $U \in (\mathcal{X} \cup \mathcal{X}^*)'$ and let $\mathfrak{A}_U$ denote the subset of $C(S)$ consisting of all $f$ such that $J_f$ commutes with $U$. If $f, g \in \mathfrak{A}_U$, then both $J_f g$ and $J_f + J_g$ commute with $U$. In view of Lemma 3.2 and relations (4), it follows that both $J_f g$ and $J_f + g$ commute with $U$, i.e., $f g \in \mathfrak{A}_U$ and $f + g \in \mathfrak{A}_U$. Hence, $\mathfrak{A}_U$ is an algebra. Since $\mathfrak{A}_U$ obviously contains $\mathcal{C} \cup \bar{\mathcal{C}}$ and all constant functions, we have $\mathfrak{A}_U \supset \mathfrak{A}$. Thus, every element of $(\mathcal{X} \cup \mathcal{X}^*)'$ commutes with any operator $J_f$ with $f \in \mathfrak{A}$.

Given $f \in C(S)$ and a compact set $K \subset S$, we set $B_{f,K} = J_f E(K)$. Let $\Psi \in \mathcal{H}$ and $\Phi = E(K) \Psi$. Then for any measurable set $A$, we have $E_\Phi(A) = E_\Psi(A \cap K)$ and, therefore, $E_\Phi$ is a finite measure supported by $K$. In view of (3), this implies that $\Phi \in D_{J_f}$, i.e., the range of $E(K)$ is contained in the domain of $J_f$. Since $E(K) = J_{\chi_K}$, where $\chi_K$ is the characteristic function of $K$, it follows from (4) that $B_{f,K} = J_{f \chi_K}$. Hence, $B_{f,K} \in L(\mathcal{H})$ and (5) implies that

$$\|B_{f,K}\| \leq \sup_{s \in K} |f(s)|. \tag{16}$$

If $f, g \in C(S)$, then (14) implies that $B_{f + g,K} = B_{f,K} + B_{g,K}$.

We now show that

$$\left(\mathcal{X} \cup \mathcal{X}^*\right)' \subset \mathcal{Y}', \tag{17}$$

where $\mathcal{Y}$ is the set of all $J_f$ with $f \in C(S)$. Let $U \in (\mathcal{X} \cup \mathcal{X}^*)'$. We first prove that $\mathcal{Y}'$ contains all operators $U_K = E(K) U E(K)$, where $K$ is a compact subset of $S$. Fix $f \in C(S)$ and let $\varepsilon > 0$. Since $\mathcal{C} \subset \mathfrak{A}$, the algebra $\mathfrak{A}$ separates points of $S$, and the Stone–Weierstrass theorem implies that there is $g \in \mathfrak{A}$ such that $|f(s) - g(s)| < \varepsilon$ for any $s \in K$. Since $U_K$ commutes with both $J_g$ and $E(K)$, it follows that $U_K$ commutes with $B_{g,K}$. In view of (16), we have

$$\|B_{f,K} U_K - U_K B_{f,K}\| \leq \|B_{f,K} U_K - B_{g,K} U_K\| + \|U_K B_{g,K} - U_K B_{f,K}\| \leq 2 \|B_{f - g,K}\| \|U\| < 2\varepsilon \|U\|.$$

Because $\varepsilon$ is arbitrary, this means that $U_K$ commutes with $B_{f,K}$. This implies that $U_K$ commutes with $J_f$ because $B_{f,K} U_K = J_f U_K$ and $U_K B_{f,K}$ is an extension of $U_K J_f$ by the commutativity of $E(K)$ and $J_f$. This proves that $U_K \in \mathcal{Y}'$. By Lemma 3.3, $\mathcal{Y}'$ is a von Neumann algebra and, in particular, is strongly closed. Hence, inclusion (17) will be proved if we demonstrate that every strong neighborhood of $U$ contains $U_K$ for some compact set $K$. To this end, it suffices to show that for every $\Psi \in \mathcal{H}$ and $\varepsilon > 0$, there is a compact set $K_{\Psi,\varepsilon}$ such that

$$\|(U - U_K)\Psi\| \leq \varepsilon \tag{18}$$
for any compact set $K \supset K_{\Phi, \varepsilon}$. Since

$$U - U_K = E(K)UE(S \setminus K) + E(S \setminus K)U,$$

we have

$$\|U - U_K\| \leq \|U\| E_\Phi(S \setminus K)^{\frac{1}{2}} + E_\Phi(S \setminus K)^{\frac{1}{2}},$$

where $\Phi = U \Psi$. As $S$ is a Polish space, Theorem 1.3 in [1] ensures that there is a compact set $K_{\Phi, \varepsilon}$ such that both $E_\Phi(S \setminus K_{\Phi, \varepsilon})$ and $E_\Phi(S \setminus K_{\Phi, \varepsilon})$ do not exceed $\varepsilon^2/4$ and, therefore, (18) holds for any $K \supset K_{\Phi, \varepsilon}$. Inclusion (17) is thus proved.

We next show that

$$\mathcal{Y}' \subset \mathcal{P}_E'.$$

Let $U \in \mathcal{Y}'$. For any closed set $F \subset S$, it is easy to construct a uniformly bounded sequence of functions $f_n \in C(S)$ that converges pointwise to $\chi_F$. Then $J_{f_n}$ strongly converge to $J_{\chi_F} = E(F)$ (see Theorem 2 of Sec. 5.3 in [2]). Since $J_{f_n}$ commute with $U$ for all $n$, this implies that $E(F)$ commutes with $U$. We have proved that $\Sigma^U$ contains all closed sets. If $A \in \Sigma^U$, then $E(S \setminus A) = 1 - E(A)$ commutes with $U$ and, hence, $S \setminus A \in \Sigma^U$. If $A_1, A_2 \in \Sigma^U$, then both $E(A_1 \cup A_2) = E(A_1)E(A_2)$ and $E(A_1 \cap A_2) = E(A_1) + E(A_2) - E(A_1)E(A_2)$ commute with $U$ and, therefore, $A_1 \cup A_2$ and $A_1 \cap A_2$ belong to $\Sigma^U$. Let $A_n$ be a sequence of elements of $\Sigma^U$ and $A = \bigcup_{n=1}^\infty A_n$. For all $n = 1, 2, \ldots$, we set $B_n = \bigcup_{j=1}^n A_j$. Then $B_n \in \Sigma^U$ for all $n$, and the $\sigma$-additivity of $E$ implies that $E(B_n)$ converge strongly to $E(A)$. Hence, $E(A)$ commutes with $U$, i.e., $A \in \Sigma^U$. We thus see that $\Sigma^U$ is a $\sigma$-algebra containing all closed sets. This implies that $\Sigma^U$ coincides with the Borel $\sigma$-algebra, and (19) is proved.

Inclusions (17) and (19) imply that $(\mathcal{X} \cup \mathcal{X}^*)' \subset \mathcal{P}_E'$ and, hence, $A(\mathcal{P}_E) \subset A(\mathcal{X})$. The lemma is proved.

The next lemma summarizes the facts about Polish and standard Borel spaces that are needed for the proof of Theorem 1.2.

**Lemma 4.4.**

1. Let $S$ and $S'$ be standard Borel spaces and $f: S \to S'$ be a one-to-one measurable mapping. Then $f(S)$ is a measurable subset of $S'$ and $f$ is a measurable isomorphism from $S$ onto $f(S)$.

2. Let $S$ be a Polish space and $B$ be its Borel subset. Then there are a Polish space $P$ and a continuous one-to-one mapping $g: P \to S$ such that $B = g(P)$.

3. If $S$ is a standard Borel space, then there exists a one-to-one function from $S$ to the segment $[0, 1]$.

**Proof.** Statement 1 follows from Theorem 3.2 in [7], which, in its turn, is a reformulation of a theorem by Souslin (see [5], Chapter III, Sec. 35.IV). For the proof of statement 2, see Lemma 6 of Sec. IX.6.7 in [3]. To prove statement 3, we recall that every standard Borel space is either countable or isomorphic to the segment $[0, 1]$ (see [10], Appendix, Corollary A.11). In the latter case, any isomorphism between $S$ and $[0, 1]$ gives us the required function. If $S$ is countable, then we can just choose any one-to-one map from $S$ to $[0, 1]$ because all functions on $S$ are measurable. The lemma is proved.

**Lemma 4.5.** Let $E$ be a spectral measure on a standard Borel space $S$, $I$ be a countable set and $\{f_i\}_{i \in I}$ be a family of measurable complex-valued functions on
S that separates the points of S. Then the von Neumann algebra generated by all operators \( J_{f_i} \) with \( i \in I \) coincides with \( \mathcal{A}(\mathcal{P}_E) \).

**Proof.** Let \( f \) denote the map \( s \to \{ f_i(s) \}_{i \in I} \) from \( S \) to \( \mathbb{C}^I \). The space \( \mathbb{C}^I \) endowed with its natural product topology is a Polish space, and the measurability of \( f_i \) implies that of \( f \). Since \( f_i \) separate the points of \( S \), the map \( f \) is one-to-one. By statement 1 of Lemma 4.4, \( f(S) \) is a Borel subset of \( \mathbb{C}^I \) and \( f \) is a measurable isomorphism of \( S \) onto \( f(S) \). By statement 2 of Lemma 4.4 there are a Polish space \( P \) and a continuous one-to-one map \( g: P \to \mathbb{C}^I \) such that \( f(S) = g(P) \). Hence, \( h = f^{-1} \circ g \) is a measurable one-to-one map from \( P \) onto \( S \). By statement 1 of Lemma 4.4 \( h \) is a measurable isomorphism from \( P \) onto \( S \). We now use \( h \) to transfer the topology from \( P \) to \( S \), i.e., we say that a set \( O \subset S \) is open if and only if \( h^{-1}(O) \) is open in \( P \). Once \( S \) is equipped with this topology, \( h \) becomes a homeomorphism between \( P \) and \( S \), and, hence, \( S \) becomes a Polish space. Since \( h \) is a measurable isomorphism, the Borel measurable structure generated by the topology of \( S \) coincides with its initial measurable structure. Because \( f = g \circ h^{-1} \) is continuous, all \( f_i \) are continuous. Hence, the statement follows from Lemma 4.3. The lemma is proved.  

The proof of Theorem 4.2.

Suppose conditions (1) and (2) hold. By Lemma 3.6 condition (1) implies that \( \mathcal{A}(\mathcal{X}_0) \subset \mathcal{A}(\mathcal{P}_E) \). Let the family \( \{ f_i \}_{i \in I} \) be as in condition (2) and \( \mathcal{X}_0 \) be the set of all \( J_{f_i}^E \) with \( i \in I \). Since \( E \) is standard, there is a measurable subset \( \hat{S} \) of \( S \) such that \( E(S \setminus \hat{S}) = 0 \) and \( \hat{S} \), considered as a measurable subspace of \( S \), is a standard Borel space. Let \( \hat{E} \) denote the restriction of \( E \) to \( \hat{S} \). For each \( i \in I \), we choose a measurable function \( f_i \) on \( \hat{S} \) that is equal \( E \)-a.e. (or, which is the same, \( \hat{E} \)-a.e.) to \( f_i \). Then we have \( J_{f_i}^E = J_f^E \) for all \( i \in I \) and it follows from Lemma 4.5 that \( \mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{P}_E) \). As \( \mathcal{P}_E = \mathcal{P}_E \), this implies that \( \mathcal{A}(\mathcal{P}_E) \subset \mathcal{A}(\mathcal{X}) \) and, hence, \( \mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{X}) \).

Conversely, let \( \mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{X}) \). Then condition (1) is ensured by Lemma 3.6. By Lemma 3.3 there is a countable set \( \mathcal{X}_0 \subset \mathcal{X} \) such that \( \mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{X}) \). Choose a countable family \( \{ f_i \}_{i \in I} \) of measurable complex functions on \( S \) such that each \( T \in \mathcal{X}_0 \) is equal to \( J_{f_i}^E \) for some \( i \in I \). It suffices to show that \( \{ f_i \}_{i \in I} \) is \( E \)-separating. Let \( f \) be the measurable map \( s \to \{ f_i(s) \}_{i \in I_0} \) from \( S \) to \( \mathbb{C}^I \). For each \( i \in I \), let \( \pi_i: \mathbb{C} \to \mathbb{C} \) be the function taking \( \{ z_\kappa \}_{\kappa \in I} \) to \( z_i \). For any \( i \in I \), we have \( \pi_i \circ f = f_i \), and it follows from \( J_{f_i}^E = J_{f_i}^E \) for all \( i \in I \), i.e., the set of all \( J_{f_i}^E \) coincides with \( \mathcal{X}_0 \). Since the family \( \{ \pi_i \}_{i \in I} \) obviously separates the points of \( \mathbb{C}^I \), Lemma 4.3 implies that \( \mathcal{A}(\mathcal{X}_0) = \mathcal{A}(\mathcal{P}_f,E) \) and, hence,

\[
\mathcal{A}(\mathcal{P}_E) = \mathcal{A}(\mathcal{P}_f,E).
\]

By statement 3 of Lemma 4.4 there exists a one-to-one measurable function \( g \) on \( \hat{S} \). Clearly, \( g \) is \( E \)-measurable on \( \hat{S} \) and it follows from Lemma 3.6 and (20) that \( \mathcal{A}(J_g^E) \in \mathcal{A}(\mathcal{P}_f,E) \). Now Lemma 3.6 implies that there exists a measurable function \( h \) on \( \mathbb{C}^I \) such that \( J_g^E = J_h^E \). In view of (6), this means that \( J_g^E = J_h^E \) and, hence, \( g \) and \( h \circ f \) are equal \( E \)-a.e. Since \( g \) is one-to-one on \( \hat{S} \), it follows that \( f \) is one-to-one on \( \hat{S} \setminus N \), where \( N \) is the set of all \( s \in \hat{S} \) such that \( g(s) \neq h(f(s)) \). This means that \( \{ f_i \}_{i \in I} \) is \( E \)-separating and the theorem is proved.  

\( \square \)
Example 4.6. Let \( \Lambda \subset \mathbb{C} \) be a set having an accumulation point in \( \mathbb{C} \) and let \( f_\lambda(z) = e^{\lambda z} \) for \( \lambda \in \Lambda \) and \( z \in \mathbb{C} \). Clearly, we can choose a countable set \( \Lambda_0 \subset \Lambda \) that has an accumulation point in \( \mathbb{C} \). If \( f_\lambda(z) = f_\lambda(z') \) for some \( z, z' \in \mathbb{C} \) and all \( \lambda \in \Lambda_0 \), then we have \( z = z' \) by the uniqueness theorem for analytic functions and, therefore, the family \( \{f_\lambda\}_{\lambda \in \Lambda_0} \) separates points of \( \mathbb{C} \). Theorem 4.2 therefore implies that \( P_E \) is equivalent to \( \{f_\lambda\}_{\lambda \in \Lambda} \) for any spectral measure \( E \) on \( \mathbb{C} \). In view of (10), it follows that any normal operator \( T \) is equivalent to the set of all operators \( e^{\lambda T} \) with \( \lambda \in \Lambda \).

5. Diagonalizations

Let \( \nu \) be a measure on a measurable space \( S \) and \( \mathcal{S} \) be a \( \nu \)-a.e. defined map on \( S \) such that \( \mathcal{S}(s) \) is a Hilbert space for \( \nu \)-a.e. \( s \) (such a \( \mathcal{S} \) will be called a \( \nu \)-a.e. defined family of Hilbert spaces on \( S \)). A \( \nu \)-a.e. defined map \( \xi \) on \( S \) is said to be a \( \nu \)-a.e. defined section of \( \mathcal{S} \) if \( \xi(s) \in \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \). Let \( \mathcal{F}(\mathcal{S}, \mathcal{S}, \nu) \) denote the set of all equivalence classes whose representatives are \( \nu \)-a.e. defined sections of \( \mathcal{S} \). Clearly, \( \mathcal{F}(\mathcal{S}, \mathcal{S}, \nu) \) has a natural structure of a vector space. A family \( \mathcal{S} \) is called \( \nu \)-measurable on \( S \) if a subspace \( \mathcal{M}(\mathcal{S}) \) of \( \mathcal{F}(\mathcal{S}, \mathcal{S}, \nu) \) is chosen such that

(I) The function \( s \to (\xi(s), \eta(s)) \) on \( S \) is \( \nu \)-measurable for any sections \( \xi, \eta \) of \( \mathcal{S} \) such that \( [\xi]_\nu, [\eta]_\nu \in \mathcal{M}(\mathcal{S}) \).

(II) If \( \xi \) is a section of \( \mathcal{S} \) and the function \( t \to (\xi(t), \eta(t)) \) is \( \nu \)-measurable for any section \( \eta \) of \( \mathcal{S} \) such that \( [\eta]_\nu \in \mathcal{M}(\mathcal{S}) \), then \( [\xi]_\nu \in \mathcal{M}(\mathcal{S}) \).

(III) There is a sequence \( \xi_1, \xi_2, \ldots \) of sections of \( \mathcal{S} \) such that \( [\xi_j]_\nu \in \mathcal{M}(\mathcal{S}) \) for all \( j \) and the linear span of the sequence \( \xi_1(s), \xi_2(s), \ldots \) is dense in \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \in S \).

Given a \( \nu \)-measurable family \( \mathcal{S} \) of Hilbert spaces, a section \( \xi \) of \( \mathcal{S} \) is called \( \nu \)-measurable if \( [\xi]_\nu \in \mathcal{M}(\mathcal{S}) \). The direct integral \( \int_S^\mathcal{S} \mathcal{S}(s) d\nu(s) \) of a \( \nu \)-measurable family \( \mathcal{S} \) is, by definition, the vector subspace of \( \mathcal{M}(\mathcal{S}) \) consisting of all \([\xi]_\nu\), where the section \( \xi \) is \( \nu \)-measurable and

\[ \int_S \|\xi(s)\|^2 d\nu(s) < \infty. \]

The scalar product of \([f]_\nu, [g]_\nu \in \int_S^\mathcal{S} \mathcal{S}(s) d\nu(s)\) is defined by the relation

\[ \langle [f]_\nu, [g]_\nu \rangle = \int_S \langle f(s), g(s) \rangle d\nu(s). \]

This scalar product makes \( \int_S^\mathcal{S} \mathcal{S}(s) d\nu(s) \) a Hilbert space. A family \( \mathcal{S}' = \{\mathcal{S}'(s)\}_{s \in S} \) of Hilbert spaces is called a \( \nu \)-measurable family of subspaces of \( \mathcal{S} \) if \( \mathcal{S}'(s) \) is a subspace of \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \) and \( \mathcal{M}(\mathcal{S}) \cap \mathcal{F}(\mathcal{S}, \mathcal{S}', \nu) \) is a measurable structure for \( \mathcal{S}' \) (i.e., satisfies conditions (I)-(III) above). In this case, \( \mathcal{S}' \) will be always assumed to be endowed with \( \mathcal{M}(\mathcal{S}') = \mathcal{M}(\mathcal{S}) \cap \mathcal{F}(\mathcal{S}, \mathcal{S}', \nu) \). A family \( \mathcal{S}' = \{\mathcal{S}'(s)\}_{s \in S} \) of Hilbert spaces is a \( \nu \)-measurable family of subspaces of \( \mathcal{S} \) if and only if \( \mathcal{S}'(s) \) is a subspace of \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \) and there is a sequence \( \xi_1, \xi_2, \ldots \) of sections of \( \mathcal{S}' \) such that \([\xi_j]_\nu \in \mathcal{M}(\mathcal{S})\) for all \( j \) and the linear span of the sequence \( \xi_1(s), \xi_2(s), \ldots \) is dense in \( \mathcal{S}'(s) \) for \( \nu \)-a.e. \( s \in S \).

Example 5.1. Let \( h \) be a separable Hilbert space and \( \nu \) be a measure on a measurable space \( S \). Let the family \( \mathcal{I}_{h, \nu} = \{\mathcal{I}_{h, \nu}(s)\}_{s \in S} \) be such that \( \mathcal{I}_{h, \nu}(s) = h \) for all \( s \in S \) and \( \mathcal{M}(\mathcal{I}_{h, \nu}) \) is the set of all \([\xi]_\nu\), where \( \xi \) is a \( \nu \)-measurable map from
Let $\mathcal{M}(\mathcal{I}_{\mathfrak{h},\nu})$ satisfies conditions (I)-(III) and, therefore, $I_{\mathfrak{h},\nu}$ is a $\nu$-measurable family of Hilbert spaces.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\nu$-a.e. defined families of Hilbert spaces on $S$, then the family $\mathcal{G}_1 \oplus \mathcal{G}_2$ is the map defined on $D_{\mathcal{G}_1} \cap D_{\mathcal{G}_2}$ and taking $s$ to $\mathcal{G}_1(s) \oplus \mathcal{G}_2(s)$. If both $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\nu$-measurable, then a $\nu$-a.e. defined section $\xi(s) = (\xi_1(s),\xi_2(s))$ of $\mathcal{G}_1 \oplus \mathcal{G}_2$ is called $\nu$-measurable if $\xi_1$ and $\xi_2$ are $\nu$-measurable sections of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Defining $\mathcal{M}(\mathcal{G}_1 \oplus \mathcal{G}_2)$ as the set of all $[\xi]_\nu$, where $\xi$ is a $\nu$-measurable section of $\mathcal{G}_1 \oplus \mathcal{G}_2$, we make $\mathcal{G}_1 \oplus \mathcal{G}_2$ a $\nu$-measurable family of Hilbert spaces on $S$.

Let $\mathcal{G}$ be a $\nu$-measurable family of Hilbert spaces on $S$ and $a$ be a $\nu$-a.e. defined map on $S$ such that $a(s)$ is an operator in $\mathcal{G}(s)$ for $\nu$-a.e. $s$ (such a map will be called a $\nu$-a.e. defined family of operators in $\mathcal{G}$). A family $a$ of operators in $\mathcal{G}$ is called $\nu$-measurable if $a(s)$ is closable for $\nu$-a.e. $s$ and the graphs $G_{a(s)}$ of $a(s)$ constitute a $\nu$-measurable family of subspaces of $\mathcal{G} \oplus \mathcal{G}$. The family $a(s)$ is measurable if there is a sequence $\xi_1,\xi_2,\ldots$ of $\nu$-measurable sections of $\mathcal{G}$ such that $\xi_n(s) \in D_{a(s)}$ for all $n$ and $\nu$-a.e. $s$ and the linear span of the vectors $(\xi_n(s),a(s)\xi_n(s))$ is dense in $G_{a(s)}$ for $\nu$-a.e. $s$. As shown in [9], if $a(s)$ is a $\nu$-measurable family of operators in $\mathcal{G}$, then the map $s \mapsto a(s)\xi(s)$ is a $\nu$-measurable section of $\mathcal{G}$ for any $\nu$-measurable section $\xi(s)$ of $\mathcal{G}$ such that $\xi(s) \in D_{a(s)}$ for $\nu$-a.e. $s$.

Let $\mathcal{G}$ be a $\nu$-measurable family of Hilbert spaces on $S$, and $g$ be a complex-valued $\nu$-measurable function on $S$. Then $g$ determines a linear operator $T_g$ in $\int_S^\oplus \mathcal{G}(s) \, d\nu(s)$ as follows. The domain $D_{T_g}$ consists of all $[f]_\nu \in \int_S^\oplus \mathcal{G}(s) \, d\nu(s)$, where $f$ is such that the equivalence class of the section $s \mapsto g(s)f(s)$ belongs to $\int_S^\oplus \mathcal{G}(s) \, d\nu(s)$, and the vector $T_g[f]_\nu$ is defined as the equivalence class of the section $s \mapsto g(s)f(s)$.

**Definition 5.2.** Let $\mathfrak{h}$ be a Hilbert space and $\mathcal{X}$ be a set of closed densely defined operators in $\mathfrak{h}$. Let $S$ be a measurable space, $\nu$ be a measure on $S$, $\mathcal{G}$ be a $\nu$-measurable family of Hilbert spaces on $S$, and $V : \mathfrak{h} \to \int_S^\oplus \mathcal{G}(s) \, d\nu(s)$ be a unitary operator. We say that the quadruple $(S,\mathcal{G},\nu,V)$ is a diagonalization for $\mathcal{X}$ if every $T \in \mathcal{X}$ is equal to $V^{-1}T_gV$ for some $\nu$-measurable complex function $g$ on $S$. A diagonalization $(S,\mathfrak{h},\nu,V)$ is called exact if $\nu$ is standard and $V^{-1}T_gV \in A(\mathcal{X})$ for any $\nu$-measurable $\nu$-essentially bounded function $g$ on $S$.

It is easy to see that the above definition of an exact diagonalization is just a reformulation of the condition (E) given in Introduction in terms of von Neumann algebras.

For any $\nu$-measurable family $\mathcal{G}$ of Hilbert spaces on $S$, we can define a spectral measure $\Pi_{\mathcal{G}}$ on $S$ by setting

$$
\Pi_{\mathcal{G}}(A) = T_{\chi_A}
$$

for any measurable set $A$, where $\chi_A$ is the characteristic function of $A$. It is easy to see that $J_{T_g}^{\Pi_{\mathcal{G}}} = T_g$ for any $\nu$-measurable function $g$ on $S$.

**Theorem 5.3.** Let $\mathfrak{h}$ be a separable Hilbert space and $\mathcal{X}$ be a set of closed densely defined operators in $\mathfrak{h}$. A quadruple $(S,\mathcal{G},\nu,V)$, where $S$, $\nu$, $\mathcal{G}$, and $V$ are as in **Definition 5.2** is a diagonalization for $\mathcal{X}$ if and only if it is a diagonalization for $A(\mathcal{X})$. A diagonalization $(S,\mathcal{G},\nu,V)$ for $\mathcal{X}$ is exact if and only if $\nu$ is standard and there is a $\nu$-separating family of $\nu$-measurable complex functions on $S$ such that

$$
V^{-1}T_gV \in \mathcal{X}, \quad g \in I.
$$
Proof. Let $S$, $\mathcal{G}$, $\nu$, and $V$ be as in Definition 5.2 and let the spectral measure $E$ for $(\mathcal{G}, S)$ be defined by the relation $E(A) = V^{-1}\Pi_\mathcal{G}(A)V$, where $A$ is a measurable subset of $S$. Then $E$-measurability coincides with $\nu$-measurability, and we have

\[(22) \quad J^E_g = V^{-1}J^\Pi_{g\nu} V = V^{-1}T_gV\]

for any $\nu$-measurable $g$ on $S$. By Lemmas 5.6 and 5.7 every $T \in \mathcal{X}$ is equal to $J^E_g$ for some $\nu$-measurable $g$ on $S$ if and only if

\[(23) \quad \mathcal{A}(\mathcal{X}) \subset \mathcal{A}(\mathcal{P}_E).\]

In view of (22), this means that the quadruple $(S, \mathcal{G}, \nu, V)$ is a diagonalization for $\mathcal{X}$ if and only if (23) holds. Since $\mathcal{A}(\mathcal{A}(\mathcal{X})) = \mathcal{A}(\mathcal{X})$, it follows that the quadruple $(S, \mathcal{G}, \nu, V)$ is a diagonalization for $\mathcal{X}$ if and only if it is a diagonalization for $\mathcal{A}(\mathcal{X})$.

Let $(S, \mathcal{G}, \nu, V)$ be a diagonalization for $\mathcal{X}$ and $\nu$ be standard. The condition that $V^{-1}T_gV \in \mathcal{A}(\mathcal{X})$ for any $\nu$-essentially bounded $\nu$-measurable function $g$ on $S$ is equivalent to the equality

\[(24) \quad \mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{P}_E),\]

as follows from Lemma 5.6 (22), and (23). Since $(S, \mathcal{G}, \nu, V)$ is a diagonalization for $\mathcal{X}$, it follows from (22) that every $T \in \mathcal{X}$ is equal to $J^E_g$ for some $\nu$-measurable $g$ on $S$. Now Theorem 4.2 implies that (24) holds if and only if there is a $\nu$-separating family $\{g_i\}_{i \in I}$ such that $J^E_{g_i} \in \mathcal{X}$ for all $i \in I$. By (22), the latter condition is equivalent to (23). The theorem is proved.

**Corollary 5.4.** Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{X}$ and $\mathcal{Y}$ be equivalent sets of closed densely defined operators in $\mathcal{H}$. Then every (exact) diagonalization for $\mathcal{X}$ is also an (exact) diagonalization for $\mathcal{Y}$.

**Theorem 5.5.** Every set of pairwise commuting normal operators in a separable Hilbert space admits an exact diagonalization.

**Proof.** Let $\mathcal{X}$ be a set of pairwise commuting normal operators in a separable Hilbert space $\mathcal{H}$. By Lemma 5.10 the algebra $\mathcal{A}(\mathcal{X})$ is Abelian. By Théorème 2 of Sec. II.6.2 in [4], there are a finite measure $\nu$ on a compact metrizable space $S$, a $\nu$-measurable family $\mathcal{G}$ of Hilbert spaces, and a unitary operator $V : \mathcal{H} \to \int_S^\nu \mathcal{G}(s) d\nu(s)$ such that $\mathcal{A}(\mathcal{X})$ coincides with the set of all operators $V^{-1}T_gV$, where $g$ is a $\nu$-measurable $\nu$-essentially bounded function on $S$. It now follows from Theorem 5.3 and Definition 5.2 that $(S, \mathcal{G}, \nu, V)$ is an exact diagonalization for $\mathcal{X}$. The theorem is proved.

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6. **Symmetry preserving extensions**

**Definition 6.1.** Let $\nu$ be a measure on a measurable space $S$, $\mathcal{G}$ be a $\nu$-measurable family of Hilbert spaces on $S$, and $a$ be a $\nu$-a.e. defined family of operators in $\mathcal{G}$. Let $D$ be the subspace of $\int_S^\nu \mathcal{G}(s) d\nu(s)$ consisting of all $[\xi]_\nu$, where $\xi$ is a square-integrable section of $\mathcal{G}$ such that $\xi(s) \in D_{a(s)}$ for $\nu$-a.e. $s$ and the section $s \to a(s)\xi(s)$ of $\mathcal{G}$ is square-integrable. The direct integral $\int_S^\nu a(s) d\nu(s)$ of the family $a(s)$ is defined as the linear operator in $\int_S^\nu \mathcal{G}(s) d\nu(s)$ with domain $D$ taking $[\xi]_\nu \in D$ to the $\nu$-equivalence class of the map $s \to a(s)\xi(s)$.
It is easy to see that \( \int_S a(s) \, dv(s) \) is closed if \( a(s) \) are closed for almost all \( s \). Clearly, the operator \( \int_S a(s) \, dv(s) \) commutes with \( T_S \) for any \( \nu \)-essentially bounded function \( g \) on \( S \). The next statement was proved in [9].

**Lemma 6.2.** Let \( \nu \) be a measure a measurable space \( S \) and \( \mathcal{S} \) be a \( \nu \)-measurable family of Hilbert spaces on \( S \). Let \( A \) be a closed operator in \( \int_S \mathcal{S}(s) \, dv(s) \) that commutes with \( T_{S} \) for any \( \nu \)-essentially bounded function \( g \) on \( S \). Then there is a unique (up to \( \nu \)-equivalence) \( \nu \)-measurable family \( a(s) \) of closed operators in \( \mathcal{S} \) such that

\[
A = \int_S a(s) \, dv(s).
\]

The operator \( A \) is self-adjoint if and only if \( a(s) \) is self-adjoint for almost every \( s \). If \( a(s) \) and \( \tilde{a}(s) \) are \( \nu \)-measurable families of closed operators in \( \mathcal{S} \), then \( \int_S \tilde{a}(s) \, dv(s) \) is an extension of \( \int_S a(s) \, dv(s) \) if and only if \( \tilde{a}(s) \) is an extension of \( a(s) \) for almost all \( s \).

**Definition 6.3.** Let \( \nu \) be a measure a measurable space \( S \) and \( \mathcal{S} \) be a \( \nu \)-measurable family of Hilbert spaces on \( S \). A \( \nu \)-a.e. defined family \( a(s) \) of operators in \( \mathcal{S} \) is said to be compatible with a subspace \( \Delta \) of \( \int_S \mathcal{S}(s) \, dv(s) \) if

(a) for any \( [\xi]_\nu \in \Delta \), the relation \( \xi(s) \in D_{a(s)} \) holds for \( \nu \)-a.e. \( s \),

(b) there is a sequence \( \xi_1, \xi_2, \ldots \) of \( \nu \)-measurable sections of \( \mathcal{S} \) such that \( [\xi_j]_\nu \in \Delta \) for all \( n \) and the linear span of \( \langle \xi_j(s), a(s)\xi_j(s) \rangle \) is dense in the graph of \( a(s) \) for almost all \( s \).

**Definition 6.4.** Let \( H \) be an operator in a Hilbert space \( \mathcal{H} \), \( \nu \) be a measure a measurable space \( S \), and \( \mathcal{S} \) be a \( \nu \)-measurable family of Hilbert spaces on \( S \). Let \( V \) be a unitary operator from \( \mathcal{H} \) to \( \int_S \mathcal{S}(s) \, dv(s) \). A \( \nu \)-a.e. defined family \( a(s) \) of operators in \( \mathcal{S} \) is called a reduction of \( H \) with respect to the quadruple \( (S, \mathcal{S}, \nu, V) \) if the family \( a(s) \) is compatible with \( V(D_H) \) and \( \int_S a(s) \, dv(s) \) is an extension of \( VH^{-1} \).

Clearly, a \( \nu \)-a.e. defined family \( a(s) \) of operators in \( \mathcal{S} \) is a reduction of \( H \) with respect to \( (S, \mathcal{S}, \nu, V) \) if and only if \( a(s) \) is compatible with \( V(D_H) \) and the relation

\[
a(s)\xi(s) = \eta(s)
\]

holds \( \nu \)-a.e. for any square-integrable sections \( \xi \) and \( \eta \) of \( \mathcal{S} \) such that

\[
[\xi]_\nu = V\Psi, \quad [\eta]_\nu = VH\Psi
\]

for some \( \Psi \in D_H \).

**Theorem 6.5.** Let \( \mathcal{X} \) be a set of closed densely defined operators in a Hilbert space \( \mathcal{H} \) and \( (S, \mathcal{S}, \nu, V) \) be an exact diagonalization for \( \mathcal{X} \). Let \( H \) be an operator in \( \mathcal{H} \) with the dense domain \( D_H \) and \( a(s) \) be a reduction of \( H \) with respect to \( (S, \mathcal{S}, \nu, V) \). Then the following statements are valid:

1. Let \( \tilde{a}(s) \) be a \( \nu \)-measurable family of closed extensions of \( a(s) \). Then

\[
\tilde{H} = V^{-1} \int_S \tilde{a}(s) \, dv(s) V
\]

is a closed extension of \( H \) that commutes with all elements of \( A(\mathcal{X}) \). If \( \tilde{a}(s) \) are self-adjoint for \( \nu \)-a.e. \( s \), then \( \tilde{H} \) is self-adjoint and commutes with all normal operators \( T \) such that \( A(T) \subset A(\mathcal{X}) \), in particular, with all elements of \( \mathcal{X} \).
2. Let $\mathcal{Y}$ be a set of closed densely defined operators that is equivalent to $\mathcal{X}$ and $H$ be a closed extension of $H$ such that both $H$ and $H^*$ commute with all elements of $\mathcal{Y}$. Then there is a unique (up to $\nu$-equivalence) $\nu$-measurable family $\tilde{a}(s)$ of closed extensions of $a(s)$ such that formula (27) holds. If $H$ is self-adjoint, then $\tilde{a}(s)$ are self-adjoint for $\nu$-a.e. $s$.

3. Suppose the family $a(s)$ is $\nu$-measurable and there is an involutive set $\mathcal{Y} \subset L(\mathcal{H})$ that is equivalent to $\mathcal{X}$ and leaves $D_H$ invariant (i.e., $T\Psi \in D_H$ for any $\Psi \in D_H$ and $T \in \mathcal{Y}$). Then $H$ is closable and commutes with all elements of $\mathcal{Y}$. Moreover, the operators $a(s)$ are closable for $\nu$-a.e. $s$ and the closure $\hat{H}$ of $H$ is given by

$$
\hat{H} = V^{-1} \int a(s) \, d\nu(s) \, V.
$$

Proof.

1. Let $A = \int a(s) \, d\nu(s)$ and $\tilde{A} = \int \tilde{a}(s) \, d\nu(s)$. By the hypothesis, $A$ is an extension of $VHV^{-1}$. Since $\tilde{A}$ is a closed extension of $A$, we conclude that $\tilde{A}$ is a closed extension of $H$. Theorem 5.3 implies that $(S, \mathcal{S}, \nu, V)$ is a diagonalization for $\mathcal{A}(\mathcal{X})$. This means that every $T \in \mathcal{A}(\mathcal{X})$ has the form $T = V^{-1} \mathcal{T}_g V$ for some $\nu$-essentially bounded $g$ on $S$. Since $\mathcal{T}_g$ commutes with $\tilde{A}$, it follows that $T$ commutes with $\tilde{H}$. If $\tilde{a}(s)$ are self-adjoint for $\nu$-a.e. $s$, then $\tilde{A}$ is self-adjoint by Lemma 6.2 and, therefore, $\hat{H}$ is self-adjoint. By Lemma 5.4, $\tilde{H}$ commutes with all normal operators $T$ such that $\mathcal{A}(T) \subset \mathcal{A}(\mathcal{X})$.

2. Let $A = VH V^{-1}$. Since both $\tilde{H}$ and $\tilde{H}^*$ commute with all elements of $\mathcal{Y}$, it follows from Lemma 3.8 that $\tilde{H}$ commutes with all elements of $\mathcal{A}(\mathcal{Y}) = \mathcal{A}(\mathcal{X})$. As the diagonalization $(S, \mathcal{S}, \nu, V)$ is exact, we have $V^{-1} \mathcal{T}_g V \in \mathcal{A}(\mathcal{X})$ for any $\nu$-measurable $\nu$-essentially bounded function $g$ on $S$. Hence, $\tilde{A}$ commutes with $\mathcal{T}_g$ for any $\nu$-measurable $\nu$-essentially bounded $g$, and it follows from Lemma 6.2 that there is a unique (up to $\nu$-equivalence) $\nu$-measurable family $\tilde{a}$ of closed operators in $\mathcal{S}$ such that

$$
\tilde{A} = \int \tilde{a}(s) \, d\nu(s).
$$

If $\hat{H}$ is self-adjoint, then $\tilde{A}$ is also self-adjoint, and Lemma 6.2 ensures that $\tilde{a}(s)$ are self-adjoint for $\nu$-a.e. $s$. For any $\nu$-measurable section $\xi$ of $\mathcal{S}$ such that $[\xi]_\nu \in V(D_H)$, we have $\xi(s) \in D_{a(s)} \cap D_{\tilde{a}(s)}$ and the relation

$$
a(s)(\xi(s)) = \tilde{a}(s)\xi(s)
$$

holds for $\nu$-a.e. $s$. Indeed, since the family $a(s)$ is compatible with to $V(D_H)$, we have $\xi(s) \in D_{a(s)}$ for $\nu$-a.e. $s$. Let $\eta$ be a $\nu$-measurable section of $\mathcal{S}$ such that $[\eta]_\nu = VH V^{-1} [\xi]_\nu$. Then obviously there is $\Psi \in D_H$ such that equalities (26) hold and, therefore, relation (25) holds for $\nu$-a.e. $s$. Since $\hat{H}$ is an extension of $H$, $\tilde{A}$ is an extension of $VHV^{-1}$ and, in view of (29), we conclude that $\xi(s) \in D_{\tilde{a}(s)}$ and

$$
\eta(s) = \tilde{a}(s)\xi(s)
$$

for $\nu$-a.e. $s$. Together with (25), this equality implies (30).

Let $\xi_1, \xi_2, \ldots$ be a sequence of sections of $\mathcal{S}$ such that $[\xi_j]_\nu \in V(D_H)$ for all $j$ and the linear span of $(\xi_j(s), a(s)\xi_j(s))$ is dense in the graph of $a(s)$ for $\nu$-a.e. $s$.
(such a sequence exists because the family of operators \(a(s)\) is compatible with to \(V(D_H)\)). Let \(L_s\) denote the linear span of \(\xi_j(s)\). By (30), \(L_s \subset D_{a(s)} \cap D_{\bar{a}(s)}\) and \(\bar{a}(s)\psi = a(s)\psi\) for any \(\psi \in L_s\). For any \(\psi \in D_{a(s)}\), there is a sequence \(\psi_n\) of elements of \(L_s\) such that \(\psi_n \rightarrow \psi\) and \(a(s)\psi_n \rightarrow a(s)\psi\) as \(n \rightarrow \infty\). Since \(\bar{a}(s)\psi_n = a(s)\psi_n\) for all \(n\) and \(\bar{a}(s)\) is closed, we conclude that \(\psi \in D_{\bar{a}(s)}\) and \(\bar{a}(s)\psi = a(s)\psi\). Hence, \(\bar{a}(s)\) is an extension of \(a(s)\) for \(\nu\text{-a.e. } s\).

3. The \(\nu\)-measurability of the family \(a(s)\) implies that \(a(s)\) are closable for \(\nu\text{-a.e. } s\) and the family \(\bar{a}(s)\) is \(\nu\)-measurable. Let \(A = \int a(s)\,d\nu(s)\) and \(B = \int \bar{a}(s)\,d\nu(s)\). Since \(A\) is an extension of \(VHV^{-1}\) and \(B\) is a closed extension of \(A\), we conclude that \(V^{-1}BV\) is a closed extension of \(H\). Hence, \(H\) is closable. By statement 1, \(V^{-1}BV\) commutes with all elements of \(\mathcal{A}(\mathcal{X})\) and, in particular, with all elements of \(\mathcal{Y}\). Since \(V^{-1}BV\) is an extension of \(H\) and \(D_H\) is invariant under \(\mathcal{Y}\), it follows that \(H\) commutes with all elements of \(\mathcal{Y}\), and Lemma 6.2 implies that \(\bar{H}\) commutes with all elements of \(\mathcal{Y}\). By statement 2, there is a \(\nu\)-measurable field \(\bar{a}(s)\) of closed extensions of \(a(s)\) such that \(\bar{H} = V^{-1} \int \bar{a}(s)\,d\nu(s)\,V\). Since \(\bar{a}(s)\) are closed for \(\nu\text{-a.e. } s\), it follows that \(\bar{a}(s)\) are extensions of \(a(s)\) for \(\nu\text{-a.e. } s\). On the other hand, since \(V^{-1}BV\) is a closed extension of \(H\), \(B\) is an extension of \(VHV^{-1} = \int \bar{a}(s)\,d\nu(s)\), and Lemma 6.2 implies that \(\bar{a}(s)\) is an extension of \(a(s)\) for \(\nu\text{-a.e. } s\). Hence, \(\bar{a}(s) = \bar{a}(s)\) for \(\nu\text{-a.e. } s\). The theorem is proved.

Note that if the conditions of statement 3 of Theorem 6.5 are satisfied and \(a(s)\) is essentially self-adjoint for almost all \(s\), then \(H\) is essentially self-adjoint by (28) and Lemma 6.2.

**Lemma 6.6.** Let \(H\) be a closable densely defined operator in a Hilbert space \(\mathfrak{H}\) and \(\mathcal{X} \subset L(\mathfrak{H})\) be an involutive set of operators. Let \((S, \mathfrak{S}, \nu, V)\) be an exact diagonalization for \(\mathcal{X}\). Then \(H\) commutes with all elements of \(\mathcal{X}\) if and only if there exists a \(\nu\)-measurable reduction of \(H\) with respect to \((S, \mathfrak{S}, \nu, V)\) and \(D_H\) is left invariant by all elements of \(\mathcal{X}\).

**Proof.** If there is a \(\nu\)-measurable reduction of \(H\) and \(D_H\) is left invariant by all elements of \(\mathcal{X}\), then \(H\) commutes with all elements of \(\mathcal{X}\) by statement 3 of Theorem 6.5. If \(H\) commutes with all elements of \(\mathcal{X}\), then \(\bar{H}\) also commutes with them by Lemma 6.2. In view of the involutivity of \(\mathcal{X}\), Lemma 6.8 implies that \(\bar{H}\) commutes with all elements of \(\mathcal{A}(\mathcal{X})\). As the diagonalization \((S, \mathfrak{S}, \nu, V)\) is exact, we have \(V^{-1}\mathcal{T}_gV \in \mathcal{A}(\mathcal{X})\) for any \(\nu\)-measurable \(\nu\text{-essentially} \) bounded function \(g\) on \(S\). Hence, \(VHV^{-1}\) commutes with \(\mathcal{T}_g\) for any \(\nu\)-measurable \(\nu\text{-essentially} \) bounded \(g\), and it follows from Lemma 6.2 that there is a unique (up to \(\nu\)-equivalence) \(\nu\)-measurable family \(a(s)\) of closed operators in \(\mathfrak{S}\) such that

\[
VHV^{-1} = \int a(s)\,d\nu(s).
\]

It suffices to show that \(a(s)\) is compatible with \(V(D_H)\). Let \(\xi_1, \xi_2, \ldots\) be square-integrable sections of \(\mathfrak{S}\) such that \(\{\xi_j\}_\nu \in V(D_H)\) for all \(j\) and the linear span of the sequence \(\{\xi_j\}_\nu, VHV^{-1}\{\xi_j\}_\nu\) is dense in the graph of \(VHV^{-1}\). Then this sequence is also dense in the graph of \(VHV^{-1}\), and it follows from Proposition 8 of Sec. II.1.6 in [14] that \(\xi_j\) satisfy the conditions of Definition 6.3. The lemma is proved.

Let \(\mathfrak{h}\) be a separable Hilbert space, \(\nu\) be a measure on a measurable space \(S\), and the \(\nu\)-measurable family \(I_{\mathfrak{h}, \nu}\) of Hilbert spaces on \(S\) be as in Example 5.1. In
respectively. Similarly, we shall say that \((S, h, \nu)\) is a diagonalization for a set of operators \(X\) if \((S, I_{h,\nu}, \nu, V)\) is a diagonalization for \(X\).

**Definition 6.7.** Let \(h\) be a Hilbert space, \(\nu\) be a measure on a set \(S\), and \(D\) be a linear subspace of \(h\). We say that a \(\nu\)-a.e. defined family \(a(s)\) of operators in \(h\) is \(\nu\)-regular with respect to \(D\) if \(D_{a(s)} = D\) for \(\nu\)-a.e. \(s\) and there is a countable subset \(Y\) of \(D\) such that the linear span of the elements \((\psi, a(s)\psi)\) with \(\psi \in Y\) is dense in the graph of \(a(s)\) for \(\nu\)-a.e. \(s\).

For any \(\psi \in h\) and \(f \in L^2(S, dv)\), we define \(\Phi_{\psi,f} \in L^2(S, h, dv)\) by the relation

\[\Phi_{\psi,f}(s) = f(s)\psi\]

for almost all \(s \in S\).

We say that a sequence \(g_1, g_2, \ldots\) of \(\nu\)-a.e. defined complex-valued functions on \(S\) is \(\nu\)-nonvanishing if there are a \(\nu\)-null set \(N\) such that \(S \setminus N\) is contained in the domains of definition of all \(g_j\) and for any \(s \in S \setminus N\), the condition \(g_j(s) \neq 0\) is satisfied for some \(j\).

**Lemma 6.8.** Let \(h\) be a Hilbert space, \(D\) be a linear subspace of \(h\), \(\nu\) be a measure on a set \(S\), and \(a(s)\) be a \(\nu\)-a.e. defined family of operators in \(h\) which is \(\nu\)-regular with respect to \(D\). Then the following statements hold:

1. If the map \(s \rightarrow a(s)\psi\) is \(\nu\)-measurable for every \(\psi \in D\), then the family \(a(s)\) is \(\nu\)-measurable.

2. Let \(\Delta\) be a subspace of \(L^2(S, h, dv)\). Suppose for any \(\psi \in D\), there is a \(\nu\)-nonvanishing sequence \(g_1, g_2, \ldots\) of square-integrable functions such that the \(\nu\)-equivalence classes of maps \(s \rightarrow g_j(s)\psi\) belong to \(\Delta\) for all \(j\). Then \(a(s)\) is compatible with \(\Delta\).

**Proof.**

1. Let \(Y \subset D\) satisfy the conditions of Definition 6.7. We enumerate the elements of \(Y\) as a sequence \(\psi_1, \psi_2, \ldots\). For each \(n = 1, 2, \ldots\), let the map \(\xi_n\) on \(S\) be defined by the relation \(\xi_n(s) = (\psi_n, a(s)\psi_n)\). Clearly, \(\xi_n\) are \(\nu\)-measurable maps from \(S\) to \(h \oplus h\) for all \(n\), and Definition 6.7 implies that the linear span of \(\xi_n(s)\) is dense in \(h\) for \(\nu\)-a.e. \(s\). Hence, \(a(s)\) is \(\nu\)-measurable.

2. For each \(n = 1, 2, \ldots\), we choose a \(\nu\)-nonvanishing sequence \(g_j^{(n)}\) of square-integrable functions such that the \(\nu\)-equivalence classes of all maps \(\eta_j^{(n)}(s) = g_j^{(n)}(s)\psi_n\) on \(S\) belong to \(\Delta\). Then for \(\nu\)-a.e. \(s\), the elements \((\eta_j^{(n)}(s), a(s)\eta_j^{(n)}(s))\) have the same linear span as \((\psi_n, a(s)\psi_n)\), which is dense in the graph of \(a(s)\) by the \(\nu\)-regularity of the family \(a\). The lemma is proved. \(\square\)

7. MEASURABLE FAMILIES OF ONE-DIMENSIONAL SCHröDINGER OPERATORS

Let \(-\infty \leq a < b \leq \infty\) and \(\lambda\) be the Lebesgue measure on \((a, b)\). Let \(q\) be a locally \(\lambda\)-square-integrable real function on \((a, b)\). Let \(D\) denote the space of all absolutely continuous functions on \((a, b)\) whose derivative is also absolutely continuous. For \(f \in D\), we denote by \(l_q f\) the \(\lambda\)-equivalence class of the function

\[x \rightarrow -f''(x) + q(x)f(x)\].
Clearly, $l_q$ is a linear operator from $\mathcal{D}$ to the space of complex $\lambda$-equivalence classes on $(a,b)$. Let $\mathcal{D}_q = \{ f \in \mathcal{D} : [f] \}$ and $l_q f$ are both in $L^2(a,b)$ and $D_q$ be the space of all equivalence classes $[f]$ with $f \in \mathcal{D}_q$. Let $D_0$ be the space of all $[f]$, where $f$ belongs to the space $C_0^{\infty}(a,b)$ of smooth functions whose support is compact and contained in $(a,b)$. We obviously have $D_0 \subset D_q \subset L^2(a,b)$. We define the operator $L_q$ in $L^2(a,b)$ by the relations

\[ D_{L_q} = D_q, \]
\[ L_q^*[f] = l_q f, \quad f \in D_q. \]

The operator $L_q$ in $L^2(a,b)$ is defined as the restriction of $L_q^*$ to $D_0$. Then $L_q$ is a symmetric operator and its adjoint is $L_q^*$ (this justifies our notation). For any $f, g \in \mathcal{D}$, their Wronskian $W(f, g)$ is an absolutely continuous function on $(a,b)$ defined by the relation

\[ W(f, g)(x) = f(x)g'(x) - f'(x)g(x). \]

A $\lambda$-measurable function $f$ on $(a,b)$ is said to be left (right) square-integrable if $\int_a^c |f(x)|^2 \, dx < \infty$ (resp., $\int_b^c |f(x)|^2 \, dx < \infty$) for any $c \in (a,b)$. If $f, g \in \mathcal{D}$ are left square-integrable functions such that $l_q f$ and $l_q g$ are also left square-integrable, then the following limit exist:

\[ W(f, g)(a) = \lim_{x \downarrow a} W(f, g)(x). \]

Similarly, the limit

\[ W(f, g)(b) = \lim_{x \uparrow b} W(f, g)(x) \]

exists for any right square-integrable $f, g \in \mathcal{D}$ such that $l_q f$ and $l_q g$ are also right square-integrable. The closure $\bar{L_q}$ of $L_q$ is the restriction of $L_q^*$ to the subspace

\[ D_{\bar{L_q}} = \{ [f] : f \in \mathcal{D}_q \text{ and } W(f, g)(a) = W(f, g)(b) = 0 \text{ for any } g \in \mathcal{D}_q \}. \]

We now consider the homogeneous equation

\[ l_q f = 0, \quad f \in \mathcal{D}. \]

There are two possibilities

1. All solutions of (33) are left square-integrable (the limit circle case (lcc) at $a$).
2. There is a solution of (34) that is not left square-integrable (the limit point case (lpc) at $a$).

The analogous alternative holds for the right end $b$ of the interval. If $f$ and $g$ are solutions of (34), then the function $W(f, g)(x)$ does not depend on $x$. It is nonzero if and only if $f$ and $g$ are linearly independent. The lpc holds at $b$ (at $a$) if and only if the condition

\[ W(f, g)(b) = 0 \quad \text{(resp., } W(f, g)(a) = 0) \]

is satisfied for any $f, g \in \mathcal{D}_q$.

The description of the self-adjoint extensions of $L_q$ depends on whether we have the limit point or limit circle case at the ends of the interval. In what follows, we assume that lpc holds at $b$. If lpc holds at $a$, then $L_q$ is essentially self-adjoint. If lcc holds at $a$, then the self-adjoint extensions of $L_q$ are parametrized by the real numbers $\lambda$.

\[ \lambda \in (a, b), \quad \lambda \in \mathbb{R}. \]

Throughout this section, all equivalence classes will be taken with respect to the restriction $\lambda_{(a,b)}$ of $\lambda$ to $(a, b)$. We shall drop the subscript and write $[f]$ instead of $[f]_{\lambda_{(a,b)}}$. 
nontrivial solutions of \((34)\) and can be described as follows. Given a real nontrivial solution \(f\) of \((34)\), let
\[
D_{\bar{q}}^f = \{ g \in D_q : W(f, g)(a) = 0 \}
\]
and let \(D_{\bar{q}}^f\) denote the space of all equivalence classes \([f]\) with \(f \in D_{\bar{q}}^f\). Then the restriction \(L_q^f\) of \(L_q\) to \(D_{\bar{q}}^f\) is a self-adjoint extension of \(L_q\). Moreover, all self-adjoint extensions of \(L_q\) can be obtained in this way. Given two real nontrivial solutions \(f\) and \(\tilde{f}\) of \((34)\), we have \(L_q^f = L_q^{\tilde{f}}\) if and only if \(f = Cf\), where \(C\) is a real number. In the lcc at \(a\), the deficiency indices are \((1, 1)\). This implies, in particular, that the orthogonal complement \(G_T \ominus G_{L_q}\) of the graph \(G_{L_q}\) of \(L_q\) in the graph \(G_T\) of \(T\) is one-dimensional for any self-adjoint extension \(T\) of \(L_q\). Let \(f_1\) and \(f_2\) be linearly independent solutions of \((34)\) and let \(g \in D_q\). Let \(\varphi\) be a \(\lambda\)-measurable function such that \([\varphi] = l_qg\). Then the function
\[
(36) \quad \rho_g(x) = \frac{1}{W(f_1, f_2)} \left[ f_1(x) \int_a^x \varphi(x') f_2(x') \, dx' - f_2(x) \int_a^x \varphi(x') f_1(x') \, dx' \right]
\]
belongs to \(D\) and satisfies the equation
\[
l_q \rho_g = l_qg.
\]
Hence, the function
\[
\sigma_g = g - \rho_g
\]
is a solution of \((34)\). It is straightforward to check that \(\rho_g\) and \(\sigma_g\) do not depend on the choice of the solutions \(f_1\) and \(f_2\). In particular, we can choose \(f_1\) and \(f_2\) to be real. Hence, if \(g\) is real, then \(\rho_g\) and \(\sigma_g\) are real.

**Lemma 7.1.** Suppose lcc holds at \(a\). Let \(g \in D_q\) be a real function and \(T\) be a self-adjoint extension of \(L_q\). Then \([g] \in D_T \setminus D_{\bar{q}}\) if and only if \(\sigma_g\) is nontrivial and \(T = L^s_q\).

**Proof.** It follows easily from \((36)\) that
\[
W(\rho_g, h)(x) = \frac{1}{W(f_1, f_2)} \left[ W(f_1, h)(x) \int_a^x \varphi(x') f_2(x') \, dx' - W(f_2, h)(x) \int_a^x \varphi(x') f_1(x') \, dx' \right]
\]
for any \(h \in D\), where \([\varphi] = l_qg\) and \(f_1, f_2\) are linearly independent solutions of \((34)\). This implies that
\[
(37) \quad W(\rho_g, h)(a) = 0
\]
for any \(h \in D\) such that \(h\) and \(l_qh\) are left square-integrable and, therefore,
\[
(38) \quad W(g, h)(a) = W(\sigma_g, h)(a).
\]
Hence, \(\sigma_g\) is trivial if and only if \(W(g, h)(a) = 0\) for any \(h \in D_q\). In view of \((35)\) (recall that lpc is assumed to hold at \(b\)), the latter condition is satisfied if and only if \([g] \in D_{\bar{q}}\).

Suppose now that \([g] \in D_T \setminus D_{\bar{q}}\). By the above, \(\sigma_g\) is nontrivial. Let \(f\) be a real solution of \((34)\) such that \(T = L^s_q\). Then we have \(W(f, g)(a) = 0\), and it follows from \((37)\) that \(W(f, \sigma_g) = 0\). This means that \(\sigma_g = Cf\) for some real \(C \neq 0\) and, therefore, \(T = L^s_q\). Conversely, suppose \(\sigma_g\) is nontrivial and \(T = L^s_q\). Since \(\sigma_g\) is
nontrivial, we have \([g] \notin D_{\bar{L}}\). Setting \(h = \sigma_g\) in (38), we obtain \(W(\sigma_g, g)(a) = 0\) and, hence, \([g] \in D_T\). The lemma is proved. \(\square\)

Given a function \(f(s, x)\) of two variables, we denote by \(f_{[s]}\) the partial function determined by \(f\) for a fixed first argument, i.e., the domain \(D_{f_{[s]}}\) of \(f_{[s]}\) consists of all \(x\) such that \((s, x) \in D_f\) and

\[
f_{[s]}(x) = f(s, x), \quad x \in D_{f_{[s]}}.
\]

**Lemma 7.2.** Let \(\nu\) be a measure on a measurable space \(S\) and \(v\) be a \((\nu \times \lambda)\)-measurable real function on \(S \times (a, b)\) such that \(v_{[s]}\) is locally square-integrable for \(\nu\)-a.e. \(s\). Then the family \(s \to L_{v_{[s]}}\) on \(S\) of operators in \(L^2(a, b)\) is \(\nu\)-measurable and \(\nu\)-regular with respect to \(D_0\).

**Proof.** Let \(C_0^\infty(a, b)\) be endowed with the topology defined by the norms

\[
\|f\|_{K,n} = \sup_{x \in K, 0 \leq j \leq n} |f^{(j)}(x)|,
\]

where \(n = 0, 1, \ldots, K\) is a compact subset of \((a, b)\) and \(f^{(j)}\) is the \(j\)-th derivative of \(f\). Then \(C_0^\infty(a, b)\) becomes a separable metrizable space such that \(L_{v_{[s]}}\) induce continuous linear maps from \(C_0^\infty(a, b)\) to \(L^2(a, b)\) for \(\nu\)-a.e. \(s\). Since the map \(f \to [f]\) puts \(C_0^\infty(a, b)\) and \(D_0\) in a one-to-one correspondence, we can transfer the topology from \(C_0^\infty(a, b)\) to \(D_0\). This makes \(D_0\) a separable metrizable space such that \(L_{v_{[s]}}\) are continuous maps from \(D_0\) to \(L^2(a, b)\) for \(\nu\)-a.e. \(s\). It follows that \(D = D_0\), \(a(s) = L_{v_{[s]}}\), and an arbitrary countable dense subset \(Y\) of \(D_0\) satisfy the conditions of Definition 6.7 and, therefore, the family \(L_{v_{[s]}}\) is \(\nu\)-regular with respect to \(D_0\). If \(f \in C_0^\infty(a, b)\), then the function \((s, x) \to -f''(x) + v(s, x)f(x)\) on \(S \times (a, b)\) is \((\nu \times \lambda)\)-measurable. Lemma A.1 hence implies that the map \(s \to L_{v_{[s]}}[f]\) is \(\nu\)-measurable. The \(\nu\)-measurability of \(L_{v_{[s]}}\) now follows from statement 1 of Lemma 6.8. The lemma is proved. \(\square\)

Let \(\nu\) be a measure on a measurable space \(S\). Given a \((\nu \times \lambda)\)-measurable real function \(v\) on \(S \times (a, b)\) such that \(v_{[s]}\) is locally square-integrable for \(\nu\)-a.e. \(s\), we can consider the homogeneous equation

\[
l_{v_{[s]}}f_{[s]} = 0.
\]

Let \(A\) be a \(\nu\)-measurable subset of \(S\). A \((\nu \times \lambda)\)-a.e. defined function \(f\) on \(A \times (a, b)\) will be called a solution of (39) on \(A\) if \(f_{[s]}\) belongs to \(D\) and satisfies (39) for \(\nu\)-a.e. \(s \in A\). A solution \(f\) on \(A\) is called nontrivial if \(f_{[s]} \neq 0\) for \(\nu\)-a.e. \(s \in A\). Two solutions \(f_1\) and \(f_2\) on \(A\) are called linearly independent if \((f_1)_{[s]}\) and \((f_2)_{[s]}\) are linearly independent for \(\nu\)-a.e. \(s \in A\).

**Lemma 7.3.** Let \(\nu\) be a measure on a measurable space \(S\) and \(v\) be as in Lemma 7.2. Let \(f\) be a solution of (39) on \(S\). Suppose there is \(x_0 \in (a, b)\) such that the functions \(s \to f_{[s]}(x_0)\) and \(s \to f'_{[s]}(x_0)\) are \(\nu\)-measurable. Then \(f\) is \((\nu \times \lambda)\)-measurable.

**Proof.** We shall show that \(f\) is \((\nu \times \lambda)\)-measurable on \(S \times [x_0, b)\) by proving that it is \((\nu \times \lambda)\)-measurable on \(S \times [x_0, c]\) for any \(c \in (x_0, b)\). We first assume that there is \(0 < C < \infty\) such that

\[
\int_{x_0}^c |v(s, x)| \, dx < C
\]
for \( \nu \)-a.e. \( s \). We define the functions \( f_0, f_1, \ldots \) on \( S \times (a, b) \) by the relations

\[
f_0(s, x) = f_{[s]}(x_0) + f'_{[s]}(x_0)(x - y),
\]

(41) \( f_n(s, x) = f_0(s, x) + \int_{x_0}^{x} dx' \int_{x_0}^{x'} v(s, \xi)f_{n-1}(s, \xi) d\xi, \quad n = 1, 2, \ldots
\)

By the hypothesis, the function \( f_0 \) is \((\nu \times \lambda)\)-measurable, and it follows from Lemma A.2 that \( f_n \) are \((\nu \times \lambda)\)-measurable for all \( n \). Let \( S' \subset S \) be a measurable set with a \( \nu \)-null complement in \( S \) such that \( v_{[s]} \) is locally square-integrable, (40) holds, and \( f_{[s]} \) is a solution of (39) for all \( s \in S' \). Since \( f_{[s]} \) satisfies (39), we have

\[
f(s, x) = f_0(s, x) + \int_{x_0}^{x} dx' \int_{x_0}^{x'} v(s, \xi)f(s, \xi) d\xi
\]

for any \( s \in S' \) and any \( x \in (a, b) \). Let \( x_1 \in (x_0, c] \) be such that \( x_1 - x_0 < 1/C \) and let

\[
M_n(s) = \sup_{x_0 \leq x \leq x_1} |f(s, x) - f_n(s, x)|, \quad n = 0, 1, \ldots
\]

It follows from (41) and (42) that

\[
M_n(s) \leq C(x_1 - x_0)M_{n-1}(s), \quad n = 1, 2, \ldots,
\]

for any \( s \in S' \). We hence have

\[
M_n(s) \leq [C(x_1 - x_0)]^nM_0(s), \quad s \in S'.
\]

Since \( C(x_1 - x_0) < 1 \), this means that \( f_n \) converge to \( f \) pointwise on \( S' \times [x_0, x_1] \) and, therefore, \( f \) is \((\nu \times \lambda)\)-measurable on \( S \times [x_0, x_1] \). Moreover, it follows from (41) and (42) that

\[
\sup_{x_0 \leq x \leq x_1} |f'_{[s]}(x) - (f_n)'_{[s]}(x)| \leq CM_n(s)
\]

for any \( s \in S' \). Hence, \( (f_n)'_{[s]}(x) \) converge to \( f'_{[s]}(x) \) for any \( s \in S' \) and \( x \in [x_0, x_1] \). In particular, the functions \( s \rightarrow (f_n)'_{[s]}(x_1) \) and \( s \rightarrow (f_n)'_{[s]}(x_1) \) converge \( \nu \)-a.e. to the functions \( s \rightarrow f'_{[s]}(x_1) \) and \( s \rightarrow f'_{[s]}(x_1) \) respectively. This implies that the latter two functions are \( \nu \) measurable because the functions \( s \rightarrow (f_n)'_{[s]}(x_1) \) and \( s \rightarrow (f_n)'_{[s]}(x_1) \) are \( \nu \)-measurable by the Fubini theorem. We therefore can repeat the above arguments replacing \( x_0 \) with \( x_1 \) and choosing some \( x_2 \in (x_1, c] \) such that \( C(x_2 - x_1) < 1 \). As a result, we shall prove that \( f \) is \((\nu \times \lambda)\)-measurable on \( S \times [x_0, x_1] \). Obviously, after a finite number of such steps we shall establish the \((\nu \times \lambda)\)-measurability of \( f \) on \( S \times [x_0, c] \).

In the general case (when (40) does not necessarily hold), we consider, for any \( N > 0 \), the set \( A_N \) of all \( s \in S \) such that \( \int_{x_0}^{c} |v(s, x)| dx < N \). By the Fubini theorem \( A_N \) is \( \nu \)-measurable. The above arguments show that \( f \) is \((\nu \times \lambda)\)-measurable on \( A_N \times [x_0, c] \). Since \( S \setminus \bigcup_{N=1}^{\infty} A_N \) is a \( \nu \)-null set, we conclude that \( f \) is \((\nu \times \lambda)\)-measurable on \( S \times [x_0, c] \). Hence \( f \) is \((\nu \times \lambda)\)-measurable on \( S \times [x_0, b] \).

Repeating the same proof with obvious changes, we make sure that \( f \) is \((\nu \times \lambda)\)-measurable on \( S \times (a, x_0] \) and, hence, on \( S \times (a, b) \). The lemma is proved.

**Corollary 7.4.** Let \( \nu \), \( S \), and \( v \) be as in Lemma 7.3. Then there are \((\nu \times \lambda)\)-measurable real solutions \( f_1 \) and \( f_2 \) of (39) on \( S \) such that \( (f_1)'_{[s]} \) and \( (f_2)'_{[s]} \) are linearly independent elements of \( D \) for \( \nu \)-a.e. \( s \).
Clearly, both \( G \) and \( \xi \) are \((\nu \times \lambda)\)-measurable on \( S \). Hence, the map \( H(s) = L^f_{\nu[s]} \) is \((\nu \times \lambda)\)-measurable on \( S \) such that \( H(s) = L^f_{\nu[s]} \) for \( \nu \)-a.e. \( s \).

**Proof.** Let \( S' \subset S \) be a set with a \( \nu \)-null complement in \( S \) such that \( v_{\nu[s]} \) is locally square integrable for all \( s \in S' \). Choose \( x_0 \in (a, b) \). By Theorem 2 of Chapter V, Sec. 16 in [8], there are real functions \( f_1 \) and \( f_2 \) on \( S' \times (a, b) \) such that (39) holds and the conditions

\[
(f_1)_{|s}(x_0) = (f_2)'_{|s}(x_0) = 1, \quad (f_1)'_{|s}(x_0) = (f_2)_{|s}(x_0) = 0
\]

are satisfied for all \( s \in S' \). Obviously, \((f_1)_{|s}\) and \((f_2)_{|s}\) are linearly independent, and Lemma 7.3 implies that \( f_1 \) and \( f_2 \) are \((\nu \times \lambda)\)-measurable. The corollary is proved. \( \square \)

**Lemma 7.5.** Let \( \nu \) be a measure on a measurable space \( S \) and let \( \nu \) be a \((\nu \times \lambda)\)-measurable real function on \( S \times (a, b) \) such that \( v_{\nu[s]} \) is locally square-integrable, \( lcc \) holds for \( \nu_{\nu[s]} \) at \( a \), and \( lpc \) holds for \( \nu_{\nu[s]} \) at \( b \) for \( \nu \)-a.e. \( s \). If \( f \) is a nontrivial \((\nu \times \lambda)\)-measurable solution of (39) on \( S \), then \( L^f_{\nu[s]} \) is a \( \nu \)-measurable family of self-adjoint extensions of \( L^f_{\nu[s]} \). If \( H(s) \) is a \( \nu \)-measurable family of self-adjoint extensions of \( L^f_{\nu[s]} \), then there is a nontrivial \((\nu \times \lambda)\)-measurable solution \( f \) of (39) on \( S \) such that \( H(s) = L^f_{\nu[s]} \) for \( \nu \)-a.e. \( s \).

**Proof.** By Lemma 7.2, the the operators \( L^f_{\nu[s]} \) constitute a \( \nu \)-measurable family on \( S \). This means that there is a sequence \( \zeta_1, \zeta_2, \ldots \) of \( \nu \)-measurable maps from \( S \) to \( L^2(a, b) \oplus L^2(a, b) \) such that the linear span of \( \zeta_1(s), \zeta_2(s), \ldots \) is dense in the graph \( G_{L^f_{\nu[s]}} \) for \( \nu \)-a.e. \( s \).

Let \( f \) be a real nontrivial \((\nu \times \lambda)\)-measurable solution of (39) on \( S \) and \( L_f(s) = L^f_{\nu[s]} \) for all \( s \in S \). Let \( \tau \) be a smooth function on \( (a, b) \) that is equal to unity in a neighborhood of \( a \) and vanishes in a neighborhood of \( b \). Let the functions \( g \) and \( h \) on \( S \times (a, b) \) be defined by the relations

\[
g(s, x) = \tau(x) f(s, x), \quad h(s, x) = -\tau''(x) f(s, x) - 2\tau'(x) f'_s(x).
\]

Clearly, both \( g \) and \( h \) are \((\nu \times \lambda)\)-measurable, \( g_{\nu[s]} \in D_{L^f_{\nu[s]}} \) for \( \nu \)-a.e. \( s \), and

(43) \[ l_{\nu[s]} g_{\nu[s]} = |h|_{\nu[s]} \]

for \( \nu \)-a.e. \( s \). Let the maps \( \xi \) and \( \eta \) from \( S \) to \( L^2(a, b) \) be defined by the relations

\[
\xi(s) = |g|_{\nu[s]}, \quad \eta(s) = |h|_{\nu[s]}.
\]

By Lemma 7.1 \( \xi \) and \( \eta \) are \( \nu \)-measurable. Since \( \sigma_{\xi_{\nu[s]}} = f_{\nu[s]} \) for \( \nu \)-a.e. \( s \), it follows from Lemma 7.4 that \( \xi(s) \in D_{L_f(s)} \setminus D_{L^f_{\nu[s]}} \). In view of (43), this implies that \( L_f(s) \xi(s) = \eta(s) \) for \( \nu \)-a.e. \( s \). Hence, the map \( \zeta : s \mapsto (\xi(s), \eta(s)) \) from \( S \) to \( L^2(a, b) \oplus L^2(a, b) \) is \( \nu \)-measurable and \( \zeta(s) \in G_{L_f(s)} \setminus G_{L^f_{\nu[s]}} \) for \( \nu \)-a.e. \( s \). Since \( G_{L_f(s)} \cap G_{L^f_{\nu[s]}} \) is one-dimensional for \( \nu \)-a.e. \( s \), this implies that the linear span of the sequence \( \zeta(s), \zeta_1(s), \zeta_2(s), \ldots \) is dense in \( G_{L_f(s)} \) for \( \nu \)-a.e. \( s \). This means that \( L_f(s) \) constitute a measurable family of operators on \( S \).

Conversely, let \( H(s) \) be a \( \nu \)-measurable family of self-adjoint extensions of \( L^f_{\nu[s]} \). Then both \( G_{H(s)} \) and \( G_{L_{\nu[s]}} \) form \( \nu \)-measurable families of subspaces of \( L^2(a, b) \oplus L^2(a, b) \) and, therefore, \( G_{H(s)} \cap G_{L_{\nu[s]}} \) is also a \( \nu \)-measurable family of subspaces of \( L^2(a, b) \oplus L^2(a, b) \). Since \( G_{H(s)} \cap G_{L_{\nu[s]}} \) is one-dimensional for \( \nu \)-a.e. \( s \), there is a
\[\nu\text{-measurable map } \zeta(s) = (\xi(s), \eta(s)) \text{ from } S \text{ to } G_{H(s)} \cap G_{L_{\nu}} \text{ such that } \zeta(s) \neq 0 \text{ for } \nu\text{-a.e. } s. \] We obviously have
\[\xi(s) \in D_{H(s)} \setminus D_{L_{\nu}}\]
for \(\nu\text{-a.e. } s. \) Let \(g\) be a function on \(S \times (a, b)\) such that \(g_s \in D_{L_{\nu}}\) and \([g_s] = \xi(s)\) for \(\nu\text{-a.e. } s. \) Since \(\xi\) is \(\nu\text{-measurable}, \) Lemma \(A.1\) implies that \(g\) is \((\nu \times \lambda)\text{-measurable.}\)

Let \(Q\) be the set of all \(s \in S\) such that \(g_s\) has a nonzero real part. We define the function \(\tilde{g}\) on \(S \times (a, b)\) by the relation
\[\tilde{g}(s, x) = \begin{cases} \frac{g(s, x) + g(x, s)}{2^1}, & s \in Q, \\ \frac{g(s, x) - g(x, s)}{2^1}, & s \in S \setminus Q. \end{cases}\]
Then \(\tilde{g}\) is \((\nu \times \lambda)\text{-measurable and } \tilde{g}_s\) is a real element of \(D_{L_{\nu}}\) for \(\nu\text{-a.e. } s. \) In view of \((44)\), we have \([\tilde{g}_s] \in D_{H(s)} \setminus D_{L_{\nu}}\) for \(\nu\text{-a.e. } s, \) and it follows from Lemma \(A.1\) that \(\sigma_{\tilde{g}_s}\) is nontrivial and
\[H(s) = L_{\tilde{g}_s}^{\sigma_{\tilde{g}_s}}\]
for \(\nu\text{-a.e. } s. \) By Corollary \(A.1\) there are \((\nu \times \lambda)\text{-measurable solutions } f_1\) and \(f_2\) of \((39)\) on \(S\) such that \((f_1)_s\) and \((f_2)_s\) are linearly independent elements of \(D\) for \(\nu\text{-a.e. } s. \)

Let the function \(f\) on \(S \times (a, b)\) be given by
\[f(s, x) = \tilde{g}(s, x) - \frac{1}{W(s)} \left[ f_1(s, x) \int_a^x \varphi(s, x') f_2(s, x') \, dx' - f_2(s, x) \int_a^x \varphi(s, x') f_1(s, x') \, dx' \right],\]
where \(W(s)\) denotes the Wronskian of \((f_1)_s\) and \((f_2)_s\) and the \((\nu \times \lambda)\text{-measurable function } \varphi\) on \(S \times (a, b)\) is defined by the relation
\[\varphi(s, x) = -\tilde{g}'_s + u(s, x) \tilde{g}(s, x).\]
Since the function \(s \to W(s)\) on \(S\) is \(\nu\text{-measurable}, \) Lemma \(A.2\) implies that \(f\) is \((\nu \times \lambda)\text{-measurable.}\) As \(f_s = \sigma_{\tilde{g}_s}\) for \(\nu\text{-a.e. } s, \) it follows from \((45)\) that \(H(s) = L_{f_s}^{\sigma_{\tilde{g}_s}}\) for \(\nu\text{-a.e. } s. \) The Lemma is proved.

Let \(\nu\) be a measure on a measurable space \(S\) and \(\nu\) be as in Lemma \(A.3\) Let \(f_1\) and \(f_2\) be real \((\nu \times \lambda)\text{-measurable solutions of } (39)\) on \(S\) such that \((f_1)_s\) and \((f_2)_s\) are linearly independent for \(\nu\text{-a.e. } s\) (such solutions always exist by Corollary \(A.4\).)

For any \(\nu\text{-measurable map } \theta\) from \(S\) to \([0, \pi), \) we define the real \((\nu \times \lambda)\text{-measurable solution } \hat{\theta}\) of \((39)\) on \(S\) by the relation
\[\hat{\theta}(s, x) = f_1(s, x) \cos \theta(s) + f_2(s, x) \sin \theta(s).\]
Let \(f\) be a real \((\nu \times \lambda)\text{-measurable solution of } (39)\) on \(S. \) Then there are \(\nu\text{-a.e. }\) defined real functions \(C_1\) and \(C_2\) on \(S\) such that the equality
\[f(s, x) = C_1(s) f_1(s, x) + C_2(s) f_2(s, x)\]
holds for \(\nu\text{-a.e. } s \in S\) and all \(x \in (a, b). \) For \(\nu\text{-a.e. } s, \) we have
\[C_1(s) = \frac{W(f_1, f_2)}{W((f_1)_s, (f_2)_s)}, \quad C_2(s) = \frac{W(f_1, f_2)}{W((f_2)_s, (f_1)_s)},\]
and, therefore, both \(C_1\) and \(C_2\) are \(\nu\text{-measurable on } S. \) Let \(U \subset \mathbb{R}^2\) be the set of all points of the form \((r \cos \varphi, r \sin \varphi)\) with \(r \geq 0\) and \(\varphi \in [0, \pi)\) and let \(\Sigma\) denote
the intersection of $U$ with the unit circle (in other words, $\Sigma$ is the set of all points of the form $(\cos \varphi, \sin \varphi)$ with $\varphi \in [0, \pi)$). We define the $\nu$-measurable functions $\tilde{C}_1$ and $\tilde{C}_2$ on $S$ by the equalities

$$\tilde{C}_1(s) = C_1(s)/C(s), \quad \tilde{C}_2(s) = C_2(s)/C(s),$$

where the $\nu$-measurable function $C$ is given by

$$C(s) = \begin{cases} \sqrt{C_1(s)^2 + C_2(s)^2}, & s \in U, \\ -\sqrt{C_1(s)^2 + C_2(s)^2}, & s \in S \setminus U. \end{cases}$$

We then have $(\tilde{C}_1(s), \tilde{C}_2(s)) \in \Sigma$ for $\nu$-a.e. $s$. Let $\chi$ be the map $\varphi \to (\cos \varphi, \sin \varphi)$ from $[0, \pi)$ to $\Sigma$. Clearly, $\chi$ is a bijection and both $\chi$ and $\chi^{-1}$ are continuous. We now define the $\nu$-measurable function $\theta$ from $S$ to $[0, \pi)$ by setting

$$\theta(s) = \chi^{-1}(\tilde{C}_1(s), \tilde{C}_2(s)).$$

We then have

$$f(s, x) = C(s)\hat{\theta}(s, x)$$

for $\nu$-a.e. $s \in S$ and all $x \in (a, b)$. Relation (47) determines $C$ and $\theta$ uniquely up to $\nu$-equivalence. Indeed, suppose there are functions $\tilde{C}$ and $\tilde{\theta}$ such that (47) holds with $C$ and $\theta$ replaced with $\tilde{C}$ and $\tilde{\theta}$ respectively. Then we have

$$C(s)\tilde{\theta}(s, x) = \tilde{C}(s)\tilde{\theta}(s, x)$$

for $\nu$-a.e. $s \in S$ and all $x \in (a, b)$. Since $(f_1)_s$ and $(f_2)_s$ are linearly independent for $\nu$-a.e. $s$, it follows that

$$C(s)\cos \theta(s) = \tilde{C}(s)\cos \tilde{\theta}(s), \quad C(s)\sin \theta(s) = \tilde{C}(s)\sin \tilde{\theta}(s)$$

for $\nu$-a.e. $s$. Because both $(\cos \theta(s), \sin \theta(s))$ and $(\cos \tilde{\theta}(s), \sin \tilde{\theta}(s))$ belong to $\Sigma$ for $\nu$-a.e. $s$, this implies that $C(s) = \tilde{C}(s)$ and $\theta(s) = \tilde{\theta}(s)$ for $\nu$-a.e. $s$.

By Lemma 7.5, $L_{v[s]}^\theta$ constitute a $\nu$-measurable family of self-adjoint extensions of $L_{v[s]}$ for any $\nu$-measurable map $\theta$ from $S$ to $[0, \pi)$. Conversely, let $H(s)$ be a $\nu$-measurable family of self-adjoint extensions of $L_{v[s]}$. Then it follows from Lemma 7.5 and the above arguments concerning representation (47) that there is a unique (up to $\nu$-equivalence) $\nu$-measurable map $\theta$ from $S$ to $[0, \pi)$ such that $H(s) = L_{v[s]}^\theta$ for $\nu$-a.e. $s$.

We now consider a more general case, when a $(\nu \times \lambda)$-measurable function $v$ on $S \times (a, b)$ is such that both lpc and lcc may hold for $l_{v[s]}$ at $a$. Let $A_{lc}$ denote the set of all $s \in S$ such that $v[s]$ is locally square-integrable on $(a, b)$ and lcc holds for $l_{v[s]}$ at $a$. Let $f_1$ and $f_2$ be as in Corollary 7.4. For any $c \in (a, b)$, the set of all $s$ such that

$$\int_a^c (f_1(s, x)^2 + f_2(s, x)^2) \, dx < \infty$$

differs from $A_{lc}$ by at most a $\nu$-null set. It follows from the Fubini theorem that $A_{lc}$ is $\nu$-measurable.

Let $f_1$ and $f_2$ be $(\nu \times \lambda)$-measurable solutions of (39) on $A_{lc}$ such that $(f_1)_s$ and $(f_2)_s$ are linearly independent for $\nu$-a.e. $s \in A_{lc}$. Given a $\nu$-measurable map
θ from \( A_{lc} \) to \([0, \pi)\), we define the family \( L_\theta(s) \) of self-adjoint extensions of \( L_{v|s} \) on \( S \) by the relation
\[
L_\theta(s) = \begin{cases} 
L_{\hat{v}|s}, & s \in A_{lc}, \\
L_{v|s}, & s \in S \setminus A_{lc},
\end{cases}
\]
where the solution \( \hat{\theta} \) of (39) on \( A_{lc} \) is given by formula (40) for \( s \in A_{lc} \). The family \( L_\theta \) is \( \nu \)-measurable on \( A_{lc} \) and \( S \setminus A_{lc} \) by Lemmas 7.5 and 7.2 respectively. Suppose now that \( H(s) \) is a \( \nu \)-measurable family of self-adjoint extensions of \( L_{v|s} \). Replacing \( S \) with \( A_{lc} \) in the above consideration, we conclude that there is a \( \nu \)-measurable map \( \theta \) from \( A_{lc} \) to \([0, \pi)\) such that \( H(s) = L_{\hat{\theta}|s} \) for \( \nu \)-a.e. \( s \in A_{lc} \). Since \( L_{v|s} \) is essentially self-adjoint for \( \nu \)-a.e. \( s \in S \setminus A_{lc} \), it follows that \( H(s) = L_{\hat{\theta}|s} \) for \( \nu \)-a.e. \( s \in S \setminus A_{lc} \) and, therefore, \( H(s) = L_\theta(s) \) for \( \nu \)-a.e. \( s \in S \). We thus have proved the next theorem.

**Theorem 7.6.** Let \( \nu \) be a measure on a measurable space \( S \) and \( v \) be a \((\nu \times \lambda)\)-measurable real function on \( S \times (a, b) \) such that \( v_{[s]} \) is locally square-integrable and lpc holds for \( l_{v|s} \) at \( b \) for \( \nu \)-a.e. \( s \). Let \( A_{lc} \) be the set of all \( s \in S \) such that \( v_{[s]} \) is locally square-integrable on \((a, b)\) and lcc holds for \( l_{v|s} \) at \( a \). Let \( f_1 \) and \( f_2 \) be \((\nu \times \lambda)\)-measurable solutions of (39) on \( A_{lc} \) such that \((f_1|s)\) and \((f_2|s)\) are linearly independent for \( \nu \)-a.e. \( s \in A_{lc} \). If \( \theta \) is a \( \nu \)-measurable map from \( A_{lc} \) to \([0, \pi)\), then the family \( L_\theta(s) \) of self-adjoint extensions of \( L_{v|s} \) defined by (42), where \( \theta \) is given by (42), is \( \nu \)-measurable. Conversely, if \( H(s) \) is a \( \nu \)-measurable family of self-adjoint extensions of \( L_{v|s} \), then there is a unique up to \( \nu \)-equivalence \( \nu \)-measurable map \( \theta \) from \( A_{lc} \) to \([0, \pi)\) such that \( H(s) = L_\theta(s) \) for \( \nu \)-a.e. \( s \).

8. **Self-adjoint extensions of the three-dimensional Aharonov–Bohm Hamiltonian**

The Hamiltonian for an electron moving in the magnetic field of an infinitely thin solenoid is formally given by the differential expression
\[
\frac{\hbar^2}{2m_e} \left( i\nabla + \frac{e}{\hbar c}A \right)^2,
\]
where \( e \) and \( m_e \) are the electron charge and mass respectively, \( c \) is the velocity of light, and the vector potential \( A = (A^1, A^2, A^3) \) has the form
\[
A^1(x, y, z) = -\frac{\Phi_y}{2\pi(x^2 + y^2)}, \quad A^2(x, y, z) = \frac{\Phi_x}{2\pi(x^2 + y^2)}, \quad A^3(x, y, z) = 0.
\]
Here, \( \Phi \) is the flux of the magnetic field through the solenoid. The vector potential \( A \) is smooth outside the \( z \)-axis \( Z = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} \). Hence, (49) naturally determines an operator \( H \) on the space \( C_0^\infty(\mathbb{R}^3 \setminus Z) \) of smooth functions on \( \mathbb{R}^3 \) with compact support contained in \( \mathbb{R}^3 \setminus Z \),
\[
H\Psi = \left( i\nabla + \frac{e}{\hbar c}A \right)^2\Psi = \left( -\Delta + \frac{2i\phi}{x^2 + y^2}(y\partial_x - x\partial_y) + \frac{\phi^2}{x^2 + y^2} \right)\Psi, \quad \Psi \in C_0^\infty(\mathbb{R}^3 \setminus Z),
\]
where
\[
\phi = -\frac{e\Phi}{2\pi\hbar c}.
\]
(to simplify notation, we have dropped the factor $\hbar^2/2m_e$ in (49)). Lifting $\hat{H}$ to $\Lambda$-equivalence classes, where $\Lambda$ is the Lebesgue measure on $\mathbb{R}^3$, yields a symmetric operator $H$ in $L^2(\mathbb{R}^3)$:

$$D_H = \{ |\Psi|_\Lambda : \Psi \in C_0^\infty(\mathbb{R}^3 \setminus \mathbb{Z}) \},$$

$$H|\Psi|_\Lambda = [\hat{H}\Psi|_\Lambda].$$

Let $G$ be the Abelian group of linear operators in $\mathbb{R}^3$ generated by translations along the $z$-axis and rotations around the $z$-axis. Given $G \in G$, we denote by $U_G$ the unitary operator in $L^2(\mathbb{R}^3)$ taking $|\Psi|_\Lambda$ to $|\Psi \circ G^{-1}|_\Lambda$ for any square-integrable function $\Psi$ on $\mathbb{R}^3$. It is straightforward to check that $H$ commutes with $U_G$ for any $G \in G$. We shall see that $H$ is not essentially self-adjoint. Hence, there are different quantum dynamics that can be associated with differential expression (49) via constructing different self-adjoint extensions of $H$. In this section, we shall describe all self-adjoint extensions of $H$ commuting with $U_G$ for any $G \in G$.

We begin by constructing an exact diagonalization for the operators $sU_G$. Let $\mu$ be the counting measure on $\mathbb{Z}$, which assigns to each set of integers the number of points in the set. We define the measure $\nu$ on $S = \mathbb{Z} \times \mathbb{R}$ by setting $\nu = \mu \times \lambda$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. For any $\nu$-integrable $f$, we have

$$\int_S f(m, p) d\nu(m, p) = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(m, p) dp.$$ 

For $\Psi \in C_0^\infty(\mathbb{R}^3)$, let the function $\hat{\Psi}$ on $S \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, be defined by

$$\hat{\Psi}(s, r) = \sqrt{r} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi \Psi(\rho \cos \varphi, \rho \sin \varphi, z) e^{imz+i\rho \varphi}, \quad s = (m, p) \in S.$$ 

Let $\mathfrak{h} = L^2(\mathbb{R}_+)$. For $\Psi \in C_0^\infty(\mathbb{R}^3)$, we define the map $\hat{\Psi}$ from $S$ to $\mathfrak{h}$ by setting

$$\hat{\Psi}(s) = [\hat{\Psi}_s]_{\lambda_+},$$

where $\lambda_+$ is the restriction to $\mathbb{R}_+$ of the Lebesgue measure $\lambda$ on $\mathbb{R}$ and $\hat{\Psi}_s \in C_0^\infty(\mathbb{R}_+)$ denotes, as in Sec. 7, the partial function on $\mathbb{R}_+$ determined by $\Psi(s, r)$ for fixed $s$,

$$\hat{\Psi}_s(r) = \hat{\Psi}(s, r), \quad r \in \mathbb{R}_+.$$ 

The next lemma follows easily from the Fubini theorem and the unitarity of the Fourier transformation and the Fourier series expansion.

**Lemma 8.1.** There is a unique unitary operator $V : L^2(\mathbb{R}^3) \to L^2(S, \mathfrak{h}, \nu)$ such that

$$V|\Psi|_\Lambda = [\hat{\Psi}]_{\nu}, \quad \Psi \in C_0^\infty(\mathbb{R}^3).$$

Given $\alpha, \beta \in \mathbb{R}$, let the function $g_{\alpha, \beta}$ on $S$ be given by

$$g_{\alpha, \beta}(m, p) = e^{im\alpha+i\beta p}.$$ 

If $G \in G$ is the composition of the rotation by the angle $\alpha$ around $z$-axis and the translation by $\beta$ along $z$-axis, then it is easy to see that

$$V U_G V^{-1} = T_{g_{\alpha, \beta}}.$$
where $T_{g_{\alpha,\beta}}$ is the operator of multiplication by $g_{\alpha,\beta}$ in $L^2(S, h, \nu)$. We now show that $\{g_{\alpha,\beta}\}_{(\alpha,\beta)\in \mathbb{Q}^2}$, where $\mathbb{Q}$ is the set of rational numbers, is a $\nu$-separating family of functions on $S$. Suppose $(m, p)$ and $(m', p')$ are such that $g_{\alpha,\beta}(m, p) = g_{\alpha,\beta}(m', p')$ for all $(\alpha, \beta) \in \mathbb{Q}^2$. Then we have
\[
e^{i\alpha(m-m')+i\beta(p-p')} = 1
\]
for all $(\alpha, \beta) \in \mathbb{Q}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, it follows that $m = m'$ and $p = p'$, i.e., the family $\{g_{\alpha,\beta}\}_{(\alpha,\beta)\in \mathbb{Q}^2}$ is $\nu$-separating. Theorem $5.3$ now implies that $(S, h, \nu, V)$ is an exact diagonalization for $U_G$.

It easily follows from (50) that
\[(53) \quad (\hat{H}\Psi)(r \cos \varphi, r \sin \varphi, z) = \left(-\partial_r^2 - \partial^2 \varphi - \frac{1}{r} \partial_r - \frac{1}{r^2}(\partial^2 \varphi + 2i\phi \partial_r - \phi^2)\right) F_{\Psi}(\varphi, z, r)
\]
for any $\Psi \in C^\infty_0(\mathbb{R}^3 \setminus Z)$, where $F_{\Psi}$ is the smooth function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ which represents $\Psi$ in the cylindrical coordinates,
\[F_{\Psi}(\varphi, z, r) = \Psi(r \cos \varphi, r \sin \varphi, z).
\]
Substituting (53) in (51) and integrating by parts yields
\[(54) \quad \widetilde{H}\Psi(s, r) = (\hat{h}_{m-\phi}\Psi)(s, r) + p^2\Psi(s, r), \quad s = (m, p) \in S,
\]
where the operator $\hat{h}_\kappa$ from $C^\infty_0(\mathbb{R}_+)$ to itself is given by
\[\hat{h}_\kappa\psi)(r) = -\psi''(r) + \frac{\kappa^2 - 1/4}{r^2}\psi(r), \quad \psi \in C^\infty_0(\mathbb{R}_+),
\]
for any $\kappa \in \mathbb{R}$. Let $h_\kappa$ denote the operator in $h$ obtained by lifting $\hat{h}_\kappa$ to $\lambda$-equivalence classes,
\[D_{h_\kappa} = D_0 = \left\{[\psi]_{\lambda_+} : \psi \in C^\infty_0(\mathbb{R}_+)\right\},
\]
\[h_\kappa[\psi]_{\lambda_+} = [\hat{h}_\kappa\psi]_{\lambda_+}.
\]
It follows from (52) and (53) that
\[(55) \quad \widetilde{H}\Psi(s) = a(s)\Psi(s), \quad s \in S,
\]
where
\[(56) \quad a(m, p) = h_{m-\phi} + p^21_h
\]
for any $(m, p) \in S$ and $1_h$ is the identity operator in $h$. Let the function $v$ on $S \times \mathbb{R}_+$ be defined by the relation
\[(57) \quad v(s, r) = \frac{(m - \phi)^2 - 1/4}{r^2}, \quad s = (m, p) \in S.
\]
Clearly, $v$ is $(\nu \times \lambda)$-measurable on $S \times \mathbb{R}_+$ and, in the notation of Sec. $7$, we have
\[(58) \quad L_{v,s} = h_{m-\phi}
\]
for any $s = (m, p) \in S$. Hence, Lemma $7.2$ implies that the family $(m, p) \rightarrow h_{m-\phi}$ of operators on $S$ is $\nu$-measurable and $\nu$-regular with respect to $D_0$. Since $(m, p) \rightarrow p^21_h$ is a $\nu$-measurable family of bounded operators on $S$, it follows from (50) that the family $a(m, p)$ is also $\nu$-measurable and $\nu$-regular with respect to $D_0$. Fix a
nonzero function \( \chi \in C^\infty_0(\mathbb{R}) \). For \( \psi \in C^\infty_0(\mathbb{R}_+) \) and \( n \in \mathbb{Z} \), let \( \Psi_{n,\psi} \in C^\infty_0(\mathbb{R}^3 \setminus Z) \) be the function whose cylindrical coordinate representation has the form

\[
F_{\Psi_{n,\psi}}(\varphi, z, r) = \frac{1}{\sqrt{r}} e^{-in\varphi} \chi(z) \psi(r).
\]

Then we have

\[
\hat{\Psi}_{n,\psi}(m, p) = \delta_{m,n} F\chi(p) [\psi]_{\lambda+}, \quad (m, p) \in S,
\]

where \( F\chi \) is the Fourier transform of \( \chi \),

\[
F\chi(p) = \int \chi(z) e^{izp} \, dz,
\]

and \( \delta_{m,n} \) is, as usual, equal to 1 for \( m = n \) and equal to 0 for \( m \neq n \). Since \( F\chi \) admits the analytic continuation to \( C \), the set of its zeros has Lebesgue measure zero. It follows that the functions \( g_n(m, p) = \delta_{m,n} F\chi(p) \) constitute a \( \nu \)-nonvanishing sequence of square-integrable functions on \( S \) (see the paragraph before Lemma 6.8).

By (59), the \( \nu \)-equivalence class of the map \( s \rightarrow g_n(s)[\psi]_{\lambda,\nu} \) is equal to \( V[\Psi_{n,\psi}]\lambda \) and, therefore, belongs to \( V(D_H) \) for any \( \psi \in C^\infty_0(\mathbb{R}_+) \). Hence, statement 2 of Lemma 6.8 implies that the family \( a(s) \) is compatible with \( V(D_H) \).

By (59), we have

\[
a(s)\xi(s) = \eta(s)
\]

for \( \nu \)-a.e. \( s \in S \), whenever \( \xi \) and \( \eta \) are \( \nu \)-measurable maps from \( S \) to \( \mathfrak{h} \) such that

\[
[\xi]_\nu = V[\Psi]_\lambda, \quad [\eta]_\nu = VH[\Psi]_\lambda
\]

for some \( \Psi \in C^\infty_0(\mathbb{R}^3 \setminus Z) \). Taking (60), (61), the compatibility of \( a(s) \) with \( V(D_H) \), and the exactness of the diagonalization \( (S, \mathfrak{h}, \nu, V) \) for \( U_G \) into account and applying Theorem 6.5, we arrive at the next result.

**Lemma 8.2.** Let \( \tilde{a}(s) \) be a \( \nu \)-measurable family of self-adjoint extensions of \( a(s) \) on \( S \). Then the operator

\[
\tilde{H} = V^{-1} \int G \tilde{a}(s) \, d\nu(s) \, V
\]

is a self-adjoint extension of \( H \) commuting with \( U_G \) for all \( G \in \mathcal{G} \). Conversely, if \( \tilde{H} \) is a self-adjoint extension of \( H \) commuting with \( U_G \) for all \( G \in \mathcal{G} \), then there is a unique (up to \( \nu \)-equivalence) \( \nu \)-measurable family \( \tilde{a}(s) \) of self-adjoint extensions of \( a(s) \) on \( S \) such that (62) holds.

It follows from (59) that \( \tilde{a}(s) \) is a \( \nu \)-measurable family of self-adjoint extensions of \( a(s) \) if and only if

\[
\tilde{a}(m, p) = \tilde{h}(m, p) + p^2 1_{\mathfrak{h}}
\]

for \( \nu \)-a.e. \( (m, p) \in S \), where \( \tilde{h}(s) \) is a \( \nu \)-measurable family of operators on \( S \) such that \( \tilde{h}(m, p) \) is a self-adjoint extension of \( h_{m-\phi} \) for \( \nu \)-a.e. \( (m, p) \in S \). In view of Lemma 8.2, this implies that the problem of describing self-adjoint extensions of \( H \) reduces to describing all such families \( \tilde{h}(s) \).

As in Sec. 7, let \( D \) denote the space of all absolutely continuous complex functions on \( \mathbb{R}_+ \) whose first derivative is also absolutely continuous. For \( \kappa \in \mathbb{R} \), let \( l_\kappa \) be the
linear map from $\mathcal{D}$ to the space of complex-valued $\lambda_+$-equivalence classes taking $\psi \in \mathcal{D}$ to the $\lambda_+$-equivalence class of the map
\[ r \to -\psi''(r) + \frac{\kappa^2 - 1/4}{r^2} \psi(r). \]

Let the subspace $\mathcal{D}_\kappa$ of $\mathcal{D}$ and the subspace $\mathcal{D}_\kappa$ of $\mathfrak{h}$ be defined by the relations
\[ \mathcal{D}_\kappa = \{ \psi \in \mathcal{D} : [\psi]_{\lambda_+} \text{ and } l_\kappa \psi \text{ are both in } \mathfrak{h} \}, \]
\[ \mathcal{D}_\kappa = \{ [\psi]_{\lambda_+} : \psi \in \mathcal{D}_\kappa \}. \]

In the notation of Sec. 7, we have
\[ (63) \]
\[ h_\kappa = L_{q_\kappa}, \quad l_\kappa = l_{q_\kappa}, \quad \mathcal{D}_\kappa = D_{\mathcal{D}_\kappa}, \quad \mathcal{D}_\kappa = D_{q_\kappa}, \]
where the function $q_\kappa$ on $\mathbb{R}_+$ is given by
\[ q_\kappa(r) = \frac{\kappa^2 - 1/4}{r^2}. \]

Hence, the adjoint $h_\kappa^*$ of $h_\kappa$ has the form
\[ D_{h_\kappa^*} = D_{\mathcal{D}_\kappa}, \]
\[ h_\kappa^*[\psi]_{\lambda_+} = l_\kappa \psi, \quad \psi \in \mathcal{D}_\kappa. \]

The equation
\[ (64) \]
\[ l_\kappa \psi = 0 \]
has two linearly independent solutions
\[ \psi_\kappa^{(1)}(r) = r^{1/2 + \kappa}, \quad \psi_\kappa^{(2)}(r) = r^{1/2 - \kappa}, \quad \kappa \neq 0, \]
\[ \psi_\kappa^{(1)}(r) = r^{1/2}, \quad \psi_\kappa^{(2)}(r) = r^{1/2} \ln r, \quad \kappa = 0. \]

Hence, lpc holds at $r = \infty$ for all $\kappa$, while lpc holds at $r = 0$ for $|\kappa| > 1$ and lcc holds at $r = 0$ for $|\kappa| < 1$. This implies that $h_\kappa$ is essentially self-adjoint for $|\kappa| \geq 1$ and its unique self-adjoint extension is $h_\kappa^*$. For $\vartheta \in \mathbb{R}$, let the solution $\psi_{\kappa, \vartheta}$ of (64) be given by
\[ (65) \]
\[ \psi_{\kappa, \vartheta}(r) = \psi_\kappa^{(1)}(r) \cos \vartheta + \psi_\kappa^{(2)}(r) \sin \vartheta. \]

For $|\kappa| < 1$, we define the self-adjoint extension $h_{\kappa, \vartheta}$ of $h_\kappa$ by setting
\[ (66) \]
\[ h_{\kappa, \vartheta} = L_{q_\kappa}^{\psi_{\kappa, \vartheta}}, \]
i.e., $h_{\kappa, \vartheta}$ is the restriction of $h_\kappa^*$ to
\[ D_{h_{\kappa, \vartheta}} = \{ [\psi]_{\lambda_+} : \psi \in \mathcal{D}_\kappa \text{ and } \lim_{r \downarrow 0} W(\psi, \psi_{\kappa, \vartheta})(r) = 0 \}, \]
where the Wronskian $W$ is given by (31).

For each $\phi \in \mathbb{R}$, there is a unique $m(\phi) \in \mathbb{Z}$ such that $m(\phi) - \phi \in (-1, 0]$ (note that $m(\phi) = \phi$ for $\phi \in \mathbb{Z}$). The operator $h_{m_\phi}$ is not essentially self-adjoint for $m = m(\phi)$ if $\phi \in \mathbb{Z}$ and for $m = m(\phi), m(\phi) + 1$ if $\phi \notin \mathbb{Z}$. Hence, families of self-adjoint extensions of $h_{m_\phi}$ are defined differently for $\phi \in \mathbb{Z}$ and $\phi \notin \mathbb{Z}.

1. Let $\phi \notin \mathbb{Z}$ and let $\tau_1$ and $\tau_2$ be $\lambda$-measurable maps from $\mathbb{R}$ to $[0, \pi)$. We define the family $\hat{h}_{\tau_1, \tau_2}(s)$ of self-adjoint operators on $S$ by setting
\[ (67) \]
\[ \hat{h}_{\tau_1, \tau_2}(m, p) = \begin{cases} h_{m-\phi}^*, & m < m(\phi) \text{ or } m > m(\phi) + 1, \\ h_{m-\phi, \tau_1(p)}, & m = m(\phi), \\ h_{m-\phi, \tau_2(p)}, & m = m(\phi) + 1. \end{cases} \]
2. Let \( \phi \in \mathbb{Z} \) and let \( \tau \) be a \( \lambda \)-measurable map from \( \mathbb{R} \) to \([0, \pi)\). We define the family \( \tilde{h}_\tau(s) \) of self-adjoint operators on \( S \) by setting

\[
\tilde{h}_\tau(m, p) = \begin{cases} 
  h_{m-\phi}^*, & m \neq \phi, \\
  h_{0,\tau(p)}, & m = \phi.
\end{cases}
\]

**Theorem 8.3.**

1. Let \( \phi \notin \mathbb{Z} \). Suppose \( \tau_1 \) and \( \tau_2 \) are \( \lambda \)-measurable maps from \( \mathbb{R} \) to \([0, \pi)\). Then the family \( \tilde{h}_{\tau_1, \tau_2}(s) \) on \( S \) of self-adjoint operators in \( \mathfrak{h} \) defined by (67) is \( \nu \)-measurable and

\[
\tilde{H} = V^{-1} \int_{S} (\tilde{h}_{\tau_1, \tau_2}(m, p) + p^2 \mathbb{1}_h) \, d\nu(m, p) \, V
\]

is a self-adjoint extension of \( H \). Conversely, for any self-adjoint extension \( \tilde{H} \) of \( H \), there are unique (up to \( \lambda \)-equivalence) \( \lambda \)-measurable maps \( \tau_1 \) and \( \tau_2 \) from \( \mathbb{R} \) to \([0, \pi)\) such that (69) holds.

2. Let \( \phi \in \mathbb{Z} \). Suppose \( \tau \) is a \( \lambda \)-measurable map from \( \mathbb{R} \) to \([0, \pi)\). Then the family \( \tilde{h}_\tau(s) \) on \( S \) of self-adjoint operators in \( \mathfrak{h} \) defined by (68) is \( \nu \)-measurable and

\[
\tilde{H} = V^{-1} \int_{S} (\tilde{h}_\tau(m, p) + p^2 \mathbb{1}_h) \, d\nu(m, p) \, V
\]

is a self-adjoint extension of \( H \). Conversely, for any self-adjoint extension \( \tilde{H} \) of \( H \), there is a unique (up to \( \lambda \)-equivalence) \( \lambda \)-measurable map \( \tau \) from \( \mathbb{R} \) to \([0, \pi)\) such that (70) holds.

**Proof.**

1. Let the functions \( f_1 \) and \( f_2 \) on \( S \times \mathbb{R}_+ \) be defined by the formulas

\[
f_1(s, r) = \psi_m^{(1)}(r), \quad f_2(s, r) = \psi_m^{(2)}(r), \quad s = (m, p) \in S.
\]

In view of (68), the functions \( f_1 \) and \( f_2 \) are real linearly independent solutions of (39) for \( \nu \) given by (67) because \( v_m^{[s]} = q_m-\phi \) for all \( (m, p) \in S \). The set \( A_{lc} \) of all \( s \in S \) such that \( lc \) holds at \( r = 0 \) for \( l_{v_m^{[s]}} \) has the form

\[
A_{lc} = \{ m(\phi), m(\phi) + 1 \} \times \mathbb{R}.
\]

If \( \tau_1 \) and \( \tau_2 \) are \( \lambda \)-measurable maps from \( \mathbb{R} \) to \([0, \pi)\), then (65), (66), and (67) imply that

\[
\tilde{h}_{\tau_1, \tau_2}(s) = \mathcal{L}_\phi(s)
\]

for \( \nu \)-a.e. \( s \in S \), where \( \mathcal{L}_\phi(s) \) is given by (48) and the \( (\nu \times \lambda) \)-measurable map \( \theta \) from \( A_{lc} \) to \([0, \pi)\) is defined by the relations

\[
\theta(m(\phi), p) = \tau_1(p), \quad \theta(m(\phi) + 1, p) = \tau_2(p).
\]

Theorem 7.6 now implies that \( \tilde{h}_{\tau_1, \tau_2}(s) \) is a \( \nu \)-measurable family of operators on \( S \). Since \( \tilde{h}_{\tau_1, \tau_2}(m, p) \) is a self-adjoint extension of \( h_{m-\phi} \) for all \( (m, p) \in S \), the operators

\[
\tilde{a}(m, p) = \tilde{h}_{\tau_1, \tau_2}(m, p) + p^2 \mathbb{1}_h
\]

constitute a \( \nu \)-measurable family of self-adjoint extensions of \( a(m, p) \). Lemma 8.2 hence implies that the operator \( \tilde{H} \) defined by (69) is a self-adjoint extension of \( H \).

Conversely, let \( \tilde{H} \) be a self-adjoint extension of \( H \). By Lemma 8.2, there is a unique (up to \( \nu \)-equivalence) \( \nu \)-measurable family \( \tilde{a}(s) \) of self-adjoint extensions of
a(s) on S such that (62) holds. Hence, the operators $\hat{a}(m,p) - p^21_b$ constitute a $\nu$-measurable family of self-adjoint extensions of $h_{m-\phi}$. In view of (58), Theorem 7.3 implies that there is a unique (up to $\nu$-equivalence) $\nu$-measurable map $\theta$ from $A_{ic}$ to $[0, \pi)$ such that

$$\hat{a}(m,p) - p^21_b = L_{\theta}(m,p).$$

(73)

Let $\tau_1$ and $\tau_2$ denote the maps $p \rightarrow \theta(m(\phi),p)$ and $p \rightarrow \theta(m(\phi) + 1, p)$ respectively. Then both $\tau_1$ and $\tau_2$ are $\nu$-measurable maps from $\mathbb{R}$ to $[0, \pi)$ and (71) holds for $\nu$-a.e. $s \in S$. Now substituting (71) in (73) yields (72), and substituting (72) in (62) yields (63). It remains to note that (63) determines $\tau_1$ and $\tau_2$ uniquely up to $\lambda$-equivalence.

2. The proof of statement 2 is the same as the proof of statement 1 with the only difference that we have

$$A_{ic} = \{\phi\} \times \mathbb{R}$$

in this case. The theorem is proved. \hfill \Box

APPENDIX A. SOME MEASURABILITY QUESTIONS

Lemma A.1. Let $\nu$ be a measure on a measurable space $S$ and $\lambda$ be a standard measure on a measurable space $T$. Let $g$ be a $(\nu \times \lambda)$-a.e. defined complex-valued function on $S \times T$. Let $h$ be a $\nu$-a.e. defined map from $S$ to $L^2(T, \lambda)$ such that $h(s) = [g(s)]_\lambda$ for $\nu$-a.e. $s$. Then $g$ is $(\nu \times \lambda)$-measurable if and only if $h$ is $\nu$-measurable.

Proof. Let $g$ be $(\nu \times \lambda)$-measurable. We have to show that the function $s \rightarrow ([f]_\lambda, h(s))$ is $\nu$-measurable for any square-integrable function $f$ on $T$. Then $|g|^2$ is a $(\nu \times \lambda)$-measurable map from $S \times T$ to the extended positive semi-axis, and the Fubini theorem implies that $s \rightarrow \int_T |g(s,t)|^2 d\lambda(t)$ is a $\nu$-measurable function on $S$. For $N = 1, 2, \ldots$, let $A_N$ be the set of all $s \in S$ such that $\int_T |g(s,t)|^2 d\lambda(t) \leq N$. Then the function $(s, t) \rightarrow g(s,t)\chi_{A_N}(s)f(t)$, where $\chi_{A_N}$ is the characteristic function of $A_N$, is $(\nu \times \lambda)$-integrable, and it follows from the Fubini theorem that the function $s \rightarrow \chi_{A_N}(s)f(t)g(s,t)\lambda(t)$ is $\nu$-measurable. This means that the function $s \rightarrow \chi_{A_N}(s)([f]_\lambda, h(s))$ is $\nu$-measurable. Since $S \setminus \bigcup_N A_N$ is a $\nu$-null set, it follows that the function $s \rightarrow ([f]_\lambda, h(s))$ is $\nu$-measurable.

Conversely, let $h$ be $\nu$-measurable. We first suppose that $h$ is square-integrable. Since $\lambda$ is standard, $L^2(T, \lambda)$ is separable. Let $e_1, e_2, \ldots$ be a sequence of $\lambda$-measurable functions on $T$ such that $[e_1]_\lambda, [e_2]_\lambda, \ldots$ is a basis in $L^2(T, \lambda)$. For each $n = 1, 2, \ldots$, we set

$$a_n(s) = \int_T \overline{e_n(t)}g(s,t)\lambda(t), \quad b_n(s, t) = \sum_{j=1}^n a_j(s)e_j(t).$$

Since $h$ is $\nu$-measurable, all $a_n$ are $\nu$-measurable and, therefore, all $b_n$ are $(\nu \times \lambda)$-measurable. For $\nu$-a.e. $s$, we have $||h(s)||^2 = \sum_{n=1}^\infty |a_n(s)|^2$, and it follows from the monotone convergence theorem that

$$\int_S||h(s)||^2 d\nu(s) = \sum_{n=1}^\infty \int_S |a_n(s)|^2 d\nu(s).$$

(75)
This implies that all $b_n$ are square-integrable and

$$
\int_{S \times T} |b_n(s, t) - b_m(s, t)|^2 d(\nu \times \lambda)(s, t) = \sum_{j=m+1}^{n} \int_{S} |a_j(s)|^2 d\nu(s), \quad n \geq m.
$$

In view of (76), it follows that $[b_n]_{\nu \times \lambda}$ is a Cauchy sequence in $L^2(S \times T, \nu \times \lambda)$ and, therefore, we can choose a subsequence $b_{n_k}$ that converges (\(\nu \times \lambda\))-a.e. to some (\(\nu \times \lambda\))-measurable function $\tilde{g}$. On the other hand, $[b_{n_k}(s, \cdot)]_{\lambda}$ converge to $h(s)$ in $L^2(T, \lambda)$ for \(\nu\)-a.e. $s$. Hence, for \(\nu\)-a.e. $s$, there is a subsequence of $b_{n_k}(s, \cdot)$ that converges \(\lambda\)-a.e. to $g(s, \cdot)$. This means that $g$ and $\tilde{g}$ coincide (\(\nu \times \lambda\))-a.e. and, therefore, $g$ is (\(\nu \times \lambda\))-measurable. In the general case, we denote by $A_N$ the set of all $s \in S$ such that $\|h(s)\|^2 \leq N$. Then the map $s \mapsto \chi_{A_N}(s)h(s)$ is square-integrable and by the above, the function $(s, t) \mapsto \chi_{A_N}(s)g(s, t)$ is (\(\nu \times \lambda\))-measurable. This implies that $g$ is (\(\nu \times \lambda\))-measurable. The lemma is proved. \(\square\)

**Lemma A.2.** Let $\nu$ be a measure on a measurable spaces $S$ and $\lambda$ be the Lebesgue measure on an interval $(a, b)$. Let $f$ be a $(\nu \times \lambda)$-measurable complex-valued function on $S \times (a, b)$ such that $f_{[s]}$ is locally $\lambda$-integrable for $\nu$-a.e. $s$. Then for any $x_0 \in (a, b)$, the function

$$
g(s, x) = \int_{x_0}^{x} f(s, x') \, dx'
$$

on $S \times (a, b)$ is $(\nu \times \lambda)$-measurable. If $f_{[s]}$ is left $\lambda$-integrable for $\nu$-a.e. $s$, then the statement also holds for $x_0 = a$.

**Proof.** For each $N = 1, 2, \ldots$, we choose a partition of $(a, b)$ into measurable sets $A_1^N, \ldots, A_k^N$ such that the diameter of every $A_j^N$ is less than $1/N$. For each $j = 1, \ldots, k_N$, we choose a point $x_j^N \in A_j^N$ and set

$$
g_N(s, x) = \sum_{j=1}^{k_N} \chi_{A_j^N}(s)g(s, x_j^N),
$$

where $\chi_{A_j^N}$ is the characteristic function of $A_j^N$. By the Fubini theorem, the function $s \mapsto g(s, x)$ on $S$ is $\nu$-measurable for any $x \in (a, b)$ and, therefore, $g_N$ are $(\nu \times \lambda)$-measurable for all $N$. Since $g_{[s]}$ is continuous on $(a, b)$ for $\nu$-a.e. $s$, the sequence $(g_{[s]}|_{[s]})$ converges pointwise to $g_{[s]}$ for $\nu$-a.e. $s$. This implies that $g_N$ converge $(\nu \times \lambda)$-a.e. to $g$ and, hence, $g$ is $(\nu \times \lambda)$-measurable. The lemma is proved. \(\square\)

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