Topological algebras of rapidly decreasing matrices and generalizations

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Abstract

It is well-known fact in K-theory that the rapidly decreasing matrices of countable size form a locally m-convex associative topological algebra whose set of quasi-invertible elements is open, and such that the quasi-inversion map is continuous. We generalize these conclusions to further algebras of weighted matrices with entries in a Banach algebra.

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If \( (\mathcal{A}, \|\cdot\|) \) is a Banach algebra over \( \mathbb{R} \) or \( \mathbb{C} \) and \( \mathcal{W} \) a non-empty set of monotonically increasing functions \( f: \mathbb{N} \to [0, \infty[ \), we define \( M(\mathcal{A}, \mathcal{W}) \) as the set of all \( T = (t_{ij})_{i,j \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N} \times \mathbb{N}} \) such that
\[
\|T\|_f := \sup_{i,j \in \mathbb{N}} f(i \vee j) \|t_{ij}\| < \infty
\]
for all \( f \in \mathcal{W} \), where \( i \vee j \) denotes the maximum of \( i \) and \( j \). It is clear that \( M(\mathcal{A}, \mathcal{W}) \) is a vector space; we give it the locally convex Hausdorff vector topology defined by the set of norms \( \{\|\cdot\|_f : f \in \mathcal{W}\} \). We show:

**Theorem.** Assume there exists \( g \in \mathcal{W} \) such that \( C_g := \sum_{n=1}^{\infty} \frac{1}{g(n)} < \infty \). If \( R = (r_{ij})_{i,j \in \mathbb{N}}, S = (s_{ij})_{i,j \in \mathbb{N}} \in M(\mathcal{A}, \mathcal{W}) \) and \( i, j \in \mathbb{N} \), then the series
\[
t_{ij} := \sum_{k=1}^{\infty} r_{ik}s_{kj}
\]
converges absolutely in \( \mathcal{A} \). Moreover, \( RS := (t_{ij})_{i,j \in \mathbb{N}} \in M(\mathcal{A}, \mathcal{W}) \), and the multiplication defined in this way makes \( M(\mathcal{A}, \mathcal{W}) \) a locally m-convex, associative topological algebra which is complete as a topological vector space, has an open set of quasi-invertible elements, and whose quasi-inversion map is continuous.

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Recall that an element $x$ in an associative (not necessarily unital) algebra $A$ is called *quasi-invertible* if there exists $y \in A$ such that $xy = yx$ and $x + y - xy = 0$. The element $q(x) := y$ is then unique and is called the *quasi-inverse* of $x$. Locally convex topological algebras $A$ with an open set $Q(A)$ of quasi-invertible elements and continuous quasi-inversion map $q: Q(A) \to A$ are called *continuous quasi-inverse algebras* (and *continuous inverse algebras* if they have, moreover, a unit element). See [7] for information on such algebras as well as [2] and [4], where such algebras are inspected due to their usefulness in infinite-dimensional Lie theory. Also recall that a topological algebra $A$ is called *locally m-convex* if its vector topology can be defined using a set of seminorms $p: A \to [0, \infty]$ which are sub-multiplicative, i.e., $p(xy) \leq p(x)p(y)$ for all $x, y \in A$. If $A$ is, moreover, complete as a topological vector space, this means that $A$ is a projective limit of Banach algebras [5].

If we take $W := \{f_m: m \in \mathbb{N}_0\}$ with $f_m(n) := n^m$, and $A := \mathbb{C}$, then $M(\mathbb{C}, W)$ is the so-called algebra of rapidly decreasing matrices, which plays an important role in the $K$-theory of Fréchet algebras [6]. It is known that this algebra (and its counterpart for general $A$) has an open group of quasi-invertible elements [6, 4.6] and is a locally m-convex Fréchet algebra [6, 2.4 (1)]. Our discussion recovers these facts, but applies to larger classes of weighted matrix algebras. As we realized, only a simple condition (the existence of $g$ with $C_g < \infty$) needs to be imposed on the set of weights.

**Proof of the theorem.** Step 1. Let $R = (r_{ij})_{i,j \in \mathbb{N}}$ and $S = (s_{ij})_{i,j \in \mathbb{N}}$ be in $M(A, W)$. We show that the series $t_{ij} := \sum_{k=1}^{\infty} r_{ik} s_{kj}$ converge absolutely in $A$, and that $T := (t_{ij})_{i,j \in \mathbb{N}} \in M(A, W)$. To this end, let $f, g \in W$ with $C_g < \infty$. If $i \geq j$, we have $i \vee j = i \leq i \vee k$ for all $k \in \mathbb{N}$, hence $f(i \vee j) \leq f(i \vee k)$ by monotonicity and thus

$$f(i \vee j) \sum_{k=1}^{\infty} \|r_{ik}\| \|s_{kj}\| = \sum_{k=1}^{\infty} f(i \vee j)\|r_{ik}\| \|s_{kj}\| \leq \sum_{k=1}^{\infty} f(i \vee k)\|r_{ik}\| \|s_{kj}\|$$

$$\leq \|R\|_f \sum_{k=1}^{\infty} \|s_{kj}\| \leq \|R\|_f \sum_{k=1}^{\infty} g(k \vee j)\|s_{kj}\| \frac{1}{g(k \vee j)} \leq \|R\|_f \sum_{k=1}^{\infty} g(k \vee j) \frac{1}{g(k \vee j)} \leq C_g \|R\|_f \|S\|_g < \infty, \quad (1)$$

using that $g$ is monotonically increasing for the penultimate inequality. If
\[ i \leq j, \text{the same argument shows that} \]
\[ f(i \lor j) \sum_{k=1}^{\infty} \|r_{ik}\| \|s_{kj}\| \leq C_g \|R\|_g \|S\|_f < \infty. \quad (2) \]

In particular, in either case \( \sum_{k=1}^{\infty} \|r_{ik}\| \|s_{kj}\| < \infty \), whence indeed \( \sum_{k=1}^{\infty} r_{ik} s_{kj} \) converges absolutely. Now (1) and (2) show that \( SR := T := (t_{ij})_{i,j \in \mathbb{N}} \in M(\mathcal{A}, \mathcal{W}) \), with
\[ \|SR\|_f \leq C_g (\|R\|_f \|S\|_g \lor \|R\|_g \|S\|_f). \quad (3) \]

Step 2: We show that the multiplication just defined is associative. To this end, let \( R = (r_{ij})_{i,j \in \mathbb{N}}, S = (s_{ij})_{i,j \in \mathbb{N}} \) and \( T = (t_{ij})_{i,j \in \mathbb{N}} \) be in \( M(\mathcal{A}, \mathcal{W}) \). Let \( R', S' \) and \( T' \) be the matrices with entries \( \|r_{ij}\|, \|s_{ij}\| \) and \( \|t_{ij}\| \), respectively. Then \( R', S', T' \in M(\mathbb{R}, \mathcal{W}) \), as is clear from the definitions. Hence
\[ \sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} \|r_{\ell\ell}\| \|s_{\ell k}\| \|t_{k j}\| = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \|r_{\ell\ell}\| \|s_{\ell k}\| \|t_{k j}\| = \sum_{\ell=1}^{\infty} \|r_{\ell\ell}\| (S'T')_{ij} = (R'(S'T'))_{ij} \in \mathbb{R} \]
(where the first equality is a well-known elementary fact, which can also be inferred by applying Fubini’s Theorem to the counting measures on \( \mathbb{N}^2 \) and \( \mathbb{N} \)). Thus \( \sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} \|r_{\ell\ell}s_{\ell k}t_{k j}\| < \infty \), showing that the family \( (r_{\ell\ell}s_{\ell k}t_{k j})_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} \) of elements of \( \mathcal{A} \) is absolutely summable. As a consequence,
\[ ((RS)T)_{ij} = \sum_{k=1}^{\infty} (RS)_{ik} t_{kj} = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} r_{\ell\ell}s_{\ell k}t_{kj} = \sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} r_{\ell\ell}s_{\ell k}t_{kj} = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} r_{\ell\ell}s_{\ell k}t_{kj} = \sum_{\ell=1}^{\infty} (ST)_{\ell j} = (R(ST))_{ij} \]
using [1, 5.3.6] for the third and fourth equalities. Thus \( (RS)T = R(ST) \).

Step 3. The locally convex space \( M(\mathcal{A}, \mathcal{W}) \) is complete. To see this, note first that \( M(\mathcal{A}, \{f\}) \) (with the norm \( \|\cdot\|_f \)) is a Banach space isomorphic to the space \( \ell^\infty(\mathcal{A}) \) of bounded \( \mathcal{A} \)-valued sequences, for each \( f \in \mathcal{W} \). Next, after replacing \( \mathcal{W} \) with the set of finite sums of elements of \( \mathcal{W} \) (which changes neither \( M(\mathcal{A}, \mathcal{W}) \) as a set, nor its topology), we may assume henceforth that \( \mathcal{W} + \mathcal{W} \subseteq \mathcal{W} \) and hence that \( \mathcal{W} \) is upward directed. Then \( M(\mathcal{A}, \mathcal{W}) \) is
the projective limit of the complete spaces $M(A, \{f\})$ ($f \in W$) and hence complete.

Step 4. We show that the set $Q$ of quasi-invertible elements in $M(A, W)$ is open. By [2, Lemma 2.6], we need only check that $Q$ is a 0-neighbourhood. To this end, choose $g \in W$ such that $C_g < \infty$. Then

$$\left\{ T \in M(A, W) : \|T\|_g < \frac{1}{C_g} \right\} \subseteq Q.$$  

Indeed, pick $T$ in the left hand side. We claim that $(\forall n \in \mathbb{N}) \|T^n\|_f \leq (C_g\|T\|_g)^{n-1}\|T\|_f$ for each $f \in W$. If this is true, then $\sum_{n=1}^{\infty} T^n$ converges in each of the Banach spaces $(M(A, \{f\}), \|\cdot\|_f)$ and hence also in the projective limit $M(A, W)$. Now the usual argument shows that $-\sum_{n=1}^{\infty} T^n$ is the quasi-inverse of $T$.

To prove the claim, we proceed by induction. If $n = 1$, then $\|T\|_f = (C_g\|T\|_g)^0\|T\|_f$. If the claim holds for $n-1$ in place of $n$, writing $T^n = T^{n-1} T$ we deduce from (3) that

$$\|T^n\|_f \leq C_g (\|T^{n-1}\|_f \|T\|_g \vee \|T^{n-1}\|_g \|T\|_f).$$  

Now

$$C_g\|T^{n-1}\|_f \|T\|_g \leq C_g (C_g\|T\|_g)^{n-2}\|T\|_f \|T\|_g = (C_g\|T\|_g)^{n-1}\|T\|_f$$  

by induction. Likewise,

$$C_g\|T^{n-1}\|_g \|T\|_f \leq C_g (C_g\|T\|_g)^{n-2}\|T\|_g \|T\|_f = (C_g\|T\|_g)^{n-1}\|T\|_f,$$

applying the inductive hypothesis to $g$ and $g$ in place of $f$ and $g$. Combining (4), (6) and (7), we see that $\|T^n\|_f \leq (C_g\|T\|_g)^{n-1}\|T\|_f$, which completes the inductive proof.

Step 5. $M(A, W)$ is locally m-convex. To see this, pick $g \in W$ with $C_g < \infty$. After replacing $W$ with $\{f + g : f \in W\}$ (which changes neither $M(A, W)$ as a set nor its topology), we may assume henceforth that $C_f < \infty$ for each $f \in W$. We may therefore choose $g := f$ in (3) and obtain

$$\|RS\|_f \leq C_f\|R\|_f\|S\|_f.$$
Let \( h := C_f \cdot f \). Then \( C_h = C_f \cdot \sum_{n=1}^{\infty} \frac{1}{f(n)} \) = 1 and \( \| f \|_{h} \) is equivalent to the norm \( \| f \|_{h} \), which is submultiplicative as \( \| RS \|_{h} \leq C_h \| R \|_{h} \| S \|_{h} = \| R \|_{h} \| S \|_{h} \).

Step 6. Continuity of quasi-inversion. Since we assume that \( C_f < \infty \) for each \( f \in \mathcal{W} \), we know from Step 5 that \( M(\mathcal{A}, \{ f \}) \) is a Banach algebra, with respect to a submultiplicative norm \( \| . \|_{h} \) which is equivalent to \( \| . \|_{f} \). Now, as we assume that \( \mathcal{W} + \mathcal{W} \subseteq \mathcal{W} \) (see Step 3), \( M(\mathcal{A}, \mathcal{W}) \) is the projective limit of the Banach algebras \( M(\mathcal{A}, \{ f \}) \) (\( f \in \mathcal{W} \)). Because quasi-inversion is continuous in each of the Banach algebras, and continuity of maps into projective limits can be checked componentwise, it follows that quasi-inversion is continuous on \( Q \subseteq M(\mathcal{A}, \mathcal{W}) \). \( \Box \)

**Remark.** Our results were first recorded in the unpublished thesis \[3\].

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