Research Article

A Note on Constant Mean Curvature Foliations of Noncompact Riemannian Manifolds

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We aimed to study constant mean curvature foliations of noncompact Riemannian manifolds, satisfying some geometric constraints. As a byproduct, we answer a question by M. P. do Carmo (see Introduction) about the leaves of such foliations.

1. Introduction

Consider a codimension-one foliation of a Riemannian manifold, whose leaves have a constant mean curvature. When the ambient manifold is compact, there are a bunch of results stating that such a foliation is totally geodesic, provided some geometric assumption is satisfied [1, 2]. This kind of phenomenon is sometimes true in the noncompact case as well. Meeks in [3] proved that any codimension-one constant mean curvature foliation of the three-dimensional Euclidean space is totally geodesic. Oshikiri [4] proved the analogous result in a Riemannian manifold with nonnegative Ricci curvature, provided the leaves have quadratic volume growth.

In this article, we prove that the leaves of any codimension-one constant mean curvature foliation of a Riemannian manifold with nonnegative Ricci curvature are totally geodesic, provided they are parabolic (Theorem 4). Theorem 4 generalizes Oshikiri’s result, as quadratic volume growth implies they are parabolic.

Then, we consider a Riemannian manifold $\mathcal{N}$ with zero volume entropy (Section 2), and we prove that, if $\mathcal{F}$ is a codimension-one foliation of $\mathcal{N}$ such that any leaf $L$ has constant mean curvature $H_L$, then $\inf |H_L| = 0$ (Theorem 2). Notice that having zero volume entropy is a weaker assumption than having nonnegative Ricci curvature, by Bishop’s comparison theorem [5] (Corollary 2.1.1).

Finally, we point out that if the leaves of a foliation $\mathcal{F}$ have the same constant mean curvature, then the leaves are stable as shown in [6], Proposition 3, and Section 3, where a sketch of the proof is given.

We recall that do Carmo in [7] asked the following question: is a noncompact, complete, stable, constant mean curvature hypersurface of $\mathbb{R}^{n+1}$, $n \geq 3$, necessarily minimal?

Our result yields a positive answer to do Carmo’s question, provided the hypersurface is a leaf of a foliation such that the leaves have the same constant mean curvature.

The answer was already known to be positive for $n = 2$ [8] ([9], when the ambient manifold is the hyperbolic space). Later, the answer was proved to be positive for $n = 3$ and 4 by Elbert et al. [10] and independently by Cheng [11], using a Bonnet–Myers’ type method. Moreover, the authors gave a positive answer to do Carmo’s question in the following cases: (1) a hypersurface with zero volume entropy of a space form of any dimension [12] (Corollary 8). (2) a hypersurface of $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, $n \leq 5$ with total curvature with polynomial growth [13] (Corollary 6.3).

2. Basic Notions

Let $N$ be a complete, noncompact Riemannian manifold. We define the entropy associated with the volume of geodesic balls in $N$ [14, 15].
Definition 1. Let $B_{\sigma}^N(R)$ be a geodesic ball in $N$ of radius $R$, centered at a fixed point $\sigma \in N$, and denoted by $|B_{\sigma}^N(R)|$ its volume. The volume entropy of $N$ is
$$
\mu_N := \limsup_{R \to \infty} \left( \frac{\ln |B_{\sigma}^N(R)|}{R} \right).
$$

The notion of volume entropy does not depend on the center $\sigma$ of the balls. It is worthwhile to notice that $\mu_N = 0$ is equivalent to
$$
\limsup_{R \to \infty} \left| \frac{|B_{\sigma}^N(R)|}{e^{\alpha R}} \right| = 0, \quad \forall \alpha > 0.
$$

Then, it is natural to say that $N$ has subexponential growth if $\mu_N = 0$ and exponential growth if $\mu_N > 0$.

We observe that having subexponential growth is a weaker assumption than being bounded by a polynomial of any degree. For example, if $|B_{\sigma}^N(R)| = e^{\beta R}$ and $\beta < 1$, then $N$ has subexponential growth.

The volume entropy of a manifold $N$ is strictly related to the bottom of its essential spectrum and to the Cheeger constant. Let us be more precise about this.

Let $\Delta$ be the Laplacian on $N$, then the bottom of the spectrum $\sigma(N)$ of $-\Delta$ is
$$
\lambda_0(N) = \inf_{f \in C^0_c(N), f \neq 0} \left( \frac{\int_N |\nabla f|^2}{\int_N f^2} \right).
$$

The bottom of the essential spectrum $\sigma_{ess}(N)$ of $-\Delta$ is
$$
\lambda_0^{ess}(N) = \inf_{K} \left( \frac{\lambda_0(N_K)}{K} \right),
$$
where $K$ runs through all compact subsets of $N$.

The Cheeger constant $h_N$ of a Riemannian manifold $N$ is defined as $h_N = \inf_{|\Omega| \neq 0} |\Omega|/|\partial \Omega|$, where $\Omega$ runs over all compact domains of $N$ with piecewise smooth boundary $\partial \Omega$.

Cheeger [16] and Brooks [14] proved the following important comparison result between the bottom of the essential spectrum, the volume entropy, and the Cheeger constant.

Theorem 1 (Brooks–Cheeger’s Theorem). If $N$ has infinite volume, then
$$
\frac{h_N^2}{4} \leq \lambda_0(N) \leq \lambda_0 \left( \frac{N}{K} \right) \leq \lambda_0^{ess}(N) \leq \frac{\mu_N^2}{4},
$$
where $K$ is any compact subset of $N$.

We finally recall that a Riemannian manifold is called parabolic if it does not admit a nonconstant positive superharmonic function. Parabolicity is strictly related to the volume growth of a manifold. In fact, quadratic area growth implies parabolicity [17] (Corollary 7A) [6].

3. Foliations of Manifolds with Zero Volume Entropy

In this section, we study foliations, with constant mean curvature leaves, of manifolds with zero volume entropy. In the article, $\mathcal{F}$ will be a $C^3$ codimension-one foliation of a manifold $N$ and $N$ will be a $C^3$ unit vector field of $N$ normal to the leaves of $\mathcal{F}$.

We have the following result.

Theorem 2. Let $N$ be a manifold with zero volume entropy. Let $\mathcal{F}$ be a codimension-one $C^3$ foliation of $N$ such that any leaf $L$ has constant mean curvature $H_L \geq 0$. Then, $\inf H_L = 0$.

Before doing the proof of Theorem 2, let us state a consequence in the case of a constant mean curvature foliation.

Corollary 1. Let $N$ be a manifold with zero volume entropy. Let $\mathcal{F}$ be a codimension-one $C^3$ foliation of $N$ by leaves of constant mean curvature $H$. Then, $H = 0$.

As we remarked in Introduction, the leaves of a constant mean curvature foliation are stable, provided all the leaves have the same mean curvature ([2], Proposition 3). We give a sketch of the proof of the latter for the sake of completeness.

By [2] (Proposition 1), one has
$$
\text{Ric}(N, N) + |A|^2 + |\theta|^2 = \text{div}_L (\nabla_N N),
$$
where $N$ is a unit vector field defined in $N$, perpendicular to $\mathcal{F}$ at any point, $A$ is the second fundamental form of $L$, $\theta$ is defined by $\theta(X) = \langle \nabla_N N, X \rangle$ for every vector field $X$ tangent to $L$, and $|\theta|^2 = \sum_{i=1}^n |\theta(E_i)|^2$, where $\{E_1, \ldots, E_n\}$ is a local orthonormal base of the tangent space to $L$.

Recall that the second variation of the volume of a leaf $L$ is
$$
V''(0) = \int_L [\nabla f]^2 - f^2 (\text{Ric}(N, N) + |A|^2),
$$
where $f$ is any smooth function on $N$ with compact support. Then, by using equality (6) multiplied by $f$, one gets
$$
V''(0) = \int_L [\nabla f]^2 - f^2 (\text{Ric}(N, N) + |A|^2) = \int_L [\nabla f]^2 + f^2 |\theta|^2 - f^2 \text{div}_L (\nabla_N N)
= \int_L [\nabla f]^2 + f^2 |\theta|^2 - \text{div}_L (f^2 \nabla_N N) + \nabla_N N (f^2)
= \int_L \nabla f + f \theta|^2 \geq 0.
$$
Then, \( V'' (0) \geq 0 \); this means that the leaves are stable.

The stability of the leaves of such a foliation suggests to inquire if we can answer Carmon’s question [7]: is a non-compact, complete, stable, constant mean curvature hypersurface of \( \mathbb{R}^n \), \( n \geq 3 \), necessarily minimal?

Corollary 1 yields a positive answer to Carmon’s question, provided the hypersurface is a leaf of a foliation with constant mean curvature leaves.

**Proof of Theorem 2.** If there is a minimal leaf \( L \), we have nothing to prove. Then, we may assume that \( \inf H_L > 0 \), and let \( N \) be the unit vector field normal to the leaves of \( \mathcal{F} \) given by the normalized mean curvature vector. Then, there exists \( c > 0 \) such that \( H_L \geq c \) for every leaf \( L \). By straightforward computation, we obtain the following:

\[
\frac{\partial}{\partial t} \left( \frac{nH_p}{\text{vol} (\partial B_p)} \right) = \frac{\text{vol} (\partial B_p)}{\text{vol} (B_p)} \left( \frac{\partial nH_p}{\partial t} \right)
\]

where \( H \) is the mean curvature of the leaf passing through \( p \).

Then, integrating (9) on a ball of radius \( R \) of \( N \), say \( B_R \),

\[
\int_{B_R} nH_p = \int_{\partial B_R} \langle N, \nu \rangle \leq \text{vol} (\partial B_R)
\]

where the last equality is by the divergence theorem. Keeping into account that \( H(p) \geq c \) at any point \( p \in N \), (10) yields

\[
nc \leq \frac{\text{vol} (\partial B_p)}{\text{vol} (B_p)}
\]

In particular,

\[
nc \leq \frac{\text{vol} (\partial B_p)}{\text{vol} (B_p)} = h_N.
\]

Then, the Brooks–Cheeger’s theorem stated in Section 2 yields \( h_N \geq h_N > 0 \) that is a contradiction.

Using the Gauss formula, one can prove the analogous of Theorem 2 for foliations of constant scalar curvature.

**Theorem 3.** Let \( N \) be a manifold with zero volume entropy and nonpositive sectional curvatures. Let \( \mathcal{F} \) be a codimension-one \( C^3 \) foliation of \( N \) such that any leaf \( L \) has constant scalar curvature \( S_L \geq 0 \). Then, \( \text{inf} S_L = 0 \).

**Proof.** By the Gauss formula, one has

\[
S_L = \sum_{i<j} R (e_i, e_j) + H_L^2 - |A|^2,
\]

where \( e_1, \ldots, e_n \) is a base of the tangent space to a leaf \( L \), \( R (e_i, e_j) \) is the sectional curvature of \( N \) along span \( \{ e_i, e_j \} \), \( |A| \) is the norm of the second fundamental form of \( L \), and \( H_L \) is its mean curvature.

Then, being the sectional curvatures of \( N \), \( \nu \) nonpositive, one has that \( S_L < H_L^2 \); hence, \( S_L > c \) implies \( H_L > \sqrt{c} \), and one can apply Theorem 2.

An immediate consequence of Theorem 2 is the following result.

**Corollary 2.** Let \( N \) be a manifold with zero volume entropy and nonpositive sectional curvatures. Let \( \mathcal{F} \) be a codimension-one \( C^3 \) foliation of \( NN \) by leaves of constant scalar curvature \( S \). Then, \( S = 0 \).

**Remark 1.** It is worthwhile noticing that all the results of this section hold for foliations given by complete graphs of constant mean or scalar curvature, when they exist.

### 4. Minimal Parabolic Foliations of Manifolds with Nonnegative Ricci Curvature

In this section, we prove a kind of Bernstein’s theorem for the leaves of a minimal foliation of a Riemannian manifold with nonnegative Ricci curvature, provided the leaves are parabolic.

**Theorem 4.** Let \( N \) be a Riemannian manifold with nonnegative Ricci curvature and let \( \mathcal{F} \) be a codimension-one foliation of \( N \) by minimal leaves. If the leaves of the foliation are parabolic, then they are totally geodesic.

**Proof of Theorem 4.** We first need a definition. We say that \( v \) is an exhaustion function on \( L \) if \( v \) is continuous on \( L \) and such that all the sets \( \mathcal{B}_r := \{ p \in L : v(x) \leq r \} \) are precompact. Notice that the latter is equivalent to say that \( v(p) \to \infty \) as \( p \to \infty \) (that is, \( p \) leaves every compact).

By [12] (Theorem 7.6), a manifold \( L \) is parabolic if and only if there exists a smooth exhaustion function \( v \) on \( L \) such that

\[
\int_{\mathcal{B}_r} \frac{\partial v}{\mathcal{V}_{\mathcal{B}_r}} dr = \infty,
\]

where \( \mathcal{V}_{\mathcal{B}_r} \) is the outward unit normal to \( \mathcal{B}_r \).

 Equality (6) in Section 3 holds for the minimal leaf \( L \), and integrating it on \( \mathcal{B}_r \), yields

\[
\int_{\mathcal{B}_r} \text{Ric} (N, N) + |A|^2 + |\theta|^2 = \int_{\mathcal{B}_r} \text{div}_L (\nabla N, N) = \int_{\mathcal{B}_r} \theta (v),
\]

and for the last equality, we used the divergence theorem.

As \( \text{Ric} (N, N) \) and \( |A|^2 \) are nonnegative, we have

\[
\int_{\mathcal{B}_r} |\theta|^2 \leq \int_{\mathcal{B}_r} \theta (v) \left( \int_{\mathcal{B}_r} |\theta|^2 / |\nabla v| \right)^{1/2} \left( \int_{\mathcal{B}_r} |\nabla v| \right)^{1/2},
\]

where the last inequality is by the Cauchy–Schwarz inequality.

By defining \( f (r) = \int_{\mathcal{B}_r} |\theta|^2 / |\nabla v| ds \), the coarea formula yields

\[
f (r) = \int_0^r f (r) dr / |\nabla v| ds.
\]

Assume that \( f (r) \neq 0 \). With this notation, the square of (16) is written as
\[
\frac{1}{\text{Flux}_{\partial \Omega, v}} \leq \frac{f'(r)}{f^2(r)} \tag{17}
\]

We integrate inequality (17) between a fixed \( r_0 \) and \( R \) (where \( f \) is nonzero), and we get

\[
\int_{r_0}^{R} \frac{dr}{\text{Flux}_{\partial \Omega, v}} \leq \frac{1}{f(r_0)} - \frac{1}{f(R)} \tag{18}
\]

By letting \( R \) go to \( \infty \), inequality (18) gives a contradiction. In fact, as \( f \) is a nondecreasing function, then the right-hand side is bounded, while by hypothesis, the left-hand side tends to infinity.

Then, \( f \equiv 0 \), that is \( \nabla N, N \equiv 0 \), on the leave \( L \), and equality (6) yields \( R(N, N) + |A|^2 \equiv 0 \). As the Ricci curvature is nonnegative, we get \( |A|^2 \equiv 0 \), i.e., \( L \) is totally geodesic. □

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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