On the Performance of Channel Statistics-Based Codebook for Massive MIMO

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Abstract—The channel feedback overhead for massive MIMO systems with a large number of base station (BS) antennas is very high, since the number of feedback bits of traditional codebooks scales linearly with the number of BS antennas. To reduce the feedback overhead, an effective codebook based on channel statistics has been designed, where the required number of feedback bits only scales linearly with the rank of the channel correlation matrix. However, this attractive conclusion was only proved under a particular channel assumption in the literature, while no rigorous theoretical proof under a general channel assumption has been provided. To fill in the blank of both theoretical proof and simulation results under a general channel assumption, in this paper, we quantitatively analyze the performance of the channel statistics-based codebook. Specifically, we firstly introduce the rate gap between the ideal case of perfect channel state information at the transmitter and the practical case of limited channel feedback, where we find that the rate gap is dependent on the quantization error of the codebook. Then, we derive an upper bound of the quantization error, based on which we prove that the required feedback bits to ensure a constant rate gap only scales linearly with the rank of the channel correlation matrix. Finally, numerical results are provided to verify this conclusion. To the best of our knowledge, our work is the first one to provide a rigorous proof of this conclusion under a general channel assumption.

I. INTRODUCTION

Massive multiple-input multiple-output (MIMO) using hundreds of antennas at the base station (BS) is one of the key technologies for future 5G wireless communication systems. To achieve the expected high spectrum efficiency and energy efficiency of massive MIMO, accurate channel state information at the transmitter (CSIT) is crucial [1]. Utilizing the channel reciprocity, the CSIT can be obtained from uplink channel estimation in time division duplexing (TDD) systems. While most of existing works on massive MIMO consider the TDD mode due to this reason, frequency division duplexing (FDD) has many benefits over TDD (especially in delay-sensitive or traffic-symmetric applications [2], [3]) and thus still dominates current cellular networks. To this end, it is important to study the CSIT acquisition problem for FDD massive MIMO.

However, the channel reciprocity does not exist in FDD systems, so accurate channel feedback from users to the BS is required in FDD massive MIMO. Traditional codebooks for channel feedback such as Grassmannian codebook [4], [5] and random vector quantization (RVQ) based codebook [6], [7] have been extensively investigated. For these codebooks, there is an important conclusion that, to maintain a constant capacity degradation due to channel quantization error, the required number of feedback bits (quantization bits) approximately scales linearly with the number of BS antennas [7]. For massive MIMO, with a large number of BS antennas, the channel feedback overhead will be overwhelming. Thus, several new codebooks have been proposed to reduce the channel feedback overhead [2], [8], [9]. Among these, an effective codebook based on channel statistics is designed by multiplying each vector of an original codebook (e.g., RVQ-based codebook) by the channel correlation matrix [9]. In this way, the distribution of the resulted quantization vectors is closer to the distribution of the actual channel vectors. Thus, the channel statistics-based codebook has better quantization performance. It has been verified through extensive simulation results that the required number of feedback bits by this channel statistics-based codebook only scales linearly with the rank of the channel correlation matrix [2], [10]. As the rank of the channel correlation matrix is much smaller than the number of BS antennas in massive MIMO [11], the feedback overhead can be reduced significantly. The authors of [10] only provided a proof of this attractive conclusion under a particular channel assumption, while no rigorous theoretical proof under a general channel assumption has been provided.

In this paper, to fill in the blank of both theoretical proof and simulation results under a general channel assumption, we quantitatively analyze the performance of the channel statistics-based codebook. Specifically, we firstly introduce the rate gap between the ideal case of perfect CSIT and the practical case of limited channel feedback. We find that the rate gap is dependent on the quantization error of the codebook. Then, we analyze the quantization error of the channel statistics-based codebook. Although the quantization error is difficult to obtain, we are able to derive an upper bound of the quantization error by utilizing several skills of magnifying and shrinking of inequation. After that, by substituting the upper bound of quantization error into the rate gap expression, we can obtain the upper bound of the rate gap. Finally, we prove that the required number of feedback bits to ensure a constant rate gap scales linearly with the rank of the channel correlation matrix. To the best of our knowledge, our work is the first one to provide a rigorous proof of this conclusion.

Notation: Lower-case and upper-case boldface letters denote vectors and matrices, respectively; $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$ and $(\cdot)^{-1}$ denote the transpose, conjugate, conjugate transpose, and in-
verse of a matrix, respectively; \( I_K \) denotes the identity matrix of size \( K \times K \); \( ||h|| \) and \( |s| \) are the norm of a vector and the absolute value of a scalar, respectively. \( E\{\cdot\} \) denotes the expectation operator. \( \Pr\{A\} \) denote the probability of \( A \).

II. System Model

In this section, we firstly introduce the massive MIMO channel model. Then, we present the limited channel feedback based on the channel statistics-based codebook for massive MIMO. Finally, we discuss the downlink precoding and per user rate for the performance analysis of the channel statistics-based codebook.

A. Massive MIMO Channel Model

In this paper, we consider a massive MIMO system with \( M \) antennas at the BS and \( K \) single-antenna users \( (M \gg K) \). The downlink channel vector \( h_k \in \mathbb{C}^{M \times 1} \) for the \( k \)-th user can be described as \[ h_k = \mathbf{R}_t^{1/2}h_{w,k}, \] where \( \mathbf{R}_t^{1/2} \in \mathbb{C}^{M \times M} \) is the square root of the channel correlation matrix \( \mathbf{R}_t \in \mathbb{C}^{M \times M} \), and \( h_{w,k} \in \mathbb{C}^{M \times 1} \) is a vector whose elements are i.i.d. complex Gaussian distributed with zero mean and unit variance. The channel correlation matrix \( \mathbf{R}_t \) is long-term channel statistics and assumed to be known at both the BS and users \( 9 \). Furthermore, the matrix \( \mathbf{R}_t^{1/2} \) can be decomposed as \( \mathbf{U}_t \mathbf{\Lambda}_t^{1/2} \mathbf{U}_t^H \), where \( \mathbf{U}_t \) is an unitary matrix, and \( \mathbf{\Lambda}_t^{1/2} \) is a diagonal matrix with the diagonal elements denoted by \( \{\sigma_1, \sigma_2, \cdots, \sigma_r, 0, \cdots\} \) in a decreasing order of magnitude, where \( r \) is the rank of the channel correlation matrix. The concatenation of channel vectors for all \( K \) users can be denoted by \( \mathbf{H} = [h_1, h_2, \cdots, h_K] \in \mathbb{C}^{M \times K} \).

B. Limited Channel Feedback

Each user is assumed to know its own channel vector \( h_k \), and such information is required at the BS via limited feedback channel. Each user quantizes its channel to \( B \) bits and then feeds back these \( B \) bits to the BS. The quantization is performed by using a quantization codebook, which is known to the BS and all users.

For the traditional RVQ-based codebook \( \mathbf{W} = [w_1, w_2, \cdots, w_{2^B}] \in \mathbb{C}^{M \times 2^B} \), the unit-norm column vectors \( w_i \) are randomly generated by selecting \( 2^B \) vectors independently from the uniform distribution on the complex unit sphere. The required number of feedback bits scales linearly with the number of BS antennas to ensure a constant capacity degradation \( 7 \). Thus, the channel feedback overhead becomes overwhelming for massive MIMO with a large number of BS antennas. In order to reduce the channel feedback overhead, a more effective codebook has been designed based on the channel statistics \( 9 \). Specifically, the quantization vector \( c_i \) can be obtained by multiplying the vector \( w_i \in \mathbb{C}^{M \times 1} \) of a traditional RVQ-based codebook \( \mathbf{W} \) by the square root of the channel correlation matrix \( \mathbf{R}_t^{1/2} \), i.e., \( c_i = \mathbf{R}_t^{1/2}w_i \). Note that to ensure the unit-norm vector requirement of \( c_i \in \mathbb{C}^{M \times 1} \), \( c_i \) should be normalized as

\[
\mathbf{c}_i = \frac{\mathbf{R}_t^{1/2}w_i}{||\mathbf{R}_t^{1/2}w_i||}.
\]

The distribution of the resulted quantization vector \( \mathbf{c}_i \) is closer to the distribution of actual channel vector \( h_k \) in \( 10 \), thus, the channel statistics-based codebook has better quantization performance \( 2, 9, 10 \). Thus, we mainly consider the channel statistics-based codebook in this paper.

The \( k \)-th (\( k = 1, 2, \cdots, K \)) user quantizes its own channel vector \( h_k \) to a quantization vector \( \mathbf{c}_{F_k} \) that is closest to \( h_k \), where “closeness” is measured by the angle between two vectors. Thus, user \( k \) computes the quantization index \( F_k \) according to

\[
F_k = \arg\min_{i=1,2,\cdots,2^B} \sin^2(\angle(h_k, c_i)) = \arg\max_{i=1,2,\cdots,2^B} |\mathbf{h}_k^H \mathbf{c}_i|^2,
\]

where the normalized channel vector \( \tilde{h}_k = \frac{h_k}{||h_k||} \) denotes the direction of channel vector. Note that only the direction of channel vector is quantized, while the channel magnitude \( ||h_k|| \) is not quantized by using codebook \( C \). Magnitude information can be used to allocate power and rate across multiple channels, but it is just a scalar that can be easily fed back. In this paper, we assume the channel magnitude can be fed back to the BS perfectly, and focus on the quantization of channel direction which requires more feedback bits. After that, the BS can search the codebook with the received index \( F_k \) and obtain the feedback channel vector \( \mathbf{h}_k = ||h_k|| \mathbf{c}_{F_k} \). The concatenation of the feedback channel vectors can be denoted as \( \tilde{\mathbf{H}} = [\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_K] \in \mathbb{C}^{M \times K} \).

C. Downlink Precoding and Per User Rate

We can utilize the widely used zero-forcing (ZF) precoding at the BS based on the feedback channel matrix \( \tilde{\mathbf{H}} \) to eliminate interferences among multiple users. The transmitted signal \( \mathbf{x} \in \mathbb{C}^{M \times 1} \) after precoding at the BS is given by

\[
\mathbf{x} = \sqrt{\frac{\gamma}{K}} \mathbf{V} \mathbf{s},
\]

where \( \gamma \) is the transmit power, \( \mathbf{s} = [s_1, s_2, \cdots, s_K] \in \mathbb{C}^{K \times 1} \) is the symbol vector intended for \( K \) users with the normalized power \( E\{||s_i||^2\} = 1 \), and \( \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_K] \in \mathbb{C}^{M \times K} \) is the precoding matrix consisting of \( K \) different \( M \)-dimensional unit-norm precoding vector \( \mathbf{v}_i \in \mathbb{C}^{M \times 1} \). We denote \( \mathbf{U} = \tilde{\mathbf{H}} (\tilde{\mathbf{H}}^H \tilde{\mathbf{H}})^{-1} \), then the precoding vectors \( \mathbf{v}_i \) can be obtained as the normalized \( i \)-th column of \( \mathbf{U} \), i.e. \( \mathbf{v}_i = \frac{\mathbf{U}(:,i)}{||\mathbf{U}(:,i)||} \).

The received signal \( y_k \) at the \( k \)-th user can be described as

\[
y_k = \mathbf{h}_k^H \mathbf{x} + n_k
\]

\[
= \sqrt{\frac{\gamma}{K}} \mathbf{h}_k^H \mathbf{v}_k s_k + \sqrt{\frac{\gamma}{K}} \sum_{i=1, i \neq k}^K \mathbf{h}_k^H \mathbf{v}_i s_i + n_k.
\]
where $n_k$ is the complex Gaussian noise at the $k$-th user with zero mean and unit variance. Thus, the signal-to-interference-plus-noise ratio (SINR) at the $k$-th user is [7]

$$\text{SINR}_k = \frac{\frac{\gamma}{K} |h_k^H v_k|^2}{1 + \frac{\gamma}{K} \sum_{i=1, i \neq k}^{K} |h_i^H v_i|^2}.$$  \hspace{1cm} (6)

Accordingly, the per user rate $R$ is

$$R = E \{ \log_2 (1 + \text{SINR}_k) \} = E \{ \log_2 (1 + \frac{\gamma}{K} |h_k^H v_k|^2) \}. \hspace{1cm} (7)$$

Clearly, the per user rate depends on the precoding matrix $V$, which is significantly affected by the quality of feedback channel $\hat{H}$. In the next section, we will analyze the per user rate when the channel statistics-based codebook is considered.

### III. Performance Analysis

In this section, we firstly introduce the per user rate and the rate gap between the ideal case of perfect CSIT and the practical case of limited channel feedback. Then, we analyze the quantization error of the channel statistics-based codebook. Finally, we derive an upper bound of the required feedback bits to ensure a constant rate gap.

#### A. Rate Gap

Providing the ideal case of perfect CSIT at the BS, i.e., $\hat{H} = H$, the ZF precoding vector $v_{\text{ideal},i}$ is obtained as the normalized $i$-th column of $H(\hat{H}^H \hat{H})^{-1}$. Thus, $v_{\text{ideal},i}$ is orthogonal to the $k$-th user’s channel vector $h_k$ for any $i \neq k$, i.e., the inter-user interference $|h_k^H v_{\text{ideal},i}| = 0$. Therefore, the BS eliminates inter-user interference and obtains the ideal per user rate as

$$R_{\text{ideal}} = E \{ \log_2 (1 + \frac{\gamma}{K} |h_k^H v_{\text{ideal},k}|^2) \}. \hspace{1cm} (8)$$

However, in the practical case of limited channel feedback, the BS can only obtain the feedback channel $\hat{H}$ using the channel statistics-based codebook. The ZF precoding is performed based on $\hat{H}$, and the precoding vector $v_i$ is obtained as the normalized $i$-th column of $\hat{H}(\hat{H}^H \hat{H})^{-1}$. Thus, the inter-user interference $|h_k^H v_i| \neq 0$, which degrades the per user rate as

$$R_{\text{practical}} = E \{ \log_2 (1 + \frac{\gamma}{K} |h_k^H v_k|^2) \}. \hspace{1cm} (9)$$

We define the rate gap $\Delta R(\gamma)$ as the difference between per user rate achieved by ideal CSIT and limited channel feedback using the channel statistics-based codebook:

$$\Delta R(\gamma) = R_{\text{ideal}} - R_{\text{practical}}. \hspace{1cm} (10)$$

Following the results from [2], [7] and using Jensen’s inequality, the rate gap $\Delta R(\gamma)$ can be upper bounded as:

$$\Delta R(\gamma) \leq \log_2(1 + \frac{\gamma}{K} (K - 1) E \{ |h_k^H v_k|^2 \}), \hspace{1cm} (11)$$

where the multi-user interference $E \{ |h_k^H v_i|^2 \}$ can be upper bounded in Lemma 1.

**Lemma 1:** The upper bound of multi-user interference $E \{ |h_k^H v_i|^2 \}$ is dependent on the channel quantization error $E \{ \sin^2(\angle(h_k, \hat{h}_k)) \}$, i.e.,

$$E \{ |h_k^H v_i|^2 \} \leq E \{ |h_k|^2 \} E \{ \sin^2(\angle(h_k, \hat{h}_k)) \}. \hspace{1cm} (12)$$

**Proof:** Denote the quantization error $X = \sin^2(\angle(h_k, \hat{h}_k)) = 1 - |\hat{h}_k^H c_{F_k}|^2$. Since the size of codebook is limited, $X \neq 0$. Thus, the normalized channel vector $\hat{h}_k$ can be decomposed along two orthogonal direction, one is the direction of quantization vector $c_{F_k}$, and the other one is in the nullspace of $c_{F_k}$. Mathematically,

$$\hat{h}_k = \sqrt{1 - X} c_{F_k} + \sqrt{X} s,$ \hspace{1cm} (13)$$

where $s$ is an unit vector distributed in the null space of $c_{F_k}$. Combining $h_k = |\hat{h}_k| \hat{h}_k$ and (13), we have

$$|h_k^H v_i|^2 \leq |h_k|^2 (1 - X) |c_{F_k}^H v_i|^2 + X |s^H v_i|^2. \hspace{1cm} (14)$$

Since the precoding vector $v_i$ is obtained as the normalized $i$-th column of $\hat{H}(\hat{H}^H \hat{H})^{-1}$, $v_i$ is orthogonal to the $k$-th user’s feedback channel vector $\hat{h}_k = |\hat{h}_k| c_{F_k}$, i.e., $c_{F_k}^H v_i = 0$. Thus, we have $|h_k^H v_i|^2 = |h_k|^2 |X| |s^H v_i|^2$. Since $|s^H v_i|^2 = \cos^2(\angle(s, v_i)) \leq 1$, we have

$$|h_k^H v_i|^2 \leq |h_k|^2 X. \hspace{1cm} (15)$$

As the norm of a vector is independent of its direction [7], $|h_k|^2$ is also independent of $X$. Therefore, we have that

$$E \{ |h_k^H v_i|^2 \} \leq E \{ |h_k|^2 \} E \{ \sin^2(\angle(h_k, \hat{h}_k)) \}. \hspace{1cm} (16)$$

Combining (11) and Lemma 1, we can further obtain

$$\Delta R(\gamma) \leq \log_2(1 + \frac{\gamma}{K} (K - 1) E \{ |h_k|^2 \} E \{ \sin^2(\angle(h_k, \hat{h}_k)) \}). \hspace{1cm} (17)$$

### B. Quantization Error

In this subsection, we discuss the quantization error $E \{ \sin^2(\angle(h_k, \hat{h}_k)) \}$ in [7] when the channel statistics-based codebook is considered. For the rest of this paper, we omit the subscript $k$ for simplicity but without loss of generality.

**Lemma 2:** The quantization error $E \{ \sin^2(\angle(h, \hat{h})) \}$ of $\hat{h}$ can be upper bounded as

$$E \{ \sin^2(\angle(h, \hat{h})) \} < 2^{-\alpha}. \hspace{1cm} (18)$$

**Proof:** Denote that $Z = \cos^2(\angle(h, \hat{h})) = |\hat{h}^H c_k|^2$. Since $F_k = \arg \max_{i=1,2,\ldots,2^\theta} |\hat{h}^H c_i|^2$, we have

$$\text{Pr}\{ |\hat{h}^H c_k|^2 < z \} = \text{Pr}\{ |\hat{h}^H c_1|^2 < z, |\hat{h}^H c_2|^2 < z, \ldots |\hat{h}^H c_{2^\theta}|^2 < z \} = \text{Pr}\{ |\hat{h}^H c_1|^2 < z \} \cdot \ldots \cdot \text{Pr}\{ |\hat{h}^H c_{2^\theta}|^2 < z \}. \hspace{1cm} (19)$$

Therefore,}

$$\text{Pr}\{ Z < z \} = \text{Pr}\{ |\hat{h}^H c_k|^2 < z \} \cdot \ldots \cdot \text{Pr}\{ |\hat{h}^H c_{2^\theta}|^2 < z \}. \hspace{1cm} (19)$$
Using (1) and (2), we can rewrite \( |\tilde{H}^H c_i|^2 \) as
\[
|\tilde{H}^H c_i|^2 = \frac{||R^{1/2} h_m^i||^2 R^{1/2} w_i^2}{||R^{1/2} h_m^i||^2 ||R^{1/2} w_i||^2} \]
\[(20)\]
where (a) is obtained by using \( U^H_i U_i = I_M \). Since the unitary matrix \( U_i \) does not change the distribution of an isotropically distributed vector, \( h_m^i \), \( d \) and \( w_i \), \( i \) denotes the equality in terms of distribution. Thus,
\[
|\tilde{H}^H c_i|^2 = \frac{|h_m^i|^2 |A_j^{1/2} H A_j^{1/2} w_i|^2}{||A_j^{1/2} h_m^i||^2 ||A_j^{1/2} w_i||^2},
\]
(21)
where the denominator is accordingly changed to ensure the unit-norm requirement of \( \tilde{h} \) and \( c_i \). Therefore, we have
\[
\Pr(|\tilde{H}^H c_i|^2 < z) = \Pr\{\frac{|h_m^i|^2 |A_j^{1/2} H A_j^{1/2} w_i|^2}{||A_j^{1/2} h_m^i||^2 ||A_j^{1/2} w_i||^2} < z\}. \quad (22)
\]
Since \( A_j^{1/2} \) is an diagonal matrix with \( r \) non-zero diagonal elements \( \{\sigma_1, \sigma_2, \ldots, \sigma_r\} \), we rewrite (22) as
\[
\Pr(|\tilde{H}^H c_i|^2 < z) = \Pr\{\frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} < z\}, \quad (23)
\]
where \( g \in C^{r \times 1} \) with \( g(j) = h_m^i(j) \), \( v \in C^{r \times 1} \) with \( v(j) = w_i(j) \), and \( \Gamma \in C^{r \times r} \) with \( \Gamma(j,j) = A_j^{1/2}(j,j), j = 1,2, \ldots, r \). The non-diagonal elements \( \Gamma(i,j) = 0 \). Actually, \( g \) and \( v \) are random vectors in the \( r \)-dimensional hyper-sphere, while \( \Gamma g \) and \( \Gamma v \) are randomly distributed in the \( r \)-dimensional hyper-ellipse, which is obtained by stretching the hyper-sphere according to the diagonal elements of \( \Gamma \). As we known, the squared cosine of the angle between two vectors \( g \) and \( v \) in hyper-sphere, whose cumulative distribution function (CDF) is given by \( \Pr\{\frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} \leq z\} = 1 - (1 - z)^{-r-1} \) for \( z \in [0,1] \) [6]. In (23), \( \frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} \) is the squared cosine of the angle between two vectors \( \Gamma g \) and \( \Gamma v \) in the hyper-ellipse, whose CDF is very difficult to be obtained. However, we can prove in the Appendix A that \( \Pr\{\frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} \leq z\} < \Pr\{\frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} < z\} \). Thus, we have
\[
\Pr\{\frac{|g^H \Gamma^H \Gamma v|^2}{||g||^2 ||\Gamma v||^2} < z\} < 1 - (1 - z)^{-r-1} \quad (24)
\]
Then, by combining (19), (23), and (24), we can obtain the upper bound of \( Z \)'s CDF as
\[
\Pr\{Z < z\} < (1 - (1 - z)^{-r-1})2^b \quad (25)
\]
Utilizing the fact that \( E\{Z\} = \int_0^1 \Pr\{Z \geq z\} dz \), the expectation \( E\{Z\} \) can be expressed as
\[
E\{Z\} = \int_0^1 (1 - \Pr\{Z < z\}) dz = 1 - \int_0^1 \Pr\{Z < z\} dz. \quad (26)
\]
Combining (25) and (26), we have
\[
E\{Z\} > 1 - \int_0^1 (1 - (1 - z)^{-r-1})2^b dz\quad (27)
\]
\[(a)\]
\[
1 - \int_0^1 (1 - s^{-r-1})2^b ds\quad (28)
\]
\[(b)\]
\[
1 - 2^b \beta(2^b, \frac{r}{r-1})\quad (29)
\]
\[(c)\]
\[
1 - 2^{-\frac{b}{r-1}}.\quad (30)
\]
where (a) is obtained by setting \( s = 1 - z \), (b) and (c) are obtained from [7, Appendix I and II], respectively. Finally, we can obtain the upper bound of quantization error as
\[
E\{\sin^2(2\angle(\hat{h}, \hat{h}))\} = 1 - E\{Z\} < 2^{-\frac{b}{r-1}}. \quad (31)
\]

C. Feedback Bits

In this subsection, we discuss the required number of feedback bits \( B \) to ensure a constant rate gap \( \Delta R(\gamma) \). By combining (17) and (28), we can easily obtain the upper bound of the rate gap
\[
\Delta R(\gamma) \leq \log_2(1 + \frac{\gamma}{K}(K-1)E\{||h_k||^2\})^2 - \frac{b}{r-1}. \quad (32)
\]
Let the rate gap \( \Delta R(\gamma) = \log_2(b) \) bps/Hz, then the number of feedback bits \( B \) should scale according to
\[
B = \frac{r - 1}{3} P_{dB} + (r - 1) \log_2(K - 1) - (r - 1) \log_2(b - 1), \quad (33)
\]
where the SNR at the receiver is defined as \( P_{dB} = 10 \log_{10} E\{||h_k||^2\} \). We can observe that the slope of the required number of feedback bits is \( r \) when \( P_{dB} \) increases. Thus, we have proved the attractive conclusion that the required number of feedback bits only scales linearly with the rank of the channel correlation matrix to maintain a constant rate gap.

IV. Simulation Verification

In this section, simulation results are provided to verify the derived theoretical result of the required number of feedback bits \( B \) for channel statistics-based channel feedback. The simulation setup is as follows: the number of BS antennas, the number of users, and the rank of the channel correlation matrix to maintain a constant rate gap. In this section, simulation results are provided to verify the derived theoretical result of the required number of feedback bits \( B \) for channel statistics-based channel feedback. The simulation setup is as follows: the number of BS antennas, the number of users, and the rank of the channel correlation matrix to maintain a constant rate gap.

Fig. 1 investigates the per user rate using the channel statistics-based codebook for channel feedback. We observe that the rate gap between the ideal case of perfect CSIT and the practical case using the channel statistics-based codebook remains constant when SNR at the receiver \( P_{dB} \) increases. It is consistent with our theoretical analysis in Section III. Note that the gap is less than \( \Delta R(\gamma) = 1 \) bps/Hz due to the error of magnifying and shrinking of inequation in (11), (15) and (24). In addition, the rate gap between the cases using the channel
statistical codebook and RVQ-based codebook becomes larger and larger, which verifies the superior performance of the channel statistical codebook.

V. CONCLUSIONS

The channel statistical codebook was designed to significantly reduce the channel feedback overhead, where the number of feedback bits only scales linearly with the rank of the channel correlation matrix. However, this attractive conclusion was only proved under a particular channel assumption in the literature, while no rigorous theoretical proof was provided under a general channel assumption. To fill in the blank of both theoretical proof and simulation results under a general channel assumption, in this paper, we quantitatively analyzed the performance of the channel statistics-based codebook. We found that the rate gap between the ideal case of perfect CSIT and using the channel statistics-based codebook is dependent on the channel quantization error. Then, we derived an upper bound of the quantization error, based on which we proved that the required feedback bits scales linearly with the rank of the channel correlation matrix. To the best of our knowledge, our work is the first one to provide a rigorous proof of this conclusion. The proof techniques in this paper can also be used to analyze the performance of other codebooks.

VI. APPENDIX A

The CDF of the squared cosine of the angle between two vectors $\mathbf{G} \mathbf{g}$ and $\mathbf{G} \mathbf{v}$ in the hyper-ellipse is upper bounded as:

$$\Pr\left\{ \frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\} < \Pr\left\{ \frac{|g^Hv|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\}. \quad (31)$$

**Proof:** Firstly, we consider the squared cosine of the angle $|g^H\Gamma g|^2$ between two isotropic vectors $g$ and $v$ uniformly distributed in a hyper-sphere. For $z \in [0, 1]$, the CDF of $\frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2}$ is given by (32)

$$\Pr\left\{ \frac{|g^Hv|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\} = 1 - (1 - z)^{r-2}. \quad (32)$$

For $|g^H\Gamma g|^2$ in (31), the elements of $g$ and $v$ are multiplied by the diagonal elements $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ of $\Gamma$. The uniformly distributed vectors $g$ and $v$ in hyper-sphere are projected onto non-uniformly distributed vectors $\mathbf{G}g$ and $\mathbf{G}v$ in hyper-ellipse, which is generated by stretching the hyper-sphere. That means the angle between two vectors in hyper-ellipse tends to be smaller, i.e., the squared cosine of the angle between two vectors in hyper-ellipse tends to be larger. Thus, the CDF of $\frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2}$ is smaller than that of $\frac{|g^Hv|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2}$, i.e.,

$$\Pr\left\{ \frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\} < \Pr\left\{ \frac{|g^Hv|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\}. \quad (33)$$

Now we give an intuitive example of an extreme case where $\sigma_1 = \sigma_2 = \cdots, \sigma_{r-1}$ and $\sigma_r \ll \sigma_2$. The $r$-dimensional hyper-ellipse is generated by compress the last dimension $r$-dimensional hyper-sphere. Thus, the vectors in hyper-sphere are projected onto a compressed hyper-sphere, i.e., the hyper-ellipse. The angle between two vectors in hyper-ellipse becomes smaller. Mathematically, the numerator of the squared cosine can be approximately expressed as

$$|g^H\Gamma g|^2 \approx |\sigma_1|^2 |\Sigma_{i=1}^{r-1} g_i^* v_i|^2. \quad (34)$$

The denominator of the squared cosine can be approximately expressed as

$$||\mathbf{G}g||^2 ||\mathbf{G}v||^2 \approx |\sigma_1|^2 |\Sigma_{i=1}^{r-1} g_i^2| |\Sigma_{i=1}^{r-1} v_i^2|. \quad (35)$$

Thus, we have

$$\frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \approx \frac{|\Sigma_{i=1}^{r-1} g_i v_i^2|}{(\Sigma_{i=1}^{r-1} g_i^2)(\Sigma_{i=1}^{r-1} v_i^2)}. \quad (36)$$

Considering (32), we can get

$$\Pr\left\{ \frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\} \approx 1 - (1 - z)^{r-2}. \quad (37)$$

Utilizing the fact that $1 - (1 - z)^{r-2} < 1 - (1 - z)^{r-1}$, we have

$$\Pr\left\{ \frac{|g^H\Gamma g|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\} < \Pr\left\{ \frac{|g^Hv|^2}{||\mathbf{G}||^2 ||\mathbf{G}v||^2} \leq z \right\}. \quad (38)$$

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