Factor Analysis of Moving Average Processes

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Abstract—The paper considers an extension of factor analysis to moving average processes. The problem is formulated as a rank minimization of a suitable spectral density. It is shown that it can be adequately approximated via a trace norm convex relaxation.

I. INTRODUCTION

Factor models are used to compress information contained in a high dimensional data vector into a small number of common factors. Those common factors represent nonobserved variables influencing the observations. Such models have been initially developed by psychologists for statistical tests of mental abilities, [25], [3], [27] and successively in econometrics and control engineering, [18], [17], [24], [20], [12], [5].

The “standard” factor model is a zero mean Gaussian $n$-dimensional random vector whose covariance matrix $X$ can be decomposed as the sum of a low rank covariance matrix $Y$ plus a diagonal covariance matrix $Z$, i.e. $X = Y + Z$. Here $X, Y$ and $Z$ belong to the vector space of symmetric matrices of dimension $n$, say $Q_n$, and are positive semidefinite, say $X,Y,Z \succeq 0$. $X$ encodes the observed information from data, $Y$ the compressed information through $r := \text{rank}(Y)$ independent common factors, and $Z$ the information which cannot be compressed. The estimation of $Y$ from $X$, leads to a constrained rank minimization of $Y$, a nonconvex problem and computationally NP hard. It admits a tight convex relaxation which minimizes the trace of $Y$ and was introduced in factor analysis by Jackson and Agunwamba, [16], and independently Bentler and Woodward, [2]. Likewise, convex relaxations of rank minimization problems have been a topic of active research in the recent years [10], [11], [4], [22].

In the above formulation data are modeled as originating from independent, identically distributed Gaussian random vectors and factor analysis consists only in dimension reduction in the cross-sectional dimension (i.e. the number of observed variables). A generalization is to assume that data originate from a stochastic process, thus compressing information in the cross-sectional and in the time dimension,

Although the estimation of an MA processes is relatively simple, it is not clear how to extract the compressible information from it. The present paper considers the case of a dynamic factor model generated by an MA process. We show that the convex relaxation for the static case then admits a natural generalization wherein the covariance matrix $X$ is replaced by the spectral density of the MA process. In Section II we recall the standard factor analysis. In Section III we introduce its dynamic MA generalization. In Section IV we analyze the constrained convex optimization problem which relaxes the corresponding minimum rank problem, and in Section V we propose a matricial SDP algorithm for computing a solution to the problem. Finally, in Section VI we present some simulation studies.

Throughout the paper we use the following notation. Functions defined on the unit circle are denoted by capital Greek letters, e.g. $\Psi(e^{i\vartheta})$ with $\vartheta \in [-\pi,\pi]$. If $\Psi$ is positive semidefinite on the unit circle we write $\Psi \succeq 0$. $A_n$ is the linear space of $C^{n\times n}$-valued analytic functions defined on the unit circle. The (normal) rank of $\Psi \in A_n$ is defined as

$$\text{rank}(\Psi) = \max_{\vartheta \in [-\pi,\pi]} \text{rank}(\Psi(e^{i\vartheta})).$$

We define the norm

$$||\Psi|| := \max_{\vartheta \in [-\pi,\pi]} \sigma(\Psi(e^{i\vartheta}))$$

where $\sigma(X)$ denotes the maximum singular value of the matrix $X$, which is equal to its maximum eigenvalue when $X \succeq 0$.

II. STANDARD FACTOR ANALYSIS

The standard factor model is a static linear model

$$x = Aw_y + Bw_z$$

where $A \in \mathbb{R}^{n \times r}$ with $r \ll n$, $B \in \mathbb{R}^{n \times n}$ diagonal, $w_y$ and $w_z$ are Gaussian random vectors with zero mean and covariance matrix equal to the identity of dimension $r$.
and \( n \), respectively. Moreover, \( w_y \) and \( w_z \) are independent, i.e. \( \mathbb{E}[w_y w_z^T] = 0 \). Let \( y := Aw_y \) and \( z := Bw_z \). The \( n \)-dimensional random vector \( x \) is called observed vector because some statistics of it are available. To explain the reason why (1) is referred to as factor model, we need some further notation. Let \( x = [x_1 \ldots x_n]^T, w_y = [w_{y,1} \ldots w_{y,r}]^T, w_z = [w_{z,1} \ldots w_{z,n}]^T, a_{jk} \) denote the entry in position \((j,k)\) of the matrix \( A \), and \( b_j \) denote the \( j \)-th entry in the main diagonal of \( B \). Therefore, \( x_j = \sum_{k=1}^{r} a_{jk} w_{y,k} + b_j w_{z,j} \)

namely the \( j \)-th observed variable \( x_j \) is generated by \( r \) independent common factors \( w_{y,1}, \ldots, w_{y,r} \) and by the specific factor \( w_{z,j} \).

In view of (1), \( x \) is a Gaussian random vector with zero mean and covariance matrix denoted by \( X \). Since \( w_y \) and \( w_z \) are independent, we get

\[
X = Y + Z
\]

where \( Y = AA^T \), with rank equal to \( r \), and \( Z = BB^T \), diagonal, are the covariance matrices of \( y \) and \( z \), respectively. Therefore, the covariance matrix of \( x \) is the sum of a low rank covariance matrix, describing the common factors, and a diagonal covariance matrix, describing the specific factors.

The purpose of factors analysis consists in characterizing common factors, representing the compressed information, and specific factors from some statistics of the observed vector. This problem can be formalized as follows: find the decomposition low rank plus diagonal (2) from an estimate of \( X \). A natural strategy for factor analysis is to solve the following minimum rank problem

\[
\min_{Y,Z \in \mathbb{Q}_n} \text{rank}(Y) \quad \text{subject to} \quad \begin{array}{l} Y, Z \succeq 0 \\ Z \text{ diagonal} \\ Y = Y + Z. \end{array}
\]

This problem, however, is computationally NP-hard. Then, the following minimum trace problem has been proposed as approximation of (3)

\[
\min_{Y,Z \in \mathbb{Q}_n} \text{tr}(Y) \quad \text{subject to} \quad \begin{array}{l} Y, Z \succeq 0 \\ Z \text{ diagonal} \\ Y = Y + Z. \end{array}
\]

It turns out that the relaxed problem recovers the correct decomposition under weak assumptions. Moreover, its solution is unique, [7]. The reason why the approximation is effective is because the convex hull of \( \text{rank}(Y) \) over the set \( \{ Y \in \mathbb{Q}_n \text{ s.t. } Y \succeq 0, \|Y\|_2 \leq 1 \} \) is the trace of \( Y \), here \( \|Y\|_2 \) denotes the spectral norm of \( Y \), see [10].

III. Moving Average Factors Analysis

In this paper we consider the following dynamic factor model

\[
x(t) = \sum_{k=0}^{m} A_k w_y(t - k) + \sum_{k=0}^{m} B_k w_z(t - k), \quad t \in \mathbb{Z}
\]

where \( A_k \in \mathbb{R}^{n \times r} \) with \( r \ll n \), \( B_k \in \mathbb{R}^{n \times n} \) diagonal, \( w_y \) and \( w_z \) are \( r \) and \( n \)-dimensional white Gaussian noise with zero mean and variance equal to the identity, respectively, and such that

\[
\mathbb{E}[w_y(t)w_z(s)^T] = 0 \quad \forall \ t, s.
\]

Note that, (4) is a linear combination of two white Gaussian noises, whereas a standard MA process involves just one noise term. On the other hand, only with (4) one characterizes the compressible information, and thus the common factors, in the data. Similarly to the static case, we define the stochastic processes

\[
y(t) := \sum_{k=0}^{m} A_k w_y(t - k),
\]

\[
z(t) := \sum_{k=0}^{m} B_k w_z(t - k).
\]

Model (4) is completely described by its spectral density

\[
\Psi_x(e^{i\vartheta}) = \sum_{k=-m}^{m} e^{-ik\vartheta} R_k
\]

where \( R_k := \mathbb{E}[x(t)x(t+k)^T] \) is the \( k \)-th covariance lag of \( x \). In view of (5), we get

\[
\Psi_x(e^{i\vartheta}) = \Psi_y(e^{i\vartheta}) + \Psi_z(e^{i\vartheta})
\]

where \( \Psi_y \) is the spectral density of \( y \) and \( \Psi_z \) the one of \( z \). Moreover, from (6) and (7) we obtain

\[
\Psi_y(e^{i\vartheta}) = \left( \sum_{k=0}^{m} e^{-ik\vartheta} A_k \right) \left( \sum_{k=0}^{m} e^{-ik\vartheta} A_k^* \right)
\]

\[
\Psi_z(e^{i\vartheta}) = \left( \sum_{k=0}^{m} e^{-ik\vartheta} B_k \right) \left( \sum_{k=0}^{m} e^{-ik\vartheta} B_k^* \right).
\]

Therefore, \( \Psi_x, \Psi_y \) and \( \Psi_z \) belong to the following family of pseudo-polynomial matrices in \( e^{i\vartheta} \):

\[
Q_{n,m} = \left\{ \sum_{k=-m}^{m} e^{-ik\vartheta} Q_k, \ Q_k = Q_k^T \in \mathbb{R}^{n \times n} \right\}.
\]

Moreover spectral densities must be positive semidefinite on the unit circle, hence \( \Psi_x, \Psi_y, \Psi_z \succeq 0 \). Since \( A_k \in \mathbb{R}^{n \times r} \) and \( B_k \) is diagonal, \( \Psi_y \) has (normal) rank \( r \) and \( \Psi_z \) is diagonal. We conclude that the observed process \( x \) of the MA factor model (4) has spectral density \( \Psi_x \) which is given by the sum of a low rank and a diagonal matrix function belonging to \( Q_{n,m} \) and positive semidefinite on the unit circle.

Factor analysis of the model (4) can be formulated as follows.
**Problem 1:** Let \( x(1), x(2), \ldots, x(N) \) be a finite-length sequence extracted from a realization of \( x \) and assume that \( m \) is given. Find the decomposition low rank plus diagonal \( \Psi \) from \( x(1), x(2), \ldots, x(N) \).

In the above problem we assumed to know \( m \). If not, one can estimate \( m \) from the data by using model order selection techniques, see for instance [14].

We propose the following identification procedure for finding \( \Psi \):

1. Estimate \( \Psi_y(e^{i\vartheta}) = \sum_{k=-m}^{m} e^{-i k \vartheta} R_k \), such that \( \Psi_y \geq 0 \), from \( x(1), x(2), \ldots, x(N) \);
2. Compute \( \Psi_y \) and \( \Psi_z \) by solving the following minimum rank problem

\[
\min_{\Psi_y, \Psi_z \in \mathbb{Q}_{n,m}} \text{rank}(\Psi_y) \quad \text{subject to} \quad \begin{cases} \Psi_y, \Psi_z \geq 0 \quad & \Psi_z \text{ diagonal} \\ \Psi_x = \Psi_y + \Psi_z. \end{cases}
\]

**Step 1.** This is an MA parameter estimation problem. One would compute the correlogram of \( x \) and then truncate it with a \( m \)-length rectangular window according to the Blackman-Tukey method, [26, page 38]. However, the truncated estimate may fail to be positive semidefinite over the unit circle, especially when \( m \ll N \), that is it is not a spectral density. One could overcome this problem designing a window which preserves the positivity of the windowed correlogram. The design of this window depends on the specific application. Since this is not the main issue we address in the paper, we consider the Durbin’s method, [8]. The sketch of this procedure is as follows:

- Fit an autoregressive (AR) model of order \( \hat{m} = 2m \) from \( x(1), x(2), \ldots, x(N) \);
- Approximate the AR model with an MA model, of order \( m \), via the least square method;
- Let \( x(t) = \sum_{k=0}^{m} C_k e(t-k) \) be the estimated MA model where \( e \) is white Gaussian noise with zero mean and covariance matrix equal to the identity, then

\[
\Psi_x(e^{i\vartheta}) = \left( \sum_{k=0}^{m} C_k e^{-i k \vartheta} \right)^* \left( \sum_{k=0}^{m} C_k e^{-i k \vartheta} \right) = (\sum_{k=0}^{m} \sum_{\vartheta} \Psi_y(e^{i\vartheta}) e^{-i k \vartheta}),
\]

**Step 2.** The minimum rank Problem \( \Psi \) is computationally NP-hard. This lead us to relax it in a such way to obtain a tractable convex optimization-based method. More precisely, we would like to approximate the rank function with a convex function. Next section is devoted to this task.

**Remark 3.1:** We assumed that the MA processes \( \Psi_y \) and \( \Psi_z \) have the same order, that is \( m \), for simplicity. Let \( \Psi_y \in \mathbb{Q}_{n,m} \), \( \Psi_y \in \mathbb{Q}_{n,m} \) and \( \Psi_z \in \mathbb{Q}_{n,m} \). Clearly, \( m_x \) is set in Step 1 in the above procedure. If we choose \( m_x > \min\{m_y, m_z\} \), the semi-definite decomposition \( \Psi_x = \Psi_y + \Psi_z \), with \( \Psi_z \) diagonal, may not exist or the solution for \( \Psi_y \) may be trivially full rank. If we choose \( m_x = \min\{m_y, m_z\} \) such decomposition does exists, but it implies that \( \Psi_y \) and \( \Psi_z \) belong to \( \mathbb{Q}_{n,m_x} \). The unique interesting case is \( m_x < \min\{m_y, m_z\} \). Note that, it is not difficult to construct examples where the low rank plus diagonal semi-definite decomposition \( \Psi_x = \Psi_y + \Psi_z \) is such that \( m_x < \min\{m_y, m_z\} \). Without loss of generality assume that \( m_x \leq m_y \leq m_z \). Then, \( \Psi_x \) and \( \Psi_y \) can be understood as elements in \( \mathbb{Q}_{n,m_x} \), setting the last \( m_z - m_x \) lags of \( \Psi_x \) equal to zero and with the linear constraint that the last \( m_z - m_y \) lags of \( \Psi_y \) are equal to zero. Accordingly, the results we will present can be easily adapted to this case. It is worth noting condition \( m_x < \min\{m_y, m_z\} \) means that we permit an “expansion” in the time dimension, with respect to \( x \), to allow a more effective compression in the cross-sectional dimension.

**IV. RELAXED MINIMUM RANK PROBLEM**

By replacing \( \text{rank}(\Psi_y) \) in \( \Psi \) with a convex function we obtain a constrained convex optimization problem. The tightest convex lower approximation of a nonconvex function is defined as follows.

**Definition 4.1:** Given \( f : \mathcal{D} \to [-\infty, \infty] \), the convex hull \( \text{co} f \) is defined as the greatest convex function such that

\[
\text{co} f(x) \leq f(x), \quad \forall x \in \mathcal{D}.
\]

In [29], we prove the following result.

**Proposition 4.1:** Let \( \Psi_y \in \mathcal{A}_n \) be an arbitrary analytic function such that \( \Psi_y \geq 0 \), we define the following restricted rank function

\[
\text{rank}'(\Psi_y) := \begin{cases} \text{rank}(\Psi_y), & \|\Psi_y\| \leq 1 \\ +\infty, & \text{otherwise}. \end{cases}
\]

Then, the convex hull of \( \text{rank}'(\Psi_y) \) is

\[
\text{co} \text{rank}'(\Psi_y) := \begin{cases} \text{tr} \int_{-\pi}^{\pi} \Psi_y(e^{i\vartheta}) \frac{d\vartheta}{2\pi}, & \|\Psi_y\| \leq 1 \\ +\infty, & \text{otherwise}. \end{cases}
\]

Consider the following convex optimization problem:

\[
\min_{\Psi_y, \Psi_z \in \mathbb{Q}_{n,m}} \text{tr} \int_{-\pi}^{\pi} \Psi_y(e^{i\vartheta}) \frac{d\vartheta}{2\pi} \quad \text{subject to} \quad \begin{cases} \Psi_y, \Psi_z \geq 0 \\ \Psi_z \text{ diagonal} \\ \Psi_x = \Psi_y + \Psi_z. \end{cases}
\]

**Proposition 4.2:** Assume that \( \Psi_x \in \mathbb{Q}_{n,m} \) is positive semidefinite and bounded on the unit circle. Then Problem \( \text{[10]} \) does admit solution.

**Proof:** In \( \text{[10]} \), \( \Psi_z = \Psi_x - \Psi_y \) does not appear in the objective function, thus the optimization problem is equivalent to

\[
\min_{\Psi_y \in \mathbb{Q}_{n,m}} \text{tr} \int_{-\pi}^{\pi} \Psi_y(e^{i\vartheta}) \frac{d\vartheta}{2\pi} \quad \text{subject to} \quad \begin{cases} 0 \leq \Psi_y \leq \Psi_x \\ \Psi_x - \Psi_y \text{ diagonal}. \end{cases}
\]

Hence, it is sufficient to show that \( \text{[11]} \) admits solution for proving the statement.
The point \( \Psi_y = \Psi_x \) is feasible for Problem \([\text{II}]\) because \( \Psi_x \succeq 0 \) by assumption. Thus, the feasibility set
\[
K = \{ \Psi_y \in \mathbb{Q}_{n,m} \text{ s.t. } 0 \preceq \Psi_y \preceq \Psi_x, \Psi_x - \Psi_y \text{ diagonal} \}
\]
is nonempty. Moreover, \( K \) is bounded, closed and contained in the finite dimensional space \( \mathbb{Q}_{n,m} \). Accordingly, \( K \) is a compact set. Since the objective function in \([\text{II}]\) is a continuous function, by Weierstrass’ theorem Problem \([\text{II}]\) admits a minimum over \( K \).

By Proposition \([3.2]\), Problem \([10]\) with \( \Psi_z \) estimated as explained in Section \([\text{III}]\) admits a solution. Define \( c := \|\Psi_x\| \). Then, Problem \([10]\) is equivalent to
\[
\begin{align*}
\min_{\Psi_y \in \mathbb{Q}_{n,m}} & \quad \frac{1}{c} \text{tr} \int_{-\pi}^{\pi} \Psi_y(e^{i\theta}) \frac{d\theta}{2\pi} \\
\text{subject to} & \quad 0 \preceq \Psi_y \preceq \Psi_x \\
& \quad \Psi_x - \Psi_y \text{ diagonal}
\end{align*}
\]
Note that the feasibility set is contained in \( \bar{K} := \{ \Psi_y \in \mathbb{Q}_{n,m} \text{ s.t. } \|\Psi_y\| \leq c \} \) and \( \frac{1}{c} \text{tr} \int_{-\pi}^{\pi} \Psi_y(e^{i\theta}) \frac{d\theta}{2\pi} \) is the convex hull of \( \text{rank}(\Psi_y) \) over \( \bar{K} \). We conclude that \( \|\Psi_y\| \) is the convex relaxation of the minimum rank Problem \([9]\).

V. A MATRICAL SDP ALGORITHM

The computation of a solution to Problem \([10]\) requires a matrix parametrization of the problem. To this end, we consider \( Y \in \mathbb{Q}_{n(m+1)} \) partitioned as follows
\[
Y = \begin{bmatrix}
Y_{00} & Y_{01} & \ldots & Y_{0m} \\
Y_{10} & Y_{11} & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
Y_{m0} & \ldots & \ldots & Y_{mm}
\end{bmatrix}
\]
and define the shift operator
\[
\Delta(e^{i\theta}) = \begin{bmatrix}
I_n & e^{i\theta}I_n & \ldots & e^{im\theta}I_n
\end{bmatrix}
\]
Then
\[
\Delta(e^{i\theta})Y \Delta(e^{i\theta})^* = D_0(Y) + \sum_{k=1}^{m} D_k(Y)e^{-ik\theta} + D_k(Y)^T e^{ik\theta}
\]
where
\[
\begin{align*}
D_0 & : \mathbb{Q}_{n(m+1)} \to \mathbb{Q}_n \quad Y \mapsto \sum_{j=0}^{m} Y_{jj} \\
D_k & : \mathbb{Q}_{n(m+1)} \to \mathbb{R}^{n \times n} \quad Y \mapsto \sum_{j=0}^{m} Y_{j,j+k}
\end{align*}
\]
where \( k = 1 \ldots m \). Therefore, \( \Delta(e^{i\theta})Y \Delta(e^{i\theta})^* \in \mathbb{Q}_{n,m} \). Moreover, any element in \( \mathbb{Q}_{n,m} \) admits the representation \([12]\) because \( D_k \)s are surjective maps and \( D_j(Y), D_k(Y) \) with \( j \neq k \) depend on different subblocks of \( Y \).

This leads us to parameterize \( \Psi_y, \Psi_z \in \mathbb{Q}_{n,m} \) as
\[
\begin{align*}
\Psi_y(e^{i\theta}) &= \Delta(e^{i\theta})Y \Delta(e^{i\theta})^* \\
\Psi_z(e^{i\theta}) &= \Delta(e^{i\theta})Z \Delta(e^{i\theta})^*
\end{align*}
\]
with \( Y, Z \in \mathbb{Q}_{n(m+1)} \), and translate \([10]\) with respect to such matrices:

- **Objective function.** We have
\[
\begin{align*}
\text{tr} \int_{-\pi}^{\pi} & \Psi_y(e^{i\theta}) \frac{d\theta}{2\pi} = \text{tr}(\Delta(e^{i\theta})Y \Delta(e^{i\theta})^*) \\
& = \text{tr} \left( Y \int_{-\pi}^{\pi} \Delta(e^{i\theta})^* \Delta(e^{i\theta}) \frac{d\theta}{2\pi} \right) = \text{tr}(Y)
\end{align*}
\]
where we exploited the fact that
\[
\int_{-\pi}^{\pi} e^{ik\theta} \frac{d\theta}{2\pi} = \begin{cases} 
1, & k = 0 \\
0, & k \neq 0.
\end{cases}
\]

- **Conditions.** \( \Psi_y, \Psi_z \succeq 0. \) The condition \( Y \succeq 0 \) implies that \( \Psi_y(e^{i\theta}) = \Delta(e^{i\theta})Y \Delta(e^{i\theta})^* \succeq 0 \) for each \( \theta \in [-\pi, \pi] \). On the other hand if \( \Psi_y \succeq 0 \), there exists \( \Gamma(e^{i\theta}) = \sum_{k=0}^{m} e^{-ik\theta} C_k \), with \( C_k \in \mathbb{R}^{n \times l} \), such that \( \Psi_y(e^{i\theta}) = \Gamma(e^{i\theta}) \Gamma(e^{i\theta})^* \). Hence, \( \Psi_y(e^{i\theta}) = \Delta(e^{i\theta})Y \Delta(e^{i\theta})^* \) with
\[
Y = \begin{bmatrix}
C_0 \\
C_1 \\
\vdots \\
C_m
\end{bmatrix} = C_0^T C_1^T \cdots C_m^T
\]
which is positive semidefinite. Thus, we can replace \( \Psi_y \succeq 0 \) with \( Y \succeq 0 \), and similarly \( \Psi_z \succeq 0 \) with \( Z \succeq 0 \).

- **Condition \( \Psi_y = \Psi_y + \Psi_z \).** Let \( \Psi_z(e^{i\theta}) = \sum_{k=-m}^{m} e^{-ik\theta} R_k \), \( \Psi_y(e^{i\theta}) = \sum_{k=-m}^{m} e^{-ik\theta} P_k \), \( \Psi_z(e^{i\theta}) = \sum_{k=-m}^{m} e^{-ik\theta} Q_k \) with \( R_k = R_k^T \), \( P_k = P_k^T \), \( Q_k = Q_k^T \). Thus, the equality constraint may be rewritten as \( \sum_{k=-m}^{m} e^{-ik\theta} (P_k + Q_k - R_k) = 0 \) which is equivalent to
\[
P_k + Q_k = R_k, \quad k = 0 \ldots m.
\]

Note that \( R_k \)s, \( P_k \)s and \( Q_k \)s are the coefficients of the pseudo-polynomial matrices \( \Psi_y \) and \( \Psi_z \), respectively. In view of \([12]\), the \( k \)-th coefficients of \( \Psi_y \) and \( \Psi_z \) are given by \( D_k(Y) \) and \( D_k(Z) \), respectively. Thus, \([13]\) is equivalent to
\[
D_k(Y) + D_k(Z) = R_k, \quad k = 0 \ldots m.
\]
Finally, by exploiting the linearity of \( D_k \)s, we obtain
\[
D_k(Y + Z) = R_k, \quad k = 0 \ldots m.
\]

- **Condition \( \Psi_z \) diagonal.** By exploiting argumentations similar to the ones of the previous point, we get that the condition is equivalent to
\[
D_k(Z) \text{ diagonal}, \quad k = 0 \ldots m.
\]
Hence, Problem (10) is equivalent to
\[
\min_{Y,Z \in Q_{n,(m+1)}} \text{tr}(Y)
\]
subject to
\[
D_k(Z) \text{ diagonal } k = 0 \ldots m \\
D_k(Y + Z) = R_k \text{ } k = 0 \ldots m
\]
and a solution to (10) is given by \( \hat{\Psi} \) and a solution to (10) is given by (10). We compute the relative estimation error, \( n \) \[6\]. In our case \( \hat{\Psi} \) and \( \hat{\Psi} \) are generated. These models differ by the number of common factors, i.e. \( r \). For each model we solve problem (10) by using the true spectral density \( \Psi \) and \( \Psi \) the estimate provided by (10). We compute the relative estimation error, averaged on the unit circle, of \( \hat{\Psi} \):
\[
m_{\hat{\Psi},y} = \int_{-\pi}^{\pi} \frac{\|\Psi_y(e^{i\theta}) - \hat{\Psi}_y(e^{i\theta})\|_2}{\|\Psi_y(e^{i\theta})\|_2} \ 2\pi
\]
In the following table the relative error \( m_{\hat{\Psi},y} \) for the 10 different models is shown.

| \( r \) | \( m_{\hat{\Psi},y} \) |
|---|---|
| 1 | 8.72 \times 10^{-10} |
| 2 | 4.56 \times 10^{-10} |
| 3 | 1.76 \times 10^{-9} |
| 4 | 1.25 \times 10^{-9} |
| 5 | 1.63 \times 10^{-9} |

The estimated decomposition is exact when the number of common factors is small, \( r \leq 5 \), whereas it is not for a large number of common factors, \( r \geq 6 \). At this point it is worth recalling that the decomposition (8) is generically unique, and thus identifiable from \( \Psi_x \), for \( r \leq n - \sqrt{n}, \) [6]. In our case \( n - \sqrt{n} \approx 6.84 \). We conclude that the relaxed formulation is able to recover all the identifiable decompositions except for the case \( r = 6 \).

B. Factor Analysis
We consider the dynamic factor model of Section VI-A with \( r = 3 \). In Figure 1 we depict the spectral norm of \( \Psi_x \), \( \Psi_y \) and \( \Psi_z \) at each frequency. From (4) we generate \( N = 6000 \) samples of \( x: x(1), x(2), \ldots, x(6000) \). We apply then the identification procedure of Section III for characterizing common and specific factors from the data. We denote by \( \hat{\Psi}_x \) the estimate of \( \Psi_x \) computed by Durbin’s method, and \( \hat{\Psi}_y \) the estimate of \( \Psi_y \) obtained by solving Problem (10). We define the relative estimation errors at each frequency:
\[
e_{\Psi_x}(e^{i\theta}) = \frac{\|\Psi_x(e^{i\theta}) - \hat{\Psi}_x(e^{i\theta})\|_2}{\|\Psi_x(e^{i\theta})\|_2}
\]
\[
e_{\Psi_y}(e^{i\theta}) = \frac{\|\Psi_y(e^{i\theta}) - \hat{\Psi}_y(e^{i\theta})\|_2}{\|\Psi_y(e^{i\theta})\|_2}
\]
The errors graph is displayed in Figure 2. We note that \( e_{\Psi_x} \) and \( e_{\Psi_y} \) take similar values for \( \theta \in [-\pi, \pi] \). This means that the estimation error is mainly imputable to Durbin’s method for estimating \( \Psi_x \) from the data. Finally, we define
\[
s_j := \max_{\theta \in [-\pi, \pi]} \frac{\sigma_j(\Psi_y(e^{i\theta}))}{\sigma_1(\Psi_y(e^{i\theta}))}
\]
where \( \sigma_j(\Psi_y(e^{i\theta})) \) is the \( j \)-th largest singular value of \( \hat{\Psi}_y \) at \( \theta \). Hence, \( s_j \) can be understood as the \( j \)-th largest normalized singular value over the unit circle of \( \hat{\Psi}_y \). Those quantities are plotted in Figure 3. The plot suggests that we can...
safely approximate \( \text{rank}(\hat{\Psi}_y) = 3 \). Accordingly, we recover the exact number of common factors. Finally, we obtained similar results with different samples and by changing the factor model.

VII. Conclusion

In this paper we proposed an identification procedure for factor analysis of MA processes. Here, the challenging step is to solve a minimum rank problem. We proposed a convex optimization problem approximating the NP-hard problem. Simulation studies point out that the convex problem is able to recover a correct solution in most of the cases. Finally, we tested the identification procedure; simulations show this method is able to identify common and specific factors with satisfactory accuracy.

REFERENCES

[1] B. Anderson and M. Deistler. Identification of dynamic systems from noisy data: Single factor case. Mathematics of Control, Signals and Systems, 6(1):10–29, 1993.
[2] P. Bentler and J. Woodward. Inequalities among lower bounds to reliability: With applications to test construction and factor analysis. Psychometrika, 45(2):249–267, 1980.
[3] C. Burt. Experimental tests of general intelligence. British Journal of Psychology, 1904-1920, 3(12):94–177, 1909.
[4] E. Candès and B. Recht. Exact matrix completion via convex optimization. Found. Comput. Math., 9(6):717–772, Jun. 2009.
[5] M. Deistler, W. Scherrer, and B. Anderson. The structure of generalized linear dynamic factor models. In Empirical Economic and Financial Research, pages 379–400. Springer, 2015.
[6] M. Deistler and C. Zinner. Modelling high-dimensional time series by generalized linear dynamic factor models: An introductory survey. Communications in Information & Systems, 7(2):153–166, 2007.
[7] G. Della Riccia and A. Shapiro. Minimum rank and minimum trace of covariance matrices. Psychometrika, 47:443–448, 1982.
[8] J. Durbin. Efficient estimation of parameters in moving-average models. Biometrika, 46(3/4):306–316, 1959.
[9] R. Engle and M. Watson. A one-factor multivariate time series model of metropolitan wage rates. Journal of the American Statistical Association, 76(376):774–781, 1981.
[10] M. Fazel. Matrix rank minimization with applications. Elec. Eng. Dept. Stanford University, 54:1–130, 2002.
[11] M. Fazel, H. Hindi, and S. Boyd. Rank minimization and applications in system theory. In Proceedings of the American Control Conference, volume 4, pages 3273–3278, Jun. 2004.
[12] M. Forni and M. Lippi. The generalized dynamic factor model: representation theory. Econometric theory, 17(06):1113–1141, 2001.
[13] J. Geweke. The dynamic factor analysis of economic time series models. In Latent Variables in Socio-Economic Models, SSRI workshop series, pages 365–383. North-Holland, 1977.
[14] G. Goodwin, M. Gevers, and B. Ninness. Quantifying the error in estimated transfer functions with application to model order selection. IEEE Trans. Autom. Control, 37(7):913–928, Jul 1992.
[15] C. Heij, W. Scherrer, and M. Deistler. System identification by dynamic factor models. SIAM Journal on Control and Optimization, 35(6):1924–1951, 1997.
[16] P. Jackson and C. Agunwamba. Lower bounds for the reliability of the total score on a test composed of non-homogeneous items: I: Algebraic lower bounds. Psychometrika, 42(4):567–578, 1977.
[17] R. Kalman. Identifiability and problems of model selection in econometrics. Cambridge University Press, 1983.
[18] W. Ledermann. On the rank of the reduced correlational matrix in multiple-factor analysis. Psychometrika, 2(2):85–93, Jun. 1937.
[19] D. Pena and G. Box. Identifying a simplifying in time series. Journal of the American Statistical Association, 82(399):836–843, Sept. 1987.
[20] G. Picci. Parametrization of factor analysis models. Journal of Econometrics, 41(1):17–38, 1989.
[21] G. Picci and S. Pinzoni. Dynamic factor-analysis models for stationary processes. IMA Journal of Mathematical Control and Information, 3(2-3):185–210, 1986.
[22] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev., 52(3):471–501, Mar. 2010.
[23] T. Sargent and C. Sims. Business cycle modeling without pretending to have too much a priori economic theory. Technical Report 55, Federal Reserve Bank of Minneapolis, 1977.
[24] J. Schuppen. Stochastic realization problems motivated by econometric modeling. In C. Byrnes and A. Lindquist, editors, Modeling Identification and Robust Control, pages 259–275. North-Holland, 1986.
[25] C. Spearman. “General Intelligence.” Objectively Determined and Measured. American Journal of Psychology, 15:201–293, 1904.
[26] P. Stoica and R. Moses. Introduction to spectral analysis. Prentice Hall, New Jersey, 1997.
[27] L. Thurstone. The vectors of the mind. Psychological Review, 41:1–12, 1934.
[28] M. Watson and R. Engle. Alternative algorithms for the estimation of dynamic factor, mimic and varying coefficient regression models. Journal of Econometrics, 23(3):385–400, Dec. 1983.
[29] M. Zorzi and R. Sepulchre. AR identification of Latent-variable Graphical models. [http://arxiv.org/abs/1405.0027] 2014.