Relativistic covariance of Ohm’s law

R. Starke
Department of Computational Materials Physics, University of Vienna,
Sensengasse 8/12, 1090 Vienna, Austria and
Institute for Theoretical Physics, TU Bergakademie Freiberg, Leipziger Straße 23, 09596 Freiberg, Germany

G. A. H. Schober
Institute for Theoretical Physics, University of Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany

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The derivation of relativistic generalizations of Ohm’s law has been a long-term issue in theoretical physics with deep implications for the study of relativistic plasmas in astrophysics and cosmology. Here we propose an alternative route to this problem by introducing the most general Lorentz covariant first order response law, which is written in terms of the fundamental response tensor \( \chi_{\mu\nu} \) relating induced four-currents to external four-potentials. We show the equivalence of this description to Ohm’s law and thereby prove the validity of Ohm’s law in every frame of reference. We further use the universal relation between \( \chi_{\mu\nu} \) and the microscopic conductivity tensor \( \sigma_{kl} \) to derive a fully relativistic transformation law for the latter, which includes all effects of anisotropy and relativistic retardation. In the special case of a constant, scalar conductivity this transformation law reproduces a standard textbook generalization of Ohm’s law.

Introduction.—Formulated as a simple, yet ingenious mathematical identity, capable of explaining a plethora of experimental data at the time of its discovery, Ohm’s law [1] soon found its technological application in the engineering of nineteenth century telegraph systems [2, 3]. Since then it has been used in nearly every branch of physical sciences to describe such different systems as neuron cells in medical physiology [4, 5], black hole membranes in astrophysics [6, 7], and recently also strongly interacting 2 + 1-dimensional conformal field theories with anti-de Sitter space duals [8]. While it was long believed that Ohm’s law would break down at the atomic scale, it was demonstrated in 2012 to hold in silicon wires only four atoms wide [9], thus raising the prospect of further applications in atomic-scale logic circuits [10–12]. On the theoretical side, the problem of deriving Ohm’s law from microscopic models (as formulated e.g. by Peierls [13]) is attracting continuous interest [14]. In plasma physics, a generalized Ohm’s law is used to describe an electrically conducting moving medium in the presence of magnetic fields [15–17]. Indeed, in the magnetohydrodynamic description, where Maxwell equations govern the electromagnetic fields while the fluid is subject to energy and momentum conservation, Ohm’s law expresses the coupling between the electromagnetic fields and the fluid variables [18]. This becomes especially relevant in astrophysics and cosmology, where relativistic plasmas are used to describe the formation of black holes, the generation of jets and gravitational waves [19] as well as the early universe [20]. Consequently, an intense research activity has been focusing on the derivation of relativistic generalizations of Ohm’s law [18, 19, 21–28], a problem which has gradually reached the modern textbook literature (see e.g. [29, Sec. 13.14]).

The present paper aims at delivering an alternative route to this problem: We will show that, contrary to the naive intuition, Ohm’s law is—on the microscopic scale—already relativistically covariant, hence it has the same form in every frame of reference. In particular, this implies that Ohm’s law has to be complemented by a suitable (non-tensorial) transformation law for the conductivity tensor, which we will derive explicitly below. In fact, on a macroscopic scale Ohm’s law relates the electric current density \( \mathbf{j} \) to an externally applied electric field \( \mathbf{E} \) by means of

\[
\mathbf{j} = \sigma \mathbf{E}.
\]

Microscopically, this has to be interpreted in the most general case as a non-local convolution

\[
j_{kl}(x) = \int \, dx' \, \sigma_{kl}(x, x') E_l(x'),
\]

where \( x \equiv x^\mu = (ct, \mathbf{x}) \) and \( dx^4 = dx^0 dx^3 dx^3 \). We choose the Minkowski metric as \( \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) such that all spatial indices can be written as lower indices, and we sum over all doubly appearing indices. From the relativistic point of view, the apparent problem with Ohm’s law is that it relates the spatial part \( j^\mu \) of the four-vector \( j^\mu = (c \rho, \mathbf{j}) \) to the spatial three vector \( E_l = c F_{0l} \), which is part of the second rank field strength tensor \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \). Hence, it is not obvious how Eq. 2 squares with the usual relativistic transformation laws. To clarify this issue on a fundamental level, we will start instead from the linear relation

\[
j^\mu(x) = \int \, dx \, \chi^\mu_{\nu}(x, x') A^\nu(x'),
\]

with the fundamental response tensor \( \chi^\mu_{\nu} \) relating the induced four-current \( j^\mu \) to the applied four-potential \( A^\nu = (\varphi/c, \mathbf{A}) \). In fact, as the four-potential contains the complete information about the externally applied fields,
Eq. (3) constitutes the most general first order response relation, which incorporates all effects of inhomogeneity, anisotropy and relativistic retardation[31]. This relation is relativistically covariant per constructionem because it relates the relativistic four-vectors $j^\mu$ and $A^\nu$. In the following, we will first establish the connection between $\chi^{\mu\nu}$ and $\sigma_{kl}$ and then derive Ohm’s law (2) in a relativistic setting from Eq. (3). Based on the tensorial transformation law for the fundamental response tensor, we will further deduce the most general relativistic transformation law for the conductivity tensor. In particular, this will allow us to express Ohm’s law in a moving frame in terms of the conductivity tensor of the rest frame of the medium. In the special case of a constant, scalar conductivity, we will show that the resulting equation coincides with a standard textbook generalization of Ohm’s law.

Relativistic derivation of Ohm’s law.—As mentioned in the introduction, all (linear) electromagnetic response properties can be derived from the fundamental response tensor, which can be defined as the functional derivative of the induced four-current with respect to the external four-potential [30],

$$\chi^{\mu\nu}(x, x') = \frac{\delta j^\mu(x)}{\delta A^\nu(x')}.$$  

(4)

The current $j^\mu$ has to be invariant under gauge transformations $A^\mu \rightarrow A^\mu + \partial^\mu f$ and fulfill the continuity equation $\partial_\mu j^\mu = 0$, which implies that the fundamental response functions obey the constraints [31, 32],

$$\partial_\mu \chi^{\mu\nu}(x, x') = 0, \quad \partial^\nu \chi^{\mu\nu}(x, x') = 0.$$  

(5)

Assuming homogeneity in time, such that $\chi^{\mu\nu}(x, x') = \chi^{\mu\nu}(x', x; t - t')$, we can write these constraints alternatively as

$$\chi^0_\ell(x, x'; \omega) = \frac{c}{i \omega} \partial_{x_k} \chi_{\ell k}(x, x'; \omega),$$  

(6)

$$\chi_{k0}(x, x'; \omega) = \frac{c}{i \omega} \partial_{x_\ell} \chi_{\ell k}(x, x'; \omega),$$  

(7)

$$\chi^0_0(x, x'; \omega) = \frac{c^2}{\omega^2} \partial_{x_k} \partial_{x_\ell} \chi_{k \ell}(x, x'; \omega).$$  

(8)

Thus, the relations (6)–(8) allow for the reconstruction of the complete fundamental response tensor $\chi^{\mu\nu}$, from its spatial part $\chi_{\ell k}$ only.

In contrast to the fundamental response tensor, the conductivity tensor in Eq. (2) relates the spatial current to the observable electric field. Since a time dependent electric field is in general also accompanied by a magnetic field, the effect of the latter is already contained in the microscopic conductivity tensor, which can hence be characterized as the total functional derivative (see [30] for a discussion of this concept)

$$\sigma_{kl}(x, x'; t - t') = \frac{d j_k(x, t)}{d E_l(x', t')}.$$  

(9)

$$= \frac{d j_k(x, t)}{d E_l(x', t')} + \int d^4 y \int ds \frac{d j_k(x, t)}{d B_l(y, s)} \frac{d B_l(y, s)}{d E_l(x', t')}.$$  

(10)

We now rederive the standard relation between the conductivity tensor $\sigma_{kl}$ and the spatial part of the fundamental response tensor $\chi_{kl}$ (see e.g. [33, 34]),

$$\chi_{kl}(x, x'; \omega) = i \omega \sigma_{kl} \delta A_j(y, x) \chi_{j0}.$$  

(11)

Our purpose here is to provide a fully relativistic and gauge independent derivation of this universal relation. In fact, the functional chain rule (see [30]) implies

$$\sigma_{kl}(x, x') = \int d^4 y \frac{d j_k(x)}{d A_l(y)} \frac{1}{c} \frac{d \phi(y)}{d E_l(x')} + \int d^4 y \frac{d j_k(x)}{d A_j(y)} \frac{d A_j(y)}{d E_l(x')}.$$  

(12)

By Fourier transformation with respect to the time variables, expressing $\chi_{k0}$ through Eq. (7) and using partial integration, we obtain

$$\sigma_{kl}(x, x'; \omega) = \frac{1}{i \omega} \int d^3 y \left\{ \chi_{kj}(x, y; \omega) \frac{d A_j(y, \omega)}{d E_l(x', \omega)} \right\}$$

$$+ \left\{ \frac{1}{i \omega} \frac{\partial}{\partial y_j} \chi_{kj}(x, y; \omega) \frac{d \phi(y, \omega)}{d E_l(x', \omega)} \right\}.$$  

(13)

$$= \frac{1}{i \omega} \int d^3 y \left\{ i \omega \chi_{kj}(x, y; \omega) \frac{d}{d E_l(x', \omega)} \right\}.$$  

(14)

The term in brackets equals $E_j(y, \omega)$, hence

$$\sigma_{kl}(x, x'; \omega) = \frac{1}{i \omega} \int d^3 y \chi_{kj}(x, y; \omega) \delta_{jk} \delta(y - x')$$  

$$= \frac{1}{i \omega} \chi_{k0}(x, x'; \omega) \chi_{00}(x, x'; \omega),$$  

(15)

(16)

which is the desired relation. Using this, we now find for the response of the spatial part of the current.
which obviously coincides with the microscopic Ohm’s law. A similar calculation leads to the response law for the charge density

$$\rho(x, \omega) = \frac{1}{i \omega} \int d^3x' \frac{\partial}{\partial x_k} \sigma_{k\ell}(x, x'; \omega) E_\ell(x', \omega),$$

(21)

which also follows from Eq. (21) by the continuity equation. We have thus shown that Ohm’s law can be derived from a covariant response theory. The seemingly paradoxical result that the induced current can be completely expressed in terms of the applied electric field (such that the magnetic field does not appear) stems from the fact that the conductivity tensor corresponds to a total functional derivative with respect to the external electric field (see the discussion in [31]).

**Transformation law for the conductivity tensor.**—As we have derived Ohm’s law from the fundamental, Lorentz covariant response relation [3], it follows that Ohm’s law holds in every Minkowski frame. Of course, in this statement it is understood that the conductivity tensor itself obeys a transformation law, exactly as the fundamental response tensor. For deriving this transformation law, we now assume the medium to be spatially homogeneous such that $\chi^{\mu\nu}(x, x') = \chi^{\mu\nu}(x - x')$, or in Fourier space $\chi^{\mu\nu}(k, k') = \chi^{\mu\nu}(k) \delta(k - k')$ with $k = (\omega/c, k)$. Then Eq. (3) simply reads

$$j^{\mu}(k) = \chi^{\mu\nu}(k) A^\nu(k),$$

(22)

and by the constraints (10)–(13) the fundamental response tensor can be written explicitly as

$$\chi^{\mu\nu}(k, \omega) = \begin{pmatrix}
-\frac{\epsilon^2}{\omega^2} k^T \hat{\chi} \\
\frac{\epsilon}{\omega} k^T \hat{\chi} \\
-\frac{\epsilon}{\omega} \hat{\chi} k \\
\hat{\chi}
\end{pmatrix}.$$  

(23)

We consider a general Lorentz transformation $x' = \Lambda x$ where $\Lambda \in O(3, 1)$, and assume e.g. that we are given the conductivity tensor $\sigma_{ij}(k, \omega)$ in the unprimed coordinate system. The conductivity tensor in the primed coordinate system can then be derived in three steps as follows (see Fig. 1): (i) By means of Eq. (11) one obtains the spatial part $\chi_{ij}(k, \omega)$ of the fundamental response tensor from the conductivity tensor, and by the constraints (10)–(13) one reconstructs from this the whole fundamental response tensor $\chi_{\mu\nu}(k, \omega) \equiv \chi(k, \omega)$. (ii) The fundamental response tensor is transformed like an ordinary second rank Lorentz tensor, i.e.,

$$\chi'(k', \omega) = \Lambda \chi(k, \omega) \Lambda^{-1}.$$  

(24)

(iii) In the primed coordinate system, one invokes again Eq. (11) to read out the conductivity tensor $\sigma_{ij}'(k', \omega')$. The concatenation of these operations then leads to a complicated (i.e. non-tensorial) transformation law for the microscopic conductivity tensor.

Before deriving this transformation law explicitly, we note that under spatial rotations the conductivity tensor transforms like an ordinary spatial second rank tensor,

$$\hat{\sigma}^{\mu\nu}(k, \omega) = \hat{R} \hat{\sigma}^{\mu\nu}(k, \omega) \hat{R}^{-1},$$  

(25)

where $\hat{R} \in O(3)$. On the other hand, every Lorentz transformation can be factorized into a spatial rotation and a Lorentz boost [32, 33]. Hence it suffices to study the transformation properties of the conductivity tensor under boosts, which have the general form

$$\Lambda(v) = \begin{pmatrix}
\gamma & -\gamma v^T/c \\
-v^T/c & \epsilon \Lambda
\end{pmatrix}.$$  

(26)

Here $v$ is the velocity of the primed coordinate frame relative to the unprimed frame, $\gamma = 1/\sqrt{1 - |v|^2/c^2}$ and

$$\epsilon \Lambda = 1 + (\gamma - 1) \frac{vv^T}{|v|^2}.$$  

(27)

In particular, the momenta and frequencies transform as

$$k' = \Lambda k - \frac{\gamma \omega v}{c^2}, \quad \omega' = (\omega - v \cdot k).$$  

(28)

From Eq. (21) we obtain after some algebra the relation between the spatial components

$$\chi'(k', \omega') = \Lambda \left(1 - \frac{vk^T}{\omega} \right) \chi(k, \omega) \left(1 - \frac{kv^T}{\omega} \right) \Lambda.$$  

(29)
By applying Eq. (11) in both coordinate systems and taking into account the resulting factor $\omega/\omega'$ through

$$\sigma'(k', \omega') = \frac{1}{\gamma} \left( 1 - \frac{v \cdot k}{\omega} \right)^{-1} \Lambda \left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right) \sigma(k, \omega) \left( \frac{1 - \frac{k v^T}{\omega}}{1 - \frac{k v}{\omega}} \right) \Lambda. \tag{30}$$

In the following, we will also need the converse relation,

$$\sigma(k, \omega) = \gamma \left( 1 - \frac{v \cdot k}{\omega} \right) \left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right)^{-1} \Lambda^{-1} \sigma'(k', \omega') \Lambda^{-1} \left( \frac{1 - \frac{k v^T}{\omega}}{1 - \frac{k v}{\omega}} \right)^{-1} \tag{31}$$

$$= \gamma \left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right)^{-1} \Lambda^{-1} \sigma'(k', \omega') \Lambda^{-1} \left( \frac{1 - \frac{v \cdot k}{\omega}}{1 + \frac{k v^T}{\omega}} \right), \tag{32}$$

where in the last step we have used the identity

$$\left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right)^{-1} = 1 + \frac{kv^T}{\omega - v \cdot k}. \tag{33}$$

Note that Eq. (30) represents a non-tensorial transformation law in contrast to the ordinary transformation law \[25\] for rotations.

Rest frame versus moving frame.—Finally, we investigate in how far our findings agree with the textbook wisdom about the relativistic generalization of Ohm’s law. Assume e.g. that the primed coordinate system corresponds to the rest frame of the medium. Then formula \[30\] allows us to calculate the conductivity tensor for any coordinate frame moving with velocity $-v$ relative to the rest frame. In particular, we can reexpress Ohm’s law in the moving frame, $j_\ell(k, \omega) = \sigma_{ij}(k, \omega)E_\ell(k, \omega)$, in terms of the conductivity tensor of the rest frame:

$$j(k, \omega) = \gamma \left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right)^{-1} \Lambda^{-1} \sigma'(k', \omega') \Lambda^{-1} \cdot \left( \frac{1 - \frac{v \cdot k}{\omega}}{1 + \frac{k v^T}{\omega}} \right) E(k, \omega). \tag{34}$$

To simplify this expression, we use that

$$\left( \frac{1 - \frac{v k^T}{\omega}}{1 - \frac{v k}{\omega}} \right) j(k, \omega) = j(k, \omega) - v \rho(k, \omega) \tag{35}$$

by the continuity equation, and moreover,

$$\left( 1 - \frac{v \cdot k}{\omega} \right) E(k, \omega) + k \frac{v \cdot E(k, \omega)}{\omega} = E(k, \omega) + \frac{v \times (k \times E(k, \omega))}{\omega} = E(k, \omega) + \frac{v \times B(k, \omega)}{\omega} \tag{36}$$

$$= E(k, \omega) + v \times B(k, \omega), \tag{37}$$

where we have employed Faraday’s law. Thus we obtain

$$j(k, \omega) - v \rho(k, \omega) = \gamma \Lambda^{-1} \sigma'(k', \omega') \Lambda^{-1} \cdot \left( E(k, \omega) + v \times B(k, \omega) \right). \tag{38}$$

Now consider the special case where the conductivity tensor of the rest frame is scalar and constant, i.e., $\sigma'_{ij}(k', \omega') = \delta_{ij} \sigma'$. Then Eq. (38) simplifies as

$$j - v \rho = \gamma \sigma' \Lambda^{-2} (E + v \times B). \tag{39}$$

Using that $(\gamma - 1)(\gamma + 1) = \gamma^2 |v|^2 / c^2$, we find

$$\Lambda^{-2} = \left( 1 + \frac{1 - \gamma \frac{v v^T}{c^2}}{\gamma} \right)^{-1} = \frac{1 - \frac{v v^T}{c^2}}{c^2}, \tag{40}$$

and consequently

$$j - v \rho = \gamma \sigma' \left( E - \frac{v}{c} \left( \frac{v}{c} \cdot E \right) + v \times B \right). \tag{41}$$

This formula is usually referred to as the relativistic generalization of Ohm’s law in the textbook literature (see e.g. \[37\] Problem 11.16 and \[38\] Problem 9-15). We have shown that it is a special case of Eq. (38), which in turn is a direct consequence of the relativistic transformation law for the conductivity tensor (30). Finally, we remark that in the non-relativistic limit where terms of order $|v|^2 / c^2$ are neglected, Eq. (11) reduces to

$$j - v \rho = \sigma' (E + v \times B), \tag{42}$$

which is also a standard textbook generalization of Ohm’s law \[38\]. In fact, this last equation is obviously equivalent to $j' = \sigma' E'$ provided one uses the non-relativistic transformation laws $j' = j - v \rho$ and $E' = E + v \times B$, respectively.

Conclusion.—Starting from the Lorentz covariant microscopic response relation \[33\], we have established the
validity of Ohm’s law in every frame of reference and derived the ensuing non-tensorial, relativistic transformation law for the microscopic conductivity tensor. Besides providing a new perspective on the problem of generalizing Ohm’s law in a relativistic setting, we have shown how the standard result [11] emerges as a limiting case with a constant, scalar conductivity from our general formula [30]. Given the conductivity tensor in the rest frame of the medium, this formula can be used to compute the conductivity tensor in any moving frame. Thus it provides a direct link to the experiment, where the relativistic current response is not only relevant for plasmas in large-scale astrophysics but also for tabletop experiments with electronic quantum liquids.

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[1] G. S. Ohm, *Die galvanische Kette, mathematisch bearbeitet* (T. H. Riemann, Berlin, 1827).
[2] S. F. B. Morse, in *Shaffner’s Telegraph Companion*, Vol. 2, edited by T. P. Shaffner (Pudney & Russell, New York, 1855) pp. 6–128.
[3] A. J. Butrica, in *Beyond History of Science. Essays in Honor of Robert E. Schofield*, edited by E. Garber (Associated University Presses, Inc., London, 1990) pp. 204–219.
[4] A. L. Hodgkin and A. F. Huxley, J. Physiol. 117, 500 (1952).
[5] W. W. Lytton and C. C. Kerr, in *Neuroscience in the 21st Century*, edited by D. W. Pfaff (Springer, New York, 2013) pp. 2275–99.
[6] M. K. Parikh and F. Wilczek, *Phys. Rev. D* **58**, 064011 (1998).
[7] K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black Holes: The Membrane Paradigm* (Yale University Press, 1986).
[8] S. A. Hartnoll and C. P. Herzog, *Phys. Rev. D* **76**, 106012 (2007).
[9] B. Weber, S. Mahapatra, H. Ryu, S. Lee, A. Fuhrer, T. C. G. Reusch, D. L. Thompson, W. C. T. Lee, G. Klimeck, L. C. L. Hollenberg, and M. Y. Simmons, *Science* **335**, 64 (2012).
[10] H. Ryu, S. Lee, B. Weber, S. Mahapatra, L. Hollenberg, M. Y. Simmons, and G. Klimeck, Nanoscale 5, 8666 (2013).
[11] D. K. Ferry, *Science* **319**, 579 (2008).
[12] M. Fuechsle, J. A. Miwa, S. Mahapatra, H. Ryu, S. Lee, O. Warschkow, L. C. L. Hollenberg, G. Klimeck, and M. Y. Simmons, Nature Nanotech **7**, 242 (2012).
[13] B. E. Peierls, in *Theoretical Physics in the Twentieth Century: A Memorial Volume to Wolfgang Pauli*, edited by M. Fierz and V. F. Weisskopf (Interscience Publishers, New York, 1960) pp. 140–160.
[14] N. I. Chernov, G. L. Eyink, J. L. Lebowitz, and Y. G. Sinai, *Phys. Rev. Lett.* **70**, 2209 (1993).
[15] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill Book Company, New York, 1973).
[16] P. Lorrain, F. Lorrain, and S. Houle, *Magneto-Fluid Dynamics. Fundamentals and Case Studies of Natural Phenomena* (Springer Science+Business Media, LLC, 2006).
[17] R. D. Blandford and K. S. Thorne, “Applications of Classical Physics,” (2013) Chap. 18.
[18] C. Palenzuela, L. Lehner, O. Reula, and L. Rezzolla, Mon. Not. Roy. Astron. Soc. **394**, 1727 (2009).
[19] D. L. Meier, The Astrophysical Journal **605**, 340 (2004).
[20] K. A. Holcomb and T. Tajima, *Phys. Rev. D* **40**, 3809 (1989).
[21] H. Ardavan, Astrophys. J. **203**, 226 (1976).
[22] J. D. Bekenstein and E. Oron, *Phys. Rev. D* **18**, 1809 (1978).
[23] E. G. Blackman and G. B. Field, *Phys. Rev. Lett.* **71**, 3181 (1993).
[24] M. Gedalin, *Phys. Rev. Lett.* **76**, 3340 (1996).
[25] R. Khanna, Mon. Not. R. Astron. Soc. **294**, 673 (1998).
[26] G. M. Kremer and C. H. Patsko, Physica A: Statistical Mechanics and its Applications **322**, 329 (2003).
[27] A. Kandus and C. G. Tsagas, Mon. Not. Roy. Astron. Soc. **385**, 883 (2008).
[28] M. Sherlock, *Phys. Rev. Lett.* **104**, 205004 (2010).
[29] M. Tsamparlis, *Special Relativity. An Introduction with 200 Problems and Solutions* (Springer-Verlag, Berlin, Heidelberg, 2010).
[30] R. Starke and G. Schober, arXiv:1401.6800v2 [cond-mat.mtrl-sci] (2014).
[31] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, 2010).
[32] E. H. Fadkin, “Linear Response Theory, in: Condensed Matter Physics II, Lecture Notes, University of Illinois at Urbana-Champaign,” [http://eduardo.physics.illinois.edu/phys561/LRT.pdf](http://eduardo.physics.illinois.edu/phys561/LRT.pdf) (2010).
[33] G. F. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press, 2005).
[34] S. B. Nam, *Phys. Rev. Lett.* **156**, 470 (1967).
[35] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific Publishing, New Jersey, 2008).
[36] G. Scharf, *Finite Quantum Electrodynamics: The Causal Approach*, 2nd ed. (Springer-Verlag, Berlin, New York, 1995).
[37] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1999).
[38] T. Tsang, *Classical Electrodynamics* (World Scientific Publishing Co. Pte. Ltd., Singapore, 1997).