Recurrence of biased quantum walks on a line

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Abstract. The Pólya number of a classical random walk on a regular lattice is known to depend solely on the dimension of the lattice. For one and two dimensions it equals one, meaning unit probability of returning to the origin. This result is extremely sensitive to the directional symmetry, and any deviation from the equal probability of travelling in each direction results in a change of the character of the walk from recurrent to transient. Applying our definition of the Pólya number to quantum walks on a line we show that the recurrence character of quantum walks is more stable against bias. We determine the range of parameters for which biased quantum walks remain recurrent. We find that there exist genuine biased quantum walks that are recurrent.
1. Introduction

Random walks are a popular topic in physics [1, 2]. The popularity stems from several sources. First, random walks are rather simple in their formulation yet powerful in their application and allow one to pinpoint the essential physics involved in the studied processes. Next, random walks are one of the tools that allow one to connect the microdynamics with the macrobehaviour of large systems. Finally, random walks are quite flexible and popular also outside physics to describe various phenomena. Hence, it comes not as a surprise that the first random walks have not been formulated within physics but to describe the alternation of share prices on the stock exchange or the spreading of insects in a forest [3, 4].

The study of random walks obtained a new stimulus when they were combined with quantum mechanics [5, 6]. Here the walker is considered to be a non-classical object enriched with wave attributes. The novel features of quantum walks have been shown not only to be of theoretical interest but also to have practical implications, especially for quantum algorithms [7]–[10]. An important concept is the hitting time [11]–[14], which helps to point out the fundamental difference between classical and quantum walks allowing for algorithmic speed-up. One of the simplest non-trivial examples for a quantum walk is the one on a line [15], which is closely related to the so-called optical Galton board [16]. Various aspects of one-dimensional quantum walks have been analysed [17]–[22]. Additional interesting effects, e.g. localization, arise when one considers multi-state quantum walks [23]–[26].

One of the characteristics of the random walk on an infinite lattice is expressed by the probability of the walker to return to its starting position, called the Pólya number [27]. If the Pólya number equals one the walk is called recurrent; otherwise there is a non-zero probability that the walker never returns to its starting position. Such walks are called transient. The recurrence nature of the random walk is determined by the asymptotic behaviour of the probability at the origin [28]. One finds that a random walk is transient if the probability at the origin decays faster than \( t^{-1} \). The recurrent behaviour has been studied in great detail for classical random walks in dependence on the dimension and the topology of the lattice [29, 30].

Recently, we have extended the concept of the Pólya number to quantum walks [31]. In our definition we proposed a particular measurement scheme to minimize disturbance: each
measurement in a series is carried out on a different member of an ensemble of equally prepared quantum systems. We have shown that the recurrence nature of the quantum walk, according to the above definition, is determined by the asymptotic behaviour of the probability at the origin in a similar way as in classical random walks. However, due to interference the asymptotics of the probability at the origin does not depend solely on the dimension of the lattice, but also on the coin operator and the initial coin state. Hence, one can find strikingly different recurrence behaviour for quantum walks compared to their classical counterparts [32]. Note that recurrence is meant here as the return to the origin, which can be considered as a fractional recurrence from the point of view of the whole quantum state [33, 34]. So far, we have considered balanced walks, i.e. ones where there is no preference in direction for the walker and the step lengths are equal. For a large class of quantum walks this assumption does not hold, and we wish to study the implications of unbalanced coins and unequal step lengths for the recurrence properties.

In the present paper, we study biased quantum walks on the line and compare their properties with their classical counterparts. As we briefly review in appendix A, recurrence of classical random walks is a consequence of the walk’s symmetry. They are recurrent if and only if the mean value of the position of the particle vanishes. This is due to the fact that the spreading of the probability distribution of the position is diffusive while the mean value of the position propagates with a constant velocity. In contrast, for quantum walks both the spreading of the probability distribution and the propagation of the mean value are ballistic. We show that this allows for maintaining recurrence even when the symmetry is broken.

Our paper is organized as follows: in section 2 we describe the biased quantum walk on a line. In section 3 we solve the time evolution equations with the help of the Fourier transformation. We find that the probability amplitudes can be expressed in terms of integrals where time enters only in the rapidly oscillating phase factor. This fact allows a straightforward asymptotic analysis of the probability amplitudes by means of the method of stationary phase. We perform this analysis in section 4. Since the recurrence of the quantum walk is determined by the asymptotics of the probability at the origin, we find a condition under which the biased quantum walk on a line is recurrent. In section 5 we analyse the recurrence of biased quantum walks from a different perspective. We find that the recurrence is related to the velocities of the peaks of the probability distribution generated by the quantum walk. The explicit form of the velocities leads us to the same condition derived in section 4. Finally, in section 6 we analyse the formula for the mean value of the position of the particle derived in appendix B in dependence of the parameters of the walk and the initial state. We find that there exist genuine biased quantum walks that are recurrent. Conclusions and outlook are left for section 7.

2. Description of the walk

Let us consider biased quantum walks on a line where the particle has two possibilities—jump to the right or to the left. Without loss of generality, we restrict ourselves to biased quantum walks where the jump to the right is of the length \( r \) and the jump to the left has a unit size. We depict the biased quantum walk schematically in figure 1.

The Hilbert space of the particle has the form of the tensor product

\[
\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C
\]

(1)

of the position space

\[
\mathcal{H}_P = \ell^2(\mathbb{Z}) = \text{Span} \{ |m\rangle \mid m \in \mathbb{Z} \},
\]

(2)
Figure 1. Schematics of the biased quantum walk on a line. If the coin is in the state $|R\rangle$ the particle moves to the right to a point at distance $r$. With the coin state $|L\rangle$ the particle makes a unit length step to the left. Before the step itself the coin state is rotated according to the coin operator $C(\rho)$.

and the two-dimensional coin space
\[ \mathcal{H}_C = \mathbb{C}^2 = \text{Span} \{ |R\rangle, |L\rangle \}. \]  
(3)

A single step of the quantum walk is given by the propagator
\[ U = S \left( I_P \otimes C \right). \]  
(4)

Here $I_P$ denotes the unit operator acting on the position space $\mathcal{H}_P$. The displacement operator $S$ has the form
\[ S = \sum_{m=-\infty}^{+\infty} |m+r\rangle \langle m| \otimes |R\rangle \langle R| + \sum_{m=-\infty}^{+\infty} |m-1\rangle \langle m| \otimes |L\rangle \langle L|. \]  
(5)

The coin flip $C$ is, in general, an arbitrary unitary operator acting on the coin space $\mathcal{H}_C$ and is applied on the coin state before the displacement $S$ itself. However, as has been discussed in [17], the probability distribution is not affected by the complex phases of the coin operator. Hence, it is sufficient to consider the one-parameter family of coins
\[ C(\rho) = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{pmatrix}. \]  
(6)

From now on we restrict ourselves to this family of coins. The value of $\rho = \frac{1}{2}$ corresponds to the well-known case of the Hadamard walk.

We write the initial state of the particle in the form
\[ |\psi(0)\rangle \equiv \sum_{m=-\infty}^{+\infty} \sum_{i=R}^{L} \psi_i(m, 0)|m\rangle \otimes |i\rangle. \]  
(7)

The state of the walker after $t$ steps is given by successive application of the time evolution operator given by equation (4) on the initial state
\[ |\psi(t)\rangle \equiv U^t|\psi(0)\rangle = \sum_{m=-\infty}^{+\infty} \sum_{i=R}^{L} \psi_i(m, t)|m\rangle \otimes |i\rangle. \]  
(8)

The state of the particle is fully determined by the set of two-component vectors
\[ \psi(m, t) \equiv (\psi_R(m, t), \psi_L(m, t))^T. \]  
(9)
Here $\psi_{R(L)}(m, t)$ is the probability amplitude to find the particle at position $m$ after $t$ steps with the coin state $|R(L)\rangle$. The probability distribution generated by the quantum walk is given by

$$P(m, t) = |\langle m, R | \psi(t) \rangle|^2 + |\langle m, L | \psi(t) \rangle|^2 = |\psi_R(m, t)|^2 + |\psi_L(m, t)|^2 = \|\psi(m, t)\|^2. \quad (10)$$

3. Time evolution of the walk

To obtain explicit and closed-form expressions for the time-dependent state vector, we rewrite the time evolution equation (8) for the state vector $|\psi(t)\rangle$ into a set of difference equations

$$\psi(m, t) = C_+(\rho)\psi(m - r, t - 1) + C_-(\rho)\psi(m + 1, t - 1), \quad (11)$$

for the probability amplitude vectors $\psi(m, t)$. The form of the matrices $C_\pm(\rho)$ follows from the matrix $C(\rho)$:

$$C_+(\rho) = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ 0 & 0 \end{pmatrix}, \quad (12)$$

$$C_-(\rho) = \begin{pmatrix} 0 & 0 \\ \sqrt{1-\rho} & -\sqrt{\rho} \end{pmatrix}. \quad (13)$$

The time evolution equations (11) are greatly simplified with the help of the Fourier transformation

$$\tilde{\psi}(k, t) \equiv \sum_{m=-\infty}^{+\infty} \psi(m, t)e^{imk}, \quad (14)$$

where the momentum $k$ is a continuous parameter ranging from $-\pi$ to $\pi$. The new function $\tilde{\psi}(k, t)$ is square integrable on a unit circle.

The time evolution in the Fourier picture turns into a single difference equation

$$\tilde{\psi}(k, t) = \tilde{U}(k)\tilde{\psi}(k, t - 1), \quad (15)$$

where the propagator has the form

$$\tilde{U}(k) \equiv \begin{pmatrix} \sqrt{\rho}e^{ik} & \sqrt{1-\rho}e^{ik} \\ \sqrt{1-\rho}e^{-ik} & -\sqrt{\rho}e^{-ik} \end{pmatrix}. \quad (16)$$

The solution of (14) is straightforward. We find

$$\tilde{\psi}(k, t) = \tilde{U}^t(k)\tilde{\psi}(k, 0), \quad (17)$$

where $\tilde{\psi}(k, 0)$ is the Fourier transformation of the initial state. We restrict ourselves to the situation where the particle is initially localized at the origin as dictated by the nature of the problem we wish to study. As follows from (13), the Fourier transformation $\tilde{\psi}(k, 0)$ of such an initial condition is equal to the initial state of the coin

$$\tilde{\psi}(k, 0) = \begin{pmatrix} \psi_R(0, 0) \\ \psi_L(0, 0) \end{pmatrix}. \quad (18)$$
which we denote by $\psi$. Since $\psi$ can be an arbitrary normalized complex two-component vector we parameterize it by two parameters $a \in [0, 1]$ and $\varphi \in [0, 2\pi)$ in the form

$$\psi = \left( \frac{\sqrt{a}}{\sqrt{1-ae^{i\varphi}}} \right).$$ (18)

To evaluate the powers of the propagator $\tilde{U}(k)$ it is convenient to diagonalize it. Since the propagator is unitary its eigenvalues have the form $e^{i\omega_{1,2}}$ where the phases read

$$\omega_1(k) = \frac{r-1}{2}k + \arcsin\left( \sqrt{\rho} \sin\left( \frac{r+1}{2}k \right) \right),$$
$$\omega_2(k) = \frac{r-1}{2}k - \pi - \arcsin\left( \sqrt{\rho} \sin\left( \frac{r+1}{2}k \right) \right).$$ (19)

We denote the corresponding eigenvectors by $v_{1,2}(k)$. We give their explicit form in appendix B. With this notation, we write the solution of the time evolution equation in the Fourier picture in the form

$$\tilde{\psi}(k, t) = \sum_{j=1}^{2} e^{i\omega_j(k)t} \left( v_j(k), \psi \right) v_j(k).$$ (20)

Here $(\cdot, \cdot)$ denotes the scalar product in the coin space. Finally, we obtain the solution in position representation by performing the inverse Fourier transformation

$$\psi(m, t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-imk} \tilde{\psi}(k, t) = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(\omega_j(k)t-mk)} \left( v_j(k), \psi \right) v_j(k).$$ (21)

4. Asymptotics of the quantum walk and recurrence

To determine the recurrence nature of the biased quantum walk we have to analyse the asymptotic behaviour of the probability at the origin [31]. Exploiting (21) the amplitude at the origin reads

$$\psi(0, t) = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\omega_j(k)t} \left( v_j(k), \psi \right) v_j(k),$$ (22)

which allows us to find the asymptotics of the probability at the origin with the help of the method of stationary phase [35]. The important contributions to the integrals in (22) arise from the stationary points of phases (19). We find that the derivatives of the phases $\omega_{1,2}(k)$ are

$$\omega'_1(k) = \frac{r-1}{2} + \frac{\sqrt{\rho}(r+1) \cos \left( \frac{k(r+1)}{2} \right)}{\sqrt{4+2\rho} \left[ \cos(k(r+1)) - 1 \right]},$$
$$\omega'_2(k) = \frac{r-1}{2} - \frac{\sqrt{\rho}(r+1) \cos \left( \frac{k(r+1)}{2} \right)}{\sqrt{4+2\rho} \left[ \cos(k(r+1)) - 1 \right]}.$$ (23)
Figure 2. The existence of stationary points of the phases $\omega_{1,2}(k)$ in dependence on the parameter $\rho$ and a fixed step length $r$. We plot the implicit functions $\omega'_{1,2}(k) \equiv 0$ for $r = 3$. The plot indicates that for $\rho < \rho_R(3) = \frac{1}{2}$ the phases $\omega_{1,2}(k)$ do not have any saddle points. Consequently, the probability amplitude at the origin decays fast and such biased quantum walk on a line is transient. For $\rho \geq \rho_R(3)$ the saddle points exist and the quantum walk is recurrent.

Using the method of stationary phase we find that the amplitude will decay slowly like $t^{-\frac{1}{2}}$, if at least one of the phases has a vanishing derivative inside the integration domain. Solving the equations $\omega'_{1,2}(k) = 0$, we find that the possible saddle points are

$$k_0 = \pm \frac{2}{r+1} \arccos \left( \pm \sqrt{\frac{(1-\rho)(r-1)^2}{4\rho r}} \right).$$

(24)

The saddle points are real valued provided the argument of the arcus-cosine in (24) is less or equal to unity

$$\frac{(1-\rho)(r-1)^2}{4\rho r} \leq 1.$$  

(25)

This inequality leads us to the condition for the biased quantum walk on a line to be recurrent

$$\rho_R(r) \geq \left( \frac{r-1}{r+1} \right)^2.$$  

(26)

We illustrate this result in figure 2 for a particular choice of the walk parameter $r = 3$.

Our simple result proves that there is an intimate non-trivial link between the length of the step of the walk and the bias of the coin. The parameter of the coin $\rho$ has to be at least equal to a factor determined by the size of the step to the right $r$ for the walk to be recurrent. We note that the recurrence nature of the biased quantum walk on a line is determined only by the parameters of the walk itself, i.e. the coin and the step, not by the initial conditions. The parameters of the initial state $\alpha$ and $\phi$ have no effect on the rate of decay of the probability at the origin.
5. Recurrence of a quantum walk and the velocities of the peaks

We can determine the recurrence nature of the biased quantum walk on a line from a different point of view. This approach is based on the following observation. The well-known shape of the probability distribution generated by the quantum walk consists of two counterpropagating peaks. In between the two dominant peaks the probability is roughly independent of \( m \) and decays like \( t^{-1} \). On the other hand, outside the decay is exponential as we depart from the peaks. As it has been found in [15] the positions of the peaks vary linearly with the number of steps. Hence, the peaks propagate with constant velocities, say \( v_L \) and \( v_R \). For the biased quantum walk to be recurrent the origin of the walk has to remain in between the two peaks for all times. In other words, the biased quantum walk on a line is recurrent if and only if the velocity of the left peak is negative and the velocity of the right peak is positive.

The velocities of the left and right peaks are easily determined. We rewrite formula (21) for the probability amplitude \( \psi(m, t) \) into the form

\[
\psi(m, t) = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(\omega_j(k) - \alpha k)t} \langle v_j(k), \psi \rangle v_j(k),
\]

where we have introduced \( \alpha = m/t \). Due to the fact that we now concentrate on the amplitudes at the positions \( m \sim t \) we have to consider modified phases

\[
\tilde{\omega}_j(k) = \omega_j(k) - \alpha k.
\]  

The peak occurs at such a position \( m_0 \) where both the first and the second derivatives of \( \tilde{\omega}_j(k) \) vanishes. The velocity of the peak is thus \( \alpha_0 = m_0/t \). Hence, solving the equations

\[
\begin{align*}
\tilde{\omega}_1'(k) &= \frac{r - 1}{2} + \frac{\sqrt{\rho}(r + 1) \cos \left( \frac{k}{2} + \frac{1}{2} \right)}{4 + 2\rho [\cos(k(r + 1)) - 1]} - \alpha = 0, \\
\tilde{\omega}_2'(k) &= \frac{r - 1}{2} - \frac{\sqrt{\rho}(r + 1) \cos \left( \frac{k}{2} + \frac{1}{2} \right)}{4 + 2\rho [\cos(k(r + 1)) - 1]} - \alpha = 0,
\end{align*}
\]

(29)

\[
\begin{align*}
\tilde{\omega}_1''(k) &= -\tilde{\omega}_2''(k) = \frac{(r - 1)\sqrt{\rho}(r + 1)^2 \sin \left( \frac{k}{2} + \frac{1}{2} \right)}{\sqrt{2} \left( 2 - \rho + \rho \cos(k(r + 1)) \right)^{3/2}} = 0,
\end{align*}
\]

for \( \alpha \) determines the velocities of the left and right peak \( v_{L,R} \). The third equation is independent of \( \alpha \) and we easily find the solution

\[
k_0 = \frac{4n\pi}{r + 1}, \quad k_0 = \frac{2\pi(2n + 1)}{r + 1}, \quad n \in \mathbb{Z}.
\]

(30)

Inserting this \( k_0 \) into the first two equations we find the velocities of the left and right peak

\[
\begin{align*}
v_L &= \frac{r - 1}{2} - \frac{r + 1}{2} \sqrt{\rho}, \\
v_R &= \frac{r - 1}{2} + \frac{r + 1}{2} \sqrt{\rho}.
\end{align*}
\]

(31)

We illustrate this result in figure 3 where we show the probability distribution generated by the quantum walk for the particular choice of the parameters \( r = 3 \) and \( \rho = 1/\sqrt{2} \). The initial state was chosen according to \( a = 1/\sqrt{2} \) and \( \varphi = \pi \). Since the velocity of the left peak \( v_L \) is negative this biased quantum walk is recurrent.
Figure 3. Velocities of the left and right peak of the probability distribution generated by the biased quantum walk on a line and the recurrence. We have chosen the parameters $r = 3$, $a = \rho = 1/\sqrt{2}$ and $\varphi = \pi$. The left peak propagates with the velocity $v_L \approx -0.68$, the velocity of the right peak is $v_R \approx 2.68$. In between the two peaks the probability distribution behaves like $t^{-1}$ while outside the decay is exponential. Since the velocity $v_L$ is negative the origin of the walk remains in between the left and right peak. Consequently, this quantum walk is recurrent.

The peak velocities have two contributions. One is identical and independent of $\rho$; the second is a product of $r$ and $\rho$ and differs in sign for the two velocities. The obtained results indicate that biasing the walk by having the size of the step to the right equal to $r$ results in dragging the whole probability distribution towards the direction of the larger step. This is manifested by the term $(r-1)/2$ which appears in both velocities $v_{L,R}$ with the same sign. On the other hand the parameter of the coin $\rho$ does not bias the walk. As we can see from the second terms entering the velocities it rather influences the rate at which the walk spreads.

As discussed above the biased quantum walk on a line is recurrent if and only if $v_L$ is negative and $v_R$ is positive. The form of velocities (31) implies that this condition is satisfied if and only if criterion (26) is fulfilled.

6. Mean value of the biased quantum walk and recurrence

As we discuss in appendix A, the classical random walks are recurrent if and only if the mean value of the position vanishes. We now show that this is not true for biased quantum walks, i.e. there exist biased quantum walks on a line that are recurrent but cannot produce probability distributions with zero mean value. This is another unique feature of quantum walks compared with the classical ones.

In appendix B, we derive the following formula for the position mean value:

$$\left\langle \frac{x}{t} \right\rangle \approx (1 - \sqrt{1 - \rho})(a(r + 1) - 1) + \sqrt{a(1-a)(1-\sqrt{1-\rho})(1-\rho)}(r+1)\cos \varphi \sqrt{\rho(1-\rho)}$$

$$+ \frac{r-1}{2}\sqrt{1-\rho} + O(t^{-1}).$$

(32)
We see that for quantum walks the mean value is affected by both the fundamental walk parameters through $r$ and $\rho$ and the initial state parameters $a$ and $\varphi$. The mean value is typically non-vanishing even for unbiased quantum walks (with $r = 1$). However, one easily finds [17] that the initial state with the parameters $a = \frac{1}{2}$ and $\varphi = \pi/2$ results in a symmetric probability distribution with zero mean independent of the coin parameter $\rho$. Indeed, the quantum walks with $r = 1$, i.e. with equal steps to the right and left, do not intrinsically distinguish left from right. On the other hand, the quantum walks with $r > 1$ treat the left and right direction in a different way. Nevertheless, one can always find for a given $r$ a coin parameter $\rho_0$ such that for all $\rho \geq \rho_0$, the quantum walk can produce a probability distribution with zero mean value. This is impossible for quantum walks with $\rho < \rho_0$ and we will call such quantum walks genuine biased.

Let us now determine the minimal value of $\rho$ for a given $r$ for which mean value vanishes. We first find the parameters of the initial state $a$ and $\varphi$ which minimizes the mean value. Clearly the term on the second line in ($32$) reaches the minimal value for $\varphi_0 = \pi$. Differentiating the resulting expression with respect to $a$ and setting the derivative equal to zero gives us the condition

$$2 + \frac{(2a - 1)\sqrt{\rho(1 - \rho)}}{\rho\sqrt{a(1 - a)}} = 0,$$

(33)

on the minimal mean value with respect to $a$. This relation is satisfied for $a_0 = \frac{1}{2}(1 - \sqrt{\rho})$. The resulting formula for the mean value reads

$$\langle x \rangle_{a_0, \varphi_0} = \frac{r - 1}{2} + \frac{(1 - \sqrt{1 - \rho})}{2\sqrt{(1 - \rho)\rho}}(1 + r).$$

(34)

This expression vanishes for

$$\rho_0(r) = \left(\frac{r^2 - 1}{r^2 + 1}\right)^2.$$

(35)

Since (34) is a decreasing function of $\rho$ the mean value is always positive for $\rho < \rho_0$ independent of the choice of the initial state. For $\rho > \rho_0$ one can achieve zero mean value for different combination of the parameters $a$ and $\varphi$.

Formula (35) is reminiscent of condition (26) for the biased quantum walk on a line to be recurrent. However, $r$ is in (35) replaced by $r^2$. Therefore, we find the inequality $\rho_R < \rho_0$. Hence, the quantum walks with the coin parameter $\rho_R < \rho < \rho_0$ are recurrent but cannot produce a probability distribution with zero mean value. We conclude that there are genuine biased quantum walks which are recurrent in contrast to situations found for classical walks.

7. Conclusions

We have analysed one-dimensional biased quantum walks. Classically, the bias leading to a non-zero mean value of the particle’s position can be introduced in two ways—unequal step lengths or unfair coin. In contrast, for quantum walks on a line the initial state can introduce bias for any coin. On the other hand, for a symmetric initial state, modifying only the unitary coin operator while keeping equal step lengths will not introduce bias. Finally, the bias due to unequal step lengths may be compensated for by the choice of the coin operator for some initial
Figure 4. ‘Phase diagram’ of biased quantum walks on a line. The horizontal axis represents the length of the step to the right $r$ and the vertical axis shows the coin parameter $\rho$. The dotted line corresponds to recurrence criterion (26), while the squares represent condition (35) on the zero mean value of the particle’s position. The quantum walks in the white area are transient and genuinely biased. In between the two curves (light grey area), we find quantum walks which are recurrent but still genuinely biased. The quantum walks in the dark grey area are recurrent and for a particular choice of the initial state they can produce a probability distribution with vanishing mean value.

conditions. For this reason, we have introduced the concept of the genuinely biased quantum walk for which no initial state exists that leads to vanishing mean value of the position.

We have determined the conditions under which one-dimensional biased quantum walks are recurrent. This together with the condition of being genuinely biased give rise to three different regions in the parameter space which we depict as a ‘phase diagram’ in figure 4.

The presented results allow for generalization to biased quantum walks in higher dimensions assuming we keep the coin operator in a tensorial form. For non-factorizable coin operators in higher dimensions when they are recurrent or transient remains an open question.

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Appendix A. Recurrence of classical biased random walk on a line

Classical random walks on a line can be biased in two ways—the step in one direction is greater than in the other one and the probability of the step to the right is different from the probability of the step to the left (see figure A.1).

Consider a random walk on a line such that the particle can make a jump of length $r$ to the right with probability $p$ or make a unit size step to the left with probability $1 - p$. The random
Figure A.1. Schematics of the biased random walk on a line. The particle can move to the right by a distance $r$ with the probability $p$. The length of the step to the left is unity and the probability of this step is $1 - p$.

The walk is recurrent if and only if the probability to find the particle at the origin at any time instant $t$ does not decay faster than $t^{-1}$. This probability is easily found to be expressed by the binomial expression

$$P_0(t) = (1 - p)^{r/(r+1)} p^{1/(r+1)} \left( \frac{t}{r} \right)^{t/(r+1)}. \quad (A.1)$$

With the help of Stirling’s formula

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad (A.2)$$

we find the asymptotic behaviour of the probability at the origin

$$P_0(t) \approx \frac{r + 1}{\sqrt{2\pi rt}} \left( (1 - p)^{r/(r+1)} p^{1/(r+1)} \frac{r + 1}{r^{r/(r+1)}} \right)^t. \quad (A.3)$$

The asymptotics of the probability $P_0(t)$ therefore depend on the value of

$$q = (1 - p)^{r/(r+1)} p^{1/(r+1)} \frac{r + 1}{r^{r/(r+1)}}. \quad (A.4)$$

Since $q \leq 1$ the probability $P_0(t)$ decays exponentially unless the inequality is saturated. Hence, the random walk is recurrent if and only if $q$ equals unity. This condition is satisfied for

$$p = \frac{1}{r + 1}. \quad (A.5)$$

i.e. the probability of the step to the right has to be inversely proportional to the length of the step.

This result can be well understood from a different point of view, as we illustrate in figure A.2. The spreading of the probability distribution is diffusive, i.e. $\sigma \sim \sqrt{t}$. The probability in the $\sigma$ neighbourhood of the mean value $\langle x \rangle$ behaves like $t^{-1/2}$ while outside this neighbourhood the probability decays exponentially. Therefore, for the random walk to be recurrent the origin must lie in this $\sigma$ neighbourhood for all times $t$. However, if the random walk is biased the mean value of the position $\langle x \rangle$ varies linearly in time; thus it is a faster process than the spreading of the probability distribution. In such a case, the origin would lie outside the $\sigma$ neighbourhood of the mean value after a finite number of steps leading to the exponential asymptotic decay of the probability at the origin $P_0(t)$. Hence, the random walk is recurrent if and only if the mean value of the position equals zero. Since the individual steps are
Figure A.2. Spreading of the probability distribution versus the motion of the mean value of a biased classical random walk on a line. While the spreading is diffusive ($\sigma \sim \sqrt{t}$) the mean value propagates with a constant velocity ($\langle x \rangle \sim t$). The probability inside the $\sigma$ neighbourhood of the mean value $\langle x \rangle$ behaves like $t^{-1/2}$. On the other hand, as we go away from the $\sigma$ neighbourhood the decay is exponential. Hence, if the mean value $\langle x \rangle$ does not vanish the origin of the walk leaves the $\sigma$ neighbourhood of the mean value. In such a case the probability at the origin decays exponentially and the walk is transient.

independent of each other the mean value after $t$ steps is simply a $t$ multiple of the mean value after single step, i.e.

$$\langle x(t) \rangle = t \langle x(1) \rangle = t \left[p(r + 1) - 1\right].$$  \hspace{1cm} (A.6)

We find that the mean value equals zero if and only if condition (A.5) holds.

Appendix B. Mean value of the particle’s position for a quantum walk on a line

In this appendix, we find the explicit form of the position mean value of the particle. With the help of the weak limit theorem [36] we express the mean value after $t$ steps in the form

$$\frac{\langle x \rangle}{t} \approx \sum_{j=1}^{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \omega_j'(k) \left(v_j(k), \psi\right) v_j(k),$$  \hspace{1cm} (B.1)

up to the corrections of the order $O(t^{-1})$. Here $v_j(k)$ are eigenvectors of the unitary propagator $\tilde{U}(k)$, $\omega_j'(k)$ are the derivatives of the phases of the corresponding eigenvalues and $\psi$ is the initial state expressed in (18). The derivatives of the phases are given in (23). We express the eigenvectors in the form

$$v_1(k) = n_1(\rho, k) \left(\sqrt{1-\rho}, -\sqrt{\rho} + e^{i(\omega_1(k)-r_k)}\right)^T,$$

$$v_2(k) = n_2(r, k) \left(\sqrt{1-\rho}, -\sqrt{\rho} + e^{i(\omega_2(k)-r_k)}\right)^T.$$  \hspace{1cm} (B.2)
The normalization factors of the eigenvectors read

\[ n_1(u) = 2 - 2\sqrt{\rho} \cos(u - \arcsin[\sqrt{\rho} \sin u]), \]

\[ n_2(u) = 2 + 2\sqrt{\rho} \cos(u + \arcsin[\sqrt{\rho} \sin u]), \]

where we denote \( u = [k(r + 1)]/2 \) to shorten the notation. The mean value is thus given by the following integral:

\[ \langle x \rangle \approx \int_0^{(r+1)\pi} \frac{f(a, \varphi, \rho, r, u) \, du}{2(r+1)\pi [1 + \sqrt{\rho} \cos u_1][1 - \sqrt{\rho} \sin u_2]} + O(t^{-1}), \]

where

\[ u_1 = u + \arcsin(\sqrt{\rho} \sin u), \quad u_2 = u + \arccos(\sqrt{\rho} \sin u), \]

and the numerator reads

\[ f(a, \varphi, \rho, r, u) = (1 - \rho) [r - 1 + \rho (a + r(a - 1)) (1 + \cos(2u)) + \sqrt{a(1-a)}\sqrt{1 - \rho}(r + 1) (\cos \varphi + \cos(\varphi + 2u))]. \]

Performing the integrations we arrive at the result

\[ \langle x \rangle \approx (1 - \sqrt{1 - \rho})(a(r + 1) - 1) + \frac{\sqrt{a(1-a)}(1 - \sqrt{1 - \rho})(1 - \rho)(r + 1) \cos \varphi}{\sqrt{\rho}(1 - \rho)} + \frac{r - 1}{2} \sqrt{1 - \rho} + O(t^{-1}). \]

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