 Seeing Things From Others’ Points of View: Collaboration in Undergraduate Mathematics

Greg Oates, Judy Paterson, Ivan Reilly, and Grant Woods

Abstract: We report on three approaches taken to incorporate collaborative activities into undergraduate mathematics classes. There is strong evidence from research in K-12 classrooms that these, and similar, approaches support a range of positive learning outcomes for students. Despite the potential benefits the cited studies have shown, research into the use of such methods at the tertiary level is limited. We describe the ways in which we have implemented research projects, collaborative tutorials, and team-based learning in a range of undergraduate mathematics classes in two countries. We present quantitative and qualitative evidence from these teaching experiences to support our claim that there is a definite mandate for significant opportunities within our courses for students to work cooperatively, talk together, and argue about mathematics.

Keywords: Collaboration, undergraduate mathematics.

1. INTRODUCTION

Various styles of group learning are widely accepted and have long been advocated as effective learning strategies in mathematics (e.g., [8, 11, 19]). Belief in the value of collaborative learning is evidenced at the secondary level. For example, five key competencies are the backbone of the 2010 New Zealand school curriculum: thinking; using language, symbols, and texts; managing self; relating to others; participating and contributing [20]. Relating to others, in particular, highlights the importance of: interacting effectively with a diverse range of people in a variety of contexts. This competency includes the ability to listen actively, recognise different points of view, negotiate, and share ideas.

In common with Olson et al. [26] and Rasmussen et al. [29], we believe that research in K-12 classrooms can inform teaching at the tertiary level. This research strongly suggests that approaches that foster discourse among students...
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and between students and the instructors offer the most promise in improving student achievement (see, for example [15]). In particular, two meta-analyses have shown the positive benefits obtained from the use of cooperative learning in mathematics [10, 32]. Springer, et al. [33] conducted a meta-analysis that integrates research on undergraduate science, mathematics, engineering, and technology (SMET) education since 1980. Their analysis demonstrated that various forms of small-group learning are effective in promoting greater academic achievement, more favourable attitudes toward learning, and increased persistence through SMET courses and programs. They concluded that the magnitude of the effects reported in their study exceeded most findings in comparable reviews of research on educational innovations and supported more widespread implementation of small-group learning in undergraduate SMET. Grouws and Cebulla’s meta-analysis of 80 studies that compared student achievement in whole-class settings with that in collaborative groups very strongly endorsed the use of collaborative approaches [10]. The importance of collaboration in tertiary studies in general is recognized in recent changes to the Assessment of Student Learning Policy at the University of Auckland [35], which actively encourages group assessment and publishes clear guidelines to ensure that the assessment is fair to all students and appropriately matched to the learning objectives of a course.

Smith described similar collaborative objectives, specifically for mathematics at the tertiary level [31]. Academically, collaborative learning approaches are claimed to enhance higher-order cognitive abilities [8, 28, 32]; support generalization as a “dynamic, socially situated process” [9, p. 308]; stimulate creative and critical thinking, and allow the focus to shift from covering content, to applying the content to solve real and meaningful problems. It supports an environment of active exploratory learning, fosters oral communication skills, and helps students clarify ideas through discussion with others [2, 19]. It often leads to an extension of “students” personal example space [27, 38] and the ability to ask themselves critical questions [27]. Collaborative learning can greatly assist in facilitating the transition from school to university, supporting students as they adjust to different learning environments and different teaching practices such as large-scale impersonal lectures [1, 24].

From a psychological perspective, collaborative learning has been shown to significantly reduce test anxiety and build self-esteem in students [3, 23]. It encourages students to seek help and accept tutoring from their peers [19, 21], enhances the satisfaction of both students and academic staff with the learning experience, and fosters team building and a team approach to problem-solving while maintaining individual accountability [11, 19]. It also develops social interaction skills, creates a stronger social support system, and fosters a greater ability in students to view situations from the perspective of others [2, 3, 23]. A study of student performance in a medical gross anatomy and embryology course indicated that the students who benefited most from team-based learning (TBL) were the academically at-risk students “who are
forced to study more consistently, are provided regular feedback on their preparedness and given the opportunity to develop higher reasoning skills” [22, p. 56]. Technological advances have even given students studying at a distance opportunities to interact with each other collaboratively using tablet-PCs and screen-recording software, in much the same way as research mathematicians now often work in international research teams [25].

However, not all experiences are positive. A descriptive study by Das Carlo et al. [7] found that the most prolific tutorial groups consisted of female group members, by comparison the male groups were less productive and even, in some cases, unproductive. Another study by Wolfe and Powell [39] found that mixed-gender team projects might contribute to female students’ negative experiences and increased attrition rates from their courses. Their study found that differences in male–female discourse often drew negative responses from males towards female contributions, and their study concluded that some coaching might be necessary to ensure teams worked more positively. Evidence exists of other possible negative aspects of working collaboratively. These include inappropriate interactions, conflicts, and disruptive group dynamics, hangers-on benefitting from others’ work, and ineffective tasks [11, 24, 39].

Despite the potential benefits shown by the cited studies, and books describing ways to effectively foster collaborative learning in tertiary mathematics (e.g., [11]), more research into the use of such methods at the tertiary level is needed [8, 33]. Opportunities for students to work together in groups or teams are still not common in undergraduate mathematics courses, where lecture-style delivery methods still dominate [8, 41]. Given the evidence in favor of collaborative methods, we believe we should continue to investigate ways of effectively utilizing these approaches. Ways of minimizing the potential negative aspects described should also be considered in any such studies. TBL, for example, offers a partial solution to student concerns about sharing marks with others who do not pull their weight, by providing a component of individual assessment within the collaborative environment [16–19, 28].

This paper arose out of a series of discussions between the four authors who work at universities in Canada and New Zealand. Two of the authors are topologists, the other two are mathematics educators. All four are involved in teaching tertiary mathematics courses at both undergraduate and graduate levels. Thus, while we draw on research from mathematics education, the findings and experiences reported in this paper are directly aimed at mathematicians who are interested in extending their teaching approaches. We present three examples of collaborative approaches used in undergraduate mathematics courses with which we have been associated. Consequently, a number of “voices” may be discerned in the writing.

The first example discusses collaborative “research projects” (see [6]) for small classes of highly motivated and talented students at the University of Manitoba in Winnipeg, Canada. The other examples are based on courses at The University of Auckland. The first of these examines the use of small-group
tutorials in a large foundation calculus course. The second describes the use of TBL in higher-level courses, specifically, third-year courses in mathematics education and combinatorics and a post graduate course in dynamical systems. The locale and the sophistication of the content and students vary significantly between the three studies. Nonetheless, there is a significant common thread in the requirement for students to talk to each other about the mathematics that they are learning and using. Indeed, it is important for students to do more than talk about the mathematics: they need to learn to make and articulate conjectures [14]; to argue with their colleagues; and to apply their knowledge as they come to decisions, as a team, about the best solution to the problem they are trying to solve or the choices they are making [16].

We suggest that working collaboratively enhances their understanding of the material and the enjoyment they experience in the process of acquiring that understanding. Moreover, they are behaving as mathematicians. As Smith declares in his discussion of the calculus reforms at college level in the United States:

It is important for a learner to develop a repertoire of learning styles, and it is important for teachers to encourage that development. One way we encourage students to learn new learning styles is by placing them in heterogeneous groups to solve problems. Each student brings to the group activity her or his prior problem solving experiences, and the mix is often richer than the sum of the parts. As students learn from each other, they also learn the value of others’ learning styles, and they begin to add aspects of those styles to their own view of learning [31, p. 37].

2. COLLABORATIVE RESEARCH PROJECTS AT THE UNIVERSITY OF MANITOBA

Collaborative research projects have been used in two mathematics courses at the University of Manitoba, the first being a two-semester course (September to April) in introductory single-variable calculus, intended for students with a strong aptitude for, and interest in, mathematics; the second, a third year (“junior”, in American parlance) one-semester course in the theory of metric spaces; for students in the Mathematics Major or Honors program.

2.1. What is a research project?

For the purpose of this paper, a research project is a problem, based on the course curriculum, but often going beyond it, that is too large and complex for the average strong student to solve in only a few hours. Note that in the strict mathematical sense these may be regarded more as explorations than
research, however, this is the way they are promoted to students. An excellent description of the use of research projects in teaching calculus courses at New Mexico State University from 1986 to 1989 is given in Cohen et al. [6]. A research project is intended to be undertaken by a group of students, ideally three (see [11]), who work together in a designated time frame to solve the problem and prepare a detailed written description of the solution. Similar group projects are common in engineering courses [5, 39]. The exact design of the project depends on the nature of the course, for example, courses with an “applied” component may stress mathematical modeling, courses in “pure” mathematics may stress the making and (dis)proving of conjectures and generalizations. According to Cohen et al. [6] a good research project will usually have most of the following features:

1. It will be based on the course curriculum, but will introduce new concepts.
2. It will stress conceptual understanding more than algorithmic processes.
3. It will contain both specific questions and “open-ended” queries.
4. It will involve combining distinct topics and concepts in novel ways.
5. It will involve creating a clear, complete, and grammatically correct written solution, explaining what was done and why it works.
6. It will lend itself to subdivision into pieces that individual group members can pursue.

The last criterion differs from the approach taken in the two examples at the University of Auckland that are in-line with the assertions of Hagelgans et al. [11] and the TBL literature [19] who argue that tasks should ideally be non-divisible and involve the whole group. If the course in which a project is used has some applied aspects (e.g., introductory calculus), it is also desirable that the project presents a real-world situation in non-mathematical terms. Translating the problem into mathematical language and translating the solution into the non-mathematical terms of the original problem are important parts of the project. Research projects aim to duplicate for students, to some extent, the experiences of research mathematicians; the clarification of thought processes gained by trying to explain a half-understood idea or difficulty to someone else; and the gaining of additional insight into a research problem through the process of writing up one’s partial research results and thereby having to organize one’s ideas. Intense concentration on a difficult problem, with frustration followed (sometimes!) by the elation of apparently sudden progress, is another hallmark of research. We also want students to learn how to write a lengthy mathematical document clearly, concisely, and in proper English, using the appropriate mix of words, symbols, and (perhaps) diagrams. Finally, particularly for students in more advanced undergraduate courses, we want to expose them to what one might call the manic-depressive aspects of doing research. With luck this will raise their frustration threshold.
2.2. Research Projects in an Introductory Calculus Course

This section reports on the first example from Manitoba, namely, the use of research projects in a 26-week introductory single-variable calculus course (Theory and Applications of Differentiation and Integration). Two classes of 20 or 25 students of above average background and aptitude were involved. Four research projects make up 30% of the course assessment, the remaining assessment being from the more traditional tests and exam. About two weeks into the course, we divided the class into groups of two, three, or four. We warned the class ahead of time that we were going to do this, and told them that they could have input into the group formation. We pointed out that they would need to meet frequently and that ease of doing so was important. (Note the difference from TBL where the teams are constituted by the lecturer with the aim of distributing useful attributes fairly between teams, and in both the other approaches where all team work is completed in class time.) On the day that the groups were formed we helped “orphans” get organized into groups. Each group was given a week to complete two simple “pre-calculus” research projects. These were graded and returned with comments (no course credit involved). The purpose was to let the students know the standards of exposition that were expected. We then distributed the first assessed project, with a two-week deadline for completion. We indicated that late projects would not be accepted; as a result, all were submitted on time. The intention was to form constant groups through the course, but some rearrangement of groups proved necessary. We needed to accommodate students who withdrew (three students dropped out after the first project!), or sort-out dysfunctional groups. In one instance we split a group of two weaker students and adjoined them to other stronger groups.

If a group became stuck (and inevitably groups did, as the projects were challenging), it could send one of its number to consult us. That member had to report back to his or her group, thereby placing importance on listening skills and expository skills among group members. If a second consultation were necessary, a different group member had to consult us. In this way most group members had the experience of listening and relaying our suggestions to others.

The written projects were graded and returned about a week to 10 days after being submitted. In grading them we paid attention to the clarity and completeness of their exposition, as well as to mathematical shortcomings. To give them some guidance we distributed the pamphlet *Guidelines for Written Homework* [6], which gives advice to students on how to write solutions to mathematical problems.

After the marked projects were returned we interviewed each group member individually. This had two purposes: to learn whether they knew how to rectify the shortcomings that we had mentioned when we marked the project, and to assess how thoroughly they understood the write-up that they had supposedly helped to create. On a seven-mark project, four or five marks would be based on
the common written solution and two or three would be based on the individual interview. This helped to alleviate the problem of lazy “hangers-on” coasting on the efforts of the more industrious and knowledgeable group members.

On the first occasion that we used research projects in this course, we assigned one “pure” mathematics problem and three “modeling” problems. A poll of students’ attitudes at the end of the course revealed that the modeling problems were much more popular than the “pure” problems, and on the second occasion that we used research projects in this course we used nothing but “modeling” problems. This experience agrees with that noted by Cohen et al. [6]. Examples of the modeling problems that we used (and some others) can be found in *Calculus Mysteries and Thrillers* [40].

Towards the end of one course, students were surveyed anonymously to find their opinion of the use of research projects. The majority thought that doing the projects had helped them understand calculus and how it is applied. Half thought that two weeks between distribution and collection of projects was about right; the rest thought more time would have been better. Although most thought that the projects were at about the right level of difficulty; a few thought that they were too difficult. Almost all liked working in groups, thought that research projects enhanced the course, and would like research projects in subsequent math courses. They appreciated the social interaction, the support and work-sharing, and the practice in explaining and listening to others’ explanations. One student’s response was especially interesting. When asked “If you were to take another math course at this university, would you want it to contain a research project component?” they responded yes and no: “Yes – it helps me learn better. I’d be glad when the course was over that it had research projects” and “No – more work.”

The process described above is obviously labor-intensive for the instructor. A class of 25 yields eight or nine research groups, which probably is the upper limit that one person can handle. Once when we taught the course, we had the assistance of an extremely able Teaching Assistant (a mathematics major in his final undergraduate year who subsequently earned a Ph.D. in mathematics). We trusted his competence and judgment sufficiently to subcontract the interviewing of some group members to him (he enjoyed the responsibility). Our observations concur with student feedback that the research projects are effective in developing student understanding and thus warrant the necessary level of commitment.

### 2.3. Research projects in an advanced metric space course

Collaborative learning can also be very useful in higher-level courses (see [21, 27, 32]). The second example of the use of research projects at Manitoba is one such course. It is a one-semester (13-week) course on the topology of metric spaces, including such topics as the definition of an abstract metric
space, Cauchy sequences, and the Banach contraction theorem. Students in this course were in their third or fourth year of a math major program (either single, or joint with one other subject, usually physics). All were reasonably strong students, and some later earned doctorates. They had all taken a rigorous introductory course in analysis, and were familiar with the notions of continuity, convergence of sequences, and $\varepsilon - \delta$ proofs in the setting of Euclidean $n$-space.

We have used projects several times when teaching this course, with similar organization methods to that described previously, except that in contrast with our practice with introductory calculus, we gave different projects to different groups, giving each group only one project to do, as projects in this course tended to be time-consuming and demanding. We also gave students slightly longer, 18 days, from their receipt of the project to its due date, which was about the end of week 9 of a 13-week course. The project was worth 15% of the final grade (split between the written project and individual interviews).

One class of 12 students was partitioned into three groups of four students each for their research projects, and they were surveyed anonymously at the end of the course. Students generally regarded the experience as positive (or at least said so to their professor), however, they all expressed difficulty in assembling four people with busy schedules for meetings to discuss the project. We chose three groups of four, rather than four groups of three, due to the perceived difficulty of finding more than 3 different research projects of the appropriate level of difficulty. In retrospect it may have been better to create four three-person groups and assign one of the projects to more than one group. Students’ opinions varied on whether they had the appropriate amount of time to complete the project, with some feeling it was about right and others feeling that there was not enough time. There was a general undertone of concern about the heaviness of the project. However, most valued the personal interaction and the stimulation of a “real” research experience. We should note, however, that the students had been told at the beginning that these were the benefits we expected, so they may just have repeated this back to us. Samples of the research projects used in this course are available from the fourth-named author for interested readers.

3. SMALL-GROUP TUTORIALS AT THE UNIVERSITY OF AUCKLAND

The next section discusses the first of two different approaches to collaborative learning at the University of Auckland. Small group tutorials were introduced into the teaching of a large foundation algebra and calculus course at the University of Auckland in 1993, and since that time they have become an integral part of the teaching and assessment of most undergraduate mathematics courses at the university. The initial use of collaborative tutorials was based more on the personal beliefs of course coordinators in the importance of
offering students a variety of learning opportunities and communicating in mathematics, than it was on following overseas or national curriculum trends reflecting socio-cultural learning theories. In addition to the focus on communicating mathematically, these tutorials also involve assessment of the group’s work, and are supported by the simultaneous development of peer-tutoring within the department. In the research projects, exposition is a key focus, here students are assessed equally as a group on their communication and solutions. Effective means of assessing group projects and team-based activities are a key element of all discussions about collaborative methods. There are a number of sources that provide useful suggestions for designing and assessing group projects at the tertiary level and mathematics in particular (see for example [4] and [36, Chapter 5]). Note that TBL provides its own assessment structure inherent in its design that has been successfully adapted for use in mathematics (see later discussion). A more in-depth discussion of the assessment techniques developed in the small-group tutorials and peer-tutoring in general at the University of Auckland is provided by Oates et al. [25].

3.1. Small-group Tutorials in a Foundation Calculus Course

Mathematics Two is an introductory one-semester (12 weeks) calculus and algebra course for students considered under-prepared for the department’s principal entry-level mathematics course. Course data shows that about two thirds of the students who take this course continue on in the department’s main first-year, non-mathematics major course Maths 108, largely to support their mathematics in other science and commerce courses. A small number of top students enrol in the department’s mathematics major course Maths 150. Experience has indicated that the remainder of students who do not continue in mathematics are taking the course for a variety of reasons, including supporting their return to study, language support, or to support other courses for which the level of mathematics in Maths 102 is sufficient. It is predominantly taught in a large lecture room (150–250 students) for three hours per week. In addition, we divided students once a week into smaller classes of about 25–30 students for tutorials. Five of the tutorials are of a more traditional nature, providing assistance with the standard written assignments, although even in these we encourage the students to work in groups and help each other. The remaining six collaborative tutorials are held every two weeks, where students are divided into self-selected groups of preferably three students, to work cooperatively on solving a mathematics problem, or a series of connected problems. The problems are drawn from real-life contexts employing mathematical applications. An example of a collaborative task, marking schedule and cover-sheet is given in Appendix A and is also available at https://www.math.auckland.ac.nz/people/goat001. These examples show how the emphasis is on discussion, and communication of ideas. The task, in this case, an exploration of the behavior
of the derivative of a periodic function and phase space based on Parkinson's Disease, asks students to interpret real-life findings in the light of the mathematical examples they develop. Assessment, as shown in the marking schedule and cover sheet, gives equal weighting to oral and cooperative skills, and written solutions, with the emphasis in their written solutions as much on the clarity and depth of their answers, as it is on the exact mathematical accuracy. Similar to the research projects used at Manitoba, the students are first offered a practice tutorial, which is not assessed and gives students experience in the nature of the problems they will encounter and the assessment process, as well as breaking the ice for those a little nervous about working in groups. At the time of the study reported on here, 173 students were enrolled in the course.

Although anecdotal feedback from both students and lecturers involved in the collaborative tutorials (colloquially called “collabs”) suggested that the tutorials had been a great success, it was felt that more rigorous evidence of the educational and social worth of the collaborative tutorials was needed to justify the department’s interest in extending the tutorials to other courses. This was especially so given that like the research projects described earlier, operating small-group tutorials across all courses is very resource-intensive (see by contrast the discussion on TBL later). We thus instigated a departmental investigation into the effectiveness and appropriateness of collaborative problem-solving activities in small-group tutorials. It was hypothesized that these tutorials would provide an effective and enjoyable complement to the existing traditional methods, in which all lecturer–student contact was in large lectures, albeit that one of the lectures was designated a “tutorial.”

We administered two surveys, one before and one after students’ exposure to the collabs. Some questions sought demographic information (e.g., age, gender, ethnicity, degree for which enrolled, future intentions in mathematics); others asked students to assess their own mathematical competency and their previous experience of collaborative problem-solving. Further questions aimed at determining students’ expectations and prior feelings towards the tutorials, and their subsequent experiences in them. These latter questions were repeated in both questionnaires to detect any changes in the students’ perceptions of the collabs. The second questionnaire also sought information about the frequency of students’ participation in the collabs, and gave them the opportunity to note any features that they particularly liked or disliked about them. A total of 118 students responded to the first survey, and 65 to the second, of which 55 had also responded to the first survey to allow for some matching of responses. Although these numbers may seem low, and suggest a dramatic drop-off in tutorial attendance, in fact nearly 70% of the 173 students attended all five of the collabs, with more than 85% attending three or more tutorials. Thus, although the drop-off in responses from the first to the second survey was of concern, it is likely other factors, such as impending examinations, lay behind the drop-off in response rate for the second survey, as opposed to lack of interest in the collaborative tutorials. Thus, we consider the overall response
rate of almost 74% to at least one of the surveys and 32% to both sufficient for the conclusions we draw from the study. The 55 students (32%) who responded to both surveys appeared evenly distributed in terms of demographics and question responses, so there is no obvious suggestion that they were in some way non-representative of the wider class. A more comprehensive discussion of the statistical analysis from this part of the study is given in [21].

3.2. Survey Results

We examined the data across a wide range of demographic factors including gender, age, and mathematical background. No evidence that these factors influenced responses was found, either in or between each of the surveys. However, the responses to the two questions repeated exactly in both surveys and the percentage responses were especially revealing, as compared in Table 1. Chi-squared tests were conducted on the total numbers of responses to each question in both surveys, to see if there were any significant differences between the numbers of students between surveys.

One notable feature of the figures presented in Table 1 is the approximately even spread of student preferences across the three options for their preferred style of learning. There was a significant increase in the number of students opting for working in groups after exposure to the collabs (increase of 15 students, \( \chi^2 = 4.74 \)), but overall, the split between the options remained remarkably stable and relatively evenly split between the three options. Again, the students’ beliefs about how they learn mathematics best remained stable over the course, 69% of the 55 students answering this question in each survey maintained their beliefs. The biggest change came from those switching from on their own to

| Statement | Percent of total responses (%) |
|-----------|-------------------------------|
| I prefer to solve mathematical problems: | |
| —on my own | 38 | 31 |
| —in pairs | 37 | 29 |
| —in groups | 25 | 40* |
| My understanding of mathematics is best helped by: | |
| —listening to explanations and practicing on my own | 66 | 54 |
| —trying unfamiliar problems on my own | 6 | 6 |
| —discussing and working in groups | 28 | 40 |

*indicates statistical significance at the the \( \alpha = 0.05 \) level, df = 1; \( \chi^2 = 4.74 \).
groups (18%), however, this increase towards groups was not statistically significant. The support for groups was more evident in a separate question in the second survey, when 66% of the students indicated that they would like to have more opportunities for collaborative work in future mathematics papers (only 15% disagreed, 19% neutral).

Some insight into the students’ reasons for their stated preferences can be gained from examining their responses to what they most liked or disliked about the collabs. As is common in surveys, these last two open-ended questions received fewer responses than the simple marking of boxes required for the other questions in the surveys. The most common reason students gave for liking the tutorials was having the opportunity to talk about mathematics and share ideas in a friendly environment. Sample student responses included “The ability to talk with others about problems and learn off others is good,” and “It helps in understanding things from other people’s point of view.” Indeed, 50% of the students gave responses that could be classified in this category. One student was particularly effusive in their praise, stating that “the collabs were the best thing that’s happened to me in maths.” The remaining responses were more concerned with specific features of the collabs, for example, the practical and interesting nature of the problems (16% made a statement of this nature), or the extra help available in a small class compared with a large lecture room (8%).

When asked what they disliked about the collabs, 33% of the students gave no response. Nearly all of these students had indicated earlier in the survey that they found the collabs interesting and enjoyable, so it is probably a fair assumption that these students could find no particular aspects that they disliked. Most negative responses could be associated with factors of group dynamics, such as dominance, or language barriers (15%), and the nature of the assessment (16%). Sample student responses included “Giving all people in the group the same mark is not fair.” This is an important concern. It is countered in TBL through using peer evaluation of contribution in the allocation of grades and in the research projects through individual interviews. Others complained that “people in your group who don’t get the assignment hold you up, especially when they don’t speak English properly.” One student, who attended only the first of the tutorials, gave a strongly negative response. This student noted that they “didn’t want (their) own mistakes to influence a whole group” and that they “didn’t want to look stupid in front of a group,” feeling so strongly about this that they then stopped attending the tutorials. This anxiety about exposing one’s ignorance is an issue highlighted in the literature [2, 3].

The results of the survey certainly support the effectiveness of the collaborative style of learning for meeting a large proportion of students’ preferred style of learning, as shown in Table 1. There was an increase in the number of students stating that working in groups was their preferred style after exposure to the collabs; however, a sizeable group (31%) still indicated that they preferred to work on their own when solving mathematical problems.
Other studies have shown similar positive effects in changing students’ opinions toward collaborative learning, e.g., Alexander and De Alba [2] reported that 85% of their students described the collaborative assignments as enjoyable and beneficial, compared with 41% who were looking forward to them at the start. However, the strength of some students’ negative reactions to the collabs suggests some students will maintain their preferences for individual learning. No other studies examined in the review of the literature revealed the differentiation in student preferences and their stability that we found. As shown in Table 1, each of the three learning preferences remained remarkably stable at about one-third over the course of the study.

One source of potential bias was raised earlier concerning the response rate to the surveys. It seems eminently plausible that students who have a real fear or dislike of the collabs will fail to attend and thus not be surveyed. Anderson, for example, suggested that students with low self-confidence in their mathematical ability feel uncomfortable in a group setting, and may drop a course primarily because they do not wish to be involved in collaborative learning [3]. However, since 74% of the students responded to at least one of the surveys, the finding that a majority of the students do not prefer to work on their own when solving problems is still valid. Indeed, Anderson claimed that these same reluctant students often show a complete change of heart towards enjoying the collaborative experience, if they can be persuaded to participate [3], which seems to be supported by the change towards collaborative learning shown by some respondents in Table 1, and is likewise similarly supported by students responses to TBL (see later). We draw attention to the fact that the shift of 15% towards group work occurred over a comparatively small exposure in 11 tutorials, only six of which specifically focused on collaborative tasks.

4. TBL AT THE UNIVERSITY OF AUCKLAND

This section discusses the second of the two different approaches to collaborative learning at the University of Auckland. TBL is used in three courses at Auckland University – two third year and one post-graduate.

4.1. What is TBL?

TBL is not just “group work” of any kind. It is a pedagogical model that shifts responsibility for learning to the students. It motivates students to prepare for class and increasingly hold each other accountable for doing so. Although the use of TBL is comparable to ‘flipping the classroom’ [37] it is the particular use of collaborative task work that is the focus of this paper, not the entire TBL approach described in [27, 28].

The constitution of the teams is one of the key characteristics of this model. They usually have five to seven students, bigger groups than in the research
projects and the collaborative tutorials. In addition, unlike the collabs, teams are fixed for the duration of the course and work together on all team tests and tasks. These teams are not friendship groupings, but resemble the types of teams businesses construct to maximize productivity. The teams are constructed by the lecturer to distribute as fairly as possible the skills, knowledge, and attributes needed to solve problems in the context of the course. Readers who are unfamiliar with TBL may wish to view the 12-minute video at the TBL website for more detail on the structure of TBL [34].

Lecturers in the quantitative sciences are yet to make much use of the strategy, despite its increasingly widespread use in the health sciences and business courses [18]. This is in line with the D’Souza and Wood’s observation that lecture-style delivery still predominates in mathematics [8]. The impetus for implementing TBL at the University of Auckland came from the second author’s attendance at workshops given by Larry Michaelsen and Dee Fink in 2008.

4.2. TBL at the University of Auckland

In this section we focus on two phases of TBL and the learning opportunities these provide. The first phase of each module in a TBL course requires students to prepare for lectures by studying carefully selected pre-readings. Their readiness for the module is established in two opportunities to do the same multiple-choice test, once individually and once in their fixed teams. This is known as the readiness assurance process (RAP). When doing the readiness assurance test (RAT) as a team, the students receive immediate feedback on their answers using immediate feedback assessment technique cards. Each question has four options shown with the correct answer indicated by a star when the covering is scratched off. If students are not correct the first time they return to their discussion to gain part marks for being correct the second or even the third time they scratch and win. This immediate feedback means students always know the correct answer by the end of the RAP. The power of immediate, goal-directed feedback has been identified in large-scale meta-studies by Hattie and Timperley [13]. An example of a pre-reading, an RAT and tasks are given in Appendix B and can also be accessed at https://www.math.auckland.ac.nz/people/goat001. These examples are taken from a third-year Combinatorics course Maths 326. The documents show the significantly complex level of mathematics the students are asked to cope with in the RATs and tasks. Readers can also see how easy the RATs and tasks are to mark for instructors. In a class of 50 students, there would be eight to nine teams with only 10 multiple-choice questions to mark for each team.

The lecturer also gets immediate feedback from the students’ performance on the multiple-choice tests. Hattie [12] claims this feedback is crucial to effective teaching and it is feedback to teachers that is the most powerful and that makes learning visible. He claims, on the basis of a very large meta-study,
that it is when teachers know what students do and do not know and where they make errors that teaching and learning can be powerful and synchronized.

Lecturers can then use lecture time to work on areas identified as problematic or challenging, through low scores on particular questions, instead of spending class time delivering concepts the students have shown they are able to learn for themselves by reading and discussing with peers. There is also time for more traditional delivery of mathematical ideas that require exposition. This feedback is similar to the feedback a lecturer gets when students arrive with a group question in the research projects or the insights gained by a lecturer observing and questioning students as they work on a collaborative learning assignment.

TBL is also used in the third-year mathematics education course Maths 302; Teaching and Learning Mathematics. Responses to a final examination question about TBL provide insightful feedback on their learning experiences. The question asked:

In MATHS 302 you worked together in teams for the entire course. In addition you were required to pre-read papers for the RATs that you then answered both as individuals and as a team. Discuss these teaching strategies in the light of the research on learning theories and understanding you have studied in this course.

Their responses demonstrated very clearly that they valued the experiences and could argue that their learning had been positively affected. They commented on a number of features of the course:

- From a constructivist point of view the instant feedback makes sense – to learn from our mistakes and adapt to our environment this kind of testing is far more beneficial than a number out of 10 you receive a week later
- At first I wondered why I needed to answer the same question twice. Now I realize that this enabled me to see the difference when I worked with and without help
- Although at first it seemed that the groups might hold back students working at higher levels, they appear to have worked by encouraging these students to study the material in more depth – developing the breadth and depth of their knowledge rather than accelerating them through the curriculum.

A second important phase in the learning cycle in TBL involves students working in teams, during class on application tasks. The class time is earned by the pre-reading and discussion process that ensures students are familiar with material that would ordinarily have been delivered in a lecture. The students valued these tasks. One student remarked: “This is like working with a team in the real world where we very seldom work on our own – co-operation is vital to any career.”
In the MATHS 302 course one module focuses on comparing international curricula. The students read articles on the subject and examined curricula on the web. Their task was to write a letter to a family who were about to leave New Zealand for a year to live with their grand-parents in a foreign country. They could assume that the children were fluent in the foreign language. They had to explain to the family what differences and similarities to learning mathematics in New Zealand they could expect to find in the new country. The students saw their learning while doing this, and similar tasks, in a very positive light:

Each of the tasks was small enough that you could focus on what was trying to be taught and not on trying to get a good mark. Although, because of this interest focus in the topic the group would still get a good mark. This idea of getting a good mark because you understand the material and not solely because you were aiming for a good mark is what all assessments should aim for.

Lecturer observations support this positive student feedback about their learning – both in terms of developing understanding and ability to communicate.

The creation of significant tasks is a key component of TBL [17, 19]. Commonly lecturers find that they can set students more demanding tasks than they previously considered appropriate for the course. Paterson and Sneddon [28] have described how the nature of task questions and problems in the MATHS 326 combinatorics course evolved over a period of 4 years. They found evidence that the questions became more demanding and open-ended, encouraging exploration; that the students had to think more and do less computation; and that they were asked to give explicit explanations for the choices they made. The lecturer said:

I wanted to shift from asking them to construct a known Basic Incomplete Block Design (BIBD) to asking them to construct a BIBD with less well-defined properties, so that some investigation is required. I like the hands-on part of the task, but I wanted to also give them a chance to explore. [28, p. 886]

The students also learn how to work together to maximize their output on the application tasks. In the post-graduate course on dynamical systems two shifts in team behaviors were apparent. Initially, the students checked and re-checked one another’s computations and did not complete the more demanding, synthesizing aspects of the task. In subsequent tasks, they each checked only one other result and then used the information to create the phase portraits. They also developed a good understanding of one another’s strengths: one was an excellent programmer, another more skilled analytically, and a third had a facility with language. Together they were a formidable team.

Shaw and Barrett–Power [30] emphasised the importance of responding positively to diversity in work groups “Diversity is an increasingly important
factor in organizational life as organizations worldwide become more diverse” [30, p. 1307]. The following quote from a Mathematics Education student illustrates the value they attached to working closely, and for an extended period, with a diverse team:

Each group had a mixture of different ethnic and educational backgrounds. This difference was a benefit to all in the group as different viewpoints were being expressed at all times. This developed to a point where students would ask questions to other people in reference to how they would learn or do something in their culture while still being culturally sensitive.

In the combinatorics course students were interviewed about their feelings towards TBL. In common with the research projects and the collabs many students showed a shift in perspective. For example:

I actually thought I was going to hate it. I hate team work. But I actually did like it. And I did actually enjoy it in the end. So maybe it is my style of learning, even though I didn’t know it was. I liked it because actually you could talk about stuff.

5. SUMMARY

We have reported here on the use of three styles of collaborative learning techniques employed in the teaching of tertiary-level mathematics courses. Although the form of the collaboration, the level of the courses, and the countries involved differ in the three studies, a number of commonalities and differences are apparent.

The findings of all three studies suggest there are considerable benefits to be derived in both conceptual understanding and expository ability from learning mathematics in small collaborative groups or teams. When describing their rationale behind the research projects used in Case Study A, Woods states four objectives that they aim to achieve in teaching their calculus course [40]:

a. Algorithmic and computational knowledge (e.g. differentiation formulae, integration techniques, finding equations of tangent lines etc.).
b. An understanding of the “big ideas”: derivatives measure rates of change, definite integrals are used to compute how much of “something” is there by “adding up” an infinitely large number of infinitely small amounts of that “something”, and how derivatives and definite integrals are related.
c. Theoretical ideas (e.g. continuity, proofs of differentiation formulae, mean value theorems, the fundamental theorem of calculus).
d. How to apply calculus (optimization problems, related-rate problems, velocity and acceleration, exponential growth and decay).
In a typical mathematics course, much of the time is spent on (a) and (c). Doing exercises and checking answers at the back of the book, for example, mostly reinforces just (a). Woods argues in fact that (b) and (d) are more important for 90% of our students, and he notes that although lecturers do try to achieve (b) and (c), it is perhaps with limited success, and they usually rush through (d), often giving students a merely algorithmic approach to such problems. Students receive very little feedback about how well their solutions are written, except through summative marking of assignments, tests, and examinations. The collaborative strategies described here offer opportunities to put more emphasis on (b) and (d).

Many studies suggest, or are motivated by, the belief that students working in teams are able to tackle more complex real-life problems because of the support they receive from team members (e.g., [5, 21, 24, 28]). Paterson and Sneddon [28] certainly concluded that this is the case in the latter author’s use of TBL in the combinatorics course, as can be witnessed in the examples provided in Appendix B. Although there are so far only limited studies that provide concrete evidence to support such claims [33], all lecturers involved with the three collaborative approaches described here emphasised the value of being able to set more demanding and realistic tasks as a significant motivation for their use.

Students frequently commented that they valued the opportunities to communicate and share ideas, and the development of a collegial work ethic. This is in-line with the literature studies (e.g., [8, 11]), and reflects several elements of the key competencies described in the 2010 New Zealand Mathematics Curriculum. Positive collaborative learning experiences seem to shift students preferences towards such methods, with many students in all three modes of delivery noting that they would like similar opportunities in future mathematics courses. Significant common benefits are also seen in the provision of more immediate feedback about students’ learning, both from the nature of the assessment itself, and from the interactions within the group. In all three modes students receive both group and individual grades that is seen as a necessary requirement for effective group learning. Some induction of students into the new style of learning is also needed. In the research projects and collabs, this is achieved through the use of non-assessed practice tasks and in TBL through an introductory RAP.

Differences between the approaches include the structuring of groups, where the tasks are done and the level of resourcing required. In the research projects and the collabs the students generally self-select their groups, often based on existing friendships, with three to four students per group. However, in TBL the teams are created by the lecturer to create balanced teams of five to seven students. The research projects differ from both the others in that the students need to do most of their tasks outside of class. All three approaches require additional resourcing, for example, in interviewing each student in the research projects, monitoring group participation and marking solutions.
in the collabs, and developing suitable RAPs and application tasks in TBL. However, once established TBL requires substantially less marking than other approaches, as detailed in [27, 28].

Some notes of caution should be raised. For the research projects, Woods notes that one should take care not to make the experience too logistically complex for students (e.g., a group’s ideal size is three to facilitate meetings) and it should be timed so that it minimally interferes with their other obligations in this and other courses [40]. For the small-group tutorials, almost one-third of students continued to show preferences for working on their own with a small number of students expressing an extreme dislike of cooperative learning methods. The feelings of these students must also be recognized on adopting cooperative methods as a teaching and learning strategy.

Finding and developing tasks of the right level of complexity is a challenge for all. One needs to do them completely ahead of time to avoid unanticipated complications and difficulties. Students’ feelings about hangers-on in shared assessments should be considered in any collaborative approach, with preferably some component of individual appraisal included.

However, despite such cautionary notes, the finding from the small-group study that around two thirds of the students consistently showed preferences for working in groups (as opposed to individually) and the qualitative data from the TBL study [27] clearly indicate that the standard lecture delivery method is not catering to the needs and/or the preferences of the majority of our students. This does not mean that we should change unilaterally or exclusively to collaborative learning methods, but there is a definite mandate for significant opportunities within our courses for students to work cooperatively, talk together, and argue about mathematics. Indeed, even though nearly a third of the students in this study still preferred to work on their own, if they are to realize the value of using different strategies [31] it is vital they are given opportunities to interact. A quote from one of the TBL students sums up the most important benefit of working collaboratively:

Someone else always has a different view on the mathematical searches than you have, it really makes you understand what you’re talking about because sometimes you’re so limited to your own way of thinking that if you look at it from another way, the solution is really easy but if you focus from your way then you get stuck at some point, if you do it with four or five people solving exercises it’s quite easy because a lot of knowledge and a lot of different views on the same thing and it really helps you to expand your way of thinking.

DEDICATION

Judy Paterson sadly passed away in early 2015 after a long illness. We dedicate this paper to Judy’s memory, in particular her pioneering work in developing the use of TBL in undergraduate and postgraduate mathematics which we
report here. TBL has permeated courses at all levels in our department and has received critical acclaim internationally in mathematics education circles. Judy has left an enduring legacy of promoting and examining the effectiveness of alternative learning and assessment practices in our department.

SUPPLEMENTAL MATERIALS

Supplemental data for this article can be accessed on the publisher’s website.

ORCID

Greg Cates  http://orcid.org/0000-0001-7132-3198

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BIOGRAPHICAL SKETCHES

Greg Oates is a Senior Lecturer in the Mathematics Department at the University of Auckland. He received his Ph.D. in Mathematics Education from Auckland University, investigating the pedagogical use of technology in undergraduate mathematics. His current interests include collaborative methods, peer tutoring, and the effects of technology on curricula content, course delivery, and assessment.

Judy Paterson was a Senior Lecturer in the Mathematics Department at the University of Auckland. She received her Ph.D. in Mathematics Education from Auckland University. Her recent research focussed on teaching and learning in undergraduate mathematics. She was particularly interested in content-based professional development both for secondary teachers and university lecturers.

Ivan Reilly is Emeritus Professor of Mathematics and Mathematics Education at the University of Auckland. His Ph.D. is in Topology from the University of Illinois, Urbana-Champaign. His research achievements have been honoured with the award of a DSc by his New Zealand Alma Mater, Victoria University of Wellington.

Grant Woods is Emeritus Professor of Mathematics at the University of Manitoba, Winnipeg. He received his Ph.D. from McGill University, Montreal, and his major research interests are in topology. He has (co-)authored books in topology and in undergraduate mathematics education.