Abstract

We investigate the connection between translation bases for Paley-Wiener spaces and exponential Fourier bases for a domain. We apply these results to the characterization of vector-valued time-frequency translates of a Paley-Wiener “window” signal.

Keywords: Paley-Wiener spaces, Riesz bases, frames, vector-valued Gabor system.

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1. Introduction

The main goal of this paper is to develop a correspondence between exponential basis on a domain in the Euclidean space and translation bases for Paley-Wiener spaces. Let \( \Omega \) be a Borel subset of \( \mathbb{R}^d, d \geq 1 \), with positive and finite measure. We define the Paley-Wiener space \( \text{PW}_\Omega(\mathbb{R}^d) \) on domain \( \Omega \) to be the subspace of all \( L^2(\mathbb{R}^d) \) functions \( f \) whose Fourier transform are supported in \( \Omega \). More precisely,

\[
\text{PW}_\Omega(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \in \Omega^c \}.
\]

For \( f \in \text{PW}_\Omega(\mathbb{R}^d) \), we say \( \hat{f} \) is nowhere zero on \( \Omega \) if \( \hat{f}(\xi) \neq 0 \) for all \( \xi \in \Omega \). Given a countable set \( A \) in \( \mathbb{R}^d \) and \( \phi \in L^2(\mathbb{R}^d) \), we say the system of translates \( \{ \phi(\cdot - a) \}_{a \in A} \) is a Riesz basis for \( \text{PW}_\Omega(\mathbb{R}^d) \) if there exist positive and finite numbers \( C_1, C_2 \) such that for any \( f \in \text{PW}_\Omega(\mathbb{R}^d) \) there is \( \{ c_a \}_{a \in A} \in l^2(A) \) such that

\[
f(x) = \sum_{a \in A} c_a \phi(x - a)
\]

and

\[
C_1 \sum_{a \in A} |c_a|^2 \leq ||f||_2^2 \leq C_2 \sum_{a \in A} |c_a|^2.
\] (1.1)

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In this case, the constants $C_1$ and $C_2$ are called Riesz constants.

For the rest of this paper, we denote by $A$ a countable set in $\mathbb{R}^d$ and by $\Omega$ a Borel set of positive and finite measure. And, by $\hat{\phi}$ we mean the Fourier transform of $\phi$, defined, as usual by

$$
\hat{\phi}(\xi) = \int e^{-2\pi ix \cdot \xi} \phi(x) dx.
$$

We say set $A$ is a Riesz basis spectrum for $L^2(\Omega)$ if there exist positive and finite constants $C_1, C_2$ such that for any $f \in L^2(\Omega)$, there is $\{c_a\}_{a \in A} \in l^2(A)$ such that

$$
f = \sum_{a \in A} c_a e_a,
$$

and

$$
C_1 \sum_{a \in A} |c_a|^2 \leq ||f||^2_{L^2(\Omega)} \leq C_2 \sum_{a \in A} |c_a|^2.
$$

Here, $e_a(x) := e^{-2\pi i \langle x, a \rangle}$, $a \in A$, $x \in \Omega$, and $\langle x, a \rangle$ is the inner product in $\mathbb{R}^n$. In this case we call $\Omega$ a Riesz basis spectral set. Analogously, we say $A$ is an orthonormal basis spectrum and $\Omega$ is orthonormal basis spectral set if $\{e_a\}_{a \in A}$ is an orthonormal basis for $L^2(\Omega)$. See the seminal paper by Fuglede ([14]) for background on orthonormal exponential bases.

In many problems in harmonic analysis, approximation theory, wavelet theory, sampling theory, Gabor systems, and signal processing ([4, 2, 16, 23, 24, 25]), it is interesting to construct a function $\phi \in L^2(\mathbb{R}^d)$, such that the set $\{\phi(\cdot - a)\}_{a \in A}$ is an orthonormal or a Riesz sequence. The structure of $\phi$ for which $\{\phi(\cdot - j)\}_{j \in \mathbb{Z}^d}$ is an orthonormal basis or Riesz basis for the closure of its spanned space in $L^2(\mathbb{R}^d)$ topology has been extensively studied in, for example, ([4, 5, 17, 6]). The techniques used in these papers strongly rely on the periodic tiling property of $[0, 1]^d$ for $\mathbb{R}^d$. In general, the same techniques can not be used for the characterization of general translates of $\phi$ when $\mathbb{Z}^d$ and $[0, 1]^d$ are replaced by any countable set $A$ and any Borel set $\Omega$, respectively. In this paper we extend the results of the characterization of translate bases where $A$ does not necessarily have a group structure. In our paper, we shall focus our attention only on the class of Paley-Wiener functions $\phi \in PW_{\Omega}(\mathbb{R}^d)$.

This paper is structured as follows. The first main result is stated in Theorem 1.3 in Section 1. In Section 2 we prove Theorem 1.3 and we establish a correspondence between bases of the type $\{\phi(\cdot - a)\}_{a \in A}$ in $L^2(\mathbb{R}^d)$ and spectrum $A$ for $\Omega$. Moreover, we prove that under some mild assumptions on $\phi$, the necessary and sufficient conditions for $\{\phi(\cdot - a)\}_{a \in A}$ to be a Riesz basis for $PW_{\Omega}(\mathbb{R}^d)$ is that $A$ is a Riesz basis spectrum for $\Omega$. In Section 3, we extend the results of Section 2 to general bases, i.e., frames. In Section 4, we use our
results to characterize the structure of Paley-Wiener valued Gabor systems. In particular, we use the result of Theorem 1.3 on the characterization of exponential functions in the proof of Theorem 4.3, thus illustrating the connection between vector-valued Gabor frames and exponential bases.

We shall recall the following result about the orthonormal bases of exponentials on a domain.

**Theorem 1.1.** Let $|\Omega| = 1$. For $\phi \in L^2(\mathbb{R}^d)$, the system $\{\phi(\cdot - a)\}_{a \in A}$ is an orthonormal basis for $PW_\Omega(\mathbb{R}^d)$ if and only if $\{e_a\}_{a \in A}$ is an orthonormal basis for $L^2(\Omega)$, provided that $|\hat{\phi}(x)| = \chi_\Omega$ for a.e. $x \in \Omega$.

Note that by the above theorem we can recover the Whittaker-Shannon-Kotel’nikov Theorem which states that the sequence 
$$\left\{ \text{sinc}(x - n) = \frac{\sin \pi(x - n)}{\pi(x - n)} : n \in \mathbb{Z} \right\}$$
is an orthonormal basis for $PW_{[-1/2, 1/2]}(\mathbb{R})$, the space of all functions in $L^2(\mathbb{R})$ with Fourier support in $[-1/2, 1/2]$.

The following result proved by Lagarias, Reed and Wang ([21]) and, independently, by A. Iosevich and S. Pederson ([18]) characterizes all orthonormal basis spectrum $A$ for $d$-dimensional cube $Q^d = [0, 1]^d$. Their results are equivalent to those in Theorem 1.1 when $\Omega$ is the cube.

**Theorem 1.2** ([18]). Let $A$ be a subset of $\mathbb{R}^d$. Then $A$ is an orthonormal basis spectrum for the $d$-dimensional unite cube $Q^d$ if and only if $A$ is a tiling set for the cube $Q^d$.

For a function $u$, we let $\|u\|_\infty$ and $\|u\|_0$ denote the supremum and infinitum of $|u|$ on $\Omega$, respectively. Our next result can be viewed as a general version of Theorem 1.1.

**Theorem 1.3.** Let $\phi \in PW_\Omega(\mathbb{R}^d)$ and $0 < \|\hat{\phi}\|_0 \leq \|\hat{\phi}\|_\infty < \infty$. Then $\{\phi(x - a)\}_{a \in A}$ is Riesz basis for $PW_\Omega(\mathbb{R}^d)$ if and only if $\{e_a\}_{a \in A}$ is Riesz basis for $L^2(\Omega)$. In this case, the associated Riesz constants for $\{e_a\}_{a \in A}$ and $\{\phi(x - a)\}_{a \in A}$ are equal and $C_1 = \|\hat{\phi}\|_0$ and $C_2 = \|\hat{\phi}\|_\infty$.

As a consequence of Theorem 1.3 we have the following results.

**Corollary 1.4.** Let $\phi \in PW_\Omega(\mathbb{R}^d)$ such that $\hat{\phi}$ is continuous and nowhere zero on $\Omega$. If $\Omega$ is compact, then $\{\phi(x - a)\}_{a \in A}$ is Riesz basis for $PW_\Omega(\mathbb{R}^d)$ if and only if $\{e_a\}_{a \in A}$ is Riesz basis for $L^2(\Omega)$.

**Corollary 1.5.** Let $A$ be a Riesz spectrum for $\Omega$. Then for any $u \in L^2(\Omega)$, the set $\{ue_a\}_a$ is a Riesz basis for $L^2(\Omega)$ if and only if $0 < \|u\|_0 \leq \|u\|_\infty < \infty$. 

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2. Proofs of Theorems 1.1 and 1.3

**Proof of Theorem 1.1.** The proof is straightforward, but we write it down for the sake of completeness. Assume that \( \{e_a\} \) is an orthonormal basis for \( L^2(\Omega) \). Let \( \phi \in PW_\Omega(\mathbb{R}^d) \) such that \( |\hat{\phi}(x)| = 1 \) a.e. \( x \in \Omega \). We prove that \( \{L_a \phi\} \) is an orthonormal basis for \( PW_\Omega(\mathbb{R}^d) \).

By the Parseval identity, the orthogonality of \( \{\phi(\cdot - a)\} \) is obtained from the relation \((2.1)\). Let \( u \in L^2(\Omega) \). Then for \( \hat{u} \), the inverse Fourier transform of \( u \), we have

\[
\hat{u} = \sum_a \langle \hat{u}, L_a \phi \rangle L_a \phi,
\]

hence by the Fourier transform

\[
u = \sum_a \langle u, e_a \rangle e_a,
\]

as desired. \( \square \)

For a compact and symmetric convex domain \( \Omega \) in the plane, in \([19]\), the first author, with Katz and Tao, proved that a set \( A \) is an orthonormal basis spectrum for \( \Omega \) if and only if \( \Omega \) tiles \( \mathbb{R}^2 \) by translations. As corollary of their result along Theorem 1.1, in the following corollary we illustrate the relation between orthonormal translation bases and tiling property in \( \mathbb{R}^2 \).

**Corollary 2.1.** Let \( \Omega \) be any compact convex domain in the plane and \( A \) be a set in \( \mathbb{R}^2 \). Then for any \( \phi \in PW_\Omega(\mathbb{R}^d) \) with Fourier transform support in \( \Omega \), the system of translations \( \{\phi(\cdot - a)\}_A \) is orthonormal and complete in \( PW_\Omega(\mathbb{R}^2) \) if and only if \( A \) is a tiling set for \( \Omega \), provided that \( |\Omega| = 1 \) and \( \hat{\phi} = \chi_\Omega \) a.e.
Proof of Theorem 1.3. Assume that $A$ is a Riesz spectrum for $\Omega$ and $C_1$ and $C_2$ are the associated Riesz constants. Let $\phi \in PW_\Omega(\mathbb{R}^d)$ for which $0 < \|\hat{\phi}\|_0 \leq \|\hat{\phi}\|_\infty < \infty$, and let $f \in PW_\Omega(\mathbb{R}^d)$. Then put

$$u := \frac{\hat{f}}{\hat{\phi}}.$$

$u$ is in $L^2(\Omega)$ and for some $\{c_a\}_{a \in A} \in l^2(A)$

$$u = \sum_{a \in A} c_a e_a,$$

such that

$$C_1 \sum_a |c_a|^2 \leq \|u\|_{L^2(\Omega)}^2 \leq C_2 \sum_a |c_a|^2. \quad (2.2)$$

We will show that $f(x) = \sum_{a} c_a \phi(x - a)$ in $L^2(\mathbb{R}^d)$ and the inequality (1.1) holds for some other constants $0 < C'_1 \leq C'_2 < \infty$. We proceed as following. Let $B$ be a finite subset of $A$. Then

$$\|f - \sum_{a \in B} c_a \phi(x - a)\|_{PW_\Omega(\mathbb{R}^d)}^2 = \|f - \sum_{a \in B} c_a \hat{\phi} e_a\|_{L^2(\Omega)}^2$$

$$= \int_{\Omega} \left| \hat{f}(x) - \sum_{a \in B} c_a \hat{\phi}(x)e_a(x) \right|^2 dx$$

$$= \int_{\Omega} \left| \hat{\phi}(x) \left( u(x) - \sum_{a \in B} c_a e_a(x) \right) \right|^2 dx$$

$$\leq \|\hat{\phi}\|_\infty^2 \|u - \sum_{a \in B} c_a e_a\|^2.$$

Since the above inequality holds for any finite subset of $A$ and since $u = \sum_{a \in A} c_a e_a$, then $f = \sum_{a \in A} c_a L_a \phi$ holds. It remains to show that the inequality (1.1) holds for $f$. Note that $\|f\|^2 = \|\hat{f}\|^2 = \|u\phi\|^2$. Therefore by the upper estimation in (2.2) we will have

$$\|f\|^2 \leq \|\hat{\phi}\|_\infty^2 \|u\|^2 \leq C_2 \|\hat{\phi}\|_\infty^2 \sum_a |c_a|^2$$

With a similar argument, one can show that

$$\|f\|^2 \geq C_1 \|\hat{\phi}\|_0^2 \sum_a |c_a|^2.$$

Consequently we have
\[ C'_1 \sum_a |c_a|^2 \leq \|f\|^2 \leq C'_2 \sum_a |c_a|^2 \]

for \( C'_1 = C_1 \|\hat{\phi}\|^2_0 \) and \( C'_2 = C_2 \|\hat{\phi}\|^2_\infty \).

Conversely, suppose that \( \{\phi(\cdot - a)\}_{a \in A} \) is a Riesz basis for \( PW_\Omega(\mathbb{R}^d) \) with constants \( m_\phi, M_\phi > 0 \). We prove that \( \{c_a\}_{a \in A} \) is a Riesz basis for \( L^2(\Omega) \). Let \( u \in L^2(\Omega) \) and define \( f = \hat{\bar{u}} * \phi \). (Here, \( \hat{\bar{u}} \) is the Fourier inverse function of \( u \).) The Fourier transform \( \hat{\bar{f}} \) has support in \( \Omega \) and by the assumptions we have \( f = \sum_a c_a \phi(\cdot - x) \) for some \( \{c_a\}_{a \in A} \in l^2(\mathbb{A}) \), such that

\[
m_\phi \sum_a |c_a|^2 \leq \|f\|_2 \leq M_\phi \sum_a |c_a|^2. \tag{2.3}
\]

To show that \( u = \sum_a c_a e_a \) and the Riesz inequality \((1.2)\) holds for \( u \), once again we let \( B \) be a finite subset of \( A \). Then

\[
\|u - \sum_{a \in B} c_a e_a\| = \int_\Omega |u(x) - \sum_{a \in B} c_a e_a(x)|^2 \, dx
\]

\[
= \int |\hat{\phi}(x)|^{-2} \left| u(x) \hat{\phi}(x) - \sum_{a \in B} c_a \hat{\phi}(x) e_a(x) \right|^2 \, dx
\]

\[
\leq \|\hat{\phi}\|_0^{-2} \|u\hat{\phi} - \sum_{a \in B} \hat{\phi} e_a\|^2
\]

\[
= \|\hat{\phi}\|_0^{-2} \|\hat{\bar{u}} * \phi - \sum_{a \in B} \phi(\cdot - a)\|^2
\]

\[
= \|\hat{\phi}\|_0^{-2} \|f - \sum_{a \in B} \phi(\cdot - a)\|^2.
\]

The above estimation holds for any finite subset of \( A \). Therefore, since \( f = \sum_{a \in A} \phi(\cdot - a) \), we conclude that \( u = \sum_{a \in A} c_a e_a \).

To prove the inequality \((1.2)\) for \( u \), first we apply the upper estimation in \((2.3)\) as follows.

\[
\|u\|^2 \leq \|\hat{\phi}\|_0^{-2} \|u\hat{\phi}\|^2 = \|\hat{\phi}\|_0^{-2} \|\hat{\bar{u}} \phi\|^2 \leq M_\phi \|\hat{\phi}\|_0^{-2} \sum_a |c_a|^2.
\]
(Recall that \( \hat{u} \ast \phi = f \) and \( \ast \) denotes the convolution operation.) With a similar calculation, one can show that
\[
\|u\|^2 \geq m_\phi \|\hat{\phi}\|_\infty^{-2} \sum_a |c_a|^2.
\]
This completes the proof of (1.2) for the function \( u \) with the Riesz basis constants \( M_\phi \|\hat{\phi}\|_0^{-2} \) and \( m_\phi \|\hat{\phi}\|_\infty^{-2} \). Hence, the proof of the theorem is completed.

2.1. Application

In Theorem 1.1 we observed that for \( \hat{\psi} = \chi_\Omega \) with \( |\Omega| = 1 \), the Paley-Wiener space \( PW_\Omega(\mathbb{R}^d) \) is exactly the reproducing kernel Hilbert space \( L^2(\mathbb{R}) \ast \psi \) and \( A \)-translates of \( \psi \) form an ONB (orthonormal basis) for this space if and only if \( \{e_a\} \) is an ONB for \( L^2(\Omega) \). Motivated by this observation, here we study the cases where for \( \psi \in L^2(\mathbb{R}^d) \), the Hilbert space \( L^2(\mathbb{R}^d) \ast \psi \) has a translate orthonormal basis or a translate Riesz basis. We will narrow our attention to the situation where \( \psi \) is a band-limited function (i.e., it has bounded Fourier transform support), and \( L^2(\mathbb{R}^d) \ast \psi \) is a Paley-Wiener space. First we need the following lemma.

Lemma 2.2. For any \( \psi \in PW_\Omega \), there holds \( PW_\Omega = L^2(\mathbb{R}^d) \ast \psi \), provided \( 0 < \|\hat{\psi}\|_0 < \infty \).

Proof. Since \( \psi \) has Fourier support in \( \Omega \), then by the Fourier transform \( L^2(\mathbb{R}^d) \ast \psi \subseteq PW_\Omega \). For the converse, let \( f \in PW_\Omega \). By our assumption on \( \hat{\psi} \), we have \( \|\hat{f}/\hat{\psi}\| \in L^2(\Omega) \) with \( \|f/\hat{\psi}\| \leq \|\hat{f}/\hat{\psi}\|_0 \). This implies that for some \( g \in L^2(\mathbb{R}^d) \), \( f = g \ast \psi \), and hence the proof of the lemma is completed.

The following result is a direct consequence of the preceding lemma.

Corollary 2.3. Let \( \phi, \psi \in PW_\Omega \) such that \( 0 < \|\hat{\phi}\|_0 \leq \|\hat{\phi}\|_\infty < \infty \) and \( 0 < \|\hat{\psi}\|_0 < \infty \). Then set \( A \) is a Riesz basis spectrum for \( \Omega \) if and only if \( \{L_a \phi\}_{a \in A} \) is a Riesz basis for \( L^2(\mathbb{R}^d) \ast \psi \).

Remark 2.4. A trivial consequence of the Corollary 2.3 is that for any \( \phi \in PW_\Omega(\mathbb{R}^d) \) with \( 0 < \|\hat{\phi}\|_0 \leq \|\hat{\phi}\|_\infty < \infty \), the system \( \{L_a \phi\}_{a \in A} \) is a Riesz basis for \( L^2(\mathbb{R}^d) \ast \phi \), if the set \( A \) is a Riesz spectrum for \( \Omega \).

3. Exponential frames

The study of exponential frames was initiated by Duffin and Schaeffer in their work on non-harmonic Fourier series ([13]). The existence of exponential frames on \( L^2(\Omega) \), for Lebesgue measurable set \( \Omega \) in \( \mathbb{R} \), is also known to be equivalent to the sampling and interpolation problems on the Paley-Wiener space \( PW_\Omega(\mathbb{R}) \) (see e.g. [26, 22, 20]). For the most recent work for sampling and interpolation of Paley-Wiener (band-limited) functions on the locally compact abelian groups see e.g. [15, 1], and for the results on non-commutative settings see e.g. [11]. In this section, we show that the existence of exponential frames on \( L^2(\Omega) \) is equivalent to the existence of translate basis for Paley-Wiener spaces.
Definition 3.1. Given $A \subset \mathbb{R}^d$ and $\phi \in L^2(\mathbb{R}^d)$, we say that the translation system \( \{ \phi(\cdot - a) \}_{a \in A} \) is a frame for $PW_\Omega(\mathbb{R}^d)$ if there exist positive and finite constants $C_1 \leq C_2$ such that for any $f \in PW_\Omega(\mathbb{R}^d)$ there holds
\[
C_1 \| f \|^2 \leq \sum_{a \in A} |\langle f, \phi(\cdot - a) \rangle|^2 \leq C_2 \| f \|^2.
\]

Similarly, \( \{ e_a \}_{a \in A} \) is a frame for $L^2(\Omega)$ if there exist positive and finite constants $C_1 \leq C_2$ such that for any given $f \in L^2(\Omega)$
\[
C_1 \| f \|^2 \leq \sum_{a \in A} |\langle f, e_a \rangle|^2 \leq C_2 \| f \|^2.
\]

A frame is called Parseval if $C_1 = C_2$.

For a necessary and sufficient conditions for translate frames (resp. Riesz basis) with the spectrum set $A \subseteq \mathbb{Z}$ see e.g. [7].

Notice that any Riesz basis is a frame, but the converse does not always hold. For examples of frames containing no Riesz bases we invite the reader to see the monograph [10] by Christensen. To state the first result of this section, we need the following definition.

For $\phi \in PW_\Omega(\mathbb{R}^d)$ let
\[
E_\phi = \{ x \in \Omega : \hat{\phi}(x) \neq 0 \}.
\]

Theorem 3.2. Given $A \subset \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$, the system $\{ e_a \}_{a \in A}$ is a frame for $L^2(\Omega)$ if and only if for any $\phi \in PW_\Omega(\mathbb{R}^d)$ the system $\{ \phi(x - a) \}_{a \in A}$ is frame for $PW_\Omega(\mathbb{R}^d)$, provided that there exist positive and finite constants $m \leq M$ such that $m \leq |\hat{\phi}(x)|^2 \leq M$ for a.e. $x \in E_\phi$.

Proof. For $f \in PW_\Omega(\mathbb{R}^d)$, put $u := \hat{\phi} \chi_{E_\phi}$. By the assumption on $\hat{\phi}$, we will then have
\[
m \| f \|^2 \leq \| u \|^2 \leq M \| f \|^2,
\]
and by the Parseval identity,
\[
\sum_a |\langle f, L_a \phi \rangle|^2 = \sum_a |\langle u, e_a \rangle|^2.
\]

If $\{ e_a \}_{a \in A}$ is a frame for $L^2(\Omega)$ with constants $C_1$ and $C_2$, then by (3.2)
\[
C_1 \| u \|^2 \leq \sum_a |\langle f, L_a \phi \rangle|^2 \leq C_2 \| u \|^2.
\]

(3.3)
A combination of the inequalities in (3.3) and (3.1) proves that \( \{L_n \phi\} \) is a frame for \( PW_\Omega(\mathbb{R}^d) \). Conversely, assume that \( \{L_n \phi\} \) is a frame for \( PW_\Omega(\mathbb{R}^d) \) and \( u \in L^2(\Omega) \). Put \( h := \chi_{E_\phi} \frac{u}{\hat{\phi}} \). By the assumptions, \( h \) belongs to \( L^2(\Omega) \). If we let \( f \) denote the inverse Fourier transform of \( h \), i.e., \( \hat{f} = h \), then
\[
\sum_a |\langle f, L_a \phi \rangle|^2 = \sum_a |\langle h, \hat{\phi} e_a \rangle|^2 = \sum_a |\langle u, e_a \rangle|^2.
\]
(3.4)

If \( C_1 \) and \( C_2 \) are the frame constants for \( \{L_n \phi\} \), then by (3.4) we will have
\[
C_1 \|f\|^2 \leq \sum_a |\langle u, e_a \rangle|^2 \leq C_2 \|f\|^2.
\]
(3.5)

From the other hand we have \( \|f\| = \|h\| = \left\| \chi_{E_\phi} \frac{u}{\hat{\phi}} \right\| \) and \( m \leq |\hat{\phi}(x)|^2 \leq M \) for a.e. \( x \in E_\phi \). Therefore
\[
M^{-1} \|u\|^2 \leq \|f\|^2 \leq m^{-1} \|u\|^2.
\]
(3.6)

By interfering (3.6) in (3.5) the frame condition holds for \( \{e_a\} \), and this completes the proof of the theorem.

**Remark 3.3.** Note that what distinguishes Theorem 3.2 from 1.3 is that for frames we only require \( \hat{\phi} \) to be non-vanishing on a subset of \( \Omega \).

In the following we give an example of a frame with a smooth generator.

**Example 3.4.** Let \( u \) be a bump function on \( \mathbb{R} \) with compact support \([0,1]\). Take any compact set \( \Omega \subset [0,1] \) such that \( u \) is away from zero on \( \Omega \) (i.e., \( \inf_{x \in \Omega} |u(x)| > 0 \)). Let \( \hat{u} \) denote the inverse Fourier transform of \( u \). Put \( \phi := \hat{u} \ast \chi_\Omega \). Then \( \phi \) is smooth with \( E_\phi = \Omega \), and \( \{\phi(\cdot - n)\}_{n \in \mathbb{Z}} \) forms a frame for \( PW_\Omega(\mathbb{R}) \) if and only if \( A \) is a frame spectrum for \( \Omega \).

We conclude this section with the following corollary.

**Corollary 3.5.** Let \( \Omega \subset \mathbb{R}^d \) be a set with finite positive measure. Then a set \( A \) is a Parseval frame spectrum for \( L^2(\Omega) \) if and only if for any \( \phi \in PW_\Omega(\mathbb{R}^d) \) the system of translates \( \{\phi(\cdot - a)\}_A \) is a Parseval frame, provided that \( \hat{\phi} \) is nowhere zero on \( \Omega \).
4. Paley-Wiener valued Gabor systems

In this section, we shall apply the results of previous sections to characterize vector-valued Gabor systems that are orthonormal basis for a class of vector-valued signals. Before we state our results, we need to introduce some notations here.

For given measurable space \((X, \mu)\) and Hilbert space \(Y\), the space \(L := L^2(X, Y, \mu)\) is defined as class of all equivalent and measurable functions \(F : X \to Y\) for which

\[
\|F\|_L^2 = \int_X \|F(x)\|_Y^2 \, d\mu(x) < \infty.
\]

\(L\) is a Hilbert space with the norm \(\| \|_L\) and the inner product

\[
\langle F, G \rangle_L = \int_X \langle F(x), G(x) \rangle_Y \, d\mu(x).
\]

To avoid any confusion, in the sequel, we shall use subscripts for all inner products for the Hilbert spaces.

Lemma 4.1. Let \((X, \mu)\) be a measurable space, and \(\{f_n\}_n\) be an orthonormal basis for \(L^2(X) := L^2(X, d\mu)\). Let \(Y\) be a Hilbert space and \(\{g_m\}_m\) be a family in \(Y\). For any \(m, n\) and \(x \in M\) define \(G_{m,n}(x) := f_n(x)g_m\). Then \(\{G_{m,n}\}_{m,n}\) is an orthonormal basis for the Hilbert space \(L\) if and if \(\{g_m\}_m\) is an orthonormal basis for \(Y\).

Proof. For any \(m, n\) and \(m', n'\) we have the following.

\[
\langle G_{m,n}, G_{m',n'} \rangle_L = \int_X \langle f_m(x)g_n, f_{m'}(x)g_{n'} \rangle_Y \, d\mu(x)
\]

\[
= \langle f_m, f_{m'} \rangle_{L^2(X)} \langle g_n, g_{n'} \rangle_Y
\]

\[
= \delta_{m,m'} \langle g_n, g_{n'} \rangle_Y
\]

This shows that the orthogonality of \(\{G_{m,n}\}_{m,n}\) is equivalent to the orthogonality of \(\{g_m\}_m\). And, \(\|G_{m,n}\| = 1\) if and only if \(\|g_n\| = 1\).

Let \(\{g_m\}_m\) be an orthonormal basis for \(Y\). To prove the completeness of \(\{G_{m,n}\}\) in \(L\), let \(F \in L\) such that \(\langle F, G_{m,n} \rangle_L = 0, \forall m, n\). We show that \(F = 0\). By the definition of inner product we have
\[ 0 = \langle F, G_{m,n} \rangle_L = \int_X \langle F(x), G_{m,n}(x) \rangle_Y d\mu(x) \]

\[ = \int_X \langle F(x), f_m(x)g_n \rangle_Y d\mu(x) \]

\[ = \int_X \langle F(x), g_n \rangle_Y \overline{f_m(x)} d\mu(x) \]

\[ = \langle A_n, f_m \rangle \]

where

\[ A_n : X \to \mathbb{C}; \quad x \mapsto \langle F(x), g_n \rangle_Y. \]

\[ A_n \] is a measurable function and lies in \( L^2(X) \) with \( \| A_n \| \leq \| F \| \). Since \( \langle A_n, f_m \rangle_{L^2(X)} = 0 \) for all \( m \) (see above), then \( A_n = 0 \) by the completeness of \( \{ f_m \} \). From the other hand, by the definition of \( A_n \) we have \( \langle F(x), g_n \rangle_Y = 0 \) for a.e. \( x \in X \). Since \( \{ g_n \} \) is complete in \( Y \), then \( F(x) = 0 \) for a.e. \( x \in X \), as we desired.

Conversely, assume that \( \{ G_{m,n} \}_{m,n} \) is an orthonormal basis for the Hilbert space \( L \). Therefore by (4.1), \( \{ g_m \} \) is an orthonormal system. We prove that if for \( g \in Y \) and \( \langle g, g_m \rangle = 0 \) for all \( m \), then \( g \) must be zero. For this, for any \( n \) let us define the map

\[ B_n : X \to Y; \quad x \mapsto f_n(x)g. \]

Then \( B_n \) is measurable and it belongs to \( \mathcal{L} \) with \( \| B_n \|_{\mathcal{L}} = \| g \|_Y. \) By expanding \( B_n \) in terms of \( \{ G_{m,n} \} \),

\[ B_n = \sum_{n',m} \langle B_n, G_{m,n'} \rangle_L G_{m,n'} \]

\[ = \sum_{n',m} \langle f_n, f_{n'} \rangle_{L^2(X)} \langle g, g_m \rangle_Y G_{m,n'} \]

\[ = \sum_m \langle g, g_m \rangle_Y G_{n,m} \]

By the assumption that \( \langle g, g_m \rangle_Y = 0 \), we will have \( B_n = 0 \). This implies that \( B_n(x) = f_n(x)g = 0 \) for a.e. \( x \). Since, \( f_n \neq 0 \), then \( g \) must be zero, and hence we are done.

\[ \square \]

We conclude this paper with another description of translate orthonormal bases in terms of vector-valued Gabor systems introduced in Theorem 4.3. The result is a consequence of Lemma 4.1 along Theorem 1.1. First we need a definition here.
Definition 4.2. Given $A$, $\Omega$, and $u \in PW_\Omega(\mathbb{R}^d)$, for any $a, b \in A$, the vector-valued time-frequency translate of $u$ along $a, b \in A$ is given by $u_{a,b} : \Omega \to PW_\Omega(\mathbb{R}^d)$, $\alpha \mapsto e_b(\alpha)T_a u$. The system
\[ \mathcal{G}(A, \Omega, u) := \{G_{a,b} : G_{a,b}(\alpha) := e_b(\alpha)T_a u, a, b \in A\} \quad (4.4) \]
is called the corresponding vector-valued Gabor or Weyl-Heisenberg system for the “window” signal $u$. For more on traditional Weyl-Heisenberg frames we refer the reader to [8, 9].

Theorem 4.3. Let $|\Omega| = 1$. Define $\mathcal{L} = L^2(\Omega, PW_\Omega)$ to be the Hilbert space of all measurable functions $F : \Omega \to PW_\Omega$ with finite norm $\|F\|_{\mathcal{L}} := \int_\Omega \|F(\alpha)\|_{L^2(\mathbb{R}^d)}^2 d\alpha$. Then for any $u \in PW_\Omega$, $\{T_a u\}_{a \in A}$ is an orthonormal basis for $PW_\Omega$ if and only if the vector-valued Gabor system $\mathcal{G}(A, \Omega, u)$ is an orthonormal basis for $\mathcal{L}$, provided that $u$ is nowhere zero on $\Omega$.

Proof. Let $u \in PW_\Omega$ with $\dot{u}$ nowhere zero on $\Omega$. Let $\{T_a u\}_{a \in A}$ be an orthonormal basis for $PW_\Omega$. Then by Theorem 1.3, the exponentials $\{e_a\}$ form an orthonormal basis for $L^2(\Omega)$, and hence $\mathcal{G}(A, \Omega, u)$ given in $(4.4)$ is an orthonormal basis for $\mathcal{L}$ by Lemma 4.1. To prove the converse, let $\mathcal{G}(A, \Omega, u)$ be an orthonormal basis for $\mathcal{L}$. The orthogonality of the family $\{T_a u\}$ follows from the following simple relation: For $a, a' \in A$ and any $b \in A$
\[ \delta_{a,a'} = \langle G_{a,b}, G_{a',b} \rangle_{\mathcal{L}} = \langle T_a u, T_{a'} u \rangle_{L^2(\mathbb{R}^d)} . \]

To show $\{T_a u\}$ spans $PW_\Omega$, let $g \in PW_\Omega$ such that $\langle g, T_a u \rangle_{L^2(\mathbb{R}^d)} = 0$ for all $a \in A$. We show that $g = 0$. For this, fix $b$ and define
\[ W_b : \Omega \to PW_\Omega; \quad \alpha \mapsto e_b(\alpha)g. \]

It is clear that $W_b \in \mathcal{L}$ with $\|W_b\|_{\mathcal{L}} = \|g\|_{L^2(\mathbb{R}^d)}$; and for any $a, b'$ we have
\[ \langle W_b, G_{a,b'} \rangle_{\mathcal{L}} = \int_\Omega \langle ge_b(\alpha), e_{b'}(\alpha)T_a u \rangle_{L^2(\mathbb{R}^d)} d\alpha = \langle g, T_a u \rangle_{L^2(\mathbb{R}^d)} \int_\Omega e_b(\alpha)\overline{e_{b'}(\alpha)} d\alpha. \quad (4.5) \]

Since $\langle g, T_a u \rangle_{L^2(\mathbb{R}^d)} = 0$ and the system $\mathcal{G}(A, \Omega, u)$ is complete in $\mathcal{L}$, then $(4.5)$ implies that $W_b(\alpha) = e_b(\alpha)g = 0$ for almost every $\alpha$. But $e_b \neq 0$, therefore $g$ must be zero, and hence we finish the proof.

We conclude this paper with the following remark.

Remark 4.4. A special case of Theorem 4.3 is when $A = \mathbb{Z}^d$. This case has already been studied, e.g., in [6]. The author showed that a necessary and sufficient condition for $\{T_j u\}_{j \in \mathbb{Z}^d}$ to be an orthonormal basis for its span is that for almost every $x \in [0, 1]^d$
\[ \sum_{j \in \mathbb{Z}^d} |\hat{u}(x + j)|^2 = 1 \]  \hspace{1cm} (4.6)

As we mentioned earlier, the similar techniques used in [6] do not apply to our situation where \( A \) is any random set. Therefore in our paper we used different techniques to overcome the problem and could prove a characterization of translate bases (ONB, Riesz, frame) in terms of exponential bases on a domain. For the equivalent conditions for orthonormal bases, frame, and Riesz bases of discrete translations in terms of the periodic function \( x \mapsto \sum_{j \in \mathbb{Z}^d} |\hat{u}(x + j)|^2 \) on \( \mathbb{R}^n \) and locally compact abelian groups we refer to [6] respectfully [3]. Relevant results for these bases in non-commutative settings, in particular, the Heisenberg group, have recently been shown by the second author et al. in [3, 12].

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