Tensor-multi-scalar theories from multidimensional cosmology

M. Rainer§†, A. Zhuk§‡

§Gravitationsprojekt, Institut für Mathematik
Universität Potsdam, PF 601553
D-14415 Potsdam, Germany
†Institute for Studies in Physics and Mathematics
Farmanieh Bld., P.O.Box 5531
Tehran 19395, Iran
‡Department of Physics
University of Odessa, 2 Petra Velikogo St.
Odessa 270100, Ukraine

Abstract

Inhomogeneous multidimensional cosmological models with a higher dimensional space-time manifold $M = M_0 \times \prod_{i=1}^{n} M_i$ ($n \geq 1$) are investigated under dimensional reduction to tensor-multi-scalar theories. In the Einstein conformal frame, these theories take the shape of a flat $\sigma$-model. For the singular case where $M_0$ is 2-dimensional, the dimensional reduction to dilaton gravity is preformed with different distinguished representations of the action.

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1 Introduction

In recent years, scalar-tensor theories have received a renewed interest. There are two reasons for it. First, extended inflation [1, 2] which originally was based on standard Brans-Dicke (BD) theory [3] revives the scenario of inflation via a first-order transition. It provides a natural (non-fine-tuned) way to restore the original ideas of inflation while avoiding the cosmological difficulties coming from vacuum-dominated exponential expansion obtained in general relativity. Second, scalar tensor theories, generalizing standard BD theory, can satisfy the solar system criteria [4] and other present observations [5] to arbitrary accuracy, but still diverge from general relativity in the strong field limit. Thus, in future, such theories may provide an important test of general relativity. It should be noted in that context that, via conformal transformation of the metric, we can write scalar-tensor models equally well in the Jordan-Brans-Dicke frame or in the Einstein-Pauli frame, and the question which frame is a physical one is still open.

Several ways to generalize standard BD theory were proposed. These theories can be split into three main groups. First, there are the theories with BD parameter $\omega$ depending on the dilatonic scalar field [6]-[14]. The second class is represented by theories with more than one dilaton field [4, 15, 16]. All other possible variations of the standard BD model form the third group, containing e.g. models in which the dilaton couples with different strengths to both, visible matter and conjectured ”dark” matter [5].

Scalar-tensor theories follow naturally as the low-energy limit of various Kaluza-Klein theories. Among them, multidimensional cosmological models with a space-time consisting of $n \geq 2$ Einstein spaces are most popular. Usually, theories with one internal Einstein space ($n = 2$) are considered [17]-[19]. The dimensional reduction of these theories yields only one dilaton field (like in the original BD theory). Here we show that this case is exceptional, with a midisuperspace metric of degenerate signature. The model with $n \geq 2$ was considered in the paper [20], where the emphasis was on the problem of the internal dimensions compactification. In our paper we shall give a more elegant way of dimensional reduction to tensor-multi-scalar theories which reveals explicitly the nature of the dilaton fields. For example, the dilaton field with opposite sign in the kinetic term of the Lagrangian is connected with a dynamical volume of the whole internal space. In a conformal Einstein Pauli frame, a $\sigma$-model representation of the theory under consideration can easily be obtained.

In Sec. 2, it is shown, for space-time dimension $D_0 > 2$ after dimensional reduction, that, in the $\sigma$-model representation of our model, the metric on the space of scalar fields is the flat Euclidean one. If $D_0 = 2$, there is no conformal Einstein (Pauli) frame. This is actually no fault of the theory here, because in two dimensions the Einstein action is a topological invariant, whence it does not yield a dynamics of the 2-geometry.

Nevertheless it is worthwhile to consider here 2-dimensional dilaton gravity, which is a subject of intensive investigations recently [21, 22, 23]. In Sec. 3 we obtain the action of 2-dimensional dilaton gravity under dimensional reduction from cosmological models. Different representations are given, which correspond to different choices of conformal frames.
2 Effective BD Models from Multidimensional Models

Let us now consider a multidimensional space-time manifold

\[ M = M_0 \times \prod_{i=1}^{n} M_i. \] (2.1)

The metric on \( M \) can be decomposed as

\[ g = g^{(0)} + \sum_{i=1}^{n} e^{2\beta(x)} g^{(i)}, \] (2.2)

where \( x \) are some coordinates of \( M_0 \), and

\[ g^{(0)} = g^{(0)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu. \] (2.3)

With the Laplace-Beltrami operator on \( M_0 \) defined by

\[ \Delta[g^{(0)}] = \frac{1}{\sqrt{|\det g^{(0)}|}} \frac{\partial}{\partial x^\mu} \left( \sqrt{|\det g^{(0)}|} g^{(0)\mu\nu} \frac{\partial}{\partial x^\nu} \right), \] (2.4)

we get the Ricci curvature scalar \[24\]

\[ R[g] = R[g^{(0)}] + \sum_{i=1}^{n} e^{-2\beta} R[g^{(i)}] \]
\[ - \sum_{i,j=1}^{n} (D_i \delta_{ij} + D_i D_j) g^{(0)\mu\nu} \frac{\partial \beta^i}{\partial x^\mu} \frac{\partial \beta^j}{\partial x^\nu} - 2 \sum_{i=1}^{n} D_i \Delta[g^{(0)}] \beta^i. \] (2.5)

With total dimension \( D := \sum_{i=0}^{n} D_i, \kappa^2 \) a \( D \)-dimensional gravitational coupling constant, and \( S_{GH} \) the standard Gibbons-Hawking boundary term, we consider an action of the form

\[ S = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|\det g|} R[g] + S_{GH}, \] (2.6)

In the following we assume that \( R[g^{(i)}] \) is finite on \((M_i, g^{(i)})\). Let us now consider the volumes \( \mu_i \) of \((M_i, g^{(i)})\) and the total internal space volume \( \mu \), satisfying

\[ \mu_i = \int_{M_i} d^D y \sqrt{|\det g^{(i)}|}, \quad \mu = \prod_{i=1}^{n} \mu_i. \] (2.7)

If all of the spaces \( M_i \) \((i = 1, \ldots, n)\) are compact, then the volumes \( \mu_i \) and \( \mu \) are finite, and so are also the numbers

\[ \rho_i = \int_{M_i} d^D y \sqrt{|\det g^{(i)}|} R[g^{(i)}]. \] (2.8)

However, a non-compact \( M_i \) might have infinite volume \( \mu_i \) or infinite \( \rho_i \). Nevertheless, for the following we have to assume only that all ratios \( \frac{\rho_i}{\mu_i}, i = 1, \ldots, n \) are finite. This is in particular the case, when \( M_i \) is homogeneous. Then

\[ \frac{\rho_i}{\mu_i} = R[g^{(i)}] \]
is always constant and finite. In the special case, where $M_i$ is a Einstein manifold $R^k_j[g^{(i)}] = \lambda^i \delta^k_j$, with constant $\lambda^i$, it is
\[
\frac{\rho_i}{\mu_i} = R[g^{(i)}] = \lambda^i D_i, 
\]
and, more specially, when $M_i$ has constant curvature $k$,
\[
\frac{\rho_i}{\mu_i} = R[g^{(i)}] = k D_i (D_i - 1). 
\]
However, here we do not restrict apriori to Einstein or constant curvature spaces. For convenience and beauty, in the following we will exemplify the dimensional reduction just for the case of homogeneous spaces $M_1, \ldots, M_n$, although the procedure could be easily generalized for the case of inhomogeneous $M_1, \ldots, M_n$. (Then one has to work with $\frac{\rho_i}{\mu_i}$ in place of $R[g^{(i)}]$.)

The bare gravitational coupling constant
\[
\kappa^2 = \kappa_0^2 \cdot \mu 
\]
might become infinite, while the true $D_0$-dimensional coupling constant $\kappa_0^2$ is always finite. If $D_0 = 4$, then $\kappa_0^2 = 8\pi G_N$, where $G_N$ is the Newton constant. Then the action (2.6) remains well defined, even when some of the volumes $\mu_i$ are infinite. After rewriting
\[
\frac{1}{\kappa^2} \int_M d^D x \sqrt{|\det g|} \sum_{i=1}^n D_i \Delta [g^{(0)}] \beta^i
\]
\[
= \frac{\mu}{\kappa^2} \sum_{i=1}^n D_i \int_{M_0} d^D_0 x \sqrt{|\det g^{(0)}|} \prod_{l=1}^n e^{D_l \beta^l} \frac{1}{\sqrt{|\det g^{(0)}|}} \frac{\partial}{\partial x^\lambda} \left( \sqrt{|\det g^{(0)}|} g^{(0)\lambda\nu} \frac{\partial}{\partial x^\nu} \beta^i \right)
\]
\[
- \sqrt{|\det g^{(0)}|} g^{(0)\lambda\nu} \frac{\partial \beta^i}{\partial x^\nu} \prod_{i=1}^n \sum_{j=1}^n D_j \frac{\partial \beta^j}{\partial x^\lambda}
\]
\[
= S_{GH} - \frac{1}{\kappa_0^2} \int_{M_0} d^D_0 x \sqrt{|\det g^{(0)}|} \prod_{l=1}^n e^{D_l \beta^l} \sum_{i,j=1}^n D_i D_j g^{(0)\lambda\nu} \frac{\partial \beta^i}{\partial x^\lambda} \frac{\partial \beta^j}{\partial x^\nu}, 
\]
the action is
\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^D_0 x \sqrt{|\det g^{(0)}|} \prod_{l=1}^n e^{D_l \beta^l} \left\{ R[g^{(0)}] - \sum_{i,j=1}^n G_{ij} g^{(0)\lambda\nu} \frac{\partial \beta^i}{\partial x^\lambda} \frac{\partial \beta^j}{\partial x^\nu} + \sum_{i=1}^n R[g^{(i)}] \right\}, 
\]
where
\[
G_{ij} := D_i \delta_{ij} - D_i D_j. 
\]
Let us first consider the exceptional case $n = 1$.
\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^D_0 x \sqrt{|\det g^{(0)}|} e^{D_1 \beta^1} \left\{ R[g^{(0)}] + D_1 (D_1 - 1) g^{(0)\lambda\nu} \frac{\partial \beta^1}{\partial x^\lambda} \frac{\partial \beta^1}{\partial x^\nu} + \sum_{i=1}^n R[g^{(i)}] \right\}, 
\]
Here, for $D_1 > 1$ the kinetic term has a different sign than usual, and for $D_1 = 1$ there is no kinetic term at all. Setting 
\[ \phi := e^{D_1 \beta_1}, \] (2.14)

it is \[ \frac{\partial \beta_1}{\partial x^\lambda} = \frac{1}{D_1} \frac{\partial \phi}{\partial x^\lambda}, \] and hence

\[
S = \frac{1}{2\kappa_0^2} \int_M d^{D_0}x \sqrt{\det g^{(0)}} \left\{ \phi R[g^{(0)}] - \omega g^{(0)} \frac{1}{\phi} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\nu} + R[g^{(1)}] \phi^{1 - \frac{D_1}{2}} \right\},
\] (2.15)

with BD parameter \( \omega = \omega(D_1) = \frac{1}{D_1} - 1 \), depending on the present extra dimension \( D_1 \). It is remarkable that, the conformal coupling constant \( \xi_{c,d+1} \) in dimension \( d + 1 \) determines the BD parameter for general extra dimension \( d \) as

\[
\omega(d) := \frac{1}{d} - 1 \equiv -4\xi_{c,d+1}.
\] (2.16)

Let us now examine the general case \( n > 1 \). Here it is useful to diagonalize the metric tensor (2.12). For the midisuperspace metric

\[
G := G_{ij} \, d\beta^i \otimes d\beta^j = \eta_{kl} dz^k \otimes dz^l = -dz^1 \otimes dz^1 + \sum_{i=2}^{n} dz^i \otimes dz^i,
\] (2.17)

the diagonalizing transformation

\[
z^1 = T^{i_1}_{j_1} \beta^i, \quad i = 1, \ldots, n
\] (2.18)

is given by (see also [25])

\[
z^1 = q^{-1} \sum_{j=1}^{n} D_j \beta^j, \\
z^i = [D_{i-1} / \Sigma_{i-1} \Sigma_i]^{1/2} \sum_{j=1}^{n} D_j (\beta^j - \beta^{i-1})
\] (2.19)

\[
i = 2, \ldots, n,
\]

where

\[
q := [D' / (D' - 1)]^{1/2} = \frac{1}{2\sqrt{\xi_{c,D'+1}}}, \quad D' := D - D_0, \quad \Sigma_k := \sum_{i=k}^{n} D_i.
\]

Especially, we have

\[
T^{1}_{i} = \frac{D_i}{q}, \quad i = 1, \ldots, n
\] (2.20)

Let us determine \( U = T^{-1} \) inverting Eq. (2.18) to

\[
\beta^i = U^{i}_{j} z^j, \quad i, j = 1, \ldots, n.
\] (2.21)

Eqns. (2.17) and (2.18) imply \( G_{ij} = \eta_{kl} T^k_i T^l_j, i, j = 1, \ldots, n \), and hence

\[
U^{i}_{j} = G^{ik} T^k_{l} \eta_{lj}, \quad i, j = 1, \ldots, n,
\] (2.22)
where the tensor components of the inverse midisuperspace metric are given as

\[ G^{ij} = \delta^{ij} \frac{D_i}{1 - D'}, \]  

(2.23)

With (2.20), we obtain especially

\[ U^i_1 = G^{ij} T^k_j \eta_{k1} = -G^{ij} T^1_j = \frac{1}{q(D' - 1)} = \frac{1}{\sqrt{D'(D' - 1)}}, \quad i = 1, \ldots, n. \]  

(2.24)

Using that, we can rewrite the action (2.11) as

\[ S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0}x \sqrt{|\det g^{(0)}|} \prod_{l=1}^{n} e^{D_l \beta_l} \left\{ \frac{1}{q} \frac{\partial R[g^{(0)}]}{\partial x^\lambda} \partial x^\lambda \partial x^\nu - \sum_{i=2}^{n} g^{(0)\lambda\nu} \frac{\partial z^{i \lambda}}{\partial x^\lambda} \partial z^{i \nu} + \left( e^{qz^1} \right)^{-\frac{1}{q^2}} \sum_{i=1}^{n} R[g^{(i)}] e^{-2 \sum_{k=2}^{n} U^i_k z^k} \right\}. \]  

(2.25)

Let define the BD field as

\[ \phi := e^{qz^1} = \prod_{l=1}^{n} e^{D_l \beta_l} \equiv v_{\text{int}}, \]  

(2.26)

where

\[ v_{\text{int}} := V_{\text{int}} / \mu \]  

(2.27)

is a scale which renormalizes the internal space volume \( V_{\text{int}} := \int_{M_1 \times \ldots \times M_n} d^{D'} y \sqrt{\det (g - g^{(0)})} \). Its corresponding logarithmic scale factor is the dilaton field \( z^1 \). The derivative of the latter is

\[ \frac{\partial}{\partial x^\mu} z^1 = \frac{1}{q\phi} \partial_{\nu} \phi. \]  

(2.28)

So we can write the action (2.25) as

\[ S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0}x \sqrt{|\det g^{(0)}|} \left\{ \frac{1}{q} \frac{\partial R[g^{(0)}]}{\partial x^\lambda} \partial x^\lambda \partial x^\nu - \phi \sum_{i=2}^{n} g^{(0)\lambda\nu} \frac{\partial z^{i \lambda}}{\partial x^\lambda} \partial z^{i \nu} \right\}, \]  

(2.29)

where now \( \omega = \omega(D') = \frac{1}{D'} - 1 \equiv -4\xi_{c,D'+1} \) is the BD parameter, depending now on the total extra dimension \( D' \). In the action (2.11) all scalar fields \( \beta^i, i = 1, \ldots, n \) couple to the curvature \( R[g^{(0)}] \). After the diagonalization (2.18) only one of the scalar fields, namely the BD field \( \phi \), is coupled to the curvature. In the action (2.29) scalar fields \( z^i \) play the role of normal scalar matter fields coupled to the dilaton BD scalar \( \phi \). Note that the kinetic terms for the fields \( z^i, i = 2, \ldots, n \), have the usual normal sign. In contrast to the action (2.13) w.r.t. its field \( \phi \), Eqn. (2.23) contains no selfinteraction terms for any of its fields \( \phi \) and \( z^i, i = 2, \ldots, n \). Rather it contains \( \phi - z^i \) cross terms \((i = 2, \ldots, n)\). These cross terms are, like the fields \( z^i \) and \( \phi \) themselves, of purely geometric nature. The exceptional case (2.13) corresponds formally to the case \( z^k \equiv 0, D_k \equiv 0 \) \((k = 2, \ldots, n)\) of (2.29).
For $D_0 \neq 2$, the action (2.29) can be written in a $\sigma$-model representation [15]. We define a new metric $\hat{g}^{(0)}_{\mu\nu}$, which yields the so-called Einstein conformal frame, and new scalar fields $\varphi^i (i = 1, \ldots, n)$ by

$$
\hat{g}^{(0)}_{\mu\nu} = \phi^{\frac{2}{D_0-1}} g^{(0)}_{\mu\nu}, \quad \varphi^1 = -A \ln \phi, \quad \varphi^i = z^i, \quad i = 2, \ldots, n,
$$

(2.30)

where $A := \pm \left[ \omega(D') + \frac{D_0-1}{D_0-2} \right]^\frac{1}{2}$. Note that, this transformation is regular for $\omega(D') \neq \omega_{c,D_0}$, where $\omega_{c,D_0} := -\frac{D_0-1}{D_0-2} \equiv -\frac{1}{4} \xi_{c,D_0}$ is the conformal parameter for dimension $D_0$. Taking into account that $-1 < \omega(D') \leq 0$ for $D' \geq 1$ and $\omega(0) = \infty$, one obtains: If $D_0 > 2$, (2.30) is regular for any $D' > 0$, with $A^2 > 0$. For $D_0 = 2$ or $D' = 0$ (2.30) is singular. It is singular with $A^2 = 0$ if $(D_0, D') = (0, 2)$. If $D_0 = 1$, (2.30) is singular for $D' = 1$, and regular for any $D' > 1$. In the latter case $A^2 < 0$, and a real redefinition of the complex field, e.g. $\varphi^1 \rightarrow |\varphi^1|$, yields again a Minkowskian metric in the space of scalar fields.

For $D_0 > 2$, with the flat $\sigma$-model metric

$$
d\sigma = \sigma_{ij} \, d\varphi^i \otimes d\varphi^j, \quad (\sigma_{ij}) = \text{diag}(+1, \ldots, +1),
$$

(2.31)

where $i, j = 1, \ldots, n$, and the potential

$$
V(\varphi) = -e^{-\frac{B}{A} \varphi^1} \sum_{i=1}^{n} R[g^{(i)}] e^{-2 \sum_{k=2}^{n} U_k \varphi^k},
$$

(2.32)

where $B := 1 - \frac{2}{D_0} - \frac{D_0-1}{D_0-2}$, the action (2.29) then reads

$$
S = \frac{1}{2\kappa_0^2} \int_{\mathcal{M}_0} d^D x \sqrt{|\text{det} \hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] - \sum_{i,j=1}^{n} \sigma_{ij} \hat{g}^{(0)}_{\lambda\mu} \frac{\partial \varphi^i}{\partial x^\lambda} \frac{\partial \varphi^j}{\partial x^\mu} - V(\varphi^i) \right\}.
$$

(2.33)

Note that the $\sigma$-model metric (2.31) is flat like the midisuperspace metric (2.17), however, while (2.17) is Minkowskian, (2.31) is Euclidean. So we found equivalent representations (2.11) and (2.33) of the same action $S$, but with different signature in their respective space of scalar fields.

In the case $n = 1$, with just one dilaton $\varphi$, the action (2.33) is equal to

$$
S = \frac{1}{2\kappa_0^2} \int_{\mathcal{M}_0} d^D x \sqrt{|\text{det} \hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] \hat{g}^{(0)}_{\mu\nu} \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu} + R[g^{(1)}] e^{-2\varphi} \right\}.
$$

(2.34)

This action can be written in the 'string-like' form (see e.g. [26, 27, 28] and refs. therein)

$$
S = \frac{1}{2\kappa_0^2} \int_{\mathcal{M}_0} d^D x \sqrt{|\text{det} \hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] \hat{g}^{(0)}_{\mu\nu} \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu} - 2\Lambda e^{-2\varphi} \right\}.
$$

(2.35)
where the constants are fixed by the conditions

\[
\begin{align*}
2\Lambda & := -R[g^{(1)}] \\
\lambda^2 & := \lambda_c^2 = \frac{D - 2}{D_1(D_0 - 2)}.
\end{align*}
\] (2.36)

In equation (2.35), \( \lambda \) is the dilatonic coupling constant. For \( D_0 = 10 \) and \( \Lambda = 0 \) (e.g. for a Ricci flat internal space), this action describes the scalar-tensor (i.e. Yang-Mills free) part of the bosonic sector from the 10-dimensional Einstein-Yang-Mills supergravity that occurs as low energy limit from superstring theory.

For arbitrary \( \Lambda \neq 0 \), the action (2.35) corresponds to the scalar-tensor sector of an effective string action in dimension \( D_0 \), only if the dilatonic coupling is fixed to

\[
\lambda^2 := \lambda_s^2 = \frac{1}{D_0 - 2}.
\] (2.37)

The coupling (2.37) is obtained for our models with (2.36) only in the limit of infinite internal dimension,

\[
\lambda_c^2 \rightarrow \lambda_s^2 = \frac{1}{D_0 - 2} \quad \text{for} \quad D_1 \rightarrow \infty.
\] (2.38)

Especially for the 10-dimensional effective action, the required value of \( \lambda^2 = \frac{1}{8} \) is obtained just in this limit, while above for \( \Lambda = 0 \) the value of \( \lambda \) was completely arbitrary. Indeed, \( \Lambda = 0 \) is a critical value for the string theories, whence \( \Lambda \neq 0 \) occurs just for non-critical string theories.

The action (2.33) can equivalently be obtained from a multidimensional cosmological model with a usual cosmological term \( \Lambda \), if the internal space \( M_1 \) is a Ricci flat Einstein space, i.e. \( R[g^1] = 0 \). Then the equivalence to our previous model is given by exchanging \( D - 1 \leftrightarrow 1 - D_1 \), which obviously leaves \( D_0 \) invariant. In this case \( \lambda_c^2 = \frac{D_1}{(D-2)(D_0-2)} \), and the correspondence to non-critical string theories is again given in the limit (2.38).

Finally note that, for \( D_0 \rightarrow 2 \), both couplings \( \lambda_s^2 \) and \( \lambda_c^2 \) become asymptotically equal to \((D_0 - 2)^{-1}\). Hence, in the limit \( D_0 \rightarrow 2 \) our models become independently from the internal dimension \( D_1 \) equivalent to effective low energy models of string theory, for any scalar curvature \(-\Lambda/2\) of the internal space.

3 2d dilaton gravity from inhomogeneous cosmology

Let us now consider in more details the dimensional reduction to a space-time of dimension \( D_0 = 2 \). In this case the conformal transformation (2.30) is singular, whence the model of (2.29) can not be expressed in a conformal Einstein Pauli frame. This is not a fault of the theory, but rather corresponds to the well known fact that 2-dimensional Einstein equations are empty, i.e. they do not imply a dynamics [21, 22]. Thus we shall consider 2-dimensional dilaton gravity only.
We start with the case with one dilaton, \( n = 1 \). The action (2.13) can be written in the 'string-like' form [29, 30, 31]

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det g^{(0)}|} e^{-2\sigma} \left\{ R[g^{(0)}] + 4mg^{(0)}_{\lambda\nu} \frac{\partial \sigma}{\partial x^\lambda} \frac{\partial \sigma}{\partial x^\nu} - 2\Lambda e^{-2(\frac{1}{k} + m)\sigma} \right\},
\]

(3.1)

where

\[
\sigma := -\frac{1}{2} D_1 \beta^1,
\]
\[
m := \frac{D_1 - 1}{D_1},
\]
\[
k := -\frac{D_1}{D_1 + 1},
\]
\[
2\Lambda := -R[g^{(1)}].
\]

(3.2)

By a conformal transformation of \( g^{(0)}_{\mu\nu} \) to

\[
\tilde{g}^{(0)}_{\mu\nu} = e^{-2m\sigma} g^{(0)}_{\mu\nu},
\]

(3.3)

we can formulate the action without kinetic dilation term, as

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det \tilde{g}^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}[\tilde{g}^{(0)}] - 2\tilde{\Lambda} e^{-\frac{k}{k+1} \sigma} \right\}.
\]

(3.4)

The 2d actions (3.1) and (3.4) are invariant under homogeneous conformal transformations

\[
\tilde{g}^{(0)}_{\mu\nu} := \Omega^{-2} g^{(0)}_{\mu\nu},
\]
\[
\tilde{g}^{(1)}_{\mu\nu} := \Omega^{-2} g^{(1)}_{\mu\nu},
\]

(3.5)

where \( \Omega \) is constant. Applying (3.3) with

\[
\Omega^2 := -\frac{D_1}{(D_1 + 1)^{1+\frac{1}{k}}} \frac{1}{2\Lambda}
\]

yields

\[
2\tilde{\Lambda} := -\tilde{R}[\tilde{g}^{(1)}] = -\frac{D_1}{(D_1 + 1)^{1+\frac{1}{k}}} = \frac{k}{(k + 1)^{1+\frac{1}{k}}}
\]

(3.6)

and the action (3.4) now reads

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det \tilde{g}^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}[\tilde{g}^{(0)}] - 2\tilde{\Lambda} e^{-\frac{k}{k+1} \sigma} \right\}.
\]

(3.7)

If we assume that the dilaton field is specifically given through the geometry on \( M_0 \) and the dimension \( D_1 \) of \( M_1 \), according to

\[
e^{-2\sigma} := (k + 1) \left( \tilde{R}[\tilde{g}^{(0)}] \right)^k,
\]

(3.8)
then the action (3.7) takes the form \[ S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det \tilde{g}^{(0)}|} \left( \tilde{R}[\tilde{g}^{(0)}] \right)^{k+1} \]

\[ = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det \tilde{g}^{(0)}|} \left( \tilde{R}[\tilde{g}^{(0)}] \right)^{\frac{1}{m+1}}. \quad (3.9) \]

In the general case of multi-scalar fields, the kinetic term of the dilaton can be removed by an obvious analogous procedure. The ‘string-like’ form of the action (2.23) is

\[ S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det g^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}[g^{(0)}] + 4mg^{(0)\lambda\nu} \frac{\partial \sigma}{\partial x^\lambda} \frac{\partial \sigma}{\partial x^\nu} - \sum_{i=2}^{n} g^{(0)\lambda\nu} \frac{\partial z^i}{\partial x^\lambda} \frac{\partial z^i}{\partial x^\nu} - e^{-2(\frac{1}{k+1})\sigma} \sum_{i=1}^{n} 2\Lambda_i e^{-2\sum_{j=2}^{n} U_{ij} z^j} \right\}, \quad (3.10) \]

where now

\[ \sigma := -\frac{1}{2}qz^1, \]

\[ m := \frac{1}{q^2} = D' - 1, \]

\[ k := -\frac{D'}{D' + 1}, \]

\[ 2\Lambda_i := -\tilde{R}[g^{(i)}]. \quad (3.11) \]

With (3.11), the conformal transformation (3.3) yields

\[ S = \frac{1}{2\kappa_0^2} \int_{M_0} d^2x \sqrt{|\det g^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}[g^{(0)}] - \sum_{i=2}^{n} \tilde{g}^{(0)\lambda\nu} \frac{\partial z^i}{\partial x^\lambda} \frac{\partial z^i}{\partial x^\nu} - e^{-2\sigma} \sum_{i=1}^{n} 2\Lambda_i e^{-2\sum_{j=2}^{n} U_{ij} z^j} \right\}. \quad (3.12) \]

In (3.12) there is no kinetic term of the dilaton field. The kinetic terms of all extra scalar fields $z^i$ have the normal sign. The extra fields $z^i$ play the role of usual matter, coupling to the dilaton field $\sigma$.

### 4 Conclusions and Discussion

We started from multidimensional cosmology. The corresponding metric is, from one side, a generalization of the Friedmann metric, which corresponds here to the special case where all $M_0, M_1, \ldots, M_n$ are spaces of constant curvature. From another side, our metric generalizes the anisotropic Kasner metric. In contrast to the (spatially) homogeneous Friedmann and Kasner metrics, our multidimensional metric is in general a (spatially) inhomogeneous one with scale factors depending on spatial coordinates of $M_0$. We obtained effective BD formulations
for multidimensional models via dimensional reduction on $M_0$. Selfinteraction terms appear exclusively in the degenerate case (2.15) where there is only one scalar field. For $n \geq 2$ scalar fields, the BD like effective action (2.29) contains $\phi - z^i$ cross terms, between the BD field $\phi$ and the other scalar fields $z^i$, $i = 2, \ldots, n$, instead.

In the case of only one internal space, $M_1$, the actions obtained after dimensional reduction of the multidimensional Einstein-Hilbert one may be written in string-like form. Thus, the associated field equations have the same form as for the (scalar) bosonic sector of the superstring theory in the low energy limit. The corresponding effective models of string theory are obtained from our models in the limit of infinite internal dimension $D_1 \rightarrow \infty$.

The BD like effective action (2.29), which has a Minkowskian metric in the space of scalar fields, is (with few exceptional cases) equivalent to a conformal Einstein $\sigma$-model action (2.33), which has an Euclidean metric in the space of scalar fields. The case of a 1-dimensional universe is exceptional: There the metric in the space of (real) scalar fields of a conformal Einstein $\sigma$-model is also Minkowskian.

With the effective dimension of the universe $D_0$ and the total extra dimension $D'$, the singular cases of the conformal transformation are given by $D_0 = 2$ or $D' = 0$, where (2.30) is undefined, or by $D = D_0 + D' = 2$, where $A$ in (2.31) is zero. In these exceptional cases our model is not conformal to an Einsteinian one. However, one should also keep in mind that Einstein equations in a 2-dimensional space-time do not imply any dynamics (see [21, 22]). For a space-time with $D_0 = 2$ the dimensional reduction of the multidimensional model can be written as a ‘string-like’ dilaton gravity, representable in the form (3.12), where the dilaton appears without kinetic term, and all extra fields couple to the dilaton with normal signs of their kinetic terms. If there are no fields besides the dilaton, then the action can be represented in the form (3.9) (see also [23, 31, 32]), which has a non-trivial variation only for nonvanishing extra dimension $D_1 > 0$.

A conformal equivalence transformation between two scalar-tensor Lagrangian models becoming singular at specific parameters (here given by the exceptional dimensions) is a familiar effect. Such singularities yielding inequivalent models were also discussed in [33].

Although, in the exceptional dimensions, the models (2.29) and (2.33) are mathematically inequivalent, the question remains, as for all other dimensions, which model is the physical one. The difference in the exceptional cases is that, in principle this question could be decided by experiments on a classical level. For the dimension $D_0 > 2$ the two models are mathematically equivalent; so on the classical level it can not be decided which is the physical one. However, if one demands that the gravitational interaction is generated by a pure massless spin-2 graviton (without scalar spin-0 admixture), then, reasoning similar as in [34], (2.33) rather than (2.29) has to be taken as the physical model.

Taking into account the conformal relation of scalar-tensor theories to fourth (or higher) order gravity (see e.g. [35]), the recent debate on the physical metric [36–39] concerns also the corresponding scalar-tensor theories. The result of this purely classical debate was rather poor: It mainly confirms Brans [40], who pointed out that, once the weak equivalence principle holds...
true in a given frame (in [40] it is the frame of the original higher order gravity), it will be violated in any non-trivially conformally related frame. However the choice of the frame with respect to which a test particle of ordinary matter moves along geodesics remains arbitrary for classical scalar-tensor theories. So the frame of Brans and Dicke [3] might be the physical one, giving geodesic paths for minimally coupled test matter, or likewise the Einstein Pauli frame might be the physical one. In [41] Hawking argued that, black holes might follow geodesics in the Einstein Pauli frame, but violate the strong equivalence principle in the BD frame, while the latter provides geodesic paths for usual test matter. For massive objects like black holes, this phenomenon is known as the Nordvedt effect [12]. Furthermore Cho [13] showed that in the BD frame quantum corrections enforce also a violation of the weak equivalence principle. We believe therefore, that the issue of the physical frame will be resolved finally only by a quantum theory of gravity. Since such a theory might not be subject to any equivalence principle, the latter might no longer serve as guiding principle for the physical metric. However, generalized arguments of Cho [34, 43], give some hint that the Einstein Pauli frame (when quantum corrections are small enough not to destroy any frame at all !) might then nevertheless be taken as the physical one.

It should however be noted that our multi-scalar-tensor theories differ essentially from usual scalar-tensor theories: There, some ”ordinary” matter field is minimally coupled to the geometry, either in the Jordan Brans Dicke frame or, equally well, in the Einstein Pauli frame (see also [15]). We saw above that, arguing on the basis of a classical equivalence principle for the ordinary matter only, there is no way select the physical frame. However, in our models all scalar fields are derived from a multidimensional geometry, which determines all couplings of all scalar fields to the geometry and among each other. These couplings can be tested in principle by experiments, thus selecting the physically admissible multi-scalar-tensor theories and their corresponding multidimensional counterpart. Because of this predictive power, it is tempting to postulate that any multi-scalar-tensor model should derive its (scalar) fields from a higher dimensional geometry, i.e. all (scalar) matter should have some geometric origin.

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