The subharmonic bifurcation of Stokes waves on vorticity flow

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Abstract

Until 1980 one of the main subjects of study in the theory of nonlinear water waves were the Stokes and solitary waves (regular waves). To that time small amplitude regular waves were constructed and the existence of large amplitude water waves of the same type was proved by using branches of water waves starting from a trivial (horizontal) wave and ending at extreme waves. Then in papers Chen & Saffman [7] and Saffman [32] numerical evidence was presented for existence of other type of waves as a result of bifurcations from a branch of ir-rotational Stokes waves on flow of infinite depth. It was demonstrated that the Stokes branch has infinitely many bifurcation points when it approaches the extreme wave and periodic waves with several crests of different height on the period bifurcate from the main branch. The only theoretical works dealing with this phenomenon are Buffoni, Dancer & Toland [4,5] where it was proved the existence of sub-harmonic bifurcations bifurcating from the Stokes branch for the ir-rotational flow of infinite depth approaching the extreme wave.

The aim of this paper is to develop new tools and give rigorous proof of existence of subharmonic bifurcations in the case of rotational flows of finite depth. The whole paper is devoted to the proof of this result formulated in Theorem 5.4.

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1. Introduction

1.1. Background

In this paper we study two-dimensional flows of an inviscid, incompressible, heavy fluid, say, water under the assumption that it is bounded above by a free surface, where the pressure is constant, and by a horizontal rigid bottom from below. Both irrotational and vortical flows are of interest and considered here.

It was Stokes [35], who had initiated mathematical studies of steady water waves as early as 1847. On the basis of approximations developed for periodic waves with a single crest per wavelength he made conjectures about the behavior of such waves now referred to as Stokes waves. These conjectures, to a large extent, determined the line of investigations of steady waves in the 20th century (see the paper [29] by Plotnikov and Toland and references cited therein). Another line of investigations originated from the observation made by Scott Russell [31] also in the 1840s concerns solitary waves. Since the 1940s, much attention was given to proving the existence of such a wave and investigation of its properties (see, for example, the survey paper [13] by Groves). In particular in Amick & Toland [1,2] the existence of global branches of Stokes/solitary ir-rotational water waves was proved. Investigation of these classical free-boundary problems requires most of modern methods in nonlinear functional analysis and nonlinear dispersive wave theory, in particular, bifurcation theory, complex variable methods, PDE methods, etc.

Extreme waves (or, equivalently, “waves of greatest height” according to Stokes) are a unique phenomenon in the mathematical theory of water waves. These are large-amplitude travelling waves with sharp crests of included angle 120 degrees. It is remarkable that while being a highly nonlinear phenomenon they were predicted by Sir George Stokes already in 1880s, see [36]. Stokes also argued that the stagnation by itself forces the surface profile to have a sharp crest of included angle $2\pi/3$. This property is known as the Stokes conjecture about waves of greatest height, which stimulated the development of the theory for many years.

The first existence of Stokes waves that are arbitrary close to the stagnation is due to Keady & Norbury [16], who used a global bifurcation theory for positive operators applied to the Nekrasov equation. Proof of existence of extreme waves by passing to the limit along a sequence of waves approaching stagnation was done by Toland [39] in 1978 for the infinite depth case and by Amick and Toland [1] for waves of finite depth. The Stokes conjecture concerning the angle $2\pi/3$ was verified independently by Amick, Fraenkel & Toland in [3] and by Plotnikov [28]. Structure of water waves with a stagnation point was studied by Varvaruca & Weiss in [40], who proved that if the stagnation point is isolated then the wave is of class $C^1$ from both sides of it and the angle can be $2\pi/3$ or $\pi$. All previously mentioned results concerned irrotational water waves, while the case of waves with vorticity is much less studied. There is also a qualitative difference. In their study Varvaruca & Weiss [41] found (without proving the existence) that surface profiles near stagnation points are either Stokes corners, horizontally flat, or horizontal cusps, though it is not known if the last two options are possible. It was shown in Varvaruca [42] that there exists a family of periodic solutions to the water wave problem with “negative” vorticity converging to an extreme wave enjoying stagnation at every crest. Unfortunately, it was not possible to show that the limiting wave is not “trivial”, that is not a laminar flow with surface stagnation. This difficulty was resolved in Kozlov & Lokharu [22] by using a different approach and extreme waves were found. A further analysis was made in Kozlov & Lokharu [25,24], where authors obtained higher-order asymptotics for the surface profile near stagnation points. We note also
the papers [27] by Plotnikov, where global branches of solitary waves in ir-rotational case was constructed and bifurcation points were studied.

Until 2000, only a few works treated steady water waves with vorticity, but activity in this area became very intensive during the last twenty years; see the survey [37] by Strauss, which covers works that had been published in the field of steady waves during the past nine decades. This review contains almost 100 references on both irrotational and vortical waves. Existence of small amplitude Stokes waves is established in [9] (unidirectional water waves), [19] (water waves with counter-currents) and existence of solitary waves for near-critical values of Bernoulli’s constant is proved in Groves & Wahlen [14] and Hur [15]. We mention here the construction of a global branch of Stokes waves without counter-currents in Constantin & Strauss [9,11] and a global branch of solitary waves in Chen, Walsh & Wheeler [8].

Stokes and solitary waves (regular waves) were the main subject of study up to 1980. In 1980 (see Chen [7] and Saffman [32]) it was discovered numerically and in 2000 (see [4,5]) this was supported theoretically for the ir-rotational case for flow of infinite depth that there exist new types of periodic waves with several crests on the period (the Stokes wave has only one crest). These waves appear as a result of bifurcation from a branch of Stokes waves when they approach the wave of greatest amplitude. The only theoretical investigation is [4,5] which was devoted to subharmonic bifurcations from a branch of Stokes waves on an ir-rotational flow of infinite depth.

1.2. The method

Starting point of our study is a branch of Stokes waves approaching an extreme wave. This is based on our papers [22] and [24], where it is proved existence of branches of Stokes waves approaching an extreme wave in the rotational case and for a flow of finite depth. We choose the period of Stokes wave as a parameter for constructed branches. This is important to guarantee that an extreme periodic wave will appear as the limit configuration for the branch.

In further analysis we need certain smoothness properties of extreme waves, which was the subject of the study in the paper [25] (see also a short presentation of the results in [24]). In previous papers (see Varvaruca [42], Varvaruca & Weiss [40,41]) it was proved that the extreme wave is $C^1$ from both sides of the stagnation point. We find an optimal asymptotics for the limit wave near the stagnation in [25]. Such properties of the extreme wave are important in order to show that there are infinitely many bifurcations at the branch of Stokes waves approaching an extreme wave.

The above results are first steps in our study of subharmonic bifurcation. Our study of bifurcations is based on the variational structure of operators. Earlier such structure was used by Plotnikov in [27], where he investigated bifurcations of solitary waves, by Buffoni, Dancer & Toland in [4,5] and by Shargorodsky & Toland in [33], devoted to irrotational Stokes waves. Since our family of operators consists of potential operators we apply the bifurcation theorem for potential family of operators whose coefficients come from the branch of Stokes waves connecting the stream solution and extreme wave. In order to find subharmonic bifurcations we are looking for solutions with period multiple of the period of coefficient of the main branch of Stokes waves. The principle difficulties here are to separate bifurcation points corresponding to the Stokes waves and to the waves with multiple period (subharmonic waves) and to study the change of the Morse index. This is done by using the results from [23] and new study of negative spectrum of the first variation of water waves close to the extreme wave presented here. Two properties of negative spectrum are important: when we approaching the extreme wave the
negative eigenvalues can be arbitrary large in absolute value, the corresponding eigenfunctions are concentrated near the stagnation point.

1.3. Formulation of the problem

We consider steady surface waves in a two-dimensional channel bounded below by a flat, rigid bottom and above by a free surface that does not touch the bottom. The surface tension is neglected and the water motion can be rotational. In appropriate Cartesian coordinates \((X, Y)\), the bottom coincides with the \(X\)-axis and gravity acts in the negative \(Y\)-direction. We choose the frame of reference so that the velocity field is time-independent as well as the free-surface profile which is supposed to be the graph of \(Y = \xi(X), \ X \in \mathbb{R}\), where \(\xi\) is a positive and continuous unknown function. Thus

\[
\mathcal{D}_\xi = \{X \in \mathbb{R}, 0 < Y < \xi(X)\}, \ \mathcal{B}_\xi = \{X \in \mathbb{R}, Y = \xi(X)\}
\]

is the water domain and the free surface respectively. We will use the stream function \(\Psi\), which is connected with the velocity vector \((u, v)\) as \(u = -\Psi_Y\) and \(v = \Psi_X\).

We assume that \(\xi\) is a positive, periodic function having period \(\Lambda > 0\) and that \(\xi\) is even and strongly monotonically decreasing on the interval \((0, \Lambda/2)\). Since the surface tension is neglected, \(\Psi\) and \(\xi\) after a certain scaling satisfy the following free-boundary problem (see for example [19]):

\[
\begin{align*}
\Delta \Psi + \omega(\Psi) &= 0 \text{ in } \mathcal{D}_\xi, \\
\frac{1}{2} |\nabla \Psi|^2 + \xi &= R \text{ on } \mathcal{B}_\xi, \\
\Psi &= 1 \text{ on } \mathcal{B}_\xi, \\
\Psi &= 0 \text{ for } Y = 0,
\end{align*}
\]

(1.1)

where \(\omega \in C^{1,\gamma}, \gamma \in (0, 1)\), is a vorticity function and \(R\) is the Bernoulli constant. The Frechet derivative of the non-linear operator is evaluated in [23] (see also Sect. 2.3 in this paper) and it is represented by the left-hand side in the following spectral problem

\[
\begin{align*}
\Delta w + \omega'(\Psi)w &= -\frac{\theta}{\Psi_Y} w \text{ in } \mathcal{D}_\xi, \\
\partial_v w - \rho w &= 0 \text{ on } \mathcal{B}_\xi, \\
\Psi &= 0 \text{ for } Y = 0,
\end{align*}
\]

(1.2)

where \(\theta\) is a real number,

\[
\rho(X, Y) = \frac{1 + \Psi_X \Psi_{XY} + \Psi_Y \Psi_{YY}}{\Psi_Y (\Psi_X^2 + \Psi_Y^2)^{1/2}}
\]

and \(w\) is an even, \(\Lambda\)-periodic function.

In what follows we will study branches of Stokes waves \((\Psi(X, Y; t), \xi(X; t))\) of period \(\Lambda(t), t \geq 0\), started from the uniform stream at \(t = 0\). It is convenient to make the following change of variables
in order to deal with the problem with a fixed period. As the result we get

\[
\left( \lambda^2 \partial_x^2 + \partial_y^2 \right) \psi + \omega(\psi) = 0 \quad \text{in} \quad D_\eta,
\]

\[
\frac{1}{2} \left( \lambda^2 \psi_x^2 + \psi_y^2 \right) + \eta = R \quad \text{on} \quad B_\eta,
\]

\[
\psi = 1 \quad \text{on} \quad B_\eta,
\]

\[
\psi = 0 \quad \text{for} \quad y = 0,
\]

where

\[
\psi(x, y; t) = \Psi(\lambda^{-1}x, y; t) \quad \text{and} \quad \eta(x; t) = \xi(\lambda^{-1}x; t).
\]

Here all functions have the same period \( \Lambda_0 := \Lambda(t) \), \( D_\eta \) and \( B_\eta \) are the domain and the free surface after the change of variables (1.3).

1.4. Uniform stream solution, dispersion equation

The uniform stream solution \( \psi = U(y) \) with the constant depth \( \eta = d \) and \( \lambda = 1 \), satisfies the problem

\[
U'' + \omega(U) = 0 \quad \text{on} \quad (0; d),
\]

\[
U(0) = 0, \quad U(d) = 1,
\]

\[
\frac{1}{2} U'(d)^2 + d = R.
\]

Let \( s = U'(0) \) and \( s > s_0 := 2 \max_{\tau \in [0, 1]} \varphi(\tau) \), where

\[
\varphi(\tau) = \int_0^\tau \omega(p) dp.
\]

Then the problem (1.5) has a solution \((U, d)\) with a strongly monotone function \( U \) if

\[
\frac{1}{2} s^2 + d(s) - \varphi(1) = R.
\]

In this case \((U, d)\) is found from the relations

\[
y = \int_0^U \frac{d\tau}{\sqrt{s^2 - 2\varphi(\tau)}}, \quad d = d(s) = \frac{1}{\sqrt{s^2 - 2\varphi(\tau)}}.
\]

The equation (1.6) is solvable if \( R > R_c \).
\[ R_c = \min_{s \geq s_0} \left( \frac{1}{2} s^2 + d(s) - \varphi'(1) \right). \]  

(1.8)

and it has two solutions if \( R \in (R_c, R_0) \), where

\[ R_0 = \frac{1}{2} s_0^2 + d(s_0) - \varphi'(1). \]  

(1.9)

We denote by \( s_c \) the point where the minimum in (1.8) is attained.

Existence of small amplitude Stokes waves is determined by the dispersion equation (see, for example, [19]). It is defined as follows. The strong monotonicity of \( U \) guarantees that the problem

\[ \gamma'' + \omega'(U)\gamma - \tau^2 \gamma = 0, \quad \gamma(0, \tau) = 0, \quad \gamma(d, \tau) = 1 \]  

(1.10)

has a unique solution \( \gamma = \gamma(y, \tau) \) for each \( \tau \in \mathbb{R} \), which is even with respect to \( \tau \) and depends analytically on \( \tau \). Introduce the function

\[ \sigma(\tau) = \kappa \gamma'(d, \tau) - \kappa^{-1} + \omega(1), \quad \kappa = \Psi'(d). \]  

(1.11)

It depends also analytically on \( \tau \) and it is strongly increasing with respect to \( \tau > 0 \). Moreover it is an even function. The dispersion equation (see, for example [19]) is the following

\[ \sigma(\tau) = 0. \]  

(1.12)

It has a positive solution if

\[ \sigma(0) < 0. \]  

(1.13)

By [19] this is equivalent to \( s + d'(s) < 0 \) or what is the same

\[ 1 < \int_0^d \frac{dy}{U''^2(y)}. \]  

(1.14)

The left-hand side here is equal to \( 1/F^2 \) where \( F \) is the Froude number (see [43] and [26]). Therefore (1.14) means that \( F < 1 \), which is well-known condition for existence of water waves of small amplitude. Another equivalent formulation is given by requirement (see, for example [20])

\[ s \in (s_0, s_c) \text{ and satisfies } (1.6). \]  

(1.15)

The existence of such \( s \) is guaranteed by \( R \in (R_c, R_0) \).

The function \( \sigma \) has the following asymptotic representation

\[ \sigma(\tau) = \kappa \tau + O(1) \text{ for large } \tau. \]
and equation (1.12) has a unique positive root, which will be denoted by $\tau_\ast$. It is connected with $\Lambda_0$ by the relation $\tau_\ast = \frac{2\pi}{\Lambda_0}$. The function $\gamma(y, \tau)$ is positive in $(0, d]$ for $\tau > \tau_\ast$.

Let

$$\rho_0 = \frac{1 + \Psi'(d)\Psi''(d)}{\Psi'(d)^2}.$$  \hfill (1.16)

We note that

$$\frac{1 + \Psi'(d)\Psi''(d)}{\Psi'(d)^2} = \kappa^{-2} - \frac{\omega(1)}{\kappa}$$

and hence another form for (1.11) is

$$\sigma(\tau) = \kappa \gamma'(d, \tau) - \kappa \rho_0.$$  \hfill (1.17)

1.5. Partial hodograph transform

When we consider a branch of solutions to the problem (1.4) it is convenient to work with a domain independent of $t$. One set to do this is to use the partial hodograph transform, see [9] or [19].

We assume that

$$\psi_y > 0 \text{ in } \overline{D_\eta}$$

and use the variables

$$q = x, \ p = \psi.$$  

Then

$$q_x = 1, \ q_y = 0, \ p_x = \psi_x, \ p_y = \psi_y,$$

and

$$\psi_x = -\frac{h_q}{h_p}, \ \psi_y = \frac{1}{h_p}, \ dx dy = h_p dq dp.$$  \hfill (1.18)

System (1.4) in the new variables takes the form

$$
\left( \frac{1 + \lambda^2 h_q^2}{2h_p^2} + \varphi(p) \right)_p - \lambda^2 \left( \frac{h_q}{h_p} \right)_q = 0 \text{ in } Q, \\
\frac{1 + \lambda^2 h_q^2}{2h_p^2} + h = R \text{ for } p = 1, \\
h = 0 \text{ for } p = 0.
$$  \hfill (1.19)

Here
\[ Q = \{(q, p) : q \in \mathbb{R}, \ p \in (0, 1)\}. \]

The uniform stream solution corresponding to the solution \( U \) of (1.5) is

\[
H(p) = \int_0^p \frac{d\tau}{\sqrt{s^2 - 2\phi(\tau)}}, \ s = U'(0) = H_p^{-1}(0). \tag{1.20}
\]

One can check that

\[
H_{pp} - H_p^3 \omega(p) = 0
\]

or equivalently

\[
\left(\frac{1}{2H_p^2}\right)_p + \omega(p) = 0.
\]

Moreover it satisfies the boundary conditions

\[
\frac{1}{2H_p^2(1)} + H(1) = R, \ H(0) = 0. \tag{1.21}
\]

The problem (1.19) has a variational formulation (see [10]) and the potential is given by

\[
f(h; \lambda) = \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^1 \left(\frac{1}{2h_p^2} - h + R - (\phi(p) - \phi(1))\right) h_p dq dp. \tag{1.22}
\]

We introduce spaces \( C_{pe}^{k, \alpha}(\Omega) \) and \( C_{pe}^{k, \alpha}(\mathbb{R}) \), \( k = 0, 1, 2, \ldots, \alpha \in (0, 1) \), which are subspaces of \( C^{k, \alpha}(\Omega) \) and \( C^{k, \alpha}(\mathbb{R}) \) respectively and consist of even, periodic functions of the period \( \Lambda_0 \).

The following theorem is proved in [22].

**Theorem 1.1.** Assume that \( \omega \in C^{1, \gamma}([0; 1]) \). Then for any \( R \in (R_c, R_0) \) there exist functions \( C : [0; +\infty) \to C_{pe}^{2, \gamma}(\Omega) \) and \( \lambda : [0; +\infty) \to (0; +\infty) \) solving the problem (1.19) for each \( t \geq 0 \) with the following properties:

(i) \( C(0) \) is the subcritical stream solution \( H_p \) given by (1.20);

(ii) \( C(t) = h(q, p; t); t > 0 \) is a solution to (1.19) with \( \lambda = \lambda(t) \); the corresponding solution \((\psi; \eta)\) represents a Stokes wave with the period \( \Lambda(t) = \Lambda_0 \lambda(t)^{-1} \);

(iii) \( (\psi(t), \lambda(t)) \) has a real analytic reparametrization locally around each \( t \geq 0 \).

Furthermore, there exists a sequence \( \{t_j\}_{j=1}^{\infty}, t_j \to \infty \) as \( j \to \infty \), such that we either have

(I) \( \max_{x \in \mathbb{R}} \eta(x; t_j) \to R \) and \( \Lambda(t_j) \to \Lambda < \infty, \Lambda \neq 0 \), as \( j \to \infty \) (stagnation at every crest);

(II) \( \lim_{j \to \infty} \max_{x \in \mathbb{R}} \eta(x; t_j) = R_1 \leq R \) and \( \Lambda(t_j) \to \infty \) as \( j \to \infty \) (solitary wave, possibly with stagnation).

(III) \( \lim_{j \to \infty} \max_{x \in \mathbb{R}} \eta'(x; t_j) = \infty \) as \( j \to \infty \).

Here \( \eta(x; t) \) is the surface profile corresponding to \( h(q, p; t) \), i.e. \( \eta(x; t) = h(x, 1; t) \).
1.6. Assumptions

Here we collect basic assumptions which will be used in this paper for proving our main result on subharmonic bifurcations.

We assume that $R \in (R_c, R_0)$ and (1.14) holds. The last assumption is equivalent to (1.13) and to (1.15). Then according to Theorem 1.1 there exists a branch of solutions to (1.19)

$$h = h(q, p; t) : [0, \infty) \to C^2_{pe}(\bar{Q}), \quad \lambda = \lambda(t) : [0, \infty) \to (0, \infty),$$

which has a real analytic reparametrization locally around each $t \geq 0$.

We assume that

$$|h_q(q, 1; t)| \leq B \text{ for } q \in \mathbb{R} \text{ and } t \geq 0,$$

where $B$ is a positive constant and that the branch satisfies Theorem 1.1 (I), i.e. there exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $t_j \to \infty$ as $j \to \infty$ and

$$h(0, 1; t_j) \to R, \quad \lambda(t_j) \to \lambda_\ast \text{ as } j \to \infty,$$

where $\lambda_\ast$ is a positive constant.

We introduce the sequence $(\Psi_j(X, Y), \xi_j(X), \Lambda_j)$, which corresponds to $h_j = h(q, p; t_j)$ and solves the problem (1.1). In particular,

$$\xi_j(X) = h_j(\lambda_j X, 0), \quad \lambda_j = \frac{\Lambda_0}{\Lambda_j}$$

and $\Psi_j$ can be found from

$$\Delta \Psi_j + \omega(\Psi_j) = 0 \text{ in } \mathcal{D}_{\xi_j}$$

$$\Psi_j = 1 \text{ on } \mathcal{B}_{\xi_j}$$

$$\Psi_j = 0 \text{ for } Y = 0.$$  

The remaining boundary condition for $\Psi_j$ holds because of a similar equation for $h_j$. Both functions $\xi_j$ and $\Psi_j$ have period $\Lambda_j$ with respect to $X$. Furthermore the consequence of (1.24) is the inequality

$$|\xi_jX(X)| \leq \lambda_j B.$$

By Proposition 4.2, [21],

$$|\nabla \Psi_j(X, Y)| \leq C(R, \omega_0), \quad \text{for } (X, Y) \in \overline{\mathcal{D}_{\xi_j}},$$

where $C$ depends only on $R$ and $\omega_0$, $\omega_0 = \max_{0 \leq p \leq 1} \omega(p)$. Due to (1.24) one can choose a subsequence of $\{\xi_j\}$ which is convergent in $C^0_{pe}(\mathbb{R})$ for any $\alpha \in (0, 1)$, to a function $\xi_* \in C^0_{pe}(\mathbb{R})$. By (1.29) we can assume also that the sequence $\{\Psi_j\}$, where $\Psi_j$ is the extension of $\Psi_j$ by 1 for $Y > \xi(X)$ and by 0 for $Y < 0$, is also convergent in $L^\infty(\mathcal{Q}_{R, a, b})$ to a function $\tilde{\Psi}_*$, where
\[ Q_{R,a,b} = \{(X, Y) : X \in (a, b), Y \in (0, R)\}. \]

Moreover
\[ \nabla \tilde{\Psi}_j \text{ weak* } \nabla \tilde{P}_{\xi_\ast} \text{ in } L^\infty(Q_{R,a,b}). \]

Here \( a < b \) are arbitrary real numbers. Then the limit functions \( (\xi_\ast, \Psi_\ast, \Lambda_\ast) \), where \( \Lambda_\ast = \Lambda_0/\lambda_\ast \) and \( \xi_\ast \) together with \( \Psi_\ast \) are even periodic functions of period \( \Lambda_\ast \), satisfying (1.1) in a weak sense, see [42]. One can verify that all conditions of Theorem 5.2, Varvarua [42], are satisfied and according to that theorem there are two options for the limit function \( \xi_\ast \):

\[
\lim_{X \to 0^+} \frac{\xi_\ast(X)}{X} = -\frac{1}{\sqrt{3}} \quad \text{or} \quad \lim_{X \to 0} \frac{\xi_\ast(X)}{X} = 0. \tag{1.30}
\]

We assume that the first option holds, i.e. the limit Stokes wave has an opening angle 120 degree at the stagnation points.

Below we give sufficient conditions for validity of the above assumptions.

**Proposition 1.2.** (Existence of highest waves, Kozlov & Lokharu [24]) Let \( \omega \geq 0 \) be sufficiently small. Then there exists a constant \( R_\ast \in (R_c, R_0) \) such that only waves of type (I) with the opening angle 120 degree can occur for \( R \in (R_\ast, R_0) \).

**Remark 1.3.** Validity of (1.24) is analysed in Strauss & Wheeler [38].

2. Bifurcation analysis

Here we present two equations for finding bifurcation points in \( q, p \) variables. One of them is defined by a boundary value problem in a two-dimensional domain and another one including a Dirichlet-Neumann operator is defined one a part of one-dimensional boundary. It was proved in Sect. 2.5 that the Frechet derivatives of operators in these two formulations have the same number of negative eigenvalues and the same dimension of the kernels. So both of them can be used in analysis of the Morse index and corresponding crossing number. The first bifurcation 2D problem is useful in analysis of the number of negative eigenvalues and the second one is more convenient for application of general bifurcation results.

Both formulations can be easily extended for study of subharmonic bifurcations, see Sect. 4.1.

In Sect. 2.3 we give an explicit connection between the Frechet derivatives in \((x, y)\) and \((q, p)\) variables.

2.1. First formulation of bifurcation equation

In order to find bifurcation points and bifurcating solutions we put \( h + w \) instead of \( h \) in (1.19) and introduce the operators

\[
F(w; h, t) = \left(1 + \lambda^2 (h_q + w_q)^2 \right) p - \left(1 + \frac{\lambda^2 h_q^2}{2h_p^2} \right) p
\]

\[
- \lambda^2 \left( \frac{h_q + w_q}{h_p + w_p} \right) q + \lambda^2 \left( \frac{h_q}{h_p} \right) q.
\]

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and

\[ G(w; h, t) = \frac{1 + \lambda^2(h_q + w)^2}{2(h_p + w_p)^2} - \frac{1 + \lambda^2 h_q^2}{2h_p^2} + w \]

acting on \(\Lambda_0\)-periodic functions \(w\) defined in \(Q\). After some cancellations we get

\[ F = \lambda^2 h_p^2 (2h_q + w_q)w_q - \frac{(2h_p + w_p)(1 + \lambda^2 h_q^2)w_p}{2h_p^2(h_p + w_p)^2} \]

\[ - \lambda^2 \left( \frac{h_p w_q - h_q w_p}{h_p(h_p + w_p)} \right)_q \]

and

\[ G = \frac{\lambda^2 h_p^2 (2h_q + w_q)w_q - (2h_p + w_p)(1 + \lambda^2 h_q^2)w_p}{2h_p^2(h_p + w_p)^2} + w. \]

Both these functions are well defined for small \(w_p\). Then the problem for finding solutions close to \(h\) is the following

\[ F(w; h, t) = 0 \text{ in } Q \]

\[ G(w; h, t) = 0 \text{ for } p = 1 \]

\[ w = 0 \text{ for } p = 0. \]  

The above problem also has a variational formulation. The corresponding potential is

\[ f(h + w; \lambda) - f(h; \lambda), \]

where \(f\) is defined by (1.22). Furthermore, the Frechet derivative (the linear approximation of the functions \(F\) and \(G\)) is the following

\[ Aw = A(t)w = \left( \frac{\lambda^2 h_q w_q}{h_p^2} - \frac{(1 + \lambda^2 h_q^2)w_p}{h_p^3} \right)_p - \lambda^2 \left( \frac{w_q}{h_p} - \frac{h_q w_p}{h_p^2} \right)_q \]  

(2.2)

and

\[ Nw = N(t)w = (Nw - w)|_{p=1}, \]  

(2.3)

where

\[ Nw = N(t)w = \left( - \frac{\lambda^2 h_q w_q}{h_p^2} + \frac{(1 + \lambda^2 h_q^2)w_p}{h_p^3} \right)|_{p=1}. \]  

(2.4)

The eigenvalue problem for the Frechet derivative, which is important for the analysis of bifurcations of the problem (2.1), is the following
\[ A(t)w = \theta w \quad \text{in} \quad Q, \]
\[ N(t)w = 0 \quad \text{for} \quad p = 1, \]
\[ w = 0 \quad \text{for} \quad p = 0. \tag{2.5} \]

**Remark 2.1.** Differentiating equations in (1.19) with respect to \( q \) we get

\[ A(t)h_q = 0 \quad \text{in} \quad Q, \]
\[ N(t)h_q = 0 \quad \text{for} \quad p = 1, \]
\[ h_q = 0 \quad \text{for} \quad p = 0. \tag{2.6} \]

Therefore \( h_q \) always solves the problem (2.5) for \( \theta = 0 \) but we note that the function \( h_q \) is odd with respect to \( q \).

2.2. **Second formulation of bifurcation equation**

There is another formulation for the bifurcating solutions, which is used the Dirichlet-Neumann operator. Let us consider the problem

\[ F(w; h, t) = 0 \quad \text{in} \quad Q, \]
\[ w = g \quad \text{for} \quad p = 1, \]
\[ w = 0 \quad \text{for} \quad p = 0. \tag{2.7} \]

We define the operator \( \mathcal{S} = \mathcal{S}(g; h, t) \) by

\[ \mathcal{S}(g; t) = \mathcal{G}(w; h, t)|_{p=1}, \tag{2.8} \]

where \( w \) is the solution of the problem (2.7). Then the equation for bifurcating solutions is

\[ \mathcal{S}(g; t) = 0. \tag{2.9} \]

We will prove the solvability of the problem (2.7) in Sect. 2.4. Here we note that spectral problem for the Frechet derivative of the left-hand side in (2.9) is given by

\[ A(t)w = 0 \quad \text{in} \quad Q, \]
\[ N(t)w - w = \mu w \quad \text{for} \quad p = 1, \]
\[ w = 0 \quad \text{for} \quad p = 0. \tag{2.10} \]

More exactly if we introduce the problem

\[ A(t)w = 0 \quad \text{in} \quad Q, \]
\[ w = g \quad \text{for} \quad p = 1, \]
\[ w = 0 \quad \text{for} \quad p = 0, \tag{2.11} \]
then the operator

\[ S_g = (Nw - w)|_{p=1} \]  

(2.12)
is the Frechet derivative of the operator (2.9). The corresponding spectral problem is

\[ S_g = \mu g. \]  

(2.13)

2.3. Spectral problems (2.5) and (2.10) in \((x, y)\) variables

In this section we write the spectral problems (2.5) and (2.10) in \((x, y)\) variables. This will be useful in our study of negative eigenvalues of the spectral problem (2.13).

Let \(\Gamma = \Gamma(x, y)\) be a \(\Lambda_0\)-periodic function in \(D_\eta\). Consider the function

\[ F(q, p; \lambda) = \Gamma(q, h)p. \]  

(2.14)

Let \(A\) be the operator (2.2). Then

\[ Ah_p = \left( \frac{\lambda^2 h_q h_{qp}}{h_p^2} - \frac{(1 + \lambda^2 h_q^2)h_{pp}}{h_p^3} \right)_p - \lambda^2 \left( \frac{h_{qp}}{h_p} - \frac{h_q h_{pp}}{h_p^2} \right)_q \]

\[ = \left( \frac{1 + \lambda^2 h_q^2}{2h_p^2} \right)_pp - \lambda^2 \left( \frac{h_q}{h_p} \right)_qq = -\omega'(p). \]

(2.15)

Next we have

\[ F_q = \Gamma h_{qp} + \Gamma_x h_p + \Gamma_y h_q h_p, \quad F_p = \Gamma h_{pp} + \Gamma_y h_{q2}. \]

(2.16)

Using (2.16) together with (2.15), we get

\[ A(\Gamma h_p) = -\omega'(p)\Gamma + \left( \frac{\lambda^2 h_q (\Gamma_x + \Gamma_y h_q)}{h_p} - \frac{(1 + \lambda^2 h_q^2)\Gamma_y}{h_p} \right)_p - \lambda^2 \left( \Gamma_x \right)_q \]

\[ + \Gamma_y \left( \frac{\lambda^2 h_q h_{qp}}{h_p} - \frac{(1 + \lambda^2 h_q^2)h_{pp}}{2h_p^2} \right) - \lambda^2 (\Gamma_x + \Gamma_y h_q) \left( \frac{h_{qp}}{h_p} - \frac{h_q h_{pp}}{h_p^2} \right) \]

\[ = -\omega'(p)\Gamma - \lambda^2 \Gamma_{xx} - \Gamma_{yy}. \]

(2.17)

Thus

\[ AF = -\omega'(p)\Gamma - \lambda^2 \Gamma_{xx} - \Gamma_{yy}. \]

(2.18)

Since

\[ \psi_{xy} = -\frac{h_{qp} + h_{pp}\psi_X}{h_p^2}, \quad \psi_{yy} = -\frac{h_{pp}\psi_y}{h_p^2}, \]

we have
\[
\frac{1 + \lambda^2 \psi_x \psi_{xy} + \psi_y \psi_{yy}}{\psi_y} = h_p \left( 1 - \lambda^2 \frac{\psi_x}{h_p} h_{qp} + \frac{\psi_x}{h_p} h_{pp} \frac{\psi_x}{h_p^2} - \frac{\psi_y}{h_p} h_{pp} \right)
\]
\[
= h_p + \lambda^2 \frac{h_q h_{qp}}{h_p^2} \left( 1 + \lambda^2 \frac{h_q^2}{h_p^3} \right). 
\]

Furthermore
\[
-\mathcal{N}F = \left( \frac{\lambda^2 h_q (\Gamma h_{qp} + \Gamma_x h_p + \Gamma_y h_q h_p)}{h_p^2} - \frac{(1 + \lambda^2 h_q^2)(\Gamma h_{pp} + \Gamma_y h_{pp}^2)}{h_p^3} \right)_{p=1}
\]
\[
= \left( \lambda^2 \frac{h_q h_{qp}}{h_p^2} - \frac{(1 + \lambda^2 h_q^2) h_{pp}}{h_p^3} \right) \Gamma + \lambda^2 \frac{h_q}{h_p} \Gamma_x - \frac{1}{h_p} \Gamma_y. 
\]

Therefore
\[
-\mathcal{N}F + F = \sqrt{\psi_x^2 + \psi_y^2} \rho \Gamma - \lambda^2 \psi_x \Gamma_x - \psi_y \Gamma_y, 
\]
where
\[
\rho = \rho(x; t) = \left. \frac{(1 + \lambda^2 \psi_x \psi_{xy} + \psi_y \psi_{yy})}{\psi_y (\psi_x^2 + \psi_y^2)^{1/2}} \right|_{y=\eta(x; t)}. 
\]

**Corollary 2.2.** If \(\Gamma = \Gamma(x, y)\) satisfies the problem
\[
(\lambda^2 \partial_x^2 + \partial_y^2) \Gamma + \omega'(\psi) \Gamma + \theta \frac{1}{\psi_y} \Gamma = 0 \text{ in } D_\eta, 
\]
\[
(\lambda^2 v_x \Gamma_x + v_y \Gamma_y - \rho \Gamma)_{p=1} = 0 \text{ on } B_\eta, 
\]
\[
\Gamma = 0 \text{ for } x = 0, 
\]
where \(v = (v_x, v_y) = \nabla \psi / |\nabla \psi|\) is the unite outward normal to \(y = \eta(x)\), then
\[
AF = \theta F, 
\]
\[
NF - F = 0 \text{ for } p = 1, 
\]
\[
F = 0 \text{ for } p = 0. 
\]

**Corollary 2.3.** If \(\Gamma = \Gamma(x, y)\) satisfies the problem
\[
(\lambda^2 \partial_x^2 + \partial_y^2) \Gamma + \omega'(\psi) \Gamma = 0 \text{ in } D_\eta, 
\]
\[
(\lambda^2 v_x \Gamma_x + v_y \Gamma_y - \rho \Gamma)_{p=1} = \mu \frac{\psi_y}{|\nabla \psi|} \Gamma \text{ on } B_\eta, 
\]
\[
\Gamma = 0 \text{ for } x = 0, 
\]
then
\[ AF = 0, \]
\[ NF - F = \mu F, \text{ for } p = 1 \]
\[ F = 0 \text{ for } p = 0. \] (2.24)

2.4. Solvability of the Dirichlet problem (2.7)

The next proposition deals with the following Dirichlet problem

\[ Aw = f, \]
\[ w = g, \text{ for } p = 1 \]
\[ w = 0 \text{ for } p = 0. \] (2.25)

**Proposition 2.4.** Let \( g \in C^{2,\alpha}_{pe}(\mathbb{R}) \) and \( f \in C^{0,\alpha}_{pe}(Q), \alpha \in (0, \gamma]. \) Then the problem (2.25) has solution \( w \in C^{2,\alpha}_{pe}(Q), \) which satisfies the estimate

\[ \|w\|_{C^{2,\alpha}_{pe}(Q)} \leq C(\|f\|_{C^{0,\alpha}_{pe}(Q)} + \|g\|_{C^{2,\alpha}_{pe}(\mathbb{R})}). \]

**Proof.** We start from the problem

\[ \lambda^2 \Gamma_{xx} + \Gamma_{yy} + \omega'(\psi)\Gamma = f \text{ in } D_\eta, \]
\[ \Gamma = g \text{ for } y = \eta(x), \]
\[ \Gamma(x,0) = 0. \]

This problem is elliptic. Let us show that it has a trivial kernel. Differentiating the first equation in (1.4) with respect to \( y \) we get

\[ \lambda^2 \psi_{xxx} + \psi_{xyy} + \omega'(\psi)\psi_y = 0 \text{ in } D_\eta. \]

Since \( \psi_y > 0 \) in \( \overline{D_\eta} \) by maximum principle the kernel of the problem is trivial (see [30]). Moreover the corresponding operator is symmetric and therefore the cokernel is also trivial. These facts and required smoothness of coefficients and standard solvability results for elliptic boundary value problems prove the proposition. \( \square \)

Next consider the nonlinear problem

\[ \mathcal{F}(w; h, t) = f \text{ in } Q, \]
\[ w = g \text{ for } p = 1, \]
\[ w = 0 \text{ for } p = 0. \] (2.26)

**Proposition 2.5.** There exist positive number \( \delta_0 \) depending on \( M, R - \max h(p, 1; t) \) such that if
\[ \| f \|_{C_{pe}^{2,\gamma}(\Omega)} + \| g \|_{C_{pe}^{2,\gamma}(\mathbb{R})} \leq \delta \leq \delta_0, \]

then there exists a unique solution \( w \in C_{pe}^{2,\gamma}(Q) \) such that
\[ \| w \|_{C_{pe}^{2,\gamma}(Q)} \leq C\delta. \]

**Proof.** Since the problem (2.26) is a small perturbation of (2.25) and the proof is quite standard. \( \Box \)

2.5. Comparison of two spectral problems

Here we will compare the negative spectrum of the following spectral problems

\[
A(t)u = 0 \quad \text{in} \quad Q \\
N(t)u - u = \mu u \quad \text{for} \quad p = 1 \\
u = 0 \quad \text{for} \quad p = 0
\]

(2.27)

and

\[
A(t)u = \theta u \quad \text{in} \quad Q \\
N(t)u - u = 0 \quad \text{for} \quad p = 1 \\
u = 0 \quad \text{for} \quad p = 0.
\]

(2.28)

**Proposition 2.6.** The spectral problems (2.27) and (2.28) have the same number of negative eigenvalues (accounting their multiplicities).

**Proof.** We introduce the bilinear form
\[
a(u, v) = \int_0^{\Lambda_0/2} \int_{-\Lambda_0/2}^{\Lambda_0/2} \left( \frac{(1 + \lambda^2 h^2_q) h_p^3}{h_p^2} - \frac{\lambda^2 h_q w_q h_p^2}{h_p^2} \right) v_p + \lambda^2 \left( \frac{w_q h_p - h_q w_p}{h_p^2} \right) \bar{v}_p dq dp
\]

\[- \int_{-\Lambda_0/2}^{\Lambda_0/2} w \bar{v} dq.
\]

We introduce several spaces. The first one \( H_{pe}^1(Q) \) consists of \( \Lambda_0 \)-periodic, even functions in \( Q \) from the usual Sobolev space \( H^1(Q) \) which are equal to zero for \( p = 0 \). The second space, denoted by \( \tilde{H}_{pe}^1(\Omega) \) is a subspace of \( H_{pe}^1(Q) \) satisfying \( Au = 0 \) in \( Q \). One more space, denoted by \( \tilde{H}_{pe}^1(\Omega) \), is the subspace of \( H_{pe}^1(Q) \) consisting of functions satisfying \( u = 0 \) for \( p = 1 \). One can verify that
\[ a(u, v) = 0 \quad \text{for} \quad u \in \tilde{H}_{pe}^1(Q) \quad \text{and} \quad v \in \tilde{H}_{pe}^1(Q). \]
Spectra of both spectral problems consist of eigenvalues bounded from below with the only accumulation point at $\infty$. Denote by $N_1$ the number (accounting their multiplicities) of negative eigenvalues of the problem (2.28) and similar number for the problem (2.27) will be denoted by $N_2$. As is known

$$N_1 = \max_{X \subset H^1_{pe}(Q), a(w, w) < 0, w \in X \setminus \{0\}} \dim X$$

and

$$N_2 = \max_{X \subset \tilde{H}^1_{pe}(Q), a(w, w) < 0, w \in X \setminus \{0\}} \dim X.$$ 

Here $X$ is a finite dimensional subspace of $H^1_{pe}(Q)$ or $\tilde{H}^1_{pe}(Q)$ respectively. Since $\tilde{H}^1_{pe}(Q)$ is a subspace of $H^1_{pe}(Q)$, $N_1 \geq N_2$. Furthermore, one can verify that

$$a(w, w) > 0 \text{ for } w \in \tilde{H}^1_{pe}(Q) \setminus \{O\}. \quad (2.29)$$

This follows from the positivity of the form

$$\int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^{\eta(x)} \left( \lambda^2 |u_x|^2 + |u_y|^2 - \omega'(\psi)|u|^2 \right) dx dy$$

considered on $\Lambda_0$-periodic, even functions from $H^1(D_\eta)$, vanishing on the boundary of $D_\eta$. This implies (2.29), after the change of variables.

Let $X$ be a finite dimensional subspace of $H^1_{pe}(Q)$ such that $\dim X = N_1$ and $a(w, w) < 0$ for $w \in X \setminus \{0\}$. We represent $w \in X$ as

$$w = U + V, \quad U \in \tilde{H}^1_{pe}(Q) \quad \text{and} \quad U \in \tilde{H}^1_{pe}(Q).$$

We denote the set of such $U$ by $X_0$. Its dimension is equal to $\dim X$. Indeed if $U = 0$ for a certain $w \in X$ then $w = U$ and hence $w$ and $w_p$ is equal to 0 on the boundary $p = 1$. This implies that $w = 0$. Furthermore, we have

$$a(U + V, U + V) = a(U, U) + a(V, V) < 0,$$

therefore

$$a(U, U) < 0 \text{ for } U \in X_0 \setminus \{0\}.$$ 

This proves that $N_1 = N_2$. \(\square\)

Combination of Proposition 2.6 and Corollaries 2.2, 2.3 gives

**Corollary 2.7.** The number of negative eigenvalues (counting together with their multiplicities) is the same for the spectral problems (2.27), (2.28) and (2.21), (2.23).
3. On negative eigenvalues of the spectral problem (2.13)

Let

\[ \Omega_\xi = \{ (X, Y) : X \in (-\Lambda/2, \Lambda/2), \ 0 < Y < \xi(X) \} \]

and let \( H^1_{pe}(\Omega_\xi) \) be the subspace of \( H^1(\Omega_\xi) \) consisting of even in \( X \) functions \( u \) satisfying \( u(-\Lambda/2, Y) = (\Lambda/2, Y) \). Introduce the form

\[
a_\xi(w, w) = \int_{\Omega_\xi} (|\nabla w|^2 - \omega'(\Psi)|w|^2)dXdY - \int_{-\Lambda/2}^{\Lambda/2} \rho|w|^2 \sqrt{1 + (\xi')^2}dX. \tag{3.1}\]

Then the spectral problem (1.2) admits the following variational formulation: find nonzero \( u \in H^1_{pe}(\Omega_\xi) \) and \( \theta \in \mathbb{R} \) such that

\[
a_\xi(u, w) = \theta \int_{\Omega_\xi} \frac{1}{\Psi_Y} u \bar{w}dXdY \text{ for all } w \in H^1_{pe}(\Omega_\xi). \tag{3.2}\]

The important observation, which is essential in analysis of subharmonic bifurcations, concerns the spectral problem (1.2) and is contained in the following theorem, where \((\Psi_j(X, Y), \xi_j(X), \Lambda_j)\) are defined in Sect. 1.6.

**Theorem 3.1.** Consider the spectral problem (1.2) (or equivalently (3.2) with \((\Psi, \xi, \Lambda) = (\Psi_j, \xi_j, \Lambda_j)\). For any positive \( \varepsilon \) there are subspaces \( X_j \subset H^1_{pe}(\Omega_{\xi_j}) \) such that

(i) sup \( u \in B_\varepsilon ((0, \xi(0))) \cap \Omega_{\xi_j} \) for \( u \in X_j \setminus \{O\} \);

(ii) \( a_{\xi_j}(u, u) < 0 \) for \( u \in X_j \setminus \{O\} \);

(iii) \( \dim X_j \to \infty \) as \( j \to \infty \).

(iv) If we denote by \( N_j \) the number of negative eigenvalues of the problem (1.2) then

\[ N_j \geq \dim X_j. \]

The proof of this theorem consists of several steps and presented in the remaining part of this section. In Sect. 3.1 and 3.2 we prove a convergence of \((\Psi(X, Y; t_j), \xi(X; t_j), \Lambda_j)\) to a Stokes extreme wave. The negative spectrum of the Frechet derivative corresponding to the extreme waves is studied in [23] and using this result we complete the proof of Theorem 3.1 in Sect. 3.3.

3.1. Some estimates for solutions to (1.4)

Let

\[ \omega_1 = \max_{0 \leq p \leq 1} (|\omega(p)| + |\omega'(p)|). \]

The following proposition is a local version of Proposition 3 from [21].
Proposition 3.2. Let $\delta > 0$ be given as well as a disc $B$ of radius $\rho > 0$ and let $C_B = \sup_{(X,\tilde{\xi}(X)) \in B} |\tilde{\xi}''(X)|$. Let also $\inf_B \Psi_Y \geq \delta$. Then there exists constants $\bar{\gamma} \in (0, 1]$ and $C > 0$, depending only on $R$, $\delta$, $\omega_1$, $\rho$ and $C_B$ such that any solution $(\Psi, \xi) \in C^1(D_\xi \cap B) \times C^{0,1}(\mathbb{R})$ of (1.1) satisfies

$$||\Psi|| \leq C, \ ||\xi|| \leq C,$$

where $\frac{1}{2}B$ is the disc with the same center and radius $\frac{1}{2}\rho$ and $I$ is the projection of $\frac{1}{2}B$ on $x$-axis.

We recall that the limit functions $(\xi_*, \Psi_*, \Lambda_*)$ satisfy

$$\Delta \Psi_* + \omega(\Psi_*) = 0 \text{ in } D_{\xi_*},$$
$$\frac{1}{2} |\nabla \Psi_*|^2 + \xi_* = R \text{ on } B_{\xi_*},$$
$$\Psi_*(X, 0) = 0, \ \Psi_*(X, \xi_*(X)) = 1$$

in a weak sense. Due to the assumptions (see Sect. 1.6)

$$\lim_{X \to 0^+} \frac{\xi_*(X) - R}{X} = -\frac{1}{\sqrt{3}}$$

and by Proposition 3.4 $\Psi_*> 0$ in $D_{\xi_*} \setminus \{(0, R)\}$. Using Proposition 3.5, we get

$$\xi_* \in C^{2,\bar{\gamma}}, \ \Psi_* \in C^{2,\bar{\gamma}} \text{ outside stagnation points.}$$

The following theorem is proved in [23].

Theorem 3.3. Assume that $\Psi_* > 0$ for $(X, Y) \in D_{\xi_*}$ and $Y \neq R$ and $\xi_*$ satisfies (3.5). Then

$$\xi_*(X) = -\frac{1}{\sqrt{3}} + \frac{2}{3} \omega(1) \sqrt{X} + a_1 \omega^2(1) X + f(X)$$

for some explicit $a_1 > 0$, where $f = O(X^{\frac{3}{2}(\tau_1-1)})$, $f_X = O(X^{\frac{3}{2}(\tau_1-1)-1})$ as $X \to 0^+$ and $\tau_1 \approx 1.8$ is the smallest root of $\tau_1 = -\frac{1}{\sqrt{3}} \cot(\frac{\pi}{2} \tau_1)$.

Using (1.28), we derive from the Bernoulli equation

$$\Psi_{jY}(X, \xi_j(X))^2 \geq \frac{2}{B_j^2} (R - \xi_j(X)), \ B_j = \frac{A_j}{\Lambda_j}.$$
Proposition 3.4. Let $\delta > 0$ be a small number and $X_* = \delta$. Then there exist positive constants $c_k$, $k = 1, \ldots, 5$, and the index $j(\delta)$ such that $|X - X_*| \leq c_1 \delta$ and $j \geq j(\delta)$ implies

$$|R - \xi_j(X)| \geq c_2 \delta$$

and

$$\Psi_j Y(X, Y) \geq c_4 \delta \text{ for } |Y - \xi_j(X)| \leq c_5 \delta. \quad (3.10)$$

Here the constants $c_k$, $k = 1, \ldots, 5$, are independent of $\delta$ and $j$.

Proof. Since $\xi_j \to \xi_*$ in $C^0_{p(x)}(\mathbb{R})$ as $j \to \infty$, for every $\varepsilon > 0$ there exists $j(\varepsilon)$ such that

$$|\xi_j(X) - \xi_*(X)| + \left| \frac{\xi_j(X) - \xi_*(X')}{|X - X'|^\alpha} - \frac{\xi_*(X) - \xi_*(X')}{|X - X'|^\alpha} \right| \leq \varepsilon \quad (3.11)$$

for $j \geq j(\varepsilon)$ and $0 \leq X', X \leq 2\delta$. We choose $\alpha = \frac{3}{2}(\tau_1 - 1) - 1$. Applying the asymptotic formula (3.7), we get

$$|\xi_*(X) - \xi_*(X') + c_*(X - X')| \leq c_6 \delta \alpha |X - X'|, \quad c_* = \frac{1}{\sqrt{3}}. \quad (3.12)$$

Now using (3.11) and (3.12), we obtain

$$R - \xi_j(X) = \xi_*(0) - \xi_*(X) + \xi_*(X) - \xi_j(X) \geq c_6 \delta - \varepsilon - c_6 \delta^{1 + \alpha}. \quad (3.13)$$

By choosing $\varepsilon = c_6 \delta/2$ and $c_1$ and $c_6$ sufficiently small we arrive at (3.9).

Let us turn to proving (3.10). We shall omit the index $j$ in the proof of proposition. Introduce function $u = e^{-s Y} \Psi Y$. Then

$$\Delta u + 2s \partial_Y u = -(s^2 + \omega'(\Psi))u \text{ and } \Psi Y > 0 \text{ in } D_\xi.$$ 

Choosing $s$ to be non-negative and $s^2 \geq -\min_{p \in [0,1]} \omega'(p)$, we get $\Delta u + 2s \partial_Y u \leq 0$. We will compare the function $u$ with the barrier function

$$U(X, Y) = \cos(s_*(X - X_*)) \sinh(s_*(Y - a)).$$ 

Then

$$\Delta U + 2s \partial_Y U = 2s \cos(s_*(X - X_*)) \cosh(s_*(Y - a)).$$

Furthermore, $U = 0$ for $X - X_* = \pm X_*$, where $s_* X_* = \pi/2$, and $U = 0$ for $Y = a$. Consider the function

$$V = u - \alpha U \text{ with } \alpha > 0.$$ 

Then
$$\Delta V + 2s \partial_Y V \leq 0 \text{ for } |X - X_*| \leq X_* \text{ and for } a \leq Y \leq \xi(X)$$

Furthermore, $V > 0$ for $X = \pm X_*$ and for $Y = a$. From (3.8) and (3.9) it follows that

$$\Psi_Y (X, \xi(X)) \geq c_7 \delta \text{ for } |X - X_*| \leq c_1 \delta.$$ 

Choosing $\alpha$ to satisfy

$$e^{-s_*\xi(X)}c_7 \delta \geq \alpha \cosh(s_*(\xi(X) - a)) \text{ for } |X - X_*| \leq c_1 \delta$$

we arrive at (3.10). \( \square \)

Combination of Propositions 3.2 and 3.4 gives the following

**Proposition 3.5.** Let the assumptions of Proposition 3.4 be satisfied and $\delta = |X_*|$. Then the inequality (3.3) is valid with $B = B_{c\delta}(X_*, \xi_j(X_*))$, where $c$ is independent of $j$ and $\delta$.

**Corollary 3.6.** Theorem 3.3 together with the Theorem 3.5 implies that $\xi_* \in C^{2, \gamma'}(I_c)$ for certain $\gamma' \in (0, 1)$, where $I_c$ is any closed interval which does not contain stagnation points inside.

3.2. **On convergence of the branch of Stokes waves (1.23)**

Due to Proposition 3.5 and the assumption (3.5), we conclude that

$$||\Psi_j||_{C^{2, \gamma'}(D_{\xi_j} \cap \frac{1}{2}B)} \leq C, \quad ||\xi_j||_{C^{2, \gamma'}(I)} \leq C, \quad j = 1, \ldots, (3.14)$$

for every disc $B = B_{c\delta}(X, \xi_j(X))$, where $c$ is independent of $j$ and $\delta$ and $C$ depending only on $\delta$, $R$ and $\omega_1$.

For $\delta > 0$ we introduce

$$\Omega^\delta_{\xi_j} = \{(X, Y) : X \in [-\Lambda_j/2, \Lambda_j/2], Y \in [0, \min(\xi_j(X), R - c\delta)]\}$$

and

$$I^\delta_j = \{X : X \in [-\Lambda_j/2, -\delta] \cup [\delta, \Lambda_j/2]\}. (3.15)$$

In the next proposition we present an important additional information on convergence of the sequence $(\Psi_j, \xi_j, \Lambda_j)$.

**Proposition 3.7.** Let $\delta$ be a positive number. Then

$$||\Psi_j||_{C^{2, \gamma'}(\Omega^\delta_{\xi_j})} \leq C \text{ and } ||\xi_j||_{C^{2, \gamma'}(I^\delta_j)} \leq C, (3.16)$$

where the constant $C$ depends on $\delta$, $R$, $\omega_1$ and $B_*$. 

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Proof. Let $B$ be a disc with the center located at a distance $\geq \delta$ from the stagnation point $(0, R)$ and with a radius $c\delta$ with a certain constant $c$ independent of $j$ and $\delta$. Then using the local estimates (3.3) and verifying conditions of Proposition 3.2 by using Proposition 3.4 we obtain estimates (3.3) for functions $\Psi_j$ and $\xi_j$ with constant $C$ depending on $\delta$, $R$, $\omega_1$ and $B_\ast$. These local estimates leads to the estimate (3.16).

By choosing a subsequence of the indexes $\{j\}$ (and numerating it by the same indexes) we can assume that for every $\delta > 0$

$$\xi_j \to \xi_\ast \text{ in } C^{2,\alpha}(\mathring{I}_\ast^{\delta})$$

(3.17)

and

$$\rho_j \to \rho_\ast \text{ in } C^{0,\alpha}(\mathring{I}_\ast^{\delta}) \text{ as } j \to \infty,$$

(3.18)

where $I_\ast^{\delta}$ is defined by (3.15) where $\Lambda_j$ is replaced by $\Lambda_\ast$—the period of $\xi_\ast$. Furthermore $\alpha \in (0, \overline{\gamma})$, $\rho_j(X) = \rho(X; t_j)$ and $\rho = \rho(x; t)$ is defined by (2.20) with $\lambda = 1$, i.e.

$$\rho = \rho(X; t) = \frac{(1 + \Psi_X \Psi_{XY} + \Psi_Y \Psi_{YY})}{\Psi_Y (\Psi_X^2 + \Psi_Y^2)^{1/2}} \bigg|_{Y = \xi(X; t)}.$$

(3.19)

Remark 3.8. As it was shown in [23] the following formula is valid for $\rho_\ast$:

$$\rho_\ast(X) = r^{-1} \frac{1}{\sqrt{3}} + O(r^{-1+\varepsilon}), \text{ as } r \to 0,$$

(3.20)

where $r^2 = (R - X)^2 + Y^2$.

3.3. Proof of Theorem 3.1 on negative eigenvalues

Proof. Let $a_j(u, w) = a_{\xi_j}(u, w)$.

Let also $(r, \theta)$ be the polar coordinates with center at $(0, R)$, i.e. $r = \sqrt{(R - X)^2 + Y^2}$ and $\theta$ is the angle measured from the ray $(X, R)$, $X > 0$. We introduce $\widehat{\xi}(X) = R - |R - X|/\sqrt{3}$. We will compare the negative eigenvalues of the problem (1.2) with the negative spectrum of the problem

$$-\Delta u = -\tau^2 u \text{ in } D_{\widehat{\xi}}$$

$$\partial u - \frac{\sqrt{3}}{2} u = 0 \text{ for } Y = \widehat{\xi}(X)$$

(3.21)

considered for even functions and supplied with a condition

$$u = \sin \left( \kappa \log \left( \frac{1}{2} r \right) + \gamma \cosh(\kappa \theta) + O(r^\epsilon) \right) \text{ near zero},$$

where $\gamma \in (0, \pi]$ is a fixed number and $\epsilon$ is a small positive number. Here $\kappa$ is the positive root of
\[ \kappa \tanh \left( \frac{\kappa \pi}{3} \right) = \frac{\sqrt{3}}{2} \]  

(3.22)

Let \( K_{ik}(z) \) be Bessel’s function of imaginary order, see [12]. According to [23] the spectral problem (3.21) has infinitely many negative eigenvalues \( \lambda = -\tau_k^2 \),

\[ \tau_k = 2e^{(\gamma_k + \gamma)/\kappa} e^{k\pi/\kappa}, \quad j \in \mathbb{Z}. \]

Corresponding eigenfunction is

\[ u_k(X, Y) = K_{ik}(\tau_k r) \cos(\kappa \theta). \]

We recall that

\[ K_{ik}(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} \left( 1 + O \left( \frac{1}{z} \right) \right), \]

for large \( z \) and

\[ K_{ik}(z) = -\left( \frac{\pi}{\kappa \sinh(\pi \kappa)} \right)^{1/2} \sin \left( \kappa \ln \left( \frac{1}{2z} \right) - \gamma_k \right) + O(z^2), \]

(3.23)

for small \( z \). Here \( \gamma_k \) is a real constant defined by

\[ \Gamma(1 + i\kappa) = \left( \frac{\pi \kappa}{\sinh(\pi \kappa)} \right)^{1/2} e^{i\gamma_k}. \]

(3.24)

We need the following asymptotic formula for small roots of \( K_{ik}(z) = 0 \):

\[ z_j = 2e^{-j\pi/\kappa} (1 + O(e^{-2(j\pi/\kappa)})) \text{ as } j \to \infty. \]

(3.25)

We note also that the function \( K_{ik}(z) \) is bounded and

\[ \int_0^\infty |K_{ik}(\tau r)|^2 r dr = c\tau^{-2}. \]

For small \( \varepsilon > 0 \) we define the function

\[ u_{k,\varepsilon}(x, y) = K_{ik}(\tau_k r) \cos(\kappa \theta) \text{ for } r > z_n/\tau_k, \]

where \( z_n \) is the smallest root of \( K_{ik}(z) = 0 \) such that \( z_n \tau_k^{-1} > \varepsilon \). If \( r < z_n/\tau_j \) then \( u_{k,\varepsilon}(X, Y) = 0 \). We choose small parameters

\[ \varepsilon_1 < \varepsilon < \sigma < \delta. \]

According to (3.17) and (3.18) for every \( \varepsilon_1 > 0 \) we have

\[ |\xi_j(X) - \xi_\ast(X)| \leq \varepsilon_1 \text{ for } X \in (\varepsilon, \delta) \]

(3.26)
and

\[ |\rho_j(X) - \rho_\ast(X)| \leq \varepsilon_1 \quad \text{for } X \in (\varepsilon, \delta) \]  

(3.27)

for \( j \geq j(\varepsilon_1) \), where \( j(\varepsilon_1) \) is sufficiently large.

We introduce also a smooth cut-off function \( \zeta = \zeta(r), \zeta(r) = 1 \) for \( r \leq 1 \) and \( \zeta(r) = 0 \) for \( r \geq 2 \) and put

\[ \zeta_\delta(r) = \zeta(r/\delta). \]

To verify that there are many negative eigenvalues of the problem (2.21), we will use the finite dimensional space \( \mathcal{X} \) of test functions

\[ w(X, Y) = \zeta_\delta(r) \sum a_k u_{k, \varepsilon}(X, Y). \]  

(3.28)

We assume in what follows that \( k \) is chosen to satisfy

\[ \sigma^{-1} \leq \tau_k \leq \sigma \varepsilon^{-1}, \]  

(3.29)

where \( \sigma \) is a small number. We use the norm

\[ ||w||_{\mathcal{X}} := |a| = \left( \sum |a_k|^2 \right)^{1/2} \]

in the space \( \mathcal{X} \). Here \( a \) denotes the vector of coefficient \( a_k \).

One can verify that \( N := \text{dim} \mathcal{X} \approx \log \varepsilon^{-1} \).

We represent the form (3.1) as

\[ a_j(w, w) = a(w, w) + b_j(w, w) \]

where

\[ a(w, w) = \int_{\Omega_{\xi_\ast}} |\nabla w|^2 dX dY - \int_{-\Lambda_0/2}^{\Lambda_0/2} \rho_a |w|^2 \sqrt{1 + (\eta'_j)^2} dX \]  

(3.30)

and

\[ b_j(w, w) = \int_{\Omega_{\eta_j} \setminus \Omega_{\xi_\ast}} |\nabla w|^2 dX dY - \int_{D_{\xi_j}} \omega'(\Psi_j) |w|^2 dX dY \]

\[ + \int_{-\Lambda/2}^{\Lambda/2} (\rho_a |w(X, \xi_\ast(X))|^2 \sqrt{1 + \xi'_\ast(x)^2} - \rho_j |w(X, \xi_j(X))|^2 \sqrt{1 + \eta'_j(x)^2}) dX \]

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Using (3.26) and (3.27), we get

\[ |b_j(w, w)| \leq cN|\alpha|^2 \left( \sigma^2 + \frac{\varepsilon_1}{\varepsilon} + \varepsilon_1 \log \frac{\delta}{\varepsilon} \right), \]

where $|\alpha|$ is the euclidian norm of the vector $\alpha$ whose components are the coefficients in (3.28). Next, we represent the form $a$ as

\[ a(w, w) = \hat{a}(w, w) + \hat{b}(w, w), \]

where

\[ \hat{a}(w, w) = \int_{\Omega_{\xi}} |\nabla w|^2 dX dY - \int_{-\Lambda_0/2}^{\Lambda_0/2} \hat{\rho}|w|^2 \sqrt{1 + (\hat{\xi}')^2} dX \]

(3.31)

and

\[ \hat{b}(w, w) = \int_{\Omega_{\xi} \setminus \Omega_{\hat{\xi}}} |\nabla w|^2 dX dY \]

\[ + \int_{-\Lambda_0/2}^{\Lambda_0/2} (\hat{\rho}|w(X, \hat{\xi}(X)|^2 \sqrt{1 + \hat{\xi}'(X)^2} - \rho_*|w(X, \xi_*(X)|^2 \sqrt{1 + \xi_*(X)^2}) dX \]

(3.32)

where, according to [23],

\[ \hat{\rho} = \left( \frac{\sqrt{3}}{2} + O(X^{1/2}) \right) r^{-1}. \]

This implies, in particular (see also [23])

\[ \rho_* - \hat{\rho} = O(X^{-1/2}). \]

Therefore

\[ |\hat{b}(w, w)| \leq C|\alpha|^2 N \sigma^{-1}. \]

Next, we introduce

\[ W_{\varepsilon}(X, Y) = \sum a_k u_{k\varepsilon}(X, Y) \]

(3.33)

and put

\[ \tilde{a}(W_{\varepsilon}, W_{\varepsilon}) = \int_{\Omega_{\hat{\xi}}} |\nabla W_{\varepsilon}|^2 dX dY - \int_{-\Lambda_0/2}^{\Lambda_0/2} \hat{\rho}|W_{\varepsilon}|^2 \sqrt{1 + (\hat{\xi}')^2} dX. \]
Then
\[ \tilde{a}(w, w) = \tilde{a}(W_\varepsilon, W_\varepsilon) + \tilde{b}(w, W_\varepsilon), \]
where
\[ \tilde{b}(w, W_\varepsilon) = \tilde{a}(w, w) - \tilde{a}(W_\varepsilon, W_\varepsilon). \]
Then
\[ |\tilde{b}(w, W_\varepsilon)| \leq C |a|^2 N \left( e^{-\delta/\sigma} + \sigma \right) \]
Furthermore
\[ a(W_\varepsilon, W_\varepsilon) = -c_0 |a|^2 \left( 1 + O(N\sigma^2) \right), \quad c_0 = \int_0^\infty K_{ik}(z)^2 z dz. \]
Choosing \( \varepsilon_1 \) sufficiently small and
\[ \sigma = \varepsilon^{1/4}, \quad \delta = \varepsilon^{1/8} \quad \text{and} \quad N = \log \varepsilon^{-1/2}, \]
we arrive at
\[ a_j(w, w) \geq -c_0 |a|^2/2 \quad (3.34) \]
if \( \varepsilon \) is sufficiently small. This proves the required result. \( \square \)

**Remark 3.9.** We note that the test function \( w \), used for estimating of negative spectrum, has support in a \( 2\sigma \) neighborhood of the crest \( (0, R) \) of the extreme Stokes wave and so it is zero below the trough.

**Corollary 3.10.** Due to Proposition 2.6 and Corollaries 2.2, 2.3 all spectral problems (2.21), (2.23), (2.27) and (2.28) have the same number of negative eigenvalues. Moreover, one can see directly that they have the same multiplicity of zero eigenvalue.

Our study of bifurcations will be based on the equation (2.9), where \( S \) is given by (2.8). Using Proposition 2.5, we conclude that
\[ S(t) : C^{2,\gamma}_{pe}(\mathbb{R}) \cap \mathcal{U} \to C^{1,\gamma}_{pe}(\mathbb{R}), \quad (3.35) \]
where \( \mathcal{U} \) is a neighborhood of origin in \( C^{2,\gamma}_{pe}(\mathbb{R}) \). This branch consists of Fredholm potential operators of index zero analytically depending on the parameter \( t \in [0, \infty) \) (we write here and in what follows “analytic” having in mind it has a real analytic re-parametrization near every point \( t \geq 0 \)). The operator (3.35) has potential
\[ G(q; h + w, \lambda) = f(h + w(g); \lambda) - f(h; \lambda), \quad (3.36) \]
where \( w = w(g) \) is the solution of the problem (2.7). Consider the equation (2.9):

\[
S(g; t) = 0.
\]  

(3.37)

Important role in analysis of bifurcations is played by the Frechet derivative, which is given by the operator (see (2.12)):

\[
Sg = (Nw - w)|_{p=1},
\]

where \( w \) solves the problem (2.11) and \( N \) is defined by (2.4). Clearly

\[
S(t) : C^{2,y}_{pe}(\mathbb{R}) \rightarrow C^{1,y}_{pe}(\mathbb{R}).
\]  

(3.38)

As is known every point \( t \) where the operator \( S(t) \) has non-trivial kernel is isolated. The following result directly follows from Theorem 3.1 and Corollary 3.10

**Corollary 3.11.** There are infinitely many points \( \{T_j\}_{j=1}^{\infty} \) such that \( T_j \rightarrow \infty \), as \( j \rightarrow \infty \), the kernel of \( S(T_j) \) is nontrivial and the crossing number of the family \( S(t) \) through 0 at \( T_j \) is non-zero.

4. **Subharmonic bifurcations**

Unfortunately Corollary 3.11 does not guarantee that the bifurcation points \( T_j \) generates subharmonic bifurcations. They can generate bifurcations which are described by Stokes waves of the same period. In this section we formulate an equation for finding subharmonic bifurcation and present some important properties for the Frechet derivative of operators involved in the equation.

4.1. **Equation for subharmonic bifurcations**

For \( M = 2M_1 + 1 \), \( M_1 = 1, 2, \ldots \), and \( \alpha \in (0, 1] \) let us introduce the subspaces \( C^{k,\alpha}_{Me}(\overline{Q}) \) and \( C^{k,\alpha}_{Me}(\mathbb{R}) \) of \( C^{k,\alpha}(\overline{Q}) \) and \( C^{k,\alpha}(\mathbb{R}) \) respectively consisting of even functions of period \( M\Lambda_0 \). A similar space for the domain \( D_\xi \) we denote by \( C^{2,\alpha}_{Me}(\overline{D_\xi}) \).

Equation (2.9) can be considered also on functions of period \( M\Lambda_0 \). Having this in mind we write as before \( h + w \) instead of \( h \) but now we assume that \( w \in C^{2,y}_{Me}(\overline{Q}). \) Now the operator \( \mathcal{F}(w; h, t), \mathcal{G}(w; h, t), A(t) \) and \( N(t) \) from Sect. 2.1 are considered on functions from \( C^{2,y}_{Me}(\overline{Q}) \).

To indicate this difference we will use the notations \( \mathcal{F}_M(w; h, t), \mathcal{G}_M(w; h, t), A_M(t) \) and \( N_M(t) \) for corresponding operators. Moreover we can define analogs of operators \( S \) and \( S \) and denote them by \( S_M \) and \( S_M \) correspondently. New operators are acting on functions from \( C^{1,y}_{Me}(\mathbb{R}) \). The equation for subharmonic bifurcations is

\[
S_M(g; t) = 0.
\]  

(4.1)

where

\[
S_M(g; t) : C^{2,y}_{Me}(\mathbb{R}) \rightarrow C^{1,y}_{Me}(\mathbb{R}).
\]  

(4.2)
The operator $S_M$ is also potential and for the definition of the potential the integration in (1.22) must be taken over the interval $(-M\Lambda_0/2, M\Lambda_0/2)$. The corresponding eigenvalue problem for the Frechet derivative $S_M(t)$ is

$$S_M(t)g = -\mu g, \quad (4.3)$$

where $S_M$ is defined by the same formulas as $S$ but now the functions $g$ and $w$ are $M\Lambda_0$-periodic. Clearly,

$$S_M(t) : C^{2,\gamma}_{M_e}(\mathbb{R}) \to C^{1,\gamma}_{M_e}(\mathbb{R}). \quad (4.4)$$

### 4.2. Estimates of negative eigenvalues of $S_M(t)$

Let

$$\Omega_{M,t} = \{(X,Y) : X \in (-M\Lambda/2, M\Lambda/2), 0 < Y < \xi(X)\}$$

We introduce the bilinear form

$$a_{M,t}(w, v) = \int_{\Omega_{M,t}} (\nabla w \nabla \bar{v} - \omega'(\Psi)w\bar{v})dYdX$$

$$- \int_{-M\Lambda/2}^{M\Lambda/2} \rho(X; t)w(X, \xi(X; t))\bar{v}(X, \xi(X; t))\sqrt{1 + \xi'\xi}dX, \quad (4.5)$$

where the functions $\xi = \xi(X; t)$, $\Psi = \Psi(X, Y; t)$, $\Lambda = \Lambda(t)$ and $\rho = \rho(X; t)$ depends on $t$. The form (4.5) defined on functions from $H^1_{M_e}(\Omega_{M,t})$, where the index $M_e$ means that only even functions are taken from the Sobolev space $H^1(\Omega_{M,t})$ which have the same values for $X = \pm M\Lambda/2$. The form $a_{M,t}$ corresponds to the self-adjoint operator in the spectral problem

$$(\partial^2_X + \partial^2_Y)w + \omega'(\Psi)w + \theta \frac{1}{\Psi_y}w = 0 \text{ in } D_\xi,$$

$$v_x w_x + v_y w_y - \rho w = 0 \text{ on } B_\xi,$$

$$w = 0 \text{ for } X = 0,$$

defined on $M\Lambda$-periodic, even functions. We decompose the space $H^1_{M_e}(\Omega_{M,t})$ in the orthogonal sum

$$H^1_{M_e}(\Omega_{M,t}) = H^1_e(\Omega_{M,t}) + X_M, \quad (4.6)$$

where $H^1_e(\Omega_{M,t})$ consists of restrictions of functions from $H^1_{pe}(D_\xi)$ onto $\Omega_{M,t}$. The restriction of the form $a_{M,t}$ to the subspace $H^1_e(\Omega_{M,t})$ corresponds to the spectral problem (2.21) define on even $\Lambda$ periodic functions and so this is invariant subspace for the operator defined on $H^1_{M_e}(\Omega_{M,t})$. Therefore the space $X$ is orthogonal to the subspace $H^1_e(\Omega_{M,t})$ with respect to the inner product.
\[ b_{M,t}(w, v) = \int_{\Omega_{M,t}} w \frac{1}{\Psi_Y} dX dY \]

and with respect to the form \( a_{M,t} \). So the space \( \mathcal{X}_M \) is invariant for the operator \( S_M(t) \) and we denote the restriction operator \( S_M(t) \) on \( \mathcal{X}_M \) by \( \hat{S}_M(t) \).

Similar to Proposition 2.6 and Corollary 2.7 one can show that

**Proposition 4.1.** The number of negative eigenvalues of the operator \( S_M(t) \) and of the operator corresponding to the form \( a_{M,t} \) defined on \( H^1_{me}(\Omega_{M,t}) \) is the same (accounting their multiplicities).

**Proposition 4.2.** Let \( \hat{S}_M(t) \) be the operator defined above on \( \mathcal{X}_M \). Then

(i) the number of negative eigenvalues of the operator \( \hat{S}_M(t) \) is estimated from above by \( C_\ast(t)M \), where \( C_\ast(t) \) depends on \( t \) but it is independent of \( M \);

(ii) for every \( N > 0 \) there exist the index \( j(N) \) such that the number of negative eigenvalue of the operator \( \hat{S}_M(t_{j(N)}) \) is estimated from below by \( MN \). Moreover \( j(N) \to \infty \) as \( N \to \infty \).

**Proof.** (i) Consider the problem (for \( M = 1 \))

\[
(\partial^2_X + \partial^2_Y)w + \omega'(\Psi) w + \theta \frac{1}{\Psi_Y} w = 0 \quad \text{in} \ \Omega_t, \\
\nu_X w_X + \nu_Y w_Y - \rho w = 0 \quad \text{for} \ Y = \xi(X), \ X \in (-\Lambda/2, \Lambda/2),
\\
w = 0 \quad \text{for} \ X = 0,
\\
w_X = 0 \quad \text{for} \ X \pm \Lambda/2 \text{ and } 0 < Y < \xi(\pm \Lambda/2),
\]  

(4.7)

where \( (\Psi, \xi, \Lambda) \) depends on \( t \). Denote by \( C_\ast = C_\ast(t) \) the number of negative eigenvalue of this operator. Then the number of negative eigenvalues of the operator \( \hat{S}_M(t) \) is estimated from above by \( C_\ast M \) which proves (i).

(ii) Represent \( M \) as \( 2M_1 + 1 \) and introduce domains

\[ \Omega_{j,k} = \{ (X, Y) : X \in (\Lambda j/2 + (k - 1) \Lambda j, \Lambda j/2 + k \Lambda j), \ 0 < Y < \xi_j(X) \} \ k = 1, \ldots, M_1, \]

and

\[ \Omega_{j,-k} = \{ (X, Y) : X \in (-\Lambda j/2 + (-k + 1) \Lambda j, -\Lambda j/2 - k \Lambda j), \ 0 < Y < \xi_j(X) \} \ k = 1, \ldots, M_1. \]

Let \( w \) be the same function as in the proof of Theorem 3.1. Consider the functions

\[ w_k(X, Y) = w(X - (2k - 1) \Lambda j, Y) + w(x + (2k - 1) \Lambda j, Y) - (w(x - 2k \Lambda j, Y) - w(x + 2k \Lambda j, Y), \]

where \( k = 1, 3, 5, \ldots, M_2, \) and \( 2M_2 \leq M_1 \). Then \( w_k \in \mathcal{X}_M \) (here we can refer to Remark 3.9 concerning smoothness of this function) and \( w_k \) are pairwise orthogonal. Due to the calculation performed in the proof of Theorem 3.1, we get

\[ a_{M,t_j} \left( \sum_k \alpha_k w_k, \sum_k \alpha_k w_k \right) < 0 \]
provided not all coefficients $\alpha_k$ are zero. This implies that the number of negative eigenvalues is estimated by

$$N_j L_2,$$

where $N_j$ the number of coefficient in the definition of the function $w$, which tends to infinity when $j \to \infty$. This implies (ii). □

5. Main theorem

5.1. Bifurcation theorem for potential operators

We will use the following bifurcation theorem for potential operator, see [17], [18]. Let $X$ and $Z$ be real Banach spaces and let $X$ be continuously embedded into $Z$. We assume that $Z$ is supplied with an inner product which is continuous with respect to the norm in $Z$. Consider a map

$$F : U \times V \to Z,$$

where $U$ is a neighborhood of 0 in $X$ and $V$ is a neighborhood of $t_n$ in $\mathbb{R}$. We assume that

(i) $F \in C(U \times V, Z)$, $D_x F \in C(U \times V, L(X, Z))$ and $A(t) := D_x F(0, t)$ is a family of Fredholm operators of index zero such that the operator $A(t)$ considered as unbounded operator in $Z$ with domain of definition $X$ is closed for each $t \in V$.

(ii) $F(\cdot, t)$ is a potential operator from $U$ into $Z$.

(iii) $F(0, t) = 0$ for $s \in V$.

We note that (i) implies that the operator $A(t)$ is self-adjoint. We recall the definition of the crossing number. Let 0 be an isolated eigenvalue of $A(t_n)$ and let $A(t)$ be invertible $t \in (t_n - \epsilon, t_n) \cup (t_n + \epsilon)$ for a small positive $\epsilon$. Denote by $\sigma_-(t)$ ($\sigma_+(t)$) the sum of multiplicities of perturbed eigenvalues of $A(t)$ near zero on the negative (positive) real axis. Then the limit

$$\chi(A(t), t_n) := \lim_{\epsilon \to 0} \left(\sigma_-(-\epsilon) - \sigma_+(\epsilon)\right)$$

exists and it is called the crossing number of the family $A(t)$ through 0 at $t_n$.

**Theorem 5.1.** ([17], Theorem II.7.3) Let the family (5.1) satisfy (i)–(iii). Let 0 be an isolated eigenvalue of $A(t_n)$. If $A(t)$ is invertible for $V \setminus \{0\}$ and the crossing number of the family $A(t)$ at $t_n$ is nonzero then $(0, t_n)$ is a bifurcation point of $F(x, t) = 0$ in the following sense: $(0, t_n)$ is a cluster point of nontrivial solution $(x, t) \in U \times V, x \neq 0$ of $F(x, t) = 0$.

5.2. Properties of the operator $S_M(t)$

Our aim is to apply Theorem 5.1 to the problem (4.1), where we have the same differential operator and the same boundary conditions as in (3.37) but solutions have period $M \Lambda_0$. So we want to describe bifurcation points for equation (4.1), the kernel of the operator $S_M(t)$ when it is non-zero and study when the crossing number is non-trivial.
The first step in the above program is to describe bounded solutions to the problem

\[ A(t)u = 0 \text{ in } Q, \]
\[ N(t)u - u = 0 \text{ for } p = 1, \]
\[ u = 0 \text{ for } p = 0. \]

Usually this can be done by using Floquet exponents to the problem

\[ A(\tau, t)w = 0 \text{ in } Q \]
\[ N(\tau, t)w - w = 0 \text{ for } p = 1 \]
\[ w = 0 \text{ for } p = 0, \]

where

\[ A(\tau, t)w = e^{-i\tau q} A(t)(e^{i\tau q} w) \]
\[ N(\tau, t)w = e^{-i\tau q} N(t)(e^{i\tau q} w) \]

Here \( A(t) \) and \( N(t) \) are given by (2.2) and (2.4) respectively. The spectral parameter \( \tau \in \mathbb{C} \) for which the problem has non-trivial \( \Lambda_0 \)-periodic solutions is called the Floquet exponent. Real \( \tau \) produces bounded solution and complex \( \tau \) corresponds to unbounded solutions at \( +\infty \) or \(-\infty\). As it is known (see [34], Sect. 3.3) the \( \tau \) spectrum of the problem (5.3) consists of isolated Floquet exponents having finite algebraic multiplicity. We note that in study of problem (5.3) we do not assume the corresponding eigenfunctions are even but certainly they have period \( \Lambda_0 \). Bounded solutions of (5.2) are given by

\[ u(q, p) = e^{i\tau q} w(q, p), \]

where \( \tau \) is a real Floquet exponent and \( w \) is a corresponding \( \Lambda_0 \)-periodic eigenfunction. If we are looking for \( M\Lambda_0 \) periodic solutions then \( \tau M\Lambda_0 = 2k\pi \), i.e.

\[ \tau = \frac{2k\pi}{M\Lambda_0} = \frac{k}{M} \tau^*. \]

Moreover, solutions, even in \( q \), are given by

\[ u_e(q, p) = e^{i\tau q} w(q, p) + e^{-i\tau q} w(-q, p). \]

The above analysis implies, in particular, that if \( M_1 \) and \( M_2 \) are two prime numbers then the intersection of the kernels of \( S_{M_1} \) and \( S_{M_2}(t) \) coincides with the kernel of the operator \( S(t) \).

Another problem which is useful in study the crossing number of zero eigenvalue of the operator (4.4) is the following
A(\tau,t)w = 0 \text{ in } Q, \\
N(\tau,t)w - w = \mu w \text{ for } p = 1, \\
w = 0 \text{ for } p = 0, \quad (5.6)

where \( t \geq 0 \) and \( \tau \) is real. Indeed, the study of crossing number is connected with eigenvalues of the operator \( S_M(t) \) and these eigenvalues and corresponding eigenfunctions can be found with the help of the problem (5.6).

5.3. Eigenvalues of (5.6) for \( t = 0 \)

For \( t = 0 \) we have \( h = H(p) \), \( \lambda = 1 \) and the spectral problem (5.6) takes the form

\[
\left( \frac{w_p}{H_p^3} \right) + (\partial_q + i\tau) \left( \frac{w_q + i\tau w}{H_p} \right) = 0 \text{ in } Q \\
\frac{w_p}{H_p^3} - w = \mu w \text{ for } p = 1 \\
w = 0 \text{ for } p = 0, \quad (5.7)
\]

where \( H \) is given by (1.20). To find solutions to this spectral problem we put

\[ w = \Gamma(q, H(p); \tau)H_p(p). \]

Then according to Corollary 2.3 \( \Gamma(x, y; \tau) \) satisfies

\[
((\partial_x + i\tau)^2 + \partial_y^2)\Gamma + \omega'(H)\Gamma = 0 \text{ for } 0 < y < d := H(1) \quad (5.8)
\]

and

\[ \Gamma(x, 0; \tau) = 0. \quad (5.9) \]

Solutions to (5.8), (5.9) with the period \( \Lambda_0 \) are given by

\[ \Gamma_n = e^{in\tau_s y} \gamma(y; \tau + n\tau_s), \quad n \text{ is integer } \]

where \( \gamma \) is the solution to (1.10). The boundary condition at \( p = 1 \) in (5.7) takes the form

\[
\frac{\gamma'(d, \tau + n\tau_s)}{H_p(1)} + \frac{H_{pp}(1)}{H_p^3(1)} - H_p(1) = H_p(1)\mu \\
\]

or

\[ \mu = \kappa^2 \gamma'(d; \tau + n\tau_s) - 1 + \kappa \omega(1), \]

where the relation \( H_p(1) = \kappa^{-1} \) is applied. Using the relation (1.11), we get
\[ \mu = \mu_n = \kappa \sigma (\tau + n \tau_n), \quad n = 0, \pm 1, \pm 2, \ldots \]  
(5.10)

Since \( \sigma = \sigma (\tau) \) is an even function strongly increasing for \( \tau > 0 \), we obtain

**Proposition 5.2.** For every \( \tau \in \mathbb{R} \) such that \( 2\tau \neq k\tau_n \), \( k \) is an integer, the eigenvalues \( \mu_n \) of the problem (5.7) are simple and given by (5.10). Moreover, \( \mu_n \neq 0 \) for \( \tau \in (0, \tau_n) \).

**Corollary 5.3.** For every \( \tau \in \mathbb{R} \) such that \( 2\tau \neq k\tau_n \), the problem (5.7) has only simple eigenvalues except isolated points on the axis \( t \geq 0 \).

### 5.4. Main result

Let \( \{ \tilde{t}_j \}, \ j = 1, \ldots, \infty \), be singular points of the operator \( S(t) \), i.e. the points where the kernel operator \( S(\tilde{t}_j) \) is non-trivial. As is known the points \( \tilde{t}_j \) are isolated (see [6]), \( \tilde{t}_j \to \infty \) as \( j \to \infty \) and the operator \( A(t) \) is invertible for \( t \neq \tilde{t}_j \). We will use the numeration of these points such that

\[ 0 \leq \tilde{t}_1 < \tilde{t}_2 < \ldots \]

**Theorem 5.4.** (i) There exists a sequence \( \{ \tilde{t}_j, M_j \} \), where \( \tilde{t}_j \neq \tilde{t}_k \) for all \( j \) and \( k \); \( M_j \) are prime numbers, and

\[ \tilde{t}_j \to \infty, \quad M_j \to \infty \quad \text{as} \quad j \to \infty. \]

Moreover, \( \tilde{t}_j \) is a bifurcation point of the equation

\[ S_{M_j}(g; t) = 0. \]  
(5.11)

(ii) There exists a sequence \( \{ \tilde{t}_j, M_j \} \), where \( \tilde{t}_j \neq \tilde{t}_k \) for all \( j \) and \( k \); \( M_j \) are prime numbers, the sequence \( \{ \tilde{t}_j \} \) is bounded and

\[ M_j \to \infty \quad \text{as} \quad j \to \infty. \]

Furthermore, \( \tilde{t}_j \) is a bifurcation point of the equation (5.11).

According to Sect. 4.1 the numbers \( \tilde{t}_j \) are pairwise different in both cases.

### 5.5. Proof

Let as before \( \tilde{t}_j, \ j = 1, \ldots, \infty \), be singular points for the operator function \( S(t) \). From Theorem 3.1 it follows that \( \tilde{t}_j \to \infty \) as \( j \to \infty \). By [34] (see Proposition 3.4), the \( \tau \) spectrum of the operators \( S(t, \tau) \) consists of isolated eigenvalues of finite algebraic multiplicities for every \( t \geq 0 \).

Let \( M \) be a prime number. Consider the problem (4.1) and the corresponding Frechet derivative \( S_M(t) \) of the operator \( S_M(t) \). We use the splitting (4.6) involving invariant subspaces of the self-adjoint operator \( S_M(t) \) and the restriction operator \( S_M(t) \) on \( X \), which was denoted by \( S_M(t) \). Introduce the singular points of the operator function \( \hat{S}_M(t) \) and denote them by \( \hat{t}_k, \ k = 1, 2, \ldots \). Let us show that \( \hat{t}_k \) are isolated points. Indeed the operator \( \hat{S}_M(t) \) is self-adjoint and its eigenvalues \( \mu_j(t) \) depend analytically on \( t \). If a certain \( \mu_j \) has infinitely many zeros in a
neighborhood of $\hat{t}_k$ then it is identically zero for all $t$ which contradicts to Proposition 5.2, which implies that $\mu_j(0) \neq 0$.

Let $t_\tau > 0$ and $t_\tau \neq \tilde{t}_j$, $j = 1, \ldots, \infty$. Denote by $\mathcal{N}(t_\tau)$ the set of all prime numbers $M$ such that the $\tau$-spectrum of the operators $S(\tilde{t}_j)$ for all $\tilde{t}_j \leq t_\tau$ does not contain the points $k \tau_n/M$, $k = 1, 2, \ldots, M - 1$. Then the kernel of the operator $\hat{S}_M(\tilde{t}_j)$ for $\tilde{t}_j \leq t_\tau$ is trivial. This means that

$$\hat{t}_k \neq \tilde{t}_j \text{ for } \tilde{t}_j \leq t_\tau, \hat{t}_k \leq t_\tau \text{ and } M \in \mathcal{N}(t_\tau). \quad (5.12)$$

(ii) We choose $t_\tau$ and $M \in \mathcal{N}(t_\tau)$. By Proposition 5.2 all eigenvalues of the operators $\hat{S}_M(t_\tau)$ are positive for small $t$. According to Proposition 4.2, the operator $\hat{S}_M(t_\tau)$ has negative eigenvalues for sufficiently large $t_\tau$ lying near $t_j$ from Theorem 3.1. By Proposition 4.2(ii) it can be chosen in such a way that all operators $\hat{S}_M(t_\tau)$ have negative eigenvalues with arbitrary $M$. This implies that for every $M \in \hat{S}_M(t_\tau)$ there exists $\hat{t}_k$ satisfying (5.12) with the crossing number different from zero. We denote this number by $k(M)$. Now reference to Theorem 5.1 proves existence of subharmonic bifurcations with period $M \in \mathcal{N}(t_\tau)$ at the bifurcation points $\hat{t}_k(M)$.

(i) Now let us choose $t_\tau^{(n)}$, $n = 1, 2, \ldots$, in the following way. First, they satisfy

$$t_\tau^{(n)} \neq \tilde{t}_j \text{ for all } n \text{ and } j, \quad 0 < t_\tau^{(1)} < t_\tau^{(2)} < \cdots < t_\tau^{(n)} \to \infty \text{ as } n \to \infty.$$

Second, we assume that the number of negative eigenvalues of the operators $\hat{S}_M(t_\tau^{(n)})$ is positive. Applying Proposition 4.2, we conclude that if $t_\tau^{(n-1)}$ is chosen then $t_\tau^{(n)}$ can be chosen to satisfy

$$N_-(\hat{S}_M(t_\tau^{(n-1)})) < N_-(\hat{S}_M(t_\tau^{(n)})) \quad \text{for all } M, \quad (5.13)$$

where $N_-$ denotes the number of negative eigenvalues.

Next we choose the prime numbers $M_n \in \mathcal{N}(t_\tau^{(n)})$. Due to (5.13) there exists $\hat{t}_k$, $k = k(M_n)$, satisfying

$$t_\tau^{(n-1)} < \hat{t}_k(M_n) < t_\tau^{(n)}$$

with the crossing number different from zero. The reference to Theorem 5.1 proves existence of subharmonic bifurcations with period $M_n \in \mathcal{N}(t_\tau^{(n)})$ at the bifurcation point $\hat{t}_k(M_n)$, $n = 1, 2, \ldots$.

Data availability

No data was used for the research described in the article.

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