S₃ Symmetry and Neutrino Masses and Mixings

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Abstract

Based on a universal seesaw mass matrix model with three scalars φᵢ, and by assuming an S₃ flavor symmetry for the Yukawa interactions, the lepton masses and mixings are investigated systematically. In order to understand the observed neutrino mixing, the charged leptons (e, µ, τ) are regarded as the 3 objects (e₁, e₂, e₃) of S₃, while the neutrino mass-eigenstates are regarded as the irreducible representation (ν₀, ν₀, ν₀) of S₃, where (ν₀, ν₀) and νᵣ are a doublet and a singlet, respectively, which are composed of the 3 objects (ν₁, ν₂, ν₃) of S₃.

1 Introduction

It is generally considered that masses and mixings of the quarks and leptons will obey a simple law of nature, so that we expect that we will find a beautiful relation among those values. However, even if there is such a simple relation in the quark sector, it is hard to see such a relation in the quark sector, because the relation will be spoiled by the gluon cloud. We may expect that such a beautiful relation will be found just in the lepton sector. Therefore, in the present paper, we will confine ourselves to the investigation of the lepton masses and mixings. Here, we would like to emphasize that we should search a model which gives a reasonable description of not only the masses, but also the mixings. Especially, we should direct our attention to the mixing pattern rather than to the mass spectrum in the neutrino sector.

It is also considered that the mass matrices of the fundamental particles will be governed by a symmetry. In the present paper, we take notice of a permutation symmetry S₃ [1]. Let us begin with giving a short review how useful a description based on the S₃ symmetry is in the lepton masses and mixings.

The observed neutrino data have strongly suggested that the neutrino mixing is approximately described by the so-called tribimaximal mixing [2]

\[
U_{TB} = \begin{pmatrix}
\frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}
\end{pmatrix}.
\]

The tribimaximal mixing is interpreted in the framework of S₃: We define the doublet (ψᵦ, ψᵢ) and singlet ψᵣ of the permutation symmetry S₃ as

\[
\begin{pmatrix}
\psiᵦ \\
ψᵣ \\
ψᵣ
\end{pmatrix} = \begin{pmatrix}
0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix} \begin{pmatrix}
ψ₁ \\
ψ₂ \\
ψ₃
\end{pmatrix},
\]

(1.2)
where \((\psi_1, \psi_2, \psi_3)\) are three objects of \(S_3\). When we assume that the mass-eigenstates in the charged lepton sector are \((e_1, e_2, e_3) = (\tau, \mu, e)\), while those in the neutrino sector are \((\nu_\pi, \nu_\eta, \nu_\sigma)\) with the mass hierarchy
\[
m_{\nu_\eta}^2 < m_{\nu_\sigma}^2 < m_{\nu_\pi}^2, \tag{1.3}
\]
the neutrino mixing matrix \(U_\nu\) of the basis \((\nu_\eta, \nu_\sigma, \nu_\pi)\) to the basis \((e_1, e_2, e_3) = (e, \mu, \tau)\) is given by the form \((1.1)\), because the basis \((\nu_\eta, \nu_\sigma, \nu_\pi)\) is given by
\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
\equiv
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\nu_\eta \\
\nu_\sigma \\
\nu_\pi
\end{pmatrix}. \tag{1.4}
\]
Here, the weak iso-doublets are given by \((\nu_i, e_i)_L\) \((i = 1, 2, 3)\) [and also \((\nu_a, e_a)_L\) \((a = \pi, \eta, \sigma)\)]. In other words, in order to obtain the tribimaximal mixing, we must build a model where the mass-eigenstates are \((e_1, e_2, e_3) = (e, \mu, \tau)\) and \((\nu_\pi, \nu_\eta, \nu_\sigma)\) with the mass hierarchy \((1.3)\).

On the other hand, it is well-known that the observed charged lepton mass spectrum \([3]\) satisfies the relation \([4, 5]\)
\[
m_e + m_\mu + m_\tau = \frac{2}{3} \left( \sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} \right)^2, \tag{1.5}
\]
with remarkable precision. The mass formula \((1.5)\) is invariant under any exchange \(\sqrt{m_i} \leftrightarrow \sqrt{m_j}\) \((i, j = e, \mu, \tau)\). This, too, suggests that a description by \(S_3\) may be useful for a mass matrix model.

As an explanation of the mass formula \((1.5)\), the author has proposed a model \([5, 6, 7]\) with 3 flavor scalars \(\phi_i\) in the framework of the universal seesaw model \([5]\): A fermion mass matrix \(M_f\) is given by
\[
M_f = m_f^L M_F^{-1} m_f^R, \tag{1.6}
\]
where \(M_F\) is a mass matrix of hypothetical heavy fermions \(F_i\) \((i = 1, 2, 3)\). For example, for the charged lepton sector, we assume
\[
m_L^{i} = \frac{1}{\kappa} m_R^{i} = y_e \text{diag}(v_1, v_2, v_3), \tag{1.7}
\]
(\(\kappa\) is a constant with \(\kappa \gg 1\)) and \(M_E \propto 1 \equiv \text{diag}(1, 1, 1)\), where \(v_i \equiv \langle \phi^0_{Li} \rangle = \langle \phi^0_{Ri} \rangle / \kappa\), and \(m_L^{e}\) (and also \(m_R^{e}\)) is defined by \(\bar{e}_L m_L^{e} E_R\)

If we assume that the vacuum expectation values (VEV) \(v_i\) satisfy the relation
\[
v_1^2 + v_2^2 + v_3^2 = \frac{2}{3} (v_1 + v_2 + v_3)^2, \tag{1.8}
\]
we can obtain the relation \((1.5)\). Of course, here, we have assumed that the Yukawa interaction in the charged lepton sector is given by an \(S_3\) invariant form
\[
H_e = y_e (\bar{\ell}_L \phi_{L1} E_{R1} + \bar{\ell}_L \phi_{L2} E_{R2} + \bar{\ell}_L \phi_{L3} E_{R3}), \tag{1.9}
\]
(and also a similar interaction for $\bar{\ell}_R \phi_R E_L$), where $\ell_{L/R} = (\nu_{L/R}, e_{L/R})$, and $\phi_{L/R} = (\phi^0_{L/R}, \phi^0_{L/R})$. The form (1.9) is not a general form under the $S_3$ symmetry. We have assumed the universality of the coupling constants in addition to the $S_3$ symmetry.

The relation among the VEVs $v_i$, (1.8), can read

$$v^2_\pi + v^2_\eta = v^2_\sigma,$$

(1.10)
in terms of $S_3$, because

$$v^2_1 + v^2_2 + v^2_3 = v^2_\pi + v^2_\eta + v^2_\sigma = 2v^2_\sigma = 2\left(\frac{v_1 + v_2 + v_3}{\sqrt{3}}\right)^2,$$

(1.11)

where $v_a = \langle \phi^0_a \rangle$ ($a = \pi, \eta, \sigma$), and $(\phi_\pi, \phi_\eta, \phi_\sigma)$ have been defined by Eq.(1.4). For a Higgs potential model based on an $S_3$ symmetry which leads to the relation (1.10), for example, see Ref. [9]. The $S_3$ symmetry is again related to the lepton masses and mixings.

Thus, it is likely that the $S_3$ symmetry (or a higher symmetry which include $S_3$) plays an essential role on a unified description of the lepton mass matrices. In the present paper, we will assume that, in the universal seesaw model with three scalars $\phi_i$, the Yukawa interactions are exactly invariant under the $S_3$ symmetry, and the $S_3$ symmetry is broken only by the VEVs $v_i$ of the three scalars $\phi_i$. For the seesaw mass matrix model (1.6), by inheriting the formulation in charged lepton sector, we assume as follows: (i) $M_F$ have a unit matrix structure, at least, for the charged lepton and neutrino sectors, i.e.

$$M_E \propto 1, \quad M_N \propto 1.$$  

(1.12)

(ii) $m^f_L$ (and $m^f_R$) have sector-dependent ($f$-dependent) structures. We still assume the diagonal form

$$m^c_L = y_e \text{diag}(v^d_1, v^d_2, v^d_3),$$  

(1.13)
in the charged lepton sector, but we consider that $m^c_L$ in the neutrino sector is not diagonal. Therefore, in the present model (1.6), the neutrino mixing is caused by the structure of $m^c_L$. We also assume that the VEVs $v^u_i$ satisfy the relation (1.8) as well as $v^d_i$ in the charged lepton sector. However, note that in spite of the assumption (1.8) for $v^u_i$, the eigenvalues of the matrix $m^u_L$, in general, do not satisfy a relation similar to Eq. (1.8). The purpose of the present paper is to investigate what structure of the Dirac mass matrix $m^u_L$ in the seesaw mass matrix model (1.6) is required in order to fit the model for the neutrino oscillation data.

By the way, the seesaw-type model (1.6) with 3 scalars $\phi_{Li}$ (and $\phi_{Ri}$) causes some trouble, for example, the flavor changing neutral currents (FCNC) problem, the spoiling of the asymptotic freedom of the SU(3) color, and so on. Therefore, instead of the Yukawa interaction (1.9), we may consider a Frogatt-Nielsen [10] type model with five dimensional operators $\bar{\ell}_L H_L \phi_L E_R$, where $H_L$ is the conventional SU(2)$_L$-doublet Higgs scalar $H_L = (H^+_L, H^0_L)$, and $\phi_i$ are 3-family SU(2)$_L$-singlet scalars:

$$H_{eff} = y_e \bar{\ell}_L H^d_L \phi^d_{Ld} E_R + y_\nu \bar{\ell}_L H^u_L \phi^u_{Lu} N_R,$$

(1.14)
where $\Lambda_f$ are scales of the effective theory. We consider $m_{W}^2 = \frac{1}{2} g_w^2 (\langle H_u^0 \rangle^2 + \langle H_d^0 \rangle^2)$ and $\langle \phi^f \rangle / \Lambda_f \sim 1$ ($f = u, d$). Since we interest only in the flavor structure, for convenience, hereafter, we will drop the Higgs scalars $H_f^L$ from Eq. (1.14) and we will call $\ell_L H_f^R \phi^a N_R$ the Yukawa interaction $\ell_L \phi^a N_R$ simply.

2 Mass eigenvalues

In general, an $S_3$ invariant Yukawa interaction with 3 scalars $\phi_a$ ($a = \pi, \eta, \sigma$) is given by

$$H = \left( y_0 \bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta + \bar{\psi}_\sigma \psi_\sigma + y_1 \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta - 2 \bar{\psi}_\sigma \psi_\sigma}{\sqrt{6}} \right) \phi_\sigma$$

$$+ y_2 \left( \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta}{\sqrt{2}} \phi_\pi + \frac{\bar{\psi}_\pi \psi_\pi - \bar{\psi}_\eta \psi_\eta}{\sqrt{2}} \phi_\eta \right)$$

$$+ y_3 \frac{\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta}{\sqrt{2}} \phi_\pi + y_4 \frac{\bar{\psi}_\sigma \phi_\pi + \bar{\psi}_\sigma \phi_\eta}{\sqrt{2}},$$

(2.1)

where we read $\bar{\psi} = \ell_L \equiv (\nu_L, e_L)$, $\psi = E_R$ and $\phi_a = \phi^d_a$ for the charged lepton sector, $\bar{\psi} = \bar{\ell}_L$, $\psi = N_R$ (or $\nu_R$) and $\phi_a = \phi^u_a$ for the neutrino sector, and we have dropped $H_f^L / \Lambda_f$ for convenience. For example, the interaction (1.9) in the charged lepton sector corresponds to the case

$$y_0 = y_e, \quad y_1 = 0, \quad y_2 = \frac{1}{\sqrt{3}} y_e, \quad y_3 = y_4 = \frac{\sqrt{2}}{3} y_e.$$

(2.2)

The Yukawa interaction (2.1) gives the mass matrix $m_f^L$ for the basis $(\psi_\pi, \psi_\eta, \psi_\sigma)$,

$$m_f^L = \begin{pmatrix}
\frac{2y_0}{\sqrt{3}} v_\pi + \frac{2y_1}{\sqrt{6}} v_\eta & \frac{2y_2}{\sqrt{2}} v_\pi & \frac{2y_3}{\sqrt{2}} v_\pi \\
\frac{2y_3}{\sqrt{2}} v_\pi & \frac{2y_0}{\sqrt{3}} v_\eta - \frac{2y_2}{\sqrt{2}} v_\eta & \frac{2y_3}{\sqrt{2}} v_\eta \\
y_4 \frac{2y_1}{\sqrt{6}} v_\pi & y_4 \frac{2y_1}{\sqrt{6}} v_\pi & \left( \frac{2y_0}{\sqrt{3}} - 2 \frac{y_1}{\sqrt{6}} \right) v_\sigma
\end{pmatrix}.$$  

(2.3)

Hereafter, for simplicity, we confine ourselves to investigating a case with a symmetric mass matrix form $(m_f^L)^T = m_f^L$, i.e. with $y_3 = y_4$. Then, we have still 5 parameters, $y_0 v_\sigma, y_1 v_\sigma, y_2 v_\pi, y_3 v_\pi$ and $v_\pi / v_\eta$, in the model, so that the model has no predictability. In the present paper, we do not impose a further symmetry on the model. Alternatively, we will investigate what constraints on the mass matrix parameters (or specific relations among those) are required from the phenomenological studies.

Now let us return to the subject on the neutrino Dirac mass matrix $m_{\nu}^L$ which is, in general, given by the form (2.3) on the basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$. (Hereafter, for convenience, we will denote $y_{\nu}^a$ as $y_i$ simply.) As we discussed in the previous section, the present neutrino oscillation data favor to the tribimaximal mixing, so that the neutrino states are approximately in the mass eigenstates $(\nu_\eta, \nu_\sigma, \nu_\pi)$ with $m_\eta^2 < m_\sigma^2 < m_\pi^2$. Therefore, for convenience, we investigate a case in the limit of
\[ y_3 = 0. \] (Since the observed neutrino mixing is not the exact tribimaximal mixing, the condition \( y_3 = 0 \) is only an approximate requirement for convenience.) The mass matrix with \( y_3 = 0 \) is diagonalized by a rotation

\[
R(\theta_{\pi\eta}) = \begin{pmatrix}
    c_{\pi\eta} & s_{\pi\eta} & 0 \\
    -s_{\pi\eta} & c_{\pi\eta} & 0 \\
    0 & 0 & 1
\end{pmatrix},
\]

(2.4)

where \( c_{\pi\eta} = \cos \theta_{\pi\eta} \) and \( s_{\pi\eta} = \sin \theta_{\pi\eta} \), and

\[
\tan 2\theta_{\pi\eta} = -\frac{v_\pi}{v_\eta},
\]

(2.5)
as

\[
R^T(\theta_{\pi\eta})m^\nu_L R(\theta_{\pi\eta}) = \text{diag}(m_\pi, m_\eta, m_\sigma).
\]

(2.6)

The mass eigenvalues \( m_\pi \), \( m_\eta \) and \( m_\sigma \) are given by

\[
\begin{align*}
    m_\pi &= \left( \frac{3y_0}{\sqrt{3}} + \frac{y_1}{\sqrt{6}} \right) \left( v_\pi \pm \frac{|y_2|}{\sqrt{2}} \sqrt{v_\pi^2 + v_\eta^2} \right), \\
    m_\eta &= \left( \frac{3y_0}{\sqrt{3}} + \frac{y_1}{\sqrt{6}} \right) \left( v_\pi \mp \frac{|y_2|}{\sqrt{2}} \sqrt{v_\pi^2 + v_\eta^2} \right), \\
    m_\sigma &= \left( \frac{3y_0}{\sqrt{3}} - 2 \frac{y_1}{\sqrt{6}} \right) v_\sigma,
\end{align*}
\]

(2.7)

where we have defined

\[
\sqrt{2}y_0 + y_1 > 0,
\]

(2.8)
and the upper and lower signs in \( \pm |y_2| \) (and also \( \mp |y_2| \)) correspond to the cases \( y_2 v_\eta > 0 \) and \( y_2 v_\eta < 0 \), respectively.

In the previous section, we have assumed that the VEVs \( v_i^d \) of the scalars \( \phi_i^d \), which couple to the charged leptons, satisfy the relation (1.10). Therefore, we also assume that the VEVs \( v_i^u \) of the scalar \( \phi_i^u \), which couple to the neutrino sector, satisfy the relation

\[
(v_\pi^u)^2 + (v_\eta^u)^2 = (v_\sigma^u)^2 \equiv \frac{1}{2} v_u^2,
\]

(2.9)
where we do not always consider \( \langle \phi_i^u \rangle = \langle \phi_i^d \rangle \). Then, the mass eigenvalues (2.7) lead to

\[
\begin{align*}
    m_\pi &= \left( \frac{1}{\sqrt{6}} y_0 + \frac{1}{2\sqrt{3}} y_1 \pm \frac{1}{2} |y_2| \right) v_u, \\
    m_\eta &= \left( \frac{1}{\sqrt{6}} y_0 + \frac{1}{2\sqrt{3}} y_1 \mp \frac{1}{2} |y_2| \right) v_u, \\
    m_\sigma &= \left( \frac{1}{\sqrt{6}} y_0 - \frac{1}{\sqrt{3}} y_1 \right) v_u.
\end{align*}
\]

(2.10)

Note that the mass spectrum is independent of the parameters \( v_\pi^u/v_\sigma^u \) and \( v_\eta^u/v_\sigma^u \), and only depends on the parameters \( y_1/y_0 \) and \( |y_2|/y_0 \). On the other hand, as seen in Eq.(2.5), the mixing angle \( \theta_{\pi\eta} \) is independent of the parameters \( y_i \) and only depends on the parameter \( v_\pi^u/v_\eta^u \).
As we discussed in Sec. 1, the observed tribimaximal mixing suggests that the neutrino mass eigenstates are \((\nu_\eta, \nu_\sigma, \nu_\pi)\). If the mass hierarchy is a normal type, it demands \(m_\eta^2 < m_\sigma^2 \ll m_\pi^2\), and if it is an inverse type, it demands \(m_\pi^2 \ll m_\eta^2 < m_\sigma^2\). The conditions for \(m_\pi^2 < m_\eta^2 < m_\sigma^2\) and \(m_\pi^2 < m_\sigma^2 < m_\eta^2\) are given in Appendix.

By the way, we have still two adjustable parameters \(y_1/y_0\) and \(y_2/y_0\) to predict the neutrino mass spectrum. In the following sections, we will investigate two typical cases by putting assumptions for the coupling constants \(y_0, y_1\) and \(y_2\). Of course, the assumptions must also be applicable to the charged lepton coupling constants (2.2).

3 Case with \(y_2^2 = y_1^2 + y_3^2\)

In the mass matrix (2.3), the \(y_1\)- and \(y_2\)-terms are traceless, while the trace of the \(y_0\)-term is not zero. This suggests that the \(y_0\)-term may be distinguished from the other terms under a higher symmetry. Therefore, by way of trial, we put the following normalization condition for the coupling constants

\[
y_0^2 = y_1^2 + y_2^2 + y_3^2, \tag{3.1}
\]

which is satisfied by the coupling constants (2.2) in the charged lepton sector. Since we have assumed that \(y_3 = 0\) in the neutrino sector, we can explicitly write the condition (3.1) as

\[
y_1 = y_0 \sin \alpha, \quad y_2 = y_0 \cos \alpha. \tag{3.2}
\]

Then, we can rewrite Eqs.(2.10) as

\[
m_\pi = \left[ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \alpha \pm \frac{2}{3} \pi \right) \right] y_0 v_u,
\]

\[
m_\eta = \left[ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \left( \alpha \pm \frac{2}{3} \pi \right) \right] y_0 v_u, \tag{3.3}
\]

\[
m_\sigma = \left[ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \alpha \right] y_0 v_u,
\]

where \(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\) \((\cos \alpha > 0)\), and, for \(\frac{\pi}{2} \leq \alpha < \frac{3}{2} \pi\), we substitute \(\pi - \alpha\) for \(\alpha\) in Eq.(3.3).

Note that the case with the condition (3.1) which leads to Eq.(3.3) gives the relation

\[
m_\pi^2 + m_\eta^2 + m_\sigma^2 = \frac{2}{3} (m_\pi + m_\eta + m_\sigma)^2. \tag{3.4}
\]

Since these masses \((m_\pi, m_\eta, m_\sigma)\) are Dirac masses in the neutrino seesaw mass matrix \(M_\nu = m_\nu^T M_N^{-1} (m_\nu^L)^T\), if we take the heavy Majorana mass matrix \(M_N\) with the unit matrix form, we obtain the neutrino masses which are proportional to \(m_\pi^2, m_\eta^2\) and \(m_\sigma^2\), respectively. Therefore, the neutrino masses will satisfy a relation similar to the charged lepton mass relation (1.1).

The differences among \(m_\pi^2, m_\eta^2\) and \(m_\sigma^2\) are given as follows:

\[
m_\pi^2 - m_\eta^2 = \pm \frac{1}{\sqrt{3}} \cos \alpha (\sqrt{2} + \sin \alpha) y_0^2 v_u^2, \tag{3.5}
\]

\[
m_\pi^2 - m_\sigma^2 = \pm \frac{1}{\sqrt{3}} \cos \left( \alpha \pm \frac{\pi}{3} \right) \left( \sqrt{2} - \sin \left( \alpha \pm \frac{\pi}{3} \right) \right) y_0^2 v_u^2, \tag{3.6}
\]
\[ m_\eta^2 - m_\sigma^2 = \pm \frac{1}{\sqrt{3}} \cos \left( \alpha \pm \frac{\pi}{3} \right) \left[ \sqrt{2} - \sin \left( \alpha \pm \frac{\pi}{3} \right) \right] y_0 v_u^2, \] (3.7)

where \(|\alpha| < \pi/2\). For a case with a normal hierarchy, we should read the upper signs in Eqs.(3.5)-(3.7), so that we obtain

\[ m_\eta^2 < m_\sigma^2 < m_\pi^2 \quad \text{for} \quad \frac{\pi}{6} < \alpha < \frac{\pi}{6}. \] (3.8)

For a case with an inverse hierarchy, since we should read the lower signs in Eqs.(3.5)-(3.7), we obtain

\[ m_\pi^2 < m_\sigma^2 < m_\eta^2 \quad \text{for} \quad \frac{-\pi}{2} < \alpha < -\frac{\pi}{6}. \] (3.9)

Next, let us seek for the numerical value of \( \alpha \) which gives the ratio of the observed values \( \Delta m_{solar}^2 = (7.9^{+0.6}_{-0.5}) \times 10^{-5} \text{ eV}^2 \) \cite{11} to \( \Delta m_{atm}^2 = (2.74^{+0.44}_{-0.26}) \times 10^{-3} \text{ eV}^2 \) \cite{12},

\[ R_{obs} = \frac{\Delta m_{solar}^2}{\Delta m_{atm}^2} = (2.9 \pm 0.5) \times 10^{-2}. \] (3.10)

The predicted ratio \( R(\alpha) \) is given by

\[ R(\alpha) \equiv \frac{m_\eta^4(\alpha) - m_\sigma^4(\alpha)}{m_\pi^4(\alpha) - m_\eta^4(\alpha)}, \] (3.11)

for a normal hierarchy \( m_\eta^2 < m_\sigma^2 \ll m_\pi^2 \). From \( R(\alpha) = R_{obs} \), we find

\[ \alpha = (3.0^{+1.2}_{-1.4})^{\circ}, \] (3.12)

where the sign \( \mp \) corresponds to the sign \( \pm \) of the experimental error in Eq.(3.10). Similarly, we seek for the case with an inverse hierarchy, but, we find that there is no solution with an inverse hierarchy.

The solution \( \alpha = (3.0^{+1.2}_{-1.4})^{\circ} \) gives

\[ m_\eta = -(0.076^{+0.006}_{-0.008}) y_0 v_u, \]
\[ m_\sigma = (0.38 \pm 0.01) y_0 v_u, \]
\[ m_\pi = (0.923 \mp 0.006) y_0 v_u. \] (3.13)

The result \( m_\eta < 0 \) leads to the change of the sign \( \sqrt{m_\nu_1} \to -\sqrt{m_\nu_1} \) in a relation similar to Eq.(1.5):

\[ m_\nu_1 + m_\nu_2 + m_\nu_3 = \frac{2}{3} (-\sqrt{m_\nu_1} + \sqrt{m_\nu_2} + \sqrt{m_\nu_3})^2. \] (3.14)

The relation (3.14) for the neutrino masses has recently speculated by Brannen \cite{13} based on an algebraic method (however, the algebraic method is highly mathematical, and the physical meaning of the method is somewhat not clear in the “masses and mixings”).
The values (3.13) predicts the following neutrino masses

\begin{align*}
m_{\nu1} &= (3.5 \pm 0.5) \times 10^{-4} \text{ eV}, \\
m_{\nu2} &= (8.7 \pm 0.2) \times 10^{-3} \text{ eV}, \\
m_{\nu3} &= (5.23^{+0.25}_{-0.30}) \times 10^{-2} \text{ eV},
\end{align*}

from the input value \( m_{\nu3} = \sqrt{\Delta m_{\text{atm}}^2} \).

Generally, the masses \( m_{f_i} \) which satisfy the relation (1.5) [or (3.14)] are expressed by a bilinear form

\[ m_{f_i} = (z_{fi})^2 m_{f0}, \]

where the sector-dependent parameters \( z_{fi} \) are normalized as \((z_{f1})^2 + (z_{f2})^2 + (z_{f3})^2 = 1\). Then, the parameters \( z_{fi} \) can always be expressed by the form

\begin{align*}
z_{f1} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin \xi_f, \\
z_{f2} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin(\xi_f + \frac{2}{3} \pi), \\
z_{f3} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \sin(\xi_f + \frac{4}{3} \pi),
\end{align*}

where we have taken \( z_{f1}^2 < z_{f2}^2 < z_{f3}^2 \). From the observed charged lepton mass values [3], we obtain the numerical value of \( \xi_e \)

\[ \xi_e = \frac{\pi}{4} - \varepsilon = 42.7324^\circ \quad (\varepsilon = 2.2676^\circ). \]

Note that, in the limit of \( \varepsilon \to 0 \), the electron mass becomes zero. We consider that the parameter \( \varepsilon \) is a fundamental parameter which governs the charged lepton mass spectrum.

Comparing the expression (3.3) (with the upper signs) with the expression (3.17), we find that the parameter \( \alpha \) is connected to \( \xi_\nu \) by the relation

\[ \alpha = \frac{\pi}{3} - \xi_\nu. \]

Therefore, we obtain

\[ \xi_\nu - \xi_e = \left( \frac{\pi}{3} - \alpha \right) - \left( \frac{\pi}{4} - \varepsilon \right) = \frac{\pi}{12} + \varepsilon - \alpha. \]

Since the value of \( \alpha \), (3.12), which is a solution of \( R(\alpha) = R_{\text{obs}} \), is very close to the value \( \varepsilon = 2.27^\circ \), (3.18), from the observed charged lepton masses, we can regard \( \alpha \) as \( \alpha = \varepsilon \). Then, we obtain a phenomenological relation

\[ \xi_\nu = \xi_e + \frac{\pi}{12}. \]

The relation (3.21) has also been speculated by Brannen [13], but the reason is still controversial. (Of course, in the present model, there is no theoretical reason for \( \alpha = \varepsilon \).)

\section*{4 Case with \( y_0^2 + y_1^2 = y_2^2 \)}
In the previous section, we have assumed a constraint (3.1) on the Yukawa coupling constants \( y_0, y_1 \) and \( y_2 \). However, the theoretical basis of the constraint is not clear. In the present section, instead of the constraint (3.1), we assume another constraint

\[
y_0^2 + y_1^2 = y_2^2 + y_3^2, \tag{4.1}
\]

which is again satisfied by the Yukawa coupling constants (2.2) in the charged lepton sector. The condition (4.1) means a requirement of the universality of the coupling constants in an extended meaning: the coupling constants of \( \bar{\psi}_a \psi_a \) \((a = \pi, \eta, \sigma)\) to the scalars are normalized with the equal weights for the scalars \( \phi_\sigma \) and \( \phi_\pi \) \((\phi_\eta)\).

In the neutrino sector, since we have assumed \( y_3 = 0 \), we can denote the condition (4.1) as

\[
y_0 = y_2 \cos \beta, \quad y_1 = y_2 \sin \beta. \tag{4.2}
\]

Then, the mass eigenvalues (2.10) are expressed as follows:

\[
m_\pi = \frac{1}{2} \lfloor \sin(\beta + \phi_0) \pm 1 \rfloor |y_2| v_u,
m_\eta = \frac{1}{2} \lfloor \sin(\beta + \phi_0) \mp 1 \rfloor |y_2| v_u, \tag{4.3}
m_\sigma = \frac{1}{\sqrt{2}} \cos(\beta + \phi_0) y_2 v_u,
\]

where

\[
\sin \phi_0 = \sqrt{\frac{2}{3}}, \quad \cos \phi_0 = \frac{1}{\sqrt{3}}, \quad (\phi_0 = 54.74^\circ), \tag{4.4}
\]

we have again taken the condition (2.8), i.e.

\[
\sin(\beta + \phi_0) > 0 \quad (-\phi_0 < \beta < \pi - \phi_0), \tag{4.5}
\]

and the upper and lower signs in Eq.(4.3) correspond to the cases \( y_2 v_\eta > 0 \) (a normal hierarchy case) and \( y_2 v_\eta < 0 \) (an inverse hierarchy case), respectively.

From the expression (4.3), we find

\[
m_\pi^2 + m_\eta^2 + m_\sigma^2 = y_2^2 v_\eta^2, \tag{4.6}
\]

\[
m_\eta + m_\sigma + m_\pi = \sqrt{\frac{3}{2}} y_2 v_u \cos \beta. \tag{4.7}
\]

Therefore, we obtain

\[
\frac{2}{3}(m_\pi + m_\eta + m_\sigma)^2 = \frac{2}{3} m_\pi^2 + m_\eta^2 + m_\sigma^2 = \cos^2 \beta = 1 - \sin^2 \beta. \tag{4.8}
\]

Thus, the parameter \( \beta \) in the present model denotes a deviation from the mass formula (3.14) \((3.4)\).

Note that if we find a solution \( \beta = \beta_1 \) which gives \( R(\beta) = R_{\text{obs}} \) \([R(\beta) \text{ is given by Eq.(3.11) with } \alpha \to \beta, \text{ and } R_{\text{obs}} \text{ is given by Eq.(3.10)}]\), the value \( \beta_2 = 2\phi_0 - \beta_1 \) \([\phi_0 \text{ is defined by Eq.(4.4)}\]
is also a solution of \( R(\beta) = R_{\text{obs}} \). From the expression (4.3), it is obvious that the solutions \( \beta_1 \) and \( \beta_2 \) give the same values for \( m_\pi \) and \( m_\eta \), but they give the values with the opposite signs to each other for \( m_\sigma \). We list those solutions of \( R(\beta) = R_{\text{obs}} \) in Table 1, together with the values of \( m_\eta \), \( m_\sigma \) and \( m_\pi \).

In Table 1, we also list the predicted values of the neutrino masses \( m_\nu_1 = m_\eta^2/M_N \), \( m_\nu_2 = m_\sigma^2/M_N \) and \( m_\nu_3 = m_\pi^2/M_N \) (\( M_N \) is a Majorana mass \( M_N \equiv M_{N1} = M_{N2} = M_{N3} \) of the heavy neutrinos \( N_i \)). Here, as the input value, we have used \( m_\nu_3 = \sqrt{\Delta m_{\text{atm}}^2} = 0.0523 \text{ eV} \) for the normal hierarchy case, and \( m_\nu_2 = \sqrt{\Delta m_{\text{atm}}^2} = 0.0523 \text{ eV} \) for the inverse hierarchy case. At present, the numerical values of \( m_\nu_i \) should not be taken rigidly. Therefore, we have omitted the error values from Table 1.

5 Neutrino mixing matrix

As we discussed in Sec.2, the additional rotation \( R(\theta_{\pi\eta}) \) from the tribimaximal mixing, (2.4), depends only on the value \( v_\pi/\nu_\eta \), and it is independent of the values of \( y_0 \), \( y_1 \) and \( y_2 \). In order to see the effects of the additional rotation \( R(\theta_{\pi\eta}) \), we change from the basis \( (\nu_\pi, \nu_\eta, \nu_\sigma) \) defined by Eq.(1.4) into the basis \( (\nu_\eta, \nu_\sigma, \nu_\pi) \) given by

\[
\begin{pmatrix}
\nu_\eta \\
\nu_\sigma \\
\nu_\pi
\end{pmatrix} = U_{TB}^T \begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix},
\]

where \( U_{TB} \) is the tribimaximal mixing matrix defined by Eq.(1.1). If \( v_\pi/\nu_\eta \neq 0 \), i.e. \( R(\theta_{\pi\eta}) \neq 1 \), the neutrino mixing matrix \( U_\nu \) is given by

\[
U_\nu = U_{TB} \begin{pmatrix}
c_{\pi\eta} & 0 & s_{\pi\eta} \\
0 & 1 & 0 \\
-s_{\pi\eta} & 0 & c_{\pi\eta}
\end{pmatrix} = \begin{pmatrix}
-\frac{2}{\sqrt{6}}c_{\pi\eta} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}s_{\pi\eta} \\
\frac{1}{\sqrt{6}}c_{\pi\eta} + \frac{1}{\sqrt{2}}s_{\pi\eta} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}s_{\pi\eta} - \frac{1}{\sqrt{2}}c_{\pi\eta} \\
\frac{1}{\sqrt{6}}c_{\pi\eta} - \frac{1}{\sqrt{2}}s_{\pi\eta} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}s_{\pi\eta} + \frac{1}{\sqrt{2}}c_{\pi\eta}
\end{pmatrix},
\]

where \( s_{\pi\eta} = \sin \theta_{\pi\eta} \) and \( c_{\pi\eta} = \cos \theta_{\pi\eta} \), i.e.

\[
\tan^2 \theta_{\text{solar}} = \frac{1}{2c_{\pi\eta}^2},
\]

\[
\sin^2 2\theta_{\text{atm}} = \left(1 - \frac{4}{3}s_{\pi\eta}^2\right)^2,
\]

\[
(U_\nu)_{13}^2 = \frac{2}{3}s_{\pi\eta}^2.
\]

For convenience, we define the following \( z_i \)-parameters

\[
\langle \phi_i^u \rangle = z_i^u v_u, \quad \langle \phi_i^d \rangle = z_i^d v_d,
\]
with the normalizations \( \sum_i (z_i^u)^2 = \sum_i (z_i^d)^2 = 1 \). Here, note that in Eq. (3.10), we have already defined the \( z_f \)-parameters similar to the present \( z_i^u \) and \( z_i^d \)-parameters. In the charged lepton sector, since \( \sqrt{m_{ei}} \propto v_i^d \), the relation \( z_{ei} = z_i^d \) holds. However, in the neutrino sector, since \( m_L^\nu \) is not diagonal, the values \( z_{\nu i} \) are not identical with \( z_i^u \).

For the \( z_i^d \)-parameters, from the relation (1.5), we obtain

\[
\frac{z_1^d}{\sqrt{m_e}} = \frac{z_2^d}{\sqrt{m_\mu}} = \frac{z_3^d}{\sqrt{m_\tau}} = \frac{1}{\sqrt{m_e + m_\mu + m_\tau}}, \tag{5.7}
\]

i.e.

\[
z_1^d = 0.016473, \quad z_2^d = 0.23678, \quad z_3^d = 0.97140. \tag{5.8}
\]

If we assume \( z_i^u = z_i^d \), we obtain \( z_1^u = 0.51939, \quad z_2^u = 0.47982 \) and \( z_3^u = 1/\sqrt{2} \) from the definition (1.4). Then, the rotation angle \( \theta_{\pi\eta} = -(1/2) \tan^{-1}(v_2^u/v_1^u) = -23.63^\circ \) is too large to explain the observed neutrino mixings [see Eqs.(5.3)-(5.5)], so that the case \( z_i^u = z_i^d \) is ruled out.

By the way, it is well known that the so-called \( 2 \leftrightarrow 3 \) symmetry [14] is promising for neutrino mass matrix description. Therefore, the simplest assumption is to require the \( 2 \leftrightarrow 3 \) symmetry for the VEV values \( \langle \phi_i^u \rangle \), i.e. \( v_2^u = v_3^u \), which leads to

\[
v_1^u = 0. \tag{5.9}
\]

The case gives \( \theta_{\pi\eta} = 0 \) from Eq. (2.5), so that the neutrino mixing is exactly given by the tribimaximal mixing (1.1). Note that if we required the \( 2 \leftrightarrow 3 \) symmetry for the fields \( \ell_L = (\nu_L, e_L) \), the symmetry would affect the charged lepton sector, too. Here, we have required the \( 2 \leftrightarrow 3 \) symmetry only for \( \langle \phi_i^u \rangle \), not for \( \langle \phi_i^d \rangle \), so that the symmetry does not affect the charged lepton mass matrix.

Of course, the \( 2 \leftrightarrow 3 \) symmetry is a phenomenological requirement, and the constraint may be broken. From the observed constraint [15] \( (U_\nu)^2 < 0.03 \), we obtain the constraint \( |\theta_{\pi\eta}| < 12.2^\circ \), i.e.

\[
\left| \frac{v_1^u}{v_1^u} \right| < 0.46. \tag{5.10}
\]

6 Concluding remarks

In conclusion, based on a universal seesaw mass matrix model (1.6) with three scalars \( \phi_i \), and by assuming an \( S_3 \) flavor symmetry for Yukawa interactions, we have investigated the neutrino masses and mixings. For the VEV values of \( \phi_i^f \) \((f = u, d)\), stimulated from a Higgs potential model [9] for \( \phi_i \), we have assumed the constraint

\[
\langle \phi_i^f \rangle^2 + \langle \phi_i^d \rangle^2 = \langle \phi_i^\sigma \rangle^2, \tag{6.1}
\]

where \( (\phi_{\pi}, \phi_{\eta}, \phi_{\sigma}) \) are defined by Eq.(1.4). However, since we have 4 independent Yukawa coupling constants \( y_0, y_1, y_2 \) and \( y_3 \), which are defined by Eq.(2.1), the model does not have predictability. Therefore, in the present paper, suggested by the observed neutrino mixing (the
tribimaximal mixing), we have investigated only a simple case with $y_3 = 0$ where only the $\nu_\pi - \nu_\eta$ mixing is caused. (Since the observed neutrino mixing is not the exact tribimaximal mixing, the condition $y_3 = 0$ is only an approximate requirement for convenience.) In the case with $y_3 = 0$ together with the assumption (6.1), our conclusion is as follows: the mass eigenvalues depend only on the values of the coupling constants $y_0$, $y_1$ and $y_2$, while the $\nu_\pi - \nu_\eta$ mixing angle $\theta_{\pi\eta}$ depends only on the value of $\langle \phi^u_\pi \rangle/\langle \phi^u_\eta \rangle$. Therefore, we can discuss the topic of the neutrino mass spectrum independently from that of the deviation from the tribimaximal mixing.

For the neutrino mass spectrum, from the economical point of view of the parameter number, we have investigated two typical cases with the constraints $y_0^2 = y_1^2 + y_2^2$ and $y_0^2 + y_1^2 = y_2^2$. The former case leads to a case which satisfies Brannen's relation (3.14) for the neutrino masses. Although the relation (3.14) is very interesting, the theoretical basis of the constraint $y_0^2 = y_1^2 + y_2^2$ is not clear. On the other hand, the later case is likely from the viewpoint of the universality of the coupling constants. The later case does not satisfy the relation (3.14). Only for a small value of the parameter $\beta$, the deviation from the relation (3.14) can become negligibly small. For example, for the solution $\beta = 2.94^\circ$ given in Table 1, the deviation from the relation (3.14) is very small, $\sin^2 \beta = 0.003$, as seen in Eq.(4.8), so that the relation (3.14) is approximately satisfied.

The neutrino mixing matrix $U_\nu$ can become the tribimaximal mixing (1.1) in the limit of $v_u^2 = v_u^3$ as given in Eq.(5.2). If we require the $2 \leftrightarrow 3$ symmetry for $v_i^u$ (not for $v_i^d$), we can obtain the tribimaximal mixing (1.1) without affecting the charged lepton mass spectrum. Of course, the $2 \leftrightarrow 3$ symmetry is a phenomenological requirement, and the constraint may be broken. From the observed constraint [15] $(U_\nu)^2 < 0.03$, we obtain the constraint $|v_\pi/v_\eta| < 0.46$.

The numerical predictions of $m_{\nu i}$ shown in Eq.(3.15) and in Table 1 were obtained by adjusting the parameter $\alpha$ ($\beta$) for the observed ratio $\Delta m^2_{\text{solar}}/\Delta m^2_{\text{atm}}$. Finally, we would like to give a speculation of neutrino masses by assuming a simple Yukawa interaction form [16] (and without using the observed value of $\Delta m^2_{\text{solar}}/\Delta m^2_{\text{atm}}$):

$$H_\nu = y_\nu \left( \frac{\ell_\pi N_\pi + \ell_\eta N_\eta + \ell_\sigma N_\sigma}{\sqrt{3}} \phi^u_\sigma + \frac{\ell_\pi N_\eta + \ell_\eta N_\pi}{\sqrt{2}} \phi^u_\pi + \frac{\ell_\pi N_\pi - \ell_\eta N_\eta}{\sqrt{2}} \phi^u_\eta \right). \quad (6.2)$$

Here, in the charged lepton sector, we have assumed the universality of the coupling constants on the basis ($e_1, e_2, e_3$), while, in the neutrino sector, we have assumed the universality of those on the $S_3$ irreducible basis ($\nu_\pi, \nu_\eta, \nu_\sigma$). Then, the neutrino masses are predicted as

$$m_{\nu 1} = \left( \frac{1}{\sqrt{6}} - \frac{1}{2} \right)^2 m^0_\nu,$$

$$m_{\nu 2} = \frac{1}{6} m^0_\nu,$$

$$m_{\nu 3} = \left( \frac{1}{\sqrt{6}} + \frac{1}{2} \right)^2 m^0_\nu,$$

without an adjustable parameter. The case predicts

$$R = \frac{\Delta m^2_{31}}{\Delta m^2_{32}} = \frac{4\sqrt{6} - 9}{4\sqrt{6} + 9} = 0.0425. \quad (6.4)$$
The value (6.4) is somewhat large comparing with the observed value (3.10), but, at present, the case is not ruled out within three sigma. Again, regarding $m_{\nu 3}$ as $m_{\nu 3} = \sqrt{\Delta m_{\text{atm}}^2}$, we predict the explicit neutrino mass values as follows:

\begin{align*}
  m_{\nu 1} &= (5.3^{+0.4}_{-0.3}) \times 10^{-4} \text{ eV}, \\
  m_{\nu 2} &= (1.05^{+0.07}_{-0.05}) \times 10^{-2} \text{ eV}, \\
  m_{\nu 3} &= (5.22^{+0.35}_{-0.25}) \times 10^{-2} \text{ eV}. \\
\end{align*}

The case (6.2) is also interesting because of the simpleness of its structure. The predictions (6.4) and (6.5) should be taken as results in an ideal limit.

In conclusion, the present model (a lepton mass matrix model with a bilinear form) based on the $S_3$ symmetry has given many interesting features for the mass spectra and mixings. However, the model still includes some adjustable parameters. Further investigation based on another symmetry which gives stronger constraints on the parameters than those in the $S_3$ symmetry will be desired.

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Appendix

In order to see whether the mass hierarchy is $m_\pi^2 < m_\sigma^2 < m_\eta^2$ or $m_\pi^2 < m_\eta^2 < m_\sigma^2$, we estimate the differences among those masses as follows:

\begin{align*}
  m_\pi^2 - m_\eta^2 &= \pm \frac{1}{\sqrt{3}} |y_2| (\sqrt{2} y_0 + y_1) v_u^2, \\
  m_\sigma^2 - m_\eta^2 &= \frac{1}{4\sqrt{3}} (\sqrt{3} y_1 \pm |y_2|) (2\sqrt{2} y_0 - y_1 \pm \sqrt{3} |y_2|) v_u^2, \\
  m_\eta^2 - m_\sigma^2 &= \frac{1}{4\sqrt{3}} (\sqrt{3} y_1 \pm |y_2|) (2\sqrt{2} y_0 - y_1 \mp \sqrt{3} |y_2|) v_u^2. \\
\end{align*}

Since we have defined the factor $(\sqrt{2} y_0 + y_1)$ as positive in Eq.(2.8), Eq.(A.1) means that, for the case of the normal hierarchy with $m_\pi^2 > m_\eta^2$, we must take the upper signs in Eqs.(A.2)-(A.3), i.e.

\begin{align*}
  m_\pi^2 - m_\sigma^2 &= \frac{1}{4\sqrt{3}} (\sqrt{3} y_1 + |y_2|) (2\sqrt{2} y_0 - y_1 + \sqrt{3} |y_2|) v_u^2 > 0, \\
  m_\eta^2 - m_\sigma^2 &= \frac{1}{4\sqrt{3}} (\sqrt{3} y_1 - |y_2|) (2\sqrt{2} y_0 - y_1 - \sqrt{3} |y_2|) v_u^2 < 0.
\end{align*}
and, for the case of the inverse hierarchy with $m_\pi^2 < m_\eta^2$, we must take the lower signs in Eqs.(A.2)-(A.3), i.e.

$$m_\pi^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}(\sqrt{3} y_1 - |y_2|)(2\sqrt{2} y_0 - y_1 - \sqrt{3}|y_2|) v_u^2 < 0,$$

(A.6)

$$m_\eta^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}(\sqrt{3} y_1 + |y_2|)(2\sqrt{2} y_0 - y_1 + \sqrt{3}|y_2|) v_u^2 < 0.$$  

(A.7)

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Table 1 Solutions of $R(\beta) = R_{obs}$

| $\beta$  | $m_\eta/y_{2u}$ | $m_\sigma/y_{2u}$ | $m_\pi/y_{2u}$ | $m_{\nu_1}$ [eV] | $m_{\nu_2}$ [eV] | $m_{\nu_3}$ [eV] |
|----------|-----------------|-----------------|----------------|-----------------|----------------|-----------------|
| 2.94°    | −0.0775         | 0.3781          | 0.9225         | 0.000368        | 0.00877        | 0.0523          |
| 67.59°   | −0.0775         | −0.3781         | 0.9225         | 0.000368        | 0.00877        | 0.0523          |
| −35.64°  | 0.6636          | 0.6682          | −0.3364        | 0.0515          | 0.0523         | 0.0132          |
| 106.17°  | 0.6636          | −0.6682         | −0.3364        | 0.0515          | 0.0523         | 0.0132          |