Data characterization in dynamical inverse problem for the 1d wave equation with matrix potential

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Abstract

The dynamical system under consideration is

\[ u_{tt} - u_{xx} + V u = 0, \quad x > 0, \quad t > 0; \]
\[ u|_{t=0} = u_t|_{t=0} = 0, \quad x \geq 0; \quad u|_{x=0} = f, \quad t \geq 0, \]

where \( V = V(x) \) is a matrix-valued function (potential); \( f = f(t) \) is an \( \mathbb{R}^N \)-valued function of time (boundary control); \( u = u(t,x) \) is a trajectory (an \( \mathbb{R}^N \)-valued function of \( x \) and \( t \)). The input/output map of the system is a response operator \( R : f \mapsto u_f(0, \cdot), \quad t \geq 0. \)

The inverse problem is to determine \( V \) from given \( R \). To characterize its data is to provide the necessary and sufficient conditions on \( R \) that ensure its solvability.

The procedure that solves this problem has long been known and the characterization has been announced (Avdonin and Belishev, 1996). However, the proof was not provided and, moreover, it turned out that the formulation must be corrected. Our paper fills this gap.

Key words: 1d wave equation with matrix potential, reachable sets, controllability, propagation of singularities, characterization of inverse data.

MSC: 35R30, 46-XX, 47-XX.
1 Introduction

About paper

The subject of this work is the characterization of data in the dynamical inverse problem for the one-dimensional vector wave equation on semi-axis with matrix potential. To characterize the data is to provide the necessary and sufficient conditions ensuring the solvability of the inverse problem.

The inverse problem under consideration is to recover the matrix potential from dynamical data (the response operator); it has long been solved (see [1, 2]). This is one of the first problems solved by the BC-method. The issue is exhausted if one needs only a procedure that determines the potential from the data. However, in the understanding of specialists, the inverse problem is completely solved if, in addition to the procedure, the data characterization is provided. If the potential is self-adjoint, the solvability conditions are well known and, in fact, are reduced to positive definiteness of the so-called connecting operator (CO) of the dynamical system with boundary control, the evolution of which is governed by the Sturm-Liouville operator with the given potential [8, 10, 11]. There was a conjecture that, in the general (non-self-adjoint) case, solvability is ensured by the isomorphism of a relevant analogue of CO, and, moreover, this result was announced in [1]. However, the proof was not given and, moreover, certain doubts arose about sufficiency of this condition. In particular, it was unclear what properties of the CO provide the locality of the potential, i.e., the absence of nonlocal Volterra additives in it. The question remained open and the main purpose of our paper is to fill this gap in the theory of one-dimensional dynamical inverse problems.

Statement and results

All spaces, classes of functions and matrices in the paper are real. We denote \( \Omega := [0, \infty) \) and \( \Omega^T := [0, T] \subset \Omega \).

• The forward problem is an initial-boundary value problem of the form

\[
\begin{align*}
&u_{tt} - u_{xx} + V u = 0, \quad x > 0, \ 0 < t < T \\
&u|_{t=0} = u_t|_{t=0} = 0, \quad x \geq 0 \\
&u|_{x=0} = f, \quad 0 \leq t \leq T,
\end{align*}
\]

(1)

where \( V \in C^1_{\text{loc}}(\Omega; \mathbb{M}^N) \) is a (real) matrix valued function (potential), defined on the semi-axis \( x \geq 0, \ T > 0 \) the final moment of time; \( f \in L_2(([0, T]; \mathbb{R}^N) \)
a boundary control; \( u = u^T(x, t) \) is a solution (wave) - an \( \mathbb{R}^N \)-valued function of variables \( x \) and \( t \). Due to the hyperbolicity of the problem (1), the relation \( \text{supp} u^f(\cdot, t) \subset \Omega^T \) holds for all \( t \).

Let \( \mathcal{F}^T := L_2([0, T]; \mathbb{R}^N) \) be the space of controls. The waves \( u^f(\cdot, t) \) are time-dependent elements of the space \( \mathcal{H}^T := L_2(\Omega^T; \mathbb{R}^N) \). Considering the problem (1) as a dynamical system, we introduce a control operator \( W^T : \mathcal{F}^T \to \mathcal{H}^T \), acting by the rule:

\[
(W^T f)(x) := u^T(x, T), \quad x \in \Omega^T.
\]

Owing to the hyperbolicity of problem (1), its extension of the form

\[
\begin{align*}
  u_{tt} - u_{xx} + V u &= 0, \quad 0 < x < T, \quad 0 < t < 2T - x \\
  u|_{t=x} &= 0 \\
  u|_{x=0} &= f, \quad 0 \leq t \leq 2T
\end{align*}
\]

is a well-posed initial boundary-valued problem, with which one associates the so-called extended response operator

\[
(R^{2T} f)(t) := u^T_T(0, t), \quad 0 \leq t \leq 2T,
\]

acting in the space \( \mathcal{F}^{2T} \). Like all system (1) attributes, the operator \( R^{2T} \) is determined by the potential \( V|_{\Omega^T} \) (does not depend on the values of \( V|_{x>T} \)).

- The problem

\[
\begin{align*}
  u_{tt} - u_{xx} + V^b u &= 0, \quad x > 0, \quad 0 < t < T \\
  u|_{t=0} &= u|_{t=0} = 0, \quad x \geq 0 \\
  u|_{x=0} &= f, \quad 0 \leq t \leq T
\end{align*}
\]

with the potential \( V^b(x) := (V(x))^b \), where \((...)^b : \mathbb{M}_N \to \mathbb{M}_N\) is the matrix transposition, is said to be dual to problem (1). Its solution \( u = u^b_T(x, t) \) possesses the same properties as \( u^T \); the control operator is

\[
(W^b_T f)(x) := u^b_T(x, T), \quad x \in \Omega^T.
\]

The map \( C^T : \mathcal{F}^T \to \mathcal{F}^T \),

\[
C^T := (W^b_T)^* W^T
\]
is called a connecting operator. It is expressed via the operator $R^{2T}$ by a simple and explicit relation established in [1].

• The inverse problem is to recover the potential $V|_{\Omega^T}$ from the given operator $R^{2T}$. Such a local statement was originated by A.S.Blagovestchenskii in [10]; it is relevant to the hyperbolicity of the problem (1).

The main result of the paper is as follows. Along with problems (1) and (3), we consider a family of "shortened" problems with final moments $t = \xi \leq T$, each of which has its own connecting operator $C_\xi$, acting in the corresponding space $\mathcal{F}_\xi$. All $C_\xi$ are defined by $R^{2T}$. We show that $R^{2T}$ is the response operator of a system (1) if and only if all operators $C_\xi$ are isomorphisms. The necessity is known: it is established in [1] in course of analysis of the forward problem. The sufficiency was announced in the same paper, but the proof still has not been provided. Our work fills this gap. At the same time, the mistake made in [1] is corrected: the assertion that for the solvability of the inverse problem it is enough only $C_T$ to be isomorphism, turns out to be wrong. All $C_\xi$ have to be isomorphisms.

2 Forward problem

Properties of waves

Here the known properties of the solutions to problem (1) are listed. They are provided or easily extracted from the results of [8, 11].

Convention 1. All time-dependent functions are extended to $t < 0$ by zero.

• Introduce the class of smooth controls

$$\mathcal{M}^T := \{f \in C^2([0,T]; \mathbb{R}^N) \mid \text{supp} f \subset (0,T)\},$$

which vanish near $t = 0$. For $f \in \mathcal{M}^T$ the problem (1) has a unique classical solution $u^f$ and the representation

$$u^f(x,t) = f(t-x) + \int_x^t w(x,s)f(t-s)\,ds, \quad x \in \Omega^T, 0 \leq t \leq T$$

(4)

holds with the kernel $w$ that solves the Goursat matrix problem

$$\begin{cases}
    w_{tt} - w_{xx} + V\,w = 0, & 0 < x < t < T \\
    w(0,t) = 0, & 0 \leq t \leq T \\
    w(x,x) = -\frac{1}{2} \int_0^x V(s)\,ds, & x \in \Omega^T 
\end{cases}$$

(5)
and is $C^2$-smooth in the domain $\{(x, t) \mid x \in \Omega^T, \ 0 \leq x \leq t \leq T\}$.

For $f \in \mathcal{F}^T := L_2([0, T]; \mathbb{R}^N)$ the right-hand side of (4) is well defined and regarded as a (generalized) solution to the problem (1) of the class $C([0, T]; L_2(\Omega^T))$. In the subsequent, we use the following of its properties.

1. The relation
   \[ \text{supp } u^f(\cdot, t) \subset \Omega^t, \quad t \geq 0 \] holds and shows that the waves propagate in the semi-axis $x \geq 0$ with the speed 1.

2. For the controls $f_\tau(t) := f(t - \tau)$, which act with the delay $\tau > 0$, one has
   \[ u^f_\tau(\cdot, t) = u^f(\cdot, t - \tau), \quad t \geq 0 \] (recall the Convention 1); as a consequence, for smooth controls the relations
   \[ u^f_t = u^f \frac{dt}{dt}, \quad u^f_{tt} = u^f \frac{dt^2}{dt^2} \] see (1) are valid.

3. As it is seen from (4), owing to the continuity of the integral term, the following is valid. If the control $f$ is piecewise continuous and has a jump at the moment $t = T - \xi$ ($0 < \xi \leq T$), the wave $u^f(\cdot, t)$ is also piecewise continuous and has a jump at a point $t = \xi$, and the equality
   \[ u^f(x, T) \bigg|_{x=\xi+0}^{x=\xi-0} = -f(t) \bigg|_{t=T-\xi-0}^{t=T-\xi+0} \] holds in $\mathbb{R}^N$. This is a simplest geometrical optics relation: it shows that the wave discontinuity initiated by the jump of control propagates along the semi-axis $x \geq 0$ with the unit velocity, and the ‘amplitude’ of the discontinuity remains constant.

4. All the above properties and relations are valid for the solution $u^f_\#$ to the dual problem (3) and the solution $u^f$ to the extended problem (2).

**Dynamical system**

Here problem (1) is endowed with standard attributes of dynamical system: spaces and operators. The system is denoted by $\alpha^T$. 
The space of controls is \( \mathcal{F}^T = L_2([0, T], \mathbb{R}^N) \) with the inner product

\[
(f, g)_{\mathcal{F}^T} = \int_0^T \langle f(t), g(t) \rangle dt,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^N \), is said to be the outer space of the system \( \alpha^T \). It contains an increasing family of subspaces

\[
\mathcal{F}^{T, \xi} := \{ f \in \mathcal{F}^T \mid \text{supp } f \subset [T - \xi, T] \}, \quad 0 \leq \xi \leq T
\]

\( (\mathcal{F}^{T, 0} = \{0\}, \mathcal{F}^{T,T} = \mathcal{F}^T) \) formed by delayed controls: \( T - \xi \) is the delay, \( \xi \) is the action time.

The space \( \mathcal{H}^T := L_2(\Omega^T, \mathbb{R}^N) \) with inner product

\[
(u, v)_{\mathcal{H}^T} := \int_{\Omega^T} \langle u(x), v(x) \rangle dx
\]

is called the inner space, the waves \( u^f(\cdot, t) \) are its elements. It contains an increasing family of subspaces

\[
\mathcal{H}^\xi := \{ y \in \mathcal{H}^T \mid \text{supp } y \subset \Omega^\xi \}, \quad 0 \leq \xi \leq T
\]

\( (\mathcal{H}^0 = \{0\}) \). By (6), we have \( u^f(\cdot, t) \in \mathcal{H}^\xi \) for \( 0 \leq t \leq \xi \).

The operator \( W^T : \mathcal{F}^T \to \mathcal{H}^T \),

\[
(W^T f)(x) := u^f(x, T), \quad x \in \Omega^T
\]

is said to be the control operator. By (10), the representation

\[
(W^T f)(x) = f(T - x) + \int_x^T w(x, s)f(T - s)ds, \quad x \in \Omega^T. \quad (10)
\]

holds. The control operator is an isomorphism of the space \( \mathcal{F}^T \). Indeed, the equation \( W^T f = y \) is a second kind Volterra equation, which is solvable for any \( y \in \mathcal{H}^T \). Moreover, (10) implies

\[
W^T \mathcal{F}^{T, \xi} = \mathcal{H}^\xi, \quad 0 \leq \xi \leq T. \quad (11)
\]

The second equality in (8) can be written as:

\[
W^T \frac{d^2}{dt^2} = -LW^T, \quad (12)
\]
where \( L = -\frac{d^2}{dx^2} + V \) is Sturm-Liouville operator, which governs the evolution of system \( \alpha^T \).

- The operator \( R^T : \mathcal{D}^T \to \mathcal{D}^T \), \( \text{Dom } R^T = \{ f \in \mathcal{D}^T | \frac{df}{dt} \in \mathcal{D}^T, f(0) = 0 \} \),
  \[
  (R^T f)(t) := u_x^f(0, t), \quad 0 \leq t \leq T
  \]
is called the response operator of the system \( \alpha^T \). Differentiation in (4) leads to the representation
  \[
  (R^T f)(t) = -\frac{df}{dt}(t) + \int_0^t r(t-s)f(s)ds, \quad 0 \leq t \leq T,
  \]
where \( r(s) := w_x(0, s) \) is a \( C^1 \)-smooth matrix-valued function called the response function.

Also, with the system \( \alpha^T \) one associates the extended response operator \( R^{2T} : \mathcal{D}^{2T} \to \mathcal{D}^{2T} \), \( \text{Dom } R^{2T} = \{ f \in \mathcal{D}^{2T} | \frac{df}{dt} \in \mathcal{D}^{2T}, f(0) = 0 \} \),
  \[
  (R^{2T} f)(t) := u_x^f(0, t), \quad 0 \leq t \leq 2T,
  \]
where \( u_x^f \) is a solution to the extended problem (2). The representation
  \[
  (R^{2T} f)(t) = -f'(t) + \int_0^t r(t-s)f(s)ds, \quad 0 \leq t \leq 2T,\quad (13)
  \]
holds with the matrix-valued reply function \( r \). The important fact is that the operator \( R^{2T} \) and its response function \( r|_{[0,2T]} \) are determined by the potential \( V|_{\Omega^T} \) (do not depend on its values outside \( \Omega^T \)).

- The problem (3) describes a dynamical system, which is called dual to \( \alpha^T \) and denoted by \( \alpha^T_\flat \). Obviously, the dual system has the same attributes and properties as the original one. The inner and outer spaces of both systems are the same, the corresponding operators \( W_T^\flat, R^T_\flat \) and \( R^{2T}_\flat \) possess the same properties and representations. As shown in the [1], the response functions are related by the equality
  \[
  r_\flat(t) = (r(t))^\flat, \quad 0 \leq t \leq 2T.
  \]
Note that the operator \( W_T^\flat \) maps \( \mathcal{D}^T \) onto \( \mathcal{H}^T \) isomorphically and
  \[
  W_T^\flat \mathcal{D}^{T,\xi} = \mathcal{H}^\xi, \quad 0 \leq \xi \leq T\quad (14)
  \]
holds. Its adjoint \( (W_T^\flat)^* : \mathcal{H}^T \to \mathcal{D}^T \) is also an isomorphism.
• The operator $C^T : \mathcal{F}^T \to \mathcal{F}^T$

$C^T := (W_\delta^T)^* W^T$

(15)

is said to be the connecting operator. By the definition, we have

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W_\delta^T g)_{\mathcal{H}^T} = (u^I(\cdot, T), u^O(\cdot, T))_{\mathcal{H}^T},$$

so that operator $C^T$ connects metrics of spaces $\mathcal{F}^T$ and $\mathcal{H}^T$. As a composition of two isomorphisms, it is an isomorphism of the outer space $\mathcal{F}^T$.

The key fact of the BC-method as an approach to inverse problems is a simple and explicit relation that expresses the connecting operator via the response operator and response function. As is shown in the [1], the representation

$$(C^T f)(t) = f(t) + \int_0^T C^T(t, s)f(s)\, ds, \quad 0 \leq t \leq T \tag{16}$$

with the matrix kernel

$$C^T(t, s) = \frac{1}{2} \int_{|t-s|}^{2T-t-s} r(\eta)\, d\eta, \quad 0 \leq s, t \leq T$$

holds. Thus, the connecting operator is determined by the reply function $r|_{0 \leq t \leq 2T}$.

The dual system $\alpha^T_\delta$ has connecting operator

$$C^T_\delta := (W^T)^* W^T_\delta = (C^T)^*,$$

for which the representation [16] is valid, replacing the kernel $C^T(t, s)$ by $C^T_\delta(t, s) = [C^T(t, s)]^\delta$.

**Systems $\alpha^\xi$**

Consider the family of ‘shortened’ systems

$$\begin{cases}
  u_{tt} - u_{xx} + V(x)u = 0, & x > 0, \quad 0 < t < \xi \\
  u|_{t=0} = u_t|_{t=0} = 0, & x \geq 0 \\
  u_x|_{x=0} = f, & 0 \leq t \leq \xi
\end{cases}$$
indexed by the parameter $0 < \xi \leq T$. Systems $\alpha^\xi$ are equipped with their spaces $\mathcal{F}^\xi$ and $\mathcal{H}^\xi$ and operators $W^\xi, R^\xi, R^{2\xi}, C^\xi$. Each $C^\xi$ is an isomorphism in $\mathcal{F}^\xi$, the representation (16)

$$(C^\xi f)(t) = f(t) + \int_0^\xi C^\xi(t, s) f(s) \, ds, \quad 0 \leq t \leq \xi$$

(17)

with matrix kernel

$$C^\xi(t, s) = \frac{1}{2} \int_{|t-s|}^{2\xi-t-s} r(\eta) \, d\eta, \quad 0 \leq s, t \leq T$$

being valid.

- The operators $C^\xi$ are related to the operator $C^T$ as follows. Recall Convention II and introduce the auxiliary operators

$$e^{T,\xi} : \mathcal{F}^\xi \to \mathcal{F}^T, \quad (e^{T,\xi} f)(t) := f(t - (T - \xi)), \quad 0 \leq t \leq T;$$

the adjoint operators are

$$(e^{T,\xi})^* : \mathcal{F}^T \to \mathcal{F}^\xi, \quad ((e^{T,\xi})^* f)(t) := f(t + (T - \xi)), \quad 0 \leq t \leq \xi.$$  

It is easy to check that

$$(e^{T,\xi})^* e^{T,\xi} = \mathbb{I}_{\mathcal{F}^\xi}; \quad e^{T,\xi}(e^{T,\xi})^* = X^{T,\xi},$$

where $\mathbb{I}_{\mathcal{F}^\xi}$ is the unit operator, and $X^{T,\xi}$ is the orthogonal projector in $\mathcal{F}^T$ onto $\mathcal{F}^{T,\xi}$, which cuts off the $\mathbb{R}^N$-valued controls to the interval $[T - \xi, T]$:

$$(X^{T,\xi} f)(t) := \begin{cases} 0, & 0 \leq t < T - \xi \\ f(t), & T - \xi \leq t \leq T. \end{cases}$$

(18)

By the use of (16), one easily derives

$$C^\xi = (e^{T,\xi})^* C^T e^{T,\xi}, \quad 0 < \xi \leq T.$$  

(19)

- As easily follows from (13), the relationship of the extended response operator with the ‘shortened’ response operators is of the form

$$R^\xi = (e^{2T,\xi})^* R^{2T} e^{2T,\xi}, \quad 0 < \xi \leq 2T.$$
3 Inverse problem

Statement

As noted above, the operator $R^{2T}$ is determined by the values of the potential $V^T$ on the segment $\Omega^T$. Hence, the relevant statement of the inverse problem, which respects such a locality, is as follows: given $R^{2T}$ to recover $V|_{\Omega^T}$. Also, since to give $R^{2T}$ is to know the response matrix-function, one needs to determine $V|_{\Omega^T}$ from the given $v|_{0 \leq t \leq 2T}$.

In such a statement, the problem is solved in [1] and below we briefly describe a simplified version of the procedure for solving it. The procedure is preceded with a description of its instruments: projectors and the so-called amplitude formula.

Projectors

- Fix a positive $\xi \leq T$. In the system $\alpha^T$, the subspace

$$\mathcal{H}_\xi := W^T \mathcal{F}^{T,\xi} = \{ u^f(\cdot, T) \mid f \in \mathcal{F}^{T,\xi} \} = \{ u^f(\cdot, \xi) \mid f \in \mathcal{F}^T \}$$

formed by waves, is called reachable (at the moment $t = \xi$).

The orthogonal projector in $H^T$ onto $\mathcal{H}_\xi$ is said to be the wave projector. By (11), it coincides with the projector in $H^T$ onto the subspace $H^{\xi}$, which cuts off the $\mathbb{R}^N$-valued functions to $\Omega^{\xi}$. So, we have:

$$P^\xi y = \begin{cases} y & \text{in } \Omega^{\xi} \\ 0 & \text{in } \Omega^T \setminus \Omega^{\xi} \end{cases}, \quad 0 \leq \xi \leq T.$$  

- In the space $\mathcal{F}^T$, define the operator

$$\mathcal{P}^{T,\xi} := [W^T]^{-1} P^\xi W^T.$$  

Since $W^T$ acts isomorphically, whereas $(\mathcal{P}^{T,\xi})^2 = \mathcal{P}^{T,\xi}$ obviously holds, it is a bounded projector. Let us describe in more detail how it acts. Begin with a general operator lemma.

Let $\mathcal{F}$ and $\mathcal{H}$ be the Hilbert spaces, $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{H}' \subset \mathcal{H}$ the (closed) subspaces; $e : \mathcal{F}' \to \mathcal{F}$ the embedding, which satisfies $e^* e = I_{\mathcal{F}'}$ and $ee^* = X$, where $X$ projects orthogonally in $\mathcal{F}$ onto $\mathcal{F}'$. Denote $\mathcal{F}'_\perp := \mathcal{F} \ominus \mathcal{F}'$ and $\mathcal{H}'_\perp := \mathcal{H} \ominus \mathcal{H}'$; let $P$ be the orthogonal projector in $\mathcal{H}$ onto $\mathcal{H}'$.  

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Let \( W : \mathcal{F} \to \mathcal{H} \) and \( V : \mathcal{F} \to \mathcal{H} \) be isomorphisms provided \( W \mathcal{F}' = V \mathcal{F}' = \mathcal{H}' \). Introduce the isomorphism \( C := V^*W : \mathcal{F} \to \mathcal{F} \) and the subspace \( C^{-1} \mathcal{F}'_\perp \subset \mathcal{F} \). The operator \( \mathcal{P} := W^{-1}PW \) acts in \( \mathcal{F} \).

**Lemma 1.** Let \( C' := e^*C e \) act isomorphically in \( \mathcal{F}' \). Then the decomposition in direct sum \( \mathcal{F} = \mathcal{F}' + C^{-1} \mathcal{F}'_\perp \) holds, whereas \( \mathcal{P} \) is the (skew) projector in \( \mathcal{F} \) onto \( \mathcal{F}' \) in parallel to \( C^{-1} \mathcal{F}'_\perp \). The representation

\[
\mathcal{P} = e [C']^{-1} e^* C
\]  
(21)

holds.

**Proof.** 1. The operators \( W \) and \( V \) act isomorphically, and we have \( W \mathcal{F}' = V \mathcal{F}' = \mathcal{H}' \). The latter equality implies \( V^* \mathcal{H}'_\perp = \mathcal{F}'_\perp \) and leads to

\[
\mathcal{F} = W^{-1}[\mathcal{H}' \oplus \mathcal{H}'_\perp] = W^{-1} \mathcal{H}' + W^{-1} \mathcal{H}'_\perp = \mathcal{F}' + C^{-1} \mathcal{F}'_\perp.
\]

2. One has \( \mathcal{P}^2 = \mathcal{P} \) just by the definition of \( \mathcal{P} \).

If \( f \in \mathcal{F}' \) then \( Wf \in \mathcal{H}' \) and, hence,

\[
\mathcal{P} f = W^{-1}PWf = W^{-1}Wf = f.
\]

If \( f \in C^{-1} \mathcal{F}'_\perp \) then \( f = C^{-1}g \) with \( g \in \mathcal{F}'_\perp \) and one has

\[
\mathcal{P} f = W^{-1}PW^TC^{-1}g = W^{-1}P[V^*]^{-1}g = 0
\]

in view of \( [V^*]^{-1}g \in \mathcal{H}'_\perp \).

Thus, \( \mathcal{P} \) is an idempotent, which acts identically on \( \mathcal{F}' \) and annuls \( C^{-1} \mathcal{F}'_\perp \). Therefore, it projects in \( \mathcal{F} \) onto \( \mathcal{F}' \) in parallel to \( C^{-1} \mathcal{F}'_\perp \).

3. Let \( Q \) be the right hand side of (21). Then we have

\[
Q^2 = e [C']^{-1} e^* C e [C']^{-1} e^* C = e [C']^{-1} C' [C']^{-1} e^* C = Q.
\]

If \( f \in \mathcal{F}' \) then \( Xf = ee^*f = f \) and, hence,

\[
Qf = e [C']^{-1} e^* C f = e [C']^{-1} e^* Ce e^* f = e [C']^{-1} C'e^* f = f.
\]

If \( f \in C^{-1} \mathcal{F}'_\perp \) then \( f = C^{-1}g \) with \( g \in \mathcal{F}'_\perp \) and one has

\[
Qf = e [C']^{-1} e^* CC^{-1}g = 0
\]

in view of \( e^* \mathcal{F}'_\perp = \{0\} \).

Thus, \( Q \) is an idempotent, which acts identically on \( \mathcal{F}' \) and annuls \( C^{-1} \mathcal{F}'_\perp \). Therefore, it projects in \( \mathcal{F} \) onto \( \mathcal{F}' \) in parallel to \( C^{-1} \mathcal{F}'_\perp \) and, hence, coincides with \( \mathcal{P} \).
Return to the definition (20). Applying the Lemma 1 to \( F = F^T, \ F' = F'^T, \ F' = F'^T, \ H = H^T, \ H' = H', \ W = W^T, \ V = W^T, \ P = P^T, \) and referring to (19), we arrive at the following.

**Corollary 1.** Operator \( P^T \) is the (skew) projector in \( F^T \) onto \( F^T = F^T, \) the representation \( P^T = e^T[(C^*)^{-1}(e^T)^*C^T], \ 0 < \xi \leq T \)

holds.

• Due to the complete equality of systems \( \alpha^T \) and \( \alpha^T, \) the operator

\[
P_y^T := [W^T]^{-1}P_w^T
\]

has the same properties as \( P^T. \) Namely, it is the projector in \( F^T \) onto \( F^T, \) in parallel of the subspace \([C^T]^{-1}F^T, \), and is represented in the form.

\[
P_y^T = e^T[(C^*)^{-1}(e^T)^*(C^*)^*], \ 0 < \xi \leq T.
\]

**Amplitude formula**

For a positive \( \xi \leq T \) and control \( f \in M^T, \) one has

\[
W^T P^T f \equiv \begin{cases} P_w^T f = P_w^T u^f(\cdot, T) = \begin{cases} u^f(\cdot, T) \text{ in } \Omega^\xi \end{cases} \\ 0 \text{ in } \Omega^T \setminus \Omega^\xi \end{cases}
\]

The control \( P^T f \in F^T \) vanishes at \( 0 \leq t < T - \xi \) and, in the generic case, has an \( \mathbb{R}^N \)-valued ‘jump’ at the moment \( t = T - \xi. \) The wave \( u^T f = P^T \in \mathcal{F} \) vanishes outside \( \Omega^\xi \) and also has a jump at the point \( x = \xi. \)

The values (amplitudes) of these jumps are related by the equality (9). Since the wave \( u^f(\cdot, T) \) is continuous in \( \Omega^T, \) we have the relation

\[
(P^T f)(T - \xi + 0) = (P_w^T u^f(\cdot, T))(\xi - 0) = u^f(\xi, T).
\]

Writing it in the form

\[
(W^T f)(\xi) = (P^T f)(T - \xi + 0), \quad \xi \in \Omega^T,
\]

we get the so-called amplitude formula, which is a simplest example of the geometrical optics relations describing the propagation of singularities in the system \( \alpha^T. \) Formulas of this type play the key role in all basic versions of the BC-method \[3, 5, 7].

12
Recovering the potential

Let \( r|_{0 \leq t \leq 2T} \) be the response matrix-function of a dynamical system \( \alpha^T \) of the form (11). The following procedure recovers the potential \( V \) on the segment \( \Omega^T \).

A. Given \( r \) determine the operators \( C^\xi \) for all \( 0 \leq \xi \leq T \) by the use of representation (17).

B. Find the projectors \( P^{T,\xi} \) for \( 0 \leq \xi \leq T \) by (22).

C. Determine the control operator \( W^T \) by means of the formula (25) and then find its kernel \( w \) (see (10)). Knowing the kernel, recover the potential \( V|_{\Omega^T} \) by \( V(x) = -2 \frac{dw(x,x)}{dx} \) (see (5)).

4 Data characterization

To characterize the data of the inverse problem under consideration is to provide the necessary and sufficient conditions on a matrix-function \( r \), which guarantee that it is the response function of some system \( \alpha^T \) and thereby ensure the solvability of the inverse problem. As will be shown, these conditions are that all operators defined (via \( r \)) by the right-hand side of (17) act isomorphically in the corresponding spaces. The necessity is already established. Indeed, if \( r \) is the reply function, the right-hand side coincides with the connecting operator \( C^\xi \) of the system \( \alpha^\xi \), which is isomorphism in \( F^\xi \). Sufficiency is more complicated and the rest of the paper is devoted to its proof.

The proof of sufficiency is constructive. In fact, it reduces to applying the procedure A – C to the given function \( r \). As the result, we construct some dynamical system \( \alpha^T \). In course of the construction, it is verified that the isomorphism of all \( C^\xi \) ensures that all steps of the procedure are realizable. At the final step, we show that the response function of the constructed system coincides with the function \( r \), with which we has began.

We proceed to the implementation of this program, starting with an exact statement of the main result.

**Theorem 1.** A matrix-function \( r \in C^1([0, 2T]; \mathbb{M}^N) \) is the response function of a dynamical system of the form (7) if and only if for every positive \( \xi \leq T \)
the operator $C^\xi$ defined by

$$(C^\xi f)(t) = f(t) + \int_0^\xi C^\xi(t, s) f(t - s) \, ds, \quad 0 \leq t \leq \xi$$

(26)

with the kernel

$$C^\xi(t, s) = \frac{1}{2} \int_{|t-s|}^{2\xi-t-s} r(\eta) \, d\eta.$$  

(27)

is an isomorphism of the space $\mathcal{F}^\xi$.

Let us make a remark about the notation in the forthcoming proof of sufficiency. In the above statement, the symbol $C^\xi$ does not assume that this operator is the connecting operator of some system $\alpha^\xi$: it is only a candidate for this role. Other symbols $\mathcal{P}^{T,\xi}, W^T, R^{2T},$ etc, are also used in this way. Such a trick simplifies the notation and, on the other hand, clarifies the meaning of the introduced objects.

**Projectors**

• Realizing the plan outlined above, we introduce the relevant analogues of the objects belonging to the system $\alpha^T$. Begin with the projectors $\mathcal{P}^{T,\xi}$. However, to introduce them by (20) is not possible because, at the moment, no $W^T$ is given. Therefore, focusing on the representation (22), we define

$$\mathcal{P}^{T,\xi} := e^{T,\xi} [C^\xi]^{-1} (e^{T,\xi})^* C^T, \quad 0 < \xi \leq T$$

(28)

that is correct since all $C^\xi$ are isomorphisms by assumption of the Theorem. Then, by perfect analogy with (28), we put

$$\mathcal{P}_\flat^{T,\xi} := e^{T,\xi} [(C^T)^*]^{-1} (e^{T,\xi})^* (C^T)^*, \quad 0 < \xi \leq T$$

(29)

(compare with (23), (24)) that is correct since $(C^\xi)^*$ are isomorphisms.

**Proposition 1.** Operator $\mathcal{P}^{T,\xi}$ is the projector in $\mathcal{F}^T$ onto $\mathcal{F}^{T,\xi}$ in parallel to $[C^T]^{-1} \mathcal{F}^{T,\xi}_\perp$. Operator $\mathcal{P}_\flat^{T,\xi}$ projects in $\mathcal{F}^T$ onto $\mathcal{F}^{T,\xi}_\perp$ in parallel to $[(C^T)^*]^{-1} \mathcal{F}^{T,\xi}_\perp$. The equalities

$$C^T \mathcal{P}^\xi = (\mathcal{P}_\flat^\xi)^* C^T, \quad \mathcal{P}^\xi (C^T)^{-1} = (C^T)^{-1} (\mathcal{P}_\flat^\xi)^*$$

(30)

are valid.
Indeed, to verify the first and second assertions, one needs just to repeat the arguments of the item 3. from the proof of Lemma 1. Then the equalities \((30)\) easily follow from \((28)\) and \((29)\).

- Here we derive an efficient representation for the projectors \(P_{T,\xi}\) and \(P_{\flat, T,\xi}\).

Recall that the projector \(X_{T,\xi}\) is defined by \((18)\).

**Lemma 2.** For any \(0 < \xi \leq T\), the representation

\[
(P_{T,\xi} f)(t) = (X_{T,\xi} f)(t) + \int_0^{T-\xi} m_{T,\xi}(t, s) f(s) \, ds, \quad 0 \leq t \leq T
\]

holds with a piecewise \(C^2\)-smooth kernel \(m_{\xi}\), which obeys \(m_{\xi}(t, s)|_{t > T - \xi} \equiv 0\).

**Proof.** (sketch) By the general Fredholm integral equation theory, the inverse to the isomorphism \(C_{\xi}\) is takes the form

\[
((C_{\xi})^{-1} f)(t) = f(t) - \int_0^\xi l_{\xi}(t, s) f(s) \, ds, \quad 0 \leq t \leq \xi
\]

with a matrix kernel \(l_{\xi}\) of the same smoothness as the kernel \(C_{\xi}\): it is continuous in square \([0, \xi] \times [0, \xi]\) twice continuously differentiable outside the diagonal \(t = s\).

Substituting \((26)\) (with \(\xi = T\)) and \((32)\) to the right of the definition \((28)\), as a result of cumbersome calculations (integration by parts, changing the order of integration, etc.), we arrive at \((31)\). Also, the calculations provide

\[
m_{\xi}(t, s) = X_{T,\xi} C_T(t, s) - \int_0^\xi l_{\xi}(t - (T - \xi), \eta) C_T(\eta + (T - \xi), s) \, d\eta,
\]

where \(C_T(\cdot, \cdot)\) is the kernel \((27)\) for \(\xi = T\).

Note in addition that the integration limits in \((31)\) correspond to how the projector \(P_{T,\xi}\) acts. If \(f \in \mathcal{F}_{T,\xi}\), then \(f|_{0 \leq t \leq T - \xi} = 0\); therefore, the integral vanishes, which provides \(P_{T,\xi} f = X_{T,\xi} f = f\). Also, since \(P_{\xi, T} f|_{0 \leq t \leq T - \xi} = 0\) for any \(f\), the kernel \(m_{\xi}\) must vanish identically for \(t > T - \xi\).

By the use of \((29)\), quite analogous arguments lead to the representation

\[
(P_{\flat, T,\xi} f)(t) = (X_{T,\xi} f)(t) + \int_0^{T-\xi} m_{\flat, T,\xi}(t, s) f(s) \, ds, \quad 0 \leq t \leq T
\]

with kernel \(m_{\flat, T,\xi}\) having the same properties as \(m_{\xi}\).
Operators $W^T$ and $W^T_b$ 

- The next definition is motivated by the amplitude formula (25). Let us recall that $\mathcal{H}^T = L^2(\Omega^T; \mathbb{R}^N)$ and introduce the operator $W^T : \mathcal{F}^T \to \mathcal{H}^T$,

\[(W^T f)(x) := (\mathcal{P}^{T,x} f)(T - x + 0), \quad x \in \Omega^T. \tag{34}\]

Note that, at the moment, it is just an operator defined (in several steps) by the given function $r|_{[0,2T]}$. However, later on, it will turn out to be the control operator of some system $\alpha^T$. The construction of this system is the main storyline of the proof of sufficiency.

To obtain a representation of $W^T$ it suffices to put $t = T - x + 0$ in (31) and take into account the form of the kernel $m^\xi$ in (33). As a result of simple calculations, one gets

\[(W^T f)(x) = f(T - x) + \int_x^T w(x,s)f(T - s) \, ds, \quad x \in \Omega^T \tag{35}\]

with $C^2$-smooth matrix kernel of the form

\[
w(x,s) = C^T(T - x,T - s) - \int_0^x l^x(0,\eta) C^T(\eta + (T - x), T - s) \, d\eta, \quad 0 \leq x \leq s \leq T, \tag{36}\]

where $l^x$ is taken from (32). Putting $x = 0$ in (36) and taking into account $C^T(T, T)$ see (27) = 0, we get

\[w(0,s) = 0, \quad 0 \leq s \leq T.\]

It is recommended to compare (35) with (10).

As it easily follows from (35), $W^T$ is an isomorphism from $\mathcal{F}^T$ on $\mathcal{H}^T$ and, moreover, the relation

\[W^T \mathcal{F}^T_\xi = \mathcal{H}^\xi, \quad 0 \leq \xi \leq T \tag{37}\]

holds. Also, by standard arguments of the theory of 2nd-order Volterra integral equations, for $y \in \mathcal{H}^T$ one has

\[( [W^T]^{-1} y)(t) = y(T - t) - \int_0^t w^{-1}(t,s) y(T - s) \, ds, \quad 0 \leq t \leq T \tag{38}\]
with kernel $w^{-1}$, twice continuously differentiable for $0 \leq s < t \leq T$ obeying
\[
w^{-1}(0, s) = 0, \quad 0 \leq s \leq T.
\]

• Quite analogously, the operator $W_b^T : \mathcal{F}^T \to \mathcal{H}^T$,
\[(W_b^T)(x) := (P_{\gamma}^{T,x}f)(T - x + 0), \quad x \in \Omega^T.
\]
possesses the same properties as $W^T$. Namely, the representation:
\[(W_b^T)(x) = f(T - x) + \int_x^T w_b(x, s) f(T - s) \, ds, \quad x \in \Omega^T \quad (39)
\]
holds with a kernel $w_b$, which is of the same smoothness as $w$ and obeys $w_b(0, s) = 0, \quad 0 \leq s \leq T$. It is an isomorphism, which provides
\[W_b^T, \mathcal{F}^T,\xi = \mathcal{H}^\xi, \quad 0 \leq \xi \leq T. \quad (40)
\]
Its inverse has the form
\[
([W_b^T]^{-1}y)(t) = y(T - t) - \int_0^t w_b^{-1}(t, s) y(T - s) \, ds, \quad 0 \leq t \leq T
\]
with a kernel $w_b^{-1}$ obeying $w_b^{-1}(0, s) = 0, \quad 0 \leq s \leq T$.

• Let $Y^\xi$ be (orthogonal) projector in $\mathcal{H}^T$ on $\mathcal{H}^\xi$, which cuts off $\mathbb{R}^N$-valued functions onto the segment $\Omega^\xi$:
\[
Y^\xi y = \begin{cases} y, & \text{in } \Omega^\xi \\ 0, & \text{in } \Omega^T \setminus \Omega^\xi \end{cases}.
\]

**Lemma 3.** For every $0 < \xi \leq T$ the relation
\[W^T P^\xi = Y^\xi W^T; \quad W_b^T P^\xi = Y^\xi W_b^T. \quad (41)
\]
holds.

**Proof.** We use two facts:
1) since $\mathcal{F}^{T,\xi} \subset \mathcal{F}^{T,\xi'}$ for $\xi < \xi'$, the projectors satisfy $P^{T,\xi} < P^{T,\xi'}$, which implies
\[
P^{T,x} P^{T,\xi} = \begin{cases} P^{T,x} & \text{for } x < \xi \\ P^{T,\xi} & \text{for } x > \xi \end{cases};
\]
2) if \( f \in \mathcal{F}^T \), then \( \mathcal{P}^{T,\xi} f \in \mathcal{F}^{T,\xi} \); hence \( \text{supp} \mathcal{P}^{T,\xi} f \subset [T - \xi, T] \) and, therefore, \( (\mathcal{P}^{T,\xi} f)(T - x - 0) = 0 \) for \( x > \xi \).

As a consequence, according to the definition (34), we have
\[
(W^T \mathcal{P}^{T,\xi} f)(x) = \begin{cases} 
(\mathcal{P}^{T,x} \mathcal{P}^{T,\xi} f)(T - x - 0) = (\mathcal{P}^{T,x} f)(T - x - 0) = (W^T f)(x), & x < \xi \\
(\mathcal{P}^{T,x} \mathcal{P}^{T,\xi} f)(T - x - 0) = (\mathcal{P}^{T,\xi} f)(T - x - 0) = 0, & x > \xi 
\end{cases}
\]
\[
= (Y^{\xi} W^T f)(x).
\]

The second equality in (41) is proved in the same way.

- Here a relation, which connects the operators \( W^T, W^T_\flat \) and \( C^T \), is established. In its form, it duplicates the definition (15). However, at the moment, \( W^T \) and \( W^T_\flat \) are just some operators constructed via the function \( r \) and we do not claim that \( C^T \) is the connecting operator of some system \( \alpha^T \). This remains to be proved.

**Lemma 4.** The relation
\[
C^T = (W^T_\flat)^* W^T
\]
holds.

**Proof.** 1. Let us denote \( A := W^T[C^T]^{-1}(W^T_\flat)^* \) and verify equality
\[
AY^{\xi} = Y^{\xi} A. \tag{43}
\]

Multiplying the first equality in (41) on the right by \( [C^T]^{-1}(W^T_\flat)^* \) we have
\[
W^T \mathcal{P}^{\xi} [C^T]^{-1}(W^T_\flat)^* = Y^{\xi} W^T [C^T]^{-1}(W^T_\flat)^* = Y^{\xi} A. \tag{44}
\]
In the second equality (41), passing to the adjoint operators, one has \( (\mathcal{P}^{\xi}_\flat)^*(W^T_\flat)^* = (W^T_\flat)^* Y^{\xi} \). Multiplying both parts on the left by \( W^T[C^T]^{-1} \), we obtain:
\[
W^T[C^T]^{-1}(\mathcal{P}^{\xi}_\flat)^*(W^T_\flat)^* = W^T[C^T]^{-1}(W^T_\flat)^* Y^{\xi} = AY^{\xi}. \tag{45}
\]
In this way, we get
\[
Y^{\xi} A \overset{44}{=} W^T \mathcal{P}^{\xi} [C^T]^{-1}(W^T_\flat)^* \overset{41}{=} W^T[C^T]^{-1}(\mathcal{P}^{\xi}_\flat)^*(W^T_\flat)^* \overset{45}{=} AY^{\xi}.
\]
So (43) is valid.

2. Representations (32) (with $\xi = T$), (35) and (39) easily imply that the operator $A = W^T [C^T]^{-1}(W_s^T)^*$ has the form
\[ A = I + K, \]
where $K$ is a compact integral operator in $H$. The commutation (43) leads to $KY^\xi = Y^\xi K$ and, then, to $K^*Y^\xi = Y^\xi K^*$. As a result, we have
\[ K^*KY^\xi = K^*Y^\xi K = Y^\xi K^*K, \]
so that a self-adjoint operator $K^*K$ commutes with the family of projectors $\{Y^\xi\}_{0 < \xi < T}$. By the well-known arguments of the spectral theory, the latter is possible if and only if $K^*K$ is the multiplication by a bounded positive measurable matrix-function. However, since $K^*K$ is compact, this is possible if and only if $K^*K = 0$, which is equivalent to $K = 0$. Thus, we arrive at $A = I$.

3. By the definition of $A$, we have
\[ A = W^T[C^T]^{-1}(W_s^T)^* = I. \]
Since $W^T$ and $W_s^T$ are isomorphisms, the latter leads to (42).

**Operator $L$**

Recall that $\mathcal{M}^T \subset \mathcal{E}^T$ is the class of $C^2$-smooth controls vanishing near $t = 0$. Also, note that the set $\mathcal{M}^T \cap \mathcal{F}^{T,\xi}$ is dense in $\mathcal{F}^{T,\xi}$ for all positive $\xi < T$.

Focusing on (12), let us define operator in $H$:
\[ L := -W^T \frac{d^2}{dt^2} [W^T]^{-1}, \quad \text{Dom } L = W^T \mathcal{M}^T. \tag{46} \]
Using representation (35) of $W^T$ and smoothness of $w$, it is easy to make sure that $\text{Dom } L = W^T \mathcal{M}^T = \{ y \in C^2(\Omega_T; \mathbb{R}^N) \mid \text{supp } y \subset [0, T) \}$.

The following result shows that $L$ is a Sturm-Liouville operator.

**Lemma 5.** The representation
\[ L = - \frac{d^2}{dx^2} + V(x) \tag{47} \]
holds with potential $V(x) := -2 \frac{dw(x,x)}{ds} \in C^1(\Omega_T; \mathbb{R}^N)$, where $w$ is the kernel of the integral part of $W^T$ in (35).
Proof. (sketch)
1. At first, an auxiliary relation is derived. To simplify the notation, we use \( \dot{()} = \frac{d}{dt} \).

Let controls \( f, g \in M_T \) obey \( f(T) = g(T) = 0 \). Then the following equality holds:
\[
(C^T \dot{g}, f)_{\mathcal{F}^T} = (C^T g, \dot{f})_{\mathcal{F}^T}.
\] (48)

It can be verified by integration by parts, with regard to the boundary conditions imposed of the controls and a specific form \( (27) \) (with \( \xi = T \)) of kernel of \( C^T \).

2. Show that the operator \( L \) is local, i.e. satisfies \( \text{supp} Ly \subset \text{supp} y \).

Let \( y \in \text{Dom} L \) and \( \text{supp} y \subset \Omega^T \). Then, due to (37), for \( f := [W^T]^{-1}y \) we have \( f \in \mathcal{M}^T \cap \mathcal{F}^{T,\xi} \) and, consequently, \( \ddot{f} \in \mathcal{H}^{T,\xi} \). Referring again to (37), we get:
\[
Ly = -W^T \ddot{f} \in \mathcal{H}^T,\xi, \text{ which means } \text{supp} Ly \subset \Omega^T. \text{ Thus, } L \text{ does not extend the support of functions to the right.}
\]

Let \( y \in \text{Dom} L \) and \( \text{supp} y \subset \Omega^T \setminus \Omega^\xi = [T - \xi, T] \). Then for \( f := [W^T]^{-1}y \in \mathcal{M}^T \) we obtain \( f(T) = y(0) = 0 \). Let \( g \in \mathcal{M}^T \cap \mathcal{F}^{T,\xi} \) and \( g(T) = 0 \). Thus, \( f \) and \( g \) satisfy the conditions, which provide (48). Then we have:
\[
-(Ly, W^T_y g)_{\mathcal{F}^T} = -(LW^T f, W^T_y g)_{\mathcal{F}^T} = (W^T \ddot{f}, W^T_y g)_{\mathcal{F}^T} \overset{42}{=} (C^T \ddot{f}, g)_{\mathcal{F}^T} = (y, W^T_y \ddot{g})_{\mathcal{F}^T} = 0,
\]

because \( \ddot{g} \in \mathcal{F}^{T,\xi} \), and therefore \( W^T_y \ddot{g} \in \mathcal{H}^\xi \), whereas \( y \in \mathcal{H}^T \ominus \mathcal{H}^\xi \) by assumption on its support. In the meantime, the set \( \{ g \in \mathcal{F}^{T,\xi} \mid g(T) = 0 \} \) is dense in \( \mathcal{F}^{T,\xi} \). Owing to (40), the images \( W^T_y \ddot{g} \) constitute a dense set in \( \mathcal{H}^\xi \). Therefore, the established equality \( (Ly, W^T_y g)_{\mathcal{F}^T} = 0 \) implies \( Ly \in \mathcal{H}^T \ominus \mathcal{H}^\xi \). The latter is equivalent to \( \text{supp} Ly \subset \Omega^T \setminus \Omega^\xi \). As a result, \( L \) does not extend the support of functions to the left.

So, \( L \) does not extend the support of functions, i.e., acts locally.

3. Let us show that (47) does hold. By (35), for \( f \in \mathcal{M}^T \) one easily derives
\[
(W^T f)''(x) - (W^T \ddot{f})(x) =
= V(x)f(T - x) + \int_x^T [w_{xx}(x, s) - w_{ss}(x, s)] f(T - s) \, ds =
= V(x)(W^T f)(x) + \int_x^T [w_{xx}(x, s) - w_{ss}(x, s) - V(x)w(x, s)] f(T - s) \, ds
\]
(49)
where \((\ldots)’ = \frac{d}{dx}\) and \(V := -2 \frac{dw(x,x)}{dx}\). Note that all operations, which are applied in course of the derivation (differentiation of integrals, integration by parts, etc) are justified owing to \(C^2\)-smoothness of the kernels of the integrals under consideration. Next, substituting \(f = [W^T]^{-1}y\) and using (38), the derived relation is transformed to
\[
(Ly)(x) = -y''(x) + V(x)y(x) + \int_x^T k(x,s)y(s)\,ds, \quad x \in \Omega^T (50)
\]
with a continuous kernel \(k\).

4. We omit a simple proof of the following fact: an operator of the form (50) is local if and only if the integral summand is absent. Thus, we arrive at (47).

Operator \(L_{\flat}\)

• By the use of the same scheme, it is established that the operator
\[
L_{\flat} := -W_{\flat}^T \frac{d^2}{dt^2} [W_{\flat}^T]^{-1}, \quad \text{Dom} \, L_{\flat} = W_{\flat}^T \mathcal{M}^T
\]
is of the form
\[
L_{\flat} = -\frac{d^2}{dx^2} + V_{\flat}(x)
\]
with potential \(V_{\flat}(x) := -2 \frac{dw_{\flat}(x,x)}{dx} \in C^1(\Omega^T; \mathbb{R}^N)\), where \(w_{\flat}\) is the kernel of the integral part of \(W_{\flat}^T\) in (39).

• Let us show that operators \(L\) and \(L_{\flat}\) are adjoint by d’Alembert, i.e., for \(y, v \in C^\infty_0(\Omega^T; \mathbb{R}^N)\) the equality
\[
(Ly, v)_{\mathcal{H}^T} = (y, L_{\flat}v)_{\mathcal{H}^T}.
\]
is valid. Indeed, by the choice of \(y\) and \(v\), the controls \(f = [W^T]^{-1}y\) and \(g = [W_{\flat}^T]^{-1}v\) vanish at \(t = 0\) and obey \(f(0) = g(0) = 0\). Hence, we have:
\[
(Ly, v)_{\mathcal{H}^T} = (W^T \tilde{f}, W_{\flat}^T g)_{\mathcal{H}^T} = (C^T \tilde{f}, g)_{\mathcal{H}^T} = (C^T f, \tilde{g})_{\mathcal{H}^T} = (W^T f, W_{\flat}^T \tilde{g})_{\mathcal{H}^T} = (y, L_{\flat}v)_{\mathcal{H}^T}
\]
As a consequence, one easily concludes that the potentials are connected by the equality
\[
V_{\flat}(x) = V^\flat(x), \quad x \in \Omega^T.
\]
Completion of the proof of Theorem 1

The operator (47) determines a dynamical system $\alpha^T$ of the form

$$\begin{cases}
  u_{tt} + Lu = 0, & x > 0, \ 0 < t < T \\
  u|_{t=0} = u_t|_{t=0} = 0, & x \geq 0 \\
  u_x|_{x=0} = f, & 0 \leq t \leq T.
\end{cases} \tag{51}$$

As is seen from (46), the operator $W^T$ is the control operator of this system.

Quite analogously, system $\alpha^T$ of the form

$$\begin{cases}
  u_{tt} + L'u = 0, & x > 0, \ 0 < t < T \\
  u|_{t=0} = u_t|_{t=0} = 0, & x \geq 0 \\
  u_x|_{x=0} = f, & 0 \leq t \leq T
\end{cases}$$

is controlled by the operator $W^T$.

As it follows from (42), the connecting operator of the system (51) defined by (15) coincides with the operator $C^T$ introduced by (26) (for $\xi = T$). Therefore, the integral parts of these operators also coincide. The latter obviously implies that the response matrix-function of the system (51) is identical to the function $r|_{0 \leq t \leq 2T}$, with which our considerations have started.

Thus, $r|_{0 \leq t \leq 2T}$ is the response function of a dynamical system of the form (1). The sufficiency of the conditions of Theorem 1 is proved.

Comments

- The deep connection between inverse problems and the problem of triangular factorization of operators is well known. It can be also traced in this work.

Recall the definitions. Let a monotone family (nest) of subspaces $\mathfrak{f} = \{\mathcal{F}^\xi\}_{0 \leq \xi \leq T} : \mathcal{F}^\xi \subset \mathcal{F}^{\xi'}$ for $\xi < \xi'$ be given in a Hilbert space $\mathcal{F}$. An operator $Z$ is called triangular with respect to this nest if $Z\mathcal{F}^\xi \subset \mathcal{F}^\xi$, i.e. all the subspaces $\mathcal{F}^\xi$ are invariant with respect to $Z$. We say that operators $Z$ and $Z_0$, which are triangular with respect to $\mathfrak{f}$, provide triangular factorization of an operator $C$ if $C = Z_0^*Z$ holds.

In our paper, in the space $\mathcal{F}^T$, there is the nest of the subspaces $\mathfrak{f} = \{\mathcal{F}^{T,\xi}\}_{0 \leq \xi \leq T}$. Let us introduce an isometry

$$I^T : \mathcal{H}^T \to \mathcal{F}^T, \quad (I^T y)(t) := y(T - t), \quad 0 \leq t \leq T$$
and note that \((I^T)^*I^T = I_{pt}\). Following from (11) and (14), operators 
\[ Z^T := I^T W^T \text{ and } Z^T_\flat := I^T W^T_\flat \]
are triangular with respect to \(f\). According to (15), we have:
\[ C^T = (W^T_\flat)^*W^T = (I^T W^T_\flat)^*I^T W^T = (Z^T_\flat)^*Z^T. \]  
Consequently, the pair \(Z^T, Z^T_\flat\) provides triangular factorization of the connecting operator of the system \(\alpha^T\) with respect to the nest of subspaces, formed by delayed controls.

Thus, solving the inverse problem by the procedure \(A.-C.\) described in the end of the section 3, we solve the triangular factorization problem for the operator \(C^T\) by (52). These problems are equivalent.

The general factorization problem for operators of the form \(I + \text{compact}\) is solved in [12]: see Theorem 2.1, which provides the necessary and sufficient conditions for its solvability. The conditions on the family of operators \(C^\xi\) adopted in our Theorem 1 are quite adequate to the mentioned classical ones.

Let us return to the question raised in the Introduction: why does the operator \(L\) given by (46) turn out to be local? The explanation is in a very specific form of the kernel of operator \(C^T\): see (27). Such a specifics is used, in particular, in the calculations (48) and (49).

- A substantial difference, which distinguishes the problem with a non-self-adjoint potential \(V\) from the problem with \(V^* = V\), is as follows. In the second case, the connecting operator \(C^T = (W^T)^*W^T\) is positive definite and, hence, all the shortened operators \(C^\xi\) turn out to be such. Therefore, for characterization it suffices to require only \(C^T\) to be isomorphism: this implies isomorphism of all \(C^\xi\). In the general case, isomorphism of \(C^T\) does not ensure isomorphism of \(C^\xi\). This is the mistake made in the statement of the conditions of Theorem 3.2 in [1].

There is a case when the isomorphism of all \(C^\xi\) certainly takes place. If \(T > 0\) is small, then the integral parts of the operators \(C^\xi\) have a small norm and all \(C^\xi\) turn out to be isomorphisms. With this reservation, the statement of Theorem 3.2 becomes true.

- The scheme, which provides the data characterization in this work, is traditional for the BC-method. Its core is, first, to elaborate an efficient procedure, which solves the inverse problem and, then, to provide the conditions, which ensure its realizability. In one-dimensional problems such a scheme
works quite successfully: see [4, 9]. There are certain results on the multi-dimensional case but the list of the characteristic conditions turns out to be rather long [6].

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