Topological complexity and the homotopy cofibre of the diagonal map

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Abstract

In this paper we analyze some relationships between the topological complexity of a space $X$ and the category of $C_{\Delta_X}$, the homotopy cofibre of the diagonal map $\Delta_X : X \to X \times X$. As a consequence of this work and of a result by M. Farber, S. Tabachnikov and S. Yuzvinsky, we obtain that the immersion problem for the real projective space $\mathbb{RP}^n$ is equivalent to the computation of the L.-S. category of $C_{\Delta_{\mathbb{RP}^n}}$.

Introduction

The topological complexity of a space $X$, $TC(X)$ is the sectional category (or Schwarz genus) of the end-points evaluation fibration $\pi_X : X^I \to X \times X, \pi_X(\alpha) = (\alpha(0), \alpha(1))$. This homotopical invariant was defined by M. Farber [11], [12], giving a topological approach to the robot motion planning problem. In robotics if one regards the topological space $X$ as the configuration space of a mechanical system, the motion planning problem consists of constructing a program or a devise, which takes pairs of configurations $(A, B) \in X \times X$ as an input and produces as an output a continuous path in $X$, which starts at $A$ and ends at $B$. Broadly speaking, $TC(X)$ measures the discontinuity of any motion planner in the space. Further developments of the topological complexity have proved to be a very interesting homotopical invariant. It not only interacts with robotics but also with deep problems arising in algebraic topology. In this sense it is shown that the problem of computing the number $TC(\mathbb{RP}^n)$ of the $n$-th real projective space is equivalent to a classical problem of manifold topology (see [16]) which asks for the minimal dimension of the Euclidean space $N$ such that there exists an immersion...
$\mathbb{RP}^n \to \mathbb{R}^N$. More exactly M. Farber, S. Tabachnikov and S. Yuzvinsky proved the following theorem

**Theorem 1** ([16]). For $n \neq 1, 3, 7$ the (normalized) topological complexity of the $n$-dimensional real projective space $\mathbb{RP}^n$ coincides with the least integer $k$ such that $\mathbb{RP}^n$ can be immersed in $\mathbb{R}^k$.

Based on a Whitehead-type formulation of sectional category, in [19] we developed the notion of weak sectional category, which is an analogous notion to that of weak category, $\text{wcat}(-)$, in the sense of Berstein-Hilton. The weak sectional category of any map $p : E \to B$ turns out to be a better lower bound than the classical nil ker $p^*$, where nil ker $p^*$ denotes the nilpotency of the kernel of the morphism $p^*$ which is induced by $p$ in cohomology. Considering the weak sectional category of the fibration $\pi_X : X^I \to X \times X$ we immediately obtain the notion of weak topological complexity $wTC(-)$, a better lower bound for topological complexity than nil ker $\cup$, the index of nilpotency of the kernel of the cup product homomorphism considered by Farber. If $C_{\Delta_X}$ denotes the homotopy cofibre of the diagonal map $\Delta_X : X \to X \times X$, then it is proved in [19] that $wTC(X) = \text{wcat}(C_{\Delta_X})$. At the sight of this equality a natural question arises: is it true that $TC(X) = \text{cat}(C_{\Delta_X})$ holds for any space $X$? We do not know the answer to this general question. However, in this paper we will see that this equality holds in many cases.

The goal of this paper is to give some partial results on the comparison of the two invariants $TC(X)$ and $\text{cat}(C_{\Delta_X})$. In particular, we obtain inequalities between the two invariants under some restrictions on the dimension and connectivity of $X$ and we establish the equality for several examples including the spheres, the $H$-spaces, the real and complex projective spaces and the standard lens spaces. As a consequence we obtain that the immersion problem of $\mathbb{RP}^n$ is equivalent to the computation of $\text{cat}(C_{\Delta_{\mathbb{RP}^n}})$. We note that, along this paper, we work with the normalized version of all the invariants related to the topological complexity and Lusternik-Schnirelmann category.

## 1 Preliminary notions and results.

The aim of this section is to recall the most important notions and results that will be used in this paper. We assume that the reader is familiarized with the notions of homotopy commutative diagrams, homotopy pushouts and homotopy pullbacks; we refer the reader to [24], [8] or [6]. We also point out that the category in which we shall work throughout this paper is the category of well-pointed compactly generated Hausdorff spaces. Therefore all categorical constructions are carried out in this category.

Recall that, given any pair of maps $A \xrightarrow{f} C \xleftarrow{g} B$, the *join of $f$ and $g$*, $A \ast_C B$, is the homotopy pushout of the homotopy pullback of $f$ and $g$.
being the dotted arrow the corresponding co-whisker map, induced by the weak universal property of homotopy pushouts. When \( C = * \) we use the notation \( A \ast B \).

Notice that the map \( A \ast_C B \to C \) is only defined up to homotopy equivalence. Any representative is homotopy equivalent to the canonical co-whisker map obtained by considering first the standard homotopy pullback of \( f \) and \( g \) and then the standard homotopy pushout of the projections on \( A \) and \( B \). When \( f \) (or \( g \)) is a fibration our preferred representative of the join map \( A \ast_C B \to C \) will be given by the following explicit construction: we first take the honest pullback of \( f \) and \( g \) (which is a homotopy pullback) and then we take the standard homotopy pushout of the projections \( A \times_C B \to A \) and \( A \times_C B \to B \). In this way we obtain the following map:

\[
A \ast_C B = A \amalg (A \times_C B \times [0,1]) \amalg B / \sim \to C
\]

where \( \sim \) is given by \((a,b,t) \sim a\) if \( t = 0 \) and \((a,b,t) \sim b\) if \( t = 1 \). When \( f \) and \( g \) are both fibrations, this explicit construction coincides, up to homeomorphism, with the notion of sum of two fibrations used by Schwarz [26] and is a fibration whose fibre is the ordinary join of the fibres. Given a fibration \( p : E \to B \) we will call \( n \)-fold fiber join of \( p \) and denote it by \( j^n_p : \ast^n_B E \to B \) the fibration obtained by taking the join of \( n + 1 \) copies of \( p \). More precisely, we set \( j^0_p = p \), \( \ast^0_B E = E \) and \( j^n_p : \ast^n_B E \to B \) is the join of \( j^{n-1}_p \) and \( p \). If we denote by \( F \) the fibre of \( p \), the fibre of \( j^n_p \), denoted by \( \ast^n_F \), is the join of \( n + 1 \) copies of \( F \).

The following result, which is a consequence of the Join Theorem ([8]), will be useful for our purposes.

**Lemma 2.** Consider a (homotopy) pullback of a fibration \( p : E \to B \) along a map \( f : B' \to B \)

\[
\begin{array}{c}
E' \\
\downarrow^p \quad \downarrow^p \\
B' \quad \quad B
\end{array}
\]

\[
f'
\]

\[
\]

3
Then, for each \( n \geq 0 \) there is a (homotopy) pullback

\[
\begin{array}{ccc}
* \wedge^n B' & \rightarrow & * \wedge^n B
\\ \downarrow & & \downarrow
\\ B' & \rightarrow & B
\end{array}
\]

**Remark.** We point out that there is a similar result replacing (homotopy) pullback for just commutative square in the statement of the above lemma.

### 1.1 L.-S. category and topological complexity.

The (normalized) Lusternik-Schnirelmann category of a space \( X \), or L.-S. category for short, is the smallest integer \( n \) such that there is an open cover of \( X \) of \( n + 1 \) elements, each of which is contractible in \( X \). We write \( \text{cat}(X) = n \) if such integer exists; otherwise we state \( \text{cat}(X) = \infty \).

A useful lower bound for \( \text{cat}(X) \) is the cup-length of \( X \), \( \text{cup}(X) \). This is just the index of nilpotency of the reduced cohomology \( \tilde{H}^*(X) \) with coefficients in any ring \( R \).

On the other hand the (normalized) topological complexity of \( X \), \( \text{TC}(X) \), is the smallest integer \( n \) such that there is an open cover of \( X \times X \) of \( n + 1 \) elements, on each of which the fibration \( \pi_X : X^I \rightarrow X \times X \), \( \pi_X(\alpha) = (\alpha(0), \alpha(1)) \) admits a (homotopy) section. In [11], Farber established that, for any path-connected space \( X \), one has

\[
\text{cat}(X) \leq \text{TC}(X) \leq \text{cat}(X \times X) \leq 2\text{cat}(X) \tag{1}
\]

Also, if \( \text{nil ker } \cup \) denotes the index of nilpotency of \( \ker \cup \), where \( \cup : \tilde{H}^*(X) \otimes \tilde{H}^*(X) \rightarrow \tilde{H}^*(X) \) is the usual cup-product of the reduced cohomology with coefficients in a field \( K \), then Farber proved that \( \text{nil ker } \cup \) is a lower bound of \( \text{TC}(X) \).

L.-S. category and Topological Complexity are both special cases of the notion of sectional category (or genus as first introduced by A. Schwarz in [26]) of a fibration \( p : E \rightarrow B \). This invariant, written \( \text{secat}(p) \), is the smallest integer \( n \) such that there is an open cover of \( B \) of \( n + 1 \) elements, on each of which the fibration \( p \) admits a (homotopy) section. We have \( \text{cat}(X) = \text{secat}(ev : PX \rightarrow X) \) and \( \text{TC}(X) = \text{secat}(\pi_X : X^I \rightarrow X \times X) \). Here, \( PX \subset X^I \) is the space of paths beginning at the base point * and \( ev = ev_1 \) is the evaluation map at the end of the path.

The sectional category of a fibration \( p : E \rightarrow B \) can be characterized in terms of the \( n \)-fold fiber join of \( p \) (see [22]):

**Theorem 3.** If \( B \) is a paracompact space, then \( \text{secat}(p) \leq n \) if and only if 

\[
\begin{array}{ccc}
* \wedge^n B & \rightarrow & * \wedge^n B
\\ \downarrow & & \downarrow
\\ B & \rightarrow & B
\end{array}
\]

In particular, considering \( p = ev : PX \rightarrow X \) we have

\[
\text{cat}(X) \leq n \iff j^*_p : * \wedge^n PX \rightarrow X \text{ admits a homotopy section};
\]
and considering \( p = \pi : X^I \to X \times X \) we have

\[
\text{TC}(X) \leq n \iff j^n_\pi : *_{X \times X}^X \to X \times X \text{ admits a homotopy section.}
\]

Both these fibrations have as fibre the join \(*^n\Omega X\), where \( \Omega X \) denotes the loop space of \( X \). Note that the fibration \( j^n_\pi : *^nPX \to X \) is equivalent to the so-called \( n\)-th Ganea fibration \( g_n : G_n(X) \to X \). Recall that \( g_0 = ev : PX \to X \) and \( g_n \)

is the fibration associated to the join of \( g_{n-1} \) with the trivial map \(* \to X\). These fibrations fit in a homotopy commutative diagram of the following form

\[
\begin{array}{cccccc}
F_0(X) & F_1(X) & F_2(X) & \ldots \\
\downarrow & \downarrow & \downarrow & \\
G_0(X) & G_1(X) & G_2(X) & \ldots \\
\downarrow g_0 & \downarrow g_1 & \downarrow g_2 & \\
X & X & X & \ldots
\end{array}
\]

where each column is a fibration and for each \( F_n(X) \to G_n(X) \to G_{n+1}(X) \) is a homotopy cofibration sequence, the so-called fibre-cofibre construction. Numerous lower bounds that approximate the L.-S. category have been defined in terms of Ganea fibrations. We refer the reader to [7] for more information on L.-S. category and its approximations. In this work we will also use the following lower bound, denoted by \( \sigma^1\text{cat}(X) \):

\[
\sigma^1\text{cat}(X) \leq n \iff \Sigma g_n \text{ admits a homotopy section}
\]

where \( \Sigma \) denotes the usual suspension functor.

### 1.2 Weak L.-S. category and weak topological complexity.

Now we recall the Whitehead characterization of L-S category for a reasonable non-restrictive class of spaces. Consider \( B \) a well-pointed, path-connected normal space and \( n \geq 0 \). Then \( \text{cat}(B) \leq n \) if and only if there is, up to homotopy, a lift of the \((n+1)\)-diagonal map

\[
\begin{array}{c}
T^n(B) \\
\downarrow \\
\Lambda \delta_{n+1} \\
\Delta \\
B \Delta_{n+1} \nabla_{n+1}
\end{array}
\]

where \( T^n(B) = \{(b_0, b_1, \ldots, b_n) \in B_{n+1}^n : b_i = *, \text{ for some } i\} \) is the \( n \)-th fatwedge. Take \( \delta_{n+1} = B_{n+1} \text{ the } (n+1)\)-fold smash-product and \( \Delta_{n+1} = \nabla_{n+1} : B \to B_{n+1} \text{ the reduced diagonal. Then the weak category of } B, \text{ weat}(B), \)
(in the sense of Berstein and Hilton [5]), is the least integer \( n \) such that \( \Delta_{n+1} \) is homotopically trivial. Obviously this is a lower bound for the L.-S. category of \( B \). Analogously, in [19] we considered for any map \( p : E \to B \) what we call the \( n \)-sectional fatwedge of \( p, \kappa_n : T^n(p) \to B^{n+1} \), which is a generalization of the classical \( n \)-th fatwedge. For more details we refer the reader to [19]. Then we prove that \( \text{scat}(p) \leq n \) if and only if \( \Delta_{n+1} : B \to B^{n+1} \) factorizes, up to homotopy, through \( T^n(p) \). If we consider \( T^n(p) \xrightarrow{\kappa_n} B^{n+1} \xrightarrow{l_n} C_{\kappa_n} \) the homotopy cofiber of \( \kappa_n \), then the weak sectional category of \( p \), \( w\text{scat}(p) \), is the least integer \( n \) (or \( \infty \)) such that the composition \( B \xrightarrow{\Delta_{n+1}} B^{n+1} \xrightarrow{l_n} C_{\kappa_n} \) is homotopically trivial.

By direct arguments, the weak sectional category is a lower bound of the sectional category. The notion of weak topological complexity, \( w\text{TC}(X) \), arises naturally as the weak sectional category of \( \pi_X : X^I \to X \times X \). It is proved in [19] the following result:

**Theorem 4.** Let \( X \) be any space. Then

(a) \( w\text{TC}(X) \leq \text{wcat}(X \times X) \)

(b) \( w\text{TC}(X) \geq \text{nil ker } \cup = \text{cup}(C_{\Delta_X}) \)

(c) \( w\text{TC}(X) = \text{wcat}(C_{\Delta_X}) \)

Here \( C_{\Delta_X} \) denotes the homotopy cofiber of the diagonal map \( \Delta_X : X \to X \times X \).

**2 Some majorations of \( \text{TC}(X) \).**

As we have recalled in (1) of the previous section, there is an inequality

\[
\text{TC}(X) \leq \text{cat}(X \times X).
\]

In many cases, this inequality turns out to be strict. For instance, for \( S^{2n+1} \) an odd-dimensional sphere, \( \text{TC}(S^{2n+1}) = 1 \) while \( \text{cat}(S^{2n+1} \times S^{2n+1}) = 2 \). One of the objectives of many works on Topological Complexity (see, for instance [15], [21]) is to provide alternative ways to get an upper bound for \( \text{TC} \). Here we will give two results in this direction. We begin with an adaptation for \( \text{TC} \) of a result established in [18] for the L.-S. category:

**Theorem 5.** Let \( f : X \to Y \) be a map between \((q-1)\)-connected CW-complexes \((q \geq 1)\). If \( f \) is an \( r \)-equivalence and \( 2 \dim X \leq r + q \text{TC}(Y) - 1 \), then \( \text{TC}(X) \leq \text{TC}(Y) \).

**Proof.** We denote by \( f^I : X^I \to Y^I \) and \( \Omega f : \Omega X \to \Omega Y \) the maps induced by \( f \). Suppose that \( \text{TC}(Y) \leq n \) and construct the \( n \)-fold join (recall that this means the
join with \( n + 1 \) factors) of the fibrations \( \pi_X : X^f \to X \times X \) and \( \pi_Y : Y^f \to Y \times Y \).

We obtain the following commutative diagram:

\[
\begin{array}{ccc}
*^n \Omega X & \overset{*^n \Omega f}{\longrightarrow} & *^n \Omega Y \\
\downarrow & & \downarrow \\
*^n X \times X X^f & \overset{*^n f \times f}{\longrightarrow} & *^n Y \times Y Y^f \\
\downarrow & & \downarrow \\
X \times X & \overset{f \times f}{\longrightarrow} & Y \times Y.
\end{array}
\]

Taking the pullback \( Q \) of the fibration \( *^n f \times f Y^f \to Y \times Y \) along the map \( f \times f \) we obtain:

\[
\begin{array}{ccc}
*^n \Omega X & \overset{*^n \Omega f}{\longrightarrow} & *^n \Omega Y \\
\downarrow & & \downarrow \\
*^n X \times X X^f & \overset{Q}{\longrightarrow} & *^n Y \times Y Y^f \\
\downarrow & & \downarrow \\
X \times X & \overset{f \times f}{\longrightarrow} & Y \times Y.
\end{array}
\]

By the homotopy exact sequence of a fibration one can check that the map \( *^n f \times f X^f \to Q \) has the same connectivity as \( *^n \Omega f \). (We point out that \( Q \) is connected. Indeed, when \( n \geq 1 \), this is certainly true as the fibre \( *^n \Omega Y \) is connected; when \( n = 0 \) we have \( \text{TC}(Y) = 0 \) so \( Y \) is contractible and therefore \( Q \simeq X \times X \) is connected.) The map \( *^n \Omega f \) is an \((r+gn-1)\)-equivalence since \( f \) is an \( r \)-equivalence (see, for example, [18]).

Now, considering \( \text{TC}(Y) \leq n \) and the pullback property we have that the fibration \( Q \to X \times X \) admits a section \( \sigma \). Finally, since the integer \( \dim(X \times X) \) is less or equal to the connectivity of \( *^n \Omega f \), the section \( \sigma \) lifts in a homotopy section of \( *^n f \times f X^f \to X \times X \). Hence \( \text{TC}(X) \leq n \).

We give the following examples of applications to Theorem 5.

**Example 6.** Let \( \mathbb{R}P^n = S^n / \mathbb{Z}_2 \) be the real \( n \)-dimensional projective space. If \( n \leq k \), then \( \text{TC}(\mathbb{R}P^n) \leq \text{TC}(\mathbb{R}P^k) \).

This result already appears in [16]. Here we give a proof, which does not depend on Theorem 1.

**Proof.** The inclusion \( f : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^k \) is an \( n \)-equivalence. If \( k \geq n+1 \), then we have \( \text{TC}(\mathbb{R}P^k) \geq \text{cat}(\mathbb{R}P^k) = k \geq n + 1 \). Then, taking \( q = 1 \), we have \( 2 \dim(\mathbb{R}P^n) = 2n \leq n+k-1 \) and Theorem 5 permits to conclude that \( \text{TC}(\mathbb{R}P^n) \leq \text{TC}(\mathbb{R}P^k) \).

**Example 7.** Similarly, consider \( L_p^{2n+1} = S^{2n+1} / \mathbb{Z}_p \) the \((2n+1)\)-dimensional lens space. If \( n \leq k \), then \( \text{TC}(L_p^{2n+1}) \leq \text{TC}(L_p^{2k+1}) \).
Proof. Analogous to the previous example. Observe that \( \text{cat} (L_p^{2k+1}) = 2k + 1 \) (see lemma 21 below). In actual fact, by lemma 21 one can prove that \( \text{TC} ((L_p^\infty)^{(n)}) \leq \text{TC} ((L_p^\infty)^{(k)}) \), for \( n \leq k \), where \( (L_p^\infty)^{(m)} \) denotes the \( m \)-skeleton of the infinite lens space \( L_p^\infty \).

The following example shows how Theorem 5 could help to compute the topological complexity of a space in a concrete situation:

**Example 8.** Let \( X = S^3 \cup \alpha e^7 \) where \( \alpha \) is the Blakers-Massey element of \( \pi_6 (S^3) \). Then we have that \( \text{TC} (X) = 3 \).

Proof. Recall that the symplectic group \( Sp(2) \) admits a cellular decomposition of the form \( S^3 \cup e^7 \cup e^{10} \). Then \( X = S^3 \cup e^7 \) is a subcomplex of \( Sp(2) \) such that the inclusion \( f : X \hookrightarrow Sp(2) \) a 9-equivalence. By [27] we know that \( \text{cat} (Sp(2)) = 3 \), and Farber showed in [12] that \( \text{TC} (G) = \text{cat} (G) \), for any Lie group \( G \). Therefore \( \text{TC} (Sp(2)) = 3 \), and Theorem 5 with \( q = 3 \) permits us to obtain the inequality \( \text{TC} (X) \leq 3 \). On the other hand, in [19] we proved that \( \text{wTC} (X) = 3 \). Since \( \text{wTC} (X) \leq \text{TC} (X) \) we conclude that \( \text{TC} (X) = 3 \).

Actually in the above example we have obtained an inequality of the form \( \text{TC} (X) \leq \text{cat} (Y) \). We shall now give a variation of Theorem 5, in which we consider in its statement the L.-S. category of \( Y \) instead of its topological complexity. This result will be useful in order to get upper bounds for TC.

If \( X \) is a space with base point \( * \), then \( i_1, i_2 : X \to X \times X \) will denote the inclusions \( i_1(x) = (x,*) \), \( i_2(x) = (*,x) \).

**Theorem 9.** Let \( X \) be a \((q-1)\)-connected finite dimensional CW-complex \((q \geq 1)\). Suppose that there exists a map \( g : X \times X \to Y \) such that

(i) The composition \( X \xrightarrow{\Delta_X} X \times X \xrightarrow{g} Y \) is homotopically trivial;

(ii) the map \( f : X \xrightarrow{i_1} X \times X \xrightarrow{g} Y \) is an \( r \)-equivalence with \( r \geq q \) and \( 2 \dim X \leq r + q \text{cat} (Y) - 1 \);

Then \( \text{TC} (X) \leq \text{cat} (Y) \).

**Proof.** Since \( g\Delta_X \) is homotopically trivial there exists a commutative diagram

\[
\begin{array}{ccc}
X^I & \xrightarrow{g} & PY \\
\downarrow{\pi} & & \downarrow{ev_1} \\
X \times X & \xrightarrow{g} & Y.
\end{array}
\]

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Consider the following diagram in which the left-hand square is a pullback and a homotopy pullback

\[
\begin{array}{ccc}
PX & \xrightarrow{i_1} & X^I \\
\downarrow{ev_1} & & \downarrow{\pi} \\
X & \xrightarrow{\iota_1} & X \times X
\end{array}
\xrightarrow{g} \begin{array}{ccc}
PY \\
\downarrow{ev_1} & & \\
X \times X & \xrightarrow{g} & Y.
\end{array}
\]

By taking the fibres of the vertical maps, we see that the map \(\hat{g} : \Omega X \to \Omega Y\) induced by \(g\) and \(\bar{g}\) between the fibres is the same as the map \(\hat{f}\) induced by \(f = g\iota_1\) and \(\bar{f} = \bar{g}\iota_1\). On the other hand, it is possible to construct a fibrewise homotopy between \(\bar{f}\) and \(Pf : PX \to PY\) so that \(\hat{f}\) is homotopic to \(\Omega f\) and hence \(\hat{g} \simeq \Omega f\). Now we follow the same strategy as the one exhibited in the proof of Theorem 9. We suppose that \(\text{cat}(Y) \leq n\) and construct the \(n\)-fold join of the fibrations \(\pi : X^I \to X \times X\) and \(ev_1 : PY \to Y\). Then we take the pullback \(Q\) of the fibration \(*_Y^nPY \to Y\) along the map \(g\) and we obtain:

\[
\begin{array}{ccc}
*_n\Omega X & \xrightarrow{*^n\hat{g}} & *_n\Omega Y \\
\downarrow{} & & \downarrow{}
\end{array}
\xrightarrow{} \begin{array}{ccc}
*_nX \times X^I \\
\downarrow{} & & \downarrow{}
\end{array}
\xrightarrow{} \begin{array}{ccc}
Q \\
\downarrow{} & & \downarrow{}
\end{array}
\xrightarrow{} \begin{array}{ccc}
*_Y^nPY \\
\downarrow{} & & \downarrow{}
\end{array}
\xrightarrow{} \begin{array}{ccc}
X \times X \\
\downarrow{} & & \downarrow{}
\end{array}
\xrightarrow{} \begin{array}{ccc}
\rightarrow Y.
\end{array}
\]

Since \(*^n\hat{g} \simeq *^n\Omega f\) and \(f\) is an \(r\)-equivalence, the map \(*^n_{X \times X}X^I \to Q\) is an \((r + qn - 1)\)-equivalence. Now from \(\text{cat}(Y) \leq n\) and the condition on \(\dim(X \times X)\) we deduce the existence of a homotopy section of \(*^n_{X \times X}X^I \to X \times X\), that is, \(\text{TC}(X) \leq n\).

The first application of Theorem 9 will be given on subcomplexes \(X\) of a CW-complex \(H\) that it is also an \(H\)-space. When \(X = H\) we will obtain \(\text{TC}(H) \leq \text{cat}(H)\) which, together with the general inequality \(\text{cat} \leq \text{TC}\) gives \(\text{TC}(H) = \text{cat}(H)\). We note that this fact is also proved in [23] (in the context of higher analogs of Topological Complexity as defined in [25]) and generalizes the result by Farber that \(\text{TC}(G) = \text{cat}(G)\) for any Lie group \(G\) ([12]).

**Corollary 10.** Let \(H\) be a connected CW-complex which is an \(H\)-space (with \(*\) as unit element). Then

1. If \(X\) is a \((q - 1)\)-connected subcomplex of \(H\) such that the inclusion \(X \hookrightarrow H\) is an \(r\)-equivalence and \(2 \dim X \leq r + q \text{cat}(H) - 1\), then \(\text{TC}(X) \leq \text{cat}(H)\).

2. \(\text{TC}(H) = \text{cat}(H)\).
Proof. Since $H$ is a $H$-space that it is a connected CW-complex, the shear map
\[ \phi : H \times H \to H \times H \] given by \( \phi(x, y) = (x, xy) \) is a homotopy equivalence ([28, p. 461]). Let \( \psi \) a homotopy inverse of \( \phi \), which preserves the base point. The map
\[ j : H \to H \] given by \( j(x) = p_2 \psi(x, *) \) is a right inverse of the identity, where \( p_2 \) denotes the projection onto the second factor. This means that the composite
\[ H \xrightarrow{\varDelta} H \times H \xrightarrow{id \times j} H \times H \xrightarrow{\mu} H \]
where \( \mu \) is the multiplication, is homotopically trivial ([28, p. 119]). We note that the (homotopy) associativity is not required for this fact. Now consider the map \( g : X \times X \to H \) given by \( g(x, y) = \mu(x, j(y)) \). The condition (i) of Theorem 9 is satisfied. On the other hand, since the map \( f = g_1 \) is, up to homotopy, the inclusion \( X \hookrightarrow H \) the condition (ii) of Theorem 9 is also satisfied, so \( TC(X) \leq \text{cat}(H) \). If \( X = H \) the map \( f \) is homotopic to the identity and we obtain \( TC(H) \leq \text{cat}(H) \). The general inequality \( \text{cat} \leq TC \) allows us to conclude that \( TC(H) = \text{cat}(H) \).

As a second application of Theorem 9 we give a new proof of the following result, which has been established by Farber, Tabachnikov and Yuzvinsky in [16]. We note that Grant also gives in [21] a different proof of this result.

**Corollary 11.** Consider integers \( k > n \). If there is an axial map \( g : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^k \), then \( TC(\mathbb{R}P^n) \leq k \).

Recall that given integers \( k > n \) an axial map \( g : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^k \) of type \((n, k)\) is defined by the fact that the composites \( \mathbb{R}P^n \times \ast \to \mathbb{R}P^n \to \mathbb{R}P^k \) and \( \ast \times \mathbb{R}P^n \to \mathbb{R}P^n \to \mathbb{R}P^k \) are homotopic to the natural inclusion \( \mathbb{R}P^n \to \mathbb{R}P^k \). The real infinite-dimensional projective space \( \mathbb{R}P^\infty \), which is the colimit of the filtration
\[ \ast = \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \ldots \subset \mathbb{R}P^n \subset \mathbb{R}P^{n+1} \subset \ldots \]
(or, equivalently, the orbit space \( S^\infty/\mathbb{Z}_2 \)) is an Eilenberg-MacLane space of type \((\mathbb{Z}_2, 1)\) and therefore it is a loop space. Then an axial map of type \((n, k)\) is nothing else than a deformation into \( \mathbb{R}P^k \) of the restriction to \( \mathbb{R}P^n \times \mathbb{R}P^n \) of the product of \( \mathbb{R}P^n \). We also recall that any map \( h : \mathbb{R}P^n \to \mathbb{R}P^k \) with \( k > n \) is either homotopic to the inclusion or homotopically trivial because it is completely determined by the image of the generator \( x \in H^1(\mathbb{R}P^k, \mathbb{Z}_2) \) via \( h^* \). Namely, since \( j_k : \mathbb{R}P^k \to \mathbb{R}P^\infty \) is a \( k \)-equivalence and \( k > n \), the map \( h \) is completely determined by the map \( j_k h \), and through the correspondence \([A, \mathbb{R}P^\infty] = [A, K(\mathbb{Z}_2, 1)] = H^1(A, \mathbb{Z}_2)\) the maps \( j_k \) and \( j_k h \) correspond to the classes \( x \in H^1(\mathbb{R}P^k, \mathbb{Z}_2) \) and \( h^*(x) \in H^1(\mathbb{R}P^n, \mathbb{Z}_2) \).

**Proof of Corollary 11.** First we check that \( g_\Delta : \mathbb{R}P^n \to \mathbb{R}P^k \) is homotopically trivial. In order to do that it suffices to see that the image by \((\Delta_{\mathbb{R}P^n})^* \) of the generator \( x \in H^1(\mathbb{R}P^k, \mathbb{Z}_2) \) is 0. Consider the identification of \( H^1(\mathbb{R}P^k \times \mathbb{R}P^n, \mathbb{Z}_2) \) with \( \mathbb{Z}_2 \cdot x \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \otimes \mathbb{Z}_2 \cdot x \) and \((\Delta_{\mathbb{R}P^n})^* \) with the cup-product. Since \( g \) is an
axial map, we have that \( g^*(x) = x \otimes 1 + 1 \otimes x \). Therefore (\( \Delta_{\mathbb{R}P^n} \))\(^*\)\( g^* \)(\( x \)) = 2x = 0 and the condition (i) of Theorem 9 is satisfied. Taking into account that \( g \) is an axial map, the map \( f = gi_1 \) is homotopic to the inclusion \( \mathbb{R}P^n \hookrightarrow \mathbb{R}P^k \), so it is an \( n \)-equivalence. Finally, since \( \text{cat}(\mathbb{R}P^k) = k \) we have \( 2 \text{dim}(\mathbb{R}P^n) = 2n \leq n + k - 1 \) so, by Theorem 9, we conclude that \( \text{TC}(\mathbb{R}P^n) \leq k \).

3 \quad \text{TC}(X) \text{ and } \text{cat}(C_{\Delta X})

For a space \( X \), we shall denote by \( C_{\Delta X} \) the homotopy cofiber of the diagonal map \( \Delta_X : X \rightarrow X \times X \), that is \( C_{\Delta X} = X \times X \cup_{\Delta X} CX \). We also denote by \( \alpha : X \times X \rightarrow C_{\Delta X} \) the induced map. We observe that, for \( \text{locally equiconnected} \) spaces (i.e., those spaces for which \( \Delta_X \) is a cofibration) \( C_{\Delta X} \) is, up to homotopy equivalence, the quotient space \( C_{\Delta X} = (X \times X)/\Delta_X(X) \) being \( \alpha \) the projection map. The class of \( \text{locally equiconnected} \) spaces is not restrictive. For instance, the CW-complexes and the metrizable topological manifolds fit on such a class of spaces ([10, 9]).

In this section we will analyze the relationship between \( \text{TC}(X) \) and \( \text{cat}(C_{\Delta X}) \). Then we give some first examples of when the equality holds.

3.1 \quad \text{General results on } \text{TC}(X) \text{ and } \text{cat}(C_{\Delta X})

In many examples, that will be developed later, Theorem 9 will permit us to establish the inequality \( \text{TC}(X) \leq \text{cat}(C_{\Delta X}) \). However, in the general case, we have to consider a rather strong condition on the dimension of \( X \). Namely, taking \( g = \alpha : X \times X \rightarrow C_{\Delta X} \) and using the fact that the map \( gi_1 \) is a \((2q-1)\)-equivalence, we obtain the following statement as a direct consequence of Theorem 9:

**Corollary 12.** Let \( X \) be a \((q-1)\)-connected \((q \geq 1)\) finite dimensional CW-complex. If \( 2 \text{dim}(X) \leq 2q - 2 + q \text{cat}(C_{\Delta X}) \), then \( \text{TC}(X) \leq \text{cat}(C_{\Delta X}) \).

Now we turn to the inequality \( \text{cat}(C_{\Delta X}) \leq \text{TC}(X) \). First we note that in [2] M. Arkowitz and J. Strom proved that if \( A \xrightarrow{h} Y \xrightarrow{f} C_h \) is a cofibre sequence and \( f : X \rightarrow Y \) any map, then \( \text{secat}(af) \leq \text{secat}(f) + 1 \); in particular \( \text{cat}(C_h) = \text{secat}(ah) \leq \text{secat}(h) + 1 \). Specializing this result to \( X \xrightarrow{\Delta_X} X \times X \xrightarrow{\alpha} C_{\Delta X} \) we obtain the inequality

\[
\text{cat}(C_{\Delta X}) \leq \text{TC}(X) + 1
\]

without any assumptions on the space \( X \). This is essentially equivalent to Lemma 18.3 of [13]. In terms of open covers, this inequality can be thought of as follows. Suppose we have an open set \( U \subset X \times X \) together with a local section \( s \) of \( \pi : X^I \rightarrow X \times X \). For each \( (x,y) \in U \), \( s(x,y) \) gives rise to a path in \( X \times X \) from \( (x,y) \) to \( (x,x) \), which will be denoted by \( w(x,y) \). Then we can consider

\[
\text{cat}(C_{\Delta X}) \leq \text{TC}(X) + 1
\]
the open set \( \tilde{U} = U \cup_{\Delta_X} \Delta^{-1}_X(U) \times [0, \frac{1}{2}) \subset X \times X \cup_{\Delta_X} CX = C_{\Delta X} \), which is contractible in \( C_{\Delta X} \) (for \((x, y) \in U\), consider the concatenation of the paths \( w(x, y) \) and \( t \mapsto [x, t] \) and, for \([x, s] \in \Delta_X^{-1}(U) \times [0, \frac{1}{2})\), consider the concatenation of the three paths \( t \mapsto [x, (1-t)s] \), \( w(x, x) \) and \( t \mapsto [x, t] \)). Thus, a cover of \( X \times X \) by open sets of the type of \( U \) to together with the open set \( \{[x, t] \in CX \mid t \in (\frac{1}{3}, 1] \} \) gives a cover of \( C_{\Delta X} \) by contractible open sets in this space. Note that if the local section \( s \) satisfies the additional condition that \( s(x, x) \) is the constant path in \( x \), then we can consider the open set \( \tilde{U} \cup_{\Delta X} \Delta^{-1}_X(U) \), instead of \( \tilde{U} \). In these conditions, this set is contractible in \( C_{\Delta X} \) (for \([x, s] \in \Delta_X^{-1}(U) \) we can now simply consider the path \( t \mapsto [x, s + (1-s)t] \) and we do not need to add the open set \( \{[x, t] \in CX \mid t \in (\frac{1}{3}, 1] \} \) to get a cover of \( C_{\Delta X} \) by contractible open sets. As we will see, the additional condition on the local sections can be performed when \( X \) is an \( H \)-space but, in general, we need a condition on the dimension and connectivity of \( X \) in order to establish \( TC(X) \geq cat(C_{\Delta X}) \). In contrast it is easy to see that \( TC(X) \geq wc(C_{\Delta X}) \) and \( TC(X) \geq \sigma_1cat(C_{\Delta X}) \) without any extra condition. In summary, we shall prove the following theorem:

**Theorem 13.** Let \( X \) be a \((q-1)\)-connected CW-complex \((q \geq 1)\). The following statements hold:

1. \( wc(C_{\Delta X}) \leq TC(X) \).
2. \( \sigma_1cat(C_{\Delta X}) \leq TC(X) \).
3. If \( \text{dim}(X) \leq q(\text{TC}(X) + 1) - 2 \), then \( \text{cat}(C_{\Delta X}) \leq \text{TC}(X) \).
4. If \( X \) is an \( H \)-space, then \( \text{cat}(C_{\Delta X}) \leq \text{TC}(X) \).

**Proof.** (1) is a direct consequence from Theorem 4. For the remaining items we consider the following pushout diagram defining \( C_{\Delta X} \):

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times X \\
\downarrow & & \downarrow \alpha \\
CX & \xrightarrow{\chi} & C_{\Delta X}
\end{array}
\]

and the following commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\xi} & PC_{\Delta X} \\
\downarrow \pi & & \downarrow ev_1 \\
X \times X & \xrightarrow{\alpha} & C_{\Delta X}
\end{array}
\]

where the vertical maps are evaluation fibrations and the map \( \xi \) comes from the homotopy triviality of \( \alpha \pi \). Taking the \( n \)-th fibre join of the two vertical fibrations
we obtain the following diagram in which $j^n_{ev_1}$ is equivalent to the $n$-th Ganea fibration of $C_{\Delta X}$:

This diagram permits us to prove (2). Indeed, since $\Delta_X$ admits a retraction, the cofibre sequence

splits after suspension. Thus $\Sigma \alpha$ admits a homotopy section, and a homotopy section of $j^n_\pi$ will induce, after suspension, a homotopy section of $\Sigma j^n_\pi$. As $\Sigma j^n_\pi$ is equivalent to the suspension of the $n$-th Ganea fibration we get $\sigma^+ \text{cat}(C_{\Delta X}) \leq TC(X)$.

Now we pull back the fibrations $j^n_\pi$ and $j^n_{ev_1}$ along $\Delta_X$ and $\chi$ respectively. Observe that, by lemma 2, this operation is equivalent to pulling back the fibrations $\pi$ and $ev_1$ along $\Delta_X$ and $\chi$ and then taking the $n$-join of the induced fibrations.

Then we can obtain the following commutative cube, where $F$ is homotopically equivalent to $\Omega C_{\Delta X}$ and $\lambda$ is induced by $\xi$ by the pullback property:

Suppose that $TC(X) \leq n$ and let $s$ be a section of $j^n_\pi$. By the pullback along $\Delta_X$, $s$ induces a section $\bar{s}$ of $j^n_\pi$. First we prove that, if the map $\lambda \bar{s}$ is homotopically trivial, then $\text{cat}(C_{\Delta X}) \leq n$. Later we will see that the hypothesis given in the statements of 3 and 4 items permit to assume this fact. Being $\lambda \bar{s}$ homotopically trivial, it is possible to factorize $\lambda \bar{s}$ through the cone $CX$ giving the following
The maps $s$, $\bar{s}$ and $\tilde{s}$ induce a map $\sigma : C_{\Delta_X} \to *_{C_{\Delta_X}}^n PC_{\Delta_X}$, and the Gluing Lemma [3, II.1.2] implies that the composite $j_{\ev_1}^n \sigma$ is a homotopy equivalence. The composition of $\sigma$ with an inverse of this homotopy equivalence produces a homotopy section of $j_{\ev_1}^n$, concluding that $\text{cat}(C_{\Delta_X}) \leq n$.

It remains to see that the hypothesis given in the statements of 3 and 4 items ensure that $\lambda \bar{s}$ is homotopically trivial. For 3, since $X$ is $(q-1)$-connected we have that $*_{C_{\Delta_X}}^n F \simeq *_{\Omega C_{\Delta_X}}^n$ is $q(n+1) - 2$ connected. Therefore, the hypothesis on $\dim(X)$ implies immediately that the map $\lambda \bar{s}$ is homotopically trivial.

Before considering the last case (when $X$ is an $H$-space) we note that in general the canonical section $s_0$ of the fibration $j_{\pi}^n$, that is the section induced by the canonical section of $\pi$ that sends $x$ to the constant loop at $x$, satisfies $\lambda s_0 \simeq *$. It suffices to check that property at the first stage, for $n = 0$, and this is easily done through an explicit description of the map $\xi$. Now, when $X$ is an $H$-space, we consider, as in the proof of Corollary 10, a right inverse $j : X \to X$ of the identity and the map $g : X \times X \to X$ given by $g(x, y) = \mu(x, j(y))$ where $\mu$ is the multiplication. Since $g_{\Delta_X}$ is homotopically trivial we have the two following commutative diagrams:

$$
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times X \\
\downarrow & & \downarrow g \\
CX & \xrightarrow{H} & X \\
\end{array} \quad \quad \begin{array}{ccc}
X^I & \xrightarrow{g} & PX \\
\downarrow \pi & & \downarrow \ev_1 \\
X \times X & \xrightarrow{g} & X \\
\end{array}
$$
where $H$ is the null-homotopy and $\bar{g}$ is given, for $\beta \in X^I$, by
\[
\bar{g}(\beta)(t) = \begin{cases} 
H([\beta(1), 1 - 2t]) & 0 \leq t \leq \frac{1}{2} \\
\mu(\beta(2 - 2t), j(\beta(1))) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]
Here $[x, s]$ stands for the class of $(x, s)$ in $CX = X \times I / X \times 1$. Using the fact that $\bar{g}$ induces a homotopy equivalence between the fibres and that $g = p_1 \circ \theta$ where $p_1 : X \times X \to X$ is the projection on the first factor and $\theta : X \times X \to X \times X$ is the homotopy equivalence given by $\theta(x, y) = (g(x, y), y)$ we can check that the right-hand diagram is a homotopy pullback. We denote by $W$ the pullback of the fibration $ev_1 : PX \to X$ along the homotopy $H$. Then we have the following commutative cube in which $\gamma$ is induced by $\bar{g}$ and the vertical faces are homotopy pullbacks:

Since $CX$ is contractible the fibration $j^n_\pi W$ is homotopically equivalent to the trivial one $X \times *_{CX}^n W \to X$ and $*_{CX}^n W \simeq *^n \Omega X$. The canonical section $s_0$ of $j^n_\pi$ satisfies $\gamma s_0 \simeq *$ (it suffices to check it at the first level $n = 0$) and is actually the unique section, up to homotopy, having this property. Now we come back to our sections $s$ and $\bar{s}$. Since $X$ is an $H$-space we know that $TC(X) = \text{cat}(X)$ and thus we can suppose that $s$ is induced by a section of $j^n_{ev_1} : *_{PX}^n \to X$. Using the commutative cube above we see that this fact implies that $\bar{s}$ is induced by a section of the fibration $*_{CX}^n W \to CX$. This shows in particular that $\gamma \bar{s}$ factors through the cone $CX$ and therefore is homotopically trivial. As a consequence $\bar{s}$ must be homotopic to the canonical section $s_0$ and $\lambda \bar{s} \simeq \lambda s_0$ is homotopically trivial. 

**Remark.** When $X$ is a topological group (with unit element as the base point) we can give a more direct argument of point 4. Indeed, we can consider $g(x, y) = x \cdot y^{-1}$. Assuming that $TC(X) \leq n$ we know that $\text{cat}(X) \leq n$. If $\zeta : V \to PX$ is a local pointed section of $ev_1 : PX \to X$, then we obtain on $U = g^{-1}(V)$ a local section $\zeta'$ of $\pi : X^I \to X \times X$ given by $\zeta'(x)(t) = \zeta(x \cdot y^{-1})(1 - t) \cdot y$.

This local section sends $(x, x)$ to the constant path in $x$. Therefore, as mentioned before the statement of Theorem 13, the open set $U \cup_{\Delta_x} C\Delta_X^{-1}(U)$ is contractible.
in $C_{\Delta X}$, and a cover of $X$ by $n+1$ open sets, each of which is contractible in $X$, induces a cover of $C_{\Delta X}$ by $n+1$ open sets, each of which is contractible in $C_{\Delta X}$. Hence $\text{cat}(C_{\Delta X}) \leq n$. Note that this argument uses the associativity of the multiplication while the argument given in the proof below does not require any associativity condition.

### 3.2 First examples of the equality $\text{TC}(X) = \text{cat}(C_{\Delta X})$

(i) The first example is quite trivial. If $X$ is contractible, then $\text{TC}(X) = \text{cat}(C_{\Delta X})$. It suffices to note that in this case $C_{\Delta X}$ is also contractible.

(ii) For any $n \geq 1$, $\text{TC}(S^n) = \text{cat}(C_{\Delta S^n})$.

As Farber showed in [11], $\text{TC}(S^n)$ is 1 if $n$ is odd and 2 if $n$ is even. In [19] we established that $C_{\Delta S^n}$ is homotopy equivalent to $S^n \cup \{\iota_n, \iota_n\} \times S^n$ where $\iota_n$ denotes the homotopy class of the identity of $S^n$ and $[\iota_n, \iota_n]$ the Whitehead product. Using the results of [5] together with the classical results on the Hopf invariant of the Whitehead product $[\iota_n, \iota_n]$, it is easy to see that the category of this space is also 1 if $n$ is odd and 2 if $n$ is even. Thus we have the equality $\text{TC}(S^n) = \text{cat}(C_{\Delta S^n})$. Note that Theorem 9 together with the knowledge of $\text{cat}(C_{\Delta S^n})$ and of $\text{nil ker} \cup$ for the spheres permit us to recover the value of $\text{TC}$ for these spaces.

(iii) If $X$ is a $H$-space that is a connected CW-complex, then $\text{TC}(X) = \text{cat}(C_{\Delta X})$.

As we have previously done, we consider a right inverse $j: X \to X$ of the identity and the map $g: X \times X \to X$ given by $g(x, y) = \mu(x, j(y))$ where $\mu$ is the multiplication. Since $g\Delta X$ is trivial $g$ factors, up to homotopy, through $C_{\Delta X}$:

$$\begin{align*}
X & \xrightarrow{i_1} X \times X \xrightarrow{g} X \\
\alpha \downarrow & \quad g \\
C_{\Delta X} & \xrightarrow{\tilde{g}} X
\end{align*}$$

Since $gi_1 \simeq id_X$ we obtain $\tilde{g}\alpha i_1 \simeq id_X$. In other words, $X$ is a homotopy retract of $C_{\Delta X}$ and therefore $\text{cat}(X) \leq \text{cat}(C_{\Delta X})$. Since $\text{cat}(X) = \text{TC}(X)$ by Corollary 10, we get the inequality $\text{TC}(X) \leq \text{cat}(C_{\Delta X})$. The other inequality follows from Theorem 13.

(iv) Let $\mathbb{C}P^n$ denote the $n$-th complex projective space. For any $n \geq 1$ we have that $\text{TC}(\mathbb{C}P^n) = \text{cat}(C_{\Delta \mathbb{C}P^n})$. Indeed, by Farber ([11]) we know that $\text{TC}(\mathbb{C}P^n) = \text{nil ker} \cup = 2n$.

In [19] we established that, for any space $X$, $\text{nil ker} \cup \leq w\text{TC}(X) = w\text{cat}(C_{\Delta X})$. Thus we obtain $w\text{cat}(C_{\Delta \mathbb{C}P^n}) \geq 2n$ and therefore $\text{cat}(C_{\Delta \mathbb{C}P^n}) \geq 2n$. Since $\mathbb{C}P^n$ is 1-connected and $2n$-dimensional we have, by Theorem 13, $\text{cat}(C_{\Delta \mathbb{C}P^n}) \leq \text{TC}(\mathbb{C}P^n) = 2n$. In conclusion, $\text{cat}(C_{\Delta \mathbb{C}P^n}) = \text{TC}(\mathbb{C}P^n) = 2n$. 

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(v) In general, $\text{TC}(X) = \text{cat}(C_{\Delta X})$ for $X$ any closed, $2n$-dimensional, 1-connected symplectic manifold. Again $\text{TC}(X) = \text{nil ker} \cup = 2n$ ([16]) and by Theorem 13, $\text{cat}(C_{\Delta X}) = \text{TC}(X) = 2n$.

The argument in the previous two examples can be summarized as follows: if $X$ is a $(q - 1)$-connected CW-complex such that $\text{TC}(X) = \text{nil ker} \cup$ and $\dim(X) \leq q(\text{TC}(X) + 1) - 2$, then one has $\text{TC}(X) = \text{cat}(C_{\Delta X})$. Using this observation, we obtain the following results:

(vi) For any compact orientable surface of genus $g$, $X = \Sigma_g$, one has $\text{TC}(X) = \text{cat}(C_{\Delta X})$.

Here we use the result of [11] that, $\text{TC}(\Sigma_g) = \text{nil ker} \cup = \begin{cases} 2 & \text{if } g \leq 1 \\ 4 & \text{if } g > 1. \end{cases}$

Notice that the condition $\dim(X) \leq q(\text{TC}(X) + 1) - 2$ is not satisfied when $g = 1$ but, in this case, the result follows from Example (iii) since $\Sigma_1 = S^1 \times S^1$ is an $H$-space.

(vii) For any connected finite graph $X$, one has $\text{TC}(X) = \text{cat}(C_{\Delta X})$.

Here we use the result of [12] that $\text{TC}(X) = \text{nil ker} \cup = \begin{cases} 0 & \text{if } b_1(X) = 0 \\ 1 & \text{if } b_1(X) = 1 \\ 2 & \text{if } b_1(X) > 1, \end{cases}$

where $b_1(X)$ is the first Betti number. Notice that the condition $\dim(X) \leq q(\text{TC}(X) + 1) - 2$ is not satisfied in the first two cases but, in these cases, the result follows from Examples (i) and (ii) since $X$ is either contractible or homotopy equivalent to $S^1$.

(viii) Let $X = F(\mathbb{R}^m, n)$ be the space of configurations of $n$ distinct points in $\mathbb{R}^m$ with $m \geq 2$ and $n \geq 2$. One has $\text{TC}(X) = \text{cat}(C_{\Delta X})$.

Here we use the fact that $X$ is an $(m-2)$-connected space, which is homotopy equivalent to a finite CW-complex of dimension less or equal to $(m-1)(n-1)$, and the result by Farber-Yuzvinsky [17] and Farber-Grant [15] that

$$\text{TC}(X) = \text{nil ker} \cup = \begin{cases} 2n - 2 & \text{if } m \text{ odd} \\ 2n - 3 & \text{if } m \text{ even}. \end{cases}$$

The only case in which the condition $\dim(X) \leq q(\text{TC}(X) + 1) - 2$ is not satisfied is when $n = m = 2$ but, in this case $X = F(\mathbb{R}^2, 2) \simeq S^1$ and the result follows from Example (ii). We observe that, in general, $F(\mathbb{R}^2, n)$ is homotopy equivalent to $X \times S^1$, where $X$ is a finite polyhedron of dimension less or equal to $n - 2$ ([17]).
4 The case of real projective spaces and standard lens spaces.

This section is devoted to establishing the equality $\text{TC}(X) = \text{cat}(\Delta_X)$ for these special cases. In the case of the lens spaces we have to consider a slight restriction.

The infinite real projective space $\mathbb{R}P^\infty$ and the infinite $p$-torsion lens space $L^\infty_p$ (see below) can be identified with the Milnor classifying spaces $B\mathbb{Z}_2$ and $B\mathbb{Z}_p$, respectively. When $X$ is the classifying space $BG$ of a group $G$, the $n$-th Ganea fibration of $X = BG$ and the inclusion $B_nG \hookrightarrow BG$ of the $n$-th stage of the classifying construction are related. More precisely we will use the following result:

**Theorem 14.** Let $G$ be a group such that the Milnor universal $G$-bundle $EG \to BG$ is equivalent, as a principal $G$-bundle, to a fibration. Then there exists, for each $n \geq 0$, a homotopy commutative diagram:

$$
\begin{array}{ccc}
B_nG & \xrightarrow{g_n} & B_nG \\
\downarrow & & \uparrow \\
BG & \xrightarrow{} & BG \\
\end{array}
$$

For the convenience of the reader we include a proof of this result in an appendix (see Section 5). Here we just recall that the $G$-principal bundle $EG \to BG$ can be constructed as follows: consider the cone $CG = G \times I/G \times 0$ and, for each $n \geq 0$, consider the subspace $E_nG \subset (CG)^{n+1}$ consisting of the elements $((g_0, t_0), \ldots, (g_n, t_n))$ such that $\sum t_i = 1$. Thus $E_nG \subset E_{n+1}G$ (through the identification $((g_0, t_0), \ldots, (g_n, t_n)) = ((e, 0), (g_0, t_0), \ldots, (g_n, t_n))$ where $e$ is the unit element and the base point of $G$) and $EG$ is defined as $\bigcup_n E_nG$ equipped with the weak topology. The elements of $E_nG$ (resp. $EG$) are written as $\sum_{i=0}^n g_i t_i$ (resp. $\sum_{i \geq 0} g_i t_i$). The spaces $B_nG$, $BG$ are defined as the orbits spaces $E_nG/G$, $EG/G$ with respect to the action $g \cdot \sum g_i t_i = \sum g \cdot g_i t_i$. There is, for each $n$, a commutative diagram of $G$-principal bundles

$$
\begin{array}{ccc}
E_nG & \xrightarrow{} & EG \\
\downarrow & & \downarrow \\
B_nG & \xrightarrow{} & BG \\
\end{array}
$$

which is a pullback.

4.1 Real projective spaces

We first recall that the covering maps $S^n \to S^n/\mathbb{Z}_2 = \mathbb{R}P^n$ and $S^\infty \to S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$ coincide with the $\mathbb{Z}_2$-principal bundles $E_n\mathbb{Z}_2 \to B_n\mathbb{Z}_2$ and $E\mathbb{Z}_2 \to B\mathbb{Z}_2$,.
which turn out to be fibrations. Since the inclusion $B_n \hookrightarrow B\mathbb{Z}_2$ coincides with the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$, Theorem 14 ensures the existence of a homotopy commutative diagram of the form:

$$
\begin{array}{ccc}
\mathbb{R}P^n & \xrightarrow{G_n} & \mathbb{R}P^n \\
& \downarrow{g_n} & \\
\mathbb{R}P^\infty & \xrightarrow{j_n} & \mathbb{R}P^\infty
\end{array}
$$

**Remark 15.** Actually, the inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ is, up to homotopy equivalence, the Ganea fibration $G_n(\mathbb{R}P^\infty) \rightarrow \mathbb{R}P^\infty$ since $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ can be obtained from $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^\infty$ through the fibre-cofibre construction.

For our main result we shall also deal with $\mathbb{Z}_2$-axial maps.

**Definition 16.** A map $a: \mathbb{R}P^n \times \mathbb{R}P^n \rightarrow \mathbb{R}P^k$ is said to be $\mathbb{Z}_2$-axial when the composite

$$
\begin{array}{ccc}
\mathbb{R}P^n \times \mathbb{R}P^n & \xrightarrow{a} & \mathbb{R}P^k \\
f \downarrow{f} & & \downarrow{f} \\
\mathbb{R}P^\infty \times \mathbb{R}P^\infty & \xrightarrow{m} & \mathbb{R}P^\infty
\end{array}
$$

classifies the $\mathbb{Z}_2$-cover $S^n \times S^n \rightarrow \mathbb{R}P^n \times \mathbb{R}P^n$. In other words, there is a homotopy commutative diagram of the form

$$
\begin{array}{ccc}
\mathbb{R}P^n \times \mathbb{R}P^n & \xrightarrow{a} & \mathbb{R}P^k \\
& \downarrow{f} & \downarrow{f} \\
\mathbb{R}P^\infty \times \mathbb{R}P^\infty & \xrightarrow{m} & \mathbb{R}P^\infty
\end{array}
$$

where $m$ stands for the multiplication coming from the $H$-group structure of $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1) = \Omega K(\mathbb{Z}_2, 2)$.

By easy integral homological arguments we have that, necessarily, $k \geq n$. For $k > n$ a $\mathbb{Z}_2$-axial map is nothing else but an axial map of type $(n, k)$. We are interested in the case $k = n$. We know that $\mathbb{R}P^n$ has an $H$-space structure for $n = 1, 3, 7$, so there exists a $\mathbb{Z}_2$-axial map $\mathbb{R}P^n \times \mathbb{R}P^n \rightarrow \mathbb{R}P^n$ in these cases. More is true.

**Lemma 17.** There are not $\mathbb{Z}_2$-axial maps $a: \mathbb{R}P^n \times \mathbb{R}P^n \rightarrow \mathbb{R}P^n$, for $n \neq 1, 3, 7$.

**Proof.** Suppose that there exists such map for some $n \neq 1, 3, 7$. We consider $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$ the mod 2 cohomology ring of $\mathbb{R}P^n$, being $x$ the generator of degree 1. We also denote by $x_i$ the corresponding element in the $i$-th factor of $\mathbb{R}P^n \times \mathbb{R}P^n$. From $a^*(x) = x_1 + x_2$ we obtain $0 = a^*(x^{n+1}) = (x_1 + x_2)^{n+1}$ so that 2 divides $\binom{n+1}{i}$, for $i = 1, \ldots, n$. Then $n + 1$ must be a power of 2 and therefore $n$ is odd.

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Now take the following homotopy commutative diagram for each $k = 1, 2$:

\[
\begin{array}{cccccc}
\mathbb{R}P^n & \xrightarrow{i_k} & \mathbb{R}P^n \times \mathbb{R}P^n & \xrightarrow{a} & \mathbb{R}P^n \\
\downarrow{j_n} & & \downarrow{j_n} & & \downarrow{j_n} \\
\mathbb{R}P^\infty & \xrightarrow{i_k} & \mathbb{R}P^\infty \times \mathbb{R}P^\infty & \xrightarrow{m} & \mathbb{R}P^\infty
\end{array}
\]

Since $mi_k \simeq id$ then we obtain a homotopy commutative diagram

\[
\begin{array}{c}
\mathbb{R}P^n \\
\downarrow{j_n} \\
\mathbb{R}P^\infty
\end{array} \xrightarrow{a_{i_k}}
\begin{array}{c}
\mathbb{R}P^n \\
\downarrow{j_n} \\
\mathbb{R}P^\infty
\end{array}
\]

Passing to integral $n$-homology $H_n(\mathbb{R}P^n)$ we obtain the commutativity

\[
\begin{array}{cccc}
H_n(\mathbb{R}P^n) & \xrightarrow{(a_{i_k})_*} & H_n(\mathbb{R}P^n) \\
(j_n)_* & & (j_n)_* \\
H_n(\mathbb{R}P^\infty) & & Z & \xrightarrow{=} & Z & \xrightarrow{q} Z_2
\end{array}
\]

In particular we have that, necessarily, $\deg(a_{i_k}) \equiv 1 \pmod{2}$, and therefore $\deg(a_{i_k}) = d_k$ is an odd integer, $k = 1, 2$. Being $q : S^n \to \mathbb{R}P^n$ a covering map and $S^n \times S^n$ a simply connected space, we can consider a lift

\[
\begin{array}{ccc}
S^n \times S^n & \xrightarrow{\varphi} & S^n \\
q \times q & & q \\
\mathbb{R}P^n \times \mathbb{R}P^n & \xrightarrow{a} & \mathbb{R}P^n
\end{array}
\]

But then $\deg(\varphi_{i_k}) = d_k$ since we have the commutativity

\[
\begin{array}{ccc}
S^n & \xrightarrow{\varphi_{i_k}} & S^n \\
q & & q \\
\mathbb{R}P^n & \xrightarrow{a_{i_k}} & \mathbb{R}P^n
\end{array} \xrightarrow{(\varphi_{i_k})_*} \begin{array}{ccc}
H_n(S^n) & \xrightarrow{(\varphi_{i_k})_*} & H_n(S^n) \\
q_* & & q_* \\
H_n(\mathbb{R}P^n) & \xrightarrow{(a_{i_k})_*} & H_n(\mathbb{R}P^n)
\end{array}
\]

and $q_* = H_n(q)$ injective. Hence, the bidegree of $\varphi$ is $(d_1, d_2)$. Observe that, in order to check that $q_*$ is injective we just have to consider the exact homology sequence associated to the cofibration $S^n \xrightarrow{q} \mathbb{R}P^n \xrightarrow{i} \mathbb{R}P^{n+1}$ and take into account that $H_{n+1}(\mathbb{R}P^{n+1}) = 0$. 

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On the one hand, from the existence of $\varphi$ we have that, if $i \in \pi_n(S^n)$ is the generator, the following Whitehead product vanishes

$$0 = [d_1i, d_2i] \in \pi_{2n-1}(S^n)$$

On the other hand, since $n$ is odd with $n \neq 1, 3, 7$, then $[i, i] \neq 0$ and has order two by J.F. Adams [1]. Being odd the product $d_1d_2$, we have that $[d_1i, d_2i] = d_1d_2[i, i] \neq 0$, which is a contradiction.

**Theorem 18.** For any $n \geq 1$ we have the equality

$$TC(\mathbb{R}P^n) = \text{cat}(C_{\Delta_{\mathbb{R}P^n}}).$$

**Proof.** For $\mathbb{R}P^1 = S^1$, $\mathbb{R}P^3$ and $\mathbb{R}P^7$, the result follows from Section 3.2, examples (ii) and (iii). Suppose that $n \neq 1, 3, 7$. By [16] we know that $TC(\mathbb{R}P^n) \geq n + 1$. Therefore, $\dim(\mathbb{R}P^n) = n \leq TC(\mathbb{R}P^n) + 1 - 2$ and Theorem 13 permits us to conclude that $TC(\mathbb{R}P^n) \geq \text{cat}(C_{\Delta_{\mathbb{R}P^n}})$. Now suppose that $\text{cat}(C_{\Delta_{\mathbb{R}P^n}}) = k$. Consider the restriction of the product $m : \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ to $\mathbb{R}P^n \times \mathbb{R}P^n$. As in the proof of Corollary 11 we can see that $m_{\Delta_{\mathbb{R}P^n}}$ is homotopically trivial and that $m$ induces a map $\tilde{m} : C_{\Delta_{\mathbb{R}P^n}} \to \mathbb{R}P^\infty$. Now, since $\text{cat}(C_{\Delta_{\mathbb{R}P^n}}) = k$, $\tilde{m}$ factors through the $k$-th Ganea fibration of $\mathbb{R}P^\infty$ and then through the inclusion $j_k : \mathbb{R}P^k \hookrightarrow \mathbb{R}P^\infty$. Therefore we obtain the following homotopy commutative diagram:

We observe from this diagram that $\tilde{m} \alpha$ is a $Z_2$-axial map. As there are not any $Z_2$-axial maps $\mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^k$ with $k \leq n$ except for $k = n = 1, 3$ or 7 (see lemma 17) then $k \geq n + 1$. Now consider Theorem 9 with $g = \tilde{m} \alpha$. Since $\alpha \Delta_{\mathbb{R}P^k}$ is homotopically trivial then so is $g \Delta_{\mathbb{R}P^k}$. Taking into account that $\tilde{m} \alpha i_1$ is an $n$-equivalence and $\text{cat}(\mathbb{R}P^k) = k \geq n + 1$, we have $2 \dim(\mathbb{R}P^n) \leq r + qk - 1$ where $q = 1$ and $r = n$. Therefore $TC(\mathbb{R}P^n) \leq k$. 

As a corollary of this result and of Theorem 1 we obtain the following characterization of the immersion problem of $\mathbb{R}P^n$ in terms of the Lusternik-Schnirelmann category.

**Theorem 19.** The immersion problem of $\mathbb{R}P^n$ is equivalent to the computation of $\text{cat}(C_{\Delta_{\mathbb{R}P^n}})$, for $n \neq 1, 3, 7$. 

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4.2 Standard lens spaces.

We will essentially follow the same strategy as the one given for the real projective spaces. Firstly, we recall the definition of lens spaces and develop the ingredients that will be used in the proof of Theorem 26 below.

Let \( p > 2 \) be an odd integer. For each \( n \geq 0 \), the cyclic group \( \mathbb{Z}_p \), identified with the multiplicative group \( \{1, \omega, \ldots, \omega^{p-1}\} \subset \mathbb{C} \) of \( p \)-th roots of unity, acts freely on \( S^{2n+1} \subset \mathbb{C}^{n+1} \) by pointwise multiplication. The corresponding orbit space is the lens space

\[
L_p^{2n+1} = S^{2n+1} / \mathbb{Z}_p
\]

We also consider the infinite-dimensional lens space

\[
L_\infty p = S_\infty / \mathbb{Z}_p
\]

There is a CW-decomposition of \( S_\infty \) (see [28]) inducing a CW-decomposition of \( S^{2n+1} \) and \( L_p^{2n+1} \), which is as follows. Consider for any \( k \geq 0 \),

\[
E^{2k} = \{(z_0, \ldots, z_k) \in S^{2k+1} : z_k \in \mathbb{R} \text{ and } z_k \geq 0\}
\]

\[
E^{2k+1} = \{(z_0, \ldots, z_k) \in S^{2k+1} : 0 \leq \arg(z_k) \leq \frac{2\pi}{p}\}
\]

Then the cells \( E^k \) and their images under the action of \( \mathbb{Z}_p \) give a CW-decomposition of \( S_\infty \), those of dimension \( \leq 2n + 1 \) giving one of \( S^{2n+1} \). The images of the cells \( E^k \) under the covering map \( q : S^{\infty} \to L_\infty p \) give a CW-decomposition of \( L_\infty p \). Hence, \( L_\infty p \) is a CW-complex with just one cell on each dimension. For each \( n \geq 0 \) we have that

\[
(L_\infty p)^{(2n+1)} = L_p^{2n+1}.
\]

Since, in this CW-decomposition of \( S_\infty \), the \( 2n + 2 \) skeleton can be identified to a cone over the \( 2n + 1 \) skeleton (which is \( S^{2n+1} \)), we can identify the covering map \( S^{2n+1} \to L_p^{2n+1} \) with the attaching map of the \( 2n + 2 \) dimensional cell in \( L_\infty p \). In other words, the following commutative diagram is a homotopy pushout:

\[
\begin{array}{ccc}
S^{2n+1} & \xrightarrow{q} & S^{2n+2} \\
\downarrow & & \downarrow \\
L_p^{2n+1} & \xrightarrow{q} & L_p^{(2n+2)}
\end{array}
\]

We also observe that the covering maps \( S^{2n+1} \to L_p^{2n+1} \) and \( S_\infty \to L_\infty p \) are fibrations and fit in a commutative diagram which is a pullback and a homotopy pullback:

\[
\begin{array}{ccc}
S^{2n+1} & \xrightarrow{q} & S_\infty \\
\downarrow & & \downarrow \\
(L_\infty p)^{(2n+1)} & \xrightarrow{q} & L_\infty p
\end{array}
\]
We now relate these maps to the Milnor universal $\mathbb{Z}_p$ principal bundle. For $n \geq 0$, the map
\[ E_{2n+1}^{2n+1} \mathbb{Z}_p \to \sum_{i=0}^{2n+1} g_it_i \mapsto (t_0\omega^g_0 + t_1\omega^g_1, \ldots, t_{2n}\omega^g_{2n} + t_{2n+1}\omega^g_{2n+1}) \]
is well-defined (since $p$ is odd, the vector $(t_0\omega^g_0 + t_1\omega^g_1, \ldots, t_{2n}\omega^g_{2n} + t_{2n+1}\omega^g_{2n+1})$ does not vanish) and is $\mathbb{Z}_p$-equivariant.

This map allows us to obtain the following commutative diagrams of principal $\mathbb{Z}_p$ bundles which are compatible with the inclusions:

\[ E_{2n+1} \mathbb{Z}_p \to S^{2n+1} \leftarrow B_{2n+1} \mathbb{Z}_p \to L_p^{2n+1} \]
\[ E\mathbb{Z}_p \to S^\infty \leftarrow B\mathbb{Z}_p \to L^\infty \]

Because these diagrams are pullbacks, the map $E\mathbb{Z}_p \to B\mathbb{Z}_p$ is a fibration. The homotopy equivalences in the right diagram come from the fact that $E\mathbb{Z}_p$ and $S^\infty$ are both contractible.

**Lemma 20.** For each $k \geq 0$ there is a homotopy commutative diagram

\[ (L^\infty)^{(k)} \to G_k(L^\infty_p) \to (L^\infty_p)^{(k)} \]

\[ L^\infty_p \leftarrow \gamma \leftarrow \gamma \]

**Proof.** The left part comes from the fact that $\text{cat}((L^\infty_p)^{(k)}) \leq \dim((L^\infty_p)^{(k)}) = k$. For $k = 0$, we also have the right part since $(L^\infty_p)^{(0)} = \ast$. Suppose now that $k = 2n + 1$ with $n \geq 0$. Using the map constructed above, the fact that $L^\infty_p$ and $B\mathbb{Z}_p$ are homotopy equivalent and Theorem 14, we obtain, for each $n \geq 0$, a homotopy commutative diagram of the following form, where the two first vertical maps are the Ganea fibrations and the two others are the inclusions.

\[ G_{2n+1}(L^\infty_p) \to G_{2n+1}(B\mathbb{Z}_p) \to B_{2n+1} \mathbb{Z}_p \to L_p^{2n+1} \]
\[ L^\infty_p \leftarrow \mathbb{Z}_p \leftarrow \mathbb{Z}_p \leftarrow L^\infty_p \]

By inverting the homotopy equivalences we obtain a homotopy commutative diagram
as follows

\[ G_{2n+1}(L_p^\infty) \longrightarrow L_p^{2n+1} = (L_p^\infty)^{(2n+1)} \]

\[ \downarrow g_{2n+1} \quad \downarrow \quad \downarrow \]

\[ L_p^\infty \]

which establishes the lemma for \( k = 2n + 1 \). Let \( F_{2n+1} \) be the homotopy fibre of \( g_{2n+1} \) and let \( \gamma_{2n+1} : F_{2n+1} \to G_{2n+1}(L_p^\infty) \) be the induced map. Recall that the \((2n+2)\)-th Ganea fibration is equivalent to the map which extends \( g_{2n+1} \) to the homotopy cofibre of \( \gamma_{2n+1} \) by sending the cone over \( F_{2n+1} \) on a point. On the other hand, from diagrams 3 and 2, we know that the homotopy fibre of the inclusion \( L_p^{2n+1} \to L_p^\infty \) is \( S^{2n+1} \), the induced map \( S^{2n+1} \to L_p^{2n+1} \) is the covering map and the homotopy cofibre of this map is equivalent to \((L_p^\infty)^{(2n+2)}\). Then we can see that the previous diagram induces a homotopy commutative diagram of the following form

\[ G_{2n+2}(L_p^\infty) \longrightarrow (L_p^\infty)^{(2n+2)} \]

\[ \downarrow g_{2n+2} \quad \downarrow \quad \downarrow \]

\[ L_p^\infty \]

We have used in the previous proof that \( \text{cat}((L_p^\infty)^{(k)}) \leq k \). Actually we have:

**Lemma 21.** \( \text{cat}((L_p^\infty)^{(k)}) = k \), for all \( k \geq 0 \).

**Proof.** It is a well-known fact that \( \text{cat}(L_p^{2n+1}) = 2n + 1 \), for all \( n \geq 0 \) (see for instance [7]). Since \( 2n + 1 = \text{cat}(L_p^{2n+1}) \leq \text{cat}((L_p^\infty)^{(2n)}) + 1 \) we have that \( \text{cat}((L_p^\infty)^{(2n)}) \geq 2n \) and therefore \( \text{cat}((L_p^\infty)^{(2n)}) = 2n \). □

As for the real projectives spaces we will use the notion of what are called \( Z_p \)-axial maps. This kind of maps appears in [20] for lens spaces.

**Definition 22.** [20] A continuous map \( a : L_p^{2n+1} \times L_p^{2n+1} \to L_p^{2r+1} \) is said to be \( Z_p \)-axial when the composite

\[ L_p^{2n+1} \times L_p^{2n+1} \xrightarrow{a} L_p^{2r+1} \hookrightarrow L_p^\infty \]

classifies the \( Z_p \)-cover \( S^{2n+1} \times_{Z_p} S^{2n+1} \to L_p^{2n+1} \times L_p^{2n+1} \).

**Remark 23.** Similarly to the case of real projective spaces, \( L_p^\infty = K(Z_p, 1) = \Omega K(Z_p, 2) \) is given with an \( H \)-group structure and a \( Z_p \)-axial map should be
thought of as a deformation of the restriction to $L_{2n+1}^{2n+1} \times L_{p}^{2n+1}$ of the product $m(x, y) = x \cdot y$ in $L_{p}^{\infty}$. That is, there exists a homotopy commutative diagram

\[
\begin{array}{ccc}
L_{p}^{2n+1} \times L_{p}^{2n+1} & \xrightarrow{a} & L_{p}^{2r+1} \\
\downarrow & & \downarrow \\
L_{p}^{\infty} \times L_{p}^{\infty} & \xrightarrow{m} & L_{p}^{\infty}
\end{array}
\]

Again, by considering easy (integral) homology arguments we have that necessarily $n \leq r$ holds.

The proof of the following proposition can be found in [20, Lemma 4.1]. It is quite similar to the first part of the proof in Lemma 17.

**Proposition 24.** If $n + 1$ and $p$ are not powers of a common odd prime, then there are not $\mathbb{Z}_{p}$-axial maps $a : L_{p}^{2n+1} \times L_{p}^{2n+1} \rightarrow L_{p}^{2n+1}$.

Unlike the case of the real projective spaces, in general the product of $L_{p}^{\infty}$ precomposed by the diagonal map is not homotopically trivial. So we will consider the map $\mu : L_{p}^{\infty} \times L_{p}^{\infty} \rightarrow L_{p}^{\infty}$, $\mu(x, y) = x \cdot y^{-1}$ induced by the loop space structure in $L_{p}^{\infty} = \Omega K(\mathbb{Z}_{p}, 2)$ and its restriction $\mu' : L_{p}^{2n+1} \times L_{p}^{2n+1} \rightarrow L_{p}^{\infty} \times L_{p}^{\infty}$ $\mu \rightarrow L_{p}^{\infty}$. As a consequence of the previous result we have:

**Corollary 25.** The map $\mu' : L_{p}^{2n+1} \times L_{p}^{2n+1} \rightarrow \mu : L_{p}^{\infty} \times L_{p}^{\infty}$ cannot be deformed in $L_{p}^{2r+1}$, in the following cases

(i) $n > r$; or

(ii) $n = r$, and $n + 1$ and $p$ are not powers of a common odd prime.

**Proof.** The inversion in $L_{p}^{\infty} = \Omega K(\mathbb{Z}_{p}, 2)$ can be deformed in a map $j : L_{p}^{2n+1} \rightarrow L_{p}^{2n+1}$. If there exists a map $a : L_{p}^{2n+1} \times L_{p}^{2n+1} \rightarrow L_{p}^{2r+1}$ such that the composite

$L_{p}^{2n+1} \times L_{p}^{2n+1} \xrightarrow{a} L_{p}^{2r+1} \xrightarrow{\mu} L_{p}^{\infty}$

is homotopic to $\mu'$ then the map

$L_{p}^{2n+1} \times L_{p}^{2n+1} \xrightarrow{id \times j} L_{p}^{2n+1} \times L_{p}^{2n+1} \xrightarrow{a} L_{p}^{2r+1}$

is a $\mathbb{Z}_{p}$-axial map. The conclusion follows from Proposition 24.

**Theorem 26.** If $n + 1$ and $p$ are not powers of a common odd prime, then

$\text{TC}(L_{p}^{2n+1}) = \text{cat}(C_{\Delta L_{p}^{2n+1}})$. 

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Proof. $L_p^{2n+1}$ is a 0-connected CW-complex of dimension $2n+1$. On the other hand Farber and Grant ([14]) have proved that $\text{TC}(L_p^{2n+1}) \geq 2(k+l)+1$ provided that $0 \leq k, l \leq n$ and $p$ does not divide the binomial coefficient $\binom{k+l}{k}$. We take integers $k, l$ such that $0 \leq k, l \leq n$ and $k+l = n+1$. If $p$ does not divide $n+1 = \binom{n+1}{1}$ we can consider $k = 1$ and conclude that $\text{TC}(L_p^{2n+1}) \geq 2(n+1)+1 = 2n+3$. Suppose that $p$ divides $n+1 = \binom{n+1}{1}$. We use the fact that the maximal common divisor of the binomial coefficients $\binom{n+1}{j}$, $1 \leq j \leq n$, is either a prime $a$ or 1 according whether $n+1$ is a power of $a$ or not $\not| n$ to conclude that, in our case, this maximal common divisor must be 1. Therefore there exists $2 \leq j \leq n-1$ such that $p$ does not divide $\binom{n+1}{j}$ and once again we can conclude that $\text{TC}(L_p^{2n+1}) \geq 2n+3$. Hence in any case $\text{TC}(L_p^{2n+1}) \geq 2n+3$ and $\text{dim}(L_p^{2n+1}) \leq \text{TC}(L_p^{2n+1})-2$ so we have that $\text{cat}(\Delta_{L_p^{2n+1}}) \leq \text{TC}(L_p^{2n+1})$. We now turn to the other direction. We consider the map $\mu : L_p^{2n+1} \times L_p^{2n+1} \hookrightarrow L_p^\infty \times L_p^\infty \xrightarrow{\mu \times \mu} L_p^\infty$ where $\mu(x, y) = xy^{-1}$. The composite $\mu' \Delta_{L_p^{2n+1}}$ is homotopically trivial and there is a map $\check{\mu} : \Delta_{L_p^{2n+1}} \rightarrow L_p^\infty$ such that $\check{\mu} \alpha$ is homotopic to $\mu'$. Now suppose that $\text{cat}(\Delta_{L_p^{2n+1}}) = k$. Then the map $\check{\mu} : \Delta_{L_p^{2n+1}} \rightarrow L_p^\infty$ factors through the $k$-th Ganea fibration of $L_p^\infty$ and hence, by Lemma 20, through the inclusion $(L_p^\infty)^{(k)} \hookrightarrow L_p^\infty$. We have a homotopy commutative diagram of the following form:

\[
\begin{array}{ccc}
L_p^{2n+1} & \xrightarrow{\Delta_{L_p^{2n+1}}} & L_p^\infty \\
\downarrow^{\iota_1} & & \downarrow^{\check{\mu}} \\
L_p^{2n+1} \times L_p^{2n+1} & \xrightarrow{\alpha} & L_p^\infty \times L_p^\infty \\
\downarrow^{\mu} & & \downarrow^{j_k} \\
C_{\Delta_{L_p^{2n+1}}} & \xrightarrow{\check{\mu}} & (L_p^\infty)^{(k)} \\
\end{array}
\]

We take $g = \check{\mu} \alpha$. The composite $g \Delta_{L_p^{2n+1}}$ is homotopically trivial. There are two possibilities for $k$. If $k = 2r+1$ is odd then $(L_p^\infty)^{(k)} = L_p^{2r+1}$ and by Corollary 25 we have the inequality $r \geq n+1$, that is $k \geq 2n+3$. On the other hand if $k = 2r$ is even, then $(L_p^\infty)^{(k+1)} = L_p^{2r+1}$ and by Corollary 25 we obtain $r \geq n+1$, or $k \geq 2n+2$. In any case we have $k \geq 2n+2$. As a consequence the inclusion $j_k$ is at least a $(2n+1)$-equivalence and hence the map $f = gi_1$ is a $(2n+1)$-equivalence since the top row is homotopic to the inclusion $j_n : L_p^{2n+1} \hookrightarrow L_p^\infty$, which is a $(2n+1)$-equivalence. Since $2 \dim(L_p^{2n+1}) = 4n+2 \leq (2n+1) + (2n+2) - 1$ we conclude, by Theorem 9, that $\text{TC}(L_p^{2n+1}) \leq \text{cat}((L_p^\infty)^{(k)}) = k = \text{cat}(\Delta_{L_p^{2n+1}})$.

\[\square\]

Remark 27. Note that in the proof of the above theorem, the integer $k = \text{cat}(\Delta_n)$ satisfies $k > 3$. Indeed, suppose that $k \leq 3$; then there is a similar
By Corollary 25, \(1 \geq n + 1 \) (or \( n \leq 0 \)), which is impossible.

5 Appendix: Proof of Theorem 14

We use the notations of Section 4.

For \(0 \leq k \leq n\) the set \(U_k = \{[\sum_{i=0}^{n} g_i t_i], t_k \neq 0\} \subset B_n G\) is open and contractible in \(B_n G\) through the homotopy:

\[
U_k \times I \to B_n G
\]

\[
\frac{[\sum_{i=0}^{n} g_i t_i], s}{[\sum_{i=0}^{n} g_i t'_i]} \text{ with } t'_j = \begin{cases} t_j (1 - s) & j \neq k \\ t_k (1 - s) + s & j = k \end{cases}
\]

Since \(U_0, ..., U_n\) cover \(B_n G\) we have \(\text{cat}(B_n G) \leq n\) which implies the existence of a map \(B_n G \to G_n BG\) such that the diagram

\[
\begin{array}{ccc}
B_n G & \longrightarrow & G_n (BG) \\
\downarrow & & \downarrow g_n \\
BG & \longrightarrow & BG
\end{array}
\]

is homotopically commutative.

Next we inductively construct a map in the other direction. Since \(G_0(BG)\) and \(B_0 G\) are both contractible, this is trivial for \(n = 0\). Suppose that there exists a homotopy commutative diagram of the following form:

\[
\begin{array}{ccc}
G_{n-1}(BG) & \longrightarrow & B_{n-1} G \\
\downarrow g_{n-1} & & \downarrow \\
BG & \longrightarrow & BG
\end{array}
\]

We also consider

\[
\begin{array}{ccc}
e & \longrightarrow & EG \\
\downarrow & & \\
BG & \longrightarrow & BG
\end{array}
\]
where $e$ is the neutral element of $G$. Taking the join of the vertical maps of these two diagrams we obtain a homotopy commutative diagram:

$$\begin{array}{cccccc}
G_{n-1}(BG) \ast BG e & \longrightarrow & B_{n-1}G \ast BG EG \\
\downarrow & & \downarrow \\
BG & \longrightarrow & BG
\end{array}$$

Since we can assume that $EG \to BG$ is a fibration, the pullback

$$\begin{array}{cccccc}
E_{n-1}G & \longrightarrow & EG \\
\downarrow & & \downarrow \\
B_{n-1}G & \longrightarrow & BG
\end{array}$$

is a homotopy pullback and the join of $B_{n-1}G \to BG$ and $EG \to BG$ can be written as

$$B_{n-1}G \ast BG EG = B_{n-1}G \amalg E_{n-1}(BG) \ast EG/ \sim \to BG$$

where the equivalence relation is given by $$(\sum_{i=0}^{n-1} g_i t_i, 0) \sim (\sum_{i=0}^{n-1} g_i t_i, s) \sim (\sum_{i=0}^{n-1} g_i t_i, 1)$$

$\sum_{i \geq 0} g_i t'_i$ with $t'_i = t_i$ for $i \leq n - 1$ and $t'_i = 0$ for $i > n - 1$. The homotopy

$$\begin{array}{cccccc}
B_{n-1}G \ast BG EG \times I & \to & BG \\
([\sum_{i=0}^{n-1} g_i t_i], t) & \mapsto & [\sum_{i=0}^{n-1} g_i t_i] \\
(\sum_{i=0}^{n-1} g_i t_i, s, t) & \mapsto & [\sum_{i=0}^{n-1} g_i t'_i]
\end{array}$$

shows that the join map $B_{n-1}G \ast BG EG \to BG$ factors, up to homotopy, through the inclusion $B_n G \to BG$. Since the $n$-th Ganea map is the associated fibration of the map $G_{n-1}(BG) \ast BG e \to BG$ we finally obtain a homotopy commutative
diagram of the following form:

\[ G_n(BG) \twoheadrightarrow B_nG \]
\[ g_n \downarrow \]
\[ BG \hookrightarrow BG \]

concluding the proof.

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