Matrices, Fermi Operators and Applications

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Abstract. We consider the vector space of $n \times n$ matrices over $\mathbb{C}$, Fermi operators and operators constructed from these matrices and the Fermi operators. The properties of these operators are studied with respect to the underlying matrices. The commutators, anticommutators, and the eigenvalue problem of such operators are also discussed. Other matrix functions such as the exponential function are studied. Density operators and Kraus operators are also discussed.

1 Introduction

We consider the vector space of $n \times n$ matrices over $\mathbb{C}$. Let $c_1^\dagger, \ldots, c_n^\dagger, c_1, \ldots, c_n$ be Fermi creation and annihilation operators, respectively. The anticommutation relations are (see, for example, [1])

\[ [c_j^\dagger, c_k]_+ = \delta_{jk}I, \quad [c_j, c_k]_+ = 0, \quad [c_j^\dagger, c_k^\dagger]_+ = 0 \quad (1) \]

where $I$ is the identity operator and 0 the zero operator. Hence $c_j^2 = 0$ and $(c_j^\dagger)^2 = 0$. Furthermore we have the commutator

\[ [c_j^\dagger c_k, c_\ell^\dagger c_m] = \delta_{k\ell}c_j^\dagger c_m - \delta_{jm}c_\ell^\dagger c_k, \quad j, k, \ell, m = 1, \ldots, n. \quad (2) \]

Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. Then we can form the operator $\hat{A}$ defined by

\[ \hat{A} := (c_1^\dagger \cdots c_n^\dagger) A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sum_{j,k=1}^{n} a_{jk} c_j^\dagger c_k, \quad (3) \]

i.e. a quadratic form in the Fermi operators. See [2] for a motivation of the study of quadratic forms in Fermi operators. The connection between the quadratic form in
and matrices permits the use of numerous techniques and results from matrix theory [3]. For example, consider the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$. Then we have

$$\hat{\sigma}_1 = c_1^\dagger c_2 + c_2^\dagger c_1, \quad \hat{\sigma}_2 = -ic_1^\dagger c_2 + ic_2^\dagger c_1, \quad \hat{\sigma}_3 = c_1^\dagger c_1 - c_2^\dagger c_2.$$ 

and

$$(\hat{\sigma}_1)^2 = (\hat{\sigma}_2)^2 = (\hat{\sigma}_3)^2 = c_1^\dagger c_1 + c_2^\dagger c_2 - 2c_1^\dagger c_1 c_2^\dagger c_2.$$ 

We study the properties of the operator $\hat{A}$ for given matrix $A$. We consider normal, nonnormal, hermitian, unitary, density matrices etc. The commutator and anticommutator of two matrices $A, B$ and the corresponding operators $\hat{A}$ and $\hat{B}$ are also investigated.

## 2 Properties of Matrices

Let $I_n$ be the $n \times n$ identity matrix. Then

$$\hat{N} = (c_1^\dagger \cdots c_n^\dagger) I_n \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sum_{j=1}^{n} c_j^\dagger c_j$$

is the number operator. We note that

$$[c_j^\dagger c_k, c_j^\dagger c_j + c_k^\dagger c_k] = 0.$$ 

Let $A$ be an normal matrix, i.e. $AA^* = A^*A$. Then $\hat{A}$ is a normal operator. Let $A$ be a nonnormal matrix, i.e. $AA^* \neq A^*A$. Then $\hat{A}$ is a nonnormal operator. An example for $n = 2$ is the nonnormal matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Then we obtain the operator $\hat{A} = c_1^\dagger c_2$ and hence $\hat{A}^\dagger = c_2^\dagger c_1$ with

$$\hat{A} \hat{A}^\dagger = c_1^\dagger c_1 (I - c_2^\dagger c_2), \quad \hat{A}^\dagger \hat{A} = c_2^\dagger c_2 (I - c_1^\dagger c_1)$$

If $H$ is a hermitian matrix, then $\hat{H}$ is a self-adjoint operator. If $C$ is a skew-hermitian matrix, then $\hat{C}$ is a skew-hermitian operator. Let $U$ be a unitary matrix. Then we cannot conclude that $\hat{U}$ is a unitary operator. As an example consider

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Then \( \hat{U} = c_1^\dagger c_2 + c_2^\dagger c_1 \), \( \hat{U} = \hat{U}^\dagger \) and

\[
\hat{U}\hat{U}^\dagger = c_1^\dagger c_1 + c_2^\dagger c_2 - 2c_1^\dagger c_1 c_2^\dagger c_2 = \hat{N} - 2c_1^\dagger c_1 c_2^\dagger c_2.
\]

Next we mention that applying the anticommutation relations we obtain

\[
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{pmatrix}
\begin{pmatrix}
c_1^\dagger & c_2^\dagger & \cdots & c_n^\dagger
\end{pmatrix}
= \begin{pmatrix}
c_1 c_1^\dagger & c_1 c_2^\dagger & \cdots & c_1 c_n^\dagger \\
c_2 c_1^\dagger & c_2 c_2^\dagger & \cdots & c_2 c_n^\dagger \\
\vdots & \vdots & \ddots & \vdots \\
c_n c_1^\dagger & c_n c_2^\dagger & \cdots & c_n c_n^\dagger
\end{pmatrix}
= \begin{pmatrix}
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I
\end{pmatrix}
- \begin{pmatrix}
c_1^\dagger c_1 & c_2^\dagger c_1 & \cdots & c_n^\dagger c_1 \\
c_1^\dagger c_2 & c_2^\dagger c_2 & \cdots & c_n^\dagger c_2 \\
\vdots & \vdots & \ddots & \vdots \\
c_1^\dagger c_n & c_2^\dagger c_n & \cdots & c_n^\dagger c_n
\end{pmatrix}.
\]

Hence we cannot expect that \( (\hat{A})(\hat{A}) = (\hat{A}^2) \) in general. Let \( A \) be a \( 2 \times 2 \) matrix. Then we obtain

\[
\hat{A}\hat{A} = (c_1^\dagger c_2^\dagger) A^2 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + 2 \det(A)c_1^\dagger c_1 c_2^\dagger c_2.
\]

Thus if \( \det(A) = 0 \), then \( \hat{A}\hat{A} = (\hat{A}^2) \). For a \( 3 \times 3 \) matrix we obtain

\[
(\hat{A})^2 = (\hat{A}^2) - 2 \sum_{i<k} \sum_{j<l} c_i^\dagger c_j^\dagger c_j c_l \det \begin{pmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{pmatrix}.
\]

Let \( A, B \) be \( n \times n \) matrices. Then we have

\[
\hat{A}\hat{B} = \hat{A}\hat{B} - \sum_{j,k,l,m=1 \atop l>j,m>k}^n g \left( \sigma_2 \begin{pmatrix} a_{jk} & a_{jm} \\ a_{lk} & a_{lm} \end{pmatrix} \sigma_2 \begin{pmatrix} b_{jk} & b_{jm} \\ b_{lk} & b_{lm} \end{pmatrix}^T \right) c_j^\dagger c_k c_m.
\]

(4)

The coefficients are given in terms of the symmetric indefinite form

\[
g(X, Y) = g(Y, X) = \text{tr} \left( \sigma_2 X \sigma_2 Y^T \right) = \det(X + Y) - \det(X) - \det(Y)
\]

over the \( 2 \times 2 \) complex matrices, where \( \sigma_2 \) is the Pauli spin matrix

\[
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]
Clearly, \( g(A, A) = 2 \det(A) \). The sum in (4) is taken over \( g(X, Y) \) for all corresponding \( 2 \times 2 \) submatrices of \( A \) and \( B \). With \( B = A \) we obtain
\[
(\hat{A})^2 = \hat{A}^2 - 2 \sum_{j, k, l, m=1 \atop \ell > j, m > k}^{n} \det([A]_{j, l, k, m}) c^\dagger_j c^\dagger_k c_k c_m 
\]
where \([A]_{j, l, k, m}\) is the \( 2 \times 2 \) submatrix of \( A \) taken from rows \( j \) and \( l \) and columns \( k \) and \( m \). Now \( \det([A]_{j, l, k, m}) = 0 \) for all \( j, k, l, m \) only if \( A \) is a rank-0 or rank-1 matrix. Let \( \mathbf{v} \) be a normalized (column) vector in \( \mathbb{C}^2 \). Then \( \rho = \mathbf{v} \mathbf{v}^* \) is a density matrix (pure state). Now
\[
\hat{\rho} = (c_1^\dagger \ c_2^\dagger) \mathbf{v} \mathbf{v}^* \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = v_1 \mathbf{v}_1 c_1^\dagger c_1 + v_1 \mathbf{v}_2 c_2^\dagger c_2 + v_2 \mathbf{v}_1 c_2^\dagger c_1 + v_2 \mathbf{v}_2 c_2^\dagger c_2 .
\]
Obviously \( \hat{\rho} \) is a self-adjoint operator. We have \( \hat{\rho} = (\hat{\rho})^2 \). Hence \( \hat{\rho} \) is a density operator. Obviously this also holds for a normalized vector \( \mathbf{v} \) in \( \mathbb{C}^n \) and \( \rho = \mathbf{v} \mathbf{v}^* \), since \( \rho \) is a rank-1 matrix.

Let \( \Pi \) be a projection matrix, i.e. \( \Pi = \Pi^* \) and \( \Pi = \Pi^2 \). Then obviously \( \hat{\Pi} = \hat{\Pi}^\dagger \) and \( \hat{\Pi}^2 = \hat{\Pi} \) if and only if \( \Pi \) is a projection onto a one-dimensional subspace (i.e. has rank 1). We note that \( \det(\Pi) = 0 \) except when \( \Pi \) is the identity matrix.

The symmetry of \( g(X, Y) \) yields that \([\hat{A}, \hat{B}] = [\hat{A}, \hat{B}] \). Finally, we note that \( A \mapsto \hat{A} \) is trace preserving, i.e.
\[
\text{tr}(\hat{A}) = \sum_{k=1}^{n} \langle 0 | c_k \hat{A} c_k^\dagger | 0 \rangle = \text{tr}(A)
\]
where \(|0\rangle\) is the vacuum state, \( c_k |0\rangle = 0 \) and \( \langle 0|c_k^\dagger = 0 \).

## 3 Exponential Function

Let \( C = (C_{jk}), C_1, C_2, \) be \( n \times n \) skew-hermitian matrices \( (j, k = 1, \ldots, n) \). Then \( V = \exp(C), V_1 = \exp(C_1), V_2 = \exp(C_2) \) are unitary matrices and
\[
\hat{U}(V) := \exp(\hat{C}) = \exp \left( \sum_{j=1}^{n} \sum_{k=1}^{n} C_{jk} c_j^\dagger c_k \right)
\]
is a unitary operator with \( V = \exp(C) \). If \( C \) is a rank-1 matrix we have that \( (e^{\hat{C}}) = e^{\hat{C}} \). Owing to the commutator given in equation (2) we obtain
\[
\hat{U}(V_1)\hat{U}(V_2) = \hat{U}(V_1 V_2)
\]
\[ \hat{U}(V^{-1}) = \hat{U}^{-1}(V) = \hat{U}^\dagger(V) \]
\[ \hat{U}(I_n) = I \]

where \( I \) is the identity operator.

We also note that the Baker-Campbell-Hausdorff formula can be expressed in terms of repeated commutators [4]

\[ \log(e^X e^Y) = \sum_{p+q = k} \frac{1}{p_1!q_1! \cdots p_k!q_k!} [X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}] \]

where the sum is taken over all \( k \in \mathbb{N} \) and \( p, q, p_1, q_1, \ldots, p_k, q_k \in \mathbb{N}_0 \) such that

\[ \sum_{i=1}^k p_i = p, \quad \sum_{i=1}^k q_i = q, \quad p_i + q_i > 0, \quad i \in \{1, 2, \ldots, k\} \]

and the repeated commutators are given by

\[ [X^{p_1} Y^{q_1} \cdots X^{p_k} Y^{q_k}] := [X \cdots X, Y \cdots Y, [X \cdots X, [Y \cdots Y, B,A]] \cdots B,A] \]

Since \( \hat{[A, B]} = [A, B] \), we have

\[ \log(e^X e^Y) = \log(e^{\hat{X}} e^{\hat{Y}}). \]

### 4 Commutators, Lie Algebras and Anticommutators

Let \( A, B \) be \( n \times n \) matrices and \( \hat{A}, \hat{B} \) be the corresponding operators. Then a straightforward calculation shows that

\[ [\hat{A}, \hat{B}] = (c_1^\dagger \cdots c_n^\dagger) [A, B] \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = [A, B] \]

As an example consider the \( 2 \times 2 \) matrices

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then \( [A, B] = A \) and \( A, B \) form a basis of a two-dimensional non-abelian Lie algebra. Now

\[ \hat{A} = c_1^\dagger c_2, \quad \hat{B} = c_2^\dagger c_2 \]
and

\[ [\hat{A}, \hat{B}] = c_1^\dagger c_2 = \hat{A}. \]

Hence the operators \( \hat{A}, \hat{B} \) form a basis of a two-dimensional non-abelian Lie algebra. For two arbitrary \( 2 \times 2 \) matrices \( A, B \) we have

\[
[A, B] = \begin{pmatrix}
  a_{12}b_{21} - a_{21}b_{12} & \text{tr}(\sigma_3 A)b_{12} - \text{tr}(\sigma_3 B)a_{12} \\
  -\text{tr}(\sigma_3 A)b_{21} + \text{tr}(\sigma_3 B)a_{21} & a_{21}b_{12} - b_{21}a_{12}
\end{pmatrix}.
\]

Now

\[ \hat{A} = a_{11}c_1^\dagger c_1 + a_{12}c_1^\dagger c_2 + a_{21}c_2^\dagger c_1 + a_{22}c_2^\dagger c_2 \]
\[ \hat{B} = b_{11}c_1^\dagger c_1 + b_{12}c_1^\dagger c_2 + b_{21}c_2^\dagger c_1 + b_{22}c_2^\dagger c_2. \]

Then

\[
[A, \hat{B}] = (c_1^\dagger c_1 - c_2^\dagger c_2)(a_{12}b_{21} - a_{21}b_{12})
\]
\[+ (a_{11}b_{12} - a_{12}b_{11} + a_{12}b_{22} - a_{22}b_{12})c_1^\dagger c_2 + (a_{21}b_{11} - a_{21}b_{22} + a_{22}b_{21} - a_{11}b_{21})c_2^\dagger c_1. \]

Investigating the anticommutator between \( \hat{A} \) and \( \hat{B} \), we find

\[
[\hat{A}, \hat{B}]_+ = [\hat{A}, \hat{B}] - 2 \sum_{j, k, i, m=1}^n g ([A]_{j, l; k, m}, [B]_{j, l; k, m}) c_j^\dagger c_k c_m
\]
where

\[ [A]_{j, l; k, m} := \begin{pmatrix}
  a_{jk} & a_{jm} \\
  a_{lk} & a_{lm}
\end{pmatrix}. \]

For example, with \( n = 2 \),

\[
[\hat{A}, \hat{B}]_+ = [\hat{A}, \hat{B}] + 2 (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12}) c_1^\dagger c_1 c_2^\dagger c_2,
\]

with

\[
[A, B]_+ = \begin{pmatrix}
  2a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} & \text{tr} (A) b_{12} + \text{tr} (B) a_{12} \\
  \text{tr} (A) b_{21} + \text{tr} (B) a_{21} & 2a_{22}b_{22} + a_{21}b_{12} + a_{12}b_{21}
\end{pmatrix}.
\]

Consider the Pauli spin matrices. Then we have

\[ [\sigma_1, \sigma_2]_+ = 0, \quad [\sigma_2, \sigma_3]_+ = 0, \quad [\sigma_3, \sigma_1]_+ = 0. \]

and

\[ [\hat{\sigma}_1, \hat{\sigma}_2]_+ = 0, \quad [\hat{\sigma}_2, \hat{\sigma}_3]_+ = 0, \quad [\hat{\sigma}_3, \hat{\sigma}_1]_+ = 0. \]

It follows that

\[
[A, B]_+ = \text{tr}(A)B + \text{tr}(B)A - \frac{1}{2} \left( \text{tr}(A)\text{tr}(B) - \sum_{j=1}^3 \text{tr}(\sigma_j A)\text{tr}(\sigma_j B) \right) I_2.
\]
5 Eigenvalue Problem

Consider first the $2 \times 2$ case with
\[ \hat{A} = a_{11} c_1^\dagger c_1 + a_{12} c_1^\dagger c_2 + a_{21} c_2^\dagger c_1 + a_{22} c_2^\dagger c_2. \]

With the basis element $|0\rangle$ and $c_j|0\rangle = 0|0\rangle$ we obtain $\langle 0|\hat{A}|0\rangle = 0$. With the basis element $c_2^\dagger c_1^\dagger|0\rangle$ we obtain
\[ \hat{A} c_2^\dagger c_1^\dagger|0\rangle = a_{11} c_2^\dagger c_1^\dagger|0\rangle + a_{22} c_2^\dagger c_1^\dagger|0\rangle = (a_{11} + a_{22}) c_2^\dagger c_1^\dagger|0\rangle = \text{tr}(A) c_2^\dagger c_1^\dagger|0\rangle. \]

With the basis $c_1^\dagger|0\rangle$, $c_2^\dagger|0\rangle$ we obtain
\[ \hat{A} c_1^\dagger|0\rangle = a_{11} c_1^\dagger|0\rangle + a_{21} c_2^\dagger|0\rangle \]
\[ \hat{A} c_2^\dagger|0\rangle = a_{11} c_1^\dagger|0\rangle + a_{21} c_2^\dagger|0\rangle \]
Together with the dual basis $\langle 0|c_1$, $\langle 0|c_2$ we obtain that the matrix representation of $\hat{A}$ is $A$.

For the $3 \times 3$ case we have
\[ \hat{A} = \sum_{j,k=1}^{3} a_{j,k} c_j^\dagger c_k. \]

With the basis $|0\rangle$ we obtain $\langle 0|\hat{A}|0\rangle = 0$. For the basis $c_1^\dagger|0\rangle$, $c_2^\dagger|0\rangle$, $c_3^\dagger|0\rangle$ and the respectively dual basis $\langle 0|c_1$, $\langle 0|c_2$, $\langle 0|c_3$ we obviously obtain $A$ as the matrix representation of $\hat{A}$. With the basis
\[ c_1^\dagger c_2^\dagger|0\rangle, \quad c_1^\dagger c_3^\dagger|0\rangle, \quad c_2^\dagger c_3^\dagger|0\rangle \]
and the corresponding dual one
\[ \langle 0|c_2 c_1$, $\langle 0|c_3 c_1$, $\langle 0|c_3 c_2$ we obtain the matrix representation of $\hat{A}$
\[ \begin{pmatrix} 
    a_{11} + a_{22} & a_{23} & -a_{13} \\
    a_{32} & a_{11} + a_{33} & a_{12} \\
    -a_{31} & a_{21} & a_{22} + a_{33}
\end{pmatrix}. \]
Note that the trace of this matrix is twice the trace of $A$. In general we have
\[ \hat{A} = \sum_{j,k=1}^{n} a_{j,k} c_j^\dagger c_k \]
and with the basis
\[ \{ c_j^\dagger |0\rangle : j = 1, \ldots, n \} \]
the matrix representation of \( \hat{A} \) is given by \( A \), and the matrix representation of \( \hat{A}\hat{B} \) is \( AB \). The trace of \( A \) is an eigenvalue of \( \hat{A} \):
\[ \hat{A}c_n^\dagger c_{n-1}^\dagger \cdots c_1^\dagger |0\rangle = \text{tr}(A) c_n^\dagger c_{n-1}^\dagger \cdots c_1^\dagger |0\rangle. \]

6 Kraus Operators

Consider the Kraus operators \( K_1 \) and \( K_2 \)
\[
K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad K_1^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad K_2^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
and an arbitrary \( 2 \times 2 \) matrix \( A = (a_{jk}) \). Then
\[
K_1 AK_1^\dagger + K_2 AK_2^\dagger = \begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix}.
\]
So the trace of \( A \) is preserved under this transformation. Let \( c_1^\dagger, c_2^\dagger, c_1, c_2 \) be Fermi creation and annihilation operators, respectively. Then
\[
\hat{K}_1 = (c_1^\dagger c_2^\dagger) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^\dagger c_2, \quad \hat{K}_1^\dagger = c_2^\dagger c_1
\]
\[
\hat{K}_2 = (c_1^\dagger c_2^\dagger) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_2^\dagger c_1, \quad \hat{K}_2^\dagger = c_1^\dagger c_2
\]
and
\[
\hat{A} = (c_1^\dagger c_2^\dagger) A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = a_{11} c_1^\dagger c_1 + a_{12} c_1^\dagger c_2 + a_{21} c_2^\dagger c_1 + a_{22} c_2^\dagger c_2.
\]
It follows that
\[
\hat{K}_1 \hat{A} \hat{K}_1^\dagger + \hat{K}_2 \hat{A} \hat{K}_2^\dagger = a_{22} c_1^\dagger c_1 + a_{11} c_2^\dagger c_2 + (a_{11} + a_{22}) c_1^\dagger c_2^\dagger c_1 c_2.
\]
Thus the embedding preserves this map when \( \text{tr}(A) = 0 \). Let \( K_1, \ldots, K_r \) be matrix Kraus operators. As we noted in the previous section, in the basis
\[ \{ c_j^\dagger |0\rangle : j = 1, \ldots, n \} \]
the matrix representation of \( \hat{A} \) is given by \( A \), and the matrix representation of \( \hat{K}_1 \hat{A} \hat{K}_1^\dagger + \cdots + \hat{K}_r \hat{A} \hat{K}_r^\dagger \) is precisely \( K_1 AK_1^\dagger + \cdots + K_r AK_r^\dagger \).
7 Extensions

In order to model all quadratic forms in the Fermi operators we need to add operators of the form

\[ \hat{B} = (c_1^\dagger \ c_2^\dagger \ \cdots \ c_n^\dagger) B \begin{pmatrix} c_1^\dagger \\ c_2^\dagger \\ \vdots \\ c_n^\dagger \end{pmatrix} \]

and

\[ \hat{D} = (c_1 \ c_2 \ \cdots \ c_n) D \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \]

where we have to take into account that \( c_j^2 = 0, (c_j^\dagger)^2 = 0 \). For \( n=2 \) we have

\[ \hat{B} = (b_{12} - b_{21}) c_1^\dagger c_2^\dagger \]

\[ \hat{D} = (d_{12} - d_{21}) c_1 c_2. \]

Now the commutator of \( \hat{B} \) and \( \hat{D} \) is given by

\[ [\hat{B}, \hat{D}] = (b_{12} - b_{21})(d_{12} - d_{21})(I - c_1^\dagger c_1 - c_2^\dagger c_2). \]

Finally, we may consider Fermi-Bose coupled quadratic forms

\[ \hat{M} = ((c_1^\dagger \ c_2^\dagger \ \cdots \ c_n^\dagger) \otimes (b_1^\dagger \ b_2^\dagger \ \cdots \ b_m^\dagger)) M \left( \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right). \]

A decomposition

\[ M = \sum_{j=1}^{r} M_{c,j} \otimes M_{b,j} \]

of the \( mn \times mn \) matrix \( M \) over the \( n \times n \) matrices \( M_{c,r} \) and \( m \times m \) matrices \( M_{b,r} \) yields a sum of quadratic forms in the Fermi operators coupled with quadratic forms in the Bose operators. However, we no longer have the straightforward relationship between the matrix commutator and the commutator of Fermi/Bose operators.
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