Cosmological matching conditions and galilean genesis in Horndeski’s theory

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Abstract. We derive the cosmological matching conditions for the homogeneous and isotropic background and for linear perturbations in Horndeski’s most general second-order scalar-tensor theory. In general relativity, the matching is done in such a way that the extrinsic curvature is continuous across the transition hypersurface. This procedure is generalized so as to incorporate the mixing of scalar and gravity kinetic terms in the field equations of Horndeski’s theory. Our matching conditions have a wide range of applications including the galilean genesis and the bounce scenarios, in which stable, null energy condition violating solutions play a central role. We demonstrate how our matching conditions are used in the galilean genesis scenario. In doing so, we extend the previous genesis models and provide a unified description of the theory admitting the solution that starts expanding from the Minkowski spacetime.

Keywords: modified gravity, cosmological perturbation theory, alternatives to inflation

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1 Introduction

Scalar fields are ubiquitous in cosmology. Inflation [1–3] is considered to be driven by one or multiple scalar fields, which can seed the large-scale structure of the Universe as well. The current cosmic acceleration may also be caused by a scalar field dominating the energy content of the Universe as dark energy (see e.g. [4] for a review). A great variety of modified gravity models have been proposed as an alternative to dark energy (see e.g. [5] and references therein), many of which involve an additional scalar degree of freedom in the gravity sector. Early-universe scenarios other than inflation have also been explored (see e.g. [6] for a recent review), such as bounce models, and they are often based on some scalar-field theory.

Almost forty years ago, Horndeski constructed the most general theory composed of the metric $g_{\mu\nu}$ and the scalar field $\phi$ with second-order field equations [7], which has long been ignored until recently [8]. In the course of generalizing the galileon scalar-field theory, Horndeski’s theory was rediscovered in its modern form [9–11]. (The equivalence of the generalized galileon and Horndeski’s theory was first shown in ref. [12].) The action is given by

$$S_{\text{Hor}} = \int d^4x \sqrt{-g} \left( L_2 + L_3 + L_4 + L_5 \right),$$  

with

$$L_2 = G_2(\phi, X),$$

$$L_3 = -G_3(\phi, X)\Box \phi,$$

$$L_4 = G_4(\phi, X)R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right],$$

$$L_5 = G_5(\phi, X)G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{6} G_{5X} \left[ (\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3 \right],$$  

where $X := -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi/2$, $R$ is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor, and $G_2, G_3, G_4,$ and $G_5$ are arbitrary functions of $\phi$ and $X$. (Here and hereafter we use the
notation $G_{iX} := \partial G_i / \partial X$, $G_{i\phi} := \partial G_i / \partial \phi$, and so on.) Since this theory contains all the single-field inflation models and modified gravity models with one scalar degree of freedom as specific cases, it is of great importance in cosmology and hence considerable attention has been paid in recent years to various aspects of Horndeski’s theory (see the nonexhaustive list of references [13–32]).

In this paper, we will address the following issue: suppose that the Universe undergoes a sharp transition caused, for example, by sudden halt of the scalar field or by a discontinuous jump in matter pressure, and then what are the continuous quantities across the transition hypersurface in Horndeski’s theory? In general relativity, it is known that the induced metric on the surface and its extrinsic curvature must be continuous. This implies that the Hubble parameter $H$ is continuous. As for linear cosmological perturbations, the matching conditions in general relativity are clarified in refs. [33, 34]. In Horndeski’s theory, however, scalar and gravity kinetic terms are mixed due to second derivatives on $\phi$ in the Lagrangian [35], and as a result the matching conditions would be nontrivial both for the homogeneous and isotropic background and for cosmological perturbations. This point was raised in the context of galilean genesis [36] and was studied based on specific Lagrangians [37, 38]. In this paper, we start from the boundary terms in Horndeski’s theory [39] and derive rigorously the cosmological matching conditions in their most general form.

The matching conditions obtained in this paper have a wide range of applications. In particular, Horndeski’s theory allows for stable violation of the null energy condition (NEC), leading to interesting possibilities such as galilean genesis mentioned above and non-singular bounce models [40–45]. Our matching conditions provide a generic algorithm to follow the evolution of the cosmological background and perturbations, which is applicable to those scenarios.

This paper is organized as follows. In the next section we summarize the boundary terms in Horndeski’s theory [39], which are the basis of the present work. Then, in section 3, we derive the cosmological matching conditions both for the background and perturbations. To present an example, we develop a unified Lagrangian accommodating all the previous models of galilean genesis, and apply our matching conditions to this general model in section 4. We draw our conclusions in section 5.

2 Boundary terms for Horndeski’s theory

We begin with summarizing the result of ref. [39]. The action we are going to study is given by

$$S = S_{\text{Hor}} + S_m + S_B,$$

(2.1)

where $S_{\text{Hor}}$ is Horndeski’s action (1.1), $S_m$ is the action for usual matter, and $S_B$ is the boundary term. This last term is necessary when one considers a spacetime $\mathcal{M}$ divided into two domains, $\mathcal{M}_\pm$, by a surface $\Sigma$. In what follows $\Sigma$ is supposed to be spacelike.

Let us take a look at the case of general relativity. The variation of the Einstein-Hilbert term with respect to the metric involves a normal derivative of the metric variation,

$$\delta_g \left( \int d^4 x \sqrt{-g} R \right) \supset - \int_{\Sigma_\pm} d^3 x \sqrt{\gamma} \gamma^{\mu\nu} n^\lambda \nabla_\lambda \delta g_{\mu\nu},$$

(2.2)
where $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the induced metric on $\Sigma_\pm$ and $n^\mu$ is the future directed unit normal. Here and hereafter we write

$$\int_{\Sigma_\pm} (\cdots) := \int_{\Sigma_+} (\cdots) - \int_{\Sigma_-} (\cdots),$$

with $\Sigma_\pm$ denoting the two sides of $\Sigma$. The presence of the normal derivative of the metric variation is problematic; to obtain a well-defined variational problem, one has to add a boundary term that cancels the contribution (2.2). By noticing that the variation of the trace of the extrinsic curvature, $K_{\mu\nu} := \gamma_\mu^a \gamma_\nu^b \nabla_{(a} n_{b)}$, gives rise to the same contribution,

$$\delta g (\sqrt{\gamma} K) \supset \frac{1}{2} \sqrt{\gamma} \gamma^\mu_\nu n^\lambda \nabla_\lambda \delta g_{\mu\nu},$$

we are lead to add the well-known Gibbons-Hawking term on the boundary [46].

Since the most general scalar-tensor Lagrangian having second-order field equations contains second derivatives of the scalar field which is nonminimally coupled to gravity, as well as the second derivatives of the metric, the corresponding boundary action is not simply given by the Gibbons-Hawking term. For example, $G_3 \Box \phi$ produces the following problematic normal derivative:

$$\delta \phi \left( \int d^4 x \sqrt{-g} G_3 \Box \phi \right) \supset \int_{\Sigma_\pm} d^3 x \sqrt{\gamma} G_3 n^\mu \nabla_\mu \delta \phi.$$

This can be canceled by adding

$$B_3 = \int_{\Sigma_\pm} d^3 x \sqrt{\gamma} F_3,$$

where

$$F_3(\phi, X_0, \tilde{X}) := \int_0^{X_0} \frac{du}{\sqrt{2u}} G_3(\phi, u + \tilde{X}),$$

with $X_0 := (n^\mu \nabla_\mu \phi)^2/2$ and $\tilde{X} := -\gamma^\mu_\nu \partial_\mu \phi \partial_\nu \phi/2$. (Note that $X = X_0 + \tilde{X}$.) Similarly, one can obtain the boundary contributions corresponding to $L_4$ and $L_5$. The boundary term for the galileon Lagrangian was considered in ref. [47], and then the complete boundary term in Horndeski’s theory, which is composed of three different parts, $S_B = B_3 + B_4 + B_5$, was derived for the first time in ref. [39]. The latter two terms are given by

$$B_4 = 2 \int_{\Sigma_\pm} d^3 x \sqrt{\gamma} \left( G_4 K - F_4 \tilde{X} D_2 \phi \right),$$

$$B_5 = \int_{\Sigma_\pm} d^3 x \sqrt{\gamma} \left\{ \frac{1}{2} G_5 \left( K^2 - K_{\mu\nu} K^{\mu\nu} \right) n^\lambda \nabla_\lambda \phi + G_5 \left( K D^2 \phi - K^{\mu\nu} D_\mu D_\nu \phi \right) + \frac{1}{2} F_5 R^{(3)} \right\},$$

where each $F_i$ ($i = 4, 5$) is defined similarly to $F_3$ as

$$F_i(\phi, X_0, \tilde{X}) := \int_0^{X_0} \frac{du}{\sqrt{2u}} G_i(\phi, u + \tilde{X}),$$

$D_\mu$ is the covariant derivative on the boundary, and $R^{(3)}$ is the boundary Ricci scalar.
Having found the boundary term, one can obtain the junction conditions that describe discontinuity across the hypersurface \( \Sigma \), as a generalization of Israel’s conditions [48]. The variational principle for (2.1) yields the equations of motion and

\[
\delta S \supset \int_{\Sigma_\pm} d^3x \sqrt{\gamma} \left( J^\mu\nu \delta \gamma^\mu\nu + J^\phi \delta \phi \right),
\]  
(2.11)

Here, \( J^\mu\nu = J_3^{\mu\nu} + J_4^{\mu\nu} + J_5^{\mu\nu} \), with

\[
J_3^{\mu\nu} = -\frac{1}{2} \gamma^{\mu\nu} \int_0^{X_0} du \sqrt{2u} G_{3u}(\phi, u + \bar{X}) + \frac{1}{2} F_3 \bar{X} D^\mu \phi D^\nu \phi,
\]  
(2.12)

and lengthy expressions for \( J_4^{\mu\nu} \) and \( J_5^{\mu\nu} \), for which we refer the reader to ref. [39].

In the case of general relativity \((G_4 = \text{const}, G_3 = 0 = G_5)\), one finds \( J^{\mu\nu} = -G_4(K^{\mu\nu} - \gamma^{\mu\nu} K) \). A concrete expression for \( J^\phi \) is also found in ref. [39].

We allow for a localized source on \( \Sigma \) whose action is denoted by \( S_\Sigma \). Variation of the action \( S_\Sigma \) will take the form

\[
\delta S_\Sigma = \int_\Sigma d^3x \sqrt{\gamma} \left( \frac{1}{2} \tau^{\mu\nu} \delta \gamma^\mu\nu - \Delta^\phi \delta \phi \right),
\]  
(2.13)

where \( \tau^{\mu\nu} \) is the surface stress tensor from the localized source, giving the jump in \( J^{\mu\nu} \). The surface action also gives rise to the source \( \Delta^\phi \) for the jump in \( J^\phi \). From eqs. (2.11) and (2.13), we obtain [39]

\[
[J^{\mu\nu}]^+_- = -\frac{1}{2} \tau^{\mu\nu},
\]  
(2.14)

and

\[
[J^\phi]^-_+ = \Delta^\phi,
\]  
(2.15)

where \([\cdots]^+_- := (\cdots)|_{\Sigma_+} - (\cdots)|_{\Sigma_-} \). The above junction conditions together with the continuity \([\gamma^\mu\nu]^-_+ = 0 \) and \([\phi]^+_- = 0 \) determine how the metric and the scalar field are matched across the surface \( \Sigma \). It is now clear from those conditions that the first time derivatives of the metric and \( \phi \) can be discontinuous, and hence the second time derivatives can be singular at \( \Sigma \).

3 Cosmological matching conditions

We consider a slightly perturbed universe whose metric is given by

\[
ds^2 = -(1 + 2A)dt^2 + 2B_i dt dx^i + a^2 [(1 - 2\psi)\delta_{ij} + 2E_{ij} + h_{ij}] dx^i dx^j,
\]  
(3.1)
where $A$ and $\psi$ are scalar perturbations, $h_{ij}$ is a traceless and transverse tensor perturbation, and $B_i$ and $E_{ij}$ are decomposed into scalar and transverse vector parts as
\[
B_i = \partial_i B + B_i^V, \quad E_{ij} = \partial_i \partial_j E + \partial_i (E_j^V).
\] (3.2)

The scalar field also has a homogeneous part and a small inhomogeneous perturbation as $\phi(t, x) = \bar{\phi}(t) + \delta\phi(t, x)$. We will omit the bar on the homogeneous part when there is no worry about confusion.

Let the matching surface be specified by $q(t, x) = 0$. This equation can be decomposed as $\bar{q}(t) + \delta q(t, x) = 0$. The cosmological matching conditions on this hypersurface are derived by calculating $\bar{J}_j = \bar{J}_j^i(t) + \delta J^i_j(t, x)$ and $\bar{J}_\phi = \bar{J}_\phi^i(t) + \delta J^i_\phi(t, x)$. By using the temporal gauge transformation $t \rightarrow \tilde{t} = t + \xi^0$, one can move to the uniform $q$ gauge, i.e., the coordinate system satisfying
\[
\delta q \rightarrow \tilde{\delta} q = \delta q - \dot{q} \xi^0 = 0.
\] (3.3)

Then, the matching surface is determined simply by the equation $\bar{q}(\tilde{t}) = 0$, or, equivalently, $\tilde{t} = \text{const} = t_*$. Although the choice of the temporal gauge has no relevance to the matching conditions for the homogeneous background and tensor and vector perturbations, this coordinate system is convenient for the computation of $\delta J^i_\phi$ and the scalar part of $\delta J^i_j$. A particular example of $q$ is $q = \phi(t, x) - \phi_\ast$, where $\phi_\ast$ is some constant. Another example is $q = \rho(t, x) - \rho_\ast$.

3.1 Matching conditions for the homogeneous background

Let us first consider the matching conditions for a homogeneous and isotropic background. The homogeneous part of $J_{ij}$ is of the form $J^i_j = (1/3)J^i_j(\phi, \dot{\phi}, H)$, where
\[
\frac{1}{3} \bar{J}(\phi, \dot{\phi}, H) = -\frac{1}{2} f_3 + 2G_4H - 4HXG_4X + \phi G_4\phi - H^2X\dot{\phi}G_5X + 2HXG_5\phi,
\] (3.4)

with
\[
f_3(\phi, X) := \int_0^X \sqrt{2u}G_3u(\phi, u)du.
\] (3.5)

Assuming that there are no localized sources on $\Sigma$, the matching conditions for the background are given by $[a]^+ = 0$ and
\[
\left[ \bar{J}(\phi, \dot{\phi}, H) \right]^+ = 0.
\] (3.6)

In general relativity eq. (3.6) reduces to the standard matching condition $[H]^+ = 0$.

The same condition can be derived by integrating the background equation $\mathcal{P} = \rho$ (see appendix) from $t = t_* - \epsilon$ to $t = t_* + \epsilon$. Isolating the second time derivatives and denoting them with the subscript $\ddot{\bullet}$, one gets
\[
\mathcal{P} = (-2XG_3X + \cdots) \ddot{\phi} + (4G_4 + \cdots) \dot{H} = \left( \frac{2}{3} \partial_i \bar{J} \right)_{\ddot{\bullet}},
\] (3.7)

which implies eq. (3.6).
It is worth emphasizing that even if $G_4 = \text{const}$ and $G_5 = 0$, i.e., even if $\phi$ is minimally coupled to gravity, $G_3X$ gives rise to a nonstandard term $f_3$ in the junction condition (3.4). This is because the gravitational field equations contain second derivatives of $\phi$ in the presence of $G_3X$.

Similarly, it is straightforward to get

$$-\mathcal{J}^\phi(\phi, \dot{\phi}, H) = J + f_3 \phi - 6H G_4 \dot{\phi},$$

where

$$J := \dot{\phi} G_2 X + 6 H X G_3 \dot{\phi} + 6 H^2 \dot{\phi} (G_4 X + 2 X G_4 X) - 12 H X G_4 \ddot{\phi} X$$

$$+ 2 H^3 (3 G_5 X + 2 X G_5 X) - 6 H^2 \dot{\phi} (G_5 \phi + X G_5 \phi X).$$

(3.9)

In the absence of a localized source, we obtain the scalar-field matching condition

$$[ \mathcal{J}^\phi(\phi, \dot{\phi}, H) ]^+ = 0,$$

(3.10)

with the continuity $[\phi]^+ = 0$. In general relativity with a scalar field whose kinetic term is canonical, we have $\mathcal{J}^\phi = - \dot{\phi}$. The same equation as eq. (3.10) can be derived as well by integrating the scalar-field equation of motion (see appendix) from $t = t_* - \epsilon$ to $t = t_* + \epsilon$, noting that the second derivatives in the scalar-field equation are given by

$$\left( J - P_\phi \right)_{\dot{\phi}} = (G_2 X + \cdots) \dot{\phi} + (6 X G_3 X + \cdots) \dot{H},$$

$$= \left( - \partial_t \mathcal{J}^\phi \right)_{\dot{\phi}}.$$

(3.11)

The matching conditions (3.6), (3.10), and $[\phi]^+ = 0$ admit the solution satisfying the same conditions as in general relativity:

$$[H]_+^+ = 0, \quad [\dot{\phi}]^+ = 0.$$  

(3.12)

The second derivatives, $\dot{H}$ and $\ddot{\phi}$, can however be discontinuous (but not singular) across $\Sigma$. Obviously, $H^+$ and $\dot{\phi}^+$ determined from eq. (3.12) satisfy the Hamiltonian constraint, $E(\phi^+, \dot{\phi}^+, H^+) = - \rho^+$. There could be other nontrivial solutions, $H^+ \neq H^-$, $\dot{\phi}^+ \neq \dot{\phi}^-$, to the matching conditions (3.6) and (3.10). However, in contrast to the trivial solution (3.12), such solutions never satisfy the Hamiltonian constraint. Thus, in the absence of any localized sources, the first derivatives, $H$ and $\dot{\phi}$, must be continuous across $\Sigma$, i.e., there is no essential modification compared with the result of general relativity. This is indeed the case if the matter equation of state undergoes a sudden transition, $p = p_-(\rho) \rightarrow p_+(\rho)$, at some $\rho = \rho_* = \text{const}$ hypersurface. Another example is the model where the nonsingular bounce is caused by some scalar-field dynamics: in the scenario of [45], $\dot{H}$ and $\ddot{\phi}$ are continuous while $\dot{H}$ and $\ddot{\phi}$ can be approximated to be discontinuous at the beginning and end of the bounce phase.

To see the situation where the matching conditions do not reduce simply to eq. (3.12), let us investigate the model with a step-like potential for the scalar field, $G_2 \supset - V_0 \theta(\phi - \phi_*)$. In deriving the above scalar matching condition, we have implicitly assumed that the singular part in the scalar-field equation of motion comes only from the second derivatives $\dot{H}$ and $\ddot{\phi}$.
However, variation with respect to $\phi$ now gives $\delta \phi G_2 \supset -V_0 \delta (\phi - \phi_*) \delta \phi$, leading to a non-vanishing localized source of the jump in the right hand side of eq. (3.10). Equivalently, one can collect the singular part of the scalar-field equation of motion,

$$
\left( \dot{J} - P_\phi \right)_{\text{sing}} = \left( -\partial_t \mathcal{J}^\phi \right)_{\text{sing}} + V_0 \delta (\phi - \phi_*),
$$

(3.13)

to see that the scalar matching condition in the form (3.10) cannot be used due to the extra singular contribution $V_0 \delta (\phi - \phi_*)$. In this case, both $H$ and $\dot{\phi}$ are discontinuous in general, but $\mathcal{J}$ is continuous. In the genesis scenario which will be discussed in the next section [36–38, 49], such a step in the potential will cause an instantaneous change in $\dot{\phi}$ to end the genesis phase.

We present our numerical result in figure 1 corresponding to the situation in which $\phi$ suddenly slows down. As a simple example containing only $G_2$ and $G_3$ plus the Einstein-Hilbert term [35, 50], the Lagrangian

$$
\mathcal{L} = \frac{R}{2} + X - cX \Box \phi - \frac{V_0}{2} \left[ 1 + \tanh(\xi \phi) \right]
$$

(3.14)

with $c, V_0 = \text{const}$, and $\xi \gg 1$ is employed for the numerical calculation to mimic the case of the step-like potential. It can be seen that $H$ and $\dot{\phi}$ experience a sharp jump, but the matching condition (3.6) still holds.
3.2 Matching conditions for cosmological perturbations

We are now in position to consider matching of cosmological perturbations. The matching conditions for scalar, vector, and tensor modes can be studied separately.

(i) Tensor perturbations. The transverse and traceless part of $\tilde{\mathcal{F}}_i^j$ is

$$\tilde{\mathcal{F}}_i^j = -\frac{1}{4} \left( G_T \dot{h}_i^j + \frac{f_5}{a^2} \partial^2 h_i^j \right),$$

where

$$G_T := 2 \left[ G_4 - 2XG_4X - X \left( H\dot{\phi}G_5X - G_5\phi \right) \right],$$

and $f_5$ is defined similarly to $f_3$ as

$$f_5(\phi, X) := \int_0^X \sqrt{2u}G_5u(\phi, u)du.$$

Thus, the matching conditions for the tensor perturbations are given by

$$[h_{ij}]^+_\Sigma = 0, \quad \left[ G_T \dot{h}_{ij} + \frac{f_5}{a^2} \partial^2 h_{ij} \right]^+_\Sigma = 0.$$

(ii) Vector perturbations. The vector part of $\tilde{\mathcal{F}}_i^j$ is of the form

$$\tilde{\mathcal{F}}_i^j = \delta^{jk} \partial_i \delta^V_k,$$

$$\delta^V_i := \frac{G_T}{2} \left( \frac{B_i^V}{a^2} - \dot{E}_i^V \right).$$

Note that $B_i^V/a^2 - \dot{E}_i^V$ is the gauge-invariant combination. The matching condition for the vector perturbations is therefore given in any coordinates by

$$\left[ G_T \left( \frac{B_i^V}{a^2} - \dot{E}_i^V \right) \right]^+_\Sigma = 0.$$
(iii) Scalar perturbations. As mentioned above, we will work in the uniform $q$ gauge, $\delta q = q(t, x) - \bar{q}(t) = 0$. In this gauge, the continuity implies that

$$[\tilde{\psi}]^+_\psi = 0, \quad [\tilde{E}]^+_E = 0, \quad [\tilde{\phi}]^+_\phi = 0,$$

(3.22)

where the second condition can always be satisfied by choosing appropriately the spatial coordinates. The junction conditions are derived from

$$\tilde{J}_i^j = -\left( G_T \tilde{\dot{\psi}} + \Theta A \right) + \frac{\Theta - H G_T}{\phi} \tilde{\phi} \delta q + \frac{1}{3} \frac{\partial \tilde{J}}{\partial \phi} \tilde{\phi} \delta q + \frac{1}{2a^2} \left( \partial_i \delta q - \delta_i \partial^2 \right) \left[ -G_T \tilde{\dot{\phi}} - f_5 \tilde{\psi} \right],$$

(3.23)

$$\tilde{\phi} \tilde{J} = -\frac{2}{\phi} \left[ 3(\Theta - H G_T) \tilde{\dot{\psi}} - (\Sigma + 3H \Theta) A - \frac{\Theta - H G_T}{a^2} \partial^2 \tilde{\phi} \right] + \frac{\partial \tilde{J}}{\partial \phi} \tilde{\phi} \delta q - \frac{\Sigma}{X} \left( 6H \Theta + 3H^2 G_T \tilde{\dot{\phi}} \right) \sigma,$$

(3.24)

where we defined the shear as

$$\sigma := a^2 \dot{E} - B,$$

(3.25)

and

$$W := \dot{\phi} G_{4X} + H X G_{5X} - \dot{\phi} G_{5\phi} + \frac{f_5 \phi}{2},$$

(3.26)

$$Z := \dot{\phi} G_{3X} + F_{3\bar{X}} - 4F_{\dot{\phi} \bar{X}} + 4H G_{4X} + 8H X G_{14XX} - 2\dot{\phi} G_{4\phi X} + 2\dot{\phi} H^2 G_{5X} + 2\dot{\phi} H^2 G_{5\phi X}.$$ 

(3.27)

The above equations yield the uniform $q$ gauge expression of the matching conditions. However, using the following formulas,

$$\tilde{A} = A - \partial_i \left( \frac{\delta q}{\dot{q}} \right), \quad \tilde{\psi} = \psi + H \frac{\delta q}{\dot{q}},$$

$$\tilde{\phi} = \sigma - \frac{\delta q}{\dot{q}}, \quad \tilde{\phi} = \phi - \frac{\delta \phi}{\dot{\phi}},$$

(3.28)

one can undo the gauge fixing to move from one gauge to the other.

Let us first consider the case where the equation of state of matter experiences a sudden jump, $p = p_-(\rho) \rightarrow p_+(\rho)$, at the time when $\rho = \rho_* = \text{const}$, so that $\Sigma$ is determined by the equation

$$q(t, x) = \rho(t, x) - \rho_* = 0.$$ 

(3.29)

We assume that the localized source is absent on $\Sigma$, and so can use the matching conditions for the background in the form $[\tilde{J}]^+_{\tilde{J}} = [\tilde{J}]^+_{\tilde{\phi}} = 0$. From the discussion in the previous subsection, it turns out in the end that $H$ and $\dot{\phi}$ are continuous. From eq. (3.28) we see that the continuity (3.22) can be written in an arbitrary gauge as

$$[\psi + H \frac{\delta p}{\dot{p}}]_{\psi}^+ = 0,$$

(3.30)

$$[\delta \phi - \dot{\phi} \frac{\delta \rho}{\dot{\rho}}]_{\phi}^+ = 0.$$ 

(3.31)
The trace and traceless parts of the equations \([\delta J^i] = 0\) reduce in an arbitrary gauge to
\[
\left[ G_T \dot{\psi} + \Theta A - \frac{\Theta}{\phi} H G T \delta \dot{\phi} + \left( G_T \dot{H} + \frac{\Theta}{\phi} H G T \right) \frac{\delta \rho}{\rho} \right]^+ = 0,
\] (3.32)
and
\[
\left[ \sigma - \frac{\delta \rho}{\rho} \right]^+ = 0,
\] (3.33)
respectively. Using eq. (3.32), the matching condition \([\delta J^\phi] = 0\) in an arbitrary gauge reads
\[
\left[ -3 \Theta \dot{\psi} + \Sigma A - \frac{1}{\phi} (\Sigma + 3H \Theta) \delta \phi + \left( -3 \Theta \dot{H} + \frac{\Sigma + 3H \Theta}{\phi} \right) \frac{\delta \rho}{\rho} \right]^+ = 0.
\] (3.34)
The two conditions (3.32) and (3.34) can be rearranged to give
\[
\left[ \dot{\psi} + \frac{H}{\phi} \delta \phi + \left( \dot{H} - \frac{H \dot{\phi}}{\phi} \right) \frac{\delta \rho}{\rho} \right]^+ = 0,
\] (3.35)
\[
\left[ A - \frac{\delta \phi}{\phi} + \frac{\dot{\phi}}{\phi} \frac{\delta \rho}{\rho} \right]^+ = 0.
\] (3.36)
Interestingly, these matching conditions are independent of the concrete form of \(G_i(\phi, X)\), and hence are the same as those in general relativity with a conventional scalar field. Note, however, that the matching procedure requires the use of the constraint equations presented in appendix, which depend on the concrete form of \(G_i(\phi, X)\).

Having thus obtained the matching conditions in an arbitrary gauge, let us see how one can consistently determine the perturbation variables at \(t = t_* + \epsilon\) in the unitary gauge \((\delta \phi = 0)\). In this gauge, eq. (3.31) reads \([\delta \rho_u / \dot{\rho}]^+ = 0\), and then eq. (3.30) implies that
\[
[\mathcal{R}]^+_u = 0,
\] (3.37)
where \(\mathcal{R}\) is the curvature perturbation in the unitary gauge,
\[
\mathcal{R} := \psi + H \frac{\delta \phi}{\phi}.
\] (3.38)
Here and hereafter the subscript \(u\) refers to the unitary gauge variable. Equation (3.33) simply becomes \([\sigma_u]^+_u = 0\). Equations (3.35) and (3.36) can be used to determine \(\mathcal{R}\) and \(A_u\) at \(t = t_* + \epsilon\). The Hamiltonian constraint is consistent with eq. (3.34), while the momentum constraint is used to fix the velocity perturbation \(\delta u_u\). Thus, all the perturbation variables at \(t = t_* + \epsilon\) can be determined.

The matching procedure in the Newtonian gauge \((\sigma = 0)\) is slightly different from that in the unitary gauge. In the Newtonian gauge, eq. (3.33) reads \([\delta \rho_N / \dot{\rho}]^+_N = 0\), where the subscript \(N\) stands for the Newtonian gauge variable. In terms of the metric potentials in the Newtonian gauge,
\[
\Phi := A - \dot{\sigma}, \quad \Psi := \psi + H \sigma,
\] (3.39)
eqs. (3.30) and (3.31) are rewritten as
\[ [\Psi]^\pm = 0, \quad [\delta \phi_N]^\pm = 0, \tag{3.40} \]
while eqs. (3.35) and (3.36) yields the two relations among \( \dot{\Psi}, \Phi, \) and \( \dot{\delta \phi}_N. \) We then invoke
\[ G_T \Phi - F_T \Psi + \frac{\dot{G}_T}{\dot{\phi}} \delta \phi_N = 0, \tag{3.41} \]
to remove \( \Phi, \) and thus determine \( \dot{\Psi} \) and \( \dot{\delta \phi}_N \) at \( t = t_* + \epsilon. \)

The next example we would like to study is the transition that occurs when \( \phi \) reaches some value \( \phi_* \):
\[ q(t, x) = \phi(t, x) - \phi_. \tag{3.42} \]
In the previous example of \( q = \rho(t, x) - \rho_* \), we considered the case where \( H \) and \( \dot{\phi} \) are continuous. In present example, however, we allow for discontinuous \( H \) and \( \dot{\phi} \), because such a situation can easily be realized at the moment when \( \phi(t, x) \) passes a step in the potential at \( \phi_* \), as already demonstrated. In this case, it is convenient to stay in the uniform \( q \) gauge since it coincides with the uniform \( \phi \) gauge. Then, the curvature perturbation on uniform \( \phi \) hypersurfaces is given by \( R = \psi + H \delta \phi / \dot{\phi} = \psi, \) and the matching conditions \( [\tilde{\psi}]^\pm = 0 \) and \( [\tilde{\delta J}_i]^\pm = 0 \) reduce to
\[ [R]^\pm = 0, \tag{3.43} \]
\[ [G_T \tilde{R} + \Theta \tilde{A}]^\pm = 0, \tag{3.44} \]
\[ [G_T \tilde{\sigma} - f_5 \tilde{R}]^\pm = 0. \tag{3.45} \]
Let us first assume for simplicity that usual matter is absent. Equation (3.44) automatically holds thanks to the momentum constraints. Combining the Hamiltonian and momentum constraints, we find
\[ G_S \tilde{R} - \frac{1}{a^2} G_T \Theta \partial^2 \tilde{R} - \frac{G_T}{a^2} \partial^2 \tilde{\sigma} = 0. \tag{3.46} \]
The matching condition (3.45) then reads
\[ \left[ G_S \tilde{R} - \frac{1}{a^2} \left( \frac{G_T^2}{\Theta} + f_5 \right) \partial^2 \tilde{R} \right]^+_\pm = 0. \tag{3.47} \]
Using eqs. (3.43) and (3.47) one can do the matching of \( R \) and \( \tilde{R}. \) In the case where \( \dot{\phi} \) and \( H \) are continuous, the latter condition is simplified to \( [R]^\pm = 0. \) However, if the second derivatives diverge and hence \( \dot{\phi} \) and \( H \) are discontinuous, one must employ the full equation (3.47).

In the presence of usual matter, the matching condition (3.47) is modified as
\[ \left[ G_S \tilde{R} - \frac{1}{a^2} \left( \frac{G_T^2}{\Theta} + f_5 \right) \partial^2 \tilde{R} \right]^+ - \left[ \frac{G_T}{2 \Theta} \left( \delta \rho + \frac{\Sigma}{\Theta} (\rho + p) \tilde{u} \right) \right]^+ = 0, \tag{3.48} \]
while eq. (3.43) remains unchanged. Since the continuity and Euler equations for matter do not contain second derivatives of the metric, all the matter-related quantities are continuous across the matching surface specified by \( q = \phi(t, x) - \phi_* = 0. \) If the transition is such that \( \dot{\phi} \) and \( H \) are continuous, then eq. (3.48) implies that we are still allowed to use the condition \( [R]^\pm = 0. \)
4 Genesis models from Horndeski’s theory

In this section, we demonstrate how the matching conditions are used at the transition from the galilean genesis phase to the standard radiation-dominated Universe. This is probably the most illustrative example because stable galilean genesis is realized thanks to the terms \( \mathcal{L}_i \) with \( i \geq 3 \), which give rise to the new boundary terms. The matching procedure has been carried out in specific examples of galilean genesis in refs. [37, 38]. We will extend those previous models and present a unified analysis of the theory admitting galilean genesis. To do so, we generalize the Lagrangian of ref. [51] and study a subclass of Horndeski’s theory defined by

\[
G_2 = e^{4\lambda\phi}g_2(Y), \quad G_3 = e^{2\lambda\phi}g_3(Y), \quad G_4 = \frac{M_{Pl}^2}{2} + e^{2\lambda\phi}g_4(Y), \quad G_5 = e^{-2\lambda\phi}g_5(Y),
\]

(4.1)

where each \( g_i \) \( (i = 2, 3, 4, 5) \) is a function of

\[
Y := e^{-2\lambda\phi}X,
\]

(4.2)

and \( \lambda \) and \( M_{Pl} \) are constants. We assume that \( g_4(0) = 0 \).

Let us look for a solution of the form

\[
e^{\lambda\phi} \simeq \frac{1}{\lambda \sqrt{2Y_0}} (-t), \quad H \simeq \frac{h_0}{(-t)^3} \quad (-\infty < t < 0),
\]

(4.3)

where \( Y_0 \) and \( h_0 \) are positive constants. Note that \( Y \simeq Y_0 \) for this background. Equation (4.3) should be regarded as an approximate solution valid for \( |t| \gg \sqrt{h_0} \), and in this section we only consider the case where this approximation is good. The spacetime is close to Minkowski for \( |t| \gg \sqrt{h_0} \) and expands as \( a \simeq 1 + h_0 (t)^{-2}/2 \). Since \( H = 3h_0 (t)^{-4} > 0 \), one can interpret this solution to be NEC violating. The above solution is essential for the galilean genesis scenario [36–38, 49]. The Lagrangian defined by eq. (4.1) contains different models of galilean genesis as specific cases, and allows us to study the genesis scenario in a unified manner.\(^3\)

The background equations read

\[
\mathcal{E} \simeq 2XG_{2X} - G_2 - 2XG_{3\phi} = e^{4\lambda\phi} \dot{\rho}(Y_0) = 0, \quad (4.4)
\]

\[
\mathcal{P} \simeq 4(G_4 + XG_{5\phi}) \dot{H} + G_2 - 2X \left(G_{3\phi} + \ddot{\phi} G_{3X}\right) + 2\ddot{\phi}G_{4\phi} + 4XG_{4\phi\phi} + 4X\ddot{\phi}G_{4\phi X} + 4HX\ddot{\phi}G_{5\phi X} + 4H\dot{X}G_{5\phi} + 4HX\ddot{\phi}G_{5\phi\phi} = 2\mathcal{G}(Y_0) \dot{H} + e^{4\lambda\phi} \dot{\rho}(Y_0) = 0, \quad (4.5)
\]

where

\[
\dot{\rho}(Y) := 2Y g_2' - g_5 - 4\lambda Y \left( g_4 - Y g_4' \right), \quad (4.6)
\]

\[
\dot{\rho}(Y) := g_2 - 4\lambda Y g_3 + 24\lambda^2 Y \left( g_4 - Y g_4' \right), \quad (4.7)
\]

\[
\mathcal{G}(Y) := M_{Pl}^2 - 4AY \left( g_5 + Y g_5' \right), \quad (4.8)
\]

\(^3\)The DBI conformal galileons used in ref. [38] can be reproduced from \( G_5 = \tilde{g}_5(Y) \), rather than \( G_5 = e^{-2\lambda\phi}g_5(Y) \). However, for the genesis background (4.3), the contribution from \( \tilde{g}_5(Y) \) is subleading for \(|t| \gg \sqrt{h_0}\) compared to the other terms, and hence has no effect on any equations.
and a prime stands for differentiation with respect to $Y$. The constant $Y_0$ is determined as a positive root of

$$\hat{\rho}(Y_0) = 0,$$

(4.9)

and then $h_0$ is determined from eq. (4.5) as

$$h_0 = -\frac{1}{24\lambda^4 Y_0^2 \hat{G}(Y_0)}. \tag{4.10}$$

As will be seen shortly, this background is stable for $\hat{G}(Y_0) > 0$. Hence, the above NEC violating solution is possible provided that

$$\hat{\rho}(Y_0) < 0. \tag{4.11}$$

For tensor perturbations, it is straightforward to compute

$$\mathcal{G}_T \simeq \hat{G}(Y_0), \quad \mathcal{F}_T \simeq M_{\text{Pl}}^2 + 4\lambda Y_0 g_5(Y_0), \tag{4.12}$$

and therefore the background is stable against tensor perturbations if

$$\hat{G}(Y_0) > 0, \quad M_{\text{Pl}}^2 + 4\lambda Y_0 g_5(Y_0) > 0. \tag{4.13}$$

For scalar perturbations, we find

$$\mathcal{G}_S \simeq \left(\frac{\hat{G}}{\Theta}\right)^2 \Sigma, \quad \mathcal{F}_S \simeq \left(\frac{\hat{G}}{\Theta}\right)^2 \left(-\hat{\Theta}\right), \tag{4.15}$$

where

$$\Sigma \simeq e^{4\lambda \phi} Y_0 \hat{\rho}'(Y_0), \quad \Theta \simeq \left[\hat{G}(Y_0) + 2Y_0 \hat{G}'(Y_0)\right] H + \frac{2\lambda Y_0}{12Y_0} \left[2Y_0 \hat{\rho}'(Y_0) - \hat{\rho}(Y_0)\right]. \tag{4.16}$$

From eq. (4.17) it is easy to show

$$-\hat{\Theta} \simeq 2\dot{H} \left(\frac{\hat{G}}{\hat{p}}\right)' \bigg|_{Y_0}, \tag{4.18}$$

so that the background is stable against scalar perturbations if

$$\hat{\rho}'(Y_0) > 0, \quad \left(\frac{\hat{G}}{\hat{p}}\right)' \bigg|_{Y_0} > 0. \tag{4.19}$$

Since $\mathcal{G}_S \propto (-t)^2$ and $\mathcal{F}_S \propto (-t)^2$, the sound speed, $c_s = \sqrt{\mathcal{F}_S / \mathcal{G}_S}$, stays constant during the genesis phase.

The genesis phase is supposed to be followed by the standard radiation-dominated phase. As in [37], we consider the model in which the transition is caused by sudden halt of the scalar field due to some upward lift of its potential, $G_2 \supset -V_0 \theta(\phi - \phi_*)$. Then, the second
derivatives $\dot{\phi}$ and $\dot{H}$ diverge at $t = t_*$. We neglect the contribution from the scalar field to the expansion rate in the radiation-dominated phase, assuming $\dot{\phi}_{\text{rad}} \simeq 0$. It is then found that

$$\frac{1}{3} \mathcal{J}_{\text{gen}} \simeq \mathcal{G}(Y_0)H_{\text{gen}} - \frac{e^{3\lambda\phi}}{2} \int_0^{Y_0} \sqrt{2gy_3'(y)}dy + 2\lambda \dot{\phi}e^{2\lambda\phi} (g_4 - Y_0g'_4), \quad (4.21)$$

in the genesis phase and $(1/3)\mathcal{J}_{\text{rad}} \simeq M_{\text{Pl}}^2H_{\text{rad}}$ in the radiation-dominated phase. The radiation-dominated universe is required to be expanding, $H_{\text{rad}} > 0$. The matching condition $\mathcal{J}_{\text{gen}} - \mathcal{J}_{\text{rad}} = 0$ therefore reads $\mathcal{J}_{\text{gen}} > 0$. Using eq. (4.10), this condition can be written as

$$-g_2 - 2\lambda Y_0g_3(Y_0) + 3\lambda \sqrt{Y_0} \int_0^{Y_0} \frac{g_3(y)}{\sqrt{y}} dy > 0. \quad (4.22)$$

Note that this can be derived without relying on what the dominant component in the post-genesis phase is; we only require that the post-genesis universe is just expanding.

We have thus arrived at the generic conclusion based on the Lagrangian defined by eq. (4.1) without specifying its further concrete form: a consistent genesis scenario is realized provided $\hat{\rho}(Y_0) = 0$ has a positive root and eqs. (4.11), (4.13), (4.14), (4.19), (4.20), and (4.22) are satisfied. It is easy to fulfill all of these conditions simultaneously even in the simple Lagrangian with [36, 37]

$$g_2 = -Y + c_2Y^2, \quad g_3 = c_3Y, \quad g_4 = g_5 = 0, \quad (4.23)$$

where $c_2$ and $c_3$ are some constants. Indeed, all the requirements are satisfied for $4\lambda c_3 > c_2 > 0$.

Let us then investigate the matching of the perturbation variables. On superhorizon scales, the general solution to the tensor perturbation equation in Fourier space is given by

$$h_{ij} = C_{-k}^{q-} - \frac{C_{-k}^{d-}}{\mathcal{G}(Y_0)} \int_t^{t_*} \frac{dt'}{a^3(t')} \quad (t < t_*), \quad (4.24)$$

$$h_{ij} = C_{+k}^{q+} + \frac{C_{+k}^{d+}}{M_{\text{Pl}}^2} \int_{t_*}^t \frac{dt'}{a^3(t')} \quad (t > t_*), \quad (4.25)$$

where $C_{-k}^{q\pm}$ and $C_{-k}^{d\pm}$ are integration constants that depend on the wavenumber $k$. From the matching conditions (3.18), the integration constants in the post-genesis phase are determined as

$$C_{-k}^{q+} = C_{-k}^{q-}, \quad C_{+k}^{d+} = C_{-k}^{d-}. \quad (4.26)$$

Note, however, that tensor perturbations generated during the genesis phase are observationally irrelevant, because $a \sim 1$, $\mathcal{G}_T, \mathcal{F}_T \sim \text{const}$, so that the vacuum fluctuations of $h_{ij}$ are not amplified.

As for the scalar perturbations, it is most convenient to use the curvature perturbation on uniform $\phi$ slices, $\mathcal{R}$. The central quantity for the matching of $\mathcal{R}$ is $\mathcal{G}_S$, because from the matching conditions we see that $\mathcal{R}$ and $\mathcal{G}_S \dot{\mathcal{R}}$ are continuous on superhorizon scales. In the present case, $\mathcal{G}_S$ is of the form $\mathcal{G}_S = \mathcal{A}(Y_0)(-t)^2$ for $t < t_*$, and $\mathcal{G}_S = \mathcal{G}_S^+ = \text{const}$ for $t > t_*$, where the concrete expression for $\mathcal{A}(Y_0)$ is not so illuminating. The superhorizon solution
for $\mathcal{R}$ is given by

$$
\mathcal{R} = C^-_k - \frac{D^-_k}{\mathcal{A}(Y_0)} \int_t^{t^*} \frac{dt'}{(-t')^2a^3(t')}(t < t^*), \tag{4.27}
$$

$$
\mathcal{R} = C^+_k + \frac{D^+_k}{G_5} \int_{t_s}^t \frac{dt'}{a^3(t')}(t > t^*), \tag{4.28}
$$

where it follows from the matching conditions that

$$
C^+_k = C^-_k, \quad D^+_k = D^-_k. \tag{4.29}
$$

The second term in eq. (4.27) is matched to the second one in eq. (4.28), i.e., the decaying mode in the post-genesis Universe. Thus, we have $\mathcal{R} \simeq C^-_k$ at sufficiently late times. Following the usual quantization procedure we determine $|C^-_k| = (2\sqrt{c_s k A(Y_0)|t_s|}^{-1} \sim k^{-1/2}$, and hence we cannot get the scale-invariant fluctuations. See also refs. [52, 53] for discussions about the spectrum of fluctuations from galilean genesis.

5 Conclusions

In this paper, we have obtained the matching conditions for the homogeneous and isotropic Universe and for cosmological perturbations in Horndeski’s most general second-order scalar-tensor theory, starting from the generalization of Israel’s conditions [39]. In the absence of any localized sources at the transition hypersurface, we have shown that the first derivatives of the metric and the scalar field, $H(t)$ and $\dot{\phi}(t)$, must be continuous as in the case of general relativity. This is the case where the equation of state of matter undergoes a sharp change. In the case where $\phi$ suddenly lose its velocity due to a step in the potential, some combination $\mathcal{F}$ of $H$ and $\dot{\phi}$, defined in eq. (3.4), is continuous across the transition hypersurface. For cosmological perturbations we have obtained the junction equations that can be used in any gauge.

Horndeski’s theory can accommodate exotic but stable cosmologies such as galilean genesis [36]. The cosmological matching conditions we have presented in this paper can be applied to such interesting scenarios. To demonstrate this, we have developed a generic Lagrangian admitting the genesis solution that starts expanding from the Minkowski spacetime in the asymptotic past, and presented the conditions under which a stable genesis background is consistently joined to an expanding universe.

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A Field equations

In this appendix, we summarize the background and linearized equations used in the main text. More details can be found in refs. [12, 54].
A.1 Background equations

The evolution of the homogeneous and isotropic background is determined from

\[
\mathcal{E} = -\rho, \quad \mathcal{P} = -p, \tag{A.1}
\]

where

\[
\mathcal{E} := 2XG_{2X} - G_2 + 6X\dot{\phi}HG_{3X} - 2XG_{3\phi} - 6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX})
- 12HX\dot{\phi}G_{4\phi} + 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi\phi}),
\]

\[
\mathcal{P} = G_2 - 2X\left(G_{3\phi} + \ddot{\phi}G_{3X}\right) + 2\left(3H^2 + 2\dot{H}\right)G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X}
- 8\dot{H}XG_{4X} - 8HX\dot{X}G_{4XX} + 2\left(\ddot{\phi} + 2H\dot{\phi}\right)G_{4\phi} + 4XG_{4\phi\phi} + 4X\left(\ddot{\phi} - 2H\dot{\phi}\right)G_{4\phi X}
- 2X\left(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi}\right)G_{5X} - 4H^2X^2\dot{\phi}G_{5\phi X} + 4HX\left(\dot{X} - HX\right)G_{5\phi X}
+ 2\left[2(HX) + 3H^2X\right]G_{5\phi} + 4HX\ddot{\phi}G_{5\phi\phi},
\]

and \(\rho\) and \(p\) are the energy density and pressure of usual matter, respectively. The first equation corresponds to the Friedmann equation (the Hamiltonian constraint), and the second one to the evolution equation containing the second derivatives of the metric and the scalar field. The equation of motion for \(\phi\) is given by

\[
\dot{J} + 3HJ = P_\phi \tag{A.2}
\]

where

\[
J := \dot{\phi}G_{2X} + 6HXG_{3X} - 2\dot{\phi}G_{3\phi} + 6H^2\dot{\phi}(G_{4X} + 2XG_{4XX}) - 12HGX_{4\phi X}
+ 2H^3X(3G_{5X} + 2XG_{5XX}) - 6H^2\dot{\phi}(G_{5\phi} + XG_{5\phi\phi}), \tag{A.3}
\]

and

\[
P_\phi = G_{2\phi} - 2X\left(G_{3\phi} + \ddot{\phi}G_{3X}\right) + 6\left(2H^2 + \dot{H}\right)G_{4\phi} + 6H\left(\dot{X} + 2HX\right)G_{4\phi X}
- 6H^2XG_{5\phi\phi} + 2H^3X\dot{\phi}G_{5\phi X}. \tag{A.4}
\]

A.2 Linear perturbations

In the main test we use the following equations for scalar cosmological perturbations: (i) the Hamiltonian constraint,

\[
-6\Theta \dot{\psi} + 2G_T \frac{\partial^2 \psi}{a^2} + 2\Sigma A + \frac{2}{a^2} \Theta \partial^2 \sigma - \partial \mathcal{E} \frac{\dot{\phi}}{\dot{\phi}} \delta \phi - \frac{2}{\phi} (\Sigma + 3H\Theta) \delta \phi - \frac{2}{\phi} (\Theta - Hg_T) \frac{\partial^2 \delta \phi}{a^2} = \delta \rho, \tag{A.5}
\]

(ii) the momentum constraint,

\[
-2\left(G_T \dot{\psi} + \Theta A\right) + \frac{2}{\phi} (\Theta - Hg_T) \delta \phi + J \delta \phi
- 2\left(\Theta G_{4\phi} - \dot{\phi}G_{4\phi\phi} + 4H\dot{X}G_{4\phi X} - 2H\dot{X}G_{5\phi\phi} + H^2X\dot{\phi}G_{5\phi X}\right)\delta \phi = (\rho + p)\delta u, \tag{A.6}
\]

and (iii) the traceless part of the \((i, j)\) equations,

\[
G_T A = F_T \psi + \frac{\dot{G}_T + H(G_T - F_T)}{\dot{\phi}} \delta \phi - G_T \dot{\phi} - (\dot{G}_T + Hg_T) \sigma = 0, \tag{A.7}
\]
where we have included the perturbations of the matter energy-momentum tensor: \( \delta T_0^0 = -\partial \rho, \delta T_i^0 = (\rho + p)\partial_i \delta u \), and \( \delta T_i^j = \delta \phi \delta^j_i \). Energy-momentum conservation implies

\[
\partial \delta \rho + 3H(\delta \rho + \delta p) - 3(\rho + p)\psi + \frac{\rho + p}{a^2}\partial^2(\delta u + \sigma) = 0, \tag{A.8}
\]

\[
\partial_t [(\rho + p)\delta u] + 3H(\rho + p)\delta u + (\rho + p)A + \delta p = 0. \tag{A.9}
\]

In the above we defined

\[
F_T := 2 \left[ G_4 - X \left( \delta G_{5X} + G_{5\phi} \right) \right], \tag{A.10}
\]

\[
G_T := 2 \left[ G_4 - 2XG_{4X} - X \left( H\phi G_{5X} - G_{5\phi} \right) \right], \tag{A.11}
\]

\[
\Sigma := XG_{2X} + 2X^2G_{2XX} + 12H\phi XG_{3X} + 6H\phi X^2G_{3XX} - 2XG_{3\phi} - 2X^2G_{3\phi X} - 6H^2G_4 + 6 \left[ H^2(7XG_{4X} + 16X^2G_{4XX} + 4X^3G_{4XXX}) - H\phi \left( G_{4\phi} + 5XG_{4\phi X} + 2X^2G_{4\phi XX} \right) \right]
+ 30H^2\phi XG_{5X} + 26H^3\phi X^2G_{5XX} + 4H^3\phi X^3G_{5XXX}
- 6H^2X(6G_{5\phi} + 9XG_{5\phi X} + 2X^2G_{5\phi XX}),
\]

\[
\Theta := -\delta \phi XG_{3X} + 2HG_{4} - 8HXG_{4X} - 8H^2X^2G_{4XX} + \phi G_{4\phi} + 2X\phi G_{4\phi X}
- H^2\phi \left( 5XG_{5X} + 2X^2G_{5XX} \right) + 2H X \left( 3G_{5\phi} + 2XG_{5\phi X} \right).
\]

The unitary gauge \( \delta \phi = 0 \) is convenient in the absence of usual matter. The evolution equation for the curvature perturbation in the unitary gauge, \( \mathcal{R} \), follows from the quadratic action

\[
S^{(2)}_R = \int d^3x a^3 \left[ G_5 \mathcal{R}^2 - \frac{F_S}{a^2}(\partial \mathcal{R})^2 \right], \tag{A.12}
\]

where

\[
F_S := \frac{1}{a} \frac{d}{dt} \left( \frac{a}{\Theta} G_T^2 \right) - F_T, \tag{A.13}
\]

\[
G_S := \frac{\Sigma}{\Theta^2} G_T^2 + 3G_T. \tag{A.14}
\]

Similarly, the quadratic action for the tensor perturbation is given by

\[
S^{(2)}_h = \frac{1}{8} \int d^3x a^3 \left[ G_T \dddot{h}_{ij}^2 - \frac{F_T}{a^2}(\partial \dddot{h}_{ij})^2 \right]. \tag{A.15}
\]

From those actions we see that the scalar and tensor perturbations are stable if

\[
F_T > 0, \quad G_T > 0, \quad F_S > 0, \quad G_S > 0. \tag{A.16}
\]

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