THE COHOMOLOGICAL BRAUER GROUP OF WEIGHTED
PROJECTIVE SPACES AND STACKS

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Abstract. We compute the cohomological Brauer groups of twists of weighted projective spaces and weighted projective stacks.

1. Introduction

For any scheme $S$, we denote $\text{Br}'(S) := H^2_{\text{ét}}(S, \mathbb{G}_m)_{\text{tors}}$ the cohomological Brauer group of $S$. In this paper, we are interested in the cohomological Brauer groups of twists of weighted projective spaces and weighted projective stacks.

Let $n \geq 1$ and let $\rho = (\rho_0, \ldots, \rho_n)$ be an $(n+1)$-tuple of positive integers. There is a $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ sending $u \cdot (t_0, \ldots, t_n) \mapsto (u^\rho_0 t_0, \ldots, u^\rho_n t_n)$; let $P_{\mathbb{Z}}(\rho) := [(\mathbb{A}^{n+1}_{\mathbb{Z}} \setminus \{0\})/\mathbb{G}_m]$ denote the quotient stack; for any scheme $S$, the base change $P_S(\rho) := P_{\mathbb{Z}}(\rho) \times_{\text{Spec} \mathbb{Z}} S$ is called the weighted projective stack associated to $\rho$ over $S$.

The assumption that each $\rho_i$ is positive implies that the inertia stack of $P_{\mathbb{Z}}(\rho)$ is finite; hence $P_{\mathbb{Z}}(\rho)$ admits a coarse moduli space which may be described as follows [AH11, §2.1]. We view the polynomial ring $R := \mathbb{Z}[t_0, \ldots, t_n]$ as a $\mathbb{Z}$-graded ring where $\deg(t_i) = \rho_i$, and set $P_{\mathbb{Z}}(\rho) := \text{Proj} R$ and $P_S(\rho) := P_{\mathbb{Z}}(\rho) \times_{\text{Spec} \mathbb{Z}} S$ for any scheme $S$. The scheme $P_S(\rho)$ is the weighted projective space associated to $\rho$ over $S$. Weighted projective spaces often arise in the construction of moduli spaces. For example, we may naturally view the moduli space of elliptic curves (in characteristic 0) as an open subscheme of $P(4, 6)$. The moduli space of cubic surfaces is isomorphic to $P(1, 2, 3, 4, 5)$, see [Rei12].

Theorem 1.1. Let $f_X : X \to S$ be a morphism of schemes such that there exists an etale surjection $S' \to S$ such that $X \times_S S' \simeq P_{S'}(\rho)$. Then there is an exact sequence

\begin{equation}
\Gamma(S, \mathbb{Z}) \to \text{Br}'(S) \xrightarrow{f^*_X} \text{Br}'(X) \to 0
\end{equation}

of groups.

Theorem 1.2. Let $S$ be a scheme, let $f_X : \mathcal{X} \to S$ be a morphism of algebraic stacks such that there exists an etale surjection $S' \to S$ such that $\mathcal{X} \times_S S' \simeq \mathcal{P}_{S'}(\rho)$. Then there is an exact sequence

\begin{equation}
\Gamma(S, \mathbb{Z}) \to \text{Br}'(S) \xrightarrow{f^*_X} \text{Br}'(\mathcal{X}) \to 0
\end{equation}

Date: October 11, 2021.
of groups. Let \( \pi : X \rightarrow X \) be the coarse moduli space of \( X \). Then \( X \) satisfies the hypothesis of Theorem 1.1 and (1.1.1) and (1.2.1) fit into a commutative diagram

\[
\begin{array}{ccc}
\Gamma(S, \mathbb{Z}) & \xrightarrow{f_*} & \text{Br}'(S) \\
\times \text{lcm}(\rho) & \downarrow & \downarrow \pi^* \\
\Gamma(S, \mathbb{Z}) & \xrightarrow{f_*} & \text{Br}'(X) \\
\end{array}
\]

(1.2.2)

where the leftmost vertical map denotes multiplication-by-lcm(\( \rho \)).

As we recall in 3.1, a weighted projective space \( \mathbb{P}(\rho) \) is a toric variety. In Section 3, we use the results of DeMeyer, Ford, Miranda [DFM93] on the Brauer group of toric varieties to compute the Brauer group of \( \mathbb{P}(\rho) \) over an algebraically closed field; taking the prime-to-\( p \) limit of dilations of the toric variety reduces us to computing the \( p \)-torsion when each weight \( \rho_i \) is a power of \( p \). A deformation theory argument of Mathur [Mat19], which uses a Tannaka duality result of Hall, Rydh [HR19], allows us to deduce Theorem 1.1 over a strictly henselian local ring (see Lemma 4.2); then a Leray spectral sequence argument 4.4 gives the result for arbitrary schemes.

The proof of Theorem 1.2 is analogous to that of Theorem 1.1, except in case \( S \) is the spectrum of a field, which we deduce using that the \( \mathbb{G}_m \)-action on \( \mathbb{A}^{n+1} \) extends to an action of the multiplicative monoid \( A^1 \) on \( \mathbb{A}^{n+1} \).

1.3 (Acknowledgements). I am grateful to Tim Ford, Daniel Huybrechts, Max Lieblich, and Siddharth Mathur for helpful conversations. I also thank the Max Planck Institute for Mathematics in Bonn for their hospitality and financial support.

2. Weighted projective spaces

For general background on weighted projective spaces, see e.g. [Dol82], [RT13]. The \( k \)-rational points of \( \mathbb{P}(\rho) \) may be viewed as the quotient of \( k^{n+1} \setminus \{(0, \ldots, 0)\} \) by the equivalence relation \( (x_0, \ldots, x_n) \sim (u^{\rho_0}x_0, \ldots, u^{\rho_n}x_n) \); thus weighted projective spaces are also called twisted projective spaces.

The weighted projective space \( \mathbb{P}(\rho) \) is projective [EGA II, (2.1.6), (2.4.7)]. It is not necessarily true that the \( (\text{lcm}(\rho)) \)-th Veronese subring of \( \mathbb{Z}[t_0, \ldots, t_n] \) is generated in degree 1; for example, if \( \rho = (1, 6, 10, 15) \), the monomial \( t_1^4 t_2^3 t_3^1 \) has degree 60 = 2 \( \text{lcm}(\rho) \) but is not a product of two monomials of degree 30 [Del75, 2.6].

2.1. Suppose a positive integer \( d \) divides all \( \rho_i \) and set \( \rho/d := (\rho_0/d, \ldots, \rho_n/d) \). Then \( \mathbb{P}(\rho/d) \simeq \mathbb{P}(\rho/d) \) by [EGA II, (2.4.7) (i)], and under this isomorphism \( O_{\mathbb{P}(\rho/d)}(\ell) \) corresponds to \( O_{\mathbb{P}(\rho/d)}(d\ell) \) for all \( \ell \in \mathbb{Z} \).

By Lemma 2.2 we may also assume that \( \gcd(\{\rho_j\}_{j \neq i}) = 1 \) for all \( i \):

**Lemma 2.2** (Reduction of weights). [Del75 1.3], [Dol82 1.3.1], [AA89 1.3, 1.4] Suppose \( \gcd(\rho) = 1 \) and set

\[
\begin{align*}
d_i &:= \gcd(\{\rho_j\}_{j \neq i}) \\
s_i &:= \text{lcm}(d_j)_{j \neq i} \\
s &:= \text{lcm}(s_0, \ldots, s_n)
\end{align*}
\]
and let $R' := \mathbb{Z}[t'_0, \ldots, t'_n]$ be the ring with the $\mathbb{Z}$-grading determined by $\deg(t'_i) = \rho'_i := \rho_i/s_i$. The ring homomorphism $R' \to R$ sending $t'_i \mapsto t_i^{s_i}$ (which multiplies the degree by $s$) induces an isomorphism

$$\varphi : \mathbb{P}_\mathbb{Z}(\rho) \to \mathbb{P}_\mathbb{Z}(\rho')$$

of schemes. We have

$$(2.2.1) \quad \text{lcm}(\rho) = s \cdot \text{lcm}(\rho')$$

since $v_p(\text{lcm}(\rho)) = \alpha_i$ and $v_p(\text{lcm}(\rho')) = \alpha_{i0} - \alpha_{i-1}$ for any prime $p$, in the notation of \cite{AA89} 1.2.

For any integer $\ell$, there exists a unique pair $(b_1(\ell), c_\ell(\ell)) \in \mathbb{Z}^2$ satisfying $0 \leq b_1(\ell) < d_i$ and $\ell = b_1(\ell)\rho_i + c_\ell(\ell)d_i$; set $\ell' := \ell - \sum_{i=0}^n b_i(\ell)\rho_i$. The multiplication-by-$(t_0^{b_1(\ell)} \cdots t_n^{b_1(\ell)})$ map $R(\ell') \to R(\ell)$ induces an isomorphism $\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(\ell') \simeq \mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(\ell)$ of $\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}$-modules. Furthermore $\ell'$ is divisible by $s$ and we obtain an isomorphism

$$\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(\ell'/s) \simeq \varphi_*\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(\ell)$$

of $\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}$-modules. In particular, we have

$$(2.2.2) \quad \varphi^*(\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(\ell)) \simeq \mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(s\ell)$$

for all $\ell \in \mathbb{Z}$ since $b_i(s\ell) = 0$.

2.3. By \cite[Lemma 2.2]{Lemma 2.2} all weighted projective lines $\mathbb{P}_\mathbb{Z}(q_0, q_1)$ are isomorphic to $\mathbb{P}_{\mathbb{P}_\mathbb{Z}}$; thus, for \cite[Theorem 1.1]{Theorem 1.1}, we may assume $n \geq 2$.

**Definition 2.4.** \cite{AA89} §2 We say that $\rho$ satisfies (N) if $\gcd(\{\rho_j\}_{j \neq i}) = 1$ for all $i$.

2.5. \cite{AA89} §8 Let $\rho, \sigma$ be two weight vectors satisfying (N). We have $\mathbb{P}_\mathbb{Z}(\rho) \simeq \mathbb{P}_\mathbb{Z}(\sigma)$ if and only if $\rho$ is a permutation of $\sigma$.

**Lemma 2.6.** The sheaf $\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(r)$ is reflexive for any $r \in \mathbb{Z}$. If $\rho$ satisfies (N), the sheaf $\mathcal{O}_{\mathbb{P}_\mathbb{Z}(\rho)}(r)$ is invertible if and only if $\text{lcm}(\rho)$ divides $r$.

**Lemma 2.7** (Picard group of $\mathbb{P}(\rho)$). \cite{AA89} §6.1 For any connected locally Noetherian scheme $S$, the map

$$\mathbb{Z} \oplus \text{Pic}(S) \to \text{Pic}(\mathbb{P}_S(\rho))$$

sending

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathbb{P}_S(\rho)}(\ell \cdot \text{lcm}(\rho)) \otimes f_S^*\mathcal{L}$$

is an isomorphism. (See also \cite{Noo} §6.)

**Proof.** By \cite[Lemma 2.1]{Lemma 2.1} we may assume $\gcd(\rho) = 1$. In \cite{AA89} the desired claim is proved assuming that $\rho$ satisfies (N). If $\rho$ does not satisfy (N), then we conclude using \cite[(2.2.1)]{2.2.1} and \cite[(2.2.2)]{2.2.2}. \qed

**Lemma 2.8** (Cohomology of $\mathcal{O}_{\mathbb{P}(\rho)}(\ell)$). \cite{Del75} §3 Let $A$ be a ring and set $X := \mathbb{P}_A(\rho)$.

1. For $\ell \geq 0$, the $A$-module $H^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{c_0} \cdots t_n^{c_n}$ such that $c_0, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0c_0 + \cdots + \rho nc_n = \ell$.
2. For $\ell < 0$, the $A$-module $H^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{c_0} \cdots t_n^{c_n}$ such that $c_0, \ldots, c_n \in \mathbb{Z}_{< 0}$ and $\rho_0c_0 + \cdots + \rho nc_n = \ell$.
3. If $(i, \ell) \notin \{(0) \times \mathbb{Z}_{\geq 0}\} \cup \{(n) \times \mathbb{Z}_{< 0}\}$, then $H^i(X, \mathcal{O}_X(\ell)) = 0$.
4. For any $A$-module $M$ and any $(i, \ell)$, the canonical map

$$H^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to H^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

is an isomorphism.
Remark 2.9. Since $\mathbb{P}_\mathbb{Z}(\rho) \to \text{Spec} \mathbb{Z}$ is projective, if $S$ is quasi-compact and admits an ample line bundle, then so does $\mathbb{P}_S(\rho)$; thus $\text{Br} = \text{Br}'$ for $\mathbb{P}_S(\rho)$ by Gabber [J03] (i.e. the Azumaya Brauer group coincides with its cohomological Brauer group).

Remark 2.10. The projection $\mathbb{P}_\mathbb{Z}(\rho) \to \text{Spec} \mathbb{Z}$ is a flat morphism of relative dimension $n$, and its geometric fibers are normal. By [Dol82 1.3.3. (iii)], we have that $\mathbb{P}_S(\rho) \to S$ is smooth if and only if $\mathbb{P}_S(\rho) \cong \mathbb{P}^n_S$. If $\rho$ satisfies (N), then [2.5] implies that $\mathbb{P}_S(\rho) \cong \mathbb{P}^n_S$ if and only if $\rho = (1, \ldots, 1)$.

3. OVER AN ALGEBRAICALLY CLOSED FIELD

In this section, we prove [Theorem 1.1] in the case when $S = \text{Spec} k$ for an algebraically closed field $k$.

3.1 (Presentation as a toric variety). We recall from [Ful93 §2.2, CLS11 Example 3.1.17] how to view a weighted projective space as a toric variety (i.e. what the fan is).

Let $U \in \text{GL}_{n+1}(\mathbb{Z})$ be an invertible matrix which has $\rho$ as its 1st row (using the Euclidean algorithm, do column operations on $\rho$ to reduce to $(1, 0, \ldots, 0)$, then apply the inverse column operations in the reverse order on the identity matrix $\text{id}_{n+1}$); let $Y \in \text{Mat}_{(n+1) \times n}(\mathbb{Z})$ be the matrix obtained by removing the leftmost column of $U^{-1}$; let $v_0, \ldots, v_n \in \mathbb{Z}^n$ be the rows of $Y$; then $\mathbb{P}(\rho)$ is isomorphic to the toric variety associated to the fan $\Delta$ whose maximal cones are generated by the $n$-element subsets of $\{v_0, \ldots, v_n\}$.

3.2 (Reduce to computing the subgroup of Zariski-locally trivial Brauer classes). Let $\Delta' \to \Delta$ be a nonsingular subdivision of $\Delta$, and let $X'$ be the toric variety associated to $\Delta'$. The map $X' \to X$ is a toric resolution of singularities for $X$. As in [DFM93], we set $H^2(K/X, \mathbb{G}_m) := \ker(H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(K, \mathbb{G}_m));$ since $X'$ is regular, the restriction $H^2_{\text{ét}}(X', \mathbb{G}_m) \to H^2_{\text{ét}}(K, \mathbb{G}_m)$ is injective; hence there is an exact sequence

$$0 \to H^2(K/X, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(X', \mathbb{G}_m)$$

of abelian groups. Here $X'$ is a smooth, proper, geometrically connected, rational $k$-scheme; hence $H^2_{\text{ét}}(X', \mathbb{G}_m) = 0$ by birational invariance of the Brauer group (see [Gro68 Corollaire (7.3)] in characteristic 0 and [CTS19 Corollary 5.2.6] in general); thus it remains to compute $H^2(K/X, \mathbb{G}_m)$. By [DFM93 4.3, 5.1], there are natural isomorphisms

$$(3.2.1) \quad \tilde{H}^2(\mathcal{X}, \mathbb{G}_m) \simeq H^2_{\text{zar}}(X, \mathbb{G}_m) \simeq H^2(K/X, \mathbb{G}_m)$$

where $\mathcal{X} = \{U_{\sigma_0}, \ldots, U_{\sigma_n}\}$ is the Zariski cover of $X$ corresponding to the set of maximal cones of $\Delta$.

3.3 (Limit of dilations). Let $A$ be a ring and let $X$ be the toric variety (over $A$) associated to a fan $\Delta$ of cones in $\mathbb{N}G$. For any positive integer $d$, the multiplication-by-$d$ map $\times d : \mathbb{N} \to \mathbb{N}$ induces a finite $A$-morphism

$$\theta_d : X \to X$$

which is equivariant for the $d$th power map on tori. This is called a dilation [CHWW09 §6] (or toric Frobenius [HMP10 Remark 4.14]). For a cone $\sigma$ of $\Delta$, this is the $A$-algebra endomorphism of $\Gamma(U_{\sigma}, \mathcal{O}_{U_{\sigma}}) = A[\sigma^\vee \cap \mathcal{M}]$ sending $\chi^m \mapsto \chi^{dm}$ for $m \in \sigma^\vee \cap \mathcal{M}$. If the fan $\Delta$ is smooth, then $\theta_d$ is flat for any $d$.

We view $N$ as a category whose objects correspond to positive integers $m \in \mathbb{N}$ and there is a morphism $m_1 \to m_2$ if $m_1$ divides $m_2$. Let $S \subset \mathbb{N}$ be a multiplicatively closed subset; there is a functor $S^{\text{op}} \to (\text{Sch})$ sending $m \to X$ and $\{m_1 \to m_2\} \mapsto \theta_{m_2/m_1}$; the limit

$$X^{1/S} := \lim_{\longleftarrow} (\theta_{m_2/m_1} : X \to X)$$
of the resulting projective system is representable by a scheme since all the transition maps are affine. The scheme $X^{1/S}$ is isomorphic to the monoid scheme obtained by the usual construction with the finite free $\mathbb{Z}$-module $N$ and its dual $M$ replaced by the $S^{-1}\mathbb{Z}$-module $S^{-1}N$ and its dual $S^{-1}M = \text{Hom}_{S^{-1}\mathbb{Z}}(S^{-1}N, S^{-1}\mathbb{Z})$. More precisely, set $U^{1/S}_\sigma := \text{Spec} A[\sigma^\vee \cap S^{-1}M]$; for any face $\tau$ of $\sigma$, the canonical map $U^{1/S}_\sigma \to U^{1/S}_{\sigma \cap \tau}$ is an open immersion; then $U^{1/S}_{\sigma_1}$ and $U^{1/S}_{\sigma_2}$ are glued along the common open subscheme $U^{1/S}_{\sigma_1 \cap \sigma_2}$.

If $A$ is reduced, then we have

\[ \Gamma(U_\sigma, \mathbb{G}_m) = (A[\sigma^\vee \cap M])^\times = A^\times \cdot (\sigma^\perp \cap M) \]

for any cone $\sigma \in \Delta$; hence, by (3.2.1) the pullback

\[ \theta^*_{\sigma} : H^p_{\text{zar}}(X, \mathbb{G}_m) \to H^p_{\text{zar}}(X, \mathbb{G}_m) \]

is multiplication-by-$d$. In the limit, we obtain a natural isomorphism

\[ S^{-1}(H^p_{\text{zar}}(X, \mathbb{G}_m)) \cong H^p_{\text{zar}}(X^{1/S}, \mathbb{G}_m) \]

of $S^{-1}\mathbb{Z}$-modules.

**Lemma 3.4.** Let $d$ be a positive integer dividing $\rho_i$, and set $\rho' := (\rho_0, \ldots, \rho'_n)$ where $\rho'_i := \rho_i/d$ and $\rho'_j := \rho_j$ for $j \neq i$. If $d \in S$, then $\mathbb{P}_2(\rho^{1/S}) \cong \mathbb{P}_2(\rho'^{1/S})$.

**Proof.** As in 3.1 let $U, U' \in \text{GL}_{n+1}(\mathbb{Z})$ be invertible matrices whose first rows are $\rho, \rho'$ respectively. Let $U^0 \in \text{GL}_{n+1}(S^{-1}\mathbb{Z})$ be the matrix obtained by dividing the $i$th column of $U$ by $d$; then $(U^0)^{-1}$ is obtained by multiplying the $i$th row of $U^{-1}$ by $d$; this does not change the cones since we are just replacing $v'_i$ by $\frac{d}{d}v'_i$. Set $V := U' \cdot (U^0)^{-1} \in \text{GL}_{n+1}(S^{-1}\mathbb{Z})$; since the first rows of $U^0, U'$ are the same, the matrix $V$ has the form

\[
V = \begin{bmatrix}
1 & 0 \\
v' & V''
\end{bmatrix}
\]

for some $V' \in \text{Mat}_{n \times 1}(S^{-1}\mathbb{Z})$ and $V'' \in \text{GL}_n(S^{-1}\mathbb{Z})$. Let $Y^0, Y' \in \text{Mat}_{(n+1) \times n}(S^{-1}\mathbb{Z})$ be the matrices obtained by removing the leftmost column of $(U^0)^{-1}, (U')^{-1}$ respectively; then $(U')^{-1} \cdot V = (U^0)^{-1}$ implies $Y' \cdot V'' = Y^0$; then $V'' : S^{-1}\mathbb{N} \to S^{-1}\mathbb{N}$ induces the desired isomorphism $\mathbb{P}_2(\rho^{1/S}) \to \mathbb{P}_2(\rho'^{1/S})$. \hfill $\square$

**3.5.** We show that $H^2_{\text{zar}}(X, \mathbb{G}_m) = 0$ by showing that the localizations $H^2_{\text{zar}}(X, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Z}(p)$ are $0$ for every prime $p$. By (3.3.2) and Lemma 3.4 we may thus assume that

\[ \rho = (1, p^{e_1}, \ldots, p^{e_n}) \]

for some nonnegative integers $e_1 \leq \cdots \leq e_n$. In this case, in 3.1 we may take the 1st row of $U$ to be $\rho$ and the other rows to coincide with the identity $\text{id}_{n+1}$, so that

\[
Y = \begin{bmatrix}
-p^{e_1} & \cdots & -p^{e_n} \\
1 & & \\
& \ddots & \\
& & 1
\end{bmatrix}
\]

and thus $v_0 = (-p^{e_1}, \ldots, -p^{e_n})$ and $v_i$ is the $i$th standard basis vector of $\mathbb{Z}^n$.

**3.6 (Definition of $A^{\bullet, \bullet}$).** For convenience, we set $[n] := \{0, 1, \ldots, n\}$; we will use $I$ to denote a subset of $[n]$. We construct a double complex

\[
\{\{A^{p,q}, \{d^p_{i,q} : A^{p,q} \to A^{p,q+1}\}, \{d^p_{i,q} : A^{p,q} \to A^{p+1,q}\}\}
\]
as follows: for \(-1 \leq p \leq n\), we set
\[
A_{p,-1}^p = \bigoplus_{|I| = n-p} \mathbb{Z}^{n-p}
\]
(3.6.1)
\[
A_p^0 = \bigoplus_{|I| = n-p} \mathbb{Z}^n
\]
and \(A_{p,q} = 0\) if \((p, q) \notin \{-1, \ldots, n\} \times \{0, 1\} \).

For the vertical differential \(d^v_{p,0} : A_p^0 \to A_p^1\), the \(I\)th component (with \(|I| = n-p\))
of this map is the group homomorphism \(\mathbb{Z}^n \to \mathbb{Z}^{n-p}\) whose corresponding matrix
has rows \(\mathbf{v}_i\) for \(i \in I\).

The horizontal differentials \(d^h_{p,q} : A_{p,q} \to A_{p,q+1}\) are
defined with the sign conventions as follows: if \(I = \{i_0, \ldots, i_{n-p-1}\} \subset [n]\) is
a subset of size \(|I| = n-p\) and \(I'\) is obtained by removing the \(i\)th element
of \(I\) (where \(0 \leq i \leq n-p-1\)), then the restriction from the \(I\)th to \(I'\)th components
has sign \((-1)^i\).

The subcomplex of \(A^{\bullet, \bullet}\) obtained by restricting to \(p \geq 1\) is
isomorphic to the morphism of Cech cohomology \(\check{C}^*(\Delta, \mathcal{M}) \to \check{C}^*(\Delta, \mathcal{S}F)\), in
the notation of [DFM98 (5.0.1)].

3.7 (Diagram of \(A^{\bullet, \bullet}\)). Here is a diagram of the double complex \(A^{\bullet, \bullet}\):

\[
\begin{array}{cccccccc}
A_{-1,1} & \rightarrow & A_{0,1} & \rightarrow & A_{1,1} & \rightarrow & A_{2,1} & \rightarrow & \cdots & \rightarrow & A_{n-1,1} & \rightarrow & A_{n,1} \\
\downarrow d_{-1,0}^{-1,1} & & \downarrow d_{0,0}^{0,1} & & \downarrow d_{1,0}^{1,1} & & \downarrow d_{2,0}^{2,1} & & \cdots & & \downarrow d_{n,0}^{n-1,1} & & \downarrow d_{n,0}^{n,1} \\
A_{-1,0} & \rightarrow & A_{0,0} & \rightarrow & A_{1,0} & \rightarrow & A_{2,0} & \rightarrow & \cdots & \rightarrow & A_{n-1,0} & \rightarrow & A_{n,0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \\
\end{array}
\]

For a weighted projective surface (i.e. \(n = 2\)), this looks like

\[
\begin{array}{cccccccc}
\mathbb{Z}^3 & \rightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \rightarrow & \mathbb{Z}^1 \oplus \mathbb{Z}^1 \oplus \mathbb{Z}^1 & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& [v_0] & \rightarrow & [v_0] & \rightarrow & [v_0] & \rightarrow & 0 \\
& [v_1] & \rightarrow & [v_1] & \rightarrow & [v_1] & \rightarrow & 0 \\
& [v_2] & \rightarrow & [v_2] & \rightarrow & [v_2] & \rightarrow & 0 \\
\end{array}
\]

3.8. Let \(C^{\bullet}_n\) be the complex with \(C^k_n = \mathbb{Z}^{(i)}\) and such that the differentials \(C_n^k \rightarrow C^{k+1}_n\) have
sign conventions as above. Then \(C^{\bullet}_n\) is isomorphic to a direct sum of shifts of \(\text{id} : \mathbb{Z}^n \to \mathbb{Z}^n\),
hence is exact. The complex \(A^{\bullet, 0}\) is isomorphic to the direct sum \((C^{\bullet}_n)^n\), hence is exact.
The complex \(A^{\bullet, 1}\) is isomorphic to the direct sum \((C^{\bullet}_{n-1})^{n+1}\), hence is exact.

Let
\[
([E_r^{p,q}], \{d^r_{p,q} : E^{p,q}_r \to E^{p+r,q-r+1}_r\})
\]
denote the spectral sequence corresponding to the horizontal filtration on \(A^{\bullet, \bullet}\), so that
\(E_0^{p,q} = A^{p,q}\) and \(d_0^{p,q} = d_{p,q}^v\). Then \(E_2^{p,0}\) is identified with our Cech cohomology groups
\(\check{H}^p(\mathcal{U}, G_m)\), where \(\mathcal{U}\) is the Zariski open cover of \(X\) corresponding to the
maximal cones of \(\Delta\). Since there are only two nonzero rows, the differentials
\(d_2^{p,1} : E_2^{p,1} \to E_2^{p+2,0}\)
are isomorphisms for all \(p\). We are interested in \(\check{H}^2(\mathcal{U}, G_m) \cong E_2^{2,0} \simeq E_2^{0,1}\).
3.9. For the differential $d_{v_i}^0 : A_{i=0}^0 \to A_{i=1}^0$, the $I_1$th component (with $|I| = n$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$ whose corresponding matrix is obtained by removing the $i$th rows from $Y$ for $i \notin I$; hence

\[
E_{i=1}^0 \simeq \bigoplus_{p \in [n]} \mathbb{Z}/(p^{e_i})
\]

where a generator of the $i$th component $\mathbb{Z}/(p^{e_i})$ is given by the image of the 1st standard basis vector of $\mathbb{Z}^n$ (see §3.1).

For the differential $d_{v_i}^1 : A_{i=1}^1 \to A_{i=2}^1$, the $I_2$th component (with $|I| = n-1$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ whose corresponding matrix is obtained by removing the $i$th rows from $Y$ for $i \notin I$; hence

\[
E_{i=2}^1 \simeq \bigoplus_{I_2 \subset I_1} \mathbb{Z}/(p^{\min(e_{i_1}, e_{i_2})})
\]

where a generator of the $i$th component $\mathbb{Z}/(p^{\min(e_{i_1}, e_{i_2})})$ is given by the image of the 1st standard basis vector of $\mathbb{Z}^{n-1}$ (see §3.1).

3.10. We compute $E_{i=1}^2 = \ker d_{i=1}^1 / \im d_{i=2}^1$. With identifications as in (3.9.1) and (3.9.2), the image of $(x_0, x_1, \ldots, x_n) \in E_{i=1}^1$ under the differential $d_{i=1}^1 : E_{i=1}^1 \to E_{i=2}^1$ has $(i_1, i_2)$th coordinate $(-1)^{i_1-1}x_{i_1} + (-1)^{i_2-1}x_{i_2}$. Suppose $(x_0, x_1, \ldots, x_n) \in \ker d_{i=1}^1$; using the differential $d_{i=2}^1 : E_{i=2}^1 \to E_{i=3}^1$, we may assume that $x_n = 0$ in $\mathbb{Z}/(p^{e_n})$. Since $e_{n-1} \leq e_n$, the condition $(-1)^{n-1}x_{n-1} + (-1)^{n-2}x_{n-2} = 0$ in $\mathbb{Z}/(p^{e_{n-1}})$ forces $x_{n-1} = 0$ in $\mathbb{Z}/(p^{e_{n-1}})$. Using downward induction on $i$, we conclude that $x_{i} = 0$ in $\mathbb{Z}/(p^{e_{i}})$ for all $i$. Thus $E_{i=1}^2 = 0$.

Remark 3.11 (Assumptions on the base field). In [DFM93], it seems there are two implicit assumptions regarding the base field $k$:

1. It is assumed that $k$ is algebraically closed. This is used to conclude that all closed points are $k$-points and to identify the henselization and the strict henselization at a closed point of a variety. In the proof of Lemma 4.1, the reference to [ZS60, VIII, §13, Theorem 32] (in showing that an affine toric variety is analytically normal) requires $k$ to be perfect (here we may also use [Mat70, (31.1) Theorem 79]).

2. It is assumed that $k$ has characteristic 0. This is used to conclude that (5.1.1) is a split surjection; we only use their Lemmas 4.3 and 5.1, which do not depend on the characteristic of $k$. There are potential subtleties when considering the Brauer group of (affine) toric varieties in positive characteristic; for example, if $k$ is an algebraically closed field of characteristic $p$, the Brauer group of $k[x_1, x_2]$ has nontrivial $p$-torsion [AG60, 7.5], and these are not cup products (since $H^1_{\text{ét}}(A^2, \mu_p) = 0$).

4. Over a General Base Scheme

The following lemma [Lemma 4.1] combined with Section 3 proves Theorem 1.1 when $S = \text{Spec } k$ for an arbitrary field $k$.

Lemma 4.1. Let $A$ be a local ring, set $X := \mathbb{P}_A(\rho)$, let $P \in X(A)$ be an $A$-rational point and let $\alpha \in H^2_\text{ét}(X, \mathbb{G}_m)$ be a class such that $\alpha_P = 0$. If there exists a finite faithfully flat $A$-algebra $A'$ such that $\alpha_{A'} = 0$, then $\alpha = 0$.

Proof. Let $\mathcal{G} \to X$ be the $\mathbb{G}_m$-gerbe corresponding to $\alpha$. Since $\mathcal{G}_{A'}$ is trivial, there is a 1-twist bundle $\mathcal{L}'$ on $\mathcal{G}_{A'}$: set $A'' := A' \otimes A$ and $A''' := A' \otimes A' \otimes A'$; then there exists a line bundle $L''$ on $X_{A''}$ such that $L''_{|\mathcal{G}_{A'}} \simeq (p_1^*L')^{-1} \otimes p_2^*L'$; this line bundle $L''$ satisfies $p_{13}^*L'' \simeq p_{23}^*L'' \otimes p_{12}^*L''$; hence $L''$ is trivial since $p_{12}, p_{13}, p_{23} : \text{Pic}(X_{A''}) \to \text{Pic}(X_{A''})$ are the same maps $\mathbb{Z} \to \mathbb{Z}$ (see Lemma 2.7). Choose an isomorphism $\varphi : p_1^*L' \to p_2^*L'$ of $\mathcal{O}_{\mathcal{G}_{A''}}$-modules; the isomorphisms $p_{13}^*\varphi$ and $p_{23}^*\varphi \circ p_{12}^*\varphi$ differ by an element $u_{\alpha} \in \Gamma(X_{A''}, \mathbb{G}_m) \simeq$
Theorem 1.1 over strictly henselian local rings:

**Lemma 4.2.** Let $A$ be a strictly henselian local ring. Then $H^2_{\text{ét}}(\mathbb{P}_A(\rho), \mathbb{G}_m) = 0$.

**Proof.** By standard limit techniques, we may assume that $A$ is the strict henselization of a localization of a finite type $\mathbb{Z}$-algebra; in particular, $A$ is excellent \cite[5.6 iii]{Gre76}. Let $\mathfrak{m}$ be the maximal ideal of $A$ and let $k := A/\mathfrak{m}$ be the residue field.

We first consider the case when $A$ is complete. Set $X := \mathbb{P}_A(\rho)$ and let $\pi : G \to X$ be a $\mathbb{G}_m$-gerbe corresponding to a class $[G] \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. The class $[G]$ is trivial if and only if $\pi$ admits a section. We have that $G_0$ is a $\mathbb{G}_m$-gerbe over $X_0 = \mathbb{P}_k(\rho)$, which is a trivial gerbe by [Lemma 4.1] since $k$ is separably closed. For $\ell \in \mathbb{N}$, set $X_\ell := X \times_{\text{Spec } A} \text{Spec } A/\mathfrak{m}^{\ell+1}$ and $G_\ell := G \times_X X_\ell$. We have equivalences of categories

$$\text{Mor}(X, G) \cong \text{Hom}_{\mathcal{C}oh}(\text{Coh}(G), \text{Coh}(X))$$

where the equivalences marked 1 are by \cite[Theorem 1.1]{HR19} (here we use that $A$ is excellent), the equivalence marked 2 is Grothendieck existence \cite[III, 5.1.4]{EGA}, the equivalence marked 3 is \cite[Lemma 3.8]{HR19}.

It remains now to construct a compatible system of morphisms $X_\ell \to G$. A morphism $X_\ell \to G$ over $\mathbb{P}_A(\rho)$ corresponds to a 1-twisted line bundle on $G_\ell$; the obstruction to lifting a line bundle via $G_\ell \to G_{\ell+1}$ lies in $H^2_{\text{ét}}(G_\ell, m^\ell \mathcal{O}_{G_\ell})$; by Lemma 4.3 this is isomorphic to $H^2_{\text{ét}}(X_\ell, m^\ell \mathcal{O}_{X_\ell})$, which is 0 by Lemma 2.8.

In general, if $A$ is not complete, we use Artin approximation to descend a 1-twisted line bundle from $G^\vee$ to $G$.

**Lemma 4.3.** Let $X$ be an algebraic stack, let $A$ be a finitely generated abelian group, let $G = D(A)$ be the diagonalizable group scheme associated to $A$, and let $\pi : G \to X$ be a $G$-gerbe. For any quasi-coherent $\mathcal{O}_X$-module $F$, the pullback maps

$$(4.3.1) \quad H^0_{\text{ét}}(X, F) \to H^0_{\text{ét}}(G, \pi^* F)$$

are isomorphisms for all $i$.

**Proof.** Set $\mathcal{F} := \pi^* F$. We first assume that $X$ is a scheme. We have a Leray spectral sequence

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q\pi_* (\mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(G, \mathcal{F})$$

with differentials $E_2^{p,q} \to E_2^{p+2,q-1}$. Since $\pi_* \mathcal{F} \simeq F$, it suffices to show that $R^q\pi_* (\mathcal{F}) = 0$ for $q \geq 1$. Here the stalks of $R^q\pi_* (\mathcal{F})$ are the cohomology $H^0_{\text{ét}}(BG_A, \mathcal{F}|_A)$ of strictly henselian local rings $A := \mathcal{O}^\wedge_{X,x}$. Set $G := BG_A$; we show that if $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_G$-module then $H^0_{\text{ét}}(G, \mathcal{F}) = 0$ for all $p > 0$. The category of quasi-coherent $\mathcal{O}_G$-modules corresponds to the category $\mathcal{C}$ of $\mathbb{Z}$-graded $A$-modules. Denoting by $\pi : G \to \text{Spec } A$ the

---

1This is an argument of Siddharth Mathur \cite{Mat19}.
structure map, the pushforward functor $\pi_* : \text{QCoh}(\mathcal{G}) \to \text{QCoh}(A)$ corresponds to sending a $\mathbb{Z}$-graded module $M_* = \bigoplus_{n \in \mathbb{Z}} M_n$ to the degree zero component $M_0$. Since this is an exact functor, we have that $\pi$ is cohomologically affine [Alp13, Definition 3.1]. Since $\pi$ has affine diagonal, we have the desired result by [Alp13, Remark 3.5].

In case $X$ is an algebraic space, let $U \to X$ be an etale surjection where $U$ is a scheme, and let $U^p := U \times_X \cdots \times_X U$ be the $(p+1)$-fold fiber product; then we have descent spectral sequences

$$E_2^{p,q} = H^{p+q}_{\text{et}}(U^p, F|_{U^p}) \Rightarrow H^{p+q}_{\text{et}}(X, F)$$

$$E_2^{p,q} = H^{p+q}_{\text{et}}(\mathcal{G}|_{U^p}, F|_{\mathcal{G}|_{U^p}}) \Rightarrow H^{p+q}_{\text{et}}(\mathcal{G}, F)$$

where, by the first paragraph, the pullback

$$H^{q}_{\text{et}}(U^p, F|_{U^p}) \to H^{q}_{\text{et}}(\mathcal{G}|_{U^p}, F|_{\mathcal{G}|_{U^p}})$$

is an isomorphism for all $p, q$ since each $U^p$ is a scheme; hence $[4.3.1]$ is an isomorphism. In case $X$ is an algebraic stack, we take $U \to X$ to be a smooth surjection where $U$ is a scheme and argue as above, noting that each $U^p$ is an algebraic space.

\[\square\]

4.4 (Proof of Theorem 1.1). Set $f := f_X$. The Leray spectral sequence associated to the map $f$ and sheaf $\mathcal{G}_m$ is of the form

\[
E_2^{p,q} = H^p_{\text{et}}(S, \mathcal{R}^q f_* \mathcal{G}_m) \Rightarrow H^{p+q}_{\text{et}}(X, \mathcal{G}_m)
\]

with differentials $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2q-1}$. For any strictly henselian local ring $A$, we have $H^2_{\text{et}}(\mathbb{P}^1_A, \mathcal{G}_m) = 0$ by Lemma 4.2; hence $\mathcal{R}^2 f_* \mathcal{G}_m = 0$ since its stalks vanish. The sheaf $\mathcal{R}^1 f_* \mathcal{G}_m$ is the sheaf associated to $T \to \text{Pic}(X_T)$; by Lemma 2.7 every line bundle on $\mathbb{P}^1_T$ is, locally on $T$, isomorphic to one pulled back from $\mathbb{P}^1_\mathbb{Z}$; hence $\mathcal{R}^1 f_* \mathcal{G}_m$ is isomorphic to the constant sheaf $\mathbb{Z}$. Hence we have an exact sequence

\[
H^0_{\text{et}}(S, \mathbb{Z}) \xrightarrow{\delta} H^2_{\text{et}}(S, \mathcal{G}_m) \xrightarrow{\beta} H^2_{\text{et}}(X, \mathcal{G}_m) \to H^1_{\text{et}}(S, \mathbb{Z})
\]

and we may argue as in [Shi19, 5.4] to show that $f^*$ restricts to a surjection on the torsion subgroups, inducing an exact sequence $[1.1.1]$ as desired.

\[\square\]

Remark 4.5. From [4.3] we see that the map $\Gamma(S, \mathbb{Z}) \to \text{Br}'(S)$ in $[1.1.1]$ corresponds to the differential $d_2^{0,1}$ in the Leray spectral sequence. The Brauer class corresponding to the image of $1 \in \Gamma(S, \mathbb{Z})$ may be described as follows. Set $R := \mathbb{Z}[t_0, \ldots, t_n]$ with $\text{deg}(t_i) = 1$, and let $\text{Aut}_{\text{gr.alg}}(R)$ denote the group sheaf sending a scheme $T$ to the set of $\mathbb{Z}$-graded $\mathcal{O}_T$-algebra automorphisms of $R \otimes \mathcal{O}_T$. By [AA89, 8.8], we have an exact sequence

\[
1 \to \mathcal{G}_m \to \text{Aut}_{\text{gr.alg}}(R) \to \text{Aut}_{\text{sch}}(\mathbb{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z})))) \to 1
\]

of sheaves of groups for the etale topology on the category of schemes, where the image of $\mathcal{G}_m$ is contained in the center of $\text{Aut}_{\text{gr.alg}}(R)$. By definition, $X$ is an $\text{Aut}_{\text{sch}}(\mathbb{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}))))$-torsor over $S$, and the class of $[X]$ under the coboundary map

\[
H^1_{\text{et}}(S, \text{Aut}_{\text{sch}}(\mathbb{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z}(\mathcal{P}_\mathbb{Z})))))) \to H^2_{\text{et}}(S, \mathcal{G}_m)
\]

is the desired Brauer class.

Alternatively, fix an etale surjection $S' \to S$ and set $S'' := S' \times_S S'$ and $S''' := S' \times_S S' \times_S S'$; the choice of an isomorphism $X \times_S S' \simeq \mathbb{P}^r_S(\mathcal{P}_S')$ yields an automorphism $\varphi : \mathbb{P}^r_S(\mathcal{P}_S') \to \mathbb{P}^r_S(\mathcal{P}_S')$ satisfying the cocycle condition $p_{13}^* \varphi = p_{23}^* \varphi \circ p_{12}^* \varphi$ over $S'''$. Choose $\ell \gg 0$ so that $\mathcal{O}_{\mathbb{P}^r_S(\mathcal{P}_S')}(\ell)$ is very ample; fixing a $\mathbb{Z}$-basis of $\Gamma(S'', \mathcal{O}_{\mathbb{P}^r_S(\mathcal{P}_S')(\ell)})$ gives an invertible matrix $\varphi^\sharp \in \text{GL}_r(\Gamma(S'', \mathcal{O}_{\mathbb{P}^r_S(\mathcal{P}_S')}(\ell)))$, where $r = \text{rank}_2 \Gamma(S'', \mathcal{O}_{\mathbb{P}^r_S(\mathcal{P}_S')(\ell)})$; here the invertible matrices
Remark 4.6 (Comparison to the argument of Gabber). In [Gab78], Gabber computes the Brauer group of Brauer-Severi schemes over an arbitrary base scheme by combining the following two facts to reduce to the \( \mathbb{P}^1 \) case:

1. Suppose \( Y \to X \) is a closed immersion locally defined by a regular sequence, and let \( B \to X \) be the blowup of \( X \) at \( Y \); then \( H^2_{\text{et}}(X, \mathbb{G}_m) \to H^2_{\text{et}}(B, \mathbb{G}_m) \) is injective.

2. The blowup of \( \mathbb{P}^n \) at a point is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^{n-1} \).

In our case, we may ask whether the analogous statement to (2) holds – namely, whether a (weighted) blowup of \( \mathbb{P}^2(\rho) \) at a (torus-invariant) local complete intersection subscheme is isomorphic to a \( \mathbb{P}(\rho' \prime) \)-bundle over \( \mathbb{P}(\rho' \prime) \) for some \( \rho', \rho'' \) such that \( |\rho| - 1 = |\rho' | - 1 + |\rho''| - 1 \).

Indeed, the blowup of the weighted projective surface \( \mathbb{P}(1, 1, q_2) \) at its unique singular point gives the \( q_2 \)-th Hirzebruch surface \( \mathbb{F}_{q_2} \) (see [Dol72 1.2.3], [Gau99]). Such a result for arbitrary \( \rho \) would give an alternative proof of [Theorem 1.1]. This seems unlikely, however, as it (with \([2.3]\)) would imply that every weighted projective surface \( \mathbb{P}(\rho_0, \rho_1, \rho_2) \) is an \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \), which has Picard group \( \mathbb{Z}^2 \), but \( \mathbb{P}(2, 3, 5) \) has three isolated singular points and blowing up these points increases the Picard rank by 3.

5. Weighted Projective Stacks

In this section we assume \( n \geq 1 \).

The weighted projective stack \( \mathcal{P}_\mathbb{Z}(\rho) \) is smooth for any \( \rho \) (hence \( \mathcal{P}_\mathbb{Z}(\rho) \to \mathbb{P}_\mathbb{Z}(\rho) \) is not an isomorphism if \( \rho \neq (1, \ldots, 1) \)). A Deligne-Mumford stack \( \mathcal{X} \) and its coarse moduli space \( \mathcal{X} \) may have different Brauer groups (and Picard groups) in general.

**Lemma 5.1.** For any field \( k \), the pullback map

\[
H^2_{\text{et}}(\text{Spec } k, \mathbb{G}_m) \to H^2_{\text{et}}(\mathcal{P}_k(\rho), \mathbb{G}_m)
\]

is an isomorphism.

**Proof.** We have a descent spectral sequence

\[
E_1^{p, q} = \frac{H^p_{\text{et}}(\mathcal{P}_{m,k} \times_k (\mathbb{A}_k^{n+1} \setminus \{0\}), \mathbb{G}_m)}{E_1^{p+q}(\mathcal{P}_k(\rho), \mathbb{G}_m)}
\]

with differentials \( d_1^{p, q} : E_1^{p, q} \to E_1^{p+1, q} \). Each \( \mathcal{P}_{m,k} \times_k (\mathbb{A}_k^{n+1} \setminus \{0\}) \) is an open subscheme of \( \mathbb{A}_k^{n+p+1} \), hence has trivial Picard group; hence \( E_1^{1,1} = 0 \) for all \( p \). The pullback \( B\mathbb{G}_{m,k} \to \mathcal{P}_k(\rho) \) induces an isomorphism of complexes \( H_{\text{et}}^0(\mathcal{P}_{m,k}, \mathbb{G}_m) \to E_1^{0,0} \); hence, by the proof of [Shi19 Lemma 4.2], we have \( E_2^{2,0} = 0 \).

It remains to compute \( E_2^{1,2} \), which is isomorphic to the equalizer of the two pullback maps

\[
a^*, p_2^* : H^2_{\text{et}}(\mathbb{A}_k^{n+1} \setminus \{0\}, \mathbb{G}_m) \to H^2_{\text{et}}(\mathcal{P}_{m,k} \times_k \mathbb{A}_k^{n+1} \setminus \{0\}, \mathbb{G}_m)
\]

corresponding to the action map and second projection, respectively; by purity for the Brauer group (see Gabber [Gab02] and Česnavičius [Ces19]), this is isomorphic to the equalizer of \( a^*, p_2^* : H^2_{\text{et}}(\mathbb{A}_k^{n+1}, \mathbb{G}_m) \to H^2_{\text{et}}(\mathbb{G}_m \times_k \mathbb{A}_k^{n+1}, \mathbb{G}_m) \), and also to the equalizer of \( a^*, p_2^* : H^2_{\text{et}}(\mathbb{A}_k^{n+1}, \mathbb{G}_m) \to H^2_{\text{et}}(\mathbb{A}_k^{n+1}, \mathbb{G}_m) \) since the restriction \( H^2_{\text{et}}(\mathbb{A}_k^{n+1}, \mathbb{G}_m) \to \)
$H^2_{\mathbb{Z}}(G_m \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$ is injective. With coordinates $A^1_k = \text{Spec } k[u]$, let $f : A^1_k \times_k \mathbb{A}^{n+1}_k \to \mathbb{A}^{n+1}_k$ be the morphism of $k$-schemes obtained by setting $u = 0$; note that $p_2f = \text{id}$ and $af$ factors through $\text{Spec } k$. Let $\alpha \in H^2_{\mathbb{Z}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m)$ be a Brauer class such that $a^*\alpha = p_2^*\alpha$ in $H^2_{\mathbb{Z}}(A^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$; then $f^*a^*\alpha = f^*p_2^*\alpha = \alpha$; hence $\alpha$ is in the image of $H^2_{\mathbb{Z}}(\text{Spec } k, \mathbb{G}_m)$.

**Lemma 5.2.** [Noo 4.3] For any connected scheme $S$, the map

$$\mathbb{Z} \oplus \text{Pic}(S) \to \text{Pic}(\mathcal{P}_S(\rho))$$

sending

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathcal{P}_S(\rho)}(\ell) \otimes \pi_2^*\mathcal{L}$$

is an isomorphism.

**Lemma 5.3** (Cohomology of $\mathcal{O}_{\mathcal{P}_S(\rho)}(\ell)$). [Mei15, 2.5] Let $A$ be a ring and set $X := \mathcal{P}_A(\rho)$.

1. For $\ell \geq 0$, the $A$-module $H^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^a \cdots t_n^m$ such that $a_0, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0a_0 + \cdots + \rho_na_n = \ell$.
2. For $\ell < 0$, the $A$-module $H^n(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^a \cdots t_n^m$ such that $a_0, \ldots, a_n \in \mathbb{Z}_{< 0}$ and $\rho_0a_0 + \cdots + \rho_na_n = \ell$.
3. If $(i, \ell) \notin \{\{0\} \times \mathbb{Z}_{\geq 0}\} \cup \{\{n\} \times \mathbb{Z}_{< 0}\}$, then $H^i(X, \mathcal{O}_X(\ell)) = 0$.
4. For any $A$-module $M$ and any $(i, \ell)$, the canonical map

$$H^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to H^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

is an isomorphism.

**Lemma 5.4.** Let $A$ be a strictly henselian local ring. Then $H^2_{\mathbb{Z}}(\mathcal{P}_A(\rho), \mathbb{G}_m) = 0$.

**Proof.** The proof is the same as that of Lemma 4.2 with the following modifications: the gerbe $G_0$ over the special fiber follows from Lemma 5.1 to obtain the equivalence marked 2, we use Grothendieck existence for stacks [Ols05, 1.4] (using that $\mathcal{P}(\rho)$ is proper [Mei15, 2.1]); to conclude that $H^2_{\mathbb{Z}}(X_{\ell}, \mathfrak{m}^i\mathcal{O}_{X_{\ell}}) = 0$, we use Lemma 5.3.

**Lemma 5.5.** Let

$$\pi_\rho : \mathcal{P}_\mathbb{Z}(\rho) \to \mathbb{P}_\mathbb{Z}(\rho)$$

denote the coarse moduli space morphism. For any $\ell \in \mathbb{Z}$, there is a canonical $\mathcal{O}_{\mathcal{P}_\mathbb{Z}(\rho)}$-linear map

$$(5.5.1) \quad \pi_\rho^*(\mathcal{O}_{\mathcal{P}_\mathbb{Z}(\rho)}(\ell)) \to \mathcal{O}_{\mathcal{P}_\mathbb{Z}(\rho)}(\ell)$$

which is an isomorphism if $\ell$ is divisible by $\text{lc}(\rho)$.

**Proof.** Set $R := \mathbb{Z}[\ell_0, \ldots, \ell_n]$ with the $\mathbb{Z}$-grading determined by $\deg(\ell_i) = \rho_i$. The restriction of $\mathcal{O}_{\mathcal{P}_\mathbb{Z}(\rho)}(\ell)$ to the open substack $[(\text{Spec } R[t_i^{-1}])/\mathbb{G}_m]$ corresponds to the graded homomorphism

$$(5.5.2) \quad R(\ell)[t_i^{-1}]_{\ell} \otimes_{R[t_i^{-1}]_{\ell}} R[t_i^{-1}]_{\ell} \to R(\ell)[t_i^{-1}]_{\ell}$$

of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules; the $m$th component of $R(t_i^{-1})_{\ell} \otimes_{R[t_i^{-1}]_{\ell}} R[t_i^{-1}]_{\ell}$ is isomorphic to the $R[t_i^{-1}]_{\ell}$-linear map

$$(5.5.3) \quad R[t_i^{-1}]_{\ell} \otimes_{R[t_i^{-1}]_{\ell}} R[t_i^{-1}]_{\ell} \to R[t_i^{-1}]_{\ell}$$

induced by multiplication. If $\ell$ is divisible by $\rho_i$, then the multiplication-by-$t_i^{\ell/\rho_i}$ map $R[t_i^{-1}] \to R[t_i^{-1}]_{\ell}$ is an isomorphism of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules, thus $R(t_i^{-1})_{\ell} \otimes_{R[t_i^{-1}]_{\ell}} R[t_i^{-1}]_{\ell}$ is an isomorphism for all $m \in \mathbb{Z}$, in other words the restriction of $\mathcal{O}_{\mathcal{P}_\mathbb{Z}(\rho)}(\ell)$ to $[(\text{Spec } R[t_i^{-1}])/\mathbb{G}_m]$ is an isomorphism. □
Lemma 5.6. The pullback $\pi^*_ρ : \text{Pic}(\mathbb{P}^m_\rho) \to \text{Pic}(\mathbb{P}^n_\rho)$ is multiplication by $\text{lcm}(ρ)$.

Proof. We have that $\text{Pic}(\mathbb{P}^m_\rho) \simeq \mathbb{Z}$ is generated by the class of $O_{\mathbb{P}^m_\rho}(\text{lcm}(ρ))$, and that $\text{Pic}(\mathbb{P}^n_\rho) \simeq \mathbb{Z}$ is generated by the class of $O_{\mathbb{P}^n_\rho}(\text{lcm}(1))$ by Lemma 2.7 and Lemma 5.2 respectively. We have the desired claim by Lemma 5.5. □

Remark 5.7. There exist $ρ, ℓ$ for which the natural map $[5.5.1]$ is not an isomorphism. For example, in case $ρ = (1, 2)$ and $ℓ = 1$, the element $t_0 \in \mathbb{Z}[t_0^{−1}]_2$ is not in the image of the map $[5.5.3]$ for $m = 1$ and $i = 0$. We have $O_{\mathbb{P}^1_ρ}(1) \simeq O_{\mathbb{P}^1_π}(1)$, and the pullback $[5.5.1]$ is multiplication by $t_1 \in \Gamma(\mathbb{P}_ρ, O_{\mathbb{P}^1_ρ}(1))$; see Lemma 2.2 for details. Furthermore, the natural map $O_{\mathbb{P}^1_ρ}(1) \otimes O_{\mathbb{P}^1_ρ}(1) \to O_{\mathbb{P}^1_π}(2)$ is not an isomorphism; here [EGA II, 2.5.13] does not apply since $R$ is not generated in degree 1. (See also [Del75 4.8], [Del82 1.5.3].)

5.8 (Proof of Theorem 1.2). The proof of the exactness of $[1.2.1]$ is the same as in [4.4] with the following modifications: to show $R^2π_*G_m = 0$, we use Lemma 5.4 to show $R^1π_*G_m \simeq \mathbb{Z}_π$. An automorphism of the weighted projective stack induces an automorphism of the weighted projective space by universal properties of a coarse moduli space morphism; hence, if $X$ is etale-locally isomorphic to $\mathbb{P}^n_ρ$, then $X$ is etale-locally isomorphic to $\mathbb{P}^n_π$.

The commutativity of $[1.2.2]$ follows from Lemma 5.6. □

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