Blocks in the category of finite-dimensional representations of principal \( W \)-algebra for \( Q(2) \)

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Abstract. We describe the blocks in the category of finite-dimensional representations of the principal finite \( W \)-algebra for the Lie superalgebra \( Q(2) \).

Introduction
In the classical case when \( g \) is a complex semi-simple Lie algebra and \( e \) is a nilpotent element in \( g \), a finite \( W \)-algebra for \( g \) is a quantization of the Poisson structure on the Slodowy slice (a transversal slice to the orbit of \( e \) in the adjoint representation). The general definition of a finite \( W \)-algebra was given by A. Premet in [16]. For a Lie superalgebra \( g = g_0 \oplus g_1 \) with a reductive even part \( g_0 \), the finite \( W \)-algebra is associated with an even nilpotent element \( e \in g_0 \). It is denoted by \( W \).

In the case when \( g = gl(m|n) \) and \( e \) is the even principal nilpotent, J. Brown, J. Brundan and S. Goodwin classified irreducible representations of \( W \) and explored the connection with the category \( O \) for \( g \) using coinvariants functor [1, 2].

In [13] we considered \( W \) associated with an even nilpotent element \( \varphi \in g_0^* \subset g^* \) in the coadjoint representation (this means that for the algebraic reductive group \( G_0^* \) of \( g_0^* \), the closure of the \( G_0 \)-orbit of \( \varphi \) in \( g_0^* \) contains zero). We proved that if \( \varphi \) is the principal nilpotent, i.e. the dimension of the even part of the annihilator of \( \varphi \) in \( g \) is minimal, and \( g \) is isomorphic to \( sl(m|n) \), \( \mathfrak{usp}(2|2n) \) or \( Q(n) \), then every simple \( W \)-module is finite-dimensional. In [15] we classified simple \( W \)-modules for \( g = Q(n) \) associated with the principal nilpotent coadjoint orbits (Theorem 4.6). The technique we used is completely different from one used in [1] due to the lack of triangular decomposition of \( W \) in our case. Instead, we described the restriction of simple \( U(\mathfrak{g}) \)-modules to \( W \) and proved that any simple \( W \)-module occurs as a constituent of this restriction.

We consider the category \( W - \text{mod} \) of finite-dimensional \( W \)-modules. The natural problem is to describe the blocks in this category. In this work we make a step in this direction and describe the blocks in the case when \( g = Q(2) \) (Theorem 8). Our results should have applications to the study of primitive ideals of \( U(\mathfrak{g}) \) in the sense of I. Losev (see [8, 9, 10]). In the super case the theory of the primitive ideals is even more complicated (see [3]). We also intend to apply these results to classify simple modules for super-Yangians of type \( Q \).

All results are joint work with V. Serganova.
1. Finite $W$-algebra for $Q(n)$

In this paper we consider the Lie superalgebra $\mathfrak{g} = Q(n)$ defined as follows (see [6]). Equip $\mathbb{C}^{n|n}$ with the odd operator $\zeta$ such that $\zeta^2 = -\text{Id}$. Then $Q(n)$ is the centralizer of $\zeta$ in the Lie superalgebra $\mathfrak{g}(n|n)$. It is easy to see that $Q(n)$ consists of matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$, where $A, B$ are $n \times n$ matrices. We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal $A$ and $B$. By $n^+$ (respectively, $n^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, lower triangular) $A$ and $B$. The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$ and we set $\mathfrak{b} = n^+ \oplus \mathfrak{h}$.

Denote by $W$ the finite $W$-algebra associated with a principal even nilpotent element $\varphi$ in the coadjoint representation of $Q(n)$. Let us recall the definition (see [16]). Let $\{e_{i,j}, f_{i,j} \mid i, j = 1, \ldots, n\}$ denote the basis consisting of elementary even and odd matrices:

$$
e_{i,j} = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}, \quad f_{i,j} = \begin{pmatrix} 0 & E_{ij} \\ -E_{ij} & 0 \end{pmatrix}.
$$

Choose $\varphi \in \mathfrak{g}^*$ such that $\varphi(f_{i,j}) = 0$, $\varphi(e_{i,j}) = \delta_{i,j+1}$.

Let $I_\varphi$ be the left ideal in $U(\mathfrak{g})$ generated by $x - \varphi(x)$ for all $x \in n^-$. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W = \{\pi(y) \in U(\mathfrak{g})/I_\varphi \mid \text{ad}(x)y \in I_\varphi \text{ for all } x \in n^-\}.
$$

Using identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g}) \otimes_{U(n)} \mathbb{C}_\varphi \simeq U(\mathfrak{b}) \otimes \mathbb{C}$ we can consider $W$ as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$ with the kernel $n^+U(\mathfrak{b})$ is called the Harish-Chandra homomorphism. It is proven in [13] that the restriction of $\vartheta$ to $W$ is injective.

Set

$$\xi_i := (-1)^{i+1}f_{i,i}, \quad x_i := \xi_i^2 = e_{i,i},$$

then

$$U(\mathfrak{h}) \simeq \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i)_{1 < j \leq n}.
$$

We will identify $W$ with $\vartheta(W)$ and use the generators in $W$ introduced in [14] (Corollary 5.15):

$$u_k(0) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})|_{\text{even}},$$

$$u_k(1) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})|_{\text{odd}}.
$$

For convenience we assume $u_k(0) = u_k(1) = 0$ for $k > n$.

2. Simple modules over associative superalgebras

We work in the category of vector superspaces over $\mathbb{C}$. We denote the parity of a homogeneous vector $v$ of a superspace by $\hat{v} \in \mathbb{Z}_2$. All tensor products are over $\mathbb{C}$ unless specified otherwise.

Let $A$ be a superalgebra. By an $A$-module $M$ we mean a $\mathbb{Z}_2$-graded left $A$-module. A submodule of $M$ is a $\mathbb{Z}_2$-graded submodule. By $\Pi$ we denote the functor of parity switch $\Pi(M) = M \otimes \mathbb{C}^{0|1}$. For a module $M$ over an associative superalgebra $A$, $\Pi M$ has the same underlying vector space but with the opposite $\mathbb{Z}$-grading. The new action of $a \in A$ on $m \in \Pi M$ is given in terms of the old action by $a \cdot m := (-1)^{\hat{a}\hat{m}}am$.

Recall that if $M$ is a simple finite-dimensional $A$-module over some associative superalgebra $A$, then by Schur’s Lemma $\text{End}_A(M)$ is either one-dimensional, or two-dimensional and has
basis \( \{\text{Id}_M, \epsilon_M\} \), where \( \epsilon_M \) is a (unique up to a sign) odd involution on \( M \): \( \epsilon_M^2 = \text{Id}_M \). Note that \( \epsilon_M \) provides an \( A \) isomorphism \( M \rightarrow \Pi(M) \). We say that \( M \) is an \textit{irreducible of M-type} in the former case and an \textit{irreducible of Q-type} in the latter (see \cite{7, 4}).

Let \( A \) and \( B \) be two superalgebras. The tensor product \( A \otimes B \) is again a superalgebra, where multiplication is given by

\[
(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\tilde{b}_1 \tilde{a}_2} a_1 a_2 \otimes b_1 b_2
\]

for \( a_i \in A, b_i \in B \). Let \( M \) and \( N \) be two modules over associative superalgebras \( A \) and \( B \). Then \( M \otimes N \) is naturally a module over \( A \otimes B \) where

\[
(a \otimes b)(m \otimes n) = (-1)^{bn am} a \otimes b m,
\]

where \( a \in A, b \in B \) and \( m \in M, n \in N \). If \( M \) and \( N \) are two simple finite-dimensional modules over associative superalgebras \( A \) and \( B \), then the module \( M \otimes N \) might be not simple. In fact, if \( M \) and \( N \) are both of M-type, then \( M \otimes N \) is simple of M-type. If one of these modules is of M-type, and the other is of Q-type, then \( M \otimes N \) is simple of Q-type. However, if \( M \) and \( N \) are both of Q-type, then \( M \otimes N \) is not simple. Let \( \epsilon_M \) and \( \epsilon_N \) be odd involutions of \( M \) and \( N \), respectively. Then the map \( \epsilon_M \otimes \epsilon_N \) defined by

\[
(\epsilon_M \otimes \epsilon_N)(m \otimes n) = (-1)^{mn} \epsilon_M(m) \otimes \epsilon_N(n)
\]

is an even \( A \otimes B \)-automorphism of \( M \otimes N \), and its square is \(-\text{Id}_{M \otimes N}\). In this case \( M \otimes N \) decomposes into a direct sum of two \( A \otimes B \)-submodules, which are formed by the \pm 1-eigenspaces of \( \epsilon_M \otimes \epsilon_N \). We can choose either submodule and denote it by \( M \otimes N \). Then

\[
M \otimes N \simeq M \otimes N \oplus \Pi(M \otimes N).
\]

Both submodules are simple and of M-type.

3. Irreducible representations of \( W \)

3.1. Representations of \( U(\mathfrak{h}) \)

Let \( s = (s_1, \ldots, s_n) \in \mathbb{C}^n \). We call \( s \) regular if \( s_i \neq 0 \) for all \( i \leq n \) and typical if \( s_i + s_j \neq 0 \) for all \( i \neq j \leq n \).

It follows from the representation theory of Clifford algebras that all irreducible representations of \( U(\mathfrak{h}) \) up to change of parity can be parameterized by \( s \in \mathbb{C}^n \). Indeed, let \( M \) be an irreducible representation of \( U(\mathfrak{h}) \). By Schur’s lemma every \( x_i \) acts on \( M \) as a scalar operator \( s_i \text{Id} \). Let \( I_\delta \) denote the ideal in \( U(\mathfrak{h}) \) generated by \( x_i - s_i \), then the quotient algebra \( U(\mathfrak{h})/I_\delta \) is isomorphic to the Clifford superalgebra \( C_\delta \) (we consider Clifford algebras as superalgebras with the natural \( \mathbb{Z}_2 \)-grading) associated with the quadratic form:

\[
B_\delta(\xi_i, \xi_j) = \delta_{ij}s_i.
\]

Then \( M \) is a simple \( C_\delta \)-module.

The radical \( R_\delta \) of \( C_\delta \) is generated by the kernel of the form \( B_\delta \). Let \( m(s) \) be the number of non-zero coordinates of \( s \), then \( C_\delta/R_\delta \) is isomorphic to the matrix superalgebra \( M(2^{\frac{m-1}{2}}|2^{\frac{m-1}{2}}) \) for even \( m \) and to the superalgebra \( M(2^{\frac{m-1}{2}}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2 - 1) \) for odd \( m \).

Therefore \( C_\delta \) has one (up to isomorphism) simple \( \mathbb{Z}_2 \)-graded module \( V(s) \) of type Q for odd \( m(s) \), and two simple modules \( V(s) \) and \( \Pi V(s) \) of type M for even \( m(s) \) (see \cite{11}). In the case when \( s \) is regular, the form \( B_\delta \) is non-degenerate and the dimension of \( V(s) \) equals \( 2^k \), where \( k = \lfloor n/2 \rfloor \). In general, \( \dim V(s) = 2^{m(s)/2} \).
Let \( i + j = n \). We have the natural embedding of the Lie superalgebras \( Q(i) \oplus Q(j) \hookrightarrow Q(n) \). If \( \mathfrak{h}_r \) denotes the Cartan subalgebra of \( Q(r) \), the above embedding induces the isomorphism

\[
U(\mathfrak{h}) \simeq U(\mathfrak{h}_i) \otimes U(\mathfrak{h}_j).
\]

It induces an isomorphism of \( U(\mathfrak{h}) \)-modules

\[
V(s) \simeq V(s_1, \ldots, s_i) \otimes V(s_{i+1}, \ldots, s_n).
\]

### 3.2. Restriction from \( U(\mathfrak{h}) \) to \( W \)

We denote by the same symbol \( V(s) \) the restriction to \( W \) of the \( U(\mathfrak{h}) \)-module \( V(s) \). We proved the following two statements in [15].

**Proposition 1** ([15], Proposition 4.1). Let \( S \) be a simple \( W \)-module. Then \( S \) is a simple constituent of \( V(s) \) for some \( s \in \mathbb{C}^n \).

**Theorem 2** ([15], Theorem 4.2). If \( s \) is typical, then \( V(s) \) is a simple \( W \)-module.

### 3.3. Simple \( W \)-modules for \( n = 2 \)

Let \( n = 2 \). Let \( s_1 \neq -s_2 \) and assume that \( s_1, s_2 \neq 0 \). Then by Theorem 2, \( V(s) \) is simple as \( W \)-module. First, we describe the action of \( U(\mathfrak{h}) \) in \( V(s_1, s_2) \simeq V(s_1) \otimes V(s_2) \). Note that

\[
U(\mathfrak{h}) \simeq U(\mathfrak{h}_1) \otimes U(\mathfrak{h}_1),
\]

where \( \mathfrak{h}_1 \) is the Cartan subalgebra of \( Q(1) \). Clearly, \( U(\mathfrak{h}_1) \cong \mathbb{C}[\xi] \). Let \( x = \xi^2 \). For some suitable bases in \( V(s_1) \) and \( V(s_2) \), namely, \( V(s_1) = \langle v_1 | v_2 \rangle \), \( V(s_2) = \langle w_1 | w_2 \rangle \), where \( \bar{v}_1 = \bar{w}_1 = 0 \) and \( \bar{v}_2 = \bar{w}_2 = 1 \), the action of \( U(\mathfrak{h}_1) \) in \( V(s_i) \) is given by

\[
\xi \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_i} \\ \sqrt{s_i} & 0 \end{array} \right), \quad x \mapsto \left( \begin{array}{cc} s_i & 0 \\ 0 & s_i \end{array} \right) \quad \text{for } i = 1, 2.
\]

We identify the elements \( \xi_i, x_i \) of \( U(\mathfrak{h}) \) as follows:

\[
\xi_1 \leftrightarrow \xi \otimes 1, \quad \xi_2 \leftrightarrow 1 \otimes \xi, \quad x_1 \leftrightarrow x \otimes 1, \quad x_2 \leftrightarrow 1 \otimes x.
\]

Then \( V(s_1) \otimes V(s_2) = V(s_1, s_2) \oplus \Pi V(s_1, s_2) \), where

\[
V(s_1, s_2) = \langle v_1 \otimes w_1 + iv_2 \otimes w_2 | v_2 \otimes w_1 + iv_1 \otimes w_2 \rangle, \quad (1)
\]

\[
\Pi V(s_1, s_2) = \langle -v_1 \otimes w_1 + iv_2 \otimes w_2 | v_2 \otimes w_1 - iv_1 \otimes w_2 \rangle. \quad (2)
\]

Hence the action of \( U(\mathfrak{h}) \) in \( V(s_1, s_2) \) is given by the following formulas in basis (1):

\[
\xi_1 \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_1} \\ \sqrt{s_1} & 0 \end{array} \right), \quad \xi_2 \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_2} \\ \sqrt{s_2} & 0 \end{array} \right).
\]

Note that \( W \) is generated by \( \phi_0, \phi_1, z_0 \) and \( z_1 \), where

\[
\phi_0 := u_1(1) = \xi_1 + \xi_2, \quad \phi_1 := -u_2(1) = x_2 \xi_1 - x_1 \xi_2, \quad z_0 = u_1(0) = x_1 + x_2, \quad z_1 = u_2(0) = x_1 x_2 - \xi_1 \xi_2.
\]

Then we obtain the following formulas for the action of the generators of \( W \):

\[
\phi_0 \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_1} + \sqrt{s_2}i \\ \sqrt{s_1} - \sqrt{s_2}i & 0 \end{array} \right), \quad \phi_1 \mapsto \sqrt{s_1s_2} \left( \begin{array}{cc} 0 & \sqrt{s_2} - \sqrt{s_1}i \\ \sqrt{s_2} + \sqrt{s_1}i & 0 \end{array} \right), \quad (3)
\]

\[
\phi_0 \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_1} + \sqrt{s_2}i \\ \sqrt{s_1} - \sqrt{s_2}i & 0 \end{array} \right), \quad \phi_1 \mapsto \sqrt{s_1s_2} \left( \begin{array}{cc} 0 & \sqrt{s_2} - \sqrt{s_1}i \\ \sqrt{s_2} + \sqrt{s_1}i & 0 \end{array} \right), \quad (3)
\]

\[
\phi_0 \mapsto \left( \begin{array}{cc} 0 & \sqrt{s_1} + \sqrt{s_2}i \\ \sqrt{s_1} - \sqrt{s_2}i & 0 \end{array} \right), \quad \phi_1 \mapsto \sqrt{s_1s_2} \left( \begin{array}{cc} 0 & \sqrt{s_2} - \sqrt{s_1}i \\ \sqrt{s_2} + \sqrt{s_1}i & 0 \end{array} \right), \quad (3)
\]
\( z_0 \mapsto (s_1 + s_2)\text{Id}, \quad z_1 \mapsto \begin{pmatrix} s_1s_2 + \sqrt{s_1s_2i} & 0 \\ 0 & s_1s_2 - \sqrt{s_1s_2i} \end{pmatrix}. \) (4)

Note that formulas (3) and (4) hold when \( s_1 \neq -s_2 \).

Assume that \( s_1 = -s_2 \). If \( s_1, s_2 = 0 \) then \( V(s) \) is isomorphic to \( \mathbb{C} \oplus \mathbb{C}^2 \), where \( \mathbb{C} \) is the trivial module. If \( s_1 \neq 0 \), we choose \( \sqrt{s_1}, \sqrt{s_2} \) so that \( \sqrt{s_2} = -\sqrt{s_1} \). Note that the choice of sign controls the choice of the parity of \( V(s) \). The following exact sequence easily follows from (3) and (4):

\[ 0 \to \Pi \Gamma_{-s_1^2 + s_1} \to V(s) \to \Gamma_{-s_1^2 - s_1} \to 0, \]  

where \( \Gamma_t \) is the simple module of dimension \((1|0)\) on which \( \phi_0, \phi_1 \) and \( z_0 \) act by zero and \( z_1 \) acts by the scalar \( t \). The sequence splits only in the case \( s_1 = 0 \), when \( \Gamma_0 \cong \mathbb{C} \) is trivial. Thus, using Proposition 1, Theorem 2 and (5) we obtain

**Lemma 3.** If \( n = 2 \), then every simple \( W \)-module is isomorphic to one of the following:

1. \( V(s_1, s_2) \) or \( \Pi V(s_1, s_2) \) for \( s_1 \neq -s_2, s_1, s_2 \neq 0 \);
2. \( V(s, 0) \) if \( s \neq 0 \);
3. \( \Gamma_t \) or \( \Pi \Gamma_t \).

### 3.4. Invariance under permutations

**Theorem 4** ([15], Theorem 4.4). Let \( s' = \sigma(s) \) for some permutation of coordinates.

1. If \( s \) is typical, then \( V(s) \) is isomorphic to \( V(s') \) as a \( W \)-module.
2. If \( s \) is arbitrary, then \( [V(s)] = [V(s')] \) or [\( \Pi V(s') \)], where \([X] \) denotes the class of \( X \) in the Grothendieck group.

**Proof.** We will prove the statement for \( n = 2 \). Assume first that \( s_2 \neq -s_1 \). In this case \( V(s_1, s_2) \) is a \((1|1)\)-dimensional simple \( W \)-module.

Let

\[ D = \begin{pmatrix} \sqrt{s_2} + \sqrt{s_1}i & 0 \\ 0 & \sqrt{s_1} + \sqrt{s_2}i \end{pmatrix}. \]

Then by direct computation we have

\[ D\phi_0D^{-1} = \begin{pmatrix} 0 & \sqrt{s_2} + \sqrt{s_1}i \\ \sqrt{s_2} - \sqrt{s_1}i & 0 \end{pmatrix} \quad \text{and} \quad D\phi_1D^{-1} = \begin{pmatrix} 0 & \sqrt{s_1} \sqrt{s_2}i \\ \sqrt{s_1} + \sqrt{s_2}i & 0 \end{pmatrix}. \]

Therefore \( D \) defines an isomorphism between \( V(s_1, s_2) \) and \( V(s_2, s_1) \).

Now consider the case \( s_1 = -s_2 \). Then the structure of \( V(s_1, -s_1) \) is given by the exact sequence (5). Let \( V(s') = V(-s_1, s_1) \), then analogously we have the exact sequence

\[ 0 \to \Pi \Gamma_{-s_1^2 - s_1} \to V(s') \to \Gamma_{-s_1^2 + s_1} \to 0. \]  

The statement (2) now follows directly from comparison of (5) and (6). The proof for an arbitrary \( n \) see in [15].

\[ \square \]

### 4. Central characters

The center of \( U(q) \) for \( q = Q(n) \) is described in [18]. The center of \( U(\mathfrak{h}) \) coincides with \( \mathbb{C}[x_1, \ldots, x_n] \) and the image of the center of \( U(q) \) under the Harish-Chandra homomorphism is generated by the polynomials \( p_k = x_1^{2k+1} + \ldots + x_n^{2k+1} \) for all \( k \in \mathbb{N} \). These polynomials are called \( Q \)-symmetric polynomials.

In [13] we proved that the center \( Z \) of \( W \) coincides with the image of the center of \( U(q) \) and hence can be also identified with the ring of \( Q \)-symmetric polynomials.
Every \( s \) defines the central character \( \chi_s : Z \to \mathbb{C} \). Furthermore, it follows from the description of simple \( W \)-modules in \([15]\) (Theorem 4.6) that every simple \( W \)-module admits central character \( \chi_s \) for some \( s \). For every \( s = (s_1, \ldots, s_n) \) we define the core \( c(s) = (s_i, \ldots, s_m) \) as a subsequence obtained from \( s \) by removing all \( s_j = 0 \) and all pairs \((s_i, s_j)\) such that \( s_i + s_j = 0 \). Up to a permutation this result does not depend on the order of removing. Thus, the core is well defined up to permutation. We call \( m \) the length of the core. The notion of core is very useful for describing the blocks in the category of finite-dimensional \( Q(n) \)-modules, see \([12, 17]\).

**Example 5.** Let \( s = (1, 0, 3, -1, -1) \), then \( c(s) = (3, -1) \).

The following is a reformulation of the central character description in \([18]\).

**Lemma 6.** Let \( s, s' \in \mathbb{C}^n \). Then \( \chi_s = \chi_{s'} \) if and only if \( s \) and \( s' \) have the same core (up to permutation).

It follows from Lemma 6 that the core depends only on the central character \( \chi_s \), we denote it \( c(\chi) \).

**5. The category of finite-dimensional \( W \)-modules and blocks**

Let \( W - \text{mod} \) be the category of finite-dimensional \( W \)-modules. A \( W \)-module \( M \) has generalized central character \( \chi \), if for any \( z \in Z \) and \( m \in M \), there exists \( n \in \mathbb{Z}_{\geq 0} \) such that \((z - \chi(z))^n \cdot m = 0\). Let \( W^x - \text{mod} \) be the full subcategory of modules admitting generalized central character \( \chi \). The category \( W - \text{mod} \) is the direct sum of the subcategories \( W^x - \text{mod} \), as \( \chi \) ranges over the central characters.

The blocks in the category \( W - \text{mod} \) are equivalence class of linked objects. Each block lies in a single \( W^x - \text{mod} \), however, different blocks can belong to the same \( W^x - \text{mod} \), see \([5]\).

**5.1. Blocks in the category of finite-dimensional \( W \)-modules for \( Q(2) \)**

**Lemma 7.** Let \( n = 2 \). A simple \( W \)-module \( S \) belongs to \( W^x - \text{mod} \) if and only if one of the following three cases takes place:

1. \( S \simeq V(s_1, s_2) \) for \( s_1 \neq s_2, s_1, s_2 \neq 0 \) and \( c(\chi) = (s_1, s_2) \),
2. \( S \simeq V(s, 0) \) for \( s \neq 0 \) and \( c(\chi) = (s) \),
3. \( S \simeq \Gamma_t \) or \( \Pi^t \) and \( \chi = 0 \).

**Proof.** We have to compute the central character of the simple \( W \)-module. For a \( Q \)-symmetric polynomial \( p_k = x_1^{2k+1} + x_2^{2k+1} \) we have that

\[
p_k(S) = \begin{cases} 
  s_1^{2k+1} + s_2^{2k+1} & \text{if (1)} \\
  s_2^{2k+1} & \text{if (2)} \\
  0 & \text{if (3)} 
\end{cases}
\]

Since \( p_k \) generate the center of \( W \) the statement follows.

\[\square\]

**Theorem 8.** (1) Each simple \( W \)-module \( V(s_1, s_2) \) for \( s_1 \neq s_2, s_1, s_2 \neq 0 \) forms a block in \( W^x \)-mod, where \( c(\chi) = (s_1, s_2) \).

2. Each simple \( W \)-module \( V(s, 0) \) for \( s \neq 0 \) forms a block in \( W^x \)-mod, where \( c(\chi) = (s) \).

3. The blocks in the subcategory \( W^x \)-mod, where \( \chi = 0 \), are described as follows. Let \( a \in \mathbb{C} \). Define

\[
a_n = a - n^2 + n\sqrt{1-4a} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots
\]

Then \( \Gamma_a \) lies in the block formed by \( \Gamma_{a_n} \) if \( n \) is even and \( \Pi^a \), if \( n \) is odd. \( \Pi^a \) lies in the block formed by \( \Pi^a \), if \( n \) is even and \( \Gamma_{a_n} \), if \( n \) is odd.
Proof. Statements (1) and (2) follow from Lemma 6 and Lemma 7. To prove (3), first we will show that $\Gamma_a$ is linked to $\Pi\Gamma_b$ if and only if

$$b = a - 1 \pm \sqrt{1 - 4a}.$$  \hfill (8)

Recall that $W$ is generated by $\phi_0, \phi_1, z_0$ and $z_1$, where

$$\phi_0 = \xi_1 + \xi_2, \quad \phi_1 = x_2\xi_1 - x_1\xi_2, \quad z_0 = x_1 + x_2, \quad z_1 = x_1x_2 - \xi_1\xi_2.$$  

We have

$$[z_1, \phi_1] = 2z_1\phi_0 + 2\phi_1, \quad (9)$$

$$[z_1, \phi_0] = -2\phi_1. \quad (10)$$

Suppose that $\Gamma_a$ is linked to $\Pi\Gamma_b$. The generators $\phi_0$ and $z_1$ act in the vector superspace $\Gamma_a \oplus \Pi\Gamma_b$ as follows:

$$\phi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad z_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$  

Then from (10)

$$\phi_1 = \begin{pmatrix} 0 & b-a \\ 0 & 2 \\ 0 \end{pmatrix}$$

and hence $[z_1, \phi_1] = \begin{pmatrix} a & 0 \\ 0 & b-a \end{pmatrix}$.  

On the other hand, from (9)

$$[z_1, \phi_1] = 2\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & b-a \\ 0 & a+b \end{pmatrix}.$$  

Hence

$$\frac{(a-b)(b-a)}{2} = 2a + b - a.$$  

Then

$$b^2 + (2 - 2a)b + (a^2 + 2a) = 0.$$  \hfill (11)

This implies (8).

Conversely, if $a \neq 0$, set $s_1 = \frac{1-\sqrt{1-4a}}{2}$ and consider $V(-s_1, s_1)$ ($s_1 \neq 0$). Recall that we have the non-split exact sequence (6):

$$0 \rightarrow \Pi\Gamma_{-s_1^2} \rightarrow V(-s_1, s_1) \rightarrow \Gamma_{-s_1^2+s_1} \rightarrow 0,$$

which becomes

$$0 \rightarrow \Pi\Gamma_b \rightarrow V(-s_1, s_1) \rightarrow \Gamma_a \rightarrow 0,$$  \hfill (12)

with $b = a - 1 + \sqrt{1 - 4a}$. If we set $s_1 = \frac{1+\sqrt{1-4a}}{2}$, we obtain (12) with $b = a - 1 - \sqrt{1 - 4a}$.

If $a = 0$, set $s_1 = 1$ in (12). Then

$$0 \rightarrow \Pi\Gamma_{-2} \rightarrow V(-1, 1) \rightarrow \Gamma_0 \rightarrow 0.$$  

Let $V$ be a $(1|1)$-dimensional module on which $z_0, z_1$ and $\phi_1$ act by zero and $\phi_0$ acts by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$0 \rightarrow \Gamma_0 \rightarrow V \rightarrow \Pi\Gamma_0 \rightarrow 0.$$
Finally, the sum of the roots of equation (11) is $2a - 2$. This gives the relation
\[ a_{n-1} + a_{n+1} = 2a_n - 2 \quad (a_n = a). \] (13)

Then (8) and (13) imply (7).

Example 9.
(1) $a = 0$, then $a_n = n(1 - n)$ and $\Gamma_0$ lies in the block
\[ \ldots, \Gamma_{-30}, \Pi_{-20}, \Gamma_{-12}, \Pi_{-6}, \Gamma_{-2}, \Pi_0, \Gamma_0, \Pi_{-2}, \Gamma_{-6}, \Pi_{-12}, \Gamma_{-20}, \Pi_{-30}, \ldots \]

(2) $a = \frac{1}{4}$, then $a_n = \frac{1}{4} - n^2$ and $\frac{\Gamma_1}{4}$ lies in the block
\[ \frac{\Gamma_1}{4}, \Pi \frac{-\frac{1}{4}}{4}, \frac{\Gamma_{-\frac{1}{4}}}{4}, \ldots \]

(3) $a = 1$, then $a_n = 1 - n^2 + n\sqrt{-3}$ and $\Gamma_1$ lies in the block
\[ \ldots, \Pi_{-3\sqrt{-3}}, \Pi_{-2\sqrt{-3}}, \Pi_{-\sqrt{-3}}, \frac{\Gamma_1}{2}, \Pi_{\sqrt{-3}}, \frac{\Gamma_2}{2}, \Pi_{3\sqrt{-3}}, \Pi_{2\sqrt{-3}}, \Pi_{\sqrt{-3}}, \ldots \]

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