A purely infinite AH-algebra and an application to AF-embeddability

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Abstract

We show that there exists a purely infinite AH-algebra. The AH-algebra arises as an inductive limit of $C^*$-algebras of the form $C_0([0,1), M_k)$ and it absorbs the Cuntz algebra $O_\infty$ tensorially. Thus one can reach an $O_\infty$-absorbing $C^*$-algebra as an inductive limit of the finite and elementary $C^*$-algebras $C_0([0,1), M_k)$.

As an application we give a new proof of a recent theorem of Ozawa that the cone over any separable exact $C^*$-algebra is AF-embeddable, and we exhibit a concrete AF-algebra into which this class of $C^*$-algebras can be embedded.

1 Introduction

Simple $C^*$-algebras are divided into two disjoint subclasses: those that are stably finite and those that are stably infinite. (A simple $C^*$-algebra $A$ is stably infinite if $A \otimes K$ contains an infinite projection, and it is stably finite otherwise.) All simple, stably finite $C^*$-algebras admit a non-zero quasi-trace, and all exact, simple, stably finite $C^*$-algebras admit a non-zero trace.

A (possibly non-simple) $C^*$-algebra $A$ is in [12] defined to be purely infinite if no non-zero quotient of $A$ is abelian and if for all positive elements $a, b$ in $A$, such that $b$ belongs to the closed two-sided ideal generated by $a$, there is a sequence $\{x_n\}$ of elements in $A$ with $x_n^*ax_n \to b$. Non-simple purely infinite $C^*$-algebras have been investigated in [12], [13], and [3]. All simple purely infinite $C^*$-algebras are stably infinite, but the opposite does not hold, cf. [14].

The condition on a (non-simple) $C^*$-algebra $A$, that all projections in $A \otimes K$ are finite, does not ensure existence of (partially defined) quasi-traces. There are stably projectionless purely infinite $C^*$-algebras—take for example $C_0(\mathbb{R}) \otimes O_\infty$, where $O_\infty$ is the Cuntz algebra generated by a sequence of isometries with pairwise orthogonal range projections—and purely infinite $C^*$-algebras are traceless.

That stably projectionless purely infinite $C^*$-algebras can share properties that one would expect are enjoyed only by finite $C^*$-algebras was demonstrated in a recent paper.
by Ozawa, [16], in which is it shown that the suspension and the cone over any separable, exact $C^*$-algebra can be embedded into an AF-algebra. (It seems off hand reasonable to characterize AF-embeddability as a finiteness property.) In particular, $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ is AF-embeddable and at the same time purely infinite and traceless. It is surprising that one can embed a traceless $C^*$-algebra into an AF-algebra, because AF-algebras are well-supplied with traces. If $\varphi: C_0(\mathbb{R}) \otimes \mathcal{O}_\infty \to A$ is an embedding into an AF-algebra $A$, then $\text{Im}(\varphi) \cap \text{Dom}(\tau) \subseteq \text{Ker}(\tau)$ for every trace $\tau$ on $A$. This can happen only if the ideal lattice of $A$ has a sub-lattice isomorphic to the interval $[0, 1]$ (see Proposition 4.3). In particular, $A$ cannot be simple.

Voiculescu’s theorem, that the cone and the suspension over any separable $C^*$-algebra is quasi-diagonal, [19], is a crucial ingredient in Ozawa’s proof.

By a construction of Mortensen, [15], there is to each totally ordered, compact, metrizable set $T$ an AH-algebra $A_T$ with ideal lattice $T$ (cf. Section 2). A $C^*$-algebra is an AH-algebra, in the sense of Blackadar [11], if it is the inductive limit of a sequence of $C^*$-algebras each of which is a direct sum of $C^*$-algebras of the form $M_n(C_0(\Omega)) = C_0(\Omega, M_n)$ (where $n$ and $\Omega$ are allowed to vary). We show in Theorem 3.2 (in combination with Proposition 5.2) that the AH-algebra $A_{[0,1]}$ is purely infinite (and hence traceless)—even in the strong sense that it absorbs $\mathcal{O}_\infty$, i.e., $A_{[0,1]} \cong A_{[0,1]} \otimes \mathcal{O}_\infty$—and $A_{[0,1]}$ is an inductive limit of $C^*$-algebras of the form $C_0([0,1], M_{2^n})$. We can rephrase this result as follows: Take the smallest class of $C^*$-algebras, that contains all abelian $C^*$-algebras and that is closed under direct sums, inductive limits, and stable isomorphism. Then this class contains a purely infinite $C^*$-algebra (because it contains all AH-algebras).

A word of warning: In the literature, an AH-algebra is often defined to be an inductive limit of direct sums of building blocks of the form $pC(\Omega, M_n)p$, where each $\Omega$ is a compact Hausdorff space (and $p$ is a projection in $C(\Omega, M_n)$). With this definition, AH-algebras always contain non-zero projections. The algebras we consider, where the building blocks are of the form $C_0(\Omega, M_n)$ for some locally compact Hausdorff space, should perhaps be called AH$_0$-algebras to distinguish them from the compact case, but hoping that no confusion will arise, we shall not distinguish between AH- and AH$_0$-algebras here.

Every AH-algebra is AF-embeddable. Our Theorem 3.2 therefore gives a new proof of Ozawa’s result that there are purely infinite—even $\mathcal{O}_\infty$-absorbing—AF-embeddable $C^*$-algebras. Moreover, just knowing that there exists one AF-embeddable $\mathcal{O}_\infty$-absorbing $C^*$-algebra, in combination with Kirchberg’s theorem that all separable, exact $C^*$-algebras can be embedded in $\mathcal{O}_\infty$, immediately implies that the cone and the suspension over any separable, exact $C^*$-algebra is AF-embeddable (Theorem 4.2). This observation yields a new proof of Ozawa’s theorem referred to above.

Section 5 contains some results with relevance to the classification program of Elliott. In Section 6 we show that $A_{[0,1]}$ can be embedded into the AF-algebra $A_\Omega$, where $\Omega$ is the Cantor set, and hence that the the cone and the suspension over any separable, exact $C^*$-
algebra can be embedded into this AF-algebra. The ordered $K_0$-group of $A_\omega$ is determined.

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2 The $C^*$-algebras $A_T$

We review in this section results from Mortensen’s paper [15] on how to associate a $C^*$-algebra $A_T$ with each totally ordered, compact, metrizable set $T$, so that the ideal lattice of $A_T$ is order isomorphic to $T$. Where Mortensen’s algebras are inductive limits of $C^*$-algebras of the form $C_0(T \setminus \{\max T\}, M_{2^n}(\mathcal{O}_2))$, we consider plain matrix algebras $M_{2^n}$ in the place of $M_{2^n}(\mathcal{O}_2)$. It turns out that Mortensen’s algebras and those we consider actually are isomorphic when $T = [0, 1]$ (see the second paragraph of Section 5).

Any totally ordered set, which is compact and metrizable in its order topology, is order isomorphic to a compact subset of $\mathbb{R}$ (where subsets of $\mathbb{R}$ are given the order structure inherited from $\mathbb{R}$). We shall therefore assume that we are given a compact subset $T$ of $\mathbb{R}$. Put $t_{\max} = \max T$, $t_{\min} = \min T$, and put $T_0 = T \setminus \{t_{\max}\}$. Choose a sequence $\{t_n\}_{n=1}^{\infty}$ in $T_0$ such that the tail $\{t_k, t_{k+1}, t_{k+2}, \ldots\}$ is dense in $T_0$ for every $k \in \mathbb{N}$. Let $A_T$ be the inductive limit of the sequence

$$C_0(T_0, M_2) \xrightarrow{\varphi_1} C_0(T_0, M_4) \xrightarrow{\varphi_2} C_0(T_0, M_8) \xrightarrow{\varphi_3} \cdots \rightarrow A_T,$$

where

$$\varphi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(\max\{t, t_n\}) \end{pmatrix} = \begin{pmatrix} f(t) & 0 \\ 0 & (f \circ \chi_{t_n})(t) \end{pmatrix},$$

and where we for each $s$ in $T$ let $\chi_s \colon T \to T$ be the continuous function given by $\chi_s(t) = \max\{t, s\}$. The algebra $A_T$ depends a priori on the choice of the dense sequence $\{t_n\}$. The isomorphism class of $A_T$ does not depend on this choice when $T$ is the Cantor set (as shown in Section 6) or when $T$ is the interval $[0, 1]$ (as will be shown in a forthcoming paper, [14]). It is likely that $A_T$ is independent on $\{t_n\}$ for arbitrary $T$.

For the sake of brevity, put $A_n = C_0(T_0, M_{2^n}) = C_0(T_0) \otimes M_{2^n}$. Let $\varphi_{\infty, n} \colon A_n \to A_T$ and $\varphi_{m,n} : A_n \to A_m$, for $n < m$, denote the inductive limit maps, so that $A_T$ is the closure of $\bigcup_{n=1}^{\infty} \varphi_{\infty, n}(A_n)$. 

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Proposition 2.1 (cf. Mortensen, [15, Theorem 1.2.1])

Let $R$ be a closed two-sided ideal in $A$ (with the convention $\max\{\}$). We have $\varphi_{n+k,n}(f) = \begin{pmatrix} f \circ \chi_{\max F_1} & 0 & \cdots & 0 \\ 0 & f \circ \chi_{\max F_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \circ \chi_{\max F_{n+k}} \end{pmatrix}$, \hspace{1cm} (2.3)

(with the convention $\max\emptyset = t_{\min}$), where $F_1, F_2, \ldots, F_{2k}$ is an enumeration of the subsets of $\{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$. Note that $\chi_{\min}$ is the identity map on $T$.

For each $t \in T$ and for each $n \in \mathbb{N}$ consider the closed ideal

$$I_t^{(n)} \overset{\text{def}}{=} \{ f \in A_n \mid f(s) = 0 \text{ when } s \geq t \} \cong C_0(T \cap [t_{\min}, t), M_{2^n}) \hspace{1cm} (2.4)$$

of $A_n$. Observe that $I_{t_{\min}}^{(n)} = \{0\}, I_{t_{\max}}^{(n)} = A_n,$ and $I_t^{(n)} \subset I_s^{(n)}$ whenever $t < s$ for all $n \in \mathbb{N}$. We have $\varphi_n^{-1}(I_t^{(n+1)}) = I_t^{(n)}$ for all $t$ and for all $n$, and so

$$I_t \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \varphi_{\infty,n}(I_t^{(n)}), \hspace{1cm} t \in T, \hspace{1cm} (2.5)$$

is a closed two-sided ideal in $A_T$ such that $I_t^{(n)} = \varphi_n^{-1}(I_t)$. Moreover, $I_{t_{\min}}^{(n)} = \{0\}, I_{t_{\max}}^{(n)} = A_T$, and $I_t \subset I_s$ whenever $s, t \in T$ and $t < s$.

**Proposition 2.1 (cf. Mortensen, [15, Theorem 1.2.1])** Let $T$ be a compact subset of $\mathbb{R}$. Then each closed two-sided ideal in $A_T$ is equal to $I_t$ for some $t \in T$. It follows that the map $t \mapsto I_t$ is an order isomorphism from the ordered set $T$ onto the ideal lattice of $A_T$.

**Proof:** Let $I$ be a closed two-sided ideal in $A_T$. Put $I^{(n)} = \varphi_{\infty,n}(I) \preceq C_0(T_0, M_{2^n}) = A_n$, and put

$$T_n = \bigcap_{f \in I^{(n)}} f^{-1}(\{0\}) \subset T, \hspace{1cm} n \in \mathbb{N}.$$ 

Then $I^{(n)}$ is equal to the set of all continuous functions $f : T \rightarrow M_{2^n}$ that vanish on $T_n$. It therefore suffices to show that there is $t$ in $T$ such that $T_n = T \cap [t, t_{\max}]$ for all $n$, cf. (2.4) and (2.5). Now,

$$T_n = T_{n+1} \cup \chi_{t_n}(T_{n+1}) = \bigcup_{F \subseteq X_{n,k}} \chi_{\max F}(T_{n+k}), \hspace{1cm} n, k \in \mathbb{N}, \hspace{1cm} (2.6)$$

where $X_{n,k} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$; because if we let $T_{n,k}'$ denote the right-hand side of

| T_{n+1} \cup \chi_{t_n}(T_{n+1}) = \bigcup_{F \subseteq X_{n,k}} \chi_{\max F}(T_{n+k}) | n, k \in \mathbb{N} |
\( (2.6) \), then for all \( f \in C_0(T_0, M_{2^n}) = A_n \),
\[
  f|_{T_n} \equiv 0 \iff f \in I^{(n)} \iff \varphi_{n+k,n}(f) \in I^{(n+k)}
\]
\[
  \iff \forall s \in T_{n+k} \colon \varphi_{n+k,n}(f)(s) = 0
\]
\[
  \iff \forall F \subseteq X_{n,k} \forall s \in T_{n+k} : f(\chi_{\max F}(s)) = 0
\]
\[
  \iff f|_{T_{n,k}} \equiv 0.
\]

It follows from \( (2.6) \) that \( \min T_n \leq \min T_{n+1} \) for all \( n \); and as
\[
  \min \chi_{t_n}(T_{n+1}) = \max \{t_n, \min T_{n+1}\} \geq \min T_{n+1},
\]
we actually have \( \min T_n = \min T_{n+1} \) for all \( n \). Let \( t \in T \) be the common minimum. Because \( t \) belongs to \( T_{n+k} \) for all \( k \), we can use \( (2.6) \) to conclude that \( T_n \) contains the set \( \{t_n, t_{n+1}, t_{n+2}, \ldots\} \cap [t, t_{\text{max}}] \); and this set is by assumption dense in \( T \cap (t, t_{\text{max}}] \). This proves the desired identity: \( T_n = T \cap [t, t_{\text{max}}] \), because \( T_n \) is a closed subset of \( T \cap [t, t_{\text{max}}] \) and \( t \) belongs to \( T_n \).

**Proposition 2.2** \( A_T \) is stable for every compact subset \( T \) of \( \mathbb{R} \).

**Proof:** Let \( f \) be a positive element in the dense subset \( C_\epsilon(T_0, M_{2^n}) \) of \( A_n \) and let \( m > n \) be chosen such that \( f(t) = 0 \) for all \( t \geq t_{m-1} \). Then \( f \circ \chi_{\max F} = 0 \) for every subset \( F \) of \( \{t_n, t_{n+1}, \ldots, t_{m-1}\} \) that contains \( t_{m-1} \). In the description of \( \varphi_{m,n}(f) \) in \( (2.6) \) we see that \( f \circ \chi_{\max F_j} = 0 \) for at least every other \( j \). We can therefore find a positive function \( g \) in \( A_m = C_0(T_0, M_{2^n}) \) such that \( g \perp \varphi_{m,n}(f) \) and \( g \sim \varphi_{m,n}(f) \) (the latter in the sense that \( x^*x = g \) and \( xx^* = \varphi_{m,n}(f) \) for some \( x \in A_m \)). It follows from \( [8] \) Theorem 2.1 and Proposition 2.2 [10] that \( A_T \) is stable.

\[
  \square
\]

3. A purely infinite AH-algebra

We show in this section that the \( C^* \)-algebra \( A_{[0,1]} \) is traceless and that \( B = A_{[0,1]} \otimes M_{2^\infty} \) is purely infinite. (In Section 5 it will be shown that \( A_{[0,1]} \cong B \).)

Following [13] Definition 4.2 we say that an exact \( C^* \)-algebra is **traceless** if it admits no non-zero lower semi-continuous trace (whose domain is allowed to be any algebraic ideal of the \( C^* \)-algebra). (By restricting to the case of exact \( C^* \)-algebras we can avoid talking about quasi-traces; cf. Haagerup [7] and Kirchberg [10].)

If \( \tau \) is a trace defined on an algebraic ideal \( \mathcal{I} \) of a \( C^* \)-algebra \( B \), and if \( I \) is the closure of \( \mathcal{I} \), then \( \mathcal{I} \) contains the Pedersen ideal of \( I \). In particular, \( (a - \varepsilon)_+ \) belongs to \( \mathcal{I} \) for every positive element \( a \) in \( I \) and for every \( \varepsilon > 0 \). (Here, \( (a - \varepsilon)_+ = f_\varepsilon(a) \), where \( f_\varepsilon(t) = \max\{t - \varepsilon, 0\} \). Note that \( \|a - (a - \varepsilon)_+\| \leq \varepsilon \).)
Proposition 3.1 The $C^*$-algebra $\mathcal{A}_{[0,1]}$ is traceless.

Proof: Assume, to reach a contradiction, that $\tau$ is a non-zero, lower semi-continuous, positive trace defined on an algebraic ideal $\mathcal{I}$ of $\mathcal{A}_{[0,1]}$, and let $I_t$ be the closure of $\mathcal{I}$, cf. Proposition 2.1. Since $\tau$ is non-zero, $I_t$ is non-zero, and hence $t > 0$.

Identify $I_t^{(n)} = \varphi_{\infty,n}^{-1}(I_t)$ with $C_0([0,t), M_{2^n})$. Put $\mathcal{I}^{(n)} = \varphi_{\infty,n}^{-1}(\mathcal{I})$. If $x$ is a positive element in $I_t^{(n)}$ and if $\varepsilon > 0$, then

$$
\varphi_{\infty,n}((x - \varepsilon)_+) = (\varphi_{\infty,n}(x) - \varepsilon)_+ \in \mathcal{I},
$$

and so $(x - \varepsilon)_+ \in \mathcal{I}^{(n)}$. This shows that $\mathcal{I}^{(n)}$ is a dense ideal in $I_t^{(n)}$, and hence that $\mathcal{I}^{(n)}$ contains $C_c([0,t), M_{2^n})$.

Let $\tau_n$ be the trace on $\mathcal{I}^{(n)}$ defined by $\tau_n(f) = \tau(\varphi_{\infty,n}(f))$. We show that

$$
\tau_n(f) = \int_0^t \text{Tr}(f(s)) \, d\mu_n(s), \quad f \in C_c([0,t), M_{2^n}),
$$

(3.1)

for some Radon measure $\mu_n$ on $[0,t)$ (where $\text{Tr}$ denotes the standard unnormalized trace on $M_{2^n}$). Use Riesz’ representation theorem to find a Radon measure $\mu_n$ on $[0,t)$ such that $\tau_n(f) = 2^n \int_0^t f(s) \, d\mu_n(s)$ for all $f$ in $C_c([0,t), \mathbb{C}) \subseteq C_c([0,t), M_{2^n})$. Let $E: C_c([0,t), M_{2^n}) \to C_c([0,t), \mathbb{C})$ be the conditional expectation given by $E(f)(t) = 2^{-n}\text{Tr}(f(t))$. Then

$$
E(f) \in \overline{\{ufu^* \mid u \text{ is a unitary element in } C([0,t], M_{2^n})\}}, \quad f \in C_c([0,t), M_{2^n}),
$$

(3.2)

from which we see that $\tau_n(f) = \tau_n(E(f))$. This proves that (3.1) holds. Because $\mu_n$ is a Radon measure, $\mu_n([0,s]) < \infty$ for all $s \in [0,t)$ and for all $n \in \mathbb{N}$.

Let $\{t_n\}_{n=1}^\infty$ be the sequence in $T$ used in the definition of $\mathcal{A}_T$. For each $n$ and $k$ in $\mathbb{N}$ we have $\tau_n = \tau_{n+k} \circ \varphi_{n+k,n}$. Set $X_{k,n} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$ and use (2.3) and (3.1) to see that

$$
\int_0^t \text{Tr}(f(s)) \, d\mu_n(s) = \tau_n(f) = \tau_{n+k}(\varphi_{n+k,n}(f))
$$

$$
= \int_0^t \text{Tr}(\varphi_{n+k,n}(f)(s)) \, d\mu_{n+k}(s)
$$

$$
= \sum_{F \subseteq X_{k,n}} \int_0^t \text{Tr}(f \circ \chi_{\text{max}(F)})(s) \, d\mu_{n+k}(s)
$$

$$
= \sum_{F \subseteq X_{k,n}} \int_0^t \text{Tr}(f(s)) \, d(\mu_{n+k} \circ \chi_{\text{max}(F)}^{-1})(s)
$$

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for all $f \in C_c([0, t), M_{2^n})$. This entails that

$$\mu_n = \sum_{F \subseteq X_{k,n}} \mu_{n+k} \circ \chi_{\text{max}(F)}^{-1}, \quad (3.3)$$

for all natural numbers $n$ and $k$.

We prove next that $\mu_n([0, s]) = 0$ for all natural numbers $n$ and for all $s$ in $[0, t)$. Choose $r$ such that $0 < s < r < t$. Put $Y_{k,n} = X_{k,n} \cap [0, s]$ and put $Z_{k,n} = X_{k,n} \cap [0, r]$. Observe that

$$\chi_u^{-1}([0, v]) = \begin{cases} \emptyset, & \text{if } v < u, \\ [0, v], & \text{if } v \geq u, \end{cases} \quad (3.4)$$

whenever $u, v \in [0, 1]$. Use (3.3) and (3.4) to obtain

$$\mu_n([0, r]) = \sum_{F \subseteq Z_{k,n}} \mu_{n+k}([0, r]) = 2^{[Z_{k,n}]} \mu_{n+k}([0, r]). \quad (3.5)$$

Use (3.3), (3.4), and (3.5) to see that

$$\mu_n([0, s]) = \sum_{F \subseteq Y_{k,n}} \mu_{n+k}([0, s]) = 2^{[Y_{k,n}]} \mu_{n+k}([0, s])$$

$$\leq 2^{[Y_{k,n}]} \mu_{n+k}([0, r]) = 2^{-([Z_{k,n}] - [Y_{k,n}])} \mu_n([0, r]).$$

As

$$\lim_{k \to \infty} (|Z_{k,n}| - |Y_{k,n}|) = \lim_{k \to \infty} |X_{k,n} \cap (s, r)| = \infty,$$

(because $\bigcup_{k=1}^{\infty} X_{k,n} = \{t_n, t_{n+1}, \ldots\}$ is dense in $[0, 1)$, and as $\mu_n([0, r]) < \infty$, we conclude that $\mu_n([0, s]) = 0$. It follows that $\mu_n([0, t]) = 0$, whence $\mu_n$ and $\tau_n$ are zero for all $n$.

However, if $\tau$ is non-zero, then $\tau_n$ must be non-zero for some $n$. To see this, take a positive element $e$ in $I$ such that $\tau(e) > 0$. Because $\tau$ is lower semi-continuous there is $\varepsilon > 0$ such that $\tau((e - \varepsilon)_+) > 0$. Now, $I^{(n)}$ is dense in $I_t^{(n)}$ and $\bigcup_{n=1}^{\infty} \varphi_{\infty,n}(I_t^{(n)})$ is dense in $I_t \supseteq I$. It follows that we can find $n \in \mathbb{N}$ and a positive element $f$ in $I^{(n)}$ such that $||\varphi_{\infty,n}(f) - e|| < \varepsilon$. Use for example [13 Lemma 2.2] to find a contraction $d \in A$ such that $d^* \varphi_{\infty,n}(f)d = (e - \varepsilon)_+$. Put $x = \varphi_{\infty,n}(f)^{1/2}d$. Then

$$\tau_n(f) = \tau(\varphi_{\infty,n}(f)) \geq \tau(\varphi_{\infty,n}(f)^{1/2}dd^* \varphi_{\infty,n}(f)^{1/2})$$

$$= \tau(xx^*) = \tau(x^*x) = \tau((e - \varepsilon)_+) > 0,$$

and this shows that $\tau_n$ is non-zero. \hfill \Box

In the formulation of the main result below, $M_{2^n}$ denotes the CAR-algebra, or equivalently the UHF-algebra of type $2^n$. 

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It is shown in [13, Corollary 9.3] that the following three conditions are equivalent for a separable, stable (or unital), nuclear C*-algebra B:

(i) $B \cong B \otimes \mathcal{O}_\infty$.

(ii) B is purely infinite and approximately divisible.

(iii) B is traceless and approximately divisible.

The C*-algebra $\mathcal{O}_\infty$ is the Cuntz algebra generated by a sequence \( \{s_n\}_{n=1}^{\infty} \) of isometries with pairwise orthogonal range projections. Pure infiniteness of (non-simple) C*-algebras was defined in [12] (see also the introduction). A (possibly non-unital) C*-algebra $B$ is said to be *approximately divisible* if for each natural number $k$ there is a sequence of unital *-homomorphisms $\psi_n : M_k \oplus M_{k+1} \to \mathcal{M}(B)$ such that $\psi_n(x)b - b\psi_n(x) \to 0$ for all $x \in M_k \oplus M_{k+1}$ and for all $b \in B$, cf. [12, Definition 5.5]. The tensor product $A \otimes M_{2\infty}$ is approximately divisible for any C*-algebra $A$.

**Theorem 3.2** Put $\mathcal{B} = \mathcal{A}_{[0,1]} \otimes M_{2\infty}$, where $\mathcal{A}_{[0,1]}$ is as defined in (2.1). Then:

(i) $\mathcal{B}$ is an inductive limit

$$C_0([0,1), M_{k_1}) \to C_0([0,1), M_{k_2}) \to C_0([0,1), M_{k_3}) \to \cdots \to \mathcal{B},$$

for some natural numbers $k_1, k_2, k_3, \ldots$. In particular, $\mathcal{B}$ is an AH-algebra.

(ii) $\mathcal{B}$ is traceless, purely infinite, and $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{O}_\infty$.

It is shown in Proposition 5.2 below that $\mathcal{A}_{[0,1]} \cong \mathcal{B}$. We stress that this fact will not be used in the proof of Theorem 4.2 below.

**Proof:** Part (i) follows immediately from the construction of $\mathcal{A}_{[0,1]}$ and from the fact that $M_{2\infty}$ is an inductive limit of matrix algebras.

(ii). The property of being traceless is preserved after tensoring with $M_{2\infty}$, so $\mathcal{B}$ is traceless by Proposition 3.1. As remarked above, $\mathcal{B}$ is approximately divisible, $\mathcal{A}_{[0,1]}$ and hence $\mathcal{B}$ are stable by Proposition 2.2 and as $\mathcal{B}$ is also nuclear and separable it follows from [13, Corollary 9.3] (quoted above) that $\mathcal{B}$ is purely infinite and $\mathcal{O}_\infty$-absorbing. \(\square\)

The C*-algebra $\mathcal{B}$ is stably projectionless, and, in fact, every purely infinite AH-algebra is (stably) projectionless. Indeed, any projection in an AH-algebra is finite (in the sense of Murray and von Neumann), and any non-zero projection in a purely infinite C*-algebra is (properly) infinite, cf. [12, Theorem 4.16].

It is impossible to find a *simple* purely infinite AH-algebra, because all simple purely infinite C*-algebras contain properly infinite projections.
4 An application to AF-embeddability

We show here how Theorem 3.2 leads to a new proof of the recent theorem of Ozawa that the cone and the suspension over any exact separable $C^*$-algebra are AF-embeddable. [16].

It is well-known that any ASH-algebra, hence any AH-algebra, and hence the $C^*$-algebras $A_{[0,1]}$ and $B$ from Theorem [3,2] are AF-embeddable. For the convenience of the reader we include a proof of this fact—the proof we present is due to Kirchberg. (An ASH-algebra is a $C^*$-algebra that arises as the inductive limit of a sequence of $C^*$-algebras each of which is a finite direct sum of basic building blocks: sub-$C^*$-algebras of $M_n(C_0(\Omega))$—where $n$ and $\Omega$ are allowed to vary.)

An embedding of $A_{[0,1]}$ into an explicit AF-algebra is given in Section 6.

Proposition 4.1 (Folklore) Every ASH-algebra admits a faithful embedding into an AF-algebra.

Proof: Note first that if $A$ is a sub-$C^*$-algebra of $M_n(C_0(\Omega))$, then its enveloping von Neumann algebra $A^*$ is isomorphic to $\bigoplus_{k=1}^n M_k(C_k)$ for some (possibly trivial) abelian von Neumann algebras $C_1, C_2, \ldots, C_n$. If $C$ is an abelian von Neumann algebra and if $D$ is a separable sub-$C^*$-algebra of $M_k(C)$, then there is a (separable) sub-$C^*$-algebra $D_1$ of $M_k(C)$ that contains $D$ and such that $D_1 \cong M_k(C(X))$, where $X$ is a compact Hausdorff space of dimension zero. In particular, $D_1$ is an AF-algebra.

To see this, let $D_0$ be the separable $C^*$-algebra generated by $D$ and the matrix units of $M_k \subseteq M_k(C)$. Then $D_0 = M_k(D_0)$ for some separable sub-$C^*$-algebra $D_0$ of $C$. Any separable sub-$C^*$-algebra of a (possibly non-separable) $C^*$-algebra of real rank zero is contained in a separable sub-$C^*$-algebra of real rank zero. (This is obtained by successively adding projections from the bigger $C^*$-algebra.) Hence $D_0$ is contained in a separable real rank zero sub-$C^*$-algebra $D_1$ of $C$. It follows from [4] that $D_1 \cong C(X)$ for some zero-dimensional compact Hausdorff space $X$. Hence $D_1 = M_k(D_1)$ is as desired.

Assume now that $B$ is an ASH-algebra, so that it is an inductive limit

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} \cdots \xrightarrow{\psi_n} B,$$

where each $B_n$ is a finite direct sum of sub-$C^*$-algebras of $M_m(C_0(\Omega))$. Passing to the bi-dual we get a sequence of finite von Neumann algebras

$$B_1^{**} \xrightarrow{\psi_1^{**}} B_2^{**} \xrightarrow{\psi_2^{**}} B_3^{**} \xrightarrow{\psi_3^{**}} \cdots.$$

Use the observation from in the first paragraph (now applied to direct sums of basic building blocks) to find an AF-algebra $D_1$ such that $B_1 \subseteq D_1 \subseteq B_1^{**}$. Use the observation again to
find an AF-algebra $D_2$ such that $C^*(\psi_1^*)(D_1), B_2) \subseteq D_2 \subseteq B_2^{**}$. Continue in this way and find, at the $n$th stage, an AF-algebra $D_n$ such that $C^*(\psi_{n-1}^*)(D_{n-1}), B_n) \subseteq D_n \subseteq B_n^{**}$. It then follows that the inductive limit $D$ of

$$D_1 \xrightarrow{\psi_1^*} D_2 \xrightarrow{\psi_2^*} D_3 \xrightarrow{\psi_3^*} \cdots \xrightarrow{} D$$

is an AF-algebra that contains $B$. □

**Theorem 4.2 (Ozawa)** The cone $CA = C_0([0, 1), A)$ over any separable exact $C^*$-algebra $A$ admits a faithful embedding into an AF-algebra. 

**Proof:** By a renowned theorem of Kirchberg any separable exact $C^*$-algebra can be embedded into the Cuntz algebra $\mathcal{O}_2$ (see [11]), and hence into $\mathcal{O}_\infty$ (the latter because $\mathcal{O}_2$ can be embedded—non-unitally—into $\mathcal{O}_\infty$). It therefore suffices to show that $C\mathcal{O}_\infty = C_0([0, 1)) \otimes \mathcal{O}_\infty$ is AF-embeddable. It is clear from the construction of $\mathcal{B}$ in Theorem 3.2 that $C_0([0, 1))$ admits an embedding into the $C^*$-algebra $\mathcal{B}$. (Actually, one can embed $C_0([0, 1))$ into any $C^*$-algebra that absorbs $\mathcal{O}_\infty$.) As $\mathcal{B} \cong \mathcal{B} \otimes \mathcal{O}_\infty$, we can embed $C\mathcal{O}_\infty$ into $\mathcal{B}$. Now, $\mathcal{B}$ is an AH-algebra and therefore AF-embeddable, cf. Proposition 4.1, so $C\mathcal{O}_\infty$ is AF-embeddable. □

Ozawa used his theorem in combination with a result of Spielberg to conclude that the class of AF-embeddable $C^*$-algebras is closed under homotopy invariance, and even more: If $A$ is AF-embeddable and $B$ is homotopically dominated by $A$, then $B$ is AF-embeddable.

The suspension $SA = C_0((0, 1), A)$ is a sub-$C^*$-algebra of $CA$, and so it follows from Theorem 4.2 that also the suspension over any separable exact $C^*$-algebra is AF-embeddable.

No simple AF-algebra contains a purely infinite sub-$C^*$-algebra. In fact, any AF-algebra, that has a purely infinite sub-$C^*$-algebra, must have uncountably many ideals:

**Proposition 4.3** Suppose that $\varphi : A \to B$ is an embedding of a purely infinite $C^*$-algebra $A$ into an AF-algebra $B$. Let $a$ be a non-zero positive element in $\text{Im}(\varphi)$. For each $t$ in $[0, \|a\|]$ let $I_t$ be the closed two-sided ideal in $B$ generated by $(a - t)_+$. Then the map $t \mapsto I_t$ defines an injective order embedding of the interval $[0, \|a\|]$ into the ideal lattice of $B$.

**Proof:** Since $A$ is traceless (being purely infinite, cf. [12]), $\text{Im}(\varphi) \cap \text{Dom}(\tau) \subseteq \text{Ker}(\tau)$ for every trace $\tau$ on $B$.

Let $0 \leq t < s \leq \|a\|$ be given. We show that $I_s$ is strictly contained in $I_t$. Find a projection $p$ in $(a - t)_+ B(a - t)_+$ such that $\|(a - t)_+ - p(a - t)_+ p\| < s - t$. There is a trace $\tau$, defined on the algebraic ideal in $B$ generated by $p$, with $\tau(p) = 1$. We claim that $I_s \subseteq \text{Ker}(\tau) \subseteq \text{Dom}(\tau) \subseteq I_t$,
and this will prove the proposition. To see the first inclusion, there is $d$ in $B$ such that $(a - s)_+ = d^* p(a - t)_+ p d$ (use for example Lemma 2.2 and (2.1)). Therefore $(a - s)_+$ belongs to the algebraic ideal generated by $p$, whence $(a - s)_+ \in \text{Im}(\varphi) \cap \text{Dom}(\tau) \subseteq \text{Ker}(\tau)$. This entails that $I_s$ is contained in the kernel of $\tau$.

The strict middle inclusion holds because $0 < \tau(p) < \infty$. The last inclusion holds because $p$ belongs to $(a - t)_+ B(a - t)_+ \subseteq I_t$.

It follows from Proposition 5.1 below that no AF-algebra can have ideal lattice isomorphic to $[0, 1]$, and so the order embedding from Proposition 4.3 can never be surjective. In Section 6 we show that one can embed a (stably projectionless) purely infinite $C^*$-algebra into the AF-algebra $A_\Omega$, where $\Omega$ is the Cantor set. The ideal lattice of $A_\Omega$ is the totally ordered and totally disconnected set $\Omega$.

5 Further properties of the algebras $A_T$

Nuclear separable $C^*$-algebras that absorb $O_\infty$ have been classified by Kirchberg in terms of an ideal preserving version of Kasparov’s $KK$-theory, see [9]. It is not easy to decide when two such $C^*$-algebras with the same primitive ideal space are $KK$-equivalent in this sense. There is however a particularly well understood special case: If $A$ and $B$ are nuclear, separable, stable $C^*$-algebras that absorb the Cuntz algebra $O_2$, then $A$ is isomorphic to $B$ if and only if $A$ and $B$ have homeomorphic primitive ideal spaces (cf. Kirchberg, [9]).

We show in this section that $A_{[0, 1]} \cong A_{[0, 1]} \otimes O_\infty$ and that $A_{[0, 1]}$ is isomorphic to the $C^*$-algebra $B$ from Theorem 5.2. It is shown in a forthcoming paper, [14], that $A_{[0, 1]} \cong A_{[0, 1]} \otimes O_2$ (using an observation that $A_{[0, 1]}$ is zero homotopic in an ideal-system preserving way, i.e., there is a $*$-homomorphism $\Psi: A_{[0, 1]} \to C_0([0, 1], A_{[0, 1]})$ such that $ev_0 \circ \Psi = \text{id} A_{[0, 1]}$ and $\Psi(J) \subseteq C_0([0, 1], J)$ for every closed two-sided ideal $J$ in $A_{[0, 1]}$). Thus it follows from Kirchberg’s theorem that $A_{[0, 1]}$ is the unique separable, nuclear, stable, $O_2$-absorbing $C^*$-algebra whose ideal lattice is (order isomorphic to) $[0, 1]$. It seems likely (but remains open) that any separable, nuclear, traceless $C^*$-algebra with ideal lattice isomorphic to $[0, 1]$ must absorb $O_2$ and hence be isomorphic to $A_{[0, 1]}$.

Not all nuclear, separable $C^*$-algebras, whose ideal lattice is isomorphic to $[0, 1]$, are purely infinite (or traceless) as shown in Proposition 5.4 below.

We derive below a couple of facts about $C^*$-algebras that have ideal lattice isomorphic to $[0, 1]$:

**Proposition 5.1** Let $D$ be a $C^*$-algebra with ideal lattice order isomorphic to $[0, 1]$. Then $D$ stably projectionless. If $D$ moreover is purely infinite and separable, then $D$ is necessarily stable.
Proof: Since $D$ and $D \otimes K$ have the same ideal lattice it suffices to show that $D$ contains no non-zero projections. Let $\{I_t : t \in [0, 1]\}$ be the ideal lattice of $D$ (such that $I_t \subset I_s$ whenever $t < s$). Suppose, to reach a contradiction, that $D$ contains a non-zero projection $e$. Let $I_s$ be the ideal in $D$ generated by $e$. The ideal lattice of the unital $C^*$-algebra $eDe$ is then $\{eI_t e : t \in [0, s]\}$ and $eI_t e \subset eI_r e$ whenever $0 \leq t < r \leq s$. This is in contradiction with the well-known fact that any unital $C^*$-algebra has a maximal proper ideal.

Suppose now that $D$ is purely infinite and separable. To show that $D$ is stable it suffices to show that $D$ has no (non-zero) unital quotient, cf. [12, Theorem 4.24]. Now, the ideal lattice of an arbitrary quotient $D/I_s$ of $D$ is equal to $\{I_t/I_s : t \in [s, 1]\}$, and this lattice is order isomorphic to the interval $[0, 1]$ (provided that $I_s \neq I_1 = D$). It therefore follows from the first part of the proposition that $D/I_s$ has no non-zero projection and $D/I_s$ is therefore in particular non-unital. □

Proposition 5.2 $\mathcal{A}_T \cong \mathcal{A}_T \otimes M_{2^\infty} \otimes K$ for every compact subset $T$ of $\mathbb{R}$.

Proof: It was shown in Proposition 2.2 that $\mathcal{A}_T$ is stable. We proceed to show that $\mathcal{A}_T$ is isomorphic to $\mathcal{A}_T \otimes M_{2^\infty}$. Recall that $A_n = C_0(T_0, M_{2^n})$, put $\tilde{A}_n = C(T, M_{2^n})$, and consider the commutative diagram:

$$
\begin{array}{cccccc}
A_1 & \stackrel{\varphi_1}{\rightarrow} & A_2 & \stackrel{\varphi_2}{\rightarrow} & A_3 & \stackrel{\varphi_3}{\rightarrow} & \cdots & \rightarrow & A_T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{A}_1 & \stackrel{\tilde{\varphi}_1}{\rightarrow} & \tilde{A}_2 & \stackrel{\tilde{\varphi}_2}{\rightarrow} & \tilde{A}_3 & \stackrel{\tilde{\varphi}_3}{\rightarrow} & \cdots & \rightarrow & \tilde{A}_T \\
\end{array}
$$

where $\varphi_n$ is as defined in (2.2), and where $\tilde{\varphi}_n : \tilde{A}_n \rightarrow \tilde{A}_{n+1}$ is defined using the same recipe as in (2.2). The inductive limit $C^*$-algebra $\tilde{A}$ is unital, each $A_n$ is an ideal in $\tilde{A}_n$, and $\mathcal{A}_T$ is (isomorphic to) an ideal in $\tilde{A}$.

We show that $\tilde{A} \cong \tilde{A} \otimes M_{2^\infty}$. This will imply that $\mathcal{A}_T$ is isomorphic to an ideal of $\tilde{A} \otimes M_{2^\infty}$. Each ideal in $\tilde{A} \otimes M_{2^\infty}$ is of the form $I \otimes M_{2^\infty}$ for some ideal $I$ in $\tilde{A}$. As $M_{2^\infty} \cong M_{2^\infty} \otimes M_{2^\infty}$ it will follow that $\mathcal{A}_T \cong \mathcal{A}_T \otimes M_{2^\infty}$.

By [2, Proposition 2.12] (and its proof) to prove that $\tilde{A} \cong \tilde{A} \otimes M_{2^\infty}$ it suffices to show that for each finite subset $G$ of $\tilde{A}$ and for each $\varepsilon > 0$ there is a unital $^*$-homomorphism $\lambda : M_2 \rightarrow \tilde{A}$ such that $\|\lambda(x)g - g\lambda(x)\| \leq \varepsilon\|x\|$ for all $x \in M_2$ and for all $g \in G$. We may assume that $G$ is contained in $\tilde{\varphi}_{\infty,n}(A_n)$ for some natural number $n$. Put $H = \tilde{\varphi}_{\infty,n}(G)$.

It now suffices to find a natural number $k$ and a unital $^*$-homomorphism $\lambda : M_2 \rightarrow \tilde{A}_{n+k}$ such that

$$
\|\lambda(x)\tilde{\varphi}_{n+k,n}(h) - \tilde{\varphi}_{n+k,n}(h)\lambda(x)\| \leq \varepsilon\|x\|, \quad x \in M_2, \quad h \in H. \quad (5.1)
$$

Put $t_{\text{min}} = \min T$, and find $\delta > 0$ such that $\|h(t) - h(t_{\text{min}})\| \leq \varepsilon/2$ for all $h$ in $H$. 

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and for all \( t \) in \( T \) with \( |t - t_{\min}| < \delta \). Let \( \{t_n\} \) be the dense sequence in \( T_0 \) used in the definition of \( A_T \). Find \( m \geq n \) such that \( |t_m - t_{\min}| < \delta \). Put \( k = m + 1 - n \), and organize the elements in \( X = \{t_n, t_{n+1}, \ldots, t_{m+1}\} \) in increasing order and relabel the elements by \( s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_k \). Let \( F_1, F_2, \ldots, F_{2k} \) be the subsets of \( X \) ordered such that \( F_1 = \emptyset \) and

\[
\max F_2 = s_1, \quad \max F_3 = \max F_4 = s_2, \ldots, \quad \max F_{2k-1} + 1 = \cdots = \max F_{2k} = s_k.
\]

Then \( |s_1 - t_{\min}| < \delta \), and so \( \|h \circ \chi_{\max F_1} - h \circ \chi_{\max F_2}\| \leq \varepsilon \) for \( h \in H \) (we use the convention \( \max \emptyset = t_{\min} \)); and \( h \circ \chi_{\max F_{2j-1}} = h \circ \chi_{\max F_{2j}} \) when \( j \geq 2 \) for all \( h \).

We shall use the picture of \( \varphi_{n+k,n} \) given in (2.3), which is valid also for \( \tilde{\varphi}_{n+k,n} \). However, since the sets \( F_1, F_2, \ldots, F_k \) possibly have been permuted, \( \varphi_{n+k,n} \) and the expression in (2.3) agree only up to unitary equivalence. Let \( \lambda: M_2 \to \tilde{A}_{n+k} \) be the unital \( * \)-homomorphism given by \( \lambda(x) = \text{diag}(x, x, \ldots, x) \) (with \( 2^{k-1} \) copies of \( x \)). Use (2.3) and the estimate

\[
\| x \left( \begin{array}{cc} h \circ \chi_{\max F_{2j-1}} & 0 \\ 0 & h \circ \chi_{\max F_{2j}} \end{array} \right) - \left( \begin{array}{cc} h \circ \chi_{\max F_{2j-1}} & 0 \\ 0 & h \circ \chi_{\max F_{2j}} \end{array} \right) x \| \leq \| x \| \| h \circ \chi_{\max F_{2j-1}} - h \circ \chi_{\max F_{2j}} \| \leq \varepsilon \| x \|,
\]

that holds for \( j = 1, 2, \ldots, 2^{k-1} \), for \( h \in H \), and for all \( x \in M_2(\mathbb{C}) \subseteq C(T, M_2) \), to conclude that \( \{5.1\} \) holds, and hence that \( \tilde{A} \cong \tilde{A} \otimes M_2^\infty \). \hfill \Box

Propositions 5.2 together with Theorem 3.2 yield:

**Corollary 5.3** The \( C^* \)-algebra \( A_{[0,1]} \) is purely infinite and \( A_{[0,1]} \cong A_{[0,1]} \otimes O_\infty \).

We conclude this section by showing that the tracelessness of the \( C^* \)-algebras \( A_{[0,1]} \) (established in Proposition 3.1) is not a consequence of its ideal lattice being isomorphic to \([0,1]\).

**Proposition 5.4** Let \( \{l_n\}_{n=1}^\infty \) be a sequence of positive integers, and let \( \{t_n\}_{n=1}^\infty \) be a dense sequence in \([0,1]\). Put \( k_1 = 1 \), put \( k_{n+1} = (l_n + 1)k_n \) for \( n \geq 1 \), and put \( D_n = C_0([0,1], M_{k_n}) \). Let \( D \) be the inductive limit of the sequence

\[
D_1 \xrightarrow{\psi_1} D_1 \xrightarrow{\psi_2} D_2 \xrightarrow{\psi_3} \cdots \xrightarrow{\psi_n} D,
\]

where \( \psi_n(f) = \text{diag}(f, f, \ldots, f, f \circ \chi_{t_n}) \) (with \( l_n \) copies of \( f \)), and where \( \chi_{t_n}: [0,1] \to [0,1] \) as before is given by \( \chi_{t_n}(s) = \max\{s, t_n\} \).

It follows that the ideal lattice of \( D \) is isomorphic to the interval \([0,1]\). Moreover, if \( \prod_{n=1}^\infty l_n / (l_n + 1) > 0 \), then \( D \) has a non-zero bounded trace, in which case \( D \) is not stable and not purely infinite.
Proof: An obvious modification of the proof of Proposition 2.1 shows that the ideal lattice of $\mathcal{D}$ is isomorphic to $[0, 1]$. As in the proof of Proposition 5.2 we construct a unital C*-algebra $\tilde{\mathcal{D}}$, in which $\mathcal{D}$ is a closed two-sided ideal, by letting $\tilde{\mathcal{D}}$ be the inductive limit of the sequence

$$
\begin{array}{cccccc}
D_1 & \xrightarrow{\psi_1} & D_2 & \xrightarrow{\psi_2} & D_3 & \xrightarrow{\psi_3} & \cdots & \xrightarrow{\psi} & \mathcal{D} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{D}_1 & \xrightarrow{\psi_1} & \tilde{D}_2 & \xrightarrow{\psi_2} & \tilde{D}_3 & \xrightarrow{\psi_3} & \cdots & \xrightarrow{\psi} & \tilde{\mathcal{D}},
\end{array}
$$

where $\tilde{D}_n = C([0, 1], M_{k_n})$ and $\tilde{\psi}_n(f) = \text{diag}(f, \ldots, f, f \circ \chi_{t_n})$. Remark that $\tilde{\mathcal{D}}/\mathcal{D} \cong \lim_{\rightarrow} \tilde{D}_n/\mathcal{D} \cong \lim_{\rightarrow} M_{k_n}

is a UHF-algebra. If $\tau$ is a tracial state on $\tilde{\mathcal{D}}$ that vanishes on $\mathcal{D}$, then $\tau$ is the composition of the quotient mapping $\tilde{\mathcal{D}} \to \tilde{\mathcal{D}}/\mathcal{D}$ and the unique tracial state on the UHF-algebra $\tilde{\mathcal{D}}/\mathcal{D}$. It follows that there is only one tracial state $\tau$ on $\tilde{\mathcal{D}}$ that vanishes on $\mathcal{D}$.

Suppose now that $\prod_{n=1}^{\infty} l_n/(l_n + 1) > 0$. It then follows, as in the construction of Goodearl in [6], that the simplex of tracial states on $\tilde{\mathcal{D}}$ is homeomorphic to the simplex of probability measures on $[0, 1]$ and hence that $\tilde{\mathcal{D}}$ has a tracial state that does not vanish on $\mathcal{D}$. The restriction of this trace to $\mathcal{D}$ is then the desired non-zero bounded trace. (Goodearl constructs simple C*-algebras; and where $f \circ \chi_{t_n}$ appears in our connecting map $\tilde{\psi}_n$, Goodearl uses a point evaluation, i.e., the constant function $t \mapsto f(t_n)$. Goodearl’s proof can nonetheless and without changes be applied in our situation.)

\[\square\]

6 An embedding into a concrete AF-algebra

Let $T$ be a compact subset of $\mathbb{R}$ and set $T_0 = T \setminus \{\max T\}$. Then $C_0(T_0, M_{2^n})$ is an AF-algebra if and only if $T$ is totally disconnected. It follows that the C*-algebra $\mathcal{A}_T$ (defined in (2.1)) is an AF-algebra whenever $T$ is totally disconnected. Let $\Omega$ denote the Cantor set (realized as the “middle third” subset of $[0, 1]$, and with the total order it inherits from its embedding in $\mathbb{R}$). Actually any totally disconnected, compact subset of $\mathbb{R}$ with no isolated points is order isomorphic to $\Omega$.

We show here that the AF-algebra from Theorem 4.2, into which the cone over any separable exact C*-algebra can be embedded, can be chosen to be $\mathcal{A}_\Omega$. The ideal lattice of $\mathcal{A}_\Omega$ is order isomorphic to $\Omega$ (by Proposition 2.1). In the light of Proposition 4.3 and by the fact that the ideal lattice of an AF-algebra is totally disconnected (in an appropriate sense) the AF-algebra $\mathcal{A}_\Omega$ has the least complicated ideal lattice among AF-algebras that admit embeddings of (stably projectionless) purely infinite C*-algebras.
We begin by proving a general result on when $A_S$ can be embedded into $A_T$:

**Proposition 6.1** Let $S$ and $T$ be compact subsets of $\mathbb{R}$. Set $T_0 = T \setminus \{\text{max} T\}$ and $S_0 = S \setminus \{\text{max} S\}$. Suppose there is a continuous, increasing, surjective function $\lambda: T \to S$ such that $\lambda(T_0) = S_0$. Let $\{t_n\}_{n=1}^\infty$ be a sequence in $T_0$ such that $\{t_n\}_{n=k}^\infty$ is dense in $T_0$ for every $k$, and put $s_n = \lambda(t_n)$. Then $\{s_n\}_{n=k}^\infty$ is dense in $S_0$ for every $k$, and there is an injective $\ast$-homomorphism $\lambda^\sharp: A_S \to A_T$, when $A_T$ and $A_S$ are inductive limits as in (2.1) with respect to the sequences $\{t_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$, respectively. If $\lambda$ moreover is injective, then $\lambda^\sharp$ is an isomorphism.

**Proof:** There is a commutative diagram:

\[
\begin{array}{cccccc}
C_0(S_0, M_2) & \xrightarrow{\phi_1} & C_0(S_0, M_4) & \xrightarrow{\phi_2} & C_0(S_0, M_8) & \xrightarrow{\phi_3} & \cdots & \xrightarrow{\chi} & A_S \\
\hat{\lambda} & & \hat{\lambda} & & \hat{\lambda} & & & & \\
C_0(T_0, M_2) & \xrightarrow{\psi_1} & C_0(T_0, M_4) & \xrightarrow{\psi_2} & C_0(T_0, M_8) & \xrightarrow{\psi_3} & \cdots & \xrightarrow{\chi} & A_T \\
\end{array}
\]

where $\lambda^\sharp(f) = \hat{\lambda}(f) = f \circ \lambda$, and where

\[
\phi_n(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ \chi_{s_n} \end{pmatrix}, \quad \psi_n(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ \chi_{t_n} \end{pmatrix},
\]

cf. (2.1). Note that $\lambda(t_{\text{max}}) = s_{\text{max}}$ (because $\lambda$ is surjective), and so $\hat{\lambda}(f)(t_{\text{max}}) = f(\lambda(t_{\text{max}})) = f(s_{\text{max}}) = 0$. To see that the diagram (6.1) indeed is commutative we must check that $\lambda^\sharp \circ \phi_n = \psi_n \circ \hat{\lambda}$ for all $n$. By (6.2),

\[
(\lambda^\sharp \circ \phi_n)(f) = \begin{pmatrix} f \circ \lambda & 0 \\ 0 & f \circ \chi_{s_n} \circ \lambda \end{pmatrix}, \quad (\psi_n \circ \hat{\lambda})(f) = \begin{pmatrix} f \circ \lambda & 0 \\ 0 & f \circ \lambda \circ \chi_{t_n} \end{pmatrix},
\]

for all $f \in C_0(S_0, M_{2^n})$, so it suffices to check that $\chi_{s_n} \circ \lambda = \lambda \circ \chi_{t_n}$. But

\[
(\chi_{s_n} \circ \lambda)(x) = \max\{\lambda(x), s_n\} = \max\{\lambda(x), \lambda(t_n)\} = \lambda\left(\max\{x, t_n\}\right) = (\lambda \circ \chi_{t_n})(x),
\]

where the third equality holds because $\lambda$ is increasing.

Each map $\hat{\lambda}$ in the diagram (6.1) is injective (because $\lambda$ is surjective), so the $\ast$-homomorphism $\lambda^\sharp: A_S \to A_T$ induced by the diagram is injective.

If $\lambda$ also is injective, then each map $\hat{\lambda}$ in (6.1) is an isomorphism in which case $\lambda^\sharp$ is an isomorphism. 

Combine (the proof of) Theorem 4.2 with Proposition 5.2 to obtain:
Proposition 6.2 The cone and the suspension over any separable exact $C^*$-algebra admits an embedding into the AH-algebra $\mathcal{A}_{[0,1]}$.

Lemma 6.3 There is a continuous, increasing, surjective map $\lambda: \Omega \to [0,1]$ that maps $[0,1)$ into $\Omega_0$, where $\Omega$ is the Cantor set and where $\Omega_0 = \Omega \setminus \{1\}$.

Proof: Each $x$ in $\Omega$ can be written $x = \sum_{n \in F} 2 \cdot 3^{-n}$ for a unique subset $F$ of $\mathbb{N}$. We can therefore define $\lambda$ by

$$\lambda\left(\sum_{n \in F} 2 \cdot 3^{-n}\right) = \sum_{n \in F} 2^{-n}, \quad F \subseteq \mathbb{N}.$$ 

It is straightforward to check that $\lambda$ has the desired properties. □

Corollary 6.4 The cone and the suspension over any separable exact $C^*$-algebra admits an embedding into the AF-algebra $\mathcal{A}_{\Omega}$.

Proof: It follows from Proposition 6.1 and Lemma 6.3 that $\mathcal{A}_{[0,1]}$ can be embedded into $\mathcal{A}_{\Omega}$. The corollary is now an immediate consequence of Proposition 6.2. □

By a renowned theorem of Elliott, [5], the ordered $K_0$-group is a complete invariant for the stable isomorphism class of an AF-algebra. We shall therefore go to some length to calculate the ordered group $K_0(\mathcal{A}_{\Omega})$.

As $K_0(\mathcal{A}_{\Omega})$ does not depend on the choice of dense sequence $\{t_n\}_{n=1}^{\infty}$ used in the inductive limit description of $\mathcal{A}_{\Omega}$, (2.1), it follows in particular from Proposition 6.5 below that the isomorphism class of $\mathcal{A}_{\Omega}$ is independent of this sequence.

The Cantor set $\Omega$ is realized as the “middle-third” subset of $[0,1]$ (so that $0 = \min \Omega$ and $1 = \max \Omega$). Consider the countable abelian group $G = C_0(\Omega_0, \mathbb{Z}_{[1/2]})$ where the composition is addition, and where the group of Dyadic rationals $\mathbb{Z}_{[1/2]}$ is given the discrete topology. Equip $G$ with the lexicographic order, whereby $f \in G^+$ if and only if either $f = 0$ or $f(t_0) > 0$ for $t_0 = \sup\{t \in \Omega \mid f(t) \neq 0\}$. (The set $\{t \in \Omega \mid f(t) \neq 0\}$ is clopen because $\mathbb{Z}_{[1/2]}$ is discrete, and so $f(t_0) \neq 0$.) It is easily checked that $(G, G^+)$ is a totally ordered abelian group, and hence a dimension group.

Proposition 6.5 The group $K_0(\mathcal{A}_{\Omega})$ is order isomorphic to the group $C_0(\Omega_0, \mathbb{Z}_{[1/2]})$ equipped with the lexicographic ordering.

Proof: Let $\{t_n\}_{n=1}^{\infty}$ be any sequence in $\Omega_0 = \Omega \setminus \{1\}$ such that $\{t_k, t_{k+1}, t_{k+2}, \ldots\}$ is dense in $\Omega_0$ for all $k$. Write $\mathcal{A}_{\Omega}$ as an inductive limit with connecting maps $\varphi_n$ as in (2.1).

By continuity of $K_0$ and because $K_0(C_0(\Omega_0, M_{2^n})) \cong C_0(\Omega_0, \mathbb{Z})$ (as ordered abelian groups) (see eg. [18, Exercise 3.4]), the ordered abelian group $K_0(\mathcal{A}_{\Omega})$ is the inductive limit of the sequence

$$C_0(\Omega_0, \mathbb{Z}) \xrightarrow{\alpha_1} C_0(\Omega_0, \mathbb{Z}) \xrightarrow{\alpha_2} C_0(\Omega_0, \mathbb{Z}) \xrightarrow{\alpha_3} \cdots \rightarrow K_0(\mathcal{A}_{\Omega}),$$

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where \( \alpha_n(f) = K_0(\varphi_n)(f) = f + f \circ \chi_{t_n} \).

Choose for each \( n \in \mathbb{N} \) a partition \( \{A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}\} \) of \( \Omega \) into clopen intervals (written in increasing order) such that

(a) \( A_j^{(n)} = A_{2j-1}^{(n+1)} \cup A_{2j}^{(n+1)} \),

(b) \( t_n \in A_1^{(n)} \) for infinitely many \( n \),

(c) \( \bigcup_{n=1}^{\infty} \{A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}\} \) is a basis for the topology on \( \Omega \).

Set \( \mathcal{F} = \bigcup_{n=1}^{\infty} \{A_1^{(n)}, A_2^{(n)}, \ldots, A_{2^n-1}^{(n)}\} \), and set

\[
H_n = \text{span}\{1_{A_j^{(n)}} \mid j = 1, 2, \ldots, 2^n - 1\} \subseteq C_0(\Omega_0, \mathbb{Z}).
\]

Note that \( 1_{A_n^{(n)}} \) does not belong to \( C_0(\Omega_0, \mathbb{Z}) \) because \( 1 \in A_2^{(n)} \).

We outline the idea of the rather lengthy proof below. We show first that \( \alpha_n(H_n) \subseteq H_{n+1} \) for all \( n \) and that \( \bigcup_{n=1}^{\infty} \alpha_{\infty,n}(H_n) = K_0(\mathcal{A}_\Omega) \), where \( \alpha_{\infty,n} = K_0(\varphi_{\infty,n}) \) is the inductive limit homomorphism from \( C_0(\Omega_0, \mathbb{Z}) \) to \( K_0(\mathcal{A}_\Omega) \). We then construct positive, injective group homomorphisms \( \beta_n : H_n \to C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}]) \) that satisfy \( \beta_{n+1} \circ \alpha_n = \beta_n \) for all \( n \), and which therefore induce a positive injective group homomorphism \( \beta : K_0(\mathcal{A}_\Omega) \to C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}]) \). It is finally proved that \( \beta \) is onto and that \( K_0(\mathcal{A}_\Omega) \) is totally ordered, and from this one can conclude that \( \beta \) is an order isomorphism.

For each interval \([r, s] \cap \Omega\) and for each \( t \in \Omega \),

\[
1_{[r, s] \cap \Omega} \circ \chi_t = \begin{cases} 
1_{[r, s] \cap \Omega} & t < r, \\
1_{[0, s] \cap \Omega} & r \leq t \leq s, \\
0 & t > s.
\end{cases} \tag{6.3}
\]

Suppose that \( A_1, A_2, \ldots, A_m \) is a partition of \( \Omega \) into clopen intervals, written in increasing order, and that \( t \in A_{j_0} \). Then, by (6.3),

\[
1_{A_j} + 1_{A_j} \circ \chi_t = \begin{cases} 
1_{A_j} & j < j_0, \\
2 \cdot 1_{A_j} + 1_{A_{j-1}} + \cdots + 1_{A_1} & j = j_0, \\
2 \cdot 1_{A_j} & j > j_0.
\end{cases} \tag{6.4}
\]

The lexicographic order on \( G = C_0(\Omega_0, \mathbb{Z}[\frac{1}{2}]) \) has the following description: If \( k \leq n \) and if \( r_1, r_2, \ldots, r_k \) are elements in \( \mathbb{Z}[\frac{1}{2}] \) with \( r_k \neq 0 \), then

\[
r_k 1_{A_k} + r_{k-1} 1_{A_{k-1}} + \cdots + r_1 1_{A_1} \in G^+ \iff r_k > 0. \tag{6.5}
\]

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It follows from (6.4) that \( \alpha_n(H_n) = H_n \subseteq H_{n+1} \). As \( \mathcal{F} \) is a basis for the topology of \( \Omega \), the set \( \{1_A \mid A \in \mathcal{F}\} \) generates \( C_0(\Omega, \mathbb{Z}) \). To prove that \( \bigcup_{n=1}^{\infty} \alpha_{\infty,n}(H_n) = K_0(\mathcal{A}_\Omega) \) it suffices to show that \( \alpha_{\infty,m}(1_A) \) belongs to \( \bigcup_{n=1}^{\infty} \alpha_{\infty,n}(H_n) \) for every \( A \) in \( \mathcal{F} \) and for every \( m \) in \( \mathbb{N} \). Take \( A \in \mathcal{F} \) and find a natural number \( n \geq m \) such that \( 1_A \) belongs to \( H_n \). Let \( A' \) be the clopen interval in \( \Omega \) consisting of all points in \( \Omega \) that are smaller than \( \min A \). Then \( 1_{A'} \) belongs to \( H_n \), and \( \alpha_{\infty,m}(1_A) \) belongs to \( \text{span}\{1_{A'},1_A\} \subseteq H_n \) by (6.4). Hence \( \alpha_{\infty,m}(1_A) = \alpha_{\infty,n}(\alpha_{\infty,m}(1_A)) \) belongs to \( \alpha_{\infty,n}(H_n) \).

The next step is to find a sequence of positive, injective group homomorphisms \( \beta_n : H_n \to G \) such that \( \beta_{n+1} \circ \alpha_n = \beta_n \). (This sequence will then induce a positive, injective group homomorphism \( \beta : K_0(\mathcal{A}_\Omega) \to G \).) Each function \( \{1_{A_1^{(n)}},1_{A_2^{(n)}},\ldots,1_{A_{2^n-1}^{(n)}}\} \to G^+ \) extends uniquely to a positive group homomorphism \( H_n \to G \), and so it suffices to specify \( \beta_n \) on this generating set. We do so by setting

\[
\beta_n(1_{A_j^{(n)}}) = \delta(j,j,n)1_{A_j^{(n)}} + \sum_{i=1}^{j-1} \delta(j,i,n)1_{A_i^{(n)}}, \quad j = 1,2,\ldots,2^n-1, \quad (6.6)
\]

for suitable coefficients, \( \delta(j,i,n) \), in \( \mathbb{Z}[\frac{1}{2}] \)—to be contructed—such that \( \delta(j,j,n) = 2^{-k} > 0 \) for some \( k \in \mathbb{N} \), and such that \( 1_{A_j^{(n)}} \) belongs to the image of \( \beta_n \) for \( j = 1,2,\ldots,2^n-1 \). Positivity of \( \beta_n \) will follow from (6.5), (6.6), and the fact that \( \delta(j,j,n) > 0 \).

For \( n = 1 \) set \( \beta_1(1_{A_1^{(1)}}) = 1_{A_1^{(1)}} \), so that \( \delta(1,1,1) = 1 \). Suppose that \( \beta_n \) has been found. The point \( t_n \) belongs to \( A_{j_0}^{(n)} \) for some \( j_0 \). The equation \( \beta_{n+1}(\alpha_n(1_{A_j^{(n)}})) = \beta_n(1_{A_j^{(n)}}) \) has by (6.4) the solution:

\[
\beta_{n+1}(1_{A_j^{(n)}}) = \begin{cases} 
\beta_n(1_{A_j^{(n)}}), & j < j_0, \\
\frac{1}{2}\beta_n(1_{A_j^{(n)}}) - \frac{1}{2} \sum_{i=1}^{j-1} \beta_n(1_{A_i^{(n)}}), & j = j_0, \\
\frac{1}{2}\beta_n(1_{A_j^{(n)}}), & j > j_0.
\end{cases} \quad (6.7)
\]
Extend $\beta_{n+1}$ from $H_n$ to $H_{n+1}$ as follows:

$$
\begin{align*}
\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) &= \delta(j, j, n)1_{A_{2j-1}^{(n+1)}} + \sum_{i=1}^{j-1} \delta(j, i, n)1_{A_{i}^{(n)}}, \quad j = 1, \ldots, j_0 - 1, \\
\beta_{n+1}(1_{A_{2j}^{(n+1)}}) &= \delta(j, j, n)1_{A_{2j}^{(n+1)}}, \quad j = 1, \ldots, j_0 - 1, \\
\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) &= \frac{1}{2}\delta(j, j, n)1_{A_{2j-1}^{(n+1)}} + \frac{1}{2}\sum_{i=1}^{j-1} (\delta(j, i, n) - \sum_{k=1}^{j-1} \delta(k, i, n))1_{A_{i}^{(n)}}, \quad j = j_0, \\
\beta_{n+1}(1_{A_{2j}^{(n+1)}}) &= \frac{1}{2}\delta(j, j, n)1_{A_{2j}^{(n+1)}}, \quad j = j_0, \\
\beta_{n+1}(1_{A_{2j-1}^{(n+1)}}) &= \frac{1}{2}\delta(j, j, n)1_{A_{2j-1}^{(n+1)}} + \frac{1}{2}\sum_{i=1}^{j-1} \delta(j, i, n)1_{A_{i}^{(n)}}, \quad j = j_0 + 1, \ldots, 2^n - 1, \\
\beta_{n+1}(1_{A_{2j}^{(n+1)}}) &= \frac{1}{2}\delta(j, j, n)1_{A_{2j}^{(n+1)}}, \quad j = j_0 + 1, \ldots, 2^n - 1, \\
\beta_{n+1}(1_{A_{n+1}^{(n+1)}}) &= 1_{A_{n+1}^{(n+1)}}.
\end{align*}
$$

The coefficients, implicit in these expressions for $\beta_{n+1}(1_{A_{j}^{(n+1)}})$, will be our $\delta(j, i, n + 1)$.

It follows by induction on $n$ that each coefficient $\delta(j, i, n)$ belongs to $\mathbb{Z}[(\frac{1}{2})]$ and that $\delta(j, j, n) = 2^{-k}$ for some $k \in \mathbb{N}$ (that depends on $j$ and $n$). The formula above for $\beta_{n+1}$ is consistent with (6.7), and so $\beta_{n+1} \circ \alpha_{n} = \beta_{n}$. It also follows by induction on $n$ that $1_{A_{j}^{(n)}}$ belongs to $\text{Im}(\beta_{n})$ for $j = 1, 2, \ldots, 2^n - 1$. This clearly holds for $n = 1$. Assume it holds for some $n \geq 1$. Then $1_{A_{j}^{(n)}}$ belongs to $\text{Im}(\beta_{n}) \subseteq \text{Im}(\beta_{n+1})$ for $j = 1, 2, \ldots, 2^n - 1$, and hence $1_{A_{2j-1}^{(n+1)}}$, $1_{A_{2j}^{(n+1)}} = 1_{A_{j}^{(n)}} - 1_{A_{2j-1}^{(n+1)}}$, and $1_{A_{n+1}^{(n+1)}}$ belong to $\text{Im}(\beta_{n+1})$. It is now verified that each $\beta_{n}$ is as desired.

To complete the proof we must show that the positive, injective, group homomorphism $\beta : K_{0}(A_{\Omega}) \to G$ is surjective and that $\beta(K_{0}(A_{\Omega}))^{+} = G^{+}$. The former follows from the already established fact that $1_{A}$ belongs to the image of $\beta$ for all $A \in \mathcal{F}$, and from the fact, which follows from Proposition 5.2, that if $f$ belongs to $\text{Im}(\beta)$, then so does $\frac{1}{2}f$. The latter identity is proved by verifying that $K_{0}(A_{\Omega})$ is totally ordered.

To show that $K_{0}(A_{\Omega})$ is totally ordered we must show that either $f$ or $-f$ is positive for each non-zero $f$ in $K_{0}(A_{\Omega})$. Write $f = \alpha_{\infty, n}(g)$ for a suitable $n$ and $g \in C_{0}(\Omega_{0}, \mathbb{Z})$. Let $r$ be the largest point in $\Omega$ for which $g(r) \neq 0$. Upon replacing $f$ by $-f$, if necessary, we can assume that $g(r)$ is positive. There is a (non-empty) clopen interval $A = [s, r] \cap \Omega$ for which $g(t) \geq 1$ for all $t$ in $A$. Put $X_{k,n} = \{t_{n}, t_{n+1}, \ldots, t_{n+k-1}\}$, $Y_{k,n} = X_{k,n} \cap [0, r]$, and
\[ Z_{k,n} = X_{k,n} \cap [0, s) \]. By \([6.3]\) and an analog of \([2.3]\) we get

\[
\alpha_{n+k,n}(g) = \sum_{F \subseteq X_{k,n}} g \circ \chi_{\max F} = \sum_{F \subseteq Y_{k,n}} g \circ \chi_{\max F} \\
\geq \sum_{F \subseteq Z_{k,n}} \min g(\Omega_0) + \sum_{F \subseteq Y_{k,n}, F \not\subseteq Z_{k,n}} 1_A \circ \chi_{\max F} \\
= 2|Z_{k,n}| \cdot \min g(\Omega_0) + (2|Y_{k,n}| - 2|Z_{k,n}|) \cdot 1_{[0,r] \cap \Omega}.
\]

Now,

\[
\lim_{k \to \infty} (|Y_{k,n}| - |Z_{k,n}|) = \lim_{k \to \infty} |X_{k,n} \cap [r, s]| = \infty,
\]

so \( \alpha_{n+k,n}(g) \geq 0 \) for some large enough \( k \). But then \( f = \alpha_{\infty,n+k}(\alpha_{n+k,n}(g)) \) is positive. □

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