SUPPORT VARIETIES FOR QUANTUM GROUPS.

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1. Introduction.

Given a root datum \((Y, X, \ldots)\) and an odd integer \(l\) G.Lusztig defined a finite-dimensional Hopf algebra \(u\) over the cyclotomic field \(k := \mathbb{Q}(\xi)\) where \(\xi\) is a primitive \(l\)-th root of unity, see [Lu1]. Suppose that \(l > h\) where \(h\) is the Coxeter number of the split reductive Lie algebra \(g\) over \(k\) associated to the root datum \((Y, X, \ldots)\). In this case the cohomology \(H^*(u, k)\) of algebra \(u\) with trivial coefficients were computed by V.Ginzburg and S.Kumar, see [GK]. For any \(u\)–module \(M\) one defines the support variety \(|u|_M\) (copying the corresponding constructions from the theory of modular representation of finite groups and from the theory of restricted Lie algebras), see 3.1 below. It follows from the result of Ginzburg and Kumar that \(|u|_M\) is a closed conical subvariety of the nilpotent cone \(N \subset g\).

Let \(U_\xi\) be the quantum group with divided powers associated with root datum \((Y, X, \ldots)\). Consider the category \(C\) of finite-dimensional \(U_\xi\)–modules of type \(1\). Any \(M \in C\) is a \(u\)–module in natural way. So we can consider the support variety \(|u|_M\). It is a \(G\)–stable subset of nilpotent cone, where \(G\) is a split algebraic group over \(k\) associated with \((Y, X, \ldots)\). Let \(2a\) denote the codimension of \(|u|_M\) in \(N\). Our first main result (Theorem 4.1) relates the integer \(a\) and the dimension of \(M\). Namely, we prove that if \(l\) is prime then \(\dim M\) is divisible by \(l^a\).

Recall that the category \(C\) is a direct sum of some its subcategories called linkage classes, see [APW]. Our second main result (Theorem 5.2) gives an a priori ‘estimate’ of support variety for a module in a fixed linkage class.

As an application we compute (combining Theorems 4.1 and 5.2) the support varieties of Weyl modules, see Theorem 5.1. Another application (and third main result of this note) is the computation of support varieties for tilting modules over the quantum group of type \(A_n\), see Theorem 6.3. This computation verifies the remarkable Humphreys’ Conjectures (in quantum setting) on support varieties of tilting modules and Lusztig’s bijection between two-sided cells in affine Weyl group and nilpotent orbits, see [H] and 3.6 below. In fact, the attempts to handle these Conjectures were the starting point of this work.

Our considerations (except for the computation of support varieties of tilting modules) are applicable as well to the situation in characteristic \(p > 0\). In this context the algebra \(u\) becomes the restricted enveloping algebra of the Lie algebra \(g\) (considered over a field of characteristic \(p\)). In this case our Theorem 6.4 gives a positive answer to Jantzen’s Question [J2] 2.7 (1) for \(p > h\).

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2. Preliminaries.

2.1. We fix a ground field \( k \). From now on all objects and tensor products will be considered over \( k \) if not indicated otherwise.

2.2. We fix a simply connected root datum \((Y, X, \ldots)\) corresponding to an indecomposable Cartan matrix. Let \( G, \mathfrak{g}, \mathfrak{b}, \mathfrak{n}, W, R, R_+, \tilde{R} \) denote the corresponding split algebraic group, its Lie algebra, its Borel subalgebra, nilpotent radical of Borel subalgebra, Weyl group, root system, the set of positive roots, and the dual root system respectively. For any \( \alpha \in R \) let \( \tilde{\alpha} \) denote the corresponding coroot. Let \( \langle , \rangle : X \times Y \to Z \) denote the canonical pairing. Let \( \rho \in X \) denote the halfsum of all positive roots. We will denote by dot (e.g. \( w \cdot \lambda \)) the action of \( W \) on \( X \) centered in \((-\rho)\). The set of simple coroots is denoted by \( I \). For any \( i \in I \) let \( \alpha_i \) denote the corresponding simple root. Let \( X^+ = \{ \lambda \in X|\langle \lambda, \tilde{\alpha}_i \rangle \geq 0, i \in I \} \) denote the set of dominant weights. Let \( h \) denote the Coxeter number of \( g \).

2.3. We fix an odd integer \( l > h \), for \( g \) of type \( G_2 \) we suppose that \( l \) is not divisible by 3. Let \( \xi \) be a primitive \( l^{-1} \)-th root of unity. Let \( U_\xi \) denote the quantum group with divided powers corresponding to \((Y, X, \ldots)\) as defined in \([Lu]\) and let \( C \) denote the category of finite-dimensional \( U_\xi \)-modules of type 1. Let \( E_i^{(s)}, F_i^{(s)}, K_i^{\pm 1}, i \in I, s > 0 \) denote the standard generators of algebra \( U_\xi \). We set

\[
U := U_\xi / \text{Ideal generated by} \{ K_i^l - 1 \}.
\]

Let \( B \subset U \) denote the Borel subalgebra. For any \( \lambda \in X, k \geq 0 \) let \( H^k(\lambda) \in C \) denote the higher induced module from one-dimensional representation of \( B \) corresponding to \( \lambda \), see \([APW]\). For \( \lambda \in X_+ \) the socle \( L(\lambda) \) of \( H_0(\lambda) \) is simple and nonzero. The modules \( L(\lambda) \) are distinct and any simple module in \( C \) is isomorphic to some \( L(\lambda) \).

2.4. Let \( W_a \) denote the affine Weyl group: it is generated by \( W \) and by translations by \( l\lambda \) for all \( \lambda \in Y \). Let \( X/W_a \) denote the set of \( W_a \)-orbits in \( X \) with respect to the dot action. For any \( \Omega \in X/W_a \) let \( C(\Omega) \) denote the full subcategory of \( C \) consisting of modules with composition factors \( L(\lambda) \) where \( \lambda \in \Omega \). The Linkage Principle (see \([APW]\)) asserts that

\[
C = \bigoplus_{\Omega \in X/W_a} C(\Omega).
\]

Moreover all composition factors of any module \( H^i(\lambda) \) are of the form \( L(\mu) \) where \( \mu \in W_a \cdot \lambda \).

2.5. We refer the reader to \([An]\) for the definition and properties of tilting modules over \( U \). Recall that

(i) Any tilting module is a sum of indecomposable ones.
(ii) Indecomposable tilting modules \( Q(\lambda) \) are naturally labelled by the dominant weights \( \lambda \in X_+ \).
(iii) Tensor product of tilting modules is tilting.
(iv) The module \( H^0(\lambda) \) is tilting iff it is simple.
2.6. Let \( \mathcal{Q} \) denote the category of tilting modules over \( \mathbf{U} \). The full subcategory \( \mathcal{Q}' \subset \mathcal{Q} \) is called tensor ideal if for any \( M \in \mathcal{Q}' \) and \( N \in \mathcal{Q} \)

(i) any direct summand of \( M \) lies in \( \mathcal{Q}' \).

(ii) \( M \otimes N \in \mathcal{Q}' \).

All tensor ideals in \( \mathcal{Q} \) were described in [Os] using results of [S1] and [S2] (these results are proved for simply laced root systems; the proof for non-simply-laced case is not yet written down).

We will say that weights \( \lambda, \mu \in X_+ \) lie in the same weight cell if the tensor ideals generated by \( Q(\lambda) \) and \( Q(\mu) \) coincide, i.e. \( Q(\lambda) \) is a direct summand of \( Q(\mu) \otimes Q \) for some \( Q \in \mathcal{Q} \) and conversely. Thus \( X_+ \) is partitioned into weight cells (this partition depends on \( l \)). It follows from [Os] that the set of weight cells is bijective to the set of canonical right cells in an affine Weyl group of \( \mathfrak{g} \).

2.7. Let \( \mathfrak{g} \) denote the quantized restricted enveloping algebra, see [L1]. It is a Hopf algebra of dimension \( t^{\dim \mathfrak{g}} \). Let \( \mathfrak{b} \) denote its Borel subalgebra.

2.8. G.Lusztig defined the Frobenius map \( \mathbf{U} \to U(\mathfrak{g}) \), see [L1]. Its kernel is \( \mathbf{U} \cdot \mathfrak{u}_+ = \mathfrak{u}_+ \cdot \mathbf{U} \) where \( \mathfrak{u}_+ \) is the augmentation ideal of \( \mathfrak{u} \). In particular, there is a natural \( U(\mathfrak{g}) \)-module structure on the cohomology \( H^*(\mathfrak{u}, M) \) of a \( \mathbf{U} \)-module \( M \).

3. Support varieties.

In this section we define support varieties for Hopf algebras (repeating the definitions for finite groups or for restricted Lie algebras) and recall their basic properties.

3.1. Let \( \mathfrak{a} \) be a finite dimensional Hopf algebra. It is known that the cohomology algebra with trivial coefficients \( H^*(\mathfrak{a}) = H^*(\mathfrak{a}, \mathbb{C}) \) is commutative, see [GR]. Suppose that \( H^*(\mathfrak{a}) \) is a finitely generated algebra. Denote by \( |\mathfrak{a}| \) the affine algebraic variety associated to the algebra \( H^*(\mathfrak{a}) \) (cohomology of even degrees). The variety \( |\mathfrak{a}| \) is equipped with natural \( \mathbb{C}^* \) action and distinguished point \( 0 \in |\mathfrak{a}| \). Let \( M \) be a finite dimensional \( \mathfrak{a} \)-module. Let \( |\mathfrak{a}|_M \subset |\mathfrak{a}| \) be the zero locus of the kernel of the natural morphism \( H^*(\mathfrak{a}) \to \text{Ext}^*_\mathfrak{a}(M, M) \).

**Definition.** The variety \( |\mathfrak{a}|_M \) is called the support variety of the module \( M \).

It is clear that \( |\mathfrak{a}|_M \) is conical with respect to the natural \( \mathbb{C}^* \) action and \( M \neq 0 \) implies that \( 0 \in |\mathfrak{a}|_M \).

**Remark.** Let \( \{L_k\}, k \in K \) be a collection of representatives of all isomorphism classes of simple \( \mathfrak{a} \)-modules. Then \( |\mathfrak{a}|_M \) is the union of supports of sheaves \( H^*(\mathfrak{a}, L_k \otimes M) \) in \( |\mathfrak{a}| \). The proof repeats word by word the arguments in [FP2] 1.4.

3.1.1. **Examples.** (i) Consider the case of the algebra \( \mathfrak{a} = \mathfrak{u} \). In this case it is known that \( H^*(\mathfrak{u}) = H^*(\mathfrak{u}, \mathbb{C}) = \mathbb{C}[\mathcal{N}] \) where \( \mathbb{C}[\mathcal{N}] \) denote the algebra of functions on the nilpotent cone \( \mathcal{N} \subset \mathfrak{g} \), see [GR]. Hence, \( |\mathfrak{u}| = \mathcal{N} \).

(ii) Let \( \mathfrak{a} = \mathfrak{b} \). In this case \( H^*(\mathfrak{b}) = H^*(\mathfrak{b}, \mathbb{C}) = S^*(\mathfrak{n}^+) \), see loc. cit. Hence, \( |\mathfrak{b}| = \mathfrak{n} \).

(iii) Let \( M \) be the trivial one-dimensional \( \mathfrak{a} \)-module. Then obviously \( |\mathfrak{a}|_M = |\mathfrak{a}| \).

(iv) If \( M \) is projective then \( |\mathfrak{a}|_M = 0 \).

3.2. The following Lemma is a Hopf-algebra analogue of the properties of support varieties established in [J1] in the case of restricted Lie algebras.

**Lemma.** (cf. [J1] 3.3(3), 2.2(1), 3.3(1)) (i) Let \( M, N \) be \( \mathfrak{a} \)-modules. Then \( |\mathfrak{a}|_{M \oplus N} = |\mathfrak{a}|_M \cup |\mathfrak{a}|_N \).
(ii) Let $a_1, a_2$ be Hopf algebras. Let $\phi : a_1 \rightarrow a_2$ be a homomorphism of algebras. It defines the morphism $\phi : |a_1| \rightarrow |a_2|$. For any $a_2$–module $M$ we have $|a_1|_M \subset \phi^{-1}|a_2|_M$.

(iii) Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of $a$–modules. Then for any permutation $(i, j, k)$ of $(1, 2, 3)$ we have $|a|_M \subset |a|_{M_i} \cup |a|_{M_k}$.

**Proof.** (i) is clear from the Remark 3.1.
(ii) is clear from the functoriality of cohomology.
(iii) is clear from the long exact sequence for cohomology. \hfill \Box

3.3. Let $M$ be a $U$–module. We will write $|u|_M$ for $|u|_{Res^U_M}$.

**Lemma.** For any $U$–module $M$ the variety $|u|_M$ is a $G$–invariant subset of the nilpotent cone $N$.

**Proof.** See [GL]. \hfill \Box

3.4. **Lemma.** (cf. [I] 3.3(5)) Suppose the map $H^{ev}(a) \otimes H^{ev}(a) \rightarrow H^{ev}(a)$ induced by the coproduct $\delta : a \rightarrow a \otimes a$ corresponds to the diagonal map $|a| \rightarrow |a| \times |a|$. Then for any $a$–modules $M$ and $N$ we have $|a|_{M \otimes N} \subset |a|_M \cap |a|_N$.

**Proof.** Clear. \hfill \Box

**Remarks.** (i) For all the algebras we consider below the assumption of Lemma is satisfied, and we will omit the checking.
(ii) It can be conjectured that equality holds: $|u|_{M \otimes N} = |u|_M \cap |u|_N$, cf. [PP] 2.1(c).

3.5. **Corollary.** Suppose $\lambda, \mu \in X_-$ lie in the same weight cell. Then $|u|_{Q(\lambda)} = |u|_{Q(\mu)}$.

**Proof.** $Q(\lambda)$ is a direct summand of $Q(\mu) \otimes Q$ for some $Q \in Q$. By Lemmas 3.2 (i) and 3.4 we obtain an inclusion $|u|_{Q(\lambda)} \subset |u|_{Q(\mu)}$. By symmetry we have the inverse inclusion. \hfill \Box

3.5.1. **Corollary.** There is a well-defined order-preserving map $H$ from the set of canonical right cells to the set of closed $G$–invariant subsets of $N$.

3.6. Now let us formulate the Humphreys’ Conjectures. Recall that the set of canonical right cells is bijective to the set of all two-sided cells in the affine Weyl group $W_\alpha$, see [LuX]. Furthermore, G. Lusztig constructed a bijection between the set of two-sided cells in $W_\alpha$ and the set of nilpotent orbits in $g$, see [Lu2]. Composing these bijections we obtain a bijection between the set of canonical right cells and the set of closed irreducible $G$–equivariant subsets of $N$. The Conjecture 1 below is a quantum version of Hypothesis [I] 12, and the Conjecture 2 modulo Conjecture 1 is close to discussion in the last paragraph of [I] 12.

**Conjecture 1.** The map $H$ coincides with Lusztig’s bijection.
Suppose $l$ is prime.

**Conjecture 2.** Suppose $|u|_{Q(\lambda)}$ has codimension $2a$ in $N$. Then $\dim Q(\lambda)$ is divisible by $l^a$.

**Remark.** In fact, J. Humphreys conjectured a stronger relation between the dimensions of tilting modules with regular highest weight and the dimension of their support varieties.

4. **Dimensions of $U$–modules.**

4.1. Let $d_R = \prod_{\alpha \in R_+} (\alpha, \alpha)$ (denominator of the Weyl dimension formula). This section is devoted to the proof of the following

**Theorem.** Let $M \in C$. Suppose the codimension of $|u|_M$ in $|u|$ is equal to $2a$. Then $d_R \dim M$ is divisible by $l^a$.

**Remarks.** (i) The codimension of a closed $G$–invariant subset of $N$ is always even since the number of nilpotent orbits is finite and any orbit is a simplectic variety.
(ii) If \( l \) is prime then we get divisibility of \( \dim M \) by \( l^a \) (since in this case \( d_R \) is not divided by \( l > h \)).

(iii) The Theorem implies the Conjecture 2 in [3.6]

4.2. **Proof.** We will write \( |b|_M \) for \( |b|_{\text{Res}_b^u(M)} \). It is a subset of \( n = |b| \), see [3.1.1]

**Lemma.** \( \dim |b| - \dim |b|_M \geq a. \)

**Proof.** By Lemma [3.3] we have \( |b|_M \subset n \cap |u|_M \). Further, \( |u|_M \) is a union of a finite number of nilpotent orbits. Let \( O \) be a nilpotent orbit. Then \( O \cap n \) is a lagrangian subvariety in \( O \), see [CG]. In particular \( \dim (O \cap n) = \frac{1}{2} \dim O \). Since \( \dim n = \frac{1}{2} \dim N \) we have \( \dim n - \dim (O \cap n) = \frac{1}{2}(\dim N - \dim O) \). The result follows. \( \Box \)

4.3. Now we use an interpretation of the dimension of the support variety as *complexity* of the module, see [FP2] 3.1.

**Lemma.** For any \( b \)-module \( N \) there exists a projective resolution \( P^* \rightarrow N \) and a constant \( C \) such that \( \dim(P^i) \leq C t^{\dim |b|_{N^1} - 1}, i \geq 1. \)

**Proof.** repeats word by word [FP2] 3.2. \( \Box \)

4.4. Recall that the algebra \( b \) has a natural \( Y \)-grading, see [Lu]. We consider the induced \( Z \)-grading such that any simple root has degree 1 (principal grading). The module \( M \) obviously has a \( Z \)-grading compatible with the principal grading on \( b \) since \( M \) is a restriction of \( U \)-module (this grading comes from the weight grading of \( M \)). We will say that \( b \)-module is graded if it has \( Z \)-grading compatible with the principal grading of \( b \).

4.5. For any finite dimensional \( Z \)-graded space \( N = \bigoplus_{i \in Z} N_i \) let \( \dim_i N \) denote its graded dimension \( \dim_i N = \sum_{i \in Z} \dim N_i t^i \). Set \( p(t) = \prod_{\alpha \in R^+} \frac{(1-t^{\langle \rho, \alpha \rangle})^s}{(1-t^{\langle \rho, \alpha \rangle})} \). The following result is well-known.

**Lemma.** Let \( P \) be an indecomposable projective \( b \)-module. It admits grading such that \( \dim_i P = t^c p(t) \) for some integer \( c \).

We will refer to the number \( c \) as to height of the graded module \( P \).

4.6. **Lemma.** There exists a projective resolution \( P^* \rightarrow M \) and a constant \( C \) such that

(i) it is graded;

(ii) indecomposable projectives with height \( c \) can occur with nonzero multiplicity only in \( P^i \) with \( 0 \leq i \leq c \);

(iii) \( \dim(P^i) \leq C t^{\dim |b|_{M^1} - 1}, i \geq 1. \)

**Proof.** Clear. \( \Box \)

4.7. Let us return to our module \( M \). We may (and will) suppose that \( \dim_i M \) is a polynomial with nonzero constant term (shifting the grading).

**Lemma.** There is a constant \( C \) such that \( \dim_i M = p(t)s(t) \) where \( s(t) = \sum_{i=0}^{\infty} s_i t^i \) is a Taylor series with integral coefficients and \( |s_i| \leq C t^{\dim |b|_{M^1} - 1}, i \geq 1. \)

**Proof.** Consider the function \( s(t) = \frac{\dim_i M}{p(t)} \). It is a rational function with poles in certain roots of unity of degree \( d_R \) since \( p(t) = 0 \) implies \( t^{d_R} = 1 \). Let \( s(t) = \sum_{i=0}^{\infty} s_i t^i \) be the Taylor expansion of \( s(t) \). Suppose the inequality in Lemma does not hold. For any \( 1 \leq k \leq d_R \) consider the sequence \( p_k(i) = s_{k+id_R} \). We claim that all these sequences are polynomials in \( i \) for \( i \) large enough. Moreover, for some \( k \) the sequence \( p_k(i) \) is polynomial of degree \( \geq \dim |b|_{M^1} \) by our assumption. It follows that \( \sum_{i=1}^{j} |s_i| \) increases faster than \( C t^{\dim |b|_{M^1} + 1} \). But this is impossible since \( s(t) \) can be obtained by computing Euler characteristic of the resolution given by Lemma 4.6. The Lemma is proved. \( \Box \)
Corollary. The series $s(t)$ is a rational function with poles of order less than or equal to $\dim |b|_M$.

4.8. Set $d_R(t) = \prod_{\alpha \in R,} \frac{1-t^{\langle \rho, \alpha \rangle}}{1-t}$. Note that the polynomial $d_R(t)p(t)$ has zero of order $\dim n$ at any nontrivial $l$–th root of unity. It follows from the Lemma and Corollary [11] that the polynomial $d_R(t)\dim M$ has zero of order at least $\dim n - \dim |b|_M \geq a$ at any nontrivial $l$–th root of unity. Hence the polynomial $d_R(t)\dim M$ with integral coefficients is divisible by $(\frac{d}{1-t})^a$. Consequently, $d_R\dim M$ is divisible by $l^a$. The Theorem is proved. □

4.9. Corollary (of the proof). In notations of Theorem the polynomial $\dim_i(M)$ has a zero of order $\geq a$ at any primitive $l$–th root of unity. Conversely, let $\dim_i(M)$ have a zero of order $\leq a$ at any primitive $l$–th root of unity. Then the codimension of $|u|_M$ in $|u|$ is less than or equal to $2a$.

Example. Let $M = H^0(\lambda)$ for $\lambda \in X_+$. By the Weyl character formula (see [APW]) we have

$$\dim_i(M) = \prod_{\alpha \in R_+} \frac{t^{\langle \lambda+\rho, \alpha \rangle} - 1}{t^{\langle \rho, \alpha \rangle} - 1}.$$  

In particular the order of zero at $l$–th primitive root of unity is equal to the number of positive roots $\alpha$ such that $\langle \lambda+\rho, \alpha \rangle \in l\mathbb{Z}$.

5. Estimation of the support variety.

In this section we estimate the support variety of a $U$–module in a given linkage class. This is a quantum analogue of Jantzen’s Conjecture [J2] 2.7(1).

5.1. For any $\lambda \in X$ consider the following root subsystem of $R$:

$$R_\lambda = \{ \alpha \in R | \langle \lambda + \rho, \alpha \rangle \in l\mathbb{Z} \}.$$  

It is easy to see that if $\lambda$ and $\mu$ lie in the same linkage class, then $R_\lambda$ and $R_\mu$ are conjugate with respect to the Weyl group action. Fix a linkage class $\Omega$ containing a weight $\lambda$. Choose $J \subset I$ such that $R_\lambda$ is $W$–conjugate to the parabolic root subsystem $R_J$ (this is possible by [J2] 2.7). Let $p_J \subset g$ denote the corresponding parabolic subalgebra, and let $u_J \subset p_J$ (resp. $l_J$) denote its nilpotent radical (resp. Levi subalgebra).

5.2. This section is devoted to the proof of the following

Theorem. The support variety of any module in the linkage class $\Omega$ is contained in $G \cdot u_J$.

Remark. $G \cdot u_J$ is an irreducible $G$–equivariant subvariety of $\mathcal{N}$. Hence, $G \cdot u_J$ is the closure of a nilpotent orbit $\mathcal{O}$. This orbit is called a Richardson orbit (corresponding to $J$). It is known that $\dim G \cdot u_J = \dim \mathcal{O} = 2 \dim u_J$.

Before we start the proof we need some preparations.

5.3. For any $J \subset I$ we consider the parabolic quantum group $P_J$: it is a Hopf subalgebra of $U$ generated by generators $F_i^{(s)}, K_i^{\pm 1}, i \in I, s > 0$ and $E_i^{(s)}, i \in J, s > 0$ (in notations of [APW] $P_J = U(J)$). Also consider the ‘Levi subgroup’ $L_J$: it is a Hopf subalgebra of $U$ generated by $K_i^{\pm 1}, i \in I$ and $E_i^{(s)}, F_i^{(s)}, i \in J, s > 0$. We have a canonical surjection $P_J \rightarrow L_J$ which sends $F_i^{(s)}, i \notin J, s > 0$ to zero and other generators into themselves. Let $\text{Inf}_{L_J}$ denote the corresponding inflation functor from $L_J$–modules to $P_J$–modules. Obviously, this functor is exact.
We define the parabolic and Levi subalgebras \( p_J \) and \( l_J \) of \( u \) in the same way. There is the Frobenius map \( P_J \to U(p_J) \) (resp. \( L_J \to U(l_J) \)) with kernel \( P_J \cdot (p_J)_+ = (p_J)_+ \cdot P_J \) (resp. \( L_J \cdot (l_J)_+ = (l_J)_+ \cdot L_J \)) where \( (p_J)_+ \subset p_J \) (resp. \( (l_J)_+ \subset l_J \)) is the augmentation ideal.

5.4. Let \( H = L_\emptyset \) denote the subalgebra of \( U \) generated by \( K_{i}^{\pm 1}, i \in I \). We have natural surjection \( B = P_\emptyset \to H \). For any parabolic subalgebra \( P \) with Levi subalgebra \( L \) consider the functors \( F_1, F_2 : \{ \text{integrable } H\text{-modules} \} \to \{ \text{integrable } P\text{-modules} \}, F_1(M) = \text{Ind}^P_B(\text{Inf}_H^B(M)), F_2(M) = \text{Ind}_L^P(\text{Inf}_L^B(\text{Inf}_H^B(M))) \). Here we use induction functors defined in [APW].

Lemma. The functors \( F_1, F_2 \) are isomorphic. In particular, for any integrable \( H\text{-module} M \) and \( i \geq 0 \) we have:

\[
R^i\text{Ind}^P_B(\text{Inf}_H^B(M)) = \text{Ind}_L^P(R^i\text{Ind}_L^B(\text{Inf}_H^B(M)))
\]

Proof. It is enough to compare the right adjoint functors. \( \square \)

5.5. Lemma. Let \( p \subset u \) be a parabolic subalgebra. Then \( H^\bullet(p) = H^{ev}(p) \) is the algebra of regular functions on \( N \cap p \) and the natural morphisms \( p| \to |u| \) and \( |p| \to |l| \) are the natural inclusion and projection maps respectively.

Proof is the same as in [CK]. Recall it very briefly. First step is the computation of cohomology of \( b \). There exists a \( b \)-equivariant isomorphism of algebras \( H^\bullet(b) = S^\bullet(n^*) \), see [CK] §2. Further, there are two spectral sequences \( E_2 \) and \( E_1 \), converging to the same limit, with \( E_2 \)-terms

\[
E_2^{p,q} = H^p(p, R^q\text{Ind}^P_B(k))
\]

and

\[
E_1^{p,q} = R^p\text{Ind}^U_{U(b)}(H^q(b, k)).
\]

Both sequences collapse at \( E_2 \) and the second one converges to the algebra \( \mathbb{C}[p \cap N] \) of regular functions on \( p \cap N \). The Lemma follows from this easily. Let us explain the equality

\[
\text{Ind}^U_{U(b)}(S^\bullet(n^*)) = \mathbb{C}[p \cap N]
\]

and vanishing of \( R^i\text{Ind}^U_{U(b)}(S^\bullet(n^*)) \), \( i \geq 0 \). From an exact sequence of \( U(b) \)-modules \( 0 \to (n/u)^* \to n^* \to u^* \to 0 \) we get that \( S^i(n^*) \) has a filtration with quotients of the form \( S^i((n/u)^*) \otimes S^{n-i}(u^*) \). Note that \( u^* \) is a restriction of \( U(p) \)-module and \( (n/u)^* \) is an inflation of \( U(b \cap n) \)-module \( (n \cap l)^* \). It follows that \( \text{Ind}(S^\bullet((n/u)^*) \otimes S^\bullet(u^*)) = \mathbb{C}[n \cap l] \otimes S^\bullet(u^*) \) and the higher inductions vanish. Hence, the natural inclusion \( \mathbb{C}[p \cap N] \hookrightarrow \text{Ind}^U_{U(b)}(S^\bullet(n^*)) \) induced by the restriction map \( \mathbb{C}[p \cap N] \to S^\bullet(n^*) \) is an isomorphism (since we have an isomorphism of affine algebraic varieties \( p \cap N \cong u \times (l \cap N) \) and the higher inductions vanish. \( \square \)

5.6. Lemma. Let \( M \) be a projective \( l \)-module. Consider \( M \) as \( p \)-module via canonical surjection \( p \to l \). Then \( p| \to \) is contained in \( u \) (unipotent radical of \( p \)).

Proof. Follows from the naturality of support variety, see Lemma 3.2 (ii). \( \square \)

5.7. Let \( L \) be a \( U \)-module.

Lemma. (cf. [I], 1.6.12) For any \( P \)-module \( M \) there are two spectral sequences converging to the same limit, with \( E_2 \)-terms:

\[
E_2^{p,q} = H^p(u, L \otimes R^q\text{Ind}_p^U(M))
\]

and

\[
E_2^{p,q} = R^p\text{Ind}_{U(b)}^U(H^q(p, L \otimes M)).
\]
Moreover, the algebra \( H^\bullet(u) \) acts on both spectral sequences commuting with all differentials and the actions of \( H^\bullet(u) \) on \( ^tE_\infty \) and \( E_\infty \) coincide.

**Proof.** Consider the following functors \( F_1, F_2 \) from the category of integrable \( P \)-modules to integrable \( U(g) \)-modules: \( F_1 = (\alpha)^u \circ \text{Ind}^U_P(\alpha), F_2 = \text{Ind}^{U(g)}_{U(p)}(\alpha) \circ (\alpha)^P \). We claim that these functors are isomorphic. Indeed, the left adjoint functor of \( F_1 \) is isomorphic to the left adjoint functor of \( F_2 \), since \( \text{Res}_P^U(\alpha) \circ \text{Ind}_U^U(\alpha) = \text{Ind}_P^P(\alpha) \circ \text{Res}_P^P(\alpha) \); it follows that for any \( P \)-module \( N \) there are two spectral sequences \( ^tE \) and \( ^uE \) converging to the same limit, with \( E_2 \)-terms:

\[
^tE_{2}^{p,q} = H^p(u, R^q\text{Ind}_P^U(N))
\]

and

\[
^uE_{2}^{p,q} = R^q\text{Ind}_P^U(H^q(p, N)).
\]

We set \( N = L \otimes M \). Then \( R^q\text{Ind}_P^U(L \otimes M) = L \otimes R^q\text{Ind}_P^U(M) \) by the generalized tensor identity, see [APW] 2.16 and [J] I.4.8. Thus, we obtain required sequences.

Now set \( N = k \). Then both spectral sequences collapse at \( E_2 \)-terms and their \( E_\infty \)-terms are equal to \( H^\bullet(u) \) (for \( ^uE \) this follows from the transitivity of induction and the proof of Lemma 5.7). The second assertion of the Lemma follows. □

5.8. **Lemma.** Let \( \lambda \in X_+ \) and \( w \in W \). All composition factors of \( H^i(w \cdot \lambda) \) are irreducibles \( L(\mu) \) with \( \mu \) linked to \( \lambda \) and \( \mu \leq \lambda \). Moreover the composition factor \( L(\lambda) \) occurs exactly once in the module \( H^i(w \cdot \lambda) \), and it does not occur in \( H^i(w \cdot \lambda), i \neq l(w) \).

**Proof.** See [APW] and [J] II.6.15-16. □

5.9. Let \( p \) be a parabolic subalgebra of \( g \) and let \( S^\bullet(p^*) \) be the algebra of functions on \( p \). Consider the variety \( G \times_P p \) (where \( P \) is parabolic subgroup of \( G \) with Lie algebra \( p \)). Let \( C[G \times_P p] \) denote the algebra of functions on \( G \times_P p \).

**Lemma.** There is an isomorphism of algebras \( \text{Ind}^{U(g)}_{U(p)}(S^\bullet(p^*)) = C[G \times_P p] \).

**Proof.** Let \( p : G \times_P p \to G/P \) denote the natural projection. Then the l.h.s. equals the global sections of the quasicoherent sheaf \( p_*O[G \times_P p] \). The Lemma follows. □

5.10. Consider a graded vector space \( M^\bullet \) such that

1) \( M^\bullet \) is a graded finitely generated \( S^\bullet(p^*) \)-module,

2) \( M^\bullet \) is a graded integrable \( p \)-module,

3) structures in 1) and 2) are compatible with respect to the coadjoint action of \( p \) on \( S^\bullet(p^*) \).

Such a datum is equivalent to a datum of \( P \times C^* \)-equivariant coherent sheaf \( S(M^\bullet) \) on \( p \). Let \( \text{supp}(M^\bullet) \) denote the support of this sheaf. For any \( i \geq 0 \) consider the graded vector space \( N_i^\bullet := R^i\text{Ind}^{U(g)}_{U(p)}(M^\bullet) \). It satisfies all the conditions 1)-3) (for \( p \) replaced by \( g \)). Let us explain how this looks geometrically. Let \( \phi : G \times_P p \to G \times_P p \) be the natural projection. There exists a sheaf \( S' \) on \( G \times_P p \) such that \( \phi^*S' = O[G] \otimes S(M^\bullet) \). By Lemma 5.9 we have \( S(N_i^\bullet) = R^i\phi_*S' \) where \( \phi : G \times_P p \to g, (g, p) \mapsto gpg^{-1} \).

**Lemma.** We have \( \text{supp}(N_i^\bullet) \subset G \cdot \text{supp}(M^\bullet) \).

**Proof.** Clear. □

5.11. **Proof of the Theorem.** It is enough to check the simple modules \( L(\lambda) \). We proceed by induction in \( \lambda \) with respect to standard ordering on \( X \) (i.e. \( \lambda \geq \mu \) iff \( \lambda - \mu \) is a sum of simple roots with nonnegative coefficients). We assume that the Theorem holds for all modules \( L(\mu) \) where \( \mu \in \Omega, \mu < \lambda \). Choose \( w \in W \) such that \( \langle w(\lambda + \rho), \alpha_i \rangle \in \mathbb{N} \) for all \( i \in J \). Consider the \( P_J \)-module \( M = \text{Ind}^P_B(w \cdot \lambda) \). Note that our choice of \( w \) gives vanishing of \( R^i\text{Ind}^P_B(w \cdot \lambda) \)
for $i > 0$ (by Kempf vanishing, see [APW]). In particular, $R^i\text{Ind}_{\mathcal{B}}^{\mathcal{P}}(M) = H^i(w \cdot \lambda)$. Further, note that $\text{Ind}_{\mathcal{B}}^{\mathcal{P}}(w \cdot \lambda)$ is an inflation from $L_J$–module by Lemma 5.4 and as $L_J$–module it is projective.

Consider the spectral sequences given by Lemma 5.7 for a $\mathcal{P}_J$–module $M$, and $L$ running through the set of simple $u$–modules (note that any simple $u$–module can be lifted to a $U$–module). Suppose that $|u|_{L(\lambda)}$ is not contained in $G \cdot u_J$. Then by Lemmas 5.8 and 3.2 (iii) for some $L$ the $\mathbb{C}[N]$–module $H^*(u, H^{(w)}(w \cdot \lambda) \otimes L)$ has support not contained in $G \cdot u_J$. It follows that the same is true for $E_\infty$. But this is impossible since $E_\infty = "E_\infty$ and by Lemmas 5.3, 5.10 the support of $"E_\infty$ is contained in $G \cdot u_J$. The Theorem is proved. □

Remark. The same proof is applicable as well to the modular situation with $p > h$.

6. Applications.

6.1. Let $\lambda \in X_+$. 

Theorem. In the notations of 5.2 we have $|u|_{H^0(\lambda)} = G \cdot u_J$.

Proof. The inclusion $|u|_{H^0(\lambda)} \subset G \cdot u_J$ is proved in Theorem 5.2. On the other hand we know from Example 4.9 and Remark 5.2 that $\dim |u|_{H^0(\lambda)} \geq \dim G \cdot u_J$. The variety $G \cdot u_J$ is irreducible. The Theorem follows. □

Remark. This gives also the support varieties for Weyl modules since the support varieties of a module and its dual coincide.

6.2. From now on we consider the case of quantum $SL_{n+1}$, i.e. we suppose that our root datum $(Y, X, \ldots)$ is of type $A_n$. We identify $R_+$ with the set of pairs $(j_1, j_2) \in [n+1] \times [n+1], j_1 < j_2$ (where $[n+1]$ denotes the set of natural numbers $j$ such that $1 \leq j \leq n+1$).

6.3. Let us fix a partition $p = \{p_1, \ldots\}$ of $n+1$. Let $p' = \{p'_1, \ldots\}$ denote the dual partition.

6.3.1. Lemma. There exists a decreasing sequence of numbers $x_1 > x_2 > \ldots > x_{n+1}$ such that $x_i - x_j \in \mathbb{Z}$ and for $\sum_{i=1}^k p'_i < j \leq \sum_{i=1}^{k+1} p'_i$ we have $x_j = x_{j+p'_k+1} = l$.

Proof. Choose any integers $x_1 > x_2 > \ldots > x_{p'_1} > x_1 - l$ (This is possible since $l > h = n$). Now for any $1 \leq j \leq p'_1$ set $x_{j+\sum_{i=1}^k p'_i} = x_j - kl$. □

6.3.2. Let us consider a weight $\lambda = (x_1-x_2-1, \ldots, x_n-x_{n+1}-1)$ (in coordinates corresponding to the basis of fundamental weights) where $x_i$ are the numbers given by 6.3.1. Obviously, $\lambda \in X_+$.

Lemma. The module $Q(\lambda)$ coincides with the induced module $H^0(\lambda)$.

Proof. It is enough to show that module $H^0(\lambda)$ is irreducible by 2.3 (iv). This follows from the quantum analogue of irreducibility criterion [1], II.8.21. □

6.3.3. Lemma. For $\lambda$ as above $|u|_{H^0(\lambda)} = \{ \text{the closure of orbit consisting of nilpotent elements with Jordan blocks of sizes } \{p'_1, \ldots\} \}$.

Proof. It is easy to see that in this case $R_\lambda$ is of type $A_{p_1-1} \times A_{p_2-1} \times \ldots$. Let $R_J$ be a parabolic root subsystem which is $W$–conjugate to $R_\lambda$. By [12] 2.6 we have $G \cdot u_J = \text{closure}$ of the orbit consisting of nilpotent elements with Jordan blocks of sizes $(p'_1, \ldots)$. The result follows from Theorem 6.1. □

6.4. In our proof we will use the explicit description of cells in the affine Weyl group of type $A_n$ given by J.Shi in terms of admissible sign types. We consider only the case of dominant admissible sign types, for the general case see [SH].
6.4.1. Definition. ([Sh]) The map \( f : R_+ \to \{+0\} \), is called dominant admissible sign type (dast for short) if for any \( \alpha, \beta, \alpha + \beta \in R_+ \) the equality \( f(\alpha) = + \) yields \( f(\alpha + \beta) = + \).

Let \( F \) denote the set of dasts, let \( P_n \) denote the set of partitions of \( n + 1 \). Let us define the map \( \pi : F \to P_n \). For any \( f \in F \) we say that a subset \( K \subset [n + 1] \) is connected if \( f(j_1, j_2) = + \) for any \( j_1, j_2 \in K \), \( j_1 < j_2 \). Now let \( p_1 \) be the maximal cardinality of a connected subset of \([n + 1] \), \( p_1 + p_2 \) be the maximal cardinality of a subset of \([n + 1] \) which is a disjoint union of two connected subsets and so on (see [Sh]). Then by C.Greene’s Theorem \( p_1, \ldots, p_k \) \((k \text{ is the largest integer such that } p_k \text{ is nonzero})\) is a partition of \( n + 1 \), i.e. \( p_1 \geq p_2 \geq \ldots \geq p_k \). We set \( \pi(f) = (p_1, \ldots, p_k) \in P_n \).

6.4.2. For any \( \lambda \in \mathbb{X}_+ \) we define \( f = f(\lambda) \in F \) as follows: for any \( \alpha \in R_+ \) we set \( f(\alpha) = + \) if \( \langle \lambda + \rho, \dot{\alpha} \rangle \geq l \) and \( f(\alpha) = 0 \) otherwise. We get a map \( f : \mathbb{X}_+ \to F \).

6.4.3. Consider the composition \( \pi f : \mathbb{X}_+ \to P_n \). The fibers of this map are unions of alcoves (possibly not closed). According to [Sh] §18.2 and §7.2 the partitions of the set of dominant alcoves into fibers of this map and into weight cells coincide. Moreover the orders on \( P_n \) and on the set of canonical right cells also coincide. Further, let \( \pi_1 \) denote the map from the set of weight cells to \( P_n \) induced by \( \pi f \). Identify \( P_n \) with the set of nilpotent orbits in \( SL_{n+1} \) as follows: to a partition \( (p_1, \ldots, p_k) \) we associate the orbit consisting of elements with Jordan blocks of sizes \( (p'_1, \ldots) \) (dual partition). It is well known that the map \( \pi_1 \) coincides with Lusztig’s bijection.

6.4.4. Examples. (i) For zero weight \( \lambda = 0 \) we have \( \pi f(0) = (1, 1, \ldots) \) (recall that \( l > n \)).

(ii) For the weight \( \lambda \) from Lemma 3.3.2 we have \( \pi f(\lambda) = (p_1, \ldots, p_k) \).

6.5. Theorem. For the quantum group of type \( A_n \) the map \( H \) (see 3.5.4) coincides with Lusztig’s bijection.

Proof. For \( \lambda \) of 6.3.2 and 6.3.3 we have \( |u|_Q(\lambda) = |u|_{H^0(\lambda)} = \text{the closure of orbit of type } (p'_1, \ldots) \). The Theorem is proved. \( \Box \)

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