FLOOD-IT ON AT-FREE GRAPHS

A body of clay, a mind full of play, a moment’s life – that’s me.*

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Abstract. Solitaire Flood-IT, or Honey-Bee, is a game played on a colored graph. The player resides in a source vertex. Originally his territory is the maximal connected, monochromatic subgraph that contains the source. A move consists of calling a color. This conquers all the nodes of the graph that can be reached by a monochromatic path of that color from the current territory of the player. It is the aim of the player to add all vertices to his territory in a minimal number of moves. We show that the minimal number of moves can be computed in polynomial time when the game is played on AT-free graphs.

1 Introduction

As Oscar Wilde already observed, ‘Life is far too important a thing ever to talk seriously about,’ so, today, let’s play!

Flood-IT is a popular game which can be played solitaire or together with other people or with machines. We define the solitaire game in terms of a graph as follows. The input is a graph $G$ of which the vertices are colored. Let $c : V(G) \rightarrow C$ and $C = \{1, \ldots, k\}$ denote the coloring of the vertices of $G$ with colors from a set $C$ of $k$ colors. Originally, the player occupies one vertex, say $x_0$, and his ‘territory’ consists of $x_0$ plus all the vertices that can be reached from $x_0$ by a monochromatic path of vertices of color $c(x_0)$. A move consists of the player’s calling of a color, say $i$, which, one may assume, is not the current color of his territory. The result of the move is that the territory of the player is recolored with the color $i$ and that those vertices are added to it that can be reached from $x_0$ by a monochromatic path of color $i$. The aim of the player is to increase his territory to $V(G)$ in as few moves as possible.

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In another version, called Free-Flood-It, the player may grow his territory from different starting points at any move. In a 2-player version, two players play against each other. The player who grows his territory to at least half of the vertices wins the game.

In this paper we confine ourselves to the analysis of the solitaire game described above. In the remainder of this introduction we briefly mention some of the known complexity results. The problem is polynomial for paths, cycles, and cocomparability graphs. It is NP-complete for splitgraphs and on trees, even when the number of colors is restricted to 3. Furthermore, the problem remains NP-complete on $3 \times n$ boards, even when the number of colors is only 4. The free version is NP-complete on trees and even on caterpillars. When parameterized by the number of colors, the free problem becomes tractable again for interval graphs and for splitgraphs. For $2 \times n$ boards the free problem is fixed-parameter tractable, when parameterized by the number of colors. The problem remains NP-complete on such boards when the number of colors is unbounded. Notice that the solitaire game is trivial in case there are only two colors.

2 Preliminaries on AT-free graphs

Asteroidal sets were introduced by Walter in 1978 to characterize certain subclasses of chordal graphs. They were rediscovered and put to use in [5,13]. The elements of asteroidal sets of cardinality 3 are called asteroidal triples. They were introduced by Lekkerkerker and Boland in their pioneering paper on the characterization of interval graphs [18]. To be precise, the authors of the underlying work characterize interval graphs as those chordal graphs without asteroidal triples. For a somewhat different proof, and an extension to the infinite, we refer to [11]. See also [15] for another description of the proof and for definitions of basic concepts that we assume here familiarity with.

Definition 1. In a graph, an asteroidal triple is an independent set of three vertices such that every pair of them is connected by a path that avoids the closed neighborhood of the third.

In an attempt to find an asteroidal triple in a graph one can examine every triple; remove, in turn, the closed neighborhood of a triple’s element, and check whether the remaining pair is contained in one component of the truncated graph. To check if a graph is AT-free this is, at the moment, basically the best known method, since it can be shown that finding an asteroidal triple in a graph is at least as hard as finding a triangle in a graph (see, eg, [16]).

Graphs without asteroidal triples, that is, AT-free graphs, generalize cocomparability graphs in a natural way. AT-free graphs properly contain cocomparability graphs and these, in turn, contain valued classes such as interval graphs and permutation graph. Notice however, that the class of AT-free graphs differs
from the smaller ones by the fact that its elements are, on the whole, not perfect. Recall that a graph is the complement of a comparability graph if it has an intersection model in which each vertex \( x \) is represented by a continuous function \( f_x : [0, 1] \to \mathbb{R} \). Two vertices are adjacent in the graph if their functions intersect \([10,15]\). To see that they are indeed AT-free, observe that for any three, pairwise non-intersecting, continuous functions, one must lie between the other two. Then every path that runs between the outer pair, must have a vertex in the closed neighborhood of the one in the middle.

Another property satisfied by cocomparability graphs is that the class is closed under edge contractions.

**Definition 2.** Let \( G \) be a graph and let \( \{x, y\} \in E(G) \). A contraction of the edge \( \{x, y\} \) replaces the pair by a single, new vertex. The neighborhood of the new vertex is the union of the neighborhoods of the endpoints of the edge, with the omission of the deleted endpoints \( x \) and \( y \):

\[
N(x) \cup N(y) \setminus \{x, y\}.
\]

Edge contractions play an important role in the theory of graph minors. A minor of a graph \( G \) is a graph obtained from \( G \) by a series of edge- and vertex deletions and edge contractions. Obviously, the class of cocomparability graphs is not closed under taking minors, since that would imply the erroneous conclusion that all graphs are cocomparability, because all graphs are obtainable from a large enough clique by taking subgraphs.

To see that cocomparability graphs are closed under edge contractions, consider a function model, as described above. To contract an edge \( \{x, y\} \), replace the functions \( f_x \) and \( f_y \) by one new function which swiftly zig-zags between the two functions \( f_x \) and \( f_y \). Clearly, the new function serves as a contraction since any vertex intersects \( f_x \) or \( f_y \) if and only if it intersects the new function (see, e.g., \([8]\)).

We show that the class of AT-free graphs is closed under edge contractions. Actually, even the larger class of 'hereditary dominating pair graphs' has the property \([21]\).

**Lemma 1.** The class of AT-free graphs is closed under edge contractions.

**Proof.** Let \( G \) be AT-free and let \( H \) be obtained from \( G \) by contracting an edge \( \{x, y\} \in E(G) \). Assume \( H \) has an AT, say \( \{p, q, r\} \). Any path between \( p \) and \( q \) in \( H - N[r] \) corresponds with a path in \( G \), possibly containing the edge \( \{x, y\} \). If \( r \) is the contracted vertex, then the path avoids the neighborhood of both \( x \) and \( y \) in \( G \). This shows that \( G \) must also contain an AT, which is a contradiction.

Consider contracting every connected, monochromatic subgraph to a single vertex. Then, according to the lemma above, the resulting graph is still AT-free,
and, furthermore, the outcome is properly colored, that is, each set of colors induces an independent set. By the definition of the game, the minimal number of moves needed to conquer the graph is the same in both graphs. This proves the following corollary.

**Corollary 1.** The complexity of solitaire Flood-It on AT-free graphs is equivalent to that of the game played on AT-free graphs with a proper vertex coloring, that is, where each color class induces an independent set.

AT-free graphs decompose quite gracefully into blocks and intervals.

**Definition 3.** Let \( G \) be a graph and let \( x \in V(G) \). A block at \( x \) is a component of \( G - N[x] \).

**Definition 4.** Let \( G \) be a graph and let \( x \) and \( y \) be nonadjacent vertices in \( G \). A vertex \( z \) is between \( x \) and \( y \) if \( x \) and \( z \) are contained in a common component of \( G - N[y] \) and \( y \) and \( z \) are contained in a common component of \( G - N[x] \). The interval between \( x \) and \( y \) is the set of all vertices that are between \( x \) and \( y \).

The close relationship between interval graphs and AT-free graphs is illustrated by the fact that a graph is AT-free if and only if every minimal triangulation is an interval graph [12]. This is demonstrated by the following two decomposition theorems, which, incidentally, lie at the heart of the algorithm that computes the independence number in AT-free graphs [4].

**Theorem 1.** Let \( G \) be AT-free and let \( I(x, y) \) be a nonempty interval between two nonadjacent vertices \( x \) and \( y \). Then, for any vertex \( z \in I(x, y) \), the removal of \( N[z] \) partitions \( I(x, y) \) into two intervals \( I(x, z) \) and \( I(z, y) \) and a collection of blocks at \( z \).

**Theorem 2.** Let \( G \) be AT-free. Let \( B \) be a block at a vertex \( x \). Then, for any vertex \( y \in B \), the removal of \( N[y] \) partitions \( B - N[y] \) into an interval \( I(x, y) \) and some blocks at \( y \).

We end this section with one more definition; that of an extreme.

**Definition 5.** Let \( G \) be a connected graph. A vertex \( x \) is an extreme if the largest component of \( G - N[x] \) has, among all vertices of \( G \), the maximal cardinality. In a disconnected graph a vertex is extreme if it is extreme in a component of the graph.

So, when \( G \) is \( P_3 \)-free, that is, when \( G \) is a clique, or a disjoint union of cliques, then every vertex of \( G \) is an extreme.

**Lemma 2.** Let \( G \) be a connected graph and let \( x \) be an extreme in \( G \) and let \( C \) be the largest component of \( G - N[x] \). Let \( S = N(C) \), that is, \( S \) is the set of vertices in \( V \setminus C \) that have a neighbor in \( C \). Let

\[
X = V(G) \setminus (N(C) \cup C).
\]

Then \( x \in X \) and every vertex of \( X \) is adjacent to every vertex of \( S \).
Proof. Assume that some vertex \( x' \in X \) is not adjacent to some vertex \( y \in N(C) \). This contradicts the assumption that \( x \) is extreme, since \( C \cup \{y\} \) is contained in a component of \( G - N[x'] \), that is, the largest component of \( G - N[x'] \) is larger than the largest component of \( G - N[x] \).

Corollary 2. All vertices of the set \( X \) defined as in Lemma 2 are extreme.

The set \( X \) is a module and, when \( G \) is AT-free, it induces an, otherwise unrestricted, AT-free graph. Anyway, AT-free or not, in turn \( G[X] \) generally contains an extreme.

Definition 6. A global extreme is,

1. an element of \( X \) when \( G[X] \) is a union of cliques, or,
2. defined recursively, as a global extreme of any component of \( G[X] \), otherwise.

3 The case where the source vertex is a global extreme

Following the stratagem of Fleischer and Woeginger in [8] we start with an analysis of the case where the source vertex, denoted by \( x_0 \), is a global extreme.

Throughout this section we assume that the graph \( G \) is a connected AT-free graph with a proper vertex coloring.

Lemma 3. Let \( x \) and \( y \) be two vertices of the same color and assume that \( y \in I(x, x_0) \). Then conquering \( x \) simultaneously conquers \( y \). By that we mean that when \( x \) is added to \( x_0 \)'s territory, either \( y \) was already in that territory or else it gets added in the same round as \( x \).

Proof. Conquering \( x \) implies, by definition, that the territory of \( x_0 \) is colored with color \( c(x) \) and that, subsequently, there is a \( x_0 \), \( x \)-path with all its vertices of the color \( c(x) \).

By Theorem 1, the neighborhood \( N[y] \) separates \( x \) and \( x_0 \) into different components of \( G - N[y] \). Since the set \( N[y] \) separates \( x_0 \) and \( x \), the \( x_0 \), \( x \)-path goes through a neighbor of \( y \). Then, calling the color \( c(x) \) by \( x_0 \) will add \( y \) to the territory as well.

Lemma 4. Let \( x \) and \( y \) be two vertices of the same color. Assume that \( y \) and \( x_0 \) are in a common block \( B \) at \( x \). Then conquering \( x \) simultaneously conquers \( y \).

Proof. By Theorem 2, \( G - N[y] \) partitions the component \( B \) of \( G - N[x] \) into an interval \( I(x, y) \) and some blocks at \( y \). The vertex \( x_0 \notin I(x, y) \) since this would contradict that \( x_0 \) is a global extreme.

To see that, assume that \( x_0 \in I(x, y) \) and let \( C \) be the largest component of \( G - N[x_0] \). If \( x \) and \( y \) were both in \( C \) then \( \{x, y, x_0\} \) would be an AT. If \( x \in C \) and \( y \notin C \),
then \(x_0\) and \(y\) cannot be in one component of \(G - N[x]\), since \(N(C) \subseteq N[y]\) by Lemma 2. Finally, assume that neither \(x\) nor \(y\) is in \(C\). Then \(x_0\) is in the interval between \(x\) and \(y\) in \(G - C - N(C)\). However, by induction, this is a contradiction, since \(x_0\) is a global extreme in \(G - C - N(C)\).

Thus, the closed neighborhood of \(y\) separates \(x\) and \(x_0\) or \(x_0 \in N[y]\), which implies the claim, as in the proof of Lemma 3.

**Theorem 3.** The vertices of a color class can be linearly ordered such that, conquering a vertex \(x\) simultaneously conquers all the vertices in the color class that precede \(x\) in the ordering.

**Proof.** Let \(x\) and \(y\) be two vertices of the same color. We show that the order in which \(x\) and \(y\) are conquered depends on the graph, and not on the conquering strategy of \(x_0\).

If \(y\) is in the block at \(x\) that contain \(x_0\), or if \(y\) is between \(x\) and \(x_0\), the conquering of \(x\) simultaneously conquers \(y\) by Lemmas 3 and 4.

If one of \(x\) and \(y\), say \(x\), is adjacent to \(x_0\) then it is easily perceived that conquering \(y\) simultaneously conquers \(x\). Assume that \(x\) is not adjacent to \(x_0\) and let \(C\) be the component of \(G - N[x]\) that contains \(x_0\). Denote

\[ S = N(C) \quad \text{then} \quad S \subseteq N(x). \]

We may assume that \(y \notin C\). If \(y\) is not adjacent to all vertices of \(S\), then there is a \(x_0, x\)-path that avoids \(N[y]\), that is, \(x\) is in the block at \(y\) that contains \(x_0\).

Assume that \(y\) is adjacent to all vertices of \(S\). Then, conquering one of \(x\) or \(y\) simultaneously conquers the other.

\[ \square \]

*Fig. 1.* This graph contains an AT, namely \([x_0, x, y]\).
Theorem 4. The solitaire Flood-It game starting at an extreme vertex can be solved in polynomial time on AT-free graphs.

Proof. By Theorem 3 each color class can be linearly ordered such that conquering a vertex implies the conquest of all preceding vertices in the same color class. Assume all color classes have been linearly ordered like that and denote by $\text{Max}(c)$ the maximal element of color $c$. Let $\mathcal{C}$ be the set of colors and let

$$M = \{ \text{Max}(c) \mid c \in \mathcal{C} \}.$$

Consider a vertex $x$ for which the cardinality of the component of $G - N[x]$ that contains the source $x_0$ is as large as possible. Let $D$ be the component. Let $\Delta = N(D)$ and $\Omega = V \setminus (D \cup \Delta)$.

Then $x \in \Omega$ and all vertices of $\Omega$ are adjacent to all vertices of $\Delta$. Notice that $\{x_0, x\}$ is a dominating pair, that is, each path running between $x_0$ and $x$ is a dominating set.

If a color $c$ appears at least once in $\Omega$, then $\text{Max}(c) \in \Omega$. Let $\Omega^* = \Omega \cap M$.

Give vertices of $M$ a weight zero and all other vertices a weight 1. Let $\text{Opt}$ denote the cost of a cheapest path to reach at least one vertex in $\Omega^*$. Then the optimal solution to solve the game has a cost $\text{Opt} + k$, where $k = |\mathcal{C}|$ is the number of colors. To see that, consider any $x_0, x$-path $P$. Add each vertex of $M$, which is not already in the path, to $P$ at a maximal distance from $x_0$; this constructs a caterpillar. The strategy which adds the colors to the territory of $x_0$ in the order of $P$, adding the vertices of $M$ when they are met, eventually adds all vertices to the territory. This proves the claim.

4 An algorithm for Flood-It on AT-free graphs

Notation. Let $\alpha$ and $\omega$ be two vertices such that the interval $I(\alpha, \omega)$ contains a maximal number of vertices. Let $C_{\alpha}(\omega)$ denote the component of $G - N[\alpha]$ that contains $\omega$ and define $C_{\omega}(\alpha)$ similarly. Let

$$S_{\alpha} = N(C_{\alpha}(\omega)) \quad A = V(G) \setminus (S_{\alpha} \cup C_{\alpha}(\omega)) \quad \text{and} \quad S_{\omega} = N(C_{\omega}(\alpha)) \quad \Omega = V(G) \setminus (S_{\omega} \cup C_{\omega}(\alpha)).$$

Notice that, possibly $S_{\alpha} \cap S_{\omega} \neq \emptyset$.

Lemma 5. Under the restrictions set out above, one may choose $\alpha$ such that all vertices of $A$ are adjacent to all vertices of $S_{\alpha}$.

Proof. Assume that there exists a vertex $a \in A$ such that $a$ is not adjacent to some vertex $\epsilon \in S_{\alpha}$. Then, the component of $G - N[a]$ which contains $\omega$ contains $C_{\alpha}(\omega) \cup \{\epsilon\}$. It follows that

$$I(\alpha, \omega) \subseteq I(a, \omega).$$

Continuing the process proves the claim.
Henceforth, we assume that \( \alpha \in A \) and that all vertices of \( A \) are adjacent to all vertices of \( S_\alpha \) and, similarly, \( \omega \in \Omega \) and all vertices of \( \Omega \) are adjacent to all vertices of \( S_\omega \).

A strategy is a sequence of colors, called by the source vertex \( v_0 \), which ultimately adds all vertices to its territory. For the case where \( v_0 \) is in one of the sets \( A, \Omega, S_\alpha \) or \( S_\omega \), the analysis is similar to where \( v_0 \) is an extreme. So, in this section we concentrate on the case where

\[ v_0 \in I(\alpha, \omega). \]

Assuming that is the case, \( N[v_0] \) separates \( \alpha \) and \( \omega \) and, by Theorem 3, each color class is partitioned into two linearly ordered sets. In other words, each color class has a ‘maximal’ and a ‘minimal’ element. The minimal element is the last one visited on the path from \( v_0 \) that contains some element of \( A \) and the maximal element is the last one visited on the path from \( v_0 \) that contains some element of \( \Omega \).

Notice that \( A \) may contain more than one element of some color, but conquering one of them, conquers them all. To see that, observe that any strategy induces a path from \( A \) to \( \Omega \). The path passes through \( S_\alpha \) and the join between \( A \) and \( S_\alpha \) proves the claim. Contrary to cocomparability graphs, the sets \( A \) and \( \Omega \) need not be cliques.

Analogous the Fleischer & Woeginger’s stratagem for cocomparability graphs, we define the essential length of a strategy as follows.

**Definition 7.** The length of a strategy \( \gamma \) is the number of colors in it. The essential length of \( \gamma \) is the length minus the number of steps where the second extremal vertex of some color class is conquered.

For a strategy \( \gamma \), let \( |\gamma| \) denote its length and let \( \text{ess}(\gamma) \) denote its essential length. Let \( \text{Opt} \) denote the solution of the solitaire Flood-It game, then we have

\[ \text{Opt} = \min \{ \text{ess}(\gamma) + k \mid \gamma \text{ is a strategy} \}, \]

where \( k \) is the number of colors.

**Notation.** For a vertex \( x \), let \( \text{Min}_x(c) \) denote the minimal element of color \( c \) which is adjacent to \( x \) and let \( \text{Max}_x(c) \) denote the maximal element of color \( c \) which is adjacent to \( x \).

**Notation.** For a pair of vertices \( x \) and \( y \) let \( D(x,y) \) denote the essential length of a strategy that conquers \( x \) and \( y \). We set \( D(v_0,v_0) = 0 \).

**Definition 8.** Two vertices \( x \) and \( x' \) that are not adjacent to \( v_0 \) are incomparable if the component of \( G - N[x'] \) that contains \( v_0 \) is the same as the component of \( G - N[x] \) that contains \( v_0 \).
Theorem 5. Let $x$ be a vertex in a component of $G - N[v_0]$ which contains some vertex of $A$ and let $y$ be a vertex in a component of $G - N[v_0]$ which contains a vertex of $\Omega$. Then

$$D(x, y) = \min \{ D(x, \text{Min}_y(x)) + \delta_y(x), D(\text{Max}_x(y), y) + \delta_x(y) \mid c \text{ is a color} \},$$

where $\delta_x(y) = 0$ if $y$ is the maximal element of some color $c$ and the minimal element is not in $I(x, y)$, or adjacent to $x$, or incomparable to $x$ or $y$ or both and $\delta_x(y) = 1$ otherwise. In other words, $\delta_x(y) = 0$ if $y$ is a maximal element of a color class and the minimal element is conquered earlier or in the same step.

As usual, we let $\infty$ be the minimal value of a set which is empty, so, for example, if $y$ is not adjacent to a vertex of color $c$, then we let $D(x, \text{Min}_y(c)) = \infty$.

Proof. In analogy to the proof in [8], let $\gamma$ be an optimal strategy and let $\alpha \in A$ and $\omega \in \Omega$ be the first two extremal vertices that are conquered. After the conquest of $\alpha$ and $\omega$, only the remaining maximal elements need to be conquered. Thus $|\gamma| \geq D(\alpha, \omega) + k$. \qed

5 Concluding remarks

We have shown that the solitaire Flood-Ir game can be solved in polynomial time on AT-free graphs. A graph is a hereditary dominating pair graph if each of its connected induced subgraphs has a dominating pair [21]. AT-free graphs are hereditary dominating pair graph. That the latter is actually a larger class of graphs is exemplified by $C_6$. As far as we know, the recognition of hereditary dominating pair graphs is still open. An interesting open question is whether solitaire Flood-Ir remains polynomial for this class of graphs.

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