An Approach for Finding Permutations
Quickly: Fusion and Dimension matching

Aravind Acharya
Department of Computer Science
and Automation
Indian Institute of Science
Bangalore 560012, India
aravind@iisc.ac.in

Uday Bondhugula
Department of Computer Science
and Automation
Indian Institute of Science
Bangalore 560012, India
udayb@iisc.ac.in

Albert Cohen
INRIA and DI
Ecole Normale Superieure
45 Rue d’Ulm
Paris 75230, France
Albert.Cohen@inria.fr

Abstract
Polyhedral compilers can perform complex loop optimizations that improve parallelism and cache behaviour of loops in the input program. These transformations result in significant performance gains on modern processors which have large compute power and deep memory hierarchies. The paper, Polyhedral Auto-transformation with No Integer Linear Programming, identifies issues that adversely affect scalability of polyhedral transformation frameworks; in particular the Pluto algorithm. The construction and solving of a complex Integer Linear Programming (ILP) problem increases the time taken by a polyhedral compiler significantly. The paper presents two orthogonal ideas, which together overcome the scalability issues in the affine scheduling problem. It first relaxes the ILP to a Linear Programming (LP) problem, thereby solving a cheaper algorithm. To overcome the sub-optimalities that arise due to this relaxation, the affine scheduling problem is decomposed into following three components: (1) Fusion and dimension matching, (2) Loop scaling and shifting, and (3) Loop skewing. This new auto-transformation framework, pluto-lp-dfp, significantly improves the time taken by the Pluto algorithm without sacrificing performance of the generated code. This report first provides proofs for the theoretical claims made in the paper surrounding relaxed LP formulation of the Pluto algorithm. The second part of the report describes an approach to find good loop fusion (or distribution) and loop permutations that enable tileability. This short report serves as the supplementary material for the paper.

1. Background
In this section, we introduce terminology used in the report. We also provide background on the current ILP formulation used in Pluto to find good transformations. This is included for the purpose of completeness.

1.1 Affine Transformations
A polyhedral compiler framework has a statement-centric view of the program. Each statement in an iteration space is modeled with integer sets called index sets or the domain of the statement. Let $S$ be the set of all statements. Let $I_S$ denote the index set of a particular statement $S$. For example, if $S$ has a two-dimensional index set, then:

$$I_S = \{ [i, j] \mid 0 \leq i, j \leq N - 1 \}$$

(1)

defines an index set where $i$ and $j$ are the original loop iterator variables of the statement $S$ and $N$ is a program parameter. These index sets represent the set of statement instances that are executed by the program. An instance of $S$ is given by the iteration vector of $S$ referred to as $\vec{s}$. Let $m_S$ denote the dimensionality of $I_S$, i.e., $I_S$ has $m_S$ components corresponding to loops surrounding the statement $S$ from outermost to innermost.

Data dependences are precisely represented in a polyhedral auto-transformation framework using dependence polyhedra, which are a conjunction of constraints. These constraints can also be viewed as a relation between source and target iterations. These relations include affine combinations of loop iterator variables of the source and target iterations, program parameters and existentially quantified variables. If $D_e$ is the dependence polyhedron associated with an edge $e$ of the data dependence graph, then an iteration $\vec{t}$ of a statement $S_1$ is dependent on an iteration $\vec{s}$ of a statement $S_i$ if and only if $(\vec{s}, \vec{t}) \in D_e$.

Formally, an affine transformation in the polyhedral model is a multi-dimensional affine function of the loop iterators and program parameters. A one-dimensional affine transformation $\phi_S$ for the statement $S$ (corresponding to a particular level or loop depth roughly speaking) can be expressed as:

$$\phi_S (i\vec{s}) = (c_1, c_2, \ldots, c_{m_S}) \cdot (i\vec{s}) + (d_1, \ldots, d_p). (\vec{p}) + c_0, c_1, \ldots, c_{m_S}, d_1, \ldots, d_p \in Z.$$  

Each statement has its own set of $c_i$‘s and $d_i$‘s, and these are called transformation coefficients corresponding to loop iterator variables and program parameters (denoted by $\vec{p}$) respectively. The transformation $\phi_S$ at a level $i$ for a statement $S_i$ can also be viewed as a hyperplane, denoted by $h_S^i$. For simplicity, we drop the statement identifier $S_i$ in places where the meaning is clear from context. The set of consecutive hyperplanes that can be permuted form a permutable band. These hyperplanes (from outermost to innermost) form the rows of the transformation matrix. The term schedule and transformation are also used interchangeably, since a transformation specifies a new schedule.

1.2 ILP Formulation in Pluto
Pluto is a polyhedral optimizer that finds affine transformations to maximize locality and parallelism. Given the index sets of the statements in the program and the dependences in the form of dependence polyhedra, the Pluto algorithm iteratively finds linearly independent hyperplanes based on an objective that minimizes dependence distances. This objective is modeled using an ILP, which we describe in the rest of this section.

The Pluto algorithm iteratively finds hyperplanes from outermost to innermost looking for tileable bands, i.e., the hyperplanes that satisfy the tiling validity constraint below for every dependence $(\vec{s}, \vec{t}) \in D_e$:

$$\phi_{S_1} (\vec{t}) - \phi_{S_i} (\vec{s}) \geq 0.$$  

(2)
where $S_i$ and $S_j$ represent the source and target statements of $D_e$ respectively.

The objective used by the Pluto algorithm is then to minimize the dependence distances using a bounding function:

$$\phi_{S_i}(\vec{t}) - \phi_{S_j}(\vec{s}) \leq \vec{u} \vec{p} + w. \quad (3)$$

The intuition behind this upper bound on dependence distances is as follows: the dependence distances are bounded by loop iterator variables, which are further bounded by program parameters. Therefore, one can choose large enough values for $\vec{u}$ to obtain an upper bound. In order to minimize dependence distances, the Pluto algorithm minimizes coefficients of this upper bound by finding the lexicographic minimum (lexmin) of $(\vec{a}, w)$ as the objective:

$$\text{lexmin} (\vec{u}, w, \ldots, c_i^S, d_i^S, \ldots), \quad (4)$$

where $c_i^S$ and $d_i^S$ are the transformation coefficients of $S$.

Note that well-known ILP solvers like GLPK, Gurobi and CPLEX do not provide a lexmin function. However, lexmin can be implemented in practice as a weighted sum objective, in which the coefficients of $\vec{u}$ (in the objective function) will be significantly higher than the coefficient of $w$ and so on.

### 1.2.1 Avoiding the zero solution

The tiling validity constraints and the dependence bounding constraints from (2) and (3) have a trivial zero vector solution. The Pluto algorithm restricts all transformation coefficients ($\phi_{S_i}$s) to non-negative integers. This restriction allows us to avoid the trivial solution for the coefficients of $\phi_{S_i}$ with the constraint:

$$\sum_{i=0}^{m_g} c_i \geq 1. \quad (5)$$

### 1.2.2 Linear Independence

Affine transformations have to be one-to-one mappings in order for them to specify a complete schedule. The Pluto algorithm thus enforces linear independence of hyperplanes, statement-wise. This is modeled by finding a basis for the null space of hyperplanes already found. The next hyperplane to be found must have a component in this null space. The exact modelling of this constraint is described in (1). It will be a constraint of the form:

$$\sum_{i=0}^{m_a} a_i \times c_i \geq 1, \quad (6)$$

where $a_i \in \mathbb{Z}$. These $a_i$’s are from the subspace that is orthogonal to the subspace of currently found hyperplanes. For the rest of this paper, we refer to the above formulation as \textit{pluto-ilp}.

We denote the set of statements in the program with $S$, and $|S|$ is its cardinality. Similarly, $C$ denotes the set of connected components in the data dependence graph (DDG). We use $\psi$ to denote affine constraints on transformation coefficients, loop bounds, dependence distances, and program parameters.

### 2. Proofs

The constraints shown in Equation (7) model the full space of non-negative rational solutions. Even though these constraints cannot be implemented in the solver, we use these constraints to prove certain interesting results that exist when linear independence and constraints are modelled precisely.

$$\sum_{i=0}^{m_g} c_i > 0, \quad \sum_{i=0}^{m_a} a_i \times c_i > 0. \quad (7)$$

In \textbf{Lemma 2.1} and \textbf{Theorem 2.2}, we prove properties of solutions of the relaxed formulation that hold irrespective of the way linear independence and trivial solution avoiding constraints are modeled (either using Equation 5, 6 or 7).

\textbf{Lemma 2.1.} The optimal solution to \textit{pluto-lp}, when it exists, is rational.

The above lemma follows from the fact that all coefficients in the LP formulation of Pluto are integers; thus the solutions of \textit{pluto-lp} are rational. For the rest of this paper, we refer to the optimal rational solution of \textit{pluto-lp} as the solution of \textit{pluto-lp}.

\textbf{Theorem 2.2.} If $\vec{z}$ is a solution to the relaxed Pluto formulation (\textit{pluto-lp}), then for any constant $k \geq 1$, $k \times \vec{z}$ is also a valid solution to \textit{pluto-lp}.

\textit{Proof.} From \textbf{Lemma 2.1} we know that, if a solution exists, then the optimal value of the objective corresponds to a rational solution of \textit{pluto-lp}. Now, we need to prove that, scaling the solutions of \textit{pluto-lp} will not violate the constraints. Consider the tiling validity constraints in (2). $\phi_{S_i}$ and $\phi_{S_j}$ are one dimensional affine transformations. Therefore,

$$\phi_{S_i}(\vec{t}) - \phi_{S_j}(\vec{s}) \geq 0 \iff k \times \phi_{S_i}(\vec{t}) - k \times \phi_{S_j}(\vec{s}) \geq 0$$

where $k \geq 1$. Therefore any hyperplane found by the \textit{pluto-lp} will not violate the tiling validity constraints after scaling the solutions. The dependence bounding constraints in (3) are bounded above by $\vec{u}$ and $w$, which are variables in the \textit{pluto-lp} formulation. The values of $\vec{u}$ and $w$ are can also be scaled up without violating the constraints. That is,

$$k \times \phi_{S_i}(\vec{t}) - k \times \phi_{S_j}(\vec{s}) \geq k \times \vec{u} + \vec{w}.$$

The trivial solution avoiding constraints and the linear independence avoiding constraints given in (7) cannot be violated by scaling the solutions. That is, if $c_i$’s are the solutions to \textit{pluto-lp} and $k \geq 1$, then from (7) it follows that for each statement $S$,

$$\sum_{i=0}^{m_g} k \times c_i > 0, \quad \sum_{i=0}^{m_a} a_i \times k \times c_i > 0.$$

Therefore scaling the solutions of \textit{pluto-lp} with a factor $k \geq 1$, will not violate the constraints.

\textbf{Theorems 2.3} and \textbf{2.4} refer to the constraints that hold only in cases where the linear independence and trivial solution avoiding constraints model the full space of rational solutions as given in Equation (7).

\textbf{Theorem 2.3.} The optimal solution to the relaxed Pluto algorithm (\textit{pluto-lp}) can be scaled to an integral solution to \textit{pluto-lp} such that the objective of the scaled (integral) solution will be equal to the objective of optimal solution of \textit{pluto-lp}.

\textit{Proof.} By \textbf{Theorem 2.2} we know that after scaling the real solutions, the resulting solution does not violate any constraints. Let $z_i$ and $z_r$ be the value of the optimal objective values for solutions to \textit{pluto-ilp} and \textit{pluto-lp} respectively. Let $c_r$ be the smallest scaling factor that scales solutions of \textit{pluto-lp} to integers. Let $z'_r = c_r \times z_r$ be the value obtained from by scaling the optimal real solution of \textit{pluto-lp} to an integral one. Note that $c_r \geq 1$; otherwise, the real solution would not be optimal. Now we prove that $z'_r \leq z_r$. Consider $z'_r$ given by

$$z'_r = z_r / c_r \implies z_r \leq z'_r \iff z_r = z'_r \iff c_r \times z_r \leq c_r \times z'_r \iff c_r \geq 1 \text{ and } z_r, z'_r \geq 0 \implies z'_r \leq z_r.$$
This proves that the optimal (minimum) objective of pluto-lp after scaling to integer coefficients will be less than or equal to that of pluto-ilp. However, the objective of the relaxed formulation after scaling cannot be strictly less than that of pluto-ilp (otherwise, pluto-ilp’s solution, z, would not be an optimal one). Therefore the optimal objective of pluto-lp after scaling up, is equal to the optimal objective of pluto-ilp.

**Theorem 2.4.** Let \( \vec{h} = (c_1, \ldots, c_n) \) be the optimal solution for pluto-ilp. Then, the optimal solution to pluto-lp, \( \vec{h}_r \), is such that \( \vec{h}_r = \vec{h}/c_s \) where \( c_s \geq 1 \).

**Proof.** Let \( z_r \) be the optimal values of the objective found by pluto-ilp and pluto-lp respectively. Let \( c_s \) be the smallest scaling factor that scales every component of \( \vec{h} \) to an integer. Note that \( c_s \geq 1 \) (otherwise, \( \vec{h} \) would not have been optimal solution). Let \( z_i \) be the solution obtained by the hyperplane \( \vec{h} \). Let \( z'_i = z_i/c_s \). Note that \( z'_i \) can be obtained by dividing all the components of \( \vec{h} \) by \( c_s \). Now we have the following cases:

- **Case 1:** If \( z_r = z'_r \) then we have nothing to prove.
- **Case 2:** Consider the case \( z_r < z'_r \). Let \( \vec{h}' = \vec{h}_r \times c_s \), and let \( z'_r \) be the objective value with \( \vec{h}' \). \( z'_r = z_r \times c_s \) (due to the nature of \( \vec{h}' \)). Since the optimal objective value for pluto-ilp was found to be \( z_i \), \( z'_r \geq z_i \). Now if we scale down each component of \( \vec{h}_r \) by \( c_s \), we get a solution that has an objective lower value than \( z_r \). This is a contradiction.

Therefore, \( z_r = z'_r \) in all cases, and \( z'_r = z_i \). Since both \( \vec{h}_r \) and \( \vec{h}' \) have the same optimal objective value and given that the lexmin provides a unique optimal solution, \( \vec{h}' = \vec{h}_r \), and \( \vec{h}_r = \vec{h}_r/c_s \).

The properties of the solutions of pluto-ilp that hold even when linear independence constraints and trivial solution avoiding constraints model the space of rational solutions imprecisely are stated in Theorems 2.3 and 2.6.

**Theorem 2.5.** The relaxed formulation, pluto-lp (in each permutable band), finds a outer parallel hyperplane if and only if pluto-ilp finds a outer parallel hyperplane.

**Proof.** There exists a parallel hyperplane if and only if \( \vec{u} = 0 \) and \( w = 0 \) in the ILP formulation of Pluto. Note that \( \vec{u} + w \) gives an upper bound on the dependence distance and therefore \( u + w \) is the smallest value of \( \vec{u} + w \). The objective of the relaxed LP is to minimize the values of \( \vec{u} \) and \( w \), and there exists an integer solution which is also present in the real space. Therefore the solution found by pluto-lp will fall into one of the two cases:

1. The solution found by pluto-lp is same as the solution found by pluto-ilp. In this case, there is nothing to prove.
2. pluto-lp finds a fractional solution with \( \vec{u} = 0 \) and \( w = 0 \). In this case, by Theorem 2.3 one can scale the real (fractional) solution to an integral one without violating any constraints. This scaling up will neither change the value of \( \vec{u} \) nor \( w \) because they were found to be equal to zero.

The “only if” part of the proof follows from Case 2 in the above argument.

**Theorem 2.6.** Given a loop nest of dimensionality \( m \), if pluto-ilp finds \( d \leq m \) permutable hyperplanes, then pluto-lp also finds \( d \) permutable hyperplanes.

**Proof.** Let us assume that pluto-lp finds \( k \) hyperplanes and let \( k \neq d \). We prove Theorem 2.6 by contradiction. Let us assume that \( k > d \). The \( k \) linearly independent hyperplanes found by pluto-lp can be scaled to integers. These scaled solutions will continue to be linearly independent as scaling transformations will not affect linear independence. Therefore these correspond to \( k \) linearly independent in the integer space. This means that there existed \( k \) linearly independent solutions in the integer space. Since the validity constraints remain the same at each level, there exits only \( d \) linearly independent solutions as found by pluto-ilp. This is a contradiction to the assumption that \( k > d \).

Suppose \( k < d \), then we know that there are \( d \) linearly independent solutions to the tiling validity constraints in the integer space. These are valid linearly independent solutions in the rational space. Therefore, it is a contradiction to our assumption \( k < d \). Therefore, pluto-lp will find \( d \) linearly independent solutions to the tiling validity constraints.

### 2.1 Proofs corresponding to routines in pluto-lp-dfp

In this section we state and prove theorems that establish the correctness of routines in pluto-lp-dfp framework. Algorithm 1 refers to the scaling MIP, SCALE, presented in Section 4; Algorithm 2 refers to the scaling and shifting routine, SCALEANDSHIFTPERMUTATIONS presented in Section 5 and Algorithm 3 refers to the skewing routine, INTRODUCESKREW, presented in Section 6 of the PLDI paper describing the framework of pluto-lp-dfp.

**Theorem 2.7.** If loop skewing and shifting transformations are disabled, the relaxed Pluto algorithm will find the transformation coefficients that are scaled down versions of the transformation coefficients of the pluto-ilp. The values of \( \vec{u} \) and \( w \) in pluto-lp will be scaled down by the same scaling factor as the transformation coefficients.

**Proof.** When loop skewing and shifting transformations are disabled, then only one of the transformation coefficients is non-zero. Without loss of generality, let us assume that \( c_i \) is the non zero coefficient in each statement. The real space of non-zero solutions is modeled imprecisely. Therefore we can assume that the lower bound of \( c_i \) of every statement to be 1. Let \( \vec{u} + w = z_r \) for pluto-ilp and pluto-lp respectively. Let us normalize the values of \( c_i \) to 1. Let \( c_i' \) be the normalizing factor. The normalized coefficients, \( c_i/c_i' \) for each statement, \( s \), will be in the space of real solutions. This is because the lower bound of each \( c_i \) is 1. The value of other \( c_i \)’s will also correspond to the lower bounds that are obtained Gaussian elimination and Fourier-Motzkin elimination of the variables from validity and dependence bounding constraints. These correspond to the lowest possible values each of these \( c_i \)’s can take. Therefore the value of \( z_i \) is equal to that of \( z_r \). Hence the values of the variables in the ILP, including the objective, will be scaled down by the same scaling factor in the relaxed LP formulation.

**Theorem 2.8.** Given a valid transformation \( T \) for a program, the output transformation that is obtained by Algorithm 3 does not violate any dependences.

**Proof.** The proof can be split into two cases: (1) If the algorithm did not introduce a skew, then we return the input transformation itself. Since the input transformation did not violate any dependences, the returned transformation is valid. (2) If Algorithm 3 introduced a skew, then for each level \( i \), it only uses the transformation coefficients from the outer levels. Note that the algorithm proceeds level by level. Let \( i \) be the dimension at which we are introducing a skew. All other hyperplanes which are outer to \( i \) can be permuted to the outer level because, \( i \) is the first dimension that does...
a negative component for some dependence. The dimensions that are used to skew will have coefficients from the outer levels and none of these levels have a negative component. Since the newly introduced skew satisfies pluto-lp, it does not violate any dependences. All dependences that were previously satisfied by level \( i \) will still continue to be satisfied at level \( i \) after skewing. Hence all the dependences will be satisfied by the transformation obtained from Algorithm 3.

**Theorem 2.9.** Given a valid transformation \( T \) for a program, Algorithm 3 does not introduce any skewing transformations in cases where \( T \) was tileable.

**Proof.** The algorithm tries to introduce a skew only when there is a negative component for one of the dependences at a level \( d \). This means that the level \( d \) cannot be permuted to the outermost level. Hence the input transformation \( T \) would result in a loop nest which can not be tiled. Therefore, Algorithm 3 would not be introducing skewing, if the original loop nest was not tileable.

**Theorem 2.10.** Given a program \( P \), the transformation hyperplanes obtained by pluto-lp-dfp are linearly independent and do not violate any dependences.

**Proof.** The correctness claim of pluto-lp-dfp follows from correctness of Algorithm 2 and 3. Both algorithms find valid affine transforms and these transformations can be composed together without violating any dependences. In order the prove that the found affine transformation hyperplanes are linearly independent, we prove that each step in pluto-lp-dfp preserve linear independence of hyperplanes. In the first step, linear independence of hyperplanes is first guaranteed by the initial permutation. Then scaling and shifting transformations introduced by Algorithm 2 does not affect linear independence. The skew introduced by Algorithm 3 will not affect linear independence as well because a skew introduced at each level will have a new component which does not exist either in outer or inner levels. Therefore, the transformations hyperplanes obtained from pluto-lp-dfp will be linearly independent.

### 3. An approach for finding permutations quickly: Fusion and dimension matching

In this section we describe our approach to find a valid permutation. This is the first step in pluto-lp-dfp after polyhedral dependences are obtained. A permutation \( P \) is said to be valid if there are no loop scaling and loop shifting factors for (each dimension in \( P \)), such that the resulting transformation will not violate any dependences. The objective of finding a good permutation is to enable loop tiling. Since loop fusion/distribution decisions are also made at this stage, we would want to model all possible fusion opportunities that enable tiling and pick one of them. It is a part of our future work, to come up with a cost function that decides a good fusion / distribution strategy.

#### 3.1 Definitions

We first provide some definitions and properties of data structures that we use in order to model the space of all permutations for fusion and tileability. The central data structure that we use in modelling all possible fusion opportunities that enable tiling is the fusion conflict graph. A fusion conflict graph (FCG), \( G = (V, E) \), where the set of vertices is given by \( V = \{S_1^1, S_2^1, \ldots, S_{\dim(S_1)}^1, S_2^2, \ldots, S_{\dim(S_n)}^n\} \), has a vertex corresponding to each dimension of a statement in the program. The vertices of the dependence graph are the statements in the program. Therefore, for every vertex \( v \) in the FCG, there exists a statement (vertex) \( S \) in the dependence graph such that \( v \) corresponds to a dimension of \( S \). Hence, one can define a function \( f : FCG \rightarrow DDG \), from the vertices of the fusion conflict graph to the vertices of the dependence graph.

The edges in the FCG represents the dimensions of statements that can not be fused together and permuted to the outermost level. That is, if there exists an edge between \( S_1^i \) and \( S_2^j \), then the \( i^{th} \) dimension of \( S_1 \) and \( j^{th} \) dimension of \( S_2 \) can not be fused together and permuted to the outermost level. Note that, if the loop nest can be fully permuted, then it can be tiled as well. Hence an edge in the fusion conflict graph encodes violation of fusion and tileability. Once the graph is constructed, the objective is to group vertices that are not connected by edges, without violating any dependences. Independent sets group vertices in a graph that are not connected by edges. However, in order to not violate any dependences, these independent sets have to be convex. Given a fusion conflict graph, we say that an independent set \( I \) of the fusion conflict graph is convex, if the \( I \) is an independent set and for each \( v \in I \), the following condition holds:

\[
S = f(v) \land \forall S_1 \in Pred(S) \exists v_1 \in I. f(v_1) = S.
\] (8)

That is, if a vertex \( v \) of the FCG corresponding to a statement \( S \) is present in \( I \), then there must be a vertex \( v_1 \) corresponding to every predecessor \( S_1 \) of \( S \) in \( I \). This condition is required to encode transitive dependences across vertices (statements) in the DDG.

We obtain a convex independent set by a convex coloring of the FCG. Given a fusion conflict graph, we say that the coloring of the fusion conflict graph is convex, if the vertices that have the same color form a convex independent set. Note that there can exist many convex colorings for the given FCG. We pick one of them. Coming up with a cost model that picks a good coloring is a part of our future work. In the rest of this section, we provide an approach that performs convex coloring of the FCG. A convex coloring of the FCG is bounded by the maximum dimensionality of the loop nest. This enforces a mapping of colors to dimensions of the loop nest. The colors are ordered; the ordering of the colors give the permutation for a statement from the outermost level to the innermost.

#### 3.2 Approach

In this section, we provide an algorithm which to find a valid permutation. Note that in the space of all valid permutations, one might want to explore different cost models that capture fusion strategies that maximize performance. For example, a fusion that does not inhibit parallelism might be desired. In such a case, one would want a cost model where the loop nests are fused only if the resulting loop nest is parallel. Coming up with a cost model to enable optimal fusion, is a focus of our future work. In this paper, we only present an approach that models the of all possible loop permutations that enable fusion and tiling and picks one of them.

We propose a two stage approach as shown in Algorithm 4. The first step models the space of all permutations that enable fusion and permutation by constructing a fusion conflict graph. The routine BuildFCG in Line 1 of the algorithm constructs the fusion conflict graph. The description of this routine is given in Section 3.3. The number of colors used to color the FCG is bounded by the maximum dimensionality of the loop nest.

The routine ColorFCG in Line 3 performs a convex coloring of the FCG. A convex coloring of the FCG. If the loop nest can be completely fused, then the graph can be colored with \( m_S \) colors, where \( m_S \) is the maximum depth of a loop nest in the program. The vertices that obtain the same color represent the dimensions that can be fused together and permuted to the outermost level. If the graph is not colorable with \( m_S \) colors, then we find a subgraph of the FCG which can be colored with \( m_S \) colors. If this subgraph is a maximal
subgraph of FCG that can be colored with \( m_S \) colors, then by fusing the dimensions with the same color, we obtain a maximally fused loop nest. However, finding the maximally colored subgraph is costly. Therefore, as a trade-off, we do not aim at finding the maximal subgraph which is colorable with \( m_S \) colors. We employ a SCC based coloring algorithm which colors SCC by SCC for a given color. If coloring fails, we cut the DDG, update the FCG and then continue coloring. The details of this coloring step is given in Section 3.4. Assuming there is a mapping from a color to a dimension of the loop nest, the permutations for a statement \( S \) at a given level can be obtained based on the colors of the vertices in the FCG that correspond to \( S \).

**Algorithm 4 PERMUTEANDFUSE(P,G)**

Require: Program \( P \) and DDG \( G \) of \( P \)
Ensure: A valid permutation \( T \) for each statement in \( P \)

1: \( F \leftarrow \text{BUILDFCG}(G) \)
2: maxColours \( \leftarrow \) Maximum dimensionality of a loop nest in \( P \)
3: \( \text{COLORFCG}(F,G,\text{maxColours}) \)

Algorithm 4 is sound, that is, the permutation found for fusion and tileability is valid and does not violate any dependences. The algorithm is also complete in the sense that all possible permutations that enable fusion and tiling are present in the space that we model. Therefore any valid fusion can be chosen by using an appropriate cost model.

### 3.3 Construction of the fusion conflict graph

In this section, we describe the construction of the fusion conflict graph. Recall that, each dimension of a statement in the program has a corresponding vertex in the fusion conflict graph. An edge in the fusion conflict graph represents the dimensions that can not be fused together and permuted to the outermost level.

**Algorithm 5 BUILDFCG(DDG G)**

Require: Dependence Graph \( G(G_e, G_t) \)
Ensure: Fusion Conflict Graph \( F(F_e, F_t) \)

1: for all \( S \in G_e \) do
2: \( \psi \leftarrow \) All intra statement dep constraints for \( S \)
3: for all \( i \in 1 \ldots m_S \) do
4: if \( (c^S_e \geq 1) \land \psi \) is infeasible then
5: \( F_e \leftarrow F_e \cup \{c^S_e \rightarrow c^S_e\} \)
6: end if
7: end for
8: end for
9: for all pair of statements \((S_s, S_t)\) such that \( i > j \) do
10: \( \psi \leftarrow \) Dep constraints for all deps between \( S_s \) and \( S_t \)
11: for all \( i \in 1 \ldots m_{S_s} \) do
12: for all \( j \in 1 \ldots m_{S_t} \) do
13: if \( (c^S_s \geq 1) \land (c^S_t \geq 1) \land \psi \) is infeasible then
14: \( F_e \leftarrow F_e \cup \{c^S_s \leftarrow c^S_t\} \)
15: end if
16: end for
17: end for
18: end for
19: for all \( S \in G_e \) do
20: for all \( i \in 1 \ldots m_S \) do
21: \( F_e \leftarrow F_e \cup \{c^S_e \rightarrow c^S_t \mid i \neq j \land 1 \leq j \leq m_S \} \)
22: end for
23: end for
24: return \( G \)

Algorithm 5 builds a fusion conflict graph. It incrementally adds edges between vertices of the FCG by analyzing dependences between every pair of statements in the program. The edges are added in two stages: (a) the first stage adds intra statement permute preventing edges (b) the second stage that adds inter statement permute and fusion preventing edges.

**Adding intra statement edges**

Given a DDG, the algorithm for each statement \( S \), collects all intra statement dependences as shown in Equation 9

\[
D_S \equiv \bigwedge_{e \in \text{DDG}} (D_e | S_{\text{Src}}(e) = D_{\text{Dest}}(e) = S). \tag{9}
\]

If a dimension \( i \) is not permutable to the outermost level, then it violates at least one of the intra statement dependences when permuted. To find if \( i \) is permutable to the outermost level, we set the lower bound of the corresponding coefficient, \( c^S_i \), to 1. Other coefficients corresponding to the statement \( S \) except \( c^S_j \), which corresponds to the shift are set to zero. Note that all variable corresponding to the transformation coefficients are not constrained to be integers. We then solve pluto-lp for the dependence polyhedron \( D_S \). If these constraints are unsatisfiable, then the dimension \( i \) is not permutable. Therefore we add a self edge on the vertex \( c^S_i \) in the FCG (Lines 11). Adding a self edge prevents coloring the vertex \( i \) indicating that the dimension can not be permuted. The self edge will removed only when the permute preventing dependence(s) are satisfied at some outer level.

**Adding inter statement edges**

Algorithm 5 adds inter statement permute and fusion preventing edges in Lines 9, 11. For each pair of statements \( S_s \) and \( S_t \) that are connected in the DDG, it collects all dependence constraints (both intra and inter statement dependences) between them as in Equation 10

\[
D \equiv \bigwedge_{e \in \text{DDG}} (D_e | S_{\text{Src}}(e), D_{\text{Dest}}(e) \in \{S^s, S^t\}). \tag{10}
\]

The algorithm adds an edge between \( S^s_i \) and \( S^t_j \) if fusing and permuting the \( i \) and \( j \) dimensions of \( S^s \) and \( S^t \) does not violate any dependence between \( S^s \) and \( S^t \). Let \( c^s_i \) and \( c^t_j \) be the coefficients corresponding to \( S^s_i \) and \( S^t_j \) respectively. After adding the constraints, \( c^s_i \geq 1 \) and \( c^t_j \geq 1 \) and setting all the transformation coefficients corresponding to other dimensions to zero, if Pluto’s LP formulation with \( D \) as the dependence polyhedron is unsatisfiable, then fusing \( i^{th} \) dimension of \( S^s \) with \( j^{th} \) dimension of \( S^t \) will violate a dependence. Hence an edge is added between \( S^s_i \) and \( S^t_j \).

Note that during the addition of edges, Algorithm 5 uses the relaxed LP formulation, for a pair of statements. This is an application of pluto-lp on a pair of statements. Finally, the algorithm adds edges between the vertices of the FCG that correspond to the same statement (line 21). This ensures that we do not assign the same color to two dimensions of a statement.

#### 3.3.1 Conflicting Shifts

A pairwise loop shifting transformation for a particular dimension \( i \) is said to be conflicting for a set of statements \( S \) if there are valid shifts for the \( i^{th} \) dimension for each pair of statements in \( S \) that allow pairwise fusion and permutability but there does not exist a valid shift for fusion and permutation for all the statements in \( S \).

**Theorem 3.1.** Given a program \( P \), the DDG \( G \) of \( P \), there does not exist a set of statements \( S \), such that there is a conflicting loop shifting transformation for statements in \( S \).

Before presenting the proof of the Theorem 3.1, we make the following observation about a loop shifting transformation. Consider a dependence \( S_s \rightarrow S_t \) in \( P \). Let a loop shifting transfor-
motion \( [i] \rightarrow [i + 1] \) be applied for \( S_i \). This transformation will execute the \( i^{th} \) iteration in the original space at the \( i + 1^{th} \) iteration. That is, the execution is delayed by a factor of 1 in the transformed space. This leads to an observation that, any positive loop shifting transformation at the target of a dependence will not violate the dependence.

**Proof.** We prove Theorem 3.1 by contradiction. We prove split the proof into three cases.

1. All the statements considered are from different SCCs. We assume that the statements are pairwise fusible and permutable for the dimension \( i \). Since the statements are in different SCCs, and pairwise fusion of these statements is valid, possibly with a loop shift, all the statements can be incrementally fused together by increasing the shifting factors of the targets of dependences. Therefore, there will exist a shifting factor, which will allow us to fuse the \( i^{th} \) dimension of all the statements considered. This large positive number will be the shift required to fuse all the statements and can be obtained by a call to pluto-lp. This is guaranteed to have a solution in this case, and the resulting rational solution can be scaled to an integer. Thus there can not exist a conflicting which will prevent the statements in different SCCs to be fused.

2. Suppose the statements are in the same SCC, and the \( i^{th} \) dimension does not carry the backward dependences. In this case, any loop shifting transformation to the \( i^{th} \) will not violate a backward dependence. Therefore, for the \( i^{th} \) dimension, all statements can be seen as target of the forward dependences. This is similar to case 1.

3. Let the dimension \( i \) is the source of some dependences and the target of few others. That is, the dimension \( i \) carries both loop independent and loop carried dependences. Since the statement is in an SCC, there exists a dimension which carries the dependence. Without loss of generality let us assume that dimension \( i \) carries these dependences. These dependences will be satisfied by the dimension \( i \) and any loop shifting transformation will not violate the dependences. Therefore, there exists a shifting factors in the original schedule which will now violate any of these dependences and fuse all the statements within this SCC. Therefore the case of conflicting shifts do not arise.

A similar argument can be provided for the absence of conflicting loop scaling factors.

### 3.4 Colouring the FCG

The routine provided in Algorithm 5 builds the fusion conflict graph. In this section, we provide a routine that implements a convex coloring of the FCG. Coloring is driven by the topological ordering on the SCCs of the DDG.

Algorithm 6 colors the vertices of the FCG, one color at a time. Coloring starts from the first SCC in the topological order of the SCCs. Given an SCC, there exists at least one dimension which fuses all the vertices of the SCC without violating any dependences. This observation has formally been stated in Theorem 3.2.

**Theorem 3.2.** Given a DDG \( G \) of a program \( P \), and a FCG \( F \) of \( P \), then for each SCC \( G \) in \( G \), there exists at least one dimension corresponding to vertices of the SCC, such that the vertices corresponding to these dimensions in \( F \) can be given the same color.

**Proof.** We prove Theorem 3.2 by contradiction. Let us assume that there existed no dimensions for every vertex in the SCC which can be given the same color. This means to say that there exists no dimensions that can be fused together and permuted to the outermost level. However, in the input program, since the vertices are part of an SCC, there must exist a loop, which carries a dependence along the back edge (Otherwise, these statements would not have been in an SCC). Therefore, when the pair of statements are analyzed using pluto-lp, at least one dimension corresponding to the loop that fuses all the statements will have a solution to pluto-lp. Therefore there will not be any edges added corresponding to the dimensions that carry the dependences for all the statements in the FCG. Therefore, all the vertices corresponding to the at least one dimension can be given the same color.

In the outermost level, the coloring of the first SCC will succeed. The coloring of a subsequent SCCs might fail, if fusing it with one of the SCCs that has already been colored violates a dependence. Note that in such a case there will be an edge between the vertices of the FCG corresponding to the statements of two SCCs that can not be fused. As soon as coloring fails, the algorithm cuts between the two SCCs and updates the DDG. FCG is also updated to remove edges corresponding to the dependences that have already been satisfied. Note that at the inner levels, the coloring of the first SCC might fail because of a permute preventing dependence. This dependence must be satisfied at some outer level. In such cases, we update the DDG by removing the edges corresponding to the dependences that have been satisfied at the outer levels. We also rebuild the FCG in order to remove edges that correspond to the dependences that have been satisfied at outer levels.

Algorithm 6 **COLORFCG**(FCG, DDG, maxColors)

1: for all \( c \in 1 \ldots \) maxColors do
2: for all \( i = 1 \) to \(|SCCs(DDG)|\) do
3: if \( \neg \text{COLORSCC}(i,c,FCG) \) then
4: \( \text{if } i == 1 \text{ then} \)
5: Update DDG by removed deps satisfied at outer levels
6: \( \text{else} \)
7: \( \text{CUTBETWEENSCCs}(i,i-1,DDG) \)
8: Update DDG by removing dependencies satisfied by the cut and other outer levels.
9: \( \text{end if} \)
10: \( \text{BUILDFCG}(DDG) \)
11: \( \text{COLOR}(i,c,DDG) \)
12: \( \text{end if} \)
13: \( \text{end for} \)
14: for all \( j = 1 \ldots |S| \) do
15: Update the \( T_S \) at level \( c \) for based on the vertices of the FCG that have been colored with \( c \)
16: \( \text{end for} \)
17: **end for**
3.5 Illustration:
Consider the example shown in Figure 1. Note that the statements cannot be fused directly because of the RAW dependence from statement S1 to S2. But if the dimensions of S1 are interchanged, which corresponds to loop interchange, then S1, S2 and S3 can be fused completely. The FCG corresponding to the program is shown in the right. There are six vertices each corresponding to a dimension of a statement of the loop nest. There are no intra statement dependences and hence no permute preventing edges are added to the FCG in Lines 1-5. Thick lines in the figure represent the edges added by the intra statement dependences. Edges added between the dimensions of a statement are shown as dashed lines.

```plaintext
for (i=0; i<N; i++)
    for (j=0; j<N; j++)
        A[i][j] = i+j; //S1
```
```plaintext
for (i=0; i<N; i++)
    for (j=0; j<N; j++)
        B[i][j] = A[j][i]; //S2
```
```plaintext
for (i=0; i<N; i++)
    for (j=0; j<N; j++)
        C[i][j] = A[i][j]; //S3
```

Figure 1. Illustration of loop interchange to enable fusion

The coloring done by the Algorithm is shown in the Figure. The vertices in red form the outer most level and the ones in green form the inner level. This corresponds to identity transformation for statements S1 and S3 and a loop interchange for the statement S2 thus resulting in a fully fused loop nest.

3.6 Correctness
The correctness of the approach of finding a valid transformation depends on the transitivity of fusion of dimensions while satisfying the tileability criterion. The fusion conflict graph is constructed by analyzing pairwise and hence in the following theorem, we formally prove that dimension wise fusion and permutability is transitive.

**Theorem 3.3.** If a dimension i of statement S1 can be fused together with a dimension j of a statement S2 and dimension j of S2 can be fused and permuted with a dimension k of statement S3 then dimensions i, j and k of statements S1, S2 and S3 can be fused together and permuted provided there are no loop skewing transformations.

**Proof.** We will prove Theorem 3.3 by contradiction. Let us assume that the dimensions i and j of S1 and S2 can be fused together and dimensions j and k of S2 and S3 can be fused together, but all the three statements cannot be fused. Then there exists at least one dependence which is violated. Let this dependence be d. The dependence d cannot be a direct dependence between S1 and S3 because, if this was a direct dependence, there would have been a fusion conflict edge between dimensions i and k of statements S1 and S3 in the fusion conflict graph. This fusion does not violate the transitive dependence between S1, S2 and S3 because the satisfaction of relaxed-LP formulation of Pluto implies that each of the dimensions can be fused and permuted to the outer most level. This implies that once the loops i and j form a permutable band and the loops j and k form a permutable band. Therefore, the loops i, j and k of statements S1, S2 and S3 can be fused together to form a permutable band. This will not violate any dependences between statements S1, S2 and S3. This violates our assumption that there is a dependence that is violated. Hence S1, S2 and S3 can be fused together and permuted to the outermost level.

References
[1] Uday Bondhugula, Aravind Acharya, and Albert Cohen. The pluto+ algorithm: A practical approach for parallelization and locality optimization of affine loop nests. ACM Trans. Program. Lang. Syst., 38(3):12:1–12:32, April 2016.
[2] GNU. GLPK (GNU Linear Programming Kit), 2000. https://www.gnu.org/software/glpk/.
[3] Inc. Gurobi Optimization. Gurobi optimizer reference manual, 2016.