Optimal Control Synthesis of Semi Active Vehicle Suspension

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Abstract. The paper discusses the problems related with the synthesis of a linear quadratic (LQ) regulator of a semi active vehicle suspension. The synthesis combines the control through output variables with such by the input excitation i.e. classical LQ regulator with Compensator. The physical nonlinearities of the controllable magneto-rheological semi-active damper are taken into an account with the including in the feedback control circuit an inverse damper model. A reduced suspension model is used to illustrate the synthesis (quarter car model).

1. Introduction

The quality criteria of the vibroisolation usually lay contradictory requirements for the vehicle's suspension synthesis. The ensuring enough ride comfort as generally requires a softer suspension, while the better vehicle stability and respectively better road-tire friction require a harder suspension. The compromise solutions in the conventional passive implementations are with limited possibilities. The usage of active or semi-active elements in vehicle suspensions gives considerable bigger potential in the solution of this problem. The semi-active vibroisolation is widely used because of its reliability, low price, and long exploitation period. It is known that due to the phase shifting, during some intervals of the oscillation time period, the damper directs energy into the object of vibroisolation instead of dissipating it. The main idea of the semi-active control is the reduction of the damping coefficient in these time period, which will decrease the amount of the energy transferred to the object [1-4].

There are many types of the control laws [5-8], but the main problem consists in the fact that, when the control is targeted to the comfort criterion the stability makes worse and vice versa. For this reason the synthesis of the optimal control requires a compromise solution satisfying both the quality factors simultaneously [9, 10, 11]. Generally the control based on a linear quadratic regulator - LQR is consider as an optimal and minimizes an integral of a time function formed by scaled variables describe the quality criteria [12-16].

2. Dynamic model of the object

2.1. Quarter car model

On figure 1 is shown the quarter car suspension model (related to one of the wheels) with semi-active suspension, achieved by damper with controlled rheology. This model is enough adequate in case of constructive selection of the suspension performance, securing approximate independence of the motion of the front and the rear axis of the vehicle. At the same time due to its simplicity it is very appropriate for a real-time control. The results can be easily adapted for more complex models.
Parameters of the model are:
- \( z_{us} \) and \( z_s \) - vibro-displacements in the vertical plane of the sprung and unsprung masses.
- \( m_{us}, m_s \) - sprung and unsprung masses;
- \( k_{us}, k_s, c_{us}, c_s (i) \) - tire and suspension spring and damping coefficients;
- \( s(t) = \begin{cases} 0.5a_0t^2, & 3a t < V_{max}^2 / a_0 \\ \frac{V_{max}^2}{2a_0}, & 3a t \geq V_{max}^2 / a_0 \end{cases} \) - motion law, \( a_0 \) - acceleration, \( V_{max} \) - maximal velocity;
- \( \tilde{\xi}(t) = \tilde{\xi}_h(s) + \tilde{\xi}_r(s) \) - kinematical excitation from the road roughness, presented as a sum of determined and stochastic component:
  - \( \tilde{\xi}_h = H_h \sin \left(2\pi s / L\right) \) models the road roughness as a harmonic function with amplitude \( H_h \) (m) and wave length \( L \) (m),
  - \( \tilde{\xi}_r(s) \) kinematical excitation modeling the road roughness according ISO 8608 [19];
- \( F_{us}, F_s \) - forces transmitted through the tire to the unsprung mass and respectively from it to the sprung mass: \( F_{us} = k_{us} (\tilde{\xi} - z_{us}) \), \( F_{ds} = c_{us} (\tilde{\xi} - z_{us}) \); \( F_{s} = k_s (z_{us} - z_s) \), \( F_{s} = f(i, z_{us}, z_s, z_{us}, z_s) \)
where the indexes ‘s’ and ‘d’ show respectively the spring and damping components and ‘i’ is the control signal.

2.2. Model of the semi-active damper

The essence of the semi-active suspension consists in damper with variable rheology. The damping force depends on the kinematical performance on its deformation, and from the control signal.

The magneto-rheological (MR) dampers – figure 2 are more appropriate solution for the considered application. At them by feeding a control current signal is creates magnetic field which forms domains
from the ferrite micro particles, contained in silicon based suspension. In this way the damping coefficient becomes a function of the control signal and varying in definite boundaries.

The experimental performance, that show the dependency of the force vs. deformation velocity for different values of the control current, for MR damper type RD-1005-3 manufactured by „LORD Corporation“-USA are shown on the figure 3.

[20] proposes a dynamic model of controllable MR damper based on the approximation of the hysteresis suggested by Bouc-Wen [21, 22]. Its mechanical analogue is shown on figure 4.

The damper force is calculated by the following equation:

$$F^d = k_0 (z_0 - z_s) + c_0 \left( \dot{z}_0 - \dot{z}_s \right) - k_h \zeta + k_0' (z_{us} - z_s + \delta_0) - c_0' ( \dot{z}_0 - \dot{z}_{us} ),$$  

(1)

where $k_h$ (N/m) is the elastic coefficient connected with the value on which the angle of the characteristic is changed and $\delta_0$ (m) takes into account the hydraulic accumulator.

From the force balance upon the level $z_0$ is obtained:

$$\dot{z}_0 = \left[ c_0' z_{us} + c_0 z_s + k_0 (z_0 - z_s) + k_h \zeta \right] / (c_0 + c_0').$$  

(2)

The evolutionary variable is given by the dependency:

$$\dot{\zeta} = -\gamma |z_0 - z_s| \zeta |\dot{\zeta}|^{\gamma - 1} + \beta (\dot{z}_0 - \dot{z}_s) \dot{\zeta} + \nu (\dot{z}_0 - \dot{z}_s) = \left[ \beta \text{-sign}(\dot{z}_0 - \dot{z}_s) \right] |\dot{\zeta}|^{\gamma - 1} \dot{\zeta} + \nu (\dot{z}_0 - \dot{z}_s),$$  

(3)

where $\gamma$ (m), $\beta$ (m), $\nu$ and $\chi$ are coefficients forming the hysteresis.

The values of $k_h$, $c_0$ and $c_0'$ as functions of the control current are given by the first order approximations:

$$k_h = k_0' + k_h' i , \ c_0 = c_0' + c_0' i , \ c_0' = c_0'' + c_0'' i.$$

(4)

Here $i'$ is function taking into account the process of rheological equilibrium after the aperiodic dependency from first order:

$$i' = \left( i - \dot{i} \right) / \tau_s , \ \tau_s - \text{time-sample of the process}, \ i \in \left[ 0, i_{max} \right].$$

(5)

2.3. Inverse model of MR damper

The MR damper has essential nonlinear characteristics and physical limitations. For the control purpose an inverse dynamical model is required. For a particular state of the system the model assures corresponding optimal damper force, calculated by the controller. It is obvious that due to the physical
limitations, the optimum force will not be generated in some situations and generally the control will be quasi-optimal.

Due to the complexity of the problem, a neural network model is used [12, 23, 24]. Its structure is shown on figure 5.

In the input layer the value of the deformation of the damper and its first and second derivatives with respect to time are entered, as well as the control signal $U$ (the damper force). The output layer represents a linear activation function and determines the control current, which generates the current damper force $U_i$.

The hidden layer consists of 12 neurons with non-linear logical sigmoid activation functions $g_i(x) = \frac{\theta}{1 - e^{\sigma x}} - 1$, where $\theta = 2$, and $\sigma$ is positive constant. The determination of the weight coefficients for the hidden layer $w_{ij}, i=1,..12, j=1,..10$ and the output layer $W_{ij}, j=1,..13$, is done by the Levenberg-Margquardt method and the Jacobian is calculated by back propagation method [25].

2.4. State space presentation

\[
\begin{align*}
\dot{X} &= AX + B_A A + B_U U, \quad X(0) = X_0 \\
Y &= CX + D_A A + D_U U
\end{align*}
\] (6)

where:

- $X = [z_{us}, z_s, \dot{z}_{us}, \dot{z}_s]^T$ - state vector; $A = [\xi, \hat{\xi}]^T$ - input excitation; $U = F^d_z$ - the control;

- $Y = [z_{us}, z_s, -F_u, \dot{z}_s]^T$ - output vector which consists of characteristics that define the quality of the suspension – the motions $z_{us}$ and $z_s$, the dynamical component of the normal force $-\bar{F}_{us}$, which determines the friction tire-road and the acceleration of the sprung mass $\dot{z}_s$ which determines the ride comfort;

- matrices of the state $A$, the control $B_u$ and the input $B_\lambda$ are:

\[
A = \begin{bmatrix} 0_{2x2} & I_2 \\ -m^{-1}k & -m^{-1}c \end{bmatrix}, \quad B_u = \begin{bmatrix} 0_{2x1} \\ m^{-1}d_u \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 0_{2x2} \\ m^{-1}d_\lambda \end{bmatrix}
\]

- and the matrices of the output variables are:
\[
C = \begin{bmatrix}
I_2 & 0 \\
k_{as} & c_{as} \\
k_s/m_s & -k_s/m_s \\
0 & 0
\end{bmatrix},
\quad D_u = \begin{bmatrix}
0_{2 \times 2}
\end{bmatrix},
\quad D_A = \begin{bmatrix}
0_{2 \times 2}
\end{bmatrix};
\]

• Mass, spring and damping matrixes and matrix of the excitation and control:

\[
m = \begin{bmatrix}
m_{as} & 0 \\
0 & m_s
\end{bmatrix},
\quad k = \begin{bmatrix}
k_{as} + k_s & -k_s \\
-k_s & k_s
\end{bmatrix},
\quad c = \begin{bmatrix}
c_{as} & 0 \\
0 & 0
\end{bmatrix},
\quad d_A = \begin{bmatrix}
k_{as} & c_{as} \\
0 & 0
\end{bmatrix},
\quad d_u = \begin{bmatrix}
-1
\end{bmatrix};
\]

\(\mathbf{T}\) – transpose symbol, \(\theta\) and \(I\) are zero and eye matrix with the shown dimensions.

### 3. Linear quadratic regulator synthesis [12, 13, 15, 18]

The synthesis of the control aims minimizing of a square function of the characteristics \(Y\), proving the quality of the suspension and from the control \(U\) representing the MR damper force:

\[
J = X^T(T_r)F(X(T_r)) + \int_0^{T_r} \left[ (CX + D_u U + D_A A)^T Q (CX + D_u U + D_A A) + U^T R U \right] dt
\]

(7)

where:

- \(T_r\) is the time interval for the control process,
- \(Q, F \in \mathbb{R}^{4 \times 4}\) are positive semi-defined
- \(R \in \mathbb{R}\) is positive defined (in this particular case - constant) weight matrices.

There it is accepted that \(F=0_{4 \times 4}\) and \(Q\) diagonal matrix.

After opening the brackets is obtained:

\[
J = \int_0^{T_r} \left[ (X^T Q U X + 2X^T N_{XU} U + U^T R U) + (2X^T N_{XA} A + 2U^T N_{UA} A + \Lambda^T R_A A) \right] dt
\]

(8)

where \(N_{XA} = C^T Q D_A \in \mathbb{R}^{4 \times 2}, N_{UA} = D_u^T Q D_A \in \mathbb{R}^{1 \times 2}, R_A = D_A^T Q D_A \in \mathbb{R}^{2 \times 2}\).

The Hamiltonian and the conjugated system are:

\[
\tilde{H} = (X^T Q U X + 2X^T N_{XU} U + U^T R U) + (2X^T N_{XA} A + 2U^T N_{UA} A + \Lambda^T R_A A) + P^T (AX + B_u U + B_A A),
\]

\[
\dot{P} = -\frac{\partial \tilde{H}}{\partial X} = \left(2Q U X + 2N_{XU} U + \Lambda^T P + 2N_{XA} A \right).
\]

(10)

From the Maximum principles

\[
\frac{\partial \tilde{H}}{\partial U} \bigg|_{U=U_{\text{extr}}}=0 \Rightarrow U_{\text{extr}} = U_{\text{opt}} = -R_u^{-1} \left[ N_{XU}^T X + B_u^T P + N_{UA} A \right]
\]

(11)

For the conjugated system is searches solution in the form:

\[
P = 2(KX - \Psi) \Rightarrow \dot{P} = 2(KX + K\dot{X} - \dot{\Psi}).
\]

(12)
Combining (10)-(12) are obtained two Riccati equations (for \( K = \text{const} \) and \( \Psi = \text{const} \)):

\[
\begin{align*}
\dot{\bar{K}} + A^T \bar{K} + Q_U - (\bar{K}B_U + N_{XU})R_U^{-1}(N_{XU}^T + B_U^T \bar{K}) & = 0, \\
\dot{\Psi} = -\left[ A^T - (\bar{K}B_U + N_{XU})R_U^{-1}B_U^T \right]\left[ (\bar{K}B_U + N_{XU})R_U^{-1}N_{UA} - (\bar{K}B_A + N_{X_A}) \right] \Lambda & = 0
\end{align*}
\]

Matrix \( \bar{K} \) is defined by (13) based on the numerical computation of the vectors of Hamiltonian matrix [26].

Combining (11), (12) and second equation of (13) is obtained the optimal control:

\[
\begin{align*}
U_{opt} & = -G_X \tilde{X} - G_A \Lambda \\
G_X & = R_U^{-1}(N_{XU}^T + B_U^T \bar{K}) \\
G_A & = R_U^{-1}\left[ N_{UA} + B_U^T \left[ A^T - G_X \bar{X}^T B_U \right]^{-1} \left[ G_X^T N_{UA} - (\bar{K}B_A + N_{X_A}) \right] \right].
\end{align*}
\]

The kinematic excitation due to the road roughness may be identified using the accelerations of the sprung and unsprung masses obtained from accelerometers:

\[
\tilde{e} = \omega_{us}^2(s) \tilde{z}_{us} + \omega_{us}(s) \tilde{z}_{su},
\]

where transfer functions are: \( \omega_{us}(s) = \frac{m_s s^2 + c_{us} s + k_{us}}{s^2(c_{us} s + k_{us})} \), \( \omega_{us}(s) = \frac{m_s}{c_{us} s + k_{us}} \) and with \( \sim \) is denoted the Laplace transform image.

On figure 6 is given the structural scheme of the control.

**Figure 6.** Structural scheme of the control

### 4. Numerical example

#### 4.1. Description of the task

- It is consider a model with the following characteristics:
  
  \( m_1 = 40 \text{ kg}; \ m_2 = 350 \text{ kg}; \ k_1 = 1.6 \times 10^5 \text{ N/m}; \ k_2 = 0.19 \times 10^5 \text{ N/m}; \ c_1 = 110 \text{ Ns/m}. \)

- The parameters of the MR damper model are given in table 1
Table 1. Parameters of the MR damper model

| $c_0^c$, Ns/m | $c_0^v$, Ns/mA | $c_1^c$, Ns/m | $c_1^v$, Ns/mA | $k_0^c$, N/m | $k_0^v$, N/mA | $c_2^c$, Ns/m | $c_2^v$, Ns/mA | $k_2^c$, N/m | $k_2^v$, N/mA |
|---------------|---------------|---------------|---------------|-------------|-------------|---------------|---------------|-------------|-------------|
| 780           | 2230          | 1400          | 14500         | 71500       | 28000       | 500           | 540           |             |             |
| $\chi$        | $\tau_r$, s   | $\gamma$, m$^2$ | $\beta$, m$^2$ | $\delta$    | $\delta_0$, m | $i_{\text{max}}$, A |
| 2             | 0.0053        | 0.5 $10^6$    | 0.8 $10^6$    | 170         | 0           | 1.2           |

The parameters of the motion law are: $a_0=1.1$ m/s$^2$, $v_{\text{max}}=130$ km/h, $T_k=25$s.

The kinematic excitation due to the road roughness as function of the time is shown on figure 7.

![Figure 7. Kinematic excitation vs. time](image)

The weighted matrix in the object function are

\[ Q = \text{diag}([1.34e4, 0.258e4, 2.462e-6, 3.453]), R=1.38e-6. \]

4.2. Results

| Characteristics | Passive suspension $b_2=1505$Ns/m | With semi active LQR control |
|-----------------|------------------------------------|------------------------------|
| $\int_0^T Z(t) dt$, ms | 0.346 | 0.23 | 0.378 | 0.0658 |
| Max($||Z||$, $||Z_0||$, $||Z_2||$, $||Z_3||$, $||Z_4||$, $||Z_5||$, $||Z_6||$) | [0.027;1,03;79,3][0,036;0,27;4,67][0,031;1,32;104,5][0,005;0,12;7,84] | [0.011;0,29;16,4] [0,01;0,082;1,5] [0,012;0,34;24,1] [0,022;0,04;1,66] |
| RMS($F_{\text{ud}}$, N) | 797 | 768,9 |
| J | 0.01602 | 0,01322 |

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