EFFICIENT GEODESICS AND AN EFFECTIVE ALGORITHM FOR DISTANCE IN THE COMPLEX OF CURVES

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ABSTRACT. We give an algorithm for determining the distance between two vertices of the complex of curves. While there already exist such algorithms, for example by Leasure, Shackleton, and Webb, our approach is new, simple, and more effective for small distances. Our method gives a new preferred finite set of geodesics between any two vertices of the complex, called efficient geodesics, which are different from the tight geodesics introduced by Masur and Minsky.

1. INTRODUCTION

The complex of curves $\mathcal{C}(S)$ for a compact surface $S$ is the simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves in $S$ and whose edges connect vertices with disjoint representatives. We can endow the 0-skeleton of $\mathcal{C}(S)$ with a metric by defining the distance between two vertices to be the minimal number of edges in any edge path between the two vertices.

The geometry of $\mathcal{C}(S)$—especially the large-scale geometry—has been a topic of intense study over the past two decades, as there are deep applications to the theories of 3-manifolds, mapping class groups, and Teichmüller space; see, e.g., [13]. The seminal result, due to Masur and Minsky in 1996, states that $\mathcal{C}(S)$ is $\delta$-hyperbolic [12]. Recently, several simple proofs of this fact have been found, and it has been shown that $\delta$ can be chosen independently of $S$; see [2, 5, 9].

In 2002, Leasure showed there is an algorithm to compute the distance between two vertices of $\mathcal{C}(S)$ [10, Section 3.2], and since then other algorithms have been devised by Shackleton [15] and Webb [17]. About his algorithm, Leasure says:

We do not mention this in the belief that anyone will ever implement it. The novelty is that finding the exact distance between two curves in the curve complex should be so awkward.

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One goal of this paper is to give an algorithm for distance—the efficient geodesic algorithm—that actually can be implemented, at least for small distances. The third author and Glenn, Morrell, and Morse [8] have in fact applied an implementation of our algorithm, called Metric in the Curve Complex [7], to the following effects:

1. to show that the minimal geometric intersection number for vertices of \( \mathcal{C}(S_2) \) with distance four is 12,
2. to explicitly list all examples of vertices of \( \mathcal{C}(S_2) \) with distance four and intersection number at most 25, and
3. to produce an explicit example of two vertices of \( \mathcal{C}(S_3) \) that have distance four and intersection number 29.

In particular the second item gives all examples with the minimal intersection number as in the first item. The highly symmetric example of a minimally intersecting pair shown in Figure 1 was discovered using their computer program. The first example of a pair of vertices of \( \mathcal{C}(S_2) \) with distance four and intersection number 12 was found by the authors of this paper; see Section 2 for a discussion of this example and a proof using the methods of this paper that the distance is actually 4.

We are only aware of one other explicit picture in the literature of a pair of vertices of \( \mathcal{C}(S_2) \) that have distance four, namely, the example of Hempel that appears in the notes of Saul Schleimer [14, Figure 2] (see [8, Example 1.6] for a proof that the distance is 4). His example has geometric intersection number 25.

Using the bounded geodesic image theorem [11, Theorem 3.1] of Masur and Minsky (as quantified by Webb [19]) it is possible to explicitly construct examples of vertices with distance four; see [15, Section 6]. We do not know how to keep the intersection numbers close to the minimum with this method, but Aougab and Taylor did in fact use this method to give examples of vertices of arbitrary distance whose intersection numbers are close to the minimum in an asymptotic sense; see their paper [4] for the precise statement.

**Efficient geodesics.** One reason why computations with the complex of curves are so difficult is that it is locally infinite and in particular there are infinitely many geodesics (i.e. shortest paths) between most pairs of vertices. Masur and Minsky addressed this issue by finding a preferred set of geodesics, called tight geodesics, and proving that between any two vertices there are finitely many tight geodesics.

A geodesic \( v_0, \ldots, v_n \) in \( \mathcal{C}(S) \) is tight if for each \( 1 \leq i \leq n-1 \) there are minimally-intersecting representatives \( \alpha_{i-1}, \alpha_i, \) and \( \alpha_{i+1} \) of \( v_{i-1}, v_i, \) and \( v_{i+1} \) so that \( \alpha_i \) is contained in a regular neighborhood of \( \alpha_{i-1} \cup \alpha_{i+1} \) (equivalently, any vertex of \( \mathcal{C}(S) \) adjacent to \( v_{i-1} \) and \( v_{i+1} \) is also adjacent to \( v_i \)). Our first goal is to give a new class of geodesics that still has finitely many elements connecting any two vertices but that is more amenable to certain computations.

Suppose \( v \) and \( w \) are vertices of \( \mathcal{C}(S) \) with \( d(v, w) \geq 3 \). We say that an oriented geodesic \( v = v_0, \ldots, v_n = w \) in \( \mathcal{C}(S) \) is initially efficient if there are representatives \( \alpha_0, \alpha_1, \) and \( \alpha_n \) of \( v_0, v_1, \) and \( v_n \) with all three pairs of curves in minimal position and with

\[
|\alpha_1 \cap \gamma| \leq n - 2
\]
for all arcs $\gamma$ of $\alpha_n \setminus \alpha_0$. We say that $v = v_0, \ldots, v_n = w$ is efficient if $v_k, \ldots, v_n$ is initially efficient for each $0 \leq k \leq n - 3$ and $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ is also initially efficient.

It is easy to see that there are at most finitely many efficient geodesics from $v$ to $w$. Indeed, $\alpha_0$ and $\alpha_n$ decompose $S$ into a collection of polygons and by the definition of initial efficiency, there are finitely many possibilities for the intersection of $\alpha_1$ with each polygon; hence there are finitely many possibilities for $v_1$. By induction, there are finitely many possibilities for $v_1, \ldots, v_{n-2}$. By the efficiency of the oriented geodesic $v_n, v_{n-1}, v_{n-2}, v_{n-3}$, there are finitely many possibilities for $v_{n-1}$ (this explains why the last part of the definition of efficiency is needed).

**Statement of the main theorem.** Let $S_g$ denote a closed, connected, orientable surface of genus $g$. While we just argued there are most finitely many efficient geodesics from one vertex of $\mathcal{C}(S_g)$ to another there is no reason a priori to believe that efficient geodesics exist.

**Theorem 1.1.** Let $g \geq 2$, and let $v$ and $w$ be two vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. There exists an efficient geodesic from $v$ to $w$, and in fact there are finitely many.

We emphasize that our theorem is only for closed surfaces; it is not immediately clear how to amend our proof for surfaces with boundary.

**Discussion of the proof.** Our method for proving Theorem 1.1 is detailed in Section 3. Briefly, the idea is to show that if some geodesic $v = v_0, \ldots, v_n = w$ is not initially efficient then we can modify $v_1, \ldots, v_{n-1}$ by surgery in order to reduce the intersection of $v_1$ with $v_n$. The basic surgeries we use in our proof are not new. The crucial point—and our new idea—is that it is usually not possible to reduce intersection by modifying a single vertex; rather, it is often the case that we can reduce intersection by modifying a sequence of vertices all at the same time.

One way that our basic surgeries have been used is to produce quasi-geodesics in the complex of curves; see, for instance, the work of Leasure [10], Hensel–Webb–Przytycki [9], and Sisto [16]. Leasure [10, Section 3.1] used the surgeries to modify an arbitrary geodesic $v = v_0, \ldots, v_n = w$ to a quasigeodesic $v = v'_0, v'_1, \ldots, v'_{k-1}, v'_n = w$ where each $v'_i$ crosses each polygon in $S_g$ determined by $v \cup w$ at most once and where $d(v'_i, v'_{i+1}) \leq 2$. In particular, the resulting path has length at most $2d(v, w)$ and so this gives a very fast algorithm for computing distance up to a factor of two.

**The efficient path algorithm.** We now explain how to use Theorem 1.1 in order to give an algorithm for distance in $\mathcal{C}(S_g)$, which we call the efficient path algorithm. Using the well-known bigon criterion [6, Proposition 1.7] it is straightforward to determine if the distance between two vertices is 0, 1, or 2. So assume that for some $k \geq 2$ we have an algorithm for determining if two vertices of $\mathcal{C}(S_g)$ have distance 0, \ldots, $k$. We would like to give an algorithm for determining if the distance between two vertices is $k + 1$.

To this end, let $v$ and $w$ be two vertices of $\mathcal{C}(S_g)$. By induction we can check if $d(v, w) \leq k$. If not, then consider an efficient geodesic $v = v_0, \ldots, v_n = w$, which exists by Theorem 1.1. As above, there are finitely many possibilities for the vertex
v_1 and in fact we can list them all. If d(v_1, w) = k for some choice of v_1, then
d(v, w) = k + 1; otherwise d(v, w) \neq k + 1.

**Corollary 1.2.** The efficient path algorithm computes distance in \( \mathcal{G}(S_g) \).

The special case of the efficient path algorithm when the distance is four was explained to us by John Hempel and served as inspiration for the cases of larger distance.

**Comparison with previously known algorithms.** Our efficient path algorithm is in the same spirit as the algorithms of Leasure and Shackleton for computing distance in \( \mathcal{G}(S_g) \). Both Leasure and Shackleton show that there is a function \( F \) of three variables so that for any two vertices \( v \) and \( w \) of \( \mathcal{G}(S_g) \) there is a geodesic \( v = v_0, \ldots, v_n = w \) with \( i(v_1, w) \) bounded above by \( F(g, d(v, w), i(v, w)) \). This gives an algorithm in the same way as our efficient path algorithm, except there is the added (computationally expensive) step of partitioning the intersections into the various components of \( \alpha_n \setminus \alpha_0 \).

Leasure’s function \( F \) is

\[
(6(6g - 2) + 2)^{d(v, w)} \cdot i(v, w).
\]

We can illustrate the improvement of our algorithm over Leasure’s with the example in \( \mathcal{G}(S_2) \) from Figure 1. To prove the distance is 4, we can suppose for contradiction that it is 3. According to Leasure, if \( v_1 \) is the first vertex we meet on a length 3 geodesic from \( v \) to \( w \), then we can choose \( v_1 \) so that is satisfies

\[
i(v_1, w) \leq (6(6g - 2) + 2)^3 i(v, w) = 62^3 \cdot 12 = 2,859,936.
\]

By contrast, any \( v_1 \) on an efficient geodesic satisfies \( i(v_1, w) \leq 12 \) and, what is more, we know there is at most one intersection of \( v_1 \) along each edge of the polygonal decomposition of \( S_2 \) determined by \( v \) and \( w \). Because of these strong restrictions, the computation can be carried out by hand, and in fact we apply the algorithm by hand to a similar example in Section 2.

Shackleton’s function \( F \) depends only on \( i(v, w) \) but superexponentially so. Ultimately this function depends on \( d(v, w) \) as well since \( i(v, w) \) grows exponentially in \( d(v, w) \); see [15, Lemma 1.21] (and again his method does not give information as to how the points of intersection between \( v_1 \) and \( v_n \) are distributed along \( v_0 \)). Shackleton does not make his function explicit, although he states that it can be made explicit by following through his proof.

Finally, Webb’s algorithm for computing distance in \( \mathcal{G}(S_g) \) is a procedure for listing all tight geodesics between two vertices; see Section 5.2 for a detailed description. As explained to us by Webb [18], one can use his methods to prove that for every tight geodesic \( v = v_0, \ldots, v_n = w \) there is a set of representatives \( \alpha_0, \ldots, \alpha_n \) so that the number of intersections of \( \alpha_1 \) with each segment of \( \alpha_n \setminus \alpha_0 \) is bounded above by a function \( E(g) \), independent of \( i(v, w) \) and \( d(v, w) \). In other words, if our geodesics are \( (n - 2) \)-efficient, then tight geodesics are \( E(g) \)-efficient. Thus for fixed \( g \) and large \( n \) Webb’s algorithm will be more efficient than ours, and vice versa.
Efficiency versus tightness. We already mentioned that there are finitely many tight geodesics between two vertices of \( \mathcal{C}(S_g) \) and so Theorem 1.1 gives a second finite class of geodesics connecting two vertices of \( \mathcal{C}(S_g) \). It is natural to ask whether or not these two classes are distinct.

Proposition 1.3. Let \( g \geq 2 \). In \( \mathcal{C}(S_g) \) there are geodesics of length three that are efficient but not tight and also geodesics of length three that are tight but not efficient.

The proposition is proved by explicit construction; see Section 5. Also in Section 2 we give an example of a geodesic of length four that is efficient but is not tight.

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2. AN EXPLICIT EXAMPLE

In this section we illustrate the efficient path algorithm by applying it to a specific example. Consider the pair of curves shown in the left-hand side of Figure 2. In this picture the surface is illustrated as a disc with three bands attached. After identifying the two boundary curves using the orientations and the given basepoint as guides, we obtain a closed surface of genus two. The identified boundary components become a simple closed curve that represents the first vertex \( v \) and the collection of arcs become another simple closed curve that represents the second vertex \( w \). The distance between \( v \) and \( w \) in \( \mathcal{C}(S_2) \) can be computed with the computer program Metric in the Curve Complex, but here we explain how to apply our algorithm by hand.

We would like to show that \( d(v, w) = 4 \). First, we will show that \( d(v, w) \leq 4 \). We can do this by explicitly giving a path of length 4. Indeed, the curves \( \alpha_1, \)
\(\alpha_2\), and \(\alpha_3\) shown on the right-hand side of Figure 2 give rise to such a path \(v = v_0, v_1, v_2, v_3, v_4 = w\). It is readily checked that \(v\) and \(w\) fill \(S_2\), for instance because there are arcs of \(w\) cutting across each band. Thus \(d(v, w) \geq 3\). It remains to use the efficient path algorithm to show that \(d(v, w) \geq 4\).

By Theorem 1.1, if \(d(v, w)\) were equal to 3, there would be a path \(v = v_0, v_1, v_2, v_3 = w\) where \(v_1\) is represented by a curve that is disjoint from \(v\) and intersects each arc of \(w\) at most once each. In the right-hand side of Figure 2 this would be a closed curve disjoint from the boundary that intersects each arc at most once each. Such a curve can clearly traverse each band at most once, and there is no simple curve that crosses all three bands exactly once, so we conclude that there are \(\binom{3}{1} + \binom{3}{2} = 6\) possibilities for \(v_1\). It is straightforward (but tedious) to check that each of these fills with \(w\) and so there is no such \(v_1\) with \(d(v_1, w) = 2\). Thus \(d(v, w) = 4\).

The geodesic \(v_0, v_1, v_2, v_3, v_4\) given above is quite special. Each pair of vertices of the geodesic realizes the minimum possible intersection number for two vertices of \(\mathcal{C}(S_2)\) with that distance. Indeed, the nonzero intersection numbers are \(i(v_0, v_2) = i(v_2, v_4) = 1\), which is the smallest possible intersection number for vertices of distance two, \(i(v_0, v_3) = i(v_1, v_4) = 4\), which is the smallest possible intersection number for vertices of distance three in \(\mathcal{C}(S_2)\) (see, e.g., [3, Theorem 2.16]), and \(i(v_0, v_4) = 12\), which we already said was minimal.

3. Existence of efficient paths

In this section we prove the main result of this paper, Theorem 1.1. Let \(g \geq 2\) be fixed throughout.

3.1. Setup: a reducibility criterion. Our first goal is to recast our problem of finding initially efficient paths in terms of sequences of numbers; see Proposition 3.1 below.

Standard representatives and intersection sequences. Let \(v\) and \(w\) be vertices of \(\mathcal{C}(S_g)\) with \(d(v, w) \geq 3\). Let \(v = v_0, \ldots, v_n = w\) be an arbitrary path from \(v\) to \(w\). We can choose representatives \(\alpha_i\) of the \(v_i\) with the following properties:

1. all intersections \(\alpha_i \cap \alpha_{i+1}\) are empty,
2. all triple intersections of the form \(\alpha_i \cap \alpha_j \cap \alpha_k\) are empty, and
3. each \(\alpha_i\) is in minimal position with \(\alpha_0\).

To do this, we take the \(\alpha_i\) to be geodesics with respect to some hyperbolic metric on \(S_g\) and then perform small isotopies to remove triple intersections. We say that such a collection of representatives for the \(v_i\) is standard.

Let \(\gamma\) be an oriented arc contained in \(\alpha_n \setminus \alpha_0\). Since \(v_{n-1}\) and \(v_n\) span an edge in \(\mathcal{C}(S)\), we have \(\alpha_{n-1} \cap \gamma = \emptyset\).

Denote the cardinality of \(\gamma \cap (\alpha_1 \cup \cdots \cup \alpha_{n-2})\) by \(N\). Traversing \(\gamma\) in the direction of the chosen orientation, we record the sequence of natural numbers \(\sigma = (j_1, j_2, \ldots, j_N) \in \{1, \ldots, n-2\}^N\) so that the \(i\)th intersection point of \(\gamma\) with \(\alpha_1 \cup \cdots \cup \alpha_{n-2}\) lies in \(\alpha_{j_i}\). We refer to \(\sigma\) as the intersection sequence of the \(\alpha_i\) along \(\gamma\).
We remark that for our purposes it suffices to consider arcs $\gamma$ that are as large as possible. Such an arc arises as the interior of an edge of a polygon in the decomposition of $S_g$ given by $\alpha_0 \cup \alpha_n$. What is more, it suffices to consider polygons with more than four sides since intersection sequences along a rectangle get repeated on the opposite edge of the rectangle. In the example of Section 2, for instance, we would choose $\gamma$ as the interior of one of the edges of a hexagon.

**Complexity of paths and reducible sequences.** We define the complexity of an oriented path $v_0, \ldots, v_n$ in $\mathcal{C}(S)$ to be

$$\sum_{i=1}^{n-2} i(v_i, v_n).$$

We say that a sequence $\sigma$ of natural numbers is reducible under the following circumstances: whenever $\sigma$ arises as an intersection sequence for a path $v_0, \ldots, v_n$ in $\mathcal{C}(S_g)$ there is another path $v'_0, \ldots, v'_n$ with $v'_0 = v_0$, $v'_{n-1} = v_{n-1}$, $v'_n = v_n$ and with smaller complexity. With this terminology in hand, the existence of initially efficient paths is a consequence of the following proposition.

**Proposition 3.1.** Suppose $\sigma$ is a sequence of elements of $\{1, \ldots, n-2\}$. If $\sigma$ has more than $n-2$ entries equal to 1, then $\sigma$ is reducible.

We can deduce Theorem 1.1 easily from Proposition 3.1.

**Proof of Theorem 1.1 assuming Proposition 3.1.** Let $v$ and $w$ be vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$, and let $v = v_0, \ldots, v_n = w$ be a geodesic with the following two properties:

1. $v_0, \ldots, v_n$ has minimal complexity among all geodesics from $v$ to $w$, and
2. among all geodesics satisfying the first property, the complexity of $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ is minimal.

It follows from the first property that each geodesic $v_k, \ldots, v_n$ with $k \leq n-3$ is initially efficient. Indeed, otherwise by Proposition 3.1 we could replace $v_k, \ldots, v_n$ with another geodesic so that the resulting geodesic from $v$ to $w$ has smaller complexity. Similarly, it follows from the second property that $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ is initially efficient. Again, if not, we could replace $v_{n-1}$ (and only this vertex) with another vertex so as to reduce complexity (this does not affect the first property).

Notice that the approach established in Proposition 3.1 disregards all information about a path in $\mathcal{C}(S_g)$ except its intersection sequences. For instance, we will not need to concern ourselves with how the strands of the $\alpha_i$ are connected outside of a neighborhood of $\gamma$.

We will prove Proposition 3.1 in three stages. First, in Section 3.2 we describe a normal form for sequences of natural numbers (Lemma 3.2 below) and also describe an associated diagram for the normal form called the dot graph. Next in Section 3.3 we will show that if the dot graph exhibits certain geometric features—empty boxes and hexagons—then the sequence is reducible. Finally in Section 4 we will show that any sequence in normal form that does not satisfy Proposition 3.1 has a dot graph exhibiting either an empty box or an empty hexagon.
3.2. **Stage 1: Sawtooth form and the dot graph.** The main goal of this section is to give a normal form for sequences of natural numbers that interacts well with our notion of reducibility. We also describe a way to diagram sequences in normal form called the dot graph.

**Sawtooth form.** We say that a sequence \((j_1, j_2, \ldots, j_k)\) of natural numbers is in **sawtooth form** if
\[
j_i < j_{i+1} \implies j_{i+1} = j_i + 1.
\]

An example of a sequence in sawtooth form is \((1, 2, 2, 3, 4, 3, 4, 2, 3, 4, 5)\). If a sequence of natural numbers is in sawtooth form, we may consider its **ascending sequences**, which are the maximal subsequences of the form \(k, k+1, \ldots, k+m\). In the previous example, the ascending sequences are \((1, 2)\), \((2, 3, 4)\), \((3, 4)\), \((3, 4)\), and \((2, 3, 4, 5)\).

**Lemma 3.2.** Let \(\sigma\) be an intersection sequence. There exists an intersection sequence \(\tau\) in sawtooth form so that \(\tau\) differs from \(\sigma\) by a permutation of its entries and so that \(\sigma\) is reducible if and only if \(\tau\) is.

**Proof.** Suppose \(\sigma = (j_1, \ldots, j_N)\) is the intersection sequence for a set of standard representatives \(\alpha_0, \ldots, \alpha_n\) along an arc \(\gamma \subseteq \alpha_n \setminus \alpha_0\). The basic idea we will use is that if \(|j_i - j_{i+1}| > 1\), then we can modify \(\alpha_{j_i}\) and \(\alpha_{j_{i+1}}\) to new curves \(\alpha'_{j_i}\) and \(\alpha'_{j_{i+1}}\) so that the new curves still form a set of standard representatives for the same path and so that the new intersection sequence along \(\gamma\) differs from \(\sigma\) by a transposition of the consecutive terms \(j_i\) and \(j_{i+1}\); see Figure 3. We call this the resulting modification of \(\sigma\) a commutation.

It suffices to show that if a sequence \(\sigma\) is not in sawtooth form, then it is possible to perform a finite sequence of commutations so that the resulting sequence \(\tau\) is in sawtooth form. Indeed, the sequence \(\tau\) appears as an intersection sequence for a particular path in \(\mathcal{E}(S_g)\) if and only if \(\sigma\) does.

We say that \(\sigma\) fails to be in sawtooth form at the index \(i\) if \(j_{i+1} > j_i + 1\). Let \(k = k(\sigma)\) be the highest index at which \(\sigma\) fails to be in sawtooth form, and say that \(k\) is zero if \(\sigma\) is in sawtooth form. Assuming \(k > 0\), we will show that we can modify \(\sigma\) by a sequence of commutations so that the highest index where the resulting sequence fails to be in sawtooth form is strictly less than \(k\).

We decompose \(\sigma\) into a sequence of subsequences of \(\sigma\), namely,
\[
(\sigma_1, \sigma_2, \sigma_3, \sigma_4)
\]
where $\sigma_2$ is the singleton $(j_k)$ and $\sigma_3$ is the longest subsequence of $\sigma$ starting from the $(k+1)$st term so that each term is greater than $j_k + 1$. The sequences $\sigma_1$ and $\sigma_4$ are thus determined, and one or both might be empty.

By a series of commutations, we can modify $\sigma$ to the sequence

$$\sigma' = (\sigma_1, \sigma_3, \sigma_2, \sigma_4).$$

We claim that $k(\sigma') < k(\sigma)$. Since the length of $\sigma_1$ is $k - 1$, it is enough to show that the subsequence $(\sigma_3, \sigma_2, \sigma_4)$ is in sawtooth form.

By the definition of $k$, we know that $\sigma_3$ is in sawtooth form. Next, the last term of $\sigma_3$ is greater than $j_k + 1$ and the first (and only) term of $\sigma_2$ is $j_k$, and so these terms satisfy the definition of sawtooth form. We know $\sigma_2 = (j_k)$ and the first term of $\sigma_4$, call it $j$, is at most $j_k + 1$, and so these terms are also in sawtooth form. Finally, the subsequence $\sigma_4$ is in sawtooth form by the definition of $k$. This completes the proof. □

Dot graphs. It will be useful to draw the graph in $\mathbb{R}_{\geq 0}^2$ of a given sequence of natural numbers, where the sequence is regarded as a function $\{1, \ldots, N\} \rightarrow \mathbb{N}$. The points of the graph of a sequence $\sigma$ will be called dots. We decorate the graph by connecting the dots that lie on a given line of slope 1; these line segments will be called ascending segments. The resulting decorated graph will be called the dot graph of $\sigma$ and will be denoted $G(\sigma)$; see Figure 4.

![Figure 4. Example of dot graph of a sequence in sawtooth form](image)

3.3. **Stage 2: Dot graph polygons and surgery.** The goal of this section is to describe certain geometric shapes than can arise in a dot graph, and then to prove that if the dot graph $G(\sigma)$ admits one of these shapes then the sequence $\sigma$ is reducible (Lemma 3.3).

**Dot graph polygons.** We say that a polygon in the plane is a dot graph polygon if

1. the edges all have slope 0 or 1,
2. the edges of slope 0 have nonzero length, and
3. the vertices all have integer coordinates.
The edges of slope 1 in a dot graph polygon are called *ascending edges* and the edges of slope 0 are called *horizontal edges*.

Let $\sigma$ be a sequence of natural numbers in sawtooth form. A dot graph polygon is a *$\sigma$-polygon* if:

1. the vertices are dots of $G(\sigma)$ and
2. the ascending edges are contained in ascending segments of $G(\sigma)$.

![Figure 5](image)

**Figure 5.** A box, a hexagon of type 1, and a hexagon of type 2; the red (darker) dots are required to be endpoints of ascending segments, while the blue (lighter) dots may or may not be endpoints.

A *box* in $G(\sigma)$ is a *$\sigma$-quadrilateral* $P$ with the following two properties:

1. the leftmost ascending edge contains the highest point of some ascending segment of $G(\sigma)$ and
2. the rightmost ascending edge contains the lowest point of some ascending segment of $G(\sigma)$.

We will also need to deal with hexagons. Up to translation and changing the edge lengths, there are four types of dot graph hexagons; two have an acute exterior angle, and we will not need to consider these. Notice that a dot graph hexagon necessarily has a leftmost ascending edge, a rightmost ascending edge, and a middle ascending edge. This holds even for degenerate hexagons since horizontal edges are required to have nonzero length.

A *hexagon of type 1* in $G(\sigma)$ is a *$\sigma$-hexagon* where:

1. no exterior angle is acute,
2. the middle ascending edge is an entire ascending segment of $G(\sigma)$, and
3. the minimum of the middle ascending edge equals the minimum of the leftmost ascending edge,
4. the leftmost ascending edge contains the highest point of an ascending segment of $G(\sigma)$.

Similarly, a *hexagon of type 2* in $G(\sigma)$ is a *$\sigma$-hexagon* that satisfies the first two conditions above and the following third and fourth conditions:

1. the maximum of the middle ascending edge equals the maximum of the rightmost ascending edge,
2. the rightmost ascending edge contains the lowest point of an ascending segment of $G(\sigma)$.

See Figure 5 for pictures of boxes and hexagons of types 1 and 2.
The following lemma is the main goal of this section. We say that a horizontal edge of a $\sigma$-polygon is **pierced** if its interior intersects $G(\sigma)$. Also, we say that a $\sigma$-polygon is **empty** if it there are no points of $G(\sigma)$ in its interior.

**Lemma 3.3.** Suppose that $\sigma$ is a sequence of natural numbers in sawtooth form and that $G(\sigma)$ has an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2. Then $\sigma$ is reducible.

Before we prove Lemma 3.3, we need to introduce another topological tool, surgery on curves.

**Surgery.** Let $\alpha$ and $\beta$ be two curves in a surface that are in minimal position. Let $\gamma$ be an oriented arc of $\beta$ that intersects $\alpha$ in at least two points. We can form a new curve $\alpha'$ from $\alpha$ by performing surgery along $\gamma$ as follows. We first remove from $\alpha$ small open neighborhoods of two points of $\alpha \cap \gamma$ that are consecutive along $\gamma$. What remains of $\alpha$ is a pair of arcs; we can connect the endpoints of either arc by another arc $\delta$ that lies in a small neighborhood of $\gamma$ in order to create the new simple closed curve $\alpha'$ (the other arc of $\alpha$ is discarded); see Figure 6.

We draw a neighborhood of $\gamma$ in the plane so that $\gamma$ is a horizontal arc oriented to the right. We say that $\alpha'$ is obtained from $\alpha$ by ++, +−, −+, or −− surgery along $\gamma$; the first symbol is + or − depending on whether the first endpoint of $\delta$ (as measured by the orientation of $\gamma$) lies above $\gamma$ or below, and similarly for the second symbol.

![Figure 6](image.png)

**Figure 6.** The four types of surgery on a curve along an arc

In general, for a given pair of intersection points of a curve $\alpha$ with $\gamma$, exactly two of the four possible surgeries result in a simple closed curve. If we orient $\alpha$, then the two intersection points of $\alpha$ with $\gamma$ can either agree or disagree. If they agree, then the +− and −+ surgeries, the **odd surgeries**, result in a simple closed curve, and if they disagree, the ++ and −− surgeries, the **even surgeries**, result in a simple closed curve.

These surgeries will of course only be of use to us if the curve $\alpha'$ is an essential simple closed curve in $S$. One variant of the well-known bigon criterion is that two transverse simple closed curves $\alpha$ and $\beta$ in a compact surface are in minimal position if and only if every closed curve formed from one arc of $\alpha$ and one arc of $\beta$ is essential; see [1, Proposition 3.10]. This immediately implies that our $\alpha'$ is essential.

We now use the surgeries described above to prove that a dot graph with an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2 corresponds to a sequence that is reducible.
Proof of Lemma 3.3. Suppose that $\sigma$ appears as an intersection sequence along $\gamma$ for a set of standard representatives $\alpha_0, \ldots, \alpha_n$, where $\gamma$ is an oriented arc of $\alpha_n \setminus \alpha_0$. We need to replace the $\alpha_i$ with new curves $\alpha'_i$ as in the definition of a reducible sequence. We treat the three cases in turn, according to whether $G(\sigma)$ has an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2.

Suppose $G(\sigma)$ has an empty, unpierced box $P$. By the definitions of sawtooth form and empty boxes there are no ascending edges of $G(\sigma)$ in the vertical strip between the two ascending edges of $P$, that is, the dots of $P$ correspond to a consecutive sequence of intersections along $\gamma$:

$$\alpha_k, \ldots, \alpha_{k+m}, \quad \alpha_k, \ldots, \alpha_{k+m}$$

where $k \geq 1$ and $m \geq 0$.

First, for $i \notin \{k, \ldots, k+m\}$ we set $\alpha'_i = \alpha_i$. We then define $\alpha'_k, \ldots, \alpha'_{k+m}$ inductively: for $i = k, \ldots, k+m$, the curve $\alpha'_i$ is obtained by performing surgery along $\gamma$ between the two points of $\alpha_i \cap \gamma$ corresponding to dots of $P$ and the surgeries are chosen so that they form a path in the directed graph in Figure 8.

Figure 7. An example of a set of surgeries as in the box case of Lemma 3.3

Figure 8. The directed graph used in the proof of Lemma 3.3
The vertices of the graph in Figure 8 correspond to the four types of surgeries: ++, +−, −+, and −−, and the rule is that the second sign of the origin of a directed edge is the opposite of the first sign of the terminus. Since every vertex has one outgoing arrow pointing to an even surgery and one outgoing arrow pointing to an odd surgery, the desired sequence of surgeries exists; in fact it is completely determined by the choice of surgery on \( \alpha_k \), and so there are exactly two possible sequences. See Figure 7 for an example of this procedure. There we perform +− surgery on \( \alpha_3 \), then ++ surgery on \( \alpha_4 \), then −+ surgery on \( \alpha_5 \).

For \( 0 \leq i \leq n \), let \( v'_i \) be the vertex of \( C(S_0) \) represented by \( \alpha'_i \). We need to check that the \( v'_i \) satisfy the definition of a reducible sequence, namely that

1. \( v'_0 = v_0, v'_{n-1} = v_{n-1}, \) and \( v'_n = v_n \), and
2. the complexity of \( v'_0, \ldots, v'_n \) is strictly smaller than that of \( v_0, \ldots, v_n \).

The first condition holds because \( 1 \leq k \leq k + m \leq n - 2 \). The second condition holds because \( i(v'_i, v_n) \leq i(v_i, v_n) \) for all \( i \) and \( i(v'_k, v_n) < i(v_k, v_n) \).

![Figure 9](image_url)

*Figure 9. An example of a set of surgeries as in the hexagon case of Lemma 3.3*

The cases of empty, unpierced hexagons of types 1 and 2 are similar, but one new idea is needed. These two cases are almost identical, and so we will only treat the first case, that is, we suppose \( G(\sigma) \) has an empty, unpierced hexagon \( P \) of type 1. By the definition of sawtooth form and the definition of an empty, unpierced hexagon of type 1, there are no ascending segments of \( G(\sigma) \) in the vertical strip between the leftmost and middle ascending edges of \( P \) and any ascending segments of \( G(\sigma) \) that lie in the vertical strip between the middle and rightmost ascending segments have their highest point strictly below the lower-right horizontal edge of \( P \). It follows that the dots of \( P \) correspond to a sequence of intersections along \( \gamma \) of the following form:

\[ \alpha_k, \ldots, \alpha_{k+m}, \alpha_k, \ldots, \alpha_k+\ell, \alpha_{j_1}, \ldots, \alpha_{j_p}, \alpha_k+\ell, \ldots, \alpha_{k+m} \]
where \( k \geq 1, m \geq 0, 0 \leq \ell \leq m, p \geq 0, \) and each \( j_i < \alpha_{k+\ell} \). See Figure 9 for an example where \( k = 3, m = 4, p = 0, \) and \( \ell = 2 \).

Again, for each \( \alpha_i \) with \( i \notin \{k, \ldots, k+m\} \) we set \( \alpha'_i = \alpha_i \). Each of the remaining \( \alpha_i \) corresponds to exactly two dots in \( P \) except for \( \alpha_{k+\ell} \), which corresponds to three. Let \( \alpha'_{k+\ell} \) be the curve obtained from \( \alpha'_{k+\ell} \) via surgery along \( \gamma \) between the first two (leftmost) points of \( \alpha'_{k+\ell} \cap \gamma \) corresponding to dots of \( P \) and satisfying the following property: \( \alpha'_{k+\ell} \) does not contain the arc of \( \alpha_{k+\ell} \) containing the third (rightmost) point of \( \alpha_{k+\ell} \cap \gamma \) corresponding to a dot of \( P \). As always, there are two choices of surgery given two consecutive points of \( \alpha_{k+\ell} \cap \gamma \); one contains this third intersection point and one does not.

We then define \( \alpha'_k, \ldots, \alpha'_m \) inductively as before using the diagram above (notice the reversed order), and finally we define \( \alpha'_k, \ldots, \alpha'_m \) inductively as before.

By our choice of \( \alpha'_{k+\ell} \), we have that \( \alpha'_{k+\ell} \cap \alpha'_{k+\ell+1} = \emptyset \), as required; indeed, we eliminated the strand of \( \alpha'_{k+\ell} \) that was in the way between the two strands of \( \alpha_{k+\ell+1} \) being surgered. Also, since each \( j_i \) is strictly less than \( k + \ell \), the curves \( \alpha'_{k+\ell+1}, \ldots, \alpha'_{k+m} \) satisfy the condition that \( \alpha_i \cap \alpha'_{i+1} = \emptyset \). The other conditions in the definition of a reducible sequence are easily verified as before. This completes the proof of the lemma.

4. Stage 3: Innermost Polygons

In this section we will put together Lemmas 3.2 and 3.3 in order to prove Proposition 3.1. We begin with two lemmas.

**Lemma 4.1.** If a dot graph \( G(\sigma) \) contains a box \( P \) pierced in exactly one edge, then it contains an unpierced box.

**Proof.** Denote the ascending edges of \( P \) by \( e \) and \( f \). There is an ascending segment \( e' \) intersecting the interior of exactly one of the two horizontal edges of \( P \); we choose \( e' \) to be rightmost if it intersects the bottom edge of \( P \) and leftmost if it intersects the top edge. Either way, we find a box \( P' \) pierced in at most one edge and where one ascending edge is contained in \( e' \) and the other ascending edge is contained in \( P \). The box \( P' \) has horizontal edges strictly shorter than those of \( P \). Therefore, we may repeat the process until it eventually terminates, at which point we find the desired unpierced box. \( \square \)

**Lemma 4.2.** Among all unpierced boxes and hexagons of type 1 and 2 in a dot graph \( G(\sigma) \), an innermost unpierced box or hexagon of type 1 or 2 is empty.

**Proof.** We treat the three cases separately. First suppose that \( P \) is an unpierced box that is not empty. We will show that \( P \) either contains another unpierced box or an unpierced hexagon of type 1. Let \( e \) be an ascending segment contained in the interior of \( P \). We choose \( e \) so that \( \max(e) \) is maximal among all such ascending segments, and we further choose \( e \) to be rightmost among all ascending segments with maximum equal to \( \max(e) \).
There is a unique (possibly degenerate) hexagon $P'$ of type 1 with one edge equal to $e$, and the other two edges contained in the ascending edges of $P$; see the left-hand side of Figure 10. If $P'$ is unpierced, we are done, so assume that $P'$ is pierced. By construction, the top horizontal edge of $P'$ and the lower-right horizontal edge of $P'$ are unpierced. Suppose that the interior of the lower-left horizontal edge of $P'$ were pierced. Let $e'$ be the rightmost ascending segment of $G(\sigma)$ that pierces this edge of $P'$. By the choice of $e$, we have that $\max(e') \leq \max(e)$, and so there is a box pierced in at most one edge whose ascending edges are contained in $e'$ and $e$. By Lemma 4.1, there is an unpierced box contained in this pierced box, and so $P$ is not innermost.

The second case is $P$ is an unpierced hexagon of type 1. Again suppose that $P$ is not empty. Let $e$ be an ascending segment contained in the interior of $P$ that has the largest maximum $\max(e)$ over all such segments and further is rightmost among all such ascending segments. Let $m$ denote the middle ascending edge of $P$. It follows from the fact that $\sigma$ is in sawtooth form that there are no ascending segments of $G(\sigma)$ that lie inside $P$ and to the right of $m$; in particular, $e$ lies to the left of $m$. We now treat two subcases, depending on whether $\max(e) > \max(m)$ or not.

If $\max(e) > \max(m)$, there is a maximal hexagon $P'$ of type 1 with ascending edges contained in $P \cup e$ as in the middle picture of Figure 10. By the same argument as in the previous case, $P'$ is either unpierced or it contains an unpierced box.

If $\max(e) \leq \max(m)$, the argument is similar. There is a hexagon $P'$ of type 2 as shown in the right-hand side of Figure 10. The topmost edge of $P'$ is unpierced by the choice of $e$. The bottom edge of $P'$ is unpierced since it is a horizontal edge for $P$, which is unpierced. And if the third horizontal edge of $P'$ were pierced, we could find a box pierced in at most one edge, hence an unpierced box, as in the previous cases. It follows that $P'$ is unpierced and again $P$ is not innermost.

The third and final case is where $P$ is an unpierced hexagon of type 2. This is completely analogous to the previous case; in fact, if we rotate the two pictures from the type 1 case by $\pi$ we obtain the required pictures for the type 2 case.

We can now use the two previous lemmas to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\sigma$ be a sequence of elements of $\{1, \ldots, n-2\}$. By Lemma 3.2 we may assume that $\sigma$ is in sawtooth form without changing the number of entries equal to 1; call this number $k$. Let $e_1, \ldots, e_k$ denote the ascending segments of $G(\sigma)$ with minimum equal to 1, ordered from left to right.
If \( \max(e_{i+1}) < \max(e_i) \) for all \( i \), then since \( \max(e_1) \leq n - 2 \) it follows that \( k \leq n - 2 \). Therefore, it suffices to show that if \( \max(e_{i+1}) \geq \max(e_i) \) for some \( i \) then \( \sigma \) is reducible.

Suppose then that \( \max(e_{i+1}) \geq \max(e_i) \) for some \( i \). The first step is to show that \( G(\sigma) \) has an unpierced box. Let \( e \) be the first ascending segment (from left to right) that appears after \( e_i \) and has \( \max(e) \geq \max(e_i) \); such \( e \) exists because \( \max(e_{i+1}) \geq \max(e_i) \). Because \( \min(e) \geq \min(e_i) = 1 \), there is evidently a (possibly degenerate) box \( P \) with two edges contained in \( e_i \) and \( e \) and two horizontal edges with heights \( \min(e) \) and \( \max(e) \). By the definition of \( e \), the interior of the upper horizontal edge of \( P \) is disjoint from \( G(\sigma) \), so \( P \) is pierced in at most one edge. By Lemma 4.1, \( P \) contains an unpierced box.

Let \( P \) now be an innermost unpierced box or hexagon of type 1 or 2; such \( P \) exists because each \( \sigma \)-polygon contains a finite number of dots of \( G(\sigma) \) and a polygon contained inside another polygon contains a fewer number of dots. By Lemma 4.2, \( P \) is empty. By Lemma 3.3, \( \sigma \) is reducible. □

5. Efficiency versus tightness

The goal of this section is to prove Proposition 1.3—that the classes of tight geodesics and oriented geodesics are distinct—and to describe Webb’s algorithm for distance in \( \mathcal{G}(S) \). Both of these require the notion of tightness.

**Tight geodesics.** A path \( v_0, \ldots, v_n \) in \( \mathcal{G}(S) \) is tight at \( v_i \) if there are representatives \( \alpha_{i-1}, \alpha_i, \) and \( \alpha_{i+1} \) of \( v_{i-1}, v_i, \) and \( v_{i+1} \) so that \( \alpha_i \) is contained in a regular neighborhood of \( \alpha_{i-1} \cup \alpha_{i+1} \). Equivalently, any vertex of \( \mathcal{G}(S) \) that is adjacent to \( v_{i-1} \) and \( v_{i+1} \) is also adjacent to \( v_i \). The path is tight if it is tight at each \( v_i \) with \( 1 \leq i \leq n - 1 \).

Any geodesic can be converted to a tight geodesic. Indeed, if the geodesic \( v_0, \ldots, v_n \) is not tight at \( v_i \) we replace \( v_i \) with another vertex satisfying the definition of tight. Applying this process iteratively to \( v_1, \ldots, v_{n-1} \) will result in a tight geodesic (one must check that tightening at \( v_{i-1} \) does not affect the tightness at \( v_i \); see [11, Lemma 4.5]).

5.1. Examples. The examples we use to prove Proposition 1.3 will be presented in terms of the branched double cover of \( S_g \) over the sphere, which we now explain.

**The branched double cover.** Let \( S_{0,2g+2} \) denote a sphere with \( 2g + 2 \) marked points. The double cover branched over the marked points is the closed surface \( S_g \). The preimage of a simple arc in \( S_{0,2g+2} \) connecting two marked points is a nonseparating simple closed curve in \( S_g \), and the preimage of a simple closed curve that surrounds \( 2k + 1 \) marked points is a separating simple closed curve in \( S_g \) that cuts off a subsurface of genus \( k \).

**Proof of Proposition 1.3.** We will explain the proof for the case \( g = 4 \). The other cases of \( g \geq 2 \) can be handled by a straightforward modification of the given examples.

Consider the arcs \( \alpha_0, \alpha_1, \alpha_2, \) and \( \alpha_3 \) shown in the left-hand side of Figure 11. As above, each arc \( \alpha_i \) represents a vertex \( v_i \) of \( \mathcal{G}(S_4) \). We can see that \( d(v_0, v_3) \geq 3 \)
since $\alpha_0$ and $\alpha_3$ fill $S_{0,10}$. We claim that the path $v_0, v_1, v_2, v_3$ is efficient but not tight. The path is certainly efficient because $|\alpha_1 \cap \alpha_3| = |\alpha_0 \cap \alpha_2| = 1$ and each of these intersection points lifts to one point of intersection in $S_4$. On the other hand, the path is not tight since we can easily find an arc in $S_{0,10}$ that intersects $\alpha_2$ without intersecting $\alpha_1$ or $\alpha_3$ (or, because a regular neighborhood of $\alpha_1 \cup \alpha_3$ is a disk with three marked points and $\alpha_2$ is not homotopic into this neighborhood).

Next consider the curves and arcs $\alpha_0$, $\alpha_1$, and $\alpha_3$ shown in the right-hand side of Figure 11. Let $\alpha_2$ be the boundary component of a regular neighborhood of $\alpha_1 \cup \alpha_3$. Let $v_0, v_1, v_2, v_3$ be the corresponding path in $\mathcal{C}(S_4)$; again we have $d(v_0, v_3) = 3$. We claim that the given path is tight but not efficient. By definition, the path is tight at $v_2$ and it is straightforward to see that it is tight at $v_1$. The path is not efficient because each intersection of $\alpha_3$ with $\alpha_0$ gives rise to two (essential) intersections of $\alpha_2$ with $\alpha_0$ and this persists in the cover $S_4$. \hfill $\Box$

There are other examples of efficient geodesics that are not tight readily available in the literature. For instance, for any $g \geq 3$, Aougab and Huang [3] gave examples of curves $\alpha_0$ and $\alpha_3$ that fill $S_g$ and so that the surface obtained by cutting $S_g$ along $\alpha_0 \cup \alpha_3$ is a single polygon with $8g - 4$ sides; in particular the distance is three and the intersection number is $2g - 1$, the minimum possible intersection number for vertices with distance three in $\mathcal{C}(S_g)$. As shown by Aougab and Taylor [4, Lemma 3.2] we can then obtain we we are calling an efficient geodesic by choosing two disjoint arcs in the polygon that descend to simple closed curves $\alpha_1$ and $\alpha_2$ in $S_g$, one intersecting $\alpha_0$ and one intersecting $\alpha_3$. Since each of these curves intersects only one of $\alpha_0$ and $\alpha_3$, and does so in a single point, the geodesic given by $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is efficient.

5.2. Webb’s algorithm. We now describe Webb’s algorithm for computing distance in $\mathcal{C}(S)$. First we need the definition of the arc complex.

Arc complex. Let $S$ be a compact surface with nonempty boundary. The arc complex $\mathcal{A}(S)$ is the simplicial complex with $k$-simplices corresponding to $(k+1)$-tuples of homotopy classes of essential arcs in $S'$ with pairwise disjoint representatives. Here, homotopies are allowed to move the endpoints of an arc along $\partial S$, and an arc is essential if it is not homotopic into $\partial S$.

The algorithm. Now we are ready to describe Webb’s algorithm. Given a path $v = v_0, \ldots, v_n = w$, we choose representatives $\alpha_i$ of the $v_i$ so that $\alpha_i \cap \alpha_{i+1} = \emptyset$ for
all $i$ and so that each $\alpha_i$ is in minimal position with $\alpha_0$. If we cut $S$ along $\alpha_0$, we obtain a compact surface $S'$, two of whose boundary components correspond to $\alpha_0$.

For each $i > 1$, the curve $\alpha_i$ gives a collection of disjoint arcs in $S'$ and hence a simplex $\sigma_i$ of $A(\alpha_i(S'))$ (some arcs of $\alpha_i$ might be parallel and these get identified in $A(\alpha_i(S'))$). For $i \geq 3$, the collection of arcs is filling, which means that when we cut $S'$ along these arcs we obtain a collection of disks and boundary-parallel annuli, and we say that the corresponding simplex of $A(\alpha_i(S'))$ is filling.

Since there is a unique configuration for $\alpha_n$ and $\alpha_0$ in minimal position, there is a unique choice for $\sigma_n$. Since $\sigma_n \cup \sigma_{n-1}$ is contained in a simplex of the arc complex of $S'$ and since $\sigma_n$ is filling, there are finitely many choices for $\sigma_{n-1}$. This is the key point: there are infinitely many vertices of $\mathcal{C}(S)$ that correspond to any given simplex in the arc complex, but there are finitely many choices for the simplex itself.

Because $\sigma_i$ is filling whenever $i \geq 3$, we can continue this process inductively, and explicitly list all possibilities for $\sigma_2$. Now, by the definition of a tight path in $\mathcal{C}(S)$, the vertex $v_1$ is represented by a boundary component of a regular neighborhood of $\alpha_0 \cup \alpha_2$. Equivalently, any such $v_1$ is given by a regular neighborhood of the union of $\partial S'$ with a representative of $\sigma_2$. Hence there are finitely many choices for $v_1$. Continuing inductively, we can list all tight paths from $v$ to $w$ of length $n$. So the algorithm is obtained by finding all tight paths of length 0, 1, 2, 3, etc. until we find the first one.

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