Rational L-space surgeries on satellites by algebraic links

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ABSTRACT

Given an \( n \)-component link \( L \) in any 3-manifold \( M \), the space \( \mathcal{L} \subset (\mathbb{Q} \cup \{\infty\})^n \) of rational surgery slopes yielding L-spaces is already fully characterized in joint work by the author when \( n = 1 \) and \( L \) is nontrivial. For \( n > 1 \), however, there are no previous results for \( \mathcal{L} \) as a rational subspace, and only limited results for integer surgeries \( \mathcal{L} \cap \mathbb{Z}^n \) on \( S^3 \). Herein, we provide the first nontrivial explicit descriptions of \( \mathcal{L} \) for rational surgeries on multi-component links. Generalizing Hedden’s and Hom’s L-space result for cables, we compute both \( \mathcal{L} \), and its topology, for all satellites by torus-links in \( S^3 \). For fractal-boundaried \( \mathcal{L} \) resulting from satellites by algebraic links or iterated torus links, we develop arbitrarily precise approximation tools. We also extend the provisional validity of the L-space conjecture for rational surgeries on a knot \( K \subset S^3 \) to rational surgeries on such satellite-links of \( K \). These results exploit the author’s generalized Jankins–Neumann formula for graph manifolds.

1. Introduction

A connected, closed, oriented 3-manifold is called an L-space if its reduced Heegaard Floer homology vanishes. The present work focuses on the following relative notion of L-space.

DEFINITION 1.1. For a compact oriented 3-manifold \( Y \) with boundary a disjoint union of \( n \) tori, the L-space region \( \mathcal{L}(Y) \subset \prod_{i=1}^n P(H_1(\partial_i Y; \mathbb{Z})) \cong (\mathbb{Q} \cup \{\infty\})^n \), with complement \( \mathcal{N}\mathcal{L}(Y) \), is the space of (rational) Dehn-filling slopes of \( Y \) which yield L-space Dehn-fillings.

Prior Results. Until now, studies of multi-component L-space surgery slopes have been confined to integer surgeries on links in \( S^3 \). These primarily include numerical methods of Liu to plot individual points in \( \mathcal{L} \cap \mathbb{Z}^2 \) for 2-component links [23], Gorsky and Hom’s identification of torus-link satellites with integer L-space surgery slopes in the positive orthant [11], and Gorsky and Némethi’s work on integer torus-link surgeries [12] and on a partial characterization (complete for algebraic links) of which 2-component links have \( \mathcal{L} \cap \mathbb{Z}^2 \) bounded from below [13].

Present Motivation. As subsets of \( (\mathbb{Q} \cup \{\infty\})^n \), L-space regions exhibit qualitative features invisible to the set of integer L-space surgery slopes, such as nontrivial topological properties, fractal behaviors at the boundary of \( \mathcal{L} \), and symmetries such as the action of \( \Lambda \) in Theorem 1.2.

Since non-L-space regions chart the silhouette of Heegaard Floer complexity as a function of varying surgery slope, this creates a rich template to compare against the surgery regions supporting any candidate geometric structure potentially responsible for nontrivial HF classes. Such comparisons for Seifert fibered spaces led to the L-space conjecture that non-L-spaces are characterized by the existence of left orders on fundamental groups and/or co-oriented taut foliations [5, 20]. Both L-spaces and \( \mathcal{L} \) also constrain complex singularities; see Section 1.3.
The author’s joint result with Rasmussen [28] characterizing nontrivial $\mathcal{L}$ for knot exteriors in 3-manifolds led both to our toroidal gluing theorem for L-spaces [28] and to the author’s independent proof of the L-space conjecture for graph manifolds [29]. Our joint work on $\mathcal{L}$ combined with Hanselman and Watson’s studies of combinatorial properties of certain bordered Floer algebras [16] gave rise to a topological realization of bordered Floer homology for single-torus-boundary boundaries [15]. A multiple-boundary-component version of this should also exist.

**Methods:** Classification formula. Despite reliance on an enhanced L-space gluing tool proved in Theorem 3.6,¹ this paper was primarily made possible by the author’s classification of graph manifolds admitting co-oriented taut foliations, with proof of the graph-manifold L-space conjecture as by-product [29]. (This is not to be confused with the author’s joint work with Hanselman et al [14].)² This classification combines a new classification formula (Theorem 4.3), generalizing that of Jankins and Neumann for Seifert fibered spaces [19], with a structure theorem (Theorem 4.4) prescribing the interpretation of outputs of this formula.

This classification tool also governs L-space regions for unions of graph manifolds with single-torus-boundary manifolds. In particular, it gives a complete abstract characterization of $\mathcal{L}$ for any graph-manifold-exterior satellite of any knot in any 3-manifold. The classification formula alternately composes a linear-fractional transformation $\phi^P_{c_e}$, induced by a gluing map $\phi_e$ for each edge $e$, with a pair $y^v_\pm$, for each vertex $v$, of extremizations of locally finite collections of piecewise-constant functions of slopes in a certain Seifert-data-compatible basis.

**Results**. Herein, we analyze the intricate behavior of solutions $\mathcal{L}$ to the classification formulae for exteriors of such satellites. The bounded-chaotic behavior of these $y^v_\pm$ generically leads to fractal-boundaried $\mathcal{L}$, but we develop precise tools for local approximation and topological characterization. As sample applications of these tools, Theorems 1.6 and 1.7 construct global inner approximations of $\mathcal{L}$ for satellites by algebraic links and iterated-torus-links, respectively.

Moreover, for a satellite in $S^3$ by an $n$-component torus link, the chaotic behavior of $y^v_\pm$ generically degenerates, and we provide an exact explicit description of $\mathcal{L}$ and its various possible topologies, in Theorems 1.2 and 1.3, respectively. Finally, in Theorem 1.4 and Corollary 1.5, we promote L-space conjecture results for knot surgeries to results for satellite surgeries.

### 1.1. Torus-link satellites

The $T(np,nq)$-torus-link satellite $K^{(np,nq)}(K)\subset M$ of a knot $K\subset M$ in a 3-manifold $M$ embeds the torus link $T(np,nq)$ in the boundary of a neighborhood $\nu(K)$ of $K\subset M$. The exterior $Y^{(np,nq)}$ of $K^{(np,nq)}$ splices $K\subset M$ to the multiplicity-$q$ fiber of the Seifert fibered exterior of $T(np,nq)$. This Seifert structure also prescribes three distinguished subsets $\Lambda, \mathcal{R}, \mathcal{Z} \subset \prod_{i=1}^n \mathbb{P}(H_1(\partial Y^{(np,nq)}))$ of slopes. The lattice $\Lambda$ acts on slopes by reparametrization of Seifert data, and $\mathcal{R} \setminus \mathcal{Z}$ catalogs reducible surgeries with no $S^1 \times S^2$ connected summand.

**Theorem 1.2.** Suppose that $K\subset S^3$ is a positive L-space knot of genus $g(K)$, and that $n, p, q \in \mathbb{Z}$, with $n, p > 0$ and $\gcd(p,q) = 1$. Then the $T(np,nq)$ torus-link satellite $K^{(np,nq)}(K)\subset S^3$ of $K$ has L-space surgery region given by the union of $\Lambda$-orbits $L_{S^3} = \Lambda \cdot \mathcal{L}^{(np,nq)}_{S^3}$, where

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¹Seven months after the current article’s appearance on the arXiv, Hanselman, Rasmussen, and Watson posted a revised version of [15] with a new L-space gluing theorem subsuming the current paper’s Theorem 3.6.

²The author conceived this foliation-classification project [29] shortly before her summons to collaborate with Hanselman et al [14]. These two proofs of the graph-manifold L-space conjecture make contact with foliations via disparate mechanisms. The classification result itself is exclusive to the author’s independent work.
(i) If \( N := 2g(K) - 1 > \frac{2}{p} \) and \( K \subset S^3 \) is nontrivial, then
\[
\mathcal{L}^*_S := \begin{cases} 
\{(\infty, \ldots, \infty)\} & p > 1 \\
\bigcup_{i=1}^n \{ (\infty)^{i-1} \times [N, +\infty) \times \{\infty\}^{n-i}\} & p = 1.
\end{cases}
\]

(ii) If \( 2g(K) - 1 \leq \frac{2}{p} \), and \( K \subset S^3 \) is nontrivial, or if \( p, q > 1 \) and \( K \subset S^3 \) is the unknot (so that \( K^{(np, nq)} = T(np, nq) \)), then for \( N_{pq} := pq - p - q + 2g(K)p \), we have
\[
\mathcal{L}^*_S := \mathcal{L}^*_S \cup (\mathcal{R}_S \setminus \mathcal{Z}_S) \cup \mathcal{L}^+_S;
\]
\[
\mathcal{R}_S \setminus \mathcal{Z}_S = \prod_{i=1}^n \left( [\infty, pq) \cup (pq, +\infty) \right)^{i-1} \times \{pq\} \times ([\infty, pq) \cup (pq, +\infty))^{n-i},
\]
\[
\mathcal{L}^{-}_S = [-\infty, pq) \setminus [-\infty, N_{pq}]^n, \quad \mathcal{L}^+_S = (pq, +\infty)^n.
\]

Remarks. Positive L-space knots \( K \subset S^3 \) have \( \mathcal{L}_S = [2g(K) - 1, +\infty) \) [27], Theorem 4.5 and its remark cover the remaining (redundant or less interesting) cases of negative L-space or non-L-space knots \( K \), and the fractal-boundaried case of \( K^{(np, nq)} = T(np, nq) \) (for \( p = 1 \) and \( K \) the unknot). We use ‘<’ for open endpoints and implicitly intersect intervals with \( \mathbb{Q} \cup \{\infty\} \).

Example. Cables. For \( n = 1 \), \( \Lambda \) is trivial and \( \mathcal{R} \setminus \mathcal{Z} = \{pq\}. \) Thus Theorem 1.2 yields
\[
\mathcal{L}^*_S(Y^{(p,q)}) = [N_{pq}, pq) \cup \{pq\} \cup \{pq, +\infty\} = [N_{pq}, +\infty) = [2g(K^{(p,q)}) - 1, +\infty]
\]
for \( K \subset S^3 \) a nontrivial positive L-space knot with \( 2g(K) - 1 \leq \frac{2}{p} \), and (i) \( \mathcal{L}^*_S(Y^{(p,q)}) = \{\infty\} \) for \( 2g(K) - 1 > \frac{2}{p} \) recovering well-known results of Hedden [17] and Hom [18] for cables.

Topology of \( \mathcal{L}(Y^{(np,nq)}) \). An appropriate real completion \( \mathcal{L}^\mathbb{R} \) of \( \mathcal{L} \) has interesting topology.

Theorem 1.3. Take \( K, K^{(np,nq)} \subset S^3, \mathcal{L}, \mathcal{N}, \mathcal{L}_N, N, \) and \( \Lambda \) as in Theorem 1.2, with \( K \) nontrivial in case (i), let \( B \) be the set of rational longitudes (as in (4)) of the exterior \( Y^{(np,nq)} \) of \( K^{(np,nq)} \), and let \( \mathcal{L}^\mathbb{R} := \overline{\mathcal{L}} \subset \bigcap_{i=1}^n \mathbb{P}(H_1(\partial Y^{(np,nq)}; \mathbb{R})) \) as discussed in Section 5.1.

(i.a) If \( N > \frac{2}{p} \) and \( p > 1 \), or if \( N > \frac{2}{p} + 1 \) and \( p = 1 \), then \( \mathcal{L}^\mathbb{R} \) deformation retracts onto \( \Lambda \).

(i.b) If \( N = \frac{2}{p} + 1 \) and \( n > 2 \), then \( \mathcal{B}^{(2)} \) is contractible, of dimension \( 1 \) or \( n \).

(ii) If \( N < \frac{2}{p} \) or if \( K^{(np,nq)} = T(np,nq) \) with \( p, q > 1 \), then \( \mathcal{L}^\mathbb{R} \) deformation retracts onto an \( (n-1) \)-torus \( T^{n-1} \) parallel to \( \mathcal{B} \approx T^{n-1} \) in \( \bigcap_{i=1}^n \mathbb{P}(H_1(\partial Y^{(np,nq)}; \mathbb{R})) \approx T^n \).

S^3 and SF slope bases. The \( S^3 \) subscript on, for example, \( \mathcal{L}^*_S \), specifies the conventional \( S^3 \) surgery basis for slopes. On Seifert fibered \( M \), the \( \text{SF-slopes} \) \( \mathcal{S}^{SF} := \bigcap_{i=1}^n \mathbb{P}(H_1(\partial M; Z)^{SF}) \) mimic the Seifert-data fractions \( \frac{2}{3} \). For \( Y^{(np,nq)} \), we change slope basis via
\[
\psi : (\mathbb{Q} \cup \{\infty\})^{n}_{SF} \rightarrow (\mathbb{Q} \cup \{\infty\})^{n}_{S^3}, \quad y \mapsto \left(pq + \frac{1}{y_1}, \ldots, pq + \frac{1}{y_n}\right).
\]

We briefly pause here to elaborate on the distinguished subsets \( \mathcal{R}, \mathcal{Z}, \mathcal{B}, \) and \( \Lambda \) of SF-slopes. Reducible slopes \( \mathcal{R} \) and \( \mathcal{Z} \). Dehn filling along the smooth-fiber slope \( \infty \in (\mathbb{Q} \cup \{\infty\})_{SF} \) decomposes a Seifert fibered space as a connected sum, with one summand for each exceptional fiber or boundary component. Once this has occurred, any additional \( \infty \)-fillings create \( S^1 \times S^2 \) summands. Thus, since \( \psi : \infty \mapsto pq + \frac{1}{\infty} = pq \), our reducible slopes \( \mathcal{R} \) and their exceptional
subset $\mathcal{Z} = \mathcal{R} \cap B$ yielding $S^1 \times S^2$ summands (see Definitions 2.4, 2.5, and 2.6) satisfy

$$\mathcal{R}_S(Y^{(np,nq)}) = \bigcup_{i=1}^{n} \{ \alpha \in (\mathbb{Q} \cup \{ \infty \})^3_{\mathcal{R}} | \alpha_i = pq \},$$

$$\mathcal{Z}_S(Y^{(np,nq)}) = \bigcup_{i<j} \{ \alpha \in (\mathbb{Q} \cup \{ \infty \})^3_{\mathcal{Z}} | \alpha_i = \alpha_j = pq \},$$

where the permutation group $\mathfrak{S}_n$ acts on $S^3$-slopes by reordering boundary components: $\sigma \cdot \alpha = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$.

Rational longitudes $B$. Let $B(M)$ denote the set of rational longitudes of $M$, that is, the slopes yielding Dehn fillings with $b_1 > 0$. In particular, for $\partial M$ with $n$ components, we have

$$Z = \mathcal{R} \cap B, \quad B^R \cong \mathbb{T}^{n-1} \hookrightarrow \bigsqcup_{i=1}^{n} \mathbb{P}(H_1(\partial_i M; \mathbb{R})) \cong (\mathbb{R} \cup \{ \infty \})^n \cong \mathbb{T}^n. \quad (3)$$

For the exterior $Y^{(np,nq)}$ of $K^{(np,nq)} \subset S^3$, Proposition 2.7 tells us $B_{sf}$ is the closure

$$B_{sf}(Y^{(np,nq)}) = \left\{ y \in (\mathbb{Q} \cup \{ \infty \})^n_{sf} \left| \frac{1}{pq} + \sum_{i=1}^{n} y_i = 0 \right. \right\} \quad (4)$$

of the linear subspace $\{ y \in \mathbb{Q}^{n}_{sf} \left| \frac{1}{pq} + \sum_{i=1}^{n} y_i = 0 \right. \} \subset (\mathbb{Q} \cup \{ \infty \})^n_{sf}$. The change of slope-basis $\psi$ transforms $B_{sf}(Y^{(np,nq)})$ into a degree-$n$ hypersurface in $(\mathbb{Q} \cup \{ \infty \})^n_{\psi}$. For example, the $n = 2$ case yields the conic $B_{sf}(Y^{(2p,2q)}) = \{ \alpha \in (\mathbb{Q} \cup \{ \infty \})^3_{\psi} | \alpha_1 \alpha_2 = (pq)^2 \}$, as in Figure 1.

Symmetry by $\Lambda$. The lattice $\Lambda_{sf}(Y^{(np,nq)}) \subset (\mathbb{Q} \cup \{ \infty \})^n_{sf}$ of Seifert-data reparametrizations,

$$\Lambda_{sf}(Y^{(np,nq)}) := \{ l \in \mathbb{Z}^n \subset (\mathbb{Q} \cup \{ \infty \})^n_{sf} \left| \sum_{i=1}^{n} l_i = 0 \right. \}, \quad (5)$$

is...
acts on $\mathbf{sf}$-slopes by addition, $y \mapsto l + y$, thereby determining an action of $\Lambda$ on slopes in any basis. In particular, in Theorem 1.2, $\psi$ induces an action of $\Lambda_{\mathbf{sf}}(Y(\frac{p}{q}, nq))$ on $(\mathbb{Q} \cup \{\infty\})^n_{\mathbb{S}^3}$, via

$$l \cdot \alpha = \psi(l + \psi^{-1}(\alpha)), \quad l \in \Lambda_{\mathbf{sf}}(Y(\frac{p}{q}, nq)), \quad \alpha \in (\mathbb{Q} \cup \{\infty\})^n_{\mathbb{S}^3}. \quad (6)$$

The action of $\Lambda$ induces homeomorphism on Dehn surgeries: $S^3_{l, \alpha}(K(\frac{p}{q}, nq)) \cong S^3_{\alpha}(K(\frac{p}{q}, nq))$ for $l \in \Lambda(Y(\frac{p}{q}, nq))$ and $\alpha \in (\mathbb{Q} \cup \{\infty\})^n_{\mathbb{S}^3}$. Thus $\Lambda$ preserves $\mathcal{R}$, $\mathcal{Z}$, $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{NL}$ as sets, and since $S^3_{\alpha}(K(\frac{p}{q}, nq)) \cong S^3_{\alpha}(K(\frac{p}{q}, nq))$, $\mathcal{L}_{\mathbf{sf}}^*$ completely catalogs the L-space surgeries on $K(\frac{p}{q}, nq)$.

The L-space surgery slopes for $K(\frac{p}{q}, nq)$ must retain their full $\Lambda$-orbits, but $\mathbf{sf}$-slopes do this automatically: the expression of $\mathcal{L}_{\mathbf{sf}}(Y(\frac{p}{q}, nq))$ in Theorem 4.5 is naturally $\Lambda$-invariant. Moreover, $\mathcal{L}_{\mathbf{sf}}^*(Y(\frac{p}{q}, nq))$ is almost $\Lambda$-invariant. When $\mathcal{L}_{\mathbf{sf}}^* = \{\infty\}$, $\mathcal{L}_{\mathbf{sf}}^* \setminus \mathcal{L}_{\mathbf{sf}}^*$ is given by

$$(\Lambda_{\mathbf{sf}} \cdot \{\infty\}) \setminus \{\infty\} = \Lambda_{\mathbf{sf}} \cap (\mathbb{Q}^n_{\mathbb{S}^3})^n \subset (\{pq - 1, pq\} \cup \{pq, pq + 1\})^n. \quad (7)$$

For any $K(\frac{p}{q}, nq)$ in Theorem 1.2, the $\Lambda$-mismatch $\mathcal{L}_{\mathbf{sf}}^* \setminus \mathcal{L}_{\mathbf{sf}}^*$ lies inside a radius-1 neighborhood,

$$\mathcal{L}_{\mathbf{sf}}^* \setminus \mathcal{L}_{\mathbf{sf}}^* \subset \bigcup_{i=1}^n \{\alpha \mid \alpha_i \in [pq - 1, pq] \cup \{pq, pq + 1\} \} =: U_{pq}^*(1), \quad (8)$$

of $\mathcal{R}_{\mathbf{sf}} = \bigcup_{i=1}^n \{\alpha \mid \alpha_i = pq\}$, and $\mathcal{L}_{\mathbf{sf}}^*$ consists of a finite union of rectangles outside any positive-radius neighborhood of $\mathcal{R}_{\mathbf{sf}}$. Moreover, (8) implies the integer slopes in $\mathcal{L}_{\mathbf{sf}}^*(Y(\frac{p}{q}, nq))$ satisfy

$$\mathcal{L}_{\mathbf{sf}}^* \cap (\mathbb{Z} \cup \{\infty\})^n \subset \mathcal{L}_{\mathbf{sf}}^* \cup \mathcal{S}_i^* \left\{\{pq - 1, pq + 1\} \times (\mathbb{Z} \cup \{\infty\})^n\right\}, \quad (9)$$

and it is a simple exercise to determine $\alpha \in (\mathcal{L}_{\mathbf{sf}}^* \setminus \mathcal{L}_{\mathbf{sf}}^*) \cap (\mathbb{Z} \cup \{\infty\})^n$ with $\alpha_i \in \{pq - 1, pq + 1\}$.

New features: Torus-link satellites versus One-strand Cables. L-space regions for satellites by $n > 1$ torus links introduce qualitatively new phenomena not present for $n = 1$ cables.

(a) For $n > 1$, the action of $\Lambda$ becomes nontrivial, although as discussed before, this action does not impact actual L-spaces resulting from surgery.

(b) For $n > 1$, the codimension-1 subspace $\mathcal{R}_{\mathbf{sf}} \subset (\mathbb{Q} \cup \{\infty\})^n_{\mathbb{S}^3}$ acquires positive dimension and the codimension-2 subspace $\mathcal{Z}_{\mathbf{sf}} \subset (\mathbb{Q} \cup \{\infty\})^n_{\mathbb{S}^3}$ becomes nonempty, although the set of reducible L-space slopes $\mathcal{R}_{\mathbf{sf}} \setminus \mathcal{Z}_{\mathbf{sf}}$ remains a disjoint union of hyperplanes $\cong \mathbb{Q}^{n-1}$ for all $n$.

(c) For $n = p = 1$, both the L-space regions $\mathcal{L}_{\mathbf{sf}}(S^3(\tilde{\nu}(K))) = [N, \infty] = [N_{1q}, \infty] = \mathcal{L}_{\mathbf{sf}}(Y(1, q))$ and the spaces of resulting L-space surgeries $S^3_L(K(\frac{p}{q}, nq))$ and $K(\frac{p}{q}, nq)$ are identical, since the $p = 1$ cable affects framing without changing the knot. For $n > 1$, however, the relationship between $S^3_L(K)$ and $S^3_L(K(\frac{n}{q}))$ depends on the difference

$$(2g(K) - 1) - \frac{2}{p};$$

$$\begin{align*}
(\approx) \quad & S^3_L(K(\frac{n}{q})) = S^3_L(K) \text{ when } 2g(K) - 1 > q, \\
(\subset) \quad & S^3_L(K(\frac{n}{q})) \subset S^3_L(K) \text{ when } 2g(K) - 1 = q, \\
(\supset) \quad & S^3_L(K(\frac{n}{q})) \supset S^3_L(K) \text{ when } 2g(K) - 1 < q.
\end{align*}$$

(d) For $n = 1$, $\mathcal{L}(Y(\frac{p}{q}, q)) := \mathcal{Z}(Y(\frac{p}{q}, q)) \setminus \mathcal{B}(Y(\frac{p}{q}, q))$ is contractible and of dimension 0 or 1. For $n > 1$, however, Theorem 1.3 catalogs six distinct topologies that occur for $\mathcal{L}(Y(\frac{p}{q}, q))^\mathbb{R}$, including

1. an infinite disjoint union of points — (i.a), $p \neq 1$;
2. an infinite disjoint union of contractible 1-dimensional spaces — (i.a), $p = 1$;
3. a connected 1-dimensional space with $b_1 = \infty$ — (i.b);
4. a contractible, 1-dimensional space with $S^1$ closure — (i.c), $2g(K) - 1 = \frac{2}{p} + 1$, $n = 2$;
5. a contractible $n$-dimensional space — (i.e), $2g(K) - 1 = \frac{2}{p};$
6. an $n$-dimensional space that deformation retracts onto $\mathbb{T}^{n-1}$ — (ii).
1.2. The L-space Conjecture

The L-space conjectures, stated formally by Boyer–Gordon–Watson [5] and Juhász [20], posit the existence of left-invariant orders on fundamental groups and of co-oriented taut foliations, respectively, for all prime, compact, oriented non-L-spaces.

For $Y$ a compact oriented 3-manifold with torus boundary, let $\mathcal{F}(Y) \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$ denote the space of slopes $\alpha \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$ for which $Y$ admits a co-oriented taut foliation (CTF) restricting to a product foliation of slope $\alpha$ on $\partial Y$. Along a similar vein, we define $\mathcal{LO}(Y) := \{ \alpha \in \mathbb{P}(H_1(\partial Y; \mathbb{Z})) | \pi_1(Y(\alpha)) \text{ is LO} \}$, where LO stands for left-orderable.

**Theorem 1.4.** Take $K, K^{(np,nq)} \subset S^3$ as in Theorem 1.2, with $K$ nontrivial and $p > 1$.

(LO) Suppose $\mathcal{LO}(Y) \supset \mathcal{NL}(Y)$, for $Y := S^3 \setminus \mathring{o}(K).

(LO.i) If $2g(K) - 1 > \frac{q+1}{p}$, then $\mathcal{LO}(Y^{(np,nq)}) = \mathcal{NL}(Y^{(np,nq)})$.

(LO.ii) If $2g(K) - 1 < \frac{q}{p}$, then $\mathcal{LO}(Y^{(np,nq)}) \supset \mathcal{NL}(Y^{(np,nq)}) \setminus \Lambda(Y^{(np,nq)}) \cdot (\{ -\infty, N_{pq} \}^n \setminus \{ -\infty, N_{pq} - p \}^n)$.

(CTF) Suppose $\mathcal{F}(Y) = \mathcal{NL}(Y)$.

(CTF.i) If $2g(K) - 1 > \frac{q+1}{p}$, then $\mathcal{F}(Y^{(np,nq)}) = \mathcal{NL}(Y^{(np,nq)}) \setminus \mathcal{R}(Y^{(np,nq)})$.

(CTF.ii) If $2g(K) - 1 < \frac{q}{p}$, then $\mathcal{F}(Y^{(np,nq)}) \supset (\mathcal{NL}(Y^{(np,nq)}) \setminus \mathcal{R}(Y^{(np,nq)})) \setminus \Lambda(Y^{(np,nq)}) \cdot (\{ -\infty, N_{pq} \}^n \setminus \{ -\infty, N_{pq} - p \}^n)$.

One might notice that our sharper results here lie in case (i) $2g(K) - 1 > \frac{q}{p}$ of Theorem 1.2. While this case is the less interesting one from the standpoint of L-space production, it is the more nontrivial one from the standpoint of the L-space conjecture, since in this case every non-$S^3$ surgery on $K^{(np,nq)}$ has nontrivial reduced Heegaard Floer homology.

For the $2g(K) - 1 < \frac{q}{p}$ case, the difficulty with slopes $\alpha \in (\{ -\infty, N_{pq} \}^n \setminus \{ -\infty, N_{pq} - p \}^n)$ is that the existence of a CTF on $Y^{(np,nq)}(\alpha)$ depends on the family of suspension foliations on $\partial Y$ — necessarily of nontrivial holonomy — that arise from taking a CTF $F$ of slope $2g(K) - 1$ on $Y$ and restricting $F$ to $\partial Y$. Such $F$ can only be extended over the union $Y^{(np,nq)}$ if it matches with the boundary restriction of some CTF of $S^3$-slope $\frac{p^2 - N_{pq}^2}{q - N_{pq}}$ on the Seifert fibered space glued to $Y$ to form the satellite. A similar phenomenon occurs for LOs on the fundamental group of $Y^{(np,nq)}(\alpha)$. See Boyer and Clay [4] for more on this subtlety in gluing behavior.

In Theorem 8.1 of Section 8, we prove a result analogous to the one above, but for satellites by algebraic links or iterated torus-links. Instead of restating this theorem here, we state a:

**Corollary 1.5.** For $K \subset S^3$ a positive L-space knot with exterior $Y$, suppose $K^\Gamma \subset S^3$ is an algebraic link satellite or iterated torus-link satellite of $K \subset S^3$, with $2g(K) - 1 \neq \frac{q_{+1} {p}}{p}$ at the root torus-link-satellite operation of $\Gamma$, such that $K^\Gamma \subset S^3$ has no L-space surgeries besides $S^3$.

(LO) If $\mathcal{LO}(Y) = \mathcal{NL}(Y)$, then every non-$S^3$ surgery on $K^\Gamma \subset S^3$ has LO fundamental group.

(CTF) If $\mathcal{F}(Y) = \mathcal{NL}(Y)$, then every irreducible non-$S^3$ surgery on $K^\Gamma \subset S^3$ admits a CTF.

In [28], Rasmussen and the author conjectured that our L-space gluing theorem (see Theorem 3.5) also holds without the hypothesis of admitting more than one L-space Dehn filling. Hanselman, Rasmussen, and Watson recently announced a proof of this conjecture in [15], implying that the above corollary also holds for any non-L-space knot $K \subset S^3$.

1.3. Satellites by algebraic links

In the context of negative definite graph manifolds, the distinction between L-space and non-L-space has consequences for algebraic geometry.
Némethi recently showed that the unique negative-definite graph manifold \( \text{Link}(X, \circ) \) bounding the germ of a normal complex surface singularity \((X, \circ)\) is an L-space if and only if \((X, \circ)\) is rational \([24]\). Due to results of the author in \([29]\), we can promote this statement to a relative version: the subregion \( \mathcal{L}^{\circ} \subset \mathcal{L}(Y^\Gamma) \) of negative-definite L-space Dehn filling slopes for a graph manifold \( Y^\Gamma \) parameterizes, up to equisingular deformation, the rational surface singularities \((X, \circ)\) admitting ‘end curves’ \((C, \circ) \subset (X, \circ)\) (see \([25]\)), such that \( \text{Link}(X \setminus C) = Y^\Gamma \). If one such \((X, \circ)\) is \((\mathbb{C}^2, 0)\), then \( Y^\Gamma \) is the exterior of an algebraic link, motivating the following study.

**Setup.** Although a \( T(np, nq)\)-satellite operation is specified by \((\text{an unknot complement in})\) the Seifert fibered exterior of \( T(np, nq) \) determined by the triple \((p, q, n)\), a sequence of torus-link-satellite operations is specified by a rooted tree \( \Gamma \) determining the graph manifold exterior of \( T \in E \) set \( j \lambda \). Since we direct the edges of \( \Gamma \) rootward, each vertex \( v \in \text{Vert}(\Gamma) \) specifies the Seifert fibered \( T_v := T(n_v p_v, n_v q_v) \)-exterior in \( S_v^3 \), determined by the triple \((p_v, q_v, n_v)\), so that \( S_v^3 \) has two exceptional fibers \( \lambda_v^{-1} \) and \( \lambda_v^0 \) of respective multiplicities \( p_v \) and \( q_v \), and components of \( T_v \) are regular fibers in \( S_v^3 \).

Since we direct the edges of \( \Gamma \) rootward, each vertex \( w \) has a unique outgoing edge \( e_w \), corresponding to the incompressible torus in whose neighborhood \( T_w \) is embedded, or equivalently, to a gluing map \( \phi_{e_w} \) splicing the multiplicity-\(q_v\) fiber \( \lambda_v^0 \in S_v^3 \) to a Seifert fiber in \( S_v^3 \), for \( u := v(e_w) \), where we write \( v(e) \) to denote the vertex on which an edge \( e \in \text{Edge}(\Gamma) \) terminates. (A ‘splice’ is a type of toroidal connected sum exchanging meridians with longitudes.)

There are only two types of fiber in \( S_v^3 \) available for splicing: a regular fiber, which we then regard as one of the \( n_v \) components \( f_j^v \in S_v^3 \) of \( T_v \), or the multiplicity-\( p_v \) fiber \( \lambda_v^0 \in S_v^3 \). When \( \phi_{e_w} \) splices \( \lambda_v^0 \in S_v^3 \) to some \( j \)-th component \( f_j^u \in S_u^3 \) of \( T_u \), we call \( \phi_{e_w} \) a smooth splice, set \( j(e) := j \in \{1, \ldots, n_v\} \), and declare the JSJ component \( Y_u \) at \( u \) to be the exterior of \( \lambda_u^0 \mathbin{\#} T_u \) in \( S_u^3 \). When \( \phi_{e_w} \) splices \( \lambda_v^0 \in S_v^3 \) to \( \lambda_v^{-1} \in S_v^3 \), we call \( \phi_{e_w} \) an exceptional splice, set \( j(e) = -1 \), and define \( Y_u \) to be the exterior of \( \lambda_u^{-1} \mathbin{\#} \lambda_u^0 \mathbin{\#} T_u \) in \( S_u^3 \). Since this latter splice could be redefined as a smooth one if \( p_u = 1 \), we demand \( p_u > 1 \) without loss of generality.

If we define \( J_v := j(E_{in}(v)) \cap \{1, \ldots, n_v\} \) and its complement \( I_v \) by

\[
J_v := j(E_{in}(v)) \cap \{1, \ldots, n_v\}, \quad I_v := \{1, \ldots, n_v\} \setminus J_v,
\]

then \( I_v \) catalogs the boundary components of \( Y_u \) left unfilled, forming the exteriors of link components, so that the total satellite \( K^\Gamma \subset S^3 \) of \( K \subset S^3 \) has \( \sum_{v \in \text{Vert}(\Gamma)} |I_v| \) components. The pattern link specified by \( \Gamma \) is then an algebraic link if and only if its graph manifold exterior is negative definite, which, by straightforward calculations as appear, for example, in Eisenbud and Neumann’s book \([8]\), is equivalent to the condition that \( \Gamma \) is a tree, and that

\((i)\) \( p_v, q_v, n_v > 0 \) for all \( v \in \text{Vert}(\Gamma) \),

\((ii)\) \( \Delta_e > 0 \) for all \( e \in \text{Edge}(\Gamma) \),

\[
\Delta_e := \begin{cases} p_v q_v - p_v q_v & j(e) = -1, \\ q_v - p_v q_v & j(e) \neq -1. \end{cases}
\]

Conversely, given an isolated planar complex curve singularity \((\circ, C) \subset (0, \mathbb{C}^2)\), one can obtain such a tree \( \Gamma \) from Newton–Puisseux expansions for the defining equations of \( C \), or alternatively from the amputated splice diagram of the dual plumbing graph of \( \check{X} \) for a good embedded resolution \((X, \check{C}) \to (\mathbb{C}^2, C)\). Again, see \([8]\) for details.

If \( Y^\Gamma \) denotes the exterior of the \( \Gamma \)-satellite \( K^\Gamma \subset S^3 \) of \( K \subset S^3 \), then for each JSJ component \( Y_v \) of \( Y^\Gamma \), we again have reducible and exceptional subsets \( R_v \), \( Z_v \in \prod_{v \in J_v} \mathbb{P}(H_1(\partial Y_v; \mathbb{Z})) \), along with a lattice \( \Lambda_v \) acting on \( \prod_{v \in I_v} \mathbb{P}(H_1(\partial Y_v; \mathbb{Z})) \) by addition of \( \mathbb{Z} \)-v-slopes.
Theorem 1.6. Suppose $Y^\Gamma := S^3 \setminus \mathcal{N}(K^\Gamma)$ is the exterior of an algebraic-link satellite $K^\Gamma \subset S^3$ of a (possibly trivial) positive L-space knot $K \subset S^3$, and suppose the triple $(p_r, q_r, n_r)$ specifies the initial torus-link satellite operation, occurring at the root vertex $r \in \text{Vert}(\Gamma)$.

(i.a) If $K$ is nontrivial, $\frac{q_r}{p_r} < 2g(K) - 1$, $p_r > 1$, and $-1 \notin j(E_{in}(r))$, then

$$\mathcal{L}(Y^\Gamma) = \Lambda_{\Gamma}; \quad Y^\Gamma(\mathcal{L}) = S^3 \text{ for all } \mathcal{L} \in \mathcal{L}(Y^\Gamma).$$

(i.b) If $K$ is nontrivial, $\frac{q_r}{p_r} < 2g(K) - 1 =: N$, and $p_r = 1$, then

$$\mathcal{L}_{S^3}(Y^\Gamma) = \left(\Lambda_{\Gamma} \cdot \mathcal{S}_{|L_r|}(\{N, +\infty\} \times \{\infty\}) \times \prod_{e \in E_{in}(r)} \Lambda_{\Gamma_{v(-e)}}\right) \prod_{e \in E_{in}(r)} \left(\mathcal{L}_{S^3}(Y^\Gamma_{v(-e)}) \times \Lambda_{\Gamma \setminus \Gamma_{v(-e)}}\right),$$

where $Y^\Gamma_{v(-e)}$ denotes the exterior of the $\Gamma_{v(-e)}$-satellite of $K \subset S^3$.

(ii) If $K$ is trivial, or if $K$ is nontrivial with $\frac{q_r}{p_r} \geq 2g(K) - 1$ and $q_r > 2g(K) - 1$, then

$$\mathcal{L}_{S^3}(Y^\Gamma) \supset \prod_{v \in \text{Vert}(\Gamma)} \left(\mathcal{L}_{S^3}^{\text{min}} - \cup R_v \cup Z_v \cup \mathcal{L}_{S^3}^{\text{min} +}\right),$$

where

$$\mathcal{L}_{S^3}^{\text{min} +} := \left\{ \mathcal{L} \in \mathcal{L} \mid \sum_{e \in I_v} |y^v_e| \geq 0 \right\},$$

$$\mathcal{L}_{S^3}^{\text{min} -} := \left\{ \mathcal{L} \in \mathcal{L} \mid \sum_{e \in I_v} |y^v_e| \leq m^v_e \right\} \setminus \left\{ \begin{array}{l} \sum |y^v_e| = 0, \\
\sum [y^v_e] = 0 \end{array} \right\} J_v = 0; \quad j(e_v) \neq -1$$

otherwise,

$$m^v_e := -\sum_{e \in E_{in}(v)} \left\{ \begin{array}{cl} 1 & j(e) \neq -1 \\
\frac{p_{v(-e)}}{p_r \Delta_e} + 1 & j(e) = -1 \end{array} \right\} - \left\{ \begin{array}{cl} 1 & J_v = 0; \quad j(e_v) \neq -1 \\
\frac{p_{v(e_v)}}{p_r \Delta_{e_v}} + 1 & j(e_v) = -1 \end{array} \right\} 0$$

otherwise.

This is not as horrible as it looks. Part (i.a) describes the case in which all L-space surgeries yield $S^3$. For part (i.b), either we trivially refill all components of $Y^\Gamma$ except for one exterior component in the root, effectively replacing $K^\Gamma$ with $K$; or, we trivially refill all exterior components but those of $Y^\Gamma_{v(-e)}$ for some incoming edge $e$, replacing $K^\Gamma$ with some $K^\Gamma_{v(-e)}$.

The notation $\Lambda_{\Gamma_v}$ and the term ‘trivially refill’ hide a subtlety, however. For both the above theorem and for Theorem 1.7 for iterated torus-link satellites, we define $\Lambda_{\Gamma_v}$ to mean

$$\Lambda_{\Gamma_v} = \left\{ \mathcal{L} \mid Y^\Gamma_0(\mathcal{L}) = S^3 \right\}, \quad Y^\Gamma_0 := \text{exterior of the } \Gamma_v\text{-satellite of the unknot.} \quad (11)$$

While $\Lambda_{\Gamma_v} \supset \prod_{v \in \text{Vert}(\Gamma_v)} \Lambda_u$, the two sets need not be equal. Similarly, if $N > \frac{q_r}{p_r}$ and $j(e') = -1$ for some $e' \in E_{in}(r)$, then $\mathcal{L}(Y^\Gamma) \supset \mathcal{L}(Y^\Gamma_{v(-e')}) \times \Lambda_{\Gamma \setminus \Gamma_{v(-e')}}$, but the containment can be proper. Section 8.2 provides a more explicit characterization of $\mathcal{L}(Y^\Gamma)$ in this case, along with a concrete description of $\Lambda_{\Gamma_v}$ in the case of iterated torus-link satellites.

Part (ii) of Theorem 1.6 is analogous to Theorem 1.2(ii) for torus-link satellites, but this similarity is masked by our transition from $S^3$-slopes to sf-slopes. For example, if $e$ is a leaf and its emanating edge $e_v$ does not correspond to an exceptional splice, then we have

$$\mathcal{L}_{S^3}^{\text{min} +} = \Lambda_v \cdot (p_v q_v, +\infty)_{|L_e|}, \quad \mathcal{L}_{S^3}^{\text{min} -} = \Lambda_v \cdot (-\infty, p_v q_v)_{|L_e|} \setminus (-\infty, p_v q_v - q_v + p_v)_{|L_e|}.$$
In all other cases, we still have $L_{SFv}^{\min+} = \Lambda_v \cdot <p_vq_v, +\infty>^{[J_v]}$, but $L_{SFv}^{\min-}$ now sits inside the unit-radius neighborhood of $R_v$, with $m_v^-$ providing a measure of how deeply inside it sits.

In the set $\psi_v^-([\infty, p_vq_v - q_v + p_v > [J_v])$ removed from $L_{SFv}^{\min-}$ at a smoothly spliced leaf $v$ (in part (ii) of the theorem), the notation $[\cdot] : \mathbb{Q} \to \mathbb{Z}, [x] := x - \lfloor x \rfloor$ gives the fractional part of a rational number $x$, whereas the notation $[a]_b := a - \lfloor \frac{a}{b}\rfloor b$ picks out the smallest nonnegative representative of $a \mod b$ for any integers $a, b \in \mathbb{Z}$. For a suitable L-space region approximation when $q_v = 2(K) - 1$ and $p_v = 1$, see line (197) and the associated remark.

1.4. Iterated torus-link-satellites

For the case of iterated torus-link satellites, we only allow ‘smooth splice’ edges, corresponding to the original type of torus-link satellite operation. We also drop the algebraicity condition that $\Delta_\circ > 0$, and while we keep all $p_v, n_v > 0$ without loss of generality, we allow $q_v < 0$ but demand $q_v \neq 0$, for each $v \in \text{Vert} (\Gamma)$.

**Theorem 1.7.** Suppose $Y^{\Gamma} := S^3 \setminus \nu(K^{\Gamma})$ is the exterior of an iterated torus-link satellite $K^{\Gamma} \subset S^3$ of a (possibly trivial) positive L-space knot $K \subset S^3$, and suppose the triple $(p_v, q_v, n_v)$ specifies the iterated torus-link satellite operation occurring at the root vertex $r \in \text{Vert} (\Gamma)$.

(i) If $K$ is nontrivial, $\frac{p_v}{q_v} < 2g(K) - 1$, $p_r > 1$, then $\mathcal{L}(Y^{\Gamma}) = \Lambda_r$.

(ii) If $K$ is nontrivial, $\frac{p_v}{q_v} < 2g(K) - 1 := N$, and $p_r = 1$, then

$$
\mathcal{L}_{SFv}^{S^3}(Y^{\Gamma}) = \Lambda_v \cdot \mathcal{E}_{|L_v|}([N, +\infty] \times [\infty, |J_v| - 1]) \times \prod_{e \in E_{in}(r)} \Lambda_{\Gamma_v(e)} \times \prod_{e \in E_{in}(r)} \mathcal{L}_{SFv}^{S^3}(Y^{\Gamma_v(e)}) \times \Lambda_{\Gamma_v \setminus \Gamma_{v(e)}},
$$

where $Y^{\Gamma_v(e)}$ denotes the exterior of the $\Gamma_v(e)$-satellite of $K \subset S^3$.

If $K$ is trivial, or if $K$ is nontrivial with $\frac{p_v}{q_v} \geq 2g(K) - 1$ and $q_r > 2g(K) - 1$, then

$$
\mathcal{L}_{SFv}^{S^3}(Y^{\Gamma}) \supset \prod_{v \in \text{Vert} (\Gamma)} \left( \mathcal{L}_{SFv}^{\min-} \cup R_v \setminus Z_v \cup \mathcal{L}_{SFv}^{\min+} \right),
$$

where

$$
\mathcal{L}_{SFv}^{\min+} := \left\{ \mathcal{L}_{SFv}^{\min+} : \sum_{i \in L_v} |y^i_v| \geq m_{v^+}^+ \right\}
$$

and

$$
\mathcal{L}_{SFv}^{\min-} := \left\{ \mathcal{L}_{SFv}^{\min-} : \sum_{i \in L_v} |y^i_v| \leq m_{v^-}^- \right\}
$$

with $m_{v^+}^+ := - \sum_{e \in E_{in}(e)} \left( \frac{p_v(e)}{q_v(e)} \right) - 1 + \left\{ \begin{array}{ll} 2 & q_v = -1 \\ 1 & J_v \neq 0; q_v < -1 or \frac{p_v}{q_v} > 1, \\ 0 & otherwise \end{array} \right.$

and $m_{v^-}^- := - \sum_{e \in E_{in}(e)} \left( \frac{p_v(e)}{q_v(e)} \right) + 1 - \left\{ \begin{array}{ll} 2 & q_v = +1 \\ 1 & J_v \neq 0; q_v > 1 or \frac{p_v}{q_v} < -1, \\ 0 & otherwise \end{array} \right.$
Monotonicity. In both of the above theorems, \( \prod_{v \in \text{Vert}(\Gamma)}(L_{\text{mono}}^{\min} - \mathcal{R}_v \cup \mathbb{Z}_v \cup L_{\text{mono}}^{\min}) \) is a component of what we call the monotone stratum \( L_{\text{mono}}^{\text{mono}}(Y^\Gamma) \) of \( L_{\text{mono}}(Y^\Gamma) \), as discussed in Section 7.6. We say that \( y^\Gamma \in L_{\text{mono}}(Y^\Gamma) \) is monotone at \( v \in \text{Vert}(\Gamma) \) if

\[
\in \phi^v_{\mathcal{E}_{v(-e)}(Y^\Gamma_v_{v(-e)}(\Gamma v_{v(-e)}^{(-e)}))} \forall e \in E_{\text{in}}(v) \quad \text{and} \quad \in \phi^v_{L_{\text{mono}}(Y^\Gamma_v v_{v}^{(-e)}),}
\]

or more prosaically (when the above interval interiors are nonempty) is monotone at \( v \) if

\[
y^v_{j(e)+} \leq y^v_{j(e)}\quad \forall e \in E_{\text{in}}(v), \quad y^v_{j(e)+} \leq y^v_{j(e)-},
\]

as these are the respective endpoints of the above intervals. The monotone stratum \( L_{\text{mono}}^{\text{mono}}(Y^\Gamma) \) of \( L_{\text{mono}}(Y^\Gamma) \) is the set of slopes \( y^\Gamma \in L_{\text{mono}}(Y^\Gamma) \) such that \( y^\Gamma \) is monotone at all \( v \in \text{Vert}(\Gamma) \).

Specifying different collections of local monotonicity conditions allows one to decompose an L-space region into strata of disparate topologies. For example, for the (globally) monotone stratum, we have the following topological result, proved in Section 7.6.

**Theorem 1.8.** Suppose that \( K^\Gamma \subset S^3 \) is an algebraic link satellite, specified by \( \Gamma \), of a positive L-space knot \( K \subset S^3 \), where either \( K \) is trivial, or \( K \) is nontrivial with \( \frac{|K|}{|v|} > 2g(K) - 1 \).

Let \( V \subset \text{Vert}(\Gamma) \) denote the subset of vertices \( v \in V \) for which \( |I_v| > 0 \).

Then the (\( \mathcal{B}_{\text{SR}^L} \)-corrected) completed monotone stratum \( L_{\text{mono}}(Y^\Gamma)^{\mathcal{B}_{\text{SR}^L}} := L_{\text{mono}}(Y^\Gamma) \setminus \mathcal{B}_{\text{SR}^L}, \) is of dimension \( |I_v| \) and deformation retracts onto an \((|I_v| - |V|)\)-dimensional embedded torus,

\[
L_{\text{mono}}(Y^\Gamma)^{\mathcal{B}_{\text{SR}^L}} \xrightarrow{\text{def. retract}} \prod_{v \in V} T^{(|I_v| - 1)} \hookrightarrow \prod_{v \in V} (\mathbb{R} \cup \{\infty\})^{(|I_v|)},
\]

where \( T^{(|I_v| - 1)} \hookrightarrow (\mathbb{R} \cup \{\infty\})^{(|I_v|)} \) is a torus parallel to \( \mathcal{B}_{\text{SR}^L} \subset (\mathbb{R} \cup \{\infty\})^{(|I_v|)} \).

Nonmonotone strata, when they exist, change the topology of the total L-space region and have implications for ‘boundedness from below’ in the sense of Némethi and Gorsky [13], but we defer the study of nonmonotone regions to later work, whether by this author or others.

**New tools.** In fact, the propositions in Sections 6 and 7 provide many tools for analyzing questions not addressed in this paper. For example, in the absence of an exceptional splice at \( v \), Proposition 6.2(+)iii precisely characterizes when \( y^v_{\mathcal{E}_{v(-e)}} = \frac{p_v}{q_v} \), defining the right-hand boundary of the nonmonotone stratum at that component. These tools can also be used to characterize the nonproduct components of the monotone stratum more explicitly.

**New features.** Even so, Theorems 1.6 and 1.7 already reveal more interesting behavior than appears for nondegenerate torus-link satellites. In particular, the boundary of the \( S^3 \)-slope L-space region need not occur at infinity. For example, if

(a) \( \Gamma = \emptyset \) specifies a single torus-link satellite of the unknot, and \( p_v = q_v = 1 \);

(b) \( \Gamma \) specifies an iterated torus-link satellite, and \( m_v^+ < 0 \) or

(c) \( \Gamma \) specifies an iterated or algebraic-link satellite, and we restrict to an appropriate piece of \( L_{\text{mono}}(Y^\Gamma)^{\mathcal{B}_{\text{SR}^L}} \) outside the product region,

then we encounter regions of the form

\[
\psi \left( \left\{ y^v \in \mathbb{Q}^{(|I_v|)} \left| \sum_{i \in I_v} |y^v_i| \geq m_v^+ \right. \right\} \right) = \Lambda_v \cdot (p_v q_v, +\infty)^{|I_v|} \cup \Lambda_v \cdot \mathcal{E}_{|I_v|} \left( (-\infty, p_v q_v - 1)^{|I_v|} \times (p_v q_v, +\infty)^{|I_v| - |m_v^+|} \right)
\]

for \(-|I_v| \leq m_v^+ < 0\), with additional components added onto the unit-radius neighborhood of \( \mathcal{R}_v \) if \( m_v^+ < -|I_v| \). (An analogous phenomenon occurs in the negative direction when \( m_v^- > 0 \).) Thus, in such cases, the L-space region ‘wraps around’ infinity. In fact, given any \( n-, n_+ \in \mathbb{Z}_{\geq 0}, \)
it is possible to construct an iterated satellite by torus links for which the $S^3_n$ component of some stratum of the L-space region fills up the quadrant

$$\langle -\infty, p_v q_v - 1 \rangle^{n-} \times \langle p_v q_v + 1, +\infty \rangle^{n+}, \quad n_- + n_+ = |I_v|.$$ 

There likewise exist iterated torus-link satellite exteriors $Y^T$ with $u, v \in \text{Vert}(\Gamma)$ for which the projections of $L_{S^3}(Y^T)$ to the positive quadrant $< p_u q_u + 1, +\infty > |I_v|$ and to the negative quadrant $< -\infty, p_v q_v - 1 > |I_v|$ are both empty.

We therefore feel that the notion of 'L-space link' should be broadened to encompass any link whose L-space surgery region contains an open neighborhood in the space of slopes, rather than defining this notion in terms of large positive slopes in the L-space surgery region.

1.5. Organization

Section 2 establishes basic Seifert fibered space conventions and elaborates on the distinguished slope subsets $\Lambda$, $\mathcal{R}$, and $\mathcal{Z}$. Section 3 introduces notation for L-space intervals and proves a new gluing theorem for knot exteriors with graph manifolds. Section 4 introduces machinery developed by the author in [29] to compute L-space intervals for fiber exteriors in graph manifolds, and applies this to prove Theorem 4.5, an SF-slope version of the torus-link satellite results in Theorem 1.2. Section 5 addresses the topology of L-space regions and proves Theorem 1.3. Section 6 describes the graph $\Gamma$ associated to an iterated torus-link satellite, computes various estimates useful for bounding L-space surgery regions for iterated-torus-link and algebraic link satellites, and proves Theorem 1.7 for iterated-torus-link satellites. Section 7 describes adaptations of this graph $\Gamma$ to accommodate algebraic link exteriors, discusses monotonicity, and proves Theorems 1.6 and 1.8. Section 8 proves results related to conjectures of Boyer–Gordon–Watson and Juhász, including generalizations of Theorem 1.4.

Readers interested in constructing their own L-space regions for algebraic link satellites or iterated torus-link satellites should refer to the L-space interval technology introduced for graph manifolds in Section 4, and to the analytical tools developed in Section 6.

2. Basis conventions and the slope subsets $\Lambda$, $\mathcal{R}$, and $\mathcal{Z}$

Since all L-spaces are rational homology spheres, any manifold with a positive-genus Seifert fibered JSJ component is necessarily not an L-space. Thus, we shall only ever be interested in genus zero Seifert fibered spaces, and our most important convention is the following one.

**Conjecture 2.1.** All Seifert fibered spaces in this paper are of genus zero. Any usage of the term “Seifert fibered” should be taken to mean “genus zero Seifert fibered.”

Suppose $Y := M \setminus \overset{\circ}{\nu}(L)$, with boundary $\partial Y = \coprod_{i=1}^n \partial_i Y$, $\partial_i Y = \partial\nu(L_i)$, is the link exterior of an $n$-component link $L = \coprod_{i=1}^n L_i \subset M$ in a closed oriented 3-manifold $M$. Then, up to choices of sign, the Dehn filling $M$ of $Y$ specifies a (multi-)meridional class $(\mu_1, \ldots, \mu_n) \in H_1(\partial Y; \mathbb{Z}) = \bigoplus_{i=1}^n H_1(\partial_i Y; \mathbb{Z})$, where each meridian $\mu_i \in H_1(\partial_i Y; \mathbb{Z})$ is the class of a curve bounding a compressing disk of the solid torus $\nu(L_i)$. Any choice of classes $\lambda_1, \ldots, \lambda_n \in H_1(\partial Y; \mathbb{Z})$ satisfying $\mu_i \cdot \lambda_i = 1$ for each $i$ then produces a surgery basis $(\mu_1, \lambda_1, \ldots, \mu_n, \lambda_n)$ for $H_1(\partial Y; \mathbb{Z})$. We call these $\lambda_i$ surgery longitudes, or just longitudes if the context is clear.

When $M = S^3$, $H_1(\partial Y; \mathbb{Z})$ has a conventional basis given by taking each $\lambda_i$ to be the rational longitude; that is, each $\lambda_i$ generates the kernel of the homomorphism $i_\ast : H_1(\partial Y; \mathbb{Q}) \rightarrow H_1(M \setminus \overset{\circ}{\nu}(L_i); \mathbb{Q})$ induced by the inclusion $i : \partial Y \hookrightarrow M \setminus \overset{\circ}{\nu}(L_i)$. For $M = S^3$, the rational longitude coincides with Seifert-framed longitude.
It is important to keep in mind that for knots and links in $S^3$, the conventional homology basis is not always the most natural surgery basis. In particular, any cable or satellite of a knot in $S^3$ determines a surgery basis for which the surgery longitude corresponds to the Seifert longitude of the associated torus knot or companion knot. This cable surgery basis or satellite surgery basis does not coincide with the conventional basis for $S^3$.

For $Y$ a compact oriented 3-manifold with boundary $\partial Y = \coprod_{i=1}^n \partial_i Y$ a disjoint union of tori, any basis $B = \prod_{i=1}^n (m_i, l_i)$ for $H_1(\partial Y; \mathbb{Z})$ determines a map

$$
\pi_B : \prod_{i=1}^n \mathbb{P}(H_1(\partial_i Y; \mathbb{Z})) \rightarrow (\mathbb{Q} \cup \{\infty\})_B^n,
$$

which associates $B$-slopes $\frac{a_i}{b_i} \in \mathbb{Q} \cup \{\infty\}$ to nonzero elements $a_i, m_i + b_i l_i \in H_1(\partial Y; \mathbb{Z})$. Each $B$-slope $(\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}) \in (\mathbb{Q} \cup \{\infty\})_B^n$ specifies a Dehn filling $Y_B((\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}))$, which is the closed 3-manifold given by attaching a compressing disk, for each $i$, to a simple closed curve in the primitive homology class corresponding to $[a_i, m_i + b_i l_i] \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$, and then gluing in a 3-ball to complete this solid torus filling of $\partial_i Y$. Notationally, we write

$$A_B(Y) := \pi_B(A(Y))$$

for any subset $A(Y) \subset \prod_{i=1}^n \mathbb{P}(H_1(\partial_i Y; \mathbb{Z}))$ of slopes for $Y$. Thus, $L_{S^3}(Y^{(np,nq)}) \subset (\mathbb{Q} \cup \{\infty\})_B^n$ realizes $L(Y^{(np,nq)})$ with respect to the conventional homology basis for link exteriors in $S^3$.

2.1. Seifert fibered basis

For $Y$ Seifert fibered over an $n$-times punctured $S^2$, there is a conventional Seifert fibered basis $\mathbf{F} = (f_1, -h_1, \ldots, f_n, -h_n)$ for $H_1(\partial Y; \mathbb{Z})$ which makes slopes correspond to Seifert data for Dehn fillings of $Y$. That is, each $-h_i$ is the meridian of the ith excised regular fiber, and each $\tilde{f}_i$ is the lift of the regular fiber class $f_i \in H_1(Y; \mathbb{Z})$ to a class $\tilde{f}_i \in H_1(\partial Y; \mathbb{Z})$ satisfying $-h_i \cdot \tilde{f}_i = 1$. Note that this makes $(\tilde{f}_i, -h_i)$ a reverse-oriented basis for each $H_1(\partial Y; \mathbb{Z})$, but this choice is made so that if $Y$ is trivially Seifert fibered, then with respect to our Seifert fibered basis, the Dehn filling $Y_{\mathbf{F}}((\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}))$ coincides with the genus zero Seifert fibered space $M := M_{S^2}(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ with (nonnormalized) Seifert invariants $(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n})$ and first homology

$$H_1(M; \mathbb{Z}) = \{f, h_1, \ldots, h_n | \sum_{i=1}^n h_i = 0; \ t_{i+} (\mu_i) = \cdots = t_{n+} (\mu_n) = 0\}, \quad \text{where} \quad (16)$$

$$\mu_i := \beta_i \tilde{f}_i - \alpha_i \tilde{h}_i \in H_1(\partial Y; \mathbb{Z}), \quad \nu_i : \partial Y \rightarrow Y, \quad h_i := \nu_i(\tilde{h}_i), \quad \text{and} \quad \nu_i(\tilde{f}_i) = f. \quad (17)$$

2.2. Action of $\mathbf{A}$

The relation $\sum_{i=1}^n h_i = 0$ in (16) comes from regarding the meridian images $-h_i \in H_1(Y; \mathbb{Z})$ as living in some global section $S^2 \setminus \coprod_{i=1}^n D_i^2 \rightarrow Y$ of the $S^1$ fibration, so that each $-h_i = -\partial D_i^2$ can be regarded as $-h_i = \partial_i (S^2 \setminus \coprod_{j \neq i} D_j^2)$, making the total class $-\sum_{i=1}^n h_i$ bound a disk in $S^2 \setminus \coprod_{i=1}^n D_i^2$. This choice of global section is not canonical, however. Any new choice of global section would correspond to a new choice of meridians,

$$\tilde{h}_i \leftrightarrow \tilde{h}_i' := \tilde{h}_i - l_i \tilde{f}_i, \quad i \in \{1, \ldots, n\} \quad \text{for some} \quad l \in \mathbb{Z}^n, \quad \sum_{i=1}^n l_i = 0. \quad (18)$$

Writing $\mu_i = \beta_i' \tilde{f}_i - \alpha_i' \tilde{h}_i$ to express $\mu_i$ in terms of this new basis yields

$$\beta_i' = \beta_i + l_i \alpha_i, \quad \alpha_i' = \alpha_i, \quad \frac{\beta_i'}{\alpha_i} = \frac{\beta_i}{\alpha_i} + l_i, \quad i \in \{1, \ldots, n\}. \quad (19)$$
In other words, the lattice of global section reparameterizations
\[ \Lambda := \{ l \in \mathbb{Z}^n \mid \sum_{i=1}^{n} l_i = 0 \} \]  
acts on Seifert data, hence on SF-slopes, by addition, without changing the underlying manifold or \( S^1 \) fibration. Moreover, for any choice of boundary-homology basis \( B \), the change of basis from SF-slopes to B slopes induces an action of \( \Lambda \) on B-slopes.

2.3. Action of \( \Lambda \) on torus-link-exterior slopes
As occurs in the case when \( Y = Y_{(p,q)}^n \) is the exterior of the torus link \( T(np, nq) \subset S^3 \) (see (43)), the \( \Lambda \)-action on \( S^3 \)-slopes
\[ \alpha \mapsto l \cdot \alpha = \psi(l + \psi^{-1}(\alpha)), \quad l \in \Lambda_{SF}(Y_{(p,q)}^n), \quad \alpha \in (Q \cup \{ \infty \})_S^3, \]  
induced by the transformation
\[ \psi : (Q \cup \{ \infty \})_{SF}^n \rightarrow (Q \cup \{ \infty \})_{S^3}^n, \quad y \mapsto \left( pq + \frac{1}{y_1}, \ldots, pq + \frac{1}{y_n} \right), \]  
is of particular interest to us. To aid in the introduction’s discussion of the role of \( \Lambda \) in Theorem 1.2, we temporarily introduce the sets \( L_0, L_1, L_2 \subset (Q \cup \{ \infty \})_{S^3}^n \) of \( S^3 \)-slopes, as follows:
\[ L_0 := \{ \infty \} \subset (Q \cup \{ \infty \})_{S^3}^n, \quad \infty := (\infty, \ldots, \infty), \]  
\[ L_1 := \mathfrak{S}_n \cdot \left( [N, +\infty] \times \{ \infty \}^{n-1} \right) \subset (Q \cup \{ \infty \})_{S^3}^n, \]  
\[ L_2 := \left( [-\infty, pq] \setminus [-\infty, N]^n \right) \cup \hat{R}_{S^3} \cup \langle pq, +\infty \rangle^n \subset (Q \cup \{ \infty \})_{S^3}^n, \]  
for some \( N \in \mathbb{Z} \), where we have temporarily introduced the notation \( \hat{R}_{S^3} \) to denote the set
\[ \hat{R}_{S^3} := R_{S^3} \setminus Z_{S^3} = \mathfrak{S}_n \cdot \left( \{ pq \} \times [N, +\infty) \} \setminus [-\infty, pq] \right) \subset (Q \cup \{ \infty \})_{S^3}^n. \]  
Finally, for \( \varepsilon > 0 \), we take \( U_{pq}(\varepsilon) \) to be the radius-\( \varepsilon \) punctured neighborhood
\[ U_{pq}(\varepsilon) := \bigcup_{i=1}^{n} \{ \alpha \mid \alpha_i \in [pq - \varepsilon, pq] \} \subset (Q \cup \{ \infty \})_{S^3}^n \]  
of the union of hyperplanes \( R_{S^3} = \bigcup_{i=1}^{n} \{ \alpha \mid \alpha_i \in pq \} \subset (Q \cup \{ \infty \})_{S^3}^n \) discussed in Section 2.4.

**Proposition 2.1.** The action of \( \Lambda \) on \( (Q \cup \{ \infty \})_{S^3}^n \) in (21) satisfies the following properties.

(a) \( \Lambda_S \cap Q_{S^3}^n \subset \{ [pq - 1, pq > U < pq, pq + 1] \} \); \( \Lambda_S = \Lambda \cdot \{ \infty \}_{S^3} = \Lambda \cdot L_0 \).

(b) \( (\Lambda \cdot L_1) \setminus L_1 \subset U_{pq}(1) \) for \( i = 0 \) always, for \( i = 1 \) when \( N > pq \), and for \( i = 2 \) when \( N \leq pq \).

(c) For \( \varepsilon > 0 \), each of the following sets of \( S^3 \)-slopes can be realized as a union of finitely many rectangles of dimensions 0, 1, and \( \{ n - 1, n \} \), respectively.
\[ (\Lambda \cdot L_0) \setminus U_{pq}(\varepsilon), \quad (\Lambda \cdot L_1) \setminus U_{pq}(\varepsilon) \text{ for } N > pq, \quad and \quad (\Lambda \cdot L_2) \setminus U_{pq}(\varepsilon) \text{ for } N \leq pq. \]

**Proof.** Part (a). The first statement follows from the fact that \( \frac{1}{m} \in [-1, 0] \cup [0, 1] \cup \{ \infty \} \) for all \( m \in \mathbb{Z} \). The second statement is due to the fact that \( 0 \in \Lambda_{SF} \) implies \( \infty = \psi(0) \in S_{S^3} \).

Part (b). First note that the action of \( \mathbb{Z} \) on \( (Q \cup \{ \infty \}) \) by addition fixes both \( \infty = \psi^{-1}(pq) \) as a point and its complement \( Q = \psi^{-1}((\infty, pq > U < pq, +\infty)) \) as a set, for any \( j \in \{ 1, \ldots, n \} \).

Since \( \hat{R}_{S^3} \) is a product of such sets, we have \( (\Lambda \cdot \hat{R}_{S^3}) \setminus \hat{R}_{S^3} = \emptyset \). We therefore reduce the problem to working with \( \hat{L}_0 := \hat{L}_0, \hat{L}_1 := \hat{L}_1, \) and \( \hat{L}_2 := \hat{L}_2 \setminus \hat{R}_{S^3} \), noting that \( L_2 = \hat{L}_2 \Pi \hat{R}_{S^3} \) when \( N \leq pq \). Since \( (\Lambda \cdot X) \times X \subset (\Lambda_{SF} \setminus \{ 0 \}) \cdot X \) for any subset \( X \subset (Q \cup \{ \infty \})_{S^3}^n \) of \( S^3 \)-slopes, and since \( \hat{L}_i \subset \{ [\infty, pq > n] \} \setminus \{ pq, +\infty \} \) for \( i = 0 \) always, for \( i = 1 \) when \( N > pq \), and for \( i = 2 \) when \( N \leq pq \), it is sufficient to show that \( (\Lambda_{SF} \setminus \{ 0 \}) \cdot \{ pq, +\infty \} \) for \( i = 0 \) always, for \( i = 1 \) when \( N > pq \), and for \( i = 2 \) when \( N \leq pq \). To see this, we first note that since \( \sum_{i=1}^{n} l_i = 0, \ l \in (\Lambda_{SF} \setminus \{ 0 \}) \) must have at least one positive
and at least one negative component, say $l_{+} \in [1, +\infty \cap \mathbb{Z}$ and $l_{-} \in (-\infty, -1] \cap \mathbb{Z}$, implying $(l + [0, +\infty))_{l_{+}} \subset [1, +\infty)$ and $(l + (-\infty, 0])_{l_{-}} \subset (-\infty, -1)$. We then have
\[ (l \cdot (pq, +\infty^{n}))_{l_{+}} = \psi(l \cdot \psi^{-1}((pq, +\infty^{n}))_{l_{+}} = \psi(l \cdot (0, +\infty^{n}))_{l_{+}} \subseteq (pq, pq + 1), \]
(28)
\[ (l \cdot (-\infty, pq)^{n})_{l_{-}} = \psi(l \cdot \psi^{-1}((-\infty, pq)^{n})_{l_{-}} = \psi(l \cdot (0, -\infty))_{l_{+}} \subseteq [pq - 1, pq), \]
(29)
and so we conclude that $l \cdot (pq, +\infty^{n}) \cap (pq, +\infty^{n}) \cap \mathbb{Z} \subset U_{pq}^{n}(1)$, completing the proof of (b).

Part (c). In the SF basis, the complement of $U_{pq}^{n}(\varepsilon)$ within the set of $S^{3}$-slopes is given by
\[ \psi^{-1}((Q \cup \{\varepsilon\})^{n}_{SF} \setminus U_{pq}^{n}(\varepsilon)) = \{(-\frac{1}{2}, +\frac{1}{2})^{n} \cup \bigcup_{i=1}^{n} \{y : y_{i} = \infty\} \subset (Q \cup \{\varepsilon\})^{n}_{SF}. \]
(30)
Since both $\bigcup_{i=1}^{n} \{y : y_{i} = \infty\}_{SF} = \psi^{-1}(\mathcal{R}_{S^{3}})$ and $\psi^{-1}(\widehat{\mathcal{R}}_{S^{3}})$ are fixed setwise by $\Lambda$, and since $\psi^{-1}(L_{i}) \cap \psi^{-1}(\mathcal{R}_{S^{3}}) = \emptyset$ for $i = 0$ and for $i = 1$ with $N > pq$, but $\psi^{-1}(L_{i}) \cap \psi^{-1}(\mathcal{R}_{S^{3}}) = \psi^{-1}(\widehat{\mathcal{R}}_{S^{3}})$ when $i = 2$ and $N \leq pq$, it follows that
\[ (\Lambda_{SF} + \psi^{-1}(L_{i})) \cap \bigcup_{i=1}^{n} \{y : y_{i} = \infty\} = \left\{ \begin{align*} &\emptyset \quad i = 0 \text{ or } (i = 1 \text{ and } N > pq) \\
&\psi^{-1}(\widehat{\mathcal{R}}_{S^{3}}) \quad i = 2 \text{ and } N \leq pq. \end{align*} \right. \]
(31)
Thus, since $\psi^{-1}(\widehat{\mathcal{R}}_{S^{3}})$ is already a finite union of $(n - 1)$-dimensional rectangles, it suffices to show that $(\Lambda_{SF} + \psi^{-1}(L_{i})) \cap \psi^{-1}(\mathcal{R}_{S^{3}})$ is a union of finitely many rectangles of dimensions $0, 1, n$ in the respective cases that $i = 0, i = 1$ with $N > pq$, or $i = 2$ with $N \leq pq$. The proof of this latter statement is straightforward, however, since $\psi^{-1}(L_{i})$ is already a finite union of rectangles of dimensions $0, 1, n$, respectively, for the three above respective cases, and only finitely many distinct rectangles can be formed by intersecting $\mathbb{Z}^{n}$ translates of these rectangles with $< -\frac{1}{2}, +\frac{1}{2} >_{SF} \subset (Q \cup \{\varepsilon\})^{n}_{SF}$. \hfill \qed

2.4. Reducible and exceptional sets $\mathcal{R}$ and $\mathcal{Z}$

Like the above action of $\Lambda$, the following facts about reducible fillings are well known in low-dimensional topology, but for the benefit of a diverse readership we provide some details.

Proposition 2.2. Let $\hat{Y}$ denote the trivial $S^{3}$ fibration over $S^{2} \setminus \bigsqcup_{i=0}^{n} D_{i}^{2}$, and let $Y$ denote the Dehn filling of $\hat{Y}$ along the $S^{1}$-fiber lift $f_{0} \in H_{1}(\partial\hat{Y}; \mathbb{Z})$, that is, along the $\infty$ SF-slope of $\partial\hat{Y}$. Then $Y$ is a connected sum $Y = \#_{i=1}^{n}(S^{1} \setminus S^{1}) \times D_{i}^{2}$ (where $S^{1}$ is the fiber), and each exterior $S^{3} \setminus S^{1} \times D_{i}^{2}$ has meridian $-\hat{h}_{i}$ and rational longitude $\hat{f}_{i}$.

Proof. Choose a global section $S^{2} \setminus \bigsqcup_{i=0}^{n} D_{i}^{2} \rightarrow \hat{Y}$ which respects the SF basis. We shall stretch the disk $S^{2} \setminus D_{i}^{2}$ into a (daisy) flower shape, with one $D_{i}^{2}$ contained in each petal. Embed $2n$ points $p_{1}^{1}, p_{1}^{2}, \ldots, p_{n}^{1}, p_{n}^{2} \leftarrow \partial D_{2}^{2}$, in that order with respect to the orientation of $\partial D_{2}^{2}$. For each $i \in \{1, \ldots, n\}$, let $\delta_{i}$ and $\varepsilon_{i}$ denote the respective arcs from $p_{i}^{-}$ to $p_{i}^{+}$ and from $p_{i}^{+}$ to $p_{i+1}^{+}$ (mod $n$) along $-\partial D_{0}^{2}$, and properly embed an arc $\gamma_{i} \rightarrow S^{2} \setminus \bigsqcup_{i=0}^{n} D_{i}^{2}$ from $p_{i}^{+}$ to $p_{i}^{-}$ which winds once positively around $D_{i}^{2}$ and winds zero times around the other $D_{j}^{2}$, without intersecting any of the other $\gamma_{j}$ arcs. Holding the $p_{i}^{\pm}$ points fixed while stretching the $\delta_{i}$ arcs outward and pulling the $\gamma_{i}$ arcs tight realizes our global section as the punctured flower shape
\[ S^{2} \setminus \bigsqcup_{i=0}^{n} D_{i}^{2} = \hat{D}_{0}^{2} \sqcup \bigsqcup_{i=1}^{n} (\hat{D}_{i}^{2} \setminus D_{i}^{2}), \]
(32)
where $\hat{D}_{0}^{2}$ denotes the central disk of the flower, bounded by $\partial \hat{D}_{0}^{2} = (\bigsqcup_{i=1}^{n} -\gamma_{i}) \cup (\bigsqcup_{i=1}^{n} \varepsilon_{i})$, and each $\hat{D}_{i}^{2}$ denotes the petal-shaped disk bounded by $\partial \hat{D}_{i}^{2} = \delta_{i} \cup p_{i}^{\pm} \gamma_{i}$.
The Dehn filling $Y$ is formed by multiplying the above global section with the fiber $S^1_f$ and then gluing a solid torus $D^2_f \times \partial D^2_0$ (with $\partial D^2_f = S^1_f$) along $S^1_f \times \partial D^2_0$. Since

$$-\partial D^2_0 = (\prod_{i=1}^n \delta_i) \cup (\prod_{i=1}^n \varepsilon_i), \quad \partial D^2_0 = (\prod_{i=1}^n -\gamma_i) \cup (\prod_{i=1}^n \varepsilon_i), \quad \text{and} \quad \partial D^2_i = \delta_i \cup \gamma_i,$$

we can decompose the solid torus $D^2_f \times \partial D^2_0$ along the disks $D^2_f \times p^+_i$, and distribute these solid-torus components among the boundaries of $\partial D^2_0$ and the $\partial D^2_i$, so that

$$D^2_f \times \partial D^2_0 = \big( D^2_f \times \partial D^2_0 \big) \setminus \big( \prod_{i=1}^n (D^2_f \times -\gamma_i) \cup \prod_{i=1}^n \big( (D^2_f \times \partial D_i) \cup (D^2_f \times \gamma_i) \big) \big), \quad \text{(34)}$$

where the union is along the boundary 2-spheres

$$S^2_i := (D^2_f \times p^-_i) \cup (S^1_f \times \gamma_i) \cup (D^2_f \times p^+_i) = \partial(D^2_f \times \gamma_i) \quad \text{(35)}$$

of the balls $D^2_f \times \pm \gamma_i$. Thus, if we set $S^3_i := (D^2_f \times \partial D^2_i) \cup (S^1_f \times \hat{D}^2_i)$ for $i \in \{0, \ldots, n\}$, then

$$Y = S^3_0 \# \prod_{i=1}^n (S^3_i \setminus S^1_f) \times D^2_i, \quad \text{(36)}$$

with the connected sum taken along the spheres $S^2_i$.

\[\square\]

**Corollary 2.3.** If $\hat{Y}$ is as above, and if $(y_1, \ldots, y_n) \in \{\infty\}^k \times \mathbb{Q}^{n-k}$ for some $k \in \{0, \ldots, n\}$, then $\hat{Y}_{SF}(\infty, y_1, \ldots, y_n) = (\#_{i=1}^n S^1 \times S^2) \# (\#_{i=k+1}^n M_{S^2}(y_i))$.

This motivates the following terminology.

**Definition 2.4.** Suppose that $\partial Y = \prod_{i=1}^n \mathbb{T}^2_i$, and that $Y$ has a (genus zero) Seifert fibered JSJ component containing $\partial Y$. Then the reducible slopes $\mathcal{R}(Y)$ and exceptional slopes $\mathcal{Z}(Y) \subset \mathcal{R}(Y)$ are given by $\mathcal{R}_{SF}(Y) := \mathcal{S}_n \cdot (\{\infty\} \times (\mathbb{Q} \cup \{\infty\})^{n-1})$ and $\mathcal{Z}_{SF}(Y) := \mathcal{S}_n \cdot (\{\infty\}^2 \times (\mathbb{Q} \cup \{\infty\})^{n-2})$, where $\mathcal{S}_n$ acts by permutation of slopes.

Note that occasionally, slopes in $\mathcal{R}(Y)$ yield Dehn fillings which are connected sums of a lens space with 3-spheres, hence are not reducible.

**Definition 2.5.** Suppose $Y$ is as above. If the Seifert fibered component containing $\partial Y$ has no exceptional fibers, then the false reducible slopes $\mathcal{R}^0(Y) \subset \mathcal{R}(Y)$ of $Y$ are given by $\mathcal{R}^0_{SF}(Y) = \mathcal{S}_n \cdot (\{\infty\} \times \mathbb{Q} \times \{0\}^{n-2})$. Any reducible slopes which are not false reducible are called truly reducible. Equivalently, the truly reducible slopes are those slopes which yield reducible Dehn fillings.

**2.5. Rational longitudes $\mathcal{B}$**

Our last distinguished slope set of interest, the set $\mathcal{B}$ of rational longitude slopes, makes sense for $Y$ of any geometric type.

**Definition 2.6.** Suppose $Y$ is a compact oriented 3-manifold with $\partial Y$ a union of $n > 0$ toroidal boundary components. The set of rational longitudes $\mathcal{B} \subset \prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$ of $Y$ is the set of slopes

$$\mathcal{B} := \{ \beta \in \prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \mid b_1(Y(\beta)) > b_1(Y) - n \}, \quad \text{(37)}$$
Note that this implies $Z = R \cap B$. Moreover, when $Y$ is Seifert fibered, $\mathcal{B}_{\text{sf}}(Y)$ is the closure of a linear subspace of $\text{sf}$-slopes. That is, by [28, Section 5] (for example), we have

$$b_1 \left( M_{S^2} \left( \frac{p_1}{n_1}, \ldots, \frac{p_n}{n_n} \right) \right) > 0 \iff \sum_{i=1}^n \frac{p_i}{n_i} = 0. \tag{38}$$

In particular, since the $T(np, nq)$-exterior $Y_{(p, q)}^n := M_{S^2}(-\frac{q}{p}, \frac{1}{p}, *, \ldots, *)$, constructed in (43), has $\frac{p_i}{n_i} = \frac{1}{pq}$, the next proposition follows immediately from line (38).

**Proposition 2.7.** If $Y = S^3 \setminus \mathring{\nu}(T(np, nq))$ is the exterior of the $(np, nq)$ torus link, then

$$\mathcal{B}_{\text{sf}}(Y) = \left\{ y \in (\mathbb{Q} \cup \{ \infty \})_{\text{sf}}^n \left\lfloor \frac{1}{pq} + \sum_{i=1}^n y_i = 0 \right\},$$

is the closure in $(\mathbb{Q} \cup \{ \infty \})_{\text{sf}}^n$ of the hyperplane $\{ y \in \mathbb{Q}^n | \frac{1}{pq} + \sum_{i=1}^n y_i = 0 \}$. Moreover, if $\overline{\mathcal{B}}_{\text{sf}}$ denotes the real closure of $\mathcal{B}_{\text{sf}}$ in $\prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{R})) \rightrightarrows (\mathbb{Q} \cup \{ \infty \})_{\text{sf}}^n$, then

$$\overline{\mathcal{B}}_{\text{sf}} \cong T^{n-1} \cong (\mathbb{R} \cup \{ \infty \})_{\text{sf}}^{n-1} \leftrightarrow (\mathbb{R} \cup \{ \infty \})_{\text{sf}}^n \cong T^n.$$

### 3.1. L-space interval notation

For the following discussion, $Y$ denotes a compact oriented 3-manifold with a one-component torus boundary $\partial Y$, and $B$ is a basis for $H_1(\partial Y; \mathbb{Z})$, inducing an identification $\pi_B : \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \rightarrow (\mathbb{Q} \cup \{ \infty \})_B =: \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B$.

**Definition 3.1.** We introduce the notation $[\cdot, \cdot]$ so that for $y_-, y_+ \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B$, the subset $[y_-, y_+] \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B$ is defined as follows.

$$[y_-, y_+] := \begin{cases} \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B \setminus \{ y_- \} = \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B \setminus \{ y_+ \} & y_- = y_+ \\ \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B \cap I_{[y_-, y_+]} & y_- \neq y_+ \\ \end{cases}$$

for $I_{[y_-, y_+]} \subset \mathbb{P}(H_1(\partial Y; \mathbb{R}))_B$ the closed interval with left-hand endpoint $y_-$ and right-hand endpoint $y_+$.

By [28, Proposition 1.3 and Theorem 1.6], the L-space interval $L(Y) \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$ of L-space Dehn filling slopes of $Y$ can only take certain forms.

**Proposition 3.2 [28].** One of the following is true.

(i) $L(Y) = \emptyset$.
(ii) $L(Y) = \{ \eta \}$, for some $\eta \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$.
(iii) $L_n(Y) = [l, \ell]$, with $l \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B$ the rational longitude of $Y$.
(iv) $L_n(Y) = [y_-, y_+]$ with $y_- \neq y_+$.

It is for this reason that we refer to the space of L-space Dehn filling slopes as an interval. It also makes sense to speak of the interior of this interval.

**Definition 3.3.** The L-space interval interior $L_n(Y) \subset L(Y)$ of $Y$ satisfies

$$L_n(Y) := \begin{cases} \emptyset & L(Y) = \emptyset \text{ or } L(Y) = \{ \eta \} \\ [l, \ell] & L_n(Y) = [l, \ell] \\ \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_B \cap I_{(y_-, y_+)} & L_n(Y) = [y_-, y_+] \text{ with } y_- \neq y_+, \end{cases}$$
where \( \hat{I}_{(y_-, y_+)} \subset \mathbb{P}(H_1(\partial Y; \mathbb{R}))_b \) indicates the open interval with left-hand endpoint \( y_- \) and right-hand endpoint \( y_+ \).

This gives us a new way to characterize the property of Floer simplicity for \( Y \).

**Proposition 3.4** [28]. The following are equivalent.

- \( Y \) has more than one L-space Dehn filling.
- \( \mathcal{L}_n(Y) = [[y_-, y_+] \in \mathbb{P}(H_1(\partial Y; \mathbb{Z}))_b \]
- \( \mathcal{L}^\circ(Y) \neq \emptyset \).

In the case that any, and hence all, of these three properties hold, we say that \( Y \) is Floer simple.

Both Floer simple manifolds and graph manifolds have predictably behaved unions with respect to the property of being an L-space.

**Theorem 3.5** [14, 28, 29]. If the manifold \( Y_1 \cup_{\varphi} Y_2 \), with gluing map \( \varphi : \partial Y_1 \to -\partial Y_2 \), is a closed union of 3-manifolds, each with incompressible single-torus boundary, and with \( Y_1 \) both Floer simple or both graph manifolds, then

\[
Y_1 \cup_{\varphi} Y_2 \text{ is an L-space } \iff \varphi^*_p(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2; \mathbb{Z})).
\]

Unfortunately, this theorem fails to encompass the case in which an L-space knot exterior is glued to a non-Floer simple graph manifold, and our study of surgeries on iterated or algebraic satellites will certainly require this case. We therefore prove the following result.

**Theorem 3.6.** If \( Y_1 \cup_{\varphi} Y_2 \), with gluing map \( \varphi : \partial Y_1 \to -\partial Y_2 \), is a closed union of 3-manifolds, such that \( Y_1 \) is the exterior of a nontrivial L-space knot \( K \subset S^3 \), and \( Y_2 \) is a graph manifold, or connected sum thereof, with incompressible single torus boundary, then

\[
Y_1 \cup_{\varphi} Y_2 \text{ is an L-space } \iff \varphi^*_p(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2; \mathbb{Z})). \tag{39}
\]

Moreover, if \( Y_2 \) is not Floer simple, then (39) holds for \( K \subset S^3 \) an arbitrary nontrivial knot.

**Proof.** We first reduce to the case prime \( Y_2 \). Let \( Y'_2 \) denote the connected summand of \( Y_2 \) containing \( \partial Y_2 \), and recall that hat Heegaard Floer homology tensors over connected sums. Thus, if \( Y_2 \) has any non-L-space closed connected summands, then \( Y_1 \cup_{\varphi} Y_2 \) is a non-L-space and \( \mathcal{L}(Y'_2) = \emptyset \), regardless of the union \( Y_1 \cup_{\varphi} Y'_2 \). On the other hand, if all the closed connected summands of \( Y_2 \) are L-spaces, then \( \mathcal{L}^\circ(Y'_2) = \mathcal{L}^\circ(Y'_2) \), and \( Y_1 \cup_{\varphi} Y'_2 \) is an L-space if and only if \( Y_1 \cup_{\varphi} Y'_2 \) is an L-space. We therefore henceforth assume \( Y_2 \) is prime.

If \( Y_2 \) is Floer simple, then when \( K \subset S^3 \) is an L-space knot, \( Y_1 \) is Floer simple, and so the desired result is already given by Theorem 3.5. We therefore assume \( Y_2 \) is not Floer simple, implying \( \mathcal{N}(\mathcal{L}(Y'_2)) = \mathbb{P}(H_1(\partial Y'_2; \mathbb{Z})) \) or \( \mathcal{N}(\mathcal{L}(Y'_2)) = \mathbb{P}(H_1(\partial Y'; \mathbb{Z})) \setminus \{y\} \) for some single slope \( y \). In either case, \( \mathcal{L}(Y'_2) = \emptyset \), and so it remains to show that \( Y_1 \cup_{\varphi} Y'_2 \) is not an L-space.

In \[29]\), the author showed for any (prime) graph manifold \( Y_2 \) with single torus boundary that if \( F(Y'_2) \) (called \( F(Y'_2) \) in that paper’s notation) denotes the set of slopes \( \alpha \in \mathbb{P}(H_1(\partial Y'_2; \mathbb{Z})) \) for which \( Y_2 \) admits a co-oriented taut foliation restricting to a product foliation of slope \( \alpha \) on \( \partial Y'_2 \), then \( F(Y'_2) = \mathcal{N}(\mathcal{L}(Y'_2)) \setminus \mathcal{R}(Y'_2) \). Since \( Y_2 \) is prime, \( \mathcal{R}(Y'_2) = \{\{\alpha\} \in \mathbb{P}(H_1(\partial Y'_2; \mathbb{Z})) \}_{\alpha} \). In particular, \( F(Y'_2) \) is the complement of a finite set in \( \mathbb{P}(H_1(\partial Y'_2; \mathbb{Z})) \).

On the other hand, Li and Roberts show in [22] that for the exterior of an arbitrary nontrivial knot in \( S^3 \), such as \( Y_1 \), one has \( F_{S^3}(Y_1) \supset \{a, b\} > \) for some \( a < 0 < b \). In particular, \( F(Y'_1) \) is infinite, implying \( \varphi^*_p(F(Y'_1)) \cap F(Y'_2) \) is nonempty. Thus we can construct a co-oriented taut
foliation $F$ on $Y_1 \cup_Y Y_2$ by gluing together co-oriented taut foliations restricting to a matching product foliation of some slope $\alpha \in \varphi_t^\infty(F(Y_1)) \cap F(Y_2)$ on $\partial Y_2$.

Eliashberg and Thurston showed in [9] that a $C^2$ co-oriented taut foliation can be perturbed to a pair of oppositely oriented tight contact structures, each with a symplectic semi-filling with $b_2^+ > 0$. Ozsváth and Szabó [26] showed that one can associate a nonzero class in reduced Heegaard Floer homology to such a contact structure. This result was recently extended to $C^0$ co-oriented taut foliations by Kazez and Roberts [21] and independently by Bowden [3]. Thus, our co-oriented taut foliation $F$ on $Y_1 \cup_Y Y_2$ implies that $Y_1 \cup_Y Y_2$ is not an L-space. \qed

4. Torus-link satellites

4.1. $T(np,nq) \subset S^3$ and Seifert structures on $S^3$

Since $S^3$ is a lens space, any Seifert fibered realization of $S^3$ can have at most two exceptional fibers:

$$S^3 = M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q}), \quad p^* p - q^* q = 1,$$  \hspace{1cm} (40)

where the right-hand constraint on $p,q,p^*,q^* \in \mathbb{Z}$ is necessary (and sufficient) to achieve $H_1(M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q}); \mathbb{Z}) = 0$. The one-exceptional-fiber Seifert structures for $S^3$ are exhausted by the cases $\{\frac{q^*}{p}, \frac{p^*}{q}\} = \{\frac{1}{n}, \frac{1}{m}\}$, $n \in \mathbb{Z}$. The above Seifert structure exhibits $S^3$ as a union

$$S^3 = \nu(\lambda_{-1}) \cup (S^2 \setminus (D_0^2 \amalg D_1^2)) \times S^1 \cup \nu(\lambda_0)$$  \hspace{1cm} (41)

$$= (D_2^1 \times S^1) \cup [-\varepsilon, \varepsilon] \times T^2 \cup (D_0^2 \times S^1),$$  \hspace{1cm} (42)

where $\lambda_{-1}$ and $\lambda_0$ are exceptional fibers of meridian-slopes $[\mu_{-1}]_{sf} = -\frac{q^*}{p}$ and $[\mu_0]_{sf} = \frac{p^*}{q}$, respectively, forming a Hopf link $\lambda_{-1}, \lambda_0 \subset S^3$.

Regular fibers in this Seifert fibration are confined to some neighborhood $[-\varepsilon, \varepsilon] \times T^2$ of a torus $T^2$, and they foliate this $T^2$ with fibers all of the same slope. Since $\lambda_{-1}$ and $\lambda_0$ are of multiplicities $p$ and $q$, respectively, any regular fiber $f$ wraps $p$ times around the core $\lambda_{-1}$ of the solid torus neighborhood $\nu(\lambda_{-1})$, and wraps $q$ times around the core $\lambda_0$ of $\nu(\lambda_0)$, or equivalently, winds $q$ times along the core of $\nu(\lambda_{-1})$. That is, any regular fiber $f$ is a $(p,q)$ curve in the boundary $T^2 = \partial \nu(\lambda_{-1})$ of the solid torus $\partial \nu(\lambda_{-1})$ of core $\lambda_{-1}$. (See Proposition 4.2 for a more careful treatment of framings and orientations.) Thus, any collection $f_1, \ldots, f_n$ of regular fibers in $M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q})$ allows us to realize the exterior

$$Y_{(p,q)}^n := M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q}) \setminus \nu(\prod_{i=1}^n f_i) =: M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q}, 1, \ldots, n) = S^3 \setminus \tilde{\nu}(T(np,nq))$$  \hspace{1cm} (43)

of $T(np,nq) \subset S^3$. As a link in the solid torus, $T(np,nq) \subset \nu(\lambda_{-1})$ inhabits the exterior

$$\tilde{Y}_{(p,q)} := M_{S^2}(-\frac{q^*}{p}, \frac{p^*}{q}) \setminus \nu(\lambda_0) =: M_{S^2}(-\frac{q^*}{p}, 0, \ast, \ldots, \ast) = \nu(\lambda_{-1})$$  \hspace{1cm} (44)

of the fiber $\lambda_0$ of meridian-slope $\frac{p^*}{q}$. This solid-torus link $T(np,nq) \subset \tilde{Y}_{(p,q)}$ then has exterior

$$\hat{Y}_{(p,q)} := \hat{Y}_{(p,q)} \setminus \tilde{\nu}(\prod_{i=1}^n f_i) =: M_{S^2}(-\frac{q^*}{p}, 0, 1, \ast, \ldots, \ast) = \hat{Y}_{(p,q)} \setminus \tilde{\nu}(T(np,nq)).$$  \hspace{1cm} (45)

To make this association $(p,q) \mapsto \hat{Y}_{(p,q)}$ well defined, we adopt the following convention.

**Definition 4.1.** To any $(p,q) \in \mathbb{Z}^2$ with $\gcd(p,q) = 1$, we associate the pair $(p^*, q^*) \in \mathbb{Z}^2$:  \hspace{1cm} (46)

$$(p,q) \mapsto (p^*, q^*) \in \mathbb{Z}^2, \quad p^* p - q^* q = 1, \quad q^* \in \{0, \ldots, p-1\},$$  \hspace{1cm} (46)
where we demand \( p > 0 \) without loss of generality (since \( p = 0 \)-satellites are unlinks).

### 4.2. Construction of satellites

For a knot \( K \subset M \) in a closed oriented 3-manifold \( M \), we define the \( T(np, nq) \)-torus-link satellite \( K^{(np, nq)} \subset M \) to be the image of the torus link \( T(np, nq) \) embedded in the boundary of \( \nu(K) \), composed with the inclusion \( \nu(K) \hookrightarrow M \).

Thus, if we write \( Y := M \setminus \overset{\circ}{\nu}(K) \) for the exterior of \( K \subset M \) and take \( \hat{Y}^n_{(p,q)} \) as defined in (45), then for an appropriate choice of gluing map \( \bar{\varphi} : \partial Y \to -\partial_0 \hat{Y}^n_{(p,q)} \), we expect the union

\[
Y^{(np, nq)} := Y \cup_{\bar{\varphi}} \hat{Y}^n_{(p,q)}, \quad \bar{\varphi} : \partial Y \to -\partial_0 \hat{Y}^n_{(p,q)}
\]

(47) to be the exterior of \( K^{(np, nq)} \subset M \).

**Proposition 4.2.** Suppose \( p, q, n \in \mathbb{Z} \) with \( n, p > 0 \) and \( \gcd(p, q) = 1 \). Choose a surgery basis \((\mu, \lambda)\) for the boundary homology \( H_1(\partial Y; \mathbb{Z}) \) of the knot exterior \( Y := M \setminus \overset{\circ}{\nu}(K) \), and take \( \hat{Y}^n_{(p,q)} \) as in (45) and \( Y^{(np, nq)} \) as in (47). If the gluing map \( \bar{\varphi} : \partial Y \to -\partial_0 \hat{Y}^n_{(p,q)} \) induces the homomorphism \( \bar{\varphi}_* : H_1(\partial Y; \mathbb{Z}) \to H_1(\partial_0 \hat{Y}^n_{(p,q)}; \mathbb{Z}) \),

\[
\bar{\varphi}_*(\mu) := -q^* f_0 + ph_0, \quad \bar{\varphi}_*(\lambda) := p^* f_0 - qh_0,
\]

(48) on homology, and hence the orientation-preserving linear fractional map

\[
\bar{\varphi}^\mu : \mathbb{P}(H_1(\partial Y; \mathbb{Z})_{\text{surg}}) \to \mathbb{P}(H_1(\partial_0 \hat{Y}^n_{(p,q)}; \mathbb{Z}))_{\text{sF}},
\]

\[
\bar{\varphi}^\mu \left( \frac{a}{b} \right) = \frac{aq^* - bp^*}{ap - bq} = \frac{q^*}{p} - \frac{b}{p(ap - bq)},
\]

(49) on slopes, then \( Y_{sf}^{(np, nq)}(0) = M \), and \( Y^{(np, nq)} \) is the exterior of of the \( T(np, nq) \) satellite \( K^{(np, nq)} \subset M \) of \( K \subset M \).

**Proof.** The Dehn filling \( Y_{sf}^{(np, nq)}(0) \) is given by the union \( Y_{sf}^{(np, nq)}(0) = Y \cup_{\bar{\varphi}} \hat{Y}^n_{(p,q)} \), for the solid torus \( \hat{Y}^n_{(p,q)} = M_{S^2}(-\frac{2\mu}{p}, \ast) \) defined in (44). The boundary of the compressing disk of \( \hat{Y}^n_{(p,q)} \) is given by the rational longitude \( l = -\sum_{i=1}^{n-1} \frac{\beta_i}{\alpha_i} = \frac{q^*}{p} \) of \( \hat{Y}^n_{(p,q)} \) (see the last line of Theorem 4.3). Thus, since \( [\bar{\varphi}(\mu)]_{\text{sF}} = \frac{q^*}{p}, Y_{sf}^{(np, nq)}(0) \) is in fact the Dehn filling \( Y_{sf}^{(np, nq)}(0) = Y(\mu) = M \).

Since \( Y^{(np, nq)} = M \setminus \bigsqcup_{i=1}^n \overset{\circ}{\nu}(f_i) \) is the exterior of \( n \) regular fibers from \( \hat{Y}^n_{(p,q)} = M_{S^2}(-\frac{2\mu}{p}, \ast) \), we must verify that our lift \( 0 f_0 \in H_1(\partial \hat{Y}^n_{(p,q)}; \mathbb{Z}) \) of a regular fiber class to the boundary \( \partial \hat{Y}^n_{(p,q)} = \partial_0 \hat{Y}^n_{(p,q)} \) of the solid torus \( \hat{Y}^n_{(p,q)} \) is represented by a \((p, q)\) torus knot on \( \partial \hat{Y}^n_{(p,q)} \) relative to the framing specified by \( \mu \) and \( \lambda \). Indeed, from (48), we have

\[
\tilde{f}_0 = (pp^* - qq^*) f_0 = p\bar{\varphi}_*(\lambda) + q\bar{\varphi}_*(\mu),
\]

as required. The induced map \( \bar{\varphi}^\mu \) on slopes preserves orientation, because the map \( \bar{\varphi} \) is orientation reversing, but the surgery basis and Seifert fibered basis are positively oriented and negatively oriented, respectively. \( \square \)

### 4.3. Computing L-space intervals

The primary tool we shall use is a result of the author which computes the L-space interval for the exterior of a regular fiber in a closed 3-manifold with a Seifert fibered JSJ component.

**Theorem 4.3** [29, Theorem 1.6]. Suppose \( M \) is a closed oriented 3-manifold with some JSJ component \( \hat{Y} \) which is Seifert fibered over an \( n_{\text{irr}} \)-times-punctured \( S^2 \), so that we may express
M as a union

\[ M = \dot{\mathcal{Y}} \cup_\phi \prod_{j=1}^{n_m} Y_j, \quad \phi_j : \partial Y_j \to -\partial_j \dot{\mathcal{Y}}, \]

where each \( Y_j \) is boundary incompressible (that is, is not a solid torus or a connected sum thereof). Write \((y_1, \ldots, y_n)\) for the Seifert slopes of \( \dot{\mathcal{Y}} \), so that \( \dot{\mathcal{Y}} \) is the partial Dehn filling of \( S^1 \times (S^2 \setminus \bigcup_{i=1}^{n_m} D_i^3) \) by \((y_1, \ldots, y_n)\) in our Seifert fibered basis. Further suppose that each \( Y_j \) is Floer simple, so that we may write

\[ \varphi_j^\delta(\mathcal{L}(Y_j))_{sf} = [\{y^m_{j-}, y^m_{j+}\}] \subset \mathbb{P}(H_1(\partial_j \dot{\mathcal{Y}}; \mathbb{Z}))_{sf} \]

for each \( j \in \{1, \ldots, n_m\} \). Let \( Y \) denote the exterior \( Y = M \setminus \overset{\circ}{\nu}(f) \) of a regular fiber \( f \subset \dot{\mathcal{Y}} \). If \( \mathcal{L}(Y) \) is nonempty, then

\[ \mathcal{L}_{sf}(Y) = \begin{cases} \{y_-=\} = \{y_+\} & Y \text{ Floer not simple} \\ \{[y_-, y_+]\} & Y \text{ Floer simple}, \end{cases} \]

where

\[ y_- := \sup_{k>0} -\frac{1}{k} \left( 1 + \sum_{i=1}^{n} [y_i k] + \sum_{j=1}^{n_m} \left( [y^m_{j-} k] - 1 \right) \right), \]

\[ y_+ := \inf_{k>0} -\frac{1}{k} \left( -1 + \sum_{i=1}^{n} [y_i k] + \sum_{j=1}^{n_m} \left( [y^m_{j-} k] + 1 \right) \right). \]

The above extrema are realized for finite \( k \) if and only if \( Y \) is boundary incompressible. When \( Y \) is boundary compressible, \( y_- = y_+ = l = -\sum_{i=1}^{n} y_i \) is the rational longitude of \( Y \).

**Remarks.** In the above, we define \( y_- := -\infty \) or \( y_+ := \infty \), respectively, if any infinite terms appear as summands of \( y_- \) or \( y_+ \), respectively. For \( x \in \mathbb{R} \), the notations \([x]\) and \( \lceil x \rceil \), respectively, indicate the greatest integer less than or equal to \( x \) and the least integer greater than or equal to \( x \), as usual. In addition, we always take \( k \) to be an integer. Thus the expression \( 'k > 0' \) always indicates \( k \in \mathbb{Z}_{>0} \).

In order to use the above theorem, we first have to know whether \( \mathcal{L}(Y) \) is nonempty and whether \( Y \) is Floer simple. The author provides a complete (and lengthy) answer to this question in [29]. Here, we restrict to the cases of most relevance to the current question.

**Theorem 4.4** [29]. Assuming the hypotheses of Theorem 4.3, set

\[ n^\infty := |\{i \in \{1, \ldots, n\} | y_i = \infty\}|, \]

\[ n^<_{\text{bi}} := |\{j \in \{1, \ldots, n_{\text{bi}}\} | -\infty < y^m_{j-} < y^m_{j+} < +\infty\}|. \]

If \( \infty \notin \{y^m_{1-}, y^m_{1+}, \ldots, y^m_{n_{\text{bi}}-}, y^m_{n_{\text{bi}}+}\} \), then the following are true.

(i) If \( n^\infty > 1 \), then any Dehn filling of \( Y \) is a connected sum with \( S^1 \times S^2 \), and \( \mathcal{L}_{sf}(Y) = \emptyset \).

(ii) If \( n^\infty = 1 \), then \( \mathcal{L}_{sf}(Y) = \begin{cases} < -\infty, +\infty > & n^<_{\text{bi}} = 0 \\ \emptyset & n^<_{\text{bi}} > 0 \end{cases} \).

(iii) If \( n^\infty = 0 \), then \( \mathcal{L}_{sf}(Y) = \begin{cases} \{[y_-, y_+]\} \text{ with } y_- > y_+ & n^<_{\text{bi}} = 0 \\ \{y_-, y_+\} & n^<_{\text{bi}} = 1 \text{ and } y_- < y_+ \\ \{y_+\} & n^<_{\text{bi}} = 1 \text{ and } y_- = y_+ \end{cases} \).
Suppose instead that $\infty \in \{ y_{i_{-}}, y_{i_{+}}, \ldots, y_{n_{i_{-}}}, y_{n_{i_{+}}} \}$.

(iv) If either $\infty \not\in \{ y_{i_{-}}, \ldots, y_{n_{i_{-}}} \}$ or $\infty \not\in \{ y_{i_{1}}, \ldots, y_{n_{i_{+}}} \}$, then $\mathcal{L}_{sf}(Y) = \begin{cases} [y_{-}, y_{+}] & n_{i_{-}} = 0 \text{ and } n^{\infty} = 0 \\ < -\infty, +\infty > & n_{i_{-}} < 0 \text{ and } n^{\infty} = 1 \\ \emptyset & n_{i_{-}} < 0 \text{ and } n^{\infty} > 1 \\ \emptyset & n_{i_{-}} \neq 0 \end{cases}$.

Proof. See [29, Proposition 4.7].

To state the below theorem efficiently, we need to introduce one last notational convention.

\textbf{Notation.} When the brackets $[\cdot]$ are applied to a real number, they always indicate the map $[\cdot] : \mathbb{R} \rightarrow [0, 1), [x] := x - [x]$. \hspace{1cm} (50)

Note that the maps $[\cdot], [\cdot], \text{ and } [\cdot] \text{ satisfy the useful identities,}$

$$ -[-x] = [x], \quad x = [x] + [x] = [x] - [-x] \quad \text{for all } x \in \mathbb{R}. \hspace{1cm} (51) $$

We are now ready to classify L-space surgeries on torus-link satellites of L-space knots.

\textbf{Theorem 4.5.} Let $Y := S^{3} \setminus \hat{\nu}(K)$ denote the exterior of a positive L-space knot $K \subset S^{3}$ of genus $g(K)$, and let $Y^{(np, nq)} := S^{3} \setminus \hat{\nu}(K^{(np, nq)})$ denote the exterior of the $(np, nq)$-torus-link satellite $K^{(np, nq)} \subset S^{3}$ of $K \subset S^{3}$, for $n, p, q \in \mathbb{Z}$ with $n, p > 0$, and $\gcd(p, q) = 1$.

Construct $\hat{Y}_{(p, q)}^{n}, \hat{Y}^{n}_{(p, q)}$, and $Y^{(np, nq)} := Y \cup_{p} \hat{Y}^{n}_{(p, q)}$ as in Proposition 4.2, with the Seifert structure $sf$ on $\hat{Y}^{n}_{(p, q)}$ as specified by Proposition 4.2.

There is a change of basis map $\psi : (\mathbb{Q} \cup \{ \infty \})^{n}_{sf} \rightarrow (\mathbb{Q} \cup \{ \infty \})^{n}_{S^{3}}, \quad y \mapsto \big( pq + \frac{1}{y_{1}}, \ldots, pq + \frac{1}{y_{n}} \big)$, which converts the above-specified Seifert basis slopes $(\mathbb{Q} \cup \{ \infty \})^{n}_{sf}$ into conventional link exterior slopes in $S^{3}$, so that $\mathcal{L}_{S^{3}}(Y^{(np, nq)}) = \psi(\mathcal{L}_{sf}(Y^{(np, nq)})).$

(i.a) If $N := 2g(K) - 1 \not\geq \frac{p}{q}$, $K \subset S^{3}$ is nontrivial, and $p > 1$, then $\mathcal{L}_{sf}(Y^{(np, nq)}) = \Lambda_{sf}(Y^{(np, nq)}) := \{ l \in \mathbb{Z}^{n} \in (\mathbb{Q} \cup \{ \infty \})^{n} | \sum_{i=1}^{n} l_{i} = 0 \}$.

(i.b) If $2g(K) - 1 \geq \frac{p}{q}$, $K \subset S^{3}$ is nontrivial, and $p = 1$, then $\mathcal{L}_{sf}(Y^{(np, nq)}) = \Lambda_{sf}(Y^{(np, nq)}) + \mathcal{E}_{n} \left[ \left\lfloor 0, \frac{1}{N_{q}} \right\rfloor \times \{ 0 \}^{n-1} \right]$. 

(ii) If $2g(K) - 1 \leq \frac{p}{q}$ with $K \subset S^{3}$ nontrivial, or if $p, q > 1$ with $K \subset S^{3}$ trivial, then $\mathcal{N}_{sf}(Y^{(np, nq)}) = \mathcal{Z}_{sf}(Y^{(np, nq)}) \cup \{ y \in \mathbb{Q}^{n} | y_{+} < 0 < y_{-} \}$, where

$$ \mathcal{Z}_{sf}(Y^{(np, nq)}) = \{ y \in (\mathbb{Q} \cup \{ \infty \})^{n} | \# \{ i \in \{ 1, \ldots, n \} | y_{i} = \infty \} > 1 \}, $$

$$ y_{-} = -\sum_{i=1}^{n} [y_{i}], $$

$$ y_{+} = -\sum_{i=1}^{n} [y_{i}] - \begin{cases} \frac{1}{p+q-2g(K)p} \sum_{i=1}^{n} [y_{i}] \{ p+q-2g(K)p \} = 0, & \text{otherwise} \\
\end{cases} $$
(iii) If $K \subset S^3$ is the unknot, with $p = 1$ and $q > 0$, then

$$\mathcal{L}_S(Y^{(n,nq)}) = R_S(Y^{(n,nq)}) \setminus Z_S(Y^{(n,nq)}) \cup \{y \in \mathbb{Q}^n \mid M_{S^2}(\frac{1}{q},y) \text{ a SF L-space}\},$$

or equivalently,

$$\{y \in \mathbb{Q}^n \mid 1 - \left[\frac{k}{q}\right] - \sum_{i=1}^{n} [y,k] < 0 < 1 - \left[\frac{k}{q}\right] - \sum_{i=1}^{n} [y,k] \forall k \in \mathbb{Z}_{>0}\}.$$ 

REMARKS. A knot $K \subset S^3$ is called a positive (respectively negative) L-space knot if $K$ admits an L-space surgery for some finite $S^3$-slope $m > 0$ (respectively $m < 0$). Since $\mathcal{L}_S(Y^{(np,nq)}) = -\mathcal{L}_S(Y^{(np,-nq)})$ for $Y^{(np,-nq)}$, the $(np,-nq)$-torus-link satellite of the mirror knot $\bar{K} \subset S^3$, the above theorem, and Theorem 1.2 are easily adapted to satellites of negative L-space knots or to negative torus links. Any with 0 satellite is just the n-component unlink, with $\mathcal{L}_S^3 = \prod_{i=1}^{n} [0, \infty, 0 \cup 0, \infty]$. Note that while Theorem 1.2 excludes the case of torus links proper (satellites of the unknot) which are ‘degenerate’, that is, which have 1 in $\{p, q\}$, this case is treated in (iii) above, setting $p = 1$ without loss of generality. If $q = 0$ in this case, we again have the n-component unlink. For any nontrivial degenerate torus link, part (iii) above implies that the boundary of $\mathcal{L}_S$ follows a piece-wise-constant chaotic pattern, similar to the boundary of the region of Seifert fibered L-spaces. This is unsurprising, since the irreducible surgeries on $T(n, n)$ consist of all oriented Seifert fibered spaces over $S^3$ of $n$ or fewer exceptional fibers. Finally, if $K^{(np,nq)} \subset S^3$ is any nontrivial-torus-link satellite of a non-L-space knot in $S^3$, then the L-space gluing result conjectured in [28] for arbitrary closed oriented 3-manifolds with single-torus boundary — which the authors of [15] have announced they expect to prove in the near future — would imply that $L(Y^{(np,nq)}) = \Lambda(Y^{(np,nq)})$.

Proof of (ψ). Let $Y_j := Y^{(np,nq)}_{SF}(0,\ldots,0,\ast,0,\ldots,0)$ denote the partial Dehn filling of $Y^{(np,nq)}$ which fills in all $n$ boundary components except $\partial Y_j = \partial_j Y^{(np,nq)}$ with regular fiber neighborhoods, so that, by the definitions of $Y^{(np,nq)}$ and $Y^{(np,nq)}_{(p,q)}$ in (47) and (45), we have

$$Y_j = Y \cup \tilde{\varphi} M_{S^2}(\frac{-q^*}{p^*},0,\ldots,0,\ast,0,\ldots,0), \quad \tilde{\varphi} : \partial Y \to -\partial_0 \hat{Y}^{(p,q)},$$

with $\tilde{\varphi}$ as defined in Proposition 4.2, where we recall from Definition 4.1 that $pp^* - qq^* = 1$ with $0 \leq q^* < p$. Observing that $Y_j$ has the Dehn filling $Y_j(\tilde{h}_j) = S^3$, we take $\mu_j := \tilde{h}_j \in H_1(\partial Y_j; \mathbb{Z})$ for the meridian in our $S^3$ surgery basis for $H_1(\partial Y_j; \mathbb{Z})$.

As shown, for example, in [29], any homology class $\lambda_j \in H_1(\partial Y_j; \mathbb{Z})$ representing the rational longitude of $Y_j$ has SF-slope $\pi_{SF}(\lambda_j)$ given by the negative sum of the SF-slope images of the rational longitudes of the manifolds glued into the boundary components of $Y^{(p,q)}_{(p,q)}$, plus the negative sum of Seifert-data slopes (which are just the SF-slope images of the rational longitudes of the corresponding fiber neighborhoods), as follows:

$$\pi_{SF}(\lambda_j) = - \left(\frac{q^*}{p} + \varphi^\circ(\pi_{S^3}(\lambda)) \sum_{i \neq j} 0\right) = - \left(\frac{q^*}{p} + \frac{p^*}{q}\right) = - \frac{1}{pq},$$

This uses the definition in (49) of the induced map $\tilde{\varphi}^\circ : \mathbb{P}(H_1(\partial Y_j; \mathbb{Z}); S^3) \to \mathbb{P}(H_1(\partial_0 \hat{Y}^{(p,q)}; \mathbb{Z}); S^3)$ on slopes, to calculate the slope $\varphi^\circ(\pi_{S^3}(\lambda)) = \varphi^\circ(\pi_{S^3}(0)) = \frac{e^*}{q} \in \mathbb{P}(H_1(\partial_0 \hat{Y}^{(p,q)}; \mathbb{Z}); S^3)$.

To obtain $\mu_j \cdot \lambda_j = 1$, we are constrained by the choice $\mu_j = -\tilde{h}_j$ to select the representative $\lambda_j := \tilde{f}_j + pq \tilde{h}_j$ for the SF-slope $\pi_{SF}(\lambda_j) = - \frac{1}{pq}$. The resulting homology change of basis

$$\mu_j \mapsto 0 \tilde{f}_j - \tilde{h}_j, \quad \lambda_j \mapsto \tilde{f}_j + pq \tilde{h}_j$$

(54)
for $H_1(\partial_j Y^{(np,nq)}; \mathbb{Z})$ then induces a map on slopes with inverse

$$
\psi_j : \mathbb{P}(H_1(\partial_j Y^{(np,nq)}; \mathbb{Z}))_{sf} \to \mathbb{P}(H_1(\partial_j Y^{(np,nq)}; \mathbb{Z}))_{S^3}, \quad y_j \mapsto pq + \frac{1}{y_j}.
$$

(55)

Setup for (i) and (ii). We begin with the case in which $K \subset S^3$ is nontrivial, so that its exterior $Y = S^3 \setminus \hat{\nu}(K)$ is boundary incompressible. It is easy to show (see ‘example’ in [28, Section 4]) that such $Y$ has L-space interval

$$
\mathcal{L}_{S^3}(Y) = [N, +\infty], \quad N := \deg(\Delta(K)) - 1 = 2g(K) - 1,
$$

(56)

where $\Delta(K)$ and $g(K)$ are the Alexander polynomial and genus of $K$. Writing

$$
\varphi^0_\nu(\mathcal{L}(Y))_{sf} = [[y_0^-, y_0^+]],
$$

(57)

we then use (49) to compute that

$$
y_0^- := \frac{Nq^* - p^*}{Np - q} = \frac{q^*}{p} + \frac{1}{p(q - Np)}, \quad y_0^+ := \frac{q^*}{p}.
$$

(58)

For a given $sf$-slope $y := (y_1, \ldots, y_n) \in (\mathbb{Q} \cup \{\infty\})_{sf}^n$, we verify whether the Dehn filling $Y_{sf}^{(np,nq)}(y)$ is an L-space by examining the L-space interval, computed via Theorem 4.3, of a regular fiber exterior in $Y_{sf}^{(np,nq)}(y)$. That is, if we let $Y^{(np,nq)}$ denote the regular fiber exterior

$$
Y^{(np,nq)} := Y^{(np,nq)} \setminus \hat{\nu}(f)
$$

(59)

for a regular fiber $f \subset Y^{(np,nq)}$, then $Y^{(np,nq)}(y)$ is an L-space if and only if the meridional slope $0 \in \mathbb{P}(H_1(Y_{sf}^{(np,nq)}(y); \mathbb{Z}))_{sf}$ satisfies $0 \in \mathcal{L}_{sf}(Y_{sf}^{(np,nq)}(y))$. Since $Y$ is Floer simple and boundary incompressible, Theorem 4.3 tells us that if $\mathcal{L}(Y_{sf}^{(np,nq)}(y))$ is nonempty, then it is determined by $y_-, y_+ \in \mathbb{P}(H_1(Y_{sf}^{(np,nq)}(y); \mathbb{Z}))_{sf}$, where

$$
y_- := \max_{k>0} y_-(k), \quad y_+ := \min_{k>0} y_+(k),
$$

(60)

$$
y_-(k) := -\frac{1}{k}\left(1 + \left\lceil -\frac{q^* k}{p} \right\rceil + \sum_{i=1}^n \lceil y_i k \rceil + ([y_{0^+} k] - 1) \right)
$$

(61)

$$
= \frac{1}{k}\left(\left\lceil \frac{q^* k}{p} \right\rceil - \lfloor y_{0^+} k \rfloor - \sum_{i=1}^n \lfloor y_i k \rfloor \right),
$$

$$
y_+(k) := -\frac{1}{k}\left(-1 + \left\lceil -\frac{q^* k}{p} \right\rceil + \sum_{i=1}^n \lfloor y_i k \rceil + ([y_{0^-} k] + 1) \right)
$$

(62)

$$
= \frac{1}{k}\left(\left\lceil \frac{q^*}{p} \right\rceil - \lfloor y_{0^-} k \rfloor - \sum_{i=1}^n \lfloor y_i k \rfloor \right).
$$

Thus, since $y_{0^+} = \frac{q^*}{p}$, we have

$$
y_-(k) = -\frac{1}{k}\sum_{i=1}^n ([\lfloor y_i \rfloor + [y_i]) k]
$$

(63)

$$
= -\frac{1}{k}\sum_{i=1}^n \lfloor y_i k \rfloor - \sum_{i=1}^n [y_i]
$$

$$
\leq -\sum_{i=1}^n [y_i]
$$
for all $k > 0$, which, since $y_-(1) = -\sum_{i=1}^n [y_i]$, implies
\begin{equation}
y_+ = -\sum_{i=1}^n [y_i].
\end{equation}
(64)

For $y_+$, there are multiple cases.

Proof of (i): $N = 2g(K) - 1 > \frac{q}{p}$. Since $q - Np < 0$, we have $y_{0-} < \frac{q}{p} = y_{0+}$, which, by Theorem 4.4, implies $L(\hat{Y}^{(np,nq)}(y)) \neq \emptyset$ if and only if $y \in \mathbb{Q}^n$ and $y_- \leq y_+$.

Case (a): $p > 1$. Since $0 < y_{0-} < \frac{q}{p} < 1$, we have
\begin{equation}
y_+ \leq y_+(1) = -\sum_{i=1}^n [y_i],
\end{equation}
so that $y_- \leq y_+$ if and only if $\sum_{i=1}^n ([y_i] - [y_i]) \leq 0$, which, for $y \in \mathbb{Q}^n$, occurs if and only if $y \in \mathbb{Z}^n$. If $y \in \mathbb{Z}^n$, then $y_+(k) \geq y_+(1)$ for all $k > 0$, implying
\begin{equation}
y_+ = y_+(1) = -\sum_{i=1}^n y_i = y_-.
\end{equation}
(66)

so that Theorem 4.4 tells us
\begin{equation}
L_{sp}(\hat{Y}^{(np,nq)}(y)) = \{y_-\} = \{y_+\} = \{-\sum_{i=1}^n y_i\}.
\end{equation}
(67)

Thus, since $Y^{(np,nq)}(y)$ is an L-space if and only if $0 \in L_{sp}(\hat{Y}^{(np,nq)}(y))$, we have
\begin{equation}
L_{sp}(Y^{(np,nq)}) = \Lambda_{sp}(Y^{(np,nq)}) := \{y \in \mathbb{Z}^n \subset (\mathbb{Q} \cup \{\infty\})^n | \sum_{i=1}^n y_i = 0\}.
\end{equation}
(68)

Case (b): $p = 1$. In this case, $\frac{q}{p} = 0$ and $y_{0-} = -\frac{1}{N-q}$, so that
\begin{equation}
y_+(k) = \frac{1}{k} \left( \left\lceil \frac{k}{N-q} \right\rceil + \sum_{i=1}^n [\left\lfloor y_i \right\rfloor] \right) - \sum_{i=1}^n [y_i].
\end{equation}
(69)

In particular, since $y_+(1) = 1 - \sum_{i=1}^n [y_i]$, the condition $y_- \leq y_+ \leq y_+(1)$ implies
\begin{equation}
\sum_{i=1}^n ([y_i] - [y_i]) \leq y_+ + \sum_{i=1}^n [y_i] \leq 1,
\end{equation}
(70)

which, for $y \in \mathbb{Q}^n$, occurs only if for some $j \in \{1, \ldots, n\}$ we have $y_i \in \mathbb{Z}$ for all $i \neq j$. For such a $y$, we then have
\begin{equation}
y_+(k) = \frac{1}{k} \left( \left\lceil \frac{k}{N-q} \right\rceil + \left\lfloor y_j \right\rfloor \right) - \sum_{i=1}^n [y_i].
\end{equation}
(71)

For $k > 0$, set $s := \left\lceil \frac{k}{N-q} \right\rceil$ and write $k = s(N-q) - t$ with $0 \leq t < N-q$.

If $\left\lfloor y_i \right\rfloor \geq \frac{N-q-1}{N-q}$, then
\begin{equation}
y_+(k) + \sum_{i=1}^n \left\lfloor y_i \right\rfloor \geq \frac{s(N-q) - \left\lfloor y_j \right\rfloor}{s(N-q) - t} \geq 1
\end{equation}
(72)

for all $k > 0$, which, since $y_+(1) + \sum_{i=1}^n \left\lfloor y_i \right\rfloor = 1$, implies
\begin{equation}
y_+ = 1 - \sum_{i=1}^n [y_i] = -\sum_{i=1}^n [y_i] = y_-,
\end{equation}
(73)

so that $Y^{(np,nq)}(y)$ is an L-space if and only if $\sum_{i=1}^n [y_i] = 0$.

If $\left\lfloor y_i \right\rfloor < \frac{N-q-1}{N-q}$, then
\begin{equation}
y_+ + \sum_{i=1}^n [y_i] \leq y_+(N-q) + \sum_{i=1}^n [y_i] \leq \frac{N-q-1}{N-q} < 1,
\end{equation}
(74)

The left half of (70) then tells us that $\sum_{i=1}^n ([y_i] - [y_i]) < 1$, implying $y \in \mathbb{Z}^n$ and $\sum_{i=1}^n [y_i] = \sum_{i=1}^n [y_i]$. Thus, since $z \geq y_- \leq y_+ < y_+ - 1$, we have $y_- \leq y_+$ if and only if $\sum_{i=1}^n [y_i] = 0$.

In total, we have learned that $y \in L_{sp}(Y^{(np,nq)})$ if and only if $\sum_{i=1}^n [y_i] = 0$ and there exists $j \in \{1, \ldots, n\}$ such that $y_i \in \mathbb{Z}$ for all $i \neq j$ and $\left\lfloor y_j \right\rfloor \in \left[\frac{N-q-1}{N-q}, 1 \right) \cup \{0\}$, or equivalently,
\([y_j] \in [0, \frac{1}{N-p}]\). In other words,

\[
\mathcal{L}_{sf}(Y^{(np,nq)}) = \mathcal{S}_n \cdot \left\{ \left[ l_1 + 0, l_1 + \frac{1}{N-p} \right] \right\} \cap \{(l_2, \ldots, l_n) \mid l_i \in \mathbb{Z}^n, \sum_{i=1}^n l_i = 0 \}
\]

\[= \Lambda_{sf}(Y^{(np,nq)}) + \mathcal{S}_n \left( \left[ 0, \frac{1}{N-q} \right] \times \{0\}^{n-1} \right). \tag{75}\]

Proof of (ii): \(N = 2g(K) - 1 \leq \frac{q}{p}\) and \(q > 0\). We divide this section into three cases: \(N = \frac{q}{p}\), \(N < \frac{q}{p}\) with \(K \subset S^3\) nontrivial, and \(K \subset S^3\) the unknot \(p,q > 1\).

Case \(N = \frac{q}{p}\). Here, \(N > 0\) implies \(K \subset S^3\) is nontrivial, and \(NP = q\) implies \(p = 1\), so that \(y_0+ = \frac{q}{p} = 0\) and \(y_0- = -\frac{1}{N-q} = \infty\). Theorem 4.4(iv) then implies that \(Z_{sf}^{\infty}(Y^{(np,nq)}) \subset \mathcal{N}_{sf}(Y^{(np,nq)})\), but that

\[
\left(\left(\mathbb{Q} \cup \{\infty\}\right)^n \setminus \mathcal{S}_n(Y^{(np,nq)})\right) \subset \mathcal{L}_{sf}(Y^{(np,nq)}), \tag{76}\]

since these are the slopes with \(n^\infty = 1\), and since \(0 < -\infty, +\infty\). For \(y \in \mathbb{Q}^n\), Theorem 4.4(iv) tells us \(\mathcal{L}_{sf}(Y^{(np,nq)}(y)) = [[y-, y_+]] = \{-\sum_{i=1}^n [y_i], +\infty\}, \tag{80}\)

so that \(\mathcal{N}_{sf}(Y^{(np,nq)}) \subseteq \mathcal{N}_{sf}(Y^{(np,nq)})\). For \(y \in \mathbb{Q}^n\), Theorem 4.4(iii) tells us that \(\mathcal{L}_{sf}(Y^{(np,nq)}(y)) = [[y-, y_+]] \cap y_- \geq y_+\), so that \(Y^{(np,nq)}(y)\) is not an L-space if and only if \(y_+ < 0 < y_-\). We therefore have

\[
\mathcal{N}_{sf}(Y^{(np,nq)}) \subseteq \mathcal{N}_{sf}(Y^{(np,nq)}(y)) \cup \{y \in \mathbb{Q}^n \mid y_+ < 0 < y_-\}, \tag{78}\]

where the definitions of \(y_-\) and \(y_+\) are appropriately adjusted in the case that \(K \subset S^3\) is the unknot.

Case \(N < \frac{q}{p}\) with \(K \subset S^3\) nontrivial. We already know that \(y_- = -\sum_{i=1}^n [y_i]\) when \(K \subset S^3\) is nontrivial and \(y \in \mathbb{Q}^n\). Thus, it remains to compute \(y_+\) for \(y \in \mathbb{Q}^n\).

Since \(x = \lfloor x \rfloor - \lceil -x \rceil\) for all \(x \in \mathbb{R}\), we have

\[
y_+(k) = \frac{1}{k} \left( \left\lfloor \frac{q}{p} \right\rfloor k - \lfloor y_0 - k \rfloor + \sum_{i=1}^n \left\lfloor -y_i k \right\rfloor - \sum_{i=1}^n [y_i] \right) \tag{79}\]

for all \(k > 0\). Write \(k = s(q - NP) + t\) for \(s,t \in \mathbb{Z}_{\geq 0}\) with \(s := \lfloor \frac{k}{q - NP} \rfloor\) and \(t < q - NP\). Using the facts that \(q^*(q - NP) = q^*q - Np - 1 - Npq = q^*p - 1 - Npq\) and \(\lfloor w \rfloor \geq \lfloor x \rfloor\) for all \(w, x \in \mathbb{R}\) (in (81)), we obtain

\[
y_+(k) = \frac{1}{k} \left( \left\lfloor \frac{s}{p} \right\rfloor + \left\lfloor \frac{q^*}{p} \right\rfloor - \lfloor y_0 - k \rfloor \right)
\]

\[= \frac{1}{k} \left( s(p^* - Nq^*) + \frac{s + t q^*}{p} \right) - \left( s(p^* - Nq^*) + \frac{t q^* + t/(q - NP)}{p} \right) \tag{80}\]

\[\geq \frac{-1}{s(q - NP) + t} \left\lfloor \frac{s}{p} \right\rfloor \tag{81}\]
If \( \sum_{i=1}^{n} \lfloor -y_i \rfloor (q - Np) = 0 \), then, writing \( k = 1(q - Np) + 0 \), we can use line (80) to compute \( \bar{y}_+(q - Np) \), so that we obtain
\[
y_+(q - Np) = \bar{y}_+(q - Np) + 0 - \sum_{i=1}^{n} \lfloor y_i \rfloor = -\frac{1}{q - Np} - \sum_{i=1}^{n} \lfloor y_i \rfloor.
\] (83)

Thus, since (82) implies \( y_+(k) \geq -\frac{1}{q - Np} - \sum_{i=1}^{n} \lfloor y_i \rfloor \) for all \( k > 0 \), we conclude that
\[
y_+ = -\frac{1}{q - Np} - \sum_{i=1}^{n} \lfloor y_i \rfloor.
\] (84)

On the other hand, if \( \sum_{i=1}^{n} \lfloor -y_i \rfloor (q - Np) > 0 \), then we know there exists \( i_* \in \{1, \ldots, n\} \) for which \( y_{i_*} \geq (q - Np)^{-1} \). Thus, writing \( k = s(q - Np) + t \) and using line (81), we obtain the lower bound
\[
y_+(k) \geq \bar{y}_+(k) + \frac{1}{k} \left( \frac{1}{q - Np} \right) - \sum_{i=1}^{n} \lfloor y_i \rfloor
\geq \frac{1}{k} \left( \frac{s}{p} \right) + \frac{1}{k} s - \sum_{i=1}^{n} \lfloor y_i \rfloor
\geq -\sum_{i=1}^{n} \lfloor y_i \rfloor
\] (85)

Since this bound is realized by \( y_+(1) = -\sum_{i=1}^{n} \lfloor y_i \rfloor \), we deduce that \( y_+ = -\sum_{i=1}^{n} \lfloor y_i \rfloor \). Thus, since \( q - Np = p + q - 2g(K)p \), the last line of Theorem 4.5(ii) holds.

Case \( N < \frac{2}{p} \) with \( p, q > 1 \) and \( K \subset S^3 \) the unknot. Since \( Y := S^3 \setminus \nu(K) \) satisfies
\[
\mathcal{L}_{S^3}(Y) = [[0, 0]] = \mathbb{Q} \cup \{ \infty \} \setminus \{ 0 \},
\] (86)

we use (49) to compute, for \( \varphi^p_*(\mathcal{L}(Y))_{\nu^p} = [[y_{0-}, y_{0+}]] \), that
\[
y_{0-} = y_{0+} = \frac{0q^* - 1p^*}{0p - 1q} = \frac{p^*}{q} = \frac{q^*}{p} + \frac{1}{pq}.
\] (87)

Thus, applying Theorem 4.3 and mildly simplifying, we obtain that
\[
y _{-} = \sup_{k>0} y_-(k) \quad \text{and} \quad y_+ = \inf_{k>0} y_+(k), \quad \text{for} \quad (88)
\]

\[
y_-(k) := \frac{1}{k} \left( -1 + \left[ \frac{q^*}{p} k \right] - \left[ \frac{p^*}{q} k \right] - \sum_{i=1}^{n} \lfloor [y_i] k \rfloor \right) - \sum_{i=1}^{n} \lfloor y_i \rfloor
\] (89)

\[
y_+(k) := \frac{1}{k} \left( 1 + \left[ \frac{q^*}{p} k \right] - \left[ \frac{p^*}{q} k \right] + \sum_{i=1}^{n} \lfloor [-y_i] k \rfloor \right) - \sum_{i=1}^{n} \lfloor y_i \rfloor.
\] (90)

For \( y_-(k) \), we (again) obtain the bound
\[
y_-(k) = \frac{1}{k} \left( \left[ \frac{q^*}{p} k \right] - \left( \left[ \frac{q^*}{p} + \frac{1}{pq} \right] k \right) + 1 \right) - \sum_{i=1}^{n} \lfloor [y_i] k \rfloor) - \sum_{i=1}^{n} \lfloor y_i \rfloor
\leq -\sum_{i=1}^{n} \lfloor y_i \rfloor \quad \text{for all} \quad k \in \mathbb{Z}_{>0},
\] (91)

which, for \( p, q > 1 \) is realized by \( y_-(1) = -\sum_{i=1}^{n} \lfloor y_i \rfloor \), so that \( y_- = -\sum_{i=1}^{n} \lfloor y_i \rfloor \).
To compute $y_+$, we note that since $p, q > 1$, we can invoke Lemma 4.7 (below), so that

$$1 + \left\lfloor \frac{q^*}{p} \frac{k}{q} \right\rfloor + \left\lfloor \frac{k}{p + q} \right\rfloor \geq \left\lceil \frac{p^*}{q} \frac{k}{q} \right\rceil$$

for all $k \in \mathbb{Z}_{>0}$. (92)

(Here, we multiplied the original inequality by $k$ and then observed that the integer on the left-hand side must be bounded by an integer.) In particular,

$$\frac{1}{k} \left( 1 + \left\lfloor \frac{q^*}{p} \frac{k}{q} \right\rfloor - \left\lfloor \frac{p^*}{q} \frac{k}{q} \right\rfloor - \frac{k}{p + q} \right) - \sum_{i=1}^{n} [y_i] \geq -\sum_{i=1}^{n} [y_i]$$

for all $k \in \mathbb{Z}_{>0}$. (93)

Thus, when $\sum_{i=1}^{n} [(-y_i)(p + q)] > 0$, so that at least one $y_i$ satisfies $[-y_i] \geq \frac{1}{p+q}$, line (93) tells us that $y_+(k) \geq -\sum_{i=1}^{n} [y_i]$, a bound which is realized by $y_+$ (1) when $p, q > 1$. On the other hand, if $\sum_{i=1}^{n} [(-y_i)(p + q)] = 0$, then (93) implies

$$y_+(k) \geq -\frac{1}{k} \left\lfloor \frac{k}{p + q} \right\rfloor - \sum_{i=1}^{n} [y_i] \geq -\frac{1}{p + q} - \sum_{i=1}^{n} [y_i]$$

for all $k \in \mathbb{Z}_{k>0}$, (94)

a bound which is realized by $y_+(p + q)$. We therefore have

$$y_+ = -\sum_{i=1}^{n} [y_i] - \begin{cases} \frac{1}{p+q} & \sum_{i=1}^{n} [(-y_i)(p + q)] = 0 \\ 0 & \sum_{i=1}^{n} [(-y_i)(p + q)] > 0 \end{cases}$$

(95)

completing the proof of part (ii).

Proof of (iii): $K \subset S^3$, $p = 1, q > 0$. Here, we have the same case as above, but with $p = 1$ and $q > 0$, implying $\frac{q}{p} = 0$ and $\frac{p}{q} = \frac{1}{q}$. Thus, the Dehn filling $Y_{sf}^{(n, nq)}(y)$ is the Seifert fibered space $M_{S^2}(\frac{1}{q}, y)$, and we have

$$\mathcal{L}_{sf}(Y^{(n, nq)}) = \mathcal{L}_{sf}(Y^{(n, nq)}) \cup \{y \in \mathcal{Q}^n | y_+ < 0 < y_- \}$$

$$= \mathcal{L}_{sf}(Y^{(n, nq)}) \cup \{y \in \mathcal{Q}^n | 1 - \left\lfloor \frac{k}{q} \right\rfloor - \sum_{i=1}^{n} [y_i k] < 0 < -1 - \left\lfloor \frac{k}{q} \right\rfloor - \sum_{i=1}^{n} [y_i k] \ \forall k \in \mathbb{Z}_{>0} \}.$$

The above theorem leads to the following

Corollary 4.6. Theorem 1.2 holds.

Proof. For the $p > 1$ case of part (i), we simply replace $\Lambda_{sf}$ with $\Lambda_{S^3}$, which contains the $S^3$-slope $(\infty, \ldots, \infty)$ in its orbit. For part (ii) and for the $p=1$ case of part (i), it is straightforward to show that both the expression in the bottom line of part (ii) of Theorem 1.2 and the expression $\mathcal{S}_n \cdot (\mathcal{N}, +\infty) \times \{\infty\}^{n-1}$ in part (i) contain fundamental domains (under the action of $\Lambda$) of the respective L-space regions specified above.

We now return to the lemma cited in the proof of Theorem 4.5(ii).

Lemma 4.7. If $p, q > 1$, then

$$\frac{1}{k} \left( 1 + \left\lfloor \frac{q^*}{p} \frac{k}{q} \right\rfloor - \left\lfloor \frac{p^*}{q} \frac{k}{q} \right\rfloor - \frac{k}{p + q} \right) \geq \frac{p^*}{q}$$

for all $k \in \mathbb{Z}_{>0}$. (97)
we define \( z(k) \in \mathbb{Q} \) for all \( k \in \mathbb{Z}_{>0} \), as follows:

\[
z(k) := k \left( \frac{1}{k} \left( 1 + \left\lfloor \frac{q^*}{p} \right\rfloor + \left\lfloor \frac{k}{p+q} \right\rfloor - \frac{p^*}{q} \right) - 1 \right)
\]

\[
= 1 + \frac{kq^* - kp^*}{pq} - \frac{[kq^*]_p}{p} + \left\lfloor \frac{k}{p+q} \right\rfloor
\]

\[
= \frac{[kq^{-1}]_p}{p} - \frac{k}{pq} + \left\lfloor \frac{k}{p+q} \right\rfloor + \begin{cases} 1 & [k]_p = 0 \\ 0 & [k]_p \neq 0 \end{cases}
\]

(98)

where we note that

\[
pq - (p + q) = (p - 1)(q - 1) - 1 \geq 0 \quad \text{for} \quad p, q > 1.
\]

(100)

We next claim that \( z(k) \geq 0 \) if \( z(k - (p + q)) \geq 0 \). First, for \( [kq^{-1}]_p \notin \{0, 1\} \), we have

\[
z(k) - z(k - (p + q)) = \frac{q - 1}{pq} + \frac{pq - (p + q)}{pq} > 0,
\]

(101)

so that \( z(k) > z(k - (p + q)) \geq 0 \). If \( [kq^{-1}]_p = 0 \), implying \( [k]_p = 0 \), then line (98) gives

\[
z(k) = -\frac{k}{pq} + \left\lfloor \frac{k}{p+q} \right\rfloor + 1 \geq \frac{k}{p+q} - \frac{k}{pq} \geq 0.
\]

(102)

This leaves us with the case in which \( [kq^{-1}]_p = 1 \), so that line (98) yields

\[
z(k) = \left\lfloor \frac{k}{p+q} \right\rfloor - \frac{k - q}{pq}.
\]

(103)

Since \( [kq^{-1}]_p = 1 \) implies \( k \equiv q \pmod{p} \), we can write \( k = (sq + t)p + q \), with \( s = \left\lfloor \frac{k - q}{p} \right\rfloor \) and \( t \in \{0, \ldots, q - 1\} \). When \( t = 0 \), we obtain

\[
z(k) = z(spq + q) = \left\lfloor \frac{spq + q}{p+q} \right\rfloor - s = \left\lfloor \frac{spq + q - s}{p+q} \right\rfloor \geq 0 \quad \text{for all} \quad p, q > 1.
\]

(104)

On the other hand, when \( t \geq 1 \), we have

\[
z(k) \geq [z(k)] = \left\lfloor \frac{spq + tp + q}{p+q} \right\rfloor - (s + 1) = \left\lfloor \frac{s(pq - (p + q)) + (t - 1)p}{p+q} \right\rfloor \geq 0,
\]

(105)

completing the proof of our claim. Since the case \( k = p + q \) is subsumed in the case \( [kq^{-1}]_p = 1 \), we also have \( z(p + q) \geq 0 \), and so by the induction, it suffices to prove the lemma for \( k < p + q \).

Suppose that \( 0 < k < p + q \) and \( k \notin p\mathbb{Z} \) (since \( z(k) \geq 0 \) for \( [k]_p = 0 \)), so that we now have

\[
z(k) = \frac{1}{pq} (q[kq^{-1}]_p - k).
\]

(106)

Since \( z(aq) = \frac{1}{pq} (q \cdot a - aq) = 0 \) for \( a \in \mathbb{Z} \), we may also assume \( k \notin q\mathbb{Z} \). Now, the Chinese Remainder Theorem tells us that

\[
k = [q[kq^{-1}]_p + p[kp^{-1}]_q]_{pq},
\]

(107)

but since \( 0 < k < p + q \) and \( k \notin p\mathbb{Z} \cup q\mathbb{Z} \), we also have

\[
k < p + q \leq q[kq^{-1}]_p + p[kp^{-1}]_q < 2pq,
\]

(108)
requiring that \( q[kq^{-1}]_p + p[kp^{-1}]_q = k + pq \),

so that \( z(k) = \frac{1}{pq} (pq - p[kp^{-1}]_q) > 0 \).

\[ \square \]

5. L-space region topology

5.1. Topologizing L-space regions

Before examining the topology of various L-space and non-L-space regions, we must first determine a meaningful way to topologize these spaces.

We first note that many of the L-space regions we have constructed so far arose as the intersection of the set of rational slopes with a connected space of real slopes. That is not a coincidence. In the Fukaya-category-theoretic description of bordered (Heegaard) Floer homology, as discussed abstractly by Auroux in [1] and realized explicitly in the case of a single torus boundary component by Hanselman, Rasmussen, and Watson in [15], the mechanism that associates a Heegaard Floer group to a choice of Dehn filling slope and spin\(^c\) structure does not care whether that Dehn filling slope and spin\(^c\) structure are rational. I do not yet know the correct topological/geometric interpretation for these groups or irrational spin\(^c\) structures in the case of irrational Dehn fillings, but the groups themselves — along with whether they satisfy the L-space condition — are perfectly well defined.

This has topological consequences for L-space slope regions. Suppose \( Y \) is a compact oriented 3-manifold with \( n \)-component toroidal boundary \( \partial Y = \bigcup_{i=1}^{n} \partial_i Y \), and let \( \mathcal{L}(Y), B(Y) \subset \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{Z})) \), respectively, denote the L-space region and the set of rational longitudes for \( Y \); that is, \( B(Y) \) is the set of slopes \( y \) such that \( b_1(Y(y)) > b_1(Y) - n \). We know that \( \mathcal{L}(Y) \cap B(Y) = \emptyset \) because all L-spaces have \( b_1 = 0 \), and we know \( B(Y) \) is closed because it is locally the vanishing locus of the determinant of a matrix. We also know from Proposition 3.2, first proved in [28], that when \( n = 1 \), the set \( \mathcal{L}(Y) \cup B(Y) \) is closed in \( \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \).

Our proof of [28, Proposition 3.2] used intrinsically rational methods, but when Hanselman et al. proved the analogous result using Fukaya-category-theoretic bordered Floer homology methods in [15], the closedness of \( \mathcal{L}(Y) \cup B(Y) \) resulted from a closed condition on the space of real slopes, involving Lagrangian intersections with lines of these slopes. Although an \( n > 1 \) version of these Fukaya-category-theoretic methods has not been written down explicitly, from its abstract description I still expect the space \( \mathcal{L}(Y) \cup B(Y) \) to arise as the intersection of \( \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{Z})) \) with a space of real slopes cut out by a closed condition.

All this is to motivate the idea that the closure \( \overline{\mathcal{L}(Y) \cup B(Y)} \subset \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{R})) \) has intrinsic mathematical meaning and is a natural object of study; hence the following notation.

**Definition 5.1.** If \( Y \) is a compact oriented 3-manifold with \( n \)-component toroidal boundary, with rational longitude set \( B(Y) \), L-space region \( \mathcal{L}(Y) \), and non-L-space region \( \mathcal{N}(Y) \) in \( \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{Z})) \), we define the respective \( \mathcal{B}(Y) \)-corrected \( \mathbb{R} \)-completions of these sets to be

\[
\mathcal{B}(Y)^R := \overline{\mathcal{B}(Y)}, \quad \mathcal{L}(Y)^R := \overline{\mathcal{L}(Y) \setminus \mathcal{B}(Y)}, \quad \mathcal{N}(Y)^R := (\mathcal{L}(Y)^R)^c \subset \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{R})),
\]

(111)

where \( {}^c \) and \( {}^R \), respectively, denote closure and complement in \( \prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y; \mathbb{R})) \).

We shall also sometimes use the notation \( A^R \) for a subset of \( \mathcal{L}(Y)^R \) deriving from a subset \( A \subset \mathcal{L}(Y) \) or for a subset of \( \mathcal{N}(Y)^R \) deriving from a subset \( A \subset \mathcal{N}(Y) \), but in such cases we shall define \( A^R \) explicitly. We furthermore use the notation \( [a, b]^R \) for closed real intervals.
5.2. L-space region topology for torus links

There are five qualitatively different topologies possible for the L-space region of a torus-link-satellite of a knot in $S^3$.

**Theorem 5.2.** For $n, p, q \in \mathbb{Z}$ with $n, p \geq 0$ and $\gcd(p, q) = 1$, let $K^{(np,nq)} \subset S^3$, with exterior $Y^{(np,nq)} := S^3 \setminus \tilde{B}(K^{(np,nq)})$, be the $T(n,p,q)$-satellite of a positive L-space knot $K \subset S^3$, with $K$ nontrivial in cases (i) and (ii.a) below. Associate $\Lambda$, $L$, and $NL$ to $Y^{(np,nq)}$ as in Theorem 4.5, let $B$ be the set of rational liftings of $Y^{(np,nq)}$ as in Section 2.5, and let $L^R = \overline{\mathcal{L}} \setminus B, \mathcal{N}L^R = \mathcal{L}^R)$, and $B^R = B$ as above.

(i.a) If $2g(K) - 1 > \frac{2}{p}$ and $p > 1$, or if $2g(K) - 1 > \frac{2}{p} + 1$ and $p = 1$, then $L^R$ deformation retracts onto $\Lambda$.

(i.b) If $2g(K) - 1 = \frac{2}{p} + 1$ and $n > 2$, then $L^R$ is connected, $\pi_1(L^R) \simeq \ker(\delta)$ as in (118), and rank $H_1(L^R) = \infty$.

(i.c) If $2g(K) - 1 = \frac{2}{p} + 1$ and $n \in \{1, 2\}$, then $L^R$ is contractible.

(ii.a) If $2g(K) - 1 = \frac{2}{p}$, then $L^R$ is contractible.

(ii.b) If $2g(K) - 1 < \frac{2}{p}$, including the case of $K^{(np,nq)} = T(n,p,q)$ with $p, q > 1$, then $NL^R$ deformation retracts onto the $(n-1)$-torus $B^R = T_n^1 \subset (\mathbb{R} \cup \{\infty\})^n = T^n$, and $L^R$ deformation retracts onto a $T_{n-1}^1$ parallel to $B^R$.

**Remark.** The reader may have noticed that in all cases considered in this paper, we have

$$\overline{\mathcal{L}} \cap B \subset \mathcal{N} \cap B = \mathcal{L}, \text{ implying } L^R = \overline{\mathcal{L}} \setminus \mathcal{N}. \quad (112)$$

This is a consequence of the general fact that open L-space intervals only ever arise as exteriors of knots in 3-manifolds with $S^1 \times S^2$ connected summands [10]. Note that $\mathcal{L}^R_{SF} \cap \mathcal{N}^R_{SF} = \emptyset$.

**Proof of (i.a).** Since we already have $\mathcal{L} = \Lambda$ for $p > 1$, assume $p = 1$. Theorem 4.5 then implies that $L^R_{SF} = \Lambda_{SF} + P(N, q)$, where $N := 2g(K) - 1$ and

$$P(N, q) := \bigcup_{i=1}^n \left( \left\{ 0 \right\}^{i-1} \times \left( 0, \frac{1}{N-q} \right)^R \times \left\{ 0 \right\}^{n-i} \right) \subset (\mathbb{R} \cup \{\infty\})^n, \quad (113)$$

with $[\cdot]^R$ denoting a real interval. Clearly $P(N, q)$ deformation retracts onto $0 \in (\mathbb{Q} \cup \{\infty\})^n$. Since $2g(K) - 1 \geq \frac{2}{p} + 1$, we have $P(N, q) \subset \bigcup_{i=1}^n \left[ 0, \frac{1}{2} \right]^R$. Thus all of the translates $\{ l + P(N, q) \}_{l \in \Lambda_{SF}}$ are pairwise disjoint, and $L^R_{SF}$ deformation retracts onto $\Lambda_{SF}$.

**Proof of (i.b).** When $2g(K) - 1 = \frac{2}{p} + 1$, and $n > 1$, line (113) still holds, but this time with $[0, \frac{1}{N-q}] = [0, 1]$. To see that $|\pi_0(L^R_{SF})| = 1$, first note that $\Lambda_{SF}$ is generated by the elements

$$\varepsilon_{ij} := \varepsilon_i - \varepsilon_j, \quad i, j \in \{1, \ldots, n\}, \quad \text{with } \varepsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n, \quad (114)$$

the standard basis element for $\mathbb{Z}^n$. Then for any such $\varepsilon_{ij}$ and any $l \in \Lambda_{SF}$, the origin $l$ of the translate $P_l := l + P(N, q)$ is path-connected to the origin $\varepsilon_{ij} + l$ of the translate $P_{\varepsilon_{ij} + l}$, via the path $\gamma^l_{\varepsilon_{ij}}(t) : [0, 1] \rightarrow L^R_{SF}$,

$$\gamma^l_{\varepsilon_{ij}}|_{[0, \frac{1}{2}]}(t) = l + 2t\varepsilon_i, \quad \gamma^l_{\varepsilon_{ij}}|_{[\frac{1}{2}, 1]}(t) = (\varepsilon_{ij} + l) + 2(1 - t)\varepsilon_j, \quad \text{satisfying} \quad (115)$$

$$\gamma^l_{\varepsilon_{ij}}(0) = l, \quad \gamma^l_{\varepsilon_{ij}}(1) = \varepsilon_i = P_l \cap P_{\varepsilon_{ij} + l} = (\varepsilon_{ij} + l) + \varepsilon_j, \quad \gamma^l_{\varepsilon_{ij}}(1) = \varepsilon_{ij} + l. \quad (116)$$

Thus $L^R_{SF}$ is path-connected (hence connected), and in fact, these basic paths $\gamma^l_{\varepsilon_{ij}}$ from $l \in P_l$ to $\varepsilon_{ij} + l \in P_{\varepsilon_{ij} + l}$ generate the groupoid $G$ of homotopy classes of paths in $L^R_{SF}$ between elements.
of $\Lambda_{sf}$. Let $G_0 \subset G$ denote the subset of homotopy classes of paths starting at 0, so that elements of $G_0$ are uniquely represented by reduced words

$$g = \left( \gamma_{i_0j_0} \frac{e_{i_0}}{e_{j_0}} \gamma_{i_1j_1} \frac{e_{i_1}}{e_{j_1}} \cdots \frac{e_{i_m}}{e_{j_m}} \gamma_{i_mj_m} \right)_{e_m} \in G$$

(117)

for $i_k < j_k$, $e_k \in \{\pm 1\}$, where $l_0 := 0$ and $l_{k+1} := l_k + e_k \epsilon_{i_kj_k} \forall k$,

with right-multiplication corresponding to concatenation of paths. Note that since $\epsilon_{ji} = -\epsilon_{ij}$, we have replaced all instances of $\gamma_{ij}$ with $(\gamma_{ji}^{-1} - 1) = \gamma_{ij}$ whenever $i > j$.

If we introduce the free group $\mathfrak{F}(\mathfrak{g})$ and epimorphism $\delta : \mathfrak{g} \rightarrow \Lambda_{sf}$,

$$\mathfrak{g}(\mathfrak{g}) := \left\{ x_{ij} \right\}_{1 \leq i < j \leq n}$$

$\delta : \mathfrak{g} \rightarrow \Lambda_{sf}$, $\delta : x_{ij} \mapsto \epsilon_{ij}$,

(118)

then an inductive argument on word length shows that the forgetful map

$$\rho : G_0 \rightarrow \mathfrak{g}(\mathfrak{g})$$

(119)

on words is invertible. That is, starting with $\rho^{-1}(1) = 1$, we can use the inductive rule

$$\rho^{-1}(w \cdot (x_{ij})^e) = \rho^{-1}(w) \cdot \left( \delta(w) + (\frac{e_{ij}}{2}) \epsilon_{ij} \right)^e$$

(120)

for any $i < j$, $e \in \{\pm 1\}$, and word $w \in x_{ij} > 1$ with known $\rho^{-1}(w)$, to reconstruct the map $\rho^{-1}$. Thus, $G_0$ inherits the structure of a free group on $\binom{n}{2}$ generators. Since $\delta \circ \rho(g)$ is the endpoint of any path $g \in G_0$, we then have

$$\pi_1(\mathcal{L}_{sf}^R) \simeq \ker \left( \delta : \mathfrak{g}(\mathfrak{g}) \rightarrow \Lambda_{sf} \right), \text{ rank } H_1(\mathcal{L}_{sf}^R) = \begin{cases} \infty & n > 2 \\ 0 & n \leq 2 \end{cases}$$

(121)

Proof of (i.c). When $n = 2$, the discussion in (i.b) still holds, so $| \pi_0(\mathcal{L}_{sf}^R) | = 1$ and $\pi_1(\mathcal{L}_{sf}^R) = \ker \delta = 1$. Thus, just as for $n = 1$, we have $\mathcal{L}_{sf}^R$ contractible and of dimension 1.

Proof of (ii.a). Case $N = \frac{2}{p}$ of part (ii) of the proof of Theorem 4.5 states in (77) that

$$\mathcal{N} \mathcal{L}_{sf} = \mathcal{Z}_{sf} \cup \left\{ y \in \mathbb{Q}^n_{sf} \mid -\infty < y_-(y) \right\}$$

(122)

with $y_-(y) = -\sum_{i=1}^n |y_i|$,

where we recall that

$$\mathcal{Z}_{sf} = \bigcup_{i < j} \left\{ y \in (\mathbb{Q} \cup \{\infty\})^n_{sf} \mid y_i = y_j = \infty \right\}$$

(123)

We therefore have

$$\mathcal{L}_{sf}^R = \mathcal{Z}_{sf} \setminus \mathcal{Z}_{sf} = \left\{ y \in \mathbb{R}^n_{sf} \mid \sum_{i=1}^n |y_i| > 0 \right\} \setminus \mathcal{Z}_{sf},$$

(124)

which is contractible and of dimension $n$.

Proof of (ii.b). Since the result is trivial for $n = 1$, we henceforth assume $n > 1$. Note that our hypotheses imply $q > 0$. The statement and proof of Theorem 4.5 then tell us that

$$\mathcal{N} \mathcal{L}_{sf} = \mathcal{Z}_{sf} \cap \mathcal{N}$$

(125)

$$y_-(y) := \max_{k > 0} y_-(y, k), \quad y_-(y, k) := -\frac{1}{k} \sum_{i=1}^n |y_i| - c_-(k)$$

(126)

$$y_+(y) := \min_{k > 0} y_+(y, k), \quad y_+(y, k) := -\frac{1}{k} \sum_{i=1}^n |y_i| - c_+(k)$$

(127)

for all $y \in \mathbb{Q}^n_{sf}$ and for certain $c_-(k), c_+(k) \in \mathbb{Z}$ bounded above and below by linear functions in $k$, and determined by $p, q$, and $2g(K) - 1$, and on whether $K \subset S^3$ is trivial. In particular, each of $c_-(k)$ and $c_+(k)$ are independent of $y \in \mathbb{Q}^n_{sf}$. 


Since $\mathcal{NL}_{sf} \setminus \mathbb{R}_{sf}^{n} = \mathcal{Z}_{sf} = \overline{B}_{sf} \setminus \mathbb{R}_{sf}^{n}$, it remains to show that $\mathcal{N}_{sf}^{\mathbb{R}} := \mathcal{N} \cap \mathbb{R}_{sf}^{n}$ deformation retracts to $\overline{B}_{sf}^{\mathbb{R}} \cap \mathbb{R}_{sf}^{n}$, where we recall that $\overline{B}_{sf}^{\mathbb{R}} = \overline{B}_{sf}$, so that

$$B_{sf}^{\mathbb{R}} \cap \mathbb{R}_{sf}^{n} = \left\{ y \in \mathbb{R}_{sf}^{n} : \frac{1}{pq} + \sum_{i=1}^{n} y_i = 0 \right\}.$$ (128)

To construct such a deformation retraction, we first define

$$1 := (1, \ldots, 1) \in \mathbb{Z}_{sf}^{n}, \quad l(y) := -\frac{1}{pq} - \sum_{i=1}^{n} y_i \in \mathbb{R},$$ (129)

so that for $y \in \mathbb{Q}_{sf}, l(y)$ is the rational longitude of the exterior $\hat{Y}^{(np,nq)}(y) := Y^{(np,nq)}(y) \setminus \nu(f)$ of a regular fiber $f$ in $Y^{(np,nq)}(y)$. We then claim that the homotopy

$$z : [0, 1] \times \mathbb{R}_{sf}^{\mathbb{R}} \to \mathbb{R}_{sf}^{n}, \quad z_t(y) := y + t \cdot \frac{1}{n} l(y) 1$$ (130)

provides a deformation retraction from $\mathcal{N}_{sf}^{\mathbb{R}}$ to $\overline{B}_{sf}^{\mathbb{R}} \cap \mathbb{R}_{sf}^{n} \subset \mathcal{N}_{sf}^{\mathbb{R}}$.

First note that (125) also implies that $\mathcal{NL}_{sf}^{\mathbb{R}}(Y^{(np,nq)}(z)) = y_+(z), y-(z) > 0$ for all $z \in \mathbb{Q}_{sf}$. Thus, for all $z \in \mathbb{Q}_{sf}^{n}$, we have $l(z) \in \mathcal{NL}_{sf}^{\mathbb{R}}(Y^{(np,nq)}(z))$, so that $l(z) \in y_+(z), y-(z) >$. Thus,

$$0 < (1-t)l(y) = l(z_t(y)) < y_-(z_t(y))$$ for all $t \in [0, 1) \cap \mathbb{Q}$ if $0 < l(y);$$ (131)

$$0 > (1-t)l(y) = l(z_t(y)) > y_+(z_t(y))$$ for all $t \in [0, 1) \cap \mathbb{Q}$ if $0 > l(y),$$ (132)

for all $y \in \mathbb{Q}_{sf}^{n}$, where the equivalence $(1-t)l(y) = l(z_t(y))$ follows quickly from the definitions of $l$ and $z$. Whether by Theorem 4.5 or by Calegari and Walker’s studies of ‘zigzags’ [6], we know that as functions on $\mathbb{R}_{sf}^{n}$, $y_-$ and $y_+$ are piecewise constant in each coordinate, with rational endpoints. Thus, since $l$ and $z$ are linear and the above inequalities are strict, we have

$$0 < y_-(z_t(y))$$ if $0 < l(y),$$ (133)

for all $y \in \mathbb{R}_{sf}^{n}$ and $t \in [0, 1) \cap \mathbb{R}$, where again, $[\cdot]_{\mathbb{R}}$ and $[\cdot]_{\mathbb{R}}^{\mathbb{R}}$ denote real intervals.

On the other hand, our definitions of $y_{\pm}$ and $z$ imply that for any $y \in \mathcal{N}_{sf}^{\mathbb{R}}$, we have

$$y_+(z_t(y)) \leq y_+(y) < 0$$ for all $t \in [0, 1]_{\mathbb{R}}$ if $0 < l(y);$$ (134)

$$y_-(z_t(y)) \geq y_-(y) > 0$$ for all $t \in [0, 1]_{\mathbb{R}}$ if $0 > l(y).$$ (135)

Combining these three lines of inequalities tells us that for any $y \in \mathcal{N}_{sf}^{\mathbb{R}}$, we have $z_t(y) \in \mathcal{N}_{sf}^{\mathbb{R}}$ for all $t \in [0, 1]_{\mathbb{R}}$. Thus $z$ provides a deformation retraction from $\mathcal{N}_{sf}^{\mathbb{R}}$ to $\overline{B}_{sf}^{\mathbb{R}} \cap \mathbb{R}_{sf}^{n} \subset \mathcal{N}_{sf}^{\mathbb{R}}$.

5.3. Topology of monotone strata

Sections 6 and 7 analyze the L-space surgery regions for satellites by iterated torus-links and by algebraic links, respectively. While these sections primarily focus on approximation tools, Section 7.6 returns to the question of exact L-space regions for such satellites, and describes how to decompose these L-space regions into strata according to monotonicity criteria, which govern where the endpoints of local L-space intervals lie, relative to asymptotes of maps on slopes induced by gluing maps.

Local monotonicity criteria also help determine the topology of these strata, a phenomenon we illustrate with Theorem 7.5 in Section 7, where we show that the $\overline{B}$-corrected $\mathbb{R}$-completion of the monotone stratum of the L-space surgery region of an appropriate satellite link admits a deformation retraction onto an embedded torus analogous to that in Theorem 5.2(ii.b) above.
6. Iterated torus-link satellites

Just as one can construct a torus-link-satellite exterior from a knot exterior by gluing an appropriate Seifert fibered space to the knot exterior (as described in Proposition 4.2), one constructs an iterated torus-link-satellite exterior by gluing an appropriate rooted (tree-)graph manifold to the knot exterior, where this graph manifold is formed by iteratively performing the Seifert-fibered-gluing operations associated to individual torus-link-satellite operations.

6.1. Construction of iterated torus-link-satellite exteriors

An iterated torus-link-satellite of a knot exterior \( Y = M \setminus \tilde{\nu}(K) \) is specified by a weighted, rooted tree \( \Gamma \), corresponding to the minimal JSJ decomposition of the graph manifold glued to the knot exterior to form the satellite. We weight each vertex \( v \in \text{Vert}(\Gamma) \) by the 3-tuple \((p_v, q_v, n_v) \in \mathbb{Z}^3\) corresponding to the pattern link \( T_v := T(n_v p_v, n_v q_v) \). As usual, we demand that \( p_v, n_v > 0 \) and that \( q_v \neq 0 \). (If any vertex had \( p_v = 0 \) or \( q_v = 0 \), then our satellite-link exterior would be a nontrivial connected sum, in which case we might as well have considered the irreducible components of the exterior separately. Moreover, the links of complex surface singularities are irreducible, so the algebraic links we consider later on will necessarily be irreducible.)

The weight \((p_v, q_v, n_v)\) also specifies the JSJ component \( Y_v \) as the Seifert fibered exterior
\[
Y_v := \hat{Y}_{(p_v, q_v)} = M_{S^2}(-\frac{q_v}{p_v}, 0, \frac{1}{n_v}, \ldots, \frac{n_v}{n_v}) = \hat{Y}_{(p_v, q_v)} \setminus \tilde{\nu}(T_v)
\] (136)
of \( T_v \subset \hat{Y}_{(p_v, q_v)} \) as a link in the solid torus
\[
\hat{Y}_{(p_v, q_v)} := M_{S^2}(-\frac{q_v}{p_v}, \frac{p_v}{q_v}) \setminus \tilde{\nu}(\lambda_0) = S^3 \setminus \nu(\lambda_0) = \nu(\lambda_{-1}),
\] (137)
for \( \lambda_{-1} \) the multiplicity-\( p_v \) fiber of meridional \( y^+ \)-slope \( y^+_{-1} = -\frac{q_v}{p_v} \), and \( \lambda_0 \) the multiplicity-\( q_v \) fiber of meridional \( y^+ \)-slope \( y^+ = \frac{p_v}{q_v} \), as in Section 4.1. As usual, \((p_v^*, q_v^*) \in \mathbb{Z}^2\) denotes the unique pair of integers satisfying \( p_v p_v^* - q_v q_v^* = 1 \) with \( q_v^* \in \{0, \ldots, p_v - 1\} \).

Specifying a root \( r \) for the root \( r \) determines an orientation on edges, up to over-all sign. We choose to direct edges toward the root \( r \), and write \( E_{in}(v) \) for the set of edges terminating on a vertex \( v \). On the other hand, each nonroot vertex \( v \) has a unique edge emanating from it, and we call this outgoing edge \( e_v \). We additionally declare one edge \( e_v \) to emanate from the root vertex \( r \) toward a null vertex \( \text{null} \notin \text{Vert}(\Gamma) \), which we morally associate (with no hat) to our original knot exterior, \( Y_{\text{null}} := Y = M \setminus \tilde{\nu}(K) \). For any (directed) edge \( e \in \text{Edge}(\Gamma) \), we write \( v(e) \) for the destination vertex of \( e \), so that \( v = v(\text{null}) \) for all \( v \in \text{Vert}(\Gamma) \).

For notational convenience, we also associate an ‘index’ \( j(e) \in \{1, \ldots, n_v(e)\} \) to each edge \( e \), specifying the boundary component \( \partial_j Y_{v(e)} \) of \( Y_{v(e)} \) to which \( \partial_j Y_{v(e)} \) is glued when we embed the pattern link \( T_{v(-e)} \) in a neighborhood of \( \partial_j Y_{v(e)} \). As such, each edge \( e \in \text{Edge}(\Gamma) \) corresponds to a gluing map
\[
\varphi_e : \partial_0 Y_{v(-e)} \to -\partial_j Y_{v(e)},
\] (138)
along the incompressible torus joining \( Y_{v(-e)} \) to \( Y_{v(e)} \). This \( \varphi_e \) is the inverse of the map \( \varphi \) used in the satellite construction of Proposition 4.2. We express its induced map on slopes
\[
\varphi_e : \mathbb{P}(H_1(\partial_0 Y_{v(-e)}; \mathbb{Z}))_{\text{SF}} \to \mathbb{P}(H_1(\partial_j Y_{v(e)}; \mathbb{Z}))_{\text{SF}}, \quad y \mapsto \frac{yp_{v(-e)} - q_{v(-e)}}{yp_{v(e)} - q_{v(e)}},
\] (139)
in terms of $sf$-slopes on both sides. Thus $\varphi^p_{v,e}$ is orientation reversing. Note that for any $v \in \mathrm{Vert}(\Gamma)$, the map $\varphi^p_{v,e}$ is determined by $(p_v, q_v, n_v)$ and $j(e_v)$. We additionally define

$$J_v := \{j(e) | e \in E_{in}(v)\}, \quad I_v := \{1, \ldots, n_v\} \setminus J_v$$

for each $v \in \mathrm{Vert}(\Gamma)$, so that the space of Dehn filling slopes of $Y^\Gamma$ is given by

$$\prod_{v \in \mathrm{Vert}(\Gamma)} \prod_{i \in I_v} \mathbb{P}(H_1(\partial_i Y_v; \mathbb{Z})).$$

Writing $\Gamma_v$ for the subtree of $\Gamma$ of which $v$ is the root, let $Y_{\Gamma_v}$ denote the graph manifold with JSJ decomposition given by the Seifert fibered spaces $Y_u$ and gluing maps $\varphi_{v(e)}$ for $u \in \mathrm{Vert}(\Gamma_v)$, so that $Y_{\Gamma_v}$ is constructed recursively as

$$Y_{\Gamma_v} = Y_v \cup_{\varphi_{v(e)}} Y_{\Gamma_v(-e)}.$$  \hfill (142)

The exterior $Y^\Gamma := M \setminus Spec(K^\Gamma)$ of the iterated torus-link-satellite $K^\Gamma \subset M$ of $K \subset M$ specified by $\Gamma$ is then given by $Y^\Gamma = Y \cup_{\varphi_{v(e)}} Y_{\Gamma}$.  

6.2. Dehn fillings of $Y^\Gamma$

For $v \in \mathrm{Vert}(\Gamma)$, any $sf_{\Gamma_v}$-slope

$$y^\Gamma_v := \prod_{v \in \mathrm{Vert}(\Gamma_v)} y^v$$

$$y^v \in (Q \cup \{\infty\})_{sf_v} := \prod_{i \in I_v} \mathbb{P}(H_1(\partial_i Y_v; \mathbb{Z}))_{sf_v},$$

determines an L-space interval $L(Y_{\Gamma_v}(y^\Gamma_v))$ for the $sf_{\Gamma_v}$-Dehn-filling $Y_{\Gamma_v}(y^\Gamma_v)$. If $Y_{\Gamma_v}(y^\Gamma_v)$ is Floer simple, then we write

$$[[y_{v(-e)}^v, y_{v(+)}^v]] := L_{sf_v}(Y_{\Gamma_v}(y^\Gamma_v)), \quad [[y_{j(e_v)}^v, y_{j(e_v)+}^v]] = \varphi^p_{v(e)}(L_{sf_v}(Y_{\Gamma_v}(y^\Gamma_v)))$$

for all $e_v \in E_{in}(v) \setminus J_v$, according to whether they are boundary compressible (BC) — a solid torus or connected sum thereof — or boundary incompressible (BI):

$$J_{v}^{bc} := \{j(e) | Y_{\Gamma_v(-e)} \text{ is } BC, e \in E_{in}(v)\}$$

$$J_{v}^{bi} := J_v \setminus J_{v}^{bc}.\hfill (146)$$

Recalling that Theorem 4.3 implies $y^v_{j(e_v)} = y^v_{j(+)}$ for any $j \in J_{v}^{bc}$, we set $y^v_{j(e_v)} := y^v_{j(+)}$ for all $j \in J_{v}^{bc}$. We additionally define the sets $J_{v2}^{bi}, J_{v2}^{bi}$ and $J_{v2}^{bc}$:

$$J_{v2}^{bi} := \{j \in J_v | y^v_{j(e_v)} \in \mathbb{Z}\}, \quad J_{v2}^{bi} := \{j \in J_{v}^{bc} | y^v_{j} \in \mathbb{Z}\}.$$  \hfill (148)

For $v \in \mathrm{Vert}(\Gamma)$, $k \in \mathbb{Z}_{>0}$, define $\bar{y}_{0+\Sigma}(k) := 0$ if $\infty \in \{y^v_{j(e_v)} \mid j \in J_v \cup \{y^v_{j(e)} \mid e \in I_v\} \}$, and otherwise set

$$\bar{y}_{0-\Sigma}(k) := \sum_{j \in J_v^b} \left[ \left| y^v_{j} \right| - 1 \right] + \sum_{i \in I_v \cup J_v^{bc}} \left[ y^v_{i(e)} \right],$$

$$\bar{y}_{0+\Sigma}(k) := \sum_{j \in J_v^b} \left[ \left| y^v_{j} \right| - 1 \right] + \sum_{i \in I_v \cup J_v^{bc}} \left[ -y^v_{i(e)} \right].\hfill (149, 150)$$
In addition, define \( \bar{y}_{0^-} = \sup_{k > 0} y_{0^-}(k) \), \( \bar{y}_{0^+} = \inf_{k > 0} y_{0^+}(k) \), where
\[
\bar{y}_{0^-}(k) := \frac{1}{k} \left( -1 + \left\lceil \frac{q^*_v}{p_v} k \right\rceil - \bar{y}_{0^-}(k) \right) - |J_{vZ}^{+}|,
\]
\[
\bar{y}_{0^+}(k) := \frac{1}{k} \left( 1 + \left\lceil \frac{q^*_v}{p_v} k \right\rceil + \bar{y}_{0^+}(k) \right) + |J_{vZ}^{-}|.
\]

The ‘sup’ and ‘inf’ account for cases in which \( \bar{y}_{0^\pm}(k) = 0 \). The above notation provides a convenient way to repackage our computation of L-space interval endpoints.

**Proposition 6.1.** If \( y_{0^-}^v, y_{0^+}^v \in \mathbb{P}(H_1(\partial_0 Y_{v}; Z))_{\text{sf}} \) are the (potential) L-space interval endpoints for \( Y_{v^v} \) as defined in Theorem 4.3, then
\[
y_{0^-}^v = \bar{y}_{0^-} - \sum_{j \in J_{v}^+} (\lfloor y_{j+}^v \rfloor - 1) - \sum_{j \in J_{v}^-} \lceil y_{j-}^v \rceil - \sum_{i \in I_{v}} |y_{i}^v|,
\]
\[
y_{0^+}^v = \bar{y}_{0^+} - \sum_{j \in J_{v}^-} (\lfloor y_{j-}^v \rfloor + 1) - \sum_{j \in J_{v}^+} \lceil y_{j+}^v \rceil - \sum_{i \in I_{v}} |y_{i}^v|.
\]

Moreover, \( \bar{y}_{0^\pm} = \frac{q^*_v}{p_v} \) when \( \infty \in \{ y_{j+}^v \}_{j \in J_{v}} \cup \{ y_{i}^v \}_{i \in I_{v}} \), but
\[
\bar{y}_{0^-} = \left\lfloor \frac{q^*_v}{p_v} \right\rfloor - 1 \quad \text{if} \quad J_{vZ}^{+} \neq \emptyset \quad \text{and} \quad \infty \notin \{ y_{j+}^v \}_{j \in J_{v}} \cup \{ y_{i}^v \}_{i \in I_{v}};
\]
\[
\bar{y}_{0^+} = 1 \quad \text{if} \quad J_{vZ}^{-} \neq \emptyset \quad \text{and} \quad \infty \notin \{ y_{j-}^v \}_{j \in J_{v}} \cup \{ y_{i}^v \}_{i \in I_{v}}.
\]

**Proof.** The displayed equations in Proposition 6.1 come directly from the definitions of \( y_{0^\pm}^v \) specified by Theorem 4.3, but subjected to some mild manipulation of terms using the facts that \( x = [x] + [x] \) and \( x = [x] - [-x] \) for all \( x \in \mathbb{R} \), and that
\[
|J_{vZ}^{+}| = \sum_{j \in J_{v}^+} (\lfloor y_{j+}^v \rfloor - (\lfloor y_{j+}^v \rfloor - 1)), \quad -|J_{vZ}^{-}| = \sum_{j \in J_{v}^-} (\lceil y_{j-}^v \rceil - (\lceil y_{j-}^v \rceil + 1)).
\]

For the second half of the proposition, first note that the \( \bar{y}_{0^\pm} = \frac{q^*_v}{p_v} \) result follows directly from taking the \( k \to \infty \) limit. In the case of \( \infty \notin \{ y_{j+}^v, y_{j-}^v, y_{i}^v \}_{j \in J_{v}, i \in I_{v}} \), we temporarily set \( \bar{y}_{0^\pm}(k) := \bar{y}_{0^\pm}(k) \pm |J_{vZ}^{\pm}| \), so that
\[
\bar{y}_{0^-}(k) \leq \frac{1}{k} \left( \left\lceil \frac{q^*_v}{p_v} k \right\rceil + (|J_{vZ}^{+}| - 1) \right) \leq \left\lceil \frac{q^*_v}{p_v} \right\rceil + \frac{1}{k} (|J_{vZ}^{+}| - 1),
\]
\[
\bar{y}_{0^+}(k) \geq \frac{1}{k} \left( \left\lfloor \frac{q^*_v}{p_v} k \right\lfloor - (|J_{vZ}^{-}| - 1) \right) \geq - \frac{1}{k} (|J_{vZ}^{-}| - 1)
\]
for all \( k \in \mathbb{Z}_{>0} \). If \( J_{vZ}^{+} \neq \emptyset \) (respectively, \( J_{vZ}^{-} \neq \emptyset \)), then the above bound for \( \bar{y}_{0^-}(k) \) (respectively, \( \bar{y}_{0^+}(k) \)) is nonincreasing (respectively, nondecreasing) in \( k \), so that
\[
\bar{y}_{0^-}(k) \leq \left\lfloor \frac{q^*_v}{p_v} k \right\rfloor + 1 + |J_{vZ}^{+}|, \quad \bar{y}_{0^+}(k) \geq 1 - |J_{vZ}^{-}|
\]
for all \( k \in \mathbb{Z}_{>0} \). Since these bounds are each realized when \( k = 1 \), this completes the proof of the bottom line of the proposition.

The above method of computation for L-space interval endpoints helps us to prove some useful bounds for these endpoints.
Proposition 6.2. Suppose $\bar{y}_{v^+}^0 + \Sigma(k)$, $\bar{y}_{v^+}^-$, and $\bar{y}_{v^+}^+$ are as defined in (150), (151), and (152), and that $\infty \notin \{y^v_{j^+}, y^v_{j^-}, y^v_i \mid j \in J_v, i \in I_v\}$. Then $\bar{y}_{v^+}^-$ and $\bar{y}_{v^+}^+$ satisfy the following properties.

\[
\begin{align*}
(\rightarrow) \quad \bar{y}_{v^+}^0 &= \frac{q_v^g}{p_v} \iff \bar{y}_{v^+}^0 + \Sigma(q_v) = 0 \quad \text{and} \quad \{y^v_{j^+}, y^v_{j^-}, y^v_i \mid j \in J_v, i \in I_v\} \subset \mathbb{Z} \\
&\Rightarrow Y_{\Gamma_v}(y^v) \text{ is BC.}
\end{align*}
\]

\[
\begin{align*}
(\leftarrow) \quad \text{If } Y_{\Gamma_v}(y^v) \text{ is BI, then } \bar{y}_{v^+}^- \in \left[\left[\frac{q_v^-}{p_v}\right] - 1, \frac{q_v^-}{p_v}\right] = \left\{0, \frac{q_v^-}{p_v} : p_v \neq 1 \right\}.
\end{align*}
\]

\[
\begin{align*}
(\downarrow) \quad \text{If } Y_{\Gamma_v}(y^v) \text{ is BI, then } \bar{y}_{v^+}^0 \in \left(\frac{q_v^-}{p_v}, \frac{q_v^+}{p_v}\right). \quad \text{If in addition, } J_{v^+}^0 = \emptyset, \quad \text{then for } q_v > 0 \text{ and } m \in \mathbb{Z} \text{ (possibly negative or zero) with } m < \frac{q_v}{p_v}, \quad \bar{y}_{v^+}^0 \text{ satisfies}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{ll}
\bar{y}_{v^+}^0 + \Sigma(q_v) = 0 \\
\bar{y}_{v^+}^0 + \Sigma(q_v) > 0
\end{array} \right.
\end{align*}
\]

where we note that for $a, b \in \mathbb{Z}$ with $a, b < \frac{q_v}{p_v}$, one has

\[
\begin{align*}
\frac{p_v^a - aq_v^a}{q_v - ap_v} < \frac{p_v^a - bq_v^a}{q_v - bp_v} \iff a < b.
\end{align*}
\]

Finally, if $q_v > p_v > 1$, then

\[
\begin{align*}
\bar{y}_{v^+}^0 = \frac{p_v^a}{q_v} \iff \bar{y}_{v^+}^0 + \Sigma(q_v) = 0, \quad \bar{y}_{v^+}^0 + \Sigma(q_v) > 0, \quad \text{and } J_{v^+}^0 = \emptyset.
\end{align*}
\]

To aid in the proof of ($\rightarrow$), we first prove the following

Claim. If the hypotheses of Proposition 6.2 hold, then

\[
J_{v^+}^0 = \emptyset \quad \text{and} \quad \{y^v_{j^+}, y^v_{j^-}, y^v_i \mid j \in J_v, i \in I_v\} \subset \mathbb{Z} \iff Y_{\Gamma_v}(y^v) \text{ is BC and } \bar{y}_{v^+}^0 = \frac{q_v^a}{p_v}.
\]

Proof of Claim. For the $\Rightarrow$ direction, the left-hand side implies the irreducible component of $Y_{\Gamma_v}(y^v)$ containing $\partial Y_{\Gamma_v}(y^v)$ is Seifert fibered over the disk with one or fewer exceptional fibers, hence is BC, and direct computation shows that $\bar{y}_{v^+}^0 = \bar{y}_{v^+}^0 = \frac{q_v^a}{p_v}$.

For the $\Leftarrow$ direction, suppose the right-hand side holds. Then $J_{v^+}^0 = \emptyset$, and the irreducible component of $Y_{\Gamma_v}(y^v)$ containing $\partial Y_{\Gamma_v}(y^v)$ is Seifert fibered over the disk with one or fewer exceptional fibers, implying $|\frac{q_v^a}{p_v}, y^v_{j^+}, y^v_{j^-}, y^v_i|_{j \in J_v, i \in I_v} \subset \mathbb{Z}| \leq 1$. If $\frac{q_v^a}{p_v} \notin \mathbb{Z}$, then we are done, but if $\frac{q_v^a}{p_v} \in \mathbb{Z}$ and $|\frac{q_v^a}{p_v}, y^v_{j^+}, y^v_{j^-}, y^v_i|_{j \in J_v, i \in I_v} \subset \mathbb{Z}| = 1$, then the longitude $l$ satisfies $l = -\sum_{i \in J_v \cup J_v} y^v_i \notin \mathbb{Z}$, contradicting the fact that $l = \bar{y}_{v^+}^0 = \bar{y}_{v^+}^0 = 1$.

Proof of ($\leftarrow$) and part of ($\rightarrow$). When $J_{v^+}^0 = \emptyset$, ($\leftarrow$) follows from Proposition 6.1, which tells us $\bar{y}_{v^+}^0 = \frac{q_v^a}{p_v} - 1$. When $J_{v^+}^0 = \emptyset$, we have the bounds

\[
\bar{y}_{v^+}^0 \geq \frac{q_v^a}{p_v} - 1 = \left\{ \begin{array}{ll}
0 & p_v \neq 1 \\
-1 & p_v = 1
\end{array} \right.
\]

\[
\frac{q_v^a}{p_v} k + 1 < \frac{q_v^a}{p_v} \forall k \in \mathbb{Z} > 0.
\]

Thus, either $\bar{y}_{v^+}^0 \in \left[\frac{q_v^a}{p_v} - 1, \frac{q_v^a}{p_v}\right]$ or $\bar{y}_{v^+}^0 = \frac{q_v^a}{p_v}$. The latter case implies $\sup_{k \to +\infty} y_{v^+}^0 - (k)$ is not attained for finite $k$, and so Theorem 4.3 tells us that $Y_{\Gamma_v}(y^v)$ is BC and $\bar{y}_{v^+}^0 = \bar{y}_{v^+}^0$. \qed
Proof of (+) and remainder of (⇒). Proposition 6.1 tells us that \( \bar{y}_{0+} = 1 \) when \( J_{vZ}^{m-} \neq \emptyset \), so we henceforth assume \( J_{vZ}^{m-} = \emptyset \). In this case, we have

\[
\bar{y}_{0+}^v < \bar{y}_{0+}^v(1) = 1, \quad \bar{y}_{0+}^v(k) \geq \frac{1}{k} \left( \left\lfloor \frac{q_v^* k}{p_v} \right\rfloor - 1 \right) > \frac{q_v^*}{p_v} \quad \forall k \in \mathbb{Z}_{>0}.
\]  

(160)

Thus, either \( \bar{y}_{0+}^v \in \left< \frac{q_v^*}{p_v}, 1 \right> \) or \( \bar{y}_{0+}^v = \frac{q_v^*}{p_v} \). The latter case implies \( \inf_{k \to +\infty} \bar{y}_{0+}^v(k) \) is not attained for finite \( k \), which Theorem 4.3 tells us implies that \( Y_{vZ}(y_0^v) \) is BC and \( \bar{y}_{0+}^v = \bar{y}_{0-}^v \).

For (+i), fix some \( m \in \mathbb{Z} \) with \( m < \frac{q_v}{p_v} \). If \( \bar{y}_{0+}^v + \Sigma(v_0 - mp_v) = 0 \), then

\[
\bar{y}_{0+}^v = \begin{cases} 
1 & \text{if } \left\lfloor \frac{q_v^*}{p_v} \right\rfloor = 1 \\
\frac{q_v^*}{p_v} & \text{if } \left\lfloor \frac{q_v^*}{p_v} \right\rfloor = 0 
\end{cases}
\]

(161)

Next, suppose that \( (q_0 - mp_v) > 0 \), so that either \( [−y_{0-}^v] > (q_0 - mp_v)^{-1} \) for some \( j \in J_v^m \), or \( [−y_{0-}^v] \geq (q_0 - mp_v)^{-1} \) for some \( j \in I_v \cup J_v^m \) (or both occur). Since for any rational \( x > (q_0 - mp_v)^{-1} \), we have \( [x]\kappa = 1 \geq \left\lfloor \frac{q_0 - mp_v}{q_0} \right\rfloor \) for all \( k \in \mathbb{Z}_{>0} \), the condition \( \bar{y}_{0+} + \Sigma(v_0 - mp_v) > 0 \) therefore implies \( \bar{y}_{0+} + \Sigma(k) \geq \left\lfloor \frac{k}{q_0 - mp_v} \right\rfloor \) for all \( k \in \mathbb{Z}_{>0} \). We then have

\[
k \left( \bar{y}_{0+}^v(k) - \frac{p_v^* - mq_v^*}{q_v - mp_v} \right) \geq k \left( 1 + \frac{q_v^*}{p_v} \right) \left( 1 + \frac{k}{(q_0 - mp_v)^2} \right) - \frac{p_v^* - mq_v^*}{q_v - mp_v} \\
= 1 + \frac{k}{(q_0 - mp_v)^2} - \frac{k}{q_0 - mp_v} + \frac{q_v^*}{p_v} \frac{k}{(q_0 - mp_v)^2} - \frac{p_v^* - mq_v^*}{q_v - mp_v} \\
= 1 - \frac{k}{q_0 - mp_v} + \frac{k}{q_0 - mp_v} - \frac{k}{q_0 - mp_v} \\
= 1 - \frac{k}{q_0 - mp_v} + \frac{k}{q_0 - mp_v} + \left( \frac{p_v - 1}{p_v} \right) \frac{k}{q_0 - mp_v} \\
> 0
\]

(162)

for all \( k \in \mathbb{Z}_{>0} \), and so \( \bar{y}_{0+}^v > \frac{p_v^* - mq_v^*}{q_0 - mp_v} \).

Statement (+.ii) is a simple consequence of the fact that

\[
\frac{p_v^* - bq_v^*}{q_v - bp_v} - \frac{p_v^* - aq_v^*}{q_v - ap_v} = \frac{b - a}{(q_v - bp_v)(q_v - ap_v)}.
\]

(163)

This leaves us with (+.iii). Since \( q_v > p_v > 1 \) implies \( \frac{q_v}{p_v} < 1 \), but Proposition 6.1 tells us \( \bar{y}_{0+}^v = 1 \) if \( J_{vZ}^{m-} \neq \emptyset \), we henceforth assume \( J_{vZ}^{m-} = \emptyset \). Setting \( m = 0 \) in (+.i) then gives us

\[
\bar{y}_{0+}^v \leq \frac{p_v^*}{q_v} \iff \bar{y}_{0+}^v(\Sigma(q_v)) = 0.
\]

(164)

Similarly, setting \( m = -1 \) in (+.i) yields the relation

\[
\bar{y}_{0+}^v \leq \frac{p_v^* + q_v^*}{q_v + p_v} \iff \bar{y}_{0+}^v(\Sigma(q_v + p_v)) = 0,
\]

(165)

where we used (+.ii) for the right-hand inequality. Finally, suppose \( \bar{y}_{0+}^v(\Sigma(q_v + p_v)) > 0 \). By reasoning similar to that used in the proof of (+.i), this implies that \( \bar{y}_{0+}^v(k) \geq \left\lfloor \frac{k}{p_v + q_v} \right\rfloor \) for all \( k \in \mathbb{Z}_{>0} \). We then have

\[
\bar{y}_{0+}^v(k) \geq \frac{1}{k} \left( 1 + \frac{q_v^*}{p_v} \right) \geq \frac{p_v^*}{q_v} \quad \forall k \in \mathbb{Z}_{>0},
\]

(166)

with the right-hand inequality coming from Lemma 4.7, and so \( \bar{y}_{0+}^v \geq \frac{p_v^*}{q_v} \). \(\square\)
There is one more collection of estimates that will be particularly useful in the case of general iterated torus-link satellites.

**Proposition 6.3.** The following bounds hold.

(i) If \( q_v > 0 \) and \( \bar{y}_{0-}^{v-\Sigma}([p_v]_{q_v}) > 0 \), then \( \bar{y}_{0-}^{v} < \frac{p_v^*}{q_v} - \frac{1}{[p_v]_{q_v}(q_v)} \), \( \varphi_{v, *}^p(\bar{y}_{0-}^{v}) > \left[ \frac{p_v}{q_v} \right] - 1 \);

(ii) If \( q_v < 0 \) and \( \bar{y}_{0-}^{v-\Sigma}([-p_v]_{q_v}) > 0 \), then \( \bar{y}_{0-}^{v} < \frac{p_v^*}{q_v} + \frac{1}{[-p_v]_{q_v}(q_v)} \), \( \varphi_{v, *}^p(\bar{y}_{0-}^{v}) > \left[ \frac{p_v}{q_v} \right] - 1 \);

(iii) If \( q_v > 0 \) and \( \bar{y}_{0+}^{v+\Sigma}([-p_v]_{q_v}) > 0 \), then \( \bar{y}_{0+}^{v} > \frac{p_v^*}{q_v} + \frac{1}{[-p_v]_{q_v}(q_v)} \), \( \varphi_{v, *}^p(\bar{y}_{0+}^{v}) < \left[ \frac{p_v}{q_v} \right] + 1 \);

(iv) If \( q_v < 0 \) and \( \bar{y}_{0+}^{v+\Sigma}([p_v]_{q_v}) > 0 \), then \( \bar{y}_{0+}^{v} > \frac{p_v^*}{q_v} - \frac{1}{[p_v]_{q_v}(q_v)} \), \( \varphi_{v, *}^p(\bar{y}_{0+}^{v}) < \left[ \frac{p_v}{q_v} \right] + 1 \).

**Proof of (i).** If \( [p_v]_{q_v} \in \{0, 1\} \), then \( \bar{y}_{0-}^{v-\Sigma}([p_v]_{q_v}) \leq 0 \) and the claim holds vacuously, so we assume \( [p_v]_{q_v} \geq 2 \), implying \( p_v \geq 2 \) and \( q_v \geq 3 \), so that

\[
\varphi_{v, *}^p\left( \frac{p_v^*}{q_v} - \frac{1}{[p_v]_{q_v}(q_v)} \right) = \left[ \frac{p_v}{q_v} \right] - 1. \tag{167}
\]

By reasoning similar to that used in the proof of (+,i) above, the hypothesis \( \bar{y}_{0-}^{v-\Sigma}([p_v]_{q_v}) > 0 \) implies that \( \bar{y}_{0-}^{v}(k) \geq \left[ \frac{k}{[p_v]_{q_v}} \right] \) for all \( k \in \mathbb{Z}_{>0} \). Thus, if we set

\[
m := \left\lfloor \frac{p_v}{q_v} \right\rfloor, \text{ so that } [p_v]_{q_v} = p_v - mq_v \text{ and } \frac{p_v^*}{q_v} - \frac{1}{[p_v]_{q_v}(q_v)} = \frac{q_v^* - mp_v^*}{p_v - mq_v}, \tag{168}
\]

then it suffices to prove negativity, for all \( k \in \mathbb{Z}_{>0} \), of the difference

\[
k\left( \bar{y}_{0-}^{v}(k) - \frac{q_v^* - mp_v^*}{p_v - mq_v} \right) \leq k \left( \frac{1}{k} \left( -1 + \left\lfloor \frac{q_v^*}{p} \right\rfloor \right) - \frac{k}{[p_v]_{q_v}(q_v)} - \frac{q_v^* - mp_v^*}{p_v - mq_v} \right) \]

\[
= -1 + \frac{-[q_v^* k]_{p_v}}{p_v} - \frac{k}{[p_v]_{q_v}(q_v)} + \frac{q_v^*}{p_v} - \frac{q_v^* - mp_v^*}{p_v - mq_v} \]

\[
= \frac{-p_v q_v + q_v [q_v^{-1} k]_{p_v}}{p_v q_v} - \frac{k}{[p_v]_{q_v}(q_v)} + \left( \frac{m}{p_v} \right) \frac{k}{[p_v]_{q_v}}. \tag{169}
\]

Now, \( \frac{m}{p_v} \) already satisfies the bound

\[
\frac{m}{p_v} = \frac{[p_v]_{q_v}}{p_v} = \frac{p_v - [p_v]_{q_v}}{p_v} \leq \frac{p_v - 2}{p_v} < \frac{1}{q_v} \leq \frac{1}{3}. \tag{170}
\]

Thus, if \( \left\lfloor \frac{k}{[p_v]_{q_v}} \right\rfloor \geq 1 \), then

\[
\left\lfloor \frac{k}{[p_v]_{q_v}} \right\rfloor \geq \frac{1}{2} \left( \left\lfloor \frac{k}{[p_v]_{q_v}} \right\rfloor + 1 \right) > \frac{1}{2} \left( \frac{k}{[p_v]_{q_v}} \right) > \left( \frac{m}{p_v} \right) \frac{k}{[p_v]_{q_v}}, \tag{171}
\]

making the right-hand side of (169) negative.

We therefore henceforth assume that \( \left\lfloor \frac{k}{[p_v]_{q_v}} \right\rfloor = 0 \), implying \( k < [p_v]_{q_v} \). Thus, since

\[
q_v [q_v^{-1} k]_{p_v} + p_v [p_v^{-1} k]_{q_v} = p_v q_v + k, \tag{172}
\]
and since $mq_v = p_v - \lfloor p_v \rfloor_{q_v}$, we obtain
\begin{equation}
  k \left( \bar{y}_{0-}^v (k) - \frac{q_v - mp_v^*}{p_v - mq_v} \right) \leq \frac{k - p_v \lfloor p_v^{-1} k \rfloor_{q_v} + (p_v - \lfloor p_v \rfloor_{q_v})}{p_v |q_v|} k \left[ \frac{k}{q_v} \right]_{q_v} = \frac{-[p_v^{-1} k]_{q_v} + \frac{k}{q_v}}{q_v} < 0.
\end{equation}

Proof of (ii). The claim holds vacuously for $[-p_v]_{q_v} \in \{0, 1\}$, so we assume $[-p_v]_{q_v} \geq 2$ and $q_v \leq -3$, in which case
\begin{equation}
  \varphi_{c_v}^v \left( \frac{p_v^*}{q_v} + \frac{1}{[-p_v]_{q_v}(q_v)} \right) = \left[ \frac{p_v}{q_v} \right] - 1.
\end{equation}
Since $\bar{y}_{0-}^v([-p_v]_{q_v}) > 0$ implies $\bar{y}_{0-}^v(k) = \lfloor \frac{k}{[-p_v]_{q_v}} \rfloor$ for all $k \in \mathbb{Z}_{>0}$, we set
\begin{equation}
  m := \left[ \frac{p_v}{q_v} \right] - \left[ \frac{p_v}{q_v} \right], \quad [p_v]_{q_v} = -p_v - mq_v \text{ and } \frac{p_v^*}{q_v} + \frac{1}{[-p_v]_{q_v}(q_v)} = \frac{-q_v + mp_v^*}{-p_v - mq_v}.
\end{equation}

Using arguments similar to those in part (i), it is straightforward to derive the bound
\begin{equation}
  k \left( \bar{y}_{0-}^v (k) - \frac{-q_v^* - mp_v^*}{-p_v - mq_v} \right) \leq \frac{-p_v |q_v| + |q_v| \lfloor \frac{1}{|q_v|} (-k) \rfloor_{q_v}}{p_v |q_v|} - \left( \frac{k}{[-p_v]_{q_v}} \right) \left( \frac{m}{[-p_v]_{q_v}} \right) \frac{k}{[-p_v]_{q_v}}
\end{equation}
for all $k \in \mathbb{Z}_{>0}$, and to show that the right-hand side is negative if $\lfloor \frac{k}{[-p_v]_{q_v}} \rfloor = 1$, allowing us to assume that $\left[ \frac{k}{[-p_v]_{q_v}} \right] = 0$ and $k < [-p_v]_{q_v}$. Thus, since
\begin{equation}
  |q_v| \lfloor \frac{1}{|q_v|} (-k) \rfloor_{q_v} + p_v \lfloor p_v^{-1} (-k) \rfloor_{q_v} = p_v q_v - k,
\end{equation}
and since $mq_v = p_v + [-p_v]_{q_v}$, we obtain
\begin{equation}
  k \left( \bar{y}_{0-}^v (k) - \frac{-q_v^* - mp_v^*}{-p_v - mq_v} \right) \leq \frac{-k - p_v \lfloor p_v^{-1} (-k) \rfloor_{q_v} + (p_v + [-p_v]_{q_v}) \frac{k}{[-p_v]_{q_v}}}{p_v |q_v|} \left[ \frac{k}{q_v} \right]_{q_v} < 0.
\end{equation}

Proofs of (iii) and (iv). Respectively similar to proofs of (ii) and (i). \hfill \Box

6.3. L-space surgery regions for iterated satellites: Proof of Theorem 1.7

We have finally done enough preparation to prove Theorem 1.7 from the introduction.

Proof of Theorem 1.7. The bulk of part (i) is proven in the Claim in the proof of Theorem 8.1. Since the right-hand condition of (265) is equivalent to the condition that $Y^\Gamma (y^\Gamma)$ be an L-space, the Claim proves that $Y^\Gamma (y^\Gamma)$ is an L-space if and only if $Y^\Gamma (y^\Gamma) = S^3$. Thus, if we define $\Lambda^\Gamma$ as in (11), then the statement $\mathcal{L}(Y^\Gamma) = \Lambda^\Gamma$ holds tautologically.

The proof of part (ii) begins similarly to the proof of Theorem 4.5(i.b), except that instead of deducing that $\sum_{i \in I_v} ([y_i^+] - [y_i^-]) \leq 1$, we deduce that
\begin{equation}
  \sum_{i \in I_v} ([y_i^+] - [y_i^-]) + \sum_{j \in J_v^c} (([y_j^+] - 1) - ([y_j^-] - 1)) - \sum_{j \in J_v^c} ([y_j^+] - [y_j^-]) \leq 1,
\end{equation}
with \( y^v_j \geq y^v_{j+} \) for all \( j \in J^v_{bc} \). In the case that \( \sum_{i \in I_v} ([y^v_i] - [y^v_{i+}]) = 1 \) and the other sums vanish, we are reduced to the original case of Theorem 4.5(i.b), obtaining the component
\[
\Lambda \cdot \mathcal{G}_{I_v}([N, +\infty] \times \{\infty\}^{j_{bc}-1}) \times \prod_{e \in \mathcal{E}_{in}(v)} \Lambda \mathcal{G}_{e}(-e) \subset \mathcal{L}_{S^3}(Y^{\Gamma}).
\]
(180)

In the case that \( \sum_{i \in I_v} ([y^v_i] - [y^v_{i+}]) = 0 \), we have that \( y^v \in \mathbb{Z}^{j_{bc}}_v \), and all but one incoming edge of \( v \), say \( e \), descend from trees with trivial fillings. Performing these trivial fillings reduces \( Y^{\Gamma} \) to the exterior of a \( \Gamma_{e}(-e) \)-satellite of the \((1, q_e)\)-cable of \( K \in S^3 \), but the \((1, q_v)\)-cable is just the identity operation, so we are left with the exterior \( Y^{\Gamma_{e}(-e)} \) of the \( \Gamma_{e}(-e) \)-satellite of \( K \in S^3 \). Considering this for all edges \( e \in \mathcal{E}_{in}(v) \) then gives the remaining component
\[
\prod_{e \in \mathcal{E}_{in}(v)} (\mathcal{L}_{S^3}(Y^{\Gamma_{e}(-e)}) \times \Lambda \mathcal{G}_{e}(-e)) \subset \mathcal{L}_{S^3}(Y^{\Gamma}).
\]
(181)

**Part (iii).** Before proceeding with the main inductive argument in this section, we attend to some bookkeeping issues. In particular, our inductive proof requires each \( I_v \) to be nonempty. For any \( w \in \text{Vert}(\Gamma) \) with \( I_w = \emptyset \), we repair this situation artificially, as follows. First, redefine \( I_w := \{1\} \). Next, if \( 0 \geq m^+_w \), then set \( \mathcal{L}_{S^w_u}^{\min} = \emptyset \), and declare \( \mathcal{R}_{S^w_u} \setminus \mathcal{Z}_{S^w_u} = \mathcal{L}_{S^w_u}^{\min} = \emptyset \). Finally, if \( 0 < m^+_w < m_w \), then \( 0 < m^+_w < m_w \), so set \( \mathcal{L}_{S^w_u}^{\min} = \emptyset \), and declare \( \mathcal{R}_{S^w_u} \setminus \mathcal{Z}_{S^w_u} = \mathcal{L}_{S^w_u}^{\min} + \emptyset \).

For a vertex \( v \in \text{Vert}(\Gamma) \), inductively assume, for each incoming edge \( e \in \mathcal{E}_{in}(v) \), that for any \( y^{\Gamma_{e}(-e)} \in \prod_{e \in \text{Vert}(\Gamma_{e}(-e))} (\mathcal{L}_{S^w_u}^{\min} \cup \mathcal{R}_{S^w_u} \setminus \mathcal{Z}_{S^w_u} \cup \mathcal{L}_{S^w_u}^{\min}) \), we have
\[
\begin{align*}
\frac{p_{v}(-e)}{q_{v}(-e)} - 1 < y^{v}_{j(e)+} & \leq y^{v}_{j(e)-} \leq \frac{p_{v}(-e)}{q_{v}(-e)} + 1, \\
\frac{p_{v}(-e)}{q_{v}(-e)} - 1 < y^{v}_{j(e)+} & \leq y^{v}_{j(e)-} < \frac{p_{v}(-e)}{q_{v}(-e)} + 1 \quad \text{if } Y^{\Gamma_{e}(-e)}(y^{\Gamma_{e}(-e)}) \text{ is BI},
\end{align*}
\]
(182)
(183)
where, again, \( y^{v}_{j(e)+} := \varphi^{p}_{e,e}([y^{v}_{j(e)-}]) \), with \( \mathcal{L}_{S^w_u}(Y^{\Gamma_{e}(-e)}(y^{\Gamma_{e}(-e)})) = [([y^{v}_{j(e)+}], y^{v}_{j(e)-})]. \) Note that this inductive assumption already holds vacuously if \( v \) is a leaf.

If \( y^{v} \in \mathcal{R}_{S^w_u} \setminus \mathcal{Z}_{S^w_u} \), then \( y^{v}_0 = y^{v}_0 = \infty \), implying that
\[
y^{v}_{j(e)+} = \varphi^{p}_{e,e}([y^{v}_{j(e)-}]) = \frac{p_{v}}{q_{v}} \in \left\{ \left[ \frac{p_{v}}{q_{v}} \right] - 1, \left[ \frac{p_{v}}{q_{v}} \right] + 1 \right\}.
\]
(184)

If \( y^{v} \in \mathcal{L}_{S^w_u}^{\min} \cup \mathcal{L}_{S^w_u}^{\min} \cup \mathcal{Q}^{j_{bc}}_v \), then applying (182) to Proposition 6.1 yields
\[
\begin{align*}
y^{v}_{0-} & \leq y^{v}_{0-} - \sum_{e \in \mathcal{E}_{in}(v)} \left( \left[ \frac{p_{v}(-e)}{q_{v}(-e)} \right] - 1 \right) - \sum_{i \in I_v} [y^v_i], \\
y^{v}_{0+} & \geq y^{v}_{0+} - \sum_{e \in \mathcal{E}_{in}(v)} \left( \left[ \frac{p_{v}(-e)}{q_{v}(-e)} \right] + 1 \right) - \sum_{i \in I_v} [y^v_i],
\end{align*}
\]
(185)
(186)
and assuming (182) for Theorem 4.4 implies
\[
y^{v}_0 = y^{v}_0.
\]
(187)

Suppose \( y^{v} \in \mathcal{L}_{S^w_u}^{\min} \), so that the bound \( \sum_{i \in I_v} [y^v_i] \geq m^+_v \), together with (185), implies
\[
\begin{align*}
y^{v}_{0-} & \leq y^{v}_{0-} - \begin{cases} 2 & q_v = -1, \\
1 & I_v \neq \emptyset; q_v < -1 \text{ or } \frac{p_v}{q_v} > 1, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
(188)
Since Proposition 6.2 tells us \( \bar{y}_{0}^{\nu} \leq \frac{q_{v}^*}{p_{v}} \), we then have
\[
y^{\nu}_{0}^{*} \leq \bar{y}_{0}^{\nu} \leq \frac{q_{v}^*}{p_{v}} < \frac{p_{v}}{q_{v}} = (\phi_{c,v}^*)^{-1}(\infty).
\] (189)
Since \( \phi_{c,v}^* \) is locally monotonically decreasing in the complement of its vertical asymptote at 
\( (\phi_{c,v}^*)^{-1}(\infty) = \frac{p_{v}}{q_{v}} \), line (189) implies
\[
y_{j(e_{v})}^{\nu(e_{v})} := \phi_{c,v}^*(\bar{y}_{0}^{\nu}) \leq y_{j(e_{v})}^{\nu(e_{v})} := \phi_{c,v}^*(\bar{y}_{0}^{\nu}) < \phi_{c,v}^*(\infty) = \frac{p_{v}}{q_{v}} < \left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor + 1,
\] (190)
where \( \phi_{c,v}^*(\infty) = \frac{p_{v}}{q_{v}} \) is the location of the horizontal asymptote of \( \phi_{c,v}^* \). Thus, since \( \left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor - 1 < \frac{p_{v}}{q_{v}} = \phi_{c,v}^*(\infty) \), we deduce that to finish establishing our inductive hypotheses for \( e_{v} \) in the 
\( y' \in L_{S_{v}^{\min}}^{\text{out}} \) case, it suffices to show that
\[
y_{0}^{\nu} \leq (\phi_{c,v}^*)^{-1}\left(\left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor - 1\right), \quad \text{with equality only if } Y_{T_{c}}(y_{T_{v}}) \text{ is BC.} \] (191)
If \( |q_{v}| = 1 \), it is straightforward to compute that
\[
(\phi_{c,v}^*)^{-1}\left(\left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor - 1\right) = \frac{p_{v}}{q_{v}} - 1 = \begin{cases} -2 & (p_{v}, q_{v}) = (1, -1) \\ -1 & (p_{v}, q_{v}) \neq (1, -1), q_{v} = -1. \\ 0 & q_{v} = 1 \end{cases}
\] (192)
Proposition 6.2 tells us \( \bar{y}_{0}^{\nu} \leq \frac{q_{v}}{p_{v}} \), with equality only if \( Y_{T_{c}}(y_{T_{v}}) \) is boundary compressible. Thus, since \( \frac{q_{v}}{p_{v}} < 1 \), with \( \frac{q_{v}}{p_{v}} = 0 \) when \( p_{v} = 1 \), it follows from (188) and (192) that (191) holds.

Next suppose that \( |q_{v}| > 1 \), so that
\[
(\phi_{c,v}^*)^{-1}\left(\left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor - 1\right) = \frac{p_{v}}{q_{v}} - 1 = \pm \frac{1}{|\pm p_{v}, q_{v}|} \quad \text{for } \pm q_{v} > 1.
\] (193)
If \( 0 < \frac{p_{v}}{q_{v}} < 1 \), this makes \( (\phi_{c,v}^*)^{-1}\left(\left\lfloor \frac{p_{v}}{q_{v}} \right\rfloor - 1\right) = \frac{q_{v}}{p_{v}} \), so that (191) follows from (188) and the fact that \( \bar{y}_{0}^{\nu} \leq \frac{q_{v}}{p_{v}} \), with equality only if \( Y_{T_{v}}(y_{T_{v}}) \) is boundary compressible. This leaves the cases in which \( \frac{p_{v}}{q_{v}} > 1 \) or \( q_{v} < -1 \). If \( J_{v} = \emptyset \), then \( L_{S_{v}^{\min}}^{\text{out}} \), respectively, excludes \( \{\sum_{v} y_{i}^{v} = \sum_{v} \lfloor y_{i}^{v} \rfloor - p_{v}, q_{v} \rfloor = 0\} \) or \( \{\sum_{v} y_{i}^{v} = \sum_{v} \lfloor y_{i}^{v} \rfloor |p_{v}, q_{v} \rfloor = 0\} \), and so (191) follows from part (i) or (ii), respectively, of Proposition 6.3. If \( J_{v} \neq \emptyset \), then \( y_{0}^{v} \leq \frac{q_{v}}{p_{v}} - 1 \), and it is easy to show that
\[
\frac{q_{v}}{p_{v}} - 1 < \frac{p_{v}}{q_{v}} = \pm \frac{1}{|\pm p_{v}, q_{v}|} \quad \text{for } \pm q_{v} > 1,
\] (194)
completing our inductive step for the case of \( y^{v} \in L_{S_{v}^{\min}}^{\text{out}} \).

The proof of our inductive step for the case of \( y^{v} \in L_{S_{v}^{\min}}^{\text{in}} \) follows from symmetry under orientation reversal.

Recall that we regard the root vertex \( r \) at the bottom of the tree \( \Gamma = \Gamma_{r} \) as having an outgoing edge \( e_{r} \) pointing to the empty vertex \( v(e_{r}) := \text{NULL} \), where this null vertex \( v(e_{r}) \) corresponds to the exterior \( Y := S^{3} \setminus \hat{\nu}(K) \) of the companion knot \( K \subset S^{3} \), from which the satellite exterior \( Y^{T} := Y_{T} \cup Y \) is formed. Recursing down to this null vertex \( v(e_{r}) \), our above induction shows that for any \( y^{r} \in \prod_{e \in \text{Vert}(\Gamma_{r})} (L_{S_{v}^{\min}}^{\text{out}} \cup R_{S_{v}^{\text{in}}}, Z_{S_{v}^{\text{in}}} \cup L_{S_{v}^{\text{in}}}) \), we have
\[
\left\lfloor \frac{p_{r}}{q_{r}} \right\rfloor - 1 \leq y_{j(e_{v})}^{v(e_{v})} < y_{j(e_{v})}^{v(e_{v})} \leq \left\lfloor \frac{p_{r}}{q_{r}} \right\rfloor + 1.
\] (195)
This final L-space interval \( \varphi_{e,s}^N(\mathcal{L}_{S^3}(Y_s, (y^r))) \) = \([y_{j(e^r)}^+, y_{j(e^r)}^-]\), expresses slopes in terms of the reversed \( S^3 \)-slope basis, \( \mathcal{S}^3 \). That is, the meridian and longitude of \( K \subset S^3 \) have respective \( \mathcal{S} \)-slopes \( 0 = \frac{1}{2} \) and \( \infty = \frac{1}{3} \). If \( K \subset S^3 \) is the unknot, then its \( \mathcal{S} \)-longitude \( \infty \) satisfies \( \infty \in \left[ y_{j(e^r)}^+, y_{j(e^r)}^- \right] \), so we deduce that \( y^r \in \mathcal{L}_{SF_r}(Y^r) \) in this case.

If \( K \subset S^3 \) is nontrivial, then its exterior \( Y \) has L-space interval \( \mathcal{L}_{S^3}(Y) = [0, \frac{1}{N}] \). Assume that \( \Gamma \) satisfies hypothesis (iii) of the theorem. Then since \( \frac{p}{q} \geq N := 2g(K) - 1 \) implies \( 0 < \frac{p}{q} \leq 1 \), (182) and (183) tell us that \( 0 \leq y_{j(e^r)}^+ \leq y_{j(e^r)}^- \), with \( y_{j(e^r)}^- = 0 \) only if \( Y_r^r(y^r) \) is bc. Thus, it remains to show that either \( y_{j(e^r)}^- > \frac{1}{N} \), or \( y_{j(e^r)}^- \leq \frac{1}{N} \) and \( Y_r^r(y^r) \) is bc.

If \( y^r|_r \in \mathcal{L}_{SF_r}^{\min,+} \), then (190) tells us \( y_{j(e^r)}^- < \frac{p}{q} < \frac{1}{N} \). If \( y^r|_r \in \mathcal{R}_v \setminus \mathcal{Z}_v \), then \( Y_r^r(y^r) \) is bc, and since \( y_{j(e^r)}^- = y_{j(e^r)}^+ = \infty \), we have \( y_{j(e^r)}^- = y_{j(e^r)}^+ = \frac{p}{q} \leq 0 < \frac{1}{N} \). This leaves us with the case of \( y^r|_r \in \mathcal{L}_{SF_r}^{\min,-} \), for which we have

\[
y_{0,+}^r = \frac{p}{q} = 1 + \frac{1}{2} \begin{cases} \frac{1}{p} & q^* = 1 \\ \frac{1}{2} p = 1 & q^* > 1 \\ \frac{1}{2} p = 1 \end{cases} + \frac{1}{p(q - pN)} = (\varphi_{e,s}^r)^{-1} \left( \frac{1}{N} \right).
\]

implying \( y_{j(e^r)}^- < \frac{1}{N} \), and thereby completing the proof of the theorem. \( \square \)

**Remark.** It is only in (196) that we use the hypothesis of part (iii) that \( q_r > 2g(K) - 1 \). If \( N \). In the case that we do have \( p_r = 1 \) and \( q_r = N \), this implies \( (\varphi_{e,s}^r)^{-1}(\frac{1}{N}) = \infty \), requiring \( \mathcal{L}_{SF_r}^{\min,-} \) to be empty, but we still have the modified result that

\[
\mathcal{L}_{SF_r}(Y^r) \supset \left( \prod_{v \in \text{Vert}(\Gamma)} \left( \mathcal{L}_{SF_v}^{\min,-} \cup \mathcal{R}_v \setminus \mathcal{Z}_v \cup \mathcal{L}_{SF_v}^{\min,+} \right) \right) \setminus \mathcal{L}_{SF_r}^{\min,-}.
\]

### 7. Satellites by algebraic links

#### 7.1. Smooth and exceptional splices

As mentioned in the introduction, the class of algebraic link satellites is slightly more general than the class of iterated torus link satellites, in that the JSJ decomposition graph of the exterior must allow one extra type of edge.

To describe how this new type of edge is different, we first address the notion of *splice* maps. Suppose \( K_1 \subset M_1 \) and \( K_2 \subset M_2 \) are knots in compact oriented 3-manifolds \( M_1 \) and \( M_2 \), with each \( \partial M_i \) a possibly-empty disjoint union of tori. Let \( Y_i := M_i \setminus \hat{N}(K) \) denote the exterior of each knot \( K_i \subset M_i \), with \( \partial Y_i := -\partial \hat{N}(K_i) \), and choose a surgery basis \((\mu_i, \lambda_i) \in H_1(\partial Y_i; \mathbb{Z}) \) for each exterior, with \( \lambda_i \), the Seifert longitude if \( M_i \) is either an integer homology sphere or a link exterior in a specified integer homology sphere.

A gluing map \( \phi : \partial Y_1 \rightarrow \partial Y_2 \) is then called a *splice* if the induced map on homology sends \( \mu_1 \mapsto \lambda_2 \) and \( \mu_2 \mapsto \lambda_1 \). Gluing via splice maps are minimally disruptive to homology. For instance, if \( M_2 \) is an integer homology sphere, then \( H_1(Y_1 \cup_{\phi} Y_2; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \). If \( M_2 = S^3 \) and \( K_2 \) is an unknot, then we in fact have \( Y_1 \cup_{\phi} Y_2 = M_1 \). In particular, if \( M_2 \) is the exterior \( M_2 = S^3 \setminus \hat{N}(L_2) \) of some link \( L_2 \subset S^3 \), and if \( K_2 \subset M_2 \) is an unknot in the composition \( K_1 \hookrightarrow M_1 \rightarrow S^3 \), then \( Y_1 \cup_{\phi} Y_2 \) is the exterior of the satellite link of the companion knot \( K_1 \subset M_1 \) by the pattern link \( L_2 \subset (S^3 \setminus \hat{N}(K_2)) \). For satellites by \( T(np, nq) \), this unknotted \( K_2 \subset M_2 \) is the multiplicity-\( q \) fiber \( \lambda_0 \subset Y_{np,nq} \) in the \( T(np, nq) \)-exterior

\[
Y_{np,nq} := S^3 \setminus \hat{N}(T(np, nq)) = M_{S^2}(-\frac{q}{p}, \frac{p}{q}^*) \setminus \hat{N}(\prod_{i=1}^n f_i), \quad \text{c.f. (43).}
\]
In an iterated torus-link satellite, we only perform satellites on components of the companion link we are building. That is, for an edge \( e \in \text{Edge}(\Gamma) \) from \( v(-e) \) to \( v(e) \), we always form a \( T_{v(e)} \)-torus-link satellite that splices the multiplicity-\( q_{v(-e)} \) fiber \( \lambda_0^{v(-e)} \), with exterior

\[
Y_{v(-e)} := Y_{v(-e)}^{n_{v(-e)}} \cup_{(p_{v(-e)}, q_{v(-e)})} \hat{\nu} \circ (\lambda_0^{v(-e)}),
\]

(199)
to the \( j(e) \)th component of the \( T_{v(e)} \) torus link. Since this \( j(e) \)th link component is represented by the smooth fiber \( f_{j(e)} \subseteq M_{S^2}(-\frac{p_v}{p_v}, \frac{q_v}{q_v}) \), we call this operation a smooth splice.

In an algebraic link exterior, however, an edge \( e \) can also specify an exceptional splice map, in which we splice the \( q_{v(-e)} \)-fiber \( \lambda_0^{v(-e)} \) to the exceptional \( p_{v(e)} \)-fiber \( \lambda_{v(e)} \) \( Y_{v(e)} \). This multiplicity-\( p_{v(e)} \) fiber \( \lambda_{v(e)} \) is not a component of our original companion link or its iterated satellites, but is rather the core of the solid torus \( \nu(\lambda_{v(e)}) \) hosting \( T_{v(e)} \). Thus an exceptional splice satellite embeds the solid torus hosting \( T_{v(-e)} \) inside the solid torus hosting \( T_{v(e)} \). Since an exceptional splice at \( e \) takes the satellite of the \( p_{v(e)} \)-fiber \( \lambda_{v(e)} \) \( Y_{v(e)} \), we set \( j(e) = -1 \) in this case.

7.2. Slope maps induced by splices

For the induced maps on slopes, we have

\[
[\varphi_{v(e)}] \begin{pmatrix} p_{v(-e)} & -q_{v(-e)} \\ q_{v(-e)} & -p_{v(-e)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \varphi_{v(e)}(y) = \frac{p_{v(-e)}y - q_{v(-e)}}{q_{v(-e)}y - p_{v(-e)}}
\]

(200)
for an edge \( e \) corresponding to a smooth splice, and

\[
[\sigma_{v(e)}] \begin{pmatrix} p_{v(e)} & -q_{v(e)} \\ q_{v(e)} & -p_{v(e)} \end{pmatrix} = \begin{pmatrix} p_{v(e)} & -q_{v(e)} \\ -q_{v(e)} & p_{v(e)} \end{pmatrix} \Rightarrow [\sigma_{v(e)}] = \begin{pmatrix} p_{v(e)} & -q_{v(e)} \\ -q_{v(e)} & p_{v(e)} \end{pmatrix}
\]

(201)
for an edge \( e \) corresponding to an exceptional splice. To accommodate our notation to these two different types of maps, we define

\[
\phi_e := \begin{cases} \sigma_e & j(e) = -1 \\ \varphi_e & j(e) \neq -1 \end{cases}
\]

(202)
In addition, since the exceptional fiber at \( \partial_{-1} Y_v \) is only exceptional if \( p_v > 1 \), we adopt the convention that exceptional splice edges only terminate on vertices \( v \) with \( p_v > 1 \).

7.3. Algebraic links

Eisenbud and Neumann show in \cite{EisenbudNeumann} that a graph \( \Gamma \) with such edges and vertices specifies an algebraic link exterior if and only if \( \Gamma \) satisfies the algebraicity conditions

\[(i) \quad p_v, q_v, n_v > 0 \text{ for all } v \in \text{Vert}(\Gamma), \]
\[(ii) \quad \Delta_e > 0 \text{ for all } e \in \text{Edge}(\Gamma), \]

(203)

\[
\Delta_e := \begin{cases} p_{v(e)}q_{v(-e)} - p_{v(-e)}q_{v(e)} & j(e) = -1 \\ q_{v(-e)} - p_{v(e)}p_{v(-e)}q_{v(e)} & j(e) \neq -1 \end{cases}
\]
ensuring negative definiteness. Eisenbud and Neumann also prove that any algebraic link exterior can be realized by such a graph. Note that the above algebraicity conditions imply

\[ 0 < \frac{p_v(-e)}{q_v(-e)} < 1 \quad \text{if} \quad j(e) \neq -1. \] (204)

For notational convenience, we adopt the convention that \( J_v \) remains the same, only indexing incoming edges corresponding to smooth splices. That is, we define

\[ J_v := \{ j(e) | e \in E_{in}(v) \} \cap \{ 1, \ldots, n_v \}, \] (205)

and its complement \( I_v \) still indexes the remaining boundary components,

\[ I_v := \{ 1, \ldots, n_v \} \cap J_v. \] (206)

Finally, since \( \phi_e \) is always orientation-reversing, the induced map \( \phi^p_e \) is still decreasing with respect to the circular order on \( sf \)-slopes, and the impact of \( \phi^p_e \) on the linear order of finite \( sf \)-slopes still depends on the positions of the horizontal and vertical asymptotes

\[ \xi_{\phi(e)} := \phi^p_e(\infty) \in (\mathbb{Q} \cup \{ \infty \})_{sf_{\phi(e)}} \quad \text{and} \quad \eta_{\phi(e)} := (\phi^p_e)^{-1}(\infty) \in (\mathbb{Q} \cup \{ \infty \})_{sf_{\phi(e)}}, \] (207)

respectively, of the graph of \( \phi^p_e \). More explicitly, we have

\[ \xi_{\phi(e)} = \begin{cases} \frac{p_v(-e)}{q_v(-e)} & j(e) \neq -1, \\ -\frac{q_v(-e)}{p_v(-e)} + \frac{p_v(-e)}{p_v(e)} \Delta_v & j(e) = -1 \end{cases}, \quad \eta_{\phi(e)} = \begin{cases} \frac{p_v(-e)}{q_v(-e)} & j(e) \neq -1, \\ \frac{q_v(-e)}{p_v(-e)} + \frac{p_v(e)}{p_v(-e)} \Delta_v & j(e) = -1. \end{cases} \] (208)

7.4. Adapting Propositions 6.1 and 6.2 for algebraic link exteriors

In the case of algebraic link exteriors, we must incorporate the possibility of exceptional splices into the expressions \( \bar{y}_0^v(k) \) and \( \bar{y}_0^+(k) \) originally defined in (151) and (152), from which

\[ \bar{y}_0^v := \sup_{k > 0} \bar{y}_0^v(k) \quad \text{and} \quad \bar{y}_0^+ := \inf_{k > 0} \bar{y}_0^v(k) \] (209)

are defined. The only changes that arise are localized to the summands \([\frac{y_v}{p_v}]\) and \([\frac{y_v}{p_v}]\) in \( \bar{y}_0^v(k) \) and \( \bar{y}_0^+(k) \), respectively. We perform such modifications as follows.

First, for \( v \in \text{Vert}(\Gamma) \), with \( \Gamma \) specifying an algebraic link exterior, set

\[ y_{-1}^v := \begin{cases} \frac{y_v}{p_v} & j(E_{in}(v)) = -1, \\ -1 & j(E_{in}(v)) \neq -1 \end{cases}, \] (210)

and for \( k \in \mathbb{Z}_{>0} \), define \( y_{-1}^v(k) \) and \( y_{-1}^v(k) \) by setting

\[ y_{-1}^v(k) := \begin{cases} -[y_{-1}^v + k] - 1 & Y_{\Gamma_{v(-e')}}(y_{-1}^v(-e')) \text{ BI}, \\ -[y_{-1}^v + k] & Y_{\Gamma_{v(-e')}}(y_{-1}^v(-e')) \text{ BC} \end{cases}, \] (211)

\[ y_{-1}^v(k) := \begin{cases} -[y_{-1}^v + k] + 1 & Y_{\Gamma_{v(-e')}}(y_{-1}^v(-e')) \text{ BI}, \\ -[y_{-1}^v + k] & Y_{\Gamma_{v(-e')}}(y_{-1}^v(-e')) \text{ BC} \end{cases}, \] (212)

where again, \( e' \in E_{in}(v) \) is the unique incoming edge with \( j(e') = -1 \), if such \( e' \) exists. If \( -1 \notin j(E_{in}(v)) \), then we take \( Y_{\Gamma_{v(-e')}}(y_{-1}^v(-e')) \) to be boundary-compressible.
Next, we define $\bar{y}_{0-}^v(k)$ and $\bar{y}_{0+}^v(k)$ to be respective results of replacing the summand $K_{\frac{q_v}{p_v}} k$ with $y_{-1-}^v(k)$ in the definition of $\bar{y}_{0-}^v(k)$ in (151), and replacing the summand $K_{\frac{q_v}{p_v}} k$ with $y_{-1+}^v(k)$ in the definition of $\bar{y}_{0+}^v(k)$ in (152). That is, we set
\begin{align}
\bar{y}_{0-}^v(k) &:= \bar{y}_{0-}^v(k) + \frac{1}{k} (y_{-1+}^v(k) - \left\lfloor \frac{q_v}{p_v} k \right\rfloor), \quad (213) \\
\bar{y}_{0+}^v(k) &:= \bar{y}_{0+}^v(k) + \frac{1}{k} (y_{-1-}^v(k) - \left\lfloor \frac{q_v}{p_v} k \right\rfloor), \quad (214)
\end{align}
and by analogy with the definition of $\bar{y}_{0-}^v$ in (209), we define
\begin{align}
\bar{y}_{0-}^v := \sup_{k > 0} \bar{y}_{0-}^v(k) \quad \text{and} \quad \bar{y}_{0+}^v := \inf_{k > 0} \bar{y}_{0+}^v(k). \quad (215)
\end{align}
We are now ready to state and prove an analog of Proposition 6.1 and a supplement to Proposition 6.2.

**Proposition 7.1.** Suppose $v \in \text{Vert}(\Gamma)$ for a graph $\Gamma$ specifying the exterior of an algebraic link. If $y_{0-}^v, y_{0+}^v \in \mathbb{P}(H_1(\partial_0 Y_v; \mathbb{Z}))_{\text{str}}$ are the (potential) L-space interval endpoints for $Y_{\Gamma_v}(y^{v+})$ as defined in Theorem 4.3, then
\begin{align}
y_{0-}^v &= \bar{y}_{0-}^v - \sum_{j \in J_v} (y_{j+}^v - 1) - \sum_{j \in J_v} [y_{j-}^v] - \sum_{i \in I_v} [y_{i-}^v], \\
y_{0+}^v &= \bar{y}_{0+}^v - \sum_{j \in J_v} (y_{j-}^v + 1) - \sum_{j \in J_v} [y_{j+}^v] - \sum_{i \in I_v} [y_{i+}^v],
\end{align}
for $J_v^{bc}$ and $J_v^{mt}$ as defined in (146) and (147).

**Proof.** This follows directly from Theorem 4.3. \hfill \Box

**Proposition 7.2.** Suppose that $\Gamma$ specifies the exterior of an algebraic link, and that $v \in \text{Vert}(\Gamma)$ has an incoming edge $e'$ with $j(e') = -1$. If
\begin{align}
-\frac{q_v}{p_v} \leq y_{j(e')}^- &= y_{j(e')-} \leq -\frac{q_v}{p_v} + m^{e'-}, \quad (216)
\end{align}
for some $m^{e'} \in \mathbb{Z}_{>0}$, then $\bar{y}_{0-}^v$ and $\bar{y}_{0+}^v$ satisfy
\begin{align}
\bar{y}_{0}^v &\leq \bar{y}_{0-}^v, \quad \bar{y}_{0+}^v \geq \bar{y}_{0+}^v + m^{e'-}. \quad (217)
\end{align}

**Proof.** It is straightforward to show that the bounds in (216) imply that
\begin{align}
y_{-1+}^v(k) &\leq \left\lfloor \frac{q_v}{p_v} k \right\rfloor, \quad y_{-1-}^v(k) \geq \left\lceil \frac{q_v}{p_v} k \right\rceil + m^{e'-}, \quad (218)
\end{align}
for all $k \in \mathbb{Z}_{>0}$. Thus, since $m^{e'} \in \mathbb{Z}$ implies $\frac{1}{k} \left\lfloor m^{e'} - k \right\rfloor = \frac{1}{k} \left\lfloor m^{e'} - k \right\rfloor = m^{e'}$, the desired result follows directly from (213), (214), and the definitions of $\bar{y}_{0-}^v$ in (215). \hfill \Box

7.5. L-space surgery regions for algebraic link satellites: Proof of Theorem 1.6

If $\Gamma$ specifies a one-component algebraic link, that is, a knot, then the L-space region is just an interval, determined by iteratively computing the genus of successive cables. For multi-component links, we can bound the L-space region as described in Theorem 1.6 in the introduction.
Proof of Theorem 1.6. The proofs of parts (i and (ii) are the same as those in the iterated torus link satellite case, if one keeps in mind that the \( p_r = 1 \) condition for (ii.b) and an explicit hypothesis for (ii.a) each rule out the possibility of an incoming exceptional splice at the root vertex.

The proof of part (iii) also adapts the proof used for iterated torus satellites, but we provide more details in this case. Again, for bookkeeping convenience, we redefine \( I_w := \{ 1 \} \) and set \( \mathcal{L}_{\mathcal{S}v}^{\min+} := \{ 0 \} \) and \( \mathcal{R}_{\mathcal{S}v} \setminus \mathcal{Z}_{\mathcal{S}v} := \mathcal{L}_{\mathcal{S}v}^{\min-} := \emptyset \) for any \( v \in \text{Vert}(\Gamma) \) with \( I_w = \emptyset \).

For a vertex \( v \in \text{Vert}(\Gamma) \), we inductively assume, for each incoming edge \( e \in E_{in}(v) \), that for any \( y^{\Gamma_{v}(-e)} \in \prod_{w \in \text{Vert}(\Gamma_{v}(-e))} (\mathcal{L}_{\mathcal{S}v}^{\min+} \cup \mathcal{R}_{\mathcal{S}v} \setminus \mathcal{Z}_{\mathcal{S}v} \cup \mathcal{L}_{\mathcal{S}v}^{\min-}) \), we have

\[
\mu_e + m_e^+ \leq y^{v}_{j(e)+} \leq y^{v}_{j(e)} \leq \mu_e + m_e^-, \quad (219)
\]

\[
\mu_e + m_e^+ < y^{v}_{j(e)+}, \quad y^{v}_{j(e)} < \mu_e + m_e^- \quad \text{if } Y_{\Gamma_{v}(-e)}(y^{\Gamma_{v}(-e)}) \text{ is BI}, \quad (220)
\]

where

\[
\mu_e := \frac{p_e}{q_e} \left( -q_e^*(-e) \right) = \begin{cases} 0 & j(e) \neq -1 \\ \frac{-q_e^*}{p_e} & j(e) = -1 \end{cases}
\]

is the meridian slope of the fiber of \( Y_v \) to which \( Y_v(-e) \) is spliced along \( e \) (so the image of the longitude of slope \( \frac{-q_e^*}{p_e}(-e) \) paired with the meridian of slope \( \frac{p_e^*}{q_e}(-e) \)), and where

\[
m_e^+ := \begin{cases} \frac{p_e}{q_e}(-e) & j(e) \neq -1 \\ 0 & j(e) = -1 \end{cases} = 0,
\]

\[
m_e^- := \begin{cases} \frac{p_e}{q_e}(-e) & j(e) \neq -1 \\ \frac{p_e}{q_e}(-e) + 1 & j(e) = -1 \end{cases},
\]

We then set

\[
m_e^+ := \sum_{e \in E_{in}(v)} m_e^+ + 0 = 0,
\]

\[
m_e^- := \sum_{e \in E_{in}(v)} m_e^- - \begin{cases} 1 & j(e) \neq 1 \\ \frac{p_e}{q_e}(-e) & j(e) = 1 \end{cases}.
\]

The statement of Theorem 1.6 makes the substitution \( \frac{p_e}{q_e}(-e) + 1 \mapsto 1 \) for the \( j(e) \neq -1 \) case of \( m_e^- \), in its role as a summand of \( m_v^- \). However, this substitution is an equivalence for all \( v \in \text{Vert}(\Gamma) \) and \( e \in E_{in}(v) \), since the algebraicity condition (203) implies \( 0 < \frac{p_e}{q_e}(-e) < 1 \) for \( j(e) \neq -1 \). Note that this does not imply \( 0 < \frac{p_e}{q_e} < 1 \) for the root vertex \( v = v(-e_r) \), because of our declared convention that \( v(e_r) = \text{NULL} \notin \text{Vert}(\Gamma) \). The algebraicity condition (203) also implies that the conditions \( q = -1 \), \( q < 0 \), and \( \frac{p_e}{q_e} > +1 \) are never met for \( j(e) \neq -1 \) and \( v \neq r \). Thus, the \( j(e) \neq -1 \) cases of our definitions of \( \mathcal{L}_{\mathcal{S}v}^{\min-} \) and \( \mathcal{L}_{\mathcal{S}v}^{\min+} \) in Theorem 1.6(iii) coincide with the respective definitions of \( \mathcal{L}_{\mathcal{S}v}^{\min-} \) and \( \mathcal{L}_{\mathcal{S}v}^{\min+} \) in Theorem 1.7(ii).

If \( y^v \in \mathcal{R}_{\mathcal{S}v} \setminus \mathcal{Z}_{\mathcal{S}v} \), then \( y^v_0 = y^v_\infty = \infty \). Thus, \( y^{v(e_r)} = \phi_{e_r}(y^{v(e_r)}_0) = \phi_{e_r}(\infty) =: \xi_{v(e_r)} \), and referring to (208) for the computation of \( \xi_{v(e_r)} \), we have

\[
y^{v(e_r)}_{j(e)} = \xi_{v(e_r)} = \mu_{v(e_r)} + \left\{ \begin{array}{ll} \frac{p_e}{q_e} & j(e) \neq -1 \\ \frac{p_e}{q_e} \phi_{e_r}(\infty) & j(e) = -1 \end{array} \right\} \in \langle \mu_{v(e_r)} + m_{v(e_r)}^+, \mu_{v(e_r)} + m_{v(e_r)}^- \rangle.
\]
We assume \( y^v \in L_{\text{max}}^{-} \cup L_{\text{max}}^{+} \) for the remainder. This assumption, together with our inductive assumptions, makes Theorem 4.4 yield

\[
y_0^{-} \leq \bar{y}_0^{-},
\]
and Proposition 6.2 tells us

\[
\bar{y}_0^{-} \leq \frac{q_v}{p_v} \leq \bar{y}_0^{+}, \text{ with equality only if } Y_{\Gamma_v}(y^{\Gamma_v}) \text{ is BC.}
\]

We furthermore already know that \( \bar{y}_0^{+} = \bar{y}_0^{+} \) when \(-1 \notin j(E_{\text{in}}(v))\). Combining this fact with Proposition 7.2, given our inductive assumptions, yields

\[
\bar{y}_0^{-} \leq \bar{y}_0^{-} \leq \bar{y}_0^{+} + \left\{ \begin{array}{ll}
m_v^- & \exists e' \in E_{\text{in}}(v), j(e') = -1 \\
0 & -1 \notin j(E_{\text{in}}(v)) \end{array} \right.
\]

Suppose \( y^v \in L_{\text{max}}^{+} \). Then from Proposition 7.1, we have

\[
y_0^{-} \leq \bar{y}_0^{-} \leq \sum_{j \in J_v} y_j^{+} - \sum_{j \in J_v} |y_j^{-}|
\]

\[
\leq \bar{y}_0^{-} - \sum_{j \in J_v} m_v^+ - m_v^+
\]

\[
= \bar{y}_0^{-} \leq \bar{y}_0^{-} \leq \frac{q_v}{p_v}.
\]

Thus, altogether we have

\[
y_0^{-} \leq y_0^{-} \leq \frac{q_v}{p_v} < \eta_v, \text{ with equality only if } Y_{\Gamma_v}(y^{\Gamma_v}) \text{ is BC.}
\]

Here, \( \eta_v := (\phi_{e_+v}^p)^{-1}(\infty) \) is the location of the vertical asymptote of \( \phi_{e_+v}^p \). The inequality \( \frac{q_v}{p_v} < \eta_v \) follows directly from the computation of \( \eta_v \) in (208), plus the fact that \( \frac{q_v}{p_v} < \frac{q_v}{y_v} \). Since \( \phi_{e_+v}^p \) is locally monotonically decreasing on the complement of \( \eta_v \), this implies that all the expressions on the left-hand side of (233) have \( \phi_{e_+v}^p \)-images below the horizontal asymptote at \( \xi_{v(e_v)} \), but in reverse order. That is, we have

\[
\phi_{e_+v}^p \left( \frac{q_v}{p_v} \right) := \mu_{e_v} \leq y_{j(e_v)+} := \phi_{e_+v}^p(y_0^{-}) \leq y_{j(e_v)+} := \phi_{e_+v}^p(y_0^{+}) < \xi_{v(e_v)}.
\]

Thus, since \( m_v^+ = 0 \) and since (226) shows that \( \xi_{v(e_v)} < \mu_{e_v} + m_v^+ \), we obtain

\[
\mu_{e_v} + m_v^+ \leq y_{j(e_v)+} \leq y_{j(e_v)+} < \mu_{e_v} + m_v^-, \text{ with equality only if } Y_{\Gamma_v}(y^{\Gamma_v}) \text{ is BC.}
\]

Finally, suppose \( y^v \in L_{\text{max}}^{-} \). Then combining Proposition 7.1 (for line (236)) with the right-hand inequality of (229), the inductive upper bounds on \( y_j^{+} \) for \( J_v := j(E_{\text{in}}(v)) \), and the upper bound \( \sum_{i \in I_v} [y_i^{+}] \leq m_v^{-} \) for \( y^v \in L_{\text{max}}^{-} \) (for line (237)), we obtain

\[
y_0^{+} \geq \bar{y}_0^{+} - \sum_{j \in J_v} y_j^{-} - \sum_{i \in I_v} |y_i^{-}|,
\]

\[
\geq \bar{y}_0^{+} - \sum_{e \in E_{\text{in}}(v)} m_v^- - m_v^{-}.
\]
Combining (228) from Proposition 6.2 with the definition (225) of $m_v$ then gives

$$ y_{0+}^v \geq \frac{q_v^*}{p_v} + \begin{cases} 1 & J_v \neq \emptyset; \ j(e_v) \neq -1 \\
\left\lfloor \frac{p_v(e_v)}{p_v \Delta_{e_v}} \right\rfloor + 1 & j(e_v) = -1 \\
0 & \text{otherwise} \end{cases} ,$$

with equality only if $Y_{e_v}(\Gamma^r)$ is BC. When $j(e_v) \neq -1$, the desired inductive result is established in the $y^r \in \mathcal{L}_{S^3}^{\min}$ case of the proof of Theorem 1.7. We henceforth assume $j(e_v) = -1$.

Since $j(e_v) = -1$ implies $\frac{q_v^*}{p_v} + \left\lfloor \frac{p_v(e_v)}{p_v \Delta_{e_v}} \right\rfloor + 1 > \eta_v$, we have $y_{0-}^v \geq y_{0+}^v > \eta_v$, that is, to the right of the vertical asymptote of $\phi_{e_v}^+$ at $\eta_v$. The respective $\phi_{e_v}^+$-images $y_{j(e_v)+}^v$ and $y_{j(e_v)-}^v$ of $y_{0-}^v$ and $y_{0+}^v$ therefore lie above the horizontal asymptote at $\xi_{e_v}^v$, but with reversed order:

$$ \mu_{e_v} + m_{e_v}^v = -\frac{q_v(e_v)}{p_v(e_v)} < \xi_{e_v}^v < y_{j(e_v)+}^v \leq y_{j(e_v)-}^v. $$

(239)

It remains to show that $\mu_{e_v} + m_{e_v}^v \geq y_{j(e_v)-}^v := \phi_{e_v}^{\Delta_{e_v}^v} \left( y_{0+}^v \right)$ (with equality only if $Y_{e_v}(\Gamma^r)$ is BC), for which it suffices to show that $(\mu_{e_v} + m_{e_v}^v) - \phi_{e_v}^{\Delta_{e_v}^v} = \phi_{e_v}^{\Delta_{e_v}^v} = 0$.

Recall that any edge $e$ with $j(e) = -1$ has $\phi_{e}^p = \sigma_{e}^p$. If we write $[\sigma_{e}^p] = (\alpha_{e}^p \ \Delta_{e}^p)$ for the entries of the matrix $[\sigma_{e}^p]$ as computed in (201), then the relations $p_{e}^u q_{e}^u - q_{e}^u p_{e}^u = 1$ for each $u \in \text{Vert}(\Gamma)$, particularly for $u \in \{v, v(e)\}$, produce simplifications, including the identities

$$ p_v(e_v) \alpha_{e_v} + q_v(e_v) \Delta_{e_v} = p_v, \quad p_{v} \beta_{e_v} + q_{v} \Delta_{e_v} = -p_v(e_v), \quad q_{v} \alpha_{e_v} + p_{v} \Delta_{e_v} = q_{v}(e_v), $$

(240)

used in the intermediate steps suppressed in the following calculation. We compute that

$$ (\mu_{e_v} + m_{e_v}^v) - \phi_{e_v}^{\Delta_{e_v}^v} \left( \frac{q_v^*}{p_v} + \left\lfloor \frac{p_v(e_v)}{p_v \Delta_{e_v}} \right\rfloor + 1 \right) $$

$$ = \left( -\frac{q_v(e_v)}{p_v(e_v)} + \left\lfloor \frac{p_v}{p_v(e_v) \Delta_{e_v}} \right\rfloor + 1 \right) - \frac{\alpha_{e_v} \left( p_v \Delta_{e_v} + \left\lfloor \frac{p_v(e_v)}{p_v \Delta_{e_v}} \right\rfloor \right) + p_v}{\Delta_{e_v} \left( p_v \Delta_{e_v} + \left\lfloor \frac{p_v}{p_v \Delta_{e_v}} \right\rfloor \right)} $$

(241)

$$ = 1 + \left\lfloor \frac{p_v}{p_v(e_v) \Delta_{e_v}} \right\rfloor - \Delta_{e_v}^2 \left( 1 + \left\lfloor \frac{p_v(e_v)}{p_v \Delta_{e_v}} \right\rfloor \right)^{-1} $$

(242)

$$ \geq 0, $$

(243)

since $[x] := x - |x|$ implies $0 \leq |x| < 1$ for $x \in \mathbb{Q}$, and this completes our inductive argument.

Since $j(e_v) \neq -1$, the proof that these inductive bounds cause the Dehn-filled satellite exterior $Y^r(\Gamma^r) := Y_{l}(\Gamma^r) \cup (S^3 \setminus \nu(K))$ to form an L-space whenever

$$ y^r \in \prod_{v \in \text{Vert}(\Gamma^r)} \left( \mathcal{L}_{\mathcal{S}_v}^{\min} + \mathcal{R}_{\mathcal{S}_v} \setminus \mathcal{Z}_{\mathcal{S}_v} \cup \mathcal{L}_{\mathcal{S}_v}^{\min} \right) $$

is the same as the corresponding argument in the proof of Theorem 1.7. \(\square\)

### 7.6. Monotone strata

Whether for iterated torus satellitess and for algebraic link satellites, our inner approximations $\mathcal{L}_{\mathcal{S}_v}^{\min} (Y^r) := \prod_{v \in \text{Vert}(\Gamma)} \left( \mathcal{L}_{\mathcal{S}_v}^{\min} + \mathcal{R}_v \setminus \mathcal{Z}_v \cup \mathcal{L}_{\mathcal{S}_v}^{\min} \right)$ each involve inductive bounds, namely,
Observe that for any torus, each particular substratum of space intervals. We call this condition ‘monotonicity’ because of its preservation of this corresponding to the endpoint-ordering consistent with that for generic Seifert fibered L-edge.

Let \( y^\Gamma \) be classified according to whether 2 into strata according to monotonicity conditions, similar to how torus link satellites must first (182) and (219), respectively, which make \( y_j^{V(e)} \leq y_j^{V(e)} \), as a function of \( y^\Gamma_{|\Gamma_{V(e)}} \), for each edge \( e \in \text{Edge}(\Gamma) \) and slope \( y^\Gamma \in L^\text{min}_{\text{SFT}}(Y^\Gamma) \). This implies that \( L^\text{min}_{\text{SFT}}(Y^\Gamma) \) is confined to a particular substratum of \( L^\text{mono}_{\text{SFT}}(Y^\Gamma) \), called the monotone stratum.

**Definition 7.3.** For any \( y^\Gamma \in (\mathbb{Q} \cup \{\infty\})^{|I_r|} \) and \( v \in \text{Vert}(\Gamma) \), we call \( y^\Gamma \) monotone at \( v \) if

\[
\begin{align*}
\infty \in \phi^p_{e,v} L^0_{\Gamma_{V(e)}}(y) \quad &\forall e \in E_{in}(v) \quad \text{and} \quad \infty \in \phi^p_{e,v} L^0_{\Gamma_{V(e)}}(y).
\end{align*}
\]

The monotone stratum \( L^\text{mono}_{\text{SFT}}(Y^\Gamma) \) of \( L^\text{SFT}_{\Gamma}(Y^\Gamma) \) is then the set of slopes \( y^\Gamma \in L^\text{SFT}_{\Gamma}(Y^\Gamma) \) such that \( y^\Gamma \) is monotone at all \( v \in \text{Vert}(\Gamma) \).

In the above, for brevity, we have adopted the following the following.

**Notation 7.4.** For any slope \( y^\Gamma \in (\mathbb{Q} \cup \{\infty\})^{|I_r|} \) and vertex \( v \in \text{Vert}(\Gamma) \), we shall write

\[
L^v_{\Gamma}(y) := L^\text{SFT}_{\Gamma_{V(e)}}(Y^\Gamma_{|\Gamma_{V(e)}}), \quad L^v_{\Gamma}(y) := L^\text{SFT}_{\Gamma_{V(e)}}(Y^\Gamma_{|\Gamma_{V(e)}}).
\]

**Remark.** Note that if \( L^v_{\Gamma}(y) \neq \emptyset \) for all \( e \in E_{in}(v) \) and \( L^v_{\Gamma}(y) \neq \emptyset \), then

\[
\phi^p_{e,v} L^v_{\Gamma_{V(e)}}(y) = \left( y_j^{v(e)} \right)_{j=1,\ldots,n} \quad \forall e \in E_{in}(v), \quad \phi^p_{e,v} L^v_{\Gamma}(y) = \left( y_j^{v(e)} \right)_{j=1,\ldots,n},
\]

and the monotonicity condition (245) at \( v \) is equivalent to the condition that

\[
y_j^{v(e)} \leq y_j^{v(e)} \quad \forall e \in E_{in}(v), \quad y_j^{v(e)} \leq y_j^{v(e)} - y_j^{v(e)},
\]

corresponding to the endpoint-ordering consistent with that for generic Seifert fibered L-space intervals. We call this condition ‘monotonicity’ because of its preservation of this ordering.

The tools developed in Sections 6 and 7 can be used in much more general settings than that of the inner approximation theorems we proved, so long as one first decomposes \( L^\text{SFT}_{\Gamma}(Y^\Gamma) \) into strata according to monotonicity conditions, similar to how torus link satellites must first be classified according to whether \( 2g(K) - 1 \leq \frac{q}{p} \). Monotonicity conditions also impact the topology of strata.

**Theorem 7.5.** Suppose that \( K^\Gamma \subset S^3 \) is an algebraic link satellite, specified by \( \Gamma \), of a positive L-space knot \( K \subset S^3 \), where either \( K \) is trivial, or \( K \) is nontrivial with \( \frac{q}{p} > 2g(K) - 1 \). Let \( V \subset \text{Vert}(\Gamma) \) denote the subset of vertices \( v \in V \) for which \( |I_v| \leq 0 \).

Then the \( \overline{B}_{\text{SFT}} \)-corrected \( \mathbb{R} \)-completed monotone stratum \( L^{\text{mono}}_{\text{SFT}}(Y^\Gamma)^{\mathbb{R}} := L^{\text{mono}}_{\text{SFT}}(Y^\Gamma) \setminus \overline{B}_{\text{SFT}} \) is of dimension \( |I_v| \) and deformation retracts onto an \( (|I_v| - |V|) \)-dimensional embedded torus,

\[
L^{\text{mono}}_{\text{SFT}}(Y^\Gamma)^{\mathbb{R}} \xrightarrow{\text{def. retract}} \prod_{v \in V} \mathbb{T}^{|I_v| - 1} \xrightarrow{} \prod_{v \in V} (\mathbb{R} \cup \{\infty\})^{|I_v|}_{|V|},
\]

projecting to an embedded torus \( \mathbb{T}^{|I_v| - 1} \xrightarrow{} (\mathbb{R} \cup \{\infty\})^{|I_v|}_{|V|} \) parallel to \( \overline{B}_{\text{SFT}} \subset (\mathbb{R} \cup \{\infty\})^{|I_v|}_{|V|} \) at each \( v \in V \).

**Proof.** We argue by induction, recursing downward from the leaves of \( \Gamma \) towards its root. Observe that for any \( v \in \text{Vert}(\Gamma) \), we have the fibration

\[
L^{\text{mono}}_{\text{SFT}_{\Gamma_v}}(Y^\Gamma_v) \xrightarrow{} \prod_{e \in E_{in}(v)} L^{\text{mono}}_{\text{SFT}_{\Gamma_{V(e)}}}(Y^\Gamma_{V(e)-e})
\]
the slope subsets parallel to fiber

\[ \mathcal{T}_{y_*} := \left\{ y_* \in \mathcal{L}_{\text{CTF}}^{\text{mono}}(Y_1) \mid y_* \bigg|_{\prod_{e \in E_{\text{in}}(v)} \Gamma_{e(-e)} = y_*} \right\} \]  \hspace{1cm} (249)

over \( y_* \in \prod_{e \in E_{\text{in}}(v)} \mathcal{L}_{\text{CTF}}^{\text{mono}}(Y_1) \) for \( I_v \neq \emptyset \), with \( \mathcal{T}_{y_*} \) regarded as a point when \( I_v = \emptyset \).

For \( v \in \text{Vert}(\Gamma) \), inductively assume the theorem holds for \( \Gamma_{e(-e)} \) for all \( e \in E_{\text{in}}(v) \). (Note that this holds vacuously when \( v \) is a leaf, in which case we declare \( \mathcal{T}_v := \mathcal{L}_{\text{CTF}}^{\text{mono}}(Y_1) \).

If \( I_v = \emptyset \), then the fibration in (248) is the identity map, making the theorem additionally hold for \( \Gamma_v \). Next assuming \( I_v \neq \emptyset \), we claim that \( (\mathcal{T}_{y_*})^k := \mathcal{T}_{y_*} \setminus \mathcal{B}_{\text{SF}_{y_*}} \) is of dimension \( |I_v| \) and deformation retracts onto an embedded torus \( \mathbb{T}^{|I_v|-1} \hookrightarrow (\mathbb{R} \cup \{\infty\})^{|I_v|} \) parallel to \( \mathcal{B}_{\text{SF}_{y_*}}^R = \mathcal{B}_{\text{SF}_{y_*}} \subset (\mathbb{R} \cup \{\infty\})^{|I_v|} \). In fact, the proof of this statement is nearly identical to the proof of Theorem 5.2(ii.b) in Section 5, but with the replacement

\[ \mathcal{N} := \{ y \in \mathcal{Q}_{\text{SF}}^u | y_u(y) < 0 < y_u(-y) \} \rightarrow \mathcal{N}_v := \{ y_v \in \mathcal{Q}_{\text{SF}_{y_*}} | y_v(y_v) < y_v < y_v(y_v) \} \]  \hspace{1cm} (250)

in line (125) (where \( \eta_v \), computed in (208), is the position of the vertical asymptote of \( \phi_{e,v,s}^P \)), along with a few minor analogous adjustments corresponding to this change.

It remains to show that the fibration in (248) is trivial, but this follows from the fact that

\[ \mathcal{L}_{\text{SF}_{y_*}}^{\text{min}}(Y_1) := \prod_{u \in V \cap \text{Vert}(\Gamma_v)} \left( \mathcal{L}_{\text{SF}_{y_*}}^{\text{min}} - \mathcal{R}_u \setminus \mathcal{Z}_u \cup \mathcal{L}_{\text{SF}_{y_*}}^{\text{min}} + \right) \]  \hspace{1cm} (251)

is a product over \( u \in V \cap \text{Vert}(\Gamma_v) \) which embeds into \( \mathcal{L}_{\text{CTF}}^{\text{mono}}(Y_1) \), and for reasons again similar to the proof of Theorem 5.2(ii.b), the \( \mathcal{B}_{\text{SF}_{y_*}}^R \)-corrected \( \mathbb{R} \)-completion of each factor \( \mathcal{L}_{\text{SF}_{y_*}}^{\text{min}} - \mathcal{R}_u \setminus \mathcal{Z}_u \cup \mathcal{L}_{\text{SF}_{y_*}}^{\text{min}} + \) also deformation retracts onto an embedded torus \( \mathbb{T}^{|I_v|-1} \hookrightarrow (\mathbb{R} \cup \{\infty\})^{|I_v|} \) parallel to \( \mathcal{B}_{\text{SF}_{y_*}}^R = \mathcal{B}_{\text{SF}_{y_*}} \subset (\mathbb{R} \cup \{\infty\})^{|I_v|} \), completing the proof. \( \square \)

8. Extensions of L-space Conjecture Results

As mentioned in the introduction, Boyer–Gordon–Watson [5] conjectured several years ago that among prime, closed, oriented 3-manifolds, L-spaces are those 3-manifolds whose fundamental groups do not admit a left orders (LO). Similarly, Juhász [20] conjectured that prime, closed, oriented 3-manifold are L-spaces if and only if they fail to admit a co-oriented taut foliation (CTF). Procedures which generate new collections of L-spaces or non-L-spaces, such as surgeries on satellites, provide new testing grounds for these conjectures.

For \( Y \) a compact oriented 3-manifold with boundary a disjoint union of \( n > 0 \) tori, define the slope subsets \( \mathcal{F}(Y), \mathcal{LO}(Y) \subset \prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \) so that

\[ \mathcal{F}(Y) := \left\{ \alpha \in \prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \bigg| Y \text{ admits a CTF } \mathcal{F} \text{ such that } \mathcal{F}|_{\partial Y} \text{ is the product foliation of slope } \alpha \right\}, \]  \hspace{1cm} (252)

\[ \mathcal{LO}(Y) := \left\{ \alpha \in \prod_{i=1}^n \mathbb{P}(H_1(\partial Y; \mathbb{Z})) \bigg| \pi_1(Y(\alpha)) \text{ is LO.} \right\}. \]  \hspace{1cm} (253)

Note that \( \alpha \in \mathcal{F}(Y) \) implies that \( Y(\alpha) \) admits a CTF, but the converse, while true for \( Y \) a graph manifold, is not known in general.

8.1. Proof of Theorem 1.4 and Generalizations

The proof of Theorem 1.4 relies on the related gluing behavior of co-oriented taut foliations, left orders on fundamental groups, and the property of being an non-L-space, for a pair
RATIONAL L-SPACE SURGERIES ON SATELLITES BY ALGEBRAIC LINKS

Y_1, Y_2 of compact oriented 3-manifolds with torus boundary glued together via a gluing map \( \varphi : \partial Y_1 \to \partial Y_2 \).

That is, the contrapositives of Theorems 3.5 and 3.6 tell us that if \( Y_i \) have incompressible boundaries and are both Floer simple manifolds, both graph manifolds, or an L-space knot exterior and a graph manifold, then

\[
\varphi^* (\mathcal{NL}(Y_1)) \cap \mathcal{NL}(Y_2) \neq \emptyset \iff Y_1 \cup_\varphi Y_2 \text{ not an L-space.}
\]  

(254)

The analogous statements for CTFs and LOs, while true for graph manifolds (once an exception was made for reducible slopes in the case of CTFs), are not established in general. However, we still have weak gluing statements in the general case. Since product foliations of matching slope can always be glued together, we have

\[
\varphi^* (\mathcal{F}(Y_1)) \cap \mathcal{F}(Y_2) \neq \emptyset \implies Y_1 \cup_\varphi Y_2, \text{ if prime, admits a CTF.}
\]  

(255)

Moreover, Clay, Lidman, and Watson [7] built on a result of Bludov and Glass [2] to show that

\[
\varphi^* (\mathcal{LO}(Y_1)) \cap \mathcal{LO}(Y_2) \neq \emptyset \implies \pi_1(Y_1 \cup_\varphi Y_2) \text{ is LO.}
\]  

(256)

**Theorem 8.1.** Suppose \( Y^\Gamma \) is the exterior of an algebraic link satellite or (possibly-iterated) torus-link satellite of a nontrivial positive L-space knot \( K \subset S^3 \) of genus \( g(K) \) and exterior \( Y \), with \( p_r > 1 \) and \(-1 \not\in J(E_{in}(r))\).

(LO) Suppose \( \mathcal{LO}(Y) \supset \mathcal{NL}(Y) \).

(lo.i) If \( 2g(K) - 1 > \frac{g + 1}{p_r} \), then \( \mathcal{LO}(Y^\Gamma) = \mathcal{NL}(Y^\Gamma) \).

(lo.ii) If \( 2g(K) - 1 < \frac{g}{p_r} \) and \( \Gamma = r \) specifies a torus link satellite, then \( \mathcal{LO}(Y^\Gamma) \supset (\mathcal{NL}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma)) \setminus \Lambda(Y^\Gamma) \cdot (-\infty, N_1)^n \setminus [-\infty, \mathcal{NI} - p_r)^n) \),

where \( N_1 := p_r q_r - q_r - p_r + 2g(K) p_r \).

(c.tf) Suppose \( \mathcal{F}(Y) = \mathcal{NL}(Y) \).

(c.tf.i) If \( 2g(K) - 1 > \frac{g + 1}{p_r} \), then \( \mathcal{F}(Y^\Gamma) = \mathcal{NL}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma) \).

(c.tf.ii) If \( 2g(K) - 1 < \frac{g}{p_r} \) and \( \Gamma = r \) specifies a torus link satellite, then \( \mathcal{F}(Y^\Gamma) \supset (\mathcal{NL}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma)) \setminus \Lambda(Y^\Gamma) \cdot (-\infty, N_1)^n \setminus [-\infty, \mathcal{NI} - p_r)^n) \),

Note that the requirement that \( K \) be nontrivial is just to simplify the theorem statement. If \( K \) is trivial, then any surgery on \( Y^\Gamma \) is a graph manifold or a connected sum thereof, in which case the L-space conjectures written down by Boyer–Gordon–Watson and Juhász are already proven to hold, through the work of Boyer and Clay [4] and of Hanselman, Rasmussen, Watson, and the author [14]. In addition, the author explicitly shows in [29] that any graph manifold \( Y^\Gamma \) always satisfies

\[
\mathcal{LO}(\Gamma) = \mathcal{NL}(Y^\Gamma), \quad \mathcal{F}(Y^\Gamma) = \mathcal{NL}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma).
\]  

(257)

**Proof of Theorem.** Suppose \( \mathcal{LO}(Y) \supset \mathcal{NL}(Y) \) (respectively, \( \mathcal{F}(Y) = \mathcal{NL}(Y) \)). Since

\[
Y^\Gamma = Y^\Gamma \cup_\varphi Y \quad \text{and} \quad (\varphi^*)^{-1}(\mathcal{L}(Y)) = \left[ \frac{q_r^*}{p_r} - \frac{1}{p_r(N - q_r)} \right]_{SF} \cdot \left[ \frac{q_r^*}{p_r} \right]_{SF},
\]  

(258)

where \( N := 2g(K) - 1 \), it follows from (256) (respectively, (255)) that in order to prove that \( y^\Gamma \in \mathcal{LO}(Y^\Gamma) \) (respectively, \( y^\Gamma \in \mathcal{F}(Y) \cup \mathcal{Z}(Y^\Gamma) \)), it suffices to show that

\[
\mathcal{NL}_{SF}(Y^\Gamma(y^\Gamma)) \cap \left[ \left( -\infty, \frac{q_r^*}{p_r} - \frac{1}{p_r(N - q_r)} \right) \cup \left( \frac{q_r^*}{p_r} + \infty \right) \right] \neq \emptyset.
\]  

(259)
On the other hand, (254) implies that $y^\Gamma \in \mathcal{N}(Y^\Gamma)$ if and only if
\[
\mathcal{N}(Y^\Gamma) \cap \left( \left[ -\infty, \frac{q^*_r}{p_r} - \frac{1}{p_r(p_rN - q_r)} \right] \cup \left[ \frac{q^*_r}{p_r} , +\infty \right] \right) \neq \emptyset
\] (260)
when $Y^\Gamma$ is $\mathcal{B}$L, and if and only if (259) holds when $Y^\Gamma$ is $\mathcal{B}$C.

Fix some slope $y^\Gamma \in (Q \cup \{\infty\})^{\sum_{v \in \text{Vert}(\Gamma)} |I_v|}$ and write
\[
\mathcal{L}(y^\Gamma) = \left[ [y_{0-}, y_{0+}] \right],
\] (261)
as usual. It is straightforward to show that (260) fails to hold if and only if
\[
\frac{q^*_r}{p_r} - \frac{1}{p_r(p_rN - q_r)} < y_{0+} \leq y_{0-} < \frac{q^*_r}{p_r},
\] (262)
and that (259) fails to hold if and only if
\[
\frac{q^*_r}{p_r} - \frac{1}{p_r(p_rN - q_r)} \leq y_{0+} \leq y_{0-} \leq \frac{q^*_r}{p_r}.
\] (263)
Note that $p_r > 1$ implies
\[
0 \leq \frac{q^*_r}{p_r} - \frac{1}{p_r(p_rN - q_r)} < \frac{q^*_r}{p_r} < 1,
\]
with $\frac{q^*_r}{p_r} - \frac{1}{p_r(p_rN - q_r)} = 0 \iff N = \frac{q^*_r}{p_r} + 1$. (264)

We begin by proving the following claim.

**Claim.** If $N := 2g(K) - 1 > \frac{q^*_r}{p_r}, p_r > 1,$ and $-1 \notin j(E_{in}(r)),$ then
\[
Y^\Gamma(y^\Gamma) = S^\Gamma \iff \begin{cases} (262) \text{ holds if } Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}L, \\ (263) \text{ holds if } Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}C, \end{cases} \] (265)
If, in addition, $N := 2g(K) - 1 \neq \frac{q^*_r+1}{p_r},$ then
\[
(263) \text{ holds } \iff \begin{cases} (262) \text{ holds if } Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}L, \\ (263) \text{ holds if } Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}C, \end{cases} \] (266)

**Proof of Claim.** Suppose $N > \frac{q^*_r}{p_r}, p_r > 1,$ and $-1 \notin j(E_{in}(r)).$ Then Proposition 6.1 together with Proposition 6.2(=) imply
\[
y_{0+} \in \frac{q^*_r}{p_r} + Z \iff y_{0-} \in \frac{q^*_r}{p_r} + Z \iff Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}C \Rightarrow y_{0-} = y_{0+},
\] (267)
so that
\[
(263) \text{ holds and } Y^\Gamma(y^\Gamma) \text{ is } \mathcal{B}C \iff y_{0+} = \frac{q^*_r}{p_r} \iff y_{0-} = \frac{q^*_r}{p_r} \Rightarrow Y^\Gamma(y^\Gamma) = S^\Gamma.
\] (268)
Thus, since the fact that $S^\Gamma$ is an $L$-space makes the $\Rightarrow$ implication of (265) automatically hold, this exhausts the case when $Y^\Gamma(y^\Gamma)$ is $\mathcal{B}C$. Next suppose that $Y^\Gamma(y^\Gamma)$ is $\mathcal{B}L$, so that Proposition 6.2(+) implies $y_{0+} \in \left( \frac{q^*_r}{p_r}, 1 \right] + Z.$ Then (264) implies that (262) always fails to hold, and that (263) fails to hold if $N \neq \frac{q^*_r+1}{p_r},$ completing the proof of the claim.

Continuing with the proof of the theorem, since the right-hand condition of (265) and (266) is equivalent to $Y^\Gamma(y^\Gamma)$ being an $L$-space, and since (263) is the negation of (259), we have shown that (259) holds if and only if $y^\Gamma \in \mathcal{N}(Y^\Gamma)$, proving that $\mathcal{L}(Y^\Gamma) \supset \mathcal{N}(Y^\Gamma)$ (respectively, $\mathcal{F}(Y^\Gamma) \supset \mathcal{N}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma)$) if $\mathcal{L}(Y) = \mathcal{N}(Y)$ (respectively, $\mathcal{F}(Y) = \mathcal{N}(Y)$).
\(\mathcal{N}(Y)\), with \(N > \frac{q_r}{p_r}\), \(p_r > 1\), and \(-1 \notin j(E_{in}(r))\). Since \(S^3\) has no co-oriented taut foliations or left-orders on its fundamental group, we then have \(\mathcal{L}(Y^\Gamma) = \mathcal{N}(Y^\Gamma)\) (respectively, \(\mathcal{F}(Y^\Gamma) = \mathcal{N}(Y^\Gamma) \setminus \mathcal{R}(Y^\Gamma)\)).

This leaves the case in which we have a single \(T(np, nq)\) torus-link satellite, with \(N := 2g(K) - 1 < \frac{q}{p}\). Arguments similar to those above then show that if \(\mathcal{L}(Y) = \mathcal{N}(Y)\) (respectively, \(\mathcal{F}(Y) = \mathcal{N}(Y)\)), then \(y^\Gamma \in \mathcal{N}(Y^\Gamma)\) implies that \(y^\Gamma \in \mathcal{L}(Y^\Gamma)\) (respectively, \(y^\Gamma \in \mathcal{F}(Y^\Gamma) \cup \mathcal{Z}(Y^\Gamma)\)), provided that

\[
y_{0+} \neq \frac{q^r}{p} + \frac{1}{p(q-pN)} \left(1 - \frac{p^r - q^r N}{q-pN}\right).
\]

(269)

Propositions 6.1 and 6.2(+) then tell us that \(y_{0+} = \frac{p^r - q^r N}{q-pN}\) implies

\[
\sum_{i=1}^{n} \left[ y_{i} \right] = 0, \sum_{i=1}^{n} \left[ -y_{i} (q - Np) \right] = 0, \sum_{i=1}^{n} \left[ -y_{i} (q - (N-1)p) \right] > 0,
\]

(270)

which, under change of basis to \(S^3\)-slopes, becomes

\[
\Lambda(Y^\Gamma) \cdot (\mathcal{L}(\Gamma))^{-1} \subset \mathcal{N}(\Gamma) \setminus \mathcal{R}(\Gamma).
\]

(271)

Since \(pq - q + pN = pq - q - p + 2g(K)p\), the theorem follows. \(\square\)

8.2. Exceptional symmetries

As mentioned in the introduction, there are instances, for exteriors of iterated torus-link satellites or algebraic link satellites, in which the \(\Lambda\)-type symmetries for Seifert fibered components have their influence extend across edges. This phenomenon is more relevant in the context of exceptional splices.

**Proposition 8.2.** Suppose \(Y^\Gamma\) is the exterior of an algebraic link satellite \(K^\Gamma \subset S^3\) of a nontrivial positive \(L\)-space knot \(K \subset S^3\) of genus \(g(K)\), with \(N := 2g(K) - 1 > \frac{q}{p}\) and \(-1 \in j(E_{in}(r))\). Then \(\mathcal{L}(Y^\Gamma) = \bigcup_{e \in E_{in}(r)} \mathcal{L}_e\), where

\[
\mathcal{L}_e := \left\{ y^\Gamma \in \mathcal{L}(Y^\Gamma) \middle| \begin{array}{l}
y_{Y_{(e)}(y^\Gamma_{(e)})}(y^\Gamma_{(e)}) \text{ is BC, with } y_{e} \in E_{in}(r), \\
y_{e} \in \mathbb{Z}\text{, for all } e' \in E_{in}(r), \\
0 < y_{0+} \leq y_{0-} < \frac{\frac{q^r}{p}}{2} \text{ if } Y_{(e)}(y^\Gamma_{(e)}) \text{ is BI}, \\
0 < y_{0+} \leq y_{0-} < \frac{\frac{q^r}{p}}{2} \text{ if } Y_{(e)}(y^\Gamma_{(e)}) \text{ is BC}.
\end{array} \right\}
\]

Proof. This is established by a straightforward but tedious adaptation of the arguments used to prove the Claim in the proof of Theorem 8.1. \(\square\)

Note that the latter two conditions place strong constraints on \(y^\Gamma\) as well. In particular, we must have \(y^\Gamma \in \mathbb{Z}\) for all but at most one \(i \in I_e\).

This phenomenon also affects \(\Lambda\) as defined in (11). While \(\Lambda \supset \prod_{e \in \text{Vert}(\Gamma)} \Lambda_e\), this containment is proper if there is an edge \(e \in \text{Edge}(\Gamma)\) for which one can have \(Y_{(e)}(y^\Gamma_{(e)}) \text{ BC with } 0 \neq y_{(e)} \in \mathbb{Z}\) if \(j(e) \neq -1\), or any integer value \(y_{(e)} \in \mathbb{Z}\) if \(j(e) = -1\). For a satellite without exceptional splices, the situation is still relatively simple. It is straightforward to show that the above edge condition can hold only if \((p_{(e)}, q_{(e)}) = (1, 2)\). For such an edge, one must locally replace the product \(\Lambda_{(e)} \times \Lambda_{(e)}\) with the product \((\Lambda_{(e)} \times \Lambda_{(e)}) \cup (\Lambda_{(e)} \times \Lambda_{(e)})\), where \(\Lambda_{(e)} := \{ y^\Gamma \in \mathbb{Z} | \sum y^\Gamma_i = 1\}\). If \(\Gamma\) has no exceptional splice edges and no vertices with \((p_{(e)}, q_{(e)}) = (1, 2)\), then \(\Lambda_{(e)} \supset \prod_{e \in \text{Vert}(\Gamma)} \Lambda_e\).
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