Some Investigations on a Class of Analytic and Univalent Functions Involving $q$-Differentiation

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ABSTRACT. We use the concept of $q$-differentiation to define a class $E_q(\beta, \delta)$ of analytic and univalent functions. The investigations thereafter includes coefficient estimates, inclusion property and some conditions for membership of some analytic functions to be in the class $E_q(\beta, \delta)$. Our results generalize some known and new ones.

1. Introduction and Definitions

We let $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ represent the unit disk and $\mathcal{A}$ represent the class of normalized analytic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathcal{U}$$

where $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ represent a subset of $\mathcal{A}$ containing functions univalent in $\mathcal{U}$. A function $f$ in $\mathcal{S}$ is a member of class $BT(\delta)$ of bounded turning functions of order $\delta$ if it satisfies the geometric condition

$$\mathcal{R}e f'(z) > \delta \in [0, 1), \quad z \in \mathcal{U}.$$ 

Let $BT(0) = BT$ represent the class of bounded turning functions. It is known (see [1]) that $f \in BT$ are univalent functions. Also, a function $f$ in $\mathcal{S}$ is a member of class $CV(\delta)$ of convex functions of order $\delta$ if it satisfies the geometric condition

$$\mathcal{R}e \left( z \frac{f''(z)}{f'(z)} + 1 \right) > \delta \in [0, 1), \quad z \in \mathcal{U}.$$ 

Let $CV(0) = CV$ represent the class of convex functions.

The importance of operators in geometric function theory cannot be underrated. For instance see [2, 13, 15] for some known ones.

In 1908, Jackson [7] (see also [3, 4, 8–11]) initiated the concept of $q$-calculus as follows.
**Definition 1.1.** For $q \in (0, 1)$, the $q$-differentiation of function $f \in \mathcal{A}$ is defined by

$$D_qf(0) = f'(0), \quad D_qf(z) = \frac{f(z) - f(qz)}{z(1-q)} \quad (z \neq 0) \quad \text{and} \quad D_q^2f(z) = D_q(D_qf(z)). \tag{2}$$

Obviously, applying (2) in (1) gives us

$$D_qf(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1} \quad \text{and} \quad zD_q^2f(z) = \sum_{m=2}^{\infty} [m-1]_q [m]_q a_m z^{m-1} \tag{3}$$

where $[m]_q = \frac{1-q^m}{1-q}$ and $\lim_{q\uparrow 1}[m]_q = m$.

For example if $f(z) = z^m$, then by using (2),

$$D_qf(z) = D_q(z^m) = \frac{1-q^m}{1-q} z^{m-1} = [m]_q z^{m-1}$$

and observe that

$$\lim_{q\uparrow 1} D_qf(z) = \lim_{q\uparrow 1} ([m]_q z^{m-1}) = mz^{m-1} = f'(z)$$

where $f'(z)$ is the classical differentiation.

In this work, the $q$-differential operator was used to define a class of analytic functions and generalize some results.

### 2. Relevant Lemmas

We represent by $\mathcal{P}$ the well-known class of analytic functions of the form

$$p(z) = 1 + \sum_{m=1}^{\infty} c_m z^m, \quad \Re p(z) > 0, \ z \in \mathcal{U}D \tag{4}$$

and by $\mathcal{P}(\delta) \subseteq \mathcal{P}(0) = \mathcal{P}$ the class whose members are of the form

$$p_\delta(z) = 1 + \sum_{m=1}^{\infty} (1-\delta)c_m z^m, \quad \Re p(z) > \delta \in [0, 1), \ z \in \mathcal{U}D. \tag{5}$$

The following lemmas shall be required to proof our results.

**Lemma 2.1** ([14]). Let $g(z) = \sum_{m=1}^{\infty} a_m z^m < G(z) = \sum_{m=1}^{\infty} b_m z^m, \ z \in \mathcal{U}D$ where $G(z)$ is univalent in $\mathcal{U}D$ and $G(\mathcal{U}D)$ is a convex domain, then $|a_m| \leq |b_1|, \ m \in \mathbb{N}$. Equality holds for the function $g(z) = G(\tau z^m), \ |	au| = 1$.

The lemmas that follow are the $q$-analogous versions of the original ones as referenced.

**Lemma 2.2** ([6]). Let $p(z)$ be analytic in $\mathcal{U}D$ such that $p(0) = 1$. If

$$\Re \left( \frac{zD_q(p(z))}{p(z)} + 1 \right) > \frac{3\delta - 1}{2\delta}, \ z \in \mathcal{U}D,$$

then for $\alpha = (\delta - 1)/\delta$ $(\delta \in [1/2, 1)), \ \Re p(z) > 2^\alpha$. The constant $2^\alpha$ is the best possible.

**Lemma 2.3** ([5]). Let $u = u_1 + u_2i$ and $v = v_1 + v_2i$ such that $\gamma(u, v) : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a complex-valued function such that

$$...$$
(1) \( \gamma(u, v) \) is continuous in \( \Pi \subset \mathbb{C}^2 \),
(2) \((1, 0) \in \Pi \) and \( \Re(\gamma(1, 0)) > 0 \) and
(3) \( \Re(\gamma(\xi + (1 - \xi)u_2i, v_1)) \leq \xi \) (0 \( \leq \xi < 1 \)) if \((\xi + (1 - \xi)u_2i, v_1) \in \Pi \) and
\( v_1 \leq -\frac{1}{2}(1 - \xi)(1 + u_2^2) \) and \( \Re(\gamma(\xi + (1 - \xi)u_2i, v_1)) \geq \xi \) (\( \xi > 1 \)) if \((\xi + (1 - \xi)u_2i, v_1) \in \Pi \) and
\( v_1 \geq \frac{1}{2}(1 - \xi)(1 + u_2^2) \).

If \( p(z) \in \mathcal{P} \) for \((p(z), zD_q p(z)) \in \Pi \) and \( \Re(\gamma(p(z), zD_q p(z))) > \xi \), \( z \in \mathcal{U} \mathcal{D} \), then \( \Re p(z) > \xi \) in \( \mathcal{U} \mathcal{D} \).

3. Main Results

The definition of the investigated class is as follows.

A function \( f(z) \in \mathcal{A} \) is a member of the class \( \mathcal{E}_q(\beta, \delta) \) if the condition
\[
\Re \left( D_q f(z) + \frac{1 + e^{i\beta}}{2} zD_q^2 f(z) \right) > \delta, \quad \delta \in [0, 1), \ \beta \in (-\pi, \pi], \ z \in \mathcal{U} \mathcal{D} \tag{6}
\]
holds.

When parameters in (6) are varied, the class \( \mathcal{E}_q(\beta, \delta) \) reduces to some well-known classes of analytic functions that have been studied by some authors. These are cited in our corollaries and remarks.

The following are the proved results.

**Theorem 3.1.** Let \( \beta \in (-\pi, \pi] \) and \( \delta \in [0, 1) \), if condition (6) holds, then
\[
\mathcal{E}_q(\beta, \delta) \subset \mathcal{B}T_q(\delta).
\]

\( \mathcal{B}T_q(\delta) \) is the class of \( q \)-bounded turning function of order \( \delta \).

**Proof.** Let \( p(z) = D_q f(z) \) so that \( D_q p(z) = D_q^2 f(z) \) and for \( \kappa = (1 + e^{i\beta})/2 \), then (6) can be expressed as
\[
\Re(p(z) + \kappa zD_q p(z)) > \delta. \tag{7}
\]
In view of the conditions in Lemma 2.3 and for \( p(z) \) in (7), we define the function
\[
\gamma(u, \nu) = u + \kappa \nu
\]
on the domain \( \Pi \) of \( \mathbb{C}^2 \), then
(i) clearly, \( \gamma(u, \nu) \) satisfies the condition (1) in Lemma 2.3,
(ii) for \((1, 0) \in \Pi \), \( \gamma(1, 0) = 1 \implies \Re(\gamma(1, 0)) > 0 \) and
(iii) \( \gamma(\delta + (1 - \delta)u_2i, \nu_1) = \delta + \frac{1 + \cos \delta}{2} \nu_1 + \left( (1 - \delta)u_2 + \frac{\sin \delta}{2} \nu_1 \right) i \), thus,
\[
\Re(\gamma(\delta + (1 - \delta)u_2i, \nu_1)) = \delta + \frac{1 + \cos \beta}{2} \nu_1 \leq \delta
\]
for \( \nu_1 \leq -\frac{1}{2}(1 - \delta)(1 + u_2^2) \).
Now since $\gamma(u, \nu)$ satisfies all the conditions $(1-3)$ in Lemma 2.3, then it implies that

$$\Re e(p(z)) = \Re e(\Delta_q f(z)) > \delta, \quad z \in \mathcal{U}$$

hence the proof is complete. \(\square\)

**Corollary 3.2** ([1]). Since class $\mathcal{B}_q(\delta)$ is well-known to consist of univalent functions, then $\mathcal{E}_q(\beta, \delta) \subset \mathcal{B}_q(\delta)$ consists of univalent functions.

**Corollary 3.3.** $\lim_{q \uparrow 1} \mathcal{E}_q(\beta, \delta) \subset \mathcal{B}_q(\delta), \ z \in \mathcal{U}$.\(\mathcal{D}_q\)

**Theorem 3.4.** If $f \in \mathcal{A}$ is such that

$$\Re e(\Delta_q f(z) + \kappa \Delta_q^2 f(z)) > \delta - \frac{1}{2\delta}, \quad \delta \in [1/2, 1), \ z \in \mathcal{U}$$

and $\kappa = (1 + e^{i\beta})/2$.

**Proof.** From (6), let $p(z) = \Delta_q f(z) + \kappa \Delta_q^2 f(z)$, then by logarithmic $q$-differentiation we obtain

$$z \Delta_q p(z) + 1 = z \Delta_q (\Delta_q f(z) + \kappa \Delta_q^2 f(z)) \Delta_q f(z) + \kappa \Delta_q f(z) + 1.$$

Now applying Lemma 2.2 gives

$$\Re e \left( \frac{z \Delta_q p(z)}{p(z)} + 1 \right) = \Re e \left( \frac{z \Delta_q (\Delta_q f(z) + \kappa \Delta_q^2 f(z)) \Delta_q f(z) + \kappa \Delta_q f(z) + 1}{\Delta_q f(z) + \kappa \Delta_q^2 f(z)} + 1 \right) > \frac{3\delta - 1}{2\delta}$$

implies that

$$\Re e \left( \frac{z \Delta_q (\Delta_q f(z) + \kappa \Delta_q^2 f(z)) \Delta_q f(z) + \kappa \Delta_q f(z) + 1}{\Delta_q f(z) + \kappa \Delta_q^2 f(z)} \right) > \frac{\delta - 1}{2\delta}$$

and by the same Lemma 2.2 the proof in complete. \(\square\)

**Corollary 3.5.** If $f \in \mathcal{A}$ satisfies condition (8), then $f \in \mathcal{E}_q(\beta, 2^{(\delta-1)/\delta})$.

**Corollary 3.6.** If $f \in \lim_{q \uparrow 1} \mathcal{E}_q(\beta, 1/2)$ is such that

$$\Re e \left( \frac{z(1 + \kappa) f''(z) + \kappa z^2 f'''(z)}{f'(z) + \kappa z f''(z)} \right) > -\frac{1}{2},$$

then

$$\Re e(f'(z) + \kappa z f''(z)) > 1/2, \quad z \in \mathcal{U}.$$
Corollary 3.7. If $f \in E_q(\pi, 1/2)$ is such that
\[ \Re \left( \frac{zD_q(D_q f(z))}{D_q f(z)} \right) > -\frac{1}{2}, \] (9)
then
\[ \Re(D_q f(z)) > \frac{1}{2}. \]
This means that if condition (9) holds, then $f$ is a $q$-bounded turning function of order $1/2$. Now if $q \uparrow 1$, then
\[ \Re \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \] (10)
implies
\[ \Re(f'(z)) > \frac{1}{2}, \quad z \in \mathcal{U} \mathcal{D}. \]
This means that if condition (10) holds, then $f$ is a bounded turning function of order $1/2$.

Corollary 3.8. If $f \in E_q(0, 1/2)$ is such that
\[ \Re \left( \frac{zD_q(D_q f(z) + zD_q^2 f(z))}{D_q f(z) + zD_q^2 f(z)} \right) > -\frac{1}{2}, \] (11)
then
\[ \Re(D_q f(z) + zD_q^2 f(z)) > \frac{1}{2} \]
and if $q \uparrow 1$,
\[ \Re \left( \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} \right) > -\frac{1}{2} \]
implies that
\[ \Re(f'(z) + zf''(z)) > 1/2, \quad z \in \mathcal{U} \mathcal{D}. \]

Theorem 3.9. Let $\beta \in (-\pi, \pi]$ and $\delta \in [0, 1)$, then the function
\[ f(z) = z + a_m z^m \in E_q(\beta, \delta), \quad m = \{2, 3, \ldots\} \] (12)
if
\[ |a_m| \leq \frac{2}{[m]_q \{ |X_m| - ((2 + [m - 1]_q) \cos \theta + [m - 1]_q \cos(\beta + \theta_0)) \}} \] (13)
where
\[ X_m = 2 + [m - 1]_q (1 + e^{i\beta}) \]
\[ |X_m| = \sqrt{2 \left\{ 2 + [m - 1]_q (2 + [m - 1]_q(1 + \cos \beta)) \right\}} \geq 2 \] (14)
and $\theta_0$ attains minimum at
\[ \theta_0 = \pi + \arctan \left( -\frac{[m - 1]_q \sin \beta}{2 + [m - 1]_q (1 + \cos \beta)} \right). \] (15)
Proof. Firstly, applying (2) in (12) gives
\[
D_q f(z) = 1 + [m]_q a_m z^{m-1} \\
zD_q^2 f(z) = [m-1]_q [m]_q a_m z^{m-1}
\] (16)
Note that it suffices to study the condition that for \(|z| = 1\),
\[
\left| D_q f(z) + \frac{1 + e^{i\beta}}{2} zD_q^2 f(z) - 1 \right| < \Re \left\{ D_q f(z) + \frac{1 + e^{i\beta}}{2} zD_q^2 f(z) \right\}
\] (17)
so that by putting (16) into (17) we obtain
\[
\left| [m]_q a_m z^{m-1} + \frac{1}{2} [m-1]_q [m]_q (1 + e^{i\beta}) a_m z^{m-1} \right| < \Re \left\{ 1 + [m]_q a_m z^{m-1} + \frac{1}{2} [m-1]_q [m]_q (1 + e^{i\beta}) a_m z^{m-1} \right\}.
\]
Now letting \(|a_m| = r, a_m z^{m-1} = r e^{i\theta}\) and using (14) we obtain
\[
\left| \frac{1}{2} [m]_q r e^{i\theta} X_m \right| < \Re \left\{ 1 + [m]_q r e^{i\theta} + \frac{1}{2} [m-1]_q [m]_q (1 + e^{i\beta}) r e^{i\theta} \right\}
\] (18)
so that
\[
\frac{1}{2} [m]_q r |X_m| \leq \Re F
\] (19)
where
\[
F = 1 + [m]_q r \cos \theta + \frac{1}{2} [m-1]_q [m]_q r \cos \theta + \frac{1}{2} [m-1]_q [m]_q r \cos(\beta + \theta) + \Re F
\]
in (18). Further simplification gives
\[
F = 1 + [m]_q r \cos \theta + \frac{1}{2} [m-1]_q [m]_q r \cos \theta + \frac{1}{2} [m-1]_q [m]_q r \cos(\beta + \theta) + \Im F
\]
so that
\[
\Re F = 1 + \frac{1}{2} [m]_q r \{ 2 \cos \theta + [m-1]_q \cos \theta + [m-1]_q \cos(\beta + \theta) \} = \psi.
\] (20)
Now (19) becomes
\[
\frac{1}{2} [m]_q r |X_m| \leq 1 + \frac{1}{2} [m]_q r \{ (2 + [m-1]_q) \cos \theta + [m-1]_q \cos(\beta + \theta) \}
\]
and by simplification we obtain (13).

To know the values of \(\theta\) where (20) attains minimum implies that
\[
\frac{\partial \psi}{\partial \theta} = -\frac{r [m]_q}{2} \left\{ (2 + [m-1]_q) \sin \theta + [m-1]_q \sin(\beta + \theta) \right\}
\]
implies that
\[
(2 + [m-1]_q) \sin \theta + [m-1]_q \sin(\beta + \theta) = 0
\]
so that
\[
\tan \theta = \frac{-[m-1]_q \sin \beta}{2 + [m-1]_q (1 + \cos \beta)}
\]
which simplifies to (15). □
Corollary 3.10. Let \( f(z) = z + a_m z^m \in \mathcal{E}_q(0, \delta) \) and \( m = \{2, 3, \ldots \} \), then

\[
|a_m| \leq \frac{1}{[m]_q \left\{ \sqrt{1 + 2[m - 1]_q + [m - 1]^2_q + 1 + [m - 1]_q} \right\}}
\]

and if \( q \uparrow 1 \), then

\[
|a_m| \leq \frac{1}{2m^2}.
\]

Corollary 3.11. Let \( f(z) = z + a_m z^m \in \mathcal{E}_q(\pi, \delta) \) and \( m = \{2, 3, \ldots \} \), then

\[
|a_m| \leq \frac{1}{2[m]_q}
\]

and if \( q \uparrow 1 \), then

\[
|a_m| \leq \frac{1}{2m}.
\]

Remark 3.12. Let \( q \uparrow 1 \), then Theorem 3.9 becomes the result in [18].

Theorem 3.13 (Coefficient Estimates). Let \( \beta \in (-\pi, \pi], \delta \in [0, 1) \) and let \( G(z) = 1 + b_1 z + b_2 z^2 + \cdots \in CV(\delta) \). If \( f \in \mathcal{A} \) belongs to \( \mathcal{E}_q(\beta, \delta) \), then

\[
|a_m| \leq \frac{2(1 - \delta) |b_1|}{[m]_q |X_m|}, \quad m = \{2, 3, \ldots \}
\]

(21)

where \( |X_m| \) is defined in (14).

Proof. Let \( f(z) \in \mathcal{E}_q(\beta, \delta) \), therefore from (6) and using (5),

\[
D_q f(z) + \frac{1 + e^{i\beta}}{2} z D_q^2 f(z) = \delta + (1 - \delta)p(z), \quad z \in UD.
\]

(22)

Now putting (3) and (4) into (22) and simplifying gives

\[
1 + \sum_{m=2}^{\infty} \left\{ 1 + [m - 1]_q \left( \frac{1 + e^{i\beta}}{2} \right) \right\} [m]_q a_m z^{m-1} = 1 + \sum_{m=2}^{\infty} (1 - \delta) c_{m-1} z^{m-1}
\]

which implies that

\[
\{2 + [m - 1]_q(1 + e^{i\beta})\} [m]_q a_m = (1 - \delta) c_{m-1}, \quad m = \{2, 3, \ldots \}
\]

where by applying (14) we obtain

\[
X_m \frac{[m]_q}{2(1 - \delta)} a_m = c_{m-1}, \quad m = \{2, 3, \ldots \}.
\]

(23)

Since \( G(UD) \) is a convex domain, then from Lemma 2.1, (23) becomes

\[
\left| X_m \frac{[m]_q}{2(1 - \delta)} a_m \right| = |c_{m-1}| \leq |b_1|
\]

and simplifying further we obtain (21).
Corollary 3.14. Let $f(z) \in \mathcal{E}_q(0, \delta)$, then
\[ |a_m| \leq \frac{(1 - \delta)|b_1|}{\sqrt{1 + 2[m - 1]_q + [m - 1]^2_q}} \]
and if $q \uparrow 1$, then
\[ |a_m| \leq \frac{(1 - \delta)|b_1|}{m}, \quad m = \{2, 3, \ldots\}. \]

Corollary 3.15. Let $f \in \mathcal{E}_q(\pi, \delta)$, then
\[ |a_m| \leq \frac{(1 - \delta)|b_1|}{[m]_q} \]
and if $q \uparrow 1$, then
\[ |a_m| \leq \frac{(1 - \delta)|b_1|}{m}, \quad m = \{2, 3, \ldots\}. \]

Remark 3.16. Let $p(z) \in \mathcal{P}$ and $\phi(z) = 1 + \frac{2}{\pi^2} \left( \ln \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2$. If $q \uparrow 1$,
1. $\beta = \pi$ and $G(z) = p(z)$, then Theorem 3.13 becomes the result in [12].
2. and $G(z) = p(z)$, then Theorem 3.13 becomes the result in [16].
3. and $G(z) = \phi(z)$, then Theorem 3.13 becomes the result in [18].
4. and $\beta = 0$, then Theorem 3.13 becomes the result in [17].

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