THE WEYL TENSOR AND EQUILIBRIUM CONFIGURATIONS OF SELF–GRAVITATING FLUIDS

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Abstract

It is shown that (except for two well defined cases), the necessary and sufficient condition for any spherically symmetric distribution of fluid to leave the state of equilibrium (or quasi–equilibrium), is that the Weyl tensor changes with respect to its value in the state of equilibrium (or quasi–equilibrium).

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1 Introduction

Since the publication of Penrose’ work \cite{1}, there has been an increasing interest in the possible role of Weyl tensor (or some function of it) in the evolution of self-gravitating systems \cite{2} and references therein. This interest is reinforced by the fact that (at least) for spherically symmetric distributions of fluid, the Weyl tensor may be expressed exclusively in terms of the density contrast and the local anisotropy of the pressure \cite{3}, which in turn are known to affect the fate of gravitational collapse \cite{4}.

In this work we shall consider spherically symmetric fluid configurations which are initially in equilibrium (or quasi–equilibrium), and we shall look for necessary and sufficient conditions to leave such regime. Usually, these conditions are expressed through the adiabatic index, (or some function of it) which indicates the variation of pressure with density, for a given fluid element (see \cite{5} and references therein).

As we shall see here, such departure from equilibrium (or quasi–equilibrium) is allowed if and only if, the Weyl tensor within the fluid distribution, changes with respect to its value in the initial (equilibrium or quasi–equilibrium) state.

There is however two possible exceptions for this. One is represented by the “inflationary” equation of state $\rho + p = 0$. The other situation when our result does not apply, corresponds to the case when then system leaving the equilibrium enters into dissipative regime and $\kappa T/\tau(\rho + p) = 1$, where $\kappa$, $T$ and $\tau$ denote respectively the thermal conduction, the temperature and the thermal relaxation time. Both situations imply from the “dynamical” point of view, that the effective inertial mass of any fluid element vanish (see \cite{6} and references therein).

The above mentioned result is reminiscent, in some sense, of a one recently obtained, regarding the conditions for a transition from a non-dissipative to a dissipative regime in a FRW-flat model\cite{7}. Indeed in that case, the Weyl tensor is always zero, and such transition is only allowed if $\kappa T/\tau(\rho + p) = 1$.

The manuscript is organized as follows. In the next section we give the expressions for the field and transport equations, and for the Weyl tensor. In Section III the conditions for the departure from equilibrium (and quasiequilibrium) are found. Finally in the last Section the results are discussed.
2 Field and Transport equations

2.1 Field equations

We consider spherically symmetric distributions of collapsing fluid, which for sake of completeness we assume to be anisotropic, undergoing dissipation in the form of heat flow, bounded by a spherical surface $\Sigma$.

The line element is given in Schwarzschild-like coordinates by

$$ds^2 = e^{\nu} dt^2 - e^\lambda dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

where $\nu(t, r)$ and $\lambda(t, r)$ are functions of their arguments. We number the coordinates: $x^0 = t; \ x^1 = r; \ x^2 = \theta; \ x^3 = \phi$.

The metric (1) has to satisfy Einstein field equations

$$G_{\mu}^{\nu} = -8\pi T_{\mu}^{\nu}$$

which in our case read [8]:

$$-8\pi T^0_0 = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right)$$

$$-8\pi T^1_1 = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right)$$

$$-8\pi T^2_2 = -8\pi T^3_3 = -\frac{e^{-\nu}}{4} \left( 2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu}) \right) + \frac{e^{-\lambda}}{4} \left( 2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r} \right)$$

$$-8\pi T^0_1 = -\frac{\dot{\lambda}}{r}$$

where dots and primes stand for partial differentiation with respect to t and r respectively.

In order to give physical significance to the $T_{\mu}^{\nu}$ components we apply the Bondi approach [3].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates $(\tau, x, y, z)$.
\[ d\tau = e^{\nu/2} dt \quad dx = e^{\lambda/2} dr \quad dy = rd\theta \quad dz = r\sin\theta d\phi \]

Then, denoting the Minkowski components of the energy tensor by a bar, we have

\[ \bar{T}_0^0 = T_0^0 \quad \bar{T}_1^1 = T_1^1 \quad \bar{T}_2^2 = T_2^2 \quad \bar{T}_3^3 = T_3^3 \quad \bar{T}_{01} = e^{-(\nu+\lambda)/2} T_{01} \]

Next, we suppose that when viewed by an observer moving relative to these coordinates with velocity \( \omega \) in the radial direction, the physical content of space consists of an anisotropic fluid of energy density \( \rho \), radial pressure \( P_r \), tangential pressure \( P_\perp \) and radial heat flux \( \hat{q} \). Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

\[
\begin{pmatrix}
\rho & -\hat{q} & 0 & 0 \\
-\hat{q} & P_r & 0 & 0 \\
0 & 0 & P_\perp & 0 \\
0 & 0 & 0 & P_\perp \\
\end{pmatrix}
\]

Then a Lorentz transformation readily shows that

\[ T_0^0 = \bar{T}_0^0 = \rho + P_r \omega^2 \left( \frac{1}{1 - \omega^2} \right)^{1/2} + \frac{2Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (7) \]

\[ T_1^1 = \bar{T}_1^1 = -P_r + \rho \omega^2 \left( \frac{1}{1 - \omega^2} \right)^{1/2} - \frac{2Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (8) \]

\[ T_2^2 = T_3^3 = \bar{T}_2^2 = \bar{T}_3^3 = -P_\perp \quad (9) \]

\[ T_{01} = e^{(\nu+\lambda)/2} \bar{T}_{01} = -\frac{(\rho + P_r) \omega e^{(\nu+\lambda)/2}}{1 - \omega^2} - \frac{Q e^{\nu/2} e^\lambda}{(1 - \omega^2)^{1/2}(1 + \omega^2)} \quad (10) \]

with

\[ Q \equiv \frac{\hat{q} e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (11) \]
Note that the velocity in the \((t, r, \theta, \phi)\) system, \(dr/dt\), is related to \(\omega\) by

\[
\omega = \frac{dr}{dt} e^{(\lambda - \nu)/2}
\]  

At the outside of the fluid distribution, the spacetime is that of Vaidya, given by

\[
ds^2 = \left(1 - \frac{2M(u)}{R}\right) du^2 + 2dudR - R^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)
\]  

where \(u\) is a time-like coordinate such that \(u = \text{constant}\) is (asymptotically) a null cone open to the future and \(R\) is a null coordinate \((g_{RR} = 0)\). It should be remarked, however, that strictly speaking, the radiation can be considered in radial free streaming only at radial infinity.

The two coordinate systems \((t, r, \theta, \phi)\) and \((u, R, \theta, \phi)\) are related at the boundary surface and outside it by

\[
u = t - r - 2M \ln \left(\frac{r}{2M} - 1\right)
\]  

\[R = r
\]  

In order to match smoothly the two metrics above on the boundary surface \(r = r_{\Sigma}(t)\), we have to require the continuity of the first fundamental form across that surface. As result of this matching we obtain

\[
[P_r]_{\Sigma} = \left[Q e^{\lambda/2} \left(1 - \omega^2\right)^{1/2}\right]_{\Sigma} = [\hat{q}]_{\Sigma}
\]  

expressing the discontinuity of the radial pressure in the presence of heat flow, which is a well known result \([9]\).

Next, it will be useful to calculate the radial components of the conservation law

\[
T^{\mu}_{\nu;\mu} = 0
\]  

After tedious but simple calculations we get

\[
\left(-8\pi T^1_1\right)' = \frac{16\pi}{r} \left(T^1_1 - T^2_2\right) + 4\pi \nu' \left(T^1_1 - T^0_0\right) + \frac{e^{-\nu}}{r} \left(\dot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda} \dot{\nu}}{2}\right)
\]  

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which in the static case becomes

\[ P_r' = -\frac{\nu'}{2} (\rho + P_r) + \frac{2(P_\perp - P_r)}{r} \]  

(19)

representing the generalization of the Tolman-Oppenheimer-Volkof equation for anisotropic fluids [11].

### 2.2 Transport equations

As it is well known, the Maxwell-Fourier law for the radiation flux, usually assumed in the study of stars interiors, leads to a parabolic equation (diffusion equation) which predicts propagation of perturbation with infinite speed (see [11]–[13] and references therein). This simple fact is at the origin of the pathologies [14] found in the approaches of Eckart [15] and Landau [16] for relativistic dissipative processes.

To overcome such difficulties, different relativistic theories with non-vanishing relaxation times have been proposed in the past [17]–[19]. The important point is that all these theories provide a heat transport equation which is not of Maxwell-Fourier type but of Cattaneo type [20], leading thereby to a hyperbolic equation for the propagation of thermal perturbation.

Accordingly we shall describe the heat transport by means of a relativistic Israel-Stewart equation [17], which reads

\[
\frac{\tau}{\partial_\beta} \frac{Dq^\alpha}{\partial_\beta} + q^\alpha = \kappa P^\alpha{}_{\beta} (T_\beta - T a_\beta) - \tau u^\alpha q_\beta a^\beta - \frac{1}{2} \kappa T^2 \left( \frac{\tau}{\kappa T^2} u^\beta \right) \frac{\partial}{\partial_\beta} q^\alpha
\]

(20)

where \( \kappa, \tau, T, q^\beta \) and \( a^\beta \) denote thermal conductivity, thermal relaxation time, temperature, the heat flow vector and the components of the four acceleration, respectively. Also, \( P^\alpha{}_{\beta} \) is the projector onto the hypersurface orthogonal to the four velocity \( u^\alpha \).

In our case this equation has only two independent components, which read, for \( \alpha = 0 \)

\[
\tau e^{(\lambda - \nu)/2} \left( Q\dot{\omega} + \dot{Q}\omega + Q\omega \dot{\lambda} \right) + \tau \left( Q'\omega^2 + Q\omega' + \frac{Q\omega^2\lambda'}{2} \right)
\]

\[
+ \frac{\tau Q\omega^2}{r} + Q\omega e^{\lambda/2} \left( 1 - \omega^2 \right)^{1/2} = -\frac{\kappa\omega^2 T e^{-\nu/2}}{(1 - \omega^2)^{1/2}} - \frac{\kappa\omega T' e^{-\lambda/2}}{(1 - \omega^2)^{1/2}}
\]
\[ -\frac{\nu'}{2} \kappa T \omega e^{-\lambda/2} - \frac{1}{2} Q \omega \left( e^{(\lambda-\nu)/2} \dot{\tau} + \omega \tau' \right) \]

\[ -\frac{1}{2} \tau Q \omega \left[ e^{(\lambda-\nu)/2} \left( \frac{\omega \dot{\omega}}{1-\omega^2} + \frac{\dot{\lambda}}{2} \right) + \left( \frac{\omega'}{1-\omega^2} + \frac{\nu' \omega}{2} \right) \right] \]

\[ + \frac{1}{2} \tau Q \omega \left[ \frac{1}{\kappa} \left( e^{(\lambda-\nu)/2} \kappa + \omega \kappa' \right) + \frac{2}{T} \left( e^{(\lambda-\nu)/2} \dot{T} + \omega T' \right) \right] \]

\[ + \left( \tau Q e^{(\lambda-\nu)/2} - \frac{\kappa T \omega e^{-\nu/2}}{(1-\omega^2)^{1/2}} \right) \times \left( \frac{\omega \dot{\lambda}}{2} + \frac{\omega}{1-\omega^2} \right) \]

\[ + \left( \tau Q - \frac{\kappa T \omega e^{-\lambda/2}}{(1-\omega^2)^{1/2}} \right) \times \frac{\omega \omega'}{1-\omega^2} \] (21)

and for \( \alpha = 1 \)

\[ \tau e^{(\lambda-\nu)/2} \left( \dot{Q} + \frac{Q \dot{\lambda}}{2} + \frac{Q \omega^2 \dot{\lambda}}{2} \right) + \tau \omega \left( \dot{Q}' + \frac{Q \lambda'}{2} \right) \]

\[ + \tau Q \omega \frac{r}{r} + Q e^{\lambda/2} \left( 1 - \omega^2 \right)^{1/2} = -\frac{\kappa \omega \dot{T} e^{-\nu/2}}{(1-\omega^2)^{1/2}} - \frac{\kappa T' e^{-\lambda/2}}{(1-\omega^2)^{1/2}} \]

\[ -\frac{\nu'}{2} \frac{\kappa T e^{-\lambda/2}}{(1-\omega^2)^{1/2}} - \frac{1}{2} Q \left( e^{(\lambda-\nu)/2} \dot{\tau} + \omega \tau' \right) \]

\[ -\frac{1}{2} \tau Q \left[ e^{(\lambda-\nu)/2} \left( \frac{\omega \dot{\omega}}{1-\omega^2} + \frac{\dot{\lambda}}{2} \right) + \left( \frac{\omega'}{1-\omega^2} + \frac{\nu' \omega}{2} \right) \right] \]

\[ + \frac{1}{2} \tau Q \left[ \frac{1}{\kappa} \left( e^{(\lambda-\nu)/2} \kappa + \omega \kappa' \right) + \frac{2}{T} \left( e^{(\lambda-\nu)/2} \dot{T} + \omega T' \right) \right] \]

\[ + \left( \tau Q e^{(\lambda-\nu)/2} - \frac{\kappa T \omega e^{-\nu/2}}{(1-\omega^2)^{1/2}} \right) \times \left( \frac{\omega \dot{\lambda}}{2} + \frac{\omega}{1-\omega^2} \right) \]

\[ + \left( \tau Q - \frac{\kappa T \omega e^{-\lambda/2}}{(1-\omega^2)^{1/2}} \right) \times \frac{\omega \omega'}{1-\omega^2} \] (22)

where the expressions
\[ u^\mu = \left( \frac{e^{-\nu/2}}{(1 - \omega^2)^{1/2}}, \frac{\omega e^{-\lambda/2}}{(1 - \omega^2)^{1/2}}, 0, 0 \right) \quad (23) \]

\[ q^\mu = Q \left( \omega e^{(\lambda-\nu)/2}, 1, 0, 0 \right) \quad (24) \]

have been used.

2.3 The Weyl tensor

For the next section we shall need the components of the Weyl tensor. Using Maple V, it is found that all non-vanishing components are proportional to

\[ W \equiv r^2 C_{232}^3 = W_{(s)} + \frac{r^3 e^{-\nu}}{12} \left( \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda} \dot{\nu}}{2} \right) \quad (25) \]

where

\[ W_{(s)} = \frac{r^3 e^{-\lambda}}{6} \left( \frac{e^\lambda}{r^2} - \frac{1}{r^2} + \frac{\nu' \lambda'}{4} - \frac{\nu'^2}{4} - \frac{r''}{2r} + \frac{\nu'}{2r} \right) \quad (26) \]

corresponds to the contribution in the static (and quasi-static) case. Also, the following expression relating the Weyl tensor through the source terms, may be found [3]

\[ W = -\frac{4\pi}{3} \int_0^r r^3 (T_0^0)' \, dr + \frac{4\pi}{3} r^3 \left( T_2^2 - T_1^1 \right) \quad (27) \]

3 Leaving the equilibrium (quasiequilibrium)

Let us now consider a spherically symmetric fluid distribution which initially may be in either hydrostatic and thermal equilibrium (i.e. \( \omega = Q = 0 \)), or slowly evolving and dissipating energy through a radial heat flow vector. Before proceeding further with the treatment of our problem, let us clearly specify the meaning of “slowly evolving”. That means that our sphere changes on a time scale which is very large as compared to the typical time in which it reacts on a slight perturbation of hydrostatic equilibrium. This typical time is called hydrostatic time scale. Thus a slowly evolving system
is always in hydrostatic equilibrium (very close to), and its evolution may be regarded as a sequence of static models linked by (13).

As we mentioned before, this assumption is very sensible, since the hydrostatic time scale is usually very small. Thus, it is of the order of 27 minutes for the sun, 4.5 seconds for a white dwarf and $10^{-4}$ seconds for a neutron star of one solar mass and 10 Km radius [5].

In terms of $\omega$ and metric functions, slow evolution means that the radial velocity $\omega$ measured by the Minkowski observer, as well as metric time derivatives are so small that their products and second order time derivatives may be neglected (an invariant characterization of slow evolution may be found in [21]).

Thus

$$\ddot{\nu} \approx \ddot{\lambda} \approx \dot{\lambda} \dot{\nu} \approx \dot{\nu}^2 \approx \omega^2 \approx \dot{\omega} = 0 \quad (28)$$

As it follows from (6) and (10), $Q$ is of the order $O(\omega)$. Thus in the slowly evolving regime, relaxation terms may be neglected and (20) becomes the usual Landau-Eckart transport equation.

Then, using (28) and (18) we obtain (19), which as mentioned before is the equation of hydrostatic equilibrium for an anisotropic fluid. This is in agreement with what was mentioned above, in the sense that a slowly evolving system is in hydrostatic equilibrium.

Let us now return to our problem. Before perturbation, the two possible initial states of our system are characterized by:

1. Static

$$\dot{\omega} = \dot{Q} = \omega = Q = 0 \quad (29)$$

2. Slowly evolving

$$\dot{\omega} = \dot{Q} = 0 \quad (30)$$

$$Q \approx O(\omega) \neq 0 \quad (small) \quad (31)$$

where the meaning of “small” is given by (28).

Let us now assume that our system is submitted to perturbations which force it to depart from hydrostatic equilibrium but keeping the spherical symmetry. We shall study the perturbed system on a time scale which is small as compared to the thermal adjustment time.
Then, immediately after perturbation ("immediately" understood in the sense above), we have for the first initial condition (static)

$$\omega = Q = 0 \quad (32)$$

$$\dot{\omega} \approx \dot{Q} \neq 0 \quad (small) \quad (33)$$

whereas for the second initial condition (slowly evolving)

$$Q \approx O(\omega) \neq 0 \quad (small) \quad (34)$$

$$\dot{Q} \approx \dot{\omega} \neq 0 \quad (small) \quad (35)$$

As we shall see below, both initial conditions lead to the same final equations.

Let us now write explicitly eq. (18). With the help of (7)–(10), we find after long but trivial calculations

$$\frac{P_r^\prime}{1 - \omega^2} + \frac{\rho' \omega^2}{1 - \omega^2} + \frac{2 \omega \omega' \rho}{1 - \omega^2} + \frac{2 \omega \omega' P_r}{(1 - \omega^2)^2}$$

$$+ \frac{2 \omega^3 \omega' \rho}{(1 - \omega^2)^2} + \frac{2 Q' \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \frac{2 Q \omega' e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \frac{2 Q \omega^2 \omega' e^{\lambda/2}}{(1 - \omega^2)^{3/2}}$$

$$+ \frac{2}{r} \left[ \frac{4 \pi r^3}{r - 2m} \left( \rho + P_r \omega^2 \right) \frac{Q \omega e^{\lambda/2}}{(1 - \omega^2)^{3/2}} + \frac{12 \pi r^3}{r - 2m} \left( \frac{Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \right)^2 \right]$$

$$+ \left( \rho + P_r \right) \frac{\omega^2}{1 - \omega^2} + \left( P_r - P_L \right) + \frac{2 Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \left( \rho + P_r \right) \frac{1 + \omega^2}{2} \frac{m}{1 - \omega^2}$$

$$+ \frac{Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \frac{m}{r - 2m} + \frac{2 \pi r^3}{r - 2m} \left( \rho + P_r \omega^2 \right) \left( \rho + P_r \right) \frac{1 + \omega^2}{(1 - \omega^2)^2}$$

$$+ \frac{8 \pi r^3}{r - 2m} \left( P_r + \rho \omega^2 \right) \frac{Q \omega e^{\lambda/2}}{(1 - \omega^2)^{3/2}} + \frac{4 \pi r^3}{r - 2m} Q \omega e^{\lambda/2} \left( \rho + P_r \right) \frac{1 + \omega^2}{(1 - \omega^2)^{3/2}} \right]$$

$$= e^{-\nu} \left( \dot{\lambda} + \frac{\lambda^2}{2} - \frac{\dot{\nu}}{2} \right) \quad (36)$$

which, when evaluated immediately after perturbation, reduces to
\[ P'_r + \frac{(\rho + P_r)m}{r^2(1 - 2m/r)} + \frac{4\pi r}{(1 - 2m/r)} \left( P_r \rho + P^2_r \right) + \frac{2(P_r - P_\perp)}{r} = \frac{e^{-\nu} \ddot{\lambda}}{8\pi r} \] (37)

for both initial states.

On the other hand, an expression for \( \ddot{\lambda} \) may be obtained by taking the time derivative of (6)

\[ \ddot{\lambda} = -8\pi r e^{(\nu + \lambda)/2} \left[ (\rho + P_r) \frac{\omega}{1 - \omega^2} \frac{\dot{\nu}}{2} + Q e^{\lambda/2} \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} \frac{\dot{\nu}}{2} \right. \]

\[ \left. + \frac{(\rho + P_r)\omega \dot{\lambda}}{2} + Q e^{\lambda/2} \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} \dot{\lambda} + \left( \dot{\rho} + \dot{P_r} \right) \frac{\omega}{1 - \omega^2} \right] \] (38)

which, in its turn, when evaluated after perturbation, reads

\[ \ddot{\lambda} = -8\pi r e^{(\nu + \lambda)/2} \left[ (\rho + P_r) \dot{\omega} + \dot{Q} e^{\lambda/2} \right] \] (39)

replacing \( \ddot{\lambda} \) by (39) en (37), we obtain

\[ -e^{(\nu - \lambda)/2} R = (\rho + P_r) \dot{\omega} + \dot{Q} e^{\lambda/2} \] (40)

where \( R \) denotes the left-hand side of the TOV equation, i.e.

\[ R \equiv \frac{dP_r}{dr} + \frac{4\pi r P^2_r}{1 - 2m/r} + \frac{P_r m}{r^2(1 - 2m/r)} + \frac{4\pi r \rho P_r}{1 - 2m/r} + + \frac{\rho m}{r^2(1 - 2m/r)} \frac{2(P_\perp - P_r)}{r} \]

\[ = P'_r + \frac{\nu'}{2} (\rho + P_r) - \frac{2}{r} (P_\perp - P_r) \] (41)

The physical meaning of \( R \) is clearly inferred from (41). It represents the total force (gravitational + pressure gradient + anisotropic term) acting on
a given fluid element. Obviously, $R > 0/R < 0$ means that the total force is directed *inward/outward* of the sphere.

Let us now turn back to thermal conduction equation (20). Evaluating its $t$-component (given by Eq. (21)) immediately after perturbation, we obtain for the first initial configuration (static), an identity. Whereas the second case (slowly evolving) leads to

$$\omega \left( T' + \frac{T'\nu'}{2} \right) = 0 \quad (42)$$

which is to be expected, since before perturbation, in the slowly evolving regime, we have according to Eckart-Landau (valid in this regime)

$$Q = -\kappa e^{-\lambda} \left( T' + \frac{T'\nu'}{2} \right) \quad (43)$$

Therefore, the quantity in bracket is of order $Q$. Then immediately after perturbation this quantity is still of order $O(\omega)$, which implies (42).

The corresponding evolution of the $r$-component of the equation (20) yields, for the initially static configuration

$$\tau \dot{Q} e^{\nu/2} = -\kappa T \dot{\omega} \quad (44)$$

where the fact has been used that after perturbation

$$Q = 0 \implies T' = -\frac{T\nu'}{2} \quad (45)$$

For the second case, the $r$-component of heat transport equation yields also (44), since after perturbation the value of $Q$ is still given by (43), up to $O(\omega)$ terms.

Finally, combining (40) and (44) we obtain

$$\dot{\omega} = -\frac{e^{(\nu-\lambda)/2} R}{(\rho + P_r)} \times \frac{1}{1 - \frac{\kappa T}{\tau(\rho + P_r)}} \quad (46)$$

or,

$$- e^{(\nu-\lambda)/2} R (\rho + P_r) \dot{\omega} (1 - \alpha) \quad (47)$$

with $\alpha$ defined by
\[ \alpha \equiv \frac{\kappa T}{\tau (\rho + P_r)} \]  

Let us first consider the \( \alpha = 0 \) case. Then, (47) has the obvious “Newtonian” form

\[
\text{Force} = \text{mass} \times \text{acceleration}
\]
since, as it is well known, \((\rho + P_r)\) represents the inertial mass density and by “acceleration” we mean the time derivative of \( \omega \). In this case \((\alpha = 0)\), an outward/inward acceleration \((\dot{\omega} > 0/\dot{\omega} < 0)\) is associated with an outwardly/inwardly \((R < 0/R > 0)\) directed total force (as one expects!).

However, in the general case \((\alpha \neq 0)\) the situation becomes quite different. Indeed, the inertial mass term is now multiplied by \((1 - \alpha)\), so that if \(\alpha = 1\), we obtain that \(\dot{\omega} \neq 0\) even though \(R = 0\). This decreasing of the “effective inertial mass density” (vanishing when \(\alpha = 1\)), has been shown to occur in the general (non–spherical symmetric case) (see [6] and references therein).

Next, evaluating (25) immediately after the system leaves the equilibrium (or quasi–equilibrium) and using (39) and (40), (47) may be written as

\[
-3e^{(\nu-\lambda)/2} \frac{[W - W(s)]}{2\pi r^4} = (\rho + P_r) \dot{\omega} (1 - \alpha)
\]

for any \(n > 1\). Thus, unless either \((\rho + P_r) = 0\) (in the non–dissipative case) or \(\alpha = 1\) (in the dissipative case), if the Weyl tensor does not change with respect to its value in the equilibrium (or quasi–equilibrium) state, the system will not abandon such state.

### 4 Conclusions

We have seen that any departure from equilibrium (or quasi–equilibrium) of a spherically symmetric distribution of matter, is tightly controlled by
changes in the Weyl tensor with respect to its value in equilibrium (or quasi–
equilibrium), as indicated by (49) and (50). As an obvious consequence of
this, it follows that a conformally flat fluid distribution in state of equilibrium
(or quasi–equilibrium) will depart from such state, only if it ceases to be
conformally flat (except for the two cases already mentioned).
Finally, it is worth mentioning that the last term in (20) is frequently omitted
(the so-called “truncated” theory) [22]. In the context of this work both
components of this term vanish and therefore all results found above are
independent of the adopted theory (Israel-Stewart or truncated).

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