COHERENT STATE REPRESENTATIONS
OF THE HOLOMORPHIC AUTOMORPHISM GROUP
OF THE TUBE DOMAIN
OVER THE DUAL OF THE VINBERG CONE

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ABSTRACT. We classify all irreducible coherent state representations of the holomorphic automorphism group of the tube domain over the dual of the Vinberg cone.

1. Introduction

Let $G_0$ be a connected Lie group, and let $(\pi, \mathcal{H})$ be a unitary representation of $G_0$. We regard the projective space $\mathbb{P}(\mathcal{H})$ as a (possibly infinite-dimensional) Kähler manifold. We call a $G_0$-orbit of $\mathbb{P}(\mathcal{H})$ a coherent state orbit (CS orbit for short) if it is a complex submanifold of $\mathbb{P}(\mathcal{H})$, and we call $\pi$ a coherent state representation (CS representation for short) if there exists a CS orbit in $\mathbb{P}(\mathcal{H})$ that does not reduce to a point (see [13, Definition 4.2]). In this case, we say that $\pi$ is generic if $\pi$ is irreducible and $\ker \pi$ is discrete. By Lisiecki [11], the generic CS representations coincide with the irreducible highest weight representations with discrete kernels for a semisimple Lie group. Thus CS representations can be considered as generalizations of the highest weight representations of semisimple Lie groups to a wider class of groups. Also the generic CS representations of connected unimodular Lie groups were studied and classified by Lisiecki [12]. After this remarkable advance, CS representations were also studied in the setting of Lie groups which have compactly embedded Cartan subalgebras by Neeb [14].

The purpose of the present article is to give classifications of irreducible CS representations and generic CS representations for a Lie group which has not been considered. Let $\Omega_5$ be the dual cone of the Vinberg cone, and let $D_5$ be the tube domain over $\Omega_5$. Let $G$ be the identity component of the holomorphic automorphism group of $D_5$.

Key words and phrases. Coherent state representation; homogeneous bounded domain; momentum mapping; reproducing kernel; multiplier representation.
In Section 2, we review the theory of CS representations studied in [11, 12, 13]. In Section 3, we review the explicit description of $G$ studied in [5, 8]. In Section 4, we show that every generic CS representation of $G$ is unitarily equivalent with a unitarization of a holomorphic multiplier representation of $G$ over $D_5$ or the complex conjugate representation of it. In Section 5, we review the classification of the unitarizations of holomorphic multiplier representations of $G$ over $D_5$ studied in [1]. In Section 6, we classify all generic CS representations of $G$. In Section 7, we classify all irreducible non-generic CS representations of $G$. In Section 8, we consider intertwining operators between the external tensor product of a one-dimensional unitary representation of $\mathbb{R}_{>0}$ and an irreducible highest weight representation of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and the unitarizations of holomorphic multiplier representations of $G$ over $D_5$.

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2. General theory of CS representations

Throughout this paper, for a Lie group, we denote its Lie algebra by the corresponding Fraktur small letter.

Let $G_0$ be a connected Lie group. For a $G_0$-equivariant holomorphic line bundle $L_0$ over a complex manifold $M_0$, let us denote the natural representation of $G_0$ on the space $\Gamma^{\text{hol}}(M_0, L_0)$ of holomorphic sections $L_0$ by $\tau_{L_0}$. We introduce a notion of unitarizability for $\tau_{L_0}$.

**Definition 2.1.** We say that the representation $\tau_{L_0}$ of $G_0$ is unitarizable if there exists a nonzero Hilbert space $\mathcal{H} \subset \Gamma^{\text{hol}}(M_0, L_0)$ satisfying the following conditions:

(i) the inclusion map $\iota : \mathcal{H} \hookrightarrow \Gamma^{\text{hol}}(M_0, L_0)$ is continuous with respect to the open compact topology of $\Gamma^{\text{hol}}(M_0, L_0)$,

(ii) $\tau_{L_0}(g)\mathcal{H} \subset \mathcal{H}$ \hspace{1em} ($g \in G_0$) and $\|\tau_{L_0}(g)s\|_{\mathcal{H}} = \|s\|_{\mathcal{H}}$ \hspace{1em} ($g \in G_0, s \in \mathcal{H}$).

In this case, we call the subrepresentation $(\tau_{L_0}, \mathcal{H})$ a unitarization of the representation $(\tau_{L_0}, \Gamma^{\text{hol}}(M_0, L_0))$ of $G_0$.

A Hilbert space $\mathcal{H}$ satisfying the condition (i) is a reproducing kernel Hilbert space. We note that a Hilbert space giving a unitarization of $\tau_{L_0}$ is unique if it exists, and any unitarization is irreducible (see [9, 10]). Thus we write $\pi_{L_0}$ instead of $(\tau_{L_0}, \mathcal{H})$. Let $(\pi, \mathcal{H})$ be a CS representation of $G_0$, and let $L$ be the natural holomorphic line bundle over $\mathbb{P}(\mathcal{H})$ such that the fiber over $[v] = \mathbb{C}v \in \mathbb{P}(\mathcal{H})$ is given by the dual space $[v]^*$. Then we can identify the dual space $\mathcal{H}^*$ with $\Gamma^{\text{hol}}(\mathbb{P}(\mathcal{H}), L)$. 

By the following proposition, we can see that if \( \pi \) is irreducible, then \( \pi \) is equivalent with \( \pi_{L_0} \) for a \( G_0 \)-equivariant holomorphic line bundle \( L_0 \) over a CS orbit in \( \mathbb{P}(\mathcal{H}) \), where \( \overline{\pi} \) denotes the complex conjugate representation.

**Proposition 2.2** ([12, Proposition 2]). Suppose that \( \pi \) is irreducible, and let \( M \subset \mathbb{P}(\mathcal{H}) \) be a CS orbit. Then the map \( \mathcal{H}^* \to \Gamma^{hol}(M, L) \) given by the composition of the map \( \mathcal{H}^* \to \Gamma^{hol}(\mathbb{P}(\mathcal{H}), L) \) and the restriction map \( \Gamma^{hol}(\mathbb{P}(\mathcal{H}), L) \to \Gamma^{hol}(M, L) \) is injective.

Let \( M \) be a CS orbit, let \( \alpha_0 : G_0 \times M \to M \) be the action of \( G_0 \) on \( M \), and let \( Z_{g_0} \) be the center of \( g_0 \). When \( \pi \) is generic, it holds that

\[
\text{Lie}(\ker \alpha_0) = Z_{g_0},
\]

where \( \ker \alpha_0 = \{ g \in G_0; \alpha_0(g, x) = x \text{ for all } x \in M \} \).

Next let us see the relationship between CS orbits and coadjoint orbits. Let \( \mu_\pi : \mathbb{P}(\mathcal{H}^\infty) \to g_0^* \) be a moment map defined by

\[
\langle x, \mu_\pi([v]) \rangle = -i \frac{\langle d\pi(x)v, v \rangle_{\mathcal{H}}}{(v, v)_{\mathcal{H}}} \quad (v \in \mathcal{H}^\infty \setminus \{0\}, x \in g_0).
\]

Then the image of \( M \) under \( \mu_\pi \) coincides with a coadjoint orbit. We note that \( M \) has the natural structure of a Kähler manifold which is induced by the Fubini-Study metric on \( \mathbb{P}(\mathcal{H}) \). As a consequence of this property, we have the following theorem.

**Theorem 2.3** ([16, Theorem 2.17]). The isotropy subgroup of \( G_0 \) at any point of \( \mu_\pi(M) \) is connected. In particular, the coadjoint orbit \( \mu_\pi(M) \) is simply connected, and \( \mu_\pi \) defines a diffeomorphism of \( M \) onto the coadjoint orbit.

3. The holomorphic automorphism group of the tube domain over the dual of the Vinberg cone

Let

\[
V = \left\{ \begin{bmatrix} x^1 & 0 & x^4 \\ 0 & x^2 & x^5 \\ x^4 & x^5 & x^3 \end{bmatrix} \in M_3(\mathbb{R}); x^1, \ldots, x^5 \in \mathbb{R} \right\},
\]

and let \( \Omega_5 = V \cap \mathbb{P}(3, \mathbb{R}) \), where \( \mathbb{P}(3, \mathbb{R}) \) denotes the homogeneous convex cone consists of all 3-by-3 real positive-definite symmetric matrices. We consider the following Siegel domain \( D_5 \) in \( V_\mathbb{C} \):

\[
D_5 = \left\{ z = \begin{bmatrix} z^1 & 0 & z^4 \\ 0 & z^2 & z^5 \\ z^4 & z^5 & z^3 \end{bmatrix} \in V_\mathbb{C}; \text{Im } z \in \Omega_5 \right\}.
\]
Let $\text{Aut}_{hol}(\mathcal{D}_5)$ be the holomorphic automorphism group of $\mathcal{D}_5$. We note that $\mathcal{D}_5$ is holomorphically equivalent to a complex bounded domain, and $\text{Aut}_{hol}(\mathcal{D}_5)$ has the unique structure of a Lie group compatible with the compact open topology. Let $G$ be the identity component of $\text{Aut}_{hol}(\mathcal{D}_5)$.

**Theorem 3.1** ([5], [8] Theorem 2.2). The following linear group is isomorphic to $G$:

$$
\left\{ \begin{array}{cccc}
    a_1 & 0 & 0 & b_1 \\
    0 & a_2 & 0 & b_2 \\
    \lambda_1 & \lambda_2 & a_3 & \mu_1 \\
    c_1 & 0 & 0 & a_1 \end{array} \right| \in M_6(\mathbb{R}); \quad \left[ \begin{array}{cc}
    a_i, b_i, c_i, d_i, \lambda_i, \mu_i, \mu'_i, \kappa \in \mathbb{R}, \\
    a_3 \in \mathbb{R} > 0, \quad a_i d_i - b_i c_i = 1,
\end{array} \right]
$$

\[
\lambda_i \mu_i = a_3 \left[ \lambda'_i \mu'_i \right] \left[ \begin{array}{cc}
    a_i & b_i \\
    c_i & d_i \\
\end{array} \right] \quad (i = 1, 2)
\]

In more detail the linear group acts on $\mathcal{D}_5$ by linear fractional transformations, and the natural map from the linear group to $G$ gives rise to an isomorphism between the Lie groups.

Let us denote by $G'$ the linear group given in Theorem 3.1. Let $E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_3, A_{3,1}, A_{3,2}, W_1,$ and $W_2$ be the elements of $M_6(\mathbb{R})$ satisfying

$e_1 E_1 + e_2 E_2 + e_3 E_3 + e_{3,1} E_{3,1} + e_{3,2} E_{3,2}
+ a_1 A_1 + a_2 A_2 + a_3 A_3 + a_{3,1} A_{3,1} + a_{3,2} A_{3,2} + k_1 W_1 + k_2 W_2
= \left[ \begin{array}{cccccc}
    a_1 & 0 & 0 & e_1 - k_1 & 0 & e_{3,1} \\
    0 & a_2 & 0 & 0 & e_2 - k_2 & e_{3,2} \\
    a_{3,1} & a_{3,2} & a_i \mu_i & e_{3,1} & e_{3,2} & e_3 \\
    k_1 & 0 & 0 & -\frac{a_i \mu_i}{2} & 0 & -a_{3,1} \\
    0 & k_2 & 0 & 0 & -\frac{a_i \mu_i}{2} & -a_{3,2} \\
    0 & 0 & 0 & 0 & \frac{a_i \mu_i}{2} & \frac{a_i \mu_i}{2}
\end{array} \right]
$

for $e_1, e_2, e_3, e_{3,1}, e_{3,2}, a_1, a_2, a_3, a_{3,1}, a_{3,2}, k_1, k_2 \in \mathbb{R}$. Then $\{E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_3, A_{3,1}, A_{3,2}, W_1, W_2\}$ form a basis of $\mathfrak{g'}$, and we use the same symbols $E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_3, A_{3,1}, A_{3,2}, W_1,$ and $W_2$ for the corresponding elements of $\mathfrak{g}$. Let $G_{i I_3}$ be the isotropy subgroup of $G$ at $i I_3 \in \mathcal{D}_5$.

**Theorem 3.2** ([5]). We have $G_{i I_3} = \exp(W_1, W_2)$. 

We have the following bracket relations:

\[
\begin{align*}
[E_1, A_1] &= -E_1, & [E_{3,1}, A_1] &= -\frac{1}{2}E_{3,1}, & [A_1, A_{3,1}] &= -\frac{1}{2}A_{3,1}, \\
[E_1, A_{3,1}] &= -E_{3,1}, & [E_{3,1}, A_3] &= -\frac{1}{2}E_{3,1}, & [A_1, W_1] &= -(W_1 + 2E_1), \\
[E_1, W_1] &= 2A_1, & [E_{3,1}, A_{3,1}] &= -2E_3, & [A_2, A_{3,2}] &= -\frac{1}{2}A_{3,2}, \\
[E_2, A_2] &= -E_2, & [E_{3,1}, W_1] &= A_{3,1}, & [A_2, W_2] &= -(W_2 + 2E_2), \\
[E_2, A_{3,2}] &= -E_{3,2}, & [E_{3,2}, A_2] &= -\frac{1}{2}E_{3,2}, & [A_3, A_{3,1}] &= \frac{1}{2}A_{3,1}, \\
[E_2, W_2] &= 2A_2, & [E_{3,2}, A_3] &= -\frac{1}{2}E_{3,2}, & [A_3, A_{3,2}] &= \frac{1}{2}A_{3,2}, \\
[E_3, A_3] &= -E_3, & [E_{3,2}, A_{3,2}] &= -2E_3, & [A_{3,1}, W_1] &= -E_{3,1}, \\
& & [E_{3,2}, W_2] &= A_{3,2}, & [A_{3,2}, W_2] &= -E_{3,2}.
\end{align*}
\]

4. CS ORBITS OF GENERIC CS REPRESENTATIONS

Let \( M \) be a CS orbit of a generic CS representation \( \pi \) of \( G \), and let \( K \) be the isotropy subgroup of \( G \) at some point \( m_0 \) of \( M \). For a connected Riemannian manifold, every isotropy subgroup of the isometry group is compact. Thus \( \exp \text{ad} \mathfrak{k} \subset \text{Int} \mathfrak{g} \) is a compact subgroup, where for a Lie algebra \( \mathfrak{g}_0 \), we denote by \( \text{Int} \mathfrak{g}_0 \) the subgroup \( \exp \text{ad} \mathfrak{g}_0 \subset \text{GL}(\mathfrak{g}_0) \). It is known \([15]\) that \( G \) has trivial center and that \( G_{it_3} = \exp(W_1, W_2) \) is a maximal compact subgroup of \( G \). Thus \( \text{Int} \mathfrak{g} \) is isomorphic to \( G \). Moreover, any two maximal compact subgroups of \( G \) are conjugate (see \([15]\) Chapter 4, Theorem 3.5)), so that we may and do assume that \( \mathfrak{k} \subset \langle W_1, W_2 \rangle \). We then have \( \mathfrak{k} = 0 \) or \( \langle W_1, W_2 \rangle \) because \( M \) is an even-dimensional differentiable manifold.

We shall show that \( \mathfrak{k} \) must equal \( \langle W_1, W_2 \rangle \). Arguing contradiction, assume that \( \mathfrak{k} = 0 \). Then \( M \) is diffeomorphic to \( G \). We have the following theorem.

**Theorem 4.1** ([15] Chapter 4, Proposition 4.4 and Theorem 4.7])

- (a) Let \( G_0 \) be a linear Lie group. If \( G_0 \) equals \( K_0D_0 \) for some compact subgroup \( K_0 \) of \( G_0 \) and for some connected real split solvable Lie subgroup \( D_0 \) of \( G_0 \), then \( K_0 \) is a maximal compact subgroup of \( G_0 \).

- (b) Let \( G_0 \) be a real algebraic linear group. Then the identity component of \( G_0 \) can be topologically decomposed into the direct product of the groups \( K_0 \) and \( D_0 \), where \( K_0 \) is a maximal compact subgroup of \( G_0 \) and \( D_0 \) a maximal real split solvable Lie subgroup of \( G_0 \).

Thus it follows from Theorem 4.1(b) that \( G \) is homeomorphic to \( D_5 \times G_{it_3} \). Hence \( \pi_1(G, e) = \pi_1(G_{it_3}, e) = \mathbb{Z}^2 \), which contradicts that \( M \) is simply connected. Therefore we conclude that \( \mathfrak{k} = \langle W_1, W_2 \rangle \).
Now we have a $G$-equivariant diffeomorphism $\mathcal{D}_5 \to M$. Let us consider the Kähler structure $(j, g)$ on $\mathcal{D}_5$ which is the pullback, by the diffeomorphism, of the Kähler structure on $M$. Also we can regard $\mathcal{D}_5$ as a Kähler manifold by means of the Bergman metric on $\mathcal{D}_5$. Then it follows from [1, Theorem 6.1] that there exists a biholomorphism $\mathcal{D}_5 \to M$ since $G$ acts on $(\mathcal{D}_5, j, g)$ by holomorphic isometries. Thus the action of $G$ on $M$ induces an action of $G$ on $\mathcal{D}_5$ by holomorphic automorphisms, and the action is given by $G \times \mathcal{D}_5 \ni (g, z) \mapsto \varphi(g)z \in \mathcal{D}_5$ for some automorphism $\varphi$ of $G$. Let $\psi$ be the automorphism of $\mathfrak{g}$ satisfying $\psi^2 = \text{id}_\mathfrak{g}$, $\psi(E_1) = E_2$, $\psi(E_3) = E_3$, $\psi(E_{3,1}) = E_{3,2}$, $\psi(A_1) = A_2$, $\psi(A_3) = A_3$, $\psi(A_{3,1}) = A_{3,2}$, and $\psi(W_1) = W_2$, and let $\sigma$ be the automorphism of $\mathfrak{g}$ satisfying $\sigma(E_1) = -E_1$, $\sigma(E_2) = -E_2$, $\sigma(E_3) = -E_3$, $\sigma(E_{3,1}) = -E_{3,1}$, $\sigma(E_{3,2}) = -E_{3,2}$, $\sigma(A_1) = A_1$, $\sigma(A_2) = A_2$, $\sigma(A_3) = A_3$, $\sigma(A_{3,1}) = A_{3,1}$, $\sigma(A_{3,2}) = A_{3,2}$, $\sigma(W_1) = -W_1$, and $\sigma(W_2) = -W_2$. For an automorphism $\varphi$ of $\mathfrak{g}$, let $\varphi^0 = \text{id}_\mathfrak{g}$, and let $\varphi^1 = \varphi$.

**Proposition 4.2.** Any automorphism $\varphi$ of $\mathfrak{g}$ can be written as $\varphi = \psi^2 \circ \sigma^2 \circ \text{Ad}(g)$ for some $g \in G$ and for some $\varepsilon, \varepsilon' \in \{0, 1\}$.

We postpone the proof to Section 7. The automorphisms $\psi$ and $\sigma$ lift to the automorphisms of $G$. To simplify the notation, we use the same symbols $\psi$ and $\sigma$ for the lifts. Then we see that $\psi$ induces a biholomorphism

$$\mathcal{D}_5 \ni (z^1, z^2, z^3, z^4, z^5) \mapsto (z^2, z^1, z^3, z^5, z^4) \in \mathcal{D}_5$$

and $\sigma$ a biholomorphism

$$\mathcal{D}_5 \ni (z^1, z^2, z^3, z^4, z^5) \mapsto (-\overline{z^1}, -\overline{z^2}, -\overline{z^3}, -\overline{z^4}, -\overline{z^5}) \in \overline{\mathcal{D}_5},$$

where $\overline{\mathcal{D}_5}$ denotes the conjugate manifold. Thus by Proposition 2.2 $\pi$ or $\overline{\pi}$ is unitarily equivalent with $\pi_{L_0}$ for some $G$-equivariant holomorphic line bundle $L_0$ over $\mathcal{D}_5$. Here for a subgroup $G_0 \subset G$, by a $G_0$-equivariant bundle over $\mathcal{D}_5$, we mean a $G_0$-equivariant bundle over $\mathcal{D}_5$ such that the action of $G_0$ on the base space $\mathcal{D}_5$ is given by $G_0 \times \mathcal{D}_5 \ni (g, z) \mapsto gz \in \mathcal{D}_5$. We note that $\mathcal{D}_5$ is a Stein manifold (see [2]), and hence every holomorphic line bundle over $\mathcal{D}_5$ is trivial by the Oka-Grauert principle. Hence the representation $\tau_{L_0}$ can be realized on the space $\mathcal{O}(\mathcal{D}_5)$ of holomorphic functions on $\mathcal{D}_5$. We call such a representation of $G$ on $\mathcal{O}(\mathcal{D}_5)$ a holomorphic multiplier representation of $G$ over $\mathcal{D}_5$.

**Theorem 4.3.** Let $\pi$ be a generic CS representation of $G$. Then $\pi$ or $\overline{\pi}$ is unitarily equivalent with a unitarization of a holomorphic multiplier representation of $G$ over $\mathcal{D}_5$. 
5. Holomorphic multiplier representations over $D_5$

Let $g_-$ be the complex subalgebra of $g_\mathbb{C}$ given by

$$g_- = \left\{ x + iy \in g_\mathbb{C}; \left. \frac{d}{dt} \right|_t \xi = e^{tx}iI_3 + i \left. \frac{d}{dt} \right|_t e^{ty}iI_3 \in T_{iI_3}^{0,1}D_5 \right\},$$

where $T_{iI_3}^{0,1}D_5$ denotes the antiholomorphic tangent vector space at $iI_3$. By Tirao and Wolf [17], the isomorphism classes of $G$-equivariant holomorphic line bundles over $D_5$ stand in one-one correspondence with the one-dimensional complex representations of $g_-$ whose restrictions to $g_{iI_3}$ lift to representations of $G_{iI_3}$. For a basis $\{x_\lambda\}$ of $g_-$, we shall denote the dual basis by $\{x_\lambda^*\}$. Let $\mathcal{M}$ be the set consists of all linear forms $\xi$ on $g$ given by

$$\xi = \xi(\xi_3, \eta_3, n, n') = \xi_3E_3^* + \eta_3A_3^* + \frac{n}{2}(2W_1^* - E_1^*) + \frac{n'}{2}(2W_2^* - E_2^*),$$

with $\xi_3, \eta_3 \in \mathbb{R}$ and $n, n' \in \mathbb{Z}$. If $\xi$ is extended to a complex linear form on $g_\mathbb{C}$, then $i\xi|_{g_-}$ ($\xi \in \mathcal{M}$) defines a one-dimensional complex representation of $g_-$ whose restriction to $g_{iI_3}$ lifts to a representation of $G_{iI_3}$. For $\xi \in \mathcal{M}$, let $L_0$ be a $G$-equivariant holomorphic line bundle over $D_5$ whose isomorphism class corresponds to $i\xi|_{g_-}$, and put $\tau_\xi = \tau_{L_0}$. Also we put $\pi_\xi = \pi_{L_0}$ when $\tau_{L_0}$ is unitarizable. Let

$$\Theta^G(n, n') = \{\xi(\xi_3, \eta_3, n, n'); \xi_3 < 0, \eta_3 \in \mathbb{R}\} \quad (n, n' \in \mathbb{Z}_{>0}),$$

$$\Theta^G(\eta_3, n, n') = \{\xi(0, \eta_3, n, n')\} \quad (\eta_3 \in \mathbb{R}, n, n' \in \mathbb{Z}_{\geq 0}).$$

Then we have the following theorem.

**Theorem 5.1** ([1], [7] Theorem 13(i) and (iii)).

(a) For $\xi \in \mathcal{M}$, the representation $\tau_\xi$ is unitarizable if and only if $\xi$ belongs to any of the sets in (5.1).

(b) For $\xi, \xi' \in \mathcal{M}$ with $\tau_\xi, \tau_{\xi'}$ unitarizable, the representations $\pi_\xi$ and $\pi_{\xi'}$ are unitarily equivalent if and only if $\xi$ and $\xi'$ belongs to the same set in (5.1).

(c) Every holomorphic multiplier representation of $G$ over $D_5$ is unitarily equivalent with $\pi_\xi$ for some $\xi \in \mathcal{M}$.

From now on, for $\xi \in \mathcal{M}$ such that $\tau_\xi$ is unitarizable, we think of $\pi_\xi$ as any of the holomorphic multiplier representations of $G$ over $D_5$. We shall mention the converse of Theorem 4.3. Let $\mathcal{H}^\xi$ be the representation space of $\pi_\xi$, let $\mathcal{K}^\xi : D_5 \times D_5 \rightarrow \mathbb{C}$ be the reproducing kernel of $\mathcal{H}^\xi$, and let $\mathcal{K}^\xi_{iI_3} \in \mathcal{H}^\xi$ be the function given by $\mathcal{K}^\xi_{iI_3}(z) = \mathcal{K}^\xi(z, iI_3) (z \in D_5)$. If the representation $d\pi_\xi$ of $g$ is extended to a
complex representation, then we have
\begin{equation}
    d\pi_\xi(x)K_\xi^x = i\xi(x)K_\xi^x \quad (x \in g_-),
\end{equation}
which implies that \( \pi_\xi \) is an irreducible CS representation of \( G \) if \( \dim H^\xi > 1 \) (see [13, Proposition 4.1]).

6. Generic CS representations

For \( n, n' \in \mathbb{Z}_{>0} \), let \( \xi_{n,n'} \) be any of the elements of \( \Theta^G(n,n') \).

**Proposition 6.1.** For any \( n, n', l, l' \in \mathbb{Z}_{>0} \), the representations \( \pi_{\xi_{n,n'}} \) and \( \pi_{\xi_{l,l'}} \) are not unitarily equivalent.

**Proof.** Let \( b = \langle E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_3, A_{3,1}, A_{3,2} \rangle \), and let \( B = \exp b \subset G \). It is enough to show that \( \pi_{\xi_{n,n'}} \) and \( \pi_{\xi_{l,l'}} \) are not equivalent as unitary representations of \( B \). Note that \( B \) is an exponential solvable Lie group, so that the equivalence classes of irreducible unitary representations of \( B \) are in one-one correspondence with the coadjoint orbits of \( B \) in \( b^* \) (see [2]). By [7, Theorem 13(ii)], the equivalence classes of \( \pi_{\xi_{n,n'}}|_B \) and \( \pi_{\xi_{l,l'}}|_B \) correspond to the coadjoint orbit through \( -g_{i} + g_{2} + g_{3} \) \( b \in b^* \) (see Remark 6.2 below for more detail). Then we see that the equivalence class of \( \pi_{\xi_{l,l'}} \) corresponds to the coadjoint orbit through \( g_{1} + g_{2} + g_{3} \) \( b \).

Let \( \eta \) be a linear form on \( b \), and suppose that \( \langle E_3, \eta \rangle > 0 \). We have \( \text{Ad}(e^{\epsilon A_3})E_3 = e^{\epsilon}E_3 \ (t \in \mathbb{R}) \), and \( E_3 \) commutes with \( E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_{3,1}, A_{3,2} \). Thus \( \langle E_3, \text{Ad}^*(b)\eta \rangle = \langle \text{Ad}(b^{-1})E_3, \eta \rangle > 0 \) for \( b \in B \). This implies that the coadjoint orbit through \( -(g_{1} + g_{2} + g_{3})|_{b} \) and the one through \( (g_{1} + g_{2} + g_{3})|_{b} \) are different. The proof is complete. \( \square \)

**Remark 6.2.** For \( \xi = \xi(\zeta, \eta_3, n, n') \in \Theta^G(n,n') \), we shall show that the equivalence class of \( \pi_\xi|_B \) corresponds to the coadjoint orbit through \( -(g_{1} + g_{2} + g_{3})|_{b} \). For \( s = (s_1, s_2, s_3) \in \mathbb{C}^3 \), let \( \alpha_s = \sum_{k=1}^{3} s_k A_k^*|_{b} \in (b^*)_{C} \), and let \( \chi^s \) be the character of \( B \) given by \( \chi^s(\exp x) = \exp \alpha_s(x) \ (x \in b) \). Let us consider the action of \( B \) on the holomorphic line bundle \( \mathcal{D}_5 \times \mathbb{C} \) given by
\[ B \times \mathcal{D}_5 \times \mathbb{C} \ni (b, z, \zeta) \mapsto (bz, \zeta^{-s/2}(b) \zeta) \in \mathcal{D}_5 \times \mathbb{C}, \]
and we denote the \( B \)-equivariant holomorphic line bundle by \( L^\xi \). Now the isomorphism classes of \( B \)-equivariant holomorphic line bundles over \( \mathcal{D}_5 \) stand in one-one correspondence with the one-dimensional complex representations of \( b_{C} \cap g_- \), and \( L^\xi \) corresponds to \( \frac{1}{2}(\sum_{k=1}^{3} \text{Re} s_k E_k^* + \text{Im} s_k A_k^*)|_{b_{C} \cap g_-} \), where we extend \( \sum_{k=1}^{3} \text{Re} s_k E_k^* + \text{Im} s_k A_k^* \) to a complex linear form on \( g_{C} \). Hence for \( s = (n, n', -2(\xi_3 + i\eta_3)) \), the representation \( \pi_{L^\xi} \) of \( B \) is unitarily equivalent with \( \pi_\xi|_B \). For \( \alpha \in g^* \),
let $b_\alpha = \{ x \in b; [y, x] = \alpha(y)x \text{ for all } y \in \langle A_1, A_2, A_3 \rangle \}$. Put $q_k = \sum_{3 \leq l > k \geq 1} \dim b_{(A_l^* - A_k^*)/2} (k = 1, 2, 3)$. Then we have $q_1 = \dim \langle A_{3,1} \rangle = 1$, $q_2 = \dim \langle A_{3,2} \rangle = 1$, $q_3 = 0$, and hence

$$
(6.1) \quad \Re s_k > q_k/2 \quad (k = 1, 2, 3).
$$

According to [7], Theorem 13(ii)], we can obtain the desired result from (6.1).

Let us consider the set of equivalence classes of irreducible unitary representations of $G$. For a unitary representation $\pi_0$ of a Lie group $G_0$, we denote the equivalence class of $\pi_0$ by $[\pi_0]$.

**Theorem 6.3.** The set of equivalence classes of generic CS representations of $G$ is given by

$$
\{ [\pi_{\xi, n, n'}]; n, n' \in \mathbb{Z}_{>0} \} \sqcup \{ [\pi_{\xi, n, n'}]; n, n' \in \mathbb{Z}_{>0} \}.
$$

**Proof.** By Theorems 4, 3 and 5, it is enough to show that

(a) For any $n, n' \in \mathbb{Z}_{>0}$, and $\xi \in \Theta^G(n, n')$, the representation $\pi_\xi$ is generic,

(b) For any $\eta \in \mathbb{R}, n, n' \in \mathbb{Z}_{>0}$, and $\xi \in \Theta^G(\eta, n, n')$, the representation $\pi_\xi$ is not generic.

For $\xi \in \mathcal{M}$ with $\tau_\xi$ unitarizable, we have $\mu_{\pi_\xi}(\mathcal{K}_{G_0}^\xi) = \xi$, and hence we can identify the coadjoint orbit through $\xi \in \mathfrak{g}^*$ with the CS orbit through $[\mathcal{K}_{G_0}^\xi] \in \mathbb{P}(\mathcal{H}^\xi)$. We denote by $\alpha$ the action of $G$ on the coadjoint orbit through $\xi$. Let $G_\xi$ be the isotropy subgroup of $G$ at $\xi$. We note that $\mathfrak{g}_\xi = \{ x \in \mathfrak{g}; \xi([x, y]) = 0 \text{ for all } y \in \mathfrak{g} \}$. The matrix of the skew-symmetric bilinear form $\xi([x, y])$ with respect to the basis $\{ E_1, E_2, E_3, E_{3,1}, E_{3,2}, A_1, A_2, A_3, A_{3,1}, A_{3,2}, W_1, W_2 \}$ is given by

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \frac{n}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & \frac{n'}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\xi_3 & 0 & 0 & 0 & 0 & 0 & 0
-\frac{n}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & -\frac{n'}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & \xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 2\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 2\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

(a) Since $\xi_3 < 0$ and $n, n' \in \mathbb{Z}_{>0}$, it follows that $\mathfrak{g}_\xi = \langle W_1, W_2 \rangle$. Now we have $\text{Lie}(\ker \pi_\xi) \subset \text{Lie}(\ker \alpha) = 0$, and hence $\pi_\xi$ is generic.
(b) We have \( E_3 \in \mathfrak{g}_\xi \). Thus \( \dim \ker \alpha \geq 1 \). We see from (2.1) that \( \pi_\xi \) is not generic. \( \square \)

7. IRREDUCIBLE NON-GENERIC CS REPRESENTATIONS

Let \( \mathfrak{h}_5 = \langle E_3, E_{3,1}, E_{3,2}, A_{3,1}, A_{3,2} \rangle \), \( \mathfrak{h}_3 = \langle E_3, E_{3,1}, A_{3,1} \rangle \), \( \mathfrak{h}_3' = \langle E_3, E_{3,2}, A_{3,2} \rangle \), \( a_1 = \langle A_3 \rangle \), \( s_3 = \langle E_1, A_1, W_1 \rangle \), \( s_3' = \langle E_2, A_2, W_2 \rangle \). Then we have the following lemma.

Lemma 7.1. (a) Every nontrivial ideal in \( \mathfrak{g} \) contains \( \langle E_3 \rangle \).
(b) Let \( \mathfrak{h} \) be an ideal in \( \mathfrak{g} \) such that \( \langle E_3 \rangle \subseteq \mathfrak{h} \). Then \( \mathfrak{h} \) contains \( \mathfrak{h}_3 \) or \( \mathfrak{h}_3' \).
(c) Let \( \mathfrak{h} \) be an ideal in \( \mathfrak{g} \) such that \( \mathfrak{h}_3' \subseteq \mathfrak{h} \). Then \( \mathfrak{h} \) contains \( \mathfrak{h}_5 \) or \( \mathfrak{h}_5 \oplus s_3' \).
(d) Let \( \mathfrak{h} \) be an ideal in \( \mathfrak{g} \) such that \( \mathfrak{h}_3' \oplus s_3' \subseteq \mathfrak{h} \). Then \( \mathfrak{h} \) contains \( \mathfrak{h}_5 \oplus s_3' \).

Proof. Let \( x = e_1 E_1 + e_2 E_2 + e_3 E_3 + e_{3,1} E_{3,1} + e_{3,2} E_{3,2} + a_1 A_1 + a_2 A_2 + a_3 A_3 + a_{3,1} A_{3,1} + a_{3,2} A_{3,2} + k_1 W_1 + k_2 W_2 \) with \( e_1, e_2, e_3, e_{3,1}, e_{3,2}, a_1, a_2, a_3, a_{3,1}, a_{3,2}, k_1, k_2 \in \mathbb{R} \).

(a) Suppose that \( x \) is contained in an ideal \( \mathfrak{h} \) of \( \mathfrak{g} \) such that \( E_3 \notin \mathfrak{h} \).

Since \( [E_3, x] = -a_3 E_3 \), we have \( a_3 = 0 \). Then we have

\[
[E_3, x] = -\frac{a_1}{2} E_{3,1} - 2a_{3,1} E_3 + k_1 A_{3,1},
\]

\[
[E_{3,1}, x] = -2k_1 E_3, \quad [A_{3,1}, [E_{3,1}, x]] = -a_1 E_3,
\]

\[
[E_{3,2}, x] = -\frac{a_2}{2} E_{3,2} - 2a_{3,2} E_3 + k_2 A_{3,2},
\]

\[
[E_{3,2}, [E_{3,2}, x]] = -2k_2 E_3, \quad [A_{3,2}, [E_{3,2}, x]] = -a_2 E_3,
\]

so that \( a_1 = a_2 = a_{3,1} = a_{3,2} = k_1 = k_2 = 0 \). Next,

\[
[A_{3,1}, x] = e_1 E_{3,1} + 2e_{3,1} E_3, \quad [A_{3,1}, [A_{3,1}, x]] = 2e_1 E_3,
\]

\[
[A_{3,2}, x] = e_2 E_{3,2} + 2e_{3,2} E_3, \quad [A_{3,2}, [A_{3,2}, x]] = 2e_2 E_3,
\]

which imply that \( e_1 = e_2 = e_{3,1} = e_{3,2} = 0 \). Thus \( x = e_3 E_3 = 0 \) and \( \mathfrak{h} = 0 \). Therefore, every nontrivial ideal of \( \mathfrak{g} \) contains \( E_3 \).

(b) Let \( \mathfrak{h}' = \langle E_3 \rangle \). It is enough to show that \( \mathfrak{h} = \mathfrak{h}/\mathfrak{h}' \) contains \( s E_{3,1} + t A_{3,1} + \mathfrak{h}' \) with \( s^2 + t^2 \neq 0 \) or \( s E_{3,2} + t A_{3,2} + \mathfrak{h}' \) with \( s^2 + t^2 \neq 0 \).

Arguing contradiction, assume that \( \mathfrak{h} \) does not contain either of them. Let \( x \in \mathfrak{h} \). We have

\[
[E_{3,1}, x] = -\frac{1}{2}(a_1 + a_3) E_{3,1} + k_1 A_{3,1}, \quad [E_{3,2}, x] = -\frac{1}{2}(a_2 + a_3) E_{3,2} + k_2 A_{3,2},
\]

\[
[A_3, x] = \frac{1}{2}(e_{3,1} E_{3,1} + e_{3,2} E_{3,2} + a_{3,1} A_{3,1} + a_{3,2} A_{3,2}),
\]

\[
[W_1, [A_3, x]] = \frac{1}{2}(a_{3,1} E_{3,1} - e_{3,1} A_{3,1}) \pmod{\mathfrak{h}'},
\]
so that $e_{3,1} = e_{3,2} = a_1 + a_3 = a_2 + a_3 = a_{3,1} = a_{3,2} = k_1 = k_2 = 0$. We also have

\[ [A_{3,1}, x] = e_1 E_{3,1} + \frac{\alpha_1 - \alpha_2}{2} A_{3,1}, \quad [A_{3,2}, x] = e_2 E_{3,2} + \frac{\alpha_2 - \alpha_3}{2} A_{3,2} \quad (\text{mod } h'), \]

so that $e_1 = e_2 = a_1 = a_2 = a_3 = 0$. Hence, $h' = 0$, which contradicts the assumption.

(c) Let $h' = h'_3$. It is enough to show that $\tilde{h} = h/\mathfrak{h}'$ contains $s E_{3,1} + t A_{3,1} + h'$ with $s^2 + t^2 \neq 1$ or $s E_2 + t A_2 + h'$ with $s^2 + t^2 \neq 0$. Arguing contradiction, assume that $\tilde{h}$ does not contain either of them. Let $x \in h$. We have

\[ (7.2) \quad [E_1, x] = -a_1 E_1 - a_{3,1} E_{3,1} + 2k_1 A_1, \quad [A_{3,1}, [E_1, x]] = -a_1 E_{3,1} + k_1 A_{3,1}, \]

\[ [A_3, x] = \frac{1}{2} [e_{3,1} E_{3,1} + a_{3,1} A_{3,1}], \quad [E_2, x] = -a_2 E_2 + 2k_2 A_2 \quad (\text{mod } h'), \]

so that $e_{3,1} = a_1 = a_2 = a_{3,1} = k_1 = k_2 = 0$. Next,

\[ [A_3, x] = e_2 E_2, \quad [A_{3,1}, x] = e_1 E_{3,1} - \frac{\alpha_2}{2} A_{3,1} \quad (\text{mod } h'), \]

which implies that $e_1 = e_2 = a_3 = 0$. Hence, $x \in h'$, which contradicts the assumption.

(d) Let $h' = h'_3 \oplus s'_3$. It is enough to show that $\tilde{h} = h/\mathfrak{h}'$ contains $s E_{3,1} + t A_{3,1} + h'$ with $s^2 + t^2 \neq 0$. Arguing contradiction, assume that $\tilde{h}$ does not contain such an element. Let $x \in h$. From (7.1) and (7.2), we see that $e_{3,1} = a_1 = a_3 = a_{3,1} = k_1 = 0$. Since

\[ [W_1, x] = -2e_1 A_1, \quad [A_{3,1}, [W_1, x]] = -e_1 A_{3,1} \quad (\text{mod } h'), \]

we obtain $e_1 = 0$. Hence, $x \in h'$, which contradicts the assumption. \qed

If we take into account Lemma 7.1 and that $s_3 \oplus s'_3$ is semisimple, it is not hard to determine all ideals of $\mathfrak{g}$. Figure 1 gives the Hasse diagram of the set of all nontrivial ideals of $\mathfrak{g}$, ordered by inclusion.

**Proof of Proposition 7.2**. We have $\varphi(h_3) = h_3$ or $\varphi(h_3) = h'_3$, and it is enough to show that if $\varphi(h_3) = h_3$, then $\varphi$ can be written as $\varphi = \sigma^{e'} \circ \text{Ad}(g)$ for some $g \in G$ and for some $e' \in \{0, 1\}$. Let $\varphi(h_3) = h_3$. Let us consider the adjoint action of $G$ on $\mathfrak{g}$. The subgroups $\exp a_1, \exp s_3 \subset G$ act on the ideal $h_3$ of $\mathfrak{g}$ by dilations

\[ h_3 \ni e_3 E_3 + e_{3,1} E_{3,1} + a_{3,1} A_{3,1} \mapsto r^2 e_3 E_3 + r e_{3,1} E_{3,1} + r a_{3,1} A_{3,1} \in h_3 \quad (r > 0) \]

and symplectic maps

\[ h_3 \ni e_3 E_3 + e_{3,1} E_{3,1} + a_{3,1} A_{3,1} \mapsto e_3 E_3 + (e_{3,1} \alpha + a_{3,1} \beta) E_{3,1} + (e_{3,1} \gamma + a_{3,1} \delta) A_{3,1} \in h_3 \]

\[(\alpha \delta - \beta \gamma = 1),\]
respectively. It is well known that the automorphism group of \( h_3 \) is generated by inner automorphisms, symplectic maps, dilations, and inversion

\[
h_3 \ni e_3E_3 + e_{3,1}E_{3,1} + a_{3,1}A_{3,1} \mapsto -e_3E_3 + a_{3,1}E_{3,1} + e_{3,1}A_{3,1} \in h_3.
\]

Thus we have \( \varphi \circ \text{Ad}(g)|_{h_3} = \text{id}_{h_3} \) or \( \varphi \circ \text{Ad}(g)|_{h_3} = \sigma|_{h_3} \) for some \( g \in \exp h_3 \oplus a_1 \oplus s_3 \subset G \).

Now it is enough to show that if \( \varphi|_{h_3} = \text{id}_{h_3} \), then \( \varphi \circ \text{Ad}(g) = \text{id}_g \) for some \( g \in G \). Let \( \varphi|_{h_3} = \text{id}_{h_3} \). We have \( \varphi \circ \text{Ad}(g)|_{h_5} = \text{id}_{h_5} \) for some \( g \in G \). Hence we may and do assume that \( \varphi|_{h_5} = \text{id}_{h_5} \). Let us consider the subrepresentation \( (\text{ad}\ h_5) \) of the adjoint representation \( \text{ad} \) of \( g \). Then the kernel of the subrepresentation equals \( \langle E_3 \rangle \) (see Remark 7.2 below), and hence it follows that \( \varphi(A_3) = A_3 + e_3E_3 \) with \( e_3 \in \mathbb{R} \).

Moreover we see that \( \varphi \circ \text{Ad}(g)|_{h_5 \oplus a_1} = \text{id}_{h_5 \oplus a_1} \) for some \( g \in \exp \langle E_3 \rangle \subset G \). Let us consider the subrepresentation \( (\text{ad}\ h_5 \oplus a_1) \) of the adjoint representation \( \text{ad} \) of \( g \). Then the kernel of the subrepresentation equals \( \langle 0 \rangle \) (see Remark 7.2 below), and hence it follows that an automorphism of \( g \) which is the identity on \( h_5 \oplus a_1 \) is the identity on \( g \). The proof is complete.

**Remark 7.2.** For \( x = e_1E_1 + e_2E_2 + e_3E_3 + e_{3,1}E_{3,1} + e_{3,2}E_{3,2} + a_1A_1 + a_2A_2 + a_3A_3 + a_{3,1}A_{3,1} + a_{3,2}A_{3,2} + k_1W_1 + k_2W_2 \) with \( e_1, e_2, e_3, e_{3,1}, e_{3,2}, a_1, a_2, a_3, a_{3,1}, a_{3,2}, k_1, k_2 \in \mathbb{R} \), let us see the matrix of \( \text{ad}(x) : h_5 \oplus a_1 \rightarrow h_5 \oplus a_1 \) with respect to the basis \( \{E_3, E_{3,1}, E_{3,2}, A_3, A_{3,1}, A_{3,2}\} \). The matrix is given by

\[
\begin{bmatrix}
a_3 & 2a_{3,1} & 2a_{3,2} & -e_3 & -2e_{3,1} & -2e_{3,2} \\
0 & a_{3,1}/2 + a_{3,2}/2 & 0 & -e_{3,1}/2 & k_1 - e_1 & 0 \\
0 & 0 & a_{3,1}/2 + a_{3,2}/2 & -e_{3,2}/2 & 0 & k_2 - e_2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -k_1 & 0 & -a_{3,1}/2 & a_3/2 - a_{1,2}/2 & 0 \\
0 & 0 & -k_2 & -a_{3,2}/2 & 0 & a_{3,2}/2 - a_{1,2}/2
\end{bmatrix}.
\]

By the definition of CS representation, if all generic CS representations of all the quotient groups of \( G \) by connected closed normal subgroups are given, then we can obtain all irreducible CS representations of \( G \) by composing the quotient maps. Let \( G \) be the quotient group by a connected closed normal subgroup of \( G \), and let \( G \neq G \), \( \{e\} \).

According to Figure 11, it is enough to consider the following cases:

1. \( \tilde{g} \simeq \mathbb{R} \),  
2. \( \tilde{g} \simeq \mathfrak{s}l(2, \mathbb{R}) \),  
3. \( \tilde{g} \simeq \mathbb{R} \oplus \mathfrak{s}l(2, \mathbb{R}) \),  
4. \( \tilde{g} \simeq \mathfrak{s}l(2, \mathbb{R}) \oplus \mathfrak{s}l(2, \mathbb{R}) \),  
5. \( \tilde{g} \simeq \mathbb{R} \oplus \mathfrak{s}l(2, \mathbb{R}) \oplus \mathfrak{s}l(2, \mathbb{R}) \),  
6. \( \tilde{g} \simeq \mathfrak{g}/h'_3 \oplus s'_3 \),  
7. \( \tilde{g} = \mathfrak{g}/h'_3 \),  
8. \( \tilde{g} = \mathfrak{g}/\langle E_3 \rangle \).

However \( \tilde{G} \) does not admit generic CS representations in the cases (vi)-(viii). We shall prove this. Suppose that \( M \) is a CS orbit of a
Proposition 7.3. (a) We have the following isomorphisms: \( \text{Int} \tilde{\mathfrak{g}} \simeq G/\exp \mathfrak{h}'_3 + \mathfrak{s}'_3 \) in the case (vi), \( \text{Int} \tilde{\mathfrak{g}} \simeq G/\exp \mathfrak{h}'_3 \) in the case (vii), and \( \text{Int} \tilde{\mathfrak{g}} \simeq G/\exp \langle E_3 \rangle \) in the case (viii).

(b) The maximal compact subgroups of \( G/\exp \mathfrak{h}'_3 + \mathfrak{s}'_3 \), \( G/\exp \mathfrak{h}'_3 \), and \( G/\exp \langle E_3 \rangle \) are conjugate to the images of \( \exp \langle W_1, W_2 \rangle \subset G \) under the quotient maps.

(c) We have \( \pi_1(\text{Int} \tilde{\mathfrak{g}}, e) = \mathbb{Z} \) in the case (vi) and \( \pi_1(\text{Int} \tilde{\mathfrak{g}}, e) = \mathbb{Z}^2 \) in the cases (vii) and (viii).

Proof. We shall prove (a) and (b) for the case (viii) and (c) for the case (vi). For the other cases, this can be proved in the same way.

(a) It is enough to show that \( G/\exp \langle E_3 \rangle \) has trivial center. Let

\[
g = \begin{bmatrix}
a_1 & 0 & 0 & 0 & b_1 & 0 & \mu'_1 \\
0 & a_2 & 0 & 0 & b_2 & \mu'_2 \\
(c_1 \mu'_1 + a_1 \lambda'_1) a_3 & (c_2 \mu'_2 + a_2 \lambda'_2) a_3 & a_3 & (d_1 \mu'_1 + b_1 \lambda'_1) a_3 & (d_2 \mu'_2 + b_2 \lambda'_2) a_3 & k & -\lambda'_1 \\
c_1 & 0 & 0 & d_1 & 0 & -\lambda'_2 \\
0 & c_2 & 0 & 0 & d_2 & 0 & 1/a_3
\end{bmatrix} \in G'.
\]
Then
\[
g^{-1} = \begin{bmatrix}
    d_1 & 0 & 0 & -b_1 & 0 \\
    0 & d_2 & 0 & 0 & -b_2 \\
    -\lambda'_1 & -\lambda'_2 & 1/a_3 & -\mu'_1 & -\mu'_2 \\
    -c_1 & 0 & 0 & a_1 & 0 \\
    0 & -c_2 & 0 & 0 & a_2 \\
    0 & 0 & 0 & 0 & a_3
\end{bmatrix}.
\]

We have
\[
\begin{align*}
\text{Ad}(g^{-1}) E_1 &= (d_1^2 - c_1^2) E_1 + \lambda'_1^2 E_3 - d_1 \lambda'_1 E_{3,1} + 2 c_1 d_1 A_1 - c_1 \lambda'_1 A_{3,1} - c_1^2 W_1, \\
\text{Ad}(g^{-1}) E_2 &= (d_2^2 - c_2^2) E_2 + \lambda'_2^2 E_3 - d_2 \lambda'_2 E_{3,2} + 2 c_2 d_2 A_2 - c_2 \lambda'_2 A_{3,2} - c_2^2 W_2, \\
\text{Ad}(g^{-1}) A_{3,1} &= (2 \mu'_1 E_3 + b_1 E_{3,1} + a_1 A_{3,1})/a_3, \\
\text{Ad}(g^{-1}) A_{3,2} &= (2 \mu'_2 E_3 + b_2 E_{3,2} + a_2 A_{3,2})/a_3, \\
\text{Ad}(g^{-1}) W_1 &= (-d_1^2 + c_1^2 - b_1^2 + a_1^2) E_1 + (-\mu'_1^2 - \lambda'_1^2) E_3 + (d_1 \lambda'_1 - b_1 \mu'_1) E_{3,1} \\
    &\quad + (-2 c_1 d_1 - 2 a_1 b_1) A_1 + (c_1 \lambda'_1 - a_1 \mu'_1) A_{3,1} + (c_1^2 + a_1^2) W_1, \\
\text{Ad}(g^{-1}) W_2 &= (-d_2^2 + c_2^2 - b_2^2 + a_2^2) E_2 + (-\mu'_2^2 - \lambda'_2^2) E_3 + (d_2 \lambda'_2 - b_2 \mu'_2) E_{3,2} \\
    &\quad + (-2 c_2 d_2 - 2 a_2 b_2) A_2 + (c_2 \lambda'_2 - a_2 \mu'_2) A_{3,2} + (c_2^2 + a_2^2) W_2.
\end{align*}
\]

Suppose that Ad\((g^{-1})\) induces the identity map of \(\mathfrak{g}/\langle E_3 \rangle\) onto itself. Then we have
\[
\begin{align*}
    d_1^2 - c_1^2 &= 1, \quad d_1 \lambda'_1 = c_1^2 = 0, \\
    d_2^2 - c_2^2 &= 1, \quad d_2 \lambda'_2 = c_2^2 = 0, \\
    b_1 &= 0, \quad a_1/a_3 = 1, \\
    b_2 &= 0, \quad a_2/a_3 = 1, \\
    c_1 \lambda'_1 - a_1 \mu'_1 &= 0, \quad c_1^2 + a_1^2 = 1, \\
    c_2 \lambda'_2 - a_2 \mu'_2 &= 0, \quad c_2^2 + a_2^2 = 1.
\end{align*}
\]

Hence it follows that \(g^{-1} \in \exp(E_3) \subset G\), which implies that \(G/\exp(E_3)\) has trivial center.

(b) By (a), we see that \(G/\exp(E_3)\) is linearizable. By Theorem 4.1 we conclude that the image of the subgroup \(\exp(W_1, W_2)\) of \(G\) under the quotient map is a maximal compact subgroup of \(G/\exp(E_3)\). Note that the image of a compact subgroup or a real split solvable Lie subgroup under a homomorphism of a Lie groups is also a compact subgroup or real split solvable Lie subgroup, respectively.

(c) The group \(\exp \mathfrak{h}'_3 \oplus \mathfrak{s}'_3 \subset G\) is a topological product of \(H_3(\mathbb{R})\) and \(SL(2, \mathbb{R})\). By (a), it follows that \(\pi_1(\text{Int} \ \tilde{\mathfrak{g}}, e) = \mathbb{Z}^2/\mathbb{Z} = \mathbb{Z}\). \(\square\)
In the case (vi), we have $\pi_1(\text{Int} \tilde{g}, e) = \mathbb{Z}$. We may assume that $\mathfrak{e} \subset \langle E_2, E_3, E_{3.2}, A_2, A_{3.2}, W_1, W_2 \rangle / \langle E_2, E_3, E_{3.2}, A_2, A_{3.2}, W_2 \rangle$, and we then have $\dim \mathfrak{e} = 0$. Since $M$ is diffeomorphic to a coadjoint orbit, the group $\text{Int} \tilde{g}$ acts transitively on $M$, and the isotropy subgroup $(\text{Int} \tilde{g})_{m_0}$ at $m_0$ equals $\{e\}$. Thus $\pi_1((\text{Int} \tilde{g})_{m_0}, e) = \{e\}$. This contradicts that $M$ is simply connected. Similarly, we have $\pi_1(\text{Int} \tilde{g}, e) = \mathbb{Z}^2$, $\pi_1((\text{Int} \tilde{g})_{m_0}, e) = \mathbb{Z}$ in the case (vii), and we have $\pi_1((\text{Int} \tilde{g}, e) = \mathbb{Z}^2$, $\pi_1(\text{Int} \tilde{g})_{m_2, e} = \mathbb{Z}$ in the case (viii). These results contradict that $M$ is simply connected. We obtain the following theorem.

**Theorem 7.4.** Every irreducible non-generic CS representation of $G$ is given by the composition of the external tensor product of a one-dimensional unitary representation of $\mathbb{R}_{>0}$ and a nontrivial irreducible highest weight representation of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ with the map $G \simeq G' \to \mathbb{R}_{>0} \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ given by

$$G' \ni \begin{bmatrix}
 a_1 & 0 & 0 & b_1 & 0 & \mu_1' \\
 0 & a_2 & 0 & b_2 & \mu_2' \\
 \lambda_1 & \lambda_2 & a_3 & \mu_1 & \mu_2 & \kappa \\
 c_1 & 0 & 0 & d_1 & 0 & -\lambda_1' \\
 0 & c_2 & 0 & d_2 & -\lambda_2' \\
 0 & 0 & 0 & 0 & a_3^{-1}
\end{bmatrix} \mapsto \begin{pmatrix}
 a_3, [a_1, b_1] \\
 c_1 
\end{pmatrix}, \begin{pmatrix}
 a_2 & b_2 \\
 c_2 & d_2
\end{pmatrix} \in \mathbb{R}_{>0} \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}).$$

8. **Intertwining Operators**

For $n, n' \in \mathbb{Z}$, let $(\pi_{n,n'}, \mathcal{H}^{n,n'})$ be any irreducible unitary representation of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ such that there exists $v \in (\mathcal{H}^{n,n'})^\infty \setminus \{0\}$ satisfying

$$d\pi_{n,n'}(x, y) v = 0 \quad \text{for} \quad (x, y) \in \mathbb{C} \begin{bmatrix}
 -i & 1 \\
 1 & i
\end{bmatrix} \times \mathbb{C} \begin{bmatrix}
 -i & 1 \\
 1 & i
\end{bmatrix}$$

and

$$\pi_{n,n'} \left( \begin{bmatrix}
 \cos \theta & -\sin \theta \\
 \sin \theta & \cos \theta
\end{bmatrix}, \begin{bmatrix}
 \cos \tau & -\sin \tau \\
 \sin \tau & \cos \tau
\end{bmatrix} \right) v = e^{i(n\theta + n'\tau)} v \quad (\theta, \tau \in \mathbb{R}).$$

Then the set of equivalence classes of irreducible highest weight representations of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is given by $\{[\pi_{n,n}]; n, n' \in \mathbb{Z}\}$. Let $\tilde{G} = \mathbb{R}_{>0} \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. For $\eta_3 \in \mathbb{R}$, let $(\pi_{\eta_3,n,n'}, \mathcal{H}^{n,n'})$ be the external tensor product of the one dimensional representation of $\mathbb{R}_{>0}$ given by $\mathbb{R}_{>0} \ni \gamma \mapsto \gamma^{2\eta_3} \in \mathbb{C}^\times$ and the representation $\pi_{n,n'}$ of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Composing with the map $G \to \tilde{G}$ given in Theorem 7.4, we regard $\pi_{\eta_3,n,n'}$ as a representation of $G$. Let $n, n' \in \mathbb{Z}_{\geq 0}$. 
By [3.2], we can take \( \pi_{\xi(0,0_3,n,n')} |_{SL(2,\mathbb{R}) \times SL(2,\mathbb{R})} \) to be the irreducible unitary representation \( \pi_{n,n'} \), and hence \( \pi_{\xi(0,0_3,n,n')} \) is unitarily equivalent with \( \pi_{0_3,n,n'} \) as representations of \( G \). Therefore we get the following theorem.

**Theorem 8.1.** The set of unitary equivalence classes of irreducible non-generic CS representations of \( G \) is given by
\[
\{ \pi_{0_3,n,n'}; (\eta_3, n, n') \in \mathbb{R} \times \mathbb{Z} \times \mathbb{R} \times \{0\} \times \{0\} \}.
\]

For \( (\eta_3, n, n') \in \mathbb{R} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), we have \( \pi_{\eta_3,n,n'} \simeq \pi_{\xi(0,0_3,n,n') \circ \pi_{-\eta_3,-n,-n'}} \).

We fix a triple \( (\eta_3, n, n') \) with \( \eta \in \mathbb{R} \) and \( n, n' \in \mathbb{Z}_{\geq 0} \). We shall give an explicit description of an intertwining operator between the unitary representations \( \pi_{\eta_3,n,n'} \) and \( \pi_{\xi(0,0_3,n,n')} \) of \( G \). Using the realization of \( G \) as a linear group in Section 3, we shall define a holomorphic multiplier representation of \( G \). Let \( m : G \times D_5 \to \mathbb{C}^\times \) be the holomorphic multiplier given by
\[
m(g, z) = (c_1 z^1 + d_1)^n (c_2 z^2 + d_2)^{n'} a_3 2^{i\eta_3} \quad (g \in G, z \in D_5),
\]
and let \( \tau_m \) be the holomorphic multiplier representation given by
\[
\tau_m(g)f(z) = m(g^{-1}, z)^{-1} f(g^{-1} z) \quad (g \in G, z \in D_5, f \in \mathcal{O}(D_5)).
\]
Then \( \pi_{\xi(0,0_3,n,n')} \) can be considered as a unitarization of \( \tau_m \).

Next we see a natural holomorphic multiplier representation of \( G \) in which \( \pi_{\eta_3,n,n'} \) is realized. Let \( D_1 \) be the unit disc in \( \mathbb{C} \), and let \( \tilde{m} : \tilde{G} \times D_1 \times D_1 \to \mathbb{C}^\times \) be the holomorphic multiplier given by
\[
\tilde{m}(\gamma, g_1, g_2, (w^1, w^2)) = (c_1 w^1 + d_1)^n (c_2 w^2 + d_2)^{n'} \gamma^{2i\eta_3} \\
((\gamma, g_1, g_2) \in \tilde{G}, (w^1, w^2) \in D_1 \times D_1),
\]
where \( g_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in SL(2,\mathbb{R}) \) for \( i = 1, 2 \). We denote by \( D_1 \ni w^i \mapsto g_i w^i \in D_1 \) the action of \( SL(2,\mathbb{R}) \) by linear fractional transformations for \( i = 1, 2 \). Then we can define the following holomorphic multiplier representation \( \tau_{\tilde{m}} \) of \( \tilde{G} \) on the space \( \mathcal{O}(D_1 \times D_1) \) of holomorphic functions on \( D_1 \times D_1 \):
\[
\tau_{\tilde{m}}(g)f(w^1, w^2) = m(g^{-1}, (w^1, w^2))^{-1} f(g_1^{-1} w^1, g_2^{-1} w^2) \\
(g = (\gamma, g_1, g_2) \in \tilde{G}, (w^1, w^2) \in D_1 \times D_1, f \in \mathcal{O}(D_1 \times D_1)).
\]
We regard \( \tau_{\tilde{m}} \) as a representation of \( G \) which \( \exp \mathfrak{h}_5 \subset G \) acts by the trivial representation. Then we have the following theorem.
Theorem 8.2. The map $F : \mathcal{O}(D_1 \times D_1) \ni f \mapsto F_f \in \mathcal{O}(D_3)$ defined by $F_f(z) = f(z^1, z^2)$ ($z \in D_3$) intertwines $\tau_m$ with $\tau_m$, and hence $F$ gives rise to an intertwining operator between the unitarizations.

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