Piecewise-Stationary Off-Policy Optimization

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Abstract

Off-policy learning is a framework for evaluating and optimizing policies without deploying them, from data collected by another policy. Real-world environments are typically non-stationary and the offline learned policies should adapt to these changes. To address this challenge, we study the novel problem of off-policy optimization in piecewise-stationary contextual bandits. Our proposed solution has two phases. In the offline learning phase, we partition logged data into categorical latent states and learn a near-optimal sub-policy for each state. In the online deployment phase, we adaptively switch between the learned sub-policies based on their performance. This approach is practical and analyzable, and we provide guarantees on both the quality of off-policy optimization and the regret during online deployment. To show the effectiveness of our approach, we compare it to state-of-the-art baselines on both synthetic and real-world datasets. Our approach outperforms methods that act only on observed context.

1 Introduction

Contextual bandits have been applied to many core machine learning systems, including search engines, recommender systems, and ad placement [20, 4]. In the contextual bandit, a policy observes a context, takes an action given the context, and observes the associated reward. For example, in search engines, the system receives queries, recommends sites, and observes the click history.

The contextual bandit can be solved online, or on-policy, where a policy learns from real interactions while deployed in the real-world [20, 1, 19]. However, in applications where suboptimal interactions are costly, it is more effective to leverage offline logged data from previously deployed policies [25]. Offline, or off-policy, learning allows evaluating and optimizing new policies from logged data without being deployed online.

Off-policy learning traditionally assumes a stationary environment. In most applications, however, the environment is non-stationary, such as evolving user preferences or sudden events that shift interests. Shifts in the environment can either be smooth, or abrupt at certain points in time. The latter is also called a piecewise-stationary environment [14, 13].

Our work is the first paper on computationally-efficient off-policy optimization in a non-stationary environment with guarantees. Prior works in non-stationary off-policy learning considered learning temporal dynamics via time series forecasting [29] and weighting past observations [18]; and focused on off-policy evaluation. In principle, any estimator can be used in optimization, but it does not come with guarantees on the quality of learned policies or computational efficiency.

Our approach is based on the idea that piecewise-stationary environments can be viewed as induced by a latent state. The offline data can be partitioned by latent state, and a policy conditioned on each state can be learned. Two method of partitioning are considered – change-point detector and hidden
Markov model (HMM). The resulting set of policies, one per state, is the policy for the piecewise-stationary environment. We derive high-probability bounds on off-policy estimates and optimization and analyze how our set of policies can be deployed online as a mixture of experts. Finally, we show the effectiveness of our approach in both synthetic and real-world datasets.

2 Background

Let $\mathcal{X}$ be the set of contexts and $\mathcal{A} = [K]$ be the set of actions. A typical contextual bandit setting consists of an agent interacting with a stationary environment over rounds $t \in [T]$ as follows. In round $t$, context $x_t \in \mathcal{X}$ is drawn from unknown distribution $P^x$. Then, conditioned on $x_t$, the agent chooses an action $a_t \in \mathcal{A}$. Finally, conditioned on $x_t$ and $a_t$, a reward $r_t \in [0, 1]$ is drawn from unknown $P^r(\cdot | x_t, a_t)$.

Now, we formalize the notion of policies and their expected reward. Let $P$ be the set of 

Now, we formalize the notion of policies and their expected reward. Let $H$ be the set of stochastic stationary policies $H = \{\pi : \mathcal{X} \to \Delta^{K-1}\}$, where $\Delta^{K-1}$ is the $K$-dimensional simplex. We use shorthand $x, a, r \sim P, \pi$ to denote a triplet sampled as $x \sim P^x$, $a \sim \pi(\cdot | x)$, and $r \sim P^r(\cdot | x, a)$.

We define $E_{x,a,r \sim P,\pi}[r] = E_{x \sim P^x, a \sim \pi(\cdot | x), r \sim P^r(\cdot | x,a)}[r]$. With this notation, the expected reward of policy $\pi \in H$ in round $t \in [T]$ can be written as $V_t(\pi) = E_{x_t,a_t,r_t \sim P_t,\pi}[ r_t ]$.

In non-stationary environments, the context and reward distributions change with round $t$. Prior works on non-stationary bandits either studied environments with gradual changes [3], or piecewise-stationary environments, where the changes are abrupt at a fixed number of unknown change-points [14, 13]. In this work we focus on the latter environment.

We consider an extended contextual bandit setting where the context and reward distributions also depend on a discrete latent variable $z \in \mathcal{Z}$, where $\mathcal{Z} = [k]$ is the set of $k$ latent states. We denote by $z_t \in \mathcal{Z}$ the latent state at round $t$, and $z_{1:T}$ its sequence over the logged data. We consider $z_{1:T}$ to be fixed but unknown. We assume that the latent state is unaffected by the actions of the agent -- a key difference from reinforcement learning (RL).

We can modify our earlier notation to account for the latent state. Let $P^x_z$ and $P^r_z$ the corresponding context and reward distributions conditioned on $z$. Then the expected reward of policy $\pi$ at round $t$ changes to $V_t(\pi) = E_{x_t,a_t,r_t \sim P_t,\pi}[r_t]$. Let $S$ be the number of stationary segments in $z_{1:T}$, where the latent state is constant over a segment, and $\tau_1 < \ldots < \tau_{S-1}$ be the change-points. To simplify exposition, we let $\tau_0 = 1$ and $\tau_S = T$.

The relation between all variables can be summarized in a graphical model in Figure 1. In search engines, we could have $\mathcal{Z} = \{\text{news}, \text{shopping}, \ldots\}$ be different user intents. If a system knew the user was shopping, it should be more likely to recommend products to buy. So, instead of policies that only act on observed context, we consider policies that also act according to latent state, which belong to the policy class $\mathcal{H}^Z = \{ (\pi_z)_{z \in \mathcal{Z}} : \pi_z \in H \}$.

3 Piecewise-Stationary Off-Policy Evaluation and Optimization

In off-policy learning, actions are drawn according to a known stationary logging policy $\pi_0 \in H$. Data are collected in the form of tuples, $D = \{(x_1, a_1, r_1, p_1), \ldots, (x_T, a_T, r_T, p_T)\}$, where $x_t, a_t, r_t \sim P, \pi_0$ and $p_t \equiv P_0(a_t | x_t)$ is the probability that the logging policy takes action $a_t$ under context $x_t$. For simplicity, we assume that $\pi_0$ is known. If not known, $\pi_0$ can be estimated from logged data [25, 32, 8]. Off-policy learning focuses on two tasks: evaluation and optimization.

In off-policy evaluation, the goal is to estimate the expected reward of a target policy $\pi \in H$, $V(\pi) = \sum_{t=1}^T V_t(\pi)$, from logged data $D$. One popular approach is inverse propensity scoring (IPS) [15], which reweights observations with importance weights as $\tilde{V}(\pi) = \sum_{t=1}^T (p_t(a_t | x_t)/p_0) r_t$. The IPS estimator is unbiased, that is $E_{x,a,r \sim P,\pi_0}[\tilde{V}(\pi)] = V(\pi)$. But its variance could be unbounded if the target and logging policies differ substantially. One common solution is to clip the importance weight with some $M \geq 0$ [17, 4], $\tilde{V}^M(\pi) = \sum_{t=1}^T \min\{ M, p_t(a_t | x_t)/p_0 \} r_t$. The clipped objective trades off variance for bias from underestimating the reward, and there are meth-
In off-policy optimization, the goal is to find a policy with the maximum reward, $\pi^* = \arg \max_{\pi \in \mathcal{H}} V(\pi)$. One popular solution is to directly maximize the off-policy IPS estimate, $\hat{\pi} = \arg \max_{\pi \in \mathcal{H}} \hat{V}(\pi)$ [9]. For stochastic policies, one often optimizes an entropy-regularized estimate [9], $\hat{\pi} = \arg \max_{\pi \in \mathcal{H}} \hat{V}(\pi) - \sum_{t=1}^{T} \sum_{a \in A} \pi(a | x_t) \log \pi(a | x_t)$, where $\tau \geq 0$ is the temperature parameter that controls the determinism of the learned policy. That is, as $\tau \to 0$, the policy chooses the maximum. Following prior work [26, 27], one class of policies that solves this entropy-regularized problem is the linear soft categorical policy: $\pi(a | x; \theta) \propto \exp(\theta^T f(x, a))$, where $\theta \in \mathbb{R}^d$ is the weight of the linear function approximation w.r.t. the joint feature maps of context and action $f(x, a) \in \mathbb{R}^d$.

To extend off-policy learning to the piecewise-stationary latent setting, we consider IPS estimator

$$\hat{V}_z^M(\pi) = \sum_{z \in Z} \hat{V}_z^M(\pi_z), \text{ where } \hat{V}_z^M(\pi_z) := \sum_{t=1}^{T} \mathbb{1}[\hat{z}_t = z] \cdot \min \left\{ M, \frac{\pi_z(a_t | x_t)}{p_t} \right\} r_t$$

(1)

that corresponds to $\pi \in \mathcal{H}^Z$, where $\hat{V}_z^M(\pi_z)$ is the IPS estimator that corresponds to the part of the logged data whose latent state is $z$, and $\hat{z}_{1:T}$ is a sequence of latent states predicted by some oracle $O$. This IPS estimator partitions the logged data by estimated latent state.

If the oracle accurately predicts all the ground-truth latent states, i.e., $\hat{z}_t = z_t, \forall t$, and if $M = \infty$, then the off-policy estimator $\hat{V}(\pi)$ is unbiased. However, in reality this estimation is difficult because the latent states $z_{1:T}$ are not observed in logged data $\mathcal{D}$. Therefore, in general, this latent IPS estimator is biased due to the latent prediction error and the clipping weight.

**Offline Optimization of Latent Sub-Policies.** Leveraging the fact that the logged-data is partitioned into $k$ sub-datasets, each corresponds to a particular latent state, and the separable structure of the IPS estimator $\hat{V}_z^M(\pi)$, the policy optimization problem can also be broken down into learning the best policy at each individual latent state $z$, i.e., for policy $\pi = (\pi_z)_{z \in Z}$, each component is learned via $\pi_z = \arg \max_{\pi \in \mathcal{H}} \hat{V}_z^M(\pi)$. Suppose the sub-policy $\pi_z = \pi(\cdot | \theta_z) \in \mathcal{H}$ is linear soft categorical, then at each latent state $z$ we solve the following problem:

$$\hat{\theta}_z = \arg \max_{\theta \in \mathbb{R}^d} \left\{ \sum_{t=1}^{T} \mathbb{1}[\hat{z}_t = z] \cdot \min \left\{ M, \frac{\pi_z(a_t | x_t; \theta)}{p_t} \right\} r_t \right\}$$

(2)

In practice, following prior work [26], we iteratively solve for this policy using standard off-the-shelf gradient ascent algorithms. Algorithm 1 summarizes the procedures for learning $\hat{\pi} \in \mathcal{H}^Z$.

**Online Latent Sub-Policy Selection.** As a result of our off-policy optimization algorithm, we learn a vector of sub-policies $\hat{\pi} = (\hat{\pi}_z)_{z \in Z}$, one for each latent state. During online deployment, however, the latent state is still unobserved, and we cannot query an oracle as in the offline case. We need an online algorithm that switches between the $k$ learned sub-policies based on past rewards.

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1Following our earlier notation, we write $\hat{V}_z^M(\pi)$ as $\hat{V}(\pi)$ and $\hat{V}_z^M(\pi)$ as $\hat{V}_z(\pi)$ when $M = \infty$. 

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**Algorithm 1: Piecewise off-policy learning**

**Input:** number of latent states $k \in \mathbb{N}$, logged data $\mathcal{D}$, and oracle $O$

Run $O$ on $\mathcal{D}$ to get estimates $\hat{z}_{1:T} \in Z^T$.

for $z \leftarrow 1$ to $k$

- Solve for $\hat{\theta}_z$ in (2)
- Create sub-policy $\hat{\pi}_z$ from $\hat{\theta}_z$ using linear soft parameterization.

end

**Algorithm 2: Piecewise policy deployment**

**Input:** learned policy $\hat{\pi} \in \mathcal{H}^Z$, and mixture-of-experts algorithm $B$

Initialize algorithm $B_1$.

for $t \leftarrow 1$ to $T$ do

- Given $x_t$, choose action $a_t \sim B_t(x_t, (\hat{\pi}_z)_{z \in Z})$.
- Update $B_{t+1}$ from $B_t$ with reward $r_t$.

end
Our solution is to treat each learned sub-policy as an “expert,” and select which sub-policy to execute each round via a mixture-of-experts algorithm $B$. This is because we can treat how well each sub-policy performs on the online data as a surrogate predictor of the unknown latent state. Algorithm 2 outlines how this is done. One such $B$ is EXP4.S [24], which has near-optimal regret guarantees.

### 4 Analysis

In the previous section we introduced the piecewise-stationary off-policy optimization algorithm, which consists of two parts: (i) an offline policy optimization (2) that solves for the latent-space policy $\hat{\pi} = (\hat{\pi}_z)_{z \in Z}$, where $\hat{\pi}_z = \pi(\cdot|\hat{z}_z) \in \mathcal{H}$; and (ii) an online sub-policy selection procedure.

Theorem 1 states that the sub-optimality performance bound of the learned policy $\hat{\theta}$ introduces an intermediate off-policy estimator $\tilde{\pi}(z) = \arg\max_{\pi \in \mathcal{H}_{z}} \tilde{V}(\pi)$, which consists of two parts: (i) an offline policy optimization (2) that solves for the latent-space policy $\hat{\pi} = (\hat{\pi}_z)_{z \in Z}$, where $\hat{\pi}_z = \pi(\cdot|\hat{z}_z) \in \mathcal{H}$; and (ii) an online sub-policy selection procedure.

In the following we provide both the performance sub-optimality analysis of the offline optimization, as well as the regret analysis of the online selection algorithm.

#### 4.1 Performance Analysis of Offline Policy Optimization

For simplicity we restrict our performance analysis to the following refined set of policies in which the clipping condition in $\tilde{V}$ is automatically satisfied so that the propensity score does not need to be clipped, i.e., $\mathcal{H}_M \triangleq \{ \pi \in \mathcal{H} : \frac{\pi(a|x)}{\pi_0(a|x)} \leq M, \ \forall a \in A, x \in X \}$. Correspondingly, we define the set of latent policies associated with $\mathcal{H}_M$ as $\mathcal{H}_{M}^\pi \triangleq \{ (\pi_z)_{z \in Z} : \pi_z \in \mathcal{H}_M \}$. Extending the following analysis to include the effect of clipping [17, 22] is straight-forward and will be omitted for the sake of brevity.

To ensure the analysis is well-posed, we assume the latent state oracle $O$ has the following property and will provide more details on how to construct such an oracle in Section 5.

**Assumption 1.** For any latent states $z_{1:T}$ and $\delta \in (0, 1]$, oracle $O$ estimates $\hat{\pi}_{1:T}$ such that $\sum_{t=1}^{T} I[\hat{z}_{t} \neq z_{t}] \leq \varepsilon(T, \delta)$ with probability at least $1 - \delta$, where $\varepsilon(\cdot, \cdot)$ is some function of $T$ and $\delta$ such that $\varepsilon(T, \delta) = o(T)$.

This assumption ensures that the number of incorrect oracle predictions is bounded (with high probability). We now analyze the performance of $\hat{\pi} \in \mathcal{H}_M^\pi$ that maximizes $\tilde{V}$. When $X$ is finite, $f(x,a)$ is an indicator vector for each pair $(x,a)$, and when $\tau \to 0$, the optimization problem in (2) for solving each $\hat{\pi}_z$ reduces to an LP [22]. The following main technical result provides an sub-optimality performance bound to the solution latent policies.

**Theorem 1.** Let $\hat{\pi} = \arg\max_{\pi \in \mathcal{H}_M^\pi} \tilde{V}(\pi)$ and $\pi^* = \arg\max_{\pi \in \mathcal{H}_M^\pi} V(\pi)$ be the optimal latent policies w.r.t. the off-policy estimated value and the true value respectively. Then for any $\delta_1, \delta_2 \in (0, 1]$, we have that $\tilde{V}(\hat{\pi}) \geq V(\pi^*) - 2M\varepsilon(T, \delta_1/2) - 4M\sqrt{T\log(4/\delta_2)}$ holds with probability at least $1 - \delta_1 - \delta_2$.

Theorem 1 states that the sub-optimality performance bound of the learned policy $\hat{\pi}$ can be decomposed into that of oracle $O$ and randomness of logged data $D$. The latter is sublinear in $T$, and we will provide an oracle that has a sub-linear $\varepsilon(T, \delta)$ in Section 5.

**Proof Sketch of Theorem 1.** (See Appendix A for more details.) To derive this result, we first introduce an intermediate off-policy estimator $\tilde{V}(\pi) = \sum_{t=1}^{T} (\pi(a_t | x_t, z_t)/p_t)r_t$, which is an IPS estimator if $z_{1:T}$ were known. By definition $\tilde{V}(\pi)$ is an unbiased estimate of $\tilde{V}(\pi)$. We first show that the error between the two IPS estimators $\tilde{V}(\pi)$ and $\hat{V}(\pi)$ is bounded by prediction error of $O$.

**Lemma 1.** For any $\pi \in \mathcal{H}_M^\pi$ and $\delta \in (0, 1]$, $|\tilde{V}(\pi) - \hat{V}(\pi)| \leq M\varepsilon(T, \delta)$ with probability $1 - \delta$.

Next, we bound the estimation error of $\tilde{V}(\pi)$ w.r.t. $V(\pi)$. This error is due to the randomness in $D$.

**Lemma 2.** For any $\pi \in \mathcal{H}_M^\pi$, $\delta \in (0, 1]$, $|\tilde{V}(\pi) - V(\pi)| \leq 2M\sqrt{\log(2/\delta)}$ with probability $1 - \delta$.

Notice that the environment is piecewise-stationary which makes the $D$ non-independent w.r.t. random latent contexts. Therefore, instead of Hoeffding’s inequality this proof relies on Azuma’s inequality. Combining Lemmas 1 and 2, we obtain an error bound between $\tilde{V}(\pi)$ and $V(\pi)$ as follows.

**Lemma 3.** For any $\pi \in \mathcal{H}_M^\pi$ and $\delta_1, \delta_2 \in (0, 1]$, $|\tilde{V}(\pi) - V(\pi)| \leq M\varepsilon(T, \delta_1/2) + 2M\sqrt{T\log(2/\delta_2)}$ with probability $1 - \delta_1 - \delta_2$.

Utilizing the property $\tilde{V}(\hat{\pi}) \geq \tilde{V}(\pi^*)$ (by definition), we have $V(\pi^*) - V(\hat{\pi}) \leq (V(\pi^*) - \tilde{V}(\pi^*)) + (\tilde{V}(\hat{\pi}) - \tilde{V}(\pi))$. The proof of Theorem 1 is thus completed by applying Lemma 3 to both $\pi^*$ and $\hat{\pi}$.  

4
4.2 Analysis of Online Regret

We have learned a set of conditional policies \( \hat{\pi} = (\hat{\pi}_z)_{z \in \mathcal{Z}} \) offline, and aim to deploy the policies online as a mixture-of-experts. Recall that we have a fixed sequence of latent states \( z_{1:T} \). For simplicity, we consider the same latent state sequence in online analysis; we can adapt our analysis to arbitrary latent state sequences by accounting for the discrepancy between the latent state frequencies in the offline and online datasets.

Recall the online deployment algorithm in Algorithm 2, with mixture-of-experts algorithm \( B \). At each round \( t \), actions are sampled \( a_t \sim B_t(x_t, (\hat{\pi}_z)_{z \in \mathcal{Z}}) \), where \( B_t \) is a function of the history of rewards up to round \( t \). To simplify the exposition, we introduce the shorthand \( \mathbb{E}_{z, \pi} \cdot \) as \( \mathbb{E}_{z, \pi} \cdot = \mathbb{E}_{x_t \sim \hat{\pi}_z} \cdot \). We define \( R(T; B, \hat{\pi}) = \sum_{t=1}^{T} \mathbb{E}_{z_t, \hat{\pi}_{z_t}} [r_t] - \sum_{t=1}^{T} \mathbb{E}_{z_t, B_t} [r_t] \), as the \( T \)-period regret. The first term is the optimal policy \( \pi^* \) acting according to the true latent state, and the second term is our offline-learned policies \( \hat{\pi} \) acting according to \( B \). In this section, we give a brief outline of how to bound the online regret, and defer details to the Appendix B.

Recall that \( S \) is the number of stationary segments, and \( \tau_0 = 1 < \tau_1 < \ldots < \tau_{S-1} < T = \tau_S \) are the change-points. We can show that the regret decomposes as,

\[
R(T; B, \hat{\pi}) \leq \left[ \sum_{t=1}^{T} \mathbb{E}_{z_t, \pi_{z_t}} [r_t] - \sum_{t=1}^{T} \mathbb{E}_{z_t, \hat{\pi}_{z_t}} [r_t] \right] + \left[ \sum_{s=1}^{S} \max_{t = \tau_{s-1}}^{\tau_{s-1}} \sum_{t=1}^{T} \mathbb{E}_{z_t, \hat{\pi}_z} [r_t] - \sum_{t=1}^{T} \mathbb{E}_{z_t, B_t} [r_t] \right],
\]

where we use that the latent state is constant over a stationary segment. The first term is bounded by our offline analysis, which shows near-optimality of \( \pi \) when the latent state is known. The second term is bounded by the regret of mixture-of-experts algorithm \( B \) over \( S - 1 \) switches.

Prior work has shown an optimal \( T \)-period switching regret with \( S - 1 \) switches of \( O(\sqrt{STK}) \) [24]. One such algorithm that is optimal up to log factors is EXP4.S [24]; we adapt EXP4.S to stochastic experts in Algorithm 4. Using this algorithm for \( B \) gives us the following bound on online regret,

**Theorem 2.** Let \( \hat{\pi} \) and \( \pi^* \) be defined as in Theorem 1, and \( B \) be EXP4.S in Algorithm 4. For horizon \( T \), let \( z_{1:T} \) be the same underlying latent states as in the offline dataset, and let \( S \) be the number of stationary segments. Then for any \( \delta_1, \delta_2 \in (0, 1] \), we have that \( R(T; B, \hat{\pi}) \leq 2M \varepsilon(T, \delta_1/2) - 4M \sqrt{T \log(4/\delta_2) + 2 \sqrt{STK \log(k)}} \), holds with probability at least \( 1 - \delta_1 - \delta_2 \).

5 Oracles for Latent State Prediction

In this section we introduce two oracles for latent state prediction. The first one is based on change-point detection, which we will show in Section 5.1 that it satisfies Assumption 1 and the corresponding policy has a sub-optimality performance guarantee. The second one is based on hidden Markov models (HMMs), which generally does not have theoretical guarantees on latent prediction error, but yield better performing bandit policies for complex problems.

5.1 Change-point Detector

In this section, we propose and analyze a change-point detector oracle required by Assumption 1. First, we assume a one-to-one mapping between the latent states and stationary segments, or \( S = k \). We let \( z_{1:T} \) form a non-decreasing sequence of integers that satisfies \( z_1 = 1, z_T = k \), with \( |z_{t+1} - z_t| \leq 1, \forall t \in [T - 1] \), and change-points \( \tau_0 = 1 < \tau_1 < \ldots < \tau_{k-1} < T = \tau_k \). This assumption is only used for analysis. In practice, this could over-segment the offline data, so we found clustering the segments by value to be helpful. We also assume changes are detectable.

**Assumption 2.** There exists a threshold \( \Delta > 0 \) such that \( |V_{\tau_i}(\pi_0) - V_{\tau_{i-1}}(\pi_0)| \geq \Delta, \forall i \in [k - 1] \). Similar assumptions are common in piecewise-stationary bandit problems, for which the state-of-the-art solution algorithms [23, 6] use an online change-point detector to detect non-stationarities (changes) and reset the parameters of the bandit algorithm upon a change. In this work, we utilize a similar idea but in an offline, off-policy setting. We construct a change-point detector oracle \( O \) with window size \( w \) and detection threshold \( c \), and deployed as detailed in Algorithm 3.

On a high-level, \( O \) computes difference statistics for each round in the offline data and iteratively chooses the round with the highest statistic, declaring that a change-point, and removing any nearby rounds from consideration. This procedure continues until there is no statistic that lies above threshold \( c \). In the following we provide a latent prediction error bound for this oracle.
Theorem 3. Let $\tau_i - \tau_{i-1} > 4w$ for all $i \in [k]$. For any $\delta \in (0, 1]$, and $c$ and $w$ in Algorithm 3 such that $\Delta/2 \geq c \geq \sqrt{2 \log(8T/\delta)}/w$, Algorithm 3 predicts $\hat{z}_{1:T}$ such that $\sum_{i=1}^{T} 1[\hat{z}_i \neq z_i] \leq kw$ holds with probability at least $1 - \delta$.

Theorem 3 implies that the oracle $O$ can correctly (without false positives) detect change-points within a window $w$ with high probability. Notice that both $w$ and $c$ in Theorem 3 depend on $\Delta$, which may not be exactly known, but rather its lower bound $\tilde{\Delta}$ is usually more available. In the following technical result (which is a corollary to Theorem 1), we derive the sub-optimality performance bound of the policy learned via Algorithm 1 using change-point detector oracle $O$.

Corollary 1. Fix any $\tilde{\Delta} \leq \Delta$ and $\delta_1, \delta_2 \in (0, 1]$. Let oracle $O$ be Algorithm 3 with $w = 8\log(16T/\delta_1)/\tilde{\Delta}^2$, $c = \Delta/2$; and $\pi^*$ and $\hat{\pi}$ be defined as in Theorem 1. Then $V(\pi^*) - V(\hat{\pi}) = 16M \left( k\log(16T/\delta_1)/\tilde{\Delta}^2 \right) - 4M \sqrt{T}\log(4/\delta_2)$ with probability at least $1 - \delta_1 - \delta_2$.

Corollary 1 directly follows by applying Theorem 3 to Theorem 1. This implies that if the estimated latent states $\hat{z}_{1:T}$ is generated by Algorithm 3, and the policy $\hat{\pi} \in \mathcal{H}^L$ is learned via Algorithm 1, then the difference in the expected rewards of $\pi^*$ and $\hat{\pi}$ is sublinear in $T$, specifically in $O(\sqrt{T})$.

Proof Sketch of Theorem 3. (See Appendix C for more details.) For any $i \in [k-1]$, let $W_i = [\tau_i - w, \tau_i + w]$ be $w$-close rounds to change-point $\tau_i$. We also define $W = \bigcup_i W_i$ to be the $w$-close rounds to any change-point. We first bound the probability that the detector declares any round $t \notin W$ as a change-point (false-positive). This occurs when the difference statistic at round $t$ exceeds threshold $c$.

Lemma 4. For any round $t \notin W$, the probability of a false detection is bounded from above as $\mathbb{P}(\mu_t^1 - \mu_t^2 \geq c) \leq 4 \exp(-wc^2/2)$.

Next we bound the probability of failing to detect a change-point that is in $W$ (false negative). This occurs if none of the difference statistics in the window exceeds threshold $c$.

Lemma 5. For any positive $c \leq \Delta/2$ and $W_i$, a change-point is not detected in $W_i$ with probability at most $\mathbb{P}(\forall t \in W_i: |\mu_t^1 - \mu_t^2| \leq c) \leq 4 \exp(-wc^2/2)$.

The results in Lemmas 4 and 5 follow by applying Hoeffding’s inequality to the noisy rewards in the windows of length $w$. Using these results, we can readily prove Theorem 3 by applying Lemma 4 to all rounds $t \notin W$, applying Lemma 5 to all change-points, combining these results with a union bound, and choosing the values of $w, c$ accordingly so that the error probability is at most $\delta$.

5.2 Graphical Model

Another natural way of partitioning the data is via a latent variable model. In this work, we specifically model the temporal evolution of $z_{1:T}$ with a HMM over $Z$ [2]. Let $A = [A_{i,j}]_{i,j=1}^k$ be the transition matrix with $A_{i,j} = P(z_t = j \mid z_{t-1} = i)$, and $P_0$ be the initial distribution over $Z$. The latent states evolve according to $z_1 \sim P_0$, and $z_{t+1} \sim \text{Categorical}(A_{z_t})$. Recall that we have joint feature maps of context and action $f(x, a) \in \mathbb{R}^d$. We assume the rewards are sampled according to the conditional distribution $P(\cdot \mid x, a, z) \sim \mathcal{N}(\beta_{i}^T f(x, a), 1)$, where $\beta = (\beta_z)_{z \in Z}$ are regression weights; though we use Gaussian, in general any choice of distributions can be incorporated. Let $\mathcal{M} = \{P_0, A, \beta\}$ be the model parameters. The parameters can be estimated by EM [2].

Oracle $O$ uses the estimated HMM $\hat{M}$ to compute the posterior over latent states. For each round $t$ and $z \in Z$, the oracle estimates the conditional posterior $Q_t(z) \simeq P(z_t = z \mid x_{1:T}, a_{1:T}, r_{1:T})$, after marginalizing over latent states of other rounds. Without the true underlying model $\mathcal{M}$, $Q_t(z)$ is computed using $\hat{M}$. The oracle $O$ predicts $\hat{z}_t = \text{max}_{z \in Z} Q_t(z)$ at each round $t$.

Though the described HMM oracle is practical, currently no guarantees similar to Corollary 1 are known to be provided. Any analysis similar to Theorem 3 would require parameter recovery guarantees on the HMM, which is non-existent for the EM nor spectral methods$^2$ [16]. Nevertheless, the HMM oracle has several appealing properties. First, unlike the change-point detect, the HMM can map multiple stationary segments into a single latent state, which potentially reduces the size of the latent space. Second, the learned reward model $\hat{r}_z(x, a) = \hat{\beta}_z^T f(x, a) \simeq \mathbb{E}_{x, r \sim P_I} [r \mid x, a, z]$ can be incorporated into more advanced off-policy estimators, e.g., DR, instead

$^2$HMM guarantees only exist on the marginal probability of observations.
of the IPS estimator in (1), which reduces the variance. Finally, the HMM can be leveraged for a Thompson sampling mixture-of-experts algorithm for online deployment, which we show has better empirical performance than EXP4.S. We detail the algorithm in Algorithm 5.

6 Experiments

In this section, we evaluate our algorithm on a synthetic and real-world datasets to demonstrate that our learning approach outperforms learning a stationary policy. We compare the following methods: (i) IPS: single policy trained on IPS objective; (ii) DR: Single policy trained on DR objective, with reward model $\hat{r}(x, a) = \beta^T f(x, a)$ fit using least squares; (iii) POEM: single policy trained on CRM objective [26]; (iv) k-CD: $k$ sub-policies trained using our method and change-point detector oracle; (v) k-HMM: $k$ sub-policies trained using our method and HMM oracle. The first three are baselines in stationary off-policy optimization, and the last two are our approach.

6.1 Synthetic Dataset

We construct a semi-synthetic contextual bandit setting using the LastFM dataset [5]. It consists of users connected in a social graph, artists that each user has listened to, and genre tags for each artist. Following the setup in Cesa-Bianchi et al. [7], we give reward 1 to an artist that the user has listened to, and reward 0 to other artists. For each user, we sample actions by sampling an artist the user listened to and 24 the user has not. Each artist is described by a TF-IDF feature vector of the artists’s tags, reduced to 25 dimensions via PCA [7]. Context is the concatenation of artist features.

We introduce non-stationarity by clustering users into 10 user groups via spectral clustering on the social graph as in Wu et al. [31]. User groups are latent states. We construct the logged data by sampling from a Markov chain over user groups; at each round, a user is sampled from the user group, then context and reward is drawn as described earlier. We make a stochastic logging policy with full support by selecting according to the LinUCB strategy with probability $1 - \epsilon$ and uniformly otherwise. We chose $\epsilon = 0.2$ in our experiments [18].

For logged data, we sampled $T = 200,000$ rounds using the sequence of latent states. First, we evaluated our learned policy assuming it knows the latent state conditional distribution; to do so, we compute the policy’s expected reward using (1): k-CD used the change-point detector, and k-HMM used the HMM. Next, we evaluate our methods “online”, by sampling 100,000 rounds using a different sequence of latent states. The learned policy was deployed using Algorithm 2; k-CD used EXP4.S as in Algorithm 4, and k-HMM used TS as in Algorithm 5.

In Figure 2, we reported reward for all the methods over 10 runs. We also plot the effect of the number of latent states, $k$, on the methods; only k-CD and k-HMM use latent states. Both of our approaches k-CD and k-HMM significantly outperformed learning a stationary policy.

6.2 Yahoo! Dataset

We also evaluate on the Yahoo! clickstream dataset [20]. The dataset consists of offline interactions: in each round, a document was uniformly sampled from a pool to show to a user, and whether the document was clicked by the user was logged. Document and user features were provided, and the click-through-rates (CTR) of documents change over time [31]. We chose a 6-day horizon, and randomly subsampled one interaction per second; for each interaction we chose a random subset of 10 documents, creating a logged dataset with $T = 86,400 \times 6 = 518,400$, and $K = 10$. The context consists of all document vectors; user preference features were ignored [20].
We used evaluated both offline and online performance the policies. In the former, we sub-sampled from the same 6-day horizon, which should have the same underlying latent states as the logged data. Then, we directly use the estimated latent states to compute the policy’s CTR using (1). We report the relative CTR, or the learned policy’s CTR divided by the logging policy’s. In the latter, we “deployed” algorithms k-CD and k-HMM on the next 4 days of data. Because our mixture-of-experts strategies depend on past interactions, offline evaluation of the policy is nontrivial [21]. For each action at each round, we compute the mean CTR for the chosen document over a half-day window around that round, and sampled Bernoulli rewards from the computed mean. The window approximately ensures that the sampled rewards come from the same latent state.

In Figure 3, we reported relative CTR for all the methods over 10 runs. We also plot the effect of the number of latent states, $k$, on the reward for k-CD and k-HMM methods. Both our approaches performed the best, with k-HMM better due to learning a full environment model. Our methods outperformed stationary baselines by up to 10%. The results show that even in situations with non-obvious latent state structure, our approach still improves on methods that ignore latent states.

7 Related Work

Off-policy Evaluation. A plethora of literature deals with building counterfactual estimators for evaluating policies. The unbiased IPS estimator has optimal theoretical guarantees when we have a good model of the logging policy [25, 32]. Various techniques have been employed to reduce the variance of IPS estimators as importance weight clipping [17, 4], or introducing a model of reward feedback, improving on the MSE of the estimator [10, 12, 30, 9]. Though we focus on IPS estimators, DR estimators can be directly co-opted into our approach.

Off-policy Optimization. Off-policy estimators can be directly applied to learning policies by optimizing the estimated value. Recent work in off-policy optimization additionally regularizes the estimated value with its empirical standard deviation [26], or uses self-normalization as control variates [27]. There is also work in handling combinatorial actions [28, 22, 8].

Non-stationary Bandits. The problem of non-stationary rewards is well-studied in bandit literature [3, 13]. Recent work in piecewise-stationary bandits has explored the idea of monitoring changes with a change-point detector. The detection works by examining differences in distributions [23] or empirical means [6]. Such algorithms have state-of-the-art theoretical and empirical performance, and can be extended with similar guarantees to the contextual case [24, 31].

Prior work in non-stationary off-policy learning has only dealt with evaluation of a fixed target policy. They use methods such as time-series forecasting of future values [29], or passively reweighing past observations [18]. There is also orthogonal work in offline evaluation of history-dependent policies in stationary environments [21, 11]. We are the first to provide a comprehensive method for both off-policy optimization and online policy selection in piecewise-stationary environments.

8 Conclusions

In this work, we take the first steps in off-policy optimization when the environment is piecewise-stationary. We propose algorithms that partition the offline dataset by latent state, and optimize latent sub-policies conditioned on the partitions. We provide two techniques to partition the data – change-point detector and HMM. We prove high-probability bounds on both the quality of off-policy optimized sub-policies, and regret during online deployment. Finally, we empirically validate our approach in a synthetic dataset and in the Yahoo! dataset.
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A Proofs for Offline Policy Optimization

Lemma 1. For any \( \pi \in \mathcal{H}_M^Z \) and \( \delta \in (0, 1] \), \( |\hat{V}(\pi) - \hat{V}(\pi)| \leq M \varepsilon(T, \delta) \) holds with probability at least \( 1 - \delta \).

Proof. The claim is proved as
\[
|\hat{V}(\pi) - \hat{V}(\pi)| \leq \sum_{t=1}^{T} \frac{\pi(a_t | x_t, \hat{z}_t)}{B_t} r_t - \frac{\pi(a_t | x_t, z_t)}{B_t} r_t \leq M \sum_{t=1}^{T} 1[\hat{z}_t \neq z_t]
\]
holds with probability at least \( 1 - \delta \).

The second inequality is by \( \pi \in \mathcal{H}_M^Z \). The third inequality is by Assumption 1 and holds with probability at least \( 1 - \delta \).

Lemma 2. For any \( \pi \in \mathcal{H}_M^Z \), logged data \( D \), and \( \delta \in (0, 1] \), \( |\hat{V}(\pi) - V(\pi)| \leq 2M \sqrt{T \log(2/\delta)} \) holds with probability at least \( 1 - \delta \).

Proof. We define a martingale sequence \( \{U_t | t \in [T] \cup \{0\}\} \) over rounds \( t \) and then use Azuma’s inequality. Let \( U_0 = 0 \) and
\[
U_t = U_{t-1} + \frac{\pi(a_t | x_t, z_t)}{B_t} r_t - V_t(\pi)
\]
for \( t > 0 \). It is easy to verify that this is a martingale. In particular, since
\[
\mathbb{E}_{x_t, a_t, r_t \sim P_{xt}, \pi_0} \left[ \frac{\pi(a_t | x_t, z_t)}{B_t} r_t - V_t(\pi) \middle| U_0, \ldots, U_{t-1} \right] = \mathbb{E}_{x_t, a_t, r_t \sim P_{xt}, \pi_0} [r_t - V_t(\pi)] = 0,
\]
we have \( \mathbb{E} [U_t | U_0, \ldots, U_{t-1}] = U_{t-1} \) for all rounds \( t \). Also, since \( \pi \in \mathcal{H}_M^Z \), we have
\[
\left| \frac{\pi(a_t | x_t, z_t)}{B_t} r_t - V_t(\pi) \right| \leq M.
\]
Finally, by Azuma’s inequality, we get
\[
\mathbb{P} \left( |\hat{V}(\pi) - V(\pi)| \geq 2M \sqrt{T \log(2/\delta)} \right) = \mathbb{P} \left( |U_T - U_0| \geq 2M \sqrt{T \log(2/\delta)} \right) \leq 2 \exp \left[ -\frac{4M^2 T \log(2/\delta)}{2M^2 T} \right] \leq \delta.
\]
This concludes the proof.

Lemma 3. For any \( \pi \in \mathcal{H}_M^Z \) and \( \delta_1, \delta_2 \in (0, 1] \), \( |\hat{V}(\pi) - V(\pi)| \leq M \varepsilon(T, \delta_1) + 2M \sqrt{T \log(2/\delta_2)} \) holds with probability at least \( 1 - \delta_1 - \delta_2 \).

Proof. We have,
\[
|\hat{V}(\pi) - V(\pi)| \leq |\hat{V}(\pi) - \hat{V}(\pi)| + |\hat{V}(\pi) - V(\pi)|,
\]
from the triangle inequality. The result follows from Lemma 2 and Lemma 3.

Theorem 1. Let
\[
\hat{\pi} = \arg\max_{\pi \in \mathcal{H}_M^Z} \hat{V}(\pi), \quad \pi^* = \arg\max_{\pi \in \mathcal{H}_M^Z} V(\pi)
\]
be the optimal latent policies w.r.t. the off-policy estimated value and the true value respectively. Then for any \( \delta_1, \delta_2 \in (0, 1] \), we have that
\[
V(\hat{\pi}) \geq V(\pi^*) - 2M \varepsilon(T, \delta_1/2) - 4M \sqrt{T \log(4/\delta_2)}
\]
holds with probability at least \( 1 - \delta_1 - \delta_2 \).
Proof. We have,

\[
V(\pi^*) - V(\bar{\pi}) = \left[ V(\pi^*) - \bar{V}(\bar{\pi}) \right] + \left[ \bar{V}(\bar{\pi}) - V(\bar{\pi}) \right]
\]

where the inequality comes from \( \bar{\pi} \in \mathcal{H}_M^Z \) maximizing \( \bar{V} \). Applying Lemma 4 on any \( \pi \in \mathcal{H}_M^Z \) yields,

\[
|\bar{V}(\pi) - V(\pi)| \leq M \varepsilon(T, \delta_1/2) + 2M \sqrt{T \log(4/\delta_2)},
\]

holds with probability \( 1 - \frac{\delta_1}{2} - \frac{\delta_2}{2} \). Doing so on both \( \bar{\pi} \) and \( \pi^* \) yields the desired result. \( \square \)

B Proofs for Online Regret

Recall that we have a mixture-of-experts algorithm \( B \) and experts/sub-policies \( \bar{\pi} = (\bar{\pi})_{z \in Z} \), such that for each round \( t \), actions are sampled according to \( a_t \sim B_t(x_t, \bar{\pi}) \). Let \( B \) be EXP4.S as described in Algorithm 4; this is similar to one proposed in Luo et al. [24], but for stochastic experts. Our first result is the following regret guarantee over any stationary segment,

**Lemma 6.** Let \( B \) be EXP4.S as in Algorithm 4. Also, let \( \gamma = 0, \eta = \sqrt{\log(k) / (LK)} \), and \( \beta = 1/k \). Then, for any stationary segment \( [\tau_{s-1}, \tau_s - 1] \) of length at most \( L \), and any latent state \( z \in Z \), the regret is bounded by,

\[
\sum_{t=\tau_{s-1}}^{\tau_s - 1} E_{x_t, \bar{\pi}_s}[r_t] - E_{x_t, B_t}[r_t] \leq \sqrt{2LK \log(k)}
\]

**Proof.** The proof of this is similar to that done by Luo et al. [24], except our EXP4.S allows for stochastic experts.

First, we have the following upper-bound,

\[
\log \left( \sum_{z' \in Z} w_t(z') \exp(-\eta \hat{c}_t(z')) \right) \leq \log \left( \sum_{z' \in Z} w_t(z') \left( 1 - \eta \hat{c}_t(z') + \eta^2 \hat{c}_t(z')^2 \right) \right)
\]

\[
\leq -\eta \sum_{z' \in Z} w_t(z') \hat{c}_t(z') + \eta^2 \sum_{z' \in Z} w_t(z') \hat{c}_t(z')^2,
\]

where we use that \( \exp(-x) \leq 1 - x + x^2 \), and \( \log(1 + x) \leq x \) for all \( x \geq 0 \). Meanwhile, for any \( z \in Z \), we can also bound the same quantity from below,

\[
\log \left( \sum_{z' \in Z} w_t(z') \exp(-\eta \hat{c}_t(z')) \right) = \log \left( \frac{w_t(z) \exp(-\eta \hat{c}_t(z))}{\hat{w}_{t+1}(z)} \right)
\]

\[
= \log \left( \frac{w_t(z)(1 - \beta)}{w_{t+1}(z) - \beta} \right) - \eta \hat{c}_t(z)
\]

\[
\geq \log \left( \frac{w_t(z)}{w_{t+1}(z)} \right) - 2\beta - \eta \hat{c}_t(z),
\]

where for the last inequality, we use that \( \log(1 - \beta) \geq -\beta/(1 - \beta) \geq -2\beta \). Combining the two inequalities, summing over all \( t \in [\tau_{s-1}, \tau_s - 1] \), and telescoping yields,

\[
\sum_{t=\tau_{s-1}}^{\tau_s - 1} \sum_{z' \in Z} w_t(z') \hat{c}_t(z') - \hat{c}_t(z) \leq \frac{1}{\eta} \log \left( \frac{w_{\tau_s}(z)}{w_{\tau_{s-1}}(z)} \right) + \frac{2\beta L}{\eta} + \eta \sum_{t=\tau_{s-1}}^{\tau_s - 1} \sum_{z' \in Z} w_t(z') \hat{c}_t(z')^2
\]

\[
\leq \frac{\log(1/\beta) + 2\beta L}{\eta} + \eta \sum_{t=\tau_{s-1}}^{\tau_s - 1} \sum_{z' \in Z} w_t(z') \hat{c}_t(z')^2,
\]

where we use that \( w_t(z) \in [\beta, 1] \) for all rounds \( t \).
When $\gamma = 0$ we know that $\hat{c}_t(a_t)$ is unbiased, or $E_{z_t, B_t} [\hat{c}_t(a_t)] = 1 - E_{z_t, B_t} [r_t]$. We also have that for any $z' \in \mathcal{Z}$,

$$E_{z_t, B_t} [\hat{c}_t(z')] = E_{z_t, B_t} \left[ \sum_{a \in A} \tilde{\pi}_{z'}(a | x_t) \hat{c}_t(a) \right] = 1 - E_{z_t, \tilde{\pi}_x} [r_t].$$

Taking the expectation of both sides leads to,

$$\sum_{t=1}^{T-1} E_{z_t, \tilde{\pi}_x} [r_t] - E_{z_t, B_t} [r_t] \leq \frac{\log(1/\beta) + 2\beta L}{\eta} + \eta \sum_{t=1}^{T-1} \sum_{z' \in \mathcal{Z}} \sum_{t=1}^{z'} E_{z_t, B_t} [w_t(z') \hat{c}_t(z')^2].$$

Next, we have that for any $z' \in \mathcal{Z}$,

$$E_{z_t, B_t} [\hat{c}_t(z')^2] = E_{z_t, B_t} \left[ \left( \frac{\hat{\pi}_{z'}(a_t | x_t)(1-r_t)}{B_t(a_t)} \right)^2 \right] \leq \sum_{a \in A} \left( \frac{1}{B_t(a)} \sum_{z' \in \mathcal{Z}} w_t(z') \pi_{z'}(a_t | x_t) \right),$$

where we use that $a_t \sim B_t$ and $r_t \in [0,1]$. Substituting this result yields,

$$\sum_{z' \in \mathcal{Z}} \sum_{t=1}^{T-1} E_{z_t, B_t} [w_t(z') \hat{c}_t(z')^2] \leq \sum_{a \in A} \left( \frac{1}{B_t(a)} \sum_{z' \in \mathcal{Z}} w_t(z') \pi_{z'}(a_t | x_t) \right) \leq K,$

where we again use that $a_t \sim B_t$. Substituting into the regret bound and using the values for $\eta, \beta$ yields

$$\sum_{t=1}^{T-1} E_{z_t, \tilde{\pi}_x} [r_t] - E_{z_t, B_t} [r_t] \leq \frac{\log(1/\beta) + 2\beta L}{\eta} + \eta KL \leq \sqrt{2LK \log(k)},$$

as desired. \hfill \Box

In practice, we do not know the lengths of stationary segments, and may not be able to find a tight upper-bound $L$ on the lengths of stationary segments. However, in our analysis, we can further partition stationary segments so that they do not exceed length $L$ at the cost of increasing the number of change-points. This is formalized in the following corollary:

**Lemma 7.** Let $B$ be EXP4.S as in Algorithm 4. Also, let $\gamma = 0, \eta = \sqrt{\log(k)/(LK)}$, and $\beta = 1/k$. Then, the total regret is bounded by,

$$\sum_{s=1}^{S} \max_{z \in \mathcal{Z}} \sum_{t=1}^{T} E_{z_t, \bar{\pi}_s} [r_t] - \sum_{t=1}^{T} E_{z_t, B_t} [r_t] \leq \left( \frac{T}{\sqrt{L}} + S \sqrt{L} \right) \sqrt{2K \log(k)}.$$

**Proof.** First, we divide the $T$ rounds equally into $T/L$ intervals. Then, we additionally divide intervals that contain change-points, so that each interval has a distinct latent state and has length bounded by $L$. This leads to at most $T/L + S$ stationary segments. Then, we can use Lemma 6 on each interval and sum the regrets to get the desired result. \hfill \Box

**Theorem 2.** Let $\bar{\pi}$ and $\pi^*$ be defined as in Theorem 1, and $B$ be EXP4.S in Algorithm 4. For horizon $T$, let $z_{1:T}$ be the same underlying latent states as in the offline dataset, and let $S$ be the number of stationary segments. Then for any $\delta_1, \delta_2 \in (0,1)$, we have that

$$R(T; B, \bar{\pi}) \leq 2M \varepsilon(T, \delta_1/2) - 4M \sqrt{T \log(4/\delta_2)} + 2 \sqrt{STK \log(k)},$$

holds with probability at least $1 - \delta_1 - \delta_2$.

**Proof.** We have the following regret decomposition,

$$R(T; B, \bar{\pi}) = \sum_{t=1}^{T} E_{z_t, \pi^*_{z_t}} [r_t] - \sum_{t=1}^{T} E_{z_t, B_t} [r_t]$$

$$= \sum_{t=1}^{T} E_{z_t, \pi^*_{z_t}} [r_t] - \sum_{t=1}^{T} E_{z_t, \tilde{\pi}_x} [r_t] + \sum_{t=1}^{T} E_{z_t, \tilde{\pi}_x} [r_t] - \sum_{t=1}^{T} E_{z_t, B_t} [r_t].$$
where we introduce \( \hat{\pi} \) playing according to the true latent state. Then, using that there are \( S \) stationary segments, we have,

\[
= \sum_{t=1}^{T} E_{z_t, \pi_{t-1}[r_t]} - \sum_{t=1}^{T} E_{z_t, \hat{\pi}_{t-1}[r_t]} + \sum_{s=1}^{S} \sum_{t=\tau_{s-1}}^{\tau_s-1} E_{z_t, \pi_{t-1}[r_t]} - \sum_{t=1}^{T} E_{z_t, B_t[r_t]},
\]

The first term can be bounded using our offline analysis, which shows near-optimality of EXP4.S, and is bounded by choosing \( L \) and is bounded by Theorem 1 w.p. at least \( t \) latent state known. In the case where \( \pi \), we have,

\[
\text{Proof.}
\]

\[ \text{Fix } s = \tau_i. \text{ From } s \in W_i, \text{ we have,} \]

\[
\text{Lemma 4. For any round } t \notin W, \text{ the probability of a false detection is bounded from above as}
\]

\[ \mathbb{P} \left( \left| \mu_t^s - \mu_t^+ \right| \geq c \right) \leq 4 \exp \left( -\frac{wc^2}{2} \right). \]

\[ \text{Proof.} \text{ Since } t \notin \bigcup W_i, \text{ we have } \mathbb{E} \left[ \mu_t^s \right] = \mathbb{E} \left[ \mu_t^+ \right]. \text{ By Hoeffding’s inequality, we get}
\]

\[ \mathbb{P} \left( \left| \mu_t^s - \mu_t^+ \right| \geq c \right) \leq \mathbb{P} \left( \left| \mu_t^s - \mathbb{E} \left[ \mu_t^s \right] \right| \geq c/2 \right) + \mathbb{P} \left( \left| \mu_t^+ - \mathbb{E} \left[ \mu_t^+ \right] \right| \geq c/2 \right) \leq \exp \left( -\frac{wc^2}{2} \right). \]

This concludes the proof. \( \square \)

\[ \text{Lemma 5. For any positive } c \leq \Delta/2 \text{ and } W_i, \text{ a change-point is not detected in } W_i \text{ with probability at most}
\]

\[ \mathbb{P} \left( \forall t \in W_i : \left| \mu_t^s - \mu_t^+ \right| \leq c \right) \leq 4 \exp \left( -\frac{wc^2}{2} \right). \]

\[ \text{Proof.} \text{ Fix } s = \tau_i. \text{ From } s \in W_i, \text{ we have,} \]

\[ \mathbb{P} \left( \forall t \in W_i : \left| \mu_t^s - \mu_t^+ \right| \leq c \right) = 1 - \mathbb{P} \left( \exists t \in W_i : \left| \mu_t^s - \mu_t^+ \right| > c \right) \leq 1 - \mathbb{P} \left( \mu_t^s - \mu_t^+ > c \right)
\]

\[ = \mathbb{P} \left( \left| \mu_t^s - \mu_t^+ \right| \leq c \right). \]

Note that \( \left| \mu_t^s - \mu_t^+ \right| \leq c \) implies that either \( \mu_t^s \) or \( \mu_t^+ \) is not close to its mean. More specifically, since \( \mathbb{E} \left[ \mu_t^s \right] = V_{s-1}(\pi_0), \mathbb{E} \left[ \mu_t^+ \right] = V_s(\pi_0), \) and \( V_s(\pi_0) - V_{s-1}(\pi_0) \geq \Delta, \) we have

\[ \mathbb{P} \left( \left| \mu_t^s - \mu_t^+ \right| \leq c \right) \leq \mathbb{P} \left( \left| \mu_t^s - \mathbb{E} \left[ \mu_t^s \right] \right| \geq \frac{\Delta - c}{2} \right) + \mathbb{P} \left( \left| \mu_t^+ - \mathbb{E} \left[ \mu_t^+ \right] \right| \geq \frac{\Delta - c}{2} \right). \]

From \( 2c \leq \Delta \) and by Hoeffding’s inequality, the first term is bounded as

\[ \mathbb{P} \left( \left| \mu_t^s - \mathbb{E} \left[ \mu_t^s \right] \right| \geq \frac{\Delta - c}{2} \right) \leq \mathbb{P} \left( \left| \mu_t^s - \mathbb{E} \left[ \mu_t^s \right] \right| \geq c/2 \right) \leq 2 \exp \left( -\frac{wc^2}{2} \right). \]

The second term is bounded analogously. Finally, we chain all inequalities and get our claim. \( \square \)

\[ \text{Theorem 3. Let } \tau_i - \tau_{i-1} > 4w \text{ for all } i \in [k]. \text{ For any } \delta \in (0, 1], \text{ and } c \text{ and } w \text{ in Algorithm 3 such that,}
\]

\[ \frac{\Delta}{2} \geq c \geq \sqrt{\frac{2 \log(8T/\delta)}{w}}. \]

\[ \text{Algorithm 3 predicts } \hat{z}_{1:T} \text{ such that } \sum_{t=1}^{T} 1[\hat{z}_t \neq z_t] \leq kw \text{ holds with probability at least } 1 - \delta. \]
Proof. Define $\delta \in (0, 1]$. We see that given $w$, setting $c$ as described satisfies,

$$4T \exp \left[ -\frac{wc^2}{2} \right], \quad 4k \exp \left[ -\frac{wc^2}{2} \right] \leq \frac{\delta}{2}.$$  

We know that $\varepsilon(T, \delta) = kw$ when all the estimated changepoints are in $W$ (at most $w$ rounds from a true change-point), and every $W_i \in W$ contains exactly one estimated change-point. This cannot happen if (1) a change-point is falsely detected outside $W$, and (2), no change-point is detected in some $W_i \in W$.

We can bound from above the probability of any error occurring with the union bound. Lemma 5 applied to every round upper-bounds the probability of (1) by $4T \exp \left( -\frac{wc^2}{2} / 2 \right)$. Meanwhile, Lemma 6 applied to every change-point upper-bounds the probability of (2) by $4k \exp \left( -\frac{wc^2}{2} / 2 \right)$. From Algorithm 3, we remove a $4w$-window around each detected changepoint, and under the assumption that $\tau_i - \tau_{i-1} > 4w$ for all $i \in [k]$, we guarantee that exactly one changepoint is detected in each $W_i$ for true changepoint $\tau_i$. Combining yields the total probability of an error,

$$4T \exp \left[ -\frac{wc^2}{2} \right] + 4k \exp \left[ -\frac{wc^2}{2} \right] \leq \delta,$$

which is the desired result.  \qed
D Pseudocode of Change-point Detector

Algorithm 3: Change-point Detector

Input: \( w \in \mathbb{N} \), detection threshold \( c \in \mathbb{R}^+ \), and logged data \( D \)

for \( t \leftarrow 1 \) to \( T \) do
  \( \mu_t^- \leftarrow w^{-1} \sum_{i=t-w}^{t-1} r_i \)
  \( \mu_t^+ \leftarrow w^{-1} \sum_{i=t}^{t+w-1} r_i \)
end

Initialize candidates \( S \leftarrow \{ t : |\mu_t^- - \mu_t^+| \geq c \} \)

while \( S \neq \emptyset \) do
  Find change-point \( \hat{\tau} \leftarrow \arg \max_{t \in S} \{ |\mu_t^- - \mu_t^+| \} \)
  \( S \leftarrow S \setminus \{ t \in S : t \in [\hat{\tau}-2w, \hat{\tau}+2w] \} \)
end

for \( t \leftarrow 1 \) to \( T \) do \( \hat{z}_t \leftarrow i \) such that \( t \in [\hat{\tau}_{i-1}, \hat{\tau}_i - 1] \)

E Pseudocode of Mixture-of-experts Algorithms

Algorithm 4: EXP4.S

Input: \( \pi \): vector of experts \(( \pi_z )_{z \in Z} \) with \( |Z| = k \).
\( \beta, \eta > 0, \gamma \in (0, 1) \): hyperparameters

Initialize \( w_1 = (1/k, \ldots, 1/k) \in [0, 1]^k \).

for \( t \leftarrow 1, 2, \ldots, T \) do
  Observe \( x_t \), and expert feedback \( \tilde{\pi}_z ( \cdot \mid x_t ) \), \( \forall z \in Z \).
  Choose \( a_t \sim B_t \), where for each \( a \in A \),

  \[
  B_t(a) = (1 - \gamma) \sum_{z \in Z} w_t(z) \pi_z(a \mid x_t) + \frac{\gamma}{k}.
  \]

  Observe \( r_t \). Estimate the action costs under full feedback \( \hat{c}_t(a) = \mathbb{I}[a_t = a] \frac{1-r_t}{B_t(a)} \), \( \forall a \in A \).
  Propagate the cost to the experts \( \hat{c}_t(z) = \hat{c}_t(a_t) \pi_z(a_t \mid x_t) \), \( \forall z \in Z \).
  Update the distribution weights, \( \hat{w}_{t+1}(z) \propto w_t(z) \exp(-\eta \hat{c}_t(z)) \), \( \forall z \in Z \).
  Mix with uniform weights, \( w_{t+1}(z) = (1 - \beta) w_t(z) + \beta, \forall z \in Z \).
end

Algorithm 5: HMM Thompson Sampling

Input: \( \hat{\pi} \): vector of experts \(( \hat{\pi}_z )_{z \in Z} \) with \( |Z| = k \).
\( \hat{\pi} = (\hat{P}_0, \hat{A}, \hat{\beta}) \): estimated HMM parameters.

Initialize \( w_1 = \hat{P}_0 \).

for \( t \leftarrow 1, 2, \ldots, T \) do
  Observe \( x_t \in X \), and expert feedback \( \hat{\pi}_z ( \cdot \mid x_t ) \), \( \forall z \in Z \).
  Choose action \( a_t \sim B_t \), where for each \( a \in A \),

  \[
  B_t(a) = \sum_{z \in Z} w_t(z) \pi_z(a \mid x_t).
  \]

  Observe \( r_t \). Update the distribution weights, \( \forall z \in Z \),

  \[
  w_{t+1}(z) \propto \sum_{z' \in Z} w_t(z') P(r_t \mid x_t, a_t, z'; \hat{\beta}) P(z \mid z'; \hat{A})
  \]
end