Entanglement transformation at absorbing and amplifying four-port devices

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March 31, 2022

Abstract

Quantum communication schemes widely use dielectric four-port devices as basic elements for constructing optical quantum channels. Since for causality reasons the permittivity is necessarily a complex function of frequency, dielectrics are typical examples of noisy quantum channels in which quantum coherence will not be preserved. Basing on quantization of the phenomenological electrodynamics, we construct the transformation relating the output quantum state to the input quantum state without placing frequency restrictions. Knowledge of the full transformed quantum state enables us to compute the entanglement contained in the output quantum state. We apply the formalism to some typical examples in quantum communication.

1 Introduction

Quantum communication experiments widely use dielectric four-port devices such as beam splitters or optical fibers as basic elements for constructing optical quantum channels. Since any frequency-dependent dielectric function describing an optical element, by virtue of the Kramers-Kronig relations, is necessarily a complex function of frequency, absorption is always present which leads to well-known phenomena as decoherence and entanglement degradation. In order to study the problem, quantization of the electromagnetic field in the presence of dielectric media is needed. A consistent formalism of quantum electrodynamics in absorbing media is reviewed in [1]. It is based on the Green function expansion of the electromagnetic field with respect to the fundamental variables of the system composed of the field, the dielectric matter and the reservoir. All relevant information about the dielectric and geometric properties are contained in the classical Green function of the corresponding scattering problem.

The formalism is especially suited for deriving input-output relations of the field at dielectric slabs [2] on the basis of measurable quantities as transmission and absorption coefficients. From the input-output relations we can then derive closed formulas for calculating the output quantum state from the (known) input quantum state [3]. That is, the complete density matrix after the transformation is known, which makes the theory most suitable for studying entanglement properties of quantum states of light. The theory has also been extended to cover amplifying media.

In this article we will proceed as follows. The quantum-state transformation at dielectric four-port devices is shortly reviewed in Sec. 2. An application to entanglement degradation of Bell states as well as the derivation of separability criteria for the two-mode squeezed vacuum state are given in Sec. 3 followed by a summary in Sec. 4.
2 Quantum-state transformation

We shortly review the basic formulas needed for the following considerations. Suppose the electromagnetic field has already been quantized and the Green function for a four-port device has been rewritten in the form of transmission and absorption matrices \( T(\omega) \) and \( A(\omega) \) \[2\]. The amplitude operators of the incoming and outgoing damped waves at frequency \( \omega \), \( \hat{a}_i(\omega) \) and \( \hat{b}_i(\omega) \), respectively, can then be connected by the quantum-optical input-output relations in the following way:

\[
\begin{pmatrix}
\hat{b}_1(\omega) \\
\hat{b}_2(\omega)
\end{pmatrix} = \begin{pmatrix}
T(\omega) & \Lambda(\omega)
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1(\omega) \\
\hat{a}_2(\omega)
\end{pmatrix}, \tag{1}
\]

with

\[
T(\omega)T^+(\omega) + \sigma A(\omega)A^+(\omega) = I, \tag{2}
\]

where \( \hat{a}_i(\omega) \) represent either device annihilation operators \( \hat{b}_i(\omega) \) for absorbing devices (\( \sigma = +1 \)) or creation operators \( \hat{b}_i^\dagger(\omega) \) for amplifying devices (\( \sigma = -1 \)). Equation (2) is nothing but current (or energy) conservation. The formulas are valid for any chosen frequency.

By defining the “four-vector” operators

\[
\hat{\alpha}(\omega) = \begin{pmatrix}
\hat{a}(\omega) \\
\hat{b}(\omega)
\end{pmatrix}, \quad \hat{\beta}(\omega) = \begin{pmatrix}
\hat{a}^\dagger(\omega) \\
\hat{b}^\dagger(\omega)
\end{pmatrix}, \tag{3}
\]

where \( \hat{f}(\omega) = \hat{h}(\omega) \) for an absorbing device, and \( \hat{f}(\omega) = \hat{h}^\dagger(\omega) \) for an amplifying device, with \( \hat{h}(\omega) \) being some auxiliary bosonic (“two-vector”) operator. The input-output relation \[2\] can then be extended to the four-dimensional transformation

\[
\hat{\beta}(\omega) = \Lambda(\omega)\hat{\alpha}(\omega) \tag{4}
\]

with

\[
\Lambda(\omega)J\Lambda^+(\omega) = J, \quad J = \begin{pmatrix}
I & 0 \\
0 & \sigma I
\end{pmatrix}. \tag{5}
\]

The matrix \( \Lambda(\omega) \) is either an element of the compact group SU(4) (for absorbing devices) or of the non-compact group SU(2,2) (for amplifying devices). By introducing the (commuting) positive Hermitian matrices

\[
C(\omega) = \sqrt{T(\omega)T^+(\omega)}, \quad S(\omega) = \sqrt{A(\omega)A^+(\omega)}, \tag{6}
\]

which, by Eq. \[2\], obey the relation \( C^2(\omega) + \sigma S^2(\omega) = I \), the matrix \( \Lambda(\omega) \) can be written in the form \[3\]

\[
\Lambda(\omega) = \begin{pmatrix}
T(\omega) & A(\omega) \\
-\sigma S(\omega)C^{-1}(\omega)T(\omega) & C(\omega)S^{-1}(\omega)A(\omega)
\end{pmatrix}. \tag{7}
\]

It can then be shown that the density operator of the quantum state of the outgoing field is given by

\[
\rho_{\text{out}}^{(D)} = \text{Tr}^{(D)}\{\hat{\rho}_{\text{in}}[J\Lambda^+(\omega)J\hat{\alpha}(\omega), J\Lambda^T(\omega)J\hat{\alpha}^\dagger(\omega)]\}, \tag{8}
\]

where \( \text{Tr}^{(D)} \) means trace with respect to the device. Equivalently, the Wigner function (for arbitrary \( s \)-parametrized phase-space functions, see \[3\]) transforms as

\[
W_{\text{out}}[\alpha(\omega)] = W_{\text{in}}[J\Lambda^+(\omega)J\alpha(\omega)]. \tag{9}
\]

3 Entanglement transformation

Let us make first a general remark on entanglement transformation. One of the requirements to be satisfied by any entanglement measure \( E \) is that a CP map does not increase entanglement \[3\], that is,

\[
E\left(\sum_i \hat{V}_i\hat{\varrho}_i\hat{V}_i^\dagger\right) \leq E(\hat{\varrho}), \quad \sum_i \hat{V}_i^\dagger\hat{V}_i = I. \tag{10}
\]

Looking at Eq. \[8\], we see that a quantum-state transformation is in fact a CP map for both absorbing and amplifying four-port devices because in both cases an ancilla (the device) is coupled to the Hilbert space of the field, a unitary transformation is performed in the product space \( \mathcal{H}_{\text{field}} \otimes \mathcal{H}_{\text{device}} \), and the trace with respect to the device is taken at the end. An obvious consequence is that amplification does not help to increase entanglement.

Now we turn to the problem of transmitting two light beams prepared in an entangled quantum state through absorbing optical fibers represented by their transmission coefficients \( T_i \) (Fig. \[1\]). As an example let us consider the two types of Bell states

\[
|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \tag{11}
\]

\[
|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{12}
\]
Applying Eq. (8), after some algebra we get

\[ E(\sigma) = \frac{1}{4} \left[ (2 - |T_1|^2 - |T_2|^2)|00\rangle\langle 00| + \frac{1}{2} (T_2|01\rangle + T_1|10\rangle) (T_2^*|01\rangle + T_1^*|10\rangle), \right. \]

\[ \hat{\rho}^{(F)}(|\Psi^\pm\rangle) = \frac{1}{2} \left[ (1-|T_1|^2)(1-|T_2|^2)|00\rangle\langle 00| + |T_1|^2(1-|T_2|^2)|10\rangle\langle 10| + |T_2|^2(1-|T_1|^2)|01\rangle\langle 01| + \frac{1}{2} (|00\rangle \pm T_1 T_2|11\rangle) (|00\rangle \pm (T_1 T_2)^*|11\rangle). \]

The density matrices in Eqs. (13) and (14) have already been written as a sum of separable states and a single pure state such that the convexity property (see, e.g., [7])

\[ E[\lambda \hat{a}_1 + (1 - \lambda) \hat{a}_2] \leq \lambda E(\hat{a}_1) + (1 - \lambda) E(\hat{a}_2) \quad (15) \]

can readily be used. Since the entanglement of a pure state is given by its reduced von Neumann entropy, the inequality (15) reduces for equal fibers to

\[ E[\hat{\rho}^{(F)}(|\Psi^\pm\rangle)] \leq |T|^2 \ln 2, \]

\[ E[\hat{\rho}^{(F)}(|\Phi^\pm\rangle)] \leq \frac{1}{2} \left[ 1 + |T|^2 \right] \ln \left[ 1 + |T|^2 \right] - |T|^2 \ln |T|^2. \quad (17) \]

The numerically calculated relative entropies of the output quantum states (13) and (14) are shown in Fig. 2 for equal transmission coefficients of the fibers satisfying the Lambert-Beer law \( T = e^{i\omega l/c} = e^{i n_R(\omega)/c_e L} \) with \( l \) and \( L = c/(n_l \omega) \) being, respectively, the propagation length through the fibers and the absorption length of the fibers with complex refractive index \( n(\omega) = n_R(\omega) + i n_I(\omega) \). It is seen that the entanglement of the states \( \hat{\rho}^{(F)}(|\Phi^\pm\rangle) \) decays considerably faster than that of the states \( \hat{\rho}^{(F)}(|\Psi^\pm\rangle) \), which can be understood from the argument that in the former case absorption acts on both photons simultaneously. Since the states considered here live in Hilbert spaces of dimension 2×2 it would also be possible to compute the exact amount of entanglement by applying the theorem of Lewenstein and Sanpera [7], which states that in that case the equality sign in (15) is realized for a decomposition of the density matrix in the form \( \lambda \hat{\rho}_{\text{wep}} + (1 - \lambda) \hat{\rho}_{\text{pure}} \) with maximal \( \lambda \).

As a second example, let us consider the two-mode squeezed vacuum state (TMSV)

\[ |\text{TMSV} \rangle = \exp[\zeta (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2)]|00\rangle \]

\[ = \sqrt{1 - q^2} \sum_{n=0}^{\infty} q^n |nn\rangle \quad (18) \]

\([q = \tanh \zeta] \), which can be used in continuous-variable quantum teleportation [1]. Being difficult to follow the lines presented above for the Bell states, it is instructive to apply the separability criterion in Refs. [1], [10] to this class of states. Since the criterion is based on properties of the Wigner function for Gaussian states, we can readily use Eq. (1) to obtain...
a relation for the bound between separability and inseparability. Assuming again equal optical fibers with transmission and reflection coefficients $T$ and $R$ and some thermal photon number $n_{th}$, we obtain

$$n_{th} \geq \frac{(1-\sigma)(1-|R|^2)+|T|^2(\sigma-\exp[-2|\zeta|])}{2\sigma(1-|R|^2-|T|^2)}.$$  \hspace{1cm} (19)

Specifically in the absorbing case ($\sigma = +1$) we arrive at the formula for the maximal fiber length $l_{\text{max}}$ after which the TMSV is still nonseparable

$$l_{\text{max}} = \frac{1}{2} \ln \left[ 1 + \frac{1}{2n_{th}} (1 - \exp[-2|\zeta|]) \right],$$  \hspace{1cm} (20)

where we have set $R = 0$ and, according to the Lambert-Beer law, $|T| = \exp[-l/L]$, with $L$ being again the absorption length. On the other hand, from Eq. (19) it follows that for an amplifier in the low-temperature limit ($n_{th} = 0$) the maximal square of the absolute value of the transmission coefficient, $|T_{\text{max}}|^2$, which corresponds to the maximal gain $g_{\text{max}} = |T_{\text{max}}|^2 - 1$, is given by

$$|T_{\text{max}}|^2 = \frac{2(1-|R|^2)}{1 + \exp[-2|\zeta|]},$$  \hspace{1cm} (21)

which for $R = 0$ reduces to

$$|T_{\text{max}}|^2 - 1 = \tanh |\zeta| = |q|.$$  \hspace{1cm} (22)

It essentially says that an amplifier that doubles the intensity of a signal destroys any initially given entanglement.

4 Conclusions

We have shown how the quantum-optical input-output relations at absorbing and amplifying four-port devices, which follow from quantum electrodynamics in linear, causal media, can easily be used to derive bounds on inseparability lengths for optical fields prepared in Gaussian states and transmitted through noisy quantum channels such as absorbing or amplifying fibers. For finite dimensional Hilbert spaces the quantum-state transformation can be used to derive upper bounds on the entanglement content of a given quantum state of the incoming field. An obvious statement is that amplifying four-port devices are not able to increase entanglement since they represent a CP map as in the case of absorbing devices.

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