BINOMIAL IDEALS ATTACHED TO FINITE COLLECTIONS OF CELLS

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ABSTRACT. We consider the ideal of inner 2-minors $I_P$ of a finite set of cells $P$, which we call the cell ideal of $P$. A nice interpretation for the height of an unmixed ideal $I_P$, in terms of the number of cells of $P$ is given. Moreover, the coordinate rings of cell ideals with isolated singularities are determined.

INTRODUCTION

Combinatorial descriptions of height of polyomino ideals have been studied in several works. Qureshi [4] proved that for a convex polyominoe $P$ the height of the polyomino ideal $I_P$ is the number of cells of $P$. Herzog and Madani [14] extended this result to simple polyominoes, which by definition are the polyominoes with no holes, see [3] and [5]. Such polyomino ideals are in particular prime. However, not all polyomino ideals are prime ideals and it is still an open problem to identify the polyominoes $P$ for which $I_P$ is a prime ideal. In [1] the same description for height in terms of the number of cells of $P$ was proved for closed path polyominoes. In this paper we consider more generally cell ideals, i.e., ideals of inner 2-minors which are attached to finite collections of cells. When any two cells of $P$ are connected in $P$, this ideal is just the polyomino ideal. In Theorem 1.1 it is shown that height $I_P \leq c \leq \text{bigheight} I_P$, where $c$ is the number of cells of $P$. In particular, if $I_P$ is an unmixed ideal, then height $I_P = c$. To this aim we use Lemma 1.2 which determines the height of an unmixed binomial ideal $I \subset S$ in terms of the dimension of the $\mathbb{Q}$-vector space spanned by the set of integer vectors \{ $v - w \in \mathbb{Q}^n : x^v - x^w \in I$ \}.

In the next section of this paper it is shown that when $K$ is a perfect field, and $P$ is a finite set of cells such that $I_P \subset S$ is a prime ideal, then the ring $S/I_P$ has an isolated singularity if and only if $P$ is an inner interval.

1. ON THE HEIGHT OF CELL IDEALS

Consider $(\mathbb{Z}^2, \leq)$ as a partially ordered set with $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. Let $a, b \in \mathbb{Z}^2$. Then the set $[a, b] = \{ c \in \mathbb{Z}^2 : a \leq c \leq b \}$ is called an interval. The interval with $a = (i, j)$ and $b = (i', j')$ is called proper, if $i < i'$ and $j < j'$. A cell is an interval of the form $[a, b]$, where $b = a + (1, 1)$. The cell $C = [a, a + (1, 1)]$ consists of the elements $a, a + (0, 1), a + (1, 0)$ and $a + (1, 1)$, which are called the vertices

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Lemma 1.2. Let $I \subset S$ be a binomial ideal, and let $V_I$ be the $\mathbb{Q}$-vector space spanned by the set of integer vectors $\{v - w \in \mathbb{Q}^n : x^v - x^w \in I\}$. Then

$$\text{height } I \leq \dim_{\mathbb{Q}} V_I \leq \text{bighight } I.$$ 

In particular, height $I = \dim_{\mathbb{Q}} V_I$, if $I$ is unmixed.

Proof. Let $x = x_1 \cdots x_n$. Then $S_x = K[x_1^\pm, \ldots, x_n^\pm]$ is the Laurent polynomial ring, and we have height $I \leq \text{height } IS_x$. Hence, for the first inequality it suffices to show that $\text{height } IS_x \leq \dim_{\mathbb{Q}} V_I$. 

Note that

$$IS_x = (1 - x^v : v \in V_I).$$

We observe that

$$(1 - x^v) - (1 - x^w) = (x^v - x^w) = x^w (1 - x^{v-w}).$$

This shows that with $1 - x^v$ and $1 - x^w$, also $(1 - x^{v-w}) \in S_x$, since $x^w$ is a unit in $S_x$. Similarly, one sees that $(1 - x^{v_1+v_2}) \in S_x$. Hence the integer vectors $v$, which span $V_I$, form an abelian subgroup $G$ of $\mathbb{Z}^n$. Any abelian subgroup of $\mathbb{Z}^n$ is free. Let $v_1, \ldots, v_r$ be a basis of $G$. Then this basis is also a $\mathbb{Q}$-basis of $V_I$, and

$$IS_x = (1 - x^{v_1}, \ldots, 1 - x^{v_r}).$$
Now, we apply Krull's generalized principle ideal theorem, to deduce that height $IS_x \leq r = \dim \mathbb{Q} V_I$, as desired.

For the second inequality we notice that height $IS_x \leq \text{bigheight } I$. Thus it suffices to show that height $1 - x^{v_1}, \ldots, 1 - x^{v_r}$ of $\mathbb{Q} V$. Observe that $S_x$ can be identified with the group ring $K[Z^n]$, whose $K$-basis consists of all monomials $x^a$ with $a \in \mathbb{Z}^n$. By the elementary divisor theorem there exists a basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ and positive integers $a_1, \ldots, a_r$ such that $v_i = a_i e_i$ for $i = 1, \ldots, r$. In these coordinates

$$IS_x = (1 - x_1^{a_1}, \ldots, 1 - x_r^{a_r})S_x.$$ 

Now, consider the ideal $J = (1 - x_1^{a_1}, \ldots, 1 - x_r^{a_r})S_r$, where $S_r = K[x_1, \ldots, x_r]$. Let $R = S_r/J$. Since dim $R = 0$, it follows that $R[x_{r+1}, \ldots, x_n]$ is Cohen-Macaulay of dimension $n - r$, and since $R[x_{r+1}, \ldots, x_n] \cong S/JS$, this implies that $JS$ is an unmixed ideal of height $r$. Because $JS$ is unmixed, we then have

$$r = \text{height } JS = \text{height } JS_x = \text{height } IS_x,$$

as desired. \qed

Proof of Theorem 1.1. Note that $V_{I_p}$ is a subspace of the $\mathbb{Q}$-vector space $W := \mathbb{Q}^V(\mathcal{P})$. We denote by $v_a \in W$ the vector, whose $a$'s component is 1, while its other components are 0. The set of vectors $\{v_a : a \in V(\mathcal{P})\}$ is the canonical basis of $W$.

For each inner interval $[a, b]$ of $\mathcal{P}$ with anti-diagonals $c$ and $d$ we define the vector

$$v_{[a, b]} = v_a + v_b - v_c - v_d.$$

It follows from the definition of $V_{I_p}$ that the vectors $v_{[a, b]}$ span $V_{I_p}$.

If $C = [a, b]$ is a cell of $\mathcal{P}$, then we write $v_C$ for the vector $v_{[a, b]}$ and claim that the vectors $v_C$ form a $\mathbb{Q}$-basis of $V_{I_p}$. Together with Theorem 1.2 this claim implies the desired conclusion.

If $[a, b]$ is an arbitrary inner interval of $\mathcal{P}$, then it is readily seen that

$$v_{[a, b]} = \sum_C v_C,$$

where the sum is taken over all cells in $[a, b]$. This shows that the vectors $v_C$ generate $V_{I_p}$.

It remains to be shown that the set of vectors $v_C$ with $C$ a cell of $\mathcal{P}$ is linearly independent. For this purpose we choose any total order on $\mathbb{Z}^2$, extending the partial order $\leq$ on $\mathbb{Z}^2$ which is defined by componentwise comparison. We set $v_a \preceq v_b$ when $a \leq b$. Then for any cell $C = [a, b]$, the leading vector in the expression of $v_C$ is $v_b$. Since the leading vectors of all the vectors $v_C$ are pairwise distinct, it follows that the vectors $v_C$ are linearly independent. \qed

2. THE COORDINATE RING OF CELL IDEALS WITH ISOLATED SINGULARITY

Let $I = (f_1, \ldots, f_m)$ be an ideal in $S$, and let

$$A = (\partial f_i/\partial x_j)_{i=1, \ldots, m, j=1, \ldots, n}$$

...
be the corresponding Jacobian matrix. Let $h$ be the height of $I$. The Jacobian ideal of the ring $R = S/I$ is the ideal $J \subset R$ generated by the $h \times h$-minors of $A$. When $K$ is a perfect field, the ideal $J$ defines the singular locus of $R$. In other words, $R_P$ is not regular for $P \in \text{Spec}(R)$ if and only if $J \subseteq P$, see [2, Corollary 16.20].

In the following result we investigate when the ring $K[P]$ has an isolated singularity.

**Theorem 2.1.** Let $K$ be a perfect field, and let $P$ be a finite set of cells such that $I_P \subset S$ is a prime ideal. Then $S/I_P$ has an isolated singularity if and only if $P$ is an inner interval.

**Proof.** We set $u_a = x_a \mod I_P$ for all $a \in V(P)$. Let $J \subset R$ be the Jacobian ideal of $R = S/I_P$. By [2, Corollary 16.20] the assumption on $K$ guarantees that the $K$-algebra $R$ has an isolated singularity if and only $\dim R/J = 0$. The latter is the case if and only if suitable powers of the $K$-algebra generators $u_a$ of $R$ belong to $J$.

Let $a \in V(P)$. Then $\pm x_a$ appears as an entry of the Jacobian matrix, if and only if there exists $b \in V(P)$ such that $a$ and $b$ are the diagonal or anti-diagonal corners of an inner interval $D$ of $P$. Let $B_a$ be the set of such elements $b$. Thus, if $u^k_a$ appears as a monomial generator of the Jacobian ideal $J$, then there should exists at least $h$ such elements $b$ so that $a$ and $b$ are the diagonal or anti-diagonal corners of an inner interval of $P$. Hence, $h \leq |B_a|$.

For each $b \in B_a$ there exists a unique cell $C_b \subseteq D$ for which $b$ is a corner of $D$.

![Figure 1. Inside cell](image)

It follows that $|B_a| \leq h$ (which is the number of cells of $P$). Thus we have shown that $|B_a| = h$ for all $a \in V(P)$. Assume that $P$ is not an interval. We claim that in this case there exists $a \in P$ such that $|B_a| < h$, which then leads to a contradiction.

Proof of the claim: choose $a \in V(P)$, and take the subset $\{b_1, \ldots, b_r\}$ of the elements in $B_a$ for which the inner interval $I_j$ with corners $a$ and $b_j$ (as displayed in Figure 2) is maximal in the sense that if $b \in B_a$, then the inner interval with corners
\[ \textbf{Figure 2.} \]

\( a \) and \( b \) is contained in one of the intervals \( I_j \). Since \(|B_a| = h\) and since the cells \( C_b\) are pairwise distinct, and since they are cells of \( \bigcup_{j=1}^{r} I_j \), it follows that \( \bigcup_{j=1}^{r} I_j \) contains \( h \) cells. By Theorem 1.1, \( P \) has exactly \( h \) cells. Hence we see that \( P \) is equal to the set of the cells of \( \bigcup_{j=1}^{r} I_j \). Let \([c, d]\) be the smallest interval containing \( P \), see Figure 3.

\[ \textbf{Figure 3.} \]

Since we assume that \( P \) is not an interval, it follows that not all corners of \([c, d]\) belong to \( V(P) \). We may assume that \( c \notin V(P) \), and in order to simplify our discussion we may further assume that \( c = (0, 0) \). Let \( b \) be the smallest element on the \( x \)-axis and \( b' \) be the smallest element on the \( y \)-axis which belongs to \( V(P) \). In our picture these are the elements \( b = b_1 \) and \( b' = b_5 \). Then \( b' + (1, 1) \notin B_b \), which implies that \( |B_b| < h \). This proves the claim and completes the proof of the theorem. \( \square \)

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