Random matrices and Riemann hypothesis

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Abstract

The curious connection between the spacings of the eigenvalues of random matrices and the corresponding spacings of the nontrivial zeros of the Riemann zeta function is analyzed on the basis of the geometric dynamical global program of Langlands whose fundamental structures are shifted quantized conjugacy class representatives of bilinear algebraic semigroups. The considered symmetry behind this phenomenology is the differential bilinear Galois semigroup shifting the product, right by left, of automorphism semigroups of cofunctions and functions on compact transcendental quanta.
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Introduction

It became more and more evident over the years that there must exist a connection between the eigenvalues of random matrices and the nontrivial zeros of the Riemann zeta function $\zeta(s)$ in such a way that the distribution of the nontrivial zeros of $\zeta(s)$ can be approached by techniques developed in random matrix theory.

And, as the distributions over ensembles of random matrices may be related to the statistical properties of waves in chaotic dynamical systems [Yau], the Riemann dynamics should be chaotic.

Studying numerically the nontrivial zeros of $\zeta(s)$, Odlyzko [Odl] found that the mean spacings between their imaginary part $\gamma_j$ and the spacings between the eigenvalues of large unitary matrices correspond. This work was based on researches of Montgomery [Mon] who found that the pair correlation of the $\gamma_j$ is equal to the Gaussian unitary ensemble (GUE) pair correlation.

In their famous paper [K-S], M. Katz and P. Sarnak put the big questions: “Why is this so and what does it tell us about the nature (e.g. spectral) of the zeroes? Also, what is the symmetry behind this “GUE” law?”

It is the purpose of this paper to propose a satisfying solution to these questions based on the following considerations:

1) The fundamental structures behind this problem are those of the geometric-dynamical global program of Langlands, i.e. double symmetric towers of shifted conjugacy class representatives of bilinear algebraic semigroups over sets of increasing finite algebraic extensions of number fields.

2) The quantization of these conjugacy class representatives, which are abstract subsemivarieties [Har], by algebraic (resp. transcendental) quanta being equivalently: a) irreducible closed algebraic (resp. transcendental) subsets characterized by an extension degree $N$ (which is the Artin conductor); b) unitary irreducible representations of these semigroups.

3) The considered symmetry behind this phenomenology is the one of the differential bilinear Galois semigroup shifting the product, right by left, of automorphism semigroups of cofunctions and functions on compact transcendental quanta.

4) The Riemann dynamics then corresponds to the wave chaotic dynamics described by random matrix theory.
The general proposed solution is organized in three parts (or chapters):

1) Reminder of universal algebraic structures of the global program of Langlands [Pie2], [Lan].

2) Reminder of universal dynamical structures of the global program of Langlands [Pie3].

3) Responses to five questions allowing finding a solution to this problem:
   a) What is behind random matrices leading to the Gaussian orthogonal ensemble (GOE) as well as the Gaussian unitary ensemble (GUE)?
   b) What is behind the partition and correlation functions between eigenvalues of random matrices?
   c) What interpretation can we give to the local spacings between the eigenvalues of large random matrices?
   d) What interpretation can we give to the spacings between the non-trivial zeros of the Riemann zeta function \( \zeta(s) \)?
   e) What is the curious connection between c) and d)?

Let us analyse more concretely these three parts.

In chapter 1, the bilinear global version of the Langlands program is recalled to be based on:

a) bisemiobjects \((o_R \times o_L)\) composed of the products of right semiobjects \(o_R\), localized on the lower half space or on \(IR^-\), and of left symmetric semiobjects \(o_L\), localized on the upper half space or on \(IR^+\).

b) the product, right by left, of two symmetric towers of increasing algebraic or compact transcendental extensions composed of an increasing number of algebraic or transcendental compact quanta and defining a (bisemi)lattice of biquanta, i.e. the product, right by left, of two symmetric (semi)lattices of algebraic or transcendental quanta.

c) Abstract bisemivarieties \(G^{2n}(F_v \times F_v)\) over products, right by left, \((F_v \times F_v)\) of sets of increasing compact transcendental extensions (or archimedean completions) \(F_v = \{F_{v1}, \ldots, F_{vj}, \ldots, F_{vr}\}\) and \(F_v = \{F_{v1}, \ldots, F_{vj}, \ldots, F_{vr}\}\) in such a way that they are (functional) representation spaces of the algebraic bilinear
semigroups \( GL_{2n}(\widetilde{F}_{\tau} \times \widetilde{F}_{v}) = T_{2n}^{t}(\widetilde{F}_{\tau}) \times T_{2n}(\widetilde{F}_{v}) \) being the 2n-dimensional bilinear representations of the products, right by left, \( \text{Gal}(\widetilde{F}_{\tau}/k) \times \text{Gal}(\widetilde{F}_{v}/k) \) of global Weil semigroups.

d) The bilinear abstract parabolic semigroups \( P^{(2n)}(F_{\tau} \times F_{v}) \), over products \((F_{\tau} \times F_{v})\) of unitary transcendental pseudoramified extensions, in such a way that they are unitary representation spaces of the algebraic bilinear semigroup of matrices \( GL_{2n}(\widetilde{F}_{\tau} \times \widetilde{F}_{v}) \).

The abstract bisemivarieties \( G^{(2n)}(F_{\tau} \times F_{v}) \), covered by their algebraic equivalents \( G^{(2n)}(\widetilde{F}_{\tau} \times \widetilde{F}_{v}) \), are separated bisemischemes or quasiprojectives bisemivarieties \( [\text{Har}] \) and constitute the corner stone of the global program of Langlands since, by an isomorphism of toroidal compactification, \( G^{(2n)}(\widetilde{F}_{\tau} \times \widetilde{F}_{v}) \) (and \( G^{(2n)}(F_{\tau} \times F_{v}) \)) is transformed into the cuspidal representation \( \Pi(GL_{2n}(\widetilde{F}_{\tau} \times \widetilde{F}_{v})) \) of \( GL_{2n}(\widetilde{F}_{\tau} \times \widetilde{F}_{v}) \).

Remark that, as abstract (bisemi)varieties are quantized, their associated (bisemi)schemes, referring to quasi-projective (bisemi)varieties, are also quantized.

The differentiable functional representation space \( \text{FREPSP}(GL_{2n}(F_{\tau} \times F_{v})) \) of the complete bilinear semigroup \( GL_{2n}(F_{\tau} \times F_{v}) \) is a bisemisheaf \( (\widehat{M}^{(2n)}_{v} \otimes \widehat{M}^{(2n)}_{v}) \) of differential bifunctions on the abstract bisemivariety \( G^{(2n)}(F_{\tau} \times F_{v}) \) which is given by the product, right by left, of a right semisheaf \( \widehat{M}^{(2n)}_{v} \) of differentiable cofunctions on the increasing conjugacy class representatives of the abstract right semivariety \( G^{(2n)}(F_{\tau}) \equiv T^{(2n)}(F_{\tau}) \) by a left semisheaf \( \widehat{M}^{(2n)}_{v} \) of symmetric differentiable functions on the increasing conjugacy class representatives of the abstract left semivariety \( G^{(2n)}(F_{v}) \equiv T^{(2n)}(F_{v}) \).

Chapter 2 deals with dynamical (geometric) \( GL(2n) \)-bisemistructures of the global program of Langlands generated from the action of the elliptic bioperator \( (D_{R}^{(2k)} \otimes D_{L}^{(2k)}) \) (i.e. the product of a right linear differential elliptic operator \( D_{R}^{(2k)} \) acting on \( 2k \) variables by its left equivalent \( D_{L}^{(2k)} \), \( k \leq n \)) on the bisemisheaf \( (\widehat{M}^{(2n)}_{v} \otimes \widehat{M}^{(2n)}_{v}) \) according to:

\[
D_{R}^{(2k)} \otimes D_{L}^{(2k)} : \text{FREPSP}(GL_{2n}(F_{\tau} \times F_{v})) \longrightarrow \text{FREPSP}(GL_{2n}[2k](F_{\tau} \times IR) \times (F_{v} \times IR))
\]

where \( \text{FREPSP}(GL_{2n}[2k](F_{\tau} \times IR) \times (F_{v} \times IR)) \) is the functional representation space of \( GL_{2n}(F_{\tau} \times F_{v}) \) fibered or shifted in \( 2k \) bilinear geometric dimensions with \( F_{v}^{S_{k}} = (F_{v} \times IR) \) the set of increasing left compact transcendental extensions fibered or shifted by real numbers.

\( \text{FREPSP}(GL_{2k}(F_{\tau} \times IR) \times (F_{v} \times IR)) \) is isomorphic to the total bisemispace of the tangent bibundle \( \text{TAN}(\widehat{M}^{(2k)}_{v} \otimes \widehat{M}^{(2k)}_{v}) \) to the bisemisheaf \( (\widehat{M}^{(2k)}_{v} \otimes \widehat{M}^{(2k)}_{v}) \subset \)
\((\hat{M}_{v_R}^{(2n)} \otimes \hat{M}_{v_L}^{(2n)})\) of which bilinear fibre \(\mathcal{F}_{R \times L}^{(2k)}(\text{TAN}) = \text{(F)REPSP}(\text{GL}_{2k}(IR \times IR))\) is the (functional) representation space of \(\text{GL}_{2k}(IR \times IR)\) corresponding to the action of the bioperator \((D_{R}^{(2k)}) \otimes (D_{L}^{(2k)})\).

It is then proved that the bilinear semigroup of matrices \(\text{GL}_{r}(IR^{(2k)} \times IR^{(2k)})\) of “algebraic” order \(r\), associated with the bilinear fibre \(\text{GL}_{2k}(IR \times IR)\) and referring to the action of the bioperator \((D_{R}^{(2k)} \otimes D_{L}^{(2k)})\) on the bisemisheaf \((\hat{M}_{v_R}^{(2k)} \otimes \hat{M}_{v_L}^{(2k)})\), corresponds to the \(2k\)-dimensional “geometric” bilinear real representation of the product, right by left, of “differential” Galois (or global Weil) semigroups \(\text{Aut}_k(\phi_R(IR)) \times \text{Aut}_k(\phi_L(IR))\), shifting or fiberizing the product, right by left, of automorphism semigroups \(\text{Aut}_k(\phi_R(F_\varphi)) \times \text{Aut}_k(\phi_L(F_\varphi))\) of cofunctions \(\phi_R(F_\varphi)\) and functions \(\phi_L(F_\varphi)\) respectively on the compact transcendental real extensions \(F_\varphi\) and \(F_\varphi\) by \(\text{Aut}_k(\phi_R(F_\varphi \times IR)) \times \text{Aut}_k(\phi_L(F_\varphi \times IR))\) such that

\[
\text{GL}_r(\phi_R(IR^{(2k)}) \times \phi_L(IR^{(2k)})) = \text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(IR)) \times \text{Aut}_k(\phi_L(IR))) , \quad 1 \leq j \leq r .
\]

Similarly, the unitary parabolic bilinear semigroup \(\text{P}_r(IR^{(2k)} \times IR^{(2k)}) \subset \text{GL}_r(IR^{(2k)} \times IR^{(2k)})\), referring to the action of the bioperator \((D_{R}^{(2k)} \otimes D_{L}^{(2k)})\) on the unitary bisemisheaf \((\hat{M}_{v_R}^{(2k)} \otimes \hat{M}_{v_L}^{(2k)}) \subset (\hat{M}_{v_R}^{(2k)} \otimes \hat{M}_{v_L}^{(2k)})\), corresponds to the \(2k\)-dimensional bilinear real representation of the product, right by left, of “differential” inertia Galois (or global Weil) semigroups \(\text{Int}_k(\phi_R(IR)) \times \text{Int}_k(\phi_L(IR))\) such that

\[
\text{P}_r(\phi_R(IR^{(2k)}) \times \phi_L(IR^{(2k)})) = \text{Rep}^{(2k)}(\text{Int}_k(\phi_R(IR)) \times \text{Int}_k(\phi_L(IR))) .
\]

Let \(O_r(IR)\) denote the orthogonal group of (algebraic) order \(r\) with entries in \(IR\) and let \(U_r(\mathfrak{C})\) denote the unitary group with entries in \(\mathfrak{C}\). Then, the orthogonal bilinear semigroup \(O_r(IR^{(2k)} \times IR^{(2k)})\) corresponds to the real parabolic bilinear semigroup \(P_r(IR^{(2k)} \times IR^{(2k)})\) and the unitary bilinear semigroup \(U_r(\mathfrak{C}^k \times \mathfrak{C}^k)\) corresponds to the complex parabolic bilinear semigroup \(P_r(\mathfrak{C}^k \times \mathfrak{C}^k)\).

Chapter 3 deals with the connection between large random matrices and the solution of the Riemann hypothesis by responding to the five above mentioned questions. From now on, the geometric dimension \(2k\) will be taken to be \(1\).

1. The first question “What is behind random matrices leading to GOE and GUE?” leads to the eigenvalue problem in the frame of the geometric dynamical program of Langlands recalled in chapter 2.
The symmetric group at the origin of the bilinear global program of Langlands is the bilinear semigroup of automorphisms $\text{Aut}_k(F_v) \times \text{Aut}_k(F_v)$ (resp. Galois automorphisms $\text{Gal}((\overline{F_v}/k) \times \text{Gal}(\overline{F_v}/k)$) of compact transcendental (resp. algebraic) quanta generating a bisemilattice of compact transcendental (resp. algebraic) quanta while the symmetry group at the origin of the dynamical (geometric) bilinear global program of Langlands is the bilinear semigroup of shifted automorphisms $\text{Aut}_k(\phi_R((F_v \times IR)) \times \text{Aut}_k(\phi_L((F_v \times IR))$ of bifunctions on compact transcendental biquanta generating a bisemilattice of compact transcendental quanta.

It then results that the bilinear semigroup of matrices $\text{GL}_r(IR \times IR)$ constitutes the “r-dimensional algebraic” representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator $(D_R \otimes D_L)$ on the bisemisheaf $(\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)})$.

Let

$$(D_R \otimes D_L)(\phi(G^{(1)}(F_v \times F_v))) = E_{R \times L}(j)(\phi(G^{(1)}(F_v \times F_v))) , \quad 1 \leq j \leq r ,$$

be the eigenbivalue equation related to the bisemisheaf $(\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)})$. Then, we have that

(a) the $j$-th eigenbifunction $\phi(G^{(1)}(F_v \times F_v))$ on the $j$ transcendental compact biquanta, being the $j$-th bisection of $(\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)})$, corresponds to the $j$-th eigenbivalue $E_{R \times L}(j)$ which is the shift of this bifunction and the shift of the global Hecke character associated with this subbisemilattice.

(b) the eigenbivalues of the matrix of $\text{GL}_r(IR \times IR)$, constituting a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R \otimes D_L)$, are the eigenbivalues of the above eigenbivalue equation.

2. The second question “What is behind the partition and correlation functions between eigenvalues of random matrices?” concerns the distribution of eigenvalues [D-H] of random matrices of the Gaussian unitary (GUE) and orthogonal ensemble (GOE) having $r$ quantum states and characterized by a Hamiltonian matrix of order $r$ whose entries are Gaussian random variables [D-S].

In the bilinear context envisaged in this paper, we are mostly interested by the $m$-point correlation function for the bilinear Gaussian unitary (or or-
orthogonal) ensemble BCUE (resp. BCOE) given by:

\[
R_{m, r \times L}(x_1^2, \ldots, x_m^2) = \frac{r!}{(r-m)!} \int_{R^{r-m}} P_{r \times L}(x_1^2, \ldots, x_r^2) \, dx_{m+1} \ldots dx_r
\]

\[
= \det(K_r(x_k, x_\ell)_{k, \ell=1}^m)
\]

with

\[
K_r(x_k, x_\ell) = \sum_{i=0}^{r-1} \psi_i(x_k) \psi_i(x_\ell)
\]

and

\[
\psi_i(x) = h^{-1/2} P_i(x) \, e^{-r[(TG^T \times TG)(R \times R)]/2}.
\]

\(P_i(x)\) being an orthogonal polynomial of degree \(i\) corresponding to the weight function \(e^{-r[(TG^T \times TG)(R \times R)]/2}\) where \(G(\mathbb{R} \times \mathbb{R}) = TG^T(\mathbb{R} \times \mathbb{R})\) is the bilinear Gauss decomposition of the matrix \(G\) of BCOE.

\(R_{m, r \times L}(x_1^2, \ldots, x_m^2)\) is the probability of finding a level around each of the bipoints (i.e. entries in \(G\)) \(x_1^2, \ldots, x_m^2\), the positions of the remaining levels being unobserved.

Let

\[
K_r(x, x) = \sum_{i=0}^{r-1} \psi_i(x) \psi_i(x)
\]

be the energy level density with \(\psi_i(x)\) given above.

Then, we have found that:

(a) the squares of the roots of the polynomial \(P_i(x)\) correspond to the eigenbivalues of the product, right by left, \((U_{rR} \times U_{rL})\) of Hecke operators.

(b) the weight \(e^{-r[(TG^T \times R) \times (TG^T \times R)])/2}\) is a measure of the eigenbivalues of the random matrix \(G \in \text{GL}_r(\mathbb{R} \times \mathbb{R})\) being a representation of the differential bilinear Galois semigroup.

The orthogonal polynomials \(P_i(x)\) satisfy the three term recurrent relation

\[
\beta_{i+1} P_{i+1}(x) = (x - \alpha_i) P_i(x) - \beta_i P_{i-1}(x)
\]

leading to a tower whose matricial form is

\[
x P = J P + \beta_i P_i
\]

where \(J\) is the Jacobi matrix and \(P\) a column vector of polynomials of increasing degrees.
It was then proven that the roots of $P_i(x)$ are the eigenvalues of the Jacobi symmetric matrix being a representation of the Hecke operator $U_{ri}$.

We are then led to the conclusions:

(a) The probabilistic interpretation of quantum (field) theories is related to the bilinear semigroup of automorphisms $\text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v)$ of compact transcendental biquanta generating a bisemilattice of these.

(b) The $m$-point correlation function for BCUE (or BCOE) $R_{m}(x_1^2, \ldots, x_m^2)$ constitutes a representation of the bilinear semigroup of automorphisms $\text{Aut}_k(\phi_R(F_\tau \times \mathbb{R})) \times \text{Aut}_k(\phi_L(F_v \times \mathbb{R}))$ of bifunctions on shifted compact transcendental biquanta.

3. The third question “What interpretation can we give to the local spacings [Gau] between the eigenvalues of large random matrices” depends on the dynamical global program of Langlands developed in chapter 2 and summarized in the first question: it results from the $r$-dimensional algebraic representation of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ associated with the action of the differential bioperator $(D_R \otimes D_L)$ on the bisemisheaf $(\hat{M}^{(1)}_{r_R} \otimes \hat{M}^{(1)}_{r_L})$ and leading to the above mentioned eigenbivalue equation in such a way that:

(a) the consecutive spacings

$$\delta E_{R \times L}(j) = E_{R \times L}(j+1) - E_{R \times L}(j), \quad 1 \leq j \leq r \leq \infty,$$

between the eigenbivalue of the random matrix $G \in \text{GL}_r(\mathbb{R} \times \mathbb{R})$ are infinitesimal bigenerators of one biquantum of the Lie subbisemialgebra $\text{gl}_1(F_\tau^{*} \times F_v^{*})$ of the bilinear parabolic unitary semigroup $P_1(F_\tau^{*} \times F_v^{*}) \in \text{GL}_1(F_\tau \times F_v)$ and correspond to the energies of one free biquantum from subbisemilattices of $(j+1)$ biquanta.

(b) the $k$-th consecutive spacings

$$\delta E^{(k)}_{R \times L}(j) = E_{R \times L}(j+k) - E_{R \times L}(j)$$

between the eigenbivalue of the random matrix $G \in \text{GL}_r(\mathbb{R} \times \mathbb{R})$ are the infinitesimal bigenerators on $k$ biquanta of the Lie subbisemialgebra $\text{gl}_1(F_\tau^{*} \times F_v^{*})$ of the bilinear $k$-th semigroup $\text{gl}_1(F_\tau^{*} \times F_v^{*})$ and correspond to the energies of $k$ free biquantum from subbisemilattices of $(j+k)$ biquanta.
The consecutive spacings \( \delta E_{R \times L}(j) \) between the eigenvalues of the matrix \( G \) of \( GL_r(I R \times I R) \) decompose into:

\[
\delta E_{R \times L}(j) = \delta EF_{R \times L}(j) + \delta EV_{R \times L}(j)
\]

where \( \delta EF_{R \times L}(j) \) and \( \delta EV_{R \times L}(j) \) denote respectively the fixed (or constant) and variable consecutive spacings between the \( r \) eigenvalues of \( G \).

Then, we have that:

(a) the consecutive spacings

\[
E^{BCUE}_{R \times L}(j + 1) - E^{BCUE}_{R \times L}(j) = \delta EV^{BCUE}_{R \times L}(j)
\]

between the eigenvalues \( E^{BCUE}_{R \times L}(j + 1) \) and \( E^{BCUE}_{R \times L}(j) \) of a unitary random matrix of \( U_r(\mathcal{C} \times \mathcal{C}) \) (or \( O_r(I R \times I R) \)) are the variable (unitary) infinitesimal bigenerators on one biquantum on the envisaged Lie subsemialgebra or the variable (unitary) energies \( \delta E^{BCUE}_{R \times L}(j) \) of one biquantum in subsemilattices of \( (j + 1) \) biquanta.

(b) the \( k \)-th consecutive spacings

\[
E^{BCUE}_{R \times L}(j + k) - E^{BCUE}_{R \times L}(j) = \delta EV^{(k)BCUE}_{R \times L}(j)
\]

between the eigenvalues of a unitary random matrix of \( U_r(\mathcal{C} \times \mathcal{C}) \) (or \( O_r(I R \times I R) \)), are the variable energies \( \delta E^{(k)BCUE}_{R \times L}(j) \) on \( k \) biquanta in subsemilattices of \( (j + k) \) biquanta.

4. The fourth question: “What interpretation can we give to the spacings between the nontrivial zeros [Zag], [Pol], of the Riemann zeta function \( \zeta(s) \)” depends on the solution of the Riemann hypothesis [Bom] proposed in [Pie7] and briefly recalled now.

The 1D-pseudounramified simple global elliptic \( \Gamma^{(1)}_{\mathcal{M}^{(1)}_{vR \times L}} \)-bisemimodule

\[
\phi_{(1),(vnr)}^{(1)}(x) = \sum_n \left( \lambda^{(nr)}(n) e^{-2\pi i n x} \otimes_D \lambda^{(nr)}(n) e^{+2\pi i n x} \right), \quad x \in I R,
\]

where \( \lambda^{(nr)}(n) \) is a global Hecke character,

can be interpreted as the sum of products, right by left, of semicircles of level “\( n \)” on \( n \) transcendental compact quanta and constitutes a cuspidal representation of the bilinear semigroup \( GL_2(F^{nr}_v \times F^{nr}_v) \).
Let \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \), \( s_- = \sigma - i\tau \) and \( s_+ = \sigma + i\tau \), be the two zeta functions defined respectively in the lower and upper half planes.

They are (distribution) inverse space functions, i.e. energy functions on the variables \( s_- \) and \( s_+ \) conjugate to the complex space variables \( z^* \in \mathcal{C} \) and \( z \in \mathcal{C} \) of the cusp forms \( f_L(z) \) and \( f_R(z^*) \) submitted respectively to the transform maps \( f_L(z) \to \zeta_L(s_+) \) and \( f_R(z^*) \to \zeta_R(s_-) \) [Pie8].

Then, the kernel \( \text{Ker}(H_{\phi_R \times L} \to \zeta_R \times L) \) of the map:
\[
H_{\phi_R \times L} : 2\phi^{(1)}_{R \times L}(x) \to \zeta_R(s_-) \otimes D \zeta_L(s_+)
\]
is the set of squares of the trivial zeros of \( \zeta_R(s_-) \), \( \zeta_L(s_+) \) and \( \zeta(s) \), corresponding to the degeneracies of the products, right by left, of circles \( 2\lambda^{(nr)}(n)e^{2\pi inx} \) on \( 2n \) transcendental quanta.

By this way, we have a one-to-one correspondence between the trivial zeros of \( \zeta(s) \) and the degeneracies of circles belonging to symmetric towers of cuspidal conjugacy class representatives of bilinear complete algebraic semigroups.

Then, it is proved that the products of the pairs of the trivial zeros of the Riemann zeta functions \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \) are mapped into the products of the corresponding pairs of the nontrivial zeros according to:
\[
\{D_{4n^2,i^2} \cdot \varepsilon_{4n^2}\} : \{\det(\alpha_{4n^2})\}_n \to \{\det(D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2})_{ss}\}_n \\
\{(-2n) \times (-2n)\}_n \to \{\lambda_{+}^{(nr)}(4n^2,i^2,E_{4n^2}) \times \lambda_{-}^{(nr)}(4n^2,i^2,E_{4n^2})\}_n
\]
where:

- \( \alpha_{4n^2} \) is the split Cartan subgroup element associated with the integer \( 2n \);
- \( D_{4n^2,i^2} \) is the coset representative of the Lie (bisemi)algebra of the decomposition bisemigroup acting on \( \alpha_{4n^2,i^2} \);
- \( \varepsilon_{4n^2} \) is the infinitesimal bigenerator of the considered bisemialgebra.

Every root of this Lie bisemialgebra is determined by the eigenvalues
\[
\lambda_{\pm}^{(nr)}(4n^2,i^2,E_{4n^2}) = \frac{1 \pm i\sqrt{16n^2 \cdot E_{4n^2} - 1}}{2}
\]
of \( D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2} \) which are the nontrivial zeros of \( \zeta(s) \) written compactly according to \( (\frac{1}{2} + ijj) \) and \( (\frac{1}{2} - ijj) \), \( j \leftrightarrow n \).

Then, we have that:
1) The consecutive spacings

\[ \delta_{\gamma_j} = \gamma_{j+1} - \gamma_j , \quad j = 1, 2, \ldots , \]

between the nontrivial zeros of \( \zeta(s) \) are equivalently:

a) the infinitesimal generators on one quantum of the Lie subsemialgebra \( \text{gl}_1(F_{v_1}^{(nr)}) \) (or \( \text{gl}_1(F_{\mathfrak{m}_1}^{(nr)}) \)) where \( F_{v_1}^{(nr)} \) is a pseudounramified compact transcendental extension (see section 1.1) of the linear parabolic unitary semigroup \( P_1(F_{v_1}^{(nr)}) \subset \text{GL}_1(F_{v_1}^{(nr)}) \equiv T_1(F_{v_1}^{(nr)}) \) (or \( P_1(F_{\mathfrak{m}_1}^{(nr)}) \));

b) the energies of one free quantum in subsemilattices of \( (j + 1) \) quanta.

2) The \( k \)-th consecutive spacings

\[ \delta_{\gamma_j}^{(k)} = \gamma_{j+k} - \gamma_j \]

between the nontrivial zeros of \( \zeta(s) \) are equivalently:

a) the infinitesimal generators on \( k \) quanta of the Lie subsemialgebra

\[ \text{gl}_1(F_{v_k}^{(nr)}) \) (or \( \text{gl}_1(F_{\mathfrak{m}_k}^{(nr)}) \)) of the \( k \)-th semigroup \( \text{GL}_1(F_{v_k}^{(nr)}) \) (or \( \text{GL}_1(F_{\mathfrak{m}_k}^{(nr)}) \));

b) the energies of \( k \) free quanta in subsemilattices of \( (j + k) \) quanta.

The fifth question “What is the curious connection between the spacings of the nontrivial zeros of \( \zeta(s) \) and the corresponding spacings between the eigenvalues of random matrices?” [Ke-S] finds response in the following statements (propositions of chapter 3):

- The consecutive spacings

\[ \delta_{\gamma_j} = \gamma_{j+1} - \gamma_j , \quad j \in \mathbb{N} , \quad 1 \leq j \leq r < \infty , \]

between the nontrivial zeros of the Riemann zeta function \( \zeta(s) \) correspond to the consecutive spacings

\[ \delta E_{R,L}^{(nr)}(j) = E_{R,L}^{(nr)}(j + 1) - E_{R,L}^{(nr)}(j) , \]

(\( R,L \) means right of left),

between the square roots of the pseudounramified eigenbivalues of a (large) random matrix of \( \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) (or of \( \text{GL}_r(\mathbb{C} \times \mathbb{C}) \)) and are equivalently:
1) the infinitesimal generators on one quantum of the Lie subsemialgebra $\text{gl}_1(F_{v_1}^{nr})$ of the linear parabolic unitary semigroup $P_1(F_{v_1}^{nr})$;

2) the energies of one transcendental compact pseudounramified ($N = 1$) quantum in subsemilattices of $(j + 1)$ transcendental compact pseudounramified quanta.

- The set $\{\delta E_{R,L,j}^{(nr)}(j)\}_{j=1}^r$ of consecutive spacings between the square roots of the eigenbivalues of a random matrix of $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$ as well as the set $\{\delta \gamma_j\}_{j=1}^r$ of consecutive spacings between the nontrivial zeros of $\zeta(s)$ constitutes a representation of the differential inertia Galois semigroup associated with the action of the differential operator $D_L$ or $D_R$.

- Let $\{\delta E_{R,L,j}^{(nr)}(j)\}_j$ be the set of consecutive spacings between the square roots of the eigenbivalues of a random matrix of $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$ or between the eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$.

Then, there is a surjective map:

$$IM_{E \rightarrow \gamma} : \{\delta E_{R,L}\}_j \longrightarrow \{\delta \gamma_j\}_j$$

of which kernel $\text{Ker}[IM_{E \rightarrow \gamma}]$ is the set $\{\delta E_{R,L,j}^{(nr)}(j) - \delta E_{R,L,j}^{(nr)}(j)\}_j$ of differences of consecutive spacings between the square roots of the pseudoramified and pseudounramified eigenbivalues of a random matrix of $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$, i.e. the energies of one compact transcendental pseudoramified ($N > 2$) quantum in subsemilattices of $(j + 1)$ transcendental pseudoramified quanta.

- Finally, we can gather the results of this paper in the following proposition (see 3.33).

Let $\delta \gamma_j^{(k)} = \gamma_{j+k} - \gamma_j$ denote the $k$-th consecutive spacings between the nontrivial zeros of $\zeta(s)$.

Let:

- $\delta E_{R,L}^{(k)}(j) = E_{R,L}(j + k) - E_{R,L}(j)$;
- $\delta E_{R,L}^{(nr),(k)}(j) = E_{R,L}^{(nr)}(j + k) - E_{R,L}^{(nr)}(j)$;
- $\delta E_{R,L}^{(nr),BCOE}(j) = E_{R,L}^{(nr),BCOE}(j + k) - E_{R,L}^{(nr),BCOE}(j)$, $1 \leq j \leq r$, $k \leq j$,

be the $k$-th consecutive spacings between respectively:

- the pseudoramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$;
- the pseudounramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$;
the pseudounramified eigenvalues of a random unitary matrix of $O_r(\mathbb{R})$.

Then, we have:

1) $\delta\gamma^{(k)}_j = \delta E^{(nr),(k)}_{R,L}(j)$ which are equivalently:
   a) the infinitesimal generators of $k$ quanta of the Lie subsemialgebra $gl_1(F^{(nr)}_{v^k})$ of the linear $k$-th semigroup $GL_1(F^{(nr)}_{v^k}) \subset GL_1(F^{(nr)})$;
   b) the energies of $k$ transcendental compact pseudounramified ($N = 1$) quanta in subsemilattices in $(j + k)$ transcendental pseudounramified quanta;
   
   c) a representation of the differential Galois semigroup associated with the action of the differential operator $D_L$ or $D_R$ on a function on $k$ transcendental pseudounramified quanta;

2) a surjective map:
   \[ IM^{(k)}_{E \rightarrow \gamma} : \{ \delta E^{(k)}_{R,L}(j) \}_j \rightarrow \{ \delta \gamma^{(k)}_j \}_j \]
   of which kernel is the set $\{ \delta E^{(k)}_{R,L}(j) - \delta E^{(nr),(k)}_{R,L}(j) \}_j$ of difference of $k$-th consecutive spacings between the pseudounramified and pseudounramified eigenvalues of a random matrix of $GL_r(\mathbb{R})$;

3) a bijective map:
   \[ IM^{(k)}_{\gamma \rightarrow E^{(nr),BCOE}_{(v)}} : \{ \delta \gamma^{(k)}_j \}_j \rightarrow \{ \delta EV^{(k),(nr),BCOE}_{R,L}(j) \}_j \]
   where $\delta \gamma^{(k)}_j$ denotes a $k$-th (variable) consecutive spacing verifying $\delta \gamma^{(k)}_j = \delta E^{(k),(nr),BCOE}_{R,L}(j)$.

- It has been inferred for a long time that the nontrivial zeros of the Riemann zeta function are probably related to the eigenvalues of some wave dynamical system. The connection between these two fields at the light of the global program of Langlands can be summarised by the proposition 3.35:

The squares of the nontrivial zeros $\gamma_j$ of the Riemann zeta function $\zeta(s)$ are pseudounramified eigenbivales of the eigenbivalue biwave operation:

\[ (D_R \otimes D_L)(\phi(G^{(1)}(F^{(nr)}_{v^j} \times F^{(nr)}_{v^j}))) = \gamma_j^2(\phi(G^{(1)}(F^{(nr)}_{v^j} \times F^{(nr)}_{v^j}))) \]

of which eigenbifunctions are the sections of the bisemisheaf $(\hat{M}^{(1)}_{v_R} \times \hat{M}^{(1)}_{v_L})$ being interpreted as the internal stringfield of an elementary (bisemi)particle [Pie8].
1 Universal algebraic structures of the global program of Langlands

1.1 Symmetric structures of the program of Langlands

In analogy with the fundamental theorem of the Galois theory which establishes a one-to-one correspondence between the set of closed intermediate fields of a given finite extension $F$ of a number field $k$ and the set of all closed normal subgroups of the Galois group $\text{Aut}_k F$, we consider a set of increasing finite algebraic extensions of $k$ in one-to-one correspondence with the corresponding Galois (sub)groups. The envisaged global program of Langlands is then constructed on $n$-dimensional representations of such Galois (sub)groups and is recalled in this chapter.

Since symmetric semiobjects have to be considered in any generality [Pie4], we take into account the bilinear global version of the Langlands program concerning the generation of general symmetric structures, i.e. double symmetric towers of conjugacy class representatives of (bilinear) algebraic (semi)groups.

1.2 Algebraic and transcendental symmetric extensions

Let then the set $\tilde{F}$ of finite algebraic extensions of a number field $k$ of characteristic 0 be a set of symmetric splitting fields composed of the left and right real symmetric splitting semifields $\tilde{F}_L^+$ and $\tilde{F}_R^+$ given respectively by the sets of positive and symmetric negative simple real roots.

Assume that the set of all increasing left (resp. right) splitting semifields [Wei]:

$$\tilde{F}_{v_1} \subset \cdots \subset \tilde{F}_{v_j,m_j} \subset \cdots \subset \tilde{F}_{v_r,m_r}$$

(resp. $\tilde{F}_{\tau_1} \subset \cdots \subset \tilde{F}_{\tau_j,m_j} \subset \cdots \subset \tilde{F}_{\tau_r,m_r}$)

is a set of increasing left (resp. right) real algebraic extensions characterized by degrees:

$$[\tilde{F}_{v_j} : k] = [\tilde{F}_{\tau_j} : k] = * + j \cdot N , \quad 1 \leq j \leq r < \infty ,$$

which are integers modulo $N$, where

- $*$ denotes an integer inferior to $N$ ($* = 0$ for the zero class);
- $m_j$ labels the multiplicity of the envisaged extension.
These algebraic extensions are then said to be pseudoramified in contrast with the pseudounramified extensions \( \{ \tilde{F}_{v,j,m_j}^{(nr)} \}_{j,m_j} \) (resp. \( \{ \tilde{F}_{\pi,j,m_j}^{(nr)} \}_{j,m_j} \)) which are characterized by their global residue degrees \( f_{v,j} \) (resp. \( f_{\pi,j} \)):

\[
f_{v,j} = [\tilde{F}_{v,j,m_j}^{(nr)} : k] = [\tilde{F}_{\pi,j,m_j}^{(nr)} : k] = j \quad \text{(case } N = 1)\]

**The smallest left (resp. right) (pseudoramified) splitting (sub)semifield \( \tilde{F}_{v_1} \) (resp. \( \tilde{F}_{\pi_1} \)) characterized by an extension degree**

\[
[\tilde{F}_{v_1} : k] = [\tilde{F}_{\pi_1} : k] = N \quad \text{(case } j = 1)\]

was interpreted as being a left (resp. right) algebraic quantum, i.e. an irreducible closed left (resp. right) algebraic subset.

According to the fundamental theorem of Galois, there could exist a one-to-one correspondence between the intermediate subsemifields

\[
\tilde{F}_{v_1} \supset \cdots \supset \tilde{F}_{v,j} \supset \cdots \supset \tilde{F}_{v_1} \supset k \quad \text{(resp. } \tilde{F}_{\pi_1} \supset \cdots \supset \tilde{F}_{\pi,j} \supset \cdots \supset \tilde{F}_{\pi_1} \supset k)\]

and the corresponding closed subsemigroups of the Galois semigroup \( \text{Gal}(\tilde{F}_L^+/k) \) (resp. \( \text{Gal}(\tilde{F}_R^+/k) \)):

\[
\{1\} \subseteq \text{Gal}(\tilde{F}_{v_1}/k) \subseteq \cdots \subseteq \text{Gal}(\tilde{F}_{v,j}/\tilde{F}_{v,j-1}) \subseteq \cdots \subseteq \text{Gal}(\tilde{F}_{r}/\tilde{F}_{r-1}).
\]

But, the extension degree \([\tilde{F}_{v_1} : k] = [\tilde{F}_{\pi_1} : k]\) would then be the product of the above intermediate algebraic extension degrees which is generally not verified by the given extension degree \([\tilde{F}_{v_1} : k] = [\tilde{F}_{\pi_1} : k] = r \cdot N\) which belongs to the zero class of integers modulo \(N\) required by the searched one-to-one correspondence between the representations of the associated Galois (sub)groups and the corresponding cuspidal representations [Pic2]. However, every algebraic extension \( \tilde{F}_{v,j} \) (resp. \( \tilde{F}_{\pi,j} \)), \( 1 \leq j \leq r \leq \infty \), characterized by the degree \([\tilde{F}_{v,j} : k] = [\tilde{F}_{\pi,j} : k] = j \cdot N\), can be decomposed according to the fundamental theorem of Galois.

Let then \( \{ \tilde{F}_{v,j,m_j} \}_{j,m_j} \) (resp. \( \{ \tilde{F}_{\pi,j,m_j} \}_{j,m_j} \)) be the set of increasing algebraic extensions characterized by degrees \( \{[\tilde{F}_{v,j,m_j} : k] = j \cdot N\}_{j,m_j} \) (resp. \( \{[\tilde{F}_{\pi,j,m_j} : k] = j \cdot N\}_{j,m_j} \)) and whose global Weil (or Galois) subgroups [Pic2] are \( \text{Gal}(\tilde{F}_{v,j,m_j}/k) \) (resp. \( \text{Gal}(\tilde{F}_{\pi,j,m_j}/k) \)).

By an **isomorphism of compactification**

\[
c_{v,j,m_j} : \tilde{F}_{v,j,m_j} \rightarrow F_{v,j,m_j} \quad \text{(resp. } c_{\pi,j,m_j} : \tilde{F}_{\pi,j,m_j} \rightarrow F_{\pi,j,m_j})\),
\]
each algebraic extensions \( \widetilde{F}_{v,j,m_j} \) (resp. \( \widetilde{F}_{\tau, j,m_j} \)) is sent into its compact image \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)) which is a closed compact subset of \( IR_+ \) (resp. \( IR_- \)).

Each compact image \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)) of the algebraic extension \( \widetilde{F}_{v,j,m_j} \) (resp. \( \widetilde{F}_{\tau, j,m_j} \)) is thus a transcendental extension [Hun] of which transcendence degree \( \text{tr} \cdot d \cdot F_{v,j,m_j} \) (resp. \( \text{tr} \cdot d \cdot F_{\tau, j,m_j} \)) is given by:

\[
\text{tr} \cdot d \cdot F_{v,j,m_j} = [\widetilde{F}_{v,j,m_j} : k] = j \cdot N \quad \text{(resp. } \text{tr} \cdot d \cdot F_{\tau, j,m_j} = [\widetilde{F}_{\tau, j,m_j} : k] = j \cdot N).\]

Every transcendental extension \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)) is also an archimedean completion which can be viewed as resulting from the semigroup of automorphisms Aut\(_k(F_{v,j,m_j})\) (resp. Aut\(_k(F_{\tau, j,m_j})\)) which is a semigroup of reflections (or permutations) of a transcendental quantum \( F_{v,j,m_j} \subset F_{v,m_j} \) (resp. \( F_{\tau, j,m_j} \subset F_{\tau,m_j} \)) characterized by the corresponding extension degree \( [\widetilde{F}_{v,j,m_j} : k] = [\widetilde{F}_{\tau, j,m_j} : k] = N \) [Pie5].

This implies that the compact transcendental extension \( F_{v_h,m_h} \) (resp. \( F_{\tau_h,m_h} \)), composed of \( h \) left (resp. right) transcendental quanta, of transcendental degree \( h \cdot N \) over \( k \), is included into the transcendental extension \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)), composed of \( j \) left (resp. right) compact transcendental quanta, in the sense of the fundamental theorem of Galois theory. Indeed, the transcendental quantum \( F_{v,h} \subset F_{v,m_j} \) (resp. \( F_{\tau,h} \subset F_{\tau,m_j} \)) of the transcendental extension \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)) is homeomorphic to the transcendental quantum \( F_{v,h} \subset F_{v,m_j} \) (resp. \( F_{\tau,h} \subset F_{\tau,m_j} \)) of the transcendental extension \( F_{v_h,m_h} \) (resp. \( F_{\tau_h,m_h} \)) since, by Galois automorphism and isomorphisms of compactification, they are the compact images either of the nonunits of the algebraic quantum \( \widetilde{F}_{v,j} \) (resp. \( \widetilde{F}_{\tau, j} \)) of the algebraic extension \( F_{v,j,m_j} \) (resp. \( F_{\tau, j,m_j} \)) or of the nonunits of the algebraic quantum \( \widetilde{F}_{v,h} \) (resp. \( \widetilde{F}_{\tau, h} \)) of the algebraic extension \( F_{v_h,m_h} \) (resp. \( F_{\tau_h,m_h} \)).

Thus, in the case of transcendental extensions or compact archimedean completions, we have that:

1.3 Proposition

If \( F_{v_r} \) (resp. \( F_{\tau_r} \)) is a finite dimensional compact transcendental extension of \( k \), there is a one-to-one correspondence between the set of intermediate semifields

\[
F_{v_r} \supset \cdots \supset F_{v_j} \supset \cdots \supset F_{v_1} \supset k
\]

( resp. \( F_{\tau_r} \supset \cdots \supset F_{\tau_j} \supset \cdots \supset F_{\tau_1} \supset k \) )
and the set of corresponding subsemigroups of the semigroup of automorphisms $\text{Aut}_k(F_{v_r})$ (resp. $\text{Aut}_k(F_{\tau_r})$):

$$\{1\} \subseteq \text{Aut}_k(F_{v_1}) \subset \cdots \subset \text{Aut}_k(F_{v_j}) \subset \cdots \subset \text{Aut}_k(F_{v_r})$$

(resp. $$\{1\} \subseteq \text{Aut}_k(F_{\tau_1}) \subset \cdots \subset \text{Aut}_k(F_{\tau_j}) \subset \cdots \subset \text{Aut}_k(F_{\tau_r})$$)

in such a way that:

a) the relative degree (or transcendental dimension) of two intermediate semifields must be an integer;

b) $F_{v_r}$ (resp. $F_{\tau_r}$) is transcendental over every intermediate semifield $F_{v_j}$ (resp. $F_{\tau_j}$) if their transcendence degrees or global residue degrees verify [Hun]:

$$\text{tr} \cdot d \cdot F_{v_r}/k = (\text{tr} \cdot d \cdot F_{v_j}/F_{v_r}) + (\text{tr} \cdot d \cdot F_{v_j}/k)$$

(resp. $$\text{tr} \cdot d \cdot F_{\tau_r}/k = (\text{tr} \cdot d \cdot F_{\tau_j}/F_{\tau_r}) + (\text{tr} \cdot d \cdot F_{\tau_j}/k)$$).

Proof: In the Galois (i.e. algebraic) case, we would have that $\text{Gal}(\bar{F}_{v_r}/\bar{F}_{v_j})$, corresponding to $\text{Aut}_{F_{v_j}} F_{v_r}$ in the transcendental case, is normal in $\text{Gal}(\bar{F}_{v_r}/k)$ and that

$$\text{Gal}(\bar{F}_{v_j}/k) = \text{Gal}(\bar{F}_{v_r}/k)/(\bar{F}_{v_r}/F_{v_j})$$

implying the relative degree $[\bar{F}_{v_r} : \bar{F}_{v_j}] = \frac{r}{j}$, being the relative index of the corresponding semigroup, is not an integer unless $j$ divides $r$.

1.4 Bisemilattices of algebraic and transcendental quanta

According to section 1.1. and [Pie2], only symmetric semiobjects have to be considered in any generality in such a way that a bisemiobject $O_R \times O_L$ is composed of the product of the right semiobject $O_R$, localized in the lower half space or on $\mathbb{R}_-$, and of the left symmetric semiobject $O_L$, localized in the upper half space or on $\mathbb{R}_+$. So, instead of envisaging the set of left (resp. right) increasing transcendental (or algebraic) extensions:

$$F_v = \{F_{v_1}, \ldots, F_{v_j, m_j}, \ldots, F_{v_r, m_r}\}$$

(resp. $F_{\tau} = \{F_{\tau_1}, \ldots, F_{\tau_j, m_j}, \ldots, F_{\tau_r, m_r}\}$);

we shall take into account their (diagonal) product

$$F_{\tau} \times F_v : \{F_{\tau_1} \times F_{v_1}, \ldots, F_{\tau_j, m_j} \times F_{v_j, m_j}, \ldots, F_{\tau_r, m_r} \times F_{v_r, m_r}\}$$
in one-to-one correspondence with their semigroups of automorphisms:

\[
\text{Aut}_k(F_{\pi}) \times \text{Aut}_k(F_v) = \\
\{ \text{Aut}_k F_{\pi_1} \times \text{Aut}_k F_{v_1}, \ldots, \text{Aut}_k F_{\pi_j,m_j} \times \text{Aut}_k F_{v_j,m_j}, \ldots, \}
\]

\[
\text{Aut}_k F_{\pi_r,m_r} \times \text{Aut}_k F_{v_r,m_r} \}.
\]

Let

\[
\text{Gal}(\widetilde{F}_{\pi}/k) \times \text{Gal}(\widetilde{F}_v/k) = \\
\{ \text{Gal}(\widetilde{F}_{\pi_1}/k) \times \text{Gal}(\widetilde{F}_{v_1}/k), \ldots, \text{Gal}(\widetilde{F}_{\pi_j,m_j}/k) \times \text{Gal}(\widetilde{F}_{v_j,m_j}/k), \ldots, \\
\{ \text{Gal}(\widetilde{F}_{\pi_r,m_r}/k) \times \text{Gal}(\widetilde{F}_{v_r,m_r}/k) \}
\]

denote the products, right by left, of the Galois semigroups of the sets of increasing algebraic extensions characterized by increasing degrees belonging to the zero class of integers modulo \(N\): they are thus, Weil global semigroups [Pie2]. We thus have the isomorphism:

\[
\text{Gal}(\widetilde{F}_{\pi}/k) \times \text{Gal}(\widetilde{F}_v/k) \xrightarrow{\sim} \text{Aut}_k(F_{\pi}) \times \text{Aut}_k(F_v)
\]

associated with the isomorphism:

\[
\widetilde{F}_{\pi} \times \widetilde{F}_v \xrightarrow{\sim} F_{\pi} \times F_v
\]

between the product, right by left, of sets of symmetric algebraic and transcendental extensions (or archimedean completions), which consists in the the product, right by left, of two symmetric towers of increasing algebraic or compact transcendental extensions composed of an increasing number of algebraic or transcendental compact quanta: this defines a bi(semi)lattice of biquanta, i.e. the product, right by left, of two symmetric (semi)lattices of algebraic or transcendental quanta.

### 1.5 Abstract bismivarieties

Let \(B_{\widetilde{F}_v}\) (resp. \(B_{\widetilde{F}_{\pi}}\)) be a left (resp. right) division semialgebra of real dimension \(2n\) over the set \(\widetilde{F}_v\) (resp. \(\widetilde{F}_{\pi}\)) of increasing real pseudoramified extensions:

\(B_{\widetilde{F}_v}\) (resp. \(B_{\widetilde{F}_{\pi}}\)) is then isomorphic to the semialgebra of Borel upper (resp. lower) triangular matrices:

\[
B_{\widetilde{F}_v} \simeq T_{2n}(\widetilde{F}_v) \quad \text{(resp. } B_{\widetilde{F}_{\pi}} \simeq T_{2n}^t(\widetilde{F}_{\pi})\text{)}
\]

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allowing introducing the algebraic bilinear semigroup of matrices by:

\[ B_{\tilde{F}_\tau} \otimes B_{\tilde{F}_v} \simeq T_{2n}^d(\tilde{F}_\tau) \times T_{2n}(\tilde{F}_v) \simeq \text{GL}_{2n}(\tilde{F}_\tau \times \tilde{F}_v) \]

in such a way that its representation (bisemi)space is given by the tensor product \( \tilde{M}_{v_R}^{(2n)} \otimes \tilde{M}_{v_L}^{(2n)} \) of a right \( T_{2n}(\tilde{F}_\tau) \)-semimodule \( \tilde{M}_{v_R}^{(2n)} \), localized in the lower half space, by a left \( T_{2n}(\tilde{F}_v) \)-semimodule \( \tilde{M}_{v_L}^{(2n)} \), localized in the upper half space.

The \( \text{GL}_{2n}(\tilde{F}_\tau \times \tilde{F}_v) \)-bisseimodule \( \tilde{M}_{v_R}^{(2n)} \otimes \tilde{M}_{v_L}^{(2n)} \) is an algebraic bilinear (affine) semigroup noted \( G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) \) whose bilinear semigroup of Galois automorphisms is \( \text{GL}(\tilde{M}_{v_R}^{(2n)} \otimes \tilde{M}_{v_L}^{(2n)}) \).

\( \text{GL}(\tilde{M}_{v_R}^{(2n)} \otimes \tilde{M}_{v_L}^{(2n)}) \) constitutes the \( 2n \)-dimensional equivalent of the product \( \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \) of Galois semigroups in such a way that \( G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) \) becomes the \( 2n \)-dimensional (irreducible) representation space \( \text{Irr Rep}_{\text{Gal}_{\tilde{F}_\tau \times \tilde{F}_v}} \)(\( \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \)) of \( \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \).

\[ G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) = \text{Irr Rep}_{\text{Gal}_{\tilde{F}_\tau \times \tilde{F}_v}} \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \]

implies the monomorphisms:

\[ \sigma_{\tilde{F}_\tau} \times \sigma_{\tilde{F}_v} : (\text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k)) \rightarrow \text{GL}(\tilde{M}_{v_R}^{(2n)} \otimes \tilde{M}_{v_L}^{(2n)}) \approx G^{(2n)}(\tilde{F}_\tau \otimes \tilde{F}_v) . \]

The isomorphism

\[ \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \rightarrow \text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v) \]

between the products, right by left, of Weil global semigroups and corresponding automorphism semigroups of transcendental extensions leads to the commutative diagram

\[ \text{Gal}(\tilde{F}_\tau/k) \times \text{Gal}(\tilde{F}_v/k) \xrightarrow{\sigma_{\tilde{F}_\tau} \times \sigma_{\tilde{F}_v}} G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) \]

\[ \text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v) \xrightarrow{\sigma_{\tilde{F}_\tau} \times \sigma_{\tilde{F}_v}} G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) \]

where the monomorphism \( \sigma_{\tilde{F}_\tau} \times \sigma_{\tilde{F}_v} \) generates the abstract bisemivariety \( G^{(2n)}(F_\tau \times F_v) \) on the product of sets of symmetric compact transcendental extensions (or archimedean completions).

The abstract bisemivariety \( G^{(2n)}(F_\tau \times F_v) \) is covered by the algebraic bilinear (affine) semigroup \( G^{(2n)}(\tilde{F}_\tau \times \tilde{F}_v) \) and is thus a complete (locally) compact (algebraic) bilinear semigroup.
At every infinite biplace $\mathfrak{v}_j \times \mathfrak{v}_j$ of $F_{\mathfrak{v}} \times F_{\mathfrak{v}}$ corresponds a conjugacy class $g_{v_{R \times L}}^{(2n)}[j]$ of the abstract bisemivariety $G^{(2n)}(\tilde{F}_{\mathfrak{v}} \times \tilde{F}_{\mathfrak{v}})$. The number of representatives of $g_{v_{R \times L}}^{(2n)}[j]$ corresponds to the number of equivalent extensions of $\tilde{F}_{\mathfrak{v}_j} \times \tilde{F}_{\mathfrak{v}_j}$.

Let $\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}$ (resp. $\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}$) be the maximal order of $\tilde{F}_{\mathfrak{v}}$ (resp. $\tilde{F}_{\mathfrak{v}}$). Then, $\Lambda_{\mathfrak{v}} = \mathcal{O}_{B_{\tilde{F}_{\mathfrak{v}}}}$ (resp. $\Lambda_{\mathfrak{v}} = \mathcal{O}_{B_{\tilde{F}_{\mathfrak{v}}}}$) in the division semialgebra $B_{\tilde{F}_{\mathfrak{v}}}$ (resp. $B_{\tilde{F}_{\mathfrak{v}}}$) is a pseudo-ramified $\mathbb{Z}/N\mathbb{Z}$-lattice, in the left (resp. right) $B_{\tilde{F}_{\mathfrak{v}}}$-semimodule $\tilde{M}_{v_{L}}^{(2n)}$ (resp. $B_{\tilde{F}_{\mathfrak{v}}}$-semimodule $M_{v_{R}}^{(2n)}$).

So, we have that

$$
\Lambda_{\mathfrak{v}} \simeq T_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}) \simeq T_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}) \quad (\text{resp.} \quad \Lambda_{\mathfrak{v}} \simeq T_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}) \simeq T_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}) )
$$

leading to

$$
\Lambda_{\mathfrak{v}} \times \Lambda_{\mathfrak{v}} = T_{2n}^t(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}}) \times T_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}})
= GL_{2n}(\mathcal{O}_{\tilde{F}_{\mathfrak{v}}} \times \mathcal{O}_{\tilde{F}_{\mathfrak{v}}})
= GL_{2n}(\mathbb{Z}/N\mathbb{Z})^2.
$$

Then, the representation space $\text{Repsp}(GL_{2n}(\mathbb{Z}/N\mathbb{Z})^2)$ of $GL_{2n}(\mathbb{Z}/N\mathbb{Z})^2$ decomposes according to:

$$
\text{Repsp}(GL_{2n}(\mathbb{Z}/N\mathbb{Z})^2) = \bigoplus_{j,m_j} \left( \Lambda_{\mathfrak{v}_j,m_j} \otimes \Lambda_{\mathfrak{v}_j,m_j} \right)
$$

where $\Lambda_{\mathfrak{v}_j,m_j}$ (resp. $\Lambda_{\mathfrak{v}_j,m_j}$) is the $(j, m_j)$-th subsemilattice referring to the conjugacy class representative $g_{v_{L}}^{(2n)}[j, m_j] \in G^{(2n)}(F_{\mathfrak{v}})$ (resp. $g_{v_{R}}^{(2n)}[j, m_j] \in G^{(2n)}(F_{\mathfrak{v}})$).

The pseudo-ramified Hecke bisemialgebra $H_{R \times L}(2n)$ of all Hecke bioperators $T_{R}(2n; r) \otimes T_{L}(2n; r)$, having a representation in the arithmetic subgroup of matrices $GL_{2n}(\mathbb{Z}/N\mathbb{Z})^2$, generates the endomorphisms of the $B_{F_{\mathfrak{v}}} \times B_{F_{\mathfrak{v}}}$-bisemimodule $(M_{v_{R}}^{(2n)} \otimes M_{v_{L}}^{(2n)})$ decomposing it according to the bisubsemilattices $(\Lambda_{\mathfrak{v}_j,m_j} \otimes \Lambda_{\mathfrak{v}_j,m_j})$ [Pie3]:

$$
M_{v_{R}}^{(2n)} \otimes M_{v_{L}}^{(2n)} = \bigoplus_{j,m_j} (M_{\mathfrak{v}_j,m_j}^{(2n)} \otimes M_{\mathfrak{v}_j,m_j}^{(2n)}) .
$$

Let

$$
F_{v}^{(nr)} = \{ F_{v_1}^{(nr)}, \ldots, F_{v_j,m_j}^{(nr)}, \ldots, F_{v_{nr},m_{nr}}^{(nr)} \}
$$

(resp. $F_{\mathfrak{v}}^{(nr)} = \{ F_{\mathfrak{v}_1}^{(nr)}, \ldots, F_{\mathfrak{v}_j,m_j}^{(nr)}, \ldots, F_{\mathfrak{v}_{nr},m_{nr}}^{(nr)} \}$).
be the set of left (resp. right) increasing pseudounramified transcendental extensions homeomorphic to the corresponding pseudounramified algebraic extensions introduced in section 1.2.

Let \( G^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)}) \) be the complete bilinear semigroup with entries in \( (F_{F_v}^{(nr)} \times F_v^{(nr)}) \).

Then, the kernel \( \text{Ker}(G^{(2n)}_{F \to F^{(nr)}}) \) of the map:

\[
G^{(2n)}_{F \to F^{(nr)}} : G^{(2n)}(F_{F_v} \times F_v) \rightarrow G^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)})
\]

is the smallest bilinear normal pseudounramified subgroup of \( G^{(2n)}(F_{F_v} \times F_v) \):

\[
\text{Ker}(G^{(2n)}_{F \to F^{(nr)}}) = P^{(2n)}(F_{F_v} \times F_{v_1}),
\]

i.e. the parabolic bilinear subsemigroup over the product \( F_{F_v} \times F_{v_1} \) of sets

\[
F_{F_v} = \{ F_{F_v,1}, \ldots, F_{F_v,j,m_j}, \ldots, F_{F_v,r,m_r} \} \quad \text{and} \quad F_{v_1} = \{ F_{v_1,1}, \ldots, F_{v_1,j,m_j}, \ldots, F_{v_1,r,m_r} \}
\]

of “unitary” transcendental pseudounramified extensions.

### 1.6 Proposition

The bilinear abstract parabolic semigroup \( P^{(2n)}(F_{F_v} \times F_{v_1}) \) is the unitary (irreducible) representation space of the complete bilinear semigroup \( GL_2(F_{F_v} \times F_v) \) of matrices.

**Proof:**

1) The abstract bisemivariety \( G^{(2n)}(F_{F_v} \times F_v) \), covered by the algebraic (affine) bisemivariety \( G^{(2n)}(\tilde{F}_{F_v} \times \tilde{F}_v) \), acts by conjugation on the bilinear parabolic subsemigroup \( P^{(2n)}(F_{F_v} \times F_{v_1}) \) in such a way that the number of conjugates of \( P^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)}) \) in the conjugacy class representative \( G^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)}) \equiv \tilde{g}_{r \times l}[j,m_j] \in G^{(2n)}(F_{F_v} \times F_v) \) is the index

\[
\left| G^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)}) : P^{(2n)}(F_{F_v}^{(nr)} \times F_v^{(nr)}) \right| = j
\]

of the normalizer \( P^{(2n)}(F_{F_v} \times F_v) \) in \( G^{(2n)}(F_{F_v} \times F_v) \).

2) Let \( \text{Out}(G^{(2n)}(F_{F_v} \times F_v)) = \text{Aut}(G^{(2n)}(F_{F_v} \times F_v))/\text{Int}(G^{(2n)}(F_{F_v} \times F_v)) \) be the bisemigroup [Pie4] of transcendental automorphisms of the complete bilinear semigroup \( G^{(2n)}(F_{F_v} \times F_v) \) where \( \text{Int}(G^{(2n)}(F_{F_v} \times F_v)) \) is the bisemigroup of transcendental inner automorphisms.

As, we have that:

\[
\text{Int}(G^{(2n)}(F_{F_v} \times F_v)) = \text{Aut}(P^{(2n)}(F_{F_v} \times F_{v_1}))
\]

and considering 1), it appears that \( P^{(2n)}(F_{F_v} \times F_{v_1}) \) is the unitary representation bisemispace with respect to the abstract bisemivariety \( G^{(2n)}(F_{F_v} \times F_v) \).
1.7 Corollary

The rank of the bilinear parabolic semigroup $P^{(2n)}(F_{v_j} \times F_{v_j})$ restricted to the $j$-th conjugacy class is

$$r_{P^{(2n)}(F_{v_j} \times F_{v_j})} \simeq (m_j \cdot N)^n \times (D) (m_j \cdot N)^n$$

and the rank of the conjugacy class representative $G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j})$ is

$$r_{G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j})} \simeq (j \cdot m_j \cdot N)^n \times (D) (j \cdot m_j \cdot N)^n$$

where $\times_D$ is the notation for a diagonal product [Pie4].

Proof: As the bilinear parabolic semigroup $P^{(2n)}(F_{v_j} \times F_{v_j})$ restricted to the $j$-th conjugacy class is the $2n$-dimensional representation space of the product, right by left, of inertia subgroups

$$I_{F_{v_j}} \times I_{F_{v_j}} = [\text{Gal}(F_{v_j}/k) \times \text{Gal}(F_{v_j}/k)] \div [\text{Gal}(F_{v_j}^{(nr)}/k) \times \text{Gal}(F_{v_j}^{(nr)}/k)]$$

according to [Pie2] and as the order of $I_{F_{v_j}}$ or $I_{F_{v_j}}$ is $N$, we have that

$$r_{P^{(2n)}(F_{v_j} \times F_{v_j})} = (m_j \cdot N)^n \times (D) (m_j \cdot N)^n$$

if it is taken into account that “$m_j$” real equivalent conjugacy class representatives $g_{v_{R,L}}^{(n)}[j]$ of dimension “$n$” cover one complex conjugacy class representative $g_{\omega_{R,L}}^{(2n)}(j)$ [Pie5] of dimension $2n$:

$$g_{\omega_{R,L}}^{(2n)}[j] = \{g_{v_{R,L}}^{(n)}[j,m_j]\}_{m_j}.$$

It is then immediate to see that the rank of $G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j}) = g_{R\times L}^{(2n)}[j,m_j]$ is

$$r_{G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j})} \simeq (j \cdot m_j \cdot N)^n \times (D) (j \cdot m_j \cdot N)^n$$

1.8 Corollary

1) The number of transcendental quanta of the bilinear parabolic semigroup $P^{(2n)}(F_{v_j} \times F_{v_j}), \forall j \in \mathbb{N}$, is

$$n(P^{(2n)}(F_{v_j} \times F_{v_j})) = \mathbb{I}^n \times (D) \mathbb{I}^n \longrightarrow (m_j)^n \times (D) (m_j)^n.$$

2) The number of transcendental quanta of the conjugacy class representative $G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j})$ is

$$n(G^{(2n)}(F_{v_j,m_j} \times F_{v_j,m_j})) = (j \cdot m_j)^n \times (D) (j \cdot m_j)^n.$$

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Proof: This results from corollary 1.7 by taking into account that the degree of a transcendental quantum is N. According to section 1.5, we thus have a bisemilattice of Σ(j x m_j)^n transcendental biquanta.

1.9 Covering of complex abstract bisemivarieties by real abstract bisemivarieties

Let M_{ω_R}^{(2n)} ⊗ M_{ω_L}^{(2n)} denote a complex GL_n( F_v × F_ω) -bsemimodule being the representation space of the complete bilinear semigroup of matrices GL_n( F_v × F_ω) of the product ( F_v × F_ω) , right by left, of complex transcendental extensions covered by their real equivalents ( F_v × F_ω) as developed in [Pie2].

If each conjugacy class representative G^{(2n)}( F_v × F_ωj) of G^{(2n)}( F_v × F_ω) ≡ M_{ω_R}^{(2n)} ⊗ M_{ω_L}^{(2n)} is unique in this j-th class, then G^{(2n)}( F_v × F_ωj) is covered by m_j real equivalent conjugacy class representatives G^{(n)}( F_vj,m_j × F_vj,m_j) of G^{(n)}( F_v × F_ω) [Pie2].

So, the complex bipoints of G^{(2n)}( F_v × F_ω) are in one-to-one correspondence with the real bipoints of G^{(2n)}( F_v × F_ω) and we have the inclusion:

\[
\frac{G^{(2n)}(F_v × F_ω)}{G^{(n)}(F_v × F_ω)} \cong M_{v_R}^{(2n)} ⊗ M_{v_L}^{(2n)} → M_{ω_R}^{(2n)} ⊗ M_{ω_L}^{(2n)}
\]

of the real abstract bisemivariety M_{v_R}^{(2n)} ⊗ M_{v_L}^{(2n)} into the complex abstract bisemivariety M_{ω_R}^{(2n)} ⊗ M_{ω_L}^{(2n)}.

1.10 Corner stone of the global program of Langlands

The real abstract bisemivariety M_{v_R}^{(2n)} ⊗ M_{v_L}^{(2n)} ≡ G^{(2n)}( F_v × F_ω) , being the representation space of the bilinear algebraic semigroup GL_{2n}( F_v × F_ω) of matrices, constitutes the 2n-dimension representation space of the product, right by left, Gal( F_v/k) × Gal( F_ω/k) of Galois semigroups according to section 1.5. So, we get the isomorphism GL(M_{v_R}^{(2n)} ⊗ M_{v_L}^{(2n)}) ≃ GL_{2n}( F_v × F_ω).

So, G^{(2n)}(F_v × F_ω) constitutes the corner stone of the real global correspondence of Langlands recalled in [Pie5] since, by an isomormophism of toroidal compactification, G^{(2n)}(F_v × F_ω) is transformed into the cuspidal representation Π,GL_{2n}( F_v × F_ω) of the algebraic bilinear semigroup of matrices GL_{2n}( F_v × F_ω).

1.11 Reducible representations of GL_{2n}( F_v × F_ω)

We have envisaged until now that the representation of the general bilinear semigroup GL_{2n}( F_v × F_ω) was irreducible. But, in consideration of the future developments of this
paper, it is useful to take into account its reducibility.
Let $2n = 2_1 + 2_2 + \cdots + 2_k + \cdots + 2_\ell$ be a partition of $2n$ labeling the reducible representation of $T_{2n}(F_v)$ (resp. $T_{2n}^t(F_\overline{\tau})$).

Then, we have that:

1) the representation $\text{Rep}(GL_{2n}(F_{\overline{\tau}} \times F_v))$ of the general bilinear semigroup $GL_{2n}(F_{\overline{\tau}} \times F_v) = T_{2n}^t(F_{\overline{\tau}}) \times T_{2n}^t(F_v)$ is orthogonally completely reducible if it decomposes diagonally according to the direct sum of 2-dimensional irreducible representations $\text{Rep}(GL_{2k}(F_\overline{\tau} \times F_v))$:

$$\text{Rep}(GL_{2n}(F_{\overline{\tau}} \times F_v)) = \bigoplus_{2k=2}^{2n} \text{Rep}(GL_{2k}(F_\overline{\tau} \times F_v))$$

and it is nonorthogonally completely reducible if it decomposes diagonally according to the direct sum of irreducible 2-dimensional bilinear representations $\text{Rep}(GL_{2k}(F_\overline{\tau} \times F_v))$ and offdiagonally according to the direct sum of irreducible mixed bilinear representations $\text{Rep}(T_{2k}^t(F_\overline{\tau}) \times T_{2\ell}(F_v))$.

Indeed, taking into account the existence of cross products in the definition of bilinear semigroups [Pie4], we have that the representation of $GL_{2n}(F_{\overline{\tau}} \times F_v)$ can be reduced to:

$$\text{Rep}(GL_{2n}(F_{\overline{\tau}} \times F_v)) = \left( \bigoplus_{2k=2}^{2n} \text{Rep}(T_{2k}^t(F_{\overline{\tau}})) \otimes \bigoplus_{2\ell=2}^{2n} \text{Rep}(T_{2\ell}(F_v)) \right)$$

$$= \bigoplus_{2k=2}^{2n} \text{Rep}(GL_{2k}(F_{\overline{\tau}} \times F_v)) \bigoplus_{2k \neq 2\ell}^{2n} \text{Rep}(T_{2k}^t(F_{\overline{\tau}})) \times T_{2\ell}(F_v))$$

If the mixed bilinear representations $\bigoplus_{2k \neq 2\ell}^{2n} \text{Rep}(T_{2k}^t(F_{\overline{\tau}})) \times T_{2\ell}(F_v)$ are equal to 0, then the above completely reducible nonorthogonal representation of $GL_{2n}(F_{\overline{\tau}} \times F_v)$ reduces to the orthogonal case.
Universal dynamical structures of the global program of Langlands

2.1 Dynamical functional representation spaces of abstract bisemivarieties

In order to generate dynamical $\text{GL}(2n)$-bisemistuctures $[\text{Pie}3]$ referring to the geometric program of Langlands, we have to take into account the differentiable functional representation space $\text{FREPSP}(\text{GL}_{2n}(F_v \times F_v))$ of the complete bilinear semigroup $\text{GL}_{2n}(F_v \times F_v)$, i.e., a bisemisheaf $\hat{M}_{v_R}^{(2n)} \otimes \hat{M}_{v_L}^{(2n)}$ of differentiable bifunctions on the abstract bisemivariety $G^{(2n)}(F_v \times F_v)$ which is given by the product, right by left, of a right semisheaf $\hat{M}_{v_R}^{(2n)}$ of differentiable (co)functions on the abstract right semivariety $G^{(2n)}(F_v)$ $\equiv T^{(2n)}(F_v)$ by a left semisheaf $\hat{M}_{v_L}^{(2n)}$ of symmetric differentiable functions on the abstract left semivariety $G^{(2n)}(F_v)$ $\equiv T^{(2n)}(F_v)$.

This functional representation space $\text{FREPSP}(\text{GL}_{2n}(F_v \times F_v))$ of bilinear geometric dimension $2n$ splits into:

$$\text{FREPSP}(\text{GL}_{2n}(F_v \times F_v)) = \text{FREPSP}(\text{GL}_{2k}(F_v \times F_v)) \oplus \text{FREPSP}(\text{GL}_{2n-2k}(F_v \times F_v))$$

in such a way that $\text{FREPSP}(\text{GL}_{2k}(F_v \times F_v))$, $k \leq n$, be the functional representation space of bilinear geometric dimension $2k$ of the bilinear semigroup $\text{GL}_{2k}(F_v \times F_v)$ on which acts the elliptic bioperator $D^{2k}_R \otimes D^{2k}_L$, i.e. the product of a right linear differential elliptic operator $D^{2k}_R$ acting on $2k$ variables by its left equivalent $D^{2k}_L$, by its biaction:

$$D^{2k}_R \otimes D^{2k}_L : \text{FREPSP}(\text{GL}_{2n}(F_v \times F_v)) \longrightarrow \text{FREPSP}(\text{GL}_{2n|2k}(F_v \times \mathbb{IR}) \times (F_v \times \mathbb{IR}))$$

where $\text{FREPSP}(\text{GL}_{2n|2k}(F_v \times \mathbb{IR}) \times (F_v \times \mathbb{IR}))$ is the functional representation space of $\text{GL}_{2n}(F_v \times F_v)$ fibered or shifted into “$2k$” bilinear geometric dimensions. $F_v^{SR} = (F_v \times \mathbb{IR})$ (resp. $F_v^{SR} = (F_v \times \mathbb{IR})$) denotes the set of increasing left (resp. right) transcendental extensions (or archimedean completions) fibered or shifted by real numbers

$$F_v \times \mathbb{IR} = \{F_v^{S_{v_1}}, \ldots, F_v^{S_{v_j,m_j}}, \ldots, F_v^{S_{v_r,m_r}}\}$$

(resp. $F_v \times \mathbb{IR} = \{F_v^{S_{\tau_1}}, \ldots, F_v^{S_{\tau_j,m_j}}, \ldots, F_v^{S_{\tau_r,m_r}}\}$)

from their unshifted equivalents, i.e. fibre bundles with vertical fibres $\mathbb{IR}$ and basis $F_v$ (resp. $F_\tau$).

Remark that the shifted transcendental extension $F_v^{SR}_{v_j,m_j}$ (resp. $F_v^{SR}_{\tau_j,m_j}$) is composed of $j$ left (resp. right) fibered or shifted transcendental quanta.
\section*{2.2 Bilinear fibre of tangent bibundle}

According to chapter 3 of [Pie3], the shifted functional representation space \( \text{FREPSP}(\text{GL}_{2n}[2k](F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R})) \) decomposes into:

\[
\text{FREPSP}(\text{GL}_{2n}[2k](F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R})) = \text{FREPSP}(\text{GL}_{2k}(F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R})) \oplus \text{FREPSP}(\text{GL}_{2n-2k}(F_{\tau} \times F_{v}))
\]

in such a way that \( \text{FREPSP}(\text{GL}_{2k}(F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R})) \) is the total bisemispace \( (\Delta_{R}^{2k} \times \Delta_{L}^{2k}) \) of the tangent bibundle \( \text{TAN}(\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}) \) to the bisemisheaf \( (\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}) \equiv \text{FREPSP}(\text{GL}_{2k}(F_{\tau} \times F_{v})) \) and is isomorphic to the adjoint (functional) representation space of \( \text{GL}_{2k}(F_{\tau} \times F_{v}) \) corresponding to the action of the bioperator \( (D_{R}^{2k} \times D_{L}^{2k}) \) on \( (\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}) \) which maps \( \text{TAN}(\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}) \simeq \text{Ad}(F) \text{REPS}(\text{GL}_{2k}(F_{\tau} \times F_{v})) \) into itself.

We have thus that:

\[
\Delta_{R}^{2k} \times \Delta_{L}^{2k} \simeq \text{Ad}(F) \text{REPS}(\text{GL}_{2k}(F_{\tau} \times F_{v})) \simeq (F) \text{REPS}(\text{GL}_{2k}(F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R}))
\]

where \( (F) \text{REPS}(\text{GL}_{2k}(\mathbb{R} \times \mathbb{R})) \) is the bilinear fibre \( \mathcal{F}_{R \times L}(\text{TAN}) \) of \( \text{TAN}(\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}) \).

\section*{2.3 Symbol of the bioperator \( (D_{R}^{2k} \times D_{L}^{2k}) \)}

Referring to the classical definition [A-S] of the symbol \( \sigma(D) \) of a differential linear operator \( D \) dealing with the unit sphere bundles in the cotangent vector bundle \( T^{*}(X) \) of the compact smooth manifold \( X \), we introduce the symbol \( \sigma(D_{R}^{2k} \times D_{L}^{2k}) \) of the differential bioperator \( (D_{R}^{2k} \times D_{L}^{2k}) \) by:

\[
\sigma(D_{R}^{2k} \times D_{L}^{2k}) \simeq \text{REPS}(P_{2k}(\mathbb{R} \times \mathbb{R})|_{F_{\tau} \times F_{v}})
\]

i.e. the unitary functional representation space of \( \text{GL}_{2k}((\mathbb{R} \times \mathbb{R})|_{F_{\tau} \times F_{v}}) \) given by the functional representation space of the fibering or shifting bilinear parabolic semigroup \( P_{2k}(\mathbb{R} \times \mathbb{R})|_{F_{\tau} \times F_{v}} \).

So, the differential bioperator \( D_{R}^{2k} \times D_{L}^{2k} \) maps from the bisemisheaf

\[
(\hat{M}_{vR}^{2n}) \otimes (\hat{M}_{vL}^{2n}) = \text{FREPSP}(\text{GL}_{2n}(F_{\tau} \times F_{v}))
\]

into the corresponding perverse bisemisheaf

\[
(\hat{M}_{vR}^{2n}[2k]) \otimes (\hat{M}_{vL}^{2n}[2k]) = \text{FREPSP}(\text{GL}_{2n}[2k](F_{\tau} \times \mathbb{R}) \times (F_{v} \times \mathbb{R}))
\]
shifted in $2k$ geometric dimensions, according to:

$$D_R^{2k} \times D_L^{2k} : \ (\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)}) \rightarrow (\widehat{M}_{v_R}^{(2n)}[2k] \otimes \widehat{M}_{v_L}^{(2n)}[2k]) ,$$

while $\sigma(D_R^{2k} \times D_L^{2k})$ maps the “unitary” bisemisheaf

$$(\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)}) = \text{FREPSP}(P_{2n}(F_{v^1} \times F_{v^1}))$$

$\equiv \text{FREPSP}(\text{GL}_{2n}(F_{v^1} \times F_{v^1}))$

into the corresponding “unitary” perverse bisemisheaf

$$(\widehat{M}_{v_R}^{(2n)}[2k] \otimes \widehat{M}_{v_L}^{(2n)}[2k]) = \text{FREPSP}(P_{2n[2k]}(F_{v^1} \times IR) \times (F_{v^1} \times IR))$$

$$\equiv \text{FREPSP}(\text{GL}_{2n[2k]}(F_{v^1} \times IR) \times (F_{v^1} \times IR))$$

according to:

$$\sigma(D_R^{2k} \times D_L^{2k}) : \ (\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)}) \rightarrow (\widehat{M}_{v_R}^{(2n)}[2k] \otimes \widehat{M}_{v_L}^{(2n)}[2k]) .$$

### 2.4 Proposition

1. The bilinear semigroup of matrices $\text{GL}_r(\mathbb{R}^{(2k)} \times \mathbb{R}^{(2k)})$ with algebraic orders “$r$”, referring to the biaction of the bioperator $(D_R^{2k} \times D_L^{2k})$ on the real bisemisheaf $(\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)})$ over the abstract real bisemivariety $G^{(2k)}(F_{v^1} \times F_{v^1}) \equiv \text{REPS}(\text{GL}_{2k}(F_{v^1} \times F_{v^1}))$, corresponds to the $2k$-dimensional bilinear real representation of the product, right by left, of “differential” Galois (or global Weil) semigroups $\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R}))$ fibering or “shifting” the product, right by left, of automorphism semigroups $\text{Aut}_L(\phi_R(F_{v^1})) \times \text{Aut}_k(\phi_L(F_{v^1}))$ of cofunctions $\phi_R(F_{v^1})$ and functions $\phi_L(F_{v^1})$ respectively on the compact transcendental real extensions $F_{v^1}$ and $F_{v^1}$ by $\text{Aut}_k(\phi_R(F_{v^1} \times \mathbb{R})) \times \text{Aut}_k(\phi_L(F_{v^1} \times \mathbb{R}))$.

So we have that:

$$\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R}))) = \text{GL}_r(\phi_R(\mathbb{R}^{(2k)}) \times \phi_L(\mathbb{R}^{(2k)})) .$$

2. The unitary parabolic bilinear semigroup $P_r(\mathbb{R}^{(2k)} \times \mathbb{R}^{(2k)}) \subset \text{GL}_r(\mathbb{R}^{(2k)} \times \mathbb{R}^{(2k)})$, referring to the biaction of the bioperator on the unitary bisemisheaf $(\widehat{M}_{v_R}^{(2n)} \otimes \widehat{M}_{v_L}^{(2n)})$ over the unitary abstract bisemivariety $P^{(2k)}(F_{v^1} \times F_{v^1}) = \text{REPS}(P_{2k}(F_{v^1} \times F_{v^1}))$, corresponds to the $2k$-dimensional bilinear representation of the product, right by left, of “differential” inertia Galois (or global Weil) semigroups $\text{Int}_k(\phi_R(\mathbb{R})) \times$
Proof:

1. The abstract bisemivariety \( G^{(2k)}(F_\pi \times F_v) \equiv M_{\pi R}^{(2k)} \otimes M_{vL}^{(2k)} \) decomposes according to the increasing filtration of its conjugacy class representatives:

\[
G^{(2k)}(F_{\pi_1} \times F_{v_1}) \subset \cdots \subset G^{(2k)}(F_{\pi_{j,m_j}} \times F_{v_{j,m_j}}) \subset \cdots \subset G^{(2k)}(F_{\pi_{r,m_r}} \times F_{v_{r,m_r}})
\equiv M_{\pi_1}^{(2k)} \otimes M_{v_1}^{(2k)} \subset \cdots \subset M_{\pi_{j,m_j}}^{(2k)} \otimes M_{v_{j,m_j}}^{(2k)} \cdots \subset M_{\pi_{r,m_r}}^{(2k)} \otimes M_{v_{r,m_r}}^{(2k)},
\]

in one-to-one correspondence with the filtration of the product \( (F_\pi \times F_v) \) of sets of archimedean transcendental extensions (see section 1.2).

Similarly, the unitary abstract bisemivariety \( P^{(2k)}(F_\pi \times F_v) \equiv M_{\pi R}^{(2k)} \otimes M_{vL}^{(2k)} \), which is a bilinear parabolic subsemigroup, decomposes according to the increasing set of its representatives:

\[
P^{(2k)}(F_{\pi_1} \times F_{v_1}) \subset \cdots \subset P^{(2k)}(F_{\pi_{j,m_j}} \times F_{v_{j,m_j}}) \subset \cdots \subset P^{(2k)}(F_{\pi_{r,m_r}} \times F_{v_{r,m_r}})
\equiv M_{\pi_1}^{(2k)} \otimes M_{v_1}^{(2k)} \subset \cdots \subset M_{\pi_{j,m_j}}^{(2k)} \otimes M_{v_{j,m_j}}^{(2k)} \cdots \subset M_{\pi_{r,m_r}}^{(2k)} \otimes M_{v_{r,m_r}}^{(2k)},
\]

in one-to-one correspondence with the filtration of the product \( (F_\pi \times F_v) \) of sets of unitary transcendental extensions.

2. Referring to the “infinite” general bilinear semigroup given by \( \text{GL}(F_\pi \times F_v) = \varinjlim \text{GL}_m(F_\pi \times F_v) \) in such a way that \( \text{GL}_m(F_\pi \times F_v) \) embeds in \( \text{GL}_{m+1}(F_\pi \times F_v) \) according to the geometric dimension “m”, it is possible to introduce as in [Pie5] an “infinite quantum” general bilinear semigroup by

\[
\text{GL}^{(Q)}(F_\pi^{(2k)} \times F_v^{(2k)}) = \varinjlim_{j=1}^{r} \text{GL}_j^{(Q)}(F_{\pi_1}^{(2k)} \times F_{v_1}^{(2k)}),
\]

in such a way that:
3. Referring to section 2.3, we see that the set \( \{ \phi(G^{(2k)}(F_v^{(2k)} \times F_{v_1}^{(2k)})) \}_{j,m_j} \) of conjugacy class representatives of the bisemivariety \( G^{(2k)}(F_v \times F_{v_1}) \), generated from the bilinear semigroup of matrices \( GL_{2k}(F_v \times F_{v_1}) \), can be rewritten according to the set \( \{ G^{(j,m_j)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \}_{j,m_j} \), where the integer “\( j \)” denotes a global residue degree, i.e. an algebraic dimension, while the integer “\( 2k \)” refers to a geometric dimension.

It was then proved in [Pie5] that the “infinite quantum” general bilinear semigroup \( GL^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \), defined with respect to the unitary parabolic bilinear semigroup \( GL_1^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) = P_{2k}(F_v \times F_{v_1}) \), is the general bilinear semigroup \( GL_{2k}(F_v \times F_{v_1}) \).

So, the set \( \{ G^{(2k)}(F_{v_{j,m_j}} \times F_{v_{j,m_j}}) \}_{j,m_j} \) of conjugacy class representatives of the bisemivariety \( G^{(2k)}(F_v \times F_{v_1}) \), generated from the bilinear semigroup of matrices \( GL_{2k}(F_v \times F_{v_1}) \), can be rewritten according to the set \( \{ G^{(j,m_j)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \}_{j,m_j} \) of increasing conjugacy class representatives of the bisemivariety \( G^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) = G^{(2k)}(F_v \times F_{v_1}) \) generated from \( GL^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \).

And, thus, the filtration

\[
G^{(1)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \subset \cdots \subset G^{(j,m_j)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \subset \cdots \subset G^{(r,m_r)}(F_v^{(2k)} \times F_{v_1}^{(2k)})
\]

of conjugacy class representatives of \( G^{(2k)}(F_v \times F_{v_1}) \) is now written with reference to the increasing algebraic dimensions of the bilinear subsemigroups of matrices:

\[
GL_1^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \subset \cdots \subset GL^{(Q)}_{j,m_j}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \subset \cdots \subset GL^{(Q)}_{r,m_r}(F_v^{(2k)} \times F_{v_1}^{(2k)})
\]

3. Referring to section 2.3, we see that the set \( \{ \phi(G^{(2k)}(F_{v_{j,m_j}} \times F_{v_{j,m_j}})) \}_{j,m_j} \) of conjugacy class representatives, or bissections, of the bisem sheaf \( (\hat{M}^{(2n)}_{v_R} \otimes \hat{M}^{(2n)}_{v_L}) = FREPSP(GL_{2k}(F_v \times F_{v_1})) \), which can be rewritten according to \( \{ \phi(G^{(j,m_j)}(F_v^{(2k)} \times F_{v_1}^{(2k)})) \}_{j,m_j} \) with respect to \( GL^{(Q)}(F_v^{(2k)} \times F_{v_1}^{(2k)}) \), are fibered or shifted, as tangent bibundles, under the action of the bioperator \( (D_{2k}^{(2k)} \otimes D_{2k}^{(2k)}) \) into:

\[
D_{2k}^{(2k)} \otimes D_{2k}^{(2k)} : (\hat{M}^{(2n)}_{v_R} \otimes \hat{M}^{(2n)}_{v_L}) \rightarrow (\hat{M}^{(2n)}_{v_R}[2k] \otimes \hat{M}^{(2n)}_{v_L}[2k])
\]

\[
\{ \phi(G^{(j,m_j)}(F_v^{(2k)} \times F_{v_1}^{(2k)})) \}_{j,m_j}
\]

\[
\rightarrow \{ \phi(G^{(j,m_j)}(F_v^{(2k)} \times I^{(2k)}_v \times (F_{v_1}^{(2k)})) \times I^{(2k)}_{v_1}) \}_{j,m_j}
\]

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Similarly, the set \( \{ \phi(P^{(2k)}(F_{\pi_{j,m}} \times F_{v_{1,m}})) \}_{j,m} \) of bisections of the unitary bismisheaf \( (\hat{\mathcal{M}}_{v_{1,k}}^{(2k)} \otimes \hat{\mathcal{M}}_{v_{1,k}}^{(2k)}) = \text{FREPSP}(P_{2k}(F_{\pi_{1}} \times F_{v_{1}})) \), which can be rewritten according to \( \{ \phi(P^{(j,m)}(F_{\pi_{1}} \times F_{v_{1}})) \}_{j,m} \) are fibered or shifted under the action of the bioperator \( (D_{R}^{2k} \otimes D_{L}^{2k}) \) into:

\[
D_{R}^{2k} \otimes D_{L}^{2k} : (\hat{\mathcal{M}}_{v_{1,k}}^{(2n)} \otimes \hat{\mathcal{M}}_{v_{1,k}}^{(2n)}) \rightarrow (\hat{\mathcal{M}}_{v_{1,k}}^{(2n)}[2k] \otimes \hat{\mathcal{M}}_{v_{1,k}}^{(2n)}[2k]) \rightarrow \{ \phi(P^{(j,m)}(F_{\pi_{1}} \times F_{v_{1}})) \}_{j,m} \rightarrow \{ \phi(P^{(j,m)}(F_{\pi_{1}} \times IR^{(2k)})) \times (F_{v_{1}}^{(2k)}) \times IR^{(2k)}) \}_{j,m}.
\]

4. Let the geometric dimension \( 2k \) be equal to \( 2k = 1 \). The tower of shifted real transcendental biextensions is:

\[
(F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR) \subset \cdots \subset (F_{\pi_{1,m}} \times IR) \times (F_{v_{1,m}} \times IR)
\]

\[
\subset \cdots \subset (F_{\pi_{r,m}} \times IR) \times (F_{v_{r,m}} \times IR),
\]

i.e. a tower of (real) shifted transcendental biquanta by vertical bifibres \( (IR \times IR) \).

They are the conjugacy class representatives generated under the action of representatives of \( GL_{1}(F_{\pi} \times IR) \times (F_{v} \times IR) \) and they can be rewritten as the filtration

\[
G_{(Q)}^{(1)}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR) \subset \cdots \subset G_{(Q)}^{(j,m)}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR)
\]

\[
\subset \cdots \subset G_{(Q)}^{(r,m)}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR))
\]

of the representatives of the bilinear subsemigroups of the (infinite) quantum general bilinear semigroup \( GL_{(Q)}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR)) \).

The bilinear semigroup of automorphisms of these fibered or shifted transcendental extensions is:

\[
\text{Aut}_{k}(F_{\pi} \times IR) \times \text{Aut}_{k}(F_{v} \times IR) = \{ \ldots, \text{Aut}_{k}(F_{\pi_{j,m}} \times IR) \times \text{Aut}_{k}(F_{v_{j,m}} \times IR), \ldots \}.
\]

So

\[
[\text{Aut}_{k}(F_{\pi} \times IR) \times \text{Aut}_{k}(F_{v} \times IR)]/[\text{Aut}_{k}(F_{\pi}) \times \text{Aut}_{k}(F_{v})] = \text{Aut}_{k}(IR)_{F_{\pi}} \times \text{Aut}_{k}(IR)_{F_{v}}
\]

corresponds to the bilinear semigroup of fibering or shifting automorphisms.

The functional representatives of \( GL_{(Q)}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR)) \), or equivalently of \( GL_{1}(F_{\pi} \times IR) \times (F_{v} \times IR) \), are

\[
F_{G_{(Q)}^{(1)}}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR)) \subset \cdots \subset F_{G_{(Q)}^{(j,m)}}((F_{\pi_{1}} \times IR) \times (F_{v_{1}} \times IR)),
\]
i.e. bifunctions on the birepresentatives \( \{ G_{(Q)}^{(j,m)} \left( (F_{\pi} \times IR) \times (F_{v^1} \times IR) \right) \}_{j,m} \).

And, thus,

\[
\text{Aut}_k(\phi_R(IR)) \times \text{Aut}_k(\phi_L(IR)) = \{ \ldots, \text{Aut}_k(\phi_{j,m}(IR)) \times \text{Aut}_k(\phi_{j,m}(IR)), \ldots \}_{j,m},
\]

where \( \phi_{j,m}(IR) \) is the \((j,m)\)-th function over \( IR \) acting on the function \( \phi_{j,m}(F_{v_j,m_j}) \) over the transcendental extension \( F_{v_j,m_j} \) or over the conjugacy class representative of \( \text{GL}_1(F_{v_j,m_j}) \).

So, \( \text{Aut}_k(\phi_R(IR)) \times \text{Aut}_k(\phi_L(IR)) \) is the bilinear differential Galois semigroup where \( \text{Aut}_k(\phi_R(IR)) \) is the set of linear differential Galois (semi)-subgroups [Car], [And], acting on the set of sections of the considered \( 1D \)-differential equation and \( \text{Aut}_k(\phi_L(IR)) \) is the set of linear differential Galois (semi)subgroups acting on the symmetric set of sections.

Referring to the fundamental theorem of Galois theory, we see that the bilinear differential Galois semigroup \( \text{Aut}_k(\phi_R(IR)) \times \text{Aut}_k(\phi_L(IR)) \) corresponds to the upper bilinear differential Galois semigroup \( \text{Aut}_k(\phi_R(IR))|_{F_{v^1,m^1}} \times \text{Aut}_k(\phi_L(IR))|_{F_{v^1,m^1}} \) with respect to the upper algebraic dimension “\( r \)”.

Similarly, the bilinear semigroup of unitary, i.e. internal, shifting automorphisms:

\[
\text{Int}_k(\phi_R(IR))|_{F_{v^1}} \times \text{Int}_k(\phi_L(IR))|_{F_{v^1}} = \left[ \text{Int}_k(F_{\pi} \times IR) \times \text{Int}_k(F_{v^1} \times IR) \right] / \left[ \text{Int}_k(F_{\pi} \times IR) \times \text{Int}_k(F_{v^1} \times IR) \right]
\]

has for differential Galois bilinear semigroup \( \text{Int}_k(\psi_R(IR)) \times \text{Int}_k(\psi_L(IR)) \) where \( \text{Int}_k(\psi_L(IR)) \) is the unitary linear differential Galois (semi)group, i.e. the inertia linear differential (semi)group, acting on a “unitary section” of the envisaged \( 1D \)-differential equation, and \( \text{Int}_k(\psi_R(IR)) \) is the inertia linear differential (semi)group acting on the symmetric section.

5. If the geometric dimension is \( 2k \), then we have a tower of fibered or shifted real conjugacy class representatives

\[
G^{(2k)}((F_{\pi^1} \times IR) \times (F_{v^1} \times IR)) \subset \cdots \subset G^{(2k)}((F_{\pi^{j,m}} \times IR) \times (F_{v^{j,m}} \times IR)) \subset \cdots
\]

generated under the (bi)action of \( \text{GL}_{2k}(F_{\pi} \times F_{v}) \) or, equivalently, a tower of class representatives

\[
G_{(Q)}^{(1)}((F_{\pi^1}^{(2k)} \times IR^{(2k)}) \times (F_{v^1}^{(2k)} \times IR^{(2k)})) \subset \cdots \subset G_{(Q)}^{(j,m)}((F_{\pi^{j,m}}^{(2k)} \times IR^{(2k)}) \times (F_{v^{j,m}}^{(2k)} \times IR^{(2k)})) \subset \cdots
\]
of the fibered or shifted bisemivariety $G^{(\tau \times v)}((F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1})$ generated from $GL(Q)((F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1})$.

Referring to 4), we see that the $2k$-dimensional representation of $(\text{Aut}_k(\phi_R(F_{\mathbb{R}} \times \mathbb{R})) \times \text{Aut}_k(\phi_L(F_v \times \mathbb{R})))$ given by $\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(F_{\mathbb{R}} \times \mathbb{R})) \times \text{Aut}_k(\phi_L(F_v \times \mathbb{R})))$ decomposes into:

$$\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(F_{\mathbb{R}} \times \mathbb{R})) \times \text{Aut}_k(\phi_L(F_v \times \mathbb{R})))$$

which generates (and is isomorphic to) the fibered or shifted bisemivariety $G^{(2k)}(F_{\mathbb{R}} \times \mathbb{R}) \times (F_v \times \mathbb{R})$, rewritten according to $G^{(\tau \times v)}((F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1}) \times (F^{(2k)}_{\mathbb{R}^1})$.

$\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R})))$ is thus the $2k$-dimensional representation of the differential bilinear Galois semigroup $(\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R})))$ and is isomorphic to the shifting bisemivariety $G^{(2k)}(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R}))$ or $G^{(\tau \times v)}(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R}))$.

So, we have a tower of $2k$-dimensional representations of the differential bilinear Galois subsemigroups

$$\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R})))$$

which are respectively in one-to-one correspondence with the bilinear semigroups:

$$\text{GL}_1(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R})) = \text{GL}_2k(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R}))$$

$$\text{GL}_2k(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R}))$$

$$\text{GL}_r(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R})) = \text{GL}_2k(\phi_R(\mathbb{R}) \times \phi_L(\mathbb{R}))$$

So, we get the thesis by considering the upper algebraic dimension “$r$”:

$$\text{Rep}^{(2k)}(\text{Aut}_k(\phi_R(\mathbb{R})) \times \text{Aut}_k(\phi_L(\mathbb{R})))$$

where $\phi_R(\mathbb{R}) \simeq \mathbb{R}$ and $\phi_L(\mathbb{R}) \simeq \mathbb{R}$ are generally constant functions.
The unitary case referring to the $2k$-dimensional representation of the bilinear inertia semigroup $\text{Int}_k(\phi_R(\mathbb{IR})) \times \text{Int}_k(\phi_L(\mathbb{IR}))$ can be reached similarly, i.e.

$$\text{Rep}^{(2k)}(\text{Int}_k(\phi_R(\mathbb{IR}|_{F_v^1}) \times \text{Int}_k(\phi_L(\mathbb{IR}|_{F_v^1}))$$

$$= P_r(\phi_R(\mathbb{IR}^{(2k)}) \times \phi_L(\mathbb{IR}^{(2k)}))$$

$$= \text{GL}_r(\phi_R(\mathbb{IR}^{(2k)}) \times \phi_L(\mathbb{IR}^{(2k)}))|_{F_v \times F_v^1} \quad \blacksquare$$

2.5 Corollary

The complex bilinear representation of differential Galois (semi)groups can be found similarly as it was done for the real case.

Let $F_\omega = \{F_{\omega_1}, \ldots, F_{\omega_{j,m}}, \ldots, F_{\omega_{r,m}}\}$ (resp. $F_{\overline{\omega}} = \{F_{\overline{\omega}_1}, \ldots, F_{\overline{\omega}_{j,m}}, \ldots, F_{\overline{\omega}_{r,m}}\}$) be the set of complex transcendental extensions (or infinite complex archimedean completions) covered by its real equivalent $F_v$ (resp. $F_{\overline{\omega}}$).

Then, the complex bilinear semigroup of matrices $\text{GL}_r(\mathbb{C}^k \times \mathbb{C}^k)$, of algebraic order “$r$”, referring to the action of the bioperator $(D_R^{2k} \otimes D_L^{2k})$ on the complex bisemisheaf $(\hat{M}_{\mathbb{IR}}^{(2k)} \otimes \hat{M}_{\mathbb{IL}}^{(2k)})$ on the abstract complex bisemivariety $G^{(2k)}(F_v \times F_\omega) \equiv \text{REPSP}(\text{GL}_k(F_v \times F_\omega))$, corresponds to the $k$-dimensional complex bilinear representation of the product, right by left, of “differential” Galois (or global Weil) semigroups $(\text{Aut}_k(\psi_R(\mathbb{C})) \times \text{Aut}_k(\psi_L(\mathbb{C})))$ shifting the product, right by left, of automorphism semigroups $(\text{Aut}_k(\psi_R(F_{\overline{\omega}})) \times \text{Aut}_k(\psi_L(F_\omega)))$ of cofunctions $\psi_R(F_{\overline{\omega}})$ and $\psi_L(F_\omega)$ by $(\text{Aut}_k(\psi_R(F_{\overline{\omega}} \times \mathbb{C})) \times \text{Aut}_k(\psi_L(F_\omega \times \mathbb{C}))).$

So, we have that:

$$\text{GL}_r(\phi_R(\mathbb{C}^k) \times \phi_L(\mathbb{C}^k)) = \text{Rep}^{(2k)}(\text{Aut}_k(\psi_R(\mathbb{C})|_{F_{\overline{\omega}}}) \times \text{Aut}_k(\psi_L(\mathbb{C})|_{F_v^1})))$$

And, in the unitary case, we have:

$$P_r(\psi_R(\mathbb{C}^k) \times \psi_L(\mathbb{C}^k)) = \text{Rep}^{(2k)}(\text{Int}_k(\psi_R(\mathbb{C})|_{F_{\overline{\omega}}}) \times \text{Int}_k(\psi_L(\mathbb{C})|_{F_v^1})))$$

where:

- $P_r(\ldots)$ is the bilinear parabolic subsemigroup;
- $\text{Rep}^{(2k)}(\ldots)$ is the $k$-dimensional complex representation of $(\ldots)$. 

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2.6 Corollary

Let $O_r(\mathbb{I} \mathbb{R})$ denote the orthogonal group of algebraic order $r$ with entries in the reals $\mathbb{I} \mathbb{R}$ and let $U_r(\mathbb{I} \mathbb{C})$ denote the unitary group of algebraic order $r$ in the complexes $\mathbb{I} \mathbb{C}$.

Then, the orthogonal bilinear semigroup $O_r(\mathbb{I} \mathbb{R}^{(2k)} \times \mathbb{I} \mathbb{R}^{(2k)})$ corresponds to the real parabolic bilinear semigroup $P_r(\mathbb{I} \mathbb{R}^{(2k)} \times \mathbb{I} \mathbb{R}^{(2k)})$ and the unitary bilinear semigroup $U_r(\mathbb{I} \mathbb{C}^k \times \mathbb{I} \mathbb{C}^k)$ corresponds to the complex parabolic bilinear semigroup $P_r(\mathbb{I} \mathbb{C}^k \times \mathbb{I} \mathbb{C}^k)$.

Proof: This results from the definition of a bilinear semigroup recalled in section 1.5 and from proposition 2.4 and corollary 2.5.

We have more particularly that:

$$O_r(\mathbb{I} \mathbb{R}^{(2k)} \times \mathbb{I} \mathbb{R}^{(2k)}) = \text{Rep}^{(2k)}(\text{Int}_k(\phi_R(\mathbb{I} \mathbb{R}))|_{F_v^{1r}} \times \text{Int}_k(\phi_L(\mathbb{I} \mathbb{R}))|_{F_v^{1r}})$$

and that

$$U_r(\mathbb{I} \mathbb{C}^k \times \mathbb{I} \mathbb{C}^k) = \text{Rep}^{(2k)}(\text{Int}_k(\psi_R(\mathbb{I} \mathbb{C}))|_{F_v^{1r}} \times \text{Int}_k(\psi_L(\mathbb{I} \mathbb{C}))|_{F_v^{1r}}).$$

2.7 Corollary

In the one-dimensional geometric case, i.e. when $2k = 1$, we have that

1) the orthogonal bilinear semigroup of algebraic order $r$

$$O_r(\mathbb{I} \mathbb{R} \times \mathbb{I} \mathbb{R}) = P_r(\mathbb{I} \mathbb{R} \times \mathbb{I} \mathbb{R})$$

corresponds to the product, right by left, of differential inertia Galois semigroups $(\text{Int}_k(\phi_R(\mathbb{I} \mathbb{R}))|_{F_v^{1r}} \times \text{Int}_k(\phi_L(\mathbb{I} \mathbb{R}))|_{F_v^{1r}})$ shifting the product, right by left,

$$(\text{Int}_k(\phi_R(F_v^{1r})) \times \text{Int}_k(\phi_L(F_v^{1r})))$$

of internal automorphism semigroups of cofunctions $\phi_R(F_v^{1r})$ and functions $\phi_L(F_v^{1r})$ respectively on the unitary transcendental lower and upper real extensions $F_v^{1r}$ and $F_v^{1r}$.

2) the unitary bilinear semigroup of algebraic order $r$

$$U_r(\mathbb{I} \mathbb{C} \times \mathbb{I} \mathbb{C}) = P_r(\mathbb{I} \mathbb{C} \times \mathbb{I} \mathbb{C})$$

corresponds to the product, right by left, of differential inertia Galois semigroups $(\text{Int}_k(\psi_R(\mathbb{I} \mathbb{C}))|_{F_v^{1r}} \times \text{Int}_k(\psi_L(\mathbb{I} \mathbb{C}))|_{F_v^{1r}})$ shifting the product, right by left,

$$(\text{Int}_k(\psi_R(F_v^{1r})) \times \text{Int}_k(\psi_L(F_v^{1r})))$$

of internal automorphism semigroups of cofunctions $\psi_R(F_v^{1r})$ and functions $\psi_L(F_v^{1r})$ respectively on the unitary complex transcendental upper extensions $F_v^{1r}$ and $F_v^{1r}$.  

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Proof:

1) \((\text{Int}_k(\phi_R(F_{\pi_r}))) \times \text{Int}_k(\phi_L(F_{\pi_l})))\) corresponds to the bilinear semigroup of internal automorphisms of bifunctions on “unitary” biquanta in a bisection \(\phi_R(F_{\pi_r}) \times \phi_L(F_{\pi_l})\) at “\(r\)” biquanta \((F_{\pi_r} \times F_{\pi_l})\) while \((\text{Int}_k(\phi_R(F_{\pi_r} \times \mathbb{I}))) \times \text{Int}_k(\phi_L(F_{\pi_l} \times \mathbb{I})))\) corresponds to the bilinear semigroup of shifted internal automorphisms (under the action of a bioperator \((D_R \otimes D_L)\) ) of bifunctions on shifted “unitary” biquanta \(((F_{\pi_r} \times \mathbb{I}) \times (F_{\pi_l} \times \mathbb{I}))\) in a bisection \(\phi_R(F_{\pi_r} \times \mathbb{I}) \times \phi_L(F_{\pi_l} \times \mathbb{I})\) at “\(r\)” shifted biquanta \((F_{\pi_r} \times \mathbb{I}) \times (F_{\pi_l} \times \mathbb{I})\).

2) The complex unitary case referring to \(U_r(\mathcal{C} \times \mathcal{C})\) can be handled similarly as the real unitary case referring to \(O_r(\mathbb{I} \times \mathbb{I})\) by taking into account that a complex bisection \(\psi_R(F_{\pi_r}) \times \psi_L(F_{\pi_l})\) is covered by real bisections \(\{\phi_R(F_{\pi_r,m_r}) \times \phi_L(F_{\pi_l,m_r})\}_{m_r}\) as developed in section 1.9.

\[ \text{2.8 Bilinear Hilbert semispaces and Von Neumann bisemialgebras} \]

Let \((\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)})\) be the bisemisheaf of differentiable bifunctions on the abstract bisemivariety \(G^{(2n)}(F_{\pi_r} \times F_{\pi_l})\).

The set
\[
\{\phi_{j,m_j}^{(2n)}(g_{vR \times L}^{(2n)}[j,m_j])\}_{j,m_j} = \{\phi_{j,m_j}^{(2n)}(g_{vR}^{(2n)}[j,m_j]) \otimes \phi_{j,m_j}^{(2n)}(g_{vL}^{(2n)}[j,m_j])\}_{j,m_j}
\]
of differentiable bifunctions, i.e. bisections of \(\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)}\),
forms an increasing filtration with respect to the algebraic dimension “\(j\)”:
\[
\phi_{1}^{(2n)}(g_{vR \times L}^{(2n)}[1]) \subset \cdots \subset \phi_{j,m_j}^{(2n)}(g_{vR \times L}^{(2n)}[j,m_j]) \subset \cdots \subset \phi_{r,m_r}^{(2n)}(g_{vR \times L}^{(2n)}[r,m_r]), \quad j \leq r \leq \infty,
\]
on the corresponding filtration of conjugacy class representatives of the abstract bisemivariety \(G^{(2n)}(F_{\pi_r} \times F_{\pi_l})\):
\[
g_{vR \times L}^{(2n)}[1] \subset \cdots \subset g_{vR \times L}^{(2n)}[j,m_j] \subset \cdots \subset g_{vR \times L}^{(2n)}[r,m_r].
\]

This bisemisheaf \((\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)})\) is transformed into an extended internal left bilinear Hilbert semispace \(H_{(\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)})}^+\) by taking into account

1) a map [Pie6]:
\[
B_L \circ p_L : \hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)} \rightarrow \hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)} \equiv H_{(\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)})}^+ \equiv H_{(\hat{\mathcal{M}}_{vR}^{(2n)} \otimes \hat{\mathcal{M}}_{vL}^{(2n)})}^+,
\]
where:

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• $p_L$ is a projective linear map projecting the right semisheaf $\hat{M}_{vR}^{(2n)}$ onto the left semisheaf $\hat{M}_{vL}^{(2n)}$;

• $B_L$ is a bijective linear isometric map from the projected right semisheaf $\hat{M}_{vR}^{(2n)}$ to $\hat{M}_{vL}^{(2n)}$ mapping each covariant element of $\hat{M}_{vR}^{(2n)}$ into a contravariant element.

Due to the structure of the bisemisheaf ($\hat{M}_{vR}^{(2n)}$ and $\hat{M}_{vL}^{(2n)}$), we can construct a tower of direct sums of embedded extended bilinear Hilbert subsemispaces:

$$H^+_\hat{M}_{vR\times L}^{(2n)} 1 \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} (j) \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} (r)$$

where $H^+_\hat{M}_{vR\times L}^{(2n)}$ is given by the set $\{\phi^{(2n)}_{j,m_j} (g_{vR\times L}^{(2n)} [j, m_j]) \otimes \phi^{(2n)}_{j,m_j} (g_{vL}^{(2n)} [j, m_j])\}_{j,m_j}$ of bisections.

But, we can also construct a tower of direct sums of embedded extended bilinear Hilbert subsemispaces:

$$H^+_\hat{M}_{vR\times L}^{(2n)} \{j\} \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} \{j\} \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} \{r\}$$

where $H^+_\hat{M}_{vR\times L}^{(2n)} \{j\}$ is defined by:

$$H^+_\hat{M}_{vR\times L}^{(2n)} \{j\} = \bigoplus_{\nu=1}^j H^+_\hat{M}_{vR\times L}^{(2n)} (\nu)$$

2) An internal bilinear form defined from $H^+_\hat{M}_{vR\times L}^{(2n)} (\hat{M}_{vR}^{(2n)} \otimes \hat{M}_{vL}^{(2n)})$ into $\mathcal{C}'$ for every bisection $\phi^{(2n)}_{j,m_j} (g_{vR\times L}^{(2n)} [j, m_j])$ by:

$$\left(\phi^{(2n)}_{j,m_j} (g_{vR\times L}^{(2n)} [j, m_j]), \phi^{(2n)}_{j,m_j} (g_{vL}^{(2n)} [j, m_j])\right) \rightarrow \mathcal{C}'$$

This bilinear Hilbert subsemispace $H^+_\hat{M}_{vR\times L}^{(2n)} (\hat{M}_{vR}^{(2n)} \otimes \hat{M}_{vL}^{(2n)})$, noted in condensed form $H^+_\hat{M}_{vR\times L}^{(2n)}$, is a natural representation (bisemi)space for the bialgebra of elliptic biorators $(D_{2k} \otimes D_{2k}^*)$ as noticed in [Pie6].

A bisemialgebra of von Neumann $\mathbb{M}_{R\times L}(H^+_\hat{M}_{vR\times L}^{(2n)})$ in $H^+_\hat{M}_{vR\times L}^{(2n)}$ is an involutive subbisemialgebra of the bisemialgebra $(\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H^+_\hat{M}_{vR\times L}^{(2n)})$ of bounded biorators having a closed norm topology.

Due to the structure of the bisemisheaf ($\hat{M}_{vR}^{(2n)} \otimes \hat{M}_{vL}^{(2n)}$), the bilinear Hilbert subsemispace $H^+_\hat{M}_{vR\times L}^{(2n)}$ is “solvable” in the sense that we have a tower of embedded bilinear Hilbert subsemispaces:

$$H^+_\hat{M}_{vR\times L}^{(2n)} \{1\} \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} \{j\} \subset \cdots \subset H^+_\hat{M}_{vR\times L}^{(2n)} \{r\}$$

where $H^+_\hat{M}_{vR\times L}^{(2n)} \{j\}$ is defined by:

$$H^+_\hat{M}_{vR\times L}^{(2n)} \{j\} = \bigoplus_{\nu=1}^j H^+_\hat{M}_{vR\times L}^{(2n)} (\nu)$$

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2.9 Random bioperators

Let \((D^{2k}_R \otimes D^{2k}_L)\) be the differential (elliptic) bioperator acting on the set \(\{\phi(G^{(2n)}(F_{v,j,m_j} \times F_{v,j,m_j}))\}_{j,m_j}\) (also written \(\{\phi_j,m_j((v_{R\times L}[j,m_j])\}_{j,m_j}\)) of differentiable bifunctions or bisections of the bisemisheaf \(\hat{M}^{2n}_{v_R} \otimes \hat{M}^{2n}_{v_L}\) according to:

\[
\begin{align*}
D^{2k}_R \otimes D^{2k}_L : (\hat{M}^{(2n)}_{v_R} \otimes \hat{M}^{(2n)}_{v_L}) & \longrightarrow (\hat{M}^{(2n)}_{v_R}[2k] \otimes \hat{M}^{(2n)}_{v_L}[2k]) \\
\{\phi_{j,m_j}(G^{(2n)}(F_{v,j,m_j} \times F_{v,j,m_j}))\}_{j,m_j} & \longrightarrow \{\phi_{j,m_j}(G^{(2n)}_{[2k]}((F_{v,j,m_j} \times \mathbb{I}R) \times (F_{v,j,m_j} \times \mathbb{I}R)))\}_{j,m_j}
\end{align*}
\]

where \(\phi_{j,m_j}(G^{(2n)}_{[2k]}((F_{v,j,m_j} \times \mathbb{I}R) \times (F_{v,j,m_j} \times \mathbb{I}R)))\) is the bifunction on the \((j,m_j)\)-th conjugacy class representative \(G^{(2n)}_{[2k]}((F_{v,j,m_j} \times \mathbb{I}R) \times (F_{v,j,m_j} \times \mathbb{I}R))\) fibered or shifted in \(2k\) bilinear geometric dimensions. The bioperator \((D^{2k}_R \otimes D^{2k}_L)\) is a random bioperator in the sense that, for every bifunction

\[
\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j} \times F_{v,j,m_j})) = \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \times \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j}))
\]

belonging to the bilinear Hilbert semispace \(H^+_{\hat{M}^{(2n)}_{v_R \times L}}\) the bilinear form

\[
(D^{2k}_R(\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j}))), (D^{2k}_L(\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j}))))
\]

is measurable.

The random bioperator \((D^{2k}_R \otimes D^{2k}_L)\) acting on \(H^+_{\hat{M}^{(2n)}_{v_R \times L}}\) is a set \(\{(D^{2k}_R( j, m_j) \otimes D^{2k}_L( j, m_j))\}_{j,m_j}\) of bioperators acting on the bisemisheaf \(\hat{M}^{(2n)}_{v_R \times L}\).

2.10 Towers of embedded von Neumann subbisemialgebras

Referring to the tower of embedded bilinear Hilbert subsemispace associated with the bisemisheaf \(\hat{M}^{(2n)}_{v_R \times L}\) and to the definition of a bisemialgebra of von Neumann \(\mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}})\) given in section 2.8, it appears that there exists a tower of embedded von Neumann subbisemialgebras:

\[
\mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}(1)) \subset \cdots \subset \mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}(j)) \subset \cdots \subset \mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}(r))
\]

according to the algebraic dimensions \(1 \leq j \leq r\), as well as a tower of sums of embedded von Neumann subbisemialgebras:

\[
\mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}\{1\}) \subset \cdots \subset \mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}\{j\}) \subset \cdots \subset \mathbb{M}_{v_R \times L}(H^+_{\hat{M}^{(2n)}_{v_R \times L}}\{r\})
\]
The bisemisheaf $\left(\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}\right)$ gives rise to the extended internal left bilinear Hilbert semispace $H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^+$ according to section 2.8.

Similarly, the diagonal bisemisheaf $\left(\hat{M}_{vR}^{2n} \otimes D \hat{M}_{vL}^{2n}\right)$, whose bisections are diagonal bisections

$$\phi_{j,mj}(G^{(2n)}(F_{\eta_{j,mj}} \times_D F_{\eta_{j,mj}}) = \phi_{j,mj}(G^{(2n)}(F_{\eta_{j,mj}}) \times_D \phi_{j,mj}(G^{(2n)}(F_{\eta_{j,mj}}))$$

characterized by a diagonal bilinear basis (the offdiagonal bilinear basis elements being null) [Pie6], gives rise to the diagonal internal left bilinear Hilbert semispace $H_{\hat{M}_{vR}^{2n} \otimes D \hat{M}_{vL}^{2n}}^+$ by taking into account a $(B_L \circ p_L)$ map and the existence of an internal diagonal bilinear form, i.e. a scalar product, as in section 2.8.

### 2.11 Proposition

Let $M_{R \times L}(H_{{\hat{M}_{vR}^{2n}}}^{+})$ be the bisemialgebra of von Neumann on the extended internal left bilinear Hilbert semispace $H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^+$ and let $M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes D \hat{M}_{vL}^{2n}}^+)$ be the von Neumann bisemialgebra on the diagonal internal left bilinear Hilbert semispace $H_{\hat{M}_{vR}^{2n} \otimes D \hat{M}_{vL}^{2n}}^+$.

Then, the discrete spectrum $\Sigma(D_R^{2k} \otimes D_L^{2k})$ of the bioperator $(D_R^{2k} \otimes D_L^{2k})$ is obtained by the composition of morphisms:

$$i_{\{j\}}_{R \times L}^D \circ i_{\{j\}}_{R \times L} : M_{R \times L}(H_{{\hat{M}_{vR}^{2n}}}^{+}) \rightarrow \left[ M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]}) \oplus \Sigma(D_R^{2k} \otimes D_L^{2k}) \right]_j$$

where $i_{\{j\}}_{R \times L}$ and $i_{\{j\}}_{R \times L}^D$ are given by:

$$i_{\{j\}}_{R \times L} : M_{R \times L}(H_{{\hat{M}_{vR}^{2n}}}^{+}) \rightarrow \left[ M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]}) \right]_j$$

$$i_{\{j\}}_{R \times L}^D : \left[ M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]}) \right]_j \rightarrow \left[ M_{R \times L}(H_{{\hat{M}_{vR}^{2n} \otimes D \hat{M}_{vL}^{2n}}}^{+[2k]}) \right]_j.$$

$\left[ M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]}) \right]_j$ is the increasing tower, over the running algebraic index “$j$”, of sums of von Neumann subbisemialgebras:

$$M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]})_j = \bigoplus_{\nu=1}^{j} M_{R \times L}(H_{\hat{M}_{vR}^{2n} \otimes \hat{M}_{vL}^{2n}}^{+[2k]})_j(\nu).$$

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over respectively the tower:

\[ \mathcal{H}^+_{M_{vR}[2k] \otimes D M_{vL}[2k]} \{1\} \subset \cdots \subset \mathcal{H}^+_{M_{vR}[2k] \otimes D M_{vL}[2k]} \{j\} \subset \cdots \subset \mathcal{H}^+_{M_{vR}[2k] \otimes D M_{vL}[2k]} \{r\} \]

of sums of diagonal internal left bilinear Hilbert subsemispace

\[ \mathcal{H}^+_{M_{vR}[2k] \otimes D M_{vL}[2k]} \{j\} = \bigoplus_{\nu=1}^{j} \left( \mathcal{H}^+_{M_{vR}[2k] \otimes D M_{vL}[2k]} \left(\nu\right) \right) \]

shifted in \((2k)\) bilinear geometric dimensions.

**Proof:** The morphism \(i(j)_{R\times L}\)

\[ i(j)_{R\times L} : \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \rightarrow \left[ \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \{j\} \right]_j \]

is in fact implicit depending on the decomposition of the bisemisheaf \((\overset{\wedge}{M}_{vR}^{(2k)} \otimes \overset{\wedge}{M}_{vL}^{(2k)})\) into bisections on the conjugacy class representatives \(\{g_{vR \times L}[j, m]\}_{j, m} \) of the abstract bisemivariety \(G^{(2n)}(F_+ \times F_v)\).

\(\mathbb{M}_{R \times L}(\mathcal{H}^+_{M_{vR \times L}} \{j\})\) is the subbisemialgebra of von Neumann on the shifted extended internal left bilinear Hilbert subsemispace \(\mathcal{H}^+_{M_{vR \times L}} \{j\}\).

The morphism

\[ i(j)^D_{R \times L} : \left[ \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \{j\} \right]_j \rightarrow \left[ \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \{j\} \right]_j \]

sends each von Neumann subbisemialgebra

\[ \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \{j\} = \bigoplus_{\nu=1}^{j} \mathbb{M}_{R \times L}(H^+_{M_{vR \times L}}) \left(\nu\right) \]

on the shifted extended internal left bilinear Hilbert subsemispace

\[ H^+_{M_{vR \times L}} \{j\} = \bigoplus_{\nu=1}^{j} H^+_{M_{vR \times L}} \left(\nu\right) \]

onto the corresponding von Neumann diagonal subbisemialgebra

\[ \mathbb{M}_{R \times L}(\mathcal{H}^+_{M_{vR \times L}} \{j\}) = \bigoplus_{\nu=1}^{j} \mathbb{M}_{R \times L}(\mathcal{H}^+_{M_{vR \times L}} \{j\}) \]

on the diagonal internal left bilinear Hilbert subsemispace:

\[ \mathcal{H}^+_{M_{vR \times L}} \times \mathbb{M}_{R \times L}(\mathcal{H}^+_{M_{vR \times L}} \{j\}) = \bigoplus_{\nu=1}^{j} \mathcal{H}^+_{M_{vR \times L}} \otimes D M_{vL}[2k] \{j\} \]

shifted in \((2k)\) bilinear geometric dimensions.
2.12 Cuspidal representations of the global program of Langlands

The differential bioperator \( (D^2_R \otimes D^2_L) \) maps the bisemisheaf \( \widehat{\mathcal{M}}^{(2n)}_{v_R} \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L} \) into the corresponding perverse bisemisheaf \( \widehat{\mathcal{M}}^{(2n)}_{v_R}[2k] \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L}[2k] \) according to:

\[
(D^2_R \otimes D^2_L) : \quad \widehat{\mathcal{M}}^{(2n)}_{v_R} \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L} \longrightarrow \widehat{\mathcal{M}}^{(2n)}_{v_R}[2k] \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L}[2k]
\]

in such a way that \( \widehat{\mathcal{M}}^{(2n)}_{v_R}[2k] \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L}[2k] \) decomposes into a tower of fibered or shifted bisections or bifunctions \( \{ \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \times \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \} \) (see section 2.10).

On the other hand, referring to the global program of Langlands [Pie2], there is a one-to-one correspondence between the bisemisheaf \( \widehat{\mathcal{M}}^{(2n)}_{v_R} \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L} \) over the abstract bisemivariety \( G^{(2n)}(F_{v} \times F_{v'}) \) and its cuspidal counterpart \( \widehat{\mathcal{M}}^{(2n)}_{v_R} \otimes \widehat{\mathcal{M}}^{(2n)}_{v_L} \) on the toroidal abstract bisemivariety \( G^{(2n)}(F_{v}^T \times F_{v}^T) \) over the sets \( F_{v}^T = \{ F_{v_1}, \ldots, F_{v,j,m_j}, \ldots, F_{v,m_r}^T \} \) and \( F_{v'}^T = \{ F_{v_1}^T, \ldots, F_{v,j,m_j}^T, \ldots, F_{v,m_r}^T \} \) of toroidal real archimedean completions or transcendental extensions.

The toroidal bisemisheaf has for bisections the bifunctions

\[
\phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \times \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) = (\lambda(2n, j, m_j) e^{-2\pi i j x}) \times (\lambda(2n, j, m_j) e^{2\pi i j x})
\]

where:

- \( \vec{x} = \sum_{c=1}^{2n} x_c \vec{e}_c \), \( x \in \mathbb{R}^{2n} \);

- \( \lambda^2(2n, j, m_j) = \prod_{c=1}^{2n} \lambda^2_c(2n, j, m_j) \) is a product of eigenvalues \( \lambda^2_c(2n, j, m_j) \) of the Hecke bioperator \( (T_R(2n;r) \otimes T_L(2n;r)) \) whose representation is \( GL(2n, \mathcal{O}_{F_{v}^T} \times \mathcal{O}_{F_{v'}^T}) \) referring to section 1.5.

Let \( \Gamma_{\widehat{\mathcal{M}}^{(2n)}_{v_R \times L}} = \{ \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \times \phi_{j,m_j}(G^{(2n)}_{[2k]}(F_{v,j,m_j})) \} \) denote the set of increasing bisections of the bisemisheaf \( \widehat{\mathcal{M}}^{(2n)}_{v_R \times L} \).

Then, a global elliptic \( (\Gamma_{\widehat{\mathcal{M}}^{(2n)}_{v_R \times L}}) \)-bisemimodule \( \phi^{(2n)}_{R \times L}(x) \), referring to the bihomomorphism

\[
\phi^{(2n)}_{R \times L}(x) : \quad \Gamma_{\widehat{\mathcal{M}}^{(2n)}_{v_R \times L}} \longrightarrow \text{End}(\Gamma_{\widehat{\mathcal{M}}^{(2n)}_{v_R \times L}}),
\]

is given by:

\[
\phi^{(2n)}_{R \times L}(x) = \sum_j \sum_{m_j} \left( \lambda(2n, j, m_j) e^{-2\pi i j x} \right) \times \sum_j \sum_{m_j} \left( \lambda(2n, j, m_j) e^{2\pi i j x} \right)
\]
in such a way that \( \phi_{R \times L}^{(2n)}(x) \) constitutes a real cuspidal representation of bilinear geometric dimension \( 2n \), of the product, right by left, of Weil global semigroups \( \text{Gal}(\tilde{F}_{\pi}/k) \times \text{Gal}(\tilde{F}_v/k) \) according to the global program of Langlands. Remark that \( \phi_{R \times L}^{(2n)}(x) \) covers the corresponding “complex” cuspidal representation [Pie2].

2.13 Proposition

The global elliptic \( (\Gamma_{\tilde{M}^{(2n)}_{vT}})_{R \times L} \)-bisemimodule \( \phi_{R \times L}^{(2n)}(x) \) is the functional representation space \( \text{FREPSP}(\text{GL}_{2n}(F_T^{\pi} \times F_T^{v})) \) of the bilinear semigroup \( \text{GL}_{2n}(F_T^{\pi} \times F_T^{v}) \), over the product, right by left, of toroidal real archimedean completions \( F_T^{\pi} \) and \( F_T^{v} \), under the \((bi)\)action of the \((bi)\)monomorphisms:

\[
\sigma_{vT} \times \sigma_{vL} : \quad \text{Aut}_k(F_T^{\pi}) \times \text{Aut}_k(F_T^{v}) \longrightarrow G^{(2n)}(F_T^{\pi} \times F_T^{v}),
\]

where \( \text{Aut}_k(F_T^{\pi} \times \text{Aut}_k(F_T^{v})) \) is the bilinear semigroup of automorphisms of toroidal transcendental extensions associated with a 1D-bisemilattice of transcendental biquanta.

Proof: This results from the bimonomorphism

\[
\sigma_{vR} \times \sigma_{vL} : \quad \text{Aut}_k(F_v^{\pi}) \times \text{Aut}_k(F_v^{v}) \longrightarrow G^{(2n)}(F_v^{\pi} \times F_v^{v})
\]

introduced in section 1.5 and generating the abstract bisemivariety \( G^{(2n)}(F_v^{\pi} \times F_v^{v}) \) from \( \text{GL}_{2n}(F_v^{\pi} \times F_v^{v}) \) as well as from the definition of the global elliptic \( (\Gamma_{\tilde{M}^{(2n)}_{vT}})_{R \times L} \)-bisemimodule \( \phi_{R \times L}^{(2n)}(x) \) given in section 2.12. \( \blacksquare \)

2.14 Proposition

The shifted global elliptic bisemimodule \( \text{ELLIP}_{R \times L}(2n[2k], r) \) resulting from the action

\[
(D_{R}^{2k} \otimes D_{L}^{2k}) : \quad \text{ELLIP}_{R \times L}(2n, r) \longrightarrow \text{ELLIP}_{R \times L}(2n[2k], r)
\]

of the bioperator \( (D_{R}^{2k} \otimes D_{L}^{2k}) \) on the global elliptic \( (\Gamma_{\tilde{M}^{(2n)}_{vT}})_{R \times L} \)-bisemimodule \( \phi_{R \times L}^{(2n)}(x) \), noted here \( \text{ELLIP}_{R \times L}(2n, r) \), and generated under the \((bi)\)monomorphism:

\[
\sigma_{vT} \otimes \sigma_{vL} : \quad \text{Aut}_k(F_T^{\pi} \times IR) \times \text{Aut}_k(F_T^{v} \times IR) \longrightarrow G^{(2n)}((F_T^{\pi} \times IR) \times (F_T^{v} \times IR))
\]

gives rise to the eigenbivalue equation:

\[
(D_{R}^{2k} \otimes D_{L}^{2k})(\text{ELLIP}_{R \times L}(2n, r)) = E_{2kR \times L}(2n, j)(\text{ELLIP}_{R \times L}(2n, r)), \quad 1 \leq j \leq r,
\]

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where the eigenbivalues $E_{2k_R \times L}(2n, j)$ are shifts in $2k$ real dimensions of the
global Hecke bicharacters $\lambda(2n, j, m_j)$ associated with the subbisemilattices
characterized by the global residue degrees $j$.

**Proof:** Similarly as in proposition 2.13, the (bi)map:

$$\sigma_{v_1} \circ \sigma_{v_2} \circ \ldots \circ \sigma_{v_L} \circ \ldots : \text{Aut}_k(F^T_{v_1} \times IR) \times \text{Aut}_k(F^T_{v_2} \times IR) \longrightarrow G^{(2n)}((F^T_{v_1} \times IR) \times (F^T_{v_2} \times IR)),$$

where $\text{Aut}_k(F^T_{v_1} \times IR) \times \text{Aut}_k(F^T_{v_2} \times IR)$ is the bilinear semigroup of automorphisms of
fibered or shifted toroidal transcendental extensions, is responsible for the generation of the
toroidal abstract fibered or shifted bisemivariety $G^{(2n)}((F^T_{v_1} \times IR) \times (F^T_{v_2} \times IR))$, referring to
section 2.1, of which functional representation space is the shifted global elliptic bisemimodule
$\text{ELLIP}_{R \times L}(2n[2k], r)$ obtained from the global elliptic bisemimodule $\text{ELLIP}_{R \times L}(2n, r)$
under the action of the bioperator $(D^{2k}_R \otimes D^{2k}_L)$ giving rise to the eigenbivalue equation:

$$(D^{2k}_R \otimes D^{2k}_L)(\text{ELLIP}_{R \times L}(2n, r)) = E^{2k}_{R \times L}(2n, j)(\text{ELLIP}_{R \times L}(2n, r)), \quad \forall \ j, \ 1 \leq j \leq r.$$  

The functional representation space $\text{FREPSP}(\text{GL}_{2n}(F^T_{v_1} \times IR) \times (F^T_{v_2} \times IR))$ of the bilinear semigroup of matrices $\text{GL}_{2n}((F^T_{v_1} \times IR) \times (F^T_{v_2} \times IR))$ responsible for the generation of the
abstract bisemivariety $G^{2n}((F^T_{v_1} \times IR) \times (F^T_{v_2} \times IR))$ is given by the set of embedded
bisemifunctions:

$$\phi_1(G^{(2n)}(F^T_{v_1} \times IR)) \times \phi_1(G^{(2n)}(F^T_{v_1} \times IR))$$

$$\subset \ldots \subset \phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}} \times IR)) \times \phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}} \times IR))$$

$$\subset \ldots \subset \phi_{r,m_r}(G^{(2n)}(F^T_{v_{r,m_r}} \times IR)) \times \phi_{r,m_r}(G^{(2n)}(F^T_{v_{r,m_r}} \times IR))$$

introduced in section 5.12. of [Pie3].

Each bisemifunction $\phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}} \times IR)) \times \phi_{j,m_j}(G^{(2n)}(F^T_{v_{j,m_j}} \times IR))$ is the product,
right by left, $T_R^{(2n)}([2k], (j, m_j)) \times T_L^{(2n)}([2k], (j, m_j))$ of a $2n$-dimensional real semitorus
$T_R^{(2n)}([2k], (j, m_j))$ shifted in $2k$ real dimensions and localized in the lower half space by
its symmetric equivalent $T_L^{(2n)}([2k], (j, m_j))$ localized in the upper half space.

They have for analytic development:

$$T_L^{(2n)}([2k], (j, m_j)) \simeq E_{2k_L}(2n, j, m_j) \lambda(2n, j, m_j) e^{2\pi ijx}$$

(reps. $T_R^{(2n)}([2k], (j, m_j)) \simeq E_{2k_R}(2n, j, m_j) \lambda(2n, j, m_j) e^{-2\pi ijx}$)

referring to section 2.12, where $E_{2k}(2n, j, m_j)$ is the shift in $2k$ real dimensions of the
global Hecke character $\lambda(2n, j, m_j)$ being also a product of eigenvalues of the Hecke operator
as described in section 2.12.
On the other hand, referring to proposition 2.11, the toroidal spectral representation of $(D_R^{2k} \otimes D_L^{2k})$ is given by the set of $r$-tuples:

$$\text{ellip}_{R \times L}(2n, 1) \subset \cdots \subset \text{ellip}_{R \times L}(2n, j) \subset \cdots \subset \text{ellip}_{R \times L}(2n, r)$$

where $\text{ellip}_{R \times L}(2n, j)$ is given by

$$\text{ellip}_{R \times L}(2n, j) = (\lambda(2n, j, m_j) e^{-2\pi i j x}) \times (\lambda(2n, j, m_j) e^{+2\pi i j x})$$

and to which corresponds the set of increasing eigenbivales

$$\mathcal{E}_{2k_{R \times L}}(2n, 1) \subset \cdots \subset \mathcal{E}_{2k_{R \times L}}(2n, j) \subset \cdots \subset \mathcal{E}_{2k_{R \times L}}(2n, r)$$

where $\mathcal{E}_{2k_{R \times L}}(2n, j)$ is the shift in $2k$ real dimensions of the Hecke bicharacter $\lambda^2(2k, j, m_j) \subset \lambda^2(2n, j, m_j)$ taking into account that

$$\lambda^2(2n, j, m_j) = \prod_{c=1}^{2k} \lambda_c^2(2k, j, m_j) \times \prod_{d=2n-2k}^{2n} \lambda_d^2(2n, j, m_j)$$

with

$$\lambda^2(2k, j, m_j) = \prod_{c=1}^{2k} \lambda_c^2(2k, j, m_j)$$

and with

$$\lambda^2(2n-2k, j, m_j) = \prod_{d=2n-2k}^{2n} \lambda_d^2(2n, j, m_j)$$
3 Large random matrices and Riemann zeta function

3.1 Five questions to find a solution to this problem

Chapters 1 and 2 have been devoted to the mathematical tools necessary to clarify the conceptual framework behind the random matrices and, more particularly, the closed numerical connection between the spacings of the nontrivial zeros of the Riemann zeta function and the spacing of the eigenvalues of typical large random matrices.

It will be shown in this chapter that the symmetry behind the Gaussian unitary ensemble (GUE) is the symmetric (bisemi)group of “Galois” automorphisms fibered or “shifted” algebraic and transcendental (bi)quanta.

The constant reference to the global program of Langlands in chapter 1 and to the geometric-shifted global program of Langlands, as well as to von Neumann (bisemi)algebras, in chapter 2, is thus not fortuitous.

In order to find a solution to this problem, it will be answered in this chapter to the five following questions:

1) What is behind random matrices leading to GOE (Gaussian orthogonal ensemble), as well as GUE (Gaussian unitary ensemble)?

2) What is behind the partition and correlation function(s) between eigenvalues of random matrices?

3) What interpretation can we give to the local spacings between the eigenvalues of large random matrices?

4) What interpretation can we give to the spacings between the nontrivial zeros of $\zeta(s)$?

5) What is the curious connection between 3) and 4)?

But, first, we would like to outline that the geometric dimension envisaged in this chapter will be one real, and, possibly, one complex, i.e. that $n = 1$, in the sense of the reducible global program of Langlands developed in [Pie2]. Thus, only curves, covering possibly surfaces, will be considered in order to meet the conditions of question 4).

3.2 The first question “What is behind random matrices leading to GOE and GUE?"

This question reflects the importance of large random matrices in the present development of mathematics and of physics.
These random matrices lead to the eigenvalue problem in the frame of the geometric shifted global program of Langlands, recalled in section 1.11 and in chapter 2, in order to find a response to question 3).

### 3.3 Bilinear differential Galois semigroup

The symmetry group behind or at the origin of the bilinear global program of Langlands is the bilinear semigroup of automorphisms $\text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v)$ (resp. Galois automorphisms $\text{Gal}(F_\tau/k) \times \text{Gal}(F_v/k)$) of compact transcendental (resp. algebraic) quanta, generating a bisemilattice of compact transcendental (resp. algebraic) quanta referring to section 1.4.

Similarly, the symmetry group at the origin of the geometric bilinear global program of Langlands is the bilinear semigroup of fibered or shifted automorphisms $\text{Aut}_k(F_\tau \times \mathbb{R}) \times \text{Aut}_k(F_v \times \mathbb{R})$ of compact transcendental quanta generating a bisemilattice of compact shifted transcendental quanta according to proposition 2.4.

Referring to this same proposition, the one-dimensional shifted functional representation space $\text{FREPSP}(\text{GL}_1(F_\tau \times \mathbb{R}) \times (F_v \times \mathbb{R}))$ of $\text{GL}_1(F_\tau \times \mathbb{R}) \times (F_v \times \mathbb{R})$ is given by the shifted bisemisheaf $\widehat{\mathcal{M}}_{R\tau}[1] \otimes \widehat{\mathcal{M}}_{v\tau}[1]$ whose set $\Gamma(\mathcal{M}_{R\tau}[1] \otimes \mathcal{M}_{v\tau}[1])$ of bisections is the set $\{\phi_{(j,m_j)}(F_\tau \times \mathbb{R}) \times (F_v \times \mathbb{R}))\}_{j,m_j}$ of fibered or shifted differentiable bifunctions obtained from the set $\{\phi_{(j,m_j)}(F_\tau \times F_v))\}_{j,m_j}$ of differentiable bifunctions under the action of the elliptic bioperator $D_R \otimes D_L$, $2k = 1$.

The set

$$\{\phi_{(j,m_j)}(F_\tau \times \mathbb{R}) \times (F_v \times \mathbb{R}))\}_{j,m_j} = \{\phi_{R}(G_{(Q)}^{(j,m_j)}(F_\tau \times \mathbb{R})) \otimes \phi_{L}(G_{(Q)}^{(j,m_j)}(F_v \times \mathbb{R}))\}_{j,m_j}$$

of differentiable bifunctions, being equivalent to the set $\{\phi_{(1)}(F_{\tau,j,m_j} \times \mathbb{R}) \times (F_{v,j,m_j} \times \mathbb{R}))\}_{j,m_j}$ of differentiable bifunctions, is isomorphic to the set $\{\phi_{(Q)}^{(j,m_j)}(F_\tau \times F_v)) \otimes \phi_{(Q)}^{(j,m_j)}(F_v \times \mathbb{R}))\}_{j,m_j}$ of bifunctions in such a way that $\{\phi_{(Q)}^{(j,m_j)}(I \times \mathbb{R}))\}_{j,m_j=1}$ is the fibering or shifting functional representation bisemispace obtained under (and being isomorphic to) the biaction of the bilinear differential Galois semigroup

$$\text{Aut}_k(\phi_R(I \mathbb{R})) \times \text{Aut}_k(\phi_L(I \mathbb{R})) \simeq \{\text{Aut}_k(\phi_R(I \mathbb{R}))|_{F_{\tau,j,m_j} \times \mathbb{R}} \times \text{Aut}_k(\phi_L(I \mathbb{R}))|_{F_{v,j,m_j} \times \mathbb{R}}\}_{j=1} \simeq \text{GL}_{r}(\phi_R(I \mathbb{R}) \otimes \phi_L(I \mathbb{R})) \simeq \text{GL}_{r}(I \mathbb{R} \times I \mathbb{R})$$

according to proposition 2.4.
Thus, the bilinear semigroup $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$ of matrices constitutes the representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator $(D_R \otimes D_L)$ on the bisemisheaf $(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})$ whose bisections are the set 
$\{\phi((G^{(j,m_j)}_{(Q)}(F_{v_l} \times F_{v^1})))\}_{j=1}^r = \{\phi((G^{(1)}_{(Q)}(F_{v_{j,m_j}} \times F_{v_{j,m_j}})))\}_{j,m}$ of differentiable bifunctions.

Let us set

$\phi_j(G_{(Q)}(F_{v^1} \times F_{v^1})) = \sum_{j=1}^{r} \phi(G^{(j,m_j)}_{(Q)}(F_{v^1} \times F_{v^1}))$

used in the following proposition.

3.4 Proposition

If

$\phi_r(G_{(Q)}(F_{v^1} \times F_{v^1})) = \sum_{j=1}^{r} \phi(G^{(j,m_j)}_{(Q)}(F_{v^1} \times F_{v^1}))$

let

$(D_R \otimes D_L)(\phi_r(G_{(Q)}(F_{v^1} \times F_{v^1})) = E_{R \times L}(j) \phi_R(G_{(Q)}(F_{v^1} \times F_{v^1}))$,  

be the eigenbivalue equation related to the bisemisheaf $(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})$ and associated with the tower of shifted differentiable bifunctions 
$\{\phi((G^{(j,m_j)}_{(Q)}(F_{v^1} \times \mathbb{IR}) \times (F_{v^1} \times \mathbb{IR})))\}_{j=1}^r$.

Then, the eigenvalues of the matrix of $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$, constituting a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R \otimes D_L)$, are the eigenvalues 
$\{E_{R \times L}(j)\}_{j=1}^r$ of the above eigenbivalue equation.

Proof:

1) $\text{GL}_r(\mathbb{IR} \times \mathbb{IR})$, being:

1) the bilinear fibre $F_{R \times L}(\text{TAN})$ of the tangent bibundle $\text{TAN}(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L}) \simeq \text{Ad}(\text{F}) \text{REPS}(\text{GL}_1(F_{v^1} \times F_{v^1}))$ according to section 2.2,

2) a representation of the bilinear differential Galois semigroup associated with the biaction of $(D_R \otimes D_L)$,

constitutes a representation of the bioperator $(D_R \otimes D_L)$ because it generates endomorphisms of $\text{TAN}(\widehat{M}^{(1)}_{v_R} \otimes \widehat{M}^{(1)}_{v_L})$.

2) Referring to propositions 2.11 and 2.14, we see that the set of differentiable bifunctions 
$\{\phi((G^{(j,m_j)}_{(Q)}(F_{v^1} \times F_{v^1})))\}_{j}$ constituting the $r$-bituple

$\langle \phi(G^{(1)}_{(Q)}(F_{v^1} \times F_{v^1})), \ldots, \phi(G^{(r,m_r)}_{(Q)}(F_{v^1} \times F_{v^1})), \ldots, \phi(G^{(r,m_r)}_{(Q)}(F_{v^1} \times F_{v^1})) \rangle$

used in the following proposition.
is the spectral representation of \((D_R \otimes D_L)\) (and the basis) associated with the \(r\)-tuple of eigenbivalues:

\[ \langle E_{R \times L}(1), \ldots, E_{R \times L}(j), \ldots, E_{R \times L}(r) \rangle. \]

\[ \square \]

### 3.5 Corollary

As the \(j\)-th eigenbifunction \(\phi(G^{(j,m_j)}_Q(F_{\pi_1} \times F_{\pi_1}))\) is a bifunction on “\(j\)” transcendental compact biquanta, the corresponding eigenbivalue \(E_{R \times L}(j)\) will be the shift of this bifunction corresponding to the biaction of the bioperator \((D_R \otimes D_L)\).

**Proof**: We refer to section 2.10 and proposition 2.11 concerning;

1) the bisemialgebra of von Neumann \(\hat{M}_{R \times L}(H^+_v)\) on the tower of embedded bilinear Hilbert semispaces associated with the bisemisheaf \(\hat{M}_{R \times L}^{(1)} \equiv \hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)}\) and

2) the discrete spectrum \(\Sigma(D_R \otimes D_L)\) of the differential bioperator \((D_R \otimes D_L)\).

\[ \square \]

### 3.6 Bilinear Gaussian orthogonal (resp. unitary) ensemble

**BGOE** (resp. **BGUE**)

The Gaussian orthogonal (resp. unitary) ensemble **GOE** (resp. **GUE**) is defined in the space of real symmetric (resp. hermitian) matrices by two requirements [Meh]:

- a) the ensemble is invariant under every transformation:

\[ H \longrightarrow W^T H W \quad \text{(resp. } H \longrightarrow U^{-1} H U \text{)} \]

where:

- \(W\) (resp. \(U\)) is any real orthogonal (resp. unitary) matrix;

- \(H\) is a real symmetric (resp. hermitian) matrix, generally related to the hamiltonian matrix invariant (resp. not invariant) under time reversal.

- b) the various elements \(H_{ij}\) are statistically independent.
In its simplified bilinear version, the Hamiltonian $H$ corresponds to the differential (random) bioperator $(D_R \otimes D_L)$ acting on the bisemisheaf $\hat{\mathcal{M}}_{vR \times L}^{(1)}$ and belonging to the bisemialgebra of von Neumann $\mathcal{M}_{R \times L}(H^+_{\hat{\mathcal{M}}_{vR \times L}^{(1)}})$ according to sections 2.8 to 2.11 and [Pie1]. Let then:

$$D_R \otimes D_L : \hat{\mathcal{M}}_{vR}^{(1)} \otimes \hat{\mathcal{M}}_{vL}^{(1)} \to \hat{\mathcal{M}}_{vR}^{(1)[1]} \otimes \hat{\mathcal{M}}_{vL}^{(1)[1]}$$

be the biact of $(D_R \otimes D_L)$ on the bisemisheaf $(\hat{\mathcal{M}}_{vR}^{(1)} \otimes \hat{\mathcal{M}}_{vL}^{(1)})$ generating the perverse bisemisheaf $(\hat{\mathcal{M}}_{vR}^{(1)[1]} \otimes \hat{\mathcal{M}}_{vL}^{(1)[1]})$ whose bisections $\{\phi_{j,m}(G_{[1]}(F_{vj,m} \times \mathbb{R}) \times (F_{vj,m} \times \mathbb{R}))\}_{j,m}$ are differentiable bifunctions on the $(j,m)$-th conjugacy class representatives of the shifted or fibered bilinear semigroup $G^{(1)}((F_{v} \times \mathbb{R}) \times (F_{v} \times \mathbb{R}))$.

Referring to [Pie1], the perverse bisemisheaf $(\hat{\mathcal{M}}_{vR}^{(1)[1]} \otimes \hat{\mathcal{M}}_{vL}^{(1)[1]})$ is the operator-valued stringfield of an elementary (bisemi)particle.

According to section 3.3, the bilinear semigroup of matrices $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ constitutes the representation of the bilinear differential Galois semigroup associated with the biact of $(D_R \otimes D_L)$ on $(\hat{\mathcal{M}}_{vR}^{(1)} \otimes \hat{\mathcal{M}}_{vL}^{(1)})$.

And, thus, $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ (and $O_r(\mathbb{R} \times \mathbb{R})$), or its complex equivalent $\text{GL}_r(\mathbb{C} \times \mathbb{C})$ (and $U_r(\mathbb{R} \times \mathbb{R})$), is the new bilinear Gaussian real (orthogonal) (resp. complex (“unitary”)) ensemble labeled BGOE (resp. BGUE) corresponding to GOE (resp. GUE).

### 3.7 Mixed higher bilinear $KK$-theory

To be complete, the deformations of random matrices have to be envisaged in the light of the new interpretation of homotopy-cohomotopy [Pie5] viewed as deformations of Galois representations in the context of mixed higher bilinear $KK$-theory related to the Langlands dynamical bilinear global program [Pie5].

Referring to proposition 2.4, we see that the 1D-geometric infinite quantum general bilinear semigroup $\text{GL}^{(Q)}(F_{v^1} \times F_{v^1})$ is defined by:

$$\text{GL}^{(Q)}(F_{v^1} \times F_{v^1}) = \lim_{j \to r} \text{GL}^{(Q)}_{j}(F_{v^1} \times F_{v^1})$$

in such a way that $\text{GL}^{(Q)}_{1}(F_{v^1} \times F_{v^1})$ is the parabolic, i.e. unitary bilinear semigroup $P_{1}(F_{v^1} \times F_{v^1})$, and that its shifted equivalent $\text{GL}^{(Q)}((F_{v^1} \times \mathbb{R}) \times (F_{v^1} \times \mathbb{R}))$ is given by

$$\text{GL}^{(Q)}((F_{v^1} \times \mathbb{R}) \times (F_{v^1} \times \mathbb{R})) = \lim_{j \to r} \text{GL}^{(Q)}_{j}((F_{v^1} \times \mathbb{R}) \times (F_{v^1} \times \mathbb{R}))$$

leading to a filtration

$$G^{(1)}_{(Q)}((F_{v^1} \times \mathbb{R}) \times (F_{v^1} \times \mathbb{R})) \subset \cdots \subset G^{(r,mr)}_{(Q)}((F_{v^1} \times \mathbb{R}) \times (F_{v^1} \times \mathbb{R}))$$

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of its representatives.

So, the bilinear version of the algebraic $K$-theory restricted to 1D-geometric dimension is given by:

$$K^1(G^{(1)}(F_{v^1} \times F_{v^i})) = \Pi_1(BGL^{(Q)}(F_{v^1} \times F_{v^i})^+)$$

where the quantum classifying bisemispace $BGL^{(Q)}(F_{v^1} \times F_{v^i})$ is the base bisemispace of all equivalence classes of deformations of the Galois representations of $GL^{(Q)}(\widetilde{F}_{v^1} \times \widetilde{F}_{v^i})$ given by the kernels $GL^{(Q)}(\delta F_{v^1+\ell} \times \delta F_{v^i+\ell})$ of the maps:

$$GD_{\ell}^{(Q)} : EGL^{(Q)}(F_{v^1+\ell} \times F_{v^i+\ell}) \rightarrow BGL^{(Q)}(F_{v^1} \times F_{v^i})$$

where $GD_{\ell}^{(Q)}$ is a universal principal $GL^{(Q)}(F_{v^1} \times F_{v^i})$-bibundle.

Referring to chapter 3 of [Pie5], it is easy to see that

$$BGL^{(Q)}(F_{v^1} \times F_{v^i}) \equiv GL_1(F_v \times F_v)$$

and that the maps $GD_{\ell}^{(Q)}$ become

$$GD_{\ell}^{(1)} : GL_1(F_{v+\ell} \times F_{v+\ell}) \rightarrow GL_1(F_v \times F_v)$$

having the same interpretation as $GD_{\ell}^{(Q)}$.

In order to recall the bilinear version of the mixed higher $KK$-theory of Quillen adapted to the Langlands dynamical global program in 1D-geometric dimension [Pie5], we have to take into account:

a) the bisemisheaf $(\hat{M}^{(1)}_{v_R} \otimes \hat{M}^{(1)}_{v_L})$, noted here $FG^{(1)}(F_v \times F_v)$ and being the functional representation space of $GL_1(F_v \times F_v)$;

b) the “plus” classifying bisemisheaf $BFGL^{(Q)}(J \times J)^+ = BFGL_{\delta}(J \times J)^+$, being the base bisemisheaf of all equivalence classes of one-dimensional inverse deformations of the Galois differential representation of $FG_{\delta}(J \times J)$ due to the action of the bioperator $(D_R \otimes D_L)$ on $FG^{(1)}(F_v \times F_v)$, and corresponding to one-dimensional deformations of the Galois representation of $GL_1(F_v \times F_v)$ given by the kernel $\{GL_1(\delta F_{v+\ell} \times \delta F_{v+\ell})\}_{\ell}$ of $GD_{\ell}^{(1)}$.

The higher (algebraic) $KK$-theory is then given by:

$$K_1(FG^{(1)}(J \times J)) \times K^1(FG^{(1)}(F_v \times F_v)) = \Pi^1(BFGL_1(J \times J)^+) \times \Pi_1(BFGL_1(F_v \times F_v)^+)$$
in such a way that the bilinear contracting $K$-theory $K_1(FG^{(1)}(IR \times IR))$ responsible for a differentiable biaction acts on the $K$-theory $K^1(FG^{(1)}(F_\tau \times F_\nu))$ of the bisemisheaf $FG^{(1)}(F_\tau \times F_\nu)$ in one-to-one correspondence with the biaction of the cohomotopy bisemigroup $\varPi^1(BFGL_1(IR \times IR)^{+})$ of the “plus” classifying bisemisheaf $BFGL_1(IR \times IR)^{+}$.

### 3.8 Proposition

The deformation

$$GD^{(1)}_\ell : \quad GL_1(F_{v+\ell} \times F_{v+\ell}) \rightarrow GL_1(F_{\tau} \times F_{\nu})$$

induces the following deformation:

$$GL_{r+\delta r_\ell \rightarrow r} : \quad GL_{r+\delta r_\ell}(IR \times IR) \rightarrow GL_r(IR \times IR)$$
on random matrices $GL_r(IR \times IR)$.

**Proof:** Indeed, the matrix of $GL_{r+\delta r_\ell}(IR \times IR)$ of order $(r+\ell)$ constitutes a representation of the deformed bioperator or $((D_R+\delta D_R) \times (D_L+\delta D_L))$ acting on the deformed bifunction $\phi_{r+\delta r_\ell}(G_{Q}(F_{\tau}^v \times F_{\nu}^v))$ and has for spectral representation the $(r+\delta r_\ell)$-bituple:

$$\langle \phi(G^{(1+\ell)}_{(Q)}(F_{v+\ell} \times F_{v+\ell})), \ldots, \phi(G^{(r+\ell,m+\ell)}_{(Q)}(F_{v+\ell} \times F_{v+\ell})) \rangle$$
don decomposing into the sum of the $r$-bituple and the $\delta r_\ell$-bituple according to:

$$\langle \phi(G^{(1+\ell)}_{(Q)}(F_{v+\ell} \times F_{v+\ell})), \ldots, \phi(G^{(r+\ell,m+\ell)}_{(Q)}(F_{v+\ell} \times F_{v+\ell})) \rangle = \langle \phi(G^{(1)}_{(Q)}(F_{v} \times F_{v})), \ldots, \phi(G^{(r,m)}_{(Q)}(F_{v} \times F_{v})) \rangle$$

$$+ \langle \phi(G^{(\ell)}_{(Q)}(F_{\tau}^\ell \times F_{\nu}^\ell)), \ldots, \phi(G^{(\ell)}_{(Q)}(F_{\tau}^\ell \times F_{\nu}^\ell)) \rangle .$$

The deformation $GL_{r+\delta r_\ell}(IR \times IR)$ of the random matrix of $GL_r(IR \times IR)$, constituting a deformation of the Galois differential representation $GL_r(IR \times IR)$, corresponds to the cohomotopy bisemigroup $\varPi^1(BFGL_1(IR \times IR)^{+})$ according to section 3.7.

### 3.9 The second question

The second question “What is behind the partition and correlation functions between eigenvalues of random matrices?” concerns the distribution of eigenvalues of random matrices. It will be seen that this problem is based on the (bisemi)group of “Galois” automorphisms of shifted transcendental compact (bi)quanta which leads to a reevaluation of the probabilistic interpretation in quantum theories.
3.10 Distribution of eigenvalues of GUE ensembles

Wigner introduced the idea of statistical mechanics of nuclei based on a Gaussian ensemble (GUE) having “r” quantum states and characterized by a Hamiltonian symmetric matrix of order “r” whose entries are Gaussian random variables and to which a Gaussian statistical weight is associated [Wig].

Unsatisfied by the impossibility of defining a uniform probability distribution on an infinite range, F. Dyson introduced the circular unitary (resp. orthogonal) ensemble CUE (resp. COE) [Dys] in such a way that the Hamiltonian is now described by a unitary matrix of order “r” whose eigenvalues are complex numbers exp(iθj), 1 ≤ j ≤ r, distributed around the unit circle.

This circular unitary (resp. orthogonal) ensemble corresponds to the Riemann symmetric space \(U(r)/O(r)\) which “lives” in \(\text{GL}_r(\mathbb{C})/U(r)\) [Mez].

Let then \(M = (M_{ij})_{i,j=1}^r\) be a random hermitian matrix to which is assigned the probability distribution

\[
d\mu_r^{\text{GUE}}(M) = \frac{1}{Z_r^{\text{GUE}}} e^{-r \text{tr} M^2} dM
\]

where:

- \(\text{tr} M^2 = \sum_{i,j=1}^r M_{ji} M_{ij} = \sum_{i=1}^r M_{ii}^2 + 2 \sum_{i>j} |M_{ij}|^2\);

- \(\mu_r^{\text{GUE}}(dM) = \frac{1}{Z_r^{\text{GUE}}} \prod_{i=1}^r (e^{-rM_{ii}^2}) \prod_{i>j} (e^{-2r|M_{ij}|^2}) dM\).

The distribution of eigenvalues of \(M\) with respect to the ensemble \(\mu_r^{\text{GUE}}\) is [Meh], [A-VM], [B-Z]:

\[
d\mu_r^{\text{GUE}}(\lambda) = \frac{1}{Z_r^{\text{GUE}}} \prod_{i>j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^r e^{-r\lambda_i^2} d\lambda
\]

where the partition function [Rue] of GUE is given by:

\[
Z_r^{\text{GUE}} = \int \prod_{i>j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^r e^{-r\lambda_i^2} d\lambda_i
\]

\[
= \frac{(2r)^{r/2}}{(2r)^{r^2/2}} \prod_{i=1}^r i! .
\]
3.11 \( m \)-point correlation function for GUE and Jacobi matrix

The joint probability density function for the eigenvalues of matrices from a Gaussian orthogonal, unitary or symplectic ensemble is given by:

\[
P_{r\beta}(x_1, \ldots, x_r) = c_{r\beta} \exp \left( -\frac{\beta}{2} \sum_{i=1}^r x_i^2 \right) \prod_{i>j} (x_i - x_j)^\beta, \quad -\infty < x_i < +\infty,
\]

where \( \beta = 1, 2 \) or 4 according as the ensemble is orthogonal, unitary or symplectic and 
\[c_{r\beta} = \frac{1}{Z_{r\text{GUE}}}.\]

The \( m \)-point correlation function for the Gaussian unitary ensemble is defined by [Dys]:

\[
R_{mr}(x_1, \ldots, x_m) = \frac{r!}{(r-m)!} \int_{\mathbb{R}^{r-m}} P_r(x_1, \ldots, x_r) \, dx_{m+1} \ldots \, dx_r
\]

which is the probability density of finding a level around each of the points (i.e. entries of \( M \)) \( x_1, \ldots, x_m \), the positions of the remaining levels being unobserved.

The Dyson determinantal formulas for correlation functions is [Meh], [Ble], [B-H]:

\[
R_{mr}(x_1, \ldots, x_m) = \det(K_r(x_k, x_\ell))_{k,\ell=1}^m
\]

where \( K_r(x_k, x_\ell) \) is given by:

\[
K_r(x_k, x_\ell) = \sum_{i=0}^{r-1} \psi_i(x_k) \psi_i(x_\ell) \quad \text{with} \quad \psi_i(x) = h_i^{\frac{1}{2}} P_i(x) \, e^{-rM^2(x)/2}
\]

where \( P_i(x) \) is an orthogonal polynomial or degree \( i \) corresponding to the weight function \( e^{-rM^2(x)/2} \) and verifying:

\[
\int_{-\infty}^{+\infty} P_i(x) \, P_j(x) \, e^{-rM^2(x)} \, dx \simeq \delta_{ij} \, h_i.
\]

These orthogonal polynomials, being sometimes Hermite polynomials, satisfy the three term recurrent relation:

\[
\beta_{i+1} P_{i+1}(x) = (x - \alpha_i) \, P_i(x) - \beta_i \, P_{i-1}(x)
\]

or

\[
x \, P_i = \beta_i \, P_{i-1} + \alpha_i \, P_i + \beta_{i+1} \, P_{i+1}, \quad \beta_i = \frac{h_i}{h_i - 1}, \quad \alpha_i = (x \, P_i, P_i).
\]
If we set $P_{-1}(x) = 0$, we get the tower:

\begin{align*}
x P_0 &= \alpha_0 P_0 + \beta_1 P_1, \\
x P_1 &= \beta_1 P_0 + \alpha_1 P_1 + \beta_2 P_2, \\
x P_2 &= 0 + \beta_2 P_1 + \alpha_2 P_2 + \beta_3 P_3 + 0, \\
&\vdots \\
x P_{i-1} &= 0 + \beta_{i-1} P_{i-2} + \alpha_{i-1} P_{i-1} + \beta_i P_i,
\end{align*}

which, in matricial form, is:

$$x \mathcal{P} = J \mathcal{P} + \beta_i P_i.$$ 

The matrix $J$ is symmetric and is the Jacobi matrix such that the $i$ roots of $P_i(x)$ verifying

$$P_i(x_j) = 0, \quad 1 \leq j \leq i,$$

lead to

$$x_j [\mathcal{P}(x_j)] = J [P(x_j)].$$

The $i$ roots of $P_i(x)$ are then the eigenvalues of the Jacobi matrix $J$.

### 3.12 Joint probability density function for the eigenvalues of matrices from BGUE and BGOE

In the bilinear case, the random matrix corresponding to $M$ is

$$G = T G^T \times T G \in \text{GL}_r(\mathbb{R} \times \mathbb{R}) \quad \text{or} \quad \text{GL}_r(\mathbb{C} \times \mathbb{C})$$

where $TG \in T_r(\mathbb{R})$ (resp. $TG^T \in T^t_r(\mathbb{R})$) is an upper (resp. lower) triangular matrix of order $r$ with entries in $\mathbb{R}$ (or $\mathbb{C}$).

The BGUE (or BGOE) probability distribution corresponding to $G$ is:

$$d_{\mu_r}^{\text{BGUE}}(G) = \frac{1}{Z_r^{\text{BGUE}}} e^{-r \text{Tr}(TG^T \times TG)} \, dG$$

leading to

$$\mu_r^{\text{BGUE}}(G) = \frac{1}{Z_r^{\text{BGUE}}} \prod_{i=1}^{r} e^{-r G_{ii}^2} \prod_{i>j} e^{-2r |G_{ij}|^2}.$$ 

The joint probability density function for the eigenvalues of matrices from a
Gaussian bilinear orthogonal or unitary ensemble is thus:

\[ P_{R \times L}(x_1, \ldots, x_r) = c_r \exp \left( - \sum_{i=1}^r r x_i^2 \right) \prod_{i>j} (x_i - x_j)^2 \]

which corresponds to the distribution of eigenvalues \( d_{\mu_r}^{\text{GUE}}(\lambda) \) of a random matrix \( M \) with respect to GUE, the eigenvalues \( (x_1, \ldots, x_r) \) being in fact eigenvalues \( (x_1^2, \ldots, x_r^2) \).

The \( m \)-point correlation function for the bilinear Gaussian ("unitary") (or real ("orthogonal")) ensemble BGUE (or BGOE):

\[ R_{mr \times L}(x_1^2, \ldots, x_m^2) = \frac{r!}{(r-m)!} \int_{\mathbb{R}^{r-m}} P_{R \times L}(x_1^2, \ldots, x_r^2) \, dx_{m+1} \ldots dx_r \]

then corresponds to the \( m \)-point correlation function for GUE \( R_{mr}(x_1, \ldots, x_m) \) developed in section 3.1.1.

\( R_{mr \times L}(x_1^2, \ldots, x_m^2) \) is thus also given by:

\[ R_{mr \times L}(x_1^2, \ldots, x_m^2) = \det(K_r(x_k, x_\ell)_{k, \ell=1}^m) \]

with:

- \( K_r(x_k, x_\ell) = \sum_{i=0}^{r-1} \psi_i(x_k) \psi_i(x_\ell) \)
- \( \psi_i(x) = h^{-\frac{1}{2}} P_i(x) \ e^{-r[(TG^T \times TG)(\mathbb{R} \times \mathbb{R})]/2} \).

As in section 3.11, \( P_i(x) \) is an orthogonal polynomial of degree \( i \) associated with the weight function \( e^{-r[(TG^T \times TG)(\mathbb{R} \times \mathbb{R})]/2} \) where \( TG^T(\mathbb{I} \mathbb{R}) \times TG(\mathbb{I} \mathbb{R}) \equiv G(\mathbb{I} \mathbb{R} \times \mathbb{I} \mathbb{R}) \) is the bilinear Gauss decomposition of the matrix \( G \).

These orthogonal polynomials \( P_i(x) \) also satisfy the three term recurrent relation leading to the Jacobi matrix.

**3.13 Proposition**

Let

\[ K_r(x, x) = \sum_{i=0}^{r-1} \psi_i(x) \psi_i(x) \]

be the energy level density with \( \psi_i(x) \) given by:

\[ \psi_i(x) = h^{-\frac{1}{2}} P_i(x) \ e^{-r[(TG^T \times TG)(\mathbb{R} \times \mathbb{R})]/2} \].

Then, we have that:
1) the squares of the roots \( (x_j) \) of the polynomial \( P_i(x) \) correspond to the eigenbivalues of the product, right by left, \((U_{rR} \times U_{rL})\) of the Hecke operators;

2) the weight \( e^{-r[(TG^T \times TG)(R \times F)])} \) is a measure on the eigenbivalues of the matrix \( G \in \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) being a representation of the differential bilinear Galois semigroup.

**Proof:**

1) According to section 3.11, the roots of the orthogonal polynomial \( P_i(x) \) are the eigenvalues of the Jacobi matrix \( J \).

On the other hand, the set \( \{P_j(x)\}_{j=1}^{i} \) of orthogonal polynomials envisaged in the tower of the three term recurrent relations leading to the matricial form

\[ x \cdot P = J \cdot P + \beta_i \cdot P_i \]

is in one-to-one correspondence with the set \( \{\phi(G^{(j)}(F_{v_R} \times F_{v_L}))\}_{j=1}^{i} \) of differentiable eigenbifunctions, being the spectral representation of the bioperator \((D_R \otimes D_L)\) according to proposition 3.4 and constituting a representation of the bilinear semigroup of automorphisms \( \text{Aut}_k(F_{v_R}) \times \text{Aut}_k(F_{v_L}) \) of compact transcendental biquanta generating a bisemilattice according to section 3.3.

This bisemilattice results from the action of Hecke bioperators \((U_{iR} \otimes U_{iL})\) generating the endomorphisms of the bisemisheaf \((\hat{M}_{vR}^{1} \times \hat{M}_{vL}^{1})\) referring to sections 1.5 and 3.6.

Consequently, the Jacobi matrix \( J \) must be a representation of the Hecke operator \( U_{iL} \), and, thus, the square of the roots of the polynomial \( P_i(x) \) correspond to the eigenbivalues of the Hecke bioperator \((U_{iR} \otimes U_{iL})\) and are global Hecke (extended) bicharacters referring to [Pie2].

2) The weight \( e^{-r[(TG^T \times TG)(R \times F)])/2} \) is thus a representation of the differential bilinear Galois semigroup \( \text{Aut}_k(\phi_R(F_{v})) \times \text{Aut}_k(\phi_L(F_{v})) \) shifting the product, right by left, of automorphism semigroups \( \text{Aut}_k(\phi_R(F_{v})) \times \text{Aut}_k(\phi_L(F_{v})) \) of cofunctions and functions on compact transcendental quanta according to proposition 2.4 and developments of N. Katz [Kat].

This representation of the bilinear differential Galois semigroup is then associated with the biaction of the differential bioperator \((D_R \otimes D_L)\) of which eigenbivalues are
shifts of global Hecke (extended) bicharacters referring to section 3.4 and proposition 3.5.

3.14 Corollary

The Jacobi matrix $J$ is a representation of the Hecke operator.

Proof: This results directly from proposition 3.13.

3.15 Proposition

The probabilistic interpretation of quantum (field) theories is thus related to the bilinear semigroup of automorphisms $\text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v)$ (resp. $\text{Aut}_k(\tilde{F}_\tau) \times \text{Aut}_k(\tilde{F}_v)$) of compact transcendental (resp. algebraic) biquanta generating a bisemilattice of these.

Proof: The probabilistic interpretation in QFT is given by the function density

$$ P_r(x,x) = \sum_{i=0}^{r-1} P_i(x) P_i(x) $$

where:

- $P_i(x)$ is an orthogonal polynomial of degree $i$;
- $P_r(x,x) \, dx \in K_r(x,x)$ being the (wave) function density gives the probability of finding a (bisemi)particle in a volume element $(x,x+dx)$.

As $P_r(x) P_i(x)$ or more exactly $P_i(-x) P_i(x)$, $x \in IR_+$, constitutes a representation of the bilinear semigroup of automorphisms $\text{Aut}_k(F_\tau) \times \text{Aut}_k(F_v)$ according to sections 1.5 to 2.12, we get the thesis.

3.16 Proposition

The $m$-point correlation function for BGUE (or BGOE) $R_{mr}(x_1^2, \ldots, x_m^2)$ constitutes a representation of a the bilinear semigroup of automorphisms $\text{Aut}_k(\phi_R(F_\tau \times IR)) \times \text{Aut}_k(\phi_L(F_v \times IR))$ of fibered or shifted compact transcendental biquanta:

$$ \text{Rep}_{\text{Aut}_k(\phi_{R\times L}(F_\tau \times IR))} : \text{Aut}_k(\phi_R(F_\tau \times IR)) \times \text{Aut}_k(\phi_L(F_v \times IR)) \rightarrow R_{mr}(x_1, \ldots, x_m). $$
Proof: We refer to proposition 2.4 and section 3.12 showing that, in

\[ K_r(x_k, x_\ell) = \sum_{i=0}^{r-1} \psi_i(x_k) \psi_i(x_\ell) \]

with

\[ \psi_i(x) = h^{-\frac{i}{2}} P_i(x) \exp(-r[TG^T \times TG](I \times I)) / 2 \]

a) the polynomials \( P_i(x) \) constitute a representation of the linear semigroup of automorphisms \( \text{Aut}_k(F_v) \) of compact transcendental quanta;

b) the weight \( \exp(-r[TG^T \times TG](I \times I)) \) is a representation of the differential bilinear Galois semigroup \( \text{Aut}_k(\phi_R(I)) \times \text{Aut}_k(\phi_L(I)) \);

c) the products \( \psi_i(x_k) \psi_i(x_\ell) \) constitute a representation of the bilinear semigroup of automorphisms \( \text{Aut}_k(\phi_R(F \times I) \times \text{Aut}_k(\phi_L(F \times I)) \) of shifted compact transcendental biquanta.

3.17 The third question “What interpretation can we give to the local spacings between the eigenvalues of large random matrices?”

This question [A-B-K], [Joh] is a direct consequence of the responses given to the two first questions and shows the central importance of the biquanta in this field as proved in the following sections.

3.18 Proposition

1) The consecutive spacings

\[ \delta E_{R \times L}(j) = E_{R \times L}(j+1) - E_{R \times L}(j) , \quad 1 \leq j \leq r < \infty \]

between the eigenvalues of the random matrix \( G \) of \( \text{GL}_r(I \times I) \) are the infinitesimal bigenerators on one biquantum of the Lie subbisemialgebra \( \mathfrak{gl}_1(F_{v^1} \times F_{v^1}) \) of the bilinear parabolic unitary semigroup \( P_1(F_{v^1} \times F_{v^1}) \subset \text{GL}_1(F_{v^1} \times F_{v^1}) \).

2) The \( k \)-th consecutive spacings

\[ \delta E_{R \times L}^{(k)}(j) = E_{R \times L}(j+k) - E_{R \times L}(j) , \quad k \leq j \]

between the eigenvalues of the random matrix \( G \) of \( \text{GL}_r(I \times I) \) are the infinitesimal bigenerators on \( k \) biquanta of the Lie subbisemialgebra \( \mathfrak{gl}_1(F_{v^k} \times F_{v^k}) \) of the
bilinear $k$-th semigroup $\text{GL}_1(F_{\pi} \times F_{\nu}) \subset \text{GL}_1(F_{\pi} \times F_{\nu})$, where $F_{\pi}$ (and $F_{\nu}$) denotes the set of transcendental subextensions characterized by a transcendence degree $\text{tr} \cdot d \cdot F_{\pi} = k \cdot N$

referring to $k$ biquanta included in transcendental extensions having higher or equal transcendence degree.

**Proof:** Referring to proposition 3.4 and corollary 3.5, it appears that the set $\{E_{R\times L}(j)\}_{j=1}^r$ of eigenvalues of a random matrix $G$ of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$, being shifts of eigenbifunctions of the eigenvalue equation

$$(D_R \otimes D_L)(\phi_r(G(Q)(F_{\pi} \times F_{\nu}))) = E_{R\times L}(\phi_r(G(Q)(F_{\pi} \times F_{\nu})))$$

is the set of infinitesimal bigenerators on $j$ biquanta of the Lie bismialgebra $\text{gl}_1(F_{\pi} \times F_{\nu})$ of the Lie bilinear semigroup $\text{GL}_1(F_{\pi} \times F_{\nu})$.

And, thus, the consecutive spacings:

$$\delta E_{R\times L}(j) = E_{R\times L}(j + 1) - E_{R\times L}(j),$$

between the eigenvalues of a random matrix $G$ of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ are the infinitesimal bigenerators on the biquantum on the Lie subbismialgebra $\text{gl}_1(F_{\pi} \times F_{\nu})$ of the bilinear parabolic unitary semigroup $P_1(F_{\pi} \times F_{\nu})$ on the product, right by left, of sets $F_{\pi}$ and $F_{\nu}$ of transcendental subextensions characterized by a transcendence degree $\text{tr} \cdot d \cdot F_{\nu} = N$.

Similarly, the $k$-th consecutive spacings

$$\delta E_{R\times L}^{(k)}(j) = E_{R\times L}(j + k) - E_{R\times L}(j)$$

refer to the infinitesimal bigenerators on $k$ biquanta of the Lie subbismeialgebra $\text{gl}_1(F_{\pi} \times F_{\nu})$ of the Lie bilinear $k$-th semigroup $\text{GL}_1(F_{\pi} \times F_{\nu}) \subset \text{GL}_1(F_{\pi} \times F_{\nu})$.

### 3.19 Corollary

1) The consecutive spacings

$$\delta E_{R\times L}(j) = E_{R\times L}(j + 1) - E_{R\times L}(j), \quad 1 \leq j \leq r < \infty,$$

between the eigenvalues of the random matrix $G$ of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ correspond to the energies of one free biquantum from subbisemilattices of $(j + 1)$ biquanta.
2) The $k$-th consecutive spacings

$$\delta E_{R\times L}^{(k)}(j) = E_{R\times L}(j + k) - E_{R\times L}(j)$$

between the eigenbivalues of the random matrix $G$ of $GL_r(\mathbb{R} \times \mathbb{R})$ correspond to the energies of $k$ free biquanta from subbisemilattices of $(j + k)$ biquanta.

**Proof:** As the infinitesimal bigenerators $E_{R\times L}(j)$ of the Lie bisemialgebra $gl_1(F_v \times F_v)$ are shifts of eigenbifunctions on $j$ transcendental biquanta according to section 3.3, they correspond to the energies of these $j$ transcendental biquanta and thus to $j$ transcendental biquanta of energy.

### 3.20 Lemma

Let $\delta E_{R\times L}(j)$ denote the consecutive spacings between the eigenbivalues of the matrix $G$ of $GL_r(\mathbb{R} \times \mathbb{R})$.

Then, $\delta E_{R\times L}(j)$ decomposes into:

$$\delta E_{R\times L}(j) = \delta EF_{R\times L}(j) + \delta EV_{R\times L}(j)$$

where $\delta EF_{R\times L}(j)$ and $\delta EV_{R\times L}(j)$ denote respectively the fixed (or constant) and variable consecutive spacings between the $r$ eigenbivalues of the matrix $G$ of $GL_r(\mathbb{R} \times \mathbb{R})$.

**Proof:** There are surjective maps

$$\delta E_{R\times L}(j) : \delta E_{R\times L}(j) \rightarrow \delta EV_{R\times L}(j)$$

between the consecutive spacings $\delta E_{R\times L}(j)$, referring to the infinitesimal generators of Lie subbisemialgebras or energies of one biquantum in a lattice of $(j + 1)$ biquanta, and the respective “variable” consecutive spacing $\delta EV_{R\times L}(j)$ in such a way that their kernels are the fixed consecutive spacings $\delta EF_{R\times L}(j)$ being equal for every integer $j$, $1 \leq j \leq r < \infty$.

Consequently, the energy $\delta E_{R\times L}(j)$ of one biquantum in a lattice of $(j + 1)$ biquanta decomposes into a fixed part common to all considered bisections and into a variable part differing from one bisection to another.

### 3.21 Proposition

1) The consecutive spacings

$$\delta EV_{R\times L}^{BCUE}(j) = E_{R\times L}^{BCUE}(j + 1) - E_{R\times L}^{BCUE}(j)$$

$1 \leq j \leq r$,
between the eigenvalues $E_{R \times L}^{BCUE}(j+1)$ and $E_{R \times L}^{BCUE}(j)$ of the unitary random matrix $u_r(\mathcal{C} \times \mathcal{C}) \in U_r(\mathcal{C} \times \mathcal{C})$ (or $o_r(\mathcal{R} \times \mathcal{R}) \in O_r(\mathcal{R} \times \mathcal{R})$) are the variable (unitary) infinitesimal bigenerators on one biquantum of the envisaged Lie subbisemialgebra or variable (unitary) energies $\delta EV_{R \times L}^{BCUE}(j)$ of one biquantum in subbisemilattices of $(j+1)$ biquanta.

2) The $k$-th consecutive spacings

$$\delta EV_{R \times L}^{(k)BCUE}(j) = E_{R \times L}^{BCUE}(j + k) - E_{R \times L}^{BCUE}(j), \quad k \leq j,$$

between the eigenvalues of the unitary random matrix $u_r(\mathcal{C} \times \mathcal{C})$ (or $o_r(\mathcal{R} \times \mathcal{R})$) are the variable (unitary) infinitesimal bigenerators on $k$ biquanta of the envisaged Lie subbisemialgebra or variable (unitary) energies $\delta EV_{R \times L}^{(k)BCUE}(j)$ on $k$ biquanta in subbisemilattices of $(j+k)$ biquanta.

**Proof:** Similarly as it was developed in proposition 3.18, the set $\{E_{R \times L}^{BCUE}(j)\}_{j=1}^r$ of eigenvalues of $u_r(\mathcal{C} \times \mathcal{C})$, being shifts of eigenbifunctions of the eigenvalue equation:

$$(D_R \otimes D_L)(\phi_r(P_Q)(F_{v_{R}} \times F_{v_{L}}))) = E_{R \times L}^{BCUE}(j)(\phi_r(P_Q)(F_{v_{R}} \times F_{v_{L}})))$$

is the set of unitary infinitesimal bigenerators of the Lie bisemialgebra $\text{gl}_1(F_{v_{R}} \times F_{v_{L}})$ of the Lie bilinear parabolic semigroup $P_1(F_{v_{R}} \times F_{v_{L}})$.

Indeed, the bilinear semigroup of matrices $U_r(\mathcal{C} \times \mathcal{C})$ (or $O_r(\mathcal{R} \times \mathcal{R})$), of the bilinear circular unitary (or orthogonal) ensemble BCUE (resp. BCOE), constitutes the representation of the unitary bilinear differential Galois semigroup associated with the biaction of the differential bioperator $(D_R \otimes D_L)$ on the unitary bismisheaf $(\widehat{M}_{v_{R}}^{(1)} \otimes \widehat{M}_{v_{L}}^{(1)}) \subset (\widehat{M}_{v_{R}}^{(1)} \otimes \widehat{M}_{v_{L}}^{(1)})$ according to proposition 2.4 to corollary 2.7.

And, thus, the consecutive spacings

$$E_{R \times L}^{BCUE}(j+1) - E_{R \times L}^{BGUE}(j) = \delta EV_{R \times L}^{BCUE}(j)$$

between the eigenvalues of $U_r(\mathcal{C} \times \mathcal{C})$ (or $O_r(\mathcal{R} \times \mathcal{R})$) are the variable (unitary) infinitesimal bigenerators of Lie subbisemialgebras $\text{gl}_1(F_{v_{R}} \times F_{v_{L}})$ or (unitary) variable energies $\delta EV_{R \times L}^{BCUE}(j)$ of one biquantum in sublattices of $(j+1)$ biquanta.

The case of $k$-th consecutive spacings $\delta EV_{R \times L}^{(k)BCUE}(j)$ can be handled similarly by taking into account the content of proposition 3.18.

Remark finally that

$$\delta EV_{R \times L}(j) = \delta EV_{R \times L}^{BCUE}(j) \equiv \delta E_{R \times L}^{BCUE}(j).$$
3.22 The fourth question “What interpretation can we give to the spacings between the nontrivial zeros of the zeta function $\zeta(s)$?”

This question depends on the solution of the Riemann hypothesis proposed in [Pie7] and is briefly recalled in the following sections [Edw], [Rie], [Tit].

3.23 Cuspidal representation given by global elliptic bisemimodule

Let $S_L(2, N = 1)$ (resp. $S_R(2, N = 1)$) denote the (semi)algebra of cusp forms $f_L(z)$ (resp. $f_R(z)$) of weight 2 and level $N = 1$ holomorphic in the upper (resp. lower) half plane as developed in [Pie7]. $f_L(z)$ (resp. $f_R(z^*)$), expanded in Fourier series according to:

$$f_L(z) = \sum_n a_n q^n, \quad q = e^{2\pi iz}, \quad z \in \mathcal{C},$$

(resp. $f_R(z^*) = \sum_n a^*_n q^n, \quad q^* = e^{-2\pi iz^*}$)

is the functional representation space of $G^{(2)}(F^T_\sigma) \equiv T_2(F^T_\sigma)$ (resp. $G^{(2)}(F^T_\sigma^*) \equiv T_2(F^T_\sigma^*)$) where $F^T_\sigma$ (resp. $F^T_\sigma^*$) is the set of increasing toroidal complex transcendental extensions referring to section 1.9.

Then, we have that:

$$f_R(z^*) \times f_L(z) = \text{FREPSP}(\text{GL}_2(F^T_\sigma \times F^T_\sigma^*))$$

$$= G^{(2)}(F^T_\sigma \times F^T_\sigma)$$

$$= M_{\omega \overline{\tau}}^{(2)} \otimes M_{\omega \overline{\tau}}^{(2)}$$

is a cusp biform of weight 2 and level 1.

On the other hand, we can consider the map:

$$M_{f \to \zeta_L} : \quad f_L(z) \mapsto \zeta_L(s_+) \quad \text{(resp.} \quad M_{f \to \zeta_R} : \quad f_R(z^*) \mapsto \zeta_R(s_-))$$

of the cusp form $f_L(z)$ (resp. $f_R(z^*)$) into the corresponding zeta function $\zeta_L(s_+)$ (resp. $\zeta_R(s_-)$) in such a way that $s_+ = \sigma + i\tau$ (resp. $s_- = \sigma - i\tau$) be an “energy” variable conjugate to the spatial variable $z$ (resp. $z^*$).

Referring to section 2.12 introducing a $2n$-dimensional global elliptic $(\Gamma_{\overline{v^{(2n)}_{R \times L}}})$-bisemimodule $\phi^{(2n)}_{R \times L}(x)$, we see that it can be reduced to a **$1D$-pseudounramified simple global elliptic $(\Gamma_{\overline{M^{(2n)}_{v^{(2n)}_{R \times L}}}})$-bisemimodule**:

$$\phi^{(1)}_{R \times L}(x) = \sum_n (\lambda^{(nr)}(n) e^{-2\pi inx} \otimes_D \lambda^{(nr)}(n) e^{+2\pi inx}), \quad x \in \mathbb{R},$$
if:

a) pseudounramification is concerned, i.e. the conductor \( N = 1 \);

b) simplicity is supposed, i.e. the multiplicity \( m_n \) is equal to one on each level “\( n \”).

This global elliptic \( (\Gamma^1_{\mathcal{M}})_{\mathcal{v}^{T}}^{\otimes} )\)-bisemimodule

\[
\phi_{R \times L}^{(1),nr}(x) = \phi_{R}^{(1),nr}(x) \otimes_D \phi_{L}^{(1),nr}(x) \quad (\otimes_D : \text{diagonal tensor product})
\]

can be interpreted geometrically as the sum of products, right by left, of semicircles of level “\( n \)” on \( n \) transcendental compact quanta in such a way that \( \phi_{R \times L}^{(1),nr}(x) \) be the cuspidal automorphic representation of the complete bilinear semigroup \( \text{GL}_2(F_{\mathcal{v}^{(nr)}}^\otimes \times F_{\mathcal{v}}^{(nr)}) \) according to sections 1.5 and 2.12 and cover the weight 2 cusp biform \( f_R(z^*) \otimes f_L(z) \).

On the other hand, as we are dealing with bisemiobjects, the zeta functions \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \), defined respectively on the lower and upper half planes, are considered.

### 3.24 Proposition

Let

\[
H_{\phi_{R \times L} \rightarrow \zeta_{R \times L}} : \quad 2\phi_{R}^{(1),nr}(x) \otimes_D \phi_{L}^{(1),nr}(x) \rightarrow \zeta_R(s_-) \otimes_D \zeta_L(s_+)
\]

be the map between the double 1D-pseudounramified simple global elliptic bisemimodule \( \phi_{R \times L}^{(1),nr}(x) \) and the product, right by left, of zeta functions given by:

\[
\zeta_R(s_-) \otimes_D \zeta_L(s_+) = \sum_n (n^{-s_-} \otimes_D n^{-s_+}).
\]

Then, the kernel \( \text{Ker}(H_{\phi_{R \times L} \rightarrow \zeta_{R \times L}}) \) of the map \( H_{\phi_{R \times L} \rightarrow \zeta_{R \times L}} \) is the set of squares of trivial zeros of \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \).

**Proof:** The kernel \( \text{Ker}(H_{\phi_{R \times L} \rightarrow \zeta_{R \times L}}) \) of \( H_{\phi_{R \times L} \rightarrow \zeta_{R \times L}} \) maps into the set of “trivial” zeros of \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \) which are the negative integers \(-2, -4, \ldots\)

Consequently, this kernel must be the set of bipoints:

\[
\{ \sigma_{nr} \times \sigma_{nl} = 2\lambda^{(nr)}(n)(e^{-2\pi i nx} \mid x = 0) \times 2\lambda^{(nr)}(n)(e^{-2\pi i nx} \mid x = 0) \}
\]

\[
= 4(\lambda^{(nr)}(n^2))^2 \}
\]

\[
= 4f^2_{\mathcal{v}^{-}} = 4n^2 \}
\]

where \( (\lambda^{(nr)}(n^2))^2 \) is the square of the global residue degree \( f_{\mathcal{v}^{-}} = n \) as proved in [Pie7],
in such a way that the left (resp. right) point \( \sigma_{nL} = 2\lambda^{(nr)}(n) e^{2\pi inx} \mid x = 0 \) (resp. \( \sigma_{nR} = 2\lambda^{(nr)}(n) e^{-2\pi inx} \mid x = 0 \)) corresponds to the degeneracy of the (irreducible) circle \( 2\lambda^{(nr)}(n) e^{2\piinx} \) (resp. \( 2\lambda^{(nr)}(n) e^{-2\piinx} \)) on \( 2n \) left (resp. right) transcendental quanta.

Remark that the one-to-one correspondence between the global elliptic semimodule \( 2\phi_L^{(1),nr}(x) \) (resp. \( 2\phi_R^{(1),nr}(x) \)) and the left (resp. right) zeta function \( \zeta_L(s_+) = \sum_n n^{-s_+} \) (resp. \( \zeta_R(s_-) = \sum_n n^{-s_-} \)) is clear if it is noted that \( n^s = n^{\sigma+i\tau} \) can be written

\[
e^{(\sigma+i\tau)n} = e^{\sigma n} \cdot e^{i\tau n}
\]

where \( e^{\sigma n} \) can correspond to the radius of a circle with phase \( e^{i\tau n} \) in such a way that \( 1/e^{\sigma n} \) maps into \( 2\lambda^{(nr)}(n) \) and \( 1/e^{i\tau n} \) maps into \( e^{2\piinx} \) for each term of \( \zeta_L(s_+) \) and of \( 2\phi_L^{(1),nr}(x) \).

\[\text{3.25 Proposition}\]

Let \( D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \) be a coset representative of the Lie (bisemi)algebra of the decomposition (bisemi)group \( D_4(Z)_{4n^2} \) and let \( \alpha_{4n^2} \) be the split Cartan subgroup (bi)element. Then, the products of the pairs of the trivial zeros of the Riemann zeta functions \( \zeta_R(s_-) \) and \( \zeta_L(s_+) \) are mapped into the products of the corresponding pairs of the nontrivial zeros according to:

\[
\begin{align*}
\{D_{4n^2,i^2} \cdot \varepsilon_{4n^2}\} : & \quad \{\det(\alpha_{4n^2})\}_n \quad \rightarrow \quad \{\det(D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2})_{ss}\}_n \\
\{(−2n) \times (−2n)\}_n & \quad \rightarrow \quad \{\lambda^{(nr)}_+(4n^2,i^2,E_{4n^2}) \times \lambda^{(nr)}_- (4n^2,i^2,E_{4n^2})\}_n ,
\end{align*}
\]

\[\forall \quad n \in \mathbb{N}\]

where “ss denotes the semisimple form.

\[\text{Proof: Let} \]

\[
\alpha_{4n^2} = \begin{pmatrix} 4n^2 & 0 \\ 0 & 1 \end{pmatrix}
\]

be the (split) Cartan subgroup element associated with the square of the global residue degree \( f_{v_2n} = 2n \).
Let

\[ D_{4n^2,i^2} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \]

be the coset representative of the Lie (bisemi)algebra \([\Pi_3]\) of the decomposition (bisemi)group acting on \(\alpha_{4n^2}\):

It corresponds to the coset representative of an unipotent Lie (bisemi)algebra mapping in the topological Lie (bisemi)algebra \(gl_1(F_{nr}^{(nr)} \times F_v^{(nr)})\) consisting in vector fields on the Lie subgroup \(GL_1(F_{nr}^{(nr)} \times F_v^{(nr)})\).

Let

\[ \varepsilon_{4n^2} = \begin{pmatrix} E_{4n^2} & 0 \\ 0 & 1 \end{pmatrix} \]

be the infinitesimal (bi)generator of this Lie (bisemi)algebra corresponding to the square of the global residue degree \(f_{v_{2n}} = 2n\).

Every root of the Lie (bisemi)algebra is determined by the (equivalent) eigenvalues \(\lambda_{\pm}^{(nr)}(4n^2,i^2,E_{4n^2})\) of

\[ D_{4n^2,i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2} = \begin{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} E_{4n^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4n^2 & 0 \\ 0 & 1 \end{pmatrix} \]

given by

\[ \lambda_{\pm}^{(nr)}(4n^2,i^2,E_{4n^2}) = \frac{1 \pm i\sqrt{(16n^2 \cdot E_{4n^2}) - 1}}{2} \]

where \(D_{4n^2,i^2} \cdot \varepsilon_{4n^2}\) is the coset representative of the Lie (bisemi)algebra \(\text{Lie}(D_{4n^2}(\mathbb{Z})_{4n^2})\) of the decomposition group \(D_{i^2}(\mathbb{Z})_{4n^2}\).

According to proposition 3.24, the squares \((-2n)^2\) of the trivial zeros of \(\zeta_R(s_-)\) and \(\zeta_L(s_+)\) are the squares of the global residue degrees \(f_{v_{2n}} = 2n\).

As \(D_{4n^2} \cdot \varepsilon_{4n^2}\) is of Galois type, it maps squares of trivial zeros \((-2n)^2\) into products of corresponding pairs of other zeros \(\lambda_{\pm}^{(nr)}(4n^2,i^2,E_{4n^2}) = \lambda_{\pm}^{(nr)}(4n^2,i^2,E_{4n^2})\) which are nontrivial zeros since they have real parts localized on the line \(\sigma = \frac{1}{2}\).

So, \(D_{4n^2,i^2} \cdot \varepsilon_{4n^2}\) maps a lattice of transcendental biquanta on the considered Lie (bisemi) group into a lattice of energies of these biquanta on the associated Lie (bisemi)algebra. \(\blacksquare\)
3.26 Corollary

The eigenvalues $\lambda_{+}^{(nr)}(4n^2, i^2, E_{4n^2})$ and $\lambda_{-}^{(nr)}(4n^2, i^2, E_{4n^2})$ of $(D_{4n^2, i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2})$ for all $n \in \mathbb{N}$ are the nontrivial zeros of the Riemann zeta function $\zeta(s) = \sum n^{-s}$.

**Proof:** The set $\{-2n\}_n$, being the trivial zeros of the right and left zeta functions $\zeta_R(s-)$ and $\zeta_L(s+)$, constitutes also the set of trivial zeros of the classical Riemann zeta function $\zeta(s) = \sum n^{-s}$.

So, the eigenvalues $\lambda_{+}^{(nr)}(4n^2, i^2, E_{4n^2})$ and $\lambda_{-}^{(nr)}(4n^2, i^2, E_{4n^2})$ of $(D_{4n^2, i^2} \cdot \varepsilon_{4n^2} \cdot \alpha_{4n^2})$, generated from the corresponding trivial zeros “$-2n$”, are:

a) the nontrivial zeros of $\zeta(s)$ since they are localized on the line $\sigma = \frac{1}{2}$ and disposed symmetrically on this line with respect to $\tau = 0$, $s = \sigma + i\tau$;

b) the relevant zeros of the energy density function $\zeta(s)$ which is an inverse space function, i.e. an “energy” function, on transcendental quanta labelled by the integers “$n$”.

3.27 Proposition

Let the nontrivial zeros of $\zeta(s)$, $\lambda_{+}^{(nr)}(4n^2, i^2, E_{4n^2})$ and $\lambda_{-}^{(nr)}(4n^2, i^2, E_{4n^2})$ be written compactly and classically according to $\frac{1}{2} + i\gamma_j$ and $\frac{1}{2} - i\gamma_j$, $j \leftarrow n$.

1) Then, the consecutive spacings

$$\delta \gamma_j = \gamma_{j+1} - \gamma_j, \quad j = 1, 2, \ldots$$

between the nontrivial zeros of $\zeta(s)$ are equivalently:

a) the infinitesimal generators on one quantum of the Lie subsemialgebra $gl_1(F_{\mathfrak{g}}^{(nr)})$ (or $gl_1(F_{\mathfrak{g}}^{(nr)})$) of the linear parabolic unitary semigroup $P_1(F_{\mathfrak{g}}^{(nr)}) \subset GL_1(F_{\mathfrak{g}}^{(nr)}) \equiv T_1(F_{\mathfrak{g}}^{(nr)})$ (or $P_1(F_{\mathfrak{g}}^{(nr)}) \subset GL_1(F_{\mathfrak{g}}^{(nr)}) \equiv T_1^{t}(F_{\mathfrak{g}}^{(nr)})$).

b) the energies of one (free) quantum from subsemilattice of $(j + 1)$ quanta in such a way that $\delta \gamma_j > \frac{\gamma_{j+1}}{j+1}$ where $\frac{\gamma_{j+1}}{j+1}$ is the energy of one quantum bound to $j$ quanta.

2) The $k$-th consecutive spacings

$$\delta_j^{(k)} = \gamma_{j+k} - \gamma_j,$$

between the nontrivial zeros of $\zeta(s)$ are equivalently:
a) the infinitesimal generators on \( k \) quanta of the Lie subsemialgebra \( \text{gl}_1(F^{(nr)}_{v^k}) \) (or \( \text{gl}_1(F^{(nr)}_{v^k}) \)) of the \( k \)-th semigroup \( \text{GL}_1(F^{(nr)}_{v^k}) \subset \text{GL}_1(F^{(nr)}_{v^k}) \) (or \( \text{GL}_1(F^{(nr)}_{v^k}) \subset \text{GL}_1(F^{(nr)}_{v^k}) \)).

b) the energies of \( k \) (free) quanta from subsemilattice of \((j+k)\) quanta.

**Proof:** According to proposition 3.25, every nontrivial zero \( \left( \frac{1}{2} \pm \gamma_{j+1} \right) \) of \( \zeta_R(s_-), \zeta_L(s_+) \) or \( \zeta(s) \) is the infinitesimal generator of the Lie semialgebra \( \text{gl}_1(F^{(nr)}_{v^j}) \) (or \( \text{gl}_1(F^{(nr)}_{v^j}) \)) on \((j+1)\) compact transcendental quanta or, physically, the energy of \((j+1)\) compact transcendental quanta.

So, the consecutive spacing

\[ \delta \gamma_j = \gamma_{j+k} - \gamma_j , \]

between \( \gamma_{j+1} \) and \( \gamma_j \), is the infinitesimal generator on one quantum on the Lie subalgebra \( \text{gl}_1(F^{(nr)}_{v^j}) \) of the parabolic unitary semigroup \( P_1(F^{(nr)}_{v^j}) \) or the energy of one free quantum in a subsemilattice of \((j+1)\) transcendental compact quanta.

The \( k \)-th consecutive spacings

\[ \delta^{(k)}_j = \gamma_{j+k} - \gamma_j \]

can be handled similarly.

### 3.28 The fifth question “What is the curious connection between the spacings of the nontrivial zeros of \( \zeta(s) \) and the corresponding spacings between the eigenvalues of random matrices?”

This question finds a response in the following propositions [B-K], [G-M].

### 3.29 Proposition

The consecutive spacings

\[ \delta \gamma_j = \gamma_{j+1} - \gamma_j , \quad j \in \mathbb{N}, \]

between the nontrivial zeros of the Riemann zeta function \( \zeta(s) \) as well as the consecutive spacings

\[ \delta E^{(nr)}_{R,L}(j) = E^{(nr)}_{R,L}(j+1) - E^{(nr)}_{R,L}(j) \]

between the square roots of the pseudounramified eigenvalues of a random matrix (of) \( \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) (or of \( \text{GL}_r(\mathbb{C} \times \mathbb{C}) \)) or between the pseudounramified eigenvalues of a random matrix of \( \text{GL}_r(\mathbb{R}) \), are equivalently:
a) the infinitesimal generators on one quantum of the Lie subsemialgebra $\text{gl}_1(F_{v_1})$ (or $\text{gl}_1(F_{\tau v})$) of the linear parabolic unitary semigroup $P_1(F_{v_1})$ (or $P_1(F_{\tau v})$);

b) the energies of one transcendental pseudounramified $(N = 1)$ quantum in subsemilattices of $(j + 1)$ transcendental pseudounramified quanta.

(R, L means R (right) or L (left)).

**Proof:** The equivalence between the consecutive spacings $\delta\gamma_j$ of $\zeta(s)$ and the consecutive spacings $\delta E_{R,L}^{(nr)}(j)$ between the square roots of pseudounramified eigenbivalues of a random matrix of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ or between the pseudounramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{R})$, results from propositions 3.18 and 3.27. The equivalence is exact if the eigenbivalues of a matrix of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ are pseudounramified, i.e. if they are eigenbivalues of the eigenvalue equation (see proposition 3.18):

$$(D_R \otimes D_L)(\phi_r(G_Q(F^{(nr)}_{\tau v} \times F^{(nr)}_{v_1}))) = E_{R\times L}^{(nr)}(j)(\phi_r(G_Q(F^{(nr)}_{\tau v} \times F^{(nr)}_{v_1})))$$

where $F^{(nr)}_{v_1}$ (and $F^{(nr)}_{\tau v}$) are unitary transcendental pseudounramified extensions (case $N = 1$) referring to section 1.5.

Finally, remark that

$$|\delta E_{R,L}^{(nr)}| = \sqrt{\delta E_{R\times L}^{(nr)}(j)}$$

where $\delta E_{R,L}^{(nr)}(j)$ is a consecutive spacing between pseudounramified eigenvalues and $\sqrt{\delta E_{R\times L}^{(nr)}(j)}$ is a consecutive spacing between square roots of pseudounramified eigenbivalues.

**3.30 Corollary**

*The set $\{\delta E_{R,L}^{(nr)}(j)\}_j$ of consecutive spacings between the square roots of the eigenbivalues of a random matrix of $\text{GL}_r(\mathbb{R} \times \mathbb{R})$ (or of $\text{GL}_r(\mathbb{C} \times \mathbb{C})$) or between the eigenvalues of a random matrix of $\text{GL}_r(\mathbb{R})$, as well as the set $\{\delta\gamma_j\}_j$ of consecutive spacings between the nontrivial zeros of $\zeta(s)$ constitutes a representation of the differential inertia Galois (semi)group associated with the action of the differential operator $D_L$ or $D_R$.***

**Proof:** This is immediate since the sets $\{\delta E_{R,L}^{(nr)}(j)\}_j$ and $\{\delta\gamma_j\}_j$ are infinitesimal generators of the Lie subsemialgebra of the linear parabolic unitary semigroup $P_1(F_{v_1}^{(nr)})$ (or $P_1(F_{\tau v}^{(nr)})$) and as the unitary parabolic bilinear semigroup $P_r(\mathbb{R} \times \mathbb{R}) \subset \text{GL}_r(\mathbb{R} \times \mathbb{R})$ corresponds to the “bilinear” representation of the product, right by left, $\text{Int}_k(\phi_R(\mathbb{R})) \times$
\[ P_r(\mathbb{R} \times \mathbb{R}) = \text{Rep}[(\text{Int}_k(\phi_R(\mathbb{R})))_{|F_0^{(n_r)}} \times (\text{Int}_k(\phi_L(\mathbb{R})))_{|F_0^{(n_l)}}] . \]

3.31 Proposition

Let \( \{ \delta \gamma_j \}_j \) be the set of consecutive spacings between the nontrivial zeros of \( \zeta(s) \) and let \( \{ \delta E_{R,L}(j) \}_j \) be the set of consecutive spacings between the square roots of the eigenbivalues of a random matrix of \( \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) (or of \( \text{GL}_r(\mathbb{C} \times \mathbb{C}) \)) or between the eigenbivalues of a random matrix of \( \text{GL}_r(\mathbb{R}) \).

Then, there is a surjective map:

\[ IM_{E \rightarrow \gamma} : \{ \delta E_{R,L}(j) \}_j \longrightarrow \{ \delta \gamma_j \}_j \]

of which kernel \( \text{Ker}[IM_{E \rightarrow \gamma}] \) is equivalently the set

\[ \{ \delta E_{R,L}(j) - \delta E_{R,L}^{(nr)}(j) \}_j , \quad \forall \ j , \quad 1 \leq j \leq r , \]

a) of differences of consecutive spacings between the square roots of the pseudoramified and pseudounramifed eigenbivalues of a random matrix of \( \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) (or of \( \text{GL}_r(\mathbb{C} \times \mathbb{C}) \)), or between the pseudoramified and pseudounramified eigenvalues of a random matrix of \( \text{GL}_r(\mathbb{R}) \);

b) of the energies of one compact transcendental pseudoramified \((N > 2)\) quantum in subsemilattices of \((j + 1)\) transcendental pseudoramified quanta.

Proof: First, remark that the surjective map

\[ IM_{E \rightarrow \gamma} : \{ \delta E_{R,L}(j) \}_j \longrightarrow \{ \delta \gamma_j \}_j \]

leads precisely to the map (or to the equality) between the spacing distribution between eigenvalues of a random matrix and the pair correlation of the nontrivial zeros of \( \zeta(s) \) [Mon].

If the kernel \( \text{Ker}[IM_{E \rightarrow \gamma}] \) of the map \( IM_{E \rightarrow \gamma} \) is null, then

\[ \delta E_{R,L}(j) = \delta E_{R,L}^{(nr)}(j) , \quad j \in \mathbb{N} , \]

i.e. the consecutive spacings between the eigen(bi)values of a random matrix are pseudounramified, implies the thesis of proposition 3.29, i.e. the equality between the consecutive spacings \( \delta \gamma_j \) of \( \zeta(s) \) and the consecutive spacings \( \delta E_{R,L}^{(nr)}(j) \) between eigenvalues.
3.32 Proposition

Let
\[ \delta EV_{R,L}^{(nr),BCOE}(j) = E_{R,L}^{(nr),BCOE}(j + 1) - E_{R,L}^{(nr),BCOE}(j) \]
\[ = \delta E_{R,L}^{(nr),BCOE}(j) , \quad j \in \mathbb{N} , \]
be the consecutive spacings between the square roots of the pseudounramified eigenbivalues of a random unitary matrix of \( O_r(\mathbb{R} \times \mathbb{R}) \) or of \( U_r(\mathbb{C} \times \mathbb{C}) \) or between the pseudounramified eigenvalues of a random unitary matrix of \( O_r(\mathbb{R}) \).

Then, there is a surjective map:
\[ IM_{\gamma \to E_{BCOE}^{(nr)}} : \{ \delta \gamma \}_j \rightarrow \{ \delta EV_{R,L}^{(nr),BCOE}(j) \}_j \]
between the set \( \{ \delta \gamma \}_j \) of consecutive spacings between the nontrivial zeros of \( \zeta(s) \) and the set \( \{ \delta EV_{R,L}^{(nr),BCOE}(j) \}_j \).

**Proof:** According to lemma 3.20, the consecutive spacings \( \delta E_{R \times L}(j) \) between the eigenbivalues of \( GL_r(\mathbb{R} \times \mathbb{R}) \) decomposes into fixed and variable consecutive spacings
\[ \delta E_{R \times L}(j) = \delta E F_{R \times L}(j) + \delta EV_{R \times L}(j) , \]
which is also the case for the consecutive spacings \( \delta E_{R \times L}^{(nr)}(j) \) between pseudounramified eigenbivalues.

As
\[ \delta \gamma_j = \delta E_{R,L}^{(nr)}(j) \]
according to proposition 3.29, it results that the consecutive spacings between nontrivial zeros of \( \zeta(s) \) also decomposes according to fixed and variable consecutive spacings
\[ \delta \gamma_j = \delta \gamma F_j + \delta \gamma V_j , \quad \forall \ j , \quad 1 \leq j \leq r , \]
in such a way that
\[ \delta \gamma V_j = \delta EV_{R,L}^{(nr),BCOE}(j) \equiv \delta E_{R,L}^{(nr),BCOE}(j) . \]
And, the variable consecutive spacings \( \{ \delta \gamma V_j \}_j \) constitute a representation of the differential variable inertia Galois subgroup according to corollary 3.30, the differential variable inertia Galois subgroup being a subgroup of the differential inertia Galois subgroup. \[ \blacksquare \]
3.33 Proposition (main)

Let

$$\delta \gamma_j^{(k)} = \gamma_{j+k} - \gamma_j$$

denote the $k$-th consecutive spacings between the nontrivial zeros of $\zeta(s)$.

Let:

- $\delta E_{R,L}^{(k)}(j) = E_{R,L}(j + k) - E_{R,L}(j)$;
- $\delta E_{R,L}^{(nr),(k)}(j) = E_{R,L}^{(nr)}(j + k) - E_{R,L}^{(nr)}(j)$;

and

- $\delta E_{R,L}^{(k),(nr),BCOE}(j) = E_{R,L}^{(nr),BCOE}(j + k) - E_{R,L}^{(nr),BCOE}(j)$,
  \hspace{1cm} 1 \leq j \leq r, \hspace{0.3cm} k \leq j$

be the $k$-th consecutive spacings between respectively

- the pseudoramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$;
- the pseudounramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$;
- the pseudounramified eigenvalues of a random unitary matrix of $\text{O}_r(\mathbb{IR})$.

Then, we have:

1) $\delta \gamma_j^{(k)} = \delta E_{R,L}^{(nr),(k)}(j)$ which are equivalently:

   a) the infinitesimal generators on $k$ quanta of the Lie subsemialgebra $\text{gl}_1(F_{v_k}^{(nr)})$ of the linear $k$-th semigroup $\text{GL}_1(F_{v_k}^{(nr)}) \subset \text{GL}_1(F_v^{(nr)})$;

   b) the energies of $k$ transcendental pseudounramified ($N = 1$) quanta in subsemilattices in $(j+k)$ transcendental pseudounramified quanta;

   c) a representation of the differential Galois (semi)group associated with the action of the differential operator $D_L$ or $D_R$ on a function on $k$ transcendental pseudounramified quanta.

2) a surjective map:

$$IM_{E \rightarrow \gamma}^{(k)}: \{ \delta E_{R,L}^{(k)}(j) \}_j \rightarrow \{ \delta \gamma_j^{(k)} \}_j$$

of which kernel $\text{Ker}[IM_{E \rightarrow \gamma}^{(k)}]$ is the set $\{ \delta E_{R,L}^{(k)}(j) - \delta E_{R,L}^{(nr),(k)}(j) \}_j$ of difference of $k$-th consecutive spacings between the pseudoramified and pseudounramified eigenvalues of a random matrix of $\text{GL}_r(\mathbb{IR})$. 69
3) a bijective map:

\[ IM^{(k)}_{\gamma \rightarrow E_{BCOE}^{(nr)}} : \{\delta\gamma_j^{(k)}\}_j \longrightarrow \{\delta EV_{R,L}^{(k),(nr),BCOE}(j)\}_j \]

where \(\delta\gamma_j^{(k)}\) denotes a \(k\)-th (variable) consecutive spacing verifying

\[ \delta\gamma_j^{(k)} = \delta E_{R,L}^{(k),(nr),BCOE}(j). \]

**Proof:** This proposition is a generalisation of propositions 3.29, 3.31 and 3.32 to \(k\)-th consecutive spacings.

3.34 Physical interpretation of the nontrivial zeros of \(\zeta(s)\)

It was suggested for a long time that the nontrivial zeros of the Riemann zeta function are probably related to the eigenvalues of some wave dynamical system of which (Hamiltonian) operator is unknown [Kna].

Considering the new mathematical framework presented here taking into account the solution of the Riemann hypothesis, the connection between these two fields is rather evident. Indeed, referring to propositions 3.29 and 3.31, we see that there exists a surjective map:

\[ IM_{E_{R,L}^{(nr)} \rightarrow \gamma} : E_{R,L}(j) \longrightarrow \gamma_j, \quad \forall \ j, \ 1 \leq j \leq r, \]

between square roots of eigenvalues of a random matrix of GL_(IR \times IR) (or GL_(IC \times IC)) and nontrivial zeros of \(\zeta(s)\) in such a way that the kernel \(\text{Ker}[IM_{E_{R,L}^{(nr)} \rightarrow \gamma}]\) of \(IM_{E_{R,L}^{(nr)} \rightarrow \gamma}\) is given by the set \(\text{Ker}[IM_{E_{R,L}^{(nr)} \rightarrow \gamma}] = \{E_{R,L}(j) - E_{E_{R,L}^{(nr)}(j)}\}_j\) of differences between the square roots of the pseudoramified and pseudounramified eigenvalues of a matrix of GL_(IR \times IR).

And thus, if \(E_{R,L}(j) = E_{E_{R,L}^{(nr)}(j)},\) then we have that:

\[ E_{E_{R,L}^{(nr)}(j)} = \gamma_j. \]

The squares \(E_{E_{R,L}^{(nr)}(j)}\) of the pseudounramified eigenvalues \(E_{E_{R,L}^{(nr)}(j)}\) of GL_(IR \times IR) are also eigenvalues of the eigenvalue equation:

\[ (D_{R} \otimes D_{L})(\phi_r(G_{Q})(F_{v_1}^{(nr)} \times F_{v_1}^{(nr)}))) = E_{E_{R,L}^{(nr)}(j)}(\phi_r(G_{Q})(F_{v_1}^{(nr)} \times F_{v_1}^{(nr)}))) \]

(according to proposition 3.18),

of which eigenfunctions is the set of \(r\)-bituple

\[ \langle \phi(G_{Q}(1)(F_{v_1}^{(nr)} \times F_{v_1}^{(nr)})), \ldots, \phi(G_{Q}(j)(F_{v_1}^{(nr)} \times F_{v_1}^{(nr)})), \ldots, \phi(G_{Q}(r)(F_{v_1}^{(nr)} \times F_{v_1}^{(nr)})) \rangle \]
referring to proposition 3.4.

And these eigenbifunctions are the set \( \Gamma(\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)}) \) of sections of the bisemisheaf \( (\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)}) \) which was interpreted as the internal string field of an elementary (bisemi)particle according to section 3.6.

These considerations lead to the following proposition.

\[ 3.35 \text{ Proposition} \]

The squares of the nontrivial zeros \( \gamma_j \) of the Riemann zeta function \( \zeta(s) \) are the pseudounramified eigenbivalues of the eigenbivalue (biwave) equation:

\[
(D_R \otimes D_L)(\phi_r(G_Q(F_{vR}^{(nr)} \times F_{vL}^{(nr)}))) = \gamma_j^2(\phi_r(G_Q(F_{vR}^{(nr)} \times F_{vL}^{(nr)})))
\]

of which eigenbifunctions are the sections of the bisemisheaf \( (\hat{M}_{vR}^{(1)} \otimes \hat{M}_{vL}^{(1)}) \) being the internal string field of an elementary (bisemi)particle.

\[ \text{Proof:} \] Consequently, the squares \( \gamma_j^2 \) of the nontrivial zeros of \( \zeta(s) \) are the eigenbivalues of a matrix of \( \text{GL}_r(\mathbb{R} \times \mathbb{R}) \) constituting a representation of the bilinear differential Galois semigroup associated with the action of the differential bioperator \( (D_R \otimes D_L) \).

And, thus, each nontrivial zero \( \gamma_j \) of \( \zeta(s) \) is the infinitesimal generator on \( j \) quanta of the Lie subsemialgebra \( \text{gl}_1(F_{v_j}^{(nr)}) \) of the Lie subsemigroup \( \text{GL}_1(F_{v_j}^{(nr)}) \) or the energy of \( j \) compact transcendental pseudounramified quanta \( (N = 1) \).

This can also be seen from propositions 3.25 and 3.27. 

\[ \blacksquare \]
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