Abstract

The $CP^1$ model with Hopf interaction is quantised following the Batalin-Tyutin (BT) prescription. In this scheme, extra BT fields are introduced which allow for the existence of only commuting first-class constraints. Explicit expression for the quantum correction to the expectation value of the energy density and angular momentum in the physical sector of this model is derived. The result shows, in the particular operator ordering that we have chosen to work with, that the quantum effect has a divergent contribution of $O(\hbar^2)$ in the energy expectation value. But, interestingly the Hopf term, though topological in nature, can have a finite $O(\hbar)$ contribution to energy density in the homotopically nontrivial topological sector. The angular momentum operator, however, is found to have no quantum correction, indicating the absence of any fractional spin even at this quantum level. Finally, the extended Lagrangian incorporating the BT auxiliary fields is computed in the conventional framework of BRST formalism exploiting Faddeev-Popov technique of path integral method.

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1 Introduction

Constrained dynamical systems that one encounters usually in physical theories may or may not involve first class constraint(s) (FCC) (in the Dirac classification scheme [1]) which are weakly in involution. For example, the free Maxwell theory has two FCCs, whereas $O(3)$ Non-Linear Sigma Model (NLSM) has none. Systems with FCC(s) correspond to gauge theories, where the FCC(s) by themselves play the role of generator(s) of gauge transformations. In this sense, NLSM is not a gauge theory since it involves only a pair of second class constraints (SCC) which are non-commuting in the classical Poisson bracket sense. However, there is nothing very sacred in this distinction between gauge invariant and gauge non-invariant theories, since one can elevate the latter to an equivalent former type by introducing extra degrees of freedom. A well known example is the equivalence between the $O(3)$ NLSM and $CP^1$ model, where the latter is a $U(1)$ gauge theory which has an enlarged phase space. The usual prescription for quantising theories with SCCs is to implement strongly these SCCs by using Dirac brackets (DB). These DBs provide the canonical symplectic structure for the theory. Thereafter the procedure for quantisation is quite straightforward in “principle”. One has to just elevate these DBs into quantum commutators ($\{,\}_{DB} \rightarrow \frac{i}{\hbar}[,]$), where the dynamical variables now correspond to operators. For gauge theories, one gets some additional subsidiary conditions by demanding that physical Hilbert space is a gauge invariant subspace of the total Hilbert space and therefore corresponds to the kernel of the FCCs, i.e., the states belonging to physical Hilbert states are annihilated by these FCCs.

In contrast to the above mentioned Dirac quantisation, these gauge systems can also be quantised in the reduced phase space scheme, where the already existing FCC’s are rendered SCC’s by using a fresh set of DB’s. The symplectic structure therefore undergoes further modification. In this scheme, the system loses all the gauge degrees of freedom and one can isolate only the physical degrees of freedom and work with them. However, this symplectic structure, associated to this reduced phase space may become field dependent so that its elevation to quantum algebras is beset with operator ordering ambiguities. These complications also arise in models with nonlinearities. For example, in NLSM or its equivalent $CP^1$ model (to be studied here), the symplectic structure given by the basic DBs are field dependent. For $CP^1$ model, the DBs appropriate for the reduced phase space will be even more complicated.

One can bypass these problems by following the Batalin, Fradkin and Vilkovisky [2] scheme where the phase space is enlarged still further, rather than reduced, by introducing some additional fields in such a manner that even the existing SCCs are now elevated to FCCs. This is quite akin to the idea mentioned at the beginning of this Section. The advantage with this scheme is that one can just work with basic PBs where there is no operator ordering problem. A particular construction by Batalin and Tyutin (BT) [3] in this regard is very appealing since this scheme renders the FCC algebra in the extended
phase space completely Abelian. A number of applications, specifically in NLSM and $CP^N$ models [4], highlights the complexities in nonlinear models. BT have further provided a systematic way of constructing first-class operators that commute with the (converted) FCC’s. This formalism has been used by Hong, Kim and Park [5] who have constructed the first-class (improved) version of the phase space variables and have shown that the extended infinite series Hamiltonian [4] can be summed to a compact form at least for the $O(3)$ and $CP^1$ models. Furthermore, the advantage of this scheme is that the $CP^1$ constraint maintains its form in terms of the improved variables. Besides, this being true also for the Hamiltonian, the existence of solitonic configurations [6] are naturally ensured in the extended phase space by virtue of the fact that Bogomol’nyi inequality will also retain its original form.

The present paper deals with the BT quantisation of the $CP^1$ model with Hopf interaction. In a recent paper [7], the $CP^1$ model (without the Hopf term) has been studied using the BT prescription. However, the analysis is not correct since the original FCC present in the gauge invariant $CP^1$ model has been overlooked. Obviously, the equivalence between $O(3)$ NLSM and $CP^1$ model is lost even at the level of degree of freedom count (if the FCC is not taken into account). This constraint, along with other ones, has a direct bearing on the construction of the BRST charge $Q_B$ which defines the physical states ($Q_B|_{phys} >= 0$) in the extended Hilbert space of states. Indeed, the presence of the Hopf term apart from contributing to more complexities in a technical sense (as the constraints and their algebras are modified), gives rise to very interesting physical consequences. For example, the $CP^1$ model with its solitonic solutions, has major implications in the realm of condensed matter physics and the $O(3)$ NLSM describes anti-ferromagnetic systems having a linear dispersion relations. On the other hand, its non-relativistic version describes a ferromagnetic system having a quadratic dispersion relation [8]. The solitons in this ferromagnetic system may correspond to the skyrmions in a quantum Hall system. It has been shown in [9] that the Hopf term alters the spin algebra drastically. Also the inclusion of the (topological) Hopf term in the NLSM (in its usual relativistic version) has been shown in [10] to impart fractional spin to the soliton. In a quantum analysis, using the path integral method, it was shown in [11] that the system acquires a non-trivial phase upon a spatial rotation by $2\pi$. Since the Hopf term contribution, i.e., the fractional spin, appeared in $O(h^0)$, it seems that this result should be derivable in a classical (Dirac) analysis. But a canonical Dirac analysis [11] revealed fractional spin in the above model only after the model was altered using a certain identity which is not a constraint in the Dirac sense. This analysis was essentially carried out at a classical level, as the structure of the Dirac brackets were too complicated to lend themselves to be elevated into quantum commutators; the expressions were marred by the operator ordering ambiguities. This has been pointed out in [12]. It was further shown [12] that fractional spin was not induced by the Hopf term alone if the above mentioned identity was not used. The same conclusion can be drawn even at the level of collective coordinate quantisation (as we show in the next Section). It is therefore
desirable to study the model at quantum level and investigate whether any fractional spin of order $O(h)$ emerges or not (as a classical treatment did not reveal any fractional spin \[12\]). In either case, therefore, the result will be different from \[11\] as follows from a dimensional analysis. BT quantisation is adopted here to avoid of the above mentioned operator ordering ambiguities appearing in the Dirac brackets. As a first step towards this goal, in the present paper, we study the quantum correction to the energy of this model following the approach of \[13\]. One does not expect the Hopf term (being a topological term) to contribute to the energy-momentum tensor. Here we get a surprising result that the Hopf term may contribute non-trivially to the energy density in the homotopically nontrivial topological sector at the quantum level. We then take up the case of angular momentum to find that there is no quantum correction. Another reason for considering $CP^1$ model, rather than NLSM, is that the Hopf term is local in terms of $CP^1$ variables \[14\] where no gauge-fixing condition is required \textit{a priori}. The model is thus a $U(1)$ gauge theory and amenable to gauge independent Dirac quantization. As we have mentioned earlier that the structure of the Dirac brackets in the presence of SSCs is less complicated in the Dirac scheme than the corresponding gauge-fixed reduced phase space scheme. Besides, in the BT scheme, even the existing SCCs are going to be elevated to FCC, so that the symplectic structure is field independent and is canonical as it is obtained from the simple PB.

The paper is organised as follows: In Section II the collective coordinate quantisation of the $CP^1$ model with the Hopf term is discussed. It has been shown that the Hopf term does not have any effect on the energy or spin of the soliton. Section III is devoted to the classical constraint analysis. Section IV deals with the Batalin-Tyutin extension where the FCCs, First Class (FC) variables and the FC Hamiltonian are constructed in the extended phase space. The important results regarding the presence and absence of Hopf term induced quantum correction to the energy density and angular momentum respectively, are derived in Section V. Conventional BRST quantisation is outlined in Section VI. The internal consistency of the results is also checked by re-deriving the action in the unitary gauge. The paper ends with a conclusion and future perspectives in Section VII.

2 Collective co-ordinate quantisation

In this Section, we are going to provide a brief review of $CP^1$ fields and their relationship with the fields of NLSM, apart from setting up our conventions. Furthermore, we are going to show that in the $CP^1$ model, fractional spin is not induced by the (topological) Hopf term at the level of collective coordinate quantisation, unless the model is \textit{altered} by using an identity, (which is not a Hamiltonian constraint), as has been done in \[11\].

The $CP^1$ manifold is given by the set of all non-zero complex doublets $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfying the normalization condition

$$Z^\dagger Z = |z_1|^2 + |z_2|^2 = 1,$$

(2.1)
and the identification \( Z \sim e^{i\theta} Z \) where \( e^{i\theta} \in U(1) \) is any unimodular number in the complex plane. Since eqn. (2.1) represents \( S^3 \), which is homeomorphic to \( SU(2) \) group manifold, \( CP^1 \) space can be identified with the coset space \( \frac{SU(2)}{U(1)} \). Alternatively, \( SU(2) \) can be identified with a \( U(1) \) prinicipal bundle over the base \( CP^1 \). Clearly, one can make a local (in the neighbourhood \( z_1 \neq 0 \)) gauge choice

\[
z_1 - z_1^* = 0,
\]

so that (2.1) is reduced to an equation of \( S^2 \) enabling one to finally identify \( CP^1 \) with \( S^2 \). This gauge will be also discussed at the end (in Section VI).

Associated with this \( U(1) \) bundle, there is a natural \( U(1) \) connection one form \( A = -i Z^\dagger d Z \),

\[
(2.3)
\]

and can be identified with the Dirac magnetic monopole connection one-form. Now the \( CP^1 \) model in \( 2+1 \) dimensions is

\[
L_0 = \int d^2 x \left[ (D_\mu Z)^{\dagger} (D_\mu Z) - \lambda (Z^\dagger Z - 1) \right],
\]

where the covariant derivative operator \( D_\mu \) is given by

\[
D_\mu = \partial_\mu - i A_\mu,
\]

and \( A_\mu = -i Z^\dagger \partial_\mu Z \) is nothing but the pull-back of the connection one-form (2.3) onto the spacetime and \( \lambda \) is a Lagrange multiplier enforcing the constraint (2.1). The form (2.4a) can be further simplified to

\[
L_0 = \int d^2 x \left[ |\partial_\mu Z|^2 - |Z^\dagger \partial_\mu Z|^2 - \lambda (|Z|^2 - 1) \right].
\]

Formally, this \( CP^1 \) model is same as NLSM given by

\[
L'_0 = \int d^2 x \left[ \frac{1}{4} \partial_\mu n^T \partial^\mu n - \lambda (n^T n - 1) \right].
\]

where \( n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \) represents a set of three real scalar fields subjected to the constraint

\[
n^T n \equiv n_\alpha n_\alpha = n_1^2 + n_2^2 + n_3^2 = 1.
\]

* We adopt the notations in which the flat Minkowski metric is: \( g_{\mu\nu} = \text{diag} (+1, -1, -1) \) and the totally antisymmetric Levi-Civita tensor satisfies: \( \varepsilon^{012} = \varepsilon_{012} = +1 \), \( \varepsilon_{0ij} = \varepsilon_{ij} \), \( \varepsilon^{12} = \varepsilon_{12} = +1 \). Here, and in what follows, the Greek indices \( \mu, \nu, \rho, ... = 0, 1, 2 \) and the Latin indices \( i, j, k, ... = 1, 2 \) correspond to space-time and space directions respectively on the space-time manifold. The summation convention on the subscript \( \alpha (= 1, 2) \) is occasionally supressed.
The equivalence between \( CP^1 \) model and NLSM can be trivially established \([12]\) (see also Rajaraman in \([6]\)) by noting the fact that these \( n_a \)'s can be obtained from the \( CP^1 \) fields by using the Hopf map

\[
    n_a = Z^i \sigma_a Z,
\]

where \( \sigma \)'s are the Pauli matrices. Although equivalent, \( CP^1 \) model is a \( U(1) \) gauge theory but NLSM is not. Correspondingly, \( CP^1 \) fields \( Z \) transform nontrivially \( (Z \to e^{i\phi}Z) \) under \( U(1) \) gauge transformation, but the fields \( n_a \) of NLSM are invariant under \( U(1) \) transformations as is clear from (2.8).

In order to obtain a finite energy static solution, it is necessary for the fields \( n_a \) to tend to constant configuration asymptotically. With this, the two-dimensional plane \( \mathcal{D} \) gets effectively compactified to \( S^2 \) and the configuration space \( C \), which is nothing but the set of all maps \( f : S^2 \to S^2 \) (field manifold) splits into a disjoint union of path connected spaces as

\[
    \Pi_0(C) = \Pi_2(S^2) = Z.
\]

Hence there exists solitons or skyrmions in this model, characterized by the set of integers \( Q \in \mathbb{Z} \) given by

\[
    Q = \int d^2 x \ j^0,
\]

where \( j^0 \) is the time component of the identically conserved \((\partial_\mu j^\mu = 0)\) topological current \((j^\mu)\) given by

\[
    j^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda} \varepsilon_{a b c} n_a \partial_\nu n_b \partial_\lambda n_c.
\]

Note that the conservation of \( j^\mu \) holds irrespective of any equation of motion. \( Q \), referred to as the soliton number, labels the disconnected pieces of the configuration space \( C \). This topological current (2.11) can also be expressed as the curl of the \( U(1) \) gauge field (2.3) and thus, in terms of the \( CP^1 \) fields, as

\[
    j^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda,
    = -\frac{i}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu Z^i \partial_\lambda Z,
\]

In any soliton number sector, the fundamental group of \( C \) is nontrivial since

\[
    \Pi_1(C) = \Pi_3(S^2) = \mathbb{Z}.
\]

This implies that the loops based at any point in the configuration space fall into separate homotopy classes labeled by another integer \( H \). This integer can be given a representation by the so-called Hopf action \([13]\)

\[
    S_H = \int d^3 x \ j^\mu A_\mu.
\]

Although it is formally similar to the Chern-Simon (CS) action, there is subtle difference in the sense that in Hopf action, unlike in CS action, the gauge field is not an independent
variable in configuration space and is determined in terms of \( j^\mu \) using certain gauge condition to render it a generic non-local current-current interaction. Thus, in Hopf case, we do not have an enlarged phase space unlike the CS case. However, the advantage of \( CP^1 \) formulation is that the Hopf action (2.14) can be expressed in a gauge invariant manner, which is \textit{not} non-local. The hermitian form of Hopf action is

\[
S_H = -\frac{1}{4\pi} \int d^3x \, \varepsilon^{\mu
u\lambda}(Z^\dagger \partial_\mu Z - \partial_\mu Z^\dagger Z)\partial_\nu Z^\dagger \partial_\lambda Z. \tag{2.15}
\]

In the topological sector \( Q = 1 \), the \( n_a \) fields can be taken to have the profile

\[
n = \begin{pmatrix} \hat{r} \sin g(r) \\ \cos g(r) \end{pmatrix} \tag{2.16}
\]

where \( \hat{r} \) being the unit vector in the \( n_1, n_2 \) plane and \( g(r) \) satisfies

\[
g(0) = 0, \quad g(\infty) = \pi. \tag{2.17}
\]

The corresponding profile for the \( Z \)-fields can be trivially obtained by inverting (2.8) in a particular gauge. For example, in the gauge (2.2), the \( CP^1 \) fields have the following profile

\[
Z = \begin{pmatrix} \cos g(r) \\ \sin g(r) \frac{1}{2} e^{i(g''(r))} \end{pmatrix} \tag{2.18}
\]

with \( \phi = \tan^{-1}\left(\frac{n_2}{n_1}\right) \) being the polar angle in the \( n_1, n_2 \) plane. As any configuration, obtained by making an \( SO(2) \) rotation by an angle \( \alpha \) in (2.16) is energetically degenerate to (2.16), we introduce a corresponding \( U(1) \) factor \( e^{i\alpha(t)} \) in (2.18) where the phase \( \alpha(t) \) is the collective coordinate in this case.

Substituting (2.18) in (2.5), one gets after a straightforward calculation

\[
L_0 = \frac{\pi}{2} \lambda \dot{\alpha}^2 - N, \tag{2.19}
\]

where

\[
N = \frac{\pi}{2} \int_0^\infty dr \, r \left[ (g'(r))^2 + \frac{1}{\pi} \sin^2 g(r) \right],
\]

\[
\lambda = \int_0^\infty dr \, r \sin^2 g(r). \tag{2.20}
\]

This is expected to be the same as the corresponding expression one gets using NLSM \( \square \). However for the same profile of the \( Z \) fields as in (2.18), the Hopf term \( \square \)

\[
L_H = \theta \int d^2x \, \varepsilon^{\mu
u\lambda} \left[ Z^\dagger \partial_\mu Z \partial_\nu Z^\dagger \partial_\lambda Z + \partial_\mu Z^\dagger Z \partial_\nu Z^\dagger \partial_\lambda Z \right], \tag{2.21}
\]

obtained from (2.15) with the inclusion of Hopf parameter \( \theta \), vanishes. This is true without performing the space-time integration. But if the Hopf term is \textit{altered} a la’ \( \square \) using the identity

\[
\int d^2x \, A_0(x) j_0(x) = -\int d^2x \, A_i(x) j_i(x), \tag{2.22a}
\]
or written in terms of $CP^1$ variables as

$$
\int d^2x \, Z^\dagger \dot{Z} \varepsilon_{ij} (\partial_i Z)^\dagger (\partial_j Z) = \int d^2x \, \varepsilon_{ij} Z^\dagger (\partial_i Z)^\dagger [((\partial_j Z)^\dagger \dot{Z} - \dot{Z}^\dagger \partial_j Z)],
$$

(2.22b)

to get

$$
\tilde{L}_H = \frac{\theta}{\pi} \int d^2x \, \varepsilon^{i\nu\lambda} \partial_\nu Z^\dagger \partial_\lambda Z^\dagger \partial_i Z,
$$

(2.23)

then for the same profile (2.18) of the $CP^1$ field, one gets a non-vanishing $\tilde{L}_H$ as

$$
\tilde{L}_H = \theta \dot{\alpha}.
$$

(2.24)

Note that the identity (2.22) which is valid in the radiation gauge, is not a Hamiltonian constraint as it involves time derivative. Thus the dynamical consequences of $CP^1$ model coupled to $L_H$ or $\tilde{L}_H$ can be quite different. It was shown in [12] that $L_H$, coupled to $CP^1$, does not induce any fractional spin at the classical level, whereas $\tilde{L}_H$ induces fractional spin at the classical level. As we shall see now again following [11] that the same result follows even at the level of collective coordinate quantisation.

In order to compute fractional spin for the model

$$
L = L_0 + L_H,
$$

(2.25)

let us first note that the Hopf term $L_H$ being a topological term, does not contribute to the symmetric expression of energy-momentum tensor ($T_{\mu\nu} \sim \frac{i\delta S}{\delta g^{\mu\nu}}$). This tensor is given by

$$
T_{\mu\nu} = (D_\mu Z)^\dagger (D_\nu Z) + (D_\nu Z)^\dagger (D_\mu Z) - g_{\mu\nu} (D_\rho Z)^\dagger (D_\rho Z).
$$

(2.26)

The angular momentum $J = \int d^2x \, \varepsilon_{ij} x_i T_{0j}$, corresponding to the profile (2.18) of the $Z$-field, is given as

$$
J = -\pi \lambda \dot{\alpha}.
$$

(2.27)

As $L_H = 0$, corresponding to this profile, the Lagrangian (2.25) reduces by using (2.19) to

$$
L = L_0 = \frac{\pi}{2} \lambda \dot{\alpha}^2 - N.
$$

(2.28)

The canonically conjugate momenta is then the angular momentum (up to a sign)

$$
p = \frac{\delta L}{\delta \dot{\alpha}} = \pi \lambda \dot{\alpha} = -J.
$$

(2.29)

The Legendre transformed Hamiltonian, using eqn. (2.28) gives

$$
H = \pi \dot{\alpha} - L = N + \frac{1}{2\pi \lambda} J^2.
$$

(2.30)

This can be identified with the Hamiltonian of a rigid rotor with moment of inertia equal to $\pi \lambda$. 

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Clearly, the energy eigenfunctions $e^{i\alpha}$ are also the eigenfunctions of $J$. The corresponding eigen values are $-m$, namely:

$$J e^{i\alpha} = -p e^{i\alpha} = i \frac{\partial}{\partial \alpha} e^{i\alpha} = -m e^{i\alpha}.$$ (2.31)

Single valuedness in $\alpha$ space restricts $m$ to be an integer (i.e., $m \in \mathbb{Z}$). Thus, the system does not exhibit fractional spin at the level of collective coordinate quantisation. On the other hand, if the $CP^1$ model is coupled to the altered Hopf term $\tilde{L}_H$, i.e., if we consider the model

$$L' = L_0 + \tilde{L}_H,$$ (2.32)

instead of (2.25), then for the profile (2.18) of the $Z$ field, eqn. (2.32), on using eqn. (2.24), reduces to

$$L' = \frac{\pi}{2} \lambda \dot{\alpha}^2 + \theta \dot{\alpha} - N.$$ (2.33)

This is the same Lagrangian considered in [11] where the existence of fractional spin was shown at the level of collective coordinate quantisation. Note the presence of the $\theta$ dependent Hopf term here. This corroborates our earlier observation that fractional spin can not be obtained unless the model is altered. But in a gauge independent Dirac quantisation one can not use the identity (2.22) which is valid only in the radiation gauge. On the other hand, Wilczek and Zee [10] have shown in the path integral framework the existence of the fractional spin. In order to obtain any contribution of the Hopf term, related to fractional spin, in a canonical framework, one has to go for the full quantisation of this model. This motivates us for our present study of this model in the BT formalism.

### 3 Constraint analysis

The classical action ($S$) of the $CP^1$ model with the Hopf interaction is obtained by adding (2.5) and (2.21) as

$$S = \int d^3 x \mathcal{L}_{cl} = \int d^3 x \left[ \frac{\partial}{\partial \dot{Z}} \dot{Z} \dot{\alpha} - (Z^\dagger \partial Z) \right] - \lambda (Z^\dagger Z - 1) + \theta \varepsilon^\mu_{\nu\rho}(Z^\dagger \partial_\mu Z \partial_\nu Z^\dagger \partial_\rho Z).$$ (3.1)

The canonical analysis yields the following expressions for the canonically conjugate momenta

$$\Pi_\alpha = \frac{\partial S}{\partial \dot{\alpha}} = \dot{z}_\alpha - z_\alpha (\dot{Z}^\dagger Z) + \theta M^*_\alpha,$$

$$\Pi^*_\alpha = \frac{\partial S}{\partial \dot{z}_\alpha} = \dot{z}_\alpha - z_\alpha (Z^\dagger \dot{Z}) + \theta M_\alpha,$$ (3.2a)

where the expressions for $M^*_\alpha$ and $M_\alpha$ are:

$$M^*_\alpha = \varepsilon_{ij} \left[ z^*_\alpha (\partial_i Z^\dagger \partial_j Z) - \partial_j z^*_\alpha (\partial_i Z^\dagger Z) + \partial_j z_\alpha (Z^\dagger \partial_i Z) \right],$$

$$M_\alpha = \varepsilon_{ij} \left[ z_\alpha (\partial_j Z^\dagger \partial_i Z) - \partial_j z_\alpha (Z^\dagger \partial_i Z) + \partial_j z_\alpha (\partial_i Z^\dagger Z) \right].$$ (3.2b)

The Legendre transformation leads to the derivation of the classical Hamiltonian density

$$\mathcal{H}_{cl} = \Pi_\alpha \dot{z}_\alpha + \Pi^*_\alpha \dot{z}^*_\alpha - \mathcal{L}_{cl},$$

$$= |\dot{Z}|^2 - |\dot{Z}^\dagger Z|^2 + |\partial_\alpha Z|^2 - |Z^\dagger \partial_\alpha Z|^2 + \lambda (|Z|^2 - 1).$$ (3.3)
In the above Hamiltonian density, the contribution of the Hopf term apparently disappears since it consists of only linear time derivative terms. However, \( \theta \)-dependent terms reappear when the phase space variables are introduced to express the above Hamiltonian density as given below

\[
\mathcal{H}_{cl} = |\Pi - \theta M^*|^2 + |\partial_i Z|^2 - |Z^* \partial_i Z|^2 + \lambda(|Z|^2 - 1),
\]

where \( P_\alpha = (\Pi_\alpha - \theta M_\alpha^*) \) and its conjugate are nothing but the conjugate momenta in the absence of Hopf term. The composite variables \( M \)'s in (3.2b) obey the following useful identities:

\[
\begin{align*}
\dot{z}_\alpha M_\alpha^* + z_\alpha M_\alpha &= 0, \\
\dot{\bar{z}}_\alpha M_\alpha^* + \bar{z}_\alpha M_\alpha &= \mathcal{L}_H.
\end{align*}
\]

\( \theta \)-From the primary constraint \( T_0 = \Pi_\lambda \approx 0 \), the rest of the constraints \( T_i \) are derived by demanding time persistence of the constraints themselves as:

\[
\begin{align*}
\{\Pi_\lambda, H_{cl}\} &= -(|Z|^2 - 1) \equiv -T_1, \\
\{T_1, H_{cl}\} &= z_\alpha P_\alpha + z_\alpha^* P_\alpha^* \equiv -T_2, \\
\{T_2, H_{cl}\} &= -4\theta \varepsilon_{ij}(\partial_j Z^\dagger \partial_i Z)T_3 - 2\lambda T_1 + 2T_4, \\
\{T_3, H_{cl}\} &= -4i\theta \varepsilon_{ij}(\partial_j Z^\dagger \partial_i Z)T_2 - i(\partial_i \partial_i Z^\dagger Z - Z^\dagger \partial_i \partial_i Z)T_1.
\end{align*}
\]

Equations (3.6–3.9) show that the constraint algebra is closed. Before proceeding to the Dirac classification of the constraints, we note that \( T_0 \) and \( T_4 \) constitute an SCC pair.

These constraints are implemented strongly (in the Dirac sense) in \( \mathcal{H}_{cl} \), without making any changes in the Poisson brackets of the remaining fields \( z, z^*, \Pi, \Pi^* \). This leads to

\[
\begin{align*}
\mathcal{H}_{cl} &= |Z|^2|P|^2 + 2|\partial Z^\dagger Z|^2 - |\partial_i Z|^2 + 4\theta \varepsilon_{ij}(\partial_i Z^\dagger Z)(P_\alpha \partial_j z_\alpha - P_\alpha^* \partial_j z_\alpha^*), \\
\mathcal{H}_{cl} &= |Z|^2|P|^2 + 2|\partial Z^\dagger Z|^2 + (2|Z|^2 - 3)|Z^\dagger \partial_i Z|^2 + 4\theta \varepsilon_{ij}( (|Z|^2 - 1)(\partial_i Z^\dagger Z)(P_\alpha \partial_j z_\alpha - P_\alpha^* \partial_j z_\alpha^*).\]
\]
Now, the constraint algebra turns out to be

\[ \{T_1(x), T_2(y)\} = 2|Z|^2 \delta(x - y), \quad \{T_1(x), T_3(y)\} = 0, \]
\[ \{T_2(x), T_3(y)\} = 8 \theta \varepsilon_{ij} \left| [Z^2 \partial_i Z \partial_j Z + (Z^\dagger \partial_i Z)(\partial_j Z Z)] \right| \delta(x - y). \quad (3.13) \]

Let us choose the pair \( T_1 \) and \( T_2 \) to be SCC. Since the \( CP^1 \) model has a \( U(1) \) gauge invariance, the corresponding FCC, in our case, is derived as the following linear combination

\[ T = T_3 + 4 \theta \varepsilon_{ij} [(\partial_i Z^\dagger \partial_j Z) + \frac{1}{|Z|^2} (Z^\dagger \partial_i Z)(\partial_j Z Z)], \]
\[ \equiv T_3 + 4 \theta \varepsilon_{ij} [(\partial_i Z^\dagger \partial_j Z)T_1 + 4 \theta \varepsilon_{ij} \frac{1}{|Z|^2} (Z^\dagger \partial_i Z)(\partial_j T_1)T_1), \]
\[ \equiv T_3 + 4 \theta \varepsilon_{ij} (\partial_i Z^\dagger \partial_j Z)T_1. \quad (3.14) \]

The last term in the second line is dropped since it is quadratic in the constraints.

As a simple warm up exercise, one can re-derive the original action (3.1), starting from the Hamiltonian in (3.10) since \( \delta(T_1) \) has been included in the measure. We shall follow this principle later too. In the expression (3.15), \( \chi \) is the gauge-fixing condition corresponding to the FCC \( T \) and the square-root factor is the Senjanovic measure which reduces to a c-number term on the constrained manifold. To recover the co-ordinate space action (3.1), we introduce the multiplier fields \( \lambda_1, \lambda_2, \lambda_3 \) as

\[ Z = \int \mathcal{D}(z, z^*, \Pi, \Pi^*) \delta(T_1) \delta(T_2) (\det\{T_1, T_2\})^{\frac{i}{2}} \delta(\delta) \delta(\chi) \det\{T, \chi\} \]
\[ \exp \left( i \int d^3x \left[ \Pi_1 \dot{z}_1 + \Pi_2 \dot{z}_2 - \left( |P|^2 + |\partial_i Z|^2 - |Z^\dagger \partial_i Z|^2 \right) \right] \right). \quad (3.15) \]

Note that in the above expression, we do not use the more complicated form of the Hamiltonian in (3.10) since \( \delta(T_1) \) has been included in the measure. We shall follow this principle later too. In the expression (3.15), \( \chi \) is the gauge-fixing condition corresponding to the FCC \( T \) and the square-root factor is the Senjanovic measure which reduces to a c-number term on the constrained manifold. To recover the co-ordinate space action (3.1), we introduce the multiplier fields \( \lambda_1, \lambda_2, \lambda_3 \) as

\[ Z = \int \mathcal{D}(z, z^*, \Pi, \Pi^*, \lambda_1, \lambda_2, \lambda_3) (\det\{T_1, T_2\})^{\frac{i}{2}} \delta(\delta) \delta(\chi) \det\{T, \chi\} \]
\[ \exp \left( i \int d^3x \left[ \Pi_1 \dot{z}_1 + \Pi_2 \dot{z}_2 - \left( |P|^2 + |\partial_i Z|^2 - |Z^\dagger \partial_i Z|^2 \right) \right] \right. \]
\[ + \lambda_1 (|Z|^2 - 1) + \lambda_2 (P_1 \dot{z}_1 + P_2 \dot{z}_2) + i \lambda_3 (P_1 \dot{z}_1 - P_2 \dot{z}_2) \right) \]
\[ , \quad (3.16) \]

and subsequently integrate out the momenta as well as the multiplier fields. In the above action, the term proportional to \( T_1 \), occurring in the FCC of eqn. (3.14) is absorbed in the \( \lambda_1 \)-term. Since these variables appear linearly, the classical equations of motion can be used and one recovers the partition function as

\[ Z = \int \mathcal{D}(z, z^*) \delta(|Z|^2 - 1) \delta(\chi) \det\{T, \chi\} \exp\left( iS \right), \quad (3.17) \]

where the classical action \( I_{cl} \) is given by the equation (3.1).

In the next Section, we shall take up the Batalin-Tyutin extension of the above model.

4 Batalin-Tyutin extension
The nonlinear nature of the SCC (as well as the presence of the fields in the SCC constraint algebra) prompts us to rely on the Batalin-Fradkin-Vilkovisky formalism and the Batalin-Tyutin scheme. In the latter scheme, the phase space is extended by incorporating auxiliary fields (also known as B-T fields). In certain cases, like in the BT extension of the Proca model, these fields can be identified with the Stueckelberg field of the Stueckelberg formalism. In fact, the extension renders the SCC’s to FCC’s with the advantage that the Dirac brackets are not required. Furthermore, the path integral measure becomes simpler (i.e., no Senjanovic measure is needed) and the gauge freedom is enhanced so that some convenient gauge conditions can be introduced in the formalism.

Using the previous results, we extend the SCC’s as follows

\[
T_1 \rightarrow \tilde{T}_1 \equiv \mid Z \mid^2 - 1 + 2\phi_1, \\
T_2 \rightarrow \tilde{T}_2 \equiv \Pi_\alpha Z_\alpha + \Pi^*_\alpha z^*_\alpha + 2\mid Z \mid^2 \phi_2,
\]

where \(\phi_i\) are the B-T fields obeying

\[
\{\phi_i(x), \phi_j(y)\} = -\frac{1}{2}\varepsilon_{ij}\delta(x - y).
\]

Thus, modulo an overall factor, \(\phi_1\) and \(\phi_2\) fields can be treated as a canonical pair. When convenient, we shall denote them by \(\phi_1 \equiv \phi\) and \(-2\phi_2 \equiv \Pi\) respectively. This leads to the Abelianization of the SCC’s, i.e., \(\{\tilde{T}_1, \tilde{T}_2\} = 0\). In [4], the corresponding first-class ‘extended’ Hamiltonian was obtained as an infinite series in higher powers of the auxiliary variables. In the present model, the application of the BT scheme (see, e.g. [4]) will be even more complicated due to the presence of Hopf term. However, a remarkable extension of the above scheme was put forward in [2] where it was proved that there exists a one-to-one mapping between the physical variables and an ‘improved’ set of variables (appearing as a power series in B-T fields) with the property that they commute with the extended SCC’s.

The complete BT extended theory, which is now a gauge theory, can now be obtained by simply replacing the physical variables in \(H_{cl}\) and the original FCC’s by their improved counterpart. This procedure was further developed in [4] where it was shown that, at least in the \(CP^1\) and \(O(3)\) nonlinear \(\sigma\)-models, the infinite series in the extended Hamiltonian \(\tilde{H}\) can be summed to give a compact expression for the same.

Using the Batalin-Tyutin prescription, one can write the improved variables, denoted here by corresponding tildes, for the present case as

\[
\tilde{z}_\alpha(x) = z_\alpha(x)\left[1 - \sum_{n=1}^{\infty} C_n^{(z)}(\frac{\phi_i}{|Z|^2})^n\right] \equiv z_\alpha(x) A, \\
\tilde{\Pi}_\alpha(x) = \left[\Pi_\alpha(x) + z^*_\alpha(x)\phi_2(x)\right] \left[1 + \sum_{n=1}^{\infty} C_n^{(\pi)}(\frac{\phi_i}{|Z|^2})^n\right] \equiv (\Pi_\alpha + z^*_\alpha\phi_2) B,
\]

where expressions for \(C^s\)’s are

\[
C_n^{(z)} = (C_n^{(z)})^* = \frac{(-1)^n(2n-3)!!}{n!}, \\
C_n^{(\pi)} = (C_n^{(\pi)})^* = \frac{(-1)^n(2n-1)!!}{n!}, \\
(-1)!! = 1, \quad n!! = n(n - 2)(n - 4)\ldots\ldots
\]
The following useful identities [3]

\[ \mathcal{A} \mathcal{B} = 1, \quad (\mathcal{A})^2 = (\mathcal{B})^{-2} = \frac{|z|^2 + 2\phi_1}{|z|^2}, \quad (4.6) \]

show that

\[ \tilde{T}_1 = |\tilde{Z}|^2 - 1, \quad \tilde{T}_2 = \tilde{\Pi}_a z^*_a + \tilde{\Pi}^*_a z_a. \quad (4.7) \]

It is elementary to check that the improved variables commute with the modified constraints \( \tilde{T}_i, (i = 1, 2) \). A crucial property of the improved variables, proved by Batalin and Tyutin in [3], is that the “tilde” of the products is the product of tildes, i.e., \((\tilde{A}\tilde{B}) = (\tilde{A})(\tilde{B})\) for any two variables \(A\) and \(B\). This key property allows us to write the improved (or first-class) Hamiltonian as:

\[ \tilde{H} = \int d^2x \tilde{\mathcal{H}} = \int d^2x \left( |\tilde{P}|^2 + |\partial_z \tilde{Z}|^2 - |\tilde{Z}| \partial_z \tilde{Z}|^2 \right). \quad (4.8) \]

This improved Hamiltonian, by construction, commutes with the constraints \( \tilde{T}_1 \) and \( \tilde{T}_2 \). Note here that since the form of the Hamiltonian (4.8) here is just the same as that of (3.4), this Hamiltonian (4.8) too admits solitonic configurations and associated topological currents. The corresponding expression for the topological current can be obtained here by just replacing \(Z\) variables in (2.12) by their ‘images’ \(Z\). We can now do the same for the angular momentum ‘\(J\)’. The improved version \(\tilde{J}\) can be now trivially obtained from \(J\). As was observed in [12] that, the expressions for angular momentum obtained either from Noether’s prescription \((J^N)\) or through the symmetric expression for energy-momentum tensor \((J^{(e)})\) (cf. (2.26)) turn out to be identical thereby indicating the absence of any fractional spin imparted by the Hopf term at the classical level. The improved \(\tilde{J}\) is therefore given by

\[ \tilde{J} = \int d^2x \varepsilon_{ij} \left( \tilde{\Pi}_a \partial_j z^*_a + z^*_a \tilde{\Pi}^*_a \partial_j z_a \right). \quad (4.9) \]

Before introducing the original first-class constraint \(T\), as given by (3.14), it is worthwhile to comment on the algebra of the improved variables. In fact, it has been proved in [3] that the following identification between the algebra of physical and improved variables holds

\[ \{A, B\}_{DB} = G_{AB} \quad \Rightarrow \quad \{\tilde{A}, \tilde{B}\}_{PB} = \tilde{G}_{AB}, \quad (4.10) \]

where \(\{A, B\}_{DB}\) is the Dirac bracket between the physical variables \(A\) and \(B\) with the SCC’s considered as strong relation, whereas the bracket \(\{\tilde{A}, \tilde{B}\}_{PB}\) stands for the Poisson bracket in the extended phase space. With the SCC’s: \(T_1 = |Z|^2 - 1\) and \(T_2 = z^*_a \Pi_a + z_a \Pi^*_a\), it is straightforward to compute the following DB’s

\[
\begin{align*}
\{z_\alpha, \bar{z}_\beta\}_{DB} & = \{z_\alpha, \bar{z}^*_\beta\}_{DB} = 0, \\
\{z_\alpha, \Pi_\beta\}_{DB} & = -\frac{1}{2|z|^2} z_\alpha z_\beta \delta(x - y), \\
\{\bar{z}_\alpha, \Pi_\beta\}_{DB} & = (\delta_\alpha_\beta - \frac{1}{2|z|^2} z_\alpha \bar{z}^*_\beta) \delta(x - y), \\
\{\Pi_\alpha, \Pi^{*_\beta}\}_{DB} & = -\frac{1}{2|z|^2} (\Pi_\alpha \bar{z}^{*\beta} - \Pi^{*_\beta} z_\alpha^*) \delta(x - y), \\
\{\Pi_\alpha, \Pi_\beta\}_{DB} & = \frac{1}{2|z|^2} (\Pi_\alpha \bar{z}^*_\beta - \Pi_\beta z_\alpha^*) \delta(x - y), \\
\end{align*}
\]
In fact, as has been shown in [12] that these structures of the DBs are inherited from the global $SU(2)$ invariant $S^3$ model. From (4.11), as mentioned before, the algebra for the improved variables are obtained just by replacing each variable by its improved ‘image’, e.g.,
\[
\{\tilde{\Pi}_\alpha, \tilde{\Pi}_\beta\} = \frac{1}{2|\tilde{Z}|^2} (\tilde{\Pi}_\alpha \tilde{z}_\beta^{*} - \tilde{\Pi}_\beta \tilde{z}_\alpha^{*}) \delta(x - y). \tag{4.12}
\]
Now we are ready to introduce the first-class constraint $\tilde{T}$ in the extended phase space as an ‘image’ of the original FCC (3.14), using the prescription of [3], as mentioned earlier
\[
T \Rightarrow \tilde{T} = \tilde{T}_3 + 4\theta i \varepsilon_{ij} (\partial_i \tilde{Z}^\dagger \partial_j \tilde{Z}) \tilde{T}_1. \tag{4.13}
\]
Obviously, by construction, we have the following algebra
\[
\{\tilde{T}_1, \tilde{T}\} = \{\tilde{T}_2, \tilde{T}\} = 0, \tag{4.14}
\]
and, finally, it can be shown that
\[
\{T, H_{cl}\}_{DB} = 0, \Rightarrow \{\tilde{T}, \tilde{H}\} = 0, \quad \tilde{0} = 0. \tag{4.15}
\]
Thus, our gauge theory constitutes of three Abelian FCC’s $\tilde{T}_1, \tilde{T}_2$ and $\tilde{T}$ with the first-class Hamiltonian $\tilde{H}$ in the extended phase space. This Hamiltonian turns out to be in involution with the FCC’s. As has been mentioned in the introduction, the presence of the original first-class constraint $T$ (3.14) and its corresponding extension $\tilde{T}$ (4.13) has been completely ignored in [7]. These FCCs $\tilde{T}_1, \tilde{T}_2$ in (4.1) and $\tilde{T}_3$ whose explicit form is given as
\[
\tilde{T}_3 = i \left( \Pi_\alpha z_\alpha - \Pi_\alpha^* z_\alpha^* - 2\theta \varepsilon_{ij} (\mathcal{A})^4 (|\tilde{Z}|^2 \partial_i \tilde{Z}^\dagger \partial_j \tilde{Z} + \tilde{Z}^\dagger \partial_i \tilde{Z} \partial_j \tilde{Z}^\dagger \tilde{Z}) \right) \tag{4.16}
\]
In its infinitesimal form, the gauge transformations generated by $\tilde{T}_a$ ($a = 1, 2, 3$) (4.1), on a generic field $\Phi(x)$, is
\[
\delta_a \Phi(x) = \int d^2y f_a(y) \{\Phi(x), \tilde{T}_a(y)\}, \quad \text{(no summation)} \tag{4.17}
\]
where $f_a(y)$ are some arbitrary parameters of transformations and can be taken to be smooth functions. In certain cases, it may be necessary to restrict them further to functions having compact supports. A straightforward calculation yields the following results
\[
\begin{align*}
\delta_1 Z(x) &= 0, \quad \delta_1 \phi(x) = 0, \quad \delta_1 \Pi_\phi = -2 f_1(x), \\
\delta_2 Z(x) &= f_2(x) Z(x), \quad \delta_2 \phi(x) = -f_2(x) |Z|^2, \quad \delta_2 \Pi_\phi = 0, \\
\delta_3 Z(x) &= i f_3(x) Z(x), \quad \delta_3 \phi(x) = 0.
\end{align*} \tag{4.18}
\]
Here we have intentionally omitted $\delta_3 \Pi_\phi$ as it is a bit complicated and will not be very useful for our further discussions. At this stage, one can make certain observations regarding the nature of the gauge transformations generated by these FCCs. First note that only $\tilde{T}_3$ (just as the original $T_3$ (3.12)) generates $U(1)$ gauge transformation on the $Z$ fields, whereas under $\tilde{T}_2$ the $Z$ field undergoes a scale transformation but remains invariant under $\tilde{T}_1$. As
far as the BT fields are concerned, the field $\Pi_\phi$ remains invariant under $\tilde{T}_2$ but undergoes a shift under $\tilde{T}_1$. And the field $\phi(x)$ remains invariant under $\tilde{T}_1$ and $\tilde{T}_3$ but transforms non-trivially under $\tilde{T}_2$. We shall make use of these facts to restrict the form of the wave functional of the system in the following section.

Finally, the explicit expression for the Hamiltonian $\tilde{H} = \int d^2 x \tilde{\mathcal{H}}$ can be simplified to a great extent and we obtain:

$$\tilde{H} = \int d^2 x \left[ (|\Pi|^2 + \phi_2 (\Pi z + \Pi^* z^*) + |Z|^2 (\phi_2)^2) \frac{1}{\mathcal{A}} - \partial_i \mathcal{A} \partial_i (|Z|^2 \mathcal{A}) \tilde{T}_1 \right. - \theta (\Pi^* M^* + \Pi M) \mathcal{A}^2 + \theta^2 (\mathcal{A}^2)^3 |M|^2 + |\partial_i Z|^2 \mathcal{A}^2 - |Z^\dagger \partial_i Z|^2 (\mathcal{A}^2)^2 \right], \quad (4.19)$$

where the expression for $\mathcal{A}^2$ is given by eqn. (4.6). In the above expression for the Hamiltonian, the total space derivative terms have been neglected. We can now proceed in an exactly similar manner to rewrite the improved version of the angular momentum to get

$$\tilde{J} = J + \int d^2 x \varepsilon_{ij} x_i \left[ \phi_2 \partial_j (|z|^2) + \partial_j (\ln \mathcal{A}) \tilde{T}_2 \right]. \quad (4.20)$$

We would like to point out, at this stage, that we have not exploited the ad-hoc and somewhat artificial restriction like the so-called conformal gauge condition [5, 7], in the above derivation. This Hamiltonian $\tilde{H}$ and angular momentum $\tilde{J}$ will now be used to calculate the expectation value of the energy- and angular momentum operators between physical states in the next section following a method analogous to the one followed in [13].

5 Quantum correction to energy and angular momentum

In this section, we are going to compute the quantum correction to energy- and angular momentum expectation value obtained by sandwiching the Hamiltonian operator between state vectors of the physical Hilbert space $\mathcal{H}_{ph}$. We shall also discuss about its possible physical implications.

To begin with, we shall have to elevate all the three FCC’s into three hermitian operators. At the classical level, these constraints (4.1) and (4.16) are

$$\begin{align*}
\tilde{T}_1 &= z_\alpha^* z_\alpha - 1 + 2\phi_1 \approx 0, \\
\tilde{T}_2 &= \Pi_\alpha z_\alpha + \Pi^*_\alpha z^*_\alpha + 2z_\alpha^* z_\alpha \phi_2 \approx 0, \\
\tilde{T}_3 &= i \left( \Pi_\alpha z_\alpha - \Pi^*_\alpha z^*_\alpha - 2\theta \varepsilon_{ij} (A)^4 \left(|z|^2 \partial_i z'^* \partial_j z'^* z_\alpha + z'^* \partial_i \partial_j z^* z_\beta \right) \right) \approx 0. \quad (5.1)
\end{align*}$$

Note that here we have written these forms of the constraints in component form rather than in a matrix form. This is because, at the quantum level, the complex conjugation of fields will be replaced by hermitian conjugates of field operators. And this may create confusion with the hermitian conjugates $Z^\dagger = (z_1^*, z_2^*)$ of the ordinary doublet of two component field $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. These two operations of taking hermitian conjugates clearly correspond to two distinct spaces. The former is in the Hilbert space of states (i.e. Fock
remains invariant under the action of \( \tilde{\mathcal{H}} \). Then it follows, using (4.18), that

\[
\begin{align*}
\hat{T}_1 &= \hat{\beta}_\alpha \hat{\beta}_\alpha - 1 + 2 \hat{\phi}_1 \approx 0, \\
\hat{T}_2 &= \hat{\Pi}_\alpha \hat{\beta}_\alpha + \hat{\beta}_\alpha \hat{\Pi}^\dagger_\alpha + 2 \hat{\beta}_\alpha \hat{\beta}_\alpha \phi_2 \approx 0, \\
\hat{T}_3 &= i \left( \hat{\Pi}_\alpha \hat{\beta}_\alpha - \hat{\beta}_\alpha \hat{\Pi}^\dagger_\alpha - 2 \theta \varepsilon_{ij} (\hat{\mathcal{A}})^i \left( \hat{\beta}_\beta \hat{\beta}_\alpha \hat{\beta}_\epsilon \hat{\beta}_\gamma \hat{\beta}_\alpha \hat{\beta}_\epsilon \hat{\beta}_\gamma \right) \right) \approx 0.
\end{align*}
\]

The basic equal time commutation relations among the fields and their corresponding conjugate momenta variables are trivially obtained from their basic Poisson brackets to get

\[
\begin{align*}
[\hat{\beta}_\alpha(x), \hat{\Pi}_\beta(y)] &= i \hbar \delta_{\alpha\beta} \delta(x - y), \\
[\hat{\beta}^\dagger_\alpha(x), \hat{\Pi}^\dagger_\beta(y)] &= i \hbar \delta_{\alpha\beta} \delta(x - y), \\
[\hat{\phi}_\alpha(x), \hat{\phi}_\beta(y)] &= - \frac{i}{2} \hbar \varepsilon_{\alpha\beta} \delta(x - y),
\end{align*}
\]

and the rest of the brackets vanish. Note that \( \hat{\phi}_\alpha \) are the hermitian operators now. The physical Hilbert space \( \mathcal{H}_{ph} \) is now given as the kernel of the three FCC’s

\[\hat{T}_a |\Psi>_{ph} = 0, \quad \forall \ a = 1, 2, 3,\]

which means that the physical states \( |\Psi>_{ph} \) are gauge invariant since \( \hat{T}_a \)'s are the generators of the gauge transformations (4.18). This can restrict the form of the “physical” wave functional \( \Psi[z_\alpha(x), z^*_\alpha(x), \Pi_\phi(x)] = <z_\alpha(x), z^*_\alpha(x), \Pi_\phi(x)|\Psi>_{ph} \) considerably. Note that we have included here \( \Pi_\phi \) as one of the arguments in the wave functional \( \Psi \). Actually, this is a matter of choice as we could have easily chosen \( \phi_1 \) as conjugate momentum corresponding to the coordinate variable \( \phi_2 \). They are just related by a trivial canonical transformation. Also since the BT fields in the extended space commute with that of the original \( CP^1 \) phase space variables \( (z_\alpha, \Pi_\phi) \), and also due to their transformation properties in (4.18), the physical Hilbert space \( \mathcal{H}_{ph} \) (which is a subspace of the total Hilbert space), can be thought of as a direct product \( \mathcal{H}_1(z) \times \mathcal{H}_2(\phi) \) or \( \mathcal{H}_1(z) \times \mathcal{H}_2(\Pi_\phi) \) and consequently one can easily either consider “position” representation or “momentum” representation for the second Hilbert space \( \mathcal{H}_2 \) irrespective of the representation one considers for \( \mathcal{H}_1 \). Our particular choice here was dictated by the observation that only \( z(x) \) field transforms non-trivially under \( \hat{T}_2 \) but \( \Pi_\phi \) remains invariant under \( \hat{T}_2 \) (4.17). It is just just other way around for \( \hat{T}_1 \) and thus, our analysis will become simpler.

If we now make a simple demand that the “physical” wave functional \( \Psi(z_\alpha, z^*_\alpha, \Pi_\phi) \) remains invariant under the action of \( \hat{T}_1 \), i.e.,

\[\Psi(z_\alpha, z^*_\alpha, \Pi_\phi) = \Psi(z_\alpha, z^*_\alpha, \Pi_\phi + \delta_1 \Pi_\phi),\]

then it follows, using (4.18), that

\[\frac{\delta \Psi(z_\alpha, z^*_\alpha, \Pi_\phi)}{\delta \Pi_\phi} = 0.\]

\[16\]
Going to the “momentum” representation, where the operator $\hat{\phi}(x)$ is given by $i\hbar \frac{\delta}{\delta \Pi \phi(x)}$, it clearly follows that
\[ \hat{\phi} |\Psi >_{ph} = 0. \] (5.7)
It immediately follows that $|\psi >_{ph}$ must belong to $H_1 \times \{0\}$, with the second factor consisting of “zero” element of $H_2$ (i.e., $0 \in H_2$) as otherwise one can easily show that the hermitian operator $(\hat{\phi}(x)\Pi \phi(y) + \Pi \phi(y)\hat{\phi}(x))$ acting on $|\Psi >_{ph}$ will produce imaginary eigenvalue $(i\hbar \delta(x - y))$. Consequently
\[ \Pi \phi |\Psi >_{ph} = 0. \] (5.8)
We can thus identify $H_{ph}$ to be basically isomorphic to $H_1(z)$ itself. With this, the above wave functional reduces to $\Psi[z(x), z^*(x)]$, thus, depending entirely on $z(x)$ and $z^*(x)$. It is not unexpected, as the condition (5.7) and (5.8) correspond to the unitary gauge conditions (in the “weak” form) to be used in the next Section. But here we would like to make the following observations. Condition (5.4) does not necessarily imply, in general, that (5.5) has to be satisfied. It can undergo, for example, an overall scaling transformations so that $|\Psi >_{ph}$ can correspond to the same element in the projective Hilbert space. In fact, one can check that under $\hat{T}_2$, the wave functional $\Psi$ undergoes a scaling transformation. In this case, the conditions like unitary gauge (cf. (5.7, 5.8)) (in the “weak” form) may not hold. But whatever conditions are imposed to define $H_{ph}$, it must be isomorphic to what we have found. However, the unitary gauge will make our computations much simpler in this Section.

We are now in a position to compute the “quantum shift” in the energy eigenvalues and study its possible physical significance. For that, we first need to write the Hamiltonian (4.18) in a hermitian form. But before that, it will be advantageous to rewrite the classical expression (4.18) itself in a form suitable for this purpose. Using (4.1) and (4.6), the Hamiltonian can be written as
\[ \tilde{H} = \int d^2x \left[ \left( \frac{Z^2}{T_1} \right)^2 \left[ |\Pi|^2 - \frac{1}{2} T_2 \Pi \phi - \frac{1}{4} |Z|^2 (\Pi \phi)^2 \right] - \theta \frac{(\tilde{T}_1 + 1)}{|Z|^2} (\Pi^* M^* + \Pi M) + \theta^2 \left( \frac{\tilde{T}_1 + 1}{|Z|^4} \right)^3 |M|^2 + \left( \frac{\tilde{T}_1 + 1}{|Z|^4} \right) |\partial Z|^2 - \left( \frac{\tilde{T}_1 + 1}{|Z|^2} \right)^4 |Z| \partial_i Z |^2 - \partial_i A \partial_i (A |Z|^2) \tilde{T}_1 \right]. \] (5.9)

There is no unique expression for the corresponding hermitian quantum Hamiltonian as there is a natural ambiguity arising from different and inequivalent operator orderings. And, inequivalent but consistent operator orderings give rise to inequivalent quantization. The point we want to emphasize, at this stage, is the fact that the operator ordering problems have been gotten rid of by this BT scheme, as the symplectic structure is now given by the basic commutators obtained by elevating the basic PB structure and not the complicated DBs. The only point we shall not worry about is the divergent nature of the product of the field operators at the same space-time point. This is because, at the present level of rigour, we are only interested in establishing the presence and qualitative nature of

\[ ^1 \text{We are assuming that these wave functionals are “normalizable” in the functional sense.} \]
quantum corrections, without going into the quantitative estimates. For this, we shall try to work with one of the simplest and yet nontrivial operator ordering.

Now coming to the energy expectation value of the Hamiltonian operator, we follow the definition

\[
E = \frac{\text{ph} < \Psi | \hat{H} | \Psi \rangle_{\text{ph}}}{\text{ph} < \Psi | \Psi \rangle_{\text{ph}}}. \tag{5.10}
\]

As physical states |\Psi\rangle_{\text{ph}}'s are annihilated by the FCC’s $\hat{T}_a$ (5.4) and also by $\hat{\phi}(x)$ and $\hat{\Pi}_\phi$ (5.7, 5.8), we order the various factors in each term in such a manner that either $\hat{\tilde{T}}_a$, $\hat{\phi}$ or $\hat{\Pi}_\phi$ appear at the right most or left most place in each term in $\hat{H}$. Also the hermitian forms of the constraints (5.2) are kept intact, i.e., the permutations of the field operators appearing within and without $\hat{\tilde{T}}_a$ are not considered. Furthermore, the expressions in the denominators, involving field operators, are re-written in terms of FCC (5.2), so that the expressions involving the quotients of field operators can be avoided. Let us do it term by term. For the first term in the integrand of (5.9), we write

\[
\hat{H}_1 = \frac{1}{2} \int d^2x \left[ \frac{1}{\hat{T}_1 + 1} \left( (\hat{\tilde{T}}_1 + 1)^v (\hat{\tilde{T}}_1 - 2\hat{\phi} + 1) \hat{\Pi}_\beta \hat{\Pi}_\beta \right)_w - \frac{1}{2} \left( (\hat{\tilde{T}}_1 + 1) \hat{\Pi}_\beta \hat{\Pi}_\beta \right)_w + \frac{1}{4} \hat{\tilde{T}}_1^{z_a z_a} \hat{\tilde{T}}_1^{z_b z_b} (\hat{\Pi}_\phi)^2 \right] + \text{h. c.} \tag{5.11}
\]

Here the subscript $w$ stands for the Weyl ordering \cite{17} and the composite operators are hermitian by construction. Further, the Weyl ordering involving the FCC ($\hat{T}_2$) is performed by treating it as a single object as we have mentioned earlier. We could have gone for simpler ordering than Weyl one for the first term within the parenthesis in (5.11) as

\[
\frac{1}{2} \left( \hat{\tilde{T}}_1^{z_a z_a} \hat{\Pi}_\beta \hat{\Pi}_\beta + \hat{\Pi}_\beta \hat{\Pi}_\beta \hat{\Pi}_\alpha \right). \tag{5.12}
\]

But as one can easily see, this ordering turns out to be rather trivial in nature. Regarding the second and third terms within the parenthesis of (5.11), they clearly vanish by using (5.8) when sandwiched between the physical states in the numerator of (5.10). Besides, the denominator $(\hat{T}_1 + 1)$ also reduces just to unity. With all this, the contribution of the term (5.11) in the expectation value simplifies to

\[
E_1 = \frac{\text{ph} < \Psi | \hat{H}_1 | \Psi \rangle_{\text{ph}}}{\text{ph} < \Psi | \Psi \rangle_{\text{ph}}} \equiv \frac{\text{ph} < \Psi | \int d^2x \left( \hat{\tilde{T}}_1^{z_a z_a} \hat{\Pi}_\beta \hat{\Pi}_\beta \right)_w | \Psi \rangle_{\text{ph}}}{\text{ph} < \Psi | \Psi \rangle_{\text{ph}}}. \tag{5.13}
\]

Now coming to the term linear in the Hopf parameter $\theta$ in (5.9), we note that it involves a factor $\frac{\hat{T}_1 + 1}{\hat{T}_1 - 2\phi + 1}$. This can be rewritten as $\frac{\hat{T}_1 + 1}{\hat{T}_1 - 2\phi + 1}$. As is obvious from the appearances of both numerator and denominator, their mutual ordering is really irrelevant as they commute with each-other. Also since the other function involves $\Pi_\alpha$ and $\Pi_\alpha^*$, we write the following ordering for the $\theta$ dependent term:

\[
\hat{H}_\theta = -\frac{q}{2} \int d^2x \left[ \frac{(\hat{T}_1 + 1)}{\hat{T}_1 - 2\phi + 1} \left( \hat{\Pi}_\alpha^\dagger \hat{M}_\alpha^\dagger + \hat{M}_\alpha \hat{\Pi}_\alpha + \hat{M}_\alpha^\dagger \hat{\Pi}_\alpha^\dagger + \hat{\Pi}_\alpha \hat{M}_\alpha \right) + \text{h. c.} \right]. \tag{5.14}
\]

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Again using (5.4) and (5.7), it is clear that when \( \hat{H}_\alpha \) is sandwiched between two physical states, this above mentioned factor, reduces to unity effectively. The corresponding contribution to the expectation value thus becomes:

\[
E_\theta = \frac{\langle \hat{H}_\alpha | \hat{H}_\alpha \rangle + \hat{M}_\alpha \hat{M}_\alpha + \hat{\theta}^\dagger \hat{\theta} | \Psi \rangle}{\langle \hat{H}_\alpha | \Psi \rangle}.
\]

(5.15)

In rest of the terms in (5.9), there is no need for operator ordering as all the variables commute among themselves. Consequently, their contribution to the energy is given by

\[
E_{\text{rest}} = \frac{\langle \hat{H}_\alpha | \hat{H}_\alpha \rangle + \hat{\theta}^\dagger \hat{\theta} | \Psi \rangle}{\langle \hat{H}_\alpha | \Psi \rangle}.
\]

(5.16)

Thus, the total energy eigenvalue is obtained by adding (5.11), (5.13) and (5.14):

\[
E = E_1 + E_\theta + E_{\text{rest}}
\]

\[
= \frac{\langle \hat{H}_\alpha | \hat{H}_\alpha \rangle + \hat{\theta}^\dagger \hat{\theta} | \Psi \rangle}{\langle \hat{H}_\alpha | \Psi \rangle}.
\]

(5.17)

Clearly, we will have to carry out the Weyl ordering only in the first term in the integrand as indicated in the above equation. The second \( \theta \) dependent term is already ordered properly as \( \hat{M}_\alpha \) involves only \( \hat{z}_\alpha \) and \( \hat{z}_\alpha^\dagger \) variables. Let us now Weyl order the first term

\[
\int d^2x \left( \hat{z}_\alpha^\dagger \hat{z}_\alpha \hat{\alpha} \hat{\beta} \right) \delta(x - y).
\]

(5.18)

For the sake of convenience, let us denote, for the time being, the integrand in (5.18) (excluding \( \delta(x - y) \)) just as

\[
\left( \hat{z}_\alpha^\dagger \hat{z}_\alpha \hat{\alpha} \hat{\beta} \right)\]

(5.19)

where \( \alpha \) and \( \beta \) indices are now taken to include space indices \( x \) and \( y \) respectively. In this compact notation, the first two non-vanishing commutators in (5.3) can be expressed as

\[
[\hat{z}_\alpha, \hat{\alpha} \hat{\beta}] = [\hat{z}_\alpha, \hat{\alpha} \hat{\beta}] = i\hbar \delta_{\alpha\beta}.
\]

(5.20)

Considering all possible permutations of the variables in (5.19), this can be Weyl ordered as

\[
\left( \hat{z}_\alpha^\dagger \hat{z}_\alpha \hat{\alpha} \hat{\beta} \right) = \frac{1}{24} \left[ 4 \{ \hat{z}_\alpha^\dagger \hat{z}_\alpha, \hat{\alpha} \hat{\beta} \} + 4 \{ \hat{z}_\alpha^\dagger \hat{\alpha} \hat{\beta}, \hat{z}_\alpha \} + \{ \hat{z}_\alpha^\dagger, \hat{\alpha} \hat{\beta} \} \right].
\]

(5.21)

where the symmetric bracket between two operators \( \hat{A} \) and \( \hat{B} \) is defined as \( \{ \hat{A}, \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A} \). Repeated application of (5.20) allows one to simplify (5.21) as

\[
\left( \hat{z}_\alpha^\dagger \hat{z}_\alpha \hat{\alpha} \hat{\beta} \right) = \hat{z}_\alpha^\dagger \hat{z}_\alpha \hat{\alpha} \hat{\beta} + \frac{\hbar^2}{2} \delta_{\alpha\beta} \delta_{\alpha\beta} - \frac{i\hbar}{2} \delta_{\alpha\beta} \left( \hat{z}_\alpha^\dagger \hat{\alpha} \hat{\beta} + \hat{\alpha} \hat{\beta} \hat{z}_\alpha \right).
\]

(5.22)
Restoring the continuous spacetime indices, (5.20) can be expressed as
\[
\left( \hat{z}^\dagger_\alpha (x) \hat{z}_\alpha (x) \hat{\Pi}^\dagger_\beta (y) \hat{\Pi}_\beta (y) \right)_w = \hat{z}^\dagger_\alpha (x) \hat{z}_\alpha (x) \hat{\Pi}^\dagger_\beta (y) \hat{\Pi}_\beta (y) + \frac{\hbar^2}{4} \delta_{\alpha\beta} \delta(x-y) \delta(x-y)
\]
\[
- \frac{i\hbar}{2} \delta_{\alpha\beta} \delta(x-y) (\hat{z}^\dagger_\alpha (x) \hat{\Pi}^\dagger_\beta (y) + \hat{\Pi}_\beta (y) \hat{z}_\alpha (x)).
\]

which on further simplification yields
\[
\left( \hat{z}^\dagger_\alpha (x) \hat{z}_\alpha (x) \hat{\Pi}^\dagger_\beta (y) \hat{\Pi}_\beta (y) \right)_w = (\hat{T}_1(x) + 1 - 2\hat{\phi}(x)) \hat{\Pi}^\dagger_\beta (y) \hat{\Pi}_\beta (y) + \frac{\hbar^2}{2} (\delta(x-y))^2
\]
\[
- \frac{i\hbar}{2} (\hat{T}_2(x) + \hat{z}^\dagger_\alpha (x) \hat{z}_\alpha (x) \hat{\Pi}_\phi (x)).
\]

(5.23)

So when this term is sandwiched between physical states |\Psi >_{ph} in (5.17), the term linear in \hbar effectively drops out and the first term effectively reduces to \hat{\Pi}^\dagger_\beta \hat{\Pi}_\beta as can be easily seen from (5.4), (5.7) and (5.8). So far, this term (\hat{z}^\dagger_\alpha \hat{z}_\alpha \hat{\Pi}^\dagger_\beta \hat{\Pi}_\beta)_w yields an \mathcal{O}(\hbar^2) quantum correction. Now coming to the term linear in \theta in (5.17), we note that the integral \int d^2x (\hat{\Pi}_\alpha (x) \hat{M}_\alpha (x)) appearing there can be re-expressed as
\[
\int d^2x d^2y \hat{\Pi}_\alpha (y) \hat{M}_\alpha (x) \delta(x-y),
\]
(5.24)
as was done for the case of the first term in (5.18). Again repeated application of the basic commutation relation (5.3) allows one to rewrite \( \int d^2x \hat{\Pi}_\alpha (x) \hat{M}_\alpha (x) \) as
\[
\int d^2x \hat{\Pi}_\alpha (x) \hat{M}_\alpha (x) = \int d^2x \hat{M}_\alpha (x) \hat{\Pi}_\alpha (x) - i\hbar \int d^2x d^2y \delta(x-y) \varepsilon^{ij} \partial_i (\hat{\pi}^\dagger_j \hat{z}_\alpha) \partial_j (\hat{\pi}_j \hat{z}_\alpha) \delta(x-y)
\]
\[
+ i\hbar \int d^2x d^2y (\delta(x-y))^2 \varepsilon^{ij} \partial_i (\hat{\pi}^\dagger_j \hat{z}_\alpha) \partial_j (\hat{\pi}_j \hat{z}_\alpha).
\]
(5.25)

We, therefore, have
\[
\int d^2x (\hat{\Pi}_\alpha (x) \hat{M}_\alpha (x) + \hat{\Pi}^\dagger_\alpha (x) \hat{\Pi}_\alpha (x)) = \int d^2x (\hat{M}_\alpha (x) \hat{\Pi}_\alpha (x) + \hat{\Pi}^\dagger_\alpha (x) \hat{\Pi}_\alpha (x))
\]
\[
+ 2i\hbar \int d^2x d^2y (\delta(x-y))^2 \varepsilon^{ij} \partial_i (\hat{\pi}^\dagger_j \hat{z}_\alpha) \partial_j (\hat{\pi}_j \hat{z}_\alpha).
\]
(5.26)

Note that here the second term in (5.25) involving the derivative of the delta function drops out, as it is skew-hermitian. Now we can write the term involving \hbar in (5.26) in more compact form, using \( \hat{T}_1 |\Psi >_{ph} = 0 \) (5.4) and (5.7) to note that \( (\hat{z}^\dagger_\alpha \hat{z}_\alpha - 1) |\Psi >_{ph} = 0 \). Thus, as far as the actions of the second \hbar dependent term in (5.24) on |\Psi >_{ph} states, taken to be an eigen state |z_\alpha(x) > of the field operator \hat{z}_\alpha as: \hat{z}_\alpha(x)|z_\alpha(x) > = z_\alpha(x)|z_\alpha(x) >, are concerned, the integrand (up to a factor) can be identified with the topological density \( j^0 \) (2.12), and allows one, using (2.10), to rewrite (2.26) as
\[
\int d^2x (\hat{\Pi}_\alpha (x) \hat{M}_\alpha (x) + \hat{\Pi}^\dagger_\alpha (x) \hat{\Pi}_\alpha (x)) = \int d^2x (\hat{M}_\alpha (x) \hat{\Pi}_\alpha (x) + \hat{\Pi}^\dagger_\alpha (x) \hat{\Pi}_\alpha (x))
\]
\[
- 4\pi \hbar \int d^2x d^2y (\delta(x-y))^2 j^0(x),
\]
\[
= \int d^2x (\hat{M}_\alpha (x) \hat{\Pi}_\alpha (x) + \hat{\Pi}^\dagger_\alpha (x) \hat{\Pi}_\alpha (x))
\]
\[
- 4\pi \hbar Q \int d^2y (\delta(x-y))^2
\]
(5.27)

\(^{\dagger}\) A typical eigen state |z_\alpha(x) > can be thought of as given by the configuration (2.18) in the \( Q = 1 \) sector.
Thus, ultimately, using (5.17), the energy expectation value of \( E \) boils down to

\[
E = \frac{1}{\langle \psi | \psi \rangle_{ph}} \langle \phi | \psi \rangle_{ph} \int d^2x [\hat{\Pi}_{\alpha} \hat{\Pi}_{\alpha} - \theta (\hat{\Pi}_{\alpha} \hat{\hat{\Pi}}_{\alpha} + \hat{\hat{\Pi}}_{\alpha} \hat{\Pi}_{\alpha})]
+ \theta^2 \hat{\Pi}_{\alpha} \hat{\Pi}_{\alpha} + |\partial_\phi \hat{\hat{Z}}|^2 - |\hat{\hat{Z}}|^2 |\psi \rangle_{ph}
+ \frac{\hbar^2}{2} \int d^2x d^2y (\delta(x-y))^3 + 2\pi \hbar \theta Q \int d^2y (\delta(x-y))^2.
\]

(5.28)

At this stage, one can use \( \Pi_{\alpha} = (P_{\alpha} + \theta \hat{\Pi}_{\alpha}) \) and \( \Pi_{\alpha} = (P_{\alpha} + \theta \hat{\Pi}_{\alpha}) \), the quantum version of (3.4) to show that the first three terms in the integral of (5.28) simplifies considerably to yield

\[
\bar{E} = \frac{1}{\langle \psi | \psi \rangle_{ph}} \langle \phi | \psi \rangle_{ph} \int d^2x [\hat{\Pi}_{\alpha} \hat{\hat{\Pi}}_{\alpha} + |\partial_\phi \hat{\hat{Z}}|^2 - |\hat{\hat{Z}}|^2 |\psi \rangle_{ph}. \]

(5.29)

With this (5.28) can be expressed more compactly as

\[
E = \bar{E} + \frac{\hbar^2}{2} \int d^2x d^2y (\delta(x-y))^3 + 2\pi \hbar \theta Q \int d^2y (\delta(x-y))^2.
\]

(5.30)

As one can easily recognize that the integral (5.29) just corresponds to the Hamiltonian of pure \( CP^1 \) model. This also happens for vanishing Hopf term (\( \theta = 0 \)). We can therefore identify, with some justification, \( \bar{E} \) in (5.30) as the classical expression and the second (\( O(\hbar^2) \) term) and the third (\( O(\hbar) \) term) in (5.30) as quantum corrections. It should be noted, however, that these terms are highly singular and to extract any meaning from these, we have to regularize them. For this purpose, let us use the Gaussian representation of the two-dimensional delta-function

\[
\delta_\sigma(x) = \frac{1}{4\pi\sigma^2} e^{-\frac{x^2}{\sigma^2}}. \tag{5.31}
\]

This represents Dirac-delta function in the limit \( \sigma \to 0 \). Thus, the first quantum correction can be written as

\[
E^{(1)}_{\text{quan}} = \frac{\hbar^2}{2} \int d^2x d^2y (\delta(x-y))^3,
= \frac{\hbar^2}{2} \lim_{\sigma \to 0} \int d^2x d^2y \frac{1}{(4\pi\sigma^2)^2} e^{-\frac{3(x-y)^2}{\sigma^4}}. \tag{5.32}
\]

If we perform the \( y \)-integration first by translating \( y \) appropriately, then this integral becomes essentially independent of \( x \) and upon the second integration over \( x \), the integral diverges. Introducing an area cut-off \( A \) for this \( x \)-integration, (5.31) can be rewritten, after some algebra, as

\[
E^{(1)}_{\text{quan}} = \lim_{\sigma \to 0} \frac{\hbar^2 A}{96 \pi^2 \sigma^4}. \tag{5.33}
\]

Looking at it dimensionally, it is clear that a quantity having length dimension should appear in the numerator, which we have taken effectively to have unit magnitude right in
the Lagrangian (2.4). This is clearly highly divergent and the situation does not improve by considering the corresponding energy density, i.e., energy per unit area ($E_{\text{quant}}^{(1)}$). However, the situation is slightly better with the $\mathcal{O}(\hbar)$ term in (5.30); namely,

$$E_{\text{quant}}^{(2)} = 2\pi\hbar\theta Q \int d^2y (\delta(x-y))^2.$$  

Again using the same regularization (5.31), (5.34) simplifies to

$$E_{\text{quant}}^{(2)} = \lim_{\sigma \to 0} \left( \frac{1}{4\sigma^2}\hbar\theta Q \right).$$  

Although this is also divergent, the corresponding contribution of the Hopf term to the energy density

$$\lim_{A \to \infty} \frac{E_{\text{quant}}^{(2)}}{A} = \lim_{\sigma \to 0} \left( \frac{1}{4A\sigma^2}\hbar\theta Q \right),$$  

can be made finite. This is a nontrivial result considering the fact that the Hopf term is topological in nature. However, note that the quantity $\lim_{\sigma \to 0, A \to \infty} (A\sigma^2)$, which can be taken to be finite, is regularization scheme dependent. Nevertheless, this result indicates that a generic topological term may contribute non-trivially in the energy momentum tensor at the quantum level in the nontrivial topological sector. This point deserves further careful investigation if one is interested in the explicit numerical estimate of energy.

We now turn our attention to the angular momentum operator. Proceeding exactly in the manner, as we have done for energy, we can write down a hermitian form of the improved angular momentum (4.20) as

$$\hat{J} = \hat{J} + \frac{1}{2} \int d^2x \varepsilon_{ij} x_i \left[ \partial_j (\ln A) \hat{T}_i^2 + \hat{T}_i \hat{T}_j - \hat{\Pi}_i (x) \partial_j (|\hat{z}|^2) \right],$$  

where

$$\hat{J} = \int d^2x \varepsilon_{ij} x_i \left[ \hat{\Pi}_i (x) \partial_j \hat{z}_i (x) + \text{h. c.} \right],$$

is the original expression of the angular momentum for (3.1), obtained either by Noether’s prescription or through the symmetric energy momentum tensor (2.26) and is nothing but the orbital angular momentum. Here also one starts by changing all the variables to their “improved” BT extended [5] form and algebraic simplification leads to the conventional form (4.20) for which (5.37) provides the hermitian counterpart. Clearly, juxtaposed between physical states $|\Psi >_{ph}$, the extra terms in (5.37) vanishes as can be easily seen on using (5.4) and (5.8). One therefore concludes that, unlike the case of energy, there is no quantum correction in the case of angular momentum. It was argued in [12] that Hopf term does not contribute to fractional spin and, now in absence of any quantum correction, there is no contribution to fractional spin of quantum mechanical origin either. This is consistent with the collective coordinate quantization carried out in Section 2.

6 BRST quantisation
In this Section, we briefly outline the BRST quantisation for the BT extended $CP^1$ model coupled to the Hopf term in the framework of Hamiltonian formalism \[2\]. Since the extension has already converted the system into a completely first-class system, the procedure of BRST quantisation is straightforward. One has to introduce the following three canonical pairs of ghosts, anti-ghosts and multiplier fields in the Batalin-Fradkin-Vilkovisky scheme

$$
(C^i, \bar{P}_i), \quad (P^i, \bar{C}_i), \quad (q^i, p_i), \quad i = 1, 2, 3,
$$

(6.1)
satisfying the super-Poisson algebra

$$
\{C^i(x, t), \bar{P}_j(y, t)\} = \{P^i(x, t), \bar{C}_j(y, t)\} = \{q^i(x, t), p_j(y, t)\} = \delta^i_j \delta(x - y),
$$

(6.2)

where the super-bracket between two variables $A$ and $B$ is defined as

$$
\{A, B\} = \frac{\delta A}{\delta q} \bigg|_r \frac{\delta B}{\delta p} \bigg|_l - (-1)^{\eta_A \eta_B} \frac{\delta A}{\delta p} \bigg|_l \frac{\delta B}{\delta q} \bigg|_r.
$$

(6.3)

Here the subscripts $l$ and $r$ stand for the left- and right derivatives respectively and $\eta_A$ corresponds to the ghost number associated with the variable $A$. The Hamiltonian path-integral for the partition function $Z$, is finally written as

$$
Z = \int D[\mu] \exp \left( i \int d^3x \left[ \Pi \dot{z} + \Pi^* \dot{z}^* + \Pi \dot{\phi} + p_\phi \dot{q}^i + \bar{P}_i \dot{C}^i + \bar{C}_i \dot{P}^i - H_U \right] \right),
$$

(6.4)

where the measure $D[\mu]$ consists of all the phase (conjugate) variables and $H_U$ is defined as the unitarizing Hamiltonian

$$
H_U = \int d^2x \ H_U \equiv \int d^2x \ H_{BRST} + \{\Psi, Q_B\}.
$$

(6.5)

In the above equation, the gauge-fixing fermion $\Psi$, the BRST charge $Q_B$ and the BRST Hamiltonian $H_{BRST}$ are defined in a conventional way \[3\],

$$
\Psi = \int d^2x (\bar{C}_i \chi^i + \bar{P}_i q^i), \quad Q_B = \int d^2x (C^i \tilde{T}_i + P^i p_i),
$$

(6.6)

$$
\{H_{BRST}, Q_B\} = 0, \quad \{Q_B, Q_B\} = 0.
$$

(6.7)

Here $\chi^i$'s are the three gauge-fixing functions (to be specified later) with the restriction that the Poisson-Bracket matrix, consisting of PB's among $\tilde{T}_i$ and $\chi^j$, should be invertible. In the construction of the BRST invariant Hamiltonian $H_{BRST}$, the presence of the original FCC $T$ (and its improved version $\tilde{T}$) creates extra complications. Notice that the improved variables $\tilde{z}, \tilde{\Pi}, \tilde{z}^*, \tilde{\Pi}^*$ were tailored to commute with the original SCC's $\tilde{T}_1$ and $\tilde{T}_2$ and hence by construction,

$$
\{\tilde{T}_1, \tilde{\mathcal{H}}\} = \{\tilde{T}_2, \tilde{\mathcal{H}}\} = 0.
$$

However, this is not in general true for $\tilde{T}$ and, therefore, in general $\tilde{\mathcal{H}} \neq H_{BRST}$. Thus, in an arbitrary model, further modifications are required to convert $\tilde{\mathcal{H}}$ to $H_{BRST}$. But, it can be explicitly checked that in the present theory

$$
\{\tilde{T}, \tilde{\mathcal{H}}\} = 0, \quad \Rightarrow \quad \tilde{\mathcal{H}} = H_{BRST},
$$

23
which leads to

\[ H_U = \int d^2x \left[ \tilde{\mathcal{H}} + q^i \tilde{T}_i + \tilde{P}_i \dot{P}^i + p_i \chi^i + \int d^2y \ (\chi^i(x), \tilde{T}_j(y)) \ C^j(y) \right]. \]  

(6.8)

In this context, we would like to comment that the introduction of terms proportional to the FCC’s to the improved Hamiltonian \( \tilde{\mathcal{H}} \) as in \[3\] with the sole purpose of maintaining the original constraint algebra in the BT scheme is unnecessary and also seems to be redundant since these constraints already appear in the action coupled to the arbitrary multiplier fields. Furthermore, for the more complicated models such as the one presented here, the form invariance of the original constraint algebra in the extended phase space is not possible.

Our aim here is to construct the Lagrangian in the extended velocity phase space which implies that all the momenta variables should be integrated out from (5.4). Fortunately, in our case, the momenta appear either linearly or at the most quadratically, and hence the classical equations of motion can be used to eliminate them. First of all, for simplicity, let us note that the term \( \int d^2x (p_1 \dot{q}^1 + \tilde{P}_1 \dot{C}^1) \) in the action is dropped because it is a BRST exact piece:

\[ \int d^2x (p_1 \dot{q}^1 + \tilde{P}_1 \dot{C}^1) = \{Q_B, \int d^2x \tilde{C}_1 \dot{q}^1 \}. \]  

(6.9)

This allows us to trivially integrate out the variables \( q^1, p_1, P_1, \tilde{P}_1, P^2, \tilde{P}_2 \) leading to the partition function

\[ Z = \int \mathcal{D}[\mu] \exp \left( i \int d^3x \left[ \Pi z^2 + \Pi^* z^* + \Pi_\phi^* \phi + p_2 \dot{q}^2 + p_3 \dot{q}^3 - \tilde{P}_3 P^3 + \tilde{P}_3 \dot{C}^3 + \tilde{C}_3 \dot{P}^3 \right] \right) \]

\[ \mathcal{D}[\mu] = \mathcal{D}(\Pi, z, \Pi^*, z^*, \Pi_\phi, \phi, q^2, p_2, q^3, p_3, C^i, \tilde{C}_i, \tilde{P}^3, \tilde{P}_3) \delta(\tilde{T}_1) \delta(z^1). \]  

(6.10)

In the same way as the above, removal of other BRST exact terms from the action is not desirable since the other constraints contain the momenta variables. Presence of \( \delta(\tilde{T}_1) \) in the measure simplifies the subsequent calculations considerably since this allows us to write

\[ \tilde{T} \equiv \tilde{T}_3 = (\Pi z) - (\Pi^* z^*) + 2\theta \varepsilon_{ij} (A^2)^2 \left[ |Z|^2 \partial_j Z^i \partial_i Z + (\partial_i Z^i \partial_j Z) \right]. \]

So far, the choice of gauge has remained arbitrary. Let us choose a unitary gauge

\[ \chi^1 = T_1(z, z^*), \quad \chi^2 = T_2(z, z^*, \Pi, \Pi^*). \]  

(6.11)

This choice, at least, ensures that the extension related to the BRST procedure reproduces the original action (3.1) in the limit when BT auxiliary fields vanish. Notice that \( \chi^3 \) is still kept arbitrary. However, for simplicity, let us restrict \( \chi^3 \) to comprise only of \( z \) and \( z^* \) fields and \( \{\tilde{T}_2, \chi^3\} = 0 \). This finally leads to the following condition

\[ \{\chi^3(x), \tilde{T}_3(y)\} = \sigma(x, y) \neq 0. \]  

(6.12)
In this particular set up, we are allowed to remove the BRST exact term \( \int d^2x \left( p_3 \dot{q}^3 + \bar{C}_3 \dot{\bar{p}}^3 \right) = \{ Q_B, \int d^2x \bar{C}_3 \dot{q}^3 \} \) from the action and similar analysis as done earlier changes the measure to

\[
\mathcal{D}[\mu] = \mathcal{D}(z, z^*, \Pi, \Pi^*, \phi, \Pi_\phi, p_2, C_i, \bar{C}^i) \delta(\bar{T}_1) \delta(\chi^1) \delta(\chi^3). \tag{6.13}
\]

Up to this point, the effective action is still BRST invariant since we are still in the BT extended phase space. Now, the momenta variables \( \Pi \) and \( \Pi^* \) can be easily removed by exploiting the equations of motion. For instance, the following expression

\[
\Pi^*_\alpha = \dot{z}_\alpha - (q^2 + q^3 + p_2 + \frac{1}{2A_2} \Pi_\phi) z_\alpha - \theta |A^2 M^*|, \tag{6.14}
\]

arises from the equations of motion w.r.t. \( \Pi^*_\alpha \), which can be used to eliminate \( \Pi^*_\alpha \). This brings about two Gaussian path-integrals in the remaining momenta variables, i.e., \( p_2 \) and \( \Pi_\phi \). The above integrations produce a long expression for the effective action which is not given here. Let us now simplify the above action further by using the following \( \delta \)-functions:

\[
\delta(\bar{T}_1) \delta(T_1) = \delta(\phi) \delta(T_1), \tag{6.15}
\]

which reduces the partition function \( Z \) to

\[
Z = \int \mathcal{D}[\mu] \exp \left( i \int d^3x \left[ |\partial_\mu Z|^2 - |Z^\dagger \partial_\mu Z|^2 + \theta \mathcal{L}_H, \right. \right.
\]

\[
- 2\bar{C}_1 C^2 + 2\bar{C}_2 C^1 + \bar{C}^2 \bar{C}_2 - \int d^2y \bar{C}_3(x) \sigma(x, y) C^3(y) \right) \right), \tag{6.16}
\]

\[
\mathcal{D}[\mu] = \mathcal{D}(z, z^*, C_1, C^1, C_3, C^3, \bar{C}_1, \bar{C}^1) \delta(||Z||^2 - 1) \delta(|\chi^3|).
\]

Once again using the equations of motion w.r.t. \( C_1, C^1 \) and taking the integration over \( C^3 \) and \( \bar{C}_3 \), all but the last two ghost contributions are removed and we end up with

\[
Z = \int \mathcal{D}[\mu] \exp \left( i \int d^3x \mathcal{L} \right), \tag{6.17}
\]

\[
\mathcal{D}[\mu] = \mathcal{D}(z, z^*) \delta(\chi^3) \det||\sigma||.
\]

Notice that the following choice for \( \chi^3 \), mentioned in section II,

\[
\chi^3 = z_1 - z^*_1,
\]

is consistent with all the restrictions imposed on \( \chi^3 \) so far and this choice yields

\[
\sigma(x, y) = (z_1 + z^*_1) \delta(x - y).
\]

The unitary gauge is introduced to ensure the consistency of the scheme. More interesting gauge choices, such as the Coulomb gauge will be studied in a future work.

7 Conclusions
The primary motivation of this work was to construct a consistent quantum theory of the $CP^1$ model coupled to the Hopf term, so that eventually we can analyse the quantum effects induced by the Hopf term. In particular, the possibility of quantum corrections to the energy expectation value and absence of any quantum correction to fractional spin are demonstrated.

Due to the nonlinearity present in the original model associated with field dependent Dirac brackets, the canonical quantization could not be carried out because of the severe operator ordering ambiguities. We have bypassed this problem by using Batalin-Tyutin scheme, where the phase space is enlarged by incorporating additional fields in such a manner that the symplectic structure in the extended phase space is given by the usual Poisson brackets. In this regard, we follow [5], whereby the structures of the Hamiltonian and the constraints (which are all first class now) remain unaffected when written in terms of the improved variables, thereby ensuring the existence of solitonic configurations. The complete structure of the first class theory in the extended space has been provided.

Using the above mentioned extended Hamiltonian, we have computed the quantum correction to the expectation value of the energy density, which stems from operator orderings of variables occurring in the Hamiltonian. Interestingly, we have found that the Hopf term, although a topological term, can have an $O(\bar{\hbar})$ finite contribution to the expectation value of the energy density at the quantum level in the nontrivial topological sector. This is apart from $O(\hbar^2)$ divergent contribution coming from non-Hopf term in the energy expectation value.

Here we would like to emphasize the following points. Note that the canonical quantization of a field theoretical model is usually marred with operator ordering ambiguities as we mentioned earlier and these ambiguities can arise mainly from the following two possible situations. Firstly, if the symplectic structure given by the DBs are field dependent in a complicated manner, then these DBs cannot be elevated to the quantum commutators consistently. This was explicitly demonstrated in [12] in the context of this model before the Batalin-Tyutin extension was made. But this BT extension really gets one out of the problem as we have seen. The second problem, which persists here, even after BT extension of this model is the non-unique hermitian expression of any observable and Hamiltonian in particular. In equivalent operator orderings result in inequivalent quantum theories. However, once a particular form of this quantum Hamiltonian is chosen, it is a matter of straightforward algebra to isolate the quantum corrections by repeated application of the simple and canonical form of the commutation relations (5.3). This model, therefore, provides a non-trivial field theoretic example involving non-linearities, where the power of this Batalin-Tyutin quantization can be demonstrated. We should also point out that if the BT fields are set to zero, one recovers the original $CP^1$ model coupled to the Hopf term along with their complicated second-class constraints. This is the strong version of the unitary gauge condition. In the quantum analysis, however, one only sets them to weakly zero, i.e., physical states are taken to be annihilated by these BT field operators (5.7), (5.8).
We expected a similar quantum correction in the angular momentum, giving rise to fractional spin. However, the operator ordering does not yield any quantum correction. It was already shown in [12] that Hopf term does not contribute to the fractional spin at the classical level. In Section 2 of this paper, we have shown that the picture remains the same even at the level of collective coordinate quantization. Finally, quantizing the model utilizing the Batalin-Tyutin scheme, where there is no operator ordering problems anymore, we find the same picture persisting at this level too, i.e., no fractional spin is induced at $\mathcal{O}(\hbar)$ level due to quantum effects. This result is, therefore, different from Wilczek and Zee [10] and furnishes another example of inequivalent quantization.

Finally, we perform the conventional BRST quantisation of the extended model where all the constraints have become first class. The partition function has also been computed in the extended scheme whereby the equivalence with the original model can also be established.
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