AN ORBIFOLD RELATIVE INDEX THEOREM

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Abstract. In this paper we prove a relative index theorem for pairs of generalized Dirac operators on orbifolds which are the same at infinity. This generalizes to orbifolds a celebrated theorem of Gromov and Lawson.

0. Introduction.

Orbifolds, generalized manifolds that are locally the quotient of an euclidean space modulo a finite group of isometries, were first introduced by Satake. In the late seventies, Kawasaki proved an orbifold signature formula, together with more general index formulas, see [Kw1], [Kw2], [Kw3]. In [Fa1] we proved a $K$–theoretical index theorem for orbifolds with operator algebraic means, and in [Fa2] and [Fa3] we studied compact orbifold spectral theory and defined orbifold eta invariants. Other orbifold index formulas were proved in [Du], [V]. In [Ch] Chiang studied compact orbifold heat kernels and harmonic maps, while in [Stan], Stanhope established some interesting geometrical applications of orbifold spectral theory.

In [Fa4] we continued our orbifold spectral analysis started with [Fa2] and [Fa3]. In particular we showed that on a non–compact complete almost complex $Spin^c$ orbifold which is sufficiently regular at infinity (see Definition 2.1), generalized Dirac operators are closed. This generalizes to orbifolds theorems of Gaffney [Gn1], Yau [Y], and Wolf [W] for the manifold case. We also proved an orbifold Divergence/Stokes Theorem.

Here we will use the results we proved in [Fa4] to establish an orbifold Gromov-Lawson, [GL], relative index theorem for non–compact complete almost complex $Spin^c$ orbifolds which are sufficiently good at infinity (see the definition right before the statement of Theorem 3.5). Further generalizations to orbifolds of theorems in the spirit of the main result in [An1], [An2] will be considered in a later paper, [Fa5], together with an analytic proof of the cobordism invariance of the index. We will also plan on investigating the question of whether every open complete orbifold is sufficiently good at infinity, which we suspect has a negative answer.

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The relative index theorem applies to pairs of generalized Dirac operators which agree at infinity. The topological index of such a pair is defined to be the difference of the index of two natural extensions of the given operators to closed manifolds, see Definition 4.1. Kawasaki’s theorems for closed orbifolds makes possible to express the pair topological index via local traces and orbifold characteristic classes. The topological index is calculated by using pairs of parametrices that agree at infinity, c.f. Theorem 4.4.

The analytical index of a generalized Dirac operator, proved to be finite in Section 3, is defined, classically, to be the dimension of the kernel minus the dimension of the cokernel. The analytical index of a pair is defined to be the difference of the analytical indices of the pair’s constituents, see Definition 3.6.

Our main result is Theorem 6.2, which asserts the equality of the analytical and topological indices of a pair of generalized Dirac operators that agree at infinity, which generalizes to orbifolds the main theorem of [GL].

**Theorem 6.2.** Let $X$ be an even-dimensional non–compact complete almost complex Hermitian Spin$^c$ orbifold which is sufficiently good at infinity. Assume that a Hermitian connection is chosen on the dual of its canonical line bundle. Let $D_i$ and $D_i^{±}$ be the generalized Dirac operators on $X$ with coefficients in the proper Hermitian orbibundle $E_i$ (with connection $\nabla^{E_i}$), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood $\Omega$ of infinity. Assume that there exists a constant $k_0 > 0$ such that

$$R \geq k_0 Id$$

on $\Omega$, where $R$ is given as in Proposition 2.8. Then the topological index (as in Definition 4.1) and the analytical index (as in Definition 3.6) of the pair $(D_1^{±}, D_2^{±})$ coincide, that is

$$\text{ind}_t(D_1^{±}, D_2^{±}) = \text{ind}_a(D_1^{±}, D_2^{±})$$

Our proof is a generalization to orbifolds of the argument given for manifolds by Gromov and Lawson in [GL]. New techniques are mainly introduced to deal with orbifold distance functions.

More in detail, the contents of this paper are as follows. In Section 1 we recall the definition of orbifold, orbibundles, and introduce orbifold generalized Dirac operators. In Section 2 we recall properties of generalized Dirac operators on non–compact orbifolds from [Fa4]. In Section 3 we prove that generalized Dirac operators (on complete sufficiently regular orbifolds) which are positive at infinity, have finite dimensional kernels and cokernels. By using these results we can thus define the analytical index of a pair of generalized Dirac operators that agree at infinity as the difference of their analytical indices, see Definition 3.6.
In Section 4 we define the pair topological index by using a gluing technique and characterize it by using Kawasaki’s index theorem for closed orbifolds, see Definition 4.1 and Theorem 4.2. In Theorem 4.4 we show how to compute the pair topological index by using parametrices that agree at infinity. Section 5 is devoted to detailing some local properties of traces of generalized Dirac operators. Finally, in Section 6 we state and prove our main result, the orbifold relative index theorem.

In the sequel, all orbifolds and manifolds are assumed to be even dimensional, smooth, Hermitian, $Spin^c$, connected, and almost complex unless otherwise specified. All vector and orbibundles are assumed to be smooth and proper. We also assume that all of our orbifolds/manifolds are endowed with a fixed connection on the dual of their canonical line bundle $K^*$. This allows us to define a ‘canonical’ $Spin^c$ Dirac operator and, given an Hermitian orbibundle $E$ endowed with a connection, the ‘canonical’ $Spin^c$ Dirac operator with coefficients in $E$. Both of these operators depend, in the $Spin^c$ case, on the choice of the selected connections, see [Du; Chapter 14], and [LM; Appendix D]. For the Spin or complex case, the choice of the connection on $K^*$ is canonical.

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1. Orbifolds, Orbibundles and Dirac Operators.

In this section we will review some definitions and results that we will use throughout this paper. For generalities on orbifolds and operators on orbifolds, see [Kw1], [Kw2], [Kw3], [Ch], [Du].

An orbifold is a Hausdorff second countable topological space $X$ together with an atlas of charts $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$, with $\tilde{U}_i/G_i = U_i$ open and with projection $\pi_i : \tilde{U}_i \to U_i$, $i \in I$, satisfying the following properties

(1) If two charts $U_1$ and $U_2$ associated to pairs $(\tilde{U}_1, G_1)$, $(\tilde{U}_2, G_2)$ of $\mathcal{U}$, are such that $U_1 \subset U_2$, then there exists a smooth open embedding $\lambda : \tilde{U}_1 \to \tilde{U}_2$ and a homomorphism $\mu : G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \lambda$ and $\lambda \circ \gamma = \mu(\gamma) \circ \lambda$, $\forall \gamma \in G_1$.

(2) The collection of the open charts $U_i$, $i \in I$, belonging to the atlas $\mathcal{U}$ forms a basis for the topology on $X$. 
We will call an orbifold atlas as above a standard orbifold atlas.

For any $x$ point of $X$, the isotropy $G_x$ of $x$ is well defined, up to conjugacy, by using any local coordinate chart. The set of all points $x \in X$ with non-trivial stabilizer, $\Sigma(X)$, is called the singular locus of $X$. $\Sigma(X)$ is a set of codimension at least 2, see e.g. [Ch]. Note that $X - \Sigma(X)$ is a manifold.

If we now endow $X$ with a countable locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_i, G_i) | i \in \mathbb{N}\}$, then by standard theory there exists a smooth partition of unity $\eta = \{\eta_i\}_{i \in \mathbb{N}}$ subordinated to $\mathcal{F}$, [Ch]. This in particular means that, for any $i \in \mathbb{N}$, $\eta_i$ is a smooth function on $U_i$ (i.e., its lift to any chart of a standard orbifold atlas is smooth), the support of $\eta_i$ is included in an open subset $U'_i$ of $U_i$, and $\bigcup U'_i = X$. We will call any $\eta$ as above an $\mathcal{F}$-partition of unity.

Let $E$ be an Hermitian orbibundle (with connection $\nabla^E$) over the orbifold $X$. (For the precise definition see [Kw1], [Kw2], [Kw3], [Ch].) In particular $E$ is an orbifold in its own right; on an orbifold chart $U_i$ associated to a pair $(\tilde{U}_i, G_i)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i) | i \in I\}$ of $X$, $E$ lifts to a $G_i$-equivariant bundle. Standard orbifold atlases on $X$ can be used to provide standard orbifold atlases on $E$.

If $E$ is an orbibundle over the orbifold $X$, a section $s : X \rightarrow E$ is called a smooth orbifold section if for each chart $U_i$ associated to a pair $(\tilde{U}_i, G_i)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i) | i \in I\}$ of $X$, we have that $s|_{U_i} : U_i \rightarrow E|_{U_i}$ is covered by a smooth $G_i$-invariant section $\tilde{s}|_{\tilde{U}_i} : \tilde{U}_i \rightarrow \tilde{E}|_{\tilde{U}_i}$. Given an orbibundle $E$ over $X$, we will denote by $C^\infty(X, E)$ the space of all smooth sections of $E$, and by $C^\infty_c(X, E)$ the space of all smooth sections with compact support. Classical orbibundles over $X$ are the tangent bundle $TX$, and the cotangent bundle $T^*X$ of $X$. We can form orbibundle tensor products by taking the tensor products of their local expressions in the charts of a standard orbifold atlas.

Define an inner product between sections of $C^\infty(X, E)$ (or $C^\infty_c(X, E)$) by the following formula (c.f., [Ch; 2.2a])

$$ (\sigma_1, \sigma_2) = \sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{\tilde{U}_i} \tilde{\eta}_i(\tilde{x}_i) < \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) > d\nu(\tilde{x}_i), $$

where $\eta = \{\eta_i\}_{i \in \mathbb{N}}$, is a $\mathcal{F}$-partition of unity subordinated to the locally finite orbifold cover $\mathcal{F} = \{(\tilde{U}_i, G_i) | i \in \mathbb{N}\}$, and $<,>$ is a $G_i$-invariant product on $\tilde{E}$. (Note that, with slight abuse of notation, we used `~` to denote lift to $\tilde{U}_i$.)

We will now review the construction of the generalized Dirac operator with coefficients in an Hermitian orbibundle $E$ (with connection $\nabla^E$) over a (compact or not) orbifold $X$, see [Du; Sections 5 and 12], [Kw2],
First of all, $X$ admits a $\text{Spin}^c$-principal tangent orbibundle, $\text{Spin}^c(TX)$, with, in our hypotheses, a canonical $\text{Spin}^c$ orbifold connection $\nabla^c$. Let $\Delta^\pm$ be the half $\text{Spin}^c$ representations (recall that the $X$ is even dimensional). Then we have two orbibundles

$$\Delta^\pm = \text{Spin}^c(TX) \times_{\text{Spin}^c} \Delta^\pm,$$

with induced connections $\nabla^\pm$ from $\nabla^c$; $\nabla^\pm : C^\infty_c(X, \Delta^\pm c(TX)) \to C^\infty_c(X, T^*X \otimes \Delta^\mp c(TX))$.

The Clifford module structure on $\Delta^\pm$ defines Clifford multiplications

$$m^\pm : TX \otimes \mathbb{R} \Delta^\pm c(TX) \to \Delta^\mp c(TX)$$

Then the generalized $\pm$ Dirac operator with coefficients in $E$, $d^\pm E$, is defined by

$$d^\pm E : C^\infty_c(X, \Delta^\pm c(TX) \otimes E) \to C^\infty_c(X, \Delta^\mp c(TX) \otimes E)$$

is defined by

$$d^\pm E = M \circ \left(\nabla^\pm c \otimes \text{Id} + \text{Id} \otimes \nabla E\right),$$

where $M$ denotes the map induced by Clifford multiplication and $TX$ has been identified with $T^*X$ via the orbifold metric. We will also use the notation $S$ for the orbifold $\text{Spin}^c$ bundle $(\Delta^\pm + c)TX)$, and $S \otimes E$ or $\mathcal{E}$ for $(\Delta^\pm + c)TX) \otimes E$. We will define $D_E$, the generalized Dirac operator on $X$ with coefficient in $E$, to be $(d^+_E + d^-_E)$.

### 2. Dirac Operators on Non–Compact Complete Orbifolds.

On an orbifold $X$ satisfying our hypotheses (but not necessarily compact), the generalized Dirac operator $D_E$ with coefficients in the orbibundle $E$, as defined in the Section 1, is given by

$$D_E : C^\infty_c(X, (\Delta^+ c + \Delta^- c)(TX) \otimes E) \to C^\infty_c(X, (\Delta^- c + \Delta^+ c)(TX) \otimes E)$$

$$D_E = M \circ \left((\nabla^+ c + \nabla^- c) \otimes \text{Id} + \text{Id} \otimes \nabla E\right).$$

On orbifold charts, $D_E$ has the following local expression $\tilde{D}_E$. Let $U = \{(\tilde{U}_i, G_i) | i \in I\}$, with $\tilde{U}_i / G_i = U_i$ be a standard orbifold atlas. On a local chart $\tilde{U}_i$, $i \in I$ fixed, we have

$$\Delta^\pm c(T\tilde{U}_i) = \text{Spin}^c(T\tilde{U}_i) \times_{\text{Spin}^c} \Delta^\pm,c,$$
with induced $G_i$–invariant connections $\nabla^{\pm,c}$, from $\nabla^c$. The Clifford module structure on $\Delta^{\pm,c}$ defines Clifford multiplications

$$m_{\pm}: T\tilde{U}_i \otimes_{\mathbb{R}} \Delta^{\pm,c}(T\tilde{U}_i) \to \Delta^{\mp,c}(T\tilde{U}_i).$$

On $\tilde{E}$, the lift of $E$, we have the $G_i$–invariant connection $\nabla^{\tilde{E}}$. Then the generalized $\pm$ Dirac operators with coefficients in $E$, $\tilde{d}^{\pm,c}_E$,

$$\tilde{d}^{\pm,c}_E: C_c^\infty(\tilde{U}_i, \Delta^{\pm,c}(T\tilde{U}_i) \otimes_{\mathbb{C}} \tilde{E}) \to C_c^\infty(\tilde{U}_i, \Delta^{\mp,c}(T\tilde{U}_i) \otimes_{\mathbb{C}} E)$$

is given by

$$\tilde{d}^{\pm,c}_E = M \circ \left( \nabla^{\pm,c} \otimes Id + Id \otimes \nabla^{\tilde{E}} \right),$$

where $M$ is induced by Clifford multiplication and $T\tilde{U}_i$ has been identified with $T^*\tilde{U}_i$ via the $G_i$–invariant metric. Also, $\tilde{D}_E$, the generalized Dirac operator on $X$ with coefficient in $E$, is given by $\tilde{d}^{\pm,c}_E + \tilde{d}^{-,c}_E$ on $\tilde{U}_i$.

If $e_1, \ldots, e_n$ is an orthonormal local basis for the space $T\tilde{U}_i$ at a point $\tilde{x}$, then $\tilde{D}_E$ has local expression

$$\tilde{D}_E = \sum_{k=1}^n e_k \tilde{\nabla}^E_{e_k},$$

where

$$\tilde{\nabla}^E = (\nabla^{+,c} + \nabla^{-,c}) \otimes 1 + \nabla^{\tilde{E}}.$$

In analogy with the manifold case, see [GL], [W], [Gn1], [LM], [Y], one can show that $D_E$ is symmetric, whenever $X$ sufficiently regular at infinity.

**Definition 2.1.** Let $X$ be a non–compact complete orbifold. Then we say that $X$ is sufficiently regular at infinity if, for any neighborhood $\Omega \subseteq X$ of infinity, there exists a compact domain $K_\Omega$ with boundary strictly included in $\Omega$, on which the Divergence and Stokes Theorems hold.

For a compact orbifold without boundary, the Divergence Theorem holds, [Ch]. See also [C] for other results. Sufficient regularity also holds in the case of a product end, by an adaptation of Chiang’s method, see [Ch]. In general, ours seems to be a very reasonable assumption to make, which will be certainly satisfied in many cases of interest. For Sobolev inequalities of Gallot type involving domains, see [N].

**Theorem 2.2 [Fa4; Theorem 2.2].** Let $X$ be a non–compact complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle over $X$. Let $D_E$ be the generalized Dirac operator with coefficients in $E$, as defined above. Then $D_E$ is symmetric, i.e.,

$$(D_E\sigma_1, \sigma_2) = (\sigma_1, D_E\sigma_2), \quad \forall \sigma_1, \sigma_2 \in C_c^\infty(X, S \otimes_{\mathbb{C}} E),$$
where $(\cdot, \cdot)$ denotes the inner product defined in the introduction. (Actually, to obtain the above equality, it is enough to assume that only one of the sections $\sigma_1, i = 1, 2$ has compact support.)

Now complete the space $C_c^\infty(X, E), E = S \otimes_{\mathbb{C}} E, S \text{Spin}^c$–bundle on $X, E$ Hermitian orbibundle (with connection $\nabla^E$) over $X$, with respect to the norm

$$||\sigma||_X = \sqrt{\langle \sigma, \sigma \rangle} = \left(\sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{U_i} \bar{\eta}_i(\tilde{x}_i) < \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) > dv(\tilde{x}_i)\right)^{\frac{1}{2}}.$$ 

We thus obtain the $L^2$–space $L^2(X, E)$. The generalized Dirac operator

$$D_E : C_c^\infty(X, E) \to C_c^\infty(X, E)$$

has two natural extensions, min and max, see [Fa4], as an unbounded operator

$$D_E : L^2(X, E) \to L^2(X, E).$$

**Theorem 2.3 [Fa4; Theorem 3.1].** Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle (with connection $\nabla^E$) over $X$, and let $D_E$ be the generalized Dirac operator with coefficients in $E$. Let $\mathcal{D}(D_E^{MIN})$ be the domain of the min extension of $D_E$, and $\mathcal{D}(D_E^{MAX})$ be the domain of the max extension of $D_E$. Then

$$\mathcal{D}(D_E^{MIN}) = \mathcal{D}(D_E^{MAX}).$$

The following very useful proposition was proved in [Fa4]; it is a generalization of results proved by Gaffney and Yau for manifolds, see [Gn2], [Y].

**Proposition 2.4 [Fa4; Proposition 3.4].** Let $X$ be a non–compact complete orbifold which is sufficiently regular at infinity, and let $y_0 \in X - \Sigma(X)$ be a fixed point of $X$. Then there exists a sequence of continuous functions $b_k, k \in \mathbb{N}$, with

1. $b_k : X \to [0, 1]$
2. $b_k = 1$ on $B_k = \{y \in X | \rho(y) = d(y, y_0) \leq k\}$.
3. The support of $b_k$ is contained in $\overline{B}_{2k}$.
4. The function $b_k$ is differentiable almost everywhere and at points of differentiability we have

$$||\nabla(b_k)||^2 \leq \frac{M^2}{k^2}, \quad k \in \mathbb{N}$$

In [Fa4] we also proved the following results.
Theorem 2.5 [Fa4; Theorem 4.1]. Let $X$ be non-comapct complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle (with connection $\nabla^E$) over $X$. Let

$$D_E : C^\infty_c(X, \mathcal{E}) \rightarrow C^\infty_c(X, \mathcal{E}),$$

be the generalized Dirac operator on $X$ with coefficients in $E$. Then $D_E(\sigma) = 0$ if and only if $D_E^2(\sigma) = 0$ for any $\sigma \in \mathcal{D}(D_E)$.

Proposition 2.6 [Fa4; Proposition 6.1]. Let $X$ be a non-compact complete $\text{Spin}^c$ orbifold which is sufficiently regular at infinity, and let $S$ be the $\text{Spin}^c$ bundle of $X$. Then for any two sections $\sigma_j$, $j = 1, 2$ in $C^\infty(S)$, at least one of which with compact support, we have

$$\int_X \langle \Delta \sigma_1, \sigma_2 \rangle dv = \int_X \langle \nabla \sigma_1, \nabla \sigma_2 \rangle dv$$

Proposition 2.7 [Fa4; Proposition 6.3]. Let $X$ be an orbifold. Let $D$ be the Dirac operator on $X$, and let $\Delta$ be the $\text{Spin}^c$ Laplacian. Then

$$D^2 = \Delta + R$$

where $R$ is given below (c.f. [Du; Theorem 6.1], [LM; Theorem D12] for the manifold case),

$$R = \frac{1}{4} k + \frac{1}{2} c(K^*)$$

where $k$ is the scalar curvature, and $c(K^*)$ denotes the Clifford multiplication of the curvature 2form of the fixed connection on the line bundle $K^*$.

When $D_E$ is the generalized Dirac operators on $X$ with coefficients in the proper Hermitian orbibundle (with connection $\nabla^E$) $E$, then the formulas above become

$$D_E^2 = \Delta_E + R_E,$$

$$R_E = \frac{1}{4} k + \frac{1}{2} c(K^*) + c(E),$$

where $c(E)$ is the Clifford multiplication of the curvature 2form of the fixed connection $\nabla^E$ on $E$. When $X$ is $\text{Spin}$, we can assume that $c(K^*) = c(E) = 0$, and $R_E = \Delta_E + \frac{1}{4} k$.

Theorem 2.8 [Fa4; Theorem 6.4]. Let $X$ be a complete, non-compact, $\text{Spin}^c$ orbifold which is sufficiently regular at infinity. If $D$ is the Dirac operator on $X$ with coefficients in the $\text{Spin}^c$ bundle $S$, then the domain $\mathcal{D}$ of the unique self-adjoint extension of $D$ is exactly

$$L^{1,2}(X, S),$$

that is,
the completion of $C^\infty_c(X, \mathcal{S})$ in the norm

$$\|\sigma\|_1^2 = \int_X \left( \langle \sigma, \sigma \rangle + \langle \nabla \sigma, \nabla \sigma \rangle \right) dv = \int_X \left( \langle \sigma, \sigma \rangle + \langle \Delta_x, \sigma \rangle \right) dv$$

Moreover, for every $\sigma \in \mathcal{D}$,

$$\|D\sigma\|_X^2 = \|\nabla \sigma\|_X^2 + (R\sigma, \sigma),$$

where $\| \cdot \|_Y$ denotes the $L^2(Y, \mathcal{E})$, $Y \subseteq X$, norm, $R$ is given as in Proposition 2.7, and $(\cdot, \cdot)$ the $L^2$ inner product.

3. Dirac Operators and Green’s Operators.

As pointed out by Agmon in [A; Section 6], Friedrichs’ Lemma is a local result. Hence we can also assume it holds for orbifolds. (This follows from the local version of the manifold lemma applied to covers of orbifold charts of a locally finite orbifold cover.)

**Theorem 3.1 (Friedrichs’ Lemma for Orbifolds).** Let $X$ be an orbifold. Let $\mathcal{S}$ be the Spin$^c$ bundle of $X$, and let $D$ be the Dirac operator on $X$, $D : C^\infty_c(X, \mathcal{S}) \to C^\infty_c(X, \mathcal{S})$. Let $\Omega$ be an open set in $X$ and let $K$ be a compact subset of $\Omega$. Let $k \in \mathbb{N}$. Then there exists a constant $C$, depending only on $K$, $\Omega$, and $k$, such that, for any $\sigma \in C^\infty_c(X, \mathcal{S}|_\Omega)$ with $D\sigma = 0$, we have

$$\|\sigma\|_{C^k, K} \leq C \|\sigma\|_{L^2(\Omega, \mathcal{S})},$$

where $\| \cdot \|_{C^k, K}$ is the uniform $C^k$ norm for sections on $K$.

**Proof.** See [GL; Theorem 3.7] for the manifold case. Endow $X$ with a countable locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(U_i, G_i) | i \in \mathbb{N}\}$, with associated partition of unity $\eta = \{\eta_k\}_{k \in \mathbb{N}}$ subordinated to $\mathcal{F}$, [Ch]. Suppose that $U_i \cap K \neq \emptyset$ for only $i = 1, \ldots, \ell$. We can also choose our atlas so that $\bigcup_{i=1}^{\ell} U_i \subseteq \Omega$. Then by the manifold local version of Friedrichs’ Lemma [A; Section 6], modulo multiplication by $|G_i|$, $i = 1, \ldots, \ell$, we obtain the claim. □

We can now prove that Dirac operators which are positive at infinity, have finite dimensional kernels and cokernels.

**Theorem 3.2.** Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity. Let $\mathcal{S}$ be the Spin$^c$ bundle of $X$, and let $D$ be the Dirac operator on $X$, $D : C^\infty_c(X, \mathcal{S}) \to C^\infty_c(X, \mathcal{S})$. Assume that there exists a compact subset $K$ of $X$ such that

$$R \geq k_0 \text{Id}$$
in $X - K$, where $R$ is given as in Proposition 2.7. Then there exists an integer $d$ depending only on $D$ on a neighborhood of $K$ and on $k_0$, such that

$$\dim (\text{Ker} D) \leq d.$$  

In particular, if the dimension of $X$ is even, and $D = D^+ \oplus D^-$ (see Section 1), then

$$\dim (\text{Ker} D^+) + \dim (\text{Ker} D^-) \leq d.$$  

Proof. Since $K$ is compact, there exists a $k_1 > 0$ such that $k_1 \text{Id} \leq -R$ in $K$. Now let $\sigma \in \mathcal{L}^2(X, S)$, with $D\sigma = 0$. Firstly, $\sigma \in C^\infty(X, S)$, since solutions of elliptic equations on manifolds are smooth by a local argument. Then by Propositions 2.6 and 2.7, we have, for $\sigma$ with compact support,

$$\|\nabla \sigma\|_X^2 + (R\sigma, \sigma) = 0,$$

where $\| \cdot \|_Y, Y \subseteq X$, is the $L^2(Y, S)$ norm. Since $R \geq k_0$ on $X - K$, we have

$$\|\nabla \sigma\|_X^2 + \int_K < R\sigma, \sigma > + k_0 \int_{X-K} \|\sigma\|^2 \leq 0.$$  

Therefore,

$$\|\nabla \sigma\|_X^2 + k_0 \|\sigma\|_{X-K}^2 \leq \|\sigma\|_K^2.$$  

We now make the assumption $\|\sigma\|_X^2 = \|\sigma\|_K^2 + \|\sigma\|_{X-K}^2 = 1$. By adding $k_0 \|\sigma\|_K^2$ to both sides, and dividing by $k_0 + k_1$, the above inequality becomes

$$(3.1) \quad \frac{\|\nabla \sigma\|_X^2}{k_0 + k_1} + \frac{k_0}{k_0 + k_1} \leq \|\sigma\|_K^2.$$  

Fix a neighborhood $\Omega$ of $K$ and let $C$ be the constant appearing in Theorem 3.1 for $k = 1$. Fix $\epsilon > 0$ and an $\epsilon$ dense subset $\{x_s\}_{s=1,\ldots,d}$ of $K$ so that every point in $K$ is within distance $\epsilon$ of some $x_s$. Let $H = \ker(D)$ on $\mathcal{L}^2(X, S)$ and suppose by contradiction that $\dim(H) > d$. Then there exists $\sigma \in H$ such that $\|\sigma\|_X = 1$, and $\sigma(x_s) = 0$, $s = 1, \ldots, q$. By Theorem 3.1, applied to $K$ and $\Omega$, and the Mean Value Theorem applied on lifts of orbifold charts, we have

$$\|\sigma(x)\| \leq \sup_K \|\nabla(\sigma)\| \epsilon \leq C \|\sigma\|_\Omega \epsilon.$$  

But this contradicts (3.1). (Note that we can assume that $\sigma$ has compact support since we are only interested in behavior near $K$.)

By applying Glazman’s variational lemma below, see [Ku; Proposition 3.4] or [Sh], we can obtain information on the spectrum of the Dirac operator.
Proposition 3.3. Let $A$ be a self-adjoint Hilbert operator that is semi–bounded from below. Let $N_h(\lambda)$ denote the number of eigenvalues in $(-\infty, \lambda]$, with multiplicity, and counting points of the continuous spectrum as points with infinite multiplicity. Then

$$N_h(\lambda) = \sup_H \dim H,$$

where the supremum is taken over all the subspaces $H$ which are such that

$$< Ah, h > \leq < h, h >, \forall h \in H.$$

Corollary 3.4. Let $X$, $D$, and $S$ be as in Theorem 3.2. Then $D^2$, and consequently $D$, have essential spectrum separated from 0.

We will now show that Dirac operators admit Green’s operators, c.f. [GL; Theorem 3.7], and [Ku; Proposition 3.3 and 3.4] for the manifold case. We assume here that the orbifold $X$ is sufficiently good at infinity. This means that, for any neighborhood of infinity $\Omega$, we can chop off $X$ along an orbifold hypersurface, $O_\Omega$, which is the boundary of a neighborhood of infinity included in $\Omega$. We also assume that $\Omega$ is of product type near $O_\Omega$

Theorem 3.5. Let $X$ be a non–compact complete Spin$^c$ orbifold which is sufficiently good at infinity. Let $S$ be the Spin$^c$ bundle of $X$, and let $D$ be the Dirac operator on $X$, $D : C^\infty_c(X, S) \to C^\infty_c(X, S)$. Assume that there exists a compact subset $K$ such that

$$R \geq k_0 \text{Id}$$

on $X - K$, where $R$ is given as in Proposition 2.7. Assume that $H$ is the finite–dimensional kernel of $D$ on $L^2(X, S)$, and let $H^\perp$ be its orthogonal complement. Then there is a $\alpha > 0$ such that

$$\|D\sigma\|_X^2 \geq \alpha^2 \|\sigma\|^2_X, \forall \sigma \in H^\perp.$$

Thus the operator $D$, and also, since $X$ is even-dimensional, the operator $D^\pm$, admits bounded Green’s operators.

Proof. (C.f. [GL; Proof of Theorem 3.7] and [Ku; 3.3 and 3.4].) First, we claim that $D$ has only point spectrum. This is proved by the same argument used in [Ku; 3.3]. Note that Kucerovsky’s proof works verbatim in the orbifold case too, since Rellich’s lemma applies in the sufficiently good case (in fact, one can double the domain to get an orbifold, and then apply Rellich’s Lemma for closed orbifolds proved in [Fa2]).
Now, let $E_\lambda$ be the $\lambda$-eigenspace of $D$ on $L^2(X, S)$ with eigenvalue $\lambda$. It will suffice to prove that there exists an $\alpha > 0$ such that the space

$$H_\alpha = \oplus_{|\lambda| \leq \alpha} E_\lambda$$

is finite-dimensional. We will now proceed as in the proof of Theorem 3.2. By Theorem 2.8, we have

$$(D^2 \sigma, \sigma) - (R \sigma, \sigma) - \|\nabla \sigma\|^2_X = 0, \forall \sigma \in D(D).$$

Hence, for $\sigma \in H_\alpha$, $D \sigma = \lambda \sigma$, with $|\lambda| \leq \alpha$, so we get

$$\lambda^2 \|\sigma\|^2_X - (R \sigma, \sigma) - \|\nabla \sigma\|^2_X = 0.$$  

Since for some $k_1 \in \mathbb{R}$, $-R \geq k_1 I d$ on $K$, we have that, for any $\sigma \in H_\alpha$,

$$\lambda^2 \|\sigma\|^2_X + \int_K k_1 <\sigma, \sigma> dv \geq \int_{X-K} <R \sigma, \sigma> dv + \|\nabla \sigma\|^2_X$$

As $k_0 \in \mathbb{R}$ is such that $k_0 I d \leq R$ on $X - K$, we get

$$\lambda^2 \|\sigma\|^2_X + k_1 \|\sigma\|^2_K \geq k_0 \|\sigma\|^2_{X-K} + \|\nabla \sigma\|^2_X, \forall \sigma \in H_\alpha.$$  

Now replace $\|\sigma\|^2_{X-K}$ by $\|\sigma\|^2_X - \|\sigma\|^2_K$ to get,

$$\lambda^2 \|\sigma\|^2_X + k_1 \|\sigma\|^2_K \geq k_0 (\|\sigma\|^2_X - \|\sigma\|^2_K) + \|\nabla \sigma\|^2_X, \forall \sigma \in H_\alpha,$$

$$\lambda^2 \|\sigma\|^2_X + (k_0 + k_1) \|\sigma\|^2_K \geq k_0 \|\sigma\|^2_X + \|\nabla \sigma\|^2_X, \forall \sigma \in H_\alpha.$$  

Because $\sigma \in H_\alpha$, $|\lambda| \leq \alpha$, and so $\lambda^2 \leq \alpha^2$, which implies

$$(k_0 + k_1) \|\sigma\|^2_K \geq (k_0 - \alpha^2) \|\sigma\|^2_X + \|\nabla \sigma\|^2_X, \forall \sigma \in H_\alpha.$$  

If we choose $\alpha > 0$ such that $\alpha^2 << k_0$, we have

$$(3.2) \quad \frac{k_0 - \alpha^2}{k_0 + k_1} \leq \frac{\|\sigma\|^2_K}{\|\sigma\|^2_X}, \forall \sigma \in H_\alpha.$$  

Now choose a parametrix $Q$ for $D$, $[Kw2]$, so that $Q D = I d - T$, with $T$ smoothing. Let $\rho$ denote the restriction to $K$, and let $\tilde{T} = \rho \circ T$. Then for any $\sigma \in H_\alpha$, we have

$$\tilde{T} \sigma = \rho \sigma - \rho Q D \sigma.$$
Moreover, for any $\sigma \in H_\alpha$, $\|D\sigma\|_X \leq \alpha \|\sigma\|_X$. Set $q = \|Q\|$; from (3.2) we get,

$$\|\hat{T}\sigma\|_X \geq \|\rho \sigma\|_X \geq (\alpha_1 - \alpha q)\|\sigma\|_X, \forall \sigma \in H_\alpha,$$

where $\alpha_1 = \frac{k_0 - \alpha^2}{k_0 + k_1}$. Choosing $\alpha$ sufficiently small, $\|\hat{T}\sigma\|_X \geq \alpha_2 \|\sigma\|_X, \forall \sigma \in H_\alpha$. Therefore $H_\alpha$ is finite-dimensional since $\hat{T}$ is a compact operator.

We will now define the analytical index of a pair of generalized Dirac operators that agree at infinity, thus generalizing a construction of Gromov and Lawson. We start with a non-compact complete orbifold which is sufficiently good at infinity. Let $D_i$ and $D_i^\pm$ be the generalized Dirac operators on $X$ with coefficients in the Hermitian orbibundle $E_i$ (with connection $\nabla^{E_i}$), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood $\Omega$ of infinity. Assume that there exists a constant $k_0 > 0$ such that

$$R \geq k_0 \text{Id} \text{ on } \Omega.$$

Then we know from Theorem 3.2 that $\text{ker}(D_i^\pm) < +\infty$. Thus the analytical index, of $D_i^\pm$, dimension of kernel minus dimension of cokernel, is finite, i.e., $\text{ind}_a(D_i^\pm) < +\infty$, $i = 1, 2$.

**Definition 3.6.** Let $X$ be a non-compact complete orbifold which is sufficiently good at infinity. Let $D_i^\pm$, $E_i$, $i = 1, 2$ be generalized Dirac operators, with coefficients in the Hermitian orbibundles $E_i$ (with connection $\nabla^{E_i}$), that agree on a neighborhood of infinity $\Omega$. Then we define the analytical index, $\text{ind}_a(D_i^\pm, D_2^\pm)$, of the pair $(D_1^\pm, D_2^\pm)$ to be

$$\text{ind}_a(D_1^\pm, D_2^\pm) = \text{ind}_a(D_2^\pm) - \text{ind}_a(D_1^\pm),$$

where $\text{ind}_a(D_i^\pm)$, is the analytical index, dimension of kernel minus dimension of cokernel, of $D_i^\pm$, $i = 1, 2$, see Theorem 3.2.

4. The Topological Index.

We will define here the topological index of a pair of generalized Dirac operators that agree at infinity, thus generalizing a construction of Gromov and Lawson, [GL]. We start with a non-compact complete orbifold $X$ which is sufficiently regular at infinity.

Let $D_i$ and $D_i^\pm$ be the generalized Dirac operators on $X$ with coefficients in the Hermitian orbibundle $E_i$ (with connection $\nabla^{E_i}$), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood of infinity.

We assume here that the orbifold $X$ is sufficiently good at infinity. Recall that this means that, for any neighborhood of infinity $\Omega$, we can chop off $X$ along an orbifold hypersurface, $O_\Omega$, which is the boundary of
a neighborhood of infinity included in $\Omega$. We also assume that the orbifold structure of $X$ is of product type in a neighborhood of $O_\Omega$. Note that sufficiently good at infinity implies sufficiently regular at infinity.

Chop off $X$ along an hypersurface $O$, with $D_1 = D_2$ on the neighborhood of infinity of which $O$ is boundary. Glue in an orbifold along $O$, so that the resulting space is a closed orbifold $W$. (This can be achieved, for example, by gluing another copy of the chopped off orbifold along $O$). We can assume that $E_1 = E_2$ on a neighborhood of $O$. Extend $E_i$ to $W$, and call $\tilde{E}_i$ this extension, $i = 1, 2$. We can assume that on the glued in part, $\tilde{E}_1 = \tilde{E}_2$. Let $\tilde{D}_i$ and $\tilde{D}_i^\pm$ be the generalized Dirac operators on $W$ with coefficients in $\tilde{E}_i$, $i = 1, 2$. We can assume that $\tilde{E}_i^\pm$ is the topological index of Kawasaki of the operators $\tilde{D}_i^\pm$ on the closed orbifold $W$, $i = 1, 2$, [Kw2], [Kw3].

We will now give an explicit local description of the above index by using Kawasaki’s formulas, see [Kw2], [Kw3], [Du], [V]. Let $\mathcal{U} = \{ \tilde{U}_i, G_i | i \in I \}$, with $\tilde{U}_i/G_i = U_i$ and projection $\pi_i : \tilde{U}_i \to U_i$, be a standard orbifold atlas of $W$. Now $\tilde{U}_i^g$, the set of points fixed by $g$ in $\tilde{U}_i$, for any $g \in G_i$, admits the action of the centralizer $Z_{G_i}(g)$ of $g$ in $G_i$. If $g$ and $g'$ are conjugate in $G_i$, then $\tilde{U}_i^g$ and $\tilde{U}_i^{g'}$ are diffeomorphic via some element $h$ in $G_i$, with $g' = hgh^{-1}$. So we can consider only one element for each conjugacy class in $G_i$. For each point $x \in W$, let $(1), \ldots,(h_x^{\rho_x})$ be all the conjugacy classes of the stabilizer $G_x$ of $x$. Then we have a natural associated orbifold, [Kw3],

$$\hat{\Sigma}W = \{(y, (h_y^j)) | y \in W, G_y \neq 1, j = 2, \ldots, \rho_y \}.$$ 

$\hat{\Sigma}W$ is stratified by orbit types; define

$$\hat{\Sigma}W = W \cup \hat{\Sigma}W.$$ 

For short, we can rewrite $\hat{\Sigma}W$ as

$$\hat{\Sigma}W = \prod_{j=1}^q \hat{\Sigma}^jW,$$

where $\hat{\Sigma}W = W$, and so $\hat{\Sigma}^jW$ is the stratum corresponding to the $j$–th orbit type, $j = 1, \ldots, \ell$. In general, the action of $Z_{G_x}(h)$ on $\tilde{U}_x^h$ is not effective. The order of the subgroup of $Z_{G_x}(h)$ that acts trivially is called
the multiplicity of $\hat{\Sigma}W$ at $(x,h)$, and it is denoted by $m(x \in W$ and $h \in G_x$). Hence to each connected component of $\hat{\Sigma}W$ we assign a certain constant multiplicity. The following theorem, the index theorem for generalized Dirac operators on closed orbifolds, was proved by Kawasaki in [Kw1], [Kw2]; see also [Du; Theorem 14.1], [BGV], [V].

**Theorem 4.2 [Kw2], [Du].** Let $W$ be a closed orbifold. Let $E$ be a Hermitian orbibundle (with connection $\nabla^E$) on $W$ and let $D^+_E$ be a generalized Dirac operator with coefficients in $E$. Then the (topological and analytical) index of $D^+_E$ on $W$ is given in terms of local traces by

$$\text{ind}(D^+_E) = \int_{\hat{\Sigma}W} d\mu^+_{D^+_E} := \sum_{j=0}^\ell m_j \int_{\hat{\Sigma}W} d\mu^+_{D^+_E,j},$$

where $d\mu^+_{D^+_E,j}$ is the density on $j$-th stratum of $\hat{\Sigma}W$ associated (via parametrices) to the operator $D^+_E$, and $m_j$ is the corresponding multiplicity function, $j = 1, \ldots, \ell$. (For simplicity’s sake, we assumed all the strata to be connected; in the general case there will be an additional summation over the connected components of the strata.)

**Remark 4.3.** For another formulation of Theorem 4.2 in terms of orbifold characteristic classes, see [Kw3].

We will now use Theorem 4.2 to compute the topological index of a pair of generalized Dirac operators on a compact orbifold that is sufficiently good at infinity.

**Theorem 4.4.** Let $X, W, D^+_i, D^+_i, \bar{D}^+_i, E_i, \bar{E}_i, i = 1, 2$ be as in Definition 4.1. Let $Q_i$ be a semi–local parametrix for $D^+_i$, $i = 1, 2$, on $W$ such that $Q_1 = Q_2$ in a neighborhood of infinity; write

$$D^+_iQ_i = Id - T_i, \text{ and } Q_iD^+_i = Id - T'_i, i = 1, 2,$$

with $T_i$ and $T'_i$ the associated semi–local smoothing operators. Then the local trace functions of Kawasaki associated to $T_1$ and $T_2$ (and $T'_1$ and $T'_2$ ) coincide at infinity, and the topological index $\text{ind}_i(D^+_1, D^+_2)$ of the pair $(D^+_1, D^+_2)$ is given by

$$\text{ind}(D^+_E) = \int_{\hat{\Sigma}X} \left(d\mu^+_{T'_2} - d\mu^+_{T'_2}\right) - \int_{\hat{\Sigma}X} \left(d\mu^+_{T'_1} - d\mu^+_{T'_1}\right)$$

**Proof.** This theorem follows directly from Theorem 4.2. Indeed, if we denote by $\Omega$ the neighborhood of infinity on where $D_1$ and $D_2$ coincide, we can cap off $X$ along an orbifold $O$ in $\Omega$. Consider a parametrix $Q_0$ for the extension $\bar{D}_1 = \bar{D}_2$ on $\Omega$. Now splice $Q_0$ onto $Q_1$ and $Q_2$, via a smooth function $f$ which is 0 outside $\Omega, 1$ on a neighborhood of infinity, and whose gradient is bounded by 1 in norm. For the rest of the proof we can proceed exactly as in [GL; Proof of Proposition 4.6]. Note that we need to use Kawasaki’s local trace formulas. □
Remark 4.5. The topological index of the pair, \( \text{ind}_t(D^+_1, D^+_2) \), is independent of the extension.

5. The Analytical Index: Computations.

We will now detail some local properties of the trace of a generalized Dirac operator \( D \), thus extending to orbifolds some results of Gromov and Lawson, see [GL]. Assume that there exists a constant \( k_0 > 0 \) such that

\[
R \geq k_0 \text{Id} \quad \text{on } \Omega,
\]

with \( \Omega = X - K \) as before. Because of Theorem 2.8, our assumption implies

\[
\|D\sigma\|_X \geq c\|\sigma\|_X, \quad \forall \sigma \in \mathcal{C}_c^\infty(X, S) \text{ such that } \text{supp}(\sigma) \cap K = \emptyset
\]

Our goal in this section is to explicitly compute the index trace of \( D \) in terms of local data, by using techniques of Gromov and Lawson and Anghel, see [GL], [An2]. This will enable us to prove, in Section 6, the relative index theorem by localization. Also, for simplicity’s sake, in this section we will take \( E = \mathbb{C} \). So \( D \) will be defined on sections of the \( \text{Spin}^c \) bundle \( S \). Start with a parametrix \( Q_0 \) of \( D \) on \( X \). Then

\[
DQ_0 = \text{Id} - R, \quad \text{and } Q_0D = \text{Id} - R', \quad \text{on } X,
\]

where \( R \) and \( R' \) are not necessarily trace class (as instead happens in the closed orbifold case). We will now replace \( Q_0 \), outside a compact set \( K_0 \), with a Green operator associated to a suitable extension of \( D \). Specifically, set \( \Omega_0 = X - K_0 \), with \( \Omega_0 \) a domain with smooth boundary included in \( \Omega \). Let \( D_{\Omega_0} \) be the graph closure of the restriction of \( D \) to \( \mathcal{C}_c^\infty(\Omega_0, S) \) in \( L^2(X, S) \). Let \( P_{\Omega_0} \) denote the orthogonal projection from \( L^2(X, S) \) to

\[
H_{\Omega_0} = \{ \sigma \in L^2(\Omega_0, S)|D|_{\Omega_0}(\sigma|_{\Omega_0}) = 0 \text{ distributionally} \}
\]

Note that, by (5.1), \( H_{\Omega_0} \) is a closed subspace of \( L^2(X, S) \).

Theorem 5.1. Let \( X \) be a non-compact complete orbifold which is sufficiently good at infinity. Let \( D, S, D_{\Omega_0}, P_{\Omega_0} \), be as above. Then the following two equations

\[
D_{\Omega_0}G_{\Omega_0} = \text{Id} - P_{\Omega_0} \quad \text{on } L^2(X, S)
\]

\[
G_{\Omega_0}D_{\Omega_0} = \text{Id} \quad \text{on the minimal domain of } D_{\Omega_0}
\]
define a bounded operator $G_{\Omega_0} : \mathcal{L}^2(X, S) \to \mathcal{L}^2(X, S)$, the Green operator of $D_{\Omega_0}$.

**Proof.** Equation (5.1) shows that $D_{\Omega_0}$ is 1:1 and has closed range, call it $\text{ran}(D_{\Omega_0})$. Hence $\mathcal{L}^2(X, S)$ decomposes orthogonally as

$$\mathcal{L}^2(X, S) = H_{\Omega_0} \oplus \text{ran}(D_{\Omega_0}).$$

Thus $G_{\Omega_0}$ can be defined to be zero on $H_{\Omega_0}$ and $D_{\Omega_0}^{-1}$ on $\text{ran}(D_{\Omega_0})$. $G_{\Omega_0}$ is bounded because of (5.1). □

Since $H_{\Omega_0}$ is a closed subspace of $\mathcal{L}^2(X, S)$, we can define the orthogonal projection on $H_{\Omega_0}$ Bergman kernel operator

$$P_{H}(x, y) = \sum_{m} \sigma_{m}(x) \otimes \sigma_{m}^{\ast}(y), \forall x, y, \in \Omega_0,$$

where $\{\sigma_{m}\}, m \in \mathbb{N}$, is an orthonormal basis of $H_{\Omega_0}$.

**Theorem 5.2.** Let $X$ be a non–compact complete orbifold which is sufficiently good at infinity. Let $D, S, D_{\Omega_0}, P_{\Omega_0}$, be as above. Then the following Bergman kernel

$$P_{H}(x, y) = \sum_{m} \sigma_{m}(x) \otimes \sigma_{m}^{\ast}(y), \forall x, y, \in \Omega_0$$

of the projection operator $P_{\Omega_0}$ converges (on lifts of local charts; see the proof for details) uniformly in the norm $C^{k}$ on compact subsets, $k \in \mathbb{N}$.

**Proof.** (In this proof "means lifted to the local orbifold charts.) Because $X$ is sufficiently good at infinity, we can assume that there exists a smooth exhaustion function $F$ on $X$, such that $\Omega_0 = F(t_0) = \{x \in X | F(x) > t_0\}$. To show the convergence of the Bergman kernel $P_{H}$ we proceed in the following way. First of all, set $B_N = \sum_{m=1}^{N} \sigma_{m}(x) \otimes \sigma_{m}^{\ast}(y)$. Obviously $B_N$ is a finite rank operator in $\mathcal{L}^2(X, S)$. Now, for any $\phi \in \mathcal{L}^2(X, S)$, $B_N(\phi) \to B(\phi)$. In particular $B_N(\phi) \to B(\phi)$ in $\mathcal{D}'$ (as distributions). By using local charts, $B_N \to B$ weakly in $\mathcal{L}_{\mathcal{G}_{U}}(\mathcal{D}_{\tilde{y}}, \mathcal{D}_{\tilde{x}}')$ for any two points $\tilde{x}, \tilde{y}$ of an orbifold chart $(\tilde{U}, \mathcal{G}_{U})$ projecting to $x$ and $y$ respectively. (By using a partition of unity, the convergence of distributions is a local property; use a standard orbifold atlas.) Also, for simplicity’s sake, we have taken $x$ and $y$ to be in the same orbifold chart.

Now, $P_{\Omega_0}$, which in this proof we will call $P$, is a bounded operator on $\mathcal{L}^2(X, S)$, and so is also continuous as an operator $\mathcal{D} \to \mathcal{D}'$. Hence $P$ has a $\mathcal{G}_{U}$–invariant distributional kernel $p(\tilde{x}, \tilde{y})$ in the sense of Schwartz, c.f. [Schw1], [Scw2], [At1]. By the Schwartz kernel theorem, $B_N \to P_{H}$ locally as a $\mathcal{G}_{U}$–invariant distribution on $\tilde{U} \times \tilde{U}$. (This follows as in the manifold case, see [At1; pg. 51].) So

$$P_{H}(x, y) = \sum_{m=1}^{+\infty} \sigma_{m}(x) \otimes \sigma_{m}^{\ast}(y)$$
G_U-invariantly, on local charts. Now note that \( P_H \) and \( B_N \) satisfy the elliptic equation \( \tilde{D}^2 = 0 \). Therefore by applying Lemma 3.1 locally, we have the required uniform convergence. □

The kernel \( P_H \) has the strong finiteness property proved below. Let \( F : X \to \mathbb{R}^+ \) be a smooth exhaustion function as above. In particular we assume that \( \Omega_0 = F^{-1}(t_0, +\infty) \), \( K = F^{-1}[0, t_0] \). Let \( X(t) = \{ x \in X | F(x) > t \} \); then \( \Omega_0 = X(t_0) \). We have, c.f. [GL; Lemma 4.20] for the manifold case,

**Theorem 5.3.** Let \( X \) be a non-compact complete orbifold which is sufficiently good at infinity. Let \( D, S, D_{\Omega_0}, P_{\Omega_0}, P_H(x, x) \) be as above. Then for any \( t > t_0 \),

\[
\int_{X(t)} P_H(x, x) < +\infty
\]

**Proof.** Fixed \( t > t_0 \) (\( t \) near \( t_0 \)), choose \( s \) so that \( t_0 < s < t \), and consider the compact "annulus" \( A = \text{closure}[X(s) - X(t)] \). Since \( P_H(x, x) \) converges uniformly in the \( C^k \), \( k \in \mathbb{N} \) norm, on compact subsets, we can assume that it converges in the Sobolev norm on \( A \), that is,

\[
(5.4) \quad \sum_m \| \sigma_m \|_{1,A} < +\infty.
\]

We now claim that there exists a constant \( c \) so that, for each \( \sigma \in H_{\Omega_0} \),

\[
(5.5) \quad \| \sigma_m \|_{1,X(t)}^2 \leq c \| \sigma \|_{1,A}^2.
\]

To do so, choose a cut-off function \( f \in C^\infty(\Omega_0) \) such that \( 0 \leq f \leq 1 \), \( f = 1 \) on \( X(t) \), and \( f = 0 \) on \( \Omega_0 - X(s) \). Clearly there exists a \( c_0 \in \mathbb{R}^+ \) such that \( \| \nabla f \|_X < c_0 \). Applying the local identity

\[
\nabla(f \sigma) = \nabla(f) \sigma + f \nabla \sigma
\]

and \( D^2 = \nabla^* \nabla + R \), we have, for every \( \sigma \in H_{\Omega_0} \),

\[
0 = (D_{\Omega_0}^2 \sigma, f^2 \sigma) = ((\nabla^* \nabla + R) \sigma, f^2 \sigma)
\]

\[
0 = (\nabla^* \nabla \sigma, f^2 \sigma) + (R \sigma, f^2 \sigma)
\]

\[
0 = (\nabla \sigma, \nabla(f^2 \sigma)) + (R \sigma, f^2 \sigma)
\]

\[
0 = (\nabla \sigma, 2f(\nabla f) \sigma) + (\nabla \sigma, f^2(\nabla \sigma)) + (R \sigma, f^2 \sigma)
\]

\[
0 = 2(f \nabla \sigma, (\nabla f) \sigma) + (f \nabla \sigma, f(\nabla \sigma)) + (R \sigma, f^2 \sigma).
\]
Since $<R\alpha,\alpha> \geq k_0 <\alpha,\alpha>$ in $\Omega_0$, we have

$$\|\nabla \sigma\|^2_{X(t)} + k_0 \|\sigma\|^2_{X(t)} \leq 2 |(f\nabla \sigma, (\nabla f)\sigma)| \leq 2c_0 \|\nabla \sigma\|_A \|\sigma\|_A \leq c_0 (\|\nabla \sigma\|^2_A + \|\sigma\|^2_A).$$

Then

$$\|\sigma\|^2_{1,X(t)} \leq c_0 (1 + \frac{1}{k_0}) \|\sigma\|^2_{1,A}.$$

We thus proved our claim (5.5). Now the Lemma is proved by combining (5.4) and (5.5). □

6. The Relative Index Theorem.

We will prove here that the analytical index of a pair of generalized Dirac operators that agree at infinity is equal to its analytical index, thus generalizing to orbifolds the main theorem of Gromov and Lawson in [GL]. Let $X$ be a non–compact complete orbifold which is sufficiently good at infinity. Assume that there exists a constant $k_0 > 0$ such that

$$R \geq k_0 Id \text{ on } \Omega, \quad K = X - \Omega,$$

with $\Omega$ a domain with smooth boundary, and $R$ as in Proposition 2.7. Let $D_i$ and $D_i^{\pm}$ be the generalized Dirac operators on $X$ with coefficients in the Hermitian orbibundle $E_i$ (with connection $\nabla_i^E$), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood $\Omega$ of infinity. Let $G_i$, $i = 1, 2$ be the Green operator associated to $D_i^{+}$, $i = 1, 2$, see Theorem 3.5. In particular, $D_i^{+}$ and $G_i$, satisfy the following equations,

$$D_i^{+}G_i = Id - P_i^{-}, \quad \text{and} \quad G_iD_i^{+} = Id - P_i^{+}, \quad i = 1, 2,$$

where $P_i^{\pm}: \mathcal{L}^2(X,S) \to \mathcal{L}^2(X,S)$ is the projection onto the finite dimensional space $\text{ker}D_i^{\pm}$. On $\Omega$, $D_1 = D_2$.

Restrict $G_i$ to $\Omega$ by defining $\hat{G}_i = \chi G_i \chi$, $i = 1, 2$, where $\chi$ is the characteristic function of $\Omega$. The difference $\hat{G}_2 - \hat{G}_1$ satisfies

$$D^{+}(\hat{G}_2 - \hat{G}_1) = \hat{P}_1^{-} - \hat{P}_2^{-},$$

where $D_1 = D_2 = D$ on $\Omega$, and $\hat{P}_i^{\pm} = \chi P_i^{\pm} \chi$, $i = 1, 2$, is a finite range operator. This implies, as in [GL], that the range of $(\hat{G}_2^{+} - \hat{G}_1^{+})$ is nearly contained in the kernel $H_\Omega$ of $D^+$ on $\Omega$, notation as in Proposition 2.7. In other words, let $V$ be $\text{ker}(\hat{P}_2^{+} - \hat{P}_1^{+})$. Then $V$ is a subspace of finite codimension in $\mathcal{L}^2(X,S^+)$, and

$$(\hat{G}_2 - \hat{G}_1)V \subseteq \text{ker}(D^+).$$
Theorem 6.1. (f. [GL; Lemma 4.28] for the manifold case.) Let \(X, D_i, G_i, \hat{G}_i, i = 1, 2,\) be as above. Then the local trace density function of \(\hat{G}_2 - \hat{G}_1\) is integrable at infinity.

Proof. Recall that \(\Omega = X - K,\) with \(K\) compact. Choose \(\Omega' = X - L,\) with \(L\) compact such that \(K \subseteq \text{int}(L).\) By Theorem 5.3, if \(P_\Omega\) denotes the orthogonal projection onto the kernel \(H_\Omega\) of \(D^+\) on \(\Omega,\) and \(P_\Omega\) is its associated trace density, then

\[
\int_{\Omega'} P_\Omega(x) \, dv < +\infty.
\]

Set \(Z = (\hat{G}_2 - \hat{G}_1).\) Then

\[
\text{range}(Z) \subseteq H_\Omega + F,
\]

where \(F\) is a finite-dimensional subspace of \(L^2(X, S^-),\) by the discussion preceding the statement of Theorem 6.1. Let \(\{\sigma_m\}, m \in \mathbb{N},\) be an orthonormal basis of \(H_\Omega + F,\) such that \(\{\sigma_m\}, m = N, N + 1, \ldots,\) is an orthonormal basis of \(H_\Omega.\) Then the Schwartzian kernel of \(Z\) can be written as

\[
K^Z(x, y) = \sum_m \sigma_m(x) \otimes (Z^* \sigma_m)(y), \quad \forall x, y, \in \Omega,
\]

where \(Z^*\) denotes the adjoint of \(Z.\) The local trace function \(P_Z\) of \(Z\) satisfies

\[
|P_Z(x)| \leq \sum_m |< \sigma_m(x), (Z^* \sigma_m)(x) >|, \quad \forall x \in \Omega.
\]

Let \(Z' = \chi'Z\chi',\) with \(\chi'\) the characteristic function of \(\Omega',\) denote the restriction of \(Z\) to \(\Omega';\) note that \(\|Z'\| \leq \|Z\|.\) Then

\[
\int_{\Omega'} |P_Z(x)| \leq \sum_m \int_{\Omega'} |< Z\sigma_m(x), \sigma_m(x) >| \, dx
\]

\[
\leq \sum_m \|Z\sigma_m\|_{\Omega'} \|\sigma_m\|_{\Omega'}
\]

\[
\leq \sum_m \|Z'\sigma_m\|_{\Omega'} \|\sigma_m\|_{\Omega'}
\]

\[
\leq \|Z\| \sum_m \|\sigma_m\|^2_{\Omega'}
\]

\[
\leq \|Z\| \left( \sum_{m=1}^{N-1} \|\sigma_m\|^2_{\Omega'} + \int_{\Omega'} |P_Z| \right) < +\infty \quad \square
\]

We are now in a position to state and prove our relative index theorem.

Theorem 6.2. Let \(X\) be a non-compact complete orbifold which is sufficiently good at infinity. Let \(D_i, D_i^\pm\) be the generalized Dirac operators on \(X\) with coefficients in the Hermitian orbibundle \(E_i\) (with connection
∇^E_i, i = 1, 2. Suppose that \( D_1 = D_2 \) in a neighborhood \( \Omega \) of infinity with smooth boundary. Assume that there exists a constant \( k_0 > 0 \) such that

\[
\mathcal{R} \geq k_0 \text{Id on } \Omega.
\]

where \( \mathcal{R} \) is given as in Section 5. Then the topological index (as in Definition 4.1) and the analytical index (as in Definition 3.6) of the pair \( (D_1^+, D_2^+) \) coincide, that is

\[
\text{ind}_t(D_1^+, D_2^+) = \text{ind}_a(D_1^+, D_2^+)
\]

The proof of Theorem 6.2 will occupy the rest of this section.

Proof. (c.f. [GL; Proof of Theorem 4.18]) We will now construct parametrices \( Q_j \) for \( D_j^+ \), \( j = 1, 2 \), by cutting off the Green operators \( G_i, i = 1, 2 \) of Theorem 3.5 in a neighborhood of the diagonal. (Notation as at the beginning of this section.) Choose \( \psi \in C^\infty(X \times X) \) with support in a small neighborhood of the diagonal, so that \( 0 \leq \psi \leq 1 \) and \( \psi = 1 \) near the diagonal. Note that \( \psi \) can in particular be lifted, one variable at the time, to a \( G_U \)–invariant function in \( \tilde{\psi} \) in \( C^\infty(\tilde{U} \times \tilde{U}) \), for any local orbifold chart \( (\tilde{U}, G_U) \). Let \( R_i, i = 1, 2 \), be the operator whose Schwartzian kernel is locally given by

\[
K^i(\tilde{x}, \tilde{y}) = \tilde{\psi} K^G(\tilde{x}, \tilde{y})
\]

The operator \( R_i, i = 1, 2 \), is a semi–local parametrix for \( D_i^+ \) with

\[
D_i^+ R_i = \text{Id} - S_i^- , \quad R_i D_i^+ = \text{Id} - S_i^+, \quad i = 1, 2.
\]

As \( R_i = G_i \) near the diagonal, the local traces associated to \( S_i^\pm, i = 1, 2 \), satisfy

\[
t^{S_i^+} = t^+, \quad t^{S_i^-} = t^-,
\]

where \( t^{\pm} \) is the trace associated to \( P_1 : \mathcal{L}^2(X, \mathcal{S}) \to \mathcal{L}^2(X, \mathcal{S}) \), the projection onto the finite dimensional space \( \ker D_i^+ \) and \( \ker D_i^- = \text{coker} D_i^+ \), \( i = 1, 2 \). From Theorem 4.4, to compute the topological index we need a pair of semi–local parametrics that agree in a neighborhood of infinity. We will do this by splicing \( R_2 \) onto \( R_1 \) in \( \Omega \) as follows. Let \( \{b_k\} k \in \mathbb{N} \), be a sequence of functions as in Proposition 2.4. We assume that, for \( k \) sufficiently large, \( b_k = 1 \) on \( K \). Moreover, since \( \text{supp}(b_k) \subseteq \overline{\mathcal{B}}_{2k} \), we can assume that \( b_k = 0 \) in a neighborhood of infinity. We claim that each \( b_k \) can be approximated by a smooth function \( f_k \) such that \( f_k = 1 \) on \( K \), \( \text{supp}(f_k) \subseteq \overline{\mathcal{B}}_{2k} \). This claim is proven in the following way. \( \text{Supp}(b_k) \) can be covered by the
union of finitely many orbifold charts, say \( U_j \) with \( \tilde{U}_j / G_j = U_j, j = 1, \ldots, \ell \). If \( \tilde{b}_{k,s} \) is the lift of a smooth approximation of \( b_k \) on \( U_s \), obtained \( G_s \)-invariantly, then \( \tilde{f}_k = \sum_s \tilde{\eta}_s \tilde{b}_{k,s} \) is a smooth approximation of \( b_k \), \( k \in \mathbb{N} \). Now define a sequence of semi–local parametrices \( R_{2,k} \) for \( D_2^+ \) by

\[
R_{2,k} = f_k R_2 + (1 - f_k) R_1
\]

(recall that on \( \Omega, D_1^+ = D_2^+ \)). Moreover,

\[
D_2^+ R_{2,k} = Id - S_{2,k}^-, \quad R_{2,k} D_2^+ = Id - S_{2,k}^+
\]

with

\[
S_{2,k}^- = f_k S_2^- + (1 - f_k) S_1^- + \nabla (f_k) (R_2 - R_1)
\]

(6.1)

\[
S_{2,k}^+ = f_k S_2^+ + (1 - f_k) S_1^+
\]

where \( S_i^\pm, i = 1, 2 \), satisfy

\[
D_1^+ R_1 = Id - S_1^-, \quad R_1 D_1^+ = Id - S_1^+, \\
D_2^+ R_2 = Id - S_2^-, \quad R_2 D_2^+ = Id - S_2^+. 
\]

Now by Theorem 4.4 applied to the pair of parametrices \( R_{2,k}, R_1 \), we get,

\[
\text{ind}_t (D_1^+, D_2^+) = \int_{\Sigma} \left( d\mu^{S_{2,k}^+} - d\mu^{S_{2,k}^-} \right) - \int_{\Sigma} \left( d\mu^{S_1^+} - d\mu^{S_1^-} \right).
\]

By (6.1) this latter expression is equal to

\[
= \int_{\Sigma} f_k \left( d\mu^{S_2^+} - d\mu^{S_2^-} - d\mu^{S_1^+} + d\mu^{S_1^-} \right) - \int_{\Sigma} d\mu \nabla f_k (R_2 - R_1), \\
= \int_{\Sigma} f_k \left( d\mu^{P_2^+} - d\mu^{P_2^-} - d\mu^{P_1^+} + d\mu^{P_1^-} \right) - \int_{\Sigma} d\mu \nabla f_k (R_2 - R_1),
\]

where \( P_i^\pm \) is the projection onto the kernel of \( D_i^\pm, i = 1, 2 \). Now the Schwartz kernel of \( Z_k = \nabla f_k (R_2 - R_1) \) on a local chart is

\[
K Z_k (\tilde{x}, \tilde{y}) = \nabla \tilde{f}_k K^{R_2 - R_1} (\tilde{x}, \tilde{y}).
\]

Near the diagonal, \( R_i = G_i, i = 1, 2 \), and so

\[
|d\mu Z_k (x)| \leq \|\nabla f_k\| |d\mu Z (x)|,
\]
where $Z = \hat{G}_2 - \hat{G}_1$ as above. By integrating on $\hat{\Sigma} \Omega$, since $\text{supp}(\nabla f_k) \subseteq \Omega$, we have
\[
\int_{\hat{\Sigma} \Omega} d\mu \nabla f_k (R_2 - R_1) = \int_{\hat{\Sigma} \Omega'} d\mu \nabla f_k (R_2 - R_1)
\leq \int_{\hat{\Sigma} \Omega'} \| \nabla f_k \| |K^{R_2 - R_1}| = \int_{\hat{\Sigma} \Omega'} \| \nabla f_k \| |K^Z|.
\]
Since $\int_{\hat{\Sigma} \Omega'} |K^Z| < +\infty$ by Lemma 5.3, our result follows if we show that
\[
\sup_{\Omega} \| \nabla f_k \| \to 0 \text{ as } k \to +\infty.
\]
To show this latter claim, we will show that $\sup_{\Omega} \| \nabla f_k \| \leq \frac{M}{k}$ for $k \in \mathbb{N}$. For, the compact subset $\text{supp}(b_k)$ can be covered by finitely many orbifold charts. On each of these charts, we can assume that we have an approximation with $\sup \| \nabla f_k \| \leq \frac{M}{k}$, $k \in \mathbb{N}$, because we first approximate on its lift, and then average over the chart group. Each function (and each function’s gradient) in the average will have the same sup. Moreover, on the lift of each chart we can assume that the gradient of the distance function is bounded by $\frac{M}{k}$ except on a (singular) set of zero measure and codimension at least 2. Hence by possibly performing a cutting and limiting procedure around the singular locus, a smooth approximation with the required bound can be found. □

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