A NEW TREATMENT OF NEUTRINO OSCILLATIONS IN MEDIUM

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ABSTRACT

A new more rigorous and accurate method for treating neutrino oscillations in the context of the MSW effect in a medium is proposed. This leads to a new type of resonance condition which for small mixing angles puts rather stringent conditions on $E_\nu/\delta m^2$. The implications on the solar neutrino problem are discussed.
One of the most important questions of the physics beyond the standard model is the problem of neutrino masses. Furthermore if the neutrinos are massive, the neutral leptons produced in weak interactions are not stationary. They are linear combinations of the neutrino mass eigenstates (neutrino mixing). Even though at present there is no theory which can predict the mass and mixing of the neutrinos, most Grand Unified Models predict small masses and mixing. If the neutrinos are almost degenerate, neutrino oscillation experiments are the best candidates to measure small $\delta m^2$ (from $1eV^2$ down to $10^{-10}eV^2$). Furthermore, neutrino oscillations may explain the solar neutrino problem, i.e. the apparent reduction of the $\nu_e$ flux at earth compared to that predicted by the standard solar model (SSM). The mechanism of neutrino oscillations however, is not effective if the neutrino mixing is small. It has been observed, though, that under the conditions of high density encountered in the sun’s interior the oscillation can be enhanced due to the MSW effect. In other words small mixing angles can be converted into large effective mixing angles due to the resonant scattering of $\nu_e$ neutrinos by electrons. Calculations of neutrino oscillations involving the MSW effect have hitherto involved the following steps:

1) Convert the evolution equation from flavor space into neutrino mass eigenstate basis by locally diagonalizing the space dependent Hamiltonian.

2) Ignore the transitions between the mass eigenstates (adiabatic approximation) or treat such transitions in perturbation treatment. In the last case one assumes that

$$\frac{\delta E_\nu}{d\theta_m/dx} \ll 1$$

where $\delta E_\nu$ is the difference in neutrino energies and $d\theta_m/dx$ the variation of the mixing angle in the medium with distance. In particular the above equation must be true at the resonance point.

In the present paper we will provide an exact solution which does not go through the local neutrino mass eigenstates. Our method is quite simple and offers itself to a simple interpretation.

For illustration purposes we will exhibit our method in the case of two generations but it can easily be extended to any number of generations.

Following standard procedure we can write the neutrino state at any time as

$$|\nu(t) > = a_e(t)|\nu_e > + a_\alpha(t)|\nu_\alpha >$$

where $|\nu_e >$ is the electron neutrino and $\nu_\alpha$ any other flavor (e.g. $\nu_\mu$). The
amplitudes \(a_e(t)\) and \(a_\alpha(t)\) satisfy the evolution equation

\[
\frac{id}{dt}\begin{pmatrix} a_e \\ a_\alpha \end{pmatrix} = \mathcal{H}(t)\begin{pmatrix} a_e \\ a_\alpha \end{pmatrix}. \tag{2}
\]

Since \(x = ct\), the above equation can be written in terms of \(x\). In the presence of matter it can be shown that \(\mathcal{H}\) can be cast in the form

\[
\mathcal{H} = \begin{bmatrix} E_\nu - \frac{\pi}{\ell} \cos 2\vartheta + \frac{2\pi}{\ell_0(x)} & \frac{\pi}{\ell} \sin 2\vartheta \\ \frac{\pi}{\ell} \sin 2\vartheta & E_\nu + \frac{\pi}{\ell} \cos 2\vartheta \end{bmatrix} \tag{3}
\]

where

\[
\ell = \frac{4\pi E_\nu}{\delta m^2} = 2.476 km \left(\frac{E_\nu/1 MeV}{(\delta m^2/1eV^2)}\right), \quad \delta m^2 = m_\alpha^2 - m_e^2 \tag{4}
\]

\(\vartheta\) is the usual (vacuum) mixing angle and \(\ell_0(x)\) takes into account the fact that the charged current interaction between \(\nu_e\) and electrons causes a shift in the electron neutrino energy. \(\ell_0(x)\) takes the form

\[
\ell_0(x) = \frac{4\pi}{2\sqrt{2} G_F \rho_e(x)} \tag{5}
\]

where \(G_F = 1.16636 \times 10^{-5} GeV^{-2}\) is the Fermi coupling constant, and \(\rho_e(x)\) in the number of electrons per unit volume at distance \(x\) from the sun’s center which is assumed to be spherically symmetric. Equation (2) can easily be integrated to yield

\[
\begin{pmatrix} a_e \\ a_\alpha \end{pmatrix} = \exp(-i \mathcal{A}) \begin{pmatrix} a_e \\ a_\alpha \end{pmatrix}_0 \tag{6}
\]

where \(\begin{pmatrix} a_e \\ a_\alpha \end{pmatrix}_0\) is the initial solution of the differential equation and the matrix \(\mathcal{A}\) is given by

\[
\mathcal{A} = \frac{\pi(x - x_0)}{\ell} \begin{bmatrix} 2 \xi - \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{bmatrix} \tag{7}
\]

where

\[
\xi = \frac{\ell}{(x - x_0)} \int_{x_0}^{x} \frac{dx'}{\ell_0(x')}. \tag{8}
\]

The diagonal matrix with elements \(E_\nu(x - x_0)\) has been omitted as irrelevant.

Using the well known fact that the minimum polynomial associated with the matrix \(\mathcal{A}\) is of degree one, which permits \(\exp(-i \mathcal{A})\) to be written as a linear combination of the identity matrix and \(\mathcal{A}\), plus the fact that

\[
|\nu(x_0) > = |\nu_e >
\]
it is straightforward to show that
\[ P(\nu_e \rightarrow \nu_e) = 1 - \frac{\ell_m^2}{\ell^2} \sin^2 2\vartheta \sin^2 \pi \frac{(x - x_0)}{\ell_m} \] (9)

where
\[ \ell_m = \ell_m(x, x_0) \]
\[ = \ell \]
\[ = \ell \sqrt{1 + \xi^2 - 2\xi \cos 2\vartheta} \]
\[ = \ell \sqrt{(\xi - \cos 2\vartheta)^2 + \sin^2 2\vartheta} \] (10)

\( \ell_m \) can be interpreted as the oscillation length. In the presence of matter, however, \( \ell_m \) is a function not only of the combination \( (E_\nu/\delta m^2) \), but also of \( \vartheta, x \) (detector point) and \( x_0 \) (source point). One can also talk about an effective mixing angle \( \vartheta_m \) defined by
\[ \sin^2 \vartheta_m(x, x_0) = \frac{\ell_m(x, x_0)}{\ell} \sin^2 \vartheta \] (11)

From eq.(11) it can easily be seen that the maximum oscillation probability occurs when
\[ \xi = \cos 2\vartheta, \quad ("\text{resonance" condition}) \] (12)

In this case
\[ \sin^2 \vartheta_m(x, x_0) = 1, \quad \ell_m(x, x_0) = \ell/\sin^2 \vartheta \] (13)

while eq.(9) becomes
\[ P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 \{ \pi \frac{x - x_0}{\ell/\sin^2 \vartheta} \} \] (14)

The above condition (12) is reminiscent but not similar to the usual resonance condition (see eqs (25) and (26) below) which occurs at some appropriate point \( x_R \). In our exact treatment the “resonance” condition (eq. (12)) involves both the initial \( (x_0) \) and the final \( (x) \) positions. It is affected by the cumulative effect of the density of the medium and not by its value at some appropriate point \( x_R \). It is a sort of “global resonance” condition. For numerical calculations we will cast \( \xi \) in the form
\[ \xi = \frac{t_0}{(x - x_0)} \int_{x_0}^x \frac{\rho_e(x')}{\rho_0} dx' \] (15)

where
\[ t_0 = \frac{2\sqrt{2} G_F E_\nu \rho_0}{\delta m^2} \] (16)
with $\rho_0$ the sun’s density at some suitable point (e.g. at its center). The gross features of the resonance are affected by $t_0$, while its details depend on the specific form of $\rho_e(x)/\rho_0$. The value of $t_0$ which is consistent with the present solar neutrino data, lies in the range [15]

$$0.5 \leq t_0 \leq 50$$

(17)

This range maybe enlarged however ($\sim 0.2 - 60$), if also uncertainties on the density $\rho_0$ are also included [12]. The resonance condition imposes constraints on the parameters $x_0, \delta m^2$ and $E_\nu(x)$ is assumed to be fixed, i.e. $x = \text{sun-earth distance})$. The half maximum width is given by

$$\Gamma = 2\sin2\theta$$

(18)

which for small mixing angle puts stringent constraints on $t_0$ or equivalently on the allowed values of $E_\nu/\delta m^2$.

Before proceeding further with the discussion of our results we will compare the above new formulas with those which have been obtained with the traditional approach. By diagonalizing the matrix of eq. (3) for each value of $x$ we obtain the eigenvalues [11] [13]

$$\lambda_\pm = E_\nu + \frac{\pi}{\ell_0} \pm \frac{\pi}{\ell_m(x)}$$

(19)

and the eigenvectors

$$|\nu_L(x)\rangle = \cos\vartheta_m(x)|\nu_e\rangle - \sin\vartheta_m(x)|\nu_\alpha\rangle$$

(20)

$$|\nu_H(x)\rangle = \sin\vartheta_m(x)|\nu_e\rangle + \cos\vartheta_m(x)|\nu_\alpha\rangle$$

(21)

with

$$\ell_m(x) = \ell/[1 - 2\cos2\theta \frac{\ell}{\ell_0(x)} + (\frac{\ell}{\ell_0(x)})^2]^{1/2}$$

(22)

$$\sin2\vartheta_m(x) = \sin2\theta/[1 - 2\cos2\theta \frac{\ell}{\ell_0(x)} + (\frac{\ell}{\ell_0(x)})^2]^{1/2}$$

(23)

The resonance in this approach occurs when the diagonal elements of $\mathcal{H}$ of eq.(3) are equal [9] [10], i.e. at a point $x_R$ in the medium such that

$$\ell_0(x_R)\cos2\theta = \ell.$$ 

(24)

At the resonance one finds

$$\sin2\vartheta_m(x_R) = 1$$

$$\ell_m(x_R) = \ell/\sin2\theta$$

(25)

(26)
Then by writing

\[ |\nu(x)| = a_L(x)|\nu_L(x)| + a_H|\nu_H(x)| \]

we obtain the following evolution equation

\[ \frac{d}{dt} \begin{pmatrix} a_L(x) \\ a_H(x) \end{pmatrix} = \mathcal{B}(x) \begin{pmatrix} a_L(x) \\ a_H(x) \end{pmatrix} \]

where

\[ \mathcal{B} = \begin{bmatrix} \lambda_+ & -\gamma \frac{d\vartheta_m}{dx} \\ \lambda_- & \gamma \frac{d\vartheta_m}{dx} \end{bmatrix} \]

Following a procedure analogous to that of our new method outlined above, the oscillation probability takes the form

\[ \mathcal{P}(\nu_e \rightarrow \nu_e) = \frac{1}{2} \left\{ 1 + \cos(2\vartheta_m(x) - 2\vartheta_m(x_0)) \right\} - \sin^2\vartheta_m(x) \sin^2 \left\{ \frac{\pi x - x_0}{\ell_m(x)} \right\} \]

where

\[ \gamma = \vartheta_m(x) - \vartheta_m(x_0) \]

and

\[ \beta = \pi \int_{x_0}^{x_s} \frac{dx}{\ell_m(x)} \]

If the sun’s density is discontinuous at its surface, we get

\[ \beta = \pi \left[ \int_{x_0}^{x_s} \frac{dx}{\ell_m(x)} + \frac{x - x_s}{\ell} \right] \]

where \( x_s \equiv R_\odot \) is the sun’s radius. We notice that the quantities \( \vartheta_m(x) \) (and \( \gamma \)) and \( \ell_m(x) \) are not independent quantities but they are given by eqs(25 and 26). In the special case \( \gamma \ll \beta \) (adiabatic approximation) we obtain

\[ \mathcal{P}(\nu_e \rightarrow \nu_e) = \frac{1}{2}\left\{ 1 + \cos(2\vartheta_m(x) - 2\vartheta_m(x_0)) \right\} - \sin^2\vartheta_m(x) \sin^2 \left\{ \frac{\pi x - x_0}{\ell_m(x)} \right\} \]

which coincides with the old expression[12]. Furthermore in the absence of matter effects i.e. when \( \vartheta_m(x) = \vartheta_m(x_0) = \vartheta \) we obtain once again the well known formula[12]

\[ \mathcal{P}(\nu_e \rightarrow \nu_e) = 1 - \sin^2 \vartheta \sin^2 \left\{ \pi \frac{x - x_0}{\ell} \right\} \]
On the other hand if $\vartheta_m(x) = \vartheta$ and $\sin 2\vartheta_m(x_0) = \sin 2\vartheta(x_R) = 1$ (i.e. the resonance condition occurs at $x_0$) one obtains for small values of $\vartheta$

$$\mathcal{P}(\nu_e \rightarrow \nu_e) \approx \frac{1}{2} - \sin 2\vartheta \sin^2 \beta \simeq \frac{1}{2}$$

(36)

The comparison of our treatment with the non-adiabatic treatment of the earlier perturbative calculations [12, 13] is not obvious. Level crossings [17] etc do not enter in our non-perturbative treatment which avoids the intermediate step of the eigenstates of eqs.(20-21). We only notice that in the limit $\gamma \gg \beta$ our expression (30) becomes

$$\mathcal{P}(\nu_e \rightarrow \nu_e) \simeq 1$$

(37)

which means no oscillations in this limit (see also eq(91) of ref.[12]).

Returning back to our new method we repeat that in the case of small mixing angles for a resonance to occur, eqs(12,13) impose a stringent condition on the properties of the neutrinos. Since in the solar neutrino experiments the sun-earth distance $x$ is fixed we expect that the resonance will occur for special values of $x_0$. To test this we will consider a reasonable model for the solar electron density employed by Lim and Marciano [18], i.e. we take

$$\frac{\rho_e(x)}{\rho_0} = \begin{cases} 1 - a \frac{|\vec{x}|}{R_\odot}, & 0 < |\vec{x}| < k_1 R_\odot \\ (1 - ak_1) \exp\{c(k_1 - \frac{|\vec{x}|}{R_\odot})\}, & |\vec{x}| > k_1 R_\odot \end{cases}$$

(38)

with $k_1 = 0.2$, $a = 10/3$, and $c = 100/9$. The value of $\rho_0$, absorbed in the definition of $t_0$ (see 18) was chosen

$$\rho_0 = 6 \times 10^{25} cm^{-3}$$

(39)

With the above parameters we have performed calculations of $\mathcal{P}(\nu_e \rightarrow \nu_e)$ as given by eq.(3) near the resonance [12] for various values of $x_0$. The obtained results for $\sin^2 2\vartheta = 2m_e/m_\mu \approx 1.1 \times 10^{-2}$ and $x_0 = 0$, $x_0 = 0.1R_\odot$ and $x_0 = 0.2R_\odot$ are shown in figs (1-3). The corresponding values of $E_\nu/\delta m^2$ in units of $MeV - eV^{-2}$ are also indicated. It is clear that the oscillation probability becomes negligible for regions away from the resonance. Our results are similar to those obtained for $x_0 = 0$ in the context of the earlier treatment by Rosen and Gelb except that:

i) The value of $E_\nu/\delta m^2$ at resonance is in our approach about a factor of 2 smaller. This, however, can be accounted for by the fact that they use a different electron density function.

ii) The width of our plots of oscillation probability as a function of $E_\nu/\delta m^2$ is quite a bit narrower.
We are currently analysing the solar neutrino data employing our new formalism for a number of solar density profiles. Detailed results will be published elsewhere. At present we note that for suitable values of $E_\nu/\delta m^2$ the resonance condition for a small mixing angle is satisfied in a small region in the $x_0$ - space. One must appropriately integrate over $x_0$ [15]. If one does this, one expects that even though some neutrinos may arrive at earth with maximal oscillation probability as given by eqs (14) and (13) the fraction of such neutrinos will be small. A rough estimate can be given as follows

$$\frac{N_\nu(\text{res.})}{N_\nu(\text{tot.})} = \frac{\int \mathcal{P}(\nu_e \rightarrow \nu_e) \rho(x_0) dx_0}{\int \rho(x_0) dx_0}$$

which in the interesting case of small mixing, i.e. $x \sin^2 \vartheta \ll \ell$ one finds

$$\frac{N_\nu(\text{res.})}{N_\nu(\text{tot.})} \simeq 1 - \frac{x^2}{\ell^2} \sin^2 2\vartheta$$

If this is borne out by more detailed calculations in which integration over the neutrino energy is also performed [16] [17] it will imply that in fact the medium may have a negligible effect on neutrino oscillations.

In conclusion we have presented a novel way of treating neutrino oscillations in a medium which is both simpler and more accurate than the traditional perturbative approach. Our method can be easily extended in the case of the three generations. Our results given above are not expected to be drastically altered by such an extension.

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Figure Captions

Figure 1: Oscillation probability at Earth for a neutrino created at the center of the sun \((x_0 = 0)\), a) as a function of the neutrino energy \(E_{\nu}\) divided by \(\delta m^2 = m^2_{\nu_\alpha} - m^2_{\nu_e}\) and, b) as a function of the distance from the sun.

Figure 2: same as in figure 1 but for \(x_0 = 0.1R_\odot\)

Figure 3: same as in figure 1 but for \(x_0 = 0.2R_\odot\)
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