Flat 3-Brane with Tension in Cascading Gravity

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In the Cascading Gravity brane-world scenario, our 3-brane lies within a succession of lower-codimension branes, each with their own induced gravity term, embedded into each other in a higher-dimensional space-time. In the 6+1-dimensional version of this scenario, we show that a 3-brane with tension remains flat, at least for sufficiently small tension that the weak-field approximation is valid. The bulk solution is nowhere singular and remains in the perturbative regime everywhere.

An old idea to address the vexing problem of the cosmological constant is to confine the visible universe on a 3-brane in a higher-dimensional space-time: vacuum energy on the brane curves the extra dimensions, but leaves the 4d geometry flat \cite{1}. While tantalizing, this proposal fails as soon as the extra dimensions are compactified; since 4d general relativity is recovered below the compactification scale, the theory unavoidably succumbs to Weinberg’s no-go theorem \cite{2}. (An alternative strategy is to use compact extra dimensions to suppress radiative corrections to the cosmological constant \cite{3}.)

The situation is drastically different, and far more promising, if the extra dimensions have infinite volume \cite{4}. In this case, gravity is approximately 4d only at short distances, thanks to an Einstein-Hilbert term on the brane, but becomes higher-dimensional in the infrared. In the DGP scenario \cite{5} with one extra dimension, the gravitational force law on the brane scales as the usual $1/r^2$ at short distances, but asymptotes to $1/r^3$ at large distances. Gravity therefore behaves as a high-pass filter \cite{6}. This weakening of gravity suggests that vacuum energy, by virtue of being the longest-wavelength source, only appears small because it is degravitated \cite{6,7}.

The degravitation phenomenon is not realized in the original DGP model because the weakening of the force law is too shallow in the infrared \cite{7}. This motivates exploring higher-codimension branes, i.e., a higher-dimensional bulk. Realizing these higher-codimension scenarios has proven difficult. To begin with, the simplest constructions are plagued by ghost instabilities \cite{8,9}. Secondly, the 4d propagator is divergent and must be regularized \cite{10}. Furthermore, for a static bulk, the geometry for codimension $N > 2$ has a naked singularity at finite distance from the brane, for arbitrarily small tension \cite{11}. (Allowing the brane to inflate gives a Hubble rate on the brane inversely proportional to the brane tension for codimension $N > 2$ \cite{11}.)

Recently, it was argued that these pathologies are resolved by embedding our 3-brane within a succession of higher-dimensional branes, each with their own induced gravity term \cite{11,12,13}. We refer to this framework as Cascading Gravity. The induced graviton kinetic term acts as a regulator for the 3-brane propagator \cite{11,12}. In the case $N = 2$ studied in \cite{11}, consisting of a 3-brane embedded in a 4-brane within a 5+1-dimensional bulk, the ghost is cured by including a sufficiently large tension on the (flat) 3-brane \cite{11,14}. Alternatively, the ghost is also cured when considering a higher-dimensional Einstein-Hilbert term localized on the brane, \cite{9,12}.

Already with $N = 2$ the solution exhibits degravitation: a 3-brane with tension creates a deficit angle in the bulk while remaining flat \cite{13}. We stress that this self-tuning mechanism crucially relies on the extra dimensions having infinite volume: if the dimensions were compact, the brane tension would have to be tuned against other branes and/or bulk fluxes \cite{14}.

Since the deficit angle must be less than $2\pi$, the tension allowed by the solutions considered in \cite{11,14} is bounded by $M_6^4$, where the 6d Planck mass $M_6$ is itself constrained to be at most $\sim \mathrm{meV}$. Given its geometrical nature, this bound is most likely an artifact of the codimension-2 case and is expected to be absent in higher-codimension.

Motivated by these considerations, in this Letter we explore Cascading Gravity with $N = 3$, consisting of a 3-brane living on a 4-brane, itself embedded in a 5-brane, together in a 6+1-dimensional bulk, as sketched in Fig. 1. Including tension on the 3-brane, we derive a solution for which i) the bulk metric is non-singular everywhere (except, of course, for a delta-function in curvature at the 3-brane location) and asymptotically flat; and ii) the induced 3-brane geometry is exactly flat.

Since the metric depends on 3 spatial coordinates, to proceed analytically we restrict ourselves to the weak-
field approximation, corresponding to sufficiently small tension. For consistency, we check that our solution remains perturbative everywhere. We are currently working on numerically extending these solutions to the non-linear regime of large tension.

Unlike the case of a pure codimension-3 DGP brane in 6+1 dimensions, where the static bulk geometry has a naked singularity for arbitrarily small tension [4], here the bulk metric is completely smooth. This traces back to the cascading mechanism of regulating the propagator: the presence of parent branes removes the power-law divergence in the 4d propagator.

As illustrated in Fig. 1, the 3 extra spatial dimensions are denoted by \( y, z, w \), with the codimension-1 brane located at \( w = 0 \), the codimension-2 brane at \( z = w = 0 \), and the codimension-3 brane at \( y = z = w = 0 \). Indices in 7d are denoted by \( A, B, \ldots \), indices in 6d by \( a, b, \ldots \), and finally in 4d by \( \mu, \nu, \ldots \).

### I. Scalar Green’s Functions

In solving for the metric perturbations, it is useful to first consider the scalar Green’s functions, determined from the action

\[
S = \frac{1}{2} \int d^7 x \, \sqrt{g} \left[ R - \frac{1}{2} \left( \frac{\partial \Phi}{\partial x^a} \right)^2 - \frac{1}{2} \kappa^2 \sqrt{g} \Phi^2 \right],
\]

marking, respectively, the transition scale from 4d to 5d, from 5d to 6d, and finally from 6d to 7d.

In the absence of the 5d and 4d kinetic terms, the propagator on the codimension-1 brane is of the DGP form [4]

\[
G_6^{(0)}(z - z') = \frac{1}{M_6^3} \int \frac{d\omega}{2\pi} \frac{e^{i\omega(z - z')}}{\omega^2 + q^2 + m_7 \sqrt{q'^2 + \omega^2}}.
\]

(For \(|q| > m_7\) we assume analytic continuation from hyperbolic tangent to its trigonometric counterpart.)

Remarkably, the codimension-1 kinetic term makes the 5d propagator finite, thereby regulating the logarithmic divergence characteristic of pure codimension-2 branes. Indeed, \( G_5^{(0)} \rightarrow M_5^{-5} \log(m_7 q) \) as \( M_6 \rightarrow 0 \), and thus \( M_6 \) plays the role of a physical cut-off. As another check, note that in the limit \( m_7 \rightarrow 0 \) in which the bulk decouples, we recover the usual DGP result: \( G_5^{(0)} \sim 1/(q^2 + m_6^2) \).

It is straightforward to repeat the same steps to derive the induced 4d propagator on the codimension-3 brane.

### II. Cascading Gravity

We now proceed to the gravitational case. The 7d Einstein equations are given by

\[
M_7^5 G_{AB}^{(7)} = -\delta(w) \left\{ \delta_A^\alpha \delta_B^\beta M_6^4 G_{\alpha\beta}^{(6)} + \delta(z) \delta_A^\alpha \delta_B^\beta M_5^2 G_{\alpha\beta}^{(5)} \right\} + \delta(z) \delta(y) \delta(A)^\mu \delta(B)^\nu \left[ M_4^2 \mathcal{G}_{\mu\nu}^{(4)} + M_6 A_{\mu\nu} \right],
\]

The effective source therefore consists of induced gravity terms on each of the branes, as well as tension \( \Lambda \) on the codimension-3 brane.

In the weak-field approximation, the 7d line element can be written as \( ds^2 = \eta_{AB} dx^A dx^B \). As shown in Appendix I, there is enough symmetry and gauge freedom to simplify the metric to the form

\[
ds^2 = (1 + \Phi(y, z, w)) \left( dw^2 + dz^2 + dy^2 \right) - \frac{\Theta(w)}{2m_7} \partial_w \Phi(y, z) dw + \left( 1 - \frac{\Phi(y, z, w)}{4} \right) \eta_{\mu\nu} dx^\mu dx^\nu,
\]

where \( \Phi_0(y, z) \equiv \Phi(y, z, w = 0) \) is the induced profile on the codimension-1 brane. Here \( \Theta(w) \) is the theta function: \( \Theta(w) = +1 \) for \( w > 0 \), and \( -1 \) for \( w < 0 \).

Substituting this ansatz into Einstein’s equations [7], we find that \( \Phi \) satisfies

\[
\left( \Box_7 + \frac{\delta(w)}{m_7} \Box_6 - \frac{3}{5} \delta^2(z, w) \Box_5 \right) \Phi = \frac{8}{5} \delta^2(y, z, w) M_7^2 \Lambda.
\]

This equation is of the cascading form [12], as reviewed above. The asymptotically flat bulk solution is given by

\[
\Phi(y, z, w) = e^{-|w|\sqrt{-\Box_6}} \Phi_0(y, z),
\]

where the induced profile \( \Phi_0(y, z) \) satisfies

\[
\left( \Box_6 - m_7 \sqrt{-\Box_6} - \frac{3}{5} m_7 \Box_5 \right) \Phi_0 = \frac{8}{5} \delta^2(y, z) M_6^2 \Lambda.
\]

To solve [11], we Fourier transform to momentum space and use the 6d and 5d Green’s functions given respectively by [3] and [5]. The result is

\[
\Phi_0(y, z) = \int \frac{dq_y dq_z dq_w}{(2\pi)^2} \frac{e^{i q_y y} e^{i q_z z} g(q_y) \phi(q_y)}{\omega^2 + q_y^2 + m_7 \sqrt{\omega^2 + q_y^2}},
\]
where the Fourier transform of the codimension-2 profile, 
\( \phi(q_y) = \int \text{d}y e^{-iq_yy}\Phi_0(y = 0, y) \), satisfies
\[
\left( \frac{3}{5} q_y^2 - g(q_y^2) \right) \phi(q_y) = \frac{8}{5M_5^2} \Lambda. \tag{13}
\]
The solution to (13) can be expressed as the sum of a principal part \( P \) and two homogeneous modes:
\[
\phi(q_y) = \frac{8\Lambda}{5M_5^2} P \left[ \frac{1}{2 q_y^2 - g(q_y^2)} \right] + \sum_{\sigma = \pm} C_\sigma \delta(q_y - \sigma q_0),
\]
where \( \frac{3}{5} q_y^2 = g(q_y^2) \). Requiring the field \( \Phi_0 \) to be real imposes \( C_+ = C_- \equiv C \), while requiring \( \Phi_0 \) to fall as \( y \to 0 \) sets \( C = 0 \). Using the resulting expression for \( \phi(q_y) \) into (12) and then into (10), we obtain the final expression for the scalar potential \( \Phi(y, z, w) = \frac{8\Lambda}{5M_6} \hat{\Phi}(y, z, w) \):
\[
\hat{\Phi} = \int \frac{\text{d}\omega dq_y}{(2\pi)^2} e^{-|w| \sqrt{\omega^2 + q_y^2}} e^{i\omega z} \frac{g(q_y)}{q_y} \frac{1}{2 q_y^2 - g(q_y^2)} P \left[ \frac{1}{2 q_y^2 - g(q_y^2)} \right]. \tag{14}
\]
This is our main result. Thanks to the cascading mechanism, which has regularized all potential divergences, this solution is finite everywhere. Figure 2 shows that \( \hat{\Phi}(y, z, w) \) is smooth everywhere and decreases with \( w \).

As it stands, however, our framework has a ghost [8, 9], as indicated by the poles at \( q_y = \pm q_0 \). Following this route, we demonstrate sufficiently large tension on both the codimension-2 and -3 branes [11]: to remove the ghost, the codimension-2 tension should be \( \gtrsim M_5^2 m_7^2 \), whereas the corresponding bound on the codimension-3 tension is yet to be determined. Additionally, one can regularize codimension-2 and -3 branes and include 6d Einstein-Hilbert term localized on these objects [10, 12]. Following this route, we demonstrate in Appendix II that the poles do disappear, and that the profile for \( \Phi(y, z, w) \) is qualitatively unchanged. III. Discussion: In this Letter we have shown that a 3-brane with tension remains flat in the 6+1-dimensional cascading gravity framework. In the weak-field approximation, we have obtained a bulk solution which is nowhere singular and remains perturbative everywhere.

These properties crucially depend on the existence of parent branes with finite Planck masses. Indeed, our solution goes outside the perturbative regime and acquires divergences in the limit \( M_5, M_6 \to 0 \), consistent with [4].

We are currently extending our solutions to the non-linear regime through numerical analysis. For now, we view the present results as a tantalizing first step towards realizing the idea of Rubakov and Shaposhnikov.

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Appendix I: We show that the weak-field metric can be brought to the form [8] by symmetry and gauge freedom. In de Donder gauge, \( \partial_A h_{AB} = \frac{1}{2} \partial_B h^{C}_{C} \), [7] reduces to
\[
- \frac{M_7^2}{2} \Box \left( h_{AB} - \frac{1}{2} \eta_{AB} h_{C}^{C} \right) = \delta(w) \left( T^{(6)}_{ab} - M_6^4 C^{(6)}_{ab} \right),
\]
where the effective stress-energy on the codimension-1 brane, \( T^{(6)}_{ab} \), includes contributions from the 5d and 6d induced gravity terms. Since there is no stress energy along the \((a, w)\) and \((w, w)\) directions, the corresponding equations are consistently satisfied by setting \( h_{aw} = 0 \) and \( h_{ww} = h_{w} \), whereas \( h_{w} \) is the 6d trace. It follows that the induced gauge choice in 6d is given by \( \partial_a h_{ab} = \partial_b h_{w} \), hence the \((a, b)\) components of \( \delta(w) \)
\[
- \frac{M_7^2}{2} \Box (h_{ab} - \eta_{ab} h_{C}^{C}) = \delta(w) \left( T^{(6)}_{ab} - M_6^4 C^{(6)}_{ab} \right), \tag{15}
\]
To proceed further, it is convenient to decompose \( h_{ab} \) into its trace and transverse-traceless (TT) parts:
\[
h_{ab} = h^{6\text{TT}}_{ab} + \partial_a \partial_b h_{w}^{w}. \tag{16}
\]
From (15), the 6dTT components satisfy
\[
- \frac{M_7^2}{2} \left( \Box + \frac{\delta(w)}{M_7} \right) h^{6\text{TT}}_{ab} = \delta(w) \left( T^{(6)}_{ab} - \frac{1}{5} \eta_{ab} T^{(6)} + \frac{1}{5} \Box T^{(6)} \right). \tag{17}
\]
The symmetries of the problem allow a simple expression for the 5d components of the 6dTT part:

$$h^{6dTT}_{\alpha\beta} = -\frac{1}{4} \Phi \eta_{\alpha\beta} - \left( \frac{\Box_5}{\Box_6} - \frac{5}{4} \frac{\partial_5 \partial_6}{\Box_5} \Phi \right).$$

(18)

This follows from setting $h^{6dTT}_{\alpha\beta} = 0$, which is consistent with the equations of motion for the case of interest. Substituting into (17), and using $T^{(5)}_{\alpha\beta} = -\delta_5^\alpha \delta_5^\beta \eta_{\mu\nu} \delta(y)$, the resulting equation of motion for $\Phi$ agrees with [9].

We can now be explicit about the form of the various metric components. Combining (16) and (18), we get:

$$h_{\alpha\beta} = -\frac{1}{4} \Phi \eta_{\alpha\beta} - \left( \frac{\Box_5}{\Box_6} - \frac{5}{4} \frac{\partial_5 \partial_6}{\Box_5} \Phi \right) = 0 \text{ and } h_{\mu\nu} = -\frac{1}{4} \Phi \eta_{\mu\nu}. \text{ Similarly, from (16) we obtain:}$$

$$h_{yz} = \frac{\partial_y \partial_z}{\Box_6} (h^{c}_{-} - \Phi) ; h_{zz} = \frac{\partial^2}{\Box_6} (h^{c}_{-} - \Phi) + \Phi;$$

$$h_{yy} = \frac{\partial^2}{\Box_6} (h^{c}_{-} - \Phi) + \Phi. \text{ (20)}$$

The is equivalent to [8] after a small diffeomorphism.

**Appendix II:** One way to cure the ghost of higher-codimension DGP models [10] [12] is to consider a higher-dimensional Einstein-Hilbert term localized on the regularized brane [8] [12]. Following this prescription, we will show that the solution remains finite everywhere.

When adding a 6d Einstein-Hilbert term on the regularized 4-brane, on the top of the usual 5d Einstein-Hilbert term of the form $\Box_5 h_{\alpha\beta}$, we must consider excitations of transverse modes along the extra dimensions as well as the higher-dimensional mode $h_{zz}$. In the thin-brane limit, however, the excitations along the extra dimension become very massive, so that any term containing $z$-derivatives can be neglected. Meanwhile, $h_{zz}$ survives in the limit. See [12] for details.

In 7d de Donder gauge, the Einstein equations are the same as in (15). Setting $h_{aw} = 0$ and $h_{ww} = h^{c}_{w}$, we have

$$-M^2 \frac{7}{2} \left( \Box_7 + \frac{\delta(w)}{m_7} \Box_6 \right) h_{ab} = \delta(w) \left( T^{(6)}_{ab} - \frac{1}{5} T^{(6)} \eta_{ab} \right),$$

with $T^{(6)}_{zz} = 0, T^{(6)}_{zz} = M_5^2 \delta(z) R_5/2$, and

$$T^{(6)}_{\alpha\beta} = -M_5^2 \delta(z) \left[ G^{(5)}_{\alpha\beta} + \frac{1}{2} (\Box_5 h_{zz} \eta_{\alpha\beta} - \partial_\alpha \partial_\beta h_{zz}) \right] - \delta(z) \delta(y) \eta_{\alpha\beta} \partial^\alpha \partial^\beta.$$

(22)

Using this in the 6d part of the Einstein equations, we get $h_{zz} = -\psi$, $\Box_5 h_{yy} = -4 \Box_5 \psi$, and $h_{yy} = 0$ and $\Box_5 h_{\mu\nu} = \Box_5 \psi \eta_{\mu\nu} + \partial_\mu \partial_\nu h^{c}_{w}$, with

$$\left[ \Box_7 + \frac{\delta(w)}{m_7} \Box_6 + \frac{\delta(2)(w, z)}{m_7 m_6} \Box_5 \right] \psi = \frac{2}{5} \frac{\delta(3)(w, z, y)}{M^2} \Lambda.$$

(23)

We notice that the kinetic term for $\psi$ is now everywhere positive, signaling that the ghost has been cured. Equation (23) is similar to [11] for $\Phi$, except for a redefinition of $m_6$ and $M_7$. The profile for $\psi(y, z, w) = -\frac{2A}{5M^2} \tilde{\Psi}(y, z, w), \text{ is similar to that of } \tilde{\Phi}, \text{ and, in particular, is free of divergences. The static solution for a codimension-3 brane with tension remains therefore well-defined, at least in the weak field approximation, in a ghost-free set-up.}$$

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