INTEGRABILITY PROPERTIES OF QUASI-REGULAR REPRESENTATIONS OF NA GROUPS

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Abstract. Let $G = N \rtimes A$, where $N$ is a graded Lie group and $A = \mathbb{R}^+$ acts on $N$ via homogeneous dilations. The quasi-regular representation $\pi = \text{ind}^G_N(1)$ of $G$ can be realised to act on $L^2(N)$. It is shown that for a class of analysing vectors the associated wavelet transform defines an isometry from $L^2(N)$ into $L^2(G)$ and that the integral kernel of the corresponding orthogonal projector has polynomial off-diagonal decay. The obtained reproducing formula is instrumental for obtaining decomposition theorems for function spaces on nilpotent groups.

1. Introduction

Let $N$ be a connected, simply connected nilpotent Lie group and let $A = \mathbb{R}^+$ act on $N$ via automorphic dilations. The semi-direct product $G = N \rtimes A$ acts unitarily on $L^2(N)$ via the quasi-regular representation $\pi = \text{ind}^G_N(1)$ of $G$. For $g \in L^2(N)$, the associated wavelet transform $V_g : L^2(N) \to L^\infty(G)$ is defined as $V_g f(x,t) = (f, \pi(x,t) g)$, $(x,t) \in G$. A vector $g \in L^2(N)$ is said to be admissible if $V_g$ is an isometry from $L^2(N)$ into $L^2(G)$.

Given an admissible vector $g \in L^2(N)$, the orthogonal projector $P$ from $L^2(G)$ onto the closed subspace $V_g(L^2(N)) \subset L^2(G)$ is given by right convolution $P(F) = F \ast V_g g$. In particular, an element $F \in V_g(L^2(N))$, i.e., $F = V_g f$ for some $f \in L^2(N)$, satisfies the reproducing formula

$$V_g f = V_g f \ast V_g g. \quad (1.1)$$

The existence of admissible vectors for irreducible, square-integrable representations $\pi$ is automatic by the orthogonality relations [10], but a non-trivial problem for reducible representations. For $N = \mathbb{R}^d$ and general dilation groups $A \leq \text{GL}(d, \mathbb{R})$, the admissibility of quasi-regular representations is well-studied, see, e.g. [2, 20, 33] and the references therein. For non-commutative groups $N$, the admissibility problem is considered in, e.g. [8, 9, 19, 37].

This note is concerned with admissible vectors that are also integrable: A vector $g \in L^2(N)$ is said to be integrable if $\Delta_G^{-1/2} V_g g \in L^1(G)$, where $\Delta_G : G \to \mathbb{R}^+$ denotes the modular function on $G$. The significance of integrably admissible vectors is that $F := \Delta_G^{-1/2} V_g g$ forms a projection in $L^1(G)$ by (1.1), that is, $F = F \ast F = F^*$, with $F^* := \Delta_G^{-1} F(\cdot^{-1})$.

The construction of projections in $L^1(G)$ arising from matrix coefficients is an ongoing research topic, and such projections provide (if they exist) a powerful tool for studying problems in non-commutative harmonic analysis. Among others, they play a vital role in the theory of atomic decompositions in Banach spaces [12, 27].

For the affine group $G = \mathbb{R} \rtimes \mathbb{R}^+$, the construction of projections in $L^1(G)$ goes back to [11]. The papers [7, 23, 32] consider groups $G = \mathbb{R}^d \rtimes A$ and provide criteria for the explicit construction of projections in $L^1(G)$ based on the dual action of $A$ on $\mathbb{R}^d$; see also [21, 23].

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The techniques of \cite{28,32} were used in \cite{40} for the Heisenberg group \( N = \mathbb{H}_1 \) acted upon by automorphic dilations. For a stratified group \( N \) with canonical dilations, the existence of smooth admissible vectors was investigated in \cite{25}, although not linked to integrability.

The main concern of this note is the integrability of \( \pi = \text{ind}_N^A \) when \( N \) is a (possibly, non-stratified) graded Lie group. The main result obtained is the following:

**Theorem 1.1.** Let \( G = N \rtimes A \), where \( N \) is a graded Lie group and \( A = \mathbb{R}^+ \) acts on \( N \) via automorphic dilations. The quasi-regular representation \( \pi = \text{ind}_N^A(1) \) admits integrably admissible vectors, i.e., there exist vectors \( g \in L^2(N) \) satisfying \( \Delta_G^{-1/2} V_g g \in L^1(G) \) and

\[
\int_G |\langle f, \pi(x,t)g \rangle|^2 \, d\mu_G(x,t) = \|f\|^2_2, \quad \text{for all } f \in L^2(N).
\]

The integrably admissible vector \( g \) can be chosen to be Schwartz with all moments vanishing, in which case \( V_g g \in L^1_a(G) \) for any polynomially bounded weight \( w : G \to [1, \infty) \).

Admissible vectors that are Schwartz with all vanishing moments are known to exist already for stratified Lie groups \cite{25, Corollary 1}. Theorem 1.1 provides a modest extension of this result to general graded Lie groups, and complements it with integrability properties of the associated matrix coefficients. More explicit (point-wise) localisation estimates for the matrix coefficients on homogeneous groups are also obtained; see Section 3 below for details.

The proof method for Theorem 1.1 resembles the construction of Littlewood-Paley functions and Calderón-type reproducing formulae. Most techniques can already be found in some antecedent form in \cite{17} as pointed out throughout the text. Particular use is made of the (non-stratified) Taylor inequality and Hulanicki’s theorem for Rockland operators. The use of a Rockland operator instead of a sub-Laplacian is essential for the proof method as the latter are no longer always homogeneous for non-stratified groups. The exploitation of homogeneity is the reason that the strategy fails for non-graded homogeneous groups (see Remark 4.5).

The motivation for Theorem 1.1 stems from the study of function spaces, and is twofold:

(i) The question whether there exist vectors yielding a reproducing kernel with suitable off-diagonal decay on homogeneous groups was posed in \cite{27, Remark 6.6(a)], where it was mentioned that this is a representation-theoretic problem rather than one of function spaces. The use of such vectors for function space theory, however, is due to the fact that the techniques \cite{27} yield frames and atomic decompositions for Besov-Triebel-Lizorkin spaces. The same holds true for the recent sampling theorems in \cite{38}. The admissible vectors provided by Theorem 1.1 satisfy the integrability conditions assumed in \cite{27,38} (see Section 3.3), and Theorem 1.1 solves the problem mentioned in \cite{27, Remark 6.6(a)] for graded Lie groups.

(ii) The differentiability properties of functions in terms of Banach spaces are well-studied on stratified Lie groups for several classes of spaces, including Lipschitz spaces \cite{10,33}, Sobolev spaces \cite{15,39}, Besov spaces \cite{6,22,39} and Triebel-Lizorkin spaces \cite{17,39}. More recently, there has been an interest in such spaces on possibly non-stratified graded Lie groups, see, e.g. \cite{1,8,14}. This was a motivation to obtain Theorem 1.1 for graded groups, as it allows to apply the techniques \cite{27,38} discussed in (i) to these new classes of spaces. Moreover, even for stratified groups, the integrability properties provided by Theorem 1.1 allow to apply the techniques \cite{38} and bridge a gap between what has been established on the locality of the sampling expansions for stratified groups in \cite{6,22,25,27} and for the classical setting \( N = \mathbb{R}^d \) in \cite{18,26}; see \cite{26,38} for more details on the discrepancy between \cite{27} and \cite{18,26,38}.

The details on the applications of Theorem 1.1 to various functional spaces are beyond the scope of the present paper, and will be deferred to subsequent work.

**Notation.** The open and closed positive half-lines in \( \mathbb{R} \) are denoted by \( \mathbb{R}^+ = (0, \infty) \) and \( \mathbb{R}^+_0 = [0, \infty) \), respectively. For functions \( f_1, f_2 : X \to \mathbb{R}^+_0 \), it is written \( f_1 \lesssim f_2 \) if there exists
a constant $C > 0$ such that $f_1(x) \leq C f_2(x)$ for all $x \in X$. The space of smooth functions on a Lie group $G$ is denoted by $C^\infty(G)$ and the space of test functions by $C^\infty_c(G)$.

2. Preliminaries on homogeneous Lie groups

This section provides background on homogeneous groups. Standard references for the theory are the books [13, 17].

2.1. Dilations. Let $\mathfrak{n}$ be a real $d$-dimensional Lie algebra. A family of dilations on $\mathfrak{n}$ is a one-parameter family $\{D_t\}_{t > 0}$ of automorphisms $D_t : \mathfrak{n} \to \mathfrak{n}$ of the form $D_t := \exp(A \ln t)$, where $A : \mathfrak{n} \to \mathfrak{n}$ is a diagonalisable linear map with positive eigenvalues $v_1, \ldots, v_d$. If a Lie algebra $\mathfrak{n}$ is endowed with a family of dilations, then it is nilpotent.

A homogeneous group is a connected, simply connected nilpotent Lie group $N$ whose Lie algebra $\mathfrak{n}$ admits a family of dilations. The number $Q := v_1 + \ldots + v_d$ is the homogeneous dimension of $N$. The exponential map $\exp_N : \mathfrak{n} \to N$ is a diffeomorphism, providing a global coordinate system on $N$. Dilations $\{D_t\}_{t > 0}$ can be transported to a one-parameter group of automorphisms of $N$, which will be denoted by $\{\delta_t\}_{t > 0}$. The associated action of $t \in \mathbb{R}^+$ on $x \in N$ will often simply be written as $tx = \delta_t(x)$.

A graded group is a connected, simply connected nilpotent Lie group $N$ whose Lie algebra $\mathfrak{n}$ admits an $\mathbb{N}$-gradation as $\mathfrak{n} = \bigoplus_{j=1}^\infty \mathfrak{n}_j$, where $\mathfrak{n}_j$, $j = 1, 2, \ldots$, are vector subspaces of $\mathfrak{n}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{n}_j, \mathfrak{n}_{j'}] \subset \mathfrak{n}_{j+j'}$ for $j, j' \in \mathbb{N}$. If, in addition, $\mathfrak{n}_1$ generates $\mathfrak{n}$, the group $N$ is stratified. Canonical dilations $D_t : \mathfrak{n} \to N$, $t > 0$, can be defined through a gradation as $D_t(X) = t^j X$ for $X \in \mathfrak{n}_j$, $j \in \mathbb{N}$.

Henceforth, a homogeneous group $N$ will be fixed with dilations $D_t := \exp(A \ln t)$. Haar measure will be denoted by $\mu_N$. The eigenvalues $v_1, \ldots, v_d$ of $A$ will be listed in increasing order and it will be assumed (without loss of generality) that $v_1 \geq 1$. In addition, a basis $X_1, \ldots, X_d$ of $\mathfrak{n}$ such that $AX_j = v_j X_j$ for $j = 1, \ldots, d$ will be fixed throughout.

2.2. Homogeneity. A function $f : N \to \mathbb{C}$ is called $\nu$-homogeneous ($\nu \in \mathbb{C}$) if $f \circ \delta_t = t^\nu f$ for $t > 0$. For all measurable functions $f_1, f_2 : N \to \mathbb{C}$,

$$\int_N f_1(x)(f_2 \circ \delta_t)(x) \, d\mu_N(x) = t^{-Q} \int_N (f_1 \circ \delta_{1/t})(x)f_2(x) \, d\mu_N(x)$$

provided the integral is convergent. The map $f \mapsto f \circ \delta_t$ is naturally extended to distributions.

A linear operator $T : C^\infty_c(N) \to (C^\infty_c(N))'$ is said to be homogeneous of degree $\nu \in \mathbb{C}$ if $T(f \circ \delta_t) = t^\nu (Tf) \circ \delta_t$ for all $f \in C^\infty_c(N)$ and $t > 0$.

A homogeneous quasi-norm on $N$ is a continuous function $|\cdot| : N \to [0, \infty)$ that is symmetric, 1-homogeneous and definite. If $|\cdot|$ is a homogeneous quasi-norm on $N$, there is a constant $C > 0$ such that $|xy| \leq C(|x| + |y|)$ for all $x, y \in N$.

2.3. Derivatives and polynomials. A basis element $X_j \in \mathfrak{n}$ acts as a left-invariant vector field on $\mathfrak{n}$ by

$$X_j f(x) = \frac{d}{ds} \bigg|_{s=0} f(x \exp_N(sX_j))$$

for $f \in C^\infty(N)$ and $x \in N$. The first-order left-invariant differential operator $X_j$ is homogeneous of degree $v_j$. For a multi-index $\alpha \in \mathbb{N}_0^d$, higher-order differential operators are defined by $X^\alpha := X_1^{\alpha_1}X_2^{\alpha_2} \cdots X_d^{\alpha_d}$. The algebra of all left-invariant differential operators on $N$ is denoted by $\mathcal{D}(N)$.

A function $P : N \to \mathbb{C}$ is a polynomial if $P \circ \exp_N$ is a polynomial on $\mathfrak{n}$. Denoting by $\xi_1, \ldots, \xi_d$ a dual basis of $X_1, \ldots, X_d$, the system $\eta_j = \xi_j \circ \exp_N^{-1}$, $j = 1, \ldots, d$, forms a global
coordinate system on \( N \). Each \( \eta_j : N \to \mathbb{C} \) forms a polynomial on \( N \), and any polynomial \( P \) on \( N \) can be written uniquely as
\[
P = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \eta^\alpha,
\]
where all but finitely many \( c_\alpha \in \mathbb{C} \) vanish and \( \eta^\alpha := \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_d^{\alpha_d} \) for a multi-index \( \alpha \in \mathbb{N}_0^d \).

The homogeneous degree of \( \alpha \in \mathbb{N}_0^d \) is defined as \( |\alpha| := \sum_{i=1}^d \alpha_i \) and the homogeneous degree of a polynomial \( P \) written as \( \sum c_\alpha \eta^\alpha \) is
\[
d(P) := \max \{ |\alpha| : \alpha \in \mathbb{N}_0^d \text{ with } c_\alpha \neq 0 \}.
\]

For any \( k \geq 0 \), the set of polynomials \( P \) on \( N \) such that \( d(P) \leq k \) is denoted by \( \mathcal{P}_k \).

### 2.4. Schwartz space

A function \( f : N \to \mathbb{C} \) belongs to the Schwartz space \( S(N) \) if \( f \circ \exp_N \) is a Schwartz function on \( \mathfrak{n} \). A family of semi-norms on \( S(N) \) is given by
\[
\| f \|_{S, K} = \sup_{|\alpha| \leq K, x \in N} (1 + |x|)^K |X^\alpha f(x)|,
\]
where \( K \) is sometimes suppressed from the notation \( \| \cdot \|_{S, K} \) and it is simply written \( \| \cdot \|_S \). The closed subspace of \( S(N) \) of functions with all moments vanishing is defined by
\[
S_0(N) = \left\{ f \in S(N) : \int_N x^\alpha f(x) \, d\mu_N(x) = 0, \ \forall \alpha \in \mathbb{N}_0^d \right\}.
\]

For arbitrary \( f \in S(N) \), it will be written \( \hat{f}(x) := f(x^{-1}) \) and \( f_t(x) := t^{-Q} f(t^{-1}x) \) for \( t > 0 \).

The dual space \( S'(N) \) of \( S(N) \) is the space of tempered distributions on \( N \). If \( f \in S'(N) \) and \( \varphi \in S(N) \), the conjugate-linear evaluation is denoted by \( \langle f, \varphi \rangle \). If well-defined, the evaluation is also written as \( \langle f, \varphi \rangle = \int_N f(x) \overline{\varphi(x)} \, d\mu_N(x) \) and extends the \( L^2 \)-inner product. Convolution is defined by \( f * \varphi(x) := \langle f, \varphi(x^{-1}) \rangle \) and \( \varphi * f(x) := \langle f, \varphi(x^{-1}) \rangle \) for \( x \in N \).

### 3. Matrix coefficients of quasi-regular representations

This section is devoted to point-wise estimates and integrability properties of the matrix coefficients of a quasi-regular representation.

### 3.1. Quasi-regular representation

Let \( N \) be a homogeneous Lie group and let \( A = \mathbb{R}^+ \) be the multiplicative group. Then \( A \) acts on \( N \) via automorphic dilations \( A \ni t \mapsto \delta_t \in \text{Aut}(N) \). The semi-direct product \( G = N \rtimes A \) is defined with via the operations
\[
(x, t)(y, u) = (x \delta_t(y), tu), \quad (x, t)^{-1} = (\delta_{t^{-1}}(x^{-1}), t^{-1}).
\]

Identity element in \( G \) is \( e_G = (e_N, 1) \). The group \( G \) is an exponential Lie group, that is, the exponential map \( \exp_G : \mathfrak{g} \to G \) is a diffeomorphism, see, e.g. [19 Proposition 5.27].

The quasi-regular representation \( \pi = \text{ind}_G^H(1) \) of \( G \) acts unitarily on \( L^2(N) \) by
\[
\pi(x, t)f = t^{-Q/2} f(t^{-1}(x^{-1} \cdot)), \quad (x, t) \in N \times A,
\]
for \( f \in L^2(N) \). Note that \( \pi(x, t) = L_x D_t \), where \( L_x f = f(x^{-1} \cdot) \) and \( D_t f = t^{-Q/2} f(t^{-1}(\cdot)) \).

A detailed account on the representation theory of quasi-regular representations of exponential groups can be found in [9, 35, 87], but these results will not be used in this paper.
3.2. **Point-wise estimates.** For $f_1, f_2 \in L^2(N)$, denote the associated matrix coefficient by

$$V_{f_2}f_1(x, t) = \langle f_1, \pi(x, t)f_2 \rangle, \quad (x, t) \in N \times A.$$ 

The following result provides point-wise estimates for a class of matrix coefficients.

**Proposition 3.1.** Let $f_1, f_2 \in S_0(N)$ and $K, M \in \mathbb{N}$ be arbitrary.

(i) For all $(x, t) \in N \times A$ with $t \leq 1$,

$$|V_{f_2}f_1(x, t)| \lesssim t^{Q/2+M}(1 + |x|)^{-K}\|f_1\|_S\|f_2\|_S. \quad (3.1)$$

(ii) For all $(x, t) \in N \times A$ with $t \geq 1$,

$$|V_{f_2}f_1(x, t)| \lesssim t^{-(Q/2+M)}(1 + |x|)^{-K}\|f_1\|_S\|f_2\|_S. \quad (3.2)$$

The implicit constants in (3.1) and (3.2) are group constants that depend further only on $M, K$.

**Proof.** Throughout the proof, a Schwartz semi-norm $\| \cdot \|_{S, N}$ is simply denoted by $\| \cdot \|_N$.

Let $K, M \in \mathbb{N}$ and let $P = P_{x, M} \in P_M$ denote the Taylor polynomial of $f \in S(N)$ at $x \in N$ of homogeneous degree $M$. By Taylor’s inequality [13, Theorem 3.1.51], there exist constants $c, C > 0$ such that for all $x, y \in N$,

$$|f(xy) - P(y)| \leq C \sum_{|\alpha| \leq M'} |y|^{|\alpha|} \sup_{|z| \leq cM' + 1} |(X^\alpha f)(xz)|,$$

where $M' := \max\{|\alpha| : \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq M\}$. For $|\alpha| \leq M' + 1$ and $x, y \in N$,

$$\sup_{|z| \leq cM' + 1} |(X^\alpha f)(xz)| \leq \|f\|_{K+M'+1} \sup_{|z| \leq cM' + 1} (1 + |xz|)^{-K}$$

$$\lesssim \|f\|_{K+M'+1} \sup_{|z| \leq cM' + 1} (1 + |x|)^{-K}(1 + |z|)^K$$

$$\lesssim \|f\|_{K+M'+1}(1 + |x|)^{-K}(1 + |y|)^K,$$

where the second line follows from the Peetre-type inequality [14, Lemma 1.10]. Thus,

$$|f(xy) - P(y)| \lesssim \|f\|_{K+M'+1}(1 + |x|)^{-K} \sum_{|\alpha| \leq M' + 1 \atop |\alpha| > M} |y|^{|\alpha|}(1 + |y|)^K \quad (3.3)$$

for all $x, y \in N$.

(i) Let $(x, t) \in N \times A$ with $t \leq 1$. Then, using that $f_2 \in S_0(N)$,

$$|V_{f_2}f_1(x, t)| = \left| \int_N f_1(xy)D_t \hat{f}_2(y^{-1}) d\mu_N(y) \right| \leq \int_N |f_1(xy) - P(y)||D_t \hat{f}_2(y^{-1})| d\mu_N(y).$$

Applying (3.3) thus gives

$$|V_{f_2}f_1(x, t)| \lesssim \|f_1\|_{K+M'+1}(1 + |x|)^{-K} t^{Q/2} \sum_{|\alpha| \leq M' + 1 \atop |\alpha| > M} \int_N |y|^{|\alpha|} |\hat{f}_2(t^{-1}y^{-1})|(1 + |y|)^K d\mu_N(y)$$

$$= \|f_1\|_{K+M'+1}(1 + |x|)^{-K} t^{Q/2} \sum_{|\alpha| \leq M' + 1 \atop |\alpha| > M} \int_N |ty|^{|\alpha|} |\hat{f}_2(y^{-1})|(1 + |ty|)^K d\mu_N(y)$$

$$\lesssim \|f_1\|_{K+M'+1}(1 + |x|)^{-K} t^{Q/2+M} \int_N |f_2(y)|(1 + |y|)^{K+Q(M'+1)} d\mu_N(y), \quad (3.4)$$
where the last inequality uses $|\alpha| \leq Q|\alpha| \leq Q(M' + 1)$. The integral in (3.4) can be estimated by

$$\int_N |f_2(y)|(1 + |y|)^{K+Q(M' + 1)} d\mu_N(y) \leq \|f_2\|_{L^p(K+Q(M' + 1)+Q+1)} \int_N (1 + |y|)^{-Q-1} d\mu_N(y) \lesssim \|f_2\|_{L^p(K+Q(M' + 1)+Q+1)} ,$$

(3.5)

where convergence of the integral follows by using polar coordinates [17, Proposition 1.15]; see also [17, Corollary 1.17]. A combination of (3.5) and (3.4) yields the desired claim (3.1).

(ii) Note that $|V_{f_2}f_1(x,t)| = |V_{f_2}(x, t)^{-1}|$ for $(x, t) \in N \times A$. Hence, if $t \geq 1$, then it follows by part (i) with $M_0 := M + K$ that

$$|V_{f_2}f_1(x,t)| \lesssim t^{-(Q/2 + M_0)}(1 + t^{-1}|x|)^{-K} \|f_1\|_{L^p(K+M_0+1)} \|f_2\|_{L^p(K+Q(M_0' + 1)+Q+1)}$$

$$\lesssim t^{-Q/2-M} t^{-K} t^K (1 + |x|)^{-K} \|f_1\|_{L^p(K+M_0+1)} \|f_2\|_{L^p(K+Q(M_0' + 1)+Q+1)} ,$$

showing (3.2). This completes the proof.

The estimates provided by Proposition 3.1 recover the well-known polynomial localisation for wavelet transforms when $N = \mathbb{R}$, see, e.g. [29, Section 11-12]. A similar use of the Taylor inequality for (compactly supported) atoms can be found in [17, Theorem 2.9].

3.3. Analysing vectors. Left Haar measure on $G$ is given by $\mu_G(x, t) = t^{-(Q+1)} d\mu_N(x) dt$ and the modular function is given by $\Delta_G(x, t) = t^{-Q}$. The measure $\mu_G$ is used to define the Lebesgue space $L^p(G) = L^p(G, \mu_G)$ for $p \in [1, \infty]$, and $\|\cdot\|_p$ will denote the $p$-norm.

A measurable function $w : G \to [1, \infty)$ is said to be a weight if it is submultiplicative, i.e., $w((x, t)(y, u)) \lesssim w(x, t)w(y, u)$ for $(x, t), (y, u) \in G$. A weight $w$ is called polynomially bounded if

$$w(x, t) \lesssim (1 + |x|)^k (t^m + t^{-m'}) , \quad (x, t) \in G ,$$

(3.6)

for some $k, m, m' \geq 0$. Given such a weight $w$, the weighted Lebesgue space $L^1_w(G)$ consists of all $F \in L^1(G)$ satisfying $\|F\|_{L^1_w} := \|Fw\|_1 < \infty$.

In [12,27,38], the space of $w$-analysing vectors of $\pi$, defined by

$$A_w := \left\{ g \in L^2(N) : V_g g \in L^1_w(G) \right\} ,$$

plays a prominent role.

The following result provides a simple criterion for analysing vectors:

Lemma 3.2. Suppose $g \in S_0(N)$. Then $g \in A_w$ for any polynomially bounded weight function $w : G \to [1, \infty)$. In particular, the representation $\pi = \text{ind}_1^G(1)$ is integrable.

Proof. Let $k, m, m' \geq 0$ be such that $w(x, t) \lesssim (1 + |x|)^k (t^m + t^{-m'})$ for all $(x, t) \in G$. Then, choosing $K, M, M' \in \mathbb{N}$ sufficiently large, it follows by Proposition 3.1 that

$$\|V_g g\|_{L^1_w} \lesssim \int_0^{\infty} \int_N V_g g(x, t)(1 + |x|)^k (t^m + t^{-m'}) d\mu_N(x) \frac{dt}{t^{Q+1}}$$

$$\lesssim \int_0^1 t^{Q/2+M'-m'} t^{-(Q+1)} dt + \int_1^{\infty} t^{-(Q/2+M)+m} t^{-(Q+1)} dt < \infty .$$

This shows that $g \in A_w$, and thus $\pi$ is $w$-integrable.
4. Admissible vectors

A vector \( g \in L^2(N) \) is said to be admissible for the quasi-regular representation \((\pi, L^2(N))\) if the map
\[
V_g : L^2(N) \to L^\infty(G), \quad f \mapsto \langle f, \pi(\cdot)g \rangle
\]
is an isometry into \( L^2(G) \).

4.1. Reproducing formulae. The following observation relates admissibility to a Calderón-type reproducing formula.

**Lemma 4.1.** Let \( g \in \mathcal{S}(N) \) with \( \int_N g(x) \, d\mu_N(x) = 0 \). Then \( g \) is admissible if, and only if,
\[
f = \int_0^\infty f \ast \hat{g}_t \ast g_t \, \frac{dt}{t} \equiv \lim_{\varepsilon \to 0, \rho \to \infty} \int_\varepsilon^\rho f \ast \hat{g}_t \ast g_t \, \frac{dt}{t}, \quad f \in \mathcal{S}(N),
\]
with convergence in \( \mathcal{S}'(N) \).

**Proof.** Under the assumptions on \( g \), it follows by [17 Theorem 1.65] that
\[
H_{\varepsilon, \rho}(z) := \int_\varepsilon^\rho \hat{g}_t \ast g_t(z) \, \frac{dt}{t}, \quad z \in N,
\]
converges in \( \mathcal{S}'(N) \) to a distribution \( H := \lim_{\varepsilon \to 0, \rho \to \infty} H_{\varepsilon, \rho} \) which is smooth on \( N \setminus \{e_N\} \) and homogeneous of degree \(-Q\). Let \( f \in \mathcal{S}(N) \). Then
\[
\|V_g f\|_2^2 = \lim_{\varepsilon \to 0, \rho \to \infty} \int_\varepsilon^\rho \int_N \left| f \ast D_t \hat{g}(x) \right|^2 \, d\mu_G(x, t)
\]
\[
= \lim_{\varepsilon \to 0, \rho \to \infty} \int_\varepsilon^\rho \int_N \int_N f(y) \hat{g}_t(y^{-1}x) \hat{g}_t(z^{-1}x) \overline{f(z)} \, d\mu_N(z) d\mu_N(y) d\mu_N(x) \frac{dt}{t}
\]
\[
= \lim_{\varepsilon \to 0, \rho \to \infty} \int_\varepsilon^\rho \int_N \int_N f(y) \hat{g}_t \ast g_t(y^{-1}z) \overline{f(z)} \, d\mu_N(y) d\mu_N(z) \frac{dt}{t}
\]
\[
= \lim_{\varepsilon \to 0, \rho \to \infty} \int_N f \ast H_{\varepsilon, \rho}(z) \overline{f(z)} \, d\mu_N(z)
\]
\[
= \int_N f \ast H(z) \overline{f(z)} \, d\mu_N(z),
\]
where the last equality used that \( f \ast H_{\varepsilon, \rho} \to f \ast H \) in \( \mathcal{S}'(N) \) as \( \varepsilon \to 0 \) and \( \rho \to \infty \).

The map \( f \mapsto f \ast H \) is bounded on \( L^2(N) \) by [17 Theorem 6.19]. Hence \( V_g : \mathcal{S}(N) \to L^2(G) \) is well-defined, and it follows that
\[
\int_G |\langle f, \pi(x, t)g \rangle|^2 \, d\mu_G(x, t) = \langle f \ast H, f \rangle, \quad f \in L^2(N).
\]

Thus \( g \) is admissible if, and only if, \( \langle f \ast H, f \rangle = \langle f, f \rangle \) for all \( f \in L^2(N) \). Polarisation yields that this is equivalent to (4.1), which completes the proof. \( \square \)

The calculations in the proof of Lemma 4.1 are classical, see, e.g. [17 Theorem 7.7].

4.2. Rockland operators. This section provides background on spectral multipliers for Rockland operators, see, e.g. [13 Chapter 4] for a detailed account. The stated results will be used in Section 4.3 below for the construction of admissible vectors.

Let \( L \in \mathcal{D}(N) \) be positive and formally self-adjoint. Then \( L \) is essentially self-adjoint on \( L^2(N) \), and \( L \) will also denote its self-adjoint extension. Let \( E_L \) be the spectral measure of \( L \). For \( m \in L^\infty(\mathbb{R}^+_0) \), the operator
\[
m(L) := \int_{\mathbb{R}^+_0} m(\lambda) \, dE_L(\lambda)
\]
Proof. Let $K \in \mathcal{S}'(N)$. By the Schwartz kernel theorem, the action of $m(\mathcal{L})$ on $\mathcal{S}(N)$ is given by

$$m(\mathcal{L}) f = f \ast K_{m(\mathcal{L})}, \quad f \in \mathcal{S}(N),$$

where $K_{m(\mathcal{L})} \in \mathcal{S}'(N)$ is the associated convolution kernel.

A Rockland operator is a homogeneous differential operator $\mathcal{L} \in \mathcal{D}(N)$ of degree that is hypoelliptic, i.e. for every distribution $f \in (C^\infty(N))'$ and every open set $U \subseteq N$, the condition $(\mathcal{L} f)_{|U} \in C^\infty(U)$ implies that $f_{|U} \in C^\infty(U)$. Positive Rockland operators are well-known to exist on any graded Lie group.

The following theorem is the key result used to construct admissible Schwartz functions.

**Theorem 4.2** (Hulanicki [31]). Let $N$ be a graded Lie group. Let $\mathcal{L} \in \mathcal{D}(N)$ be a positive Rockland operator and let $| \cdot | : N \to [0, \infty)$ be a fixed homogeneous quasi-norm on $N$.

For any $M_1 \in \mathbb{N}$, $M_2 \geq 0$, there exist $C = C(M_1, M_2) > 0$ and $k = k(M_1, M_2), k' = k'(M_1, M_2) \in \mathbb{N}_0$ such that, for any $m \in C^k(\mathbb{R}^+_0)$, the kernel $K_{m(\mathcal{L})}$ of $m(\mathcal{L})$ satisfies

$$\sum_{|\alpha| \leq M_1} \int_G |X^\alpha K_{m(\mathcal{L})}(x)|(1 + |x|)^{M_2} \, d\mu_N(x) \leq C \sup_{\lambda > 0, \nu > 0} (1 + \lambda)^k |\partial_\nu^k m(\lambda)|.$$

**Corollary 4.3.** Let $\mathcal{L} \in \mathcal{D}(N)$ be a positive Rockland operator.

(i) If $m \in \mathcal{S}(\mathbb{R}^+_0)$, then $K_{m(\mathcal{L})} \in \mathcal{S}(N)$.

(ii) If $m \in \mathcal{S}(\mathbb{R}^+_0)$ vanishes near the origin, then $K_{m(\mathcal{L})} \in \mathcal{S}_0(N)$.

4.3. Existence of admissible vectors. The following result yields a class of Schwartz vectors that are admissible.

**Proposition 4.4.** Let $N$ be a graded Lie group and let $\mathcal{L} \in \mathcal{D}(N)$ be a positive Rockland operator of degree $\nu$. Let $K_{m(\mathcal{L})}$ be the convolution kernel of a multiplier $m \in \mathcal{S}(\mathbb{R}^+_0)$ satisfying

$$\int_0^\infty |m(t)|^2 \frac{dt}{t} = \nu. \quad (4.3)$$

Then $g := K_{m(\mathcal{L})} \in \mathcal{S}(N)$ is an admissible vector for $\pi = \text{ind}_A^{N \times A}(1)$.

**Proof.** Let $m \in \mathcal{S}(\mathbb{R}^+_0)$ be as in the statement, so that

$$\int_0^\infty |m(\lambda t^\nu)|^2 \frac{dt}{t} = \frac{1}{\nu} \int_0^\infty |m(t)|^2 \frac{dt}{t} = 1, \quad \text{for all } \lambda > 0. \quad (4.4)$$

By Corollary 4.3, $g := K_{m(\mathcal{L})} \in \mathcal{S}(N)$, and it suffices to show the reproducing formula (11). Define $H_{\varepsilon, \rho} := \int_\varepsilon^\rho \tilde{g} \ast g \, dt$ for $0 < \varepsilon < \rho < \infty$. Let $f_1, f_2 \in \mathcal{S}(N)$. Then

$$\langle f_1 \ast H_{\varepsilon, \rho}, f_2 \rangle = \int_\varepsilon^\rho \langle f_1 \ast \tilde{g} \ast g, f_2 \rangle \frac{dt}{t} = \int_\varepsilon^\rho \langle f_1 \ast (\tilde{g} \ast g), f_2 \rangle \frac{dt}{t}. \quad (4.5)$$

The spectral theorem implies that $\tilde{\tilde{g}} \ast g = K_{|m|^2(\mathcal{L})} \ast K_{m(\mathcal{L})} = K_{|m|^2(\mathcal{L})}$. In addition, the homogeneity of $\mathcal{L}$ yields that $(\tilde{g} \ast g)_t = K_{|m|^2(\mathcal{L})}$ for all $t > 0$, see, e.g. [13] Corollary 4.11.6. Combining this with (4.5) gives

$$\langle f_1 \ast H_{\varepsilon, \rho}, f_2 \rangle = \int_\varepsilon^\rho |m|^2(\mathcal{L}) f_1, f_2 \rangle \frac{dt}{t} = \int_\varepsilon^\rho \int_0^\infty |m(t\nu)|^2 \, d\langle E_{\mathcal{L}}(\lambda), f_1, f_2 \rangle \frac{dt}{t}$$

$$= \int_0^\infty \int_\varepsilon^\rho |m(t\nu)|^2 \, d\langle E_{\mathcal{L}}(\lambda), f_1, f_2 \rangle \frac{dt}{t}.$$
Hence, by the identity (4.4),
\[
\lim_{\varepsilon \to 0, \rho \to \infty} \langle f_1 \ast H_{\varepsilon,\rho}, f_2 \rangle = \int_0^\infty \int_0^\infty |m(t^\nu \lambda)|^2 \frac{dt}{t} d\langle E_{\lambda}(\lambda) f_1, f_2 \rangle = \langle f_1, f_2 \rangle.
\]
An application of Lemma 4.1 therefore yields that \( g \) is admissible. \( \square \)

Spectral multipliers for sub-Laplacians on stratified groups were used for constructing admissible vectors in [25]. See also [24] for similar discrete Littlewood-Paley decompositions.

**Remark 4.5.** The use of a *homogeneous* operator is essential in the proof of Proposition 4.4 to guarantee that the spectral dilates \( m(t \cdot), t > 0 \), of a multiplier \( m \in S(\mathbb{R}_{>0}^+) \) yield a convolution kernel \( K_{m(t \mathcal{L})} \) that is compatible with automorphic dilations \( \{ \delta_t \}_{t > 0} \). For non-homogeneous operators, other techniques seem required, see, e.g. [4, 36].

### 4.4. Proof of Theorem 1.1
Theorem 1.1 follows from combining Lemma 3.2, Corollary 4.3 and Proposition 4.4.

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