Fire-hose instability of inhomogeneous plasma flows with heat fluxes

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We study the effects of heat flows and velocity shear on the parallel firehose instability in weakly collisional plasma flow. For this purpose we apply an anisotropic 16-moments MHD fluid closure model that takes into account the pressure and temperature anisotropy, as well as the effect of anisotropic heat flux. The linear stability analysis of the firehose modes is carried out in the incompressible limit, where the MHD flow is parallel to the background magnetic field, while the velocity is sheared in the direction transverse to the flow direction. It seems that an increase of the velocity shear parameter leads to higher growth rates of the firehose instability. The increase of the instability growth rate is most profound for perturbations with oblique wave-numbers \( k_\perp / k_\parallel < 1 \). The heat flux parameter introduces an asymmetry of the instability growth in the shear plane: perturbations with wave-vectors with a component in the direction of the velocity shear grow significantly stronger as compared to those with components in the opposite direction. We discuss the implications of the presented study on the observable features of the solar wind and possible measurements of local parameters of the solar wind based on the stability constraints set by the firehose instability.

I. INTRODUCTION

It is well known that in low density, ionized, rarefied flows, magnetic field effects can dominate over the particle collision effects. Such systems can exhibit different values of temperature and pressure when measured along and normal to the direction of the background magnetic field. In such flows the anisotropic effect of the gyration of the charged particles around magnetic field lines dominates over the isotropic particle collision process leading to a plasma with anisotropic thermodynamic properties.

Anisotropic ionized flows are prone to a number of kinetic instabilities that tap energy from the magnetic anisotropy and grow due to various destabilization mechanisms. Flows where the thermal energy dominates over the magnetic energy are prone to the firehose instability. When the pressure parallel to the magnetic field is sufficiently higher than the perpendicular pressure, transverse kink perturbations of the magnetic field can become unstable and grow exponentially in time. Also, it has been shown that in viscous dissipation, anisotropy plays a significant role in the heating of the solar coronal plasma shear flows, even when other effects are abandoned.

The firehose instability has gained much attention recently, since it is believed to be one of the primary kinetic instabilities affecting the solar and stellar wind dynamics. The importance of the firehose instability in various astrophysical situations is also recognized. Still, much of the observational evidence about the development of the instability comes from the measurements of pressure anisotropies in solar wind fluctuations.

It has been shown that the nonlinear development of the microscopic firehose instability can affect the large-scale dynamics of astrophysical plasmas. A more detailed analysis of the instability development at nonlinear amplitudes can be conducted by means of numerical simulations. Moreover, direct numerical simulations of the classical firehose instability using a hybrid-kinetic numerical approximation confirm the predictions of the standard linear theory on the exponentially growing instability. In the isotropic limit, the plasma non-equilibrium thermalization manifested by periodic or rapid periodic variations of the system entropy in time, leads to strong coupling of different MHD wave modes (described in a similar mathematical framework as for waves in shear flows) and also to development of parametric instabilities.

The theoretical framework for many of the advances in the analysis of magnetically anisotropic flows using a fluid description, is adopting the Chew-Goldberger-Low (CGL) approximation. This limit exploits the simplicity of a fluid description by using a closure model that leads to two different equations of state, viz. parallel and perpendicular to the magnetic field. Thus, the CGL limit is often referred to as a double adiabatic anisotropic MHD description.

The purpose of the present paper is to study the properties of the firehose instability in weakly ionized anisotropic shear flows with heat fluxes. Waves and instabilities of anisotropic MHD shear flows have been studied in the framework of the non-modal approach in the CGL limit. The CGL closure model neglects heat fluxes as low frequency phenomena and uses the double adiabatic approximation for the analysis of higher frequency modes. On the other hand, the velocity shear of the flow is acting on the shearing time-scales: these time-scales are usually much longer than the short time scales described within the CGL limit and are comparable to the time scales of thermal processes. Therefore, we here employ a so-called 16-moments anisotropic MHD model that can be effectively used to analyze low frequency phenomena in such.
This approximation enables us to study anisotropic plasmas with heat fluxes using the fluid description. Indeed, the 16-moments formalism has already proven to be successful in analyzing waves and instabilities in weakly collisional media.

In this paper, we present a stability analysis of the low frequency incompressible perturbations to the anisotropic MHD shear flows with heat fluxes. The physical model of the problem is described in Sec. 2, where the steady-state flow, the linear perturbations and the stability analysis of both uniform and sheared flows are described, respectively. The effects of the velocity shear and anisotropic heat fluxes are summarized in Sec. 3.

II. PHYSICAL MODEL

The incompressible anisotropic MHD system can be described in the 16-moments approximation by using the following simplified fluid model:

$$
\frac{dV}{dt} + \frac{\nabla P_\perp}{\rho} + \frac{B \times (\nabla \times B)}{4\pi \rho} = \nabla \left( \frac{(P_\perp - P_\parallel)}{B^2} \right),
$$

(1)

$$
\frac{dB}{dt} - \nabla \cdot V = 0,
$$

(2)

$$
\nabla \cdot B = 0,
$$

(3)

and the incompressibility condition:

$$
\nabla \cdot V = 0,
$$

(4)

where the following notations are used for the shortness:

$$
d/dt \equiv \partial/\partial t + V \cdot \nabla,
$$

$$
\nabla || = \frac{B \cdot \nabla}{B}.
$$

The equations of state in the parallel and perpendicular to the magnetic field directions now include the heat fluxes:

$$
\frac{d}{dt} \left( \frac{P_\parallel B^2}{\rho^3} \right) = -\frac{B^2}{\rho^2} \left( \nabla || \left( \frac{S_\parallel}{B} \right) + \frac{2S_\parallel}{B^2} \nabla B \right),
$$

(5)

and

$$
\frac{d}{dt} \left( \frac{P_\perp}{\rho B} \right) = -\frac{B^2}{\rho} \nabla \left( \frac{S_\perp}{B^2} \right).
$$

(6)

The CGL MHD equations can be derived by setting the heat flux parameters to zero ($S_\perp = S_\parallel = 0$) in Eqs. (5) and (6). The full closure of the system (Eqs. (16)) can be accomplished through the 16-moments closure model that provides two more equations for the heat fluxes, namely:

$$
\frac{d}{dt} \left( \frac{S_\parallel B^3}{\rho^4} \right) = -\frac{3P_\parallel B^2}{\rho^4} \nabla || \left( \frac{P_\parallel}{\rho} \right),
$$

(7)

and

$$
\frac{d}{dt} \left( \frac{S_\perp}{\rho^2} \right) = -\frac{P_\perp}{B \rho^2} \left( \nabla || \left( \frac{P_\perp}{\rho} \right) + \frac{P_\parallel - P_\perp}{\rho} \frac{P_\perp}{P_\parallel B} \nabla B \right).
$$

(8)

A. Steady-state equilibrium flow

We consider a stationary, weakly collisional MHD flow parallel to the uniform background magnetic field in the $x-$direction, with a velocity profile that is sheared linearly in the transverse ($y-$)direction:

$$
V_0 = (Ay, 0, 0), \quad B_0 = (B_0, 0, 0),
$$

(9)

where $A$ is the transverse shear parameter of the parallel velocity. Assuming the similar physical origins of the pressure and heat flux anisotropy in the rarefied flow, we introduce the pressure anisotropy parameter as follows:

$$
\alpha = P_{\perp 0}/P_{\parallel 0} = S_{\perp 0}/S_{\parallel 0}.
$$

(10)

Such a flow matches the average configuration of the solar and stellar winds locally, where the flow convexity and turbulent component can be neglected. Indeed it is known that, while being stable global configurations, such flows can exhibit a number of micro-instabilities depending on the magnetic field, anisotropy and heat flux parameters.

B. Linear stability analysis

For the purpose of the stability analysis, we introduce linear perturbations of the background incompressible parallel shear flow embedded in a uniform magnetic field as follows:

$$
V = V_0 + V', \quad B = B_0 + B',
$$

(11)

$$
P_{\parallel} = P_{\parallel 0} + P_{\perp}', \quad P_{\perp} = P_{\perp 0} + P_{\perp}',
$$

(12)

$$
S_{\parallel} = S_{\parallel 0} + S_{\perp}', \quad S_{\perp} = S_{\perp 0} + S_{\perp}',
$$

(13)

where $V' \ll V_0$, $B' \ll B_0$, etc., and

$$
C_{\parallel 0}^2 \equiv P_{\parallel 0}/\rho_0, \quad C_{\perp 0}^2 \equiv P_{\perp 0}/\rho_0.
$$

(14)

Defining the Alfvén velocity as

$$
V_A = B_0/\sqrt{4\pi \rho},
$$

(15)

we may introduce parallel and perpendicular plasma beta parameters as follows:

$$
\beta_\parallel = 4\pi P_{\parallel 0}/B_0^2, \quad \beta_\perp = 4\pi P_{\perp 0}/B_0^2.
$$

(16)

Effects of the heat fluxes can be described by the non-dimensional heat flux parameters:

$$
\gamma_{\parallel} = \rho S_{\parallel 0}/P_{\parallel 0}^2, \quad \gamma_{\perp} = \rho S_{\perp 0}/P_{\perp 0}^2.
$$

(17)

In the present paper, we introduce parallel and perpendicular heat flux parameters that also account for the magnetic field of the plasma:

$$
q_{\parallel} = 2\gamma_{\parallel}^{1/2} \beta_{\parallel}^{1/2}, \quad q_{\perp} = 2\gamma_{\perp}^{1/2} \beta_{\perp}^{1/2}.
$$

(18)

Hence, upon substituting Eqs. (12) into the Eqs. (16) and neglecting all the nonlinear terms, we can obtain the linear
system of partial differential equations describing the dynamics of perturbations in the anisotropic MHD flow. This system can be analyzed in the shearing stress limit, where spatial Fourier expansion with time dependent wave-numbers can be employed as follows:

\[
\begin{pmatrix}
P^0_i(\mathbf{r}, t)/P_{i0}^0 \\
P^1_i(\mathbf{r}, t)/P_{i0}^1 \\
S^0_i(\mathbf{r}, t)/P_{i0}^0 \\
S^1_i(\mathbf{r}, t)/P_{i0}^1 \\
\n\dot{V}_{\parallel}(\mathbf{r}, t)/V_A \\
\n\dot{B}(\mathbf{r}, t)/B_0
\end{pmatrix}
\propto
\begin{pmatrix}
\text{ip}_i \parallel (\mathbf{k}, \tau) \\
\text{ip}_i \perp (\mathbf{k}, \tau) \\
s_i \parallel (\mathbf{k}, \tau) \\
s_i \perp (\mathbf{k}, \tau) \\
v(\mathbf{k}, \tau) \\
i_b(\mathbf{k}, \tau)
\end{pmatrix}
\exp\left(\text{i}k(\mathbf{r})\frac{\tau}{L}\right).
\]

(17)

Here, \(L\) corresponds to the characteristic length-scale of the flow, \(\tau\) is the non-dimensional time variable:

\[
\tau = t V_A / L,
\]

(18)

\(R\) is the normalized velocity shear parameter

\[
R = AL/V_A,
\]

and \(k_y(\tau)\) is the non-dimensional shearing wave-number of the perturbation harmonics:

\[
k_y(t) = k_{y0} - R k_z \tau.
\]

In this framework we may derive the system of differential equations governing the linear dynamics of perturbation harmonics of the incompressible anisotropic shear flow system:

\[
\dot{v}_x(\tau) = R v_y(\tau) - \beta_\perp \frac{k^2}{k_x} b_\perp(\tau) - \frac{\Delta \beta k^2}{k_x} b_x(\tau),
\]

(19)

\[
\dot{v}_y(\tau) = \beta_\perp k_y p_\perp(\tau) + k_y b_\perp(\tau) - (1 + \Delta \beta) k_y b_y(\tau),
\]

(20)

\[
\dot{p}_\perp(\tau) = k_x v_x(\tau) + R b_y(\tau) - k_x s_\perp(\tau) + i q \alpha k_x b_\perp(\tau),
\]

(21)

\[
\dot{s}_\parallel(\tau) = -\frac{3 i q \parallel}{2} k_x v_x(\tau) + 6 R v_y(\tau) - 3 \beta_\perp \frac{k^2}{k_x} p_\perp(\tau) - \frac{2 \Delta \beta k^2}{k_x} b_\perp(\tau) + \frac{3 i q \parallel}{2} R b_y(\tau),
\]

(22)

\[
\dot{s}_\perp(\tau) = \beta_\parallel k_x p_\perp(\tau) + \Delta \beta k_x b_\perp(\tau),
\]

(23)

\[
\dot{b}_x(\tau) = R b_y(\tau) + k_x v_x(\tau),
\]

(24)

\[
\dot{b}_y(\tau) = k_x v_y(\tau),
\]

(25)

where

\[
k^2 = k_y^2 + k_z^2,
\]

\[
k^2 = k_x^2 + k_\perp^2,
\]

\[
\Delta \beta \equiv \beta_\perp - \beta_\parallel,
\]

(27)

and the dot denotes the temporal derivative, e.g. \(\dot{\psi}(\tau) \equiv d\psi/d\tau\).

### C. Firehose instability in uniform flows

In the zero-shear limit \((R = 0)\), \(k_y\) and thus also all coefficients in the differential equations become time-independent.

\[\text{FIG. 1. Sketch illustration of the firehose instability in the flow parallel to the magnetic field. Instability occurs when the transverse effective pressure acting on the perturbed magnetic tube is less than the parallel pressure: } P_{\parallel0} + B_0^2/4\pi < P_{\parallel0} \text{ and } c^2_F < 0. \text{ In this case, transverse Alfvénic deformation of the magnetic field lines grow in time exponentially.}\]

Hence, we may use a spectral expansion of the linear perturbations in time \(\propto \exp(\text{i}\omega \tau)\) and obtain the dispersion equation of the incompressible, anisotropic MHD system, which reads as follows:

\[
\left[\omega^2 - (1 + \Delta \beta) k^2 \right] D_0 = 0,
\]

(28)

where

\[
D_0 \equiv \omega^4 - (1 + \beta_\perp) k^2 \omega^2 + \alpha \beta_\perp q_\perp k_x^2 \omega^2 + \left[1 + \Delta \beta \right] k^2 + \left(1 + \beta_\perp - \alpha^2 \beta_\parallel \right) k^2 \beta_\parallel k_x^2.
\]

(29)

The first obvious solution of this dispersion equation is given by the firehose mode:

\[
\omega_{\parallel \pm} = \pm c_F k_x,
\]

(30)

where

\[
c^2_F \equiv 1 + \Delta \beta
\]

(31)

stands for the square of the characteristic speed of the firehose mode, when positive. It seems that kink deformations of the parallel magnetic structures that are described by the firehose mode do not feel the effect of heat fluxes \((q_\parallel, q_\perp)\). Hence the linear stability criterion for firehose instability \((c^2_F < 0)\) can be set by the balance of the parallel and perpendicular plasma beta parameters:

\[
\beta_\parallel > 1 + \beta_\perp.
\]

(32)

The growth of linear perturbations is described by \(\omega_{\parallel +}\) or \(\omega_{\parallel -}\), depending on whether the streamwise wave-number \(k_x\) is positive or negative, respectively.

The mechanism of this classical microscopic anisotropic MHD instability can be illustrated using a simple sketch...
shown in Fig. (1). Transverse perturbations of the magnetic field \((B'_\perp \neq 0)\) can be shown as kink perturbations of the parallel magnetic structure. The response to this perturbation consists from a combined action of the parallel and perpendicular pressure and the magnetic field. If the perpendicular pressure is lower then the parallel pressure to the extent that even the magnetic field can not compensate for the kink deformation, the perturbation will grow and the instability will be developed. Interestingly, the heat fluxes \((S_\parallel, S_\perp)\) do not affect this process. At least not in the uniform flow limit.

D. Firehose instability in non-uniform flows

The dynamic behaviour of linear perturbations in shear flows \((R \neq 0)\) are described by Eqs. (20)-(26). Employing the low frequency limit, we can use the adiabatic approximation, when the time derivative of perturbation harmonics can be generally represented as

\[
\psi(k, \tau) \approx -i\omega \psi_k(k, \tau). \tag{33}
\]

In this limit, we can derive the adiabatic dispersion equation of the shear flow system and obtain:

\[
\left[\omega^2 - (1 + \Delta \beta)k_x^2\right] D_0 + i R k_x k_y D_1 = 0 , \tag{34}
\]

where

\[
D_1 \equiv (1 + \beta_\perp)\omega^2 - \alpha \beta_\perp q_{\perp} k_x \omega + (\beta_\perp^2 - \beta_\parallel^2 \beta_\perp^2 - \beta_\parallel^2) k_x^2 , \tag{35}
\]

shows the modification to the standard dispersion equation. The implications of the velocity shear effects on the firehose instability set by Eq. (37) are obvious. Unlike the uniform flow result, the firehose solutions now do depend on the velocity shear \(R\) as well as on the heat flux parameter \(q_{\perp}\).

To analyze the effect of shear flow on the classical firehose mode we introduce the deviation from the uniform flow solution (39) as follows:

\[
\omega = \omega_{F \pm} + R \omega_{1 \pm} . \tag{36}
\]

Substituting Eq. (36) into the Eq. (34) and neglecting terms of order higher than \(R^2\) in the low shear limit \((R < 1)\), we may derive the second-order dispersion equation with respect to \(\omega_{1 \pm}\) as follows:

\[
RA_{\pm} \omega_{1 \pm}^2 + (B_{\pm} + i RC_{\pm}) \omega_{1 \pm} + i D_{\pm} = 0 , \tag{37}
\]

where

\[
A_{\pm} = 4 c_F^2 (c_F^2 - \beta_\parallel^2 \beta_\perp^2 + (3 \Delta_\pm - 2(1 + 2 \beta_\perp)) k_x^2 , \tag{38}
\]

\[
B_{\pm} = 2 \omega_{F \pm} \Delta_\pm k_x^2 , \tag{39}
\]

\[
C_{\pm} = (1 + 2 \beta_\perp - 2 \Delta_\pm) k_x k_y , \tag{40}
\]

\[
D_{\pm} = -\omega_{F \pm} \Delta_\pm k_x k_y , \tag{41}
\]

and

\[
\Delta_\pm = \pm c_F \alpha \beta_\perp q_{\perp} + (1 - c_F^2)(1 + 2 \beta_\perp) - c_F^2 . \tag{42}
\]

The solution of the Eq. (36) should match the standard firehose solution in the zero shear limit, i.e.

\[
\omega(R = 0) = \omega_{F \pm} .
\]

This sets the following convergence requirement on the shear flow correction:

\[
\lim_{R \to 0} (R \omega_{1 \pm}) = 0 . \tag{43}
\]

Hence, from the two solutions of the reduced dispersion equation (37), we can choose the one obeying the asymptotic condition (43).

\[
\omega_{1 \pm} = \frac{B_{\pm} + i RC_{\pm}}{2RA_{\pm}} \left[ -1 + \left(1 - \frac{4i RA_{\pm} D_{\pm}}{(B_{\pm} + i RC_{\pm})^2}\right)^{1/2} \right] , \tag{44}
\]

Using the low shear limit \((R < 1)\) we may separate the real and complex parts of the solution (45) for stable, unstable and neutrally stable firehose modes:

\[
\omega = \pm c_F k_x + \delta_{1 \pm} + i \sigma_{1 \pm} \quad \text{when } c_F^2 > 0 , \tag{45}
\]

\[
\omega = 2i \sigma_A \quad \text{when } c_F^2 = 0 , \tag{46}
\]

\[
\omega = (\sigma_A + \delta_{2 \pm}) + i (\pm|c_F| k_x + \sigma_{2 \pm}) \quad \text{when } c_F^2 < 0 , \tag{47}
\]

where

\[
\sigma_A \equiv \frac{R k_x k_y}{2 k_x^2} .
\]

describes the shear flow induced transient growth of aperiodic perturbations, while the explicit forms of \(\delta_{1 \pm}, \delta_{2 \pm}, \sigma_{1 \pm}, \sigma_{2 \pm}\) are given in the Appendix A.

III. DISCUSSION

In order to show the effects of different physical factors and parameters on the firehose instability, we illustrate analytic solutions given by Eqs. (45)-(47).

Fig. (2) shows the comparison of the growth rates of the instability in a uniform flow to flows with different values of the velocity shear parameter. It seems that the velocity shear effect is negligible for strictly parallel \((k = k_x)\) or perpendicular \((k = k_\perp)\) perturbations. However, velocity shear induces a significant boost to the instability growth rates for perturbations with wave-vectors oblique to the magnetic field.

Fig. (3) shows the growth rates of the instability in uniform and sheared flows for different values of the vertical wave-number. It seems that the effect of the velocity shear is most profound on the vertically uniform perturbations with \(k_z = 0\). The increase of the instability growth rates is most profound for nearly parallel perturbations with \(k_x / k \lesssim 1\).

Fig. (4) shows the growth rates of the instability in flows with different heat flux parameters. Interestingly, the effect of heat fluxes introduces an asymmetry of the firehose instability growth in wave-number space: perturbations having a transverse component pointing in the direction of the velocity shear \((k_y > 0)\), are amplified, while the perturbations with
FIG. 2. The growth rate of the firehose instability as a function of the angle \( \phi \) of the wave vector with respect to the direction of the magnetic field. Here, \( \alpha = 0.5, \beta = 2, q_\perp = 0.2, k_x = 1, c_F^2 = -0.01 \) and \( R = 0.05, 0.2, 0.4 \) (left to right). The horizontal axis corresponds to the direction parallel and the vertical axis to the direction perpendicular to the magnetic field, respectively. The red dashed line shows the instability growth rate for uniform flows \( (R = 0) \), while the solid blue line shows the velocity shear modification. It seems that an increase of the shear parameter leads to stronger instabilities for perturbations with oblique wave-numbers: \( \phi \sim \pi/4 \).

FIG. 3. The growth rate of the firehose instability as a function of the angle \( \phi \) of the wave vector with respect to the direction of the magnetic field. Here, \( \alpha = 0.5, \beta = 2, q_\perp = 0.2, R = 0.05, c_F^2 = -0.01 \) and \( k_x = 0, 1, 2 \) (left to right). It seems that the velocity shear effect is the strongest for vertically uniform perturbations \( (k_x = 0) \), while small scale perturbations with \( k_x > 1 \) remain largely insensitive to the velocity shear effects.

wave vectors pointing in the opposite direction \( (k_y < 0) \), are somewhat suppressed. Notably weaker asymmetry is introduced in the streamwise direction: perturbations in the \( k_x > 0 \) area grow stronger in comparison to those in the \( k_x < 0 \) area.

Finally Fig. 5 shows the growth rates of the instability for different values of the firehose parameter \( c_F^2 \). The effects of the velocity shear are most profound for marginally unstable perturbations, while the violent firehose processes remain largely insensitive to the velocity shear modifications.

It seems that low frequency processes in the anisotropic flows that are marginally unstable to firehose perturbations can be significantly modified by the velocity shear as well as by heat fluxes of the MHD medium. The parallel firehose instability acquires a transverse component in shear flows and, hence, is affected by the heat fluxes. Thus the observable features of the MHD fluctuations can be significantly modified in high shear regions of anisotropic flows.

In solar and stellar winds the combined effect of the velocity shear and heat fluxes can be an additional source of turbulence and anomalous heating in rarefied magnetized outflows. The firehose instability can lead to enhanced turbulence in anisotropic MHD flows. Thus, sustained fluctuations of the solar wind can occur in areas of phase space where no instabilities are present. Therefore, observations of the fluctuations in the solar wind can be a powerful tool for analyzing the physical conditions in the wind using the stability considerations for local perturbations. Indeed, the firehose instability modified by the effects of velocity shear and heat fluxes can draw the energy of the background flow into turbulent fluctuations and, ultimately, into heat via dissipative effects which can be included in the solar/stellar models in terms of statistically proven macroscopic wave heating and pressure gradient quantities\(^{35,36}\). It is also interesting to elaborate on the role of the firehose instability in the dynamics of the smaller scale flows like solar coronal jets, for which quasi-oscillatory precursors in the mean intensity variations recently have been
observed\textsuperscript{12} that pretend to be triggers of the instability processes occurring in coronal bright points.

Thus, the analytic solutions derived in the present paper can be used to study the features of the solar wind fluctuations by deducing the physical conditions in the outflow, such as the parameters of the heat fluxes and the azimuthal velocity shear of the radial outflow.

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**REFERENCES**

\textsuperscript{1}C. L. Longmire, M. N. Rosenbluth, Phys. Rev. \textbf{103}, 507 (1956).

\textsuperscript{2}A. A. Vedenov, R. Z. Sagdeev, R. Z. Sov. Phys. Dokl. \textbf{3}, 278 (1958).

\textsuperscript{3}S. Chandrasekhar, A. N. Kaufman, K. M. Watson, Proc. Roy. Soc. Lon. A \textbf{245}, 435 (1958).

\textsuperscript{4}E. N. Parker, Phys. Rev. \textbf{109}, 1874 (1958).

\textsuperscript{5}A. A. Vedenov, E. P. Velikhov, R. Z. Sagdeev, Sov. Usp. Fiz. Nauk \textbf{73}, 701 (1961).

\textsuperscript{6}B. M. Shergelashvili, S. Poedts, A. D. Pataraya, ApJL \textbf{642}, L73 (2006).

\textsuperscript{7}M. Sarfraz, P. H. Yoon, S. Saeed, G. Abbas, and H. A. Shas, Phys. Plasmas \textbf{24}, 012907 (2017).

\textsuperscript{8}P. Astfalk, and F. Jenko, J. Geophys. Res. Space Phys. \textbf{121}, 2842 (2016).

\textsuperscript{9}M. S. dos Santos, L. F. Ziebell, and R. Gaezler, Phys. Plasmas \textbf{21}, 112102 (2014).

\textsuperscript{10}M. Lazar, S. Poedts, R. Schlickeiser, D. Ibscher Solar Phys. \textbf{289}, 369 (2014).

\textsuperscript{11}R. Schlickeiser, and T. Skoda, Astrophys. J. \textbf{716}, 1596 (2010).

\textsuperscript{12}M. Lazar, and S. Poedts Astron. Astrophys. \textbf{494}, 311 (2009).
Shear flow modification in the stable \((c_P^2 > 0)\) flow configuration to the real:
\[
\delta_{1\pm} = \frac{k_{\perp}(t)^2}{8c_P k_{\perp} A_3 \Delta_\pm} \left\{ M_{1\parallel} k_{\perp}^2 + \frac{|k_z \Delta_\pm|}{k_{\perp} A_3} \sigma_A^2 [2\beta_\perp + 1 - 2\Delta_\pm]^2 \right\},
\]
(A1)
and imaginary parts of the frequency \(\omega\):
\[
\sigma_{1\pm} = \sigma_A \frac{P_{1\parallel} k_{\perp}^2 + P_{1\perp} k_{\perp}(t)^2}{2A_3},
\]
(A2)
where
\[
\eta \equiv \alpha |c_P| \beta_\perp q_\perp,
\]
(A3)
and
\[
M_{1\parallel} = 8c_P^2 \Delta_\pm^2 \left(1 + \frac{|k_z \Delta_\pm|}{k_{\perp} A_3}\right),
\]
(A4)
\[
P_{1\parallel} = 4[(c_P^2 - 1)(c_P^2 - \beta_\parallel) + 2 - \beta_\perp + 1],
\]
(A5)
\[
P_{1\perp} = \Delta_\pm + \left(1 + \frac{|k_z \Delta_\pm|}{k_{\perp} A_3}\right) \pm [\eta - 2c_P^2 (1 + \beta_\perp)]
\]
(A6)
Shear flow modification in the unstable flow configuration \((c_P^2 < 0)\) to the real:
\[
\delta_{2\pm} = -\sigma_A^2 \alpha \beta_\perp q_\perp k_{\perp}(t)^2 \times \frac{M_{2\parallel}[1 + \eta^2 N_{2\parallel}]k_{\perp}^2 + (M_{2\perp} + \eta^2 N_{2\perp})k_{\perp}(t)^2}{T_2},
\]
\[
\sigma_{2\pm} = \frac{\sigma_A k_{\perp}(t)^2}{2|c_P|} \times \frac{(M_{2\parallel} + \eta^2 Q_{2\parallel})k_{\perp}^2 + (M_{2\perp} + \eta^2 Q_{2\perp})k_{\perp}(t)^2}{T_2},
\]
\[
\eta \equiv \alpha |c_P| \beta_\perp q_\perp,
\]
where
\[
T_2 = \left\{ \eta^2 - (2c_P^2 (1 + \beta_\perp) - (1 - 2\beta_\perp)]^2 \right\} [4c_P^2 \times (c_P^2 - \beta_\parallel)k_{\perp}^2 - 6(c_P^2 + 1 + \beta_\perp) - (-1 + 2\beta_\perp)k_{\perp}(t)^2 + 9\eta^2 k_{\perp}(t)^4]
\]
\[
M_{2\parallel} = 4[c_P^2 (c_P^2 - \beta_\parallel) - 2\beta_\perp (2c_P^2 + 1) - 4\beta_\parallel + 1] \times (2\beta_\perp (2c_P^2 + 4\beta_\parallel - 5) - 4\beta_\parallel + 3),
\]
\[
N_{2\parallel} = 16c_P^2 k_{\perp}^2 (c_P^2 - \beta_\parallel),
\]
\[
M_{2\perp} = 3[1 + 2(c_P^2 - \beta_\parallel)](2c_P^2 - 1)^2 [2(c_P^2 - 1)(c_P^2 + \beta_\parallel + 1) + 6(c_P^2 + \beta_\parallel)(c_P^2 + 2\beta_\parallel - 2) + 1 + 2\beta_\perp] \times \times [-2\beta_\perp (-2c_P^2 + 1) - 4\beta_\parallel + 1]2\beta_\perp \times \times (2c_P^2 + 4\beta_\parallel - 5) - 4\beta_\parallel + 3),
\]
\[
N_{2\perp} = 48(c_P^2 + \beta_\parallel)(\beta_\parallel - 1) + 4 + 8\beta_\perp,
\]
\[
P_{2\parallel} = 4(c_P^2 - \beta_\parallel)c_P^2 [1 + 2(c_P^2 + \beta_\parallel)(2c_P^2 - 1)]^2 \times \times [2(c_P^2 - 1)(c_P^2 + \beta_\parallel) + 1],
\]
\[
P_{2\perp} = 4(c_P^2 - \beta_\parallel)c_P^2 [1 + 2(c_P^2 + \beta_\parallel)(2c_P^2 - 1)]^2 \times \times [2(c_P^2 - 1)(c_P^2 + \beta_\parallel) + 1],
\]
\[
(T_2 = \left\{-\eta^2 - (2c_P^2 (1 - \beta_\perp) - (1 - 2\beta_\perp)]^2 \right\} [4c_P^2 \times (c_P^2 - \beta_\parallel)k_{\perp}^2 - 6(c_P^2 + 1 + \beta_\perp) - (-1 - 2\beta_\perp)k_{\perp}(t)^2 + 9\eta^2 k_{\perp}(t)^4]
\]
\[
M_{2\parallel} = 4[c_P^2 (c_P^2 - \beta_\parallel) + 2\beta_\perp (2c_P^2 + 1) - 4\beta_\parallel + 1] \times (2\beta_\perp (2c_P^2 + 4\beta_\parallel - 5) - 4\beta_\parallel + 3),
\]
\[
N_{2\parallel} = 16c_P^2 k_{\perp}^2 (c_P^2 - \beta_\parallel),
\]
APPENDIX: FIREHOSE SOLUTION IN NON-UNIFORM FLOWS

The modification of the firehose solutions in non-uniform flows can be described in low shear limit \((R < 1)\) using Eqs. (44-46). In these equations we have introduced number of notations for the shortness of the presentation in the main text of the article.
\[Q_{2\parallel} = 32c_F^4(\beta_{\parallel}^2 - c_F^4),\]
\[Q_{2\perp} = 12\eta^2 + 3(2\beta_{\perp}(-2c_F^2 + 1) - 4\beta_{\parallel} + 1)[2\beta_{\perp} \times \text{(A17)}\]
\[P_{2\perp} = [6(c_F^2 + \beta_{\parallel})(c_F^2 + 2\beta_{\parallel} - 2) + 1 + 2\beta_{\perp}] \times \text{(A16)} \times (c_F^2 + 2\beta_{\parallel} - 2) + 1 + 2\beta_{\perp}][c_F^2 + 4\beta_{\parallel} - 5] - 4\beta_{\parallel} + 3] - 8[6(c_F^2 + \beta_{\parallel})(c_F^2 + 2\beta_{\parallel} - 2) + 1 + 2\beta_{\perp}] \times \text{(A18)} \times (c_F^2 + \beta_{\parallel})c_F^2.\]