Intermediate Jacobians and the slice filtration

Doosung Park

Abstract

Let $X$ be an $n$-dimensional connected scheme smooth and projective over $\mathbb{C}$. We decompose the motive $\text{Hom}(\mathbb{L}^{n-2}, M(X))$ using intermediate Jacobians. We also construct a morphism $M_{2n-2}(X) \to M(X)$ induced by a conjectural Chow-Künneth decomposition of $M(X)$.

1. Introduction

1.1. Throughout this paper, for brevity, put

$$\text{DM}^{eff} = \text{DM}^{eff}(\text{Spec } k, \mathbb{Z}), \quad \text{DM}_Q^{eff} = \text{DM}^{eff}(\text{Spec } k, \mathbb{Q}),$$

whose definitions are in [2, 11.1.1]. Here, $k$ is an algebraically closed field, and for most part (from \[1.3\]) of this paper, we assume that $k = \mathbb{C}$. Let

(i) $1$ denote the object $\text{M}(\text{Spec } k)$ of $\text{DM}^{eff}$ or $\text{DM}_Q^{eff}$,

(ii) $L$ denote the object $1(1)[2]$ of $\text{DM}^{eff}$ or $\text{DM}_Q^{eff}$,

(iii) $\text{Hom}$ denote the internal hom of $\text{DM}^{eff}$ or $\text{DM}_Q^{eff}$.

Note that in [2, 11.1.4], we have the change of coefficients functor

$$\text{DM}^{eff} \to \text{DM}_Q^{eff}.$$

1.2. Let $X$ be an $n$-dimensional connected scheme smooth and projective over an algebraically closed field $k$. The Picard group of $X$ admits the exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \to \text{NS}(X) \to 0$$

of abelian groups. The exact sequence is related to a decomposition

$$\text{Hom}(\mathbb{L}^{n-1}, M(X)) = \text{NS}_Q(X) \oplus \text{Pic}_Q^0(X) \oplus L$$

(1.2.1)

in $\text{DM}^{eff}(\text{Spec } k, \mathbb{Q})$, where the subscript $\mathbb{Q}$ means the corresponding ones with $\mathbb{Q}$-coefficient. This should come from a conjectural Chow-Künneth decomposition [6, Definition 6.1.1] $M(X) = M_0(X) \oplus \cdots \oplus M_{2n}(X)$ like

$$\text{Hom}(\mathbb{L}^{n-1}, M_{2n-2}(X)) = \text{NS}_Q(X),$$

$$\text{Hom}(\mathbb{L}^{n-1}, M_{2n-1}(X)) = \text{Pic}_Q^0(X),$$

$$\text{Hom}(\mathbb{L}^{n-1}, M_{2n}(X)) = L.$$
What is the generalization of (1.2.1) for higher codimensions? The answer will give also a decomposition of each motive in the slice filtration [5]

\[ L^n = \text{Hom}(L^n, M(X)) \otimes L^n \to \text{Hom}(L^{n-1}, M(X)) \otimes L^{n-1} \to \cdots \to \text{Hom}(1, M(X)) = M(X) \]
of \( M(X) \).

When \( k = \mathbb{C} \), we use intermediate Jacobians to study the question as follows.

**Theorem 1.3.** Let \( X \) be an \( n \)-dimensional connected scheme smooth and projective over \( \mathbb{C} \), and let \( d \in [1, n] \) be an integer. Then

1. \( \text{NS}_{\text{hom}, \mathbb{Q}}^d(X) \oplus \text{Griff}_{\mathbb{Q}}^d(X) \) is a direct summand of \( \text{Hom}(L^{n-d}, M(X)) \) in \( \text{DM}^{\text{eff}}_\mathbb{Q} \),

2. \( J_{a, \mathbb{Q}}^d(X) \) is a direct summand of \( \text{Hom}(L^{n-d}, M(X)) \) in \( \text{DM}^{\text{eff}}_\mathbb{Q} \).

See (2.1) and (3.2) for the definitions of \( \text{NS}_{\text{hom}, \mathbb{Q}}^d(X) \), \( \text{Griff}_{\mathbb{Q}}^d(X) \), and \( J_{a, \mathbb{Q}}^d(X) \).

1.4. In particular, using this, we obtain the generalization of (1.2.1) for dimension 2 as follows.

**Theorem 1.5.** Let \( X \) be an \( n \)-dimensional connected scheme smooth and projective over \( \mathbb{C} \) with \( n \geq 2 \). Then for some motive \( M_2(X)^* \) in \( \text{DM}^{\text{eff}}_\mathbb{Q} \), there is a decomposition

\[ \text{Hom}(L^{n-2}, M(X)) = \text{NS}_{\text{hom}, \mathbb{Q}}^2(X) \oplus \text{Griff}_{\mathbb{Q}}^2(X) \oplus J_{a, \mathbb{Q}}^2(X) \oplus M_2(X)^* \oplus (\text{Pic}^0(X) \otimes L) \oplus L^2. \]

1.6. Let \( X \) be an \( n \)-dimensional connected scheme smooth and projective over \( \mathbb{C} \), and let

\[ i_{2n-2} : M_2(X)^* \otimes L^{n-2} \to M(X) \]
denote the morphism induced by the morphism \( M_2(X)^* \to \text{Hom}(L^{n-2}, M(X)) \) obtained by the above decomposition. If \( M(X) \) has a Chow-K"unneth decomposition

\[ M(X) = M_0(X) \oplus M_1(X) \oplus \cdots \oplus M_{2n}(X), \]

then the morphism \( i_{2n-2} \) is the candidate for a morphism \( M_{2n-2}(X) \to M(X) \) induced by a decomposition. Thus we have the following conjecture.

**Conjecture 1.7.** Let \( X \) be an \( n \)-dimensional connected scheme smooth and projective over \( \mathbb{C} \). Then there is a morphism \( p_{2n-2 : M(X) \to M_2(X)^* \otimes L^{n-2}} \) in \( \text{DM}^{\text{eff}}_\mathbb{Q} \) such that

1. \( p_{2n-2} i_{2n-2} = \text{id} \),
2. \( i_{2n-2} p_{2n-2} : M(X) \to M(X) \) induces the Künneth projector

\[ H^*(X, \mathbb{Q}) \to H^{2n-2}(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}), \]

3. the dual projector \( (i_{2n-2} p_{2n-2})^t \) induces the Künneth projector

\[ H^*(X, \mathbb{Q}) \to H^2(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}). \]

1.8. A successful construction of \( p_{2n-2} \) with the above properties gives the construction of projectors of \( M(X) \) defining \( M_2(X) \) and \( M_{2n-2}(X) \). In particular, since projectors of \( M(X) \) defining \( M_0(X), M_1(X), M_{2n-1}(X), \) and \( M_{2n}(X) \) are already constructed, the conjecture together with some vanishing conjectures in \( [3, 5.8] \) will prove the Künneth type standard conjecture when the dimension of \( X \) is 3.
1.9. Organization of the paper. In Section 2, we prove (1.3(1)) by constructing a morphism 
\[ \text{Hom}(L^{n-d}, M(X)) \to \text{NS}^{d}_{\text{alg}, Q}(X) \] and its section. In Section 3, we prove (1.3(2)) by constructing a morphism 
\[ \text{Hom}(L^{n-d}, M(X)) \to J^{d}_{\text{alg}, Q}(X) \] and its section. In Section 4, we prove (1.5) by constructing the other pieces and using [F 7.3.10]. In Section 5, we discuss some conjectures other than (1.7).

1.10. Conventions and notations. Alongside (1.1), we have the following.

(1) Let \( T \) be a complex analytic variety or a scheme over an algebraically closed field \( k \). We denote by \( \text{cl}(T) \) the set of closed points of \( T \).

(2) \( Sm/C \) denotes the category of smooth \( C \)-schemes.

(3) For any \( \mathbb{Q} \)-vector space \( V \), consider the constant Nisnevich sheaf with transfer on \( Sm/C \) associated to \( V \). We denote by \( V \) (by abuse of notation) its associated object in \( \text{DM}^{\text{eff}}_{\mathbb{Q}} \).

2. Proof of (1.3(1))

Definition 2.1. Let \( X \) be a scheme smooth over \( C \), and let \( d \) be a nonnegative integer. We put

\[
CH^{d}_{\text{alg}}(X) = \{ Z \in CH^{d}(X) : Z \sim_{\text{alg}} 0 \},
\]
\[
CH^{d}_{\text{hom}}(X) = \{ Z \in CH^{d}(X) : Z \sim_{\text{hom}} 0 \}
\]
\[
\text{NS}^{d}_{\text{alg}}(X) = CH^{d}(X)/CH^{d}_{\text{alg}}(X),
\]
\[
\text{NS}^{d}_{\text{hom}}(X) = CH^{d}(X)/CH^{d}_{\text{hom}}(X),
\]
\[
\text{Griff}^{d}(X) = CH^{d}_{\text{hom}}(X)/CH^{d}_{\text{alg}}(X)
\]
where \( \sim_{\text{alg}} \) (resp. \( \sim_{\text{hom}} \)) denotes the algebraic equivalence relation (resp. homological equivalence relation for the singular cohomology. We also denote by

\[
CH^{d}_{\text{alg}, Q}(X), \ CH^{d}_{\text{hom}, Q}(X), \ NS^{d}_{\text{alg}, Q}(X), \ NS^{d}_{\text{hom}, Q}(X), \ \text{Griff}^{d}_{Q}(X)
\]
the corresponding ones defined for \( \mathbb{Q} \)-coefficient.

Definition 2.2. Let \( X \) and \( Y \) be schemes smooth over \( C \), and let \( d \) be a nonnegative integer. We put

\[
CH^{d}_{X}(Y) = CH^{d}(Y \times X).
\]
When \( Y \) is connected, we put

\[
CH^{d}_{\text{alg}, X}(Y) = \{ Z \in CH^{d}(Y \times X) : i_{y}^{*}Z \sim_{\text{alg}} 0 \},
\]
\[
CH^{d}_{\text{hom}, X}(Y) = \{ Z \in CH^{d}(Y \times X) : i_{y}^{*}Z \sim_{\text{hom}} 0 \}
\]
\[
\text{NS}^{d}_{\text{alg}, X}(Y) = CH^{d}_{X}(Y)/CH^{d}_{\text{alg}, X}(Y),
\]
\[
\text{NS}^{d}_{\text{hom}, X}(Y) = CH^{d}_{X}(Y)/CH^{d}_{\text{hom}, X}(Y)
\]
where \( y \) is a closed point of \( Y \) and \( i_{y} \) denotes the closed immersion \( y \times X \to Y \times X \). Note that the above definitions are independent of \( y \) since \( i_{y}^{*}Z \) and \( i_{y'}^{*}Z \) are algebraically equivalent for two closed points \( y \) and \( y' \) of \( Y \).
When $Y$ is not necessarily connected and has the connected components $\{Y_i\}_{i \in I}$, we put

\[
CH^d_{\text{alg},X}(Y) = \bigoplus_{i \in I} CH^d_{\text{alg},X}(Y_i),
\]

\[
CH^d_{\text{hom},X}(Y) = \bigoplus_{i \in I} CH^d_{\text{hom},X}(Y_i),
\]

\[
NS^d_{\text{alg},X}(Y) = \bigoplus_{i \in I} NS^d_{\text{alg},X}(Y_i),
\]

\[
NS^d_{\text{hom},X}(Y) = \bigoplus_{i \in I} NS^d_{\text{hom},X}(Y_i).
\]

We consider $CH^d_X, CH^d_{\text{alg},X}, CH^d_{\text{hom},X}, NS^d_{\text{alg},X},$ and $NS^d_{\text{hom},X}$ as presheaves with transfer on $Sm/\mathbb{C}$.

We also denote by

\[
CH^d_X, \mathbb{Q}, CH^d_{\text{alg},X}, \mathbb{Q}, CH^d_{\text{hom},X}, \mathbb{Q}, NS^d_{\text{alg},X}, \mathbb{Q}, NS^d_{\text{hom},X}, \mathbb{Q}
\]

the corresponding ones defined for $\mathbb{Q}$-coefficient.

**Proposition 2.3.** Under the notations and hypotheses of (2.2), the homomorphisms

\[
NS^d_{\text{alg},X}(Y) \to NS^d_{\text{alg},X}(\text{Spec } \mathbb{C}),
\]

\[
NS^d_{\text{hom},X}(Y) \to NS^d_{\text{hom},X}(\text{Spec } \mathbb{C})
\]

induced by $i_*^y$ are isomorphisms.

**Proof.** The homomorphisms are surjective since $i_*^y : CH^d(Y \times X) \to CH^d(X)$ is surjective. The homomorphisms are injective since the kernels of the homomorphisms

\[
CH^d(Y \times X) \to NS^d_{\text{alg}}(X),
\]

\[
CH^d(Y \times X) \to NS^d_{\text{hom}}(X)
\]

are

\[
\{Z \in CH^d(Y \times X) : i_*^y Z \in CH^d(X)_{\text{alg}}\},
\]

\[
\{Z \in CH^d(Y \times X) : i_*^y Z \in CH^d(X)_{\text{hom}}\}
\]

respectively. \hfill \qed

**Corollary 2.4.** The presheaves $NS^d_{\text{alg},X}$ and $NS^d_{\text{hom},X}$ on $Sm/X$ are constant Nisnevich sheaves with transfer associated to $NS^d_{\text{alg}}(X)$ and $NS^d_{\text{hom}}(X)$ respectively.

**Proof.** Let $Y$ be a connected scheme smooth over $\mathbb{C}$, and let $p : Y \to \text{Spec } \mathbb{C}$ denote the structural morphism. Then let $y \in Y$ be a closed point, and let $c_y : y \to Y$ denote the closed immersion for the point $y$. Consider the homomorphisms

\[
NS^d_{\text{alg},X}(\text{Spec } \mathbb{C}) \to NS^d_{\text{alg},X}(Y) \to NS^d_{\text{alg},X}(\text{Spec } \mathbb{C})
\]
induced by $p$ and $c_y$ respectively. The composition is an isomorphism since $c_y p = \text{id}$, and the second arrow is an isomorphism by (2.3). Thus the first arrow is an isomorphism. This shows that $\text{NS}^d_{\text{alg},X}$ is a constant Zariski sheaf associated with $\text{NS}^d_{\text{alg},X}(\text{Spec } C) = \text{NS}^d_{\text{alg}}(X)$. Thus it is a constant Nisnevich sheaf with transfer.

The proof for $\text{NS}^d_{\text{hom},X}$ is the same as above. 

2.5. Note that (2.4) also holds for $\mathbb{Q}$-coefficient. Thus from now, we can use the notations $\text{NS}^d_{\text{alg},\mathbb{Q}}(X)$, $\text{NS}^d_{\text{hom},\mathbb{Q}}(X)$ instead of $\text{NS}^d_{\text{alg},X,\mathbb{Q}}$ and $\text{NS}^d_{\text{hom},X,\mathbb{Q}}$ respectively following the convention in (1.10).

**Definition 2.6.** For $i \in \mathbb{Z}$, we denote by $h_i : \text{DM}^{\text{eff}} \to \text{Sh}^{tr}(\text{Sm}/C)$ the homology functor obtained by the homotopy $t$-structure defined in [1, Definition 3.1]. Here, $\text{Sh}^{tr}(\text{Sm}/C)$ denote the category of sheaves with transfer on $\text{Sm}/C$ with coefficient $\mathbb{Q}$.

2.7. Let $X$ be an $n$-dimensional connected scheme smooth and projective over $C$. As in [3, Section A.3] we have that

$$h_i(\text{Hom}(L^{n-d}, M(X))) = 0,$$

$$h_0(\text{Hom}(L^{n-d}, M(X))) \cong CH^d_X$$

in $\text{DM}^{\text{eff}}$ for $i < 0$. Then we have the morphisms

$$\text{Hom}(L^{n-d}, M(X)) \to h_0(\text{Hom}(L^{n-d}, M(X))) \cong CH^d_X$$

in $\text{DM}^{\text{eff}}$. We also have the morphism

$$CH^d_X \to \text{NS}^d_{\text{alg},X} = \text{NS}^d_{\text{alg}}(X)$$

in $\text{DM}^{\text{eff}}$ taking the quotient of $CH^d(Y \times X)$ for $Y \in \text{Sm}/C$. In conclusion, we have the morphism

$$\text{Hom}(L^{n-d}, M(X)) \to \text{NS}^d_{\text{alg}}(X)$$

(2.7.1)

in $\text{DM}^{\text{eff}}$. Thus we get the morphism

$$\text{Hom}(L^{n-d}, M(X)) \to \text{NS}^d_{\text{alg},\mathbb{Q}}(X)$$

(2.7.2)

in $\text{DM}^{\text{eff}}$. 

**Proposition 2.8.** Under the notations and hypotheses of (2.7), the above morphism has a section in $\text{DM}^{\text{eff}}_{\mathbb{Q}}$.

**Proof.** Since $\text{NS}^d_{\text{alg},\mathbb{Q}}(X)$ is a $\mathbb{Q}$-vector space, it has a basis $\{a_i\}_{i \in I}$ for some set $I$. Then $\text{NS}^d_{\text{alg},\mathbb{Q}}(X)$ is isomorphic to $\bigoplus_{i \in I} \mathbb{Q}$. In $\text{DM}^{\text{eff}}_{\mathbb{Q}}$, we have an isomorphism

$$\text{NS}^d_{\text{alg},\mathbb{Q}}(X) \cong \bigoplus_{i \in I} \mathbb{1}.$$
Now we have bijections
\[
\begin{align*}
\Hom_{\text{DM}_Q}^\text{eff}(\text{NS}_{d\text{alg},Q}(X), \text{Hom}(L^{n-d}, M(X))) & \cong \Hom_{\text{DM}_Q}^\text{eff}(\text{NS}_{d\text{alg},Q}(X) \otimes L^{n-d}, M(X)) \\
& \cong I \times \Hom_{\text{DM}_Q}^\text{eff}(L^{n-d}, M(X)) \cong I \times \text{CH}^d_Q(X) \cong \text{Hom}_{\text{Set}}(I, \text{CH}^d_Q(X)) 
\end{align*}
\]
where Set denotes the category of sets. Choose \( \{b_i \in \text{CH}^d_Q(X)\}_{i \in I} \) such that the image of \( b_i \) in \( \text{NS}_{d\text{alg},Q}(X) \) is \( a_i \). Then via (2.8.1), the function \( I \to \text{CH}^d(X) \) given by \( i \mapsto b_i \) corresponds to a section of (2.7.2).

2.9. The quotient homomorphism \( \text{NS}_{d\text{alg},Q}(X) \to \text{NS}_{d\text{hom},Q}(X) \) has a section since they are \( \mathbb{Q} \)-vector spaces. Thus we have a decomposition
\[
\text{NS}_{d\text{alg},Q}(X) \cong \text{NS}_{d\text{hom},Q}(X) \oplus \text{Griff}^d_Q(X),
\]
and then (2.8) completes the proof of (1.3(1)).

3. Proof of (1.3(2))

**Lemma 3.1.** Let \( X \) and \( Y \) be schemes of finite type over an algebraically closed field \( k \). Assume that \( X \) is integral and that each connected component of \( Y \) is integral. If \( X \) is quasi-projective over \( k \), then for any function \( f : \text{cl}(Y) \to \text{cl}(X) \), there are at most one morphism \( Y \to X \) of schemes inducing \( f \).

**Proof.** The question is Zariski local on \( Y \), so we reduce to the case when \( Y \) is integral and affine. Then the statement follows from the classical fact that the category of varieties quasi-projective over \( k \) is a full subcategory of the category of schemes over \( k \).

3.2. We review here several facts about intermediate Jacobians and Abel-Jacobi maps. Let \( X \) be an \( n \)-dimensional connected scheme smooth and projective over \( \mathbb{C} \), and let \( d \in [1, n] \) be an integer.

1. For \( x \in \text{cl}(X) \), we have the Albanese map
\[
\text{Alb}_{X,x} : X \to \text{Alb}(X)
\]
mapping \( x \) to 0.

2. We have the intermediate Jacobian \( J^d(X) \), which is a complex torus. See [13, Definition 12.2] for the definition.

3. We have the Abel-Jacobi map
\[
AJ^d_X : \text{CH}^d_{\text{hom}}(X) \to \text{cl}(J^d(X)).
\]
See [13, p. 294] for the definition.
(4) We have $J^d_a(X)$, which is an abelian subvariety of $J^d(X)$. See [12, 2.3.2] for the definition. We have the commutative diagram

\[
\begin{array}{ccc}
CH^d_{alg}(X) & \xrightarrow{AJ^d_X} & \text{cl}(J^d_a(X)) \\
\downarrow & & \downarrow \\
CH^d_{hom}(X) & \xrightarrow{AJ^d_X} & \text{cl}(J^d(X))
\end{array}
\]

of abelian groups where the vertical arrows are the obvious inclusions, and the upper horizontal arrow is surjective. When $d = n$, we have that $J^n_a(X) = J^n_a(X) = \text{Alb}(X)$.

(5) We denote by $\text{Alb}(X)$ (resp. $J^d_a(X)$) (by abuse of notation) the element in $\text{DM}^{eff}$ associated to the abelian variety $\text{Alb}(X)$ (resp. $J^d_a(X)$). See [11] for the definition. We also denote by $\text{Alb}_Q(X)$ (resp. $J^d_a, Q$) the corresponding object in $\text{DM}_Q^{eff}$.

(6) Let $Y$ be a scheme smooth over $\mathbb{C}$. By [10, §4], there is a homomorphism $AJ^d_{X,Y} : CH^d_{alg,X}(Y) \to \text{Hom}_{\text{Sch}_C}(Y, J^d_a(X))$ of abelian groups such that for $y \in \text{cl}(Y)$ and $Z \in CH^d_{alg,X}(Y)$, $AJ^d_{X,Y}(Z)$ is the morphism $Y \to J^d_a(X)$ mapping $y$ to $AJ^d_X(i^*_y Z)$. Here, $\text{Sch}_C$ denotes the category of $\mathbb{C}$-schemes, and $i_y : y \times X \to Y \times X$ denotes the closed immersion. Note that by (3.1), $AJ^d_{X,Y}$ is uniquely determined by the above information.

(7) Let $Y$ be an $m$-dimensional connected scheme smooth and projective over $\mathbb{C}$, and let $Z \in CH^d_X(Y)$ be an element. Consider the homomorphism $\psi_Z : \text{Alb}(Y) \to J^d_a(X)$ of abelian varieties induced by the morphism of the Hodge structures

\[H^{2m-1}(Y, Z) \to H^{2d-1}(X, Z)\]

induced by $Z$ (see [13, Theorem 12.17] for detail). Then by [10, §3, §4], for $y \in \text{cl}(Y)$ and $Z \in CH^d_X(Y)$, we have the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{AJ^d_{X,Y}} & J^d_a(X) \\
\text{Alb}(Y) & \xrightarrow{\psi_Z} & \text{Alb}(Y)
\end{array}
\]

of schemes where $Z' = Z - C \times i_y^* Z$. Here, $i_y : y \times X \to Y \times X$ denote the closed immersion.

**Proposition 3.3.** Let $X$ be an $n$-dimensional connected scheme smooth and projective over $\mathbb{C}$, and let $d \in [1, n]$ be an integer. Then $AJ^d_X : CH^d_{alg,X} \to J^d_a(X)$ is a morphism of presheaves with transfer on $\text{Sm}/\mathbb{C}$.

**Proof.** Let $Y$ and $Y'$ be schemes smooth over $\mathbb{C}$, and let $V$ be a finite correspondence from $Y'$ to $Y$. The statement is that the diagram

\[
\begin{array}{ccc}
CH^d_{alg,X}(Y) & \xrightarrow{AJ^d_{X,Y}} & \text{Hom}_{\text{Sch}_C}(Y, J^d_a(X)) \\
\downarrow & & \downarrow \\
CH^d_{alg,X}(Y') & \xrightarrow{AJ^d_{X,Y'}} & \text{Hom}_{\text{Sch}_C}(Y', J^d_a(X))
\end{array}
\]

commutes.
of abelian groups commutes where Sch\textsubscript{C} denotes the category of C-schemes, and α and β denote the homomorphisms induced by V. To show this, we may assume that Y and Y' are connected and V is an elementary correspondence.

Here, we will review the definition of β given in \[11\ 3.1.2\]. Let \( f : Y' \to J'^d_a(X) \) be a morphism of schemes. If \( V \) has degree \( r \), then we have the morphism \( Y' \to Y'(r) \) induced by \( V \), and we have the morphisms

\[
Y' \to Y'(r) \xrightarrow{f(r)} (J'^d_a(X))^{(r)} \xrightarrow{\sum} J'^d_a(X)
\]

of schemes. Here, \( Y'(r) \), \( J'^d_a(X) \), and \( f(r) \) denote the symmetric powers. The composition is \( \beta(f) \).

Let \( Z \in CH\textsubscript{alg,X}(Y)^d \) be an element, and let \( y' \in cl(Y) \) be a closed point. Then via \( V \), \( y' \) corresponds to \( a_1y_1 + \cdots + a_sy_s \) for some \( a_1, \ldots, a_s \in \mathbb{N}^+ \) and \( y_1, \ldots, y_s \in cl(Y) \). By definition, \( AJ^d_X(Y,Z) \) maps \( y \in cl(Y) \) to \( AJ^d_X(i^*_y Z) \) where \( i_y : y \times X \to Y \times X \) denotes the closed immersion. Using the above description of β, we see that \( \beta(AJ^d_X(Y,Z)) \) maps \( y' \) to

\[
a_1AJ^d_X(i^*_y Z) + \cdots + a_sAJ^d_X(i^*_y Z).
\]

Since \( i^*_y(\alpha(Z)) = a_1i^*_y Z + \cdots + a_si^*_y Z \), we see that \( AJ^d_X(Y,Y,\alpha(Z)) \) maps \( y' \) to

\[
AJ^d_X(a_1i^*_y Z + \cdots + a_si^*_y Z) = a_1AJ^d_X(i^*_y Z) + \cdots + a_sAJ^d_X(i^*_y Z).
\]

Thus \( \beta(AJ^d_X(Y,Z)) \) and \( AJ^d_X(Y,Y,\alpha(Z)) \) maps \( y' \) to the same closed point of \( J'^d_a(X) \). Then by (3.1), (3.3.1) commutes.

3.4. By (3.3), we can consider

\[
AJ^d_{X,-} : CH^d_{alg,X} \to J'^d_a(X)
\]

as a morphism in DM\textsuperscript{eff}. In (2.7.1), we have the morphism

\[
\text{Hom}(\mathbb{L}^{n-d}, M(X)) \to NS_{alg}(X)^d
\]

in DM\textsuperscript{eff}. Let \( K \) denote its cocone. By (2.7), we have that \( h_0(K) \cong CH^d_{alg,X} \) and \( h_i(K) = 0 \) for \( i < 0 \). Thus we have the morphisms

\[
K \to h_0(K) \cong CH^d_{alg,X} \xrightarrow{AJ^d_{X,Y}} J'^d_a(X)
\]

in DM\textsuperscript{eff}. Consider the induced morphism

\[
\gamma : K_Q \to J'^d_{a,Q}(X)
\]

in DM\textsubscript{Q} where \( K_Q \) denotes the image of \( K \) in DM\textsubscript{Q}. Our next goal is to construct its section in DM\textsubscript{Q}.

In \[12\ 2.3.3\], it is shown that there is a curve \( C \) smooth and projective over \( C \) (not necessarily connected) and an element \( Z \in CH^d(C \times X) \) such that the induced homomorphism \( \psi_Z : \text{Alb}(C) \to J'^d_{a}(X) \) of abelian varieties is surjective. From (3.2.1), we have the commutative diagram

\[
\begin{array}{ccc}
M(C) & \longrightarrow & K_Q \\
\downarrow & & \downarrow \gamma \\
\text{Alb}_Q(C) & \xrightarrow{\psi_Z} & J'^d_{a,Q}(X)
\end{array}
\]
in $\text{DM}_{\mathbb{Q}}^{\text{eff}}$ where the left vertical arrow is induced by the Albanese map and the upper horizontal arrow is induced by $Z' = Z - C \times i'_y Z$. Here, $i'_y : y \times X \to Y \times X$ denotes the closed immersion.

The category whose objects are abelian varieties and the set of morphism from $A$ to $B$ are $\text{Hom}(A, B) \otimes \mathbb{Q}$ is semi-simple, so $\psi_{Z, \mathbb{Q}}$ has a section since $\psi_{Z}$ is surjective. Since $\text{Alb}_{\mathbb{Q}}(C)$ is a direct summand of $M(C)$ in $\text{DM}_{\mathbb{Q}}^{\text{eff}}$, the composition $M(C) \to J_{a, \mathbb{Q}}^d(X)$ has a section. Thus $\gamma$ has a section. This completes the proof of (1.3(2)) since $K_{\mathbb{Q}}$ is a direct summand of $\text{Hom}(L, M(X))$ by (1.3(1)).

4. Proof of (1.5)

Lemma 4.1. Let $M$ be an object of $\text{DM}_{\mathbb{Q}}^{\text{eff}}$, and let $\alpha, \beta : M \to M$ be projectors. We put

$$F = \text{im} \alpha, \quad G = \text{im} \beta.$$ 

Assume that $\text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(G, F) = 0$. Then $F \oplus G$ is a direct summand of $M$.

Proof. The assumption implies that $\alpha \beta = 0$. Using this, we have that

$$\alpha(\beta - \beta \alpha) = \alpha \beta - \alpha \beta \alpha = 0,$$

$$(\beta - \beta \alpha) \alpha = \beta \alpha - \alpha \beta = 0,$$

$$(\beta - \beta \alpha)^2 = \beta^2 - \beta \alpha \beta + \beta \alpha \beta \alpha = \beta - \beta \alpha.$$ 

Thus $\beta - \beta \alpha$ is a projector orthogonal to $\alpha$. Since

$$\beta(\beta - \beta \alpha) \beta = \beta^3 - \beta \alpha \beta = \beta,$$

$$\beta(\beta - \beta \alpha)(\beta - \beta \alpha) = \beta^3 - \beta \alpha \beta^2 - \beta^3 \alpha + \beta \alpha \beta^2 \alpha = \beta - \beta \alpha,$$

we have that $\text{im} \beta \cong \text{im}(\beta - \beta \alpha)$. Thus $\alpha + \beta - \beta \alpha$ is a projector whose image is isomorphic to $F \oplus G$. \hfill \square

4.2. Let $X$ be an $n$-dimensional connected scheme smooth and projective over $\mathbb{C}$ with $n \geq 2$, and let $x$ be a closed point of $X$. Note that $\mathbf{1}$ and $\text{Alb}_{\mathbb{Q}}(X)$ are direct summands of $M(X)$. Then

$$L^n \cong \text{Hom}(1, L^n), \quad L^{n-1} \otimes \text{Pic}^0_{\mathbb{Q}}(X) \cong \text{Hom}(\text{Alb}_{\mathbb{Q}}(X), L^n)$$

are direct summands of $\text{Hom}(M(X), L^n)$, which is isomorphic to $M(X)$ by [11, 16.24]. Thus using [7, 16.25], we see that $L^2$ and $L \otimes \text{Pic}^0_{\mathbb{Q}}(X)$ are direct summands of $\text{Hom}(L^{n-2}, M(X))$. We also have that

$$\text{NS}^2_{\text{hom}, \mathbb{Q}}(X) \oplus \text{Griff}^2_{\mathbb{Q}}(X) \cong \text{NS}^2_{\text{alg}, \mathbb{Q}}(X)$$

in $\text{DM}_{\mathbb{Q}}^{\text{eff}}$ by (2.9). Thus to prove (1.4), by (4.1), it suffices to show that

$$\text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(L^2, \text{Pic}^0_{\mathbb{Q}}(X) \otimes L) = 0, \quad \text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(L^2, J_{a, \mathbb{Q}}^2(X)) = 0,$$

$$\text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(L^2, \text{NS}^2_{\text{alg}, \mathbb{Q}}(X)) = 0, \quad \text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(\text{Pic}^0_{\mathbb{Q}}(X) \otimes L, J_{a, \mathbb{Q}}^2(X)) = 0,$$

$$\text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(\text{Pic}^0_{\mathbb{Q}}(X) \otimes L, \text{NS}^2_{\text{alg}, \mathbb{Q}}(X)) = 0, \quad \text{Hom}_{\text{DM}_{\mathbb{Q}}^{\text{eff}}}(J_{a, \mathbb{Q}}^2(X), \text{NS}^2_{\text{alg}}(X)) = 0.$$ 

These follow from [9, 7.3.10] because of the following reasons.
(i) The motive $L^2$ is isomorphic to $M_4(S_0)$ for some $S_0$.

(ii) The motive $Pr^0_{Q}(X) \otimes L$ is isomorphic to $M_5(S_1)$ for some $S_1$.

(iii) The motive $J^2_{eff}(X)$ is isomorphic to $M_1(S_2)$ for some $S_2$.

(iv) The motive $NS^2_{alg, Q}(X)$ is isomorphic to $M_0(S_3)$ for some $S_3$.

Here, $S_0$, $S_1$, $S_2$, and $S_3$ are (not necessarily connected) surfaces smooth and projective over $C$. This completes the proof of (1.3).

5. Conjectures

**Definition 5.1.** Let $X$ be an $n$-dimensional connected scheme smooth and projective over $C$, and let $d \in [1, n]$ be an integer. Consider the homomorphism

$$AJ^d_{X, Q} : CH^d_{hom, Q}(X) \to \text{cl}(J^d_a(X)) \otimes Z Q$$

of $Q$-vector spaces induced by $AJ^d_{X}$. We put

$$CH^2_{Jac, Q}(X) = \ker AJ^d_{X, Q}.$$  

**5.2.** Here, we give two conjectures other than (1.7).

**Conjecture 5.3.** Let $X$ an $n$-dimensional connected scheme smooth and projective over $C$ with $n \geq 2$. Then

$$CH^2_{Jac, Q}(X) \subset CH^2_{alg, Q}(X).$$

**5.4.** Let us conjecturally prove (5.3). The statement is that any element in the kernel of

$$AJ^2_{X, Q} : CH^2_{hom, Q}(X) \to \text{cl}(J^2_a(X)) \otimes Z Q$$

is algebraically equivalent to 0. Assume that $M(X)$ has a Chow-K"unneth decomposition $M_0(X) \oplus \cdots \oplus M_{2n}(X)$ in $DM^eff$. The conjectural Bloch-Beilinson filtration on $CH^2(X)$ expects that

$$\ker AJ^2_{X, Q} \cong \text{Hom}_{DM^eff}(L^{n-2}, M_{2n-2}(X)), \quad 0 = \text{Hom}_{DM^eff}(L^{n-2}, M_r(X))$$

(5.4.1)

for $r < 2n - 4$ and $r > 2n - 1$.

If some nonzero element in the kernel of $AJ^2_{X, Q}$ is not algebraically equivalent to 0, then it gives a direct summand 1 of $NS^2_{alg, Q}(X)$, which is also a direct summand of $\text{Hom}(L^{n-2}, M(X))$ in $DM^eff$ by (1.3). The induced morphism

$$1 \to \text{Hom}(L^{n-2}, M_0(X) \oplus M_1(X) \oplus \cdots \oplus M_{2n-3}(X) \oplus M_{2n-1}(X) \oplus M_{2n}(X))$$

in $DM^eff$ is 0 by (5.4.1) and the assumption that the element is in the kernel of $AJ^2_{X, Q}$. Thus we see that 1 is a direct summand of $\text{Hom}(L^{n-2}, M_{2n-2}(X))$. Conjecturally, we have that $M_{2n-2}(X) \cong L^{n-2} \otimes M_2(X)$ in $DM^eff$. Then by the cancellation law [7 16.25], we see that 1 is a direct summand of

$$\text{Hom}(L^{n-2}, M_{2n-2}(X)) \cong \text{Hom}(L^{n-2}, L^{n-2} \otimes M_2(X)) \cong \text{Hom}(1, M_2(X)) \cong M_2(X).$$

In particular, we have a nonzero morphism $M_2(X) \to 1$ in $DM^eff$. This contradicts to the conjecture [8 5.8].
Conjecture 5.5. Let $X$ be an $n$-dimensional connected scheme smooth and projective over $\mathbb{C}$, and let
$$M(X) = M_0(X) \oplus \cdots \oplus M_{2n}(X)$$
be a conjectural Chow-Künneth decomposition. Then
$$\text{Hom}(L^{n-d}, M_{2n-2d}(X)) = \text{NS}^d_{\text{hom}, \mathbb{Q}}(X),$$
$$\text{Hom}(L^{n-d}, M_{2n-2d+1}(X)) = \text{CH}^d_{\text{hom}, \mathbb{Q}}(X)/(\text{CH}^d_{\text{Jac}, \mathbb{Q}}(X) + \text{CH}^d_{\text{alg}, \mathbb{Q}}(X))) \oplus J^d_{\alpha, \mathbb{Q}}(X)$$
for any integer $d \in [1, d]$.

5.6. The meaning of the second equation is that the motive $\text{Hom}(L^{n-d}, M_{2n-2d+1}(X))$ is the direct sum of $J^d_{\alpha, \mathbb{Q}}(X)$ and the image of the homomorphism
$$\text{CH}^d_{\text{hom}, \mathbb{Q}}(X)/\text{CH}^d_{\text{alg}, \mathbb{Q}}(X) \to \text{cl}(J^d(X))/\text{cl}(J^d_{\alpha}(X))$$
of abelian groups. In particular, this implies that to study the motive, we do not need the whole complex torus $J^d(X)$.

References

[1] J. Ayoub The $n$-motivic $t$-structure for $n = 0$, 1 and 2, Advances in Mathematics 226 (2011), 111-138.

[2] D.-C. Cisinski and F. Déglise, Triangulated categories of mixed motives, preprint, arXiv:0912.2110v3, 2012.

[3] U. Janssen, Motivic Sheaves and Filtration on Chow groups, Proceedings of Symposia in Pure Mathematics 55 (I) (1994), 245–302.

[4] A. Huber, Slice filtration on motives and the Hodge conjecture (with an appendix by J. Ayoub), Math. Nachr. 281 (2008), 1764-1776.

[5] Huber and B. Kahn, The Slice filtration and mixed Tate motives, Compos. Math. 142, No. 4 (2006), 907-936.

[6] J. Murre, J. Nagel and C. Peters, Lectures on the Theory of Pure Motives, University Lecture Series 61, American Mathematical Society (2012).

[7] C. Mazza, V. Voevodsky and C. Weibel, Lecture Notes on Motivic Cohomology, Clay Monographs in Math. 2, AMS (2006)

[8] J. Murre, On the motive of an algebraic surface, J. Reine Angew. Math. 409 (1990), 190–204.

[9] B. Kahn, J. Murre and C. Pedrini, On the transcendental part of the motive of a surface, Algebraic cycles and motives. Vol. 2, 143202, Lond. Math. Soc. Lect. Notes Series, Cambridge University Press 307 (2007)

[10] D. Lieberman, Intermediate Jacobians, Algebraic Geometry Oslo 1970, Wolters-Noordhoff, Groningen (1972), 125-139

[11] F. Orgogozo, Isomotifs de dimension inférieure ou égale à 1, Manuscript a Math. 115 (2004), 339-360.

[12] C. Vial, Projectors on the intermediate algebraic Jacobians, New York J. Math. 19 (2013), 793-822.

[13] C. Voisin, Hodge Theory and Complex Algebraic Geometry I, Cambridge Studies in Advanced Mathematics 76, Cambridge University Press (2002)