THE PÄDE INTERPOLATION METHOD APPLIED TO ADDITIVE DIFFERENCE PAINLEVÉ EQUATIONS

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Abstract. We study Padé interpolation problems on an additive grid, related to additive difference (d-) Painlevé equations of type $E_8^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_3^{(1)}$. By choosing suitable Padé problems, we can derive time evolution equations, scalar Lax pairs of contiguous type and determinant formulae of special solutions given in terms of hypergeometric functions, for the corresponding d-Painlevé equations.

1. Introduction

We begin in Sections 1.1, 1.2 and 1.3 by introducing the background of time evolution equations, Lax forms and hypergeometric solutions to the additive difference (d-) Painlevé equation. Next, in Section 1.4 we introduce the background of the Padé method. In Section 1.5 we state the purpose and organization of this paper.

1.1. The background of continuous/discrete Painlevé equations. Both the second order continuous and discrete Painlevé equation has been well studied in mathematics and physics (e.g. [5, 8]). In the geometric approach, for each Painlevé equation, K. Okamoto constructed certain rational surfaces, called the “spaces of initial values”, which parametrize all the solutions [35]. Furthermore K. Takano found that the Painlevé equations are uniquely determined by the spaces of initial values [23, 41]. Extending these works, H. Sakai proposed a certain class of second order continuous/discrete Painlevé equations, as mentioned below.

In Sakai’s theory [39], the continuous/discrete Painlevé equations have been classified on the basis of the spaces of initial values connected to extended affine Weyl groups. The spaces of initial values are obtained from $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) by blowing up at 9 (resp. 8) points. In view of the configuration of 9 (resp. 8) points in $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$), there exist three types of discrete Painlevé equations in the classification: elliptic difference ($e$-), multiplicative difference ($q$-), additive difference ($d$-) and continuous (differential) types. The only $e$-Painlevé equation [34] possesses the extended affine Weyl group symmetry of type $E_8^{(1)}$ and is obtained from the most generic configuration. All the other Painlevé equations are obtained from its degeneration.

The second order continuous/discrete Painlevé equations are classified into the 22 cases as in Figure 1.

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1Some $q$-Painlevé equations, such as a second order case of the system [16] (see also [24]), do not belong to the list of discrete Painlevé equations appearing in [39].
ell. (e-)  $E_8^{(1)}$

mul. (q-)  $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_{1,0})^{1,14} \rightarrow A_1^{(1)}$

add. (d-)  $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)} \rightarrow (P_{VI}) \rightarrow (P_V) \rightarrow 2(A_1)^{(1)} \rightarrow (A_1 + A_{1,0})^{1,4} \rightarrow A_1^{(1)}$

Figure 1. degeneration diagram of affine Weyl group symmetries

Here the symbol $A \rightarrow B$ represents that $B$ is obtained from $A$ by a certain limiting procedure. The $d$-Painlevé equation of type $D_4^{(1)}$ and its degeneration arise from Bäcklund (Schlesinger) transformations of the continuous Painlevé equations \cite{1} ($P_{II}, \ldots P_{VI}$). The symbol $A_1^{(1)}$ means the root subsystem of type $A_1^{(1)}$ whose square length of roots is $l$.

In this paper we put an emphasis on studying the $d$-Painlevé equations of type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_4^{(1)}$ through Padé interpolated problems on the additive grid as in Table 2 of Section 1.4. In Sections 1.2 and 1.3 we briefly mention the background of Lax forms and hypergeometric special solutions for the $d$-Painlevé equations.

1.2. The background of Lax forms for $d$-Painlevé equations. The continuous Painlevé equations are obtained from deformation theory of linear differential equations of $2 \times 2$ matrix type (see \cite{14}). The continuous Painlevé equations are given as the forms of the compatibility conditions between the linear differential equations and the corresponding differential deformation equations.

The $d$-Painlevé equation of type $D_4^{(1)}$ and its degeneration can be characterized by the same linear differential equations as the corresponding continuous Painlevé equations and additive deformations (Schlesinger transformations). For example, some $2 \times 2$ matrix Lax pairs for types from $d$-$D_4^{(1)}$ to $d$-$A_4^{(1)}$ have been derived in \cite{11}, using a Schlesinger transformation of differential equations. We call the linear differential equation the “differential Lax form”. The $d$-Painlevé equations of type $E_8^{(1)}$, $E_7^{(1)}$ and $E_6^{(1)}$ do not correspond to any continuous Painlevé equation, and the differential $2 \times 2$ matrix Lax forms of these $d$-Painlevé equations have been unknown. However, some $6 \times 6$, $4 \times 4$ and $3 \times 3$ matrix Lax forms have been constructed as a certain Fuchsian system of differential equations in \cite{3} for types $d$-$E_8^{(1)}$, $d$-$E_7^{(1)}$ and $d$-$E_6^{(1)}$ respectively.

On the other hand, it is also known that all the $d$-Painlevé equations are obtained from a compatibility condition of a linear additive (difference) equation and its additive deformation (e.g. additive matrix type \cite{11}, additive scalar type \cite{17, 25}). For example, some $2 \times 2$ matrix Lax forms have been obtained utilizing moduli spaces of difference connections on $P^1$ in \cite{11} for type $d$-$E_8^{(1)}$ and $d$-$D_4^{(1)}$. We call the linear additive (difference) equation the “additive (difference) Lax form”.

$P_{III}^{P(1)}$ symbolizes $P_{III}$ having the surface connected to the affine root system of type $D_i^{(1)}$. 
1.3. **The background of hypergeometric special solutions to d-Painlevé equations.** The continuous Painlevé equations, namely $P_{II}, \ldots, P_{VI}$, admit special solutions expressible in terms of various hypergeometric functions. The coalescence cascade of the hypergeometric functions, from the Gauss hypergeometric function $\binom{2}{1}$ to the Airy function, corresponds to that of continuous Painlevé equations, from $P_{VI}$ to $P_{II}$ in such works as [9, 13, 17]. Therefore, the $d$-Painlevé equation of type $D^{(1)}_4$ and its degenerations admit special solutions expressed in terms of the same hypergeometric functions as the continuous Painlevé equations. Also, the special solutions to the $d$-Painlevé equations of type $E^{(1)}_8$, $E^{(1)}_7$ and $E^{(1)}_6$ have been given in terms of the generalized hypergeometric functions $\binom{k}{l}$ in [15, 17].

Let us define the additive shifted factorials and the HGF (the generalized hypergeometric series [2, 10]) as follows:

\[ (a_1, a_2, \ldots, a_i)_j = (a_1)^j a_2 \cdots a_i \]
\[ (a_i)_j = \frac{\Gamma(a_i + j)}{\Gamma(a_i)}. \]

Here, we consider the condition $k = l + 1$. If the parameters $a_i, b_i$ satisfy the relation $a_1 + \ldots + a_{l+1} + n = b_1 + \ldots + b_l$, the series are called “$n$-balanced”. If the parameters satisfy the relations $1 + a_1 = b_1 + a_2 = \ldots = b_l + a_{l+1}$ and $a_2 = 1 + a_1/2$, the series are called “very-well-poised”.

Then, the hypergeometric solutions to the $d$-Painlevé equations are summarized as in Figure 2.

**Figure 2.** Degeneration diagram of hypergeometric solutions

**Remark 1.3.1. On the transformation formulas of hypergeometric series**

The terminating $\binom{7}{6}$ can be rewritten as the terminating $\binom{4}{3}$ (e.g. [2]) and the terminating Kummer (confluent hypergeometric) function $\binom{1}{0}$ can be also rewritten as the terminating $\binom{2}{0}$ (e.g. [18]). □
1.4. The background of the Padé method. There exists a certain connection among Padé approximation/interpolation and continuous/discrete Painlevé/Garnier systems. The Padé method gives time evolution equations, scalar Lax pairs and determinant formulae of special solutions simultaneously, by starting from suitable problems of Padé approximation (of differential grid)/interpolation (of difference grid) as in Table 2. In [47] the Padé method has been applied to continuous Painlevé equations of type $P_{VI}$, $P_{V}$, $P_{IV}$ and the continuous Garnier system using differential grid (i.e. Padé approximation) by Y. Yamada.

The Padé approximation is an approximation of a given function by a rational function of given order. A typical formulation is as follows. For a given function $\psi(x)$ analytic around $x = 0$, we want to find polynomials $P(x)$ and $Q(x)$ of degree $m$ and $n \in \mathbb{Z}_{\geq 0}$ respectively, such that

$$
\psi(x) = \frac{P(x)}{Q(x)} + O(x^{m+n+1}) \quad \text{(differential grid)}.
$$

The Padé interpolation is a discrete analog of the Padé approximation as follows. For a given sequence $\psi_s$, we want to find polynomials $P(x)$ and $Q(x)$ of degree $m$ and $n \in \mathbb{Z}_{\geq 0}$, by the interpolation condition

$$
\psi_s = \frac{P(x_s)}{Q(x_s)} \quad (s = 0, 1, \ldots, m + n) \quad \text{(difference grid)}.
$$

Representative choices of the interpolating points (difference grids) are the following.

| Difference grid   | Interpolating point                  |
|-------------------|-------------------------------------|
| additive         | $x_s = s$                           |
| additive quadratic | $x_s = s^2 + \nu s$               |
| $q$              | $x_s = q^s$                         |
| $q$-quadratic    | $x_s = q^s + \kappa q^{-s}$        |
| elliptic         | $x_s = \frac{[s + \alpha][s + \beta]}{[s + \gamma][s + \delta]}$ |
|                  | $\alpha + \beta = \gamma + \delta$ |
|                  | $[x]$ is the theta function         |

Table 1. Interpolating grids

As is shown in Table 2, the Padé method has been applied to discrete Painlevé/Garnier systems using the various kinds of grids mentioned above.

In this paper, we apply Padé interpolation on the additive grid to the additive Painlevé equations of type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_3^{(1)}$.

Remark 1.4.1. On the key points of the Padé method

We have two key points on the application for the Padé approximation/interpolation method [12, 24, 26, 27, 28, 29, 30, 33, 47, 50, 51]. The first point is how to choose approximated/interpolated functions (see Table 3 and Remark 2.1.2). The second point is to consider two linear continuous/difference three term relations (e.g. (2.2), called “contiguity relations”) satisfied by the error

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3Recently it has been shown in [20, 21, 22, 30] that Hermite-Padé approximation is related to the continuous Garnier system.

4The Padé interpolation (Cauchy 1821, Jacobi 1846) is older than the Padé approximation (Padé 1892).
Table 2. Previous works of Padé method

| Grid          | discrete | continuous |
|--------------|----------|------------|
| elliptic     | [51]     | [33]       |
| $q$-quadratic| [30]     | [30, 50]   |
| $q^2$         | [28, 29, 30] | [12, 24, 27, 30] |
| additive quadratic |         | this paper |
| differential  | [28, 30] | [12, 26, 30] | [30, 47] | [30, 47] |

Remark 1.4.2. On a connection between the Padé method and the theory of semiclassical orthogonal polynomials

The connection among semiclassical orthogonal polynomials (classical orthogonal polynomials related to a suitable weight function) and Painlevé/Garnier systems has been demonstrated in [19]. It has been shown that coefficients of three term recurrence relations, satisfied by several semiclassical orthogonal polynomials, can be expressed in terms of solutions of Painlevé/Garnier systems (see [4, 31, 37, 43, 44, 45, 46] for example). Thus there exists a close connection between the Padé method and the theory of semiclassical orthogonal polynomials. Namely, using both approaches, we can obtain the evolution equations, the Lax pairs and the special solutions for the corresponding continuous/discrete Painlevé/Garnier systems. □

1.5. The purpose and the organization of this paper. The purpose of this paper is to apply the Padé interpolation method on the additive grid to type $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$. As the main results given in Section 3, the following items are presented for each type.

(a) Setting of the Padé interpolation problem on the additive grid.
(b) Contiguity three term relations.
(c) The time evolution equation of the $d$-Painlevé equation.
(d) The additive difference Lax form of scalar type.
(e) Determinant formulae of hypergeometric special solutions.

This paper is organized as follows: In Section 2 we explain the Padé interpolation method through the items (a)–(e) above. In Section 3 we present main results for type $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$. In Section 4 we give a summary and discuss some future problems.

5The theory of semiclassical orthogonal polynomials give more general solutions of Painlevé/Garnier systems and the Padé method is simpler to compute. For example, their relation was briefly proved in [47].
In this section, for the additive grid case, we explain the methods for deriving the items (a)–(e) in the main results given in Section 3. In the item (a), the interpolated functions and interpolated sequences are given as in Table 3. For the items (b)–(e), we partly change the $q$-grid case to the additive grid case.

2.1. (a) Setting of the Padé interpolation problem on the additive grid.

2.1.1. Padé interpolation problem on additive grid. Let us consider the following interpolation problem.

For a given function $Y(x)$, we look for functions $P_m(x)$ and $Q_n(x)$ which are polynomials of degree $m$ and $n \in \mathbb{Z}_{\geq 0}$, satisfying the interpolation condition

\begin{equation}
Y(s) = \frac{P_m(s)}{Q_n(s)} \quad (s = 0, 1, \ldots, m + n).
\end{equation}

We call this problem the “Padé interpolation problem on the additive grid”, since the interpolation grid $s$ is an additive sequence (see Table 1). Then we call the function $Y(x)$ and the sequences $Y_s = Y(s)$ the “interpolated function” and “interpolated sequence”, respectively (see Table 3). Correspondingly we call both the polynomials $P_m(x)$ and $Q_n(x)$ “interpolating polynomials”. The explicit expressions of the interpolating polynomials $P_m(x)$ and $Q_n(x)$ are given in the formulae (2.11) (see the item (e)).

\[\text{Remark 2.1.1. On the common normalization factor of the polynomials } P_m(x) \text{ and } Q_n(x)\]

The interpolation condition (2.1) can not determine the common normalization factor of the interpolating polynomials $P_m(x)$ and $Q_n(x)$. However, this normalization factor is not essential for our arguments, i.e. the main results in Section 3 (see Remark 2.2.1). □

2.1.2. The interpolated function and sequence. Let $a_i, b_i, c$ and $d \in \mathbb{C}^\times$ be complex parameters. As is given in Table 3, we set up the interpolation problems (2.1) by specifying the interpolated functions $Y(x)$ and the interpolated sequences $Y_s = Y(s)$. We note that $a_1 + a_2 + a_3 + m - (b_1 + b_2 + b_3 + n) = 0$ is a constraint for the parameters only in the case $d-E^{(1)}_{12}$.

| $Y(x)$ | $d-E^{(1)}_{12}$ | $d-E^{(1)}_{13}$ | $d-D^{(1)}_4$ | $d-A^{(1)}_1$ |
|--------|-----------------|-----------------|---------------|---------------|
| $\begin{array}{c} Y(x) \\ i=1 \end{array}$ | $\begin{array}{c} \Gamma(a_i) \Gamma(x + b_i) \\ \Gamma(x + a_i) \Gamma(b_i) \end{array}$ | $\begin{array}{c} \Gamma(a_i) \Gamma(x + b_i) \\ \Gamma(x + a_i) \Gamma(b_i) \end{array}$ | $\begin{array}{c} \Gamma(a_1) \Gamma(x + b_1) \\ \Gamma(x + a_1) \Gamma(b_1) \end{array}$ | $\begin{array}{c} \Gamma(x + b_1) \\ \Gamma(b_1) \end{array}$ |
| $Y_s$ | $\begin{array}{c} \Gamma(a_i) \Gamma(x + b_i) \\ \Gamma(x + a_i) \Gamma(b_i) \end{array}$ | $\begin{array}{c} \Gamma(a_i) \Gamma(x + b_i) \\ \Gamma(x + a_i) \Gamma(b_i) \end{array}$ | $\begin{array}{c} \Gamma(a_i) \Gamma(x + b_i) \\ \Gamma(x + a_i) \Gamma(b_i) \end{array}$ | $\begin{array}{c} \Gamma(x + b_1) \\ \Gamma(b_1) \end{array}$ |
| HGF | $4F_3$ | $3F_2$ | $2F_1$ | $2F_0$ |

Table 3. Interpolated functions and interpolated sequences and HGFs

\[\text{Remark 2.2.1. (b)–(e) Setting of the Padé interpolation problem on the } q \text{-grid case.}\]

The given functions $Y(x)$ are interpolated by rational functions of given order. However $Y(x)$ need not be rational functions.
equations for the corresponding any object $F$. We note that these functions are special solutions to type $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$ in Figure 2 of Section 1.3.

Remark 2.1.2. On the choice of the interpolated functions $Y(x)$ and sequences $Y_s$. One may wonder how to choose the suitable interpolated functions $Y(x)$ and sequences $Y_s = Y(s)$ in Table 3. However, there is no guiding principle, i.e. only heuristics, to choose the functions $Y(x)$ and the sequences $Y_s$ in the Padé interpolation method, as far as we know. In this paper we succeed in choosing $Y_s$ and $Y(x)$ suitably as follows.

| $Y(q^s)$ | $q$-$E_7^{(1)}$ | $q$-$E_6^{(1)}$ | $q$-$D_4^{(1)}$ | $q$-$A_3^{(1)}$ |
|---|---|---|---|---|
| $\prod_{i=1}^{3} \left( \frac{\beta_i; q_s}{\alpha_i; q_s} \right)$ | $\prod_{i=1}^{2} \left( \frac{\beta_i; q_s}{\alpha_i; q_s} \right)$ | $\gamma^s \left( \frac{\beta_1; q_s}{\alpha_1; q_s} \right)$ | $\delta^s \left( \frac{\beta_1; q_s}{\alpha_1; q_s} \right)$ |

Table 4. Interpolated sequences for $q$-Painlevé equations in [24].

Step 1: We can choose the suitable sequences $Y_s$ of type $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$ by taking the replacement replacing $q = e^r$, $\alpha_i = e^{r_i}$, $\beta_i = e^{b_i}$, $\alpha_i = e^{a_i}$, $\gamma = c$, $\delta = -d/r$ and the limit $r \to 0$ for the sequences $Y(q^s)$ of type $q$-$E_7^{(1)}$, $q$-$E_6^{(1)}$, $q$-$D_4^{(1)}$ and $q$-$A_3^{(1)}$ in Table 4, respectively. Step 2: We can guess the suitable functions $Y(x)$ in response to the chosen sequences $Y_s$, respectively. We note that $\alpha_1 \alpha_2 \alpha_3 q^m / \beta_1 \beta_2 \beta_3 q^n = 1$ is a constraint for the parameters in the case $q$-$E_7^{(1)}$. The $q$-shifted factorial is defined by $(x; q)_n = \prod_{k=0}^{n-1} (1 - x q^k)$.

2.1.3. Time evolution. We give the parameter shift operators $T$ as in Table 5. Here the operators $T$ are called the “time evolutions”, since they specify the directions of the time evolution equations for the corresponding $d$-Painlevé equations.

| parameters | shifted parameters |
|---|---|
| $d$-$E_7^{(1)}$ $(a_1, a_2, a_3, b_1, b_2, b_3, m, n)$ | $(a_1 + 1, a_2, a_3 + 1, b_1 + 1, b_2, b_3 + 1, m, n)$ |
| $d$-$E_6^{(1)}$ $(a_1, a_2, b_1, b_2, m, n)$ | $(a_1 + 1, a_2, b_1 + 1, b_2, m, n)$ |
| $d$-$D_4^{(1)}$ $(a_1, b_1, c, m, n)$ | $(a_1 + 1, b_1 + 1, c, m, n)$ |
| $d$-$A_3^{(1)}$ $(b_1, d, m, n)$ | $(b_1 + 1, d, m, n)$ |

Table 5. Directions of time evolutions

We consider yet another Padé problem $\overline{Y}(s) = \overline{P}_m(s)/\overline{Q}_n(s)$ $(s = 0, 1, \ldots, m + n)$. Here, for any object $F$ the corresponding shifts are denoted by $\overline{F} := T(F)$ and $\overline{F}^{-1} := T^{-1}(F)$.

2.2. (b) Contiguity three term relations.

2.2.1. Contiguity relations by determinant expressions. Let us consider two linear three term relations: $L_2(x) = 0$ among $y(x), y(x + 1), \overline{y}(x)$ and $L_3(x) = 0$ among $y(x), \overline{y}(x), \overline{y}(x - 1)$ satisfied
by fundamental solutions, \( y(x) = P_m(x), Y(x)Q_n(x) \), where \( L_2 \) and \( L_3 \) are given as expressions

\[
L_2(x) \propto \begin{vmatrix}
    y(x) & y(x+1) & \overline{Y}(x) \\
    P_m(x) & P_m(x+1) & \overline{P}_m(x) \\
    Y(x)Q_n(x) & Y(x+1)Q_n(x+1) & \overline{Y}(x)\overline{Q}_n(x)
\end{vmatrix},
\]

\[
L_3(x) \propto \begin{vmatrix}
    y(x) & \overline{Y}(x) & \overline{Y}(x-1) \\
    P_m(x) & \overline{P}_m(x) & \overline{P}_m(x-1) \\
    Y(x)Q_n(x) & \overline{Y}(x)\overline{Q}_n(x) & \overline{Y}(x-1)\overline{Q}_n(x-1)
\end{vmatrix}.
\]

(2.2)

Here the symbol \( \propto \) means the direct proportion. Then we call the linear relations \( L_2 = 0 \) and \( L_3 = 0 \) the "contiguity relations", and the contiguity relations are the main subject in our study.

2.2.2. Computation method. Let us show the method of computation of the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \).

We set \( y(x) := \begin{vmatrix}
    P_m(x) \\
    Y(x)Q_n(x)
\end{vmatrix} \) and define Casorati determinants \( D_i(x) \) by

\[
D_1(x) := \det[y(x), y(x+1)], \quad D_2(x) := \det[y(x), \overline{Y}(x)], \quad D_3(x) := \det[y(x+1), \overline{Y}(x)].
\]

Then the expressions (2.2) can be rewritten as follows.

\[
L_2(x) \propto D_1(x)\overline{Y}(x) - D_2(x)y(x+1) + D_3(x)y(x),
\]

\[
L_3(x) \propto \overline{D}_1(x-1)y(x) + D_3(x-1)\overline{Y}(x) - D_2(x)\overline{Y}(x-1).
\]

Let us define basic quantities \( G(x), K(x) \) and \( H(x) \) (e.g. (3.4) and (3.15)) by

\[
G(x) := Y(x+1)/Y(x), \quad K(x) := \overline{Y}(x)/Y(x), \quad H(x) := \text{L.C.M}(G_{\text{den}}(x), K_{\text{den}}(x)).
\]

Here an abbreviation L.C.M represents the lowest common multiple, and a symbol \( \chi_{\text{den}}(x) \) (resp. \( \chi_{\text{num}}(x) \)) means a polynomial of the denominator (resp. numerator) in a rational function \( \chi(x) \). For example, in the case of \( d\text{-}E_7^{(1)} \), \( G_{\text{den}}(x) = \prod_{i=1}^{3}(x+a_i) \), \( G_{\text{num}}(x) = \prod_{i=1}^{3}(x+b_i) \), \( K_{\text{den}}(x) = (x+a_1)(x+a_3) \) and \( K_{\text{num}}(x) = (x+b_1)(x+b_3) \) (see eq. (3.4)). Substituting these quantities into the determinants (2.3), we obtain the expressions

\[
D_1(x) = \frac{Y(x)}{G_{\text{den}}(x)} E_1(z), \quad E_1(x) = G_{\text{num}}(x)P_m(x)Q_n(x+1) - G_{\text{den}}(x)P_m(x+1)Q_n(x),
\]

\[
D_2(x) = \frac{Y(x)}{K_{\text{den}}(x)} E_2(x), \quad E_2(x) = K_{\text{num}}(x)P_m(x)\overline{Q}_n(x) - K_{\text{den}}(x)\overline{P}_m(x)Q_n(x),
\]

\[
D_3(x) = \frac{Y(x)}{H(x)} E_3(x), \quad E_3(x) = \frac{H(x)}{K_{\text{den}}(x)} K_{\text{num}}(x)P_m(x+1)\overline{Q}_n(x) - \frac{H(x)}{G_{\text{den}}(x)} G_{\text{num}}(x)\overline{P}_m(x)Q_n(x+1).
\]

Using the interpolation condition (2.1) and the form of the basic quantities \( G(x), K(x) \) and \( H(x) \) (e.g. eqs. (3.4) and (3.15)), we can investigate positions of zeros (e.g. \( x = 0, 1, \ldots, m + n - 1 \)) and degrees of the polynomials \( E_i(x) \) (e.g. the polynomial \( E_1(x) \) is of degree \( m + n + 1 \) in \( x \)) in the expressions (2.6). Then we can simply compute the determinants \( D_i(x) \) (e.g. eqs. (3.5) and (3.16)) except for some factors such as \( x - f, x - g \) and \( c_i \) in \( D_i(x) \), where \( f, g \) and \( c_i \) are constants depending on parameters \( a_i, b_i, c \) but independent of \( x \) (see Remark 2.2.2). In this way we obtain the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) (e.g. eqs. (3.6) and (3.17)).

\[ \]
Remark 2.2.1. On the gauge invariance of the product \( C_0 C_1 \)
Changing the common normalization factor of the interpolating polynomials \( P_m(x) \) and \( Q_n(x) \), we can make an \( x \)-independent gauge transformation of \( y(x) \) in the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \). Under the \( x \)-independent gauge transformation of \( y(x) \): \( y(x) \mapsto G y(x) \), we can change the coefficients of \( \overline{y}(x), y(x-1), y(x) \) and \( y(x), \overline{y}(x), \overline{y}(x-1) \) in \( L_2 = 0 \) and \( L_3 = 0 \) as follows.

\[
\begin{align*}
( D_1(x) : D_2(x) : D_3(x)) & \mapsto ( \overline{G} D_1(x)/G : D_2(x) : D_3(x)) \\
(\overline{D}_1(x-1) : D_3(x-1) : D_2(x)) & \mapsto (G \overline{D}_1(x-1)/G : D_3(x-1) : D_2(x)).
\end{align*}
\]

Let us define the coefficients \( C_0 \) and \( C_1 \) in \( L_2 = 0 \) and \( L_3 = 0 \) (e.g. eqs. (3.6) and (3.17)) as the normalization factors of the coefficients of \( \overline{y}(x) \) and \( y(x) \), respectively. Then \( C_0 \) and \( C_1 \) change under the gauge transformation, although the product \( C_0 C_1 \) is a gauge invariant quantity. Furthermore, \( C_0 \) and \( C_1 \) do not appear in the final form of the \( d \)-Painlevé equations (e.g. eqs. (3.7) and (3.18)). □

Remark 2.2.2. On two meanings of the variables \( f, g \) and parameters \( m, n \)
We use \( f \) and \( g \) for two different meanings. The first meaning is constants (i.e. special solutions) \( f \) and \( g \) which are explicitly determined in terms of parameters \( a_i, b_j, m \) and \( n \) by the Padé interpolation problem (e.g. eqs. (3.5), (3.6), (3.11), (3.16), (3.17) and (3.22)). The second meaning is generic variables (i.e. generic solutions) \( f \) and \( g \) in \( \mathbb{P}^1 \) apart from the Padé interpolation problem (e.g. eqs. (3.7), (3.10), (3.18) and (3.21)), namely \( f \) and \( g \) are unknown functions in the \( d \)-Painlevé equation. In the items (c), (d) (resp. in the items (b) and (e)) we consider \( f \) and \( g \) in the second meaning (resp. in the first meaning).

Similarly, we use \( m \) and \( n \) for two meanings. In the first meaning, \( m \) and \( n \in \mathbb{Z}_{>0} \) are non-negative integer parameters (e.g. eqs. (3.2), (3.5), (3.6), (3.13), (3.16) and (3.17)). In the second meaning, \( m \) and \( n \in \mathbb{C}^\times \) are generic complex parameters, namely \( m \) and \( n \) are replaced by generic complex parameters \( a_0 \) and \( b_0 \in \mathbb{C}^\times \), respectively (e.g. eqs. (3.7), (3.10), (3.18) and (3.21)). In the items (c) and (d) (resp. in the items (a), (b) and (e)), we consider \( m \) and \( n \) in the second meaning (resp. in the first meaning). Then the result of the compatibility of the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) also holds with respect to the second meaning. □

2.3. (c) The time evolution equations of the \( d \)-Painlevé equation. The computation method is as follows. Let us consider generic variables \( f, g \in \mathbb{P}^1 \) and generic parameter \( a_0, b_0 \in \mathbb{C}^\times \) as in the second meaning in Remark 2.2.2. Then we can derive the \( d \)-Painlevé equation as the necessary condition for the compatibility of the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) (e.g. eqs. (3.6) and (3.17)). Computing the compatibility condition, we determine three variables \( g, \overline{f} \) and \( C_0 C_1 \). Expressions for variables \( g \) and \( \overline{f} \) are obtained in terms of variables \( f \) and \( g \). An expression for the product \( C_0 C_1 \) is obtained in terms of variables \( f, g \) and \( \overline{f} \) (and hence in terms of variables \( f \) and \( g \)).

Finally, we note the following. The first and second expressions are the \( d \)-Painlevé equation (e.g. eqs. (3.7) and (3.18)). The third expression is a constraint for the product \( C_0 C_1 \) (e.g. eqs. (3.8) and (3.19)).
Furthermore, thanks to the method above, we also construct scalar Lax pairs and determinant formulae of hypergeometric special solutions, while simultaneously deriving these time evolution equations, as in Section 2.4 and 2.5.

### 2.4. (d) The additive difference Lax form of scalar type.

#### 2.4.1. Scalar Lax pair.

Let us consider a linear three term equation for the unknown function $y(x)$: $L_1(x) = 0$ among $y(x + 1), y(x), y(x - 1)$ and its deformation equation $L_2(x) = 0$ among $y(x), y(x + 1), y(x - 1)$, where $L_1$ and $L_2$ are given by the expressions

$$
L_1(x) = A_1(x)y(x - 1) + A_2(x)y(x) + A_3(x)y(x + 1),
$$

$$
L_2(x) = A_4(x)y(x) + A_5(x)y(x) + A_6(x)y(x - 1).
$$

We call the linear three term equation $L_1 = 0$ and its deformation equation $L_2 = 0$ (2.8) the “scalar Lax pair” if the compatibility condition of the linear equations $L_1 = 0$ and $L_2 = 0$ (2.8) is equivalent to a $d$-Painlevé equation. We note that the scalar Lax pair $L_1 = 0$ and $L_2 = 0$ is equivalent to the pair of contiguity relations $L_1 = 0$ and $L_3 = 0$.

#### 2.4.2. Computation method.

Let us show how to compute the scalar Lax pair. Similarly to the item (c), we consider generic variables $f, g \in \mathbb{P}^1$ and generic parameters $a_0, b_0 \in \mathbb{C}^\times$ as in the second meaning in Remark 2.2.2. We derive the Lax pair $L_1 = 0$ and $L_2 = 0$, which satisfies the compatibility condition, by using the results of the items (a)–(c) as follows. Firstly the Lax equation $L_2 = 0$ (e.g. eqs. (3.10) and (3.21)) in the item (d) is the same as the contiguity relation $L_2 = 0$ (e.g. eqs. (3.6) and (3.17)) in the item (c) under an $x$-independent gauge transform of $y(x)$ and changes of parameters. Secondly the Lax equation $L_1 = 0$ can be obtained as follows. Combining the contiguity relations $L_2 = 0$ and $L_3 = 0$ (e.g. eqs. (3.6) and (3.17)) under generic variables $f, g$ and generic parameters $a_0, b_0$, one obtains a linear equation among the three terms $y(x + 1), y(x)$ and $y(x - 1)$ (see Figure 3), whose coefficient functions depend on the variables $f, g, \overline{f}, C_0$ and $C_1$.

![Figure 3. Derivation of $L_1(x)$](image)

However, the variables $C_0$ and $C_1$ appear through the product $C_0C_1$. Therefore, expressing $\overline{f}$ (e.g. eqs. (3.7), (3.18)) and $C_0C_1$ (e.g. eqs. (3.8) and (3.19)) only in terms of $f$ and $g$, one obtains the Lax equation $L_1 = 0$ (e.g. eqs. (3.10) and (3.21)).

Then, under generic variables $f, g$ and generic parameters $a_0, b_0$, the $d$-Painlevé equation (e.g. eqs. (3.7) and (3.18)) is necessary and sufficient for the compatibility of the Lax pair $L_1 = 0$ and $L_2 = 0$ (e.g. eqs. (3.10) and (3.21)). The proof for the case of $d-E_7^{(1)}$ will be shown in Appendix A. The other cases $d-E_6^{(1)}, d-D_4^{(1)}$ and $d-A_4^{(1)}$ are similarly proved.
2.5. **(e) Determinant formulae of hypergeometric special solutions.** By construction, expressions for \( f \) and \( g \) as in the first meaning in Remark 2.2.2 give a special solution for the \( d \)-Painlevé equation. We present how to compute determinant formulae of the special solutions.

2.5.1. **Determinant formulae on the additive grid.** We derive the formulae (2.11), which are convenient for computing the special solutions \( f \) and \( g \). For a given sequence \( Y_s \), the polynomials \( P_m(x) \) and \( Q_n(x) \) of degree \( m \) and \( n \) for an interpolation problem

\[
(2.9) \quad Y_s = P_m(x_s)/Q_n(x_s) \quad (s = 0, 1, \ldots, m + n)
\]

are given by the determinant expression\(^8\):

\[
(2.10) \quad P_m(x) = F(x) \det \left[ \sum_{j=0}^{m+n} u_s x_s^{j+i-j} \right]_{i,j=0}^n, \quad Q_n(x) = \det \left[ \sum_{j=0}^{m+n} u_s x_s^{j+i-j}(x-x_i) \right]_{i,j=0}^{n-1},
\]

where \( u_s = Y_s/F'(x_s) \) and \( F(x) = \prod_{i=0}^{m+n} (x-x_i) \).

In the additive grid case of the problem (2.9) (i.e., the problem (2.1)), the formulae (2.10) takes the forms

\[
(2.11) \quad P_m(x) = \frac{F(x)}{(-m+n)^{m+n}} \det \left[ \sum_{j=0}^{m+n} Y_s^{(-m+n)} s^j \frac{x^j}{s!} \right]_{i,j=0}^n
\]

\[
Q_n(x) = \frac{1}{(-m+n)^{m+n}} \det \left[ \sum_{j=0}^{m+n} Y_s^{(-m+n)} s^j (x-s)^j \frac{x^j}{s!} \right]_{i,j=0}^{n-1}
\]

In the derivation of (2.11), we have used the differential coefficient

\[
(2.12) \quad F'(x_s) = (x_s - x_0) \ldots (x_s - x_{s-1})(x_s - x_{s+1}) \ldots (x_s - x_{m+n})
\]

\[
= (s-0)(s-1) \ldots (s-(s-1))(s-(s+1)) \ldots (s-(m+n))
\]

Therefore, substituting the values of \( Y_s \) (2.2) and \( F'(x_s) \) (2.12) into the formulae (2.11), one obtains the determinant formulae (2.11).

2.5.2. **Computation method.** Let us demonstrate how to compute the special solutions \( f \) and \( g \). We can derive the expressions for the special solutions \( f \) and \( g \) by comparing the determinants \( D_i(x) \) in eq. (2.6) and \( D_i(x) \) (e.g. eqs. (3.5) and (3.16)) in the item (b) as the identity with respect to the variable \( x \) and by applying the formulae (2.11).

In case of \( d-E_{ij}^{(1)} \), we make the following calculation. Firstly substituting \( x = -a_i \) \((i = 1, 2)\) into the determinants \( D_i(x) \) in eq. (2.6) and \( D_i(x) \) in (3.5) respectively, we construct an expression for the special solution \( f \) in the first equation of eq. (3.11) by comparing the two expressions for \( D_1(x) \) and by applying the formulae (2.11). Similarly substituting \( x = -b_i \) \((i = 2, 3)\) into the determinants \( D_i(x) \) in eq. (2.6) and \( D_i(x) \) in (3.5) respectively, we construct an expression for the special solution \( g \) in the second equation of eq. (3.11) by comparing the two expressions for \( D_3(x) \) and by applying the formulae (2.11).
In this section, for each case $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$, we show the main results by the method in Section 2.

We use the notations

\[ a_1a_2 \ldots a_n/b_1b_2 \ldots b_n := \frac{a_1a_2 \ldots a_n}{b_1b_2 \ldots b_n}, \quad N(x) := \prod_{i=0}^{m+n-1} (x-i), \]

\[ T_a(F) := F|_{a_i \rightarrow a_i+1}, \quad T^{-1}_a(F) := F|_{a_i \rightarrow a_i-1}, \]

for any quantity (or function) $F$ depending on variables $a_i$ and $b_i$.

### 3.1. Case $d$-$E_7^{(1)}$.

(a) Setting of the Padé interpolation problem on the additive grid

In Table (2.2) the interpolated function, the interpolated sequence and the constraint for the parameters are set up as

\[ Y(x) := \prod_{i=1}^{3} \frac{\Gamma(a_i)\Gamma(x+b_i)}{\Gamma(x+a_i)\Gamma(b_i)}, \quad Y_s = \prod_{i=1}^{3} \frac{(b_i)_s}{(a_i)_s}, \quad m + 3 \sum_{i=1}^{3} a_i = n + 3 \sum_{i=1}^{3} b_i, \]

and in Table (2.2) the time evolution is chosen as

\[ T : (a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (a_1 + 1, a_2, a_3 + 1, b_1 + 1, b_2, b_3 + 1, m, n). \]

(b) Contiguity three term relations

By Definition (2.5) we have the basic quantities

\[ G(x) = \prod_{i=1}^{3} \frac{(x+b_i)}{(x+a_i)}, \quad K(x) = \prod_{i=1,3} \frac{a_i(x+b_i)}{b_i(x+a_i)}, \quad H(x) = b_1b_3 \prod_{i=1}^{3} (x+a_i), \]

and by the expression (2.6) we obtain the Casorati determinants

\[ D_1(x) = c_0(x-f)N(x)Y(x)/G(x)_{\text{den}}, \quad D_2(x) = c_1(x-h)(x-m-n)N(x)Y(x)/K(x)_{\text{den}}, \]

\[ D_3(x) = c_1(x-g)N(x)Y(x) \prod_{i=1,3} (x+b_i)/H(x), \]

where $h = g + a_2 - b_1 - b_3 - n$. Here $f, g, c_0$ and $c_1$ are constants depending on parameters $a_i, b_i \in \mathbb{C}^\times (i = 1, 2, 3)$ and $m, n \in \mathbb{Z}_{\geq 0}$ but independent of $x$. Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by

\[ L_2(x) = C_0(f-x)\bar{y}(x) - (x+a_2)(x-m-n)(x-h)y(x+1) + (x+b_1)(x+b_2)(x-g)y(x), \]

\[ L_3(x) = C_1(x-f-1)y(x) + (x+a_1)(x+a_3)(x-g-1)\bar{y}(x) - x(x+b_2-1)(x-h)\bar{y}(x-1), \]

where $C_0 = b_1b_3c_0/c_1$ and $C_1 = a_1a_3c_0/c_1$.

Take note that in the items (c) and (d) below we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.6) for generic complex parameters $a_0, b_0$ (replacing $m, n \in \mathbb{Z}_{\geq 0}$ by $a_0, b_0 \in \mathbb{C}^\times$) and generic variables $f, g \in \mathbb{P}^1$ (depending on parameters $a_i, b_i \in \mathbb{C}^\times, i = 0, 1, 2, 3$) apart from the Padé interpolation problem (2.1) with eqs. (3.2) and (3.3) (see Remark 2.2.2).

(c) The time evolution equations of the $d$-Painlevé equation
Compatibility of the contiguity relations $L_2 = 0$ and $L_3 = 0$ \[ (3.6) \] gives the evolution equations and the product $C_0 C_1$ as follows.

\begin{align}
(3.7) \quad \frac{(f - h)(f - h + 1)}{(f - g)(f - g)} &= \frac{A_2(f)}{A_1(f)}, \quad \frac{(f - h)(f - h + 1)}{(f - g)(f - g)} = \frac{A_2(h)}{A_1(g)}
\end{align}

and

\begin{align}
(3.8) \quad C_0 C_1 &= (h - g)(h - g - 1)A_1(g)/(f - g)(f - g)
= (h - g)(h - g - 1)A_2(h)/(f - h)(f - h + 1),
\end{align}

where $A_1(x) = (x + a_2)(x + b_2)(x + 1)(x - a_0 - b_0)$ and $A_2(x) = \prod_{i=1,3}(x + a_i)(x + b_i)$.

The evolution equations \[ (3.7) \] are equivalent to the $d$-Painlevé equation of type $E_7^{(1)}$ given in \[ 15 \, 17 \, 25 \, 38 \, 40 \]. The eight singular points in coordinates $(f, g)$ are on the two lines $f = g$ and $f = h(= g + a_2 - b_1 - b_3 - b_0)$ as follows.

\begin{align}
(3.9) \quad (f, g) &= (-a_2, -a_2), (-b_2, -b_2), (-1, -1), (-a_0 - b_0, -a_0 - b_0),
(-a_1, a_3 - b_2 + a_0), (-b_3, b_1 - a_2 + b_0), (-a_3, a_1 - b_2 + a_0), (-b_1, b_3 - a_2 + b_0).
\end{align}

\textbf{(d) The additive difference Lax form of scalar type}

The contiguity relations $L_2 = 0$ and $L_3 = 0$ \[ (3.6) \] give two scalar additive Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

\begin{align}
(3.10) \quad L_1(x) &= (g - h)\left[ \frac{A_1(g)}{(f - g)(x - g - 1)} - \frac{A_2(h)}{(f - h)(x - h)} \right] y(x)
- x \prod_{i=1}^3(x + b_i - 1)\left[ y(x - 1) - \frac{(x - h - 1)(x - a_0 - b_0 - 1)(x + a_2 - 1)}{(x - g - 1)(x + b_1 - 1)(x + a_2)} y(x) \right]
- \frac{(x - a_0 - b_0) \prod_{i=1}^3(x + a_i)}{x - f} \left[ y(x + 1) - \frac{(x - g)(x + b_1)(x + b_3)}{(x - h)(x - a_0 - b_0)(x + a_2)} y(x) \right],
\end{align}

\begin{align}
L_2(x) &= (f - x)\sqrt{y(x)} - (x + a_2)(h)(x - a_0 - b_0)y(x + 1) + (x + b_1)(x + b_3)(x - g)y(x),
\end{align}

where $h$ and $\lambda_i(x)$ are given in the Casorati determinant \[ (3.5) \] and the evolution equations \[ (3.7) \].

The additive Lax form of scalar type $L_1 = 0$ and $L_2 = 0$ \[ (3.10) \] is equivalent to the scalar ones in \[ 17 \, 25 \] by using suitable gauge transformations of $y(x)$. On the other hand, the differential $4 \times 4$ matrix Lax form has been given as a certain Fuchsian system of differential equations in \[ 34 \].

\textbf{(e) Determinant formulae of hypergeometric special solutions}

The hypergeometric solutions are constructed as the explicit forms

\begin{align}
(3.11) \quad \frac{f + a_i}{f + b_j} &= \alpha T_{a_i}(\tau_{m,n})T_{a_i}^{-1}(\tau_{m+1,n-1}) \quad (i, j = 1, 2, 3), \quad \frac{g + a_2}{h + a_1} = \beta T_{a_2}(\tau_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1}).
\end{align}
where the determinant $\tau_{m,n}$ is given by

$$
\tau_{m,n} = \det [(b_1)(a_1 - j)_{j=1} F_3 \left( \begin{array}{c} b_1 + i, b_2, b_3, -(m + n) \\ a_1 - j, a_2, a_3 \\ \end{array} \right) x^{m-j} y^{n-j}, 1]_{i,j=0},
$$

(3.12)

$$
\alpha = -\frac{(a_1 + m + n)(a_1 - 1)^n(b_j - 1)^n \prod_{k=1}^3 (b_k - a_j)}{a_i^{n+1} b_j^n \prod_{k=1}^3 (a_k - b_j)}
$$

$$
\beta = \frac{b_1 b_3 (a_2 + m + n)(a_2 - 1)^n(b_2 - a_2)}{a_2^{n+1} a_3(b_1 - a_1)(b_3 - a_1)}.
$$

These determinant formulae of the generalized hypergeometric solutions (3.11) are given in terms of the hypergeometric function $4F_3$. A certain $1 \times 1$ determinant formula has been constructed in terms of the hypergeometric function $7F_6$ in [15, 17]. The terminating $7F_6$ can be transformed into the terminating $4F_3$ (see Remark 1.3.1).

3.2. Case $dF_{6}(1)$. (a) Setting of the Padé interpolation problem on the additive grid

In Table (2.2) the interpolated function and the interpolated sequence are set up as

$$
Y(x) := \prod_{i=1}^2 \frac{\Gamma(a_i) \Gamma(x + b_i)}{\Gamma(x + a_i) \Gamma(b_i)}, \quad Y_x = \prod_{i=1}^2 \frac{(b_i)_x}{(a_i)_x},
$$

(3.13)

and in Table (2.2) the time evolution is chosen as

$$
T : (a_1, a_2, b_1, b_2, m, n) \mapsto (a_1 + 1, a_2, b_1 + 1, b_2, m, n).
$$

(b) Contiguity three term relations

By Definition (2.5) we have the basic quantities

$$
G(x) = \prod_{i=1}^2 \frac{(x + b_i)(x + a_i)}{(x + a_i)}, \quad K(x) = \frac{a_1(x + b_1)}{b_1(x + a_1)}, \quad H(x) = b_1 \prod_{i=1}^2 (x + a_i),
$$

(3.15)

and by the expression (2.6) we obtain the Casorati determinants

$$
D_1(x) = c_0(x - f)N(x)Y(x)/G(x)_{d\text{en}}, \quad D_2(x) = c_1(x - m - n)N(x)Y(x)/K(x)_{d\text{en}},
$$

$$
D_3(x) = c_1(x - g)N(x)Y(x)(x + b_1)/H(x),
$$

(3.16)

where $f, g, c_0$ and $c_1$ are constants depending on parameters $a_i, b_i \in \mathbb{C}^\times (i = 1, 2)$ and $m, n \in \mathbb{Z}_{\geq 0}$ but independent of $x$. Then the contiguity relations $L_2 = 0$ and $L_3 = 0$ are expressed by

$$
L_2(x) = C_0(f - x)\overline{\gamma}(x) - (x - m - n)(x + a_2)y(x + 1) + (x + b_1)(x - g)y(x),
$$

$$
L_3(x) = C_1(x - g - 1)y(x) + (x + a_1)(x - g - 1)\overline{\gamma}(x) - x + b_2 - 1)(x - h)\overline{\gamma}(x - 1),
$$

where $C_0 = b_1 c_0 / c_1$ and $C_1 = a_1 c_0 / c_1$.

Take note that in the items (c) and (d) below we study the contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.17) for generic complex parameters $a_0, b_0$ (replacing $m, n \in \mathbb{Z}_{\geq 0}$ by $a_0, b_0 \in \mathbb{C}^\times$) and generic variables $f, g \in \mathbb{P}^1$ (depending on parameters $a_i, b_i \in \mathbb{C}^\times, i = 0, 1, 2$) apart from the Padé interpolation problem (2.11) with eqs. (3.13) and (3.14) (see Remark 2.2.2).

(e) The time evolution equations of the $d$-Painlevé equation
Compatibility of the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) (3.17) gives the evolution equations and the product \( C_0 C_1 \) as follows.

\[
(f - g)(f - g) = \frac{(f + a_2)(f + b_2)(f + 1)(f - a_0 - b_0)}{(f + a_1)(f + b_1)},
\]

\[
(f - g)(\overline{f - g}) = \frac{(g + a_2)(g + b_2)(g + 1)(g - a_0 - b_0)}{(g - a_1 + b_2 - a_0)(g + a_2 - b_1 - b_0)}
\]

and

\[
C_0 C_1 = (g + a_2)(g + b_2)(g + 1)(g - a_0 - b_0)/(f - g)(\overline{f - g}) = (g - a_1 + b_2 - a_0)(g + a_2 - b_1 - b_0).
\]

The evolution equations (3.18) are equivalent to the \( d \)-Painlevé equation of type \( E_6^{(1)} \) given in [15, 17, 25, 35, 40]. The eight singular points in coordinates \((f, g)\) are on the three lines \( f = g, \quad f = \infty \) and \( g = \infty \) as follows.

\[
(f, g) = (-a_2, -a_2), (-b_2, -b_2), (-1, -1), (a_0 + b_0, a_0 + b_0),
\]

\( \quad (\infty, a_3 - b_2 + a_0), (\infty, b_1 - a_3 + b_0), (-a_1, \infty), (-b_1, \infty). \)

(d) The additive difference Lax form of scalar type

The contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) (3.17) give two scalar additive Lax equations \( L_1 = 0 \) and \( L_2 = 0 \) expressed by

\[
L_1(x) = \left[ \frac{(g + a_2)(g + b_2)(g + 1)(g - a_0 - b_0)}{(f - g)(x - g - 1)} - (g - a_1 + b_2 - a_0)(g + a_2 - b_1 - b_0) \right] y(x)
\]

\[
- \frac{x \prod_{i=1}^2 (x + b_1 - 1)}{x - f - 1} \left[ y(x - 1) - \frac{(x - a_0 - b_0 - 1)(x + a_2 - 1)}{(x - g - 1)(x + b_1 - 1)} y(x) \right]
\]

\[
- \frac{(x - a_0 - b_0) \prod_{i=1}^2 (x + a_i)}{x - f} \left[ y(x + 1) - \frac{(x - g)(x + b_1)}{(x - a_0 - b_0)(x + a_2)} y(x) \right],
\]

\[
L_2(x) = C_0 (f - x) \overline{y}(x) - (x - a_0 - b_0)(x + a_2) y(x + 1) + (x + b_1)(x - g) y(x).
\]

The additive Lax form of scalar type \( L_1 = 0 \) and \( L_2 = 0 \) (3.21) is equivalent to the scalar ones in [17, 25] by using suitable gauge transformations of \( y(x) \). On the other hand, the differential \( 3 \times 3 \) matrix Lax form has been given as a certain Fuchsian system of differential equations in [4], and in [1] a certain additive Lax form of \( 2 \times 2 \) matrix type has been constructed utilizing moduli spaces of difference connections on \( \mathbb{P}^1 \) for type \( d-E_6^{(1)} \), called the difference Painlevé VI there.

(e) Determinant formulae of hypergeometric special solutions

The hypergeometric solutions are constructed as the explicit forms

\[
\frac{f + a_i}{f + b_j} = \frac{\alpha T_{a_i}(\tau_{m,n}) T_{a_i}^{-1}(\tau_{m+1,n-1})}{\beta T_{b_i}(\tau_{m,n}) T_{b_i}^{-1}(\tau_{m+1,n-1})} \quad (i, j = 1, 2), \quad g = -a_2 + \frac{\beta T_{a_2}(\tau_{m,n}) T_{a_2}^{-1}(\tau_{m+1,n-1})}{\alpha T_{a_1}(\tau_{m,n}) T_{a_1}^{-1}(\tau_{m+1,n-1})}.
\]
where the determinant \( \tau_{m,n} \) is given by
\[
\tau_{m,n} = \det \left[ (b_1)_i (a_1 - j) \right] \frac{F_2 \left[ b_1 + i, b_2, -(m + n) \atop a_1 - j, a_2 \right]}{1} ]_{i,j=0}^n.
\]
(3.23)

\[
\alpha = \frac{(a_1 + m + n)(a_1 - 1)^n (b_1 - 1)^n \prod_{k=1}^n (b_k - a_k)}{a_1^{e+1} b_1^e \prod_{k=1}^n (a_k - b_k)}.
\]
(3.24)

\[
\beta = \frac{b_1 (a_2 + m + n)(a_2 - 1)^n (b_2 - a_2)}{a_2^{e+1} (b_1 - a_1)}.
\]
(3.25)

These determinant formulae of the generalized hypergeometric solutions \( _3F_2 \) are given in terms of the hypergeometric function \( _3F_2 \). A certain \( 1 \times 1 \) determinant formula has been constructed in terms of the hypergeometric function \( _3F_2 \) in [15] [17].

3.3. Case \( d-D_3^{(1)} \). (a) Setting of the Padé interpolation problem on the additive grid

In Table (2.2) the interpolated function and the interpolated sequence are set up as
\[
Y(x) := c^x \frac{\Gamma(a_1) \Gamma(x + b_1)}{\Gamma(x + a_1) \Gamma(b_1)} Y_s = c^x (b_1)_s^{(a_1)_s}
\]
(3.26)

and in Table (2.2) the time evolution is chosen as
\[
T : (a_1, b_1, c, m, n) \mapsto (a_1 + 1, b_1 + 1, c, m, n).
\]
(3.27)

(b) Contiguity three term relations

By Definition (2.3) we have the basic quantities
\[
G(x) = c \frac{x + b_1}{x + a_1}, \quad K(x) = \frac{a_1 (x + b_1)}{b_1 (x + a_1)}, \quad H(x) = b_1 (x + a_1),
\]
(3.28)

and by the expression (2.6) we obtain the Casorati determinants
\[
D_1(x) = c_0 (x - f) N(x) Y(x) / G(x)_{den}, \quad D_2(x) = c_1 (x - m - n) N(x) Y(x) / K(x)_{den}.
\]
(3.29)

where \( f, c_0, c_1 \) and \( c_2 \) are constants depending on parameters \( a_1, b_1, c \in \mathbb{C}^\times \) and \( m, n \in \mathbb{Z}_{\geq 0} \) but independent of \( x \). Then the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) are expressed by
\[
L_2(x) = C_0 (f - x \gamma(x) - (x - m - n) \gamma(x) + (x + b_1) \gamma(x) / g,
\]
(3.30)

\[
L_3(x) = C_1 (f - x - 1) \gamma(x) + (x + a_1) \gamma(x) / g - c x \gamma(x - 1),
\]
where \( C_0 = b_1 c_0 / c_1, C_1 = a_1 c_0 / c_1 \) and \( g = c_1 / c_2 \).

Take note that in the items (c) and (d) below we study the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) \( (3.28) \) for generic complex parameters \( a_0, b_0 \) (replacing \( m, n \in \mathbb{Z}_{\geq 0} \) by \( a_0, b_0 \in \mathbb{C}^\times \) and generic variables \( f, g \in \mathbb{P}^1 \) (depending on parameters \( a_i, b_i, c \in \mathbb{C}^\times, i = 0, 1 \)) apart from the Padé interpolation problem \( (2.1) \) with eqs. (3.24) and (3.25) (see Remark [2.2.2].

(c) The time evolution equations of the \( d \)-Painlevé equation

Compatibility of the contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) \( (3.28) \) gives the evolution equations and the product \( C_0 C_1 \) as follows.
\[
g g = \frac{C (f + 1) (f - a_0 - b_0)}{(f + a_1) (f + b_1)}, \quad f + g = \frac{a_1 + a_0}{c g - 1} + \frac{b_1 + b_0}{g - 1} + a_0 + b_0 - 1
\]
(3.31)

and
\[
C_0 C_1 = (g - 1) (c g - 1) / g^2.
\]
(3.32)
The evolution equations (3.29) are equivalent to the d-Painlevé equation of type $D_4^{(1)}$ given in [15, 17, 25, 38, 40]. The eight singular points in coordinates $(f, g)$ are on the four lines $f = 0$, $g = 0$, $f = \infty$ and $g = \infty$ as follows.

\begin{equation}
(f, g) = (-a_1, \infty), (-b_1, \infty), (1/\varepsilon, [1 + (a_0 + a_1)e]/c)_2, \\
(1/\varepsilon, 1 + (b_0 + b_1)e)_2, (-1, \infty), (a_0 + b_0, \infty).
\end{equation}

Here, the third point is a double point at $(\infty, 1/c)$ with the gradient $f(g-1/c) = a_0 + a_1/c$ and the fourth point is a double point at $(\infty, 1)$ with the gradient $f(g-1) = b_0 + b_1$. (The meaning of the two double points is also written in [17].)

(d) The additive difference Lax form of scalar type

The contiguity relations $L_2 = 0$ and $L_3 = 0$ (3.28) give two scalar additive Lax equations $L_1 = 0$ and $L_2 = 0$ expressed by

\begin{equation}
L_1(x) = (g-1)(cg-1)\left\{ \frac{c(a_1 + a_0)}{cg-1} + \frac{b_1 + b_0}{g-1} \right\} y(x) - \frac{cx(x + b_1 - 1)}{x - f - 1} \left[ y(x-1) - \frac{g(x - a_0 - b_0 - 1)}{x + b_1 - 1} y(x) \right]
\end{equation}

\begin{equation}
L_2(x) = (f - x) y(x) - \frac{(x - a_0 - b_0)(x + a_1)}{g(x - a_0 - b_0)} y(x+1) - \frac{x + b_1}{g(x - a_0 - b_0)} y(x),
\end{equation}

The additive Lax form of scalar type (3.32) is equivalent to the scalar ones in [17, 25] by using suitable gauge transformations of $y(x)$. On the other hand, in [11] a certain additive Lax form of $2 \times 2$ matrix type has been obtained utilizing moduli spaces of difference connections on $\mathbb{P}^1$ for type $d$-$D_4^{(1)}$, called the difference Painlevé V. Concerning the differential Lax form for type $d$-$D_4^{(1)}$, the $2 \times 2$ matrix Lax pair and the scalar one have been derived respectively in [11] and [17] by using a Schlesinger transformation of differential equations.

(e) Determinant formulae of hypergeometric special solutions

The hypergeometric solutions are constructed as the explicit forms

\begin{equation}
\frac{f + a_1}{f + b_1} = \alpha \left\{ \frac{T_{a_1}(\tau_{m,n})^{-1}}{T_{b_1}(\tau_{m+1,n-1})}, \frac{1}{g} = 1 - \frac{b_1(c+1)}{b_1 - a_1} \frac{T_m(\tau_{m,n})}{T_{a_1}(\tau_{m,n}) T_{a_1}(\tau_{m+1,n-1})}, \right\}
\end{equation}

where the determinant $\tau_{m,n}$ is given by

\begin{equation}
\tau_{m,n} = \det \left[ (b_1)_{(a_1 - j)_{2F_1}} \left( \frac{b_1 + i, -(m+n)}{a_1 - j}, c \right) \right]_{i,j=0}^n, \quad \alpha = c \frac{(a_1 + m + n)(a_1 - 1)^n(b_1 - 1)^n}{a_1^{n+1} b_1^n}.
\end{equation}

These determinant formulae of the Gauss hypergeometric solutions (3.33) are given in terms of the hypergeometric function $2F_1$. A certain $1 \times 1$ determinant formula has been constructed in terms of the hypergeometric function $2F_1$ (e.g. [17]).

3.4. Case $d$-$A_1^{(1)}$. (a) Setting of the Padé interpolation problem on the additive grid

In Table (2.2) the interpolated function and the interpolated sequence are set up as

\begin{equation}
Y(x) := d^a \frac{\Gamma(x + b_1)}{\Gamma(b_1)}, \quad Y_s = d^s(b_1),
\end{equation}

and in Table (2.2) the time evolution is chosen as

\begin{equation}
T : (b_1, d, m, n) \mapsto (b_1 + 1, d, m, n).
\end{equation}
(b) Contiguity three term relations

By Definition (2.5) we have the basic quantities

\[
G(x) = d(x + b_1), \quad K(x) = \frac{x + b_1}{b_1}, \quad H(x) = b_1,
\]

and by the expression (2.6) we obtain the Casorati determinants

\[
D_1(x) = c_0(x - f)N(x)Y(x)/G(x)_{\text{den}}, \quad D_2(x) = c_1(x - m - n)N(x)Y(x)/K(x)_{\text{den}},
\]

\[
D_3(x) = c_2N(x)Y(x)(x + b_1)/H(x),
\]

where \(f, c_0, c_1\) and \(c_2\) are constants depending on parameters \(b_1, d \in \mathbb{C}^x\) and \(m, n \in \mathbb{Z}_{\geq 0}\) but independent of \(x\). Then the contiguity relations \(L_2 = 0\) and \(L_3 = 0\) are expressed by

\[
L_2(x) = C_0(f - x)\overline{Y}(x) - (x - m - n)y(x + 1) + (x + b_1)y(x)/g,
\]

\[
L_3(x) = C_1(x - f - 1)y(x) + \overline{Y}(x)/g - dx\overline{Y}(x - 1),
\]

where \(C_0 = b_1c_0/c_1, C_1 = \overline{c}_0/c_1\) and \(g = c_1/c_2\).

Take note that in the items (c) and (d) below we study the contiguity relations \(L_2 = 0\) and \(L_3 = 0\) (3.39) for generic complex parameters \(a_0, b_0\) (replacing \(m, n \in \mathbb{Z}_{\geq 0}\) by \(a_0, b_0 \in \mathbb{C}^x\) and generic variables \(f, g \in \mathbb{P}^1\) (depending on parameters \(a_0, b_0, b_1, d \in \mathbb{C}^x\)) apart from the Padé interpolation problem (2.1) with eqs. (3.35) and (3.36) (see Remark 2.2.2).

(e) The time evolution equations of the \(d\)-Painlevé equation

Compatibility of the contiguity relations \(L_2 = 0\) and \(L_3 = 0\) (3.39) gives the evolution equations and the product \(C_0C_1\) as follows.

\[
gg = \frac{f + b_1}{d(f + 1)(f - a_0 - b_0)}, \quad f + \overline{f} = \frac{1}{dg} + \frac{b_0 + b_1}{g - 1} + a_0 + b_0 - 1
\]

\[
C_0C_1 = d(g - 1)/g.
\]

The evolution equations (3.40) are equivalent to the \(d\)-Painlevé equation of type \(A_3^{(1)}\) given in [15] [17] [25] [38] [40]. The eight singular points in coordinates \((f, g)\) are on the four lines \(f = 0, g = 0, f = \infty\) and \(g = \infty\) as follows.

\[
(f, g) = (-1, 0), (a_0 + b_0, 0), (1/\varepsilon, \varepsilon/d(1 + m\varepsilon))_3, (1/\varepsilon, 1 + (b_0 + b_1)\varepsilon)_2, (-b_1, \infty).
\]

Here, the third point is a triple point at \((\varepsilon, 0)\) and the fourth point is a double point at \((\varepsilon, 1)\) with the gradient \(f(g - 1) = b_0 + b_1\). (The meaning of the triple and double points is also written in [17].)

(d) The additive difference Lax form of scalar type
The contiguity relations \( L_2 = 0 \) and \( L_3 = 0 \) \((3.39)\) give two scalar additive Lax equations \( L_1 = 0 \) and \( L_2 = 0 \) expressed by

\[
L_1(x) = (g - 1) \left[ \frac{1}{dg} + \frac{b_1 + b_0}{g - 1} - \frac{(x + f - a_0 - b_0)}{g - 1} \right] y(x) - \frac{x(x + b_1 - 1)}{x - f - 1} \left[ y(x - 1) - \frac{g(x - a_0 - b_0 - 1)}{x + b_1 - 1} y(x) \right]
\]

\[
L_2(x) = (f - x) y(x) - \frac{(x - a_0 - b_0)}{g(x - a_0 - b_0)} y(x + 1) + (x + b_1) y(x)/g.
\]

The additive Lax form of scalar type \( L_1 = 0 \) and \( L_2 = 0 \) \((3.43)\) is equivalent to the scalar ones in \([17, 25]\) by using suitable gauge transformations of \( y(x) \). Concerning the differential Lax form for type \( dA_1 \), the \( 2 \times 2 \) matrix Lax pair and the scalar one have been derived respectively in \([11]\) and \([17]\) by using a Schlesinger transformation of differential equations.

**Determinant formulæ of hypergeometric special solutions**

The hypergeometric solutions are constructed as the explicit forms

\[
f = -b_1 + \frac{db_1}{d(b_1 - 1)^n} \frac{T_{b_1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}{\tau_{m,n}\tau_{m+1,n-1}}, \quad g = 1 - db_1 \frac{\tau_{m,n}\tau_{m+1,n-1}}{\tau_{m,n}\tau_{m+1,n-1}},
\]

where the determinant \( \tau_{m,n} \) is given by

\[
\tau_{m,n} = \det \left( (b_1)_{i=0}^{-(m+n)} F_i \right)_{j=0}^{-(m+n)\ ell_j=0}.
\]

These determinant formulæ of the Kummer hypergeometric solutions \((3.44)\) are given in terms of the hypergeometric function \( _2F_0 \). A certain \( 1 \times 1 \) determinant formula has been constructed in terms of the hypergeometric function \( _1F_1 \) (e.g. \([17]\)). The terminating \( _1F_1 \) can be transformed into the terminating \( _2F_0 \) (see Remark \([1.3.1]\)).

4. Conclusions

**Summary.** In this paper for the interpolated function \( Y(x) \) and the interpolated sequence \( Y_i \) given in Table \([5]\) of Section \([2.1]\) we set up the Padé interpolation problem on the additive grid, related to the \( d \)-Painlevé equations of type \( E_7^{(1)}, E_6^{(1)}, D_4^{(1)} \) and \( A_3^{(1)} \). Then for the time evolution \( T \) given in Table \([5]\) of Section \([2.1]\) we set up another Padé interpolation problem on the additive grid. By choosing these suitable problems, we derived the evolution equations, the Lax pairs of scalar type and the determinant formulæ of the special solutions for the corresponding \( d \)-Painlevé equations. The main results were given in Section \([3]\).

**Problems.** As is shown in Table \([2]\) of Section \([1.4]\) some open problems related to the results of this paper are as follows:

1. One may be interested in studying whether the Padé interpolation method on the additive quadratic grid can be applied to the additive difference \((d-)\) Painlevé equations.
2. Differently from the additive grid, it may be interesting to investigate whether the Padé method on the differential grid (i.e. Padé approximation) can be also applied to the \( d \)-Painlevé equations by using a Schlesinger transformation of linear differential equations.
3. It may be interesting to study whether the Padé method can be further applied to other generalized \( d \)-Painlevé systems, for example an additive difference analogue of the Garnier system.
and a higher order Painlevé system, which are called “d-Garnier system” [6, 7, 36] and “higher order d-Painlevé system”, respectively.

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APPENDIX A. SUFFICIENCY FOR THE COMPATIBILITY OF THE LAX PAIR

In Section 3.1 we gave the d-E*(1) equation (3.7) as the necessary condition for the compatibility of the Lax pair (3.10). In this appendix, we prove that the d-E*(1) equation is the sufficient condition for the compatibility of the Lax pair.

As in Figure 3, eliminating \( f(x) \) and \( f(x - 1) \) from \( L_2(x) = L_2(x - 1) = L_3(x) = 0 \) (3.6), one constructs the linear equation \( L_1 = 0 \) among \( y(x + 1) \), \( y(x) \) and \( y(x - 1) \), where

\[
L_1(x) = \frac{(x - a_0 - b_0) \prod_{i=1}^{3} (x + a_i)}{x - f} y(x + 1) + \frac{x \prod_{i=1}^{3} (x + b_i - 1)}{(x - f - 1)} y(x - 1) \]

(A.1)

\[ - \frac{1}{x - h} A_2(x)(x - g) \]

and

\[
V(x) = (x - h)(x - h + 1)A_1(x) - C_0 C_1(x - f)(x - f).
\]

Here, the variable \( f \) and the product \( C_0 C_1 \) in (A.2) should be viewed as functions in terms of \( f \) and \( g \), and they are determined in (3.7) and (3.8), respectively. The expression \( L_1 \) (A.1) is rewritten into (3.10) by using (3.7) and (3.8).

Lemma A.0.1. The expression \((x - f)(x - f - 1)L_1(x) \) (A.1) (or (3.10)) has the following characterization:

(i) It is a linear equation among \( y(x + 1) \), \( y(x) \) and \( y(x - 1) \), and the coefficients of these terms are polynomials of degree 5 in \( x \).

(ii) The coefficients of \( y(x + 1) \) (resp. \( y(x - 1) \)) have zeros at \( x = -a_1, -a_2, -a_3, a_0 + b_0 \) (resp. \( x = -b_1 + 1, -b_2 + 1, -b_3 + 1, 0 \)).

(iii) Under the conditions

\[
\frac{y(x + 1)}{y(x)} = 1 + \frac{a_0}{x} + \frac{a_0(a_0 - 1)/2}{x^2} + \frac{w}{x^3} + O\left(\frac{1}{x^4}\right),
\]

(A.3)

\[
\frac{y(x - 1)}{y(x)} = 1 - \frac{a_0}{x} + \frac{a_0(a_0 - 1)/2}{x^2} - \frac{w}{x^3} + O\left(\frac{1}{x^4}\right),
\]

the terms \( x^3, \ldots, x^5 \) in the expression \((x - f)(x - f - 1)L_1(x) \) vanish, namely \((x - f)(x - f - 1)L_1(x) = O(x^1) \) around \( x = \infty \). Here, \( w \in \mathbb{C} \) is an arbitrary constant.
(iv) The equation \((x - f)(x - f - 1)L_1 = 0\) holds at the two points \(x = f, f + 1\), where
\[
(A.4) \quad \frac{y(f + 1)}{y(f)} = \frac{(f + b_1)(f + b_3)(f - g)}{(f + a_2)(f - a_0 - b_0)(f - h)}.
\]

Conversely, the expression \((x - f)(x - f - 1)L_1(x)\) is uniquely characterized by these properties (i) – (iv). □

**Proof.** The property (i) is obtained by the relations (3.8). Concretely, the expression \(\frac{V(x)}{x - f}\) reduces to a polynomial of degree 5 in \(x\) under the first relation of (3.8). Moreover, the coefficient of the term \(y(x)\) is obtained as a polynomial of degree 5 in \(x\) by using the second relation of (3.8). The property (ii) is trivial. The property (iii) can easily be checked by the condition (A.1). The property (iv) follows by substituting \(x = f, f + 1\) into the equation \(L_1(x) = 0\).

**Remark A.0.2.** Two points \(x = f, f + 1\) are apparent singularities in the sense that at those two points the equation \((x - f)(x - f - 1)L_1(x) = 0\) (A.7) is satisfied under the same condition (in this case (A.4)). □

Similarly, as in Figure 4 eliminating \(y(x)\) and \(y(x + 1)\) from \(L_2(x) = L_3(x) = L_3(x + 1) = 0\) (3.6), we obtain the linear equation \(L_1^*(x) = 0\) among \(\overline{y}(x + 1), \overline{y}(x)\) and \(\overline{y}(x - 1)\), where
\[
(A.5) \quad L_1^*(x) = \frac{(x - a_0 - b_0)(x + a_1 + 1)(x + a_2)(x + a_3 + 1)}{x - f} \overline{y}(x + 1) + \frac{x + b_1(x + b_2 - 1)(x + b_3)}{(x - f - 1)} \overline{y}(x - 1) - \frac{1}{x - h} \left\{ \frac{A_2(x)(x - g - 1)}{x - f - 1} + \frac{V(x)}{(x - f)(x - g)} \right\} y(x).
\]

**Figure 4.** Derivation of \(L_1^*(x)\)

The following Lemma (and its proof) is similar to Lemma (A.0.1).

**Lemma A.0.3.** The expression \((x - f)(x - f - 1)L_1^*(x)\) (A.5) has the following characterization:
(i) It is a linear three term expression among \(\overline{y}(x + 1)\) and \(\overline{y}(x)\) and \(\overline{y}(x - 1)\), and the coefficients of these terms are polynomials of degree 5 in \(x\).
(ii) The coefficients of \(\overline{y}(x + 1)\) (resp. \(\overline{y}(x - 1)\)) have zeros at \(x = -a_1 - 1, -a_2, -a_3 - 1, a_0 + b_0\) (resp. \(x = -b_1, -b_2 + 1, -b_3, 0\)).
(iii) Under the conditions
\[
(A.6) \quad \frac{\overline{y}(x + 1)}{\overline{y}(x)} = 1 + \frac{a_0}{x} + \frac{a_0(a_0 - 1)/2}{x^2} + \frac{w}{x^3} + O\left(\frac{1}{x^4}\right),
\]
\[
\frac{\overline{y}(x - 1)}{\overline{y}(x)} = 1 - \frac{a_0}{x} + \frac{a_0(a_0 - 1)/2}{x^2} - \frac{w}{x^3} + O\left(\frac{1}{x^4}\right),
\]
the terms \( x^5, \ldots, x^2 \) in the expression \((x-\bar{f})(x-\bar{f}-1)L_1^*(x)\) vanish, namely \((x-\bar{f})(x-\bar{f}-1)L_1^*(x) = O(x^3) \) around \( x = \infty \). Here, \( w \in \mathbb{C} \) is the same arbitrary constant as in (A.3).

(iv) The equation \((x-\bar{f})(x-\bar{f}-1)L_1^* = 0\) holds at the two points \( x = \bar{f}, \bar{f} + 1 \) where

\[
\frac{\gamma(\bar{f} + 1)}{\gamma(\bar{f})} = \frac{(\bar{f} + b_2)(\bar{f} + 1)(\bar{f} + h - 1)}{(\bar{f} + a_1 + 1)(\bar{f} + a_3 + 1)(\bar{f} - g)}.
\]

Conversely, the expression \((x-\bar{f})(x-\bar{f}-1)L_1^*\) is uniquely characterized by these properties (i) – (iv). □

The sufficiency for the compatibility means that \( T(L_1(x)) \propto L_1^*(x) \) holds when the \( d-E_7^{(1)} \) equation (3.7) is satisfied. In order to prove the sufficiency, we characterize \( L_1 \) and \( L_1^* \) as polynomials in terms of \( x \), and compare these characterizations.

**Proposition A.0.4.** The linear equations \( L_1 = 0 \) and \( L_2 = 0 \) (3.10) for the unknown function \( y(x) \) are compatible if and only if the \( d-E_7^{(1)} \) equation (3.7) is satisfied. □

**Proof.** The compatibility means that the shift operator \( T \) changes the equation \( L_1 = 0 \) into the equation \( L_1^* = 0 \), i.e. the commutativity in Figure 5

\[
L_1^* = 0 \quad \text{(Lemma A.0.3)} \iff L_1^* = 0 \quad \text{(A.5)}
\]

\[
\uparrow T\text{-shift (3.3)} \quad L_2 = L_3 = 0 \quad \text{(3.6)}
\]

\[
L_1 = 0 \quad \text{(Lemma A.0.1)} \iff L_1 = 0 \quad \text{(A.1)} \iff L_1 = 0 \quad \text{(3.10)}.
\]

**Figure 5.** Compatibility of \( L_1(x) \) and \( L_1^*(x) \)

This commutativity is almost clear from the characterizations (i), (ii) of the equation \( L_1 = 0 \) (respectively \( L_1^* = 0 \)) in Lemma A.0.1 (respectively Lemma A.0.3). The remaining task is to check that the operator \( T \) changes expression (A.4) into expression (A.7), utilizing the characterization (iii) of the equation \( L_1 = 0 \) (respectively \( L_1^* = 0 \)) and the first part of equation (3.7).

As the point of the proof, the following two are applied to type \( d-E_7^{(1)} \) together: The first is that the equation \( L_1(f, \bar{f}, g) = 0 \) in terms of \( f, \bar{f} \) and \( g \) is derived from the equations \( L_2(f, g) = 0 \) and \( L_3(f, g) = 0 \) (see [17, 33, 48, 49]). The second is that the equation \( L_1(f, \bar{f}, g) = 0 \) is characterized as a polynomial in terms of \( x \) (see [25, 28]).

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