Dilatation structures in sub-riemannian geometry

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Abstract. Based on the notion of dilatation structure [2], we give an intrinsic treatment to sub-riemannian geometry, started in the paper [4]. Here we prove that regular sub-riemannian manifolds admit dilatation structures. From the existence of normal frames proved by Bellaïche we deduce the rest of the properties of regular sub-riemannian manifolds by using the formalism of dilatation structures.

1. Introduction

Sub-riemannian geometry is the modern incarnation of non-holonomic spaces, discovered in 1926 by the romanian mathematician Gheorghe Vrăanceanu [22], [23]. The sub-riemannian geometry is the study of non-holonomic spaces endowed with a Carnot-Carathéodory distance. Such spaces appear in applications to thermodynamics, to the mechanics of non-holonomic systems, in the study of hypo-elliptic operators cf. Hörmander [14], in harmonic analysis on homogeneous cones cf. Folland, Stein [10], and as boundaries of CR-manifolds.

The interest in these spaces comes from several intriguing features which they have: from the metric point of view they are fractals (the Hausdorff dimension with respect to the Carnot-Carathéodory distance is strictly bigger than the topological dimension, cf. Mitchell [17]); the metric tangent space to a point of a regular sub-riemannian manifold is a Carnot group (a simply connected nilpotent Lie group with a positive graduation), also known classically as a homogeneous cone; the asymptotic space (in the sense of Gromov-Hausdorff distance) of a finitely generated group with polynomial growth is also a Carnot group, by a famous theorem of Gromov [11] which leads to an inverse to the Tits alternative; finally, on such spaces we have enough structure to develop a differential calculus resembling to the one proposed by Cheeger [9] and to prove theorems like Pansu’ version of Rademacher theorem [18], leading to an ingenious proof of a Margulis rigidity result.

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There are several fundamental papers dedicated to the establishment of the sub-riemannian geometry, among them Mitchell [17], Bellaïche [1], a substantial paper of Gromov asking for an intrinsic point of view for sub-riemannian geometry [13], Margulis, Mostow [15, 16], dedicated to Rademacher theorem for sub-riemannian manifolds and to the construction of a tangent bundle of such manifolds, and Vodopyanov [19] (among other papers), concerning the same subject.

There is a reason for the existence of so many papers, written by important mathematicians, on the same subject: the fundamental geometric properties of sub-riemannian manifolds are very difficult to prove. Maybe the most difficult problem is to provide a rigorous construction of the tangent bundle of such a manifold, starting from the properties of the Carnot-Carathéodory distance, and somehow allowing to generalize Pansu’ differential calculus.

In several articles devoted to sub-riemannian geometry, these fundamental results were proved using differential geometry tools, which are not intrinsic to sub-riemannian geometry, therefore leading to very long proofs, sometimes with unclear parts, corrected or clarified in other papers dedicated to the same subject.

The fertile ideas of Gromov, Bellaïche and other founders of the field of analysis in sub-riemannian spaces are now developed into a hot research area. For the study of sub-riemannian geometry under weaker than usual regularity hypothesis see for example the string of papers by Vodopyanov, among them [19], [20]. In these papers Vodopyanov constructs a tangent bundle structure for a sub-riemannian manifold, under weak regularity hypothesis, by using notions as horizontal convergence.

Based on the notion of dilatation structure [2], I tried to give a an intrinsic treatment to sub-riemannian geometry in the paper [4], after a series of articles [5], [6], [7] dedicated to the sub-riemannian geometry of Lie groups endowed with left invariant distributions.

In this article we show that normal frames are the central objects in the establishment of fundamental properties in sub-riemannian geometry, in the following precise sense. We prove that for regular sub-riemannian manifolds, the existence of normal frames (definition 3.7) implies that induced dilatation structures exist (theorems 6.3, 6.4). The existence of normal frames has been proved by Bellaïche [11], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62). From these facts all classical results concerning the structure of the tangent space to a point of a regular sub-riemannian manifold can be deduced as straightforward consequences of the structure theorems [4, 2] [4, 3] [4, 4] [4, 5] from the formalism of dilatation structures.

In conclusion, our purpose is twofold: (a) we try to show that basic results in sub-riemannian geometry are particular cases of the abstract theory of dilatation structures, and (b) we try to minimize the contribution of classical differential calculus in the proof of these basic results, by showing that in fact the differential calculus on the sub-riemannian manifold is needed only for proving that normal frames exist and after this stage an intrinsic way of reasoning is possible.
If we take the point of view of Gromov, that the only intrinsic object on a sub-riemannian manifold is the Carnot-Carathéodory distance, the underlying differential structure of the manifold is clearly not intrinsic. Nevertheless in all proofs that I know this differential structure is heavily used. Here we try to prove that in fact it is sufficient to take as intrinsic objects of sub-riemannian geometry the Carnot-Carathéodory distance and dilatation structures compatible with it.

The closest results along these lines are maybe the ones of Vodopyanov. There is a clear correspondence between his way of defining the tangent bundle of a sub-riemannian manifold and the way of dilatation structures. In both cases the tangent space to a point is defined only locally, as a neighbourhood of the point, in the manifold, endowed with a local group operation. Vodopyanov proves the existence of the (locally defined) operation under very weak regularity assumptions on the sub-riemannian manifold. The main tool of his proofs is nevertheless the differential structure of the underlying manifold. In distinction, we prove in [2], in an abstract setting, that the very existence of a dilatation structure induces a locally defined operation. Here we show that the differential structure of the underlying manifold is important only in order to prove that dilatation structures can indeed be constructed from normal frames.

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2. Metric profiles

Notations. The space $CMS$ is the collection of isometry classes of pointed compact metric spaces. The notation used for elements of $CMS$ is of the type $[X, d, x]$, representing the equivalence class of the pointed compact metric space $(X, d, x)$ with respect to (pointed) isometry. The open ball of radius $r > 0$ and center $x \in (X, d)$ is denoted by $B(x, r)$ or $B_d(x, r)$ if we want to emphasize the dependence on the distance $d$. The notation for a closed ball is obtained by adding an overline to the notation for the open ball. The distance on $CMS$ is the Gromov-Hausdorff distance $d_{GH}$ between (isometry classes of) pointed metric spaces and the topology is induced by this distance. For the Gromov-Hausdorff distance see Gromov [12]. We denote by $O(\varepsilon)$ a positive function such that $\lim_{\varepsilon \to 0} O(\varepsilon) = 0$.

To any locally compact metric space there is an associated metric profile (Buliga [6], [7]).

Definition 2.1. The metric profile associated to the locally compact metric space $(M, d)$ is the assignment (for small enough $\varepsilon > 0$)

$$(\varepsilon > 0, \ x \in M) \mapsto P^m(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon}d, x] \in CMS$$

We may define a notion of metric profile which is more general than the previous one.
Definition 2.2. A metric profile is a curve \( \mathbb{P} : [0, a] \to CMS \) such that

(a) it is continuous at 0,
(b) for any \( \mu \in [0, a] \) and \( \varepsilon \in (0, 1] \) we have

\[
\text{d}_{GH}(\mathbb{P}(\varepsilon\mu), \mathbb{P}^\mu_{d\mu}(\varepsilon, x_\mu)) = O(\mu)
\]

The function \( O(\mu) \) may change with \( \varepsilon \). We used the notations

\[
\mathbb{P}(\mu) = [\bar{B}(x, 1), d_\mu, x_\mu] \quad \text{and} \quad \mathbb{P}^\mu_{d\mu}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon}d_\mu, x_\mu]
\]

We shall unfold further the definition 2.2 in order to clearly understand what is a metric profile. For any \( \mu \in (0, a] \) and for any \( b > 0 \) there is \( \varepsilon(\mu, b) \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon(\mu, b)) \) there exists a relation \( \rho = \rho_{\varepsilon, \mu} \subset \bar{B}_{d_\mu}(x_\mu, \varepsilon) \times \bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1) \) such that:

1. \( \text{dom } \rho_{\varepsilon, \mu} \) is \( b \)-dense in \( \bar{B}_{d_\mu}(x_\mu, \varepsilon) \),
2. \( \text{im } \rho_{\varepsilon, \mu} \) is \( b \)-dense in \( \bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1) \),
3. \( (x_\mu, x_{\mu\varepsilon}) \in \rho_{\varepsilon, \mu} \),
4. for all \( x, y \in \text{dom } \rho_{\varepsilon, \mu} \) we have \( |\frac{1}{\varepsilon}d_\mu(x, y) - d_{\mu\varepsilon}(x', y')| \leq b \), for any \( x', y' \)

such that \( (x, x'), (y, y') \in \rho_{\varepsilon, \mu} \).

Therefore a metric profile gives two types of information:

- a distance estimate like the one from point 4 above,
- an ”approximate shape” estimate, like in the points 1–3, where we see that two sets, namely the balls \( \bar{B}_{d_\mu}(x_\mu, \varepsilon) \) and \( \bar{B}_{d_{\mu\varepsilon}}(x_{\mu\varepsilon}, 1) \), are approximately isometric.

The metric profile \( \varepsilon \mapsto \mathbb{P}^\mu_{d\mu}(\varepsilon, x) \) of a metric space \( (M, d) \) for a fixed \( x \in M \) is a metric profile in the sense of the definition 2.2 if and only if the space \( (M, d) \) admits a tangent space in \( x \). Here is the general definition of a tangent space in the metric sense.

Definition 2.3. A (locally compact) metric space \( (M, d) \) admits a (metric) tangent space in \( x \in M \) if the associated metric profile \( \varepsilon \mapsto \mathbb{P}^\mu_{d\mu}(\varepsilon, x) \) (as in definition 2.1) admits a prolongation by continuity in \( \varepsilon = 0 \), i.e if the following limit exists:

\[
[T_x M, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^\mu_{d\mu}(\varepsilon, x) \tag{2.1}
\]

Metric tangent spaces are metric cones.

Definition 2.4. A metric cone \( (X, d, x) \) is a locally compact metric space \( (X, d) \), with a marked point \( x \in X \) such that for any \( a, b \in (0, 1] \) we have

\[
\mathbb{P}^a_{d^a}(a, x) = \mathbb{P}^b_{d^b}(b, x)
\]
Metric cones have the simplest metric profile, which is one with the property: 
\((\bar{B}(x_b, 1), d_b, x_b) = (X, d, x)\). In particular metric cones have dilatations.

**Definition 2.5.** Let \((X, d, x)\) be a metric cone. For any \(\varepsilon \in (0, 1]\) a dilatation is a function \(\delta^\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)\) such that

(a) \(\delta^\varepsilon(x) = x\),
(b) for any \(u, v \in X\) we have

\[d(\delta^\varepsilon(u), \delta^\varepsilon(v)) = \varepsilon d(u, v)\]

The existence of dilatations for metric cones comes from the definition 2.4. Indeed, dilatations are just isometries from \((\bar{B}(x, 1), d, x)\) to \((\bar{B}, \frac{1}{\varepsilon}d, x)\).

**3. Sub-riemannian manifolds**

Let \(M\) be a connected \(n\) dimensional real manifold. A distribution is a smooth subbundle \(D\) of \(M\). To any point \(x \in M\) there is associated the vector space \(D_x \subset T_x M\). The dimension of the distribution \(D\) at point \(x \in M\) is denoted by

\[m(x) = \dim D_x\]

The distribution is smooth, therefore the function \(x \in M \mapsto m(x)\) is locally constant. We suppose further that the dimension of the distribution is globally constant and we denote it by \(m\) (thus \(m = m(x)\) for any \(x \in M\)). Clearly \(m \leq n\); we are interested in the case \(m < n\).

A horizontal curve \(c : [a, b] \to M\) is a curve which is almost everywhere derivable and for almost any \(t \in [a, b]\) we have \(\dot{c}(t) \in D_c(t)\). The class of horizontal curves will be denoted by \(\text{Hor}(M, D)\).

Further we shall use the following notion of non-integrability of the distribution \(D\).

**Definition 3.1.** The distribution \(D\) is completely non-integrable if \(M\) is locally connected by horizontal curves curves \(c \in \text{Hor}(M, D)\).

A sufficient condition for the distribution \(D\) to be completely non-integrable is given by Chow condition (C) [8].

**Theorem 3.2.** (Chow) Let \(D\) be a distribution of dimension \(m\) in the manifold \(M\). Suppose there is a positive integer number \(k\) (called the rank of the distribution \(D\)) such that for any \(x \in X\) there is a topological open ball \(U(x) \subset M\) with \(x \in U(x)\) such that there are smooth vector fields \(X_1, ..., X_m\) in \(U(x)\) with the property:

(C) the vector fields \(X_1, ..., X_m\) span \(D_x\) and these vector fields together with their iterated brackets of order at most \(k\) span the tangent space \(T_y M\) at every point \(y \in U(x)\).

Then the distribution \(D\) is completely non-integrable in the sense of definition 3.1.
**Definition 3.3.** A sub-riemannian (SR) manifold is a triple \((M, D, g)\), where \(M\) is a connected manifold, \(D\) is a completely non-integrable distribution on \(M\), and \(g\) is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) \(D\).

### 3.1 The Carnot-Carathéodory distance

Given a distribution \(D\) which satisfies the hypothesis of Chow theorem 3.2, let us consider a point \(x \in M\), its neighbourhood \(U(x)\), and the vector fields \(X_1, \ldots, X_m\) satisfying the condition (C).

One can define on \(U(x)\) a filtration of bundles as follows. Define first the class of horizontal vector fields on \(U\):

\[
\mathcal{X}^1(U(x), D) = \{ X \in \mathcal{X}^\infty(U) : \forall y \in U(x), X(y) \in D_y \}
\]

Next, define inductively for all positive integers \(j\):

\[
\mathcal{X}^{j+1}(U(x), D) = \mathcal{X}^j(U(x), D) + [\mathcal{X}^1(U(x), D), \mathcal{X}^j(U(x), D)]
\]

Here \(\cdot, \cdot\) denotes the bracket of vector fields. We obtain therefore a filtration \(\mathcal{X}^j(U(x), D) \subset \mathcal{X}^{j+1}(U(x), D)\). Evaluate now this filtration at \(y \in U(x)\):

\[
V^j(y, U(x), D) = \{ X(y) : X \in \mathcal{X}^j(U(x), D) \}
\]

According to Chow theorem there is a positive integer \(k\) such that for all \(y \in U(x)\) we have

\[
D_y = V^1(y, U(x), D) \subset V^2(y, U(x), D) \subset \ldots \subset V^k(y, U(x), D) = T_yM
\]

Consequently, to the sub-riemannian manifold is associated the string of numbers:

\[
\nu_1(y) = \dim V^1(y, U(x), D) < \nu_2(y) = \dim V^2(y, U(x), D) < \ldots < n = \dim M
\]

Generally \(k, \nu_j(y)\) may vary from a point to another.

The number \(k\) is called the step of the distribution at \(y\).

**Definition 3.4.** The distribution \(D\) is regular if \(\nu_j(y)\) are constant on the manifold \(M\). The sub-riemannian manifold \((M, D, g)\) is regular if \(D\) is regular and for any \(x \in M\) there is a topological ball \(U(x) \subset M\) with \(x \in U(M)\) and an orthonormal (with respect to the metric \(g\)) family of smooth vector fields \(\{X_1, \ldots, X_m\}\) in \(U(x)\) which satisfy the condition (C).

The length of a horizontal curve is

\[
l(c) = \int_a^b \left( g_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t)) \right)^{\frac{1}{2}} \, dt
\]

The length depends on the metric \(g\).
Definition 3.5. The Carnot-Carathéodory distance (or CC distance) associated to the sub-riemannian manifold is the distance induced by the length \(l\) of horizontal curves:
\[
d(x, y) = \inf \{ l(c) : c \in \text{Hor}(M, D), \ c(a) = x, \ c(b) = y \}
\]

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance is locally finite. The distance depends only on the distribution \(D\) and metric \(g\), and not on the choice of vector fields \(X_1, \ldots, X_m\) satisfying the condition (C). The space \((M, d)\) is locally compact and complete, and the topology induced by the distance \(d\) is the same as the topology of the manifold \(M\). (These important details may be recovered from reading carefully the constructive proofs of Chow theorem given by Bellaïche [1] or Gromov [13].)

3.2 Normal frames

In the following we stay in a small open neighbourhood of an arbitrary, but fixed point \(x_0 \in M\). All results are local in nature (that is they hold for some small open neighbourhood of an arbitrary, but fixed point of the manifold \(M\)). That is why we shall no longer mention the dependence of various objects on \(x_0\), on the neighbourhood \(U(x_0)\), or the distribution \(D\).

We shall work further only with regular sub-riemannian manifolds, if not otherwise stated. The topological dimension of \(M\) is denoted by \(n\), the step of the regular sub-riemannian manifold \((M, D, g)\) is denoted by \(k\), the dimension of the distribution is \(m\), and there are numbers \(\nu_j, j = 1, \ldots, k\) such that for any \(x \in M\) we have \(\dim V^j(x) = \nu_j\). The Carnot-Carathéodory distance is denoted by \(d\).

Definition 3.6. An adapted frame \(\{X_1, \ldots, X_n\}\) is a collection of smooth vector fields which is obtained by the construction described below.

We start with a collection \(X_1, \ldots, X_m\) of vector fields which satisfy the condition (C). In particular for any point \(x\) the vectors \(X_1(x), \ldots, X_m(x)\) form a basis for \(D_x\). We further associate to any word \(a_1 \ldots a_q\) with letters in the alphabet \(1, \ldots, m\) the multi-bracket \([X_{a_1}, \ldots, X_{a_q}]\).

One can add, in the lexicographic order, \(n - m\) elements to the set \(\{X_1, \ldots, X_m\}\) until we get a collection \(\{X_1, \ldots, X_n\}\) such that: for any \(j = 1, \ldots, k\) and for any point \(x\) the set \(\{X_1(x), \ldots, X_{\nu_j}(x)\}\) is a basis for \(V^j(x)\).

Let \(\{X_1, \ldots, X_n\}\) be an adapted frame. For any \(j = 1, \ldots, n\) the degree \(\deg X_j\) of the vector field \(X_j\) is defined as the only positive integer \(p\) such that for any point \(x\) we have
\[
X_j(x) \in V^p_d \setminus V^{p-1}(x)
\]

Further we define normal frames. The name has been used by Vodopyanov [19], but for a slightly different object. The existence of normal frames in the sense
of the following definition is the hardest technical problem in the classical establishment of sub-riemannian geometry. For the informed reader the referee pointed out that condition (a) Definition 3.7 is a part of the conclusion of Gromov approximation theorem, namely when one point coincides with the center of nilpotentization; also condition (b) is equivalent with a statement of Gromov concerning the convergence of rescaled vector fields to their nilpotentization (an informed reader must at least follow in all details the papers Bellaïche [1] and Gromov [13], where differential calculus in the classical sense is heavily used). Therefore the conditions of Definition 3.7 concentrate that part of the foundations of sub-riemannian geometry which makes use of classical differential calculus.

The key details in the Definition below are uniform convergence assumptions. This is in line with Gromov suggestions in the last section of Bellaïche [1].

**Definition 3.7.** An adapted frame \( \{ X_1, \ldots, X_n \} \) is a normal frame if the following two conditions are satisfied:

(a) we have the limit

\[
\lim_{\varepsilon \to 0_+} \frac{1}{\varepsilon} d \left( \exp \left( \sum_{i=1}^{n} \varepsilon^{\text{deg} X_i} a_i X_i \right)(y), y \right) = A(y, a) \in (0, +\infty)
\]

uniformly with respect to \( y \) in compact sets and \( a = (a_1, \ldots, a_n) \in W \), with \( W \subset \mathbb{R}^n \) compact neighbourhood of \( 0 \in \mathbb{R}^n \),

(b) for any compact set \( K \subset M \) with diameter (with respect to the distance \( d \)) sufficiently small, and for any \( i = 1, \ldots, n \) there are functions

\[
P_i(\cdot, \cdot, \cdot) : U_K \times U_K \times K \to \mathbb{R}
\]

with \( U_K \subset \mathbb{R}^n \) a sufficiently small compact neighbourhood of \( 0 \in \mathbb{R}^n \) such that for any \( x \in K \) and any \( a, b \in U_K \) we have

\[
\exp \left( \sum_{i=1}^{n} a_i X_i \right)(x) = \exp \left( \sum_{i=1}^{n} P_i(a, b, y) X_i \right) \circ \exp \left( \sum_{i=1}^{n} b_i X_i \right)(x)
\]

and such that the following limit exists

\[
\lim_{\varepsilon \to 0_+} \varepsilon^{-\text{deg} X_i} P_i(\varepsilon^{\text{deg} X_j} a_j, \varepsilon^{\text{deg} X_k} b_k, x) \in \mathbb{R}
\]

and it is uniform with respect to \( x \in K \) and \( a, b \in U_K \).

The existence of normal frames is proven in Bellaïche [1], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62).

In order to understand normal frames let us look to the case of a Lie group \( G \) endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace \( D \) of the algebra \( \mathfrak{g} = T_e G \) which bracket generates the whole algebra \( \mathfrak{g} \). Take \( \{ X_1, \ldots, X_m \} \) a collection of \( m = \text{dim} D \) left invariant independent vector fields and define with their help an adapted frame, as explained in definition 3.6. Then the adapted frame \( \{ X_1, \ldots, X_n \} \) is in fact normal.
4. Dilatation structures

In this section we review the definition and main properties of a dilatation structure, according to [2], [3].

4.1 The axioms of a dilatation structure

Further are listed the axioms of a dilatation structure $(X, d, \delta)$, starting with axiom 0, which is a preparation for the axioms which follow.

We restrict the generality from [2] to the case which is related to sub-riemannian geometry, that is we shall consider only dilatations $\delta_x^\varepsilon$ with $\varepsilon \in (0, +\infty)$.

A0. The dilatations $\delta_x^\varepsilon: U(x) \to V_\varepsilon(x)$ are defined for any $\varepsilon \in (0, 1]$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of $x$. All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x).$$

We suppose that for all $\varepsilon \in (0, 1)$, we have

$$B_d(x, \varepsilon) \subset \delta_x^\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset U(x).$$

There is a number $B \in (1, A)$ such that for any $\varepsilon \in (1, +\infty)$ the associated dilatation $\delta_x^\varepsilon: W_\varepsilon(x) \to B_d(x, B)$, is injective, invertible on the image. We shall suppose that $W_\varepsilon(x)$ is a open neighbourhood of $x$,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in (0, 1)$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}} \delta_x^\varepsilon u = u.$$ 

We have therefore the following string of inclusions, for any $\varepsilon \in (0, 1)$, and any $x \in X$:

$$B_d(x, \varepsilon) \subset \delta_x^\varepsilon B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_x^\varepsilon B_d(x, B).$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)
A1. We have $\delta^x x = x$ for any point $x$. We also have $\delta^x_1 = \text{id}$ for any $x \in X$.

Let us define the topological space

$$\text{dom} \, \delta = \{ (\varepsilon, x, y) \in (0, +\infty) \times X \times X : \begin{cases} \text{if } \varepsilon \leq 1 \text{ then } y \in U(x) , \\ \text{else } y \in W_\varepsilon(x) \end{cases} \}$$

with the topology inherited from the product topology on $(0, +\infty) \times X \times X$.

Consider also $\text{Cl}(\text{dom} \, \delta)$, the closure of $\text{dom} \, \delta$ in $[0, +\infty) \times X \times X$ with product topology. The function $\delta : \text{dom} \, \delta \to X$ defined by $\delta(\varepsilon, x, y) = \delta^x y$ is continuous. Moreover, it can be continuously extended to $\text{Cl}(\text{dom} \, \delta)$ and we have

$$\lim_{\varepsilon \to 0} \delta^x_\varepsilon y = x .$$

A2. For any $x, \in K, \varepsilon, \mu \in (0, 1)$ and $u \in \overline{B}_d(x, A)$ we have:

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon \mu} u .$$

A3. For any $x$ there is a function $(u, v) \mapsto d^x(u, v)$, defined for any $u, v$ in the closed ball (in distance $d$) $\overline{B}_d(x, A)$, such that

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) - d^x(u, v) : u, v \in \overline{B}_d(x, A) \right\} = 0$$

uniformly with respect to $x$ in compact set.

Remark that $d^x$ may be a degenerated distance: there might exist $v, w \in U(x)$ such that $d^x(v, w) = 0$.

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \overline{B}_d(x, R)$ and all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\delta^x_\varepsilon v \in W_{\varepsilon^{-1}}(\delta^x_\varepsilon u) .$$

With this assumption the following notation makes sense:

$$\Delta^x_\varepsilon(u, v) = \delta^x_{\varepsilon^{-1}} \delta^x_\varepsilon v .$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, v) = \Delta^x(u, v)$$

uniformly with respect to $x, u, v$ in compact set.

Definition 4.1. A triple $(X, d, \delta)$ which satisfies A0, A1, A2, A3, but $d^x$ is degenerate for some $x \in X$, is called degenerate dilatation structure.

If the triple $(X, d, \delta)$ satisfies A0, A1, A2, A3, A4 and $d^x$ is non-degenerate for any $x \in X$, then we call it a dilatation structure.
4.2 Metric profile of a dilatation structure

Here we describe the metric profile associated to a dilatation structure. This will be relevant further for understanding the geometry of the metric tangent spaces of regular sub-riemannian manifolds.

The following result is a reformulation of theorem 6 [2].

Theorem 4.2. Let \((X, d, \delta)\) be a dilatation structure, \(x \in X\) a point in \(X\), \(\mu > 0\) sufficiently small, and let \((\delta, \mu, x)\) be the distance on \(\bar{B}_{d^\mu}(x, 1) = \{y \in X: d^\mu(x, y) \leq 1\}\) given by

\[
(\delta, \mu, x)(u, v) = \frac{1}{\mu}d(\delta^\mu x u, \delta^\mu x v)
\]

Then the curve \(\mu > 0 \mapsto \mathbb{P}_x(\mu) = [\bar{B}_{d^\mu}(x, 1), (\delta, \mu, x), x]\) admits an extension by continuity to a metric profile, by setting \(\mathbb{P}_x(0) = [\bar{B}_{d^0}(x, 1), d^0, x]\). More precisely we have the following estimate:

\[
d_{GH}([\bar{B}_{d^\mu}(x, 1), (\delta, \varepsilon \mu, x), x], [\bar{B}_{d^\mu}(\delta^\mu \varepsilon, x), (1/\varepsilon)(\delta^\mu x, \mu, x), x]) = O(\varepsilon \mu) + \frac{1}{\varepsilon}O(\mu) + O(\mu)
\]

uniformly with respect to \(x\) in compact set.

4.3 Tangent bundle of a dilatation structure

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [2].

Theorem 4.3. Let \((X, d, \delta)\) be a dilatation structure. Then the metric space \((X, d)\) admits a metric tangent space at \(x\), for any point \(x \in X\). More precisely we have the following limit:

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \{ | d(u, v) - d^\varepsilon(u, v) | : d(x, u) \leq \varepsilon , d(x, v) \leq \varepsilon \} = 0 .
\]

Theorem 4.4. Let \((X, d, \delta)\) be a dilatation structure. Then for any \(x \in X\) the triple \((U(x), \Sigma^x, d^x)\) is a normed local conical group. This means:

(a) \(\Sigma^x\) is a local group operation on \(U(x)\), with \(x\) as neutral element and \(inv^x\) as the inverse element function;
(b) the distance \(d^x\) is left invariant with respect to the group operation from point (a);
(c) For any \(\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1\), the dilatation \(\delta^\varepsilon\) is an automorphism with respect to the group operation from point (a);
(d) the distance $d^x$ has the cone property with respect to dilatations: for any $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^x(u, v) = \frac{1}{\mu} d^x(\delta^x \mu u, \delta^x \mu v).$$

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of $(X, d, \delta)$ at $x$. Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

The following is corollary 4.7 [3].

**Theorem 4.5.** Let $(X, d, \delta)$ be a dilatation structure. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).

5. Examples of dilatation structures

In this section we give some examples of dilatation structures, which share some common features. There are other examples, typically coming from iterated functions systems, which will be presented in another paper.

The first example is known to everybody: take $(X, d) = (\mathbb{R}^n, d_E)$, with usual (euclidean) dilatations $\delta^x$, with:

$$d_E(x, y) = \|x - y\|, \quad \delta^x y = x + \varepsilon(y - x).$$

Dilatations are defined everywhere. There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d_E(\delta^x \varepsilon u, \delta^x \varepsilon v) = d_E(u, v),$$

therefore for any $x \in X$ we have $d^x = d_E$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta^x \varepsilon^{-1} \delta^x v = x + \varepsilon(u - x) + \frac{1}{\varepsilon}(x + \varepsilon(v - x) - x - \varepsilon(u - x)) =$$

$$= x + \varepsilon(u - x) + v - u$$

therefore this quantity converges to

$$x + v - u = x + (v - x) - (u - x)$$

as $\varepsilon \to 0$. The axiom A4 is verified.

We continue further with less obvious examples.
5.1 Riemannian manifolds

Take now $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a bi-Lipschitz diffeomorphism. Then we can define the dilatation structure: $X = \mathbb{R}^n,
\begin{align*}
  d_\phi(x, y) &= \|\phi(x) - \phi(y)\|, \\
  \delta_\varepsilon^x y &= x + \varepsilon(y - x),
\end{align*}

or the equivalent dilatation structure: $X = \mathbb{R}^n,
\begin{align*}
  d_\phi(x, y) &= \|x - y\|, \\
  \delta_\varepsilon^x y &= \phi^{-1}(\phi(x) + \varepsilon(\phi(y) - \phi(x))).
\end{align*}

In this example (look at its first version) the distance $d_\phi$ is not equal to $d^x$. Indeed, a direct calculation shows that
\[ d^x(u, v) = \|D\phi(x)(v - u)\|. \]

The axiom A4 gives the same result as previously.

Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [1] last section).

5.2 Snowflakes

The next example is a snowflake variation of the euclidean case: $X = \mathbb{R}^n$ and for any $a \in (0, 1]$ take
\[ d_a(x, y) = \|x - y\|^a, \quad \delta_\varepsilon^x y = x + \varepsilon^a(y - x). \]

We leave to the reader to verify the axioms.

More general, if $(X, d, \delta)$ is a dilatation structure then $(X, d_a, \delta(a))$ is also a dilatation structure, for any $a \in (0, 1]$, where
\[ d_a(x, y) = (d(x, y))^a, \quad \delta(a)_\varepsilon^x = \delta_{\varepsilon^a}^x. \]

5.3 Nonstandard dilatations in the euclidean space

Take $X = \mathbb{R}^2$ with the euclidean distance. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilatations
\[ \delta_\varepsilon^z x = \varepsilon^z x. \]

It is easy to check that $(X, \delta, +, d)$ is a conical group, equivalently that the dilatations
\[ \delta_\varepsilon^x y = x + \delta_\varepsilon(y - x). \]

form a dilatation structure with the euclidean distance.
Two such dilatation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other surprising properties of these dilatation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in $X$ which are differentiable almost everywhere. The second property is that any holomorphic and Lipschitz function from $X$ to $X$ (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from $X$ to $X$ which are not differentiable almost everywhere (suffices to take a $C^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

6. Sub-riemannian dilatation structures

To any normal frame of a regular sub-riemannian manifold we associate a dilatation structure. (Technically this is a dilatation structure defined only locally, as in the case of riemannian manifolds.)

**Definition 6.1.** To any normal frame $\{X_1, \ldots, X_n\}$ of a regular sub-riemannian manifold $(M, D, g)$ we associate the dilatation structure $(M, d, \delta)$ defined by: $d$ is the Carnot-Carathéodory distance, and for any point $x \in M$ and any $\varepsilon \in (0, +\infty)$ (sufficiently small if necessary), the dilatation $\delta_x^\varepsilon$ is given by:

$$
\delta_x^\varepsilon \left( \exp \left( \sum_{i=1}^n a_i X_i \right)(x) \right) = \exp \left( \sum_{i=1}^n a_i \varepsilon^{\deg X_i} X_i \right)(x)
$$

We shall prove that $(M, d, \delta)$ is indeed a dilatation structure. This allows us to get the main results concerning the infinitesimal geometry of a regular sub-riemannian manifold, as particular cases of theorems 4.2, 4.3, 4.4 and 4.5.

We only have to prove axioms A3 and A4 of dilatation structures. We do this in the next two theorems. Before this let us describe what we mean by "sufficiently closed".

**Convention 6.2.** Further we shall say that a property $P(x_1, x_2, x_3, \ldots)$ holds for $x_1, x_2, x_3, \ldots$ sufficiently closed if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $P(x_1, x_2, x_3, \ldots)$ is true for any $x_1, x_2, x_3, \ldots \in K$ with $d(x_i, x_j) \leq C(K)$.

In the following we prove a result similar to Gromov local approximation theorem [13], p. 135, or to Bellaïche theorem 7.32 [1]. Note however that here we take as a hypothesis the existence of a normal frame.

**Theorem 6.3.** Consider $X_1, \ldots, X_n$ a normal frame and the associated dilatations provided by definition 6.1. Then axiom A3 of dilatation structures is satisfied, that is the limit

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) - d(u, v) \right)
$$

exists and it uniform with respect to $x,u,v$ sufficiently closed.
Proof. Let \( x, u, v \in M \) be sufficiently close. We write

\[
u = \exp \left( \sum_{1}^{n} u_{i}X_{i} \right)(x), \quad v = \exp \left( \sum_{1}^{n} v_{i}X_{i} \right)(x)\]

we compute, using definition [6.1]

\[
\frac{1}{\varepsilon} d (\delta_{\varepsilon} u, \delta_{\varepsilon} v) = \frac{1}{\varepsilon} d \left( \delta_{\varepsilon} \exp \left( \sum_{1}^{n} u_{i}X_{i} \right)(x), \delta_{\varepsilon} \exp \left( \sum_{1}^{n} v_{i}X_{i} \right)(x) \right) = \frac{1}{\varepsilon} d \left( \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} u_{i}X_{i} \right)(x), \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} v_{i}X_{i} \right)(x) \right) = A_{\varepsilon}
\]

Let us denote by \( u_{\varepsilon} = \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} u_{i}X_{i} \right)(x) \). Use the first part of the property (b), definition [3.7] of a normal system, to write further:

\[
A_{\varepsilon} = \frac{1}{\varepsilon} d \left( u_{\varepsilon}, \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{j}} P_{j}(\varepsilon^{-\deg X_{j}} v_{j}, \varepsilon^{-\deg X_{k}} u_{k}, x)X_{i} \right)(u_{\varepsilon}) \right) = \frac{1}{\varepsilon} d \left( u_{\varepsilon}, \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} \left( \varepsilon^{-\deg X_{j}} P_{j}(\varepsilon^{-\deg X_{j}} v_{j}, \varepsilon^{-\deg X_{k}} u_{k}, x) \right)X_{i} \right)(u_{\varepsilon}) \right)
\]

We make a final notation: for any \( i = 1, \ldots, n \)

\[
a_{i}^{\varepsilon} = \varepsilon^{-\deg X_{i}} P_{i}(\varepsilon^{-\deg X_{j}} v_{j}, \varepsilon^{-\deg X_{k}} u_{k}, x)
\]

thus we have:

\[
\frac{1}{\varepsilon} d (\delta_{\varepsilon} u, \delta_{\varepsilon} v) = \frac{1}{\varepsilon} d \left( u_{\varepsilon}, \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} a_{i}^{\varepsilon} X_{i} \right)(u_{\varepsilon}) \right)
\]

By the second part of property (b), definition [3.7] the vector \( a^{\varepsilon} \in \mathbb{R}^{n} \) converges to a finite value \( a^{0} \in \mathbb{R}^{n} \), as \( \varepsilon \to 0 \), uniformly with respect to \( x, u, v \) in compact set. In the same time \( u_{\varepsilon} \) converges to \( x \), as \( \varepsilon \to 0 \). The proof ends by using property (a), definition [3.7]. Indeed, we shall use the key assumption of uniform convergence.

With the notations from definition [3.7] for fixed \( \eta > 0 \) the term

\[
B(\eta, \varepsilon) = \frac{1}{\varepsilon} d \left( u_{\eta}, \exp \left( \sum_{1}^{n} \varepsilon^{-\deg X_{i}} \varepsilon a_{i}^{\eta} X_{i} \right)(u_{\eta}) \right)
\]

converges to a real number \( A(u_{\eta}, a_{\eta}) \) as \( \varepsilon \to 0 \), uniformly with respect to \( u_{\eta} \) and \( a_{\eta} \). Since \( u_{\eta} \) converges to \( x \) and \( a_{\eta} \) converges to \( a^{0} \) as \( \eta \to 0 \), by the uniform convergence assumption in (a), definition [3.7] we get that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d (\delta_{\varepsilon} u, \delta_{\varepsilon} v) = \lim_{\eta \to 0} A(u_{\eta}, a_{\eta}) = A(x, a^{0})
\]
The proof is done. □

In the next Theorem we prove that axiom A4 of dilatation structures is satisfied. The referee informed us that Theorem 6.4 also follows from results of Vodopyanov and Karmanova [21], quoted in [20] p. 267; a complete version of this result will appear in a work by Karmanova and Vodopyanov “Geometry of Carnot-Carathéodory spaces, differentiability and coarea formula” in the book “Analysis and Mathematical Physics”, Birchhäuser 2008.

**Theorem 6.4.** Consider $X_1, \ldots, X_n$ a normal frame and the associated dilatations provided by definition 6.1. Then axiom A4 of dilatation structures is satisfied: as $\varepsilon$ tends to 0 the quantity

$$\Delta^\varepsilon_x(u, v) = \delta^\varepsilon_x \circ \delta^\varepsilon_x(v)$$

converges, uniformly with respect to $x, u, v$ sufficiently closed.

**Proof.** We shall use the notations from definition 3.6, 3.7, 6.1.

Let $x, u, v \in M$ be sufficiently closed. We write

$$u = \exp \left( \sum_{i=1}^{n} u_i X_i(x) \right), \quad v = \exp \left( \sum_{i=1}^{n} v_i X_i(x) \right)$$

We compute now $\Delta^\varepsilon_x(u, v)$:

$$\Delta^\varepsilon_x(u, v) = \delta^{\varepsilon u}_{\varepsilon-1} \circ \delta^{\varepsilon v}_{\varepsilon-1}(x)$$

Let us denote by $u_\varepsilon = \delta^\varepsilon u$. Thus we have

$$\Delta^\varepsilon_x(u, v) = \delta^{u_\varepsilon}_{\varepsilon-1} \exp \left( \sum_{i=1}^{n} \varepsilon \deg X_i v_i X_i(x) \right)$$

We use the first part of the property (b), definition 3.7 in order to write

$$\exp \left( \sum_{i=1}^{n} \varepsilon \deg X_i v_i X_i(x) \right) = \exp \left( \sum_{i=1}^{n} P_i(\varepsilon \deg X_j v_j, \varepsilon \deg X_k u_k, x) X_i(x) \right)$$

We finish the computation:

$$\Delta^\varepsilon_x(u, v) = \exp \left( \sum_{i=1}^{n} \varepsilon^{-\deg X_i} P_i(\varepsilon \deg X_j v_j, \varepsilon \deg X_k u_k, x) X_i(x) \right)$$

As $\varepsilon$ goes to 0 the point $u_\varepsilon$ converges to $x$ uniformly with respect to $x, u$ sufficiently closed (as a corollary of the previous theorem, for example). The proof therefore ends by invoking the second part of the property (b), definition 3.7. □
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