Unital Dilations of Completely Positive Semigroups: From Combinatorics to Continuity

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Abstract

Semigroups of completely positive maps arise naturally both in noncommutative stochastic processes and in the dynamics of open quantum systems. Since its inception in the 1970’s, the study of completely positive semigroups has included among its central topics the dilation of a completely positive semigroup to an endomorphism semigroup. In quantum dynamics, this amounts to embedding a given open system inside some closed system, while in noncommutative probability, it corresponds to the construction of a Markov process from its transition probabilities. In addition to the existence of dilations, one is interested in what properties of the original semigroup (unitality, various kinds of continuity) are preserved.

Several authors have proved the existence of dilations, but in general, the dilation achieved has been non-unital; that is, the unit of the original algebra is embedded as a proper projection in the dilation algebra. A unique approach due to Jean-Luc Sauvageot overcomes this problem, but leaves unclear the continuity of the dilation semigroup. The major purpose of this thesis, therefore, is to further develop Sauvageot’s theory in order to prove the existence of continuous unital dilations. This existence is proved in Theorem 5.4.9, the central result of the paper.

The dilation depends on a modification of free probability theory, and in particular on a combinatorial property akin to free independence. This property is implicit in some of Sauvageot’s original calculations, but a secondary goal of this paper is to present it as its own object of study, which we do in chapter 2.

The content discussed here is based on [Sau86], and the present exposition is substantially the same as in the author’s Ph.D. thesis [Gae13].
Preface: Background and Terminology

Following [Sak98], we distinguish between \(W^\ast\)-algebras, which are abstractly defined as \(C^\ast\)-algebras having a Banach-space predual (necessarily unique, as it turns out), and von Neumann algebras, which are concretely defined as weakly closed self-adjoint subalgebras of \(B(H)\) for some Hilbert space \(H\). In this convention, every von Neumann algebra is also a \(W^\ast\)-algebra (with predual equal to a quotient of the predual \(B(H)\), \(\simeq L^1(H)\)), whereas every \(W^\ast\)-algebra is isomorphic to some von Neumann algebra ([Sak98] 1.16.7). We depart somewhat from Sakai in referring to the weak-* topology on a \(W^\ast\)-algebra as the ultraweak topology, which he calls the \(\sigma\)-topology or weak topology, and the topology induced by the seminorms \(x \mapsto \phi(x^\ast x)\) for positive weak-* continuous functionals \(\phi\) as the ultrastrong topology, which he calls the strong topology or s-topology. In the case of a von Neumann algebra, these topologies coincide with the ultraweak and ultrastrong operator topologies as usually defined ([Sak98] 1.15.6), and hence also with the weak and strong operator topologies on bounded subsets ([Sak98] 1.15.2).

Because of this latter fact, we sometimes drop the “ultra” and refer merely to the weak and strong topologies when working on a bounded subset of a \(W^\ast\)-algebra. Among the properties of these topologies that we will need are the following:

- Multiplication is separately continuous in both the ultraweak and ultrastrong topologies. However, it is jointly continuous in neither. On bounded sets, multiplication is jointly strongly continuous, but not jointly weakly continuous.
- The adjoint map \(x \mapsto x^\ast\) is ultraweakly continuous, but not ultrastrongly nor even strongly continuous.
- On bounded subsets, one may relate the weak and strong topologies as follows: \(x_\nu \to x\) strongly iff \(x_\nu \to x\) weakly and \(x_\nu^\ast x_\nu \to x^\ast x\) weakly.
- The Kaplansky density theorem: If \(A\) is a \(W^\ast\)-algebra and \(A_0 \subset A\) an ultraweakly dense *-subalgebra, then the unit ball of \(A_0\) is strong-* dense in the unit ball of \(A\). In the case of a von Neumann algebra, the hypothesis of ultraweak density may be replaced by WOT-density.

A linear map between \(W^\ast\)-algebras which is continuous with respect to their ultraweak topologies is called normal; if the map in question is positive, this is equivalent to the property of preserving upward-convergent nets (in this case weak and strong convergence are equivalent) of positive elements, that is, a positive linear map is normal iff \(\phi(x_\alpha) \uparrow \phi(x)\) whenever \(x_\alpha \uparrow x\) ([Con00] Corollary 46.5). A \(C^*\)-isomorphism between two \(W^\ast\)-algebras is automatically normal, but a *-homomorphism or completely positive map need not be.

We refer to a \(W^\ast\)-algebra \(\mathcal{A}\) as separable if its predual \(\mathcal{A}_\ast\) is a separable Banach space; this can be shown to be equivalent to numerous other conditions,
including the separability of either $\mathcal{A}$ or its unit ball in either the ultraweak or ultrastrong topologies, and the existence of a faithful normal representation of $\mathcal{A}$ on a separable Hilbert space. A related but strictly weaker property is that of countable decomposability, which can be defined as the property that every mutually orthogonal family of nonzero projections in $\mathcal{A}$ is at most countable; this is equivalent to the existence of a faithful state, the existence of a faithful normal state, or the strong metrizability of the unit ball ([Bla06] III.2.2.27).

Throughout, we use the boldface symbol $\mathbf{1}$ to denote the unit of an algebra, while $1$ will denote the first natural number. The phrase “unital subalgebra” will always be taken to mean that the subalgebra contains the unit of the larger algebra.
CHAPTER 1

Introduction to Completely Positive Semigroups

1.1. Completely Positive Maps and Semigroups

In this section we introduce the basic objects of study.

Definition 1.1.1. Let $A, B$ be C*-algebras and $\phi : A \to B$ a linear map. We say that $\phi$ is

1. positive if it maps positive elements of $A$ to positive elements of $B$,
2. $n$-positive if the map $I_n \otimes \phi : M_n(\mathbb{C}) \otimes A \to M_n(\mathbb{C}) \otimes B$ is positive, and
3. completely positive if $\phi$ is $n$-positive for all $n \geq 1$.

We record here without proof some of the important properties of positive and completely positive maps. Throughout, $A, B$ denote C*-algebras and $\phi : A \to B$ a linear map.

- Every positive linear map is a *-map, that is, has the property that $\phi(a^*) = \phi(a^*)$ for all $a \in A$. ([Pau02] Exercise 2.1)
- Any two of the following properties implies the third:
  1. $\phi(1) = 1$
  2. $\|\phi\| = 1$
  3. $\phi$ is positive. ([Pau02] Cor 2.9 and Prop 2.11)
- If $\phi$ is 2-positive, then $\phi(a)^* \phi(a) \leq \|\phi(1)\| \|\phi(a^*) a\|$ for all $a \in A$. This is known as the Schwarz inequality for 2-positive maps. In particular, if $\phi$ is unital and completely positive then $\phi(a)^* \phi(a) \leq \phi(a^* a)$ for all $a \in A$. ([Pau02] Proposition 3.3 and Exercise 3.4)
- If either $A$ or $B$ is commutative, the map $\phi : A \to B$ is positive iff it is completely positive. ([Pau02] Theorems 3.9 and 3.11)
- If $A$ and $B$ are W*-algebras, a completely positive map $\phi : A \to B$ is normal iff it is strongly continuous. ([Bla06] Proposition III.2.2.2). Strong continuity is equivalent to ultrastrong because of the boundedness of the map.
- If $\phi : A \to B(H)$ is a completely positive map, there exists a triple $(K, V, \pi)$, unique up to isomorphism, such that
  1. $K$ is a Hilbert space
  2. $V : H \to K$ is a linear map such that $\|\phi\| = \|V\|^2$
  3. $\pi : A \to B(K)$ is a *-homomorphism such that $V^* \pi(a) V = \phi(a)$ for all $a \in A$

and with the additional minimality property that $\overline{\pi(A)V H} = K$. The triple $(H, V, \pi)$ is called the minimal Stinespring dilation of $\phi$. If $\phi$ is unital, $V$ is an isometry; if $\phi$ is normal, so is $\pi$. This is known
as Stinespring’s dilation theorem (Sti55, Pau02 Theorem 4.1, Bla06 Theorems II.6.9.7 and III.2.2.4).

**Definition 1.1.2.** Let $\mathcal{A}$ be a C*-algebra (resp. W*-algebra).

1. A **cp-semigroup** on $\mathcal{A}$ is a family $\{\phi_t : t \in [0, \infty)\}$ of (normal) completely positive contractive linear maps $\phi_t : \mathcal{A} \to \mathcal{A}$ such that $\phi_0 = \text{id}_\mathcal{A}$ and

$$\phi_t \circ \phi_s = \phi_{t+s}$$

for all $s, t \geq 0$.

2. An **e-semigroup** on $\mathcal{A}$ is a cp-semigroup in which each $\phi_t$ is a *-endomorphism.

3. Capital letters (CP-semigroup, E-semigroup) indicate that for each $a \in \mathcal{A}$, $t \mapsto \phi_t(a)$ is a continuous function from $[0, \infty)$ to $\mathcal{A}$, where $\mathcal{A}$ is given the norm (resp. ultraweak) topology. We refer to this continuity property of the semigroup as **strong continuity** or **point-norm continuity** in the C* case, and **point-weak continuity** in the W* case.

4. A subscript of 0 (cp$_0$-semigroup, CP$_0$-, $e_0$-, $E_0$-) indicates that $\mathcal{A}$ contains a unit $1$ and that $\phi_t(1) = 1$ for all $t \geq 0$.

**Remark 1.1.3.** The term **quantum Markov process** or **quantum Markov semigroup** is sometimes used in the literature to describe cp$_0$- or CP$_0$-semigroups.

**Remark 1.1.4.** In the case where $\mathcal{A}$ is a W*-algebra, the definitions of cp-semigroup and CP-semigroup remain unchanged when stated in terms of the strong topology rather than the weak topology. That is, each map $\phi_t$ is normal iff it is strongly continuous, as noted above; and, as we shall show in more detail below, the map $t \mapsto \phi_t(a)$ for fixed $a$ is continuous with respect to the weak topology iff it is continuous with respect to the strong topology (that is, point-weak continuity is equivalent to point-strong continuity).

**Definition 1.1.5.** Let $\phi = \{\phi_t\}$ be a cp-semigroup on $\mathcal{A}$. An **invariant state** for $\phi$ is a state $\omega : \mathcal{A} \to \mathbb{C}$ with the property

$$\forall t \geq 0 : \omega \circ \phi_t = \omega.$$  

### 1.2. Dilation

In this section we introduce the ways in which cp-semigroups and e-semigroups may be related to each other.

**Definition 1.2.1.** Let $A, B$ be C*-algebras.

1. A **conditional expectation** on $A$ is a linear map $E : A \to A$ such that $E^2 = E$, $\|E\| = 1$, and the range $E(A)$ is a C*-subalgebra.

2. An **embedding from $A$ to $B$** is an injective (equivalently, isometric) *-homomorphism from $A$ to $B$.

3. Given an embedding $i : A \to B$, a **retraction with respect to $i$** is a completely positive map $e : B \to A$ such that $e \circ i = \text{id}_A$.

**Remark 1.2.2.** A linear map $E : A \to A$ whose range is a C*-subalgebra is a conditional expectation if it is a completely positive contraction and is a bimodule map over its range, i.e. has the property that $E(E(a)x) = E(a)E(x) = E(aE(x))$ for all $a, x \in A$; this is known as **Tomiyama’s theorem** ([Tom57]). As a result, if $i : A \to B$ is an embedding and $e : B \to A$ a corresponding retraction, then $i \circ e$ is a conditional expectation on $B$ with range $i(A)$. Hence,
the distinction between a retraction and a conditional expectation is precisely the
distinction between identifying $A$ as a subalgebra of $B$, and explicitly writing an
inclusion map from $A$ to $B$. The difference is a matter of taste; we generally follow
the latter approach.

**Definition 1.2.3.** Let $\phi = \{\phi_t\}$ be a cp-semigroup on a $C^*$-algebra $A$. An
e-dilation of $(A,\phi)$ is a tuple $(A,i,E,\sigma)$ where $A$ is a $C^*$-algebra, $i : A \to A$ an
embedding, $E : A \to A$ a retraction with respect to $i$, and $\sigma = \{\sigma_t\}$ an e-semigroup
on $A$, satisfying

$$\forall t \geq 0 : \quad \phi_t = E \circ \sigma_t \circ i.$$ 

We summarize the relationship in the following diagram:

```
A ------\sigma_t ----> A
|             |        |
|             |        | E
|             |        |
A ------\phi_t ----> A
```

We call $(A, i, E, \sigma)$ a strong e-dilation if it satisfies $E \circ \sigma_t = \phi_t \circ E$, corresponding
to the diagram

```
A ------\sigma_t ----> A
|             |        |
|             |        | E
|             |        |
A ------\phi_t ----> A
```

Note that this implies

$$\phi_t = \phi_t \circ E \circ i = E \circ \sigma_t \circ i$$

so that every strong dilation is a dilation, but the converse does not always hold.
An $e_0$-dilation of a $cp_0$-semigroup is said to be unital if $i(1) = 1$.

1.3. Motivation and Examples

**Example 1.3.1 (Conjugation by Contractions).** Let $H$ be a Hilbert space and
$\{T_t\}$ a semigroup of contractions on $H$. Then the maps $\phi_t : B(H) \to B(H)$ defined
by

$$\phi_t(X) = T_t^*XT_t$$

form a cp-semigroup. It is a $cp_0$-semigroup iff all the $T_t$ are isometries, an e-
semigroup iff all the $T_t$ are coisometries, and hence an $e_0$-semigroup iff all the $T_t$
are unitaries. If $\{T_t\}$ is strongly continuous, in that $t \mapsto T_t$ is continuous with
respect to the strong operator topology on $B(H)$, then $\{\phi_t\}$ is a CP-semigroup. A
theorem of Cooper ([Coo47]) states that, given a strongly continuous contraction
semigroup $\{T_t\}$ on $H$, there exist a Hilbert space $K$, an isometry $V : H \to K$, and
a strongly continuous group $\{U_t\}$ of unitaries on $K$ such that

$$T_t = V^*U_tV.$$ 

If the $T_t$ are isometries, one obtains the stronger condition

$$VT_t = U_tV.$$ 

Given the Cooper dilation of the semigroup $\{T_t\}$, one can then define

1) the $E_0$-semigroup $\{\alpha_t\}$ on $B(K)$ by $\alpha_t(Y) = U_t^*YU_t$
2) the non-unital embedding $i : B(H) \to B(K)$ by $i(X) = VXV^*$
the retraction $E : B(K) \to B(H)$ by $E(Y) = V^* Y V$

Then $(B(K), i, E, \{\alpha_t\})$ is an $E_\sigma$-dilation of $(B(H), \{\phi_t\})$.

This example plays a role in the general theory; for instance, Evans and Lewis prove their dilation theorem ([EL77]) by relating certain more general semigroups to those of the form $X \mapsto T_t^* X T_t$, and then applying Cooper dilation.

Example 1.3.2 (Open Quantum Systems). In (one of the axiomatizations of) quantum mechanics, every physical system corresponds to a von Neumann algebra $A$, with states of the system corresponding to positive elements of $A$ of trace 1. A physical transformation of the system must map states to states and hence, in particular, must be a positive map; a continuous-time evolution of the system corresponds therefore to a semigroup of positive maps. If the system is entangled with an environment, a physical transformation of the composite system must map composite states to composite states, which implies complete positivity of the restriction to the original system; hence, a continuous-time evolution of such an open quantum system is represented by a semigroup of completely positive maps. Continuity requirements are also natural to impose in this setting as one of the physical axioms.

Actually, the representation of such a system as a completely positive semigroup is an approximation to a more general master equation, which approximation holds under various simplifying physical assumptions such as those of “weak coupling” or a “singular reservoir.” Completely positive semigroups arise, for instance, in quantum thermodynamics, where the environment is sometimes regarded as an infinite “heat bath” whose self-interactions are much faster than those of the system under study. For more on these matters see [Haa73, Dav74, GKS76, Lin76, Dav76, EL77, and AJP06]. In the thermodynamic context one typically assumes the existence of a normal $\phi$-invariant state $\omega$ on $A$, representing an equilibrium of the system; correspondingly, one is interested in dilations $(A, i, E, \sigma)$ for which there exists a normal $\sigma$-invariant state $\varpi$ on $A$, which dilates $\omega$ in the sense that $\varpi \circ i = \omega$. In the case of a strong dilation this is automatic, as one can simply define $\varpi = \omega \circ E$, and it follows that

$$\varpi \circ \sigma_t = \omega \circ E \circ \sigma_t = \omega \circ \phi_t \circ E = \omega \circ E = \varpi.$$

In this setting, dilation is a way of relating the dynamics of an open (or “dissipative”) system to the dynamics of a closed (or “non-dissipative”) system containing it.

Example 1.3.3 (Daniell-Kolmogorov Construction). Let $A$ be a commutative unital C*-algebra, and let $S$ be the maximal ideal space of $A$, so that $A \simeq C(S)$. Let $\{P_t\}$ be a CP$_0$-semigroup on $A$. By Riesz representation we obtain for each $t \geq 0$ and each $x \in S$ a measure $p_{t,x}$ characterized by the property

$$\forall f \in C(S) : \int_S f(y) \, dp_{t,x}(y) = (P_t f)(x).$$

Moreover, since $P_t f$ is a continuous function, the family $\{p_{t,x}\}$ varies weak-* continuously in $x$. The property $P_0 = id$ implies that $p_{0,x}$ is the point mass at $x$, and the semigroup property $P_{s+t} = P_s P_t$ implies the Chapman-Kolmogorov equation

$$p_{t+s,x}(E) = \int_S p_{s,y}(E) \, dp_{t,x}(y).$$
Let $\mathcal{S}$ denote the path space $S^{[0,\infty)}$, and $\mathfrak{A} = C(\mathcal{S})$. We have the embedding $i : \mathcal{A} \to \mathfrak{A}$ given by $i(f)(p) = f(p(0))$. By the Stone-Weierstrass theorem, the $*$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ consisting of finite sums of functions of the form $f^{(t_1)}_1 \cdots f^{(t_n)}_n$, where for a path $p \in \mathcal{S}$ the value of $f^{(t_i)}_i$ depends only on $p(t_i)$, is dense in $\mathfrak{A}$. We define a unital linear map $E_0 : \mathfrak{A}_0 \to \mathcal{A}$ on $\mathfrak{A}_0$ by

$$E_0[f^{(t_1)}_1 \cdots f^{(t_n)}_n] = f_n P_{t_n-t_{n-1}} \cdots P_{t_2-t_1}(f_1) \cdots.$$ 

Clearly $E_0 \circ i = \text{id}_\mathcal{A}$. We will show shortly that $E_0$ is well-defined and contractive, so that it extends to a unital contractive (hence positive, hence completely positive) linear map $E : \mathfrak{A} \to \mathcal{A}$ which satisfies $E \circ i = \text{id}_\mathcal{A}$ and is therefore a retraction with respect to $i$.

We define for each $t \geq 0$ the continuous maps $\lambda_t : \mathcal{S} \to \mathcal{S}$ by $(\lambda_t p)(s) = p(s+t)$, and the corresponding $*$-endomorphisms $\sigma_t : \mathfrak{A} \to \mathfrak{A}$ by $\sigma_t f = f \circ \lambda_t$. It is immediate from the above that $E \circ \sigma_t \circ i = P_t$, so that we have obtained a unital $e_0$-dilation of our CP$_0$-semigroup.

Given any regular Borel probability measure $\mu_0$ on $S$, we obtain through Riesz representation a regular Borel probability measure $\mu$ on $\mathcal{S}$ characterized by the property

$$\forall f \in \mathfrak{A} : \int_{\mathcal{S}} f \, d\mu = \int_S (E f) \, d\mu_0.$$ 

This then implies that

$$\forall f \in \mathcal{A} : (P_t f)(x) = E \left[ f(p(t)) \bigg| p(0) = x \right]$$ 

where $E$ denotes conditional expectation in the probabilistic sense, so that we have constructed a Markov process $\{p(t)\}$ with specified transition probabilities. We thus obtain a C$^*$-algebraic version of the classical Daniell-Kolmogorov construction, at least in the context of Feller processes rather than general Markov processes.

We now consider an alternate perspective on the same construction, which enables us easily to prove that $E_0$ is well-defined and contractive, and simultaneously offers a preview of the techniques used in this paper. For each nonempty finite subset $\gamma \subset [0,\infty)$ let $\mathcal{A}_\gamma$ denote a tensor product of $|\gamma|$ copies of $C(S)$. For $\beta \subset \gamma$ we obtain unital embeddings $\mathcal{A}_\beta \to \mathcal{A}_\gamma$ as follows: Writing $\gamma$ as a disjoint union $\beta \cup \gamma'$, identify $\mathcal{A}_\beta$, with $\mathcal{A}_\beta \otimes \mathcal{A}_{\gamma'}$ and embed via $f \mapsto f \otimes 1$. This yields an inductive system and, using the general fact that $C(X \times Y) \simeq C(X) \otimes C(Y)$ for compact Hausdorff spaces $X$ and $Y$, we see that $\lim \mathcal{A}_\gamma$ is isomorphic to $\mathfrak{A}$. The domain of $E_0$ is the union of the images of all the $\mathcal{A}_\gamma$ inside $\mathfrak{A}$, and the well-definedness and contractivity of $E_0$ reduce, by induction, to the well-definedness and contractivity of the maps $\theta_t : C(S) \otimes C(S)$ given on simple tensors by $\theta_t(f \otimes g) = (P_t f)g$. But such a map $\theta_t$ may be equivalently defined as

$$(\theta_t F)(x) = \int_S F(y, x) d\mu_t(x)$$

which obviously yields a well-defined contraction on $C(S) \otimes C(S)$.

We note that the $e_0$-semigroup $\{\sigma_t\}$ is not continuous, even when the original semigroup $\{P_t\}$ is; that is, we obtain only an $e_0$-dilation, not an $E_0$-dilation, of a CP$_0$-semigroup. We shall return to this point in chapter 5.

**Remark 1.3.4.** The last two examples represent the two major streams of thought which motivate the dilation theory of completely positive semigroups. On
the one hand, in the physics setting such a dilation corresponds to an embedding of an open quantum system inside some closed quantum system. On the other hand, we have seen that dilating a CP₀-semigroup defined on a commutative C*-algebra amounts to construction of a Markov process; hence, we may think of dilations of general CP₀-semigroups as a way of constructing “noncommutative Markov processes.”

1.4. Continuity Properties of Semigroups

In this section we examine in greater detail the continuity properties of completely positive semigroups, beginning with more general considerations regarding contraction semigroups on Banach spaces.

1.4.1. C₀-Semigroups. We recount here some of the essentials of the theory of semigroups on Banach spaces, which can be found in [HP57], [DS88], [BR87], and [EN06].

A semigroup \( \{T(t)\}_{t \geq 0} \) of bounded linear operators on a Banach space \( X \) is said to be

(1) **uniformly continuous** if \( t \mapsto T(t) \) is continuous with respect to the norm topology on \( B(X) \); that is, if \( \lim_{t \to t_0} \|T(t) - T(t_0)\|_{B(X)} = 0 \);

(2) **strongly continuous** if, for each \( x \in X \), \( t \mapsto T(t)x \) is continuous with respect to the norm topology on \( X \); that is, if \( \lim_{t \to t_0} \|T(t)x - T(t_0)x\|_X = 0 \) for each \( x \in X \);

(3) **weakly continuous** if, for each \( x \in X \), \( t \mapsto T(t)x \) is continuous with respect to the weak topology on \( X \); that is, if \( \lim_{t \to t_0} \ell(T(t)x - T(t_0)x) = 0 \) for each \( x \in X \) and each \( \ell \in X^* \).

In case \( X \) is the dual of some Banach space \( \mathcal{X}_\ast \), we define the semigroup to be

(4) **weak-* continuous** if, for each \( x \in \mathcal{X}_\ast \), \( t \mapsto T(t)x \) is continuous with respect to the weak-* topology on \( \mathcal{X} \); that is, if \( \lim_{t \to t_0} \ell(T(t)x - T(t_0)x) = 0 \) for each \( x \in \mathcal{X} \) and each \( \ell \in \mathcal{X}_\ast \).

These modes of continuity can, of course, be defined for other families \( \{T(t)\} \) of operators which do not necessarily form a semigroup; however, when they do, strong and weak continuity are equivalent ([EN06] Theorem 1.1.6). Furthermore, uniform continuity is too stringent a hypothesis to be attainable in most applications of interest, so that the bulk of semigroup theory revolves around strongly continuous semigroups, also known as **C₀-semigroups**. Many important C₀-semigroups are **contractive** (meaning \( \|T(t)\| \leq 1 \) for all \( t \)), including completely positive semigroups; as the theory is somewhat simpler for contractive C₀-semigroups, we shall focus on this case.

The most important object associated with a contractive C₀-semigroup is its **generator**, the operator \( \mathcal{L} \) on \( \mathcal{X} \) defined by the formula

\[
\mathcal{L}x = \lim_{t \to 0} t^{-1}[T(t)x - x].
\]

This is in general a closed densely defined unbounded operator, and in fact is bounded if the semigroup is uniformly continuous. Furthermore, the generator satisfies the resolvent growth condition \( \|\lambda I - \mathcal{L}^{-1}\| \leq \lambda^{-1} \) for all \( \lambda > 0 \). The **Hille-Yosida theorem** provides a converse, stating that every closed densely defined operator satisfying this resolvent growth condition is the generator of some...
contractive $C_0$-semigroup. Intuitively, this semigroup is given by $T(t) = e^{t\mathcal{L}}$, but this exponential functional cannot be defined through the usual power series when $\mathcal{L}$ is unbounded; one can, however, write

$$T(t)x = \lim_{n \to \infty} \left( 1 - \frac{t}{n} \mathcal{L} \right)^{-n} x$$

which is sometimes called the Post-Widder inversion formula for $C_0$-semigroups. We thus have a bijection between contractive $C_0$-semigroups and closed densely defined operators satisfying a resolvent growth condition, with explicit formulas for both directions of the bijection.

A notable consequence of the semigroup property is the equivalence between certain notions of continuity and measurability. We define a family $\{\mathcal{T}(t)\}$ of operators on $\mathcal{X}$, equivalently viewed as a function $T : [0, \infty) \to B(\mathcal{X})$, to be

1. uniformly measurable if $T$ is the a.e. norm limit of a sequence of countably-valued functions from $[0, \infty)$ to $B(\mathcal{X})$;
2. strongly measurable if, for each $x \in \mathcal{X}$, $t \mapsto \mathcal{T}(t)x$ is the a.e. norm limit of a sequence of countably-valued functions from $[0, \infty)$ to $\mathcal{X}$;
3. weakly measurable if, for each $x \in \mathcal{X}$ and $\ell \in \mathcal{X}^*$, $t \mapsto \ell(\mathcal{T}(t)x)$ is measurable from $[0, \infty)$ to $\mathbb{C}$.

In case $\mathcal{X}$ is the dual of another Banach space $\mathcal{X}_*$, we also define $\{\mathcal{T}(t)\}$ to be

4. weak-* measurable if, for each $x \in \mathcal{X}$ and $\ell \in \mathcal{X}_*$, $t \mapsto \ell(\mathcal{T}(t)x)$ is a measurable function from $[0, \infty)$ to $\mathbb{C}$.

One might ask why we do not instead define the different types of measurability using the Borel $\sigma$-algebras generated by the corresponding continuity types; the short answer is that a better integration theory results from the definitions given here (the Bochner integral in the case of uniform measurability, the Pettis integral for the others).

It turns out that weak and strong measurability are equivalent when $\mathcal{X}$ is separable ([HP57, Corollary 2, p. 73]) and, when $\{\mathcal{T}(t)\}$ is a contraction semigroup, both are equivalent to strong and weak continuity at times $t > 0$ ([HP57, Theorem 10.2.3]). This latter result is analogous to the fact that measurable solutions to the Cauchy functional equation $f(x + y) = f(x)f(y)$ on $\mathbb{R}$ are exponentials, and hence are continuous. However, strong measurability at $t = 0$ is not enough to infer strong continuity at $t = 0$, but requires the additional hypothesis that $\bigcup_{t > 0} \mathcal{T}(t)X$ be dense in $\mathcal{X}$ ([HP57, Theorem 10.5.5]).

A contraction semigroup $\{\mathcal{T}(t)\}$ on $\mathcal{X}$ induces an adjoint semigroup $\{\mathcal{T}(t)^*\}$ on $\mathcal{X}^*$ by the formula $(\mathcal{T}(t)^* f)(x) = f(\mathcal{T}(t)x)$. If $\mathcal{X}$ is the dual of $\mathcal{X}_*$ and if each $\mathcal{T}(t)$ is weak-* continuous, one obtains also a pre-adjoint semigroup $\{\mathcal{T}(t)_*\}$ through the same formula; since the weak-* topology is of much more interest than the weak topology for spaces having a predual, this is usually referred to in the literature as the adjoint semigroup (and of course is the restriction of the adjoint semigroup to $\mathcal{X}_* \subset \mathcal{X}^*$). Weak-* continuity and measurability of $\{\mathcal{T}(t)\}$ are equivalent to weak continuity and measurability of $\{\mathcal{T}(t)_*\}$, so that in particular they are equivalent to each other at times $t > 0$ if $\mathcal{X}_*$ is separable.

The last topic to consider for contraction semigroups is the passage from separate to joint continuity. We summarize the results in the following theorem.

**Theorem 1.4.1 (Joint Continuity of $C_0$-Semigroups).**
1. INTRODUCTION TO COMPLETELY POSITIVE SEMIGROUPS

(1) Let $X$ be a Banach space and $\{T(t)\}_{t \geq 0}$ a contractive $C_0$-semigroup. Then $T(t)(x)$ is jointly continuous in $t$ and $x$; that is, the map $[0, \infty) \times X \xrightarrow{T} X$ is continuous with respect to the norm topology on $X$.

(2) Let $X$ be a Banach space with separable predual $X_*$, and $\{T(t)\}_{t \geq 0}$ a weak-$^*$ continuous semigroup of weak-$^*$ continuous contractions on $X$. Then $T(t)(x)$ is jointly weak-$^*$ continuous in $t$ and $x$ on bounded subsets of $X$. That is, the map $[0, \infty) \times X_1 \xrightarrow{T} X_1$ is continuous with respect to the weak-$^*$ topology on $X_1$.

(3) Let $A$ be a $W^*$-algebra and $\{\phi_t\}_{t \geq 0}$ a $C_0$-semigroup of strongly continuous contractions on $A$. Then $\phi_t(a)$ is jointly strongly continuous in $t$ and $a$ at nonzero times. That is, the map $(0, \infty) \times A_1 \xrightarrow{\phi} A_1$ is continuous with respect to the strong topology on $A_1$.

Proof.

(1) By the triangle inequality and the contractivity of the semigroup,
$$
\|T(s)(y) - T(t)(x)\| \leq \|T(s)(y-x)\| + \|T(s)x - T(t)x\| \leq \|y-x\| + \|T(s)(x) - T(t)(x)\|
$$
which tends to zero as $(s,y) \to (t,x)$.

(2) By Alaoglu’s theorem, $X_1$ is weak-$^*$ compact, and since $X_*$ is assumed to be separable, another standard result implies that $X_1$ is weakly metrizable ([Con90], V.5.1). Joint weak-$^*$ continuity at $(t,a)$ with $t > 0$ is therefore a special case of Theorem 4 in [CM70]. Joint weak-$^*$ continuity at $(0,a)$ is more complicated to establish, but is a consequence of Corollary 3.3 of [Law74]. A very different proof, using less topology and more semigroup theory, appears in Lemma A.2 of [Ske11]. Notably, this proof does not require separability of the predual; although stated for $W^*$-algebras, it uses only their Banach-space structure.

(3) For strong continuity, we use the same proof, plus the fact that $A_1$ is also weakly metrizable ([Bla06], III.2.2.27). Since $A_1$ is not strongly compact, however, we cannot infer joint continuity at $(0,a)$.

\[\Box\]

1.4.2. Completely Positive Semigroups. So far we have considered semigroups of contractions on Banach spaces. When the Banach space is a $W^*$-algebra, and the contractions are normal completely positive maps, some stronger continuity results hold. Here we note three. First, recall that a CP-semigroup was defined by the property of point-weak continuity. It turns out that such a semigroup is automatically point-strongly continuous. This is Theorem 3.1 of [MS10]. Second, if a CP-semigroup on a $W^*$-algebra is point-norm continuous (which, confusingly enough, would be called “strongly continuous” in the setting of semigroups on general Banach spaces), then it is automatically uniformly continuous. This is Theorem 1 of [Ell00].

Our third continuity result which is specific to completely positive semigroups is an improved statement of joint continuity.

Theorem 1.4.2 (Joint Continuity for CP-Semigroups).

Let $A$ be a separable $W^*$-algebra and $\{\phi_t\}_{t \geq 0}$ a CP-semigroup on $A$.

(1) $\phi_t(a)$ is jointly weakly continuous in $t$ and $a$; that is, the map $[0, \infty) \times A_1 \xrightarrow{\phi} A_1$ is continuous with respect to the weak topology on $A_1$. 


1.5. Survey of Extant Results

The first results concerning the existence of dilations for cp-semigroups date from the 1970’s and pertain to uniformly continuous semigroups. Recall that a contraction semigroup is uniformly continuous iff its generator is bounded; [CE79], preceded in special cases by [GKS76] and [Lin76], showed that the generator of a uniformly continuous CP-semigroup on a W*-algebra must have the form \( a \mapsto \Psi(a) + k^*a + ak \) for some element \( k \in \mathcal{A} \) and completely positive map \( \Psi : \mathcal{A} \to \mathcal{A} \). This structure theorem was used by [EL77] to construct dilations of uniformly continuous semigroups on \( B(H) \) or, more generally, on injective von Neumann algebras.

Dilations of point-weakly continuous CP-semigroups were shown to exist in special cases (for instance, on semigroups having specific forms, on semigroups satisfying additional hypotheses such as the existence of a faithful normal invariant state, in the case of discrete-time semigroups, or using a weaker sense of the word “dilation”) by [Emc78], [AFL82], [VS84], [Küm85], and others. However, progress on the general problem required a new insight. This insight was the notion of a product system of Hilbert spaces, developed by Arveson in [Arv89a], [Arv90a], [Arv89b], and [Arv90b]. Briefly, there is an equivalence of categories between \( E_0 \)-semigroups on \( B(H) \) and product systems of Hilbert spaces, so that the problem of constructing \( E_0 \)-dilations reduces in some sense to the problem of

(2) \( \phi_t(a) \) is jointly strongly continuous in \( t \) and \( a \); that is, the map \([0, \infty) \times \mathcal{A}_1 \xrightarrow{\phi_t} \mathcal{A}_1 \) is continuous with respect to the strong topology on \( \mathcal{A}_1 \).

**Proof.**

(1) This follows from Theorem [1.4.1] we mention it here in order to observe that a considerably simpler proof is available in this special case, which appears as Proposition 2.23 of [Sel97] and as Proposition 4.1(2) of [MS02]. See also Lemma 3.2 of [AMV96]. (Although the statement of that lemma does not restrict to bounded subsets of \( \mathcal{A} \) nor assume separability of \( \mathcal{A} \), both seem to be necessary to justify the use of sequences rather than nets in the proof.)

(2) This is an improvement on Theorem 1.4.1 because of the joint continuity at time 0, which we shall need later. Assume that \( \mathcal{A} \subset B(H) \), with \( H \) separable. Let \( t_n \to t \) be a convergent sequence in \([0, \infty) \) and \( a_n \to a \) an SOT-convergent sequence in \( \mathcal{A}_1 \). (We can use sequences rather than nets because \( \mathcal{A}_1 \) is SOT-metrizable.) By the first part of this theorem, \( \phi_{t_n}(a_n) \to \phi_t(a) \) in SOT. Now for any \( h \in H \),

\[
\| \phi_{t_n}(a_n)h - \phi_t(a)h \|^2 = \| \phi_{t_n}(a_n)h \|^2 - 2 \text{Re} \langle \phi_{t_n}(a_n)h, \phi_t(a)h \rangle + \| \phi_t(a)h \|^2
\]

\[
= \langle \phi_{t_n}(a_n)^* \phi_{t_n}(a_n)h, h \rangle - 2 \text{Re} \langle \phi_{t_n}(a_n)h, \phi_t(a)h \rangle + \| \phi_t(a)h \|^2
\]

\[
\leq \langle \phi_{t_n}(a_n)^* \phi_{t_n}(a_n)h, h \rangle - 2 \text{Re} \langle \phi_{t_n}(a_n)h, \phi_t(a)h \rangle + \| \phi_t(a)h \|^2
\]

where we have used the Schwarz inequality for 2-positive maps. Taking the limsup as \( n \to \infty \), and using the fact that \( a_n^*a_n \to a^*a \) in SOT whenever \( a_n \to a \) in SOT, we see that \( \phi_{t_n}(a_n) \to \phi_t(a) \) in SOT.

This appears as Lemma 4 in [VS84] and as Lemma 6.4 in [Sha08].

\[\Box\]
building a product system out of a CP₀-semigroup. Variants of this strategy were used in [Bha96] and [Sel97] to show that every CP₀-semigroup on $B(H)$ has an $E₀$-dilation, a result known as Bhat's theorem, and the corresponding result for separable $W^*$-algebras was established in [Arv03]. Later, the more general notion of a product system of Hilbert modules was introduced, leading to new proofs of these theorems in [BS00] and [MS02]. More recently, product systems have been used to study families of completely positive maps indexed by semigroups other than $[0, \infty)$, with the existence of dilations depending on an additional hypothesis known as strong commutativity ([Sha08]).

A different approach to dilation theory, standing outside this narrative, was proposed by Jean-Luc Sauvageot in [Sau86], [Sau88], and [Sau91]. Writing during the nascent era of free probability (shortly after the publication of [Vol85], for instance), Sauvageot developed a modified version of the free product appropriate for use in dilation theory. Since the Daniell-Kolmogorov construction (Example 1.3.3) can be built using tensor products, which are the coproduct in the category of commutative unital $C^*$-algebras, and since free products play the corresponding role in the category of unital $C^*$-algebras, this is an attractively functorial way to conceptualize a noncommutative Markov process. Using his version of the free product, Sauvageot proved that every cp₀-semigroup on a $C^*$-algebra has a unital $e₀$-dilation. This dilation theorem was then used to solve a Dirichlet problem for $C^*$-algebras, much as classical Brownian motion can be used to solve the classical Dirichlet problem ([Kak45]).

Sauvageot’s theorem is unusual in that it achieves a unital dilation; at some point, all the other dilation strategies mentioned so far rely upon the non-unital embedding of $B(H)$ into $B(K)$ for Hilbert spaces $H \subset K$. And although other unital dilation techniques exist (for instance, the quantum stochastic calculus pioneered in [HP84] and expounded more recently in [SG07]), they tend to require restrictions on the algebra or the semigroup or both, in contrast to the generality of Sauvageot’s construction. However, although [Sau86] asserts that his dilation technique can be modified to yield continuous dilations on $W^*$-algebras, no detail is given as to how this modification would proceed. Hence, given a CP₀-semigroup, it seems that one may be forced to choose either a unital $e₀$-dilation or a continuous (that is, $E₀$-) dilation. The present paper will expound Sauvageot’s dilation techniques in order to demonstrate the possibility of achieving both objectives together (Theorem 5.4.9).
CHAPTER 2

Liberation

2.1. Introduction

Free probability theory was introduced by Voiculescu in [Voi85] as a tool to address the free group factor problem. Free probability has since blossomed into its own area of study; its development has been an important success, even though the free group factor problem remains unresolved. Sauvageot’s ad hoc modification of free probability, in contrast, does not appear to have inspired further pursuit beyond his first paper. This could be due in part to the relevant free independence-like property remaining implicit in that paper, appearing only in the midst of the proof of Proposition 1.7.

In the present chapter, Sauvageot’s version of free independence, hereinafter referred to as liberation (meant to suggest something similar to freeness; not to be confused with Voiculescu’s use of the same word in [Voi99]) is studied in its own right. As yet the only nontrivial liberated system known to the author is the one originally used by Sauvageot in application to dilation theory. However, it is still advantageous to separate this part of the exposition, both (i) to clarify the combinatorial aspects of dilation, in contrast to its algebraic and analytic features, and (ii) to suggest possibilities for further investigation of connections with standard free probability theory.

2.2. Background: Free Independence and Joint Moments

We recall some of the basic notions of free probability, which can be found in references such as [Voi85], [VDN92], and [NS06].

A noncommutative probability space is a pair \((\mathcal{A}, \phi)\) where \(\mathcal{A}\) is a unital complex algebra and \(\phi : \mathcal{A} \to \mathbb{C}\) a unital linear functional. Subalgebras \(\{\mathcal{A}_i\}_{i \in I}\) of \(\mathcal{A}\) are said to be freely independent with respect to \(\phi\) if \(\phi(a_{i_1}a_{i_2}\ldots a_{i_n}) = 0\) whenever

- \(i_1, \ldots, i_n\) are elements of \(I\) such that adjacent indices are not equal, i.e. for \(k = 1, \ldots, n - 1\) one has \(i_k \neq i_{k+1}\); this condition is abbreviated as \(i_1 \neq i_2 \neq \cdots \neq i_n\);
- \(a_{i_k} \in \mathcal{A}_{i_k}\) for each \(k = 1, \ldots, n\)
- \(\phi(a_{i_k}) = 0\) for each \(k = 1, \ldots, n\).

Given noncommutative probability spaces \(\{(\mathcal{A}_i, \phi_i)\}\), a construction known as the free product of unital algebras yields, in a universal (i.e. minimal) way, a noncommutative probability space \((\mathcal{A}, \phi)\) and injections \(f_i : \mathcal{A}_i \to \mathcal{A}\) satisfying \(\phi \circ f_i = \phi_i\), such that the images \(f_i(\mathcal{A}_i)\) are freely independent with respect to \(\phi\). Furthermore, this construction on unital *-algebras or C*-algebras; in the latter case it is related to the free product of Hilbert spaces.
One implication of free independence which is essential for our present purposes is that it determines the value of \( \phi \) on the subalgebra generated by \( \{A_i\} \). Given \( i_1 \neq i_2 \neq \cdots \neq i_n \) and elements \( a_{i_k} \in A_{i_k} \), one can compute the joint moment \( \phi(a_{i_1} \ldots a_{i_n}) \) as follows:

- **Center** each term \( a_{i_k} \); that is, rewrite it as \( \hat{a}_{i_k} + \phi(a_{i_k})1 \), where we define \( \hat{x} = x - \phi(x)1 \).
- **Expand** the product \( (\hat{a}_{i_1} + \phi(a_{i_1})1) \cdots (\hat{a}_{i_n} + \phi(a_{i_n})1) \), thus obtaining a sum of \( 2^n \) words.
- **Simplify** by pulling out scalars: rewrite, for instance, \( \hat{a}_{i_1} (\phi(a_{i_2})1) \hat{a}_{i_3} \) as \( \phi(a_{i_2})\hat{a}_{i_1} \hat{a}_{i_3} \).
- **After simplification**, the only remaining word of length \( n \) is the centered word \( \hat{a}_{i_1} \ldots \hat{a}_{i_n} \). Applying the procedure iteratively to all the smaller words that have been generated, one can rewrite the original word as a sum of many centered words, plus a word of length 0, i.e. a scalar. Since \( \phi \) vanishes on centered words and is unital, its value at the original word is therefore whatever scalar is left when this iteration terminates.

Using this outline, one can calculate \( \phi(a_{i_1} \ldots a_{i_n}) \) whenever \( i_1 \neq i_2 \neq \cdots \neq i_n \). Of course, no generality is lost by this hypothesis, as neighboring terms belonging to the same subalgebra can be combined.

### 2.3. Defining Liberation

We now develop two variations on free independence, which will be of use in dilation theory.

**Definition 2.3.1.** Let \( C \) be a complex algebra and \( \epsilon : C \to C \) a linear map. Given a triple \((A, B, \rho, \psi)\) consisting of subalgebras \( A, B \subseteq C \) and linear maps \( \rho : A \to B \) and \( \psi : B \to A \), we introduce the notation \( \hat{a} = a - \rho(a) \) for elements \( a \in A \) and \( \hat{b} = b - \psi(b) \) for elements \( b \in B \); note that in general \( \hat{a} \) and \( \hat{b} \) are elements neither of \( A \) nor of \( B \). We say the pair \((A, B)\) is:

1. **right-liberated** (with respect to \( \epsilon, \rho, \psi \)) if \( \epsilon \) is a \( B \)-bimodule map, i.e. \( \epsilon[b_1 \epsilon x b_2] = b_1 \epsilon(x) b_2 \) for all \( b_1, b_2 \in B \) and \( x \in C \), and for every \( n \geq 1 \), every \( a_1, \ldots, a_n \in A \), and every \( b_1, \ldots, b_{n-1} \in B \),

\[
\epsilon \left[ \hat{a}_1 \hat{b}_1 \hat{a}_2 \hat{b}_2 \cdots \hat{b}_{n-1} \hat{a}_n \right] = 0;
\]

2. **left-liberated** if \( \epsilon \) is an \( A \)-bimodule map and for every \( n \geq 1 \), every \( a_1, \ldots, a_{n-1} \in A \), and every \( b_1, \ldots, b_n \in B \),

\[
\epsilon \left[ \hat{b}_1 \hat{a}_1 \hat{b}_2 \hat{a}_2 \cdots \hat{a}_{n-1} \hat{b}_n \right] = 0.
\]

We note that the criteria in these definitions resemble free independence, in that the alternating product of centered terms is centered. The key difference, however, is that the centering takes place with respect to several different maps—elements of \( B \) are centered with respect to \( \psi \), elements of \( A \) with respect to \( \rho \), and the alternating product with respect to \( \epsilon \).

In some cases it will be useful to generalize this definition.

**Definition 2.3.2.** Let \( A, B \) be complex algebras, and \( \rho : A \to B \) and \( \psi : B \to A \) linear maps. A **right-liberating representation** of the quadruple \((A, B, \rho, \psi)\) is a quadruple \((A, f, g, \epsilon)\) where
2.4. Liberation and Joint Moments

Like free independence, liberation is a property that implies an algorithm. The idea is the same—by centering, expanding, and simplifying, one can write any word as a centered word plus shorter words—but since the centering takes place with respect to three different maps, the details of the procedure are more complicated.

Suppose \((A, B)\) are right-liberated in \(C\) with respect to \(\epsilon, \rho, \psi\) as above. Suppose also that \(C\) and \(B\) are unital (with the same unit). We continue to use the notation \(\hat{a} = a - \rho(a)\) and \(\hat{b} = b - \psi(b)\). We will show that the liberation property determines \(\epsilon\) on the subalgebra \(\langle A, B \rangle \subset C\) generated by \(A\) and \(B\). Since \(B\) is unital, \(\langle A, B \rangle\) is linearly spanned by words of the form \(b_0a_1b_1 \cdots a_\ell b_\ell\) for \(\ell \geq 0\). To determine the value of \(\epsilon\) on such a word, we write

\[
b_0a_1b_1 \cdots a_\ell b_\ell = b_0\hat{a}_1\hat{b}_1\hat{a}_2\hat{b}_2 \cdots \hat{a}_{\ell-1}\hat{a}_\ell b_\ell + x,
\]

where \(x\) is a sum of words with “fewer \(A, B\) alternations” (in a sense to be made precise) than before. One can calculate \(x\) by expanding

\[
x = b_0a_1b_1 \cdots b_{\ell-1}a_\ell b_\ell - b_0\left[ a_1 - \rho(a_1) \right] \left[ b_1 - \psi(b_1) \right] \cdots \left[ b_{\ell-1} - \psi(b_{\ell-1}) \right] \left[ a_\ell - \rho(a_\ell) \right] b_\ell.
\]
The liberation property implies that $\epsilon[b_0a_1b_1 \cdots a_kb_k] = \epsilon[x]$, which can then be computed recursively. The recursion terminates on elements of $b$, for which the bimodule property implies $\epsilon[b] = b\epsilon[1]$. To make these computations more explicit, we use the following combinatorial notation and terminology:

- For a natural number $m \geq 1$, $[m]$ denotes the set $\{1, 2, \ldots, m\}$, while $2[m]$ denotes the set $\{2, 4, \ldots, 2m\}$.
- Given $\ell$ and $S$ as above, the **alternation number** of $S$, which we denote $Alt(S)$, is

  $$Alt(S) = \frac{1}{2} \sum_{j=1}^{2\ell} |\chi_s(j) - \chi_s(j+1)|.$$  

  If one colors $S \cup \{0, 2\ell\}$ white and $[2\ell - 1] \setminus S$ black, then $Alt(S)$ is the number of times that the color changes from white to black and back again as one counts from 0 to $2\ell$. Note that the maximum value is $\ell$, achieved if $S = 2[\ell - 1]$.
- Given $\ell$ and $S$ as above, the **consecutive in-subsets** of $S$, which we denote $T_0, \ldots, T_{\text{Alt}(S)}$, are the maximal subsets of $S \cup \{0, 2\ell\}$ consisting of consecutive elements, while the **consecutive out-subsets**, denoted $U_1, \ldots, U_{\text{Alt}(S)}$, are the maximal consecutive subsets of $[2\ell - 1] \setminus S$. For example, if $\ell = 5$ and $S = \{1, 3, 4, 8\}$ then

  $$T_0 = \{0, 1\}, \quad T_1 = \{3, 4\}, \quad T_2 = \{8\}, \quad T_3 = \{10\}, \quad U_1 = \{2\}, \quad U_2 = \{5, 6, 7\}, \quad U_3 = \{9\}.$$  

We then introduce the following algebraic definitions:

(i) Given complex algebras $A, B$, let

$$W_\ell = \{(b_0, a_1, b_1, \ldots, a_\ell, b_\ell) \mid a_1, \ldots, a_\ell \in A; b_0, \ldots, b_\ell \in B\}, \quad \ell \geq 0$$

denote the alternating tuples of length $2\ell + 1$ which start and end with an element of $B$, and $W_\# = \bigcup_{\ell=0}^{\infty} W_\ell$.

(ii) Given $A, B$ as above, as well as linear maps $\rho : A \to B$ and $\psi : B \to A$, a natural number $\ell \geq 1$, a subset $S \subseteq [2\ell - 1]$, and a tuple $w \in W_\ell$, let

$$x_j = \begin{cases} \rho(a_{(j+1)/2}) & j \text{ odd} \\ b_{j/2} & j \text{ even} \end{cases}$$

for $j \in S \cup \{0, 2\ell\}$, and

$$y_k = \begin{cases} a_{(k+1)/2} & k \text{ odd} \\ \psi(b_{k/2}) & k \text{ even} \end{cases}$$

for $k \in [2\ell - 1] \setminus S$. Then the **collapse of $w$ determined by $S$** is the tuple in $W_{\text{Alt}(S)}$ given by

$$\text{Col}(w; S) = \left( \prod_{j \in T_0} x_j, \prod_{k \in U_1} y_k, \prod_{j \in T_1} x_j, \prod_{k \in U_2} y_k, \ldots, \prod_{k \in U_{\text{Alt}(S)}} y_k, \prod_{j \in T_{\text{Alt}(S)}} x_j \right).$$
(iii) Finally, we define the **moment function** $\mathcal{M} : \mathcal{W}_\# \to B$ recursively by

$$\mathcal{M}(x) = x \text{ for } x \in \mathcal{W}_0 = B, \text{ and}$$

$$\mathcal{M}(x) = \sum_{S \subseteq [2\ell - 1], S \neq \emptyset} (-1)^{\ell+|S|} \mathcal{M}(\text{Col}(x; S)), \quad x \in \mathcal{W}_\ell, \quad \ell \geq 1.$$  

The recursion for $\mathcal{M}$ is well defined because the sum is over sets with alternation number strictly less than $\ell$.

The point of these definitions is that, in expanding the expression $b_0 a_1 b_1 \cdots a_\ell b_\ell - b_0 [a_1 - \rho(a_1)] [b_1 - \psi(b_1)] \cdots [a_\ell - \rho(a_\ell)] b_\ell$, each term in the expansion corresponds to a subset $S \subseteq [2\ell - 1]$ indicating from which of the bracketed factors one has chosen an element of $B$ (either $\rho(a_i)$ or $b_i$). In the resulting term of the expansion, one then multiplies together consecutive elements of $A$ and $B$ to obtain a word with fewer $A, B$ alternations than the original word $b_0 a_1 b_1 \cdots a_\ell b_\ell$.

**Remark 2.4.1** (Complexity of the moment function). Note that evaluating $\mathcal{M}$ on a word of length $\ell$ returns a sum of $2^{2\ell - 1}$ evaluations on words of length up to $\ell - 1$. This implies that the number of terms in the evaluation of $\mathcal{M}$ on words of length $\ell$ is bounded above by the sequence $\{s_\ell\}$ determined by $s_0 = 1$ and $s_{\ell+1} = 2^{2\ell+1} s_\ell$, which has the closed form $s_\ell = 2^{2\ell}$. Of course the actual number of terms is considerably less, due both to cancellation and to the fact that this estimate treats all words of length less than $\ell$ as if they had length $\ell - 1$.

**Theorem 2.4.2.** Let $(A, B)$ be right-liberated in $A$ with respect to $\epsilon, \rho, \psi$, and suppose $A$ and $B$ are unital. Define the product function $\Pi : \mathcal{W}_\# \to A$ by $\Pi(b_0, a_1, b_1, \ldots, a_\ell, b_\ell) = b_0 a_1 b_1 \cdots a_\ell b_\ell$. Then for any $x \in \mathcal{W}_\#$,

$$\epsilon[\Pi(x)] = \mathcal{M}(x)[1].$$

**Proof.** This follows from the above discussion of $\mathcal{M}$ as a formalized way of expanding certain expressions relating to the right-liberation property. \qed

**Corollary 2.4.3.** Let $(A, B)$ be right-liberated in $A$ with respect to $\epsilon, \rho, \psi$. Then

$$\epsilon[(A, B)] = \epsilon[B].$$

We note that this holds even without assuming unitality of $A$ and $B$, although in that case it follows from the proof of Theorem (2.4.2) rather than the theorem itself.

The obvious generalizations of Theorem (2.4.2) and Corollary (2.4.3) to right-liberating representations are true as well, and are verified inductively in the same manner. We record them here without proof.

**Theorem 2.4.4.** Let $(A, f, g, \epsilon)$ be a right-liberating representation of $(A, B, \rho, \psi)$. For $x = (b_0, a_1, b_1, \ldots, a_\ell, b_\ell) \in \mathcal{W}_\#$, let $(f \times g)(x)$ denote the element $g(b_0) f(a_1) g(b_1) \cdots f(a_\ell) g(b_\ell) \in A$. Suppose $A, g$ are unital.

Then for any $x \in \mathcal{W}_\#$,

$$\epsilon[(f \times g)(x)] = g(\mathcal{M}(x)) \epsilon[1].$$

**Corollary 2.4.5.** Let $(A, f, g, \epsilon)$ be a right-liberating representation of $(A, B, \rho, \psi)$. Let $(A, B)$ denote the subalgebra of $A$ generated by $f(A)$ and $g(B)$. Then

$$\epsilon[(A, B)] = \epsilon[g(B)].$$
Later we shall be interested in the continuity properties of joint moments. We record here the following simple observation:

**Proposition 2.4.6.** Let $\mathcal{A}$ be a $W^*$-algebra, $A$ and $B$ subalgebras, and $\rho : A \to B$ and $\psi : B \to A$ normal linear maps.

1. For any $x \in \mathcal{W}_\#, M(x)$ is normal in each entry of $x$. That is, given $\ell \geq 0$, $x \in \mathcal{W}_\ell$, and $1 \leq j \leq 2\ell + 1$, let $x_k$ be fixed for all $1 \leq k \leq 2\ell + 1$ with $k \neq j$; then $M(x)$, viewed as a function of $x_j$, is a normal linear map from $A$ or $B$ (depending on the parity of $j$) to $\mathcal{A}$.

2. If $\rho$ is strongly continuous on the unit ball $A_1$, and $\psi$ strongly continuous on $B_1$, then $M(x)$ is jointly strongly continuous in the entries of $x$ on bounded subsets. That is, the corresponding map $A_1 \times B_1 \times \cdots \times A_1 \to A$ is strongly continuous.

The proof is a straightforward induction on $\ell$. Later we will need to consider moments with respect to several maps. When need arises, we use $M(x; \rho; \psi)$ in place of $M(x)$ for specificity.

**Proposition 2.4.7.** Let $\mathcal{A}$ be a $W^*$-algebra, $\psi : A \to A$ a strongly continuous linear map, and $\{\rho_t\}_{t \geq 0}$ a CP-semigroup on $A$. Then for each fixed $x \in \mathcal{W}_\#$, $M(x; \rho_t; \psi)$ and $M(x; \psi; \rho_t)$ are strongly continuous in $t$.

Here we are implicitly using $B = A$ in the definition of $\mathcal{W}_\#$. For the proof, recall that $t \mapsto \phi_t(a)$ is strongly continuous for fixed $a \in A$, as discussed in section 1.4.2. This fact plus the joint strong continuity of multiplication on $A_1$ yields a straightforward induction.

### 2.4.1. Left Liberation and Joint Moments

The calculation of joint moments given the property of left liberation is essentially the same as for right liberation. Informally, given a moment calculation based on right liberation, one may obtain a corresponding moment calculation for left liberation by interchanging the roles of $A$ and $B$, and of $\rho$ and $\psi$; for example, given that

$$
\mathcal{E}[b_0a_1a_2b_2] = b_0 \left( \rho(a_1)b_1\rho(a_2) + \rho(a_1)\rho(b_1)a_2 \right) - \rho(a_1)\rho(b_1)\rho(a_2)
$$

when $(A, B)$ are right-liberated and $\mathcal{A}, B$ unital, one can infer that

$$
\mathcal{E}[a_0b_1a_2b_2] = a_0 \left( \psi(b_1)a_1\psi(b_2) + \psi(b_1)\psi(a_1)b_2 \right) - \psi(b_1)\psi(a_1)\psi(b_2)
$$

when $(A, B)$ are left-liberated and $\mathcal{A}, A$ unital.

More precisely, we modify our algebraic definitions above as follows:

(i) Given $A, B$ we define for each $\ell \geq 0$ the set

$$
\hat{\mathcal{W}}_\ell = \{(a_0, b_1, a_1, \ldots, b_\ell, a_\ell) \mid a_0, \ldots, a_\ell \in A; b_1, \ldots, b_\ell \in B\}
$$

and the corresponding set $\hat{\mathcal{W}}_\# = \bigcup_{\ell=0}^\infty \hat{\mathcal{W}}_\ell$. 

(ii) Given \( A, B, \rho, \psi, \ell \geq 1, S \subseteq [2\ell - 1], \) and \( w \in \hat{W}_\ell, \) let
\[
\hat{x}_j = \begin{cases} 
\psi(b_{j/2}) & \text{if } j \text{ odd} \\
\psi(a_{j/2}) & \text{if } j \text{ even}
\end{cases}
\]
for \( j \in S \cup \{0, 2\ell\}, \) and
\[
\hat{y}_k = \begin{cases} 
b_{k/2} & \text{if } k \text{ odd} \\
\rho(a_{k/2}) & \text{if } k \text{ even}
\end{cases}
\]
for \( k \in [2\ell - 1] \setminus S. \) Then we let
\[
\hat{\text{Col}}(w; S) = \left( \prod_{j \in T_0} \hat{x}_j, \prod_{k \in U_1} \hat{y}_k, \prod_{j \in T_1} \hat{x}_j, \prod_{k \in U_2} \hat{y}_k, \ldots, \prod_{k \in U_{\text{A}(S)}} \hat{y}_k, \prod_{j \in T_{\text{A}(S)}} \hat{x}_j \right).
\]

(iii) Finally, we define \( \hat{M} : \hat{W}_\# \to A \) recursively by \( \hat{M}(x) = x \) for \( x \in \hat{W}_0 = A, \) and
\[
\hat{M}(x) = \sum_{S \subseteq [2\ell - 1], S \neq \emptyset} (-1)^{|S| + |\hat{\text{Col}}(x; S)|} \hat{M}(\hat{\text{Col}}(x; S)), \quad x \in \hat{W}_\ell, \quad \ell \geq 1.
\]

We arrive at the obvious analogues of our results for right liberation, presented here without proof.

**Theorem 2.4.8.** Let \((A, B)\) be left-liberated in \(A\) with respect to \(\epsilon, \rho, \psi,\) and suppose \(A\) and \(A\) are unital. Define the product function \(\hat{\Pi} : \hat{W}_\# \to A\) by \(\hat{\Pi}(a_0, b_1, a_1, \ldots, b_\ell, a_\ell) = a_0 b_1 a_1 \cdots b_\ell a_\ell.\) Then for any \(x \in \hat{W}_\#,\)
\[
\epsilon(\hat{\Pi}(x)) = \hat{M}(x) \epsilon[1].
\]

**Corollary 2.4.9.** Let \((A, B)\) be left-liberated in \(A\) with respect to \(\epsilon, \rho, \psi.\) Then
\[
\epsilon[\langle A, B \rangle] = \epsilon[A].
\]

**Theorem 2.4.10.** Let \((A, f, g, \epsilon)\) be a left-liberating representation of \((A, B, \rho, \psi).\) For \(x = (a_0, b_1, a_1, \ldots, b_\ell, a_\ell) \in \hat{W}_\#,\) let \((f \times g)(x)\) denote the element \(f(a_0)g(b_0)f(a_1) \cdots g(b_\ell)f(a_\ell) \in A.\) Suppose also that \(A, f, a_0, b_0, f, g, a_0, b_0\) are unital. Then for any \(x \in \hat{W}_\#,\)
\[
\epsilon((f \times g)(x)) = f(\hat{M}(x)) \epsilon[1].
\]

**Corollary 2.4.11.** Let \((A, f, g, \epsilon)\) be a left-liberating representation of \((A, B, \rho, \psi).\) Let \(\langle A, B \rangle\) denote the subalgebra of \(A\) generated by \(f(A)\) and \(g(B).\) Then
\[
\epsilon[\langle A, B \rangle] = \epsilon[f(A)].
\]

**Proposition 2.4.12.** Let \(A\) be a \(W^*\)-algebra, \(A\) and \(B\) subalgebras, and \(\rho : A \to B\) and \(\psi : B \to A\) normal linear maps.

1. For any \(x \in \hat{W}_\#,\) \(\hat{M}(x)\) is normal in each entry of \(x.\)
2. If \(\rho\) is strongly continuous on the unit ball \(A_1,\) and \(\psi\) strongly continuous on \(B_1,\) then \(\hat{M}(x)\) is jointly strongly continuous in the entries of \(x\) on bounded subsets.
Proposition 2.4.13. Let $A$ be a $W^*$-algebra, $\psi : A \rightarrow A$ a strongly continuous linear map, and $\{\rho_t\}_{t \geq 0}$ a CP-semigroup on $A$. Then for each fixed $x \in \hat{W}_{\#}$, $\hat{M}(x; \rho_t; \psi)$ and $\hat{M}(x; \psi; \rho_t)$ are strongly continuous in $t$.

2.4.2. Tables and the Scalar Case. We include in Appendix A values of the moment function $M$ for small $\ell$.

An important special case of right-liberation occurs when the map $\psi : B \rightarrow A$ is scalar-valued (in particular, the applications to Sauvageot’s dilation theory are built on the case where $\psi$ is a state on $B$). This allows considerable simplification of the expressions for joint moments, both because $\psi(B) \subset A \cap A'$ and because $\rho$ is a $\psi$-bimodule map. Due to its importance in applications, this special case also has a table in Appendix A.
CHAPTER 3

The Sauvageot Product

3.1. Introduction

In this chapter we develop a modification of the unital free product of C*-algebras, adapted for use in dilation theory. As mentioned in example 1.3.3, the classical Daniell-Kolmogorov construction can be reduced to the construction of maps $\theta_t : C(S) \otimes C(S) \to C(S)$ given on simple tensors by $\theta_t(f \otimes g) = (P_t f)g$. We shall return to the details of this reduction in chapter 4; at present we only describe enough of its features to see what we shall need for the appropriate noncommutative analogue.

Among the many embeddings of $C(S)$ into $C(S) \otimes C(S)$ we distinguish two, the “left” embedding $f \mapsto f \otimes 1$ and the “right” embedding $f \mapsto 1 \otimes f$. The map $\theta_t$ is a retraction with respect to the right embedding, and its composition with the left embedding is $P_t$. That is, by constructing $\theta_t$ we factor $P_t$ into an embedding followed by a retraction (with respect to a different embedding), as depicted in the following diagram:

$$
\begin{array}{c}
C(S) \otimes C(S) \\
\downarrow \theta_t \uparrow \\
C(S) \\
\downarrow P_t \\
C(S)
\end{array}
\quad
\begin{array}{c}
\theta_t(f \otimes g) = P_t(f)g
\end{array}
$$

More generally, the inductive process will work with tensor powers $C(S)^{\otimes \gamma}$ for finite sets $\gamma \subset [0, \infty)$, building for each one a retraction $\epsilon_\tau : C(S)^{\otimes \gamma} \to C(S)$. Given $\gamma' = \gamma \cup \{t_k\}$, where $\tau = t_k - \min_{t \in \gamma} t > 0$, we will seek to define $\epsilon_{\gamma'}$ such that

$$
\begin{array}{c}
C(S)^{\otimes \gamma'} \\
\downarrow \epsilon_{\gamma'} \uparrow \\
C(S)^{\otimes \gamma} \\
\downarrow P_\tau \\
C(S)
\end{array}
\quad
\begin{array}{c}
\epsilon_{\gamma'}(f \otimes g) = P_\tau(\epsilon_\gamma(f))g
\end{array}
$$

We note in passing that Stinespring dilation produces a very similar diagram: Given a unital completely map $\phi : A \to B(H)$ with minimal Stinespring triple $(K, V, \pi)$, we obtain

$$
\begin{array}{c}
B(K) \\
\downarrow \theta \uparrow \\
A \\
\downarrow \phi \\
B(H)
\end{array}
\quad
\begin{array}{c}
\theta(T) = V^*TV
\end{array}
$$
Crucially, however, the right embedding in this case is the non-unital map \( X \mapsto VXV^* \), in contrast to the unital embedding in the commutative example. Hence, we take the tensor product as our model in what follows.

In addition to constructing tensor products \( C(X) \otimes C(Y) \simeq C(X \times Y) \) of commutative unital C*-algebras, one can also form tensor products of maps between them, and the resulting maps satisfy certain functorial properties. We summarize the properties of the tensor product which we shall seek to replicate in this chapter:

1. Given unital C*-algebras \( A, B \) and a unital completely positive map \( \phi: A \to B \), we construct a unital C*-algebra \( A \ast B \) with unital embeddings of \( A \) and \( B \), the images of which generate \( A \ast B \).
2. We also construct a retraction \( A \ast B \to B \) which factors \( \phi \) in the sense of the above diagrams.
3. Given unital completely positive maps \( \phi: A \to B \) and \( \eta: C \to D \), and given unital *-homomorphisms \( f: A \to C \) and \( g: B \to D \) such that the square

\[
\begin{array}{c}
C \xrightarrow{\psi} D \\
\downarrow f & \downarrow g \\
A \xrightarrow{\phi} B
\end{array}
\]

commutes, we construct a (necessarily unique) unital *-homomorphism \( f \ast g: A \ast B \to C \ast D \) such that the squares

\[
\begin{array}{ccc}
A \ast B & \xrightarrow{f \ast g} & C \ast D \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{f} & C \\
\end{array}
\quad
\begin{array}{ccc}
A \ast B & \xrightarrow{f \ast g} & C \ast D \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{g} & D
\end{array}
\]

commute.

We now begin our development of a construction satisfying these requirements.

### 3.2. Sauvageot Products of Hilbert Spaces and Bounded Operators

Just as the free product of unital C*-algebras can be constructed from a free product of Hilbert spaces (Vol85), our product construction on C*-algebras will rely on an underlying construction on Hilbert space. Some notational preliminaries: For a Hilbert space \( H \), we use \( H^+ \) to denote \( H \oplus \mathbb{C} \), and if a unit vector has been distinguished, \( H^- \) to denote the complement of its span. A distinguished unit vector (such as \( 1 \in \mathbb{C} \) as an element of the direct sum \( H \oplus \mathbb{C} \)) is generally denoted by \( \Omega \). We also follow the convention, most common in physics and in Hilbert C*-modules, that inner products are linear in the second variable.

**Definition 3.2.1.** Let \( \mathcal{H} \) and \( \mathcal{L} \) be Hilbert spaces. The **Sauvageot product** \( \mathcal{H} \ast \mathcal{L} \) is the space

\[
\mathcal{H} \ast \mathcal{L} = H^+ \oplus \bigoplus_{n=0}^{\infty} \left[ (L^+ \otimes \mathcal{L}) \oplus (\mathcal{H} \otimes L^+) \otimes \mathcal{L} \right]
\]

with the convention \( L^+ \otimes \mathcal{L} = \mathcal{L} \).
Though defined as a direct sum, the Sauvageot product of Hilbert spaces may also be viewed as an infinite tensor product, as expressed in the following proposition.

Proposition 3.2.2. Let \( \mathcal{H} \) and \( \mathcal{L} \) be Hilbert spaces, and \( \mathcal{K} = \mathcal{H}^+ \oplus \mathcal{L} \). Denote by \( \mathcal{L}^{\otimes N} \) the infinite tensor power of \( \mathcal{L}^+ \) with respect to \( \Omega \). Then there are unitary equivalences between \( \mathcal{H} \ast \mathcal{L} \) and both \( \mathcal{H}^+ \otimes \mathcal{L}^{\otimes N} \) and \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \), under which

- the subspace \( \mathcal{H} \otimes \mathcal{L}^{\otimes N} \) of \( \mathcal{H}^+ \otimes \mathcal{L}^{\otimes N} \) is identified with the subspace
  \[ \bigoplus_{n=0}^{\infty} \mathcal{H} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L} \] of \( \mathcal{H} \ast \mathcal{L} \)
- the subspace \( \mathcal{C} \otimes \mathcal{L}^{\otimes N} \) of \( \mathcal{H}^+ \otimes \mathcal{L}^{\otimes N} \) is identified with the subspace
  \[ \bigoplus_{n=0}^{\infty} \mathcal{L}^{\otimes n} \otimes \mathcal{L} \] of \( \mathcal{H} \ast \mathcal{L} \)
- the subspace \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \) of \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \) is identified with the subspace
  \[ \bigoplus_{n=0}^{\infty} \mathcal{H} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L} \] of \( \mathcal{H} \ast \mathcal{L} \)
- the subspace \( \mathcal{C} \otimes \mathcal{L}^{\otimes N} \) of \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \) is identified with the subspace
  \[ \bigoplus_{n=0}^{\infty} \mathcal{C} \otimes \mathcal{L}^{\otimes (n-1)} \otimes \mathcal{L} \] of \( \mathcal{H} \ast \mathcal{L} \)
- the subspace \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \) of \( \mathcal{K} \otimes \mathcal{L}^{\otimes N} \) is identified with the subspace
  \[ \bigoplus_{n=1}^{\infty} \mathcal{L} \otimes \mathcal{L}^{\otimes (n-1)} \otimes \mathcal{L} \] of \( \mathcal{H} \ast \mathcal{L} \).

Proof. We will use the unitary equivalences \( \mathcal{L}^{\otimes N} \simeq \mathcal{L}^{+} \otimes \mathcal{L}^{+\otimes N} \), which is evident, and \( \mathcal{L}^{+\otimes N} \simeq \mathcal{C} \otimes \bigoplus_{n=0}^{\infty} \mathcal{L}^{\otimes n} \otimes \mathcal{L} \), which follows from the definition of infinite tensor powers. Repeated application of these equivalences plus the associative, commutative, and distributive laws

\[ H \otimes (K_1 \otimes K_2) \simeq (H \otimes K_1) \otimes K_2, \quad H \otimes K \simeq K \otimes H, \quad H \otimes (K_1 \oplus K_2) \simeq (H \otimes K_1) \oplus (H \otimes K_2) \]

and the identity \( H \otimes C \simeq H \) yield

\[ \mathcal{K} \otimes \mathcal{L}^{\otimes N} = (\mathcal{H}^+ \oplus \mathcal{C}) \otimes \mathcal{L}^{\otimes N} \simeq (\mathcal{H} \oplus \mathcal{L}^+) \otimes \mathcal{L}^{\otimes N} \simeq (\mathcal{H} \otimes \mathcal{L}^{\otimes N}) \oplus (\mathcal{L}^+ \otimes \mathcal{L}^{\otimes N}) \]

\[ \simeq (\mathcal{H} \otimes \mathcal{L}^{\otimes N}) \oplus \mathcal{L}^{\otimes N} \simeq (\mathcal{H} \otimes \mathcal{L}^{\otimes N}) \oplus (\mathcal{C} \otimes \mathcal{L}^{\otimes N}) \simeq \mathcal{H}^+ \otimes \mathcal{L}^{\otimes N} \]

and

\[ \mathcal{H} \ast \mathcal{L} = \mathcal{H}^+ \oplus \bigoplus_{n=0}^{\infty} (\mathcal{H} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L}) \]

\[ \simeq (\mathcal{H}^+ \otimes \mathcal{C}) \oplus \bigoplus_{n=0}^{\infty} (\mathcal{H} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L} \otimes \mathcal{C} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L}) \]

\[ \simeq (\mathcal{H}^+ \otimes \mathcal{C}) \oplus \bigoplus_{n=0}^{\infty} (\mathcal{H}^+ \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L}) \simeq \mathcal{H}^+ \otimes \left( \bigoplus_{n=0}^{\infty} \mathcal{L}^{\otimes n} \otimes \mathcal{L} \right) \simeq \mathcal{H}^+ \otimes \mathcal{L}^{\otimes N}. \]

The specific identifications arise by following subspaces through these equivalences.

\[ \square \]

As a simple corollary, we obtain the following identifications:

Proposition 3.2.3. Let \( \mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{L} \) be Hilbert spaces.
Our next goal is to define the product of maps between Hilbert spaces.

**Definition 3.2.4.** Let \( H_1, H_2, L_1, L_2 \) be Hilbert spaces, \( K_1 = H_1^+ \oplus L_1 \) and \( K_2 = H_2^+ \oplus L_2 \), \( S : H_1^+ \to H_2^+ \) and \( T : K_1 \to K_2 \) bounded maps, and \( V : L_1 \to L_2 \) a contraction. Define the bounded maps \( H \otimes H \to H \otimes H \) as follows: Let \( V^+ = V \otimes \text{id}_1 : L_1^+ \to L_2^+ \) and let \( \mathcal{L}^2_{\odot n} : L_2^{\odot n} \to L_1^{\odot n} \) be the limit of the contractions \( V^+ \odot n : L_1^{\odot n} \to L_2^{\odot n} \). Then \( S \otimes V : H_1 \otimes L_1 \to H_2 \otimes L_2 \) is the operator \( \mathcal{S} \otimes \mathcal{V} : \mathcal{H}_1^+ \otimes L_1 \to \mathcal{H}_2^+ \otimes L_2 \) composed with the unitary equivalences of Proposition 3.2.2. Similarly, \( T \otimes \mathcal{V} \) is the operator \( \mathcal{T} \otimes \mathcal{V} : \mathcal{H}_1^+ \otimes \mathcal{L}_1 \to \mathcal{H}_2^+ \otimes \mathcal{L}_2 \) composed with the appropriate unitary equivalences.

By following the sequence of equivalences in the proof of Proposition 3.2.2, we can calculate how product maps act on the various summands of \( H_1 \otimes L_1 \).

**Proposition 3.2.5.** Let \( H_1, H_2, L_1, L_2 \) be Hilbert spaces, and for \( i = 1, 2 \) let \( K_i = H_i^+ \oplus L_i \). Let \( H_1^+ \xrightarrow{S} H_2^+ \) and \( H_1 \xrightarrow{T} H_2 \) be bounded operators and \( L_1 \xrightarrow{V} L_2 \) a contraction. For each \( n \geq 0 \), let \( V(n) \) denote \( V^+ \odot n \otimes V : L_1^{\odot n} \otimes L_1 \to L_2^{\odot n} \otimes L_2 \).

Let \( h \in H_1^+, h_0 \in H_1, k \in K_1, \ell \in L_1^+, \) and \( \xi \in L_1^{\odot n} \otimes L_1 \) for some \( n \geq 0 \), and suppose that

\[
\begin{align*}
S\Omega_1 &= \alpha \Omega_2 + y, & \alpha \in \mathbb{C}, y \in H_2 \\
Sh_0 &= \beta \Omega_2 + z, & \beta \in \mathbb{C}, z \in H_2 \\
T\eta &= \eta + w, & \eta \in H_2, w \in L_2^+ \\
T\ell &= \zeta + u, & \zeta \in H_2, u \in L_2^+.
\end{align*}
\]

Then

\[
\begin{align*}
(S \otimes V)h &= Sh \\
(S \otimes V)\xi &= \alpha V(n)\xi + (y \otimes V(n)\xi) \\
(S \otimes V)(h_0 \otimes \xi) &= \beta V(n)\xi + (z \otimes V(n)\xi) \\
(T \otimes \mathcal{V})k &= Tk \\
(T \otimes \mathcal{V})(h_0 \otimes \xi) &= (\eta \otimes V(n)\xi) + (w \otimes V(n)\xi) \\
(T \otimes \mathcal{V})(\ell \otimes \xi) &= (\zeta \otimes V(n)\xi) + (u \otimes V(n)\xi).
\end{align*}
\]

Next, we develop some of the essential properties of this construction.

**Proposition 3.2.6.** Let \( H_1^+ \xrightarrow{S} H_2^+ \xrightarrow{S'} H_3^+ \) and \( K_1 \xrightarrow{T} K_2 \xrightarrow{T'} K_3 \) be bounded maps, and \( L_1 \xrightarrow{V} L_2 \xrightarrow{V'} L_3 \) contractions.

1. \((S' \circ S') \circ (S \circ S) = (S' \circ S) \circ (V' \circ V)\) and \((T' \otimes \mathcal{V}) \circ (T \otimes \mathcal{V}) = (T' \circ T) \otimes V\).
2. If \( S \) is the identity map on \( H_1 = H_2 \), and \( V \) the identity map on \( L_1 = L_2 \), then \( S \otimes V \) is the identity map on \( H_1 \otimes L_1 \). Similarly, if \( T \) and \( V \) are the appropriate identity maps, then so is \( T \otimes V \).
3. \(\|S \otimes V\| \leq \|S\|\|V\|\) and \(\|T \otimes \mathcal{V}\| \leq \|T\|\|\mathcal{V}\|\).
(4) If $S$ and $V$ are isometries (resp. unitaries), so is $S \ast V$; if $T$ and $V$ are isometries (resp. unitaries), so is $T \ast V$.

(5) $(S \ast V)^* = S^* \ast V^*$ and $(T \ast V)^* = T^* \ast V^*$.

(6) If $S$ decomposes as a direct sum $S_k \oplus S_R : \mathcal{H}_1 \oplus \mathbb{C} \to \mathcal{H}_2 \oplus \mathbb{C}$, then $S \ast V$ maps the summands of $\mathcal{H}_1 \ast \mathcal{L}_1$ into the corresponding summands of $\mathcal{H}_2 \ast \mathcal{L}_2$. That is, if $P_1$ is the projection from $\mathcal{H}_1 \ast \mathcal{L}_1$ onto any of $\mathcal{H}_1^\perp$, $\mathcal{L}_1^\perp \oplus \mathcal{L}$, or $\mathcal{H}_1 \oplus \mathcal{L}_1^\perp \oplus \mathcal{L}$, and $P_2$ the projection from $\mathcal{H}_2 \ast \mathcal{L}_2$ onto its corresponding subspace, then

\begin{equation}
(S \ast V) P_1 = P_2 (S \ast V).
\end{equation}

Similarly, if $T$ decomposes as a direct sum $T_k \oplus T_R : \mathcal{H}_1 \oplus \mathcal{L}_1 \to \mathcal{H}_2 \oplus \mathcal{L}_2$ and $P_1, P_2$ are as before, then

\begin{equation}
(T \ast V) P_1 = P_2 (T \ast V).
\end{equation}

(7) If $S$ decomposes as the direct sum $S_k \oplus \text{id}_k : \mathcal{H}_1 \oplus \mathbb{C} \to \mathcal{H}_2 \oplus \mathbb{C}$ and $T = S \oplus V$, then $S \ast V = T \ast V$.

**Proof.** The first five assertions are simple consequences of the corresponding properties of tensor products of operators. The sixth follows as a special case of Proposition 3.2.5 with $y = \beta = 0$ or $w = \zeta = 0$, and the seventh is also an easy corollary of Proposition 3.2.5.

**Remark 3.2.7.** The first two properties say that $- \ast -$ and $- \ast \hat{\ast} -$ are bifunctors from (Hilbert spaces, bounded maps) $\times$ (Hilbert spaces, contractions) to (Hilbert spaces, bounded maps), and the third and fourth imply that they restrict to bifunctors from (Hilbert spaces, contractions)$^2$ to (Hilbert spaces, contractions), from (Hilbert spaces, isometries)$^2$ to (Hilbert spaces, isometries), and from (Hilbert spaces, unitaries)$^2$ to (Hilbert spaces, unitaries).

**Remark 3.2.8.** Together, the fourth and seventh parts of Proposition 3.2.5 imply that, given isometries $W : \mathcal{H}_1 \to \mathcal{H}_2$ and $V : \mathcal{L}_1 \to \mathcal{L}_2$, one obtains an isometry $\mathcal{H}_1 \ast \mathcal{L}_1 \to \mathcal{H}_2 \ast \mathcal{L}_2$ which may be constructed either as $(W \oplus \text{id}_k) \ast V$ or as $(W \oplus \text{id}_k \oplus V) \ast V$. The sixth part then implies that this induced isometry maps each summand of $\mathcal{H}_1 \ast \mathcal{L}_1$ into the corresponding summand of $\mathcal{H}_2 \ast \mathcal{L}_2$. Hence, if $\mathcal{H}_1 \subset \mathcal{H}_2$ and $\mathcal{L}_1 \subset \mathcal{L}_2$, we may regard $\mathcal{H}_1 \ast \mathcal{L}_1$ as a subspace of $\mathcal{H}_2 \ast \mathcal{L}_2$.

An important special case of Proposition 3.2.5 occurs when $\mathcal{H}_1 = \mathcal{H}_2$, $\mathcal{L}_1 = \mathcal{L}_2$, and $V$ is the identity map. In this case we obtain unital representations of both $B(\mathcal{H}^+)$ and $B(\mathcal{K})$ on $\mathcal{H} \ast \mathcal{L}$, given by $S \mapsto S \ast I$ and $T \mapsto T \ast I$.

**Proposition 3.2.9.** Let $\mathcal{H}$ and $\mathcal{L}$ be Hilbert spaces and $\mathcal{K} = \mathcal{H}^+ \oplus \mathcal{L}$. Let $\Phi : B(\mathcal{H}^+) \to B(\mathcal{H} \ast \mathcal{L})$ and $\Psi : B(\mathcal{K}) \to B(\mathcal{H} \ast \mathcal{L})$ be the representations induced by the unitary equivalences of Proposition 3.2.5. Let $b \in B(\mathcal{H}^+)$, $a \in B(\mathcal{K})$, $h \in \mathcal{H}^+$, $h_0 \in \mathcal{H}$, $k \in \mathcal{K}$, $\ell \in \mathcal{L}^+$, and $\xi \in \mathcal{L}^{n+1} \oplus \mathcal{L}$ for some $n \geq 0$, and suppose that

\begin{align*}
b \Omega &= a \Omega + y, & a \in \mathcal{C}, \ y \in \mathcal{H} \\
b h_0 &= \beta \Omega + z, & \beta \in \mathcal{C}, \ z \in \mathcal{H} \\
ah_0 &= \eta + w, & \eta \in \mathcal{H}, \ w \in \mathcal{L}^+ \\
a \ell &= \zeta + u, & \zeta \in \mathcal{H}, \ u \in \mathcal{L}^+
\end{align*}
Then

\[ \Phi(b)h = h \]
\[ \Phi(b)\xi = \alpha\xi + (y \otimes \xi) \]
\[ \Phi(b)(h_0 \otimes \xi) = \beta\xi + (z \otimes \xi) \]
\[ \Psi(a)k = ak \]
\[ \Psi(a)(h_0 \otimes \xi) = (\eta \otimes \xi) + (w \otimes \xi) \]
\[ \Psi(a)(\eta \otimes \xi) = (\zeta \otimes \xi) + (u \otimes \xi). \]

The following is an immediate consequence:

**Corollary 3.2.10.** In the situation of Proposition 3.2.9, the subspaces \( \mathcal{H}^+ \) and \( (\mathcal{L}^{+\otimes n} \otimes \mathcal{L}) \oplus (\mathcal{H} \otimes \mathcal{L}^{+\otimes n} \otimes \mathcal{L}) \) of \( \mathcal{K} \) are \( \Phi \)-invariant, while the subspaces \( \mathcal{H}^+ \oplus \mathcal{L} \) and \( (\mathcal{H} \otimes \mathcal{L}^{+\otimes n}) \oplus (\mathcal{L}^{+\otimes (n+1)} \otimes \mathcal{L}) \) are \( \Psi \)-invariant.

We visualize this corollary using a stairstep diagram:

\[\begin{array}{cccccc}
\mathcal{H}^+ & \mathcal{L} & (\mathcal{H} \otimes \mathcal{L}) & (\mathcal{L}^{+\otimes 2} \otimes \mathcal{L}) & (\mathcal{H} \otimes \mathcal{L}^{+\otimes 2} \otimes \mathcal{L}) & \cdots
\end{array}\]

The rows here are \( \Phi \)-invariant, while the columns are \( \Psi \)-invariant. Equivalently, \( \Phi \) and \( \Psi \) have staggered block-diagonal decompositions:

\[ \Phi = \begin{bmatrix}
* & 0 & 0 & 0 & 0 & \cdots \\
0 & * & 0 & 0 & 0 & \cdots \\
0 & * & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & * & 0 & \cdots \\
0 & 0 & 0 & * & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad \Psi = \begin{bmatrix}
* & * & 0 & 0 & 0 & 0 & \cdots \\
* & * & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & * & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & * & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & * & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & * & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}\]

We see that a sufficiently long word \( \Phi(b_0)\Psi(a_1)\Phi(b_1)\Psi(a_2) \cdots \) applied to a vector in one of these subspaces could have a nonzero component in any other subspace. Keeping track of such components will become important later on.

**Remark 3.2.11.** For simplicity of definition, we have thus far begun with Hilbert spaces \( \mathcal{H} \) and \( \mathcal{L} \), and defined the space \( \mathcal{K} = \mathcal{H}^+ \oplus \mathcal{L} \) in terms of them. In application, however, we will begin with an inclusion \( \mathcal{H} \subset \mathcal{K} \) (obtained from Stinespring dilation), select a unit vector \( \Omega \in \mathcal{H} \), and form the Sauvageot product \( \mathcal{H}^{-*}(K\otimes\mathcal{H}) \). As noted above, \( B(\mathcal{H}^{-*}(K\otimes\mathcal{H})) \) contains unital copies of both \( B(\mathcal{H}) \) and \( B(\mathcal{K}) \). In this alone, however, it is no different from \( B(\mathcal{H} \otimes \mathcal{K}) \) or \( B(\mathcal{H} \ast \mathcal{K}) \), where \( \mathcal{H} \ast \mathcal{K} \) denotes the free product of Hilbert spaces in the sense of [Voia85]. The crucial difference is that, when both are represented on \( \mathcal{H}^{-*}(K\otimes\mathcal{H}) \), the copy of \( B(\mathcal{H}) \) is a corner of the copy of \( B(\mathcal{K}) \); if \( \mathcal{H} \subset \mathcal{K} \) is a Stinespring dilation, the compression will implement a given unital completely positive map.
3.3. Sauvageot Products of \(C^\ast\)-Algebras and \(W^\ast\)-Algebras

We now begin our construction of the Sauvageot product of unital \(C^\ast\)-algebras with respect to a unital completely positive map; the construction requires the choice of a state on one of the \(C^\ast\)-algebras, prompting the following definition.

**Definition 3.3.1.** A **CPC\(^\ast\)**-tuple (resp. **CPW\(^\ast\)**-tuple) is a quadruple \((A, B, \phi, \omega)\) where \(A\) and \(B\) are unital \(C^\ast\)-algebras (resp. \(W^\ast\)-algebras), \(\phi : A \to B\) a unital (normal) completely positive map, and \(\omega\) a (normal) state on \(B\). The term **CP-tuple** will refer to CPC\(^\ast\)- and CPW\(^\ast\)**-tuples together. A CP-tuple is said to be **faithful** if \(\omega\) is a faithful state.

**Definition 3.3.2.** A representation of a CPC\(^\ast\)**-tuple \((A, B, \phi, \omega)\) is a sextuple \((H, \Omega, \pi_R, K, V, \pi_L)\) where

1. \(H\) is a Hilbert space
2. \(\Omega \in H\) is a unit vector
3. \(\pi_R : B \to B(H)\) is a unital \(*\)-homomorphism such that \(\langle \Omega, \pi_R(\cdot)\Omega \rangle = \omega(\cdot)\)
4. \(K\) is a Hilbert space
5. \(V : H \to K\) is an isometry
6. \(\pi_L : A \to B(K)\) is a unital \(*\)-homomorphism such that \(V^*\pi_L(\cdot)V = \pi_R(\phi(\cdot))\).

For a CPW\(^\ast\)**-tuple, we also require that \(\pi_R\) and \(\pi_L\) be normal. A representation is **right-faithful** if \(\pi_R\) is injective (which is automatically the case for a representation of a faithful CP-tuple), and **left-faithful** if \(\pi_L\) is injective. We also refer to \((H, \Omega, \pi_R)\) satisfying the first three criteria as a representation of \((A, \omega)\).

From now until Definition 3.3.9 we fix a CP-tuple \((A, B, \phi, \omega)\) and a right-faithful representation \((H, \Omega, \pi_R, K, V, \pi_L)\). We introduce the following additional notation:

- \(L = K \oplus VH\)
- \(\mathfrak{H} = H^\ast \ast L\)
- \(\psi_L : A \to B(\mathfrak{H})\) and \(\psi_R : B \to B(\mathfrak{H})\) are the compositions of \(\pi_L\) and \(\pi_R\) with the representations of Proposition 3.2.9
- \(A \ast B\) is the \(C^\ast\)-subalgebra (resp. von Neumann subalgebra) of \(B(\mathfrak{H})\) generated by the images of \(\psi_L\) and \(\psi_R\)
- \(H' = H \ominus \pi_R(B)\Omega\), which could be zero
- \(q_n\) for \(n \geq 0\) is the projection from \(\mathfrak{H}\) onto the subspace \(H' \otimes L^+ \otimes L\) of \(H \otimes L^+ \otimes L\)
- \(\mathcal{C} : B(\mathfrak{H}) \to B(\mathfrak{H})\) is the non-unital conditional expectation

\[
\mathcal{C}(T) = P_H TP_H + \sum_{n=0}^{\infty} q_n T q_n.
\]

**Proposition 3.3.3.**

\[
\mathcal{C} \circ \psi_L = \mathcal{C} \circ \psi_R \circ \phi.
\]

**Proof.** For \(a \in A\) and \(h \in H\), let \(\pi_L(a)h = x + \ell\) with \(x \in h\) and \(\ell \in L\); then \(x = P_H \pi_L(a)P_H h = \pi_R(\phi(a))h\). It follows from Proposition 3.2.9 that \(\psi_L(a)h = x + \ell\), so that

\[
P_H \psi_L(a)P_H = x = P_H \pi_R(\phi(a))P_H.
\]

Similarly, \(q_n \psi_L(a)q_n = q_n \pi_R(\phi(a))q_n\) for all \(n \geq 0\). Summing over \(n\) yields the result. \(\square\)
For the next lemma and proposition we use \( E_n \) to denote the subspace \( L^+ \otimes n \otimes L \) of \( \mathcal{H} \).

**Lemma 3.3.4.** Let \( \zeta \in E_n \).

1. Let \( a \in A \) and \( b \in B \) with \( \omega(b) = 0 \). Then
   \[
   [\psi_L(a) - \psi_R(\phi(a))] \psi_R(b) \zeta = [P_{L^+} \pi_L(a) \pi_R(b) \Omega] \otimes \zeta - \omega(\phi(a)b) \zeta \in E_n \oplus E_{n+1}.
   \]

2. More generally, given \( a_1, \ldots, a_k \in A \) and \( b_1, \ldots, b_k \in B \) such that \( \omega(b_i) = 0 \) for \( i = 1, \ldots, k \), if we define
   \[
   \zeta_k = \left( \prod_{i=1}^{k} [\psi_L(a_i) - \psi_R(\phi(a_i))] \psi_R(b_i) \right) \zeta,
   \]
   then
   \[
   \zeta_k \in \bigoplus_{i=0}^{k} E_{n+i} \text{ with } PE_n(\zeta_k) = (-1)^k \prod_{i=1}^{k} \omega(\phi(a_i)b_i) \zeta.
   \]

**Proof.** The stipulation that \( \omega(b) = 0 \) implies that \( \pi_R(b) \Omega \in H^- \), so that
\[
\xi := \psi_R(b) \zeta = (\pi_R(b) \Omega) \otimes \zeta \in H^- \otimes E_n
\]
where we have used the calculations in Proposition [3.2.9]. We have now to apply two different operators to \( \xi \) and subtract the results. First, when we apply \( \psi_R(\phi(a)) \) we get
\[
\psi_R(\phi(a)) \xi = (\pi_R(\phi(a)b) \Omega) \otimes \zeta = [\omega(\phi(a)b) \Omega] \otimes \eta + [P_{H^-} \pi_R(\phi(a)b) \Omega] \otimes \zeta
\]
by virtue of the fact that \( \pi_R(\beta) \Omega = \omega(\beta) \Omega + P_{H^-} \pi_R(\beta) \Omega \) for all \( b \in B \). Secondly, we apply \( \psi_L(a) \) as follows:
\[
\psi_L(a) \xi = (\pi_L(a) \pi_R(b) \Omega) \otimes \zeta
= \left[ (P_{H^-} \pi_L(a) \pi_R(b) \Omega) \otimes (P_{L^+} \pi_L(a) \pi_R(b) \Omega) \right] \otimes \zeta
= \left[ (P_{H^-} \pi_L(a) \pi_R(b) \Omega) \otimes (P_{L^+} \pi_L(a) \pi_R(b) \Omega) \right] \otimes \zeta
= \left[ (P_{H^-} \pi_R(\phi(a)b) \Omega) \otimes \zeta \right] \oplus \left[ (P_{L^+} \pi_L(a) \pi_R(b) \Omega) \otimes \zeta \right].
\]
Subtracting yields the desired result. The second assertion of the lemma follows by induction.

We now connect the current material to chapter [2]

**Proposition 3.3.5.** \((\mathcal{B}(\mathcal{H}), \psi_L, \psi_R, \mathcal{C})\) is a right-liberating representation of \((A, B, \phi, \omega)\).

**Proof.** Since \( H \) and each \( H' \otimes E_n \) are \( \pi_R \)-invariant subspaces by Proposition [3.2.9], their projections all commute with \( \psi_R \), so that \( \mathcal{C} \) is a \( \psi_R(B) \)-bimodule map. Now let \( \xi \in H \) and let \( a_1, \ldots, a_n \in A \), \( b_1, \ldots, b_n \in B \) such that \( \omega(b_2) = \cdots = \omega(b_n) = 0 \). Define the operators
\[
T_k = \left( \psi_L(a_k) - \psi_R(\phi(a_k)) \right) \psi_R(b_k)
\]
on \( \mathcal{H} \), and the vectors
\[
\zeta_k = T_k \cdots T_1 \xi \in \mathcal{H}.
\]
We will show by induction that $\zeta_k \in \bigoplus_{j=0}^{k-1} E_j$, which is contained in the kernel of $P_H$; it will follow that $P_H T_k \ldots T_1 P_H = 0$. For the base case $k = 1$, we have $\psi_n(b_1)\xi = \pi_R(b_1)\xi$, so that $\psi_n(\phi(a_1))\psi_n(b_1)\xi = \pi_R(\phi(a_1)b_1)\xi$. We also have

$$\psi_L(a_1)\psi_R(b_1)\xi = \pi_L(a_1)\pi_R(b_1)\xi$$

and subtracting yields

$$\zeta_1 = P_L \pi_L(a_1)\pi_R(b_1)\xi \in E_0$$

as desired. The inductive step is immediate from Lemma 3.3.4.

Similarly, for $\xi \in H$ and $\eta \in E_n$, $\psi_n(b_1)(\xi \otimes \eta) = (\pi_R(b_1)\xi) \otimes \eta$ so that

$$\psi_n(\phi(a_1))\psi_n(b_1)(\xi \otimes \eta) = [\pi_R(\phi(a_1)b_1)\xi] \otimes \eta.$$

Then

$$\psi_L(a_1)\psi_R(b_1)(\xi \otimes \eta) = \psi_L(a_1)[(\pi_R(b_1)\xi) \otimes \eta]$$

$$= [\pi_L(a_1)\pi_R(b_1)\xi] \otimes \eta$$

$$= [P_L \pi_L(a_1)\pi_R(b_1)\xi] \otimes \eta$$

$$= [P_L \pi_L(a_1)\pi_R(b_1)\xi] \otimes \eta$$

$$= [P_L \pi_L(a_1)\pi_R(b_1)\xi] \otimes \eta$$

and subtracting yields

$$\zeta_1 = [P_L \pi_L(a_1)\pi_R(b_1)\xi] \otimes \eta \in E_{n+1}.$$

It follows by induction that $\zeta_k \in \bigoplus_{j=0}^{k-1} E_j$, so that $q_n\zeta_k = 0$; hence $q_n T_k \ldots T_1 q_n = 0$. Summing over $n$, we have $\mathfrak{C}(T_k \ldots T_1) = 0$. □

**Corollary 3.3.6.**

$$\mathfrak{C}(A \ast B) = \mathfrak{C}(\psi_R(B)).$$

**Proof.** This is an immediate consequence of Proposition 3.3.5 together with Corollary 2.4.3 and the norm continuity and normality of $\mathfrak{C}$. □

Before making our next definition, we note that the right-faithfulness of our representation guarantees that $b \mapsto P_H \psi_R(b)P_H$ is injective, so that $\mathfrak{C} \circ \psi_R$ is injective as well.

**Definition 3.3.7.** The *Sauvageot retraction* for the given tuple and representation is the map $\theta : A \ast B \to B$ given by

$$\theta = (\mathfrak{C} \circ \psi_R)^{-1} \circ \mathfrak{C}.$$
The Sauvageot retraction is well-defined by Corollary [3.3.6] and is a retraction with respect to $\psi_\mathcal{R}$; furthermore, as a consequence of Proposition [3.3.3] it factors $\phi$ in the sense that

$$\theta \circ \psi_\mathcal{L} = \phi. \quad (3.3)$$

Furthermore, the following is an immediate consequence of Proposition [3.3.5]:

**Corollary 3.3.8.** $(A \star B, \psi_\mathcal{L}, \psi_\mathcal{R}, \psi_\mathcal{R} \circ \theta)$ is a right-liberating representation of $(A, B, \phi, \omega)$.

**Definition 3.3.9.** Given a CP-tuple $(A, B, \phi, \omega)$ and a right-liberating representation $(H, \Omega, \pi_\mathcal{R}, K, V, \pi_\mathcal{L})$, the **Sauvageot product of the tuple realized by the representation** is the tuple $(A \star B, \psi_\mathcal{L}, \psi_\mathcal{R}, \theta)$ of objects constructed as above.

### 3.4. Induced Morphisms and Uniqueness

We pause now to consider an analogy with other product constructions. In building either the (minimal) tensor product or the free product of $C^*$-algebras (resp. $W^*$-algebras) $A$ and $B$, one can proceed as follows:

1. Represent $A$ and $B$ on Hilbert spaces $H$ and $K$
2. Form the product Hilbert space $H \otimes K$ or $H \star K$
3. Lift the representations of $A$ and $B$ to representations of each on this product Hilbert space
4. Take the $C^*$-subalgebra (resp. von Neumann subalgebra) generated by the images of these representations.

In both cases, one can show that the resulting $C^*$-algebra (resp. $W^*$-algebra) $A \otimes B$ or $A \star B$ is, up to isomorphism, independent of the choice of the representations of $A$ and $B$ provided both are faithful.

We have followed the same outline in constructing $A \star B$, and come now to the question of the independence of this object from the representations used to produce it. It turns out that we need some more complicated hypotheses on the representation, resulting from the fact that a representation of a CP-tuple is more complicated than a representation of a $C^*$-algebra or $W^*$-algebra, as well as the fact that the product $A \star B$ comes with the additional information of a retraction onto $B$.

**Definition 3.4.1.** Let $(A, B, \phi, \omega)$ be a CP-tuple, $(H, \Omega, \pi_\mathcal{R}, K, V, \pi_\mathcal{L})$ a representation, and $L = K \ominus VH$.

- A **decomposition** of the representation is a pair $(L', L'')$ of $\pi_\mathcal{L}$-invariant subspaces $L' \subset L$ and $L'' \subset L^+$, with the properties
  $$L \subset L' + \pi_\mathcal{L}(A)VH$$
  $$L^+ \subset L'' + \pi_\mathcal{L}(A)VH^{-}.$$

- A decomposition is **faithful** if the subrepresentation $\pi_\mathcal{L}|_{L'}$ is faithful.
- A representation for which there exists a faithful decomposition is **faithfully decomposable**.
- A representation is **faithful** if it is right-faithful and faithfully decomposable.

**Proposition 3.4.2.** Every CP-tuple has a faithful representation.
PROOF. We begin by letting \((H, \Omega, \pi_H)\) be the GNS construction for \((B, \omega)\); if \(\omega\) is not faithful, we replace \((H, \pi_H)\) by its direct sum with some faithful representation of \(B\). This guarantees that our representation will be right-faithful.

Next, let \((K, V, \pi_L)\) be the minimal Stinespring dilation of \(\pi_H \circ \phi\), and let \(L' = K \oplus \pi_L(A)\overline{V\mathcal{H}}\) and \(L'' = K \oplus \pi_L(A)\overline{\mathcal{H}}\). If \(\pi_L\big|_{L'}\) is not faithful, we replace \((K, \pi_L)\) by its direct sum with some faithful representation of \(A\), thereby guaranteeing faithful decomposability.

When some of these additional hypotheses are satisfied, we can define a retraction from \(A \ast B\) to \(A\) which has properties analogous to the retraction \(A \ast B \xrightarrow{\theta} B\) already discussed. We continue our standard notation for a CP-tuple, a right-faithful\(A \ast B\)-representation from \(A\ast B\) to \(B\), thereby guaranteeing faithful decomposability.

\[\]

\[\]

\[\]

We introduce additional notation:
- \(E_n\) is the subspace \(L' \subset L\)
- For \(n \geq 1\), \(E_n'\) is the subspace \(L'' \otimes L^{+ \otimes (n-1)} \otimes L \subset E_n\)
- For all \(n \geq 0\), \(p_n\) is the projection from \(\mathcal{H}\) onto \(E_n'\)
- For all \(n \geq 0\), \(F_n = H \otimes E_n\) and \(F_n' = H' \otimes E_n\); recall that \(q_n\) is the projection onto \(F_n'\)
- \(E_{-1} = \mathbb{C} \mathcal{H}\) and \(F_{-1} = H^\perp\)

**Definition 3.4.3.** The **left corner map** for the given realization and decomposition is the non-unital conditional expectation \(\mathcal{C}'\) on \(B(\mathcal{H})\) defined by

\[\mathcal{C}'(T) = \sum_{n=0}^{\infty} p_n T p_n.\]

**Lemma 3.4.4.** Let \(J \subset A \ast B\) be an ideal contained in the intersection of the kernels of \(\mathcal{C}\) and \(\mathcal{C}'\). Then \(J = \{0\}\).

**Proof.** Let \(J^0\) denote the annihilator of \(J\) in \(\mathcal{H}\), i.e., the largest (necessarily closed) subspace such that \(JJ^0 = \{0\}\).

1. For any \(\alpha \in J\), note that \(\alpha^* \alpha \in J\); then
   \[\langle \alpha P_n \rangle^* \langle \alpha P_n \rangle = P_n \alpha^* \alpha P_n \leq \mathcal{C}[\alpha^* \alpha] = 0,\]
   so that \(\alpha P_n = 0\). Similarly, \(\alpha q_n = \alpha p_n = 0\) for all \(n\). Hence \(H \subseteq J^0\), and for each \(n \geq 0\), \(E_n' \subseteq J^0\) and \(H' \otimes E_n \subseteq J^0\).

2. We will prove by induction that \(E_n \subseteq J^0\) and \(F_n \subseteq J^0\); the base case \(n = -1\) was just established, since \(E_{-1} \oplus F_{-1} = H \subset J^0\).

3. Suppose \(E_n \subseteq J^0\) and \(F_n \subseteq J^0\).
   - (a) Since \(J \psi_L(A)F_n \subseteq J F_n = \{0\}\), we have \(\psi_L(A)F_n \subseteq J^0\).
   - (b) By Proposition 3.2.9 \([\psi_L(A)H^-] \otimes E_n \subseteq \psi_L(A)F_n\), so that \([\psi_L(A)H^-] \otimes E_n \subseteq J^0\), for the case \(n \geq 0\); for \(n = -1\), we have \(\psi_L(A)H \subseteq J^0\).
   - (c) Since we already know \(L' \subseteq J^0\) (in the case \(n = -1\)) or \(L'' \otimes E_n \subseteq J^0\) (in the case \(n \geq 0\)), it follows that \(E_{n+1} = L \subseteq \psi_L(A)H + L' \subseteq J^0\) for the case \(n = -1\), or \(E_{n+1} = L' \otimes E_n \subseteq [\psi_L(A)H^-] \otimes E_n \subseteq J^0\) for the case \(n \geq 0\).
   - (d) Since \(J \psi_R(B)E_{n+1} \subseteq J E_{n+1} = \{0\}\), we have \(\psi_R(B)E_{n+1} \subseteq J^0\).
   - (e) By Proposition 3.2.9 this implies \(\psi_R(B)E_{n+1} \subseteq \psi_R(B)E_{n+1} \subseteq J^0\).
(f) Since we also have $F_n' \subseteq J^0$,
\[
F_{n+1} = H^- \otimes E_{n+1} \subseteq H \otimes E_{n+1} = (\pi_R(B)\Omega + H') \otimes E_{n+1} \subseteq \overline{\psi_R(B)E_{n+1}} + F_{n+1}' \subseteq J^0.
\]

**Remark 3.4.5.** We note that if $L'$ and $L''$ are both zero (for instance, when $K$ is given by a minimal Stinespring dilation), then $C'$ is the zero map; in this case, the lemma says that $C$ has no nontrivial ideals in its kernel. This corresponds to the fact that $A$ and $B$ together move $H$ around to all the other components of $\mathcal{H}$, in the sense that $(A \ast B)H = \mathcal{H}$. On the other extreme, if $(L', L'')$ is a faithful decomposition, one has instead that $(A \ast B)H + (A \ast B)L' + (A \ast B)L'' = \mathcal{H}$, but none of $H, L', L''$ by itself is enough to reach all of $\mathcal{H}$. As a result, $C$ and $C'$ may each contain ideals in their kernel, but these ideals are “orthogonal” in the sense of the lemma.

**Proposition 3.4.6.** $(B(\mathcal{H}), \psi_L, \psi_R, C')$ is a left-liberating representation of $(A, B, \phi, \omega)$.

**Proof.** By Proposition 3.2.9 all of the $E_n'$ are $\psi_L$-invariant, so their projections commute with $\psi_L$; hence $C'$ is a $\psi_L(A)$-bimodule map. Now let $a_1, \ldots, a_n \in A$ and $b_0, \ldots, b_n \in B$ with $\omega(b_i) = 0$. Let $m \geq 0$ and $\xi \in E_m'$. By Lemma 3.3.4
\[
\prod_{k=1}^n [\psi_L(a_k) - \psi_R(\phi(a_k))] \xi \in \bigoplus_{k=0}^m E_{n+k}.
\]

Since $\omega(b_0) = 0$, it follows that $\pi_R(b_0)\Omega \in H^-$, so that by Proposition 3.2.9 we obtain
\[
\psi_R(b_0) \prod_{k=1}^n [\psi_L(a_k) - \psi_R(\phi(a_k))] \xi \in \bigoplus_{k=0}^m H^- \otimes E_{n+k}.
\]

From this we see that
\[
p_n \psi_R(b_0) \prod_{k=1}^n [\psi_L(a_k) - \psi_R(\phi(a_k))] p_n = 0,
\]
and summing over $n$ finishes the proof. \qed

**Corollary 3.4.7.** $C'(A \ast B) = C'(A)$.

**Proof.** This is an immediate consequence of Proposition 3.4.6, Corollary 2.4.9 and the contractivity (and, in case $(A, B, \phi, \omega)$ is a CPW*-tuple, the normality) of $C'$.

In the case of a faithful decomposition, $C' \circ \psi_L$ is injective, which allows us to make the following definition:

**Definition 3.4.8.** Given a CP-tuple, a faithful representation, and a choice of faithful decomposition, the left retraction for the given tuple and representation is the map $\theta : A \ast B \to A$ given by
\[
\theta = (C' \circ \psi_L)^{-1} \circ C'.
\]
This is well-defined by Corollary 3.4.7 and is a retraction with respect to $\psi_L$.

We come now to the main result of this section.
THEOREM 3.4.9. Let \((A_1, B_1, \phi_1, \omega_1)\) and \((A_2, B_2, \phi_2, \omega_2)\) be CPC\(^*\)-tuples (resp. CPW\(^*\)-tuples), \((H_1, \Omega_1, \pi^{(1)}_l, \pi^{(1)}_r, K_1, V_1, \pi^{(1)}_l)\) a faithful representation of the former, \((H_2, \Omega_2, \pi^{(2)}_l, K_2, V_2, \pi^{(2)}_l)\) a right-faithful representation of the latter, and \((A_1 \ast B_1, \psi^{(1)}_l, \psi^{(1)}_r, \theta_1)\) and \((A_2 \ast B_2, \psi^{(2)}_l, \psi^{(2)}_r, \theta_2)\) the Sauvageot products realized by these representations. Let \(f : A_1 \rightarrow A_2\) and \(g : B_1 \rightarrow B_2\) be unital (normal) \(*\)-homomorphisms satisfying \(\phi_2 \circ f = g \circ \phi_1\) and \(\omega_2 \circ g = \omega_1\). Then there is a unique \((\text{normal})\) unital \(*\)-homomorphism \(f \ast g : A_1 \ast B_1 \rightarrow A_2 \ast B_2\) with the properties that

\[
\begin{align*}
(1) & \quad (f \ast g) \circ \psi^{(1)}_l = \psi^{(2)}_l \circ f \\
(2) & \quad (f \ast g) \circ \psi^{(1)}_r = \psi^{(2)}_r \circ g \\
(3) & \quad \theta_2 \circ (f \ast g) = g \circ \theta_1
\end{align*}
\]

If \(f\) and \(g\) are both injective and \((H_2, \Omega_2, \pi^{(2)}_l, K_2, V_2, \pi^{(2)}_l)\) is faithful, then \(f \ast g\) is injective.

PROOF. Let \(H = H_1 \oplus H_2\), \(\Omega = \Omega_1\), \(\pi_l = \pi^{(1)}_l \oplus (\pi^{(2)}_l \circ f), K = K_1 \oplus K_2, V = V_1 \oplus V_2\), \(\pi_r = \pi^{(1)}_r \oplus (\pi^{(2)}_r \circ f)\). Then \((H, \Omega, \pi_l, K, V, \pi_r)\) is another right-faithful representation of \((A_1, B_1, \phi_1, \omega_1)\). Moreover, if \((L'_1, L''_1)\) is a decomposition for \((H_1, \ldots, \pi^{(1)}_l)\), then \(L' = L'_1 \oplus L_2, L'' = L''_1 \oplus L_2\) defines a decomposition \((L', L'')\) of \((H, \ldots, \pi_l)\), and the faithfulness of \(\pi^{(1)}_l\) on \(L'_1\) implies the faithfulness of \(\pi_l\) on \(L'\).

Let \((A_1 \ast B_1, \psi, \psi, \theta)\) be the Sauvageot product realized by \((H, \ldots, \pi_l)\), on the Hilbert space \(\mathcal{M} = H^* \ast L\).

The inclusions of \(H_1\) into \(H\) and of \(L_1\) into \(L\) induce an isometry \(W : \mathcal{S}_1 \rightarrow \mathcal{S}\) as in Remark [2.2.3]. Moreover, by Equation [3.1], this isometry satisfies

\[(3.4) \quad W \circ \psi^{(1)}_l(\cdot) = \psi_l(\cdot) \circ W, \quad W \circ \psi^{(1)}_r(\cdot) = \psi_r(\cdot) \circ W.\]

Let \(\Psi\) be the restriction to \(A_1 \ast B_1\) of the (normal) unital CP map \(T \mapsto W^* TW\), which maps \(B(\mathcal{S})\) to \(B(\mathcal{S})\). It follows from Equation [3.4] that the image of \(W\) is invariant under \(\psi_l\) and \(\psi_r\), so that \(WW^*\) commutes with \(A_1 \ast B_1\). Then

\[\Psi(\cdot Y) = W^* \Psi(\cdot Y) = W^* W W^* X Y W = W^* X W W^* Y W = \Psi(\cdot X)\]

for \(X, Y \in A \ast B_1\), so that \(\Psi\) is a \(*\)-homomorphism.

Next, we show that \(\Psi\) intertwines the representations, states, and retractions:

- \(\Psi \circ \psi_l = \psi^{(1)}_l\)
- \(\Psi \circ \psi_r = \psi^{(1)}_r\)
- \(\theta_1 \circ \Psi = \theta\)
- \(\theta_1' \circ \Psi = \theta'\)

The first three are immediate consequences of Equation [3.4]. For the fourth and fifth, we have by Equation [3.1] that

\[\Psi \circ \mathcal{C}(T) = W^* P_n T^* P_n W^* + \sum_{n} W^* P_n T P_n W = P_n W^* T W P_n + \sum_{n} P_n W^* T W P_n = \mathcal{C}_1 \circ \Psi(T)\]

so that \(\Psi \circ \mathcal{C} = \mathcal{C}_1 \circ \Psi\), and similarly for \(\mathcal{C}'\) and \(\mathcal{C}'_1\). Now

\[\Psi \circ \mathcal{C} \circ \psi_r = \mathcal{C}_1 \circ \Psi \circ \psi_r = \mathcal{C}_1 \circ \psi^{(1)}_r,\]

which is invertible; then

\[(\mathcal{C} \circ \psi_r)^{-1} = (\Psi \circ \mathcal{C} \circ \psi_r)^{-1} \circ (\mathcal{C} \circ \psi_r)^{-1} = (\Psi \circ \mathcal{C} \circ \psi_r)^{-1} \circ \Psi \]

and

\[(\mathcal{C} \circ \psi_r)^{-1} = (\Psi \circ \mathcal{C} \circ \psi_r)^{-1} \circ (\mathcal{C} \circ \psi_r)^{-1} = (\Psi \circ \mathcal{C} \circ \psi_r)^{-1} \circ \Psi \]
from which it follows that

$$\theta = (c \circ \psi)_{1}^{-1} \circ c$$

$$= (\psi \circ c \circ \psi)_{1}^{-1} \circ \psi \circ c$$

$$= (c_{1} \circ \psi^{(1)})^{-1} \circ c_{1} \circ \psi$$

$$= \theta_{1} \circ \psi$$

(3.5)

and similarly

$$\theta' = \theta'_{1} \circ \psi.$$  \hspace{1cm} (3.6)

Note that $\psi$ maps into $A_{1} \star B_{1}$, as it maps both $\psi_{L}(A_{1})$ and $\psi_{R}(B_{1})$ into $A_{1} \star B_{1}$, hence also the C*-algebra (resp. von Neumann algebra) that they generate; moreover, it is onto $A_{1} \star B_{1}$, since its range is a C*-algebra (resp. von Neumann algebra) which includes both $\psi_{L}^{(1)}(A_{1})$ and $\psi_{R}^{(1)}(B_{1})$.

Next, $\psi$ is injective, because its kernel is an ideal in $A_{1} \tilde{\star} B_{1}$ which, by equations 3.5 and 3.6, is contained in the kernels of both $\theta$ and $\theta'$, therefore also in the kernels of both $c$ and $c'$, and hence is the zero ideal by Lemma 3.4.4. So $\psi$ is an isomorphism from $A_{1} \tilde{\star} B_{1}$ to $A_{1} \star B_{1}$.

We repeat the above analysis for the inclusions of $H_{2}$ into $H$ and $K_{2}$ into $K$ to obtain a unital *-homomorphism $\Xi : A_{1} \tilde{\star} B_{1} \to A_{2} \star B_{2}$ such that

- $\Xi \circ \psi_{L} = \psi_{L}^{(2)} \circ f$
- $\Xi \circ \psi_{R} = \psi_{R}^{(2)} \circ g$
- $\theta_{2} \circ \Xi = g \circ \theta$

We can now define $f \star g = \Xi \circ \psi_{L}^{-1} : A_{1} \star B_{1} \to A_{2} \star B_{2}$. Then we obtain the enumerated properties of $f \star g$ by combining the lists of properties for $\psi$ and $\Xi$, as

$$(f \star g) \circ \psi_{L}^{(1)} = \Xi \circ \psi_{L}^{-1} \circ \psi^{(1)}_{L} = \Xi \circ \psi_{L} = \psi_{L}^{(2)} \circ f$$

and similarly.

The uniqueness of $f \star g$ follows from the fact that it is contractive (resp. normal) and is determined on the dense subalgebra of $A_{1} \star B_{1}$ generated by $\psi_{L}^{(1)}(A_{1})$ and $\psi_{R}^{(1)}(B_{1})$.

Finally, if $f$ and $g$ are both injective and $(H_{2}, \ldots, \pi^{(2)}_{L})$ is faithful, we can prove the additional property $\theta'_{2} \circ \Xi = \Xi \circ f$, after which we prove $\Xi$ to be injective exactly as we did with $\psi$. Hence $f \star g$ is a composition of injective maps. □

Corollary 3.4.10. Let $(A, B, \phi, \omega)$ be a CP-tuple. Then the realizations of the Sauvageot product by any two faithful representations are isomorphic.

Here “isomorphic” refers to an isomorphism which intertwines the appropriate maps. The proof is simply to take the map $\text{id}_{A} \star \text{id}_{B}$ constructed in the theorem. Based on this corollary, we may now speak of the Sauvageot product of a CP-tuple.

Another special case of interest occurs when one, but not both, of the initial maps is the identity. The results are summarized as follows.

Corollary 3.4.11. Let $A, B$ be unital C*-algebras (resp. $W^{*}$-algebras), $A \xrightarrow{\phi} B$ a unital (normal) *-homomorphism and $C$ another unital C*-algebra (resp. $W^{*}$-algebra).
3.4. INDUCED MORPHISMS AND UNIQUENESS

(1) Let \( B \xrightarrow{\phi} C \) be a (normal) unital CP map and \( \omega \) a (normal) state on \( C \). Then, for the CP-tuples \((A, C, \phi \circ f, \omega)\) and \((B, C, \phi, \omega)\) with Sauvageot retractions \( A \ast C \xrightarrow{\theta} C \) and \( B \ast C \xrightarrow{\eta} C \), the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A \ast C & \xrightarrow{f \ast \text{id}} & B \ast C
\end{array}
\quad \begin{array}{ccc}
A \ast C & \xrightarrow{f \ast \text{id}} & B \ast C \\
\downarrow \theta & & \downarrow \eta \\
C
\end{array}
\]

commute.

(2) Let \( C \xrightarrow{\phi} A \) be a (normal) unital CP map and \( \omega \) a (normal) state on \( B \). Then, for the CP-tuples \((C, A, \phi, \omega \circ f)\) and \((C, B, f \circ \phi, \omega)\), the square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C \ast A & \xrightarrow{\text{id} \ast f} & C \ast B
\end{array}
\]

commutes.

The composition of Sauvageot products of maps obeys the obvious functorial property:

**Proposition 3.4.12.** For \( i = 1, 2, 3 \) let \((A_i, B_i, \phi_i, \omega_i)\) be CP-tuples, and for \( i = 1, 2 \) let \( A_i \xrightarrow{f_i} A_{i+1} \) and \( B_i \xrightarrow{g_i} B_{i+1} \) be (normal) unital \(*\)-homomorphisms, such that the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 \\
\downarrow \omega_1 & & \downarrow \omega_2 & & \downarrow \omega_3 \\
C
\end{array}
\]

commutes. Then

\[
(f_2 \circ f_1) \ast (g_2 \circ g_1) = (f_2 \ast g_2) \circ (f_1 \ast g_1).
\]

Next, we note that the Sauvageot retraction possesses a certain universal property:

**Proposition 3.4.13.** Let \((A, B, \phi, \omega)\) be a CP-tuple, with Sauvageot product \( A \ast B \) and retraction \( \theta : A \ast B \to B \). Suppose \( \hat{\theta} : A \ast B \to B \) is another (normal) retraction with respect to \( \psi_L \) such that \((A \ast B, \psi_L, \psi_R, \hat{\theta} \circ \psi_R)\) is a right-liberating representation for \((A, B, \phi, \omega)\). Then \( \theta = \hat{\theta} \).

**Proof.** Applying Theorem 2.4.2 to the conditional expectations \( \psi_L \circ \theta \) and \( \psi_R \circ \hat{\theta} \), we see that they agree on a dense \(*\)-subalgebra of \( A \ast B \), hence on the whole by continuity. Since \( \psi_R \) is injective, this implies \( \theta = \hat{\theta} \). \( \square \)
3.5. Trivial Cases of the Sauvageot Product

Tensor products have the property that \( A \otimes \mathbb{C} \simeq A \simeq \mathbb{C} \otimes A \) for any commutative unital \( C^* \)-algebra \( A \); similarly, unital free products have the property that \( A \ast \mathbb{C} \simeq A \simeq \mathbb{C} \ast A \) for any unital \( C^* \)-algebra \( A \). Moreover, amalgamated free products satisfy \( A \ast A \simeq A \). We now consider analogues of these properties for the Sauvageot product. These are of interest not only for their own sake, but also as the base cases in the inductive system of the next chapter.

**Proposition 3.5.1** \((\mathbb{C} \ast A \simeq A)\). Let \( A \) be any unital \( C^* \)-algebra (resp. \( W^* \)-algebra), \( \nu : \mathbb{C} \to A \) the embedding of \( \mathbb{C} \), and \( \omega \) any (normal) state on \( A \). Then the Sauvageot product \( \mathbb{C} \ast A \) of the \( CP \)-tuple \((\mathbb{C}, A, \nu, \omega)\) is isomorphic to \( A \); modulo this identification, the embedding \( \psi_L : \mathbb{C} \to \mathbb{C} \ast A \) is \( \nu \), and \( \psi_R : A \to \mathbb{C} \ast A \) and \( E : \mathbb{C} \ast A \to A \) are both the identity map.

**Proof.** One can prove this by constructing a representation of this \( CP \)-tuple; on the space \( \mathfrak{H} \), one has \( \psi_L \) mapping into \( \psi_R(A) \), so that the algebra generated by both the images together is isomorphic to \( A \). Alternatively, right-liberation becomes trivial when one of the algebras involved is \( \mathbb{C} \), so that \( E \) is multiplicative and hence is a \( \ast \)-homomorphic inverse for \( \psi_R \).

**Remark 3.5.2.** One might conjecture that, more generally, the Sauvageot product with respect to an embedding is trivial; that is, if \( A \hookrightarrow B \) is an embedding, or equivalently if \( A \subset B \) is an embedding, that \( A \ast B \simeq B \).

This turns out not to be the case. We are interested in whether \( \psi_L = \psi_R \circ \iota \); but on the subspace \( \mathfrak{H} \) in a faithful decomposition, \( \psi_L \) acts faithfully, whereas \( \psi_R \circ \iota \) acts in a trivial fashion (in particular, the component in \( \mathfrak{H} \) of \( \psi_R(\iota(a))\xi \) for \( \xi \in \mathfrak{H} \) must be a scalar multiple of \( \xi \)).

This illustrates an important feature of the Sauvageot product. If we were to start by representing \( B \) on some \( H \) through the GNS construction, then use Stinespring dilation to obtain a representation of \( A \) on \( K \), then in the special case that the map from \( A \) to \( B \) is an embedding (indeed, any homomorphism) one would have \( K = H \) and therefore \( L = \{ 0 \} \), from which it would follow that \( \mathfrak{H} \simeq H \) as well, and \( A \ast B \simeq B \). But the Sauvageot product is defined with respect to a faithful representation, which involves taking direct sums at various points in the process so as to avoid collapsing into triviality.

**Proposition 3.5.3** \((A \ast \mathbb{C} \simeq \mathbb{C})\). Let \( A \) be any unital \( C^* \)-algebra (resp. \( W^* \)-algebra), and \( \omega \) any (normal) state on \( A \). Then the Sauvageot product \( A \ast \mathbb{C} \) of the \( CP \)-tuple \((A, \mathbb{C}, \omega, \text{id}_\mathbb{C})\) is isomorphic to \( A \); modulo this identification, the left embedding \( \psi_L : A \to A \ast \mathbb{C} \) is the identity map, the right embedding \( \psi_R : \mathbb{C} \to A \ast \mathbb{C} \) is \( \nu \), and the retraction \( E : A \ast \mathbb{C} \to \mathbb{C} \) is \( \omega \).

**Proof.** As with the previous proposition.

**Remark 3.5.4.** Now given a \( CP \)-tuple \((A, B, \phi, \omega)\), one can identify \( A \) with \( \mathbb{C} \ast A \) (resp. \( A \ast \mathbb{C} \)) and \( B \) with \( \mathbb{C} \ast B \) (resp. \( B \ast \mathbb{C} \)); it is then natural to ask whether \( \phi \) is thereby identified with \( \text{id}_\mathbb{C} \ast \phi \) (resp. \( \phi \ast \text{id}_\mathbb{C} \)). The answer is yes; indeed, this is a special case of Corollary 3.4.11.
CHAPTER 4

Algebraic C*-Dilations through Iterated Products

4.1. Introduction

Having shown how to construct the Sauvageot product of a CP-tuple, we now broach the question of how to iterate this product in order to construct dilations. For motivation, we return again to the Daniell-Kolmogorov construction as viewed through the lens of the tensor product (Example 1.3.3).

Recall that we begin with a compact Hausdorff space $S$ (the state space of a Markov process), with corresponding path space $S = S^{[0, \infty)}$; we use $\mathcal{A}$ to denote $C(S)$ and $\mathfrak{A}$ to denote $C(\mathcal{S})$, though we seek here to construct $\mathfrak{A}$ only through $C^*$-algebraic means, without reference to $S$. For each finite subset $\gamma \subset [0, \infty)$, we let $\mathcal{A}_\gamma$ denote a tensor product of $|\gamma|$ copies of $C(S)$ with itself. When we have constructed $\mathcal{A}$, we will embed such an $\mathcal{A}_\gamma$ into it, corresponding to those functions on the path space which only depend on times in $\gamma$.

For $\beta \leq \gamma$ we can embed $\mathcal{A}_\beta$ into $\mathcal{A}_\gamma$ by tensoring with 1’s in all the missing coordinates. It is difficult to find notation which makes this more precise while maintaining the basic simplicity of the concept, but here are two attempts. First, an example: If $\gamma = \{t_1, \ldots, t_7\}$ with the times listed in increasing order, and $\beta = \{t_2, t_5, t_6\}$, then one embeds $\mathcal{A}_\beta$ into $\mathcal{A}_\gamma$ via

$$f \otimes g \otimes h \mapsto \mathbf{1} \otimes f \otimes \mathbf{1} \otimes g \otimes h \otimes \mathbf{1}.$$ 

Second, a general observation: Such an embedding can be built from repeated embeddings corresponding to adding a single time, so we may reduce to the case $\gamma = \{t_1, \ldots, t_n\}$ and $\beta = \{t_1, \ldots, t_k, \tau, t_{k+1}, \ldots, t_n\}$ where again we assume the times are in increasing order. In this case the embedding is

$$f_1 \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes f_k \otimes \mathbf{1} \otimes f_{k+1} \otimes \cdots \otimes f_n.$$ 

It is easy to see that the family of embeddings under consideration form an inductive system, so that we may take the limit to obtain a $C^*$-algebra $\mathfrak{A}$ generated by copies of the $\mathcal{A}_\gamma$.

Having constructed the limit algebra $\mathfrak{A}$, with the embedding $\mathcal{A} \hookrightarrow \mathfrak{A}$ corresponding to the identification of $\mathcal{A}$ with $\mathcal{A}_{[0]}$, we are left with the task of constructing the retraction $E : \mathfrak{A} \to \mathcal{A}$. We do this by first constructing a consistent family of retractions $\mathcal{A}_\gamma \to \mathcal{A}_\beta$ for $\beta \leq \gamma$, then showing how to use a limiting process to induce the retraction $\mathfrak{A} \to \mathcal{A}$. First, we reduce as before to the case where $\gamma$ contains one more point than $\beta$, then retract

$$f_1 \otimes \cdots \otimes f_k \otimes g \otimes f_{k+1} \otimes \cdots \otimes f_n \mapsto f_1 \otimes \cdots \otimes (f_k P_{\tau-t_k} g) \otimes f_{k+1} \otimes \cdots \otimes f_n.$$ 

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Note that in particular, when $\gamma$ contains 0 and one identifies $A$ with $A_{(0)}$, repeated application of this rule yields the retraction $A_\gamma \to A$ given on simple tensors by

$$f_1 \otimes \cdots \otimes f_n \mapsto f_1 P_{t_2 - t_1} \left( f_2 P_{t_3 - t_2} \left( f_3 P_{t_4 - t_3} \left( \cdots (f_n P_{t_{n+1} - t_n}) \cdots \right) \right) \right).$$

Again, one can check that this family of retraction is consistent with the inductive system, so that it yields a well-defined and contractive map onto $A$ from the dense subalgebra of $A$ generated by the images of all the $A_\gamma$; as this map is contractive, it extends to a retraction on all of $A$.

When seeking to carry this method across to the Sauvageot product, one runs into several hurdles. First, one does not form the Sauvageot product merely of two $C^*$-algebras, but rather of a CP-tuple; hence, one cannot begin by defining $A_\gamma = A \times \cdots \times A$ without specifying what maps are used between the various copies of $A$. Related, but more profound, is the failure of associativity; even when the relevant maps have been selected to make the notation well-defined, in general one does not have $(A \star A) \star (A \star A)$ isomorphic to $(A \star (A \star A)) \star A$. Hence, we are led to adopt a more laborious inductive construction, though we follow the same high-level strategy as in the commutative case.

For the remainder of the chapter, we fix a unital $C^*$-algebra (resp. $W^*$-algebra) $A$, a faithful (normal) state $\omega$ on $A$, and a cpo-semigroup $\{\phi_t\}$ on $A$. We use $F$ to denote the set of finite subsets of $[0, \infty)$. Throughout, we assume unless otherwise indicated that times within such sets are listed in increasing order; hence, writing $\gamma = \{t_1, \ldots, t_n\}$ implies $t_1 < \cdots < t_n$.

4.2. Construction of the Inductive System and Limit

4.2.1. Objects and Immediate-Tail Morphisms.

**Definition 4.2.1.** Let $\beta, \gamma \in F$ with $\gamma = \{t_1, \ldots, t_n\}$. We call $\beta$ an initial segment of $\gamma$ if $\beta = \{t_1, \ldots, t_m\}$ for some $1 \leq m \leq n$, and a tail of $\gamma$ if $\beta = \{t_\ell, \ldots, t_n\}$ for some $1 \leq \ell \leq n$. If $\ell = 2$ we call $\beta$ an immediate tail with distance $t_2 - t_1$.

We are now able to define the objects of our inductive system, as well as some of the morphisms.

**Definition 4.2.2.** For nonempty $\gamma \in F$ we define inductively

1. a unital $C^*$-algebra (resp. $W^*$-algebra) $A_\gamma$
2. a unital embedding $\iota_\gamma : A \to A_\gamma$
3. a retraction $\epsilon_\gamma : A_\gamma \to A$

as follows:

- If $\gamma$ is a singleton, then $A_\gamma = A$ and both $\iota_\gamma$ and $\epsilon_\gamma$ are the identity.
- If $\beta$ is an immediate tail of $\gamma$ with distance $\tau$, let $\Phi = \phi_\tau \circ \epsilon_\beta : A_\beta \to A$, and form the CP-tuple $(A_\beta, A, \Phi, \omega)$. Then $A_\gamma$ is the Sauvageot product $A_\beta \star A$, $\iota_\gamma$ is the embedding of $A$ into $A_\beta \star A$ (denoted $\psi_\beta$ in the previous chapter), and $\epsilon_\gamma$ is the Sauvageot retraction from $A_\beta \star A$ onto $A$ (denoted $\theta$ in the previous chapter).

We also define $A_0 = \mathbb{C}$.

Note that this construction also implicitly gives us embeddings $A_\beta \hookrightarrow A_\gamma$ in the special case where $\beta$ is an immediate tail of $\gamma$; this is just the canonical embedding of $A_\beta$ into $A_\beta \star A$, the map denoted in the previous chapter by $\psi_\beta$. 

4.3. Endomorphisms of the Limit Algebra

We turn next to the question of how to embed $A_\beta$ into $A_\gamma$ when $\beta \leq \gamma$ more generally.

4.2.2. General Morphisms. Consider now any inclusion $\beta \leq \gamma$ of nonempty elements of $\mathcal{F}$. Let $\gamma = \{t_1, \ldots, t_n\}$ and for each $\ell \in \{1, \ldots, n\}$ define subsets $\gamma(\ell) \leq \gamma$ and $\beta(\ell) \leq \beta$ by

$$\gamma(\ell) = \gamma \cap \{t_\ell, \ldots, t_n\}, \quad \beta(\ell) = \beta \cap \{t_\ell, \ldots, t_n\}.$$  

Then each $\gamma(\ell)$ is a tail of $\gamma$, with $\gamma(1) = \gamma$, and similarly for $\beta$. (Note that some of the $\beta(\ell)$ may be empty, if $t_n \notin \beta$.)

DEFINITION 4.2.3. For $\beta, \gamma$ as above, we define an embedding $A_\beta \xrightarrow{f_\ell} A_\gamma$ by recursively defining embeddings $A_{\beta(\ell)} \xrightarrow{f_\ell} A_{\gamma(\ell)}$ and letting $f = f_1$. The embeddings are as follows:

- In the base case $\ell = n$, the embedding $f_n$ is the identity map in case $t_n \in \beta$, or the canonical embedding $C \hookrightarrow A$ otherwise.
- Given $f_{\ell+1}$, let $B$ denote either $A$ in the case that $t_\ell \in \beta$, or $C$ otherwise; more succinctly, $B = A_{\beta(\ell)}$. Let $B \xrightarrow{\psi} A$ be either the identity map or the embedding of $C$, accordingly. Then

$$f_\ell = f_{\ell+1} * \psi.$$

PROPOSITION 4.2.4. The family of embeddings $A_\beta \hookrightarrow A_\gamma$ in Definition 4.2.3 is an inductive system.

PROOF. Let $\beta \leq \gamma \leq \delta$ be nonempty sets in $\mathcal{F}$. Write $\delta = \{t_1, \ldots, t_n\}$. We first prove that the embedding $A_\delta \hookrightarrow A_\delta$ is the identity map. We prove this for the embeddings $A_{\delta(\ell)} \hookrightarrow A_{\delta(\ell)}$ by reverse induction; the base case $\ell = n$ is trivial, and the inductive step is just Corollary 3.4.10.

Now for each $\ell = 1, \ldots, n$ let

$$A_{\beta(\ell)} \xrightarrow{g_\ell} A_{\gamma(\ell)}$$

$$A_{\gamma(\ell)} \xrightarrow{f_\ell} A_{\delta(\ell)}$$

$$A_{\delta(\ell)} \xrightarrow{h_\ell} A_{\delta(\ell)}$$

be the embeddings from Definition 4.2.3. We will prove by reverse induction for $\ell = n, \ldots, 1$ that $f_\ell \circ g_\ell = h_\ell$. The base case $\ell = n$ is trivial, as each of the three maps in question is either the identity map or the embedding $C \hookrightarrow A$. Supposing now the result to be established for $\ell + 1$, let $B = A_{\beta(\ell+1)}$ and $C = A_{\gamma(\ell+1)}$, and let $B \xrightarrow{\psi} C \xrightarrow{\eta} A$ be the corresponding embeddings. Then by Proposition 3.4.12

$$h_\ell = h_{\ell+1} * (\eta \circ \psi) = (f_{\ell+1} \circ g_{\ell+1}) * (\eta \circ \psi) = (f_{\ell+1} * \eta) \circ (g_{\ell+1} * \psi) = f_\ell \circ g_\ell.$$

□

4.3. Endomorphisms of the Limit Algebra

We have constructed unital C*-algebras (resp. W*-algebras) $A_\gamma$ for each $\gamma \in \mathcal{F}$, together with (normal) embeddings $A_\beta \hookrightarrow A_\gamma$ for $\beta \leq \gamma$, which we now denote $f_{\gamma, \beta}$,
satisfying the inductive properties

\[ f_{\gamma,\gamma} = \text{id}_{A_{\gamma}} \]
\[ f_{\delta,\beta} = f_{\delta,\gamma} \circ f_{\gamma,\beta} \quad \text{for } \beta \leq \gamma \leq \delta. \]

By a standard construction (see for instance section 1.23 of [Sak98], Proposition 11.4.1 of [KR86], or section II.8.2 of [Blu06]) we obtain an inductive limit, that is, a unital C*-algebra \( A \) and embeddings \( f_{\infty,\gamma} : A_{\gamma} \rightarrow A \) such that \( f_{\infty,\gamma} \circ f_{\gamma,\beta} = f_{\infty,\beta} \) for all \( \beta \leq \gamma \), and with the universal property that, given any other unital C*-algebra \( B \) and *-homomorphisms (not necessarily embeddings) \( g_{\infty,\gamma} : A_{\gamma} \rightarrow B \) satisfying \( g_{\infty,\gamma} \circ f_{\gamma,\beta} = g_{\infty,\beta} \), there is a unique unital *-homomorphism \( \Phi : A \rightarrow B \) satisfying \( g_{\infty,\gamma} = \Phi \circ f_{\infty,\gamma} \) for all \( \gamma \). We denote by \( i \) the distinguished embedding \( f_{\infty,\{0\}} : A \rightarrow A \).

We note that inductive limits do not always exist in the category of W*-algebras and normal *-homomorphisms; hence, \( A \) will not in general be a W*-algebra even when \( A \) is. We postpone until the next chapter the question of how to adapt our construction to the W*-category, and continue for the time being with a purely C*-construction.

Our next task is to define a semigroup of unital *-endomorphisms of \( A \). For this we note that for any \( \gamma \in \mathcal{F} \) and any \( \tau \geq 0 \), if \( \gamma + \tau \) denotes the set \( \{ t + \tau \mid t \in \gamma \} \), then \( A_{\gamma + \tau} = A_{\gamma} \). (This is an equality, not just an isomorphism.) This is immediate from Definition 4.2.2 by induction on \( |\gamma| \). Similarly, \( f_{\gamma,t+\tau} = f_{\gamma,t} \). But this latter equation implies that \( f_{\infty,\gamma+t} \circ f_{\gamma,\beta} = f_{\infty,\beta+t} \) for any \( \beta \leq \gamma \), allowing us to make the following definition.

**Definition 4.3.1.** For each \( t \geq 0 \) let \( \sigma_t : A \rightarrow A \) denote the unital *-endomorphism obtained through the inductive limit as the unique map for which all the diagrams

\[ \begin{array}{ccc}
A_{\gamma} & \xrightarrow{f_{\infty,\gamma}} & A \\
A_{\gamma+t} & \xleftarrow{f_{\infty,\gamma+t}} & A \\
& \sigma_t \downarrow & \\
& A & \\
\end{array} \]

commute.

The universal property of the inductive limit then immediately implies:

**Proposition 4.3.2.** The maps \( \{\sigma_t\}_{t \geq 0} \) form an \( c_0 \)-semigroup on \( A \). That is, \( \sigma_0 = \text{id}_A \), and for all \( s, t \geq 0 \),

\[ \sigma_t \circ \sigma_s = \sigma_{s+t}. \]

**4.4. The Limit Retraction**

We now turn to the construction of our retraction. In the commutative analogue, for a set \( \gamma \) with minimum time \( \tau \), the retraction \( \epsilon_\gamma \) would (when composed with the embedding \( A \hookrightarrow A \)) correspond to a conditional expectation onto the subalgebra of \( A \) consisting of functions which depend only on the location of a path at time \( \tau \). This does not form a consistent system with respect to the embeddings \( f_{\gamma,\beta} \), because for \( \beta \leq \gamma \) one could have times in \( \gamma \) earlier than any in \( \beta \). However, the restriction to time sets which contain 0 is consistent, which we now show in the noncommutative case. We first consider how to relate the retraction for a given set to the retractions for its tails.
Lemma 4.4.1. Let $\gamma = \{t_1, \ldots, t_n\} \in \mathcal{F}$ and $1 \leq \ell \leq n$. Then

$$\epsilon_\gamma \circ f_{\gamma,\gamma(t)} = \phi_{t-1} \circ \epsilon_\gamma(t).$$

Proof. We proceed by (forward) induction on $\ell$. The base case $\ell = 1$ is trivial. Now supposing the result is true for $\ell$, recall that $A_{\gamma(t)}$ is the product $A_{\gamma(t)} \times A$ with respect to the map $\phi_{\ell+1}; t \circ \epsilon_{\gamma(t+1)} : A_{\gamma(t+1)} \to A$, that $f_{\gamma(t),\gamma(t+1)}$ is the embedding of $A_{\gamma(t+1)}$ into this product, and that $\epsilon_{\gamma(t)}$ is the Sauvageot retraction. By Equation [5.3] we therefore have

$$\epsilon_{\gamma(t)} \circ f_{\gamma(t),\gamma(t+1)} = \phi_{t-1} \circ \epsilon_{\gamma(t+1)}$$

so that

$$\epsilon_{\gamma(t)} \circ f_{\gamma(t),\gamma(t+1)} = \epsilon_{\gamma(t)} \circ f_{\gamma(t),\gamma(t+1)}$$

$$= \phi_{t-1} \circ \epsilon_{\gamma(t)} \circ f_{\gamma(t),\gamma(t+1)}$$

$$= \phi_{t-1} \circ \phi_{t+1} \circ \epsilon_{\gamma(t+1)}$$

$$= \phi_{t-1} \circ \epsilon_{\gamma(t+1)}.$$

□

Proposition 4.4.2. Let $\beta \leq \gamma \in \mathcal{F}$ such that the minimum time in $\gamma$ is also in $\beta$. Then

$$\epsilon_{\gamma(t)} \circ f_{\gamma,\beta} = \epsilon_{\beta(t)}.$$

Proof. Let $\gamma = \{t_1, \ldots, t_n\}$. We will prove that

$$\epsilon_{\gamma(t)} \circ f_{\gamma(t),\beta(t)} = \epsilon_{\beta(t)}$$

for all $\ell$ such that $t_{\ell+1}$ is the next time in $\gamma$, $\beta$, and $t_{\ell+1}$ is the next time in $\beta$, so that $\beta(\ell+1) = \beta(\ell+k)$; then

$$\phi_{t_{\ell+1}-1} \circ \epsilon_{\beta(t+1)} \circ f_{\gamma(t),\beta(t+1)} \circ \phi_{t_{\ell+1}-1} \circ \epsilon_{\gamma(t+1)} \circ f_{\gamma(t),\beta(t+1)}$$

$$= \phi_{t_{\ell+1}} \circ \epsilon_{\gamma(t+1)} \circ f_{\gamma(t),\beta(t+1)}$$

$$= \phi_{t_{\ell+1}} \circ \epsilon_{\gamma(t+1)} \circ f_{\gamma(t),\beta(t+1)}$$

$$= \phi_{t_{\ell+1}} \circ \epsilon_{\gamma(t+1)} \circ f_{\gamma(t),\beta(t+1)}$$

where the equalities follow respectively from the assumption that $\beta(\ell+1) = \beta(\ell+k)$, the consistency of the $f$’s, Lemma 4.4.1 and induction. It then follows from Corollary 4.4.11 that

$$\epsilon_{\gamma(t)} \circ f_{\gamma(t),\beta(t)} = \epsilon_{\beta(t)} (f_{\gamma(t+1),\beta(t+1)} \circ \phi_{t_{\ell+1}}) = \epsilon_{\beta(t)}$$

as desired. The case $\ell = 1$ gives us the result. □

Corollary 4.4.3. The restriction of the family of retractions $\{\epsilon_{\gamma}\}$ to the subset $\mathcal{F}_0 \subset \mathcal{F}$ of sets containing $0$ is consistent.

Since $\mathcal{F}_0$ is a tail of $\mathcal{F}$, the limit $\mathcal{A}$ is generated by images of $A_{\gamma}$ with $\gamma \in \mathcal{F}_0$. Hence, Corollary 4.4.3 implies the existence of a retraction (with respect to $i$) $E : \mathcal{A} \to \mathcal{A}$ with the property that $E \circ f_{\omega,\gamma} = \epsilon_{\gamma}$ for all $\gamma \in \mathcal{F}_0$.

Definition 4.4.4. The Sauvageot dilation retraction for $(\mathcal{A}, \{\phi_t\}, \omega)$ is the map $E : \mathcal{A} \to \mathcal{A}$ characterized by

$$E \circ f_{\omega,\gamma} = \epsilon_{\gamma} \quad \text{for all } 0 \in \gamma \in \mathcal{F}.$$
We now prove that \((E, \{\sigma_t\})\) provides a strong dilation of the semigroup \(\{\phi_t\}\).

**Theorem 4.4.5.** For all \(t \geq 0\),
\[
E \circ \sigma_t = \phi_t \circ E.
\]

**Proof.** The case \(t = 0\) is trivial. Now let \(\gamma \in F\) be nonempty and \(t > 0\). Let \(\delta = (\gamma + t) \cup \{0\}\); then \(A_\delta\) is the Sauvageot product \(A_{\gamma+t} \star A\) with respect to the map \(\phi_t \circ \epsilon_\gamma\). By Equation 3.3, it follows that
\[
\epsilon_\delta \circ f_{\delta,\gamma + t} = \phi_t \circ \epsilon_\gamma.
\]
Then
\[
E \circ \sigma_t \circ f_{\infty, \gamma} = E \circ f_{\infty, \gamma + t} = E \circ f_{\infty, \delta} \circ f_{\delta, \gamma + t} = \epsilon_\delta \circ f_{\delta, \gamma + t} = \phi_t \circ \epsilon_\gamma = \phi_t \circ E \circ f_{\infty, \gamma}.
\]
So \(E \circ \sigma_t\) and \(\phi_t \circ E\) agree on the dense subalgebra of \(A\) consisting of the images of all the \(f_{\infty, \gamma}\); as both are contractive, they are equal. \(\square\)

This concludes our construction of unital \(e_0\)-dilations for \(cp_0\)-semigroups on \(C^*\)-algebras. We summarize the result in the following theorem.

**Theorem 4.4.6.** Let \(A\) be a unital \(C^*\)-algebra on which there exists a faithful state. Then every \(cp_0\)-semigroup on \(A\) has a strong unital \(e_0\)-dilation.
CHAPTER 5

Continuous $W^*$-Dilations

In the previous chapter we saw how to construct a unital $c_0$-dilation of a $c_0$-semigroup. It remains to investigate whether such a construction dilates a continuous semigroup to a continuous semigroup (that is, whether it produces a unital $E_0$-dilation of a CP0-semigroup), or, failing that, whether the construction can be modified to achieve this result. Additionally, we have not yet resolved the question of how to adapt our $C^*$ construction to the $W^*$ setting. To these issues we now turn our attention.

5.1. Introduction: The Problem of Continuity

The first question to consider is whether the existing dilation may already be continuous. It turns out that this is never the case unless $A = \mathbb{C}$. Consider a nontrivial $\mathcal{A}$ with faithful state $\omega$, and let $a$ be any nonzero element of ker $\omega$. Fixing some faithful representation $(H, \Omega, \pi)$ of $(\mathcal{A}, \omega)$, let $h = \pi_a(\Omega)$, which is orthogonal to $\Omega$. For each $t > 0$ there is a faithful representation $(H, \Omega, \pi^{(t)}, K^{(t)}, V^{(t)}, \pi_L^{(t)})$ of $(A, A, \phi_t, \omega)$. Form the Sauvageot product $\psi^{(t)} = H^{-} \star L^{(t)}$, and let $\xi$ be any unit vector in $L^{(t)}$. By Proposition 3.2.9 we see that $\psi^{(t)}(a)\xi$ is a vector in $L^{(t)}$, whereas $\psi^{(t)}(a)\xi = h \otimes \xi$ is in $H^{-} \otimes L^{(t)}$. Since these are orthogonal subspaces of $\psi^{(t)}$,

$$||\psi^{(t)}_L(a)\xi - \psi^{(t)}_R(a)\xi|| \geq ||h \otimes \xi|| = ||h|| ||\xi||$$

which implies

$$||\psi^{(t)}_L(a) - \psi^{(t)}_R(a)|| \geq ||h||.$$

Now letting $\gamma = \{0, t\}$, we have $\mathcal{A}_\gamma$ as the Sauvageot product $\mathcal{A} \star \mathcal{A}$ with respect to $\phi_t$, so that $\psi^{(t)}(a) - \psi^{(t)}(a)$ is the element $f_{\gamma, (t)}(a) - f_{\gamma, (0)}(a)$ of $\mathcal{A}_\gamma$. By the above, this element has norm at least $||h||$. Now because $f_{\gamma, (\cdot)}$ is isometric,

$$||\sigma_t(\iota(a)) - \iota(a)|| = ||f_{\gamma, (t)}(a) - f_{\gamma, (0)}(a)||$$

$$= \left\| f_{\gamma, (t)}(a) - f_{\gamma, (0)}(a) \right\|$$

$$= ||f_{\gamma, (t)}(a) - f_{\gamma, (0)}(a)|| \geq ||h||.$$

It follows that $||\sigma_t(\iota(a)) - \iota(a)|| \neq 0$ as $t \to 0^+$.

Upon further reflection, the discontinuity of $\{\sigma_t\}$ is not surprising, because it appears in the commutative dilation that the Sauvageot construction mimics. Consider again the case $\mathcal{A} = C(S)$, $\mathfrak{A} = C(\mathcal{F})$ of Example 3.3.3. Given a regular Borel probability measure $\mu_0$ on $S$, we obtain via Riesz representation a regular Borel probability measure $\mu$ on $\mathcal{F}$ characterized by

$$\forall f \in \mathfrak{A} : \int f d\mu = \int_S (E f) d\mu_0.$$
Let us call the semigroup \( \{ \sigma_i \} \) “point-pointwise” continuous if for any fixed path \( p \in \mathcal{I} \) and any \( f \in \mathcal{A} \), \( (\sigma_i f - f)(p) \to 0 \). The failure of point-pointwise continuity certainly implies the failure of point-norm continuity. Now let \( p \) be any path not continuous at time 0, let \( \phi : S \to [0, 1] \) be a continuous function such that \( \phi(p(t)) \not\rightarrow \phi(p(0)) \) as \( t \to 0^+ \) (which exists by Urysohn’s lemma), and let \( f \in \mathcal{A} \) be defined by \( f(p) = \phi(p(0)) \). Then

\[
\lim_{t \to 0^+} (\sigma_t f - f)(p) = \lim_{t \to 0^+} \phi(\lambda_t p) - \phi(p) = \lim_{t \to 0^+} \phi(p(t)) - \phi(p(0)) \neq 0.
\]

To remedy the problem, we move to the \( \mathcal{W}^* \) setting. First, however, we need more information about the retraction constructed in the previous chapter.

### 5.2. Moment Polynomials

In the Sauvageot \( \mathcal{C}^* \)-dilation of chapter 4, the inductive limit algebra \( \mathcal{A} \) is norm-generated as an algebra by elements \( \sigma_i(i(a)) \) for \( t \geq 0 \) and \( a \in \mathcal{A} \). In studying the retraction \( \mathcal{E} \), therefore, one is naturally led to consider expressions of the form

\[
(5.1) \quad \mathcal{E} \left[ \sigma_{t_1} i((a_1))\sigma_{t_2} i((a_2)) \cdots \sigma_{t_n} i((a_n)) \right], \quad t_1, \ldots, t_n \geq 0; \quad a_1, \ldots, a_n \in \mathcal{A}.
\]

In particular, it would be desirable to have a formula for the value of (5.1) in terms of the original semigroup \( \{ \phi_i \} \) and the state \( \omega \) chosen for the dilation procedure. From the construction of \( \mathcal{E} \) in previous chapters, we see that (5.1) can be evaluated as follows:

1. If all the \( t_i \) are strictly positive, let \( \tau \) denote the minimum; then, by Theorem 4.4.5,

\[
\mathcal{E} \left[ \sigma_{t_1} i((a_1))\sigma_{t_2} i((a_2)) \cdots \sigma_{t_n} i((a_n)) \right] = \phi_{\tau} \left( \mathcal{E} \left[ \sigma_{t_1 - \tau} i((a_1))\sigma_{t_2 - \tau} i((a_2)) \cdots \sigma_{t_n - \tau} i((a_n)) \right] \right).
\]

2. If some of the \( t_i \) are zero, let \( \gamma = \{ t_i \}, \gamma' = \gamma \setminus \{ 0 \} \), and, disregarding the trivial case \( \gamma = \{ 0 \}, \tau = \min \gamma' \). By definitions 1.3.1 and 1.4.4,

\[
\mathcal{E} \left[ \sigma_{t_1} i((a_1))\sigma_{t_2} i((a_2)) \cdots \sigma_{t_n} i((a_n)) \right] = \mathcal{E} \left[ f_{\infty,(t_1)}(a_1) \cdots f_{\infty,(t_n)}(a_n) \right]
\]

\[
= \mathcal{E} \left[ f_{\infty, \gamma'}(f_{\gamma,(t_1)}(a_1) \cdots f_{\gamma,(t_n)}(a_n)) \right]
\]

\[
= \epsilon_{\gamma'} \left[ f_{\gamma,(t_1)}(a_1) \cdots f_{\gamma,(t_n)}(a_n) \right].
\]

Now \( \mathcal{A}_\gamma \) is the Sauvageot product \( \mathcal{A}_{\gamma'} \star \mathcal{A} \) with respect to the map \( \phi_{\tau} \circ \epsilon_{\gamma'} \), so \( f_{\gamma,(t_1)}(a_1) \cdots f_{\gamma,(t_n)}(a_n) \) is a word in \( \mathcal{A}_{\gamma'} \) (corresponding to nonzero \( t_i \)) and \( \mathcal{A} \) (corresponding to those \( t_i \) equal to zero); the value of \( \epsilon_{\gamma'} \) at this word can be computed using the moment function from chapter 2.

In carrying out the second step, one ends up applying the map \( \epsilon_{\gamma'} \) to words of the form \( f_{\gamma',(t_{i_1})} \cdots f_{\gamma',(t_{i_m})} \), yielding expressions similar to (5.1). It is therefore convenient, for both theoretical and practical purposes, to introduce a recursive definition for such expressions, rather than relying on appropriate evaluations of the moment function of chapter 2.

We take as our basic object of study pairs of the form \( \langle t_1, \ldots, t_n; a_1, \ldots, a_n \rangle \), where \( t_1, \ldots, t_n \geq 0 \) and \( a_1, \ldots, a_n \in \mathcal{A} \). The length of such a pair is \( n \). This pair corresponds to a word form \( \sigma_{t_1} i((a_1)) \cdots \sigma_{t_n} i((a_n)) \) in \( \mathcal{A} \). We identify elements \( a \in \mathcal{A} \) with pairs \( \langle 0; a \rangle \).
We define a concatenation or “multiplication” operation on pairs by
\[ \langle \langle t_1, \ldots, t_n; a_1, \ldots, a_n \rangle \rangle \lor \langle \langle s_1, \ldots, s_m; b_1, \ldots, b_m \rangle \rangle = \langle \langle t_1, \ldots, t_n, s_1, \ldots, s_m; a_1, \ldots, a_n, b_1, \ldots, b_m \rangle \rangle \]
or, more succinctly,
\[ \langle \langle \bar{t}; \bar{a} \rangle \rangle \lor \langle \langle \bar{s}; \bar{b} \rangle \rangle = \langle \langle \bar{t} \lor \bar{s}; \bar{a} \lor \bar{b} \rangle \rangle. \]
We make simultaneous recursive definitions of an \( A \)-valued function \( \mathcal{G} \) on pairs (the \textbf{moment polynomial} function, corresponding to that defined in chapter 8 of [Arv03]), as well as a collapse function similar to that in chapter 2, as follows:

**Definition 5.2.1.** Given a pair \( \langle \langle \bar{t}; \bar{a} \rangle \rangle \), let \( \gamma = \cup \{ t_i \} \).

- If \( \gamma = \{ 0 \} \) then \( \mathcal{G} \langle \langle \bar{t}; \bar{a}; \gamma \rangle \rangle = \Pi(\bar{a}) \).
- If \( 0 < \tau = \min \gamma \),
  \[ \mathcal{G} \langle \langle \bar{t}; \bar{a} \rangle \rangle = \phi_\tau \circ \mathcal{G} \langle \langle \bar{t} - \tau; \bar{a} \rangle \rangle. \]
- If \( 0 \in \gamma \neq \{ 0 \} \), let \( \tau = \min \gamma \setminus \{ 0 \} \) and decompose
  \[ \langle \langle \bar{t}; \bar{a} \rangle \rangle = \langle \langle \bar{n}_0; \bar{z}_0 \rangle \rangle \lor \langle \langle \bar{s}_1; \bar{w}_1 \rangle \rangle \lor \langle \langle \bar{n}_1; \bar{z}_1 \rangle \rangle \lor \cdots \lor \langle \langle \bar{s}_\ell; \bar{w}_\ell \rangle \rangle \lor \langle \langle \bar{n}_\ell; \bar{z}_\ell \rangle \rangle \]
where each \( \bar{n}_i \) is a vector of zeros, each \( \bar{s}_i \) a vector of nonzero numbers, and some of the pairs may be empty. We refer to the pairs \( \langle \langle \bar{n}_i; \bar{z}_i \rangle \rangle \) and \( \langle \langle \bar{s}_i; \bar{w}_i \rangle \rangle \) as the \textbf{components} of this decomposition. Given \( S \subset \{ 2\ell - 1 \} \), let
\[
x_j = \begin{cases} 
\mathcal{G} \langle \langle \bar{s}(j+1)/2; \bar{w}(j+1)/2 \rangle \rangle & j \text{ odd} \\
\Pi(\bar{z}/2) & j \text{ even}
\end{cases}
\]
for \( j \in S \cup \{ 0, 2\ell \} \) and
\[
y_k = \begin{cases} 
\langle \langle \bar{s}(k+1)/2; \bar{w}(k+1)/2 \rangle \rangle & k \text{ odd} \\
\langle \langle \tau; \omega(\Pi(\bar{z}/2)) \rangle \rangle & k \text{ even}
\end{cases}
\]
for \( k \in (2\ell - 1) \setminus S \). Let \( T_0, \ldots, T_m \) denote the consecutive in-subsets of \( S \) and \( U_1, \ldots, U_m \) the consecutive out-subsets as in chapter 2. Then we define
\[
\text{Col}(\langle \langle \bar{t}; \bar{a} \rangle \rangle, S) = \left( \bigvee_{j \in T_0} x_j \right) \lor \left( \bigvee_{k \in U_1} y_k \right) \lor \left( \bigvee_{j \in T_1} x_j \right) \lor \cdots \lor \left( \bigvee_{j \in T_m} x_j \right)
\]
and
\[
\mathcal{G} \langle \langle \bar{t}; \bar{a} \rangle \rangle = \sum_{S \subset \{ 2\ell - 1 \}} (-1)^{|S|} \mathcal{G}(\text{Col}(\langle \langle \bar{t}; \bar{a} \rangle \rangle, S)).
\]

By the reasoning given above when introducing these pairs, we arrive at the following:

**Proposition 5.2.2.** Let \( A \) be a unital \( C^* \)-algebra, \( \{ \phi_i \} \) a \( cp_0 \)-semigroup on \( A \), \( \omega \) a faithful state on \( A \), and \( (\mathfrak{A}, i, E, \{ \sigma_i \}) \) the Sauvageot dilation. Then for every \( t_1, \ldots, t_n \geq 0 \) and \( a_1, \ldots, a_n \in A \),
\[
E \left[ \sigma_{t_1}(i(a_1)) \cdots \sigma_{t_n}(i(a_n)) \right] = \mathcal{G} \langle \langle \bar{t}; \bar{a} \rangle \rangle.
\]
5.3. Continuity Properties of Moment Polynomials

The continuity properties of $\mathcal{S}(\vec{t}; \vec{a})$ in the case where $\mathcal{A}$ is a $W^*$-algebra will be important in what follows. There are three types of continuity properties to consider: continuity in $a_1, \ldots, a_n$ with respect to both the weak and the strong topologies, and continuity in $t_1, \ldots, t_n$. It turns out that weak continuity holds with respect to $a_1, \ldots, a_n$ separately (which is the best we could hope for, as multiplication is not jointly weakly continuous), whereas strong continuity holds jointly in $a_1, \ldots, a_n$, and a restricted form of joint continuity in $t_1, \ldots, t_n$ holds as well.

**Proposition 5.3.1.** Let $\mathcal{A}$ be a $W^*$-algebra, $\{\phi_t\}$ a CP$_0$-semigroup on $\mathcal{A}$, $\omega$ a faithful normal state on $\mathcal{A}$. Fix $n \geq 1$, $t_1, \ldots, t_n \geq 0$, $j \in \{1, \ldots, n\}$, and $a_k$ for $k \in \{1, \ldots, n\} \setminus \{j\}$. Then $\mathcal{S}(\vec{t}; \vec{a})$, viewed as a function of $a_j$, is a normal linear map from $\mathcal{A}$ to itself.

**Proof.** This follows from Definition 4.2.1 by induction on the length of the pair. We also use the normality of the state $\omega$ and the maps $\phi_t$, as well as the normality of multiplication by a fixed element of $\mathcal{A}$. ~

**Definition 5.3.2.** For $n \geq 1$ and elements $\{\vec{s}_k\}$ and $\vec{t}$ of $[0, \infty)^n$, we say that $\vec{s}_k$ converges non-crossingly to $\vec{t}$ if $\vec{s}_k \to \vec{t}$ and, for all $k$, the order relations among the entries of $\vec{s}_k$ are the same as those in $\vec{t}$; that is, if

$$\forall k : \forall i, j = 1, \ldots, n : (s_k)_i \leq (s_k)_j \iff t_i \leq t_j.$$ 

**Proposition 5.3.3.** Let $\mathcal{A}$ be a separable $W^*$-algebra, $\{\phi_t\}$ a CP$_0$-semigroup on $\mathcal{A}$, $\omega$ a faithful normal state on $\mathcal{A}$. Let $n \geq 1$. Let $\vec{t}_k \to \vec{t}$ converge non-crossingly in $[0, \infty)^n$, and let $\vec{a}_k \to \vec{a}$ be a strongly convergent sequence of tuples in $(\mathcal{A}_1)^n$. Then $\mathcal{S}(\vec{t}_k; \vec{a}_k) \to \mathcal{S}(\vec{t}; \vec{a})$ strongly. That is, $\mathcal{S}(\vec{t}; \vec{a})$ is jointly strongly continuous in $\vec{t}$ and $\vec{a}$, subject to the non-crossing restriction on $\vec{t}$.

**Proof.** It is convenient to write each $\vec{t}_i = \delta_i + \vec{u}_i$ and $\vec{t} = \delta + \vec{u}$, where $\delta_i, \delta \geq 0$ and each $\vec{u}_i$ and $\vec{u}$ contains at least one zero entry. We make the following observations:

1. $\delta_i \to \delta$
2. $\vec{u}_i \to \vec{u}$ non-crossingly
3. $\mathcal{S}(\vec{t}_i; \vec{a}_i) = \phi_{\delta_i} \circ \mathcal{S}(\vec{u}_i; \vec{a}_i)$.

By the joint strong continuity of the semigroup $\phi$ (Theorem 1.4.2), it therefore suffices to consider the case where all $\vec{t}_i$ and $\vec{t}$ have zero entries. The non-crossing hypothesis implies that these occur at the same positions for all $i$, and hence that the components of the decompositions of all $\vec{t}_i$ in the definition of $\mathcal{S}$ all have constant length as $i$ varies. The time vectors in each component pair converge non-crossingly to the time vector in the corresponding component of $\vec{t}$, and the result follows by induction on $n$. ~

To illustrate the necessity of the non-crossing hypothesis, Appendix B contains an example where this hypothesis is violated and where discontinuity results. The underlying reason is that a crossing creates a change in the lengths of the components of the decomposition, so that a different branch of the recursion is followed.
5.4. The Continuous Theorem

We now return to the question of how to obtain a continuous $W^*$-dilation from an algebraic $C^*$-dilation. The technique in this section is adapted from the eighth chapter of [Arv03]. Throughout, we let $\mathcal{A}$ denote a separable $W^*$-algebra, $\{\phi_t\}$ a $CP_0$-semigroup on $\mathcal{A}$, $(\mathfrak{A}, i, E, \{\sigma_t\})$ the Sauvageot dilation from the previous chapter, $\mathcal{P} \subset \mathfrak{A}$ the subset

$$\mathcal{P} = \{\sigma_{t_1}(i(a_1)) \cdots \sigma_{t_k}(i(a_k)) \mid t_1, \ldots, t_k \geq 0; a_1, \ldots, a_k \in \mathcal{A}\},$$

$\mathfrak{A}_0 \subseteq \mathfrak{A}$ the norm-dense linear span of $\mathcal{P}, (H, \pi)$ a faithful normal representation of $\mathcal{A}$ on a separable Hilbert space, $(\mathfrak{A}, V, \psi)$ a minimal Stinespring dilation of $\pi \circ E$, $\mathfrak{A} = \psi(\mathfrak{A})''$, and $\mathfrak{E} : \mathfrak{A} \rightarrow \mathfrak{A}$ the map $\mathfrak{E}(T) = \pi^{-1}(V^*TV)$, which is well-defined because $T \mapsto V^*TV$ is normal and maps the weakly dense subspace $\psi(\mathfrak{A}) \subset \mathfrak{A}$ into the weakly closed set $\pi(\mathcal{A})$, and because $\pi$ is faithful; it satisfies $\mathfrak{E} \circ \psi = E$ and therefore is a normal retraction with respect to $\psi \circ i$.

We begin with the observation that weak-operator continuity of families of contractions can be checked on a dense subset of Hilbert space.

**Lemma 5.4.1.** Let $\mathcal{H}$ be a Hilbert space and $\{T_t\}_{t \geq 0}$ a family (not necessarily a semigroup) of contractions on $\mathcal{H}$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a dense linear subspace such that for all $\xi, \eta \in \mathcal{H}_0$, the map $t \mapsto \langle T_t\xi, \eta \rangle$ is continuous. Then $t \mapsto T_t$ is WOT-continuous.

**Proof.** Let $\xi, \eta \in \mathcal{H}$ and $t_0 \geq 0$. Given $\epsilon > 0$, choose $\xi_0, \eta_0 \in \mathcal{H}_0$ with $\|\xi - \xi_0\| < \max(1, \epsilon)$ and $\|\eta - \eta_0\| < \epsilon$. Then for any $t \geq 0$,

$$\langle (T_t - T_{t_0})\xi, \eta \rangle = \langle (T_t - T_{t_0})\xi_0, \eta_0 \rangle + \langle (T_t - T_{t_0})\xi_0, \eta_0 - \eta \rangle + \langle (T_t - T_{t_0})\xi, \eta - \xi_0 \rangle + \langle (T_t - T_{t_0})(\xi - \xi_0), \eta \rangle.$$

The first term tends to zero as $t \rightarrow t_0$ by hypothesis, so that in particular it is less than $\epsilon$ for $t$ sufficiently close to $t_0$. The second term is at most $2\|\xi_0\|\epsilon \leq 2(\|\xi\| + 1)\epsilon$ by Cauchy-Schwarz, and the third term at most $2\|\eta\|\epsilon$. Hence

$$\left|\langle (T_t - T_{t_0})\xi, \eta \rangle\right| \leq \left(3 + 2\|\eta\| + 2\|\xi\|\right)\epsilon$$

for $t$ sufficiently near $t_0$. \hfill $\Box$

The next lemma is rather technical, but it advances our study of how $E$ interacts with time translations, and in particular with translation of the middle term of a threefold product. It is based on Lemma 8.6.3 in [Arv03].

**Lemma 5.4.2.** Let $y, z \in \mathcal{P}$ and $t \geq 0$. There exist $y_0, z_0 \in \mathcal{P}$ and a normal linear map $Q : \mathcal{A} \rightarrow \mathcal{A}$ such that, for every $x \in \mathfrak{A}$,

$$E[y\sigma_t(x)z] = Q\left(E[y_0xz_0]\right).$$

(5.2)

**Proof.** Let $y = \sigma_{t_1}(a_1) \cdots \sigma_{t_p}(a_p)$ and $z = \sigma_{t_1}(b_1) \cdots \sigma_{t_n}(b_n)$. We proceed by strong induction on $m + n$. In the base case $m + n = 0$ (meaning that $y = z = 1$) the requisite map is $Q = \phi_t$, by Theorem 4.4.5. Inductively, letting $\tau = \min(t_1, \ldots, t_p, t_1, \ldots, t_n)$, the result is again trivial in case $t \leq \tau$, as then one can use $Q = \phi_t$, $y_0 = \sigma_{t_1}(a_1) \cdots \sigma_{t_p-t}(a_p)$, and $z_0 = \sigma_{t_1-t}(b_1) \cdots \sigma_{t_n-t}(b_n)$. Hence
we assume \( t > \tau \). We further assume \( \tau = 0 \), as the case \( \tau > 0 \) reduces to this by Theorem 4.4.5 again.

Let \( (v'_1, \ldots, v'_q) \) be the (possibly empty) final segment of nonzero entries from \( (v_1, \ldots, v_q) \), and \( (a'_1, \ldots, a'_p) \) the corresponding entries from \( (a_1, \ldots, a_p) \). Similarly, let \( (t'_1, \ldots, t'_h) \) be the initial segment of nonzero entries from \( (t_1, \ldots, t_n) \), and \( (b'_1, \ldots, b'_h) \) the corresponding entries from \( (b_1, \ldots, b_n) \). Let \( y_0 = \sigma_{v'_1}(a'_1) \cdots \sigma_{v'_q}(a'_q) \) and \( z_0 = \sigma_{t'_1}(b'_1) \cdots \sigma_{t'_h}(b'_h) \).

For any \( x \in \mathcal{P} \), write \( x = \sigma_{u_1}(c_1) \cdots \sigma_{u_m}(c_m) \), so that \( \sigma_t(x) = \sigma_{u_1+t}(c_1) \cdots \sigma_{u_m+t}(c_m) \).

Now \( E[y\sigma_t(x)z] = \mathcal{S}(\tilde{v} \vee (\tilde{u} + t) \vee \tilde{t} \vee \tilde{c} \vee \tilde{b}) \) by Proposition 5.2.2. In the standard decomposition \( \tilde{v} \vee (\tilde{u} + t) \vee \tilde{t} = \tilde{n}_0 \vee \tilde{s}_1 \vee \cdots \vee \tilde{n}_t \), we must have \( \tilde{n}_0 = (v'_1, \ldots, v'_q) \vee (t'_1, \ldots, t'_h) \) and \( \tilde{n}_t = (a'_1, \ldots, a'_p) \vee (b'_1, \ldots, b'_h) \). Then Propositions 5.2.2 and 2.4.6 imply that \( E[y\sigma_t(x)z] \) is the composition of \( E[y_0xz_0] \) with some normal map \( Q \), which is independent of \( x \). This gives us equation (5.2) for all \( x \in \mathcal{P} \), and since both sides are linear and norm-continuous in \( x \), it follows that (5.2) holds for all \( x \in \mathfrak{A} \).

**Theorem 5.4.3.** There exists a (necessarily unique) semigroup of normal unital \(*\)-endomorphisms \( \{\tilde{\sigma}_t\}_{t \geq 0} \) of \( \mathfrak{A} \) such that

\begin{equation}
\forall t \geq 0 : \quad \tilde{\sigma}_t \circ \psi = \psi \circ \sigma_t.\end{equation}

**Proof.** We construct \( \{\tilde{\sigma}_t\} \) and verify its properties in many small steps.

1. For each \( t \geq 0 \) and \( \xi, \eta \in \psi(\mathcal{P})VH \), we construct a normal linear functional \( \rho_{t,\xi,\eta} \) on \( \mathfrak{A} \) as follows. Let \( \xi = \psi(y)V\xi' \) and \( \eta = \psi(z)V\eta' \) for \( y, z \in \mathcal{P} \) and \( \xi', \eta' \in H \). By Lemma 5.4.2, there exists a normal linear map \( Q : \mathcal{A} \to \mathcal{A} \) and elements \( y_0, z_0 \in \mathcal{P} \) such that \( E[z^*\sigma_t(x)y] = Q(E[z_0^*xy_0]) \) for all \( x \in \mathfrak{A} \). We thus have

\[ \forall x \in \mathfrak{A} : \quad \langle \psi(\sigma_t(x))\xi, \eta \rangle = \langle \pi \circ Q \circ \psi(z_0)\xi', \eta' \rangle.\]

We now define \( \rho_{t,\xi,\eta} \) by

\[ \rho_{t,\xi,\eta}(T) = \langle \pi \circ Q \circ \psi(z_0)^*T\psi(y_0)\xi', \eta' \rangle, \quad T \in \mathfrak{A}.\]

Then the restriction to \( \psi(\mathfrak{A}) \) satisfies

\[ \forall x \in \mathfrak{A} : \quad \rho_{t,\xi,\eta}(\psi(x)) = \langle \pi \circ Q \circ \psi(z_0)\xi', \eta' \rangle = \langle \pi \circ Q \circ \psi(z_0)^*\xi', \eta' \rangle = \langle \pi \circ E[z^*\sigma_t(x)y]\xi', \eta' \rangle = \langle V^*\psi(z^*\sigma_t(x)y)V\xi', \eta' \rangle = \langle \psi(\sigma_t(x))\xi, \eta \rangle. \]

(5.4)

2. We extend the definition to \( \xi, \eta \) in the linear span of \( \psi(\mathcal{P})VH \) in the natural way; for \( \xi = \sum c_i\xi_i \) and \( \eta = \sum d_j\eta_j \) with \( \xi_i, \eta_j \in \psi(\mathcal{P})VH \), we define \( \rho_{t,\xi,\eta} = \sum_{i,j} c_i d_j \rho_{t,\xi_i,\eta_j} \). This is well-defined because, if \( \xi_i = \sum_k \bar{c}_{ik}\xi_k \) and \( \eta_j = \sum_{\ell} \bar{d}_{\ell j}\eta_{\ell} \) then equation (5.4) implies that, for \( x \in \mathfrak{A} \)}
the ultraweakly dense subspace $\psi(\mathfrak{A})$ of $\tilde{\mathfrak{A}},$

$$\rho_{t,\Sigma c_i \xi_i, \Sigma d_i \eta_i}(\psi(x)) = \left\langle \psi(\sigma_t(x)) \sum c_i \xi_i, \sum d_i \eta_i \right\rangle$$

$$= \left\langle \psi(\sigma_t(x)) \sum \tilde{c}_k \tilde{x}_k, \sum \tilde{d}_k \tilde{\eta}_k \right\rangle = \rho_{t,\Sigma \tilde{c}_k \tilde{x}_k, \Sigma \tilde{d}_k \tilde{\eta}_k}(\psi(x)).$$

(3) Next, we note that equation (5.4) also implies that $\|\rho_{t,\xi,\eta}\| \leq \|\xi\|\|\eta\|$. This allows us to extend the definition to $\xi, \eta$ in the norm closure of the linear span of $\psi(\mathcal{P})VH$, which is all of $\tilde{\mathfrak{A}}$.

(4) Having defined the family of functionals $\{\rho_{t,\xi,\eta}\}$, we now use them to define the family of endomorphisms $\{\tilde{\sigma}_t\}$. Equation (5.4) implies that, for fixed $t \geq 0$ and $x \in \mathfrak{A}$, $\rho_{t,\xi,\eta}(\psi(x))$ is a bounded sesquilinear function of $\xi$ and $\eta$, so that it corresponds to a unique operator in $B(\tilde{\mathfrak{A}})$, which we call $S_t(\psi(x))$, characterized by the property

$$\forall \xi, \eta \in \tilde{\mathfrak{A}} : \quad \rho_{t,\xi,\eta}(\psi(x)) = \langle S_t(\psi(x))\xi, \eta \rangle.$$

(5) Equations (5.4) and (5.5) together imply that

$$\forall x \in \mathfrak{A} : \quad S_t(\psi(x)) = \psi(\sigma_t(x)).$$

(6) Because $\psi$ and $\sigma_t$ are unital *-homomorphisms, equation (5.6) implies that $S_t$ is as well.

(7) Because $S_t$ is a unital *-homomorphism of a C*-algebra, it is contractive. This implies

$$\forall x \in \mathfrak{A} : \quad \|\psi(\sigma_t(x))\| \leq \|\psi(x)\|.$$

(8) Given any $z \in \tilde{\mathfrak{A}}$, we can now show that $\rho_{t,\xi,\eta}(z)$ is a bounded sesquilinear function of $\xi$ and $\eta$. For boundedness, we will show more precisely that

$$\|\rho_{t,\xi,\eta}(z)\| \leq \|z\|\|\rho\|\|\eta\|.$$

Let $z, \xi, \eta$ be given, and choose $\epsilon > 0$. By the Kaplansky density theorem and the normality of $\rho_{t,\xi,\eta}$, there exists $x \in \mathfrak{A}$ such that $\|\psi(x)\| \leq \|z\|$ and $\|\rho_{t,\xi,\eta}(z - \psi(x))\| < \epsilon$. Then

$$\|\rho_{t,\xi,\eta}(z)\| \leq \|\rho_{t,\xi,\eta}(\psi(x))\| + \|\rho_{t,\xi,\eta}(z - \psi(x))\|$$

$$\leq \epsilon + \|\psi(\sigma_t(x)\xi, \eta)\|$$

$$\leq \epsilon + \|\psi(\sigma_t(x))\|\|\xi\|\|\eta\|$$

$$\leq \epsilon + \|\psi(x)\|\|\xi\|\|\eta\|$$

$$\leq \epsilon + \|z\|\|\xi\|\|\eta\|.$$

Letting $\epsilon \to 0$, we have (5.8)

To show linearity in $\xi$, let $c_1, c_2 \in \mathbb{C}$ and $\xi_1, \xi_2, \eta \in \tilde{\mathfrak{A}}$ be given, and choose $\epsilon > 0$. By Kaplansky density and the normality of $\rho_{t,\xi,\eta}$, $\rho_{t,\xi_1,\eta}$, and $\rho_{t,\xi_1+c_1\xi_2,\eta}$, there exists $x \in \mathfrak{A}$ such that $\|\psi(x)\| \leq \|z\|$ and the three inequalities

$$|\rho_{t,\xi_1+c_2\xi_2,\eta}(z - \psi(x))| < \epsilon$$

$$|c_1| \|\rho_{t,\xi_1,\eta}(z - \psi(x))\| < \epsilon$$

$$|c_2| \|\rho_{t,\xi_2,\eta}(z - \psi(x))\| < \epsilon$$
all hold. Then

$$|\rho_{t,c_1\xi_1+c_2\xi_2\eta}(z) - c_1\rho_{t,\xi_1,\eta}(z) - c_2\rho_{t,\xi_2,\eta}(z)|$$

$$\leq |\rho_{t,c_1\xi_1+c_2\xi_2\eta}(z - \psi(x))| + |c_1| |\rho_{t,\xi_1,\eta}(z - \psi(x))| + |c_2| |\rho_{t,\xi_2,\eta}(z - \psi(x))|$$

$$+ |\rho_{t,c_1\xi_1+c_2\xi_2\eta}(\psi(x)) - c_1\rho_{t,\xi_1,\eta}(\psi(x)) - c_2\rho_{t,\xi_2,\eta}(\psi(x))|$$

$$\leq 3\epsilon + |\psi(\sigma_t(x))(c_1\xi_1 + c_2\xi_2)\eta - c_1\psi(\sigma_t(x))\xi_1,\eta - c_2\psi(\sigma_t(x))\xi_2,\eta| = 3\epsilon$$

and as this is true for all $\epsilon > 0$, we conclude that

$$\rho_{t,c_1\xi_1+c_2\xi_2\eta}(z) = c_1\rho_{t,\xi_1,\eta}(z) + c_2\rho_{t,\xi_2,\eta}(z).$$

Conjugate-linearity in $\eta$ is, of course, established in the same way.

(9) We therefore obtain an operator in $B(\mathcal{H})$, which we call $\hat{\sigma}_t(z)$, characterized by the property

$$\forall \xi, \eta \in \mathcal{H} : \quad \rho_{t,\xi,\eta}(z) = \langle \hat{\sigma}_t(z)\xi, \eta \rangle.$$ (5.9)

We now have a function (not yet known to be linear, continuous, multiplicative, or self-adjoint) $\hat{\sigma}_t : \mathfrak{A} \to B(\mathcal{H})$ which extends the unital $^*$-endomorphism $S_t : \psi(\mathfrak{A}) \to \psi(\mathfrak{A})$.

(10) The function $\hat{\sigma}_t$ is contractive, because

$$\|\hat{\sigma}_t(z)\| = \sup_{\xi,\eta \in \mathcal{H}} |\langle \hat{\sigma}_t(z)\xi, \eta \rangle| = \sup_{\xi,\eta \in \mathcal{H}} |\rho_{t,\xi,\eta}(z)| \leq \sup_{\xi,\eta \in \mathcal{H}} \|z\|\|\xi\|\|\eta\| = \|z\|$$

by equation (5.8).

(11) Weak continuity of $\hat{\sigma}_t$ is a straightforward consequence of the normality of the $\rho_{t,\xi,\eta}$. Indeed, if $z_{\nu} \to z$ weakly in the unit ball $\mathfrak{A}_1$, then for all $\xi, \eta$ it follows that

$$\langle \hat{\sigma}_t(z_{\nu})\xi, \eta \rangle = \rho_{t,\xi,\eta}(z_{\nu}) \to \rho_{t,\xi,\eta}(z) = \langle \hat{\sigma}_t(z)\xi, \eta \rangle$$

so that $\hat{\sigma}_t(z_{\nu}) \to \hat{\sigma}_t(z)$ in the weak operator topology, which agrees with the weak topology of $\mathfrak{A}$ on bounded subsets.

(12) Since $\hat{\sigma}_t$ maps the unit ball of $\psi(\mathfrak{A})$ into $\psi(\mathfrak{A})$, it follows from the previous step and the Kaplansky density theorem that it maps the unit ball of $\mathfrak{A}$ into $\mathfrak{A}$. Hence $\hat{\sigma}_t$, initially defined as a map from $\mathfrak{A}$ into $B(\mathcal{H})$, is actually a self-map of $\mathfrak{A}$.

(13) Next, we prove that $\hat{\sigma}_t$ is a $^*$-endomorphism of $\mathfrak{A}$. Let $x, y \in \mathfrak{A}$ and choose bounded nets $\{x_{\nu}\}, \{y_{\nu}\} \subset \mathfrak{A}$ with $\psi(x_{\nu}) \to x$ and $\psi(y_{\nu}) \to y$ weakly. By the weak continuity of $\hat{\sigma}_t$ and its multiplicativity on $\psi(\mathfrak{A})$,

$$\hat{\sigma}_t(xy) = \lim_{\mu,\nu} \hat{\sigma}_t\left(\lim_{\mu,\nu} \psi(x_{\nu})\psi(y_{\mu})\right)$$

$$= \lim_{\mu,\nu} \hat{\sigma}_t\left(\psi(x_{\nu})\psi(y_{\mu})\right)$$

$$= \lim_{\mu,\nu} \hat{\sigma}_t\left(\psi(x_{\nu})\right)\hat{\sigma}_t\left(\psi(y_{\mu})\right)$$

$$= \hat{\sigma}_t(x)\hat{\sigma}_t(y).$$

Linearity and self-adjointness are proved similarly.

(14) Finally, it is clear that $\hat{\sigma}_0 = \text{id}_\mathfrak{A}$, and for all $s, t \geq 0$ and all $x \in \mathfrak{A}$,

$$\hat{\sigma}_{s+t}(\psi(x)) = \psi(\sigma_{s+t}(x)) = \psi(\sigma_s(\sigma_t(x))) = \hat{\sigma}_s(\psi(\sigma_t(x))) = \hat{\sigma}_s(\hat{\sigma}_t(\psi(x)))$$
so that \( \tilde{\sigma}_{s+t} \) and \( \tilde{\sigma}_t \circ \tilde{\sigma}_t \) agree on the ultraweakly dense subset \( \psi(A) \subseteq \tilde{A} \); as both are normal, they are equal.

\[ \square \]

As one corollary, we can now find many dense subspaces of \( \mathcal{H} \). Recall that \( \psi(\mathcal{P})VH \) is dense by the standard properties of the minimal Stinespring dilation plus the fact that \( \mathcal{P} \) is norm-dense in \( \mathfrak{A} \).

**Lemma 5.4.4.** For any finite set \( F \subseteq [0, \infty) \) let \( \mathcal{P}(F) \) denote those elements of \( \mathcal{P} \) which do not use any time indices from \( F \). Then for all finite \( F \subseteq [0, \infty) \), \( \psi(\mathcal{P}(F))VH \) is dense in \( \mathcal{H} \).

**Proof.** Consider a general vector of the form \( \sigma_{t_1}(i(a_1)) \cdots \sigma_{t_n}(i(a_n))Vh \), which we already know to be total in \( \mathcal{H} \). We proceed by induction on \( n \). In the case \( n = 1 \) we have for any \( \tilde{t} \) that

\[
\| (\sigma_{t_1}(i(a)) - \sigma_{\tilde{t}_1}(i(a))) VH \|^2 = \mathcal{S} \langle t; a^* a \rangle - \mathcal{S} \langle \tilde{t}; a^* a \rangle - \mathcal{S} \langle \tilde{t}; \tilde{t}^* a \rangle + \mathcal{S} \langle \tilde{t}; \tilde{t}^* a \rangle.
\]

As \( \tilde{t} \to t \), this approaches zero by the continuity properties of \( \mathcal{S} \). Inductively, we can approximate \( \sigma_{t_1}(i(a_2)) \cdots \sigma_{t_n}(i(a_n))Vh \) by a vector in \( \psi(\mathcal{P}(F))VH \), which we then use as our \( h \) and proceed as before. \( \square \)

Before establishing our main continuity result, one more preliminary is needed.

**Proposition 5.4.5.** The Hilbert space \( \mathcal{H} \) is separable.

**Proof.** Let \( H_0 \) be a countable dense subset of \( H \), and \( \mathcal{A}_0 \) a countable ultraweakly dense subset of \( \mathcal{A} \). We may assume WLOG that \( \mathcal{A}_0 \) is a self-adjoint \( \mathbb{Q} \)-subalgebra, so that its unit ball is strongly dense in the unit ball of \( \mathcal{A} \) by Kaplansky’s theorem.

We will show that the countable set

\[
\left\{ \psi(\sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n))) VH \mid 0 \leq t_1, \ldots, t_n \in \mathbb{Q}; x_1, \ldots, x_n \in \mathcal{A}_0; h \in H_0 \right\}
\]

spans a dense subset of \( \mathcal{H} \). We already know that \( \psi(\mathcal{P})VH \) has dense span, so it suffices to show that vectors in \( \psi(\mathcal{P})VH \) can be norm-approximated by vectors of the prescribed form. Let \( t_1, \ldots, t_n \geq 0, y_1, \ldots, y_n \in \mathcal{A} \), and \( k \in H \). By the triangle inequality, we have for any \( h \in H_0 \), any \( t_1, \ldots, t_n \in \mathbb{Q}_+ \), and any \( x_1, \ldots, x_n \in \mathcal{A}_0 \) that

\[
\psi(\sigma_{t_1}(i(y_1)) \cdots \sigma_{t_n}(i(y_n))) Vh - \psi(\sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n))) Vh \leq \| \psi(\sigma_{t_1}(i(y_1)) \cdots \sigma_{t_n}(i(y_n))) - \sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n)) \| \| h - k \|
\]

The first term can be made small by choosing \( h \) sufficiently close to \( k \). For the second, note that each composition \( \psi \circ \sigma_t \circ i \) is normal, since it equals the composition \( \tilde{\sigma}_1 \circ \psi \circ i \); hence \( \psi(\sigma_t(i(A_0))) \) is weakly dense in \( \psi(\sigma_t(i(A))) \). By Kaplansky’s theorem, it follows that the unit ball of \( \psi(\sigma_t(i(A_0))) \) is strongly dense in the unit ball of \( \psi(\sigma_t(i(A))) \); this plus the joint strong continuity of multiplication implies that

\[
\left\{ \psi(\sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n))) \mid s_1, \ldots, s_n \geq 0; x_1, \ldots, x_n \in \mathcal{A}_0 \right\}
\]

is strongly dense in

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\[ \left\{ \psi(\sigma_t(i(x_1)) \cdots \sigma_{t_n}(i(x_n))) \mid t_1, \ldots, t_n \geq 0; x_1, \ldots, x_n \in A \right\} \]. Hence, once \( h \) has been fixed, an appropriate choice of \( x_1, \ldots, x_n \) makes the second term arbitrarily small. So far we have shown that vectors of the form

\[
(5.10) \quad \psi(\sigma_t(i(x_1)) \cdots \sigma_{t_n}(i(x_n)))Vh \quad \tau_1, \ldots, \tau_n \geq 0; x_1, \ldots, x_n \in A_0; h \in H_0
\]

are total in \( \mathcal{F} \). It remains to prove that such vectors remain total under the added restriction that the \( \tau_i \) be rational. Let \( \xi \in \psi(\mathcal{P})VH \) be orthogonal to all vectors of the form \( (5.10) \). That is, we let \( z_1, \ldots, z_m \in A, \eta \in H, \) and \( s_1, \ldots, s_m \geq 0 \) such that, for all \( x_1, \ldots, x_n \in A_0, \) all \( 0 \leq t_1, \ldots, t_n \in \mathbb{Q} \), and all \( h \in H_0, \)

\[
0 = \langle \psi(\sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n)))Vh, \psi(\sigma_{s_1}(i(z_1)) \cdots \sigma_{s_m}(i(z_m)))V\eta \rangle
=

\[
= \langle V^*\psi(\sigma_{s_m}(i(z_m^*)) \cdots \sigma_{s_1}(i(z_1^*))\sigma_{t_1}(i(x_1)) \cdots \sigma_{t_n}(i(x_n)))Vh, \eta \rangle
=

\[
= \langle \pi(\mathcal{S}(\langle s^* \vee t; z^* \vee \overline{x} \rangle))h, \eta \rangle
\]

where we introduce the notation \((s_1, \ldots, s_m)^* = (s_m, \ldots, s_1)\) for \( s_1, \ldots, s_m \geq 0 \) and \((z_1, \ldots, z_m)^* = (z_m, \ldots, z_1)\) for \( z_1, \ldots, z_m \in A \). Now for any \( \overline{t} \in [0, \infty)^n \), let \( \{\overline{t}_k\} \subset \mathbb{Q}^n \) such that \( \overline{s}^* \vee \overline{t}_k \to \overline{s}^* \vee \overline{t} \) non-crossingly; then by Proposition \( 5.3.3 \),

\[
\langle \pi(\mathcal{S}(\langle s^* \vee \overline{t}_k; z^* \vee \overline{x} \rangle))h, \eta \rangle = \lim_{k \to \infty} \langle \pi(\mathcal{S}(\langle s^* \vee \overline{t}; z^* \vee \overline{x} \rangle))h, \eta \rangle = 0.
\]

We thus see that \( \xi \) must be orthogonal to a known total set and hence zero. \( \square \)

**Theorem 5.4.6.** For any \( a \in \mathfrak{A}, t \mapsto \overline{\sigma}_t(a) \) is ultraweakly continuous for all \( t > 0 \).

**Proof.** We establish this in a series of steps.

1. For any \( a \in A_0 \) and \( \xi, \eta \in \psi(\mathcal{P})VH \), the value of \( \langle \overline{\sigma}_t(\psi(a))\xi, \eta \rangle = \langle \psi(\sigma_t(a))\xi, \eta \rangle \) is given by a certain Sauvageot moment polynomial; explicitly, if

\[
a = \sigma_{s_1}(i(z_1)) \cdots \sigma_{s_m}(i(z_m)); \xi = \sigma_{s_1}(i(z_1)) \cdots \sigma_{s_m}(i(z_m))V\xi_0, \text{ and}
\]

\[
\eta = \sigma_{u_1}(i(z_1)) \cdots \sigma_{u_m}(i(z_m))V\eta_0, \text{ then}
\]

\[
\langle \overline{\sigma}_t(\psi(a))\xi, \eta \rangle = \langle \pi(\mathcal{S}(\langle u^* \vee (t + \overline{t}) \vee \overline{s}; z^* \vee \overline{x} \rangle))\xi_0, \eta_0 \rangle.
\]

2. Given \( t \) and a time \( t_0 \geq 0 \), let \( F \) be the set of times in \( t + t_0 \). Taking any \( \xi_0, \eta_0 \in \psi(\mathcal{P}(F))VH \), which is dense by Lemma \( 5.4.4 \), we see by Proposition \( 5.3.3 \) that the above expression is continuous at \( t_0 \), since if \( t \to t_0 \) within a sufficiently small neighborhood of \( t_0 \) then \( u^* \vee (t + t_0) \) non-crossingly. We therefore have that \( t \mapsto \langle \overline{\sigma}_t(\psi(a))\xi, \eta \rangle \) is continuous at \( t_0 \) for all \( \xi, \eta \in \psi(\mathcal{P}(F))VH \) and all \( a \in A_0 \).

3. By Lemma \( 5.4.1 \) this implies that \( t \mapsto \langle \overline{\sigma}_t(\psi(a))\xi, \eta \rangle \) is continuous at \( t_0 \) for all \( \xi, \eta \in \mathcal{F} \) and all \( a \in A_0 \).

4. Now let \( a \in \mathfrak{A} \). By Kaplansky density, there is a sequence \( \{a_n\} \subset A_0 \) such that \( \psi(a_n) \to a \) in \( \text{SOT} \). We can use a sequence rather than a net because the separability of \( \mathcal{F} \), established in Proposition \( 5.4.3 \) implies the SOT-metrizability of \( B(\mathcal{F}) \) \( \text{[Bla06] III.2.2.27} \). Then for any \( \xi, \eta \in \mathcal{F} \),

\[
\langle \overline{\sigma}_t(a)\xi, \eta \rangle = \lim_n \langle \overline{\sigma}_t(\psi(a_n))\xi, \eta \rangle
\]

so that the left-hand side, as a function of \( t \), is a pointwise limit of a sequence of continuous functions, hence measurable. That is, \( t \mapsto \overline{\sigma}_t(a) \) is WOT-measurable; as the \( \overline{\sigma} \) are contractions and the WOT agrees with
the ultraweak topology on bounded subsets, \( t \mapsto \tilde{\sigma}_t(a) \) is ultraweakly measurable at all \( t \geq 0 \).

(5) Since each \( \tilde{\sigma}_t \) is normal, there is a corresponding preadjoint semigroup \( \{\rho_t\} \) on \( \mathfrak{A}_* \) given by \( \rho_t f = f \circ \tilde{\sigma}_t \), as discussed in section 1.4.1 such that for each \( f \in \mathfrak{A}_* \), \( t \mapsto \rho_t(f) \) is weakly measurable at all \( t \geq 0 \).

(6) Since \( \mathfrak{H} \) is separable and \( \mathfrak{A} \subset B(\mathfrak{H}) \), it follows that \( \mathfrak{A}_* \) is a separable Banach space. By section 1.4.1, the weak measurability of \( \{\rho_t\} \) is therefore equivalent to its weak continuity at times \( t > 0 \). This is then equivalent to the ultraweak continuity of \( t \mapsto \tilde{\sigma}_t \).

\[ \square \]

Theorem 5.4.7. \((\mathfrak{A}, \psi \circ i, \mathfrak{E}, \{\tilde{\sigma}_t\})\) is a strong dilation of \((A, \{\phi_t\})\).

Proof. By the definition of \( \mathfrak{E} \), equation 5.3, and theorem 4.4.5,
\[
\mathfrak{E} \circ \tilde{\sigma}_t \circ \psi = \mathfrak{E} \circ \psi \circ \sigma_t = \mathfrak{E} \circ \sigma_t \\
= \phi_t \circ \mathfrak{E} = \phi_t \circ \mathfrak{E} \circ \psi.
\]
Since both \( \phi_t \circ \mathfrak{E} \) and \( \mathfrak{E} \circ \tilde{\sigma}_t \) are normal, and since they are equal on the ultraweakly dense subset \( \psi(\mathfrak{A}) \subset \mathfrak{A}_* \), they must be equal. \( \square \)

So far, theorem 5.4.6 leaves open the question whether \( \{\tilde{\sigma}_t\} \) is point-weakly continuous at \( t = 0 \). Without resolving that question, we can still remedy the situation by taking a suitable quotient.

Lemma 5.4.8. Let \( A \) be a separable \( W^* \)-algebra and \( \{\alpha_t\} \) an \( e_0 \)-semigroup on \( A \) which is point-weakly continuous at all \( t > 0 \). Then \( \alpha_t \) is point-weakly continuous at 0 iff
\[
\bigcap_{t > 0} \ker \alpha_t = \{0\}.
\]

Proof. The point-weak continuity of \( \alpha_t \) at \( t = 0 \) is equivalent to the weak (equivalently, strong) continuity at \( t = 0 \) of the preadjoint semigroup \( \{\rho_t\} \) on \( A_* \) defined by \( \rho_t f = f \circ \alpha_t \). As mentioned in section 1.4.1 this is equivalent to the condition
\[
\bigcup_{t > 0} \rho_t A_* = A_*
\]
since \( A_* \) is assumed separable. Now the annihilator of the left-hand side is
\[
\bigcup_{t > 0} \rho_t A_*^\perp = \{a \in A \mid \forall t > 0 : \forall f \in A_* : (\rho_t f)(a) = 0\} \\
= \{a \in A \mid \forall t > 0 : \forall f \in A_* : f(\alpha_t(a)) = 0\} \\
= \{a \in A \mid \forall t > 0 : \alpha_t(a) = 0\} \\
= \bigcap_{t > 0} \ker \alpha_t
\]
because \( A_* \) separates points on \( A \). \( \square \)
Theorem 5.4.9. Let $A$ be a separable $W^*$-algebra and $\{\phi_t\}$ a CP$_0$-semigroup on $A$. Then there exists a unital strong dilation of $\{\phi_t\}$ to an E$_0$-semigroup on a separable $W^*$-algebra.

Proof. The dilation $(\widehat{A}, \psi \circ i, \widehat{\sigma}_t)$ constructed in this chapter satisfies all the requirements except possibly point-ultraweak continuity at $t = 0$.

We now let

$$\mathcal{R} = \bigcap_{t > 0} \ker \sigma_t.$$  

This is an ultraweakly closed ideal in $\widehat{A}$; we use $\hat{A}$ for the quotient $\widehat{A}/\mathcal{R}$, which is another separable $W^*$-algebra. Because $\hat{\sigma}_t(\mathcal{R}) \subset \mathcal{R}$ for each $t > 0$, we obtain for each $t > 0$ a map $\hat{\sigma}_t : \hat{A} \to \hat{A}$ characterized by the commutative diagram

$$
\begin{array}{ccc}
\hat{A} & \xrightarrow{\hat{\sigma}_t} & \hat{A} \\
\downarrow & & \downarrow \\
A & \xrightarrow{\sigma_t} & A
\end{array}
$$

Defining also $\hat{\sigma}_0 = \text{id}_\hat{A}$, we see that $\{\hat{\sigma}_t\}$ inherits from $\{\sigma_t\}$ the properties of being an E$_0$-semigroup and of point-ultraweak continuity at $t > 0$. Furthermore, $\bigcap_{t > 0} \ker \hat{\sigma}_t = \{0\}$ by construction, so that $\{\hat{\sigma}_t\}$ is point-weakly continuous at $t = 0$ and hence is an E$_0$-semigroup. Our embedding of $A$ into $\hat{A}$ is given by $q \circ \psi \circ i$, where $\hat{A} \xrightarrow{q} \hat{A}$ is the quotient map; this is injective because, if $a \in A$ is such that $q(\psi(i(a))) = 0$, then $\psi(i(a)) \in \mathcal{R}$, so that for all $t > 0$ one has

$$
\begin{align*}
\hat{\sigma}_t(\psi(i(a))) &= 0 \\
\psi(\sigma_t(i(a))) &= 0 \\
\sigma_t(i(a)) &= 0 \\
E[\sigma_t(i(a))] &= 0 \\
\phi_t(E[i(a)]) &= 0 \\
\phi_t(a) &= 0
\end{align*}
$$

and since $\phi_t(a) \to a$ as $t \to 0^+$ this implies $a = 0$. To construct our retraction, we first note that $\mathcal{R} \subset \ker \hat{E}$; indeed, if $a \in \mathcal{R}$ then for all $t > 0$ we have

$$
\begin{align*}
\hat{\sigma}_t(a) &= 0 \\
\hat{E} \circ \hat{\sigma}_t(a) &= 0 \\
\phi_t \circ \hat{E}(a) &= 0
\end{align*}
$$

and by letting $t \to 0^+$ we conclude $\hat{E}(a) = 0$. Hence, $\ker q \subset \ker \hat{E}$, so there is a unique map $\hat{E} : \hat{A} \to A$ with $\hat{E} = \hat{E} q$. This map satisfies $\hat{E} \circ q \circ \psi \circ i = \hat{E} \circ \psi \circ i = \text{id}_A$, so it is a retraction with respect to the given embedding. Finally,

$$
\hat{E} \circ \hat{\sigma}_t \circ q = \hat{E} \circ q \circ \sigma_t = \hat{E} \circ \sigma_t = \phi_t \circ \hat{E} = \phi_t \circ \hat{E} \circ q,
$$

and since the image of $q$ generates $\hat{A}$ this implies $\hat{E} \circ \sigma_t = \phi_t \circ \hat{E}$. We therefore have a strong dilation of the original semigroup. \(\square\)
APPENDIX A

Table of Moments

Here \( w_\ell \) denotes the tuple \((b_0, a_1, b_1, \ldots, a_\ell, b_\ell)\). The function \( M_0 \) in the tables is related to \( \mathcal{M} \) by the relation \( \mathcal{M}(w_\ell) = b_0 M_0(w_\ell) b_\ell \). We also let

\[
\rho(a_{i,j}) = \rho(a_i a_j) - \rho(a_i) \rho(a_j)
\]
\[
\rho(a_{i,j,k}) = \rho(a_i a_j a_k) - \rho(a_i) \rho(a_j a_k) - \rho(a_i a_j) \rho(a_k) + \rho(a_i) \rho(a_j) \rho(a_k)
\]
\[
\rho(a_{i,j,k,l}) = \rho(a_i a_j a_k a_l) - \rho(a_i) \rho(a_j a_k a_l) - \rho(a_i a_j a_k) \rho(a_l) - \rho(a_i a_j a_k a_l) \rho(a_i) + \rho(a_i a_j) \rho(a_k) \rho(a_l)
\]
\[
+ \rho(a_i) \rho(a_j a_k) \rho(a_l) + \rho(a_i) \rho(a_j) \rho(a_k a_l) - \rho(a_i) \rho(a_j) \rho(a_k) \rho(a_l)
\]

General Case

\[
M_0(w_1) = \rho(a_1)
\]
\[
M_0(w_2) = \rho(a_1) b_1 \rho(a_2) + \rho(a_1 \psi(b_1) a_2) - \rho(a_1) \rho[a_1 \psi(b_1) a_2] - \rho[a_1 \psi(b_1)] \rho(a_2) + \rho(a_1) \rho[a_1 \psi(b_1)] \rho(a_2)
\]
\[
M_0(w_3) = \rho(a_1) b_1 \rho(a_2) b_2 \rho(a_3) + \rho(a_1) \rho[a_1 \psi(b_1) a_2] b_2 \rho(a_3) + \rho(a_1) b_1 \rho(a_2) \rho[a_1 \psi(b_2)] \rho(a_3)
\]
\[
\rho(a_{i,j,k,l}) + \rho(a_{i,j,k}) \rho(a_l) + \rho(a_{i,j}) \rho(a_k a_l) - \rho(a_{i,j}) \rho(a_k) \rho(a_l)
\]

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\[ M_{0}(u_3) = \cdots + \rho(a_1)\rho[\psi(b_1)]\rho[a_2\psi(b_2)a_3] + \rho(a_1)\rho[\psi(b_1)]\rho[\psi[a_2] \psi(b_2)a_3] \\
+ \rho(a_1)\rho[\psi(b_1)\psi[\rho(a_2)]]\rho[\psi(b_2)a_3] + \rho(a_1)\rho[\psi(b_1)a_2]\rho[\psi(b_2)a_3] \\
- \rho(a_1)\rho[\psi[\psi(b_1)\rho(a_2)]]\rho[\psi(b_2)a_3] - \rho(a_1)\rho[\psi[\psi(b_1)]\rho(\psi[\rho(a_2)b_2]a_3) \\
- \rho(a_1)\rho[\psi(b_1)]\rho[\psi[\rho(a_2)]]\rho(\psi(b_2)a_3] + \rho[a_1\psi(b_1)a_2]b_2\rho(a_3) \\
- \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) + \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) \\
- \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) + \rho[a_1\psi(b_1)\psi[\rho(a_2)b_2]]\rho(a_3) \\
+ \rho[a_1\psi(b_1)]\rho[a_2\psi(b_2)]\rho(a_3) - \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)b_2)]\rho(a_3) \\
- \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) + \rho[a_1\psi(b_1)\psi[\rho(a_2)]b_2)\rho(a_3) \\
+ \rho[a_1\psi(b_1)]\rho[\psi[a_2] \psi(b_2)]\rho(a_3) - \rho[a_1\psi(b_1)]\rho[\psi[a_2] \psi(b_2)]\rho(a_3) \\
+ \rho[a_1\psi(b_1)]\rho[a_2\psi(b_2)a_3] + \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] \\
- \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) - \rho[a_1\psi(b_1)]\rho[\rho(a_2)b_2]a_3) \\
+ \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) + \rho[a_1\psi(b_1)]\rho[\rho(a_2)b_2]a_3) \\
+ \rho(a_1\psi[b_1,\rho(a_2)]b_2)\rho(a_3) - \rho[a_1\psi(b_1)]\rho[\rho(a_2)b_2]a_3] \\
- \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] - \rho(a_1\psi[\psi(b_1)]\rho[\rho(a_2)]a_3] \\
- \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] + \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] \\
- \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] + \rho[a_1\psi(b_1)]\rho[\psi[\rho(a_2)] \psi(b_2)a_3] \]
Special Case: Scalar-Valued $\psi$

We list here values of the moment function in the special case where the map from $B$ to $A$ is scalar-valued. To reflect this extra assumption, we denote that map by $\nu$ rather than $\psi$. Then

\[
\mathcal{M}_0(w_1) = \rho(a_1)
\]

\[
\mathcal{M}_0(w_2) = \nu(b_1)\rho(a_{1,2}) + \rho(a_1)b_1\rho(a_2)
\]

\[
\mathcal{M}_0(w_3) = \rho(a_1)b_1\rho(a_2)b_2\rho(a_3) + \nu(b_1)\nu(b_2)\rho(a_{1,2,3}) + \nu(b_1)\rho(a_{1,2})b_2\rho(a_3) + \nu(b_2)\rho(a_1)b_1\rho(a_{2,3})
\]

\[
+ \left[\nu(b_1)\nu(b_2)\nu(b_3) - \nu(b_1)\nu(b_2)\nu(b_3) - \nu(b_1)\nu(b_2)\nu(b_3) + \nu(b_1)\nu(b_2)\right] \rho(a_{1,3})
\]

\[
\mathcal{M}_0(w_4) = \rho(a_1)b_1\rho(a_2)b_2\rho(a_3)b_3\rho(a_4) + \nu(b_1)\nu(b_2)\nu(b_3)\rho(a_{1,2,3,4})
\]

\[
+ \nu(b_1)\nu(b_2)\nu(b_3)\left[\nu(\rho(a_{2,3})) + 2\nu(\rho(a_2))\nu(\rho(a_3)) - \nu(\rho(a_2)\rho(a_3))\right] \rho(a_{1,4})
\]

\[
+ \nu(b_1)\nu(b_2)\nu(b_3)\rho(a_{1,3})b_3\rho(a_4) - \nu(b_1)\nu(b_2)\nu(b_3)\rho(a_1)b_2\rho(a_3)\rho(a_4)
\]

\[
+ \nu(b_1)\nu(b_3)\rho(a_{1,2})b_2\rho(a_3) - \nu(b_1)\nu(b_2)\nu(\rho(a_2)b_2)\rho(a_{1,3,4})
\]

\[
+ \nu(b_1)\nu(b_3)\rho(a_1)b_2\rho(a_3)\rho(a_4) - \nu(b_1)\nu(b_2)\rho(a_{1,3})b_3\rho(a_4)
\]

\[
+ \nu(b_1)\nu(b_2)\rho(a_1)b_3\rho(a_4) - \nu(b_1)\nu(b_2)\rho(a_{1,2})b_3\rho(a_4)
\]

\[
- \nu(b_1)\left[\nu(\rho(a_2)b_2)\nu(b_3) - \nu(\rho(a_2)b_2)\nu(b_3) - \nu(\rho(a_2)b_2)\nu(b_3)\right] \rho(a_{1,4})
\]

\[
+ \nu(b_2)\rho(a_1)b_2\rho(a_{2,3})b_3\rho(a_4) - \nu(b_2)\nu(b_1)\rho(a_2)\rho(a_{1,3})b_3\rho(a_4)
\]

\[
- \nu(b_2)\rho(a_1)b_1\rho(a_{2,3})b_3\rho(a_4) - \nu(b_2)\nu(b_1)\rho(a_2)\rho(a_{1,3})b_3\rho(a_4)
\]

\[
- \nu(\rho(a_1)b_2)\rho(a_{1,2})b_3\rho(a_4) - \nu(\rho(a_1)b_2)\rho(a_{1,2})b_3\rho(a_4)
\]

\[
- \nu(b_3)\left[\nu(b_1)\rho(a_2)\rho(a_{2,3}) - \nu(b_1)\rho(a_2)\rho(a_{2,3}) - \nu(b_1)\rho(a_2)\rho(a_{2,3})\right] \rho(a_{1,4})
\]

\[
+ \nu(b_1)\nu(\rho(a_2)b_2)\rho(a_{2,3})b_3\rho(a_4) - \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4)
\]

\[
+ \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4) - \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4)
\]

\[
- \nu(b_3)\left[\nu(b_1)\rho(a_2)\rho(a_{2,3}) - \nu(b_1)\rho(a_2)\rho(a_{2,3}) - \nu(b_1)\rho(a_2)\rho(a_{2,3})\right] \rho(a_{1,4})
\]

\[
+ \nu(b_1)\nu(\rho(a_2)b_2)\rho(a_{2,3})b_3\rho(a_4) - \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4)
\]

\[
+ \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4) - \nu(b_1)\rho(a_2)\rho(a_{2,3})b_3\rho(a_4)
\]
APPENDIX B

Table of Moment Polynomials

For the sake of brevity, we use $1, 2, 3$ to denote $t_1, t_2, t_3$, with the standing assumption that $0 < t_1 < t_2 < t_3$, and omit listing $a_1, \ldots, a_n$; hence $S(1, 0, 3, 2)$ is an abbreviation for $S(t_1, 0, t_3, t_2; a_1, a_2, a_3, a_4)$, and $\phi_{2-1}$ for $\phi_{t_2-t_1}$. After the first few, we omit polynomials in which $0$ appears as the first or last index, since the bimodule property easily reduces these to others, viz.

\[
S(0, s_1, \ldots, s_k; a_0, a_1, \ldots, a_k) = a_0 \phi_\tau \left( S(s_1 - \tau, \ldots, s_k - \tau; a_1, \ldots, a_k) \right)
\]

\[
S(s_1, \ldots, s_k, 0; a_1, \ldots, a_k, a_{k+1}) = \phi_\tau \left( S(s_1 - \tau, \ldots, s_k - \tau; a_1, \ldots, a_k) \right) a_{k+1}
\]

where $\tau = \min(s_1, \ldots, s_k)$. We also omit polynomials with consecutive time indices equal, since these can be reduced by multiplying consecutive terms with the same time index; for instance,

\[
S(t_1, t_1, t_2, t_3; a_1, a_2, a_3, a_4, a_5) = S(t_1, t_2, t_3; a_1 a_2, a_3, a_4 a_5).
\]

Then

\[
S(0) = a_1
\]

\[
S(0, 1) = a_1 \phi_1(a_2)
\]

\[
S(1, 0) = \phi_1(a_1) a_2
\]

\[
S(0, 1, 0) = a_1 \phi_1(a_2) a_3
\]

\[
S(1, 0, 1) = \phi_1(a_1) a_2 \phi_1(a_3) + \omega(a_2) [\phi_1(a_1 a_3) - \phi_1(a_1) \phi_1(a_3)]
\]

\[
S(0, 1, 2) = a_1 \phi_1(a_2) \phi_{2-1}(a_3)
\]

\[
S(0, 2, 1) = a_1 \phi_1(\phi_{2-1}(a_2) a_3)
\]

\[
S(1, 0, 2) = \phi_1(a_1) a_2 \phi_2(a_3) + \omega(a_2) [\phi_1(a_1) \phi_{2-1}(a_3) - \phi_1(a_1) \phi_2(a_3)]
\]

\[
S(1, 2, 0) = \phi_1(a_1) \phi_{2-1}(a_2) a_3
\]

\[
S(2, 0, 1) = \phi_2(a_1) a_2 \phi_1(a_3) + \omega(a_2) [\phi_2(a_1) \phi_{2-1}(a_3) - \phi_2(a_1) \phi_1(a_3)]
\]

\[
S(2, 1, 0) = \phi_1(\phi_{2-1}(a_1) a_2) a_3
\]

\[
S(1, 0, 1, 2) = \phi_1(a_1) a_2 \phi_1(a_3) \phi_{3-1}(a_4) + \omega(a_2) [\phi_1(a_1) a_3 \phi_{2-1}(a_4) - \phi_1(a_1) \phi_1(a_3) \phi_{2-1}(a_4)]
\]

\[
S(1, 0, 2, 1) = \phi_1(a_1) a_2 \phi_1(\phi_{2-1}(a_3) a_4) + \omega(a_2) [\phi_1(a_1) \phi_{2-1}(a_3) a_4 - \phi_1(a_1) \phi_1(\phi_{2-1}(a_3) a_4)]
\]

\[
S(1, 2, 0, 1) = \phi_1(a_1) \phi_{2-1}(a_2) a_3 \phi_1(a_4) + \omega(a_3) [\phi_1(a_1) \phi_{2-1}(a_2) a_4 - \phi_1(a_1) \phi_{2-1}(a_2) \phi_1(a_4)]
\]

\[
S(2, 1, 0, 1) = \phi_1(\phi_{2-1}(a_1) a_2) a_3 \phi_1(a_4) + \omega(a_3) [\phi_1(\phi_{2-1}(a_1) a_2) a_4 - \phi_1(\phi_{2-1}(a_1) a_2) \phi_1(a_4)]
\]

\[
S(1, 2, 0, 2) = \phi_1(a_1) \phi_{2-1}(a_2) a_3 \phi_2(a_4) + \omega(a_3) [\phi_1(a_1) \phi_{2-1}(a_2) a_4 - \phi_1(a_1) \phi_{2-1}(a_2) \phi_2(a_4)]
\]

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\( \mathcal{S}(2, 0, 1, 2) = \phi_2(a_1) a_2 \phi_1(a_3 \phi_{2-1}(a_4)) + \omega(a_2) \left[ \phi_1(\phi_{2-1}(a_1)a_3 \phi_{2-1}(a_4)) \right] \\
\quad - \phi_2(a_1) \phi_1(a_3 \phi_{2-1}(a_4)) + \omega(a_2) \omega(a_3) \left[ \phi_2(a_1 a_4) - \phi_2(a_1) \phi_2(a_4) \right] \\
\mathcal{S}(2, 0, 2, 1) = \phi_2(a_1) a_2 \phi_1(\phi_{2-1}(a_1)a_4) + \omega(a_2) \left[ \phi_1(\phi_{2-1}(a_1)a_3 \phi_{2-1}(a_4)) - \phi_2(a_1) \phi_1(\phi_{2-1}(a_3)a_4) \right] \\
\mathcal{S}(2, 1, 0, 2) = \phi_1(\phi_{2-1}(a_1)a_2) a_3 \phi_2(a_4) + \omega(a_3) \left[ \phi_1(\phi_{2-1}(a_1)a_2 \phi_{2-1}(a_4)) \right] \\
\quad - \phi_1(\phi_{2-1}(a_1)a_2) \phi_2(a_4) + \omega(a_2) \omega(a_3) \left[ \phi_2(a_1 a_4) - \phi_2(a_1) \phi_2(a_4) \right] \\
\mathcal{S}(1, 0, 2, 3) = \phi_1(1) a_2 \phi_2(a_3 \phi_{3-2}(a_4)) + \omega(a_2) \left[ \phi_1(a_1 \phi_{2-1}(a_3 \phi_{3-2}(a_4))) - \phi_1(a_1) \phi_2(a_3 \phi_{3-2}(a_4)) \right] \\
\mathcal{S}(1, 0, 3, 2) = \phi_1(1) a_2 \phi_2(\phi_{3-2}(a_3)a_4) + \omega(a_2) \left[ \phi_1(a_1 \phi_{2-1}(a_3 \phi_{3-2}(a_4))) - \phi_1(a_1) \phi_2(a_3 \phi_{3-2}(a_4)) \right] \\
\mathcal{S}(1, 2, 0, 3) = \phi_1(a_1 \phi_{2-1}(a_2)) a_3 \phi_3(a_4) + \omega(a_3) \left[ \phi_1(a_1 \phi_{2-1}(a_2 \phi_{3-2}(a_4))) - \phi_1(a_1 \phi_{2-1}(a_2)) \phi_3(a_4) \right] \\
\mathcal{S}(1, 3, 0, 2) = \phi_1(a_1 \phi_{2-1}(a_3)) a_3 \phi_2(a_4) + \omega(a_3) \left[ \phi_1(a_1 \phi_{2-1}(a_3 \phi_{3-2}(a_4))) - \phi_1(a_1 \phi_{2-1}(a_3)) \phi_2(a_4) \right] \\
\mathcal{S}(2, 0, 1, 3) = \phi_2(a_1) a_2 \phi_1(a_3 \phi_{3-1}(a_4)) + \omega(a_2) \left[ \phi_1(\phi_{2-1}(a_1)a_3 \phi_{3-1}(a_4)) \right] \\
\quad - \phi_2(a_1) \phi_1(a_3 \phi_{3-1}(a_4)) + \omega(a_2) \omega(a_3) \left[ \phi_2(a_1 \phi_{3-2}(a_4)) - \phi_2(a_1) \phi_3(a_4) \right] \\
\mathcal{S}(2, 0, 3, 1) = \phi_2(a_1) a_2 \phi_1(a_3 \phi_{3-1}(a_4)) + \omega(a_2) \left[ \phi_1(\phi_{2-1}(a_1 \phi_{3-2}(a_4))) \right] \\
\quad - \phi_2(a_1) \phi_1(a_3 \phi_{3-1}(a_4)) + \omega(a_2) \omega(a_3) \left[ \phi_2(a_1 \phi_{3-2}(a_4)) - \phi_2(a_1) \phi_3(a_4) \right] \\
\mathcal{S}(2, 1, 0, 3) = \phi_1(\phi_{2-1}(a_1)a_2) a_3 \phi_3(a_4) + \omega(a_3) \left[ \phi_1(\phi_{2-1}(a_1)a_2 \phi_{3-1}(a_2)) \right] \\
\quad - \phi_1(\phi_{2-1}(a_1)a_2) \phi_3(a_4) + \omega(a_2) \omega(a_3) \left[ \phi_2(a_1 \phi_{3-2}(a_4)) - \phi_2(a_1) \phi_3(a_4) \right] \\
\mathcal{S}(2, 3, 0, 1) = \phi_2(a_1 \phi_{3-2}(a_2)) a_3 \phi_1(a_4) + \omega(a_3) \left[ \phi_1(\phi_{2-1}(a_1 \phi_{3-2}(a_3))) \right] \\
\quad - \phi_2(a_1 \phi_{3-2}(a_2)) \phi_1(a_4) + \omega(a_2) \phi_1(\phi_{3-2}(a_2)) \phi_1(a_4) \\
\mathcal{S}(3, 0, 1, 2) = \phi_3(a_1) \phi_1(a - 3 \phi_{2-1}(a_4)) + \omega(a_2) \left[ \phi_1(\phi_{3-1}(a_1)a_3 \phi_{2-1}(a_4)) \right] \\
\quad - \phi_3(a_1) \phi_1(a_3 \phi_{2-1}(a_4)) + \omega(a_2) \omega(a_3) \left[ \phi_2(\phi_{3-2}(a_1)a_4) - \phi_3(a_1) \phi_2(a_4) \right] \\
\mathcal{S}(3, 0, 2, 1) = \phi_3(a_1) a_2 \phi_1(\phi_{2-1}(a_3)a_4) + \omega(a_2) \left[ \phi_1(\phi_{2-1}(a_3)a_3 \phi_{2-1}(a_4)) \right] \\
\quad - \phi_3(a_1) \phi_1(\phi_{2-1}(a_3)a_4) + \omega(a_2) \omega(a_3) \left[ \phi_2(\phi_{3-2}(a_1)a_4) - \phi_3(a_1) \phi_2(a_4) \right] \\
\mathcal{S}(3, 1, 0, 2) = \phi_1(\phi_{3-1}(a_1)a_2) a_3 \phi_2(a_4) + \omega(a_3) \left[ \phi_1(\phi_{3-1}(a_1)a_2 \phi_{2-1}(a_4)) \right] \\
\quad - \phi_1(\phi_{3-1}(a_1)a_2) \phi_2(a_4) + \omega(a_2) \omega(a_3) \left[ \phi_2(\phi_{3-2}(a_1)a_4) - \phi_3(a_1) \phi_2(a_4) \right] \\
\mathcal{S}(3, 2, 0, 1) = \phi_2(\phi_{3-2}(a_1)a_2) a_3 \phi_1(a_4) + \omega(a_3) \left[ \phi_1(\phi_{3-2}(a_1)a_2 \phi_{3-2}(a_4)) \right] \\
\quad - \phi_2(\phi_{3-2}(a_1)a_2) \phi_1(a_4) + \omega(a_3) \phi_1(\phi_{3-2}(a_1)a_2) \phi_1(a_4)
To illustrate possible discontinuity in the time parameters, we consider the following for $0 < \tau < t_1 < t_2 < t_3$:

\[
\mathcal{G}(t_1, \tau, t_3, 0, t_2) = \phi_\tau (\phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3))a_4\phi_{t_2}(a_5)
+ \omega(a_2)\phi_\tau \left( \phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3) \right) - \phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3) \right)a_4\phi_{t_2}(a_5)
+ \omega(a_4)\phi_\tau \left( \phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3) \right)
+ \omega(a_2)\omega(a_4)\phi_\tau \left[ \phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3) \right] - \phi_{t_1-\tau}(a_1)a_2\phi_{t_3-\tau}(a_3) \right) \phi_{t_2}(a_5)
\]

\[
\mathcal{G}(t_1, 0, t_3, 0, t_2) = \phi_{t_1}(a_1)a_2\phi_{t_3}(a_3)a_4\phi_{t_2}(a_5)
+ \omega(a_2)\omega(a_4) \left[ \phi_{t_1}(a_1)a_2\phi_{t_3-\tau}(a_3) \right] - \phi_{t_1}(a_1)a_2\phi_{t_3-\tau}(a_3) \right) a_4\phi_{t_2}(a_5)
- \phi_{t_1}(a_1)a_2\phi_{t_3-\tau}(a_3) \right) \phi_{t_2}(a_5)
+ \omega(a_4)\phi_{t_1}(a_1)a_2 \left[ \phi_{t_2}(a_1)a_2\phi_{t_3-\tau}(a_3) \right] - \phi_{t_1}(a_1)a_2\phi_{t_3-\tau}(a_3) \right)
+ \omega(a_2)\omega(a_3)\omega(a_4) - \omega(a_2)\omega(a_3)a_4 - \omega(a_2)\phi_{t_3}(a_3) \right) \omega(a_4)
+ \omega(a_2)\phi_{t_3}(a_3) \right) \left[ \phi_{t_1}(a_1)a_2\phi_{t_3}(a_3) \right] - \phi_{t_1}(a_1)a_2\phi_{t_3}(a_3) \right)
\]

\[
\lim_{\tau \to 0^+} \mathcal{G}(t_1, \tau, t_3, 0, t_2) = \left[ \omega(a_2)\omega(a_3)\omega(a_4) - \omega(a_2)\omega(a_3)a_4 - \omega(a_2)\phi_{t_3}(a_3) \right] \omega(a_4)
+ \omega(a_2)\phi_{t_3}(a_3) \right] \phi_{t_1}(a_1)a_2\phi_{t_3}(a_3) - \phi_{t_1}(a_1)a_2\phi_{t_3}(a_3) \right]
\]
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