VIRTUAL EXTENSIONS

Sérgio F. Cortizo

Instituto de Matemática e Estatística, Universidade de São Paulo
Cidade Universitária, Rua do Matão, 1010
05508-900, São Paulo, SP, Brasil
cortizo@ime.usp.br

Abstract

A process of extending sets which can be used as foundation for an alternative organization for Differential and Integral Calculus is presented.
PACS 02.90.+p
I. Introduction

Our goal is to present an extension process that can be applied to any set. It is a relatively simple construction, which can be developed strictly inside the limits of Elementary Set Theory.

When applied to the ordered field $\mathbb{R}$ of real numbers, this process introduces infinitesimal and infinite quantities, which can be used in an alternative organization of Differential and Integral Calculus. This application was our original motivation, and will be presented in a subsequent work.

In Secs. II, III and IV we define concepts occurring in the statement of the fundamental result of this work: the theorem presented in Sec. V. In the remainder sections we discuss that result.

II. Extension of Sets and Subsets

Let $A$ be any set. We will denote the set of all infinite sequences of elements of $A$ by $\Sigma(A)$:

$$\Sigma(A) = \{(a_1, a_2, a_3, \ldots) \mid a_i \in A, \ i \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.

We will introduce now a relation on the set $\Sigma(A)$: we will say that two sequences $(a_i) = (a_1, a_2, \ldots) \in \Sigma(A)$ and $(b_i) = (b_1, b_2, \ldots) \in \Sigma(A)$ $(a_i)$ and $(b_i)$ end equals when there exists $n \in \mathbb{N}$ such that $i > n$ implies $a_i = b_i$. This is an equivalence relation on $\Sigma(A)$, which we will represent by $\equiv$. The equivalence class of $(a_i) \in \Sigma(A)$ will be denoted by $\langle a_i \rangle$, so $\langle a_i \rangle = \langle b_i \rangle$ if and only if $(a_i) \equiv (b_i)$.

The quotient $\overline{A} = \Sigma(A)/\equiv$ will be called virtual extension of set $A$, or simply extension of $A$. In other words, the members of $\overline{A}$ are the equivalence classes modulo $\equiv$:

$$\overline{A} = \{\langle a_i \rangle \mid (a_i) \in \Sigma(A)\}.$$

Let now $B \subset A$ be any subset of $A$. We will say that a sequence $(a_i) \in \Sigma(A)$ ends in $B$ when $a_i \in B$ after a certain value for the index, i.e., when there exists $n \in \mathbb{N}$ such that $a_i \in B$ for all $i > n$. It is clear that if a sequence ends in $B$ then any other equivalent sequence (by $\equiv$) will also end in $B$. So, we can define the subset $\overline{B} \subset \overline{A}$ of all classes $\langle a_i \rangle \in \overline{A}$ whose representatives sequences end in $B$:

$$\overline{B} = \{\langle a_i \rangle \in \overline{A} \mid (a_i) \text{ ends in } B \subset A\}.$$
Example: we will call virtual numbers, or just virtuals, the members of the extension \( \overline{\mathbb{R}} \) of the real numbers set \( \mathbb{R} \). Since \( \mathbb{Z} \) is a subset of \( \mathbb{R} \), we have the virtual extension \( \mathbb{Z} \subset \overline{\mathbb{R}} \), whose elements will be called virtual integers. The members of \( \mathbb{Z} \) are represented by sequences that assume, after a certain value of the index, only integer values. An example of virtual integer is the class of the sequence \( (1, 2, 3, \ldots) \in \Sigma(\mathbb{R}) \), which will be denoted simply by \( \infty \in \overline{\mathbb{Z}} \).

For any \( a \in A \), we will represent by \( \overline{\pi} \in \overline{A} \) the equivalence class of the sequence \( (a, a, a, \ldots) \in \Sigma(A) \) constant at \( a \in A \). Besides, for any subset \( B \subset A \), we will denote by \( K(B) \subset \overline{A} \) the class of all constant sequences in \( B \):

\[
K(B) = \{ \overline{b} = \langle b, b, b, \ldots \rangle \in \overline{A} \mid b \in B \}.
\]

It is easy to see that:

**II.1** For any subset \( B \subset A \), we have \( K(B) = K(A) \cap \overline{B} \).

**II.2** For any subset \( B \subset A \), we have:

(i) \( \overline{B} = \overline{A} \) if and only if \( B = A \);

(ii) \( \overline{B} = \emptyset \) if and only if \( B = \emptyset \);

(iii) \( \overline{B} \) is unitary if and only if \( B \) is unitary.

**II.3** If \( B \) and \( C \) are two subsets of \( A \) then:

(i) \( \overline{B} \subset \overline{C} \) if and only if \( B \subset C \);

(ii) \( \overline{B} = \overline{C} \) if and only if \( B = C \).

**III. Relations and Functions**

The objective of this section is to establish terminology and notation. Many definitions below are universal, but not all of them.

Let \( E \) be any set. We will identify a relation between \( n \) variables \( x_i \in E \) \( (i = 1, \ldots, n) \) with the class of \( n \)-tuples \( (x_1, \ldots, x_n) \in E^n \) which satisfy that relation, i.e., we are considering an \( n \)-ary relation on \( E \) as a subset \( P \subset E^n \) of the Cartesian product \( E^n \) of \( n \) copies of \( E \). For example, the subset \( \text{eq}_E \subset E \times E \) below is the equality relation on \( E \):

\[
\text{eq}_E = \{ (x, y) \in E \times E \mid x = y \}.
\]
We will often write
\[ P(x_1, x_2, \ldots, x_n) \]
instead of
\[ (x_1, x_2, \ldots, x_n) \in P \]
when a set \( P \subset E^n \) is being interpreted as an \( n \)-ary relation. We will also abbreviate the \( n \)-tuple \( '(x_1, x_2, \ldots, x_n)' \) to \( 'x' \), writing simply \( 'P(x)' \).

For every \( n \)-ary relation \( P \subset E^n \), we will denote its negation by \( (\text{not } P) \subset E^n \), i.e.:

(i) \( (\text{not } P) = \{ x \in E^n \mid x \notin P \} \).

Moreover, for every pair of \( n \)-ary relations \( P \subset E^n \) and \( Q \subset E^n \) (the same \( n \in \mathbb{N} \)), we will use the following notation:

(ii) \( (P \text{ and } Q) = P \cap Q \);

(iii) \( (P \text{ or } Q) = P \cup Q \);

(iv) \( (P \Rightarrow Q) = [(\text{not } P) \text{ or } Q] \);

(v) \( (P \Leftrightarrow Q) = [(P \Rightarrow Q) \text{ and } (Q \Rightarrow P)] \).

Given a \((k + n)\)-ary relation \( P \subset E^{k+n} \), with \( n \geq 1 \), we can fix its \( k \) first entries, leaving the remaining ones free, and thus construct an \( n \)-ary relation between elements of the same set \( E \). If \( a \in E^k \), then we define:

\[ Pa = \{ x \in E^n \mid P(a, x) \} . \]

It is clear that \( Pa \subset E^n \) is an \( n \)-ary relation on the same set \( E \). With this definition, the condition \( 'Pa(x)' \) is equivalent to \( 'P(a, x)' \).

We will use the symbol \( '\forall' \) as abbreviation of ‘for every’ (universal logic quantifier), the symbol \( '\exists' \) will mean ‘there exists’ (existential quantifier), and \( '\exists! ' \) will be an abbreviation for ‘there exists one and only one’. For example, the statement:

there exists \( k \in \mathbb{N} \) such that, for all \( i > k \), we have \( P(x_i) \)

will be shortened to:

\[ \exists k \in \mathbb{N}, \forall i > k, \ P(x_i). \]

Furthermore, those three symbols will be used to indicate the relations constructed quantifying the first entries of a given relation, according to the following definitions.
If $P \subset E^{k+n}$ is a $(k+n)$-ary relation ($n \geq 1$), and $D \subset E^k$ a subset of $E^k$, then we define:

(i) $(\forall D, P) = \{ x \in E^n \mid \forall y \in D, P(y, x) \}$;

(ii) $(\exists D, P) = \{ x \in E^n \mid \exists y \in D, P(y, x) \}$;

(iii) $(\exists! D, P) = \{ x \in E^n \mid \exists! y \in D, P(y, x) \}$.

Obviously, $(\forall D, P) \subset E^n$ and $(\exists D, P) \subset E^n$, as well as $(\exists! D, P) \subset E^n$, are $n$-ary relations on the same set $E$, for which we have:

(i) $(\forall D, P)(x)$ if and only if $[\forall y \in D, P(y, x)]$;

(ii) $(\exists D, P)(x)$ if and only if $[\exists y \in D, P(y, x)]$;

(iii) $(\exists! D, P)(x)$ if and only if $[\exists! y \in D, P(y, x)]$.

Let $P \subset E^{n+k}$ be any $(n+k)$-ary relation. We will say that $P$ is a functional relation from $E^n$ to $E^k$ if, for every $x \in E^n$, there exists at most one $y \in E^k$ such that $P(x, y)$. Besides, given $D \subset E^n$ and $C \subset E^k$, we will say that a functional relation $P \subset E^{n+k}$ is a function from $D$ into $C$ when, for every $x \in D$, there exists $y \in C$ such that $P(x, y)$. As usual, we will indicate that $f$ is a function from $D$ into $C$ by writing:

$$f : D \rightarrow C.$$ 

We will say that two functions $f : D_f \rightarrow C_f$ and $g : D_g \rightarrow C_g$ form a chain when $C_f \subset D_g$; in this case we define the composite function $(g \circ f) : D_f \rightarrow C_g$ by $(g \circ f)(x) = g[f(x)]$, for every $x \in D_f$. We will represent the identity function on any set $D$ by $\text{id}_D$, i.e., $(\text{id}_D)(x) = x$, for all $x \in D$.

The composition operation can also be done between a relation and a function: if $P \subset E^k$ and $f : D \rightarrow C$, with $D \subset E^n$ and $C \subset E^k$, then $(P \circ f) \subset E^n$ is the $n$-ary relation defined by:

$$(P \circ f)(x) \text{ if and only if } x \in D \text{ and } P[f(x)].$$

If $f_i : D \rightarrow C$ are functions defined from the same domain into the same set ($i = 1, \ldots, n$), then we will denote the “aggregation” of the $n$ functions $f_i$ by

$$(f_1, \ldots, f_n) : D \rightarrow C^n,$$

which means:

$$(f_1, \ldots, f_n)(x) = (f_1(x), \ldots, f_n(x)).$$
In addition, for every subset $D \subset E^n$, we will represent the function that simply selects its $i$-th entry by

$$\pi_D^i : D \to E.$$ 

That means:

$$\pi_D^i(x_1, \ldots, x_n) = x_i \quad (i = 1, \ldots, n).$$

IV. Extension of Relations and Functions

Let $U$ be a fixed non-empty set. We will now apply the virtual extension process defined in Sec. II simultaneously to $U$ and to the product $U^n$, thus obtaining the sets $\overline{U}$ and $\overline{U^n}$ respectively. In principle, the extension $\overline{U^n}$ of the product is different from the product $\overline{U}^n$ of the extensions, but in this work we will adopt a practically universal identification in mathematics: a sequence of members of the product $U^n$ is the same as the corresponding $n$-tuple of sequences in $U$. In other words, we will consider $\Sigma(U^n) = [\Sigma(U)]^n$. Thus, it is easily seen that two sequences in the product will end equal (Sec. II) if and only if its $n$ component sequences end equal, for we are working with finite products only. We then have $\overline{U^n} = \overline{U}^n$, for every $n \in \mathbb{N}$.

Let now $P \subset U^n$ be a generic relation between $n$ variables $x_i \in U \ (i = 1, \ldots, n)$. According to the extension process of subsets defined in Sec. II, the virtual extension of the set $P \subset U^n$ is a subset of $\overline{P} \subset \overline{U^n}$. With the identification above, this extended subset $\overline{P} \subset \overline{U}^n = \overline{U^n}$ defines a new relation between $n$ variables $\xi_i$ which range over the extended set $\overline{U} \ (i = 1, \ldots, n)$. We will call $\overline{P}$ the virtual extension of relation $P$, or simply the extension of $P$.

For instance, let ‘$<$’ be the ordering relation between real numbers. According to the construction above, we have defined a binary relation ‘$\prec$’ between virtual numbers. Thinking about the members of $\overline{R}$ as classes of sequences, we have $\langle a_i \rangle \prec \langle b_i \rangle$ if and only if there exists $n \in \mathbb{N}$ such that $a_i < b_i$ for every $i > n$. Then, $\overline{x} \prec \infty$ for every $x \in R$.

It is important to note that the virtual extension of $P \subset U^n$ does not depend on the interpretation of $P$ as a subset or a relation. In other words, if

$$A = \{x \in U^n \mid P(x)\}$$

then

$$\overline{A} = \{\xi \in U^n \mid \overline{P}(\xi)\}.$$
For example, if $\mathbb{R}^*_+ \subset \mathbb{R}$ is the set of real non-negative numbers:

$$\mathbb{R}^*_+ = \{ x \in \mathbb{R} \mid x > 0 \},$$

then

$$\overline{\mathbb{R}^*_+} = \{ \xi \in \mathbb{R} \mid \xi > 0 \}.$$  

It is easy to see that $P \subset U^{n+k}$ is a functional relation if and only if its virtual extension $\overline{P} \subset \overline{U}^{n+k}$ is a functional relation with respect to the same entries. Thus, if $f \subset U^{n+k}$ is a function from $D \subset U^n$ into $C \subset U^k$:

$$f: D \rightarrow C,$$

then its virtual extension $\overline{f} \subset \overline{U}^{n+k}$ is a function from the extension $\overline{D} \subset \overline{U}^n$ of the domain of $f$ into the extension $\overline{C} \subset \overline{U}^k$ of the codomain of $f$, i.e.:

$$\overline{f}: \overline{D} \rightarrow \overline{C}.$$  

If $\langle x_i \rangle \in \overline{D} \subset \overline{U}^n$ then the sequence $(x_i) \in \Sigma(U^n)$ ends in $D \subset U^n$. Hence, we can evaluate $f(x_i)$ for every $i$ greater than certain $k \in \mathbb{N}$. It is not difficult to verify that $\overline{f}(\langle x_i \rangle) \in \overline{C} \subset \overline{U}^k$ is the class of sequences $(y_i) \in \Sigma(U^k)$ for which there exists $l > k$ such that $y_i = f(x_i)$ for every $i > l$.

Examples: for $U = \mathbb{R}$, we have the operations $+ : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\times : \mathbb{R}^2 \rightarrow \mathbb{R}$, whose extensions are:

$$\overline{+} : \overline{\mathbb{R}}^2 \rightarrow \overline{\mathbb{R}} \quad \text{and} \quad \overline{\times} : \overline{\mathbb{R}}^2 \rightarrow \overline{\mathbb{R}}.$$  

If $\alpha \in \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$ are two generic virtual numbers, we can calculate the sum $\alpha + \beta \in \overline{\mathbb{R}}$ and the product $\alpha \times \beta \in \overline{\mathbb{R}}$. So, the symbols ‘$\infty + 1$’ and ‘$\infty \times (\infty + 1)$’ represent two well defined virtual numbers. Moreover, we have the function $\ln : \mathbb{R}^*_+ \rightarrow \mathbb{R}$, whose extension is:

$$\overline{\ln} : \overline{\mathbb{R}}^*_+ \rightarrow \overline{\mathbb{R}}.$$  

Since $\infty \not> 0$, there exists $\overline{\ln} \infty \in \overline{\mathbb{R}}$.

We can apply the logical operations defined in Sec. III both to relations on $U$ and to its extensions on $\overline{U}$. Thus, $\overline{P \lor Q} \subset \overline{U}^n$ is the virtual extension of relation $(P \lor Q) \subset U^n$, while $(\overline{P} \lor \overline{Q}) \subset \overline{U}^n$ is the logical disjunction of the extensions $\overline{P} \subset \overline{U}^n$ and $\overline{Q} \subset \overline{U}^n$.

The following section is a theorem which states the “commutation rules” between the virtual extension process and the operations on relations and functions defined in the last section.
V. The Virtual Extension Theorem

For any set $U$, and for every $n$-ary relation $P \subset U^n$, the condition ‘$P(x)$’ is equivalent to ‘$\overline{P}(\overline{x})$’, i.e., the statement ‘$P(x)$’ is true for certain $x \in U^n$ if and only if ‘$\overline{P}(\overline{x})$’ is true for $\overline{x} \in K(U^n) \subset \overline{U^n}$.

We also have:

(i) $\overline{\text{eq}_U} = \text{eq}_U$;
(ii) $\overline{\neg P} \subset (\neg \overline{P})$;
(iii) $\overline{P \text{ and } Q} = (\overline{P} \text{ and } \overline{Q})$;
(iv) $\overline{P \text{ or } Q} \supset (\overline{P} \text{ or } \overline{Q})$;
(v) $\overline{P \implies Q} \subset (\overline{P} \implies \overline{Q})$;
(vi) $\overline{P \iff Q} \subset (\overline{P} \iff \overline{Q})$;
(vii) $\overline{P \circ f} = \overline{P} \circ \overline{f}$;
(viii) $\overline{(\forall D, P)} = \overline{\forall D, \overline{P}}$;
(ix) $\overline{(\exists D, P)} = \overline{\exists D, \overline{P}}$;
(x) $\overline{(\exists! D, P)} = \overline{\exists! D, \overline{P}}$;
(xi) $\overline{P \circ f} = \overline{P} \circ \overline{f}$ and $\overline{\text{id}_D} = \overline{\text{id}_D}$;
(xii) $\overline{(f_1, \ldots, f_n)} = (\overline{f_1}, \ldots, \overline{f_n})$ and $\overline{\pi^i_D} = \overline{\pi^i_D}$.

Furthermore, for any $D \subset U^n$, we have:

(a) The statements ‘$\forall x \in D$, $P(x)$’ and ‘$\forall \xi \in \overline{D}$, $\overline{P}(\xi)$’ are logically equivalent, i.e., the first is true if and only if the second is true;

(b) The statements ‘$\exists x \in D$, $P(x)$’ and ‘$\exists \xi \in \overline{D}$, $\overline{P}(\xi)$’ are logically equivalent;

(c) The statements ‘$\exists! x \in D$, $P(x)$’ and ‘$\exists! \xi \in \overline{D}$, $\overline{P}(\xi)$’ are logically equivalent.

The remainder of this section will be dedicated to the proof of the VET (Virtual Extension Theorem):

The first assertion is exactly the proposition II.1 applied to subset $P \subset U^n$:

$$K(P) = K(U^n) \cap \overline{P}.$$
(i) For any classes \( \langle x_i \rangle \in \mathbb{U} \) and \( \langle y_i \rangle \in \mathbb{U} \), the following statements are equivalent:

\[
\text{eq}_U(\langle x_i \rangle, \langle y_i \rangle) \\
\exists k, \forall i > k, \text{ eq}_U(x_i, y_i) \\
\exists k, \forall i > k, \ x_i = y_i \\
\langle x_i \rangle = \langle y_i \rangle \\
\text{eq}_U(\langle x_i \rangle, \langle y_i \rangle).
\]

(ii) If \( \langle x_i \rangle \in \mathbb{U}^n \) then the condition ‘ \( \neg \overline{\text{P}}(\langle x_i \rangle) \) ’ is equivalent to:

\[
\exists k, \forall i > k, \neg \text{P}(x_i),
\]

which is sufficient for the validity of

\[
\forall k, \exists i > k, \neg \text{P}(x_i),
\]

which, in turn, is equivalent to ‘ \( \neg \overline{\text{P}}(\langle x_i \rangle) \) ’.

(iii) It is enough to note that the following statements are equivalent:

\[
(\neg \text{P} \text{ and } \neg \text{Q})(\langle x_i \rangle) \\

\overline{\text{P}}(\langle x_i \rangle) \text{ and } \overline{\text{Q}}(\langle x_i \rangle) \\
[\exists k_1, \forall i > k_1, \text{P}(x_i)] \text{ and } [\exists k_2, \forall i > k_2, \text{Q}(x_i)] \\
\exists k, \forall i > k, [\text{P}(x_i) \text{ and } \text{Q}(x_i)] \\
\exists k, \forall i > k, (\text{P and Q})(x_i) \\
\overline{\text{P}} \text{ and } \overline{\text{Q}}(\langle x_i \rangle).
\]

(iv) First, we have the equivalences:

\[
(\neg \text{P} \text{ or } \neg \text{Q})(\langle x_i \rangle) \\

\overline{\text{P}}(\langle x_i \rangle) \text{ or } \overline{\text{Q}}(\langle x_i \rangle) \\
[\exists k_1, \forall i > k_1, \text{P}(x_i)] \text{ or } [\exists k_2, \forall i > k_2, \text{Q}(x_i)].
\]

This last statement implies the first below, which is equivalent to the following ones:

\[
\exists k, \forall i > k, \text{P}(x_i) \text{ or } \text{Q}(x_i) \\
\exists k, \forall i > k, (\text{P or Q})(x_i) \\
\overline{\text{P}} \text{ or } \overline{\text{Q}}(\langle x_i \rangle).
\]
(v) We will prove that \[ \lnot (P \Rightarrow Q) \subseteq \lnot P \Rightarrow \lnot Q \]. The following statements are equivalent:

\[
\lnot (P \Rightarrow Q)((x_i)) \\
[\lnot P \text{ and } \lnot Q)((x_i)) \\
P((x_i)) \text{ and } \lnot Q((x_i)) \\
[\exists k_1, \forall i > k_1, P(x_i)] \text{ and } [\forall k_2, \exists i > k_2, \lnot Q(x_i)].
\]

From this last assertion we conclude the first one below, which is equivalent to the following:

\[
\forall k, \exists i > k, [P(x_i) \text{ and } \lnot Q(x_i)] \\
\forall k, \exists i > k, \lnot (P \Rightarrow Q)(x_i) \\
\lnot P \Rightarrow \lnot Q((x_i)).
\]

(vi) Applying items (iii) and (v) above, we have:

\[
P \iff Q = (P \Rightarrow Q) \text{ and } (Q \Rightarrow P) \\
= [(P \Rightarrow Q) \text{ and } (Q \Rightarrow P)] \\
\subseteq [(P \Rightarrow Q) \text{ and } (Q \Rightarrow P)] \\
= (P \iff Q).
\]

(vii) It is enough to note that, if \( a \in U \) and \( \langle x_i \rangle \in \overline{U}^n \), then the next following statements are equivalent:

\[
P^a((x_i)) \\
\exists k, \forall i > k, Pa(x_i) \\
\exists k, \forall i > k, P(a, x_i) \\
P(\bar{a}, \langle x_i \rangle) \\
P^a(\langle x_i \rangle).
\]

(viii) For \( \langle x_i \rangle \in \overline{U}^n \), we have the equivalences:

\[
(\forall D, P)((x_i)) \\
\exists k, \forall i > k, (\forall D, P)(x_i) \\
\exists k, \forall i > k, \forall y \in D, P(y, x_i) \\
(\forall (y_i) \in \Sigma(D), \exists k, \forall i > k, P(y_i, x_i) \\
\forall (y_i) \in \overline{D}, P((y_i), \langle x_i \rangle) \\
(\forall \overline{D}, P)(\langle x_i \rangle).
\]
(It is easier to see that the fourth statement above implies the third by negating both).

(ix) If \( \langle x_i \rangle \in \mathbb{U}^n \), then the assertions below are equivalent:

\[
\begin{align*}
&\exists D, P (\langle x_i \rangle) \\
&\exists k, \forall i > k, (\exists D, P)(x_i) \\
&\exists k, \forall i > k, \exists \ y \in D, \ P(y, x_i) \\
&\exists (y_i) \in \Sigma(D), \exists k, \forall i > k, \ P(y_i, x_i) \\
&\exists \langle y_i \rangle \in D, \ P(\langle y_i \rangle, \langle x_i \rangle) \\
& (\exists D, P)(\langle x_i \rangle).
\end{align*}
\]

(x) Also, the following statements are equivalent:

\[
\begin{align*}
&\exists! D, P (\langle x_i \rangle) \\
&\exists k, \forall i > k, (\exists! D, P)(x_i) \\
&\exists k, \forall i > k, \exists! \ y \in D, \ P(y, x_i) \\
&\exists! (y_i) \in D, \ P(\langle y_i \rangle, \langle x_i \rangle) \\
& (\exists D, P)(\langle x_i \rangle).
\end{align*}
\]

(xi) As the following:

\[
\begin{align*}
&\overline{P \circ f} (\langle x_i \rangle) \\
&\exists k, \forall i > k, (P \circ f)(x_i) \\
&\exists k, \forall i > k, P[f(x_i)] \\
&\overline{P} (\overline{f}(\langle x_i \rangle)) \\
&(\overline{P \circ f})(\langle x_i \rangle).
\end{align*}
\]

(xii) For any classes \( \langle x_i \rangle \in \overline{D_f} \) and \( \langle y_i \rangle \in \overline{C_g} \), we have the equivalences:

\[
\begin{align*}
&g \circ f (\langle x_i \rangle) = \langle y_i \rangle \\
&\exists k, \forall i > k, (g \circ f)(x_i) = y_i \\
&\exists k, \forall i > k, g[f(x_i)] = y_i \\
&\overline{g} (\overline{f}(\langle x_i \rangle)) = \langle y_i \rangle \\
&(\overline{g \circ f})(\langle x_i \rangle) = \langle y_i \rangle.
\end{align*}
\]

The equality \( \overline{id_D} = id_{\overline{D}} \) follows from item (i) proved earlier.
(xiii) If \( \langle x_i \rangle \in \overline{\Theta} \) and \( \langle y_j^i \rangle = \langle y_1^i, y_2^i, y_3^i, \ldots \rangle \in \overline{\Theta} \) \( (j = 1, \ldots, n) \), then these statements are equivalent:

\[
\left( f_1, \ldots, f_n \right)(\langle x_i \rangle) = (\langle y_1^i \rangle, \ldots, \langle y_n^i \rangle) \\
\exists k, \forall i > k, (f_1, \ldots, f_n)(x_i) = (y_1^i, \ldots, y_n^i) \\
\exists k, \forall i > k, f_j(x_i) = y_j^i \quad (j = 1, \ldots, n) \\
\left( \overline{f_1}, \ldots, \overline{f_n} \right)(\langle x_i \rangle) = (\langle y_1^i \rangle, \ldots, \langle y_n^i \rangle),
\]

and also we have:

\[
\pi^j_D(\langle y_1^i \rangle, \ldots, \langle y_n^i \rangle) = \langle x_i \rangle \\
\exists k, \forall i > k, \pi^j_D(y_1^i, \ldots, y_n^i) = x_i \\
\exists k, \forall i > k, y_j^i = x_i \\
\langle y_j^i \rangle = \langle x_i \rangle \\
\pi^j_D(\langle y_1^i \rangle, \ldots, \langle y_n^i \rangle) = \langle x_i \rangle.
\]

Finally, the last three items of the VET are corollaries of proposition II.2:

(a) By II.2(i), we have the equivalences:

\[
\forall x \in \Theta, \ P(x) \\
\Theta \subset \overline{\Theta} \\
\overline{\Theta} \subset \overline{\Theta} \\
\forall \xi \in \overline{\Theta}, \ \overline{P}(\xi).
\]

(b) Using II.2(ii) and (iii) above, we see the following assertions are equivalent:

\[
\exists x \in \Theta, \ P(x) \\
\Theta \cap \overline{\Theta} \neq \emptyset \\
\overline{\Theta} \cap \overline{\Theta} \neq \emptyset \\
\Theta \cap \overline{\Theta} \neq \emptyset \\
\exists \xi \in \overline{\Theta}, \ \overline{P}(\xi).
\]

(c) Now using II.2(iii) and item (iii), we have the equivalences:

\[
\exists! x \in \Theta, \ P(x) \\
\Theta \cap \overline{\Theta} \text{ is unitary} \\
\overline{\Theta} \cap \overline{\Theta} \text{ is unitary} \\
\Theta \cap \overline{\Theta} \text{ is unitary} \\
\exists! \xi \in \overline{\Theta}, \ \overline{P}(\xi).
\]
VI. Extension of Relation Attributes

Our objective in this section is to illustrate the application of the VET with some basic examples. For that, let \( A \) be any subset of \( U^n \), where \( U \) is the set from Sec. IV and \( n \in \mathbb{N} \) a natural number.

A binary relation \( P \subset A^2 \) is reflexive if and only if its virtual extension \( \overline{P} \subset \overline{A}^2 \) is reflexive.

Proof: By the VET, the following statements are equivalent:

\[
\forall x \in A, \; P(x, x) \\
\forall x \in A, \; [P \circ (\text{id}_A, \text{id}_A)](x) \\
\forall \xi \in \overline{A}, \; \overline{P} \circ (\text{id}_A, \text{id}_A)(\xi) \\
\forall \xi \in \overline{A}, \; \overline{P} \circ (\text{id}_\overline{A}, \text{id}_\overline{A})(\xi) \\
\forall \xi \in \overline{A}, \; \overline{P} \circ (\text{id}_\overline{A}, \text{id}_A)(\xi) \\
\forall \xi \in \overline{A}, \; \overline{P}(\xi, \xi),
\]

then \( P \) is reflexive if and only if \( \overline{P} \) is reflexive. ■

A binary relation \( P \subset A^2 \) is symmetric if and only if its virtual extension \( \overline{P} \subset \overline{A}^2 \) is symmetric.

Proof: Again by the VET, the following statements are equivalent:

\[
\forall (x, y) \in A^2, \; [P(x, y) \Rightarrow P(y, x)] \\
\forall (x, y) \in P, \; P(y, x) \\
\forall (x, y) \in P, \; [P \circ (\pi_{A2}, \pi_{A2}^{-1})](x, y) \\
\forall (\xi, v) \in \overline{P}, \; [\overline{P} \circ (\pi_{\overline{A}2}, \pi_{\overline{A}2}^{-1})](\xi, v) \\
\forall (\xi, v) \in \overline{P}, \; [\overline{P} \circ (\pi_{\overline{A}2}, \pi_{\overline{A}2}^{-1})](\xi, v) \\
\forall (\xi, v) \in \overline{P}, \; [\overline{P} \circ (\pi_{\overline{A}2}, \pi_{\overline{A}2}^{-1})](\xi, v) \\
\forall (\xi, v) \in \overline{P}, \; \overline{P}(v, \xi) \\
\forall (\xi, v) \in \overline{A}^2, \; [\overline{P}(\xi, v) \Rightarrow \overline{P}(v, \xi)],
\]

so \( P \) is symmetric if and only if \( \overline{P} \) is symmetric. ■

A binary relation \( P \subset A^2 \) is transitive if and only if its virtual extension \( \overline{P} \subset \overline{A}^2 \) is transitive.
Proof: Once more the VET gives us:
\[
\forall (x, y, z) \in A^3, \{ [P(x, y) \textrm{ and } P(y, z)] \Rightarrow P(x, z) \}
\]
\[
\forall (x, y, z) \in [P \circ (\pi^1_{A_3}, \pi^2_{A_3}) \textrm{ and } P \circ (\pi^2_{A_3}, \pi^3_{A_3})],\ [P \circ (\pi^1_{A_3}, \pi^3_{A_3})](x, y, z)
\]
\[
\forall (\xi, \upsilon, \zeta) \in P \circ (\pi^1_{A_3}, \pi^2_{A_3}) \textrm{ and } P \circ (\pi^2_{A_3}, \pi^3_{A_3}),\ [P \circ (\pi^1_{A_3}, \pi^3_{A_3})](\xi, \upsilon, \zeta)
\]
\[
\forall (\xi, \upsilon, \zeta) \in \overline{A}^3, \{ [\overline{P}(\xi, \upsilon) \textrm{ and } \overline{P}(\upsilon, \zeta)] \Rightarrow \overline{P}(\xi, \zeta) \},
\]
then \( P \) is transitive if and only if \( \overline{P} \) is transitive.

Thus we have: a binary relation \( P \subset A^2 \) is an equivalence relation on \( A \) if and only if its virtual extension \( \overline{P} \) is an equivalence relation on \( \overline{A} \).

We will say that a binary relation \( P \subset A^2 \) is antisymmetric when:
\[
\forall (x, y) \in A^2,\ [P(x, y) \textrm{ and } P(y, x)] \Rightarrow x = y.
\]
In addition, we will say that \( P \) is a partial ordering on \( A \) when it is reflexive, transitive and antisymmetric. According to that, we have:

A binary relation \( P \subset A^2 \) is antisymmetric if and only if \( \overline{P} \) is antisymmetric.

Proof: By the VET, the following statements are equivalent:
\[
\forall (x, y) \in A^2,\ [P(x, y) \textrm{ and } P(y, x)] \Rightarrow x = y
\]
\[
\forall (x, y) \in \{ P \textrm{ and } [P \circ (\pi^2_{A_2}, \pi^1_{A_2})]\},\ \text{eq}_A(x, y)
\]
\[
\forall (\xi, \upsilon) \in \overline{P} \text{ and } [P \circ (\pi^2_{A_2}, \pi^1_{A_2})],\ \text{eq}_{\overline{A}}(\xi, \upsilon)
\]
\[
\forall (\xi, \upsilon) \in \{ \overline{P} \text{ and } [\overline{P} \circ (\pi^2_{A_2}, \pi^1_{A_2})]\},\ \text{eq}_{\overline{A}}(\xi, \upsilon)
\]
\[
\forall (\xi, \upsilon) \in \overline{A}^2,\ [\overline{P}(\xi, \upsilon) \textrm{ and } \overline{P}(\upsilon, \xi)] \Rightarrow \xi = \upsilon,
\]
then \( P \) is antisymmetric if and only if \( \overline{P} \) is antisymmetric.

So, we conclude that a binary relation \( P \subset A^2 \) is a partial ordering on \( A \) if and only if its virtual extension \( \overline{P} \) is a partial ordering on \( \overline{A} \).

A binary relation \( P \subset A^2 \) will be called a total ordering when it is reflexive, transitive, antisymmetric and satisfies the trichotomy:
\[
\forall (x, y) \in A^2,\ P(x, y) \textrm{ or } P(y, x).
\]
Applying the VET to the trichotomy, as we did above, we obtain the following equivalences:
\[
\forall (x, y) \in A^2,\ [P(x, y) \textrm{ or } P(y, x)](x, y)
\]
\[
\forall (x, y) \in A^2,\ [P \circ (\pi^2_{A_2}, \pi^1_{A_2})](x, y)
\]
\[
\forall (\xi, \upsilon) \in \overline{A}^2,\ \overline{P} \circ (\pi^2_{A_2}, \pi^1_{A_2})(\xi, \upsilon).
\]
However, we cannot proceed as we did before because it is not true that the extension of a logical disjunction is the same as the disjunction of the extensions of disjunctives.

For instance, let us consider the total ordering ‘≤’ between real numbers. The extension ‘≤’ is just a partial ordering relation between virtual numbers. If \( \alpha \in \mathbb{R} \) is the class of the sequence \((-1,+1,-1,+1,\ldots) \in \Sigma(\mathbb{R})\), then the statements ‘\( \alpha \leq \emptyset \)’ and ‘\( \emptyset \leq \alpha \)’ are both false.

In the case of the connectives ‘not’, ‘⇒’, and ‘⇔’ we have a similar situation, but in the opposite direction, since inclusions present in items (ii), (v) and (vi) of the VET are opposite the one occurring in item (iv). In spite of that, the VET can establish unidirectional implications between statements involving those connectives, as shown in the following examples:

(i) If the extension \( \overline{P} \) satisfies the trichotomy then \( P \) also satisfies it, since, by item (iv) of the VET, the condition:

\[
P \text{ or } P \circ (\pi^2_{\overline{A}_2}, \pi^1_{\overline{A}_2}) (\xi, \upsilon)
\]

is necessary (although not sufficient) for the validity of:

\[
[\overline{P} \text{ or } P \circ (\pi^2_{\overline{A}_2}, \pi^1_{\overline{A}_2})](\xi, \upsilon).
\]

(ii) If \( P \) and \( Q \) are two binary relations such that:

\[
\forall y \in A, \exists x \in A, P(x, y) \Rightarrow Q(x, y),
\]

then:

\[
\forall \upsilon \in \overline{A}, \exists \xi \in \overline{A}, \overline{P}(\xi, \upsilon) \Rightarrow \overline{Q}(\xi, \upsilon)
\]

holds. To verify this, we initially have the equivalences:

\[
\forall y \in A, \exists x \in A, P(x, y) \Rightarrow Q(x, y)
\]
\[
\forall y \in A, \exists x \in A, (P \Rightarrow Q)(x, y)
\]
\[
\forall y \in A, [\exists A, (P \Rightarrow Q)](y)
\]
\[
\forall \upsilon \in \overline{A}, \exists \xi \in \overline{A}, (P \Rightarrow Q)(\upsilon)
\]
\[
\forall \upsilon \in \overline{A}, [\exists \overline{A}, P \Rightarrow Q](\upsilon)
\]
\[
\forall \upsilon \in \overline{A}, \exists \xi \in \overline{A}, \overline{P} \Rightarrow \overline{Q}(\xi, \upsilon).
\]
Furthermore, by item (v) of the VET, the condition:

\[ P \Rightarrow Q(\xi, \upsilon) \]

is sufficient to guarantee that:

\[ (P \Rightarrow Q)(\xi, \upsilon). \]

Therefore, the last statement of the series above implies

\[ \forall \upsilon \in \overline{A}, \exists \xi \in \overline{A}, (P \Rightarrow Q)(\xi, \upsilon), \]

which is equivalent to:

\[ \forall \upsilon \in \overline{A}, \exists \xi \in \overline{A}, P(\xi, \upsilon) \Rightarrow Q(\xi, \upsilon). \]

VII. Extension of Function and Operation Properties

In this section we will introduce some basic applications of the VET involving functions and operations. Let \( D \subset U^n \) and \( C \subset U^k \) be two generic subsets, where \( U \) is the set considered in Sec. IV, and \( n, k \in \mathbb{N} \) two natural numbers.

A function \( f : D \rightarrow C \) is one-to-one if and only if its virtual extension \( \overline{f} : \overline{D} \rightarrow \overline{C} \) is one-to-one.

Proof: By the VET, the following statements are equivalent:

\[ \forall (x_1, x_2) \in D^2, [f(x_1) = f(x_2) \Rightarrow x_1 = x_2] \]
\[ \forall (x_1, x_2) \in \overline{[eq_C \circ (f \circ \pi_D^1, f \circ \pi_D^2)]}, \overline{eq_D}(x_1, x_2) \]
\[ \forall (\xi_1, \xi_2) \in \overline{[eq_C \circ (f \circ \pi_D^1, f \circ \pi_D^2)]}, \overline{eq_D}(\xi_1, \xi_2) \]
\[ \forall (\xi_1, \xi_2) \in \overline{D^2}, [f(\xi_1) = f(\xi_2) \Rightarrow \xi_1 = \xi_2], \]

then \( f \) is one-to-one if and only if \( \overline{f} \) is one-to-one.

A function \( f \) maps \( D \) onto \( C \) if and only if its virtual extension \( \overline{f} \) maps \( \overline{D} \) onto \( \overline{C} \).
Proof: Again by the VET, the following statements are equivalent:

\[ \forall y \in C, \exists x \in D, f(x) = y \]
\[ \forall y \in C, \exists x \in D, \ eq_C \circ (f \circ \pi_{1_{D \times C}}, \pi_{2_{D \times C}})(x, y) \]
\[ \forall y \in C, \exists x \in D, \ eq_{C} \circ (f \circ \pi_{1_{D \times C}}, \pi_{2_{D \times C}})](y) \]
\[ \forall x \in \overline{C}, \exists y \in \overline{D}, \ eq_{\overline{C}} \circ (\overline{f} \circ \pi_{1_{D \times C}}, \pi_{2_{D \times C}})](y) \]
\[ \forall x \in \overline{C}, \exists \xi \in \overline{D}, \ eq_{\overline{C}} \circ (\overline{f} \circ \pi_{1_{D \times C}}, \pi_{2_{D \times C}}))]_2 \]
\[ \forall x \in \overline{C}, \exists \xi \in \overline{D}, \ f(\xi) = y. \]

Therefore, a function \( f: D \to C \) is invertible if and only if its virtual extension \( \overline{f}: \overline{D} \to \overline{C} \) is invertible. In this case, it follows directly from item (xii) of the VET that the extension of the inverse function of \( f \) is equal to the inverse function of its virtual extension.

Let now \( \oplus: A^2 \to A \) be a binary operation on \( A \subset U^n \). Applying the VET as we did above, we verify that \( \oplus \) is associative if and only if \( \overline{\oplus} \) is associative, and that \( \oplus \) is commutative if and only if \( \overline{\oplus} \) is commutative. Besides, if \( \odot: A^2 \to A \) is another binary operation defined on the same set \( A \), then \( \odot \) is distributive with respect to \( \oplus \) if and only if \( \overline{\odot} \) is distributive with respect to \( \overline{\oplus} \). In other words, the equality:

\[ a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c) \]

holds for every triple \( (a, b, c) \in A^3 \) if and only if

\[ \alpha \overline{\odot} (\beta \overline{\oplus} \gamma) = (\alpha \overline{\odot} \beta) \overline{\oplus} (\alpha \overline{\odot} \gamma) \]

holds for every triple \( (\alpha, \beta, \gamma) \in \overline{A}^3 \).

As to the neutral element, we have:

If \( \oplus: A^2 \to A \) is a binary operation then \( e \in A \) is a right neutral element of \( \oplus \) if and only if \( \overline{e} \in K(A) \subset \overline{A} \) is a right neutral element of \( \overline{\oplus} \).
Proof: By the VET, the following statements are equivalent:

\[ \forall a \in A, \ e \oplus a = a \]
\[ \forall a \in A, \ [eq_A \circ (\oplus, \pi^2_A)](e, a) \]
\[ \forall a \in A, \ [eq_A \circ (\oplus, \pi^2_A)]e(a) \]
\[ \forall \alpha \in \overline{A}, \ [eq_A \circ (\overline{\oplus}, \pi^2_A)]e(\alpha) \]
\[ \forall \alpha \in \overline{A}, \ [eq_A \circ (\overline{\oplus}, \pi^2_A)](\overline{\alpha}) \]
\[ \forall \alpha \in \overline{A}, \ e \oplus \overline{\alpha} = \overline{\alpha}. \]

Analogously:

If \( \oplus: A^2 \to A \) is a binary operation then \( e \in A \) is a left neutral element of \( \oplus \) if and only if \( \overline{e} \in K(A) \subset \overline{A} \) is a left neutral element of \( \overline{\oplus} \).

Proof: Once more the VET gives us:

\[ \forall a \in A, \ a \oplus e = a \]
\[ \forall a \in A, \ {eq_A \circ [\oplus \circ (\pi^2_A, \pi^1_A, \pi^2_A)]}(e, a) \]
\[ \forall a \in A, \ {eq_A \circ [\oplus \circ (\pi^2_A, \pi^1_A, \pi^2_A)]}e(a) \]
\[ \forall \alpha \in \overline{A}, \ {eq_A \circ [\overline{\oplus} \circ (\pi^2_A, \pi^1_A, \pi^2_A)]}e(\alpha) \]
\[ \forall \alpha \in \overline{A}, \ {eq_A \circ [\overline{\oplus} \circ (\pi^2_A, \pi^1_A, \pi^2_A)]}(\overline{\alpha}) \]
\[ \forall \alpha \in \overline{A}, \ a \oplus \overline{\alpha} = \overline{\alpha}. \]

Now we will consider the existence of opposites:

If \( \oplus: A^2 \to A \) and \( c \in A \), then the condition:

\[ \forall a \in A, \ \exists b \in A, \ a \oplus b = c \]

is necessary and sufficient to the validity of:

\[ \forall \alpha \in \overline{A}, \ \exists \beta \in \overline{A}, \ \alpha \overline{\oplus} \beta = \overline{c}. \]
Proof: By the VET, the following statements are equivalent:

\[ \forall a \in A, \exists b \in A, \ a \oplus b = c \]
\[ \forall a \in A, \exists b \in A, \ \{ \text{eq}_A \circ [\oplus \circ (\pi_{A^3}^2, \pi_{A^3}^3, \pi_{A^3}^1)] \}(c, a, b) \]
\[ \forall a \in A, \exists b \in A, \ \{ \text{eq}_A \circ [\oplus \circ (\pi_{A^3}^2, \pi_{A^3}^3, \pi_{A^3}^1)] \}c(a, b) \]
\[ \forall \alpha \in \overline{A}, \exists \beta \in \overline{A}, \ \{ \text{eq}_{A^\circ} \circ [\oplus \circ (\pi_{A^3}^2, \pi_{A^3}^3, \pi_{A^3}^1)] \}c(a, b) \]
\[ \forall \alpha \in \overline{A}, \exists \beta \in \overline{A}, \ \{ \text{eq}_{A^\circ} \circ [\oplus \circ (\pi_{A^3}^2, \pi_{A^3}^3, \pi_{A^3}^1)] \} \overline{c}(\alpha, \beta) \]
\[ \forall \alpha \in \overline{A}, \exists \beta \in \overline{A}, \ \{ \text{eq}_{A^\circ} \circ [\oplus \circ (\pi_{A^3}^2, \pi_{A^3}^3, \pi_{A^3}^1)] \}(\overline{c}, \alpha, \beta) \]
\[ \forall \alpha \in \overline{A}, \exists \beta \in \overline{A}, \ \alpha \overline{\oplus} \beta = \overline{c}. \]

Nevertheless, the condition:

\[ \forall a \neq d, \exists b \in A, \ a \odot b = c \]

is equivalent to:

\[ \forall \alpha \neq d, \exists \beta \in \overline{A}, \ \alpha \overline{\odot} \beta = \overline{c}, \]

which, by item (ii) of the VET, is necessary but not sufficient to assure the validity of:

\[ \forall \alpha \neq d, \exists \beta \in \overline{A}, \ \alpha \overline{\odot} \beta = \overline{c}. \]

VIII. Extension of Mathematical Structures

The results of the last two sections illustrate how the VET can be used to logically relate a statement about the set \( U \) to another statement about its virtual extension. The following syntactic rules informally describe how that “extended statement” is obtained from the original:

(i) consistently substitute the bound variables (quantified) ranging over a subset \( A \subset U^n \) by bound variables ranging over its virtual extension \( \overline{A} \subset \overline{U}^n \), keeping the corresponding quantifier;

(ii) consistently substitute the free variables (not quantified) by the corresponding element in \( K(U) \subset \overline{U} \);

(iii) substitute the functions present in the original statement by the respective virtual extensions;

(iv) selectively substitute the relations in the original statement by its virtual extensions, respecting the restrictions on the connectives ‘not’, ‘or’, ‘⇒’ and ‘⇔’.
However, it is important to note that the VET has been enunciated and proved by methods of Elementary Set Theory, not having used the formal distinction between syntactic and semantic planes which characterizes Mathematical Logic.

The results presented in the last two sections show that application of the VET also does not require more than elementary mathematical techniques. Collecting some of those results, we conclude that:

A pair \((G, \oplus)\) is a group if and only if its virtual extension \((G, \oplus)\) is a group. In this case, \((G, \oplus)\) is commutative if and only if \((G, \oplus)\) is commutative.

A triple \((A, \oplus, \odot)\) is a ring if and only if its virtual extension \((G, \oplus, \odot)\) is a ring. In this case, \(e \in A\) is a unity in \(A\) if and only if \(e \in K(A) \subset A\) is a unity in \(A\).

Nevertheless, virtual extension of a total ordering is just partial ordering, and the virtual extension of a field is just a ring with unity. For example, the virtual extension \((\mathbb{R}, +, \times, <)\) of the ordered field of real numbers is not a field, nor is it totally ordered.

On the other hand, every relation and function defined on any set \(A\) is extended to the set \(\overline{A}\). In addition, the loss of part of the mathematical structure of \(A\) during the process of virtual extension can be compensated by the VET, which allows us to transport many facts about those extended relations and functions directly to \(\overline{A}\). For instance, we have the trigonometric functions:

\[
\sin: \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \cos: \mathbb{R} \to \mathbb{R},
\]

which satisfy the identity

\[
\sin^2 \alpha + \cos^2 \alpha = 1,
\]

for any virtual number \(\alpha \in \mathbb{R}\).

An important aspect of the virtual extension process is that it must be applied “simultaneously” to every set bound by the relations which we intend to extend. We can do that just by taking the “disjoint union” of these sets as “universe”. In other words, if we intend to extend relations between the sets of a family \(A_i\) \((i \in I)\), then we should take the disjoint union of the family \((A_i)\) as the set represented by \(U\) in the previous sections.

As an illustration, to extend a vectorial space \(V\) over real scalars, we can make \(U\) equal the disjoint union of \(V\) and \(\mathbb{R}\), so that a generic member of \(U\) will be a class of sequences whose elements can be either vectors of \(V\) or real scalars. In this manner, the virtual vectors will be the members of the subset \(\overline{V} \subset U\) (i.e., the classes of sequences which end taking values only in \(V\)), whereas the virtual scalars will be the members of
the subset $\mathbf{R} \subset \mathbf{U}$ (which is the class of sequences that end taking only real values). Thus, the multiplication of scalars and vectors:

$$\times: \mathbf{R} \times \mathbf{V} \to \mathbf{V}$$

extends to an operation between virtual scalars and virtual vectors:

$$\overline{\times}: \overline{\mathbf{R}} \times \overline{\mathbf{V}} \to \overline{\mathbf{V}}.$$

Another example of mathematical structure whose definition involves more than one set is that of manifold (topological or differentiable). Proceeding as above, we can construct a virtual manifold armed with an atlas of virtual charts, which associate local virtual coordinates to the virtual points of the manifold.

The word “extension” is commonly used in mathematics to indicate that we have a “copy” of the original set $\mathbf{U}$ inside the set $\overline{\mathbf{U}}$ constructed from it, and that we intend to identify $\mathbf{U}$ with that copy. The VET guarantees that $\mathbf{K}(\mathbf{U}) \subset \overline{\mathbf{U}}$ is a faithful copy of $\mathbf{U}$, since any relation $P$ involving the members of $\mathbf{U}$ is equivalent to the restriction of its extension $\overline{P}$ to the corresponding members in $\mathbf{K}(\mathbf{U})$. This fact authorizes the identification $\mathbf{U} = \mathbf{K}(\mathbf{U})$, which allows us to consider $\mathbf{U} \subset \overline{\mathbf{U}}$. If $\mathbf{U}$ is the disjoint union of family $(\mathbf{A}_i)$, then we will have $\mathbf{A}_i = \mathbf{K}(\mathbf{A}_i) \subset \overline{\mathbf{A}_i} \subset \overline{\mathbf{U}}$, for every $i \in I$.  

21