On “model independent” cosmic determinations of $H_0$

E. Ó Colgáin$^{a,b}$ & M. M. Sheikh-Jabbari$^c$

$^a$ Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea
$^b$ Department of Physics, Sogang University, Seoul 121-742, Korea
$^c$ School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5531, Tehran, Iran

Abstract

Motivated by Hubble tension, we have recently witnessed a number of “model independent” $H_0$ determinations at cosmological scales. Here we compare two “model independent” techniques, Taylor expansion and Gaussian Processes (GP). While Taylor expansion is truly model independent in a limited range, we show that one can reduce the $H_0$ errors by increasing the range of the expansion, but the approximation suffers. For GP, we confirm for the Matérn class kernels that the errors on $H_0$ decrease as the parameter $\nu \to \infty$, where we recover the Gaussian kernel. The errors from GP are typically smaller than Taylor and by mapping the GP analysis back into the Taylor expansion, we show that GP explores a smaller portion of the parameter space. In a direct comparison of GP with the CPL model, the simplest model of dynamical dark energy, we see that correlations are suppressed by GP relative to CPL. Therefore, GP cannot be model independent. We emphasise that if a truly model independent statement of Hubble tension exists, then it will have serious consequences for the FLRW framework.
1 Introduction

The main goal in cosmology is to uncover the evolution history of the Universe. It is typically assumed that on cosmic scales the Universe is an FLRW background governed by the metric

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right),$$

with $a(t)$ being the scale factor and $k = 0, \pm 1$ denoting the curvature of constant time slices. One considers homogeneous and isotropic fluctuations on this background. In this setup the scale factor $a(t)$ and the redshift $z$ are related as $a = (1 + z)^{-1}$ and all the information in the background is encoded in the Hubble diagram $H(z) = (da/dt)a^{-1}$. So, the goal is to find $H(z)$ which, in principle, may be done directly from the cosmic data in a “model independent” way.

On the other hand, the gravity field equations, namely Einstein equations in Einstein’s General Relativity theory, relate $H(z)$ to the energy-momentum tensor of the cosmic fluid filling the Universe. More precisely, by a choice of a “cosmological model” we specify the equation of state (EoS) of different components in the cosmic fluid. One can for example assume an $N$ component fluid with EoS $w_i(z), i = 1, 2, \cdots, N$, where the typically used fluids are radiation, specified by $\omega_i = 1/3$, (dark) matter by $\omega_i = 0$, cosmological constant (dark energy) by $\omega_i = -1$, and the curvature $k$ by $\omega_i = -1/3$. Observe that in the simplest setting $w_i$ is a constant. However, more generally given $\omega_i(z)$, the functional form of $H(z)$ is fixed up to $N - 1$ values of relative energy densities of the components at $z = 0$, $\Omega_i$ and an integration constant $H_0$:

$$H(z) = H_0 \sqrt{\sum_{i=1}^{N} \Omega_i f_i(z)}, \quad f_i(z) = \exp \left( 3 \int_0^{z} \frac{1 + w_i(\tilde{z})}{1 + \tilde{z}} d\tilde{z} \right), \quad \sum_{i=1}^{N} \Omega_i = 1. \quad (1.2)$$

In cosmology it is implicitly assumed that $H(z)$ is a smooth function of $z$ and in particular $H_0 = H(z \to 0) = H(z = 0)$. Therefore, as was noted in [1], $H_0$ is conceptually different than other model parameters, $\omega_i(z), \Omega_i$, and is the only model independent parameter in (1.2).

$H_0$ is also special from the perspective of observation: it can be determined locally by extremely mild assumptions on the cosmology, essentially that it is described by a Taylor expansion at low redshift, e.g. [2]. For this reason it is agnostic about the cosmological model, as has been demonstrated [3]. Indeed, at low redshift different local values have been reported for $H_0$ from direct measurements; see [4, 5] for recent updates. The local measurements of $H_0$ have now reached percent level precision and a combined analysis of these different observations yields $H_0 = 73.27 \pm 0.76 \text{ km/s/Mpc}$ at 68% CL [5]. On the other hand, the value reported by the Planck mission [6] based on the best fit values for the ΛCDM model is $H_0 = 67.4 \pm 0.5$, which has again reached percent level precision. The difference between these two values of $H_0$, the Hubble tension, has reached $6.4\sigma$ level and if not due
to the systematic errors of either measurements/determination, it is calling for a departure from \( \Lambda \)CDM model.

There are, however, various low redshift cosmological datasets in the \( 0.1 \lesssim z \lesssim 2.5 \) range, including Type Ia supernovae, cosmic chronometers (CC), Baryon Acoustic Oscillation (BAO) and strong gravitational lensing (GLS) datasets. These datasets have been analyzed extensively in various combinations with or without inclusion of CMB and other high redshift data, or considering different priors. They have been performed either within a class of cosmological models with different dark energy models, or in a “model independent” way. They report different values of \( H_0 \) which depending on the analysis is somewhere between the local measurement or the Planck \( \Lambda \)CDM value, e.g. see [5, 7]. In this work we focus on such model independent analyses of the low redshift cosmological data.

Model independent reconstructions of \( H(z) \) are typically based on a Taylor expansion in power of \( z \) around \( z = 0 \) [8,9] or Gaussian Processes (GP) [10,11]. The former was first used at low redshift by Riess et al. [12] through the parametrisation \( H(z) = H_0(1 + (1 + q_0)z + O(z^2)) \) in order to define the deceleration parameter \( q_0 \). The same expansion by inclusion of some higher order terms was then used in the context of cosmology and was given the name “cosmography” [13]. As we will discuss in section 2.1, Taylor expansion is model independent only if we restrict the expansion close to \( z = 0 \). To increase its applicability, one can work at higher redshift, but one needs to be conscious that it only describes a class of models.\(^1\)

The GP, as described in [19], is a fully Bayesian smoothing technique that allows for the reconstruction of \( H(z) \) and its derivatives directly from observational data, without assuming a specific model or choosing a parameterization. It, however, uses a kernel for smoothing the data. As we will discuss in section 2.2, the Matérn class of kernels is commonly used and they come in a family specified by the parameter \( \nu \), where \( \nu \rightarrow \infty \) recovers a Gaussian kernel. GP has been used extensively to extract dark energy EoS or dark energy-dark matter coupled models [19–21], the contribution of the cosmic curvature term \( \Omega_k \) [22], or to read the value of \( H_0 \) [23–29].

The goal of this note is to get to grips with frequently used “model independent” techniques and to compare them. As we will explicitly demonstrate “model independence” in the context of (low redshift) cosmology and specifically in determining the value of \( H_0 \), is often no more than a misnomer. Moreover, Taylor expansion cannot yield the percent level precision, one would desire to have, to be on par with the Planck or local \( H_0 \) values. On the contrary, one can get percent level errors from GP, but this comes at a cost. We find that GP suppresses correlations to the point that only the simplest cosmological models may show similar suppression in correlations, and are therefore covered by the approach. In particular, the CPL model [30, 31] falls outside of this class, so GP, as currently implemented with the choice of kernels, appears to be restricting the parameter space of more elaborate models. In section 5 we make this statement precise using the parameter space of a more general

\(^1\)There have been other cosmographic expansions based on Taylor expansion in powers of \( y = 1 - a = z/(1 + z) \) or some given function of \( z \), e.g. log(1 + z) [14] (see however, [15,16]) or a Padé approximant [17] or Chebyshev polynomials [18].
Taylor expansion. The immediate take-home message is that some reflection on the “model independence” of GP, and Taylor expansion to a lesser extent, is required. This has knock-on implications for $H_0$ determinations and the Hubble constant.

The rest of this paper is organized as follows. In section 2 we review the basic tools that we will be using in this work, namely Taylor expansion and Gaussian Processes (GP) methods. We will also introduce the Observational Hubble Data (OHD) serving as the basis for mock realisations. In sections 3 and 4 we respectively perform Taylor and GP analyses of the (mock) OHD to determine $H_0$. In section 5 we compare Taylor and GP, along with the corresponding errors on $H_0$. In section 6 we conclude with some discussion. In two appendices A and B, we gather some technical details pertaining to the Matérn class GP kernels and $H(z)$ in the CPL dynamical dark energy model.

## 2 Methodology

### 2.1 Taylor expansion

In cosmology one is interested in the Hubble parameter $H(z)$. It is not only tacitly assumed that $H(z)$ is continuous, but also that it is differentiable, namely that derivatives of $H(z)$ are also continuous functions. Within this setting, Taylor’s theorem (see [32]) rests on the premise that one can differentiate $H(z)$ $n$ times and still get a continuous function. This allows one to expand the Hubble parameter about a given value $z = z_0$:

$$H(z) = H(z_0) + H'(z_0)(z-z_0) + \frac{H''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{H^{(n)}(z_0)}{n!}(z-z_0)^n + h_n(z)(z-z_0)^n. \quad (2.1)$$

The theorem guarantees that the remainder function $h_n(z)$ approaches zero as $z$ approaches $z_0$, i.e. $\lim_{z \to z_0} h_n(z) = 0$. While one could perform this expansion at any redshift, it is natural to consider expansions around $z = 0$, and this is the basis for cosmographic (Taylor) expansions [8, 13]. From our perspective, working in the vicinity of $z = 0$ is also sufficient for determining $H_0$.

Before proceeding, a comment on “model independence” of Taylor expansions is in order. Recalling the discussions in the introduction, one can be confident that in the immediate vicinity of $z_0$ all models are covered, so Taylor expansion is model independent in a real sense. This is essentially the regime that Riess et al. [2] operate in to make local determinations of $H_0$ (see also [3]). Unfortunately, the farther one goes from $z = z_0$, the fewer models that are accurately described by the Taylor expansion. In this sense, away from $z = z_0$, Taylor expansion is not model independent, but simply describes a class of models. As we discuss below considering more terms in the expansion, higher $n$ in (2.1), does not change this fact. We will also see in due course that Taylor expansion is “more model independent” than GP within its range of validity. Ergo, GP cannot be model independent.

Now, for Taylor expansion the practical problem is that $H(z)$ is determined by data and ab initio it is unknown. To overcome this, one can naively expand $H(z)$, but one needs to
decide on (i) the order $n$ and (ii) how far one can venture from $z = 0$ so that the remainder function $h_n(z)$ remains small. Some may regard this as an "ad hoc" choice.\footnote{See comments in section 4 of [25].} However, it is relatively easy to address these two points once one demands that the standard model, flat $\Lambda$CDM, is covered by the expansion, i.e. the expansion explores models close to $\Lambda$CDM. Since $H_0$ is an overall factor in (1.2), it is enough to consider the normalised Hubble parameter $E(z) := H(z)/H_0$ which for flat $\Lambda$CDM is,

$$E_{\text{exact}}(z) := \sqrt{1 - \Omega_m 0 + \Omega_m 0 (1 + z)^3}
\approx 1 + \frac{3\Omega_m 0}{2} z + \frac{3\Omega_m 0}{8} (4 - 3\Omega_m 0) z^2 + \cdots = E_{\text{approx}}(z), \quad (2.2)$$

where dots denote omitted terms: we have not yet specified the order of the expansion. $E_{\text{exact}}(z)$ in the first line is the exact expression for flat $\Lambda$CDM whereas, the expression on the second line, dubbed as $E_{\text{approx}}(z)$, is a low $z$ approximation. Observe that for $z \gg 1$, $E_{\text{exact}}(z) \sim z^{3/2}$, which is a non-integer power of $z$, and that the coefficients of the $z, z^2, z^3$ (and in fact all the higher powers in the expansion) in $E_{\text{approx}}(z)$ are not independent and are specified in terms of $\Omega_m 0$.

Let us begin by specifying the order $n$ and restrict our analysis to $n \leq 10$, essentially to mimic analysis on higher order polynomials in Figure 6 of [25]. Next, let us define the % difference between the exact expression for $E(z)$ and its approximation at a given $z$:

$$\Delta E(z) = \left[ E_{\text{exact}}(z) - E_{\text{approx}}(z) \right] / E_{\text{exact}}(z). \quad (2.3)$$

We also introduce the discrete sum,

$$S(z_1, \ldots z_N) = \sum_{i=1}^{N} \left[ E_{\text{exact}}(z_i) - E_{\text{approx}}(z_i) \right]^2, \quad (2.4)$$

where we take $z_1 = 0$ and $\Delta z_i = z_i - z_{i-1} = 0.001$ with $z_N$ corresponding to the maximum value of the redshift range, which we rename $z_{\text{max}}$. Note that (2.3) is defined at a given redshift $z$, but (2.4) is a summed quantity over a range of redshifts. For this reason, (2.4) is a better measure of how close $E_{\text{approx}}(z)$ is to $E_{\text{exact}}(z)$. We will use the two measures interchangeably, since usually they lead to the same conclusions, but if one wants to be more precise, the most accurate polynomial at a given order $n$ is the one that minimises (2.4). It is worth emphasising again that the analysis in this section is purely analytic with no input from data.

The initial results of this exercise are shown in Table 1 ($\Omega_m 0 = 0.3$), where we have employed (2.4) as a measure of precision. Simply put, polynomials with smaller $S$ numbers are more accurate, so this provides an easy way to rank the polynomials. Now, it is well documented that the $z$ expansion has a radius of convergence of at most $|z| = 1$ [9]. Thus, below $z_{\text{max}} = 1$ one should expect that including higher order terms in the expansion will...
| $z_{\text{max}}$ | Most precise $n$ | maximum $|\Delta E(z)|$ |
|-----------------|-----------------|-----------------|
| 2.5             | 3, 2, 5, ...    | 13.9%, 16.9%, 28%, ... |
| 2               | 5, 3, 8, ...    | 8.7%, 9.2%, 16%, ... |
| 1.5             | 10, 5, 8, ...   | 1.5%, 1.7%, 2.4%, ... |
| 1               | 10, 5, 8, ...   | 0.008%, 0.1%, 0.13%, ... |
| 0.5             | 10, 9, 8, ...   | 0.000016%, 0.00026%, 0.00056%, ... |

Table 1: The precision of different order ($n \leq 10$) polynomials in recovering flat ΛCDM with $\Omega_{m0} = 0.3$ up to redshift $z_{\text{max}}$. We highlight only the 3 best performing polynomials.

increase the agreement with the exact result. This is clearly the case when $z_{\text{max}} = 0.5$. Nevertheless, at $z_{\text{max}} = 1$ and beyond, it is no longer true that adding higher order terms improves the approximation. This is evident from the $z_{\text{max}} = 2.5$ entry, where it is in fact the polynomials of lower order that perform the best.\(^3\)

This may be a little counter-intuitive, but this notion that higher order terms improve precision is only true within the radius of convergence. In fact, within the radius of convergence, one can find cases where the addition of a term reduces precision, but the overall trend is of increasing precision. Note, these problems with convergence can be solved by expanding in the $y$-parameter, $y = z/(1 + z)$, as advocated in [9]. The $y$-parameter, however, performs worse than $z$ below $z_{\text{max}} = 1$, and one typically requires a large number of expansion parameters.\(^4\)

It is also worth noting from Table 1 that overall the $n = 10$ polynomial only performs marginally better than $n = 5$, and given that the latter has fewer parameters, this singles it out as a better choice. Thus, let us jettison the higher order terms $n > 5$ and focus on identifying the redshift where the precision dips under 1%, 3%, 5% and 10%, respectively, and stays there for matter densities in the window $0.25 \leq \Omega_{m0} \leq 0.35$ within the flat ΛCDM model. The latter matter density range is reasonably wide, so for $\Omega_{m0} = 0.3$, we will be conservatively within the precision.

The result of this second exercise is shown in Table 2, where the vagaries of these types of expansions are further exposed. Here we do not use (2.4), but simply focus on the % error, which, as we have argued, is usually just as good. If one demands less than 3%, 5% or 10% precision at a given point, the lower order polynomials perform surprisingly well, but it is only below 1% that our intuition, namely that higher orders give better approximations, is restored. Observe that the $n = 4$ polynomial performs more or less on par with the $n = 3$ polynomial. Moreover, given the quality of current cosmological data, we find that MCMC chains of a $n = 5$ polynomial take a long time to converge. For this reason, in the analysis

---

\(^3\)There is considerable confusion in the literature on this point. See [14,33] for recent examples of a $n = 5$ polynomial at $z_{\text{max}} \sim 2.5$ and $z_{\text{max}} \sim 4$, respectively.

\(^4\)See for example Figure 9 of [15].
Table 2: For polynomials of different order $n$ we highlight the maximum redshift range where the approximation with flat ΛCDM in the matter density range $0.25 \leq \Omega_{m0} \leq 0.35$ is within a given %.

| $n$ | $z_{\text{max}}(<1\%)$ | $z_{\text{max}}(<3\%)$ | $z_{\text{max}}(<5\%)$ | $z_{\text{max}}(<10\%)$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 5   | 1.14            | 1.39            | 1.53            | 1.76            |
| 4   | 0.86            | 1.13            | 1.29            | 1.55            |
| 3   | 0.8             | 1.16            | 1.4             | 1.87            |
| 2   | 0.66            | 1               | 1.24            | 1.72            |

that follows we focus on the $n = 3$ polynomial.

In Figure 1 we provide confirmation that a subset of the CPL [30,31] class of dynamical dark energy models (see appendix B for a short analysis) is covered by the Taylor expansion. More concretely, we work with the $n = 3$ polynomial through to $z_{\text{max}} = 0.8$ and the dots denote the original parameter space, green dots denote values of $(\Omega_{m0}, w_0, w_a)$ that are recovered within 1%. While the class of CPL models described by the Taylor expansion is clearly not null, there is noticeable bias towards $w_a > 0$.

### 2.2 Gaussian Processes

GP is a method to reconstruct, or identify interpolating data given a (potentially sparser) initial dataset. In essence, given $n$ observational real data points $y = (y_1, \ldots y_n)$ at redshifts $z = (z_1, \ldots z_n)$ with a covariance matrix $C$, one wishes to reconstruct a function $f^* = (f(z^*_1), \ldots f(z^*_N))$ underlying the data at $N$ new points $z^* = (z^*_1, \ldots, z^*_N)$. Note that $n$ and $N$ do not need to be the same, and in practice $N > n$. Obviously, attempting to reconstruct data far outside the range of the original data will lead to questionable results.

In implementing GP, one has to make an assumption on how the reconstructed data points are correlated, and to do so, one introduces a new covariance matrix $K(z^*, z^*)$, typically called a “kernel”. The kernel $K$ is a function of some hyperparameters, commonly taken to be two in cosmological applications ($\sigma_f$, $\ell_f$). The most commonly used kernel, from which the method derives its name, is Gaussian,

$$K(z, \tilde{z}) = \sigma_f^2 \exp \left( -\frac{(z - \tilde{z})^2}{2\ell_f^2} \right). \quad (2.5)$$

---

5Note that in our viewpoint, CPL dark energy model should be viewed as a Taylor expansion in $y = z/(1+z)$, cf. (B.2). For this reason, despite being generic at lower redshifts, at higher redshifts it corresponds to a restricted class of dark energy models. See [34] where this is spelled out for the Quintessence regime, $w(z) > -1$.

6That being said, we will be taking a slight liberty with the range of the original data in extrapolating from $z \sim 0.07$ down to $z \sim 0$ to extract $H_0$. This is in line with the analysis of [25].
The other kernels that are commonplace in cosmological settings are the Matérn covariance functions:

\[ K_{\nu}(z, \tilde{z}) = \sigma^2 \left( \frac{\sqrt{2\nu(z - \tilde{z})^2}}{\ell_f} \right)^\nu \tilde{K}_\nu \left( \frac{\sqrt{2\nu(z - \tilde{z})^2}}{\ell_f} \right), \]

where \( \Gamma \) is the gamma function and \( \tilde{K}_\nu \) is a modified Bessel function. Here \( \nu \) is a positive parameter and in the limit that \( \nu \to \infty \), one recovers the Gaussian kernel (2.5). It should be noted that the Matérn kernels are only mean square \( n \)-differentiable provided \( \nu > n \). Thus, the advantage of the Gaussian kernel is that it is infinitely differentiable. This differentiability property is important when one is interested in the derivatives of \( H(z) \), but as we work here with OHD, this is less of a concern. In addition to the Gaussian, following [19], we will largely focus on \( \nu = p + \frac{1}{2} \), where \( p = 0, 1, 2, 3, 4 \), and simplified expressions for these functions can be found in the appendix A.

Since we are only interested in \( H(z) \), the mean \( \bar{f}^* \) and the covariance \( \text{cov}(f^*) \) from the GP reconstruction can be easily constructed through a few lines of linear algebra [11]:

\[ \bar{f}^* = \mu^* + K(z^*, z)[K(z, z) + C]^{-1}(y - \mu), \]

\[ \text{cov}(f^*) = K(z^*, z^*) - K(z^*, z)[K(z, z) + C]^{-1}K(z, z^*), \]

where \( \mu^* \) is a prior that one typically sets to zero \( \mu^* = 0 \) [11]. The only problem now is to identify the hyperparameters and this is done through the following log normal likelihood:

\[ \ln L = -\frac{1}{2} y^T[K(z, z) + C]^{-1} y - \frac{1}{2} \ln |K(z, z) + C| - \frac{n}{2} \ln 2\pi. \]
In a strict Bayesian sense, one should marginalise over the hyperparameters through an MCMC routine. However, this is computationally more expensive, so in practice it is common to simply optimise (2.9). We will comment on the (minimal) difference it makes to our analysis later. As the kernels become more complicated, this process requires further information and it is useful to note that the gradient of \( \ln L \) can be easily calculated,

\[
\nabla (\ln L) = \frac{1}{2} y^T (K + C)^{-1} \nabla K (K + C)^{-1} y - \frac{1}{2} \text{tr}[(K + C)^{-1} \nabla K].
\] (2.10)

This completes a quick review of the GP reconstruction process. Finally, as we will see for the mock data realisations we study, \( K \) is typically much greater the \( C \), \( K \gg C \). As a result, the kernel choice amounts to imposing a prior on the covariance matrix and this induces implicit model dependence.

### 2.3 Data

In this paper we will use observational Hubble data or OHD, which will serve as the basis for mock realisations. More precisely, we will make use of CC [35] and homogenised BAO [37] data. It should be stressed that the CC data largely comprises statistical errors only, and the systematic errors on \( H(z) \) are a work in progress [40]. That being said, this OHD will largely serve as the basis for mock realisations of the flat \( \Lambda \)CDM cosmological model with the canonical parameters \((H_0, \Omega_{m0}) = (70, 0.3)\). Furthermore, we are not interested in the absolute value of \( H_0 \), but the errors. We present the OHD in Figure 2 and it is worth noting that there are 51 data points. From the real data, we will extract the redshifts \( z_i \) and the errors in the Hubble parameter \( \sigma_{H(z_i)} \). To perform the mocks, at each \( z_i \) we will choose a new \( H(z_i) \) value from a normal distribution about the flat \( \Lambda \)CDM value with standard deviation \( \sigma_{H(z_i)} \).

### 3 Taylor expansion analysis

Let us fix notation so that our Taylor expansion takes the form

\[
H(z) = H_0 \left(1 + h_1 z + h_2 z^2 + h_3 z^3 + O(z^4) \right),
\] (3.1)

where as we are largely interested in \( H_0 \), we do not see the need to specify the remaining parameters. One could rewrite them in terms of the usual cosmographic parameters [8, 9] but this will not interest us here. However, it is useful to get a feel for reasonable values of the \( h_i \) and we document their values in a canonical flat \( \Lambda \)CDM set-up: \( h_1 = 0.45, h_2 = 0.349 \) and \( h_3 = -0.007 \). The fact that \( h_3 \) is small explains why the \( n = 2 \) polynomial approximates flat \( \Lambda \)CDM reasonably well (see Table 2).

---

7See Table 1 of [24,25] and the original references [36].
8See Table 2 of [38] and [39] for the original results.
In section 2 we introduced “close to ΛCDM” Taylor expansions of order $n$ that are valid to a given precision in the sense that they recover flat ΛCDM in the range $0.25 \leq \Omega_m \leq 0.35$ within 1%, 3%, 5% and 10%, respectively. As we have seen, one simply has to “stretch” the Taylor expansion to larger redshifts and this reduces the precision. By stretching, we are fitting the same number of parameters against the data over an extended range. Thus, it is intuitively obvious that the errors of all the cosmological parameters will reduce. Thankfully, this is easy to quantify. First, let us adopt the polynomial (3.1) and fix the data to the original real data. Next, one runs an MCMC exploration using emcee [41] up to $z = 0.8, z = 1.16, z = 1.4$ and $z = 1.87$, respectively. From the MCMC chains, one can plot a few thousand curves and at each $z_i$ at typical separations of $\Delta z_i \sim 0.01$, one evaluates (3.1) and extracts the mean, 16$^{\text{th}}$ and 84$^{\text{th}}$ percentile. The mean and 1$\sigma$ confidence intervals are shown in Figure 3, where we have separated the four polynomials into two plots to aid presentation. We have restricted the plot to the range $z \leq 0.8$, but it is clear that the most accurate Taylor expansion ($< 1\%$) leads to the largest errors on $H_0$. This simply backs up intuition.

We can also make a further observation based on the same data, which will be helpful later. As one stretches the redshift range, or alternatively allows for less precision, it is also intuitively obvious that the Taylor expansion has less freedom to explore the parameter space. As a result, correlations between parameters become less pronounced. We show this trend in Figure 4, where we made use of getdist package [42]. It is worth noting that not only are the ellipses shrinking, but also the tilt in the confidence intervals becomes less noticeable as the precision decreases. This is particularly evident from the blue ellipses: $H_0$ is more correlated with $h_1$ than $h_2$, and the correlation with $h_2$ exceeds $h_3$. A similar pattern is also there for $h_i - h_j$ plots, which are not shown here. So, while “nearby” parameters know something about each other as the distance in redshift grows, the correlations drop off as precision is lost in moving from the green to the blue contours.

Finally, all our comments so far have been for a specific realisation, namely the real data in Figure 2, so it is instructive to perform some repeats with different realisations of the
Figure 3: The mean values and 1σ confidence intervals for the Taylor expansion (3.1) fitted to the real data over different ranges with different approximations to flat ΛCDM. To aid comparison, we have restricted the plots to $z = 0.8$ (see Table 1).

Figure 4: Correlations of the parameters in the polynomial (3.1) at different precision.

data in order to get results that do not rely on a specific mock. Concretely, we mock up flat ΛCDM data as described in section 2 so that we can confirm the average error in $H_0$ over a larger sample. The results can be found in Table 3. In performing these repeats, we find that a number of the Hubble parameters are not monotonically increasing with redshift. As we mention in the appendix B for the CPL case, a non-monotonic $H(z)$ ($H'(z) \leq 0$ and in particular $h_1 \leq 0$) is an exotic feature and we exclude these mocks. These become more prevalent as $z_{\text{max}}$ is lowered and the Taylor expansion starts to explore parameter space more. Nevertheless, we have ensured approximately 300 repeats once these mocks are eliminated.

| Precision | $< 1\%$ | $< 3\%$ | $< 5\%$ | $< 10\%$ |
|-----------|---------|---------|---------|---------|
| $H_0$ (km/s/Mpc) | $61.40_{-10.41}^{+10.07}$ | $61.98_{-7.77}^{+7.72}$ | $64.03_{-6.23}^{+6.22}$ | $66.55_{-5.26}^{+5.25}$ |

Table 3: $H_0$ for the polynomial (3.1) based on an average over $\sim 300$ mock flat ΛCDM realisations of the data.
4 GP analysis

In this section we focus on $H_0$ extracted from the GP reconstruction. This is arguably the simplest cosmological parameter that one can reconstruct from the data since it just involves an extrapolation beyond the last data point ($z = 0.07$ in our study) to $z = 0$. Technically speaking, one should be careful about implementing GP reconstruction beyond the range of the data, but in cosmology, such extrapolations are second nature. Before beginning, it is instructive to remove the BAO data and run the GP analysis for the CC data with a Gaussian kernel, just to examine our GP code. We find $H_0 = 67.55^{+4.75}_{-4.68}$ km/s/Mpc, which is completely consistent with the result of Yu, Ratra & Wang [24], $H_0 = 67.42^{±4.75}$ km/s/Mpc, so there is no indication that our GP code is doing anything unusual. In particular, the errors appear consistent. It should be stressed again that GP is simple linear algebra.

In comparison to the last section, where fits of the polynomial to mock data required MCMC, which is time consuming, here we begin exploring the GP process whereby the likelihood (2.9) is minimised. This represents a simplification, but once the results are in, we will make amends and comment on the difference marginalisation makes. Let us begin with some general comment on GP. It is expected that GP suppresses correlations as there is an exponential fall off over redshifts for all kernels in the Matérn class (see (A.1)). This means that data points that are separated by larger redshift ranges are less correlated and it is the rate of fall off of the correlations that governs the final errors.

\[
\begin{align*}
K_\nu & | H_0 \text{ (km/s/Mpc)} & \sigma_f & \lf_l \\
\nu = 1/2 & | 73.92^{+13.23}_{-13.22} & 176.52^{+180.72}_{-172.24} & 43.45^{+4.65}_{-5.52} \\
\nu = 3/2 & | 68.61^{+5.11}_{-5.11} & 284.23^{+30.90}_{-38.17} & 8.77^{+1.65}_{-1.83} \\
\nu = 5/2 & | 68.81^{+3.91}_{-3.90} & 252.51^{+30.03}_{-40.47} & 5.19^{+0.91}_{-0.92} \\
\nu = 7/2 & | 69.35^{+3.64}_{-3.64} & 241.94^{+30.06}_{-38.64} & 4.30^{+0.76}_{-0.80} \\
\nu = 9/2 & | 69.25^{+3.57}_{-3.57} & 237.34^{+30.00}_{-40.06} & 3.91^{+0.66}_{-0.75} \\
\text{Gaussian (}\nu = \infty & | 69.56^{+3.42}_{-3.42} & 230.20^{+41.48}_{-42.08} & 3.03^{+0.60}_{-0.57} \\
\end{align*}
\]

Table 4: Average values of $H_0$ and hyperparameters ($\sigma_f, \lf_l$) for different kernels and 500 mock realisations of the data. There is a definite trend where the error in $H_0$ decreases with increasing $\nu$.

In Table 4 we show how the inferred Hubble constant $H_0$ depends on the kernel for the full redshift range of the data $0 \lesssim z \lesssim 2.5$. It should be stressed that we are using OHD, namely CC and BAO data, but since we average over a large number of mocks, we should be reporting general trends for OHD. In a comparison with the Taylor results in Table 3, we see that the $\nu = 1/2$ kernel performs worse on the errors than any of the Taylor expansions, which suggests that it may be more agnostic about models than Taylor. 9 For $\nu = 3/2$,

---

9Interestingly, in [25] a weighted polynomial regression method, which is based on Taylor expansion, is
the errors are consistent with an $n = 3$ Taylor expansion that only follows models to 10% precision, while for $\nu = 5/2$ and beyond, the GP outperforms any of our Taylor expansions. Furthermore, the differences in the errors are small beyond $\nu = 5/2$. Note that $\nu = 1/2$ and $\nu = 3/2$ are rarely discussed by GP practitioners.

We find that errors on $H_0$ decrease with increasing $\nu$, and our analysis shows that the smallest $H_0$ error is achieved for the Gaussian kernel\(^{10}\). Our findings are in line with Table 1 of [23], where we have included the $\nu = 1/2$ and $\nu = 3/2$ entries just to fill out the picture. Thus, it seems only reasonable that the Gaussian kernel should exhibit the fastest fall off of the correlations with redshift. To check this, we set one of the redshifts in the kernel $K_\nu(z, \tilde{z})$ to a nominal value, which we take to be zero, and plot the normalised Kernel, $K_\nu/\sigma_f^2$ against redshift in Figure 5. We take our $l_f$ values from the central values in Table 4. As can be clearly seen from the plot, the correlations are suppressed the most by Matérn covariance matrices with larger $\nu$. As a result, we have an intuitive picture: GP provides smaller errors because it limits correlations.

![Figure 5](image_url)  
**Figure 5:** For different kernels in the Matérn class we show the fall off in correlations based on the data in Table 4.

We can test this idea in another way. So far, we have used the whole dataset, but one can limit the redshift range by imposing a $z_{\text{max}}$ and one should find that the error on $H_0$ is insensitive to the inclusion of higher redshift data. In Table 5, we record $H_0$ for different $z_{\text{max}}$. Interestingly, below $z = 0.4$, where the data is of relatively poor quality (no BAO), we still find a $H_0$ determination that is better than any of the Taylor expansions explored in the last section (see Table 3). With the inclusion of the BAO data at lower redshifts, the error improves considerably at $z_{\text{max}} = 0.8$ and beyond that we see relatively little improvement in studied and the $H_0$ errors are comparable to GP. It would be nice to study this further.

\(^{10}\)As is clear from Table 3 of [25] if we add a prior on $H_0$, then this effect may not be seen. This is easy to understand: GP will treat the $H_0$ prior essentially as input data and anchors $H_0$. 

12
the error through to \( z_{\text{max}} = 2 \). This can be contrasted with the Taylor expansion, where we see from Table 3 that the error on \( H_0 \) halves between \( z_{\text{max}} = 0.8 \) and \( z_{\text{max}} = 1.76 \). Overall, the observation that the GP \( H_0 \) is less sensitive to the higher redshift data, may be attributed to the fall off in correlations seen in Figure 5.\(^{11}\)

\[
\begin{array}{cccc}
 z_{\text{max}} & H_0 \text{ (km/s/Mpc)} & \sigma_f & I_f \\
0.4 & 69.67^{+4.75}_{-4.74} & 80.66^{+3.50}_{-3.47} & 2.01^{+2.64}_{-0.65} \\
0.8 & 66.95^{+3.73}_{-3.73} & 102.05^{+10.70}_{-10.97} & 2.20^{+0.40}_{-0.47} \\
1.2 & 67.05^{+3.74}_{-3.74} & 109.48^{+13.50}_{-15.37} & 2.30^{+0.51}_{-0.67} \\
1.6 & 67.94^{+3.64}_{-3.63} & 157.59^{+16.06}_{-15.17} & 2.70^{+0.91}_{-0.79} \\
2 & 69.06^{+3.65}_{-3.65} & 164.95^{+21.40}_{-21.65} & 2.68^{+0.94}_{-0.78} \\
\end{array}
\]

Table 5: For the Gaussian kernel we illustrate how the errors in \( H_0 \) change as we restrict the data below at given \( z_{\text{max}} \). All results are based on 500 realisations.

It is instructive to fit the same data to concrete models in order to compare the errors in \( H_0 \). The result of this exercise is reported in Table 6 for a number of repetitions of mock data. Evidently, the errors on \( H_0 \) for both \( \Lambda \)CDM and \( w \)CDM are within the GP errors, but for CPL we find that the error on \( H_0 \) is larger. Something similar was observed in [23] with poorer quality CC data (no BAO). However, there the \( w \)CDM model led to larger errors on \( H_0 \) than GP. At the time, no flags were raised, but it is a little puzzling when a “model independent” technique outperforms a specific model on errors.

In order to ascertain if this is a fluke, we replace our OHD with the DESI \( H(z) \) determination forecasts in the extended redshift range \( 0.05 \leq z \leq 3.55 \) [43],\(^{12}\) where we assume the optimistic outcome that the five-year survey covers 14,000 deg\(^2\). Repeating the exercise, we can see from Table 7 that even with the forecasted data, GP outperforms the CPL model by leading to smaller errors on \( H_0 \). We conclude that this is not an artifact of the dataset and that GP produces smaller errors on \( H_0 \) than CPL. We will return to interpret this in section 5.

\[
\begin{array}{cc|ccc}
 \text{Model} & \Lambda \text{CDM} & \text{wCDM} & \text{CPL} \\
 H_0 \text{ (km/s/Mpc)} & 69.91^{+1.17}_{-1.20} & 69.79^{+3.05}_{-2.86} & 65.64^{+5.11}_{-4.95} \\
\end{array}
\]

Table 6: Average values of \( H_0 \) for different models based on \( \sim 300 \) mock realisations of the data in Figure 2.

A relevant question now is whether marginalisation makes enough of a difference to the numbers in Table 4. Unfortunately, running an MCMC to identify the hyperparameters\(^{13}\)

\(^{11}\)This is the advantage of GP over Taylor expansion: GP is more stable over an extended redshift range.

\(^{12}\)See [44] for how related forecasts will constrain GP constraints on deceleration.
Optimisation Marginalisation

given results in the literature [19].
performs quite well and marginalisation adds little more. This may have been anticipated
errors. At least for the dataset under study, and the related mocks, it seems that optimisation
marginalisation. Table 8: A number of mocks showing the difference in $H_0$ for different models based on between 300+ mock realisations of the forecasted DESI data.

for each realisation of the mock data takes time, so we will show the results of particular realisations, thereby providing a “before and after” snapshot of the errors, once again with the focus on $H_0$. This exercise has been performed in various works, in particular [19], where it was noted that marginalisation does not necessarily increase the errors. It has also been shown in [25] that the distribution of the hyperparameters is not strongly peaked and for this reason marginalisation does not change the errors.

From our end, the observations in [19,25] should be enough, but we independently verify it through the mocks in Table 8. The first realisation represents the real data in Figure 2, so our numbers are easily reproduced. Note that the confidence intervals for $\sigma_f$ and $\ell_f$ are shifted to slightly higher values relative to their optimised counterparts. This is due to the lopsided nature of the distribution, an example of which we present in Figure 6. As a result, the mean, 16$^{\text{th}}$ and 84$^{\text{th}}$ percentile confidence intervals are slightly to the right relative to the peak. This explains the slight conflict between the hyperparameters in Table 8. As noted in [25], there also appears to be a trend whereby the central value of $H_0$ decreases and the errors increase slightly.  

| Mock | Optimisation | Marginalisation |
|------|-------------|----------------|
|      | $H_0$ (km/s/Mpc) | $\sigma_f$ | $\ell_f$ | $H_0$ (km/s/Mpc) | $\sigma_f$ | $\ell_f$ |
| 1    | 67.63$^{+3.35}_{-3.40}$ | 175.57 | 2.71 | 65.48$^{+3.69}_{-3.46}$ | 425.65$^{+516.64}_{-210.51}$ | 5.43$^{+4.61}_{-2.31}$ |
| 2    | 75.52$^{+3.41}_{-3.41}$ | 244.30 | 2.92 | 74.94$^{+3.48}_{-3.41}$ | 491.97$^{+520.84}_{-226.39}$ | 4.28$^{+2.26}_{-1.35}$ |
| 3    | 73.28$^{+3.80}_{-3.80}$ | 173.56 | 2.12 | 71.59$^{+4.06}_{-3.79}$ | 318.24$^{+356.51}_{-133.79}$ | 3.19$^{+2.19}_{-1.08}$ |
| 4    | 72.94$^{+3.47}_{-3.50}$ | 244.88 | 2.74 | 72.17$^{+3.49}_{-3.43}$ | 449.83$^{+480.92}_{-205.05}$ | 4.07$^{+2.18}_{-1.33}$ |
| 5    | 66.26$^{+3.15}_{-3.14}$ | 261.68 | 4.12 | 65.20$^{+3.34}_{-3.26}$ | 595.51$^{+598.91}_{-285.24}$ | 7.16$^{+4.38}_{-2.82}$ |

Table 8: A number of mocks showing the difference in $H_0$, $\sigma_f$ and $\ell_f$ for optimisation versus marginalisation.

Overall, we find no evidence to suggest that marginalisation can increase the size of the errors. At least for the dataset under study, and the related mocks, it seems that optimisation performs quite well and marginalisation adds little more. This may have been anticipated given results in the literature [19].

\[13\] We thank Fabrizio Renzi for bringing this to our awareness.
5 Mapping GP to Taylor

As we saw in the previous section, GP can produce tighter errors on $H_0$ than the CPL model. Naively, this contradicts the “model independence” of GP and hints at the fact that the kernels employed may be somehow limiting the parameter space of the CPL model. In this section, we attempt to substantiate this idea by mapping the GP analysis back into the language of the Taylor expansion. This has the advantage that at least in the Taylor expansion, one can make a clear statement regarding the relations among the parameters imposed by models and which models can be accommodated in the framework. On the contrary, when one works directly with the data and makes assumptions about correlations, the implications for models are never clear. One is essentially blind to the consequences.

To begin, let us recall that the typical output of a GP reconstruction is a confidence interval. As is clear from Figure 3 it is easy to convert an MCMC chain into a confidence interval in more or less the same way done with GP. This provides a way to visually compare the confidence intervals. However, here we will do the reverse and map the GP back into the Taylor expansion. At first sight there appears to be no obvious connection, but given that GP returns a mean and a covariance matrix, we are able to generate a large number of configurations, essentially parametrising curves, and map them back into the parameters of the Taylor expansion. Since the Taylor expansion’s regime of validity is limited, this means we will have to employ GP over a shorter redshift range than in usual applications.

GP is essentially a data smoothing technique, so we can expect the curves one draws from the mean and covariance to be within the scope of a Taylor expansion provided it can track models to an appropriate precision. To make this discussion more concrete, let us fix

Figure 6: The distribution of the hyperparameters $(\sigma_f, \ell_f)$ for the real data in Figure 2.
\( z_{\text{max}} = 0.8 \), which ensures that our \( n = 3 \) Taylor expansion accurately recovers a class of models to within 1%. Since the Taylor expansion is well-defined in the vicinity of flat \( \Lambda \text{CDM} \) with canonical values, we will mock up data largely in the vicinity of \((H_0, \Omega_{m0}) = (70, 0.3)\). Once again, we stress that the (truncated) Taylor expansion is *not* model independent, but simply describes a class of models. Let us also work with the original data below the cut-off redshift. To extract information from the GP, we choose the Gaussian kernel, which is the most constraining, minimise the likelihood (2.9) and extract \( \sigma_f, l_f \). This allows us to reconstruct the mean and the covariance matrix. We choose to do this at 100 data points in the range \( 0 \leq z \leq 0.8 \), so we have a \( 100 \times 100 \) covariance matrix. Finally, we generate 10,000 random configurations and for each one we fit the \( n = 3 \) Taylor expansion. This leads to an array of \((H_0, h_1, h_2, h_3)\).

\[
\begin{array}{c|c|c|c}
\text{Kernel} & H_0 & h_1 & h_2 \\
\hline
\text{Taylor} & \text{Green} & \text{Gray} & \text{Red} \\
(\nu = 3/2) & \text{Blue} & \text{Purple} & \text{Orange} \\
(\nu = 5/2) & & & \\
(\nu = \infty) & & & \\
\end{array}
\]

Figure 7: Comparison between Taylor expansion and GP in the parameter space of the Taylor expansion.

Since we are looking at a large number of configurations, one may worry that any random curve from the GP may not be well approximated by the Taylor expansion. To this end, we can define an analogous sum to (2.4), but now we are comparing the best-fit Taylor expansion against the mean of the GP for a given configuration. We accept this best-fit provided the summed error is less than the sum corresponding to a 1% error at each redshift for the GP reconstruction.\(^{14}\) This ensures that our Taylor expansion is following the GP curve. The comparison between the Taylor and GP in the parameter space of the Taylor expansion is shown in Figure 7. For comparison, we also plotted the parameter space of the \( \nu = 3/2 \) and \( \nu = 5/2 \) kernels.

Let us put Figure 7 in proper context. Previously, we had established that GP performs better than Taylor in producing \( H_0 \) with a lower error. This is a statement at \( z = 0 \), and Figure 7 extends this comparison over a finite range of redshifts up to \( z_{\text{max}} = 0.8 \), where we have confidence in the Taylor expansion. This allows us to see the correlations between \( H_0 \) and the sub-leading in \( z \) parameters in the Taylor expansion. Consistent with Figure 5, we

\(^{14}\)Only for the \( \nu = 3/2 \) kernel did we need to relax the error to 1%. For the higher order kernels 0.1\% worked well.
see that correlations are being suppressed by the kernels with higher \( \nu \). In particular, the tilt of the confidence ellipses is being affected and this can be directly traced to the underlying GP assumption that the correlations fall off exponentially.

Figure 8: Mappings of the GP, \( w_{\text{CDM}} \) and CPL model into the \( n = 3 \) Taylor series parameter space.

Now, the reader is right to complain that Figure 7 is not a fair comparison, as we have to restrict GP to the Taylor expansion’s regime of validity and GP can be applied at higher redshifts. In other words, we have deliberately set up a comparison that does not play to the strengths of GP. Here, we will make amends and return to the puzzle in the last section, namely the question why does GP, a “model independent” technique, underestimate errors in \( H_0 \) relative to the CPL model? As we have seen, future DESI data is not expected to change this conclusion. As mentioned CPL is a Taylor expansion in the dark energy equation of state (B.2). As such, at least at low redshift, it is generic, and it can cover a large class of dynamical dark energy models. For this reason, one can expect some of our observations above to transfer wholesale over to the CPL model.

To make the comparison fair, we fix the data to the original data and fit both \( w_{\text{CDM}} \) and CPL through MCMC over the entire redshift range. In tandem, we perform a GP reconstruction using the Gaussian kernel, since the other preferred kernels, namely \( \nu = 7/2 \) and \( \nu = 9/2 \), perform similarly well [19]. Now, while we would struggle to map curves from either of these techniques back into a Taylor expansion in \( z \), essentially because Taylor expansion makes little sense beyond the radius of convergence \( |z| = 1 \), we can map the low redshift region of the curves back into the Taylor expansion. In other words, we can use the Taylor expansion in its regime of validity as a setting to compare the CPL model and the GP reconstruction. This is an honest and meaningful comparison.

However, as is evident from Figure 8 in the parameter space of the Taylor expansion, the confidence intervals, in particular the 2\( \sigma \) intervals are larger. This means that the GP is not capturing all the parameter space of the CPL model, but some subset. Moreover, it is clear that this subset encompasses the \( w_{\text{CDM}} \) model. Interestingly, for the real data, despite the correlations being visibly different, when projected onto the \( H_0 \) axis, we see that the difference in \( H_0 \) for this realisation is negligible. That being said, it should be clear from
Tables 4, 6 and 7 that over a larger number of mocks that the errors in $H_0$ in the CPL model exceed the errors in $H_0$ from GP. Now, since $H_0$ is just the projection of the correlations across $(H_0, h_1, h_2, h_3)$ onto the the $H_0$ axis, it should be evident to the reader that there will also be a considerable difference in the correlations.

![Figure 9: Comparison between the CPL model and GP in the Taylor expansions parameter space. Here, we mocked the data up on the CPL model.](image)

To put this beyond doubt, we can mock up the data on a CPL model, which we choose to be $(H_0, \Omega_{m0}, w_0, w_a) = (70, 0.3, -1.1, 0.1)$. We will repeat the same steps as before and map the configurations back into the Taylor expansion at low redshift. The result is shown in Figure 9. Evidently, the GP only covers a portion of the CPL parameter space in the low redshift region described by the Taylor expansion. Note, this is also the region where one determines $H_0$. Thus, the picture that is emerging is that GP can struggle to properly sample the parameter space of specific models, and as a result produce lower errors. Note, GP itself is not at fault, but the kernel assumptions appear too strong.

6 Concluding remarks

In the cosmology literature one finds no shortage of papers discussing “model independence”. The point of this paper is that a complete model independence in precision cosmology, and particularly with the Hubble tension, may be a mirage. As is clear from the Taylor expansion in section 2, any technique that is transparent in covering a sufficiently large class of models is expected to lead to large errors for cosmological parameters.\footnote{See [45] for a “model independent” $H_0$ determination with errors that raise few suspicions.} This undermines its utility. In contrast to Taylor expansion, which is analytic, one has a host of other data reconstruction techniques that carry the label “model independent”, and GP is just one example. Often the implications for the space of cosmological models is unclear. Indeed, it can be difficult to compare and this problem motivates our work.

Thus, the main thrust of this work was to benchmark GP against a (more) “model independence”
independent” technique, Taylor expansion, where the underlying model assumptions were clearer. We made a number of observations:

- Taylor expansion is safest performed within its radius of convergence - here \( |z| = 1 \) \[9\] - and this ensures higher order terms in the expansion improve the approximation to the exact model. Thus, expansions in \( z \) beyond \( z = 1 \) are dubious.

- Taylor expansion is only model independent in the immediate vicinity of the redshift \( z = z_0 \) around which one is expanding. Away from \( z = z_0 \), Taylor expansion describes a class of models and it is imperative to check this is not the null set.

- Neglecting the \( \nu = 1/2 \) Matérn kernel, GP typically leads to smaller errors than Taylor expansions that cover flat \( \Lambda \)CDM to 1% precision. Moreover, there is a definite trend where the error in \( H_0 \) decreases with increasing \( \nu \). It can be shown that this is related to the fall off in correlations.

- As our analysis and discussions reveal, the GP kernel dominates the original covariance matrix and this introduces model dependence. This model dependence appears through a restriction in the correlations of the parameters, rather than usual model dependence, i.e. imposing relations among parameters. This model dependence especially shows up in the reduction of the errors on the value of parameters determined through GP analysis of the data.

- One can map GP back into the Taylor expansion in its range of validity to confirm that the parameter space explored by GP is smaller. Moreover, by mapping the CPL model into the Taylor expansion, one can show that GP covers the \( w \)CDM parameter space, but fails to cover the entire CPL parameter space.

There is a big picture here. In recent years, we have witnessed a number of papers, e.g. [23–29], where GP is employed to infer \( H_0 \) in a “model independent” way. One often ends up with percent level [25,28,29], or on occasion, even sub percent level determinations [26]. Bearing in mind that local determinations of \( H_0 \) are themselves independent of cosmological models, one faces a conflict, which, taken at face value, can only be resolved either by systematics in the data\[16\], or by revisiting the FLRW cosmology framework.\[17\] However, before resorting to systematics or the other drastic conclusion of a departure from FLRW cosmology, there is another simple resolution that should not be overlooked.

The currently used GP kernels, in particular the Matérn class with \( \nu \geq 5/2 \), while seemingly sufficient to generalise flat \( \Lambda \)CDM, may be a problem. As we have shown, these kernels suppress correlations among different parameters in a (truncated) Taylor expansion, and as a result, one may be confined to a reduced parameter space in a given model. Concretely,

---

\[16\] Attempts have been made to incorporate CC systematics in the analysis, e.g. [25,26].

\[17\] Note that this does not rely on Einstein field equations, or Friedmann equations, and is only based on the isotropy and homogeneity of the FLRW spacetime. So, it is true for any modified gravity too.
we have shown that this is the case for the CPL model, which itself should be viewed as a Taylor expansion in dark energy EoS, and thus, it represents the generic behaviour for a dynamical dark energy model at low redshift. The pertinent question our work raises is to what extent are the simplest GP kernels precluding dynamical dark energy? In other words, should more general kernels\(^\text{18}\) be considered, and if so, what are the implications for \(H_0\)?

Finally, it is worth asking whether one can have faith in “model independent” determinations of \(H_0\) within a cosmological framework? We have seen a difference in the errors between Taylor expansion and GP. Furthermore, even within the confines of a set technique, one can reduce the errors by increasing the range of the Taylor expansion or using a GP kernel that restricts correlations more. This introduces a degree of arbitrariness that is difficult to ignore. Ultimately, while Hubble tension is a mismatch between two numbers\(^\text{19}\), it is also a more general narrative about the errors and great care is required to make sure that they are not underestimated.

**Acknowledgements**

We thank Stephen Appleby and Tao Yang for discussions and ongoing related collaborations. We are also grateful to Chris Clarkson, Adrià Gómez-Valant, Eric Linder, Rafael Nunes and Fabrizio Renzi for sharing their expertise on GP through helpful comments on the draft. EÖC is funded by the National Research Foundation of Korea (NRF-2020R1A2C1102899). MMShJ would like to acknowledge SarAmadan grant No. ISEF/M/99131.

**A Matérn Covariance Functions**

For \(\nu = p + \frac{1}{2}, p \in \mathbb{N}^+\), the Matérn covariance matrix can be written as a product of an exponential and a polynomial of order \(p\). Here we record some simplified expressions:

\[
K_{1/2}(z, \tilde{z}) = \sigma_f^2 \exp\left(-\frac{|z - \tilde{z}|}{\ell_f}\right),
\]

\[
K_{3/2}(z, \tilde{z}) = \sigma_f^2 \exp\left(-\frac{\sqrt{3}|z - \tilde{z}|}{\ell_f}\right) \left(1 + \frac{\sqrt{3}|z - \tilde{z}|}{\ell_f}\right),
\]

\[
K_{5/2}(z, \tilde{z}) = \sigma_f^2 \exp\left(-\frac{\sqrt{5}|z - \tilde{z}|}{\ell_f}\right) \left(1 + \frac{\sqrt{5}|z - \tilde{z}|}{\ell_f} + \frac{5(z - \tilde{z})^2}{3\ell_f^2}\right).
\]

\(^\text{18}\)The Matérn class kernels have two parameters, a normalization \(\sigma_f\) and a width (length scale) \(\ell_f\). One may however consider GP covariance function with two length scales. We thank Chris Clarkson for a discussion on this.

\(^\text{19}\)However, see [1] where it is predicted that Hubble tension, if cosmological in origin, will lead to additional \(H_0\) determinations.
\[ K_{7/2}(z, \tilde{z}) = \sigma_f^2 \exp \left( -\frac{\sqrt{7}|z - \tilde{z}|}{\ell_f} \right) \left( 1 + \frac{\sqrt{7}|z - \tilde{z}|}{\ell_f} + \frac{14(z - \tilde{z})^2}{5\ell_f^2} + \frac{7\sqrt{7}|z - \tilde{z}|^3}{15\ell_f^3} \right), \]

\[ K_{9/2}(z, \tilde{z}) = \sigma_f^2 \exp \left( -\frac{3|z - \tilde{z}|}{\ell_f} \right) \times \left( 1 + \frac{3|z - \tilde{z}|}{\ell_f} + \frac{27(z - \tilde{z})^2}{7\ell_f^2} + \frac{18|z - \tilde{z}|^3}{7\ell_f^3} + \frac{27(z - \tilde{z})^4}{35\ell_f^4} \right). \]  

(A.1)

B Turning points in \( H(z) \) in CPL model

One imagines that turning points in \( H(z) \) arise when one is far from flat ΛCDM. To get an idea of how exotic a scenario this may be, it is instructive to consider the CPL model. Recall from the Friedmann equations that

\[ (1 + z)HH'(z) = \frac{1}{2} \sum_i (\rho_i + p_i) = \frac{1}{2} \sum_i (1 + w_i) \rho_i, \]  

(B.1)

where the subscript denotes a sum over sectors and we have set \( M_{pl} = 1 \) for simplicity. This means that a turning point, \( H'(z) = 0 \), is possible provided \( \sum_i (1 + w_i) \rho_i = 0 \).

Let us specialise to the CPL model which is a dynamical dark energy model with \( w(z) = w_0 + w_a \frac{z}{1+z} \),

(B.2)

and hence with (dark) matter with relative density \( \Omega_{m0} \), (1.2) takes the form,

\[ E(z) = H(z)/H_0 = \sqrt{\Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0})(1 + z)^3(1 + w_0 + w_a)e^{-3w_a z/(1+z)}}, \]

(B.3)

\[ = 1 + \frac{3}{2} (1 + w_0(1 - \Omega_{m0})) z \]

\[ + \frac{3}{4} \left[ 1 + 2(1 - \Omega_{m0})(2w_0 + 3w_0^2 + w_a) - 3w_0^2(1 - \Omega_{m0})^2 \right] z^2 + O(z^3). \]

Observations seem to require \( H' > 0 \) (absence of turning points), implying \( 1 + w_0(1 - \Omega_{m0}) > 0 \). Note, this is just the statement that \( (1 + q_0) > 0 \) and this avoids a turning point in the vicinity of \( z = 0 \). Moreover, observations have now established a positive acceleration, \( q_0 < 0 \) and hence \( 1 + 3w_0(1 - \Omega_{m0}) < 0 \),

(B.4)

which is the range of CPL parameter space that we have explored. The key point here is that one needs quite a negative \( w_0 \) to get a turning point. This is not favoured by any dataset. Also, requiring that \( H(z) \) at large \( z \) is dominated by the matter, \( H(z) \sim z^{3/2} \), necessarily implies the weak condition \( w_0 + w_a < 0 \).
References

[1] C. Krishnan, E. Ó Colgáin, M. M. Sheikh-Jabbari and T. Yang, “Running Hubble Tension and a H0 Diagnostic,” [arXiv:2011.02858 [astro-ph.CO]].

[2] A. G. Riess, L. M. Macri, S. L. Hoffmann, D. Scolnic, S. Casertano, A. V. Filippenko, B. E. Tucker, M. J. Reid, D. O. Jones and J. M. Silverman, et al. “A 2.4% Determination of the Local Value of the Hubble Constant,” Astrophys. J. 826 (2016) no.1, 56 [arXiv:1604.01424 [astro-ph.CO]].

[3] S. Dhawan, D. Brout, D. Scolnic, A. Goobar, A. G. Riess and V. Miranda, “Cosmological Model Insensitivity of Local $H_0$ from the Cepheid Distance Ladder,” Astrophys. J. 894 (2020) no.1, 54 [arXiv:2001.09260 [astro-ph.CO]].

[4] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri and D. Scolnic, “Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM,” Astrophys. J. 876 (2019) no.1, 85 [arXiv:1903.07603 [astro-ph.CO]].

[5] E. Di Valentino, “A (brave) combined analysis of the $H_0$ late time direct measurements and the impact on the Dark Energy sector,” [arXiv:2011.00246 [astro-ph.CO]].

[6] N. Aghanim et al. [Planck], “Planck 2018 results. VI. Cosmological parameters,” Astron. Astrophys. 641 (2020), A6 [arXiv:1807.06209 [astro-ph.CO]].

[7] L. Verde, T. Treu and A. G. Riess, “Tensions between the Early and the Late Universe,” Nature Astron. 3, 891 [arXiv:1907.10625 [astro-ph.CO]].

[8] M. Visser, “Jerk and the cosmological equation of state,” Class. Quant. Grav. 21 (2004), 2603-2616 [arXiv:gr-qc/0309109 [gr-qc]]; [9] C. Cattoen and M. Visser, “The Hubble series: Convergence properties and redshift variables,” Class. Quant. Grav. 24 (2007), 5985-5998 [arXiv:0710.1887 [gr-qc]].

[10] A. Shafieloo, A. G. Kim and E. V. Linder, “Gaussian Process Cosmography,” Phys. Rev. D 85 (2012), 123530 [arXiv:1204.2272 [astro-ph.CO]].

[11] M. Seikel, C. Clarkson and M. Smith, “Reconstruction of dark energy and expansion dynamics using Gaussian processes,” JCAP 06 (2012), 036 [arXiv:1204.2832 [astro-ph.CO]].

[12] A. G. Riess et al. [Supernova Search Team], “Type Ia supernova discoveries at z > 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution,” Astrophys. J. 607 (2004), 665-687 [arXiv:astro-ph/0402512 [astro-ph]]; D. H. Weinberg, M. J. Mortonson, D. J. Eisenstein, C. Hirata, A. G. Riess and E. Rozo, “Observational Probes of Cosmic Acceleration,” Phys. Rept. 530 (2013), 87-255 [arXiv:1201.2434 [astro-ph.CO]].
[13] M. Visser, “Cosmography: Cosmology without the Einstein equations,” Gen. Rel. Grav. 37 (2005), 1541-1548 [arXiv:gr-qc/0411131 [gr-qc]].

[14] G. Risaliti and E. Lusso, “Cosmological constraints from the Hubble diagram of quasars at high redshifts,” Nature Astron. 3 (2019) no.3, 272-277 [arXiv:1811.02590 [astro-ph.CO]]; E. Lusso, E. Piedipalumbo, G. Risaliti, M. Paolillo, S. Bisogni, E. Nardini and L. Amati, “Tension with the flat ΛCDM model from a high-redshift Hubble diagram of supernovae, quasars, and gamma-ray bursts,” Astron. Astrophys. 628 (2019), L4 [arXiv:1907.07692 [astro-ph.CO]].

[15] T. Yang, A. Banerjee and E. Ó Colgáin, “Cosmography and flat ΛCDM tensions at high redshift,” Phys. Rev. D 102 (2020), 123532 [arXiv:1911.01681 [astro-ph.CO]].

[16] A. Banerjee, E. Ó Colgáin, M. Sasaki, M. M. Sheikh-Jabbari and T. Yang, “On cosmography in the cosmic dark ages: are we still in the dark?,” [arXiv:2009.04109 [astro-ph.CO]].

[17] A. Aviles, C. Gruber, O. Luongo and H. Quevedo, “Cosmography and constraints on the equation of state of the Universe in various parametrizations,” Phys. Rev. D 86 (2012), 123516 [arXiv:1204.2007 [astro-ph.CO]]; C. Gruber and O. Luongo, “Cosmographic analysis of the equation of state of the universe through Padé approximations,” Phys. Rev. D 89 (2014) no.10, 103506 [arXiv:1309.3215 [gr-qc]]; A. Aviles, A. Bravetti, S. Capozziello and O. Luongo, “Precision cosmology with Padé rational approximations: Theoretical predictions versus observational limits,” Phys. Rev. D 90 (2014) no.4, 043531 [arXiv:1405.6935 [gr-qc]]; P. K. S. Dunsby and O. Luongo, “On the theory and applications of modern cosmography,” Int. J. Geom. Meth. Mod. Phys. 13 (2016) no.03, 1630002 [arXiv:1511.06532 [gr-qc]]. O. Luongo, G. B. Pisani and A. Troisi, “Cosmological degeneracy versus cosmography: a cosmographic dark energy model,” Int. J. Mod. Phys. D 26 (2016) no.03, 1750015 [arXiv:1512.07076 [gr-qc]]. S. Capozziello, Ruchika and A. A. Sen, “Model independent constraints on dark energy evolution from low-redshift observations,” Mon. Not. Roy. Astron. Soc. 484 (2019), 4484 [arXiv:1806.03943 [astro-ph.CO]]; K. Dutta, Ruchika, A. Roy, A. A. Sen and M. M. Sheikh-Jabbari, “Beyond ΛCDM with low and high redshift data: implications for dark energy,” Gen. Rel. Grav. 52 (2020) no.2, 15 [arXiv:1808.06623 [astro-ph.CO]]; S. Capozziello, R. D’Agostino and O. Luongo, “Extended Gravity Cosmography,” Int. J. Mod. Phys. D 28 (2019) no.10, 1930016 [arXiv:1904.01427 [gr-qc]]; S. Capozziello, R. D’Agostino and O. Luongo, “High-redshift cosmography: auxiliary variables versus Padé polynomials,” Mon. Not. Roy. Astron. Soc. 494 (2020) no.2, 2576-2590 [arXiv:2003.09341 [astro-ph.CO]].

[18] S. Capozziello, R. D’Agostino and O. Luongo, “Cosmographic analysis with Chebyshev polynomials,” Mon. Not. Roy. Astron. Soc. 476 (2018) no.3, 3924-3938 [arXiv:1712.04380 [astro-ph.CO]].
[19] M. Seikel and C. Clarkson, “Optimising Gaussian processes for reconstructing dark energy dynamics from supernovae,” [arXiv:1311.6678 [astro-ph.CO]].

[20] S. Yahya, M. Seikel, C. Clarkson, R. Maartens and M. Smith, “Null tests of the cosmological constant using supernovae,” Phys. Rev. D 89 (2014) no.2, 023503 [arXiv:1308.4099 [astro-ph.CO]].

[21] J. Alberto Vazquez, M. Bridges, M. P. Hobson and A. N. Lasenby, “Reconstruction of the Dark Energy equation of state,” JCAP 09 (2012), 020 [arXiv:1205.0847 [astro-ph.CO]]; T. Yang, Z. K. Guo and R. G. Cai, “Reconstructing the interaction between dark energy and dark matter using Gaussian Processes,” Phys. Rev. D 91 (2015) no.12, 123533 [arXiv:1505.04443 [astro-ph.CO]]; J. Z. Qi, M. J. Zhang and W. B. Liu, “Testing dark energy models with $H(z)$ data,” [arXiv:1606.00168 [gr-qc]]; M. J. Zhang and J. Q. Xia, “Test of the cosmic evolution using Gaussian processes,” JCAP 12 (2016), 005 [arXiv:1606.04398 [astro-ph.CO]]; R. G. Cai and T. Yang, “Estimating cosmological parameters by the simulated data of gravitational waves from the Einstein Telescope,” Phys. Rev. D 95 (2017) no.4, 044024 [arXiv:1608.08008 [astro-ph.CO]]; M. Raveri, P. Bull, A. Silvestri and L. Pogosian, “Priors on the effective Dark Energy equation of state in scalar-tensor theories,” Phys. Rev. D 96 (2017) no.8, 083509 [arXiv:1703.05297 [astro-ph.CO]]; R. G. Cai, N. Tamanini and T. Yang, “Reconstructing the dark sector interaction with LISA,” JCAP 05 (2017), 031 [arXiv:1703.07323 [astro-ph.CO]]; D. Wang and X. H. Meng, “Improved constraints on the dark energy equation of state using Gaussian processes,” Phys. Rev. D 95 (2017) no.2, 023508 [arXiv:1708.07750 [astro-ph.CO]]; S. Joudaki, M. Kaplinghat, R. Keeley and D. Kirkby, “Model independent inference of the expansion history and implications for the growth of structure,” Phys. Rev. D 97 (2018) no.12, 123501 [arXiv:1710.04236 [astro-ph.CO]]; A. Gómez-Valent and J. Solà Peracaula, “Density perturbations for running vacuum: a successful approach to structure formation and to the $\sigma_8$-tension,” Mon. Not. Roy. Astron. Soc. 478 (2018) no.1, 126-145 [arXiv:1801.08501 [astro-ph.CO]]; M. J. Zhang and H. Li, “Gaussian processes reconstruction of dark energy from observational data,” Eur. Phys. J. C 78 (2018) no.6, 460 [arXiv:1806.02981 [astro-ph.CO]]; E. Elizalde and M. Khurshudyan, “Swampland criteria for a dark energy dominated universe ensuing from Gaussian processes and $H(z)$ data analysis,” Phys. Rev. D 99 (2019) no.10, 103533 [arXiv:1811.03861 [astro-ph.CO]]; R. von Marttens, V. Marra, L. Casarini, J. E. Gonzalez and J. Alcaniz, “Null test for interactions in the dark sector,” Phys. Rev. D 99 (2019) no.4, 043521 [arXiv:1812.02333 [astro-ph.CO]]; F. Gerardi, M. Martinelli and A. Silvestri, “Reconstruction of the Dark Energy equation of state from latest data: the impact of theoretical priors,” JCAP 07 (2019), 042 [arXiv:1902.09423 [astro-ph.CO]]; M. Martinelli, N. B. Hogg, S. Peirone, M. Bruni and D. Wand, “Constraints on the interacting vacuum–geodesic CDM scenario,” Mon. Not. Roy. Astron. Soc. 488 (2019) no.3, 3423-3438 [arXiv:1902.10694 [astro-ph.CO]]; A. M. Velasquez-Toribio, M. M. Machado and J. C. Fabris, “Reconstruction of the Cosmic Equation of State for High Redshift,” Eur. Phys. J. C 79 (2019) no.12, 1010 [arXiv:1905.10492 [astro-ph.CO]]; H. N. Lin, X. Li and
L. Tang, “Non-parametric reconstruction of dark energy and cosmic expansion from the Pantheon compilation of type Ia supernovae,” Chin. Phys. C 43 (2019) no.7, 075101, [arXiv:1905.11593 [gr-qc]]; Z. Zhou, T. J. Zhang and T. P. Li, “The model-independent degeneracy-breaking point in cosmological models with interacting Dark Energy and Dark Matter,” Eur. Phys. J. C 79 (2019) no.6, 527 [arXiv:1908.06254 [astro-ph.CO]]; A. Mehrabi and S. Basilakos, “Does $\Lambda$CDM really be in tension with the Hubble diagram data?,” Eur. Phys. J. C 80 (2020) no.7, 632 [arXiv:2002.12577 [astro-ph.CO]]; M. Aljaf, D. Gregoris and M. Khurshudyan, “Constraints on interacting dark energy models through cosmic chronometers and Gaussian process,” [arXiv:2005.01891 [astro-ph.CO]]; P. Mukherjee and N. Banerjee, “Revisiting a non-parametric reconstruction of the deceleration parameter from observational data,” [arXiv:2007.15941 [astro-ph.CO]];

[22] R. G. Cai, Z. K. Guo and T. Yang, “Null test of the cosmic curvature using $H(z)$ and supernovae data,” Phys. Rev. D 93 (2016) no.4, 043517 [arXiv:1509.06283 [astro-ph.CO]]; R. G. Cai, Z. K. Guo and T. Yang, “Dodging the cosmic curvature to probe the constancy of the speed of light,” JCAP 08 (2016), 016 [arXiv:1601.05497 [astro-ph.CO]]; H. Yu and F. Y. Wang, “New model-independent method to test the curvature of the universe,” Astrophys. J. 828 (2016) no.2, 85 [arXiv:1605.02483 [astro-ph.CO]]; J. J. Wei and X. F. Wu, “An Improved Method to Measure the Cosmic Curvature,” Astrophys. J. 838 (2017) no.2, 160 [arXiv:1611.00904 [astro-ph.CO]]; G. J. Wang, J. J. Wei, Z. X. Li, J. Q. Xia and Z. H. Zhu, “Model-independent Constraints on Cosmic Curvature and Opacity,” Astrophys. J. 847 (2017) no.1, 45 [arXiv:1709.07258 [astro-ph.CO]]; [arXiv:1711.03437 [astro-ph.CO]]; J. Zheng, F. Melia and T. J. Zhang, “A Model-Independent Measurement of the Spatial Curvature using Cosmic Chronometers and the HII Hubble Diagram,” [arXiv:1901.05705 [astro-ph.CO]]. H. Zhou and Z. X. Li, “Model-independent Estimations for the Cosmic Curvature from the Latest Strong Gravitational Lensing Systems,” Astrophys. J. 899 (2020), 186 [arXiv:1912.01828 [astro-ph.CO]]; Y. Yang and Y. Gong, “Model independent measurement on the cosmic curvature,” [arXiv:2007.05714 [astro-ph.CO]]; P. Mukherjee and N. Banerjee, “Revisiting a non-parametric reconstruction of the deceleration parameter from observational data,” [arXiv:2007.15941 [astro-ph.CO]]; Y. Liu, S. Cao, T. Liu, X. Li, S. Geng, Y. Lian and W. Guo, “Model-independent constraints on cosmic curvature: implication from updated Hubble diagram of high-redshift standard candles,” Astrophys. J. 901 (2020) no.2, 129 [arXiv:2008.08378 [astro-ph.CO]]; X. Zheng, S. Cao, Y. Liu, M. Biesiada, T. Liu, S. Geng, Y. Lian and W. Guo, “Model-independent constraints on cosmic curvature: implication from the future gravitational wave observation DECIGO,” [arXiv:2012.14607 [astro-ph.CO]].

[23] V. C. Busti, C. Clarkson and M. Seikel, “Evidence for a Lower Value for $H_0$ from Cosmic Chronometers Data?,” Mon. Not. Roy. Astron. Soc. 441 (2014), 11 [arXiv:1402.5429 [astro-ph.CO]].
[24] H. Yu, B. Ratra and F. Y. Wang, “Hubble Parameter and Baryon Acoustic Oscillation Measurement Constraints on the Hubble Constant, the Deviation from the Spatially Flat $\Lambda$CDM Model, the Deceleration–Acceleration Transition Redshift, and Spatial Curvature,” Astrophys. J. 856 (2018) no.1, 3 [arXiv:1711.03437 [astro-ph.CO]].

[25] A. Gómez-Valent and L. Amendola, “$H_0$ from cosmic chronometers and Type Ia supernovae, with Gaussian Processes and the novel Weighted Polynomial Regression method,” JCAP 04 (2018), 051 [arXiv:1802.01505 [astro-ph.CO]].

[26] B. S. Haridasu, V. V. Luković, M. Moresco and N. Vittorio, “An improved model-independent assessment of the late-time cosmic expansion,” JCAP 10 (2018), 015 [arXiv:1805.03595 [astro-ph.CO]].

[27] Z. Li, J. E. Gonzalez, H. Yu, Z. H. Zhu and J. S. Alcaniz, “Constructing a cosmological model-independent Hubble diagram of type Ia supernovae with cosmic chronometers,” Phys. Rev. D 93 (2016) no.4, 043014 [arXiv:1504.03269 [astro-ph.CO]]; S. D. P. Vientiti and M. Penna-Lima, “A general reconstruction of the recent expansion history of the universe,” JCAP 09 (2015), 045 [arXiv:1505.01883 [astro-ph.CO]]; D. Wang and X. H. Meng, “Model-independent determination on $H_0$ using the latest cosmic chronometer data,” Sci. China Phys. Mech. Astron. 60 (2017) no.11, 110411 [arXiv:1610.01202 [gr-qc]]; “Determining $H_0$ with the latest HII galaxy measurements,” Astrophys. J. 843 (2017) no.2, 100 [arXiv:1612.09023 [astro-ph.CO]]; I. Tutusaus, B. Lamine and A. Blanchard, “Model-independent cosmic acceleration and redshift-dependent intrinsic luminosity in type-Ia supernovae,” Astron. Astrophys. 625 (2019), A15 [arXiv:1803.06197 [astro-ph.CO]]; C. Z. Ruan, F. Melia, Y. Chen and T. J. Zhang, “Using spatial curvature with HII galaxies and cosmic chronometers to explore the tension in $H_0$,” Astrophys. J. 881, 137 [arXiv:1901.06626 [astro-ph.CO]]; H. Zhou and Z. Li, “Testing the fidelity of Gaussian processes for cosmography,” Chin. Phys. C 43 (2019) no.3, 035103; D. Wang, W. Zhang and X. H. Meng, “ Searching for the evidence of dynamical dark energy,” Eur. Phys. J. C 79 (2019) no.3, 211 [arXiv:1903.08913 [astro-ph.CO]]; C. A. P. Bengaly, C. Clarkson and R. Maartens, “The Hubble constant tension with next-generation galaxy surveys,” JCAP 05 (2020), 053 [arXiv:1908.04619 [astro-ph.CO]]; E. K. Li, M. Du, Z. H. Zhou, H. Zhang and L. Xu, “Test the effects of $H_0$ on $f\sigma_8$ tension with Gaussian Process method,” [arXiv:1911.12076 [astro-ph.CO]]; R. C. Nunes, S. K. Yadav, J. F. Jesus and A. Bernùi, “Cosmological parameter analyses using transversal BAO data,” Mon. Not. Roy. Astron. Soc. 497 (2020) no.2, 2133-2141 [arXiv:2002.09293 [astro-ph.CO]]; C. A. P. Bengaly, C. Clarkson, M. Kunz and R. Maartens, “Null tests of the concordance model in the era of Euclid and the SKA,” [arXiv:2007.04879 [astro-ph.CO]].

[28] A. Bonilla, S. Kumar and R. C. Nunes, “Measurements of $H_0$ and reconstruction of the dark energy properties from a model-independent joint analysis,” [arXiv:2011.07140 [astro-ph.CO]].
[29] F. Renzi and A. Silvestri, “A look at the Hubble speed from first principles,” [arXiv:2011.10559 [astro-ph.CO]].

[30] M. Chevallier and D. Polarski, “Accelerating universes with scaling dark matter,” Int. J. Mod. Phys. D 10, 213 (2001) [gr-qc/0009008].

[31] E. V. Linder, “Exploring the expansion history of the universe,” Phys. Rev. Lett. 90, 091301 (2003) [astro-ph/0208512].

[32] Spivak, Michael, “Calculus”, Houston, TX: Publish or Perish, Cambridge University Press (1994).

[33] R. Arjona and S. Nesseris, “Machine Learning and cosmographic reconstructions of quintessence and the Swampland conjectures,” [arXiv:2012.12202 [astro-ph.CO]].

[34] R. J. Scherrer, “Mapping the Chevallier-Polarski-Linder parametrization onto Physical Dark Energy Models,” Phys. Rev. D 92 (2015) no.4, 043001 [arXiv:1505.05781 [astro-ph.CO]].

[35] R. Jimenez and A. Loeb, “Constraining cosmological parameters based on relative galaxy ages,” Astrophys. J. 573 (2002), 37-42 [arXiv:astro-ph/0106145 [astro-ph]].

[36] C. Zhang, H. Zhang, S. Yuan, T-J. Zhang and Y-C. Sun, Four new observational H(z) data from luminous red galaxies in the Sloan Digital Sky Survey data release seven, Res. Astron. Astrophys. 14 (2014) 1221 [arXiv:1207.4541]
R. Jiménez, L. Verde, T. Treu and D. Stern, Constraints on the equation of state of dark energy and the Hubble constant from stellar ages and the CMB, Astrophys. J. 593 (2003) 622 [arXiv:astro-ph/0302560]
J. Simon, L. Verde and R. Jiménez, “Constraints on the redshift dependence of the dark energy potential”, Phys. Rev. D71 (2005) 123001 [arXiv:astro-ph/0412269]
M. Moresco et al., “Improved constraints on the expansion rate of the Universe up to z 1.1 from the spectroscopic evolution of cosmic chronometers”, J. Cosmol. Astropart. Phys. 1208 (2012) 006 [arXiv:1201.3609]
M. Moresco et al., “6 measurement of the Hubble parameter at z 0.45: direct evidence of the epoch of cosmic re-acceleration”, J. Cosmol. Astropart. Phys. 1605 (2016) 014 [arXiv:1601.01701]
A.L. Ratsimbazafy et al., “Age-dating Luminous Red Galaxies observed with the Southern African Large Telescope”, Mon. Not. Roy. Astron. Soc. 467 (2017) 3239 [arXiv:1702.00418]
D. Stern, R. Jiménez, L. Verde, M. Kamionkowski and S.A. Stanford, “Cosmic Chronometers: Constraining the Equation of State of Dark Energy. I: H(z) Measurements”, J. Cosmol. Astropart. Phys. 1002 (2010) 008 [arXiv:0907.3149]
[37] D. J. Eisenstein et al. [SDSS], “Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies,” Astrophys. J. 633 (2005), 560-574 [arXiv:astro-ph/0501171 [astro-ph]].

[38] J. Magana, M. H. Amante, M. A. Garcia-Aspeitia and V. Motta, “The Cardassian expansion revisited: constraints from updated Hubble parameter measurements and type Ia supernova data,” Mon. Not. Roy. Astron. Soc. 476 (2018) no.1, 1036-1049 [arXiv:1706.09848 [astro-ph.CO]].

[39] E. Gaztanaga, A. Cabre and L. Hui, “Clustering of Luminous Red Galaxies IV: Baryon Acoustic Peak in the Line-of-Sight Direction and a Direct Measurement of H(z),” Mon. Not. Roy. Astron. Soc. 399 (2009), 1663-1680 [arXiv:0807.3551 [astro-ph]].

A. Oka, S. Saito, T. Nishimichi, A. Taruya and K. Yamamoto, “Simultaneous constraints on the growth of structure and cosmic expansion from the multipole power spectra of the SDSS DR7 LRG sample,” Mon. Not. Roy. Astron. Soc. 439 (2014), 2515-2530 [arXiv:1310.2820 [astro-ph.CO]].

Y. Wang et al. [BOSS], “The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: tomographic BAO analysis of DR12 combined sample in configuration space,” Mon. Not. Roy. Astron. Soc. 469 (2017) no.3, 3762-3774 [arXiv:1607.03154 [astro-ph.CO]].

C. H. Chuang and Y. Wang, “Modeling the Anisotropic Two-Point Galaxy Correlation Function on Small Scales and Improved Measurements of H(z), DA(z), and β(z) from the Sloan Digital Sky Survey DR7 Luminous Red Galaxies,” Mon. Not. Roy. Astron. Soc. 435 (2013), 255-262 [arXiv:1209.0210 [astro-ph.CO]].

S. Alam et al. [BOSS], “The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample,” Mon. Not. Roy. Astron. Soc. 470 (2017) no.3, 2617-2652 [arXiv:1607.03155 [astro-ph.CO]].

C. Blake, S. Brough, M. Colless, C. Contreras, W. Couch, S. Croom, D. Croton, T. Davis, M. J. Drinkwater and K. Forster, et al. “The WiggleZ Dark Energy Survey: Joint measurements of the expansion and growth history at z < 1,” Mon. Not. Roy. Astron. Soc. 425 (2012), 405-414 [arXiv:1204.3674 [astro-ph.CO]].

L. Anderson, E. Aubourg, S. Bailey, F. Beutler, A. S. Bolton, J. Brinkmann, J. R. Brownstein, C. H. Chuang, A. J. Cuesta and K. S. Dawson, et al. “The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: measuring DA and H at z = 0.57 from the baryon acoustic peak in the Data Release 9 spectroscopic Galaxy sample,” Mon. Not. Roy. Astron. Soc. 439 (2014) no.1, 83-101 [arXiv:1303.4666 [astro-ph.CO]].

J. E. Bautista, N. G. Busca, J. Guy, J. Rich, M. Blomqvist, H. d. Bourboux, M. M. Pieri, A. Font-Ribera, S. Bailey and T. Delubac, et al. “Measurement of baryon acoustic oscillation correlations at z = 2.3 with SDSS DR12 Ly-α-forests,” Astron. Astrophys. 603 (2017), A12 [arXiv:1702.00176 [astro-ph.CO]].
T. Delubac et al. [BOSS], “Baryon acoustic oscillations in the Lyα forest of BOSS DR11 quasars,” Astron. Astrophys. 574 (2015), A59 [arXiv:1404.1801 [astro-ph.CO]].

A. Font-Ribera et al. [BOSS], “Quasar-Lyman α Forest Cross-Correlation from BOSS DR11 : Baryon Acoustic Oscillations,” JCAP 05 (2014), 027 [arXiv:1311.1767 [astro-ph.CO]].

[40] M. Moresco, R. Jimenez, L. Verde, A. Cimatti and L. Pozzetti, “Setting the Stage for Cosmic Chronometers. II. Impact of Stellar Population Synthesis Models Systematics and Full Covariance Matrix,” Astrophys. J. 898 (2020) no.1, 82 [arXiv:2003.07362 [astro-ph.GA]].

[41] D. Foreman-Mackey, D. W. Hogg, D. Lang and J. Goodman, “emcee: The MCMC Hammer,” Publ. Astron. Soc. Pac. 125, 306 (2013) [arXiv:1202.3665 [astro-ph.IM]].

[42] A. Lewis, “GetDist: a Python package for analysing Monte Carlo samples,” [arXiv:1910.13970 [astro-ph.IM]].

[43] A. Aghamousa et al. [DESI], “The DESI Experiment Part I: Science,Targeting, and Survey Design,” [arXiv:1611.00036 [astro-ph.IM]].

[44] C. A. P. Bengaly, “Evidence for cosmic acceleration with next-generation surveys: A model-independent approach,” Mon. Not. Roy. Astron. Soc. 499 (2020) no.1, L6-L10 [arXiv:1912.05528 [astro-ph.CO]].

[45] J. C. Zhang, K. Jiao and T. J. Zhang, “Model-independent measurement of the Hubble Constant and the absolute magnitude of Type Ia Supernovae,” [arXiv:2101.05897 [astro-ph.CO]].