Global $W^{2,\delta}$ estimates for a type of singular fully nonlinear elliptic equations

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Abstract

We obtain global $W^{2,\delta}$ estimates for a type of singular fully nonlinear elliptic equations where the right hand side term belongs to $L^\infty$. The main idea of the proof is to slide paraboloids from below and above to touch the solution of the equation, and then to estimate the low bound of the measure of the set of contact points by the measure of the set of vertex points.

1 Introduction

In this paper, we obtain global $W^{2,\delta}$ estimates for viscosity solutions of the singular fully nonlinear elliptic inequalities

$$|Du|^{-\gamma}P^-(\lambda,\Lambda)(D^2u) - |Du|^{1-\gamma} \leq f \leq |Du|^{-\gamma}P^+(\lambda,\Lambda)(D^2u) + |Du|^{1-\gamma} \text{ in } B_1, \quad (1.1)$$

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where $B_1$ is the unit open ball of $\mathbb{R}^n$, $\mathcal{P}_{\lambda,\Lambda}^\pm$ are the Pucci extremal operators, $0 < \lambda \leq \Lambda < \infty$, $0 \leq \gamma < 1$ and the right hand side term $f \in C^0 \cap L^\infty(B_1)$.

The class of solutions of inequalities (1.1) includes solutions of several kinds of important equations. The most common equation is the singular fully nonlinear elliptic equation in the following type

$$|Du|^{-\gamma} F(D^2u, Du, u, x) = f \quad \text{in} \quad B_1,$$

where $0 \leq \gamma < 1$, $F(0, 0, \cdot, \cdot) \equiv 0$ and $F$ is uniformly elliptic (see [CC95] and [Win09]), that is,

$$\mathcal{P}_{\lambda,\Lambda}(M - N) - b|p - q| \leq F(M, p, r, x) - F(N, q, s, x) \leq \mathcal{P}_{\lambda,\Lambda}(M - N) + b|p - q|$$

for all $M, N \in S(n)$, $p, q \in \mathbb{R}^n$, $r, s \in \mathbb{R}$ and $x \in B_1$, where $0 < \lambda \leq \Lambda < \infty$ and $b \geq 0$. The investigation of equations of this type has made much progress in recent years. I. Birindelli and F. Demengel proved comparison principle [BD04] and $C^{1,\alpha}$ estimate [BD10]. G. Dávila, P. Felmer and A. Quaas proved Alexandroff-Bakelman-Pucci (ABP for short) estimate [DFQ09] and Harnack inequality [DFQ10]. To the best of our knowledge, $W^{2,\delta}$ estimate for this kind of equation is only known for $\gamma = 0$, that is the uniformly fully nonlinear elliptic equation. In 1986, F.-H. Lin [Lin86] first established the interior $W^{2,\delta}$ estimates for uniformly elliptic equations of non-divergent type with measurable coefficients, with the help of Fabes-Stroock type reverse Hölder inequality, estimates of Green’s function and the ABP estimates. In 1989, L. A. Caffarelli [Caf89] applied ABP estimate, Calderón-Zygmund cube decomposition technique, barrier function method and touching by tangent paraboloid method to obtain interior $W^{2,\delta}$ estimates for viscosity solutions of

$$\mathcal{P}_{\lambda,\Lambda}(D^2u) \leq f \leq \mathcal{P}_{\lambda,\Lambda}^+(D^2u),$$

and then he use such $W^{2,\delta}$ estimates to get interior $W^{2,p}$ estimates for solutions of

$$F(D^2u, x) = f(x),$$

where the oscillation of $F(M, x)$ in $x$ is small and the homogeneous equations with constants coefficients: $F(D^2v(x), x_0) = 0$ have $C^{1,1}$ interior estimates (or $F$ is concave) (see also [CC95]). $W^{2,p}$ estimates up to the boundary were proved by N. Winter [Win09] in 2009. Another famous example of singular equation which satisfies (1.1) is the singular $p$-Laplace equation

$$\Delta_p u = f \quad \text{in} \quad B_1,$$  \quad (1.2)
where $1 < p \leq 2$ and
\[
\Delta_p u(x) := \text{div} \left( |Du(x)|^{p-2} Du(x) \right) = |Du(x)|^{-(2-p)} \left( \delta_{ij} - (2 - p) \frac{D_i u(x) D_j u(x)}{|Du(x)|^2} \right) D_{ij} u(x).
\]

In [Tol84], P. Tolksdorf proved that each $W^{1,p}_1 \cap C^0(B_1)$-weak solution of (1.2) with $f \in L^\infty(B_1)$ is $W^{2,p}_\text{loc} \cap W^{1,p+2}_\text{loc}(B_1)$ (see also [Lind05]).

We give an elementary proof of $W^{2,\delta}$ estimates for viscosity solutions of singular fully nonlinear elliptic inequalities of the type (1.1). Our estimates are global but the proof does not need to flatten the boundary as usual, that is, instead of separating it into interior estimates and boundary estimates, we do it directly by using a new type of covering lemma. The basic idea is to slide paraboloids from below and above to touch the solution of the equation, and then to estimate the low bound of the measure of the set of contact points by the measure of the set of vertex points. This idea has originated in the work of X. Cabré [Cab97] and continued in the work of O. Savin [Sav07] (see also [IS13]). Following the same idea, J.-P. Daniel [Dan15] proved an estimate equivalent to local $W^{2,\delta}$ estimate for uniformly parabolic equation. For singular fully nonlinear elliptic equations, intuitively, once we have a universal control of $\|Du\|_{L^\infty}$, for instance $C^{1,\alpha}$ estimate (see [BD10]), the $W^{2,\delta}$ estimate will then be a natural corollary of the traditional results of [Caf89], [CC95] and [Win09]. But our method does not depend on any \textit{a priori} estimate of $Du$ and it does not use maximum principles, so we can deal with a large class of equations as illustrated above.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $0 \leq \gamma < 1$. Assume that $u \in C^0(\overline{B_1})$ satisfies (1.1) in the viscosity sense, with $f \in C^0 \cap L^\infty(B_1)$. Then $u \in W^{2,\delta}(B_1)$ and

\[
\|u\|_{W^{2,\delta}(B_1)} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|^{\frac{1}{1-\gamma}}_{L^\infty(B_1)} \right)
\]

for any $\delta \in (0, \sigma)$, where $\sigma = \sigma(n, \lambda, \Lambda) > 0$ and $C = C(n, \lambda, \Lambda, \delta) > 0$.

**Remark 1.1.** The above global $W^{2,\delta}$ estimates in $B_1$ can be easily extended to those in some general domain $\Omega \subset \mathbb{R}^n$. For example, $\Omega$ is bounded and
can be decomposed as the union of a collection of balls with uniformly finite overlapping and with uniformly lower bound radius.

The paper is organized as follows. We start in Section 2 by giving some notations and preliminary tools. In Section 3 we first reduce Theorem 1.1 to Lemma 3.1 by rescaling and normalization, and then the remainder of this section is devoted to the proof of Lemma 3.1.

2 Preliminaries

In this paper, $B_r(x)$ denotes $\{ y \in \mathbb{R}^n : |y - x| < r \}$ and $B_r$ denotes $B_r(0)$. $S(n)$ denotes the linear space of symmetric $n \times n$ real matrices. $I$ denotes the identity matrix.

Given two functions $u$ and $v : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ and a point $x_0 \in \Omega$, we say that $u$ touches $v$ by below at $x_0$ in $\Omega$ and denote it briefly by $u \lhd v$ in $\Omega$, if $u(x_0) = v(x_0)$ and $u(x) \leq v(x), \forall x \in \Omega$.

For a given continuous function $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we slide the concave paraboloid of opening $\kappa > 0$ and vertex $y$

$$-\frac{\kappa}{2}|x - y|^2 + C$$

from below in $U$ (by increasing or decreasing $C$) till it touch the graph of $u$ for the first time. If the contact point is $x_0$, we then have

$$C = u(x_0) + \frac{\kappa}{2}|x_0 - y|^2 = \inf_{x \in U} \left( u(x) + \frac{\kappa}{2}|x - y|^2 \right).$$

For a given closed set $V \subset \mathbb{R}^n$ and a continuous function $u : B_1 \rightarrow \mathbb{R}$, we introduce the definitions of the contact sets as follows:

$$T^-_{\kappa}(V) := T^-_{\kappa}(u, V) := \left\{ x_0 \in B_1 \mid \exists y \in V \text{ such that } u(x_0) + \frac{\kappa}{2}|x_0 - y|^2 = \inf_{x \in B_1} \left( u(x) + \frac{\kappa}{2}|x - y|^2 \right) \right\} = \left\{ x_0 \in B_1 \mid \exists y \in V \text{ such that }$$

$$u(x_0) + \frac{\kappa}{2}|x_0 - y|^2 = \inf_{x \in B_1} \left( u(x) + \frac{\kappa}{2}|x - y|^2 \right) \right\}$$

$$= \left\{ x_0 \in B_1 \mid \exists y \in V \text{ such that }$$
\[-\frac{k}{2}|x - y|^2 + u(x_0) + \frac{k}{2}|x_0 - y|^2 \lesssim u \text{ in } B_1,\]

\[T^+_{\kappa}(V) := T^+_\kappa(u, V) := T^-_{\kappa}(-u, V)\]

and

\[T_{\kappa}(V) := T_{\kappa}(u, V) := T^-_{\kappa}(u, V) \cap T^+_{\kappa}(u, V).\]

For simplicity of notation, we will write \(T^\pm_{\kappa}\) instead of \(T^\pm_{\kappa}(u, B_1)\) when there is no confusion. Note that \(T^\pm_{\kappa}\) are closed in \(B_1\).

We remark that the contact set \(T^-_{\kappa}(u, V)\) has the dual functionality of \(\{u = \Gamma_u\}\) and \(G_M(u, \Omega)\) in \([CC95]\) simultaneously. The former digs up information on the equation, while the latter measures the second derivatives of the solution.

Given \(0 < \lambda \leq \Lambda\), we define the so called maximal and minimal Pucci extremal operators \(P^+_{\lambda, \Lambda}\) and \(P^-_{\lambda, \Lambda}\) (see also \([CC95]\)) as follows

\[P^+_{\lambda, \Lambda}(X) := \lambda \sum_{e_i(X) < 0} e_i(X) + \Lambda \sum_{e_i(X) > 0} e_i(X)\]

and

\[P^-_{\lambda, \Lambda}(X) := \lambda \sum_{e_i(X) > 0} e_i(X) + \Lambda \sum_{e_i(X) < 0} e_i(X),\]

where \(X \in S(n)\) and \(e_i(X)\) denote the eigenvalues of \(X\). We will always abbreviate \(P^\pm_{\lambda, \Lambda}(X)\) to \(P^\pm(X)\).

For convenience, we state some basic properties of the Pucci extremal operators as below:

1. \(P^\pm(kX) = kP^\pm(X), P^\pm(-X) = -P^\pm(X), \forall X \in S(n), \forall k \geq 0.\)
2. \(P^- (X) + P^- (Y) \leq P^- (X + Y) \leq P^- (X) + P^+ (Y) \leq P^+ (X + Y) \leq P^+ (X) + P^+ (Y), \forall X, Y \in S(n).\)
3. \(P^+_{\lambda, \Lambda}(X) = \lambda \text{tr} X \text{ and } P^-_{\lambda, \Lambda}(X) = \lambda \text{tr} X, \text{ provided } X \geq 0, X \in S(n).\)

We recall the definition of viscosity solutions (see \([CC95]\) for more details). For example, we say that \(u \in C^0(B_1)\) satisfies

\[F(D^2u, Du, u, x) \leq f \text{ in } B_1\]
in the viscosity sense, if \( \forall \varphi \in C^2(B_1), \forall x_0 \in B_1, \)

\[ \varphi_{x_0} \preceq u \text{ in } U(x_0) \Rightarrow F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \leq f(x_0), \]

where \( U(x_0) \) is an open neighborhood of \( x_0 \) in \( B_1 \).

We need the following equivalent descriptions of \( L^p \)-integrability.

**Lemma 2.1.** (see Lemma 7.3. in [CC95]) Let \( g \) be a nonnegative and measurable function in a bounded domain \( \Omega \subset \mathbb{R}^n \). Suppose that \( \eta > 0, M > 1 \) and \( 0 < p < \infty \). Then

\[ g \in L^p(\Omega) \iff s := \sum_{k=1}^{\infty} M^{pk} |\{ x \in \Omega \mid g(x) > \eta M^k \}| < \infty \]

and

\[ C^{-1} s \leq \| g \|_{L^p(\Omega)}^p \leq C(s + |\Omega|), \]

where \( C > 0 \) is a constant depending only on \( \eta, M \) and \( p \).

In the last part of this section, we introduce the following consequence of Vitali’s covering lemma, which has similar functionality to the Calderón-Zygmund cube decomposition lemma (see [CC95]) but has the advantage of giving global estimates directly. This result is slightly different from that (growing ink-spots lemma) in [IS13], but the idea of the proof is similar, which according to [IS13], was first introduced by Krylov.

**Lemma 2.2.** Let \( 0 < \mu < 1 \). Assume that \( E \subset F \) are closed subsets of \( B_1 \) and \( E \neq \emptyset \). Suppose that for any open ball \( B \subset B_1 \), if \( B \cap E \neq \emptyset \), then \( |B \cap F| \geq \mu |B| \). Then \( |B_1 \setminus F| \leq (1 - \mu/5^n)|B_1 \setminus E| \).

**Proof.** It suffices to prove that \( |F \setminus E| \geq \frac{\mu}{n!}|B_1 \setminus E| \).

For any \( x \in B_1 \setminus E \), by the openness of \( B_1 \setminus E \), there exist open balls contained in \( B_1 \setminus E \) and containing \( x \), we choose one of the largest of them and denote it by \( B^x \).

We claim that \( |B^x \cap F| \geq \mu |B^x| \). Otherwise, since \( E \neq \emptyset \), and hence \( B^x \subset B_1 \setminus E \subsetneq B_1 \), we may enlarge \( B^x \) a little bit, denoted by \( \tilde{B}^x \), such that \( B^x \subset \tilde{B}^x \subset B_1 \) and \( |\tilde{B}^x \cap F| < \mu |\tilde{B}^x| \). By the hypothesis of the lemma,
Let \( \tilde{B}^x \cap E = \emptyset \), thus \( \tilde{B}^x \subset B_1 \setminus E \), which contradicts the definition of \( B^x \).
Furthermore, since \( B^x \cap F \setminus E = B^x \cap F \), it follows that \( |B^x \cap F \setminus E| \geq \mu |B^x| \).

Now consider the covering \( \bigcup_{x \in B_1 \setminus E} B^x \supset B_1 \setminus E \). By the Vitali covering lemma, there exists an at most countable set of points \( x_i \in B_1 \setminus E \), such that \( \{B^x_i\}_i \) are disjoint and \( \bigcup_i 5B^x_i \supset B_1 \setminus E \). Hence we have
\[
|F \setminus E| \geq \left| \left( \bigcup_i B^x_i \right) \cap F \setminus E \right| = \left| \bigcup_i (B^x_i \cap F \setminus E) \right| = \sum_i |B^x_i \cap F \setminus E| \\
\geq \mu \sum_i |B^x_i| = \frac{\mu}{5^n} \sum_i 5|B^x_i| \geq \frac{\mu}{5^n} \left| \bigcup_i 5B^x_i \right| \geq \frac{\mu}{5^n} |B_1 \setminus E|.
\]

\[\square\]

3 Global \( W^{2,\delta} \) estimates

We first give the following lemma.

**Lemma 3.1.** Let \( 0 \leq \gamma < 1 \). Assume that \( u \in C^0(\overline{B_1}) \) satisfies (1.1) in the viscosity sense, where \( f \in C^0 \cap L^\infty(B_1) \). Then there exists \( \sigma = \sigma(n, \lambda, \Lambda) > 0 \) such that for any \( \delta \in (0, \sigma) \), if \( \|u\|_{L^\infty(B_1)} \leq 1/16 \) and \( \|f\|_{L^\infty(B_1)} \leq 1 \), then
\[
\|u\|_{W^{2,\delta}(B_1)} \leq C(n, \lambda, \Lambda, \delta).
\]

To prove Theorem 1.1 it suffices to prove Lemma 3.1. Indeed, suppose \( u \) satisfies the hypothesis of Theorem 1.1. Let
\[
a := \left( 16\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}^{1/1-\gamma} + \varepsilon \right)^{-1},
\]
where \( \varepsilon > 0 \). Then the scaled function \( \tilde{u}(x) := au(x) \) solves
\[
|D\tilde{u}|^{-\gamma} P^{-\gamma}_{\lambda,\Lambda}(D^2\tilde{u}) - |D\tilde{u}|^{1-\gamma} \leq a^{1-\gamma} f := \tilde{f} \leq |D\tilde{u}|^{-\gamma} P^+_{\lambda,\Lambda}(D^2\tilde{u}) + |D\tilde{u}|^{1-\gamma} \text{ in } B_1,
\]
and satisfies \( \|\tilde{u}\|_{L^\infty(B_1)} \leq 1/16 \) and \( \|\tilde{f}\|_{L^\infty(B_1)} \leq 1 \). Therefore, if
\[
\|\tilde{u}\|_{W^{2,\delta}(B_1)} \leq C(n, \lambda, \Lambda, \delta),
\]
by scaling back to \( u \) and letting \( \varepsilon \to 0 \), we obtain
\[
\|u\|_{W^{2,\delta}(B_1)} \leq Ca^{-1} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}^{1/1-\gamma} \right),
\]

7
which is the assertion of Theorem 1.1.

We establish Lemma 3.1 by the following two lemmas. First we need the density lemma (Lemma 3.2) which is a key lemma in this paper and the strategy of its proof is modified from that in [Sav07].

**Lemma 3.2.** Let $0 \leq \gamma < 1$ and $K \geq 1$. Assume that $u \in C^0(B_1)$ satisfies

$$|Du|^{-\gamma}P_{\lambda,\Lambda}(D^2u) - |Du|^{1-\gamma} \leq f \text{ in } B_1$$

in the viscosity sense, where $f \in C^0 \cap L^\infty(B_1)$ and $\|f\|_{L^\infty(B_1)} \leq 1$. Then there exist constants $M = M(n, \lambda, \Lambda) > 1$ and $0 < \mu = \mu(n, \lambda, \Lambda) < 1$, such that if $B_r(x_0) \subset B_1$ satisfies

$$B_r(x_0) \cap T_{KM}^- \neq \emptyset,$$

then

$$|B_r(x_0) \cap T_{KM}^-| \geq \mu |B_r(x_0)|.$$

**Proof.** By assumption, there exist $x_1 \in B_r(x_0) \cap T_K^-$ and $y_1 \in \overline{B_1}$ such that

$$P_{K,y_1}^-(x) := -K \frac{|x - y_1|^2}{2} + u(x_1) + K \frac{|x_1 - y_1|^2}{2} x_1 \preceq u \text{ in } B_1.$$

The proof now will be divided into three steps.

**Step 1.** We prove that there exist $x_2 \in B_{r/2}(x_0)$ and $C_0 = C_0(n, \lambda, \Lambda) > 0$ such that

$$u(x_2) - P_{K,y_1}^-(x_2) \leq C_0 Kr^2.$$  

Set

$$\psi(x) := P_{K,y_1}^-(x) + Kr^2 \phi \left( \frac{|x - x_0|}{r} \right),$$

where $\phi(t) = e^Ae^{-At^2} - 1$ with $A = A(n, \lambda, \Lambda) > 1$ to be determined later. Let $x_2 \in \overline{B_r(x_0)}$ such that

$$(u - \psi)(x_2) = \min_{B_r(x_0)} (u - \psi).$$

Since $(u - \psi)|_{\partial B_r(x_0)} \geq 0$ and

$$(u - \psi)(x_2) \leq (u - \psi)(x_1) = -Kr^2 \phi \left( \frac{|x_1 - x_0|}{r} \right) < 0,$$
we deduce that \(x_2 \in B_r(x_0)\) and
\[
 u(x_2) < \psi(x_2) = P_{K,y_1}^-(x_2) + K r^2 \phi \left( \frac{|x_2 - x_0|}{r} \right) \leq P_{K,y_1}^-(x_0) + e^A K r^2.
\]

Hence, by letting \(C_0 := e^A\), we now only need to show that there exists an \(A = A(n, \lambda, \Lambda) > 1\) such that \(x_2 \in B_{r/2}(x_0)\).

To obtain a contradiction, suppose that \(x_2\) is outside of \(B_{r/2}(x_0)\) for any \(A > 1\). Let \(t := \frac{|x_2 - x_0|}{r}\). We have \(1/2 < t < 1\). Since
\[
 \psi + \min_{B_r(x_0)} (u - \psi) \leq u \text{ in } B_r(x_0),
\]
by applying the definition of the viscosity solution of (3.1), we have
\[
 P_{\lambda,\Lambda}(D^2 \psi(x_2)) - |D \psi(x_2)| \leq |D \psi(x_2)|^\gamma f(x_2). \tag{3.3}
\]
Since
\[
 |DP_{K,y_1}^-(x_2)| \leq CK,
\]
\[
 |D^2 P_{K,y_1}^-(x_2)| \leq CK,
\]
\[
 \left| \frac{\phi_t}{t} \right| \leq CA e^{A e^{-At^2}}
\]
and
\[
 |\phi_{tt}| \geq \frac{1}{C} A^2 e^{A e^{-At^2}},
\]
we see that
\[
 |D \psi(x_2)|^\gamma \leq C \left( K A e^{A e^{-At^2}} \right)^\gamma
\]
and
\[
 P_{\lambda,\Lambda}(D^2 \psi)(x_2) - |D \psi(x_2)| \begin{align*}
 &\geq CK (|\phi_{tt}| - 1) - CK \left( \left| \frac{\phi_t}{t} \right| + 1 \right) - CK A e^{A e^{-At^2}} \\
 &\geq CK A e^{A e^{-At^2}},
\end{align*}
\]
where \(A\) is large enough and all the constants \(C\) depend only on \(n, \lambda\) and \(\Lambda\). By (3.3), we conclude that
\[
 \left( K A e^{A e^{-At^2}} \right)^{1-\gamma} \leq 1,
\]
9
which is impossible.

Step 2. We prove that there exists \( M = M(n, \lambda, \Lambda) > 1 \) such that
\[
T_{KM}(V) \subset B_r(x_0) \cap T_{KM},
\]
where
\[
V := B_{r(M^{-1}) \frac{1}{8M}} \left( \frac{1}{M} y_1 + \frac{M - 1}{M} x_2 \right).
\]
\( \forall \tilde{x} \in T_{KM}(V), \exists y \in V \) such that
\[
P_{KM,y}^{-}(x) := -\frac{KM}{2} |x - y|^2 + u(\tilde{x}) + \frac{KM}{2} |\tilde{x} - y|^2 \leq u \text{ in } B_1.
\]
Since
\[
P_{KM,y}^{-}(x) - P_{K,y_1}^{-}(x) = -\frac{K(M - 1)}{2} |x - y|^2 + R,
\]
where
\[
y := \frac{M \tilde{y} - y_1}{M - 1}
\]
and \( R = R(\tilde{y}, K, M, y_1, \tilde{x}, u(\tilde{x}), x_1, u(x_1)) \) both do not depend on \( x \), we have
\[
P_{K,y_1}^{-}(x) - \frac{K(M - 1)}{2} |x - y|^2 + R \leq u(x), \forall \ x \in B_1.
\]
Since \( x_2 \in B_{r/2}(x_0) \subset B_1 \) and \( y \in \overline{B_{r/8}(x_2)} \), we see from (3.2) that
\[
R \leq u(x_2) - P_{K,y_1}^{-}(x_2) + \frac{K(M - 1)}{2} |x_2 - y|^2 \leq \left( C_0 + \frac{M - 1}{128} \right) Kr^2.
\]
On the other hand,
\[
0 \leq u(\tilde{x}) - P_{K,y_1}^{-}(\tilde{x}) = P_{KM,y}^{-}(\tilde{x}) - P_{K,y_1}^{-}(\tilde{x}) = -\frac{K(M - 1)}{2} |\tilde{x} - y|^2 + R.
\]
It follows that
\[
|\tilde{x} - y|^2 \leq \frac{2}{M - 1} \left( C_0 + \frac{M - 1}{128} \right) r^2 = \left( \frac{2C_0}{M - 1} + \frac{1}{64} \right) r^2.
\]
Let \( M > \frac{128C_0}{3} + 1 \), we obtain \(|\tilde{x} - y| < r/4\). Thus
\[
|\tilde{x} - x_2| \leq |\tilde{x} - y| + |y - x_2| < r/4 + r/8 = 3r/8,
\]
10
and hence
\[ T_{KM}^{-}(V) \subset B_{3r/8}(x_2) \subset B_{r/2}(x_2) \subset B_{r}(x_0) \subset B_{1}. \tag{3.6} \]

By (3.5), we see that \( \forall \tilde{y} \in V, \exists y \in B_{r/8}(x_2) \) such that
\[ \tilde{y} = \frac{1}{M}y_1 + \frac{M-1}{M}y. \]

Since \( y_1 \in B_{1} \) and \( B_{r/8}(x_2) \subset B_{r/8}(x_0) \subset B_{r}(x_0) \subset B_{1} \), we obtain
\[ V \subset B_{1}, \tag{3.7} \]
by the convexity of \( B_{1} \). Thus
\[ T_{KM}^{-}(V) \subset T_{KM}^{-}(B_{1}) = T_{KM}, \]
and hence
\[ T_{KM}^{-}(V) \subset B_{r}(x_0) \cap T_{KM}. \]

Step 3. We claim that
\[ |V| \leq C |T_{KM}^{-}(V)|, \tag{3.8} \]
where \( C = C(n, \lambda, \Lambda) > 0 \). If we prove this, then by (3.4), we will obtain
\[
\begin{align*}
|B_{r}(x_0) \cap T_{KM}| & \geq |T_{KM}^{-}(V)| \geq \frac{1}{C}|V| \\
& = \frac{1}{C} \left( \frac{M-1}{8M} \right)^{n} |B_{r}| \\
& =: \mu |B_{r}(x_0)|,
\end{align*}
\]
which proves the lemma.

Hence it remains to prove (3.8). To do this, we need to regularize \( u \) by the standard \( \varepsilon \)-envelope method of Jensen (see [CC95]). That is, for \( \varepsilon > 0 \), let
\[
\begin{align*}
u_{\varepsilon}(x) := \inf_{z \in B_{1}} \left( u(z) + \frac{1}{\varepsilon^4} |z - x|^2 \right), \quad \forall x \in B_{1}.
\end{align*}
\]

It is easy to see that \( u_{\varepsilon} \) is \( C^{1,1} \) a.e. in \( B_{1} \), \( u_{\varepsilon} \in C^{0}(B_{1}) \) and \( u_{\varepsilon} \to u \) \( (\varepsilon \to 0+) \) uniformly on compact subsets of \( B_{1} \). Furthermore, we show that there exists \( \varepsilon_0 > 0 \) sufficiently small such that for any \( \varepsilon \in (0, \varepsilon_0) \),
\[ T_{KM}^{-}(u_{\varepsilon}, V) \subset B_{1-\varepsilon}. \tag{3.9} \]
and $u_\epsilon$ satisfies

$$|Du_\epsilon|^{-\gamma} \mathcal{P}_{\lambda, \Lambda}(D^2 u_\epsilon) - |Du_\epsilon|^{1-\gamma} \leq f_\epsilon \quad \text{in } B_{1-\epsilon}$$

(3.10)

in the viscosity sense, with $f_\epsilon$ to be given later which satisfies $f_\epsilon \in C^0(B_{1-\epsilon})$, $\|f_\epsilon\|_{L^\infty(B_{1-\epsilon})} \leq 1$ and $f_\epsilon \to f$ ($\epsilon \to 0^+$) uniformly on compact subsets of $B_{1-\epsilon}$. To see (3.9), we only need to note (3.6) and (3.7). We now verify (3.10). Suppose that $\varphi \in C^2(B_1)$, $x_\ast \in B_{1-\epsilon}$ and

$$\varphi \lesssim u_\epsilon \quad \text{in } U(x_\ast),$$

where $U(x_\ast)$ is an open neighborhood of $x_\ast$ in $B_{1-\epsilon}$. Let $\tilde{x}_\ast \in \overline{B_1}$ such that

$$u_\epsilon(x_\ast) = \inf_{z \in B_1} \left( u(z) + \frac{1}{\epsilon^4} |z - x_\ast|^2 \right) = u(\tilde{x}_\ast) + \frac{1}{\epsilon^4} |\tilde{x}_\ast - x_\ast|^2.$$

In view of

$$|\tilde{x}_\ast - x_\ast|^2 = \epsilon^4 (u_\epsilon(x_\ast) - u(\tilde{x}_\ast)) \leq \epsilon^4 \cdot \text{osc}_{\overline{B_1}} u,$$

we have $\tilde{x}_\ast \in B_1$ provided $\epsilon$ is small enough. Since

$$\varphi(x - \tilde{x}_\ast + x_\ast) \leq u_\epsilon(x - \tilde{x}_\ast + x_\ast)$$

$$= \inf_{z \in B_1} \left( u(z) + \frac{1}{\epsilon^4} |z - x + \tilde{x}_\ast - x_\ast|^2 \right)$$

$$\leq u(x) + \frac{1}{\epsilon^4} |\tilde{x}_\ast - x_\ast|^2$$

$$= u(x) + u_\epsilon(x_\ast) - u(\tilde{x}_\ast)$$

$$= u(x) + \varphi(x_\ast) - u(\tilde{x}_\ast),$$

provided $x - \tilde{x}_\ast + x_\ast \in U(x_\ast)$, we deduce that

$$\varphi(x - \tilde{x}_\ast + x_\ast) - \varphi(x_\ast) + u(\tilde{x}_\ast) \lesssim u(x) \quad \text{in } U(\tilde{x}_\ast),$$

where $U(\tilde{x}_\ast)$ is a small open neighborhood of $\tilde{x}_\ast$ in $B_1$. Hence

$$|D\varphi(x_\ast)|^{-\gamma} \mathcal{P}_{\lambda, \Lambda}(D^2 \varphi(x_\ast)) - |D\varphi(x_\ast)|^{1-\gamma} \leq f(\tilde{x}_\ast) =: f_\epsilon(x_\ast),$$

which gives (3.10).
Now since \( u \in C^{1,1} \), we see that for almost every \( x \in T^{-}_{KM}(u_{\epsilon}, V) \), there exists a unique \( y \in V \) which satisfies

\[
Du_{\epsilon}(x) = -KM(x - y)
\]

and

\[
D^{2}u_{\epsilon}(x) \geq -KMI.
\]

Hence we may define the mapping \( y : x \mapsto y, T^{-}_{KM}(u_{\epsilon}, V) \rightarrow V \), given by

\[
y = x + \frac{1}{KM}Du_{\epsilon}(x).
\]

Since

\[
|Du_{\epsilon}(x)| = KM|x - y| \leq 2KM
\]

and

\[
D_{xy}y = I + \frac{1}{KM}D^{2}u_{\epsilon}(x) \geq 0,
\]

we conclude from (3.9) and (3.10) that

\[
\lambda \text{tr}(D_{xy}) = \mathcal{P}_{\lambda, \Lambda}^{-}(D_{xy})
\]

\[
\leq \mathcal{P}_{\lambda, \Lambda}^{+}(I) + \mathcal{P}_{\lambda, \Lambda}^{-}\left(\frac{D^{2}u_{\epsilon}(x)}{KM}\right)
\]

\[
\leq n\Lambda + \frac{1}{KM}(|Du_{\epsilon}(x)|^{\gamma}f_{\epsilon}(x) + |Du_{\epsilon}(x)|)
\]

\[
\leq n\Lambda + \frac{2}{(KM)^{1-\gamma}} + 2
\]

\[
\leq n\Lambda + 4 \leq C
\]

and

\[
0 \leq \det(D_{xy}) \leq \left(\frac{\text{tr}(D_{xy})}{n}\right)^{n} \leq C,
\]

where the constants \( C \) depend only on \( n, \lambda \) and \( \Lambda \). Hence, we have

\[
|V| \leq \int_{T^{-}_{KM}(u_{\epsilon}, V)} \det(D_{xy})dx \leq C \left| T^{-}_{KM}(u_{\epsilon}, V) \right|,
\]

by the area formula.
Finally, it is easy to check that

$$\lim_{\epsilon \to 0^+} T_{K_M}^{-}(u_\epsilon, V) \subset T_{K_M}^{-}(u, V),$$

where

$$\lim_{\epsilon \to 0^+} T_{K_M}^{-}(u_\epsilon, V) := \cap_{0 < \epsilon \ll \eta \ll \epsilon} T_{K_M}^{-}(u_\delta, V).$$

Hence we have

$$|V| \leq C \lim_{\epsilon \to 0^+} |T_{K_M}^{-}(u_\epsilon, V)| \leq C |T_{K_M}^{-}(u, V)| \leq C |T_{K_M}^{-}(u, V)|,$$

which gives (3.8) and completes the proof. \qed

We now give a lemma which states the decay of $|B_1 \setminus T_i| \setminus t$.

**Lemma 3.3.** Let $0 \leq \gamma < 1$ and $K \geq 1$. Assume that $u \in C^0(B_1)$ satisfies

$$|Du|^{-\gamma} \mathcal{P}_{\lambda, \Lambda}^-(D^2 u) - |Du|^{1-\gamma} \leq f \quad \text{in } B_1$$

in the viscosity sense, where $f \in C^0 \cap L^\infty(B_1)$, $\|f\|_{L^\infty(B_1)} \leq 1$ and $\|u\|_{L^\infty(B_1)} \leq 1/16$. Then there exist constants $M = M(n, \lambda, \Lambda) > 1$ and $\theta = \theta(n, \lambda, \Lambda) \in (0, 1)$ such that

$$|B_1 \setminus T_{K_M}^{-}| \leq \theta |B_1 \setminus T_{K}^{-}|.$$ 

**Proof.** We first show that

$$B_1 \cap T_{K}^{-} \neq \emptyset. \quad (3.11)$$

Let $\bar{x} \in \overline{B_1}$ such that

$$u(\bar{x}) + \frac{K}{2} |\bar{x}|^2 = \min_{x \in \overline{B_1}} \left( u(x) + \frac{K}{2} |x|^2 \right) =: m.$$

We have $-\frac{K}{2} |x|^2 + m \ll u$ in $\overline{B_1}$. In particular, we obtain $m \leq u(0)$ and $-\frac{K}{2} |\bar{x}|^2 + m = u(\bar{x})$. Subtracting one from the other, we conclude that

$$\frac{K}{2} |\bar{x}|^2 \leq u(0) - u(\bar{x}) \leq \operatorname{osc} u \leq 2\|u\|_{L^\infty(B_1)} \leq 1/8,$$
which implies $|\bar{x}| \leq 1/2$. Thus $\bar{x} \in B_1 \cap T^-_K$.

In view of (3.11) and Lemma 3.2, by applying Lemma 2.2 to $E := T^-_K$ and $F := T^-_{KM}$, we obtain

$$|B_1 \setminus T^-_{KM}| \leq (1 - \frac{\mu}{3^n}) |B_1 \setminus T^-_K| =: \theta |B_1 \setminus T^-_K|.$$  

This completes the proof of Lemma 3.3.

**Corollary 3.1.** Under the assumptions of Lemma 3.3, we have

$$|B_1 \setminus T^-_{M_k}| \leq |B_1| \theta^k, \ \forall k \geq 1.$$

**Proof.** Applying Lemma 3.3 to $K = M^{k-1}, M^{k-2}, \ldots, M$ and 1, we deduce that

$$|B_1 \setminus T^-_{M_k}| \leq \theta |B_1 \setminus T^-_{M_{k-1}}| \leq \ldots \leq \theta^k |B_1 \setminus T^-_1| \leq |B_1| \theta^k.$$  

□

Lemma 3.1 now follows easily from Corollary 3.1.

**Proof of Lemma 3.1.** (1) By definition, $T^+_k(u, B_1) = T^-_k(-u, B_1)$. Applying Corollary 3.1 to $-u$, we have $|B_1 \setminus T^+_k| \leq |B_1| \theta^k, \ \forall k \geq 1$. Therefore

$$|B_1 \setminus T^+_k| = |B_1 \setminus (T^-_{M_k} \cap T^+_k)| \leq |B_1 \setminus T^-_{M_k}| + |B_1 \setminus T^+_k| \leq 2|B_1| \theta^k, \ \forall k \geq 1.$$

It follows that

$$|B_1 \setminus T^+_t| \leq C t^{-\sigma}, \ \forall t > 0, \quad (3.12)$$

where $\sigma := -\log M \theta$ and $C := 2|B_1|/\theta$.

(2) To get $\|u\|_{W^{2,\delta}(B_1)} \leq C$, by the interpolation theorem (see Theorem 7.28 of [GT98]), we only need to show that $\|D^2 u\|_{L^\delta(B_1)} \leq C$. By (3.12), this can be obtained directly from Lemma 2.1 (cf. [CC95], pp. 6-7, 62-63, 66-67).

For heuristic purposes, we here additionally give a simple proof for the special case that $u \in C^2(B_1)$. Since

$$B_1 \cap T_t \subset \{x \in B_1 : -tI \leq D^2 u(x) \leq tI\} \subset \{x \in B_1 : |D^2 u(x)| \leq \sqrt{nt}\},$$

we have $|B_1 \setminus T^+_t| \leq C t^{-\sigma}, \ \forall t > 0.$
we see that
\[ \{ x \in B_1 : |D^2u(x)| > \sqrt{nt} \} \subset B_1 \setminus T_t. \]
Hence
\[ | \{ x \in B_1 : |D^2u(x)| > \sqrt{nt} \} | \leq |B_1 \setminus T_t| \leq Ct^{-\sigma}. \]
Applying Lemma 2.1, we obtain
\[
\| D^2u \|_{L^2(B_1)}^\delta \leq C(n, M, \delta) \left( |B_1| + \sum_k M^{\delta k} | \{ x \in B_1 : |D^2u(x)| > \sqrt{nM^k} \} | \right)
\]
\[ \leq C(n, M, \delta) \left( |B_1| + C \sum_k M^{(\delta - \sigma)k} \right) \leq C(n, \lambda, \Lambda, \delta) < \infty. \]

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