Biquaternions for analytic and numerical solution of equations of electrodynamics

Kira V. Khmelnytskaya, Vladislav V. Kravchenko
Department of Mathematics
CINVESTAV del IPN, Queretaro
Libramiento Norponiente No. 2000
Fracc. Real de Juriquilla
Queretaro, Qro.
C.P. 76230
MEXICO
e-mail: vkravchenko@qro.cinvestav.mx
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Abstract

We give an overview of recent advances in analysis of equations of electrodynamics with the aid of biquaternionic technique. We discuss both models with constant and variable coefficients, integral representations of solutions, a numerical method based on biquaternionic fundamental solutions for solving standard electromagnetic scattering problems, relations between different operators of mathematical physics including the Schrödinger, the Maxwell system, the conductivity equation and others leading to a deeper understanding of physics and mathematical properties of the equations.

1 Introduction

Application of the algebra of biquaternions to equations of electromagnetism has been subject of an important number of research articles and books (see,
The aim of this work is to present some recent results in the field concerning the usage of algebraic advantages of biquaternions for analytic and numerical solution of Maxwell’s system for chiral media as well as for inhomogeneous media. Compared to a considerable number of publications dealing with biquaternionic reformulations of Maxwell’s equations for a vacuum or for a homogeneous isotropic medium, application of biquaternions to electromagnetic models corresponding to more complicated media (a much more challenging object for studying) was discussed in relatively few sources ([13], [21], [23], [28], [31], [40], [41]). Meanwhile the possibility of representation of Maxwell’s system for a vacuum in the form of a single biquaternionic equation is known since 1919 [35], only recently it became clear how this result can be generalized for inhomogeneous media [27], [28] and for chiral media [13]. An appropriate quaternionic or biquaternionic reformulation of a first order system of mathematical physics opens the way for applying different methods which in many aspects preserve the algebraic power of complex analysis. For example, it is not easy to arrive at the Cauchy integral formula for holomorphic functions using two-component vector formalism or even more difficult task to develop using this formalism a holomorphic function into a Taylor series. No mathematician would consider such way of presenting complex function theory helpful or appropriate. However, this is precisely what is happening in the study of three or four-dimensional models of mathematical physics. Compare, e.g., the Stratton-Chu integrals written in their standard form (see, e.g., [10]) with their biquaternionic representation [26], [28], [32] which is in fact a convolution of a biquaternionic fundamental solution of the Maxwell operator with the electromagnetic field and it is quite evident that the latter is natural and elucidating. The meaning of the Stratton-Chu formulas as a Cauchy integral formula for the electromagnetic field becomes transparent and no doubt in their biquaternionic form the Stratton-Chu formulas could be included in a moderately advanced course of electromagnetic theory which is not the usual case up to now in spite of their central role in electrical engineering applications.

The results presented in this work are “essentially quaternionic” in the sense that it is not clear how they could be obtained with other techniques. In the first part (sections 2-5) we explain a numerical method for solving electromagnetic scattering problems with an unusual for three-dimensional models precision by the aid of biquaternionic fundamental solutions which to the difference of the usually utilized matrix fundamental solutions for the Maxwell
equations (see, e.g., [1] and [12]) enjoy some advantageous properties. First
of all they are not matrices but vectors (of four components). Second, they
have a clear physical meaning of fields generated by point sources. Third,
their singularity is lower than that of fundamental solutions based on a ma-
trix approach. The main idea of this work is to explain how our approach
works and how it can be used. We only formulate some necessary results like
those about the completeness of our systems of quaternionic fundamental
solutions in appropriate functional spaces referring the interested reader to
some previous publications, in particular [23] where the corresponding proofs
can be found.

In sections 6-8 we consider the time-dependent Maxwell system for chiral
media, rewrite it in a biquaternionic form as a single equation and then
construct explicitly a corresponding Green function. Section 9 is dedicated
to the time-dependent Maxwell equations for inhomogeneous media. We
show that these equations can also be written as a single biquaternionic
equation. In a static case the corresponding quaternionic operator factorizes
the stationary Schrödinger operator. We study relationship between solutions
of these important physical equations.

2 Biquaternionic fundamental solutions

Let $\mathbb{H}(\mathbb{C})$ denote the set of complex quaternions (= biquaternions). Each
element $a$ of $\mathbb{H}(\mathbb{C})$ is represented in the form $a = \sum_{k=0}^{3} a_k i_k$ where \{a_k\} $\subset \mathbb{C}$,
$i_0$ is the unit and \{i_k | k = 1, 2, 3\} are the quaternionic imaginary units:

$$i_0^2 = i_0 = -i_k^2; \quad i_0i_k = i_ki_0 = i_k, \quad k = 1, 2, 3;$$

$$i_1i_2 = -i_2i_1 = i_3; \quad i_2i_3 = -i_3i_2 = i_1; \quad i_3i_1 = -i_1i_3 = i_2.$$  

We denote the imaginary unit in $\mathbb{C}$ by $i$ as usual. By definition $i$ commutes
with $i_k$, k = 0, 3.

We will use the vector representation of complex quaternions, every $a \in 
\mathbb{H}(\mathbb{C})$ is represented as follows $a = a_0 + \overrightarrow{a}$, where $a_0$ is the scalar part of
$a$: $\text{Sc}(a) = a_0$, and $\overrightarrow{a}$ is the vector part of $a$: $\text{Vec}(a) = \overrightarrow{a} = \sum_{k=1}^{3} a_k i_k$. 
Complex quaternions of the form $a = \overrightarrow{a}$ are called purely vectorial and can
be identified with vectors from $\mathbb{C}^3$. The operator of quaternionic conjugation
we denote by $C_H$: $\overrightarrow{a} = C_H a = a_0 - \overrightarrow{a}$. 

Let us introduce the operator $D = \sum_{k=1}^{3} i_k \partial_k$, where $\partial_k = \frac{\partial}{\partial x_k}$, whose action on quaternion valued functions can be represented in a vector form as follows

$$Df = - \text{div} \overrightarrow{f} + \text{grad} f_0 + \text{rot} \overrightarrow{f}.$$ 

That is, $\text{Sc}(Df) = - \text{div} \overrightarrow{f}$ and $\text{Vec}(Df) = \text{grad} f_0 + \text{rot} \overrightarrow{f}$.

Denote $D_\alpha = D + \alpha$, where $\alpha$ is a complex constant. We have the following factorization of the Helmholtz operator [15]:

$$\Delta + \alpha^2 = - D_\alpha D_{-\alpha} = - D_{-\alpha} D_\alpha. \quad (1)$$

Using the fundamental solution of the Helmholtz operator

$$\theta_\alpha(x) = - \frac{e^{i\alpha|x|}}{4\pi |x|}$$

(we suppose that $\text{Im} \alpha \geq 0$), the fundamental solutions $\mathcal{K}_\alpha$ and $\mathcal{K}_{-\alpha}$ for the operators $D_\alpha$ and $D_{-\alpha}$ can be obtained from (1) in the following way

$$\mathcal{K}_\alpha = -(D - \alpha)\theta_\alpha \quad \text{and} \quad \mathcal{K}_{-\alpha} = -(D + \alpha)\theta_\alpha. \quad (2)$$

We have

$$D_{\pm\alpha} \mathcal{K}_{\pm\alpha} = \delta,$$

where $\delta$ is the Dirac delta function.

From (2) we obtain the explicit form of $\mathcal{K}_\alpha$ and $\mathcal{K}_{-\alpha}$:

$$\mathcal{K}_{\pm\alpha}(x) = (\pm \alpha + \frac{x}{|x|^2} - i\alpha \frac{x}{|x|})\theta_\alpha(x). \quad (3)$$

Here $x = \sum_{k=1}^{3} x_k i_k$. Note that $\mathcal{K}_\alpha$ and $\mathcal{K}_{-\alpha}$ are full biquaternions with $\text{Sc}(\mathcal{K}_{\pm\alpha}(x)) = \pm \alpha \theta_\alpha(x)$ and $\text{Vec}(\mathcal{K}_{\pm\alpha}(x)) = - \text{grad} \theta_\alpha(x) = (\frac{x}{|x|^2} - i\alpha \frac{x}{|x|})\theta_\alpha(x)$.

More information on the algebra of biquaternions and related calculus can be found in [28].

3 Biquaternionic reformulation of Maxwell’s equations in chiral media

The operators $D_\alpha$ and $D_{-\alpha}$ are closely related to the Maxwell equations. Consider the Maxwell system for a homogeneous chiral medium (see, e.g., [34, 36])
\[ \text{rot} \vec{E} (x) = -i\alpha \left( \vec{H} (x) + \beta \text{rot} \vec{H} (x) \right) \]  
(4)

and

\[ \text{rot} \vec{H} (x) = i\alpha \left( \vec{E} (x) + \beta \text{rot} \vec{E} (x) \right), \]  
(5)

where \( \alpha = \omega \sqrt{\varepsilon \mu} \). Some examples of numerical values of \( \beta \) for physical media can be found, e.g., in [36]. We notice only that when \( \beta = 0 \) we obtain the Maxwell system for a homogeneous, isotropic achiral medium with the wave number \( \alpha \).

The vectors \( \vec{E} \) and \( \vec{H} \) in (4) and (5) are complex. Consider the following purely vectorial biquaternionic functions

\[ \vec{\varphi} = \vec{E} + i\vec{H} \quad \text{and} \quad \vec{\psi} = \vec{E} - i\vec{H}. \]

It is easy to verify (see [21], [31] or [23]) that \( \vec{\varphi} \) and \( \vec{\psi} \) satisfy the following equations

\[ (D + \alpha_1) \vec{\varphi} = 0 \]

and

\[ (D - \alpha_2) \vec{\psi} = 0, \]

where

\[ \alpha_1 = \frac{\alpha}{(1 + \alpha \beta)}, \quad \alpha_2 = \frac{\alpha}{(1 - \alpha \beta)}. \]

**Remark 1** If \( \beta = 0 \) then \( \alpha_1 = \alpha_2 = \alpha \) and we arrive at the quaternionic form of the Maxwell equations in the achiral case (see [32, Sect. 9], [28]) but in general \( \alpha_1 \) and \( \alpha_2 \) are different and physically characterize the propagation of electromagnetic waves of opposing circular polarizations.

Obviously the vectors \( \vec{E} \) and \( \vec{H} \) are easily recovered from \( \vec{\varphi} \) and \( \vec{\psi} \):

\[ \vec{E} = \frac{1}{2} (\vec{\varphi} + \vec{\psi}) \quad \text{and} \quad \vec{H} = \frac{1}{2i} (\vec{\varphi} - \vec{\psi}). \]
4 Completeness of a system of biquaternionic fundamental solutions

Let \( \Gamma \) be a sufficiently smooth closed surface in \( \mathbb{R}^3 \). Here we use the term sufficiently smooth for surfaces whose smoothness allows us to introduce the corresponding Sobolev space \( H^s(\Gamma) \) for a given \( s \in \mathbb{R} \).

The interior domain enclosed by \( \Gamma \) we denote by \( \Omega^+ \) and the exterior by \( \Omega^- \).

Let \( \overrightarrow{e} \) and \( \overrightarrow{h} \) be two complex vectors defined on \( \Gamma \).

**Definition 2** We say that \( \overrightarrow{e} \) and \( \overrightarrow{h} \) are extendable into \( \Omega^+ \) if there exist such pair of vectors \( \overrightarrow{E} \) and \( \overrightarrow{H} \) defined on \( \Omega^+ \) that equations (4) and (5) are satisfied in \( \Omega^+ \) and on \( \Gamma \) we have \( \overrightarrow{E} |_{\Gamma} = \overrightarrow{e} \) and \( \overrightarrow{H} |_{\Gamma} = \overrightarrow{h} \).

**Definition 3** The vectors \( \overrightarrow{e} \) and \( \overrightarrow{h} \) are extendable into \( \Omega^- \) if there exist such pair of vectors \( \overrightarrow{E} \) and \( \overrightarrow{H} \) defined on \( \Omega^- \) that equations (4) and (5) are satisfied in \( \Omega^- \), the Silver-Müller condition

\[
\overrightarrow{E} - \left[ \frac{x}{|x|} \times \overrightarrow{H} \right] = o\left(\frac{1}{|x|}\right)
\]  

(6)

is fulfilled at infinity uniformly for all directions and on \( \Gamma \): \( \overrightarrow{E} |_{\Gamma} = \overrightarrow{e} \) and \( \overrightarrow{H} |_{\Gamma} = \overrightarrow{h} \).

With the aid of quaternionic analysis techniques these two introduced classes of vector functions can be completely described. For achiral media it was done in [25] (see also [32, Sect. 11]) and for chiral media in [21]. Here we recall these results without proof.

We will need the following operators

\[
S_\alpha f(x) = -2 \int_{\Gamma} \mathcal{K}_\alpha(x - y)\overrightarrow{n}(y)f(y)d\Gamma_y, \quad x \in \Gamma,
\]

\[
P_\alpha = \frac{1}{2}(I + S_\alpha) \quad \text{and} \quad Q_\alpha = \frac{1}{2}(I - S_\alpha)
\]

which are bounded in \( H^s(\Gamma) \) for all real \( s \). The function \( f \) in (7) is a biquaternion valued function, \( \overrightarrow{n} \) is the quaternionic representation of the outward
with respect to $\Omega^+$ unitary normal on $\Gamma$: $\overrightarrow{n} = \sum_{k=1}^{3} n_k i_k$ and all the products in the integrand in (7) are quaternionic products. From the numerous interesting properties of the operators $P_\alpha$, $Q_\alpha$ and $S_\alpha$ (see [32]) we will need only the following fact.

**Theorem 4** Let complex vectors $\overrightarrow{e}$ and $\overrightarrow{h}$ belong to $H^s(\Gamma)$, $s > 0$. Then

1. in order for $\overrightarrow{e}$ and $\overrightarrow{h}$ to be extendable into $\Omega^+$ the following condition is necessary and sufficient

$$ (\overrightarrow{e} + i \overrightarrow{h}) \in \text{im } P_{\alpha_1}(H^s(\Gamma)) \quad \text{and} \quad (\overrightarrow{e} - i \overrightarrow{h}) \in \text{im } P_{-\alpha_2}(H^s(\Gamma)) $$

or which is the same

$$ \overrightarrow{e} + i \overrightarrow{h} = S_{\alpha_1}(\overrightarrow{e} + i \overrightarrow{h}) \quad \text{and} \quad \overrightarrow{e} - i \overrightarrow{h} = S_{-\alpha_2}(\overrightarrow{e} - i \overrightarrow{h}) \quad \text{on } \Gamma. $$

2. in order for $\overrightarrow{e}$ and $\overrightarrow{h}$ to be extendable into $\Omega^-$ the following condition is necessary and sufficient

$$ (\overrightarrow{e} + i \overrightarrow{h}) \in \text{im } Q_{\alpha_1}(H^s(\Gamma)) \quad \text{and} \quad (\overrightarrow{e} - i \overrightarrow{h}) \in \text{im } Q_{-\alpha_2}(H^s(\Gamma)) $$

or which is the same

$$ \overrightarrow{e} + i \overrightarrow{h} = -S_{\alpha_1}(\overrightarrow{e} + i \overrightarrow{h}) \quad \text{and} \quad \overrightarrow{e} - i \overrightarrow{h} = -S_{-\alpha_2}(\overrightarrow{e} - i \overrightarrow{h}) \quad \text{on } \Gamma. $$

Now we will show how two systems of quaternionic fundamental solutions suitable for the approximation of the vector functions extendable into $\Omega^+$ or $\Omega^-$ can be constructed.

By $\Gamma^-$ we denote a closed surface enclosed in $\Omega^+$ and being a boundary of a bounded domain $V$, and by $\Gamma^+$ we denote a closed surface enclosing $\overline{\Omega^+}$ as shown in the figure. By $\{y^-_n\}_{n=1}^\infty$ we denote a set of points densely distributed on $\Gamma^-$, and by $\{y^+_n\}_{n=1}^\infty$ a set of points densely distributed on $\Gamma^+$. For each of these two sets we construct a corresponding pair of systems of quaternionic fundamental solutions. The pair of systems

$$ \{K_{\alpha_1,n}(x) = K_{\alpha_1}(x - y^-_n)\}_{n=1}^\infty \quad \text{and} \quad \{K_{-\alpha_2,n}(x) = K_{-\alpha_2}(x - y^+_n)\}_{n=1}^\infty $$

(10)
corresponds to \( \{y_n^+\}_{n=1}^\infty \) and the pair of systems
\[
\{ K_{\alpha_1,n}(x) = K_{\alpha_1}(x-y_n^-) \}_{n=1}^\infty \quad \text{and} \quad \{ K_{-\alpha_2,n}(x) = K_{-\alpha_2}(x-y_n^-) \}_{n=1}^\infty
\]
corresponds to \( \{y_n^-\}_{n=1}^\infty \).

The following theorems show us the possibility to apply the fundamental solutions \((10)\) for the numerical solution of interior boundary value problems for the Maxwell equations \((4), (5)\), and fundamental solutions \((11)\) for the solution of exterior problems.

**Theorem 5** \([23]\) Let two complex vectors \(\overrightarrow{e}^+\) and \(\overrightarrow{h}^+\) belong to \(H^s(\Gamma)\), \(s > 1\), be extendable into \(\Omega^+\) and both \(\alpha_1^+\) and \(\alpha_2^+\) be not eigenvalues of the Dirichlet problem in \(\Omega^+\). Then \(\overrightarrow{e}^+\) and \(\overrightarrow{h}^+\) can be approximated with an arbitrary precision (in the norm of \(H^{s-1}(\Gamma)\)) by right linear combinations of the form
\[
\overrightarrow{e}_N = \frac{1}{2} \left( \sum_{j=1}^{N} K_{\alpha_1,j}^+ a_j + \sum_{j=1}^{N} K_{-\alpha_2,j}^+ b_j \right)
\]
and
\[
\overrightarrow{h}_N = \frac{1}{2i} \left( \sum_{j=1}^{N} K_{\alpha_1,j}^+ a_j - \sum_{j=1}^{N} K_{-\alpha_2,j}^+ b_j \right).
\]
where \(a_j\) and \(b_j\) are constant complex quaternions.

**Theorem 6** [23] Let two complex vectors \(\vec{e}\) and \(\vec{h}\) belong to \(H^s(\Gamma)\), \(s > 1\), be extendable into \(\Omega^-\) and let both \(\alpha_1^2\) and \(\alpha_2^2\) be not eigenvalues of the Dirichlet problem in \(V\). Then \(\vec{e}\) and \(\vec{h}\) can be approximated with an arbitrary precision (in the norm of \(H^{s-1}(\Gamma)\)) by right linear combinations of the form

\[
\vec{e}_N = \frac{1}{2} \left( \sum_{j=1}^{N} K_{\alpha_1,j} a_j + \sum_{j=1}^{N} K_{-\alpha_2,j} b_j \right) \tag{12}
\]

and

\[
\vec{h}_N = \frac{1}{2i} \left( \sum_{j=1}^{N} K_{\alpha_1,j} a_j - \sum_{j=1}^{N} K_{-\alpha_2,j} b_j \right), \tag{13}
\]

where \(a_j\) and \(b_j\) are constant complex quaternions.

**Remark 7** The right linear combinations in Theorem 5 and Theorem 6 are in general full quaternions. In order to ensure that they will be purely vectorial additionally to a usual boundary condition for the electromagnetic field we have to add the requirement that their scalar parts be equal to zero. We show how this can be easily achieved on some examples of numerical realization considered in the next section.

5 **Numerical realization**

Consider the exterior boundary value problem for the Maxwell equations corresponding to the model of electromagnetic scattering by a perfectly conducting body with a boundary \(\Gamma\). Find two vectors \(\vec{E}\) and \(\vec{H}\) satisfying (4) and (5) in \(\Omega^-\), the condition (6) at infinity and the following boundary condition

\[
[\vec{E}(x) \times \vec{n}(x)] = \vec{f}(x), \quad x \in \Gamma, \tag{14}
\]

where \(\vec{f}\) is a given tangential field.

We look for the solutions in the form (12) and (13), applying the collocation method in order to find the coefficients \(a_j\) and \(b_j\). Substitution of the vector part of (12) in (14) gives us two linearly independent equations in
every collocation point. In each collocation point \( x \in \Gamma \) we must require also that
\[
\text{Sc}(\sum_{j=1}^{N} K_{\alpha_1,j}(x)a_j + \sum_{j=1}^{N} K_{\alpha_2,j}(x)b_j) = 0, \tag{15}
\]
and
\[
\text{Sc}(\sum_{j=1}^{N} K_{\alpha_1,j}(x)a_j - \sum_{j=1}^{N} K_{\alpha_2,j}(x)b_j) = 0, \tag{16}
\]
which gives us other two linearly independent equations. Taking into account that in (12) and (13) we have 8N unknown complex quantities we need 2N collocation points. After having solved the corresponding system of linear algebraic equations we obtain the coefficients \( a_j \) and \( b_j \) and consequently the approximate solution of the problem. A good approximation of the boundary condition (14) guarantees a good approximation of the electromagnetic field in the domain \( \Omega^- \) due to the following estimate (see [12, p. 126])
\[
\| \vec{E} \|_{\infty, \Omega_r^-} + \| \vec{H} \|_{\infty, \Omega_r^-} \leq C \| \vec{n} \times \vec{E} \|_{L_2(\Gamma)}
\]
where \( \| \cdot \|_{\infty, \Omega_r} \) stands for the supremum norm in any closed subset \( \Omega_r^- \) of \( \Omega^- \) and \( C \) is a positive constant depending on \( \Gamma \) and \( \Omega_r^- \).

The method was tested [20], [22], [23] using different exact solutions. For example, let \( \beta = 0 \) and consequently \( \alpha = \alpha_1 = \alpha_2 \). The vectors
\[
\vec{E}_m(x) = \text{rot} \vec{c} \theta_\alpha(x)
\]
and
\[
\vec{H}_m(x) = -\frac{1}{i\alpha} \text{rot} \vec{E}_m(x), \quad x \in \mathbb{R}^3 \setminus \{0\},
\]
where \( \vec{c} \in \mathbb{R}^3 \) is constant, represent the electromagnetic field of a magnetic dipole situated at the origin [10, Sect. 4.2]. They satisfy (1) and (3) (for \( \beta = 0 \)) as well as the Silver-Müller conditions at infinity.

Let \( \Gamma \) be an ellipsoid described by the equalities.
\[
x_1 = a \cos \eta \sin \nu, \quad x_2 = b \sin \eta \sin \nu, \quad x_3 = c \cos \nu, \tag{17}
\]
where \( 0 < \eta \leq 2\pi, 0 < \nu \leq \pi \). Then \( \vec{E}_m \) and \( \vec{H}_m \) give us the solution of the following boundary value problem
\[
\text{rot} \vec{E}(x) = -i\alpha \vec{H}(x), \quad x \in \Omega^-,
\]
\[
\text{rot } \overrightarrow{H}(x) = i\alpha \overrightarrow{E}(x), \quad x \in \Omega^-,
\]
\[
\left[ \overrightarrow{E}(x) \times \overrightarrow{n}(x) \right] = \overrightarrow{f}(x), \quad x \in \Gamma
\]
where
\[
\overrightarrow{f}(x) = \begin{bmatrix}
c_3 \partial_2 \theta_\alpha(x) - c_2 \partial_3 \theta_\alpha(x) \\
c_1 \partial_2 \theta_\alpha(x) - c_3 \partial_1 \theta_\alpha(x) \\
c_2 \partial_3 \theta_\alpha(x) - c_1 \partial_2 \theta_\alpha(x)
\end{bmatrix} \times \overrightarrow{n}(x).
\]

We give the numerical results for \(a = 5, \ b = 3\) and \(c = 2\) in (17). As the auxiliary surface \(\Gamma^-\) containing points \(y^-\) we have chosen an ellipsoid interior with respect to \(\Gamma\) with \(a, \ b\) and \(c\) multiplied by 0.15. In the following table we present the results for \(\alpha = 1 + 0.3i\) and for different values of \(N\). The corresponding errors represent the absolute maximum difference between the exact and the approximate solutions at the points on the ellipsoid exterior with respect to \(\Gamma\) with \(a, \ b\) and \(c\) multiplied by 5.

| \(N\) | Error for \(\overrightarrow{E}\) | Error for \(\overrightarrow{H}\) |
|-------|----------------|----------------|
| 10    | 0.441E-03     | 0.332E-03     |
| 15    | 0.693E-05     | 0.713E-05     |
| 20    | 0.162E-05     | 0.186E-05     |
| 25    | 0.245E-06     | 0.248E-06     |
| 30    | 0.113E-06     | 0.171E-06     |
| 35    | 0.522E-07     | 0.409E-07     |

A quite fast convergence of the method can be appreciated (all numerical results were obtained on a PC Pentium 4).

Let us notice that the approximation by linear combinations of quaternionic fundamental solutions can be applied to other classes of boundary value problems for the Maxwell system like for example the impedance problem with the boundary condition
\[
\left[ \overrightarrow{E}(x) \times \overrightarrow{n}(x) \right] - \xi \left[ \overrightarrow{H}(x) \times \overrightarrow{n}(x) \right] = \overrightarrow{f}(x), \quad x \in \Gamma.
\]
This implies some obvious changes in the matrix of coefficients of the system of linear algebraic equations corresponding to collocation points.

More results and analysis of numerical experiments were given in [20].
6 Time-dependent Maxwell’s equations for chiral media

Consider time-dependent Maxwell’s equations

\[
\text{rot} \, \vec{E}(t, x) = -\partial_t \vec{B}(t, x),
\]

\[
\text{rot} \, \vec{H}(t, x) = \partial_t \vec{D}(t, x) + \vec{j}(t, x),
\]

\[
\text{div} \, \vec{E}(t, x) = \rho(t, x) \varepsilon, \quad \text{div} \, \vec{H}(t, x) = 0
\]

with the Drude-Born-Fedorov constitutive relations corresponding to the chiral media (see, e.g., \[2\], \[33\], \[36\]):

\[
\vec{B}(t, x) = \mu(\vec{H}(t, x) + \beta \text{rot} \, \vec{H}(t, x)),
\]

\[
\vec{D}(t, x) = \varepsilon(\vec{E}(t, x) + \beta \text{rot} \, \vec{E}(t, x)),
\]

where \( \beta \) is the chirality measure of the medium. \( \beta, \varepsilon, \mu \) are real scalars assumed to be constants. Note that the charge density \( \rho \) and the current density \( \vec{j} \) are related by the continuity equation \( \partial_t \rho + \text{div} \, \vec{j} = 0 \).

Incorporating the constitutive relations (21), (22) into the system (18)-(20) we arrive at the time-dependent Maxwell system for a homogeneous chiral medium

\[
\text{rot} \, \vec{H}(t, x) = \varepsilon(\partial_t \vec{E}(t, x) + \beta \partial_t \text{rot} \, \vec{E}(t, x)) + \vec{j}(t, x),
\]

\[
\text{rot} \, \vec{E}(t, x) = -\mu(\partial_t \vec{H}(t, x) + \beta \partial_t \text{rot} \, \vec{H}(t, x)),
\]

\[
\text{div} \, \vec{E}(t, x) = \frac{\rho(t, x)}{\varepsilon}, \quad \text{div} \, \vec{H}(t, x) = 0.
\]

Application of \text{rot} to (23) and (24) allows us to separate the equations for \( \vec{E} \) and \( \vec{H} \) and to obtain in this way the wave equations for a chiral medium

\[
\text{rot rot} \, \vec{E} + \varepsilon \mu \partial_t^2 \vec{E} + 2\beta \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{E} + \beta^2 \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{E} = -\mu \partial_t \vec{j} - \beta \mu \partial_t \text{rot} \, \vec{j},
\]

\[
\text{rot rot} \, \vec{H} + \varepsilon \mu \partial_t^2 \vec{H} + 2\beta \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{H} + \beta^2 \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{H} = \text{rot} \, \vec{j}.
\]
It should be noted that when $\beta = 0$, (26) and (27) reduce to the wave equations for non-chiral media but in general to the difference of the usual non-chiral wave equations their chiral generalizations represent equations of fourth order.

7 Field equations in a biquaternionic form

In this section following [13] we rewrite the field equations from Section 6 in a biquaternionic form.

Let us introduce the following biquaternionic operator

$$M = \beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t - i D$$

(28)

and consider the purely vectorial biquaternionic function

$$\vec{V}(t, x) = \vec{E}(t, x) - i \sqrt{\frac{\mu}{\varepsilon}} \vec{H}(t, x).$$

(29)

Proposition 8 [13] The equation

$$M \vec{V}(t, x) = -\sqrt{\frac{\mu}{\varepsilon}} \vec{j}(t, x) - \beta \sqrt{\frac{\mu}{\varepsilon}} \partial_t \rho(t, x) + \frac{i \rho(t, x)}{\varepsilon}$$

(30)

is equivalent to the Maxwell system (23)-(25), the vectors $\vec{E}$ and $\vec{H}$ are solutions of (23)-(25) if and only if the purely vectorial biquaternionic function $\vec{V}$ defined by (29) is a solution of (30).

Proof. The scalar and the vector parts of (30) have the form

$$-\beta \sqrt{\varepsilon \mu} \partial_t \text{div} \vec{E} + \sqrt{\frac{\mu}{\varepsilon}} \text{div} \vec{H} + i (\text{div} \vec{E} + \beta \mu \partial_t \text{div} \vec{H}) = -\beta \sqrt{\frac{\mu}{\varepsilon}} \partial_t \rho + \frac{i \rho(t, x)}{\varepsilon},$$

(31)

$$\beta \sqrt{\varepsilon \mu} \partial_t \text{rot} \vec{E} + \sqrt{\varepsilon \mu} \partial_t \vec{E} - \sqrt{\frac{\mu}{\varepsilon}} \text{rot} \vec{H} - i (\text{rot} \vec{E} + \beta \mu \partial_t \text{rot} \vec{H} + \mu \partial_t \vec{H}) = -\sqrt{\frac{\mu}{\varepsilon}} \vec{j}.$$  

(32)

The real part of (32) coincides with (23) and the imaginary part coincides with (24). Applying divergence to the equation (32) and using the continuity equation gives us

$$\partial_t \text{div} \vec{H} = 0 \quad \text{and} \quad \partial_t \text{div} \vec{E} = \frac{1}{\varepsilon} \partial_t \rho.$$
Taking into account these two equalities we obtain from (31) that the vectors $\mathbf{E}$ and $\mathbf{H}$ satisfy equations (25).

It should be noted that for $\beta = 0$ from (28) we obtain the biquaternionic Maxwell operator for a homogeneous achiral medium for which the following equality is valid

$$\varepsilon \mu \partial_t^2 - \Delta x = (\sqrt{\varepsilon \mu} \partial_t + iD)(\sqrt{\varepsilon \mu} \partial_t - iD).$$

In the case under consideration ($\beta \neq 0$) we obtain a similar result. Let us denote by $M^*$ the complex conjugate operator of $M$:

$$M^* = \beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t + iD.$$

For simplicity we consider now a sourceless situation. In this case the equations (26) and (27) are homogeneous and can be represented as follows

$$M M^* \mathbf{U}(t, x) = 0,$$

where $\mathbf{U}$ stands for $\mathbf{E}$ or for $\mathbf{H}$.

### 8 Green function for the operator $M$

Here we present a procedure from [13] which gives us a Green function for the operator $M$. Consider the equation

$$(\beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t - iD)f(t, x) = \delta(t, x).$$

Applying the Fourier transform $\mathcal{F}$ with respect to the time-variable $t$ we obtain

$$(\beta \sqrt{\varepsilon \mu} i \omega D + \sqrt{\varepsilon \mu} i \omega - iD)F(\omega, x) = \delta(x),$$

where $F(\omega, x) = \mathcal{F}\{f(t, x)\} = \int_{-\infty}^{\infty} f(t, x)e^{-i\omega t}dt$. The last equation can be rewritten as follows

$$(D + \alpha)(\beta \sqrt{\varepsilon \mu} \omega - 1)iF(\omega, x) = \delta(x),$$

where $\alpha = \frac{\sqrt{\mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}$. The fundamental solution of $D_\alpha$ is given by (3), so we have

$$(\beta \sqrt{\varepsilon \mu} \omega - 1)iF(\omega, x) = K_\alpha(x) = \left(\alpha + \frac{x}{|x|^2} - i\alpha \frac{x}{|x|}\right)\Theta_\alpha(x),$$

where $\Theta_\alpha(x)$ is the fundamental solution of $D_\alpha$. The Green function $G(t, x)$ for the operator $M$ can be obtained from $F(\omega, x)$ by the inverse Fourier transform.

$$G(t, x) = \mathcal{F}^{-1}\{F(\omega, x)\} = K_{\frac{\beta \sqrt{\varepsilon \mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}}(x),$$

where $\Theta_{\frac{\beta \sqrt{\varepsilon \mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}}(x)$ is the fundamental solution of $\frac{\beta \sqrt{\varepsilon \mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}D_{\frac{\beta \sqrt{\varepsilon \mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}}$.
from where

\[ F(\omega, x) = \left[ \frac{i \sqrt{\varepsilon \omega}}{(\beta \sqrt{\varepsilon \omega} - 1)^2} \left( 1 - \frac{ix}{|x|} \right) + \frac{ix}{|x|^2} \frac{1}{\beta \sqrt{\varepsilon \omega} - 1} \right] e^{i|\rho| \sqrt{\varepsilon \omega} - 1} \frac{e^{i|\rho| \sqrt{\varepsilon \omega}}}{4\pi |\rho|}. \]

We write it in a more convenient form

\[ F(\omega, x) = \left( \frac{1}{(\omega - a)^2} A(x) + \frac{1}{\omega - a} B(x) \right) E(x) e^{i\lambda(x)} \]

where \( a = \frac{1}{\beta \sqrt{\varepsilon \mu}}, c(x) = \frac{|x|}{\beta^2 \sqrt{\varepsilon \mu}}, E(x) = \frac{e^{i|x|}}{4\pi |x|}, \)

\[ A(x) = \frac{i}{\beta^3 \varepsilon \mu} \left( 1 - \frac{ix}{|x|} \right), \quad B(x) = \frac{i}{\beta \sqrt{\varepsilon \mu}} \left( \frac{1}{\beta \left( 1 - \frac{ix}{|x|} \right) + \frac{x}{|x|^2}} \right). \]

In order to obtain the fundamental solution \( f(t, x) \) we should apply the inverse Fourier transform to \( F(\omega, x) \). Among different regularizations of the resulting integral we should choose the one leading to a fundamental solution satisfying the causality principle, that is vanishing for \( t < 0 \). Such an election is done by introducing a small parameter \( y > 0 \) in the following way

\[ f(t, x) = \lim_{y \to 0} \mathcal{F}^{-1} \{ F(z, x) \} \]  

where \( z = \omega - iy \). This regularization is in agreement with the condition \( \text{Im} \alpha \geq 0 \). We have

\[ \mathcal{F}^{-1} \{ F(z, x) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\omega - a)^2} A(x) + \frac{1}{\omega - a} B(x) \left( \frac{e^{i\lambda(x)}}{e^{\omega - ay}} e^{i\omega t} d\omega \right) \]  

where \( a_y = a + iy \). Expression (34) includes two integrals of the form

\[ I_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega y} e^{i\omega t}}{(\omega - a_y)^k} d\omega, \quad k = 1, 2 \]

where \( c = c(x) \). We have

\[ I_k = \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{(ic)^j}{j!} \int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{(\omega - a_y)^{j+k}}. \]
where the change of order of integration and summation is possible because the two necessary for this conditions are fulfilled: the series is uniformly convergent on each segment and the integrals of partial sums converge uniformly with respect to $j$. Denote

$$ I_{k,j}(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{(\omega - a_y)^{j+k}}. $$

For $k = 1$ and $j = 0$ we obtain (see, e.g., [9, Sect. 8.7])

$$ I_{1,0}(t) = 2\pi i H(t)e^{ita_y} $$

where $H$ is the Heaviside function. For all other cases, that is for $k = 1$ and $j = 0, 1, \infty$ and for $k = 2$ and $j = 0, 1, \infty$ we have that $j + k \geq 2$ and the integrand in (35) has a pole at the point $a_y$ of order $j + k$. Using a result from the residue theory [11, Sect. 4.3] we obtain

$$ I_{k,j}(t) = 2\pi i \text{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} \text{ for } t \geq 0 \text{ and } j + k \geq 2. $$

Consider

$$ \text{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} = \frac{1}{(j + k - 1)!} \lim_{\omega \to a_y} \frac{\partial^{j+k-1} e^{i\omega t}}{\partial \omega^{j+k-1}} = \frac{(it)^{j+k-1} e^{ia_y t}}{(j + k - 1)!} \text{ for } t \geq 0 $$

and $j + k \geq 2$.

For $t < 0$ we have that $I_{k,j}(t)$ is equal to the sum of residues with respect to singularities in the lower half-plane $y < 0$ which is zero because the integrand is analytic there. Thus we obtain

$$ I_{k,j}(t) = 2\pi i H(t) \frac{(it)^{j+k-1}}{(j + k - 1)!} e^{ia_y t}. $$

Substitution of this result into (35) gives us

$$ I_1 = iH(t)e^{ia_y t} \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!} \quad \text{and} \quad I_2 = -H(t)e^{ia_y t} \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!(j+1)!}. $$

Now using the series representations of the Bessel functions $J_0$ and $J_1$ (see e.g. [44, Chapter 5]) we obtain

$$ I_1 = iH(t)e^{ia_y t} J_0 \left(2\sqrt{ct}\right) \quad \text{and} \quad I_2 = -H(t) \sqrt{\frac{t}{c}} e^{ia_y t} J_1 \left(2\sqrt{ct}\right). $$
Substituting these expressions in (34) and then in (33) we arrive at the following expression for $f$:

$$f(t, x) = H(t) e^{iat} E(x) \left( -A(x) \frac{t}{c} J_1 \left( 2\sqrt{ct} \right) + iB(x) J_0 \left( 2\sqrt{ct} \right) \right).$$

Finally we rewrite the obtained fundamental solution of the operator $M$ in explicit form:

$$f(t, x) = H(t) \frac{e^{\beta \sqrt{x^2}}}{\sqrt{\varepsilon \mu}} \left( K_{\beta}(x) J_0 \left( \frac{2 \sqrt{t|\beta \mu|}}{\beta \sqrt{x^2}} \right) + i \frac{\Theta_{\beta}(x)}{\sqrt{x^2}} \left( 1 - \frac{i x}{|x|^2} \right) \sqrt{\frac{t}{|x|}} J_1 \left( \frac{2 \sqrt{t|\beta \mu|}}{\beta \sqrt{x^2}} \right) \right).$$

Let us notice that $f$ fulfills the causality principle requirement which guarantees that its convolution with the function from the right-hand side of (30) gives us the unique physically meaningful solution of the inhomogeneous Maxwell system (23)-(25) in a whole space.

## 9 Inhomogeneous media

Consider Maxwell’s equations in a nonchiral inhomogeneous medium. Thus we assume that $\varepsilon$ and $\mu$ are functions of coordinates:

$$\varepsilon = \varepsilon(x) \quad \text{and} \quad \mu = \mu(x).$$

Then the Maxwell system has the following form

$$\text{rot } H = \varepsilon \partial_t E + j,$$  \hspace{1cm} (36)

$$\text{rot } E = - \mu \partial_t H,$$  \hspace{1cm} (37)

$$\text{div}(\varepsilon E) = \rho,$$  \hspace{1cm} (38)

$$\text{div}(\mu H) = 0.$$  \hspace{1cm} (39)

In this section following the procedure exposed in [28] we show that this system of equations can be written in the form of a single biquaternionic equation.
Equations (38) and (39) can be written as follows
\[
\text{div } \mathbf{E} + \frac{\text{grad } \varepsilon}{\varepsilon} (\mathbf{E}) = \frac{\rho}{\varepsilon}
\]
and
\[
\text{div } \mathbf{H} + \frac{\text{grad } \mu}{\mu} (\mathbf{H}) = 0.
\]
Combining these equations with (36) and (37) we obtain the Maxwell system in the form
\[
D \mathbf{E} = \left< \frac{\text{grad } \varepsilon}{\varepsilon}, \mathbf{E} \right> - \mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon} \tag{40}
\]
and
\[
D \mathbf{H} = \left< \frac{\text{grad } \mu}{\mu}, \mathbf{H} \right> + \varepsilon \partial_t \mathbf{E} + \mathbf{j}. \tag{41}
\]
Let us make a simple observation: the scalar product of two vectors \( \mathbf{p} \) and \( \mathbf{q} \) can be written as follows
\[
\left< \mathbf{p}, \mathbf{q} \right> = -\frac{1}{2} \left( \mathbf{M} \mathbf{p} + \mathbf{M} \mathbf{q} \right).
\]
Using this fact, from (40) and (41) we obtain the pair of equations
\[
(D + \frac{1}{2} \frac{\text{grad } \varepsilon}{\varepsilon}) \mathbf{E} = -\frac{1}{2} \mathbf{M} \text{grad } \varepsilon \mathbf{E} - \mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon} \tag{42}
\]
and
\[
(D + \frac{1}{2} \frac{\text{grad } \mu}{\mu}) \mathbf{H} = -\frac{1}{2} \mathbf{M} \text{grad } \mu \mathbf{H} + \varepsilon \partial_t \mathbf{E} + \mathbf{j}. \tag{43}
\]
Note that
\[
\frac{1}{2} \frac{\text{grad } \varepsilon}{\varepsilon} = \text{grad } \sqrt{\varepsilon}.
\]
Then (42) can be rewritten in the following form
\[
\frac{1}{\sqrt{\varepsilon}} D(\sqrt{\varepsilon} \cdot \mathbf{E}) + \mathbf{E} \cdot \overrightarrow{\varepsilon} = -\mu \partial_t \mathbf{H} - \frac{\rho}{\varepsilon}, \tag{44}
\]
where
\[
\overrightarrow{\varepsilon} := \frac{\text{grad } \sqrt{\varepsilon}}{\sqrt{\varepsilon}}.
\]
Analogously, (43) takes the form

\[
\frac{1}{\sqrt{\mu}} D(\sqrt{\mu} \cdot \mathbf{H}) + \mathbf{H} \cdot \overrightarrow{\mu} = \varepsilon \partial_t \mathbf{E} + \mathbf{j},
\]  

(45)

where

\[\overrightarrow{\mu} := \frac{\text{grad} \sqrt{\mu}}{\sqrt{\mu}}.\]

Introducing the notations

\[\overrightarrow{E} := \sqrt{\varepsilon} \mathbf{E}, \quad \overrightarrow{H} := \sqrt{\mu} \mathbf{H},\]

multiplying (44) by \(\sqrt{\varepsilon}\) and (45) by \(\sqrt{\mu}\) we arrive at the equations

\[(D + M \overrightarrow{\varepsilon}) \overrightarrow{E} = -\frac{1}{c} \partial_t \overrightarrow{H} - \frac{\rho}{\sqrt{\varepsilon}},\]

(46)

and

\[(D + M \overrightarrow{\mu}) \overrightarrow{H} = \frac{1}{c} \partial_t \overrightarrow{E} + \sqrt{\mu} \mathbf{j},\]

(47)

where as before \(c = 1/\sqrt{\varepsilon \mu}\) is the speed of propagation of electromagnetic waves in the medium.

Equations (46) and (47) can be rewritten in an even more elegant form. Consider the function

\[\mathbf{V} := \overrightarrow{E} + i \overrightarrow{H}.\]

Let us apply to it the biquaternionic operator

\[\frac{1}{c} \partial_t + iD.\]

We obtain

\[(\frac{1}{c} \partial_t + iD) \mathbf{V} = \frac{1}{c} \partial_t \overrightarrow{E} - D \overrightarrow{H} + i(\frac{1}{c} \partial_t \overrightarrow{H} + D \overrightarrow{E}).\]

Applying (47) and (46) to the real and imaginary parts of this equation gives

\[(\frac{1}{c} \partial_t + iD) \mathbf{V} = -i(M \overrightarrow{\varepsilon} \overrightarrow{E} + iM \overrightarrow{\mu} \overrightarrow{H}) - \sqrt{\mu} \mathbf{j} - \frac{i\rho}{\sqrt{\varepsilon}},\]

(48)
Note that
\[ \mathbf{\bar{E}} = \frac{1}{2}(\mathbf{V} + \mathbf{V}^*) \quad \text{and} \quad \mathbf{\bar{H}} = \frac{1}{2i}(\mathbf{V} - \mathbf{V}^*). \]

Hence
\[ M \mathbf{\bar{E}} + iM \mathbf{\bar{H}} = \frac{1}{2}(M(\mathbf{\bar{V}} + \mathbf{\bar{V}}^*)\mathbf{V} + M(\mathbf{\bar{V}} - \mathbf{\bar{V}}^*)\mathbf{V}^*). \]

Let us notice that
\[ \mathbf{\bar{E}} + \mathbf{\bar{H}} = -\frac{\text{grad} \, c}{c} \quad \text{and} \quad \mathbf{\bar{E}} - \mathbf{\bar{H}} = -\frac{\text{grad} \, W}{W}, \]
where \( W = \sqrt{\mu}/\sqrt{\varepsilon} \) is the intrinsic wave impedance of the medium. Denote
\[ \mathbf{\bar{c}} := \frac{\text{grad} \, \sqrt{c}}{\sqrt{c}} \quad \text{and} \quad \mathbf{\bar{W}} := \frac{\text{grad} \, \sqrt{W}}{\sqrt{W}}. \]

Then
\[ M \mathbf{\bar{E}} + iM \mathbf{\bar{H}} = -(M \mathbf{\bar{V}} + M \mathbf{\bar{V}}^*). \]

From (48) we obtain the Maxwell equations for an inhomogeneous medium in the following form
\[ \left( \frac{1}{c} \partial_t + iD \right) \mathbf{V} - M \mathbf{\bar{E}} \mathbf{V} - M \mathbf{\bar{H}} \mathbf{V}^* = -\left( \sqrt{\mu} j + i\rho \sqrt{\varepsilon} \right). \]

This equation first obtained in [27], [28] is completely equivalent to the Maxwell system (36)-(39) and represents Maxwell’s equation for inhomogeneous media in a quaternionic form. We formulate this as the following statement.

**Theorem 9** Let \( c = 1/\sqrt{\varepsilon \mu}, \mathbf{\bar{c}} \text{ and } \mathbf{\bar{W}} \) be defined by (49). Then two real-valued vectors \( \mathbf{E} \text{ and } \mathbf{H} \) are solutions of the system (36)-(39) iff the purely vectorial biquaternion \( \mathbf{V} = \sqrt{\varepsilon} \mathbf{E} + i\sqrt{\mu} \mathbf{H} \) is a solution of (50).

Note that if \( \varepsilon \) and \( \mu \) are constant (a homogeneous medium), equation (50) turns into a well known (at least since the work of C. Lanczos [35]) biquaternionic reformulation of the Maxwell system in a vacuum which was rediscovered by many researchers (e.g., [18] and comments in [14]).

**Remark 10** Equation (50) can be considered as a generalization of the Vekua equation, well known in complex analysis, that describes generalized analytic functions [43]. Recently in [37] using the L. Bers approach [7] another generalization of the Vekua equation was considered. Most likely some of the interesting results discussed in [37] can be obtained for (50) also. Their physical meaning would be of great interest.
9.1 Static case and factorization of the Schrödinger operator

When the vectors of the electromagnetic field do not depend on time from (46) and (47) we obtain two independent equations

\[(D + M \vec{\varepsilon}) \vec{E} = -\frac{\rho}{\sqrt{\varepsilon}}\]

and

\[(D + M \vec{\mu}) \vec{H} = \sqrt{\mu} j.\]

Let us consider the sourceless situation, that is we are interested in the solutions for the operator \(D + M \vec{\alpha}\), where the complex quaternion \(\vec{\alpha}\) represents \(\vec{\varepsilon}\) or \(\vec{\mu}\) and has the form

\[\vec{\alpha} = \frac{\text{grad} f}{f}.\]

The scalar function \(f\) is different from zero.

Note that the study of the operator \(D + M \vec{\alpha}\) practically reduces to that of \(D\), as shown in [39]. In the case of the operator \(D + M \vec{\alpha}\) (which can be called the static Maxwell operator) the situation is quite different.

Consider the equation

\[(-\Delta + \nu) g = 0\]

in \(\Omega\) (51) where \(\nu\) and \(g\) are complex valued functions, and \(\Omega\) is a domain in \(\mathbb{R}^3\). We assume that \(g\) is twice continuously differentiable.

**Theorem 11** [29] Let \(f\) be a nonvanishing particular solution of (51). Then for any scalar twice continuously differentiable function \(g\) the following equality holds,

\[(D + M \frac{\partial f}{\partial x})(D - M \frac{\partial f}{\partial x}) g = (-\Delta + \nu) g.\] (52)

**Remark 12** The factorization (52) was obtained in [5], [6] in a form which required a solution of an associated biquaternionic Riccati equation. In [24] it was shown that the solution has necessarily the form \(Df/f\) with \(f\) being a solution of (51).

**Remark 13** Theorem 11 generalizes theorem 21 from [30].
Remark 14 As $g$ in (52) is a scalar function, the factorization of the Schrödinger operator can be written in the following form

$$(D + M^D)f D(f^{-1}g) = (-\Delta + \nu)g,$$

from which it is obvious that if $g$ is a solution of (51) then the vector $F = f D(f^{-1}g)$ is a solution of the equation

$$(D + M^D)F = 0 \quad \text{in } \Omega. \quad (53)$$

The inverse result is given by the next statement where the following notation is used

$$\mathcal{A}[G](x, y, z) = \int_{x_0}^x G_1(\xi, y_0, z_0) d\xi + \int_{y_0}^y G_2(x, \zeta, z_0) d\zeta + \int_{z_0}^z G_3(x, y, \eta) d\eta + C$$

($C$ is an arbitrary complex constant).

Theorem 15 \cite{29} Let $F$ be a solution of (53) in a simply connected domain $\Omega$. Then $g = f \mathcal{A}[f^{-1}F]$ is a solution of (51).

Moreover, a factorization of the operator $\text{div } p \text{ grad } + q$ is valid also.

Theorem 16 \cite{29} Let $u_0$ be a nonvanishing particular solution of the equation

$$(\text{div } p \text{ grad } + q)u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \quad (54)$$

with $p$, $q$ and $u$ being complex valued functions, $p \in C^2(\Omega)$ and $p \not= 0 \text{ in } \Omega$. Then for any scalar function $\varphi \in C^2(\Omega)$ the following equality holds

$$(\text{div } p \text{ grad } + q)\varphi = -p^{1/2}(D + M^D)(D - M^D)p^{1/2}\varphi \quad (55)$$

where $f = p^{1/2}u_0$.

Thus, if $u$ is a solution of equation (54) then

$$F = f D(f^{-1}p^{1/2}u) = f D(u_0^{-1}u)$$

is a solution of equation (53) (see remark 14). The inverse result has the following form.
Theorem 17 [29] Let $F$ be a solution of equation (53) in a simply connected domain $\Omega$, where $f = p^{1/2}u_0$ and $u_0$ be a nonvanishing particular solution of (54). Then

$$u = u_0 A [f^{-1} F]$$

is a solution of (54).

Notice that due to the fact that in (55) $\varphi$ is scalar, we can rewrite the equality in the form

$$(\text{div} \, p \, \text{grad} + q) \varphi = -p^{1/2} (D + M \frac{Df}{f}) (D - \frac{Df}{f} C_H) p^{1/2} \varphi.$$ 

Now, consider the equation

$$(D - \frac{Df}{f} C_H) W = 0, \quad (56)$$

where $W$ is an $\mathbb{H}(\mathbb{C})$-valued function. Equation (56) is a direct generalization of the main Vekua equation considered in [30]. Moreover, we show that it preserves some of its important properties.

Theorem 18 [29] Let $W = W_0 + W$ be a solution of (56). Then $W_0$ is a solution of the stationary Schrödinger equation

$$-\Delta W_0 + \nu W_0 = 0, \quad (57)$$

where $\nu = \Delta f / f$. Moreover, the function $u = f^{-1} W_0$ is a solution of the equation

$$\text{div}(f^2 \, \text{grad} \, u) = 0 \quad (58)$$

and the vector function $v = f \, W$ is a solution of the equation

$$\text{rot}(f^{-2} \, \text{rot} \, v) = 0. \quad (59)$$

Remark 19 [29] Observe that the functions

$$F_0 = f, \quad F_1 = \frac{i_1}{f}, \quad F_2 = \frac{i_2}{f}, \quad F_3 = \frac{i_3}{f}$$

give us a generating quartet for the equation (56). They are solutions of (56) and obviously any $\mathbb{H}(\mathbb{C})$-valued function $W$ can be represented in the form

$$W = \sum_{j=0}^{3} \varphi_j F_j,$$
where \( \varphi_j \) are complex valued functions. It is easy to verify that the function \( W \) is a solution of (56) iff

\[
\sum_{j=0}^{3} (D\varphi_j) F_j = 0
\]

in a complete analogy with the two-dimensional case. Denote

\[
w = \varphi_0 + \varphi_1 i_1 + \varphi_2 i_2 + \varphi_3 i_3.
\]

Then (60) can be written as follows

\[
D(w + \bar{w})f + D(w - \bar{w})\frac{1}{f} = 0
\]

which is equivalent to the equation

\[
Dw = \frac{1 - f^2}{1 + f^2} D\bar{w}.
\]

**Remark 20** The results of this section remain valid in the \( n \)-dimensional situation if instead of quaternions the Clifford algebra \( Cl_{0,n} \) (see, e.g., [8], [16]) is considered. The operator \( D \) is then introduced as follows \( D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \)

where \( e_j \) are the basic basis elements of the Clifford algebra.

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