Discretization of superintegrable systems on a plane

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Abstract. We construct difference analogues of so called Smorodinsky-Winternitz superintegrable systems in the Euclidean plane. Using methods of umbral calculus, we obtain difference equations for generalized isotropic harmonic oscillator on the uniform lattice, and also its solution in terms of power series. In the case of gauge-rotated Hamiltonian, the solution is a polynomial, well-defined in the whole plane.

1. Introduction
Recent development in the field of theoretical and mathematical physics leads to the idea that existing models in quantum mechanics are only a continuous approximation of discrete space-time. Discretization has shown to be a convenient tool for quantum chromodynamics and renormalization theories [1], as well as for one of the candidates for grand unification theory – loop quantum gravity [2]. This assumes an elementary length $l_P = \sqrt{\hbar k} = 10^{-33} \text{ cm}$ which is referred to as Planck length.

There have been several attempts to create models for discrete quantum mechanics. The approach to the harmonic oscillator and hydrogen atom using special functions theory has been introduced in Atakishiev, Suslov [3] and Lorente [4]. Odake and Sasaki [5], [6] construct the Hamiltonians as infinite-dimensional Jacobi matrices which can be understood as discrete quantum mechanical systems on a uniform grid or $q$-grid. An operator approach for discretization of harmonic oscillator has been used by Turbiner [7].

The problem with these methods is that one encounters issues with preserving Lorentz and Galilei invariance and symmetry algebras. This can be partially solved by using the mathematical tool called “umbral calculus”, introduced by Roman [8] and Rota [9] in 1970’s. The umbral approach for simple systems has been used in Dimakis [10] and later extended to two dimensions by Levi and Winternitz [11].

The aim of this article is to extend the application of umbral calculus and use a particular realization of difference operators that transfer some integrable systems to two-dimensional uniform grid. Thanks to the essence of umbral theory, Lie symmetries are preserved and solutions are obtained by the simple substitution.

In Section 2, we introduce the mathematics of umbral calculus and we show particular difference operators to be used. In Section 3, the superintegrable systems are defined and two classes of harmonic oscillators on the Euclidean plane are described. Section 4 is devoted to the own discretization and we find the solutions of the corresponding difference equations. Finally, in Section 5 some conclusions are drawn.
2. Discretization Method

Let $\mathbb{F}$ be a field of characteristic zero. We denote $\mathcal{P} = \mathbb{F}[x]$ a vector space of polynomials over $\mathbb{F}$ in variable $x$ and $\mathcal{L}(\mathcal{P})$ a space of linear operators on $\mathcal{P}$. Addition and scalar multiplication are defined as usual.

Let $\mathbb{F}$ be an algebra of formal power series in variable $t$, i.e. the elements of $\mathbb{F}$ are in the form $\sum_{k=0}^{\infty} a_k t^k$. For $f(t) = \sum_{k=0}^{\infty} a_k t^k$ and $g(t) = \sum_{k=0}^{\infty} b_k t^k$ we define the algebraical operations as follows:

\[
\begin{align*}
    f(t) + g(t) &= \sum_{k=0}^{\infty} (a_k + b_k) t^k, \\
    f(t) g(t) &= \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j b_{k-j} \right) t^k.
\end{align*}
\]

With these operations, $\mathbb{F}$ is an algebra with no zero divisors, and is called an umbral algebra.

Moreover we define the formal derivative on $\mathbb{F}$ naturally as

\[
f'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}.
\]

There is a certain correspondence between formal power series and linear operators on $\mathcal{P}$. For each $f(t) \in \mathbb{F}$ we define an operator $U_f \in \mathcal{L}(\mathcal{P})$ as

\[
f(t) \mapsto U_f = \sum_{k=0}^{\infty} a_k \partial_x^k,
\]

where $\partial_x = \frac{d}{dx}$ is the operator of derivative with respect to $x$. The operator $U_f$ is called a delta operator if and only if $a_0 = 0$ and $a_1 \neq 0$. For $\sigma \in \mathbb{F}$, we define a shift operator $U_f = T_\sigma \in \mathcal{L}(\mathcal{P})$ using power series $f(t) \in \mathbb{F}$ as

\[
f(t) = \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} t^k.
\]

We can easily see that the action of $T_\sigma$ on $\mathcal{P}$ is

\[
T_\sigma p(x) = p(x + \sigma),
\]

in other words, $T_\sigma$ is indeed a shift in the variable $x$. Consequently, we can define an important subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{P})$ in the following manner:

\[
\mathcal{A} = \{ S \in \mathcal{L}(\mathcal{P}) | \forall \sigma \in \mathbb{F} \ ST_\sigma = T_\sigma S \}.
\]

We call the elements of $\mathcal{A}$ shift-invariant operators. There is one-to-one correspondence between $\mathbb{F}$ and $\mathcal{A}$. We provide the following theorems without proofs, all can be found in Roman [8].

**Theorem 1.** The map $f(t) \mapsto U_f$ is an isomorphism between umbral algebra $\mathbb{F}$ and shift-invariant operators $\mathcal{A}$.

Now we establish a connection between delta operators and certain polynomial sequences.

**Theorem 2.** For each delta operator $Q \in \mathcal{A}$ there exists a unique associated sequence $p_n(x)$, where the degree of $p_n(x)$ is $n$, such that

\[
p_0(x) = 1, \quad p_n(0) = 0 \quad \text{for} \quad n = 1, 2, \ldots \\
Q p_n(x) = np_{n-1}(x).
\]
A simple example of an associated sequence is $p_n(x) = x^n$ for the delta operator $Q = \partial_x$.
However, the previous theorem shows that similar sequences can be found for every delta operator.

Let $Q \in \mathcal{A}$ be a delta operator with the associated sequence $p_n(x)$.
An operator $\theta \in \mathcal{L}(\mathcal{P})$ is called an umbral shift if for all $n \in \mathbb{Z}^+$ it holds $\theta p_n(x) = p_{n+1}(x)$.
For the operator $\partial_x$ the umbral shift is trivially $\theta = x$, that is a multiplication by $x$ in $\mathcal{P}$.
There is an important theorem that gives us the formula to find the umbral shift for any operator:

**Theorem 3.** The umbral shift for a delta operator $Q \in \mathcal{A}$ has the form

$$\theta = x\beta, \quad \text{with} \quad \beta = (Q')^{-1},$$

where $Q' = Qx - xQ$ is so called Pincherle derivative of the operator $Q$. The operator $\beta$ is called a conjugate operator to $Q$. Moreover

$$[Q, x\beta] = 1.$$  

The Pincherle derivative is defined for every shift-invariant operator and is easy to compute even without the series expansion from $\mathcal{P}$.
However, if $f(t)$ is the indicator (i.e. the defining series) for $Q$, it can be proved that the formal derivative $f'(t)$ is indeed the indicator of $Q'$.
Also, thanks to the explicit formula in Theorem 3, the umbral shift is unique.

Because $\theta = x\beta$ takes a polynomial $p_n(x)$ of a given delta operator to $p_{n+1}(x)$, it can be used to “generate” the complete associated sequence as

$$p_n(x) = \theta p_{n-1}(x) = \ldots = \theta^m p_0(x) = (x\beta)^n.$$

The discretization procedure is based on so called umbral correspondence. We use it in the particular form

$$\partial_x \leftrightarrow Q, \quad x \leftrightarrow x\beta,$$

where $Q$ is an arbitrary delta operator. This mapping, thanks to Theorem 3, preserves Heisenberg commutation relations, particularly it preserves Lie symmetries of the system.
For example, let $\mathcal{A}$ be a $m$-dimensional Lie algebra with generators \{v_1, \ldots, v_m\} that can be written in the form

$$v_i = \sum_{j=1}^{p} \xi_j(x_1, \ldots, x_p)\partial_{x_j}.$$  

Then the umbral correspondence maps $\mathcal{A}$ to the algebra $\mathcal{A}^D$ generated by the vector fields

$$v_i^D = \sum_{j=1}^{p} \xi_j(x_1\beta x_1, \ldots, x_p\beta x_p)Q_{x_j}.$$  

Because the umbral mapping preserves Heisenberg commutation relations, these two algebras are isomorphic.

Let

$$F(\partial_x, x)f(x) = 0$$

be a linear differential equation with a solution $f(x)$ that can be expanded into a power series around a nonsingular point $x_0$. Without loss of generality, we assume that $x_0 = 0$ and that the expansion is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$
Let $Q$ be a delta operator with the conjugate operator $\beta$ and the associated sequence $p_n(x)$. We perform the operator substitution in the differential equation obtaining

$$F(Q, x\beta)\tilde{f}(x) = 0.$$ 

Following the umbral correspondence, we can see that after the substitution $x^n \leftrightarrow p_n(x)$ in the solution $f(x)$ we can write down the solution of our new equation, that is

$$\tilde{f}(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{n!}{p_n(x)}.$$ 

This can be proved realizing that the pair $(Q, x\beta)$ acts on $p_n(x)$ in the same manner as $(\partial_x, x)$ acts on $x^n$.

Generally, $\tilde{f}$ does not have to be a polynomial in $x$. The action of the operators $Q$ and $x\beta$ is strictly formal in the sense that we do not have any assumptions for convergence of the series (1). In fact, the series generally converge only on certain points which will be the subject of later discussion.

A special case of delta operators, which is in our interest, is the case of difference operators. Using the formalism of shift operators, we can introduce three simple cases of delta operators, right, left and symmetric discrete derivatives. For the right discrete derivative we get

$$\Delta_+ = \frac{1}{\sigma}(T_\sigma - 1), \quad p_n^+(x) = \prod_{i=0}^{n-1} (x - i\sigma),$$

for the left discrete derivative

$$\Delta_- = \frac{1}{\sigma}(1 - T_{\sigma}^{-1}), \quad p_n^-(x) = \prod_{i=0}^{n-1} (x + i\sigma),$$

and finally for the symmetric one

$$\Delta_s = \frac{1}{2\sigma}(T_\sigma - T_{\sigma}^{-1}), \quad p_n^s(x) = x \prod_{i=1}^{n-1} [x + (n - 2i)\sigma], \quad p_1^s(x) = x.$$ 

For $\sigma \to 0$ the operators converge to the continuous derivative, whereas the associated sequences tend to the simple sequence $x^n$. The corresponding operator substitution in the differential equation leads to the difference equation which can be understood as a discrete analogue of the original system.

In the Hamiltonians and solutions of the considered systems, we need to substitute not only the positive powers of $x$, but also the negative ones. Therefore the following extension of the associated sequences will be convenient. Let $k \in \mathbb{Z}^-$, then

$$p_k(x) = (x\beta)^k \cdot 1 = [(x\beta)^{-1}]^{-k} \cdot 1 = \left(\beta^{-1} \frac{1}{x}\right)^{-k} \cdot 1.$$ 

For the difference operators $\beta = \Delta_+, \Delta_-, \Delta_s$, we get the following extensions:

$$p_k^+(x) = \frac{1}{(x+\sigma)(x+2\sigma)\ldots(x+k\sigma)} = \frac{1}{\prod_{i=k}^{-1}(x-i\sigma)},$$

$$p_k^-(x) = \frac{1}{(x-\sigma)(x-2\sigma)\ldots(x+k\sigma)} = \frac{1}{\prod_{i=k}^{-1}(x+i\sigma)},$$

$$p_k^s(x) = \frac{x}{[x+k\sigma][x+(k+2)\sigma]\ldots[x-(k+2)\sigma][x-k\sigma]} = \frac{x}{\prod_{i=-k}^{n}[x-(k+2i)\sigma]}.$$ 

However, one has to be careful with the domain of these expressions: for example $p_k^+(x)$ has singularities in negative lattice points.
3. Smorodinsky-Winternitz Systems
In this section, we introduce a class of quantum-mechanical systems that will be discretized using the methods of umbral calculus. Let \( P_i, Q_j \) be operators of canonical momenta and coordinates, \( i, j = 1, \ldots, n \). We say that a quantum mechanical system with \( n \) degrees of freedom described by the Hamiltonian
\[
\mathcal{H} = \sum_{i=1}^{n} P_i^2 + V(Q_1, \ldots, Q_n)
\]
is integrable if it allows \( n - 1 \) independent integrals of motion in involution, that is the operators \( X_1, \ldots, X_{n-1} \) such that
\[
[H, X_a] = 0, \quad [X_a, X_b] = 0.
\]
The system is called superintegrable if there exist further \( 1 \leq k \leq n - 1 \) operators \( Y_1, \ldots, Y_k \) commuting with the Hamiltonian. The operators \( X_a \) and \( Y_b \) must be also independent of the Hamiltonian \( H \).

Considerable attention is given to the superintegrable systems since 1920's, beginning with works of Jauch and Hill [12] (harmonic oscillator), Pauli [13], Fock [14] and Bargmann [15] (hydrogen atom). A complete classification of the superintegrable systems on an Euclidean plane was provided by Winternitz, Smorodinsky and collaborators [16], [17] in 1965, and it was later found that all these models are exactly solvable [18]. We are interested mainly in the following two oscillator systems:

I. Generalized isotropic harmonic oscillator
\[
\mathcal{H}_I(x, y) = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + \frac{\omega^2}{2}(x^2 + y^2) + \frac{\alpha}{2x^2} + \frac{\beta}{2y^2},
\]
with the solution of eigenvalue problem in Cartesian coordinates
\[
\psi_{n,m}(x, y) = x^p y^q L_n^{(p-\frac{3}{2})}(\omega x^2) L_m^{(q-\frac{1}{2})}(\omega y^2) e^{-\frac{\omega x^2}{2}} e^{-\frac{\omega y^2}{2}},
\]
\[
E_{n,m} = \omega(2n + 2m + p + q + 1),
\]
where \( \alpha = p(p-1) > -\frac{1}{8}, \beta = q(q-1) > -\frac{1}{8} \) are parameters, \( \omega \) is frequency of the oscillator and \( L_n^{(\alpha)} \) are Laguerre polynomials. Gauge-rotated Hamiltonian follows as
\[
h_I = \frac{1}{2\omega} \psi_{0,0}^{-1}(\mathcal{H}_I - E_{0,0}) \psi_{0,0} \bigg|_{t=x\omega x^2} = -t\partial_t^2 - u\partial_u^2 + t\partial_t + u\partial_u - (p + \frac{1}{2})\partial_t - (q + \frac{1}{2})\partial_u,
\]
having simple polynomial solution
\[
\Xi_{n,m}(t, u) = L_n^{(p-\frac{3}{2})}(t) L_m^{(q-\frac{1}{2})}(u).
\]
This system is also separable (and can be solved) in polar coordinates.

II. Generalized nonisotropic harmonic oscillator
\[
\mathcal{H}_{II}(x, y) = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + \omega^2 x^2 + \frac{\omega^2}{2} y^2 + \frac{\beta}{2y^2}.
\]
The solution of the corresponding Schrödinger equation is given by
\[
\psi_{n,m}(x, y) = y^q H_n(\sqrt{2\omega x}) L_m^{(q-\frac{1}{2})}(\omega y^2) e^{-\omega x^2} e^{-\frac{\omega y^2}{2}},
\]
\[
E_{n,m} = \omega(2n + 2m + q + \frac{3}{2}).
\]
The parameter $\beta$ is the same as in case I, $H_n(x)$ are Hermite polynomials. After the gauge rotation, we get

$$h_{II} = \frac{1}{\Sigma \omega} \psi_{0,0}^{-1}(\mathcal{H}_{II} - E_{0,0})\psi_{0,0} \bigg|_{\tau=\sqrt{\Sigma \omega}t} = -\frac{1}{2} \partial_t^2 + t\partial_t - u\partial_u^2 + u\partial_u - (q + \frac{1}{2})\partial_u. \quad (8)$$

The polynomial solution of this equation is

$$\Xi_{n,m}(t,u) = H_n(t) L_m^{(q-\frac{1}{2})}(u). \quad (9)$$

This Hamiltonian also separates in parabolic coordinates.

There are also two classes of Coulomb-type systems in the Euclidean plane which are related to the generalized oscillators through so called coupling constant metamorphosis. Basically, their Hamiltonians in parabolic coordinates coincide with the systems I and II and, therefore, they can be discretized in similar manner.

4. Results

4.1. General Discretization

For the discretization of S.-W. systems, we need to perform an umbral correspondence in the Euclidean plane $\mathcal{E}_2$, i.e. in two coordinates. In variables $x, y$, the substitution is denoted

$$\begin{align*}
\partial_x & \longrightarrow \Delta_x \\
x & \longrightarrow x\beta_x
\end{align*} \quad \begin{align*}
\partial_y & \longrightarrow \Delta_y \\
y & \longrightarrow y\beta_y
\end{align*}$$

where $\Delta_x$ and $\Delta_y$ are arbitrary difference operators in the corresponding variable and $\beta_x, \beta_y$ their conjugates. Similar notation is used for different coordinates (after a substitution).

Both models of generalized oscillators in $\mathcal{E}_2$ are separable in Cartesian coordinates and their gauge-rotated partners are separable in the substituted variables. Consequently, the difference equation obtained by the umbral correspondence is also separable and the solutions can be written as products of two functions.

Let us start with an operator substitution in gauge-rotated Hamiltonian (4). The discrete version has the form

$$h_D^{II} = -\frac{1}{2} \partial_t^2 + t\partial_t - u\partial_u^2 + u\partial_u - (q + \frac{1}{2})\partial_u.$$

Since the eigenfunctions are polynomials, we can immediately discretize the solutions (5) of the corresponding Schrödinger equation:

$$\Xi_{n,m}^D(t,u) = \Xi_{n,m} \cdot 1 = L_n^{(p-\frac{1}{2})}(t\beta_t)L_m^{(q-\frac{1}{2})}(u\beta_u) \cdot 1 = \sum_{i=0}^n l_i^{(p-\frac{1}{2})} p_i(t) \sum_{j=0}^m l_j^{(q-\frac{1}{2})} p_j(u),$$

where $l_i^{(p-\frac{1}{2})}$ is the $i$-th coefficient of Laguerre polynomial $L_n^{(p-\frac{1}{2})}$ and $p_i(t)$ is the associated sequence for the delta operator $\Delta_t$.

In case of the original Hamiltonian (2), the general discretization leads to the operator

$$\mathcal{H}_D^D = -\frac{1}{2} (\Delta_x^2 + \Delta_y^2) + \frac{\omega^2}{2} [(x\beta_x)^2 + (y\beta_y)^2] + \frac{\alpha}{2} (x\beta_x)^2 + \frac{\beta}{2} (y\beta_y)^2,$$
The eigenfunctions of this operator corresponding to the lowest eigenvalue can be written in the
terms of power series in associated polynomials. We restrict ourselves to \(p, q \in \mathbb{Z}\) which allows
us to use the extended associated sequences. For the ground state, we get

\[
\psi^{D}_{0,0}(x, y) = \sum_{k=0}^{\infty} \frac{(-\omega)^k}{2^k k!} p_{2k+p}(x) \sum_{l=0}^{\infty} \frac{(-\omega)^l}{2^l l!} p_{2l+q}(y),
\]

and the excited states can be computed as

\[
\psi^{D}_{n,m}(x, y) = L_n^{(p-\frac{1}{2})} \left( \omega(x \beta_x)^2 \right) L_m^{(q-\frac{1}{2})} \left( \omega(y \beta_y)^2 \right) \psi^{D}_{0,0}.
\]

The expressions \(x \beta_x\) and \(y \beta_y\) in the expansion of Laguerre polynomials act as the umbral shifts
for the associated polynomials in the ground state \(\psi^{D}_{0,0}\). The issue of convergence reduces to the
convergence of the infinite series of discrete Gaussians in \(\psi^{D}_{0,0}\).

Similar transfer to the lattice in the case of the Hamiltonian (8) follows as

\[
h^{D}_{II} = -\frac{1}{2} \Delta^2 + t \beta_t \Delta_t - u \beta_u \Delta^2_u + u \beta_u \Delta_u - (q + \frac{1}{2}) \Delta_u,
\]

which is solved by

\[
\Xi^{D}_{n,m}(t, u) = H_n(t \beta_t) L_m^{(q-\frac{1}{2})} (u \beta_u) \cdot 1 = \sum_{i=0}^{n} h_{i,n} p_i(t) \sum_{j=0}^{m} \binom{p-\frac{1}{2}}{j} p_j(u).
\]

The number \(h_{i,n}\) is the \(i\)-th coefficient of the \(n\)-th Hermite polynomial \(H_n(t)\), other notation as
before.

Similarly, the original operator (6) takes the form

\[
\mathcal{H}^{D}_{II} = -\frac{1}{2} (\Delta^2_x + \Delta^2_y) + 2\omega^2 (x \beta_x)^2 + \frac{\omega^2}{2} (y \beta_y)^2 + \frac{\beta}{2} (y \beta_y)^{-2}.
\]

The expression for the ground state is for \(q \in \mathbb{Z}\)

\[
\psi^{D}_{0,0}(x, y) = \sum_{k=0}^{\infty} \frac{(-\omega)^k}{k!} p_{2k}(x) \sum_{l=0}^{\infty} \frac{(-\omega)^l}{2^l l!} p_{2l+q}(y),
\]

For the excited states we get

\[
\psi^{D}_{n,m}(x, y) = H_n(\sqrt{2} \omega x \beta_x) L_m^{(q-\frac{1}{2})} (\omega(y \beta_y)^2) \psi_{0,0}(x, y).
\]

The Hermite and Laguerre polynomials of the arguments \(x \beta_x\) and \((y \beta_y)^2\) (up to constants) are
the umbral shifts acting on the appropriate parts of the wave function \(\psi^{D}_{0,0}\). The convergence
is not affected by these terms.

### 4.2. Particular Discretization

In this paragraph we show the results for the particular difference operator mentioned in
Section 2. The solutions obtained by the umbral correspondence are well-defined on the
lattice points \(\sigma \mathbb{Z}\) (at least positive) and in the case of gauge-rotated Hamiltonian they converge
everywhere. For brevity, the results will be demonstrated on the generalized isotropic harmonic
oscillator.
With the right discrete derivative, we denote the spacings on the lattice as \((\sigma_t, \sigma_u)\) or \((\sigma_x, \sigma_y)\) (according to the coordinates used). Similarly, the shift operators are denoted \(T_{\sigma_t}, T_{\sigma_u}\) etc. The operator (4) is discretized as

\[
    h^D_t = \frac{1}{\sigma_t^2} \left[ \left( (\sigma_t + 2)t + \sigma_t(p + \frac{1}{2}) \right) \left( t + \sigma_t(p + \frac{1}{2}) \right) T_{\sigma_t} - t(\sigma_t + 1)T_{\sigma_t}^{-1} \right] + \frac{1}{\sigma_u^2} \left[ \left( \left( u + \sigma_u(q + \frac{1}{2}) \right) \right) \left( (\sigma_u + 2)u + \sigma_u(q + \frac{1}{2}) \right) - u(\sigma_u + 1)T_{\sigma_u}^{-1} \right].
\]

The difference equation \(h^D_t \Xi = c\Xi\) follows as

\[
    \frac{1}{\sigma_t^2} \left[ \left( t + \sigma_t(p + \frac{1}{2}) \right) \Xi(t + \sigma_t, u) - t(\sigma_t + 1)\Xi(t - \sigma_t, u) + \left( (\sigma_t + 2)t + \sigma_t(p + \frac{1}{2}) \right) \Xi(t, u) \right] + \frac{1}{\sigma_u^2} \left[ u(\sigma_u + 1)\Xi(t, u - \sigma_u) + \left( (\sigma_u + 2)u + \sigma_u(q + \frac{1}{2}) \right) \Xi(t, u) - (u + \sigma_u(q + \frac{1}{2})) \Xi(t, u + \sigma_u) \right] = c\Xi(t, u).
\]

The solution is a polynomial and can be expressed as

\[
    \Xi^D(t, u) = \sum_{i=0}^{n} \left( \sum_{j=0}^{m} \prod_{l=0}^{\infty} \left( t - r_{ij} \right) \prod_{k=0}^{\infty} \left( u - s_{ij} \right) \right).
\]

If we return to the original problem and use the right discrete derivative, the eigenvalue problem can be formulated by the following difference equation on a lattice:

\[
    \left[ \frac{\alpha}{2(x + \sigma_x)(x + 2\sigma_x)} - \frac{1}{2\sigma_x^2} \right] \psi(x + 2\sigma_x, y) - \frac{1}{2\sigma_x^2} \psi(x, y) + \frac{1}{\sigma_x^2} \psi(x + \sigma_x, y) + \frac{\omega^2}{2} x(x - \sigma_x)\psi(x - 2\sigma_x, y) + \\
    \left[ \frac{\beta}{2(y + \sigma_y)(y + 2\sigma_y)} - \frac{1}{2\sigma_y^2} \right] \psi(x, y + 2\sigma_y) - \frac{1}{2\sigma_y^2} \psi(x, y) + \frac{1}{\sigma_y^2} \psi(x, y + \sigma_y) + \frac{\omega^2}{2} y(y - \sigma_y)\psi(x, y - 2\sigma_y) = E\psi(x, y).
\]

Using the extended associated sequence, the ground state for this eigenvalue problem can be written as

\[
    \psi^D_{0,0}(x, y) = \left[ \sum_{k=0}^{\infty} \left( \frac{-\omega}{2k} \right)^k \frac{1}{2k!} \prod_{i=0}^{\infty} \left( x - i\sigma_x \right) \right] \times \\
    \left[ \sum_{k=0}^{\infty} \left( \frac{-\omega}{2k} \right)^k \frac{1}{2k!} \prod_{i=0}^{\infty} \left( y - i\sigma_y \right) \right] \times \\
    \left[ \sum_{k=0}^{\infty} \left( \frac{-\omega}{2k} \right)^k \frac{1}{2k!} \prod_{i=0}^{\infty} \left( x - i\sigma_x \right) \right] \times \\
    \left[ \sum_{k=0}^{\infty} \left( \frac{-\omega}{2k} \right)^k \frac{1}{2k!} \prod_{i=0}^{\infty} \left( y - i\sigma_y \right) \right]
\]

where we use the blanket hypothesis that \(\sum_{k=0}^{\infty} \frac{-c}{2k} = 0\) for \(c\) positive. This function solves the lattice points \((i\sigma_x, j\sigma_y)\) for \(i, j = 0, 1, 2, \ldots\), that is for the first quadrant in \(E_2\). In other points the series diverges.

The excited states can be obtained as

\[
    \psi^D_{n,m}(x, y) = \sum_{j=0}^{n} \left( \sum_{k=0}^{\infty} \left( \frac{-\omega}{2k} \right)^k \frac{1}{2k!} p_{2k+2j+p}(x) \right) \times \sum_{l=0}^{m} \left( \sum_{l=0}^{\infty} \left( \frac{-\omega}{2l} \right)^l \frac{1}{2l!} p_{2l+2i+q}(y) \right)
\]

where \(p_n^m(x)\) is the generalized associated sequence for the right discrete derivative. For \(p, q \in \mathbb{Z}_0^+\) these polynomials can be easily substituted.

The results for the left and symmetric discrete derivatives would be obtained in similar fashion.
5. Conclusions
We have shown that certain two-dimensional quantum-mechanical superintegrable systems can be transferred to a uniform lattice by the means of umbral discretization. The difference equations for the generalized isotropic harmonic oscillator using the simple example of right discrete derivative have been found and the solutions have been obtained by substituting into the original ones. In the case of gauge-rotated Hamiltonian, the solution of the difference analogue is a well-defined polynomial in $E_2$, however, for the original system, we need to restrict ourselves to the lattice points only.

The method of umbral discretization offers an infinite number of difference operators (that approximates the derivative in an arbitrary order) and therefore this procedure can be done with various operator replacements. Moreover, there is no restriction for the dimension of the system, nor for the coordinate system. Therefore the difference analogues of quantum mechanics can be formulated on non-square lattices as well, while still preserving the symmetries.

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References
[1] Creutz M 1983 Quarks, Gluons and Lattices (Cambridge: Cambridge University Press)
[2] Ashtekar A 2001 General Relativity and Gravitation 28 (New York: Word Scientific)
[3] Atakishiev N M and Suslov S K 1991 Theoret. Math. Phys. 85 1055–1062
[4] Lorente M 2001 Phys. Lett. A 285 119–126
[5] Odake S and Sasaki R 2008 J. Math. Phys. 49 053053
[6] Odake S and Sasaki R 2005 J. Nonlin. Math. Phys. 12 507–521
[7] Turbiner A 2001 Int. J. Mod. Phys. A 16 1579
[8] Roman S 1984 The Umbral Calculus (San Diego: Academic Press)
[9] Rota G C 1975 Finite Operator Calculus (San Diego: Academic Press)
[10] Dimakis A, Müller-Hoissen F and Striker T 1996 J. Phys. A: Math. Gen. 29 6861–6876
[11] Levi D, Tempesta P and Winternitz P 2004 J. Math. Phys. 45 4077
[12] Jauch J and Hill E 1940 Phys. Rev. 57 641
[13] Pauli W 1926 Z. Phys. 36 336–363
[14] Fock V Z. Phys. 98 145
[15] Bargmann V 1936 Z. Phys. 99 578
[16] Fris J, Mandrosov V, Smorodinsky Y A and Winternitz P 1965 Phys. Lett. 16 354
[17] Winternitz P, Smorodinsky Y A, Uhlir M and Fris I 1967 Sov. J. Nucl. Phys. 4 444
[18] Tempesta P, Turbiner A and Winternitz P 2001 J. Math. Phys. 42 4248