A New Lifetime Distribution: Properties, Copulas, Applications, and Different Classical Estimation Methods

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1. Introduction

The Weibull model [1] is a very useful distribution in modeling real data exhibiting monotonic hazard rate function (HRF). But it cannot be used in modeling and studying data which have nonmonotonic HRF such as the “bathtub shape (U-HRF).” For avoiding this drawback, Bebbington et al. [2] have defined a new two-parameter distribution which is an extension of the Weibull distribution referred to as a flexible Weibull (FW) extension distribution; it has a failure function that can be “decreasing,” “increasing,” or “bathtub-shaped.” Analogously, El-Gohary et al. [3] derived the two-parameter inverse flexible Weibull (IFW) model which is the reciprocal of a random variable (RV) which has FW model. Several mathematical properties of this distribution such as the mode, moments, and moment generating function (MGF) have been discussed. El-Gohary et al. [3] proved that the hazard rate function (RRF) of the IFW model can be “upside down constant,” the cumulative distribution function (CDF) of IFW distribution is given by

\[
G_{\alpha,\theta}(z) = \exp\left[-\exp\left(\frac{\alpha}{z} - \theta z\right)\right]_{(z>0)},
\]

where the two parameters \(\alpha > 0\) and \(\theta > 0\) control the shape of the distribution. The corresponding probability density function (PDF) is

\[
g_{\alpha,\theta}(z) = \left(\theta + \frac{\alpha}{z^2}\right)\exp\left(\frac{\alpha}{z} - \theta z\right)\exp\left[-\exp\left(\frac{\alpha}{z} - \theta z\right)\right]_{(z>0)}.
\]
(TIITL-G) family of distributions. Due to [11], the CDF of the TIITL-G family is given by
\begin{equation}
F_{\lambda,\gamma}(z) = 1 - \left[1 - G_{\gamma}(z)^{\lambda}\right].
\end{equation}

The PDF is defined by
\begin{equation}
f_{\lambda,\gamma}(z) = 2\lambda g_{\gamma}(z)G_{\gamma}(z)\left[1 - G_{\gamma}(z)^{\lambda+1}\right],
\end{equation}
where \( \lambda > 0 \) is a shape parameter, \( g_{\gamma}(z) = dG_{\gamma}(z)/dz \) is the baseline PDF, and \( G_{\gamma}(z) \) is a baseline CDF.

The remainder of the paper is organized as follows: in Section 2, we define the CDF, PDF, and HRF of TIITLIFW model and provide a simple expansion of the PDF. Simple type copula is derived in Section 3. Various mathematical properties are discussed in Section 4. Non-Bayesian estimation methods under uncensored schemes are given in Section 5. Section 6 presents a comparison under the non-Bayesian estimation methods using uncensored schemes via a simulation study. Section 7 presents a comparison under uncensorship with some competitive models. Concluding remarks are contained in Section 8.

2. The New Model

Using (1) in (3), the CDF of the TIITLIFW distribution can be written as
\begin{equation}
F_{\lambda,\gamma}(z) |_{z=\alpha,\theta,\lambda} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z} - \theta z\right)\right]\right\}^{\lambda}. 
\end{equation}

The corresponding PDF is given by
\begin{equation}
f_{\lambda,\gamma}(z) = 2\lambda \left\{\frac{\theta + \alpha}{z}\right\} \exp\left[-2\exp\left((\alpha/z) - \theta z\right)\right] \exp\left((\alpha/z) - \theta z\right) \left[1 - \exp\left[-2\exp\left((\alpha/z) - \theta z\right)\right]\right\}^{1-\lambda}.
\end{equation}

The HRF can be expressed as
\begin{equation}
h_{\lambda,\gamma}(z) = 2\lambda \left\{\frac{\theta + \alpha}{z}\right\} \left[1 - \exp\left[-2\exp\left((\alpha/z) - \theta z\right)\right]\right\} \left(\frac{\theta + \alpha}{z}\right) \exp\left((\alpha/z) - \theta z\right).
\end{equation}

Figure 1 shows some plots of the PDF of the TIITLIFW distribution for some different values of the parameters. Figure 2 shows some plots of the HRF of the TIITLIFW distribution for some different parameter values. Based on Figure 1, we conclude that the new PDF can have many right skewed heavy tail shapes. Based on Figure 2, it is observed that the new HRF can be "increasing-constant," "bathtub-constant," "bathtub," "constant," "J-HRF," "upside down bathtub," "increasing," "upside down-increasing-constant," and "upside down".

The PDF of the TIITLIFW distribution can be written as
\begin{equation}
f_{\lambda,\gamma}(z) = \sum_{i,j,k=0}^\infty \psi_{i,j,k}(\theta z^{-k} + \alpha z^{-k-2}) \exp[-(j+1)\theta z].
\end{equation}

\textbf{Proof.} Supposing \( |w_1/w_2| < 1, w_3 > 0 \) is a real noninteger, we have the power series expansion
\begin{equation}
\left(1 - \frac{w_1}{w_2}\right)^{w_3} = \sum_{i=0}^{\infty} \left(\frac{w_1}{w_2}\right)^i \frac{\Gamma(1 + w_3)}{i!\Gamma(1 + w_3 - i)},
\end{equation}
using the power series (9) in equation (8), and the fact that \( 0 < \exp[-2\exp((\alpha/z) - \theta z)] < 1 \), we get
\begin{equation}
f_{\lambda,\gamma}(z) = \sum_{i=0}^{\infty} \left(-1\right)^i 2\lambda \Gamma(\lambda) \frac{\theta + \alpha}{z} \exp\left(\frac{\alpha}{z} - \theta z\right) \exp\left[-2(i + 1)\exp\left(\frac{\alpha}{z} - \theta z\right)\right],
\end{equation}
expanding \( \exp[-2(i + 1)\exp((\alpha/z) - \theta z)] \) using Taylor series
\begin{equation}
\exp[-2(i + 1)\exp((\alpha/z) - \theta z)] = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left[2(i + 1)\right]^j \exp\left[j\left(\frac{\alpha}{z} - \theta z\right)\right].
\end{equation}
again, using a series expansion of \( \exp[(j + 1)(\alpha/z)] \), and after some algebras, the PDF can be written as
\begin{equation}
f_{\lambda,\gamma}(z) = \sum_{i,j,k=0}^\infty \psi_{i,j,k}(\theta z^{-k} + \alpha z^{-k-2}) \exp[-(j + 1)\theta z],
\end{equation}
where
\begin{equation}
\psi_{i,j,k} = \frac{2^{j+1} \alpha^{j+1} (-1)^{i+j+1}(j + 1)!}{i!j!k!\Gamma(\lambda + 1)}.
\end{equation}

3. Copula

3.1. Bivariate TIITLIFW (BTIITLIFW) via Morgenstern Family. The CDF of the Morgenstern family of two random RVs \( (X_1, X_2) \) can be derived as
\begin{equation}
F_{1,2}(x_1, x_2) |_{(i,j) \leq 1} = F_1(x_1)F_2(x_2)\left[1 + \rho F_1(x_1)F_2(x_2)\right],
\end{equation}
where
\begin{equation}
F_1(x_1) = 1 - F_1(x_1),
F_2(x_2) = 1 - F_2(x_2),
\end{equation}
setting
\begin{equation}
F_{\lambda,\gamma}(x) |_{z=\alpha,\theta,\lambda} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{x_1} - \theta_1 x_1\right)\right]\right\}^{\lambda_1},
\end{equation}
\begin{equation}
F_{\lambda,\gamma}(x) |_{z=\alpha,\theta,\lambda} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{x_2} - \theta_2 x_2\right)\right]\right\}^{\lambda_2},
\end{equation}
Figure 1: PDF plots of the TIITLFW model for some different values of the parameters. (a) $\alpha = 1, \theta = 1, \lambda = 1$. (b) $\alpha = 0.0001, \theta = 4, \lambda = 0.15$. (c) $\alpha = 0.0001, \theta = 2, \lambda = 0.1$. (d) $\alpha = 5, \theta = 0.05, \lambda = 0.05$. (e) $\alpha = 2, \theta = 2, \lambda = 10$. (f) $\alpha = 0.1, \theta = 5, \lambda = 0.15$. 

Complexity
then we have

\[
H_{\rho_1,\theta_1,\lambda_1}(x_1, x_2) = \left[ 1 - \exp\left( -2 \exp\left( \frac{\alpha_1}{x_1} - \theta_1 x_1 \right) \right) \right]^{\lambda_1} \times \left[ 1 - \exp\left( -2 \exp\left( \frac{\alpha_2}{x_2} - \theta_2 x_2 \right) \right) \right]^{\lambda_2}
\times \left( 1 + \rho \left\{ 1 - \exp\left( -2 \exp\left( \frac{\alpha_1}{x_1} - \theta_1 x_1 \right) \right) \right\}^{\lambda_1} \times \left\{ 1 - \exp\left( -2 \exp\left( \frac{\alpha_2}{x_2} - \theta_2 x_2 \right) \right) \right\}^{\lambda_2} \right).
\]

(17)

Figure 2: HRFs of the TIITLIFW model for some different values of the parameters. (a) $\alpha = 1, \theta = 1, \lambda = 1$. (b) $\alpha = 0.1, \theta = 2.5, \lambda = 0.1$. (c) $\alpha = 0.001, \theta = 2, \lambda = 0.1$. (d) $\alpha = 5, \theta = 0.5, \lambda = 0.5$. (e) $\alpha = 2, \theta = 2, \lambda = 10$. (f) $\alpha = 0.1, \theta = 5, \lambda = 0.15$. (g) $\alpha = 0.0001, \theta = 4, \lambda = 0.15$. (h) $\alpha = 0.25, \theta = 5, \lambda = 0.25$. (i) $\alpha = 0.25, \theta = 0.25, \lambda = 0.25$. 

Complexity
3.2. Via Clayton Copula. The BTIITLIFW type extension: the weighted version of the Clayton copula can be expressed as

\[ H(u, v) = \left[ u^{-\rho} + v^{-\rho} - 1 \right]^{-\frac{1}{\rho}} \quad \text{for } \rho \geq 0. \tag{18} \]

Let us assume that \( X \sim \text{TITLIFW} (V_1) \) and \( Y \sim \text{TITLIFW} (V_2) \). Then, setting

\[
\begin{align*}
    u &= u_{V_1}(x) = 1 - \left\{ 1 - \exp\left[ -2 \exp\left( \frac{\alpha_1}{x} - \theta_1 x \right) \right] \right\}^{\lambda_1}, \\
    v &= u_{V_2}(y) = 1 - \left\{ 1 - \exp\left[ -2 \exp\left( \frac{\alpha_2}{y} - \theta_2 y \right) \right] \right\}^{\lambda_2},
\end{align*}
\tag{19} \]

\[ H(x, y) = \left\{ \left\{ 1 - \left\{ 1 - \exp\left[ -2 \exp\left( \frac{\alpha_1}{x} - \theta_1 x \right) \right] \right\}^{\lambda_1} \right\}^{\frac{1}{\rho}} + \left\{ 1 - \left\{ 1 - \exp\left[ -2 \exp\left( \frac{\alpha_2}{y} - \theta_2 y \right) \right] \right\}^{\lambda_2} \right\}^{\frac{1}{\rho}} - 1 \right\}^{-\frac{1}{\rho}}. \tag{20} \]

A straightforward \( M \)-dimensional extension from the above will be

\[ H(x_1, x_2, \ldots, x_M) = \left\{ \sum_{i=1}^{M} \left\{ 1 - \exp\left[ -2 \exp\left( \frac{\alpha_i}{x_i} - \theta_i x_i \right) \right] \right\}^{\lambda_i} \right\}^{\frac{1}{\rho}} - M + 1 \right\}^{-\frac{1}{\rho}}. \tag{21} \]

The associated CDF of the BTIITLIFW type distribution can be written as

\[
\frac{\partial}{\partial u} A(u) = A(u) + u \frac{\partial}{\partial u} A(u),
\]

\[
\kappa_1(u) = \left\{ u: u \in (0, 1) \left\{ \frac{\partial A(u)}{\partial u} \right\} \text{ exists} \right\}, \tag{23} \]

\[
\kappa_1(w) = \left\{ w: w \in (0, 1) \left\{ \frac{\partial V(w)}{\partial w} \right\} \text{ exists} \right\}. \tag{24} \]

Type I:

Recalling the following functional form for both \( A(u) \) and \( V(w) \). Then, the BTIITLIFW-FGM (Type-I) can be derived from

\[ H_{\rho}(u, w) = u w + \rho A(u) V(w) \mid_{\rho \in (-1, 1)}, \tag{25} \]

where

\[
\hat{A}(u) = u \left[ 1 - F_{L_1}(u) \right],
\]

\[
\hat{V}(w) = w \left[ F_{L_2}(w) \right].
\]

Therefore,
Via Renyi’s Entropy. Let $m \in (0,1) = F_{\tilde{L}}(x_1)$ and
$w \in (0,1) = F_{\tilde{L}}(x_2)$. Then, the Renyi’s entropy copula can be expressed as

$$H_y(u, w) = x_1 \left\{ 1 - \exp \left[ -2 \exp \left( \frac{a_1}{x_1} - \theta_1 u \right) \right] \right\}^{\lambda_1} + x_1 \left\{ 1 - \exp \left[ -2 \exp \left( \frac{a_2}{x_1} - \theta_2 u \right) \right] \right\}^{\lambda_1} - x_1 x_2. \quad (34)$$
4. Statistical and Reliability Measures

4.1. Quantile Function. For RV $Z$ has CDF of the TIIITLIFW distribution, the quantile function $Z_q$ of the TIIITLIFW distribution is given by the following equation:

$$Z_q = \frac{1}{2} \left\{ -d(q) + \sqrt{d^2(q) + 4\alpha} \right\}, \quad 0 < q < 1, \quad (35)$$

where

$$d(q) = \log \left\{ \frac{\log(1 - (1 - q)\sqrt{1})}{2} \right\}. \quad (36)$$

4.2. Moments and Generating Functions. The $r_{th}$ moment of the TIIITLIFW distribution is obtained using the formula

$$\mu_r(z) = \int_{0}^{\infty} z^r f(z)dz,$$  \quad (37)

hence using equation (7) we obtain

$$\mu_r(z) = \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left\{ \frac{\Gamma(r - k + 1)}{\theta_{i,j,k}^r - \theta_{i,j,k}^{r-k}} \right\} \left\{ \frac{\Gamma(r - k - 1)}{\theta_{i,j,k}^r - \theta_{i,j,k}^{r-k-1}} \right\}, \quad (38)$$

where

$$\Gamma(1 + V) = \int_{0}^{\infty} z^V \exp(-z)dz \quad (40)$$

is a gamma function. In particular, if $r = 1$ and $r = 2$, we obtain the mean and variance of the TIIITLIFW distribution. The MGF of the TIIITLIFW is given by

$$M_Z(t) = E(\exp(tZ)) \quad (41)$$

4.3. Incomplete Moments. The $s_{th}$ lower and upper incomplete moments of $Z$ are defined by

$$v_{s,Z}(u) = E\left( Z^{s} | Z < u \right) = \int_{0}^{u} z^s f(z)dz, \quad (42)$$

$$O_{s,Z}(u) = E\left( Z^{s} | Z > u \right) = \int_{u}^{\infty} z^s f(z)dz, \quad (43)$$

respectively, for any real $s > 0$. The $s_{th}$ lower incomplete moment of the TIIITLIFW distribution is

$$v_{s,Z}(t) = \int_{0}^{t} z^s f(z)dz = \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left\{ \frac{\theta_{i,j,k}^{s-k} + az^{s-k+1}}{\theta_{i,j,k}^{s-k+1} - \theta_{i,j,k}^{s-k}} \right\} \exp[-(j+1)z]dz$$

where

$$\gamma(s_1, s_2) = \int_{0}^{t} z^{s_1-1} \exp(-z)dz \quad (44)$$

is the lower incomplete gamma function. Similarly, the $s_{th}$ upper incomplete moment of the TIIITLIFW distribution is

$$O_{s,Z}(t) = \int_{t}^{\infty} z^s f(z)dz = \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left\{ \frac{\theta_{i,j,k}^{s-k} + az^{s-k+1}}{\theta_{i,j,k}^{s-k+1} - \theta_{i,j,k}^{s-k}} \right\} \exp[-(j+1)z]dz \quad (45)$$

where

$$\Gamma(1 + s, t) = \int_{t}^{\infty} z^{s} \exp(-z)dz \quad (46)$$

is the upper incomplete gamma function.

4.4. Mean Deviation, Lorenz, and Bonferroni Curves. For RV $Z$ with PDF $f(z)$, distribution function $F(z)$, mean $\mu = E(Z)$, and $M = \text{Median}(Z)$, the mean deviation about the mean and median, respectively, is given by

$$\delta_{\mu}(z) = \int_{0}^{\infty} \left| z - \mu \right| f(z)dz = 2\mu F(\mu) - 2\mu + 2O_{1}(\mu), \quad (47)$$

$$\delta_{\mu}(z) = \int_{0}^{\infty} \left| z - M \right| f(z)dz = 2MF(M) - M - \mu + 2O_{1}(M), \quad (47)$$

where

$$O_{1,Z}(d) = \int_{d}^{\infty} z f(z)dz$$

$$\delta_{\mu}(z) = \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left\{ \frac{\theta_{i,j,k}^{s-k + 2, (k+1)\theta} d}{\theta_{i,j,k}^{s-k + 2}} + \frac{\alpha^{s-k, (k+1)\theta} d}{\theta_{i,j,k}^{s-k + 2}} \right\}. \quad (48)$$

The mathematical form of the Galton skewness and Moors kurtosis of TIIITLIFW distribution can be computed using the quantile function and well-known relationships. The first four moments, skewness, and kurtosis of the TIITLIFW distribution for different values of parameters are represented in Table 1. Table 1 shows that the skewness is always positive, and kurtosis is always greater than three.
The Lorenz curve for a positive RV $Z$ is defined as

$$L(p) = \frac{1}{\mu} \int_0^p z f(z)dz = \frac{1}{\mu} F_{V1,Z}(q).$$

(49)

Then, we have

$$L(p) = \frac{1}{\mu} \sum_{i,j,k=0}^{\alpha} \nabla_{i,j,k} \left[ \frac{\partial y(-k, (j+1)\theta q)}{\theta_{q}^{k+2}} + \frac{\alpha y(-k, (j+1)\theta q)}{\theta_{q}^{k}} \right].$$

(50)

where $q = G^{-1}(p)$. Also, Bonferroni curve is defined by
\[ B(p) = \frac{1}{\mu p} \int_0^q z f(z)dz = \frac{\nu_1 z(q)}{\mu p}. \]  

(51)

Then,

\[ B(p) = \frac{1}{\mu p} \sum_{i,j,k=0}^\infty \nu_{i,j,k} \left[ \frac{\theta_i(-k+2,j+1)}{\theta_i^{k+1}} + \frac{\alpha y(-k,j+1)}{\theta_i^k} \right]. \]  

(52)

4.5. The Mean and Strong Mean Inactivity Times Functions.

The mean inactivity time (MIT) can be derived from

\[ m(t) = E[(t-Z)_{Z\leq t}] = t - \frac{1}{F(t)} \int_0^t z f(z)dz. \]  

(53)

Then, the MIT can be derived as

\[ m(t) = t - \frac{1}{F(t)} \sum_{i,j,k=0}^\infty \nu_{i,j,k} \left[ \frac{\theta_i(-k+2,j+1)}{\theta_i^{k+1}} + \frac{\alpha y(-k,j+1)}{\theta_i^k} \right]. \]  

(54)

The strong mean inactivity time (SMIT) is a new reliability function given by

\[ S(t) = \frac{1}{F(t)} \int_0^t z f(z)dz = t - \frac{1}{F(t)} \int_0^t z^2 f(z)dz. \]  

(55)

Therefore, the SMIT can be expressed as

\[ S(t) = t - \frac{1}{F(t)} \sum_{i,j,k=0}^\infty \nu_{i,j,k} \left[ \frac{\theta_i(-k+3,j+1)}{\theta_i^{k+3}} + \frac{\alpha y(-k,j+1)}{\theta_i^{k+2}} \right]. \]  

(56)

5. Non-Bayesian Estimation Methods under Uncensored Schemes

5.1. The MLE Method. Let \( Z_1, Z_2, \ldots, Z_n \) be a random sample of size \( n \) from TIITLIFW. The log-likelihood function for the vector of parameters \( \theta \) can be written as

\[ \log(L) = n \log(2\lambda) + \sum_{i=1}^n \log \left( \frac{\alpha}{z_{[i:n]}} \right) \]

\[ - \alpha \sum_{i=1}^n z_{[i:n]} - \theta \sum_{i=1}^n z_{[i:n]} - 2 \sum_{i=1}^n \exp \left( \frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]} \right) \]

\[ + (\lambda - 1) \sum_{i=1}^n \log \left( 1 - \exp \left( -2 \exp \left( \frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]} \right) \right) \right). \]  

(57)

The associated score function is given by

\[ U_n(\theta) = \left[ \frac{\partial \log(L)}{\partial \lambda} \frac{\partial \log(L)}{\partial \alpha} \frac{\partial \log(L)}{\partial \theta} \right]^T. \]  

(58)

The log-likelihood can be maximized by solving the following nonlinear likelihood equations \( U_n(\theta) = 0, U_n(\alpha) = 0, U_n(\lambda) = 0 \). Then,

\[ U_n(\theta) = \frac{n}{\lambda} + \sum_{i=1}^n \log \left( 1 - \exp \left( -2 \exp \left( \frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]} \right) \right) \right). \]  

(59)

The maximum likelihood estimation (MLE) of \( \theta \), say \( \hat{\theta} \), is obtained by solving the system of nonlinear equations

\[ U_n(\theta) = U_n(\alpha) = U_n(\lambda) = 0. \]  

(60)

5.2. The CVME Method. The CVMEs of the parameters \( \alpha, \theta \), and \( \lambda \) [12] are obtained by minimizing the following expression with respect to the parameters \( \alpha, \theta \), and \( \lambda \), respectively, where
\[ \text{CVME} (\mathcal{V}) = \frac{1}{12} n^{-1} + \sum_{i=1}^{n} \left[ F_\mathcal{V}(z_{i:n}) - P_{(i:n)} \right]^2, \]  
where \( P_{(i:n)} = \frac{2i - 1}{2n} \) (62)

refers to the empirical estimate of the CDF at \( z_{i:n} \) computed from a certain sample and

\[ \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - P_{(i:n)} \right) D_{(\alpha)}(z_{i:n}; \mathcal{V}) = 0, \]

\[ \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - P_{(i:n)} \right) D_{(\theta)}(z_{i:n}; \mathcal{V}) = 0, \]

\[ \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - P_{(i:n)} \right) D_{(\lambda)}(z_{i:n}; \mathcal{V}) = 0, \]

where

\[ D_{(\alpha)}(z_{i:n}; \mathcal{V}) = \frac{\partial F_\mathcal{V}(z_{i:n})}{\partial \alpha}, \]

\[ D_{(\theta)}(z_{i:n}; \mathcal{V}) = \frac{\partial F_\mathcal{V}(z_{i:n})}{\partial \theta}, \]

\[ D_{(\lambda)}(z_{i:n}; \mathcal{V}) = \frac{\partial F_\mathcal{V}(z_{i:n})}{\partial \lambda}. \]

5.3. The OLSE and WLSE Method. Let \( F_\mathcal{V}(z_{i:n}) \) denote the CDF of the TITF model and \( z_{1:n} < z_{2:n} < \cdots < z_{n:n} \) be the \( n \) ordered RS. The OLSEs [13] are obtained by minimizing

\[ 0 = \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - q_{(i:n)} \right) D_{(\alpha)}(z_{i:n}; \mathcal{V}), \]

\[ 0 = \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - q_{(i:n)} \right) D_{(\theta)}(z_{i:n}; \mathcal{V}), \]

\[ 0 = \sum_{i=1}^{n} \left( 1 - \left[ 1 - \text{exp} \left( -2 \text{exp} \left( \frac{\alpha}{z_{i:n}} - \theta z_{i:n} \right) \right) \right]^{\lambda} - q_{(i:n)} \right) D_{(\lambda)}(z_{i:n}; \mathcal{V}), \]

where \( D_{(\alpha)}(z_{i:n}; \mathcal{V}), D_{(\theta)}(z_{i:n}; \mathcal{V}), \) and \( D_{(\lambda)}(z_{i:n}; \mathcal{V}) \) are as defined above. The WLSE is obtained by minimizing the function WLSE \( \mathcal{V} \) with respect to \( \alpha, \theta, \) and \( \lambda. \) Then,

\[ \text{WLSE} (\mathcal{V}) = \sum_{i=1}^{n} \tau_{(i:n)} \left[ F_\mathcal{V}(z_{i:n}) - q_{(i:n)} \right]^2, \]

where
The WLSEs are obtained by solving

\[
\tau_{(i,n)} = \frac{(1 + n)^3 (2 + n)}{[i(1 + n - i)]},
\]

The WLSEs are obtained by solving

\[
0 = \sum_{i=1}^{n} \left[ 1 - \left( 1 - \exp \left[ -2 \exp \left( \frac{z[i, n]}{z[i, n]} - \theta z[i, n] \right) \right] \right)^\lambda - q_{(i,n)} \right] \tau_{(i,n)} D_{(\alpha)}(z[i, n], Y),
\]

\[
0 = \sum_{i=1}^{n} \left[ 1 - \left( 1 - \exp \left[ -2 \exp \left( \frac{z[i, n]}{z[i, n]} - \theta z[i, n] \right) \right] \right)^\lambda - q_{(i,n)} \right] \tau_{(i,n)} D_{(\theta)}(z[i, n], Y),
\]

\[
0 = \sum_{i=1}^{n} \left[ 1 - \left( 1 - \exp \left[ -2 \exp \left( \frac{z[i, n]}{z[i, n]} - \theta z[i, n] \right) \right] \right)^\lambda - q_{(i,n)} \right] \tau_{(i,n)} D_{(\lambda)}(z[i, n], Y).
\]

5.4. The ADE Method. The ADEs are obtained by minimizing the function

\[
A DE_z \left( z_{[i, n]} \right)(Y) = -n - n^{-1} \sum_{i=1}^{n} \left( 2i - 1 \right) \times \left\{ \log F(Y) (z[i, n]) + \log \left( z_{[i, n]} \right) \right\},
\]

where

\[
F(Y) (z_{[i, n]} - z_{[i+1:n]}) = \left[ 1 - F(Y) (z_{[i+1:n]}) \right].
\]

Then, the parameter estimates are derived by solving the nonlinear equations

\[
0 = \frac{\partial ADE_z \left( z_{[i, n]} \right)(Y)}{\partial \alpha},
\]

\[
0 = \frac{\partial ADE_z \left( z_{[i, n]} \right)(Y)}{\partial \theta},
\]

\[
0 = \frac{\partial ADE_z \left( z_{[i, n]} \right)(Y)}{\partial \lambda}.
\]

5.5. The ADE (R-T) Method. The ADEs (R-T) are obtained by minimizing

\[
A DE_{(R-T)}(Y) = \frac{1}{2} n - 2 \sum_{i=1}^{n} F(Y) (z[i, n]) - \frac{1}{n} \sum_{i=1}^{n} \left( 2i - 1 \right) \left\{ \log F(Y) (z_{[i+1:n]}) \right\}.
\]
Table 2: Simulation study where the parameters $\alpha = 2$, $\theta = 3$, and $\lambda = 1.1$.

|       | n      | BIAS | RMSE | BIAS | RMSE | BIAS | RMSE |
|-------|--------|------|------|------|------|------|------|
| MLE   | 3.9606664661 | 7.37129530 | 6.50297058 | 12.2786085 | 5.994434410 | 19.60201039 |
| CVM   | 1.620584171 | 4.069729958 | 2.213270370 | 6.529760670 | 12.5993180 | 30.87329914 |
| OLS   | 2.710178460 | 5.609086527 | 5.167076155 | 10.577460960 | 4.269781273 | 11.26816107 |
| WLS   | 0.5606973831 | 1.328096449 | 1.240119374 | 2.801333754 | 0.3803635751 | 1.221199984 |
| ADE   | 0.877995110 | 1.077412769 | -1.636464215 | 1.761963513 | 0.438186918 | 0.623439359 |
| ADE (R-T) | 0.805406079 | 0.893762293 | -1.636465831 | 1.709771020 | 0.467858067 | 0.699341007 |

1. Sã_he comparison is performed on the bias (BIAS) and root mean - standard error (RMSE).
2. The RMSE tends to 0 as n increases and tends to 0 which means incidence of consistency property.

6. Comparing the Non-Bayesian Estimation Methods under Uncensored Schemes via a Simulation Study

A numerical simulation study is conducted to compare the non-Bayesian estimation methods. The simulation study is based on $N = 1000$ which generated datasets from the TIITLIFW distribution where $n = 20, 60, 100, 200$, and 500 and $\alpha = 2$, $\theta = 3$, and $\lambda = 1.1$. The comparison is performed based on the bias (BIAS) and root mean - standard error (RMSE).

From Table 2, we note the following:

1) The BIAS ($V$) tends to 0 as n increases and tends to 0 which means that all estimators are nonbiased.
2) The RMSE tends to 0 as n increases and tends to 0 which means incidence of consistency property.

7. Comparing Models under Uncensorship

We illustrate the flexibility and the performance of the TIITLIFW distribution as compared to some alternative models using two real data applications. The goodness-of-fit statistics for this distribution are compared with other competitive distributions. The MLEs of the distribution parameters are determined numerically. To compare the distributions, we consider the measures of goodness-of-fit, such as Akaike information criterion ($C_1$), consistent Akaike information criterion ($C_2$), and Bayesian information criterion ($C_3$) statistic. The better distribution to fit the data corresponds to smaller values of these statistics.

We consider two uncensored datasets for comparing competitive models. The first data present the remission times (in months) of a random sample of 128 bladder cancer patients [14]: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31,
0.81, 2.69, 4.23, 5.41, 0.90, 2.69, 4.18, 5.34, 7.59, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 5.71, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 10.66, 15.96, 36.66, 1.05, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, and 22.69. 

Sãohe second data present the lifetimes of 38 devices provided by [14]: 0.1, 1, 1, 1, 2, 3, 6, 40, 45, 46, 47, 50, 55, 0.2, 7, 11, 12, 18, 18, 18, 18, 21, 32, 36, 1, 82, 83, 84, 84, 85, 85, 1, 60, 63, and 86, 86. According to [15], the total time in test (TTT) plots, box plots, quantile-quantile (QQ) plots, and kernel density estimation (KDE) plots are shown in Figure 5 and Figures 6(a) and 6(c) for bladder cancer data and lifetimes data, respectively. Based on Figures 6(a) and 6(c), the HRF of the lifetimes data is “U-shape.” The box plots (middle panels) are presented along with its corresponding normal quantile-quantile plot (right panels) in Figures 5 and 6 for discovering the outliers and normality. The following competitive models are considered in the comparison: the exponentiated IFW (Exp-IFW) [12], IFW [3], exponentiated generalized IW (ExpG-IW) [16], generalized IW (G-IW) [17], IW [18], and [2].

Tables 3 and 4 present the MLEs for the bladder cancer data and lifetimes data. Tables 5 and 6 show the statistics criteria for the bladder cancer data and lifetime data. From Tables 5 and 6, it is clear that the TIITLIFW distribution provides the best fits for the two datasets. Figures 7 and 8 show the estimated PDFs (EPDFs) (left panel) and the estimated HRF (EHRFs) (right panel) for bladder cancer data and lifetimes data, respectively. Figures 9 and 10 show the profile of the log-likelihood function for bladder cancer data and lifetimes data, respectively. From
Table 3: The MLEs for bladder cancer data.

| Model         | $\alpha$ | $\theta$ | $\lambda$ | $\beta$ |
|---------------|----------|----------|------------|---------|
| TIITLIFW      | 0.070    | 0.468    | 0.26       |         |
| Exp-IFW       | 0.080    | 0.169    | 2.47       |         |
| IFW           | 0.126    | 0.143    |            |         |
| ExpG-IFW      | 1.006    | 0.500    | 1.05       | 2       |
| G-IW          | 0.750    | 0.340    | 1.79       |         |
| IW            | 16.14    | 0.464    |            |         |
| FW            | 0.054    | 0.915    |            |         |

Table 4: The MLEs for lifetimes data.

| Model         | $\alpha$ | $\theta$ | $\lambda$ | $\beta$ |
|---------------|----------|----------|------------|---------|
| TIITLIFW      | 0.108    | 0.025    | 1.198      |         |
| Exp-IFW       | 0.099    | 0.030    | 2.187      |         |
| IFW           | 0.165    | 0.024    |            |         |
| ExpG-IFW      | 1.008    | 0.610    | 2.14       | 0.75    |
| G-IW          | 0.596    | 0.274    | 1.27       |         |
| IW            | 1.043    | 0.397    |            |         |
| FW            | 0.012    | 0.70     |            |         |
Table 5: Statistics for bladder cancer data.

| Model   | $-2\log (\mathcal{L})$ | $C_1$      | $C_2$      | $C_3$      |
|---------|--------------------------|------------|------------|------------|
| TIITLIFW| 413.85                   | 833.695    | 833.889    | 842.251    |
| Exp-IFW | 423.46                   | 852.909    | 853.104    | 861.470    |
| IFW     | 453.61                   | 911.220    | 911.310    | 916.920    |
| ExpG-IFW| 488.05                   | 984.090    | 984.420    | 995.500    |
| G-IW    | 495.18                   | 996.360    | 996.560    | 1004.92    |
| IW      | 500.12                   | 1004.25    | 1004.33    | 1009.94    |
| FW      | 525.53                   | 1055.07    | 1055.16    | 1060.77    |

Table 6: Statistics for lifetimes data.

| Model   | $-2\log (\mathcal{L})$ | $C_1$      | $C_2$      | $C_3$      |
|---------|--------------------------|------------|------------|------------|
| TIITLIFW| 233.51                   | 473.027    | 473.549    | 478.763    |
| Exp-IFW | 233.52                   | 473.029    | 473.551    | 478.765    |
| IFW     | 242.57                   | 488.914    | 489.169    | 492.738    |
| ExpG-IFW| 250.81                   | 505.620    | 505.880    | 509.448    |
| G-IW    | 254.92                   | 517.839    | 518.727    | 525.487    |
| IW      | 287.48                   | 580.951    | 581.473    | 586.687    |
| FW      | 281.07                   | 566.140    | 566.396    | 569.964    |

Figure 7: The EPDF (a) and the EHRF (b) for bladder cancer data.
Figure 8: The EPDF (a) and the EHRF (b) for lifetimes data.

Figure 9: Continued.
Figure 9: The profile of the log-likelihood function for bladder cancer data.

Figure 10: The profile of the log-likelihood function for lifetimes data.
Figures 7 and 8, we conclude that the new model can achieve a good fit.

8. Concluding Remarks

A three-parameter lifetime distribution, so-called the TII-TLIFW distribution, is introduced as an extension of the inverse flexible Weibull distribution. Some explicit expressions for mathematical quantities of the TII-TLIFW distribution are derived. The hazard rate function allows constant, decreasing, increasing, upside down bathtub, or bathtub-shaped forms. We consider six different estimation methods to estimate the parameters of the TII-TLIFW distribution. The performance of these proposed estimation methods is conducted via some simulations. A real data application proves that the TII-TLIFW model provides consistently better fits compared to some other well-known competitive models.

Data Availability

The data used to support the findings in this study are included within the paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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