Dimensional quantization and waveguide effect of Dyakonov surface waves in twisted confined media

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Abstract

We theoretically study Dyakonov-like surface waveguide modes that propagate along the planar strip interfacial waveguide between two uniaxial dielectrics. We demonstrate that due to the one-dimensional electromagnetic confinement, Dyakonov surface waveguide modes can propagate in the directions that are forbidden for the classical Dyakonov surface waves at the infinite interface. We show that this situation is similar to a waveguide effect and formulate the resonance conditions at which Dyakonov surface waveguide modes exist. We also consider a case of two-dimensional confinement, where the interface between two anisotropic dielectrics is bounded in both orthogonal directions. We show that such a structure supports Dyakonov-like surface cavity modes. Analytical results are confirmed by comparing with full-wave solutions of Maxwell’s equations. We believe that our work paves the way towards new insights in the field of surface waves in anisotropic media.

Abbreviations

DSW — Dyakonov surface wave
DSWM — Dyakonov surface waveguide mode
DSCM — Dyakonov surface cavity mode
EWM — extraordinary waveguide mode
PEC — perfect electric conductor
TE — transverse electric
TM — transverse magnetic
DCP — degree of circular polarization
FOM — figure of merit

Keywords

Dyakonov surface waves, anisotropic materials, waveguide, optical cavity, surface waves, electromagnetic confinement

Introduction

Surface electromagnetic waves, propagating along the interface of two dissimilar media, is the subject of extensive research during the last decades because they represent one of the fundamental concepts of nanophotonics. Understanding the optical properties of surface waves is of great importance for realizing their practical application.

There are several types of surface waves that differ in a material type, a domain of existence, propagation constant, decay profile, etc. Among different types of surface waves, there are surface plasmon-polariton at a metal-dielectric interface, Tamm surface states at a photonic crystal boundary, surface solitons
at a nonlinear interface⁵ and many others.

Another family of surface waves is Dyakonov surface waves (DSW) which exist at the interface of two media at least one of which is anisotropic as predicted in 1988 in Ref. [6]. In this pioneering work the first medium was considered as isotropic dielectric with the refractive index \( n_m \), while the second medium was anisotropic uniaxial dielectric with the refractive indices \( n_o \) and \( n_e \) and an optical axis is parallel to the interface. It has been shown that Dyakonov surface waves exist in such a system if the condition

\[
n_o < n_m < n_e. \tag{1}
\]

is satisfied. In 1998 Walker et al. extended the theory of Dyakonov surface waves to the case of biaxial medium⁷ with refractive indices \( n_x < n_y < n_z \). In isotropic/biaxial system the condition (1) transforms into

\[
n_x < n_y < n_m < n_z \tag{2}
\]

Later, different combinations of isotropic, uniaxial, biaxial and chiral materials have been demonstrated to support DSWs.⁸⁻¹⁴

A narrow range of propagation angles makes the experimental observation of Dyakonov surface waves rather complicated.¹⁵ As a result, the first detection of these waves has been demonstrated only in 2009.¹⁶ The authors used Otto-Kretchmann configuration to observe Dyakonov surface states at the interface of biaxial crystal and isotropic liquid. Another perspective approach to obtain Dyakonov-like surface waves experimentally is the usage of partnering thin films between anisotropic and isotropic media.¹⁷ In such systems, the direction of hybrid Dyakonov-guided modes propagation can be controlled by changing the isotropic medium’s refractive index. The results presented in Ref. ¹⁷ show that these types of waves can be used as a sensing unit.

It has been demonstrated in a number of publications that Dyakonov surface waves can exist at the interface of isotropic materials and materials with artificially designed shape anisotropy.⁹⁻¹¹,¹³⁻¹⁴ Moreover, as theoretically shown in Ref. [23,24], in the metamaterial composed of alternating layers of metals and dielectric, exotic types of surface waves such as Dyakonov plasmons and hybrid plasmons can appear. In such structures, the angular range of existence of Dyakonov surface waves can be extended up to \( \Delta \phi \sim 65^\circ \).

Recently, in 2019, a new type of surface waves, referred to as Dyakonov-Voigt surface waves, have been theoretically demonstrated at the interface of isotropic and uniaxial materials.¹⁸ Unlike conventional Dyakonov surface waves, Dyakonov-Voigt surface waves decay as the product of a linear and an exponential function of the distance from the interface in the anisotropic medium.¹⁹⁻²⁰ In contrast to Dyakonov surface waves, Dyakonov-Voigt surface waves propagate only in one direction in each quadrant of the interface plane.

Like other surface waves, the feasibility of practical use of the Dyakonov surface wave depends ultimately on whether they can exist in resonator structures of finite size. In Ref. ²₀ it has been shown that the Dyakonov surface waves can be conformally transformed into the bound states of cylindrical metamaterials. Dyakonov-like surface waves have been also theoretically predicted in anisotropic cylindrical waveguides.²¹

This paper is devoted to the theoretical study of Dyakonov surface states at a flat interface confined in one or two dimensions. We consider two anisotropic uniaxial lossless dielectrics, the optical axes of which are rotated relative to each other by \( \alpha = 90^\circ \) and are parallel to the interface plane. We analyze the dimensional quantization effect in such a system, study Dyakonov surface waveguide modes in the case of one-dimensional confinement and introduce the concept of Dyakonov surface cavity modes in the case of two-dimensional confinement.

Interface of two uniaxial crystals

We start our discussion by considering a flat infinite interface between two semi-infinite anisotropic uniaxial media shown in Fig. 1a.
We denote the dielectric permittivity tensor of the upper half-space as $\varepsilon_{\text{up}} = \text{diag}(\varepsilon_2, \varepsilon_1, \varepsilon_1)$ and of the lower half-space as $\varepsilon_{\text{low}} = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1)$. Dyakonov surface waves in such system can be obtained as a linear combination of exponentially decaying ordinary and extraordinary waves which are solutions of Maxwell’s equation in each half-space:

$$\begin{align*}
\vec{E}^+_{\text{DSW}} &= C^+_o \vec{E}^+_o + C^+_e \vec{E}^+_e \\
\vec{B}^+_{\text{DSW}} &= C^+_o \vec{B}^+_o + C^+_e \vec{B}^+_e \\
\vec{E}^-_{\text{DSW}} &= C^-_o \vec{E}^-_o + C^-_e \vec{E}^-_e \\
\vec{B}^-_{\text{DSW}} &= C^-_o \vec{B}^-_o + C^-_e \vec{B}^-_e
\end{align*}$$

(3) at $z > 0$,

$$\begin{align*}
\det \begin{pmatrix}
0 & k^2_0 \varepsilon_1 - k^2_x & k_0 k_0^- & k_x k_y \\
-k_0^2 \varepsilon_1 & k^2_x & 0 & -\varepsilon_1 k_0 k_0^- \\
k_x k_y & k_0^2 & 0 & 0 \\
k^2_0 \varepsilon_1 - k^2_y & 0 & k_x k_y & 0
\end{pmatrix}
= 0
\end{align*}$$

(4)

where $k_0 = 2\pi/\lambda$ is the vacuum wavenumber, $k_x$, $k_y$ and $k_z$ are the wavevector components, $\lambda$ is the wavelength and the $z$-components of wavevectors are:

$$\begin{align*}
k_{z}^{+} &= \sqrt{k_0^2 \varepsilon_1 - k^2_x - k^2_y} & \text{at } z > 0, \\
k_{z}^{e+} &= \sqrt{k_0^2 \varepsilon_2 - \gamma k^2_x - k^2_y} \\
k_{z}^{o-} &= -\sqrt{k_0^2 \varepsilon_1 - k^2_x - k^2_y} \\
k_{z}^{e-} &= -\sqrt{k_0^2 \varepsilon_2 - k^2_x - \gamma k^2_y}
\end{align*}$$

(5)

where $\gamma = \varepsilon_2/\varepsilon_1$ is the anisotropy factor. In Eq. (5), all $k_z$ are purely imaginary and the signs of square roots are chosen in such a way that the solution decays with distance from the interface in upper and lower half-spaces.

The numerical solution of the equation (4) for the Dyakonov wave is represented in Fig.1 by the red curve. One can see that the DSW is located near the intersection of the dispersion curves of extraordinary waves with $k_z = 0$ in upper and lower half-spaces. The $(k_x, k_y)$-range of existence of DSW determines the narrow domain of azimuthal angle $\varphi$ near the bisector between the crystals’ optical axes where the DSW can propagate (Fig. 1b). This is in agreement with the results reported in Refs. [31,32] for the two symmetrical uniaxial anisotropic crystals.

To estimate the partial contributions of ordinary and extraordinary waves to the DSWM we calculate the ratio of the coefficients $C_e/C_o$ as well as the ratio of the electric field intensities $|E_e|^2/|E_o|^2$ in the most symmetric case $\varphi = 45^\circ$ when $C_{o,e}^+ = C_{o,e}^-$ and $|E_{o,e}^+|^2 = |E_{o,e}^-|^2$. Fig. 1 demonstrates that at $\gamma \lesssim 2$ the contribution of ordinary wave is dominant.

The angular dependencies of $|\text{Im}(k_{z}^{o,e})|$ and the propagation constant $\beta(\varphi)$ for permittivities $\varepsilon_1 = 9$ and $\varepsilon_2 = 16$ are presented in Fig. 1C. One can see that the extraordinary wave decays slower than the ordinary wave. In the cut-off points $\varphi_1$ and $\varphi_2$, the imaginary part of $|\text{Im}(k_{z}^{e})|$ turns to zero and the solution is no longer localized near the interface. It is worth noting that at $\varphi = 45^\circ$, the absolute values of $k_{z}^{e+}$ and $k_{z}^{e-}$ are the same, and the DSW decays to the upper and lower half-spaces equally.

Dispersion relation of DSW can be found analytically from Eq. (1) for the symmetric case of $\varphi = 45^\circ$ ($k_x = k_y$), when the DSW propagates along the bisector. The expression for the propagation constant of DSW reads:

$$\beta(\pi/4) = k_0 \sqrt{\frac{\varepsilon_1 + \sqrt{(2\varepsilon_2 - \varepsilon_1)\varepsilon_1}}{2}}$$

(6)
Figure 1: (Color online) (a) Interface between two anisotropic materials. Red lines show the profile of electric field intensity. (b) Top view of the interface. Red shaded regions show the propagation cones of DSWs. Angles $\varphi_1$ and $\varphi_2$ are the limits of the $\varphi$-range of existence of DSWs. (c) Ordinary wave (solid black line) and extraordinary waves in anisotropic materials 1 and 2 (dashed and dotted lines) as well as DSW (red line) in reciprocal space. (d) Azimuthal angle dependencies of propagation constant $\beta$ of DSW and the absolute value of imaginary parts of $z$-projections of wavevectors of ordinary and extraordinary waves in anisotropic materials 1 and 2 with $\varepsilon_1 = 9$ and $\varepsilon_2 = 16$. Black dashed lines bound the $\varphi$-range of existence of DSWs (e) The ratio of electric field intensities of ordinary and extraordinary waves which form DSW (blue line) and the ratio of coefficients $C_e/C_o$ (red lines) as functions of anisotropy factor $\gamma = \varepsilon_2/\varepsilon_1$. (f) Degree of circular polarization (DCP) of DSW at $z = 0$ propagating at $\varphi = 45^\circ$ as a function of anisotropy factor $\gamma = \varepsilon_2/\varepsilon_1$. (g) DCP of DSW propagating at $\varphi = 45^\circ$ as a function of coordinate $z$ at $\varepsilon_2/\varepsilon_1 = 3$. (h) Thick black lines denote the limits of the $\varphi$-range existence of DSWs as functions of anisotropy factor, $\varphi_2(\varepsilon_2/\varepsilon_1)$ and $\varphi_1(\varepsilon_2/\varepsilon_1)$. Heatmap shows the anisotropy factor and azimuthal angle dependence of the DCP of DSW. Colorscale is shown on the right.

Besides, Eq. (4) can also be analytically solved in the cut-off points $\varphi_1$ and $\varphi_2$ of angular domain of existence. At $\varphi = \varphi_1$ we get

$$k_x(\varphi_1) = k_0 \sqrt{\frac{\gamma(\gamma + 2) + \sqrt{\gamma(\gamma^2 + \gamma - 1)}}{2(\gamma + 1)}},$$

(7)
\[ k_y(\varphi_1) = k_0 \sqrt{\epsilon_1} \sqrt{\frac{\gamma^2 - \sqrt{\gamma(\gamma^2 + \gamma - 1)}}{2\gamma(\gamma + 1)}}. \]  

(8)

This gives us the analytical expression for the cut-off angle \( \varphi_1 \):

\[ \tan \varphi_1 = \frac{k_y(\varphi_1)}{k_x(\varphi_1)} = \frac{\gamma^2 - \sqrt{\gamma(\gamma^2 + \gamma - 1)}}{\gamma^2 + \gamma \sqrt{\gamma(\gamma^2 + \gamma - 1)}}. \]  

(9)

Owing to the symmetry, the second cut-off angle \( \varphi_2 \) can be determined by swapping \( k_x \) and \( k_y \). This leads us to a simple relation \( \tan \varphi_2 = \cot \varphi_1 \) (Fig. 1b). One can see that the cut-off angles depend only on the anisotropy factor \( \gamma = \epsilon_2/\epsilon_1 \) and does not depend on the values of \( \epsilon_1 \) and \( \epsilon_2 \). One can also notice that the higher the anisotropy factor, the larger the angular domain of existence (Fig. 1h). Due to the structure symmetry, the DSW can propagate in identical angular domains rotated relative to the OZ axis by \( \pi/2 \). (Fig. 1b).

Like many surface waves, DSW are circularly polarized. However, the degree of circular polarization (DCP) depends on the anisotropy factor \( \gamma \), on the azimuthal angle of propagation \( \varphi \) and on the coordinate \( z \) where the electric field is considered. For the most symmetric case \( (\varphi = 0, z = 0) \) the \( \gamma \)-dependence of the DCP can be expressed analytically:

\[ \text{DCP} = 2 \sqrt{2} \frac{\sqrt{\gamma - 1}}{\gamma + 1}. \]  

(10)

From Eq. (10) we can see that at \( \gamma = 3 \) the DCP equals to 1 (Fig. 1p) which corresponds to a purely circularly polarized field. In the limit of a low anisotropy, the DCP goes to 0 and the DSW becomes almost linearly polarized. By means of full-wave electromagnetic simulations made by scattering matrix method\(^{33,35}\) we can calculate the DCP in less symmetric case when \( z \neq 0 \). We obtain that with distance from the interface the DCP decreases and changes its sign (Fig. 1g). We also vary the azimuthal angle \( \varphi \) (Fig. 1h) and find that the \( \varphi \)-dependence of the DCP is weak, however for all \( \varphi \neq 45^\circ \) DCP<1. The orientation of polarization cones in DSW changes with coordinate \( z \).

**Reflection from boundary**

Before proceeding to the study of dimensional quantization of Dyakonov modes, it is essential to analyze the scattering of a DSW on a single boundary perpendicular to the interface plane along which the DSW propagates. The results obtained in the previous section provide a possibility to perform this analysis.

We consider a DSW propagating along the interface and hitting the boundary at a varying angle of incidence \( \alpha \) (Fig. 2). In this section we use a new coordinate system where the azimuthal propagation angle \( \varphi = 45^\circ \) corresponds to the bisector between optical axes of upper and lower anisotropic dielectrics. The angle between the optical axes and the boundary is always chosen in such a way that the boundary is parallel to \( y \)-axis. We consider two cases when the boundary separates anisotropic materials from (i) air and (ii) perfect electric conductor (PEC). In reflection, the \( y \)-component of the wavevector \( k_y(\alpha) \) is conserved and, hence, can be described by the propagation constant \( \beta \) from (6) in the following way:

\[ k_y(\alpha) = \beta(\pi/4) \sin \alpha. \]  

(11)

It enables us to reduce the 3D scattering problem to a 2D problem in the XZ plane with a fixed out-of-plane wavevector component \( k_y(\alpha) \) and the corresponding orientation of optical axes of anisotropic materials (Fig. 2b). We perform the corresponding electromagnetic simulations of DSW scattering on the boundary in COMSOL Multiphysics.

The calculated angular dependencies of the specular reflection and transmission coefficients are shown in Fig. 2 for the air or PEC boundary. Since DSWs can only propagate in the limited angular domain of \( \varphi \) near the bisector between the optical axes, the reflection of this mode without significant scattering losses can occur only at the angle \( \alpha \) close to 45°. In the case of air boundary, the transmission turns to zero at \( \alpha > 17.5^\circ \) which is due to total internal reflection at incident angles exceeding critical
Figure 2: (Color online) Top view (a) and side view (b) of the interface between two anisotropic materials bounded by air or a perfect electric conductor (PEC) half-space on the right. In panel (b) the red line schematically shows the Dyakonov surface wave, which is induced by the port denoted by dashed magenta line, and then hits the boundary. (c) Angular dependence of specular reflection and transmission in such configuration. Vertical cross section of electric field intensity when Dyakonov surface wave hits the air boundary (a–g) or the PEC boundary (h–k) calculated at different incident angles $\alpha$. Arrows show the direction of scattering and transmission. All simulations are made for $\lambda = 1550$ nm, $\epsilon_1 = 9$ and $\epsilon_2 = 16$. Colorscale is shown on the right.

angle. Please note that we do not consider the transmission to PEC because it is zero by definition.

Fig. 2l–k shows the profiles of the period average electric field intensity of the DSW produced by the port and falling on the boundary at different incident angles $\alpha$. One can see from Fig. 2l that at $\alpha = 0^\circ$ the incident wave is partially reflected, scattered and transmitted to the air. Intensity modulation between the port and the boundary is due to the interference of incident and reflected DSWs. At $\alpha = 45^\circ$ (Fig. 2f) the DSW is reflected back to the interface. In the case of air boundary there
is a slight scattering which occurs only to the incident side due to the total internal reflection. The specularly reflected DSW propagates perpendicularly to the incident DSW, along the second bisector between two orthogonal optical axes. In the less symmetric cases of \( \alpha = 30^\circ \) and \( 60^\circ \) the wave is scattered out to the anisotropic media almost completely. Similar behaviour of DSW is observed for the case of PEC boundary with the exception of the fact that at \( \alpha = 45^\circ \) no scattering occurs.

**One-dimensional confinement**

Let us now study the system of upper and lower slabs, tangent to each other, confined between by two parallel boundaries located at \( x = 0 \) and \( x = d \) as shown in Figs. 3a, b. Like in the previous section, we consider the cases of air or PEC boundaries. In Fig. 2 we have demonstrated that the maximal reflectance of DSW from the boundary is reached at the incident angle \( \alpha = 45^\circ \). If the reflected wave encounters yet another boundary, parallel to the first one, then the process of multi-reflection continues and lasts until the energy is lost due to scattering. Provided that one of the optical axes is parallel to the boundary while another one is perpendicular to the boundary (Fig. 3b), one can expect the existence of a Dyakonov-like surface waveguide mode (DSWM) in the system of two anisotropic materials confined between two boundaries. While one can also consider the case of \( \alpha = 0 \) where the reflection spectrum in Fig. 2 has local maximum (see Supplemental Materials for details), in this section we are focused on \( \alpha = 45^\circ \).

Since such a strip waveguide has mirror symmetries \( x \rightarrow -x \) and \( y \rightarrow -y \), in the case of ideal reflection by PEC it is possible to treat the propagation of DSWM analytically quite easily. If a solution for DSWs in the infinite interface

\[
\vec{E}_{k_x, k_y}(x, y, z) = \vec{E}_{k_x, k_y}(z)e^{ik_xx + ik_yy} \tag{12}
\]

is known then the solution for DSWMs which satisfies mirror boundary conditions at \( x = 0 \) is the following:

\[
\begin{align*}
\vec{E}_{\text{tot}}(x, y, z) &= \vec{E}_{k_x, k_y}(z)e^{ik_xx + ik_yy} \\
&- \vec{E}_{-k_x, k_y}(z)e^{-ik_xx + ik_yy} \tag{13}
\end{align*}
\]

Moreover, it also satisfies mirror boundary conditions at \( x = d \) provided that \( k_xd = \pi n \), where \( n \in \mathbb{N} \).

These considerations enable us to find the dispersion law of DSWMs. First, we note that the dispersion law of DSW at an infinite interface in \( xy \)-plane can be written in a form

\[
F \left( \frac{k_x}{k_0}, \frac{k_y}{k_0} \right) = 0, \tag{14}
\]

where a function \( F \) does not depend on \( k_0 \) explicitly (see Eq. 11). From the quantization condition \( k_xd = \pi n \), the dispersion relation of the DSWM propagating along the \( y \)-axis system with the propagation constant \( k_y \) has the following form:

\[
F \left( \frac{\pi n}{k_0d}, \frac{k_y}{k_0} \right) = 0, \tag{15}
\]

where \( n \) is the mode order.

The dispersion curves of DSWMs \( k_0(k_y) \) calculated by Eq. (15) for the case of PEC boundary and by COMSOL for the case of air boundary are shown in Figs. 3c,g by black and red lines respectively. For comparison, Figs. 3c,g also show the dispersions of extraordinary waveguide modes (EWM) of the upper and lower slabs, which are respectively TE and TM polarized because of the specific dielectric tensor orientations and PEC boundary condition. Both \( k_y \) and \( k_0 \) are plotted in \( 1/d \) units, making the displayed dispersion curves universal in terms of the waveguide width \( d \). One can see that the DSWMs appear near the intersection of the EWMs. We note that the dispersion curves of DSWMs have cut-off points which originate from the angular cut-offs of the DSWs at the infinite interface. \(^1\)

The waveguide width dependence of the prop-

\(^1\)Please note that the cut-off points for DSWM for the air surrounding are determined approximately due to limitation of the computational domain size in COMSOL.
Figure 3: (Color online) Side view (a) and top view (b) of the interface between two anisotropic materials bounded by air or PEC half-spaces from left and right. Optical axes of anisotropic materials are parallel to coordinate axes as show by green and blue lines in panel (b). DSWM in such configuration is a superposition of DSWs reflecting from both sides of the boundary at the angle of $\alpha = 45^\circ$ as is shown in panel (b) by red arrows. (c), (e) and (g): Extraordinary waveguide modes (EWM) in upper and lower anisotropic slab (dashed green and blue lines) with PEC boundaries and DSWMs in the case of a PEC (solid black line) or air (solid red line) as surrounding medium. (d): Range of the waveguide width, $d$, in which DSWMs exist. (f): Figure of merit (FOM) calculated for the case of air as surrounding medium.

Agitation constant $k_y(d)$ can also be calculated by Eq. (15) for the case of PEC and is shown in Fig. 3e along with the same dependence calculated in COMSOL for the case of air. For comparison, $k_y(d)$ dependencies of extraordinary waveguide modes (EWM) of the upper and lower slabs are also shown in Fig. 3e. The $d$-range where DSWM can propagate is determined by the angular existence domain of the DSW at the infinite interface. With increase of the anisotropy factor, the existence range of DSWM broadens (Fig. 3d) and, at large anisotropy, the existence ranges of DSWMs with different mode numbers $n$ overlap. It is worthy to note that we compared the analytical results obtained from Eq. (15) for PEC with full-wave simulations made in COMSOL Multiphysics and observed an excellent agreement (not shown in Fig. 3).

As it has been demonstrated in Fig. 2, the reflection of DSWMs from the PEC half-space at $\alpha = 45^\circ$ is ideal, meaning that there is no
scattering and, therefore, the DSWMs are lossless. However, in the case air half-space, even at \( \alpha = 45^\circ \), the reflection coefficient \( R < 1 \). This means that in the strip waveguide surrounded by air, DSWMs can have radiation losses, which scatter out the DSWM energy to the waveguide modes of the upper and lower slabs. Like in Ref. [30], to describe the radiation losses quantitatively, we calculate the following figure of merit (FOM)

\[
FOM = \frac{\text{Re} k_y}{\text{Im} k_y},
\]

which has the meaning of a DSWM decay length expressed in units of the DSWM wave-
length. Fig. 3 shows the FOM calculated in COMSOL. One can see that the FOM tends to infinity near the cut-off points while having a local minimum between them. We will explain the presence of the local minimum after considering the field distributions in DSWMs. We also observe that for the 1-st order DSWM ($n = 1$) the FOM is infinite. To explain this fact, we calculate the overlap integrals between DSWMs and waveguide modes of the upper and lower slabs (see Supplemental materials). Our full-wave simulations in COMSOL revealed that for the 1-st order DSWM the overlap integrals vanish which indicates that the coupling of this mode with the slabs’ waveguide modes is not possible due to their symmetries mismatch. As there is no radiative leakage to the air (see Fig. 2 and its discussion), we conclude that the 1-st order DSWM has no radiative losses which results in the infinite FOM. Radiative losses of higher order DSWMs are fully attributed to the coupling with the slabs’ waveguide modes.

Let us consider the field distributions in DSWMs. Cross-sectional electric field profiles of the 2-nd order DSWM in the strip waveguide surrounded by air calculated for different waveguide widths $d$ within the range of DSWM existence are shown in Fig. 4a. One can see that at $d = 610$ nm the mode is localized near the interface almost equally penetrating into the upper and lower slabs. Whereas at widths close to the cut-offs, $d = 600$ nm and 620 nm, the mode localization appears upward or downward-biased. DSWMs inherit these peculiar properties of their localization from the classical DSWs at the infinite interface. In the case of the PEC surrounding, the waveguide widths $d_n$ corresponding to the most symmetric DSWM mode penetration into the slabs can be found from Eq. (15) by setting the azimuthal propagation angle of DSW as $\varphi = 45^\circ$:

$$d_n = \pi n/k_y.$$  

(17)

In the case of air surrounding, this condition will be more complex. The biasing of the DSWMs towards upper or lower slabs explains the local minimum in the waveguide width dependence of the FOM shown in Fig. 3.

Electric and magnetic field intensity profiles of the 1-st order and the 2-nd order DSWMs are shown in Figs. 4b,c for the widths $d$ such that the symmetry condition (17) is satisfied. For the air surrounding, the electric field intensity profile of the $n$-th order DSWM has $n + 1$ local maxima in the upper slab and $n$ local maxima in lower slab. Magnetic field intensity has $n$ local maxima in both cases. For the PEC surrounding, the situation is different: $n$ (or $n + 1$) local maxima in the upper (or lower) waveguide for electric field and $n + 1$ local maxima for the magnetic field. Projections of electric and magnetic vectors directions on $xz$ and $xy$ planes are shown in Figs. 4d-f for the 1-st order DSWM. One can see the periodic pattern in Fig. 4 which demonstrates the propagation of DSWM along the strip waveguide. It is necessary to note here that electric and magnetic fields in DSWM are circularly polarized like those in conventional DSWs, however the orientation of polarization ellipse and the degree of circular polarization depend also on the $x$ coordinate (See Supplemental Materials for details). The examples of field distribution for cases when $d_n \neq \pi n/k_y$ and for different mode orders $n$ are presented in Supplemental materials.

At the end of this Section, we conclude, that the one-dimensional electromagnetic confinement makes DSWMs traveling along the direction where classical DSWs cannot propagate. Indeed, as is shown in Fig. 1b, DSWs exist in a small angle around the bisector between optical axes of upper and lower anisotropic materials, while the DSWMs propagate along with one of these optical axes. This feature distinguishes DSWMs from DSWs.

Two-dimensional confinement

Due to symmetrical configuration of DSWs relative to the optical axes (Fig. 1b), one can confine DSWs in two dimensions using two pairs of orthogonal boundaries, as shown in Figs. 5a,b. In such a system, the DSW reflects at an angle of $\alpha = 45^\circ$ from four boundaries forming a closed Dyakonov-like surface cavity mode
Figure 5: (a) The interface between two square cylinders made of anisotropic materials (AM). (b) Orientation of optical axes is shown by blue and green lines. Dyakonov waves reflect from the air boundaries at the angle of $\alpha = 45^\circ$ forming two-dimensionally confined mode. (c) Anisotropy factor dependence of square widths $d$ at which DSCM exist for different mode orders $n$. (d) Electric and (e) magnetic fields calculated in the horizontal plane $z = 0$ in DSCM. Electric (f, g, i–k) and magnetic (h, l–n) field intensities in DSCM mode in horizontal (i–n) and vertical (f–h) cross sections. $d = 264.34\, \text{nm}$, $\varepsilon_1 = 9$, $\varepsilon_2 = 36$, $\lambda = 1550\, \text{nm}$. Black lines denote the edges of rods.

(DSCM), which exists at the interface between two adjacent anisotropic rods of square cross-section. In such configuration, optical axes of anisotropic materials have to be parallel to the square’s sides. If the rods are surrounded by PEC then the DSCMs existence condition can be expressed by a simple formula:

$$\beta(\pi/4)d/\sqrt{2} = \pi n, n \in \mathbb{N} \quad (18)$$

where $n$ is the mode order, $d$ is the side of the square, and $\beta(\pi/4)$ is the propagation constant of the DSW at the infinite interface determined by Eq. (6). Since this condition is only valid for $\alpha = \pi/4$, then, in contrast to one-dimensionally confined DSWMs, DSCMs exist only at a discrete set of the square side $d$. The calculated by Eq. (18) values of the square side supporting DSCM are shown in Fig. 5c as a function of the anisotropy factor for mode orders $n = 1$–5 and PEC boundaries. These theoretical dependencies are confirmed by full-wave simulations in COMSOL Multiphysics. One can also obtain similar curves for the air boundaries.

We note that the structure with anisotropic rods shown in Figs. 5a, b is $S_4$-symmetrical, i.e. it is invariant under 90° rotation about the $z$-
Figure 6: (Color online) Intensity (a) and z-projection (b) of electric and magnetic fields of DSCMs at \( n = 1 - 3 \) calculated for the cases of air and PEC as surrounding medium. Symbols A or B denote irreducible representations of the \( S_4 \) point group. \( \varepsilon_1 = 9, \varepsilon_2 = 36, \lambda = 1550 \text{ nm} \). Colorscales are shown on the right.

To explore this in application to our system consisting of two tangent rods surrounded by air or PEC, we simulate its eigenmodes in COMSOL. Such a system supports waveguide modes propagating along the rods, as well as DSCMs localized at the interface between the rods. We notice that the waveguide modes can be singlets or doublets while the DSCMs are always singlets. This is due to more strict selection rules for DSCM in comparison to waveguide modes. The calculated electric and magnetic field intensity profiles of the 1-st order DSCM in vertical and horizontal cross sections are shown in Figs. 5f–n. One can see that in the upper (or lower) rod the electric field is mainly localized near the boundaries \( y = \text{const} \) (or \( x = \text{const} \)), whereas at the interface between the rods (\( z = 0 \)) it is localized near the four square’s corners, thereby, having a higher degree of symmetry. Also, at the interface the electric vector takes the vortex shape, while magnetic field is maximal in the center of the vortex (Fig. 5d,e). Electric and magnetic field intensity profiles in the DSCMs of the orders \( n = 1 - 3 \) are shown in Fig. 6a. We notice that due to different boundary conditions, the number of nodes (or antinodes) in electric (or magnetic) intensity field profile equals to \( n \times n \) for the air surrounding, and to \((n + 1) \times (n + 1)\) for the PEC surrounding. To ascribe a specific irreducible representation to the obtained DSCMs, we also plot the z-projection of electric and magnetic vectors in Figs. 6b. By inspecting Figs. 6b one can see that the displayed DSCMs refer to the irreducible representations either A or B of the \( S_4 \) point group.\(^2\)

\(^2\)The difference between irreducible representations
Due to radiation losses caused by scattering of DSWs at the air boundaries, DSCMs should have a finite Q-factor when rods are surrounded by air. Generally, the Q-factor depends on the dielectric permittivities of rods and environment, as well as on the mode order. The COMSOL simulation reveals that for the case of air boundaries and components of dielectric tensors of rods $\varepsilon_1 = 9$ and $\varepsilon_1 = 36$, the Q-factor of DSCMs with $n = 1$, 2 and 3 equal to 8.83, 52.98 and 197.06, respectively. Apparently, the Q-factor increases with the mode order due to decrease of diffraction losses. Finally, due to lack of scattering and absorption losses in the rods surrounded by PEC, the corresponding Q-factor is infinite.

Conclusion

In conclusion, we have studied Dyakonov-like surface states which appear at the interface between two identical anisotropic dielectrics whose optical axes form an angle of 90 degrees to each other. First, we have studied the case of the infinite horizontal interface where Dyakonov surface waves exist in a small range of azimuthal angles. In the presence of vertical boundaries that constrain the system from two sides, electromagnetic confinement comes into play. We have demonstrated that such a one-dimensionally quantized system supports Dyakonov surface waveguide modes propagating along the direction where conventional Dyakonov surface waves do not exist. We have shown that the 1-st order Dyakonov surface waveguide mode can propagate without losses.

A and B in $S_4$ point group is whether a field changes its sign under the symmetry operation $S_4$. See Ref. [36] for details.

cavity modes. We believe that our work can open new insights in the field of surface waves in anisotropic media, which can lead to the practical application of Dyakonov surface waves in optoelectronic devices.

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Theoretical methods

To simulate the DSW reflection from a single boundary, we developed a model in COMSOL Multiphysics where the DSW is excited by a port plane. The field and the wavevector of the mode which are excited by the port are taken as a DSW solution at the infinite interface described in Sec. Interface of two uniaxial crystals. Then, we find the S-parameters of such a system by calculating the fields at the reflection and transmission sides. As a result, we obtain the total reflectance and transmittance of DSW at the boundary.

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