CONVEX INTEGRATION WITH LINEAR CONSTRAINTS AND ITS APPLICATIONS

SEONGHAK KIM

ABSTRACT. We study solutions of the first order partial differential inclusions of the form $\nabla u \in K$, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $K$ is a set of $m \times n$ real matrices, and derive a companion version to the result of Müller and Šverák [20], concerning a general linear constraint on the components of $\nabla u$. We then consider two applications: the vectorial eikonal equation and a $T_4$-configuration both under linear constraints.

1. Introduction

We study the existence of solutions to the Dirichlet problem of a homogeneous partial differential inclusion

\begin{align}
\begin{cases}
\nabla u \in K & \text{a.e. in } \Omega, \\
u = v & \text{on } \partial \Omega,
\end{cases}
\end{align}

where $m, n \geq 2$ are integers, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $v \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ is a boundary map, $K$ is a subset of the space $\mathcal{M}^{m \times n}$ of $m \times n$ real matrices, and $u \in v + W^{1,\infty}_0(\Omega; \mathbb{R}^m)$ is a solution to the problem.

Such a problem of differential inclusion (1.1) has stemmed from the study of models of crystal microstructure by Ball and James [2, 3] and Chipot and Kinderlehrer [5]. Later, Müller and Šverák [20, 21] generalized the theory of convex integration of Gromov [13] and applied the results to the two-well problem in the theory of martensite [20] and to the construction of wild solutions of some $2 \times 2$ elliptic system [21]. Constructing a suitable in-approximation and applying the result of [20], Conti, Dolzmann and Kirchheim [7] obtained Lipschitz minimizers for the three-well problem in solid-solid phase transitions. On the other hand, Dacorogna and Marcellini [10] and Dacorogna and Tanteri [11] extensively studied (1.1) and its inhomogeneous version under the Baire category framework.

The generalization of Gromov’s result by Müller and Šverák [20] was pursued in two directions. Firstly, they showed that constraints on a minor of $\nabla u$ can be imposed in solving problem (1.1) under the convex integration method. Secondly, they enlarged the set of matrices, in which $\nabla v$ can stay

2010 Mathematics Subject Classification. 35F60, 35D30.

Key words and phrases. convex integration, linear constraints, partial differential inclusions, $T_4$-configuration, Baire’s category method, vectorial eikonal equation.
for solvability of (1.1), from the lamination convex hull of $K$ to its rank-one convex hull when $K$ is open and bounded. Also, an in-approximation scheme was adopted to handle the case that $K$ is not necessarily open.

In this paper, we show that one can impose a general linear constraint on the components of $\nabla u$ to solve problem (1.1) in the spirit of [20] and provide two examples of application: the vectorial eikonal equation and a $T_1$-configuration both under linear constraints. Unlike [20], we avoid using piecewise linear approximation for rank-one connections, but instead maintain $C^1$ regularity in our approximation. This turns out to be possible in case of a linear constraint (also in the unconstrained case) although in the special case that $m = n \geq 2$ with the constraint $\text{div} u = \text{const}$, piecewise linear approximation can be constructed as mentioned in [20] and proved in [25].

To state our main results, we first introduce some definitions. A set $E \subset \mathbb{M}^{m \times n}$ is called lamination convex if $[\xi_1, \xi_2] \subset E$ for all $\xi_1, \xi_2 \in E$ with $\text{rank}(\xi_1 - \xi_2) = 1$, where $[\xi_1, \xi_2]$ denotes the closed line segment in $\mathbb{M}^{m \times n}$ joining $\xi_1$ and $\xi_2$. The lamination convex hull $E^{lc}$ of a set $E \subset \mathbb{M}^{m \times n}$ is defined to be the intersection of all lamination convex sets in $\mathbb{M}^{m \times n}$ containing $E$; that is, it is the smallest lamination convex set in $\mathbb{M}^{m \times n}$ containing $E$. A function $f : \mathbb{M}^{m \times n} \rightarrow \mathbb{R}$ is called rank-one convex if

$$f(\lambda \xi_1 + (1 - \lambda) \xi_2) \leq \lambda f(\xi_1) + (1 - \lambda) f(\xi_2)$$

for all $\xi_1, \xi_2 \in \mathbb{M}^{m \times n}$ with $\text{rank}(\xi_1 - \xi_2) = 1$ and all $\lambda \in [0, 1]$, or equivalently, if $\mathbb{R} \ni s \mapsto f(\xi + sa \otimes b)$ is convex for each $(\xi, a, b) \in \mathbb{M}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$. The rank-one convex hull $K^{rc}$ of a compact set $K \subset \mathbb{M}^{m \times n}$ is defined as

$$K^{rc} = \{ \xi \in \mathbb{M}^{m \times n} \mid f(\xi) \leq \sup_K f \quad \forall f : \mathbb{M}^{m \times n} \rightarrow \mathbb{R} \text{ rank-one convex} \}. $$

The rank-one convex hull $E^{rc}$ of a set $E \subset \mathbb{M}^{m \times n}$ is then defined to be

$$E^{rc} = \bigcup \{ K^{rc} \mid K \subset E, K \text{ is compact} \}. $$

With this definition, the rank-one convex hull $V^{rc}$ of any open set $V$ in $\mathbb{M}^{m \times n}$ is again open in $\mathbb{M}^{m \times n}$.

Throughout the paper, we reserve the following notations unless otherwise stated. Let $m, n \geq 2$ be integers, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $L \in \mathbb{M}^{m \times n} \setminus \{0\}$, and its corresponding linear function $\mathcal{L} : \mathbb{M}^{m \times n} \rightarrow \mathbb{R}$ is given by

$$\mathcal{L}(\xi) = L \cdot \xi = \sum_{1 \leq i \leq m, 1 \leq j \leq n} L_{ij} \xi_{ij} \quad \forall \xi \in \mathbb{M}^{m \times n}. $$

As an abuse of notation, we often view $L$ as the linear map $b \mapsto Lb$ from $\mathbb{R}^n$ into $\mathbb{R}^m$, which should be distinguished from $\mathcal{L}$. We fix any number $t \in \mathbb{R}$ and write

$$\Sigma_t = \{ \xi \in \mathbb{M}^{m \times n} \mid \mathcal{L}(\xi) = t \},$$

which is an $(mn - 1)$-dimensional flat manifold in $\mathbb{M}^{m \times n}$. We denote by $\partial|_{\Sigma_t}$ the relative boundary in the space $\Sigma_t$. 
A map \( v : \Omega \to \mathbb{R}^m \) is called \( \text{piecewise } C^1 \) if there exists a sequence \( \{\Omega_j\}_{j \in \mathbb{N}} \) of disjoint open subsets of \( \Omega \) whose union has measure \( |\Omega| \) and such that \( v \in C^1(\overline{\Omega}_j; \mathbb{R}^m) \) for all \( j \in \mathbb{N} \).

We now state the first main result of the paper as follows.

**Theorem 1.1.** Assume
\[
Lb \neq 0 \quad \forall b \in \mathbb{R}^n \setminus \{0\}.
\]
Let \( U \) be a bounded open set in \( \Sigma_t \), and let \( v \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) be a \( \text{piecewise } C^1 \) map satisfying
\[
\nabla v \in U^\ast \quad \text{a.e. in } \Omega.
\]
Then for each \( \epsilon > 0 \), there exists a map \( u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) such that
\[
\left\{
\begin{array}{l}
\nabla u \in U \quad \text{a.e. in } \Omega, \\
u = v \quad \text{on } \partial\Omega, \\
\|u - v\|_{L^\infty(\Omega)} < \epsilon.
\end{array}
\right.
\]

Note that hypothesis (1.2) is equivalent to saying that \( m \geq n \) and the linear map \( L : \mathbb{R}^n \to \mathbb{R}^m \) is injective.

To deal with the sets that may not be open, we adopt the following notion from [20].

**Definition 1.2.** Let \( K \subset \Sigma_t \). A sequence \( \{U_j\}_{j \in \mathbb{N}} \) of open sets in \( \Sigma_t \) is called an \( \text{in-approximation} \) of \( K \) in \( \Sigma_t \) if the following are satisfied:

(i) \( U_j \) (\( j \in \mathbb{N} \)) are uniformly bounded,
(ii) \( U_j \subset U_{j+1}^\ast \) for every \( j \in \mathbb{N} \), and
(iii) \( U_j \to K \) as \( j \to \infty \) in the following sense: If \( \xi_i \in U_i \) for all \( i \in \mathbb{N} \) and \( \xi_j \to \xi \) as \( j \to \infty \) for some \( \xi \in \Sigma_t \), then \( \xi \in K \).

The second main result of this paper is then formulated as follows.

**Theorem 1.3.** Assume (1.2) and \( K \subset \Sigma_t \). Let \( \{U_j\}_{j \in \mathbb{N}} \) be an \( \text{in-approximation} \) of \( K \) in \( \Sigma_t \), and let \( v \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) be a \( \text{piecewise } C^1 \) map satisfying
\[
\nabla v \in U_1 \quad \text{a.e. in } \Omega.
\]
Then for each \( \epsilon > 0 \), there exists a map \( u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \) such that
\[
\left\{
\begin{array}{l}

abla u \in K \quad \text{a.e. in } \Omega, \\
u = v \quad \text{on } \partial\Omega, \\
\|u - v\|_{L^\infty(\Omega)} < \epsilon.
\end{array}
\right.
\]

The first application of our results concerns the Dirichlet problem of the vectorial eikonal equation
\[
(1.3) \quad \left\{
\begin{array}{l}
|\nabla u| = 1 \quad \text{a.e. in } \Omega, \\
u = v_{\eta,\gamma} \quad \text{on } \partial\Omega,
\end{array}
\right.
\]
where \( \eta \in \mathbb{M}^{m \times n} \), \( \gamma \in \mathbb{R}^m \) and \( v_{\eta,\gamma}(x) := \eta x + \gamma \) (\( x \in \Omega \)). Here, we look for solutions \( u \) in the space \( v_{\eta,\gamma} + W^{1,\infty}_0(\Omega; \mathbb{R}^m) \). If \( u \) is a solution to problem
the vectorial case

(1.3), then

$$|\eta|_{\Omega} = \int_{\Omega} |\eta dx| = \int_{\Omega} \nabla u(x) dx \leq |\Omega|;$$

so $|\eta| \leq 1$. Thus there is no solution to (1.3) if $|\eta| > 1$. If $|\eta| = 1$, we have the trivial solution $u = v_{\eta,\gamma}$ to (1.3). So we assume $|\eta| < 1$. In case of $m = n = 1$ with $\Omega = (0, 1)$, one can trivially construct infinitely many solutions $u$ to (1.3) whose graph has slopes $\pm 1$ a.e. in $\Omega$, left-end point $(0, \gamma)$ and right-end point $(1, \eta + \gamma)$. Motivated by this simplest case, we may pose a question: For the vectorial case $m, n \geq 2$, when $\eta^\pm \in M^{m \times n}$ are two distinct matrices with $|\eta|^2 = 1$, is there a solution $u$ to (1.3) which assumes only the two gradient values $\eta^\pm$ a.e. in $\Omega$? The answer is negative when rank($\eta^+ - \eta^-$) $\geq 2$ due to the rigidity of the two gradient problem [2]. A partially positive answer is available when rank($\eta^+ - \eta^-$) $= 1$ and $\eta \in (\eta^+, \eta^-)$. In this case, one can employ either the convex integration method with an in-approximation scheme [20] or the Baire category method [10] to obtain infinitely many solutions $u$ to (1.3) such that dist($\nabla u, \{\eta^+, \eta^\pm\}$) $< \epsilon$ a.e. in $\Omega$, for any given $\epsilon > 0$. We can even impose suitable linear constraints as follows.

Theorem 1.4. Suppose $\eta \in M^{m \times n}$, $|\eta| < 1$, $\gamma \in \mathbb{R}^m$, $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$, $a \neq 0$, $Lb \neq 0$, $L(a \otimes b) = 0$ and $\epsilon > 0$. Then there are infinitely many maps $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ satisfying

$$\begin{cases}
|\nabla u| = 1 & \text{a.e. in } \Omega, \\
L(\nabla u) = t & \text{a.e. in } \Omega, \\
\text{dist}(\nabla u, \{\eta^\pm\}) < \epsilon & \text{a.e. in } \Omega, \\
u = v_{\eta,\gamma} & \text{on } \partial \Omega, \\
\|u - v_{\eta,\gamma}\|_{L^\infty(\Omega)} < \epsilon,
\end{cases}
$$

(1.4)

where $v_{\eta,\gamma}(x) := \eta x + \gamma$ ($x \in \Omega$), $t := L(\eta)$, and $s^+ > 0 > s^-$ are the unique numbers, with $\eta^\pm_{a \otimes b} := \eta + s^\pm a \otimes b$, such that $|\eta^\pm_{a \otimes b}| = 1$.

Note that the constraint $L(\nabla u) = t$ (i.e., $\nabla u \in \Sigma_t$) restricts the selection of a rank-one direction $a \otimes b$ for lamination as $L(a \otimes b) = L(a \otimes b) = 0$. This is inevitable in the convex integration method since the gradient of a map involved in approximation should always stay in the manifold of constraint $\Sigma_t$. As an application of Theorem 1.3, Theorem 1.4 can be proved directly by constructing a simple in-approximation in $\Sigma_t$ when $m \geq n \geq 2$ and the linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is injective. This additional hypothesis arises due to the general feature of a differential inclusion in Theorem 1.3 that does not single out a principal rank-one direction $a \otimes b$ for lamination. In Section 5, we explain the use of such an in-approximation for the special case of Theorem 1.4 in terms of Theorem 1.3 and also provide the complete proof of Theorem 1.4 under the Baire category framework.
The other application focuses on a $T_1$-configuration (see [21] for precise definition). Consider the set $K \subset M^{2\times 2}_{\text{diag}}$ consisting of the four matrices

\[(1.5) \quad A_1 = -A_3 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = -A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \]

where $M^{2\times 2}_{\text{diag}}$ denotes the space of $2 \times 2$ diagonal matrices. Such a set $K$ was discovered independently by [26, 11, 4, 28] as an example of a compact set $K$ for which $K^{lc} \neq K^{rc}$ and has found striking applications for constructing wild solutions in elliptic system [21], parabolic system [19], porous media equation [8] and active scalar equations [27]. Actually, it is easy to check that $K^{rc}$ contains the segments $[A_1, J_2], [A_2, J_3], [A_3, J_4], [A_4, J_1]$ and the convex hull of $\{J_1, J_2, J_3, J_4\}$, where $J_1 = -J_3 = \text{diag}(-1, -1)$ and $J_2 = -J_4 = \text{diag}(1, -1)$. On the other hand, since $K$ has no rank-one connection, we simply have $K^{lc} = K$. By the same reason, the differential inclusion $\nabla u \in K$ only admits the trivial solutions $\nabla u = A_i$ ($i = 1, 2, 3, 4$) due to the rigidity of the four gradient problem [6]. Regardless of such rigidity, it is still possible to have the gradient $\nabla u$ concentrated near the matrices $A_1, A_2, A_3, A_4$ by a nontrivial map $u \in \eta \mathcal{L} + W^{1,\infty}(\Omega; \mathbb{R}^2)$ if $\eta \in K^{rc}$ (see [20, Corollary 1.5]).

We can slightly improve this corollary by imposing linear constraints as follows.

**Corollary 1.5.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary, and let $k \in \mathbb{R} \setminus \{0\}$ and $L = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}$. Let $K \subset M^{2\times 2}_{\text{diag}}$ be the set consisting of the four matrices in (1.5), and let $\eta \in K^{rc}, \gamma \in \mathbb{R}^2$ and $\epsilon > 0$. Then there exists a map $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

\[
\begin{cases}
\text{dist}(\nabla u, K) < \epsilon & \text{a.e. in } \Omega, \\
\mathcal{L}(\nabla u) = 0 & \text{a.e. in } \Omega, \\
u = v_{\eta, \gamma} & \text{on } \partial \Omega, \\
\|u - v_{\eta, \gamma}\|_{L^\infty(\Omega)} < \epsilon,
\end{cases}
\]

where $v_{\eta, \gamma}(x) := \eta x + \gamma \ (x \in \Omega)$.

Here the constraint $\mathcal{L}(\nabla u) = 0$ also reads as $\partial_{x_1} u^2 + k \partial_{x_2} u^1 = 0$ or as $\nabla u \in \Sigma_0$. Observe that the dimensions of $M^{2\times 2}_{\text{diag}} \subsetneq \Sigma_0 \subsetneq M^{2\times 2}$ are 2, 3 and 4, respectively. Targeting the set $K \subset M^{2\times 2}_{\text{diag}}$, the gradient $\nabla u$ of our approximate solution $u$ to the differential inclusion $\nabla u \in K$ may go beyond the plane $M^{2\times 2}_{\text{diag}}$ but always stays in the 3-dimensional manifold $\Sigma_0$. Note also that $\Sigma_0$ is the space of $2 \times 2$ symmetric matrices for $k = -1$ and that of $2 \times 2$ skew-symmetric matrices if $k = 1$.

The rest of the paper is organized as follows. Section 2 concerns a functional tool, Theorem 2.2 for the passage from lamination convex hull to rank-one convex hull in the manifold of constraint $\Sigma_t$. In Section 3 we equip with the main tool, Theorem 3.1 for rank-one smooth approximation under a general linear constraint that eventually leads, with the help
of Theorem 2.2 to Lemma 3.3 which is a precursor to the main results of the paper, Theorems 1.1 and 1.3. Then the proof of Theorems 1.1 and 1.3 is provided in Section 4. Section 5 finishes the proof of the applications, Theorem 1.4 and Corollary 1.5. Lastly, the proof of Theorem 3.1 is included in Section 6.

In closing this section, we add some notations. For a vector $a \in \mathbb{R}^n$, we write $|a| = (\sum_i a_i^2)^{1/2}$ for its Euclidean norm. For a matrix $\xi \in \mathbb{M}^{m \times n}$, we denote by $|\xi| = (\sum_{i,j} \xi_{ij}^2)^{1/2}$ the Hilbert-Schmidt norm of $\xi$. For a measurable set $E \subset \mathbb{R}^n$, its Lebesgue measure is denoted by $|E|$. Some other notations will be introduced as we go along the paper if necessary.

2. **Rank-one convex functions and hulls**

This section prepares a powerful tool that enables us to handle rank-one convex hulls instead of lamination convex hulls, which can be strictly smaller than the former ones. Our version of such a tool, Theorem 2.2 for the manifold of constraint $\Sigma_t$ originates from [20, Theorem 3.1] that is dealing with the case of a constraint on a minor of $\nabla u$ and unconstrained case and that was motivated by and generalized from a result of [23]. We thus closely follow the exposition and relevant proofs from Section 3 of [20] but add more details for the reader’s convenience.

We first introduce many definitions. Let $\mathcal{O}$ be an open set in $\mathbb{M}^{m \times n}$ or in $\Sigma_t$. A function $f : \mathcal{O} \to \mathbb{R}$ is called rank-one convex if, for all $\xi_1, \xi_2 \in \mathcal{O}$ with $[\xi_1, \xi_2] \subset \mathcal{O}$ and $\text{rank}(\xi_1 - \xi_2) = 1$, we have

$$f(\lambda \xi_1 + (1 - \lambda) \xi_2) = \lambda f(\xi_1) + (1 - \lambda) f(\xi_2)$$

for all $\lambda \in [0, 1]$.

We denote by $\mathcal{P}$ the set of all probability (Borel) measures on $\mathbb{M}^{m \times n}$ with compact support. For a compact set $K \subset \mathbb{M}^{m \times n}$, let $\mathcal{P}(K)$ denote the set of all $\nu \in \mathcal{P}$ with supp($\nu$) $\subset K$. For each $\nu \in \mathcal{P}$, we write its center of mass as $\bar{\nu} = \int_{\mathbb{M}^{m \times n}} \xi \, d\nu(\xi) \in \mathbb{M}^{m \times n}$.

Let $\nu \in \mathcal{P}$. For a continuous function $f : \mathbb{M}^{m \times n} \to \mathbb{R}$, we write $\langle \nu, f \rangle = \int_{\mathbb{M}^{m \times n}} f(\xi) \, d\nu(\xi) \in \mathbb{R}$. We say that $\nu$ is a laminate if $\langle \nu, f \rangle \geq f(\bar{\nu})$ for every rank-one convex function $f : \mathbb{M}^{m \times n} \to \mathbb{R}$. For a compact set $K \subset \mathbb{M}^{m \times n}$, we define

$$\mathcal{M}^{rc}(K) = \{ \nu \in \mathcal{P}(K) \mid \nu \text{ is a laminate} \}.$$

By [23], it has been well known that $K^{rc} = \{ \bar{\nu} \mid \nu \in \mathcal{M}^{rc}(K) \}$.

Let $\xi \in \mathbb{M}^{m \times n}$, and let $\delta_\xi$ denote the Dirac mass at $\xi$. As $\text{supp}(\delta_\xi) = \{ \xi \}$, we have $\delta_\xi \in \mathcal{P}$. Also, $\delta_\xi = \xi$. Thus for any continuous function $f : \mathbb{M}^{m \times n} \to \mathbb{R}$, we have $\langle \delta_\xi, f \rangle = f(\xi) = f(\bar{\delta}_\xi)$; hence $\delta_\xi$ is a laminate.

Let $\mathcal{O}$ be an open set in $\Sigma_t$. We define a class $\mathcal{L}(\mathcal{O})$ of laminates in $\mathcal{P}$, called laminates of finite order in $\mathcal{O}$, inductively as follows:

(i) For each $\xi \in \mathcal{O}$, we have $\delta_\xi \in \mathcal{L}(\mathcal{O})$, called a laminate of order 0.

(ii) If $k \in \mathbb{N}$, $\lambda_1, \cdots, \lambda_k > 0$, $\sum_{j=1}^k \lambda_j = 1$, $\xi_1, \cdots, \xi_k \in \mathcal{O}$ are pairwise distinct, $\sum_{j=1}^k \lambda_j \delta_{\xi_j} \in \mathcal{L}(\mathcal{O})$ is a laminate of order $k - 1$, and
\[ \xi_k = s\eta_1 + (1 - s)\eta_2 \] for some \( 0 < s < 1 \) and some rank-one segment \([\eta_1, \eta_2] \subseteq \mathcal{O}\) with \( \eta_i \neq \xi_j \) \((i = 1, 2, j = 1, \ldots, k - 1)\), then
\[
\sum_{j=1}^{k-1} \lambda_j \delta_{\xi_j} + s\lambda_k \delta_{\eta_1} + (1 - s)\lambda_k \delta_{\eta_2} \in \mathcal{L}(\mathcal{O}),
\]
called a laminate of order \( k \).

From definition, it follows that \( \langle \nu, f \rangle \geq f(\bar{\nu}) \) for each \( \nu \in \mathcal{L}(\mathcal{O}) \) and each rank-one convex hull \( f : \mathcal{O} \to \mathbb{R} \).

Let \( K \subset M^{m \times n} \) be a compact set. From the definition of \( K^{rc} \), for any \( \xi \in M^{m \times n} \), we have that \( \xi \notin K^{rc} \) if and only if \( f(\xi) > 0 \) for some rank-one convex function \( f : M^{m \times n} \to \mathbb{R} \) with \( f \leq 0 \) on \( K \). Now, let \( K \) be a compact subset of \( \Sigma_t \). The rank-one convex hull \( K^{rc, \Sigma_t} \) of \( K \) relative to \( \Sigma_t \) is a subset of \( \Sigma_t \) defined as follows: For each \( \xi \in \Sigma_t \), \( \xi \notin K^{rc, \Sigma_t} \) if and only if \( f(\xi) > 0 \) for some rank-one convex function \( f : \Sigma_t \to \mathbb{R} \) with \( f \leq 0 \) on \( K \). Then the rank-one convex hull \( E^{rc, \Sigma_t} \) of a set \( E \subset \Sigma_t \) relative to \( \Sigma_t \) is defined to be
\[
E^{rc, \Sigma_t} = \bigcup \{K^{rc, \Sigma_t} | K \subset E, K \text{ is compact}\}.
\]

With this definition, the rank-one convex hull \( V^{rc, \Sigma_t} \) of any open set \( V \) in \( \Sigma_t \) relative to \( \Sigma_t \) is also open in \( \Sigma_t \). Another simple fact that is needed later is as follows.

**Proposition 2.1.** Let \( K \) be a compact subset of \( \Sigma_t \). Then \( K^{rc, \Sigma_t} \) and \( K^{rc} \) are both compact, and
\[
K^{rc, \Sigma_t} \subset K^{rc} \subset \Sigma_t.
\]

**Proof.** Compactness of \( K^{rc, \Sigma_t} \) and \( K^{rc} \) easily follows from the definitions.

To show that \( K^{rc} \subset \Sigma_t \), choose an open ball \( B \) in \( M^{m \times n} \) with \( K \subset B \), and define \( f(\xi) = \text{dist}(\xi, B \cap \Sigma_t) \) for all \( \xi \in M^{m \times n} \). As \( B \cap \Sigma_t \) is convex and compact, the function \( f : M^{m \times n} \to \mathbb{R} \) is (rank-one) convex and satisfies that \( f(\xi) > 0 \) for all \( \xi \in M^{m \times n} \setminus (B \cap \Sigma_t) \). Since \( f = 0 \) on \( K \), we have from definition that \( \xi \notin K^{rc} \) for all \( \xi \in M^{m \times n} \setminus (B \cap \Sigma_t) \); thus \( K^{rc} \subset \Sigma_t \).

Next, let \( \xi \in \Sigma_t \setminus K^{rc} \). By definition, we have \( g(\xi) > 0 \) for some rank-one convex function \( g : M^{m \times n} \to \mathbb{R} \) with \( g \leq 0 \) on \( K \). As \( g|_{\Sigma_t} : \Sigma_t \to \mathbb{R} \) is rank-one convex, we have \( \xi \notin K^{rc, \Sigma_t} \). Thus \( K^{rc, \Sigma_t} \subset K^{rc} \).

We now state the main result of this section as follows.

**Theorem 2.2.** Let \( K \) be a compact subset of \( \Sigma_t \). Then \( K^{rc} = K^{rc, \Sigma_t} \). Let \( \nu \in M^{rc}(K) \), and let \( \mathcal{O} \) be an open set in \( \Sigma_t \) containing \( K^{rc} \). Then there exists a sequence \( \nu_j \in \mathcal{L}(\mathcal{O}) \) with \( \nu_j \to \bar{\nu} \) that converges weakly* to \( \nu \) in \( \mathcal{P} \).

An immediate consequence of this theorem is as follows.

**Corollary 2.3.** If \( V \) is an open set in \( \Sigma_t \), then \( V^{rc} = V^{rc, \Sigma_t} \), and \( V^{rc} \) is open in \( \Sigma_t \).

From this corollary, we have the following.

**Corollary 2.4.** Let \( U \) be a bounded open set in \( \Sigma_t \). Then for any compact set \( K \subset U^{rc} \), there exists an open set \( V \) in \( \Sigma_t \) with \( \bar{V} \subset U \) such that
\[
K \subset V^{rc} \quad \text{and} \quad \overline{(V^{rc})} \subset U^{rc}.
\]
Proof. For each \( \epsilon > 0 \), let
\[
U_\epsilon = \{ \xi \in U \mid \text{dist}(\xi, \partial \Omega) > \epsilon \}.
\]
By the definition of \( U^{rc} \), we easily see that
\[
U^{rc} = \cup_{\epsilon>0}(U_\epsilon)^{rc} = \cup_{\epsilon>0}U^{rc}.
\]
Let \( K \subset U^{rc} \) be any compact set; then we have \( K \subset U^{rc}_{\epsilon_0} \) for some \( \epsilon_0 > 0 \) since \( \{U^{rc}_\epsilon\}_{\epsilon>0} \) is an open covering for \( K \) by Corollary 2.3. We set \( V = U^{rc}_{\epsilon_0} \).

By definition, we now have
\[
(V^{rc}) \subset (V)^{rc} \subset U^{rc}.
\]

We now pay attention to the proof of Theorem 2.2 that requires two ingredients. The first one is on the representation of the rank-one convex envelope of a continuous function in terms of laminates of finite order.

Lemma 2.5. Let \( \Omega \) be an open set in \( \Sigma_1 \), and let \( f : \Omega \to \mathbb{R} \) be a continuous function. Let \( R_{\Omega} f : \Omega \to \mathbb{R} \cup \{-\infty\} \) be the function defined by
\[
R_{\Omega} f(\xi) = \sup \{ g(\xi) \mid g : \Omega \to \mathbb{R} \text{ is rank-one convex, } g \leq f \text{ in } \Omega \}
\]
for all \( \xi \in \Omega \). Assume that there exists a rank-one convex function \( g_0 : \Omega \to \mathbb{R} \) with \( g_0 \leq f \) in \( \Omega \). Then for each \( \xi \in \Omega \), we have
\[
R_{\Omega} f(\xi) = \inf \{ \langle \nu, f \rangle \mid \nu \in \mathcal{L}(\Omega), \bar{\nu} = \xi \} \in \mathbb{R}.
\]

Proof. Let \( \tilde{f} : \Omega \to \mathbb{R} \cup \{-\infty\} \) be the function given by
\[
\tilde{f}(\xi) = \inf \{ \langle \nu, f \rangle \mid \nu \in \mathcal{L}(\Omega), \bar{\nu} = \xi \} \quad \forall \xi \in \Omega.
\]
We have to show that \(-\infty < \tilde{f} = R_{\Omega} f \) in \( \Omega \).

Let \( g : \Omega \to \mathbb{R} \) be any rank-one convex function with \( g \leq f \) in \( \Omega \). Fix any \( \xi \in \Omega \), and let \( \nu \in \mathcal{L}(\Omega) \) and \( \bar{\nu} = \xi \). Then \( \langle \nu, f \rangle \geq \langle \nu, g \rangle \geq g(\bar{\nu}) = g(\xi) \). Taking supremum on such \( g \)'s and infimum on such \( \nu \)'s, we have \(-\infty < \tilde{f} = R_{\Omega} f \) from the existence of the function \( g_0 \). In particular, \( \tilde{f} \) is real-valued in \( \Omega \).

Let us now check that \( \tilde{f} : \Omega \to \mathbb{R} \) is rank-one convex and that \( \tilde{f} \leq f \) in \( \Omega \). Once these are done, it follows from the definition of \( R_{\Omega} f \) that \( f \leq R_{\Omega} f \) in \( \Omega \); thus \( R_{\Omega} f = \tilde{f} \) in \( \Omega \), and the proof is complete.

We now turn to the remaining assertions. Let \( \xi \in \Omega \). As \( \delta_\xi \in \mathcal{L}(\Omega) \) and \( \bar{\delta}_\xi = \xi \), the definition of \( \tilde{f} \) implies \( \tilde{f}(\xi) \leq \langle \delta_\xi, f \rangle = f(\xi) \); thus, \( \tilde{f} \leq f \) in \( \Omega \). It remains to show that \( \tilde{f} : \Omega \to \mathbb{R} \) is rank-one convex. Let \([\xi_1, \xi_2] \subset \Omega \) be any rank-one segment.

Let \( 0 < s < 1 \) and \( \xi = s\xi_1 + (1-s)\xi_2 \in \Omega \). Let \( \mu, \nu \in \mathcal{L}(\Omega) \) be such that \( \bar{\mu} = \xi_1, \bar{\nu} = \xi_2 \). As \( \text{rank}(\xi_1 - \xi_2) = 1 \), it is easy to see (by double induction) that \( s\mu + (1-s)\nu \in \mathcal{L}(\Omega) \) and \( s\mu + (1-s)\nu = \xi \). So, by the definition of \( \tilde{f} \), we get \( \tilde{f}(\xi) \leq \langle s\mu + (1-s)\nu, f \rangle = s\langle \mu, f \rangle + (1-s)\langle \nu, f \rangle \).

Taking infimum on such \( \mu \)'s and \( \nu \)'s, we get \( \tilde{f}(s\xi_1 + (1-s)\xi_2) = \tilde{f}(\xi) \leq s\tilde{f}(\xi_1) + (1-s)\tilde{f}(\xi_2) \). Thus, \( \tilde{f} : \Omega \to \mathbb{R} \) is rank-one convex. □
The other lemma for the proof of Theorem 2.2 is stated as follows, which may not be so simple to verify.

**Lemma 2.6.** Let $K$ be a compact subset of $\Sigma_t$, and let $O$ be an open set in $\Sigma_t$ containing $\bar{K} := K^{rc,\Sigma_t}$. Let $f : O \to \mathbb{R}$ be a rank-one convex function. Then for each $\epsilon > 0$, there exists a rank-one convex function $F : M^{m \times n} \to \mathbb{R}$ such that $|F - f| < \epsilon$ on $\bar{K}$.

To prove this lemma, we need some auxiliary results. We begin with a simple observation on rank-one convex functions.

**Lemma 2.7.** Let $U, V$ be open sets in $\Sigma_t$ with $\bar{V} \subset U$, and let $f_1 : \Sigma_t \to \mathbb{R}$ and $f_2 : U \to \mathbb{R}$ be rank-one convex functions such that $f_2 \geq f_1$ in $V$ and that $f_1 = f_2$ on $\partial |\Sigma_t|$. Let $f : \Sigma_t \to \mathbb{R}$ be the function given by

$$f(\xi) = \begin{cases} f_2(\xi), & \xi \in V, \\ f_1(\xi), & \xi \in \Sigma_t \setminus V. \end{cases}$$

Then $f : \Sigma_t \to \mathbb{R}$ is rank-one convex. Moreover, the same result holds when $\Sigma_t$ is replaced by $M^{m \times n}$.

As a precursor of this lemma, we first deal with its one-dimensional version.

**Lemma 2.8.** Let $\{I_j\}_{j \in J}$ be a countable collection of disjoint open intervals in $(0,1)$. Let $f_1 : [0,1] \to \mathbb{R}$ be a convex function, and let $f_2 : \cup_{j \in J} I_j \to \mathbb{R}$ be a function such that it is convex on each interval $I_j$, $f_2 \geq f_1$ in $\cup_{j \in J} I_j$, and $f_1 = f_2$ on $\cup_{j \in J} \partial I_j$. Define

$$f(x) = \begin{cases} f_2(x), & x \in \cup_{j \in J} I_j, \\ f_1(x), & x \in [0,1] \setminus \cup_{j \in J} I_j. \end{cases}$$

Then $f : [0,1] \to \mathbb{R}$ is convex.

**Proof.** Let $x_0 < x_1$ be any two numbers in $[0,1]$, and let $\bar{x} = \lambda x_0 + (1-\lambda)x_1$, where $0 < \lambda < 1$ is any fixed number. We have to show that

$$f(\bar{x}) \leq \lambda f(x_0) + (1-\lambda)f(x_1).$$

If $\bar{x} \in [0,1] \setminus \cup_{j \in J} I_j$, then

$$f(\bar{x}) = f_1(\bar{x}) \leq \lambda f_1(x_0) + (1-\lambda)f_1(x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1),$$

from the definition of $f$.

Next, assume $\bar{x} \in \cup_{j \in J} I_j$. Then there is a unique index $j_0 \in J$ such that $\bar{x} \in I_{j_0}$ with $y_0$ and $y_1$ denoting the left- and right-end points of $I_{j_0}$, respectively. If $x_0 < y_0$ and $x_1 \leq y_1$, then

$$f(\bar{x}) = f_2(\bar{x}) \leq \lambda f_2(x_0) + (1-\lambda)f_2(x_1) = \lambda f(x_0) + (1-\lambda)f(x_1).$$

Suppose $x_0 < y_0$ and $x_1 \leq y_1$. As $\bar{x} \in I_{j_0}$, we have $\bar{x} = \mu y_0 + (1-\mu)x_1$ and $y_0 = \nu x_0 + (1-\nu)x_1$ for some $0 < \mu, \nu < 1$. So

$$\lambda x_0 + (1-\lambda)x_1 = \bar{x} = \mu y_0 + (1-\mu)x_1;$$
hence $\mu\nu = \lambda$, and we get
\[
  f(\bar{x}) = f_2(\bar{x}) = f_2(\mu y_0 + (1 - \mu)x_1) \leq \mu f_2(y_0) + (1 - \mu)f_2(x_1)
\]
\[
  = \mu f_1(y_0) + (1 - \mu)f_2(x_1) = \mu f_1(\nu x_0 + (1 - \nu)x_1) + (1 - \mu)f_2(x_1)
\]
\[
  \leq \mu \nu f_1(x_0) + \mu(1 - \nu)f_1(x_1) + (1 - \mu)f_2(x_1)
\]
\[
  \leq \lambda f_1(x_0) + (1 - \lambda)f_2(x_1) \leq \lambda f(x_0) + (1 - \lambda)f(x_1).
\]

The other cases can be handled similarly; we omit these.

In any case, inequality (2.1) holds, and the proof is complete. \hfill \Box

We now finish the proof of Lemma 2.7.

**Proof of Lemma 2.7.** Let $[\xi_1, \xi_2]$ be any rank-one segment in $\Sigma_t$; that is, $\xi_1, \xi_2 \in \Sigma_t$ and $\text{rank}(\xi_1 - \xi_2) = 1$. It suffices to show that the function $s \mapsto f(s\xi_1 + (1 - s)\xi_2)$ ($s \in [0, 1]$) is convex.

If $[\xi_1, \xi_2] \subset V$ or $[\xi_1, \xi_2] \subset \Sigma_t \setminus V$, there is nothing to show. So we assume both inclusions do not hold; that is,

$$[\xi_1, \xi_2] \cap V \neq \emptyset \quad \text{and} \quad [\xi_1, \xi_2] \cap (\Sigma_t \setminus V) \neq \emptyset.$$

Consider the 1-1 map $s \mapsto s\xi_1 + (1 - s)\xi_2 =: H(s)$ from $[0, 1]$ onto $[\xi_1, \xi_2]$. Let $\{I_j\}_{j \in J}$ be the countable collection of disjoint open intervals in $(0, 1)$ such that $H(\cup_{j \in J} I_j) = (\xi_1, \xi_2) \cap V$. We can now apply Lemma 2.8 to the function $f \circ H : [0, 1] \rightarrow \mathbb{R}$ to conclude that it is convex.

One can repeat the same proof for the unconstrained case by replacing $\Sigma_t$ with $M^{m \times n}$. \hfill \Box

The following lemma is on the identification of the rank-one convex hull of a compact set as the zero level set of some nonnegative rank-one convex function.

**Lemma 2.9.** Let $K$ be a compact subset of $\Sigma_t$, and let $\tilde{K} = K^{rc, \Sigma_t}$. Then there exists a nonnegative rank-one convex function $g : \Sigma_t \rightarrow \mathbb{R}$ such that

$$\tilde{K} = \{ \xi \in \Sigma_t | g(\xi) = 0 \}.$$

The same result remains true when $\Sigma_t$ and $K^{rc, \Sigma_t}$ are replaced by $M^{m \times n}$ and $K^{rc}$, respectively.

**Proof.** For each $r > 0$, set $\Sigma_{t, r} = \{ \xi \in \Sigma_t | |\xi| < r \}$. Choose an $R > 0$ so large that $\tilde{K} \subset \Sigma_{t, R/2}$. Define a function $g_1 : \Sigma_{t, R} \rightarrow \mathbb{R}$ by

$$g_1(\xi) = \sup \{ f(\xi) | f : \Sigma_{t, R} \rightarrow \mathbb{R} \text{ is rank-one convex, } f \leq \text{dist}(\cdot, K) \text{ in } \Sigma_{t, R} \}$$

for all $\xi \in \Sigma_{t, R}$. As the zero function $0 \leq \text{dist}(\cdot, K)$ in $\Sigma_{t, R}$ is rank-one convex, we have $g_1 \geq 0$ in $\Sigma_{t, R}$. It is also easy to see that $g_1 : \Sigma_{t, R} \rightarrow \mathbb{R}$ is rank-one convex and that $\tilde{K} \subset \{ \xi \in \Sigma_{t, R} | g_1(\xi) = 0 \}$.

Let us check that $g_1 > 0$ in $\Sigma_{t, R} \setminus \tilde{K}$. To see this, let $\xi \in \Sigma_{t, R} \setminus \tilde{K}$. By definition, $\alpha := \tilde{f}(\xi) > 0$ for some rank-one convex function $\tilde{f} : \Sigma_t \rightarrow \mathbb{R}$.
with \( \tilde{f} \leq 0 \) on \( K \). We now verify that

\[
0 < M := \sup_{\eta \in \Sigma_{t,R} \setminus K} \frac{\tilde{f}(\eta) - \frac{\alpha}{2}}{\text{dist}(\eta, K)} < \infty.
\]

Note first that

\[
0 < \frac{\alpha}{2 \cdot \text{dist}(\xi, K)} = \frac{\tilde{f}(\xi) - \frac{\alpha}{2}}{\text{dist}(\xi, K)} \leq M.
\]

It thus remains to show that \( M < \infty \). Suppose on the contrary that \( M = \infty \). Then we can choose a sequence \( \{\eta_j\}_{j \in \mathbb{N}} \) in \( \Sigma_{t,R} \setminus K \) so that

\[
(2.3) \quad \frac{\tilde{f}(\eta_j) - \frac{\alpha}{2}}{\text{dist}(\eta_j, K)} \to \infty \quad \text{as} \quad j \to \infty.
\]

Passing to a subsequence if necessary, we can assume \( \eta_j \to \eta_0 \) for some \( \eta_0 \in \Sigma_{t,R} \). If \( \eta_0 \not\in K \), then

\[
\frac{\tilde{f}(\eta_j) - \frac{\alpha}{2}}{\text{dist}(\eta_j, K)} \to \frac{\tilde{f}(\eta_0) - \frac{\alpha}{2}}{\text{dist}(\eta_0, K)} \in \mathbb{R} \quad \text{as} \quad j \to \infty,
\]

which is a contradiction to \((2.3)\). If \( \eta_0 \in K \), then \( \tilde{f}(\eta_0) \leq 0 \), and so

\[
\frac{\tilde{f}(\eta_j) - \frac{\alpha}{2}}{\text{dist}(\eta_j, K)} \to -\infty \quad \text{as} \quad j \to \infty,
\]

which is also a contradiction to \((2.3)\). Thus \((2.2)\) holds, and this implies that

\[
\frac{1}{M}(\tilde{f}(\eta) - \frac{\alpha}{2}) \leq \text{dist}(\eta, K) \quad \forall \eta \in \Sigma_{t,R}.
\]

As \( \frac{1}{M}(\tilde{f} - \frac{\alpha}{2}) : \Sigma_{t,R} \to \mathbb{R} \) is rank-one convex, it now follows from the definition of \( g_1 \) that

\[
\frac{1}{M}(\tilde{f}(\eta) - \frac{\alpha}{2}) \leq g_1(\eta) \quad \forall \eta \in \Sigma_{t,R}.
\]

In particular,

\[
0 < \frac{\alpha}{2M} = \frac{1}{M}(\tilde{f}(\xi) - \frac{\alpha}{2}) \leq g_1(\xi).
\]

Therefore, \( g_1 > 0 \) in \( \Sigma_{t,R} \setminus \bar{K} \).

Let \( g : \Sigma_t \to \mathbb{R} \) be the function defined by

\[
g(\xi) = \begin{cases} 
\max\{g_1(\xi), 12|\xi| - 9R\}, & \xi \in \Sigma_{t,R}, \\
12|\xi| - 9R, & \xi \in \Sigma_t, \ |\xi| \geq R.
\end{cases}
\]

Lastly, we check that \( g \) is the desired function. Clearly, \( g \geq 0 \) in \( \Sigma_t \). Since \( g_1(\xi) \leq \text{dist}(\xi, K) < \frac{3M}{2} \) for all \( \xi \in \Sigma_{t,R} \), we have \( g(\xi) = 12|\xi| - 9R \) for all \( \xi \) in some neighborhood of \( \{\eta \in \Sigma_t \mid |\eta| = R\} \) in \( \Sigma_t \). Thus it follows from Lemma \(2.7\) that \( g : \Sigma_t \to \mathbb{R} \) is rank-one convex. Next, we verify that

\[
\bar{K} = \{\xi \in \Sigma_t \mid g(\xi) = 0\}.
\]

As \( \bar{K} \subset \Sigma_{t,R/2} \), it follows from the above fact that \( g_1(\xi) > 0 \) for all \( \xi \in \Sigma_t \) with \( R/2 \leq |\xi| < R \), and so \( g(\xi) > 0 \) for all \( \xi \in \Sigma_t \) with \( |\xi| \geq R/2 \). This implies that

\[
\{\xi \in \Sigma_t \mid g(\xi) = 0\} = \{\xi \in \Sigma_t \mid |\xi| < R/2, \ g_1(\xi) = 0\} \subset \bar{K}.
\]
To show the reverse inclusion, let $\xi \in \bar{K}$. Let $f : \Sigma_{\ell,R} \to \mathbb{R}$ be any rank-one convex function such that $f \leq \text{dist}(\cdot, K)$ in $\Sigma_{\ell,R}$. If we can show that $f(\xi) \leq 0$, then the definition of $g_1$ implies that $g(\xi) = g_1(\xi) = 0$, and the proof is complete for the case of the linear constraint. Suppose on the contrary that $f(\xi) > 0$. Define

$$\tilde{f}(\eta) = \begin{cases} \max\{f(\eta), 12|\eta| - 9R\}, & \eta \in \Sigma_{\ell,R}, \\ 12|\eta| - 9R, & \eta \in \Sigma_{\ell}, |\eta| \geq R. \end{cases}$$

Then $\tilde{f} : \Sigma_\ell \to \mathbb{R}$ is rank-one convex as above. Also, $\tilde{f}(\xi) = f(\xi) > 0$. As $\tilde{f} \leq 0$ on $K$, we now have $\xi \notin \bar{K}$; a contradiction. Thus $f(\xi) \leq 0$.

For the unconstrained case, one can repeat the same proof with $\Sigma_\ell$, $\Sigma_{\ell,r}$ ($r > 0$) and $K^{rc,\Sigma_\ell}$ replaced by $M^{m \times n}$, $B_r = \{\xi \in M^{m \times n} | |\xi| < r\}$ ($r > 0$) and $K^{rc}$, respectively.

Using the previous lemma, we obtain the following.

**Lemma 2.10.** Let $K$ be a compact subset of $\Sigma_\ell$, let $\mathcal{O}$ be an open set in $\Sigma_\ell$ containing $\bar{K} := K^{rc,\Sigma_\ell}$, and let $f : \mathcal{O} \to \mathbb{R}$ be a rank-one convex function. Then there exists a rank-one convex function $F : \Sigma_\ell \to \mathbb{R}$ such that

$$F \equiv f \quad \text{in some neighborhood of } \bar{K} \text{ in } \mathcal{O}.$$ 

The same result holds when $\Sigma_\ell$ and $K^{rc,\Sigma_\ell}$ are replaced by $M^{m \times n}$ and $K^{rc}$, respectively.

**Proof.** We first use Lemma 2.9 to obtain a nonnegative rank-one convex function $g : \Sigma_\ell \to \mathbb{R}$ such that $\bar{K} = \{\xi \in \Sigma_\ell | g(\xi) = 0\}$. Set $m = \min_K f$. Choose a number $c > 0$ so that $m + c > 0$. Define $\tilde{f} = f + c$ in $\mathcal{O}$; then $\min_{\bar{K}} \tilde{f} = m + c > 0$. For each $\xi \in \bar{K}$, we thus can choose an open ball $B_\xi$ in $\Sigma_\ell$ with $\bar{B}_\xi \subset \mathcal{O}$ and center $\xi$ such that $\tilde{f} > 0$ on $\bar{B}_\xi$. As $\bar{K}$ is compact, we can choose finitely many matrices $\xi_1, \ldots, \xi_N \in \bar{K}$ such that $\bar{K} \subset \bigcup_{j=1}^N B_{\xi_j} =: U$; so $\tilde{f} > 0$ on $U \subset \mathcal{O}$.

For each $k \in \mathbb{N}$, let

$$U_k = \{\xi \in \mathcal{O} | \tilde{f}(\xi) > kg(\xi)\},$$

which is open in $\mathcal{O}$, and let $V_k$ be the union of all connected components of $U_k$ that have a nonempty intersection with $\bar{K}$; then $\bar{K} \subset V_k \subset U_k$. For each $\delta > 0$, let $S_\delta = \{\xi \in \Sigma_\ell | \text{dist}(\xi, \bar{K}) < \delta\}$. Fix a $\delta > 0$ so small that $S_\delta \subset U$. As $g > 0$ on $U \setminus S_\delta$, we can choose a number $k \in \mathbb{N}$ so large that

$$k \cdot \min_{U \setminus S_\delta} g \geq \max_{U \setminus S_\delta} \tilde{f},$$

and so $U_k \cap (U \setminus S_\delta) = \emptyset$. Thus we easily see from the definition of $V_k$ that $V_k \subset S_\delta$.

Next, define

$$\tilde{F}(\xi) = \begin{cases} \tilde{f}(\xi), & \xi \in V_k, \\ kg(\xi), & \xi \in \Sigma_\ell \setminus V_k; \end{cases}$$
then, by Lemma 2.7, \( \tilde{F} : \Sigma_t \to \mathbb{R} \) is rank-one convex. Take \( F = \tilde{F} - c \) in \( \Sigma_t \); then \( F : \Sigma_t \to \mathbb{R} \) is rank-one convex and \( F \equiv f \) in \( V_k \), where \( K \subset V_k \subset \mathcal{O} \).

For the unconstrained case, one can repeat the same proof with \( \Sigma_t \) and \( K^{rc, \Sigma_t} \) replaced by \( \mathbb{M}^{m \times n} \) and \( K^{rc} \), respectively.

Note that for each \( \xi \in \mathbb{M}^{m \times n} \), there exists a unique number \( s_\xi \in \mathbb{R} \) such that \( \pi(\xi) := \xi + s_\xi L/|L| \in \Sigma_t \); so \( \xi = \pi(\xi) + t_\xi L/|L| \), where \( t_\xi := -s_\xi \). As the last preparation for the proof of Lemma 2.6, we prove the following lemma.

**Lemma 2.11.** Let \( f : \Sigma_t \to \mathbb{R} \) be a smooth rank-one convex function. For each \( \epsilon > 0 \) and each \( k > 0 \), let \( F_{\epsilon,k} : \mathbb{M}^{m \times n} \to \mathbb{R} \) be the function defined by

\[
F_{\epsilon,k}(\xi) = f(\pi(\xi)) + \epsilon|x|^2 + k|\mathcal{L}(\xi) - t|^2 \quad \forall \xi \in \mathbb{M}^{m \times n}.
\]

Let \( K \) be a compact subset of \( \Sigma_t \). Then for each \( \epsilon > 0 \), there exists a number \( k > 0 \) such that \( F_{\epsilon,k} : U_{\epsilon,k} \to \mathbb{R} \) is rank-one convex, for some open set \( U_{\epsilon,k} \subset \mathbb{M}^{m \times n} \) containing \( K \).

**Proof.** We prove by contradiction; suppose there exists an \( \epsilon > 0 \) such that for each \( k > 0 \), if \( V \) is any open set in \( \mathbb{M}^{m \times n} \) containing \( K \), then \( F_{\epsilon,k} : V \to \mathbb{R} \) is not rank-one convex.

Let \( k \in \mathbb{N} \), and set

\[
V_k = \{ \xi \in \mathbb{M}^{m \times n} \mid \text{dist}(\xi, K) < 1/k \}.
\]

Then \( F_{\epsilon,k} : V_k \to \mathbb{R} \) is not rank-one convex. By the Legendre-Hadamard condition, there exist a matrix \( \eta^k \in V_k \) and vectors \( \lambda^k \in \mathbb{R}^m \) and \( \mu^k \in \mathbb{R}^n \) such that \( |\lambda^k| = |\mu^k| = 1 \) such that

\[
D^2 F_{\epsilon,k}(\eta^k)(\lambda^k \otimes \mu^k, \lambda^k \otimes \mu^k) = \sum_{1 \leq i,j \leq m, 1 \leq \alpha, \beta \leq n} \left| \frac{\partial^2 F_{\epsilon,k}(\xi)}{\partial \xi_{i\alpha} \partial \xi_{j\beta}} \right|_{\xi = \eta^k} \lambda^k \lambda^k \mu^k \mu^k < 0.
\]  

Passing to a subsequence if necessary, we have

\[
\eta^k \rightarrow \eta \quad \text{in} \quad \mathbb{M}^{m \times n}, \quad \lambda^k \rightarrow \lambda \quad \text{in} \quad \mathbb{R}^m \quad \text{and} \quad \mu^k \rightarrow \mu \quad \text{in} \quad \mathbb{R}^n,
\]

for some \( \eta \in K \), \( \lambda \in \mathbb{R}^m \) and \( \mu \in \mathbb{R}^n \) with \( |\lambda| = |\mu| = 1 \).

Set \( g(\xi) = |\mathcal{L}(\xi) - t|^2 \) for all \( \xi \in \mathbb{M}^{m \times n} \). Note here that \( \mathcal{L}(\eta^k + s\lambda^k \otimes \mu^k) = a_k s + b_k \) for all \( s \in \mathbb{R} \), where \( a_k = L \cdot (\lambda^k \otimes \mu^k) \) and \( b_k = L \cdot \eta^k \). So

\[
g(\eta^k + s\lambda^k \otimes \mu^k) = |a_k s + b_k - t|^2 = a_k^2 s^2 + 2a_k (b_k - t)s + (b_k - t)^2.
\]

From this, we get

\[
\frac{d^2}{ds^2} g(\eta^k + s\lambda^k \otimes \mu^k) = 2a_k^2.
\]

Differentiating \( F_{\epsilon,k}(\eta^k + s\lambda^k \otimes \mu^k) \) twice with respect to the variable \( s \) and then letting \( s = 0 \), we obtain from (2.4) and (2.5) that

\[
D^2(f \circ \pi)(\eta^k)(\lambda^k \otimes \mu^k, \lambda^k \otimes \mu^k) + 2\epsilon + 2ka_k^2 < 0.
\]
Taking the limit supremum as \( k \to \infty \), we get
\[
D^2(f \circ \pi)(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) + 2\epsilon + 2 \limsup_{k \to \infty} ka_k^2 \leq 0;
\]
thus \( \lim_{k \to \infty} a_k = 0 \) and
\[
D^2(f \circ \pi)(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) + 2\epsilon \leq 0.
\]

Let \( l > 0 \). Then
\[
D^2 F_{\epsilon,l}(\eta^k)(\lambda^k \otimes \mu^k, \lambda^k \otimes \mu^k) = D^2(f \circ \pi)(\eta^k)(\lambda^k \otimes \mu^k, \lambda^k \otimes \mu^k) + 2\epsilon + 2la_k^2.
\]
Letting \( k \to \infty \), we get
\[
D^2 F_{\epsilon,l}(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) = D^2(f \circ \pi)(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) + 2\epsilon \leq 0;
\]
that is,
\[
(2.6) \quad D^2 F_{\epsilon,l}(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) \leq 0 \quad \forall l > 0.
\]

Next, observe
\[
\frac{d}{ds} \mathcal{L}(\eta^k + s\lambda^k \otimes \mu^k) = a_k;
\]
thus letting \( k \to \infty \), we get
\[
\frac{d}{ds} \mathcal{L}(\eta + s\lambda \otimes \mu) = 0 \quad \forall s \in \mathbb{R}.
\]
Thus \( \mathcal{L}(\eta + s\lambda \otimes \mu) = \mathcal{L}(\eta) = t \) for all \( s \in \mathbb{R} \). Let \( l > 0 \). We now have
\[
F_{\epsilon,l}(\eta + s\lambda \otimes \mu) = f(\eta + s\lambda \otimes \mu) + \epsilon|\eta + s\lambda \otimes \mu|^2.
\]
Since the function \( s \mapsto f(\eta + s\lambda \otimes \mu) \quad (s \in \mathbb{R}) \) is convex, we have
\[
\frac{d^2}{ds^2} f(\eta + s\lambda \otimes \mu) \geq 0 \quad \forall s \in \mathbb{R}.
\]
Thus,
\[
D^2 F_{\epsilon,l}(\eta)(\lambda \otimes \mu, \lambda \otimes \mu) = \left. \frac{d^2}{ds^2} F_{\epsilon,l}(\eta + s\lambda \otimes \mu) \right|_{s=0} \geq 2\epsilon \quad \forall l > 0;
\]
this is a contradiction to \((2.6)\), and the proof is complete. \( \square \)

We are now ready to prove Lemma \ref{Lemma 2.6}.

**Proof of Lemma \ref{Lemma 2.6}**. Using Lemma \ref{Lemma 2.10}, we can find a rank-one convex function \( g : \Sigma_t \to \mathbb{R} \) such that \( g = f \) on \( \tilde{K} \). Then we choose an open ball \( B \) in \( \mathbb{M}^{m \times n} \) containing \( \tilde{K} \) and set \( R = \sup_{\xi \in B \cap \Sigma_t} |\xi| \).

Let \( \epsilon > 0 \). Upon on mollifying the function \( g \), we can find a smooth rank-one convex function \( \tilde{g} : \Sigma_t \to \mathbb{R} \) such that \( |\tilde{g} - g| < \epsilon/2 \) on the compact set \( B \cap \Sigma_t \).

For each \( k > 0 \), let \( \tilde{G}_{\epsilon,k} : \mathbb{M}^{m \times n} \to \mathbb{R} \) be the function defined by
\[
\tilde{G}_{\epsilon,k}(\xi) = \tilde{g}(\pi(\xi)) + \frac{\epsilon}{2R^2}|\xi|^2 + k|\mathcal{L}(\xi) - t|^2 \quad \forall \xi \in \mathbb{M}^{m \times n}.
\]
Then by Lemma 2.11, there exists a number \( k > 0 \) such that \( \tilde{G}_{\epsilon,k} : U_{\epsilon,k} \to \mathbb{R} \) is rank-one convex, for some open set \( U_{\epsilon,k} \) in \( \mathbb{M}^{m \times n} \) containing \( \tilde{B} \cap \Sigma_t \). Let us write \( G = \tilde{G}_{\epsilon,k} : U_{\epsilon,k} \to \mathbb{R} \). Note that for all \( \xi \in \tilde{K} \subset B \cap \Sigma_t \),

\[
|G(\xi) - g(\xi)| \leq |G(\xi) - \tilde{g}(\xi)| + |\tilde{g}(\xi) - g(\xi)| < \frac{\epsilon}{2R^2}R^2 + \frac{\epsilon}{2} = \epsilon.
\]

Observe \( (\tilde{B} \cap \Sigma_t)^{rc} = \tilde{B} \cap \Sigma_t \subset U_{\epsilon,k} \). Applying Lemma 2.10 we can choose a rank-one convex function \( F : \mathbb{M}^{m \times n} \to \mathbb{R} \) such that \( F = G \) on \( \tilde{B} \cap \Sigma_t \).

Thus

\[
|F - f| = |G - g| < \epsilon \quad \text{on} \quad \tilde{K}.
\]

\[ \square \]

We finally get to the proof of Theorem 2.2.

**Proof of Theorem 2.2.** Recall from Proposition 2.1 and 23 that

\[
K^{rc,\Sigma_t} \subset K^{rc} = \{ \tilde{\mu} \mid \mu \in \mathcal{M}^{rc}(K) \} \subset \Sigma_t.
\]

Let \( \nu \in \mathcal{M}^{rc}(K) \). To show the reverse inclusion \( K^{rc,\Sigma_t} \supset K^{rc} \), it suffices to check that \( \tilde{\nu} \in \tilde{K} := K^{rc,\Sigma_t} \). To prove by contradiction, suppose \( \tilde{\nu} \in \Sigma_t \setminus \tilde{K} \).

Then there exists a rank-one convex function \( g : \Sigma_t \to \mathbb{R} \) with \( g \geq 0 \) on \( K \) such that \( g(\tilde{\nu}) > 0 \); so \( \langle \nu, g \rangle \leq 0 < g(\tilde{\nu}) \). Then by Lemma 2.6 for any given \( \epsilon > 0 \) to be specified below, we get a rank-one convex function \( h : \mathbb{M}^{m \times n} \to \mathbb{R} \) such that \( |h - g| < \epsilon \) on the compact set \( K \cup \{ \tilde{\nu} \} \). This implies that \( \langle \nu, h \rangle < \langle \nu, g \rangle + \epsilon < g(\tilde{\nu}) - \epsilon < h(\tilde{\nu}) \) if \( \epsilon > 0 \) is chosen so small that \( \epsilon < \frac{g(\tilde{\nu}) - \langle \nu, g \rangle}{2} \).

In short, we have \( \langle \nu, h \rangle < h(\tilde{\nu}) \); a contradiction to the fact that \( \nu \) is a laminate. Thus \( \tilde{\nu} \in \tilde{K} \), and so \( K^{rc} = K^{rc,\Sigma_t} \).

Next, let \( \mathcal{O} \) be an open set in \( \Sigma_t \) containing \( K^{rc} \). We choose a bounded open set \( U \) in \( \Sigma_t \) such that \( K^{rc} \subset U \subset \tilde{U} \subset \mathcal{O} \). Set \( \mathcal{F} = \{ \mu \in \mathcal{L}(U) \mid \tilde{\mu} = \tilde{\nu} \} \); then \( \delta_{\tilde{\nu}} \in \mathcal{F} \neq \emptyset \). To finish the proof, it suffices to show that the weak* closure \( \mathcal{F}^* \) of \( \mathcal{F} \) in \( \mathcal{P} \) contains \( \nu \). We prove by contradiction; suppose \( \nu \notin \mathcal{F}^* \). Since \( \mathcal{F} \) is convex, it follows from the Hahn-Banach Theorem that there exists a continuous function \( f : \tilde{U} \to \mathbb{R} \) such that

\[
\langle \nu, f \rangle < \inf \{ \langle \mu, f \rangle \mid \mu \in \mathcal{F} \}.
\]

Since \( \tilde{U} \) is compact, it follows from Lemma 2.5 that

\[
R_{\tilde{U}} f(\tilde{\nu}) = \inf \{ \langle \mu, f \rangle \mid \mu \in \mathcal{F} \}.
\]

Note that \( R_{\tilde{U}} f : U \to \mathbb{R} \) is a rank-one convex function with \( R_{\tilde{U}} f \leq f \) in \( U \). From the above observation, we have \( \langle \nu, R_{\tilde{U}} f \rangle \leq \langle \nu, f \rangle < R_{\tilde{U}} f(\tilde{\nu}) \).

By Lemma 2.6 for any given \( \epsilon > 0 \) to be chosen below, we obtain a rank-one convex function \( F : \mathbb{M}^{m \times n} \to \mathbb{R} \) such that \( |F - R_{\tilde{U}} f| < \epsilon \) on \( \tilde{K} = K^{rc} \).

Since \( \tilde{\nu} \in K^{rc} \), we choose \( 0 < \epsilon < \frac{R_{\tilde{U}} f(\tilde{\nu}) - \langle \nu, R_{\tilde{U}} f \rangle}{2} \) to have \( \langle \nu, F \rangle < F(\tilde{\nu}) \); a contradiction to the fact that \( \nu \) is a laminate.

The proof is now complete. \[ \square \]
3. Rank-one smooth approximation under linear constraint

We begin this section by introducing a pivotal approximation result, Theorem 3.1, for proving the main results of the paper, Theorems 1.1 and 1.3. Its special cases have been successfully applied to some nonstandard evolution problems [15, 16, 17, 14]. Although the proof of Theorem 3.1 already appeared in [14], we include it in Section 6 for the sake of completeness as we make use of the general version of the theorem for the first time in this paper.

**Theorem 3.1.** Let $m, n \geq 2$ be integers, and let $A, B \in \mathbb{M}^{m \times n}$ be such that $\text{rank}(A - B) = 1$; hence

$$A - B = a \otimes b = (a_i b_j)$$

for some nonzero vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ with $|b| = 1$. Let $L \in \mathbb{M}^{m \times n}$ satisfy

$$Lb \neq 0 \quad \text{in} \quad \mathbb{R}^m,$$

and let $\mathcal{L} : \mathbb{M}^{m \times n} \to \mathbb{R}$ be the linear function defined by

$$\mathcal{L}(\xi) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} L_{ij} \xi_{ij} \forall \xi \in \mathbb{M}^{m \times n}.$$

Assume $\mathcal{L}(A) = \mathcal{L}(B)$ and $0 < \lambda < 1$ is any fixed number. Then there exists a linear partial differential operator $\Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying the following properties:

1. For any open set $\Omega \subset \mathbb{R}^n$, $\Phi v \in C^{k-1}(\Omega; \mathbb{R}^m)$ whenever $k \in \mathbb{N}$ and $v \in C^k(\Omega; \mathbb{R}^m)$ and

$$\mathcal{L}(\nabla \Phi v) = 0 \quad \text{in} \quad \Omega \forall v \in C^2(\Omega; \mathbb{R}^m).$$

2. Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. For each $\tau > 0$, there exist a function $g = g_\tau \in C_0^\infty(\Omega; \mathbb{R}^m)$ and two disjoint open sets $\Omega_A, \Omega_B \subset \subset \Omega$ such that

(a) $\Phi g \in C_0^\infty(\Omega; \mathbb{R}^m)$,

(b) $\text{dist}(\nabla \Phi g, [-\lambda(A - B), (1 - \lambda)(A - B)]) < \tau$ in $\Omega$,

(c) $\nabla \Phi g(x) = \begin{cases} (1 - \lambda)(A - B) & \forall x \in \Omega_A, \\ -\lambda(A - B) & \forall x \in \Omega_B, \end{cases}$

(d) $|\Omega_A| - \lambda|\Omega| < \tau, |\Omega_B| - (1 - \lambda)|\Omega| < \tau,$

(e) $\|\Phi g\|_{L^\infty(\Omega)} < \tau,$

where $[-\lambda(A - B), (1 - \lambda)(A - B)]$ is the closed line segment in $\ker \mathcal{L} \subset \mathbb{M}^{m \times n}$ joining $-\lambda(A - B)$ and $(1 - \lambda)(A - B)$.

Using this theorem, we deduce a preliminary result towards Theorems 1.1 and 1.3. We remark that Lemma 3.2 is the spot where the two major tools of the paper, Theorems 2.2 and 3.1, meet. Note also that piecewise linear approximation scheme is not used here and below in Lemma 3.3 (cf. [20, Lemma 4.1]).
Lemma 3.2. Assume that (1.2) is satisfied. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $V$ be an open set in $\Sigma_t$, and let $\xi \in V^t$. Then for each $\epsilon > 0$, there exists a map $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that

$$
\begin{cases}
\xi + \nabla \varphi \in V^t \text{ in } \Omega, \\
|\{x \in \Omega | \xi + \nabla \varphi(x) \not\in V\}| < \epsilon|\Omega|,
\end{cases}
$$

$$
\|\varphi\|_{L^\infty(\Omega)} < \epsilon.
$$

Proof. As $\xi \in V^t$, there exists a compact set $K \subset V$ such that $\xi \in K^t = \{\bar{\mu} | \mu \in \mathcal{M}^t(K)\}$. So $\xi = \bar{\nu}$ for some $\nu \in \mathcal{M}^t(K)$. From Corollary 2.3, we see that $V^t$ is open in $\Sigma_t$. We thus can apply Theorem 2.2 to extract a sequence $\nu_k \in \mathcal{L}(V^t)$ with $\nu_k = \bar{\nu} = \xi$ that converges weakly* to $\nu$ in $\mathcal{P}$.

Claim: For each $\mu \in \mathcal{L}(V^t)$ of order $N - 1 \geq 0$ with $\mu = \sum_{j=1}^N \lambda_j \delta_{\xi_j}$, there exists an $\eta_0 > 0$ such that for each $0 < \eta < \eta_0$ and each $\epsilon > 0$, there is a map $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$ satisfying

$$
\begin{cases}
\bar{\mu} + \nabla \varphi \in V^t \text{ in } \Omega, \\
|\{x \in \Omega | \bar{\mu} + \nabla \varphi(x) - \xi_j | < \eta}\} - \lambda_j|\Omega| < \epsilon|\Omega| \text{ for all } 1 \leq j \leq N, \\
\|\varphi\|_{L^\infty(\Omega)} < \epsilon.
\end{cases}
$$

Suppose for the moment that Claim holds. Choose a function $F \in C_c^\infty(\widetilde{V})$ so that $0 \leq F \leq 1$ in $\widetilde{V}$ and $F \equiv 1$ on $K$, where $\widetilde{V}$ is some open set in $\mathcal{M}^{m \times n}$ with $V = \widetilde{V} \cap \Sigma_t$. Let $0 < \epsilon \leq 2$. Since

$$
\int_{\mathcal{M}^{m \times n}} F d\nu_k \to \int_{\mathcal{M}^{m \times n}} F d\nu = 1 \text{ as } k \to \infty,
$$

we can choose an index $i \in \mathbb{N}$ so large that

$$
0 \leq 1 - \sum_{j=1}^N \lambda_j F(\xi_j) = 1 - \int_{\mathcal{M}^{m \times n}} F d\nu_i < \frac{\epsilon}{2},
$$

where $\nu_i = \sum_{j=1}^N \lambda_j \delta_{\xi_j} \in \mathcal{L}(V^t)$ is of order $N - 1 \geq 0$. Set $J_V = \{j \in \{1, \ldots, N\} | \xi_j \in V\}$. If $J_V = \emptyset$, then $1 = 1 - \sum_{j=1}^N \lambda_j F(\xi_j) < \frac{\epsilon}{2}$, a contradiction. Thus $J_V \neq \emptyset$. Now, let

$$
0 < \eta < \min \left\{ \min_{j \in J_V} \text{dist}(\xi_j, \partial \Sigma_t, V), \min_{j,k \in J_V, j \neq k} 2^{-1} |\xi_j - \xi_k|, \eta_0 \right\},
$$

where the number $\eta_0 > 0$ is from the result of Claim above. It then follows from the result of Claim that there exists a map $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that

$$
\begin{cases}
\xi + \nabla \varphi = \bar{\nu}_i + \nabla \varphi \in V^t \text{ in } \Omega, \\
|\{x \in \Omega | |\xi + \nabla \varphi(x) - \xi_j | < \eta}\} - \lambda_j|\Omega| < \frac{\epsilon}{2|\Omega|} |\Omega| \text{ for all } 1 \leq j \leq N, \\
\|\varphi\|_{L^\infty(\Omega)} < \epsilon.
\end{cases}
$$
Thus, by (3.3), (3.4) and (3.5), we have
\[
|\{x \in \Omega \mid \xi + \nabla \varphi(x) \notin V\}| = |\Omega| - |\{x \in \Omega \mid \xi + \nabla \varphi(x) \in V\}|
\leq |\Omega| - |\Omega| \sum_{j \in J_V} \lambda_j F'_{1}(\xi_j) + \sum_{j \in J_V} \lambda_j |\Omega|
- \sum_{j \in J_V} |\{x \in \Omega \mid \xi + \nabla \varphi(x) - \xi_j < \eta\}| < \frac{\epsilon |\Omega|}{2} + \frac{\epsilon |\Omega|}{2} = \epsilon |\Omega|;
\]
hence the map \( \varphi \) satisfies the required properties for the conclusion of the lemma.

It now remains to prove Claim above. Let us prove this by induction on the order \( l \geq 0 \) of a laminate \( \mu = \sum_{j=1}^{l+1} \lambda_j \delta_{\xi_j} \in \mathcal{L}(V^{rc}) \). If the order \( l = 0 \), we can simply take \( \varphi \equiv 0 \) in \( \Omega \); then (3.2) holds for all \( \eta > 0 \) and \( \epsilon > 0 \).

Next, assume that the assertion holds for the order \( l = k \), where \( k \geq 0 \) is an integer. Let \( \mu = \sum_{j=1}^{k+2} \lambda_j \delta_{\xi_j} \in \mathcal{L}(V^{rc}) \) be of a laminate of order \( l = k+1 \). Reordering the indices \( j \) in \( \mu \) if necessary and setting
\[
\tilde{\lambda}_{k+1} = \lambda_{k+1} + \lambda_{k+2}, \quad \tilde{\lambda} = \frac{\lambda_{k+1}}{\lambda_{k+1}} \in (0,1),
\]
\[
\tilde{\xi}_{k+1} = \tilde{\lambda}_{k+1} \xi_{k+1} + (1 - \tilde{\lambda}) \xi_{k+2}, \quad \tilde{\mu} = \sum_{j=1}^{k} \lambda_j \delta_{\xi_j} + \tilde{\lambda}_{k+1} \delta_{\tilde{\xi}_{k+1}},
\]
it follows that \([\xi_{k+1}, \xi_{k+2}]\) is a rank-one segment in \( V^{rc} \) and that
\[
\mu = \tilde{\mu} - \tilde{\lambda}_{k+1} \tilde{\delta}_{\tilde{\xi}_{k+1}} + \tilde{\lambda}_{k+1} \tilde{\delta}_{\tilde{\xi}_{k+1}} + (1 - \tilde{\lambda}) \lambda_{k+1} \delta_{\tilde{\xi}_{k+1}},
\]
where \( \tilde{\mu} \) is a laminate of order \( k \) in \( V^{rc} \). Let \( \epsilon > 0 \) and
\[
0 < \eta < \frac{1}{2} \min \left\{ \min_{1 \leq i,j \leq k+2, i \neq j} |\xi_i - \xi_j|, \min_{1 \leq j \leq k} |\xi_j - \tilde{\xi}_{k+1}|, \right. \]
\[
\left. \text{dist}(\{\xi_{k+1}, \xi_{k+2}\}, \partial|_{\Sigma_t V^{rc}}), \eta_{\tilde{\mu}} \right\} =: \eta_{\tilde{\mu}},
\]
where the number \( \eta_{\tilde{\mu}} > 0 \) is from the induction hypothesis. By the induction hypothesis, there exists a map \( \psi \in C_{\infty}^{0}(\Omega; \mathbb{R}^{m}) \) such that
(3.6)
\[
\begin{align*}
\bar{\mu} + \nabla \psi & = \tilde{\mu} + \nabla \psi \in V^{rc} \quad \text{in } \Omega, \\
|\{x \in \Omega \mid |\bar{\mu} + \nabla \psi(x) - \xi_j| < \eta\}| - \lambda_j |\Omega| & < \frac{\epsilon}{6(k+1)} |\Omega| \quad \text{for all } 1 \leq j \leq k, \\
|\{x \in \Omega \mid |\bar{\mu} + \nabla \psi(x) - \tilde{\xi}_{k+1}| < \eta\}| - \tilde{\lambda} |\Omega| & < \frac{\epsilon}{6(k+1)} |\Omega|, \\
|\psi|_{L^{\infty}(\Omega)} & < \frac{\epsilon}{2}.
\end{align*}
\]
Set
\[
E_j = \{x \in \Omega \mid |\bar{\mu} + \nabla \psi(x) - \xi_j| < \eta\} \quad (1 \leq j \leq k),
\]
\[
E_{k+1} = \{x \in \Omega \mid |\bar{\mu} + \nabla \psi(x) - \tilde{\xi}_{k+1}| < \eta\}, \quad F = \Omega \setminus \left( \bigcup_{j=1}^{k} E_j \cup E_{k+1} \right);
\]
then from the choice of \( \gamma \), we see that \( E_1, \ldots, E_k \) and \( E_{k+1} \) are pairwise disjoint. We now choose finitely many disjoint open cubes \( Q_1, \ldots, Q_N \subset \subset \).
It follows from (a) and (3.6) that for all open sets $\mathcal{E}_{k+1}$, parallel to the axes, so that

$$|\mathcal{E}_{k+1} \cup \bigcup_{i=1}^{N_i} Q_i| < \frac{\epsilon}{6} |\Omega|.$$  

Fix an index $i \in \{1, \cdots, N_i\}$, and set $\eta_i = \max_{x \in Q_i} |\bar{\mu} + \nabla \psi(x) - \xi_k| < \eta$. Choose finitely many disjoint dyadic cubes $Q_i^1, \cdots, Q_i^{N_i} \subset Q_i$ with $|Q_i \setminus \bigcup_{j=1}^{N_i} Q_i^j| = 0$ so small that

$$|\nabla \psi(x) - \nabla \psi(y)| < \frac{\eta - \eta_i}{2} \quad \forall x, y \in \bar{Q}_i^j, \forall j = 1, \cdots, N_i.$$  

Fix an index $j \in \{1, \cdots, N_i\}$. Let $x_i^j$ denote the center of the cube $Q_i^j$, and set $\xi_i^j = \bar{\mu} + \nabla \psi(x_i^j) \in V^\tau$; then $|\xi_i^j - \xi_k| \leq \eta_i$. Since the matrix $L$ satisfies $(1.2)$, rank$(\xi_k - \xi_k - 2) = 1$, and $L(\xi_k) = L(\xi_k)(= t)$, we can apply Theorem 3.1 to the cube $Q_i^j$ and number $0 < \lambda < 1$ to obtain that for any given $\tau > 0$, there exist a function $h_i^j \in C^\infty_c(Q_i^j; \mathbb{R}^m)$ and two disjoint open sets $Q_{i+1}, Q_{i+2} \subset Q_i^j$ satisfying

(a) $h_i^j \in C^\infty_c(Q_i^j; \mathbb{R}^m)$, $L(\nabla h_i^j) = 0$ in $Q_i^j$,
(b) dist$(\nabla h_i^j, [-\lambda(\xi_k - \xi_k - 2), (1 - \lambda)(\xi_k - \xi_k - 2)]) < \tau$ in $Q_i^j$,
(c) $\nabla h_i^j(x) = \left\{ \begin{array}{ll} (1 - \lambda)(\xi_k - \xi_k - 2) & \forall x \in Q_{i+1}, \\ -\lambda(\xi_k - \xi_k - 2) & \forall x \in Q_{i+2}, \end{array} \right.$
(d) $|Q_{i+1} - \lambda|Q_i^j| < \tau$, $|Q_{i+2} - (1 - \lambda)|Q_i^j| < \tau$,
(e) $\|h_i^j\|_{L^\infty(Q_i^j)} < \tau$.

For our purpose, we choose

$$0 < \tau < \min \left\{ \eta, \frac{\epsilon}{3}, \frac{\epsilon |\Omega|}{12(k + 1) \sum_{i=1}^{N_i} N_i} \right\}.$$  

We now define

$$\varphi = \psi + \sum_{1 \leq i \leq N_i, 1 \leq j \leq N_i} h_i^j \chi_{Q_i^j} \quad \text{in} \quad \Omega.$$  

Let us check that $\varphi : \Omega \to \mathbb{R}^m$ is a desired function. It is clear from the construction and (3.6) that

$$\varphi \in C^\infty_c(\Omega; \mathbb{R}^m).$$  

Let $1 \leq i \leq N$ and $1 \leq j \leq N_i$. Note from (3.6), (c) and (3.9) that $|\varphi| < 5\epsilon/6$ in $Q_i^j$; thus, from the definition of $\varphi$, we have

$$\|\varphi\|_{L^\infty(\Omega)} < \epsilon.$$  

It follows from (a) and (3.6) that for all $x \in Q_i^j$, we have $L(\bar{\mu} + \nabla \varphi(x)) = L(\bar{\mu} + \nabla \psi(x)) + L(\nabla h_i^j(x)) = L(\bar{\mu} + \nabla \psi(x)) = t$, i.e., $\bar{\mu} + \nabla \varphi(x) \in \Sigma_t$. In
addition, we have from (3.8), (b) and the choice \( 0 < \tau < \eta < \eta_\mu \) that for all 
\[ x \in Q_i^j, \]

\[
\text{dist}(\bar{\mu} + \nabla \varphi(x), [\xi_{k+1}, \xi_{k+2}])
= \text{dist}(\bar{\mu} + \nabla \psi(x) - \xi_i^j + \xi_i^j - \tilde{\xi}_k + \tilde{\xi}_k + \nabla h_i^j(x), [\xi_{k+1}, \xi_{k+2}])
\leq |\bar{\mu} + \nabla \psi(x) - \xi_i^j| + |\xi_i^j - \tilde{\xi}_k + \tilde{\xi}_k + \nabla h_i^j(x), [\xi_{k+1}, \xi_{k+2}])
\leq \frac{\eta - \eta_i}{2} + \eta_i + \text{dist}(\nabla h_i^j(x), [-\tilde{\lambda}(\xi_{k+1} - \xi_{k+2}), (1 - \tilde{\lambda})(\xi_{k+1} - \xi_{k+2})])
\leq \eta + \tau < 2\eta < \text{dist}([\xi_{k+1}, \xi_{k+2}], \partial|_{\Sigma_t} V^\tau). \]

Combining these two observations, we see that \( \bar{\mu} + \nabla \varphi \in V^\tau \) in \( Q_i^j \), and thus from (3.6) and the definition of \( u \), we have

\[
\bar{\mu} + \nabla \varphi \in V^\tau \quad \text{in } \Omega. 
\]

We now write

\[
G_l = \{ x \in \Omega \mid |\bar{\mu} + \nabla \varphi(x) - \xi_i| < \eta \} \quad (1 \leq l \leq k + 2).
\]

Let \( 1 \leq i \leq N \) and \( 1 \leq j \leq N_i \). By (c) and (3.8), for all \( x \in Q_{i,k+1}^j \), we have

\[
|\bar{\mu} + \nabla \varphi(x) - \xi_{k+1}| = |\bar{\mu} + \nabla \psi(x) + \nabla h_i^j(x) - \xi_{k+1}|
= |\bar{\mu} + \nabla \psi(x) - \xi_i^j + \xi_i^j - \tilde{\xi}_k + \tilde{\xi}_k + (1 - \tilde{\lambda})(\xi_{k+1} - \xi_{k+2}) - \xi_{k+1}|
= |\bar{\mu} + \nabla \psi(x) - \xi_i^j + \xi_i^j - \tilde{\xi}_k + \tilde{\xi}_k| \leq \frac{\eta - \eta_i}{2} + \eta_i < \eta.
\]

Likewise, we have \( |\bar{\mu} + \nabla \varphi(x) - \xi_{k+2}| < \eta \) for all \( x \in Q_{i,k+2}^j \). Thus it follows from (3.6), (d) and (3.9) that for \( 1 \leq l \leq k, \)

\[
|G_l| - \lambda_i|\Omega| \leq |E_l| - \lambda_i|\Omega| + \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} |Q_i^j \setminus (Q_{i,k+1}^j \cup Q_{i,k+2}^j)|
\leq \frac{e|\Omega|}{6(k + 1)} + 2\tau \sum_{i=1}^N N_i < \frac{e|\Omega|}{3(k + 1)} < e|\Omega|.
\]

It now remains to check that this inequality also holds for \( l = k + 1, k + 2. \)

Note from the above observation that

\[
|G_{k+1}| - \lambda_{k+1}|\Omega| \leq \left| \bigcup_{1 \leq i \leq N, 1 \leq j \leq N_i} Q_{i,k+1}^j - \tilde{\lambda}_k|\Omega| \right|
+ |F| + |\bar{E}_{k+1} \setminus \bigcup_{i=1}^N Q_i| + \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} |Q_i^j \setminus (Q_{i,k+1}^j \cup Q_{i,k+2}^j)|
=: I_1 + I_2 + I_3 + I_4.
\]
By (d), (3.6), (3.7) and (3.9), we can estimate:

\[
I_1 = \left| \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} Q_{i,j,k+1}^j - \lambda \big| \bigcup_{i=1}^N Q_i \big| - \tilde{\lambda} \lambda_{k+1} |\Omega| \right|
\]

\[
+ \lambda |E_{k+1}| - \tilde{\lambda} |E_{k+1}| + \tilde{\lambda} |\bigcup_{i=1}^N Q_i| \quad \leq \quad \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} \left| Q_{i,j,k+1}^j - \lambda Q_i^j + \tilde{\lambda} |E_{k+1}| - \tilde{\lambda}_{k+1} |\Omega| + \tilde{\lambda} |E_{k+1} \setminus \bigcup_{i=1}^N Q_i| \right|
\]

\[
< \tau \sum_{i=1}^N N_i + \frac{\tilde{\lambda} \epsilon |\Omega|}{6(k + 1)} + \frac{\tilde{\lambda} \epsilon |\Omega|}{6} < \frac{\epsilon |\Omega|}{2},
\]

\[
I_2 + I_3 + I_4 \leq \sum_{i=1}^k \left| E_i \right| - \lambda |\Omega| + |E_{k+1}| - \tilde{\lambda}_{k+1} |\Omega| + \frac{\epsilon |\Omega|}{6} + 2 \tau \sum_{i=1}^N N_i
\]

\[
< \frac{\epsilon |\Omega|}{6} + \frac{\epsilon |\Omega|}{6} + \frac{\epsilon |\Omega|}{6} = \frac{\epsilon |\Omega|}{2}.
\]

In all, we get \( I_1 + I_2 + I_3 + I_4 < \epsilon |\Omega| \). In a similar manner, we also see that

\[
\left| G_{k+2} \right| - \lambda_{k+2} |\Omega| < \epsilon |\Omega|.
\]

We have checked that the assertion holds for the laminate \( \mu \) of order \( k+1 \), and the proof is now complete. \( \square \)

As the last preparation for the proof of Theorems 1.1 and 1.3, we improve the above lemma to deal with \( C^1 \) boundary data.

**Lemma 3.3.** Assume (1.2). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, let \( U \) be a bounded open set in \( \Sigma_t \), and let \( v \in C^1(\bar{\Omega}; \mathbb{R}^m) \) be a map satisfying

\[
\nabla v(x) \in U^{rc} \quad \text{for all } x \in \Omega.
\]

Then for each \( \epsilon > 0 \), there exist a map \( u \in C^1(\bar{\Omega}; \mathbb{R}^m) \) and an open set \( \Omega' \subset \subset \Omega \) with \( \partial \Omega' = 0 \) such that

\[
\begin{cases}
u(x) = v(x) & \text{for all } x \text{ near } \partial \Omega, \\
\nabla u \in U^{rc} & \text{in } \Omega, \\
\nabla u \in U & \text{in } \Omega', \\
|\Omega \setminus \Omega'| < \epsilon |\Omega|, \\
\|u - v\|_{L^\infty(\Omega)} < \epsilon.
\end{cases}
\]

**Proof.** Let \( \epsilon > 0 \). Choose finitely many disjoint open cubes \( Q_1, \ldots, Q_N \subset \subset \Omega \), parallel to the axes, such that

\[
|\Omega \setminus \bigcup_{i=1}^N Q_i| < \frac{\epsilon}{3} |\Omega|.
\]

(3.10)

Fix an index \( 1 \leq i \leq N \). As \( \nabla v \in U^{rc} \) on \( Q_i \), we can use Corollary 2.1 to choose an open set \( V_i \) in \( \Sigma_t \) with \( \bar{V}_i \subset U \) such that

\[
\nabla v \in V_i^{rc} \quad \text{on } Q_i \quad \text{and} \quad \bar{V}_i^{rc} \subset U^{rc}.
\]
Set  
$$\delta_i = \min \left\{ \text{dist}(\partial|_{\Sigma_i} V_i^{rc}, \partial|_{\Sigma_i} U^{rc}), \text{dist}(\partial|_{\Sigma_i} V_i, \partial|_{\Sigma_i} U) \right\} > 0.$$  

Then divide $Q_i$ into finitely many disjoint dyadic cubes $Q_{i,1}, \cdots, Q_{i,N_i}$ whose union has measure $|Q_i|$ and such that

$$|\nabla v(x) - \nabla v(y)| < \frac{\delta_i}{2}$$

for all $x, y \in Q_{i,j}$ and $j = 1, \cdots, N_i$. Now, fix an index $1 \leq j \leq N_i$ let $x_{i,j}$ denote the center of the cube $Q_{i,j}$, and set $\xi_{i,j} = \nabla v(x_{i,j}) \in V_i^{rc}$. Then we apply Lemma 3.2 to obtain a map $\varphi_{i,j} \in C_c^\infty(Q_{i,j}; \mathbb{R}^m)$ such that

$$\left\{ \begin{array}{ll} \xi_{i,j} + \nabla \varphi_{i,j} \in V_i^{rc} & \text{in } Q_{i,j}, \\
\{x \in Q_{i,j} | \xi_{i,j} + \nabla \varphi_{i,j}(x) \notin V_i\} < \frac{\epsilon}{3}|Q_{i,j}|, \\
\|\varphi_{i,j}\|_{L^\infty(Q_{i,j})} < \epsilon. \end{array} \right. \quad (3.12)$$

Define

$$u = v + \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} \varphi_{i,j} \chi_{Q_{i,j}} \text{ in } \Omega.$$  

Then by (3.12) and the definition of $u$, we have

$$u \in C^1(\Omega; \mathbb{R}^m), \quad u(x) = v(x) \text{ for all } x \text{ near } \partial \Omega, \quad \|u - v\|_{L^\infty(\Omega)} < \epsilon.$$

Let $1 \leq i \leq N$ and $1 \leq j \leq N_i$. Then for all $x \in Q_{i,j}$, we have $\xi_{i,j} + \nabla \varphi_{i,j}(x) \in V_i^{rc}$ and

$$|\nabla u(x) - (\xi_{i,j} + \nabla \varphi_{i,j}(x))| = |\nabla v(x) - \xi_{i,j}| < \frac{\delta_i}{2} \quad \text{(by (3.11))};$$

thus from the definition of $\delta_i$, we get $\nabla u(x) \in U^{rc}$. By the definition of $u$, we now see that

$$\nabla u(x) \in U^{rc} \text{ for all } x \in \Omega.$$  

We write  

$$E_{i,j} = \{x \in Q_{i,j} | \xi_{i,j} + \nabla \varphi_{i,j}(x) \notin V_i\}, \quad G_{i,j} = Q_{i,j} \setminus E_{i,j};$$ 

then $|E_{i,j}| < \frac{\epsilon}{3}|Q_{i,j}|$, and $G_{i,j}$ is an open set in $Q_{i,j}$. If $x \in G_{i,j}$, then $\xi_{i,j} + \nabla \varphi_{i,j}(x) \in V_i$, and (3.13) holds; thus $\nabla u(x) \in U$. We now choose an open subset $H_{i,j}$ of $G_{i,j}$ such that

$$|G_{i,j} \setminus H_{i,j}| < \frac{\epsilon}{3}|Q_{i,j}| \quad \text{and} \quad |\partial H_{i,j}| = 0. \quad (3.14)$$

Let

$$\Omega' = \bigcup_{1 \leq i \leq N, 1 \leq j \leq N_i} H_{i,j};$$

then $\Omega' \subset \subset \Omega$, $|\partial \Omega'| = 0$, and

$$\nabla u(x) \in U \text{ for all } x \in \Omega'.$
Moreover, we have from (3.10), (3.12) and (3.14) that
\[ |\Omega \setminus \Omega'| = |\Omega \setminus \bigcup_{i=1}^{N} Q_i| + \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} |E_{i,j}| + \sum_{1 \leq i \leq N, 1 \leq j \leq N_i} |G_{i,j} \setminus H_{i,j}| < \epsilon |\Omega|. \]

The proof is now complete. \[\Box\]

4. PROOF OF MAIN THEOREMS

We are now ready to prove Theorems 1.1 and 1.3 by iteration of the result of Lemma 3.3 in suitable ways. We first finish the proof of Theorem 1.1 using

**Proof of Theorem 1.1**. We only consider the case that \( v \in C^1(\bar{\Omega}; \mathbb{R}^m) \) as the general case that \( v \) is piecewise \( C^1 \) can be handled by following the proof below for countably many disjoint open subsets on which \( v \) is \( C^1 \) up to the boundary. Now, by assumption, we have \( \nabla v \in \overline{U^{rc}} = U^{rc} \cup \partial \Sigma U^{rc} \in \Omega \) and \( |\Gamma| = 0 \), where \( \Gamma := \{ x \in \Omega \mid \nabla v(x) \in \partial \Sigma U^{rc} \} \). So \( \Omega' := \Omega \setminus \Gamma \) is an open subset of \( \Omega \) with \( |\Omega \setminus \Omega'| = 0 \). Fix an \( 0 < \epsilon \ll 1 \).

We write \( \Omega^{(0)} = \Omega' \) and \( \tilde{u}^{(0)} = v \) in \( \Omega^{(0)} \). Since \( \tilde{u}^{(0)} \in C^1(\bar{\Omega}^{(0)}; \mathbb{R}^m) \) and \( \nabla \tilde{u}^{(0)} \in U^{rc} \) in \( \Omega^{(0)} \), we can apply Lemma 3.3 to find a map \( \tilde{u}^{(1)} \in C^1(\bar{\Omega}^{(0)}; \mathbb{R}^m) \) and an open set \( G^{(0)} \subset \subset \Omega^{(0)} \) with \( |\partial G^{(0)}| = 0 \) such that setting \( \Omega^{(1)} = \Omega^{(0)} \setminus G^{(0)} \), we have

\[
\begin{cases}
\tilde{u}^{(1)}(x) = \tilde{u}^{(0)}(x) & \text{for all } x \text{ near } \partial \Omega^{(0)}, \\
\nabla \tilde{u}^{(1)} \in U^{rc} & \text{in } \Omega^{(0)}, \\
\tilde{u}^{(1)}(x) \in U & \text{in } G^{(0)}, \\
|\Omega^{(1)}| < \epsilon|\Omega^{(0)}|, \\
\|\tilde{u}^{(1)} - \tilde{u}^{(0)}\|_{L^\infty(\Omega^{(0)})} < \frac{\epsilon}{22}.
\end{cases}
\]

Since \( \tilde{u}^{(1)} \in C^1(\bar{\Omega}^{(1)}; \mathbb{R}^m) \) and \( \nabla \tilde{u}^{(1)} \in U^{rc} \) in \( \Omega^{(1)} \), we can also apply Lemma 3.3 to obtain a map \( \tilde{u}^{(2)} \in C^1(\bar{\Omega}^{(1)}; \mathbb{R}^m) \) and an open set \( G^{(1)} \subset \subset \Omega^{(1)} \) with \( |\partial G^{(1)}| = 0 \) such that letting \( \Omega^{(2)} = \Omega^{(1)} \setminus G^{(1)} \), we have

\[
\begin{cases}
\tilde{u}^{(2)}(x) = \tilde{u}^{(1)}(x) & \text{for all } x \text{ near } \partial \Omega^{(1)}, \\
\nabla \tilde{u}^{(2)} \in U^{rc} & \text{in } \Omega^{(1)}, \\
\tilde{u}^{(2)}(x) \in U & \text{in } G^{(1)}, \\
|\Omega^{(2)}| < \epsilon|\Omega^{(1)}|, \\
\|\tilde{u}^{(2)} - \tilde{u}^{(1)}\|_{L^\infty(\Omega^{(1)})} < \frac{\epsilon}{22}.
\end{cases}
\]

Repeating this process indefinitely, we obtain a sequence of open sets \( \Omega^{(0)} \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \cdots \), a sequence of open sets \( G^{(k)} \subset \subset \Omega^{(k)} \) with \( |\partial G^{(k)}| = 0 \) \( (k = 0, 1, 2, \cdots) \), and a sequence of maps \( \tilde{u}^{(k+1)} \in C^1(\bar{\Omega}^{(k)}; \mathbb{R}^m) \) \( (k = \cdots, 2, 1, 0) \),
such that for every integer \( k \geq 0 \),

\[
\begin{align*}
\Omega^{(k+1)} &= \Omega^{(k)} \setminus \bar{G}^{(k)}, \\
\tilde{u}^{(k+1)}(x) &= \tilde{u}^{(k)}(x) \\
\nabla \tilde{u}^{(k+1)} &\in U^{rc} \\
\nabla \tilde{u}^{(k+1)} &\in U \\
|\Omega^{(k)}| &< \varepsilon \Omega \\
\|\tilde{u}^{(k+1)} - \tilde{u}^{(k)}\|_{L^{\infty}(\Omega^{(k)})} &< \frac{\varepsilon}{2^{k+1}}.
\end{align*}
\]

Let

\[
\tilde{u}^{(1)} = \begin{cases} 
\tilde{u}^{(1)} & \text{in } \Omega^{(0)}, \\
v & \text{in } \Omega \setminus \Omega^{(0)},
\end{cases}
\]

and for each \( k \in \mathbb{N} \), define

\[
u^{(k+1)} = \sum_{j=1}^{k} \tilde{u}^{(j)} \chi_{\Omega^{(j-1)}} + \tilde{u}^{(k+1)} \chi_{\Omega^{(k)}} + v \chi_{\Omega^{(k)}} \text{ in } \Omega.
\]

Then for all \( k \in \mathbb{N} \), we have \( u^{(k)} \in C^1(\Omega; \mathbb{R}^m) \), \( u^{(k)} = v \) near \( \partial \Omega \), and \( \nabla u^{(k)} \in U^{rc} \) a.e. in \( \Omega \).

Let

\[
u = \sum_{j=1}^{\infty} \tilde{u}^{(j)} \chi_{\Omega^{(j-1)}} + v \chi_{\Omega^{(0)}} \text{ in } \Omega.
\]

Since \( u^{(k)} \) (\( k \in \mathbb{N} \)) are uniformly Lipschitz in \( \Omega \) and \( u^{(k)} \to u \) a.e. in \( \Omega \) as \( k \to \infty \), it follows that \( u \in v + W^{1,\infty}_{0}(\Omega; \mathbb{R}^m) \). Note also that \( \nabla u \in U \) in \( \cup_{j=0}^{\infty} G^{(j)} \), where \( |\cup_{j=0}^{\infty} G^{(j)}| = |\Omega| \), and that

\[
\|u - v\|_{L^{\infty}(\Omega)} \leq \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon.
\]

The proof is now complete. \( \square \)

The proof of Theorem 1.3 relies on a more subtle iteration of the result of Lemma 3.3. We remark that the result of Theorem 1.1 cannot be used directly to prove Theorem 1.3 (cf. [20, Proof of Theorem 1.3]).

**Proof of Theorem 1.3.** Again we only consider the case that \( v \in C^1(\Omega; \mathbb{R}^m) \) as the piecewise \( C^1 \) case can be adapted easily from the simpler case.

For each \( j \in \mathbb{N} \), let

\[
\Omega_j = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > 2^{-j} \}.
\]

Let \( \rho \in C^\infty_c(\mathbb{R}^n) \) denote the standard mollifier, and for each \( \varepsilon > 0 \), let \( \rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon) \) for all \( x \in \mathbb{R}^n \).

Let

\[
\Omega^{(1)} = \{ x \in \Omega \mid \nabla v(x) \in U_1 \};
\]

then \( \Omega^{(1)} \) is an open subset of \( \Omega \) with \( |\Omega \setminus \Omega^{(1)}| = 0 \). Let us write \( u^{(1)} = v \) in \( \Omega^{(1)} \), and fix any two numbers \( \varepsilon > 0 \) and \( 0 < \delta_1 < 1 \).
Choose an $0 < \epsilon_1 < 2^{-1}$ such that
\[ \|\rho_{\epsilon_1} * \nabla u^{(1)} - \nabla u^{(1)}\|_{L^\infty(\Omega_1)} < 2^{-1}. \]
Let
\[ \delta_2 = \min\{2^{-2} \epsilon, \delta_1 \epsilon_1 / 2\}. \]
Since $\nabla u^{(1)} \in U_1 \subset U_{rc}^2$ in $\Omega^{(1)}$, it follows from Lemma [3.3] that there exist a map $u^{(2)} \in C^1(\Omega^{(1)}; \mathbb{R}^m)$ and an open set $\Omega^{(2)} \subset \subset \Omega^{(1)}$ with $|\partial \Omega^{(2)}| = 0$ such that
\[
\begin{cases}
  u^{(2)}(x) = u^{(1)}(x) = v(x) & \text{for all } x \text{ near } \partial \Omega^{(1)}, \\
  \nabla u^{(2)} \in U_{rc}^2 & \text{in } \Omega^{(1)}, \\
  \nabla u^{(2)} \in U_2 & \text{in } \Omega^{(2)}, \\
  |\Omega^{(1)} \setminus \Omega^{(2)}| < \delta_2 |\Omega^{(1)}|, \\
  \|u^{(2)} - u^{(1)}\|_{L^\infty(\Omega^{(1)})} < \delta_2.
\end{cases}
\]
Next, choose an $0 < \epsilon_2 < \min\{\epsilon_1, 2^{-2}\}$ such that
\[ \|\rho_{\epsilon_2} * \nabla u^{(2)} - \nabla u^{(2)}\|_{L^\infty(\Omega_2^0)} < 2^{-2}. \]
Let
\[ \delta_3 = \min\{2^{-3} \epsilon, \delta_2 \epsilon_2 / 2\}. \]
Since $\nabla u^{(2)} \in U_{rc}^2 \subset U_{rc}^3$ in $\Omega^{(1)}$, it follows again from Lemma [3.3] that there exist a map $u^{(3)} \in C^1(\Omega^{(1)}; \mathbb{R}^m)$ and an open set $\Omega^{(3)} \subset \subset \Omega^{(1)}$ with $|\partial \Omega^{(3)}| = 0$ such that
\[
\begin{cases}
  u^{(3)}(x) = u^{(2)}(x) = v(x) & \text{for all } x \text{ near } \partial \Omega^{(1)}, \\
  \nabla u^{(3)} \in U_{rc}^3 & \text{in } \Omega^{(1)}, \\
  \nabla u^{(3)} \in U_3 & \text{in } \Omega^{(3)}, \\
  |\Omega^{(1)} \setminus \Omega^{(3)}| < \delta_3 |\Omega^{(1)}|, \\
  \|u^{(3)} - u^{(2)}\|_{L^\infty(\Omega^{(1)})} < \delta_3.
\end{cases}
\]
Repeating this process indefinitely, we obtain a sequence $(u^{(j)})_{j=2}^{\infty}$ in $C^1(\Omega^{(1)}; \mathbb{R}^m)$, a sequence of open sets $\Omega^{(j)} \subset \subset \Omega^{(1)}$ with $|\partial \Omega^{(j)}| = 0$ ($j \geq 2$), and a decreasing sequence $(\epsilon_j)_{j=1}^{\infty}$ in $(0, 1/2)$ with $0 < \epsilon_j < 2^{-j}$ such that for every integer $j \geq 2$, we have
\[
\begin{cases}
  \delta_j := \min\{2^{-j} \epsilon, \delta_{j-1} \epsilon_{j-1} / 2\}, \\
  \|\rho_{\epsilon_j} * \nabla u^{(j)} - \nabla u^{(j)}\|_{L^\infty(\Omega_j)} < 2^{-j}, \\
  u^{(j)}(x) = v(x) & \text{for all } x \text{ near } \partial \Omega^{(1)}, \\
  \nabla u^{(j)} \in U_{rc}^j & \text{in } \Omega^{(1)}, \\
  \nabla u^{(j)} \in U_j & \text{in } \Omega^{(j)}, \\
  |\Omega^{(1)} \setminus \Omega^{(j)}| < \delta_j |\Omega^{(1)}|, \\
  \|u^{(j)} - u^{(j-1)}\|_{L^\infty(\Omega^{(1)})} < \delta_j.
\end{cases}
\]
We then extend $u^{(j)} \equiv v$ on $\Omega \setminus \Omega^{(1)}$ for all $j \geq 1$.

Since $\sum_{j=2}^{\infty} \delta_j < 2^{-2} < \infty$ and $U_{rc}^j$ ($j \in \mathbb{N}$) are uniformly bounded, we have
\[ \|u^{(j)} - u\|_{L^\infty(\Omega)} \rightarrow 0 \quad \text{for some } u \in v + W_0^{1, \infty}(\Omega; \mathbb{R}^m). \]
and
\[ \|u - v\|_{L^\infty(\Omega)} \leq \frac{\varepsilon}{2} < \varepsilon. \]

It now remains to show that \( \nabla u \in K \) a.e. in \( \Omega \). Note
\[
\|\nabla u^{(j)} - \nabla u\|_{L^1(\Omega)} \leq \|\nabla u^{(j)} - \nabla u\|_{L^1(\Omega \setminus \Omega_j)} + \|\nabla u^{(j)} - \rho_{\varepsilon_j} * \nabla u\|_{L^1(\Omega_j)} + \|\rho_{\varepsilon_j} * (\nabla u^{(j)} - \nabla u)\|_{L^1(\Omega_j)} + \|\rho_{\varepsilon_j} * \nabla u - \nabla u\|_{L^1(\Omega_j)}
\]
\[ =: I_{1,j} + I_{2,j} + I_{3,j} + I_{4,j}. \]

As \( j \to \infty \),
\[
I_{1,j} \leq C|\Omega \setminus \Omega_j| \to 0,
\]
\[
I_{2,j} \leq 2^{-j}|\Omega| \to 0,
\]
\[
I_{3,j} \leq \|\nabla \rho_{\varepsilon_j} * (u^{(j)} - u)\|_{L^1(\Omega_j)} \leq \frac{C}{\varepsilon_j} \sum_{i=j+1}^{\infty} \delta_i \leq \frac{C}{\varepsilon} \sum_{i=j}^{\infty} \delta_i \to 0,
\]
\[
I_{4,j} \leq \|\rho_{\varepsilon_j} \ast \nabla u - \nabla u\|_{L^1(\Omega)} \to 0, \quad \text{with } \nabla u := 0 \text{ outside } \Omega.
\]

Thus after passing to a subsequence if necessary, we have \( \nabla u^{(j)} \to \nabla u \) a.e. in \( \Omega \). We now claim that for a.e. \( x \in \Omega \), we have \( \nabla u^{(j)}(x) \in U_j \) for infinitely many indices \( j \in \mathbb{N} \). Suppose on the contrary that there is a set \( N \subset \Omega \) of positive measure such that for each \( x \in N \), we have \( \nabla u^{(j)}(x) \notin U_j \) for only finitely many indices \( j \in \mathbb{N} \). For each \( k \in \mathbb{N} \), let
\[
N_k = \{ x \in N \mid \nabla u^{(j)}(x) \notin U_j \quad \text{for all } j > k \};
\]
then \( N_1 \subset N_2 \subset \cdots \) and \( N = \bigcup_{k \in \mathbb{N}} N_k \). Choose a \( k_0 \in \mathbb{N} \) so large that \( |N_{k_0}| \geq |N|/2 > 0 \). Then for all \( x \in N_{k_0} \), we have \( \nabla u^{(j)}(x) \notin U_j \) for all \( j > k_0 \). By the construction above, we thus have
\[
N_{k_0} \subset \Omega \setminus \Omega^{(j)} \quad \forall j > k_0.
\]

As \( |\Omega \setminus \Omega^{(j)}| < \delta_j \to 0 \) as \( j \to \infty \), we have \( |N_{k_0}| = 0 \), a contradiction. Thus, for a.e. \( x \in \Omega \), we have that \( \nabla u^{(j)}(x) \to \nabla u(x) \) in \( \Sigma \), and that there is an increasing sequence \( \{j_k\}_{k \in \mathbb{N}} \) in \( \mathbb{N} \) such that
\[
\nabla u^{(j_k)}(x) \in U_{j_k} \quad \forall k \in \mathbb{N}.
\]
Since \( \{U_j\}_{j \in \mathbb{N}} \) is an in-approximation of \( K \) in \( \Sigma \), we now have for such an \( x \in \Omega \) that
\[
\nabla u(x) \in K;
\]
hence \( \nabla u \in K \) a.e. in \( \Omega \).

The proof is now complete. \( \square \)

5. Proof of the applications

In this section, we finish the proof of Theorem 5.3 and Corollary 5.5.
5.1. Proof of Theorem 1.4. We first prove Theorem 1.4 under an additional hypothesis that the linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is injective, where $m \geq n \geq 2$. We follow the notations in the beginning of the full proof below. Fix an integer $k_0 \geq 1$ so large that $\eta \in \{ \xi \in U \mid \text{dist}(\xi, K) > 1/k_0 \} =: U_1$.

For $k = 2, 3, \cdots$, define $U_k = \{ \xi \in U \mid \frac{1}{k_0 + k + 1} < \text{dist}(\xi, K) < \frac{1}{k_0 + k} \}$.

Then it is easy to see that $\{ U_k \}_{k \in \mathbb{N}}$ is an in-approximation of $K$ in $\Sigma_t$ with $\nabla v_{\eta, \gamma} = \eta \in U_1$. Therefore, the result follows from Theorem 1.3. However, as explained in Introduction, we should perform a more careful justification for the general case without such an additional assumption. We adopt the Baire category framework for the proof below.

Proof of Theorem 1.4. Set $\Sigma_t = \{ \xi \in M^{m \times n} \mid L(\xi) = t \}$ and $B_t = \{ \xi \in \Sigma_t \mid ||\xi|| < 1 \}$; then $\eta \in B_t$ and $\eta_{a \otimes b}^+ \in \partial|_{\Sigma_t} B_t$. For each $\alpha > 0$, let $V_\alpha = \{ \xi \in \Sigma_t \mid \text{dist}(\xi, \eta_{a \otimes b}^+ \eta_{a \otimes b}^-) < \alpha \}$, where $\eta_{a \otimes b}^+ \eta_{a \otimes b}^-$ denotes the straight line in $\Sigma_t$ passing through $\eta_{a \otimes b}^\pm$. Choose an $\alpha_\epsilon > 0$ so small that $\bar{V_\alpha} \cap \partial|_{\Sigma_t} B_t$ is the disjoint union of two connected sets $K^\pm$ with $\eta_{a \otimes b}^\pm \in K^\pm$ such that $\text{diam}(K^\pm) < \epsilon/2$. Then set $K = K^+ \cup K^-$ and $U = V_{\alpha_\epsilon} \cap B_t$.

Define the admissible class $A$ as $A = \{ v \in v_{\eta, \gamma} + C^\infty_c(\Omega; \mathbb{R}^m) \mid \nabla v \in U \text{ in } \Omega, \|v - v_{\eta, \gamma}\|_{L^\infty(\Omega)} < \epsilon/2 \}$; then $v_{\eta, \gamma} \in A$ $\neq \emptyset$. For each $\delta > 0$, define the $\delta$-approximating class $A_\delta$ as $A_\delta = \{ v \in A \mid \int_{\Omega} \text{dist}(\nabla v(x), K) \, dx < \delta |\Omega| \}$.

We now divide the proof into several steps as follows.

**Claim:** For each $\delta > 0$,

$A_\delta$ is dense in $A$ with respect to the $L^\infty(\Omega; \mathbb{R}^m)$-norm.

Suppose for the moment that Claim holds. We now generate solutions to problem (1.4) under the Baire category framework.

**Baire’s category method:** Let $\mathcal{X}$ denote the closure of $A$ in the space $L^\infty(\Omega; \mathbb{R}^m)$. Then $(\mathcal{X}, \| \cdot \|_{L^\infty(\Omega)})$ is a nonempty complete metric space. As $U$ is bounded in $\Sigma_t$, we easily see that $\mathcal{X} \subset v_{\eta, \gamma} + W^{1, \infty}_0(\Omega; \mathbb{R}^m)$ and that $\|u - v_{\eta, \gamma}\|_{L^\infty(\Omega)} \leq \epsilon/2 < \epsilon$ for all $u \in \mathcal{X}$. Since the gradient operator $\nabla : \mathcal{X} \to L^1(\Omega; M^{m \times n})$ is a Baire-one map [9, Proposition 10.17],
it follows from the Baire Category Theorem [9, Theorem 10.15] that the set \( C \) of points of continuity for the operator \( \nabla \) is dense in \( X \).

**Solution set \( C \):** We now check that every map \( u \in C \) is a solution to problem (1.4). Let \( u \in C \). From the previous step, we have

\[
\| u - u_{\eta, \gamma} \|_{L^\infty(\Omega)} < \epsilon.
\]

By the definition of \( X \), we can choose a sequence \( \{ \tilde{u}_k \}_{k \in \mathbb{N}} \) in \( A \) such that \( \| \tilde{u}_k - u \|_{L^\infty(\Omega)} \to 0 \) as \( k \to \infty \). By the result of Claim above, for each \( k \in \mathbb{N} \), we can choose a map \( u_k \in A_{1/k} \) such that \( \| u_k - \tilde{u}_k \|_{L^\infty(\Omega)} < 1/k \); thus \( \| u_k - u \|_{L^\infty(\Omega)} \to 0 \). Since \( u \in C \), we now have \( \nabla u_k \to \nabla u \) in \( L^1(\Omega; \mathbb{M}^{m \times n}) \), and so \( \nabla u_k(x) \to \nabla u(x) \) in \( \mathbb{M}^{m \times n} \) for a.e. \( x \in \Omega \) after passing to a subsequence if necessary. On the other hand, from \( u_k \in A_{1/k} \),

\[
\int_\Omega \text{dist}(\nabla u_k(x), K) \, dx < \frac{1}{k} |\Omega| \to 0.
\]

Since \( K \) is compact and \( U \) is bounded, it follows from the Dominate Convergence Theorem that the map \( u \in v_{\eta, \gamma} + W^{1, \infty}_0(\Omega; \mathbb{R}^m) \) satisfies the differential inclusion

\[
\nabla u \in K \quad \text{a.e. in } \Omega.
\]

This together with (5.1) implies that \( u \) is a solution to (1.4).

**Infinitely many solutions:** To show that there are infinitely many solutions to problem (1.4), it now suffices to check that \( C \) has infinitely many elements. Suppose on the contrary that \( C \) has only finitely many elements. Since \( C \) is dense in \( X \), we thus have

\[
v_{\eta, \gamma} \in X = \overline{C} = C.
\]

By the previous step, we arrive at the conclusion that \( v_{\eta, \gamma} \) is a solution to (1.4), a contradiction. Therefore, \( C \) has infinitely many elements.

**Proof of Claim:** To finish the proof, it only remains to verify Claim above. Let \( \delta > 0 \), \( \theta > 0 \) and \( v \in A \). We will show that there exists a map \( v_\theta \in A_\delta \) such that \( \| v_\theta - v \|_{L^\infty(\Omega)} < \theta \).

Since \( v \in A \), there exists a map \( \varphi \in C_0^\infty(\Omega; \mathbb{R}^m) \) such that \( v = v_{\eta, \gamma} + \varphi \) in \( \Omega \), \( \| \varphi \|_{L^\infty(\Omega)} < \epsilon/2 \), and \( \eta + \nabla \varphi \in U \) in \( \Omega \). Set

\[
\epsilon' = 2^{-1} (2^{-1} \epsilon - \| \varphi \|_{L^\infty(\Omega)}) > 0.
\]

Choose finitely many open cubes \( Q_1, \ldots, Q_N \subset \subset \Omega \), parallel to the axes, such that

\[
|\Omega \setminus \bigcup_{i=1}^N Q_i| < \frac{\delta |\Omega|}{4}.
\]

Fix an index \( 1 \leq i \leq N \). Let us write

\[
d_i = \min_{Q_i} \text{dist}(\eta + \nabla \varphi, \partial |_{Q_i} U) > 0.
\]
Choose finitely many dyadic cubes $Q_{i,1}, \ldots, Q_{i,N_i} \subset Q_i$ with $|Q_i \setminus \bigcup_{j=1}^{N_i} Q_{i,j}| = 0$ such that

\begin{equation}
|\nabla \varphi(x) - \nabla \varphi(x)| < \min \left\{ \frac{d_i}{2}, \frac{\delta}{16} \right\}
\end{equation}

for all $x,y \in \bar{Q}_{i,j}$ and all $1 \leq j \leq N_i$. For each $1 \leq j \leq N_i$, let $x_{i,j}$ denote the center of the cube $Q_{i,j}$ and write $\xi_{i,j} = \nabla v(x_{i,j}) = \eta + \nabla \varphi(x_{i,j}) \in U$.

Define

\begin{equation}
\Lambda_i = \{ j \in \{1, \ldots, N_i\} \mid \text{dist}(\xi_{i,j}, K) > \frac{\delta}{8} \}.
\end{equation}

Fix an index $j \in \Lambda_i$. Then we can choose two numbers $s_{i,j}^+ > 0 > s_{i,j}^-$ such that

\begin{equation}
\xi_{i,j} + s_{i,j}^+ a \otimes b \in U \quad \text{and} \quad \text{dist}(\xi_{i,j} + s_{i,j}^+ a \otimes b, K^\pm) = \frac{\delta}{8}.
\end{equation}

Now, thanks to Theorem 3.1, for any given $\tau > 0$, we can choose a map $\psi_{i,j} \in C^\infty_c(Q_{i,j}; \mathbb{R}^m)$ and two disjoint open sets $Q_{i,j}^+ \subset \subset Q_{i,j}$ satisfying the following:

(a) $\mathcal{L}(\nabla \psi_{i,j}) = 0$ in $Q_{i,j}$,

(b) $\text{dist}(\nabla \psi_{i,j}, [-\lambda_{i,j}(s_{i,j}^+ - s_{i,j}^-)a \otimes b, (1 - \lambda_{i,j})(s_{i,j}^+ - s_{i,j}^-)a \otimes b]) < \tau$ in $Q_{i,j}$,

(c) $\nabla \psi_{i,j}(x) = \begin{cases} (1 - \lambda_{i,j})(s_{i,j}^+ - s_{i,j}^-)a \otimes b & \forall x \in Q_{i,j}^+, \\ -\lambda_{i,j}(s_{i,j}^+ - s_{i,j}^-)a \otimes b & \forall x \in Q_{i,j}^- \end{cases}$,

(d) $|Q_{i,j}^+| - \lambda_{i,j}|Q_{i,j}| < \tau$, $|Q_{i,j}^-| - (1 - \lambda_{i,j})|Q_{i,j}| < \tau$,

(e) $\|\psi_{i,j}\|_{L^\infty(Q_{i,j})} < \tau$,

where $\lambda_{i,j} := \frac{-s_{i,j}^-}{s_{i,j}^+ - s_{i,j}^-} \in (0, 1)$. Here, we choose

\begin{equation}
0 < \tau < \min \left\{ \frac{d_i}{2}, \frac{\delta}{16}, \theta', \frac{\delta(Q_{i,j})}{8} \right\}.
\end{equation}

Define

$$v_\theta = v_{\eta, \gamma} + \varphi + \sum_{1 \leq i \leq N_i, j \in \Lambda_i} \psi_{i,j} \chi_{Q_{i,j}} \text{ in } \Omega.$$}

We now check that $v_\theta$ is a desired map for the proof of Claim. By definition, we have

$$v_\theta \in v_{\eta, \gamma} + C^\infty_c(\Omega; \mathbb{R}^m).$$

Note from the definition of $v_\theta$, (e), (5.8) and (5.2) that

$$\|v_\theta - v_{\eta, \gamma}\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)} + \epsilon' < \frac{\epsilon}{2}.$$
Let $1 \leq i \leq N$, $j \in \Lambda_i$, and $x \in Q_{i,j}$. For all $s_{i,j}^- \leq s \leq s_{i,j}^+$, we have from (5.5), (b) and (5.8) that

\[
|\eta + \nabla \varphi(x) + \nabla \psi_{i,j}(x) - (\xi_{i,j} + sa \otimes b)| \\
\leq |\nabla \varphi(x) - \nabla \varphi(x_{i,j})| + |\nabla \psi_{i,j}(x) - sa \otimes b| \\
< \min \left\{ \frac{d_i}{2}, \frac{\delta}{16} \right\} + \text{dist}(\nabla \psi_{i,j}(x), [s_{i,j}^+ a \otimes b, s_{i,j}^- a \otimes b]) < \min \left\{ d_i, \frac{\delta}{8} \right\}.
\]

This implies that

\[
\text{dist}(\nabla v_\theta(x), [\xi_{i,j} + s_{i,j}^+ a \otimes b, \xi_{i,j} + s_{i,j}^- a \otimes b]) < \min \left\{ d_i, \frac{\delta}{8} \right\}.
\]

Also, from (a), we have $\mathcal{L}(\nabla v_\theta(x)) = \mathcal{L}(\eta + \nabla \varphi(x)) = t$; hence, $\nabla v_\theta(x) \in \Sigma_t$. Thus, from these two observations together with (5.4) and (5.7), we have $\nabla v_\theta(x) \in U$. By the definition of $v_\theta$, we now have

\[
\nabla v_\theta \in U \quad \text{in } \Omega;
\]

therefore, $v_\theta \in \mathcal{A}$. Also, from (e) and (5.8), we get

\[
\|v_\theta - v\|_{L^\infty(\Omega)} < \tau < \theta.
\]

Next, observe

\[
\int_{\Omega} \text{dist}(\nabla v_\theta(x), K) \, dx = \int_{\Omega \setminus \bigcup_{i=1}^N Q_i} \text{dist}(\nabla v(x), K) \, dx \\
+ \sum_{1 \leq i \leq N, 1 \leq j \leq N_i, j \notin \Lambda_i} \int_{Q_{i,j}} \text{dist}(\nabla v(x), K) \, dx \\
+ \sum_{1 \leq i \leq N, j \in \Lambda_i} \int_{Q_{i,j} \setminus (Q_{i,j}^+ \cup Q_{i,j}^-)} \text{dist}(\eta + \nabla \varphi(x) + \nabla \psi_{i,j}(x), K) \, dx \\
+ \sum_{1 \leq i \leq N, j \in \Lambda_i} \int_{Q_{i,j}^+ \cup Q_{i,j}^-} \text{dist}(\eta + \nabla \varphi(x) + \nabla \psi_{i,j}(x), K) \, dx \\
=: I_1 + I_2 + I_3 + I_4.
\]

Since $\nabla v_\theta \in U$ in $\Omega$, we have $\text{dist}(\nabla v_\theta, K) \leq 1$ in $\Omega$. We now estimate:

\[
I_1 \leq |\Omega \setminus \bigcup_{i=1}^N Q_i| < \frac{\delta|\Omega|}{4}, \quad \text{(by (5.3))}
\]

\[
I_2 \leq \sum_{1 \leq i \leq N, 1 \leq j \leq N_i, j \notin \Lambda_i} \frac{3\delta|Q_{i,j}|}{16} < \frac{\delta|\Omega|}{4}, \quad \text{(by (5.5) and (5.6))}
\]

\[
I_3 \leq \sum_{1 \leq i \leq N, j \in \Lambda_i} |Q_{i,j} \setminus (Q_{i,j}^+ \cup Q_{i,j}^-)| < \frac{\delta|\Omega|}{4}, \quad \text{(by (d) and (5.8))}
\]

\[
I_4 \leq \sum_{1 \leq i \leq N, j \in \Lambda_i} \frac{3\delta|Q_{i,j}|}{16} < \frac{\delta|\Omega|}{4}, \quad \text{(by (c), (5.5) and (5.7))}
\]

thus $I_1 + I_2 + I_3 + I_4 < \delta|\Omega|$. Therefore, we have $v_\theta \in \mathcal{A}_8$, and the proof of Claim is complete.
follows that the product $L$ determined; then for $\leq$ where $1$

where each blank component in (6.1) is zero. From (3.1) and rank$(L) = \nu, \gamma$.

We will find a linear differential operator $\Phi : C^1(\Omega; \mathbb{R}^n) \to C^0(\Omega; \mathbb{R}^m)$ such that

(6.2) $\mathcal{L}(\nabla \Phi v) \equiv 0 \quad \forall v \in C^2(\Omega; \mathbb{R}^m)$.

So our candidate for such a $\Phi = (\Phi^1, \cdots, \Phi^m)$ is of the form

(6.3) $\Phi^i v = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl} v^k_{x_l}$,

where $1 \leq i \leq m$, $v \in C^1(\Omega; \mathbb{R}^m)$, and $a^i_{kl}$'s are real constants to be determined; then for $v \in C^2(\Omega; \mathbb{R}^m)$, $1 \leq i \leq m$, and $1 \leq j \leq n$,

$\partial_{x_j} \Phi^i v = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl} v^k_{x_lx_j}$.
Rewriting (6.2) with this form of $\nabla \Phi v$ for $v \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, we have

$$0 \equiv \sum_{1 \leq k \leq m, 1 \leq j, l \leq n} L_{1j} a_{kl}^1 v_{x_j x_l}^k + \sum_{i=2}^r \sum_{1 \leq k \leq m, 1 \leq l \leq n} L_{ii} a_{kl}^i v_{x_j x_l}^k$$

$$= \sum_{k=1}^m \left( L_{11} a_{k1}^1 v_{x_1 x_1}^k + \sum_{j=2}^r \left( L_{1j} a_{kj}^1 + L_{jj} a_{kj}^j \right) v_{x_j x_j}^k + \sum_{l=r+1}^n L_{1j} a_{kl}^1 v_{x_j x_l}^k \right)$$

$$+ \sum_{l=2}^r \left( L_{1l} a_{k1}^1 + L_{ll} a_{k1}^l \right) v_{x_1 x_l}^k + \sum_{l=r+1}^n \left( L_{1l} a_{kl}^1 + L_{ll} a_{kl}^l \right) v_{x_l x_l}^k$$

$$+ \sum_{2 \leq j < l \leq r} (L_{1j} a_{kl}^1 + L_{lj} a_{kl}^j + L_{jl} a_{kl}^j + L_{ll} a_{kl}^l) v_{x_j x_l}^k$$

$$+ \sum_{r+1 \leq j < l \leq n} (L_{1j} a_{kl}^1 + L_{lj} a_{kl}^j) v_{x_j x_l}^k$$.

Should (6.2) hold, it is thus sufficient to solve the following algebraic system for each $k = 1, \cdots, m$ (after adjusting the letters for some indices):

(6.4) $L_{11} a_{k1}^1 = 0,$

(6.5) $L_{1j} a_{kj}^1 + L_{jj} a_{kj}^j = 0$ \quad $\forall j = 2, \cdots, r,$

(6.6) $L_{1l} a_{k1}^1 + L_{lj} a_{kj}^j + L_{jl} a_{kj}^j = 0$ \quad $\forall j = 2, \cdots, r,$

(6.7) $L_{1l} a_{k1}^1 + L_{lj} a_{kj}^j + L_{ll} a_{kj}^j = 0$ \quad $\forall j = 3, \cdots, r,$

(6.8) $L_{1j} a_{kj}^1 = 0$ \quad $\forall j = r + 1, \cdots, n,$

(6.9) $L_{1l} a_{k1}^1 + L_{lj} a_{kj}^j = 0$ \quad $\forall j = r + 1, \cdots, n,$

(6.10) $L_{1l} a_{k1}^1 + L_{lj} a_{kj}^j + L_{ll} a_{kj}^j = 0$ \quad $\forall j = r + 1, \cdots, n,$

(6.11) $L_{1l} a_{k1}^1 + L_{lj} a_{kj}^j = 0$ \quad $\forall j = r + 1, \cdots, n,$

Although these systems have infinitely many solutions, we will solve those in a way for a later purpose that the matrix $(a_{kl}^j)_{2 \leq j, k \leq m} \in \mathbb{M}^{(m-1) \times (m-1)}$ fulfills

(6.12) $a_{21}^j = a_j$ \quad $\forall j = 2, \cdots, m,$ and $a_{k1}^j = 0$ otherwise.

Firstly, we let the coefficients $a_{kl}^j$ ($1 \leq i, k \leq m, 1 \leq l \leq n$) that do not appear in systems (6.4)–(6.11) ($k = 1, \cdots, m$) be zero with an exception that we set $a_{21}^j = a_j$ for $j = r + 1, \cdots, m$ to reflect (6.12). Secondly, for $1 \leq k \leq m, k \neq 2$, let us take the trivial (i.e., zero) solution of system (6.4)–(6.11). Finally, we take $k = 2$ and solve system (6.4)–(6.11) as follows.
with (6.12) satisfied. Since $L_{11} \neq 0$, we set $a_{21}^i = 0$; then (6.4) is satisfied. So we set
\[ a_{21}^j = -\frac{L_{11}}{L_{jj}} a_{2j}^1, \quad a_{2j}^1 = -\frac{L_{jj}}{L_{11}} a_j \quad \forall j = 2, \cdots, r; \]
then (6.6) and (6.12) hold. Next, set
\[ a_{2j}^i = -\frac{L_{1j} L_{jj}}{L_{11}} a_{2j}^1 = \frac{L_{jj}}{L_{11}} a_j \quad \forall j = 2, \cdots, r; \]
then (6.5) is fulfilled. Set
\[ a_{2j}^l = \frac{L_{1l} a_{2j}^1 + L_{lj} a_{2l}^1}{L_{ll}} = \frac{L_{11} L_{jj} a_j + L_{lj} L_{ll} a_l}{L_{ll} L_{11}}, \quad a_{2j}^l = 0 \]
for $j = 3, \cdots, r$ and $l = 2, \cdots, j - 1$; then (6.7) holds. Set
\[ a_{2j}^1 = 0 \quad \forall j = r + 1, \cdots, n; \]
then (6.8) and (6.9) are satisfied. Lastly, set
\[ a_{2j}^i = 0, \quad a_{2j}^l = -\frac{L_{1j}}{L_{ll}} a_{2l}^1 = \frac{L_{lj}}{L_{11}} a_l \quad \forall j = r + 1, \cdots, n, \forall l = 2, \cdots, r; \]
then (6.10) and (6.11) hold. In summary, we have determined the coefficients $a_{kl}^i (1 \leq i, k \leq m, 1 \leq l \leq n)$ in such a way that system (6.3)–(6.11) holds for each $k = 1, \cdots, m$ and that (6.12) is also satisfied. Therefore, (1) follows from (6.2) and (6.3).

To prove (2), without loss of generality, we can assume $\Omega = (0, 1)^n \subset \mathbb{R}^n$. Let $\tau > 0$ be given. Let $u = (u^1, \cdots, u^m) \in C^\infty(\Omega; \mathbb{R}^m)$ be a function to be determined. Suppose $u$ depends only on the first variable $x_1 \in (0, 1)$. We wish to have
\[ \nabla \Phi u(x) \in \{-\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1\} \]
for all $x \in \Omega$ except in a set of small measure. Since $u(x) = u(x_1)$, it follows from (6.3) that for $1 \leq i \leq m$ and $1 \leq j \leq n$,
\[ \Phi^i u = \sum_{k=1}^m a_{k1}^i u^k_{x_1}; \quad \text{thus } \partial_{x_j} \Phi^i u = \sum_{k=1}^m a_{k1}^i u^k_{x_1 x_j}. \]
As $a_{k1}^i = 0$ for $k = 1, \cdots, m$, we have $\partial_{x_j} \Phi^i u = \sum_{k=1}^m a_{k1}^i u^k_{x_1 x_j} = 0$ for $j = 1, \cdots, n$. We first set $u^1 \equiv 0$ in $\Omega$. Then from (6.12), it follows that for $i = 2, \cdots, m$,
\[ \partial_{x_j} \Phi^i u = \sum_{k=2}^m a_{k1}^i u^k_{x_1 x_j} = a_{21}^i u^2_{x_1 x_j} = a_{21}^i u^2_{x_1 x_j} = \begin{cases} a_i u^2_{x_1 x_1} & \text{if } j = 1, \\ 0 & \text{if } j = 2, \cdots, n. \end{cases} \]
As $a_1 = 0$, we thus have that for $x \in \Omega$,
\[ \nabla \Phi u(x) = (u^2)''(x_1) a \otimes e_1. \]
For irrelevant components of $u$, we simply take $u^3 = \cdots = u^m \equiv 0$ in $\Omega$. Lastly, for a number $\delta > 0$ to be chosen later, we choose a function $u^2(x_1) \in C^\infty_c(0, 1)$ such that there exist two disjoint open sets $I_1, I_2 \subset \subset
(0, 1) satisfying \( |I_1| - \lambda | < \tau/2, \ |I_2| - (1 - \lambda) | < \tau/2, \ |u^2|_{L^\infty((0,1)} < \delta, \ \|(u^2)''\|_{L^\infty((0,1)} < \delta, \ -\lambda \leq (u^2)''(x_1) \leq 1 - \lambda \) for \( x_1 \in (0, 1) \), and
\[
(u^2)''(x_1) = \begin{cases} 
1 - \lambda & \text{if } x_1 \in I_1, \\
-\lambda & \text{if } x_1 \in I_2.
\end{cases}
\]
In particular,
\[(6.13) \quad \nabla \Phi u(x) \in [-\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1] \ \forall x \in \Omega.
\]
We now choose an open set \( \Omega''_\tau \subset \subset \Omega' := (0, 1)^{n-1} \) with \( |\Omega' \setminus \Omega''_\tau| < \tau/2 \) and a function \( \eta \in C^\infty_c(\Omega') \) so that
\[0 \leq \eta \leq 1 \text{ in } \Omega', \ \eta \equiv 1 \text{ in } \Omega''_\tau, \text{ and } |\nabla_x \eta| < \frac{C}{\tau^2} \ (i = 1, 2) \text{ in } \Omega',\]
where \( x' = (x_2, \cdots, x_n) \in \Omega' \) and the constant \( C > 0 \) is independent of \( \tau \).
Now, we define \( g(x) = \eta(x')u(x_1) \in C^\infty_c(\Omega; \mathbb R^m) \). Set \( \Omega_A = I_1 \times \Omega''_\tau \) and \( \Omega_B = I_2 \times \Omega''_{\tau'} \). Clearly, (a) follows from (1). As \( g(x) = u(x_1) = u(x) \) for \( x \in \Omega_A \cup \Omega_B \), we have
\[
\nabla \Phi g(x) = \begin{cases} 
(1 - \lambda)a \otimes e_1 & \text{if } x \in \Omega_A, \\
-\lambda a \otimes e_1 & \text{if } x \in \Omega_B;
\end{cases}
\]
hence (c) holds. Also,
\[|\Omega_A| - \lambda |\Omega| = |\Omega_A| - \lambda = |I_1| |\Omega''_\tau| - \lambda = |I_1| - |I_1| |\Omega' \setminus \Omega''_\tau| - \lambda| < \tau,\]
and likewise
\[|\Omega_B| - (1 - \lambda) |\Omega| < \tau;\]
so (d) is satisfied. Note that for \( i = 1, \cdots, m, \)
\[\Phi^i g = \Phi^i(\eta u) = \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl}(\eta u^k)_{x_l} = \eta \Phi^i u + \sum_{1 \leq k \leq m, 1 \leq l \leq n} a^i_{kl} \eta_{x_l} u^k = \eta \Phi^i u + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_l} = \eta a^i_{21} u^2_{x_1} + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_l}.
\]
So
\[|\Phi g|_{L^\infty(\Omega)} \leq C \max\{\delta, \delta \tau^{-1}\} < \tau\]
if \( \delta > 0 \) is chosen small enough; so (e) holds. Next, for \( i = 1, \cdots, m \) and \( j = 1, \cdots, n, \)
\[\partial_{x_j} \Phi^i g = \eta_{x_j} a^i_{21} u^2_{x_1} + \eta \partial_{x_j} \Phi^i u + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_l} + u^2 \sum_{l=1}^n a^i_{2l} \eta_{x_l x_j};\]
hence from (6.13),
\[\text{dist}(\nabla \Phi g, [-\lambda a \otimes e_1, (1 - \lambda) a \otimes e_1]) \leq C \max\{\delta \tau^{-1}, \delta \tau^{-2}\} < \tau \text{ in } \Omega\]
if \( \delta \) is sufficiently small. Thus (b) is fulfilled.
(Case 2): Assume that $L_{i1} = 0$ for all $i = 2, \ldots, m$, that is,

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ 0 & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & L_{m2} & \cdots & L_{mn} \end{pmatrix} \in M^{m \times n}$$

and that

$$A - B = a \otimes e_1$$

for some nonzero vector $a \in \mathbb{R}^m$; then by (3.1), we have $L_{11} \neq 0$.

Set

$$\hat{L} = \begin{pmatrix} L_{22} & \cdots & L_{2n} \\ \vdots & \ddots & \vdots \\ L_{m2} & \cdots & L_{mn} \end{pmatrix} \in M^{(m-1) \times (n-1)}.$$

As $L_{11} \neq 0$ and $\text{rank}(L) = r$, we must have $\text{rank}(\hat{L}) = r - 1$. Using the singular value decomposition theorem, there exist two matrices $\hat{U} \in O(m-1)$ and $\hat{V} \in O(n-1)$ such that

$$\hat{U}^T \hat{L} \hat{V} = \text{diag}(\sigma_2, \ldots, \sigma_r, 0, \ldots, 0) \in M^{(m-1) \times (n-1)},$$

where $\sigma_2, \ldots, \sigma_r$ are the positive singular values of $\hat{L}$. Define

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \hat{U} \end{pmatrix} \in O(m), \quad V = \begin{pmatrix} 1 & 0 \\ 0 & \hat{V} \end{pmatrix} \in O(n).$$

Let $L' = U^T LV$, $A' = U^T AV$, and $B' = U^T BV$. Let $L' : M^{m \times n} \to \mathbb{R}$ be the linear map given by

$$L' (\xi') = \sum_{1 \leq i \leq m, 1 \leq j \leq n} L'_{ij} \xi'_{ij} \quad \forall \xi' \in M^{m \times n}.$$

Then, from (6.14), (6.15) and (6.16), it is straightforward to check the following:

\[ \begin{align*} 
A' - B' &= a' \otimes e_1 \quad \text{for some nonzero vector } a' \in \mathbb{R}^m, \\
L' e_1 &\neq 0, \ L'(A) = L'(B), \text{ and} \\
L' \text{ is of the form (6.1) in Case 1 with } \text{rank}(L') = r. 
\end{align*} \]

Thus we can apply the result of Case 1 to find a linear operator $\Phi' : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m)$ satisfying the following:

1. For any open set $\Omega' \subset \mathbb{R}^n$,
   \[ \Phi' (v') \in C^{k-1}(\Omega'; \mathbb{R}^m) \quad \text{whenever } k \in \mathbb{N} \text{ and } v' \in C^k(\Omega'; \mathbb{R}^m) \]

and

\[ L'(\nabla \Phi' v') = 0 \quad \text{in } \Omega' \text{ for all } v' \in C^2(\Omega'; \mathbb{R}^m). \]

2. Let $\Omega' \subset \mathbb{R}^n$ be any bounded domain. For each $\tau > 0$, there exist a function $g' = g'_\tau \in C^\infty_c(\Omega'; \mathbb{R}^m)$ and two disjoint open sets $\Omega'_{A'}, \Omega'_{B'} \subset \subset \Omega'$ such that

a. $\Phi' g' \in C^\infty_c(\Omega'; \mathbb{R}^m)$,
(b') \( \text{dist}(\nabla \Phi' g', [-\lambda(A' - B'), (1 - \lambda)(A' - B')] ) < \tau \) in \( \Omega' \),

(c') \( \nabla \Phi' g'(x) = \begin{cases} (1 - \lambda)(A' - B') & \forall x \in \Omega'_A, \\ -\lambda(A' - B') & \forall x \in \Omega'_B, \end{cases} \)

(d') \( ||\Omega'_A' - \lambda|\Omega'| || < \tau, ||\Omega'_B' - (1 - \lambda)|\Omega'| || < \tau \),

(e') \( ||\Phi' g'||_{L^\infty(\Omega')} < \tau \).

For \( v \in C^1(\mathbb{R}^n; \mathbb{R}^m) \), let \( v'(y) = U^T v(V y) \) for \( y \in \mathbb{R}^m \). We define \( \Phi v(x) = U \Phi' v(V^T x) \) for \( x \in \mathbb{R}^n \), so that \( \Phi v \in C^0(\mathbb{R}^n; \mathbb{R}^m) \). Then it is straightforward to check that properties (1') and (2') of \( \Phi' \) imply respective properties (1) and (2) of the linear operator \( \Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m) \).

(Case 3): Finally, we consider the general case that \( A, B \) and \( L \) are as in the statement of the theorem. As \( |b| = 1 \), there exists an \( R \in O(n) \) such that \( R^T b = e_1 \in \mathbb{R}^n \). Also there exists a symmetric (Householder) matrix \( P \in O(m) \) such that the matrix \( L' := PLR \) has the first column parallel to \( e_1 \in \mathbb{R}^m \). Let

\[ A' = PAR \text{ and } B' = PBR. \]

Then \( A' - B' = a' \otimes e_1 \), where \( a' = Pa \neq 0 \). Note also that \( L' e_1 = PLR b = PLb = 0 \). Define \( L'(\xi') = \sum_{i,j} L_{ij} \xi_{ij} \quad (\xi' \in \mathbb{M}^{m \times n}) \); then \( L'(A') = L(A) = L(B) = L'(B') \). Thus by the result of Case 2, there exists a linear operator \( \Phi' : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m) \) satisfying (1') and (2') above.

For \( v \in C^1(\mathbb{R}^n; \mathbb{R}^m) \), let \( v' \in C^1(\mathbb{R}^n; \mathbb{R}^m) \) be defined by \( v'(y) = P v(R y) \) for \( y \in \mathbb{R}^m \), and define \( \Phi v(x) = P \Phi' v(V^T x) \in C^0(\mathbb{R}^n; \mathbb{R}^m) \). Then it is easy to check that the linear operator \( \Phi : C^1(\mathbb{R}^n; \mathbb{R}^m) \to C^0(\mathbb{R}^n; \mathbb{R}^m) \) satisfies (1) and (2) by (1') and (2') similarly as in Case 2.

References

[1] R. Aumann and S. Hart, Bi-convexity and bi-martingales, Israel J. Math., 54 (2) (1986), 159–180.
[2] J.M. Ball and R.D. James, Fine phase mixtures as minimizers of energy, Arch. Rational Mech. Anal., 100 (1) (1987), 13–52.
[3] J.M. Ball and R.D. James, Proposed experimental tests of a theory of fine structure and the two-well problem, Phil. Trans. Roy. Soc. London A, 338 (1992), 389–450.
[4] E. Casadio-Tarabusi, An algebraic characterization of quasi-convex functions, Ricerche Mat., 42 (1) (1993), 11–24.
[5] M. Chipot and D. Kinderlehrer, Equilibrium configurations of crystals, Arch. Rational Mech. Anal., 103 (3) (1988), 237–277.
[6] M. Chlebík and B. Kirchheim, Rigidity for the four gradient problem, J. Reine Angew. Math., 551 (2002), 1–9.
[7] S. Conti, G. Dolzmann and B. Kirchheim, Existence of Lipschitz minimizers for the three-well problem in solid-solid phase transitions, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (6) (2007), 953–962.
[8] D. Cordoba, D. Faraco and F. Gancedo, Lack of uniqueness for weak solutions of the incompressible porous media equation, Arch. Ration. Mech. Anal., 200 (3) (2011), 725–746.
[9] B. Dacorogna, “Direct methods in the calculus of variations,” Second edition. Applied Mathematical Sciences, 78. Springer, New York, 2008.
[10] B. Dacorogna and P. Marcellini, “Implicit partial differential equations,” Progress in Nonlinear Differential Equations and their Applications, 37. Birkhäuser Boston, Inc., Boston, MA, 1999.

[11] B. Dacorogna and C. Tanteri, Implicit partial differential equations and the constraints of nonlinear elasticity, J. Math. Pures Appl. (9), 81 (4) (2002), 311–341.

[12] C. De Lellis and L. Székelyhidi Jr., The Euler equations as a differential inclusion, Ann. of Math. (2), 170 (3) (2009), 1417–1436.

[13] M. Gromov, Convex integration of differential relations. I, Izv. Akad. Nauk SSSR Ser. Mat., 37 (1973), 329–343.

[14] S. Kim and Y. Koh, Weak solutions for one-dimensional non-convex elastodynamics, Preprint.

[15] S. Kim and B. Yan, Convex integration and infinitely many weak solutions to the Perona-Malik equation in all dimensions, SIAM J. Math. Anal., 47 (4) (2015), 2770–2794.

[16] S. Kim and B. Yan, On Lipschitz solutions for some forward-backward parabolic equations, Preprint.

[17] S. Kim and B. Yan, On Lipschitz solutions for some forward-backward parabolic equations. II: The case against Fourier, Preprint.

[18] N.H. Kuiper, On $C^1$-isometric embeddings. I, Nederl. Akad. Wetensch. Proc. Ser. A., 58 (1955), 545–556.

[19] S. Müller, M.O. Rieger and V. Šverák, Parabolic systems with nowhere smooth solutions, Arch. Rational Mech. Anal., 177 (1) (2005), 1–20.

[20] S. Müller and V. Šverák, Convex integration with constraints and applications to phase transitions and partial differential equations, J. Eur. Math. Soc. (JEMS), 1 (4) (1999), 393–422.

[21] S. Müller and V. Šverák, Convex integration for Lipschitz mappings and counterexamples to regularity, Ann. of Math. (2), 157 (3) (2003), 715–742.

[22] J. Nash, $C^1$ isometric imbeddings, Ann. of Math. (2), 60 (1954), 383–396.

[23] P. Pedregal, Laminates and microstructure, Europ. J. Appl. Math., 4 (2) (1993), 121–149.

[24] L. Poggiolini, Implicit pdes with a linear constraint, Ricerche Mat., 52 (2) (2003), 217–230.

[25] W. Pompe, Explicit construction of piecewise affine mappings with constraints, Bull. Pol. Acad. Sci. Math., 58 (3) (2010), 209–220.

[26] V. Scheffer, “Regularity and irregularity of solutions to nonlinear second-order elliptic systems of partial differential equations and inequalities,” Dissertation, Princeton University, 1974.

[27] R. Shvydkoy, Convex integration for a class of active scalar equations, J. Amer. Math. Soc., 24 (4) (2011), 1159–1174.

[28] L. Tartar, “Some remarks on separately convex functions, in: Microstructure and Phase Transitions,” IMA Vol. Math. Appl. 54 (D. Kinderlehrer, R. D. James, M. Luskin and J. L. Ericksen, eds.), Springer-Verlag, New York, 1993, pp. 191–204.

Institute for Mathematical Sciences, Renmin University of China, Beijing 100872, PRC

E-mail address: kimseo14@ruc.edu.cn