The macro-behavior of agents’ opinion under the influence of an external field

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Abstract
The critical behavior of Ising model on a one-dimensional network, which has long-range connections at distances \( l > 1 \) with the probability \( \Theta(l) \sim l^{-m} \), is studied by using Monte Carlo simulations. Through studying the Ising model on networks with different \( m \) values, this paper discusses the impact of the global correlation, which decays with the increase of \( m \), on the phase transition of the Ising model. Adding the analysis of the finite-size scaling of the order parameter \( \langle M \rangle \), it is observed that in the whole range of \( 0 < m < 2 \), a finite-temperature transition exists, and the critical exponents show consistence with mean-field values, which suggests a mean-field nature of the phase transition.

1 Introduction
Since first suggested by Watts-Strogatz \cite{1} in 1998, the Watts-Strogatz (WS) small world model has been widely and deeply studied. WS model is a very simple network model, but it can reflect small world effect very well. In WS model, vertices are placed on a ring with each vertex has a finite number of \( 2k \) nearest regular connections initially. The connections are then rewired with a probability \( p_r \) to form long-range connections or short paths. By varying the single parameter \( p_r \) which represents disorder from 0 to 1, WS model displays

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phase transition from a regular network to a small world, and end up with a random network at $p_r = 1$. Considering the rewiring of connections may cause isolated vertex, the addition-type WS model was considered later, where the long-range connections are added with a probability $p$ while keeping the initial regular connections unchanged. Phase transition to small world was also observed [2, 3]. Marcelo Kuperman and Guillermo Aberamson generalized WS model in [4]. Besides connections with the two nearest neighboring vertices, with a probability $p$ each of the vertices in the model chooses a non-neighboring vertex to construct a long-range connection between them. The choice of the non-neighboring vertex is governed by the distance-dependent probability distribution:

$$\Theta(l) \propto l^{-m} \quad (1)$$

The case $m = 0$ is equivalent to the addition-type WS model. Through the tune of $m$ values, the global correlation of the network is influenced. After the analysis of the topological characteristic of the network, they figured out that the network also shows a small world nature, and with $m \to \infty$ the network converges to a regular lattice [4, 5, 6, 7].

Analyzing the behavior of a small world network from the point of statistical physics view, researchers focus on its long-range connections or short-paths, which may correspond to global coherence. In networks without any short-path, the transition of information has to pass a long distance $\sim O(N)$ ($N$ is the network size), so global coherence is difficult to reach. With the increasing of the number of short-paths, or the formation of the long-range connections, the system emerges collective behavior. This global coherence and the ubiquitous small world phenomenon in real-life make the study of thermodynamic phase transition on networks with small world property significantly popular [8, 9, 10, 11] and Ising model is one of the most fascinating points.

As a comparatively simple but very important model of statistical physics, Ising model perfectly shows order-disorder transition of the system. The Ising model on a one-dimensional regular lattice does not show phase transition at any finite-temperature and the phase transition on a multi-dimensional regular lattice is not of mean-field nature. The Ising model undergoes phase transitions on the addition-type WS model has been studied in [12, 13, 14]. It has also been shown that in the small world phase the addition-type WS model has a mean-field nature [8, 10]. The critical behavior of the Ising model on one-dimensional network with distance-dependent connections given by Eq.(1) is expected to reflect the nature of the network at different values of $m$ indirectly. This issue has been widely addressed earlier [3, 6, 7, 15, 16]. But it remains controversial about the mean-field nature of the phase transition in the range of $1 \leq m < 2$ [6, 7, 16].

To detect whether a network has a small world behavior, one can calculate the shortest distances separating two vertices, take the average and analyze its behavior in different network size. Sen and Chakrabarti in [6] and Moukarzel and de Menezes in [7] got the contradictory results by analyzing the topological
structure of networks with connection probability at distance $l$ given by Eq.(1). In Ref. [6], it was found that the averaged shortest distances behaved as $\log N$ on chains of size $N$ for all $m < 2$ and hence it was concluded that small world behavior occurs for $0 < m < 2$ while in Ref. [7], it was argued that small world behavior occurs only for $m < 1$. In the section $1 < m < 2$, according to Ref. [7], the averaged shortest distance scale as $N^\delta$ with the value of the averaged shortest distance exponent $\delta(0 < \delta < 1/2)$ depending on $m$. Ref. [16] studied the critical behavior of Ising model on such a one-dimensional network, since if the network’s behavior is that of small world it should be reflected in the critical exponents of the Ising model, which will assume mean-field values, and got the results that there is a mean-field behavior for $m < 1$ and a finite-dimensional behavior for $1 < m < 2$. Thus in accordance with the conclusion in Ref. [7] that the small world behavior of the underlying network exists only for $m < 1$.

In this paper, we reconsider the critical behavior of Ising model on a one-dimensional network with distance-dependent connections in Ref. [16] by using Monte Carlo simulation. But we enlarge the underlying network sizes, and for large $m$ values, the finite-size scaling analysis of the order parameter $[\langle M \rangle]$ is added by using the plot of $[\langle M \rangle]N^{1/4}$ versus $T$ [8] (\cdot\cdot\cdot \text{ and } \langle \cdot \cdot \cdot \rangle \text{ denote the thermal average taken over Monte Carlo steps for equilibrium at each temperature, and over different network realizations, respectively.}) Thus more reliable results can be produced from our simulation, since statistical fluctuations in these large-scale networks are greatly suppressed, and the finite-size effects are much more discernible in Binder’s cumulant $U_N$ [17, 18] and in the specific heat $C_v$ than in the order parameter $[\langle M \rangle]$ [8]. The results show that the model has a mean-field nature in the whole range of $0 < m < 2$.

In the next section, we will give out the model we use and the Monte Carlo simulation, in section 3, the results and analysis, and finally in section 4, the conclusion and discussion.

## 2 Ising model and Monte Carlo simulation

The underlying network of the one-dimensional Ising model here is the generalized WS model [4]. The addition of connections considers the distance between vertices. That means the vertex $j$ to which a connection is attached to is not randomly selected, but in accordance with a distribution depending on the distance from $i$ to $j$: $\Theta(l) \sim l^{-m}$; the case $m = 0$ is equivalent to the addition-type WS model. We start with a one-dimensional regular lattice with $N = 2K + 1$ vertices and the initial degree $2k = 2$. From $i = 0$, select the $ith$ vertex and generate a random number $\varphi \in [0,1)$ from a uniform distribution. If $\varphi < p$, select one vertex from the clockwise $K$ vertices to attach to according to the probability distribution $\Theta(l) \sim l^{-m}$. Self-connections and multiple connections are prohibited. Repeat this process until all the vertices are selected resulting in a network with
$2N(1+p)$ connections on average.

Figure 1: Examples of generated networks with $N = 101$ under the same $p = 1$ and different $m$ values.

In order to investigate the influence of $m$, here we consider the case $p = 1$ for simplicity. The case $m = 0$ is equivalent to the model used by Andrzej Pekalski [9] and $m \to \infty$ results in a one-dimensional regular network with $2k = 4$. Fig.1 shows that with the increase of $m$, long-range connections are more and more difficult to form between vertices with long distance.

Figure 2: The relationship of the characteristic path length with the size of the network under different $m$ values (the case $m = 0.5$ is almost identical with $m = 0$), all the curves are averaged over 50 realizations.

Fig.2 shows that the increase of $m$ results in the increase of the averaged shortest distance $L$, but the small world property remains within the range $0 < m < 2$ as $L \propto \log N$. In Fig.3, the left plot shows that with the increase of $m$ value, esp. after $m > 1$, the network clustering coefficient shows a very slow variation with the increase of system size, which implies the characteristic of a
regular network. Furthermore, in the right plot of Fig. 3, for a certain network size, with the increase of \( m \), the clustering coefficient converges to \( C_{\text{max}} = 1/2 \) finally. \( C_{\text{max}} \) is the clustering coefficient of a one-dimensional regular network with each vertex connected with its nearest and next-nearest neighbors.

Figure 3: (a) Network clustering coefficient \( C \) versus network size for different \( m \) values. (b) Network clustering coefficient \( C \) versus \( m \) value for a network with \( N = 100001 \) vertices. The shown \( C \) values in these two plots have been averaged for 50 different network realizations.

The Ising model is described by the Hamilton:

\[
H = -J \sum_{(i,j)} \sigma_i \sigma_j
\]

where \( J \) is the coupling constant between vertex \( i \) and \( j \) if they are connected, \((i,j)\) is the collection of all the connections in the network.

Our Monte Carlo simulation of the Ising model starts on a periodic one-dimensional ring. At the beginning, all the spins on the ring are given the same value +1, then select vertex randomly to flip according to the rule of the Metropolis algorithm \[9, 19\] and Glauber dynamics \[11\] to simulate the evolution of Ising model under different temperature. And use finite-size scaling analysis to investigate the paramagnetic-ferromagnetic transition temperature and the critical exponents. During the simulation, we mainly use the random number generator in \[20\], and we also use the drand48() function in standard C library as a comparison. It is found that there is no notable influence on the simulation results. Measured in the simulations are Binder’cumulant \( U_N \), the susceptibility \( \chi \) and the specific heat \( C_v \) \[17, 18, 21, 22, 23\]:

\[
U_N = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle},
\]

\[
\chi = \frac{1}{N} \sum_{ij} \langle [\sigma_i \sigma_j] \rangle,
\]

\[
C_v = \frac{\langle (H)^2 \rangle - \langle H \rangle^2}{T^2 N}
\]

with \( M = \frac{1}{N} \sum_i \sigma_i \). \( \langle \cdots \rangle \) denotes the thermal average taken over \( 2.5 \times 10^4 \) Monte Carlo steps after discarding the initial \( 2.5 \times 10^4 \) ones for equilibrium at each
temperature, and $\langle \cdots \rangle$ denotes the average over different network realizations (taken over 30 – 80 configurations).

Besides these three physical quantities, with the increase of $m$, the finite-size scaling analysis of the order parameter $\langle M \rangle$, which exhibits the critical behavior $\langle M \rangle \sim (T_c - T)^\beta$ in the thermodynamic limit, is used. In a finite-sized system, the order parameter scales as [8]:

$$\langle M \rangle = N^{-\beta/\bar{\nu}} g((T - T_c)N^{1/\bar{\nu}})$$  \hspace{1cm} (4)

$g(x)$ is an appropriate scaling function. Eqn.(4) leads to a unique crossing point at $T_c$ in the plot of $\langle M \rangle N^{\beta/\bar{\nu}}$ versus $T$, so Beom etc.[8] suggested analyzing the finite-size scaling of the order parameter $\langle M \rangle$ for large $m$ because the finite-size effects are much more discernible in Binder’s cumulant and in the specific heat than in the order parameter $\langle M \rangle$.

3 Results and Analysis

Figure 4: For $m = 0$, (a) The Binder’s cumulant $U_N$ has a unique crossing point at $T_c = 3.10(5)$ (in units of $J/k_B$). (b) The critical exponent $\bar{\nu} = 2.08$ is obtained from the linear fit of $\log \Delta U_N$ vs. $\log N$. (c) Specific heat $C_v$ has a unique crossing point at $T_c = 3.10(7)$, suggesting $\alpha = 0$. (d) The expansion of $C_v$ near $T_c$ obtains $\bar{\nu} = 2.04$. (e) Finite-size scaling of the susceptibility again determines $T_c = 3.10(5)$ with the critical exponent $\gamma = 1$. (f) Finite-size scaling of the order parameter $\langle M \rangle$ leads to a unique crossing point at $T_c = 3.10(1)$, and yields $\beta = \frac{1}{2}$ and $\bar{\nu} = 2.0$.

For $m = 0$, in Fig.4, the finite-size scaling of the Binder’s cumulant $U_N$, the specific heat $C_v$, the susceptibility $\chi$ and the order parameter $\langle M \rangle$ all reveals unanimously $T_c = 3.10(5)$ (in units of $J/k_B$), confirming the presence of a finite-temperature transition. In particular, the obtained critical exponents suggest the mean-field nature of the transition.
Till now, the most confusing property of the phase transition of one-dimensional Ising model with distance-dependent connections given by Eq.(1) is in the range of $1 \leq m < 2$ \cite{6, 7, 10}, so we will focus on the critical behavior of Ising model for $1 \leq m < 2$.

![Figure 5](image)

**Figure 5:** The finite-size scaling of the Binder’s cumulant $U_N$ and the order parameter $\langle M \rangle$ at $m = 1.0$.

Fig.5 gives out the finite-size scaling of the Binder’s cumulant $U_N$ and the order parameter $\langle M \rangle$ at $m = 1.0$. The two measured quantities remains a unique crossing point and revealing unanimously $T_c = 3.05(3)$ (in units of $J/k_B$), confirming the presence of a finite-temperature phase transition. The obtained critical exponents assume the mean-field value also.

![Figure 6](image)

**Figure 6:** The finite-size scaling of the Binder’s cumulant $U_N$ and the order parameter $\langle M \rangle$ at $m = 1.5$.

Fig.6 is the finite-size scaling of the Binder’s cumulant $U_N$ and the order parameter $\langle M \rangle$ at $m = 1.5$. The Binder’s cumulant do not intersect to one unique crossing point very well. But the trends of the evolution suggest one unique crossing point when the underlying network size is large enough. The finite-size scaling of the order parameter $\langle M \rangle$ under different network sizes behaves better in intersecting to one unique point, revealing unanimously $T_c = 2.70(4)$ (in units of $J/k_B$), and suggesting the presence of a mean-field finite-temperature phase transition.

With the increase of $m$, the algebra decreasing of $\Theta(l) \sim l^{-m}$ will be cut off by the finite network size. This makes the normalization of the distribution function depends on the size of the network and makes it inevitable to study systems of.
very large size for obtaining correct scaling behavior \[8, 23\]. However, the finite-size effects are much more discernible in Binder’s cumulant and in the specific heat than in the order parameter \[\langle M \rangle\], so Beom etc. \[8\] suggested analyzing the finite-size scaling of the order parameter \[\langle M \rangle\] by plotting \[\langle M \rangle N^{\beta/\nu}\] versus \[T\] to get the transition temperature. With the increase of \[m\], we get the transition temperature by the plot of \[\langle M \rangle N^{1/4}\] versus \[T\], as given in Fig.7. This also suggests the fact that the phase transition still has a mean-field nature for \[m > 1\]. The phase transition temperatures here are higher than the phase transition given out by H. Hong etc. in Ref. \[21\], this is because there are more long-range connections here which may contribute to the different phase transition properties.

In Fig.7, the decrease of \[T_c\] in the range of \((0, 1)\) is not significant, but this decreasing become obvious after \[m > 1\]. This brings difficulty to the phase transition analysis of Ising model due to the following two factors: a) with the increase of \[m\], one can see from Fig.6 that the confirmation of the unique crossing point needs larger network size; b) with the increase of \[m\], the transition temperature is decreasing (Fig.7), so we need to do Monte Carlo simulation at lower temperature, and lower temperature means lower probability to flip. What’s more, the value of the order parameter \[\langle M \rangle\] has a strong correlation with its previous step. Thus the system needs more Monte Carlo steps to reach equilibrium, and we need more iteration to get the thermodynamic average. These all cause long simulation time.

4 Conclusion and Discussion

In this paper, we reconsider the phase transition of Ising model on a one-dimensional network with distance-dependent connections given by \[\Theta(l) \sim l^{-m}\] adding the finite-size scaling analysis of the order parameter \[\langle M \rangle\]. Ref. \[16\] based their model on a network with 100 vertices and got the conclusion that the phase transition has a mean-field nature only in the range of \(0 < m < 1\). But our results, based on larger network size and further finite-size scaling analysis of the order parameter \[\langle M \rangle\] as well as the Binier’s cumulant \[U_N\], the specific heat \[C_v\] and the susceptibility \[\chi\], show that the phase transition has a mean-field nature in the
whole range of $0 < m < 2$.

Because of the constraint of the long simulation time, we can not apply the analysis to larger network size at present, thus can not convince whether there is a critical value of $m$, above which there will be no mean-field nature or finite-temperature phase transition since the transition point $m = 2$ is investigated by the study of the analysis of the topological structure [6, 7, 16]. With proper solution of the long simulation time, more interesting results are supposed to come out.

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