Classification of Supersymmetries

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Abstract

In the first part of my talk I will explain a solution to the extension of Lie’s problem on classification of “local continuous transformation groups of a finite-dimensional manifold” to the case of supermanifolds. (More precisely, the problem is to classify simple linearly compact Lie superalgebras, i.e. topological Lie superalgebras whose underlying space is a topological product of finite-dimensional vector spaces). In the second part I will explain how this result is used in a classification of superconformal algebras. The list consists of affine superalgebras and certain super extensions of the Virasoro algebra. In the third part I will discuss representation theory of affine superalgebras and its relation to ”almost” modular forms. Furthermore, I will explain how the quantum reduction of these representations leads to a unified representation theory of super extensions of the Virasoro algebra. In the forth part I will discuss representation theory of exceptional simple infinite-dimensional linearly compact Lie superalgebras and will speculate on its relation to the Standard Model.

Introduction

The theory of Lie groups and Lie algebras began with the 1880 paper [L] of S. Lie where he posed the problem of classification of “local continuous transformation groups of a finite-dimensional manifold” $\mathcal{M}$ and gave a solution to this problem when $\dim \mathcal{M} = 1$ and $2$.

The most important part of Lie’s problem is the classification of the corresponding Lie algebras of vector fields on $\mathcal{M}$ up to “formal” isomorphism. A more invariant (independent of $\mathcal{M}$) formulation is to classify linearly compact Lie algebras, i.e., topological Lie algebras whose underlying space is a topological product of discretely topologized finite-dimensional vector spaces [GS], [G2]. (Of course, it is well-known that it is impossible to classify even all finite-dimensional Lie algebras. What is usually meant be a “classification” is a complete list of simple algebras (no non-trivial ideas) and a description of semisimple algebras (no abelian ideals) in terms of simple ones.)

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It turned out that a solution to this problem requires quite different methods in
the cases of finite-dimensional and infinite-dimensional groups. The most important
advance in the finite-dimensional case was made by W. Killing and E. Cartan at
the end of the 19th century who gave the celebrated classification of simple finite-
dimensional Lie algebras over $\mathbb{C}$. The infinite-dimensional case was studied by
Cartan in a series of papers written in the beginning of the 20th century, which
culminated in his classification of infinite-dimensional “primitive” Lie algebras of
vector fields on a finite-dimensional manifold.

The advent of supersymmetry in theoretical physics in the 1970s motivated
work on the “super” extension of Lie’s problem. In the finite-dimensional case the
latter problem was settled in [K2]. However, it took another 20 years before the
problem was solved in the infinite-dimensional case [K7], [CK2], [CK3].

In the first part of my talk I will explain the classification of simple linearly
compact Lie superalgebras. Remarkably, unlike in the Lie algebra case, the
approach, based on the ideas of the papers [GS], [W], [K1] and [G2], is very similar in
the finite- and infinite-dimensional cases.

The advent of conformal field theory in the mid-1980s motivated the work on
classification and representation theory of superconformal algebras. In the second
part of my talk I will explain how the classification of infinite-dimensional simple
linearly compact Lie algebras is applied to classification of “linear” simple supercon-
formal algebras. A complete list consists of the affine superalgebras, and of several
series and one exceptional example of super extensions of the Virasoro algebra [FK].
(The most famous of these super extensions is the $N = 2$ superconformal algebra,
which plays a fundamental role in the mirror symmetry theory.)

In the third part of my talk I will discuss representation theory of affine super-
algebras [KW3], [KW4]. The key property of “admissible” representations of
affine algebras is that their characters are modular functions. This is not so in the
super case—for some mysterious reason, modular functions get replaced by closely
related but more general functions, like Appell’s function [KW4].

Next, I will explain how the quantum reduction of “admissible” representations
of affine superalgebras leads to a unified representation theory of (not necessarily
linear) super extensions of the Virasoro algebra [KRW], [KW7]. This gives rise to a
large class of supersymmetric rational conformal field theories.

In the last part of my talk I will discuss representation theory of exceptional
infinite-dimensional simple linearly compact Lie superalgebras [KR1]–[KR3]. I am
convinced that this theory may have applications to “real” physics. The main reason
for this belief is the exceptional Lie superalgebra $E(3|6)$ whose maximal compact
group of automorphisms is the gauge group of the Standard Model (= a quotient of
$SU_3 \times SU_2 \times U_1$ by a cyclic group of order 6). Furthermore, representation theory
of $E(3|6)$ accurately predicts the number of generations of leptons ($= 3$), but not so
accurately the number of generations of quarks ($= 5$) [KR2]. It is also striking that
the inclusion of the gauge group of the Standard Model in $SU_5$, which is the gauge
group of the Grand Unified Model, extends to the inclusion of $E(3|6)$ in $E(5|10)$,
the largest exceptional linearly compact Lie superalgebra.
1. Classification of simple linearly compact Lie superalgebras.

1.1. First, recall some basic superalgebra terminology. A superalgebra is simply a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra:

\[
S = S_0 + S_1, \quad S_\alpha S_\beta \subset S_{\alpha + \beta}, \quad \alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}.
\]

If \( a \in S_\alpha \), one says that the parity \( p(a) \) is equal to \( \alpha \). A derivation \( D \) of parity \( p(D) \) of a superalgebra \( S \) is a vector space endomorphism satisfying condition

\[
D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db).
\]

The sum \( \text{Der}_S \) of the spaces of derivations of parity \( 0 \) and \( 1 \) is closed under the super bracket:

\[
[D, D_1] = DD_1 - (-1)^{p(D)p(D_1)}D_1D.
\]

This super bracket satisfies super analogs of anticommutativity and Jacobi identity, hence defines what is called a Lie superalgebra. (The super anticommutativity axiom is \( [a, b] = (-1)^{p(a)p(b)}[b, a] \), and the super Jacobi identity axiom means that the operator \( (\text{ad}_a)b := [a, b] \) is a derivation.)

One of the basic constructions is the superization which basically amounts to adding anticommuting indeterminates. In other words, given an algebra (associative or Lie) \( A \) we consider the Grassmann algebra \( A\langle n \rangle \) over \( A \). This algebra carries a canonical \( \mathbb{Z}/2\mathbb{Z} \)-gradation defined by letting \( p(A) = 0 \), \( p(\xi_i) = 1 \). If \( \mathcal{O}_m \langle n \rangle \) denotes the algebra of formal power series over \( \mathbb{C} \) in \( m \) indeterminates, then \( \mathcal{O}_m \langle n \rangle \) is the algebra over \( \mathbb{C} \) of formal power series in \( m \) commuting indeterminates \( x = (x_1, \ldots, x_m) \) and \( n \) anticommuting indeterminates \( \xi = (\xi_1, \ldots, \xi_n): x_i x_j = x_j x_i, \quad x_i \xi_j = \xi_j x_i, \quad \xi_i \xi_j = -\xi_j \xi_i \).

Note that the associative superalgebra \( \mathcal{O}_m \langle n \rangle \) is linearly compact with respect to the topology for which the powers of the augmentation ideal \( (x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \) form a fundamental system of neighborhoods of 0. The algebra \( A\langle n \rangle \) has odd (i.e., of parity \( 1 \)) derivations \( \partial/\partial \xi_i \) defined by

\[
\frac{\partial}{\partial \xi_i}(a) = 0 \quad \text{for} \quad a \in A, \quad \frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij},
\]

and these derivations anticommute, i.e., \( [\partial/\partial \xi_i, \partial/\partial \xi_j] = 0 \).

The first basic example of a linearly compact Lie superalgebra is the Lie superalgebra denoted by \( W(m|n) \), of all continuous derivations of the topological superalgebra \( \mathcal{O}_m \langle n \rangle \):

\[
W(m|n) = \left\{ \sum_{i=1}^m P_i(x, \xi) \frac{\partial}{\partial x_i} + \sum_{j=1}^n Q_j(x, \xi) \frac{\partial}{\partial \xi_j} \right\},
\]

where \( P_i(x, \xi), Q_j(x, \xi) \in \mathcal{O}_m \langle n \rangle \). In a more geometric language, this is the Lie superalgebra of all formal vector fields on a supermanifold of dimension \( m|n \).
1.2. Cartan’s theorem \[3\] states that a complete list of infinite-dimensional linearly compact simple Lie algebras over \( \mathbb{C} \) consists of four series: the Lie algebra \( W_m (= W(m|0)) \) of all formal vector fields on an \( m \)-dimensional manifold, and its subalgebras \( S_m \) of divergenceless vector fields (\( m > 1 \)), \( H_m \) of Hamiltonian vector fields (\( m \) even), \( K_m \) of contact vector fields (\( m \) odd).

There is a unique way to extend divergence from \( W_m \) to \( W(m|n) \) such that the divergenceless vector fields form a subalgebra:

\[
\text{div} \left( \sum_i P_i \frac{\partial}{\partial x_i} + \sum_j Q_j \frac{\partial}{\partial \xi_j} \right) = \sum_i \frac{\partial P_i}{\partial x_i} + \sum_j (-1)^p(Q_j) \frac{\partial Q_j}{\partial \xi_j},
\]

and the super analog of \( S_m \) is

\[ S(m|n) = \{ X \in W(m|n) | \text{div} X = 0 \}. \]

In order to define super analogs of the Hamiltonian and contact Lie algebras \( H_m \) and \( K_m \), introduce a super analog of the algebra of differential forms \([K2]\).

Note that \( W(0|n) \), \( S(0|n) \), and \( H(0|n) \) are finite-dimensional Lie superalgebras. The Lie superalgebras \( W(0|n) \) and \( S(0|n) \) are simple iff \( n \geq 2 \) and \( n \geq 3 \), respectively. However, \( H(0|n) \) is not simple as its derived algebra \( H'(0|n) \) has codimension 1 in \( H(0|n) \), but \( H'(0|n) \) is simple iff \( n \geq 4 \). Thus, in the Lie superalgebra
case the lists of simple finite- and infinite-dimensional algebras are much closer related than in the Lie algebra case.

These four series of Lie superalgebras are infinite-dimensional if \( m \geq 1 \), in which case they are simple except for \( S(1|n) \). The derived algebra \( S'(1|n) \) has codimension 1 in \( S(1|n) \), and \( S'(1|n) \) is simple iff \( n \geq 2 \).

Remarkably it turned out that the above four series do not exhaust all infinite-dimensional simple linearly compact Lie superalgebras (as has been suggested in [K2]). Far from it!

As was pointed out by several mathematicians, the Schouten bracket [SV] makes the space of polyvector fields on a \( m \)-dimensional manifold into a Lie superalgebra. The formal analog of this is the following fifth series of superalgebras, called by physicists the Batalin-Vilkoviski algebra (\( H \) stands here for “Hamiltonian” and \( O \) for “odd”):

\[
HO(m|m) = \{ X \in W(m|m) | X\omega_{os} = 0 \},
\]

where \( \omega_{os} = \sum_{i=1}^{m} dx_i d\xi_i \) is an “odd” symplectic form. Furthermore, unlike in the \( H(m|n) \) case, not all vector fields of \( HO(m|m) \) have zero divergence, which gives rise to the sixth series:

\[
SHO(m|m) = \{ X \in HO(m|m) | \text{div} X = 0 \}.
\]

The seventh series is the odd analog of \( K(m|n) \) [ALS]:

\[
KO(m|m+1) = \{ X \in W(m|m+1) | X\omega_{oc} = f\omega_{oc} \},
\]

where \( \omega_{oc} = d\xi_{m+1} + \sum_{i=1}^{m} (\xi_i dx_i + x_i d\xi_i) \) is an odd contact form. One can take again the divergence 0 vector fields in \( KO(m|m+1) \) in order to construct the eighth series, but the situation is more interesting. It turns out that for each \( \beta \in \mathbb{C} \) one can define the deformed divergence \( \text{div}_\beta X \) \([K2], [K7] \), so that \( \text{div} = \text{div}_0 \) and

\[
SKO(m|m+1;\beta) = \{ X \in KO(m|m+1) | \text{div}_\beta X = 0 \}
\]

is a subalgebra. The superalgebras \( HO(m|m) \) and \( KO(m|m+1) \) are simple iff \( m \geq 2 \) and \( m \geq 1 \), respectively. The derived algebra \( SHO'(m|m) \) has codimension 1 in \( SHO(m|m) \), and it is simple iff \( m \geq 3 \). The derived algebra \( SKO'(m|m+1;\beta) \) is simple iff \( m \geq 2 \), and it coincides with \( SKO(m|m+1;\beta) \) unless \( \beta = 1 \) or \( m \) when it has codimension 1.

Some of the examples described above have simple “filtered deformations”, all of which can be obtained by the following simple construction. Let \( L \) be a subalgebra of \( W(m|n) \), where \( n \) is even. Then it happens in three cases that

\[
L^\sim := (1 + \Pi_{j=1}^{n} \xi_j) L
\]

is different from \( L \), but is closed under bracket. As a result we get the following three series of superalgebras: \( S^\sim(0|n) \) \([K2]\), \( SHO^\sim(m|m) \) \([CK2]\) and \( SKO^\sim(m|m+1; m \geq 2) \) \([Ko]\) (the constructions in \([Ko]\) and \([CK2]\) were more complicated). We thus get the ninth and the tenth series of simple infinite-dimensional Lie superalgebras:

\[
SHO^\sim(m|m), \quad m \geq 2, \quad m \text{ even},
\]

\[
SKO^\sim(m|m + 1; 1 + 2/m), \quad m \geq 3, \quad m \text{ odd}.
\]
A surprising discovery was made in [Sh1] where the existence of three exceptional simple infinite-dimensional Lie superalgebras was announced. The proof of the existence along with one more exceptional example was given in [Sh2]. An explicit construction of these four examples was given later in [CK3]. The fifth exceptional example was found in the work on conformal algebras [CK1] and independently in [Sh2]. (The alleged sixth exceptional example \( E(2|2) \) of [K7] turned out to be isomorphic to \( SKO(2|3;1) \) [CK3].)

Now I can state the first main theorem.

**Theorem 1.** [K7] The complete list of simple infinite-dimensional linearly compact Lie superalgebras consists of ten series of examples described above and five exceptional examples: \( E(1|6) \), \( E(3|6) \), \( E(3|8) \), \( E(4|4) \), and \( E(5|10) \).

Here and before the notation \( X(m|n) \) means that this superalgebra can be embedded in \( W(m|n) \) and that this embedding is minimal possible; \( E \) stands for “exceptional”.

**Remark.** The local classification of transitive primitive (i.e., leaving no invariant fibrations) actions on a (super)manifold \( M \) is equivalent to the classification of all “primitive” pairs \((L, L_0)\), where \( L \) is a linearly compact Lie (super)algebra and \( L_0 \) is a maximal open subalgebra (such that \( \dim L/L_0 = \dim M \)) without non-zero ideals of \( L \). If \( L \) is simple, choosing any maximal open subalgebra \( L_0 \), we get a primitive pair \((L, L_0)\). One can show that if, in addition, \( L \) is a Lie algebra and \( \dim L = \infty \), there exists a unique such \( L_0 \). According to Cartan’s theorem, the remaining infinite-dimensional primitive pairs are the Lie algebras obtained from \( S_n \) and \( H_n \) by adding the Euler operator \( E \). Using the structure results on general transitive linearly compact Lie algebras [G1], it is not difficult to reduce the classification of infinite-dimensional primitive pairs to the classification of simple infinite-dimensional linearly compact Lie algebras (cf. [G2]). Such a reduction is possible also in the Lie superalgebra case, but it is much more complicated for two reasons: (a) a simple linearly compact Lie superalgebra may have several maximal open subalgebras (see [CK3] for a classification), (b) construction of arbitrary primitive pairs in terms of simple primitive pairs is more complicated in the superalgebra case (see [K8]).

1.3. Here I will describe the classification of finite-dimensional simple Lie superalgebras. We already have four “non-classical” series: \( W(0|n) \), \( S(0|n) \), \( H'(0|n) \) and \( S^-(0|n) \). The four “classical” series are constructed as follows. Introduce the following even and odd Euler operators \( E = \sum_i x_i \frac{\partial}{\partial x_i} + \sum_j \xi_j \frac{\partial}{\partial \xi_j} \in W(m|n) \) and \( E_0 = \sum_i x_i \frac{\partial}{\partial x_i} + \sum_j \xi_j \frac{\partial}{\partial \xi_j} \in W(m|m) \).

\[
\begin{align*}
sl(m|n) &= \{ X \in S(m|n) | [E, X] = 0 \}, \\
sp(m|n) &= \{ X \in H(m|n) | [E, X] = 0 \}, \\
p(m|m) &= \{ X \in SHO(m|m) | [E, X] = 0 \}, \\
q(m|m) &= \{ X \in W(m|m) | [E_0, X] = 0 \}.
\end{align*}
\]
The Lie algebras $\mathfrak{sl}_m = \mathfrak{sl}(m|0)$, $\mathfrak{sp}_m = \mathfrak{sp}(m|0)$, $\mathfrak{so}_n = \mathfrak{so}(0|n)$ are simple. Furthermore, $\mathfrak{sl}(m|n)$ are simple for $m \neq n$, all $\mathfrak{sp}(m|n)$ and $\mathfrak{p}(m|m)$ ($m \geq 3$) are simple. The superalgebra $\mathfrak{sl}(m|m)$ contains 1-dimensional ideal $\mathbb{C}E$ and $\mathfrak{sl}(m|m)/\mathbb{C}E$ is simple for $m \geq 2$. Finally, the derived algebra $q'(m|m)$ has codimension 1 in $q(m|m)$ and $q'(m|m)/\mathbb{C}E$ is simple for $m \geq 3$.

**Theorem 2.** The complete list of simple finite-dimensional Lie superalgebras consists of eight series of examples described above, the exceptional Lie superalgebras $F(4)$ and $G(3)$ of dimension 40 and 31, respectively, a 1-parameter family of 17-dimensional exceptional Lie superalgebras $D(2,1;α)$, and the five exceptional Lie algebras.

1.4. Plan of the proof of Theorem 1.

Step 1. Introduce Weisfeiler’s filtration $[W]$ of a simple linearly compact Lie superalgebra $L$. For that choose a maximal open subalgebra $L_0$ of $L$ and a minimal subspace $L_{-1}$ satisfying the properties: $L_{-1} \supsetneq L_0$, $[L_0,L_{-1}] \subseteq L_{-1}$. (Geometrically this corresponds to a choice of a primitive action of $L$ and an invariant irreducible differential system.) The pair $L_{-1},L_0$ can be included in a unique filtration: $L = L_{-d} \supsetneq L_{-d+1} \supsetneq \cdots \supsetneq L_{-1} \supsetneq L_0 \supsetneq L_1 \supsetneq \cdots$, called Weisfeiler’s filtration of depth $d$. (In the Lie algebra case, $d > 1$ only for $K_m$, when $d = 2$, but in the Lie superalgebra case, $d > 1$ in the majority of cases.) The associated to Weisfeiler’s filtration $\mathbb{Z}$-graded Lie superalgebra is of the form $GrL = \Pi_{j \geq -d} g_j$, and has the following properties:

(G0) $\dim g_j < \infty$ (since $\text{codim } L_0 < \infty$),
(G1) $g_{-j} = g_j^L$ for $j \geq 1$ (by maximality of $L_0$),
(G2) $[x,g_{-1}] = 0$ for $x \in g_j$, $j \geq 0$ \(\Rightarrow x = 0\) (by simplicity of $L$),
(G3) $g_0$-module $g_{-1}$ is irreducible (by choice of $L_{-1}$), and faithful (by (G2)).

Weisfeiler’s idea was that property (G3) is so restrictive, that it should lead to a complete classification of $\mathbb{Z}$-graded Lie algebras satisfying (G0)–(G3). (Incidentally, the infinite-dimensionality of $L$ and hence of $GrL$, since $L$ is simple, is needed only in order to conclude that $g_1 \neq 0$.) This indeed turned out to be the case $[K1]$. In fact, my idea was to replace the condition of finiteness of the depth by finiteness of the growth, which allowed one to add to the Lie-Cartan list some new Lie algebras, called nowadays affine Kac-Moody algebras.

However, unlike in the Lie algebra case, it is impossible to classify all finite-dimensional irreducible faithful representations of Lie superalgebras. One needed a new idea to make this approach work.

Step 2. The main new idea is to choose $L_0$ to be invariant with respect to all inner automorphisms of $L$ (meaning to contain all even ad-exponentiable elements of $L$). A non-trivial point is the existence of such $L_0$. This is proved by making use of the characteristic supervariety, which involves rather difficult arguments of Guillemin $[G2]$.

Next, using a normalizer trick of Guillemin $[G2]$, I prove, for this choice of $L_0$, the following very powerful restriction on the $g_0$-module $g_{-1}$ (at this point $\dim L = \infty$ is used):
Step 3. Consider a faithful irreducible representation of a Lie superalgebra \( \mathfrak{p} \) in a finite-dimensional vector space \( V \). This representation is called \textit{strongly transitive} if \( \mathfrak{p} \cdot x = V \) for any non-zero even element \( x \in V \). By properties (G0), (G3) and (G4), the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{-1} \) is strongly transitive.

In order to demonstrate the power of this restriction, consider first the case when \( \mathfrak{p} \) is a Lie algebra and \( V \) is purely even. Then the strong transitivity simply means that \( V \setminus \{0\} \) is a single orbit of the Lie group \( P \) corresponding to \( \mathfrak{p} \). It is rather easy to see that the only strongly transitive subalgebras \( \mathfrak{p} \) of \( \mathfrak{g}_V \) are \( \mathfrak{g}_V \), \( \mathfrak{s}_V \), \( \mathfrak{sp}_V \) and \( \mathfrak{csp}_V \). These four cases lead to \( GrL \), where \( L = W_n, S_n, H_n \) and \( K_n \), respectively.

In the super case the situation is much more complicated. First we consider the case of “inconsistent gradation”, meaning that \( \mathfrak{g}_{-1} \) contains a non-zero even element. The classification of such strongly transitive modules is rather long and the answer consists of a dozen series and a half dozen exceptions (see [K7], Theorem 3.1). Using similar restrictions on \( \mathfrak{g}_{-2}, \mathfrak{g}_{-3}, \ldots \), we obtain a complete list of possibilities for \( GrL := \bigoplus j \leq 0 \mathfrak{g}_j \), in the case when \( \mathfrak{g}_{-1} \) contains non-zero even elements. It turns out that all but one exception are not exceptions at all, but correspond to the beginning members of some series. As a result, only \( E(4|4) \) “survives”.

Step 4. Next, we turn to the case of a consistent gradation, i.e., when \( \mathfrak{g}_{-1} \) is purely odd. But then \( \mathfrak{g}_0 \) is an “honest” Lie algebra, having a faithful irreducible representation in \( \mathfrak{g}_{-1} \) (condition (G4) becomes vacuous). An explicit description of such representations is given by the classical Cartan-Jacobson theorem. In this case I use the “growth” method developed in [K1] and [K2] to determine a complete list of possibilities for \( GrL \). This case produces mainly the (remaining four) exceptions.

Step 5 is rather long and tedious [CK3]. For each \( GrL \) obtained in Steps 3 and 4 we determine all possible “prolongations”, i.e., infinite-dimensional \( Z \)-graded Lie superalgebras satisfying (G2), whose negative part is the given \( GrL \).

Step 6. It remains to reconstruct \( L \) from \( GrL \), i.e., to find all possible filtered simple linearly compact Lie superalgebras \( L \) with given \( GrL \) (such an \( L \) is called a simple filtered deformation of \( GrL \)). Of course, there is a trivial filtered deformation: \( GrL := \bigoplus j \geq -d \mathfrak{g}_j \), which is simple if \( GrL \) is. It is proved in [CK2] by a long and tedious calculation that only \( \text{SHO}(m|m) \) for \( m \) even \( \geq 2 \) and \( \text{SKO}(m|m+1; \frac{m+2}{m}) \) for \( m \) odd \( \geq 3 \) have a non-trivial simple filtered deformation, which are the ninth and tenth series. It would be nice to have a more conceptual proof. Recall that \( \text{SHO}(m|m) \) is not simple, though it does have a simple filtered deformation. Note also that in the Lie algebra case all filtered deformations are trivial.

1.5. Plan of the proof of Theorem 4

The key idea is the same as in the proof of Theorem 1. Choose a maximal subalgebra \( L_0 \) of a simple finite-dimensional Lie superalgebra \( L \) containing the even part of \( L \). It is easy to see that \( L_{-1} = L \), so that the corresponding Weisfeiler’s filtration has depth 1. Hence \( GrL \) has the form: \( GrL = \bigoplus j \leq -1 \mathfrak{g}_j \). This gradation is consistent and, of course, satisfies conditions (G0)–(G3). There are two cases.
Case 1. $N \geq 1$. Then we apply the growth method (as in Step 4 of Sec. [1.4]) to obtain a complete list of possibilities for $GrL_\leq$. Then, as in Step 5 of Sec. [1.4], we determine all prolongations of each $GrL_\leq$ (all of them will be subalgebras of $W(0, \dim g_\mathfrak{t})$, and all filtered deformations of these prolongations. This case produces all “non-classical” series, and also $sl(m|m)$, $spo(m|2)$ and $p(m|m)$.

Case 2. $N = 0$. Then $L_\mathfrak{T}$ is a semisimple Lie algebra and its representation in $L_\mathfrak{T}$ is irreducible. The Killing form on $L$ is either non-degenerate, in which case we apply the standard Killing-Cartan techniques, or it is identically zero. In the latter case one uses Dynkin’s index to find all possibilities for the $L_\mathfrak{T}$-module $L_\mathfrak{T}$.

1.6. In order to describe the construction of the exceptional infinite-dimensional Lie superalgebras (given in [K3]), I need to make some remarks. Let $\Omega_m = \Omega(m|0)$ be the algebra of differential forms over $\mathcal{O}_m$, let $\Omega_m^k$ denote the space of forms of degree $k$, and $\Omega_m^{k,cl}$ the subspace of closed forms. For any $\lambda \in \mathbb{C}$ the representation of $W_m$ on $\Omega_m^k$ can be “twisted” by letting

$$X \mapsto L_X + \lambda \text{div } X, \quad X \in W_m,$$

where $\lambda \in \mathbb{C}$ is such that $\Omega_m^k(\lambda)$ is invariant with respect to all automorphisms. There are three other maximal open subalgebras in $E(5|10)$, associated to $\mathbb{Z}$-gradations corresponding to quintuples $(1, 1, 1, 1, 2)$, $(2, 2, 2, 1, 1)$ and $(3, 3, 2, 2, 2)$, and one can show that these four are all, up to conjugacy, maximal open subalgebras (cf. [CK3]).
Another important \( \mathbb{Z} \)-gradation of \( E(5|10) \), which is, unlike the previous four, by infinite-dimensional subspaces, corresponds to the quintuple \((0, 0, 0, 1, 1)\) and has depth 1: \( E(5|10) = \bigoplus_{i \geq 1} \mathfrak{g}_i \). One has: \( \mathfrak{g}^0 \cong E(3|6) \) and the \( \mathfrak{g}^0 \) form an important family of irreducible \( E(3|6) \)-modules \( [KR2] \). The consistent \( \mathbb{Z} \)-gradation of \( E(5|10) \) induces that of \( \mathfrak{g}^0 : E(3|6) = \bigoplus_{i \geq -2} a_i \), where

\[
a_0 \cong sl_3 \oplus sl_2 \oplus gl_1, \quad a_{-1} \cong \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}, \quad a_{-2} \cong \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}.
\]

A more explicit construction of \( E(3|6) \) is as follows \( [CK3] \): the even part is \( W_3 + \Omega_3 \) \( \otimes \mathfrak{sl}_2 \), the odd part is \( \Omega_3 \cdot (-\frac{1}{2}) \otimes \mathbb{C}^2 \) with the obvious action of the even part, and the bracket of two odd elements is defined as follows:

\[
[\omega \otimes u, \omega' \otimes v] = (\omega \wedge \omega') \otimes (u \wedge v) + (d\omega \wedge \omega' + \omega \wedge d\omega') \otimes (u \cdot v).
\]

Here the identifications \( \Omega_3(-1) = W_3 \) and \( \Omega_0 = \Omega_3^2(-1) \) are used.

The gradation of \( E(5|10) \) corresponding to the quintuple \((0, 1, 1, 1, 1)\) has depth 1 and its zeroth component is isomorphic to \( E(1|6) \) (cf. \( [K3] \)).

The construction of \( E(4|4) \) is also very simple \( [CK3] \): The even part is \( W_4 \), the odd part is \( \Omega_4 \cdot (-\frac{1}{2}) \) and the bracket of two odd elements is:

\[
[\omega, \omega'] = d\omega \wedge \omega' + \omega \wedge d\omega' \in \Omega_4(-1) = W_4.
\]

The construction of \( E(3|8) \) is slightly more complicated, and we refer to \( [CK3] \) for details.

1.7. All exceptional simple finite-dimensional Lie superalgebras (including the exceptional Lie algebras) are obtained as special cases of the following important construction \( [K2] \). Let \( I = \{1, \ldots, r\} \) and let \( I_\tau \) be a subset of \( I \), \( I_\tau = I \setminus I_\tau \). Let \( A = (a_{ij})_{i,j \in I} \) be a matrix over \( \mathbb{C} \). We associate to the pair \((A, I_\tau)\) a Lie superalgebra \( \mathfrak{g}(A, I_\tau) \) as follows. Let \( \mathfrak{g}(A, I_\tau) \) be the Lie superalgebra on generators \( e_i, f_i, h_i \) \((i \in I)\) of parity \( p(h_i) = 0 \) for \( i \in I \), \( p(e_i) = p(f_i) = \alpha \in \{0,1\} \) for \( i \in I_\alpha \), and the following standard relations:

\[
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j.
\]

Define a \( \mathbb{Z} \)-gradation \( \mathfrak{g}(A, I_\tau) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j \) by letting \( \deg h_i = 0 \), \( \deg e_i = -\deg f_i = 1 \). Then \( \mathfrak{g}_0 \) is the \( \mathbb{C} \)-span of \( \{h_i\}_{i \in I} \), and we denote by \( J(A, I_\tau) \) the sum of all \( \mathbb{Z} \)-graded ideals of \( \mathfrak{g}(A, I_\tau) \) that intersect \( \mathfrak{g}_0 \) trivially. We let

\[
\mathfrak{g}(A, I_\tau) = \mathfrak{g}(A, I_\tau)/J(A, I_\tau).
\]

Of course, if \( A \) is the Cartan matrix of a simple finite-dimensional Lie algebra \( \mathfrak{g} \), then \( \mathfrak{g} \cong \mathfrak{g}(A, \emptyset) \), the ideal \( J(A, \emptyset) \) being generated by "Serre relations". Likewise, generalized Cartan matrices give rise to Kac-Moody Lie algebras \( [K3] \).

Consider the following matrices \((a \in \mathbb{C} \setminus \{0,-1\})\):

\[
D_a = \begin{bmatrix} \ 0 & -1 & -a \\ -1 & 2 & 0 \\ -a & 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} \ 0 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} \ 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}.
\]
Then $D(2,1;\alpha) \simeq \mathfrak{g}(D_{\alpha},\{1\})$, $F(4) \simeq \mathfrak{g}(F,\{1\})$ and $G(3) \simeq \mathfrak{g}(G,\{1\})$. Note, however, that unlike in the Lie algebra case, “inequivalent” pairs $(A, I_T)$ may produce isomorphic Lie superalgebras. For example, in the cases $D(2,1;\alpha)$, $F(4)$ and $G(3)$ there are 2, 6 and 4 such pairs, respectively.

Finite-dimensional simple Lie superalgebras that are isomorphic to $\mathfrak{g}(A, I_T)$ for some matrix $A$ are called basic (they will play an important role in the next parts of the talk). The remaining basic simple Lie superalgebras (that are not Lie algebras) are $\mathfrak{s}l(m|n)/\delta_{m,n}CE$ and $\mathfrak{spo}(m|n)$.

2. A classification of superconformal algebras

Superconformal algebras have been playing an important role in superstring theory and in conformal field theory. Here I will explain how to apply Theorem \[\text{[DK]}\] to the classification of “linear” superconformal algebras. By a (“linear”) superconformal algebra I mean a Lie superalgebra $\mathfrak{g}$ spanned by coefficients of a finite collection $F$ of fields such that the following two properties hold:

1. for $a, b \in F$ the singular part of OPE is finite, i.e.,

   \[ [a(z), b(w)] = \sum c_j(w)\partial_w^j \delta(z-w) \quad \text{(a finite sum)} \]

   where all $c_j(w) \in \mathbb{C}[\partial_w]F$,

2. $\mathfrak{g}$ contains no non-trivial ideals spanned by coefficients of fields from a $\mathbb{C}[\partial_w]$-submodule of $\mathbb{C}[\partial_w]F$.

(Recall that a field is a formal expression $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, where $a_n \in \mathfrak{g}$ and $z$ is an indeterminate, and $\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} (w/z)^n$ is the formal $\delta$-function. See \[\text{[K]}\] for details.)

This problem goes back to the physics paper \[\text{[RS]}\], some progress in its solution was made in \[\text{[K]}\] and a complete solution was given in \[\text{[FK]}\]. (A complete classification even in the “quadratic” case seems to be a much harder problem, see \[\text{[F]}\] and Section 4 below for some very interesting examples.) The simplest example is the loop algebra $\mathfrak{gl} = \mathbb{C}[x, x^{-1}] \otimes \mathfrak{g}$ (= centerless affine Kac-Moody superalgebra), where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra. Then $F = \{ a(z) = \sum_{n \in \mathbb{Z}} (x^a \otimes a) z^{-n-1} \}_{a \in \mathfrak{g}}$, and $[a(z), b(w)] = [a, b](w) \delta(z-w)$. The next example is the Lie algebra $\text{Vect} \mathbb{C}^\times$ of regular vector fields on $\mathbb{C}^\times$ (= centerless Virasoro algebra); $F$ consists of one field, the Virasoro field $L(z) = \sum_{n \in \mathbb{Z}} (x_0^{n+1}) z^{-n-1}$, and $[L(z), L(w)] = \partial_w L(w) \delta(z-w) + 2L(w) \delta'_w (z-w)$.

One of the main theorems of \[\text{[DK]}\] states that these are all examples in the Lie algebra case. The strategy of the proof is the following. Let $\partial = \partial_x$ and consider the (finitely generated) $\mathbb{C}[\partial]$-module $R = \mathbb{C}[\partial]F$. Define the “$\lambda$-bracket” $R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R$ by the formula: $[a \lambda b] = \sum_{j} \lambda^j c_j$. This satisfies the axioms of a conformal (super)algebra (see \[\text{[DR]}\], \[\text{[K]}\]), similar to the Lie (super)algebra axioms:

\begin{itemize}
  \item[(i)] $[\partial a \lambda b] = -\lambda [a \lambda b]$, $[a \lambda \partial b] = (\partial + \lambda)[a \lambda b]$,
  \item[(ii)] $[a \lambda b] = -(-1)^{p(a)p(b)} \delta_{b,-\lambda,0} a \lambda b$,
  \item[(iii)] $[a \lambda [b \mu c]] = [[a \lambda b] \lambda + \mu, c] + (-1)^{p(a)p(b)} [b \mu [a \lambda c]]$.
\end{itemize}
The main observation of [DK] is that a conformal (super)algebra is completely determined by the Lie (super)algebra spanned by all coefficients of negative powers of $z$ of the fields $a(z)$ from $F$, called the annihilation algebra, along with an even surjective derivation of the annihilation algebra. Furthermore, apart from the case of current algebras, the completed annihilation algebra turns out to be an infinite-dimensional simple linearly compact Lie (super)algebra of growth 1. Since in the Lie algebra case the only such example is $W_1$, the proof is finished.

In the superalgebra case the situation is much more interesting since there are many infinite-dimensional simple linearly compact Lie superalgebras of growth 1. By Theorem 1, the complete list is as follows:

\[ W(1|N), \quad S'(1|N), \quad K(1|N) \quad \text{and} \quad E(1|6). \]

In all cases, except the second, there is a unique, up to conjugacy, even surjective derivation, hence a unique corresponding superconformal algebra. They are denoted by $W(N)$, $K(N)$ if $N \neq 4$, $K'_4$, and $CK(6)$, respectively. The Lie superalgebras $W(N)$ and $K(N)$ are constructed in the same way as $W(1|N)$ and $K(1|N)$, except that one replaces $O_1(N)$ by $\mathbb{C}[x, x^{-1}](N)$. The construction of the exceptional superconformal algebra $CK(6)$ is more difficult, and may be found in [CK1] or [K6]. However, $S'(1|N)$ has two families of even surjective derivations. The corresponding superconformal algebras are derived algebras of

\[ S(N), e, a = \{ X \in W(N) | \text{div}(e^{a x}(1 + \epsilon \xi_1 \ldots \xi_N)X) = 0 \}, \quad a \in \mathbb{C}, \quad \epsilon = \pm 1. \]

Thus, one obtains the following theorem.

**Theorem 3.** [FK] A complete list of superconformal algebras consists of loop algebras $\tilde{g}$, where $\tilde{g}$ is a simple finite-dimensional Lie superalgebra, and of Lie superalgebras $(N \in \mathbb{Z}_+)$: $W(N)$, $S'(N+2), e, a$ ($N$ even and $a = 0$ if $\epsilon = 1$), $K(N)$ ($N \neq 4$), $K'_4$, and $CK(6)$.

Note that the first members of the above series are well-known superalgebras: $W(0) \simeq K(0)$ is the Virasoro algebra, $K(1)$ is the Neveu-Schwarz algebra, $K(2) \simeq W(1)$ is the $N = 2$ algebra, $K(3)$ is the $N = 3$ algebra, $S'(2), S'(2,0,0)$ is the $N = 4$ algebra, $K'_4$ is the big $N = 4$ algebra (all centerless). These algebras, along with $W(2)$ and $CK(6)$, are the only superconformal algebras for which all fields are primary with positive conformal weights [K6]. It is interesting to note that all of them are contained in $CK(6)$, which consists of 32 fields, the even ones are the Virasoro fields and 15 currents that form $\tilde{so}_6$, and the odd ones are 6 and 10 fields of conformal weight $3/2$ and $1/2$, respectively. Here is the table of (some) inclusions, where in square brackets the number of fields is indicated:

\[
\begin{align*}
CK(6)[32] & \supset W(2)[12] \supset W(1) = K(2)[4] \supset K(1)[2] \supset \text{Vir}[1] \\
K(3)[8] & \subset K'_4[16] \quad S'(2, e, a)[8]
\end{align*}
\]

All of these Lie superalgebras have a unique non-trivial central extension, except for $K'_4$ that has three [K1] and $CK(6)$ that has none. All other Lie superalgebras...
listed by Theorem 2 have no non-trivial central extensions. (The presence of a central term is necessary for the existence of interesting representations and the construction of an interesting conformal field theory.)

3. Representations of affine superalgebras and “almost” modular forms

3.1. Finite-dimensional irreducible representations of simple finite-dimensional Lie superalgebras are much less understood than in the Lie algebra case, the main reason being the occurrence of isotropic roots in the super case. (A review may be found in the proceedings of the last ICM, see [Se].) The natural analogues of these representations in the case of affine (super)algebras are the integrable highest weight modules.

Let us first recall the basic definitions in the Lie algebra case, i.e., for an affine Kac-Moody algebra $\hat{g}$. Let $g$ be a finite-dimensional simple Lie algebra and let $(\cdot, \cdot)$ be an invariant symmetric bilinear form on $g$ normalized by the condition that $(\alpha|\alpha) = 2$ for a long root $\alpha$ ((a|b) = tr ab in the case $g = sl_m$). Recall that the associated affine algebra is

$$\hat{g} = (\mathbb{C}[x, x^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}D$$

with the following commutation relations ($a, b \in g$; $m, n \in \mathbb{Z}$ and $a(m)$ stands for $x^m \otimes a$):

$$[a(m), b(n)] = [a, b] (m + n) + m\delta_{m,-n}(a|b)K, \quad [D, a(m)] = ma(m), \quad [K, \hat{g}] = 0.$$ 

Note that the derived algebra $\hat{g}'$ is a central extension (by $\mathbb{C}K$) of the loop algebra $\hat{g}$ that has made an appearance in Section 2. (It is also isomorphic to $g(\hat{A})$, where $\hat{A}$ is the extended Cartan matrix of $g$, cf. Sec. [17].) As we shall see, without central extension one loses all interesting representations. In any irreducible $\hat{g}$-module $V$ one has: $K = kI_V$; the number $k$ is called the level of $V$. The scaling element $D$ is necessary for the convergence of characters.

Choose a Cartan subalgebra $h$ of $g$ and let $\mathfrak{g} = h \oplus (\bigoplus_{\alpha \in \Delta} g_\alpha)$ be the root space decomposition, where $g_\alpha$ denotes the root space attached to a root $\alpha \in \Delta \subset h^*$. Let $\mathfrak{h} = h + \mathbb{C}K + \mathbb{C}D$ be the Cartan subalgebra of $\hat{g}$, and, as before, let $g_\alpha(m) = x^m \otimes g_\alpha$. We extend the invariant bilinear form from $h$ to a symmetric bilinear form on $\mathfrak{h}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}D) = 0, (K|K) = (D|D) = 0, (K|D) = 1$, and identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via this form.

A $\hat{g}$-module $V$ is called integrable if the following two properties hold:

(M1) $\mathfrak{h}$ is diagonizable on $V$.

(M2) For each $\alpha \in \Delta$ and $m \in \mathbb{Z}$, $g_\alpha(m)$ is locally finite on $V$.

Choose a set of positive roots $\Delta^+ \subset \Delta$, and let $n^+ = \bigoplus_{\alpha \in \Delta^+} g_\alpha$, $\hat{n}^+ = n^+ + \sum_{n \geq 1} x^n \otimes g$. For each $\lambda \in \hat{h}^*$ one defines an irreducible highest weight module
Let $L(\Lambda)$ over $\hat{\mathfrak{g}}$ as the (unique) irreducible $\hat{\mathfrak{g}}$-module for which there exists a non-zero vector $v_\Lambda$ such that

$$hv_\Lambda = \Lambda(h)v_\Lambda \text{ for all } h \in \hat{\mathfrak{h}}, \hat{\mathfrak{n}}_+v_\Lambda = 0.$$ 

Without loss of generality we shall let $\Lambda(D) = 0$; then the spectrum of $-D$ on $L(\Lambda)$ is $\mathbb{Z}_+$. Integrable highest weight modules over affine Lie algebras (they are automatically irreducible) attracted a lot of attention in the past few decades both of mathematicians and of physicists (some aspects of the theory are discussed in [K3], [Wa2].) Here I will only mention some relevant to the talk facts. First, the level $k$ of such a module is a non-negative integer (and $k = 0$ iff $\dim L(\Lambda) = 1$), and there is a finite number of them for each $k$. One of the most remarkable properties of these modules is modular invariance, which I explain below.

3.2. Let us coordinatize $\hat{\mathfrak{h}}$ by letting

$$\begin{pmatrix} \tau, z, t \end{pmatrix} = 2\pi i \begin{pmatrix} z - \tau D + tK \end{pmatrix},$$

where $z \in \mathfrak{h}$, $\tau, t \in \mathbb{C}$, and define the character of the $\hat{\mathfrak{g}}$-module $L(\Lambda)$ by:

$$ch_\Lambda(\tau, z, t) = \text{tr}_{L(\Lambda)} e^{2\pi i (z - \tau D + tK)}.$$ 

If $L(\Lambda)$ is integrable, then $ch_\Lambda(\tau, z, t)$ is a holomorphic function for $(\tau, z, t) \in \mathcal{H} \times \mathfrak{h} \times \mathbb{C}$, where $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im } \tau > 0\}$.

Recall the following well-known action of $SL_2(\mathbb{Z})$ on $\mathcal{H} \times \mathfrak{h} \times \mathbb{C}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, t) = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}, \begin{pmatrix} z \\ c\tau + d \end{pmatrix}, t - \frac{c(z)}{2(c\tau + d)}.$$ 

Then it turns out that there exists an explicit rational number $m_\Lambda$, called modular anomaly (see [K3], (12.7.5)) such that the normalized character $\chi_\Lambda = e^{2\pi im_\Lambda \tau}ch_\Lambda$ of an integrable $L(\Lambda)$ is invariant with respect to a congruence subgroup of $SL_2(\mathbb{Z})$ (see [K3], Chapter 13). If $ch_\Lambda$ has this property, one says that $L(\Lambda)$ is modular invariant.

3.3. It turns out that $L(\Lambda)$ is modular invariant for a much wider (than integrable) collection of $\Lambda$’s, called admissible, defined by the condition [KW1], [KW2], [K4]:

$$2(\Lambda + \hat{\rho}|\alpha)/(\alpha|\alpha) \in \mathbb{Q}\setminus\{0, -1, -2, \ldots\} \text{ for all } \alpha \in \hat{\Delta}^+ \text{ such that } (\alpha|\alpha) \neq 0.$$ 

(The conjecture of Wakimoto and myself is that these are all modular invariant $L(\Lambda)$.) Here $\hat{\Delta}^+$ is the set of positive roots of $\hat{\mathfrak{g}}$ corresponding to $\hat{\mathfrak{n}}^+ : \hat{\Delta}^+ = \Delta^+ \cup \{\alpha + n\mathfrak{k}\} \cup \{n\mathfrak{k}|n \geq 1\}$, and $\hat{\rho} \in \mathfrak{h}^*$ is a vector satisfying $(\hat{\rho}|\alpha_i) = \frac{1}{2}(\alpha_i|\alpha_i)$ for $i = 0, 1, \ldots, r$, where $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ are simple roots of $\Delta^+$, $\Pi = \{\alpha_0 := K - \theta\} \cup \Pi$ are simple roots of $\hat{\Delta}^+$, $\theta$ is the highest root of $\Delta^+$.

I shall describe explicitly the most important class of them, called principal admissible. Fix a positive integer $u$ and let $\Pi_u = \{uK - \theta\} \cup \Pi$. Let $k = v/u$ be
a rational number, such that $v \in \mathbb{Z}$ is relatively prime to $u$ and $u(k + h^\vee) \geq h^\vee$. Here $h^\vee$ is the dual Coxeter number (defined in a more general Lie superalgebra context further on). Let $W$ be the Weyl group of $\mathfrak{g}$ and let $P^\vee = \{ \lambda \in \mathfrak{h}^*| (\lambda|\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta \}$. For each $\alpha \in P^\vee$ define a translation $t_\alpha \in \text{End} \mathfrak{h}$ by the formula $t_\alpha(\lambda) = \lambda + (\lambda|K)\alpha - ((\lambda|\alpha) + (\frac{(\lambda|K)}{2})(\alpha|\alpha))K$. Pick an element $\hat{w} = t_\beta w$, where $w \in W$, such that $\hat{w}\Pi_u \subset \Delta^+$ (these are all subsets of $\hat{\Delta}^+$ isomorphic to $\overline{\Pi}$). Let $\Lambda^0$ be an integrable highest weight of level $(k + h^\vee) - h^\vee$. Then

$$\Lambda = \hat{w}(\Lambda^0 + \hat{\rho} - (u - 1)(k + h^\vee)D) - \hat{\rho}$$

is a principal admissible weight of level $k$. The character of the corresponding $L(\Lambda)$ is given by the following formula [KW1]–[KW2]:

$$\left( Rch_{L(\Lambda)} \right)(\tau, z, t) = \left( Rch_{L(\Lambda^0)} \right)(u\tau, w^{-1}(z + \tau\beta), u^{-1}(t + (z|\beta) + \frac{1}{2}\tau(\beta|\beta))),$$

(Ch)

where $R = e^\phi \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})^{\text{mult } \alpha}$ is the Weyl denominator function for $\mathfrak{g}$, and mult $\alpha = 1$ except for mult $nK = r$ for all $n$ (note that this formula is a tautology if $\Lambda = \Lambda^0$ is integrable; this happens iff $u = 1$). Recall that $ch_{L(\Lambda^0)}$ is given by the Weyl-Kac character formula [K3].

3.4. Let now $\mathfrak{g} = \mathfrak{g}(A)$ be a basic simple finite-dimensional Lie superalgebra (see Sec. 1.7). Then $\mathfrak{g}$ carries a unique, up to a constant factor, non-degenerate invariant bilinear form $B$ ("invariant" means that $B([a, b], c) = B(a, [b, c])$). Let $\mathfrak{h} = \sum_{i=1}^{r} Ch_i$ be the Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{n}^+$ the subalgebra of $\mathfrak{g}$ generated by all $e_i$, $\Delta^+ \subset \mathfrak{n}^+$ the set of positive roots (i.e., roots of $\mathfrak{h}$ in $\mathfrak{n}^+$), $\Delta = \Delta^+ \cup -\Delta^+$ the set of all roots, $\Delta^0$ and $\Delta^1$ the sets of even and odd roots, $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ the set of simple roots $\{\alpha_i(h_j) = \delta_{ij}\}, \theta \in \Delta^+$ the highest root. Define $\rho \in \mathfrak{h}^*$ by $B(\rho, \alpha_i) = \frac{1}{2}B(\alpha_i, \alpha_i), i = 1, \ldots, r$. Then $h^\vee_B = B(\rho, \rho) + \frac{1}{2}B(\theta, \theta)$ is the eigenvalue of the Casimir operator in the adjoint representation.

If $h^\vee_B \neq 0$, we let $\Delta^0_\# = \{\alpha \in \Delta^0| h^\vee_B B(\alpha, \alpha) > 0\}$. If $h^\vee_B = 0$, which happens for $\mathfrak{g} = sl(m|m)/CE$, spo$(2m|2m + 2)$ and $D(2, 1; a)$, we take for $\Delta^0_\#$ the sets of roots of the subalgebra $sl_m, so_{2m+2}$ and $sl_2 \oplus sl_2$, respectively. Let $W^\#$ denote the subgroup of the Weyl group of $\mathfrak{g}$ generated by reflections with respect to all $\alpha \in \Delta^0_\#$. Denote by $(\cdot, \cdot)$ the invariant bilinear form on $\mathfrak{g}$ normalized by the condition $(\alpha|\alpha) = 2$ for the longest root $\alpha \in \Delta^0_\#$. The corresponding to this form number $h^\vee = h^\vee_{(\cdot, \cdot)}$ is called the dual Coxeter number. (For example, this number equals $|m - n|$ for $sl(m|n)$, $\frac{1}{2}(m - n) + 1$ for spo$(m|n)$ with $m \geq n - 2$, 30 for $E_8$, 3 for $F(4)$, 2 for $G(3)$.)

3.5. Define the affine superalgebra $\hat{\mathfrak{g}}$ associated to the Lie superalgebra $\mathfrak{g}$ in exactly the same way as in the Lie algebra case. The highest weight $\hat{\mathfrak{g}}$-modules $L(\Lambda)$ are defined in the same way too. The integrability of a $\mathfrak{g}$-module $V$ is defined by (M1), (M2) with $\Delta$ replaced by $\Delta^0_\#$, and
Integrable $\hat{g}$-modules $L(\Lambda)$ were classified in [KW4]. However, very little is known about their characters. The following example shows that modular invariance fails already in the simplest case $g = \mathfrak{sl}(2|1)$, $\Lambda = D$. In this case

$$e^{-D}ch_{L(D)} = \prod_{n=1}^{\infty} ((1 - q^n)^{-1})(1 + z_1 q^n)(1 + z_2^{-1} q^{n-1})A(z_1^{-1}, z_2 q^{1/2}, q), \quad (A)$$

where $z_i = e^{\epsilon_i + \gamma}$ ($i = 1, 2$; $\epsilon_i$ is the standard basis of the space of $3 \times 3$ diagonal matrices), $q = e^{2\pi i \tau}$, and

$$A(x, z, q) = \sum_{n \in \mathbb{Z}} \frac{q^{n^2/2} z^n}{1 + 2q^n}.$$ 

The function $A(x, z, q)$ converges to a meromorphic function in the domain $x, y, z \in \mathbb{C}$, $|q| < 1$, and is called Appell’s function. Since the first factor in (A) has the modular invariance property and $A(x, z, q)$ doesn’t have it, we see that $L(D)$ is not modular invariant. We call the Appell function an almost modular form since it is a section of a rank 2 vector bundle on an elliptic curve for each $\tau \in \mathcal{H}$ (whereas modular forms are sections of rank 1 vector bundles on it).

We call a weight $\Lambda \in \hat{h}^*$ of the Lie superalgebra $\hat{g}$ admissible (resp. principal admissible) if $\Lambda$ is admissible (resp. principle admissible) for the affine Lie algebra associated to a semisimple Lie algebra with root system $\Delta_0^\#$, and we conjecture [KW3] that formula (Ch) still holds, where $\hat{R}$ in the super case is defined by $\hat{R} = e^{\rho} \prod_{\alpha \in \Delta^+} (1 - (-1)^{p(\alpha)} e^{-\alpha})(-1)^{p(\alpha) \text{mult} \alpha}$. Unfortunately, $ch_{L(\Lambda^0)}$ is not known in general, but in the boundary level case, i.e., when $u(k + h) = h$, the level of $\Lambda^0$ is zero, hence $ch_{L(\Lambda^0)} = 1$, and formula (Ch) gives an explicit expression for $ch_{L(\Lambda)}$. In particular, the character is modular invariant in this case.

Let me mention in conclusion of this section that the Weyl-Kac character formula in the case of 1-dimensional module over an affine Lie algebra $\hat{g}$ turns into celebrated Macdonald’s identities that express $\hat{R}$ as an infinite series, the special case for $g = \mathfrak{sl}_2$ being the Jacobi triple product identity. A sum formula for $\hat{R}$ in the super case is also known [KW3] (see also the talk [Wa] at the last ICM). In the simplest case of $\hat{g} = sl(2|1)$ one gets the identity:

$$\prod_{n=1}^{\infty} \frac{(1 - q^n)^2(1 - uvq^{n-1})(1 - u^{-1}v^{-1}q^n)}{(1 - uq^{n-1})(1 - u^{-1}q^n)(1 - uvq^{n-1})(1 - v^{-1}q^n)} = \left( \sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{\infty} \right) u^m v^n q^{mn},$$

which goes back to Ramanujan and even further back to Kronecker.

4. Quantum reduction for affine Lie superalgebras
4.1. This part of my talk is based on a joint work with S.-S. Roan and M. Waki-
moto [KW]. I will explain a general quantum reduction scheme, which
is a further development of a number of works. They include [DS], [KS] and [Kh]
on classical Drinfeld-Sokolov reduction, and [FF], [FF2], [FKW], [B], [B1] on its
quantization. As in [FF2], the basic idea is to translate the geometric Drinfeld-
Sokolov reduction, and [FF1], [FF2], [FKW], [B], [BT] on its
moto [KRW], [KW5]. I will explain a general quantum reduction scheme, w hich
in the free fermionic vertex algebras ([K5], [K5]), all related notations can be fou nd in [K5].

4.2. Let \( g \) be a basic simple finite-dimensional Lie superalgebra with a non-degenerate
invariant bilinear form \( (\cdot,\cdot) \). Fix a number \( k \) and a nilpotent even element \( f \) of \( g \). Include
\( f \) in an \( \mathfrak{sl}_2 \)-triple \( \{e,h,f\} \) so that \([h,e] = 2e, [h,f] = -2f, [e,f] = h\). We have the eigenspace decomposition of \( g \) with respect to \( \text{ad} h \): \( g = \bigoplus_{i \in \mathbb{Z}} g_i \), and we let \( g_+ = \bigoplus_{i > 0} g_i \). The element \( f \) defines a non-degenerate skew-supersymmetric bilinear form \((\cdot,\cdot)\) on \( g_1 \) by the formula \((a,b) = (f|a,b)\). Let \( A_{ne} \) denote the superspace \( g_1 \) with the form \((\cdot,\cdot)\), Denote by \( A_{ch} \) the superspace \( \pi g_+ + \pi g_+^* \), where \( \pi \) stands for the reversal of parity, with the skew-supersymmetric bilinear form defined by \((a,b^*) = b^*(a)\) for \( a \in \pi g_+, b \in \pi g_+^* \), \((\pi g_+, \pi g_+) = 0 = (\pi g_1^*, \pi g_1^* )\).

Let \( V^k(g) \) be the universal affine vertex algebra, and let \( F^1(A_{ne}), F^1(A_{ch}) \) be
the free fermionic vertex algebras ([K5], §4.7). Consider the vertex algebra
\[
C(g,f,k) = V^k(g) \otimes F^1(A_{ne}) \otimes F^1(A_{ch}),
\]
and define its charge decomposition \( C(g,f,k) = \bigoplus_{m \in \mathbb{Z}} C_m \) by letting charge \( V^k(g) = \text{charge } F^1(A_{ne}) = 0 \), charge \( \pi g_+ = 1 \) = charge \( \pi g_+^* \).

Next, we define a differential \( d \) on \( C(g,f,k) \) which makes it a homology complex. For this choose a basis \( \{u_i\}_{i \in S'} \) of \( g_1 \) and extend it to a basis \( \{u_i\}_{i \in S} \) of \( g_+ \) compatible with its \( \mathbb{Z} \)-gradation. Denote by \( \{\varphi_i\}_{i \in S} \) and \( \{\varphi_i^*\}_{i \in S} \) the corresponding dual bases of \( \pi g_+ \) and \( \pi g_+^* \), and by \( \{\Phi_i\}_{i \in S} \) the corresponding basis of \( A_{ne} \). Consider the following odd field of the vertex algebra \( C(g,f,k) \):
\[
d(z) = \sum_{i \in S} (-1)^{p(u_i)} u_i(z) \otimes \varphi_i^*(z) \otimes 1 - \frac{1}{2} \sum_{i,j,k \in S} (-1)^{p(u_i)p(u_k)} c_{ij}^k \varphi_k(z) \varphi_i^*(z) \varphi_j^*(z) \otimes 1 + \sum_{i \in S} (f|u_i) \otimes \varphi_i^*(z) \otimes 1 + \sum_{i \in S'} 1 \otimes \varphi_i^*(z) \otimes \Phi_i(z),
\]
where \([u_i,u_j] = \sum_k c_{ij}^k u_k \) in \( g_+ \), and let \( d = \text{Res}_{z=0} d(z) \). (Note that the first two summands of \( d \) form the usual differential of a Lie (super)algebra complex.) Then one checks that \([d(z),d(w)] = 0\), hence \( d^2 = 0 \). It is also clear that \( d C_m \subset C_{m-1} \).

We define the vertex algebra \( W(g,f,k) \) as the \( 0 \)-th homology of this complex, and call it the \textit{quantum reduction} of the triple \( (g,f,k) \) (actually, it depends only on \( g,k \) and the conjugacy class of \( f \) in \( g_1 \)).
One of the fields of the vertex algebra \( W(\mathfrak{g}, f, k) \) is the following Virasoro field

\[
L(z) = \frac{1}{2(k+h^\vee)} \sum_i :a_i(z)b_i(z) : + \frac{1}{2} \partial_z h(z) + \sum_{i \in S} ((1-m_i) :\partial_z \varphi_i^*(z)\varphi_i(z) : - m_i :\varphi_i^*(z)\partial_z \varphi_i(z) : + \frac{1}{2} \sum_{i \in S'} g^{ij} :\partial_z \Phi_i(z)\Phi_j(z) : ,
\]

where \( \{b_i\} \) and \( \{a_i\} \) are dual bases of \( \mathfrak{g} \): \( (b_i|a_j) = \delta_{ij} \), \( [h, u_i] = 2m_iu_i \), \( (g^{ij}) \) is the matrix inverse to \( \langle (u_i, u_j) \rangle \). The central charge of \( L(z) \) is equal to:

\[ c(k) = \frac{k \text{sdim} \mathfrak{g}}{k+h^\vee} - 3(h|h)k - \sum_{i \in S} (-1)^{p(u_i)} (12m_i^2 - 12m_i + 2) - \frac{1}{2} \text{sdim} \mathfrak{g} . \]

Here \( \text{sdim} V = \dim V^\vee - \dim V^1 \) is the superdimension of the superspace \( V \). Thus, all \( W(\mathfrak{g}, f, k) \) are super-extensions of the Virasoro algebra. Furthermore, for each ad \( h \) eigenvector with eigenvalue \( -2j \) in the centralizer of \( f \) in \( \mathfrak{g} \), \( W(\mathfrak{g}, f, k) \) contains a field of conformal weight \( 1 + j \) (so that \( L(z) \) corresponds to \( f \)), and these fields generate the vertex algebra \( W(\mathfrak{g}, f, k) \).

**Examples.** (1) \( \mathfrak{g} \) is a simple Lie algebra.

(a) \( f \) is a principal nilpotent element. Then \( W(\mathfrak{g}, f, k) \) is called the quantum Drinfeld-Sokolov reduction. These algebras and their representations were extensively studied in \( \text{[FZ]} \), \( \text{[FB]} \), \( \text{[FKW]} \) and many other papers. The simplest case of \( \mathfrak{g} = \mathfrak{sl}_2 \) produces the Virasoro vertex algebra. The case \( \mathfrak{g} = \mathfrak{sl}_3 \) gives the \( W_3 \) algebra \( \text{[Z]} \).

(b) \( f \) is a lowest root vector of \( \mathfrak{g} \). These vertex algebras were discussed from a different point of view in \( \text{[FL]} \) under the name quasi-superconformal algebras. The special case of \( \mathfrak{g} = \mathfrak{sl}_3 \) was studied from a quantum reduction viewpoint in \( \text{[Be]} \).

(2) \( \mathfrak{g} \) is a simple basic Lie superalgebra and \( f \) is an even lowest root vector.

(a) One has the following correspondence:

| \( \mathfrak{g} \) | \( \text{spo}(2|1) \) | \( \mathfrak{sl}(2|1) \) | \( \mathfrak{sl}(2|2) \) | \( \text{spo}(2|3) \) | \( D(2,1|a) \) |
|---|---|---|---|---|---|
| \( W(\mathfrak{g}, f, k) \) | Neveu-Schwarz | \( N = 2 \) | \( N = 4 \) | \( N = 3 \) | big \( N = 4 \) |

(In the last two columns one gets an isomorphism after adding one fermion, resp. four fermions and one boson.)

(b) Almost every lowest root vector of a simple component of \( \mathfrak{g}^\mathfrak{g} \) can be made equal \( f \). This gives all superconformal algebras of \( \text{[FL]} \) (by definition, they are generated by the Virasoro field, the even fields of weight 1 and \( N \) odd fields of weight 3/2), and many new examples.

### 4.3.

Let \( M \) be a highest weight module over \( \hat{\mathfrak{g}} \). It extends to a vertex algebra module over \( V^k(\hat{\mathfrak{g}}) \), and we consider the \( C(\mathfrak{g}, f, k) \)-module \( C(M) = M \otimes F^1(A_{\mathfrak{g}}) \otimes F^1(A_{\mathfrak{g}}) \). The element \( d \) acts on \( C(M) \) and again \( d^2 = 0 \), hence we can consider homology \( H(M) = \oplus_j H_j(M) \) which is a module over \( W(\mathfrak{g}, f, k) \). The \( W(\mathfrak{g}, f, k) \)-module \( H(M) \) is called the quantum reduction of the \( \hat{\mathfrak{g}} \)-module \( M \). Using the Euler-Poincaré principle one easily computes the character of \( H(M) \) in terms of \( \text{ch}_M \). The
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basic conjecture of \[FKW\], \[KRW\] is that \(H(M)\) is irreducible (in particular, at most one \(H_j(M)\) is non-zero) if \(M\) is admissible.

**Examples.** (1) \(g = \mathfrak{sl}_2\). Let \(k\) be an admissible level, i.e., \(k\) is a rational number with positive denominator \(u\) such that \(u(k + 2) \geq 2\) (recall that \(h' = 2\)). The set of principal admissible weights of level \(k\) is as follows (\(\alpha\) is a simple root of \(\mathfrak{sl}_2\) [KW], [K4]):

\[
\{\Lambda_{k,j,n} = kD + \frac{1}{2}(n - j(k + 2))\alpha \mid 0 \leq j \leq u - 1, 0 \leq n \leq u(k + 2) - 2\}.
\]

Then the quantum reduction of the \(\mathfrak{sl}_2\)-module \(L(\Lambda_{k,j,n})\) is the “minimal series” module corresponding to parameters \(p = u(k + 2), p' = u\) (cf. [BPZ], [K4]):

\[
c^{(p,p')} = 1 - 6 \frac{(p - p')^2}{pp'}, \quad h_{j+1,n+1}^{(p,p')} = \frac{(p(j + 1) - p'(n + 1))^2 - (p - p')^2}{4pp'}.
\]

The character formula (Ch) for \(L(\Lambda_{k,j,n})\) gives immediately all the characters of minimal series.

(2) \(g = \mathfrak{spo}(2\mid 1)\). We get all minimal series modules over the Neveu-Schwarz algebra and their characters by quantum reduction of all (not only principal) admissible \(\hat{g}\)-modules.

(3) \(g = \mathfrak{sl}(2\mid 1)\). Then the boundary admissible levels are \(k = m^{-1} - 1\), where \(m \in \mathbb{Z}, m \geq 2\). One has the following \(\hat{w}\Pi\)'s (see Sec. [3.3]), where \(\alpha_1\) and \(\alpha_2\) are odd: \(\{\alpha_0 + b_0K, \alpha_1 + b_1K, \alpha_2 + b_2K\}\). Here \(b_i\) are non-negative integers, \(b_0 \geq 1\) and \(\sum b_i = m - 1\). The quantum reduction of the corresponding admissible \(\hat{g}\)-modules gives all the minimal series representations of the \(N = 2\) superconformal algebra (cf. [FST] and references there). Again, formula (Ch) gives immediately their characters.

(4) \(g = \mathfrak{sl}(2\mid 1)\) (resp. \(\mathfrak{spo}(2\mid 3)\)). In a similar fashion we recover the characters of \(N = 4\) (resp. \(N = 3\)) superconformal algebras.

5. **Representations of \(E(3\mid 6)\) and the standard model**

5.1. By a representation of a linearly compact Lie superalgebra \(L\) we shall mean a continuous representation in a vector space \(V\) with discrete topology (then the contragredient representation is a continuous representation in a linearly compact space \(V^*\)). Fix an open subalgebra \(L_0\) of \(L\). We shall assume that \(V\) is locally \(L_0\)-finite, meaning that any vector of \(V\) is contained in a finite-dimensional \(L_0\)-invariant subspace (this property actually often implies that \(V\) is continuous). These kinds of representations were studied in the Lie algebra case by Rudakov [R]. It is easy to show that such an irreducible \(L\)-module \(V\) is a quotient of an induced module \(\text{Ind}_{L_0}^L U = U(L) \otimes_{U(L_0)} U\), where \(U\) is a finite-dimensional irreducible \(L_0\)-module, by a (unique in good cases) maximal submodule. The induced module \(\text{Ind}_{L_0}^L U\) is called degenerate if it is not irreducible. An irreducible quotient of a degenerate induced module is called a degenerate irreducible module.
One of the most important problems of representation theory is to determine all degenerate representations. I will state here the result for $E = E(3|6)$ with $L_0 = \Pi_{j \geq 0} a_j$ (see Sec. [L.6]). The finite-dimensional irreducible $L_0$-modules are actually $a_0 = s_{\ell_3} \otimes s_{\ell_2} \otimes gl_1$-modules (with $\Pi_{j \geq 0} a_j$ acting trivially). We shall normalize the generator $Y$ of $g\ell_1$ by the condition that its eigenvalue on $\alpha_{-1}$ is $-1/3$. The finite-dimensional irreducible $a_0$-modules are labeled by triples $(p, q; r; Y)$, where $p, q$ (resp. $r$) are labels of the highest weight of an irreducible representation of $s_{\ell_3}$ (resp. $s_{\ell_2}$), so that $0, 0$ and $0, q$ label $S^p \mathbb{C}^3$ and $S^q \mathbb{C}^3$ (resp. $r$ labels $S^r \mathbb{C}^2$), and $Y$ is the eigenvalue of the central element $Y$. Since irreducible $E(3|6)$-modules are unique quotients of induced modules, they can be labeled by the above triples as well.

**Theorem 4.** [KR1 - KR3] The complete list of irreducible degenerate $E(3|6)$-modules consists of four series: $(p, 0; r; -r + \frac{2}{3} p), (p, 0; r; r + \frac{2}{3} p + 2), (0, q; r; -r - \frac{2}{3} q - 2), (0, q; r; r - \frac{2}{3} q)$.

5.2. Remarkably, all four degenerate series occur as cokernels of the differential of a differential complex $(M, \nabla)$ constructed below (see [KR2] for details). I shall view $E(3|6)$ as a subalgebra of $E(5|10) = \Pi_{j \geq -2} g\ell_j$ as in Sect. [L.6], expressed in terms of vector fields and differential forms in the indeterminates $x_1, x_2, x_3, z_+ = x_4, z_- = x_5$. Recall that $g\ell_0$ is the algebra of divergenceless vector fields with linear coefficients. Let $Y = \frac{2}{3} \sum_i x_i \partial_i - \sum_\epsilon z_\epsilon \partial_\epsilon \in g\ell_0$. Here and further $i = 1, 2, 3, \epsilon = +, -$, and $\partial_i = \partial/\partial x_i, \partial_\epsilon = \partial/\partial z_\epsilon$. Then $a_0$ is the centralizer of $Y$ in $g\ell_0$ and $a_{-1}$ is the span of all elements $d_\epsilon = dx_\epsilon \wedge dz_\epsilon$.

Consider the following four $a_0$-modules (extended to $L_0$-modules by trivial action of $a_j$ with $j > 0$):

\[ V_I = \mathbb{C}[x_i, z_\epsilon], \quad V_{II} = \mathbb{C}[x_i, \partial_\epsilon][2], \quad V_{III} = \mathbb{C}[\partial_i, z_\epsilon][-2], \quad V_{IV} = \mathbb{C}[\partial_i, \partial_\epsilon], \]

where the subscript $[a]$ means that $Y$ is shifted by the scalar $a$. For each $R = I - IV$ introduce a bigraduation $V_R = \oplus_{m, n \in \mathbb{Z}} V_R^{(m, n)}$ by letting $\deg x_i = (1, 0)$, $\deg z_\epsilon = (0, 1)$, and let $M_R = \operatorname{Ind}_{L_0} V_R = \oplus_{m, n \in \mathbb{Z}} M_R^{(m, n)}$. Then the non-zero $M_R^{(m, n)}$ are all the degenerate $E(3|6)$-modules of the $R$th series. We let

\[ M = \left( \oplus_{(m, n) \neq (0, 0)} M_I^{(m, n)} \right) \oplus M_{II} \oplus M_{III} \oplus \left( \oplus_{(m, n) \neq (0, 0)} M_{IV}^{(m, n)} \right). \]

The differentials $\nabla_k$ introduced further are elements of $U(L) \otimes \operatorname{End} V$ that act on $U(L) \otimes_{U(L_0)} V$ by the formula:

\[ (\sum_j u_j \otimes A_j)(u \otimes v) = \sum_j u u_j \otimes A_j v. \]

**Example.** The dual to the ordinary formal de Rham complex is $\mathbb{C}[\partial/\partial x_1, \ldots, \partial/\partial x_m] \otimes \Lambda(\partial/\partial \xi_1, \ldots, \partial/\partial \xi_m)$ with the differential $d^* = \sum_j \partial/\partial x_j \otimes \xi_j$ and the $\mathbb{Z}$-gradation defined by $\deg \partial/\partial x_i = 0$, $\deg \partial/\partial \xi_i = 1$. Rudakov’s theorem [R] says that all irreducible degenerate $W_m$-modules occur as cokernels of $d^*$. 


Turning now to $\nabla_k$, we let $\Delta^{\pm} = \sum_i d_i^\pm \otimes \partial_i$, $\delta_i = d_i^+ \otimes \partial_+ + d_i^- \otimes \partial_-$. Then $\nabla_1 = \Delta^+ (1 \otimes \partial_+) + \Delta^- (1 \otimes \partial_-)$ is a well-defined operator on all $M_R$ such that $\nabla_1^2 = 0$. Furthermore there are differentials $\nabla_2 = \Delta^+ \Delta^-, \nabla_3 = \delta_1 \delta_2 \delta_3$, $\nabla_4$, $\nabla'_4$ and $\nabla_6$ (the explicit expressions of the last three can be found in [KR2]) that sew together these four complexes. These differentials are illustrated by Table M. The white nodes and black marks represent the induced modules of the $R^{th}$ series. The plain arrows represent $\nabla_1$, the dotted arrows represent $\nabla_2$, the interrupted arrows represent $\nabla_3$ and the bold arrows represent $\nabla'_4$, $\nabla'_4$ and $\nabla_6$. The white nodes denote the places with zero homology. The black marks denote the places with non-zero homology, also computed in [KR2]. For example, at the star mark the homology is $\mathbb{C}$.

Similar results for $E(3|8)$ and (to a lesser extent) for $E(5|10)$ are given in [KR4].

5.3. The first hint that the Lie superalgebra $E(3|6)$ is somehow related to the Standard Model comes from the observation that its subalgebra $a_0$ is isomorphic to the complexified Lie algebra of the group of symmetries of the Standard Model. Table P below lists all $a_0$-multiplets of fundamental particles of the Standard Model (see e.g. [3]): the upper part is comprised of three generations of quarks and the middle part of three generations of leptons (these are all fundamental fermions from which matter is built), and the lower part is comprised of the fundamental bosons (which mediate the strong and electro-weak interactions).

| multiplets | charges | particles |
|------------|---------|-----------|
| (01, 1, $\frac{1}{3}$) | 2, $-\frac{1}{3}$ | ($\tilde{u}^L_dL$) ($\tilde{c}^L_dL$) ($\tilde{t}^L_dL$) |
| (10, 1, $\frac{2}{3}$) | $-\frac{2}{3}$, $\frac{1}{3}$ | ($\tilde{u}^R_dR$) ($\tilde{c}^R_dR$) ($\tilde{t}^R_dR$) |
| (10, 0, $-\frac{2}{3}$) | $-\frac{2}{3}$ | $\tilde{u}_L$, $\tilde{c}_L$, $\tilde{t}_L$ |
| (01, 0, $\frac{1}{3}$) | $\frac{2}{3}$ | $u_R$, $c_R$, $t_R$ |
| (01, 0, $-\frac{1}{3}$) | $-\frac{1}{3}$ | $d_R$, $s_R$, $b_R$ |
| (10, 0, $\frac{2}{3}$) | $\frac{1}{3}$ | $\tilde{d}_L$, $\tilde{s}_L$, $\tilde{b}_L$ |
| (00, 1, $-1$) | 0, $-1$ | ($\nu^{\mu}_\tau L$) ($\tau^{\mu}_L$) ($\nu^{\tau}_L$) |
| (00, 1, 1) | 0, 1 | ($\tilde{\nu}^R_{\mu} R$) ($\tilde{\nu}^R_{\mu} R$) ($\tilde{\nu}^R_{\tau} R$) |
| (00, 0, 1) | 1 | $\tilde{\epsilon}_L$, $\tilde{\mu}_L$, $\tilde{\tau}_L$ |
| (00, 0, $-2$) | $-1$ | $\epsilon_R$, $\mu_R$, $\tau_R$ |
| (11, 0, 0) | 0 | | gluons |
| (00, 2, 0) | 1, $-1, 0$ | $W^+, W^-, Z$ (gauge bosons) |
| (00, 0, 0) | 0 | $\gamma$ (photon) |

Table P
It is easy to deduce from Theorem 4 that this list of multiplets (plus the multiplets \((11,0,\pm 2)\)) is characterized by the conditions:

(i) \(a_0\)-multiplet occurs in a degenerate irreducible \(E(3|6)\)-module,
(ii) when restricted to \(\mathfrak{sl}_3 \subset a_0\), this multiplet contains only 1-dimensional, the two fundamental or the adjoint representation,
(iii) \(|Q| \leq 1\) for all particles of the multiplet, where the charge \(Q\) of a particle is given by the Gell-Mann-Nishijima formula: \(Q = \frac{1}{2}(y + h)\), where \(y\) (resp. \(h\)) is the \(Y\)-eigenvalue (resp. \(H = \text{diag}(1,-1) \in \mathfrak{sl}_2\)-eigenvalue).

How can we see the number of generations of quarks and leptons? For that order the sequence subcomplexes in Table M by \(t = r - q + 3\) in sector IV (time), and replace in them the induced modules by their irreducible quotients. Then we find \([KR2]\) (based on computer calculations by Joris Van der Jeugt) that a fundamental particle multiplet appears in the \(t\)th sequence iff \(t \geq 1\). Furthermore, for \(1 \leq t \leq 7\) we get sequences with various particle contents, but for \(t \geq 8\) the particle contents remains unchanged, and it is invariant under the CPT symmetry (though for \(t \leq 7\) it is not). The explicit contents is exhibited in \([KR2]\), 659–660.

Remarkably, precisely three generations of leptons occur in the stable region \((t \geq 8)\), but the situation with quarks is more complicated: this model predicts a complete fourth generation of quarks and an incomplete fifth generation (with missing down type triplets).

In view of this discussion, it is natural to suggest that the algebra \(su_3 + su_2 + u_1\) of internal symmetries of the Weinberg-Salam-Glashow Standard Model extends to \(E(3|6)\). It is hoped that the representation theory of \(E(3|6)\) will shed new light on various features of the Standard Model. I find it quite remarkable that the \(SU_5\) Grand Unified Model of Georgi-Glashow combines the left multiplets of fundamental fermions in precisely the negative part of the consistent gradation of \(E(5|10)\) (see Sec. 1.6). This is perhaps an indication of the possibility that the extension from \(su_5\) to \(E(5|10)\) algebra of internal symmetries may resolve the difficulties with the proton decay.

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