RELATIONS ON $\overline{M}_{g,n}$ VIA THE ORBIFOLD $[C/\mathbb{Z}_r]$

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Abstract. Using an expression for the virtual cycle of components of the moduli space of stable maps to the orbifold $[C/\mathbb{Z}_r]$ in terms of twisted theory over $B\mathbb{Z}_r$, we derive tautological relations in the Chow ring of $\overline{M}_{g,n}(B\mathbb{Z}_r, 0)$. These push forward to yield tautological relations on $\overline{M}_{g,n}$. We also prove that the quantum cohomology of $[C/\mathbb{Z}_r]$ is generically semisimple, which, in light of recent results of Pandharipande-Pixton-Zvonkine, may yield another strategy for computing tautological relations.

1. Introduction

The tautological ring of the moduli space of curves— or, more precisely, the tautological rings $R^*(\overline{M}_{g,n})$— is the minimal family of subrings stable under pullbacks and pushforwards via the forgetful and attaching maps. In particular, the kappa, psi, and boundary classes, which play important roles in Gromov-Witten theory, are all elements of the tautological ring.

Relations between tautological classes in $\mathcal{M}_g$ were initially studied by Mumford [21] in the 1980s. More recently, ideas from topological string theory have been applied to develop a more complete picture. In 2000, Faber and Zagier conjectured the so-called FZ relations on $\mathcal{M}_g$, which were proved via the geometry of stable quotients in 2010 [22]. A striking generalization occurred in 2013, when Pandharipande-Pixton-Zvonkine [23] used Witten’s 3-spin theory and the method of quantization developed by Givental and Teleman [13, 25] to prove that a set of relations previously conjectured by Pixton [24] holds in the cohomology of $\overline{M}_{g,n}$. When restricted to $\mathcal{M}_g$, these relations recover the FZ relations, and indeed, all presently known tautological relations follow from those proved in [23].

The main result of the present paper is a set of tautological relations in the Chow ring of $\overline{M}_{g,n}(B\mathbb{Z}_r, 0)$ obtained by studying the virtual cycle of the moduli space of stable maps to the orbifold $[C/\mathbb{Z}_r]$, which yield tautological relations in $A^*(\overline{M}_{g,n})$ via pushforward.

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**Theorem 1.0.1.** Suppose that $a_1, \ldots, a_n \in \{0, 1, \ldots, r-1\}$ are not all zero and $\sum_{i=1}^{n} a_i \equiv 0 \mod r$. Then, for any $d$ such that
\[
\frac{1}{r} \sum_{i=1}^{n} a_i + g - 1 < d \leq 3g - 3 + n,
\]
one has the following relations in $A^d(\overline{M}_{g,a}(B\mathbb{Z}_r,0))$:
\[
0 = \sum_{d_1, \ldots, d_m = d, m \geq 1} \frac{1}{m! \prod_{i=1}^{m} d_i (d_i + 1)} \prod_{i=1}^{m} (B_{d_i+1}(0) \kappa_{d_i} - \sum_{j=1}^{n} B_{d_i+1} \left( \frac{a_j}{r} \right) \psi_{d_i}^j + \frac{r}{2} \sum_{0 \leq l \leq g, l \in \mathbb{Z}[m]} B_{d_i+1} \left( \frac{q(l,I)}{r} \right) p^* i(I)^* (\gamma_{d_i-1}) + \frac{r}{2} \sum_{q=0}^{r-1} B_{d_i+1} \left( \frac{q}{r} \right) j(\text{irr},q)^* (\gamma_{d_i-1}) \right).
\]
If $a_1 = \cdots = a_n = 0$, the same equations hold for $g > 0$ on components of $\overline{M}_{g,a}(B\mathbb{Z}_r,0)$ corresponding to curves with a nontrivial $r$-torsion line bundle.

In particular, the pushfowards of these equations via the map
\[
\rho : \overline{M}_{g,a}(B\mathbb{Z}_r,0) \to \overline{M}_{g,n}
\]
give tautological relations in the Chow ring of $\overline{M}_{g,n}$.

The definitions of the classes appearing in the theorem are reviewed in Section 3.2. One easy consequence is an expression for certain polynomials in the $\kappa$ classes on $\overline{M}_g$ in terms of classes supported on the boundary (see Corollary 3.3.2).

Although the relations in Theorem 1.0.1 lie in a strictly smaller range of degrees than those proved in [23], the theorem has one advantage in that the relations we obtain in $\overline{M}_{g,n}(B\mathbb{Z}_r,0)$ are not pulled back from the moduli space of curves. This makes them in some sense strictly stronger than their pushed-forward versions.

We also prove that the quantum cohomology of $[\mathbb{C}/\mathbb{Z}_r]$ is generically semisimple, which is suggestive of further relations. Indeed, one might expect that, following the methods of [23], semisimplicity would allow one to define a Cohomological Field Theory whose vanishing properties imply tautological relations by way of an $R$-matrix action. In fact, the degeneracy of the Poincaré pairing on $[\mathbb{C}/\mathbb{Z}_r]$ presents a serious obstacle to this program. Nevertheless, it is possible that a limiting procedure could be applied to recover the desired relations from an $R$-matrix action on a CohFT defined via a compactification of $[\mathbb{C}/\mathbb{Z}_r]$, a strategy that we discuss in the final section of the paper.
1. Outline of the paper. In Section 2, we review the basic definitions from orbifold Gromov-Witten theory that we require. Most importantly, we express the virtual cycle for most components of the moduli space of stable maps to \([\mathbb{C}/\mathbb{Z}_r]\) as the Chern class of a bundle over \([\overline{M}_{g,n}(B\mathbb{Z}_r, 0)]\). An application of the Grothendieck-Riemann-Roch Theorem, which has previously been carried out and simplified by Chiodo [3], thus expresses the virtual cycle in terms of tautological classes. We present this expression in Section 3 and explain how it implies the tautological relations of Theorem 1.0.1. Semisimplicity of the quantum cohomology is proved in Section 4, which concludes with a more detailed discussion of the possibility of applying the methods of Pandharipande-Pixton-Zvonkine to this setting.

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2. Preliminaries on \([\mathbb{C}/\mathbb{Z}_r]\) and its Gromov-Witten theory

2.1. The orbifold and its cohomology. By \([\mathbb{C}/\mathbb{Z}_r]\), we mean the stack quotient of \(\mathbb{C}\) by the action of \(\mathbb{Z}_r\) via multiplication by \(r\)th roots of unity.

The appropriate notion of orbifold cohomology for the purposes of Gromov-Witten theory is the Chen-Ruan cohomology \(H^*_{CR}(\mathbb{C}/\mathbb{Z}_r; \mathbb{Q})\). As a vector space, the Chen-Ruan cohomology is the ordinary orbifold cohomology of the inertia stack \(I[\mathbb{C}/\mathbb{Z}_r]\), whose objects are pairs \((x, g)\) with \(x \in [\mathbb{C}/\mathbb{Z}_r]\) and \(g\) an element of the isotropy group at \(x\). Explicitly,

\[
I[\mathbb{C}/\mathbb{Z}_r] = [\mathbb{C}/\mathbb{Z}_r] \sqcup \bigsqcup_{i=1}^{r-1} B\mathbb{Z}_r,
\]

in which \(B\mathbb{Z}_r\) denotes the stack consisting of a single point with \(\mathbb{Z}_r\) isotropy. Thus,

\[
H^*_{CR}([\mathbb{C}/\mathbb{Z}_r]; \mathbb{Q}) = \mathbb{Q}\{\zeta_0, \zeta_1, \ldots, \zeta_{r-1}\},
\]

where \(\zeta_0\) is the constant function 1 on \([\mathbb{C}/\mathbb{Z}_r]\) and \(\zeta_i\) is the constant function 1 on the \(i\)th copy of \(B\mathbb{Z}_r\). The component of the inertia stack
whose cohomology is generated by $\zeta_0$ is called the *untwisted sector*, while the other components are referred to as *twisted sectors*.

2.2. **Stable maps to orbifolds.** For an orbifold $\mathcal{X}$, let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli stack of $n$-pointed genus-$g$ orbifold stable maps to $\mathcal{X}$ of degree $d$. That is, this stack parameterizes the following families of objects, up to a suitable notion of isomorphism:

$$
(C, \{\Sigma_i\}) \xrightarrow{f} \mathcal{X}
$$

where

1. $C/T$ is a genus-$g$, $n$-pointed orbifold curve;
2. For $i = 1, \ldots, n$, the substack $\Sigma_i \subset C$ is a (trivial) gerbe over $T$ with a section $\sigma_i : T \to \Sigma_i$ inducing an isomorphism between $T$ and the coarse moduli of $\Sigma_i$;
3. $f$ is a representable morphism whose induced map between coarse moduli spaces is a stable map of degree $d$.

The condition of *representability* means that the induced homomorphism on isotropy groups at every point is injective.

The precise definition of an orbifold curve can be found, for example, in [1]. For the present, we recall just two particularly important features. First, orbifold curves are allowed isotropy only at marked points and nodes. Second, the orbifold structure at nodes is required to be balanced, which means that étale locally near each node, $C$ has the form

$$\{xy = 0\}/\mathbb{Z}_\ell$$

with $\mathbb{Z}_\ell$ acting by $\xi(x, y) = (\xi x, \xi^{-1} y)$.

The moduli stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ admits a perfect obstruction theory relative to the Artin stack of prestable pointed orbicurves, and this obstruction theory can be used to define a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^{\text{vir}} \in H^*(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d); \mathbb{Q}).$$

As in the non-orbifold case, the perfect obstruction theory is given by the object $(R^* \pi_* f^* \mathcal{T} \mathcal{X})^\vee$ in the derived category, where

$$
\begin{array}{ccc}
C & \xrightarrow{f} & \mathcal{X} \\
\pi \downarrow & & \\
\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)
\end{array}
$$
is the universal family over the moduli space.

Also analogously to the non-orbifold theory, there are evaluation maps, but in this case they map to the inertia stack:

\[ \text{ev}_i : \mathcal{M}_{g,n}(\mathcal{X}, d) \to I\mathcal{X}. \]

Specifically, the image of \((f : C \to \mathcal{X})\) can be viewed as the point \((x, g) \in I\mathcal{X}\), where \(x\) is the image of the \(i\)th marked point \(p_i\) and \(g\) is the image of a generator of the isotropy group at \(p_i\) under the homomorphism of isotropy groups \(G_{p_i} \to G_x\) induced by \(f\).

In particular, elements of \(H_{CR}^*(\mathcal{X}; \mathbb{Q})\) can be pulled back and used to define primary Gromov-Witten invariants:

\[
\langle \phi_1 \cdots \phi_n \rangle_{g,n,d} = \int_{[\mathcal{M}_{g,n}(\mathcal{X}, d)]^{vir}} \text{ev}_1^*(\phi_1) \cdots \text{ev}_n^*(\phi_n).
\]

There is a decomposition of \(\mathcal{M}_{g,n}(\mathcal{X}, d)\) according to the images of the evaluation maps:

\[
\mathcal{M}_{g,n}(\mathcal{X}, d) = \bigsqcup_{a_1, \ldots, a_n \in I} \text{ev}_1^{-1}(\mathcal{X}_{a_1}) \cap \cdots \cap \text{ev}_n^{-1}(\mathcal{X}_{a_n}),
\]

where

\[ I\mathcal{X} = \bigsqcup_{i \in I} \mathcal{X}_i \]

decomposes \(I\mathcal{X}\) into twisted sectors. In the case where \(\mathcal{X} = [\mathbb{C}/\mathbb{Z}_r]\), the index set \(I\) is equal to \(\{0, 1, \ldots, r-1\}\), and \(\zeta_i\) is the constant function 1 on the sector \(\mathcal{X}_i\).

We denote

\[
\mathcal{M}_{g,a}(\mathcal{X}, d) = \text{ev}_1^{-1}(\mathcal{X}_{a_1}) \cap \cdots \cap \text{ev}_n^{-1}(\mathcal{X}_{a_n}),
\]

where \(a = (a_1, \ldots, a_n)\). By pullback, each of these components obtains a virtual fundamental class, and the invariant (1) can be written as an integral against this restricted virtual class.

2.3. Stable maps to \([\mathbb{C}/\mathbb{Z}_r]\). In the special case where \(\mathcal{X} = [\mathbb{C}/\mathbb{Z}_r]\), there are no stable maps of positive degree. The decomposition (2) of the moduli space \(\mathcal{M}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0)\) is written

\[
\mathcal{M}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0) = \bigsqcup_{a_1, \ldots, a_n \in \mathbb{Z}_r} \mathcal{M}_{g,a}([\mathbb{C}/\mathbb{Z}_r], 0),
\]

where \(\mathcal{M}_{g,a}([\mathbb{C}/\mathbb{Z}_r], 0)\) is the component in which the \(i\)th marked point maps to the twisted sector indexed by \(\zeta_i\). (We identify \(\mathbb{Z}_r\) with the set \(\{0, 1, \ldots, r-1\}\) throughout this paper.)
We observe that the component $\overline{\mathcal{M}}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0)$ is:

1. Compact if and only if not every $a_i$ is equal to zero.\footnote{There will, however, be at least one compact component of $\overline{\mathcal{M}}_{g,(a_0,...,0)}(\mathbb{C}/\mathbb{Z}_r, 0)$ when $g > 0$; see Remark 2.3.1.}
2. Nonempty if and only if $\sum_{i=1}^{n} a_i \equiv 0 \mod r$.

The proof of the first of these assertions is straightforward, while the second is a standard fact in orbifold Gromov-Witten theory; alternatively, it will follow from the description of the moduli space in terms of line bundles given below.

We will always assume, unless otherwise specified, that not every $a_i$ is zero, so that the moduli space is compact. In this case, $\overline{\mathcal{M}}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0)$ is isomorphic to $\overline{\mathcal{M}}_{g,a}(B\mathbb{Z}_r, 0)$. However, their obstruction theories are different, and this leads to the following relationship between their virtual cycles:

\begin{equation}
[\overline{\mathcal{M}}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0)]^{vir} = [\overline{\mathcal{M}}_{g,a}(B\mathbb{Z}_r, 0)] \rightsquigarrow e(R^1\pi_* f^* N_{B\mathbb{Z}_r/\mathbb{C}/\mathbb{Z}_r}).
\end{equation}

Another perspective on stable maps to $B\mathbb{Z}_r$ can be used to make equation (3) more explicit. The data of an orbifold stable map $C \to B\mathbb{Z}_r$ is equivalent to an orbifold line bundle $L$ on $C$ satisfying $L^{\otimes r} \cong \mathcal{O}_C$. The map $f : C \to B\mathbb{Z}_r$ corresponds to the line bundle $f^* N$, where $N$ is the topologically trivial line bundle over $B\mathbb{Z}_r$ on which $\mathbb{Z}_r$ acts by multiplication by $r$th roots of unity.

From this point of view, $f^* N_{B\mathbb{Z}_r/\mathbb{C}/\mathbb{Z}_r}$ corresponds to the universal line bundle $\mathcal{L}$ over $\overline{\mathcal{M}}_{g,a}(B\mathbb{Z}_r, 0)$. Furthermore, the monodromy $a_i$ becomes the multiplicity of $L$ at the $i$th marked point— that is, $a_i$ is the weight of the action of the isotropy group at the marked point $p_i$ on the fiber of $L$.

Much more about orbifold line bundles and their tensor powers can be found, for example, in [2]. One crucial fact is the relationship between an orbifold line bundle $L$ and its pushforward $|L|$ under the map $\epsilon : C \to |C|$ to the coarse underlying curve. Suppose that $C_0$ is an irreducible component of $C$ with special points $x_1, \ldots, x_k$ at which $L$ has multiplicities $a_1, \ldots, a_k$. Then, if $L^{\otimes r} \cong \epsilon^* M$ for a line bundle $M$ on $|C|$, the coarse underlying line bundle $|L|$ satisfies

\[|L|^{\otimes r} \cong M \otimes \left(-\sum_{i=1}^{k} a_i[x_i]\right).\]
In the case of interest, \( M = \mathcal{O}_C \), and the above formula implies that if \( C \) has no nodes, then

\[
\text{deg}(|L|) = -\frac{1}{r} \sum_{i=1}^{n} a_i,
\]

and in general, an analogous formula holds on each component of the normalization of \( C \). There are two important consequences of (4). First, since \(|L|\) is a line bundle on a non-orbifold curve, it must have integral degree, so we recover the fact mentioned previously that \( \sum_{i=1}^{n} a_i \equiv 0 \mod r \). Second, because at least one \( a_i \) is nonzero, there is at least one component on which \( \text{deg}(|L|) < 0 \), which implies that \( H^0(L) = H^0(|L|) = 0 \).

Combining these observations, (3) gives the following expression for the virtual cycle of the moduli space of stable maps to \([C/Z_r]\) in terms of twisted theory over \( BZ_r \):

\[
[\mathcal{M}_{g,a}(C/Z_r,0)]^\text{vir} = [\mathcal{M}_{g,a}(BZ_r,0)] \sim e(-R^\bullet \pi_* \mathcal{L}).
\]

This expression will be central to the computations of the next two sections.

Remark 2.3.1. If \( a_1 = \cdots = a_n = 0 \), then one has \( \text{deg}(L) = 0 \) and the above argument that \( H^0(L) = 0 \) fails, so (4) is no longer valid in general. However, in this case, the moduli space \( \mathcal{M}_{g,0}(BZ_r,0) \) of \( r \)-torsion line bundles splits into components, on each of which \( L \) is either always trivial or always nontrivial. On nontrivial components, which exist as long as \( g > 0 \), an \( r \)-torsion line bundle can have no global sections. Thus, (3) still holds on these components of the moduli space.

3. Relations via twisted theory

In this section, we write (5) explicitly in terms of tautological classes on \( \mathcal{M}_{g,a}(BZ_r,0) \), and derive tautological relations from immediate vanishing properties of the virtual cycle.

3.1. Introducing Chern characters. Any multiplicative invertible characteristic class can be written in terms of Chern characters. In particular, as observed in [6], the \( \mathbb{C}^* \)-equivariant Euler class is a multiplicative invertible characteristic class, where \( \mathbb{C}^* \) acts trivially on the base and by multiplication on the fibers of a bundle. In terms of Chern characters, one obtains

\[
e^{\mathbb{C}^* ((-V)^\vee)} = \exp \left( \sum_{d=0}^{\infty} s_d \text{ch}_d([V]) \right)
\]
for

\[ s_d = \begin{cases} 
-\ln(\lambda) & d = 0 \\
\frac{(d-1)!}{\lambda^d} & d > 0,
\end{cases} \]

where \( \lambda \) is the equivariant parameter.

The rank of the vector bundle \(- R^* \pi_* \mathcal{L} \) can be computed using the
Riemann-Roch formula; it equals

\[ \text{rank}(- R^* \pi_* \mathcal{L}) = h^1(|L|) = -\deg(|L|) + g - 1 = \frac{1}{r} \sum_{i=1}^n a_i + g - 1 \]

for any element \((C, L)\) of \( \overline{M}_{g,a}(B\mathbb{Z}_r, 0) \). Combining this with \((6)\) yields

\[ e^{C^*}(- R^* \pi_* \mathcal{L}) = (-\lambda)^{\frac{1}{r} \sum a_i + g - 1} \exp \left( \sum_{d=1}^{\infty} \frac{(d-1)!}{\lambda^d} \text{ch}_d(R^* \pi_* \mathcal{L}) \right). \]

Taking a non-equivariant limit, \((5)\) shows that the virtual cycle of \( \overline{M}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0) \) can be expressed as

\[ \lim_{\lambda \to 0} \left( (-\lambda)^{\frac{1}{r} \sum a_i + g - 1} \exp \left( \sum_{d=1}^{\infty} \frac{(d-1)!}{\lambda^d} \text{ch}_d(R^* \pi_* \mathcal{L}) \right) \right), \]

suppressing the application of Poincaré duality.

The existence of this limit is a consequence of the fact that the total
Chern class, whose components can be expressed in terms of Chern
characters, vanishes in degrees past the rank of the bundle. On the
other hand, via the formula for the Chern characters in terms of tautological classes described in the next section, this vanishing implies tautological relations.

3.2. GRR for the universal \textit{rth root}. By applying the orbifold
Grothendieck-Riemann-Roch formula (see Appendix A of \cite{24})— or
the ordinary GRR formula to coarse underlying curves— the Chern
characters of \( R^* \pi_* \mathcal{L} \) can be expressed in terms of tautological classes.

In \cite{3}, Chiodo carries out this computation explicitly and substantially simplifies the result. Strictly speaking, his computation is carried out on a slightly different moduli space from \( \overline{M}_{g,a}(B\mathbb{Z}_r, 0) \), parameterizing \textit{rth} roots of \( \mathcal{O}(- \sum_i a_i[x_i]) \) on curves with balanced \( \mathbb{Z}_r \) orbifold structure only over the nodes. Because the representability assumption does not appear in his moduli space, it will only agree with \( \overline{M}_{g,a}(B\mathbb{Z}_r, 0) \) when all of the multiplicities at special points are coprime to \textit{r}. However, there is always a morphism between his moduli space \( \overline{M}_{g,n} \) and the moduli space \( \overline{M}_{g,a}(B\mathbb{Z}_r, 0) \) appearing here,
which is invertible on the interior and has explicitly-computable ramification indices on the boundary. Thus, we can state our formula on $\mathcal{M}_{g,n}$, but all of the classes in the formula will be pulled back from $\overline{\mathcal{M}}_{g,n}$.

Corollary 3.1.8 of [3] states, in the present situation, that

$$
\text{ch}_d(R^\bullet \pi_\ast \mathcal{L}) = \frac{B_{d+1}(0)}{(d+1)!} \kappa_d - \sum_{i=1}^n \frac{B_{d+1}(\frac{n\ast}{r})}{(d+1)!} \psi_i^d +
$$

$$
\frac{r}{2} \sum_{0 \leq l \leq g} \sum_{I \subset [n]} \frac{B_{d+1}(\frac{q(l,I)}{r})}{(d+1)!} \pi^\ast \gamma_{d-1}(\kappa_{d-1}) + \frac{r}{2} \sum_{q=0}^{g-1} \frac{B_{d+1}(\frac{q}{r})}{(d+1)!} \pi^\ast \gamma_{d-1}(\kappa_{d-1}).
$$

Let us review the notation appearing in this formula. First, $B_{d+1}(x)$ are the Bernoulli polynomials, defined by the generating function

$$
\frac{te^x}{e^x-1} = \sum_{n=0}^\infty B_n(x) \frac{t^n}{n!}.
$$

The kappa and psi classes are defined by pullback under the map

$$
\pi : \mathcal{M}_{g,n}(B\mathbb{Z}_r, 0) \to \mathcal{M}_{g,n}(r),
$$

where $\mathcal{M}_{g,n}(r)$ is the moduli space of $r$-stable, $n$-pointed, genus-$g$ curves. The notion of $r$-stability is defined precisely in Definition 2.1.1 of [3], but essentially, it refers to orbifold curves in which the marked points have trivial isotropy and the nodes are balanced with $\mathbb{Z}_r$ isotropy. On this moduli space, kappa and psi classes are defined as usual:

$$
\kappa_d = \pi_\ast (c_1(\omega_{\log})^d), \quad \psi_i = c_1(\omega|_{p_i}),
$$

where $\omega_{\log} = \omega_{C/X}(\sum_{i=1}^n [p_i])$.

Let $Z$ be the codimension-two locus inside the universal curve over $\mathcal{M}_{g,n}(r)$ consisting of nodes in singular fibers. There is a decomposition

$$
Z = \bigcup_{0 \leq l \leq g} Z_{(l,I)} \cup Z_{\text{irr}},
$$

where $Z_{(l,I)}$ consists of nodes separating the curve $C$ into a component of genus $l$ containing the marked points in $I$ and a component of genus $g-l$ containing the other marked points, while $Z_{\text{irr}}$ consists of non-separating nodes. Let $Z' \to Z$ be the two-fold cover given by nodes together with a choice of branch. The decomposition (8) induces an analogous decomposition of $Z'$, and the morphisms

$$
i_{(l,I)} : Z'_{(l,I)} \to \mathcal{M}_{g,n}(r)$$

are given by this two-fold cover, followed by the inclusion into the universal curve and the projection to $\overline{M}_{g,n}(r)$.

The index $q(l, I)$ is the multiplicity at the chosen branch of the node for any point in $Z'_{(l, I)}$. The equation (1) determines this multiplicity from $l$ and $I$; specifically, it is the unique number in $\{0, 1, \ldots, r - 1\}$ satisfying

$$q(l, I) + \sum_{i \in I} a_i \equiv 0 \pmod{r}.$$ 

If $\psi$ is the first Chern class of the line bundle over $Z'_{(l, I)}$ whose fiber is the cotangent line to the chosen branch of the node, and $\hat{\psi}$ is the first Chern class of the bundle whose fiber is the cotangent line to the opposite branch, then $\gamma_d$ is defined by

$$\gamma_d = \sum_{i + j = d} (-\psi)^i \hat{\psi}^j.$$ 

Finally, let $Z'_{(\text{irr}, q)}$ be the locus inside the universal curve over $\overline{M}_{g,n}$ consisting of irreducible nodes with a choice of branch such that the multiplicity of the line bundle $L$ at the chosen branch is equal to $q \in \{0, 1, \ldots, r - 1\}$. It should be noted that $q$ is not determined by the topology of the underlying curve alone, as the multiplicity was in the case of separating nodes, so we must view $Z'_{(\text{irr}, q)}$ as lying over $\overline{M}'_{g,n}$ rather than over $\overline{M}_{g,n}(r)$. We have morphisms

$$j_{(\text{irr}, q)} : Z'_{(\text{irr}, q)} \to \overline{M}'_{g,n}$$

given, as before, by the two-fold cover, inclusion into the universal curve, and projection. The class $\gamma_d$ is defined by exactly the same formula as previously. By pullback under the birational map discussed above, the class $j_{(\text{irr}, q)*}(\gamma_{d-1})$ is viewed as lying on $\overline{M}_{g,a}(B\mathbb{Z}_r, 0)$, and, by abuse of notation, it is still denoted $j_{(\text{irr}, q)*}(\gamma_{d-1})$.

### 3.3. Tautological relations.

Equipped with this formula, the proof of Theorem 1.0.1 is immediate from (7):

**Proof of Theorem 1.0.1.** The existence of the nonequivariant limit in (7) — or, equivalently, the vanishing of the total Chern class beyond the rank of the bundle — implies that any term with a negative power of $\lambda$ must vanish, which proves the theorem whenever not every $a_i$ is zero. The case $a_1 = \cdots = a_n = 0$ follows from the same argument after applying Remark 2.3.1. □

**Remark 3.3.1.** It is worth observing again that, while pushforward yields relations in $A^d(\overline{M}_{g,n})$, the relations in $A^d(\overline{M}_{g,a}(B\mathbb{Z}_r, 0))$ do not arise by pullback from relations in the moduli space of curves. The
reason for this is the appearance of the classes \( j_{(irr,q)}(\gamma_{d-1}) \), which are not pullbacks because the multiplicity at a nonseparating node is not determined by the topology of the coarse underlying curve.

One easy application of Theorem 1.0.1 is an expression for certain polynomials in the odd-codimension \( \kappa \) classes in terms of classes supported on the boundary, which yields some special cases of Getzler’s conjecture (Ionel’s theorem), the statement that \( R^d(M_{g,n}) = 0 \) for \( d \geq g \).

**Corollary 3.3.2.** For \( d \geq g \geq 2 \),

\[
\sum_{d_1 + \cdots + d_m = d} \frac{1}{m!} \prod_{i=1}^{m} \frac{B_{d_i+1}(0)}{d_i(d_i+1)} \kappa_{d_i} = 0 \in A^d(M_g),
\]

and the boundary classes that appear are given explicitly by Theorem 1.0.1.

**Proof.** Take \( n = 0 \) in Theorem 1.0.1 (The Bernoulli number vanishes when any \( d_i \) is even, so all the \( \kappa \) classes in the resulting polynomial will have odd codimension.) \( \square \)

To make the expression completely explicit—and, in general, to precisely determine the pushforward to \( \overline{M}_{g,n} \) of the relations appearing in Theorem 1.0.1—one must expand the product appearing in the theorem. This requires a more careful study of the intersection theory on \( \overline{M}_{g,a}(B\mathbb{Z}_r,0) \), and will be taken up in future work.

### 3.4. Examples

Let us compute the pushforward of Theorem 1.0.1 to the moduli space of curves explicitly in some easy examples.

**Example 3.4.1.** The simplest case of the formula in Theorem 1.0.1 occurs when \( g = 0 \), \( n = 4 \), and \( \mathbf{a} = (r-1,1,0,0) \). In this case, all of the tautological classes on \( \overline{M}_{0,4}(B\mathbb{Z}_r,0) \) are obtained by pullback from \( \overline{M}_{0,4} \). The degree of the pushforward \( \rho : \overline{M}_{0,(r-1,1,0,0)}(B\mathbb{Z}_r,0) \to \overline{M}_{0,4} \) is \( 1/r \), while its restriction to the boundary divisors has degree \( 1/r^2 \) due to the presence of additional “ghost” automorphisms in which the generator acts about the node as \( (x,y) \mapsto (x,e^{2\pi i/r}y) \) in the local picture of the node as \( \{xy = 0\}/\mathbb{Z}_r \).

Thus, one obtains the following relation in \( A^1(\overline{M}_{0,4}) \):

\[
0 = B_2(0)\kappa_1 - B_2(\frac{1}{r})\psi_1 - B_2(\frac{1}{r})\psi_2 - B_2(0)\psi_3 - B_2(0)\psi_4 + \sum_{\Gamma} B_2(\frac{2\pi}{r})[\Gamma].
\]

The sum is over dual graphs of the three boundary divisors, where we write \([\Gamma] = i_{r^*}(1)\), and \( q_{\Gamma} \) is the multiplicity of \( L \) at the node. Note that
$B_2(\mathcal{M})$ is independent of which branch of the node is chosen to define the multiplicity, which is why we have canceled the factor of one-half in front of the sum.

Since all of the classes appearing in this formula are equal to the class of a point, this relation is clearly valid.

**Example 3.4.2.** Let $g = 1$, $r = 2$, and $a = (1, 1)$. Then Theorem 1.0.1 implies the following relation in $A^2(\overline{M}_{1,(1,1)}(\mathbb{Z}_2, 0))$:

\[
0 = \left( \frac{1}{6} \kappa_1 + \frac{1}{12} \psi_1 + \frac{1}{12} \psi_2 + \frac{1}{3} [\Gamma] + \frac{1}{3} [\Gamma_{irr,0}] - \frac{1}{6} [\Gamma_{irr,1}] \right)^2.
\]

Here, we have again canceled the factor of one-half in front of the boundary divisors by setting $[\Gamma]$ to be the class of the unique boundary divisor with a separating node and $[\Gamma_{irr,q}]$ to be the class of the boundary divisor with a nonseparating node of multiplicity $q$, which does not depend on a choice of branch in this case.

Let

\[
\alpha = \frac{1}{6} \kappa_1 + \frac{1}{12} \psi_1 + \frac{1}{12} \psi_2 + \frac{1}{6} [\Gamma] \in A^1(\overline{M}_{1,2}),
\]

where we abuse notation slightly by using the same names for the corresponding classes in the moduli space of curves as for their analogues in $\overline{M}_{1,(1,1)}(\mathbb{Z}_2, 0)$. Let

\[
\beta = \frac{1}{3} [\Gamma_{irr,0}] - \frac{1}{6} [\Gamma_{irr,1}] \in A^1(\overline{M}_{1,(1,1)}(\mathbb{Z}_2, 0)).
\]

Then the pushforward of $[\beta]$ to $\overline{M}_{1,2}$ yields the relation

\[
0 = 2\alpha^2 + 2\alpha \rho_*(\beta) + \rho_*(\beta^2).
\]

One has $\rho_*([\Gamma_{irr,q}]) = \frac{1}{2}[\Gamma_{irr}]$, where $\Gamma_{irr}$ is the divisor on $\overline{M}_{1,2}$ corresponding to curves with a nonseparating node. To prove this claim, note that the degree-1 map

\[
\mu_{irr,q} : j_{irr,q}(Z'_{irr,q}) \to \overline{M}_{0,(1,1,q,q^{-1})}(\mathbb{Z}_2, 0)
\]

described in [5] descends to a map $\Gamma_{irr,q} \to [\overline{M}_{0,(1,1,q,q^{-1})}(\mathbb{Z}_2, 0)/\mathbb{Z}_2]$ that also has degree 1, where the involution on the target switches the third and fourth marked point. The latter maps with degree $1/2$ to $\Gamma_{irr}$ via the quotient of the forgetful map $\overline{M}_{0,4}(\mathbb{Z}_2, 0) \to \overline{M}_{0,4}$.

To evaluate $\beta^2$, one observes that

\[
[\Gamma_{irr,q}]^2 = \int_{[\overline{M}_{0,(1,1,q,q^{-1})}(\mathbb{Z}_2, 0)/\mathbb{Z}_2]} (-\psi_3 - \psi_4) = \frac{1}{4}.
\]
while
\[
\int_{[\overline{M}_{1,1}(\mathbb{P}^1,0)]} \left[ \Gamma_{\text{irr},0} \right] \left[ \Gamma_{\text{irr},1} \right] = \frac{1}{4}.
\]

The second equation follows from the fact that the intersection of the two divisors \( \Gamma_0 \) and \( \Gamma_1 \) consists of curves with two nonseparating nodes. There is only one such curve, but there are two ways to glue the line bundles on the components at the multiplicity-zero node, and each choice has isotropy of order 8.

Together, the previous two equations imply
\[
\rho_* (\beta^2) = -\frac{1}{16}.
\]

In all, then, the relation in \( \overline{M}_{g,n} \) reads
\[
0 = 2\alpha^2 + \frac{1}{6} \alpha \cdot [\Gamma_{\text{irr}}] - \frac{1}{16} [\text{point}]
\]
with \( \alpha \) as above.

4. Semisimplicity of the quantum cohomology

Another straightforward consequence of the expression for the virtual cycle of \( \overline{M}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0) \) in terms of tautological classes is a proof that the quantum cohomology of \([\mathbb{C}/\mathbb{Z}_r]\) is semisimple. We describe the proof in this section.

4.1. Quantum product. For each \( t \in H^*_{CR}([\mathbb{C}/\mathbb{Z}_r]; \mathbb{Q}) \), the quantum product \( \ast_t \) based at \( t \) is defined for \( \alpha, \beta \in H^*_{CR}([\mathbb{C}/\mathbb{Z}_r]; \mathbb{Q}) \) by
\[
\langle \alpha \ast_t \beta, \gamma \rangle = \sum_{k \geq 0} \langle \alpha \beta \gamma t \cdots t \rangle_{0,k+3,0}
\]
for any \( \gamma \) in the compactly-supported cohomology \( H^*_{CR,c}([\mathbb{C}/\mathbb{Z}_r]; \mathbb{Q}) \).

In particular, we compute the quantum product based at \( t = \zeta_1 \).

**Lemma 4.1.1.** The quantum product on \([\mathbb{C}/\mathbb{Z}_r]\) based at \( \zeta_1 \) is given by
\[
\zeta_a \ast_{\zeta_1} \zeta_b = \begin{cases} 
\zeta_{a+b} & a + b \leq r - 1 \\
-\frac{1}{r} \zeta_{a+b+1-r} & a + b \geq r.
\end{cases}
\]

**Proof.** Since \( \zeta_0 \) is the unit for the product, it suffices to prove the claim when \( a, b > 0 \). In this case, the product can be described by pairing with the usual (as opposed to the compactly-supported) cohomology, and we have
\[
\langle \zeta_a \ast_{\zeta_1} \zeta_b, \zeta_c \rangle = \sum_{k \geq 0} \langle \zeta_a \zeta_b \zeta_c \zeta_1 \cdots \zeta_1 \rangle_{0,k+3,0} \zeta_c,
\]
for any pairings of the \( \zeta_i \).
in which \( \zeta^c = r \zeta_{c-r} \) is the Poincaré dual of \( \zeta_c \) for \( c \in \{1, \ldots, r - 1\} \).

The virtual dimension of the stack \( \mc{M}_{g,(a,b,c,1,\ldots,1)}([\mathbb{C}/\mathbb{Z}_r],0) \) over which the integral in the \( k \)th term is computed is
\[
k + 1 - \frac{1}{r}(a + b + c + k),
\]
and since the integrand has degree zero, a nonvanishing contribution is only achieved when the above is zero. Given that \( k \) must be a nonzero integer, the only possibilities are:
\begin{enumerate}
  \item \( a + b + c - 1 = r - 1 \) and \( k = 0 \), which only occurs if \( a + b \leq r - 1 \);
  \item \( a + b + c - 1 = 2r - 2 \) and \( k = 1 \), which only occurs if \( a + b \geq r \).
\end{enumerate}
In the first case,
\[
\zeta_a * \zeta_1 \zeta_b = \langle \zeta_a \zeta_b \zeta_{r-a-b} \rangle_{0,3,0} \cdot r \zeta_{a+b} = \zeta_{a+b},
\]
using the fact that \( \mc{M}_{0,(a,b,r-a-b)}([\mathbb{C}/\mathbb{Z}_r],0) \cong B\mathbb{Z}_r \).

In the second case,
\[
(11) \quad \zeta_a * \zeta_1 \zeta_b = \langle \zeta_a \zeta_b \zeta_{2r-a-b-1} \zeta_1 \rangle_{0,4,0} \cdot r \zeta_{a+b+1-r}.
\]
This Gromov-Witten invariant equals
\[
- \int_{[\mc{M}_{0,(a,b,2r-a-b-1,1)}(B\mathbb{Z}_r,0)]} \text{ch}_1(R^\bullet \pi_* \mathcal{L}),
\]
using the formula (5) and the fact that the rank of \( R^\bullet \pi_* \mathcal{L} \) equals 1 in this case. In what follows, we will write \( \gamma = (a, b, 2r - a - b - 1, 1) \) for convenience.

The GRR computation in [3] gives
\[
\text{ch}_1(R^\bullet \pi_* \mathcal{L}) = \frac{B_2(0)}{2} \kappa_1 - \frac{B_2(a)}{2} \psi_1 - \frac{B_2(b)}{2} \psi_3 - \frac{B_2(2r-a-b-1)}{2} \psi_3 - \frac{B_2(1)}{2} \psi_4 + \sum_{\Gamma} \frac{rB_2(q\Gamma)}{2} \kappa_{r^\Gamma}(1),
\]
where the sum is over the three dual graphs \( \Gamma \) of boundary divisors and \( q_{\Gamma} \) is the multiplicity at the node for each such divisor. The Bernoulli polynomials are all given by the formula
\[
B_2 \left( \frac{k}{r} \right) = \frac{k^2}{r^2} - \frac{k}{r} + \frac{1}{6}.
\]
As for the tautological classes, we have
\[
\int_{[\mc{M}_{0,\gamma}(B\mathbb{Z}_r,0)]} \kappa_1 = \int_{[\mc{M}_{0,\gamma}(B\mathbb{Z}_r,0)]} \psi_i = \frac{1}{r},
\]
for all \( i \), since a general point of this moduli space has \( \mathbb{Z}_r \) isotropy given by fiberwise multiplication of the line bundle by an \( r \)th root of
unity. The boundary divisors have these automorphisms, as well as an additional \( \mathbb{Z}_r \) worth of ghost automorphisms. Thus,

\[
\int_{\overline{\mathcal{M}}_{0,\gamma}(B\mathbb{Z}_r,0)} i_{\Gamma^*}(1) = \frac{1}{r^2}
\]

for each \( \Gamma \).

It follows, after some simplification, that

\[
\langle \zeta_a \zeta_b \zeta_{2r-a-b-1} \zeta_1 \rangle_{0,4,0} = \int_{\overline{\mathcal{M}}_{0,\gamma}(B\mathbb{Z}_r,0)} \text{ch}_1(R^\bullet \pi_* \mathcal{L}) = -\frac{1}{r^2}.
\]

Combining this with (11) proves the claim. \( \square \)

**Remark 4.1.2.** In the case where \( r = 2 \), there is a somewhat simpler and more geometric proof of the four-point formula, which we mentioned to illuminate the connection to the Hodge bundle. In that case, the relevant invariant is \( \langle \zeta_1 \zeta_1 \zeta_1 \zeta_1 \rangle_{0,4,0} \) and elements of \( \overline{\mathcal{M}}_{0,(1,1,1,1)}(B\mathbb{Z}_2,0) \) can be identified with twofold covers of a genus zero curve ramified over four points. The domain of such a cover has genus one, so there is a map

\[
p : \overline{\mathcal{M}}_{0,(1,1,1,1)}(B\mathbb{Z}_2,0) \to \overline{\mathcal{M}}_{1,1}
\]

that sends a cover to its domain curve with the fourth branch point serving as a marked point.

The Hodge bundle \( p^* \mathbb{E} \) is acted on by \( \mathbb{Z}_2 \) and thus splits into sub-bundles \( \mathbb{E}_p \) on which \( \mathbb{Z}_2 \) acts by the character \( \rho \). The summand \( \mathbb{E}_1 \) corresponding to the trivial character is pulled back from the Hodge bundle on \( \overline{\mathcal{M}}_{0,4} \), which is trivial. Thus, if \( -1 \) denotes the nontrivial character, one has

\[
p^*(\mathbb{E}^\vee) = \mathbb{E}_1^\vee = R^1 \pi_*(\mathcal{L}),
\]

where the second equality follows from Serre duality. Since \( p \) has degree six (as one can verify via the \( j \)-invariant), it follows that

\[
\langle \zeta_1 \zeta_1 \zeta_1 \zeta_1 \rangle_{0,4,0} = \int_{\overline{\mathcal{M}}_{0,\gamma}(B\mathbb{Z}_2,0)} p^*(c_1(\mathbb{E}^\vee)) = 6 \int_{\overline{\mathcal{M}}_{1,1}} -\lambda_1 = -\frac{1}{4},
\]

in accordance with the Lemma.

Given the Lemma, semisimplicity of the quantum cohomology is straightforward.

**Proposition 4.1.3.** For generic \( t \in H^*_{CR}(\mathbb{C}/\mathbb{Z}_r; \mathbb{Q}) \), the quantum product \( *_t \) is semisimple.
Proof. Since semisimplicity is an open condition on \( t \), it suffices to prove semisimplicity of the quantum product at a single basepoint. By the Lemma,

\[
H^*_{CR}([C/\mathbb{Z}_r]; \mathbb{Q}), \ast_{\zeta_1} \cong \frac{\mathbb{C}[\zeta_1]}{(\zeta_1^r = -\frac{1}{r}\zeta_1)}.
\]

This algebra is semisimple, either by direct computation or using the fact that it is isomorphic to the Milnor ring of a polynomial with non-degenerate critical points. \( \square \)

4.2. Connection between semisimplicity and relations. In [23], Pandharipande-Pixton-Zvonkine derive tautological relations by studying a semisimple Cohomological Field Theory (CohFT)

\[
\Omega_{g,n} : V^{\otimes n} \to H^*(\overline{M}_{g,n}).
\]

Here, \( V = \mathbb{Q}\{e_0, e_1\} \) is a vector space and \( \Omega_{g,n}(e_{a_1}, \ldots, e_{a_n}) \) is a mixed-degree class supported in degree less than or equal to

\[
D_{g,n}(a_1, \ldots, a_n) := \frac{g - 1 + \sum_{i=1}^n a_i}{3}
\]

for any \( a_1, \ldots, a_n \in \{0, 1\} \). Semisimplicity, via the work of Givental and Teleman [12, 13, 25], implies that \( \Omega \) can be expressed in terms of an \( R \)-matrix action as

\[
\Omega = R. \omega
\]

for a topological field theory \( \omega \) determined solely by the Poincaré pairing and genus-zero, three-point invariants. By explicitly computing both \( R \) and \( \omega \), Pandharipande-Pixton-Zvonkine obtain an expression for \( \Omega \) as a graph sum involving tautological classes. The vanishing property thus implies relations in all degrees past \( D_{g,n}(a_1, \ldots, a_n) \).

Proposition 4.1.3 appears to imply that an analogous strategy can be applied to

\[
\overline{\Omega}_{g,n} : \nabla^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}),
\]

in which \( \nabla = \mathbb{Q}\{\zeta_0, \ldots, \zeta_{r-1}\} \) and

\[
\overline{\Omega}_{g,n}(\zeta_{a_1}, \ldots, \zeta_{a_n}) = \sum_{k \geq 0} \frac{1}{k!} \rho_{k*}(\overline{\mathcal{M}}_{g,(a_1,\ldots,a_n,1,\ldots,1)}([C/\mathbb{Z}_r], 0))^{vir}.
\]

Here, the \( k \)th summand has \( n + k \) markings, and the map

\[
\rho_k : \overline{\mathcal{M}}_{g,n+k}([C/\mathbb{Z}_r], 0) \to \overline{\mathcal{M}}_{g,n}
\]

forgets the orbifold structure, the map to \([C/\mathbb{Z}_r]\), and the last \( k \) marked points. Like \( \Omega \), this gives a mixed-degree class supported in degree less
than or equal to

$$\overline{D}_{g,n}(a_1, \ldots, a_n) := g - 1 + \frac{1}{r} \sum_{i=1}^{n} a_i.$$  

Unfortunately, $\overline{\Omega}_{g,n}$ does not, in fact, define a CohFT, because non-compactness makes the Poincaré pairing on $[\mathbb{C}/\mathbb{Z}_r]$ degenerate. The inverse of the pairing appears importantly in the definition of the $R$-matrix action, so this degeneracy is a serious obstacle.

One possible way to proceed would be to embed $[\mathbb{C}/\mathbb{Z}_r]$ into a compact orbifold, such as a projective line $X$ with one orbifold point. This would introduce positive-degree maps indexed by a Novikov variable $Q$ as well as a hyperplane class $H$. Taking a limit as $Q \to 0$ and $H \to 0$ would recover the original theory $\overline{\Omega}_{g,n}$. A naïve attempt to carry out this limit, however, is unsuccessful: the $R$-matrix for $X$ has negative powers of $Q$. The fact that the limit exists for $\overline{\Omega}$, then, means that cancellation must occur in the $R$-matrix action. This strategy might still be successfully carried out if an explicit expression for the $R$-matrix could be achieved that witnesses the cancellation, but so far no such expression has been computed. We hope to do so in future work.

If successful, this method would yield relations in exactly the same range of degrees as those produced in Theorem 1.0.1. It will be interesting to determine whether the same relations result from either approach.

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