A COMBINATORIAL METHOD FOR COMPUTING CHARACTERISTIC POLYNOMIALS OF STARLIKE HYPERGRAPHS

YAN-HONG BAO, YI-ZHENG FAN*, YI WANG AND MING ZHU

ABSTRACT. By using the Poisson formula for resultants and the variants of chip-firing game on graphs, we provide a combinatorial method for computing a class of of resultants, i.e. the characteristic polynomials of the adjacency tensors of starlike hypergraphs including hyperpaths and hyperstars, which are given recursively and explicitly.

1. INTRODUCTION

Here a tensor (or hypermatrix) refers to a multi-array of entries in some field, which can be viewed to be the coordinates of the classical tensor (as multilinear function) under an orthonormal basis. The eigenvalues of a tensor were introduced by Qi [12, 13] and Lim [9] independently. To find the eigenvalues of a tensor, Qi [12, 13] introduced the characteristic polynomial of a tensor, which is defined to be a resultant of a system of homogeneous polynomials. In general, there is no an explicit polynomial formula yet for resultants except some very special cases; and many fundamental questions about resultants still remain open.

As we know, there are mainly three tools to compute a concrete resultant. The first one is Koszul complex, whose terms is given by the graded tensor product of a polynomial algebra and an exterior algebra, and the differential is built from objective polynomials in the resultant. The resultant is exactly equal to a certain characteristic of the related Koszul complex. The second one is generalized trace, which is defined by Morozov and Shakirov [11]. Using the generalized traces and the Schur function, Hu et.al gave an expression of the characteristic polynomial of a tensor [7]. Shao, Qi and Hu gave a graph theoretic formula for the generalized trace [14]. As an application, Cooper and Dulte computed the characteristic polynomial of the adjacency tensor of a single edge hypergraph [3]. The third tool is Poisson formula, which may provide an inductively computing method, see [6, Chapter 13, Theorem 1.2] or [8, Proposition 2.7]. For example, Cooper and Dutle computed the spectrum of the “all ones” tensors using the Poisson formula, see [4, Theorem 3]. We refer to [6, Chapter 13] and [5, Chapter 3] for an overview of calculation of resultants.

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*The corresponding author.
Recently, spectral hypergraph theory is proposed to explore connections between the structure of a uniform hypergraph and the eigenvalues of some related symmetric tensors. Cooper and Dutle [3] proposed the concept of adjacency tensor for a uniform hypergraph. Shao et al. [15] proved that the adjacency tensor of a connected \( k \)-uniform hypergraph \( G \) has a symmetric H-spectrum if and only if \( k \) is even and \( G \) is odd-bipartite. This result gives a certification to check whether a connected even-uniform hypergraph is odd-bipartite or not.

The characteristic polynomial of a hypergraph is defined to be the characteristic polynomial of its adjacency tensor. In this paper, we mainly aim to give a lower dimension formula to compute the characteristic polynomial of hypergraphs based on Poisson formula and variants of chip-firing game, and give the characteristic polynomials of starlike hypergraphs including hyperstars and hyperpaths, recursively and explicitly.

For simplicity of notation, we denote \( [n] = \{1, 2, \cdots, n\} \) and \( [m, n] = \{m, m + 1, \cdots, n\} \) for integers \( m < n \).

2. Preliminaries

In this section, we mainly recall some basic notions and useful results on resultants and hypergraphs.

2.1. Resultants. Let \( F_1(x_1, \cdots, x_n), \cdots, F_n(x_1, \cdots, x_n) \) be \( n \) homogeneous polynomials over \( \mathbb{C} \) in variables \( x_1, \cdots, x_n \), and the degree of \( F_i \) is \( d_i > 0 \) for \( i \in [n] \). An important question is whether the system of equations

\[
\begin{align*}
F_1(x_1, \cdots, x_n) &= 0, \\
\vdots & \\
F_n(x_1, \cdots, x_n) &= 0
\end{align*}
\]

(2.1)

admits nontrivial solutions.

Generally, each \( F_i \) can be written as

\[
F_i = \sum_{|\alpha| = d_i} c_{i,\alpha} x^\alpha,
\]

where \( \alpha = (i_1, \cdots, i_n) \), \( |\alpha| = i_1 + \cdots + i_n \) and \( x^\alpha = x_1^{i_1} \cdots x_n^{i_n} \). Note that the number of \( \alpha \)'s with \( |\alpha| = d \) is \( \binom{n+d-1}{d-1} \).

For each possible pair of indices \( i, \alpha \), we introduce a variable \( u_{i,\alpha} \). Then, given a polynomial \( P \in \mathbb{C}[u_{i,\alpha} : |\alpha| = d, i \in [n]] \), we let \( P(F_1, \cdots, F_n) \) denote the value obtained by replacing each variable \( u_{i,\alpha} \) in \( P \) with the corresponding coefficient \( c_{i,\alpha} \).

**Theorem 2.1.** [5, Chapter 3, Theorem 2.3] For fixed positive degrees \( d_1, \cdots, d_n \), there exists a unique polynomial \( \text{Res} \in \mathbb{Z}[u_{i,\alpha}] \) satisfying the following properties:

(i) If \( F_1, \cdots, F_n \in \mathbb{C}[x_1, \cdots, x_n] \) are homogeneous of degrees \( d_1, \cdots, d_n \) respectively, the system (2.1) has a nontrivial solution if and only if \( \text{Res}(F_1, \cdots, F_n) = 0 \).

(ii) \( \text{Res}(x_1^{d_1}, \cdots, x_n^{d_n}) = 1 \).

(iii) \( \text{Res} \) is irreducible, even regarded as a polynomial in \( \mathbb{C}[u_{i,\alpha}] \).
Res\((F_1, \cdots, F_n)\) is called the resultant of \(F_1, \cdots, F_n\). Resultants have an important application in algebraic geometry, algebraic combinatorics and spectral hypergraph theory. However, it is very difficult to compute the resultant of general polynomials. Here, we only list some useful properties and calculation methods of resultants which will be used in this paper.

**Lemma 2.2.** \([3\text{ Lemma 3.2}]\) Let \(F_1, \cdots, F_n \in \mathbb{C}[x_1, \cdots, x_n]\) be homogeneous polynomials of degree \(d_1, \cdots, d_n\) respectively, and let \(G_1, \cdots, G_m \in \mathbb{C}[y_1, \cdots, y_m]\) be homogeneous polynomials of degree \(\delta_1, \cdots, \delta_m\) respectively. Then
\[
\text{Res}(F_1, \cdots, F_n, G_1, \cdots, G_m) = \text{Res}(F_1, \cdots, F_n) \prod_{j=1}^m \text{Res}(G_1, \cdots, G_m) \prod_{i=1}^n d_i.
\]

**Lemma 2.3.** \([5\text{ Chapter 3, Theorem 3.1}]\) For a fixed \(j \in [n]\),
\[
\text{Res}(F_1, \cdots, \lambda F_j, \cdots, F_n) = \lambda^{d_1 - d_{j-1} - d_j + \cdots + d_n} \text{Res}(F_1, \cdots, F_n),
\]
where \(d_i\) is the degree of \(F_i\) for each \(i \in [n]\).

Next, we recall the Poisson formula. Given homogeneous polynomials \(F_1, \cdots, F_n \in \mathbb{C}[x_1, \cdots, x_n]\) of degree \(d_1, \cdots, d_n\) respectively, let
\[
\begin{align*}
f_i(x_1, \cdots, x_{n-1}) &= F_i(x_1, \cdots, x_{n-1}, 1), \quad (1 \leq i \leq n) \quad (2.2) \\
F_i(x_1, \cdots, x_{n-1}) &= F_i(x_1, \cdots, x_{n-1}, 0), \quad (1 \leq i \leq n - 1) \quad (2.3)
\end{align*}
\]
Observe that \(F_1, \cdots, F_{n-1}\) are still homogeneous in \(\mathbb{C}[x_1, \cdots, x_{n-1}]\) of degree \(d_1, \cdots, d_{n-1}\) respectively, but \(f_1, \ldots, f_n\) are not homogeneous in general.

**Lemma 2.4 (Poisson formula).** Keep the above notation. If \(\text{Res}(F_1, \cdots, F_{n-1}) \neq 0\), then the quotient algebra \(A = \mathbb{C}[x_1, \cdots, x_{n-1}] / \langle f_1, \cdots, f_{n-1} \rangle\) has dimension \(d_1 \cdots d_{n-1}\) as a vector space over \(\mathbb{C}\), where \(\langle f_1, \cdots, f_{n-1} \rangle\) is the ideal of the polynomial algebra \(\mathbb{C}[x_1, \cdots, x_{n-1}]\) generated by \(f_1, \cdots, f_{n-1}\), and
\[
\text{Res}(F_1, \cdots, F_n) = \text{Res}(F_1, \cdots, F_{n-1})^d_n \det(m_{f_n} : A \to A)
\]
where \(m_{f_n} : A \to A\) is the multiplication map given by \(f_n\).

Here, the above form of Poisson formula follows from \([5\text{ Chapter 3, Theorem 3.4}]\), which is different from the original one in \([3]\).

2.2. Hypergraphs. A hypergraph \(H\) is a pair \((V, E)\), where \(V\) is the set of vertices, and \(E \subset \mathcal{P}(V)\) is the set of edges. A hypergraph \(H\) is called \(k\)-uniform for an integer \(k \geq 2\) if for each \(e \in E, |e| = k\). Clearly, a 2-uniform hypergraph is just a classical simple graph.

**Definition 2.5.** \([3\text{ Section 2.2}]\) Let \(H = (V, E)\) be a \(k\)-uniform hypergraph. The (normalized) adjacency tensor \(A(H) = (a_{i_1 \cdots i_k})_{i_1, \cdots, i_k \in V}\) is defined by
\[
a_{i_1 \cdots i_k} = \begin{cases} 
\frac{1}{(k-1)!}, & \text{if } \{i_1, \cdots, i_k\} \in E, \\
0, & \text{otherwise}.
\end{cases}
\]
For convenience, we use the following notation. Let \( V \) be a finite set and \( m \) a positive integer. For each \( \mathbf{e} = (i_1, \cdots, i_m) \in V^m \) and \( \mathbf{c} = (c_1, \cdots, c_m) \in \mathbb{N}^m \), we denote \( \mathbf{x}^{\mathbf{e}} = x^{i_1}_{c_1} \cdots x^{i_m}_{c_m} \). We also write \( \mathbf{x}^\mathbb{1}_C \) as \( \mathbf{x}_C \), where \( \mathbb{1} = (1, \cdots, 1) \in \mathbb{N}^m \). If \( V = [n] \), \( \mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{N}^n \), we write \( \mathbf{x}^{\mathbf{c}}_m \) as \( \mathbf{x}^\mathbf{c} \).

The eigenvalues of a tensor was introduced by Qi \cite{12, 13} and Lim \cite{9} independently. The adjacency tensor of a uniform hypergraph was introduced by Cooper and Dutle \cite{3}. Here we briefly give the definition of eigenvalues of uniform hypergraphs based on the above.

**Definition 2.6.** \cite{12, 3} Let \( H = (V, E) \) be a \( k \)-uniform hypergraph and \( A = (a_{i_1, \cdots, i_k}) \) be the adjacency tensor of \( H \). For some \( \lambda \in \mathbb{C} \), if there exists a nonzero vector \( \mathbf{x} \in \mathbb{C}^{|V|} \) such that for each \( j \in V \),

\[
\sum_{i_2, i_3, \cdots, i_k \in V} a_{j, i_2, i_3, \cdots, i_k} x_{i_2} x_{i_3} \cdots x_{i_k} = \lambda x^k_j,
\]

or equivalently, for each \( v \in V \),

\[
\sum_{e \in E} x_{e \setminus \{v\}} = \lambda x^k_v,
\]

then \( \lambda \) is called an *eigenvalue* of \( H \).

For each \( v \in V \), define

\[
F_v = \lambda x^k_v - \sum_{e \in E} x_{e \setminus \{v\}}.
\]

The polynomial

\[
\phi_H(\lambda) = \text{Res}(F_v; v \in V)
\]

in the indeterminant \( \lambda \) is called the *characteristic polynomial* of \( H \). Consequently, \( \lambda \) is an eigenvalue of \( H \) if and only if \( \phi_H(\lambda) = 0 \).

### 2.3. Dollar game on graph

Let \( G = (V, E) \) be a simple graph. Recall that a *configuration* \( \mathbf{c} \) on \( G \) means a function \( \mathbf{c} : V \to \mathbb{N} \), which can be understood there is a pile of \( \mathbf{c}(v) \) tokens (chips, or dollars) at each vertex \( v \). A dollar game on \( G \) starts from a configuration \( \mathbf{c} \). At each step of the game, a vertex \( v \) is *fired*, that is, dollars move from \( v \) to its adjacent vertices, one dollar going along each edge incident to \( v \). Fix a vertex \( w \) of \( G \), called the *bank vertex*. A vertex \( v \) other than \( w \) can be fired if and only if \( \mathbf{c}(v) \geq \deg(v) \), where \( \deg(v) \) is the degree of the vertex \( v \). The bank vertex \( w \) is allowed to go into debt such that \( w \) can be fired if and only if no other firing is possible.

Suppose that \( X \) is a non-empty finite sequence of (not necessarily distinct) vertices of \( G \), such that starting from a configuration \( \mathbf{c} \), the vertices can be fired in the order of \( X \). If \( v \) occurs \( x(v) \) times, we shall refer to \( x \) as the representative vector for \( X \). The configuration \( \mathbf{c}' \) after the sequence of firing \( X \) is given by

\[
\mathbf{c}' = \mathbf{c} - Lx
\]

where \( L \) is the Laplacian matrix of \( G \).

The dollar game on graph was introduced by Biggs \cite{1}, which is a variant of chip-firing game, and is often described in terms of “snowfall” and “avalanches” in the literature. A configuration \( \mathbf{c} \) is said to be *stable* if \( 0 \leq \mathbf{c}(v) < \deg(v) \) for any
A sequence of firing is \( w \)-legal if and only if each occurrence of a vertex \( v \neq w \) follows a configuration \( t \) with \( t(v) \geq \deg(v) \) and each occurrence of \( w \) follows a stable configuration. A configuration \( c \) on \( G \) is said recurrent if there is a \( w \)-legal sequence for \( c \) which leads to the same configuration. A critical configuration \( c \) means that \( c \) is both stable and recurrent. We refer to [1] for more details.

Lemma 2.7. [1, Theorem 6.2] If \( G \) is a connected graph, then the number of critical configurations is equal to the number of spanning trees of \( G \).

Example 2.8. Let \( K_k \) be a completed graph on \( k \) vertices. Then the number of critical configurations is \( k^k - 2 \).

3. Poisson formula for characteristic polynomials of hypergraphs

3.1. Poisson formula for hypergraphs. Let \( H = (V, E) \) be a \( k \)-uniform hypergraph. Recall that the characteristic polynomial of \( H \) is defined as

\[
\phi_H(\lambda) = \text{Res}(F_v : v \in V),
\]

where

\[
F_v = \lambda x_v^{k-1} - \sum_{e \in E} x_{e \setminus \{v\}} \in \mathbb{C}[x_v : v \in V].
\]

In order to use Poisson formula for the resultant \( \text{Res}(F_v : v \in V) \), we need fix a vertex \( w \) in \( V \). Denote by \( E_w \) the set of all edges containing the vertex \( w \) and \( e_w = e \setminus \{w\} \) for each \( e \in E_w \). Then we have

\[
\begin{align*}
\bar{F}_v &= \lambda x_v^{k-1} - \sum_{e \in E \setminus E_w} x_{e \setminus \{v\}} - \sum_{e \in E_w} x_{e \setminus \{v\}}, \\
\bar{F}_w &= \lambda x_w^{k-1} - \sum_{v \in V \setminus \{w\}} x_{e \setminus \{v\}},
\end{align*}
\]

(3.1)

Deleting the vertex \( w \) in \( V \) and the edges in \( E_w \), one can obtain a sub-hypergraph \( \hat{H} = (\hat{V}, \hat{E}) \). To be precise, \( \hat{V} = V \setminus \{w\} \) and \( \hat{E} = E \setminus E_w \).

Lemma 3.1. Retain the above notation. Then

\[
\phi_H(\lambda) = \phi_{\hat{H}}(\lambda)^{k-1} \det(m_{f_w} : A \to A),
\]

where \( A \) is the quotient algebra \( \mathbb{C}[x_v : v \in \hat{V}] / \langle f_v : v \in V \rangle \) and \( m_{f_w} \) is the multiplication map of \( A \) given by \( f_w \).

Proof. By Lemma 2.4, the characteristic polynomial of \( H \) is

\[
\phi_H(\lambda) = \text{Res}(\bar{F}_v : v \in \hat{V})^{k-1} \det(m_{f_w} : A \to A).
\]

Considering the subhypergraph \( \hat{H} \) of \( H \), by Equation (3.1), we have

\[
\phi_{\hat{H}}(\lambda) = \text{Res}(\bar{F}_v : v \in \hat{V}).
\]

The result follows. \( \square \)
By definition, the algebra $A$ is $(k-1)^{r-1}$-dimensional as a vector space over $\mathbb{C}$ where $r$ is the number of vertices of $H$. In general, it is difficult to compute the determinant $\det(m_{f_w}: A \to A)$. However, we can give some description for some special cases.

### 3.2. Hypergraphs with a cut vertex.

Let $H = (V, E)$ be a $k$-uniform connected hypergraph and $w \in V$. Denote by $E_w = \{e \mid e \in E_w\}$. Deleting the vertex $w$, we can get a (non-uniform) hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$, with $\tilde{V} = V \setminus \{w\}$ and $\tilde{E} = (E \setminus E_w) \cup E_w$. Recall the vertex $w$ is called a **cut vertex** if $\tilde{H}$ is not connected; see Fig. 1. Suppose that $w$ is a cut vertex and $\tilde{H}_1 = (\tilde{V}_1, \tilde{E}_1), \ldots, \tilde{H}_n = (\tilde{V}_n, \tilde{E}_n)$ ($n \geq 2$) are the connected components of $\tilde{H}$. For each $i \in [n]$, we set $V_i = \tilde{V}_i$, $E_i = \tilde{E}_i \setminus E_w$, and then obtain a subhypergraph $H_i = (V_i, E_i)$ of $H$. Note that each $H_i$ is a $k$-uniform hypergraph and may not be connected.

**Figure 1.** A $k$-uniform hypergraph $H$ with a cut vertex $w$

For each $i \in [n]$, we denote

$$E_w^i = \{e \in E_w \mid e \cap V_i \neq \emptyset\}.$$

By definition, we have

$$\phi_H(\lambda) = \text{Res}(F_v : v \in V),$$

where

$$F_{v_i} = \lambda x_{v_i}^{k-1} - \sum_{e \in E_i \setminus \{v_i\}} x_e \cdot \sum_{e \in E_i \setminus \{v_i\}} x_e, \quad v_i \in V_i, i \in [n],$$

$$F_w = \lambda x_w^{k-1} - \sum_{i=1}^n \sum_{e \in E_w^i} x_e.$$ 

Therefore,

$$f_w = \lambda - \sum_{i=1}^n \sum_{e \in E_w^i} x_e,$$

$$f_{v_i} = \lambda x_{v_i}^{k-1} - \sum_{e \in E_i \setminus \{v_i\}} x_e \cdot \sum_{e \in E_i \setminus \{v_i\}} x_e, \quad v_i \in V_i, i \in [n],$$

$$\bar{F}_{v_i} = \lambda x_{v_i}^{k-1} - \sum_{e \in E_i} x_e, \quad v_i \in V_i, i \in [n].$$

Let $A$ be the quotient algebra $\mathbb{C}[x_v : v \in \tilde{V}] / \langle f_v : v \in \tilde{V} \rangle$, and $m_{f_w}: A \to A$ is the linear map given by multiplication by $f_w$. Since $V_1, \ldots, V_n$ form a partition of $\tilde{V}$ and for each
v_i \in V_i, f_{v_i} \in \mathbb{C}[x_v: v \in V_i], we have A = A_1 \otimes \cdots \otimes A_n, where

\[ A_i = \mathbb{C}[x_v: v \in V_i], \quad i \in [n]. \]

We define and denote \( m_{i,v}: A_i \to A_i \) the linear map given by the multiplication by \( x_{e_{i,v}} \) for each \( i \in [n] \). Then

\[ m_{i,v} = \lambda \mathbb{I}_d - \sum_{j=1}^n \mathbb{I}_d A_j \otimes \cdots \otimes \mathbb{I}_d A_{i-1} \otimes m_{i,v} \otimes \mathbb{I}_d A_{i+1} \otimes \cdots \otimes \mathbb{I}_d A_n, \]

where \( \mathbb{I}_d \) denotes the identity map on certain vector space.

**Assumption 1.** For each \( i \in [n] \), there exists an ordered \( \mathbb{C} \)-basis \( x^{\alpha_{i,1}}, \ldots, x^{\alpha_{i,i'}} \) for \( A_i \) such that the matrix of \( m_{i,v} \) with respect to this basis is a lower triangular matrix with the diagonal entry \( \alpha_{i,j_i} \), \( j_i \in [d_i] \), where \( d_i = (k-1)^{r_i} \) and \( r_i = |V_i| \).

Under the Assumption 1, \( \{x^{\alpha_{i,1}}, \ldots, x^{\alpha_{i,i'}} \mid j_i \in [d_i], i \in [n] \} \) with the lexicographic order is a basis for \( A \) such that the matrix of \( m_{i,v} \) with respect to this basis is still a lower triangular matrix with diagonal entries \( \lambda - \sum_{j=1}^n \alpha_{i,j_i} \) for \( j_i \in [d_i] \) and \( i \in [n] \). In this situation, we have

\[ \det(m_{i,v}: A \to A) = \prod_{1 \leq i \leq j_i \leq \ell} (\lambda - \sum_{i=1}^n \alpha_{i,j_i}). \]  

**Corollary 3.2.** Let \( H \) be a \( k \)-uniform hypergraph with a cut vertex \( v \). Then, under the Assumption 1,

\[ \phi_H(\lambda) = \prod_{i=1}^n \phi_{H_i}(\lambda)^{(k-1)^{r_i} j_i} \prod_{1 \leq j_i \leq \ell_i} (\lambda - \sum_{i=1}^n \alpha_{i,j_i}). \]

**Proof.** By definition, the characteristic polynomial of \( H_i = (V_i, E_i) \) is \( \phi_{H_i}(\lambda) = \text{Res}(\bar{F}_{v_i}: v_i \in V_i) \).

By Lemma 2.2, we have

\[ \text{Res}(\bar{F}_{v}: v \in \bar{V}) = \prod_{i=1}^n \text{Res}(\bar{F}_{v_i}: v_i \in V_i)^{(k-1)^{r_i} j_i} = \prod_{i=1}^n \phi_{H_i}(\lambda)^{(k-1)^{r_i} j_i} \]

From Theorem 3.1 it follows that

\[ \phi_H(\lambda) = \text{Res}(\bar{F}_{v}: v \in \bar{V})^{k-1} \det(m_{i,v}: A \to A) = \prod_{i=1}^n \phi_{H_i}(\lambda)^{(k-1)^{r_i} j_i} \prod_{1 \leq j_i \leq \ell_i} (\lambda - \sum_{i=1}^n \alpha_{i,j_i}). \]

\[ \square \]

### 3.3. Hypergraphs with a cored vertex

Let \( H = (V, E) \) be a \( k \)-uniform hypergraph. Recall that a vertex \( w \in V \) is called a cored vertex if it is contained in only one edge; see Fig. [2] Deleting the cored vertex \( w \) and the edge \( e_w \) containing \( w \),
one can obtain a sub-hypergraph $\hat{H} = (\hat{V}, \hat{E})$ with $\hat{V} = V \setminus \{w\}$ and $\hat{E} = E \setminus \{e_w\}$.

Then

$$F_w = \lambda x_w^{k-1} - x_{e_w},$$

$$F_v = \lambda x_v^{k-1} - \sum_{e \in \hat{E}} x_{e \setminus \{v\}} - \sum_{v \in e_w} x_{e_w \setminus \{v\}}, \; v \neq w.$$ 

Moreover,

$$f_w = \lambda x_{e_w},$$

$$f_v = \lambda x_v^{k-1} - \sum_{e \in \hat{E}} x_{e \setminus \{v\}} - \sum_{v \in e_w} x_{e_w \setminus \{v\}}, \; v \neq w,$$

$$\bar{F}_v = \lambda x_v^{k-1} - \sum_{e \in \hat{E}} x_{e \setminus \{v\}}, \; v \neq w.$$ 

**Corollary 3.3.** Let $H$ be a $k$-hypergraph with a cored vertex $w$. Retain the above notation. Then

$$\phi_H(\lambda) = \phi_{\hat{H}}(\lambda)^{k-1} \det(m_{f_w} : A \to A),$$

where $A$ is the quotient algebra $\mathbb{C}[x_v : v \in \hat{V}] / \langle f_v : v \in \hat{V} \rangle$ and $m_{f_w}$ is the multiplication map of $A$ given by $f_w$.

4. **Hyperpaths**

Let $P^k_n$ be a $k$-uniform hyperpath with $n$ edges or of length $n$, which has vertices labelled as $0, 1, \cdots, r = n(k-1)$ from left to right as in Fig. 3 and edges $e_t = \{(k-1)(t-1), (k-1)(t-1) + 1, \cdots, (k-1)t\}$ for $t \in [n]$.

![Figure 3. A $k$-uniform hyperpath $P^k_n$ with $n$ edges](image)

In this section, we will give a recursive formula for the characteristic polynomials of hyperpaths. By the Poisson formula introduced in Theorem 3.1, it suffices to compute the related determinant. For this, we need introduce the following model of dollar game on hypergraphs.
4.1. Dollar game on hypergraph and firing graph. We now define a dollar game on a hypergraph, considered as a variant of dollar game on a graph. Let \( H = (V, E) \) be a \( k \)-uniform hypergraph with a specified bank vertex \( w \). A function \( c : V \rightarrow \mathbb{N} \) is called a configuration on \( H \). A dollar game starts from a configuration \( c \). At each of step of the game, a vertex \( v \) is fired, that is, the vertex \( v \) decreases \( k - 1 \) dollars, and each of its adjacent vertices increases 1 dollar. A vertex \( v \) other than \( w \) can be fired if and only if \( c(v) \geq k - 1 \), and the bank vertex \( w \) is allowed to go into debt such that \( w \) can be fired if and only if no other firing is possible. We say that a configuration \( c \) is stable if \( 0 \leq c(v) < k - 1 \) for any \( v \neq w \). The notions of \( w \)-legal sequence of firing, recurrent or critical configurations are same as those defined in Section 2.3. The above setting of dollar game on hypergraphs is different from that of dollar game on simple graphs in Section 2.3, but will be useful for our discussion.

Let \( \leq \) be a total ordering on the set \( \bar{V} = V \setminus \{w\} \). Let \( c \) be a configuration on \( H \). The weight of \( c \) is defined and denoted to be \( \omega(c) = \sum_{v \in \bar{V}} c(v) \). We define the left anti-lexicographical order \( \prec \) for all of configurations on \( H \). To be precise, for any configurations \( c \) and \( c' \), \( c \prec c' \) if and only if either \( \omega(c) < \omega(c') \), or \( \omega(c) = \omega(c') \), \( c(i) = c'(i) \) for any \( 1 \leq i \leq t - 1 \) and \( c(t) > c'(t) \) for some \( t \).

Suppose further that the bank vertex \( w \) is also a cored vertex of \( H \). Based on the above discussion, we now define a directed graph, called a firing graph \( G(c_0) \) associated with a stable configuration \( c_0 \) on the hypergraph \( H \), which is closely related the dollar game on \( H \) starting from \( c_0 \). Here, a vertex of \( G(c_0) \) is a configuration and a directed edge will be called an arrow.

Step 1. Initially, set \( V_0 = \{c_0\} \) and \( E_0 = \emptyset \).

Step 2. Let \( c_0 \) be a configuration given by

\[
    c_0(v) = \begin{cases} 
    c_0(v) + 1, & \text{if } v \in e_w \setminus \{w\}, \\
    c_0(v), & \text{otherwise},
    \end{cases}
\]

where \( e_w \) is the unique edge of \( H \) containing \( w \). Put \( V_1 = V_0 \cup \{c_0\} \) and \( E_1 = \{c_0 \stackrel{\omega_{w,w} \rightarrow}{\longrightarrow} c_0\} \), where the arrow in \( E_1 \) means that the configuration \( c_0 \) is obtained from \( c_0 \) by firing the bank vertex \( w \) on the edge \( e_w \).

Step 3. If all configuration \( V_1 \setminus V_{i-1} \) are stable, then we define \( V_i(G(c_0)) = V_i \) and \( E_i(G(c_0)) = E_i \). Otherwise, for each non-stable configuration \( c \in V_i \setminus V_{i-1} \), we choose the vertex \( u_c := \max\{v \in \bar{V} \mid c(v) \geq k - 1\} \). For each edge \( e \in E \) containing \( u_c \), we define an arrow \( c \xrightarrow{\omega_{u_c,e}} c_e \), where the configuration \( c_e \) is given by

\[
    c_e(v) = \begin{cases} 
    c(v) - (k - 1), & \text{if } v = u_c, \\
    c(v) + 1, & \text{if } v \in e \setminus \{u_c\}, \\
    c(v), & \text{otherwise},
    \end{cases}
\]

and

\[
    V_{i+1} = V_i \cup \{c_e \mid u_c \in e \in E, c \text{ is not stable in } V_i \setminus V_{i-1}\},
\]

\[
    E_{i+1} = E_i \cup \{c \xrightarrow{\omega_{u_c,e}} c_e \mid u_c \in e \in E, c \text{ is not stable in } V_i \setminus V_{i-1}\}.
\]

Note that \( c_e \) may have been in \( V_i \). If \( c_e \notin V_i \), we say that \( c_e \) is obtained from \( c \) by firing \( u_c \) on \( e \).
The Step 3 tells us the firing rule, that is, which vertex will be fired at next step among all non-stable vertices other than the bank vertex. In addition, from the construction of $\mathcal{G}(c_0)$, we have $0 \leq \omega(c) \leq \omega(c_0) + k - 1$ for any $c \in \mathcal{V}(\mathcal{G}(c_0))$, which forces that $\mathcal{G}(c_0)$ is a finite directed graph.

**Example 4.1.** Let $P^3_3$ be the 3-uniform hyperpath with 3 edges as in Fig. 3 by taking $n = 3$ and $k = 3$. Let 0 be the bank vertex of $P^3_3$, and let $c_0 = (*, 1, 1, 1, 0, 0)$. Here, the value of the bank vertex 0 is omitted in each configuration of the dollar game starting from $c_0$. The firing graph $\mathcal{G}(c_0)$ is drawn in Fig. 4, where each vertex within a circle means that it will be fired at the next step.

![Figure 4. The firing graph of $P^3_3$ associated with $c_0$](image)

Generally speaking, it is difficult to obtain the firing graphs $\mathcal{G}(c_0)$ for all stable configurations $c_0$ on a general hypergraph. However, for some special classes of hypergraphs, e.g. hyperpaths, we can characterize the structure of $\mathcal{G}(c_0)$.

**Example 4.2.** Let $P^k_1$ be the $k$-uniform hyperpath on vertices $0, 1, \cdots, k - 1$ with a single edge, and let 0 be the bank vertex. To determine the firing graph associated to a given configuration $c_0$, it is equivalent to consider the dollar game on the completed graph $K_i$ with the bank vertex 0. To be precise, a configuration $c_0$ is a critical stable configuration if and only if the firing graph $\mathcal{G}(c_0)$ is a directed cycle of length $k$, and $c_0$ is a non-critical stable configuration if and only if $\mathcal{G}(c_0)$ is a directed path of length less than $k - 1$.

### 4.2 Firing graphs of hyperpaths.

In this part, we characterize the structure of the firing graph of $P^k_n$ in Fig. 3 associated to arbitrary fixed stable configuration, where the vertex 0 is the bank vertex. For a configuration $c$ on $[0, n(k - 1)]$, we denote $c = (c(0), c^1, \cdots, c^n)$, where $c^i$ is the restriction of $c$ on $\hat{e}_i := e_i \setminus \{(i - 1)(k - 1)\}$, i.e.

$$c^i = (c((i - 1)(k - 1) + 1), \cdots, c(i(k - 1))), i \in [n].$$

Let $\hat{c}^i$ be the restriction of $c$ on $e_i$, i.e.

$$\hat{c}^i = (c((i - 1)(k - 1)), c((i - 1)(k - 1) + 1), \cdots, c(i(k - 1))), i \in [n].$$

Then for each $i \in [n]$, $\hat{c}^i$ can be considered as a stable configurations on the completed graph $K_k$ with vertex set $e_i$ and bank vertex $(i - 1)(k - 1)$.

Let $S_k$ be the set of all stable configurations on the completed graph $K_k$, and $C_k$ be the set of all critical stable configurations on $K_k$. Denote by $B$ the set of all
stable configurations on $P_n^k$, and for $s \in [0, n-1]$

\[ B_s = \{ (e^1, \ldots, e^n) \mid \tilde{c}_i \in C_s \text{ for } i \in [s], \tilde{c}_{i+1} \notin C_i, \text{ and } \tilde{c}_i \in S_s \text{ for } i \in [s+2, n] \}, \]

and $B_n = \{ (e^1, \ldots, e^n) \mid \tilde{c}_i \in C_n, i \in [n] \}$. Clearly, $B$ is the disjoint union of $B_0, \ldots, B_n$.

**Lemma 4.3.** Let $P_n^k$ be the $k$-uniform hyperpath as in Fig. 3. Suppose that $c_0$ is a stable configuration on $P_n^k$ with 0 as the bank vertex, and $G(c_0)$ is the firing graph of $P_n^k$ associated to $c_0$. Then

(i) for any $t \in [n]$, the vertex $t(k-1)$ is fired at $c \in V(G(c_0))$ if and only if $c(t(k-1)) = k-1$.

(ii) for any configuration $c \in V(G(c_0))$, $\omega(c) \geq \omega(c_0)$, where the equality holds if and only if the vertices $1, \ldots, k-1$ have been fired on $c_1$ exactly once along any directed path from $c_0$ to $c$.

**Proof.** (i) By the firing rule defined in Section 4.1, it suffices to show that if $c(t(k-1)) = k-1$, then $c(v) < k-1$ for any $v > t(k-1)$. According to the construction, for each $c \in V(G(c_0))$, there exists a directed path $P(c_0, c)$ from $c_0$ to $c$ (without repeated vertices on the path).

Clearly, if each configuration $c'$ in $P(c_0, c)$ except $c$ satisfies $c'(t(k-1)) < k-1$, then $c(v) = c_0(v) < k-1$ for any $v > t(k-1)$ since each vertex $v > t(k-1)$ is first fired only after $t(k-1)$ is fired. Otherwise, $t(k-1)$ is fired at some configuration say $c'_{-1}$ before $c$ along the path $P(c_0, c)$.

Assume to the contrary that $c(t(k-1)) = k-1$ and $c(u) \geq k-1$ for some $u > t(k-1)$. Consider the directed path $P(c_0, c)$:

\[
\begin{array}{cccccccc}
& c_0 & \rightarrow & \cdots & c_{-1}^{\alpha_{(k-1),1}} & c_0^{\alpha_{0,1}} & c_1^{\alpha_{1,1}} & \cdots & c_m^{\alpha_{m,1}} & \rightarrow c \\
\end{array}
\]

Furthermore, we can assume that along the path $P(c_0, c)$, $c$ first appears with $c(t(k-1)) = k-1$ and $c(u) \geq k-1$ for some $u > t(k-1)$, and $t(k-1)$ is last fired (at $c'_{-1}$). So, $c'_i(t(k-1)) < k-1$ for $i \in [0, m]$. We also have $c'_{-1}(t(k-1)) = k-1$ and $c'_v < k-1$ for any $v > t(k-1)$ by the firing rule.

If the arrow $c'_{-1} \xrightarrow{\alpha_{(k-1),1}} c_0$ satisfies $\alpha_{(k-1),1} = \alpha_{(k-1),1}$, then $u_i < t(k-1)$ for $i = 0, 1, \ldots, m$, as $t(k-1)$ is no longer fired so that any vertex $v > t(k-1)$ keep value invariant given by the configurations form $c'_{-1}$ to $c$. It follows that

\[ c(v) = c'_i(v) = c'_{-1}(v) < k-1 \]

for any $v > t(k-1)$ and $i \in [0, m]$, a contradiction.

If the arrow $c'_{-1} \xrightarrow{\alpha_{(k-1),1}} c_0$ satisfies $\alpha_{(k-1),1} = \alpha_{(k-1),1}$, then

\[
c'_0(v) = \begin{cases} 
0, & \text{if } v = t(k-1) \\
\alpha_{-1}(v) + 1, & \text{if } v \in [t(k-1) + 1, (t+1)(k-1)], \\
\alpha_{-1}(v), & \text{if } v \geq (t+1)(k-1) + 1
\end{cases}
\]

It follows that the number of configurations in $\{ c'_i \mid \alpha_{u_i} = \alpha_{u_i, i+1}, i = 0, 1, \ldots, m \}$ is $k-1$ as $c(t(k-1)) = k-1$. Observe that for any $v \in [t(k-1)+1, (t+1)(k-1) - 1]$

\[ c'_i(v) \leq c'_0(v) + (k-2) = c'_{-1}(v) + (k-1) < 2k-2. \]
It forces that each vertex in \([t(k-1)+1, (t+1)(k-1)-1]\) is fired exactly once on the edge \(e_{t+1}\) from \(c_0'\) and \(c\), implying that the vertex \((t+1)(k-1)\) is also fired exactly once on \(e_t+1\). Let \(u_i\) be the vertex in \([t(k-1)+1, (t+1)(k-1)]\) last fired on \(e_t+1\). Then \(c_{t+1}((t(k-1)) = k-1\), and \(c_{t+1}(v) < k-1\) for any \(v > t(k-1)\) by the assumption on \(u_j\) and firing rule. As \(t(k-1)\) is last fired at \(c'_{t-1}\), \(c_{t+1} = c\), a contradiction.

(ii) By definition, we have \(\omega(e_0) = \omega(c_0) + k - 1\). Observe that for any arrow \(c \xrightarrow{\alpha_{u_0, e_1}} c'\), \(\omega(c') = \omega(c) - 1\) if \(e_t = e_{1}\), and \(\omega(c') = \omega(c)\) if \(e_t \neq e_1\). Therefore, we need only show that no vertex is fired on \(e_1\) more than one time in any directed path starting from \(c_0\). Let

\[
\begin{align*}
c_0 \xrightarrow{\alpha_{0, e_1}} c_1 \rightarrow & \cdots \rightarrow c_\ell \xrightarrow{\alpha_{u_\ell, e_1}} \cdots \rightarrow c_t \xrightarrow{\alpha_{u_t, e_1}} \cdots
\end{align*}
\]

be arbitrary directed path in \(G(c_0)\). If there exists a vertex in \([k-1]\) be fired more than one time on \(e_1\), we can assume that \(u_t\) is the first occurrence of the vertex satisfying \(\alpha_{u_t, e_1} = \alpha_{u_\ell, e_1}\) for some \(\ell < t\). If \(1 \leq u_\ell = u_t \leq k - 2\), by our assumption, we have

\[
c_t(u_t) \leq c_0(u_t) - (k - 1) + (k - 2) = c_0(u_t) < k - 1,
\]

since there are at most \(k - 2\) vertices different from \(u_t\) in \(e_1 \setminus \{0\}\), a contradiction. If \(u_\ell = u_t = k - 1\), we have \(c_v(v) < k - 1\) for any \(v > k - 1\) by the firing rule, \(u_i \in [k - 2]\) for each \(i \in [\ell + 1, t - 1]\) and each of them is fired only one time by our assumption on \(u_t\). By the part (i), we have \(c_t(k - 1) = k - 1\). Therefore,

\[
c_t(k - 1) \leq c_t(k - 1) - (k - 1) + (k - 2) = k - 2.
\]

This is also contradiction.

Next, we describe the structure of the firing graph of \(P_n^k\) associated to any stable configuration.

**Proposition 4.4.** Let \(P_n^k\) be the \(k\)-uniform hyperpath as in Fig. 3 Then, for a given stable configuration \(c_0 \in B_s\) on \(P_n^k\) with \(0\) as the bank vertex, where \(s \in [n]\), the firing graph \(G(c_0)\) of \(P_n^k\) associated to \(c_0\) has the structure as in Fig. 5, where \(G'\) is the subgraph induced by the configurations obtained by first firing the vertex \(s(k-1)\) on the edge \(e_{s+1}\), each directed cycle in \(G(c_0)\setminus G'\) has length \(k\), such that

(i) \(G'\) is not empty if and only if \(s < n\);
(ii) if \(s < n\), there does not exist arrows from \(G'\) to \(G(c_0)\setminus G'\);
(iii) if \(s < n\), \(c_0 \prec c'\) for any \(c' \in G'\).

![Figure 5](image-url) The firing graph of \(P_n^k\) associated with \(c_0 \in B_s\)
Proof. (i) Let $c_0 \in B_s$ be a fixed stable configuration on $P_n^k$, where $s \in [n]$. Clearly, $\tilde{c}_0$ is a stable configuration on $K_n$ for each $i \in [n]$, where $\tilde{c}_0$ is defined at the beginning of this section. By definition, $c_0 \in B_s$ means that $\tilde{c}_0 \in C_k$ for $1 \leq i \leq s$ and $\tilde{c}_0 + 1 \in S_k \setminus C_k$. Observe that to fire the vertices of $e_i$ on the edge $e_1$, it is equivalent to consider the dollar game on a completed graph on vertices of $e_i$ with the bank vertex $(i-1)(k-1)$. Since $\tilde{c}_0$ is a critical stable configuration, we get the subgraph of $\mathcal{G}(c_0)$ by firing the vertices 0, 1, · · · , $k-1$ on $e_1$ and $k-1$ also $e_2$; see Fig. 6.

Similarly, since $\tilde{c}_0^2, \cdots, \tilde{c}_0^n$ are critical stable configurations, each $e_i$ yields a directed cycle of length $k$ by firing the vertices of $e_i$ on $e_i$, $i \in [s]$. Hence, we get $s$ directed cycles, which implies that $\mathcal{G}'$ is not empty if and only if $s < n$.

(ii) Denote

$$\forall' = \{c' \in \mathcal{G}(c_0) \mid c' \text{ is obtained from } c_s \text{ by first firing } s(k-1) \text{ on } e_{s+1}\},$$

and $\omega'(c) = \sum_{v=s(k-1)+1}^{n(k-1)} c(v)$ for any $c \in \mathcal{G}(c_0)$. We claim that $\omega'(c') > \omega'(c)$ for any $c' \in \forall'$ and $c \in \mathcal{G}(c_0) \setminus \forall'$.

In fact, we choose a configuration $c'_m \in \forall'$ such that $\omega'(c'_m) = \min\{\omega'(c') \mid c' \in \forall'\}$, and a directed path $P(c_s, c'_m)$ from $c_s$ to $c'_m$ as follows:

$$P(c_s, c'_m) : c_s \xrightarrow{\alpha_{s(k-1), s+1}} c'_1 \xrightarrow{\alpha_{s', s+1}} \cdots \xrightarrow{\alpha_{s'-1, s}} c'_m.$$ 

We may also assume that $\omega'(c'_i) > \omega'(c'_m)$ for any $i \in [m-1]$. By the firing rule, we have $\alpha_{u'_j, s+1} = \alpha_s(k-1, e_{s+1})$ for some $i \in [s-1]$, letting $u'_i$ be the last one for the vertex $s(k-1)$ fired on $e_{s+1}$, then it suffices to consider the restriction of the configurations on $[s(k-1), n(k-1)]$, which is equivalent to consider the firing graph of $P_{n-s}$ associated to the configuration $(c'_1(s(k-1)), \cdots, c'_1(n(k-1)))$ with $s(k-1)$ as the bank vertex, where $P_{n-s}$ is the sub-hypergraph of $P_n^k$ consisting of the edges $e_{s+1}, \cdots, e_n$. By Lemma 4.3 (ii), we have $\omega'(c'_n) \geq \omega'(c'_1)$, a contradiction. So, form $c'_1$ to $c'_n$, $s(k-1)$ is no longer fired on $e_{s+1}$.

By the above discussion, in order to compute the $\omega'(c'_m)$, it suffices to consider the firing graph of $P_{n-s}$ associated to the configuration $(c_s(s(k-1)), \cdots, c_s(n(k-1)))$ with $s(k-1)$ as the bank vertex. Note that $c_s(v) = c_0(v) = c(v)$ for any $v > s(k-1)$ and any $c$ in the front $s$ directed cycles of $\mathcal{G}(c_0)$. By Lemma 4.3 (ii)

![Figure 6. The directed cycle obtained by firing the vertices in $e_1$](image-url)
again, we have
\[ \omega'(c'_m) > \omega'(c_s) = \omega'(c_0) = \omega'(\hat{e}), \]
since \( c_0^{s+1} = (c_0(s(k-1)), \ldots, c_0((s+1)(k-1))) \) is a non-critical. So the claim follows, and the subgraph \( G' \) induced by \( \mathcal{V}' \) contains no configurations in the front \( s \) directed cycles of \( G(c_0) \).

(iii) If \( s = 0 \), then Lemma \( 4.3(ii) \), \( \omega(c') > \omega(c_0) \) for any stable configuration \( c' \) in \( G' \) as \( c_0^{s+1} \) is not critical. If \( s > 1 \), also by Lemma \( 4.3(ii) \), we have \( \omega(c') \geq \omega(c_0) \), and \( \omega(c') = \omega(c_0) \) if and only if the vertices \( 1, \ldots, k-1 \) have been fired on \( e_1 \) exactly once along any directed path from \( c_0 \) to \( c' \). So it suffices to show there exists a vertex \( u \in [n(k-1)] \) such that \( c_0(v) = c'(v) \) for any \( 1 \leq v < u \) and \( c_0(u) > c'(u) \) when \( \omega(c') = \omega(c_0) \).

Suppose that \( P(c_0, c') \) is a directed path from \( c_0 \) to \( c' \) as follows:

\[
P(c_0, c') : \quad c_0 \xrightarrow{\alpha_{u_1}} c_0 = c_1 \xrightarrow{\alpha_{u_1}} \cdots \xrightarrow{\alpha_{u_{k-1}}} c_{k-1} \xrightarrow{\alpha_{u_{k-1}}} c_k \xrightarrow{\alpha_{u_{k-1}}} \cdots \xrightarrow{\alpha_{u_{k-1}}} c_s \xrightarrow{\alpha_{u_{k-1}}} \cdots \xrightarrow{\alpha_{u_{k-1}}} c_t \xrightarrow{\alpha_{u_{k-1}}} c'\]

As \( \omega(c') = \omega(c_0) \), we have a unique \( \sigma(v) \in [t] \) such that \( \alpha_{u_{\sigma(v)}} = \alpha_{v,e_1} \) for each \( v \in [k-1] \) by Lemma \( 4.3(ii) \). It follows that \( c'(v) = c_0(v) \) for \( v \in [k-2] \).

By construction, there exist \( 1 \leq i_1 < j_1 \leq t \) such that \( \alpha_{u_{i_1}} = \alpha_{k-1,e_2} \) and \( \alpha_{u_{j_1}} = \alpha_{k-1,e_1} \). Note that the vertex \( k-1 \) is fired on \( e_1 \) exactly once so that the arrow from \( c_{u_{i_1}} \) to \( c_{u_{j_1}} \) must be traveled. If there exists \( i_1 \leq i_1 < j_1 \) satisfying \( 1 \leq u_{i_1} \leq k-2 \), then by Lemma \( 4.3(i) \),

\[
c_{u_{i_1}}(k-1) = c_0(k-1) + m_1 = k-1,
\]

\[
c_{u_{j_1}}(k-1) = c_{u_{i_1}}(k-1) + m_2 = k-1,
\]

where \( m_1 \) (respectively, \( m_2 \)) is the number of vertices of \([k-2]\) fired on \( e_1 \) from \( c_1 \) to \( c_{u_{i_1}} \) (respectively, from \( c_{u_{i_1}} \) to \( c_{u_{j_1}} \)). By the firing rule, \( c_{u_{i_1}}(k-1) \leq k-1 \). So,

\[
c'(k-1) = k-1 - (m_1 + m_2 + 1) = c_0(k-1) - ((k-1) - c_{u_{i_1}}(k-1)) < c_0(k-1),
\]

which implies that \( c_0 \preceq c' \). Otherwise, we have a unique \( \sigma(v) \in [i_1 + 1, j_1 - 1] \) such that \( \alpha_{u_{\sigma(v)}} = \alpha_{v,e_2} \) for each \( v \in [k-1, 2(k-1)] \). So \( c'(v) = c_0(v) \) for \( v \in [k-1, 2(k-1) - 1] \). Then we continue to compare the values of \( v \) given by \( c_0 \) and \( c' \) for \( v \geq 2k \).

Consider the firing graph of \( P_{n-1}^k \) starting from the configuration \( (c_{i_1}(k-1), \ldots, c_{i_1}(n(k-1))) = (c_0(k), \ldots, c_0(n(k-1))) \), where \( P_{n-1}^k \) is the sub-hypergraph of \( P_{n}^k \) consisting of the edge \( e_{2}, \ldots, e_{n} \) with \( k-1 \) as the bank vertex. Similar to the analysis for \( P_{n}^k \), there exist \( i_1 < i_2 < j_2 < j_1 \) such that \( \alpha_{u_{i_2}} = \alpha_{2(k-1),e_3} \) and \( \alpha_{u_{j_2}} = \alpha_{2(k-1),e_3} \). If there exists \( i_2 < j_2 < j_2 \) satisfying \( k \leq u_{i_2} \leq 2(k-1) - 1 \), then we have \( c'(2(k-1)) < c_0(2(k-1)) \). Otherwise, for each \( v \in [2(k-1) + 1, 3(k-1)] \), we have a unique \( \sigma(v) \in [j_2 + 1, j_2 - 1] \) such that \( \alpha_{u_{\sigma(v)}} = \alpha_{v,e_3} \). So \( c'(v) = c_0(v) \) for \( v \in [2(k-1), 3(k-1) - 1] \). Then we continue to compare the firing graph of \( P_{n-2}^k \) starting from the configuration \( (c_{i_2}(2(k-1)), \ldots, c_{i_2}(n(k-1))) = (c_0(2(k-1)), \ldots, c_0(n(k-1))) \), and so on.

By the above discussion, it follows that either \( c_0(v) = c'(v) \) for \( v \in [1, s'(k-1) - 1] \) and \( c_0(s'(k-1)) > c'(s'(k-1)) \) for some \( s' \in [1, s-1] \), or \( c_0(v) = c'(v) \) for \( v \in [1, s(k-1) - 1] \). If the former case occurs, then \( c_0 \preceq c' \). Otherwise, there exist
Observe that the directed graph \( G \) to obtain more explicit formulas, we need simplify the weighted firing graph \( G \) by assigning the weight 1 on the arrow for any \( c \). As \( \tilde{G}^{*+1} \) is a non-critical stable configuration, there exists \( i_s < \ell_s < j_s \) satisfying \((s-1)(k-1)+1 \leq u_{\ell_s} \leq s(k-1)-1 \), implying \( c'(s(k-1)) < c_0(s(k-1)) \) by a similar discussion. The result follows. \( \square \)

4.3. **Formulas from firing graph.** Our goal is to compute the determinant of the left multiplication map by \( f_w \) of \( A = \mathbb{C}[x_v: v \in \tilde{V}] \) associated with the hyperpath \( P_n^k \) as in Fig. 3, where \( w \) is taken to be the vertex 0. We focus on the firing graph \( \mathcal{G}(c_0) \) of the structure in Fig. 5. If a configuration \( c \) in \( \mathcal{G}(c_0) \) refers to a homogeneous polynomial \( x^e = \prod_{e \in \tilde{V}} x_v^i \) by ignoring the bank vertex, then we have

\[
m_{f_0}(x^c) = \lambda x^c - x^e_1 x^c = \lambda x^c - x^{(0)},
\]

and

\[
x^c = \sum_{(c,c') \in \mathcal{E}(\mathcal{G}(c_0))} \frac{1}{\lambda} x^{c'}
\]

for any \( c \neq c_0 \).

In view of this, we consider the weighted directed graph, still denoted by \( \mathcal{G}(c_0) \), by assigning the weight 1 on the arrow \( \alpha_{0,e_1} \) and the weight \( \frac{1}{\lambda} \) on the others. In order to obtain more explicit formulas, we need simplify the weighted firing graph \( \mathcal{G}(c_0) \). Observe that the directed graph \( \mathcal{G}(c_0) \) may contain non-stable configurations. Next, we will erase all non-stable configurations by modifying the weight until all of vertices are stable configurations.

Define a function \( g^i(x) \) in indeterminant \( \lambda \) recursively:

\[
(4.1) \quad g^{-1}(x) = 0, g^0(x) = 1, g^1(x) = g(x) = \frac{1}{1 - \frac{x}{\lambda}}, g^i(x) = g^{i-1}(g(x)) \text{ for } i \geq 2.
\]

**Lemma 4.5.** Let \( c_0 \) be a configuration in \( B_s \), \( 0 \leq s \leq n \). Then we have

\[
x^{c_0} = \frac{g^{s-1}(1)}{\lambda^{k-1}} x^{c_0} + \sum_{c' \in \mathcal{S}(\mathcal{G}')} h_{c'}(\lambda)x^{c'},
\]

where \( \mathcal{G}_0(v) = \begin{cases} c_0(v) + 1, & 1 \leq v \leq k - 1, \\ c_0(v), & k \leq v \leq n(k-1) \end{cases} \), \( g^i(x) \) is defined as in (4.1), \( h_{c'}(\lambda) \) is a function in \( \lambda \) for each \( c' \), and \( \mathcal{S}(\mathcal{G}') \) is the set of all stable configurations in \( \mathcal{G}' \) defined in Proposition 4.4.

**Proof.** Clearly, for \( s = 0 \),

\[
x^{c_0} = \sum_{c' \in \mathcal{S}(\mathcal{G}')} h_{c'}(\lambda)x^{c'} = \frac{g^{s-1}(1)}{\lambda^{k-1}} x^{c_0} + \sum_{c' \in \mathcal{S}(\mathcal{G}')} h_{c'}(\lambda)x^{c'},
\]

and for \( s = 1 \),

\[
x^{c_0} = \frac{1}{\lambda^{k-1}} x^{c_0} + \sum_{c' \in \mathcal{S}(\mathcal{G}')} h_{c'}(\lambda)x^{c'}.
\]

For \( s \geq 2 \), from the subgraph of \( \mathcal{G}(c_0) \) in Fig. 7, we get

\[
x^{c_{s-1}} = \frac{1}{\lambda} x^{c_{s-1}} + \frac{1}{\lambda^k} x^{c_{s-1}} + \frac{1}{\lambda^s} x^{c_s},
\]
where \( l_s \) is the length of the directed path from \( c_{s-1} \) to \( c'_1 \). It follows that
\[
x^{c_{s-1}} = g(1) \frac{1}{\lambda} x^{c_{s-1}} + g(1) \frac{1}{\lambda l_s} x^{c'_1}.
\]
Therefore, we can erase the \( s \)-th directed cycle by adding an arrow \((c_{s-1}, c'_1)\) with weight \( g(1) \frac{1}{\lambda l_s} \) and replacing the weight of \((c_{s-1}, \bar{c}_{s-1})\) by \( g(1) \frac{1}{\lambda} \); see Fig. 8.

\[\text{Figure 7. The weighted } s\text{-th direct cycle by firing the vertices in } e_s\]

\[\text{Figure 8. The modified weighted } (s-1)\text{-th direct cycle by erasing the } s\text{-th directed cycle}\]

Let \( l_j \) be the length of the directed path from \( c_{j-1} \) to \( c_j \) for \( 2 \leq j \leq s-1 \). From this modified weighted subgraph, we immediately get
\[
x^{c_{s-2}} = \frac{1}{\lambda} x^{c_{s-2}} + g(1) \frac{1}{\lambda^2} x^{c_{s-2}} + g(1) \frac{1}{\lambda l_{s-1} + l_s} x^{c'_1},
\]
and
\[
x^{c_{s-2}} = g^2(1) \frac{1}{\lambda} x^{c_{s-2}} + g^2(1) g(1) \frac{1}{\lambda l_{s-1} + l_s} x^{c'_1}.
\]
Assume that \( x^{c_{i-1}} = g^i(1) \frac{1}{\lambda} x^{c_{i-1}} + g^i(1) \frac{1}{\lambda l_{s-1} + \cdots + l_s} x^{c'_1} \) for \( i \geq 1 \). Then from the modified weighted subgraph (see Fig. 9), we have
\[
x^{c_{s-(i+1)}} = \frac{1}{\lambda} x^{c_{s-(i+1)}} + g^i(1) \frac{1}{\lambda^2} x^{c_{s-(i+1)}} + g^i(1) \frac{1}{\lambda l_{s-1} + \cdots + l_s} x^{c'_1},
\]
and hence
\[
x^{c_{s-(i+1)}} = g^{i+1}(1) \frac{1}{\lambda} x^{c_{s-(i+1)}} + g^{i+1}(1) \frac{1}{\lambda l_{s-1} + \cdots + l_s} x^{c'_1}.
\]
for some function $g$ where

$$g = \frac{g^{s-1}(1) \cdots g^2(1)g(1)}{\lambda^i + \lambda^{i+1} + \cdots + \lambda^s} x^{s-1}. $$

It follows that

$$x^{e_0} = \frac{g^{s-1}(1)}{\lambda^{k-1}} x^{e_0} + \prod_{i=1}^{s-1} g^{i}(1) \frac{1}{\lambda^{l}} x^{e_i},$$

where $l_1$ is the length of the directed path from $c_0$ to $c_1$. By Proposition \ref{4.4}(i), we know that there is not arrow from $G'$ to $G$ and therefore $x^{e_i} = \sum_{e' \in S(G')} h_{e'}(\lambda) x^{e'}$ for some function $h_{e'}(\lambda)$ in $\lambda$. It follows that

$$x^{e_0} = \frac{g^{s-1}(1)}{\lambda^{k-1}} x^{e_0} + \sum_{e' \in S(G')} h_{e'}(\lambda) x^{e'}.$$

$\square$

4.4. The characteristic polynomial of hyperpaths. We will give a recursive formula of the characteristic polynomial of hyperpaths. Define

$$\mu_{n,k}(s) = \begin{cases} k^{s}(k-2)((k-1)k^{s-1} - k^{s-2})((k-1)k^{s-2})^{n-s-1}(k-1)^{s}, & s \in [0, n-1], \\ k^{s}(k-2), & s = n. \end{cases}$$

**Theorem 4.6.** Let $\phi_{n,k}^{P}(\lambda)$ be the characteristic polynomial of the hyperpath $P_{n}^{k}$, where $n \geq 2$. Then

$$\phi_{n,k}^{P}(\lambda) = \lambda^{k-2}(k-1)^{n-1} \prod_{i=0}^{n} \left( \lambda - \frac{g^{s-1}(1)}{\lambda^{k-1}} \right)^{\mu_{n,k}(s)} \phi_{n-1,k}^{P}(\lambda)(k-1)^{k-1}$$

where $g'(x)$ is defined in \ref{4.4} and $\mu_{n,k}(s)$ is defined in \ref{4.2}.

**Proof.** By definition, the characteristic polynomial of $P_{n}^{k}$ is

$$\phi_{n,k}^{P}(\lambda) = \text{Res}(F_0, F_1, \cdots, F_t),$$

where

$$F_i(x_0, x_1, \cdots, x_r) = \lambda x_i^{k-1} - \sum_{i \in \mathcal{E} \in E} x^{e_0(i)}.$$
is a homogeneous polynomial in variables $x_0, x_1, \ldots, x_r$ of degree $k - 1$ for $i = 0, 1, \ldots, r = n(k - 1)$. Clearly,

$$f_i(x_1, \ldots, x_r) = \begin{cases} 
\lambda - x_1 \cdots x_{k-1}, & \text{if } i = 0, \\
\lambda x_{i-1}^{k-1} - x_1 \cdots x_i x_{i+1} \cdots x_{k-1}, & \text{if } i \in [k - 2], \\
\lambda x_{k-1} - x_1 \cdots x_{k-2} - x_k \cdots x_{2(k-1)}, & \text{if } i = k - 1, \\
F_i(x_0, x_1, \ldots, x_r), & \text{otherwise},
\end{cases}$$

and

$$F_i(x_1, \ldots, x_r) = \begin{cases} 
\lambda x_{i-1}^{k-1}, & \text{if } i \in [k - 2], \\
\lambda x_{k-1}^{k-1} - x_k \cdots x_{2(k-1)}, & \text{if } i = k - 1, \\
F_i(x_0, x_1, \ldots, x_r), & \text{otherwise}.
\end{cases}$$

By Lemma 2.2 and 2.3, we have

$$\phi_{n,k}(\lambda) = \text{Res}(F_1, \ldots, F_r)^{k-1} \det(m_{f_0})$$

$$= \text{Res}(\lambda x_1^{k-1}, \ldots, \lambda x_{k-1}^{k-1}, F_{k-1}, \ldots, F_r)^{k-1} \det(m_{f_0})$$

$$= \left( \text{Res}(\lambda x_1^{k-1}, \ldots, \lambda x_{k-2}^{k-1})^{(k-1)(n-1)(k-1)} + \text{Res}(F_{k-1}, \ldots, F_r)^{(k-1)(k-2)} \right)^{k-1} \det(m_{f_0})$$

$$= \lambda^{(k-2)(k-1)\binom{n}{k-1}} \phi_{n-1,k}(\lambda')^{k-1} \det(m_{f_0})$$

where $m_{f_0}$ is the multiplication map of the quotient algebra

$$A = C[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$$

given by $m_{f_0}(x_1^{i_1} \cdots x_r^{i_r}) = \lambda x_1^{i_1} \cdots x_r^{i_r} - x_1^{i_1+1} \cdots x_{k-1}^{i_k} \cdots x_r^{i_r}$.

We choose a $C$-basis $B = \{\mathbf{x}^c | c : [r] \to [k - 2] \}$ for $A$, where $\mathbf{x}^c = x_1^{c(1)} \cdots x_r^{c(r)}$. In fact, for any $\mathbf{x}^c \in B$, $\bar{c} = (\bar{c}(0), c(1), \ldots, c(r))$ can be viewed as a stable configuration on $P_n^k$, where 0 is the bank vertex whose value can be omitted. We denote by $\mathcal{B}$ be the set of all stable configurations on $P_n^k$. Clearly, there is a one-to-one correspondence between $B$ and $\mathcal{B}$ by ignoring the bank vertex, and the left anti-lexicographical ordering on $B$ gives a total ordering on $\mathcal{B}$. To be precise, $x_1^{i_1} \cdots x_r^{i_r} \prec x_1^{j_1} \cdots x_r^{j_r}$ if and only if $\sum_{t=1}^n i_t < \sum_{t=1}^n j_t$, or $\sum_{t=1}^n i_t = \sum_{t=1}^n j_t$, $i_s = j_s$ for $1 \leq s \leq t - 1$ and $i_t > j_t$ for some $t \in [r]$.

Retaining the notation in Section 4.2, we observe that $B$ is exactly the disjoint union of $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n$. For each $s \in [n]$, the number of configurations in $\mathcal{B}_s$ is exactly $\mu_{n,k}(s)$. Let $c_0$ be a configuration in $\mathcal{B}_s$. By Lemma 4.5, we know

$$m_{f_0}(\mathbf{x}^{c_0}) = \mathbf{x}^{c_0} = \left( \lambda - \frac{g^{s-1}(1)}{\lambda^{k-1}} \right) \mathbf{x}^{c_0} - \sum_{c' \in \mathcal{S}(\mathcal{G}')} h_{\mathcal{G}}(\lambda) \mathbf{x}^{c'},$$

where $\mathcal{G}', \mathcal{S}(\mathcal{G}')$ and $h_{\mathcal{G}}(\lambda)$ are defined in Proposition 4.3 or Lemma 4.5. On the other hand, by Proposition 4.4 (ii), we have $c_0 \prec c'$ for any $c' \in \mathcal{G}'$. It follows that the matrix of $m_{f_0}$ associated to the ordered basis $B$ is a lower triangle matrix with $\frac{g^{s-1}(1)}{\lambda^{k-1}}$ appearing on the diagonal exactly $\mu_{n,k}(s)$ times for $s \in [0, n]$. So

$$\det(m_{f_0}) = \prod_{s=0}^n \left( \lambda - \frac{g^{s-1}(1)}{\lambda^{k-1}} \right)^{\mu_{n,k}(s)},$$
and
\[
\phi_{n,k}^P(\lambda) = \lambda^{(k-2)(k-1)n(k-1)} \prod_{s=0}^{m-1} \left( \lambda - \frac{g^{s-1}(1)}{\lambda^{k-1}} \right)^{\mu_{n,k}^s(\lambda)} \phi_{n-1,k}(\lambda)^{k(k-1)^{k-1}}.
\]

**Example 4.7.** By Theorem 4.6, we get the characteristic polynomial \(\phi_{n,k}^P(\lambda)\) of \(P_n^k\) for some specified \(n\) and \(k\).

- \(\phi_{1,3}^P(\lambda) = \lambda^3(\lambda^3 - 1)^3\) of degree 12;
- \(\phi_{2,3}^P(\lambda) = \lambda^9(\lambda^3 - 1)^9(\lambda^3 - 2)^9\) of degree 80;
- \(\phi_{3,3}^P(\lambda) = \lambda^{51}(\lambda^3 - 1)^{27}(\lambda^3 - 2)^{18}(\lambda^6 - 3\lambda^3 + 1)^{27}\) of degree 448;
- \(\phi_{1,4}^P(\lambda) = \lambda^{81}(\lambda^3 - 1)^{201}(\lambda^3 - 2)^{81}(\lambda^6 - 3\lambda^3 + 1)^{81}\) of degree 2304;
- \(\phi_{2,4}^P(\lambda) = \lambda^{444}(t^4 - 1)^{16}\) of degree 108;
- \(\phi_{3,4}^P(\lambda) = \lambda^{2071}(\lambda^4 - 1)^{352}(\lambda^4 - 2)^{256}\) of degree 5103;
- \(\phi_{4,4}^P(\lambda) = \lambda^{55774}(\lambda^4 - 1)^{11440}(\lambda^4 - 2)^{5632}(\lambda^8 - 3\lambda^4 + 1)^{4096}\) of degree 196830.

5. **Starlike hypergraphs**

In the last section we will deal with a class of \(k\)-uniform hypergraphs, called *starlike hypergraph* and denoted by \(S^k_{n_1,\ldots,n_m}\), which is obtained from \(m\) hyperpaths of length \(n_1,\ldots,n_m\) by sharing a common vertex \(w\), where \(m \geq 1\) and \(n_i \geq 1\) for \(i \in [m]\); see Fig. 10. When \(m = 1\), it is a hyperpath \(P_n^k\). When \(n_i = 1\) for each \(i \in [m]\), it is called a *hyperstar* with \(m\) edges and denoted by \(S_m^k\).

![Figure 10. The starlike hypergraph \(S^k_{n_1,n_2,\ldots,n_m}\)](image)

**Theorem 5.1.** Let \(S^k_{n_1,\ldots,n_m}\) be the \(k\)-uniform starlike hypergraph with exactly \(t\) hyperpaths having length 1, where \(0 \leq t \leq m\). Then the characteristic polynomial \(\phi_{S^H_{n_1,\ldots,n_m,k}}^P(\lambda)\) of \(S^k_{n_1,\ldots,n_m}\) is

\[
\lambda^{(m(2-t)+(t-1))k} \prod_{i=1}^{m(t-1)+1} \phi_{n_i-1,k}^P(\lambda)^{(k-1)^{k-1}} \prod_{i \in [m], \sum_{j=1}^{n_i} r_j = i \leq j \leq m} \left( \lambda - \sum_{i=1}^{m} \frac{g^{i-1}(1)}{\lambda^{k-1}} \right)^{\prod_{i=1}^{m} \mu_{n_i,k}(s_i)},
\]

where \(r_i = n_i(k - 1)\) for \(i \in [m]\), \(\phi_{n_i,k}^P(\lambda)\) denotes the characteristic polynomial of \(P_n^k\), \(g^s(x)\) and \(\mu_{n,k}(s)\) are defined in (4.1) and (4.2) respectively.
3.2. The quotient algebra

So, combining (5.1) and (5.2), we have

\[ \varphi(5.2) \]

otherwise,

\[ \varphi(5.1) \]

we get

\[ r \]

\[ c \]

\[ G \]

\[ A \]

\[ H \]

For each

\[ \lambda \]

Proof.

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of configurations in

\[ B \]

as the bank vertex. As discussed in Theorem 4.6, there is a one-to-one correspondence between the set of stable configuration of \( P^k_{n_i} \) and the number of configurations in \( B_s \). Fix an \( i \in [m] \). Let \( B = \{ x^e : V_i \setminus \{ w \} \to [k - 2] \} \) be a basis of \( A_i \), where \( x^e = \prod_{v \in V_i \setminus \{ w \}} x^e(v) \) and let \( B \) be the set of stable configuration of \( P^k_{n_i} \) with \( w \) as the bank vertex. As discussed in Theorem 4.6 there is a one-to-one correspondence between \( B \) and \( B \) by ignoring the bank vertex. We also have a left anti-lexicographical ordering \( \prec \) on \( B \) arising from the order of \( B \). Retaining the notation in Section 4.2, \( B \) is the disjoint union of \( B_0, B_1, \ldots, B_{n_i} \), and the number of configurations in \( B_s \) is \( \mu_{n_i,k}(s) \) for each \( s \in [0, n_i] \).

For each \( c_0 \in B_s \), by Lemma 4.5

\[ m_{i,w}(x^{c_0}) = x^{c_0} = \frac{c_0^{s-1}(1)}{\lambda^{k-1}} x^{c_0} + \sum_{c' \in \mathcal{S}(G')} h_{c'}(\lambda) x^{c'}, \]

where \( G' \), \( \mathcal{S}(G') \) and \( h_{c'}(\lambda) \) are defined in Proposition 4.4 or Lemma 4.5. Note that \( c_0 \prec c' \) for any \( c' \in G' \). So the matrix of \( m_{i,w} \) associated with the basis \( B \) under
the above order is a lower triangle matrix with \( \frac{g^{n-1}(s)}{\lambda^{m-1}} \) appearing on the diagonal exactly \( \mu_{n,k}(s) \) times for \( s \in [0, n] \).

By the above discussion,

\[
\det(m_{f_m}) = \prod_{s_i \in [0, n] \leq m} \left( \lambda - \sum_{i=1}^{m} \frac{g^{s_i-1}(1)}{\lambda^{k-1}} \right) \prod_{i=1}^{n} \mu_{n,k}(s_i).
\]

(5.4) The result follows by Corollary 3.2 and the equalities (5.3) and (5.4).

Taking \( n_i = 1 \) for \( i \in [m] \) in Theorem 5.1 we get the characteristic polynomial of hyperstar \( S^k_m \).

**Corollary 5.2.** Let \( \phi^H_{n,k} \) be the characteristic polynomial of the \( k \)-uniform hyperstar \( S^k_m \) with \( m \) edges. Then

\[
\phi^H_{m,k}(\lambda) = \lambda^{r(k-1)} \prod_{p=0}^{m} \left( \lambda - \frac{p}{\lambda^{k-1}} \right) (\lambda^k - 2)^{k(p-1)} \left( (k-1)^{k-1} - k^{k-2} \right)^{n-\rho}
\]

where \( r = m(k-1) \).

**Corollary 5.3.** [3, Theorem 4.3] Let \( E \) be the \( k \)-uniform hypergraph with \( k \) vertices and a single edge. Then

\[
\phi_E(\lambda) = \lambda^{k(k-1)^{k-1} - k^{k-1}} (\lambda^k - 1)^{k-2}.
\]

**Corollary 5.4.** Let \( S^k_2 \) be a \( k \)-uniform hyperstar with two edges or hyperpath with two edges. Then

\[
\phi_{S^k_2}(\lambda) = \lambda^{\mu_k} (\lambda^k - 1)^{2k^{k-2}} ((k-1)^{k-1} - k^{k-2}) (\lambda^k - 2)^{k^2(k-2)},
\]

where \( \mu_k = (2k-1)(k-1)^2 - 2k^{k-1}(k-1)^{k-1} + k^{2k-3} \).

**Corollary 5.5.** The characteristic polynomial of the starlike hypergraph \( S^k_{1,1,2} \) is

\[
\phi^H_{1,1,2,k}(\lambda) = \lambda^{4(2k-3)(k-1)}((k-1)^{k-1} - 3k^{k-2}(k-1)^{k-2} + 3k^{2(k-2)}(k-1)^2 - k^{3(k-2)}(k-1)^{k-1})
\]

\[
\cdot \left( \lambda - \frac{1}{\lambda^{k-1}} \right)^{k-2}((k-1)^{k-1} - k^{k-2})^2
\]

\[
\cdot \left( \lambda - \frac{2}{\lambda^{k-1}} \right)^{k-2}((3(k-1)^{k-1} - k^{k-2})(k-1)^{k-1} - k^{k-2})
\]

\[
\cdot \left( \lambda - \frac{3}{\lambda^{k-1}} \right)^{k-2}((k-1)^{k-1} - k^{k-2})
\]

\[
\cdot \left( \lambda - \frac{\lambda}{\lambda^{k-1}} \right)^{k-2}((k-1)^{k-1} - k^{k-2})
\]

\[
\cdot \left( \lambda - \frac{1}{\lambda^{k-1}} - \frac{\lambda}{\lambda^{k-1}} \right)^{k-2}((k-1)^{k-1} - k^{k-2})
\]

\[
\cdot \left( \lambda - \frac{2}{\lambda^{k-1}} - \frac{\lambda}{\lambda^{k-1}} \right)^{k-2}((k-1)^{k-1} - k^{k-2})
\]

\[
\cdot \left( \lambda - \frac{\lambda}{\lambda^{k-1}} \right)^{k-2}.
\]
In particular, the characteristic polynomial $\phi_{S^1_{1,2,3}}^H(\lambda)$ of $S^1_{1,2,3}$ is
$$\lambda^{880}(\lambda^3 - 1)^{75}(\lambda^3 - 2)^{64}(\lambda^3 - 3)^{27}(\lambda^4 - 2\lambda)^9(\lambda^6 - 3\lambda^3 + 1)^{54}(\lambda^6 - 4\lambda^3 + 2)^{81}$$
of degree 2294.

6. Conclusion

We give an explicit and recursive formula for the characteristic polynomial of the adjacency tensor of a starlike hypergraph, which is a resultant of a system of polynomials related to the structure of the hypergraph. The variants of chip-firing game on simple graphs such as dollar game on simple graphs or hypergraphs are applied to the such kinds of resultants. So we provide a combinatorial method for computing resultants, which will have potential use for commutative algebra, algebraic geometry and physical fields.

We note that there are many numerical methods and algorithms for computing partial (real or extreme) eigenvalues of a general (symmetric) tensor; see e.g. Chang et.al [2] and the references therein. We also note the starlike hypergraph is a power hypergraph $G^k$, which is obtained from a starlike simple graph $G$ by adding $k - 2$ vertices to each of its edges. Zhou et.al [16] proved that if $\lambda$ is a nonzero eigenvalue of $G$ or any subgraph of $G$, then $\lambda^{2k}$ is an eigenvalue of $G^k$. In fact, the nonzero eigenvalues of $G^k$ are exactly those eigenvalues arising from $G$ in the above way. However we do not know the algebraic multiplicities of the eigenvalues (including the zero eigenvalue) from their result.

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Yan-Hong Bao  
School of Mathematical Sciences, Anhui University, Hefei 230601, China  
E-mail address: baoyh@ahu.edu.cn

Yi-Zheng Fan  
School of Mathematical Sciences, Anhui University, Hefei 230601, China  
E-mail address: fanyz@ahu.edu.cn

Yi Wang  
School of Mathematical Sciences, Anhui University, Hefei 230601, China  
E-mail address: wangy@ahu.edu.cn

Ming Zhu  
School of Electronics and Information Engineering, Anhui University, Hefei 230601, China  
E-mail address: zhuming@ahu.edu.cn