A short note about diffuse Bieberbach groups

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Abstract

We consider low dimensional diffuse Bieberbach groups. In particular we classify diffuse Bieberbach groups up to dimension 6. We also answer a question from [7, page 887] about the minimal dimension of a non-diffuse Bieberbach group which does not contain the three-dimensional Hantzsche-Wendt group.

1 Introduction

The class of diffuse groups was introduced by B. Bowditch in [2]. By definition a group $\Gamma$ is diffuse, if every finite non-empty subset $A \subset \Gamma$ has an extremal point, i.e. an element $a \in A$ such that for any $g \in \Gamma \setminus \{1\}$ either $ga$ or $g^{-1}a$ is not in $A$. Equivalently (see [7]) a group $\Gamma$ is diffuse if it does not contain a non-empty finite set without extremal points.

The interest in diffuse groups follows from Bowditch’s observation that they have the unique product property [1]. Originally unique products were introduced in the study of group rings of discrete, torsion-free groups. More precisely, it is easily seen that if a group $\Gamma$ has the unique product property, then it satisfies Kaplansky’s unit conjecture. In simple terms this means that the units in the group ring $\mathbb{C}[\Gamma]$ are all trivial, i.e. of the form $\lambda g$ with $\lambda \in \mathbb{C}^\ast$ and $g \in \Gamma$. For more information about these objects we refer the reader to [1], [9, Chapter 10] and [7]. In part 3 of [7] the authors prove that any torsion-free crystallographic group (Bieberbach group) with trivial center is not diffuse. By definition a crystallographic group is a discrete and cocompact subgroup of the group $O(n) \ltimes \mathbb{R}^n$ of isometries of the Euclidean space $\mathbb{R}^n$. From Bieberbach’s theorem (see [12]) the normal subgroup $T$ of all translations of any crystallographic group $\Gamma$ is a free abelian group of finite rank and the quotient group (holonomy group) $\Gamma/T = G$ is finite.

In [7, Theorem 3.5] it is proved that for a finite group $G$:

1. If $G$ is not solvable then any Bieberbach group with holonomy group isomorphic to $G$ is not diffuse.

2. If every Sylow subgroup of $G$ is cyclic then any Bieberbach group with holonomy group isomorphic to $G$ is diffuse.

3. If $G$ is solvable and has a non-cyclic Sylow subgroup then there are examples of Bieberbach groups with holonomy group isomorphic to $G$ which are examples which are not diffuse.

Using the above the authors of [7] classify non-diffuse Bieberbach groups in dimensions $\leq 4$. One of the most important non-diffuse groups is the 3-dimensional Hantzsche-Wendt group, denoted in [11] by $\Delta_P$. For the following presentation

$$
\Delta_P = \langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2} \rangle
$$

the maximal abelian normal subgroup is generated by $x^2, y^2$ and $(xy)^2$ (see [6, page 154]). At the end of part 3.4 of [7] the authors ask the following question.

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**Question 1.** What is the smallest dimension $d_0$ of a non-diffuse Bieberbach group which does not contain $\Delta_P$?

The answer for the above question was the main motivation for us. In fact we prove, in the next section, that $d_0 = 5$. Moreover, we extend the results of part 3.4 of [7] and with support of computer, we present the classification of all Bieberbach groups in dimension $d \leq 6$ which are (non)diffuse.

## 2 (Non)diffuse Bieberbach groups in dimension $\leq 6$.

We use the computer system CARAT [10] to list all Bieberbach groups of dimension $\leq 6$.

Our main tools are the following observations:

1. The property of being diffuse is inherited by subgroups (see [2, page 815]).
2. If $\Gamma$ is a torsion-free group, $N \triangleleft \Gamma$ such that $N$ and $\Gamma/N$ are both diffuse then $\Gamma$ is diffuse (see [2, Theorem 1.2 (1)]).

Now let $\Gamma$ be a Bieberbach group of dimension less than or equal to 6. By the first Betti number $\beta_1(\Gamma)$ we mean the rank of the abelianization $\Gamma/[\Gamma, \Gamma]$. Note that we are only interested in the case when $\beta_1(\Gamma) > 0$ (see [7, Lemma 3.4]). Using a method of E. Calabi [12, Propostions 3.1 and 4.1], we get an epimorphism

$$f: \Gamma \to \mathbb{Z}^k,$$

where $k = \beta_1(\Gamma)$. (1)

From the assumptions $\ker f$ is a Bieberbach group of dimension $< 6$. Since $\mathbb{Z}^k$ is a diffuse group our problem is reduced to the question about the group $\ker f$.

**Remark 1.** Up to conjugation in $\text{GL}(n+1, \mathbb{R})$, $\Gamma$ is a subgroup $\text{GL}(n, \mathbb{Z}) \rtimes \mathbb{Z}^n \subset \text{GL}(n+1, \mathbb{Q})$, i.e. it is a group of matrices of the form

$$\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix},$$

where $A \in \text{GL}(n, \mathbb{Z}), a \in \mathbb{Q}^n$. If $p: \Gamma \to \text{GL}(n, \mathbb{Z})$ is a homomorphism which takes the linear part of every element of $\Gamma$

$$p\left(\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}\right) = A \text{ for every } \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix} \in \Gamma,$$

then there is an isomorphism $\rho: G \to p(\Gamma) \subset \text{GL}(n, \mathbb{Z})$. It is known that that the rank of the center of a Bieberbach group equals the first Betti number (see [5, Proposition 1.4]). By [12, Lemma 5.2], the number of trivial constituents of the representation $\rho$ is equal to $k$. Hence without lose of generality we can assume that the matrices in $\Gamma$ are of the form

$$\begin{bmatrix} A & B & a \\ 0 & I & b \\ 0 & 0 & 1 \end{bmatrix},$$

where $A \in \text{GL}(n-k, \mathbb{Z}), I$ is the identity matrix of degree $k, B$ is an integral matrix of dimension $n-k \times k, a \in \mathbb{Q}^{n-k}$ and $b \in \mathbb{Q}^k$. Then $f$ may be defined by

$$f\left(\begin{bmatrix} A & B & a \\ 0 & I & b \\ 0 & 0 & 1 \end{bmatrix}\right) = b$$

and one can easily see that the map $F: \ker f \to \text{GL}(n-k+1, \mathbb{Q})$ given by

$$F\left(\begin{bmatrix} A & B & a \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}$$

is a monomorphism and hence its image is a Bieberbach group of rank $n-k$.  

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Now if $\Gamma$ has rank 4 we know that the only non-diffuse Bieberbach group of dimension less than or equal to 3 is $\Delta_P$. Using the above facts we obtain 17 non-diffuse groups. Note that the list from [7, section 3.4] consists of 16 groups. The following example presents the one which is not in [7] and illustrates computations given in the above remark.

**Example 1.** Let $\Gamma$ be a crystallographic group denoted by "05/01/06/006" in [3] as a subgroup of $\text{GL}(5, \mathbb{R})$. Its non-lattice generators are as follows

\[
A = \begin{bmatrix}
0 & -1 & 1 & 0 & 1/2 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1/2 \\
0 & 0 & 0 & -1 & 1/2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 1 & -1 & 0 & 1/2 \\
0 & 1 & 0 & 0 & 1/2 \\
-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Conjugating the above matrices by

\[
Q = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\in \text{GL}(5, \mathbb{Z})
\]

one gets

\[
A^Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 1/2 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1/2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
B^Q = \begin{bmatrix}
-1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 1/2 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Now its easy to see that the rank of the center of $\Gamma$ equals 1 and the kernel of the epimorphism $\Gamma \to \mathbb{Z}$ is isomorphic to a 3-dimensional Bieberbach group $\Gamma'$ with the following non-lattice generators:

\[
A' = \begin{bmatrix}
1 & 0 & 0 & 0 & 1/2 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
B' = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1/2 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

Clearly the center of $\Gamma'$ is trivial, hence it is isomorphic to the group $\Delta_P$.

Now we formulate our main result.

**Theorem 1.** The following table summarizes the number of diffuse and non-diffuse Bieberbach groups of dimension $\leq 6$.

| Dimension | Total | Non-diffuse | Diffuse |
|-----------|-------|-------------|---------|
| 1         | 1     | 0           | 1       |
| 2         | 2     | 0           | 2       |
| 3         | 10    | 1           | 9       |
| 4         | 74    | 17          | 57      |
| 5         | 1060  | 352         | 708     |
| 6         | 38746 | 19256       | 19490   |

**Proof.** If a group has a trivial center then it is not diffuse. In other case we use the Calabi [1] method and induction. A complete list of groups was obtained using computer algebra system GAP [4] and is available here [8].

Before we answer Question [1] from the introduction, let us formulate the following lemma:
Lemma 1. Let $\alpha, \beta$ be any generators of the group $\Delta_P$. Let $\gamma = \alpha \beta, a = \alpha^2, b = \beta^2, c = \gamma^2$. Then the following relations hold:

$$
\begin{align*}
[a, b] &= 1 \\
[a, c] &= 1 \\
[b, c] &= 1
\end{align*}
$$

and

$$
\begin{align*}
a^\beta &= a^{-1} \\
b^\alpha &= b^{-1} \\
c^\gamma &= c^{-1}
\end{align*}
$$

(2)

where $x^y := y^{-1}xy$ denotes the conjugation of $x$ by $y$.

The proof of the above lemma is omitted. Just note that the relations are easily checked if consider the following representation of $\Delta_P$ as a matrix group

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1/2 \\
0 & -1 & 0 & 1/2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$\gamma_1, \gamma_2, l_1, \ldots, l_5$ where

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1/2 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and $l_1, \ldots, l_5$ generate the lattice $L$ of $\Gamma$:

$$
l_i := \begin{bmatrix} I_5 & e_i \\ 0 & 1 \end{bmatrix}
$$

where $e_i$ is the $i$-th column of the identity matrix $I_5$. $\Gamma$ fits into the following short exact sequence

$$
1 \longrightarrow L \longrightarrow \Gamma \longrightarrow D_8 \longrightarrow 1
$$

where $\pi$ takes the linear part of every element of $\Gamma$:

$$
\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \mapsto A
$$

and the image $D_8$ of $\pi$ is the dihedral group of order 8.

Now assume that $\Gamma'$ is a subgroup of $\Gamma$ isomorphic to $\Delta_P$. Let $T$ be its maximal normal abelian subgroup. Then $T$ is free abelian group of rank 3 and $\Gamma'$ fits into the following short exact sequence

$$
1 \longrightarrow T \longrightarrow \Gamma' \longrightarrow C_2^2 \longrightarrow 1
$$

where $C_m$ is a cyclic group of order $m$. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T \cap L & \longrightarrow & T & \longrightarrow & H = \pi(T) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & L & \longrightarrow & \pi^{-1}(H) & \longrightarrow & H & \longrightarrow & 1
\end{array}
$$

We get that $H$ must be an abelian subgroup of $D_8 = \pi(\Gamma)$ and $T \cap L$ is a free abelian group of rank 3 which lies in the center of $\pi^{-1}(H) \subset \Gamma$. Now if $H$ is isomorphic to either to $C_4$ or $C_2^2$ then the center of $\pi^{-1}(H)$ is
of rank at most 2. Hence \( H \) must be the trivial group or the cyclic group of order 2. Note that as \( \Gamma' \cap L \) is a normal abelian subgroup of \( \Gamma' \) it must be a subgroup of \( T \):

\[
T \cap L \subset \Gamma' \cap L \subset T \cap L,
\]

hence \( T \cap L = \Gamma' \cap L \). We get the following commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & T \cap L & T & H & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \Gamma' \cap L & \Gamma' & G & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & C_2^2 & C_2^2 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1
\end{array}
\]

where \( G = \pi(\Gamma') \). Consider two cases:

1. \( H \) is trivial. In this case \( G \) is one of the two subgroups of \( D_8 \) isomorphic to \( C_2^2 \). Since the arguments for both subgroups are similar, we present only one of them. Namely, let

\[
G = \langle \text{diag}(1, -1, -1, 1, 1), \text{diag}(-1, -1, 1, 1, 1) \rangle.
\]

In this case \( \Gamma' \) is generated by the matrices of the form

\[
\alpha = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and

\[
\beta = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & y_1 + 1/2 \\
0 & -1 & 0 & 0 & 0 & y_2 - 1/2 \\
0 & 0 & 1 & 0 & 0 & y_3 \\
0 & 0 & 0 & 1 & 0 & y_4 + 1/2 \\
0 & 0 & 0 & 0 & 1 & y_5 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

where \( x_i, y_i \in \mathbb{Z} \) for \( i = 1, \ldots, 5 \). If \( c = (\alpha \beta)^2 \) then by Lemma 1 \( c^\alpha = c^{-1} \), but

\[
c^\alpha - c^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4y_5 + 4x_5 - 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Obviously solutions of the equation \( 4y_5 + 4x_5 - 2 = 0 \) are never integral and we get a contradiction.

2. \( H \) is of order 2. Then \( G = D_8 \) and \( H \) is the center of \( G \). The generators \( \alpha, \beta \) of \( \Gamma' \) lie in the cosets \( \gamma_1 \gamma_2 L \) and \( \gamma_2 L \), hence

\[
\alpha = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & x_1 \\
1 & 0 & 0 & 0 & 0 & x_2 - 1/2 \\
0 & 0 & -1 & 0 & 0 & x_3 - 1/2 \\
0 & 0 & 0 & 1 & 0 & x_4 + 1/2 \\
0 & 0 & 0 & 0 & -1 & x_5 + 1/2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and

\[
\beta = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & y_1 - 1/2 \\
0 & -1 & 0 & 0 & 0 & y_2 \\
0 & 0 & 1 & 0 & 0 & y_3 \\
0 & 0 & 0 & -1 & 0 & y_4 \\
0 & 0 & 0 & 0 & 1 & y_5 - 1/2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
where \( x_i, y_i \in \mathbb{Z} \) for \( i = 1, \ldots, 5 \), as before. Setting \( a = \alpha^2, b = \beta^2 \) we get

\[
ab - ba = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 2 - 4y_1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and again the equation \( 2 - 4y_1 = 0 \) does not have an integral solution.

The above considerations show that \( \Gamma \) does not have a subgroup which is isomorphic to \( \Delta_P \).

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Bibliography

References

[1] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones conjecture and its applications. *J. Topol.*, 1(1):57–86, 2008.

[2] B. H. Bowditch. A variation on the unique product property. *J. London Math. Soc. (2)*, 62(3):813–826, 2000.

[3] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, and H. Zassenhaus. *Crystallographic groups of four-dimensional space*. Wiley-Interscience [John Wiley & Sons], New York-Chichester-Brisbane, 1978. Wiley Monographs in Crystallography.

[4] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.8.3*, 2016. [http://www.gap-system.org/](http://www.gap-system.org/)

[5] H. Hiller and C.-H. Sah. Holonomy of flat manifolds with \( b_1 = 0 \). *Quart. J. Math. Oxford Ser. (2)*, 37(146):177–187, 1986.

[6] J. A. Hillman. *Four-manifolds, geometries and knots*, volume 5 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2002.

[7] S. Kionke and J. Raimbault. On geometric aspects of diffuse groups. *Doc. Math.*, 21:873–915, 2016. With an appendix by Nathan Dunfield.

[8] R. Lutowski. Diffuse property of low dimensional Bieberbach groups. [https://mat.ug.edu.pl/~rlutowski/diffuse/](https://mat.ug.edu.pl/~rlutowski/diffuse/)

[9] W. Lück. *\( L^2 \)-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[10] J. Opgenorth, W. Plesken and T. Schultz. *CARAT - Crystallographic Algorithms and Tables, Version 2.0*, 2003. [http://wwwb.math.rwth-aachen.de/CARAT/](http://wwwb.math.rwth-aachen.de/CARAT/)

[11] S.D. Promislov. A simple example of a torsion-free, nonunique product group. *Bull. London Math. Soc.*, 20(4):302–304, 1988.

[12] A. Szczepański. *Geometry of crystallographic groups*, volume 4 of *Algebra and Discrete Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.