Synthetic-gauge-field-induced Dirac semimetal state in an acoustic resonator system

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Abstract

Recently, a proposal of synthetic gauge field in reduced two-dimensional (2D) system from three-dimensional (3D) acoustic structure shows an analogue of the gapped Haldane model with fixed \( k_z \), and achieves the gapless Weyl semimetal phase in 3D momentum space. Here, extending this approach of synthetic gauge flux, we propose a reduced square lattice of acoustic resonators, which exhibits Dirac nodes with broken effective time-reversal symmetry. Protected by an additional hidden symmetry, these Dirac nodes with quantized values of topological charge are characterized by nonzero winding number and the finite structure exhibits flat edge modes that cannot be destroyed by perturbations.

Synthetic gauge fields in classical wave systems [1–25] have drawn a lot of attention in recent years. The bosonic systems have no intrinsic spin degree of freedom and response hardly to external magnetic fields. Thus the principle of quantum Hall effect [26] and quantum spin Hall effect [27, 28] stayed isolated from bosonic systems for a long while. However, by using gyromagnetic materials [2, 3], dynamic modulation [6], spatial-modulated waveguides [5] in photonics, laser-assisted tunneling in ultra-cold systems [9, 10] or moving fluid in acoustics [11–13], synthetic gauge fields can bend the trajectories of bosons effectively. Therefore, the non-reciprocity and topological properties of classical waves in bosonic systems such as photonics [1–8], acoustics [11–18] and mechanics [19–25] have made a rapid development in the recent years.

These methods of synthetic gauge fields for photonic, ultra-cold or acoustic systems lead to either high technical requirements or engineering complexity. Recently, a proposal [15] of synthetic gauge flux for reduced two-dimensional (2D) system in a three-dimensional (3D) acoustic structure demonstrated an analogue of gapped topological Haldane model with fixed \( k_z \) and gapless Weyl nodes in 3D momentum space. The key point of this proposal is to extend the in-plane connecting waveguide to an extra dimension and thus the experienced phase shift for acoustic waves during extra-dimensional hopping will induce a complex hopping strength [29, 30] in the reduced system. This construction of synthetic gauge field in acoustics shows a flexible method to manipulate the effective magnetic flux through each plaquette of a lattice.

In this work, by manipulating the synthetic gauge field in a reduced square lattice consisting of acoustic resonators in a 3D lattice, we construct a 2D acoustic gapless phase with Dirac nodes. The synthetic gauge field [15] is built up by the oblique coupling waveguides in the third-dimension within fixed \( k_z \). We numerically calculate the band structure and find that the acoustic system acquires 2D Dirac nodes with opposite chirality, characterized by nonzero winding numbers of \( \pm 1 \) [31, 32]. The Dirac nodes are further protected by a hidden symmetry for the 2D model. Here, the synthetic gauge field breaks effective \( T \) (time-reversal) symmetry in 2D of \( (k_x, k_y) \) with fixed \( k_z \), whereas \( P \) (parity) symmetry is preserved. Furthermore, we demonstrate the flat edge states connecting two Dirac nodes with opposite chirality in two types of finite structures for the gapless acoustic systems, being different from previous gapped systems [11–13]. These flat edge modes for gapless acoustic systems can acquire dispersion by introducing perturbations, but they cannot be destroyed.
The unit cell of the acoustic structure is shown in left panel of figure 1 (a). It consists of two complete acoustic cylindrical resonators with lattice constant \(a\) in a square lattice, which ensures the condition of two-band model. The radius and height of each acoustic cylindrical resonator are \(r = 0.4a\) and \(h = 0.4a\), respectively. In the \(z\) dimension, all the connecting waveguides have the same height \(d = a\), and a small radius \(r_c = 0.04a\). The coupling strength between two connected acoustic resonators can be tuned by changing the radius of connecting waveguides. The blue surfaces with the sound hard boundary condition and the gray-dot shaded surfaces with periodic boundary condition enclose one unit cell. The spaces inside the two acoustic resonators and connecting waveguides are all filled with air. The right panel of figure 1 (a) shows the side view of two unit cells stacked in \(z\) axis. The whole system is governed by the acoustic wave equation:

\[
\nabla \cdot \left( \frac{\nabla p}{\rho} \right) - \frac{\partial^2 p}{\partial c^2} = 0,
\]

where \(\rho\) is the air density, \(c\) is the speed of sound in air, and \(p\) is the sound pressure. In this work, we investigate the two-band model with two lowest acoustic eigen modes. The pressure field patterns in these modes are almost single valued in each acoustic resonator, as verified by simulation at a fixed wave vector \((k_x, k_y, k_z) = (0, 0, \pi/4a)\) shown in figure 1 (b).

We follow the general idea about the synthetic gauge field [15] in acoustics as shown in left panel of figure 1(a). Each coupling waveguide connects two acoustic resonators at different \(z\) coordinates, constructing the synthetic gauge field in the reduced 2D lattice. The Peierls substitution [29, 30] indicates that the hopping strength become complex with an additional phase term \(\phi = k_z d\). As long as the acoustic structure is periodic in \(z\) direction, the \(k_z\) is a good quantum number and the gauge field is preserved when \(k_z\) is fixed.

We extract the tight-binding model, similar to previous resonator systems in photonics [4–6] and acoustics [15], for the above acoustic structure, as schematically shown in figure 1 (c). Only the nearest-neighbor (NN) hopping is considered. The accompanying phase \(\phi\) of NN hopping are indicated by arrows. This complex
hopping divides the original lattice with lattice constant \(a\) into two sub-lattices denoted by A (blue) and B (red) with lattice constant \(a/2\). The new primitive lattice vectors are \(\vec{a}_1 = \vec{a}, \vec{a}_2 = -\vec{a}/2\), which correspond to the reciprocal lattice vectors of \(\vec{b}_1 = \pi/a(1, 1)\) and \(\vec{b}_2 = \pi/a(1, -1)\), as shown in figure 1(d) where the first Brillouin zone is enclosed by blue lines.

The tight-binding Hamiltonian of this acoustic system is

\[
H = \sum_{ij} \left[ -t \left( a_{ij}^\dagger b_{i+1,j} + a_{ij}^\dagger b_{i-1,j} \right) + e^{i\phi}(a_{ij}^\dagger b_{i+1,j} + b_{i,j-1}^\dagger a_{i,j-1}^\dagger) + \text{h.c.} \right],
\]

where \(a\) (\(a^\dagger\)) are the annihilation and creation operators for the sites A (B), \(t\) is the amplitude of NN hopping, and the subscripts \(i, j\) are site positions along axes \(x, y\) in units of constant \(a\). The summation term in equation (2) represents the NN hopping between nearest A and B. Hereafter, we investigate the situation where the on-site energy difference from term \(s_0\) (applying small radius of coupling waveguides) and a global shift of the dispersion spectrum from the term \(s_0 = I\) are neglected in the analysis for simplicity.

The corresponding Bloch Hamiltonian \(H(k)\) for the reduced 2D acoustic system is

\[
H(k) = 2t \cos \phi \left( \cos (k_x a) + \cos (k_y a) \right) \sigma_x + 2t \sin \phi \left( \cos (k_x a) - \cos (k_y a) \right) \sigma_y.
\]

With a fixed \(k_z\), the effective \(T\) symmetry is broken in equation (3), whereas \(P\) symmetry is preserved. We point out that the periodicity of \(\phi = k_z d\) is \(\pi\) in Hamiltonian equations (2), (3), and the band structures of \(-\pi/2 < \phi < 0\) and \(0 < \phi < \pi/2\) are the same. Hereafter we assume \(0 < \phi < \pi/2\) for further investigation.

First we calculate the band structures from the tight-binding model by assuming hopping amplitude \(t = 1\) when \(\phi = 0\) and \(\phi = \pi/4\). The results are presented in figures 2(a) and (b). It can be seen that when \(\phi = 0\), there are degenerate line nodes shown in figure 2(a) along the edges of first Brillouin zone. When synthetic gauge field is introduced with \(\phi = \pi/4\), the line degeneracies are lifted and two point degeneracies emerge at high symmetry points \(X_1\) and \(X_2\) \((\pi/2a_x, \pm \pi/2a_y)\), as shown in figure 2(b).

For the real acoustic structure, we numerically calculate, by solving equation (1) with finite element method, the acoustic bands along high symmetry lines in the first Brillouin zone. The parameter of the lattice we use here.
is a = 0.1 m. The results are presented in figure 2(c). We find the degenerate lines (red curves) along the Brillouin zone boundary for \( \phi = 0 \). When synthetic gauge field is introduced for \( \phi = \pi/4 [k_x = \pi/(4a)] \), the original degenerate lines are lifted, and two point degeneracies locating at the \( X_{1,2} \) points (blue curves) can be clearly seen. Note that these isolated degenerate points at \( X_{1,2} \) exist as long as \( 0 < \phi < \pi/2 \). These results validate the tight-binding model for the proposed acoustic structure.

Furthermore, the acoustic Hamiltonian equation (3) can be expanded by substituting \( k_x = \pi/2a + \delta k_x/a \) and \( k_y = \pm \pi/2a + \delta k_y/a \) around these degenerate points. After keeping the first order term, we arrive at

\[
h(\delta k) = -2t \cos \phi (\delta k_x \pm \delta k_y)\sigma_x - 2t \sin \phi (\delta k_x \mp \delta k_y)\sigma_y,
\]

where ‘\( \pm \)' (‘\( \mp \)’) represents the situations at points of \( X_1 \) and \( X_2 \), respectively. We can define the effective chirality as \( w = \text{sgn}[\text{det}(\nu_{ij})] \) for the Dirac nodes. Group velocity \( \nu_{ji} = \begin{pmatrix} -2t \cos \phi - 2t \sin \phi \\ \mp 2t \cos \phi \pm 2t \sin \phi \end{pmatrix} \) (\( i,j \) are \( x,y \) here) comes from equation (4). It can be obtained subsequently that the chirality is \( w = -1 \) and \( w = 1 \) for the Dirac nodes at \( X_1 \) and \( X_2 \), respectively. Note that, usually Dirac nodes as in graphene are protected by PT symmetry.

Here the Dirac nodes are under the condition of effective T symmetry breaking and P symmetry preserving. The Dirac nodes in momentum space can also be regarded as topological charges. The topological invariant is the winding number, as shown in [32],

\[
w = 1/2\pi \oint_c d\delta k \cdot [\hat{d}_x \nabla \hat{d}_y - \hat{d}_y \nabla \hat{d}_x],
\]

where \( d_{x,y} = d_{x,y}/|\delta k| \), \( d_x = 2t \cos \phi [\cos(k_xa) + \cos(k_ya)] \), \( d_y = 2t \sin \phi [\cos(k_xa) - \cos(k_ya)] \) from Hamiltonian equation (3). Numerical calculation shows that the winding number is \( w = -1 \) and \( w = 1 \) at \( X_1 \) and \( X_2 \) points for \( 0 < \phi < \pi/2 \). We show that the degenerate points correspond to vortex centers in momentum space with non-zero winding numbers. Thus we confirm that the chirality is opposite at the two degenerate nodes of \( X_1 \) and \( X_2 \). The vortex structure in momentum space manifests the topological feature of the nodal points.

Intuitively, we calculate the planar vector field [32] \( \vec{d} = (d_x, d_y) \) in the momentum space as shown in figure 2(d). At a point of \( \vec{k} \), the length of this 2D vector (\( |\vec{d}| \)) characterizes the strength of energy splitting between the two energy bands. For the band degenerate points, this vortex-like vector field possesses topological defects with non-zero winding numbers. At each vortex core, \( \vec{d} \) vanishes and the bandgap closes. The topological structure of these vortices dictates the stability of the band degeneracies against any continuous deformation.

In a two-band model, one must tune the three momentum parameters in order to obtain a band degeneracy in 3D [33, 34], which does not require any other symmetry except the translational symmetry of the crystal itself. However in 2D cases, besides time reversal and parity symmetries, there must be an additional symmetry to protect the degeneracy. These hidden symmetries could be discrete symmetries with antiunitary composite operators consisting of translation, complex conjugation, and sublattice exchange.

In the above acoustic model, the corresponding composite antiunitary symmetry operator is \( U = \sigma_z K T_x \) [35], where \( \sigma_z \), \( K \), and \( T_x \) are Pauli matrix, complex conjugation operator (time reversal operator for spinless particles) and translation operator with translation length \( a \) along x direction. We find that the Hamiltonian equation (2) is invariant under this composite transformation as \( H = U H U^{-1} \). In the Brillouin zone, it can be seen that for operator \( U \), the invariant points are high-symmetry points \( \Gamma \), \( M \) and \( X_{1,2} \). Thus as a result of derivation, we have \( U^2 = 1 \) for \( \Gamma \), \( M \) points and \( U^2 = -1 \) for \( X_{1,2} \) points. For points \( X_{1,2} \) in the Brillouin zone, since the acoustic system is invariant under the antiunitary operator and the square of this operator is not 1, there will be degenerate points protected by this antiunitary operator [35]. These degenerate points of \( X_{1,2} \) are consistent with previous analysis from numerical simulations and the tight-binding model.

Considering the locations of degenerate nodes with opposite chirality in first Brillouin zone, we investigate a ribbon-like acoustic structure that consists of 20 (or 19.5) unit cells as shown in left panel of figure 1(a). It is finite in the \( \vec{x} - \vec{y} \) direction and periodic in the \( \vec{x} + \vec{y} \) and \( \vec{z} \) direction with good quantum number \( \vec{k}_f = (k_x \pm k_y)/\sqrt{2} \) and a fixed \( k_z = \pi/4a \), which means the acoustic waves partly propagate in \( z \) direction. In this case, Dirac nodes with the same chirality will meet at the same projection point. In the following, we show that gapless acoustic structures can exhibit flat edge states [36] against effective T symmetry breaking in equation (3) with fixed \( k_x \). The existence of the flat bands is protected by \( P \) symmetry and effectively particle-hole symmetry in Hamiltonian equation (3) [36–38].

Generally, there are two possibilities to form a termination (an ‘edge’) of this acoustic structure. We select the number of unit cells as \( N = 19.5 \) and \( N = 20 \) to construct the two types of terminations, as shown in figures 3(c) and (e).

For the case of \( N = 19.5 \) unit cells, there will be two, nondispersive, edge states at the degenerate frequency, as shown in figure 3(a). These edge states connect the projected Dirac nodes in momentum \( \vec{k}_f \) from \(-1\) to \(0\) in

\[
\hat{a}_x \nabla \hat{d}_y - \hat{d}_y \nabla \hat{a}_x,
\]
unit of $\pi / \sqrt{2}a$. They are localized at one or the other edge of the ribbon. Figures 3(c) and (d) correspond to the acoustic pressure patterns for the three edge states around frequency 180 Hz, as marked by ‘(c)’ and ‘(d)’ in figure 3(a). (The two degenerate states at ‘(c)’ with $k_{//} = -0.5$ locate at two edges of the ribbon.) For $k_{//}$ near projected Dirac nodes, the edge modes will couple into the bulk bands as shown in figure 3(d).

For the case of $N = 20$ unit cells, we find in figure 3(b) that there is one single flat band of edge states around the frequency 180 Hz. It corresponds to one edge of the ribbon (both edges are B-type resonator) in momentum space from $-1$ to 0, and to the other edge in momentum space from 0 to 1. This single mode is an even–odd effect due to the chiral symmetry [36, 37]. The degeneracy of the edge states at all momentum points including the $T$-invariant points is one, which indicates that $T$ symmetry does not induce Kramers degeneracy for bosonic systems. In figure 3(e), the acoustic pressure pattern shows the localized edge state with $k_{//} = -0.5$ locating at the left edge of the ribbon.

Similar to [36–39], the topological nature of the flat band of edge states indicate their robustness property. To demonstrate this property, we add on-site modulation perturbation. The previous edges are formed by incomplete resonators. Here we add additional parts of resonators beyond the unit cell in figure 1(a), to both edges of the ribbon, such that the edges are formed by complete resonators (figures 4(c) and (e)). As shown in figures 4(a) and (b), the perturbation does not destroy the band of edge states, but only shift it away from the original degenerate frequency. In figure 4(a), there are still two degenerate edge states at each point from momentum $-1$ to 0. The acoustic pressure patterns are demonstrated in figures 4(c)–(e), similar to figures 3(c)–(e). Although the frequencies are lifted away, the existence of edge states is still topologically protected.

In conclusion, we propose a 3D acoustic resonator structure but with a 2D gapless phase, exhibiting Dirac nodes. By introducing a designed synthetic gauge field into the acoustic resonator lattice, we show that the Dirac nodes can exist under the condition of broken effective $T$ symmetry and preserved $P$ symmetry. The topological structure of the Dirac nodes in momentum space ensures the existence of nontrivial edge states, being robust against perturbations. In a 3D point of view, the band structure has quadratic and nodal line degeneracies [40–42], which can be viewed as an intermediate system between conventional metals and Weyl semimetal phases in condensed matter physics in 3D momentum space. Here we mainly focus on the synthetic gauge field and investigate the 2D systems. For experiment, by applying angle-resolved transmission measurement of the acoustic structure that can be 3D printed, it is possible to observe the projected 2D Dirac nodes in the

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**Figure 3.** Flat edge states. (a), (b) Band dispersions of two-type ribbon structures with (a) $N = 19.5$ and (b) $N = 20$ unit cells. The flat bands around frequency 180 Hz indicate the edge states. The acoustic structure is periodic in $\hat{x} + \hat{y}$ direction and finite in $\hat{x} - \hat{y}$ direction. The red points mark the band locations for (c)–(f) acoustic pressure patterns of the edge states.
transmission diagram. Note that nonsymmetric symmetries can also lead to extra degeneracies in fermionic bands \[43\] and protect Weyl nodes \[44\] without inversion \[45\]. The similar principle in acoustic systems of constructing this kind of nontrivial point degeneracy, which is beyond the scope of the current study, may open new opportunities for further exploration of acoustic Weyl nodes in 3D momentum space.

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*Figure 4.* Robust existence of the edge states. (a), (b) Band dispersions of two-type ribbon structures with (a) \( N = 19.5 \) and (b) \( N = 20 \) unit cells. The results show that the flat edge states are shifted away from degenerate frequency around 180 Hz, but on-site modulation perturbations to both edges of the ribbon cannot destroy the edge modes. (c)-(f) The acoustic pressure patterns for edge states.
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