Input-Output Theory with Quantum Pulses

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(Dated: March 1, 2019)

We present a formalism that accounts for the evolution of quantum states of travelling light pulses incident on and emanating from a local quantum scatterer such as an atom or a cavity. We assume non-dispersive asymptotic propagation of the pulses and Markovian coupling of the stationary system to input and output fields. This permits derivation of a cascaded system master equation where the input and output pulses are treated as single oscillator modes that both couple to the local system. As examples of our theory we analyse reflection by an empty cavity with phase noise, stimulated atomic emission by a quantum light pulse, and formation of a Schrödinger-cat state by the dispersive interaction of a coherent pulse and a single atom in a cavity.

Introduction.— Quantum states of light may find applications for precision sensing [1, 2] and as processing or flying qubits in quantum computers and quantum communication networks [3, 4]. While the intuition behind generation of single photon and multi-photon states and demonstration of atom-photon and photon-photon quantum gates portrays the state of light as a superposition of Fock states $|n\rangle$ of a single mode or a few modes, propagating fields in reality explore an infinite continuum of modes which prohibit a full quantum treatment by a Schrödinger picture wave function or density matrix.

If the physical setup contains guided fields and material systems with only a single shared quantum of excitation, the quantum state can be expanded on discrete excited states and one-dimensional photon wave packets — and if the Hamiltonian is time independent, the Schrödinger equation can be solved algebraically in the frequency domain (see, e.g., [5]). The introduction of further quanta of energy, however, complicates matters significantly (for a recent review of theory approaches see Ref. [6]), and while expansion of the field state on the continua of one and two-photon states [7, 8] may be adequate to describe many processes relevant to quantum information processing [9–10], a more general and more tractable theory is desired. A cascaded system approach [11] and the so-called Fock master equation approach [12] permit evaluation of the quantum state of a quantum system which is driven by a quantum pulse in a superposition of Fock states, and while these methods allow calculation of the mean values and correlation functions of the fields emitted from the system, they do not provide the quantum state of the output field.

The emission from a quantum system may not be restricted to a single mode, but we may choose to examine any particular propagating wave packet and consider the quantum state occupying just that mode after the interaction with the quantum system. Our theory thus accounts for the kind of experiment depicted in Figure 1(a), where a dispersion free wave packet is incident on an arbitrary quantum system, which we assume can be adequately described in a Hilbert space of finite dimension $d$.

The quantum state of a suitably defined outgoing wave packet is precisely the information retained by typical quantum communication or computing protocols, while the radiation which is not captured by that mode represents loss. The restriction of the dynamics from the infinite continuum to only two field modes reduces the infinite dimensional Hilbert space to one of dimension $\mathcal{N} = (N + 1) \times d \times (M + 1)$, where $N$ and $M$ are the maximum number of excitations in the incoming and outgoing modes. Our full quantum description amounts thus to a master equation for an $\mathcal{N} \times \mathcal{N}$ density matrix $\rho$.

Theory. — In Figure 1(a), a quantum system described by a possibly time-dependent Hamiltonian $\hat{H}_s$ is coupled to an input bosonic field $\hat{b}_{\text{in}}(t)$ by an interaction ($\hbar = 1$) $\hat{V}_{SB}(t) = i\sqrt{\gamma}[(\hat{c}\hat{b}_{\text{in}}(t) - \hat{c}^\dagger\hat{b}_{\text{in}}(t))]$ where
\( \hat{c} \) is a system operator. If \( \hat{c} \) is a lowering operator, \( \gamma \) is the corresponding decay rate of excitations in the system, and the outgoing field after interaction with the system is given by the input-output operator relation,

\[
\hat{b}_{\text{out}}(t) = \hat{b}_{\text{in}}(t) + \sqrt{\gamma} \hat{c}(t) \tag{1}
\]

Direct application of this expression requires knowledge of the time dependent system operator \( \hat{c}(t) \) in the Heisenberg picture which is only available if \( \hat{H}_s(t) \) is sufficiently simple (e.g., quadratic in oscillator raising and lowering operators \( \hat{c} \) and \( \hat{c}^\dagger \)).

Since we shall treat the case of a multi-mode vacuum input except for a quantum state occupying a single normalized wave packet \( u(t) \), it is natural to seek a quantum description of the input by the Fock states related to a single bosonic annihilation operator

\[
\hat{b}_u = \int dt \, u(t) \hat{b}(t). \tag{1}
\]

Similarly, we shall consider only the output radiation from the system that is carried by a particular outgoing mode function \( v(t) \), as sketched in Figure 1(a).

We introduce virtual one-sided cavities as illustrated in Figure 1(b), with time-dependent complex coupling \( v \int [g^{\dagger}(t) \delta_{in} - g(t) \delta_{in}^\dagger] \) to their respective input continuum fields such that the field ejected from an eigenmode of the first cavity occupies the temporal mode \( u(t) \) and such that the field occupying the mode \( v(t) \) is transferred into an eigenmode of the second cavity. We stress that these cavities are a purely mathematical tool to model the physical dynamics, why we do not have to be concerned about experimental realization of the complex temporal control of the respective \( g_u(t) \) and \( g_v(t) \).

If a single mode cavity is coupled to an input field with amplitude \( g(t) \), the quantum Langevin equation for the field annihilation operator \( \hat{a} \) reads

\[
\dot{a} = -\frac{\kappa(t)}{2} a - g(t) \hat{b}_{\text{in}}(t), \tag{2}
\]

where we assume a rotating frame around the carrier frequency of the field mode and note that if \( g(t) \) varies slowly compared to the continuum field spectral range, the Born-Markov approximation yields the time dependent cavity decay rate \( \kappa(t) = |g(t)|^2 \). The general solution for the intra-cavity field reads

\[
\hat{a}(t) = e^{-\frac{1}{2} \int_0^t dt' \kappa(t')} \hat{a}(0) - \int_0^t dt' g(t') e^{-\frac{1}{2} \int_0^t dt'' \kappa(t'')} \hat{b}_{\text{in}}(t'), \tag{3}
\]

and with a vacuum input with \( \langle \hat{b}_{\text{in}} \rangle = 0 \), the amplitude of the outgoing field from the cavity is given by the first factor in Eq. (3) multiplied with the complex coupling coefficient,

\[
\langle \hat{b}_{\text{out}}(t) \rangle = g(t) e^{-\frac{1}{2} \int_0^t dt' \kappa(t')} \langle \hat{a}(0) \rangle. \tag{4}
\]

The outcoupling from the cavity is a linear process, and the initial intracavity quantum state is transferred to a travelling mode given by the time dependent function in Eq. (4). This function equals \( u(t) \) if \( g_u \) assumes the same phase as \( u(t) \), \( g_u(t) = \frac{u(t)}{|u(t)|} \sqrt{\kappa_u(t)} \) and

\[
\kappa_u(t) = \frac{|u(t)|^2}{1 - \int_0^t dt' |u(t')|^2}. \tag{5}
\]

A similar result, applying a real coupling coefficient and a time dependent cavity detuning, was derived in [15].

To capture the quantum state occupying the wave packet \( v(t) \), we assume a complete asymptotic decay of the initial cavity amplitude in the second virtual cavity, i.e., that the factor on the first term in Eq. (3) vanishes and that the integral over the input field in the second term has the desired temporal weight factor. From Eq. (1) with \( u \rightarrow v \), this yields the condition

\[
v(t) = -g_u(t) e^{-\frac{1}{2} \int_0^t dt' \kappa_u(t')}. \tag{6}
\]

Due to the assumption of Markovian coupling to the continuous field degrees of freedom and dispersion free, undistorted propagation of the wave packets, we can eliminate the propagation segments and apply the cascaded system analysis by Gardiner [19] and Carmichael [20] to obtain a master equation that involves only the quantum states of the intermediate quantum system and the field states of the two cavity modes, represented by field operators \( \hat{a}_u \) and \( \hat{a}_v \).

The equations of motion for the density matrix of such cascaded systems are derived in [19, 20] and they can be obtained in a systematic manner in the so-called SLH framework [21, 22], by concatenating the Hamiltonians and damping terms according to the routing of output from one system into another. The density matrix of the total system evolves according to a master equation on the Lindblad form,

\[
\frac{d\rho}{dt} = -i[\hat{H}, \rho] + \sum_{i=0}^2 \left( \hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2} \left\{ \hat{L}_i^\dagger \hat{L}_i, \rho \right\} \right), \tag{7}
\]

where \( \{ \cdot, \cdot \} \) denotes the anticommutator, and the Hamiltonian

\[
\hat{H}(t) = \hat{H}_s(t) + i \frac{\hbar}{2} \left( \sqrt{\gamma} g_u(t) \hat{a}_u^\dagger \hat{c} + \sqrt{\gamma} g_v(t) \hat{a}_v^\dagger \hat{c} + g_u(t) g_v(t) \hat{a}_u^\dagger \hat{a}_v - \text{h.c.} \right) \tag{8}
\]

contains terms that represent coherent exchange of energy between the three components.

The damping terms in Eq. (7) include a single Lindblad operator of the form,

\[
\hat{L}_0(t) = \sqrt{\gamma} \hat{c} + g_u(t) \hat{a}_u^\dagger + g_v(t) \hat{a}_v, \tag{9}
\]
representing the output loss from the last cavity, as well as operators $L_i$ with $i > 0$, representing additional decay and loss mechanisms of the quantum scatterer. The solution to (7) yields the density matrix of the joint system of the incoming mode, the local scatterer and the outgoing mode, giving direct access to a full quantum state description of the output mode and of the potentially entangled state of the scatterer and the output mode. Our theory thus goes far beyond the study of expectation values and low order correlation functions of the output field operators. Next, we shall present a few examples of our formalism. Numerical solutions to the master equation (7) are obtained using the QuTiP toolbox [23, 24].

**Examples.** As a first example of our formalism, we consider the scattering on an empty, one-sided cavity with resonance frequency $\omega_r$. The local system Hamiltonian is $H_s = \omega_r \hat{c} \hat{c}^\dagger$ and scattering with coupling amplitude $\sqrt{\gamma}$ of the input field to the cavity field is described by a frequency dependent reflection coefficient $r(\omega) = \frac{[i(\omega - \omega_r) + \gamma/2]/[i(\omega - \omega_r) - \gamma/2]}. That is, the Fourier transformed pulse shapes obey

$$v(\omega) = r(\omega)u(\omega). \tag{10}$$

In the upper panel of Figure 2 we show how a real Gaussian pulse $u(t)$ is reflected into a mode $v(t)$ which is also real but has a sign change around the time $\tau \gamma t = 4$. The corresponding cavity loss rates $\kappa_u(\omega)$ are shown in the same panel. The lower panel shows how the average photon number in the input, cavity and output modes change with time for an initial one-photon Fock state in the input pulse. We emphasize that the complete transfer is allowed by the unitary and linear transfer to a known output mode. If we perform the calculation with any other choice of output mode, the transfer will be imperfect.

We observe that the radiative output (loss) from the last cavity has the intensity $I_{\text{out}} = \langle \hat{L}_0^\dagger \hat{L}_0 \rangle$ with the Lindblad operator $\hat{L}_0$ given in Eq. (9). As a first example of a system that scatters a single input pulse into a multi mode output, we consider phase noise in the system, e.g., due to a jittering of one of the cavity mirrors on a timescale $\tau_{\text{jit}}$. This imposes an additional Lindblad term $\hat{L}_1 = \tau_{\text{jit}}^{-1/2} \hat{c} \hat{c}^\dagger$ in the master equation (7) (see Ref. [24] for an extended discussion of this model) but poses no problem for our numerical solution of the problem.

In Figure 3 we present calculations for the same input and output modes as in Figure 2 with the incoming pulse prepared in a coherent state (\alpha = 2). The phase noise causes an imperfect transfer to the examined output mode $\langle x \rangle$ and a corresponding output flux $I_{\text{out}}(t)$ from the final virtual cavity at intermediate times. At the same time, the insert emphasizes that the quantum state of the outgoing mode is not merely a coherent state as it would be in the absence of phase noise. Instead it may be described as a statistical mixture of coherent states with a reduced amplitude, rotated by a range of different complex phases.
This is verified by our formalism in Figure 5 where for $\Gamma = 0.98$ and for $\alpha = 1.4$ and 2.0 and assume the parameters given in Ref. 28, $(g, \gamma, \Gamma, \kappa_{\text{out}}) = 2\pi \times (7.8, 2.3, 3.0, 0.2)\text{MHz}$. Left panel: Fidelity with the state (11) as a function of time. For comparison, we show also the fidelity, assuming ideal conditions with no decoherence ($\Gamma = \kappa_{\text{out}} = 0$). Color plots: Wigner function of the outgoing pulse after a $\pi/2$ spin rotation and post selection on detection of the atom in the state $|\uparrow\rangle$. According to Eq. (11), this procedure should ideally produce an even cat state, $(|\alpha\rangle + |-\alpha\rangle)/\sqrt{2}$ in the outgoing mode.

The characteristic signature of a Schrödinger cat state emerges in the output pulse. Parameters and details concerning the Wigner functions are given in the figure caption.

**Outlook.** — The formalism presented in this letter provides, in a straightforward manner, a full quantum description of a light pulse reflected by a quantum system into a possibly distorted mode function. Our theory applies equally well to light and other (dispersion free) carriers of quantum states such as microwaves and surface acoustic waves, considered in recent experiments 11 [28, 31, 36] and experimental proposals 10 [15, 37, 42].

We illustrated our theory by the solution of the cascaded master equation for the input and output field Fock space density matrices, but the theory may also employ Heisenberg picture and phase space approaches. Similarly, quantum trajectory analyses of heralded or conditional dynamics have been proposed 6 [20, 43] and follow effortlessly from our formalism.

As our examples illustrate, an ill-chosen output mode presents a loss and an impediment to retrieve the desired quantum state. To optimize the mode function $v(t)$, we may first calculate the emitter autocorrelation function $g^{(1)}(t,t') = \langle \hat{L}_1^\dagger(t)\hat{L}_2(t') \rangle$, where $\hat{L}_1 = g_n(t)\hat{a}_n(t) + \sqrt{\kappa_{\text{out}}}\hat{e}(t)$, from the cascaded master equation of the input cavity and quantum system. If the emission occurs into a single mode, the correlation function factorizes and $g^{(1)}(t,t') \propto v^*(t)v(t')$, while in the general case, an expansion $g^{(1)}(t,t') = \sum \lambda_i v_i^*(t)v_i(t')$ may be used to identify a few dominant modes for which the output quantum state can be calculated by a few-mode extension of our theory.

The chiral coupling and spatial separation of input and output fields in Figure 4 may be achieved by various means for single sided cavity systems, while two-sided cavities
should be described by two (reflected and transmitted) output modes, and more complex interferometric setups with multiple input and output ports may explore an even larger number of modes [11].

The authors acknowledge support from the European Union FETFLAG program, Grant No. 820391 (SQUARE), and the U.S. ARL-CDQI program through cooperative Agreement No. W911NF-15-2-0061.

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