Entropy Generation at the Horizon Diffuses Cosmological Constant in 2d de Sitter Space

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Abstract

We investigate a solution of the exactly renormalized Liouville action to foresee the fate of the 2-dimensional de Sitter space. We work in the semiclassical region with a large matter central charge $c$. Instead of de Sitter expansion, it performs a slow roll inflation with the parameters $\epsilon = (1/2)\eta = 6/c$. An inflaton field is induced in the effective theory to describe quantum effects of the Liouville theory. The geometric entropy increases logarithmically with the Hubble radius. We propose that de Sitter entropy is carried by super horizon modes of the metric. It can be directly estimated from the partition function as $S = \log Z$ in Liouville gravity. We formulate a gravitational Fokker-Planck equation to elucidate the Brownian process at the horizon: the super horizon modes are constantly jolted by new comers. We show that such a built-in entropy generating process diffuses the cosmological constant. We evaluate von Neumann entropy associated with the distribution function of super horizon modes. It always increases under Fokker-Planck equation in a consistent way with semiclassical estimates. The maximum entropy principle operates in quantum gravity. An analogous entropy production mechanism at the horizon might have increased the Hubble radius much beyond the microscopic physics scale in the universe.

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1 Introduction

It is often suspected that the cosmological constant problem is an infra-red problem. de Sitter space is unstable due to some shielding effects or particle productions [1]. In particular the importance of IR logarithmic effects is stressed to break de Sitter symmetry [2]. A stochastic approach is proposed to sum up leading IR logarithms [3,4]. In order to understand physics in 4-dimensional de Sitter space, 2-dimensional exactly solvable models may be instructive. In this paper we investigate 2-dimensional de Sitter space in Liouville quantum gravity. It is realized as a classical solution in the semiclassical regime with large matter central charge. By examining the exact solution, we note that the negative anomalous dimension of the cosmological constant operator makes the Hubble parameter time dependent and it fades away. The shielding effect of the cosmological constant arises due to the negative sign of the kinetic term of the conformal mode. IR logarithmic effects are important to make the Hubble parameter time dependent. We estimate semiclassical geometric entropy of the 2-dimensional de Sitter space. It grows logarithmically with the inverse Hubble parameter $H$ like $S \sim \log(1/H)$. We propose it is carried by the super horizon mode of the conformal degrees of metric. We investigate its distribution function by Fokker-Planck equations. We offer evidences that the von Neumann entropy of the distribution function can be identified with the geometric entropy of the space.

The presence of event horizons leads to thermodynamics of black holes [5]. Gibbons and Hawking showed that analogous relations hold in de Sitter space with cosmological horizons [6]. The area of the cosmological horizon is

$$A_0 = 4\pi l^2,$$

where the surface gravity is $\kappa = 1/l$. A short summary of de Sitter space thermodynamics is given in Appendix A.

The metric of de Sitter space in the global coordinate is

$$\frac{ds^2}{l^2} = -d\tau^2 + \cosh^2(\tau)d\Omega^2_3,$$

while it becomes conformally flat in the local Poincaré patch

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2d\Omega^2_2) = \left(\frac{1}{Ht}\right)^2(-d\tau^2 + dx^2).$$

The metric is periodic under the shift of time into the imaginary direction by $2\pi l$. The Green’s functions must have the identical periodicity. It implies that the temperature of the cosmological horizon is

$$T_{\text{dS}} = \frac{1}{2\pi l}.$$
We may rotate the de Sitter space in the global coordinates \((1.2)\) into Euclidean \(S^4\) with the metric
\[
\frac{ds^2}{l^2} = d\chi^2 + \sin^2(\chi)d\Omega_3^2.
\] (1.5)

The Euclidean action on \(S^4\) with the radius \(1/H\) is
\[
\int d^4x \sqrt{g} \frac{1}{8G_N} \left( \frac{1}{2} \hat{R} - 3H^2 \right) = \int d^4x \sqrt{\hat{g}} \frac{3H^2}{8G_N} = \frac{8\pi}{3} \frac{3H^2}{8G_N}.
\] (1.6)

The classical action gives us the entropy of de Sitter space once we put the solutions in it. In quantum gravity, the size of the space is a dynamical variable to be integrated over. In general the partition function \(Z\) must be stationary with respect to the change of size \(l\) or inverse temperature \(\beta = 2\pi l\) of the manifold
\[
\frac{\partial}{\partial \beta} \log Z = 0 \Rightarrow S = \left( 1 - \beta \frac{\partial}{\partial \beta} \right) \log Z = \log Z.
\] (1.7)

At the semiclassical level, the entropy of de Sitter space is given by
\[
S_0 = \frac{A_0}{4G_N} = \frac{\pi}{H^2 G_N}.
\] (1.8)

The Euclidean quantum gravity represents a possible equilibrium state with the temperature set by the scale of event horizon. It may have instability since the Einstein action is not bounded below. The kinetic term of the scale factor of the metric (conformal mode) is of the wrong sign. There is no fundamental remedy for this problem and we mostly work with Lorentz signature. The situation is better with respect to the super horizon modes as the potential term dominates over the kinetic term. The thermodynamics of super horizon modes may be studied in Euclidean gravity.

Let us start our investigation on 2d quantum gravity which has de Sitter space as a classical solution at the tree level. We focus on the dynamics of the conformal degrees of the metric \(\phi\) by choosing the conformal gauge as follows
\[
g_{\mu\nu} = e^{\phi} \hat{g}_{\mu\nu}.
\] (1.9)

In this expression, \(\hat{g}_{\mu\nu}\) denotes the classical background of the metric. In 2d quantum gravity we work with the Liouville action which is the gift of conformal anomaly
\[
\frac{Q^2}{4\pi} \int d^2x \sqrt{\hat{g}} \left( \frac{1}{4} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \phi \hat{R} - H^2 e^\phi \right).
\] (1.10)

The equation of motion with respect to constant \(\phi\) is
\[
\hat{R} = 2H^2 e^\phi.
\] (1.11)

\(^{\dagger}\) A short summary of the derivation of the Liouville action is given in Appendix A.
The 2-dimensional symmetric space $S^2$ is the solution with the scale $e^\phi = 1/H^2$. By rotating 2-dimensional de Sitter space into $S^2$, we obtain semiclassical estimate of the geometric entropy

$$\frac{Q^2}{8\pi} \int d^2x \sqrt{g}(\phi \hat{R} - \hat{R}) = Q^2 \log \left(\frac{1}{H^2}\right). \quad (1.12)$$

This should be contrasted with (1.8) in 4-dimensions. We note that the inverse Newton’s coupling constant is replaces by $Q^2 = (c - 25)/6$ in 2-dimensions. The matter central charge $c$ must be larger than 25 to correspond with positive Newton’s coupling. It implies the sign of the kinetic term of conformal mode is negative. We work in this semiclassical region. It also depends on the Hubble’s parameter $H^2$. Although the entropy increases if $H^2$ decreases in the both cases, its dependence is weak in 2-dimensions: $\log(1/H^2)$ while much stronger in 4-dimensions: $1/H^2$.

Nevertheless, this toy model may reveal us what carries the entropy of de Sitter space [7,8]. Furthermore we identify a mechanism to create entropy as more and more degrees of freedom are going out of the horizon. Simultaneously the cosmological constant is diffused away. Since we find the common prerequisite between 2 and 4-dimensional de Sitter spaces for such a mechanism to work, the cosmological constant $\Lambda$ may have decreased far beyond the scale of the microscopic physics in an analogous mechanism. Certainly explaining why the dimensionless ratio $H^2G_N \sim 10^{-120}$ is so small is a difficult problem. However the most important step in solving the problem is to correctly formulate it. It may be a more appropriate question to ask why current universe has such a huge entropy. As the above equation (1.8) shows, the entropy of 4-dimensional de Sitter space is inversely proportional to the Hubble parameter $1/H^2$. Since the entropy always increases, the Hubble parameter must decrease especially when there are huge entropy creations in the universe [9]. Such a simple trend seems to capture the essence of the history of the universe.

This concludes our introductory section to this paper. In Section 2, we investigate a solution of exactly renormalized Liouville action. 2-dimensional de Sitter space appears as a solution of nonrenormalized Liouville action. Fortunately, the effects of short distance divergences are known exactly. In 2-dimensions UV and IR divergences are closely related since the propagators do not change unlike in 4-dimensions. We can thus predict IR behavior from the UV behavior. In the semiclassical region with a large matter central charge $c$, the scaling dimension $\gamma$ of the cosmological constant operator is less than canonical $\gamma < 1$. In such a case, we show de Sitter expansion becomes slow roll inflation with the slow roll parameters inversely proportional to $c$. In Section 3, we propose that de Sitter entropy is the von Neumann entropy of super horizon modes. We show the entropy is generated at the rate $\dot{S} = 2H$ by the stochastic equations which diffuses cosmological constant. We conclude in Section 4. Basic information are summarized in 3 appendices for self-containedness: A on de Sitter thermodynamics, B on operator renormalization in Liouville theory and C on stochastic equations.

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§There is a quantum correction to this estimate due to anomalous dimension of the cosmological constant operator. $Q^2$ should be replaced by $Q^2/\gamma$. 

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2 Solutions of 2d de Sitter quantum gravity

In this section we investigate the quantum IR effects in an exactly solvable model: 2-dimensional quantum gravity. We adopt a conformal gauge and parametrize the metric as

\[ g_{\mu\nu} = e^\phi \hat{g}_{\mu\nu}, \]  

(2.1)

where \( \hat{g}_{\mu\nu} \) is a background metric. The metric is Lorenzian corresponding to real spacetime. The effective action for the conformal mode \( \phi \) is the Liouville action:

\[ \int \sqrt{-\hat{g}} d^2x \left\{ \frac{c-25}{96\pi} (\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2\phi \hat{R}) - \Lambda e^{\gamma\phi} \right\}. \]  

(2.2)

Here \( c \) denotes the central charge of the matter minimally coupled to 2-dimensional quantum gravity. In the free field case, \( c \) counts massless scalars and fermionic fields as \( c = N_s + N_f/2 \).

We consider the semiclassical regime: \( c > 25 \) where the sign of the kinetic term for the conformal mode is negative and hence time-like. It is the identical feature with 4-dimensional Einstein gravity. In the above expression \( \Lambda \) is the cosmological constant, \( e^{\gamma\phi} \) is a renormalized cosmological constant operator and \( \gamma - 1 \) denotes the anomalous dimension.

The equation of motion with respect to \( \phi \) is given by

\[ \frac{Q^2}{8\pi} \nabla^2 \phi + \Lambda e^\phi = 0, \]  

(2.3)

where we put the background scalar curvature \( \hat{R} = 0 \) anticipating to adopt conformally flat coordinates. \( Q^2 \) is the effective inverse Newton’s coupling in 2-dimensions and we also put \( \gamma \to 1 \). It is allowed in the large \( c \) limit and the classical geometry holds. In contrast, geometry is quantized when \( \gamma < 1 \).

Furthermore there is another equation of motion with respect to the traceless mode of the metric \( h_{\mu\nu} \):

\[ \frac{Q^2}{8\pi} (\nabla_\mu \phi \nabla_\nu \phi - 2\nabla_\mu \nabla_\nu \phi) = \nabla_\mu f \nabla_\nu f, \]  

(2.4)

where \( f \) denotes a free scalar field and the presence of the \( \phi \hat{R} \) term in the Liouville action results in the linear term in \( \phi \) on the left-hand side.

For the conformally flat metric \( g_{\mu\nu} = e^\phi \eta_{\mu\nu} \), the action (2.2) is simplified as

\[ \int d^2x \left\{ \frac{Q^2}{16\pi} (\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4H^2 e^\phi) \right\}. \]  

(2.5)

The effective Newton’s coupling constant \( G_N \) becomes small when \( c \) becomes large like \( 1/Q^2 \sim 6/c \). We confine our investigation of this model in the semiclassical region \( c > 25 \).
The kinetic term of the conformal made \( \phi \) is negative as we can see in (2.5). This feature is shared with 4-dimensional Einstein gravity. The expansion of the universe occurs due to such an instability. For theorists, the wrong sign of the kinetic term of \( \phi \) looks like a curse against going beyond semiclassical investigations. On the contrary it could be a blessing as we demonstrate it by investigate 2-dimensional toy model in this paper. It provides a mechanism to grow the universe so vast in comparison to the microscopic physics scale. Namely the cosmological constant is shielded by quantum fluctuations of the conformal mode \( \phi \). The shielding occurs as more entropy is generated at the cosmological horizon as conformal zero mode \( \phi_0 \) accumulates at the horizon. They perform Brownian motion as being constantly jolted by new comers. The negative sign kinetic term of \( \phi \) is crucial for shielding the cosmological constant. Note that the Hubble parameter \( H^2 \) is reduced by \( Q^2 \) when we fix the cosmological constant \( \Lambda \)

\[
H^2 = G_N \Lambda = \frac{4\pi}{Q^2} \Lambda. \tag{2.6}
\]

With large \( c \), matter fields reduce \( H^2 \) by a large factor as \( H^2 \sim \frac{24\pi}{c} \Lambda \).

The classical solution of this action (2.5) is 2-dimensional de Sitter space with the metric

\[
\phi_c = -2 \log(-H\tau) = \log a^2(t), \quad e^{\phi_c} = \left(\frac{1}{H\tau}\right)^2 = e^{2Ht}. \tag{2.7}
\]

By adopting the Poincaré coordinate, we obtain the following metric as a solution of the classical Liouville theory:

\[
ds^2 = \left(\frac{1}{-H\tau}\right)^2(-d\tau^2 + dx^2) = -dt^2 + e^{2Ht} dx^2. \tag{2.8}
\]

It covers the upper half triangle of the Penrose diagram of the global de Sitter manifold. The volume operator is of the identical form with the metric in accord with the classical geometry. We identify cosmic time with the classical solution of the conformal mode \( \phi_c(t) = 2Ht \). The distance of the cosmic horizon from the observer is \( 1/H \). She or he is situated at the center of the line segment. On the other hand if the cosmological constant can be neglected, we also have a solution with a non-trivial free matter field \( f \):

\[
\phi_c = A\tau, \quad f_c = A\sqrt{\frac{Q^2}{8\pi}}\tau, \tag{2.9}
\]

where \( A \) is an arbitrary constant. This is a 2-dimensional Friedmann spacetime. This solution should go over to the 2-dimensional dS space solution when the cosmological constant becomes dominant. Another solution is obtained by adding gas of massless scalar particles with temperature \( T \) to empty de Sitter space. We may solve (2.4) to linear order in perturbation \( \phi_c + \phi \) as

\[
\frac{Q^2}{4\pi} (\nabla_0\phi_c\nabla_0\phi - \nabla_0\nabla_0\phi) = 2(\nabla_0 f\nabla_0 f). \tag{2.10}
\]
Specifically,
\[
\frac{Q^2}{4\pi} \left( -\tau \nabla_0 \phi - \nabla_0 \nabla_0 \phi \right) = \frac{\pi}{3} T^2 \Rightarrow \phi = -\frac{\pi^2}{15Q^2} \frac{T^2}{H^2 a^2(t)}.
\] (2.11)

Since its energy density decays like \( T^2/a^2(t) \) with the expansion of the universe, it fades away in comparison to the cosmological constant at a late time. However their contribution to entropy remains constant as can be seen from (1.12). They contribute \( c = 1 \) to the coefficient of the Liouville Lagrangian. In this way the massless fields leave their legacy in reducing the Hubble parameter.

We expand the action around the classical background \( \phi_c + \phi \),
\[
\int d^2 x \frac{Q^2}{8\pi} \left\{ -\frac{1}{2} \left( 1 + h^{00} \right) \frac{\partial}{\partial \tau} \phi_c \frac{\partial}{\partial \tau} \phi_c + h^{00} \frac{\partial^2}{\partial \tau^2} \phi_c - 2H^2 e^{\phi_c} \right\}
+ \int d^2 x \frac{Q^2}{8\pi} \sqrt{-\hat{g}} \left\{ \frac{1}{2} \hat{g}^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi \frac{\partial}{\partial x^\nu} \phi + (1 + \phi) \hat{R} - 2H^2 e^\phi \right\},
\] (2.12)

where
\[
\sqrt{-\hat{g}} = e^{\phi_c}, \quad \sqrt{-\hat{g}} \hat{R} = \frac{\partial^2}{\partial \tau^2} \phi_c = 2H^2 \sqrt{-\hat{g}}.
\] (2.13)

Note that we have recovered the Liouville action on the de Sitter space (2.2) for quantum fluctuations. We then renormalize the cosmological constant operator as we review it briefly in Appendix B. The results of the investigations from various viewpoints agree that the cosmological constant operator is renormalized as
\[
e^{\phi} \rightarrow e^{\gamma \phi}.
\] (2.14)

At the short distance limit the volume operator is of the original form \( e^\phi \). It is renormalized to become \( e^{\gamma \phi} \) in the IR limit under the renormalization group equation (B.18). It is a diffusion equation which plays a crucial role in this work. The both conformal invariance and renormalization group arguments show that the scaling dimension satisfies the following relation
\[
\gamma + \gamma^2 \frac{Q^2}{2} = 1.
\] (2.15)

By solving this equation, the scaling dimension of the cosmological constant operator is determined to all orders
\[
\gamma = \frac{2}{1 + \sqrt{1 + \frac{Q^2}{2}}} = 1 - \frac{1}{Q^2} + 2 \left( \frac{1}{Q^2} \right)^2 + \cdots.
\] (2.16)

The important feature is that \( \gamma < 1 \) namely smaller than the canonical dimension in the semiclassical region.

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* This is essentially the entangled entropy. There are important quantum corrections to this classical formula which is the subject of this work.
Since the cosmological constant operator changes after the renormalization, we may look for a new classical solution of the following type action

\[
\int d^2x \left\{ \frac{Q^2}{16\pi} (\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{4H^2}{\gamma} e^{\gamma \phi}) \right\}.
\] (2.17)

It is apparent from the above action in comparison to nonrenormalized one (2.5) that a reinterpretation \( \phi \) as \( \gamma \phi \) may give us a solution of the renormalized action (2.17).

However such a candidate cannot satisfy the equation of motion with respect to \( h^{00} \) (2.4) since it is nonlinear and does not allow a simple scaling transformation. In order to circumvent this problem, we introduce an inflaton field \( f \) in such a way that (2.4) is satisfied. We argue that it is a standard strategy to enlarge field spaces in order to solve the highly nonlinear problems.

Let us postulate the following Lagrangian

\[
\int d^2x \frac{Q^2}{8\pi} \sqrt{-g} \left\{ -\frac{1}{2} \gamma \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \left( \gamma \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2 \hat{R} \phi \right) - \frac{2H^2}{\gamma} e^{\phi-(1-\gamma)f} \right\}.
\] (2.18)

The equations of motion with respect to the inflaton \( f \) and the conformal mode \( \phi \) are

\[
\nabla^2_0 \gamma f = 2H^2 e^{\phi-(1-\gamma)f}, \quad \nabla^2_0 \gamma \phi = 2H^2 e^{\phi-(1-\gamma)f},
\] (2.19)

where we assume \( \hat{R} = 0 \) in a conformally flat gauge. We can identify \( f = \phi \) and the right hand side of (2.4) can be supplied by the inflaton field \( f \) since they coincide exponentially fast with cosmic time.

After putting the inflaton field under the rug with the identification \( f = \phi \), we obtain the following action

\[
\int d^2x \frac{Q^2}{8\pi} \sqrt{-g} \left\{ \gamma \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \hat{R} \phi - \frac{2H^2}{\gamma} e^{\phi} \right\}.
\] (2.20)

We regard this theory as an effective field theory equivalent to fully quantized Liouville theory. In other words, a solution of 2d quantum gravity on de Sitter type space. We have hence no intension to quantize this theory. On the other hand, the quantum effects of \( \phi \) on the Hubble parameter can be explained by the classical motion of inflaton \( f \) (2.45) at the weak coupling or slow roll limit

\[
H^2(t) = H^2 e^{-(1-\gamma)f(t)} \sim H^2 \exp \left( -\frac{1}{Q^2} 2Ht \right).
\] (2.21)

After the rescaling the fields \( \gamma \phi \rightarrow \phi \), our action takes the following form

\[
\int d^2x \frac{Q^2}{8\pi} \left( -\frac{1}{2} \frac{\partial}{\partial \tau} \phi_c \frac{\partial}{\partial \tau} \phi_c - 2H^2 e^{\phi_c} \right).
\] (2.22)
+ \int d^2x \frac{Q^2}{8\pi} \sqrt{-\hat{g}} \{ \frac{1}{2} \hat{g}^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi \frac{\partial}{\partial x^\nu} \phi + \phi \hat{R} - 2H^2(e^\phi - 1) \}. \quad (2.23)

It is nothing but the nonrenormalized action and its solution is the original one (2.7). The important difference is that $Q^2 = Q^2/\gamma$ appears in the effective inverse Newton’s coupling $G_N$. In the end the solution of the renormalized action can be obtained from the nonrenormalized one by a simple scaling $\phi_c \to \gamma \phi_c$. The geometric objects are defined by new solutions after $\phi_c$ is reinterpreted as $\gamma \phi_c$. We confirm that this action explains the scaling relations between the Newton’s coupling $G_N$ and the Hubble parameter $H^2$ correctly:

$$\sqrt{-\hat{g}} = e^{\gamma \phi_c}, \quad \sqrt{-\hat{g}} \hat{R} = \frac{\partial^2}{\partial \gamma^2} \gamma \phi_c = 2H^2 \sqrt{-\hat{g}}. \quad (2.24)$$

It is important to recognize that the effective inverse Newton’s coupling $Q^2$ is replaced by $Q^2/\gamma$. This fact implies that the physical scale has changed by the factor $\gamma$. We thus believe this recycling an old solution as a new one is not vacuous but a scale transformation.

We can read off the scaling relations between the Hubble parameter and the topological coupling $G_T$ in front of scalar curvature $\phi \hat{R}$ term as we explain below. It is identical to Newton’s coupling $G_T = G_N$. It has been useful to consider the response of the action under $\phi_c \to \phi_c - \varphi$ and $\phi \to \phi + \varphi$. Of course the action is invariant if $\varphi$ is a local conformal transformation since we start with (2.5). In fact conformal invariance has been an effective tool to determine the renormalized Liouville action [10, 11].

The action (2.23) implies that the semiclassical entropy of 2-dimensional de Sitter space is

$$S_c = \frac{Q^2}{\gamma} \varphi = \frac{Q^2}{\gamma} \log \frac{H_0^2}{H^2(t)} = \frac{1}{G_N} \log \frac{H_0^2}{H^2(t)}. \quad (2.25)$$

The renormalization of the cosmological constant operator has introduced $\gamma$ dependence. It depends not only on the gravitational coupling but also on the Hubble parameter. We can predict the relative scaling relation between Hubble parameter $H^2$ and effective Newton’s coupling from (2.23). Let us assume that the Hubble parameter changes slowly over cosmic time evolution. We may postulate

$$H^2(t) = H_0^2 e^{-\varphi}. \quad (2.26)$$

We consider a constant shift of the quantum field

$$\phi(x) \to \phi(x) + \varphi, \quad (2.27)$$

$$\frac{Q^2}{8\pi\gamma} \int d^2x \sqrt{-\hat{g}} (\phi \hat{R} - 2H^2(t)e^\phi) \to \frac{Q^2}{8\pi\gamma} \int d^2x \sqrt{-\hat{g}} \{ (\phi + \varphi) \hat{R} - 2H_0^2 e^\phi \}. \quad (2.28)$$

The action changes as

$$i \frac{Q^2}{8\pi\gamma} \int_{dS^2} d^2x \sqrt{-\hat{g}} \varphi \hat{R} \to \frac{Q^2}{8\pi\gamma} \int_{S^2} d^2x \sqrt{\hat{g}} \varphi \hat{R} = \frac{Q^2}{\gamma} \varphi. \quad (2.29)$$
In the last step, we have compactified $dS^2$ into $S^2$.

The coefficient $Q^2/\gamma$ in front of the $\hat{R}$ term can be regarded as an effective inverse topological coupling $1/G_T$. It is equal to Newton’s coupling $G_N = G_T$. This coupling plays an important role in our estimation of the semiclassical entropy of 2-dimensional de Sitter space.

As is well-known, quantum gravity has conformal invariance due to the ambiguity in how to separate fields between the background and fluctuations. Since scale invariance is a part of the symmetry, it may not be very surprising to construct a new solution by a scale transformation. To be precise, we have solved the model with an inflaton at the classical level which reproduces many features of the solution of exactly renormalized 2d quantum gravity on de Sitter type space. The introduction of an inflaton is necessary to satisfy the equation of the motion with respect to the traceless tensor $h^{\mu \nu}$. We suspect nature also adopts a similar trick. In fact the classical motion of an inflaton reproduces the quantum effects of $\phi$ in the weak coupling limit.

Since the rescaled field $\gamma \phi_c$ obeys the same equation of motion, the cosmological constant operator keeps the identical expression in the Poincaré coordinate:

$$e^{\gamma \phi_c} = \left( \frac{1}{-H\tau} \right)^2. \tag{2.30}$$

It is the solution of (2.19) and the classical part of (2.20). It also satisfies the stationary condition, namely the coefficient of the linear $\phi$ term vanishes when the background satisfies $\hat{R} = 2H^2$. Although there may remain subtle issues in constructing the exact solution of 2-dimensional de Sitter quantum gravity, the physical picture is robust.

The most remarkable quantum effect in our solution is that the metric is modified and no longer agrees with the volume operator

$$e^{\phi_c} = \left( \frac{1}{-H\tau} \right)^2, \tag{2.31}$$

while

$$ds^2 = dt^2 + a^2(t)dx^2, \quad a(t) = \left(1 + \frac{1 - \gamma Ht}{\gamma} \right)^{1/\gamma}. \tag{2.32}$$

The Hubble parameter is

$$H(t) = \frac{\dot{a}}{a} = \frac{H}{\gamma \left(1 + \frac{1}{\gamma} Ht\right)}. \tag{2.33}$$

Note that it is no longer constant but it decreases with time. The renormalization of the cosmological constant operator with the scaling dimension $\gamma < 1$ gives rise to a remarkable result. The contribution of matter to the coefficient of the kinetic term of the Liouville field reduces the Hubble parameter by a substantial amount. Nevertheless the cosmological constant remains with a definite value. The anomalous dimension of the cosmological constant
operator has produced a more profound effect. The Hubble parameter is no longer constant but decreases with time. Let us estimate the acceleration speed of the universe.

\[ \dot{H}(t) = \frac{\partial}{\partial t} \left( \frac{\dot{a}}{a} \right) = \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 = \frac{\dddot{a}}{a} - H^2(t). \]  

(2.34)

\( -\dot{H}(t) \) must be smaller than \( H^2(t) \) when the expansion of the universe is accelerating \( \ddot{a} > 0 \).

From (2.33), these quantities are

\[ \dot{H}(t) = -H^2 \left( \frac{1 - \gamma}{\gamma^2} \right) \left( \frac{1}{1 + \frac{1}{\gamma} H t} \right)^2, \quad H^2(t) = \left( \frac{H}{\gamma} \right)^2 \left( \frac{1}{1 + \frac{1}{\gamma} H t} \right)^2. \]  

(2.35)

We thus find

\[ -\dot{H}(t) = (1 - \gamma) H^2(t). \]  

(2.36)

Since \( \gamma < 1 \) in the semiclassical region where \( c > 25 \), the expansion of this class of the universes are accelerating. Its acceleration speed is kept well for the weak coupling

\[ -\dot{H}(t) = \frac{1}{Q^2} H^2(t). \]  

(2.37)

Such a universe performs a slow roll inflation with the following slow roll parameters

\[ \epsilon = -\frac{\dot{H}(t)}{H^2(t)} = \frac{1}{Q^2}, \quad \eta = \epsilon - \frac{1}{2} \frac{\dot{H}(t)}{H(t) \dot{H}(t)} = \frac{2}{Q^2}. \]  

(2.38)

On the other hand the acceleration vanishes at the critical point like

\[ -\dot{H}(t) = (1 - Q) H^2(t). \]  

(2.39)

Note that the classical equation of motion still holds \( \ddot{R} = 2H^2 \). This equation appears in the coefficient of linear term of \( \phi \dot{R} \) in \( (2.23) \). The action for the classical field \( \phi_c \) in \( (2.22) \) also admits such a solution.

More precisely speaking, we have separated the classical and quantum part of this action as follows

\[ \int d^2x \frac{Q^2}{8\pi} \left\{ -\frac{\gamma}{2} \frac{\partial}{\partial \tau} \phi_c \frac{\partial}{\partial \phi_c} - \frac{2H^2}{\gamma} e^{\gamma \phi_c} \right\} \]  

(2.40)

\[ + \int d^2x \frac{Q^2}{8\pi} \left\{ \frac{1}{2} \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi \frac{\partial}{\partial \phi} + 2H^2 e^{\epsilon \phi} (e^\phi - \phi - 1) \right\}. \]  

(2.41)

Note that the potential term for the quantum field \( \phi \) is

\[ V(\phi) = e^{\gamma \phi_c} 2H^2 \left( \frac{Q^2}{8\pi} \right) (e^\phi - \phi - 1). \]  

(2.42)
The linear term in φ vanishes due to the equation of motion for φ:

\[ \int d^2x \sqrt{-g} \frac{Q^2}{8\pi} (\dddot{R} - 2H^2)\phi \rightarrow \dddot{R} = 2H^2. \]  

We emphasize that the potential has no flat direction as it increases when \( \phi \rightarrow \pm \infty \). The lifting of the flat direction for negatively large \( \phi \) has been achieved by demanding that the action is stationary with respect to the de Sitter solution. It originates from \( \sqrt{-g}\phi\dddot{R} \) term in the action. Being to be topological, there is no renormalization of this term while the cosmological constant operator \( \sqrt{-g} \) is renormalized. We have to pay a special attention to keep the balance of these two terms unless classical solutions are no longer valid.

We have shown that the anomalous dimension \( \gamma < 1 \) of the renormalized cosmological constant operator reduces the Hubble parameter \( H^2 = 4\pi\Lambda/\gamma Q^2 \) with the fixed cosmological constant Λ. This is a short distance effect of 2-dimensional Liouville quantum gravity. Furthermore it makes Hubble parameter \( \dot{a}/a \) to be time dependent. In fact negative anomalous dimension implies it vanishes at a late time. The weak coupling behavior to the leading order of \( 1/Q^2 \) is

\[ H(t) \sim H \frac{1}{1 + \frac{Q^2}{a}Ht} = H \frac{1}{1 + \frac{Q^2}{a} \log a(t)}. \]  

In the perturbation theory, time dependence of the Hubble parameter occurs through the IR logarithms \( \log a(t) \):

\[ H^2(t) \sim H^2(\langle e^\phi \rangle) \sim H^2(1 - \frac{2}{Q^2}Ht) = H^2(1 - \frac{2}{Q^2} \log a(t)). \]

It arises because the momentum integral is logarithmically divergent. In the case of exponential expansion of the universe, the one-loop momentum integral behaves as \( \log a(t) \) as the infrared cut-off goes like \( L/a(t) \) for a fixed UV cut-off \( L \). The negative sign of the one-loop correction is due to the negative sign of the kinetic term of \( \phi \). The shielding effect of the cosmological constant does not occur if the metric is positive. The exact solution is in accord with the perturbation theory in important issues whose significance still to be explored.

The Hubble parameter \( H(t) \) decays inversely proportional to \( \log a(t) \) namely e-folding number at a late time. It decays faster when the effective coupling \( 1/Q^2 \) is stronger. We show there is no other pure IR effects which diffuse \( H(t) \). This important conclusion is obtained from the investigation of the exactly renormalized Lagrangian. Since it is a conclusive result on the fate of the Hubble parameter \( H(t) \) in 2-dimensional Liouville quantum gravity, we briefly recall its renormalization procedure in Appendix B.

We have been suspecting that the inflaton may be a dual description of quantum effects in gravity. It is encouraging that this 2-dimensional toy model provides us a concrete example of such an idea. Let us go back to the original action before eliminating the inflaton by the
equation of motion
\[
\int d^2 x \frac{1}{8\pi} \sqrt{-\tilde{g}} \left\{ \frac{Q^2}{2} \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + 2 \tilde{R} \phi - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu f \partial_\nu f - 2H^2 Q^2 e^{\phi-(1-\gamma)f} \right\},
\]  
(2.46)
where we took the weak coupling limit. The effective Hubble parameter is induced by putting the classical solution of the inflaton into the potential
\[
H^2(f) = H^2 e^{-\frac{1}{Q^2}f} = H^2 e^{-\frac{1}{Q^2}2Ht}.
\]  
(2.47)
The slow roll parameters agree with the estimate of the exact solution (2.38)
\[
\epsilon = Q^2 \left( \frac{V'}{V} \right)^2 = \frac{-\dot{H}(t)}{H^2(t)} = \frac{1}{Q^2},
\]
\[
\eta = 2Q^2 \frac{V''}{V} = 2\epsilon + 2Q^2 \left( \frac{V'}{V} \right)' = \frac{2}{Q^2}.
\]  
(2.48)
At the weak coupling, our solution is a slow roll inflation model where the inflaton \(f\) rolls down the exponential potential
\[
2H^2 Q^2 \exp \left( -\frac{1}{Q^2} f \right).
\]  
(2.49)
We find it remarkable that this toy model underwrites a long suspected scenario that an inflaton is to provide a dual description of quantum effects in gravity.

We are not going to quantize it since it would be a double counting. It should be OK to use it as a low energy effective theory just like pions in QCD. As an example, we may examine the density perturbation in this model following the standard prescription. The inflaton field may fluctuate around the classical solution as
\[
f_c(t) + f = f_c(t + \delta t).
\]  
(2.50)
We pick a comoving gauge to eliminate the fluctuation of the inflaton
\[
\delta t = \frac{f}{f_c(t)}.
\]  
(2.51)
It then generates density perturbation
\[
-d\tau^2 + e^{2Ht} e^{2\zeta} dx^2, \quad \zeta = -H \delta t = -H \frac{f}{f_c(t)}.
\]  
(2.52)
The spectrum of the density perturbation is
\[
\langle \zeta_k \zeta_{k'} \rangle = (4\pi)^2 \delta(k + k') \frac{1}{2k} \left( \frac{H}{f_c} \right)^2.
\]  
(2.53)
In our case $f_c = H$, so there seems to be no enhancement.

\[ \langle \zeta_{\vec{k}} \zeta_{\vec{k}'} \rangle = (4\pi)^2 \delta(\vec{k} + \vec{k}') \frac{1}{2k}. \tag{2.54} \]

However it is enhanced in comparison to the conformal mode

\[ \langle \phi_{\vec{k}} \phi_{\vec{k}'} \rangle = -(4\pi)^2 \delta(\vec{k} + \vec{k}') \frac{1}{2k Q^2}. \tag{2.55} \]

So the enhancement of the density perturbation over the gravitational modes by a slow roll parameter $\epsilon$ appears to hold also in 2-dimensional de Sitter space.

The conclusion in this section is that the renormalized volume operator $e^{\gamma \phi}$ is obtained after integrating short distance degrees of freedom. It is the relevant operator to investigate long distance physics. The scaling dimension $\gamma$ is less than canonical $\gamma < 1$ in the semiclassical region where $c > 25$. We have examined a de Sitter type solution of the renormalized Liouville action. It shows that the Hubble parameter becomes not only time dependent but vanishes at a late time. This effect clearly breaks de Sitter invariance and is caused by the renormalization of the cosmological constant operator.

### 3 Entropy production at the horizon diffuses cosmological constant

We recall the semiclassical entropy in 4-dimensional de Sitter space is given by

\[ S_0 = \frac{A_0}{4G_N} = \frac{\pi}{H^2 G_N}. \tag{3.1} \]

It can be compared with our estimate (2.25) in 2-dimensional de Sitter space

\[ S_0 = \frac{1}{G_N} \log \frac{H_0^2}{H^2(t)}. \tag{3.2} \]

The semiclassical entropy of the system is given by $Q^2/\gamma \sim c/6$ which plays the role of the inverse Newton’s coupling $1/G_N$. So the increase of entropy by putting more matter reduces the Hubble parameter. Entropy also increases if the Hubble parameter decreases. It is in the same direction with 4-dimensional de Sitter space although the speed of the increase is much slower: logarithmic $\log(1/H^2(t))$ versus power law $1/H^2(t)$. In this section, we investigate IR effects on Hubble parameter from an entropic point of view.

Due to the existence of the cosmological horizon, plain waves constantly accumulate at it. They are called super horizon modes and constant in space. They do change with time over the cosmic scale. From a static observer inside, more and more constant modes are
accumulating at the horizon. As they evolve in a stochastic process, it is expected that entropy is continuously generated there. Simultaneously the Hubble parameter is diffused with the evolution of the universe. We show that such a dramatic process takes place in 2-dimensional Liouville quantum gravity. This conclusion follows from the entropy generation effects associated with the evolution of super horizon modes.

The precursor of the effect is the quantum fluctuation of the conformal mode

\[ \langle e^{\phi} \rangle = \langle e^{\phi_c + \gamma \hat{\phi}} \rangle \sim e^{\phi_c(t) + \frac{1}{2} \gamma^2 \langle \hat{\phi}^2 \rangle}. \]  

(3.3)

Here \( \phi_c(t) \) denote a classical solution while

\[ \langle \hat{\phi}^2 \rangle = -\frac{4}{Q^2} \int_{P_{\text{min}}}^{P_{\text{max}}} \frac{dP}{P} = -\frac{4}{Q^2} \log a(t). \]  

(3.4)

In this integral with respect to physical momentum, we fix the UV cut-off \( P_{\text{max}} \sim L \). We identify the IR cut-off as \( P_{\text{min}} = L/a(t) \). Here \( a(t) = \exp(\phi_c(t)/2) \) is the scale factor of the Universe and \( L \) is the initial size of the universe. Since we consider conformal zero mode \( \phi_0 \), it could only depend on time. Its characteristic time scale is the Hubble scale.

In this way the quantum IR fluctuation grows:

\[ \langle \hat{\phi}^2 \rangle \sim -\frac{2}{Q^2} \phi_c(t) \Rightarrow \langle e^{\phi} \rangle \sim e^{(1 - \frac{1}{2\gamma^2}) \phi_c(t)}. \]  

(3.5)

This effect may diminish the effective cosmological constant as the Universe expands [12]:

\[ H_{\text{eff}}^2 \sim H^2 e^{-\frac{2}{Q^2} \phi_c(t)} \sim H^2 a(t) \frac{2}{Q^2}. \]  

(3.6)

The important point here is that the quantum IR effect is time dependent and hence cannot be subtracted by a dS invariant counter term. Here we have only considered the leading order IR effect in \( H^2 \). Note that this one-loop IR effect [3.6] is consistent to the leading order with the prediction [2,33] based on the exact scaling dimension \( \gamma \) of the cosmological constant operator

\[ H^2(t) = \frac{H^2}{\gamma} \left( 1 + \frac{1 - \gamma}{\gamma} H t \right)^{-2} \sim H^2 \left( 1 - 2 \frac{1 - \gamma}{\gamma} H t \right) \sim H^2 \left( 1 - \frac{1}{Q^2} 2Ht \right). \]  

(3.7)

Let us examine what creates the entropy to reduce the Hubble parameter like the above. We conjecture that de Sitter entropy is carried by conformal zero mode. It performs a Brownian motion due to the constant disturbance by new comers who has just joined it. Such a process can be investigated by a Fokker-Planck type equation which governs the evolution of the distribution function of conformal zero mode \( \rho(\phi_0) \). To be more precise, de Sitter entropy is the von Neumann entropy of \( \rho(\phi_0) \).
We may put this formula (3.7) into the semiclassical estimate of the de Sitter entropy:

\[
\frac{Q^2}{\gamma} \log \frac{1}{\mathcal{H}^2(t)} = \frac{Q^2}{\gamma} 2 \log (1 + \frac{1 - \gamma}{\gamma} \mathcal{H} t) \\
\sim \frac{Q^2}{\gamma} \frac{1 - \gamma}{\gamma} 2 \mathcal{H} t = 2 \mathcal{H} t.
\]

(3.8)

The speed of entropy generation is given by taking the time derivative of the above

\[
\frac{2 \mathcal{H}}{1 + \frac{1 - \gamma}{\gamma} \mathcal{H} t} = 2 \gamma \mathcal{H}(t).
\]

(3.9)

It is given by the Hubble parameter and thus it also slows down with cosmic evolution. This semiclassical estimate can be compared with that of von Neumann entropy of the distribution function \(\rho(\phi_0)\).

The distribution functions of Fokker-Planck equations are well approximated by gaussian for weak coupling. (3.40) shows there is \(-1/2\) log \(\omega\) term in the von Neumann entropy. 1/\(\omega\) is the standard deviation of the gaussian and the entropy grows as \(\omega\) decreases. (3.9) implies

\[
- \frac{1}{2} \frac{\partial}{\partial t} \log \omega \sim 2 \mathcal{H},
\]

(3.10)

which is in qualitative agreement with (3.41) which is the increasing speed of the von Neumann entropy of \(\rho(\phi_0)\) under Fokker-Planck equation.

We believe carrying out the renormalization and summing IR logarithms by Fokker-Planck equations are double counting. We should not do the both. We should only perform the renormalization that is necessary anyway. We thus conclude that (3.8) is the correct semiclassical estimate. The new entropy is generated by the accumulation of conformal zero modes. They manifest as IR logarithms in perturbation theory which grows with time. Fortunately in 2-dimensions, we can decipher their physical effects through renormalization procedures since UV and IR effects are closely related.

Geometric entropy of de Sitter space arises since there are much wider world outside the cosmological horizon. It is analogous to the entangled entropy in the sense both arises after integrating out the Hilbert space of the outer world. From the view point of observers inside the cosmological horizon, they see nothing is going out of the horizon. For them only conformal zero modes are piling up. So they must carry the entire de Sitter entropy and this investigation supports such a point of view. We are able to verify that conformal zero modes contribute to shield the Hubble’s parameter due to its negative sign for the kinetic energy. Simultaneously, we can offer various evidences that they generate de Sitter entropy at the rate in accord with semiclassical estimates.

We recall here

\[
\frac{\gamma^2}{4} \langle \phi^2 \rangle = -\gamma^2 \frac{1}{Q^2} \log a \sim -\frac{1}{Q^2} \gamma \mathcal{H} t = -\frac{1 - \gamma}{\gamma} \mathcal{H} t.
\]

(3.11)
It is because of the following relations: 

\begin{equation} \frac{1 - \gamma}{\gamma} = \frac{\gamma}{Q^2}, \end{equation} 

(3.12) and

\begin{equation} \log a(t) = \frac{1}{1 - \gamma} \log(1 + \frac{1 - \gamma}{\gamma} Ht). \end{equation} 

(3.13)

The expectation value of any function of \(\gamma \phi\) must be the function of \(\frac{1 - \gamma}{\gamma} Ht\) as the following relation holds

\begin{equation} \langle (\gamma \phi)^2 \rangle = -4 \log(1 + \frac{1 - \gamma}{\gamma} Ht). \end{equation} 

(3.14)

The time dependence of the Hubble’s parameter \(H(t)\) implies that the lower cut-off of the momentum integral is of the size of the universe for the fixed UV cut-off. It clearly originates from the IR effects. Since this factor \(\frac{1 - \gamma}{\gamma} Ht\) is the leading log, we need to sum all powers of this variable. The solutions of the renormalized action are such functions. From this respect, we believe that the leading IR logs are already contained in them.

The anomalous dimensions are the short distance effect. However it also predicts long distance cut-off dependence of the operator since the short-distance and long distance cut-offs must appear together as the ratio on dimensional grounds. It is because the propagators of the minimally coupled scalars are the same in the both UV and IR regions in 2-dimensions. We have thus confirm that the time dependence of the cosmological constant operator is related to the anomalous dimension \(\gamma - 1\) is negative in the semiclassical regime \(c > 25\). Surprisingly the short distance effect alone makes Hubble parameter time dependent. \((2.33)\) further shows that 2-dimensional de Sitter space is doomed as the Hubble parameter \(H(t)\) fades away with \(\gamma < 1\). In the weak coupling limit, \(H(t)\) decays as \(Q^2/t\) with cosmic time. Nevertheless it is important to investigate if there are other source of IR logarithms in the entire Liouville theory.

The loop integral is logarithmically divergent with respect to the IR cut-off. It is also known that the \(n\)-th powers of IR logarithms may appear if the diagram contains \(n\) propagators \([13]\). Let us recall the each logarithms behaves as

\begin{equation} \frac{1}{Q^2} \log a(t) \sim \frac{1}{Q^2} Ht. \end{equation} 

(3.15)

So it becomes \(O(1)\) if the e-folding number of the universe becomes \(O(Q^2)\). We thus need to sum up all of them at a late times. The leading IR logarithms of these origins can be summed by the Langevin and Fokker-Planck equations. In 2-dimensions, the effective gravitational coupling is \(1/Q^2\). It is very large in comparison to 4-dimensions where \(1/Q^2\) is replaced by the notorious ratio \((H/M_P)^2 \sim 10^{-120}\) where \(M_P\) is the Planck mass. Nevertheless such effects
may have a significant impact on the evolution of the universe. Fortunately this problem
turns out to be solvable by renormalizing the cosmological constant operator exactly.

In order to understand the geometric entropy of the 2-dimensional de Sitter space from super
horizon degrees freedom, it is useful to investigate them in Liouville gravity. Let us recall
the definition of entropy:

$$\beta F = -\log Z, \quad S = -(1 - \beta \frac{\partial}{\partial \beta}) \beta F, \quad \text{(3.16)}$$

where $\beta = 1/T$. The 2-dimensional de Sitter space may be rotated into $S^2$ and $\beta$ corresponds
to the radius $l$ of $S^2$ as $\beta = 2\pi l$. If we assume the scale invariance, $\beta F$ cannot depend on $\beta$
since it is dimensionless. Therefore the conformal anomaly and Liouville action is the source
of the nontrivial geometric entropy. Since the size of the universe is dynamically determined
in quantum gravity, $-\beta F = \log Z$ gives us nothing but entropy. It is stationary with respect
to the change of the geometry of the manifold such as $\beta$.

Note that (2.25) is reminiscent of the entangled entropy with the central charge $c$ [14,15]:

$$S_{\text{en}} = \frac{c}{3} \log \frac{b}{a}, \quad \text{(3.17)}$$

where $a$ and $b$ denotes the short distance and long distance cut-offs of the subsystem respec-
tively. We may identify $H/H(t) = b/a$.

The geometric entropy may be the quantized version of the entangled entropy. Entangled
entropy is obtained from the density matrix of the subsystem. It is the entropy of the mixed
state after integrating local degrees of freedom belonging to the outer system. Geometric
entropy of de Sitter space is expected to be constructed in an analogous way. The density
matrix may be obtained by integrating out the states outside the horizon. The expectation
value of the operators inside the horizon can be evaluated by the density matrix. In field
theory that may be accomplished by evaluating correlation functions in the Liouville gravity.
In the case of conformal zero modes, their correlators are calculable from the Fokker-Planck
type distribution function $\rho(\phi_0)$. Understanding the relations of these various approaches
will shed light on elucidating this problem.

Let us recall how to estimate the entangled entropy. We may divide the real line into the
two sectors: positive and negative half-lines. We may change coordinates from the plane
to a cylinder $z = e^w$. The lower half-plane is mapped to a rectangular where the the lower
line segment corresponds to our section and upper line section to be outer section. We
may impose periodic boundary conditions on the remaining sections. The density matrix
is obtained by integrating out the fields on the outer segment after we glue two cylinders
together.

In this case the problem is effectively compactified on to torus while geometric entropy of
dS$^2$ is often studied by compactifying it on to $S^2$. After integrating out the localized states
outside the cosmological horizon, we are left with a half-line of the length $2l$. It becomes
a circle if we adopt periodic boundary condition on this strip. It is a natural and may be even the right choice to compactify $dS^2$ to $S^2$ with the identification of this circle and the circumference. The density matrix $\rho(\phi, \phi')$ may be obtained by performing the path integral of the fields like conformal mode on $S^2$ with specified field $\phi, \phi'$ at the both sides of the equator. The expectation value of the fields may be evaluated by inserting them on the equator and perform the path integral on the whole $S^2$ with a suitable action like Liouville quantum gravity. The geometric entropy $S$ can be evaluated by simply evaluating the partition function $Z$ since it gives the geometric entropy $S = \log Z$ right away in quantum gravity. Suppose the Hubble parameter changes slowly with cosmic time. In this case, it may be a good strategy to change the radius of the corresponding $dS^2$ and $S^2$ as $1/H(t)$. As far as conformal zero mode is concerned, there is no problem in Euclidean rotation since the potential term dominates the kinetic term.

So far we have assumed that the matter system is at the critical point, namely conformally invariant. In a more generic situation, the central charge $c$ is known to be a decreasing function with respect to the IR cut-off and hence time $\phi_c(t)$. For example a nonlinear sigma model may develop a mass gap. In such a situation, the number of massless scalar fields decreases. This effect may enhance the magnitude of the anomalous dimension and the screening effect of the cosmological constant.

The conformal zero mode performs a Brownian motion with the scale set by $1/Q^2$. As the universe expands, plane waves constantly come out of horizon to join the super horizon mode. They collide the main body just like a Brownian process of the strength $1/Q^2$. It is noteworthy that the metric of conformal mode is negative like Einstein theory in 4-dimensions. Although there could be an equilibrium distribution for conformal zero mode if there is a counter effect to diffusion. However there is no such a possibility here since we have no drift force due to the uniqueness of the classical solution. See Appendix C for its explanation.

We focus on the dynamics of super horizon mode of conformal factor of the metric. Their cosmic evolution in real spacetime can be investigated by Langevin type equation. The ensemble average of a function of $f(\phi(t))$ are a natural observable in the system governed by Langevin equation. The Langevin equation is equivalent to the Fokker-Planck equation. We define the ensemble average of a function of $f(\phi(t))$ as

$$\langle f(\phi(t)) \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\phi_i(t)), \quad (3.18)$$

where $i$ denotes the observation of the $i$-th member. In this context, it is natural to introduce a distribution function $\rho_i(\phi)$ in the following way:

$$\langle f(\phi(t)) \rangle = \int d\phi \rho_i(\phi)f(\phi). \quad (3.19)$$

The probability distribution function $\rho_i(\phi)$ obeys the Fokker-Planck equation. This formalism is close to field theory approach especially with respect to investigating the super
horizon mode. The system may approach an equilibrium state at a late time. In that case \( \rho(\phi) \) describes an equilibrium state whose temperature \( T \) is determined by the strength of the random force. We thus conclude that
\[
\rho(\phi) = e^{-\beta V(\phi)} \frac{1}{Z},
\]
where \( \log Z = -\beta F \). From this formula, we can verify that von Neumann entropy of \( \rho \) gives us the entropy of this equilibrium state:
\[
\int d\phi (-\rho \log \rho) = \int d\phi (\beta E - \beta F) \rho = S.
\]

Although our strategy is to investigate IR effects in real spacetime with Langevin equation, Fokker-Plank equations relate the problem with thermodynamics. The equilibrium state is studied very well by Euclidean field theory. We can estimate the von Neumann entropy of \( \rho_t \) even if it is not equilibrated. It is the measure of the entropy of the system which evolves according to Fokker-Planck or Langevin equation.

In order to connect shielding mechanism of the cosmological constant with entropy generation at the horizon, we employ stochastic approach [3,4]. The Langevin equation for the conformal zero mode \( \phi_0 \) with respect to the cosmic time \( t \) is derived in Appendix C. The super horizon mode of the conformal degree of metric is given by
\[
\phi_0(x) = \sqrt{\frac{8\pi}{Q^2}} \int \frac{d\vec{p}}{2\pi} \theta(Ha(t) - p)(a_{\vec{p}} \frac{1}{\sqrt{2p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger \frac{1}{\sqrt{2p}} e^{-i\vec{p}\cdot\vec{x}}),
\]
where \([a_{\vec{p}}, a_{\vec{p}}^\dagger] = -2\pi \delta(\vec{p} - \vec{p}')\). Since plane waves become constant in time, the time dependence is caused by the step function which restricts physical momenta \( P < H \). The Langevin equation is given by
\[
\dot{\phi}(x) = \dot{\phi}_0(x), \quad \langle \dot{\phi}_0(t, \vec{x}) \dot{\phi}_0(t', \vec{x}') \rangle = -\frac{4}{Q^2} H(t) \delta(t - t').
\]
We have dropped the drift term but kept the quantum fluctuation effect. Our purpose of this investigation is not to do the double counting as the renormalization of the cosmological constant operator occurs by identical quantum fluctuations. It is rather to see to what extent we can reproduce the features of the exact solutions. By doing so, we shall be able to examine the consistency of our understandings on this issue.

If \( \phi(t) \) obeys the Langevin equation, the Fokker-Planck equation for distribution function \( \rho_t(\phi) \) follows
\[
\dot{\rho} = -\frac{2}{Q^2} H(t) \frac{\partial}{\partial \phi^2} \rho.
\]
We notice the right hand-side is of the opposite sign in comparison to those appearing in the study of unitary matter systems. Of course this is due to the negative metric of the
conformal mode. So this equation is obtained by the time reversal of the former. It appears that our equation listed so far in this section runs the show backward in comparison to the standard evolution in the matter system.

However there is an important issue we have to address in quantum gravity. The distribution of conformal zero modes $\phi$ must change under the evolution. For this reason we may include the renormalization factor $\omega$ for the cosmological constant operator. Note that the linear term in $\phi$ cancels in the potential which ensures that the equation of motion $\hat{R} = 2H^2$ holds:

$$V = \sqrt{-g}H^2\frac{Q^2}{\omega}(e^{\omega\phi} - \omega\phi - 1).$$  \hspace{1cm} (3.25)

We assume under Euclidean rotation

$$\frac{i}{4\pi}\int d^2x \sqrt{-g}H^2\frac{Q^2}{\omega}(e^{\omega\phi} - \omega\phi - 1) \rightarrow \frac{Q^2}{\omega}(e^{\omega\phi} - \omega\phi - 1),$$

when we compactify $dS^2$ into $S^2$ with the radius of $1/H$. As we emphasized in the preceding section, the renormalization property of the operators $\sqrt{-g}$ and $\sqrt{-g}\phi\hat{R}$ are different while the de Sitter background is realized by balancing them. In fact $e^{\omega\phi}$ and $\phi$ terms in the potential (3.26) come from the former and latter operators respectively.\footnote{The identity is our normalization convention.} It is required to keep the balance of the two different operators. We need a formalism which let the renormalization of operators to cancel the effect of the evolution by Fokker-Planck equation.

We thus assume the following distribution containing $\omega$:

$$\rho_\omega = N_\omega e^{-\frac{Q^2}{2\omega}(e^{\omega\phi} - \omega\phi - 1)} \sim \sqrt{\frac{Q^2\omega}{2\pi}} e^{-\omega Q^2 \frac{1}{2}\phi^2},$$  \hspace{1cm} (3.27)

where $N_\omega$ is the normalization factor.

The readjustment of the background under time evolution is realized by requiring the time independence for the distribution function:

$$\dot{\rho}_\omega = -2H\frac{1}{Q^2} \frac{\partial^2}{\partial \phi^2} \rho_\omega + \dot{\omega} \frac{\partial}{\partial \omega} \rho_\omega = 0.$$  \hspace{1cm} (3.28)

In this way, the background adjusts itself automatically to cancel the evolution brought by Fokker-Planck equation.

The gravitational Fokker-Planck equation is

$$\dot{\omega} \frac{\partial}{\partial \omega} \rho_\omega = 2H\frac{1}{Q^2} \frac{\partial^2}{\partial \phi^2} \rho_\omega.$$  \hspace{1cm} (3.29)

After the dust settled, the sign of the right hand side turns back to normal. The time derivative of the distribution function is specified through its $\omega$ dependence.
From this modified Fokker-Planck equation (3.29), we obtain the following evolution equation for the background $\omega$

$$\dot{\omega} = -4H\omega^2.$$  \hspace{1cm} (3.30)

The solution is

$$\omega(t) = \frac{1}{1 + 4Ht}.$$ \hspace{1cm} (3.31)

The initial probability distribution $\rho_0$ is

$$\rho_0 = N_0 \exp \left\{-Q^2(e^\phi - \phi - 1)\right\} = N_0 \exp \left\{-\frac{4\pi}{H^2}\Lambda(e^\phi - \phi - 1)\right\}. \hspace{1cm} (3.32)$$

This formula suggests an Euclidean system on $S^2$ with a radius $1/H$. This solution coincides with our original potential before effects of IR logarithms becomes important namely at the beginning of the de Sitter expansion. In the semiclassical region where $Q^2$ is large, this potential is well approximated by a gaussian:

$$N_0 \int d\phi \exp \left\{-Q^2(e^\phi - \phi - 1)\right\} \sim Q\sqrt{2\pi} \int d\phi \exp\left(-\frac{Q^2}{2}\phi^2\right). \hspace{1cm} (3.33)$$

We have shown that there is an effect to reduce the effective cosmological constant (3.6). As long as this effect is concerned, we believe that the UV investigation (2.33) has shown that the Hubble parameter acquires time dependence and it eventually vanish. Since the propagators are identical in the both UV and IR regions, the two birds can be dealt with by a single stone. We argue there are no drift force effects since the solution of the theory is unique. There are unstable deformations if they increase the entropy of the system. As we emphasized, the system evolves toward the state with maximum entropy in quantum gravity.

What is the geometric entropy of de Sitter space? We propose that it is the entropy of super horizon conformal mode which accumulates with cosmic expansion. There are no other massless modes in 2-dimensional Liouville gravity. Furthermore the entropy could increase in a stochastic process. Let us evaluate the von Neumann entropy of the conformal zero mode with (3.32)

$$S_0 = -tr \rho_0 \log \rho_0$$

$$= \int d\phi \rho_0 \left\{Q^2(e^\phi - \phi - 1) - \log Q + \frac{1}{2} \log(2\pi)\right\}$$

$$\sim \frac{1}{2} - \log Q + \frac{1}{2} \log(2\pi). \hspace{1cm} (3.34)$$

In our view the von Neumann entropy of the super horizon mode are the identity of geometric entropy of de Sitter space. Characteristic feature of this expression is its $Q^2$ dependence. The
von Neumann entropy becomes larger if the effective gravitational coupling $1/Q^2$ becomes stronger.

Let us investigate its $Q^2$ dependence from the 2-dimensional Liouville quantum gravity point of view:

$$\frac{\partial}{\partial Q^2} \log Z = -\langle (e^\phi - \phi - 1) \rangle = -\frac{1}{2Q^2}. \quad (3.35)$$

Here $Z$ is the partition function of the super horizon sector of 2-dimensional Liouville gravity on $S^2$:

$$Z = \int d\phi e^{-Q^2(e^\phi - \phi - 1)}. \quad (3.36)$$

Note that the potential is bounded below and the expectation value of $n$-point function of super horizon mode is calculable. The potential dominates kinetic energy for super horizon conformal mode. We can safely ignore the wrong sign kinetic term in comparison to the potential term. The wrong sign problem of conformal mode may turn out be a blessing with respect to cosmological constant problem. As it is explained, $\log Z$ gives us entropy itself in quantum gravity. So $Q^2$ dependence of von Neumann entropy is consistent with geometric entropy of super horizon conformal modes of Liouville gravity (3.35). The expectation value of any function $f(\phi)$ is well defined unless $f(\phi)$ grows too rapidly at $\phi \to \pm \infty$ to spoil the convergence of the integral. The expectation value of $f(\phi)$ is real if $f$ is a real function.

We consider the change of von Neumann entropy in the stochastic process by introducing one parameter deformation of an initial distribution function by $\omega$. This factor allows to renormalize the cosmological constant operator as

$$\rho_\omega = N_\omega e^{-\frac{Q^2}{2}\frac{e^{\omega \phi} - \omega \phi - 1}{\omega}}. \quad (3.37)$$

In a gaussian approximation,

$$N_\omega = \frac{Q\sqrt{\omega}}{\sqrt{2\pi}}. \quad (3.38)$$

We rotate the Minkowski space-time potential of $dS^2$ into Euclidean $S^2$:

$$\frac{i}{4\pi} \int d^2x \sqrt{-g}H^2(Q^2)e^{\omega \phi - \omega \phi - 1} \rightarrow \frac{Q^2}{\omega}(e^{\omega \phi} - \omega \phi - 1). \quad (3.39)$$

The corresponding von Neumann entropy is

$$S_\omega = -\text{tr} \rho_\omega \log \rho_\omega$$

$$= \int d\phi \rho_\omega \left\{ \frac{Q^2}{\omega}(e^{\omega \phi} - \omega \phi - 1) - \log Q - \frac{1}{2} \log \omega - \frac{1}{2} \log(2\pi) \right\}$$

$$\sim S_0 - \frac{1}{2} \log \omega. \quad (3.40)$$

\*\* We do not exclude that a constant term like $Q^2$ is missing since it becomes negative for large $Q$.\*\*
This expression shows that entropy increases if $\omega$ decreases from the initial point $\omega = 1$. So there might be an unstable deformation of this configuration. If $\omega$ decreases, the distribution of super horizon mode spreads out. Let us examine a possible instability of this configuration in the vicinity of $\omega \sim 1$ under Fokker-Planck equation:

$$
\dot{\omega} \frac{\partial}{\partial \omega} S_{\omega} = - \text{tr} \dot{\omega} \frac{\partial}{\partial \omega} \rho_{\omega} \log \rho_{\omega} \\
= - \text{tr} \frac{2}{Q^2} H \frac{\partial^2}{\partial \phi^2} \rho_{\omega} \log \rho_{\omega} \\
\sim 2H\omega = - \frac{1}{2 \omega} \dot{\omega}.
$$

(3.41)

From the inspection of (3.41), it is clear that the von Neumann entropy of $\rho_{\omega}$ always increases. In particular its grow $\Delta S = 2Ht$ as the system evolves away from the initial distribution with $\omega = 1$.

This result is consistent with semiclassical estimates of the geometric entropy (3.8) when $\omega \sim 1$. The exact solution in the weak coupling region shows that the entropy increases as

$$
-\frac{1}{2} \log \omega = 2 \frac{Q^2}{\gamma} \log \left(1 + \frac{1 - \gamma}{\gamma} Ht\right) \sim 2Ht,
$$

(3.42)

for the weak coupling or short time limit. We can verify von Neumann entropy increases logarithmically in the evolution under the Fokker-Planck equation by using the explicit solution (3.31)

$$
S_{\omega} = \frac{1}{2} \log(1 + 4Ht) \sim 2Ht.
$$

(3.43)

All approaches agree that entropy grows like $2Ht$ away from the initial distribution function. They also agree that $H(t)$ eventually vanishes. The eventual fate of 2d de Sitter space is not agreed upon. The exact solution predicts it is $Q^2$ dependent. The slow roll parameter is given by $\epsilon = 1/Q^2 = \eta/2$. It also predicts that the acceleration stops at the critical point $Q^2 = 0$. Fokker-Planck equation predicts more rapid slow down of the acceleration.

The existence of configurations of higher von Neumann entropy implies that the potential for the super horizon modes of 2-dimensional Liouville gravity is modified also as in a process of evolution,

$$
V(\phi) = \frac{Q^2}{\omega}(e^{\omega \phi} - \omega \phi - 1).
$$

(3.44)

The partition function for the super horizon sector of conformal mode evolves as

$$
Z_{\omega} = \int d\phi e^{-\frac{Q^2}{\omega}(e^{\omega \phi} - \omega \phi - 1)}.
$$

(3.45)

It is because $\phi$ field obeys the identical Langevin equation in 2-dimensional gravity. So the potential for the conformal mode must change according Fokker-Planck equation. It must
be identical to that of distribution function (3.37). With this potential, we can reproduce \( \omega \) dependence of von Neumann entropy from Liouville gravity

\[
\log Z_\omega \sim -\log Q - \frac{1}{2} \log \omega. \tag{3.46}
\]

So geometric entropy in Liouville gravity in de Sitter space is consistent with von Neumann entropy (3.40) of super horizon modes. From these considerations, we are able to obtain consistent pictures on these IR effects on Hubble parameter. The Hubble parameter is generically suppressed by the e-folding number. The habitants of 2-dimensional de Sitter space may always wonder why the Hubble parameter is always the size of the universe.

Let us check the universes like (3.31). \( \omega \) is just like the scaling dimension of the cosmological constant \( \gamma \). This universe starts with a slow roll inflation while the slow roll parameter grows as \( \gamma = 1/(4Ht) \) decreases. Finally the accelerated expansion stops as \( \gamma \rightarrow 0 \) like arriving at the critical point. This is an interesting scenario encompassing the exact solutions all together. It is remarkable first place that UV effects predicts IR behavior of the theory. It is specific to 2-dimensions as the propagators of the minimally coupled scalars are scale invariant. UV and IR divergences are closely related since \( \log(\frac{P_{\text{max}}}{P_{\text{min}}}) \) type large logarithms are expected in 2-dimensions. It is very interesting to find out whether IR effects diminishes the scaling dimension all together down to nil.

The situation is different in 4-dimensions where only \( \log H/P_{\text{min}} \) type IR logarithms appear. The solutions of Fokker-Planck equations do not depend on \( Q^2 \) or contain all of them. It is a characteristic feature of the one-loop approximation which takes account of the leading IR logarithms. On the other hand, the exact solution of the renormalized action shows explicit \( Q^2 \) dependence indicating all-loops contributions. The picture we obtain from the exact solutions are not only more sophisticated but also very convincing.

Apparently the effective field theory for the exactly renormalized action contains inflaton like freedom. Such a freedom seems necessary to describe the solution of quantum gravity in terms of the effective theory at the tree level. It is in some sense dual description of quantum gravity. Remarkably its classical behavior reproduces quantum effects. We wonder the inflaton in our universe may be such a dual description of quantum effects.

4 Conclusions

We have investigated IR quantum effects in the 2-dimensional de Sitter space from a solution of the exactly renormalized Liouville action. We work in the semiclassical region where matter central charge \( c > 25 \). In such a region, the exact scaling dimension \( \gamma \) of the cosmological constant operator is less than \( \gamma < 1 \). This is due to the screening effect of the conformal mode with negative metric. The 2-dimensional de Sitter space is obtained as a solution of the Liouville action. The solution of the renormalized action shows that 2-dimensional de
Sitter space is doomed. The Hubble parameter is no longer constant and decreases with time. It does so slowly at the weak coupling with large $c$ and even stops acceleration at the critical point $c = 25$.

In conclusion, we have made a strong case for the instability of 2-dimensional de Sitter space. The exact solutions show that the negative anomalous dimension of the cosmological constant operator makes the Hubble parameter time dependent and vanishes at a late time. They underscore the importance of IR logarithms which becomes shielding effects due to the negative sign of the kinetic term of the conformal mode. We estimate the semiclassical entropy of 2-dimensional de Sitter space. It increases logarithmically with the Hubble radius like $\log(1/H^2(t))$ versus $1/H^2(t)$ in 4-dimensions. The cosmological constant is diffused by entropy production at the horizon. The conformal zero mode generates entropy in a Brownian diffusion process. We formulate Fokker-Planck equation in 2-dimensional quantum gravity. We take account of the change of the conformal zero mode distribution by the renormalization of the cosmological constant operator. In this way we can obtain very analogous equations with unitary matter systems despite the negative metric of the conformal mode. Nevertheless we argue that the drift term is absent due to the uniqueness of the classical solution.

We propose the de Sitter entropy is carried by the conformal zero modes. In order to verify our proposal, we have evaluated the von Neumann entropy of the distribution functions for them. Their characteristics are in agreement with semiclassical estimates. In matter systems, it is known that the equilibrium state is stable. Since we have only the quantum fluctuation term, the system is diffused away with Hubble parameter to vanish at a late time. In the matter systems, the free energy $F = E - TS$ is minimized. In low temperature, minimizing the energy is important. In a standard model, we look for the ground state with the smallest energy. On the other hand, we maximize the entropy as there is no energy in de Sitter space. It should be interesting to understand such an evolution which takes place in quantum gravity. Fine-tuning problem may be shed new lights on from this perspective since the maximum entropy principle operates in quantum gravity [16, 17]. In fact cosmological constant may turn out be such an example. There are many common features between 2-dimensional and 4-dimensional gravity such as the negative sign of the kinetic term of the conformal mode. We hope to investigate the relation between cosmological constant and generation of entropy of super horizon mode in 4-dimensional Einstein gravity.

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A  de Sitter thermodynamics

Globally de Sitter space is a hyperboloid

$$ \frac{ds^2}{l^2} = -d\tau^2 + \cosh^2(\tau)d\Omega_2^2. \quad (A.1) $$

The characteristic length $l$ is set by the Hubble parameter $H$ as $l = 1/H$. It is related to the cosmological constant $\Lambda$ as $l = \sqrt{3/\Lambda}$. Locally in the Poincaré coordinate, it corresponds to an expanding flat universe

$$ ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2d\Omega_2^2) = \left(\frac{1}{-H\tau}\right)^2(-d\tau^2 + d\vec{x}^2), \quad (A.2) $$

where we can obtain conformally flat parametrization. We often work in this coordinate where $\tau$ runs from $-\infty$ to 0. Let us draw an $S^2$ with the radius $a(t)r$ in this space where $a(t) = e^{Ht}$. The expansion velocity of this space is $Ha(t)r$. Since it coincides with the velocity of light at the apparent horizon, its radius is $\rho_h = 1/H$. In the static coordinate system

$$ \frac{ds^2}{l^2} = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2d\Omega_2^2, \quad (A.3) $$

where $V(r) = 1 - r^2$. It becomes manifest that an observer at $r = 0$ is surrounded by a cosmological horizon at $r = 1$. The radius of the horizon is $\rho_h = 1/H$ in agreement with that in the Poincaré coordinate.

The presence of event horizons leads to thermodynamics [5]. According to Bekenstein and Hawking, black holes possess finite temperature

$$ T_{\text{hor}} = \frac{\kappa}{2\pi}. \quad (A.4) $$

For a Schwarzschild black hole of mass $M$, $\kappa = 1/4M$. The first law of thermodynamics is

$$ \frac{1}{T_{\text{hor}}} = \frac{\partial S}{\partial M}. \quad (A.5) $$

The entropy is given by the area of the event horizon

$$ S_{\text{hor}} = \frac{A}{4}. \quad (A.6) $$

We list here the relationship among different coordinates on 2-dimensional de Sitter space or its Euclidean version $S^2$. The metric on $S^2$ is

$$ ds^2 = \frac{1}{H^2}(d\theta^2 + \sin^2(\theta)d\varphi^2). \quad (A.7) $$
It can be embedded into 3 Euclidean dimensions.

\[ z_0^2 + z_1^2 + z_2^2 = \left( \frac{1}{H} \right)^2. \] (A.8)

We rotate it into real spacetime: \( z_0 \to iz_0 \),

\[ -z_0^2 + z_1^2 + z_2^2 = \left( \frac{1}{H} \right)^2. \] (A.9)

We then consider the following coordinate

\[ z_0 = \frac{1}{H} \sinh(Ht) + \frac{1}{2} He^{Ht} x^2, \quad z_2 = \frac{1}{H} \cosh(Ht) - \frac{1}{2} He^{Ht} x^2, \quad z_1 = e^{Ht} x, \] (A.10)

with the line element

\[ ds^2 = \frac{1}{H^2}(-dt^2 + e^{2Ht}dx^2) = \left( \frac{1}{-H^2} \right)^2(-d\tau^2 + dx^2), \] (A.11)

which covers the upper half triangle of the Penrose diagram of the global de Sitter manifold. The metric on \( S^2 \) can be continues to global de Sitter metric more directly \( \theta = \frac{\pi}{2} + it \).

\[ ds^2 = \frac{1}{H^2}(-dt^2 + \cosh^2(t) d\varphi^2). \] (A.12)

Although we mostly work in the Poincaré coordinate, we may rotate it into \( S^2 \) by using these relations if appropriate. The coordinate transformation can be done by inspection. For example, we claim the following term is topologically quantized after Euclidean rotation into \( S^2 \):

\[ \frac{1}{8\pi} \int d\tau dx \frac{1}{(-H\tau)^2} 2H^2 = \frac{1}{8\pi} \int d^2x \sqrt{-g}R \to -i \frac{1}{4\pi} \int d\theta d\omega \sin \theta = -i. \] (A.13)

The 2-dimensional Liouville gravity may be thought of a little Einstein gravity descending from 4-dimensions to \( D = 2 + \epsilon \) dimensions,

\[ \frac{Q^2}{4\pi} \int d^Dx \sqrt{-g}\left(\frac{1}{\epsilon}R - H^2\right) = \frac{Q^2}{4\pi} \int d^Dx \sqrt{-\hat{g}}\left(\frac{1}{\epsilon}\hat{R} - e^\phi H^2\right), \] (A.14)

where we explicitly show the constant conformal mode dependence

\[ g_{\mu\nu} = e^\phi \hat{g}_{\mu\nu}. \] (A.15)

The first term with the \( 1/\epsilon \) pole is

\[ \frac{1}{\epsilon} \frac{Q^2}{4\pi} \int d^2x \sqrt{g}R = \frac{Q^2}{4\pi} \int d^2x \sqrt{\hat{g}}\left(\frac{1}{\epsilon}\hat{R} + \frac{1}{4}\hat{g}^{\mu\nu}\nabla_\mu \nabla_\nu \phi + \frac{1}{2}\hat{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2}\phi \hat{R}\right) \]
\[ = \frac{Q^2}{4\pi} \int d^2x \sqrt{\hat{g}}\left(\frac{1}{\epsilon}\hat{R} + \frac{1}{4}\hat{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2}\phi \hat{R}\right). \] (A.16)

The leading term with the \( 1/\epsilon \) pole acts as the counter term. We thus obtain

\[ \frac{Q^2}{4\pi} \int d^2x \sqrt{\hat{g}}\left(\frac{1}{4}\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2}\phi \hat{R} - H^2\right). \] (A.17)

This is the Liouville action which is the gift of conformal anomaly.

27
B Renormalization of cosmological constant operator

To renormalize the cosmological constant operator to the leading order in $1/Q^2$, we need to consider the quantum fluctuation of the cosmological constant operator:

$$\langle e^\phi \rangle = \langle e^{\phi_c + \hat{\phi}} \rangle \sim e^{\phi_c(t) + \frac{1}{2} \langle \hat{\phi}^2 \rangle}.$$  \hfill (B.1)

Here $\phi_c(t)$ denotes a classical solution while

$$\langle \hat{\phi}^2 \rangle = -\frac{4}{Q^2} \int_{P_{\min}}^{P_{\max}} \frac{dP}{P}.$$  \hfill (B.2)

The scalar propagator is both UV and IR divergent in 2-dimensions. We recall here that the physical momenta $P$ depend on the metric:

$$P_{\max} = p_{\max} e^{\frac{1}{2} \hat{\phi}(x)}.$$  \hfill (B.3)

We first consider the UV contribution in a dimensional regularization. We consider $D = 2 - \epsilon$ dimensions since $1/\epsilon$ pole can be identified with the logarithmic UV divergence. The cosmological constant operator is

$$e^{\frac{D}{2} \hat{\phi}} = e^{(1-\frac{\epsilon}{2}) \hat{\phi}}.$$  \hfill (B.4)

We evaluate the two-point function as

$$\frac{1}{2} \langle \hat{\phi}^2 \rangle = -\frac{2}{Q^2} \int dp p^{1-\epsilon} \frac{1}{p^2 + m^2} \mu^\epsilon e^{-\frac{1}{2} \epsilon \hat{\phi}}$$

$$= -\frac{1}{Q^2} \Gamma\left(\frac{\epsilon}{2}\right)m^{-\epsilon} \mu^\epsilon e^{-\frac{1}{2} \epsilon \hat{\phi}}$$

$$\sim -\frac{2}{Q^2} \left(\frac{1}{\epsilon} + \log \frac{\mu}{m} - \frac{1}{2} \hat{\phi}\right),$$  \hfill (B.5)

where $\mu$ is the renormalization scale. We put $\mu = m$ for simplicity. One of difficulties of renormalizing the operators in quantum gravity is that the both propagators and the interaction vertices depend on the metric.

To carry over this renormalization process to all-orders, we employ renormalization group [18,19]. Let us recall our dimensional regularized Lagrangian

$$\int d^Dx \frac{Q^2}{8\pi} \left\{ \frac{1}{2} g^{\mu\nu} e^{-\frac{4}{\epsilon} \psi} \frac{\partial}{\partial x^\mu} \phi \frac{\partial}{\partial x^\nu} \phi - 2H^2 e^{(1-\frac{\epsilon}{2}) \psi} \right\}. \hfill (B.6)$$

It is always a good idea to canonically normalize the kinetic term by the change of field variable $e^{-\frac{4}{\epsilon} \psi} = 1 - \frac{\epsilon}{4} \psi$,

$$\int d^Dx \frac{Q^2}{8\pi} \left\{ \frac{1}{2} g^{\mu\nu} \frac{\partial}{\partial x^\mu} \psi \frac{\partial}{\partial x^\nu} \psi - 2H^2 (1 - \frac{\epsilon}{4} \psi)^{\frac{4}{\epsilon} (1-\frac{\epsilon}{2})} \right\}. \hfill (B.7)$$
Now the two-point function is just (B.5) without $\psi$ field dependence:

$$\frac{1}{2} \langle \bar{\psi} \psi \rangle \sim -\frac{2}{Q^2} \left( 1 + \log \frac{\mu}{m} \right).$$  \hspace{1cm} (B.8)

The advantage of this approach is that we need not worry about the interaction vertices in the kinetic term. Let us investigate quantum correction to the cosmological constant operator next

$$(1 - \frac{\epsilon}{4} \psi)^{-\frac{\epsilon}{2} (1 - \frac{\epsilon}{2})} = \exp \left\{ -\frac{4}{\epsilon} (1 - \frac{\epsilon}{2}) \log \left( 1 - \frac{\epsilon}{4} \psi \right) \right\}$$

$$= \exp \left\{ (1 - \frac{\epsilon}{2} \psi + \frac{\epsilon}{4} \bar{\psi}^2 + \frac{\epsilon^2}{4^2} \bar{\psi}^3 + \cdots) \right\}. \hspace{1cm} (B.9)$$

The one-loop quantum corrections starts with the $e^\psi$ part of the operator

$$\langle \exp \left\{ (1 - \frac{\epsilon}{2} \psi) \right\} \rangle \sim 1 + (1 - \frac{\epsilon}{2}) \bar{\psi} + \frac{1}{2} (1 - \epsilon) \langle \psi^2 \rangle + \frac{1}{2} (1 - \frac{3\epsilon}{2}) \langle \psi^2 \rangle \bar{\psi}$$

$$= \left\{ 1 - (1 - \epsilon) \frac{1}{2 Q^2} \right\} \left\{ 1 + (1 - \frac{\epsilon}{2}) \bar{\psi} \right\}. \hspace{1cm} (B.10)$$

There are additional contributions

$$\langle \frac{\epsilon}{4} \bar{\psi} \psi^2 \rangle + \langle \frac{\epsilon}{4} \bar{\psi} \psi^3 \rangle = -\frac{1}{2^2 Q^2} - \frac{3}{2} \frac{1}{2 Q^2} \bar{\psi}. \hspace{1cm} (B.11)$$

By a multiplicative renormalization by $Z$, we obtain the renormalized operator at the one-loop level:

$$Z \left\{ 1 - (1 - \epsilon) \frac{1}{2 Q^2} - \frac{1}{2} \frac{1}{2 Q^2} \right\} \left\{ 1 + (1 - \frac{1}{Q^2} (1 - \frac{\epsilon}{2}) \bar{\psi} \right\}$$

$$= 1 + (1 - \frac{1}{Q^2}) (1 - \frac{\epsilon}{2}) \bar{\psi} \sim \exp \left\{ (1 - \frac{1}{Q^2}) (1 - \frac{\epsilon}{2}) \bar{\psi} \right\}. \hspace{1cm} (B.12)$$

Let us introduce a trick to determine the UV divergence of a generic operator. We consider the following integral weight

$$\sqrt{\epsilon Q^2} \frac{d^4 \psi e^{-\frac{\epsilon Q^2}{8\pi} \psi^2}}{8\pi}, \hspace{1cm} (B.13)$$

such that the average of the two-point function produces its $1/\epsilon$ pole in $2 + \epsilon$ dimensions:

$$\sqrt{\epsilon Q^2} \frac{d^4 \psi e^{-\frac{\epsilon Q^2}{8\pi} \psi^2}}{8\pi} = \frac{4}{\epsilon Q^2}. \hspace{1cm} (B.14)$$

We split the field such that $\psi \rightarrow \psi_c + \psi$ and take the average over $\psi$ field with this measure first. We can determine its UV divergences this way.
A generic proof is
\[
\langle \frac{1}{2l!}(\psi_c + \psi)^{2l} \rangle_\psi = \langle \sum \frac{1}{2m!} \psi_c^{2m} \frac{1}{2n!} \psi^{2n} \rangle_\psi \\
= \sum \frac{1}{2m!} \psi_c^{2m} \frac{1}{n!} \left( \frac{1}{2} \langle \psi^2 \rangle_\psi \right)^n,
\]
where the average denoted by \( \langle \psi^{2n} \rangle_\psi \) is with respect to the weight \( \text{(B.13)} \). It is also clear that
\[
\exp \left( -\frac{2}{\epsilon Q^2} \frac{\partial^2}{\partial \psi^2} \right) F(\psi)
\]
is the finite operator. It is because
\[
\begin{align*}
\exp \left( -\frac{2}{\epsilon Q^2} \frac{\partial^2}{\partial \psi^2} \right) \frac{1}{2l!} \langle (\psi_c + \psi)^{2l} \rangle_\psi \\
= \exp \left( -\frac{2}{\epsilon Q^2} \frac{\partial^2}{\partial \psi^2} \right) \sum \frac{1}{2m!} \psi_c^{2m} \frac{1}{n!} \left( \frac{1}{2} \langle \psi^2 \rangle_\psi \right)^n \\
= \sum \frac{1}{m!} \left( -\frac{2}{\epsilon Q^2} \right)^m \frac{1}{n!} \left( \frac{1}{2} \langle \psi^2 \rangle_\psi \right)^n \\
= \frac{1}{l!} \left\{ \left( -\frac{2}{\epsilon Q^2} \right)^l + \frac{2}{\epsilon Q^2} \right\}^l = 0.
\end{align*}
\]
Let us introduce the renormalization scale \( \mu \) according to its canonical dimension. In doing so, we have introduced an arbitrary scale \( \mu \) in the bare inverse coupling \( Q_B^2 = Q^2 \mu^{-\epsilon} \). Since the bare coupling cannot depend on how to decompose it, we conclude that \( Q^2 \sim \mu^\epsilon \). By demanding \( \mu \) independence on the bare operator \( \text{(B.16)} \), we can derive a renormalization group equation for the renormalized operator
\[
\mu \frac{\partial}{\partial \mu} F = -\frac{2}{Q^2} \frac{\partial^2}{\partial \psi^2} F.
\]
This equation doesn’t depend on the sign of \( \epsilon \). The operator \( F \) diffuses at long distance.

In fact the solution of this diffusion equation coincides with the finite cosmological constant operator constructed by the integral measure \( \text{(B.13)} \). It is the diffusion kernel where diffusion time is identified with \( 1/\epsilon \sim -\log \mu \) in \( 2 + \epsilon \) dimensions. In the 2-dimensional limit, it can be exactly calculated as follows
\[
\begin{align*}
\sqrt{\frac{\epsilon Q^2}{8\pi}} \int d\psi e^{-\frac{\epsilon Q^2}{\epsilon Q^2} \psi^2} e^\frac{1}{3} (1+\frac{1}{3}) \log(1+\frac{1}{3} \psi) \\
= \sqrt{\frac{\epsilon Q^2}{8\pi}} \int d\psi e^{-\frac{\epsilon Q^2}{\epsilon Q^2} (\psi-\psi_c)^2} e^\frac{1}{3} (1+\frac{1}{3}) \log(1+\frac{1}{3} \psi) \\
= \sqrt{\frac{\epsilon Q^2}{8\pi}} \int d\rho e^{-\frac{2\epsilon Q^2}{\epsilon Q^2} (\rho-\psi_c)^2} e^\frac{1}{3} (1+\frac{1}{3}) \log(1+\rho) \sim e^{Q^2 \rho \psi_c}.
\end{align*}
\]
In the $\epsilon \to 0$ limit, $\rho_0$ is determined by the saddle point approximation which leads (2.15) and the scaling dimension is determined as

$$Q^2 \rho_0 = \gamma.$$  \hfill (B.20)

The renormalized cosmological constant operator is determined to be $e^{\gamma \phi}$ in agreement with the conformal invariance approach (2.16). The advantage of this approach is that it demonstrates the original cosmological constant operator at short distance evolves toward the renormalized form at long distance due to quantum effects [19].

The Fokker-Planck equation is also a diffusion equation

$$\dot{\rho} = 2H \frac{1}{Q^2} \frac{\partial^2}{\partial \phi^2} \rho,$$  \hfill (B.21)

with the identification $Ht = \mu^\epsilon/\epsilon \sim -\log \mu$ to relate it to (B.18). Let us construct the diffusion kernel in the conjugate variables to $\phi$:

$$K = \frac{1}{\sqrt{2}} e^{-tQ^2 \frac{\partial}{\partial \phi}^2}, \quad \frac{\partial}{\partial t} K = -2H \frac{1}{Q^2} p^2 K.$$  \hfill (B.22)

The solution in the dual variables is given by

$$\rho(t, p) = K(t, p) \rho(p).$$  \hfill (B.23)

After the Fourier transformation, we obtain

$$\rho_t = \int d\phi' K(t, \phi - \phi') \rho(\phi')$$

$$= \int d\phi' \frac{Q}{\sqrt{8Ht}} e^{-\frac{Q^2(\phi - \phi')^2}{8Ht}} \frac{Q}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2}}$$

$$= \sqrt{\frac{Q^2}{2\pi(1 + 4Ht)}} e^{-\frac{Q^2}{2(1 + 4Ht)^2}}.$$  \hfill (B.24)

There is an alternative method to impose the conformal invariance on the bare operator as it is mentioned before. In this approach we construct the bare operator which is invariant under $\phi_e \rightarrow \phi_e - \varphi, \tilde{\phi} \rightarrow \tilde{\phi} + \varphi$ [10][11].

The one-loop long distance divergence is evaluated in a dimensional regularization (B.5)

$$\langle e^{\gamma \phi} \rangle = \exp \left( \frac{\gamma^2}{2} \langle \phi^2 \rangle \right) = \exp \left\{ -\frac{2\gamma^2}{Q^2} \left( \frac{1}{\epsilon} - \frac{1}{2} \frac{\hat{\phi}}{\phi} \right) \right\}.$$  \hfill (B.25)

The bare operator is constructed by subtracting the UV cut-off dependent part

$$e^{\phi_e} e^{\gamma \phi} Z, \quad Z = \exp \left( \frac{2\gamma^2}{Q^2} \frac{1}{\epsilon} \right).$$  \hfill (B.26)
Under the above transformation, the bare operator changes as

\[ e^{\gamma \phi} Z(\phi) \rightarrow e^{\gamma \phi + \frac{\gamma^2}{Q^2} \phi}. \]  

(B.27)

We find the condition

\[ \gamma + \frac{\gamma^2}{Q^2} = 1. \]  

(B.28)

One-loop computation is sufficient to perform the exact renormalization as the self-consistent solution is obtained. By solving this equation, the scaling dimension of the cosmological constant operator is determined to all-orders. It matches with the leading order renormalization process we carried out here

\[ \gamma = \frac{2}{1 + \sqrt{1 + \frac{4}{Q^2}}} = 1 - \frac{1}{Q^2} + 2 \left( \frac{1}{Q^2} \right)^2 + \cdots. \]  

(B.29)

\section{C No drift force for conformal mode}

In de Sitter spaces for super horizon modes, the effective viscosity might become large and \( \phi \) field moves with the velocity proportional to the potential force.

\[ \phi \rightarrow \phi + \frac{2}{Q^2} \frac{\partial}{\partial \phi} V(\phi) \log a = \phi + 2(e^\phi - 1)Ht. \]  

(C.1)

There is no suppression factor by \( 1/Q^2 \) here since it is a tree effect. They cancel between the propagator and the vertex. We thus obtain an analogous equation to that of the inflaton in inflation theory:

\[ \dot{\phi} = 2H(e^\phi - 1). \]  

(C.2)

This equation is obtained diagrammatically but it must follow from the equation of motion of conformal field \( \phi \).

The equation of motion for quantum field \( \phi \) is

\[ \frac{\partial^2}{\partial \tau^2} \phi - \frac{\partial^2}{\partial x^2} \phi - 2H^2 e^{\gamma \phi_c} (e^\phi - 1) = 0. \]  

(C.3)

Let us consider \( \tilde{\phi} = \gamma \phi_c + \phi \) where \( \phi_c \) is the classical solution which describes de Sitter space. We find that \( \tilde{\phi} \) satisfies the identical equation with \( \gamma \phi_c \)

\[ \frac{\partial^2}{\partial \tau^2} \tilde{\phi} - \frac{\partial^2}{\partial x^2} \tilde{\phi} - 2H^2 e^{\tilde{\phi}} = 0. \]  

(C.4)
The both $\gamma \phi_c$ and $\tilde{\phi}$ are the solution of the Liouville theory. As pointed out before, we have separated original field $\phi_0 = \gamma \phi_c + \phi$. We can consider the following transformation $\gamma \phi_c \rightarrow \gamma \phi_c + \varphi$ and $\phi \rightarrow \phi - \varphi$ which leaves the theory invariant as long as $\varphi$ represents local fluctuations.

However it is doubtful that there are two different solutions. In fact $\tilde{\phi}$ is the solution only when the second derivative with respect to time can be neglected namely near the origin of the field space $\tilde{\phi} \sim 0$. From the equations of $\phi_c$ and $\phi$:

$$\gamma \phi_c = 2Ht, \quad \dot{\phi} = 2H(e^\phi - 1),$$  \hspace{1cm} \text{(C.5)}

we obtain

$$e^{2Ht} \dot{\phi} = 2H^2 e^{\phi + 2Ht} = 2H^2 e^\phi,$$  \hspace{1cm} \text{(C.6)}

while

$$e^{2Ht} \ddot{\phi} = (2H)^2 e^{\phi + 2Ht}(e^\phi - 1) = 2H^2 e^\phi(e^\phi - 1).$$  \hspace{1cm} \text{(C.7)}

In other words $\tilde{\phi}$ is not the solution in other region. We conclude there is a unique de Sitter solution $\gamma \phi_c$ in this model. The free field solution without the potential is

$$\frac{1}{\sqrt{2p}} e^{-ip\tau + i\vec{p} \cdot \vec{x}}.$$  \hspace{1cm} \text{(C.8)}

We focus on the super horizon mode:

$$\phi_0(x) = \sqrt{\frac{8\pi}{Q^2}} \int \frac{d\vec{p}}{2\pi} \frac{1}{\sqrt{2p}} e^{i\vec{p} \cdot \vec{x}} \theta(Ha(t) - p)(a_{\vec{p}} \frac{1}{\sqrt{2p}} e^{ip\tau + i\vec{p} \cdot \vec{x}} + a^\dagger_{\vec{p}} \frac{1}{\sqrt{2p}} e^{-ip\tau - i\vec{p} \cdot \vec{x}}),$$  \hspace{1cm} \text{(C.9)}

where $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = -2\pi \delta(\vec{p} - \vec{p}')$. Since plane waves become constant in time, the time dependence is caused by the step function which restricts physical momenta $P < H$.

The Yang-Feldman type solution is

$$\phi(x) = \phi_0(x) + i \int d\tau' \int d\vec{x}' G_R(x, x') 2H^2 e^{\gamma \phi_c}(e^\phi - 1)(x').$$  \hspace{1cm} \text{(C.10)}

The retarded propagator may be approximated for super horizon mode

$$G_R(x, x') \sim \theta(t - t') \int \frac{d\vec{p}}{2\pi} - i(\tau - \tau') e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}$$

$$= -i \frac{1}{H} \theta(t - t') \delta(\vec{x} - \vec{x}') \left\{ \frac{1}{a(t')} - \frac{1}{a(t)} \right\}.$$  \hspace{1cm} \text{(C.11)}

We thus obtain

$$\phi(x) = \phi_0(x) + 2H \int^t dt' (e^{\phi(t', \vec{x})} - 1).$$  \hspace{1cm} \text{(C.12)}
By differentiating (C.12), we obtain Langevin equation.

\[
\dot{\phi}(x) = \dot{\phi}_0(x) + 2H(e^{\phi(x)} - 1), \quad \langle \dot{\phi}_0(t, \vec{x}) \dot{\phi}_0(t', \vec{x}) \rangle = \frac{4}{Q^2} H \delta(t - t'). \tag{C.13}
\]

However we believe there is no drift force effects in Liouville gravity since there is no acceptable solution except $\gamma \phi_c$. So we are left with random noise effects only

\[
\dot{\phi}(x) = \dot{\phi}_0(x), \quad \langle \dot{\phi}_0(t, \vec{x}) \dot{\phi}_0(t', \vec{x}) \rangle = \frac{4}{Q^2} H \delta(t - t'). \tag{C.14}
\]

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