On projective group properties of the 6D pseudo-Riemannian space

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We study the six-dimensional pseudo-Riemannian spaces with two time-like coordinates that admit non-homothetic infinitesimal projective transformations. The metrics are manifestly obtained and the projective group properties are determined. We also find a generic defining of projective motion in the 6-dimensional rigid \( h \)-space.

1 Introduction

The problem of defining 2D Riemannian manifolds which admit projective motions, i.e. continuous transformation groups preserving geodesics dates back to S.Lie who considered it in [1]. In more recent times, that was A.V.Aminova who has got a complete solution to this problem [1]. For the Riemannian manifolds of dimension greater than 2 the same problem has been solved by G.Fubini [4] and A.S.Solodovnikov [5].

Note that they essentially used in their studies positive definiteness of metrics under consideration. Without this positivity condition, considering pseudo-Riemannian spaces, the problem is much more complicated and requires absolutely new method of solution.

In later paper, [6] A.V.Aminova has classified all the Lorentzian manifolds of dimension \( \geq 3 \) that admit nonhomothetic projective or affine infinitesimal transformations. In each case, there were determined the corresponding maximal projective and affine Lie algebras.

The General problem is to classify the n-dimensional pseudo-Riemannian spaces admitting projective motions, i.e. continuous transformation groups preserving geodesics.

This problem is not solved yet for pseudo-Riemannian spaces with arbitrary signature.

Our concrete problem here is to study a projective group properties of a 6-dimensional pseudo-Riemannian space with signature \([ + + - - - ]\), which admits projective motions, i.e. continuous transformation groups preserving geodesics.

The Method we use here is the method of the skew-normal frames and general approach to investigation of projective motions of pseudo-Riemannian manifolds due to A. V. Aminova.

2 The metrics of the rigid \( h \)-spaces

Let me remind that vector field \( X \) on a pseudo-Riemannian manifold \((M, g)\) is an infinitesimal projective transformation if and only if [7]

\[
L_X g = h, \quad \text{generalized Killing equation} \tag{1}
\]

\[
\nabla h(Y, Z, W) = 2g(Y, Z)W \varphi + g(Y, W)Z \varphi + g(Z, W)Y \varphi, \tag{2}
\]

\(^1\)Another important result was obtained by A.Z.Petrov [2], who classified geodesically equivalent pseudo-Riemannian spaces \( V^3 \).
A necessary and sufficient condition of constant curvature is expressed by the formula

\[ g_{ij}dx^idx^j = e_2(f_4 - f_2)^2\Pi_\sigma(f_\sigma - f_2)\left\{2Adx^1dx^2 - A^2\Sigma_1(dx^2)^2\right\} + \]

\[ e_4(f_2 - f_4)^2\Pi_\sigma(f_\sigma - f_4)\left\{2\tilde{A}dx^3dx^4 - \tilde{A}^2\Sigma_2(dx^4)^2\right\} + \sum_\sigma e_\sigma\Pi'_\sigma(f_i - f_\sigma)(dx^\sigma)^2, \]

where

\[ A = \epsilon x^1 + \theta(x^2), \quad \tilde{A} = \tilde{\epsilon} x^3 + \omega(x^4), \]

\[ \Sigma_1 = 2(f_4 - f_2)^{-1} + \sum_\sigma(f_\sigma - f_2)^{-1}, \quad \Sigma_2 = 2(f_2 - f_4)^{-1} + \sum_\sigma(f_\sigma - f_4)^{-1}, \]

\[ f_2 = \epsilon x^2, \quad f_4 = \tilde{\epsilon} x^4 + a, \quad \epsilon, \tilde{\epsilon} = 0, 1, \quad a \text{ is a constant which is nonzero when } \tilde{\epsilon} = 0, \quad f_\sigma(x^\sigma), \theta(x^2), \omega(x^4) \]

are arbitrary functions, \( e_i = \pm 1, \quad \sigma = 5, 6. \)

The tensor \( h_{ij} \) of the \( h \)-space of \([2211]\) type is

\[ h_{ij}dx^idx^j = 2f_4g_{12}dx^1dx^2 + (f_2g_{22} + Ag_{12})(dx^2)^2 + 2f_4g_{34}dx^3dx^4 + \]

\[ (f_4g_{44} + \tilde{A}g_{34})(dx^4)^2 + \sum_\sigma f_\sigma g_{\sigma\sigma}(dx^\sigma)^2 + (2f_2 + 2f_4 + \sum_\sigma f_\sigma + c)g_{ij}, \]

For every solution \( h_{ij} \) of the geodesic Eisenhart equation, there is a quadratic first integral

\[ (h_{ij} - 4\varphi g_{ij})\dot{x}^i\dot{x}^j = \text{const}, \]

where \( \dot{x}^i \) is the tangent vector to the geodesic.

For the other rigid \( h \)-spaces we have similar results.

### 3 On projective group properties of the 6-dimensional pseudo-Riemannian space

To move further, we need a necessary and sufficient condition of constant curvature of this \( h \)-spaces. A necessary and sufficient of condition of constant curvature is expressed by the formula

\[ R^i_{jkl} = K(\delta_k^ig_{jl} - \delta_l^ig_{jk}), \quad K = \text{const}. \]
Calculating components of the curvature tensor of the rigid $h$-spaces and substituting into this equality, one obtains a necessary and sufficient conditions of constant curvature of this $h$-space.

In particular, for the $h$-space of [2211] type

$$\rho_p - \rho_{p\sigma} = \rho_p - \rho_{pq} = \epsilon = \tilde{\epsilon} = 0 \quad (p \neq q, p, q = 2, 4, \sigma = 5, 6), \quad (7)$$

where

$$\rho_p = -\frac{1}{4} \sum_{\sigma} \frac{(f'_\sigma)^2}{(f_\sigma - f_p)^2 g_{\sigma\sigma}}, \quad \rho_{pq} = -\frac{1}{4} \sum_{\sigma} \frac{(f'_\sigma)^2}{(f_\sigma - f_p)(f_\sigma - f_q) g_{\sigma\sigma}},$$

$$\rho_{p\sigma} = -\frac{1}{4} \frac{(f'_\sigma)^2}{(f_\sigma - f_p)g_{\sigma\sigma}} \left\{ \frac{2f''_\sigma}{(f'_\sigma)^2} - \frac{1}{f_\sigma - f_p} + \sum_{i, i \neq \sigma} (f_i - f_\sigma)^{-1} \right\} - \frac{1}{4} \sum_{\gamma, \gamma \neq \sigma} (f'_\gamma)^2 \frac{1}{(f_\gamma - f_p)(f_\gamma - f_\sigma) g_{\gamma\gamma}}.$$

Further, investigating the Eisenhart equations and their integrability conditions for each obtained rigid $h$-spaces we prove some theorems, when give important information about structure projective Lie algebra in the rigid $h$-spaces.

**Theorem 1.** Any defining function of projective motion in rigid $h$-spaces of nonconstant curvature can be presented as $\phi = a_1 \varphi$, where $a_1$ is a constant.

**Theorem 2.** Any covariantly constant symmetric tensor $b_{ij}$ in 6-dimensional rigid $h$-spaces of nonzero curvature is proportional to the fundamental tensor, i.e. $b_{ij} = a_2 g_{ij}$, where $a_2$ is a constant.

From this theorems, one immediately obtains

**Theorem 3.** The affine group of 6-dimensional rigid $h$-spaces of non-constant curvature consists of homothetics.

These theorems and linearity of the Eisenhart equation give the general solution of the Eisenhart equation in the rigid $h$-spaces of non-constant curvature in the form

$$a_1 h_{ij} + a_2 g_{ij} \quad (8)$$

with two arbitrary constants $a_1, a_2$.

Hence, one obtains

**Theorem 4.** All projective motions of a 6-dimensional rigid $h$-spaces of non-constant curvature are obtained by integrating the equation

$$L \xi g_{ij} = \xi_{i,j} + \xi_{j,i} = a_1 h_{ij} + a_2 g_{ij}. \quad (9)$$

It leads to the following important group characteristic of rigid $h$-spaces:

**Theorem 5.** If rigid $h$-spaces of non-constant curvature admit a nonhomothetic projective Lie algebra $P_r$, then this algebra contains the subalgebra $H_{r-1}$ of infinitesimal homothetics of dimension $r - 1$.

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