Bose condensation at high temperatures

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February 1999
PACS numbers: 05.10.Cc, 05.70.Jk, 11.10.Wx, 11.10.Kk

Abstract

Bose condensation is usually a low temperature phenomenon due to a low particle number density. When the number density is kept large compared to the inverse Compton volume, Bose condensation can occur at a temperature much higher than the mass of the particle. We can then use a three dimensional effective theory to study the thermal properties. We compute the transition temperature for a complex scalar field theory with a small interaction parameter.

In a previous letter [1] we used an effective three dimensional field theory to compute the critical temperature of the weak coupling $\phi^4$ theory. The purpose of this letter is to extend the same technique to study theories with a chemical potential coupled to a conserved particle number. We will compute the transition temperature $T_c$ for Bose condensation and the temperature dependence of the specific heat near $T_c$.

It is the low number density of particles which makes Bose condensation a low temperature phenomenon. If the number density is large compared to the inverse Compton volume $m^3$, where $m$ is the particle mass, the transition temperature $T_c$ for Bose condensation becomes much higher than $m$ so that $T_c$ is calculable using a three dimensional effective theory.

In order to see this possibility more quantitatively, we consider a free theory of spinless particles and antiparticles of mass $m$. The particle number density $n$ (or more precisely the number density of particles minus that of antiparticles) is an increasing function of the chemical potential $\mu$. At the maximum chemical potential $\mu = m$, we obtain the maximum number density without Bose condensation. For temperature $T$ much higher than
For concreteness we consider a complex scalar field theory with a global U(1) symmetry in thermal equilibrium at temperature $T$. The lagrangian density in the four dimensional euclidean space is given by

$$L_0 = \partial_\mu \phi^* \partial_\mu \phi + m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 + \text{(counterterms)},$$

where $\phi$ is a complex scalar field which is periodic in the euclidean time direction:

$$\phi(\vec{x}, \tau + 1/T) = \phi(\vec{x}, \tau).$$

The counterterms are given in the $\overline{\text{MS}}$ scheme. Hence, the renormalized parameters satisfy the renormalization group (RG) equations

$$\frac{d}{dt} \frac{\lambda}{(4\pi)^2} = -5 \left( \frac{\lambda}{(4\pi)^2} \right)^2 + 15 \left( \frac{\lambda}{(4\pi)^2} \right)^3 + \ldots,$$

$$\frac{d}{dt} m^2 = \left( 2 - 2 \frac{\lambda}{(4\pi)^2} + \frac{5}{2} \left( \frac{\lambda}{(4\pi)^2} \right)^2 + \ldots \right) m^2.$$

We can introduce the chemical potential $\mu$ as the euclidean time component of an external U(1) gauge field:

$$A_\tau = i\mu.$$

Note this is purely imaginary. The Ward identity protects it from renormalization. The total lagrangian density is therefore

$$\mathcal{L} = L_0 - \mu Z \phi^* \partial_\tau \phi - \mu^2 Z \phi^* \phi,$$
where $Z$ is the wave function renormalization constant. The average number density of particles, $n$, is the first order derivative of the free energy density $Y_4(T, \mu)$ with respect to $\mu$:

$$n = -\left(\frac{\partial Y_4}{\partial \mu}\right)_T.$$  \hfill (9)

Our task is to find the transition temperature $T_c$ so that for $T < T_c$ the field $\phi$ gets a non-vanishing expectation value. We will find $T_c$ first for a given chemical potential $\mu$, and then for a given number density $n$.

A naive loop expansion of $Y_4$ suffers from infrared divergences at two-loop and beyond [4]. This difficulty is best avoided by reducing the theory to a three dimensional effective theory [2, 3] whose infrared properties are much better understood [5, 6]. The effective theory is given by the following lagrangian density:

$$L_3 = g_3 + \partial_\mu \phi_3^* \partial_\mu \phi_3 + m_3^2 \phi_3^* \phi_3 + \frac{\lambda_3}{4} (\phi_3^* \phi_3)^2 + \text{(counterterms)},$$  \hfill (10)

where the counterterms are given in the \text{MS} scheme. The higher dimensional interaction terms are negligible within our approximation. The RG equations of the parameters are given by

$$\frac{d}{dt} g_3 = 3 g_3 + \tilde{A} \lambda_3^2, \quad \frac{d}{dt} m_3^2 = 2 m_3^2 + \tilde{C} \lambda_3^2, \quad \frac{d}{dt} \lambda_3 = \lambda_3,$$  \hfill (11)

where

$$\tilde{A} = \frac{1}{(4\pi)^2} \frac{5}{3 \cdot 2^9}, \quad \tilde{C} = -\frac{1}{(4\pi)^2} \frac{1}{2}.$$  \hfill (12)

The parameters of the two theories are related such that the free energy density $F_3(g_3, m_3^2, \lambda_3)$ of the effective theory reproduces that of the original theory:

$$Y_4(T, \mu) = T F_3(g_3, m_3^2, \lambda_3).$$  \hfill (13)

For the three dimensional reduction to be valid, the temperature must be high compared to $m$, $\mu$:

$$T \gg m \approx \mu.$$  \hfill (14)

Then, we can choose a renormalization scale $\Lambda$ such that

$$T/\Lambda = O(N), \quad m^2/\Lambda^2 = O(N), \quad \mu^2/\Lambda^2 = O(N),$$  \hfill (15)
where $O(N)$ denotes an order of $N$, a large number. We consider such a range of $T$ so that we can identify the smallness of $1/N$ with the smallness of the coupling $\lambda$:

$$\lambda = O(1/N). \quad (16)$$

Eqns. (13-16) guarantee that we can calculate the parameters of the effective theory in powers of $\lambda$, $m^2/T^2$, and $\mu^2/T^2$ which are all of order $1/N$. Since the calculation is straightforward, we omit the detail and only state the result:

$$\lambda_3 \simeq \lambda T, \quad (17)$$

$$m_3^2 \simeq m^2 - \mu^2 + \frac{\lambda}{12} T^2 + \frac{\lambda}{(4\pi)^2} \left[ 2m^2(\ln T/\Lambda + j_2) - 2\mu^2 \right]$$

$$+ \frac{\lambda^2}{(4\pi)^2} T^2 \left[ -\frac{1}{12} \ln T/\Lambda + \frac{j_2}{6} - \frac{j_3}{4} \right], \quad (18)$$

$$Tg_3 \simeq T^4 \frac{(4\pi)^2}{144} \left[ -\frac{1}{5} + \frac{1}{2} \ln T/\Lambda + \frac{1}{2} \left( \frac{3}{2} \ln T/\Lambda + j_2 + \frac{3}{2} j_4 \right) \right]$$

$$+ \frac{T^2}{12} \left[ m^2 \left( 1 + 2 \frac{\lambda}{(4\pi)^2}(\ln T/\Lambda + j_2) + \mu^2 \left( -3 - 2 \frac{\lambda}{(4\pi)^2} \right) \right) \right]$$

$$+ \frac{1}{(4\pi)^2} \left[ m^4(\ln T/\Lambda + j_2) + \frac{2}{3} \mu^4 - 2m^2\mu^2 \right], \quad (19)$$

where the constants are given by $[7]$:

$$j_2 = \ln 4\pi - \gamma, \quad j_3 = \ln 4\pi - 1 - \frac{\zeta'(-1)}{\zeta(-1)},$$

$$j_4 = \ln 4\pi - \frac{31}{30} - 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-3)}{\zeta(-3)}. \quad (20)$$

In the above we have computed $\lambda_3, m_3^2/\Lambda^2$ to order $N^0$ and $Tg_3/\Lambda^4$ to order $N^2$.

Let us determine the transition temperature $T_c$ as a function of the chemical potential $\mu$. In the effective theory the expectation value $\langle \phi_3 \rangle \propto \langle \phi \rangle$ is non-vanishing if

$$R(m_3^2, \lambda_3) < R_c,$$

where $R(m_3^2, \lambda_3)$ is an RG invariant defined by $[7]$

$$R(m_3^2, \lambda_3) \equiv \frac{m_3^2}{\lambda_3^3} - \tilde{C} \ln \lambda_3. \quad (22)$$

$^{2}$The necessary integrals have been calculated by Arnold and Zhai $[7]$. 
Only a non-perturbative calculation can determine \( R_c \), and we must leave it as an unknown constant here. By substituting eqns. (17,18) into \( R = R_c \), we obtain the transition temperature \( T_c \):

\[
\frac{\lambda}{12} T_c^2 \simeq \mu^2 - m^2 + \frac{\lambda}{(4\pi)^2} \left[ 12(\mu^2 - m^2) \left( (4\pi)^2 R_c - \frac{1}{2} \ln \lambda - \frac{5}{24} \ln \frac{12(\mu^2 - m^2)}{\lambda\Lambda^2} + \frac{j_3}{4} - \frac{j_2}{6} \right) 
- m^2 \ln \frac{12(\mu^2 - m^2)}{\lambda\Lambda^2} - 2m^2 j_2 + 2\mu^2 \right],
\]

(23)

where the constants \( j_2, j_3 \) are given by eqns. (20). The above gives \( T_c/\Lambda \) to order \( N^0 \). The dependence on \( \ln \lambda \) is the source of infrared divergences in the naive loop expansions.

Eqn. (23) gives \( T_c \) as a function of the chemical potential \( \mu \), but it is more convenient to express \( T_c \) as a function of the number density \( n \). To do this, we must express \( \mu \) as a function of \( n \) by inverting eqn. (9). The equivalence (13) implies that we must obtain the free energy density \( F_3 \) of the effective theory. Since the cosmological constant \( g_3 \) has nothing to do with interactions, we find

\[
F_3(g_3, m_3^2, \lambda_3) = g_3 + f_3(m_3^2, \lambda_3).
\]

(24)

The RG eqns. (11) imply that the function \( f_3 \) can be written as

\[
f_3(m_3^2, \lambda_3) = \lambda_3^3 \left( -\tilde{A} \ln \lambda_3 + \overline{f_3}(R) \right),
\]

(25)

where \( \tilde{A} \) is given in eqns. (12), and \( \overline{f_3} \) is a function of the RG invariant \( R \) (22). The theory of critical phenomena gives the following scaling formula near \( R = R_c \) [8, 9]:

\[
\overline{f_3}(R) = \overline{f_3}(0) + a \left| R - R_c \right|^{\frac{3}{y_E}} + O \left( \left| R - R_c \right|^{\frac{3-y'}{y_E}} \right),
\]

(26)

where \( a > 0 \) is a constant. The constants \( y_E \) and \( y' \) are the critical exponents of the three dimensional XY model: \( y_E > 0 \) is the scale dimension of the relevant parameter, and \( y' < 0 \) is that of the least irrelevant parameter. They are given approximately as [5]

\[
y_E \simeq 1.6, \quad y' \simeq -0.4.
\]

(27)
Substituting the above results into eqn. (9), we obtain the relation between \( n \) and \( \mu \)

\[
n = 2\mu T^2 \left[ \frac{1}{4} - \frac{1}{N^2} \frac{1}{(4\pi)^2} \frac{\mu^2 - 2m^2}{T^2} + \frac{\lambda}{(4\pi)^2} \left( \frac{1}{6} + (4\pi)^2 f_3'(R) \right) \right]
\]  

(28)

up to terms of order \( N^2 \Lambda^3 \). Note that the derivative \( f_3'(R) \) vanishes at criticality \( R = 0 \).

Solving eqns. (23,28), we can obtain the critical temperature \( T_c \) as a function of the number density \( n \) to the order \( N^0 \Lambda \). For simplicity, however, we will only present the result at the leading order which is \( N \Lambda \). Eqns. (23,28)

give

\[
\frac{1}{12} \lambda T_c^2 = \mu^2 - m^2, \quad n = \frac{1}{2} \mu T_c^2.
\]

(29)

These give a cubic equation for \( X \equiv \frac{mT_c^2}{2\pi} \):

\[
\nu X^3 = 1 - X^2,
\]

(30)

where \( \nu \equiv \frac{\lambda n}{6m^4} \) is a dimensionless constant of order \( N^0 \). Let \( X_0(\nu) \) be the solution that lies between 0 and 1. (See Fig. 1.) Then, we obtain

\[
T_c^2 = \frac{12m^2}{\lambda} \nu X_0(\nu), \quad \mu^2 - m^2 = m^2 \nu X_0(\nu).
\]

(31)

The function \( \nu X_0(\nu) \) increases as \( \nu \), and it behaves as \( \nu^2 \) for \( \nu \gg 1 \) and \( \nu - \nu^2 \) for \( \nu \ll 1 \). For a very small coupling such that \( \nu \ll 1 \), we find

\[
T_c^2 \simeq \frac{2n}{m} - \frac{n^2}{6m^4}.
\]

(32)

Therefore, the critical temperature decreases as the coupling increases. This tendency also exists in the low temperature Bose condensation phenomena.

Fig. 1  Solution of the cubic equation (30)  
Fig. 2  Specific heat near the critical temperature
Finally, let us compute the temperature dependence of the specific heat near the critical temperature $T_c$. Using the free energy density of the three dimensional effective theory given by eqns. (24–26), we obtain

$$C_\mu(T, \mu) \equiv -T \frac{\partial^2 Y_4(T, \mu)}{(\partial T)^2} = -T \frac{\partial^2 (T F_3)}{(\partial T)^2} \simeq C_\mu(T_c, \mu) + s \frac{T - T_c}{\lambda T_c} - A \left| \frac{T - T_c}{\lambda T_c} \right|^{-\alpha},$$

where

$$C_\mu(T_c, \mu) = \frac{(4\pi)^2}{60} T_c^3 + \frac{1}{3} m^2 T_c,$$  \hspace{1cm} (34)

$$s = \frac{(4\pi)^2}{30} \lambda T_c^3, \quad A = \frac{1}{6 y E} \left( \frac{3}{y E} - 1 \right) a \lambda T_c^3,$$  \hspace{1cm} (35)

and

$$\alpha \equiv 2 - \frac{3}{y E} \simeq -0.01.$$  \hspace{1cm} (36)

The above expression (33) is valid for $|T - T_c|/(\lambda T_c) = O(N^0)$, and we have ignored the terms of order $N \Lambda^3$. By keeping only the first two terms of eqn. (26), we have also ignored $((T - T_c)/(\lambda T_c))^{-y/y E} \ll 1$. The positive constant $a$ in $A$ is the unknown constant in eqn. (29). In Fig. 2 we show the specific heat schematically. The specific heat at constant number density, $C_n$, differs from $C_\mu$ only by a constant up to order $N^2 \Lambda^3$:

$$C_n - C_\mu \simeq -2 \mu^2 T_c \simeq -2 T_c \left( m^2 + \frac{\lambda}{12} T_c^2 \right).$$  \hspace{1cm} (37)

In conclusion we have shown how to apply the method of three dimensional effective theory to understand Bose condensation at high temperatures.

I would like to thank Prof. K. Kuboki for discussions.
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