ABSTRACT INTERPOLATION PROBLEM IN GENERALIZED SCHUR CLASSES.

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Abstract. An indefinite variant of the abstract interpolation problem is considered. Associated to this problem is a model Pontryagin space isometric operator \( V \). All the solutions of the problem are shown to be in a one-to-one correspondence with a subset of the set of all unitary extensions \( U \) of \( V \). These unitary extension \( U \) of \( V \) are realized as unitary colligations with the indefinite de Branges-Rovnyak space \( D(s) \) as a state space.

1. Introduction.

The abstract interpolation problem in the Schur class \( S \) have been posed and considered by V. Katsnelson, A. Khajefet and P. Yuditskij [15]. It contains the most classical interpolation problems such as the moment problem, the bitangential Schur-Nevanlinna-Pick problem and others (see [21], [19], [20] and [32]). The method of abstract interpolation problem contains and develops ideas of the V. P. Potapov’s approach to interpolation problems [22], the theory of unitary colligation [10], and the theory of reproducing kernel Hilbert spaces [9]. In particular, the results of D. Z. Arov and L. Z. Grossman on scattering matrices of unitary operators [4] were used in order to describe the set of solutions of this problem. These results are closely related to the M.G. Krein’s theory of \( L \)-resolvent matrices for symmetric operators [23] and [24] (see also [31] and [15]).

The present paper deals with the indefinite abstract interpolation problem \( AIP(\kappa) \), \( \kappa \in \mathbb{Z}_+ \) in generalized Schur classes (see definition below). It is shown that this problem can be reduced to the extension problem for a model Pontryagin space isometric operator \( V \), associated with the problem \( AIP(\kappa) \). As distinct from the Hilbert space case the correspondence between minimal unitary extensions \( U \) of \( V \) and their scattering matrices does not give anymore a parametrization of the solution set of the problem \( AIP(\kappa) \). The desired description is given in Section 5 by selecting of a subclass of the so-called \( L \)-regular unitary extensions of \( V \). The unitary extension \( U \) of \( V \) is realized in the paper as a unitary colligation with the indefinite de Branges–Rovnyak space \( D(s) \) as a state space. The corresponding construction is very close to that given in [1] and [12]. The statement of the abstract interpolation problem in the present paper is different from the statement of this problem in [12]. The problem data in this paper contain two different operators \( M \) and \( N \) whereas in the paper [12] one of them equals an identity operator. The description of solutions of the Problem \( AIP(\kappa) \) in the present formulation can be used for getting a description of the bitangential interpolation problem.

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2. Preliminaries.

2.1. Linear relations. Let $H_1$, $H_2$ be Hilbert spaces. A linear manifold $T \subset H_1 \oplus H_2$ is called a linear relation (shortly l.r.) in $H_1 \oplus H_2$ (from $H_1$ to $H_2$). We denote by $\tilde{C}(H_1, H_2)$ ($C(H)$) the set of all closed linear relations in $H_1 \oplus H_2$ (in $H \oplus H$). For a linear relation $T \subset H_1 \oplus H_2$ we denote by $\text{dom} T$, $\text{ran} T$, $\ker T$ and $\text{mul} T$ the domain, the range, the kernel and the multivalued part of $T$ respectively.

If $T$ is a linear relation in $H_1 \oplus H_2$, then the inverse $T^{-1}$ and adjoint $T^*$ relations are defined as

$$T^{-1} = \left\{ \left[ f' \right] : \left[ f \right] \in T \right\}, \quad T^{-1} \subset H_2 \oplus H_1$$

$$T^* = \left[ g' \right] \in H_1 \oplus H_2 : (f', g) = (f, g'), \quad \left[ f \right] \in T, \ T^* \in \tilde{C}(H_2, H_1).$$

A closed linear operator $T$ from $H_1$ to $H_2$ is identified with its graph $\text{gr} T \in \tilde{C}(H_1, H_2)$.

In the case $T \in \tilde{C}(H_1, H_2)$ we write:

- $0 \in \rho(T)$ if $\ker T = 0$ and $\text{ran} T = H_2$;
- $0 \in \tilde{\rho}(T)$ if $\ker T = 0$ and $\text{ran} T = H_2 \neq H_2$;
- $0 \in \sigma(T)$ if $\ker T = 0$ and $\text{ran} T = H_2 \neq \text{ran} T$;
- $0 \in \sigma_\rho(T)$ if $\ker T \neq 0$;
- $0 \in \sigma_\tilde{\rho}(T)$ if $\ker T \neq 0$ and $\text{ran} T \neq H_2$.

For a l.r. $T \in \tilde{C}(H)$ we denote by $\rho(T) = \{ \lambda \in \mathbb{C} : 0 \in \rho(T - \lambda) \}$ and $\tilde{\rho}(T) = \{ \lambda \in \mathbb{C} : 0 \in \tilde{\rho}(T - \lambda) \}$ the resolvent set and the set of regular type points of $T$ respectively. Next, $\sigma(T) = \mathbb{C} \setminus \rho(T)$ stands for the spectrum of $T$.

2.2. Linear relations in Pontryagin spaces. In this subsection we review some facts and notation from [5, 8]. Let $H$ be a Hilbert space and $j_H$ be a signature operator in this space, (i.e., $j_H = j_H^* = j_H^{-1}$). The space $H$ can be considered as a Krein space $(H, j_H)$ (see [5]) with the inner product $\langle \varphi, \psi \rangle_H = (j_H \varphi, \psi)_H$. The signature operator $j_H$ can be represented as $j_H = P_+ - P_-$, where $P_+$ and $P_-$ are orthogonal projections in $H$. In the case when $\dim P_+ H = \kappa < \infty$, the Krein space $(H, j_H)$ is called a Pontryagin space with the negative index $\kappa$ and is denoted by $\text{ind}_- H = \kappa$.

Let us consider two Pontryagin spaces $(H_1, j_{H_1})$, $(H_2, j_{H_2})$ and a linear relation $T$ from $H_1$ to $H_2$. Then an adjoint l.r. $T^{[*]}$ consists of pairs $\left[ \begin{array}{c} f_2 \\ g_2 \end{array} \right] \in H_2 \times H_1$ such that

$$\left[ \begin{array}{c} f_2 \\ g_2 \end{array} \right] |_{H_2} = \left[ \begin{array}{c} f_1 \\ g_1 \end{array} \right] |_{H_1} \text{ for all } \left[ \begin{array}{c} f_1 \\ f_2 \end{array} \right] \in T.$$ 

If $T^{[*]} : H_2 \rightarrow H_1$ is an adjoint linear relation of $T$ in the Hilbert spaces $H_1$ and $H_2$ then $T^{[*]} = j_{H_1} T^* j_{H_2}$.

The l.r. $T^{[*]}$ satisfies the following equations

$$\text{(dom } T^{[*]})^{[\perp]} = \text{mul} T^{[*]}, \quad (\text{ran } T^{[*]})^{[\perp]} = \ker T^{[*]},$$

where the sign $[\perp]$ denotes orthogonality in Pontryagin spaces.

**Definition 2.1.** A l.r. $T$ from a Pontryagin space $(H_1, j_{H_1})$ to a Pontryagin space $(H_2, j_{H_2})$ is called an isometry if the equality

$$\langle \varphi', \varphi \rangle_{H_2} = \langle \varphi, \varphi \rangle_{H_1},$$

holds.
holds for every $\varphi \in T$ and it is called a contraction if the sign in the (2.3) is substituted for $\leq$. A l.r. $T$ is called a unitary l.r. from $\mathcal{H}_1^j$ to $\mathcal{H}_2^j$ if $T^{-1} = T^*$. Clearly, a l.r. $T$ is an isometric l.r. if and only if $T^{-1} \subset T^*$.

We recall (5) that the sets $\mathbb{D} \setminus \tilde{\rho}(T)$ and $\mathbb{D}_c \setminus \tilde{\rho}(T)$ for an isometric operator $T$ in a Pontryagin space $\text{ind} \mathcal{H} = \kappa$ consist of at most $\kappa$ points belonging to $\sigma_p(T)$.

The definition of unitary l.r. at first was introduced in [30]. In particular, in [30] the following Proposition was proved

**Proposition 2.2.** If $T$ is a unitary relation from a Pontryagin space $(\mathcal{H}_1^j, j\mathcal{H}_1^j)$ to a Pontryagin space $(\mathcal{H}_2^j, j\mathcal{H}_2^j)$ then

1. $\text{dom} T$ is closed if and only if $\text{ran} T$ is closed;
2. the equalities $\ker T = \text{dom} T^\perp$, $\mul T = \text{ran} T^\perp$ hold.

From Proposition 2.2 we get

**Corollary 2.3.** If $T$ is a unitary l.r. in a Pontryagin space then the condition $\mul T \neq \{0\}$ is equivalent to the condition $\ker T \neq \{0\}$. Moreover, the equality $\dim \mul T = \dim \ker T$ holds.

### 2.3. The generalized Schur class.

Recall that a Hermitian kernel $K_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m \times m}$ is said to have $\kappa$ negative squares if for every positive integer $n$ and every choice of $\lambda_j \in \Omega$ and $u_j \in \mathbb{C}^m (j = 1, \ldots, n)$ the matrix

$$
[|K_{\lambda_j}(\lambda_k) u_j, u_k|]_{j,k=1}^n
$$

has at most $\kappa$ negative eigenvalues and for some choice of $\lambda_1, \ldots, \lambda_n \in \Omega$ and $u_1, \ldots, u_n \in \mathbb{C}^m$ exactly $\kappa$ negative eigenvalues. In this case we write

$$
\text{sq}_- K = \kappa.
$$

Let $\kappa$ be a nonnegative integer, $\mathcal{L}_2 = \mathbb{C}^p$, $\mathcal{L}_1 = \mathbb{C}^q$ $(p, q \in \mathbb{Z}_+)$ and let $\mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$ be the set of $p \times q$-matrices. A $\mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$ valued function holomorphic in a neighborhood of $0$ is said to belong to the generalized Schur class $S_\kappa(\mathcal{L}_2, \mathcal{L}_1)$ in the unit disc if the kernel

$$
A_\omega(\lambda) = \frac{I_p - s(\lambda) s(\omega)^*}{1 - \overline{\omega} \lambda} \quad (\lambda, w \in \Omega_s \subset \mathbb{D})
$$

has $\kappa$ negative square on $\Omega_s$ (see [24]). The class $S(\mathcal{L}_2, \mathcal{L}_1) := S_0(\mathcal{L}_2, \mathcal{L}_1)$ consists of usual Schur function. An example of a generalized Schur function is provided by the Blaschke-Potapov product

$$
b(\lambda) = \prod b_j(\lambda), \quad b_j(\lambda) = I - P_j + \frac{\lambda - \alpha_j}{1 - \sigma_j \lambda} P_j,
$$

where $\alpha_j \in \mathbb{D}$, $P_j$ orthoprojections in $\mathbb{C}^p$ $(j = 1, \ldots, k)$. The factor $b_j(\cdot)$ is called simple if $P_j$ has rank one. Although $b(\cdot)$ can be written as a product of simple factors in many ways; the number of this factors is the same for every representation (2.4). It is called the degree of the Blaschke-Potapov product $b(z)$ [25].

A theorem of Krein-Langer [24] guarantees that every generalized Schur function $s(\cdot) \in S_{\kappa}^{k \times q}(\mathcal{L}_2, \mathcal{L}_1)$ admits a factorization of the form

$$
s(\lambda) = b(\lambda)^{-1} s_1(\lambda) \quad (\lambda \in \Omega_s),
$$
where \( b_l(\cdot) \) is a Blaschke-Potapov product of degree \( \kappa \), \( s_l(\cdot) \) is in the Schur class \( S(\mathbb{L}_2, \mathbb{L}_1) \) and

\[
\ker s_l(\lambda)^* \cap \ker b_l(\lambda)^* = \{0\} \quad (\lambda \in \Omega_s).
\]

The representation (2.5) is called a left Krein-Langer factorization. The constraint (2.6) can be expressed in the equivalent form

\[
\text{rank}[b_l(\lambda), s_l(\lambda)] = p \quad (\lambda \in \Omega_s).
\]

If \( \alpha_j \in \mathbb{D} \) \((j = 1, \ldots, n)\) are all zeros of \( b_l(\cdot) \) in \( \mathbb{D} \), then the condition (2.6) ensures that \( \Omega_s = \mathbb{D} \setminus \{\alpha_1, \ldots, \alpha_n\} \). The left Krein-Langer (2.5) is essentially unique in a sense that \( b_l(\cdot) \) is defined uniquely up to a left unitary factor \( V \in \mathbb{C}^{p \times p} \).

Similarly, every generalized Schur function \( s(\cdot) \in S_{p \times q}^p(\mathbb{L}_2, \mathbb{L}_1) \) a right Krein-Langer factorization

\[
s(\lambda) = s_r(\lambda)b_r(\lambda)^{-1}, \quad (\lambda \in \Omega_s),
\]

where \( b_r(\cdot) \) is a Blaschke-Potapov product of degree \( \kappa \), \( s_r(\cdot) \) is in the Schur class \( S(\mathbb{L}_2, \mathbb{L}_1) \) and

\[
\ker s_r(\lambda) \cap \ker b_r(\lambda) = \{0\} \quad (\lambda \in \Omega_s).
\]

This condition can be rewritten in the equivalent form

\[
\text{rank}[b_r(\lambda), s_r(\lambda)] = q \quad (\lambda \in \Omega_s).
\]

Under assumption (2.8) the matrix valued function \( b_r(\cdot) \) is uniquely defined up to a right unitary factor \( V' \in \mathbb{C}^{q \times q} \).

Let \( \Pi_+ \) and \( \Pi_- \) denote the orthogonal projections from \( L^2_k \) onto \( H^k_2 \) and \( (H^k_2)^\perp \) respectively, where \( k \) is a positive integer that will be understood from the context. Let us introduce the Hilbert spaces

\[
H(b_r) := H^2_2 \ominus b_r H^2_2, \quad H_{s}(b_l) := (H^2_2)^\perp \ominus b_l^* (H^2_2)^\perp
\]

and the operators

\[
X_r : h \in H(b_r) \to \Pi_- s^* h, \quad X_l : h \in H_{s}(b_l) \to \Pi_+ s^* h
\]

based on \( s(\cdot) \).

The next operators will play an important role.

**Definition 2.4.** Let

\[
\Gamma_l : f \in L^2_k \to X_l^{-1} P_{H(b_r)} f \in H_{s}(b_l);
\]

\[
\Gamma_r : g \in L^2_k \to X_r^{-1} P_{H_{s}(b_l)} g \in H(b_r),
\]

where \( X_l \) and \( X_r \) are defined by (2.12).

This operators \( \Gamma_r \) and \( \Gamma_l \) are using for introducing a metric in some space.
2.4. Unitary colligations in Pontryagin spaces. The theory of unitary colligations in Hilbert spaces was introduced in the paper [27] and had further development in the papers [11, 19]. The theory of unitary colligations in Pontryagin spaces was built in [26] and [1], in the latter the functional models of these colligations were studied.

In the present paper we use the notation \( \hat{D}(s) \) which is different from that used in the papers [11] and [12].

Let us recall some basic notions from the theory of unitary colligations (see [10], [16]). Let \( \mathcal{H} \) be a Pontryagin space with the negative index \( \kappa \) (see [4, 5]), let \( \mathcal{L}_1, \mathcal{L}_2 \) be Hilbert spaces, and let \( U = \begin{bmatrix} T & F \\ G & H \end{bmatrix} \) be a unitary operator from \( \mathcal{H} \oplus \mathcal{L}_2 \) into \( \mathcal{H} \oplus \mathcal{L}_1 \). Then the quadruple \( \Delta = (\mathcal{H}, \mathcal{L}_2, \mathcal{L}_1, U) \), where \( \mathcal{H} \) denotes the so-called state space and \( \mathcal{L}_2, \mathcal{L}_1 \) stand for the incoming and outgoing spaces, respectively, is said to be a unitary colligation.

The colligation \( \Delta \) is said to be simple, if there is no reducing subspace \( \mathcal{H}_1 \subset \mathcal{H} \). The colligation \( \Delta \) is simple (see [16]) if and only if
\[
(2.15) \quad (\mathcal{H}_\Delta :=) \overline{\text{span}} \left\{ T^n F h_2, T^n G^* h_1 : h_1 \in \mathcal{L}_1, h_2 \in \mathcal{L}_2, n \in \mathbb{Z}_+ \right\} = \mathcal{H}.
\]

The operator valued function
\[
(2.16) \quad s(z) = H + \lambda G (I - \lambda T)^{-1} F : \mathcal{L}_2 \to \mathcal{L}_1 \quad (1/\lambda \in \rho(T))
\]
is said to be the characteristic function of the colligation (or the scattering matrix of the unitary operator \( U \) with respect to the channel subspaces \( \mathcal{L}_2, \mathcal{L}_1 \) see [4]). If the colligation \( \Delta \) is simple then \( s \in S_{\kappa}(\mathcal{L}_2, \mathcal{L}_1) \). One can rewrite the formula (2.16) in the form
\[
(2.17) \quad s(\lambda) = P_{\mathcal{L}_1} U (I - \lambda P_{\mathcal{H}} U)^{-1} P_{\mathcal{L}_2} = P_{\mathcal{L}_1} (I - \lambda U P_{\mathcal{H}})^{-1} U P_{\mathcal{L}_2},
\]

since
\[
(I - \lambda P_{\mathcal{H}} U)^{-1} = \begin{bmatrix} (I - \lambda T)^{-1} & \lambda (I - \lambda T)^{-1} F \\ 0 & I \end{bmatrix},
\]
\[
(2.18) \quad U (I - \lambda P_{\mathcal{H}} U)^{-1} = \begin{bmatrix} T (I - \lambda T)^{-1} & F + \lambda T (I - \lambda T)^{-1} F \\ G (I - \lambda T)^{-1} & H + \lambda G (I - \lambda T)^{-1} F \end{bmatrix}.
\]

Here \( P_{\mathcal{H}}, P_{\mathcal{L}_i} \) are orthogonal projections from \( \mathcal{H} \oplus \mathcal{L}_i \) onto \( \mathcal{H} \) and \( \mathcal{L}_i \) \( (i = 1, 2) \), respectively.

2.5. The de Branges-Rovnyak space \( D(s) \). The symbol \( A^{[-1]} \) stands for the Moore-Penrose pseudoinverse of the matrix \( A \) (17),
\[
(2.19) \quad \Delta_s(\mu) := \begin{bmatrix} I_p \\ -s(\mu)^* \\ s(\mu) \\ I_q \end{bmatrix}
\]
a.e. on \( \mathbb{T} \) for \( s(\cdot) \in S^{p \times q} \).

Definition 2.5. Let a matrix valued function \( s(\cdot) \in S^{p \times q} \) admit left and right Krein-Langer factorizations (2.5) and (2.5). Define \( D(s) \) as the set of vector valued functions \( f(t) = \begin{bmatrix} f_+(t) \\ f_-(t) \end{bmatrix} \) such that the following conditions hold:

1. \( b f_+ \in H^2_x \);
2. \( b f_- \in (H^2_\mathbb{R})^\perp \);
\[ (3) \quad f(t) \in \text{ran} \Delta_s(\mu) \ a.e. \ on \ \mathbb{T} \text{ and the following integral} \\
\frac{1}{2\pi} \int_{\mathbb{T}} f(\mu)^* \Delta_s(\mu)^{-1} f(\mu) d\mu \\
\text{converges.} \]

The inner product in \( \mathcal{D}(s) \) is defined by
\[ [f, g]_{\mathcal{D}(s)} = \frac{1}{2\pi} \int_{\mathbb{T}} g(\mu)^* \left( \Delta_s(\mu)^{-1} \right) f(\mu) d\mu, \]
where the operator \( \Gamma_r \) was defined in (2.14).

As has been shown in \[13\] the space \( \mathcal{D}(s) \) is a Pontryagin space with the negative index \( \kappa \). In the case when \( \kappa = 0 \) the space \( \mathcal{D}(s) \) was introduced in \[9\] (see also \[12\]).

2.6. The generalized Potapov class and generalized \( J \)-inner functions. Let \( \kappa, m \in \mathbb{N} \) and \( J \) be a \( m \times m \) signature matrix (i.e., \( J = J^* \), \( JJ^* = I_m \)).

**Definition 2.6.** Recall that a meromorphic in \( \mathbb{D} \) \( m \times m \)-valued matrix function \( W(\cdot) \) belongs the generalized Potapov class \( \mathcal{P}_\kappa(J) \) \[2\], if the kernel
\[ K^W_\omega(\lambda) = \frac{J - W(\lambda)JW(\omega)^*}{1 - \lambda\overline{\omega}} \]
has \( \kappa \) negative squares in \( \Omega_W \), where \( \Omega_W \) is the domain of holomorphy of \( W \) in \( \mathbb{D} \).

**Definition 2.7.** \[2\] \[13\] A meromorphic in \( \mathbb{D} \) \( m \times m \)-valued matrix function \( W(\cdot) \) is called \( J \)-inner matrix valued function (it is denoted by \( W \in \mathcal{U}_\kappa(J) \)), if it belongs the generalized Potapov class \( \mathcal{P}_\kappa(J) \) and
\[ J - W(\mu)JW(\mu)^* = 0 \]
a.e. \( \mu \in \mathbb{T} \). This class is denoted by \( \mathcal{U}_\kappa(J) \).

2.7. Reproducing kernel Pontryagin spaces. A Pontryagin space \( (\mathcal{H}, [\cdot, \cdot]_\mathcal{H}) \) of \( \mathbb{C}^m \)-valued functions defined in a subset \( \Omega \subset \mathbb{C} \) is called a reproducing kernel Pontryagin space if there exists a Hermitian kernel \( K^\mu_\lambda : \Omega \times \Omega \to \mathbb{C}^{m \times m} \) such that
\[ (1) \quad K^\mu_\lambda(\cdot)u \in \mathcal{H}, \text{ for every } \mu \in \Omega, u \in \mathbb{C}^m; \]
\[ (2) \quad [\mu, K^\mu_\lambda u]_\mathcal{H} = u^* f(\mu), \text{ for every } f(\cdot) \in \mathcal{H}, \mu \in \Omega, u \in \mathbb{C}^m. \]

It is known (see \[29\]) that for every Hermitian kernel \( K^\mu_\lambda(\cdot) : \Omega \times \Omega \to \mathbb{C}^{m \times m} \) with a finite number of negative squares on \( \Omega \times \Omega \) there is a unique Pontryagin space \( \mathcal{H} \) with reproducing kernel \( K^\mu_\lambda(\cdot) \), and that \( \text{ind} \mathcal{H} = \text{sq} - K = \kappa \). In the case \( \kappa = 0 \) this fact is due to Aronszajn \[3\] (see also \[12\]).

2.8. Space \( \bar{\mathcal{D}}(s) \). Let \( s \in \mathbb{S}_p^{\infty}(\mathbb{D}) \) be the characteristic function of a unitary colligation \( \Delta = (\mathcal{H}, \Sigma_2, \Sigma_1; T, F, G, H) \). Let us consider the kernel \( D_s(\lambda, \mu) \) on \( \Omega_s \times \Omega_s \) defined by the matrix
\[ D_s(\lambda, \mu) = \begin{bmatrix} \frac{I_{2s} - s(\lambda)\overline{s(\mu)}}{\lambda - \mu} & \frac{-\mu s(\lambda) - s(\mu)}{\lambda - \mu} \\
\frac{-\mu s(\lambda) - s(\mu)}{\lambda - \mu} & \frac{I_{2s} - s(\lambda)\overline{s(\mu)}}{\lambda - \mu} \end{bmatrix}, \quad (\lambda, \mu \in \Omega_s). \]

The introduced kernel is similar to the kernels from \[12\] and \[13\]. A Pontryagin space corresponding to this reproducing kernel \( D_s(\lambda, \mu) \) is denoted by \( \bar{\mathcal{D}}(s) \).
Let us define two operator functions \( G_1(z) : \mathcal{S}_1 \to \mathcal{H} \), \( G_2(z) : \mathcal{S}_2 \to \mathcal{H} \)

\[
\begin{align*}
G_1(\lambda)[^*] &= P_{\mathcal{S}_2} U (I - \lambda P_{\mathcal{H}} U)^{-1}|_{\mathcal{H}} \quad (1/\lambda \in \rho(T)), \\
G_2(\lambda) &= -P_{\mathcal{H}} U (I - \lambda P_{\mathcal{H}} U)^{-1}|_{\mathcal{S}_2} \quad (1/\lambda \in \rho(T)).
\end{align*}
\]

It follows from (2.18) that

\[
G_1(\lambda)[^*] = G(I - \lambda T)^{-1}, \quad G_2(\lambda) = -(I - \lambda T)^{-1} F \quad (1/\lambda \in \rho(T))
\]

and the formula (2.15) for the subspace \( \mathcal{H}_\Delta \) can be rewritten as

\[
\mathcal{H}_\Delta = \mathop{\text{span}} \{ G_1(\lambda) h_1, G_2(\lambda) h_2 : \ h_1 \in \mathcal{S}_1, h_2 \in \mathcal{S}_2, \ 1/\lambda \in \rho(T) \}.
\]

As is easily checked for every \( \tilde{f} = \begin{bmatrix} \tilde{f}_1 \\
\tilde{f}_2
\end{bmatrix}, \ \tilde{g} = \begin{bmatrix} \tilde{g}_1 \\
\tilde{g}_2
\end{bmatrix} \in \mathcal{S}_1 \oplus \mathcal{S}_2 \) the following identity holds

\[
(D_s(\mu, \lambda) \tilde{f}, \tilde{g})_{\mathcal{S}_1 \oplus \mathcal{S}_2} = [G_1(\mu) \tilde{f}_1 + \mu G_2(\mu) \tilde{f}_2, G_1(\lambda) \tilde{g}_1 + \lambda G_2(\lambda) \tilde{g}_2]_\mathcal{H}
\]

It follows from (2.25) that the kernel \( D_s(\mu, \lambda) \) has at most \( \kappa \), and if the colligation \( \Delta \) is simple exactly \( \kappa \), negative squares on \( \Omega(s) \).

The next theorem is the reformulation of the theorem from the paper [13] for the reproducing kernel space \( \tilde{D}(s) \).

**Theorem 2.8.** Let \( s \in S_0^{p \times q} \), then the de Branges-Rovnyak space \( \mathcal{D}(s) \) is unitarily equivalent to the reproducing kernel space \( \tilde{D}(s) \) via the mapping

\[
\mathcal{T} : \tilde{f} = \begin{bmatrix} \tilde{f}_1 \\
\tilde{f}_2
\end{bmatrix} \in \tilde{D}(s) \to f = \begin{bmatrix} f^+ \\
f^-
\end{bmatrix} \in \mathcal{D}(s),
\]

where \( \tilde{f}_1 \) is the meromorphic continuation of \( f^+ \) to \( \Omega_s \), and \( \tilde{f}^*_2 \) is the meromorphic continuation of \( f^- \) to \( \Omega_s \) such that \( \tilde{f} \) is a non-tangential limit of \( f \) from the unit disk.

3. **Functional model of a unitary colligation \( \Delta \).**

In this section we will define the Fourier representation of an unitary colligation and recall its functional model like in [12].

Recall (see [5]) that a subspace \( \mathcal{H}_1 \) of the Pontryagin space \( \mathcal{H} \) is called regular if it is orthocomplemented.

**Proposition 3.1.** Let \( \Delta = (\mathcal{H}, \mathcal{S}_2, \mathcal{S}_1; U) \) be a unitary colligation such that \( \mathcal{H} \) is a Pontryagin space and let \( s(\cdot) \) be the corresponding characteristic function. If \( \mathcal{H}_\Delta \) is a regular subspace of \( \mathcal{H} \) then the space \( \mathcal{D}(s) \) can be identified with the space \( \mathcal{D} \) of vector functions

\[
(\mathcal{F} h)(\lambda) = \frac{G_1(\lambda)[^*] h}{\lambda G_2(\lambda)[^*] h} = \begin{bmatrix} \lambda G_1(\lambda)[^*] h \\
-\lambda G_2(\lambda)[^*] h
\end{bmatrix} (h \in \mathcal{H}_\Delta),
\]

equipped with the inner product

\[
[\mathcal{F} h, \mathcal{F} g]_{\mathcal{D}(s)} = [h, g]_{\mathcal{H}} (h, g \in \mathcal{H}_\Delta).
\]
Hence the function $D_s(\mu, \lambda)$ is isometrically isomorphic to the space $h$ for every $x \in D_s(\mu, \lambda)$. Therefore, $h = 0$. Moreover, it follows from (3.2) and (3.3) that $D$ is isometrically isomorphic to the space $\mathcal{H}_\Delta$ and the following reproducing property of the kernel $D_s(\mu, \lambda)$ holds:

$$[\mathcal{F}h, D_s(\mu, \cdot)x]_{\mathcal{D}(\mathcal{H}_\Delta)} = [h, \mathcal{F}(\mu)^*x]_{\mathcal{H}_\Delta} = (\mathcal{F}(\mu)h, x)_{\mathcal{L}_1 \oplus \mathcal{L}_2}$$

for every $x \in \mathcal{L}_1 \oplus \mathcal{L}_2, \mu \in \Omega_s$. □

If the colligation $\Delta$ is simple, then the space $\mathcal{D}$ is isometrically isomorphic to the space $\mathcal{H}$ under the mapping $\mathcal{F}$. If the colligation $\Delta$ is not simple but $\mathcal{H}$ is regular, then the operator $\mathcal{F}$ can be continued by zero to the subspace $\mathcal{H} \oplus \mathcal{H}_\Delta$ and the continuation $\mathcal{F}$ is given by the same formula (3.4) for every $h \in \mathcal{H}$. The operator $\mathcal{F}$ is called the Fourier representation of the colligation $\Delta$.

**Proposition 3.2.** (see [15] for the case $\kappa = 0$) The Fourier representation $\mathcal{F}$ satisfies the relation

$$\mathcal{F}\mathcal{P}_\mathcal{H}U^*[s] + \begin{bmatrix} s(t) \\ -I_{\mathcal{L}_2} \end{bmatrix} P_{\mathcal{L}_1} U^*[s] = t \cdot \mathcal{F}\mathcal{P}_\mathcal{H} + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s(t)^* \end{bmatrix} P_{\mathcal{L}_2}.$$

Here $\mathcal{P}_\mathcal{H}$ and $\mathcal{P}_{\mathcal{L}_i}$ are orthoprojections onto $\mathcal{H}$ and $\mathcal{L}_i$ ($i = 1, 2$), respectively, $t \in \mathbb{T}$.

**Proof.** Due to (2.17) and (2.24) one can reduce the left-hand side of (3.5) to the form

$$\begin{bmatrix} P_{\mathcal{L}_1} (I - t\mathcal{P}_\mathcal{H})^{-1} U(P_{\mathcal{H}} U^*[s] + P_{\mathcal{L}_2} U^*[s]) \\
-I P_{\mathcal{L}_2} (I - t\mathcal{P}_\mathcal{H})^{-1} U^*[s] + P_{\mathcal{L}_2} U^*[s] 
\end{bmatrix} = \begin{bmatrix} P_{\mathcal{L}_1} (I - t\mathcal{P}_\mathcal{H})^{-1} \\
-I P_{\mathcal{L}_2} (I - t\mathcal{P}_\mathcal{H})^{-1} U^*[s] 
\end{bmatrix}.$$

Similarly, the right-hand side of (3.5) can be rewritten as

$$\begin{bmatrix} \mathcal{P}_\mathcal{H}(I - t\mathcal{P}_\mathcal{H})^{-1} P_{\mathcal{H}} + P_{\mathcal{L}_1} \\
-P_{\mathcal{L}_2} (I - t\mathcal{P}_\mathcal{H})^{-1} U^*[s] + P_{\mathcal{L}_2} (I - t\mathcal{P}_\mathcal{H})^{-1} U^*[s] P_{\mathcal{L}_2} 
\end{bmatrix}.$$

Now the equality (3.5) follows from two last equalities. □

**Definition 3.3.** The colligation $\Delta = (\mathcal{H}, \mathcal{F}, \mathcal{G}; T, F, G, H)$ is called the unitarily equivalent to the colligation $\Delta' = (\mathcal{H}', \mathcal{F}, \mathcal{G}; T', F', G', H')$ if there exists a mapping $Z$ from $\mathcal{H}$ to $\mathcal{H}'$ such that

$$T' = ZTZ^{-1}, \quad F' = ZF, \quad G' = GZ^{-1},$$

or in other words

$$\begin{bmatrix} Z & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} T & F \\
G & H \end{bmatrix} \begin{bmatrix} Z^{-1} & 0 \\
0 & I \end{bmatrix} = \begin{bmatrix} T' & F' \\
G' & H' \end{bmatrix}.$$
Proof. Let the operator $U$ (see Theorem 3.4, (3.8)) and let the colligation $\Delta_s$ be defined by the equality

$$T_s f = \mathcal{F} \left( f - \begin{bmatrix} I & -s \\ -s^* & I \end{bmatrix} f_h(0) \right), \quad F_s = -\mathcal{F} \begin{bmatrix} I & -s \\ -s^* & I \end{bmatrix} s(0),$$

$$G_s f = f_h(0), \quad H_s = s(0).$$

Then the colligations $\Delta$ and $\Delta_s$ are unitary equivalent via

$$U_s \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_{\mathcal{L}_2} \end{bmatrix} = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_{\mathcal{L}_1} \end{bmatrix} U.$$

Proof. The equality (3.8) can be rewritten in the form

$$\mathcal{F} P_H + \begin{bmatrix} s \\ -I_{\mathcal{L}_2} \end{bmatrix} P_{\mathcal{L}_2} = t \cdot \mathcal{F} P_H U + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} P_{\mathcal{L}_2}.$$

Hence one obtains for every $h \in \mathcal{H}$ and $x \in \mathcal{L}_2$

$$\mathcal{F} h = t \cdot \mathcal{F} T_h + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} G h \quad (h \in \mathcal{H}),$$

$$x = \mathcal{F} F x + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} H x \quad (x \in \mathcal{L}_2).$$

Let the operator $U_s = \begin{bmatrix} T_s & F_s \\ G_s & H_s \end{bmatrix}$ be defined by the equality

$$U_s = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_{\mathcal{L}_1} \end{bmatrix} U \begin{bmatrix} \mathcal{F} & 0 \\ 0 & I_{\mathcal{L}_2} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{F} \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{F} \\ 0 \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{F} F^{-1} & \mathcal{F} \end{bmatrix}.$$

Setting $f = \mathcal{F} h$, one obtains from (3.9), (3.10) and (2.11)

$$T_s f = \mathcal{F} F^{-1} f = \mathcal{F} \left( f - \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} G_s f \right) = \mathcal{F} \left( f - \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} f_h(0) \right),$$

$$F_s x = \mathcal{F} F x = \mathcal{F} \left( \begin{bmatrix} s \\ -I_{\mathcal{L}_2} \end{bmatrix} x - \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} H x \right) = \mathcal{F} \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} \begin{bmatrix} s(0) x \\ x \end{bmatrix},$$

$$G_s f = \mathcal{F} G \mathcal{F}^{-1} f = f_h(0), \quad H_s = H = s(0),$$

which prove the formula (3.8). □

Theorem 3.4 shows why the mapping $\mathcal{F}$ is called the Fourier representation of $\Delta$. In the case when the colligation $\Delta$ is simple, this mapping gives the unitary equivalence between $\Delta$ and its functional model.

4. Abstract Interpolation Problem $AIP(\tilde{\kappa})$.

Given are Hilbert spaces $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$, integer $\kappa, \tilde{\kappa} \in \mathbb{Z}_+$ and operators $M, N \in \mathcal{B}(\mathcal{H}, \mathcal{L}_1), C_1 \in \mathcal{B}(\mathcal{H}, \mathcal{L}_1), C_2 \in \mathcal{B}(\mathcal{H}, \mathcal{L}_2), P \in \mathcal{B}(\mathcal{H})$, such that

(A1) $P = P^*, 0 \in \rho(P), \quad \text{sq}_{-}\!(P) = \kappa.$

(A2) for every $f, g \in \mathcal{H}$ the following identity holds

$$PMf, Mg)_{\mathcal{H}} - (PNf, Ng)_{\mathcal{H}} = (C_1f, C_1g)_{\mathcal{L}_1} - (C_2f, C_2g)_{\mathcal{L}_2}.$$

Find an operator function $s(\cdot) \in \mathcal{L}_2(\mathcal{L}_2, \mathcal{L}_1)$ and the mapping $\Phi : \mathcal{H} \to \mathcal{D}(s)$, such that:

(i) $[\Phi h, \Phi h]_{\mathcal{D}(s)} \leq (Ph, h)_{\mathcal{H}}, \quad \text{for every } h \in \mathcal{H};$
(ii) $\Phi Mh - t\Phi Nh = \begin{bmatrix} I & -s^* \\ -s & I \end{bmatrix} Ch$, where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, $t \in \mathbb{T}$, for every $h \in H$.

Alongside with the Problem $AIP(\bar{\kappa})$ let us consider the Problem $AIP_0(\bar{\kappa})$, by replacing (i) by the generalized Parseval equality

$(\text{i}^\prime) \quad [\Phi h, \Phi \tilde{h}]_{D(s)} = (Ph, h)_{H}$, for every $h \in H$.

Let $H$ be supplied with the inner product $[\cdot, \cdot]_H := (P, \cdot)$. Then the space $(H, [\cdot, \cdot]_H)$ is a Pontryagin space with the negative index $\kappa$.

It follows from the identity (4.1) that the operator

$(4.2) \quad V : \begin{bmatrix} Mf \\ C_2 f \end{bmatrix} \rightarrow \begin{bmatrix} Nf \\ C_1 f \end{bmatrix}$

is a Pontryagin space isometric operator from $H \oplus L_2$ to $H \oplus L_1$. The problem $AIP(\bar{\kappa})$ can be reduced to the problem of extension of the isometric operator $V$ to a unitary operator

$U = \begin{bmatrix} T & F \\ G & H \end{bmatrix} : \begin{bmatrix} H \\ L_2 \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{H} \\ L_1 \end{bmatrix}, \quad (H \ni \tilde{H})$.

**Definition 4.1.** A unitary extension $U$ of $V$ will be called $(L_2, L_1)$-regular, if $\tilde{H} \ni H_\Delta$ is a Hilbert space. An extension $U$ will be called $(L_2, L_1)$-minimal, if the corresponding colligation $\Delta = (H, L_2, L_1, U)$ is simple, or in other words $H_\Delta = \tilde{H}$.

Clearly, every $(L_2, L_1)$-minimal unitary extension $U$ is $(L_2, L_1)$-regular.

In the case $\kappa = 0$ a description of the set of solutions of the Problem $AIP(\bar{\kappa})$ was given in [18], [21].

**Lemma 4.2.** Let $U$ be a unitary operator in a Pontryagin space $\tilde{H}$ with the negative index $\bar{\kappa}$ and $s(\lambda) = P_{L_1}(I - zUP_{\tilde{H}})^{-1}UP_{L_2}$ be its characteristic function. Then $s(\cdot)$ belongs $S_{\bar{\kappa}}(L_2, L_1)$ if and only if the operator $U$ is a $(L_2, L_1)$-regular.

**Proof.** If $U$ is a $(L_2, L_1)$-regular, then $\text{ind} (H_\Delta) = \bar{\kappa}$. The mapping

$(4.3) \quad P_{L_1}(I - \lambda UP_{\tilde{H}})^{-1}UP_{H_\Delta} \quad \lambda \rightarrow \begin{bmatrix} XP_{L_2}(I - \lambda U[^*]P_{\tilde{H}})^{-1}UP_{\tilde{H}} \\ XP_{L_2}(I - \lambda U[^*]P_{\tilde{H}})^{-1}UP_{\tilde{H}} \end{bmatrix}$

is isometric from $H_\Delta$ to $D(s)$. Hence, $\text{ind} (D(s)) = \bar{\kappa}$ and $s(\cdot) \in S_{\bar{\kappa}}(L_2, L_1)$.

Conversely, if $s \in S_{\bar{\kappa}}$, then since $\text{ind} (D(s)) = \bar{\kappa}$ one gets $\text{ind} (H_\Delta) = \bar{\kappa}$.

Hence, $U$ is a $(L_2, L_1)$-regular. $\square$

**Theorem 4.3.** For the Problem $AIP(\bar{\kappa})$ to be solvable it is necessary that $\kappa \leq \bar{\kappa} \in \mathbb{Z}_+$. The formulas

$(4.4) \quad s(\lambda) = P_{L_1}(I - \lambda UP_{\tilde{H}})^{-1}UP_{L_2}$

$(4.5) \quad \Phi(\lambda) = \begin{bmatrix} P_{L_1}(I - \lambda UP_{\tilde{H}})^{-1}UP_{\tilde{H}} \\ -XP_{L_2}(I - \lambda U[^*]P_{\tilde{H}})^{-1}UP_{\tilde{H}} \end{bmatrix}$

establish a one-to-one correspondence between the set of solutions $\{s, \Phi\}$ of the Problem $AIP(\bar{\kappa})$ and the set of all $(L_2, L_1)$-regular unitary operator extensions $U$ of the operator $V$, such that:

$(4.6) \quad \text{ind} (\bar{H}) = \bar{\kappa}$.

A solution $\{s, \Phi\}$ of the Problem $AIP(\bar{\kappa})$ is a solution of the Problem $AIP_0(\bar{\kappa})$, if and only if the extension $U$ is $(L_2, L_1)$-minimal.
Proof. 1) Let $U$ be a $(\mathcal{L}_2, \mathcal{L}_1)$-regular extension of the operator $V$, satisfying (4.0) and let $P_{\Delta}$ be the orthogonal projection onto $\mathcal{H}_{\Delta}$ in $\mathcal{H}$. Consider a unitary colligation

$$\Delta = (\tilde{\mathcal{H}}, \mathcal{L}_2, \mathcal{L}_1; U), \quad U = \begin{bmatrix} T & F \\ G & H \end{bmatrix}. $$

Due to Proposition 3.1 the space $\mathcal{D}(s)$ can be interpreted as the set of vector function

$$(\mathcal{F}h)(\lambda) = \begin{bmatrix} G(I - \lambda T)^{-1}h \\ -\lambda F^* \phi(I - \lambda T^* \phi)^{-1}h \end{bmatrix}, \quad h \in \tilde{\mathcal{H}}$$

with the scalar product

$$\langle \mathcal{F}h, \mathcal{F}g \rangle_{\mathcal{D}(s)} = [P_{\Delta}h, P_{\Delta}g]_{\tilde{\mathcal{H}}}. $$

Since $\tilde{\mathcal{H}} \ominus \mathcal{H}_{\Delta}$ is a Hilbert space for every $h \in \tilde{\mathcal{H}}$ the following inequality holds

$$\langle \mathcal{F}h, \mathcal{F}h \rangle_{\mathcal{D}(s)} = [P_{\Delta}h, P_{\Delta}h]_{\tilde{\mathcal{H}}} \leq [h, h]_{\tilde{\mathcal{H}}}. $$

Setting $\Phi h = \mathcal{F}h$ for $h \in \mathcal{H}$, one obtains the mapping $\Phi : h \rightarrow \mathcal{D}(s)$, which satisfies (i).

The equality (ii) is implied by relation (3.5)

$$(4.7) \quad \mathcal{F}P_{\tilde{\mathcal{H}}} U^{[s]} + \begin{bmatrix} s & -I_{\mathcal{L}_2} \\ -I_{\mathcal{L}_2} & 0 \end{bmatrix} P_{\mathcal{L}_2} U^{[s]} = t \mathcal{F}P_{\tilde{\mathcal{H}}} + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} P_{\mathcal{L}_1},$$

For a vector

$$(4.8) \quad U^{[s]} \begin{bmatrix} Nh \\ C_2h \end{bmatrix} = \begin{bmatrix} Mh \\ C_2h \end{bmatrix} \quad (h \in \mathcal{H})$$

Substituting (4.8) into (4.7) and taking account of $\mathcal{F}Mh = \Phi Mh$, $\mathcal{F}Nh = \Phi Nh$, one obtains the equality

$$\Phi Mh + \begin{bmatrix} s & -I_{\mathcal{L}_2} \\ -I_{\mathcal{L}_2} & 0 \end{bmatrix} C_2h = t \Phi Nh + \begin{bmatrix} I_{\mathcal{L}_1} \\ -s^* \end{bmatrix} C_1h, $$

which is equivalent (ii).

2) Conversely, let $\{s, \Phi\}$ be a solution of the Problem $AIP(\kappa)$ and let $\Delta_s = (\mathcal{D}(s), \mathcal{L}_2, \mathcal{L}_1; U_s)$ be a unitary colligation with the characteristic function $s(\cdot)$, built in Proposition 3.3. Since the operator $I - \Phi^{[s]} \Phi : \mathcal{H} \rightarrow \mathcal{H}$ is nonnegative in the Pontryagin space $\mathcal{H}$, it admits a Bognar-Kramli factorization [8]: $I - \Phi^{[s]} \Phi = DD^{[s]}$, where the defect operator $D$ acts from the Hilbert space $\mathcal{D} = \mathcal{H}/(I - \Phi^{[s]} \Phi)$ to the Pontryagin space $\mathcal{H}$. Let us construct a lifting $\tilde{\Phi} : \mathcal{H} \rightarrow \mathcal{D}(s) \oplus \mathcal{D} =: \tilde{\mathcal{H}}$ of the mapping $\Phi : \mathcal{H} \rightarrow \mathcal{D}(s)$, setting $\tilde{\Phi} := \begin{bmatrix} \Phi \\ D^{[s]} \end{bmatrix}$. Then for every $h, g \in \mathcal{H}$ one obtains the equality:

$$\left[ \Phi h, \tilde{\Phi} g \right]_{\tilde{\mathcal{H}}} = [\Phi f, \Phi g]_{\mathcal{D}(s)} + [D^{[s]} h, D^{[s]} g]_{\mathcal{D}} = (Ph, g)_{\mathcal{H}}, $$

which proves that the mapping $\tilde{\Phi}$ is isometric. Further it follows from (3.9) that

$$(4.9) \quad U_s \begin{bmatrix} \Phi Mh \\ C_2h \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & I_{\mathcal{L}_1} \end{bmatrix} U \begin{bmatrix} F^{-1} & 0 \\ 0 & I_{\mathcal{L}_2} \end{bmatrix} \begin{bmatrix} \Phi Mh \\ C_2h \end{bmatrix} = \begin{bmatrix} \Phi Nh \\ C_1h \end{bmatrix}, \quad h \in \mathcal{H}$$
Since the operators $U_s$, $Φ$ and $V$ are isometric, one obtains for every $h ∈ ℋ$:

\[
(D[s]Nh, D[s]Nh)_ℓ₂ = [ΦNh, ΦNh]_ℓ₂ - [ΦNh, ΦNh]_ℓ₂
\]

Thus the operator $U_D : D[s]Nh → D[s]Nh, h ∈ ℋ$ is isometric. Let $U_D$ be a unitary extension of $U_D$ in a Hilbert space $D ⊃ ℋ$. Then

\[
U = U_s ⊕ U_D : [\hat{ℋ}, ℋ]_ℓ₂ → [\hat{ℋ}, ℋ]_ℓ₂, \quad \hat{ℋ} = D(s) ⊕ D
\]

is a unitary operator. It follows from (4.9) and (4.10) that

\[
[ΦNh, C_2h]_ℓ₂ = [ΦNh, C_1h], \quad h ∈ ℋ
\]

and the operator $U$ is a unitary extension of the isometric operator

\[
\bar{V} = [Φ, 0] V [Φ, 0]^{-1}
\]

Hence, $κ ≤ \bar{κ}$ and $U$ is an $(ℓ₂, ℋ)$-regular extension of the operator $\bar{V}$ satisfying (4.10), since $\hat{ℋ} ⊗ D(s) = D$ is a Hilbert space.

3) If the extension $U$ is $(ℓ₂, ℋ)$-minimal, then $H_{\Delta} = \hat{ℋ}$ and the mapping $F : H_{\Delta} ⊗ D(s)$ is isometric. It proves the Parseval equality (ii’). Conversely, if $s, Φ$ is a solution of the Problem $\text{AIP}_0(\bar{κ})$, then the mapping $Φ$ is isometric and the operator $U_s$ is a unitary extension of the operator $\bar{V} = [Φ, 0] V [Φ, 0]^{-1}$. Since the colligation $\Delta_s$ is simple, the extension $U$ is $(ℓ₂, ℋ)$-minimal. □

5. Parametrization of solutions.

**Definition 5.1.** Recall that $λ$ is a regular point of the pencil $M − λN$ (it is denoted by $λ ∈ ρ(M, N)$) if $0 ∈ ρ(M − λN)$. Denote

\[
ρ(M, N)^# := \{λ : 0 ∈ ρ\left(M - \frac{1}{λ}N\right)^\#\}
\]

Suppose that the Problem $\text{AIP}(κ)$ data satisfy the condition (A3) There exists a point $a ∈ T \cap ρ(M, N)$. We will suppose that $ρ(M, N) ⊃ D$ except finite set of points. Recall the following definition from the paper [3].

**Definition 5.2.** We shall write $λ ∈ ρ(V, ℋ)$ if $1 ∈ \widehat{ρ}(λP_H V)$ and

\[\begin{align*}
(I - \lambda P_H V)\text{dom } V + \begin{bmatrix} 0 \\ ℋ \end{bmatrix} & = \begin{bmatrix} ℋ \\ ℋ \end{bmatrix}, \\
(I - λP_H V^{-1})\text{ran } V + \begin{bmatrix} 0 \\ ℋ \end{bmatrix} & = \begin{bmatrix} ℋ \\ ℋ \end{bmatrix}
\end{align*}\]

and $λ ∈ ρ(V^{-1}, ℋ)$ if $1 ∈ \widehat{ρ}(λP_H V^{-1})$ and

\[\begin{align*}
(I - λP_H V^{-1})\text{ran } V + \begin{bmatrix} 0 \\ ℋ \end{bmatrix} & = \begin{bmatrix} ℋ \\ ℋ \end{bmatrix}, \\
(I - \lambda P_H V)\text{dom } V + \begin{bmatrix} 0 \\ ℋ \end{bmatrix} & = \begin{bmatrix} ℋ \\ ℋ \end{bmatrix}
\end{align*}\]
Proposition 5.3. The following equivalences hold
1) \( \lambda \in \rho(V, \mathcal{L}_2) \) if and only if \( \lambda \in \rho(M, N) \);
2) \( \overline{\lambda} \in \rho(V^{-1}, \mathcal{L}_1) \) if and only if \( \overline{\lambda} \in \rho(N, M) \);
3) \( \lambda \in \rho(V, \mathcal{L}_2) \) if and only if \( \lambda \in \rho(M, N) \cap \rho(N, M)^* \).

Proof. 1) The statement \( \lambda \in \rho(V, \mathcal{L}_2) \) means that for every vector \( \begin{bmatrix} f \\ u_2 \end{bmatrix} \in \mathcal{H} \mathcal{L}_2 \) there exist uniquely determined vectors \( h \in \mathcal{H} \) and \( l_2 \in \mathcal{L}_2 \) such that
\[
\begin{bmatrix} f \\ u_2 \end{bmatrix} = (I - \lambda P_{\mathcal{H}} V) \begin{bmatrix} Mh \\ C_2h + l_2 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2 \end{bmatrix},
\]
\[
\begin{bmatrix} f \\ u_2 \end{bmatrix} = (M - \lambda N)h.
\]
The condition of the unique representation for every vector \( \begin{bmatrix} f \\ u_2 \end{bmatrix} \) gives an invertibility of \( M - \lambda N \).

Conversely, let \( 0 \in \rho(M - \lambda N) \), then for every \( \begin{bmatrix} f \\ u_2 \end{bmatrix} \in \mathcal{H} \mathcal{L}_2 \) one can define vectors \( h := (M - \lambda N)^{-1} f \) and \( l_2 := u_2 - C_2(M - \lambda N)^{-1} f \). The vector \( \begin{bmatrix} f \\ u_2 \end{bmatrix} \) can be represented in the following way
\[
\begin{bmatrix} f \\ u_2 \end{bmatrix} = (I - \lambda P_{\mathcal{H}} V) \begin{bmatrix} M(M - \lambda N)^{-1} f \\ C_2(M - \lambda N)^{-1} f \end{bmatrix} + \begin{bmatrix} 0 \\ u_2 - C_2(M - \lambda N)^{-1} f \end{bmatrix}.
\]

2) Let the statement \( \overline{\lambda} \in \rho(V^{-1}, \mathcal{L}_1) \) hold. This means that for every vector \( \begin{bmatrix} f \\ u_1 \end{bmatrix} \in \mathcal{H} \mathcal{L}_1 \) there exists uniquely determined vectors \( h \in \mathcal{H} \) and \( l_1 \in \mathcal{L}_1 \), such that
\[
\begin{bmatrix} f \\ u_1 \end{bmatrix} = (I - \overline{\lambda} P_{\mathcal{H}} V^{-1}) \begin{bmatrix} Nh \\ C_1h \end{bmatrix} + \begin{bmatrix} 0 \\ l_1 \end{bmatrix},
\]
\[
\begin{bmatrix} f \\ u_1 \end{bmatrix} = (N - \overline{\lambda} M)h.
\]
Therefore, the operator \( N - \overline{\lambda} M \) is invertible (i.e. \( \overline{\lambda} \in \rho(N, M) \)).

The converse proposition is proved in the way similar to the converse proposition of 1).

The proposition 3) follows from 1) and 2). □

Corollary 5.4. For \( a \in T \) we have \( \rho(V, \mathcal{L}_2) = \rho(V^{-1}, \mathcal{L}_1) = \rho_V(\mathcal{L}_2, \mathcal{L}_1) \).

Definition 5.5. For \( \lambda \in \rho(V, \mathcal{L}_2) \), by \( \mathcal{P}_{\mathcal{L}_2}(\lambda) \) we denote the skew projection onto \( \mathcal{L}_2 \) in the decomposition \( \mathcal{L}_2 \) and introduce the operator
\[
\mathcal{Q}_{\mathcal{L}_1}(\lambda) := P_{\mathcal{L}_1} V(I - \lambda P_{\mathcal{H}} V)^{-1}(I - \mathcal{P}_{\mathcal{L}_2}(\lambda)).
\]

Introduce the operator-function \( W(\lambda) \) defined by
\[
J - W(\lambda) JW(\mu)^* = (1 - \lambda \overline{\mu}) G(\lambda) G(\mu)^*.
\]
where
\[ J = \begin{bmatrix} I_{\mathcal{L}_1} & 0 \\ 0 & -I_{\mathcal{L}_2} \end{bmatrix}, \quad G(\lambda) := -\begin{bmatrix} Q_{\mathcal{L}_1}(\lambda) \\ IP_{\mathcal{L}_2}(\lambda) \end{bmatrix} : [\mathcal{H}] \rightarrow [\mathcal{L}_1] \rightarrow [\mathcal{L}_2]. \]

**Definition 5.6.** The operator-function
\[ W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} : [\mathcal{L}_1] \rightarrow [\mathcal{L}_1] \rightarrow [\mathcal{L}_2], \quad (\lambda \in \rho_V(\mathcal{L}_2, \mathcal{L}_1)), \]
satisfying the equality (5.3) is called the resolvent matrix for the operator \( P(\lambda) \).

**Proposition 5.7.** If the assumptions (A1)-(A3) hold, then the resolvent matrix \( W(\cdot) \) can be defined by the equality
\[ W(\lambda) = I - (1 - \lambda \pi)G(\rho)[\pi]J_{\mathcal{L}_2}, \quad \lambda \in \rho_V(\mathcal{L}_2, \mathcal{L}_1). \]

Let us find the explicit form of the operators \( P_{\mathcal{L}_2}(\lambda) \), \( Q_{\mathcal{L}_1}(\lambda) \) and \( W(\lambda) \). Let \( u, v \in \mathcal{L}_2 \) and \( f, h \in \mathcal{H} \), then
\[ \begin{bmatrix} C2h + u \\ C2h \end{bmatrix} = \begin{bmatrix} (M - \lambda N)h \\ C2h \end{bmatrix} = [f] \]
Hence, the operator \( P_{\mathcal{L}_2}(\lambda) \) is
\[ P_{\mathcal{L}_2}(\lambda) [f] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
and the adjoint operator is
\[ P_{\mathcal{L}_2}(\lambda)^* [0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Let us find the explicit form for \( Q_{\mathcal{L}_1}(\lambda) = P_{\mathcal{L}_1}V(I - \lambda P_N)V^{-1}(I - P_{\mathcal{L}_2}(\lambda)) \):
\[ (I - P_{\mathcal{L}_2}(\lambda)) [f] = C2(M - \lambda N)^{-1}f \]
hence, \( h = (M - \lambda N)^{-1}f \).
\[ Q_{\mathcal{L}_1}(\lambda) [f] = P_{\mathcal{L}_1}V \begin{bmatrix} C2h \\ C2h \end{bmatrix} = C1(M - \lambda N)^{-1}f, \]
Therefore, the \( Q_{\mathcal{L}_1}(\lambda) \) is
\[ Q_{\mathcal{L}_1}(\lambda) [f] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
and the adjoint operator is
\[ Q_{\mathcal{L}_1}(\lambda)^* [0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Then from (5.5) we get the explicit form for the resolvent matrix \( W(\lambda) \):
\[ W(\lambda) = I - (1 - a\lambda)C(M - \lambda N)^{-1}P^{-1}(M - \pi N)^{-1}C^*J, \quad a \in \mathbb{T}, \]
where \( J \) is defined by (5.4).

Consider the main properties of \( W(\cdot) \). We will need one more condition
(A4) The set of points \( \mathbb{B} \backslash \rho(M, N) \) consists of at most of countable set of isolated points.

**Proposition 5.8.** If the assumptions (A1)-(A4) are in force, then \( W(\cdot) \in P_{\kappa^*}(J) \)
for some \( \kappa^* \leq \kappa \).
\textbf{Theorem 5.11.} Let the data of the Problem $AIP(\kappa)$ satisfies the assumptions $(A1)$-$(A3)$. The the solution set of $AIP(\kappa)$ is described by the formula
\begin{equation}
(5.15) \quad s(\lambda) = (w_{11}(\lambda)\varepsilon(\lambda) + w_{12}(\lambda))(w_{21}(\lambda)\varepsilon(\lambda) + w_{22}(\lambda))^{-1},
\end{equation}
where $\varepsilon(\cdot)$ ranges over the class $S(\mathfrak{L}_2, \mathfrak{L}_1)$ and $w_{21}(0)\varepsilon(0) + w_{22}(0)$ is invertible. In this case the mapping $\Phi : \mathcal{H} \to \mathcal{D}(s)$ is uniquely defined by
\begin{equation}
\Phi(t) = \begin{bmatrix} I & -s(t) \\ -s^*(t) & I \end{bmatrix} C(M - tN)^{-1}, \quad t \in \mathbb{T}.
\end{equation}
Proof. Let \( \{ s, \Phi \} \) be a solution of the Problem \( AIP(\kappa) \), where \( s(\cdot) \in S_\kappa(\mathcal{L}_2, \mathcal{L}_1) \) and is holomorphic on a neighborhood of 0. From Theorem 4.3 we get
\[
(5.16) \quad s(\lambda) = P_{\mathcal{L}_1}(I - \lambda U P_{\mathcal{R}})^{-1} U P_{\mathcal{L}_2}.
\]
Further, using Theorem 3 from [6], we get \( s(\lambda) = T W(\lambda)[\varepsilon(\lambda)] \), where \( \varepsilon(\cdot) \in S(\mathcal{L}_2, \mathcal{L}_1) \), and \( w_{21}(\cdot)\varepsilon(\cdot) + w_{22}(\cdot) \) is invertible at 0.

Conversely, let \( \varepsilon(\cdot) \in S(\mathcal{L}_2, \mathcal{L}_1) \), \( s(\cdot) = T W[\varepsilon] \in S_\kappa(\mathcal{L}_2, \mathcal{L}_1) \) and the matrix-function be holomorphic at 0. According to Theorem 3 in [6] \( s(\cdot) \) admits the representation (5.16). Since \( s(\cdot) \in S_\kappa(\mathcal{L}_2, \mathcal{L}_1) \) (see Lemma 4.2), the unitary operator \( U \) is a \( (\mathcal{L}_2, \mathcal{L}_1) \)-regular. \( \square \)
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