FREE MONOIDS AND GENERALIZED METRIC SPACES

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To the memory of Michel-Marie Deza, with affection and admiration.

Abstract. Let $A$ be an ordered alphabet, $A^*$ be the free monoid over $A$ ordered by the Higman ordering, and let $F(A^*)$ be the set of final segments of $A^*$. With the operation of concatenation, this set is a monoid. We show that the submonoid $F \circ (A^*) = F(A^*) / \{ 0 \}$ is free. The MacNeille completion $N(A^*)$ of $A^*$ is a submonoid of $F(A^*)$. As a corollary, we obtain that the monoid $N \circ (A^*) = N(A^*) / \{ 0 \}$ is free. We give an interpretation of the freeness of $F \circ (A^*)$ in the category of metric spaces over the Heyting algebra $V = F(A^*)$, with the non-expansive mappings as morphisms. Each final segment of $A^*$ yields the injective envelope $S_F$ of a two-element metric space over $V$. The uniqueness of the decomposition of $F$ is due to the uniqueness of the block decomposition of the graph $G_F$ associated to this injective envelope.

1. Introduction and presentation of the main results

The original motivation of this paper is the work of Quilliot [16, 17]. He considers reflexive and directed graphs as metric spaces; the distance between two vertices $x$ and $y$ of a graph $G$ being, instead of a non-negative real, the set $d_G(x,y)$ of words over a two-letters alphabet $\{ +, - \}$ which code the zig-zag paths going from $x$ to $y$. Then, he uses concepts of the theory of metric spaces like balls, non-expansive maps, and Helly property. This point of view was extended to transition systems in [15]. Indeed, one may view the graph $G$ as a transition system $M$ over $\{ +, - \}$ and the distance as the language $d_M(x,y)$ accepted by the automaton $(M, \{ x \}, \{ y \})$ with initial state $x$ and final state $y$. In the case of reflexive and directed graphs, the values of the distance are final segments of the free monoid $\{ +, - \}^*$ equipped with the Higman ordering. To make the study of transitions systems over an alphabet $A$ closer to the graph case, it is convenient to suppose that the value of $d_M(x,y)$ determines the value of $d_M(y,x)$; for that, we suppose that the alphabet $A$ is equipped with an involution $- \circ$ and each transition system $M := (Q, T)$ is involutive, in

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the sense that \((p,a,q) \in T\) if and only if \((q,\overline{a},p) \in T\). Once the involution on \(A\) is extended to the free monoid \(A^*\) and then to the power set \(\mathcal{P}(A^*)\), we have \(d_M(x,y) = d_M(y,x)\). Going a step further, we say that \(M\) is reflexive if every letter occurs to every vertex, that is \((p,a,p) \in T\) for every \(p \in Q\) and \(a \in A\). In this case, distances values are final segments of the free monoid \(A^*\) equipped with the Higman ordering.

Structural properties of transition systems rely upon algebraic properties of languages and conversely. In fact, transition systems can be viewed as geometric objects interpreting these algebraic properties. This paper is an illustration of this claim.

We start with an ordered alphabet \(A\). Let \(A^*\) be the free monoid equipped with the Higman ordering. Let \(F(A^*)\) be the set of final segments of \(A^*\). The concatenation of words extends to \(\mathcal{P}(A^*)\); this operation defined by \(XY := \{\alpha \beta : \alpha \in X, \beta \in Y\}\) induces an operation on \(F(A^*)\) for which the set \(A^*\) is neutral. Hence \(F(A^*)\) is a monoid. Since it contains the empty set \(\varnothing\) and \(\varnothing\) has several decompositions (e.g. \(\varnothing = \varnothing A^* = A^* \varnothing\)), this monoid is not free. Let \(F^0(A^*) := F(A^*) \setminus \{\varnothing\}\) be the set of non-empty final segments of \(A^*\). This is submonoid of \(F(A^*)\) (see Subsection 2.2 for definitions, if needed).

**Theorem 1.** \(F^0(A^*)\) is a free monoid.

The existence (or not) of an involution on \(A\) has no effect on the conclusion.

The following illustration of Theorem 1 was proposed to us by J. Sakarovitch \[20\]. An antichain of \(A^*\) is any subset \(X\) of \(A^*\) such that any two distinct elements \(\alpha\) and \(\beta\) of \(X\) are incomparable w.r.t. the Higman ordering. The set \(\text{Ant}(A^*)\) of antichains of \(A^*\) and the set \(\text{Ant}_{\omega}(A^*)\) of finite antichains of \(A^*\) are submonoids of \(\mathcal{P}(A^*)\); the sets \(\text{Ant}^0(A^*) := \text{Ant}(A^*) \setminus \{\varnothing\}\) and \(\text{Ant}^\omega_\omega(A^*) := \text{Ant}_{\omega}(A^*) \setminus \{\varnothing\}\) of non-empty antichains are also submonoids. From Theorem 1 we deduce:

**Theorem 2.** The monoids \(\text{Ant}^0(A^*)\) and \(\text{Ant}^\omega_\omega(A^*)\) are free.

Note that if \(A\) is well-quasi-ordered (w.q.o) (that is to say that every final segment of \(A\) is finitely generated) then the monoids \(\text{Ant}(A^*)\) and \(\text{Ant}_{\omega}(A^*)\) are equal and isomorphic to the monoid \(F(A^*)\), thus Theorem 2 reduces to Theorem 1. Indeed, if \(A\) is w.q.o. then, according to a famous result of Higman \[5\], \(A^*\) is w.q.o. too, that is every final segment \(F\) of \(A^*\) is generated by Min\((F)\) the set of minimal elements of \(F\). Since Min\((F)\) is an antichain and in this case a finite one, our claim follows.

Let \(N(A^*)\) be the MacNeille completion of the poset \(A^*\), that we may view as the collection of intersections of principal final segments of \(A^*\). The MacNeille completion of \(N(A^*)\) is a submonoid of \(F(A^*)\). From Theorem 1 we derive:

**Theorem 3.** Let \(A\) be an ordered alphabet. The monoid \(N^0(A^*) := N(A^*) \setminus \{\varnothing\}\) is free.

We recall that a member \(F\) of \(F(A^*)\) is irreducible if it is distinct from \(A^*\) and is not the concatenation of two members of \(F(A^*)\) distinct of \(F\) (note
that with this definition, the empty set is irreducible. The fact that $F^\circ(A^*)$ is free amounts to the fact that each member decomposes in a unique way as a concatenation of irreducible elements. We interpret this fact by means of injective envelopes of 2-element metric spaces.

We suppose that $A$ equipped with an involution (this is not a restriction: we may choose the identity on $A$ as our involution). The category of metric spaces over $F(A^*)$, with the non-expansive maps as morphisms has enough injectives (meaning that every metric space extends isometrically to an injective one). The gluing of two injectives by a common vertex yields an injective (see Theorem 11); we will say that an injective which is not the gluing of two proper injectives is irreducible. For every final segment $F$ of $A^*$, the 2-element space metric space $E := (\{x,y\}, d)$ such that $d(x,y) = F$, has an injective envelope $S_F$ (a minimal extension to an injective metric space). To $S_F$ corresponds a transition system $M_F$ on the alphabet $A$, with transitions $(p,a,q)$ if $a \in d(p,q)$. The automaton $A_F := (M_F, \{x\}, \{y\})$ with $x$ as initial state and $y$ as final state accepts $F$. A transition system yields a directed graph whose arcs are the ordered pairs $(x,y)$ linked by a transition. Since the transition system $M_F$ is reflexive and involutive and thus the corresponding graph $G_F$ is undirected and has a loop at every vertex. For an example, if $F = A^*$, $S_F$ is the one-element metric space and $G_F$ is the two-elements metric space $E := (\{x,y\}, d)$ with $d(x,y) = \emptyset$ and $G_F$ has no edge.

With the notion of cut vertex and block borrowed from graph theory, we prove:

**Theorem 4.** Let $F$ be a final segment of $A^*$ distinct of $A^*$. Then $F$ is irreducible if and only if $S_F$ is irreducible if and only if $G_F$ has no cut vertex. If $F$ is not irreducible, the blocks of $G_F$ are the vertices of a finite path $C_0,\ldots,C_{n-1}$ with $n \geq 2$, whose end vertices $C_0$ and $C_{n-1}$ contain respectively the initial state $x$ and the final state $y$ of the automaton $A_F$ accepting $F$. Furthermore, $F = F_0\ldots F_i\ldots F_{n-1}$, the automaton $A_{F_i}$ accepting $F_i$ being isomorphic to $(M_F \mid C_i, \{x_i\}, \{y_i\})$, where $x_i := x$ if $i = 0$, $y_i = y$ if $i = n-1$ and $\{x_i\} = C_{i-1} \cap C_i$, $\{y_i\} = C_i \cap C_{i+1}$, otherwise.

From this result, the freeness of $F^\circ(A^*)$ follows.

An approach of transition systems as metric spaces was developed in [8, 9, 10, 11, 15, 18]. A study of retraction, coretraction and injective objects among transition systems was also developed by Hudry [6, 7].

This paper is organized as follows. The proof of Theorems 1 and 2 is in Section 2. The proof of Theorem 3 is in Section 3. Properties of metric spaces over a Heyting algebra and their injective envelopes are summarized in subsection 4.1. Involution and reflexive transition systems are presented in subsection 4.2. The injective envelope of a 2-element metric space over $F(A^*)$ is described in subsection 4.3. We prove Theorem 4 in subsection 4.4.
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2. THE ORDERED MONOIDS $F(A^*)$ AND $\text{Ant}(A^*)$

In this section we prove Theorems 1 and 2. The proof of Theorem 1 relies on Levi’s Lemma and a decomposition property we introduce at this occasion. The proof of Theorem 2 is a consequence.

2.1. Monoids, Ordered monoids, Heyting algebras. Let $V$ be a monoid.
We denote $\cdot$ the operation and $1$ its neutral element. The monoid $V$ is cancellative if it is cancellative on the left and on the right that is if for all $u,v,w \in V$:

(1) $w \cdot u = w \cdot v \implies u = v$

and

(2) $u \cdot w = v \cdot w \implies u = v$.

The monoid $V$ is equidivisible if for all $u_1,u_2,v_1,v_2 \in V$:

(3) $u_1 \cdot u_2 = v_1 \cdot v_2 \implies$ either $u_1 = v_1 \cdot w, u_2 = w \cdot v_2$ or $v_1 = u_1 \cdot w, u_2 = w \cdot v_2$ for some $w \in V$.

We say that $V$ is graded if there is a morphism $\gamma$ of $V$ into the additive monoid of non-negative integers such that $\gamma^{-1}(0) = 1$. Such morphism $f$ is called a graduation. We will use the following form of Levi’s lemma [13] (cf [14] p.13, 1.1.1, Section 1.1, Problems).

**Lemma 1.** A monoid $V$ is free if and only if $V$ is equidivisible and graded.

An element $x$ of a monoid $V$ is irreducible if

(4) $x \neq 1$ and $x = y \cdot z \implies x = y$ or $x = z$.

We recall that

**Fact 1.** Every element of a monoid $V$ has a decomposition into irreducible elements provided that $V$ is graded.

The proof of this fact follows the lines of E.Nether’s proof that ideals of an Noetherian ring decompose into irreducible ideals. If this fact was not true, the subset $B$ of $x \in V$ with no decomposition into irreducible elements will be non empty. Pick $x \in B$ such that $\gamma(x)$ is minimal w.r.t the graduation $\gamma$. Check that $x$ is irreducible. This yields a contradiction.

We denote by $\text{Irr}(V)$ the set of irreducible members of a monoid $V$.

**Lemma 2.** The submonoid $W$ generated by some set $I$ of irreducible members of a free monoid $V$ is free.
Indeed, each element of $W$ has a unique decomposition as a product of members of $I$.

An ordered monoid is a monoid equipped with a compatible ordering. The ordered monoid is a meet-semilattice monoid if the ordering is a meet-semilattice, that is every pair of elements $u, v \in V$ has a meet, denoted by $u \land v$, and if the monoid operation distributes with the meet, that is:

$$ (w \cdot u) \land (w \cdot v) = w \cdot (u \land v) $$

and

$$ (u \cdot w) \land (v \cdot w) = (u \land v) \cdot w. $$

The free monoid $A^*$ with the Higman ordering satisfies the following two conditions:

$$ w \leq u \cdot v \text{ implies } w = u' \cdot v' \text{ for some } u' \leq u \text{ and } v' \leq v $$

and

$$ u \cdot v \leq w \text{ implies } w = u' \cdot v' \text{ for some } u \leq u' \text{ and } v \leq v'. $$

for all $u, v, w \in A^*$.

**Lemma 3.** Let $V$ be an ordered monoid and $u, v, u', v' \in V$.

(a) If $V$ is cancellative then $u \cdot v = u' \cdot v'$, $u \leq u'$ and $v \leq v'$ imply $u = u'$ and $v = v'$.

(b) If the neutral element $1$ is the least element of $V$, if $V$ is equidivisible and satisfies Condition [7] or Condition [8] then $u \cdot v \leq u' \cdot v'$, resp. $u \cdot v < u' \cdot v'$, implies $u \leq u'$, resp. $u < u'$, or $v \leq v'$, resp. $v < v'$.

**Proof.** (a). Since $u \leq u'$ and $v \leq v'$ we have $u \cdot v \leq u' \cdot v \leq u' \cdot v'$. Since $u \cdot v = u' \cdot v'$ we have $u \cdot v = u' \cdot v$. Since $V$ is cancellative, this yields $u = u'$. Similarly, we get $v = v'$.

(b). Suppose that $V$ satisfies Condition [7]. Suppose $u \cdot v \leq u' \cdot v'$. There are $u''$ and $v''$ such that $u'' \leq u'$, $v'' \leq v'$ and $u \cdot v = u'' \cdot v''$. By equisivibility, either $u'' = u \cdot u_1$ for some $u_1$, hence $u \leq u''$ in which case $u \leq u'$ or $v'' = v_1 \cdot v$ for some $v_1$ hence $v \leq v''$ in which case $v \leq v'$. If $u \cdot v < u' \cdot v'$ we get either $u < u'$ or $v < v'$. We get the same conclusion if $V$ satisfies Condition [8].

**Definitions 4.** Let $V$ be an ordered monoid. The cartesian product $V \times V$ is ordered so that $(u_1, u_2) \leq (v_1, v_2)$ if $u_1 \leq v_1$ and $u_2 \leq v_2$. Let $(v_1, v_2) \in V \times V$. This pair is above $v \in V$ if $v_1 \cdot v_2 \geq v$. It is minimal above $v$ if it is above $v$ and there is no pair $(u_1, u_2) < (v_1, v_2)$ which is above $v$. It is minimal if it is minimal above $v := v_1 \cdot v_2$. The pair $(v_1, v_2)$ satisfies the convexity property if for every minimal pair $(u_1, u_2) \in V \times V$ above $v := v_1 \cdot v_2$ either:

$$ v_1 \leq u_1 \cdot u_2, \ u_1 \leq v_1 \text{ and } u_2 = u_1^1 \cdot v_2 \text{ for some } u_1^1 \in V $$

or:

$$ v_2 \leq u_1 \cdot u_2, \ u_2 \leq v_2 \text{ and } u_1 = v_1 \cdot u_1^2 \text{ for some } u_1^2 \in V. $$
This pair is summable if it is minimal and satisfies the convexity property. The ordered monoid \( V \) satisfies the decomposition property if every pair is summable.

**Lemma 5.** If a meet-semilattice monoid \( V \) satisfies the decomposition property, then it is cancellative and equidivisible.

**Proof.** Suppose that \( V \) satisfies the decomposition property. Then according to our definition, each pair \((v_1, v_2)\) is summable hence minimal above \( v := v_1 \cdot v_2 \). This property implies that \( V \) is cancellative. Indeed, let \( u, v, w \in V \) such that \( w \cdot u = w \cdot v \). Due to distributivity, we have \( w \cdot u = (w \cdot u) \wedge (w \cdot v) = w \cdot (u \wedge v) \). By minimality of \((w, w)\) above \( w \cdot u \) we have \( u \wedge v = u \), hence \( u \leq v \). The minimality of \((w, v)\) above \( w \cdot v \) yields similarly \( v \leq u \), hence \( u = v \), proving that \( V \) is cancellative on the left. The proof that \( V \) is cancellative on the right is similar. Hence \( V \) is cancellative.

Let \( u_1, u_2 \) and \( v_1, v_2 \) such that \( u_1 \cdot u_2 = v_1 \cdot v_2 \). Set \( v := v_1 \cdot v_2 \). Since \( u_1 \cdot u_2 = v_1 \cdot v_2 \), \((u_1, u_2)\) is above \( v \). Since \((u_1, u_2)\) is summable, it is minimal above \( u_1 \cdot u_2 \), that is minimal above \( v \). Since \((v_1, v_2)\) is summable, it satisfies the decomposition property. Hence Condition (9) or Condition (10) holds. Suppose that Condition (9) holds. Let \( u_1^2 \in V \) such that \( v_1 \leq u_1 \cdot u_1^2 \) such that \( v \) is summable, it satisfies the decomposition property. Hence Condition (9) or Condition (10) holds. Suppose that Condition (9) holds. Let \( u_1^2 \in V \) such that \( v_1 \leq u_1 \cdot u_1^2 \) and \( v_2 = u_2 \cdot v_2 \). Since \( v_1 \cdot v_2 = u_1 \cdot (u_1^2 \cdot v_2) = (u_1 \cdot u_1^2) \cdot v_2 \) and \( V \) is cancellative we have \( v_1 = u_1 \cdot u_1^2 \). With \( w := u_1^2 \), Condition (3) holds. If Condition (10) holds, we get the same conclusion. Hence \( V \) is equidivisible.

An ordered monoid \( V \) is a **Heyting algebra** if the ordering is complete (every subset has a meet and a join) and the following distributivity condition holds:

\[
\bigwedge_{\alpha \in A, \beta \in B} u_\alpha \cdot v_\beta = \bigwedge_{\alpha \in A} u_\alpha \cdot \bigwedge_{\beta \in B} v_\beta
\]

for all \( u_\alpha \in V \) (\( \alpha \in A \)) and \( v_\beta \in V \) (\( \beta \in B \)).

A Heyting algebra \( V \) is **involutive** if there is an involution \( \neg \) on \( V \) which preserves the ordering and reverses the monoid operation (that is \( \neg(\neg u) = u \) for all \( u \) in \( V \)) in particular the involution preserves the neutral element of the monoid.

In a Heyting algebra, the least element is not necessarily the neutral element for the monoid operation (in the next section, the set \( \mathcal{P}(A^*) \) of langages over an alphabet \( A \) provides such an example). However, in the Heyting algebras we work with, namely \( F(A^*) \) and \( \text{Ant}(A^*) \), the least element and the neutral element coincide.

### 2.2. The monoid of final segments

Let \( A \) be a set. Considering \( A \) as an **alphabet** whose members are **letters**, we write a word \( \alpha \) with a mere juxtaposition of its letters as \( \alpha = a_0 \ldots a_{n-1} \) where \( a_i \) are letters from \( A \) for \( 0 \leq i \leq n - 1 \). The integer \( n \) is the **length** of the word \( \alpha \) and we denote it \( |\alpha| \). Hence we identify letters with words of length 1. We denote by \( \square \) the empty word, which is the unique word of length zero. The **concatenation** of two word \( \alpha := a_0 \ldots a_{n-1} \)
and $\beta := b_0 \cdots b_{m-1}$ is the word $\alpha \beta := a_0 \cdots a_{n-1} b_0 \cdots b_{m-1}$. We denote by $A^*$ the set of all words on the alphabet $A$. Once equipped with the concatenation of words, $A^*$ is a monoid, whose neutral element is the empty word, in fact $A^*$ is the free monoid on $A$. A language is any subset $X$ of $A^*$. We denote by $\mathcal{P}(A^*)$ the set of languages. We will use capital letters for languages. If $X, Y \in \mathcal{P}(A^*)$ the concatenation of $X$ and $Y$ is the set $XY := \{ \alpha \beta : \alpha \in X, \beta \in Y \}$ (and we will use $XY$ and $xY$ instead of $X\{y\}$ and $\{x\}Y$). This operation extends the concatenation operation on $A^*$; with it, the set $\mathcal{P}(A^*)$ is a monoid whose neutral element is the set $\{\square\}$. Ordered by inclusion, this is (join) lattice ordered monoid. Indeed, concatenation distributes over arbitrary union, namely:

$$\bigcup_{i \in I} X_i Y = \bigcup_{i \in I} X_i Y.$$  

But concatenation does not distribute over intersection (for a simple example, let $A := \{a, b, c\}$, $I := \{1, 2\}$, $X_1 := \{ab\}$, $X_2 := \{a\}$, $Y := \{c, bc\}$, then $\varnothing = (X_1 \cap X_2)Y \neq X_1Y \cap X_2Y = \{abc\}$). Hence, ordered by reverse of the inclusion, the monoid $\mathcal{P}(A^*)$ becomes a Heyting algebra (while ordered by inclusion it is not). If $-$ is an involution on $A$, it extends to an involution on $A^*$, by setting $\overline{\square} := \square$, and $\overline{\alpha} = \overline{a_{n-1} \cdots a_0}$ if $\alpha = a_0 \cdots a_{n-1}$. This involution reverses the concatenation of words. Extended to $\mathcal{P}(A^*)$ by setting $\overline{X} := \{\overline{\alpha} : \alpha \in X\}$, it reverses the concatenation of languages and preserves the inclusion order on languages. In summary:

**Lemma 6.** The set $\mathcal{P}(A^*)$ equipped with the concatenation of languages and the reverse of the inclusion order is a Heyting algebra. Moreover, this is an involutive Heyting algebra if we add to it the extension of an involution on $A$.

We suppose from now that the alphabet $A$ is ordered. We order $A^*$ with the Higman ordering: if $\alpha$ and $\beta$ are two elements in $A^*$ such $\alpha := a_0 \cdots a_{n-1}$ and $\beta := b_0 \cdots b_{m-1}$ then $\alpha \leq \beta$ if there is an injective and increasing map $h$ from $\{0, \ldots, n - 1\}$ to $\{0, \ldots, m - 1\}$ such that for each $i$, $0 \leq i \leq n - 1$, we have $a_i \leq b_{h(i)}$. Then $A^*$ is an ordered monoid with respect to the concatenation of words. A final segment of $A^*$ is any subset $F \subseteq A^*$ such that $\alpha \leq \beta, \alpha \in F$ implies $\beta \in F$. Initial segments are defined dually. Let $X$ be a subset of $A^*$; then

$$\uparrow X := \{ \beta \in A^* : \alpha \leq \beta \text{ for some } \alpha \in X \}$$

is the upper set generated by $X$ and

$$\downarrow X := \{ \alpha \in A^* : \alpha \leq \beta \text{ for some } \beta \in X \}$$

is the lower set generated by $X$. For a singleton $X = \{\alpha\}$, we omits the set brackets and call $\uparrow \alpha$ and $\downarrow \alpha$ a principal upper set and a principal lower set respectively. Let $F(A^*)$ be the collection of final segments of $A^*$. The set $F(A^*)$ is stable w.r.t. the concatenation of languages: if $X, Y \in F(A^*)$, then $XY \in F(A^*)$ (indeed, if $u, v, w \in A^*$ with $wv \leq u$ then $w = u'v'$ with $u \leq u'$ and $v \leq v'$). Clearly, the neutral element is $A^*$. The set $F(A^*)$ ordered by
inclusion is a complete lattice (the join is the union, the meet is the intersec-
tion). Concatenation distributes over union. If we order \( F(A^*) \) by reverse of
the inclusion, denoting \( X \leq Y \) instead of \( X \supseteq Y \), and we set \( 1 := A^* \), we have

**Lemma 7.** The set \( F(A^*) \) equipped with the concatenation of languages and
the reverse of the inclusion order is a Heyting algebra. Moreover, this is an
involutive Heyting algebra if we add to it the extension of an involution on \( A \).

A correspondance between \( \mathcal{P}(A^*) \) and \( F(A^*) \) is given in the following lemma.

**Lemma 8.** The correspondance which associates to every subset \( X \) of \( A^* \) the
final segment \( \uparrow X \) is a morphism of ordered monoids from \( \mathcal{P}(A^*) \) onto \( F(A^*) \).

**Proof.** Clearly this correspondance preserves the ordering. Since by definition
it is surjective, to show that it is a morphism of monoid it suffices to show that

\[
(12) \quad \uparrow X \uparrow Y = \uparrow (XY)
\]

for all \( X, Y \in \mathcal{P}(A^*) \).

Let \( z \in \uparrow X \uparrow Y \). Then \( z \) decomposes as \( z = xy \) with \( x \in \uparrow X \) and \( y \in \uparrow Y \). There are \( x' \in X \) with \( x' \leq x \) and \( y' \in Y \) with \( y' \leq y \). Hence, \( x'y' \in XY \) and
\( x'y' \leq xy = z \). Thus \( z \in \uparrow (XY) \). This proves that \( \uparrow X \uparrow Y \subseteq \uparrow (XY) \).

Conversely, let \( z \in \uparrow (XY) \). Then there are \( x' \in X \) and \( y' \in Y \) such that
\( x'y' \leq z \). Thus \( z \) decomposes as \( z = xy \) with \( x' \leq x \) and \( y' \leq y \). Hence \( x \in \uparrow X \) and \( y \in \uparrow Y \). Thus \( z = xy \in \uparrow X \uparrow Y \). This proves that \( \uparrow XY \subseteq \uparrow X \uparrow Y \). The
equality holds, as claimed. \( \square \)

An **antichain** of \( A^* \) is any subset \( X \) of \( A^* \) such that any two distinct elements
\( x \) and \( y \) of \( X \) are incomparable w.r.t. the Higman ordering. Let \( \text{Ant}(A^*) \) be the
set of antichains of \( A^* \) and \( \text{Ant}_{\omega}(A^*) \) be the set of finite antichains.

**Lemma 9.** \( \text{Ant}(A^*) \) and \( \text{Ant}_{\omega}(A^*) \) are submonoids of \( \mathcal{P}(A^*) \). The morphism
\( X \mapsto \uparrow X \) from \( \mathcal{P}(A^*) \) into \( F(A^*) \) induces a one-to-one morphism from \( \text{Ant}(A^*) \)
into \( F(A^*) \). The correspondance which associates to every final segment \( X \) of
\( A^* \) the set \( \text{Min}(X) \) of its minimal elements is a morphism of monoids from
\( F(A^*) \) onto \( \text{Ant}(A^*) \).

**Proof.** We prove first that:

\[
(13) \quad \text{Min}(XY) = \text{Min}(X) \text{Min}(Y)
\]

for all \( X, Y \in F(A^*) \).

Let \( z \in \text{Min}(XY) \). Since \( z \in XY \), it decomposes as \( z = xy \) with \( x \in X \) and
\( y \in Y \). We prove that \( x \in \text{Min}(X) \) and \( y \in \text{Min}(Y) \), from which follows that \( z = xy \in \text{Min}(X) \text{Min}(Y) \) and thus the inclusion \( \text{Min}(XY) \subseteq \text{Min}(X) \text{Min}(Y) \).

If \( x \notin \text{Min}(X) \) there is some \( x' \in X \) with \( x' < x \). In this case, \( x'y < xy = z \).
Since \( x'y \in XY \) this contradicts the minimality of \( z \). Thus \( x \in \text{Min}(X) \). By
the same argument, we have \( y \in \text{Min}(Y) \).
Let $z \in \text{Min}(X)\text{Min}(Y)$. We claim that $z \in \text{Min}(XY)$ from which the inclusion $\text{Min}(X)\text{Min}(Y) \subseteq \text{Min}(XY)$ follows. This element $z$ decomposes as $z = xy$ with $x \in \text{Min}(X)$ and $y \in \text{Min}(Y)$. In particular, $z \in XY$. If $z \notin \text{Min}(XY)$, there is some $z' \in XY$ with $z' < z$. This element $z'$ decomposes as $z' = x'y'$ with $x' \in X$, $y' \in Y$. Since $x'y' < xy$ then according to (b) of Lemma 3 either $x' < x$ or $y' < y$. The first case is impossible since $x \in \text{Min}(X)$ and the second too since $y \in \text{Min}(Y)$. Thus $z \in \text{Min}(XY)$ as claimed.

Let $U, V \in \text{Ant}(A^*)$. Then $UV \in \text{Ant}(A^*)$ and

$$\tag{14} |UV| = |U||V|.$$  

We have $U = \text{Min}(\uparrow U)$ and $V = \text{Min}(\uparrow V)$ and by Equation (13) we have $UV = \text{Min}(\uparrow U \uparrow V)$, proving that $UV$ is an antichain. The map $(x, y) \mapsto xy$ is a bijection from $U \times V$ to $UV$. Indeed, if $xy = x'y'$ then by (b) of Lemma 3 either $x \leq x'$ or $y \leq y'$. If $x < x'$ then since $U$ is an antichain then $x = x'$; since $A^*$ is free it is cancellative, hence $y = y'$. Similarly, the case $y' \leq y$ yields $y = y'$ and $x = x'$. Equation (14) follows.

Concatenation preserves $\text{Ant}(A^*)$ and, according to Equation (14), also $\text{Ant}_{\omega}(A^*)$. Since for each $x \in A^*$, $\{x\}$ is an antichain, $\{\square\}$ is an antichain. Since this is the neutral element of $\mathcal{P}(A^*)$, this is the neutral element of $\text{Ant}(A^*)$ and $\text{Ant}_{\omega}(A^*)$ which are then submonoids of $\mathcal{P}(A^*)$.

Since $\text{Ant}(A^*)$ is a submonoid of $\mathcal{P}(A^*)$, the map $X \mapsto \uparrow X$ from $\mathcal{P}(A^*)$ into $F(A^*)$ induces a morphism from $\text{Ant}(A^*)$ into $F(A^*)$. This morphism is one-to-one. Indeed, if $U$ is an antichain, $U = \text{Min}(\uparrow U)$. The map $X \mapsto \text{Min}(X)$ transforms the neutral element of the monoid $F(A^*)$, namely $A^*$, into $\{\square\}$ which is the neutral element of the monoid $\text{Ant}(A^*)$. Since, according to Equation (13), this maps preserves the concatenation, it is a morphism of monoid. \hfill \Box

**Lemma 10.** The set $F^\circ(A^*) := F(A^*) \setminus \{\varnothing\}$ is a graded and cancellative submonoid of $F(A^*)$.

**Proof.** Set $\gamma(X) := \text{Min}\{|x|: x \in X\}$ for every $X \in F^\circ(A^*)$. This is a graduation.

Let $X, Y, Z$ be three elements in $F^\circ(A^*)$ Then

$$XY = XZ \implies Y = Z \quad \tag{1}$$

and

$$YX = ZX \implies Y = Z. \quad \tag{2}$$

Indeed, let $x \in X$ such that $|x| = \gamma(X)$. Let $y \in Y$. Since $xy \in XY$, there exists $x' \in X$ and $z \in Z$ such that $xy = x'z$. From the equidivisibility property of $A^*$ and the fact that $|x| \leq |x'|$, we have $z \leq y$. Since $Z$ is a final segment, it follows that $y \in Z$. Hence $Y \subseteq Z$. Similarly we get $Z \subseteq Y$ proving $Y = Z$. By the same argument we prove (2). \hfill \Box
Lemma 11. Let $X,Y,X',Y'$ be non-empty final segments of $A^*$. If $XY = X'Y'$ with $X \subseteq X'$ and $Y \subseteq Y'$ then $X = X'$ and $Y = Y'$.

**Proof.** According to the definition given Subsection 2.1, we need to prove that $\uparrow \text{Ant}$ maps the irreducibles of $\text{Ant}$ into $\text{Ant}(A^*)$. Since $\uparrow \text{Ant}$ is an antichain, $Z = \text{Min}(\uparrow Z) = \text{Min}(XY) = \text{Min}(X)\text{Min}(Y)$. Thus, if $\uparrow Z = \text{Min}(X) \uparrow \text{Min}(Y)$, we have $\uparrow Z = \text{Min}(X) \uparrow \text{Min}(Y) = XY$. 

**Corollary 5.** The one-to-one morphism $Z \mapsto \uparrow Z$ from $\text{Ant}(A^*)$ into $F(A^*)$ maps the irreducibles of $\text{Ant}(A^*)$ and $\text{Ant}_{\omega}(A^*)$ into the irreducibles of $F(A^*)$.

**Proof.** Let $Z$ be an irreducible of $\text{Ant}(A^*)$. We claim that $\uparrow Z$ is irreducible in $F(A^*)$. Indeed, suppose $\uparrow Z = XY$ with $X,Y \in F(A^*)$. Since $Z$ is an antichain, $Z = \text{Min}(\uparrow Z) = \text{Min}(XY) = \text{Min}(X)\text{Min}(Y)$. Since $Z$ is irreducible in $\text{Ant}(A^*)$, either $Z = \text{Min}(X)$ or $Z = \text{Min}(Y)$. Hence, either $\uparrow Z = \uparrow \text{Min}(X)$ or $\uparrow Z = \uparrow \text{Min}(Y)$. If $\uparrow Z = \emptyset$, then necessarily $X = \emptyset$ or $Y = \emptyset$, hence $\uparrow Z = X$ or $\uparrow Z = Y$ and thus $\uparrow Z$ is irreducible. If $\uparrow Z \neq \emptyset$, then according to Lemma 12 $\uparrow Z$ is an antichain, proving that $\uparrow Z$ is irreducible.

**Lemma 13.** The ordered monoid $F^\circ(A^*)$ satisfies the decomposition property.

**Proof.** According to the definition given Subsection 2.1, we need to prove that all pairs $(V_1,V_2)$ of $F^\circ(A^*)$ are summable, that is are minimal and have the convexity property (minimality being w.r.t. the reverse of inclusion).

- **Minimal property**: A pair $(V_1,V_2)$ means:
  \[
  V_1V_2 = U_1U_2, \quad V_1 \subseteq U_1 \text{ and } V_2 \subseteq U_2 \implies V_1 = U_1 \text{ and } V_2 = U_2.
  \] (3)

  This property readily follows from Lemma 11.

- **Convexity**: A pair (with respect to reverse of inclusion) such that $U_1U_2 \subseteq \overline{V} := V_1V_2$.

  We have either:
  
  (15) $U_1U_2 \subseteq V_1, V_1 \subseteq U_1$ and $U_2 = U_1V_2$ for some $U_2 \in F^\circ(A^*)$ or

  (16) $U_1U_2 \subseteq V_2, V_2 \subseteq U_2$ and $U_1 = V_1U_2$ for some $U_2 \in F^\circ(A^*)$.

  Let $U_1U_2 \in F^\circ(A^*)$ such that $(U_1,U_2)$ is a minimal pair (with respect to reverse of inclusion) such that $U_1U_2 \subseteq \overline{V} := V_1V_2$.

  **Claim 1.** If $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ then $U_1 = V_1$ and $U_2 = V_2$ hence both Conditions (15) and (16) hold.

  This follows directly from the minimality of the pair $(U_1,U_2)$. 

Since $F^\circ(A^*)$ is cancellative, it satisfies (a) of Lemma 3 that:

- **Lemma 3**. Let $X,Y,X',Y'$ be non-empty final segments of $A^*$. If $XY = X'Y'$ with $X \subseteq X'$ and $Y \subseteq Y'$ then $X = X'$ and $Y = Y'$.
Claim 2. Either $U_1 \subseteq V_1$ or $U_2 \subseteq V_2$.

Indeed, if $U_1 \notin V_1$ and $U_2 \notin V_2$, then let $u_1 \in U_1 \setminus V_1$ and $u_2 \in U_2 \setminus V_2$. We have $u_1u_2 \in U_1U_2 \subseteq V_1V_2$. Let $v_1 \in V_1$, $v_2 \in V_2$ such that $u_1u_2 = v_1v_2$. The equidivisibility property of $A^*$ implies that either $v_1$ is a left factor of $u_1$ or $v_2$ is a right factor of $u_2$. Therefore either $v_1 \leq u_1$ or $v_2 \leq u_2$. Since $V_1$ and $V_2$ are final segments of $A^*$, this implies $u_1 \in V_1$ or $u_2 \in V_2$ which contradicts the choice of $u_1$ and $u_2$.

Claim 3. If $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ then Condition (16) holds.

Indeed, let $u \in U_2 \setminus V_2$. We may write $U_1 = \bigcup_{i \in I} Z_i$ where for each $i \in I$, $Z_i$ is of the form $\{x \in A^* : \alpha_i \leq x\}$ for some $\alpha_i \in U_1$. Observe first, for every $i \in I$, $\alpha_i$ has a proper left factor $\gamma_i \in V_1$. Indeed, we have $\alpha_i u \in U_1 U_2 \subseteq V_1 V_2$. There are $\gamma_i \in V_1, \delta_i \in V_2$ such that $\alpha_i u = \gamma_i \delta_i$. From the equidivisibility property, either $\gamma_i$ is a left factor of $\alpha_i$ or $\alpha_i$ is a left factor of $\gamma_i$. The later relation is impossible, since we would get $\delta_i \leq u$ and then since $V_2$ is a final segment of $A^*$, this would imply $u \in V_2$. Since $A^*$ is cancellative, the choice of $u$ ensures that $\gamma_i$ is a proper left factor of $\alpha_i$, which belongs to $V_1$. Let $\xi_i \in A^*$ such that $\alpha_i = \beta_i \xi_i$. Set $U^2_i := \bigcup_{i \in I} W_i$ where

$$W_i := \{x \in A^* : \xi_i \leq x\} \quad \text{and} \quad U^2_2 := \{x \in A^* : \forall a \in U^2_2, ax \in V_2\}.$$  
First, we show that $U_1 \subseteq V_1 U^2_2$. Let $z \in U_1$. There exists $i \in I$ such that $\alpha_i \leq z$. Thus $\alpha_i = \beta_i \xi_i$. There are $\beta_i', \xi_i' \in A^*$ such that $z = \beta_i \xi_i' \beta_i \leq \beta_i'$ and $\xi_i \leq \xi_i'$. Since $V_1$ and $U^2_2$ are final segments, we have $\beta_i \in V_1$ and $\xi_i' \in U^2_2$ proving $z \in V_1 U^2_2$. Next, we have trivially $U^2_1 U^2_2 \subseteq V_2$. Finally, we prove that $U_2 \subseteq U^2_2$. Suppose that is false and let $u' \in U_2 \setminus U^2_2$. This means that there is $x \in U^2_2$ such that $xu' \notin V_2$. Let $i \in I$ such that $\xi_i \leq x$. Since $V_2$ is a final segment and $\xi_i u' \leq x u'$, we get $\xi_i u' \notin V_2$. But $\beta_i \xi_i u' = \alpha_i u' \in U_1 U_2 \subseteq V_1 V_2$. Let $z_1 \in V_1$ and $z_2 \in V_2$ such that $\beta_i \xi_i u = z_1 z_2$. Since $\xi_i u \notin V_2$, it follows from the equidivisibility property that $z_1$ is a proper left factor of $\beta_i$. This contradicts the choice of $\beta_i$. In summary, we have $U_1 \subseteq V_1 U^2_1, U_1 U^2_2 \subseteq V_2, U_2 \subseteq U^2_2$. With the minimality of the pair $(U_1, U_2)$, we prove $U_1 = V_1 U^2_1$ and $V_2 \subseteq U_2$. Indeed, we have $U_1 \subseteq V_1 U^2_1 \subseteq V_1$, hence $U_1 \subseteq V_1$. With $V_2 \subseteq U_2$ we obtain $U_1 V_2 \subseteq V_1 V_2 \subseteq V$. With $U_1 U_2 \subseteq V$ we have $V \supseteq U_1 V_2 \cup U_1 U_2 = U_1 (V_2 \cup U_2)$. The minimality of $(U_1, U_2)$ implies $U_2 = V_2 \cup U_2$, that is $V_2 \subseteq U_2$. From $U_2 \subseteq U_2$ and $U_1 \subseteq V_1 U^2_1$, minimality of $(U_1, U_2)$ yields $U^2_2 = U_2$ and $V_1 U^2_1 = U_1$. From $V_2 \supseteq U^2_1 U^2_2$ and $U^2_1 = U_1 \cup V_2 = U_2$ we have $V_2 \supseteq U^2_1 U_2$. Hence, Condition (16) holds.

Similarly we prove:

Claim 4. $U_1 \notin V_1$ and $U_2 \notin V_2$ imply that Condition (15) holds.

Convexity property follows from these claims.

\[ \square \]

2.3. Proof of Theorem 1. According to Lemma 13, $F^\circ(A^*)$ satisfies the decomposition property. From Lemma 5 it is equidivisible. It is graded by Lemma 10. From Levi’s Lemma (Lemma 11) it is free.
\( F^\circ(A^*) \) is free, the monoid generated by this subset of irreducibles is free (Lemma 2). Hence \( \text{Ant}(A^*)^\circ \) and \( \text{Ant}_{\omega}(A^*)^\circ \) are free. \( \square \)

**Remark 6.** Let \( \text{Irr}_A \) be the set of non-empty irreducible members of \( F(A^*) \). The monoid \( F(A^*) \setminus \{ \varnothing \} \) is isomorphic to the monoid \( (\text{Irr}_A)^* \). But, as an ordered monoid, \( F^\circ(A^*) \) is not isomorphic to the ordered monoid \( (\text{Irr}_A)^* \) equipped with the Higman ordering w.r.t the reverse of inclusion on \( \text{Irr}_A \). Indeed, let \( A \mathrel{:=} \{ a, b \} \), \( X \mathrel{:=} \{ a, b \} = A^* \setminus \{ \square \} \) and \( Y \mathrel{:=} \{ a, b, b \} \). Then \( X \) and \( Y \) are irreducible and \( XX \leq Y \).

**Problem 7.** Describe the irreducible of \( F(A^*) \)

For an example, if \( F = \uparrow \{ u, v \} \) with \( u \) incomparable to \( v \), then \( F \) is irreducible iff \( u \) and \( v \) do not have a common prefix nor a common suffix.

### 3. The MacNeille completion of the free monoid

The ordered monoid \( A^* \) can be extended to a complete lattice ordered monoid by applying the MacNeille completion. The necessary notation (cf. Skornjakow [21], 1973), Lemma 14 and Theorem 8 of [1] are introduced next.

Let \( X \) be a subset of \( A^* \). Then

\[
X^\Delta := \bigcap_{x \in X} \uparrow x
\]

and

\[
X^\nabla := \bigcap_{x \in X} \downarrow x
\]

are the *upper cone* and the *lower cone* respectively, generated by \( X \).

The pair \((\Delta, \nabla)\) of mappings on \( \mathcal{P}(A^*) \), the power set lattice of \( A^* \), constitutes a Galois connection, yieldings the *MacNeille completion* of \( A^* \). This completion is realized as the complete lattice

\[
\{ W \subseteq A^* : W = W^{\Delta \nabla} \}
\]

ordered by inclusion or its isomorphic copy

\[
\{ Y \subseteq A^* : Y = Y^{\nabla \Delta} \}
\]

ordered by reverse inclusion. The set \( A^* \) embeds into the former via \( x \mapsto \downarrow x \) and into the latter via \( x \mapsto \uparrow x \) \(( x \in A^* \)).

The completion of \( A^* \) inherits its monoid structure from the power set. The cone operators preserve this multiplication as the following lemma confirms.

For reader convenience, we give the proof.

**Lemma 14.** For any subsets \( X, Y \) of \( A^* \),

\[
(XY)^\nabla = X^\nabla Y^\nabla, \text{ and } (XY)^\Delta = X^\Delta Y^\Delta \text{ if } X, Y \neq \varnothing,
\]

hence

\[
(XY)^{\nabla \Delta} = X^{\nabla \Delta} Y^{\nabla \Delta} \text{ and } (XY)^{\Delta \nabla} = X^{\Delta \nabla} Y^{\Delta \nabla}.
\]
Proof. First, observe that $\varnothing \uparrow = \varnothing = A^*$ and $(A^*)\Delta = \varnothing$, while $(A^*)\uparrow := \{\square\}$. Further, $\varnothing Z = \varnothing$ for every subset $Z$ of $A^*$. The inclusions $X\uparrow Y \uparrow \subseteq (XY)\uparrow$ and $X\Delta Y \Delta \subseteq (XY)\Delta$ are thus immediate.

Suppose that there exists a word $w$ in $(XY)\uparrow$ that does not belong to $X\uparrow Y \uparrow$. Then let $u$ be the longest prefix of $w$ from $X\uparrow$, and let $v$ be the longest suffix of $w$ from $Y\uparrow$ so that $w$ is of the form

$$w = ua_1\ldots a_kv$$

for some letters $a_1, \ldots, a_k$, where $k \geq 1$. By the choice of $u$ and $v$, there are words $x \in X$ and $y \in Y$ such that

$$ua_1 \not\in x \text{ and } a_kv \not\in y.$$ 

This, however, is in conflict with $w = ua_1\ldots a_kv \leq xy$. Therefore $(XY)\uparrow$ equals $X\uparrow Y \uparrow$. Finally, suppose $z$ is a word in $(XY)\Delta$ which does not belong to $X\Delta Y \Delta$, where $X$ and $Y$ are nonempty. Then the shortest prefix of $z$ from $X\Delta$ and the shortest suffix of $z$ from $Y\Delta$ intersect in a nonempty subword

$$w := a_1\ldots a_k$$

so that $z$ can be written as

$$z = uww \text{ with } uw \in X\Delta \text{ and } wv \in Y\Delta.$$ 

By the choice of the words $u$ and $v$, we can find words $x \in X$ and $y \in Y$ with

$$x \not\in ua_1\ldots a_{k-1} \text{ and } y \not\in v.$$ 

This contradicts the hypothesis that $xy \leq z = ua_1\ldots a_{k-1}a_kv$.

We conclude that $(XY)\Delta = X\Delta Y \Delta$, completing the proof. \hfill \Box

The completion of $A^*$, realized by the upper closed sets, that we denote by $N(A^*)$, is a complete lattice in which suprema are set-theoretic intersections, whereas infima are the closures of set-theoretic unions. The closed union of a family $Z_i (i \in I)$ of upper sets in $A^*$ is given by:

$$\bigcup_{i \in I} Z_i = (\bigcup_{i \in I} Z_i)^\Delta.$$

The following result entails that the completion of $A^*$ is a complete lattice ordered monoid (in the sense of Birkhoff, 1967 [2]).

**Theorem 8.** For any ordered alphabet $A$, the collection $N(A^*)$ of all closed upper sets of words over $A^*$ is a monoid and complete lattice such that the multiplication distributes over intersection and closed unions, that is:

$$Y(\bigcap_{i \in I} Z_i) = \bigcap_{i \in I} YZ_i \text{ and } (\bigcap_{i \in I} Z_i)Y = \bigcap_{i \in I} Z_i Y,$$

$$Y(\bigcup_{i \in I} Z_i) = \bigcup_{i \in I} YZ_i \text{ and } (\bigcup_{i \in I} Z_i)Y = \bigcup_{i \in I} Z_i Y.$$
for any index set \( I \) and all closed upper sets \( Y, Z_i (i \in I) \).

According to Theorem 8, \( N(A^*) \) is a submonoid of \( F(A^*) \) and a Heyting algebra too. Also, \( N^0(A^*):= N(A^*) \setminus \{ \emptyset \} \) is a submonoid of \( F^0(A^*) = F(A^*) \setminus \{ \emptyset \} \).

3.1. Proof of Theorem 3

Lemma 15. Let \( Z \in N^0(A^*) \). If \( Z = XY \) with \( X, Y \in F(A^*) \) then \( X, Y \in N^0(A^*) \).

Proof. According to Lemma 14, \( (XY)^\vee = X^\vee Y^\vee \) and \( (XY)^\vee = X^\vee Y^\vee \). Since \( Z = XY \) and \( Z = Z \) we have \( Z = X^\vee Y^\vee \). Since \( X \subseteq X^\vee \), \( Y \subseteq Y^\vee \) and \( Z = XY \), we have \( X = X^\vee \) and \( Y = Y^\vee \) by Lemma 11. \( \square \)

From this lemma, the irreducible members of \( N^0(A^*) \) are irreducible in \( F^0(A^*) \). According to Lemma 2, \( N^0(A^*) \) is free.

4. Metric spaces over \( F(A^*) \)

4.1. Basics on metric spaces over a Heyting algebra. The following is a brief outline of \( \text{[11]} \). Let \( V \) be an ordered monoid, the operation being denoted multiplicatively, the neutral element denoted by \( 1 \) (denoted respectively + and 0 in \( \text{[11]} \)). We suppose \( V \) equipped with an involution – such that \( \overline{u \cdot v} = \overline{v} \cdot \overline{u} \) for every \( u, v \in V \). Let \( E \) be a set. A \( V \)-distance on \( E \) is a map \( d : E^2 \to V \) satisfying the following properties for all \( x, y, z \in E \):

\[
\begin{align*}
\text{d1) } & d(x, y) = 1 \iff x = y, \\
\text{d2) } & d(x, y) \leq d(x, z) \cdot d(z, y), \\
\text{d3) } & d(x, y) = d(y, x).
\end{align*}
\]

The pair \( (E, d) \) is called a \( V \)-metric space. If there is no danger of confusion we will denote it by \( E \). These notions appear in \( \text{[4]} \) (cf. p.41) under the name of generalized metric and generalized distance space (with the difference that the law is denoted additively and \( 1 \) is replaced by 0).

If \( V \) is a Heyting algebra (i.e. satisfies the distributivity condition given in equation \( \text{[11]} \)), a \( V \)-distance can be defined on \( V \). This fact relies on the classical notion of residuation. Let \( v \in V \). Given \( \beta \in V \), the sets \( \{ r \in V : v \leq r \beta \} \) and \( \{ r \in V : v \leq \beta - r \} \) have least elements, that we denote respectively by \( [v, \beta - 1] \) and \( [\beta - 1, v] \) (note that \( [\beta - 1, v] = [v, (\overline{\beta})^{-1}] \)). It follows that for all \( p, q \in V \), the set

\[
D(p, q) := \{ r \in V : p \leq q \cdot \overline{r} \quad \text{and} \quad q \leq p \cdot r \}
\]

has a least element, namely \( [p \cdot (q^{-1})] \lor [p^{-1} \cdot q] \), that we denote by \( d_V(p, q) \). As shown in \( \text{[8]} \), the map \( (p, q) \mapsto d_V(p, q) \) is a \( V \)-distance.

Let \( (E, d) \) and \( (E', d') \) be two \( V \)-metric spaces. Recall that a map \( f : E \to E' \) is a non-expansive map (or a contraction) from \( (E, d) \) to \( (E', d') \) provided that \( d'(f(x), f(y)) \leq d(x, y) \) holds for all \( x, y, \in E \). The map \( f \) is an isometry if \( d'(f(x), f(y)) = d(x, y) \) for all \( x, y, \in E \). We say that \( E \) and \( E' \) are
isomorphic, a fact that we denote by \( E \cong E' \), if there is a surjective isometry from \( E \) onto \( E' \).

Let \(((E_i, d_i))_{i \in I}\) be a family of \( V \)-metric spaces. The direct product \( \prod_{i \in I} (E_i, d_i) \), is the metric space \((E, d)\) where \( E \) is the cartesian product \( \prod_{i \in I} E_i \) and \( d \) is the "sup" (or \( \ell^\infty \)) distance defined by \( d((x_i)_{i \in I}, (y_i)_{i \in I}) = \bigvee_{i \in I} d_i(x_i, y_i) \).

For a \( V \)-metric space \( E \), let \( x \in E \) and \( r \in V \), we define the ball \( B_E(x, r) \) as the set \( \{ y \in E : d(x, y) \leq r \} \). We say that \( E \) is convex if the intersection of two balls \( B_E(x_1, r_1) \) and \( B_E(x_2, r_2) \) is non-empty provided that \( d(x_1, x_2) \leq r_1 \cdot \overline{r_2} \). We say that \( E \) is hyperconvex if the intersection of every family of balls \( (B_E(x_i, r_i))_{i \in I} \) is non-empty whenever \( d(x_i, x_j) \leq r_i \cdot \overline{r_j} \) for all \( i, j \in I \). For an example, \((V, d_V)\) is a hyperconvex \( V \)-metric space and every \( V \)-metric space embeds isometrically into a power of \((V, d_V)\) \cite{8}. This is due to the fact that for every \( V \)-metric space \((E, d)\) and for all \( x, y \in E \) the following equality holds:

\[
\text{(17)} \quad d(x, y) = \bigvee_{z \in E} d_V(d(z, x), d(z, y)).
\]

The space \( E \) is a retract of \( E' \), in symbols \( E \prec E' \), if there are two non-expansive maps \( f : E \to E' \) and \( g : E' \to E \) such that \( g \circ f = \text{id}_E \) (where \( \text{id}_E \) is the identity map on \( E \)). In this case, \( f \) is a coretraction and \( g \) a retraction. If \( E \) is a subspace of \( E' \), then clearly \( E \) is a retract of \( E' \) if there is a non-expansive map from \( E' \) to \( E \) such \( g(x) = x \) for all \( x \in E \). We can easily see that every coretraction is an isometry. A metric space is an absolute retract if it is a retract of every isometric extension. The space \( E \) is said to be injective if for all \( V \)-metric space \( E' \) and \( E'' \), each non-expansive map \( f : E' \to E \) and every isometry \( g : E' \to E'' \) there is a non-expansive map \( h : E'' \to E \) such \( h \circ g = f \). We recall that for a metric space over a Heyting algebra \( V \), the notions of absolute retract, injective, hyperconvex and retract of a power of \((V, d_V)\) coincide \cite{8}.

A non-expansive map \( f : E \to E' \) is essential if for every non-expansive map \( g : E' \to E'' \), the map \( g \circ f \) is an isometry if and only if \( g \) is isometry (note that, in particular, \( f \) is an isometry). An essential non-expansive map \( f \) from \( E \) into an injective \( V \)-metric space \( E' \) is called an injective envelope of \( E \). We will rather say that \( E' \) is an injective envelope of \( E \). The construction of injective envelopes is based upon the notion of minimal metric form. A weak metric form is every map \( f : E \to V \) satisfying \( d_V(d(x, y), f(y)) \leq f(y) \) for all \( x, y \in E \). This is a metric form if in addition \( f(x) \leq d(x, y) \cdot f(y) \) for all \( x, y \in E \). A (weak) metric form is minimal if there is no other (weak) metric form \( g \) satisfying \( g \leq f \) (that is \( g(x) \leq f(x) \) for all \( x \in E \)). Since every weak metric form majorizes a metric form, the two notions of minimality coincide. As shown in \cite{8} every \( V \)-metric space has an injective envelope, namely the space of minimal metric forms (cf. also Theorem 2.2 of \cite{11}). From this result follows that an injective envelope of a metric space \( E \) is a minimal injective \( V \)-metric space containing (isometrically) \( E \). We will use particularly the following fact:
Lemma 16. If a non-expansive map from an injective envelope of $E$ into itself fixes $E$ pointwise it is the identity map.

We also note that two injective envelopes of $E$ are isomorphic via an isomorphism which is the identity over $E$. This allows to talk about ”the” injective envelope of $E$; we will denote it by $\mathcal{N}(E)$. A particular injective envelope of $E$ will be called a representation of $\mathcal{N}(E)$.

We include the few facts we need about injective envelopes of two-element metric spaces (see Proposition 2.7 of Lemma 2.3, Proposition 2.7 of [[11]] for proofs).

Let $V$ be a Heyting algebra and $v \in V$. Let $E := \{x, y\}$ be a two-element $\mathcal{V}$-metric space such that $d(x, y) = v$. We denote by $\mathcal{N}_v$ the injective envelope of $E$. We give two representations of it. Let $C_v$ be the set of all pairs $(u_1, u_2) \in V^2$ such that $v \leq u_1 \cdot \overline{u_2}$. Equip this set with the ordering induced by the product ordering on $V^2$ and denote by $\mathcal{N}_v$ the set of its minimal elements. Each element of $\mathcal{N}_v$ defines a minimal metric form. We equip $V^2$ with the supremum distance: $$d_{V^2}((u_1, u_2), (u'_1, u'_2)) := d_V(u_1, u'_1) \lor d_V(u_2, u'_2).$$

Let $v \in V$ and $\mathcal{S}_v := \{[v \cdot (\beta)^{-1}] : \beta \in V\}$ be the subset of $V$; equipped with the ordering induced by the ordering over $V$ this is a complete lattice. According to Lemma 2.5 of [[11]], $(x_1, x_2) \in \mathcal{N}_v$ iff $x_1 = [v \cdot (x_2)^{-1}]$ and $x_2 = [(x_1)^{-1} \cdot v]$. This yields a correspondence between $\mathcal{N}_v$ and $\mathcal{S}_v$.

Lemma 17. (Lemma 2.3, Proposition 2.7 of [[11]]) The space $\mathcal{N}_v$ equipped with the supremum distance and the set $\mathcal{S}_v$ equipped with the distance induced by the distance over $V$ are injective envelopes of the two-element metric spaces $\{(1, v), (v, 1)\}$ and $\{1, v\}$ respectively. These spaces are isometric to the injective envelope of $E := \{x, y\}$.

4.2. Composition of metric spaces. Let $(E_1, d_1)$ and $(E_2, d_2)$ be two disjoint $V$-metric spaces; let $x_1 \in E_1$ and $x_2 \in E_2$. If we endow the set $\{x_1, x_2\}$ with a $V$-distance $d'$, then we can define a $V$-distance $d$ on $E := E_1 \cup E_2$ as follows:

- If $x, y \in E_i$ with $i \in \{1, 2\}$ then $d(x, y) = d_i(x, y)$;
- If $x \in E_i$, $y \in E_j$ with $i, j \in \{1, 2\}$ and $i \neq j$, then $d(x, y) = d_i(x, x_i) \cdot d'(x_i, x_j) \cdot d_j(x_j, y)$. In particular, we can identify $x_1$ and $x_2$ which amounts to set $d'(x_1, x_2) = 1$ in the above formula.

If $E_1$ and $E_2$ are not disjoint, we replace it by two disjoint copies $E'_i$, $E''_i$ (e.g. $E'_i := E_i \times \{i\}$). Identifying the corresponding elements $x'_i, x''_i$, we obtain a $V$-metric space that we denote $(E_1, d_1) \cdot (E_2, d_2)$. Alternatively, we may suppose that $E_1$ and $E_2$ have only one element in common, say $z_{1,2}$, and define the distance $d$ on $E_1 \cup E_2$ by setting $d(x, y) := d_i(x, z_{1,2}) \cdot d_j(z_{1,2}, y)$ if $x \in E_i$, $y \in E_j$, $i \neq j$, and $d(x, y) := d_i(x, y)$ if $x, y \in E_i$.

Remark 9. If $E = E_1 \cdot E_2$ is injective then $E_1$ and $E_2$ are retract of $E$ and hence they are injective. The converse holds if $V = F(A^*)$ (see Theorem [[11]]).

We say that a metric space $E$ is irreducible if it has more than one element and $E = E_1 \cdot E_2$ implies $E = E_1$ or $E = E_2$. 
4.3. Transition systems as metric spaces. We refer to [19]. Let $A$ be a set. A transition system on the alphabet $A$ is a pair $M := (Q, T)$ where $T \subseteq Q \times A \times Q$. The elements of $Q$ are called states and those of $T$ transitions. Let $M := (Q, T)$ and $M' := (Q', T')$ be two transition systems on the alphabet $A$. A map $f : Q \rightarrow Q'$ is a morphism of transition systems if for every transition $(p, a, q) \in T$, we have $(f(p), f(a), f(q)) \in T'$. When $f$ is bijective and $f^{-1}$ is a morphism from $M'$ to $M$, we say that $f$ is an isomorphism. The collection of transition systems over $A$, equipped with these morphisms, form a category and this category has products. The graph of a transition system $M := (Q, T)$ is the directed graph with vertex set $Q$ and arcs $(x, y)$ such that $(x, a, y) \in T$ for some $a \in A$.

An automaton $A$ on the alphabet $A$ is given by a transition system $M := (Q, T)$ and two subsets $I, F$ of $Q$ called the sets of initial and final states. We denote the automaton as a triple $(M, I, F)$. A path in the automaton $A := (M, I, F)$ is a sequence $c := (e_i)_{i \equiv n}$ of consecutive transitions, that is of transitions $e_i := (q_i, a_i, q_{i+1})$. The word $\alpha := a_0 \ldots a_{n-1}$ is the label of the path, the state $q_0$ is its origin and the state $q_n$ its end. One agrees to define for each state $q$ in $Q$ a unique null path of length 0 with origin and end $q$. Its label is the empty word $\Box$. A path is successful if its origin is in $I$ and its end is in $F$. Finally, a word $\alpha$ on the alphabet $A$ is accepted by the automaton $A$ if it is the label of some successful path. The language accepted by the automaton $A$, denoted by $L_A$, is the set of all words accepted by $A$. Let $A := (M, I, F)$ and $A' := (M', I', F')$ be two automata. A morphism from $A$ to $A'$ is a map $f : Q \rightarrow Q'$ satisfying the two conditions:

1. $f$ is morphism from $M$ to $M'$;
2. $f(I) \subseteq I'$ and $f(F) \subseteq F'$.

If, moreover, $f$ is bijective, $f(I) = I'$, $f(F) = F'$ and $f^{-1}$ is also a morphism from $A'$ to $A$, we say that $f$ is an isomorphism and that the two automata $A$ and $A'$ are isomorphic.

According to Lemma [7], $F(A^*)$ is a Heyting algebra (we will sometimes denote $F \cdot F'$ the concatenation of $F$ and $F'$). Hence, we may consider metric spaces over $V := F(A^*)$. To a metric space $(E, d)$ over $V := F(A^*)$, we may associate the transition system $M := (E, T)$ having $E$ as set of states and $T := \{(x, a, y) : a \notin d(x, y) \cap A\}$ as set of transitions. Notice that such a transition system has the following properties: for all $x, y \in E$ and every $a, b \in A$ with $b \geq a$:

1) $(x, a, x) \in T$;
2) $(x, a, y) \in T$ implies $(y, a, x) \in T$;
3) $(x, a, y) \in T$ implies $(x, b, y) \in T$.

We say that a transition system satisfying these properties is reflexive and involutive (cf. [18], [11]). Clearly, if $M := (Q, T)$ is such a transition system, the map $d_M : Q \times Q \rightarrow F(A^*)$ where $d_M(x, y)$ is the language accepted by the automaton $(M, \{x\}, \{y\})$ is a distance. The graph of $M$ is reflexive and symmetric. We have the following:
Lemma 18. Let $(E,d)$ be a metric space over $F(A^*)$. The following properties are equivalent:

(i) The map $d$ is of the form $d_M$ for some reflexive and involutive transition system $M := (E,T)$;
(ii) For all $\alpha,\beta \in A^*$ and $x, y \in E$, if $\alpha\beta \in d(x,y)$, then there is some $z \in E$ such that $\alpha \in d(x,z)$ and $\beta \in d(z,y)$.

The category of reflexive and involutive transition systems with the morphisms defined above identify to a subcategory of the category having as objects the metric spaces and morphisms the non-expansive maps. Indeed:

Lemma 19. Let $M_i := (Q_i,T_i)$ $(i = 1,2)$ be two reflexive and involutive transition systems. A map $f : Q_1 \rightarrow Q_2$ is a morphism from $M_1$ to $M_2$ if and only if $f$ is a non-expansive map from $(Q_1,d_{M_1})$ to $(Q_2,d_{M_2})$.

Injective objects satisfy the convexity property stated in (ii) of Lemma 18. In particular, if $F$ is a final segment of $A^*$, the distance on the injective envelope $N_F$ comes from a transition system. Moreover, if $A$ is well-quasi-ordered then from Higman theorem [5], the final segment $F$ has a finite basis, that is, there are finitely many words $\alpha_0,...,\alpha_{n-1}$ such that $F = \uparrow \{\alpha_i : i < n\}$. In particular, we get:

Theorem 10. For every $F \in F(A^*)$ there is a transition system $M := (Q,T)$, an initial state $x$ and a final state $y$ such that the language accepted by the automaton $A = (M,\{x\},\{y\})$ is $F$. Moreover, if $A$ is well-quasi-ordered then we may choose $Q$ to be finite.

Let $M_i := (Q_i,T_i)$, resp. $G_i := (Q_i,E_i)$, $(i = 1,2)$, be two transition systems, resp. graphs. Let us suppose that they have exactly one state, resp. one vertex, in common, say $x$. We denote by $M_1 \cdot M_2$, resp. $G_1 \cdot G_2$, the transition system $M := (Q,T)$, resp. graph $G := (Q,E)$, such that $Q := Q_1 \sqcup Q_2$ and $T := T_1 \sqcup T_2$, resp. $E := E_1 \sqcup E_2$.

The following lemma is immediate:

Lemma 20. Let $M_i := (Q_i,T_i)$, $(i = 1,2)$ be two transition systems having $x$ as the only state in common. If $E_i$ and $G_i$ are the metric space and graph corresponding to $M_i$ ($i = 1,2$) then $E_1 \cdot E_2$ and $G_1 \cdot G_2$ are the metric space and graph corresponding to $M_1 \cdot M_2$.

We recall the following results of [11]:

Theorem 11. (Proposition 4.7 p. 175) Let $M_i := (Q_i,T_i)$, $(i = 1,2)$, be two transition systems having $x$ as the only state in common and $M := M_1 \cdot M_2$. If the space $E_i := (Q_i,d_{M_i})$ $(i = 1,2)$ is injective, then $E_1 \cdot E_2$ is injective.

Corollary 12. (Corollary 4.9. p. 177) Let $F_1$ and $F_2$ be two final segments of $A^*$. If $F_1F_2$ is non empty then $S_{F_1F_2}$ is isomorphic to $S_{F_1} \cdot S_{F_2}$.

The reader will realize that these two results express in terms of metric spaces the fact that $F(A^*)$ satisfies the decomposition property.
4.4. **Proof of Theorem** \[4\]. We refer to \[3\] for notions of graph theory, particularly to Chapter 5, for the notions of cut vertex and block decomposition. The graphs we consider are simple, with a loop at every vertex, and can be infinite. A *cut vertex* \( x \) of a graph \( G \) is any vertex whose deletion increases the number of connected components of \( G \) (hence if \( G \) has no edge, no vertex is a cut vertex); a *block* is a maximal connected induced subgraph with no cut vertex (since our graphs are reflexive, we prefer this definition to the usual one); any two blocks have at most one vertex in common; if \( G \) is connected with more than a vertex, the blocks of \( G \) induce a decomposition of the edge set of \( G \) and are the vertices of a tree (cf. Proposition 5.3, p. 120 of \[3\]).

Let \( F \) be a final segment of \( A^* \), let \( E = (\{x, y\}, d) \) be a 2-element metric space such that \( d(x, y) = F \) and \( N_F \) be its injective envelope. With no loss of generality, we may suppose that \( x = A^* \), \( y = F \) and \( N_F = S_F \). Let \( M_F \) be the transition system associated with \( S_F \), let \( Q \) be its domain and \( G_F \) be the graph of this transition system.

We suppose that \( F \neq A^* \), hence \( x \neq y \). We prove first that \( F \) is irreducible if and only if \( S_F \) is irreducible. If \( F \) is not irreducible then there are two final segments \( F_1 \) and \( F_2 \) distinct from \( F \) such that \( F = F_1 F_2 \). Necessarily, \( F, F_1 \) and \( F_2 \) are non-empty. According to Corollary \[12\], \( S_F \) is isomorphic to \( S_{F_1} \cdot S_{F_2} \), hence \( S_F \) is not irreducible. Conversely, suppose that \( S_F \) is not irreducible. Let \( E_1 \) and \( E_2 \) such that \( S_F = E_1 \cdot E_2 \) and \( z \) in their intersection. First, \( x \) and \( y \) do not belong to the same \( E_i \), otherwise we may retract \( S_F \) onto \( E_i \) by a non-expansive map sending \( E_j \) (\( j \neq i \)) onto \( z \), contradicting Lemma \[16\]. Suppose \( x \in E_1 \) and \( y \in E_2 \). From Lemma \[20\] we have \( F = d_{S_F}(x, y) = d_{E_1}(x, z) \cdot d_{E_2}(y, z) \) hence \( F \) is not irreducible.

We prove now that \( S_F \) is irreducible if and only if \( G_F \) has no cut vertex. If \( S_F \) is not irreducible then \( S_F = E_1 \cdot E_2 \) for two proper subspaces of \( S_F \). Let \( M_i \) be the restriction of \( M_F \) to \( E_i \) (\( i = 1, 2 \)). We claim that \( M = M_1 \cdot M_2 \). Since \( S_F \) is injective, the distance \( d_{S_F} \) is equal to the distance \( d_{M_F} \) (Lemma \[18\]). By Remark \[9\] \( E_1 \) and \( E_2 \) are injective, hence the distance induced on \( E_i \) coincides with the the distance \( d_{M_i} \). Let \( z \) with \( \{z\} = E_1 \cap E_2 \), \( x' \in E_1 \setminus \{z\} \) and \( y' \in E_2 \setminus \{z\} \). Since \( S_F = E_1 \cdot E_2 \), \( d_{S_F}(x', y') = d_{S_F}(x', z) \cdot d_{S_F}(z, y') \). Hence, there is no transition, thus no edge, linking \( x' \) and \( y' \). This proves our claim. In particular, \( z \) is a cut vertex of \( G_F \).

Suppose that \( G_F \) has a cut vertex \( z \). Then \( F = d(x, y) \neq \emptyset \) otherwise \( S_F = E \) and \( G_F \) has no cut vertex. We claim that since \( F \neq \emptyset \), \( x \) and \( y \) are in the same connected component. Furthermore \( G_F \) is connected, neither \( x \) nor \( y \) is a cut vertex and every cut vertex \( z \) separates \( G_F \) into two connected components, one containing \( x \), the other \( y \). The proof of this claim uses repeatedly Lemma \[16\]. If one of these assertions is false, we can define a proper non-expansive retraction of \( S_F \) which fixes \( x \) and \( y \). According to Lemma \[16\] it fixes \( S_F \), contradicting the fact that it is proper. To illustrate, suppose that \( z \) in a cut vertex of \( G_F \) distinct from \( x \) and \( y \). Let \( D \) be the union of connected components containing \( x \) and \( y \). If there are other connected components we
can retract these components on \( z \). Since \( \mathcal{M}_F \) is reflexive this retraction is a retraction of \( \mathcal{M}_F \) onto its restriction to \( D \cup \{ z \} \). It induces a non-expansive map from \( S_F \) onto itself which fixes \( x \) and \( y \). According to Lemma 16 it fixes \( Q \), hence \( Q = D \cup \{ z \} \) contradicting the existence of other connected components. Since \( z \) is a cut vertex \( D \) consists of two connected components \( D_x \) and \( D_y \). The sets \( D_x \cup \{ z \} \) and \( D_y \cup \{ z \} \) form a covering of \( Q \) into two connected subsets with no crossing edge, hence \( \mathcal{M}_F = \mathcal{M}_{1D_x} \cdot \mathcal{M}_{1D_y} \). According to Fact 20 \( S_F = S_{1D_x} \cdot S_{1D_y} \) hence \( S_F \) is not irreducible.

Suppose that \( F \) is not irreducible. In this case \( F \) is non-empty. Hence \( \mathcal{G}_F \) is connected. Since \( \mathcal{G}_F \) has a cut vertex, it has at least two blocks. The collection of blocks forms a tree. Let \( C \) be the shortest path joining the block containing \( x \) to the block containing \( y \) and let \( \tilde{C} \) be the graph induced on the union of blocks belonging to \( C \). Since \( \mathcal{G}_F \) is a tree with a loop at every vertex, we may retract \( \mathcal{G}_F \) on \( \tilde{C} \) by a map fixing pointwise the vertices in \( \tilde{C} \) (send each vertex \( z \in \mathcal{G}_F \) on the closest vertex belonging to \( \tilde{C} \)). Since \( \mathcal{M}_F \) is reflexive, this retraction is a retraction from \( \mathcal{M}_F \) onto the transition system induced on \( \tilde{C} \) and thus a retraction of the injective envelope \( S_F \) onto the space induced on \( \tilde{C} \). Since this retraction fixes \( x \) and \( y \), it fixes \( Q \) (Lemma 16), hence \( Q = \tilde{C} \). We can enumerate the vertices of \( C \) in a sequence \( C_0, \ldots, C_{n-1} \) with \( x \in C_0 \) and \( y \in C_{n-1} \), with \( n \geq 2 \). Let \( F_i \) be the language accepted by the automaton \( ( \mathcal{M}_F \upharpoonright C_i, \{ x_i \}, \{ y_i \} ) \), where \( x_i := x \) if \( i = 0 \), \( y_i = y \) if \( i = n-1 \) and \( \{ x_i \} = C_{i-1} \cap C_i \), \( \{ y_i \} = C_i \cap C_{i+1} \), otherwise. Clearly, \( F \) is the product \( F_0 \cdots F_{n-1} \). Also, \( S_F \upharpoonright C_i \) is the metric space associated with the injective envelope of \( ( \{ x_i, y_i \}, d_i ) \) where \( d_i(x_i, y_i) = F_i \). With this the proof is complete.

\[ \square \]

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