Testing the global flow reconstruction method on coupled chaotic oscillators

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Abstract.
Irregular behaviour of pulsating variable stars may occur due to low dimensional chaos. To determine the quantitative properties of the dynamics in such systems, we apply a suitable time series analysis, the global flow reconstruction method. The robustness of the reconstruction can be tested through the resultant quantities, like Lyapunov dimension and Fourier frequencies. The latter is specially important as it is directly derivable from the observed light curves. We have performed tests using coupled Rössler oscillators to investigate the possible connection between those quantities. In this paper we present our test results.

1. Introduction
If the timescale of the modulation in semiregular data is comparable to the period, that may indicate chaotic dynamics in the background. Radially pulsating semiregular variable stars often show such modulation in their light curves due to nonlinear nonadiabatic properties of their oscillation. Using the global flow reconstruction method, evidence has been found for low dimensional chaos in several cases. These investigations suggest that irregular pulsation occurs as the result of energy exchange between coupled modes (Buchler et al. 1996, Kolláth et al. 1998, Buchler et al. 2004). To model the interaction of the modes we have generated artificial data of coupled Rössler oscillators. It is well known that Rössler oscillators have chaotic solution for certain parameter combinations. This applies for two coupled Rössler oscillators as well. So we have based our tests on a priori known chaotic time series with two interacting oscillation modes. As the chaotic semiregular stars have two groups of peaks in their Fourier spectra, our examinations concern the one dimensional output of the coupled Rössler oscillators that clearly show the presence of the two frequency groups.

2. Coupled Rössler Oscillators
We have generated our artificial data with the following set of differential equations, where the first three equations describe one oscillator and the last three equations describe the other one.

\[
\begin{align*}
\dot{x}_1 &= -a(x_2 + x_3), & \dot{x}_2 &= a(x_1 + bx_2) + ex_5, & \dot{x}_3 &= a(c + x_3(x_1 - d)) \\
\dot{x}_4 &= -f(x_5 + x_6), & \dot{x}_5 &= f(x_4 + gx_5) + jx_2, & \dot{x}_6 &= f(h + x_6(x_4 - i))
\end{align*}
\]
Ten parameters are involved, \(a\) and \(f\) refer to the oscillation frequencies, while \(e\) and \(j\) assure the coupling through \(x_5\) and \(x_2\) variables. By varying the parameters we have constructed different data sets. In our tests we have used the 1D time series of \(x_1\) variable. Gaussian noise was also added to the test signals to generate data similar to real observations.

3. The Global Flow Reconstruction Method

Our nonlinear analysis starts with the construction of a strobed representation of the trajectory of the system in the reconstruction space. Using the sequence with equal time spacing \(s(t_n)\) we produce the so called 'delay vectors' \(X(t_n) = s(t_n), s(t_n - \Delta), s(t_n - 2\Delta),..., s(t_n - (d_e - 1)\Delta)\), where \(\Delta\) is the 'time delay' and \(d_e\) is the embedding dimension of the reconstruction space. We assume that there exists a map \(F\) that evolves the trajectory in time by connecting the neighboring points, \(X(t_{n+1}) = F(X(t_n))\). We fit the map \(F\) in polynomial form. By iterating the constructed \(F\) map we can produce synthetic signals that can be compared to the original data. For comparison different representations are useful, the Fourier spectrum, Broomhead-King (BK) projection (Broomhead & King, 1987), and the data set itself. Synthetic and original data must be in accordance in all representations to be acceptable as a good robust map. By varying the 'time delay' \(\Delta\), the embedding dimension \(d_e\), the smoothing parameter \(\sigma\) and the added noise intensity \(\xi\) we can identify a good robust map region in the parameter space. Good reconstructions provide approximated quantitative information of the system, like Lyapunov exponents and dimension. For detailed description of the method we refer to Serre et al. (1996).

4. Discussion

We have applied the global flow reconstruction method on the test data. We have used 5 and 6 dimensional reconstruction space \((d_e = 5, 6)\). The order of the polynomial fit has been fixed at 4 \((p = 4)\). Smoothing parameter \((\sigma = 0.000, 0.001, 0.002)\), noise intensity \((\xi = 0.000, 0.001, 0.002)\) and 'time delay' \((\Delta = 4 - 25)\) characterize our examined parameter regime. The iterated synthetic data of the successful reconstruction maps have been compared to the original test data.

Considering we deal with chaotic data, we can not expect exact accordance comparison method, only strong similarity. We present a subsample of tests on Figure 1. We compared the test dataset, a different realisation of the test dataset and two reconstructed synthetic signals. The difference of Test D lies in the initial value of the iteration. Syn 1 is the synthetic signal of the best reconstruction and Syn 2 is a typical synthetic time series of the examined parameter space. Comparison of a fraction of datasets and the BK projections has revealed that reconstructed signals do not differ from the test data more than a different realisation of the test data itself. Figure 2 displays the Fourier spectra of the same datasets, that show two groups of frequencies, a larger group around 0.0035, and a smaller group at 0.0128. The frequency spectra of synthetic signals are in good correspondence to the test data as well, however the signals of typical reconstructions show different periodicities causing noticeable shifts of frequency peaks.

Our further investigations have focused on the frequency peaks. As chaotic oscillation shows temporal modulation, the oscillation frequencies do not appear as single strong peaks in the Fourier spectrum, but as a forest of peaks in a certain frequency range. Different realisations of the same system display different behaviour in their frequency spectra, like different location of the highest amplitude peaks. Therefore spectra can be compared only in statistical sense.
Figure 1. Comparison of Rössler test data and synthetic solution by different representations.

Figure 2. Comparison of Rössler test data and synthetic reconstructions by Fourier spectra.

We have applied Gauss curve fitting on the frequency spectra to perform a reliable comparison by determining the centre position of the Gaussian envelope of the test dataset and the synthetic signals. With this manner we have obtained $0.003469 \pm 0.000013$ and $0.012875 \pm 0.000008$ frequency values for the original coupled Rössler oscillator. The frequency values of synthetic
signals are $0.00361 \pm 0.00022$ and $0.01289 \pm 0.00016$ in 5D, and $0.00359 \pm 0.00015$ and $0.01288 \pm 0.00006$ in 6D reconstruction. The central frequency of Gaussian fit to the Fourier spectra is a well defined and useful quantity. Our analysis revealed that the original and synthetic values of the central frequencies differ significantly.

We have tested the connection between the frequencies and the Lyapunov dimensions to investigate the correlation between the errors of the quantities (Figure 3). The Lyapunov dimension value of the test Rössler data has been calculated to be $3.53 \pm 0.07$, while synthetic signals have $3.03 \pm 0.30$ (5D) and $3.11 \pm 0.31$ (6D) values. Figure 3 clearly shows the large scatter in Lyapunov dimensions and in both frequency groups. The majority of the Lyapunov dimension values are below the original ones.

![Figure 3. Lyapunov dimension versus centre position frequency of the Gaussian envelope. 5D: circle, 6D: square, test dataset: triangle.](image)

5. Conclusion
We have succeeded in reconstructing artificial datasets with Lyapunov dimension higher than 3, that was composed by coupling two oscillations. Our tests suggest however, that the global flow reconstruction method do not reconstruct the exact frequencies and the Lyapunov dimension of the original data. The errors of these quantities do not correlate with each other. The lack of correlation between the examined measures confirms that both quantities must be used to determine the robustness of the reconstruction, or to strain off the false reconstructed signals. We can not use the average values of the measures either as they show systematic deviation beside their large scatter. This results should be taken into consideration when real world observations are analysed and interpreted. Possible direction of further investigations can be the development and test of a reconstruction method which guarantees the results with smaller frequency discrepancy.

References
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