QCD at finite isospin density: from pion to quark-antiquark condensation

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Abstract

QCD at finite isospin chemical potential $\mu_I$ is studied. This theory has no fermion sign problem and can be simulated on the lattice using present-day techniques. We solve this theory analytically in two limits: low $\mu_I$ where chiral perturbation theory is applicable, and asymptotically high $\mu_I$ where perturbative QCD is at work. At low isospin density the ground state is a superfluid pion condensate. At very high density it is a Fermi liquid with Cooper pairing. The pairs carry the same quantum numbers as the pions. Motivated by this observation, we put forward a conjecture that the transition from hadron to quark matter is smooth. The conjecture passes several nontrivial tests. Our results imply a nontrivial phase diagram in the space of temperature and chemical potentials of isospin and baryon number. At asymptotically large values of $\mu_I$ and small values of baryon chemical potential the ground state is in a phase similar to Fulde-Ferrell-Larkin-Ovchinnikov phase. It is characterized by a spatially modulated superfluid order parameter $\langle \bar{u} \gamma_5 d \rangle$ and may be the asymptotic limit of the inhomogeneous pion condensation phase advocated by Migdal and others.

1 Introduction

Good knowledge of QCD in the regime of finite temperature and baryon density is crucial for understanding a wide range of physical phenomena. In cosmology, one faces the problem of understanding how the Universe has evolved through the QCD phase transition at temperature $T \sim 150$ MeV. Due to the smallness of the baryon asymmetry, finite-temperature QCD should be sufficient to deal with this problem. However, for the physics of heavy-ion collisions, one needs to know how QCD behaves when both temperature and baryon chemical potential are finite. Lastly, neutron stars require the knowledge of matter in the “dense” regime, i.e. at large baryon densities and very low temperatures. Much less is known about the last two regimes compared to that of high-temperature baryon-antibaryon-symmetric QCD.
Let us use neutron stars as an example to illustrate the range of questions one would like to have answers to. The equation of state (EOS) of nuclear matter at high densities determines the mass-radius relationship and the maximum mass of neutron stars. Walecka model of nuclear matter predicts that the EOS becomes stiffer at higher densities and approaches the Zel’dovich limit, \( \epsilon = p \) (where the velocity of sound approaches light speed) at very high densities. However, at asymptotically high densities one expects nuclear matter to become a weakly-interacting quark liquid, with a much softer EOS \( \epsilon = 3p \). At what density does the transition happen, and is it a phase transition or a crossover?

Migdal [1] and others [2] suggested that at very high densities pion condensation might happen. It is also argued that at even higher densities kaons are condensed [3]. One would like to know whether pion and kaon condensations do indeed occur in nuclear matter, before the transition to quark matter has happened.

Finally, there is a strong recent interest in the phenomenon of color superconductivity [4, 5]. One very interesting prediction is that at high enough chemical potential, the ground state of QCD is the “color-flavor-locking” state [6], which breaks chiral symmetry. However, while reliable results can be obtained at asymptotically high densities, where the strong coupling is small [7, 8, 9], it is not known how to extend these results to the region of smaller, more realistic, densities without having to rely on uncontrollable approximations.

Lacking reliable analytical means to approach QCD in the strong coupling regime, one naturally turns to numerical methods. First principle lattice numerical Monte Carlo calculations provide a solid basis for our knowledge of the finite temperature regime. However, the regime of finite baryon chemical potential \( \mu_B \) is still unaccessible by Monte Carlo, because present methods of evaluating QCD partition function require taking a path integral with a measure which includes a complex fermion determinant. At zero chemical potential one can simply ignore the determinant (as one does in the popular quenched approximation) and still find reasonable results for physical quantities. However, at finite \( \mu_B \) this procedure leads to qualitatively unacceptable answers as was realized long time ago [10]. It has been understood more recently that the quenched approximation breaks down at finite \( \mu_B \) because it describes an unphysical theory containing, beside the normal quarks, the so-called conjugate quarks with opposite baryon charge [11].

As a side remark, one notes that though the conjugate quarks are absent in real QCD, there are many theories where they are naturally present. One class of such theories is QCD with two colors, where quarks are self-conjugate [12]. Another class contains theories with quarks in the adjoint color representation [13]. In all these theories, the positivity of the fermion determinant ensures the applicability of lattice Monte Carlo methods. However, the particle content of all these theories is very different from the real world.

The failure of the quenched approximation in real QCD at finite \( \mu_B \) and our inability to include complex fermion determinant in a Monte Carlo simulation is one of the main reasons for our understanding of QCD at finite baryon density to be still rudimentary.

One way QCD at finite baryon density is different from finite-temperature QCD is that the transition from hadronic to quark degrees of freedom occurs due to the large density of a conserved charge (such as baryon number) while temperature plays no role. This is the motivation for us to turn to QCD at finite chemical potential \( \mu_I \) of isospin (more precisely, of the third component of isospin, \( I_3 \)), which is conserved by strong interaction.
Before going into details, we would like to comment on the relevance of this regime to the real world. Nature does provide us with non-zero $\mu_I$ systems in the form of isospin-asymmetric matter (e.g., inside neutron stars), however, the latter contains both isospin density and baryon number density. In contrast, the idealized system considered in this paper does not carry baryon number: the chemical potentials of the two light quarks, $u$ and $d$, are equal in magnitude, $|\mu_I|/2$, and opposite in sign. Such a system, strictly speaking, is unstable with respect to weak decays which do not conserve isospin, and, as we shall see, is also not electrically neutral and thus does not exist in the thermodynamic limit. However, since we are interested in the dynamics of strong interaction alone, one can imagine that all relatively unimportant electromagnetic and weak effects are turned off. Once this is done, we have a nontrivial regime which, as we shall see, is accessible by present lattice Monte Carlo methods, while being analytically tractable in various interesting limits. As a result, the system we consider has a potential to improve substantially our understanding of cold dense QCD. This regime carries many attractive traits of two-color QCD \cite{12,13}, but is realized in a physically relevant theory — QCD with three colors.

2 Positivity and QCD inequalities

Since in Euclidean space the fermion determinant of our theory is real and positive, some rigorous results on the low-energy behavior can be obtained from QCD inequalities \cite{14,13}. Let us recall how the inequalities are derived in vacuum QCD. The starting point is the following property of the Euclidean Dirac operator $\mathcal{D} = \gamma \cdot (\partial + iA) + m$:

$$\gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger. \quad (1)$$

which, in particular, implies positivity of the determinant, $\det \mathcal{D} \geq 0$. For the correlator of a generic meson $M = \bar{\psi}\Gamma\psi$, we can write, using (1) and Bunyakovsky-Schwartz inequality:

$$\langle M(x)M^\dagger(0) \rangle_{\psi,A} = -\langle \text{Tr} S(x,0)\Gamma S(0,x)\Gamma \rangle_A$$

$$= \langle \text{Tr} S(x,0)\Gamma i\gamma_5 S^\dagger(x,0)i\gamma_5 \Gamma \rangle_A \leq \langle \text{Tr} S(x,0) S^\dagger(x,0) \rangle_A, \quad (2)$$

where $S \equiv \mathcal{D}^{-1}$ and $\Gamma \equiv \gamma_0 \Gamma_0^{\dagger} \gamma_0$. The inequality is saturated for mesons with $\Gamma = i\gamma_5 \tau_i$, since $\mathcal{D}$ commutes with isospin $\tau_i$, which means that the pseudoscalar correlators majorate all other $I = 1$ meson correlators. As a consequence, one obtains an important restriction on the pattern of the spontaneous symmetry breaking. For example, the symmetry breaking cannot be driven by a condensate of $\langle \bar{\psi} \gamma_5 \psi \rangle$. Indeed, broken axial SU(2) symmetry generators acting on such a pseudoscalar condensate would have produced $0^+$ Goldstone bosons $\bar{\psi} \tau_i \psi$.

At finite isospin density, $\mu_I \neq 0$, positivity still holds \cite{14} and certain inequalities can be derived. However, in contrast with the case of $\mu_B \neq 0$ when there is no positivity and hence no inequality can be derived. Now $\mathcal{D} = \gamma \cdot (\partial + iA) + \frac{1}{2} \mu_I \gamma_0 \tau_3 + m$, and Eq. (1) is not true anymore, since the operation on the r.h.s. of (1) changes the relative sign of $\mu_I$. However,\footnote{It is important, as is the case for $I = 1$, that there is no disconnected piece after $\psi$ integration in (3). The proof does not apply to $\sigma$-meson correlator, $\Gamma = 1$.}
provided \( m_u = m_d \), interchanging up and down quarks compensates for this sign change (the \( u \) and \( d \) quarks play the role of mutually conjugate quarks \[1\]), i.e,

\[
\tau_1 \gamma_5 D \gamma_5 \tau_1 = D^\dagger. \tag{3}
\]

Instead of isospin \( \tau_1 \) in (3) one can also use \( \tau_2 \) (but not \( \tau_3 \)). Eq. (3) replaces the now invalid Eq. (1) and ensures that \( \det D \geq 0 \). Repeating the derivation of the QCD inequalities using (3) we find that the lightest meson, or the condensate, must be in channels \( \bar{\psi} i \gamma_5 \tau_1,2 \psi \), i.e., a linear combination of \( \pi^- \sim \bar{u} \gamma_5 d \) and \( \pi^+ \sim \bar{d} \gamma_5 u \) states. Indeed, as shown below, in both two analytically tractable regimes of small and large \( \mu_I \) the lightest mode is a massless Goldstone mode which is a linear combination of \( \bar{u} \gamma_5 d \) and \( \bar{d} \gamma_5 u \).

3 Small isospin densities: pion condensate

When \( \mu_I \) is small, chiral perturbation theory can be used to treat the problem. To have a rough sense of how small \( \mu_I \) should be, we require that no particles other than pions are excited due to the chemical potential. This gives \( \mu_I \simeq m_\rho \) as the upper limit of the applicability of the chiral perturbation theory.

For zero quark mass and zero \( \mu_I \), the pions are the massless Goldstone bosons of the spontaneously broken \( \text{SU}(2)_L \times \text{SU}(2)_R \) chiral symmetry. In reality, the quarks have small masses, which break this symmetry explicitly. Assuming equal quark masses, the symmetry of the Lagrangian is \( \text{SU}(2)^4 \). The low-energy dynamics is governed by the familiar chiral Lagrangian, which is written in terms of the matrix pion field \( \Sigma \in \text{SU}(2) \):

\[
\mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr}[\partial_\mu \Sigma \partial_\mu \Sigma^\dagger - 2 m_\pi^2 \text{Re} \Sigma].
\]

This Lagrangian contains only two phenomenological parameters: the pion decay constant, \( f_\pi \), and the pion mass in vacuum, \( m_\pi \). We will see that interesting physics occurs at \( \mu_I > m_\pi \), and since \( m_\pi \ll m_\rho \), there is a nontrivial range of \( \mu_I \) where the chiral Lagrangian is a reliable and useful treatment.

The isospin chemical potential further breaks \( \text{SU}(2)^4 \) down to \( \text{U}(1)^4 \). Its effect can be included into the effective Lagrangian to leading order in \( \mu_I \), without introducing additional phenomenological parameters. Indeed, \( \mu_I \) enters the QCD Lagrangian in the same way as the the zeroth component of a gauge potential \[13\]. Thus the finite-\( \mu_I \) chiral Lagrangian is obtained by promoting the global \( \text{SU}(2)_L \times \text{SU}(2)_R \) symmetry to a local gauge symmetry: gauge invariance completely fixes the way \( \mu_I \) enters the chiral Lagrangian \[13\]:

\[
\mathcal{L}_{\text{eff}} = \frac{f_\pi^2}{4} \text{Tr}\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger - \frac{m_\pi^2 f_\pi^2}{2} \text{Re} \text{Tr} \Sigma. \tag{4}
\]

The covariant derivative is defined as

\[
\nabla_0 \Sigma = \partial_0 \Sigma - \frac{\mu_I}{2} (\tau_3 \Sigma - \Sigma \tau_3), \quad \nabla_i \Sigma = \partial_i \Sigma. \tag{5}
\]

which follows from the transformation property of \( \Sigma \) under rotations by the isospin generator \( I_3 = \tau_3/2 \).
Using (4) it is straightforward to determine vacuum alignment of \( \Sigma \) as a function of \( \mu_I \) and the spectrum of excitations around the vacuum. We will be interested in negative \( \mu_I \), which favors neutrons over protons, as in neutron stars. The results are very similar to the two-color QCD at finite baryon density [13]. From (4), one finds the potential energy for \( \Sigma \),

\[
V_{\text{eff}}(\Sigma) = \frac{f_\pi^2 \mu_I^2}{8} \text{Tr}(\tau_3 \Sigma \tau_3 \Sigma^\dagger) - \frac{f_\pi^2 m_\pi^2}{2} \text{Re} \text{Tr} \Sigma .
\]  

(6)

The first term in (6) favors directions of \( \Sigma \) which anticommute with \( \tau_3 \), i.e., \( \tau_1 \) and \( \tau_2 \), while the second term prefers the vacuum direction \( \Sigma = 1 \). It turns out that the minima of (6) at all \( \mu_I \) are captured by the following Ansatz:

\[
\Sigma = \cos \alpha + i(\tau_1 \cos \phi + \tau_2 \sin \phi) \sin \alpha .
\]  

(7)

Substituting (7) to (6), one sees that the potential energy depends only on \( \alpha \), but not \( \phi \):

\[
V_{\text{eff}}(\alpha) = \frac{f_\pi^2 \mu_I^2}{4} (\cos 2\alpha - 1) - f_\pi^2 m_\pi^2 \cos \alpha .
\]  

(8)

Minimizing \( V_{\text{eff}}(\alpha) \) with respect to \( \alpha \), one sees that the behavior of the system is different in two distinct regimes:

(i) For \(|\mu_I| < m_\pi\), the system is in the same ground state as at \( \mu_I = 0 \): \( \alpha = 0 \), or \( \Sigma = 1 \). This result is easy to understand. The lowest lying pion state costs a positive energy \( m_\pi - |\mu_I| \) to excite, thus at zero temperature no pion is excited. The ground state of the Hamiltonian at such \( \mu_I \) coincides with the normal vacuum of QCD. The isospin density is zero in this case.

(ii) When \(|\mu_I| \) exceeds \( m_\pi \) the minimum of (8) occurs at

\[
\cos \alpha = \frac{m_\pi^2}{\mu_I^2} .
\]  

(9)

In this regime the energy to excite a \( \pi^- \) quantum, \( m_\pi - |\mu_I| \), is negative, thus it is energetically favorable to excite a large number of these quanta. Since pions are bosons, the result is a Bose condensate of \( \pi^- \). If the pions did not interact, the density of the condensate would be infinite. However, the repulsion between pions stabilizes the system at a finite value of the isospin density. This value can be found by differentiating the ground state energy with respect to \( \mu_I \):

\[
n_I = -\frac{\partial L_{\text{eff}}}{\partial \mu_I} = f_\pi^2 \mu_I \sin^2 \alpha = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4}\right) .
\]  

(10)

For \(|\mu_I| \) just above the condensation threshold, \(|\mu_I| - m_\pi \ll m_\pi \), Eq. (10) reproduces the equation of state of the dilute non-relativistic pion gas [13],

\[
n_I = 4 f_\pi^2 (\mu_I - m_\pi)
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\]

At larger \( \mu_I \), \(|\mu_I| \gg m_\pi \), the isospin density is linear in \( \mu_I \),

\[
n_I = f_\pi^2 \mu_I , \quad |\mu_I| \gg m_\pi
\]
From Eq. (10) one can find the pressure and the energy density as functions of $\mu_I$. The interesting quantity is the ratio between the two,

$$\frac{p}{\epsilon} = \frac{\mu_I^2 - m_\pi^2}{\mu_I^2 + 3m_\pi^2}. \quad (11)$$

This ratio starts from 0 at threshold and quickly approaches 1 as one increases $\mu_I$. Thus, as far as the chiral Lagrangian is still applicable, the EOS approaches the Zel’dovich limit of maximal stiffness at high densities, similar to nuclear matter in Walecka model.

The fact that the minimum of the potential (6) is degenerate with respect to the angle $\phi$ corresponds to the spontaneous breaking of the $U(1)_{L+R}$ symmetry generated by $I_3$ in the Lagrangian (4). This is not unexpected since the ground state is, in essence, a pion superfluid, with one massless Goldstone mode. Since we start from a theory with three pions in vacuum, in the superfluid there are, in addition to the massless mode, two massive modes. One can be identified with $\pi^0$. The other is a linear combination of $\pi^+$ and $\pi^-$, which we denote as $\tilde{\pi}^+$, since it coincides with $\pi^+$ at the condensation threshold. The mass (defined as the rest energy) of these modes can be obtained by expanding the Lagrangian (4) around the minimum. The result reads (cf. [13])

$$m_{\pi^0} = |\mu_I|, \quad m_{\tilde{\pi}^+} = |\mu_I|\sqrt{1 + 3(m_\pi/\mu_I)^4}. \quad (12)$$

At the condensation threshold, $m_{\pi^0} = m_\pi$ and $m_{\tilde{\pi}^+} = 2m_\pi$, while for $|\mu_I| \gg m_\pi$ both masses approach $|\mu_I|$ (see Fig.4).

The values of the chiral condensate, $\langle \bar{u}u + \bar{d}d \rangle$, and the pion condensate, $\langle \bar{u}\gamma_5 d \rangle$ follow from (9):

$$\langle \bar{u}u + \bar{d}d \rangle = 2\langle \bar{\psi}\psi \rangle_{\text{vac}} \cos \alpha \quad \text{and} \quad \langle \bar{u}\gamma_5 d \rangle + \text{h.c.} = 2\langle \bar{\psi}\psi \rangle_{\text{vac}} \sin \alpha, \quad (13)$$

i.e., the chiral condensate “rotates” into the pion condensate as a function of $|\mu_I|$.

It is also possible to find baryon masses, i.e. the energy cost of introducing a single baryon into the system. The most interesting baryons are those with lowest energy and highest isospin, i.e. neutron $n$ and $\Delta^-$ isobar. There are two effects of $\mu_I$ on the baryon masses. The first comes from the isospin of the baryons, which effectively reduces the neutron mass by $\frac{1}{2}|\mu_I|$, and the $\Delta^-$ mass by $\frac{3}{2}|\mu_I|$. If this was the only effect, the effective $\Delta^-$ mass would vanish at $|\mu_I| = \frac{2}{3}m_\Delta$. For larger $\mu_I$, baryon or antibaryon Fermi surfaces would form, which lead to a nonzero baryon susceptibility $\chi_B \equiv \partial n_B/\partial \mu_B$. However, long before that another effect turns on: the $\pi^-$’s in the condensate tend to repel the baryons, lifting up their masses.

These effects can be treated in the framework of the baryon chiral perturbation theory [16]. For example, the (Euclidean) Lagrangian describing nucleons and their interactions with the pions at finite $\mu_I$ can be written as:

$$\mathcal{L}_N = \bar{N}\gamma_\mu \nabla_\mu N + m_N \left(\bar{N}\Sigma N_R + \text{h.c.}\right), \quad (14)$$

where

$$\nabla_0 N = \left(\partial_0 - \frac{\mu_I}{2}\tau_3\right)N, \quad \nabla_i N = \partial_i N.$$
Figure 1: Schematic plot of masses (rest energies) of lowest lying excitations in QCD at finite (negative) $\mu_I$, in the regime of applicability of chiral perturbation theory: $m_\pi, \mu_I \ll m_\rho$. 
Diagonalizing this bilinear Lagrangian in the pion background given by $\Sigma = \bar{\Sigma}$ from (7) one finds the nucleon masses. The result for the neutron and the $\Delta^-$ isobar reads:

$$m_n = m_N - \frac{|\mu_I|}{2} \cos \alpha, \quad m_{\Delta^-} = m_\Delta - \frac{3|\mu_I|}{2} \cos \alpha,$$

in the approximation of nonrelativistic baryons. Equation (15) can be interpreted as follows: as a result of the rotation (7) of the chiral condensate, the nucleon mass eigenstate becomes a superposition of vacuum $n$ and $p$ states. The expectation value of the isospin in this state is proportional to $\cos \alpha$ appearing in (15). With $\cos \alpha$ given in Eq. (9), we see that the two mentioned effects cancel each other when $m_\pi \ll |\mu_I| \ll m_\rho$. Thus the baryon mass never drops to zero, and $\chi_B = 0$ at zero temperature in the region of applicability of the chiral Lagrangian.

As one forces more pions into the condensate, the pions are packed closer and their interaction becomes stronger. When $\mu_I \sim m_\rho$, the chiral perturbation theory breaks down. To find the equation of state in this regime, full QCD has to be employed. As we have seen, this can be done using present lattice techniques since the fermion sign problem is not present at finite $\mu_I$, similar to the two-color QCD [12].

4 Asymptotically high isospin densities: quark-antiquark condensate

In the opposite limit of very large isospin densities, or $|\mu_I| \gg m_\rho$, the description in terms of quark degrees of freedom applies since the latter are weakly interacting due to asymptotic freedom. In our case of large negative $\mu_I$, or $n_I$, the ground state contains an equal number of $d$ quarks and $\bar{u}$ antiquarks per unit volume. If one neglects the interaction, the quarks fill two Fermi spheres with equal radii $|\mu_I|/2$. Turning on the interaction between the fermions leads to the instability with respect to the formation and condensation of Cooper pairs, similar to BCS instability in metals or the diquark pairing at high baryon density [4]. To the leading order of perturbation theory, quarks interact via the one-gluon exchange. It is easy to see that the attraction is strongest in the color singlet channel, thus the Cooper pair consists of a $\bar{u}$ and a $d$. The ground state, hence, is a fermionic superfluid.

The perturbative one-gluon exchange, however, does not discriminate between the scalar, $\bar{u}d$, channel, and the pseudoscalar, $\bar{u}\gamma_5d$, channel: the attraction is the same in both cases. But one can expect that the instanton-induced interaction, however small, will favor the $\bar{u}\gamma_5d$ channel over the $\bar{u}d$ one. The condensate thus is pseudoscalar and breaks parity,

$$\langle \bar{u}\gamma_5d \rangle \neq 0.$$

This is consistent with our earlier observation that QCD inequalities constrain the $I = 1$ condensate to be a pseudoscalar at any $\mu_I$. Note that the order parameter in (15) has the same quantum numbers as the pion condensate at lower densities. We shall discuss this coincidence later.

As a consequence of the Cooper pairing, the fermion spectrum acquires a gap $\Delta$ at the Fermi surface, where

$$\Delta = b|\mu_I|g^{-5}e^{-c/g}, \quad c = 3\pi^2/2,$$
where \( g \) should be evaluated at the scale \( |\mu_I| \). The peculiar \( e^{-c/g} \) behavior comes from the long-range magnetic interaction, as in the superconducting gap at large \( \mu_B \) [7]. The constant \( c \) is smaller by a factor of \( \sqrt{2} \) compared to the latter case due to the stronger one-gluon attraction in the singlet \( q\bar{q} \) channel compared to the \( 3 \) diquark channel. Consequently, the gap (17) is exponentially larger than the diquark gap at comparable baryon chemical potentials. Using the methods of [8] one can estimate \( b \approx 10^4 \). As in the BCS theory, the critical temperature, at which the superfluid state is destroyed, is of order \( \Delta \).

Asymptotically, \( \Delta \) is much less than \( \mu_I \), and superfluidity has little effect on the equation of state. The ratio \( p/\epsilon \) approaches 1/3 from below in the limit \( \mu_I \to \infty \).

5 Quark-hadron continuity and confinement

Since the order parameter (16) has the same quantum numbers and breaks the same symmetry as the pion condensate in the low-density regime, it is plausible that there is no phase transition along the \( \mu_I \) axis. In this case, as one increases the density, the Bose condensate of weakly interacting pions smoothly transforms into the superfluid state of \( ud \) Cooper pairs. The situation is very similar to that of strongly-coupled superconductors with a “pseudogap” [7], and possibly of high-temperature superconductors [9]. This also parallels the continuity between nuclear and quark matter in three-flavor QCD as conjectured by Schäfer and Wilczek [17]. We hence conjecture that in two-flavor QCD one can move continuously from the hadron phase to the quark phase without encountering a phase transition. We stress here that this conjecture needs to be verified by lattice calculations.

At first sight, this conjecture seems to contradict a common wisdom that there is a “deconfinement” phase transition from the hadron phase to the quark phase. It is logically possible that there exists a first order phase transition at intermediate value of \( \mu_I \). However, there are several nontrivial arguments that make the continuity hypothesis highly plausible.

The first argument arises from considering baryons. One notices that all fermions have a gap at large \( |\mu_I| \), which means that all excitations carrying baryon number are massive. In particular, at zero temperature, the baryon number susceptibility \( \chi_B \) vanishes. This is also true at small \( \mu_I \). It is thus natural to expect that all excitations with nonzero baryon number are massive at any value of \( \mu_I \), and \( \chi_B \) remains zero at \( T = 0 \) for all \( \mu_I \). This also suggests one way to verify the continuity on the lattice.

Another argument comes from considering the limit of large number of colors \( N_c \). Let us recall that in finite-temperature QCD, there is a mismatch, at large \( N_c \), between the number of gluon degrees of freedom, which is \( \mathcal{O}(N_c^2) \), and of hadrons, which is \( \mathcal{O}(N_c^0) \). This fact is a strong hint of a first order confinement-deconfinement phase transition, at which the effective number of degrees of freedom jumps from \( \mathcal{O}(N_c^0) \) to \( \mathcal{O}(N_c^2) \). It is easy to see, however, that the \( N_c \) behavior of thermodynamic quantities is the same in the “hadronic” phase (low \( \mu_I \)) and the “quark” phase (large \( \mu_I \)). Indeed, at very large \( \mu_I \) the isospin density \( n_I \) is proportional to the number of quarks, which is \( \mathcal{O}(N_c) \):

\[
\frac{n_I}{N_c} = \frac{\mu_I^3}{\sqrt{3}} \frac{3}{8\pi^2}.
\]

In the small \( \mu_I \) region the isospin density is given by Eq. (10). At large \( N_c \) limit the pion
decay constant scales as $f_\pi^2 = O(N_c)$, and thus the isospin density in the pion gas is also proportional to $N_c$. What happens is that the repulsion between pions becomes weaker as one goes to larger $N_c$, thus more pions can be stacked at a given chemical potential. As a result, the $N_c$ dependence of thermodynamic quantities is the same in the quark and the hadronic regimes, although for seemingly very different reasons.

Now let us return to the question of confinement. Naively, one would think that at asymptotically large $\mu_I$, the $\bar{u}$ and $d$ quarks are packed at a very high density, and the system should become deconfined. At finite temperature, there is no rigorous way to distinguish the confined and deconfined phases in QCD with quarks in the fundamental representation. However, at zero temperature (and finite $\mu_I$), a sharp distinction can be made between the two phases. In the confined phase, all particle excitations carry integer baryon number; the deconfined phase can be defined as the phase where there exist finite-energy excitations carrying fractional baryon charge. The pion superfluid at small $\mu_I$ clearly is in the confined phase. The question is: is the quark matter at large $\mu_I$ confined or deconfined?

It might seem that at very large $\mu_I$ there exist excitations with fractional baryon number. These are the fermionic quasiparticles near the Fermi surface, which are related to the original quarks and antiquarks by a Bogoliubov-Valatin transformation. The opening of a BCS gap makes the energy of these excitations larger than $\Delta$, but still finite.

To see that the logic above has a fault and there are no such excitations, one needs to consider dynamics of very soft gluons. The crucial observation is that at large $\mu_I$, gluons softer than $\Delta$ are not screened neither by Meissner nor by Debye effect. Meissner effect is absent because the condensate does not break gauge symmetry (in contrast to the color superconducting condensate [4]). Debye screening is also absent, because on scales softer than $\Delta$ there are no charge excitations in the medium: the Cooper pairs are neutral, while the fermions are too heavy to be excited. Thus, the gluon sector below the $\Delta$ scale is described by pure SU(3) gluodynamics, which is a confining theory. This means there are no quark excitations above the ground state: all particles and holes must be confined in colorless objects, mesons and baryons, just like in vacuum QCD.

If there is no transition along the $\mu_I$ axis, we expect confinement at all values of $\mu_I$. At large $\mu_I$, since the running strong coupling $\alpha_s$ at the scale of $\Delta$ is small, the confinement scale $\Lambda'_{\text{QCD}}$ is much less than $\Delta$. In more detail, let us imagine following the running of the strong coupling from the UV to the IR. First, $\alpha_s$ increases until the scale $g\mu_I$ is reached when it “freezes” due to Debye screening and Landau damping. The freezing continues until we reach the scale $\Delta$, after which the coupling runs again as in pure gluodynamics. Since the coupling is still small at the scale $\Delta$, it can become large only at some scale $\Lambda'_{\text{QCD}}$ much lower than $\Delta$. Thus, at large $|\mu_I|$ there are three different scales separated by large exponential factors, $\mu_I \gg \Delta \gg \Lambda'_{\text{QCD}}$.

That the scale of confinement is much smaller than the gap at large $\mu_I$ has an important consequence for finite temperature. One can actually predict a temperature driven deconfinement.

\[2\] With physical values of $N_c$, $f_\pi$ and $m_\pi$, the values of $n_I$ given by eqs. (10) and (18), naively continued into the regime of intermediate $\mu_I$, cross at $\mu_I \approx 800 \text{MeV}$. This agrees with the value of $\mu_I \sim m_\rho$ where one would expect the crossover between the quark and hadron regimes to occur. This is a quantitative indication that a phase transition is not necessary.

\[3\] This is similar to the behavior of the unbroken SU(2)$_c$ sector of two-flavor color superconductors [3].
ment phase transition at a temperature $T'_c$ of order $\Lambda'_{QCD}$. Indeed, at such low temperatures, quarks are unimportant, so the transition must be of first order as in pure gluodynamics. In particular, one expects the baryon number susceptibility temperature dependence to change from $e^{-3\Delta/T}$ to $e^{-\Delta/T}$ around $T'_c$ due to deconfinement.

The smallness of the confinement scale $\Lambda'_{QCD}$ compared to the BCS gap $\Delta$ allows one to conclude that the binding energy of quarks and antiquarks is small and the hadronic spectrum follows the pattern of the constituent quark model, with $\Delta$ playing the role of the constituent quark mass. This means mesons weigh $2\Delta$ and baryons weigh $3\Delta$, approximately. A good analog of the large $\mu_I$ regime is vacuum QCD with only heavy quarks. As in the latter case, the string tension and string breaking are determined by parametrically different energy scales ($\Lambda'_{QCD}$ and $\Delta$, respectively). Hence the area law should work up to some distance much larger $\Lambda'_{QCD}$, even when fundamental quarks are present. For the same reason one also expects the high-spin excited states of hadrons to be narrow at large $\mu_I$.

The energy hierarchy also leads to a curious dispersions relation of hadrons in the isospin dense regime. Consider, for example, the $\rho^-$ meson, which is a bound state of a $\bar{u}$ and a $d$. At zero total momentum, the $\bar{u}$ and the $d$ are on the opposite sides of the Fermi surface. As one increases the total momentum, the two constituents move along the Fermi surface, remaining close to the latter until the total momentum becomes larger than $\mu_I$, i.e. twice the Fermi momentum. Thus, the dispersion curve of the $\rho^-$ must remain essentially flat in the interval of momentum ($0, |\mu_I|$). For baryons, energy is almost independent of the momentum in the interval ($0, 1.5|\mu_I|$). The group velocity of hadrons, thus, almost vanishes in these intervals. Above these intervals it should be equal to the speed of light. It would be interesting to follow, on the lattice, the evolution of the dispersion curves of $\rho^-$ from small to large $\mu_I$.

6 The phase diagrams on $(T, \mu_I)$ and $(\mu, \mu_I)$ planes

By considering nonzero $\mu_I$, we make the phase diagram of QCD three-dimensional: $(T, \mu_B, \mu_I)$. Two planes in this three-dimensional space are of a special interest: the $\mu_B = 0$ $(T, \mu_I)$ plane, which is completely accessible by present lattice techniques, and the $T = 0$ $(\mu_I, \mu_B)$ plane, where the neutron star matter belongs.

Let us first consider the phase diagram on the $(T, \mu_I)$ plane, which is simpler. Two phenomena determine the phase plane on this plane (Fig.2): pion condensation and confinement. At sufficiently high temperature the condensate (16) melts (solid line in Fig. 2). For large $\mu_I$, this critical temperature is proportional to the BCS gap (17). There are two phases which differ by symmetry: the high temperature phase where the explicit flavor symmetry is restored, and the low-temperature phase where this symmetry is spontaneously broken. The phase transition is in the $O(2)$ universality class. The critical temperature $T_c$ vanishes at $\mu_I = m_\pi$ and is an increasing function of $\mu_I$ in both regimes we studied: $|\mu_I| \ll m_\rho$ and $|\mu_I| \gg \Lambda_{QCD}$. Thus, it is likely that $T_c(\mu_I)$ is a monotonous function of $\mu_I$. In addition, as explained before, at large $\mu_I$, there is a first-order deconfinement phase transition at some

4 The width of the Ginzburg region is suppressed by $(\Delta/\mu_I)^4$ at large $\mu_I$ as in usual BCS superconductors and also by $1/N_c^2$ at large $N_c$ as at the QCD chiral transition [21].
temperature $T'_c$ much lower than $T_c(\mu_I)$. Since there is no phase transition at $\mu_I = 0$ (for small $m_{u,d}$) or at $T = 0$ (assuming quark-hadron continuity), this first-order line must end at some point $A$ on the $(T,\mu_I)$ plane (Fig. 2).

The phase diagram in the $(\mu_I, \mu_B)$ plane at zero temperature turns out to be quite complicated. We defer the more detail study of this plane for future work. Here we shall only consider the regime $|\mu_I| \gg \mu_B$, both much larger than $\Lambda_{QCD}$, so that perturbative QCD can be used. When $\mu_B = 0$ and $|\mu_I| \gg \Lambda_{QCD}$ we have seen that the system is a superfluid with a gap $\Delta$. Finite $\mu_B$ provides a mismatch between $\bar{u}$ and $d$ Fermi spheres. The superconducting state becomes unfavorable at some value of $\mu_B$ of order $\Delta$. It is known [22] that the destruction of this state occurs through two separate phase transitions. As one increases $\mu_B$, at $\mu_B$ slightly below $\Delta/\sqrt{2}$, a first-order phase transition takes the system to the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [22], which is characterized by a spatially modulated superfluid order parameter $\langle \bar{u} \gamma_5 d \rangle$ with a wavenumber of order $2\mu_B$. How exactly this spatial dependence looks like is still unknown, mostly because the FFLO state has not been observed in metals. The FFLO state persists only until $\mu_B = 0.754\Delta$ when it goes through a second-order phase transition to a $\langle \bar{u} \gamma_5 d \rangle = 0$ state. The latter must be a color superconductor with one-flavor diquark condensates $\langle uu \rangle$ and $\langle dd \rangle$, due to the attraction between quarks of same flavor. In the region of the $(\mu_B, \mu_I)$ phase diagram directly relevant to neutron stars, $\mu_B > \mu_I$, the color superconducting FFLO phase is studied in a recent paper of Alford, Bower and Rajagopal [23].

The most interesting feature of the FFLO state is that it has the same symmetries as the inhomogeneous pion condensation state which might form in electrically neutral nuclear matter at high densities, as argued by Migdal [1] and others [2]. The FFLO phase, thus, can be thought of as a realization of Migdal’s pion condensate in the regime of asymptotically high densities. It is also conceivable that the two phases are actually one, i.e., continuously connected on the $(\mu_I, \mu_B)$ phase diagram.
7 Conclusion

Our original and primary motivation for considering QCD at finite isospin densities is to have a dense regime of realistic, three-color QCD that can be studied on lattice. Based on analytical calculations in the two asymptotic regimes of low and high densities, we found that there is likely no phase transition along the $\mu_I$ axis at zero temperature. This conjecture should be verified on the lattice. An obvious way is to study the thermodynamics of the system. If our continuity conjecture is correct, all thermodynamic quantities should be smooth functions of $\mu_I$. In this case, we also suggest that the ratio $p/\epsilon$ is a non-monotonic function of $\mu_I$: it raises from 0 at the threshold $\mu_I = m_\pi$ to some value close to 1, then drops to some minimal value, and then approaches $1/3$ from below at large $\mu_I$. The baryon susceptibility should vanish at any $\mu_I$ at zero temperature. We also predict the existence of a line of first-order phase transition on the $(T, \mu_I)$ plane, which terminates at a second order point.

The phase diagram on the $(\mu_I, \mu_B)$ plane, which is most relevant for neutron star physics, remains inaccessible to the lattice. Based on our preliminary investigations, the phase diagram on this plane should have a rather complicated topology. The most interesting feature of this diagram appears to be the existence of the FFLO phase, which is reliably predicted at $\mu_I \gg \mu_B$, both being large. This phase has the same symmetry as the pion condensation state conjectured by Migdal, and both might be different regions of a single connected region on the phase diagram.

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Note Added

Further research [24] reveals that the physics below the scale $\Delta$ is described by “gluodynamics of continuous media” with a large dielectric constant. As a result, the scale of the confinement $\Lambda'_{\text{QCD}}$ is small and decreases exponentially with the chemical potential $\mu_I$. This means that the line of the first order deconfinement transition goes down as indicated in Fig. 4.

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