Generative Learning of Heterogeneous Tail Dependence

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Abstract

We propose a multivariate generative model to capture the complex dependence structure often encountered in business and financial data. Our model features heterogeneous and asymmetric tail dependence between all pairs of individual dimensions while also allowing heterogeneity and asymmetry in the tails of the marginals. A significant merit of our model structure is that it is not prone to error propagation in the parameter estimation process, hence very scalable, as the dimensions of datasets grow large. However, the likelihood methods are infeasible for parameter estimation in our case due to the lack of a closed-form density function. Instead, we devise a novel moment learning algorithm to learn the parameters. To demonstrate the effectiveness of the model and its estimator, we test them on simulated as well as real-world datasets. Results show that this framework gives better finite-sample performance compared to the copula-based benchmarks as well as recent similar models.

1 Introduction

Modeling the dependence of extreme events is of great importance in economics and finance, especially for high-dimensional applications like portfolio management, institutional risk management, and systematic risk measurement [Zimmer, 2012][Glasserman et al., 2002][Oh and Patton, 2018]. A large amount of literature focuses on modeling univariate heavy-tailed distribution or extreme dependence in bivariate variables. Very rare econometric or statistical models pay attention to extreme dependence modeling in multi-dimensions because it is difficult to define a proper model structure while preserving enough flexibility. However, real-world applications show great needs for model flexibility as well as the ease of computing.

For univariate distribution modeling, it is widely accepted that the financial data shows asymmetric heavy tails [Cont, 2001]. For dependence modeling, extreme dependence, or tail dependence, also presents asymmetry and heterogeneity for pairs of variables in financial markets. Different pairs will exhibit different degrees of extreme dependence and they are asymmetric on up and down sides. It is very hard to capture all these stylized facts simultaneously in one model. Moreover, in practical applications, dimensions of constituents are often relatively high, for instance, up to 100 in portfolio investment. There are two major issues for high-dimensional extreme dependence modeling. It requires parsimonious assumptions to make the model tractable while keeping the model flexible enough to capture stylized facts of extreme dependence, see [Oh and Patton, 2017]. Therefore, one
issue is to keep the balance between model parsimony and necessary flexibility. Another issue for high-dimensional modeling is the high computational burden and that the parameter estimation error will significantly increase as dimension grows.

In this paper, we propose a simple and flexible generative model to capture the extreme dependence structure of multiple variables, for example, of different financial assets. It is a specially designed combination of some known random variables. The one-dimensional form of our model shows asymmetric heavy tails and we give some analytical results on the tail heaviness of the distribution determined by our model. For the multivariate form, it is hard to give an explicit formula of the tail dependence for pairs of dimensions of our model. Therefore, we show heterogeneous and asymmetric tail dependence structure by simulation. Because of the lack of a closed-form probabilistic density function of our model, it is not feasible to use a likelihood method or other traditional methods to estimate the parameters with data. In this work, we devise a simulation-based moment learning method to learn the parameters. One technical merit of our learning method is that it is not prone to the error propagation issue (which exists in the similar work [Yan et al., 2019]), hence being highly scalable when the dimension of the model becomes large.

This paper makes two primary contributions. First, we come up with a succinct generative model which shows flexible tail dependence structure. And it is interpretable and quite easy to understand, which is important and necessary in many domains like finance and business. Moreover, it is easy to sample, and scalable to apply our model in multi-dimensions, showing a great advantage comparing to other existing models. The second contribution of this paper is that we devise a novel simulation-based moment method to conduct the parameter estimation, whose consistency and efficiency are verified in simulations. At last, our model is widely applicable not only in financial data but also in many other data science domains.

2 Literature Review

In many applications, questions involving extremal events, such as large insurance claims and large fluctuations in financial data, play an increasingly important role. Many theories and approaches have been proposed in past years, see [McNeil et al., 2015] and [Embrechts et al., 2013] for an overview.

In financial markets, classical models for investments, asset pricing, and risk management assume that the return data follows a multivariate Gaussian distribution. Also in credit markets, [Li, 2000] first modeled asset default correlations by Gaussian copula. However, multivariate Gaussian is not sufficient to model the extreme dependence between different assets because normal variables have symmetric non-heavy tails and zero tail dependence. To incorporate tail dependence, [Demarta and McNeil, 2005] presented Student’s \( t \) copula and related copulas because the tail dependence of the multivariate \( t \) distribution is non-zero. However, symmetric tail dependence of the \( t \) distribution on up and down sides is a weakness. It is observed that financial assets become more correlated in downturns than in upturns, which means lower tail dependence is greater than upper tail dependence [Yan et al., 2019]. Some variants of the \( t \) copula tried to present asymmetric tail dependence, such as skewed \( t \) copula in [Smith et al., 2012].

More critically, in multivariate \( t \) distribution, only one parameter, the degrees of freedom, determines the tail dependence for all pairs of dimensions. It means that, for different pairs, pairwise tail dependence is approximately proportional to pairwise correlation, which is not realistic in financial markets as well as in other datasets. The usual correlation and the tail dependence vary together across pairs, with no additional freedom for tail dependence. Elliptical distribution [Owen and Rabinovitch, 1983] is similarly restrictive in this regard. Because that the daily fluctuations of asset prices and the extremal movements across the whole market may be attributed to different economic reasons, we believe that the model flexibility should be increased in the way that the tail dependence should be separately modeled, with more freedoms incorporated.

Copulas are an important tool to model tail dependence, see [Nelsen, 2007]. Archimedean copulas [Savu and Trede, 2010] like Gumbel or Clayton copula are applicable when the dimension is low (\( N < 5 \)) and they present asymmetric tail dependence. However, similarly, Archimedean copulas are restrictive because they always use one or two parameters to control the tail dependence of any pair of variables. Another class of copulas called vine copulas are constructed by sequentially applying bivariate copulas to build a high-dimensional one [Aas et al., 2009]. However, vine copulas are hard to test and interpret.
To fix the issues mentioned above, [Oh and Patton, 2017] proposed a factor copula approach which creates the desired copula by a linear combination of heavy-tail factors. Although it has nice properties of extreme dependence, or is flexible enough, the possible issue of this method is the parameter estimation. Because a closed-form likelihood cannot be obtained, they adopted a simulation-based estimation method, which may have a great computational cost and cannot guarantee the estimation accuracy. [Yan et al., 2019] proposed a generative model using a special transformation of some latent normal random variables to separate the modeling of tail dependence from the correlation. The tail dependence was modeled to be totally free and was controlled by corresponding parameters. However, the transformation in their model is too complicated to be implemented in practice when the dimension is high, and even moderate. We find that the lower-triangular structure of their model will lead to error propagation issue in the parameter fitting process. Another issue is that the tail dependence is not easy to be linked to the corresponding parameters accurately, so not easy to interpret.

3 Related Works

[Yan et al., 2018] is the first work using a generative model of transforming latent normal random variable for extremal modeling. They proposed the following new random variable having asymmetric heavy tails:

\[
Y = \mu + \sigma Z \left( \frac{e^{uZ}}{A} + 1 \right) \left( \frac{e^{-vZ}}{A} + 1 \right). \tag{1}
\]

\(Z\) is a latent standard normal random variable, \(\mu\) and \(\sigma\) are location and scale parameters, \(u \geq 0\) and \(v \geq 0\) control right and left tail heaviness. \(A\) is a relatively large positive constant. Authors of this paper have shown nice properties of the distribution of \(Y\) by analyzing its quantile function, which is exactly \(\mu + \sigma \Theta^{-1}(\tau) \left( \frac{e^{u\Theta^{-1}(\tau)}}{A} + 1 \right) \left( \frac{e^{-v\Theta^{-1}(\tau)}}{A} + 1 \right)\), where \(\Theta^{-1}\) is the quantile function of \(Z\).

Later in [Yan et al., 2019], researchers extended the above one-dimensional model to multi-dimensional case, to capture not only the heavy tails of marginal distributions but also tail dependence between pairs of dimensions. They first modified Equation (1) to \(Y = \mu + \sigma Z \left( \frac{u^2}{A} + \frac{v^2}{A} + 1 \right)\), abbreviated here as \(Y = \mu + \sigma g(Z|u, v)\), then proposed a lower-triangular structure for multi-dimensional case:

\[
\begin{align*}
Y_1 &= \mu_1 + \sigma_{11} g(Z_1 | u_{11}, v_{11}), \\
Y_2 &= \mu_2 + \sigma_{21} g(Z_1 | u_{21}, v_{21}) + \sigma_{22} g(Z_2 | u_{22}, v_{22}), \\
&\vdots \\
Y_n &= \mu_n + \sigma_{n1} g(Z_1 | u_{n1}, v_{n1}) + \sigma_{n2} g(Z_2 | u_{n2}, v_{n2}) + \cdots + \sigma_{nn} g(Z_n | u_{nn}, v_{nn}). \tag{2}
\end{align*}
\]

The resulting random vector \([Y_1, \ldots, Y_n]\)' has the most freedom regarding marginal tail heaviness and pairwise tail dependence. While \(\{\sigma_{ij}\}\) determine the usual correlation or covariance structure, \(\{u_{ij}\}\) or \(\{v_{ij}\}\) determine the right/left tail heaviness and tail dependence structure. However, it is a difficult task to estimate the parameters of such a complicated generative model, because there is no density function available. [Yan et al., 2019] proposed a moment method and quantile regression based estimation algorithm tackling \(Y_1, \ldots, Y_n\) sequentially. Not surprisingly, this will lead to error propagation issue, i.e., the estimation error will accumulate in the estimation process from \(Y_1\) to \(Y_n\) because the number of terms in the expression of \(Y_i\) increases as \(i\) increases. Thus the estimated parameters for \(Y_n\) will be highly unreliable. This severely limits the application of this model in multi-dimensional cases. In this paper, we will solve this issue by proposing a more succinct model structure while preserving the model flexibility in terms of heterogeneous and asymmetric tail heaviness and tail dependence.

4 Our Model

For univariate variable, the tail heaviness of its distribution can be represented by the so-called tail index. For multivariate case, one terminology, the tail dependence index, can be used to measure the extremal co-movement of two variables. The lower and the upper tail dependence for two random
variables $Y_1$ and $Y_2$, such as two asset returns, are defined as:

$$
\lambda_{ij}^L = \lim_{\tau \to 0^+} \frac{P(Y_i \leq F^{-1}_i(\tau), Y_j \leq F^{-1}_j(\tau))}{\tau},
$$

$$
\lambda_{ij}^U = \lim_{\tau \to 1^-} \frac{P(Y_i > F^{-1}_i(\tau), Y_j > F^{-1}_j(\tau))}{1 - \tau},
$$

where $F_i(\cdot), F_j(\cdot)$ are marginal distribution functions of $Y_i$ and $Y_j$, respectively. Obviously, $\lambda_{ij}^L, \lambda_{ij}^U$ take values in $[0, 1]$. We say the two variables have upper or lower tail dependence if $\lambda_{ij}^U$ or $\lambda_{ij}^L$ is non-zero.

In finite-sample cases, it is also important how the limits in the above equations are approached. In real-world data or applications, we can never see the extremely low probability-level quantile event happened. So, except for the limits, it is also very valuable to construct a model that can capture the behavior of the dependence exhibited in the data before $\tau$ in the above equations goes to $0^+$ or $1^-$.

### 4.1 Univariate Generative Model

We first propose a new random variable $Y$ which is generated by the following latent variable structure:

$$
Y = \mu + e^{uZ_1} - e^{vZ_2} + \sigma Z_3, \quad Z_1, Z_2, Z_3 \sim N(0, 1),
$$

where $\mu, u, v, \sigma$ are parameters satisfying $u \geq 0, v \geq 0, \sigma \geq 0$. The three latent $Z_1, Z_2, Z_3$ are standard normal and mutually independent random variables. The random variable $Y$ can be decomposed into three independent parts: the log-normally distributed $e^{uZ_1}$ which controls the upper tail heaviness of $Y$ by the parameter $u$; the log-normally distributed $-e^{vZ_2}$ which controls the lower tail heaviness of $Y$ by the parameter $v$; and the normally distributed $\mu + \sigma Z_3$ which controls the overall shape of the distribution of $Y$. The following proposition gives the tail heaviness of $Y$.

**Proposition 1.** The random variable $Y$ in Equation (5) has the same right tail heaviness as log-normal distribution $e^{uZ_1}, Z_1 \sim N(0, 1)$ and the same left tail heaviness as log-normal distribution $-e^{vZ_2}, Z_2 \sim N(0, 1)$, namely,

$$
P(Y > t) \sim P(e^{uZ_1} > t) \quad \text{as } t \to \infty,
$$

$$
P(Y < t) \sim P(e^{vZ_2} > -t) \quad \text{as } t \to -\infty.
$$

Moreover, $Y$ has the following asymptotic property:

$$
P(Y > t) \sim \frac{u}{\sqrt{2\pi ln t}} \exp\left(-\frac{(ln t)^2}{2u^2}\right) \quad \text{as } t \to \infty,
$$

$$
P(Y < t) \sim \frac{v}{\sqrt{2\pi ln(-t)}} \exp\left(-\frac{(ln(-t))^2}{2v^2}\right) \quad \text{as } t \to -\infty.
$$

The proof is available in the Appendix A.

This proposition describes the asymptotic results of tail heaviness of $Y$ and supports the claim that the tail heaviness are controlled by the two log-normally distributed latent parts respectively. However, the log-normal tails are not necessary. If we consider the following variant:

$$
Y = \mu + e^{Z_1} - e^{Z_2} + \sigma Z_3, \quad Z_1 \sim \text{Exp}(\lambda_1), Z_2 \sim \text{Exp}(\lambda_2), Z_3 \sim N(0, 1),
$$

now the normally distributed $uZ_1$ and $vZ_2$ in Equation (5) are replaced by the exponentially distributed $Z_1$ and $Z_2$ in Equation (10), with parameters $\lambda_1$ and $\lambda_2$ respectively. We can prove now the tails of $Y$ are power-law tails, i.e.,

$$
P(Y > t) \sim t^{-\lambda_1} \quad \text{as } t \to \infty,
$$

$$
P(Y < t) \sim (-t)^{-\lambda_2} \quad \text{as } t \to -\infty.
$$

The proof is also included in the Appendix B. Furthermore, if we replace $Z_1$ and $Z_2$ in Equation (5) by standard $t$-distributed random variables instead (with some degrees of freedom), the tails of $Y$ will become heavier than power-law tails.
4.2 Multivariate Generative Model

Now we extend the univariate latent model to multivariate case. Consider a \( n \)-dimensional random vector \( \mathbf{Y} = [Y_1, \ldots, Y_n] \) which is generated by transforming three latent random vectors as follows:

\[
Y_1 = \mu_1 + e^{u_1}Z_{11} - e^{v_1}Z_{21} + \sigma_1 Z_{31}, \\
Y_2 = \mu_2 + e^{u_2}Z_{12} - e^{v_2}Z_{22} + \sigma_2 Z_{32}, \\
\cdots \\
Y_n = \mu_n + e^{u_n}Z_{1n} - e^{v_n}Z_{2n} + \sigma_n Z_{3n},
\]

where \( u_j \geq 0, v_j \geq 0, \sigma_j \geq 0, j = 1, \ldots, n \), \( Z_i = [Z_{i1}, Z_{i2}, \ldots, Z_{in}] \sim N(0, \Sigma_i), i = 1, 2, 3 \), and the diagonal elements of \( \Sigma_i \) are all one. \( Z_1, Z_2, Z_3 \) are three mutually independent random vectors. Each \( Y_j \) has four unknown parameters \( \mu_j, u_j, v_j, \sigma_j \) which determine the marginal distribution, named as marginal parameters. And the three correlation matrices \( \Sigma_1, \Sigma_2, \Sigma_3 \) control the dependence structure of the joint distribution of \( \mathbf{Y} \), named as joint parameters. Therefore, the total number of parameters is \( 4n + 3n(n - 1)/2 \).

To reduce the number of parameters, one can assume \( Z_1, Z_2, \) or \( Z_3 \) has a factor structure, as usual. Or, to add flexibility, one can assume \( Z_1, Z_2, \) and \( Z_3 \) have some other multivariate distributions rather than Gaussian, such as multivariate \( t \) or multivariate exponential distributions \( \{u_j\} \) and \( \{v_j\} \) may need to be deleted in Equation (13). For simplicity, in the qualitative analysis and real-data experiment in the rest of the paper, we always consider Equation (13) and multivariate Gaussian \( Z_1, Z_2, \) and \( Z_3 \).

The second-order dependence, i.e., the covariance, between \( Y_i \) and \( Y_j \) is mainly determined by \( \sigma_1, \sigma_2 \) and \( \rho_{ij}^3 = (\Sigma_3)_{ij} \). Besides, this model can exhibit great flexibility in tail dependence structure. For instance, when \( u_j = 0, v_j = 0, j = 1, \ldots, n \), \( \mathbf{Y} \) degenerates to a multivariate normal distribution. When they are not zero, upper tail dependence between \( Y_i \) and \( Y_j \) is controlled by \( u_i, u_j \) and the correlation parameter \( \rho_{ij}^1 = (\Sigma_1)_{ij} \), or, the correlation between \( Z_{1i} \) and \( Z_{1j} \). Similarly, lower tail dependence between \( Y_i \) and \( Y_j \) is controlled by \( v_i, v_j \) and the correlation parameter \( \rho_{ij}^2 = (\Sigma_2)_{ij} \), or, the correlation between \( Z_{2i} \) and \( Z_{2j} \). Thus, we give complete freedom to tail dependence structure beyond the correlation structure for \( \mathbf{Y} \). We will verify this through simulation later.

The joint distribution of the pair \( (Y_i, Y_j) \) is controlled by their respective marginal parameters \( \mu, u, v, \alpha \) and three correlation parameters \( \rho_{ij}^1, \rho_{ij}^2, \rho_{ij}^3 \). So we can treat every pair of variables separately in parameter learning. Therefore, for a high-dimensional application, we only need to consider a two-dimensional case every time and then assemble all the learned parameters together, to finish the parameter learning for \( \mathbf{Y} \). It is scalable comparing to the lower-triangular model in [Yan et al., 2019], even though they have the same number of parameters. In other words, the error propagation issue in parameter learning is overcome by us. Another issue it may incur is the possible non-positive definite correlation matrix we may obtain. One quick and easy solution is modifying the non-positive eigenvalues of the correlation matrix to be a small positive number.

Our model does not have a central parameter, like the degrees of freedom in multivariate \( t \) distribution, that determines tail dependence for all pairs of dimensions. It is a great advantage in terms of tail dependence flexibility. Besides, marginal tails can also be asymmetric and heavy. Other advantages of our model include the easy simulating, although there is no density function for the joint distribution available. And our model is much easier to interpret. It is very clear how each parameter works and is explained.

4.3 Pairwise Tail Dependence

We illustrate the flexibility of dependence structure in our model by simulation. For instance, for the pair of variables \( Y_1 \) and \( Y_2 \) (other pairs are in the same situation), by setting all other parameters in the set \( \{\mu_1, u_1, v_1, \sigma_1, \mu_2, u_2, v_2, \sigma_2, \rho_{12}^1, \rho_{12}^2, \rho_{12}^3\} \) as constants, we just vary only one parameter every time to see how the usual correlation and the tail dependence of \( Y_1 \) and \( Y_2 \) vary accordingly. Since it is hard to derive a closed-form result for the tail dependence, we simulate many data points using our model and calculate a proxy with the data, as in [Yan et al., 2019]. For example, for lower tail dependence, \( \lambda_{21}^{(\tau)}(\tau) = P(Y_1 \leq F_{\tau}^{-1}(\tau), Y_2 \leq F_{\tau}^{-1}(\tau))/\tau \) with a small \( \tau \) is calculated. We will change \( \tau \) in \([0.001, 0.1]\) and plot the proxy lower tail dependence against the varying \( \tau \). For symmetry, the upper tail dependence case is in the same situation.
In Figure 1, they are the plots of the correlation against the varying parameter. We can see $u_1$ has little impact on the correlation. For symmetry, for $u_2$, $v_1$, or $v_2$, this is also the case. $\rho^1_{12}$ (and also $\rho^2_{12}$) has some slight effect on the correlation, as expected by us. $\rho^3_{12}$ is the dominant parameter that determines the correlation between $Y_1$ and $Y_2$. In Figure 2, they are the plots of the proxy lower tail dependence against varying $\tau$. In each plot, we vary one corresponding parameter only in the set $\{0.3, 0.6, 0.9\}$ and use different line styles to present the three results. The blue solid line, the orange dashed line, and the green dotted line correspond to the cases when the varying parameter is set to be 0.3, 0.6, and 0.9 respectively. From the figure, we can obtain an intuitive conclusion that $v_1$, $\rho^2_{12}$ (and also $v_2$) are the dominant parameters that positively affect the lower tail dependence of $Y_1$ and $Y_2$. For symmetry, $u_1$, $u_2$, and $\rho^1_{12}$ are the dominant parameters that positively affect the upper tail dependence. Thus, all the parameters in our model have clear and intuitive interpretations.

5 Parameter Learning

Traditional parameter estimation method, like maximum likelihood, needs density function of the distribution of the model. However, it is hard to get the closed-form density of our multivariate generative model. Instead, we propose a moment learning algorithm to learn our parameters, some part of which is similar to McFadden’s simulated method of moments [McFadden, 1989]. The algorithm has two parts. The first part estimates the marginal parameters $\mu, u, v, \sigma$ of every variable in $Y$. The second part estimates the joint parameters $\rho^1, \rho^2, \rho^3$ for each pair of variables in $Y$ with all $\mu, u, v, \sigma$ fixed. For simplicity, we sometimes drop the subscripts $i, j$ in this section.

5.1 Marginal Parameter Learning

The $i$-th moment of any one variable $Y$ in our model, $i = 1, 2, 3, 4$, can be easily written in closed form by some basic calculations. By making them equal to corresponding sample moments $m_i$,
where \( y_k, k = 1, \ldots, K \) are \( K \) univariate observations. Solving this equation system is not straightforward numerically. It is better to solve it by alternately minimizing the following two objectives: 
\[
\min_{\mu, \sigma}(E[Y^1] - m_1)^2 + (E[Y^2] - m_2)^2
\]
and 
\[
\min_{u, v}(E[Y^3] - m_3)^2 + (E[Y^4] - m_4)^2.
\]
The reason for doing this is that \( \mu, \sigma \) and \( u, v \) control different parts of the distribution, thus should be treated differently. Moreover, in our experience, the data minus its mean shall be applied first and an initial estimation \( \hat{\mu}^*, \hat{\sigma}^* \) can be obtained by solving the equation system. Then, we subtract the constant \( \hat{\mu}^* \) from the data and solve the system again to get the final estimation. The estimated marginal parameters are used in the learning of joint parameters, treated as known numbers.

5.2 Joint Parameter Learning

It is a big challenge to train such a flexible multivariate model with many parameters. Fortunately, due to the succinct structure of our model, parameter learning can be done pair-wisely each time for each pair of dimensions. All parameters can be obtained by running the learning algorithm \( C_n^2 \) times. Here we still adopt the idea of moment method to learn \( \rho^1, \rho^2, \rho^3 \) for \( Y_i \) and \( Y_j \), but notice that the moments representing the special dependence structure of the pair should be considered.

Because \( \rho^1, \rho^2, \rho^3 \) control different aspects of the dependence structure, we can design tailor-made moments for learning them. For example, using the moment \( E[Y_i Y_j] \) is enough for the purpose of learning \( \rho^1 \), which describes the correlation mainly. For \( \rho^1 \) and \( \rho^2 \) which describe the extremal dependence, we design four moments that only take extremal cases of \( (Y_i, Y_j) \) into account. Specifically, two transformations are defined: 
\[
f(x) = \ln(\max(x - c, 1)) \quad \text{and} \quad g(x) = \ln(\max(-x - c, 1)),
\]
where \( c \geq 0 \) is a constant. And four new moments \( E[f(Y_i) f(Y_j)], E[g(Y_i) g(Y_j)], E[f(Y_i) g(Y_j)], E[g(Y_i) f(Y_j)] \) are used in our learning. They only take the four corners in the coordinate plane of \( (Y_i, Y_j) \) into the calculation. In the remaining area, \( Y_i \) or \( Y_j \) is transformed to 0. Formally, we solve the following optimization problem:
\[
\min_{\rho^1, \rho^2, \rho^3} \left( E[Y_i Y_j] - \frac{1}{K} \sum y_{ik} y_{jk} \right)^2 + \sum_{h_1 \in \{f, g\}} \sum_{h_2 \in \{f, g\}} \left( E[h_1(Y_i) h_2(Y_j)] - \frac{1}{K} \sum h_1(y_{ik}) h_2(y_{jk}) \right)^2,
\]
where \( y_{ik}, y_{jk}, k = 1, \ldots, K \) are observations of \( Y_i \) and \( Y_j \). In fact, it is better to optimize the above first term and the second term alternately, with \( \rho^3 \) being updated and \( \{\rho^1, \rho^2\} \) being updated respectively, because they control different aspects of the dependence structure.

Because it is not straightforward to write the above moments as closed-form functions of \( \rho^1, \rho^2, \rho^3 \), we take advantage of the easy sampling of our model, and sample many data points to calculate \( E[Y_i Y_j] \) and \( E[h_1(Y_i) h_2(Y_j)] \) when given the values of \( \rho^1, \rho^2, \rho^3 \). Moreover, we can first sample from two independent standard normal \( Z'_i \) and \( Z'_j \), and then apply \( Z_i = Z'_i, Z_j = \rho Z'_i + \sqrt{1 - \rho^2} Z'_j \) to obtain correlated normal variables for further sampling of \( Y_i \) and \( Y_j \). Thus, the gradient of the objective on \( \rho^1, \rho^2, \rho^3 \) is possible, for the convenience of gradient-descent based optimization. At last, to ensure the positive definiteness of the correlation matrix learned for multiple variables, we may modify the negative eigenvalues to a small positive number.

5.3 Algorithm Convergence

To illustrate the effectiveness of the whole algorithm for learning both marginal and joint parameters, we randomly specify parameters for our model, sample data from it, and implement the learning algorithm with the data to see how the true and the learned parameters are close. For a fixed sample size \( N \), we do this 20 times and calculate the average of the \( L_2 \) norm errors between the true and the learned parameters in all attempts. When \( N \) changes from \( 2^0 \times 10000 \) to \( 2^7 \times 10000 \), we plot the average \( L_2 \) norm error against \( \log_2(N/10000) \) in Figure [9]. For homogeneity of the model estimation,
Figure 3: The plots of the average $L_2$ norm errors between true and learned parameters. The x-axis is $\log_2(N/10000)$ where $N$ is the sample size. For every $N$, we try 20 times. The left subplot is the result for marginal parameters and the right one is the result for all parameters.

Table 1: Discrepancy functions of four copulas, the LT model, and our model. $\tau$ takes values in $\{0.01, 0.02, 0.03, 0.04, 0.05\} \cup \{0.95, 0.96, 0.97, 0.98, 0.99\}$.

| Model   | Stock Pair | 1 & 2 | 1 & 3 | 1 & 4 | 2 & 3 | 2 & 4 | 3 & 4 |
|---------|------------|-------|-------|-------|-------|-------|-------|
| normal copula | 0.0120 | 0.0125 | 0.0126 | 0.0189 | 0.0225 | 0.0238 |
| $t$ copula   | 7.61E-04 | 3.30E-04 | 5.13E-04 | 2.56E-04 | 2.15E-04 | 1.70E-04 |
| Clayton copula | 0.0067 | 0.0085 | 0.0076 | 0.0102 | 0.0094 | 0.0120 |
| Gumbel copula | 0.0086 | 0.0062 | 0.0074 | 0.0057 | 0.0064 | 0.0060 |
| LT model    | 0.0054 | 6.54E-04 | 0.0012 | 0.0015 | 9.87E-04 | 0.0013 |
| our model   | 5.29E-04 | 2.06E-04 | 5.01E-04 | 1.78E-04 | 1.27E-04 | 3.86E-04 |

we only consider having two variables. The left subplot in Figure 3 is the result for only marginal parameters, and the right one is the result for all parameters. Both subplots show decaying errors that approach 0, indicating the consistency and efficiency of our model estimation method.

6 Application on Stock Market

In this section, our model is applied on stock market. We fit our model with daily return data of 4 U.S. stocks: BA, JPM, CAT, and DWDP. All return time series start from Dec 15, 1980 and end on May 21, 2019 with 9,690 observations. For comparisons, we also implement 4 copula-based benchmark models: normal, $t$, Clayton [Clayton, 1978], and Gumbel [Kole et al., 2007], as well as the lower-triangular model (LT model) in [Yan et al., 2019].

To evaluate the ability of a model in modeling tail dependence, we propose a discrepancy function which can measure the difference between two tail dependence structures, for example, one from the real data and another from the estimated model. Using this discrepancy function, we can evaluate our model and compare it to other benchmark models. Before introducing it, it is better to review that if one wants to calculate the difference between two univariate distributions, he or she can compare their quantiles at many probability levels. If the tail side is the focus, one can let the probability levels close to 0 or 1. This inspires us to extend this idea to the two-dimensional case.

[Yan et al., 2019] proposed a joint-quantile of a two-dimensional distribution of random variable pair $(X, Y)$. For the left-tail side, it is defined as:

$$P(X \leq F_X^{-1}(\tau^*), Y \leq F_Y^{-1}(\tau^*)) = \tau,$$

where $\tau$ is a pre-specified small probability level close to 0 and $\tau^*$ solves the above equation. $F_X^{-1}(\tau^*)$, $F_Y^{-1}(\tau^*)$ are corresponding $\tau^*$-quantiles of $X$ and $Y$. $(F_X^{-1}(\tau^*), F_Y^{-1}(\tau^*))$ are defined as the left joint $\tau$-quantile of $(X, Y)$. We also denote $(F_X^{-1}(\hat{\tau}^*), F_Y^{-1}(\hat{\tau}^*))$ as the joint $\tau$-quantile of the real data, when we only use real observations of $X$ and $Y$ to solve Equation (16) empirically. So, given real observations of $X$ and $Y$ and for a fixed $\tau$, we can obtain a $\tau^*$ empirically. After fitting a model with the observations, we can again obtain a $\tau^*$ but with the distribution of the learned model. If $\tau^*$ and $\hat{\tau}^*$ are close, that means the learned distribution and the real data are close in terms of their left tail dependence. The right tail dependence case can be similarly handled by defining right-tail
side joint-quantile when given a fixed $\tau$ that is close to 1:

$$P(X \geq F_X^{-1}(\tau^*), Y \geq F_Y^{-1}(\tau^*)) = 1 - \tau. \quad (17)$$

Our discrepancy function is defined to describe the distance between $\tau^*$ and $\tilde{\tau}^*$. Let $\tau$ vary in \{0.95, 0.96, 0.97, 0.98, 0.99\} for upper tail dependence case and in \{0.01, 0.02, 0.03, 0.04, 0.05\} for lower tail dependence case, then ten $\tau^*$ and $\tilde{\tau}^*$ will be obtained, denoted as $\tau^*_i$ and $\tilde{\tau}^*_i$. The discrepancy function is defined as:

$$D = \sum_i (\tau^*_i - \tilde{\tau}^*_i)^2. \quad (18)$$

Now we list the discrepancy function for each pair of stocks generated by our model and by other five models in Table 1. It shows that our model gives the smallest discrepancy function comparing to other models for 5 stock pairs out of 6, indicating the advantage of our model in modeling the tail dependence structure in real data.

7 Conclusions

In this paper, we proposed a multivariate generative model to characterize flexible tail dependence structure. Our succinct model features heterogeneous and asymmetric tail dependence between all pairs of individual dimensions. It is also very easy to interpret, and easy to sample from it. The model structure has computationally scalable property, and is not prone to the error propagation issue in the parameter estimation process, as the dimensions of data grow large. We devised a moment learning algorithm to learn the parameters and further demonstrated its effectiveness. The experimental results show that our proposed model exhibits better performance compared to the copula-based benchmarks and the LT model.

Future works include studying a time-varying tail dependence model, which means that marginal parameters $\mu, u, v, \sigma$ and correlation coefficients $\Sigma_1, \Sigma_2, \Sigma_3$ can be time-varying. It is well known that the variance and covariance of financial asset returns are persistent or clustering across time. A direct idea to extend our work is that parameters $\mu, u, v, \sigma$ are coupled with a GARCH model and the correlation coefficients $\Sigma_1, \Sigma_2, \Sigma_3$ are kept constants. A more challenge extension is that one can combine our model with a dynamic conditional correlation model or a regime-switching dynamic correlation model to capture the dynamic correlation matrix or dynamic tail dependence.

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Appendix A. The Proof of Proposition

First, we give the following propositions.

**Proposition 2.** The survival function of standard normal distribution shows the following asymptotic property:

\[ \bar{F}(x) \sim f(x)/x = \frac{\exp(-x^2/2)}{\sqrt{2\pi}x} \quad \text{as } x \to +\infty \]  

**Proof.** By definition,

\[ \lim_{x \to \infty} \frac{\bar{F}(x)}{f(x)} = \lim_{x \to \infty} \frac{\bar{F}(x) - \frac{x}{\sqrt{2\pi}} \exp(-x^2/2)}{-\frac{x}{\sqrt{2\pi}} \exp(-x^2/2)} = (L’ Hospital’s Rule) \]

We only need to show \( \lim_{x \to \infty} \frac{\bar{F}(x)}{\frac{x}{\sqrt{2\pi}} \exp(-x^2/2)} = 0 \).

\[ \lim_{x \to \infty} \frac{\bar{F}(x)}{\frac{x}{\sqrt{2\pi}} \exp(-x^2/2)} = \lim_{x \to \infty} \frac{-\frac{x}{\sqrt{2\pi}} \exp(-x^2/2)}{\frac{x}{\sqrt{2\pi}} \exp(-x^2/2) + \frac{x^2}{2\sqrt{2\pi}} \exp(-x^2/2)} = (L’ Hospital’s Rule) \]

\[ = \lim_{x \to \infty} \frac{-x}{x^2 - 1} = 0 \]  

as desired. \( \square \)
Proposition 3. The survival function of log normally distributed $e^X$, where $X$ is a standard normal random variable, shows the following asymptotic property:

$$\bar{F}_{LN}(t) \sim f(ln t) / ln t = \exp\left(-\frac{(ln t)^2}{2}\right) \quad as \quad t \to +\infty$$

(24)

Proof. By definition,

$$\bar{F}_{LN}(t) = P(e^X > t) = P(X > \ln t) = \bar{F}(\ln t)$$

(25)

By Proposition 2, we can show the result directly.

Now we give the proof of Proposition 1.

Proof. For symmetry, we just show the upper tail heaviness case. By definition,

$$\lim_{t \to \infty} \frac{P(Y > t)}{P(e^{uZ_1} > t)} = \lim_{t \to \infty} \frac{P(\mu + e^{uZ_1} - e^{vZ_2} + \sigma Z_3 > t)}{P(e^{uZ_1} > t)}$$

$$= \lim_{t \to \infty} \int_{z_2} \int_{z_3} \frac{P(e^{uZ_1} > t - \mu + e^{vZ_2} - \sigma z_3)}{P(e^{uZ_1} > t)} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_3^2}{2}\right) dz_2 dz_3$$

(26)

We know

$$\lim_{t \to \infty} \frac{P(e^{uZ_1} > t - \mu + e^{vZ_2} - \sigma z_3)}{P(e^{uZ_1} > t)} = 1 \quad for \quad fixed \quad z_2, z_3$$

(27)

By dominated convergence theorem,

Equation (26) = \int_{z_2} \int_{z_3} \lim_{t \to \infty} \frac{P(e^{uZ_1} > t - \mu + e^{vZ_2} - \sigma z_3)}{P(e^{uZ_1} > t)} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_3^2}{2}\right) dz_2 dz_3$$

(28)

= 1

By Proposition 3 we can directly obtain that:

$$P(e^{uZ_1} > t) \sim \frac{u}{\sqrt{2\pi ln t}} \exp\left(-\frac{(ln t)^2}{2u^2}\right) \quad as \quad t \to +\infty$$

(29)

Now the proof is complete.

Appendix B. Power-law Tail

Now, consider the following structure:

$$Y = \mu + e^{Z_1} - e^{Z_2} + \sigma Z_3,$$  (30)

where $Z_1, Z_2$ are exponential distribution with parameter $\lambda_1, \lambda_2$, and $Z_3 \sim N(0, 1)$. This structure gives a power-law tail.
Proposition 4. The random variable \( Y \) in Equation (30) has the same right tail heaviness as \( e^{Z_1} \), where \( Z_1 \) is exponential distribution with parameter \( \lambda_1 \) and the same left tail heaviness as \( -e^{Z_2} \), where \( Z_2 \) is exponential distribution with parameter \( \lambda_2 \), namely,

\[
P(Y > t) \sim P(e^{Z_1} > t) \quad \text{as } t \to \infty \tag{31}
\]

\[
P(Y < t) \sim P(e^{Z_2} > t) \quad \text{as } t \to -\infty \tag{32}
\]

Moreover, \( Y \) has the following asymptotic property:

\[
P(Y > t) \sim t^{-\lambda_1} \quad \text{as } t \to \infty \tag{33}
\]

\[
P(Y < t) \sim (-t)^{-\lambda_2} \quad \text{as } t \to -\infty \tag{34}
\]

Proof. First, we can show that \( e^{Z_1} \), where \( Z_1 \) is exponential distribution with parameter \( \lambda_1 \) has the power-law tail:

\[
P(e^{Z_1} > t) = P(Z_1 > \ln t) = \exp(-\lambda_1 \ln t) = t^{-\lambda_1} \tag{35}
\]

Second, we show the asymptotic property:

\[
\lim_{t \to \infty} \frac{P(Y > t)}{P(e^{Z_1} > t)} = \lim_{t \to \infty} \frac{P(\mu + e^{Z_1} - e^{Z_2} + \sigma Z_3 > t)}{P(e^{Z_1} > t)} = \lim_{t \to \infty} \int_{z_2, z_3} \frac{P(e^{Z_1} > t - \mu + e^{z_2} - \sigma z_3)}{P(e^{Z_1} > t)} \lambda_2 e^{-\lambda_2 z_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) dz_2 dz_3 \tag{36}
\]

We know

\[
\lim_{t \to \infty} \frac{P(e^{Z_1} > t - \mu + e^{z_2} - \sigma z_3)}{P(e^{Z_1} > t)} = 1 \quad \text{for fixed } z_2, z_3 \tag{37}
\]

By dominated convergence theorem,

\[
\text{Equation (36)} = \int_{z_2, z_3} \lim_{t \to \infty} \frac{P(e^{Z_1} > t - \mu + e^{z_2} - \sigma z_3)}{P(e^{Z_1} > t)} \cdot \lambda_2 e^{-\lambda_2 z_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) dz_2 dz_3 \tag{38}
\]

\[
= 1
\]

This implies:

\[
P(Y > t) \sim t^{-\lambda_1} \quad \text{as } t \to \infty \tag{39}
\]

Now the proof is complete. \( \square \)