Similarities Between Classical Timelike Geodesics in a Naked Reissner-Nordstrom Singularity Background and the Behaviour of Electrons in Quantum Theory

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ABSTRACT: It is generally assumed that naked singularities must be physically excluded, as they could otherwise introduce unpredictable influences in their future null cones. Considering geodesics for a naked Reissner-Nordstrom singularity, it is found that the singularity is effectively clothed by its repulsive nature. Regarding electron as naked singularity, the size of the clothed singularity (electron) turns out to be classical electro-magnetic radius of the electron, to an observer falling freely from infinity, initially at rest. The size shrinks for an observer falling freely from infinity, with a positive initial velocity. For geodetic parameters corresponding to negative energy there are trapped geodesics. The similarity of this picture with that arising in the Quantum Theory is discussed.

KEYWORDS: Reissner-Nordstrom, Naked Singularity, Timelike Geodesics
1. Introduction

Penrose proposed the cosmic censorship hypothesis [1] so as to avoid the possibility of unpredictable influences emerging from the singularity, where physical laws break down. As he put it [2], “it is as if there is a cosmic censor board that objects to naked singularities being seen and ensures that they only appear suitably clothed by an event horizon”. Due to this conjecture, naked singularities are seldom studied seriously in themselves, though various discussions focus on the possibility of finding counter-examples to it even for singularities that arise from realistic gravitational collapse processes [3]. The geodesics of arbitrarily charged particles in a naked Reissner-Nordstrom singularity background are studied in [4]. However, it concentrates on calculating the geodesics only and not on deducing any consequences from them. While attempting to foliate the Reissner-Nordstrom geometry by flat spacelike hypersurfaces [5] for a usual black hole we found it necessary to investigate geodesics in a naked singularity background. Here we report on the striking similarity of the behaviour of these geodesics and the Quantum picture for an electron.

The fact that two such apparently different theories as General Relativity (GR) and Quantum Theory (QT) come up with unexpectedly similar features seems remarkable to us. In particular, GR dealing with point particles seems to require (as will be shown subsequently) that they acquire an extension and that their interactions involve non-local effects. This is not to say that we claim that QT derives from GR or vice versa. Rather, we note that the similarity of the pictures suggest that some more fundamental theory yields both. While the spatial extension is, in some sense apparent for a "clothed" black hole it is not directly apparent for the naked singularity. Since a GR description of an electron would be as a naked singularity, this fact is of importance.
2. Timelike Geodesics

We start by giving a brief review of unforced timelike geodesics (corresponding to the paths of uncharged test particles) in the Reissner-Nordstrom background. The metric is taken in the usual form

\[ ds^2 = e^{\nu(r)} dt^2 - e^{-\nu(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (2.1) \]

where \( e^{\nu(r)} = 1 - 2m/r + Q^2/r^2 \). The geodesic equation for \( t \) gives

\[ \frac{dt}{ds} = \dot{t} = k e^{-\nu(r)}. \quad (2.2) \]

For freely falling observers we take \( k = 1 \) so as to obtain the flat spacetime value of \( \dot{t} \) at infinity. Had there been a finite velocity at infinity, \( \dot{t} \) would have been greater than unity and consequently, we would have to take \( k > 1 \).

From Eq.(2.1), for \( \theta = \phi = \text{constant} \), we have

\[ e^{\nu(r)} \dot{t}^2 - e^{-\nu(r)} \dot{r}^2 = 1. \quad (2.3) \]

We can re-write the two requirements, Eqs.(2.2 and 2.3), in a single equation for the change of \( r \) with \( t \), as

\[ \frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \frac{\pm \sqrt{k^2 - e^{\nu(r)}}}{ke^{-\nu(r)}}. \quad (2.4) \]

It is clear that the geodesics will be defined only for \( e^{\nu(r)} < k^2 \). We see that we must take the negative root, on account of the initial attraction of the gravitational source. If we take \( k = 1 \), as required for observers falling freely from infinity (initially at rest), then there is obviously a boundary at

\[ r_b = \frac{Q^2}{2m}, \quad (2.5) \]

at which \( dr/dt \) becomes zero and after which the positive root has to be chosen.

The geodesics can be better represented in terms of a re-scaled radial parameter,

\[ r^* = \int e^{-\nu(r)} dr = \int \left( 1 - 2m/r + Q^2/r^2 \right)^{-1} dr. \quad (2.6) \]

Here, the constant of integration is chosen so that \( r^* = 0 \) at \( r = 0 \). Therefore,

\[ r^* = r + m \ln \left( \frac{r^2 - 2mr + Q^2}{Q^2} \right) + \frac{2m^2 - Q^2}{\sqrt{Q^2 - m^2}} \times \left[ \tan^{-1} \left( \frac{r - m}{\sqrt{Q^2 - m^2}} \right) - \tan^{-1} \left( \frac{m}{\sqrt{Q^2 - m^2}} \right) \right]. \quad (2.7) \]

Now, taking \( k = 1 \), Eq.(2.4) can be re-written, in \((t, r^*)\) coordinates, as

\[ \frac{dr^*}{dt} = \frac{\dot{r}^*}{\dot{t}} = \pm \sqrt{\frac{2m}{r} - \frac{Q^2}{r^2}}. \quad (2.8) \]
Notice that this would be the speed of light, if it took the value 1. It takes its extremal values at $r = 2r_b$, namely $\pm m/Q$, which are necessarily less than magnitude 1. Figure 1 shows the geodesics in a plot of $t$ against $r^*$. It is clear that the geodesics coming in from infinity turn at $r = r_b$ and go back to infinity. Thus, no timelike geodesics from infinity can enter the region $r < r_b$. That region is protected from view by its repulsive nature!

These geodesics can be displayed most meaningfully in a Carter-Penrose diagram [6]. We consider only the case $Q > m$. In this case there is only one coordinate patch required, and we only need to change coordinates to compactify them. For this purpose we define,

$$\begin{align*}
\psi &= \tan^{-1}\left\{\left(t + r^*\right)/R\right\} + \tan^{-1}\left\{\left(t - r^*\right)/R\right\}, \\
\xi &= \tan^{-1}\left\{\left(t + r^*\right)/R\right\} - \tan^{-1}\left\{\left(t - r^*\right)/R\right\},
\end{align*}$$

where $R$ is a constant with dimensions of length. Thus $r = 0$ corresponds to $\xi = 0; r = \infty$ at $t = 0$ to $\xi = \pi; t = -\infty$ to $\psi = -\pi; t = \infty$ to $\psi = \pi$. Hence the Carter-Penrose diagram representing the naked Reissner-Nordstrom singularity is an isosceles right-angled triangle standing on its vertex, with the vertical line representing the singularity. Geodesics start at $I^-$, graze the boundary, $r = r_b$, and end at $I^+$. Different geodesics correspond to different values of $\psi$ and $\xi$ at which the geodesic grazes the boundary. Some typical geodesics and the boundary, $r = r_b$, are displayed in Figure 2 (taking $Q = 2m$).
For $k > 1$, we get geodesics corresponding to an observer with a positive velocity at infinity. Putting $k^2 = 1 + \varepsilon$, we find that the geodesics will not turn back at $r = r_b$, but rather at $r_c = \left[ -m \pm \sqrt{m^2 + \varepsilon Q^2} \right] / \varepsilon$. It is easily verified that only the +ve root is valid. Again, $dr^*/dt$ takes its maximum value at $r = 2r_b$, but now the extremal values are

$$\left( \frac{dr^*}{dt} \right)_{\text{max}} = \pm \sqrt{\frac{\varepsilon + m^2/Q^2}{1 + \varepsilon}}.$$

Clearly, for small $\varepsilon$ this tends to the previous value, but for large $\varepsilon$ it tends to 1. Now the boundary moves back from $r_b$ to $r_c \approx r_b - \varepsilon Q^4/8m^3$, see Figure 3, and the “clothed” singularity appears smaller to a faster moving observer. Note that here $k$ is bounded from below by 1, but is not bounded from above. In the limit as $k$ goes to infinity, $r_c$ goes to zero. The geodesics have the same general behaviour as in the previous case. Figure 4 shows some typical geodesics in the Carter-Penrose diagram for $k > 1$.

For $k < 1$ we put $k^2 = 1 - \varepsilon$. The possible reversals are at $r_{\pm} = \left[ m \pm \sqrt{m^2 - \varepsilon Q^2} \right] / \varepsilon$. In this case the boundary will move forward from $r_b$ to $r_- \approx r_b + \varepsilon Q^4/8m^3$ while the limit at infinity moves back to $r_+ \approx 2m/\varepsilon$, see Figure 5, and the “clothed” singularity appears larger. These geodesics again start at $I^-$ going along $r = r_+$ and then grazing $r = r_-$ and going on to $I^+$ along $r_+$. Again there can be different choices of $\psi$ where the geodesic grazes the inner boundary. Clearly, $k$ is bounded from below by the requirement that $(m^2 - \varepsilon Q^2)$ be +ve. Thus, for a given $m$ and $Q$, $\varepsilon \leq m^2/Q^2$. At $\varepsilon = m^2/Q^2$ we get
Figure 5: Two geodesics in the \((t,r^*)\) co-
ordinates for \(k^2 = 0.9\). Now the geodesics
start at \(r = r_+\) (dashed line) in the infinite
past and go in to \(r = r_-\) before going back
out to \(r = r_+\). We have taken one geodesic,
g_1, touching the inner boundary at \(t = 0\),
and the other, \(g_2\), at \(t = 10\). Notice that
the inner boundary lies outside the classical
electromagnetic radius.

Figure 6: Two geodesics for \(k^2 = 0.9\) in
the Carter-Penrose diagram. Here \(g_1\) is the
same as before (touching \(r_- \) at \(\psi = 0\), \(\xi =
0.98\)) but we have chosen a new geodesic, \(g_3\),
instead of \(g_2\) (touching \(r_- \) at \(\psi = 1.43\), \(\xi =
0.55\)), to be able to show the relevant features
in both diagrams. The latter geodesic would
touch \(r_- \) at \(t = 1\) in the previous diagram.
The geodesics still start at \(I^-\), but now go
along \(r = r_+\), touch \(r = r_-\) and go on to \(I^+\)
along \(r = r_+\).

\[ r_+ = r_- = 2r_b. \] This gives the geodesic as \(r = 2r_b\). In the other limit, as \(k\) tends to 1 we
see that the outer limit, \(r_+\), tends to infinity. In general, the geodesics are bounded by the
two boundaries \(r = r_+\) and \(r = r_-\), being asymptotic to the former and tangential to the
latter. Some typical geodesics are displayed in a Carter-Penrose diagram, in Figure 6.

We need to interpret the significance of \(k\) or \(\varepsilon\). When \(k = 1\), we have the energy at
infinity equal to the rest energy. Thus these geodesics correspond to zero kinetic energy.
For \(k > 1\) there is extra energy of motion at infinity and hence these geodesics correspond
to faster moving observers. However, for \(k < 1\) the energy of motion is less than zero!
Hence these geodesics must correspond to negative energy. In fact, as \(r \to \infty\), \(i \to k\). Also,
\(i\) corresponds to the ratio of total energy to rest energy. Thus \(\varepsilon = k^2 - 1\) corresponds to
\((KE/RE)/(2+KE/RE)\), where \(KE\) is the kinetic energy and \(RE\) the rest energy, at infinity.
In the high energy limit, then, \(\varepsilon\) corresponds to \((KE/RE)^2\), while in the low energy
limit it corresponds to \(2(KE/RE)\). In the intermediate energy it comes out approximately
\(3(KE/RE)\). Now \(k < 1\) corresponds to negative \(\varepsilon\), and hence negative energy as defined at
infinity. It must be borne in mind that these geodesics are never at infinity, but remain
trapped near the singularity.
3. The Electron as a Naked Singularity

In gravitational units \((c = G = 1)\), \(Q = 1.4 \times 10^{-34}\)cm and \(m = 6.8 \times 10^{-59}\)cm for the electron. Thus, if we were to take the general relativistic (GR) description of the electron seriously, we must consider it as a naked singularity. Of course, we should really treat the electron as a charged Kerr (or Kerr-Newmann, K-N) singularity and not a Reissner-Nordstrom singularity. However, for the present purposes, the essential features are already highlighted by the simpler analysis.

We see that here \(r_b\) is the classical electromagnetic radius, \(1.4 \times 10^{-13}\) cm. This is the size the electron would appear to be, to an observer at rest at infinity. However, a faster moving observer would see it shrunk arbitrarily smaller to \(r_c\). In the high-energy limit, the size would decrease to \(Q/\sqrt{\varepsilon}\), thus decreasing inversely as the kinetic energy. This is strongly reminiscent of QT!

One is used to non-local effects in GR, where the global aspect of the theory plays a fundamental role. Here, there will be non-local effects associated with the electron corresponding to the negative energy paths. In particular, an electron passing through a single slit will behave differently from one passing through a double slit, because of the negative energy paths that may be blocked or pass through the second slit. Once again, strongly reminiscent of the QT! Remember that the further out the negative energy paths go, the less the energy associated with them. As such, they would be disturbed more readily. Hence, the non-local effects of the electron would be more difficult to maintain further out.

Since the negative energy paths can not reach out to infinity, they can not be seen by outside observers. What, then, is their significance? They may still be physically relevant when dealing with the interaction of two electrons. We have no means available of getting an exact solution for the two electrons. It seems reasonable to conjecture, however, that the approximate solution would allow the two electrons to interact through their negative energy paths as well as their positive energy paths. This would provide a correction to the usual calculation of the interaction. This is strongly reminiscent of the renormalisation calculation involving virtual particles!

It had been demonstrated earlier [7] that for the Kerr metric the total angular momentum and the axial component are well defined, but the other two components are not. This, too, is strongly reminiscent of the QT!

4. Problems — Incorporation of Angular Momentum and Dealing with Protons

It is clear that we can not neglect the spin angular momentum of the electron. The spin angular momentum per unit mass of the electron is, \(a = 1.9 \times 10^{-11}\) cm. Thus, if the \(a\) enters into the calculations like \(Q\), the scale will be literally astronomical. As we have no analysis for the naked K-N singularity so far, we are unable to say whether, or not, the very high \(a\) of the electron would create a problem for us. It seems likely to us that the dragging
of inertial frames will change the way a enters into the calculations for the geodesics, as compared with $Q$.

There is another worry. While discussing various aspects of the behaviour of electrons treated as naked singularities we have remarked on its similarity with that in Quantum Theory. However, there is no way that Planck’s constant can enter into the analysis. Of course, if we incorporate the spin angular momentum Planck’s constant will automatically enter, but will not be provided by this analysis.

If electrons can be treated as naked singularities, why not protons? If they could be so treated, they should have a size of $7 \times 10^{-16}$ cm. This does not seem reasonable. Since the proton already shows structure at a scale of $10^{-13}$ cm, it can not be thought of as a point particle. In fact, it is best described as a bound state of three quarks, which could perhaps be thought of as three point particles. It would not, then, be described by one, but three, naked singularities close together. Further, these singularities would not be Reissner-Nordstrom singularities, but Einstein-Yang-Mills singularities. Clearly, there can be no exact description of this situation. Though there is a solution available to the Einstein-Yang-Mills equations for a single source, it is not exact. (In fact it does not even have a singularity, though it is a solution for a point source.) As such, there is no reason to suppose that the existence of the proton poses a problem for our argument treating the electron as a naked singularity. It must be admitted, however, that there is much work required to verify whether this proposal would be workable in a more general context, not only including the K-N, but other naked singularities and finding methods for dealing with approximate solutions rigorously.

Acknowledgments

One of us (AQ) is most grateful to David Finkelstein for very valuable criticism of the manuscript and Prof. Remo Ruffini for hospitality at ICRA, where he got the opportunity to meet Prof. Finkelstein.

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