The universal valuation of Coxeter matroids

Christopher Eur, Mario Sanchez and Mariel Supina

Abstract

Coxeter matroids generalize matroids just as flag varieties of Lie groups generalize Grassmannians. Valuations of Coxeter matroids are functions that behave well with respect to subdivisions of a Coxeter matroid into smaller ones. We compute the universal valuative invariant of Coxeter matroids. A key ingredient is the family of Coxeter Schubert matroids, which correspond to the Bruhat cells of flag varieties. In the process, we compute the universal valuation of generalized Coxeter permutahedra, a larger family of polyhedra that model Coxeter analogues of combinatorial objects such as matroids, clusters, and posets.

1. Introduction

Let $V$ be a real finite-dimensional vector space. For a polyhedron $P \subseteq V$, let $1_P : V \to \mathbb{Z}$ be the indicator function defined by $1_P(x) = 1$ if $x \in P$ and 0 otherwise. For a family $\mathcal{P}$ of polyhedra in $V$, let its indicator group $\mathbb{I}(\mathcal{P})$ be the $\mathbb{Z}$-submodule of $\mathbb{Z}^V$ defined by

$$\mathbb{I}(\mathcal{P}) := \left\{ \sum_{P \in \mathcal{P}} a_P 1_P \in \mathbb{Z}^V \mid a_P \in \mathbb{Z}, \text{ all but finite many coefficients } a_P \text{ are zero} \right\}.$$

Definition 1.1. A function $f : \mathcal{P} \to A$ from a family of polyhedra $\mathcal{P}$ to an abelian group $A$ is valuative\footnote{There are several variants of valuative functions in the literature. Valuative functions as we defined here are sometimes called strongly valuative functions.} if there is a $\mathbb{Z}$-linear map $\tilde{f} : \mathbb{I}(\mathcal{P}) \to A$ such that $\tilde{f}(1_P) = f(P)$ for all $P \in \mathcal{P}$, or equivalently, if

$$\sum_{i=1}^k a_i f(P_i) = 0 \quad \text{whenever} \quad \sum_{i=1}^k a_i 1_{P_i} = 0 \quad \text{for} \quad a_1, \ldots, a_k \in \mathbb{Z} \text{ and } P_1, \ldots, P_k \in \mathcal{P}.$$

We say that $f$ is a valuation on $\mathcal{P}$ in this case, and abuse the notation by writing $f$ also for $\tilde{f}$.

Valuative functions are fundamental objects in convex geometry, where they function as analogues of measures on convex bodies; see \cite{28} for a survey. For (lattice) polyhedra, valuations provide a bridge between geometry and combinatorics particularly in the context of Ehrhart theory; see \cite{24} for a survey. The interaction between geometry and combinatorics is further strengthened in the study of valuations of (extended) generalized permutahedra.

The set of (extended) generalized permutahedra is a family of polyhedra that model several combinatorial objects including matroids, clusters, and posets \cite{1, 33}. Valuative functions
of these polyhedra were studied in [6, 16] as invariants that behave well with respect to subdividing the associated combinatorial objects into smaller ones. In particular, matroid subdivisions have rich connection to geometry, such as compactifications of fine Schubert cells [25–27], tropical geometry [10, 36], and the $K$-theory of Grassmannians [17, 37]. The valuativeness of many matroid invariants, such as the beta invariant and the Tutte polynomial, is a witness to such geometric connections. Most notably, the Schubert decomposition of Grassmannians appears in the $G$-invariant of matroids, an invariant that the authors of [16] establish as the universal valuative invariant of matroids.

We study valuative functions of the set of (extended) generalized Coxeter permutohedra. These polyhedra originate in Coxeter combinatorics, which recognizes that combinatorial objects associated to generalized permutohedra are inherently related to permutation groups, and accordingly studies their Coxeter analogues in arbitrary reflection groups. Examples of such Coxeter analogues include Coxeter matroids [8, 20], clusters [22], and posets [35]. The study of generalized Coxeter permutohedra as polyhedral models of these Coxeter combinatorial objects was initiated in [5]. Focusing on Coxeter matroids, we define the $G$-invariant for Coxeter matroids, and show that it is the universal valuative invariant. A key ingredient is the family of Coxeter matroids that correspond to Bruhat cells of flag varieties.

1.1. Main results

Let $W$ be a finite reflection group with the associated root system $\Phi = (V, R)$ consisting of roots $R$ in a vector space $V$. Let $\Pi_\Phi \subset V$ be a $\Phi$-permutohedron, which is the convex hull of the $W$-orbit of a general point in $V$. Its normal fan is the Coxeter complex $\Sigma_\Phi$.

**Definition 1.2** [5, Definition 4.3]. An extended generalized $\Phi$-permutohedron is an extended deformation of $\Pi_\Phi$, that is, a (possibly unbounded) polyhedron $P \subseteq V$ such that each cone of its normal fan $\Sigma_P$ is a union of cones of $\Sigma_\Phi$. Denote by $GP_\Phi^+$ the set of all extended generalized $\Phi$-permutohedra.

Our first main theorem explicitly describes the universal valuative function of $GP_\Phi^+$ in terms of the Coxeter root cones and tight containments. Let $\Sigma_\phi^\vee = \{\sigma^\vee \mid \sigma \in \Sigma_\Phi\}$ be the set of Coxeter root cones, which are the dual cones of cones in $\Sigma_\Phi$, and let $\text{tran}(\Sigma_\phi^\vee) = \{C + v \mid C \in \Sigma_\phi^\vee, v \in V\}$ be the set of all affine Coxeter root cones, which are translates of cones in $\Sigma_\phi^\vee$. An affine Coxeter root cone $C + v$ is said to tightly contain a polyhedron $P \subseteq V$ if

$$C + v \supseteq P \quad \text{and} \quad \text{lineal}(C + v) \cap P \neq \emptyset$$

(see Figure 3 for an illustration).

**Theorem A.** Write $\{e_{C + v}\}_{C + v \in \text{tran}(\Sigma_\phi^\vee)}$ for the standard basis of $\mathbb{Z}^{\oplus \text{tran}(\Sigma_\phi^\vee)}$. The function

$$F : GP_\Phi^+ \rightarrow \mathbb{Z}^{\oplus \text{tran}(\Sigma_\phi^\vee)} \quad \text{defined by} \quad P \mapsto \sum_{C + v \in \text{tran}(\Sigma_\phi^\vee) \text{ tightly containing } P} e_{C + v}$$

is the universal valuative function in the sense that for any valuative function $f : GP_\Phi^+ \rightarrow A$ to an abelian group $A$, there exists a unique map $\varphi : \mathbb{Z}^{\oplus \text{tran}(\Sigma_\phi^\vee)} \rightarrow A$ such that $\varphi \circ F = f$.

We prove Theorem A by establishing a more general statement Theorem 2.10, which provides the universal valuative function for extended deformations of an arbitrary polyhedron. Here, a useful tool we develop is Theorem 2.3, which states that tight containments define valuations on extended deformations. For a combinatorial application of Theorem A when $\Phi$ is a type
A root system, see [7], whose authors show further that $\mathcal{F}$ is a Hopf monoid morphism, and consequently construct new valuations on matroid polytopes, poset cones, and nestohedra.

A feature unique to $\text{GP}^+_{\Phi}$ not shared by extended deformations of an arbitrary polyhedron is the action of the group $W$ on $\text{GP}^+_{\Phi}$, induced by the action of $W$ on $V$. A valuative function $f$ on $\text{GP}^+_{\Phi}$ is a valuative invariant if $f(w \cdot P) = f(P)$ for every $w \in W$ and $P \in \text{GP}^+_{\Phi}$. The universal valuative invariant of $\text{GP}^+_{\Phi}$ is derived from Theorem A in Corollary 3.3.

We then turn to Coxeter matroids, which form a distinguished $W$-invariant subfamily of $\text{GP}^+_{\Phi}$. For $I$ a subset of a set of simple roots of $\Phi$, let $W_I$ be the corresponding parabolic subgroup of $W$ generated by the reflections corresponding to $I$. Write $\leq$ for the Bruhat order on $W/W_I$, and for $w \in W$ and $B, B' \in W/W_I$, write $B \leq^w B'$ to mean $w^{-1}B \leq w^{-1}B'$.

**Definition 1.3 [8, 6.1.1].** For $I$ a subset of a set of simple roots of $\Phi$, a Coxeter matroid of type $(\Phi, I)$, or a $(\Phi, I)$-matroid, is a subset $M \subseteq W/W_I$ such that for every $w \in W$ there exists a unique $\leq^w$-minimal element in $M$. The $\leq^w$-minimal element of $M$ is denoted $\text{min}^w(M)$.

By setting $W$ to be the permutation group and $W_I$ a maximal proper parabolic subgroup, one recovers the usual notion of matroids defined by Whitney [39] and independently by Nakasawa [31]. We will say ‘ordinary matroids’ to distinguish this usual notion of matroids from Coxeter matroids. Like ordinary matroids, Coxeter matroids have several descriptions arising from different perspectives [8, 19].

- (Coxeter theory) Definition 1.3 of Coxeter matroids in terms of Bruhat order generalizes the characterization of ordinary matroids by the greedy algorithm optimization.
- (Lie theory) If $\Phi$ is crystallographic, (realizations of) Coxeter matroids correspond to points on the flag varieties of $\Phi$, just as (realizations of) ordinary matroids correspond to points on Grassmannians.
- (Polyhedral geometry) Coxeter matroids admit polyhedral models that are characterized in terms of edges being parallel to the roots of $\Phi$ (see Theorem 3.4). This generalizes the theorem of [18] which characterized the base polytopes of ordinary matroids in terms of edges being parallel to $e_i - e_j \in \mathbb{R}^n$ for some $i \neq j \in \{1, \ldots, n\}$.

The last point above implies that the polyhedral models of Coxeter matroids of type $(\Phi, I)$ form a subfamily of $\text{GP}^+_{\Phi}$, which we denote by $\text{Mat}_{\Phi, I}$. Our second main theorem computes the universal valuative invariant of Coxeter matroids.

**Theorem B.** Write $\{U_B\}_{B \in W/W_I}$ for the standard basis of $\mathbb{Q}^{W/W_I}$. The function

$$
\mathcal{G} : \text{Mat}_{\Phi, I} \to \mathbb{Q}^{W/W_I} \text{ defined by } M \mapsto \sum_{w \in W} U_{w^{-1}\text{min}^w(M)}
$$

is the universal valuative invariant in the sense that for any valuative invariant $g : \text{Mat}_{\Phi, I} \to A$ to a $\mathbb{Q}$-vector space $A$, there exists a unique linear map $\psi : \mathbb{Q}^{W/W_I} \to A$ such that $\psi \circ \mathcal{G} = g$.

We prove Theorem B by establishing properties of Coxeter Schubert matroids (Definition 3.11), which correspond to Bruhat cells of flag varieties (Remark 3.12). As a corollary, we establish that Coxeter Schubert matroids form a basis for the indicator space of isomorphism classes of Coxeter matroids (Corollary 3.16). This parallels the fact that Bruhat cells give a basis of the cohomology ring of flag varieties. As another application, for a class of Coxeter matroids known as delta-matroids, we show that a well-studied invariant called the interlace polynomial (Definition 3.5) is a specialization of the $\mathcal{G}$-invariant and hence a valuative invariant (Theorem 3.6).
1.2. Relation to the work of Derksen and Fink

For an ordinary matroid $M$ of rank $r$ on a ground set $[n] = \{1, \ldots, n\}$, Derksen and Fink [16] define the $G$-invariant of $M$ as follows: For a permutation $w \in S_n$, let $X_i := \{w(1), \ldots, w(i)\}$ for $i = 1, \ldots, n$ and $X_0 := \emptyset$, and define $r_M(w) \in \{0, 1\}^n$ by

$$r_M(w)_i := \text{rk}_M(X_i) - \text{rk}_M(X_{i-1}) \quad i = 1, \ldots, n,$$

where $\text{rk}_M$ is the rank function of $M$. Let $\{U_\alpha \mid \alpha \in \{0, 1\}^n\}$ be the standard basis of $\mathbb{Q}\{0,1\}$.

Then the $G$-invariant of $M$ was defined in [16] as

$$G(M) := \sum_{w \in S_n} U_{r(w)}.$$

Let us relate this to our definition of $G$-invariant in Theorem B. For $w \in S_n$, when the ground set $[n]$ is given weights $w^{-1}(i)$ for each $i \in [n]$, the greedy algorithm for matroids implies that the set $\{w(i) \in [n] \mid r(w)_i = 1\}$ is the minimal basis of $M$. In other words, identifying elements of $\{2, 1\}^n$ with subsets of $[n]$, we have $r(w) = w^{-1} \min^w(M)$, so that the $G$-invariant in the sense of [16] is exactly our $G$-invariant

$$G(M) = \sum_{w \in S_n} U_{w^{-1} \min^w(M)}.$$

While our work is thus a generalization of the work of Derksen and Fink, several arguments in [16] fail fundamentally in the general Coxeter setting. We highlight two differences between our work and [16].

- Rank functions: The rank function characterization of polymatroids and matroids plays a central role throughout [16]. While Coxeter submodularity was defined in [5], there is no known characterization of which $(\Phi, I)$-submodular functions are rank functions of $(\Phi, I)$-matroids. We circumvent this by introducing the notion of tight containment and translating certain results of [16] into this geometric language.

- Schubert matroids: A key feature of the type A root system utilized in [16] is that Schubert matroid polytopes are the intersections of uniform matroid polytopes with the vertex cones of the standard permutahedron. In other types, however, these intersections are not necessarily Coxeter matroid polytopes, even in minuscule types (Remark 3.15). We circumvent this by borrowing tools from 0-Hecke algebras to establish properties of Coxeter Schubert matroids, along with a new argument for proving Theorem B.

2. Valuative functions of extended deformations

Since generalized Coxeter permutahedra are extended deformations of certain polytopes, we first study valuative functions of extended deformations of an arbitrary polyhedron. In §2.1, we introduce tight containments as useful valuative functions of extended deformations. In §2.2, we compute a basis for the indicator group of the set of extended deformations of a polytope. In §2.3, we establish Theorem 2.10, which includes Theorem A as a special case.

Throughout, let $V$ be a finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$.

2.1. Extended deformations and tight containments

For a hyperplane $H = \{x \in V \mid \langle y, x \rangle = a\}$ defined by $y \in V$ and $a \in \mathbb{R}$, we write $H^+ = \{x \in V \mid \langle y, x \rangle \geq a\}$ and $H^- = \{x \in V \mid \langle y, x \rangle \leq a\}$ for its two closed half-spaces. A polyhedron $Q \subseteq V$ is a finite intersection of closed half-spaces, and is a polytope if it is compact.

**NOTATION.** For a polyhedron $Q \subseteq V$ and a face $F$ of $Q$, we set:
Figure 1 (colour online). Tangent and affine tangent cones of two faces of the polygon $Q$.

Figure 2 (colour online). From left to right: A polytope $Q$, a deformation of $Q$, and an extended deformation of $Q$.

- $\Sigma_Q$ to be the outer normal fan of $Q$;
- $\sigma_F$ to be the cone of $\Sigma_Q$ corresponding to the face $F$, which is the cone of linear functionals that attain their maximum value in $Q$ exactly on the face $F$;
- $C_F$ to be the tangent cone of $Q$ at $F$, which is the cone dual to $\sigma_F$, or explicitly, is $\text{Cone}(v' - v \mid v \in F, v' \in Q)$;
- $C_F + F$ to be the affine tangent cone of $Q$ at $F$, which is the Minkowski sum of $C_F$ and $F$, or equivalently, is the translate $C_F + v$ of the tangent cone at $F$ by any $v \in F$;
- $\Sigma_Q' := \{C_F \mid F \text{ is a face of } Q\} = \{\sigma^\vee \mid \sigma \in \Sigma_Q\}$ to be the set of tangent cones of $Q$; and
- $\text{tran}(\Sigma_Q') := \{C + v \mid C \in \Sigma_Q', v \in V\}$ to be the set of all translates of tangent cones of $Q$.

**Definition 2.1.** Let $Q \subseteq V$ be a polyhedron. A (possibly unbounded) polyhedron $P \subseteq V$ is an extended deformation of $Q$ if each cone of $\Sigma_P$ is a union of cones of $\Sigma_Q$. The polyhedron $P$ is further a deformation of $Q$ if the supports of $\Sigma_P$ and $\Sigma_Q$ are equal, or equivalently, if $\Sigma_P$ is a coarsening of $\Sigma_Q$ (see Figure 2).

We write:
- $\text{Def}^+(Q)$ for the set of all extended deformations of $Q$; and
- $\text{Def}(Q)$ for the set of all deformations of $Q$.

We now introduce the notion of tight containments, which will define useful valuative functions on the set of extended deformations. For a cone $C \subseteq V$, let lineal($C$) be its lineality space, which is the maximal subspace of $V$ contained in $C$.

**Definition 2.2.** Let $C \subseteq V$ be a cone and $v \in V$. We say that the translate $C + v$ of the cone $C$ tightly contains a polyhedron $P$ if $C + v$ contains $P$ and $P \cap (\text{lineal}(C) + v) \neq \emptyset$.

See Figure 3 for illustrations of examples and nonexamples of tight containments.
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Figure 3 (colour online). Examples (3I, 3II) and non-examples (3III, 3IV) of tight containments.

Theorem 2.3. Let $F$ be a face of a polyhedron $Q \subset V$, and $v \in V$. Then the tight containment indicator function

$\overline{j_{C_F+v}} : \mathbb{I} (\text{Def}^+(Q)) \to \mathbb{Z}$ defined by $\overline{j_{C_F+v}}(P) := \begin{cases} 1 & \text{if } C_F + v \text{ tightly contains } P \\ 0 & \text{otherwise} \end{cases}$

is valuative.

Testing for tight containment in a cone does not in general define a valuative function for an arbitrary family of polyhedra. For example, consider the cone $C = \text{Cone}(e_1, e_1 + e_2) \subset \mathbb{R}^2$ along with polyhedra $P_1 = \text{Conv}(0, e_2)$ and $P_2 = \text{Conv}(0, -e_2)$. One has the relation $1_{P_1} + 1_{P_2} = 1_{P_1 \cup P_2} + 1_{P_1 \cap P_2}$, but $0 = \overline{j_C}(P_1) + \overline{j_C}(P_2) \neq \overline{j_C}(P_1 \cup P_2) + \overline{j_C}(P_1 \cap P_2) = 1$.

Our proof of Theorem 2.3 broadly consists of three steps: (i) tight containment for deformations can be expressed as a limit of containments in closed half-spaces, (ii) containment in a closed half-space is valuative, and (iii) taking a limit preserves valuativeness. Let us begin by noting that the limit of a sequence of valuations with an elementary stabilization property is again a valuation.

Lemma 2.4. Let $P$ be a family of polyhedra in $V$, and $A$ an abelian group. Suppose $(f_0, f_1, f_2, \ldots)$ is a sequence of valuations $P \to A$ with the property that for any polyhedron $P \in \mathcal{P}$ there exists an integer $N_P \geq 0$ such that $f_i(P) = f_{N_P}(P)$ for all $i \geq N_P$. Then the function $f : \mathcal{P} \to A$ defined by $f(P) := f_{N_P}(P)$ is valuative.

Proof. Suppose $a_1, \ldots, a_k \in \mathbb{Z}$ and $P_1, \ldots, P_k \in \mathcal{P}$ satisfies $\sum_{i=1}^k a_i 1_{P_i} = 0$. We need to show that $\sum_{i=1}^k a_i f_i(P_i) = 0$. Writing $N = \max\{N_{P_i} \mid 1 \leq i \leq k\}$, we have $\sum_{i=1}^k a_i f_i(P_i) = \sum_{i=1}^k a_i f_N(P_i)$ by definition of $f$, and $\sum_{i=1}^k a_i f_N(P_i) = 0$ since $f_N$ is valuative.

Notation. We denote the function $f$ in Lemma 2.4 by $\lim_{i \to \infty} f_i$.

For any family of polyhedra, containment in a closed half-space is a valuation.

Lemma 2.5. Let $H^+ \subset V$ be a closed affine half-space, and let $\mathcal{P}$ be a family of polyhedra in $V$. The function $j_{H^+} : \mathcal{P} \to \mathbb{Z}$ defined as

$\overline{j_{H^+}}(P) := \begin{cases} 1 & \text{if } P \subseteq H^+ \\ 0 & \text{otherwise} \end{cases}$

is a valuation on $\mathcal{P}$.

Proof. If a function $f$ on the family of all polyhedra is valuative, then its restriction $f|_{\mathcal{P}}$ is a valuation on $\mathcal{P}$. We thus show that $j_{H^+}$ is a valuative function on the family of all polyhedra.
in $V$. For the family of all polyhedra, [29, Propositions 3.2 & 3.3] states that a function $f$ is valuative if and only if it satisfies

$$f(P) + f(P \cap L) = f(P \cap L^+) + f(P \cap L^-)$$

for every polyhedron $P \subseteq V$ and hyperplane $L \subset V$. Now, for an arbitrary polyhedron $P \subseteq V$ and a hyperplane $L \subset V$, the relation

$$j_{H^+}(P) + j_{H^-}(P \cap L) = j_{H^+}(P \cap L^+) + j_{H^-}(P \cap L^-)$$

follows from observing that:

$$P \cap L^+ \subseteq H^+ \text{ or } P \cap L^- \subseteq H^+ \Rightarrow P \cap L \subseteq H^+, \text{ and}$$

$$P \cap L^+ \subseteq H^+ \text{ and } P \cap L^- \subseteq H^+ \Rightarrow P \subseteq H^+.$$ 

The result follows. \hfill $\square$

Lastly, we note the following alternate characterization of tight containment for extended deformations. See Figure 4 for an illustration.

**Lemma 2.6.** Let $F$ be a face of a polyhedron $Q \subset V$, and $v \in V$. Let $y \in \text{relint}(\sigma_F)$ be a linear functional that is maximized on $Q$ at the face $F$, and let $H$ be the affine hyperplane \{ $x \in V$ | $\langle y, x \rangle = \langle y, v \rangle$ \}. Then, for $P \in \text{Def}^+(Q)$, the affine cone $C_F + v$ tightly contains $P$ if and only if $P \subseteq H^-$ and $\emptyset \subseteq P \cap H \subseteq \text{lineal}(C_F) + v$.

**Proof.** First, we note that $(C_F + v) \cap H = \text{lineal}(C_F) + v$. This is equivalent to the equality of subspaces $\text{lineal}(C_F) = \{ x \in C_F | \langle y, x \rangle = 0 \}$, which is clear since both subspaces can be written as $\text{span}\{ w - w' | w, w' \in \text{vert}(F) \}$.

Next, suppose that $P \in \text{Def}^+(Q)$ is tightly contained in $C_F + v$. Then $(\text{lineal}(C_F) + v) \cap P$ is nonempty, so $H \cap P$ is also nonempty. Furthermore, $P \subseteq C_F + v$ implies $P \subseteq H^-$. Finally, $P \cap H \subseteq (C_F + v) \cap H = \text{lineal}(C_F) + v$.

For the converse, suppose that $P \subseteq H^-$ and $\emptyset \subseteq P \cap H \subseteq \text{lineal}(C_F) + v$. Since $P \cap \text{lineal}(C_F) + v \neq \emptyset$, it remains to show that $P \subseteq C_F + v$. First, since $P \subseteq H^-$, the intersection $P \cap H$ is a face $F'$ of $P$, and $y$ is contained in the normal cone of $F'$. As $P$ is a deformation of $Q$, the normal cone of $F'$ is a union of normal cones in $\Sigma_Q$ including $\sigma_F$. Since the normal cone of $F'$ in $\Sigma_P$ contains the normal cone of $F$ in $\Sigma_Q$, the tangent cone $C_F$ of $Q$ at $F$ must contain the tangent cone $C_{F'}$ of $P$ at $F'$. Let $u$ be a vertex of $F'$, so
is well defined because the normals $\mathcal{Y}$ and that $\ell$ we can construct a sequence of affine half-spaces $H^\prime_i$ are drawn as solid lines, and the translations $H^\prime_i$ are drawn as dashed lines. Observe that not only do the dashed hyperplanes $H^\prime_i$ get closer and closer to their solid counterparts $H_i$, but the intersection points $H^\prime_i \cap H$ also approach $\ell$, which is a stronger condition.

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Since the ‘translation by $v$’ sending $P \in \text{Def}^+(Q)$ to $\text{relint}(P - v)$ is valuative, we can assume without loss of generality that $v = 0$. Let $y \in \text{relint}(\sigma_F)$ be a linear functional that is maximized on $Q$ at the face $F$, and define the hyperplane $H := \{x \in V \mid \langle y, x \rangle = 0\}$ and the subspace $\ell := \text{lineal}(C_F)$. By Lemma 2.6, we can express the tight containment function $\mathcal{J}_{C_F}$ as

$$\mathcal{J}_{C_F}(P) = \begin{cases} 1, & P \subseteq H^- \quad \text{and} \quad \emptyset \subseteq P \cap H \subseteq \ell \\ 0, & \text{otherwise.} \end{cases}$$

To show that this is a valuation, we will express $\mathcal{J}_{C_F}$ as a limit of valuations in the sense of Lemma 2.4. We will construct a sequence of pairs of half-spaces $(H^-_i, H^-_i)$ where for large enough $i$, the polyhedra $P \in \text{Def}^+(Q)$ contained in $H^-_i$ but not in $H^+_i$ are exactly those with $\mathcal{J}_{C_F}(P) = 1$; in other words, $\mathcal{J}_{C_F} = \lim_{i \to \infty}(j_{H^-_i} - j_{H^-_i})$.

First, consider the case where $C_F$ itself is a half-space, that is, where the face $F$ has codimension 1. Then the half-space $H^-$ is equal to $C_F$, and $\text{lineal}(C_F) = H$. In this case, we can construct a sequence of affine half-spaces $\{H^-_i\}$ where each $H^-_i$ is contained in $H^-$ and $\lim_{i \to \infty} H^-_i = H^-$. The sequence of functions $j_{C_F} = \lim_{i \to \infty}(j_{H^-_i} - j_{H^-_i})$ is a valuation. Here the functions $j_{H^-}$ and $j_{H^+_i}$ are the half-space containment functions as defined in Lemma 2.5.

Now suppose that $C_F$ is not a half-space, and let $\{y_i\}_{i \in \mathbb{N}}$ be a sequence of vectors in $\text{relint}(\sigma_F)$ converging to $y$ that are not scalar multiples of $y$. For each $i$, define the hyperplane $H^+_i := \{x \in V \mid \langle y_i, x \rangle = 0\}$, $\mathcal{J}_{C_F}$ converges pointwise to $H^-$ and that $\ell \subseteq H^+_i$ for all $i$. Let $d(X, Y)$ denote the minimum distance between two subsets $X, Y \subseteq V$, and construct a second sequence of affine hyperplanes denoted by $H^+_i$ with the properties that $H^-_i \subseteq H^+_i$ and $\lim_{i \to \infty} d(H^+_i \cap H, \ell) = 0$ (see Figure 5). The latter condition is well defined because the normals $y_i$ were chosen not be scalar multiples of $y$ so that the intersections $H^+_i \cap H$ are nonempty. These conditions on $H^+_i$ guarantee that for any set $X \subset H$ that is bounded away from $\ell$, for sufficiently large $i$, if $H^-_i$ contains $X$, then so does $H^-_i$. 

![Figure 5](colour online). The first three elements in an example of the sequences of hyperplanes \{(H_i)\} and \{(H'_i)\}, as described in the proof of Theorem 2.3. The hyperplanes $H_i$ are drawn as solid lines, and the translations $H'_i$ are drawn as dashed lines. Observe that not only do the dashed hyperplanes $H'_i$ get closer and closer to their solid counterparts $H_i$, but the intersection points $H'_i \cap H$ also approach $\ell$, which is a stronger condition.
Lemma 2.4. We do this by partitioning $\text{Def}^+(Q)$ into four subfamilies (Figure 6) and showing convergence for each subfamily. We then invoke Lemma 2.4 to show that $\overline{j_{C_F}}$ is a valuation. Let $P \in \text{Def}^+(Q)$.

(I) $P \subseteq H^-$ and $\emptyset \subseteq P \cap H \subseteq \ell$.

By Lemma 2.6, this case can be equivalently stated as the case where $P$ is tightly contained in $C_F$; hence we want to show that $\lim_{i \to \infty}(j_{H_i^-}(P) - j_{H_i^-}(P)) = 1$. Since $y_i \in \text{relint}(\sigma_F)$, it follows that $C_F$ (and hence $P$) is contained in $H_i^-$ for all $i$. Furthermore, since $P \cap \ell \neq \emptyset$, it holds that $P \nsubseteq H_i^-$ for all $i$. Hence $j_{H_i^-}(P) - j_{H_i^-}(P) = 1$ for all $i$.

(II) $P \nsubseteq H^-$.

Since the $H_i^-$ converge to $H^-$, for sufficiently large $i$ we will have that $P \nsubseteq H_i^-$. Since $H_i^- \subseteq H^-$, we will also have $P \nsubseteq H_i'^-$. Thus $\lim_{i \to \infty}(j_{H_i^-}(P) - j_{H_i^-}(P)) = 0$.

(III) $P \subseteq H^-$ and $P \cap H = \emptyset$.

Since the $H_i^-$ converge to $H^-$, for sufficiently large $i$ we will have that $P \subseteq H_i^-$. Since $P$ is closed, $P$ is bounded away from $H$, and hence $P \nsubseteq H_i'^-$ for sufficiently large $i$. Thus $\lim_{i \to \infty}(j_{H_i^-}(P) - j_{H_i^-}(P)) = 0$.

(IV) $P \subseteq H^-$ and $\emptyset \subseteq P \cap H \nsubseteq \ell$.

If $P \nsubseteq H_i'^-$, then we also have $P \nsubseteq H_i'^- \subseteq H_i^-$. Thus $j_{H_i^-}(P) - j_{H_i^-}(P) = 0$. Suppose now that $P \subseteq H_i^-$. Since $P \in \text{Def}^+(Q)$, all edge directions of $P$ must be edge directions of $Q$. Furthermore, the only edge directions of $Q$ that are contained in $H^-$ are the ones generating $C_F$, and so the only edge directions contained in $H$ must necessarily be the ones in lineal($C_F$) = $\ell$.

Thus $P \cap \ell$ must be contained in an affine translation of $\ell$, and $P$ is tightly contained in the affine cone $(P \cap \ell) + C_F$. By construction, the collection of edge directions of $Q$ contained in $H^-$ is the same as the collection contained in $H_i^-$. Thus if $H_i'^-$ contains $P \cap H$, then it also contains $P$. Since $P \cap H$ is contained in a translation of $\ell$, it is bounded away from $\ell$. As we saw in the construction of $H_i'^-$, this means that for sufficiently large $i$, the half-space $H_i'^-$ contains $P \cap H$ if $H_i^-$ contains it. This means that $H_i'^-$ contains $P$. Hence $\lim_{i \to \infty}(j_{H_i^-}(P) - j_{H_i^-}(P)) = 0$.

This concludes the proof of Theorem 2.3. 

2.2. A basis for the indicator group of extended deformations

Recall that $\text{tran}(\Sigma_Q^\ell) = \{C + v \mid C \in \Sigma_Q^\ell, \ v \in V\}$ denotes the set of all translates of tangent cones of a polyhedron $Q$.

**Theorem 2.7.** Let $Q \subseteq V$ be a polyhedron. The collection $\{1_{C + v} \mid C + v \in \text{tran}(\Sigma_Q^\ell)\}$ of indicator functions of translations of tangent cones of $Q$ is a basis for the $\mathbb{Z}$-module $\mathbb{I}(\text{Def}^+(Q))$. 
We begin by showing that the proposed collection spans \( I(\text{Def}^+(Q)) \). Our proof closely mirrors one given in \([16, \text{Theorem 4.2}]\). We prepare by recalling the Brianchon–Gram decomposition theorem. When a polyhedron \( Q \) has lineality space \( L \), we say that a face \( F \) of \( Q \) is relatively bounded if \( F/L \) is a bounded face of \( Q/L \).

**Theorem 2.8 [11, 21].** Let \( Q \) be a polyhedron with lineality dimension \( \ell \). Then
\[
1_Q = \sum_F (-1)^{\dim F - \ell} 1_{C_F + F},
\]
where this sum is taken over all relatively bounded faces \( F \) of \( Q \).

**Proposition 2.9.** Let \( Q \subset V \) be a polyhedron. The \( \mathbb{Z} \)-module \( I(\text{Def}^+(Q)) \) is spanned by \( \{1_{C + v} \mid C + v \in \text{tran}(\Sigma_Q'^\vee)\} \), the indicator functions of translations of tangent cones of \( Q \).

**Proof.** Let \( P \in \text{Def}^+(Q) \); that is, \( P \) is a polyhedron whose normal fan coarsens a subfan of \( \Sigma_Q \). Given \( \epsilon > 0 \), consider the Minkowski sum \( P + \epsilon Q \subseteq \mathbb{R}^n \). Since the normal fan of the Minkowski sum of two polyhedra is the common refinement of the two normal fans, we have that \( \Sigma_{P+\epsilon Q} \) is a subfan of \( \Sigma_Q \). Thus, the affine tangent cones of \( P + \epsilon Q \) all have the form \( C_F + v_{F,\epsilon} \) for some \( C_F \in \Sigma_Q'^\vee \) and vertex \( v_{F,\epsilon} \) of \( P + \epsilon Q \), where \( F \) is a face of \( Q \). By the Brianchon–Gram Theorem 2.8, we have that
\[
I_{P + \epsilon Q} = \sum_{\text{Relatively bounded faces } F \text{ of } P + \epsilon Q} (-1)^{\dim F} 1_{C_F + v_{F,\epsilon}},
\]
where \( C_F + v_{F,\epsilon} \) is the affine tangent cone of \( P + \epsilon Q \) corresponding to the face \( \hat{F} \). As \( \epsilon \) goes to 0, the left side of equation (1) converges pointwise to \( I_P \). Since vertices of \( P + \epsilon Q \) converge to vertices of \( P \) as \( \epsilon \to 0 \), each affine cone \( C_F + v_{F,\epsilon} \) on the right side of equation (1) converges to a tangent cone of \( Q \) translated by a vertex of \( P \). \( \square \)

We now complete the proof of Theorem 2.7 by using tight containments to establish the linear independence of the proposed basis for \( I(\text{Def}^+(Q)) \).

**Proof of Theorem 2.7.** Proposition 2.9 states that \( \{1_{C + v} \mid C + v \in \text{tran}(\Sigma_Q'^\vee)\} \) spans \( I(\text{Def}^+(Q)) \), so it remains to show that these indicator functions are linearly independent. Let \( \{C_i + v_i \mid 1 \leq i \leq k\} \) be a finite collection of translates of cones in \( \Sigma_Q'^\vee \), and suppose that there exist \( a_i \in \mathbb{Z} \) such that
\[
\sum_{i=1}^k a_i \cdot 1_{C_i + v_i} = 0.
\]
(2)

There exists some \( i \) so that \( C_i + v_i \) contains no other \( C_j + v_j \) for \( j \neq i \). Suppose without loss of generality that \( i = 1 \). We then have
\[
\tilde{j}_{C_1 + v_1}(C_i + v_i) := \begin{cases} 1, & \text{if } C_1 + v_1 \text{ tightly contains } C_i + v_i, \\ 0, & \text{otherwise} \end{cases}
\]
\[
= \begin{cases} 1, & i = 1 \\ 0, & i \neq 1. \end{cases}
\]
Since \( \tilde{j}_{C_1 + v_1} : \text{Def}^+(P) \to \mathbb{Z} \) is a valuation by Theorem 2.3, we can apply \( \tilde{j}_{C_1 + v_1} \) to both sides of equation (2) to obtain
for any nonzero $v$. Since the translate of $C+v$ is a well-defined $\mathbb{Z}$-linear isomorphism. In other words, by abusing the notation as before to write $F$ also for the map $\text{Def}^+(Q) \to \mathbb{Z}^{\text{tran}(\Sigma_Q)}$ determined by $P \mapsto \sum_{C+v \in \text{tran}(\Sigma_Q)} e_{C+v}$ if and only if $C$ tightly contains $C'$, it suffices to check this for $P = C' + v' \in \text{tran}(\Sigma_Q)$, since any $P \in \text{Def}^+(Q)$ is a linear combination over finitely many translates of tangent cones of $Q$. Lemma 2.11 implies

$$F(1_{C'+v'}) = \sum_{C+v} e_{C+v},$$

where the sum is over all affine tangent cones $C + v$ of $C' + v'$, which is a finite sum.

2.3. **The universal valuative function of extended deformations**

We now prove Theorem A by establishing a more general statement for extended deformations of an arbitrary polyhedron. For a set $S$, denote by $\{e_i \mid i \in S\}$ the standard basis of $\mathbb{Z}^{\oplus S}$.

**Theorem 2.10.** Let $Q \subseteq V$ be a polyhedron. The map

$$F : \mathbb{I}(\text{Def}^+(Q)) \to \mathbb{Z}^{\oplus \text{tran}(\Sigma_Q)}$$

is a well-defined $\mathbb{Z}$-linear isomorphism. In other words, by abusing the notation as before to write $F$ also for the map $\text{Def}^+(Q) \to \mathbb{Z}^{\oplus \text{tran}(\Sigma_Q)}$ determined by $P \mapsto F(1_P)$, we have that for any valuative function $f : \text{Def}^+(Q) \to A$ to an abelian group $A$, there exists a unique linear map $g : \mathbb{Z}^{\oplus \text{tran}(\Sigma_Q)} \to A$ such that $f = g \circ F$.

We prepare the proof of Theorem 2.10 with an observation about tight containments of tangent cones.

**Lemma 2.11.** Let $Q \subseteq V$ be a polyhedron, and $C' + v' \in \text{tran}(\Sigma_Q)$ a translate of a tangent cone of $Q$. Another translate of a tangent cone $C + v \in \text{tran}(\Sigma_Q)$ of $Q$ tightly contains $C' + v'$ if and only if $C + v$ is an affine tangent cone of $C' + v'$.

**Proof.** First consider the $v = v'$ case, which is equivalent to the case of $v = v' = 0$ by translating both $C + v$ and $C' + v'$ by $-v$. For cones, tight containment is equivalent to containment, since every cone contains the origin in its lineality space. Moreover, for tangent cones $C$ and $C'$ of $Q$, we have $C' \subseteq C$ if and only if $C'$ is a translate of $C'$ (since both dual cones are elements of the fan $\Sigma_Q$), or equivalently, if and only if $C$ is a tangent cone of $C'$.

For the general case, suppose $C$ tightly contains $C' + v' - v$, so that there exists $x \in \text{lineal}(C) \cap (C' + v' - v)$. Since $x \in \text{lineal}(C)$ implies that at most one of $x + y$ and $x - y$ lie in $C$ for any nonzero $y \notin \text{lineal}(C)$, the smallest face of $C' + v' - v$ containing $x$ is contained in $\text{lineal}(C)$. Thus, the affine span of this face contains the origin, and hence $v' - v \in \text{lineal}(C)$. Since the translate of $C$ by the element $v' - v$ in its lineality space is equal to $C$ itself, we are thus reduced to the case of $v = v'$. \qed
To establish that \( F \) is an isomorphism, we first note that Theorem 2.7 gives an isomorphism
\[
\varphi : \mathbb{Z}_{\text{tran}(\Sigma_Q')}^+ \sim (\text{Def}^+(Q)) \text{ defined by } e_{C+v} \mapsto 1_{C+v}.
\]
Now consider the composition
\[
F \circ \varphi : \mathbb{Z}_{\text{tran}(\Sigma_Q')}^+ \rightarrow \mathbb{Z}_{\text{tran}(\Sigma_Q')}^+ \text{ defined by } e_{C+v} \mapsto \sum_{C+v \in \text{tran}(\Sigma_Q')} f_{C+v}(C' + v')e_{C+v}.
\]
We claim that for any finite subset \( S \subset \text{tran}(\Sigma_Q') \), there exists a finite subset \( S' \subset \text{tran}(\Sigma_Q') \) containing \( S \) such that the restriction of \( F \circ \varphi \) has image \( \mathbb{Z}^{S'} \) and is an isomorphism \( \mathbb{Z}^{S'} \sim \mathbb{Z}^{S'} \). Given any finite \( S \subset \text{tran}(\Sigma_Q') \), let \( S' \) be the set of affine tangent cones of the elements in \( S \). The equation (†) established by Lemma 2.11 implies that a linear order on \( S' \) that refines the containment relation makes the matrix of the map \( F \circ \varphi : \mathbb{Z}^{S'} \rightarrow \mathbb{Z}^{S'} \) triangular with 1’s on the diagonal. \( \square \)

Recall from Definition 1.2 that an extended generalized \( \Phi \)-permutohedron is an extended deformation of a \( \Phi \)-permutohedron \( \Pi_\Phi \), and
\[
\text{GP}_\Phi := \text{Def}^+(\Pi_\Phi), \quad \text{the set of all extended generalized } \Phi \text{-permutohedra.}
\]

**Proof of Theorem A.** Set \( Q = \Pi_\Phi \) in Theorem 2.10. \( \square \)

The remainder of this section will only be used in the proof of Proposition 3.9. Nonetheless, because the results may be of independent interest in polyhedral geometry, we develop them here in the setting of deformations of arbitrary polytopes. Let \( \mathcal{V} \) be the set of vertices of a deformation of a polytope \( Q \subset V \). Let
\[
\text{Def}(Q)|_{\mathcal{V}} := \{ Q' \in \text{Def}(Q) \mid Q' \subseteq Q, \text{ Vert}(Q') \subseteq \mathcal{V} \}
\]
be the set of deformations of \( Q \) that have vertices in \( \mathcal{V} \). Let \( \text{min}(\Sigma_Q') \) be the set of minimal tangent cones of \( Q \), that is, the tangent cones of the vertices of \( Q \). Denote by \( X_{\mathcal{V}} := \{ C + v \mid C \in \text{min}(\Sigma_Q'), \ v \in \mathcal{V} \} \subset \text{tran}(\Sigma_Q') \), and define a function
\[
F_{\mathcal{V}} : \mathbb{I}(\text{Def}(Q)|_{\mathcal{V}}) \rightarrow \mathbb{Z}^{X_{\mathcal{V}}} \text{ by } 1_P \mapsto \sum_{C+v \in X_{\mathcal{V}}} e_{C+v}. \tag{2.12}
\]

**Proposition 2.12.** Let \( Q \), \( \mathcal{V} \), and \( F_{\mathcal{V}} \) be as above. There is a \( \mathbb{Z} \)-linear injection \( \mathcal{E}_{\mathcal{V}} : \mathbb{Z}^{X_{\mathcal{V}}} \rightarrow \mathbb{Z}^{\oplus \text{tran}(\Sigma_Q')} \) such that the following diagram commutes:
\[
\begin{array}{ccc}
\mathbb{I}(\text{Def}(Q)|_{\mathcal{V}}) & \xrightarrow{F_{\mathcal{V}}} & \mathbb{Z}^{X_{\mathcal{V}}} \\
\downarrow \ & & \downarrow \mathcal{E}_{\mathcal{V}} \\
\mathbb{I}(\text{Def}^+(Q)) & \xrightarrow{F} & \mathbb{Z}^{\oplus \text{tran}(\Sigma_Q')}.
\end{array}
\]

**Proof.** A translate of tangent cone of \( Q \) tightly contains \( P \in \text{Def}(Q)|_{\mathcal{V}} \) only if it is of the form \( C + v \) where \( C \in \Sigma_Q \), and \( v \in \mathcal{V} \). Let us denote \( Y_{\mathcal{V}} := \{ C + v \mid C \in \Sigma_Q, \ v \in \mathcal{V} \} \). For each \( C \in \Sigma_Q \), fix a minimal cone \( \tilde{C} \in \text{min}(\Sigma_Q) \) that is contained in \( C \). We claim that \( P \) is tightly contained in \( C + v \in Y_{\mathcal{V}} \) if and only if \( P \) is tightly contained in exactly one of \( \{ \tilde{C} + \tilde{v} \mid \tilde{v} \in \mathcal{V} \} \) and \( \tilde{v} \in \text{lineal}(C + v) \subset X_{\mathcal{V}} \). Assuming the claim for now, we have an equality of functions
\[
\frac{f_{C+v}}{\tilde{v} \in \mathcal{V} \cap (\text{lineal}(C) + v)} = \sum_{\tilde{v} \in \mathcal{V} \cap (\text{lineal}(C) + v)} \frac{f_{\tilde{C}+\tilde{v}}}{\tilde{v}}.
\]
on \( \mathbb{I}(\text{Def}(Q)|_\mathcal{Y}) \) for each \( C + v \in Y_\mathcal{Y} \). Thus, since the map \( \mathcal{F}_\mathcal{Y} \) is defined by

\[
\mathcal{F}_\mathcal{Y}(1\rho) = \sum_{C + v \in X_\mathcal{Y}} \overline{f}_{C + v}(P)e_{C + v},
\]

the map \( \mathcal{E}_\mathcal{Y} : \mathbb{Z}^{X_\mathcal{Y}} \to \mathbb{Z}^{Y_\mathcal{Y}} \) \( \to \mathbb{Z}^{\mathbb{Q}^{\text{tran}(\Sigma_Q^v)}} \) defined by

\[
e_{C + v'} \mapsto \sum_{\substack{C + v \in Y_\mathcal{Y} \\
\text{such that } C = C' \text{ and } v' \in \text{lineal}(C) + v}} e_{C + v}
\]

satisfies the commutativity of the diagram stated in the proposition. The map \( \mathcal{E}_\mathcal{Y} \) is the dual map of the map \( \mathcal{E}_\mathcal{Y}^* : \mathbb{Z}^{Y_\mathcal{Y}} \to \mathbb{Z}^{X_\mathcal{Y}} \) defined by \( e_{C + v} \mapsto \sum_{\tilde{v} \in \mathcal{Y} \cap (\text{lineal}(C) + v)} e_{\tilde{C} + \tilde{v}} \). Since \( \mathcal{E}_\mathcal{Y}^* \) is identity when restricted to \( \mathbb{Z}^{X_\mathcal{Y}} \), and in particular surjective, the dual map \( \mathcal{E}_\mathcal{Y} \) is injective.

For the claim, the ‘if’ direction is immediate. For the ‘only if’ direction, suppose \( C + v \) tightly contains \( P \). Let \( y \in \text{relint}(\tilde{C}') \), and consider the face of \( P \) maximizing the linear functional \( \langle y, \cdot \rangle \). Because \( P \) is a deformation of \( Q \), this face is necessarily a vertex \( \tilde{v} \in \mathcal{Y} \) of \( P \), and moreover the tangent cone of \( P \) at \( \tilde{v} \) is contained in \( \tilde{C} \). Hence, the translate \( \tilde{C} + \tilde{v} \) tightly contains \( P \), and \( \tilde{v} \in \text{lineal}(C) + v \) since \( C' \) is a face of \( \tilde{C}' \). This translate of \( \tilde{C} \) is the unique one because any polyhedron is tightly contained in at most one translate of a cone. \( \square \)

Remark 2.13. In the proof of Proposition 2.12 above, let \( \text{St}(C) := \{ C' \in \min(\Sigma_Q^v) \mid C' \subseteq C \} \) for each \( C \in \Sigma_Q^v \). Then, by the claim in the proof, for each \( C + v \in Y_\mathcal{Y} \) we have an equality of functions

\[
|\text{St}(C)| \cdot \overline{f}_{C + v} = \sum_{C' \in \text{St}(C)} \sum_{v' \in \mathcal{Y} \cap (\text{lineal}(C) + v)} \overline{f}_{C' + v'}.
\]

Thus, if we are working with \( \mathbb{Q} \)-coefficients instead of \( \mathbb{Z} \) in the previous proposition, we can alternatively define \( \mathcal{E}_\mathcal{Y} : \mathbb{Q}^{X_\mathcal{Y}} \to \mathbb{Q}^{Y_\mathcal{Y}} \) by

\[
e_{C' + v'} \mapsto \sum_{\substack{\tilde{v} \in \mathcal{Y} \cap (\text{lineal}(C) + v) \\
|\text{St}(C)|}} \frac{1}{e_{\tilde{C} + \tilde{v}}}.
\]

This eliminates making choices of the minimal cone \( \tilde{C} \in \text{St}(C) \) for each cone \( C \in \Sigma_Q^v \) in the proof of Proposition 2.12 when one works over \( \mathbb{Q} \)-coefficients.

3. Valuative invariants

We now consider universal valuative invariants. In §3.1, we compute the universal valuative invariant of extended generalized Coxeter permutohedra. In §3.2, we review Coxeter matroids and the \( \mathcal{G} \)-invariant, and discuss an application to delta-matroids. In §3.3, we prove Theorem B.

3.1. Valuative invariants of generalized Coxeter permutohedra

Let us first fix notations and recall basic facts about root systems. See [23] for a general reference on reflection groups, and [5] for an account tailored toward generalized Coxeter permutohedra.

Notation. As before, let \( W \) be a finite reflection group with the root system \( \Phi = (V, R) \). We assume \( \Phi \) to be reduced. Let \( n = \dim V \) and write \( [n] = \{1, \ldots, n\} \). Let us fix once and for all a system of positive roots \( R^+ \subset R \), and set
by Proposition 3.1, the orbits of the action of $W$ on vector spaces; see, for instance, \[ \alpha \cdot \{ \omega \mid i \in I \} \]

For a subset $I \subset [n]$, we set

- $W_I = \langle s_i \mid i \in I \rangle$ to be the parabolic subgroup of $W$ corresponding to $I$;
- $C_I := \text{Cone}(\{ \alpha_1, \ldots, \alpha_n \} \cup \{ -\alpha_i \mid i \in I \})$, whose dual cone is
- $\sigma_{[n] \setminus I} := \text{Cone}(\omega_i \mid i \in [n] \setminus I)$; and
- $\omega_{[n] \setminus I} := \sum_{i \in [n] \setminus I} \omega_i$, whose $W$-orbit $W \cdot \omega_{[n] \setminus I}$ is identified with $W/W_I$.

The Coxeter complex $\Sigma_\Phi$ consists of the $W$-translates of the cones $\sigma_I$ as $I$ ranges over all subsets of $[n]$. See [5, §3.2] for a summary of some combinatorial properties of $\Sigma_\Phi$. We will use the following standard fact about the action of $W$ on $\Sigma_\Phi$.

**Proposition 3.1.** The $W$-orbits of the action of $W$ on $\Sigma_\Phi$ are in bijection with $2^{[n]}$, where a subset $I \subset [n]$ corresponds to the orbit $W \cdot \sigma_I$ with the stabilizer of $\sigma_I$ being $W_I$. Similarly, the $W$-orbits of $\Sigma_\Phi^+$ are the orbits of $C_I$ as $I$ ranges over all subsets of $[n]$.

The reflection group $W$ acts on $V$, inducing an action of $W$ on $Z^V$ by $(w \cdot f)(v) = f(w^{-1}v)$. If $\mathcal{P}$ is a $W$-invariant family of polyhedra in $V$, the group $W$ thus acts on $I(\mathcal{P})$ by $w \cdot 1_P = 1_{w \cdot P}$. We say that a valuative function $f : \mathcal{P} \to A$ is a valuative invariant if $f(w \cdot P) = f(P)$ for all $w \in W$ and $P \in \mathcal{P}$. We will often use the following standard facts about action of a finite group on vector spaces; see, for instance, [2, §2.5.1].

**Lemma 3.2.** Let a finite group $W$ act on a $\mathbb{Q}$-vector space $U$. The ‘average map’ $\text{avg} : U \to U$ defined by

$$\text{avg}(u) := \frac{1}{|W|} \sum_{w \in W} w \cdot u$$

has the following properties.

1. It is a projection onto the space $U^W := \{ u \in U \mid w \cdot u = u \text{ for all } w \in W \}$ of $W$-fixed points, which is identified with the space $U/W := U/\text{span}(u - w \cdot u \mid u \in U, \ w \in W)$ since $\ker \text{avg} = \text{span}(w \cdot u - u \mid u \in U, \ w \in W)$.
2. A map $f : U \to A$ to a $\mathbb{Q}$-vector space $A$ satisfies $f(w \cdot u) = f(u)$ for all $w \in W$ and $u \in U$ if and only if $f$ factors as $f = f|_{U^W} \circ \text{avg}$. In other words, a $W$-invariant map from $U$ is equivalent to a map from $U^W$.

We now compute the universal valuative invariant of $\text{GP}_\Phi^+$ over $\mathbb{Q}$-coefficients. First, note that by Proposition 3.1, the orbits of the action of $W$ on $\text{tran}(\Sigma_\Phi^+)$ are in bijection with $\{ C_I + v \mid I \subseteq [n], \ v \in V \}$. Since $C_I + v = C_{I'} + v'$ if and only if $I = I'$ and $v' - v \in \text{lineal}(C_I) = \text{span}(\alpha_i \mid i \in I)$, let us denote by $2^{[n]} \boxtimes V$ the set of equivalence classes of pairs $(I, v) \in 2^{[n]} \times V$ where $(I, v) \sim (I', v')$ if $I = I'$ and $v - v' \in \text{span}(\alpha_i \mid i \in I)$. The set $2^{[n]} \boxtimes V$ is in bijection with the $W$-orbits of $\text{tran}(\Sigma_\Phi^+)$.

**Corollary 3.3.** Write $\{ U_{I,v} \mid (I,v) \in 2^{[n]} \boxtimes V \}$ for the standard basis of $\mathbb{Q}^{\boxplus(2^{[n]} \boxtimes V)}$. The map

$$G^+ : \text{GP}_\Phi^+ \to \mathbb{Q}^{\boxplus(2^{[n]} \boxtimes V)}$$
defined by $P \mapsto \sum_{(I,v) \in 2^{[n]} \boxtimes V} \left( \sum_{w \in W} f_w(C_{I+v})(P) \right) U_{I,v}$
is the universal valuative invariant over \( \mathbb{Q} \). That is, for any valuative invariant \( g : \text{GP}_\Phi^+ \to A \) to a \( \mathbb{Q} \)-vector space \( A \), there exists a unique linear map \( \psi : \mathbb{Q}^{\boxplus(2^{[n]} \boxtimes V)} \to A \) such that \( \psi \circ \mathcal{G}^+ = g \).

**Proof.** Let us denote \( \mathbb{I}(\text{GP}_\Phi^+)_{\mathbb{Q}} := \mathbb{I}(\text{GP}_\Phi^+) \otimes \mathbb{Q} \), and again abuse notation to write \( \mathcal{G}^+ \) for the function \( \mathcal{G}^+ : \mathbb{I}(\text{GP}_\Phi^+)_{\mathbb{Q}} \to \mathbb{Q}^{\boxplus(2^{[n]} \boxtimes V)} \) defined by \( 1_P \mapsto \mathcal{G}^+(P) \). We need to show that \( \mathcal{G}^+ \) is a \( \mathbb{Q} \)-linear isomorphism.

Let \( W \) act on \( \mathbb{Z}^{\boxplus \text{tran}(\Sigma^\vee_\Phi)} \) by its action on \( \text{tran}(\Sigma^\vee_\Phi) \). Note first that the isomorphism \( \mathcal{F} : \mathbb{I}(\text{GP}_\Phi^+) \sim \mathbb{Z}^{\boxplus \text{tran}(\Sigma^\vee_\Phi)} \) in Theorem 2.10 is \( W \)-equivariant: For \( w \in W \) and a translate of a tangent cone \( C + v \), we have \( \overline{j_{C+v}}(P) = 1 \) if and only if \( \overline{j_{w}(C+v)}(w \cdot P) = 1 \) for any \( P \in \text{GP}_\Phi^+ \), and hence

\[
\sum_{C+v \in \text{tran}(\Sigma^\vee_\Phi)} \overline{j_{C+v}(w \cdot P)} e_{C+v} = \sum_{C+v \in \text{tran}(\Sigma^\vee_\Phi)} \overline{j_{C+v}(P)} e_{w \cdot (C+v)}.
\]

We thus have a commuting diagram

\[
\begin{array}{ccc}
\mathbb{I}(\text{GP}_\Phi^+)_{\mathbb{Q}} & \xrightarrow{\mathcal{F}} & \mathbb{Q}_{\text{tran}(\Sigma^\vee_\Phi)} \\
\downarrow{\text{avg}} & & \downarrow{\text{avg}} \\
\mathbb{I}(\text{GP}_\Phi^+)_{W} & \xrightarrow{\sim} & (\mathbb{Q}_{\text{tran}(\Sigma^\vee_\Phi)})^W.
\end{array}
\]

Since the \( W \)-orbits of \( \text{tran}(\Sigma^\vee_\Phi) \) are in bijection with \( 2^{[n]} \boxtimes V \), we identify \( (\mathbb{Q}_{\text{tran}(\Sigma^\vee_\Phi)})^W \) with \( \mathbb{Q}^{\boxplus(2^{[n]} \boxtimes V)} \) by identifying \( \text{avg}(e_{C_I+v}) \) with \( U_{I,v} \). Then the composition \( \text{avg} \circ \mathcal{F} \) is the map \( \mathcal{G}^+ \), since

\[
(\text{avg} \circ \mathcal{F})(P) = \text{avg} \left( \sum_{C+v \in \text{tran}(\Sigma^\vee_\Phi)} j_{C+v}(P) e_{C+v} \right)
= \text{avg} \left( \sum_{(I,v) \in 2^{[n]} \boxtimes V} \sum_{w \in W} \overline{j_{w}(C_I+v)}(P) e_{w \cdot (C_I+v)} \right)
= \sum_{(I,v) \in 2^{[n]} \boxtimes V} \sum_{w \in W} \overline{j_{w}(C_I+v)}(P) \text{avg}(e_{w \cdot (C_I+v)})
= \sum_{(I,v) \in 2^{[n]} \boxtimes V} \left( \sum_{w \in W} \overline{j_{w}(C_I+v)}(P) \right) U_{I,v}.
\]

Lemma 3.2 now implies that \( \mathcal{G}^+ \) is the universal valuative invariant. \( \square \)

### 3.2. Coxeter matroids, the \( \mathcal{G} \)-invariant, and interlace polynomials

We start with a brief treatment of Coxeter matroids. See [8] for a detailed account.

We use \( u \leq u' \) for the Bruhat order on the elements \( u, u' \) of \( W \), and write \( u \leq^w u' \) to mean \( w^{-1}u \leq w^{-1}u' \). For a parabolic subgroup \( W_I \) of \( W \) corresponding to a subset \( I \subseteq [n] \), the Bruhat order on \( W/W_I \) is given by \( B \leq^w B' \) for \( B, B' \in W/W_I \) if \( u \leq^w u' \) for some \( u \in B \) and \( u' \in B' \). Recalling that the stabilizer of \( \varpi_{[n] \setminus I} = \sum_{i \in [n] \setminus I} \) is \( W_I \), for \( B \in W/W_I \), we denote \( \delta_B := u \cdot \varpi_{[n] \setminus I} \) for any \( u \in B \).

For a subset \( S \subseteq W/W_I \), denote by \( P_S \) the polytope

\[
P_S := \text{Conv}(\delta_B \mid B \in S).
\]
The following generalization of [18] establishes Coxeter matroids as subfamilies of generalized Coxeter permutohedra.

**Theorem 3.4** [8, Theorem 6.3.1]. Let $S$ be a subset of $W/W_I$. The following are equivalent.

1. The subset $S \subseteq W/W_I$ is a Coxeter matroid. That is, for every $w \in W$, there exists a unique $\leq w$-minimal element in $S$ (Definition 1.3).
2. The polytope $P_S$ is a deformation of $\Pi_{\Phi}$, that is, $P_S \in \text{GP}_{\Phi}^+$.

For a Coxeter matroid $M \subseteq W/W_I$, we call the elements of $M$ the bases of $M$, and the $\leq w$-minimal basis of $M$ is denoted $\text{min}^w(M)$. The polytope $P_M$ is called the **base polytope** of $M$. Its vertices are $\{\delta_B \mid B \in M\}$. Theorem 3.4 states that Coxeter matroids of type $(\Phi, I)$ are exactly the polytopes in $\text{GP}_{\Phi}^+$, whose vertices are subsets of $\mathcal{V}_I := \{\delta_B \mid B \in W/W_I\}$. Hence, we consider

$$\text{Mat}_{\Phi, I} := \{\text{Coxeter matroids of type } (\Phi, I)\}$$

as a subfamily of $\text{GP}_{\Phi}^+$. Recall that the $\mathcal{G}$-**invariant** of Coxeter matroids of type $(\Phi, I)$ is the function

$$\mathcal{G} : \text{Mat}_{\Phi, I} \to \mathbb{Q}^{W/W_I} = \mathbb{Q}\{U_B \mid B \in W/W_I\} \text{ defined by } M \mapsto \sum_{w \in W} U_{w^{-1}\text{min}^w(M)}.$$  

Theorem B states that $\mathcal{G}$ is the universal valuative invariant of $\text{Mat}_{\Phi, I}$ over $\mathbb{Q}$-coefficients. For ordinary matroids, the specialization of the $\mathcal{G}$-invariant to the Tutte polynomial was computed in [16]. Here we feature an example of nonordinary $(\Phi, I)$-matroids where the $\mathcal{G}$-invariant specializes to a well-studied polynomial invariant.

Let $\Phi = B_n$ be the type $B$ root system, so that $W = S_n^B$, the signed permutation group. Let $W_I$ be the parabolic subgroup such that $S_n^B/W_I = (\mathbb{Z}/2\mathbb{Z})^n$. In other words, we have that $\delta_{W_I}$ is the fundamental weight $w_n = \frac{1}{2}(e_1 + \cdots + e_n)$, and that $\mathcal{V}_I = \{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_n)\}$. In this case, the $(\Phi, I)$-matroids are also known as **delta-matroids**, which were originally studied by Bouchet [9] and rediscovered as combinatorial abstractions of graphs embedded on surfaces [14, 15]. A well-studied invariant of delta-matroids is the interlace polynomial.

**Definition 3.5.** Identifying the positive coordinates of an element in $\mathcal{V}_I = \{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_n)\}$ with the subset of $[n] = \{1, \ldots, n\}$, consider a delta-matroid $M$ as a collection of subsets of $[n]$. The **interlace polynomial** of $M$ is a univariate polynomial defined by

$$q_M(x) := \sum_{A \subseteq [n]} x^{d_M(A)} \text{ where } d_M(A) := \min\{|(B - A) \cup (A - B)| : B \in M\} \text{ for } A \subseteq [n].$$

Interlace polynomials were originally defined in a study arising from DNA sequencing [3, 4], and were generalized to delta-matroids in [12]; see [30] for a survey. Here we show that the interlace polynomial is a specialization of the $\mathcal{G}$-invariant in the following way.

**Theorem 3.6.** Denote by $|\delta_B|$ the coordinate sum of the vector $\delta_B \in \mathcal{V}_I$. For a delta-matroid $M$, we have

$$q_M(x) = \frac{1}{n!} \sum_{w \in S_n^B} x^{-|\delta_{w^{-1}\text{min}^w(M)}| + \frac{n}{2}}.$$  

In particular, the interlace polynomial is a specialization of the $\mathcal{G}$-invariant, and hence is a valuative invariant of delta-matroids.
Proof. Again identifying the positive coordinates of an element in \( \mathcal{Y}_l = \{ \frac{1}{2} (\pm e_1 \pm \cdots \pm e_n) \} \) with the subset of \([n] = \{1, \ldots, n\}\), let us write \( \delta_A := \frac{1}{2} (\sum_{i \in A} e_i - \sum_{i \notin A} e_i) \) for a subset \( A \subseteq [n] \). We claim that if \( w \in W \) satisfies \( \delta_{wW_l} = \delta_A \), then
\[
d_M(A) = -|\delta_{w^{-1} \min^w(M)}| + \frac{n}{2}.
\]
Since \( W_l \), which is the stabilizer of \( \delta_{W_l} \), has order \( n! \), the proposition then follows immediately from the claim and the definition of the interlace polynomial.

For the proof of our claim, first observe that for two subsets \( A, B \subseteq [n] \), one has
\[
|(B - A) \cup (A - B)| = -\langle 2\delta_A, \delta_B \rangle + \frac{n}{2}.
\]
Thus, when \( A \) is fixed, the elements \( B \in M \) that achieve the minimum value \( d_M(A) \) are exactly the ones that maximize the pairing \( \langle \delta_A, \delta_B \rangle \). Such \( B \in M \) are exactly \( \min^w(M) \) as \( w \in W \) ranges over all elements of \( W \) such that \( \delta_{wW_l} = \delta_A \). Thus, we have
\[
d_M(A) = -\langle 2\delta_{wW_l}, \delta_{\min^w(M)} \rangle + \frac{n}{2} = -\langle 2\delta_{W_l}, \delta_{w^{-1} \min^w(M)} \rangle + \frac{n}{2} = -|\delta_{w^{-1} \min^w(M)}| + \frac{n}{2},
\]
as claimed. \(\square\)

3.3. Proof of Theorem B

Let us fix \( I \subseteq [n] \), and recall the notation \( \mathcal{Y}_l = \{ \delta_B \mid B \in W/W_l \} \). We first recall a standard fact relating the positions of the points \( \mathcal{Y}_l \) to the Bruhat order on \( W/W_l \).

LEMMA 3.7 [8, Lemma 6.2.4]. Let \( B, B' \in W/W_l \) be such that \( \delta_{B'} - \delta_B = \lambda \alpha \) for some root \( \alpha \in R \) and \( \lambda \geq 0 \). Then for \( w \in W \), one has \( B \leq w \ B' \) if and only if \( \alpha \in w \cdot R^+ \). In particular, if \( B \leq w \ B' \), then \( \delta_{B'} \in w \cdot C_G + \delta_B \), where \( w \cdot C_G \) is the \( w \)-permutation of the positive root cone \( C_G = \text{Cone}(\alpha_1, \ldots, \alpha_n) \).

Lemma 3.7 allows us to express the \( G \)-invariant in terms of tight containments in the following way, thereby showing that it is a valuative invariant.

PROPOSITION 3.8. For any Coxeter matroid \( M \in \text{Mat}_{\phi, I} \), we have an equality
\[
G(M) = \sum_{B \in W/W_l} \left( \sum_{w \in W} j_{w(C_G + \delta_B)}(P_M) \right) U_B.
\]
In particular, the \( G \)-invariant is valuative, and is an invariant.

Proof. For a fixed \( w \in W \), the base polytope \( P_M \) is tightly contained in at most one translate \( w \cdot C_G + \delta_B \) of the strongly convex cone \( w \cdot C_G \). Since \( \min^w(M) \) is the unique minimal element in \( M \) with respect to \( \leq^w \), Lemma 3.7 implies that \( w \cdot C_G + \delta_{\min^w(M)} = w \cdot (C_G + \delta_{w^{-1} \min^w(M)}) \) tightly contains \( P_M \). We now compute that
\[
\sum_{B \in W/W_l} \left( \sum_{w \in W} j_{w(C_G + \delta_B)}(P_M) \right) U_B = \sum_{w \in W} \sum_{B \in W/W_l} j_{w(C_G + \delta_B)}(P_M) U_B
\]
\[
= \sum_{w \in W} U_{w^{-1} \min^w(M)},
\]
as desired. The function \( G \) is valuative since the functions \( j_{w(C_G + \delta_B)} \) are valuative on \( GP_{\Phi}^+ \) by Theorem 2.3, and hence valuative on the subfamily \( \text{Mat}_{\phi, I} \). The function \( G \) is evidently \( W \)-invariant. \( \square \)
We can now prove the first half of Theorem B.

**Proposition 3.9.** For any valuative invariant \( g : \text{Mat}_{\Phi, I} \to A \) to a \( \mathbb{Q} \)-vector space \( A \), there exists a map \( \psi : \mathbb{Q}^{W/W_I} \to A \) such that \( g = \psi \circ \mathcal{G} \).

We will prove that the map \( \psi \) is unique in Proposition 3.14, thereby finishing the proof of Theorem B.

**Proof.** Using the notation as in the discussion preceding Proposition 2.12, we have \( \text{Mat}_{\Phi, I} = \text{Def}(\Pi_{\Phi})|_{Y_I} \), where \( Y_I = \{ \delta_B \mid B \in W/W_I \} \). Moreover, we have \( X_{Y_I} = \{ w \cdot C_\Phi + \delta_B \mid w \in W, B \in W/W_I \} \) since the minimal cones of \( \Sigma_\Phi \) are \( W \)-permutations of the positive root cone \( C_\Phi \). Considering the action of \( W \) on \( \mathbb{Q}^{X_{Y_I}} \), we can identify \( (\mathbb{Q}^{X_{Y_I}})_{W} \) with \( \mathbb{Q}^{W/W_I} \) by identifying \( \text{avg}(e_{C_\Phi + \delta_B}) \) with \( U_B \). Then, the map \( \mathcal{G} \) is the composition \( \text{avg} \circ F_{Y_I} \), since

\[
(\text{avg} \circ F_{Y_I})(P_M) = \text{avg}\left( \sum_{B \in W/W_I} \sum_{w \in W} J_{w \cdot (C_\Phi + \delta_B)}(P_M)e_{w \cdot (C_\Phi + \delta_B)} \right)
\]

\[
= \sum_{B \in W/W_I} \left( \sum_{w \in W} J_{w \cdot (C_\Phi + \delta_B)}(P_M) \right) U_B
\]

\[
= \mathcal{G}(M),
\]

where the last equality follows from Proposition 3.8. Now Proposition 2.12 states that there is an injection \( E_{Y_I} : \mathbb{Q}^{X_{Y_I}} \to \mathbb{Q}^{\oplus \text{trans}(\Sigma_\Phi)} \) fitting into the left square of the commuting diagram

\[
\begin{array}{ccc}
\mathbb{I}(\text{Mat}_{\Phi, I})_{\mathbb{Q}} & \xrightarrow{F_{Y_I}} & \mathbb{Q}^{X_{Y_I}} \xrightarrow{\text{avg}} \mathbb{Q}^{W/W_I} \\
\downarrow & & \downarrow \text{avg} \\
\mathbb{I}(\text{GP}_{\Phi})_{\mathbb{Q}} & \xrightarrow{F} & \mathbb{Q}^{\oplus \text{trans}(\Sigma_\Phi)} \xrightarrow{\text{avg}} \mathbb{Q}^{2^{\text{SS}}} \mathbb{G}_V.
\end{array}
\]

Moreover, Remark 2.13 implies that over \( \mathbb{Q} \)-coefficients the map \( E_{Y_I} \) can be made \( W \)-equivariant, so that the right square of the diagram above also commutes. Extend the valuative invariant \( g \) on \( \mathbb{I}(\text{Mat}_{\Phi, I})_{\mathbb{Q}} \) to any valuative invariant on \( \mathbb{I}(\text{GP}_{\Phi})_{\mathbb{Q}} \), for example, by extending \( g \) to a function on \( \mathbb{I}(\text{GP}_{\Phi})_{\mathbb{Q}} \) (which may not be \( W \)-invariant) and then precomposing with the averaging map. Then, since \( \mathcal{G} = \text{avg} \circ F_{Y_I} \), the universality of \( \text{avg} \circ F = \mathcal{G}^+ \) from Corollary 3.3 implies that there exists \( \psi \) such that \( g = \psi \circ \mathcal{G} \). \( \square \)

We remark that the map \( F_{Y_I} \) in the proof above is generally not an isomorphism. To prove the other half of Theorem B, we first consider the following family of Coxeter matroids, which are special cases of Bruhat interval polytopes studied in [13, 38].

**Proposition 3.10 [13, Theorem 4.5].** For \( B \in W/W_I \), the subset

\[
\Omega_B := \{ B' \in W/W_I \mid B \leq B' \}
\]

is a Coxeter matroid.

**Definition 3.11.** We call the Coxeter matroid \( \Omega_B \) in the proposition above the **Coxeter Schubert matroid** with respect to \( B \in W/W_I \).

**Remark 3.12.** Suppose \( \Phi \) is crystallographic. Let \( G \) be the associated Lie group with a chosen Borel subgroup arising from our fixed choice of positive system for \( \Phi \). Let \( P_I \) be the
parabolic subgroup of $G$ corresponding to the subset $I$ of simple roots, and $T$ the torus in $G$. Coxeter Schubert matroids correspond to the Bruhat cells of the flag variety $G/P_I$ in the following way. For $B \in W/W_I$, consider the torus-orbit closure $\overline{T \cdot x}$ of a general point $x$ in the Bruhat cell of $G/P_I$ corresponding to $B$. The authors of [38] show that the base polytope of $\Omega_B$ is the moment polytope of $\overline{T \cdot x}$, and hence a Coxeter matroid.

The following key lemma establishes an ‘upper triangularity’ property of Coxeter Schubert matroids that we will need for establishing uniqueness of the map $\psi$ in Proposition 3.14.

**Lemma 3.13.** Let $B \in W/W_I$. There exist scalars $c_B^{B'} \in \mathbb{Q}$ for $B' \leq B \in W/W_I$ such that

$$G(\Omega_B) = \sum_{B' \leq B} c_B^{B'} U_{B'},$$

where $c_B^{B} \neq 0$.

**Proof of Lemma 3.13.** We claim that for $B \in W/W_I$ and $w \in W$, we have

$$w^{-1} \cdot \min_w(\Omega_B) \leq B.$$

The lemma then follows immediately from the claim since

$$G(\Omega_B) = \sum_{w \in W} U_{w^{-1} \cdot \min_w(\Omega_B)},$$

and the coefficient $c_B^{B} \neq 0$ because $w^{-1} \cdot \min_w(\Omega_B) = B$ when $w = e$ the identity.

We now prove the claim by borrowing tools from 0-Hecke algebras. Let us first recall some basic properties; see [32] for general reference. The 0-Hecke algebra $H_0$ of $W$ is the unital $\mathbb{Q}$ algebra generated by $\{T_i\}_{i \in [n]}$, subject to the relations.

- $T_i^2 = -T_i$ for all simple reflections $s_i$ ($i = 1, \ldots, n$).
- $(T_iT_jT_i \cdots)_{n_{ij}} = (T_jT_iT_j \cdots)_{n_{ij}}$, where $n_{ij}$ is the order of $s_is_j$ in $W$ and $(T_iT_jT_i \cdots)_{n_{ij}}$ is the product of the first $n_{ij}$ terms in the sequence $T_i, T_j, T_i, \ldots$.

For any $w = s_{i_1} \cdots s_{i_k} \in W$, let $T_w$ denote the product $T_{i_1} \cdots T_{i_k}$. One can show that this product is the same for any reduced expression of $w$. The only result we use is the following [32, 1.3]:

$$T_w \cdot T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) > \ell(w) \\ -T_w & \text{else,} \end{cases}$$

and

$$T_{s_i} \cdot T_w = \begin{cases} T_{s_iw} & \text{if } \ell(s_iw) > \ell(w) \\ -T_w & \text{else.} \end{cases}$$

From this, it follows that for $u, v, w \in W$, one has

$$T_w \cdot T_u = \pm T_v \Rightarrow v \geq u \text{ and } v \geq w. \quad (4)$$

For the claim, it suffices to prove the result for $W_I = \{e\}$ since we have a projection map from $W$ to $W/W_I$. Fix $b \in W$. Since $w^{-1} \cdot \min_w(\Omega_b) = \min(w^{-1} \cdot \Omega_b)$, we need to show that for all $w \in W$, we have

$$\min(w^{-1} \cdot \Omega_b) \leq b.$$
Proposition 3.14. Let $g : \mathbb{I}(\text{Mat}_{\Phi, I})_Q \to A$ be a valuative invariant into a $Q$-vector space $A$. There exists at most one linear map $\psi : Q^{W/W_I} \to A$ such that
$$\psi \circ \mathcal{G}(M) = g(M).$$

Proof. Suppose $\psi$ is a map satisfying the conditions of the proposition. We will show that $\psi$ is determined by the values on the Coxeter Schubert matroids by induction over the Bruhat order. For the base case, let $B = eW I \in W/W_I$. Then the equation
$$\psi \circ \mathcal{G}(\Omega_B) = c_B^B \psi(U_B) = g(\Omega_B)$$
implies that $\psi(U_B) = \frac{g(\Omega_B)}{c_B^B}$. For any other $B \in W/W_I$, suppose we have already computed $\psi(U_{B'})$ for $B' \leq B$, then the equation
$$\psi \circ \mathcal{G}(\Omega_B) = g(\Omega_B) = c_B^B \psi(U_B) + \sum_{B' \leq B} c_{B'}^B \psi(U_{B'})$$
gives the value of $\psi(U_B)$. As the set $\{U_B\}$ is a basis for $Q^{W/W_I}$, this construction determines $\psi$. Hence, there is at most one such map.

Note that this proposition also gives an algorithm to compute the specialization map $\psi$. Combining Propositions 3.9 and 3.14 completes the proof of Theorem B.

Remark 3.15. Using Lemma 3.13 can be avoided if the pair $(\Phi, I)$ satisfies the following property: For any $v \in \mathcal{V}_I$, the intersection of $C_{\infty} + v$ with the polytope $P_{W/W_I} = \text{Conv}(\delta_B \mid B \in W/W_I)$ is a base polytope of a $(\Phi, I)$-matroid. Let us say that a pair $(\Phi, I)$ is intersection-stable if it satisfies this property. If $A_n$ is the type $A$ root system of dimension $n$, the pair $(A_n, [n] \setminus i)$ is intersection-stable for each $i \in [n]$. This feature of ordinary matroids is important in the argument of [16] establishing $\mathcal{G}$ as the universal invariant of ordinary matroids. However, intersection-stability is a rare property for general Coxeter systems.

- Even in type $A$, the pair $(A_3, \emptyset)$ is not intersection-stable.
- For $B_3$ (or $C_3$), let $\varpi_1$ and $\varpi_2$ be the fundamental weights such that their weight polytopes are not the cube. Then the pairs $(B_3, [3] \setminus 1)$ and $(B_3, [3] \setminus 2)$ are not intersection-stable (similarly for $C_3$).
- In type $D_4$, three of the four fundamental weights make the pair $(D_4, [4] \setminus i)$ not intersection-stable.

The first point reflects that nonminuscule types usually fail to be intersection-stable since the converse of Lemma 3.7 holds only for minuscule types [34, 4.1]. The latter two points show that intersection-stability can fail even in minuscule types unless $\Phi = A_n$. If $\varpi_n$ is the fundamental weight of $B_n$ so that the weight polytope is the hypercube, one can show that the pair $(B_n, [n] \setminus n)$ is intersection-stable.

Two $(\Phi, I)$-matroids $M, M'$ are said to be isomorphic if $M = w \cdot M'$ for some $w \in W$. As an application of Theorem B, we show that Coxeter Schubert matroids form a basis for the space
$$\mathbb{I}(\text{Mat}_{\Phi, I})/W := \mathbb{I}(\text{Mat}_{\Phi, I})_Q/\text{span}(1_M - 1_{w \cdot M} \mid M \in \text{Mat}_{\Phi, I}, w \in W)$$
of isomorphism classes of indicator functions of $(\Phi, I)$-matroids.

Corollary 3.16. For a Coxeter matroid $M$, let $[M]$ denote the image of $1_{P_M}$ in $\mathbb{I}(\text{Mat}_{\Phi, I})/W$. Then the set $\{[\Omega_B]\}_{B \in W/W_I}$ of the classes of Coxeter Schubert matroids is a basis for $\mathbb{I}(\text{Mat}_{\Phi, I})/W$. 


Proof. Combined with Lemma 3.2.(2), Theorem B states that the map $\mathcal{G} : I(\text{Mat}_{\Phi,1})_Q^W \to Q^{W/W_i}$ is an isomorphism, and we have $I(\text{Mat}_{\Phi,1})_Q^W \cong I(\text{Mat}_{\Phi,1})_Q^W$ by Lemma 3.2.(1). Now, the ‘upper-triangularity’ Lemma 3.13 states that we have $\mathcal{G}(\Omega_B) = \sum_{B' \subseteq B} c_{B'}^BU_{B'}$ with $c_{B'}^B \neq 0$, which shows that $\{\mathcal{G}(\Omega_B)\}$ and $\{U_B\}$ are two bases of $Q^{W/W_i}$ related by an upper-triangular matrix. 

Remark 3.17. Suppose $\Phi$ is crystallographic, and let $G/P_I$ be the flag variety as in Remark 3.12. As Coxeter Schubert matroids correspond to Bruhat cells of $G/P_I$, we note the parallelism between Corollary 3.16 and the fact that (closures of) Bruhat cells of $G/P_I$ form a basis for the cohomology ring of $G/P_I$.

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Christopher Eur
Department of Mathematics
Stanford University
450 Jane Stanford Way, Building 380
Stanford, CA 94305
USA
chriseur@stanford.edu

Mario Sanchez and Mariel Supina
Department of Mathematics
University of California Berkeley
970 Evans Hall
Berkeley, CA 94720
USA
mario_sanchez@berkeley.edu supina@math.berkeley.edu

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