TASI Lectures on Matrix Theory

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Abstract: This is a summary of key issues in Matrix Theory and its compactifications. It is emphasized that Matrix Theory is a valid Discrete Light Cone Quantization of M Theory with at least 6 noncompact asymptotically flat dimensions and 16 or 32 Supersymmetry charges. The background dependence of the quantum mechanics of M Theory, and the necessity of working in light cone frame in asymptotically flat spacetimes are explained in terms of the asymptotic density of states of the theory, which follows from the Bekenstein-Hawking entropy formula. In four noncompact dimensions one is led to expect a Hagedorn spectrum in light cone energy. This suggests the possible relevance of "little string theories" (LSTs) to the quantum description of four dimensional compactifications, because one can argue that their exact high energy spectrum has the Hagedorn form. Some space is therefore devoted to a discussion of the properties of LSTs, which were first discovered as the proper formulation of Matrix Theory on the five torus.

Keywords: M-Theory, String Duality, Superstring Vacua.
1. Introduction – Limitations on Lagrangian Quantum Mechanics

This lecture series is about Matrix Theory [5], a nonperturbative, Lagrangian formulation of M Theory. There has been a lot of confusion about this theory in the literature,
to the extent that it has been characterized as *controversial* in the popular press \[7\]. Much of this confusion has been caused by misinterpretation and misunderstanding of what the theory was supposed to do, and too little appreciation of how important it is to take the large $N$ limit in order to calculate amplitudes of interest to a Lorentz invariant theory. However, some of the difficulties of Matrix Theory are more important, and reflect crucial issues about any quantum theory of gravity. I therefore want to begin with a critical discussion of what we can expect to be the limitations of ordinary Lagrangian quantum mechanics in the description of *any* quantum theory of gravity.

Any theory which contains general relativity (GR) must be time reparametrization invariant. Mathematically this means that the time translation generator for an arbitrary definition of time is a constraint, which must vanish on physical states. Physically it means that any nonvanishing definition of energy must be conjugate to a physical clock variable which measures time. This is certain to cause problems in the quantum theory, where there must in general be variables which do not commute with the clock. Thus, the very definition of physical time translation implies some sort of semiclassical approximation in which the clock evolves classically. In a closed cosmology, we cannot expect such an approximation to be valid with arbitrary precision. However, if the universe has a boundary and is of infinite size, then the boundary conditions at infinity define frozen classical variables which can be used as clocks. Typically, we insist that the metric at infinity approaches that of a noncompact symmetric space (Minkowski or Anti-DeSitter (AdS)) and the natural time translation generators are chosen from the asymptotic symmetry group of the metric. In these lectures we will be concerned primarily with asymptotically Minkowski spaces.

Let us first consider an ordinary Lorentz frame at spacelike infinity. Then there is a special Poincaré subgroup of the asymptotic diffeomorphisms of the metric \[8\], and up to a Lorentz transformation, a unique choice of Hamiltonian. Quantum M Theory (or, for those who are still skeptical, *any* quantum theory of gravity) will have a Hilbert space on which this generator acts as a Hermitian operator. We also expect it to have a ground state $|0\rangle$, whose energy eigenvalue must, for consistency, be zero. It might in fact have a discrete or continuous ground state degeneracy, labelled by expectation values of Poincaré invariant operators. We will assume that, as in local field theory, there is a large class of interesting operators (hereafter called localizable operators) that do not disturb the boundary conditions at infinity. If we restrict attention to localizable operators, the Hilbert space breaks up into superselection sectors, each of which has a unique ground state. Given any localizable operator $O$, we can formally define the time dependent Heisenberg operator $O(t)$ by

$$O(t) \equiv e^{iHt}Oe^{-iHt}. \quad (1.1)$$
To investigate the degree of formality of this definition, we compute the two point function
\[ \langle O(t)O^\dagger(0) \rangle = \int_0^\infty dE e^{-iEt} \rho_O(E), \] (1.2)
where the spectral density is defined by
\[ \rho_O(E) \equiv \sum_n \delta(E - E_n) \langle 0|O|n \rangle^2. \] (1.3)

The crucial question is now the convergence of this integral representation, or equivalently, the high energy behavior of the spectral density. In quantum field theory, the high energy behavior of the theory is determined by a conformally invariant fixed point. The density of states in volume \( V \) behaves like
\[ \rho \sim e^{cV(E/V)^{(d-1)}} \] (1.4)
where \( d \) is the dimension of spacetime. Generic operators localized in the volume will have a spectral density \( \rho_O \) with the same behavior. However, there is a special class of local operators of fixed dimension, which connect the vacuum only to the states in a given irreducible representation of the conformal group. The spectral density of these operators grows only like a power of the energy. Note that in either case, we can define the Green’s function by analytic continuation of an absolutely convergent integral in Euclidean time.

In a quantum theory of gravity, it is extremely plausible, that for a theory with four or more asymptotically Minkowski dimensions the high energy density of states is dominated by highly metastable black holes. The existence and gross properties of these states follow from semiclassical GR. The density of black hole states is given by the Bekenstein-Hawking formula
\[ \rho \sim e^{k(E/M_P)^{(d-2)/(d-3)}} \] (1.5)
where \( M_P \) is the \( d \) dimensional Planck mass. There are two interesting features of this formula. The first is its independence of the volume. This is a consequence of the Jeans instability. If we try to construct an extended translation invariant state other than the vacuum in a theory containing gravity, we eventually get to an object whose Schwarzschild radius exceeds its physical size and the system collapses into a black hole. The only translation invariant states are those which are superpositions of a single black hole at different positions in spacetime. Secondly, any operator whose matrix elements between the vacuum and states of energy \( E \) are not drastically cut off at energies above the Planck scale, will not have a well defined two point function. Thus,
although the general formalism of quantum mechanics will be valid, we should not expect a conventional Lagrangian description of the system to be applicable. Indeed, the Lagrangian formalism produces Green’s functions of Heisenberg operators as its fundamental output, and we are supposed to deduce the energy spectrum and the structure of the Hilbert space from this more fundamental data (similar remarks are of course applicable to any more abstract formalism which takes Green’s functions as its basic starting point). In fact, implicit in the definition of the Lagrangian formalism is an assumption that the short time behavior of the system is approximately free. This assumption is not even valid for a nontrivial fixed point theory. However, in field theory we can restore the validity of Lagrangian methods by realizing the theory as the limit of a cutoff system or (in many cases) by realizing the nontrivial fixed point as an infrared limit of an asymptotically free theory. No such workarounds appear to be available for the quantum theory of gravity.

Another fascinating possibility is that the space of states of a very large black hole has a group of symmetries which partitions it into irreducible representations in much the same way that the conformal group partitions the states of charged black branes with AdS horizons. Then one could construct operators which connected the vacuum only to those states in an irrep of the group. If the density of states in an irrep had subexponential growth then these operators would have sensible correlation functions. In the absence of such a large black hole symmetry group, there will be no conventional Green’s functions in the holographic dual of asymptotically flat spacetimes.

It is interesting to contrast these results with our expectations in a light cone frame. We will discuss light cone formalism more extensively in the next section. For the moment we will need only the formula for light cone energy:

$$E \equiv P^- = \frac{P^2 + M^2}{P^+}$$

(1.6)

where $P$ (which we will set equal to zero) and $P^+$ are the transverse and longitudinal momenta respectively. Again assuming that the high energy density of states is dominated by black holes, we can write the light cone density of states as

$$\rho \sim e^{k(E/M_P)^{\frac{(d-2)}{2(d-3)}}}.$$  

(1.7)

Note that we now expect a convergent two point function in light cone time for any $d \geq 5$. Even for $d = 4$, we find only a Hagedorn spectrum (rather than the more divergent form of the black hole spectrum in ordinary energy) and the Green’s function will be defined for sufficiently long Euclidean time separations.

The conclusion that I want you to draw from this is that we should only expect a conventional Lagrangian quantum mechanics for M Theory in light cone time, and
perhaps only for 5 or more noncompact Minkowski dimensions. Another point to remember is that the high energy density of states seems to increase as the number of noncompact dimensions decreases. This suggests that compactified M Theory has more fundamental degrees of freedom than uncompactified M Theory a conclusion which we will see is borne out in the sequel.

Some readers will be curious about the case of two or three asymptotically Minkowski dimensions, where there are no black holes. Here the story is quite different, at least in those situations with enough supersymmetry (SUSY) to guarantee that there is a massless scalar field in the supergravity (SUGRA) multiplet. In these cases one can argue [10] that the system has very few states, because would be localized excitations so distort the geometry of spacetime that the asymptotic boundary conditions are not satisfied. In some sense, the resulting theory is topological.

Another interesting example of the arguments used in this section is M Theory in spaces with 3 or more asymptotically AdS dimensions (AdS2 has the same sort of problems as two or three dimensional Minkowski space [11]). Here the boundary of spacetime is timelike and there is no analog of a light cone frame. There are two natural inequivalent choices of Hamiltonian, corresponding to global and Poincaré time. The corresponding black objects are AdS Schwarchild black holes and near extremal black branes of appropriate dimension. Both have positive specific heat, which is to say that their density of states grows less rapidly than an exponential. And in precisely these cases, we expect [12] that there is an exact description of the system in terms of a conventional quantum field theory.

In the next section, we will begin an exploration of M Theory in the light cone frame, starting with the simplest case of eleven noncompact directions.

2. Matrix Theory in Eleven Dimensions

2.1 Quantum Field Theory in Light Cone Frame and Discrete Light Cone Quantization

To set the stage for our discussion of Matrix Theory we begin with a brief introduction to light cone field theory [13]. As hinted in the previous section, the basic idea is to choose a light cone frame pointing in a specific spatial direction. In this basis, the momentum takes the form \( (P^-, P^+, P) \). \( P^- \) is taken to be the time translation generator and takes the form \( E \equiv P^- = \frac{P^2 + M^2}{P^+} \), where \( M^2 \) is the mass squared operator. The subgroup of the Lorentz group which leaves the light cone frame invariant is obviously isomorphic to the Galilean group in \( d - 2 \) dimensions, with \( P^- \) transforming like the Galilean energy, \( P^+ \) like the Galilean momentum, \( P^+ \) like the Galilean mass, and \( M^2 \)}
like a Galilean invariant potential. Compared to a true nonrelativistic field theory the
new feature here is the continuous Galilean mass spectrum and the associated possibility of particles breaking up into others with smaller Galilean mass. There are also two other sets of Lorentz transformations. The first is the longitudinal boost generator, which rescales $P^\pm$ in opposite directions. The other is the set of null plane rotating transformations which rotate the direction of the null vector into the transverse directions. These are typically the most difficult symmetries to realize in building an actual Lagrangian.

In supersymmetric theories, we must also include the supertranslations. Since they are Lorentz spinors, they break up into left moving and right moving pieces under longitudinal boosts. These satisfy the following commutation relations

\[ [q_a, q_b]_+ = \delta_{ab} P^+ \tag{2.1} \]
\[ [Q_a, Q_b]_+ = \delta_{ab} P^- \tag{2.2} \]
\[ [Q_a, q_b]_+ = \gamma^i_{ab} P_i \tag{2.3} \]

It will turn out in Matrix Theory that it is relatively easy to implement these symmetries, but that they give strong constraints on the dynamics of the system.

For the purposes of this brief introduction to light cone field theory, we will restrict our attention to a simple scalar Lagrangian of the form

\[ \mathcal{L} = \partial_+ \phi \partial_- \phi - (\nabla \phi)^2 - V(\phi). \tag{2.4} \]

Standard (Dirac) quantization of this Lagrangian gives us the commutation relations

\[ [\phi(z, x, t), \partial_z \phi(z', x', t)] = \frac{1}{2} \delta(z - z') \delta(x - x'). \tag{2.5} \]

which are solved by

\[ \phi = \int_0^{\infty} \frac{dP^+}{P^+}[a(x, P^+) e^{-iP^+z} + a^\dagger(x, P^+) e^{iP^+z}] \tag{2.6} \]

where $a$ and $a^\dagger$ have the commutation relations of ordinary second quantized nonrelativistic fields. The $z$ independent part of $\phi$ has no canonical momentum and is a constraint variable. Often one solves for it at the classical level, but this procedure is unsatisfactory. A better strategy (in principle) is to derive the light cone formalism from a covariant path integral. Then one sees explicitly that the zero longitudinal momentum degrees of freedom must be integrated out [15] and that there are contributions to the effective interaction for the nonzero modes at all orders in the loop expansion.
(and furthermore that the higher order terms are larger than those obtained in the tree approximation — the semiclassical expansion is not applicable).

In the formalism developed so far, the zero mode problem is mixed up with an equally vexing problem from modes with nonzero, but arbitrarily small, longitudinal momentum. The method of Discrete Light Cone Quantization \[13\] (DLCQ) is an attempt to repair this difficulty by compactifying the lightlike longitudinal direction (studying the field theory on a space where \(x^-\) is identified with \(x^- + R\)) , thus rendering the longitudinal momentum discrete. This has another property which at first sight renders DLCQ extremely attractive. As one can see from the expansion \(2.6\), the Fock space of light cone field theory contains only particles with positive longitudinal momentum. Operators with negative longitudinal momentum are annihilation operators. If longitudinal momentum is conserved, positive and discrete, then states with \(P^+ = N/R\) can have at most \(N\) particles in them. Thus in DLCQ in the sector with \(N\) units of momentum, field theory reduces to nonrelativistic quantum mechanics with a fixed number of particles.

As there is no such thing as a free lunch, there must be a catch somewhere. In fact, in field theory there are two. First of all, in order to have a hope of recapturing Lorentz invariant physics one must take the large \(N\) limit. Field theory in a space with a periodic lightlike direction is weird, very close to a space with periodic time, which has apparent grandfather paradoxes\(^1\). If \(N\) is large, one can hope to make wave packets which are localized along the lightlike direction. Furthermore, since systems with large longitudinal momenta would be expected to Lorentz contract, their physical size in the longitudinal direction might also be small. The physics of such large \(N\) systems could very well be oblivious to the lightlike identification and reproduce that of the uncompactified, Lorentz invariant, system. The use of words like \(might\), \(could\) and \(hope\) in the last three sentences, signals that there is no rigorous argument guaranteeing that this is the case.

The second catch is integrating out the zero modes. One can get some insight into how difficult this is in field theory \[15\] by viewing lightlike compactification as an infinite boost limit of spacelike compactification on an infinitesimally small circle. If we compactify an ordinary field theory on a circle of very small radius \(R_S\) and concentrate only on the lagrangian of the zero modes, then \(R_S\) appears as a multiplicative factor. In other words, the theory of the zero modes is at infinitely strong coupling. Thus, even if the original field theory is weakly coupled, the problem of calculating the effective Lagrangian for the nonzero modes in DLCQ appears intractable in general\(^2\).

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\(^1\)Though resolved in the way first proposed by R.A. Heinlein in \[14\].

\(^2\)We will later encounter a field theory where DLCQ leads to a weakly coupled system.
Given all of these problems, why are we interested in light cone quantization in M Theory? Apart from the general motivation given in the introduction, there are many indications that M Theory is much better behaved than field theory in the light cone frame. The many successes of light cone string theory attest to this. In particular, in perturbative string theory in light cone gauge, longitudinal momentum is the spatial coordinate on the string world sheet. DLCQ is a discretization of the string world sheet. Since the uncompactified light cone string theory is a conformal field theory, the process of taking the large $N$ limit is controlled by the world sheet renormalization group and the problems with zero longitudinal momentum modes are encoded in local contact terms. Furthermore, although the actual computation of these counterterms to all orders in perturbation theory would be tedious, the form of light cone string perturbation theory leads immediately to the guess that the correct answer is given by conformal field theory on higher genus Riemann surfaces, a prescription which automatically fixes the contact terms by analytic continuation, at least in many cases.

The other reason to be somewhat more hopeful about our prospects for success, is SUSY. SUSY nonrenormalization theorems give us some control over the possible effects of integrating out the zero modes. In the cases we will study, this is probably enough to fix the effective lagrangian uniquely.

2.2 The Holographic Principle and the Matrix Theory Lagrangian

For a number of years, Charles Thorn \cite{16} championed an approach to string theory in light cone frame based on the notion of string bits, which were taken to be pieces of string carrying the lowest possible value of longitudinal momentum. Full strings, with higher values of longitudinal momenta were supposed to be bound states of these more fundamental constituents. Thorn explicitly noted that in such a formalism, one of the dimensions of spacetime appeared dynamically. The fundamental constituents propagated on a surface of one lower dimension.

In an independent development some years later, G. ’t Hooft \cite{17} proposed that the apparent paradoxes of black hole physics in local field theory might be resolved if the fundamental quantum theory of gravity had degrees of freedom which lived on hypersurfaces of dimension one lower than that of the full spacetime, with a density equal to the Planck density. The motivation for the latter restriction was the the Bekenstein-Hawking formula for the entropy of a black hole. He characterized a theory of this type as holographic. Susskind \cite{18} then realized that light cone gauge string theory embodied at least half of the holographic principle of ’t Hooft, essentially because of the properties described in the paragraph above. The Bekenstein bound is not satisfied in perturbative string theory. If we think of string bits as the fundamental degrees of freedom, then a string made up of $N$ bits has a transverse extent of order $\ln N$. 


On the other hand, the Bekenstein bound would suggest that the transverse area had to grow like $N$. It is not terribly surprising that this part of the holographic principle can only be realized in a nonperturbative manner in string theory. The Bekenstein bound is formulated in Planck units and formally goes to infinity when the string coupling is taken to zero with the string tension fixed. Morally speaking, it is similar to restrictions on operators in large $N$ field theory which stem from the fact that the traces, $\text{tr} M^k$, of an $N \times N$ matrix are not all independent. It is well known that such restrictions are nonperturbative in the $1/N$ expansion.

In nonperturbative formulations of M Theory, such as Matrix Theory and the AdS/CFT correspondence, the second half of the holographic principle is derived by explicit dynamical calculations [5], [19]. In the latter case it is rather easy to derive and one finds that the bound is saturated, while in Matrix Theory the argument is based on crude approximations and one finds the restriction on the number of degrees of freedom only as a bound.

Let us now proceed to the construction of the Matrix Theory Lagrangian in eleven flat spacetime dimensions. The original construction of [5] used the holographic principle as its starting point and used the language of the Infinite Momentum Frame rather than light cone quantization. Susskind [63] then suggested that the finite $N$ Matrix Theory lagrangian was the DLCQ of M Theory. Here we will follow a much cleaner argument due to Seiberg [20] (see also [21]) which begins from the idea that DLCQ of a Lorentz invariant theory can be obtained by a boost applied to a system compactified on a spacelike circle, if the radius, $R_S$, of the spacelike circle is taken to zero, with the rapidity $\omega$ of the boost scaling like $\ln(1/R_S^2)$. If we wish to be in the sector with $N$ units of longitudinal momentum in DLCQ, then we should work in the sector with $N$ units of spacelike momentum. This argument is a derivation of Susskind’s claim.

The crucial feature which distinguishes M Theory from most field theories, is that the limiting theory on a small spacelike circle is free. Indeed, it is free Type IIA string theory [22] (I am assuming that all the students at this school are familiar with this paper or will shortly become so). The sector with $N$ units of momentum around the circle is the sector of IIA string theory with $N$ units of D0 brane charge. In free string theory we can characterize this sector as containing $N$ D0 branes and the strings connecting them, as well as any number of closed strings and D0 brane anti D0 brane pairs. However, we are interested only in degrees of freedom with finite light cone energy. The light cone energy is of order $e^\omega(E - N/R_S) \sim E/R_S - N/R_S^2$. The lowest energy in the sector with $N$ D0 branes is $N/R_S$. We are clearly interested only in states whose splitting from this ground state is inversely proportional to the D0 brane mass. For a single D0 brane, examples of such states are the states of the D0 brane moving with (transverse from the point of view of the 11 dimensional light cone frame) momenta
fixed as $R_S \to 0$. Note that the eleven dimensional Planck scale remains fixed in this limit, so this is the same as requiring the transverse momenta to be a finite number of Planck units in the weak string coupling limit. For multiple D0 branes separated by distances of order the Planck scale, we must also include degrees of freedom which create and annihilate minimal length open strings between the branes. The Lagrangian for this system was written down in this context in [23]. It is the dimensional reduction of ten dimensional Super Yang Mills theory on a nine torus, and, as such, was first written down by [24]. We will write it both in string and eleven dimensional Planck units

$$ L = \frac{L^3}{g_S} \text{Tr} \left( \frac{\dot{\phi}^2 + [\phi^i, \phi^j]^2}{R} + i \Theta \dot{\Theta} - \Theta [\phi^i, \gamma^i \Theta] \right) $$  \hspace{1cm} (2.7)

$$ L = \text{Tr} \left( \frac{\dot{X}^2}{R} + R \frac{[X^i, X^j]^2}{L_p^2} + i \theta \dot{\theta} - R \frac{\theta [X^i, \gamma^i \theta]}{L_p^3} \right). $$  \hspace{1cm} (2.8)

Here $\phi^i$ and $X^i$ are nine Hermitian $N \times N$ matrices, the former with dimensions of mass and the latter with dimensions of length. Similarly $\Theta$ is a sixteen component $SO(9)$ spinor, which is an Hermitian $N \times N$ matrix, and has dimensions of $[m]^{3/2}$, while $\theta$ has the same transformation properties, but is dimensionless.

Witten’s motivation for this Lagrangian was that it summed up the leading infrared singularities of string perturbation theory, that are caused by zero energy open strings when D0 branes are separated by distances less than the string length. The authors of [25] gave a careful argument to all orders in string perturbation theory that this Lagrangian in fact captured all of the dynamics at energy scales equal to the kinetic energy of a D0 brane with Planck momenta. Seiberg’s argument is often criticized as being “too slick” and the work of [15] is cited as an example of the dangers of naively ignoring the integration out of the zero modes in DLCQ. In fact, the fact that M Theory on a small circle is weakly coupled Type IIA string theory, and the careful analysis of [23] (which shows that the kind of perturbative divergences of the small $R_S$ limit found by [15] in field theory are absent to all orders in perturbation theory) suggest that the latter reference is completely irrelevant in the present context. About the only loophole one could imagine in the argument is the possibility that weakly coupled IIA string theory has nonperturbative corrections to (2.7) which somehow survive the $R_S \to 0$ limit.

Even this loophole can probably be closed by proving the following conjecture: the Lagrangian (2.7) is the only Lagrangian for this set of degrees of freedom consistent with the symmetries we will list below. The italicized phrase means in particular that the Lagrangian may not contain time derivatives higher than the first, though it may contain higher powers of the first time derivatives. The conjecture has been
partially proven in [24]. In trying to give a more complete proof one should use certain facts which were not employed in this reference. In particular, the Lagrangian we have written has a thirty two generator odd subalgebra of its symmetry algebra and has translation invariance in the transverse directions, as well as Galilean boost invariance. Furthermore, its gauge group is $U(N)$ and not $SU(N) \times U(1)$, so that arbitrary separations of the trace parts of matrices from their traceless parts are not allowed. These facts seem to give a fairly straightforward argument for the conjecture if one restricts attention to Lagrangians which can be written as a single trace. I do not pretend to have a complete proof of this conjecture (although I am convinced it is correct) and leave it to an enterprising student. To my mind, the strongest arguments for Matrix Theory come from its successes in reproducing known facts and conjectures about M Theory as dynamical results of a complete Lagrangian system. There are difficult questions about whether the large $N$ limit really reproduces the Lorentz invariant dynamics which interests us. But there seems to be little doubt that in a variety of backgrounds Matrix Theory is a correct DLCQ of M Theory.

To proceed with the exposition of the results of Matrix Theory we begin with a list of its symmetries:

The most important of these are SUSYs. The full supertranslation algebra is preserved in DLCQ. Only the spectrum of the translation generators is different from that expected in the uncompactified theory. As usual in light cone frame, spinors can be decomposed as right moving and left moving under the $SO(1,1)$ group of boosts in the longitudinal direction (which is not a symmetry of DLCQ). Thus, there are two sets of spinor SUSY generators, each transforming as the $16$ of the transverse $SO(9)$ rotation group (which is preserved by DLCQ). The first of these is simply realized in terms of the 16 canonical matrix variables of Matrix Theory as

$$q_a = \sqrt{\frac{1}{R} \text{tr} \theta_a} \quad (2.9)$$

The anticommutator of these is $\delta_{ab} \frac{N}{R}$, which identifies $N$ as the integer valued, positive longitudinal momentum $P^+$ of DLCQ. The anticommutator of the left and right moving SUSY generators is

$$[q_a, Q_b]_+ = \gamma P_{ab} \quad (2.10)$$

This is realized by

$$Q_a = \sqrt{\frac{R}{N}} \text{tr} (\gamma P_{ab} \theta_b + i \gamma^{ij}_{ab} [X^i, X^j]) \theta_b \quad (2.11)$$

The second term does not contribute to (2.10) but is probably required by the final relation of the supertranslation algebra

$$[Q_a, Q_b]_+ = \delta_{ab} P^- \quad (2.12)$$
In fact, the latter is realized only on $U(N)$ invariant states of the model, which identifies $U(N)$ as a gauge group. The word probably in the penultimate sentence reflects the incompleteness of the uniqueness proof I referred to above.

In addition to these symmetries, the model is invariant under $SO(9)$ rotations and transverse Galilean boosts. The missing parts of the eleven dimensional super-Poincaré group are the longitudinal boosts and the null plane rotating parts of the spatial rotation group. These may be restored in the large $N$ limit. Note that the Galilean transformations act only on the $U(1)$ center of mass variables and restrict their Hamiltonian to be quadratic in canonical momenta.

Finally, I note a discrete symmetry under $\theta \to \theta^T$, $X^i \to -(X^i)^T$, which commutes with half of the supertranslations and with $P^\pm$. This symmetry is instrumental in the matrix theory description of Hořava-Witten domain walls [26].

### 2.3 Gravitons and Their Scattering

The classical Lagrangian of Matrix Theory has a moduli space consisting of commuting matrices. The high degree of supersymmetry of the system guarantees that this moduli space is preserved in the quantum theory. This means that if we integrate out all of the non moduli space variables, then the effective Lagrangian on the moduli space has no potential. Furthermore, the terms quadratic in time derivatives are not renormalized, and the terms quartic in time derivatives appear only at one loop [28]. Furthermore, for $N > 2$ there are other terms in the effective Lagrangian which receive only a unique loop correction and thus are exactly calculable [27].

The justification for the description by an effective Lagrangian is the Born-Oppenheimer approximation. When we go off in some moduli space direction $X_0 = \bigoplus_r r_k I_{N_k \times N_k}$, then variables which do not commute (as matrices) with $X_0$ have frequencies of order $|r_k - r_l|$. Thus, if these distances are large, these variables can be safely integrated out in perturbation theory.

For the $SU(N_k)$ variables, which commute with $X_0$ the Born-Oppenheimer argument depends on a fundamental conjecture about this system, due to Witten [23]. That is, that the $SU(N)$ version of this supersymmetric quantum mechanics has exactly one (up to an obvious spinor degeneracy to be discussed below) normalizable SUSY ground state. This conjecture lies at the heart of the M Theory - IIA duality, and the whole web of string dualities would collapse if it proved false. More impressively, the conjecture has been more or less rigorously proven for $N = 2$, and arguments exist for higher values of $N$ [28]. Finally, arguments can be given [1] that the typical scale of energy of excitation of these bound states is of order $1/N^p$ with $p < 1$. Since we will see that energies on the moduli space are of order $1/N$, this justifies the use of the Born-Oppenheimer approximation for large $N$.
If we accept the bound state conjecture it follows that the large $N$ limit of the model contains in its spectrum the Fock space of free eleven dimensional supergravitons. In fact, the theorems we have cited show that the Lagrangian along the moduli space direction $X_0$ with large separations, is

$$
\sum_{k=1}^{n} \frac{N_k}{2R} \dot{r}_k^2 + i \theta_k \dot{\theta}_k
$$

which is that of a collection of massless eleven dimensional superparticles in light cone frame. Each of the $\theta_k$ variables is a 16 component $SO(9)$ spinor. The Hamiltonian of this system is $\theta$ independent, so the fermionic variables serve merely to parametrize the degeneracy of particle states. They are quantized as 16 Clifford variables so their representation space is 256 dimensional. It decomposes under $SO(9)$ as $44 \oplus 84 \oplus 128$, which are the states of a symmetric traceless tensor, a totally antisymmetric three tensor, and a vector spinor satisfying $\gamma_i^{ab} \psi^i_b = 0$. This is precisely the content of the 11D SUGRA multiplet. The required Bose or Fermi symmetrization of multiparticle states follows from the residual $S_n$ gauge invariance on the moduli space (commuting matrices are diagonal matrices modulo permutations) and the fermionic nature of the spinor coordinates.

I want to note in particular, that SUSY was crucial to the cluster property of these multiparticle states. In the nonsupersymmetric version of the matrix model, an $|r_k - r_l|$ potential is generated on the moduli space and the whole system collapses into a single clump. I think that this may be one of the most interesting results of Matrix Theory. In perturbative string theory, explicit SUSY breaking is usually associated with the nonexistence of a stable, interacting vacuum state, and often with tachyonic excitations which violate the cluster property. Matrix Theory suggests even more strongly that SUSY may be crucial to the existence of a theory of quantum gravity in which propagation in large classical spacetimes is allowed. In fact, it appears that only asymptotic SUSY is strictly necessary for the cluster property. Indeed the cancellation of the large distance part of the potential has to do with the SUSY degeneracy between states at extremely high energy. However, simple attempts to break SUSY even softly appear to lead to disaster [31].

Before beginning our discussion of graviton scattering, I want to clarify what we can expect to extract from perturbative or finite $N$ calculations in the matrix quantum mechanics. The basic idea of the calculations that have been done is to study zero longitudinal momentum transfer scattering by concentrating on the region of configuration space where some number of blocks are very far away from each other. The intra-block wave functions are taken to be the normalizable ground states in each block. This leaves the coefficients of the unit matrix in each block and the off block diagonal
variables. In the indicated region of configuration space the frequencies of the latter are very high, and one attempts to integrate them out perturbatively. There are two rather obvious reasons why these calculations should fail to give the answers we are interested in. The first is that the nominal perturbation parameter for this expansion is \( \frac{N r^3}{\rho^3} \) where \( r \) is a transverse distance between some pair of blocks. In order to make comparisons with SUGRA we want to take \( r/L \gg 1 \), but \emph{independent of} \( N \) as \( N \) tends to infinity. Indeed the scattering amplitudes in this regime of impact parameters should become independent of \( N \) (or rather scale with very particular powers since they refer to exactly zero longitudinal momentum transfer), as a consequence of Lorentz invariance. This is evidently not true for individual terms in the perturbation expansion. It should be emphasized that this means we are \emph{not} interested in the 't Hooft limit of this theory.

In addition to this, the perturbation expansion is not even a correct asymptotic expansion of the amplitudes in the large \( r \) region. To leading order in inverse distances, the interactions between the high frequency off diagonal variables, and the \( SU(N_i) \) variables in individual blocks does not enter in the expressions for amplitudes. At higher orders this is no longer the case. The complete calculation involves expectation values of operators in the individual block wave functions. The fact that the off diagonal variables have very high frequencies allows us to make operator product expansions and limits the number of unknown expectation values that come in at a given power of \( r \). Terms like this will give fractional powers of the naive expansion parameter. However, since the short time limit of quantum mechanics is free, all the operators have integer dimensions and we are led to expect only integer powers \( L_P \) in the expansion. The \( N \) dependence of these terms is completely unknown.

The second of these problems is inescapable, but the first could be avoided if it were possible to make direct comparisons between finite \( N \) Matrix Theory and DLCQ SUGRA. It is important to realize that there is absolutely no reason to expect this to be so. The intuitive reason is that the gravitons of Matrix Theory are complicated bound states rather than structureless particles. They only behave like structureless particles when their relative velocities are very small, because then the scattering state is almost a BPS state. In the large \( N \) limit the velocities become arbitrarily small and one can argue that to all orders in energies over \( M_P \) they should behave like particles of an effective field theory. However, for finite \( N \), there is no reason for this to be so, even at low energy and momentum transfer. More mathematically, we can note that SUGRA is a limit of M Theory in which all momenta are small compared to the Planck mass. On the other hand, we have seen that the DLCQ system can be viewed as a system compactified on a tiny circle, with a finite number of units of momentum. Thus every state in the system contains momenta large compared to the Planck scale. There
is no reason to expect the limit which gives DLCQ to commute with the SUGRA limit.

The reason that some amplitudes are calculable in perturbation theory is that the system has a host of nonrenormalization theorems. That is, the large SUSY of the Matrix Theory Lagrangian so constrains certain terms in the effective Lagrangian for the relative positions that they are given exactly by their value at some order of the loop expansion. I will not give a description of the state of these calculations, but refer the reader to the literature [28], [27].

The fact that only quantities determined by symmetries are calculable gives one pause, I must admit (unless one is able to prove the conjecture above that the symmetries completely determine the Lagrangian) but one must recognize that this is a rather generic state of affairs in recent results about M Theory. However, what I consider important about Matrix Theory is that it reduces all questions about M Theory (in the backgrounds where it applies) to concrete, albeit difficult, problems in mathematical physics. We are no longer reduced to guesswork and speculation. Of course, this is no better than the statement that lattice gauge theory reduces hadron physics to a computational problem. Obviously Matrix Theory will only be truly useful if one can find analytic or numerical algorithms for efficiently extracting the S-matrix from the Lagrangian. On the other hand, it may be possible to attack certain conceptual problems before a practical calculation scheme is found.

### 2.4 General Properties of the S-Matrix and the Graviton Wave Function

We can easily write down an LSZ-like path integral formula for the S-matrix of gravitons in Matrix Theory. Simply perform the path integral with the following boundary conditions: as $t \to -\infty$, the matrices approach the moduli space

$$X \to X_0^I = \bigoplus_k r_k(t)I_{N_k \times N_k}$$

with similar formulae for the fermionic variables. $r_k(t)$ are classical solutions of the moduli space equations of motion. They are labelled by the transverse momenta of the incoming states, while the longitudinal momenta are the $N_k$. Similarly, for $t \to \infty$ we have

$$X \to X_0^F = U^\dagger \bigoplus_k r_k(t)I_{N_k \times N_k}U$$

where we must integrate over the $U(N)$ matrix, $U$, in order to impose gauge invariance. Of course, the number of outgoing particles, as well as their transverse and longitudinal momenta, will in general be different from those in the initial state $^3$.

$^3$An alternate approach to the Matrix Theory S-matrix can be found in [29].
This formula does not quite give the S-matrix since the $SU(N_k)$ variables are sent to zero asymptotically by the boundary conditions. In principle we should allow them to be free, and convolute the path integral with the bound state wave function for each external state. Thus, our path integral computes the S-matrix multiplied by a (momentum dependent) factor for each external leg equal to the bound state wave function at the origin. It is likely that in the large $N$ limit these renormalization factors will vanish, so we would have to be careful to extract them before computing the limiting S-matrix.

Several of the S-matrix elements for multigraviton scattering at zero longitudinal and small transverse momentum transfers can be computed with the help of nonrenormalization theorems. All of these computations agree precisely with the formulae from 11D SUGRA. As noted above, we cannot expect to make more detailed comparisons until we understand the large $N$ limit much better than we do at present.

We can however try to understand what could possibly go wrong with the limiting S-matrix. Assuming the bound state conjecture, we know that the large $N$ theory has the correct relativistic Fock space spectrum. Furthermore, the S-matrix exists and is unitary for every finite $N$. This implies that individual S-matrix elements cannot blow up in the limit. Furthermore, we know that some T-matrix elements are nonzero so the S-matrix cannot approach unity (I do not have an argument that amplitudes with nonzero longitudinal momentum transfer cannot all vanish in the limit). The absence of pathological behavior in which individual S-matrix elements oscillate infinitely often in the large $N$ limit is more or less equivalent to longitudinal boost invariance, which states that as the $N_k$ get large, S-matrix elements should only depend on their ratios. Thus, proving the existence of generic S-matrix elements is probably equivalent to proving longitudinal boost invariance.

Assuming the existence of limiting S-matrix elements, there is another disaster that could occur in the limit. This is an infrared catastrophe. That is, the cross section for reactions initiated by only a few particles might be dominated by production of a number of particles scaling like a positive power of $N$. Then, the S-matrix would not approach a well defined operator in Fock space. In 11D SUGRA the infrared catastrophe is prevented by Lorentz invariance. Again we see a possible connection between the mere existence of the S-matrix, and its Lorentz invariance.

A possible avenue for investigating the infrared catastrophe is to exploit the fact that the production of a large number of particles at fixed energy and momentum means that each of the produced particles has smaller and smaller energy and momentum. It is barely possible that the nonrenormalization theorems will give us sufficient information about scattering in this regime to put a bound on the multiparticle production amplitudes. Personally, I believe that a demonstration of the existence and Lorentz
invariance of the limiting S-matrix must await the development of more sophisticated tools for studying these very special large $N$ systems.

2.5 Membranes

One of the attractive features of Matrix Theory is the beautiful way in which membranes are incorporated into its dynamics. This connection has its origin in groundbreaking work on membrane dynamics done in the late 80s [32]. In that work, the Matrix Theory Lagrangian was derived as a discretization of the light cone Lagrangian for supermembranes. The idea was to build a theory analogous to string theory, with membranes as the fundamental objects. The theory appeared to fail when it was shown that the Lagrangian had continuous spectrum [33]. Today we realize that this is actually a sign that the theory exceeds its design criteria: it actually describes multibody states of membranes and gravitons, and the continuum states are simply the expected scattering states of a multibody system.

The membrane/matrix connection has been described so many times in the literature that I will only give a brief summary of it here. It is simplest to describe toroidal membranes, though in principle any Riemann surface can be treated [34]. One of the amusing results of this construction is that, for finite $N$, the topology of the membrane has no intrinsic meaning. States describing any higher genus surface can be found in the toroidal construction. It is only in the large $N$ limit that one appears to get separate spaces of membranes with different topology. The question of whether topology changing interactions (which certainly exist for finite $N$) survive the large $N$ limit, has not been studied, but there is no reason to presume that they do not.

The heart of the membrane construction is the famous Von Neumann-Weyl basis for $N \times N$ matrices in terms of unitary clock and shift operators satisfying

\begin{equation}
U^N = V^N = U^\dagger U = V^\dagger V = 1.
\end{equation}

\begin{equation}
UV = e^{2\pi i/N} VU
\end{equation}

Any matrix can be expanded in a series

\begin{equation}
A = \sum a_{mn} U^m V^n
\end{equation}

If, as $N \to \infty$ we restrict attention to matrices whose coefficients $a_{mn}$ approach the Fourier coefficients of a smooth function, $\hat{A}(p,q)$, on a two torus, then it is easy to show that

\begin{equation}
[A, B] \to \frac{i}{N}\{\hat{A}, \hat{B}\}_{P.B.}
\end{equation}
and

$$\text{tr } A \rightarrow N \int dp dq \hat{A}(p, q) \quad (2.20)$$

Using these equivalences, one can show that, on this subclass of large $N$ matrices, the Matrix Theory Lagrangian approaches that of the supermembrane.

One can extend the construction to more general Riemann surfaces (the original matrix papers cited in [32] worked on the sphere) by noting that the equations (2.16), (2.17) arise in the theory of the lowest Landau level of electrons on a torus propagating in a uniform background magnetic field of strength proportional to $N$. One can then study an analogous problem on a general Riemann surface. Note that since all of these Landau systems have finite dimensional Hilbert spaces, they can be mapped into each other. Thus, for finite $N$, the configuration spaces of membranes of general topology are included inside the toroidal case.

It is interesting to note that, at the level of the classical dynamics of the matrix model, the condition (2.19) is sufficient to guarantee that the membrane states constructed as classical solutions of the equations of motion obeying this restriction, will have energies of order $1/N$ and are thus candidates for states which survive in the (hoped for) Lorentz invariant large $N$ limit. This suggests that a more general condition, viz. that the matrices be replaced by operators whose commutator is trace class, may be a useful formulation of Matrix Theory directly in the infinite $N$ limit.

However, it is not clear that this sort of classical consideration is useful when $N$ is large. Indeed, it can be argued that in a purely bosonic matrix model, the classical energy of membrane states is renormalized by an amount which grows with $N$. By contrast, in the supersymmetric model, the infinite flat membrane is a BPS state [35] and the energies of large smooth membranes are all of order $1/N$ in the quantum theory.

The direct formulation of the infinite $N$ theory in eleven dimensions is an outstanding problem. It is clear that it is not simply the light cone supermembrane Lagrangian. But perhaps supermembranes do give us a clue to the ultimate formulation.

2.6 Fivebranes

We will attack the problem of finding the 5-brane of M Theory in Matrix Theory by applying Seiberg’s algorithm. In fact for longitudinal 5-branes, this was done by Berkooz and Douglas [36] long before Seiberg’s argument was conceived of. A longitudinal 5-brane is one which is wrapped around the longitudinal circle. In the IIA string language, it is an M5 brane wrapped around the small circle, and thus a D4 brane. Berkooz and Douglas [36] proposed that the Matrix Theory model for such a 5-brane was the large $N$ limit of the ND0-D4 system. This is a supersymmetric quantum mechanics obtained as the dimensional reduction of $\mathcal{N} = 2, d = 4$ SUSY Yang Mills, with
an adjoint and a fundamental hypermultiplet. For \( k \) such longitudinal five branes we simply introduce \( k \) fundamental hypermultiplets. Seiberg’s argument shows that this is the appropriate DLCQ of M Theory with \( k \) longitudinal 5-branes. In the large \( N \) limit, one can argue that a different procedure \cite{37} which dispenses with the fundamentals, may be a sufficiently good description of the system.

We now turn to the more problematic question of fivebranes in the transverse dimensions. After all, the longitudinal branes have infinite energy (relative to the Lorentz invariant spectrum) in the large \( N \) limit. Again, we use Seiberg’s argument and find ourselves faced with a system containing an NS 5-brane and N D0 branes in IIA string theory with vanishing coupling. This is the system described by (a certain soliton sector of) the \( k = 1 \) IIA little string theory \cite{4}. The system of \( k \) NS 5 branes in any string theory with vanishing coupling is a six dimensional Lorentz invariant quantum system which “decouples from gravity”. That is to say, although it contains states with the quantum numbers of bulk gravitons (and other closed string modes) in a ten dimensional spacetime with a linear dilaton field \cite{38}, they are described holographically \cite{39} in terms of a quantum theory with 5 + 1 dimensional Lorentz invariance. It is not a quantum field theory, because it has T-duality when compactified on circles \cite{10} and because it is an interacting theory with a Hagedorn spectrum \cite{11}.

The absence of a simple description of fivebranes in the original eleven dimensional Matrix Theory Lagrangian is probably the first indication of a general principle. In quantum field theory, the fundamental degrees of freedom are local. When we study the theory on a compact space we can encounter new degrees of freedom, like Wilson lines, but they are all implicit in the local variables which describe the dynamics in infinite flat space. In a theory of fundamental extended objects, we may expect that this will cease to be true. If there are fundamental degrees of freedom associated with objects wrapped around nontrivial cycles of a compact manifold, and if, as may be expected, the energies of all states associated with these degrees of freedom scale to infinity with the volume of the manifold, then the theory describing infinite flat space may be missing degrees of freedom.

Infinite fivebranes have infinite energy. We may imagine them to arise as limits of finite energy wrapped fivebranes on a compact space whose volume has been taken to infinity. Their description may involve degrees of freedom which decouple in the infinite volume limit. We shall see that this appears to be the case. Nonetheless, one feels a certain unease with the asymmetrical treatment of membranes and fivebranes\footnote{Which is only partly relieved by noting that the only true duality between the two \cite{30} is realized in the standard formulation of Matrix Theory on a three torus, and that this duality \textit{is} captured by the Matrix Theory formulation we will present below.}.
Perhaps there is a completely different formulation of light cone M Theory in which one somehow discretizes the light cone dynamics of the M5 brane. Indeed, since membrane charges are certainly incorporated in the world volume theory of the M5 brane\footnote{M5 branes are in a sense D branes of the M2 brane \cite{Hull} and so carry charges which measure the number of M2 branes ending on them. These couple to the two form potential on the world volume.} one might hope to obtain a more complete formalism in this way.

3. M Theory on a Circle

Let us now imagine trying to compactify one of the transverse dimensions of M Theory on a circle of size $aL_P$. We are led to study D0 branes in IIA string theory with coupling $g_S \sim (R_S/L_P)^{3/2} \to 0$ on a circle of radius $\sim ag_s^{1/3}l_s$. This situation is T dual to N D strings in Type IIB string theory with coupling $G_S \sim g_s^{2/3}$. The D strings are wound on a circle of radius $\sim g_s^{-1/3}l_s/a$. The states of this system whose energy gap above the ground state of the D0 brane system is of order $R_S$ are described by the 1 + 1 dimensional world volume theory of nonrelativistic D strings. This is 1 + 1 dimensional dimensional SYM theory with 16 SUSYs. After rescaling to light cone energy the Hamiltonian of the system is

$$RL_p \int_0^{L_p^2/R_9} ds \, dt \, Tr \left( f^2 + \left( \frac{DX}{L_p^2} \right)^2 + \frac{[X^i, X^j]^2}{L_p^4} + \theta [\gamma D + iX, \theta]L_p \right). \quad (3.1)$$

In this formula, boldface characters are $SO(8)$ vectors. $X$ has dimensions of length. The electric field strength $f$ has dimensions of $[m]^2$ and $\theta$ has dimensions of mass (so that the kinematic SUSY generator $q = \frac{1}{\sqrt{R}} \int ds \, Tr \theta$ has dimensions of $[m]^{1/2}$). $R$ is the radius of the lightlike circle. $\gamma$ are 1 + 1 dimensional Dirac matrices. $\theta$ is a sixteen component spinor which transforms as $(L, 8_c) + (R, 8_s)$ under the Lorentz and $SO(8)$ symmetries. The model has $(8, 8)$ SUSY as a two dimensional field theory.

It is obvious that as $R_9$ is taken to infinity, this system reduces to the 11 dimensional matrix theory we studied in the previous section. Indeed, the compactified theory has more degrees of freedom than the uncompactified one. In the Seiberg analog model, these correspond to strings connecting the D0 branes which wind around the torus and they obviously become infinitely massive in the limit $R_9 \to \infty$. Amusingly, in terms of the 1 + 1 dimensional field theory this decoupling is the standard one of Kaluza-Klein states when the radius of a circle compactified field theory goes to zero. A catchy phrase for describing this phenomenon is that \textit{in Matrix Theory dimensional oxidation is T dual to dimensional reduction.}

More interesting is the opposite limit $R_9/L_P \to 0$. According to the duality relation between M Theory and IIA string theory this limit is supposed to be the free IIA string
theory. This argument is based on the BPS formula which shows that the IIA string tension is the lightest scale in the theory in this limit, plus the relations between the low energy 11D and IIA SUGRA Lagrangians. It is important to realize that Matrix Theory provides us with a true derivation of this relation. In a sense the relation between duality arguments and Matrix Theory is similar to that between symmetry arguments based on current algebra and the QCD Lagrangian.

The derivation is easy. In the indicated limit, the mass scale of the SYM theory goes to infinity in Planck units and we should be left with an effective conformal field theory describing any massless degrees of freedom. The only obvious massless degrees of freedom are those on the moduli space, which is a 1+1 dimensional orbifold CFT with target space (supersymmetrized) $R^{8N}/S_N$. This is a classical statement, but the nonrenormalization theorems for a field theory with sixteen SUSYs ($(8,8)$ SUSY in the language of 1+1 field theory) assure us that the Lagrangian on the moduli space is not renormalized. Indeed, we will see in a moment that the leading perturbation of this system consistent with the symmetries is an irrelevant operator with dimension $(3/2, 3/2)$.

First we want to show that the spectrum of the orbifold quantum field theory at order $P^{-}\sim 1/N$ is precisely that of the Fock space of free Type IIA Green Schwarz string field theory [26]. To do this we note that the orbifold theory has topological sectors not contained in the $R^{8N}/S_N$ CFT in which the diagonal matrix fields are periodic only up to an orbifold gauge transformation in $S_N$. These are labelled by the conjugacy classes of the permutation group. A general permutation can be written as a product $C_{N_1}\ldots C_{N_p}$ of cyclic permutations. Within each such sector, we recognize that there is a residual $Z_{N_1}\times\ldots Z_{N_p}$ gauge symmetry of cyclic permutation within each block of the matrix.

The importance of these topological sectors is that, as the $N_k\to\infty$, they contain states of energy $1/N_k$. Indeed, a diagonal matrix function on an interval of length $2\pi$, satisfying $x_i(\sigma + 2\pi) = x_{i+1}(\sigma): \text{mod } N_k$, is equivalent to a single function $X_i(s)$ on an interval of length $2\pi N_k$. The Hamiltonian for Fourier modes of $X_i$ is $H = \frac{1}{N_k}\sum_n n\alpha^i_n\alpha^i_{-n}$. For a general topological sector, the Hamiltonian is a sum of $p$ such single string Hamiltonians. It is these long strings which are the strings of perturbative string theory.

There are two important constraints which follow from the remnants of the $U(N)$ gauge symmetry of the original matrix Lagrangian. First of all, on the subspace of states which are generated by finite Fourier modes of the $X^i$, the $Z_{N_k}$ residual gauge symmetries just become translations in the variable $s$. In the language of string theory, these are the light cone Virasoro constraints: $L_0 - \bar{L}_0 = 0$, on physical states. Secondly, for configurations in which several of the long strings are identical, there is a residual
permutation gauge symmetry which exchanges them. This is the conventional statistics
symmetry of quantum mechanics. It picks up the right minus signs because half integral
spin in the model is carried by Grassmann variables.

Finally, we want to note that the single string Lagrangians derived from this model
are Type IIA Green-Schwarz superstrings. Indeed, the $U(N)$ SYM theory from which
we began has an $SO(8)$ R symmetry group under which the left and right SUSY gener-
ators transform as the two different spinor representations. Thus, we may summarize
the results we have derived by the statement that in the $R_{10} \to 0, N \to \infty$ limits, the
Hilbert space of states of the Matrix Theory Hamiltonian with energies of order $1/N$,
is precisely the Fock space of free light cone gauge Type IIA string field theory. This
is a derivation of the famous duality conjecture relating M theory to Type IIA string
theory.

Dijkgraaf, Verlinde and Verlinde [26] went one step further, and showed that the
first correction to the free string Hamiltonian for finite $R_{10}$ was an irrelevant operator
which precisely reproduced the Mandelstam [42] three string vertex. Their argument
used effective field theory: this is the lowest dimension operator compatible with the
symmetries of the underlying SYM theory. Thus, they were unable to compute the
coefficient of the Mandelstam vertex. However, because $R_{10}$ determines the $1 + 1$ di-
mensional mass of the excitations of SYM theory which decouple in the zero radius
limit, the dimension $((3/2, 3/2))$ of the irrelevant operator determines that the string
coupling scales as $g_S \sim (R_{10}/L_p)^{3/2}$ which is the scaling anticipated from duality con-
siderations. The importance of the result is twofold. It shows that at least to leading
order in the small radius expansion the dynamics of Matrix Theory is, in the large $N$
limit, invariant under the ten dimensional Super-Poincaré group. And it shows that the
general structure of string perturbation theory will follow from matrix theory. Indeed,
up to contact terms on the long string world sheet, the Mandelstam vertex generates
the Riemann surface expansion of string perturbation theory. As yet, no one has found
an argument that Matrix Theory provides the correct contact terms to all orders in
perturbation theory.

Let us stress the important points that we have learned in this section. We have
seen in a very explicit way how Matrix Theory interpolates between the quantum
mechanics of the previous section, which describes 11D SUGRA, and free string theory.
In particular we have given a very explicit dynamical argument for why the IIA strings
are free in the zero radius limit. In previous discussions of the duality between IIA
strings and 11D SUGRA, this was more or less a postulate, supported only by the
behavior of the low energy effective Lagrangian. Another important lesson was that in
the limit of small radius, positions of objects on the compactified circle become wildly
fluctuating quantum variables. Indeed, these positions are Wilson lines in the gauge
theory, and we are going to the strong coupling limit. There is no longer any sensible geometrical meaning to the small circle, but the theory itself is perfectly smooth in the limit.

4. M Theory on a Two Torus

To study the theory on $T^2$ we apply the same set of arguments. D0 branes in weakly coupled Type IIA on an 11D Planck scale torus are treated by double T duality and related to D2 branes in a dual (but still weakly coupled) Type IIA theory. The states of finite light cone energy are described by maximally symmetric $2 + 1$ dimension SYM theory, with finite coupling, compactified on a torus which is dual to the M Theory torus. (In fact, here is an exercise: go through Seiberg’s argument for a general torus and show that the states of finite light cone energy are those whose energy scale is given by the coupling of the SYM theory obtained by doing T duality on all the radii. For $T^4$ and up this SYM theory is nonrenormalizable and we will see the implications of this in the next section.) The two Wilson lines of the SYM theory represent the coordinates of particles on the M Theory torus.

Aspinwall and Schwarz [43] argued using string duality that Type IIB theory in ten dimensional space was obtained as the zero area limit of M Theory on a two torus. The seeming contradiction that $11 - 2 = 10$ is resolved by noting that in the zero area limit a continuum of light wrapped M2 brane states appear, and play the role of momentum in a new tenth dimension. One of the puzzles of this approach is why the theory should be symmetric under rotations which rotate this new dimension into the other 9. Fundamental (F) and D strings are identified as M2 branes wrapped around the short resp. long cycle of the torus. This identifies the type IIB string coupling as the ratio of the short and long cycles (more generally, the imaginary part of the complex structure), explains the $SL(2, \mathbb{Z})$ duality of the theory and explains why there is a T duality between weakly coupled IIA and IIB theories.

If we try to take the zero area limit of the M Theory torus we are led to study the SYM theory on a torus of infinite area. Again, because the SYM coupling is relevant, this is equivalent to an infinite coupling limit in which all but conformal degrees of freedom decouple. Again, the existence of a moduli space ensures us that there is some sort of conformal limit rather than a completely trivial topological theory. Now however, there is a difference. In 2+1 dimensions there is a finite superconformal algebra which has 16 ordinary supercharges. It has an $SO(8)$ R subalgebra under which the supercharges transform as $(8,2)$ (with the 2 standing for their transformation under the $2 + 1$ dimensional Lorentz group). Furthermore, since the theory is conformal,
it depends only on the complex structure of the limiting $T^2$ and there is an obvious $SL(2, Z)$ invariance which acts on the complex structure $[44]$.

The finiteness of the superconformal algebra allows for the possibility of an interacting theory, and indeed the same interacting superconformal theory was postulated to describe the interactions of $N$ M2 branes at separations much smaller than the 11D Planck scale. Here we want the theory to be interacting because on a two torus with finite complex structure we are trying to describe interacting Type IIB string theory. We can understand the weak coupling limit, and simultaneously get a better understanding of the $SO(8)$ symmetry, by going to large complex structure. In the limit where the $a$ cycle of the M Theory torus is much smaller than the $b$ cycle, the corresponding SYM torus has cycles with the opposite ratio. Thus we can do a Kaluza Klein reduction on the cycle of the dual torus corresponding to the $b$ cycle and obtain a $1 + 1$ dimension field theory.

This is best done by going to the moduli space, which is a $U(1)^N$ SYM theory before the Aspinwall Schwarz limit. To take the limit we do an electromagnetic duality transformation, replacing the Abelian gauge fields by compact scalars $(F_{\mu \nu} \propto \epsilon_{\mu \nu \lambda} \partial^\lambda X^8)$, which decompactify in the limit. It is then obvious that there is an $SO(8)$ symmetry which rotates this boson into the seven original scalars. Indeed the Lagrangian on the moduli space is

$$\mathcal{L} = (\partial_\mu X^i)^2 + \bar{\theta}_a \Gamma^\mu \partial_\mu \theta_a$$

(4.1)

Here $i$ and $a$ each run from 1 to 8, and the $\theta$'s are two component, $2 + 1$ dimensional spinors. Of course, $2+1$ Lorentz invariance is broken by the compactification on a torus, but under the $SL(2, Z)$ which transforms the radii of the torus, the spinors transform as a doublet. There is an obvious $SO(8)$ symmetry of this Lagrangian under which the $X^i$ and $\theta_a$ could each transform in any of the eight dimensional representations. Note that both components of $\theta_a$ must transform the same way. Superconformal invariance in the Aspinwall Schwarz limit, assures us that this $SO(8)$ group survives in the interacting theory. Furthermore it tells us that the scalars must transform in the vector representation of $SO(8)$ and the fermions in one of the two chiral spinors (which one is a matter of pure convention).

When we further make the Kaluza-Klein reduction corresponding to large complex structure the moduli space theory is identical to the light cone gauge IIB Green-Schwarz string. Actually it is $N$ copies of this theory, related by a residual $S_N$ gauge symmetry (as in the previous section). We can rerun the analysis there and show that the theory is the Fock space of free strings in this limit and that the first correction to the limit is the correct Mandelstam interaction for IIB strings, with the right scaling of the couplings. The $SO(8)$ symmetry is seen to be the spacetime rotation symmetry of
IIB string theory, and superconformal invariance in $2 + 1$ dimensions has given us an understanding of the reason for emergence of this symmetry (without recourse to a weak coupling expansion) and of the chirality of the resulting spacetime physics. An alternative derivation of the $SO(8)$ symmetry using the compactification of the theory on the three torus can be found in \cite{45}.

One thing which does not work is the continuous perturbative shift symmetry of the theory under translations of the Ramond-Ramond scalar. This can be attributed to the virtual presence in the theory of various longitudinally wrapped branes which are sensitive to the value of $\theta$. Presumably this will all go away in the large $N$ limit when the energies of these states go off to infinity (relative to the Lorentz invariant states). This example show us that, at least in perturbation theory there can be many versions of DLCQ. A perturbative DLCQ of Type IIB string theory would have preserved the continuous symmetry.

5. Three and Four Tori

The theory on the three torus is somewhat less interesting. One again obtains a maximally SUSY YM theory, which is scale invariant and has an $SL(2, Z)$ Olive Montonen duality symmetry. This combines with the geometrical $SL(3, Z)$ to give the proper U duality of M Theory. There are now no new limits. Olive Montonen duality is identified with the M duality \cite{30} which identifies M2 branes with M5 branes wrapped on a three torus, giving a new version of M Theory. The four torus is much more interesting. Going through the Seiberg scaling one obtains D0 branes in weakly coupled IIA theory on a Planck scale four torus. Performing four T dualities to get to a a large manifold we come back to IIA theory, but this time with a large coupling $G_S$ (because the coupling rescales by four powers of $g^{-1/3}$). The zero branes have of course become D4 branes. Large coupling means that we are going back to M Theory a new dimension is opening up, and the D4 branes become M5 branes. We are thus led to the theory of $N$ M5 branes at distances well below the Planck length in a new copy of M Theory. This is a $5 + 1$ dimensional superconformally invariant field theory $(2, 0)_N$ \cite{46} whose moduli space is $N$ self dual tensor multiplets with Wilson surfaces lying in the self dual $U(N)$ weight lattice. This proposal was first made by \cite{47}. In terms of the original IIA theory we can understand the extra momentum quantum number in the field theory as arising from D4 branes of the (pre T duality) IIA theory wrapped around the Planck scale torus.

The $(2, 0)_N$ theory is superconformal and lives on a five torus. The smallest radius of this torus defines the Planck length of M Theory and the remaining four torus is the
dual torus to the one M Theory theory lives on. The $SL(5, Z)$ U duality of M Theory compactified on $T^4$ is manifest in this presentation of the theory.

Govindarajan and Berkooz and Rozali [48] have suggested that a very similar construction can be made for M Theory compactified on K3 manifolds. One first uses the Seiberg scaling to relate the problem to D0 branes on a K3 of scale $g_s^{1/3}l_S$ in the zero coupling limit of IIA string theory. Then one uses the K3 T duality to relate this to D4 branes wrapped on a dual K3 in a strongly coupled IIA string theory and thus to the $(2, 0)_N$ theory compactified on $S^4 \times \hat{K}3$. They show that many properties of the theory, including all of the expected dualities and the F theory [49] limit can be qualitatively understood in this formulation.

6. Five and Six (Where We Run Out of Tricks)

We now come to the five torus, where things really start to get interesting. The standard limiting procedure leads us to D0 branes in weakly coupled IIA string theory on a Planck scale 5 torus, which is T dual to D5 branes wrapped on a 5 torus in strongly coupled IIB string theory, which in turn is S dual to NS 5 branes wrapped on a five torus in weakly coupled IIB string theory. If one goes through the dualities carefully, then one sees that the scale that is being held fixed in the latter theory, is the IIB string scale. We can see this as follows. As usual, in the original picture, we are taking the DKPS limit and the scale which is held fixed is the kinetic energy of a single D0 brane with Planck scale momenta. After T duality this always leads at low enough energy to a SYM theory in which the SYM coupling is held fixed. In Type IIB theory, with string coupling $G_S$, the SYM coupling is given by

$$g_{SYM}^2 = G_S L_S^2$$  \hspace{1cm} (6.1)

Let us rewrite this in terms of the parameters of the S dual IIB theory:

$$\bar{G}_S = 1/G_S,$$ \hspace{1cm} (6.2)

$$\bar{L}_S^2 = G_S L_S^2.$$ \hspace{1cm} (6.3)

We learn that a collection of $N$ coincident NS 5 branes in weakly coupled IIB string theory, has, on its collective world volume, a SYM theory whose coupling depends only on the string tension and does not go to zero with the string coupling. As a consequence we learn (believing always in the consistency of string theory) that the $G_S = 0$ limit of a collection of $N$ coincident NS 5 branes in the zero coupling limit of Type IIB string theory is a consistent interacting quantum theory with manifest
5 + 1 dimensional Lorentz invariance. We call this the \( U(N)_B \) little string theory. The Matrix Theory for M Theory on a five torus with Planck scale radii and \( N \) units of longitudinal momentum is the \( U(N)_B \) little string theory compactified on a dual five torus. The parameter \( L_S \) and radii \( \Sigma^A \) of the little string theory are related to those of M Theory by

\[
1 = \frac{R^2 L_1 L_2 L_3 L_4 L_5}{L_S^2 L_p^9}
\]  

(6.4)

\[
\Sigma^A = \frac{L^3_p}{R L_A}
\]  

(6.5)

Here \( R \) is the lightlike compactification radius.

The little string theory retains the manifest \( O(5,5) \) T duality of compactified IIB string theory with zero coupling. This is now interpreted as the duality group of M Theory and is in fact the correct \( U \) duality group of M Theory on a five torus. This symmetry has several interesting consequences. First of all, the little string theory cannot be a quantum field theory. In quantum field theory, the variation of correlation functions with respect to the metric is given uniquely in terms of insertions of the stress tensor. But T-duality transformations change the metric of the torus without changing the theory. If we make a small variation of a radius of the torus around some T-self dual point, then we find (assuming that we are dealing with a quantum field theory) that every matrix element of the operator \( \int \delta g^{\mu \nu} \theta_{\mu \nu} \) vanishes. It is easy to argue that this is incompatible with the properties of field theory. We will find more evidence below that little string theories are not field theories.

Another consequence of T duality is the existence of another type of little string theory, called \( U(N)_A \). Indeed if we do a T duality transformation on a single radius of the five torus, we get \( N \) NS five branes in IIA theory. If we now take the infinite torus limit, we obtain a distinct theory. This could have been obtained directly by considering the zero coupling limit of NS five branes in the IIA theory in infinite ten dimensional spacetime. Indeed, one can construct similar little string theories from the zero coupling limit of NS five branes in the heterotic theories. The low energy limit of the \( U(N)_A \) little string theory is not a SYM theory but rather the \((2,0)_N\) superconformal field theory.

An interesting way of understanding the fact that interactions of NS five branes survive the limit of zero string coupling is to write down the low energy SUGRA solution corresponding to a collection of \( N \) NS fivebranes. In the string conformal frame this has the form

\[
g_{ij} = \delta_{ij} e^{2\phi}
\]

(6.6)

\[
e^{2\phi} = e^{2\phi_0} + \frac{N}{r^2}
\]

(6.7)
Here the indices span the four dimensional space transverse to the five brane and \( r \) is the Euclidean distance from the five brane in this space. \( v_{ijk} \) is the volume form of the unit three sphere in this space. \( e^{2\phi_0} = \tilde{G}_S^2 \), the square of the string coupling. The metric components along the fivebrane are Minkowskian.

Near \( r = 0 \) the transverse space has the form of a flat infinite one dimensional space times a three sphere of fixed radius. The dilaton varies linearly in this flat space. This limiting background is in fact an exact solution of the classical string equations of motion to all orders in \( \alpha' \) \(^{59} \). Indeed, the background \( H \) field converts the three sphere \( \sigma \) model into the level \( N \) \( SU(2) \) Wess-Zumino-Witten (WZW) conformal field theory. The linear dilaton is such that the value of the super central charge is \( \hat{c} = 10 \). This is called the linear dilaton background.

For finite \( \tilde{G}_S \) this infinite space is cut off on one end and merges smoothly into an asymptotically flat space with finite string coupling. However, no matter how small the asymptotic value of the string coupling, the region near the five brane is strongly coupled. The effect of taking \( \tilde{G}_S \) to zero is to make the linear dilaton background valid everywhere. As a consequence of the fact that the coupling goes to infinity at \( r = 0 \), the higher order terms in the formal genus expansion are infinite and the perturbation expansion is not useful for correlation functions which probe the \( r = 0 \) region. We will see later that for large \( N \), duality allows us to study this region in terms of a different low energy SUGRA expansion.

In addition to the failure of quantum field theory to capture the dynamics of DLCQ M Theory on \( T^5 \), this geometry presents us with another new phenomenon. In all previous cases, Seiberg’s argument allowed us to relate DLCQ M Theory to string theory, but we then found that most of the string states decoupled in the DLCQ limit. Here we have found that the scale which is kept finite is the string scale and we are left with a little string theory. Another name we might have chosen for this is a “Kondo string theory”. The famous Kondo model in condensed matter physics is a free 1+1 dimensional field theory interacting with a localized defect with a finite number of degrees of freedom. The full system is a rather nontrivial interacting quantum problem, and the degrees of freedom of the field theory cannot be thrown away even though they are free everywhere except at the position of the defect.

Maldacena and Strominger \(^{38} \) have argued that the same is true for the little string theory. If we analyze the scattering problem of string modes off a five brane in the zero coupling limit it is easy to convince oneself that every asymptotic state of the string theory maps into an asymptotic state of the linear dilaton background. Maldacena and Strominger argue that at large \( N \) any state of the fivebrane with energy
density\textsuperscript{6} above a certain cutoff can be described in a sufficiently good approximation as a $1 + 1$ dimensional black hole in the low energy effective field theory. The point is that for large enough $N$ and energy density, the black hole horizon is in the region where the coupling is still weak. Thus, the classic Hawking analysis of black hole radiation is valid and indicates that such fivebrane states decay into asymptotic string states. A standard calculation shows that the Hawking temperature is of order $1/\sqrt{NL_S^2}$. This has two interesting consequences. First it shows that in the large $N$ limit, there is a true decoupling of these string states. Second, since the temperature is independent of the energy, it implies a Hagedorn spectrum of black hole states, which must be interpreted as states of the little string theory. This is a second indication that little string theories are not field theories. We will find independent confirmation of this spectrum by a very different method below.

The large $N$ decoupling of the string states is extremely important, because it is easy to see that in the Matrix Theory context these states do not have Lorentz invariant dispersion relations. First of all, since their D0 brane charge vanishes, they have no longitudinal momentum. Secondly, in the DLCQ limit they actually have vanishing transverse momentum as well. Indeed, the typical states to which perturbative string theory applies have transverse momenta of order the string scale as the string coupling goes to zero. The weakly coupled string theory which we use in the description of M Theory on $T^5$ is an S-dual Type IIB theory, whose string length in terms of the original Type IIB theory is given by equation (6.3). In turn we have, in terms of the original IIA string coupling $G_S = o(g_s^{-5/3})$. Thus, a momentum of order $L_S^{-1}$ is of order $g_s^{5/6} L_S^{-1}$. By contrast, the D0 branes have momenta of order the eleven dimensional Planck scale, which scales like $g_s^{-1/3} L_S^{-1}$. Thus, in the limit, string states of the little string theory carry zero Planck units of M Theory transverse momentum.

This remark explains the otherwise paradoxical fact that the transverse momentum of string states in the NS 5 brane background is not conserved (note that the $O(4)$ angular momentum is conserved). From the point of view of the flat space string theory we began from, the string modes are interacting with states which carry infinitely more transverse momentum than they do, and therefore they can gain or lose arbitrary amounts of (string scale) transverse momentum.

From the M Theory point of view then, the string states of the little string theory are troublesome. They carry finite light cone energy but exactly zero transverse and longitudinal momentum. They are not consistent with M Theory Lorentz invariance.

\textsuperscript{6}“All the experts” agree that the same conclusions are valid for localized states of finite energy on the fivebrane, although no calculations have been done for this case. The case of finite energy density is directly relevant to the toroidally compactified little string theory which is our essential concern in Matrix Theory.
Fortunately, they seem to decouple in the large $N$ limit. The Maldacena Strominger calculation seems to indicate that excitations on the five branes do not excite such states (even if it were energetically possible) in the large $N$ limit\(^7\).

On the six torus, things get even more out of hand. After performing the Seiberg limit and using T duality we obtain the theory of D6 branes in a strongly coupled Type IIA string theory. We are instructed to keep the SYM coupling on the D6 brane world volume finite. It is well known, that in this limit, D6 branes can be viewed as KK monopoles of 11D SUGRA compactified on a very large circle. Be very careful to note that this is not the 11D SUGRA we are trying to model. In fact, the gravitons of this 11D SUGRA arise from the IIA DLCQ point of view, as D6 branes wrapped on the M Theory 6 torus. The SYM coupling on the KK monopole world volume is just the Planck scale of this (new) 11D SUGRA.

This is a new wrinkle. Previously the theories which described DLCQ M Theory did not contain gravity. This was an advance because the conceptual problems of quantizing gravity seemed to be avoided. This is no longer the case on $T^6$. The only saving grace here is that one can again argue that these fake gravitons had better decouple in the large $N$ limit. Indeed, the reader may verify that, just like the string states of the little string theories, they carry vanishing longitudinal and transverse momenta from the M Theory point of view. This means they had better decouple. A hand waving argument that they do in fact decouple is the following.

KK monopoles are manifolds which are circle bundles over the space transverse to a six brane. The radius of the circle is fixed at infinity (though we must take the limit in which this asymptotic radius is itself infinite) and goes to zero near the six brane. For a monopole of charge $N$ the rate at which the circle shrinks to zero as the radius is varied, is multiplied by $N$. Thus gravitons with nonzero momentum\(^8\) around the circle will be repelled from the KK monopoles, and the repulsion will set in at a larger distance for large $N$. From the point of view of M Theory we want to study the scattering of $N$ KK monopoles (wrapped on the dual 11D SUGRA $T^6$) at transverse separations much smaller than the dual Planck scale (although we want to keep energies which are of order the dual Planck scale). It seems plausible that these scattering processes will not involve graviton emission in the large $N$ limit. Obviously, we could do with a stronger argument.

The example of $T^6$ kills once and for all the idea that the finite $N$ DLCQ should

---

\(^7\)O.Aharony has argued to me that since the spectrum of stringy states appears to begin only at energies of order $1/\sqrt{N}$ times the string scale, there is an energetic argument for decoupling, independent of the Maldacena-Strominger calculation.

\(^8\)It is easy to see that gravitons with zero momentum decouple in the limit that the SUGRA circle goes to infinity.
reduce to finite $N$ DLCQ SUGRA in the limit of low energy and large transverse separations. It is clear that at finite $N$, DLCQ M Theory contains states of arbitrarily low light cone energy (wrapped D6 branes in the original description – gravitons in the T dual description) which are simply not there in DLCQ SUGRA.

One might have thought that the simple scaling arguments above go through for any compactification on a six manifold. However, Seiberg’s argument implicitly contains assumptions about the moduli space of string theory compactified on manifolds smaller than the string scale — assumptions which are valid only if there is enough SUSY to provide nonrenormalization theorems for the space and the metric on it. The argument indeed goes through for $K3 \times T^2$ but the authors of [50] have pointed out that things are quite different for a general Calabi-Yau threefold, where there are only eight supercharges. Indeed, it is well known [51] that the Kähler moduli space of string theory on CY 3-folds is corrected when the sizes of cycles reach the string scale. The exact form of the Kähler moduli space and the metric on it can be read off from the complex structure moduli space of the mirror manifold [52]. The authors of [50] suggest that the point in moduli space corresponding to a “Planck scale Calabi Yau” is a mirror CY whose complex structure is very close to the conifold point. This conjecture is based on the notion that mirror symmetry is obtained [53] by writing the CY as a $T^3$ fibration and doing T duality on the three torus. It is not precisely clear what this means since the manifold has no Killing vectors with which to perform an honest T duality transformation. Nonetheless, the idea that mirror symmetry would map a very small (real) 6 fold into a 6 fold with a shrinking three cycle sounds plausible.

If this suggestion is correct, then we know at least that the effective theory for the DLCQ will not contain gravity. Indeed, it is known since the seminal work of Strominger [9] that the effective theory of the new massless states coming from wrapping Type IIB three branes on the shrinking 3 cycle, is a 3 + 1 dimensional gauge field theory with a massless hypermultiplet. The authors of [50] suggest that the whole Matrix Theory on a CY threefold may be some sort of 3 + 1 dimensional field theory with four supercharges. This is an interesting idea, but not much follow up work has been done on it. In my opinion it is a direction which may lead to some interesting progress.

We have seen that Matrix Theory becomes more and more complicated as we compactify more and more dimensions. This is quite interesting, since it is not the way field theory behaves. When we compactify a field theory we generally lose degrees of freedom rather than gain them\(^9\). This is not completely true. In gauge field theory compactification adds Wilson lines, and in gravity, it adds the moduli of the compact-

\(^9\)To make this statement more precise, count the number of degrees of freedom below a certain energy, and ask how this number changes as we shrink the size of the compactification manifold.
ification manifold. However this addition is far outweighed by the loss of modes with nontrivial variation on small manifolds. Perhaps more importantly, these modes did exist as gauge degrees of freedom on the noncompact manifold, but with gauge functions which cannot live on the compactified space.

Some of the extra degrees of freedom we have discovered in Matrix Theory are artifacts of DLCQ. In the low energy SYM approximation, the momentum modes of the field theory represent (from the original M Theory point of view) branes wrapped around both transverse and longitudinal cycles. These states have energy of order one when $N \to \infty$ and should decouple from the hypothetical Lorentz invariant limiting theory.

Examples where this can be worked out rather explicitly are the weak coupling limits of Type II [26] and Heterotic [54] strings, as derived from various $1+1$ and $2+1$ SYM theories. There it is seen that only certain quasi-topological modes of the SYM theory, which vary at a rate $1/N$ along the SYM torus (and manage to be periodic by wandering a distance of order $N$ in the space of matrices), survive the large $N$ limit. In my opinion, the key question in the dynamics of Matrix Theory is to find a way to isolate and describe the spectrum of order $1/N$ with an effective Lagrangian, away from the weak coupling limit.

6.1 The Seven Torus and Beyond

From the point of view described in the introduction, the problems we have encountered as we increased the number of compactified dimensions beyond four are connected to the density of states of the theory at large energies. The little string theory has, as we shall see below, a Hagedorn spectrum. This is the essential feature that prevents it from being a quantum field theory. The DLCQ of M Theory on a six torus does not decouple from gravity. As a consequence, its light cone density of states grows faster than an exponential, because its high energy light cone spectrum is identical to that of SUGRA in an ordinary reference frame.

As we have emphasized, these problems should go away in the large $N$ limit. The Lorentz invariant spectral density of the models grows more slowly than an exponential. Indeed, for both the five and six tori we have suggested that the offending states decouple in the large $N$ limit.

On the seven torus we face a problem of a somewhat different nature. It has long been known that massive excitations of a Lorentz invariant vacuum in $2+1$ dimensional gravity do not preserve globally asymptotically flat boundary conditions. Worse, in theories with massless scalar fields in $2+1$ dimensions (which includes SUGRA with all but the minimal SUSY) excitations tend to have logarithmically growing scalar Coulomb fields and infinite energy. This has been argued to imply [10] that the Hilbert
space of Lorentz invariant, asymptotically flat 1 + 1 or 2 + 1 dimensional string theory is topological in nature and contains no local propagating excitations.

In DLCQ we compactify one more dimension than necessary to describe the Lorentz invariant system we are trying to model. Thus, the paucity of states with asymptotically flat boundary conditions should become a problem in compactifications to four spacetime dimensions. Indeed, following Seiberg’s argument for Matrix Theory on the seven torus we are led to a theory of seven branes in Type IIB string theory. The BPS formula tells us that these have logarithmically divergent tension. Thus, there is no sensible DLCQ of M Theory with 1 + 1, 2 + 1, or 3 + 1 dimensional Lorentz invariant asymptotics. Note that we do expect a noncompact formulation of light cone M Theory with 3 + 1 dimensional asymptotics to exist (it should have a Hagedorn spectrum, like little string theory), but it cannot be found as the large $N$ limit of DLCQ.

7. DLCQ and Holography of $(2, 0)_k$ Theories and Little String Theories

In our discussion of compactification of Matrix Theory we encountered two new types of Lorentz invariant quantum theories which seemed to be decoupled from gravity in the sense that they could be formulated on fixed spacetime manifolds. This is certainly true for the $(2, 0)_k$ theories and their less supersymmetric cousins, which are ordinary quantum field theories in six dimensions. It is likely to be true for the little string theories as well.

In this section we will introduce two complementary methods for studying these theories. At the moment, both methods make sense only in flat six dimensional Minkowski spacetime. Even toroidal compactification results in new singularities which are not well understood. Although we could treat the field theories as limits of the little string theories we will instead find it useful to introduce both methods of computation in the simpler context of field theory. We begin with DLCQ.

7.1 DLCQ of $(2, 0)_k$ Theories

We have remarked above that DLCQ is not a terribly useful tool for ordinary field theory because the theory compactified on a small spatial circle is usually strongly coupled and intractable. This is not the case for the $(2, 0)_k$ theories. Indeed, dimensional reduction on a small circle leads us to an infrared free 4 + 1 dimensional SYM theory.

One simple argument for this comes from the derivation of the $(2, 0)_k$ theory as the effective theory of $k$ coincident M5 branes. If we compactify on a small spatial circle along the brane then we are studying $k$ coincident D4 branes in weakly coupled Type IIA
theory. Things become even simpler if we ask what in the SYM theory corresponds to momentum around the small circle. The only obvious conserved quantum number is the instanton number (remember that instantons are particles in 4 + 1 dimensions). That this is indeed the right identification follows from the BPS formula for the instanton mass \( M_I = 8\pi^2/g_{SYM}^2 \). Remembering the identification of the coupling in terms of the radius of the fifth dimension, we see that this is just the formula for the mass of a KK mode, \( M_{KK} = 2\pi/R_5 \).

Since the SYM coupling is small when the radius is small, and the 4 + 1 dimensional SYM theory is infrared free, a semiclassical analysis of the dynamics of the instantons is valid. Thus, in the sector with longitudinal momentum \( N \), DLCQ of the \((2,0)_k\) theory would seem to reduce to quantum mechanics on the moduli space of \( N \) instantons in \( U(k) \) gauge theory. The fact that this moduli space and the quantum mechanics on it are calculable from classical considerations follows from the high degree of SUSY of the problem\(^{10}\).

Well, almost. The fly in the ointment is that this moduli space is singular. Fortunately, there is an elegant and unique regularization of the moduli space of instantons in four Euclidean dimensions, which appears to make the system completely finite and sensible. The \((2,0)_k\) theory has 16 ordinary SUSYs. In light cone frame we expect only half of them to be realized linearly so we expect to find a quantum mechanics with 8 SUSYs. The target space of the quantum mechanics must therefore be a hyperkähler quotient. There is a famous construction (called the ADHM construction)\(^{65}\) of instanton moduli space as a singular hyperkähler quotient. It is the solution space of the algebraic equations

\[
[X, X^\dagger] - [Y, Y^\dagger] + q_i q_i^\dagger - (p^i)^\dagger p^i = 0 \tag{7.1}
\]

and

\[
[X, Y] = q_i p^i \tag{7.2}
\]

modded out by a \( U(N) \) gauge symmetry which acts on \( X \) and \( Y \) as adjoints and the \( k \) \( q_i \) and \( k \) \( p^i \) as fundamentals and antifundamentals respectively. The products of fundamentals and adjoints appearing in these equations are tensor products of \( U(N) \) representations and are to be interpreted as matrices in the adjoint representation.

These equations also define the Higgs branch of the moduli space of \( \mathcal{N} = 2, \, d = 4 \) \( U(N) \) SYM theory with \( k \) fundamental hypermultiplets. The latter interpretation also introduces the natural regularization of the space, for we can add a Fayet Iliopoulos term by modifying the first ADHM equation to read

\[
[X, X^\dagger] - [Y, Y^\dagger] + q_i q_i^\dagger - (p^i)^\dagger p^i = \zeta I_N, \tag{7.3}
\]

\(^{10}\)All of these arguments come from the papers\(^{39} \) and\(^{55} \), while the regularization below was invented in the second of these two papers.
where $I_N$ is the $N \times N$ unit matrix, and $\zeta$ is a real number. Note that this is a regularization of the moduli space but not of the Yang Mills equations as local differential equations. Instead it corresponds to solving the Yang Mills equations on a certain noncommutative geometry [56].

An important facet of this DLCQ of the $(2, 0)_k$ theories is the fact that when they are KK reduced on a circle, the low energy effective theory is five dimensional SYM theory, which is infrared free. Thus, the difficulties encountered in [13] should be absent and the semiclassical identification of the system as quantum mechanics on the ADHM moduli space is valid. SUSY nonrenormalization theorems guarantee that the metric on this space is unique, and the regularization of the singularities by the FI term is the unique way to deform the instanton moduli space into a smooth hyperkähler manifold. The key to finding the spectrum of chiral primary operators in the $(2, 0)_k$ theory from DLCQ is the following observation of [55]. These authors observe that the DLCQ procedure preserves a subgroup of the superconformal group of the full theory. They identify these generators as explicit operators in the quantum mechanics on instanton moduli space. In particular, they show that a vertex operator is primary, only if it is concentrated on the singular submanifold of zero scale size instantons. Chiral primary operators can then be identified in terms of the cohomology with compact support of the Fayet-Iliopoulos regulated instanton moduli space, which has been investigated in the mathematical literature. We will not explore the details of these calculations. Suffice it to say that they find the correct spectrum of chiral primary operators.

The way we know this is that the spectrum calculated from DLCQ coincides with that implied by the AdS/CFT correspondence. This is not just a matter of agreement between two unrelated conjectures (which in itself would be impressive). Rather, the basis for the AdS/CFT identification of primary operators comes from a low energy analysis of the interaction of 11D SUGRA with fivebranes. There must be one primary for each SUGRA field which is in a short multiplet of $AdS_7 \times S^4$ SUSY. As usual with short multiplets, the number and properties of these multiplets are independent of parameters, and can be calculated in the low energy approximation.

### 7.2 DLCQ of the Little String Theories

The papers [39] and [57] described the DLCQ of the $U(k)_A$ little string theories and [58] performed the same task for the $U(k)_B$ theories. We will restrict attention to the $U(k)_A$ case.

One way to understand the derivation for the Type A theory is to consider the DLCQ of Type IIA string theory as derived in the Matrix String picture and add fivebranes wrapped around the longitudinal direction. The result is the $1 + 1$ dimensional field theoretical generalization of the model of Berkooz and Douglas [36] for longitudi-
nal 5 branes in Matrix-Theory. One obtains a $1+1$ dimensional field theory with $(2,2)$ SUSY. It is a $U(N)$ gauge theory with one adjoint and (in the sector with $k$ fivebranes) $k$ fundamentals. As in [20] one takes the weak string coupling limit by descending to the moduli space. Now however we want to be on the Higgs branch of the moduli space (the Higgs and Coulomb branches obviously decouple from each other in the limit) and we obtain a $\sigma$ model with target space the ADHM moduli space. Of course this moduli space is singular, but we can regularize it by adding FI terms. Thus, the DLCQ$_N$ of $U(k)_A \text{LST}$ is a sigma model on the moduli space of $N \ U(k)$ instantons on $R^4$. This moduli space can be regularized by the addition of FI terms to the ADHM equations, so that the DLCQ theory is realized as a limit of a well defined, conformally invariant $(4,4)$ supersymmetric sigma model. Note that, in contrast to the regularized quantum mechanics, the sigma model retains its conformal invariance after regularization. However, the conformal generators of the sigma model are not symmetries of the spacetime M Theory. From the M Theory point of view, the spatial momentum on the sigma model world sheet is a quantum number that counts longitudinally wrapped branes, and should decouple in the limit of large $N$.

Before regularization the ADHM moduli space is locally flat. Thus, the central charge $c$ of the SCFT is just $c = 6Nk$. Because the variation of the FI term is a marginal perturbation of the sigma model, this remains the value of $c$ in the regularized model. We can immediately turn this into a computation of the high energy density of states in the DLCQ model. The entropy is given by

$$S(P^-) \rightarrow \sqrt{2cP^-} = \sqrt{6kEl_S}$$

(7.4)

In the last equality we have used the relation between light cone and ordinary energy for vanishing transverse momentum. This is the Hagedorn spectrum that we advertised for the little string theories. In the DLCQ approach, it arises, as in perturbative string theory, because the light cone energy is identified with the ordinary energy of a $1+1$ dimensional CFT.

The only problem with this derivation is that it applies to the asymptotic density of states of the DLCQ theory, which are not actually states with $P^-$ of order $1/N$. It has been argued by Ofer Aharony [3] that the sigma model contains such states as strings which wander through of order $N$ instantons before closing (note that the instanton moduli space has an $S_N$ orbifold symmetry). As in our treatment of matrix string theory, or the Maldacena-Susskind description of fat black holes [1], these configurations should have an entropy of the Hagedorn form even at energies much lower than those at which (7.4) is naively valid in the CFT. We will see below that a completely different argument produces the same formula for the entropy.
The sigma model on regularized instanton moduli space is a fascinating CFT, which also arises in the study of D1-D5 black holes. Its properties have recently been studied in [6].

### 7.3 Holography

The AdS/CFT correspondence will be covered by other lecturers at this school. Suffice it to say that for the \((2,0)_N\) theory it provides the leading term in a large \(N\) expansion of the correlation functions of the chiral primary operators. The calculation is performed by solving the classical equations of 11D SUGRA in the presence of certain perturbations of an \(AdS_7 \times S^4\) spacetime, with \(N\) units of fivebrane flux on the \(S^4\). In order to calculate corrections to higher order in \(1/N\) one needs the higher terms in the derivative expansion of the effective action. We have only a limited amount of information about such terms. Even in leading order, few calculations have been done for this case.

Just like the DLCQ solution of these theories, the AdS/CFT calculations only work in uncompactified spacetime. The SUGRA background corresponding to toroidally compactified \((2,0)_N\) theories is singular and the derivative expansion does not seem sensible even for large \(N\). In this case we can get an inkling of the reason for the greater degree of singularity of the compactified case. We are used to the fact that in M Theory, singularities correspond to light degrees of freedom which have not been included in the effective Lagrangian. In the toroidally compactified \((2,0)_N\) theory it is obvious that there are such degrees of freedom. The zero modes along the moduli space, which are frozen expectation values in the infinite volume theory are here zero frequency quantum variables. Indeed it is precisely the scattering matrix of these variables which one would hope to compute in Matrix Theory. It seems unlikely that this dynamics will be easily captured by reliable calculations in SUGRA.

We now turn to the holographic description of little string theories. It was suggested in [41] that these are simply the exact description of string theory in the linear dilaton backgrounds. More precisely, these are Type II string theories in the following background.

\[
ds^2 = -dt^2 + dx^2 + d\phi^2 + aNl_s^2 d\Omega_3^2
\]

\[
H = \omega_{N\Omega_3} \tag{7.6}
\]

\[
g_S^2 = e^{-\phi/\sqrt{Nl_s}} \tag{7.7}
\]

11In the previous section, in order to conform to the literature on the subject we used \(k\) to signify the number of five branes, while \(N\) was reserved for the longitudinal momentum. Here we will revert to the standard use of \(N\) in the AdS/CFT correspondence. It denotes the number of fivebranes.
Here \( \mathbf{x} \) are coordinates on the worldvolume of \( N \) coincident fivebranes, \( \Omega_3 \) are coordinates on a three sphere transverse to the fivebranes, and \( \omega_{\Omega_3} \) is its volume form.

There are some interesting differences in the way holography works in this context, as compared to the AdS/CFT correspondence. Most of them stem from the fact that the asymptotic geometry of these backgrounds is Minkowski space with an exponentially vanishing string coupling. Therefore, even for finite \( N \), there is an infinite region of spacetime in which the description of the system in terms of freely propagating particles becomes exact. The linear dilaton systems have an S-matrix. By contrast, even though the geometry of AdS space has infinite volume, the boundary conditions which define a Cauchy problem in this space are reminiscent of those for a system in a box.

In the AdS/CFT correspondence we do not expect to see any sort of large spacetime unless \( N \) is large, but even for \( N = 1 \) (note that unlike other systems of this type, the \( N = 1 \) little string theory does not appear to be a trivial gaussian system) or 2 the little string theories should have asymptotic multiparticle states propagating in the weak coupling region.

Another dramatic difference between the two types of theories is that the little string theory has a Hagedorn spectrum and is not a quantum field theory. Thus, in many ways, the little string theory is much closer to string theory in Minkowski space than the AdS systems.

We have seen the Hagedorn spectrum in the DLCQ calculation above. In the holographic description one calculates the asymptotic density of states by using the Bekenstein-Hawking formula for black hole entropy. This is justified in the linear dilaton background because the mass of the black hole is inversely proportional to the string coupling at the horizon. The world outside a large mass black hole is completely contained in the weak coupling regime.

It is well known \[60\] that the Hawking temperature of linear dilaton black holes is independent of their mass. This is equivalent to the statement that the entropy is linear in the energy \( i.e. \) we have a Hagedorn spectrum\[12\]. The Hagedorn temperature can be computed in terms of the coefficient which governs the rate of increase of the dilaton. We again find that \( S = \sqrt{6N}\omega_5 \).

The Hagedorn spectrum actually solves a potential paradox in the claim of \[41\]. These authors argue that the S-matrix of string theory in the linear dilaton background can be interpreted as the correlation functions of observables in the LST. The \( p \) particle

\[12\] It should be noted in passing that this simple calculation shows that all extant nonperturbative formulations of the \( c = 1 \) string theory are wrong. The entropy in all such calculations is that of a \( 1+1 \) dimensional field theory rather than the much more degenerate Hagedorn spectrum. The \( c = 1 \) model was solved by trying to resum a divergent perturbation expansion. Clearly some nonperturbative states (Liouville “D branes”??) have been missed in this resummation.
S-matrix elements are of course symmetric under interchange of arguments (the $S_p$ group of statistics). They are also $5+1$ Lorentz invariant. If these are to be interpreted as correlation functions in a quantum theory, their $S_p$ symmetry implies that they must be Fourier transforms of time ordered products of Heisenberg operators.

Lorentz invariance would then imply that LST was a local field theory, because time ordered products are only Lorentz invariant if the operators commute at spacelike separations. However, the Hagedorn spectrum prevents us from performing the Fourier transform and this conclusion cannot be reached\textsuperscript{13}.

Indeed, by calculating two point functions of operators by analogy with AdS/CFT (solving linearized wave equations in the linear dilaton background), Peet and Polchinski\cite{61} showed explicitly that they were not Fourier transformable. Their behavior is exponential, with the Hagedorn temperature controlling the rate of growth of the exponent.

Peet and Polchinski’s calculation is easy to summarize: The scalar wave equation in the linear dilaton background is

\[
[-\partial_\phi^2 + \frac{(2l + 3)(2l + 1)}{4} + k^2 \alpha' N] e^{3\phi/2} \psi = 0.
\] (7.8)

For large $k$ and $\phi$ its solutions have the form

\[
\psi \sim e^{(\alpha' Nk^2)^{1/2} \phi}
\] (7.9)

In the holographic interpretation the S-matrix computed from these wave functions, which has the same behavior in momentum space as the wave functions themselves, is supposed to be the two point function of some operator in the LST. Thus, if the two point function is not Fourier transformable, and grows in the way we would expect from the Hagedorn spectrum.

To conclude this brief summary of our knowledge of little string theories, I want to discuss the question of what the scale of nonlocality is in these theories. What we know so far suggests two rather different answers. Seiberg’s original argument about T-duality suggests string nonlocality on a scale $l_S$. On the other hand, the length scale

\textsuperscript{13}Some readers may be confused by our apparent denial of the possibility of having a Hagedorn spectrum in local field theory. What, they will ask, about the Hagedorn spectrum of large $N$ QCD? In fact there is no contradiction. Two point functions of operators in large $N$ QCD are in fact controlled by the asymptotically free fixed point at short distances. However, the crossover scale, above which free behavior sets in, depends on the operator. At infinite $N$ there are some operators which never get to the crossover point, because it scales with a positive power of $N$. This phenomenon of operators which have rapidly vanishing matrix elements between the vacuum and most of the high energy states, appears to be connected to the fact that infinite $N$ QCD is a free theory, with an infinite number of conservation laws. I do not expect such behavior in a finite system with interaction.
defined by the Hagedorn temperature is of order \( l_s \sqrt{N} \) which is much longer. Note however that the argument for the latter scale is based on high energy asymptotics. Thus, although the Hagedorn temperature is low for large \( N \), it might be that the exponential behavior of the density of states does not set in until energies of order \( l_s^{-1} \).

The Hagedorn temperature controls the rate of growth of the asymptotic density of states, but does not tell us anything about the finite scale at which the asymptotic behavior begins to dominate. Minwalla and Seiberg have done a calculation which suggests that in fact the Hagedorn behavior does not set in until scales far above the Hagedorn temperature. They argued that if, in the \( LST_A \) theory, one takes the limit \( l_s \to 0 \) with \( l_s \sqrt{N} \) fixed, then the SUGRA approximation to scattering amplitudes with energies of order this fixed scale becomes exact. The point is that in \( LST_A \), the strong coupling behavior of the theory is described at low energy by 11D SUGRA. Minwalla and Seiberg show that in the large \( N \) limit described above, there is a SUGRA description of the scattering amplitude which is valid for arbitrary values of the dilaton. Thus, the full amplitude is calculable by solving partial differential equations. The resulting equation is complicated, but Minwalla and Seiberg obtained a qualitative understanding of its behavior and were able to solve it approximately in various regimes.

They calculated the amplitude for a single massless string to scatter off the NS 5 brane in this limit, and found a Fourier transformable answer. This suggests that for large \( N \), the density of states in the vicinity of the Hagedorn energy scale, increases more slowly than the Hagedorn formula\(^{14} \). It is tempting to suggest that the Hagedorn behavior of the spectrum sets in only above the string scale, which is the scale of nonlocality indicated by T duality. Indeed, in the spacetime picture of this system, the high energy CGHS black hole spectrum can only be computed reliably for energies above the string scale. If this conjecture is correct, there is a puzzle about the nature of the large \( N \) limiting theory defined by Minwalla and Seiberg. Naive application of the logic we applied to the full \( LST \) would suggest that it is a quantum field theory, since its correlation functions have spacetime Fourier transforms, which can then be interpreted as Lorentz invariant time ordered products. But large \( N \) limits are tricky, and I expect that if the Minwalla-Seiberg limit of all the correlation functions of \( LST_A \) exists, it does not define a quantum field theory.

Finally, I want to discuss an issue raised by the analysis of Minwalla and Seiberg, which is not particularly related to the bulk of the material in these lectures. There is some confusion in the literature, and in discussions I have participated in, about

\(^{14}\text{This is not a definitive argument against a Hagedorn spectrum because the matrix elements of operators between the vacuum and high energy spectrum might fall sufficiently rapidly to give a Fourier transformable two point function. It is however suggestive that the limiting Minwalla-Seiberg calculation shows a different behavior than that found by Peet and Polchinski.}\)
whether the AdS/CFT correspondence (and in particular the fact that the theory is formulated in terms of a Hermitian Hamiltonian in a well defined Hilbert space) says something definitive about the issue of unitarity in Hawking radiation. I would claim that it does not, because the AdS theory does not have an S-matrix\textsuperscript{15}.

The little string theories do have an S-matrix and one can begin to address the question. In particular, Minwalla and Seiberg find a nonzero absorption cross section for the black fivebrane. This could be taken as a signal of lack of unitarity. Like many extremal black holes, the extremal fivebrane metric has an analytic completion with multiple asymptotic regions. One could try to interpret the absorption cross section as matter being scattered into another asymptotic region, violating unitarity in any given region.

I would like to present a more conservative interpretation of the absorption cross section: the fivebrane absorbs only because it is infinite. There is indeed another asymptotic region, but this is the region along the infinite brane. This is most clearly seen in the IIB case, where the low energy theory on the brane is infrared free 5 + 1 dimensional SYM theory. A particle coming in from infinity in \( \phi \), can excite gluons on the brane which propagate off to asymptotic infinity in 5 dimensions without any propagation back off to infinite values of \( \phi \). In the IIA theory analyzed by Minwalla and Seiberg \textsuperscript{66}, the low energy theory is conformal and has no conventional S-matrix. However it should still be true that localized disturbances in CFT eventually dissipate out to infinity\textsuperscript{16}, so similar physics is to be expected.

The key test of this interpretation is to see what happens when the theory is compactified. Indeed, the origin of the absorption cross section is a logarithm in momentum space which comes from the continuum of low energy modes. Since these are described by field theory, we understand what happens to them upon compactification. The modes on the brane are discrete. There will be no absorption at generic energies. All that will happen is the excitation of resonant modes of the brane, which will eventually decay back to the vacuum. The only continuum in the system is that describing modes propagating far from the brane, in the weak coupling region.

On the gravitational side of the holographic correspondence, compactification causes a singularity to appear on the horizon (the compactification circles have zero radius there). One cannot conclude anything from this about unitarity, but it certainly does not contradict the conclusion based on the nongravitational side of the duality. Thus, there is no evidence against unitarity of the LST S-matrix.

\textsuperscript{15}Attempts to extract the flat space S-matrix from AdS/CFT, \textsuperscript{22}, have not progressed to the point where one can decide if the S-matrix is unitary.
\textsuperscript{16}I thank A.Zamolodchikov for a discussion of this point.
Little string theories are a fascinating area for future work. They are our only example of Lorentz invariant quantum theories which are neither quantum field theories nor theories of gravity (in $5 + 1$ dimensions). Conventional Lagrangian techniques are applicable only in the light cone frame. It would be of the utmost interest to find an alternative, manifestly Lorentz invariant, framework for formulating and solving these theories.

8. Conclusions

Matrix theory is a nonperturbative DLCQ formulation of M Theory in backgrounds with six or more asymptotically flat directions. It provides proofs of a large number of duality conjectures, and has led to a new class of Lorentz invariant, gravity free theories. It demonstrates the existence of a new class of large $N$ limits of ordinary gauge field theories, in which one concentrates on states with energies of order $1/N$. There is a lot of evidence that the theory becomes simpler in the large $N$ limit, in the sense that many of the finite $N$ degrees of freedom decouple. A Lorentz invariant formulation awaits the development of techniques to study these new kinds of large $N$ limit.

In the meantime, we can try to use Matrix Theory to study a variety of issues in gravitational physics which do not require us to compactify to low dimensions. A beginning of the study of black holes in Matrix Theory may be found in [64].

There are a number of important general lessons about M Theory that may be learned from Matrix Theory. Among these are

1. The statistical gauge symmetry of identical particles arises as a subgroup of a much larger, continuous, gauge symmetry.

2. The cluster property, and the existence of spacetime itself seems to be closely intertwined with supersymmetric cancellations.

3. The number of degrees of freedom of the theory increases as we compactify. This is quite odd from the point of view of quantum field theory.

4. Short distance divergences in the effective gravitational theory turn out to be infrared divergences caused by the neglect of degrees of freedom which become light when particles are brought together. These correspond to light branes stretched between the particles, and again are very different from the kinds of degrees of freedom encountered in field theory.
5. As in any generally covariant theory we expect a conventional Hamiltonian description only when space is asymptotically flat or AdS. In the asymptotically flat case we have argued that conventional Hamiltonian quantum mechanics will only be applicable in the light cone frame and only when there are five or more noncompact dimensions. The phenomenologically relevant case of four dimensions has a Hagedorn spectrum in light cone energy and may be describable by some kind of little string theory.

The outstanding problem in Matrix Theory is to find a way to isolate the dynamics of the states with DLCQ energy $1/N$ and to write a Lagrangian (for $d_{noncompact} \geq 5$) for the infinite $N$ system. For the phenomenologically relevant case of $d = 4$ one must obtain a sensible substitute for Lagrangian methods for systems with a Hagedorn spectrum. Another unsolved problem is the formulation of DLCQ M Theory on Calabi-Yau threefolds. Beyond this, Matrix Theory cannot go, for light cone methods do not appear to be useful for cosmology or for studying the problem of SUSY breaking (where the typical ground state of the system may not have null Killing vectors).

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