Lagrangian submanifolds foliated by $(n - 1)$-spheres in $\mathbb{R}^{2n}$

Henri Anciaux, Ildefonso Castro and Pascal Romon

Abstract

We study Lagrangian submanifolds foliated by $(n - 1)$-spheres in $\mathbb{R}^{2n}$ for $n \geq 3$. We give a parametrization valid for such submanifolds, and refine that description when the submanifold is special Lagrangian, self-similar or Hamiltonian stationary. In all these cases, the submanifold is centered, i.e. invariant under the action of $SO(n)$. It suffices then to solve a simple ODE in two variables to describe the geometry of the solutions.

Keywords: Lagrangian submanifold, special Lagrangian, self-similar, Hamiltonian stationary.

2000 MSC: 53D12 (Primary) 53C42 (Secondary).

Introduction

Lagrangian submanifolds constitute a distinguished subclass in the set of $n$-dimensional submanifolds of $\mathbb{R}^{2n}$: a submanifold is Lagrangian if it has dimension $n$ and is isotropic with respect to the symplectic form $\omega$ of $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$. Such submanifolds are locally characterized as being graphs of the gradient of a real map on a domain of $\mathbb{R}^n$, however to give a general classification of them is not an easy task.

In the case of surfaces i.e. $n = 2$, powerful techniques such as Weierstrass representation formulas may be used to gain some insight into this question (cf [HR]). In higher dimension, some work has been done in the direction of characterizing Lagrangian submanifolds with various intrinsic and extrinsic geometric assumptions, see [Ch] for a general overview and more references.

Recently, D. Blair has studied in [Bl] Lagrangian submanifolds of $\mathbb{C}^n$ which are foliated by $(n - 1)$-planes. In the present paper, we are devoted to study those Lagrangian submanifolds of $\mathbb{R}^{2n}$, which are foliated by Euclidean $(n - 1)$-spheres. In the following, we shall for brevity denote them by $\sigma$-submanifolds. We first observe that any isotropic round $(n - 1)$-sphere spans a $n$-dimensional Lagrangian subspace, a condition which breaks down in dimension 2. Hence we restrict our attention to...
the case $n \geq 3$ and we give a characterization of $\sigma$-submanifolds as images of an immersion of a particular form involving as data a planar curve and a $\mathbb{R}^n$-valued curve (see Theorem 1).

Next, we focus our attention on several curvatures equations and characterize those among which are $\sigma$-submanifolds. The most classical of these equations is the *minimal submanifold equation*, involving the mean curvature vector $\vec{H}$

$$\vec{H} = 0.$$ 

A Lagrangian submanifold which is also minimal satisfies in addition a very striking property: it is calibrated (*cf* [HL]) and therefore a minimizer; these submanifolds are called *special Lagrangian*. Many special Lagrangian submanifolds with homogeneity properties have been described in [CU2, Jo]. In this context we recover the characterization of the Lagrangian catenoid as the only (non flat) special Lagrangian $\sigma$-submanifold (see [CU1]).

We shall call *self-similar* a submanifold satisfying

$$\vec{H} + \lambda \vec{X} = 0,$$

where $\vec{X}$ stands for the projection of the position vector $\vec{X}$ of the submanifold on its normal space and $\lambda$ is some real constant (*cf* [An2]). Such a submanifold has the property that its evolution under the mean curvature flow is a homothety (shrinking to a point if $\lambda > 0$ and expanding to infinity if $\lambda < 0$). Here we shall show that there are no more self-similar $\sigma$-submanifolds than the ones described in [An2].

The third curvature equation we shall be interested in deals with the *Lagrangian angle* $\beta$ (*cf* [Wo] for a definition), a $\mathbb{R}/2\pi\mathbb{Z}$-valued function which is defined up to an additive constant on any Lagrangian submanifold and satisfies $J\nabla \beta = n\vec{H}$, where $J$ is the complex structure in $\mathbb{C}^n$ and $\nabla$ is the gradient of the induced metric on the submanifold. Following [Oh], we call *Hamiltonian stationary* a Lagrangian submanifold which is critical for the volume functional under Hamiltonian deformations (generated by vector fields $V$ such that $V \lrcorner \omega$ is exact). It turns out that the corresponding Euler-Lagrange equation is

$$\Delta \beta = 0.$$ 

In others words a Lagrangian submanifold is Hamiltonian stationary if and only if its Lagrangian angle function is harmonic (for the induced metric). Many Hamiltonian stationary surfaces in $\mathbb{C}^2$ have been described in [HR] and [An1], but in higher dimensions, very few examples were known so far (for example the Cartesian product of round circles $a_1\mathbb{S}^1 \times \ldots \times a_n\mathbb{S}^1$). In this paper we shall describe all Hamiltonian stationary $\sigma$-submanifolds.

The paper is organized as follows: the first section gives the proof of characterization of $\sigma$-submanifolds (Theorem 1). In the second one, we compute for this kind of submanifold the Lagrangian angle and the mean curvature vector. In Section 3, we
obtain as a corollary the fact that special Lagrangian or self-similar $\sigma$-submanifolds are centered (i.e. invariant under the standard action of the special orthogonal group $SO(n)$, see Example 1 in Section 1 for a precise definition), thus the only examples of such submanifolds are those described in [CU1] (for special Lagrangian ones) and in [An2] (for self-similar ones). In Section 4, we show that the only Hamiltonian stationary $\sigma$-submanifolds are also centered (Theorem 3), and in Section 5 we describe them in details (Corollary 1).

Acknowledgments: The authors would like to thank Francisco Urbano for his numerous valuable comments and suggestions and Bruno Fabre for the proof of Lemma 5.

1 Characterization of Lagrangian submanifolds foliated by $(n-1)$-spheres

On $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $n \geq 3$, with coordinates $\{z_j = x_j + iy_j, 1 \leq j \leq n\}$ equipped with the standard Hermitian form $(.,.)_{\mathbb{C}^n}$ and its associated symplectic form $\omega := \sum_{j=1}^n dx_j \wedge dy_j = \text{Im} (.,.)_{\mathbb{C}^n}$, we consider submanifolds of dimension $n$ foliated by round spheres $S^{n-1}$. Locally, they can be parameterized by immersions

$$\ell : I \times S^{n-1} \rightarrow \mathbb{C}^n \simeq \mathbb{R}^{2n} \\
(s, x) \mapsto r(s)M(s)x + V(s)$$

where

- $I$ is some interval,
- $\forall s \in I$, $r(s) \in \mathbb{R}^+$, $M(s) \in SO(2n)$, $V(s) \in \mathbb{C}^n$,
- $x \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$, where we note $\mathbb{R}^n = \{y_j = 0, 1 \leq j \leq n\}$.

Lemma 1 A necessary condition for the immersion $\ell$ to be Lagrangian is that for any $s$ in $I$, the (affine) $n$-plane containing the leaf $\ell(s, S^{n-1})$ is Lagrangian; in other words, $M \in U(n)$, where $U(n)$ is embedded as a subgroup of $SO(2n)$.

Proof. For a fixed $s \in I$, the leaf $\ell(s, S^{n-1})$ spans exactly the affine $n$-plane $V + M\mathbb{R}^n$. So we have to show that for two independent vectors $W, W' \in \mathbb{R}^n$, $\omega(MW, MW') = 0$. Since $n \geq 3$, there exists $x \in S^{n-1}$ such that $W$ and $W'$ are tangent to $S^{n-1}$ at $x$. Thus $rMW$ and $rMW'$ are tangent to the leaf $\ell(s, S^{n-1})$ at $\ell(s, x)$. The tangent space to this leaf is isotropic (being included in the tangent space to $\ell(I \times S^{n-1})$). Therefore $\omega(MW, MW') = 0$. Finally, $M$ is an isometry mapping a Lagrangian $n$-plane to another Lagrangian $n$-plane, hence a unitary transformation. \qed

Remark 1 This crucial lemma does not hold in dimension two: we can produce examples of Lagrangian surfaces foliated by circles such that the planes containing the circular leaves are not Lagrangian (cf Example 3).
Now, we know that for any $\xi \in \mathbb{R}^n$ orthogonal to $x$, $M\xi$ is tangent to the submanifold at $\ell(s, x)$ (because it is tangent to the leaf) so we have:

$$\omega(\ell_s, M\xi) = 0,$$

where $\ell_s = \partial \ell/\partial s = \dot{r}Mx + \dot{M}x + \dot{V}$. Along the paper the dot ' $\cdot$ ' will be a shorthand notation for the derivative with respect to $s$. We compute:

$$\omega(\dot{r}Mx, M\xi) + \omega(r\dot{M}x, M\xi) + \omega(\dot{V}, M\xi) = 0$$

$$\iff \text{Im} \left( \langle \dot{r}Mx, M\xi \rangle_{\mathbb{C}^n} + \langle r\dot{M}x, M\xi \rangle_{\mathbb{C}^n} + \langle \dot{V}, M\xi \rangle_{\mathbb{C}^n} \right) = 0$$

$$\iff \text{Im} \left( \dot{r} + r\dot{M} - M^{-1}\dot{M} \right) = 0,$$

because $\xi$ is real.

In the following, we shall note $b := \text{Im} (M^{-1}\dot{V}) \in \mathbb{R}^n$ and $B := \text{Im} (M^{-1}\dot{M})$. The complex matrix $M^{-1}\dot{M}$ is skew-Hermitian so $B$ is symmetric. Then we have the following equation:

$$(*) \quad \langle b, \xi \rangle_{\mathbb{R}^n} + r\langle Bx, \xi \rangle_{\mathbb{R}^n} = 0, \quad \text{for all } (x, \xi) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \text{ such that } \langle x, \xi \rangle_{\mathbb{R}^n} = 0$$

From now on, we denote the real scalar product $\langle ., . \rangle_{\mathbb{R}^n}$ by $\langle ., . \rangle$. Next we have three steps:

**Step 1:** $b$ vanishes or is an eigenvector for $B$.

If $b \neq 0$, take $x$ collinear to $b$, so that $x = \lambda b$ for some real non zero constant $\lambda$. From $(*)$, we have

$$\forall \xi \in b^\perp, \langle B\lambda b, \xi \rangle = 0.$$

Therefore $Bb$ is collinear to $b$, so there exists $\mu$ such that $Bb = \mu b$.

**Step 2:** $b$ vanishes.

Suppose $b \neq 0$, then set $\xi = b$ and $x$ orthogonal to $b$, so $\langle b, x \rangle = 0$ and $\langle Bb, x \rangle = 0$ (using step 1). Using the fact that $B$ is symmetric, we write:

$$\langle b, Bx \rangle = 0,$$

so

$$\langle \xi, Bx \rangle = 0.$$

From $(*)$ we deduce that $\langle b, \xi \rangle = ||b||^2 = 0$. So $b$ vanishes.

**Step 3:** $B$ is a homothecy.

Now, $(*)$ becomes

$$\langle Bx, \xi \rangle = 0, \quad \forall x, \xi \in \mathbb{S}^{n-1} \text{ such that } \langle x, \xi \rangle = 0.$$

This implies that any vector $x$ is an eigenvector for $B$, so $B$ is a homothecy.
We deduce that \( M^{-1}\dot{V} \in \mathbb{R}^n \), and that \( M^{-1}\dot{M} \in \mathfrak{so}(n) \oplus i\mathbb{R}Id \). This implies that \( M \in SO(n).U(1) \). In other words, we may write \( M = e^{i\phi}N \), where \( N \in SO(n) \). Moreover, we observe that \( N \) can be fixed to be \( Id \): this does not change the image of the immersion. Then \( M = e^{i\phi}Id \), so \( \dot{V} = e^{i\phi}W \), where \( W \in \mathbb{R}^n \).

Finally, we have shown:

**Theorem 1** Any Lagrangian submanifold of \( \mathbb{R}^{2n}, n \geq 3 \), which is foliated by round \((n - 1)\)-spheres is locally the image of an immersion of the form:

\[
\ell: I \times \mathbb{S}^{n-1} \longrightarrow \mathbb{C}^n \simeq \mathbb{R}^{2n}
\]

\[
(s, x) \longmapsto r(s)e^{i\phi(s)}x + \int_{s_0}^s e^{i\phi(t)}W(t)dt,
\]

where

- \( s \mapsto \gamma(s) := r(s)e^{i\phi(s)} \) is a planar curve,
- \( W \) is a curve from \( I \) into \( \mathbb{R}^n \),
- \( s_0 \in I \).

Geometrically, \( r(s) \) is the radius of the spherical leaf and \( V(s) := \int_{s_0}^s e^{i\phi(t)}W(t)dt \) its center. The fiber lies in the Lagrangian plane \( e^{i\phi(s)}\mathbb{R}^n \).

**Example 1** If \( V = W = 0 \), the center of each leaf is fixed. In the following, we shall simply call these submanifolds centered. In this case, the submanifold is \( SO(n) \)-equivariant (for the following action \( z \mapsto Az, A \in SO(n) \) where \( SO(n) \) is seen as a subgroup of \( U(n) \), itself a subgroup of \( SO(2n) \)).

**Example 2** Assume the curve \( \gamma \) is a straight line passing through the origin. Then up to reparametrization we can write \( \gamma(s) = se^{i\phi_0} \), where \( \phi_0 \) is some constant. This implies that

\[
\ell(x, s) = e^{i\phi_0} \left[ sx + \int_{s_0}^s W(t)dt \right],
\]

thus the immersion is totally geodesic and the image is simply an open subset of the Lagrangian subspace \( e^{i\phi_0}\mathbb{R}^n \).

**Example 3** Assume the centers of the leaves lie on some straight line. Then there exists some function \( u(s) \) such that \( V(s) = u(s)a + c \) with \( a, c \in \mathbb{C}^n \). Differentiating, we obtain

\[
\dot{u}(s)a = e^{i\phi(s)}W(s).
\]

As \( a \) is constant and \( W \) real, it implies that \( \phi \) is constant, so we have \( a = e^{i\phi}b \), where \( b \in \mathbb{R}^n \). Then

\[
\ell(x, s) = e^{i\phi}(r(s)x + u(s)b) + c
\]

so again the immersion is totally geodesic. We notice that this situation is in contrast with the case of dimension 2, where there exists a Lagrangian flat cylinder, which is foliated by round circles whose centers lie on a line.
Example 4 Epicycloids: Assume the centers of the leaves lie on a circle contained in a complex line. Then there exists some function $u(s)$ such that $V(s) = e^{iu(s)}a + c$ with $a, c \in \mathbb{C}^n$. Differentiating, we obtain

$$W(s) = i\dot{u}(s)e^{i(u(s) - \phi(s))}a.$$ 

As in the previous example, it implies that $\phi - u$ is constant; without loss of generality, we can take $u = \phi$ and $a = -ib$, where $b \in \mathbb{R}^n$. Then

$$\ell(x, s) = e^{i\phi(s)}(r(s)x - ib) + c.$$ 

2 Computation of the Lagrangian angle and of the mean curvature vector

2.1 The Lagrangian angle

We may assume, and we shall do so from now on, that $\gamma$ is parameterized by arclength, so there exists $\theta$ such that $\dot{\gamma} = e^{i\theta}$, and the curvature of $\gamma$ is $k = \dot{\theta}$. We also introduce, as in [An2], $\alpha := \theta - \phi$, so that $\dot{r} = \cos \alpha$ and $\dot{\phi} = \frac{\sin \alpha}{r}$.

Let $(v_2, ..., v_n)$ be an orthonormal basis of $T_xS^{n-1}$. Here and in the following, the indices $j, k$ and $l$ (and the sums) run from 2 to $n$, unless specified. We have:

$$\ell_s = e^{i\theta}x + e^{i\phi}W \quad \text{and} \quad \ell_s v_j = re^{i\phi}v_j.$$ 

The induced metric $g$ on $I \times S^{n-1}$ has the following components with respect to the basis $(\partial_s, v_2, ..., v_n)$:

$$g_{11} = 1 + |W|^2 + 2 \cos \alpha \langle W, x \rangle,$$

$$g_{1j} = g_{j1} = r \langle W, v_j \rangle,$$

$$g_{jk} = g_{kj} = \delta_{jk}r^2.$$ 

We consider the orthonormal basis $(e_1, ..., e_n)$ defined as follows:

$$e_j := \frac{v_j}{r}, \quad e_1 := A\partial_s + \sum B_jv_j,$$

where the real numbers $A$ and $B_j$ are uniquely determined by the orthonormality condition:

$$A = \left(1 + 2 \cos \alpha \langle W, x \rangle + \langle W, x \rangle^2\right)^{-1/2} \quad \text{and} \quad B_j = -\frac{A \langle W, v_j \rangle}{r},$$ 

so

$$e_1 = A \left(\partial_s - \sum \frac{\langle W, v_j \rangle}{r}v_j\right).$$
This yields an orthonormal basis of the tangent space to the submanifold and we may compute the Lagrangian angle by means of the following formula: $e^{i\beta} = \det_C (\ell_* e_1, ..., \ell_* e_n)$. Indeed we have

$$\ell_* e_j = e^{i\phi} v_j,$$

and

$$\ell_* e_1 = A (\ell_* - \sum \langle W, v_j \rangle r \ell_* v_j) = A (e^{i\theta} x + e^{i\phi} W - \sum \langle W, v_j \rangle e^{i\phi} v_j) = A (e^{i\theta} + e^{i\phi} \langle W, x \rangle) x.$$

From the above remarks we deduce that:

$$\beta = \text{Arg}(e^{i\alpha} + e^{i\phi} \langle W, x \rangle) + (n-1)\phi = \text{Arg}(e^{i\alpha} + \langle W, x \rangle) + n\phi.$$

In the centered case $W = 0$ (Example 1), we find that $\beta = n\phi + \alpha$. This is the only case where the Lagrangian angle is constant on the leaves. We recall that a necessary and sufficient condition for a Lagrangian submanifold to be a special Lagrangian one is that the Lagrangian angle be locally constant. So our computation shows that a special Lagrangian $\sigma$-submanifold must be centered. Moreover, it has been shown in [An2] that the only such submanifolds are pieces of the Lagrangian catenoid (cf [CU1] for a complete description), so we recover one of the theorems of [CU1]:

**Theorem** A (non flat) special Lagrangian submanifold which is foliated by $(n-1)$-spheres is congruent to a piece of the Lagrangian catenoid.

### 2.2 Computation of the mean curvature vector

We first calculate the second derivatives of the immersion.

$$\ell_{ss} = ike^{i\theta} x + \left( i \frac{\sin \alpha}{r} W + W \right) e^{i\phi},$$

$$v_j(\ell_s) = e^{i\theta} v_j,$$

$$\ell_{v_j v_k} = -\delta_{jk} r e^{i\phi} x.$$  

Then we obtain the following expressions:

$$\langle \ell_{ss}, J\ell_s \rangle = k + k \cos \alpha \langle W, x \rangle + \frac{\sin \alpha \cos \alpha}{r} \langle W, x \rangle + \sin \alpha \langle W, x \rangle + \frac{\sin \alpha}{r} |W|^2,$$

$$\langle \ell_{ss}, J\ell_s v_j \rangle = \langle v_j(\ell_s), J\ell_s \rangle = \sin \alpha \langle W, v_j \rangle,$$
\[ \langle \ell_{v_j}v_j, J\ell_s \rangle = \langle v_j(\ell_s), J\ell_s v_j \rangle = r \sin \alpha, \]
\[ \langle \ell_{v_j}v_k, J\ell_s v_k \rangle = 0. \]

Thus we have, using the property that the tensor \( C := \langle h(.,.), J_\ast \rangle \) is totally symmetric for any Lagrangian immersion:

\[ C_{111} = \langle h(e_1, e_1), J\ell_s e_1 \rangle \]
\[ = A^3 \left( h \left( \partial_s - \frac{1}{r} \sum (W, v_j) v_j, \partial_s - \frac{1}{r} \sum (W, v_j) v_j \right), J\ell_s - \frac{\sum (W, v_j) J\ell_s v_j}{r} \right) \]
\[ = A^3 \left( k + k \cos \alpha \langle W, x \rangle + \frac{\sin \alpha \cos \alpha}{r} \langle W, x \rangle + \sin \alpha \langle \dot{W}, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2 \right) \]
\[ + \frac{\sin \alpha}{r} |W|^2 + \frac{\sin \alpha}{r} \sum (W, v_j)^2 - 2 \sin \alpha \sum (W, v_j)^2 \]
\[ = A^3 \left( k + k \cos \alpha \langle W, x \rangle + \frac{\sin \alpha \cos \alpha}{r} \langle W, x \rangle + \sin \alpha \langle \dot{W}, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2 \right), \]

\[ C_{jj1} = \langle h(e_j, e_j), J\ell_s e_1 \rangle = \frac{A}{r^2} \left( h(v_j, v_j), J\ell_s - \frac{1}{r} \sum (W, v_k) J\ell_s v_k \right) = \frac{A \sin \alpha}{r}, \]

\[ C_{11j} = \langle h(e_1, e_1), J\ell_s e_j \rangle = \frac{A^2}{r} \left( \sin \alpha \langle W, v_j \rangle - \frac{2}{r} \sum (W, v_k) r \sin \alpha \delta_{jk} \right) \]
\[ = - \frac{A^2 \sin \alpha \langle W, v_j \rangle}{r}, \]

\[ C_{jjk} = \langle h(e_j, e_j), J\ell_s e_k \rangle = 0. \]

So finally we have the following:

\[ n\tilde{H} = A \left[ A^2 \left( k + k \cos \alpha \langle W, x \rangle + \frac{\sin \alpha \cos \alpha}{r} \langle W, x \rangle \right) \right. \]
\[ + \sin \alpha \langle W, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2 + \left( \frac{n - 1}{r} \right) \sin \alpha \right] J\ell_s e_1 \]
\[ - \sum \frac{A^2 \sin \alpha \langle W, v_j \rangle}{r} J\ell_s e_j. \]

In Section 5, we shall use the following notations: \( nJ\tilde{H} = a e_1 + \sum a_j e_j \), where

\[ (H_0) \quad a = -A^3 \left( k + \left( k \cos \alpha + \frac{\sin \alpha \cos \alpha}{r} \right) \langle W, x \rangle \right. \]
\[ + \sin \alpha \langle W, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2 \left. \right) - (n - 1) \frac{A \sin \alpha}{r} \]

\[ (H_j) \quad a_j = A^2 \frac{\sin \alpha \langle W, v_j \rangle}{r}. \]
3 Application to self-similar equations

**Theorem 2** In the class of Lagrangian submanifolds which are foliated by \((n - 1)\)-spheres, there are no self-translators and the only self-shrinkers/expanders are the centered ones described in [An2].

**Proof.** The self-translating equation is \(\vec{H} = V^\perp\) for some fixed vector \(V \in \mathbb{C}^n\). In particular,

\[
\langle \vec{H}, J\ell_\ast e_j \rangle = \langle V, J\ell_\ast e_j \rangle
\]

\[
\iff -\frac{A^2 \sin \alpha}{nr} \langle W, v_j \rangle = \langle V, Jre^{i\phi}e_j \rangle
\]

\[
\iff -\frac{A^2 \sin \alpha}{nr} \langle W, v_j \rangle = \langle V, Je^{i\phi}v_j \rangle
\]

\[
\iff -\frac{\sin \alpha}{nr} \langle W, v_j \rangle = (1 + 2 \cos \alpha \langle W, x \rangle + \langle W, x \rangle^2) \langle V, Je^{i\phi}v_j \rangle.
\]

Differentiating this last equation with respect to \(v_k, k \neq j\) (this is possible since \(n \geq 3\)), we obtain

\[
0 = \left(1 + 2\langle W, v_k \rangle (\cos \alpha + \langle W, x \rangle)\right) \langle V, Je^{i\phi}v_j \rangle.
\]

We now observe that the set of points \((x, v_k)\) for which the first factor in the r.h.s. term of the above equation vanishes is of codimension 1 at most in the unit sphere bundle over \(\mathbb{S}^{n-1}\) and apart from this set, \(\langle V, Je^{i\phi}v_j \rangle\) vanishes. Coming back to the third equality of the above equivalences, this yields that either \(\sin \alpha\) vanishes or \(\langle W, v_j \rangle\) does. In the first case the curve \(\gamma\) is a line passing through the origin. Then we know by Example 2 that the immersion is totally geodesic so the image is a Lagrangian subspace. In the other case, we know that for \(j \neq k\), \(\langle W, v_j \rangle\) vanishes on some dense open subset of points \((x, v_k)\) in the unit sphere bundle over \(\mathbb{S}^{n-1}\). This shows that \(W\) vanishes identically. Then it is clear that so does \(V^\perp\). This implies that the only solution of the equation is for vanishing \(\vec{H}\), which is the trivial, minimal case.

The self-shrinking/expanding equation is \(\vec{H} + \lambda X^\perp = 0\), where \(\lambda\) is some real constant. This implies

\[
\langle \vec{H}, J\ell_\ast e_j \rangle + \lambda \langle \ell, J\ell_\ast e_j \rangle = 0
\]

\[
\iff -\frac{A^2 \sin \alpha}{nr} \langle W, v_j \rangle + \lambda \left(\int e^{i\phi}W, Je^{i\phi}v_j\right) = 0
\]

\[
\iff \frac{A^2 \sin \alpha}{nr} \langle W, v_j \rangle = \lambda \left(\text{Im} \left(\int e^{i\phi}W\right), v_j\right).
\]

The quantity \(\text{Im} \left(\int e^{i\phi}W\right)\) depends only on \(s\), so the same argument as above holds, and we deduce that either \(\gamma\) is a line passing through the origin (totally geodesic case) or \(W\) vanishes, which is the centered case treated in [An2]. \qed
4 Hamiltonian stationary $\sigma$-submanifolds

The purpose of this section is to prove the following

**Theorem 3** Any Hamiltonian stationary Lagrangian submanifold foliated by $(n-1)$-spheres must be centered.

**Proof.** Let $\ell$ be a parametrization of such a submanifold. We follow the same notations than in Section 2. We are going to show that if $\Delta \beta$ vanishes, then either $\gamma$ is a straight line, so as we have seen in Example 2, we are in the totally geodesic case, which is in particular centered, or $W$ vanishes (centered case). The proof is based on an analysis of the quantity $f(x) := A^{-6} \Delta \beta$ which turns to be polynomial. Its expression is given in the next lemma and the computation is detailed in Appendix.

**Lemma 2** For any fixed $s$, $f := A^{-6} \Delta \beta$ is a polynomial in the three variables $\langle W, x \rangle$, $\langle \dot{W}, x \rangle$ and $\langle \ddot{W}, x \rangle$. Indeed we have: $f = (I) + (II) + (III) + (IV)$ with

$$(I) := 3B \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right)$$

$$- A^{-2} \left[ \dot{B} - (n-1) \frac{\sin \alpha}{r} \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right) \right]$$

$$- A^{-4} (n-1) \frac{\partial_s \left( \frac{\sin \alpha}{r} \right)}{s},$$

$$(II) := -3B \frac{\cos \alpha + \langle W, x \rangle}{r} (|W|^2 - \langle W, x \rangle^2)$$

$$+ \frac{A^{-2}}{r} \left[ \left( k \cos \alpha - (n-2) \frac{\sin \alpha \cos \alpha}{r} - (n-3) \frac{\sin \alpha}{r} \langle W, x \rangle \right) (|W|^2 - \langle W, x \rangle^2) \right.$$}

$$+ \sin \alpha \left( \langle \dot{W}, W \rangle - \langle \dot{W}, x \rangle \langle W, x \rangle \right),$$

$$(III) := -A^{-2} \frac{\sin \alpha}{r^2} (\cos \alpha + \langle W, x \rangle) \left( |W|^2 - \langle W, x \rangle^2 \right) - A^{-4} (n-1) \frac{\sin \alpha}{r^2} \langle W, x \rangle,$$

$$(IV) := -A^{-2} B \frac{n-1}{r} (\cos \alpha + \langle W, x \rangle) - A^{-4} \frac{(n-1)^2 \sin \alpha}{r^2} (\cos \alpha + \langle W, x \rangle)$$

and

$$B := k + \left( \frac{\sin \alpha \cos \alpha}{r} \right) \langle W, x \rangle + \sin \alpha \langle \dot{W}, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2.$$}

**Proof.** See Appendix. $\Box$

**Lemma 3**

$$f(x) = - \left( \sin \alpha + 2 \sin \alpha \cos \langle W, x \rangle + \sin \langle W, x \rangle^2 \right) \langle \dot{W}, x \rangle + R \left( \langle W, x \rangle, \langle \dot{W}, x \rangle \right),$$

where $R(.,.)$ is a polynomial in its two variables.
Proof. In the previous expression of \( f \), we see that there are no contributions to the terms in \( \langle \bar{W}, x \rangle \) apart from the \( \dot{B} \) term in (I). Then we compute

\[
\dot{B} = \sin \alpha \langle \bar{W}, x \rangle + S \left( \langle W, x \rangle, \langle \bar{W}, x \rangle \right),
\]

where \( S(\ldots) \) is a polynomial in its two variables. We deduce that the only term of \( \langle \bar{W}, x \rangle \) in \( f \) is the following:

\[-A^{-2} \sin \alpha \langle \bar{W}, x \rangle,
\]

so we have our claim. \( \square \)

**Lemma 4** The polynomial \( f \) has total degree at most 5, and in that case its leading term is \( -\frac{n^2 + n - 2}{r^2} \sin \alpha \langle W, x \rangle^5 \).

Proof. Remembering that \( A^{-2}, B \) and \( \dot{B} \) are polynomials of degree at most 2 in \( \langle W, x \rangle \) and \( \langle \dot{W}, x \rangle \), we first observe that there are no terms of order 5 in (I). Then we check that in (II), (III) and (IV) there are no terms in \( \langle \bar{W}, x \rangle \langle W, x \rangle^4 \) (this has already been observed in the previous lemma) or in \( \langle W, x \rangle \langle \dot{W}, x \rangle^4 \). Finally, the coefficient of \( \langle W, x \rangle^2 \) in \( A^{-2} \) and \( B \) are respectively 1 and \( \sin \alpha \), so summing up, we obtain that the coefficients of \( \langle W, x \rangle^5 \) in (II), (III) and (IV) are respectively

\[
3 \frac{\sin \alpha}{r^2} + (n - 3) \frac{\sin \alpha}{r^2},
\]

\[
\frac{\sin \alpha}{r^2} - (n - 1) \frac{\sin \alpha}{r^2}
\]

\[-(n - 1) \frac{\sin \alpha}{r^2} - (n - 1)^2 \frac{\sin \alpha}{r^2},
\]

from which we conclude the proof. \( \square \)

**Lemma 5** Let \( P(x_1, x_2, x_3) \) be an irreducible polynomial with real coefficients. Assume the set of zeroes of \( P \) is non empty and that \( f \in \mathbb{R}[x_1, x_2, x_3] \) vanishes on it. Then \( f \) is of the form \( PQ \) with \( Q \in \mathbb{R}[x_1, x_2, x_3] \).

Proof. We embed \( \mathbb{R}^3 \) into \( \mathbb{C}^3 \). Assume by contradiction that \( f \) is not of the form \( PQ \) with \( Q \in \mathbb{R}[x_1, x_2, x_3] \). Then also \( f \) is not of the form \( PQ \) with \( Q \in \mathbb{C}[x_1, x_2, x_3] \), so the set \( Y := \{ x/f(x) = P(x) = 0 \} = \{ f(x) = 0 \} \cap \{ P(x) = 0 \} \) has complex codimension 2, so real dimension \( 6 - 4 = 2 \). Now our assumption is that \( Y \cap \mathbb{R}^3 \) contains \( \{ P = 0 \} \cap \mathbb{R}^3 \), so in particular, \( Y \) which has real dimension 2 contains \( \{ P = 0 \} \cap \mathbb{R}^3 \), also of real dimension 2. We conclude that it is an irreducible component of \( Y \), a contradiction because \( \mathbb{R}^3 \) does not contain any complex curve. \( \square \)
End of the proof of Theorem 3.

First case: the three vectors $W, \dot{W}$ and $\ddot{W}$ do not span $\mathbb{R}^n$ (it is in particular always the case when $n > 3$).

In this case, there exist coordinates $(y_1, \ldots, y_n)$ on $\mathbb{R}^n$ such that $f$ does not depend on $y_n$. In particular $\{y, f(y) = 0\}$ contains some straight line $\{y_j = \text{Const}, 1 \leq j \leq n - 1\}$. However by assumption $\{y, f(y) = 0\}$ contains also the hypersphere $S^{n-1}$. As it is an algebraic set, it is necessarily the whole $\mathbb{R}^n$, thus $f$ vanishes identically. This implies that either $W$ vanishes (this is the centered case), or that so does $f$ as a polynomial of independent variables of $\mathbb{R}^n$ : the vectors $W, \dot{W}$ and $\ddot{W}$ might be non independent and in this case we should rewrite $f$ as a polynomial of less variables. Anyway, we know from Lemma 4 that the only term of order 5 in $f$ is $\sin \alpha$ times a non negative constant, thus we deduce in any case that $\sin \alpha$ should vanish, which means that the curve $\gamma$ would be a line passing through the origin. Then we know that by Example 2 the immersion is totally geodesic.

Second case: $n = 3$ and $\text{Span}(W, \dot{W}, \ddot{W}) = \mathbb{R}^3$.

Here we apply the algebraic Lemma 5 with $P(x_1, x_2, x_3) = |x|^2 - 1$, thus obtaining that either $f$ is a multiple of $|x|^2 - 1$, or it vanishes. In the first case, there should be at least one term of degree 2 in $(\dot{W}, x)$, which is not the case (cf Lemma 3). So we deduce again that $f$ vanishes and the conclusion is the same as above. □

5 Centered Hamiltonian stationary $\sigma$-submanifolds

5.1 The differential system

In this case, the induced metric is diagonal: $g = \text{diag}(1, r^2, \ldots, r^2)$, and $\det g = r^{2(n-1)}$. Moreover, $\beta$ depends only on $s : \beta = \alpha + n \phi$, thus

$$\Delta \beta = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial s} \left( \sqrt{\det g} \frac{\partial \beta}{\partial s} \right).$$

Hence the submanifold is Hamiltonian stationary if and only if

$$\sqrt{\det g} \frac{\partial \beta}{\partial s} = C,$$

which amounts to

$$r^{n-1} (n \phi + \dot{\alpha}) = C,$$

for some non vanishing constant $C$ (the case of vanishing $C$ implies $\beta$ to be constant, so this is the special Lagrangian case). We add that we could have used the computations of the previous section as well with $W = 0$ to find the same equation.

Thus we are reduced to studying the following differential system (using that $\dot{\phi} = \frac{\sin \alpha}{r}$):

$$\begin{cases} \dot{\alpha} = \frac{C}{r^{n-1}} - \frac{n \sin \alpha}{r} \\ \dot{r} = \cos \alpha. \end{cases}$$
There are fixed points \((\bar{\alpha} = \pm \frac{\pi}{2} \text{ mod } 2\pi, \bar{r} = \left( \frac{|C|}{n} \right)^{1/(n-2)} \), corresponding to the case of \(\gamma\) being a circle centered at the origin. Up to a sign change of parameter \(s\), we may assume that \(C\) is non negative and then \(\bar{r} = \frac{\pi}{2} \text{ mod } 2\pi\). Moreover, the system admits a first integral: \(E = 2r^n \sin \alpha - Cr^2\), so we can draw the phase portrait easily. As the system is periodic in the variable \(\alpha\), it is sufficient to study integral lines in a strip of length \(2\pi\). Moreover integral lines are symmetric with respect to the vertical lines \(\alpha = \pi/2 \text{ mod } \pi\). The energy of the fixed points is \(E_0 := \left( \frac{|C|}{n} \right)^{n/(n-2)} (2 - n)\). There is a critical integral line of energy \(E_0\) with bounded pieces connecting two fixed points and unbounded pieces starting from (or ending to) a fixed point and having a branch asymptotic to a vertical line \(\alpha = k\pi, k \in \mathbb{Z}\). We observe that the integral lines corresponding to a non negative energy (resp. strictly less than \(E_0\)) are unbounded, we call them \textit{type I} (resp. \textit{type II}) curves. On the other hand, integral lines of energy \(E \in (E_0, 0)\) have two connected components (up to periodicity), one

Figure 1: Phase diagram
of them being bounded in the variable \( r \); this will be called in the following a type III curve. As the unbounded component shares the same features as the curves of non negative energy, it is also called a type I curve.

In order to have a better picture of the corresponding curves \( \gamma \) and of Hamiltonian stationary Lagrangian submanifolds they generate, we shall discuss in the next two paragraphs the inflection points of \( \gamma \) and the quantity \( \Phi(E) := \int \dot{\phi} = \int \frac{\sin \alpha}{r} \) that we call total variation of phase.

### 5.2 Inflection points of \( \gamma \)

They correspond to the vanishing of the curvature \( k = \dot{\theta} = \frac{C}{r^{n-1}} - \frac{(n-1) \sin \alpha}{r} \). This implies the relation \( r = \left( \frac{C}{(n-1) \sin \alpha} \right)^{1/(n-2)} \).

We observe that in the case of dimension 3, this is exactly the equation of the integral curve of level \( E = 0 \) : so in this dimension there is a solution which is of curvature zero, i.e. a straight line, and the other ones don’t have any inflection point and are locally convex.

In higher dimension, let us compute the energy \( E \) at points of vanishing curvature. At such a point \( (\alpha, r) \), we have \( r^{n-2} = \frac{C}{(n-1) \sin \alpha} \) so we deduce that \( E = Cr^2 \left( \frac{2}{n-1} - 1 \right) \), which has range \((-\infty, E_1)\) where \( E_1 \) is the energy level defined by \( E_1 = 2r_1^n - Cr_1^2 \) (\( r_1 \) is the least radius on the curve of points \( (\alpha, r) \) corresponding to vanishing curvature: \( r_1 = \left( \frac{C}{(n-1) \sin \alpha} \right)^{1/(n-2)} \)). As \( r \mapsto E(r) = 2r^n - Cr^2 \) is increasing when \( r > r_1 \), \( E_1 \) belongs to the interval \((E_0, 0)\). We conclude that when \( n > 3 \) every type II curve has two (symmetric) inflection points. This is also the case of some type I curves, while the remaining ones are locally convex.

### 5.3 Study of the total variation of phase

We first compute \( \Phi(E) \) for type I curves. Let \( r_0 \) be the minimal value taken by \( r \). We have the relation \( E = 2r_0^n - Cr_0^2 \). We shall use the fact that on the curve \( \sin \alpha > 0 \); moreover, by symmetry we may restrict ourselves to half of the curve, thus we may also assume that \( \cos \alpha > 0 \). Thus

\[
\Phi = \int \frac{\dot{\phi}}{r} dr = 2 \int_{r_0}^{\infty} \frac{\sin \alpha}{r \cos \alpha} dr = 2 \int_{r_0}^{\infty} \frac{dr}{r} \left( \frac{1}{\sin^2 \alpha} - 1 \right)^{-1/2}
\]

\[
= 2 \int_{r_0}^{\infty} \frac{dr}{r} \left( \frac{2r^n}{Cr^2 + E} \right)^{1/2}
\]

\[
= 2 \int_{r_0}^{\infty} \frac{dr}{r} \left( \frac{2r_0^n}{C(r^2 - r_0^2) + 2r_0^2} \right)^{1/2}
\]
Making the change of variable \( r = xr_0 \), we infer
\[
\Phi = 2 \int_1^\infty \frac{dx}{x} \left( \left( \frac{x^n}{\lambda(x^2 - 1) + 1} \right)^2 - 1 \right)^{-1/2},
\]
where \( \lambda = \frac{C}{2r_0^{n-2}} \). We observe that this integral equals \( \pi/n \) when \( \lambda \) vanishes, is divergent for \( \lambda = n/2 \), i.e. for \( E = E_0 \) and is decreasing in the variable \( r_0 \), so \( \Phi(E) \) has range \((\pi/n, +\infty)\). Notice that the sign of \( \dot{\phi} = \sin \alpha/r \) does not change, so that \( \gamma = r e^{i\phi} \) is embedded whenever the total variation of phase is small enough, i.e. whenever \( E \) is large enough.

On the other hand a unbounded piece of the integral curve of energy level \( E_0 \) is singular, with an infinite spiral branch asymptotic to the unit circle.

We now look at the case of type II integral curves. Let \( r_1 \) be the value of \( r \) at \( \alpha = 0 \mod \pi \), so that \( E = -Cr_1^2 \). We shall calculate separately the contributions of the curve when the integrand is positive (resp. negative), corresponding to the part of the curve lying in the region \( \{ \sin \alpha > 0 \} \) (resp. \( \{ \sin \alpha < 0 \} \)).

\[
\Phi_+ := 2 \int_{r_1}^\infty \frac{\sin \alpha}{r \cos \alpha} dr = 2 \int_{r_1}^\infty \frac{dr}{r} \left( \frac{1}{\sin^2 \alpha} - 1 \right)^{-1/2}
\]
\[
= 2 \int_{r_1}^\infty \frac{dr}{r} \left( \left( \frac{2r_1^n}{C r^2 + E} \right)^2 - 1 \right)^{-1/2}
\]
\[
= 2 \int_{r_1}^\infty \frac{dr}{r} \left( \frac{2r_1^n}{C(r^2 - r_1^2)} \right)^2 - 1 \right)^{-1/2}
\]

Making the change of variable \( r = xr_1 \), we obtain
\[
\Phi_+ = 2 \int_1^\infty \frac{dx}{x} \left( \left( \frac{x^n}{\lambda(x^2 - 1)} \right)^2 - 1 \right)^{-1/2},
\]
where \( \lambda = \frac{C}{2r_1^{n-2}} \).

One calculates that \( \Phi_+ \) is increasing in the variable \( E \) and that \( \lim_{E \to -\infty} = \pi/n \).

In the computation of \( \Phi_- \), we may use \( \alpha \) as a parameter:

\[
\Phi_- := \int_\pi^{2\pi} \frac{\dot{\phi}}{\alpha} d\alpha = \int_\pi^{2\pi} \frac{\sin \alpha}{r} \frac{r^n - n \sin \alpha}{r^{n-1}} d\alpha
\]
\[
= \int_\pi^{2\pi} \frac{r^{n-2} \sin \alpha}{C - nr^{n-2} \sin \alpha} d\alpha
\]
Lagrangian submanifolds foliated by \((n-1)\)-spheres

It is an easy computation to show that \(-\pi/n < \Phi_- < 0\) and that \(\lim_{E \to -\infty} \Phi_- = -\pi/n\).

Next we show that \(\gamma\) has always a self-intersection. Let \(s_0\) be the parameter value corresponding to the point of the curve of least radius (in particular \(\alpha(s_0) = 3\pi/2 \mod 2\pi\)). The fact that \(\Phi_+ > |\Phi_-|\) and the intermediate value theorem imply that there exists \(s_1\) such that \(\phi(s_1) = \phi(s_0)\); moreover by the symmetry of the phase portrait, there exists \(s_2 \neq s_1\) with the same property, the corresponding points in the phase portrait satisfying \(r(s_1) = r(s_2)\) and \(\frac{1}{2}(\alpha(s_1) + \alpha(s_2)) = 3\pi/2 \mod 2\pi\). Thus \(\gamma(s_1) = \gamma(s_2)\).

We end this section by looking at the type III curves. As \(\alpha\) is always increasing on the curve, we shall use it as a parameter and consider the piece of curve \(\gamma([\pi/2, 2\pi + \pi/2])\).

As \(\phi\) is decreasing on \(\alpha \in [\pi, 2\pi]\) and increasing elsewhere, we have:

\[
\Phi_+ := \int_{\pi/2}^{\pi} \dot{\phi} + \int_{2\pi}^{2\pi + \pi/2} \dot{\phi} > 0
\]

and

\[
\Phi_- := \int_{\pi}^{2\pi} \dot{\phi} < 0.
\]

It is easy to show (the calculations are left to the Reader) that \(\Phi_+ > |\Phi_-|\) and that \(\lim_{E \to 0} \Phi_+ = \lim_{E \to 0} \Phi_- = 0\). Moreover, \(\lim_{E \to E_0} \Phi_+ = +\infty\) and \(\lim_{E \to E_0} \Phi_-\) is finite. This implies that the range of the total variation of phase for type III curves is \((0, \infty)\). In particular, to the limiting case \(E = E_0\) corresponds a bounded, complete curve spiraling asymptotically to the unit circle.

With a similar argument as above, we show that here again \(\gamma\) has always self-intersections. We conclude that for a type III curve such that \(\Phi(E) \in 2\pi\mathbb{Q}\), the corresponding curve \(\gamma\) is bounded, closed and non-embedded.

5.4 Conclusion (Corollary 1)

We are now in position to describe the whole family of Hamiltonian stationary \(\sigma\)-submanifolds:

**Corollary 1** Any Hamiltonian stationary Lagrangian submanifold which is foliated by \((n-1)\)-spheres is locally congruent to one of the following:

- The standard embedding
  \[\mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{C}^n, \quad (e^{is}, x) \mapsto e^{is}x.\]

- A singular, bounded immersion of \(\mathbb{R} \times \mathbb{S}^{n-1}\) "spiraloid" asymptotic to \(\mathbb{S}^1 \times \mathbb{S}^{n-1}\).
• A singular, unbounded “spiraloid” with a smooth end and asymptotic to $S^1 \times S^{n-1}$.

• A family of smooth “catenoid-type” immersions of $\mathbb{R} \times S^{n-1}$, some of them being embedded. In dimension 3, one of them takes the particular following form:

$$\mathbb{R} \times S^2 \rightarrow \mathbb{C}^3,$$

$$(s, x) \mapsto (C + is)x,$$

where $C$ is some real constant.

• A family of non-standard smooth immersions of $S^1 \times S^{n-1}$. They always have self-intersections.

**Appendix: computation of $\Delta \beta$ (proof of Lemma 2)**

Here we are going to use the same notations as in Section 2 and 4. We first compute the following

$$\langle \ell_{ss}, \ell_s \rangle = \langle W, \dot{W} \rangle + \left( \frac{\sin \alpha}{r} - k \sin \alpha \right) \langle W, x \rangle + \cos \alpha \langle \dot{W}, x \rangle$$

$$\langle \ell_{ss}, \ell_s v_j \rangle = r \langle \dot{W}, v_j \rangle$$

$$\langle v_j(\ell_s), \ell_s \rangle = \cos \alpha \langle W, v_j \rangle$$

$$\langle v_j(\ell_s), \ell_s v_k \rangle = \delta_{jk} r \cos \alpha$$

$$\langle \ell_{v_jv_k}, \ell_s \rangle = -\delta_{jk} r (\cos \alpha + \langle W, x \rangle)$$

$$\langle \ell_{v_jv_k}, \ell_s v_l \rangle = 0.$$

We shall make use of the following formula:

$$\Delta \beta = \text{div} Jn \tilde{H} = \langle \nabla_{e_1} Jn \tilde{H}, e_1 \rangle + \sum_{j=2}^n \langle \nabla_{e_j} Jn \tilde{H}, e_j \rangle$$

Writing $Jn \tilde{H} = a e_1 + \sum_k a_k e_k$, we have

$$\nabla_{e_j} Jn \tilde{H} = e_j(a) e_1 + a \nabla_{e_j} e_1 + \sum_k e_j(a_k) e_k + \sum_k a_k \nabla_{e_j} e_k,$$

and an analogous expression for $\nabla_{e_1} Jn \tilde{H}$.

Then we obtain

$$\Delta \beta = e_1(a) + a \langle \nabla_{e_1} e_1, e_1 \rangle + \sum_k a_k \langle \nabla_{e_1} e_k, e_1 \rangle$$

$$+ \sum_j \left[ a \langle \nabla_{e_j} e_1, e_j \rangle + e_j(a_j) + \sum_k a_k \langle \nabla_{e_j} e_k, e_j \rangle \right].$$
We give the expressions of the coefficients of $J \tilde{H}$:

$$a_j = A^2 \frac{\sin \alpha \langle W, v_j \rangle}{r}$$

and

$$a = -A^3 B - A(n-1) \frac{\sin \alpha}{r},$$

where

$$A = \left( 1 + 2 \cos \alpha \langle W, x \rangle + \langle W, x \rangle^2 \right)^{-1/2}$$

and

$$B = k + \left( k \cos \alpha + \frac{\sin \alpha \cos \alpha}{r} \right) \langle W, x \rangle + \sin \alpha \langle \dot{W}, x \rangle + \frac{\sin \alpha}{r} \langle W, x \rangle^2.$$ 

We start with some easy computations:

**Lemma 6 a)** $\dot{A} = -A^3 \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right)$.

**b)** $\langle \partial_s, e_1 \rangle = A^{-1}$.

**c)** $v_j(A) = -A^3 \left( \cos \alpha + \langle W, x \rangle \right) \langle W, v_j \rangle$.

**d)** $v_j(B) = \left( k \cos \alpha + \frac{\sin \alpha \cos \alpha}{r} + \frac{2 \sin \alpha}{r} \langle W, x \rangle \right) \langle W, v_j \rangle + \sin \alpha \langle \dot{W}, v_j \rangle$.

**e)** $\nabla_{v_j} v_k = -\delta_{jk} Ar (\cos \alpha + \langle W, x \rangle) e_1$.

**f)** $\nabla_{v_j} \partial_s = \cos \alpha e_j$.

**Proof.** a), b), c) and d) are easy and left to the reader.

e) We write $\nabla_{v_j} v_k = b_l \partial_s + \sum_m b_m v_m$. In particular, we have

$$0 = \langle \nabla_{v_j} v_k, v_l \rangle = b_1 \langle \partial_s, v_l \rangle + \sum_m b_m \langle v_m, v_l \rangle = b_1 g_{11} + \sum_m b_m g_{ml} = b_1 g_{11} + b_1 r^2,$$

so we obtain the relation $b_l = -\frac{\langle W, v_l \rangle}{r} b_1$.

When $j \neq k$, it yields

$$0 = \langle \nabla_{v_j} v_k, \partial_s \rangle = b_1 \left( g_{11} - \sum_l \frac{\langle W, v_l \rangle}{r} g_{1l} \right),$$

so all the coefficients $b_1, b_l$ vanish, and we obtain e).

If $j = k$, we have

$$-r (\cos \alpha + \langle W, x \rangle) = \langle \nabla_{v_j} v_j, \partial_s \rangle = b_1 \left( g_{11} - \sum_l \frac{\langle W, v_l \rangle}{r} g_{1l} \right),$$

so we obtain e).
so \( b_1 = -A^2 r (\cos \alpha + \langle W, x \rangle) \). We deduce that

\[
\nabla_{v_j} v_j = b_1 \left( \partial_s - \sum_l \frac{\langle W, v_l \rangle}{r} v_l \right) = b_1 e_1 = -A r (\cos \alpha + \langle W, x \rangle) e_1,
\]

which implies e).

f) We write \( \nabla_{v_j} \partial_s = b_1 \partial_s + \sum_l b_l v_l \). Then we have

\[
\delta_{jk} r \cos \alpha = \langle \nabla_{v_j} \partial_s, v_k \rangle = b_1 g_{1k} + \sum_l b_l g_{kl}.
\]

When \( j \neq k \), we deduce that

\[
b_k = -\frac{g_{1k}}{g_{kk}} b_1 = -\frac{\langle W, v_k \rangle}{r} b_1.
\]

When \( j = k \), we obtain

\[
r \cos \alpha = b_1 r \langle W, v_j \rangle + r^2 b_j,
\]

so

\[
b_j = r^{-2} (r \cos \alpha - b_1 r \langle W, v_j \rangle) = r^{-1} (\cos \alpha - b_1 \langle W, v_j \rangle).
\]

From these relations we can write

\[
\cos \alpha \langle W, v_k \rangle = \langle \nabla_{v_k} \partial_s, \partial_s \rangle = b_1 g_{11} + \sum_j b_j g_{1j}
\]

\[
= b_1 \left( g_{11} - \sum_j r \langle W, v_j \rangle \frac{\langle W, v_j \rangle}{r} \right) + r \langle W, v_k \rangle r^{-1} \cos \alpha = b_1 A^{-2} + \cos \alpha \langle W, v_k \rangle.
\]

We deduce that \( b_1 \) vanishes and also \( b_l \) for \( l \neq k \), and finally \( b_k = \frac{\cos \alpha}{r} \), so we obtain f).

\[
\square
\]

We now compute the different terms of \( \Delta \beta \):

**Lemma 7 a)**

\[
e_1(a) = A \dot{a} - \sum_j A \frac{\langle W, v_j \rangle}{r} v_j(a)
\]

with

\[
A \dot{a} = 3 A^6 B \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right)
\]
\[-A^4 \left[ \dot{B} - (n-1) \frac{\sin \alpha}{r} \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right) \right] \]

and

\[-A^2(n-1) \partial_s \left( \frac{\sin \alpha}{r} \right) \]

\[\sum_j \frac{A(W, v_j)}{r} v_j(a) = 3A^6 B \frac{\cos \alpha + \langle W, x \rangle}{r} (|W|^2 - \langle W, x \rangle^2) \]

\[-\frac{A^4}{r} \left[ \left( k \cos \alpha - (n-2) \frac{\sin \alpha \cos \alpha}{r} - (n-3) \frac{\sin \alpha}{r} \langle W, x \rangle \right) (|W|^2 - \langle W, x \rangle^2) \right. \]

\[+ \sin \alpha \left( \langle \dot{W}, W \rangle - \langle \dot{W}, x \rangle \langle W, x \rangle \right) \],

b) \[\langle \nabla_{e_1} e_1, e_1 \rangle = 0,\]

c) \[\sum_k a_k \langle \nabla_{e_k} e_1, e_1 \rangle + \sum_j e_j(a_j) = -\frac{\sin \alpha}{r^2} \left( A^4 (\cos \alpha + \langle W, x \rangle)(|W|^2 - \langle W, x \rangle^2) \right. \]

\[+A^2(n-1)\langle W, x \rangle \),\]

d) \[\sum_j \langle \nabla_{e_j} e_1, e_j \rangle = \frac{A(n-1)}{r} (\cos \alpha + \langle W, x \rangle),\]

e) \[\langle \nabla_{e_j} e_k, e_j \rangle = 0, \quad \forall j, k.\]

**Proof.**

a) We compute

\[\dot{a} = -3A^2 \dot{A} B - A^3 \dot{B} - \dot{A}(n-1) \frac{\sin \alpha}{r} - A \partial_s \left( (n-1) \frac{\sin \alpha}{r} \right) \]

\[= A^3 \left( 3A^2 B + (n-1) \frac{\sin \alpha}{r} \right) \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right) \]

\[-A^3 \dot{B} - A(n-1) \partial_s \left( \frac{\sin \alpha}{r} \right)\]
from which we deduce the first equality. Next we have

\[
v_j(a) = -3A^2 v_j(A)B - A^3 v_j(B) - v_j(A)(n-1)\frac{\sin \alpha}{r}
\]

\[
= A^3 \left( 3A^2 B + (n-1)\frac{\sin \alpha}{r} \right) \left( \cos \alpha + \langle W, x \rangle \right) \langle W, v_j \rangle \\
- A^3 \left[ \left( k \cos \alpha + \frac{\sin \alpha \cos \alpha}{r} \right) + 2\frac{\sin \alpha}{r} \langle W, x \rangle \right] \langle W, v_j \rangle + \sin \alpha \langle \hat{W}, v_j \rangle 
\]

\[
= 3A^5 (\cos \alpha + \langle W, x \rangle) B \langle W, v_j \rangle \\
- A^3 \left[ \left( k \cos \alpha + \frac{\sin \alpha \cos \alpha}{r} \right) - (n-1)\frac{\sin \alpha \cos \alpha}{r} \right. \\
+ \left. \left( \frac{2\sin \alpha}{r} - (n-1)\frac{\sin \alpha}{r} \langle W, x \rangle \right) \right] \langle W, v_j \rangle + \sin \alpha \langle \hat{W}, v_j \rangle 
\]

\[
= 3A^5 (\cos \alpha + \langle W, x \rangle) B \langle W, v_j \rangle \\
- A^3 \left[ \left( k \cos \alpha - (n-2)\frac{\sin \alpha \cos \alpha}{r} \right) - (n-3)\frac{\sin \alpha}{r} \langle W, x \rangle \right] \langle W, v_j \rangle \\
+ \sin \alpha \langle \hat{W}, v_j \rangle 
\].

Thus

\[
A \sum \frac{v_j(a) \langle W, v_j \rangle}{r} = 3A^6 B \frac{\cos \alpha + \langle W, x \rangle}{r} \sum \langle W, v_j \rangle^2 \\
- A^4 \left[ \left( k \cos \alpha - (n-2)\frac{\sin \alpha \cos \alpha}{r} \right) - (n-3)\frac{\sin \alpha}{r} \langle W, x \rangle \right] \sum \langle W, v_j \rangle^2 \\
+ \sin \alpha \sum \langle W, v_j \rangle \langle \hat{W}, v_j \rangle 
\]


b) We first compute

\[ \nabla_{e_1}e_1 = A \left[ \nabla_{\partial_s} \left( A \partial_s - \frac{A}{r} \sum (W, v_j) v_j \right) \right. \]

\[ - \sum \frac{(W, v_j)}{r} \nabla_{v_j} \left( A \partial_s - \frac{A}{r} \sum (W, v_k) v_k \right) \]

\[ = A \left[ A \nabla_{\partial_s} \partial_s + \dot{A} \partial_s - \sum \partial_s \left( \frac{A(W, v_j)}{r} \right) v_j - \frac{A}{r} \sum (W, v_j) \nabla_{\partial_s} v_j \right. \]

\[ - \sum \frac{(W, v_j)}{r} A \nabla_{v_j} \partial_s - \sum \frac{(W, v_j) v_j(A)}{r} \partial_s \]

\[ + \frac{1}{r^2} \sum_{j,k} (W, v_j) \left( v_j(A)(W, v_k) v_k + A(-\delta_{jk}(W, x)v_k + A(W, v_k) \nabla_{v_j} v_k) \right). \]

Making the scalar product with \( e_1 \), many terms vanish and we obtain

\[ \langle \nabla_{e_1}e_1, e_1 \rangle = A^2 \langle \nabla_{\partial_s} \partial_s, e_1 \rangle + A \dot{A} \langle \partial_s, e_1 \rangle - \frac{A}{r} \left( \sum (W, v_j) v_j(A) \right) \langle \partial_s, e_1 \rangle \]

\[ - \frac{A^3}{r} (\cos \alpha + \langle W, x \rangle) \sum (W, v_j)^2 \]

\[ = A^3 \left( \dot{A} \sin \alpha \langle W, x \rangle + \frac{\sin^2 \alpha \langle W, x \rangle}{r} - k \sin \alpha \langle W, x \rangle \right) \]

\[ + \frac{1}{r} \left( \sum (W, v_j)^2 A^3 (\cos \alpha + \langle W, x \rangle) \right) \]

\[ - \frac{A^3}{r} (\cos \alpha + \langle W, x \rangle) \sum (W, v_j)^2. \]

Using the fact that \( \dot{\alpha} = \dot{\theta} - \dot{\phi} = k - \frac{\sin \alpha}{r} \) we see that the first term in the last sum vanishes, and so does the last two ones together, so we obtain b).

c) It will be a consequence of the fact that \( |W|^2 = \langle W, x \rangle^2 + \sum (W, v_j)^2 \) and of the two following equalities:

(1) \[ \sum e_j(a_j) = -\frac{\sin \alpha}{r^2} \left( 2A^4 (\langle W, x \rangle + \cos \alpha) \sum (W, v_j)^2 \ight. \]

\[ \left. + (n - 1)A^2 \langle W, x \rangle \right), \]

(2) \[ \sum a_k \langle \nabla_{e_1}e_k, e_1 \rangle = \frac{A^4}{r^2} \sin \alpha (\cos \alpha + \langle W, x \rangle) \sum (W, v_k)^2. \]

To prove (1) and (2) we proceed as follows:

\[ e_j(a_j) = \frac{1}{r^2} v_j \left( A^2 \frac{\sin \alpha \langle W, v_j \rangle}{r} \right) = \frac{\sin \alpha}{r^2} \left( 2A v_j \langle A, W, v_j \rangle - A^2 \langle W, x \rangle \right) \]
\[ \frac{\sin \alpha}{r^2} (2A^4(\langle W, x \rangle + \cos \alpha)\langle W, v_j \rangle^2 + A^2\langle W, x \rangle), \]

from which we obtain (1). Then we compute

\[
\nabla_{e_1} e_k = A \left( \nabla_{\partial_s} e_k - \sum_l \frac{\langle W, v_l \rangle}{r} \nabla_{v_l} e_k \right) \\
= A \left( \frac{1}{r} \nabla_{\partial_s} v_k + \partial_s (r^{-1}) v_k + \sum_l \frac{\langle W, v_l \rangle}{r^2} \nabla_{v_l} v_k \right) \\
= A \left( \frac{\cos \alpha}{r} e_k - \frac{\cos \alpha}{r^2} v_k + \sum_l \frac{\langle W, v_l \rangle}{r^2} \nabla_{v_l} v_k \right).
\]

The first two terms of the last expression vanish and thanks to the e) of Lemma 6, we deduce

\[ \langle \nabla_{e_1} e_k, e_1 \rangle = -\frac{A^2 \langle W, v_j \rangle}{r} (\cos \alpha + \langle W, x \rangle), \]

which implies (2).

d) We have

\[
\nabla_{e_j} e_1 = \frac{1}{r} \nabla_{v_j} \left( A\partial_s - \sum_k \frac{A\langle W, v_k \rangle}{r} v_k \right) \\
= \frac{1}{r} \left[ v_j(A)\partial_s + A \nabla_{v_j} \partial_s - \sum_k v_j \left( \frac{A\langle W, v_k \rangle}{r} \right) v_k - \sum_k \frac{A\langle W, v_k \rangle}{r} \nabla_{v_j} v_k \right] \\
= \frac{1}{r} v_j(A)\partial_s + A \frac{\cos \alpha}{r} e_j - \frac{1}{r} \sum_k v_j \left( \frac{A\langle W, v_k \rangle}{r} \right) v_k \\
+ \frac{A\langle W, v_j \rangle}{r^2} r (\cos \alpha + \langle W, x \rangle) e_1
\]

so

\[ \langle \nabla_{e_j} e_1, e_j \rangle = \frac{1}{r} v_j(A)\langle \partial_s, e_j \rangle + A \cos \alpha - v_j \left( \frac{A\langle W, v_j \rangle}{r} \right) \\
= \frac{1}{r} v_j(A)\langle W, v_j \rangle + A \cos \alpha - v_j(A) \frac{\langle W, v_j \rangle}{r} + \frac{A\langle W, x \rangle}{r}. \]

The first and the third terms cancel out, and we obtain d).

e) This is an obvious consequence of part e) the Lemma 6.
Summing these computations, and using the formula for $\Delta \beta$ given at the beginning of this Appendix, we conclude that $f := A^{-6} \Delta \beta = (I) + (II) + (III) + (IV)$ with

\[
(I) := A^{-5} \dot{a} = 3B \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right) \\
- A^{-2} \left[ \dot{B} - (n-1) \frac{\sin \alpha}{r} \left( \cos \alpha \langle \dot{W}, x \rangle - \dot{\alpha} \sin \alpha \langle W, x \rangle + \langle W, x \rangle \langle \dot{W}, x \rangle \right) \right] - A^{-4} (n-1) \partial_s \left( \frac{\sin \alpha}{r} \right)
\]

\[
(II) := -A^{-6} \sum \frac{\langle W, v_j \rangle v_j(a)}{r} \\
= -3B \frac{\cos \alpha + \langle W, x \rangle}{r} (|W|^2 - \langle W, x \rangle^2) \\
+ A^{-2} \frac{1}{r} \left[ \left( k \cos \alpha - (n-2) \frac{\sin \alpha \cos \alpha}{r} - (n-3) \frac{\sin \alpha}{r} \langle W, x \rangle \right) (|W|^2 - \langle W, x \rangle^2) \\
+ \sin \alpha \left( \langle W, \dot{W} \rangle - \langle \dot{W}, x \rangle \langle W, x \rangle \right) \right]
\]

\[
(III) := A^{-6} \left( \sum_k a_k \langle \nabla e_1 e_k, e_1 \rangle + \sum_j e_j(a_j) \right) \\
= -A^{-2} \frac{\sin \alpha}{r^2} (\cos \alpha + \langle W, x \rangle) \left( |W|^2 - \langle W, x \rangle^2 \right) - A^{-4} (n-1) \frac{\sin \alpha}{r^2} \langle W, x \rangle
\]

\[
(IV) := A^{-6} a \sum \langle \nabla e_j e_1, e_j \rangle \\
= -A^{-2} B \frac{n-1}{r} (\cos \alpha + \langle W, x \rangle) - A^{-4} \frac{(n-1)^2 \sin \alpha}{r^2} (\cos \alpha + \langle W, x \rangle).
\]

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Henri Anciaux,
IMPA, Estrada Dona Castorina, 110
22460–320 Rio de Janeiro, Brasil
email address: henri@impa.br

Ildefonso Castro
Departamento de Matemáticas
Universidad de Jaén,
23071 Jaén, Spain
email address: icastro@ujaen.es

Pascal Romon
Université de Marne-la-Vallée
5, bd Descartes, Champs-sur-Marne,
77454 Marne-la-Vallée cedex 2, France
email address: romon@univ-mlv.fr