Dispersion relations in differential form

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Abstract

Various forms of derivative dispersion relations, in which the dispersion integral is replaced by a series of derivatives of the imaginary part of a scattering amplitude, are reviewed. Conditions of their validity and practical applicability as well as their relevance to high-energy small-angle hadron-hadron scattering are discussed.

1 Introduction

There have been attempts since the mid 1970’s to adapt the existing dispersion relation technique to the energy range which is far enough from resonance peaks and in which changes are slow and cross sections smooth. Present experimental projects proposing to measure small-angle high-energy hadron-hadron scattering at LHC energy make this subject, after a certain time of silence, again topical. Time therefore seems to be ripe to discuss the high-energy status of dispersion relations, to point out a number of remarkable merits of the differential approach, and also to remind the reader of its limits and dangerous points which may emerge at a careless application.

We cannot give here a comprehensive review of the subject. We will just select several typical theorems to illustrate the variety of results obtained in the past, referring for mathematical and technical details to original papers. To discuss the issue, we choose the example of $F(s, t)$, the crossing-even amplitude of a generic hadron-hadron scattering process.

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Let us consider a fixed-$t$ dispersion relation for $F(s,t)$,

$$
R(s) = \frac{2s^2}{\pi} \frac{ds'}{s'(s'^2 - s^2)} I(s') \quad (1)
$$

(with poles and subtraction constants removed for simplicity), where $R(s)$ and $I(s)$ is a shorthand for $\text{Re} F(s,t)$ and $\text{Im} F(s,t)$ respectively the $t$-dependence being suppressed in the notation. It was proposed [1] to replace (1) by the “quasilocal” relation

$$
\frac{R(s)}{s^\alpha} = \tan \left[ \frac{\pi}{2} \left( \alpha - 1 + \frac{d}{d \ln s} \right) \right] \frac{I(s)}{s^\alpha} \quad (2)
$$

with $\alpha$ real and $s$ the c.m. scattering energy squared. As $s$ is linear in $E$, the laboratory scattering energy, $s$ and $s'$ in (1) and (2) can be replaced by $E$ and $E'$ respectively, with inessential changes in the form of these equations.

Choosing $\alpha = 1$ for simplicity, we obtain from (2)

$$
g(x) = T(x), \quad (3)
$$

where we use the notation $x = \ln s$, $g(x) = R(s)/s$, and $f(x) = I(s)/s$. The “tangent series” $T(x)$ is defined by the relation

$$
T(x) = \tan \left( \frac{\pi}{2} \frac{d}{dx} \right) f(x) = \sum_{n=1}^{\infty} a_n f^{(2n-1)}(x) \quad (4)
$$

with

$$
a_n = \frac{2\pi^{2n-1}(2^{2n} - 1)}{(2n)!} |B_{2n}|,
$$

where $B_{2n}$ are the Bernoulli numbers.

The method of the derivative dispersion relations (called also derivative analyticity relations) is again becoming topical. It is therefore worth emphasizing their interesting merits as well as categorical caveats, in particular

- the rather restrictive conditions of their validity
- the problem of how to give (4) precise mathematical meaning, and
- problems of their practical applicability.

After a short historical survey, we shall discuss these subjects. In discussing the relations (2) to (4), we have carefully to distinguish two very different formulations of the problem: (i) either we keep the energy fixed and push the approximation order (i.e. the number of terms in (4)) to infinity or (ii) the
order is kept fixed and the energy tends to infinity. Needless to say, the latter does not require so many restrictive assumptions.

2 History

Applications of the “derivative dispersion relation” are very wide. Already in 1968, Gribov and Migdal [2] made use of this relation in the context of Regge theory. Later Bronzan, Kane and Sukhatme [1] introduced the method into the phenomenology of high-energy small-angle hadron-hadron scattering. In 1975 Kang and Nicolescu [3] proposed a model based on the derivative relation to analyze the rising total cross sections for hadron-proton scattering.

Soon after [1] was published it was shown that the relation (2) is restricted to certain mathematical models; it was proved by Eichmann and Dronkers [4] that relation (2) is exactly valid only on some class of entire functions of ln s. Bronzan, Kane and Sukhatme made the crucial step towards applications, approximating (4), the infinite series for $T(x)$, with a finite number of terms at a fixed energy $s = e^x$. As Eichmann and Dronkers [4] showed, however, the mathematical condition for the convergence of the series excludes many cases of practical interest.

In 1976 G. Höhler [5] and A. Bujak and O. Dumbrajs [6] published critical comments on the use of the derivative dispersion relations, showing that the difference between a dispersion relation and its “differential form” (2) may grow in an uncontrollable way, e.g. due to low-energy contributions. In a series of papers P. Kolář, J. Fischer and I. Vrkoč [7–11] gave the derivative dispersion relations precise meaning and found conditions of their validity and practical applicability.

In 1986 J.C. Pumplin, W.W. Repko, G.L. Kane and M.J. Duncan [12] used the method to study the gluonic production of vector bosons and boson pairs in the Standard Model.

In 1990 and 1999 M.N. Mnatsakanova and Yu.S. Vernov [13] applied the method to weakly oscillating amplitudes and established validity conditions of the derivative relations. M.J. Menon and co-authors [14] used the method for the case of an arbitrary number of subtractions, and made a systematic comparison of derivative relations with experimental data.
3 Convergence of the tangent series $T(x)$ at a fixed, finite energy

**Theorem 1** [9]: Let $f : \mathbb{R}^1 \to \mathbb{R}^1$. The series $T(x)$, (4), is convergent at a point $x \in \mathbb{R}^1$ if and only if the series

$$D(x) = \sum_{n=0}^{\infty} f^{(2n+1)}(x)$$

(5)

is convergent.

This relatively simple theorem is of fundamental importance for many subsequent results. As for practical applications, we have to parameterize a scattering amplitude in an energy interval. So we need the following

**Theorem 2** [10]: Let $f : I \to \mathbb{R}^1$ have all derivatives at every $x \in I$, $I \subset \mathbb{R}^1$ (i.e., let $f \in C^\infty(I)$). If $T(x)$ converges for every $x \in I \subset \mathbb{R}^1$, then an entire function of complex $x$ exists which assumes the values of $f(x)$ on $I$.

These results show an extraordinarily restricted validity of the “derivative dispersion relations” at finite energy: it follows that $T(x)$ is convergent on an energy interval $I \subset \mathbb{R}^1$ only if $f(x)$ is an entire function of complex $x$ and if two series, (5) for $D(x)$ and $E(x) = \sum_{n=0}^{\infty} f^{(2n)}(x)$, also converge.

4 Link to dispersion relations

If $T(x)$ is convergent, we can derive the corresponding dispersion relation:

$$T(x) = \tan \left( \frac{\pi}{2} \frac{d}{dx} \right) f(x) = \int_{0}^{\infty} a(t) e^{-t} [f(x + t) - f(x - t)] \, dt$$

(6)

with

$$a(t) = \frac{2}{\pi} \left( 1 - e^{-2t} \right)^{-1}$$

(7)

and

$$D(x) = \sum_{n=0}^{\infty} f^{(2n+1)}(x) = \frac{1}{2} \int_{0}^{\infty} e^{-t} [f(x + t) - f(x - t)] \, dt.$$ 

(8)
Setting $x = \ln s$ in (6), we obtain

$$\tan\left(\frac{\pi}{2} \frac{d}{d \ln s}\right) f(\ln s) = \frac{2s}{\pi} \int_{0}^{\infty} \frac{f(\ln s')}{s'^2 - s^2} ds'.$$

(9)

This is to be confronted with the ordinary dispersion relation, which is obtained by putting $f(x) = \text{Im} F(s)/s$.

5 Practical applicability of the derivative dispersion relations

The results discussed in the previous sections indicate that the class of “amplitudes” to which the derivative relations may be applied is very narrow. Problems of their practical applicability were studied in detail in [5,6,9], and we shall briefly discuss some of the results obtained.

Let us consider two fits of the imaginary part, $\text{Im} F^D$ and $\text{Im} F^B$, which are used in the dispersion relation approach and in the derivative approach respectively. Can one impose an upper bound on the modulus of their difference? (Note that $\text{Im} F^D$ belongs to a much wider class of functions than $\text{Im} F^B$.)

It is pointed out in [5,6,9] that $\text{Im} F^D - \text{Im} F^B$ may, in certain situations, grow in an uncontrollable way due to low-energy contributions. Bounds can be obtained only if bounds on low-energy contributions are known; see the cited papers for details. A simple example to illustrate the situation is as follows [6]. Let us add the term $cs^\alpha$ to a parametrization of the imaginary part of the scattering amplitude, with $0 < \alpha \leq 1$ and $|c|$ very small, so as not to change the fit. Then the real part acquires the term $-cs^\alpha \cot(\alpha \pi/2)$, which becomes arbitrarily large for $\alpha$ near zero. This is an argument against the derivative relations, but not only against them, because similar problems can arise also in an ordinary dispersion relation [9].

We emphasize once again that the class of functions for which the derivative dispersion relations are applicable is very narrow; moreover, predictions based on them may be unstable.

A dramatic change takes place when we pass from a finite energy relation to the high-energy limit. A wide spectrum of derivative relations are valid in the high-energy limit for a large class of functions. More than that, the infinite series (4) for $T(x)$ can be safely replaced by its first term!
6 Derivative dispersion relations in the high-energy limit

The validity of such relations has been proved [7,8] for a class of functions which is almost as large as the class of scattering amplitudes described by “first principles”, i.e., functions satisfying analyticity, crossing symmetry, polynomial boundedness, etc. Then it is sufficient to retain the first term for \( T(x) \), \( 3\pi |B_2|'f'(x) \), and take the limit \( s \to \infty \). In this way, a number of high-energy \( (s \to \infty) \) derivative relations are obtained. Some of them correlate the real with the imaginary part of \( F(s) \),

\[
\frac{\text{Re} F(s)}{s} \to \frac{\pi}{2} \frac{d}{d \ln s} \frac{\text{Im} F(s)}{s},
\]

(10)
others the phase with the modulus of the amplitude,

\[
\frac{d}{d \ln s} \ln \left| \frac{F(s)}{s} \right| \to \frac{2}{\pi} \arctan \left( \frac{\text{Re} F(s)}{\text{Im} F(s)} \right).
\]

(11)
The arrow \( \to \) means either that the ratio of the left to the right-hand side tends to unity, or that their difference tends to zero with \( s \to \infty \).

Let us give one example to illustrate the results.

**Theorem 3 [8]:** Let \( f(s) \) satisfy, apart from the properties of analyticity, crossing symmetry, polynomial boundedness, positivity of the imaginary part and the Froissart-Martin bound, the following conditions:

\[
\int_{s_1}^{\infty} \text{Im} F(s) \frac{ds}{s} = \infty
\]

(12)
for some \( s_1 > 0 \) and

\[
\lim_{s \to \infty} \ln \left| \frac{F(s)}{s^{1-a}} \right| = \infty
\]

(13)
for some real \( a \). If the limit \( \lim_{s \to \infty} A(s) \) exists, where

\[
A(s) = \frac{d}{d \ln s} \ln \left| \frac{F(s)}{s^{1-a}} \right| / \left[ a + \frac{2}{\pi} \arctan \left( \frac{\text{Re} F(s)}{\text{Im} F(s)} \right) \right],
\]

(14)
then it is equal to 1.
Taking $a = 0$, we obtain
\[
\frac{d}{d\ln s} \ln \left| \frac{F(s)}{s} \right| \arctan \left( \frac{\text{Re} F(s)}{\text{Im} F(s)} \right) \xrightarrow{s \to \infty} \frac{\pi}{2}.
\] (15)

As was mentioned above, there are a number of analogous asymptotic relations connecting the real with the imaginary part, phase with modulus, both for the crossing-even and the crossing-odd scattering amplitude. Details can be found in [8] and [11].

7 Conclusions

1. The derivative dispersion relations again become topical, even more than ever before, because of the extraordinarily high energy on the LHC.

2. In using derivative dispersion relations, there are essentially two different approaches. One is to keep the energy fixed and to approximate the tangent series $T(x)$ with a finite, possibly increasing, number of terms. The other is to keep fixed the number of terms approximating the series $T(x)$ and let the scattering energy tend to infinity. The class of applicability in the latter approach is considerably wider than that in the former.

3. If the tangent series $T(x)$ converges on an interval $I$, $f(x)$ must be extendable to an entire function of $x = \ln s$. Then the sum of $T(x)$ is equal to the corresponding dispersion integral. Conclusion: This class of functions is too narrow to contain the true amplitude.

4. If $f \in C^\infty$ and the dispersion integral converges, the latter is equal to the generalized sum (6) (modified Borel summation). This class is wider, but the true scattering amplitude is not necessarily included.

5. In the high energy limit, $s \to \infty$, derivative dispersion relations are valid in which the “tangent” operator $T(x)$ is replaced by its first expansion term. The class of applicability includes the majority of physically interesting functions.

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