On series $\sum c_k f(kx)$ and Khinchin’s conjecture

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Abstract

We prove the optimality of a criterion of Koksma (1953) in Khinchin’s conjecture, settling a long standing open problem in analysis. Using this result, we also give a near optimal condition for the a.e. convergence of series $\sum_{k=1}^{\infty} c_k f(kx)$ for $f \in L^2$.

1 Introduction

Let $T = \mathbb{R}/\mathbb{Z} \simeq [0,1)$ denote the circle endowed with Lebesgue measure, $e(x) = \exp(2i\pi x)$, $e_n(x) = e(nx)$, $n \in \mathbb{Z}$. Let

$$f \in L^2(T), \quad \int_T f(x)dx = 0, \quad f(x) \sim \sum_{\ell \in \mathbb{Z}} a_{\ell} e_{\ell}, \quad a_0 = 0. \quad (1)$$

Two closely related classical problems of analysis are the almost everywhere convergence of series

$$\sum_{k=1}^{\infty} c_k f(kx) \quad (2)$$

and the a.e. convergence of averages

$$\frac{1}{N} \sum_{k=1}^{N} f(kx). \quad (3)$$

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Khinchin [16] conjectured (assuming only $f \in L^1(T)$ in (1)) that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = 0 \quad \text{a.e.} \quad (4)$$

This conjecture remained open for nearly 50 years. Koksma [17] proved that (4) holds if the Fourier coefficients of $f$ satisfy

$$\sum_{k=1}^{\infty} |a_k|^2 (\log \log k)^3 < \infty,
(5)$$

and in [18] he weakened the condition to

$$\sum_{k=1}^{\infty} |a_k|^2 \sigma_{-1} (k) < \infty,
(6)$$

where

$$\sigma_{s}(k) = \sum_{d|k} d^s.
(7)$$

The function $\sigma_{-1}(k)$ is multiplicative and by Gronwall’s estimate [15] we have

$$\limsup_{k \to \infty} \frac{\sigma_{-1}(k)}{\log \log k} = e^\lambda,$$

where $\lambda$ is Euler’s constant. Thus condition (6) is satisfied if

$$\sum_{k=1}^{\infty} |a_k|^2 \log \log k < \infty.
(8)$$

Note the difference between (6) and (8): by a theorem of Wintner ([23], p. 180) the averages $\frac{1}{J} \sum_{j=1}^{J} \sigma_{-1}(j)$ remain bounded, which easily implies that for any function $\omega(k) \to \infty$ we have $\sigma_{-1}(k) \ll \omega(k)$ on a set of $k$’s with asymptotic density 1. Thus (6) is only slightly stronger than $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, showing that relation (4) holds for a very large class of functions $f \in L^2(T)$. However, Marstrand [19] showed that there exist functions $f \in L^2(T)$ (and even bounded functions $f$) satisfying (1) such that (4) fails, thereby disproving Khinchin’s conjecture. In his seminal paper [9], Bourgain used his entropy method to construct a new counterexample and mentioned, concerning conditions (5), (8), that “... A more detailed analysis of the previous construction shows that Koksma’s double logarithmic condition is essentially best possible.” See the remark on p. 89 of [9]. The purpose of the present paper is to verify Bourgain’s claim in a slightly modified form; we will namely prove the following

**Theorem 1.** Let $w(n)$ be a nonnegative function of a natural argument, which is sub-multiplicative and bounded in mean. Assume that

$$w(n) = o(\log \log n).$$
Then there exists a function $f$ satisfying (1) with
\[ \sum_{k=1}^{\infty} |a_k|^2 w(k) < \infty \]
such that (4) is not valid.

A function $w$ is called sub-multiplicative if
\[ w(nm) \leq w(n)w(m) \]
for all $m, n$ with $(m, n) = 1$. Clearly, for any $0 < \varepsilon < 1$ the function $w(n) = \sigma_{-1}(n)^{1-\varepsilon}$ satisfies the assumptions of Theorem 1 and thus it follows that (1) and
\[ \sum_{k=1}^{\infty} |a_k|^2 \sigma_{-1}(k)^{1-\varepsilon} < \infty \]
do not generally imply (4). In other words, Koksma’s condition (6) is optimal for the a.e. convergence relation (4). Whether Theorem 1 remains valid without the assumption of sub-multiplicativity and bounded means of $w$ remains open.

Theorem 1 shows that $\sigma_{-1}(k)$ is an optimal Weyl factor in the Fourier series of $f$ for the validity of (1), but it does not mean that in the absence of this Weyl factor relation (4) is always false. In fact, we will see that under mild regularity conditions on the Fourier coefficients of $f$, relation (4) holds under assuming only $f \in L^2$, i.e. $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Also, the norming factor in (4) can be substantially diminished, see Corollaries 1-3.

The previous results give a fairly satisfactory picture on the validity of the convergence relation (4). Results concerning the convergence of sums (2) are much less complete. Note that, by the Kronecker lemma, the a.e. convergence of (2) with $c_k = 1/k$ implies (4), so the two problems are closely connected. By Carleson’s theorem [11], in the case $f(x) = \sin 2\pi x$, $f(x) = \cos 2\pi x$ the series (2) converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 \sigma_1 < \infty$. Gaposhkin [14] showed that this remains valid if the Fourier series of $f$ converges absolutely; in particular this holds if $f$ belongs to the Lip ($\alpha$) class for some $\alpha > 1/2$. However, Nikishin [20] showed that the analogue of Carleson’s theorem fails for $f(x) = \text{sgn} \sin 2\pi x$ and fails also for some continuous $f$. Recently, Aistleitner and Seip [2] and Berkes [5] showed that for $f \in BV$ the series (2) converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 (\log \log k)^{\gamma} < \infty$ for $\gamma > 4$, but generally not for $\gamma < 2$. A similar, slightly weaker result holds for the Lip ($1/2$) class, showing, in particular, that Gaposhkin’s result above is sharp. For the class Lip ($\alpha$), $0 < \alpha < 1/2$ and other classical function classes like $C(0, 1)$, $L^p(0, 1)$ or function classes defined by the order of magnitude of their Fourier coefficients, the results are much less complete: there are several sufficient criteria (see e.g. [11], [6], [7], [10], [12], [22]; see also [13] for the history of the subject until 1966) and necessary criteria (see [4]), but there are large gaps between the sufficient and necessary conditions. For $f \in \text{Lip} (\alpha)$, $0 < \alpha < 1/2$ Weber [22] proved that the series (2) converges a.e. provided
\[ \sum_{k=1}^{\infty} c_k^2 d(k) (\log k)^2 < \infty, \]
where \( d(n) = \sum_{d|n} 1 \) is the divisor function. For functions \( f \) with Fourier coefficients of order \( O(k^{-s}) \), \( 1/2 < s \leq 1 \) Berkes and Weber [7] obtained the convergence criterion

\[
\sum_{k=1}^{\infty} c_k^2 \sigma_{1-2s}(k)(\log k)^2 < \infty. \tag{11}
\]

In view of the role played by the function \( \sigma_{-1}(k) \) in Khinchin’s conjecture, the appearance of the arithmetic functions \( d(k), \sigma_{1-2s}(k) \) in (10), (11) is not surprising. However, unlike relation (4), there are no comparable necessary conditions for the convergence of (2) for these function classes: for example, in the case \( \text{Lip} (\alpha) \), \( 0 < \alpha < 1/2 \) we know only that

\[
\sum_{k=1}^{\infty} c_k^2 (\log k)^\gamma < \infty, \quad 0 < \gamma < 1 - 2\alpha \tag{12}
\]

is in general not sufficient for the a.e. convergence of (2) (see Berkes [4]). Note the large gap between (10) and (12): while the average order of magnitude of \( d(n) \) is \( \log n \), we have \( d(n) \geq \exp(c \log n/\log \log n) \) for infinitely many \( n \). The purpose of this paper is to give a sufficient condition for the a.e. convergence of (2) which is optimal up to a logarithmic factor. Specifically, we will prove the following result.

**Theorem 2.** Let \( f \) satisfy (7) and put

\[
g(r) = \sum_{k=1}^{\infty} |a_{rk}|^2, \quad G(r) = \sum_{j \leq 2r} g(j), \quad h(n) = \sum_{d|n} (dg(d) + G(d)). \tag{13}
\]

Then \( \sum_{k=1}^{\infty} c_k f(kx) \) converges a.e. provided

\[
\sum_{k=1}^{\infty} c_k^2 h(k)(\log k)^2 < \infty. \tag{14}
\]

On the other hand, for any \( \delta > 0 \) there exists an \( f \in L^2 \) and coefficients \( c_k \) such that

\[
\sum_{k=1}^{\infty} c_k^2 h(k)(\log k)^{-\delta} < \infty \tag{15}
\]

but \( \sum_{k=1}^{\infty} c_k f(kx) \) does not converge a.e.

Theorem 2 is the series analogue of Koksma’s theorem, providing a nearly optimal Weyl factor for the a.e. convergence of \( \sum_{k=1}^{\infty} c_k f(kx) \) for general \( f \in L^2(T) \). Theorem 2 shows that the convergence of \( \sum_{k=1}^{\infty} c_k f(kx) \) is intimately connected with the behavior of the function \( g(d) = \sum_{k=1}^{\infty} |a_{dk}|^2 \) which, in turn, depends on the distribution of the numbers \( |a_k| \) on \( [0, \infty) \). Under mild regularity conditions on the \( |a_k| \), \( g(d) \) will be small for large \( d \), leading to a small Weyl factor \( h \) and better convergence properties. See Corollaries 1-3 below, where the Weyl factor \( h(n) \) reduces
to classical arithmetic functions like \(d(n), \sigma_s(n)\). For general \(f \in L^2(\mathbb{T})\), \(g(d)\) can behave rather irregularly which, combined with the summation \(d|n\) in the definition of \(h(n)\), leads to very irregularly changing functions \(h(n)\). Clearly, \(h(n)\) will be small for numbers \(n\) with few prime factors, showing, e.g., that the a.e. convergence properties of dilated sums \(\sum_{p \leq d} c_p f(px)\) extended for primes are considerably better that those of general series \((2)\)

By the Kronecker lemma, Theorem 2 implies for any \(f \in L^2(\mathbb{T})\) that
\[
\left| \sum_{k=1}^{N} f(kx) \right| \ll \sqrt{N} (\log N)^{3/2+\varepsilon} \hat{h}(N)^{1/2} \quad \text{a.e.} \quad (16)
\]

where \(\hat{h}(n) = \max_{1 \leq k \leq n} h(k)\) is the smallest monotone majorant of the function \(h\). Using the Kronecker lemma leads to some loss of accuracy, but the corollaries below show that under mild monotonicity or regularity conditions on the \(|a_k|\), the right hand side of \((16)\) will be \(O(N^{1/2+\varepsilon})\), and thus the Khinchin conjecture is valid under such conditions.

**Corollary 1.** Let \(f\) satisfy \((1)\) where \((a_k)\) satisfies one of the following conditions:

(a) \(|a_k|\) is regularly varying as \(k \to \infty\)
(b) \(k^{-\gamma}|a_k|\) is non-increasing for some \(\gamma > 0\)
(c) There exists a \(C > 0\) such that for any integer \(d \geq 1\) we have \(\sum_{k=1}^{\infty} |a_{dk}|^2 \leq C/d\).

Then the series \(\sum_{k=1}^{\infty} c_k f(kx)\) converges a.e. provided \(\sum_{k=1}^{\infty} c_k^2 d(k)(\log k)^2 < \infty\) and consequently we have
\[
\left| \sum_{k=1}^{N} f(kx) \right| \ll \sqrt{N} (\log N)^{3/2+\varepsilon} \hat{d}(N)^{1/2} \quad \text{a.e.}
\]

for any \(\varepsilon > 0\), where \(\hat{d}(N) = \max_{1 \leq k \leq N} d(k)\).

Since \(d(k) \ll k^\varepsilon\) for any \(\varepsilon > 0\), we get

**Corollary 2.** Let \(f\) satisfy \((1)\). Under any of the regularity conditions (a), (b), (c) we have
\[
\left| \sum_{k=1}^{N} f(kx) \right| \ll N^{1/2+\varepsilon} \quad \text{a.e.}
\]

for any \(\varepsilon > 0\).

It is easily seen that in Corollary 1 both (a) and (b) imply (c), so (c) is the weakest condition of the three. It is also quite natural: it requires that all subsums \(\sum_{k=1}^{\infty} |a_{dk}|^2\) carry at most their “fair share” in the sum \(\sum_{k=1}^{\infty} |a_k|^2\). However, in a number of important cases the estimate \(\sum_{k=1}^{\infty} |a_{dk}|^2 \ll d^{-1}\) can be improved, leading to a convergence theorem for \(\sum_{k=1}^{\infty} c_k f(kx)\) with a smaller Weyl factor.

**Corollary 3.** Let \(f\) satisfy \((1)\), where
\[
|a_k| \ll k^{-1/2} \varphi(k) \quad k = 1, 2, \ldots
\]
with a non-increasing function \( \varphi \) satisfying \( \sum_{k=1}^{\infty} k^{-1} \varphi^2(k) < \infty \). Let

\[
\psi(r) = \sum_{k \geq r} k^{-1} \varphi^2(k), \quad h(N) = \sum_{d|N} \psi(d)
\]

and assume that \( \psi \) is regularly varying with exponent \( > -1 \). Then the series \( \sum_{k=1}^{\infty} c_k f(kx) \) converges a.e. provided \( \sum_{k=1}^{\infty} c_k^2 h(k)(\log k)^2 < \infty \) and consequently (16) holds.

For example, for \( g(k) = k^{-\gamma}, 0 < \gamma \leq 1/2 \) we get \( h(k) \ll \sigma(k)^{-2\gamma} \), leading to the convergence condition (11) in Berkes and Weber [7]. For \( g(k) = (\log k)^{-\gamma}, \gamma > 1/2 \) we get

\[
h(k) \ll \sum_{d|k} (\log d)^{-(2\gamma-1)}.
\]

2 Proof of Theorem 1

Let \( w = \{w_n, n \in \mathbb{Z}\} \) be a sequence of positive reals. Let \( L^2_w \) be the associated Sobolev space on the circle, namely the subspace of \( L^2 \) consisting with functions \( f \) such that

\[
\|f\|_w^2 := \sum_{n \in \mathbb{Z}} w_n a_n^2(f) < \infty.
\]

This is a Hilbert space with scalar product defined by \( \langle f, g \rangle = \sum_{n \in \mathbb{Z}} w_n a_n(f) a_n(g), f, g \in L^2_w \).

The proof of Theorem 1 is based on an adaptation to the Sobolev space \( L^2_w \) of the method elaborated by Bourgain in [9]. Let \( f(x) = \sum_{\ell \in \mathbb{Z}} a_\ell e^{i \ell x}, a_0 = 0 \) and consider the dilation operators \( T_j f(x) = f(jx) \). These are positive isometries on \( L^p, p \geq 1 \), such that \( T_j 1 \equiv 1 \) for all \( j \), and for all \( f \in L^2 \)

\[
\frac{1}{J} \sum_{j \leq J} T_j f \overset{L^2}{\to} f, \quad J \to \infty,
\]

To \( f \in L^2 \) we associate

\[
F_{J,f} = \frac{1}{\sqrt{J}} \sum_{1 \leq j \leq J} g_j T_j f, \quad (J \geq 1)
\]

where \( g_1, g_2, \ldots \) are i.i.d. standard Gaussian random variables.

**Proposition.** Let \( S_n : L^2 \to L^2, n = 1, 2, \ldots \) be continuous operators commuting with \( T_j \) on \( L^2 \), \( S_n T_j = T_j S_n \) for all \( n \) and \( j \). Assume that the following property is fulfilled:

\[
\left\{ \sup_{n \geq 1} |S_n(f)| < \infty \right\} = 1, \quad \forall f \in L^2_w,
\]

then there exists a constant \( C \) depending on \( \{S_n, n \geq 1\} \) only, such that

\[
\sup_{\varepsilon > 0} \varepsilon \sqrt{\log N_f(\varepsilon)} \leq C \limsup_{J \to \infty} \left( \mathbb{E} \|F_{J,f}\|_w \right)^{1/2}, \quad \forall f \in L^2_w,
\]
where \( N_f(\varepsilon) \) is the entropy number associated with the set \( C_f = \{ S_n f, n \geq 1 \} \), namely the minimal number of \( L^2 \) open balls of radius \( \varepsilon \), centered in \( C_f \) and enough to cover \( C_f \).

**Proof.** By the Banach principle, there exists a non-increasing function \( C : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\forall \varepsilon > 0, \forall g \in L^2_w(T), \quad \left\{ \sup_{n \geq 1} |S_n(g)| \geq \| g \|_w C(\varepsilon) \right\} \leq \varepsilon.
\]

Let \( 0 < \varepsilon < 1/4 \). Let \( f \in L^2(T) \). Taking \( g = F_{J,f} \) and using Fubini’s theorem, gives

\[
\int_T \mathbf{P} \left\{ \sup_{n \geq 1} |S_n(F_{J,f})| \geq C(\varepsilon)\|F_{J,f}\|_w \right\} \, d\varepsilon \leq \varepsilon.
\]

It follows that

\[
\left\{ x \in T : \mathbf{P} \left\{ \omega : \sup_{n \geq 1} |S_n(F_{J,f}(\omega,.))(x)| \geq C(\varepsilon)\|F_{J,f}(\omega,.))\|_w \right\} \geq \sqrt{\varepsilon} \right\} \leq \sqrt{\varepsilon},
\]

which is better rewritten under the following form

\[
\left\{ x \in T : \mathbf{P} \left\{ \omega : \sup_{n \geq 1} |S_n(F_{J,f}(\omega,.))(x)| \leq C(\varepsilon)\|F_{J,f}(\omega,.))\|_w \right\} \geq 1 - \sqrt{\varepsilon} \right\} \geq 1 - \sqrt{\varepsilon}.
\]

By Tchebycheff’s inequality, \( \mathbf{P} \left\{ \|F_{J,f}\|^2_w > \mathbf{E} \|F_{J,f}\|^2_w / \varepsilon \right\} \leq \varepsilon. \) We deduce that the set

\[
X_{\varepsilon,J,f} = \left\{ x \in T : \mathbf{P} \left\{ \omega : \sup_{n \geq 1} |S_n(F_{J,f}(\omega,.))(x)| \leq C(\varepsilon)(\mathbf{E} \|F_{J,f}\|^2_w / \varepsilon)^{1/2} \right\} \geq 1 - 2\sqrt{\varepsilon} \right\}
\]

has measure greater than \( 1 - \sqrt{\varepsilon} \). The classical estimate of Gaussian semi-norms implies

\[
\forall x \in X_{\varepsilon,J,f}, \quad \mathbf{E} \sup_{n \geq 1} |S_n(F_{J,f}(\omega,.))(x)| \leq \frac{4}{(1 - 2\sqrt{\varepsilon})} \frac{C(\varepsilon)}{\sqrt{\varepsilon}} \left( \mathbf{E} \|F_{J,f}\|^2_w \right)^{1/2}.
\]

Now let \( I \) be a finite set of integers such that \( \|S_n(f) - S_m(f)\|_2 \neq 0, \) for all distinct elements \( m,n \in I \). By the commutation property, \( S_n(F_{J,f}) = F_{J,S_n,f}; \) so that

\[
\mathbf{E} \left| S_n(F_{J,f}) - S_m(F_{J,f}) \right|^2 = \mathbf{E} \left| F_{J,S_n,f} - S_m(f) \right|^2 = \frac{1}{J} \sum_{j \leq J} (T_j(S_n f - S_m f))^2
\]

\[
= \frac{1}{J} \sum_{j \leq J} T_j(S_n f - S_m f)^2 \to \|S_n f - S_m f\|^2_2,
\]

in \( L^2 \) as \( J \) tends to infinity. We have used the fact that \( (T_j f)^2 = T_j f^2, \) if \( f \in L^2 \).

By proceeding by extraction, we can find a partial index \( J \) such that the set

\[
A(I) = \left\{ \forall J \in \mathcal{J}, \forall n, m \in I, m \neq n, \quad \frac{\left( \mathbf{E} \left| S_n(F_{J,f}) - S_m(F_{J,f}) \right|^2 \right)^{1/2}}{\|S_n(f) - S_m(f)\|_2} \geq \sqrt{1 - \varepsilon} \right\},
\]

\[7\]
has measure greater that \( 1 - \sqrt{\varepsilon} \).

Let \( J \in \mathcal{J} \), then \( (A(I) \cap X_{\varepsilon,J,f}) \geq 1 - 2\sqrt{\varepsilon} > 0 \), and for any \( x \in A(I) \cap X_{\varepsilon,J,f} \)

\[
C(\varepsilon) \left( \mathbf{E} \|F_{J,f}\|_w^2 \right)^{1/2} \geq \mathbf{E} \sup_{n \geq 1} |S_n(F_{J,f})(x)| \geq \mathbf{E} \sup_{n \in I} S_n(F_{J,f})(x) \geq \sqrt{1 - \varepsilon} \mathbf{E} \sup_{n \in I} Z(S_n(f)) \geq (1 - 2\sqrt{\varepsilon}) \mathbf{E} \sup_{n \in I} Z(S_n(f)).
\]

Therefore,

\[
\mathbf{E} \sup_{n \in I} Z(S_n(f)) \leq C \left( \limsup_{J \to \infty} \mathbf{E} \|F_{J,f}\|_w^2 \right)^{1/2}.
\]

Sudakov’s minoration implies

\[
\sup_{\rho > 0} \rho \sqrt{\log N_f(\rho)} \leq C \left( \limsup_{J \to \infty} \mathbf{E} \|F_{J,f}\|_w^2 \right)^{1/2}.
\]

**Proof of Theorem 1.** Let \( P_1, P_2, \ldots \) denote the sequence of prime numbers. Fix some positive integer \( s \) and let \( d \) be some other integer such that \( 2^d \leq P_s \). There exists an integer \( T \) such that if

\[
A_T = \{ n = P_1^{\alpha_1} \cdots P_s^{\alpha_s} : 2^T \leq n < 2^{T+1}, \, \alpha_i \geq 0, \, i = 1, \ldots, s \},
\]

then \( \sharp(A_{T+d}) \leq 2\sharp(A_T) \). Put

\[
f = f_T = \frac{1}{\sharp(A_T)^{1/2}} \sum_{n \in A_T} e_n.
\]

It follows from Bourgain’s proof [9] p. 88-89, (or [21] p. 239-240 for details) that

\[
N\left( (S_{4i}(f), i \leq \left[ \frac{d}{2} \right]), \frac{1}{8} \right) \geq T.
\]

So that

\[
\sqrt{\log T} \leq C \left( \limsup_{J \to \infty} \mathbf{E} \|F_{J,f}\|_w^2 \right)^{1/2}. \tag{17}
\]

Now as

\[
F_{J,f} = \frac{1}{J^{1/2}} \sum_{j \leq J} g_j \frac{1}{\sharp(A_T)^{1/2}} \sum_{n \in A_T} e_{nj} = \frac{1}{(J \sharp(A_T))^{1/2}} \sum_{\nu \geq 1} \sum_{\nu \leq J \cap \nu} g_{j
u} \mathbf{E} \left( \sum_{\nu \leq J \cap \nu} g_{j\nu} \right)
\]

we have

\[
\|F_{J,f}\|_w^2 = \frac{1}{J \sharp(A_T)} \sum_{\nu \geq 1} w_{\nu} \left( \sum_{\nu \leq J \cap \nu} g_{j \nu} \right)^2.
\]
These sums are finite sums. Further,
\[
E \|F_{J,f}\|_w^2 = \frac{1}{J\#(A_T)} \sum_{\nu=1}^{\infty} w_{\nu} \left( \sum_{1 \leq j \leq J} 1 \right) = \frac{1}{J\#(A_T)} \sum_{j \leq J} \sum_{m \in A_T} w_{mj}
\]
\[
\leq \left( \frac{1}{J} \sum_{j \leq J} w_j \right) \left( \frac{1}{\#(A_T)} \sum_{m \in A_T} w_m \right)
\]
\[
\leq \left( \frac{1}{J} \sum_{j \leq J} w_j \right) \max_{m \in A_T} w_m
\]

Therefore,
\[
\limsup_{J \to \infty} E \|F_{J,f}\|_w^2 \leq \left( \limsup_{J \to \infty} \frac{1}{J} \sum_{j \leq J} w_j \right) \max_{m \in A_T} w_m \leq M \max_{m \in A_T} w_m,
\]
where \( M < \infty \) and further
\[
\max_{m \in A_T} w_m = o \left( \max_{m \in A_T} \log \log m \right) = o(\log T),
\]
by assumption. Consequently
\[
\limsup_{J \to \infty} \left( E \|F_{J,f}\|_w \right)^{1/2} = o(\sqrt{\log T}). \tag{18}
\]
But this contradicts (17), completing the proof of Theorem 1.

3 Proof of Theorem 2.

Clearly it suffices to prove Theorem 2 for real valued \( f \), when for the Fourier coefficients we have \( a_{-\ell} = \overline{a_\ell} \), \( \ell \in \mathbb{Z} \). We first prove the following lemma.

Lemma. Let \( f \) satisfy (1). Then for any \( r \geq 1 \) and any real coefficients \( c_j \) we have
\[
\int_0^1 \left( \sum_{\ell=2^r+1}^{2^{r+1}} c_\ell f(\ell x) \right)^2 dx \leq \sum_{\ell=2^r+1}^{2^{r+1}} c_\ell^2 h(\ell), \tag{19}
\]
where the arithmetic function \( h \) is defined by (13).

Proof. Fix \( m, n \geq 1 \) and put \( m' = m/d, n' = n/d \), where \( d = (m, n) \). Using (1) and \( a_{-\ell} = \overline{a_\ell} \), we get
\[
\lambda_{m,n} := \left| \int_0^1 f(mx) f(nx) dx \right| = \left| \sum_{mk=nl, k,l \geq 1} a_k \overline{a_l} \right| \leq \sum_{mk=nl, k,l \geq 1} |a_k||a_l|
\]
\[
= 2 \sum_{mk=nl, k,l \geq 1} |a_k||a_l| = 2 \sum_{m'k=n'l, k,l \geq 1} |a_k||a_l|. \tag{20}
\]
Since \((m', n') = 1\), the equation \(m'k = n'l\) implies that \(m'\) is a divisor of \(l\), i.e. \(l = m'i\) and consequently \(k = n'i\) for some \(i \geq 1\). Thus the last expression in (20) equals

\[
2 \sum_{i=1}^{\infty} |a_{m'i}| |a_{n'i}| \leq \sum_{i=1}^{\infty} (|a_{m'i}|^2 + |a_{n'i}|^2) = g(m') + g(n').
\] (21)

Now for any \(r \geq 1\) and any coefficients \(c_\ell\),

\[
\int_0^1 \left( \sum_{\ell=2^r+1}^{2^r+1} c_\ell f(\ell x) \right)^2 dx \leq \sum_{i,j=2^r+1}^{2^r+1} \lambda_{i,j} |c_i||c_j| \leq \frac{1}{2} \sum_{i,j=2^r+1}^{2^r+1} \lambda_{i,j}(c_i^2 + c_j^2) = \sum_{i,j=2^r+1}^{2^r+1} \lambda_{i,j} c_i^2 = \sum_{i=2^r+1}^{2^r+1} c_i^2 \rho(i)
\] (22)

where

\[
\rho(i) = \sum_{j=2^k+1}^{2^{k+1}} \lambda_{i,j} \quad \text{for } 2^k < i \leq 2^{k+1}.
\] (23)

Thus using (20), (21) we get for \(2^k < i \leq 2^{k+1}\),

\[
\rho(i) = \sum_{j=2^k+1}^{2^{k+1}} \lambda_{i,j} \leq \sum_{j=2^k+1}^{2^{k+1}} \left( g(i/(i,j)) + g(j/(i,j)) \right).
\]

Fix \(i\) and \(d|i\) and sum here for all \(j\) with \((i, j) = i/d\). Then \(j = ri/d\) for some \(r \leq 2d\) and thus the contribution of these terms is

\[
\ll \sum_{r \leq 2d} (g(d) + g(r)) \ll dg(d) + G(d).
\]

Thus summing now for \(d|i\), we get

\[
\rho(i) \ll \sum_{d|i} (dg(d) + G(d)) = h(i).
\]

The lemma now follows from (22).

**Proof of Theorem 2.** Using the Lemma, the proof of the sufficiency part can be completed by using the method of Rademacher and Mensov, see e.g. [3], pp. 80–81. By the Lemma and (14) we have

\[
\sum_{r=1}^{\infty} \int_0^1 r^2 \left( \sum_{j=2^r+1}^{2^r+1} c_j f(jx) \right)^2 dx \ll \sum_{r=1}^{\infty} r^2 \sum_{j=2^r+1}^{2^r+1} c_j^2 h(j)
\]

\[
\ll \sum_{r=1}^{\infty} \sum_{j=2^r+1}^{2^r+1} c_j^2 (\log j)^2 h(j) < \infty.
\]
Thus
\[ \sum_{r=1}^{\infty} r^2 \left[ \sum_{j=2^r+1}^{2^{r+1}} c_j f(j) \right]^2 \leq \infty \quad \text{a.e.} \]

and the Cauchy-Schwarz inequality yields for any \( 1 \leq M < N \)
\[
\left| \sum_{j=2^M+1}^{2N} c_j f(j) \right|^2 \leq \left( \sum_{k=M}^{N-1} k^2 \sum_{j=2^k+1}^{2^{k+1}} c_j f(j) \right)^2 \leq \sum_{k=M}^{N-1} k^2 \sum_{j=2^k+1}^{2^{k+1}} c_j f(j) \to 0
\]
as \( M \to \infty \). This implies that \( \sum_{j=1}^{2^m} c_j f(j) \) converges a.e. as \( m \to \infty \). Now the Lemma and standard maximal inequalities (see e.g. [21], Lemma 8.3.4) imply that
\[
\sum_{k=1}^{\infty} \max_{2^k+1 \leq i \leq j \leq 2^{k+1}} \left| \sum_{\ell=i}^{j} c_\ell f(\ell x) \right|^2 \ll \sum_{k=1}^{\infty} k^2 \left( \sum_{\ell=2^k+1}^{2^{k+1}} c_\ell^2 h(\ell) \right) \ll \sum_{\ell=1}^{\infty} c_\ell^2 (\log \ell)^2 h(\ell) < \infty
\]
which yields
\[
\max_{2^k+1 \leq i \leq j \leq 2^{k+1}} \left| \sum_{\ell=i}^{j} c_\ell f(\ell x) \right| \to 0 \quad \text{a.e.} \tag{24}
\]
proving the first part of Theorem 2.

To prove the second statement of Theorem 2, note that for \( c_k = 1/k \) and for any positive non-increasing, slowly varying sequence \((\varepsilon_k)\) we have \( \sum_{k=1}^{\infty} c_k^2 h(k) \varepsilon_k = \sum_1 + \sum_2 \), where
\[
\sum_1 = \sum_{k=1}^{\infty} k^{-2} \varepsilon_k \sum_{d|k} d g(d) = \sum_{d=1}^{\infty} d g(d) \sum_{j=1}^{\infty} (dj)^{-2} \varepsilon_{dj} \ll \sum_{d=1}^{\infty} d g(d) d^{-2} \varepsilon_d = \sum_{d=1}^{\infty} \varepsilon_d \sum_{k=1}^{\infty} |a_{dk}|^2 = \sum_{j=1}^{\infty} |a_j|^2 \sum_{d|j} \varepsilon_d \sum_{d=1}^{\infty} d \varepsilon_d = \sum_{j=1}^{\infty} |a_j|^2 \tilde{\sigma}(j), \tag{25}
\]
with
\[
\tilde{\sigma}(k) = \sum_{d|k} \varepsilon_d / d. \tag{26}
\]
Similarly,
\[
\sum_2 = \sum_{k=1}^{\infty} k^{-2} \varepsilon_k \sum_{d|k} G(d) = \sum_{d=1}^{\infty} G(d) \sum_{j=1}^{\infty} (dj)^{-2} \varepsilon_{dj} \ll \sum_{d=1}^{\infty} G(d) d^{-2} \varepsilon_d \ll \sum_{d=1}^{\infty} g(d) d^{-1} \varepsilon_d, \tag{27}
\]
which is the same bound as the last expression in the first line of (25) and thus continuing, we get the same estimate as in (25). To justify the last step in (27), set \( G(0) = 0 \), \( S_d = \sum_{j=d}^{\infty} \varepsilon_j j^{-2} \) and note that

\[
\sum_{d=1}^{\infty} G(d) d^{-2} \varepsilon_d = \sum_{d=1}^{\infty} G(d) (S_d - S_{d+1}) = \sum_{\ell=1}^{\infty} (G(\ell) - G(\ell - 1)) S_{\ell} = \sum_{\ell=1}^{\infty} (g(2\ell) + g(2\ell - 1)) S_{\ell} \ll \sum_{\ell=1}^{\infty} (g(2\ell) + g(2\ell - 1)) \frac{\varepsilon_{\ell}}{\ell}
\]

\[
\ll \sum_{\ell=1}^{\infty} g(2\ell) \frac{\varepsilon_{2\ell}}{2\ell} + \sum_{\ell=1}^{\infty} g(2\ell - 1) \frac{\varepsilon_{2\ell - 1}}{2\ell - 1} = \sum_{r=1}^{\infty} g(r) \frac{\varepsilon_r}{r}.
\]

Thus we proved

\[
\sum_{k=1}^{\infty} c_k^2 h(k) \varepsilon_k \ll \sum_{j=1}^{\infty} a_j^2 \tilde{\sigma}(j).
\]

Now choosing \( \varepsilon_k = (\log k)^{\delta} \) we have

\[
\tilde{\sigma}(k) = \sum_{d|k} \frac{\varepsilon_d}{d} = \sum_{d|k, d \leq \exp(\sigma(k)^{\delta})} \frac{\varepsilon_d}{d} + \sum_{d|k, d > \exp(\sigma(k)^{\delta})} \frac{\varepsilon_d}{d} \ll \sum_{d \leq \exp(\sigma(k)^{\delta})} \frac{1}{d} + \varepsilon_{\exp(\sigma_k^\delta)} \sum_{d|k} \frac{1}{d} \ll (\sigma_k^\delta) + (\sigma_k^{-\delta})_{\sigma_k} \ll (\sigma_k^\delta)^{1-\delta^2}.
\]

By Theorem 1 we can choose a function \( f \) satisfying (1) such that \( \sum_{j=1}^{\infty} |a_j|^2 \sigma_k^{-\delta^2} \) converges, but \( N^{-1} \sum_{k=1}^{N} f(kx) \) does not converge a.e. But then by relations (28) and (29) we have \( \sum_{k=1}^{\infty} c_k^2 h(k) \varepsilon_k < \infty \) for \( c_k = 1/k \) and \( \sum_{k=1}^{\infty} c_k f(kx) \) cannot converge a.e., since then by the Kronecker lemma we would have \( N^{-1} \sum_{k=1}^{N} f(kx) \to 0 \) a.e.

**Proof of Corollary 3.** Assuming the regular variation of \( g(n) \), the statement is an easy consequence of Theorem 2. However, specializing the proof of Theorem 2 to this case instead, we get the statement without any additional assumptions.

Extend \( \varphi \) to \([1, +\infty)\) in a monotone non-increasing fashion. Then the contribution of the terms in the first sum in (21) for \( i \geq 2 \) can be estimated for \( m \leq n \) as follows:

\[
2 \sum_{i=2}^{\infty} |a_{m^i}| a_{m^i} \ll (m'^n)^{-1/2} \sum_{i=2}^{\infty} i^{-1} \varphi(m'^i) \varphi(n'^i) \ll (m'^n)^{-1/2} \sum_{i=2}^{\infty} i^{-1} \varphi^2(m'^i)
\]

\[
\ll (m'^n)^{-1/2} \int_{1}^{\infty} x^{-1} \varphi^2(m'x) \, dx = (m'^n)^{-1/2} \int_{m'}^{\infty} y^{-1} \varphi^2(y) \, dy
\]

\[
\ll (m'^n)^{-1/2} \sum_{k \geq m'} k^{-1} \varphi^2(k).
\]
On the other hand, well known properties of regularly varying functions (see e.g. [3], Theorem 1.5.11 (ii) with \( f(x) = \varphi^2(x) \), \( \sigma = -1 \) imply that the ratios
\[
\varphi^2(r)/ \int_r^{\infty} t^{-1} \varphi^2(t) \, dt \quad \text{and} \quad \varphi^2(r)/ \sum_{k=r}^{\infty} k^{-1} \varphi^2(k)
\]
(31)
converge, as \( r \to \infty \), to a finite limit \( c \geq 0 \). Thus for \( m \leq n \) we have
\[
|a_{m'}||a_{n'}| \ll (m'n')^{-1/2} \varphi(m') \varphi(n') \ll (m'n')^{-1/2} \varphi^2(m') \ll (m'n')^{-1/2} \sum_{k=m'}^{\infty} k^{-1} \varphi^2(k).
\]

Hence (30) implies for \( m \leq n \), adding the term for \( i = 1 \),
\[
2 \sum_{i=1}^{\infty} |a_{m'i'}| |a_{n'i'}| \ll (m'n')^{-1/2} \sum_{k \geq m'} k^{-1} \varphi^2(k) = (m'n')^{-1/2} \psi(m')
\]
(32)
and the same estimate holds for \( m \geq n \). Recall now that for any \( r \geq 1 \) and any coefficients \( c_\ell \) we have (22), where \( \rho(i) \) is defined by (23). Since \( \lambda_{m,n} \ll \sum_{i=1}^{\infty} |a_{m'i'}| |a_{n'i'}| \), using (32) we get for \( 2^k < i \leq 2^{k+1} \), using the monotonicity and regular variation of \( \psi \),
\[
\rho(i) = \sum_{j=2^k+1}^{2^{k+1}} \lambda_{i,j} \ll \sum_{j=2^k+1}^{2^{k+1}} (i,j)^{-1/2} \psi((i \wedge j)/(i,j)) \ll 2^{-k} \sum_{j=2^k+1}^{2^{k+1}} (i,j) \psi(j/(i,j)).
\]

Fix \( i \) and \( d|i \) and sum here for all \( j \) with \( (i,j) = i/d \). Then \( j = ri/d \) for some \( r \leq 2d \) and thus the contribution of these terms is
\[
\ll 2^{-k} \sum_{r \leq 2d} (i/d) \psi \left( \frac{ri/d}{i/d} \right) \ll (1/d) \sum_{r \leq 2d} \psi(r) \ll \psi(d),
\]
where we used the regular variation of \( \psi \) with index \( > -1 \) and Theorem 1.5.11 of [3], p. 28 with \( \sigma = 0 \). Thus summing now for \( d|i \), we get
\[
\rho(i) \ll \sum_{d|i} \psi(d) = h(i),
\]
and finally by (22)
\[
\int_0^1 \left( \sum_{\ell=2^k+1}^{2^{k+1}} c_\ell f(\ell x) \right)^2 \, dx \leq \sum_{\ell=2^k+1}^{2^{k+1}} c_\ell^2 h(\ell).
\]
(33)
The proof can now be completed as in Theorem 2.

**Proof of Corollary 1.** Assume condition (c) of the Corollary. Then the first expression in (21) can be bounded as
\[
2 \sum_{i=1}^{\infty} |a_{m'i'}| |a_{n'i'}| \leq 2 \left( \sum_{i=1}^{\infty} |a_{m'i'}|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |a_{n'i'}|^2 \right)^{1/2} \leq C(m'n')^{-1/2}
\]
and thus in this case $\lambda_{m,n}$ can be bounded by the last expression of (32) with $\psi = 1$. Hence the rest of the proof of Corollary 3 applies with $\psi = 1$, showing that (58) holds with $h(n) = \sum_{d|n} 1 = d(n)$. Following the proof of Theorem 2, the statement of Corollary 1 follows in the case (c).

Next we show that in Corollary 1 we have (a) $\implies$ (c) and (b) $\implies$ (c). Assume first that (b) holds, then

$$|a_{n+1}/a_n| \leq |(n + 1)/n|^j \leq 1 + C/n \quad (n \geq 1)$$

for some constant $C > 0$. Let now $k \geq 1, d \geq 2$ and $0 \leq j \leq d/2$. Then we get, setting $C_1 = e^C$,

$$|a_{kd}/a_{kd-j}| = \prod_{r=kd-j}^{kd-1} |a_{r+1}/a_r| \leq \prod_{r=kd-j}^{kd-1} (1 + C/r) \leq \exp \left( \sum_{r=kd-j}^{kd-1} C/r \right) \leq \exp(Cj/(d/2)) \leq C_1$$

and consequently

$$\sum_{n=1}^{\infty} |a_n|^2 \geq \sum_{k=1}^{\infty} \sum_{j=1}^{[d/2]} |a_{kd-j}|^2 \geq [d/2] C_1^{-2} \sum_{k=1}^{\infty} |a_{kd}|^2,$$

proving the validity of condition (c). If condition (a) holds, then by $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ its exponent of regularity is negative, i.e. there exists a $\rho > 0$ such that $n^\rho |a_n|$ is slowly varying. But then by the remark in [8], p. 23 preceding Theorem 1.5.4, there exists a non-increasing sequence $b_n \sim |a_n|$. Clearly, $\sum_{n=1}^{\infty} b_n^2 < \infty$ and by monotonicity, $(b_n)$ satisfies condition (c). But then $(a_n)$ also satisfies condition (c).

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