A Unified Approach to Singularly Perturbed Quasilinear Schrödinger Equations

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Abstract. In this paper we present a unified approach to investigate existence and concentration of positive solutions for the following class of quasilinear Schrödinger equations,

\[-\varepsilon^2 \Delta u + V(x)u \mp \varepsilon^2 \gamma u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N,\]

where \(N \geq 3, \varepsilon > 0, V(x)\) is a positive external potential, \(h\) is a real function with subcritical or critical growth. The problem is quite sensitive to the sign changing of the quasilinear term as well as to the presence of the parameter \(\gamma > 0\). Nevertheless, by means of perturbation type techniques, we establish the existence of a positive solution \(u_{\varepsilon,\gamma}\) concentrating, as \(\varepsilon \to 0\), around minima points of the potential.

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1. Introduction

In this paper, we consider the following class of quasilinear Schrödinger equations,

\[-\varepsilon^2 \Delta u + V(x)u \mp \varepsilon^2 \gamma u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N,\]  

where \(N \geq 3, \varepsilon, \gamma > 0, V\) is the external Schrödinger potential, \(h\) is a real function. Solutions of (1.1) are related to standing wave solutions for the time-dependent quasilinear Schrödinger equations

\[i\varepsilon \dot{z}_t = -\varepsilon^2 \Delta z + W(x)z + \kappa z \Delta |z|^2 - f(|z|^2)z, \quad x \in \mathbb{R}^N,\]

where \(z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, W : \mathbb{R}^N \to \mathbb{R}\) is a given potential, \(f : \mathbb{R} \to \mathbb{R}\) is a real function, \(\varepsilon > 0, \) and \(\kappa\) is a parameter. An equation of type (1.2) appears in

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the literature in the context of plasma physics and the continuum limit of discrete molecular structures. In particular, $\kappa$ turns out to be small and negative when (1.2) represents the weak nonlocal limit of some general nonlocal models, whereas $\kappa$ has no prescribed sign in plasma physics, see for instance [6,7,20,26,28] for more details on the Physics involved.

A well-known standard tool in the investigation of the elliptic side of Schrödinger equations, is the ansatz $z(t,x) = \exp(-iEt/\varepsilon)u(x)$ where $E \in \mathbb{R}$ and $u$ is a real function, by which equation (1.2) turns into the following quasilinear elliptic equation,

$$-\varepsilon^2 \Delta u + V(x)u - \kappa u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N,$$

where $V(x) = W(x) - E$ and $h(u) = f(|u|^2)u$.

On the one hand, the study of the existence of ground states (minimal energy solutions) and bound states (finite energy) solutions to (1.3) with $\varepsilon = 1$ has received considerable attention in recent years, see [12,23,24,27,30,34] for arbitrary fixed $\kappa > 0$ and [4,32] for $\kappa < 0$ and $|\kappa|$ small. On the other hand, a lot of work has been done on the existence of semi-classical states for (1.3), namely $\kappa = \varepsilon^2$ and $\varepsilon \to 0^+$. Assume $V : \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous and satisfies the following condition ($V$):

$$0 < V_0 < \inf_{x \in \mathbb{R}^N} V(x)$$

and there is a bounded open set $\mathcal{O}$ such that

$$0 < m := \inf_{x \in \mathcal{O}} V(x) < \min_{x \in \partial \mathcal{O}} V(x).$$

With the assumption ($V$), the existence of a localized solution concentrating around $\mathcal{M} := \{x \in \mathcal{O} : V(x) = m\}$ has been studied under various conditions on the nonlinearity $h$. When $N \geq 3$, we refer to [11,16,36] for the subcritical case and [18,33] for the critical case. The case of dimension $N = 2$ has been addressed in [13,14].

Recently, S. Adachi et al. in [1–3] have studied asymptotic properties of the ground state of the quasilinear Schrödinger equation

$$-\Delta u + \lambda u - \kappa u \Delta u^2 = |u|^{p-1}u, \quad x \in \mathbb{R}^N.$$

As $\kappa \to 0^+$, they have proved that the ground state $u_\kappa$ convergences to a ground state solution of a limit equation for $2 < p < 2^*$ whence it blows up, in a suitable sense, for $p > 2^*$.

Therefore, it is a legitimate and somewhat interesting question what happens when the two parameters $\varepsilon$ and $\kappa$ in (1.3) tend to zero at different speed rates.

The purpose of this paper is to present a unified approach to study the asymptotic properties of the solutions to (1.3) for $\kappa = \pm \varepsilon^{2+\gamma}$ with $\gamma > 0$ and $\varepsilon \to 0^+$. Let us stress the fact that both signs for $\kappa$, which is assumed to be small in absolute value, turn out to be physically relevant as developed in [31]. We will show that beyond the expected concentration phenomenon, the limit equations for the case $\gamma > 0$ and $\gamma = 0$ turn out to be different when $\kappa = \varepsilon^{2+\gamma}$. Moreover, to the best of our knowledge, no result is known in the case $\kappa = -\varepsilon^{2+\gamma}$.
Observe that (1.1) is the Euler-Lagrange equation associated to the energy functional

\[ I_{\varepsilon,\gamma}(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( (\varepsilon^2 + \kappa u^2)|\nabla u|^2 + V(x)u^2 \right) dx - \int_{\mathbb{R}^N} H(u) dx, \quad (1.4) \]

where \( H(t) = \int_0^t h(s) ds \). From the variational point of view, the first difficulty to handle is to find a proper function space setting where (1.4) is well defined. For \( \kappa = \varepsilon^{2+\gamma} \), this difficulty can be overcome by minimization on spheres (mass constraint) or on the Nehari manifold, see [22, 24]. Another important tool for this type of equations is to perform a suitable change of variable [23, 29] which reduces (1.1) to a semilinear elliptic equation for which variational methods are available, see [12]. Unfortunately, the methods which work in the case \( \kappa = \varepsilon^{2+\gamma} \) can not be directly adapted to deal with the case \( \kappa = -\varepsilon^{2+\gamma} \). Indeed, here the principle part of the corresponding energy functional does change sign, see [4, 32].

As we look for positive solutions, let us assume \( h(t) = 0 \) for \( t \leq 0 \) and we require \( h \in C(\mathbb{R}^+, \mathbb{R}) \) to enjoy the following conditions, which were introduced by Byeon and Jeanjean in [10], see also [9]:

\begin{align*}
(h_1) & \lim_{t \to 0} \frac{h(t)}{t} = 0; \\
(h_2) & \lim_{t \to +\infty} \frac{h(t)}{tp} = 0, \quad p \in (1, 2^* - 1); \\
(h_3) & \text{there exists } T > 0 \text{ such that } \frac{1}{2} mT^2 < H(T).
\end{align*}

Our main result is the following:

**Theorem 1.1 (Subcritical case).** Let \( \gamma > 0 \) and assume that (V) and (h_1)-(h_3) hold. Then, for sufficiently small \( \varepsilon > 0 \), there exists a positive solution \( u_{\varepsilon,\gamma} \) of (1.1) satisfying the following:

1. There exists a local maximum point \( x_\varepsilon \) of \( u_{\varepsilon,\gamma} \) such that \( \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0 \) and \( u_{\varepsilon,\gamma}(\varepsilon \cdot -\varepsilon x_\varepsilon) \) converges locally uniformly to a positive ground state of

\[ -\Delta u + mu = h(u). \quad (1.5) \]

2. There exist \( C, c > 0 \) such that

\[ u_{\varepsilon,\gamma}(x) \leq C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right) \]

for all \( x \in \mathbb{R}^N \).

**Critical case.** Assume in place of (h_2) the following critical growth,

\[ (h_4) \quad \lim_{t \to +\infty} \frac{h(t)}{t^{2^*-1}} = 1 . \]
The above results extend to cover the critical case provided we further assume
\[
(h_3) \begin{cases}
\lim_{t \to +\infty} \frac{H(t) - \frac{1}{t^2} v^2}{t^2} = +\infty, & \text{if } N = 3; \\
\lim_{t \to +\infty} \frac{H(t) - \frac{1}{t^2} v^2}{t^2 \ln t} = +\infty, & \text{if } N = 4; \\
\lim_{t \to +\infty} \frac{H(t) - \frac{1}{t^2} v^2}{t^2} = +\infty, & \text{if } N \geq 5.
\end{cases}
\]
In [16], similar results to those of Theorem 1.1 are obtained for the equation (1.3) with \( \gamma = 0 \), namely \( \kappa = \varepsilon^2 \) and \( p \in (1, 2(2^*) - 1) \) in \((h_2)\). The limit equation for \( \kappa = \varepsilon^2 \) being
\[-\Delta u + mu - u\Delta u^2 = h(u).\]
In our case, where \( \kappa = \varepsilon^2 + \gamma \) with \( \gamma > 0 \), the situation is different and we can not cover the range \( p \in (2^* - 1, 2(2^*) - 1) \) as the solution can blow up. Closely related problems were considered in [4], where the authors study the following equation,
\[-\Delta u + V(x)u + \alpha u\Delta u^2 = |u|^{p-1}u, \quad x \in \mathbb{R}^N \tag{1.6}\]
and assuming \( V \) is a trapping potential, they have proved the existence of a positive solution for \( 2 < p < 2^* - 1 \) and \( \alpha > 0 \) small enough. Moreover, in [32] the authors prove that if \( V \in C^1(\mathbb{R}, [0, +\infty)) \) is such that \( x \cdot \nabla V(x) + 2V(x) \geq 0 \) for all \( x \in \mathbb{R}^N \), then the equation (1.6) has only trivial \( C^2 \) solutions for any \( p \geq 2^* - 1 \) and any \( \alpha > 0 \).
Finally, let us point out that difficulties in (1.3) with \( \kappa = -\varepsilon^2 \) are due to the fact that it is still an open problem if there exists a solution to equation (1.6) for all \( \alpha > 0 \).
The outline of the paper is as follows: in Section 2, we modify (1.1) by introducing an auxiliary function. In Section 3, we prove the existence of a positive solution for the modified problem. Then, Sections 4 and 5 are devoted to prove Theorem 1.1 respectively in the subcritical and critical case where we use a truncation approach.

**Notation.** Without loss of generality, we may assume \( 0 \in \mathcal{M} \). For any set \( \Omega \subset \mathbb{R}^N \) and \( \varepsilon, \delta > 0 \), we define \( \Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\} \) and \( \Omega^\delta := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq \delta\} \). Let
\[
E_\varepsilon := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)v^2dx < \infty \right\}
\]
the Hilbert space endowed with the norm \( \|v\|_\varepsilon := \sqrt{\int_{\mathbb{R}^N}(|\nabla v|^2 + V(\varepsilon x)v^2)dx}. \)
We denote by \( C \) a positive constants whose exact value may change from line to line without affecting the overall result.

## 2. An auxiliary problem

Let \( w(x) = u(\varepsilon x) \), equation (1.1) is equivalent to
\[-\Delta w + V_\varepsilon(x)w \pm \varepsilon^2 w\Delta w^2 = h(w), \quad x \in \mathbb{R}^N, \tag{2.1}\]
where \( V_\varepsilon(x) = V(\varepsilon x) \).
By definition of $g$, we define an auxiliary function $g : \mathbb{R} \to \mathbb{R}^+$ as follows:

\[ g_{\varepsilon,\gamma}(t) = \sqrt{1 + \varepsilon^2 t^2} \eta(t), \]

where $\eta(t) \in C^\infty_0(\mathbb{R}, [0, 1])$ is a cut-off function such that $\eta(t) = \eta(-t)$, $\eta(t) = 1$ for $0 \leq t < \sqrt{1/\varepsilon^2}$ and $\eta(t) = 0$ for $t \geq 2\sqrt{1/\varepsilon^2}$. Moreover, there exists some $C > 0$ independent of $\varepsilon$ and $\gamma$ such that $-C\sqrt{\eta(t)} \leq \eta'(t)t \leq 0$, for all $t \in \mathbb{R}$.

**Remark 2.1.** The cut-off function $\eta(t)$ can be constructed as follows. Let

\[ \zeta(t) = \frac{1}{1 + e^{1 - \frac{1}{1-t}}}, \]

for $0 < t < 1$, $\zeta(t) = 1$ for $t \geq 1$ and $\zeta(t) = 0$ for $t \leq 0$. For $t \in \mathbb{R}^+$ let

\[ \rho(t) = \zeta^2 \left[ 2 - \sqrt{\varepsilon^2 t} \right]. \]

Then, the function $\eta(t) = \rho(t)$ for $t \geq 0$ and $\eta(t) = \rho(-t)$ for $t < 0$, enjoys the required properties.

Let

\[ G_{\varepsilon,\gamma}(t) = \int_0^t g_{\varepsilon,\gamma}(s) ds, \]

then $G_{\varepsilon,\gamma}(t)$ is an odd and smooth function as well as the inverse function $G_{\varepsilon,\gamma}^{-1}(t)$. For any fixed $\gamma > 0$ and small $\varepsilon > 0$, next we establish a few properties of $G_{\varepsilon,\gamma}^{-1}(t)$.

**Lemma 2.2.** The following properties hold:

(i) $\lim_{t \to 0} \frac{G_{\varepsilon,\gamma}(t)}{t} = 1$;

(ii) $\lim_{t \to +\infty} \frac{G_{\varepsilon,\gamma}^{-1}(t)}{t} = 1$;

(iii) $\frac{1}{\sqrt{1 + 4\varepsilon^2}} |t| \leq |G_{\varepsilon,\gamma}^{-1}(t)| \leq |t|$ and $|t| \leq |G_{\varepsilon,\gamma}^{-1}(t)| \leq \frac{1}{\sqrt{1 + 4\varepsilon^2}} |t|$ for all $t \in \mathbb{R}$;

(iv) there exists $C > 0$ such that $|G_{\varepsilon,\gamma}^{-1}(t) - t| \leq C\sqrt{\varepsilon^2 |t|}$;

(v) $|G_{\varepsilon,\gamma}^{-1}(t)|^2$ is convex in $t$ for small $\varepsilon > 0$.

**Proof.** (i) and (ii) are consequences of L’Hôpital’s rule:

\[ \lim_{t \to 0(+\infty)} \frac{G_{\varepsilon,\gamma}^{-1}(t)}{t} = \lim_{t \to 0(+\infty)} \frac{1}{g_{\varepsilon,\gamma}(G_{\varepsilon,\gamma}^{-1}(t))} = 1. \]

By definition of $g_{\varepsilon,\gamma}(t)$ and $G_{\varepsilon,\gamma}(t)$, (iii) follows.

Let us prove (iv): for $t > 0$ we have

\[ |G_{\varepsilon,\gamma}(t) - t| \leq \int_0^t |g_{\varepsilon,\gamma}(s) - 1| ds \leq \varepsilon^\gamma \int_0^t s^2 \eta(s)ds \leq 4\sqrt{\varepsilon^2} t. \]
From (iii), we have \( |G_{\varepsilon,\gamma,\pm}(t) - t| \leq 4\varepsilon |G_{\varepsilon,\gamma,\pm}(t)| \leq C\varepsilon |t| \).

Finally let \( T_\pm(t) := |G_{\varepsilon,\gamma,\pm}(t)|^2 \). Then, for small \( \varepsilon > 0 \), by (iii) we have

\[
\frac{1}{2}g^4(G_{\varepsilon,\gamma,\pm}(t))T_\pm''(G_{\varepsilon,\gamma,\pm}(t)) = 1 - \varepsilon^\gamma |G_{\varepsilon,\gamma,\pm}(t)|^2 \eta(G_{\varepsilon,\gamma,\pm}(t)) \\
- \varepsilon^\gamma |G_{\varepsilon,\gamma,\pm}(t)|^2 \eta'(G_{\varepsilon,\gamma,\pm}(t))G_{\varepsilon,\gamma,\pm}(t) > 0
\]

and

\[
\frac{1}{2}g^4(G_{\varepsilon,\gamma,-}(t))T_\mp''(G_{\varepsilon,\gamma,-}(t)) = 1 + \varepsilon^\gamma |G_{\varepsilon,\gamma,-}(t)|^2 \eta(G_{\varepsilon,\gamma,-}(t)) \\
+ \varepsilon^\gamma |G_{\varepsilon,\gamma,-}(t)|^2 \eta'(G_{\varepsilon,\gamma,-}(t))G_{\varepsilon,\gamma,-}(t) > 0,
\]

which imply (v).

Next we consider the following modified quasilinear Schrödinger equation,

\[-div(g_{\varepsilon,\gamma,\pm}(w)\nabla w) + g_{\varepsilon,\gamma,\pm}(w)g'_{\varepsilon,\gamma,\pm}(w)|\nabla u|^2 + V_\varepsilon(x)w = h(w).
\]  

(2.4)

Direct calculations show that (2.4) turns into (2.1) when \( g_{\varepsilon,\gamma,\pm}(t) = \sqrt{1 + \varepsilon^\gamma t^2} \) and the energy functional related to (2.4) is given by

\[
\bar{I}_{\varepsilon,\gamma,\pm}(w) = \frac{1}{2}\int_{\mathbb{R}^N} [g_{\varepsilon,\gamma,\pm}(w)|\nabla w|^2 + V_\varepsilon(x)w^2]dx - \int_{\mathbb{R}^N} H(w)dx
\]  

(2.5)

which is well defined on \( E_\varepsilon \), though it is still non-smooth. As in [29], we exploit the change of variable \( w = G_{\varepsilon,\gamma,\pm}(v) \) to replace the functional \( \bar{I}_{\varepsilon,\gamma,\pm} \) by the following smooth functional:

\[
P_{\varepsilon,\gamma,\pm}(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V_\varepsilon(x)|G_{\varepsilon,\gamma,\pm}^{-1}(v)|^2) dx - \int_{\mathbb{R}^N} H(G_{\varepsilon,\gamma,\pm}^{-1}(v))dx.
\]  

(2.6)

Indeed, under conditions (h1)–(h3) and (i)–(ii) of Lemma 2.2, it is standard to check that \( P_{\varepsilon,\gamma,\pm} \in C^1(E_\varepsilon, \mathbb{R}) \).

The Euler-Lagrange equation associated to \( P_{\varepsilon,\gamma,\pm} \) is

\[-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon,\gamma,\pm}^{-1}(v)}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))} = \frac{h(G_{\varepsilon,\gamma,\pm}^{-1}(v))}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))}, \quad x \in \mathbb{R}^N.
\]  

(2.7)

Now notice that if \( v \in E_\varepsilon \) is a critical point of \( P_{\varepsilon,\gamma,\pm}(v) \), then it satisfies for all \( \phi \in E_\varepsilon \),

\[
\int_{\mathbb{R}^N} [\nabla v \nabla \phi + V_\varepsilon(x) \frac{G_{\varepsilon,\gamma,\pm}^{-1}(v)}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))}\phi] dx - \int_{\mathbb{R}^N} \frac{h(G_{\varepsilon,\gamma,\pm}^{-1}(v))}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))}\phi dx = 0.
\]

Let \( w = G_{\varepsilon,\gamma,\pm}^{-1}(v) \in E_\varepsilon, \varphi \in C_0^\infty(\mathbb{R}^N) \), then \( \phi := \varphi g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v)) \in E_\varepsilon \) and

\[
\int_{\mathbb{R}^N} [\nabla w \nabla \varphi + g_{\varepsilon,\gamma,\pm}(w)g'_{\varepsilon,\gamma,\pm}(w)|\nabla w|^2 \varphi + V_\varepsilon(x)w \varphi] dx - \int_{\mathbb{R}^N} h(w)\varphi dx = 0,
\]

which implies that \( w \) is a weak solution of (2.4).

Define the functional

\[
\Gamma_{\varepsilon,\gamma,\pm}(v) = P_{\varepsilon,\gamma,\pm}(v) + Q_\varepsilon(v), \quad (2.8)
\]
where
\[ Q_\varepsilon(v) = \left( \int_{\mathbb{R}^N} \chi_\varepsilon v^2 \, dx - 1 \right)_+^2 \tag{2.9} \]
with \( \chi_\varepsilon(x) = 0 \) for \( x \in \mathcal{O}_\varepsilon \) and \( \chi_\varepsilon(x) = \varepsilon^{-1} \) for \( x \not\in \mathcal{O}_\varepsilon \). The functional \( Q_\varepsilon \) will have a penalization effect which forces concentration phenomena to occur inside \( \mathcal{O} \). This type of penalization was first introduced in [8]. Clearly, \( \Gamma_{\varepsilon, \gamma, \pm} \subset C^1(E_\varepsilon, \mathbb{R}) \).

### 3. Existence of solutions for the modified problem

Let us recall some well known results about the limit equation (1.5) under conditions \((h_1)\)–\((h_3)\). The energy functional is defined by
\[ L_m(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + mv^2) \, dx - \int_{\mathbb{R}^N} H(v) \, dx. \]
In [5], Berestycki and Lions proved that there exists a ground state (least energy solution) of (1.5) and each solution \( v \) of (1.5) satisfies Pohozaev’s identity
\[ \frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \, dx - \int_{\mathbb{R}^N} H(v) \, dx = 0. \tag{3.1} \]
Let \( S_m \) be the set of ground states \( U \) of (1.5) satisfying \( U(0) = \max_{x \in \mathbb{R}^N} U(x) \). Then, \( S_m \) is compact and
\[ |U(x)| + |\nabla U(x)| \leq C \exp(-c|x|), \quad x \in \mathbb{R}^N, \tag{3.2} \]
for some \( C, c > 0 \) independent of \( U \), see Proposition 1 in [10]. In [19], Jeanjean and Tanaka proved that the ground state has a mountain pass characterization, i.e.,
\[ L_m(U) = C_m = \inf_{\eta \in \Phi} \max_{t \in [0,1]} L_m(\eta(t)), \]
where \( \Phi = \{ \eta \in C([0,1], H^1(\mathbb{R}^N)) : \eta(0) = 0, L_m(\eta(1)) < 0 \} \).

Now set \( U_t(x) = U(\frac{x}{t}) \) for \( t > 0 \). By (3.1), for any \( t > 0 \), we get
\[ L_m(U_t) = \left( \frac{t^{N-2}}{2} - \frac{t^N}{2^*} \right) \int_{\mathbb{R}^N} |\nabla U|^2 \, dx. \]
So, there exists \( t_0 > 1 \) such that \( L_m(U_t) < -2 \) for all \( t \geq t_0 \).

Choose a positive number \( \beta < \frac{\text{dist}((M_\varepsilon \mathbb{R}^N \setminus \mathcal{O}), [0,1])}{100} \) and a cut-off function \( \varphi(x) \in C^\infty_0(\mathbb{R}^N, [0,1]) \) such that \( \varphi(x) = 1 \) for \( |x| \leq \beta \) and \( \varphi(x) = 0 \) for \( |x| \geq 2\beta \). We define \( \varphi_\varepsilon(x) := \varphi(\varepsilon x) \) and \( U_{\varepsilon}^y(x) := \varphi_\varepsilon(x - \frac{y}{\varepsilon})U(x - \frac{y}{\varepsilon}) \) for each \( y \in \mathcal{M}^\beta \), \( U \in S_m \). For sufficiently small \( \varepsilon > 0 \), we will find a solution near the set \( X_\varepsilon := \{ U_{\varepsilon}^y(x) : y \in \mathcal{M}^\beta, U \in S_m \} \).

Let \( W_{\varepsilon,t}(x) = \varphi_\varepsilon(x)U_t(x) \). Note that for fixed \( x \in \mathbb{R}^N \), \( W_{\varepsilon,t}(x) \to 0 \) as \( t \to 0 \). Thus, we set \( W_{\varepsilon,0}(x) = 0 \).

**Lemma 3.1.** \( \lim_{t \to 0} \max_{t \in (0, t_0]} |\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon,t}) - L_m(U_t)| = 0 \).
Proof. Since $0 \in \mathcal{M}$, we get $\text{supp}(W_{\varepsilon,t}(x)) \subset \{x \in \mathbb{R}^N : |x| \leq 2\beta\} \subset (\mathcal{M}^{2\beta})_{\varepsilon} \subset \mathcal{O}_{\varepsilon}$. Thus, $Q_{\varepsilon}(W_{\varepsilon,t}(x)) = 0$. For $t \in (0, t_0]$, we have

$$\Gamma_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}) - L_m(U_t) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla W_{\varepsilon,t}|^2 - |\nabla U_t|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (V_{\varepsilon}|G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t})|^2 - m U_t^2) \, dx + \int_{\mathbb{R}^N} [H(U_t) - H(G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}))] \, dx . \quad (3.3)$$

Next we prove that the three integrals on the right-hand side of (3.3) tend to zero as $\varepsilon \to 0$. By (3.2) we have, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} ||\nabla W_{\varepsilon,t}||^2 - |U_t|^2| \, dx \leq C \int_{\mathbb{R}^N} [C\varepsilon + (1 - \varphi_{\varepsilon}^2)] e^{-c|x|/t_0} \, dx \to 0 . \quad (3.4)$$

By Lemma 2.2-(iv) we have

$$\int_{\mathbb{R}^N} ||G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t})||^2 - |U_t|^2| \, dx \leq C\sqrt{\varepsilon^\gamma} \int_{\mathbb{R}^N} U^2 \, dx \leq C\sqrt{\varepsilon^\gamma} . \quad (3.5)$$

Thus, by (3.5) we have

$$\int_{\mathbb{R}^N} V_{\varepsilon}|G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t})|^2 - m U_t^2 \, dx \leq C \int_{\mathbb{R}^N} V_{\varepsilon} - m |U_t|^2 \, dx + C \int_{\mathbb{R}^N} |\xi_{\varepsilon,t}(\varphi_{\varepsilon} - 1) U_t| \, dx + C\sqrt{\varepsilon^\gamma} \quad (3.6)$$

as $\varepsilon \to 0$, where $\xi_{\varepsilon,t} = \tau W_{\varepsilon,t} + (1 - \tau) U_t$, $0 < \tau < 1$, $|\xi_{\varepsilon,t}| \leq C$. By ($h_1$)–($h_2$) we have

$$\int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}(U_t))| \, dx \leq C \int_{\mathbb{R}^N} (|U| + |U|^p) |U - G_{\varepsilon,\gamma,\pm}(U)| \, dx \leq C\sqrt{\varepsilon^\gamma} . \quad (3.7)$$

From (3.7) we get

$$\int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}))| \, dx \leq \int_{\mathbb{R}^N} |H(G_{\varepsilon,\gamma,\pm}(U_t)) - H(G_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}))| \, dx + \int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}(U_t))| \, dx \leq \int_{\mathbb{R}^N} h(G_{\varepsilon,\gamma,\pm}(\eta)) |U_t - W_{\varepsilon,t}| \, dx + C\sqrt{\varepsilon^\gamma} \leq C \int_{\mathbb{R}^N} (|U_t| + |W_{\varepsilon,t} + U_t^p + W_{\varepsilon,t}^p| |U_t - W_{\varepsilon,t}| \, dx + C\sqrt{\varepsilon^\gamma} \leq C \int_{\mathbb{R}^N} |\varphi_{\varepsilon} - 1| \exp(-c|x|/t_0) \, dx + C\sqrt{\varepsilon^\gamma} \to 0 , \text{ as } \varepsilon \to 0. \quad (3.8)$$

Combining (3.3)–(3.8) we obtain the result. \qed
From Lemma 3.1, there exists \( \varepsilon_0 > 0 \), such that for \( \varepsilon \in (0, \varepsilon_0) \),
\[
\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, t_0}) \leq L_m(U_{t_0}) + 1 < -1.
\]

Fix \( \varepsilon \in (0, \varepsilon_0) \) and define the minimax level
\[
C_{\varepsilon, \gamma, \pm} = \inf_{\eta_\varepsilon \in \Phi_\varepsilon} \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(s)),
\]
where \( \Phi_\varepsilon = \{ \eta_\varepsilon \in C([0, 1], E_\varepsilon) : \eta_\varepsilon(0) = 0, \eta_\varepsilon(1) = W_{\varepsilon, t_0} \} \).

**Lemma 3.2.** \( \lim_{\varepsilon \to 0} C_{\varepsilon, \gamma, \pm} = C_m. \)

**Proof.** Let \( \eta_\varepsilon(s) = W_{\varepsilon, s t_0}, s \in [0, 1] \), then \( \eta_\varepsilon(s) \in \Phi_\varepsilon \). Since \( t_0 \to 1 \), from Lemma 3.1 we have
\[
\limsup_{\varepsilon \to 0} C_{\varepsilon, \gamma, \pm} = \limsup_{\varepsilon \to 0} \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, s t_0})
\]
\[
= \limsup_{\varepsilon \to 0} \max_{t \in [0, t_0]} \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, t})
\]
\[
\leq \max_{t \in [0, t_0]} L_m(U_t) = C_m.
\]
The result follows if we prove that \( \liminf_{\varepsilon \to 0} C_{\varepsilon, \gamma, \pm} \geq C_m \). By the definition of \( C_{\varepsilon, \gamma, \pm} \), for any \( \varepsilon > 0 \), there exists \( \tilde{\eta}_\varepsilon \in \Phi_\varepsilon \) such that
\[
\max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) < C_{\varepsilon, \gamma, \pm} + \varepsilon. \tag{3.9}
\]
Since \( P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(0)) = 0 \) and \( P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(1)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(1)) = \Gamma_{\varepsilon, \gamma, +}(W_{\varepsilon, t_0}) < -1 \), there exists \( s_0 \in (0, 1) \) such that
\[
P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s_0)) = -1 \quad \text{and} \quad P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) \geq -1, \quad s \in [0, s_0).
\]
Then we have
\[
Q_\varepsilon(\tilde{\eta}_\varepsilon(s)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) + 1 < C_{\varepsilon, \gamma, \pm} + \varepsilon + 1, \quad s \in [0, s_0].
\]

By Lemma 2.2-(iii) we also have
\[
\int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |G_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s))|^2 dx \leq C \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |\tilde{\eta}_\varepsilon(s)|^2 dx
\]
\[
\leq C \varepsilon \left[ \sqrt{Q_\varepsilon(\tilde{\eta}_\varepsilon(s)) + 1} \right] \leq C \varepsilon \left[ \sqrt{C_{\varepsilon, \gamma, \pm} + \varepsilon + 1 + 1} \right], \quad s \in [0, s_0].
\]

Let us estimate \( P_{\varepsilon, \gamma, +} \) and \( P_{\varepsilon, \gamma, -} \) separately:

(i) Since \( g_{\varepsilon, \gamma, +}^2(t) \geq 1 \), we get
\[
P_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(s))
\]
\[
\geq L_m(G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))) - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))|^2 dx
\]
\[
\geq L_m(G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))) - \frac{1}{2} m \varepsilon C \left[ \sqrt{C_{\varepsilon, \gamma, +} + \varepsilon + 1 + 1} \right], \quad s \in [0, s_0], \tag{3.10}
\]
which yields
\[ L_m(G^{-1}_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(0))) \leq \frac{1}{2} m\varepsilon C \left[ \sqrt{C_{\varepsilon, \gamma, +} + \tilde{\varepsilon} + 1} + 1 \right] - 1 < 0, \]
for small \( \varepsilon > 0 \).

Thus, \( G^{-1}_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(t_0)) \in \Phi \) and \( \max_{t \in [0, 1]} L_m(G^{-1}_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(t_0))) \geq C_m \). So, by (3.9) and (3.10), we get
\[ C_{\varepsilon, \gamma, +} + \tilde{\varepsilon} > \max_{s \in [0, s_0]} \Gamma_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(s)) \]
\[ \geq C_m - \frac{1}{2} m\varepsilon C \left[ \sqrt{C_{\varepsilon, \gamma, +} + \tilde{\varepsilon} + 1} + 1 \right], \]
which yields \( \liminf_{\varepsilon \to 0} C_{\varepsilon, \gamma, +} \geq C_m \) since \( \tilde{\varepsilon} \) is arbitrary.

(ii) For simplicity, let \( u_s = G_{\varepsilon, \gamma, -}^{-1}(\tilde{\eta}_\varepsilon(s)) \), \( \omega_s(x) = u_s(\sqrt{1 - 4\sqrt{\varepsilon}}x) \), to have
\[ P_{\varepsilon, \gamma, -}(\tilde{\eta}_\varepsilon(s)) \]
\[ \geq \frac{1}{2} \int_{\mathbb{R}^N} (g_{\varepsilon, \gamma, -}(u_s)|\nabla u_s|^2 + mu_s^2)dx - \int_{\mathbb{R}^N} H(u_s)dx - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \Omega} u_s^2 dx \]
\[ \geq (1 - 4\sqrt{\varepsilon}) \left[ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_s|^2 + 1 - 4\sqrt{\varepsilon} mu_s^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} H(u_s)dx \right] \]
\[ - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \Omega} u_s^2 dx \]
\[ \geq (1 - 4\sqrt{\varepsilon})^{-\frac{\gamma}{2}} L_m(\omega_s) - \frac{1}{2} m\varepsilon C \left( \sqrt{C_{\varepsilon, \gamma, -} + \tilde{\varepsilon} + 1} + 1 \right), \quad s \in [0, s_0]. \]

Hence
\[ L_m(\omega_{s_0}) \leq (1 - 4\sqrt{\varepsilon})^{-\frac{\gamma}{2}} \left[ \frac{1}{2} m\varepsilon C \left( \sqrt{C_{\varepsilon, \gamma, -} + \tilde{\varepsilon} + 1} + 1 \right) - 1 \right] < 0, \]
which implies \( \omega_{t_{s_0}}(x) \in \Phi \) and \( \max_{t \in [0, 1]} L_m(\omega_{t_{s_0}}) \geq C_m \). So, by (3.9) and (3.11), we get
\[ C_{\varepsilon, \gamma, -} + \tilde{\varepsilon} > \max_{s \in [0, s_0]} \Gamma_{\varepsilon, \gamma, -}(\tilde{\eta}_\varepsilon(s)) \]
\[ \geq (1 - 4\sqrt{\varepsilon})^{-\frac{\gamma}{2}} C_m - \frac{1}{2} m\varepsilon C \left[ \sqrt{C_{\varepsilon, \gamma, -} + \tilde{\varepsilon} + 1} + 1 \right], \]
which yields \( \liminf_{\varepsilon \to 0} C_{\varepsilon, \gamma, -} \geq C_m \).

Combine (i) and (ii) to have \( \liminf_{\varepsilon \to 0} C_{\varepsilon, \gamma, \pm} \geq C_m \) which completes the proof.

\[ \square \]

**Remark 3.3.** Let \( D_{\varepsilon, \gamma, \pm} = \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(w_{\varepsilon, s_{t_0}}) \); from Lemma 3.2 one has \( \lim_{\varepsilon \to 0} D_{\varepsilon, \gamma, \pm} = C_m \).
Next we borrow some ideas from Byeon and Jeanjean [10] in order to prove the existence of critical points for the modified problem. This method has been used to deal with the case \( \gamma = 0 \) in [16]. Here the situation is quite different as the limit equation is different from the case \( \gamma = 0 \) and we set the problem in a Sobolev space setting instead of the Orlicz space setting used in [16]. We set the problem in the space \( E^R_\varepsilon := H^1_0(B_{R/\varepsilon}(0)) \) endowed with the norm

\[
\|v\|_{\varepsilon,R} = \left[ \int_{B_{R/\varepsilon}(0)} (|\nabla v|^2 + V_\varepsilon(x)v^2)dx \right]^\frac{1}{2}.
\]

Note that any \( v \in E^R_\varepsilon \) can be regarded as an element in \( E_\varepsilon \) by defining \( v = 0 \) on \( \mathbb{R}^N \setminus B_{R/\varepsilon}(0) \), so that we may assume the two norms \( \| \cdot \|_\varepsilon \) and \( \| \cdot \|_{\varepsilon,R} \) coincide.

Define the level set

\[
\Gamma^c_{\varepsilon,\gamma,\pm} := \{ u \in E^R_\varepsilon : \Gamma_{\varepsilon,\gamma,\pm} \leq c \}
\]

and

\[
A^d := \{ u \in E^R_\varepsilon : \inf_{v \in A} \| u - v \|_{\varepsilon,R} < d \}, \quad d > 0.
\]

Let \( v_n \in X^d_{\varepsilon_n} \cap E^R_{\varepsilon_n} \) with \( \varepsilon_n \to 0 \) and \( R_n \to +\infty \) such that

\[
\lim_{n \to \infty} \Gamma_{\varepsilon_n,\gamma,\pm}(v_n) \leq C_m \quad \text{and} \quad \lim_{n \to \infty} \| \Gamma'_{\varepsilon_n,\gamma,\pm}(v_n) \|_{(E^R_{\varepsilon_n})'} = 0.
\]

From the definition of \( X^d_{\varepsilon_n} \), we can find \( \{U_n\} \subset S_m \) and a sequence of points \( \{y_n\} \subset \mathcal{M}^\beta \) such that

\[
\left\| v_n - \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n})U_n(\cdot - \frac{y_n}{\varepsilon_n}) \right\|_{\varepsilon_n,R_n} \leq d. \tag{3.12}
\]

Since \( S_m \) and \( \mathcal{M}^\beta \) are compact, there exist \( U \in S_m \) and \( y_0 \in \mathcal{M}^\beta \) such that \( U_n \to U \) in \( H^1(\mathbb{R}^N) \), \( U_n(x) \to U(x) \) a.e. in \( \mathbb{R}^N \) and \( y_n \to y_0 \). It is easy to check that for \( d \in (0, d_0) \) with \( d_0 > 0 \) sufficiently small, we have

\[
\left\| v_n - \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n})U(\cdot - \frac{y_n}{\varepsilon_n}) \right\|_{\varepsilon_n,R_n} \leq 2d \tag{3.13}
\]

provided \( n \) is large enough and in particular \( \{v_n\} \) stays bounded in \( E_\varepsilon \).

**Lemma 3.4.**

\[
\lim_{n \to \infty} \sup_{z \in \{ z \in \mathbb{R}^N : \beta/2 \leq |\varepsilon_n z - y_n| \leq 3\beta \}} \int_{B_R(z)} v_n^2 dx = 0, \quad \forall R > 0.
\]

**Proof.** Suppose by contradiction that there exist \( R > 0 \) and a sequence \( \{z_n\} \subset \{ z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |\varepsilon_n z - y_n| \leq 3\beta \} \) such that

\[
\lim_{n \to \infty} \int_{B_R(z_n)} v_n^2 dx > 0. \tag{3.14}
\]
Let $\varepsilon_n z_n \to z_0 \in \{ z \in \mathbb{R}^N : \frac{1}{2} \beta \leq |z - y_0| \leq 3 \beta \}$, as $n \to \infty$. Set $\tilde{v}_n := v_n (\cdot + z_n)$ and, up to a subsequence, we may assume $\tilde{v}_n \to \tilde{v}$ in $H^1 (\mathbb{R}^N)$ and $\tilde{v}_n \to \tilde{v}$ in $L^p_{loc} (\mathbb{R}^N)$, $p \in [2, 2^*)$. Then, (3.14) yields

$$\int_{B_R (0)} |\tilde{v}|^2 d x = \lim_{n \to \infty} \int_{B_R (0)} \tilde{v}_n^2 d x > 0, \quad (3.15)$$

which implies $\tilde{v} \neq 0$.

From $\lim_{n \to \infty} \| \Gamma_{\varepsilon_n, \gamma, \pm} (v_n) \| (E_{\varepsilon_n}) = 0$, we have $\langle \Gamma'_{\varepsilon_n, \gamma, \pm} (v_n), \phi \rangle = o_n (1) \| \phi \|$ for any $\phi \in C_0^\infty (\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} \nabla \tilde{v}_n \nabla \phi d x + \int_{\mathbb{R}^N} V_{\varepsilon_n} (x + z_n) \frac{G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n)}{g_{\varepsilon_n, \gamma, \pm} (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))} \phi - \frac{h (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))}{g_{\varepsilon_n, \gamma, \pm} (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))} \phi d x$$

$$- 4 \left( \int_{\mathbb{R}^N} \chi_{\varepsilon_n} \tilde{v}_n^2 d x - 1 \right) + \int_{\mathbb{R}^N} \chi_{\varepsilon_n} (x + z_n) \tilde{v}_n \phi d x = o_n (1) \| \phi \|. \quad (3.16)$$

Clearly,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \chi_{\varepsilon_n} (x + z_n) \tilde{v}_n \phi d x = 0.$$

Now let us prove

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} V_{\varepsilon_n} (x + z_n) \frac{G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n)}{g_{\varepsilon_n, \gamma, \pm} (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))} \phi d x = V (z_0) \int_{\mathbb{R}^N} \tilde{v} \phi d x \quad (3.17)$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{h (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))}{g_{\varepsilon_n, \gamma, \pm} (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n))} \phi d x = \int_{\mathbb{R}^N} h (\tilde{v}) \phi d x. \quad (3.18)$$

Indeed, by Lemma 2.2-(iv), one has $G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n) - \tilde{v}_n \to 0$, a.e. in $\mathbb{R}^N$. Since $\tilde{v}_n \to \tilde{v}$, a.e. in $\mathbb{R}^N$, we get $G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n) \to \tilde{v}$, a.e. in $\mathbb{R}^N$. Moreover, $g_{\varepsilon_n, \gamma, \pm} (G_{\varepsilon_n, \gamma, \pm}^{-1} (\tilde{v}_n)) \to 1$, a.e. in $\mathbb{R}^N$. Thus, (3.17) and (3.18) are consequences of the Lebesgue dominated convergence theorem.

Combine (3.16)–(3.18) to obtain

$$\int_{\mathbb{R}^N} [\nabla \tilde{v} \nabla \phi + V (z_0) \tilde{v} \phi - h (\tilde{v}) \phi] d x = 0, \quad \forall \phi \in C_0^\infty (\mathbb{R}^N),$$

so that $\tilde{v}$ turns out to be a nontrivial solution to the following equation,

$$-\Delta \tilde{v} + V (z_0) \tilde{v} = h (\tilde{v}), \quad x \in \mathbb{R}^N.$$

By the maximum principle $\tilde{v} > 0$. From $V (z_0) \geq m$, we have $C_{V (z_0)} \geq C_m$.

Next choose $R > 0$ sufficiently large and apply Pohozaev’s identity to get, on the one hand,

$$2 \lim_{n \to \infty} \int_{B_R (z_n)} |\nabla v_n |^2 d x \geq \int_{\mathbb{R}^N} |\nabla \tilde{v} |^2 d x = NLV (z_0) (\tilde{v}) \geq NC_{V (z_0)} \geq NC_m,$$
on the other hand, for large \( n \), it follows from (3.13) that
\[
\int_{B_R(z_n)} |\nabla v_n|^2 \, dx \leq 4d^2 + 2\varepsilon_n^2 \int_{B_R(z_n)} |\nabla \varphi_\varepsilon(x - \frac{y_n}{\varepsilon_n})U(x - \frac{y_n}{\varepsilon_n})|^2 \, dx
\]
\[
+ 2 \int_{B_R(z_n)} |\varphi_\varepsilon(x - \frac{y_n}{\varepsilon_n})\nabla U(x - \frac{y_n}{\varepsilon_n})|^2 \, dx
\]
\[
\leq d + C\varepsilon_n^2 + C \int_{B_R(0)} \exp(-c|x + z_n - \frac{y_n}{\varepsilon_n}|) \, dx.
\]
Note that \( \lim_{n \to \infty} |z_n - \frac{y_n}{\varepsilon_n}| = +\infty \) since \( \frac{1}{2\varepsilon_n}\beta \leq |z_n - \frac{y_n}{\varepsilon_n}| \leq \frac{3\beta}{\varepsilon_n} \). Thus, for \( n \) large, we get
\[
\frac{1}{2}NC_m \leq \lim_{n \to \infty} \int_{B_R(z_n)} |\nabla v_n|^2 \, dx \leq 3d,
\]
which is a contradiction for small \( d > 0 \). This completes the proof of Lemma 3.4. □

Next choose \( \eta \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \eta \leq 1 \), \( \eta(z) = 1 \) if \( z \in \{ z \in \mathbb{R}^N : \beta \leq |z| \leq 2\beta \} \), \( \eta(z) = 0 \) if \( z \in \mathbb{R}^N \setminus \{ z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |z| \leq 3\beta \} \). Setting \( \eta_n(z) = \eta(\varepsilon_n z - y_n)v_n \), clearly, \( \eta_n \) is bounded in \( H^1(\mathbb{R}^N) \). Thus, from Lemma 3.7, we have
\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\eta_n|^2 \, dx = 0. \tag{3.19}
\]
This fact, together with Lions’ lemma [21], gives \( \eta_n \to 0 \) in \( L^{p+1}(\mathbb{R}^N) \). So that we obtain
\[
\lim_{n \to \infty} \int_{\{ x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta \}} |v_n|^{p+1} \, dx
\]
\[
= \lim_{n \to \infty} \int_{\{ x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta \}} |\eta_n|^{p+1} \, dx \tag{3.20}
\]
\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\eta_n|^{p+1} \, dx = 0.
\]
Set \( v_{n,1} = \varphi_\varepsilon_n(\cdot \frac{y_n}{\varepsilon_n})v_n \) and \( v_{n,2} = v_n - v_{n,1} \). Then, we can prove the following lemma.

**Lemma 3.5.** \( \Gamma_{\varepsilon_n,\gamma,\pm}(v_n) \geq \Gamma_{\varepsilon_n,\gamma,\pm}(v_{n,1}) + \Gamma_{\varepsilon_n,\gamma,\pm}(v_{n,2}) + o(n(1)) \).

**Proof.** Note that \( supp(v_{n,1}) \subset \{ z \in \mathbb{R}^N : |\varepsilon_n z - y_n| \leq 2\beta \} \subset \mathcal{M}_\varepsilon^{3\beta} \subset O_z \), thus, \( Q_{\varepsilon_n}(v_{n,1}) = 0 \) and \( Q_{\varepsilon_n}(v_{n,2}) = Q_{\varepsilon_n}(v_n) \). On the other hand, we note that \( supp(v_{n,2}) \subset \{ z \in \mathbb{R}^N : |\varepsilon_n z - y_n| \geq \beta \} \). Therefore, by the convexity of \( |G_{\varepsilon_n,\gamma,\pm}(t)|^2 \)
and since $G^{-1}_{\varepsilon, n, \gamma, \pm}(0) = 0$, for large $n$, we get

$$
\Gamma_{\varepsilon, n, \gamma, \pm}(v_{n, 1}) + \Gamma_{\varepsilon, n, \gamma, \pm}(v_{n, 2})
= \Gamma_{\varepsilon, n, \gamma, \pm}(v_{n}) + \int_{\mathbb{R}^N} \varphi_n(x - \frac{y_n}{\varepsilon}) \left[ \varphi_n(x - \frac{y_n}{\varepsilon}) - 1 \right] |\nabla v_{n}|^2 dx
+ \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) \left[ |G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 1})|^2 + |G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 2})|^2 - |G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n})|^2 \right] dx
+ \int_{\mathbb{R}^N} \left[ H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 1})) - H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 2})) \right] dx + o_n(1)
\leq \Gamma_{\varepsilon, n, \gamma, \pm}(v_{n}) + \int_{\mathbb{R}^N} \left[ H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n})) - H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 1})) - H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 2})) \right] dx + o_n(1). \tag{3.21}
$$

By $(h_1)-(h_2)$, given any $\sigma > 0$, there exists some $C_{\sigma} > 0$ such that

$$
\int_{\mathbb{R}^N} \left[ H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n})) - H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 1})) - H(G^{-1}_{\varepsilon, n, \gamma, \pm}(v_{n, 2})) \right] dx
\leq \sigma \int_{\mathbb{R}^N} v_n^2 dx + C_{\sigma} \int_{\{x \in \mathbb{R}^N : |x - y_n| \leq 2\varepsilon\}} |v_n|^{p+1} dx. \tag{3.22}
$$

Since $\sigma$ is arbitrary, by (3.21), (3.22) and (3.20) we obtain the result. \hfill \Box

**Lemma 3.6.** $\Gamma_{\varepsilon, n, \gamma, \pm}(v_{n, 2}) > 0$.

*Proof.* From (3.13), we get

$$
\int_{\{x \in \mathbb{R}^N : |x - y_n| \geq 2\varepsilon\}} |\nabla v_{n, 2}|^2 dx = \int_{\{x \in \mathbb{R}^N : |x - y_n| \geq 2\varepsilon\}} |\nabla v_n|^2 dx
= \int_{\{x \in \mathbb{R}^N : |x - y_n| \geq 2\varepsilon\}} \left| \nabla \left[ v_n - \varphi_n \left( \cdot - \frac{y_n}{\varepsilon} \right) \right] - \nabla \left[ \varphi_n \left( \cdot - \frac{y_n}{\varepsilon} \right) \right] \right|^2 dx
\leq \left\| v_n - \varphi_n \left( \cdot - \frac{y_n}{\varepsilon} \right) \right\|^2 \leq d^2. \tag{3.23}
$$

Similarly, by (3.13), we get

$$
\int_{\{x \in \mathbb{R}^N : |x - y_n| \leq 2\varepsilon\}} |\nabla v_{n, 2}|^2 dx
\leq C \int_{\{x \in \mathbb{R}^N : |x - y_n| \leq 2\varepsilon\}} (|\nabla v_n|^2 + v_n^2) dx
\leq Cd^2 + \left\| \varphi_n \left( \cdot - \frac{y_n}{\varepsilon} \right) \right\|^2 \left| \{x \in \mathbb{R}^N : |x - y_n| \leq 2\varepsilon\} \right| \tag{3.24}
\leq Cd^2 + o_n(1). \n$$
Consequently, as $n$ is large enough, we have $\|\nabla v_{n,2}\|_2 \leq Cd$. For small $d > 0$, using (h1)–(h2) again, we get
\[
\Gamma_{\varepsilon_n,\gamma,\pm}(v_{n,2}) \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x)|G_{\varepsilon_n,\gamma,\pm}^{-1}(v_{n,2})|^2 dx - C \int_{\mathbb{R}^N} |v_{n,2}|^2^* dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x)|G_{\varepsilon_n,\gamma,\pm}^{-1}(v_{n,2})|^2 dx - C \left( \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2^* dx \right)^{2^*} \\
\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x)|G_{\varepsilon_n,\gamma,\pm}^{-1}(v_{n,2})|^2 dx > 0 .
\] (3.25)

This completes the proof of Lemma 3.6. \(\square\)

**Lemma 3.7.** For $d > 0$ sufficiently small, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$, $y_0 \in \mathcal{M}$, and $U_0 \in S_m$ satisfying, up to a subsequence, the following:

\[
\lim_{n \to \infty} |\varepsilon_n z_n - y_0| = 0, \quad \lim_{n \to \infty} \|v_n - \varphi_{\varepsilon_n}(\cdot - z_n)U_0(\cdot - z_n)\|_{\varepsilon_n,R_n} = 0.
\]

**Proof.** Let $w_n := v_{n,1}(\cdot + \frac{y_n}{\varepsilon_n})$. Then, by (3.13), we get $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, up to a subsequence if necessary, we may assume $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $v_n \to v$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \in [2, 2^*]$ and $w_n \to w$ a.e. in $\mathbb{R}^N$. From (3.13), for given $R > 0$, when $n$ large, we get
\[
4d^2 \geq \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \leq \beta\}} V_{\varepsilon_n}(x) \left| v_{n,1} - \varphi_{\varepsilon_n}(x - \frac{y_n}{\varepsilon_n})U(x - \frac{y_n}{\varepsilon_n}) \right|^2 dx \\
= \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x| \leq \beta\}} V(\varepsilon_n x + y_n) |w_n - U(x)|^2 dx \\
\geq m \int_{B_R(0)} |w_n - U(x)|^2 dx.
\]

Thus, we have
\[
\int_{B_R(0)} w^2 dx \geq \lim_{n \to \infty} \int_{B_R(0)} w_n^2 dx \\
\geq \int_{B_R(0)} |U(x)|^2 dx - \lim_{n \to \infty} \int_{B_R(0)} |w_n - U(x)|^2 dx \\
\geq C - \frac{4}{m} d^2,
\]

which yields $w \neq 0$.

Now let $\phi \in C_0^\infty(\mathbb{R}^N)$ and note that $w_n(x) = v_{n,1}(x + \frac{y_n}{\varepsilon_n}) = v_n(x + \frac{y_n}{\varepsilon_n})$ for $x \in \text{supp}(\phi)$ and large $n$. Moreover, $\text{supp}(w_n(x)) \subset \{x \in \mathbb{R}^N : |\varepsilon_n x| \leq 2\beta\} \subset \mathcal{O}$. Thus, from $\langle \Gamma_{\varepsilon_n,\gamma,\pm}(v_n), \phi \rangle = o_n(1)\|\phi\|$ and analogous to the proof of (3.17) and (3.18), $w$ is a nontrivial solution of equation
\[
-\Delta w + V(y_0)w = h(w), \quad x \in \mathbb{R}^N.
\] (3.26)

By the maximum principle, $w > 0$. 

Now we claim:

\[ \lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |w_n - w|^2 \, dx = 0. \tag{3.27} \]

In fact, if by contradiction (3.27) does not occur, then there exists a sequence \( \{z_n\} \subset \mathbb{R}^N \) with \( |z_n| \to +\infty \) such that

\[ \lim_{n \to \infty} \int_{B_1(z_n)} |w_n - w|^2 \, dx > 0. \]

Thus, we have

\[ \lim_{n \to \infty} \int_{B_1(z_n)} |w|^2 \, dx = 0, \quad \lim_{n \to \infty} \int_{B_1(z_n)} |w_n|^2 \, dx > 0. \]

We have \( |\varepsilon_n z_n - y_n| < 1/2 \beta \). In fact, if \( |\varepsilon_n z_n - y_n| \geq 1/2 \beta \), by (3.14), we have

\[ 0 < \lim_{n \to \infty} \int_{B_1(z_n)} |w_n|^2 \, dx \leq \lim_{n \to \infty} \sup_{z \in \{z \in \mathbb{R}^N : 1/2 \beta \leq |\varepsilon_n z - y_n| \leq 3/2 \beta \}} \int_{B_1(z)} |v_n|^2 \, dx = 0, \]

which is impossible. Thus, up to a subsequence, we may assume that \( \varepsilon_n z_n \to z_0 \in \{z \in \mathbb{R}^N : |z| \leq 2 \beta \} \). Assume that \( v_{n,1}(\cdot + z_n + \frac{y_n}{\varepsilon_n}) \to v_1 \) in \( H^1(\mathbb{R}^N) \). Analogously to the proof of (3.26), we have

\[ -\Delta v_1 + V(z_0 + y_0) v_1 = h(v_1), \quad x \in \mathbb{R}^N. \]

By the maximum principle, \( v_1 > 0 \).

Thus, for large \( R \), we get

\[ \frac{1}{2} N C_m \leq \int_{B_R(0)} |\nabla v_{n,1}(x + z_n + \frac{y_n}{\varepsilon_n})|^2 \, dx \]

\[ = \int_{B_R(z_n + \frac{y_n}{\varepsilon_n})} |\nabla v_{n,1}(x)|^2 \, dx \]

\[ \leq C \varepsilon_n^2 + C \int_{B_R(z_n + \frac{y_n}{\varepsilon_n})} |\nabla v_n|^2 \, dx \]

\[ \leq C \varepsilon_n^2 + C d + C \int_{B_R(0)} e^{-c|x+z_n|} \, dx. \]

We get a contradiction for large \( n \) and small \( d \) since \( |z_n| \to +\infty \).

Therefore, (3.27) holds. By Lions’ Lemma, we have \( w_n \to w \) in \( L^{p+1}(\mathbb{R}^N) \). Then, for given \( \sigma > 0 \), there exists \( C_\sigma > 0 \) such that

\[ \int_{\mathbb{R}^N} |H(G_{\varepsilon_n,\gamma,\pm}^{-1}(w_n)) - H(G_{\varepsilon_n,\gamma,\pm}^{-1}(w))| \, dx \]

\[ = \int_{\mathbb{R}^N} \left| \frac{h(G_{\varepsilon_n,\gamma,\pm}(w))}{g_{\varepsilon_n,\gamma,\pm}(G_{\varepsilon_n,\gamma,\pm}^{-1}(w))} \right| |w_n - w| \, dx \]

\[ \leq \sigma \|w_n - w\|_2^2 + C_\sigma \|w_n - w\|_{p+1}^{p+1}, \]
where \( \varpi(x) = tw_n(x) + (1 - t)w(x) \), \( t \in (0, 1) \). Thus, by the Lebesgue dominated convergence theorem,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} H(G_{\varepsilon_n, \gamma}^{-1}(w_n)) \, dx = \int_{\mathbb{R}^N} H(w) \, dx.
\]

By Fatou’s Lemma, Lemmas 3.5, 3.6 and (3.25), up to a subsequence, we have

\[
C_m \geq \lim_{n \to \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_n)
\geq \lim_{n \to \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) + \lim_{n \to \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2})
\geq \lim_{n \to \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1})
= \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + y_n)|G_{\varepsilon_n, \gamma}^{-1}(w_n)|^2) \, dx 
\quad - \int_{\mathbb{R}^N} H(G_{\varepsilon_n, \gamma}^{-1}(w_n)) \, dx \right]
\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + V(y_0)w^2) \, dx - \int_{\mathbb{R}^N} H(w) \, dx
\geq C_{V(y_0)}
\geq C_m.
\]

Hence, we get \( \lim_{n \to \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) = C_m \). Moreover, we get \( V(y_0) = m \) and as a consequence we see that \( w \) is a ground state to (1.5). Thus, there exists some \( z \in \mathbb{R}^N \) such that \( U_0 := w(\cdot + z) \in S_m \). By (3.28), we have

\[
\lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|G_{\varepsilon_n, \gamma}^{-1}(w_n)|^2 \, dx \right]
= \left[ \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + m \int_{\mathbb{R}^N} w^2 \, dx \right].
\]

This together Fatou’s Lemma again give

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla w|^2 \, dx,
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|G_{\varepsilon_n, \gamma}^{-1}(w_n)|^2 \, dx = m \int_{\mathbb{R}^N} w^2 \, dx.
\]

Moreover, for any set \( A \subset \mathbb{R}^N \), we get

\[
\lim_{n \to \infty} \int_A V(\varepsilon_n x + y_n)|G_{\varepsilon_n, \gamma}^{-1}(w_n)|^2 \, dx = m \int_A w^2 \, dx.
\]
By (3.29), we have
\[
\lim_{n \to \infty} \left\| \nabla \left[ v_{n,1} - \varphi_{\varepsilon_n} \left( \cdot - z - \frac{y_n}{\varepsilon_n} \right) w \left( \cdot - \frac{y_n}{\varepsilon_n} \right) \right] \right\|_2 = \lim_{n \to \infty} \left\| \nabla \left[ w_n - \varphi_{\varepsilon_n} \left( \cdot - z \right) w \right] \right\|_2 \\
\leq \lim_{n \to \infty} \left\| \nabla \left( w_n - w \right) \right\|_2 + \lim_{n \to \infty} \left\| \nabla \left[ \left( 1 - \varphi_{\varepsilon_n} \left( \cdot - z \right) \right) w \right] \right\|_2 = 0.
\]

According to (3.30) and (3.31), buying the line of Proposition 2.1-(3) in [24], for any \( \sigma > 0 \), we can take a bounded set \( A_\sigma \subset \mathbb{R}^N \) and \( n_0 \in \mathbb{N} \) such that \( |G_{\varepsilon_n,\gamma,\pm}^{-1}(t)|^2 \) is convex and for \( n > n_0 \),
\[
\int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n) \left[ |G_{\varepsilon_n,\gamma,\pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w)|^2 + |G_{\varepsilon_n,\gamma,\pm}(w_n)|^2 \right] dx \leq \frac{1}{2} \sigma.
\]

Thus, we have
\[
\int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n)|G_{\varepsilon_n,\gamma,\pm}(w_n - \varphi_{\varepsilon_n}(x - z)w)|^2 dx \leq C\sigma, \quad n > n_0.
\]

Thus, for small \( \delta > 0 \), we get
\[
\int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n) \left| G_{\varepsilon_n,\gamma,\pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w) \right|^2 dx \\
- |w_n - \varphi_{\varepsilon_n}(x - z)w|^2 dx \\
\leq \int_{\{x \in \mathbb{R}^N \setminus A_\sigma: 0 < |(w_n - w)(x)| \leq \delta\}} V(\varepsilon_n x + y_n)|w_n - \varphi_{\varepsilon_n}(x - z)w|^2 \left| G_{\varepsilon_n,\gamma,\pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w) \right|^2 dx \\
+ C \int_{\{x \in \mathbb{R}^N \setminus A_\sigma: \delta \leq |(w_n - \varphi_{\varepsilon_n}(x - z)w)(x)|\}} |w_n - \varphi_{\varepsilon_n}(x - z)w|^{p+1} dx \\
\leq C\sigma \|w_n - \varphi_{\varepsilon_n}(x - z)w\|^2_2 + C\|w_n - \varphi_{\varepsilon_n}(x - z)w\|^{p+1}_{p+1}.
\]

On the other hand, since \( w_n \to w \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), we have
\[
\lim_{n \to \infty} \int_{A_\sigma} V(\varepsilon_n x + y_n)|w_n - \varphi_{\varepsilon_n}(x - z)w|^2 dx = 0.
\]

Combining (3.32) and (3.33), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x) \left[ v_{n,1} - \varphi_{\varepsilon_n}(x - z - \frac{y_n}{\varepsilon_n})w(x - \frac{y_n}{\varepsilon_n}) \right]^2 dx = 0.
\]

Let \( z_n = z + \frac{y_n}{\varepsilon_n} \), then
\[
\|v_{n,1} - \varphi_{\varepsilon_n}(\cdot - z_n)U_0(\cdot - z_n)\|_{\varepsilon_n, R_n} \to 0.
\]

Finally, using (3.28) again,
\[
0 = \lim_{n \to \infty} \Gamma_{\varepsilon_n,\gamma,\pm}(v_{n,2}) \geq \frac{1}{4} \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ \left| \nabla v_{n,2} \right|^2 + V(\varepsilon_n x)|G_{\varepsilon_n,\gamma,\pm}^{-1}(v_{n,2})|^2 \right] dx,
\]
which yields \( \lim_{n \to \infty} \|v_{n,2}\|_{\varepsilon_n, R_n} = 0 \) by Lemma 2.2-(iii). The result follows. \( \square \)
Let \( d \in (0, d_0) \) such that Lemma 3.7 holds.

**Lemma 3.8.** For any \( d \in (0, d_0) \), there exist positive constants \( \delta_d, R_d \) and \( \varepsilon_d \) such that

\[
\| \Gamma'_{\varepsilon, \gamma, \pm}(v) \|_{(E^R)'_\nu} \geq \delta_d
\]

for any \( v \in E^R_\varepsilon \cap \Gamma^D_{\varepsilon, \gamma, \pm} \cap (X^d_\varepsilon \setminus X^d_d), \) \( R \geq R_d, \) and \( \varepsilon \in (0, \varepsilon_d). \)

**Proof.** By contradiction, we assume that for some \( d \in (0, d_0), \) there exists \( \varepsilon_n < \frac{1}{n}, \)
\( R_n > n \) and \( v_n \in E^R_{\varepsilon_n} \cap \Gamma^D_{\varepsilon_n, \gamma, \pm} \cap (X^d_{\varepsilon_n} \setminus X^d_d) \) such that

\[
\| \Gamma'_{\varepsilon_n, \gamma, \pm}(v_n) \|_{(E^R_{\varepsilon_n})'_{\nu}} < \frac{1}{n}.
\]

By Lemma 3.7, there exist a sequence \( \{z_n\} \subset \mathbb{R}^N, \) \( y_0 \in \mathcal{M} \) and \( U_0 \in S_m \) satisfying

\[
\lim_{n \to \infty} |\varepsilon_n z_n - y_0| = 0, \quad \lim_{n \to \infty} \| v_n - \varphi_{\varepsilon_n}(\cdot - z_n) U_0(\cdot - z_n) \|_{\varepsilon_n, R_n} = 0
\]

up to a subsequence. Thus, for large \( n, \varepsilon_n z_n \in \mathcal{M}^B, \varphi_{\varepsilon_n}(\cdot - z_n) U_0(\cdot - z_n) \in X_{\varepsilon_n} \) and \( v_n \in X^d_{\varepsilon_n}, \) which contradicts the fact \( v_n \in X^d_{\varepsilon_n} \setminus X^d_d. \)

**□**

**Lemma 3.9.** For any given \( \nu > 0, \) there exist small positive constants \( \varepsilon_1 \) and \( d \) such that \( \Gamma'_{\varepsilon, \gamma, \pm}(v) > C_m - \nu \) for any \( v \in X^d_\varepsilon \) and \( \varepsilon \in (0, \varepsilon_1). \)

**Proof.** See [15].

**□**

**Lemma 3.10.** For sufficiently small \( \varepsilon > 0 \) and large \( R, \) there exists a sequence \( \{v^R_{\varepsilon, n}\} \subset E^R_\varepsilon \cap \Gamma^D_{\varepsilon, \gamma, \pm} \cap X^d_\varepsilon \) such that \( \| \Gamma'_{\varepsilon, \gamma, \pm}(v^R_{\varepsilon, n}) \|_{(E^R)'_\nu} \to 0 \) as \( n \to \infty. \)

**Proof.** The proof follows [15, 16]. However, for convenience the reader, we give a detailed proof. By contradiction, for small \( \varepsilon > 0 \) and \( R > R_0, \) there exists \( C(\varepsilon, R) > 0 \) such that

\[
\| \Gamma'_{\varepsilon, \gamma, \pm}(v) \|_{(E^R)'_\nu} \geq C(\varepsilon, R), \quad v \in E^R_\varepsilon \cap \Gamma^D_{\varepsilon, \gamma, \pm} \cap X^d_\varepsilon.
\]

On the other hand, by Lemma 3.8, there exists \( \delta > 0 \) independent of \( \varepsilon \in (0, \varepsilon_0) \) and \( R > R_0 \) such that

\[
\| \Gamma'_{\varepsilon, \gamma, \pm}(v) \|_{(E^R)'_\nu} \geq \delta, \quad v \in E^R_\varepsilon \cap \Gamma^D_{\varepsilon, \gamma, \pm} \cap (X^d_\varepsilon \setminus X^d_d).
\]

Thus, there exists a pseudo-gradient vector field \( \Upsilon^R_\varepsilon \) on neighborhood \( N^R_\varepsilon \subset E^R_\varepsilon \) of \( E^R_\varepsilon \cap \Gamma^D_{\varepsilon, \gamma, \pm} \cap (X^d_\varepsilon \setminus X^d_d). \) Let \( \tilde{N}^R_\varepsilon \subset N^R_\varepsilon \) such that

\[
\| \Gamma'_{\varepsilon, \gamma, \pm}(v) \|_{(E^R)'_\nu} \geq \frac{1}{2} C(\varepsilon, R), \quad v \in \tilde{N}^R_\varepsilon.
\]

We choose two positive Lipschitz continuous functions \( \zeta^R_\varepsilon \) and \( \xi \) satisfying

\[
\zeta^R_\varepsilon(v) = 1, \quad \text{if } v \in \Gamma^D_{\varepsilon, \gamma, \pm} \cap X^d_\varepsilon \quad \text{and} \quad \zeta^R_\varepsilon(v) = 0, \quad \text{if } v \in E^R_\varepsilon \setminus \tilde{N}^R_\varepsilon,
\]

\[
0 \leq \zeta^R_\varepsilon \leq 1, \quad \xi \leq 1, \quad \xi(a) = 1, \quad \text{if } |a - C_m| \leq \frac{1}{2} C_m \quad \text{and} \quad \xi(a) = 0, \quad \text{if } |a - C_m| \geq C_m.
\]

Now, we define

\[
\psi^R_\varepsilon = \begin{cases} 
\zeta^R_\varepsilon(v)\xi(v)\Upsilon^R_\varepsilon, & \text{if } v \in N^R_\varepsilon, \\
0, & \text{if } v \not\in E^R_\varepsilon \setminus N^R_\varepsilon. 
\end{cases}
\] (3.34)
Thus, \( \lim \) which implies that \( \eta = \varphi U \). Let \( \eta \), \( t > 0 \), \( t > 0 \), \( \varphi U \), \( s \in \mathbb{R} \) as before. Then, for the small \( d_1 > 0 \), there exists some \( \mu > 0 \) such that if \( |s - a_1| \leq \mu \), we have

\[
\| \eta - \varphi U \| = \| \varphi U - U \| \leq C \| U - U \| \leq d_1,
\]

which implies that \( \eta \in X_{\varepsilon, s} \) since \( s \in \mathbb{R} \). On the other hand, if \( |s - a_1| \geq \mu \), since \( t = 1 \) is the unique maximum point of \( L_{\varepsilon, s} \) and \( \max_{t \geq 0} L_{\varepsilon, s} = L_{\varepsilon, s} = C_m \), there exists \( \rho > 0 \) such that \( L_{\varepsilon, s} < C_m - 2\rho \) for \( |s - a_1| \geq \mu \). From Lemma 3.1, there exists \( \varepsilon > 0 \) such that

\[
\max_{s \in (0, 0)} | \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, s}) - L_{\varepsilon, s} | < \rho, \quad \varepsilon \in (0, \varepsilon_1).
\]

So, for \( |s - a_1| \geq \mu \), we have

\[
\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, s}) < | \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, s}) - L_{\varepsilon, s} | + L_{\varepsilon, s} < C_m - \rho, \quad \varepsilon \in (0, \varepsilon_1).
\]

Define \( \eta_{\varepsilon}^R(s, t) := F_{\varepsilon}^R(\eta_{\varepsilon}(s, t), (s, t) \in [0, 1] \times \mathbb{R} \). Since \( \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}(0)), \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}(1)) \notin (0, 2C_m) \), we get \( \eta_{\varepsilon}^R(s, t) \in \Phi_{\varepsilon} \) for any \( t > 0 \).

If \( |s - a_1| \geq \mu \), we have

\[
\Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, t)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}(s)) < C_m - \rho,
\]

which is a contradiction.

If \( |s - a_1| \leq \mu \), we get \( \eta_{\varepsilon}(s) \in X_{\varepsilon, s} \). In this case, one has (a) \( \eta_{\varepsilon}^R(s, t) \in X_{\varepsilon, s} \) for all \( t > 0 \) or (b) there exists some \( t_s > 0 \) such that \( \eta_{\varepsilon}^R(s, t) \notin X_{\varepsilon, s} \). If (a) holds, we have

\[
\Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, t)) = \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}(s)) + \int_0^t \frac{d}{dt} \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, \tau)) d\tau
\]

\[
\leq D_{\varepsilon, \gamma, \pm} - \min\{\delta^2, C(\varepsilon, R)^2\} t.
\]

Thus, \( \lim_{t \to +\infty} \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, t)) = -\infty \), which contradicts to Lemma 3.9. So, (b) holds.

For any fixed \( s \) with \( |s - a_1| \leq \mu \), we find \( t_1^1, t_1^2 > 0 \) such that \( \eta_{\varepsilon}^R(s, t) \in X_{\varepsilon, s} \) for \( t \in [t_1^1, t_1^2] \subset (0, t_s) \) for \( |t_1^1 - t_1^2| > \sigma \) for some \( \sigma > 0 \) dependent of \( d_0 \) and \( d_1 \). Thus,

\[
\Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, t_s)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}(s)) + \int_{t_1^1}^{t_1^2} \frac{d}{dt} \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, \tau)) d\tau
\]

\[
\leq D_{\varepsilon, \gamma, \pm} - \delta^2(t_1^2 - t_1^1)
\]

\[
\leq C_m - \frac{1}{2} \delta^2 \sigma, \quad t \in [t_1^1, t_1^2], \quad \text{if } |s - a_1| \leq \mu.
\]

Therefore, since \( [0, 1] \) is compact, using infinite covering theorem, for all \( s \in [0, 1] \), we can find \( t_{\varepsilon}^R \) such that \( \Gamma_{\varepsilon, \gamma, \pm}(\eta_{\varepsilon}^R(s, t_{\varepsilon}^R)) < C_m - \frac{1}{2} \delta^2 d_1 \). On the other hand, we
note that $\eta_\varepsilon^R(s, t_\varepsilon^R) \in \Phi_\varepsilon$. Thus,

$$C_{\varepsilon, \gamma, \pm} \leq \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t_\varepsilon^R)) < C_m - \rho,$$

which is a contradiction. □

**Lemma 3.11.** For sufficiently small $\varepsilon > 0$, there exists a critical point $v_\varepsilon \in X_{\varepsilon}^{d_0} \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_\varepsilon, \gamma, \pm}$ of $\Gamma_{\varepsilon, \gamma, \pm}$.

**Proof.** By Lemma 8, there exists $\varepsilon_0$ and $R_0 > 0$ such that there exists a sequence \( \{v_\varepsilon^{R_n}\} \subset E_{\varepsilon}^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_\varepsilon, \gamma, \pm} \cap X_{\varepsilon}^{d_0} \), such that \( \|\Gamma_{\varepsilon, \gamma, \pm}^{D_\varepsilon, \gamma, \pm}(v_\varepsilon^{R_n})\|_{(E_{\varepsilon}^R)^\prime} \to 0 \) as $n \to \infty$ for $\varepsilon \in (0, \varepsilon_0)$, and $R \in (R_0, +\infty)$. Clearly, \( \{v_\varepsilon^{R_n}\} \) is bounded in $H_0^1(B_{\varepsilon}^R(0))$ since $v_\varepsilon^{R_n} \in X_{\varepsilon}^{d_0}$. By defining $v_\varepsilon^{R_n} = 0$ on $\mathbb{R}^N \setminus B_{\varepsilon}^R(0)$, $v_\varepsilon^{R_n}$ can be regarded as an element in $E_{\varepsilon}$. Up to a subsequence if necessary, we may assume $v_\varepsilon^{R_n} \rightharpoonup v_\varepsilon^R$ in $H_0^1(B_{\varepsilon}^R(0))$, $v_\varepsilon^{R_n} \to v_\varepsilon^R$ in $L^p(B_{\varepsilon}^R(0))$, $p \in [1, 2^*)$ and $v_\varepsilon^{R_n} \to v_\varepsilon^R$ a.e. in $\mathbb{R}^N$. Thus $v_\varepsilon^R$ is a solution of

$$-\Delta v = \frac{h(G_{\varepsilon, \gamma, \pm}^{-1}(v)) - V(\varepsilon x)G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} - 4\left(\int_{B_{\varepsilon}^R(0)} \chi_\varepsilon|v|^2 dx - 1\right) \chi_\varepsilon v, \quad x \in B_{\varepsilon}^R(0). \tag{3.36}$$

From (3.36) we have $v_\varepsilon^{R_n} \rightharpoonup v_\varepsilon^R$ in $H_0^1(B_{\varepsilon}^R(0))$ and $v_\varepsilon^R \in X_{\varepsilon}^{d_0} \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_\varepsilon, \gamma, \pm}$. By the maximum principle, $v_\varepsilon^R > 0$. In consequence of $(h_1)-(h_3)$ and $|G_{\varepsilon, \gamma, \pm}^{-1}(t)| \leq \lambda_2 |t|$ for all $t \in \mathbb{R}$, any positive solutions of (3.36) satisfies

$$-\Delta v \leq C v^p, \quad x \in B_{\varepsilon}^R(0),$$

where $C > 0$ depends only on $h$. In particular, $-\Delta v_\varepsilon^R \leq C(v_\varepsilon^R)^p, \quad x \in B_{\varepsilon}^R(0)$. By applying the standard Moser iteration argument (see [17]), \( \{v_\varepsilon^R\} \) is bounded in $L_{loc}^p(\mathbb{R}^N)$ uniformly on $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$ for any $p < \infty$ and is bounded in $L^\infty$. Thus, since $\|v_\varepsilon^R\|_\infty$ and $\{\Gamma_{\varepsilon, \gamma, \pm}(v_\varepsilon^R)\}$ are bounded, we get $\{Q_\varepsilon(v_\varepsilon^R)\}$ is uniformly bounded on $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. So, we have

$$\int_{\mathbb{R}^N \setminus B_{R_0}(0)} |v_\varepsilon^R|^2 dx \leq \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |v_\varepsilon^R|^2 dx + \varepsilon \int_{\mathbb{R}^N} \chi_\varepsilon|v_\varepsilon^R|^2 dx \leq \varepsilon C$$

for any $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Thus, for $|x| \geq \frac{R_0}{\varepsilon}$ and $R \geq R_0$, we have

$$h(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon^R)) \leq \frac{V_0}{2} G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon^R).$$

Similarly to Proposition 3 in [8], we get

$$\lim_{A \to +\infty} \int_{\mathbb{R}^N \setminus B_{A}(0)} \|\nabla v_\varepsilon^R|^2 + V_\varepsilon(x)|G_{\varepsilon}^{-1}(v_\varepsilon^R)|^2 \| dx = 0 \tag{3.37}$$

uniformly on $R \geq R_0$. Let $v_k = v_{\varepsilon_k}^R$ and $R_k \to +\infty$ as $k \to \infty$. Then $\{v_k\}$ is bounded in $E_{\varepsilon_k}$ and we may assume that $v_k \rightharpoonup v_{\varepsilon_k}$ in $H_1(\mathbb{R}^N)$, $v_k \to v_{\varepsilon_k}$ a.e. in $\mathbb{R}^N$. By iterating the above argument, we get $\|v_{\varepsilon_k}\|_\infty \leq C$. Since $v_k$ satisfies (3.36) and
using (3.37), we get \( \|v_k - v_\varepsilon\|_\varepsilon \to 0 \) as \( k \to \infty \). Thus, \( v_\varepsilon \in X^d_{\varepsilon} \cap \Gamma_{\varepsilon, \gamma, \pm} \). Moreover, \( \Gamma'_{\varepsilon, \gamma, \pm}(v_\varepsilon) = 0 \). \( \square \)

4. Proof of Theorem 1.1: the subcritical case

By Lemma 3.11, for small \( \varepsilon > 0 \), there exists a nonnegative solution \( v_\varepsilon \) to

\[
-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}(v))} = \frac{h(G_{\varepsilon, \gamma, \pm}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}(v))} - 4 \left( \int_{\mathbb{R}^N} \chi_{\varepsilon} v^2 dx - 1 \right) + \chi_{\varepsilon} v, \tag{4.1}
\]

in \( \mathbb{R}^N \). By the maximum principle we have \( v_\varepsilon > 0 \). Since \( v_\varepsilon \in X^d_{\varepsilon} \), by Moser’s iteration and Lemma 2.2, \( \{v_\varepsilon\} \) is uniformly bounded in \( L^\infty(\mathbb{R}^N) \) for small \( \varepsilon > 0 \). Moreover, there exists \( \alpha > 0 \) such that \( \|v_\varepsilon\|_\infty \geq \alpha \), see e.g. [35]. Thus, by choosing \( \varepsilon > 0 \) sufficiently small, we get

\[
g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}(v_\varepsilon)) = 1 \pm \varepsilon \gamma G_{\varepsilon, \gamma, \pm}(v_\varepsilon)^2.
\]

As in the proof of Lemma 3.7, there exist a sequence \( \{\varepsilon_n\} \subset \mathbb{R}^N \) such that \( \varepsilon \varepsilon_n \in \mathcal{M}^2 \beta \) and for any sequence \( \varepsilon_n \to 0 \) there exist \( y_0 \in \mathcal{M} \) and \( U_0 \in S_m \) satisfying

\[
\lim_{n \to \infty} |\varepsilon_n z_{\varepsilon_n} - y_0| = 0,
\]

\[
\lim_{n \to \infty} \|v_{\varepsilon_n} - \varphi_{\varepsilon_n}(-z_{\varepsilon_n})U_0(-z_{\varepsilon_n})\|_{\varepsilon_n} = 0
\]

up to a subsequence. Thus, we get

\[
\lim_{n \to \infty} \|v_{\varepsilon_n}(\cdot + z_{\varepsilon_n}) - U_0\|_{H^1(\mathbb{R}^N)} = 0,
\]

which implies that for given \( \sigma > 0 \), there exist \( R > 0 \) and \( \varepsilon_0 > 0 \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{\mathbb{R}^N \setminus B_R(0)} v_{\varepsilon_n}^2(x + z_{\varepsilon_n}) dx \leq \sigma.
\]

Setting \( w_\varepsilon = v_{\varepsilon}(\cdot + z_{\varepsilon}) \), we have \( -\Delta w_\varepsilon \leq M w_\varepsilon \). Hence, from [17], there exists a constant \( C_0 = C_0(N, C) \) such that

\[
\sup_{B_1(y)} w_\varepsilon \leq C_0 \|w_\varepsilon\|_{L^2(B_2(y))} \quad \text{for all } y \in \mathbb{R}^N.
\]

Since \( \|w_{\varepsilon}\|_{L^2(B_2(y))} \to 0 \), as \( |y| \to \infty \), we conclude that \( w_{\varepsilon}(x) \to 0 \), as \( |x| \to \infty \). Let \( y_{\varepsilon} \) be a maximum point of \( w_{\varepsilon}(x) \), then \( \{y_{\varepsilon}\} \) is bounded. By the comparison principle we have

\[
w_{\varepsilon}(x) \leq C \exp(-c|x|) \quad \text{for all } x \in (0, \varepsilon_0).
\]

Hence, \( x_{\varepsilon} := y_{\varepsilon} + z_{\varepsilon} \) is a maximum point of \( v_{\varepsilon} \) and

\[
v_{\varepsilon}(x) = w_{\varepsilon}(x - x_{\varepsilon}) \leq C \exp(-c|x - x_{\varepsilon}|).
\]

Therefore, \( Q_{\varepsilon}(v_{\varepsilon}) = 0 \) for small \( \varepsilon > 0 \) and \( v_{\varepsilon} \) is a positive critical point of \( P_{\varepsilon, \gamma, \pm} \). Hence, \( w_{\varepsilon, \gamma} = G^{-1}_{\varepsilon, \gamma, \pm}(v_{\varepsilon}) \) is a positive solution of (2.1). Let \( y_{\varepsilon} \to y_0 \) and \( x = y_0 + x_0 \), then

\[
\|v_{\varepsilon, \gamma}(\cdot + x_{\varepsilon}) - U(\cdot + y_0)\|_{H^1} \to 0.
\]

Furthermore,

\[
\|w_{\varepsilon, \gamma}(\cdot + x_{\varepsilon}) - U(\cdot + y_0)\|_{H^1}^2 \leq C \|v_{\varepsilon, \gamma}(\cdot + x_{\varepsilon}) - U(\cdot + y_0)\|_{H^1}^2 \to 0.
\]

The results follow by letting \( u_{\varepsilon, \gamma}(x) = w_{\varepsilon, \gamma}(\frac{x}{\varepsilon}) \).
5. Proof of Theorem 1.1: the critical case

In this Section, we deal with the existence and concentration of solutions to (1.1). We use a truncation approach, see also [37]. In the following, we first show the existence and qualitative properties of ground state solutions to the associated limit problem (1.5).

In [25], the authors proved that, if \((h_1)\) and \((h_4)-(h_5)\) hold, then there exists a ground state solution of (1.5) and the least energy \(E_m\) is strictly less than \(\frac{S_N}{2}/N\), where \(S\) is the Sobolev best constant, namely
\[
S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.
\]

Moreover, let
\[
\Upsilon(u) = \int_{\mathbb{R}^N} \left( H(u) - \frac{m}{2} u^2 \right) \, dx, \quad u \in H^1(\mathbb{R}^N)
\]
and
\[
M := \inf \left\{ T(u) : \Upsilon(u) = 1, \quad u \in H^1(\mathbb{R}^N) \right\}.
\]

Similarly to [38], we have
\[
E_m = \frac{2}{N} M^\frac{N}{2} \left( \frac{N-2}{N} \right)^\frac{N-2}{2}.
\]

and then
\[
0 < M < \frac{1}{2} \left( 2^* \right)^\frac{N-2}{N} S.
\]

Let \(S_m\) be the set of ground state solutions \(U\) of (1.5) satisfying \(U(0) = \max_{x \in \mathbb{R}^N} U(x)\). Then \(S_m \neq \emptyset\) and analogously to [9], the following result holds true:

**Proposition 5.1.**

(1) \(S_m\) is compact in \(H^1(\mathbb{R}^N)\);
(2) \(0 < \inf \{ \|U\|_\infty : U \in S_m \} \leq \sup \{ \|U\|_\infty : U \in S_m \} < \infty\).

By Proposition 5.1 there exists \(\kappa > 0\) such that
\[
\sup_{U \in S_m} \|U\|_\infty < \kappa. \tag{5.2}
\]

In the following, for any fixed \(k > \max_{t \in [0,\kappa]} h(t)\), we consider the truncated problem
\[
- \Delta u + mu = h_k(u), \quad u \in H^1(\mathbb{R}^N), \tag{5.3}
\]
where
\[
h_k(t) := \min \{ h(t), k \}, \quad t \in \mathbb{R}.
\]

Let \(S_m^k\) be the set of positive ground state solutions \(U\) of (5.3) satisfying \(U(0) = \max_{x \in \mathbb{R}^N} U(x)\). Then, similarly to [39], we have

**Lemma 5.2.**

\(S_m^k = S_m\) if \(k > \max_{t \in [0,\kappa]} h(t)\).
Now we are in the position to prove the critical part of Theorem 1.1 by means of a truncation approach. Precisely, let us fix $k > \max_{t \in [0, \kappa]} f(t)$ and consider the truncated problem

$$- \Delta w + V_\varepsilon(x)w \equiv \varepsilon^\gamma w \Delta w^2 = h_k(w), \quad x \in \mathbb{R}^N,$$

Similarly to Section 2, by the change of variables $w = G_{\varepsilon, \gamma, \pm}(v)$, $v$ satisfies

$$- \Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} = \frac{h_k(G_{\varepsilon, \gamma, \pm}^{-1}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (5.5)$$

In view of Lemma 5.2, $S_m^k = S_m$. For some fixed $\beta > 0$ small and a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$ and $\varphi(x) = 0$ for $|x| \geq 2\beta$. Let $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$, $y \in \mathbb{R}^N$ and for each $x \in \mathcal{M}^\beta$ and $U \in S_m$, similarly as above, we define a set of approximating solutions by

$$X_\varepsilon = \{U_{\varepsilon}^x(y) \mid x \in \mathcal{M}^\beta, U \in S_m\},$$

where

$$U_{\varepsilon}^x(y) = \varphi_\varepsilon \left(y - \frac{x}{\varepsilon}\right) U \left(y - \frac{x}{\varepsilon}\right).$$

Moreover, it follows from Theorem 1.1 that for some small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, (5.5) admits a positive solution $v_\varepsilon$ satisfying the following property: there exist $U \in S_m$ and a maximum point $x_\varepsilon \in \mathbb{R}^N$ of $v_\varepsilon$, such that $\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$ and $v_\varepsilon(\varepsilon \cdot + x_\varepsilon) \to U(\cdot + z_0)$ as $\varepsilon \to 0$ in $H^1(\mathbb{R}^N)$ for some $z_0 \in \mathbb{R}^N$. Clearly, $w_\varepsilon(\cdot) = v_\varepsilon(\varepsilon \cdot + x_\varepsilon)$ satisfies

$$- \Delta w_\varepsilon + V_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))} = \frac{h_k(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))}, \quad x \in \mathbb{R}^N.$$

Without loss of generality, we can choose $\varepsilon_0 > 0$ small enough such that $\frac{1}{2} \leq G_{\varepsilon, \gamma, \pm}(t) \leq \frac{3}{2}$ uniformly for $t \in \mathbb{R}$ and $\varepsilon < \varepsilon_0$. Thanks to the fact that $h_k(t) \leq k$ for all $t \in \mathbb{R}$ and Lemma 2.2, in the weak sense, $w_\varepsilon$ satisfies $- \Delta w_\varepsilon + \frac{3}{2} V_0 w_\varepsilon \leq 2k$, $x \in \mathbb{R}^N$. Then it follows from standard regularity estimates (see [17]) that $w_\varepsilon(\cdot) \to U(\cdot + z_0)$ in $C^{1, \alpha}_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$. In particular, $w_\varepsilon(0) \to U(z_0)$ as $\varepsilon \to 0$. Then it follows from (5.2) that $\|v_\varepsilon\|_\infty = w_\varepsilon(0) < \kappa$ holds for sufficiently small $\varepsilon > 0$. Recalling that $0 \leq G_{\varepsilon, \gamma, \pm}(t) \leq t$ for $t \geq 0$ and sufficiently small $\varepsilon > 0$, we know $h_k(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon(x))) \equiv h(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon(x)))$, $x \in \mathbb{R}^N$ for sufficiently small $\varepsilon > 0$. Therefore, $v_\varepsilon$ turns out to be a positive solution of the original problem

$$- \Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} = \frac{h(G_{\varepsilon, \gamma, \pm}^{-1}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))}, \quad x \in \mathbb{R}^N.$$

The proof of Theorem 1.1 is now complete. \hfill \Box

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