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A Theory of Hyperfinite Sets

Abstract

We develop an axiomatic set theory — the Theory of Hyperfinite Sets THS, which is based on the idea of existence of proper subclasses of big finite sets. We demonstrate how theorems of classical continuous mathematics can be transferred to THS, prove consistency of THS and present some applications.

Introduction

Many applications of nonstandard analysis are based on the simulation of infinite structures by hyperfinite ones. When translated into the language of standard mathematics such simulation means an approximation of infinite structures by finite ones. Thus, nonstandard analysis provides us with a machinery that allows to obtain new results about infinite structures using such approximations and corresponding results about finite structures. The latter are often much easier to obtain. This approach is implemented in the famous monograph [14] for the construction of probability theory on infinite probability spaces. In the monograph [7] it was shown how this approach can be used for systematic construction of harmonic analysis on locally compact abelian groups starting from harmonic analysis on finite abelian groups.

The results obtained on this way allow to look at this approach from another point of view.

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According to this point of view Mathematics should be developed on the base of the hypothesis that all sets are finite (some kind of the ancient Greeks’ atomism).

The historically first approach due to A. Yessenin-Volpin [20], [21] to develop this idea on the base of modern logic is called ultraintuitionism. It assumes the existence of the maximal natural number. This approach stimulated investigations of the notion of feasible numbers. The first mathematically rigorous formalization of the notion of feasibility of natural numbers was introduced by R. Parikh [15]. Many papers develop R. Parikh’s approach as well as some other approaches to the notion of feasibility (see e.g. [8], [4], [16]). We do not discuss them here. A very interesting discussion of correlation between the Real Analysis and the Discrete Analysis is contained in [22]. The main idea of this paper is as follows: ”Continuous analysis and geometry are just degenerate approximations to the discrete world... While discrete analysis is conceptually simpler ... than continuous analysis, technically it is usually much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics....”. In some sense, our paper together with the paper [6] contributes to this idea.

In this paper we develop an axiomatic theory of finite sets, which we call the Theory of Hyperfinite Sets (THS), by the reasons explained below. Similarly to Kelley-Morse theory or von Neumann - Bernays - Gödel theory (NBG), THS is a theory of classes in the $\in$-language, where sets are defined as elements of classes. The universe of sets satisfies all the axioms of $\text{ZF}^{\text{fin}}$ – the theory obtained by replacing in $\text{ZF}$ the axiom of infinity by its negation and adding a suitable form of regularity (for instance, an axiom saying that every set has a transitive closure; see [17]).

However, the properties of classes differ essentially from those of NBG. For example, the Separation Axiom fails in THS: there exist sets that contain proper subclasses (subclasses that are not subsets). The reason why we need to include the last statement in our theory is that we want to consider in THS such properties as feasibility discussed above. Indeed, let $F(x)$ be the statement ”$x$ is a feasible number” and $N$ be a non-feasible number. Then the set $A = \{x \leq N \mid F(x)\}$ satisfies the following inconsistent conditions: 1) $0 \in A$, 2) $\forall x(x \in A \rightarrow x + 1 \in A)$, 3) $N \notin A$. The only way to avoid this paradox, if one wants to keep the induction principle for sets, is to assume that $A$ is not a set, and thus the separation axiom fails for the finite set $\{0, 1, \ldots, N\}$.

The paradox discussed in the previous paragraph is a version of the well-known paradox about a pile of sand, due to Eubilides, IV century B.C.: since one grain of sand is not a pile and if $n$ grains of sand do not form a pile of sand, then $n + 1$ grains do not form a pile of sand also, then how can we get a pile of sand? The
paradoxes of these type can not be considered in the framework of classical
set theory since the objects, like a pile of sand, have a very vague description
and, thus, cannot be considered as any objects of classical mathematics, i.e.
as sets. On the other hand there are many examples that show that such
notions arise very naturally in mathematics (see, e.g., the example concerning
the feasibility above). The first mathematician who realized the importance
of the notions of this type was P. Vopěnka. In [18] he introduced the first theory
of finite sets, the Alternative Set Theory (AST), where the existence of finite
sets containing subclasses that are not sets was postulated. Such subclasses of
sets are called *semisets*.

The main defect of P. Vopěnka’s approach is the opposition of his theory to
classical mathematics. As it was mentioned above (see the quotation from [22])
the advantage of the continuous mathematics in comparison with the discrete
one is its simplicity that often allows to solve problems concerning discrete
objects.

**THS** introduced here is also based on the idea of existence of proper subclasses
of big finite sets. Finite sets that contain proper subclasses are called hyper-
finite sets. This term is borrowed from nonstandard analysis. The primary
model for **THS** is the collection of all subclasses of the set of hereditarily finite
sets in the Nonstandard Class Theory NCT [1]. The central notion of a thin
class is defined by a formulation equivalent in NCT to the definition of a class
of standard size: a class is thin if any subset of it does not contain proper
subclasses. Sets that do not contain proper subclasses are called *small*. The
class of all small natural numbers is a thin class. It coincides with the set $\omega$
in **ZF**. Under our approach the class of all small numbers can be interpreted
as the class of feasible numbers.

We prove that all results of classical mathematics that can be formalized
in Zermelo set theory can be proved for thin structures in **THS**. This is a
substantial difference between **THS** and AST. It allows to formalize within **THS**
those proofs of theorems about finite sets that use continuous mathematics
and, hence, it is not necessary to invent any new proofs for such theorems.

In the discrete world continuous objects have their place as well: they origi-
nate from hyperfinite sets or their $\sigma$-subclasses as quotient ”sets” by some
*indiscernibility* relation. An indiscernibility relation $\rho$ is an equivalence rela-
tion that is a $\pi$-class and satisfy some special condition (see section 7). A class
is called a $\sigma$-class ($\pi$-class) if it can be represented by the union (the intersec-
tion) of a thin class. We prove that there exists a thin class of representatives
of all $\rho$-equivalence classes, which represent the quotient ”set” (more exactly,
the quotient system of classes) by $\rho$.

For example, to obtain the field of reals $\mathbb{R}$ in **THS** one should consider a
computer arithmetic implemented in an idealized computer with a hyperfinite memory for simulation of the field of reals. It may be the usual computer arithmetic, based on the representation of reals in the form with floating point. Let \( \langle R; \oplus, \odot \rangle \) be this system. It is well-known that because of the rounding off the operations \( \oplus \) and \( \odot \) are neither associative, nor distributive. Let \( R_b = \{ x \in R \mid \exists \text{small } n (|x| < n) \} \), where \( \exists \text{small } n \) means "there exists a small natural number \( n \)". Since the class of all small natural numbers is a thin class, it is easy to see that \( R_b \) is a \( \sigma \)-class. We can interpret the elements of \( R_b \) as the computer numbers that are far enough from the boundary of the computer’s memory, so that doing computations with these numbers one can never get overfilling of memory. Indeed, it can be proved that the class \( R_b \) is closed under the operations \( \oplus \) and \( \odot \). The indiscernibility relation \( \rho \) is defined by the condition \( x \rho y \iff \forall \text{small } n (|x-y| < \frac{1}{n}) \). Obviously \( \rho \) is a \( \pi \)-class. It also has the natural interpretation: we identify those numbers that differ on a number close enough to the computer zero. It can be proved (in \( \text{THS} \)) that the quotient system \( R_b/\rho \) is isomorphic to the field \( \mathbb{R} \).

Certainly, there are many other systems, from which one can obtain \( \mathbb{R} \) in the way similar to one described in the previous paragraph. The system based on representation of reals in the form with floating point is discussed in details in [5]. In this paper we introduce a hyperfinite system that is a little bit simpler and has some better properties - it is an abelian group for addition. However, we proved that it is impossible to obtain the field \( \mathbb{R} \) from a hyperfinite system that is an associative ring [6]. Similar facts hold also for many locally compact non-commutative groups. It is shown in [6] that we cannot find the hyperfinite groups that have approximate properties of many important Lie groups such as \( SO(3) \). The hyperfinite objects with the best properties, from which all unimodular locally compact groups can be constructed, are quasigroups (latin squares) [6]. These facts demonstrate that the continuous world has better properties than the discrete one. The theory \( \text{THS} \) introduced here allows to formalize not only all classical mathematics, but also the statements about the connection between the discrete world and its continuous approximation.

In [3] the Non Standard Regular Finite Set Theory was formulated by S. Baratella and R. Ferro. Based on a countably saturated universe of hereditarily hyperfinite sets, NRFST contains a rich structure of external sets over it. We believe that a theory of classes of higher levels over the universe of hyperfinite sets of \( \text{THS} \) can be formulated in the pure \( \in \)-language and simulated \textit{within} \( \text{THS} \).

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1 Axioms

1.1 Remark. In [1, §6] we announced a theory of hyperfinite sets \( \text{THS} \). The theory presented here is a result of further development of the idea; the implementation is very different though and, we believe, much more interesting than the one described in [1].

1.2 \( \text{THS} \) is a first-order theory. Semantical objects of the theory are classes. Its language contains only one non-logical symbol — the binary predicate symbol \( \in \) of the membership.

1.3 Sets are defined as members of classes:

\[
\text{Set}(X) \equiv \exists Y \ (X \in Y).
\]

We accept the convention to use small letters for sets and capital letters for classes.

1.4 Formulas where all quantifiers range over set variables are called normal formulas.

1.5 Set formulas are normal formulas where no class constants or variables occur.

Axiom of Extensionality:

\[
\Box \text{Ext} \forall X \forall Y \left( \forall x \ (x \in X \iff x \in Y) \rightarrow X = Y \right).
\]

Axioms of class formation (arbitrary formulae are allowed):

\[
\Box \text{Class} \forall X_1, \ldots, X_n \exists Y \forall y \ (y \in Y \iff \Phi(y, X_1, \ldots, X_n)).
\]

Axiom of set formation:

\[
\Box \text{Set} \quad \text{Set} \emptyset \& \forall x \forall y \ \text{Set}(x \cup \{y\}).
\]

Axioms of induction and regularity (only set formulae are allowed):

\[
\Box \text{Ind} \quad \varphi(\emptyset) \& \forall x \forall y \ (\varphi(x) \& \varphi(y) \rightarrow \varphi(x \cup \{y\}) \rightarrow \forall x \ \varphi(x).
\]

1.6 The class of all sets is denoted by \( \mathbb{H} \).

1.7 Subclasses of sets are called semisets.
Sets in THS may contain proper subsemisets, i.e. subclasses which are not sets.

1.8 A set is called small if it does not contain proper subsemisets:
\[
\text{small } x \iff \forall Y \subseteq x \ (\text{Set } Y).
\]

1.9 A class \( X \) which is not small is called infinitely large or simply infinite (inf \( X \)).
\[
\text{inf } X \iff \exists Y \subseteq X \ (\neg \text{Set } Y).
\]

1.10 A class is called thin iff every subset of it is small:
\[
\text{thin } X \iff \forall a \subseteq X \forall C \subseteq a \ (\text{Set } C).
\]

Thus, thin set is the same as small set.

In formulas we use quantifiers with superscripts thin, small and inf in a natural way.

**Axiom of Thin Semisets:**

\[
\text{Thin} \quad \forall X \ (\text{thin } X \to \exists x \ (X \subseteq x)).
\]

**Axiom of Compactness:**

\[
\text{Comp} \quad \forall \text{thin } X \forall u \ (u \subseteq \bigcup X \to \exists x \subseteq X \ (u \subseteq \bigcup x)).
\]

**Axiom of Exponentiation:**

\[
\text{Exp} \quad \forall \text{thin } X \exists \text{thin } P \forall y \exists p \in P \ (y \cap X = p \cap X).
\]

1.11 We define the ordered pair \( \langle x, y \rangle \) def \{ \{ x \}, \{ x, y \} \} and the operations \( \times \) (cartesian product), \( \text{dom} \) (domain), " (image: \( X" A = \{ b : \exists a \in A \ (\langle a, b \rangle \in X) \} \) in the usual way. We also use \( \text{Fnc } F \) as a shorthand for \( \forall x \forall y \forall z \ (\langle x, y \rangle \in F \ & \langle x, z \rangle \in F \to y = z) \).

**Axiom of Choice:**

\[
\text{Choice} \quad \forall X \ (\text{thin } \text{dom}(X) \to \exists F \ (\text{Fnc } F \ & \ \text{dom}(F) = \text{dom}(X) \ & \ F \subseteq X)).
\]
1.12 The class $\text{SN}$ of small natural numbers is defined as the smallest class which contains the empty set and is closed under the von Neumann successor operation:

$$\text{SN} = \{ x : \forall N \left( \exists \varnothing \in N \land \forall n \in N \left( n \cup \{ n \} \in N \right) \rightarrow x \in N \right) \}. $$

Axioms of Dependent Choices (arbitrary formulae are allowed):

$$\text{DC} \quad \forall X \exists Y \Phi(X, Y) \rightarrow \forall X_0 \exists Z \left( Z'' \{ \varnothing \} = X_0 \land \forall n \in \text{SN} \Phi(Z'' \{ n \}, Z'' \{ n \cup \{ n \} \}) \right).$$

1.13 Similarly to small natural numbers, the class $\text{S}$ of all standard sets is defined as the smallest class containing the empty set and closed under the operation of adjoining one element:

$$\text{S} = \{ x : \forall S \left( \exists \varnothing \in S \land \forall a \in S \forall b \in S \left( a \cup \{ b \} \in S \right) \rightarrow x \in S \right) \}. $$

Axioms of Transfer (only set formulae are allowed):

$$\text{T} \quad \forall t_1 \in \text{S} \cdots \forall t_n \in \text{S} \left( \exists x \varphi(x, t_1, \ldots, t_n) \rightarrow \exists x \in \text{S} \varphi(x, t_1, \ldots, t_n) \right).$$

1.14 We denote

$$\text{TFS} = \text{Ext} + \text{Class} + \text{Set} + \text{Ind}$$

$$\text{THS}_0 = \text{TFS} + \text{Thin} + \text{Comp}$$

$$\text{THS} = \text{THS}_0 + \text{Exp} + \text{Choice} + \text{DC} + \text{T}$$

1.15 Remark. Axioms of $\text{TFS}$ are borrowed from Vopěnka’s AST. Thin and Comp are true in AST for countable classes. See also 2.11, 2.14 and 3.3.

1.16 Remark. The axioms of transfer are not as important in $\text{THS}$ as in other non-standard frameworks because standard sets are not the primary object of investigation here. The main reason for including them into the list of axioms is Theorem 3.8.

2 Basic facts and notions

2.1 Natural numbers are defined the same way as ordinals are defined in $\text{ZF}$: they are transitive sets linearly ordered by the membership relation. The class
of all natural numbers is denoted by \( \mathbb{N} \). **Ind** implies induction over \( \mathbb{N} \) for any set-formula \( \varphi \):

\[
[ \varphi(\emptyset) \land \forall n \in \mathbb{N} ( \varphi(n) \rightarrow \varphi(n \cup \{ n \} ) ) ] \rightarrow \forall n \in \mathbb{N} \varphi(n).
\]

2.2 **Ind** implies also that for any set \( x \) its size \( \#(x) \) is uniquely defined as a natural number \( k \) such that there is a set-bijection from \( x \) onto \( k \).

2.3 Theorem.

(1) All axioms of TFS hold in \( \mathcal{S} \).

(2) The universe \( \mathbb{H} \) of all sets and the universe \( \mathcal{S} \) of standard sets both satisfy the axioms of \( \mathsf{ZF}^\text{fin} \).

*Proof.* It follows from the definition of \( \mathcal{S} \) that the axioms of TFS are true in \( \mathcal{S} \). Sochor[17] proved that \( \mathsf{ZF}^\text{fin} \) is equivalent to the theory with the axioms **Set**, **Ind** and extensionality for sets. \( \square \)

2.4 Proposition (TFS). There exists a bijective mapping \( \text{ac} \) from the universe \( \mathbb{H} \) of all sets onto the class \( \mathbb{N} \) of natural numbers, definable by a set formula and such that \( x \in y \) implies \( \text{ac}(x) < \text{ac}(y) \) for all sets \( x \) and \( y \).

*Proof.* It can be proved in \( \mathsf{ZF}^\text{fin} \) that the Ackermann encoding of finite sets defined inductively by the conditions \( \text{ac}(\emptyset) = 0 \) and \( \text{ac}(x) = \sum_{a \in x} 2^{\text{ac}(a)} \) is a total bijection. \( \square \)

2.5 Proposition (TFS).

(1) \( \mathcal{S} \) is a thin class;

(2) \( \mathcal{S} \) coincides with the class of hereditarily small sets;

(3) \( \mathcal{S} \mathbb{N} = \{ n \in \mathbb{N} : \text{small } n \} = \mathcal{S} \cap \mathbb{N} \).

Many properties of small sets and thin classes can be proved in TFS already.

2.6 Proposition (TFS).

(1) \( \text{small } x \land y \subseteq x \rightarrow \text{small } y \); \( \text{thin } X \land Y \subseteq X \rightarrow \text{thin } Y \);

(2) \( \text{small } x \leftrightarrow \text{small } \#(x) \);

(3) \( \text{small } x \rightarrow \text{small } F \upharpoonright x \land \text{small } F''x \), for any function \( F \);

(4) \( \forall x \in \mathbb{N} \exists x ( X \subseteq x \land \#(x) \leq n ) \rightarrow \text{thin } X \);

(5) \( \text{small } \cup x \rightarrow \text{small } x \); \( \text{inf } x \rightarrow \text{inf } \cup x \);

(6) \( \text{thin } X \rightarrow \text{thin } \{ y : y \subseteq X \} \);

(7) \( [ \text{thin } X \land \forall x \in X \ \text{thin } Y'' \{ x \} ] \rightarrow \text{thin } Y \upharpoonot X \);

(8) \( \text{thin } X \land \text{thin } Y \rightarrow \text{thin } X \cup Y \land \text{thin } X \times Y \);
Proof. (2). \[\rightarrow\] Assume \(\inf\sharp(x)\). Let \(f\) be some set-bijection from \(\sharp(x)\) onto \(x\). Then the class \(Y = \{ f(n) : n \in \mathbb{N} \}\) is a proper semiset, since otherwise \(\mathbb{N} = f^{-1}''Y\) would be a set. Thus, \(x\) is infinite.

\[\leftarrow\] One should proceed by induction on \(\sharp(x)\) over \(\mathbb{N}\). Due to \textbf{Set} adjoining one element to a small set gives a small set again.

(3). We use (2) and proceed by induction over \(\sharp(x)\).

(4). Let \(X\) be not thin. Then, by definition of thin class, there exists an infinite subset \(y \subseteq X\). Therefore, every superset \(x \supseteq X\) cannot contain less than \(\sharp(y)\) elements.

(5). Let \(y = \cup x\) be a small set. Then \(\sharp(x) \leq 2^\sharp(y)\). Since, by (2), \(\sharp(y)\) is small, \(\sharp(x)\) is small and \(x\) is small.

(6). Denote \(Y = \{ y : y \subseteq X \}\) and assume \(u \subseteq Y\) is infinite. Then, according to (5), \(\cup u\) is infinite as well. But \(\cup u \subseteq X\), in contradiction with the fact that \(X\) is thin.

(7). Let the left hand side of the implication holds. Take any set \(a \subseteq Y \upharpoonright X\). Then \(\text{dom } a\) is small, since \(\text{thin } X \supseteq \text{dom } a\). By (2) the numbers \(p = \sharp(\text{dom } a)\) and \(q = \max\{ \sharp(a"u) : u \in \text{dom } a \}\) are small. Hence, \(\sharp(a) \leq p \cdot q\) is also small, and \(a\) is small. This proves that \(Y \upharpoonright X\) is thin. By (8), \(Y"X\) is also thin. \(\square\)

2.7 Proposition ([THS\(_0\) ]). The following statements are equivalent for any class \(X\):

(1) \(X\) is a thin class (all subsets of \(X\) are small);
(2) \(\forall\inf n \in \mathbb{N} \exists a \ ( X \subseteq a \ & \ \sharp(a) = n )\);
(3) \(\forall Y \subseteq X \exists y \ ( Y = y \cap X )\).

Proof. Assume \(X\) is thin, \(Y \subseteq X\) and \(n \in \mathbb{N}\) is infinitely large. Taking into account item (4) of Proposition 2.6, it is enough to show that

\[\exists y \subseteq x \ ( y \cap X = Y \ & \ \sharp(y) \leq n ).\]  \hfill (1)

By \textbf{Thin} \(X \subseteq x\) for some infinite set \(x\). Denote \(s = \{ y \subseteq x : \sharp(y) \leq n \}\), \(D = \{ y \in s : a \notin y \} : a \in Y\}\), \(D = \{ y \in s : b \in y \} : b \in X \setminus Y\}.\) \(D\) is thin since for every set \(t \subseteq D\) there is a set \(\cup \{ x \setminus d : d \in t \} \subseteq Y\) of the same size. Similarly, \(\bar{D}\) is thin as well. Suppose (1) does not hold. Then \(\cup(D \cup \bar{D}) \supseteq s\). Hence, by \textbf{Comp}, \(\cup t \supseteq s\) for some small \(t \subseteq D \cup D\) which is impossible because \(\sharp(t) < \sharp(x)\). \(\square\)

As an immediate corollary we get the following proposition.
2.8 Proposition (THS\(_0\) + Exp). \(\forall\, \text{thin} \, X \, \exists\, \text{thin} \, P \, \forall Y \subseteq X \, \exists p \in P \, (Y = p \cap X).\)

2.9 Proposition (THS\(_0\)).

(1) \(\text{thin} \, X \rightarrow \text{thin} (\text{dom} \, X);\)
(2) \(\text{thin} \, X \rightarrow \big[\, \text{thin} \, F'' \, X \, \& \, \forall y \subseteq F'' \, X \, \exists x \subseteq X \, (y = F'' \, x) \big],\) for any function \(F;\)
(3) \(\text{thin} \, X \rightarrow \text{Sms} \cup X.\)

Proof. (1) follows immediately from the previous proposition since \(\sharp (\text{dom}(x)) < \sharp (x)\) for any \(x.\)

(2). Let \(X\) be thin. It follows from Proposition 2.6,(7) that \(F \upharpoonright X\) is also thin. By 1 \(F'' \, X\) is thin. Using induction over \(\mathbb{SN}\) on the cardinality of \(y \subseteq F'' \, X\) one proves the existence of a set \(x\) such that \(F'' \, x = y.\)

2.10 A class is called countable iff it can be bijectively mapped onto the class of small natural numbers.

2.11 It follows from the previous proposition that the theory AST + Thin + Comp + ”there exists an uncountable thin class” is inconsistent. Indeed, in AST, due to the axiom of two cardinalities saying that there is a bijection between any two uncountable classes, an uncountable thin semiset can be bijectively mapped onto an infinite set which is not thin.

2.12 The property of being a thin infinite class behaves as a cardinality lying between the cardinalities of small sets and those of infinite sets. This fact can be expressed in the following way.

We define inner cardinality of a class:

\[ \text{ICard} \, X \overset{\text{def}}{=} \{ \, n : \exists x \, (x \subseteq X \, \& \, \sharp (x) = n + 1) \, \}. \]

Then for sets we have \(\text{ICard} \, x = \sharp (x),\) and infinite thin classes are exactly the classes \(X\) such that \(\text{ICard} \, X = \mathbb{SN}.\)

2.13 Proposition (TFS). Axioms Thin and Comp together are equivalent to the following statement:

Prolongation principle:

\[ \forall \, \text{thin} \, X \, \big( \forall \, \text{small} \, x \subseteq X \, \varphi (x) \rightarrow \exists y \, \big( \, \text{Set}(y) \, \& \, X \subseteq y \, \& \, \varphi (y) \, \big) \big) \]

where \(\varphi\) is any set-formula with set-parameters.

2.14 Remark. The prolongation principle formulated here is a generalization of prolongation axiom of AST, which says that every countable function is a
subclass of a set-function.

The next proposition lists counterparts of statements which became customary tools in non-standard analysis. All of them are just special cases of the prolongation principle formulated above.

2.15 Proposition (THS₀).

Saturation: \( \forall \text{thin } Y \ ( \forall y \subseteq Y \ ( \cap y \neq \emptyset ) \rightarrow \cap Y \neq \emptyset ) \);

Extension: \( \forall F \ ( \text{thin } F \ & \ Fnc F \rightarrow \exists f \ ( Fnc f \ & \ F \subseteq f ) ) \);

Nelson's idealization principle:

\[ \forall \text{thin } A \ ( \forall a_0 \subseteq A \ \exists x \ \forall a \in a_0 \ \varphi(a, x) \rightarrow \exists x \ \forall a \in A \ \varphi(a, x) ), \]

for any set–formula \( \varphi \) with set parameters.

2.16 As we said already in the introduction, the simplest and the most important proper classes are \( \sigma \)–classes and \( \pi \)–classes which are defined in THS as follows: a \( \sigma \)–class is a union of a thin class and a \( \pi \)–class is an intersection of a thin class.

Both \( \pi \)–classes and \( \sigma \)–classes are semisets (see Proposition 2.9).

2.17 Together with sets and classes one can consider in THS also systems of classes defined as collections of classes satisfying a certain formula and written as terms of the form

\[ \{ X : \Phi(X) \}, \]

where \( \Phi \) is an arbitrary formula with some class– or set–parameters.

2.18 A system of classes \( \mathcal{X} = \{ X : \Phi(X) \} \) is called codable if there exists a class \( C \) such that

\[ \mathcal{X} = \{ C^n \{ d \} : d \in \text{dom}(C) \}. \]

Such coding by a class \( C \) is called extensional iff \( C^n \{ d \} \neq C^n \{ d' \} \) for distinct elements \( d, d' \in \text{dom}(C) \).

If a system \( \mathcal{X} \) is coded by a class \( C \) one can use quantification over sub-systems of \( \mathcal{X} \). For instance, \( \forall \mathcal{Y} \subseteq \mathcal{X} \ \Phi(\mathcal{Y}) \) can be interpreted as \( \forall D \subseteq \text{dom} C \ \Phi(\{ C^n \{ d \} : d \in D \} ). \)

2.19 If there exists a coding \( C \) such that \( \text{dom}(C) \) is a thin class, the system \( \mathcal{X} \) is called a thin system of classes.

If \( \mathcal{X} \) is a thin system we can always assume without loss of generality, due to axiom of Choice, that a given coding of \( \mathcal{X} \) is extensional.

If, furthermore, \( \mathcal{X} \) is a system of thin classes we can speak of systems of systems of classes and so on, using an appropriate encoding for higher levels.
Such an encoding can always be chosen to be a thin class, due to the axiom of exponentiation.

2.20 Proposition (THS).

(1) For every thin class $X$ there exists a strong well-ordering of $X$ (a well-ordering is strong iff every subclass of $X$ has a least element).

(2) Every infinite class contains a countable infinite subclass.

Proof. If an infinite class $X$ is thin then one can build a strong well-ordering of $X$, applying Proposition 2.8 and Choice, very much like the way one gets a well-ordering of a set using the axiom of choice. The least infinite initial segment of $X$ under that ordering will be countable.

If $X$ is not thin it contains an infinite subset $x$. Hence, there exists a bijective mapping $h$ from an infinite natural number onto $x$. The class $h^\mathbb{N}$ will be countable and infinite. $\square$

2.21 Remark. The statement converse to Proposition 2.20,1) is true if we accept an additional axiom analogous to the axiom of chromatic classes of NCT.

The following proposition describes small sets in a way similar to the classical Dedekind’s characterization of finite sets (a set is finite iff it is not of equal cardinality with any its proper subset).

2.22 Proposition (THS). $\forall X \left( \text{small } X \longleftrightarrow \forall Y \subseteq X \left[ Y \neq X \rightarrow \forall F : X \to Y \left( \text{"F is not injective"} \right) \right] \right)$.

3 $\in$-structures and Zermelo universes

It is known (see [9,10]) that saturation principles allow to simulate structures satisfying the axioms of Zermelo set theory or even ZFC within nonstandard models of arithmetic. From the other hand, in ”fully saturated” nonstandard set theories (such as E. Nelson’s Internal Set theory, NCT or Hrbacek Set Theory of V. Kanovei and M. Reeken [12]) every $\in$–structure of standard size is isomorphic to an $\in$–substructure of some hereditarily hyperfinite set.

In accordance with the above mentioned facts, the main result of this section states that every thin semiset can be embedded in a thin subuniverse that satisfies axioms of Zermelo set theory with choice, subclasses in the sense of THS corresponding to subsets in the sense of the Zermelo subuniverse.
3.1 Theorem (THS). For any thin class \( X \) there exists a thin class \( Z \) such that

1. \( X = Z \cap x \) for some \( x \in Z \);
2. \( \forall x \in Z \forall C \subseteq x \cap Z \exists q \in Z \ ( q \cap Z = C ) \);
3. \( \forall x \ ( x \subseteq Z \rightarrow x \in Z ) \);
4. All axioms of \( ZC^- \) (the Zermelo theory with the axiom of choice and without the axiom of regularity) are true in \( Z \).

3.2 A class satisfying the conditions (2), (3) and (4) of Theorem 3.1 is called a Zermelo universe.

3.3 Remark. The definition of Zermelo universe and Theorem 3.1 are very close in formulation to \( ZF \)–classes and Cantorian axioms in \( AST \) (given in chapter 12 of [19]). But the important condition (2) does not hold in \( AST \).

This theorem becomes a theorem of \( THS \) if we give a formal meaning to (4) using encoding of formulas.

We fix some explicit coding, by standard sets, for symbols of logical connectives, quantifiers, membership relation, punctuation signs and a countable set of variables, a coding for sets as parameters. Formal formulas are naturally defined within \( THS \) by induction as special (well-formed) sequences of codes. Every formula \( \varphi \) of \( THS \) gets its formal counterpart \([\varphi]\) — the code of \( \varphi \). Any formal formula of small length with standard parameters is standard (as a set).

The language \( SL(P) \) is defined as the class of all formal formulas of small length with parameters from the class \( P \).

Evidently, the language \( SL(P) \) is thin for any thin \( P \).

3.4 Proposition (TFS). For any class \( X \) there exists a unique class \( T \) which consists of closed formulas of \( SL(X) \) and satisfies the following properties:

1. \([x_1 = x_2]\in T \iff x_1, x_2 \in X \& x_1 = x_2\);
2. \([x_1 \in x_2]\in T \iff x_1, x_2 \in X \& x_1 \in x_2\);
3. \([\theta_1 \lor \theta_2]\in T \iff \theta_1 \in T \lor \theta_2 \in T\);
4. \([-\theta_1]\in T \iff \theta_1 \notin T\);
5. \([\exists v\theta]\in T \iff \exists x \in X \ ( \theta_{v\rightarrow x} \in T )\),

where \( \theta_1, \theta_2 \) are closed formulas of \( SL(X) \), \( \theta \) is a formula of \( SL(X) \) with the only free (symbol of) variable \( v \) and \( \theta_{v\rightarrow x} \) is obtained from \( \theta \) by replacing \( v \) with the code of the set \( x \).

We denote as \( \text{True}(X) \) the class, the existence of which is stated by Proposi-
For any closed formula $\varphi$ with parameters from some class $X$ it is provable in THS that

$$\varphi^X \leftrightarrow [\varphi] \in \text{True}(X),$$

where $\varphi^X$ is a relativization of $\varphi$ to the class $X$.

We denote for any $\theta \in \text{SL}(X)$

$$X \models_f \theta \iff \theta \in \text{True}(X).$$

Now we formalize (4) from Theorem 3.1:

$$(4) \quad Z \models_f \theta \text{ for each } \theta \text{ such that } "\theta \text{ is an axiom of } ZC^-", $$

where the phrase in quotes is appropriately expressed as a formula of THS.

We define the class $\text{Def}(X)$ of sets definable with a formula from $\text{SL}(X)$ as follows:

$$\text{Def}(X) \overset{\text{def}}{=} \{ x : x = \{ y : \mathbb{H} \models \theta(y) \} : \theta \in \text{SL}(X) \text{ has exactly one free variable} \}$$

Obviously, $\text{Def}(X)$ is a class. If $X$ is thin, $\text{Def}(X)$ is also thin.

We will say that a class $C$ is an $f$-elementary submodel of a class $M \supseteq C$ (notation: $C \preceq_f M$) iff

$$C \models_f \varphi \iff M \models_f \varphi$$

for any $\varphi \in \text{SL}(C)$.

3.5 Theorem (TFS). For any class $X$ the class $\text{Def}(X)$ is an $f$-elementary submodel of $\mathbb{H}$.

Proof. Note that $\text{ac}^{-1}(\min \{ \text{ac}(a) : \theta(a) \}) \in \text{Def}(X)$ for $\theta \in \text{SL}(X)$. \qed

3.6 Corollary. In the theory THS without the transfer axioms, the transfer axioms follow from the statement $\text{Def}(\emptyset) = S$.

Proof of Theorem 3.1. Using the axiom of dependent choices we construct the sequence of structures $S_n$ as follows.

We start from some thin class $S_0 \supseteq X$ such that $S_0 \preceq \mathbb{H}$. Such a class does exist by Theorem 3.5.
Given a class $S_n$, using the axioms of exponentiation and choice, we can choose a thin class $S_{n+1}$ to satisfy the following properties:

1. $\forall C \subseteq S_n \exists y \in S_{n+1} (y \cap S_n = C \land (\text{Set}(C) \rightarrow y = C))$
2. $\forall x, y \in S_{n+1} (x \neq y \rightarrow x \cap S_n \neq y \cap S_n)$
3. $\forall x \in S_{n+1} \setminus S_n \forall y \in S_n (x \notin y)$

We put $Z = \bigcup_{n \in \mathbb{N}} S_n$.

It is easy to see that the axioms of extensionality, union, power set, separation, infinity and choice hold in $Z$.

3.7 Corollary. (1) THS is not a conservative extension of $\text{ZF}^{\text{fin}}$; 
(2) THS is strictly stronger than $\text{ZC}^{-}$.

If we take $X = \mathbb{S}$ in the conditions of the previous Theorem and apply transfer, we get immediately the following theorem.

3.8 Theorem. Every statement about finite sets provable in $\text{ZC}^{-}$ holds in THS as well.

3.9 Remark. Non-standard extensions of superstructures over a thin class can be constructed easily in THS as thin $\in$–structures which allows to use ”essentially external” methods of nonstandard analysis such as nonstandard hulls of Banach spaces or Loeb measures.

4 Real numbers

Real numbers can be introduced in THS in a quite usual and straightforward way — as elements of a complete linearly ordered field.

We introduce explicitly the rational numbers first.

4.1 First of all define operations $+$ and $\cdot$ on $\mathbb{N}$ by the formulas

\[ x + y = \sharp(x \cup \{0\} \times y); \quad x \cdot y = \sharp(x \times y) \]

Obviously the introduced operations satisfy the the classical recursive definitions of addition and multiplication of natural numbers. It is easy to see that the subclass $\mathbb{S}\mathbb{N}$ of $\mathbb{N}$ is closed under these operations.
4.2 Usually the ring of integers is defined as quotient set of $\mathbb{N} \times \mathbb{N}$ under the equivalence relation

\[ (a, b) \sim (a_1, b_1) \iff a + b_1 = a_1 + b. \]

Since in our case $\mathbb{N}$ is a class we must define the quotient class by a system of representatives.

Thus, the class $\mathbb{Z}$ of integers can be defined e.g. by the formula

\[ \mathbb{Z} = \{0\} \times \mathbb{N} \cup \mathbb{N} \times \{0\} \]

with obviously defined addition, multiplication and linear order relation. It is easy to prove also that the thin class $\mathbb{S}\mathbb{Z}$ of standard elements of $\mathbb{Z}$ is a subring of $\mathbb{Z}$.

4.3 The field $\mathbb{Q}$ is defined as the quotient field of the integral domain $\mathbb{Z}$. As before we must define this quotient field by a system of representatives, e.g. by the formula

\[ \mathbb{Q} = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0, \gcd(a, b) = 1 \} \]

Once again it is easy to prove that the thin class $\mathbb{S}\mathbb{Q}$ of standard elements of $\mathbb{Q}$ is a subfield of $\mathbb{Q}$.

4.4 A class $\langle R; +, \cdot, \leq \rangle$ is called a field of real numbers iff it satisfies the axioms of linearly ordered field and the following completeness property:

every bounded above subclass of $R$ has a supremum.

4.5 Theorem. (THS)

(1) There exists a thin class that is a field of real numbers.
(2) The field $\mathbb{S}\mathbb{Q}$ is dense in a field of real numbers.
(3) Any two fields of real numbers are isomorphic.

Proof sketch. The proof quite repeats the classical one. We can choose any usual way of constructing real numbers. Take, for example, Dedekind cuts. Due to axiom of exponentiation $\{ C : C \subseteq \mathbb{S}\mathbb{Q} \} = \{ P' \{ c \} : c \in \text{dom}(P) \}$ for some thin class $P$. Every Dedekind cut can be identified then with an element of $\text{dom}(P)$ and we build the field of real numbers as a subclass of $\text{dom}(P)$.

The classical proofs of (2) and (3) can also be transferred easily to THS (see also Theorem 5.7). \qed
4.6 Remark. In every Zermelo subuniverse $Z$, the field of reals in the sense of $Z$ is a field of reals in the global sense.

4.7 In what follows we fix some field of real numbers $\langle \mathbb{R}; +, \cdot, \leq \rangle$, and call it the field of reals.

4.8 Remark. There is no definable field of real numbers in THS (see Proposition 6.3).

As in non-standard analysis, every bounded rational number has a standard part in $\mathbb{R}$.

Put $\mathbb{Q}_b = \{ x \in \mathbb{Q} : \exists r \in \mathbb{S}Q (|x| < r) \}$. We call elements of $\mathbb{Q}_b$ bounded rationals. Obviously $\mathbb{Q}_b$ is a subring of $\mathbb{Q}$ and $\mathbb{S}Q \subseteq \mathbb{Q}_b$.

Let $\mu(0) = \{ \alpha \in \mathbb{Q}_b : \forall r \in \mathbb{S}Q (r > 0 \rightarrow |\alpha| < r) \}$. Then $\mu(0) \subseteq \mathbb{Q}_b$ is an ideal in $\mathbb{Q}_b$.

4.9 Theorem. There exists a unique surjective homomorphism $\text{st} : \mathbb{Q}_b \rightarrow \mathbb{R}$. The kernel $\text{ker}(\text{st}) = \mu(0)$.

The real number $\text{st}(x)$ is called the standard part of a bounded rational $x$.

5 Ordinary mathematics in THS

Intuitively, thin classes behave exactly as usual infinite sets. We would like to transfer notions and results of ordinary mathematics to systems of thin classes. The informal principle is:

Everything that is true in ordinary mathematics about sets, their subsets, powersets and so on is true in THS about thin classes, their subclasses, systems of their subclasses and so on.

Note that cartesian products are implemented within iterated powersets; finiteness can be expressed as Dedekind finiteness and is equivalent to smallness by Proposition 2.22.

In what follows we will give a formal account of the formulated principle.

5.1 As an example, we would like to say whether a system $\mathcal{T}$ of subclasses of a thin class $X$ is a topology on $X$. If $\mathcal{T} = \{ T^d : d \in D \}$ this can be
expressed in the following way:

\[ T''D = X \& \forall d_1, d_2 \in D \ ( T''\{ d_1 \} \cap T''\{ d_2 \} \neq \emptyset ) \& \forall D' \subseteq D \exists d \big( \bigcup_{e \in D'} T''\{ e \} = T''\{ d \} \big). \quad (2) \]

If \( X \) is represented in a Zermelo universe \( Z \) by an element \( x \in Z \ ( Z \cap x = X ) \) then \( \mathcal{T} \) is also represented in \( Z \) by some \( t \in Z \):

\[ \forall Y \ ( Y \in \mathcal{T} \leftrightarrow \exists y \in t \cap Z \ ( y \cap Z = Y ) ), \]

and (2) is true iff \( Z \models "t \text{ is a topology on } x". \)

5.2 In a more generic setting we may need to refer to higher levels of cumulative hierarchy over some thin class. Some encoding is necessary for that. To describe a general situation and abstract from a particular encoding of systems of classes (as we did in 5.1) we consider extensional systems over thin classes.

5.3 A system of classes (see 2.17) \( \mathcal{X} \) equipped by a system of pairs of classes \( \mathcal{E} \) is called a (thin) extensional system over a thin class \( A \) iff the following conditions hold:

(1) \( X \mathcal{E} Y \imp X \in \mathcal{X} \& Y \in \mathcal{X} ; \)
(2) \( A \in \mathcal{X} \& A_\mathcal{E} = A \), \( \forall a \in A \ ( a_\mathcal{E} = a \cap A ) \), where \( Y_\mathcal{E} \overset{\text{def}}{=} \{ Z : Z \mathcal{E} Y \} \);
(3) \( \forall X, Y \in \mathcal{X} \setminus A \ ( \forall Z \ ( Z \mathcal{E} X \longleftrightarrow Z \mathcal{E} Y ) \imp X = Y ) ; \)
(4) \( \forall X \in \mathcal{X} \exists k \small Y \in X_\mathcal{E} \big( \bigcup_{\mathcal{E}} \cdots \bigcup_{\mathcal{E}} X_\mathcal{E} \subseteq A \big) \times \times \times \big) \)

where \( \cup_{\mathcal{E}} S \overset{\text{def}}{=} \{ Y : \exists Z \ ( Z \in S \& Y \mathcal{E} Z ) \} \).

We put

\( \mathcal{X}_0 = A ; \mathcal{X}_1 = \{ X : X_\mathcal{E} \subseteq A \} ; \mathcal{X}_k = \{ X \in \mathcal{X} : \bigcup_{\mathcal{E}} \cdots \bigcup_{\mathcal{E}} X_\mathcal{E} \subseteq A \} \), \( k > 1 \).

A system \( \mathcal{X} \) is called \( k \)-full iff

\[ \forall Y \subseteq \mathcal{X}_k \exists Y \in \mathcal{X} \ ( Y_\mathcal{E} = \mathcal{Y} ) \).

5.4 We define a formula of ordinary mathematics (o.m.–formula) to be an \( \in \)–formula \( \varphi(A, X_1, \ldots, X_n) \) where all quantifiers have the form \( \exists x \in \mathcal{P}^k(A) \) or \( \forall x \in \mathcal{P}^k(A) \) where each quantifier has its own natural number \( k \) of iterations of the powerset operation \( \mathcal{P} \). (Formally, "\( \forall x \in \mathcal{P}(A) \ldots " \) is to be read as \( \forall x \ ( \forall z \ ( z \in x \longrightarrow z \in A \) \longrightarrow \ldots ) \), and so on). The maximal number of iterations of \( \mathcal{P} \) in the bounding terms of \( \varphi \) is called the height of \( \varphi \).
For any o.m.-formula $\varphi$ of height $k$ the truth of $\varphi(A, X_1, \ldots, X_n)$ in a $k$-full extensional system $\mathcal{X}$ over $A$ ($X_i \in \mathcal{X}$) is defined in a straightforward way (we omit the obvious details). We write $\mathcal{X} \models \varphi$ if $\varphi$ is true in $\mathcal{X}$.

5.5 Theorem. Let $\mathcal{X}$ be an extensional system over a thin class $A$. Suppose $A$ is represented in a Zermelo universe $Z$: $A = Z \cap a$ for some $a \in Z$. Then there exists a unique embedding $\mathcal{J} : \mathcal{X} \rightarrow Z$ such that

$$\mathcal{J}(A) = a \& \forall X,Y \in \mathcal{X} \ (X \in Y \iff \mathcal{J}(X) \in \mathcal{J}(Y)).$$

Moreover, if $\mathcal{X}$ is $k$–full then for any $X_1, \ldots, X_n \in \mathcal{X}$ and any o.m.-formula $\varphi(A, X_1, \ldots, X_n)$ of height $\leq k$ we have

$$\mathcal{X} \models \varphi(A, X_1, \ldots, X_n) \iff Z \models \varphi(\mathcal{J}(A), \mathcal{J}(X_1), \ldots, \mathcal{J}(X_n)).$$

The proof is quite straightforward.

5.6 Theorem 5.5 allows to extend the definition of truth of o.m.-formulas so that it can be applied to extensional systems that not necessarily are full to the height of the formula.

Let $\mathcal{X}$ be an extensional system over a thin class $A$, $X_1, \ldots, X_n \in \mathcal{X}$ and $\varphi(A, X_1, \ldots, X_n)$ be an o.m.-formula of height $k$. Then we say that $\varphi(A, X_1, \ldots, X_n)$ is true for $\mathcal{X}$ iff $\mathcal{X}' \models \varphi(A, X_1, \ldots, X_n)$ for some (and then for any) $k$–full extensional system $\mathcal{X}' \supseteq \mathcal{X}$ over $A$.

5.7 Theorem. Suppose $\forall A \forall X_1 \in \mathcal{P}^{k_1} \ldots \forall X_n \in \mathcal{P}^{k_n}$ $\varphi(A, X_1, \ldots, X_n)$ is a theorem of ordinary mathematics provable in $\mathcal{ZC}^-$. Let $X_1 \in \mathcal{X}_{k_1}, \ldots, X_n \in \mathcal{X}_{k_n}$ in an extensional system $\mathcal{X}$ over a thin class $A$. Then $\varphi(A, X_1, \ldots, X_n)$ is true for $\mathcal{X}$.

It is easy to see, for example, that the statement (3) of Theorem 4.5 follows from this theorem.

6 Interpretaions of THS

6.1 Theorem. The collection of all subclasses of the set $V_\omega$ of hereditarily finite sets together with the original membership relation gives an interpretation of THS in NCT. Moreover, under this interpretation

1. sets are exactly finite subsets of $V_\omega$;
2. thin classes are exactly subsemisets of $V_\omega$ of standard size.
Proof. By Theorem 3.12 of [1] a set is S-finite in NCT (that is having a standard finite cardinality) iff every subclass of it is a set. Corollary 4.12 of [1] states that a semiset $X$ has a standard size iff every subset of $X$ is S-finite. So we have that small sets are interpreted as S-finite, and thin classes are interpreted as semisets of standard size. Extent holds obviously. Axiom Class follows from Corollary 4.17 of [1] stating that any formula in which only semisets are quantified is equivalent to a normal formula. Axioms Set and Ind can be derived from the theorem of Sochor mentioned in the proof of Theorem 2.3. Proposition 4.13 of [1] says that any semiset of standard size can be embedded into a set of any given infinitely large cardinality. Axiom of thin semisets follows. The truth of Comp can be derived easily from the Saturation Theorem 4.7 of [1].

The Choice Theorem 4.20 of [1] implies Choice. The axiom of exponentiation Exp also follows from the Choice theorem because if $\kappa$ is the ”standard size” of a semiset $X$ then $2^\kappa$ is the ”standard size” of the class of its subclasses.

Theorem 1.10 of Kanovei and Reeken [11] shows that the scheme of dependent choices for sets holds in BST (Bounded Set Theory). Since NCT is a conservative extension of BST ([1, Theorem 5.1]) and semisets are uniformly parameterized by sets ([1, Theorem 4.16]), the scheme DC is also true. □

6.2 Remark. In a model of E. Nelson’s IST [13] the collection of all subclasses of $V_\omega$ in that model gives a model of THS$^0$ but does not give a model of the full THS. The reason is that there exists a subclass $O$ of $V_\omega$ that can be one-to-one mapped onto the class of all standard sets and therefore will be thin but neither Choice nor Exp can be proved for $O$.

6.3 Proposition. There is no formula $\Phi$ with one free variable such that $\exists X \Phi(X) \ & \forall X \ (\Phi(X) \rightarrow "X \ is \ an \ uncountable \ thin \ class")$ would be a theorem of THS.

Proof. Under the interpretation of THS in NCT described above every formula of THS gets translated to a normal formula of NCT. By Proposition 7 from [2] any class of standard size defined by a formula without parameters consists of standard elements\(^2\). Therefore any thin class definable by a formula in such a model has to consist of hereditarily small sets, and hence cannot be uncountable. □

For any class $X$, denote

$$\text{ADef}(X) = \bigcap_{n \in \text{Def}(X) \setminus S} \{ x : \text{ac}(x) < n \}.$$\(^2\)

\(^2\) The proof is given in [2] for BST but can be transferred literally to NCT.
Due to compactness, ADef(X) is nonempty for any thin class \( X \neq \mathbb{S} \).

6.4 Proposition. For any thin class \( X \neq \mathbb{S} \), the collection of all subclasses of the class ADef(X) together with the original membership relation forms an interpretation of THS in THS.

Thus, there are interpretations of THS in THS of any size: ADef(\( l \)) is a subclass of a set having less than \( l \) elements. Moreover, the universe \( \mathbb{H} \) of all sets can be thought of as a subclass of some highly unfeasibly large hyperfinite set.

7 Indiscernibility equivalences and locally compact topological spaces

In this section we discuss how the approach to continuous structures that considers them as the images of accessible parts of certain hyperfinite structures under identifying indiscernible elements can be formalized in THS.

Let \( x \) be a set and \( \approx \) be a \( \pi \)-equivalence relation on \( x \) which means that \( \approx = \cap E \) where \( E \) is a thin class of subsets of \( x \times x \).

7.1 Let \( X \subseteq x \) be a \( \sigma \)-subset of \( X \), which means that \( X \) is the union of a thin class of subsets of \( X \). The relation \( \approx \) is called an indiscernibility equivalence on \( X \) iff

\[
\forall \inf u \subseteq X \exists a, b \in u \ (a \neq b \&\ a \approx b).
\]

7.2 Example To consider an example of an indiscernibility relation we introduce the following notation.

Let \( \alpha, \beta \in \mathbb{Q} \). Then

(1) \( \alpha \approx \beta \iff \alpha - \beta \in \mu(0) \) (cf. Theorem 4.9);
(2) \( \alpha \sim \infty \iff \alpha \in \mathbb{Q} \setminus \mathbb{Q}_b \).

Note that \( \mathbb{N} \ni n \sim \infty \iff n \in \mathbb{N} \setminus \mathbb{SN} \).

Fix \( n, m \in \mathbb{N} \setminus \mathbb{SN} \) such that \( \frac{n}{m} \sim \infty \). Let \( x = \{ \pm \frac{k}{r} \mid 0 \leq k \leq n, \ 0 < l \leq m \} \).

Consider the restriction of the relation \( \approx \) to \( x \). We denote this restriction also by \( \approx \) in this example. Let \( X = x \cap \mathbb{Q}_b \). Obviously \( X \) is a \( \sigma \)-class. Let us show that \( \approx \) is an indiscernibility relation on \( X \) \(^3\). Indeed, let \( u \subseteq X \). Put \( v = \{ |a - b| \mid a, b \in u, \ a \neq b \} \). If \( \min v \approx 0 \) then for every \( n \in \mathbb{SN} \) the set \( u_n = \{ a, b \in u \mid a \neq b, \ |a - b| < \frac{1}{n} \} \neq \emptyset \). Since the decreasing countable sequence \( \{ u_n \mid n \in \mathbb{SN} \} \) consists of nonempty sets, its intersection is also

\(^3\) This is obvious for those, who are familiar with nonstandard analysis.
nonempty by Proposition 2.15. Thus there exist \(a, b \in u\) such that \(a \neq b\) but \(a \approx b\). Therefore, in this case our statement is proved.

Let now \(\min v > \delta > 0\), where \(\delta \in SQ\). This implies that the map \(st : \mathbb{Q}_b \rightarrow \mathbb{R}\) defined in Theorem 4.9 is injective on \(u\) and \(\min\{\left|st(a) - st(b)\right| : a \neq b, a, b \in U\} \geq \delta\).

On the other hand the set \(st''u\) is bounded. Indeed, since \(u \subseteq \mathbb{Q}_b\), we have \(st''u \subseteq [st(\min u) - 1, st(\max u) + 1]\). Applying the theorem of ordinary mathematics, which states that every infinite bounded set of reals has an accumulation point we obtain, that \(st'u\) is finite and, thus, \(u\) is small.

Using Theorem 5.7 we can easily formalize this consideration in THS.

**7.3 Theorem.** The following conditions are equivalent:

1. \(\approx_e^E\) is an indiscernibility equivalence;
2. \(\forall e \in E \forall y \subseteq X \exists \text{ small } v \subseteq X(y \subseteq e''v)\);
3. \(\exists \text{ thin } N \subseteq X(\forall a \in X \exists n \in N(a \approx n) \& \forall n, m \in N(n \approx m \rightarrow n = m))\).

**Proof.** (1) \(\rightarrow\) (2). If (2) does not hold then there exist \(e \in E\) and \(y \subseteq X\) such that \(a \setminus (y''v) \neq \emptyset\) for any small \(v \subseteq y\). Using induction for \(SN\), we will find for any small \(n\) an injective function from \(n\) into \(y\) such that its range consists of pairwise \(e\)-non-equivalent elements. Applying prolongation, we get an infinite set of pairwise \(e\)-non-equivalent elements.

(2) \(\rightarrow\) (3) Assume \(X = \bigcup D\). Using choice and prolongation we can assign to any \(e \in E\) and \(d \in D\) a set \(v_{ed}\) such that \(d \subseteq e''v_{ed}\). Take any \(d \in D\) and elements \(x, y \in d\). Suppose \(\forall e \in E \exists z \in v_{ed} (x \in e''\{z\} \& y \in e''\{z\})\). Then \(x \approx_e^E y\). Indeed, take any \(e \in E\). Since \(E\) is an equivalence relation there is an \(e_1 \in E\) such that \(e_1 \circ e_1^{-1} \subseteq e\). Since, for some \(z, \langle x, z \rangle \in e_1 \& \langle y, z \rangle \in e_1\), we have \(\langle x, y \rangle \in e\).

Now, since the class \(B = \{ e''\{z\} : z \in v_{ed} \& e \in E \& d \in D \}\) is thin, we can choose \(N\) in such a way that \(N\) contains exactly one element in the intersection of each centered subfamily of \(B\) (\(C \subseteq B\) is centered iff any small subset of \(B\) has a non-empty intersection). It can be checked easily that \(N\) is as required in (3).

The implication (3) \(\rightarrow\) (1) is straightforward.

 Obviously, a class \(N\) satisfying the condition 3 of Theorem 7.3, generally speaking, is not unique. Fix any such \(N\). For \(u \subseteq X\) define \(\hat{u} = \{ a \in u : \forall b \in X(a \approx_e^E b \rightarrow b \in u)\}\) and \(\hat{u}^\# = \hat{u} \cap N\).
7.4 Proposition (THS).

\[
\left[ \forall n \in N \exists u \subseteq X (n \in u^\#) \right] \& \forall n \in N \forall u_1, u_2 \subseteq X \\
\left( n \in u_1^\# \cap u_2^\# \rightarrow \exists u \subseteq X \left( n \in u^\# \subseteq u_1^\# \cap u_2^\# \right) \right) \quad (3)
\]

Semantically, this proposition means that the system of classes \( \mathcal{T} = \{ u^\# \mid u \subseteq X \} \) forms a base of topology on \( N \).

It can be proved that this topology is locally compact. Let us show how this statement can be formulated explicitly in THS. Let \( C \subseteq N \). We say that \( F \) is an open covering of \( C \) if \( F \) is a function, \( \text{dom} \ F = I \) is a thin class and \( \forall i \in IF(i) \subseteq X \) and \( N \subseteq \bigcup_{i \in I} \overset{o}{\sim} F(i)^\# \). A class \( C \subseteq N \) is compact iff \( \forall F(F \text{ is an open covering of } C) \rightarrow \exists \text{small } p \subseteq \text{dom} \ F(C \subseteq \bigcup_{i \in p} \overset{o}{\sim} F(i)^\#) \) (cf. Remark 2.22). Now, "\((N, \mathcal{T}) \text{ is a locally compact space}\)" is equivalent to the following THS-formula:

\[
\forall n \in N \exists u \subseteq X (n \in u^\# \& \exists C \subseteq N ((C \text{ is compact}) \& (u^\# \subseteq C)). \quad (LC)
\]

Similar approach to the construction of locally compact spaces was developed in [7] in terms of nonstandard analysis for the case of locally compact abelian groups. The proof of Theorem 2.2.4 of [7] can be easily transformed to a proof of (LC) in THS.

It is easy to see that any locally compact space can be represented as a quotient class \( N \) constructed by an appropriate triple \( \langle x, X, \overset{E}{\sim} \rangle \), where \( X \) is a thin subclass of a set \( x \) and \( \overset{E}{\sim} \) is an indiscernibility relation on \( X \).

Let us consider such representations of the field \( \mathbb{R} \) in more detail. As it was mentioned in the Introduction they can be considered as numerical systems that simulate reals in an idealized computer of infinite (i.e. hyperfinite non-small) memory. This point of view gives a motivation for the following definition.

7.5 Consider a tuple \( R = (\langle r; \oplus, \odot \rangle; R_b, \rho) \), where \( r \) is a hyperfinite set with binary operations \( \oplus, \odot \) on it. We say that \( R \) is a \textit{hyperfinite computer arithmetic} if \( R_b \subseteq r \) is a \( \sigma \)-class and \( \rho \) is a \( \pi \)-equivalence relation on \( r \) such that

1. \( \rho \) is an indiscernibility relation on \( R_b \);
2. \( R_b \) is closed under the operations \( \oplus \) and \( \odot \);
3. \( \rho \) is a congruence relation on \( R_b \);
4. the quotient algebra \( R_b/\rho \) is topologically isomorphic to the field \( \mathbb{R} \).
The previous considerations show that this definition can be formalized in THS.

We interpret the elements of \( R_b \) the same way as it was discussed in the Introduction. Elements of \( R_b \) are computer reals that are not too big, i.e. not too close to the boundary of the computer memory. Obviously, this is not a definition in the framework of the classical mathematics. That is why \( R_b \) is a proper semiset. Operating with these numbers does not imply the overfilling of the memory. Thus the computer operations restricted to \( R_b \) approximate the corresponding operations on reals. This fact is formalized in the statement 3 of definition 7.5.

**Example 2.** Let \( 0 < e \in \mu(0) \) and \( \omega \in N \setminus SN \) be such that \( \omega e \sim \infty \). Consider a tuple \( R(\omega, e) = (\langle r_\omega; \oplus, \odot \rangle; R_b, \rho) \), where

1. \( r_\omega = \{-\omega, \ldots, \omega\} \);
2. the operation \( \oplus \) is the addition modulo \( 2\omega + 1 \);
3. the operation \( \odot \) is defined by the formula
   \[
   k \odot m = \lfloor km \rfloor \mod (2\omega + 1),
   \]
   where \( k, m \in r_\omega \) and \( \lfloor \alpha \rfloor \) is the integral part of a real number \( \alpha \);
4. \( R_b = \{ k \in r_\omega \mid ke \in Q_b \} \);
5. an equivalence relation \( \rho \subseteq r_\omega \times r_\omega \) is such that
   \[
   k \rho m \iff ke \approx me
   \]

**7.6 Proposition.** The tuple \( R(\omega, e) \) is a nonstandard computer arithmetic.

\( \triangleright \) It is easy to see that if \( k, m \in R_b \) then \( k \oplus m = k + m \) and \( k \odot m = n \), where \( ne \leq km \varepsilon^2 < (n + 1)e \).

Define the map \( F : R_b \rightarrow R \) by the formula \( F(k) = st(ke) \). It is easy to see that for all \( k, m \in R_b \) holds

1. \( F(k) = F(m) \iff k \rho m \);
2. \( \forall q \in Q_b \exists k \in R_b (q \approx ke) \);
3. \( (k \odot m) e \approx ke \cdot me \).

These properties prove that \( F \) is a surjective homomorphism and that \( R_b / \rho \) is isomorphic to \( \mathbb{R} \).

The nonstandard computer arithmetic \( R(\omega, e) \) discussed in Example 2 is not a hyperfinite version of the computer arithmetic, which is implemented in existing computers. The last one is based on the floating point representation of reals. We will call it FP-arithmetic. Its hyperfinite version was discussed in [5] in terms of nonstandard analysis. The computer arithmetic \( R(\omega, e) \) (no
matter standard or nonstandard $\omega$ and $e$ are considered) has some better than FP-arithmetic algebraic properties. Indeed, it is well-known that the addition and the multiplication in FP-arithmetic are neither associative, nor distributive, while $\langle r, \oplus \rangle$ is an abelian group. However, multiplication in $R(\omega, e)$ is not associative and the law of distributivity also fails [5].

It is not quite clear how the good algebraic properties of numerical systems would affect on the quality of numerical computations. It was shown [7] that the convergence properties of approximation of the Fourier Transformation on $\mathbb{R}$ by sampling of its kernel are better when the result of this sampling is the matrix of Finite Fourier Transformation, i.e. when we approximate the additive group $\mathbb{R}$ is by finite abelian groups. The theory of approximation of locally compact groups by finite abelian groups was developed in [7]. It can be proved (cf. [5], where similar questions where discussed in terms of nonstandard analysis) that the problem of approximation of locally compact algebraic systems by finite ones can be reduce to a problem of representation of locally compact systems by the quotients of $\sigma$-subsystems of hyperfinite systems under indiscernibility relations.

The following theorem demonstrates the restrictions that occur on the way of construction of computer arithmetics with the best possible algebraic properties.

**7.7 Theorem.** There does not exist a hyperfinite computer arithmetic $R = \langle \langle r; \oplus, \odot; R_b, \rho \rangle; R \rangle$ such that $\langle r; \oplus, \odot \rangle$ is an associative ring (even non-commutative).

A similar theorem about finite approximations of locally compact fields was proved in [5] (see also [6]). Theorem 7.7 is a little bit more general than those of [5]. However, the proof presented in [5] can be easily adjusted to Theorem 7.7. This proof can be formalized in THS.

Let $L$ be the first order language in the signature $\sigma_1 = \langle +, \cdot \rangle$ and $L_h$ - the first order language in the signature $\sigma_2 = \langle \oplus, \odot; R_b, \rho \rangle$. Here $\oplus$ and $\odot$ are symbols of binary operations, $R_b$ is a symbol of a unary predicate and $\rho$ - a symbol of binary predicate.

Formulas of $L_h$ have the natural interpretation in any hyperfinite computer arithmetic $R = \langle \langle r; \oplus, \odot; R_b, \rho \rangle; R \rangle$. We use notations $\forall^h$ and $\exists^h$ for the universal and the existential quantifiers restricted to $R_b$.

Let $t$ be a term in the signature $\sigma_1$. Replace each occurrence of the operation $+ (\cdot)$ in $t$ by $\oplus (\odot)$. The obtained term in the signature $\sigma_2$ will be denoted by $t_h$.

Let $\varphi(x_1, \ldots, x_n)$ be an $L$-formula. Denote by $\varphi_h$ the $L_h$-formula obtained from $\varphi$ by replacement of each atomic formula $t = s$ by $t_h \rho s_h$ and each quantifier
Let $R = (\langle r; \oplus, \odot \rangle; R_b, \rho)$ be a hyperfinite computer arithmetic. By Definition 7.5 (4) there exists a homomorphism $\psi : R_b \to \mathbb{R}$ such that $\psi(a) = \psi(b) \iff a \rho b$. This homomorphism may not be unique. We call $\psi$ a canonical isomorphism. It may not be unique. The following theorem follows immediately from the definitions.

**7.8 Theorem.** For any $L$-formula $\varphi(x_1, \ldots, x_n)$ the following statement holds.

If $R = (\langle r; \oplus, \odot \rangle; R_b, \rho)$ is a hyperfinite computer arithmetic, $\psi : R_b \to \mathbb{R}$, and $a_1, \ldots a_n \in R_b$ then

$$R \models \varphi_h(a_1, \ldots, a_n) \iff \mathbb{R} \models \varphi((\psi(a_1), \ldots, \psi(a_n)))$$

This theorem gives some qualitative formalization of the fact that if we operate with relatively small numbers, so that the memory overfilling cannot occur during the computations, we obtain results that are approximately true at least, when we deal with algebraic statements that can be formalized in the language $L$.

A version of Theorem 7.8 can be formulated in the language of classical mathematics only for some specific $L$-formulas - the *positive bounded formulas* [5].

The investigation of the correlation between continuous and computer mathematics in terms of THS (i.e. on the qualitative level) for higher order properties (e.g. formulated in the language of type theory) is an interesting problem. It can help to discover some new phenomena concerning numerical investigation of some more complicated structures.

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