Quantum cryptography based on photon “frequency” states: example of a possible realization
S. N. Molotkov
Institute of Solid-State Physics, Russian Academy of Sciences, 142432 Chernogolovka, Moscow District
E-mail: molotkov@issp.ac.ru

Abstract
A quantum cryptosystem is proposed using single-photon states with different frequency spectra as information carriers. A possible experimental implementation of the cryptosystem is discussed.

1. Introduction

The idea of quantum cryptography was first proposed in Ref. [1], initially inaccessible. In its current form the protocol for propagation of a key (a secret random sequence of zeros and ones) was proposed in Ref. [2]. A new qualitative jump in the understanding of secrecy in quantum cryptography arose after Ref. [3], in which a protocol was proposed for exchange using nonorthogonal states, and Ref. [4] where a protocol was described using the Einstein–Podolsky–Rosen effect.

Various versions of cryptosystems using nonorthogonal states and the Einstein–Podolsky–Rosen effect were proposed subsequently. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] Experimental prototypes of quantum cryptosystems have been implemented using nonorthogonal polarization states of photons for coding [5, 11, 12] and also using the principle of phase coding based on a fiber-optic Mach–Zehnder time-division interferometer [8, 9, 10, 19] (an improved version of the system from Ref. [10] was implemented in Ref. [19]). The longest communication channel achieved under laboratory conditions is 30 km (Ref. [10]). The operation of a prototype quantum cryptosystem has been demonstrated under natural conditions using a 23 km long optical cable below Lake Geneva [18] and also between various buildings at the Los Alamos National Laboratory [13].

Most of these systems use the interference principle which roughly involves “splitting” a photon at the transmitting end of the line and “collimating” it at the receiving end. Here we propose a quantum cryptosystem using different photon frequency states which utilizes the “internal interference” of the different photon frequency components. Such a system may well prove more stable in operation than direct interference systems although this can only be confirmed by means of an experimental implementation.

The secrecy of the key in quantum cryptography is based on two facts: 1) the impossibility of coding (cloning) a previously unknown quantum state [23] and 2) the impossibility of extracting information on quantum states without perturbing them if they belong to a nonorthogonal basis. [3] Formally, any pair of nonorthogonal states corresponding to logic 0 and 1 can be used as information carriers. The detection procedure (quantum-mechanical measurement) at the receiving end should be set up such that any attempts at intervention within the communication channel, i.e., changes in states, can be identified from the results of the measurements. If the pair of nonorthogonal states \( |\psi_0\rangle \) and \( |\psi_1\rangle \) are used as carriers, the formal measurements are given by the projectors [3]

\[
\mathcal{E}_0 = 1 - |\psi_0\rangle\langle\psi_0|, \quad \mathcal{E}_1 = 1 - |\psi_1\rangle\langle\psi_1|,
\]

whose action reduces to the projection of states orthogonal to the vectors \( |\psi_0\rangle \) and \( |\psi_1\rangle \), respectively, on the subspaces. [3] The result of the action of the projectors is treated as a statement and the probability of the measurement results is given by the expressions:

\[
\begin{align*}
\text{Pr}_0 &= \text{Tr}\{\rho_0 \mathcal{E}_0\} = \text{Tr}\{\rho_1 \mathcal{E}_1\} = 0, \\
\text{Pr}_1 &= \text{Tr}\{\rho_0 \mathcal{E}_1\} = \text{Tr}\{\rho_1 \mathcal{E}_0\} = 1 - |\langle\psi_0|\psi_1\rangle|^2 \neq 0.
\end{align*}
\]

Measurements using \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) in an ideal communication channel (without noise) can detect any attempts at eavesdropping, i.e., changes in states. The first nonzero outcome of a control measurement
2. Cryptography using photon “frequency” states

A quantum cryptographic system using the Einstein–Podolsky–Rosen effect for a biphoton field was proposed in an earlier study[20] and this idea is developed here. Subsequently, the use of frequency states which do not use interference over large distances is suggested. Let us first analyze a formal system and then discuss an experimental realization. Three single-photon states are used as carriers: two information states corresponding to logic zero and one and one control state. The information states are mutually orthogonal. The control state is pairwise nonorthogonal to the information states. The use of only two information states is inadequate for secrecy because they can be reliably distinguished as a result of their orthogonality. The information states comprise pure stationary states with the density matrices

\[
\hat{\rho}_0 = |e_0\rangle\langle e_0|, \quad \hat{\rho}_1 = |e_1\rangle\langle e_1|, \quad \langle e_1|e_0\rangle = 0,
\]

where \(|e_0\rangle\) and \(|e_1\rangle\) are certain basis states assigned to the energies \(\omega_0\) and \(\omega_1\), respectively. The control state is nonstationary and contains both basis components \(|e_0\rangle\) and \(|e_1\rangle\):

\[
|\psi_c(t_0)\rangle = e^{-i\omega_0 t_0} f_0 |e_0\rangle + e^{-i\omega_1 t_0} f_1 |e_1\rangle,
\]

\[
\hat{\rho}_c(t_0) = |\psi_c(t_0)\rangle\langle \psi_c(t_0)|,
\]

and the normalization condition

\[
|f_0|^2 + |f_1|^2 = 1.
\]

The time \(t_0\) describes the beginning of the time measurement — the state preparation time (see below). The density matrix at times \(t > t_0\) is obtained by substituting into the argument \(\hat{\rho}_c(t) = |\psi_c(t - t_0)\rangle\langle \psi_c(t - t_0)|\). The introduction of two orthogonal information states reduces the number of “idle” outcomes because of their distinguishability if no eavesdropping is detected in the exchange process. This scheme uses two types of measurements. The measurements of the frequency spectrum are described by an orthogonal resolution of identity in space spanned on states \(|e_0\rangle, |e_1\rangle\):

\[
E_0 + E_1 = I, \quad E_0 = |e_0\rangle\langle e_0|, \quad E_1 = |e_1\rangle\langle e_1|,
\]

where \(I\) is the unit operator. The second family of measurements involves measuring the time and is given by a nonorthogonal resolution of identity (see Ref. [24], for example), which in our case has the form

\[
\int_0^T E(dt) = I, \quad \int_0^T = \frac{2\pi}{|\omega_1 - \omega_0|},
\]

\[
E(dt) = \left(e^{-i\omega_0 t}|e_0\rangle + e^{-i\omega_1 t}|e_1\rangle\right) \left(\langle e_0|e^{i\omega_0 t} + \langle e_1|e^{i\omega_1 t}\right) \frac{dt}{T}.
\]
In accordance with the general philosophy of quantum-mechanical measurements, the measurements were made at a certain time. The probability of the outcome of the measurements using the projectors $E_0$ and $E_1$ does not depend on time and is given by

$$\Pr = \text{Tr}\{\hat{\rho}_t E_0\} = 1, \quad \Pr = \text{Tr}\{\hat{\rho}_t E_1\} = 1, \quad \Pr = \text{Tr}\{\hat{\rho}_{0,1} E_{1,0}\} = 0,$$

Measurements of the time give the probability distribution of the outcomes in the range $(t, t + dt)$:

$$\Pr(dt) = \text{Tr}\{\hat{\rho}_{0,1} E(dt)\} = 1 \cdot \frac{dt}{T},$$

$$\Pr(dt) = \text{Tr}\{\hat{\rho}_c(t) E(dt)\} = |f_0| |\exp[-i\omega_0(t-t_0)] + f_1 \exp[-i(\omega_1(t-t_0))]|^2 \left(\frac{dt}{T}\right) =$$

$$= \{1 + 2\Re[f_0 f_1^* \exp[-i(\omega_0 - \omega_1)(t-t_0)]]\} \left(\frac{dt}{T}\right).$$

For the control state the probability is an oscillating function with the period $T = 2\pi/|\omega_1 - \omega_0|$. This set of measurements can completely reconstruct information on the states — no other density matrices can reproduce the statistics of the measurements so that any attempts at eavesdropping can be detected (for further details see Ref. [22]).

The key generation protocol is as follows. We assume that all the parameters of the states are known to everybody, including any potential eavesdropper. User A (henceforth “Alice”) randomly sends into the communication channel states $\hat{\rho}_c$, $\hat{\rho}_0$, or $\hat{\rho}_1$. User B (henceforth “Bob”) randomly and independently of Alice selects measurement type $E_0$, $E_1$, or $E(dt)$. After making a series of measurements, Alice transmits through the open channel (accessible to all including an eavesdropper — “Eve”) the numbers of some measurements when $\hat{\rho}_0$ and $\hat{\rho}_1$ were sent and all the numbers when control state $\hat{\rho}_c$ was sent. Bob sorts the measurements into three groups according to when $\hat{\rho}_c$, $\hat{\rho}_0$, or $\hat{\rho}_1$ were transmitted. In each of these three groups, three subgroups are identified according to measurement procedures $E_0$, $E_1$, or $E(dt)$. For instance, for those messages when Alice transmitted state $\hat{\rho}_c$, the relative fraction of the measurement outcomes when the projectors $E_0$ and $E_1$ were used should be $|f_0|^2/|f_1|^2$ regardless of the measurement time. For the $E(dt)$ measurements the probability of the measurement results at various times should converge toward the probability distribution (8). The convergence of the distribution function for a finite sample should be checked by using some statistical criterion such as the Kolmogorov criterion [27] (see also Ref. [22]). The convergence is checked similarly for the measurements when states $\hat{\rho}_0$ or $\hat{\rho}_1$ were sent.

For example, for state $\hat{\rho}_0$ the measurements using $E_0$ should give the same outcome in all attempts, which does not depend on the measurement time. For the $E_1$ measurements in all attempts the outcome should be zero regardless of the measurement time. For the $E(dt)$ measurements the probability of the outcome is only determined by the duration of the time interval $dt$ and does not depend on the time $t$.

The secrecy of the protocol is guaranteed by the nonorthogonality of the information states to the control state and by the fact that a set of measurements is information-complete so that any attempts at eavesdropping, i.e., changes in states, can be detected. In other words, no other density matrices can reproduce the statistics of the measurements at the receiving end (for further details see Ref. [22]).

After having established that no eavesdropping is taking place, Alice transmits the numbers of those measurements when the control state was sent. All the idle measurements when the detector was not triggered are discarded. Then, for the remaining numbers Bob only transmits the numbers of those measurements in which he used $E_0$ or $E_1$ but does not communicate which measurement, $E_0$ or $E_1$, was used in each specific attempt (this information is now known only to Alice and Bob).

These remaining measurements give the secret key (an identical random sequence of zeros and ones for Alice and Bob).

We shall illustrate why an eavesdropper will inevitably introduce errors. In order to obtain information on the key, Eve must distinguish states $\hat{\rho}_0$ and $\hat{\rho}_1$. To do this, she must make measurements
with a narrow-band detector (measurements of $E_0$ or $E_1$). If there were no control state $\hat{\rho}_c$ containing both spectral components with frequencies $\omega_0$ and $\omega_1$, as a result of the mutual orthogonality of the information states, it would be possible to determine uniquely which state is present in the line. However, their nonorthogonality to the control state will inevitably lead to errors since there will always be measurements with an undetermined result. For instance, if $\hat{\rho}_c$ is present in the line and Eve measured $E_0$ and obtained a nonzero result, it is impossible to uniquely determine which state, $\hat{\rho}_c$ or $\hat{\rho}_0$, gave this result. Resending $\hat{\rho}_0$ instead of the true state $\hat{\rho}_c$ leads to a change in Bob’s measurement statistics. It is also impossible to discern in one measurement that both spectral components with energies $\omega_0$ and $\omega_1$ are present simultaneously in a state because of the orthogonality of the components since this requires measurements by means of more general (non-von Neumann) measurements, which is guaranteed by the theorem in Ref. [3].

No unique information can be obtained on the simultaneous presence of spectral components using more general (non-von Neumann) measurements, which is guaranteed by the theorem in Ref. [3].

3. Possible implementation of a cryptosystem

We shall now discuss a possible experimental implementation in which the carriers are three single-photon states of the type

$$
|1_{\omega_0}\rangle = a_{\omega_0}^{\pm}|0\rangle, \quad |1_{\omega_1}\rangle = a_{\omega_1}^{\pm}|0\rangle,
$$

$$
|1_c\rangle = f_0 e^{-i\omega_0 t_0}a_{\omega_0}^{+}|0\rangle + f_1 e^{-i\omega_1 t_0}a_{\omega_1}^{+}|0\rangle
$$

with the corresponding density matrices

$$
\hat{\rho}_{0,1} = |1_{\omega_0,1}\rangle \langle 1_{\omega_0,1}|, \quad \hat{\rho}_c = |1_c\rangle \langle 1_c|,
$$

where $a_{\omega_i}^{\pm}$ is the creation operator of a Fock monochromatic state with the frequency $\omega_i$ ($i, 1$) and polarization $\epsilon$, and $|0\rangle$ is the vacuum state. Quite clearly, a strictly monochromatic state is an idealization. However, there are no fundamental constraints on the formation of states arbitrarily close to monochromatic.

The measurement procedures described above may be implemented by using a fast (fairly wide-band) photodetector operated in a waiting regime, and two narrow-band filters at frequencies $\omega_0$ and $\omega_1$. From standard photodetection theory, the detection probability is proportional to the first-order correlation function of the field

$$
\Gamma^{(1)}(t) = \text{Tr}\left\{\hat{\rho}_i \hat{E}^{(-)}(x,t) \hat{E}^{(+)}(x,t)\right\},
$$

where

$$
\hat{E}^{(+)}(x,t) = i \sum_{n} \sqrt{\frac{\hbar \omega_n}{2 V}} a_{\epsilon_n \omega_n} \exp(-i\omega_n t + ik_n x),
$$

$$
\hat{E}^{(-)}(x,t) = -i \sum_{n} \sqrt{\frac{\hbar \omega_n}{2 V}} a_{\epsilon_n \omega_n}^{+} \exp(i\omega_n t - ik_n x),
$$

and $V$ is the normalization volume. At this stage it is more convenient to use a formal normalization of the states in a finite volume (see below). We can even use unnormalized states. With this definition the probabilities of the measurement outcomes will also be unnormalized, but since only the relative probability is important for the different measurements, this lack of normalization is unimportant. Measurements of the correlation function of the field (the instantaneous intensity) $\Gamma^{(1)}(t)$ are a realization of the $E_{0,1}$ and $E(dt)$ measurements described above in the sense that the statistics of the outcomes gives the same information on the states as the statistics of the $E_{0,1}$ and $E(dt)$ measurements. A combination of measurements using a fast photodetector and measurements using two narrow-band filters and the same photodetector can provide information on the amplitude $|f_{0,1}|$ and
relative phase of the components $f_0$ and $f_1$, which exhausts the information on the states (see also Ref. [22]).

The probability $p$ of a photon being recorded in the time interval $(t, t+dt)$ by an ideal photodetector is proportional to the field intensity $I(t) \propto \Gamma^{(1)}(t)$ (Ref. [28]):

$$p(t)dt \propto I(t)dt = \Gamma^{(1)}(t)dt. \quad (11)$$

If the photodetector trigger time is $\tau_{\text{det}} \ll |\omega_1 - \omega_0|$, this photodetector implements $E(dt)$ measurements in the sense indicated above. It can be seen from Eq. (10) that for state (9) the recording probability with allowance for Eqs. (9)–(11) has the form:

$$p(t)dt \propto I(t)dt = \Gamma^{(1)}(t)dt = \sqrt{\omega_0 f_0} \exp \left[ -i \omega_0 (t - t_0) + \frac{i k_0 L}{c} \right] +$$

$$+ \sqrt{\omega_1 f_1} \exp \left[ -i \omega_1 (t - t_0) + \frac{i k_1 L}{c} \right] \frac{dt}{2}, \quad (12)$$

where $k_{0,1}$ are the wave vectors corresponding to the frequencies $\omega_{0,1}$ and $L$ is the length of the communication channel (we assume that the measurement is made at a distance $L$ from the transmitting end).

Measurements of the amplitude of the spectral components $f_{0,1}$ are made using a pair of narrow-band filters which cut out the frequencies $\omega_{0,1}$ prior to photodetection, and the same photodetector. The recording probability, in accordance with Eqs. (9)–(11), does not depend on time:

$$p_c(t)dt \propto \Gamma^{(1)}(t)dt = \begin{cases} \frac{\hbar \omega_0}{V} |f_0|^2 dt, & E_0 \text{ measured}, \\ \frac{\hbar \omega_1}{V} |f_1|^2 dt, & E_1 \text{ measured}, \end{cases}$$

$$p_{0,1}(t)dt \propto \Gamma^{(1)}(t)dt = \begin{cases} \frac{\hbar \omega_0}{V} \times 1 dt, & E_0 \text{ measured for } \hat{\rho}_0, \\ 0 & E_1 \text{ measured for } \hat{\rho}_0, \\ \frac{\hbar \omega_1}{V} \times 1 dt, & E_1 \text{ measured for } \hat{\rho}_1, \\ 0 & E_0 \text{ measured for } \hat{\rho}_1. \end{cases} \quad (13)$$

A qualitative version of a cryptosystem is shown in Fig.1. Before entering the line, the signal from a single-photon source is expanded into a spectrum from which are cut either one of the frequencies ($\omega_0$ or $\omega_1$), or both spectral components with the frequencies $\omega_0$ and $\omega_1$. The $E(dt)$ measurements are made using a fast photodetector operating in a waiting mode. In this mode the occurrence of an event (its recording) will take place at a random time, not chosen by the experimentalist. This differs from the $E(dt)$ measurements made at a time preselected by the experimentalist in the range $(t, t+dt)$; the probability of recording at this time is described by the density $p_c(t)$. In this case, the $E(dt)$ measurement cannot be understood as a measurement by the fast photodetector which has an input diaphragm in front of it which is opened in the interval $(t, t+dt)$. This procedure also corresponds to some measurement, but not to an $E(dt)$ measurement. The integrated recording probability at time $T$ is given by

$$P(T) = \int dt p_i(t), \quad i = c, 0, 1,$$

from which the probability density in the time interval $(t, t+dt)$ can be obtained by differentiating with respect to the upper limit. The $E(dt)$ measurement essentially contains information on the “interference” of the different spectral component within a single quantum state (information on the relative phase of the components with the frequencies $\omega_0$ and $\omega_1$). Thus, it is fundamentally important that in different messages the state $\hat{\rho}_c$ is prepared so that the relative phase of the spectral components is the same. Otherwise, the time-oscillating (interference) component with the frequency $\omega_1 - \omega_0$ in
the probability $p_c(t)$ will not be reproduced at the same times in different attempts. The problem of preparing a single-photon state where the relative phase of the components is fixed can be solved as follows. Let us assume that a two-level system exists with a spin-nondegenerate electron spectrum (for example, a quantum dot with Coulomb interaction, see details given in Ref. [29]).

Resonant illumination with a square-wave pulse can transfer the system to the excited (quasisteady) state. We shall assume that the duration of the square-wave pulse is much shorter than the radiative recombination time ($\tau_\pi \ll \tau_R$). The duration of the square-wave pulse can be made substantially shorter than $\tau_R$ with some margin for not impairing the condition of resonant illumination. This implies that the time of excitation $t_0$ is determined to within $\sim \tau_\pi \ll \tau_R$. After the square-wave pulse has been switched off, the free evolution of the system comprising an electron in the excited state plus the electromagnetic field in the vacuum state leads to recombination of the electron and the appearance of a single-photon packet with the characteristic spectral width $\Delta \omega \approx 1/\tau_R$. The single-photon packet is defined as [30, 31]

$$|1_f\rangle = \sum_k f_k a_\omega^+ |0\rangle, \quad \sum_k |f_k|^2 = 1. \quad (14)$$

The average number of photons in the packet is

$$n = \langle 1_f | a_\omega^+ a_\omega | 1_f \rangle = 1, \quad (15)$$

and physically this implies that recording by an ideal wide-band photodetector (which captures all the spectral components) leads to triggering with a probability of unity. Recording by an ideal narrow-band detector at the frequency $\omega_n$ leads to a probability $|f_n|^2 < 1$ of triggering.

Since each system at time $t_0$ starts from the same state, in different messages the single-photon packets are the same (the phase of all the spectral components determined by the factors $\exp(-i\omega t_0)$ is the same in different messages). Cutting out two narrow spectral components from the spectrum conserves their relative phase. In fact, the cutting of the spectral components is formally described as the action of the projector [30, 31]

$$E_0 + E_1 = (|1_{\omega_0}\rangle \langle 1_{\omega_0}| + |1_{\omega_1}\rangle \langle 1_{\omega_1}|),$$

after which the density matrix of the single-photon wave packet is transferred to a new state (see Refs. [25, 26], and [32])

$$\hat{\rho}_{in}(t) = \left\{ \sum_k \exp [-i\omega_k(t-t_0)] f_k |1_{\omega_k}\rangle \right\} \times$$

$$\times \left\{ \sum_{k'} \langle 1_{\omega_{k'}} | f_k^* \exp [i\omega_k(t-t_0)] \right\} \rightarrow \frac{1}{\text{Tr} \{\hat{\rho}_{in}(t)(E_0 + E_1)\}} (E_0 + E_1) \times$$

$$\times \left\{ \sum_k \exp [-i\omega_k(t-t_0)] f_k |1_{\omega_k}\rangle \right\} \left\{ \sum_{k'} \langle 1_{\omega_{k'}} | f_k^* \exp [i\omega_k(t-t_0)] \right\} (E_0 + E_1) \rightarrow$$

$$\rightarrow \frac{1}{|f_0|^2 + |f_1|^2} \{ \exp [-i\omega_0(t-t_0)] f_0 |1_{\omega_0}\rangle + \exp [-i\omega_1(t-t_0)] f_1 |1_{\omega_1}\rangle \} \times$$

$$\times \{ \langle 1_{\omega_0} | f_0^* \exp [i\omega_0(t-t_0)] \rangle + \langle 1_{\omega_1} | f_1^* \exp [i\omega_1(t-t_0)] \rangle \}. \quad (16)$$

Physically this implies that if an ideal wide-band photodetector is placed after the filters, in a large number of repeated tests it will only be actuated in the fraction $\text{Tr} \{\hat{\rho}_{in}(t)(E_0 + E_1)\}$ of the total number of cases.

The relative phase of the components with frequencies $\omega_0$ and $\omega_1$ is determined by their phase at the preparation time which is attainable in principle, as described above. Thus, provided that $\tau_\pi \ll \tau_R \ll 1/|\omega_1 - \omega_0|$ we can assume that in different messages the temporal interference pattern stays

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\(^1\)Strictly speaking, an ideal filter corresponds to a projector on the subspace of states with frequencies $\omega_0$, $\omega_1$ and all the occupation numbers $E_{\omega_0} + E_{\omega_1} = \sum_{n=0}^{\infty} (|n_{\omega_0}\rangle \langle n_{\omega_0}| + |n_{\omega_1}\rangle \langle n_{\omega_1}|)$, but this does not alter the results.
in place. In different messages the interference pattern only “floats” to the extent of the inaccuracy of the initial preparation time \( \delta t_0 \) by the amount \( \delta t_0 \leq \tau_\pi \ll T = 2\pi/|\omega_1 - \omega_0| \) which is substantially less than the period of the temporal interference pattern.

Assuming that the square-wave pulse duration is \( \tau_\pi \sim 10^{-12} \text{s} \) (see Ref. [29]) and the radiative recombination time \( \tau_R \sim 10^{-10} \text{s} \) (in this case the spectral width of the initial single-photon state is \( \Delta \omega \sim 10^{10} \text{Hz} \)), and spectral components of width \( \sigma \approx 10^7 \text{Hz} \) separated by the distance \( \delta \omega = |\omega_1 - \omega_0| \sim 10^8 \text{Hz} \) are cut out (which is still very far from the limits now attainable), the required photodetector response speed is satisfied for \( \tau_{\text{det}} \sim 10^{-9} \text{s} \). The chain of inequalities \( \tau_\pi \ll \tau_R \ll \tau_{\text{det}} \ll 1/\delta \omega \) is then satisfied with some margin. The efficiency as a result of cutting out narrow spectral components of width \( \sigma \sim 10^7 \text{Hz} \) from a spectrum of width \( \Delta \omega \sim 10^{10} \text{Hz} \) is \( \sim \sigma/\Delta \omega \sim 10^{-3} \).

However, at least the problem of a strictly single-photon source can be solved in principle. We shall estimate the accuracy in fixing the length of the communication channel. Changes in the length of the fiber-optic line also lead to blurring of the interference pattern as a result of the presence of terms with \( k_0 L \) in the exponent in formula (12). The changes in the relative phase of the spectral components as a result of variation of the line length \( \delta L \) should satisfy the condition

\[
|k_1 - k_0| \delta L \approx |\omega_1 - \omega_0| \delta L/c \ll 2\pi,
\]

and permissible variations of the line length should lead to a relative phase shift much less than \( 2\pi \). This gives the estimate

\[
\delta L \ll 2\pi c/\delta \omega \approx 10^2 \text{cm},
\]

which is a fairly soft condition.

The interference pattern may also become blurred as a result of the polarization vector rotating at different speeds in the different frequency components. However, if the position of the cable is fixed, the interference pattern may be precalibrated. In this case, any changes will only be attributed to the different optical paths of the frequency components, i.e., variation of the line length. This condition is clearly noncritical. The frequency dispersion of the dielectric constant of the optical fiber can also lead to smoothing of the amplitude of the oscillations of the interference pattern. The longer the line, the stronger this smoothing. However, estimates[21] show that if the width of the spectral components is \( \sigma \approx 10^7 \text{Hz} \), the dispersion has an influence at far greater lengths than the attenuation.

Attenuation does not influence the secrecy of the system and only reduces its efficiency by increasing the fraction of idle measurements.

The states with infinitely narrow spectral components analyzed above are an idealization and are unsuitable for transmission along a communication channel because of their formally infinite duration. In real experiments we can only prepare states with a finite line width (the preparation of strictly monochromatic photon states would require a formally infinite time). The information states can be single-photon states of the type (14) with Gaussian spectral densities

\[
|\omega_{0,1},c\rangle = \int_0^{\infty} f_{0,1,c}(\omega) a^+(\omega)|0\rangle, \quad [a(\omega), a^+(\omega')] = \delta(\omega - \omega')\hat{I},
\]

\[
E^{(+)}(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp \left[ -i\omega \left( t - \frac{x}{c} \right) \right] a(\omega) d\omega,
\]

\[
f_{0,1}(\omega) = \frac{1}{(2\pi \sigma^2)^{1/4}} \exp \left[ \frac{-(\omega - \omega_{0,1})^2}{4\sigma^2} \right] \exp(-i\omega t_0)
\]

and a control state containing both narrow-band Gaussian components with amplitudes \( f_0 \) and \( f_1 \):

\[
f_c(\omega) = \frac{\text{const}}{(2\pi \sigma^2)^{1/4}} \left\{ f_0 \exp \left[ \frac{-(\omega - \omega_0)^2}{4\sigma^2} \right] + f_1 \exp \left[ \frac{-(\omega - \omega_1)^2}{4\sigma^2} \right] \right\} \exp(-i\omega t_0),
\]

where the normalization constant is

\[
\text{const} = \left\{ |f_0|^2 + |f_1|^2 + \sqrt{2} \text{Re} f_0 f_1^* \exp \left[ \frac{(\omega_0^2 + \omega_1^2 - \omega_0 \omega_1)}{2\sigma^2} \right] \right\}^{-1}.
\]
Measurements of narrow spectral components using suitable Gaussian filters yield a weak time dependence of the measurement outcomes unlike the previous analysis of strictly monochromatic states where the probability of the outcome did not depend on time. The corresponding probability density of the results has the form

$$p(t)dt \propto I(t)dt = 2\sqrt{2\pi}\sigma^2 \exp \left[-2\sigma^2(t - t_0 - L/c)^2\right] dt,$$

from which it follows in particular that the probability of recording by a photodetector in the waiting mode only tends to unity if the waiting time $T$ exceeds the reciprocal width of the spectrum ($T \geq 1/\sigma$). This factor is consistent with intuitive ideas on the prolonged recording time of a narrow-band state.

The probability density of the measurement outcomes for the control state has the form

$$p_c(t)dt \propto I(t)dt = \text{const} \cdot 2\sqrt{2\pi}\sigma^2 \exp \left[-2\sigma^2(t - t_0 - L/c)^2\right] \times$$

$$\left\{|f_0|^2 + |f_1|^2 + 2\text{Re}(f_0 f_1^* \exp \left[-i(\omega_0 - \omega_1)(t - t_0 - L/c)\right])\right\} dt.$$

The interference oscillating component is well-defined under the condition $\sigma \ll |\omega_1 - \omega_0|$.

4. Conclusions

Although various versions of quantum cryptosystems have been proposed, in the author’s view there is some indeterminacy associated with the following. The evidence of secrecy in quantum cryptography using two nonorthogonal states proposed in Ref. [3] implies that the states are stationary and belong to the same energy. Otherwise, for nonstationary states the projectors $E_0$ and $E_1$ would differ at different times. The nonorthogonality of the stationary states implies that they correspond to the same energy. Otherwise the stationary states belonging to different energies would automatically be orthogonal. In this sense, the protocol for the stationary states exists as it were outside time. Attempts to introduce the time explicitly in the exchange protocol [14, 17] still use reasoning from Ref. [3] for stationary states as evidence of secrecy (see, for example, Ref. [17]). The stationary state are infinite in time. A similar situation arises here. Evidence of secrecy for states with infinitely narrow spectral densities, which are thus infinitely extended in time, is also based on the reasoning given in Ref. [3]. For the case where the state space of the system is infinite-dimensional (described by a continuous variable), evidence of secrecy of the same degree of rigoroussness as in Ref. [3] is not obtained, as far as we are aware. In this sense, the author takes the view that evidence of secrecy is not obtained for real-time quantum cryptosystems.

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Figure 1: Qualitative version of cryptosystem. The signal from a single-photon source is directed to a “prism” beyond which is a screen which transmits the signal either with the frequency $\omega_0$ (logic zero, upper diaphragm open) or $\omega_1$ (logic one, lower diaphragm open), or both frequencies (control signal, both diaphragms open). At the receiving end the measurement procedure is arranged similarly. Upper diaphragm open — measurement of $E_0$, lower open — measurement of $E_1$, and both open — measurement of $E(dt)$. The diaphragms are open or shut during an entire specific message.
