Entropy and Poincaré recurrence from a geometrical viewpoint

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Abstract
We study Poincaré recurrence from a purely geometrical viewpoint. In (Downarowicz and Weiss 2004 Illinois J. Math. 48 59–69) it was proven that the metric entropy is given by the exponential growth rate of return times to dynamical balls. Here we use combinatorial arguments to provide an alternative and more direct proof of this result and to prove that minimal return times to dynamical balls grow linearly with respect to its length. Some relations using weighted versions of recurrence times are also obtained for equilibrium states. Then we establish some interesting relations between recurrence, dimension, entropy and Lyapunov exponents of ergodic measures.

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1. Introduction

Since it was introduced in dynamical systems more than fifty years ago, entropy has become an important ingredient in the characterization of the complexity of dynamical systems in both topological and measure theoretical senses. From the measure theoretical viewpoint the metric entropy of invariant measures turned out to be a surprisingly universal concept in ergodic theory since it appears in the study of different subjects such as information theory, Poincaré recurrence and in the analysis of either local or global complexities. Just as an illustration of its universal nature, metric entropy is characterized as the exponential growth rate of the measure of decreasing partition elements and dynamical balls (see e.g. [12] for the Shannon–McMillan–Breiman’s theorem and [2]), the number of dynamical balls and partition elements necessary to cover a relevant part of the phase space (see e.g. [9]) and the recurrence rate to elements of a given partition (see, e.g. [15]). We refer the reader to [10] for a very complete survey on the notion of entropy in dynamical systems.
A particularly interesting and deep connection is the one established between metric entropy and Poincaré recurrence. Given a measurable dynamical system \( f \), it follows by the pioneering work of Poincaré that the set of recurrent points has full probability. This means that the iterates of almost every point (with respect to an arbitrary invariant probability measure \( \mu \)) will return arbitrarily close to itself. In particular, for any positive measure set \( A \) the function

\[ R_A(x) = \inf \{ k \geq 1 : f^k(x) \in A \} \]

is finite almost everywhere in \( A \). Given a decreasing sequence of partitions \( U_n \) it is natural to look for a limiting behaviour of the return times \( R_{U_n} \) in finer scales. Such a limiting behaviour turned out to exist for ergodic stationary processes and it coincides with the metric entropy of the system. More precisely, Ornstein and Weiss [15] proved that the entropy \( h_\mu(f, Q) \) of an ergodic measure \( \mu \) with respect to a partition \( Q \) is given by the (almost everywhere) well-defined limit

\[ h_\mu(f, Q) = \lim_{n \to \infty} \frac{1}{n} \log R_n(x, Q), \]  

where \( R_n(x, Q) = \inf \{ k \geq 1 : f^k(x) \in Q^{(n)}(x) \} \) is the \( n \)th return time (with respect to the partition \( Q \)), \( Q^{(n)} = \bigvee_{j=0}^{n-1} f^{-j}Q \) is the dynamically generated partition and \( Q^{(n)}(x) \) denotes the element of \( Q^{(n)} \) that contains the point \( x \). Consequently, the metric entropy is the supremum of the exponential growth rates of Poincaré recurrences over all possible choices of partitions. Moreover, when return times are weighted with respect to some potential we can recover estimates concerning the pressure. Some interesting formulae concerning pressure and return times were also obtained in [13].

Also very important is the notion of minimal return times that we now describe. By Poincaré recurrence theorem, for every invariant probability measure \( \mu \) the minimal return time \( S(A) \) to any positive measure set \( A \) defined by

\[ S(A) = \inf \{ k \geq 1 : f^{-k}(A) \cap A \neq \emptyset \} \]

is finite. Heuristically it is natural to expect the limiting behaviour of \( S(U_n) \) in finer scales \( U_n \), if it exists, to be simpler than the the one presented by regular return times \( R_{U_n} \). In fact, Afraimovich et al [1] proved that maps with a specification property satisfy

\[ \lim_{n \to \infty} \frac{S_n(x, Q)}{n} = 1 \quad \text{for } \mu\text{-almost every } x \]

for every ergodic measure \( \mu \) (provided that \( h_\mu(f, Q) > 0 \)), where \( S_n(x, Q) \) denotes the \( n \)th minimal return time to the partition element \( Q^{(n)}(x) \). Hence, contrary to the exponential growth presented by regular return times, minimal return times grow linearly with \( n \), i.e. the time needed for a cylinder to return to itself is asymptotically given by its size.

To the best of our knowledge, the class of dynamical systems for which return time statistics are studied are mostly those that present some (finite or countable) reference partition with some Markovian property or such that the bounded distortion property holds. We refer the reader to [5–7, 16, 23] just to quote some recent contributions. However, the existence of such partitions itself constitutes a problem even in a context of non-uniform hyperbolicity. We refer the reader to [18] for a recent important contribution on the construction of such partitions for non-uniformly expanding maps.

So, we turn our attention to return times to purely geometrical objects as (regular and dynamically defined) balls. In fact, not only regular and dynamically defined balls arise naturally in a non-uniformly hyperbolic context as the study of Poincaré recurrence to these purely geometrical objects encloses much information about invariant measures. Given an
invariant measure $\mu$ the upper and lower pointwise dimensions $d^{u}_{\mu}(x)$ and $d^{l}_{\mu}(x)$ are defined by the limits

$$d^{u}_{\mu}(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad d^{l}_{\mu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$ 

Its Hausdorff dimension $\dim_{H}(\mu)$, defined as the infimum of the Hausdorff dimension of sets of full $\mu$-measure, satisfies $d^{u}_{\mu}(x) \leq \dim_{H}(\mu) \leq d^{l}_{\mu}(x)$ (see, e.g. [17]). Barreira et al [3] proved that any hyperbolic and ergodic measure $\mu$ of a $C^{1+\alpha}$ diffeomorphism is exact dimensional, i.e. the upper and lower pointwise dimensions $d^{u}_{\mu}(x)$ and $d^{l}_{\mu}(x)$ do exist and coincide almost everywhere. By [25] the limit is equal to $\dim_{H}(\mu)$. Moreover, Barreira and Saussol [4] proved that the pointwise recurrence rates

$$\mathcal{R}(x) = \limsup_{r \to 0} \frac{\log R_{B(x,r)}(x)}{-\log r} \quad \text{and} \quad \mathcal{R}(x) = \liminf_{r \to 0} \frac{\log R_{B(x,r)}(x)}{-\log r}$$

satisfy $\mathcal{R}(x) \leq d^{u}_{\mu}(x)$ and $\mathcal{R}(x) \leq d^{l}_{\mu}(x)$ in general, and conjectured that any $C^{1+\alpha}$ diffeomorphism $f$ and any hyperbolic ergodic measure $\mu$ should satisfy

$$\mathcal{R}(x) = \mathcal{R}(x) = \dim_{H}(\mu), \quad \mu\text{-almost everywhere.} \quad (2)$$

We note that (2) was proved to hold for dynamical systems that either present some hyperbolicity (e.g. axiom A diffeomorphisms [4] and piecewise monotone interval maps whose derivative has $p$-variation [21]) or that satisfy a rapidly mixing property as in [20]. In [21, 22] the minimal recurrence rates

$$\mathcal{S}(x) = \limsup_{r \to 0} \frac{S(B(x,r))}{-\log r} \quad \text{and} \quad \mathcal{S}(x) = \liminf_{r \to 0} \frac{S(B(x,r))}{-\log r}$$

are studied. In the case of endomorphisms it is shown that if $\mu$ is a positive entropy ergodic measure and $\lambda_{\mu}, \Lambda_{\mu} > 0$ denote, respectively, the smallest and the largest Lyapunov exponents of $(f, \mu)$ then $\mathcal{S}(x) \geq 1/\Lambda_{\mu}$ and, under some specification property, that $\mathcal{S}(x) \leq 1/\lambda_{\mu}$. In particular, a wide family of piecewise monotone interval maps with $p$-variation satisfy

$$\mathcal{S}(x) = \mathcal{S}(x) = 1/\lambda_{\mu} \quad \text{for $\mu$-almost every $x$,} \quad (3)$$

where $\lambda_{\mu}$ denotes the Lyapunov exponent of $\mu$.

Our purpose is to study return times to more natural topological objects than partitions. Indeed, we characterize the metric entropy as the exponential growth rate of return times to dynamical balls and show that minimal return times to dynamical balls grow linearly with respect to their length. These constitute geometrical counterparts to some results in [1, 15]. Afterwards these results are used to establish some new results relating recurrence, dimension and Lyapunov exponents. Although our first result appeared previously in [8] as a consequence of a generalization of Shannon–McMillan–Breiman's theorem, computing the exponential decreasing rate of the measure of partition elements determined when a point enters a given set, we believe that the combinatorial arguments used here can be applied in some different contexts as in the study of hitting time statistics and fluctuations of return times. In fact, one expects the fluctuations of the return times in theorem A to be log-normal with respect to any measure satisfying a weak Gibbs property as in [23, 24, 26]. Using [19] this is the case provided exponential hitting time statistics. However, to the best of our knowledge, there are no known examples where exponential return time statistics to dynamical balls have been obtained for (multidimensional) dynamical systems outside the uniformly hyperbolic setting. It seems that the combinatorial arguments used here can be useful to recover exponential return time statistics for dynamical balls from the corresponding result for elements of some relevant partition. In particular, this should apply to the non-uniformly hyperbolic maps considered in [14, 23].
This paper is organized as follows. In section 2 we present the main result. Some definitions and preliminaries are presented in section 3. In section 4 we study regular and return times to dynamical balls. The proofs of theorems A and B are given in sections 4.1 and 4.2, respectively. Finally, in section 5 we apply the previous results to study dimension of ergodic measures and prove proposition A and theorem C.

2. Statement of the main results

In this section we introduce some necessary definitions and state our main results. Throughout, assume that \(X\) is a compact metric space and let \(f : X \to X\) be a continuous transformation. Given \(\varepsilon > 0\) and \(n \geq 1\) the dynamical ball \(B(x, n, \varepsilon)\) is the set \(B(x, n, \varepsilon) = \{y \in X : d(f^j(x), f^j(y)) < \varepsilon\text{ for every }0 \leq j \leq n - 1\}\). We define the \(n\)th return time \(R_n(x, \varepsilon)\) to the dynamical ball \(B(x, n, \varepsilon)\) by

\[
R_n(x, \varepsilon) = \inf\{k \geq 1 : f^k(x) \in B(x, n, \varepsilon)\}.
\]

We recall the following result that follows from the more general result in [8].

**Theorem A.** Let \(\mu\) be an ergodic \(f\)-invariant probability measure. The limits

\[
\overline{h}(f, x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R_n(x, \varepsilon) \quad \text{and} \quad \underline{h}(f, x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log R_n(x, \varepsilon)
\]

exist for \(\mu\)-almost every \(x\) and coincide with the metric entropy \(h_\mu(f)\).

Let us comment on the assumption of ergodicity in the above theorem. By ergodic decomposition every \(f\)-invariant probability measure \(\mu\) can be decomposed as a convex combination of ergodic measures \(\mu_x\), i.e., \(\mu = \int \mu_x \, d\mu(x)\). Moreover, since the metric entropy map is affine then \(h_\mu(f) = \int h_{\mu_x}(f) \, d\mu(x)\). So, applying theorem A to each ergodic component \(\mu_x\) and integrating with respect to \(\mu\) we obtain the following immediate consequence.

**Corollary A.** If \(\mu\) is an \(f\)-invariant probability measure then the limits \(\overline{h}(f, x)\) and \(\underline{h}(f, x)\) defined above do exist for \(\mu\)-almost every \(x\). Moreover, the metric entropy \(h_\mu(f)\) satisfies

\[
h_\mu(f) = \int \overline{h}(f, x) \, d\mu(x) = \int \underline{h}(f, x) \, d\mu(x).
\]

Given a continuous potential \(\phi : X \to \mathbb{R}\) the metric pressure \(P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu\) of the invariant measure \(\mu\) with respect to \(f\) and \(\phi\) can also be written using weighted recurrence times. This is a consequence of Birkhoff’s ergodic theorem and theorem A as we now explain. Indeed, given an \(f\)-invariant and ergodic probability measure \(\mu\) there exists a full measure set \(\mathcal{R}\) such that

\[
\lim_{n \to \infty} \frac{1}{n} S_n \phi(x) = \int \phi \, d\mu \quad \text{and} \quad h_\mu(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R_n(x, \varepsilon)
\]

for every \(x \in \mathcal{R}\). Given any \(\delta > 0\) and \(x \in \mathcal{R}\) it follows from uniform continuity of \(\phi\) the existence of \(\varepsilon_\delta > 0\) such that \(|S_n \phi(B(x, n, \varepsilon)) - S_0 \phi(x)| < \delta n\) for every \(n \geq 1\) and every \(0 < \varepsilon < \varepsilon_\delta\), where \(S_n \phi(B(x, n, \varepsilon)) = \sup\{\sum_{j=0}^{n-1} \phi(f^j(y)) : y \in B(x, n, \varepsilon)\}\). In consequence,

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} S_n \phi(B(x, n, \varepsilon)) + \frac{1}{n} \log R_n(x, \varepsilon) \right] - \left( h_\mu(f) + \int \phi \, d\mu \right) < 2\delta
\]

for every small \(\varepsilon > 0\). Hence we deduced the following result, relating the metric pressure to appropriate weighted return times.
Corollary B. Let $\mu$ be an $f$-invariant and ergodic probability measure. Then

$$P_n(f, \phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \epsilon S_n(\phi, \epsilon) + \mathcal{R}_n(\epsilon)$$

for $\mu$-a.e. $x$.

Now we turn our attention to minimal return times. We define the $n$th minimal return time $S_n(x, \epsilon)$ to the dynamical ball $B(x, n, \epsilon)$ by

$$S_n(x, \epsilon) = \inf\{k \geq 1 : f^{-k}(B(x, n, \epsilon)) \cap B(x, n, \epsilon) \neq \emptyset\}.$$

Clearly $S_n(x, \epsilon) \leq R_n(\epsilon)$ and so these minimal return times are finite in a set of total probability. Moreover, we will prove that minimal return times $S_n$ to dynamical balls grow linearly with $n$. First we recall a definition. We say that $f$ satisfies the specification property if, given $\delta > 0$ there is an integer $N \geq 1$ such that the following holds: for any $k \geq 1$, any points $x_1, \ldots, x_k$, any integers $0 = a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ satisfying $a_i + 1 - b_i > N(\delta)$ and any integer $p \geq b_k + N(\delta)$ there exists a point $x \in X$ such that $f^p(x) = x$ and $d(f^j(x), f^j(x)) < \delta$ for every $a_i \leq j \leq b_i$ and $1 \leq i \leq k$. Our second main result is as follows.

Theorem B. Assume that $f$ has the specification property. If $\mu$ is an $f$-invariant, ergodic probability measure such that $h_{\mu}(f) > 0$, the limits

$$\overline{S}(x) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} S_n(x, \epsilon)$$

and

$$\underline{S}(x) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} S_n(x, \epsilon)$$

exist and are equal to one for $\mu$-almost every $x$.

It is not hard to check that this result also holds true if $\mu$ satisfies the non-uniform specification property of [22]. However, we shall not use or prove this fact. The final part of this section is devoted to the discussion of the relation between entropy, dimension and Lyapunov exponents.

Proposition A. Assume that $f : X \to X$ is a continuous transformation and that there exist constants $\delta, \lambda, \Lambda > 0$ such that $\lambda d(x, y) \leq d(f(x), f(y)) \leq \Lambda d(x, y)$ for every $x, y \in X$ so that $d(x, y) < \delta$. If $\mu$ is an $f$-invariant ergodic probability measure with positive entropy then

$$\frac{h_{\mu}(f)}{\log \lambda} \leq \overline{S}(x)$$

and

$$\frac{h_{\mu}(f)}{\log \lambda} \leq \overline{S}(x)$$

exist and are equal to one for $\mu$-almost every $x$. If, in addition, $f$ satisfies the specification property then $\overline{S}(x) \leq 1/\log \lambda$ for $\mu$-a.e. $x$.

If $f$ is a linear, conformal expanding tori endomorphism it satisfies the specification property and there exists $\lambda > 1$ so that $\lambda d(x, y) = d(f(x), f(y))$ for every close $x, y \in X$. Moreover, if $\mu$ is an ergodic measure its Lyapunov exponent is $\lambda_{\mu} = \log \lambda$. Using that $\dim_{\mu}(\mu) = h_{\mu}(f)/\lambda_{\mu}$ (see e.g. [25]) we obtain:

Corollary C. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a linear, conformal expanding tori endomorphism. If $\mu$ is a positive entropy ergodic $f$-invariant probability measure then (2) and (3) hold.

The following result is an asymptotic version of proposition A above for differentiable endomorphisms.

Theorem C. Assume that $f : M \to M$ is a $C^{1+\alpha}$ endomorphism, $\mu$ is a positive entropy $f$-invariant, ergodic probability measure and that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ are the Lyapunov exponents of $(f, \mu)$. Then $h_{\mu}(f)/\lambda_d \leq \overline{S}(x)$, $h_{\mu}(f)/\lambda_1 \geq \overline{S}(x)$ and $1/\lambda_d \leq \overline{S}(x)$ for $\mu$-almost every $x$. If, in addition, $f$ satisfies the specification property then $\overline{S}(x) \leq 1/\lambda_1$ for $\mu$-almost every $x$. 

Entropy and Poincaré recurrence from a geometrical viewpoint 2369
Since topologically mixing continuous interval maps satisfy the specification property then we get:

**Corollary D.** Let \( f : I \to I \) be a topologically mixing \( C^{1+\alpha} \) interval map and assume that \( \mu \) is a positive entropy, ergodic, \( f \)-invariant probability measure. Then (2) and (3) hold.

### 3. Preliminaries

#### 3.1. Metric entropy

We recall some characterizations of metric entropy. The first one is due to Katok [9, theorem I.1]. Given \( 0 < c < 1 \), denote by \( N(n, \varepsilon, c) \) the minimum number of dynamical balls necessary to cover a set of measure \( c \). Indeed, if \( \mu \) is ergodic Katok proved that for every \( c \in (0, 1) \)

\[
  h_{\mu}(f) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, c) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, c) \tag{4}
\]

Using Shannon–McMillan–Breiman’s theorem and arguments analogous to the ones used in the proof of (4) it is straightforward to check the following property.

**Lemma 3.1.** Let \( Q \) be a partition on \( X \) and \( c \in (0, 1) \) be given. Then

\[
  h_{\mu}(f, Q) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, c) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, c) \tag{5}
\]

where \( N(n, \varepsilon, c) \) denotes the minimum number of n-cylinders of the partition \( Q^{(n)} \) necessary to cover a set of measure \( c \).

#### 3.2. Combinatorial lemma

In this subsection we prove the following covering lemma for dynamical balls associated with points with slow recurrence to the boundary of a given partition.

**Lemma 3.2.** Let \( Q \) be a finite partition of \( X \) and consider \( \varepsilon > 0 \) arbitrary. Let \( V_\varepsilon \) denote the \( \varepsilon \)-neighbourhood of the boundary \( \partial Q \). For any \( \alpha > 0 \) there exists \( \gamma > 0 \) (depending only on \( \alpha \)) such that for every \( x \in X \) satisfying \( \sum_{j=0}^{n-1} 1_{V_\varepsilon}(f^j(x)) < \gamma n \) the dynamical ball \( B(x, n, \varepsilon) \) can be covered by \( e^{\alpha n} \) cylinders of \( Q^{(n)} \).

**Proof.** Fix an arbitrary \( \alpha > 0 \). Since \( B(z, \varepsilon) \subset Q(z) \) for every \( z \notin V_\varepsilon \), the itinerary of any point \( y \) in the dynamical ball \( B(x, n, \varepsilon) \) centred at a point \( x \in X \) satisfying \( \sum_{j=0}^{n-1} 1_{V_\varepsilon}(f^j(x)) < \gamma n \) will differ from the one of \( x \) by at most \( [\gamma n] \) choices of partition elements. Since there are at most \( (\#Q)^{\gamma n} \) such choices, this can be made smaller than \( e^{\alpha n} \) provided that \( \gamma > 0 \) is small enough. This completes the proof of the lemma. \( \square \)

### 4. Dynamical balls and recurrence

In this section our purpose is to prove theorems A and B that relate entropy with the usual and minimal return times to dynamical balls.
4.1. Proof of theorem A

We begin the proof of the theorem by noting that the limits in the statement of theorem A are indeed well defined almost everywhere. Given \( n \geq 1, \varepsilon > 0 \) and \( x \in X \) it holds that \( R_n(x, \varepsilon) \geq R_{n-1}(f(x), \varepsilon) \). Indeed, \( f^{R_n(x,\varepsilon)}(x) \in B(x, n, \varepsilon) \) implies that \( f^{R_n(x,\varepsilon)}(f(x)) \in f(B(x, n, \varepsilon)) \subset B(f(x), n-1, \varepsilon) \), which proves our claim. Define

\[
\underline{h}(x, \varepsilon) = \liminf_{n \to \infty} \frac{1}{n} \log R_n(x, \varepsilon) \quad \text{and} \quad \overline{h}(x, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log R_n(x, \varepsilon).
\]

It follows from the discussion in the previous paragraph that \( \underline{h}(f(x), \varepsilon) \leq h(x, \varepsilon) \) and \( h(f(x), \varepsilon) \leq \overline{h}(x, \varepsilon) \). Since \( \mu \) is ergodic these functions are almost everywhere constant and their values will be denoted by \( \underline{h}(\varepsilon) \) and \( \overline{h}(\varepsilon) \), respectively. Denote by \( \underline{h}(f) \) and by \( \overline{h}(f) \) the limits when \( \varepsilon \to 0 \) of the functions \( \underline{h}(\varepsilon) \) and \( \overline{h}(\varepsilon) \). Such limits do exist by monotonicity of the previous functions on \( \varepsilon \). Hence, to prove the theorem it is enough to show that

\[
\overline{h}(f) \leq h_{\mu}(f) \leq \overline{h}(f).
\]

To deal with the left-hand side inequality in (6), let \( \varepsilon > 0 \) be fixed and pick any partition \( \mathcal{Q} \) satisfying \( \mu(\partial \mathcal{Q}) = 0 \) and \( \text{diam}(\mathcal{Q}) < \varepsilon \). By construction we get that \( B(x, n, \varepsilon) \supset \mathcal{Q} n(x) \) for \( \mu \)-almost every \( x \) and every \( n \geq 1 \). Consequently, \( \underline{R}(\cdot, \mathcal{Q}) \geq \underline{R}(\cdot, \varepsilon) \) and, using Ornstein–Weiss’s theorem,

\[
\underline{h}(f) \leq \underline{h}(f, \mathcal{Q}) \geq \limsup_{n \to \infty} \frac{1}{n} \log \underline{R}_n(x, \varepsilon)
\]

for \( \mu \)-a.e. \( x \). Since \( \varepsilon \) was chosen arbitrary one gets that \( \underline{h}(f) \geq \overline{h}(f) \) as claimed.

We are left to prove the second inequality in (6). Assume, by contradiction, that \( \overline{h}(f) > \overline{h}(f, \mathcal{Q}) \) and pick a finite partition \( \mathcal{Q} \) such that \( \mu(\partial \mathcal{Q}) = 0 \) and \( \overline{h}(f) > b > a > \overline{h}(f) \). Fix \( 0 < \gamma < (b - a)/6 \) small such that lemma 3.2 holds for \( \alpha = (b - a)/2 \).

For every sufficiently small \( \varepsilon > 0 \), if \( V_{\varepsilon} \) denotes the \( \varepsilon \)-neighbourhood of the boundary \( \partial \mathcal{Q} \) then \( \mu(V_{\varepsilon}) < \gamma/2 \). By ergodicity and Birkhoff’s ergodic theorem we may choose \( N_0 \geq 1 \) large such that

\[
A = \left\{ x \in X : \sum_{j=0}^{n-1} 1_{V_{\varepsilon}}(f^j(x)) < \gamma n, \forall n \geq N_0 \right\}
\]

has measure larger than \( 1 - \gamma \). By lemma 3.2 each dynamical ball \( B(z, \ell, \varepsilon) \) of length \( \ell \geq N_0 \) centred at any point \( z \in A \) can be covered by \( e^{a\ell} \) cylinders of \( \mathcal{Q}(\ell) \). Furthermore, provided that \( N_1 \geq N_0 \) is large enough, the measure of the set

\[
B = \{ x \in X : \exists 0 \leq n \leq N_1 \text{ s.t. } R_n(x, \varepsilon) \leq e^{an} \}
\]

is also larger than \( 1 - \gamma \). For notational simplicity we shall omit the dependence of the sets \( A \) and \( B \) on the integers \( N_0 \) and \( N_1 \). Using once more Birkhoff’s ergodic theorem, we may take \( N_2 \geq 1 \) large enough so that

\[
\Lambda = \left\{ x \in X : \sum_{j=0}^{k-1} 1_{A \cap B}(f^j(x)) > (1 - 3\gamma)k, \forall k \geq N_2 \right\}
\]

has measure at least \( 1/2 \). We claim that there exists a constant \( C > 0 \) such that \( \Lambda \) is covered by \( C e^{bk} \) cylinders of \( \mathcal{Q}(k) \), for every large \( k \). This will imply that

\[
\underline{h}(f, \mathcal{Q}) = \lim_{n \to \infty} \frac{1}{n} \log \underline{N}(k, \mathcal{Q}, 1/2) < b,
\]

leading to a contradiction that will complete the proof of the theorem.
Fix \( x \in \Lambda \) and \( k \gg N_2 \). We proceed to divide the set \([0, 1, 2, \ldots, k]\) into blocks according to the recurrence properties of the orbit of \( x \). If \( x \not\in A \cap B \) then we consider the block \([0]\). Otherwise, we take the first integer \( N_0 \leq m \leq N_1 \) such that \( R_m(x, \varepsilon) \leq e^{|m|} \) and consider the block \([0, 1, \ldots, m - 1]\). We proceed recursively and, if \( \{1, \ldots, k'\} \) \((k' < k)\) is partitioned into blocks then the next block is \([k' + 1]\) if \( f^{k'+1}(x) \not\in A \cap B \) and it will be \([k' + 1, k' + 2, \ldots, k' + m']\) if \( f^{k'+1}(x) \in A \cap B \) and \( m' \) is the first integer in \([N_0, N_1]\) such that \( R_m(x, \varepsilon) \leq e^{|m'|} \). This process will finish after a finite number of steps and partitions \([1, 2, \ldots, k]\) according to the recurrence properties of the iterates of \( x \), except possibly the last block which has size at most \( N_1 \). We write the list of sequence of block lengths determined above as \( x = [m_1, m_2, \ldots, m_{(k)}] \). By construction there are at most \( 3yk \) blocks of size one. This enables us to give an upper bound on the number of \( k \)-cylinders \( Q^{(k)} \) necessary to cover \( \Lambda \). First note that since each \( m_i \) is either one or larger than \( N_0 \) then there are at most \( k/N_0 \) blocks of size larger than \( N_0 \). Hence there are at most

\[
\sum_{j \leq 3yk} \left( \frac{k}{N_0} + 3yk \right) \leq 3yk \left( \frac{k}{N_0} + 3yk \right) \frac{1}{3yk}
\]

possibilities to arrange the blocks of size one. Now, we give an estimate on the number of possible combinatorics for every prefixed configuration \( i = [m_1, m_2, \ldots, m_{(k)}] \), satisfying \( \sum m_j = k \) and \( \# \{ j : m_j = 1 \} < 3yk \). This will be done fixing elements from the right to the left. Define \( M_j = \sum_{i < j} m_i \). If \( x \in \Lambda \) is such that \( i(x) = i \) there are at most \( Q^\gamma \) possibilities to choose a symbol for each block of size one. Moreover, if \( 1 \leq \kappa \leq \ell \) is the first integer such that \( \sum_{j = 1}^{\ell} m_j < N_1 + e^{|N_1|} \) then there are at most \( (Q^\gamma)^{\kappa(N_1 + e^{|N_1|})} \) possibilities for choices of \( (m_1 + m_2 + \ldots + m_\ell) \)-cylinders with combinatorics \([m_1, \ldots, m_\ell] \). Recall that \( R_{m_{(i-1)}}(f^{m_{(i-1)}}(x), \varepsilon) \leq e^{m_{(i-1)}} \leq e^{\kappa(N_1 + e^{|N_1|})} \) and, by lemma 3.2, the dynamical ball \( B(f^{m_{(i-1)}}(x), m_{(i-1)}, \varepsilon) \) is contained in at most \( e^{m_{(i-1)}} \) cylinders in \( Q^{m_{(i-1)}} \). Hence the possible itineraries for the \( m_{(i-1)} \) iterates \( [f^{m_{(i-1)}}(x), \ldots, f^{m_{(i-1)}}(x)] \) may be chosen among \( e^{m_{(i-1)}} \) options corresponding to each of the \( e^{m_{(i-1)}} \) previously possibly distinct and fixed blocks of size \( m_{(i-1)} \) \([m_1, \ldots, m_{(i-1)}] \). This shows that there are at most \( e^{(\alpha m_{(i-1)})m_{(i-1)}} \) possible itineraries for the \( m_{(i-1)} \) iterations of \( f^{m_{(i-1)}}(x) \). Proceeding recursively for \( m_{(i-2)}, \ldots, m_2, m_1 \) we conclude, after some finite number of steps, that there exists \( C > 0 \) (depending only on \( N_1 \)) such that if \( \gamma \) was chosen small then \( \Lambda \) can be covered by

\[
3yk \left( \frac{k}{N_0} + 3yk \right) (Q^\gamma)^{N(1 + N_1 + e^{|N_1|})} \leq C e^{|k|} \]

cylinders in \( Q^{(k)} \). This proves the claim and finishes the proof of the theorem.

4.2. Proof of theorem B

The proof of the theorem is divided into two steps. On the one hand, the specification property guarantees that for every small \( \varepsilon > 0 \) there exists an integer \( N(\varepsilon) \geq 1 \) such that for any \( x \in X \) and \( n \geq N(\varepsilon) \) there is some periodic point of period smaller or equal to \( n + N(\varepsilon) \) in \( B(x, n, \varepsilon) \). Consequently, \( \limsup_{n \to \infty} \frac{1}{n} S_n(x, \varepsilon) \leq 1 \) for every small \( \varepsilon > 0 \), and proves that \( S(x, \varepsilon) \leq 1 \) almost everywhere.

So, to prove the theorem it remains to show that \( S(x, \varepsilon) \geq 1 \) for \( \mu \)-almost every \( x \). We claim that for any \( \eta < 1 \) there exists a measurable set \( E_\eta \) such that \( \mu(E_\eta) > 1 - \eta \) and \( \mu(x \in E_\eta : S_n(x, \varepsilon) < \eta n) \) is summable for every small \( \varepsilon \). Using Borel–Cantelli lemma it will follow that any point \( x \in E_\eta \) satisfies \( S_n(x, \varepsilon) > \eta n \) for all but finitely many values of
of $n$ and every small $\varepsilon$. The result will follow from the arbitrariness of $\eta$. The rest of this paragraph is devoted to the proof of the previous claim. Let $\eta \in (0, 1)$ be arbitrary and fix a small $0 < \alpha < \frac{1}{3} (1 - \eta) h_\mu(f)$. Consider a finite partition $Q$ satisfying $\mu(\partial Q) = 0$ and $3\alpha < (1 - \eta) h$, where $h = h_\mu(f, Q) > 0$. If $\varepsilon_0 > 0$ is small enough then $\mu(V_\varepsilon) < \gamma / 2$ for every $0 < \varepsilon < \varepsilon_0$, for $\gamma = \gamma(\alpha) > 0$ given by lemma 3.2. Using Birkhoff’s ergodic theorem, Shannon–McMillan–Breiman’s theorem and lemma 3.2, for almost every $x$ there exists an integer $N(x) \geq 1$ such that for every $n \geq N(x)$

$$\sum_{j=0}^{n-1} 1_{V_\varepsilon}(f^j(x)) < \gamma n$$

and, consequently, any dynamical ball $B(x, n, \varepsilon)$ is covered by a collection of $e^{\alpha n}$ cylinders of the partition $Q^{(n)}$. Pick $N \geq 1$ large such that set $E_\eta$ of points $x \in X$ satisfying (8) for every $n \geq N$ has measure greater than $1 - \eta$. Since $Q$ is finite there is $K > 0$ such that

$$K^{-1} e^{-(h+\alpha)n} \leq \mu(Q^{(n)}(x)) \leq Ke^{-(h-\alpha)n}$$

for every $x \in E_\eta$ and every $n \geq 1$. For $n \geq N$ we denote by $E_\eta(n, k)$ the set of points $x \in X$ satisfying (8) for every $n \geq N$ and the dynamical ball $B(x, n, \varepsilon)$ is contained in the subcollection of cylinders $Q_n \in Q^{(n)}(x, \varepsilon)$ whose iteration by $f^k$ intersects any of the $n$-cylinders of $Q^{(n)}(x, \varepsilon)$. Any such cylinder $Q_n$ is determined by its first $k$ symbols and by the at most $e^{\alpha n}$ possible strings following them. So, the number of those cylinders is bounded by $e^{\alpha n}$ times the number of cylinders in $Q^{(k)}$ that intersect $E_\eta$, that is, $e^{\alpha n} Ke^{(h+\alpha)k}$. Hence, if $n \geq N$

$$\mu(x \in E_\eta : S_n(x, \varepsilon) < \eta n) \leq \sum_{k=0}^{\eta n} \sum_{Q_n \in Q^{(n)}} \mu(Q_n) \leq K \eta n e^{-(h-2\alpha)n} e^{(h+\alpha)\eta n},$$

which is summable because $(h - 2\alpha) - \eta (h + \alpha) > (1 - \eta) h - 3\alpha > 0$. This proves our claim and completes the proof of theorem B.

5. Local recurrences and applications to dimension theory

This section is devoted to the proof of proposition A and theorem C.

5.1. Proof of proposition A

Our assumptions guarantee that $B(x, n, \varepsilon^{\Lambda^{-n}}) \subset B(x, n, \varepsilon) \subset B(x, \varepsilon^{\Lambda^{-n}})$ for every $x \in X$, $n \geq 1$ and every small $\varepsilon > 0$. Hence

$$\mathcal{R}(x) \geq \liminf_{\varepsilon \to 0} \left[ \lim_{n \to \infty} \frac{\log R_{B(x, \varepsilon^{\Lambda^{-n}})}(x)}{-\log(\varepsilon^{\Lambda^{-n}})} \right] \geq \frac{h_\mu(f)}{\log \Lambda}$$

for $\mu$-almost every $x$, using theorem A. The proof of the inequality $\mathcal{R}(x) \leq h_\mu(f) / \log \lambda$ is analogous. Moreover, using the specification property and theorem B it also follows similarly that $1 / \log \Lambda \leq S(x)$ and $S(x) \leq 1 / \log \lambda$ in a set of total probability. This finishes the proof of the proposition.

5.2. Proof of theorem C

We make use of Pesin’s local charts (see, e.g. [11, Appendix]). Given $\eta > 0$, for $\mu$-almost every $x$ there exists $q_\eta(x) \geq 1$ and an embedding $\Phi_x$ of the neighbourhood $R_x \subset \mathbb{R}^d$ of size
$R(x)$ around 0 onto a neighbourhood $U_x \subset M$ of $x$ such that

1. $e^{-\eta}q_0(x) \leq q_0(f(x)) \leq e^{\eta}q_0(x)$;
2. $C^{-1}d(\Phi_x(z), \Phi_x(z')) \leq |z - z'| \leq q_0(x) d(\Phi_x(z), \Phi_x(z'))$ for every $z, z' \in B_x$, for some universal constant $C$;
3. the map $f_\hat{x} = \Phi^{-1}_x \circ f \circ \Phi_x$ satisfies
   (a) $e^{\lambda d - \eta} |v| \leq |Df_\hat{x}(0)v| \leq e^{\lambda d + \eta} |v|, \forall v \in T_x M$, and
   (b) $\text{Lip}(f_{\hat{x}} - Df_\hat{x}(0)) < \eta$.

We claim that the dynamical ball $B(x, n, \epsilon)$ contains the ball of radius $r_n(x, \epsilon) = \epsilon e^{-(\lambda d + 3\eta)n}/(Cq_0(x)^2)$ centred at $x$ for every small $\epsilon$. Given $x \in M$ set $\hat{x} = \Phi^{-1}_x(x)$ and $f_k^\hat{x} = f_k^\hat{x}(\hat{x}) \circ \cdots \circ f_1^\hat{x}(\hat{x}) \circ f_\hat{x}$. Indeed, if $d(x, y) < r_n(x, \epsilon)$ then $\hat{y} = \Phi^{-1}_x(y) \in B_x$ and $|f_\hat{x}(\hat{x}) - f_k^\hat{x}(\hat{y})| \leq e^{\lambda d + 2\eta} |\hat{x} - \hat{y}| \ll 1/q_0(f(x))$. Recursively, we get that $f_k^\hat{x}(\hat{y}) \in R_{f^k}^{\hat{x}(\hat{y})}$ and

$$d(f^k(x), f^k(y)) \leq C|f_k^\hat{x}(\hat{x}) - f_k^\hat{x}(\hat{y})| \leq \frac{\epsilon}{q_0(f^k(x))} e^{(\lambda d + 3\eta)(k-n)} < \epsilon$$

for every $0 \leq k \leq n$, which proves our claim. Since $R_n(x, \epsilon) \leq R_{B(x, r_n(x, \epsilon))}(x)$ for every $n$ and every $\epsilon$, using theorems A and B we conclude that

$$R(x) = \liminf_{\epsilon \to 0} \log \frac{R_n(x, \epsilon)\eta}{-\log r_n(x, \epsilon)} = \liminf_{\epsilon \to 0} \left( \lim_{n \to \infty} \log \frac{R_n(x, \epsilon)}{-\log r_n(x, \epsilon)} \right) \geq \frac{h_\mu(f)}{\lambda d + 3\eta}.$$ 

and, analogously, $S(x) \geq \frac{1}{\lambda d + 3\eta}$. Hence $\frac{R(x)}{\lambda d + 3\eta} \geq h_\mu(f)/\lambda_d$ and $\frac{S(x)}{\lambda_d} \geq 1/\lambda_d$, because $\eta$ was chosen arbitrary.

On the other hand, if one assumes that for almost every $x$ and every $n$ there exists a radius $\epsilon_0(x, n) > 0$ so that the dynamical ball $B(x, n, \epsilon)$ is contained in the ball of radius $r_n(x, \epsilon) = C\epsilon q_0(x)^{-1}e^{-(\lambda_d - 3\eta)n}$ around $x$ for every $0 < \epsilon < \epsilon_0$ then

$$\overline{R}(x) = \limsup_{\epsilon \to 0} \frac{\log R_n(x, \epsilon)}{-\log r_n(x, \epsilon)} \leq \limsup_{\epsilon \to 0} \left( \lim_{n \to \infty} \frac{R_n(x, \epsilon)}{-\log r_n(x, \epsilon)} \right) \leq \frac{h_\mu(f)}{\lambda_1 - 3\eta}.$$ 

and, similarly, $\overline{S}(x) \leq 1/(\lambda_1 - 3\eta)$. So, the result will follow by arbitrariness of $n$. To prove the previous claim note that if $\epsilon_0(x, n) = e^{-(\lambda_d - 3\eta)n}/(Cq_0(x)^2)$ then any $y \in B(x, n, \epsilon)$ satisfies $f_k^\hat{x}(\hat{y}) \in R_{f^k}^{\hat{x}(\hat{y})}$ for every $0 \leq k \leq n$ and $0 < \epsilon < \epsilon_0$. Moreover, 

$$d(x, y) \leq C\epsilon e^{-(\lambda_d - 2\eta)n} |f_k^\hat{x}(\hat{x}) - f_k^\hat{x}(\hat{y})| < C\epsilon q_0(x)^{-1}e^{-(\lambda_d - 3\eta)n}.$$ 

This proves the claim and completes the proof of the theorem.

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