GENERAL POLYNOMIALS OVER DIVISION ALGEBRAS AND
LEFT EIGENVALUES*

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Abstract. In this paper, an isomorphism between the ring of general polynomials over a division ring of degree \( p \) over its center \( F \) and the group ring of the free monoid with \( p^2 \) variables is presented. Using this isomorphism, the characteristic polynomial of a matrix over any division algebra is defined, i.e. a general polynomial with one variable over the algebra whose roots are precisely the left eigenvalues. Plus, it is shown how the left eigenvalues of a \( 4 \times 4 \) matrices over any division algebra can be found by solving a general polynomial equation of degree 6 over that algebra.

Key words. General Polynomials, Characteristic Polynomial, Determinants, Left Eigenvalues, Quaternions.

AMS subject classifications. 12E15, 16S10, 11R52

1. Introduction.

1.1. Polynomial rings over division algebras. Let \( F \) be a field and \( D \) be a division ring over \( F \) of degree \( p \). We adopt the terminology in [2]. Let \( D_L[z] \) denote the usual ring of polynomials over \( D \) where the variable \( z \) commutes with every \( d \in D \). When substituting a value we consider the coefficients as though they are placed on the left-hand side of the variable. The substitution map is not a ring homomorphism in general. For example, for non-central \( d \in D \) and the substitution \( S_d : D_L[z] \to D \), if \( f(z) = az \) and \( ad \neq da \) then \( S_d(f^2) = S_d(a^2z^2) = a^2d^2 \) while \( S_d(f)^2 = S_d(az)^2 = (ad)^2 \neq S_d(f^2) \).

The ring \( D_G[z] \) is, by definition, the (associative) ring of polynomials over \( D \), where \( z \) is assumed to commute with every \( d \in F = Z(D) \), but not with arbitrary elements of \( D \). For example, if \( d \in D \) is non-central, then \( dz^2, zdz \) and \( z^2d \) are distinct elements of this ring. There is a ring epimorphism \( D_G[z] \to D_L[z] \), defined by \( z \mapsto z \), whose kernel is the ideal generated by the commutators \([d, z] \ (d \in D)\). Unlike the situation for \( D_L[z] \), the substitution maps from \( D_G[z] \) to \( D \) are all ring homomorphisms. Polynomials from \( D_G[z] \) are called “general polynomials”, for example \( ziz + jzi + zij + 5 \in H_G[z] \).

*Received by the editors on Month x, 200x. Accepted for publication on Month y, 200y Handling Editor:.
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Polynomials in $D_L[z]$ or and polynomials in $D_G[z]$ which “look like” polynomials in $D_L[z]$, i.e. the coefficients are placed on the left-hand side of the variable, are called “left” or “standard polynomials”, for example $z^2 + iz + j \in H_G[z]$.

Let $D\langle x_1, \ldots, x_N \rangle$ be the ring of multi-variable polynomials, where for all $i$, $x_i$ commutes with every $d \in D$ and is not assumed to commute with $x_j$ for $i \neq j$. This is the group algebra of the free monoid $\langle x_1, \ldots, x_N \rangle$ over $D$. The commutative counterpart is $D_L[x_1, \ldots, x_N]$, which is the ring of multi-variable polynomials where for all $i$, $x_i$ commutes with every $d \in D$ and with every $x_j$ for $i \neq j$.

For further reading on what is generally known about polynomial equations over division rings see [3].

1.2. Left eigenvalues of matrices over division algebras. Given a matrix $A \in M_n(D)$, a left eigenvalue of $A$ is an element $\lambda \in D$ for which there exists a nonzero vector $v \in D^{n \times 1}$ such that $Av = \lambda v$.

For the special case of $D = H$ and $n = 2$ it was proven by Wood in [6] that the left eigenvalues of $A$ are the roots of a standard quadratic quaternion polynomial. In [5] proved that for $n = 3$, the left eigenvalues of $A$ are the roots of a general cubic quaternion polynomial.

In [4], Macías-Virgós and Pereira-Sáez gave another proof to Wood’s result. Their proof makes use of the Study determinant.

Given a matrix $A \in M_n(H)$, there exist unique matrices $B, C \in M_n(C)$ such that $A = B + Cj$. The Study determinant of $A$ is $\det \begin{bmatrix} B & -C \\ C & B \end{bmatrix}$. The Dieudonné determinant is (in this case) the square root of the Study determinant. For further information about these determinants see [1].

2. The isomorphism between the ring of general polynomials and the group ring of the free monoid with $[D : F]$ variables. Let $N = p^2$, i.e. $N$ is the dimension of $D$ over its center $F$. In particular there exist invertible elements $a_1, \ldots, a_{N-1} \in D$ such that $D = F + a_1 F + \ldots + a_{N-1} F$.

Let $h : D_G[z] \to D(x_1, \ldots, x_N)$ be the homomorphism for which $h(d) = d$ for all $d \in D$, and $h(z) = x_1 + a_1 x_2 + \ldots + a_{N-1} x_N$. $D_L[x_1, \ldots, x_N]$ is a quotient ring of $D(x_1, \ldots, x_N)$. Let $g : D(x_1, \ldots, x_N) \to D_L[x_1, \ldots, x_N]$ be the standard epimorphism.

In [2, Theorem 6] it says that if $D$ is a division algebra then homomorphism
$g \circ h : D_G[z] \to D_L[x_1, \ldots, x_N]$ is an epimorphism. The next theorem is a result of this fact.

**Theorem 2.1.** The homomorphism $h : D_G[z] \to D(x_1, \ldots, x_N)$ is an isomorphism, and therefore $D_G[z] \cong D(x_1, \ldots, x_N)$.

**Proof.** $h$ is well-defined because $z$ commutes only with the center.

Both $D_G[z]$ and $D(x_1, \ldots, x_N)$ can be graded, $D_G[z] = G_0 \oplus G_1 \oplus \ldots$ and $D(x_1, \ldots, x_N) = H_0 \oplus H_1 \oplus \ldots$ such that for all $n$, $G_n$ and $H_n$ are spanned by monomials of degree $n$.

For all $n$, $h(G_n) \subseteq H_n$. Furthermore, the basis of $G_n$ as a vector space over $F$ is $\{b_1z b_2 \ldots b_n z b_{n+1} : b_1, \ldots, b_n, b_{n+1} \in 1, a_1, \ldots, a_{N-1}\}$, which means that $[G_n : F] = N^{n+1}$. Plus, the basis of $H_n$ as a vector space over $F$ is $\{b x_k_1 x_k_2 \ldots x_k_n : b \in \{1, a_1, \ldots, a_{N-1}\}, \forall j, k_j \in \{1, \ldots, N\}\}$, hence $[H_n : F] = N \cdot N^n = N^{n+1} = [G_n : F]$.

Consequently, it is enough to prove that $h|_{G_n} : G_n \to H_n$ is an epimorphism. For that, it is enough to prove that for each $1 \leq k \leq N$, $x_k$ has a co-image in $G_1$. The reduced epimorphism $g|_{G_1}$ is an isomorphism and since $g \circ h|_{G_1}$ is an epimorphism, $h|_{G_1}$ is also an epimorphism, and that finishes the proof. \[\square\]

Here is a suggested algorithm for finding the co-image of $x_k$ for any $1 \leq k \leq N$:

**Algorithm 2.2.** Let $p_1 = z$, therefore $h(p_1) = x_1 + a_1 x_2 + \ldots + a_{N-1} x_N$. We shall define a sequence $\{p_j : j = 1, \ldots, n\} \subseteq G_1$ as follows: If there exists a monomial in $h(p_j)$ whose coefficient $a$ does not commute with the coefficient of $x_k$, denoted by $c$, then we shall define $p_{j+1} = ap_j a^{-1} - p_j$, by which we shall annihilate at least one monomial (the one whose coefficient is $a$), and yet the element $x_k$ will not be annihilated, because $c x_k$ does not commute with $a$.

If $c$ commutes with all the other coefficients then we shall pick some monomial which we want to annihilate. Let $b$ denote its coefficient. Now we shall pick some $a \in D$ which does not commute with $cb^{-1}$ and define $p_{j+1} = bap_j b^{-1} a^{-1} - p_j$.

The element $x_k$ is not annihilated in this process, because if we assume that it does at some point, let us say it is annihilated in $h(p_{j+1})$, then $bacb^{-1} a^{-1} = c = 0$. Therefore $c^{-1}bacb^{-1} a^{-1} = 1$, hence $cb^{-1} a^{-1} = (c^{-1}ba)^{-1} = a^{-1} b^{-1} c$ and, since $b$ commutes with $c$, a monomials with $cb^{-1}$ and that creates a contradiction.

In each iteration the length of $h(p_j)$ (the number of monomials in it) decreases by at least one, and yet the element $x_k$ always remains, and since the length of $h(p_j)$ is finite, this process will end with some $p_m$ for which $h(p_m)$ is a monomial. In this case, $h(q_m) = cx_k$ and consequently $x_k = h(c^{-1} q_m)$.

\[\text{An easy exercise}\]
2.1. Real Quaternions. Let \( D = \mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + ij\mathbb{R} \). Now \( h(z) = x_1 + x_2i + x_3j + x_4ij \)
\[
h(z - jzj^{-1}) = h(z + jzj) = 2x_2i + 2x_4ij
\]
\[
h((z + jzj) - ij(z + jzj)(ij)^{-1}) = 2x_2i + 2x_4ij - ij(2x_2i + 2x_4ij)(ij)^{-1} = 4x_2i
\]
therefore \( h^{-1}(x_2) = -\frac{1}{4}((z + jzj) + ij(z + jzj)ij)) = -\frac{1}{4}(iz + jzz - jzij + zi) \).

Similarly, \( h^{-1}(x_1) = \frac{1}{4}(z - izi - jjzj - jzij) \), \( h^{-1}(x_3) = \frac{1}{4}(jz - ijz + izij + zj) \) and
\[
h^{-1}(x_4) = -\frac{1}{4}((z - izi - jjzj + jzij). Consequently, \( \lambda = h^{-1}(x_1 + x_2i + x_3j + x_4ij) = h^{-1}(x_1 - x_2i - x_3j - x_4ij) = -\frac{1}{2}(z + izi + jzj + izij) \).

3. The characteristic polynomial. Let \( D, F, p, N \) be the same as they were in the previous section.

There is an injection of \( D \) in \( M_p(K) \) where \( K \) is a maximal subfield of \( D \). (In particular, \([K : F] = p\).) More generally, there is an injection of \( M_k(D) \) in \( M_{kp}(K) \) for any \( k \in \mathbb{N} \). Let \( \hat{A} \) denote the image of \( A \in M_{kp}(K) \) for any \( A \in M_k(D) \).

The determinant of \( \hat{A} \) is equal to the Dieudonné determinant of \( A \) to the power of \( p \). \( \square \)

Therefore \( \lambda \in D \) is a left eigenvalue of \( A \) if and only if \( \det(\hat{A} - \lambda I) = 0 \). Considering \( D \) as an \( F \)-vector space \( D = F + Fa_1 + \ldots + Fa_{N-1} \), we can write \( \lambda = x_1 + x_2a_1 + \ldots + x_Na_{N-1} \) for some \( x_1, \ldots, x_N \in F \). Then \( \det(\hat{A} - \lambda I) \in F[x_1, \ldots, x_N] \) and can also be considered as an polynomial in \( D(x_1, \ldots, x_N) \). Now, there is an isomorphism \( h : D_G[z] \to D(x_1, \ldots, x_N) \), and so \( \det(\hat{A} - \lambda I) \in D_G[z] \).

Defining \( p_A(z) = h^{-1}(\det(\hat{A} - \lambda I)) \) to be the characteristic polynomial of \( A \), we get that the left eigenvalues of \( A \) are precisely the roots of \( p_A(z) \).

The degree of the characteristic polynomial of \( A \) is therefore \( kp \).

Remark 3.1. If one proves that the Dieudonné determinant of \( A - \lambda I \) is the absolute value of some polynomial \( q(x_1, \ldots, x_N) \in D_L[x_1, \ldots, x_N] \) then we will be able to define the characteristic polynomial to be \( h^{-1}(q(x_1, \ldots, x_N)) \) and obtain a characteristic polynomial of degree \( k \).

4. The left eigenvalues of a \( 4 \times 4 \) quaternion matrix. Let \( Q \) be a quaternion division \( F \)-algebra. Calculating the roots of the characteristic polynomial as defined in Section \( \ref{sec:quaternion} \) is not always the best way to obtain the left eigenvalues of a given matrix.

The reductions Wood did in \( \ref{sec:wood} \) and So did in \( \ref{sec:so} \) suggest that in order to obtain the left eigenvalues of a \( 2 \times 2 \) or \( 3 \times 3 \) matrix one can calculate the roots of a polynomial of degree 2 or 3 respectively, instead of calculating the roots of the characteristic polynomial.
In the next proposition we show how (under a certain condition) the eigenvalues of a $4 \times 4$ quaternion matrix can be obtained by calculating the roots of three polynomials of degree 2 and one of degree 6.

**Proposition 4.1.** If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A, B, C, D \in M_2(Q)$ and $C$ is invertible then $\lambda$ is a left eigenvalue of $M$ if and only if it is either $e(\lambda) = f(\lambda)g(\lambda) = 0$ or $e(\lambda) \neq 0$ and $e(\lambda)e(\lambda)h(\lambda) - g(\lambda)e(\lambda)f(\lambda) = 0$ where $C(A - \lambda I)C^{-1}(D - \lambda I) - CB = \begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix}$.

**Proof.** Let $M$ be a $4 \times 4$ quaternion matrix. Therefore $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A, B, C, D$ are $2 \times 2$ quaternion matrices.

An element $\lambda$ is a left eigenvalue if $\det(M - \lambda I) = 0$. Assuming that $\det(C) \neq 0$, we have $\det(M - \lambda I) = \det(C(A - \lambda I)C^{-1}(D - \lambda I) - CB)$ (This is an easy result of the Schur complements identity for complex matrices extended to quaternion matrices in).

The matrix $C(A - \lambda I)C^{-1}(D - \lambda I) - CB$ is equal to $\begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix}$ for some quadratic polynomials $e, f, g, h$.

Now, if $e(\lambda) \neq 0$ then $\det\begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix} = 0$ if and only if $h(\lambda) - g(\lambda)e(\lambda)^{-1}f(\lambda) = 0$.

This happens if and only if $e(\lambda)e(\lambda)h(\lambda) - g(\lambda)e(\lambda)f(\lambda) = 0$.

As we saw in Subsection 2.1, $e(\lambda)$ is also a quadratic polynomial, which means that $e(\lambda)e(\lambda)h(\lambda) - g(\lambda)e(\lambda)f(\lambda)$ is a polynomial of degree 6, while the characteristic polynomial of $M$ as defined in Section 3 is of degree 8.

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