Limit stable objects on Calabi-Yau 3-folds

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Abstract

In this paper, we introduce new enumerative invariants of curves on Calabi-Yau 3-folds via certain stable objects in the derived category of coherent sheaves. We introduce the notion of limit stability on the category of perverse coherent sheaves, a subcategory in the derived category, and construct the moduli spaces of limit stable objects. We then define the counting invariants of limit stable objects using Behrend’s constructible functions on that moduli spaces. It will turn out that our invariants are generalizations of counting invariants of stable pairs introduced by Pandharipande and Thomas. We will also investigate the wall-crossing phenomena of our invariants under change of stability conditions.

1 Introduction

The purpose of this paper is to introduce new enumerative invariants of curves on Calabi-Yau 3-folds from certain stable objects in the derived category of coherent sheaves. The notion of stability conditions on derived categories, more generally on triangulated categories, is introduced by Bridgeland [6], motivated by Douglas’s work on II-stability [11], [12]. However at this time, there are some issues in studying Bridgeland’s stability conditions on projective Calabi-Yau 3-folds. Instead, we consider a generalized notion of stability conditions which we call limit stability, and study their stable objects. The limit stability is considered as the “large volume limit” in the stringy Kähler moduli space. We construct the moduli spaces of limit stable objects, and introduce the enumerative invariants of such objects.

On the other hand, a kind of enumeration problem of objects in the derived category is studied by Pandharipande and Thomas [27], [28], [29]. We will see how our invariants relate to the invariants of stable pairs introduced by them [27]. We will also investigate the wall-crossing phenomena of our invariants under change of stability conditions, and propose a conjectural wall-crossing formula which is related to the rationality conjecture proposed in [27].

1.1 Background

Let $X$ be a non-singular projective Calabi-Yau 3-fold over $\mathbb{C}$. The Gromov-Witten (GW) invariants of $X$ are counting invariants of curves on $X$, integrating over the virtual class of the moduli space of stable maps $\overline{M}(X)$,

$$\overline{M}(X) = \{(C, f) \mid f : C \to X \text{ is a stable map from a curve } C\}.$$

Since stable maps have non-trivial automorphisms, $\overline{M}(X)$ is in general a Deligne-Mumford stack and the GW invariants are rational numbers. Another kind of counting invariants of curves on $X$, called Donaldson-Thomas (DT) invariants, are defined as the integration over the virtual class of the moduli space of the ideal sheaves,

$$I(X) = \{I_C \subseteq \mathcal{O}_X \mid C \subseteq X \text{ is a subscheme with } \dim C \leq 1\}.$$
Since $I(X)$ is nothing but the Hilbert scheme, the resulting invariants are integer valued. The *GW-DT correspondence* \cite{25} is a conjectural relationship between two generating functions involving GW invariants, DT invariants respectively. More precisely, one dimensional subschemes $C \subset X$ contain zero dimensional subschemes, hence the DT theory does not directly count curves. Instead by dividing by the generating series of counting invariants of zero dimensional subschemes, we can define the reduced DT theory which should correspond to the GW theory in GW-DT correspondences.

The notion of stable pairs on $X$ is introduced in \cite{27} in order to give a geometric interpretation to the reduced DT theory. By definition a stable pair consists of data $(F, s)$,

$$s: \mathcal{O}_X \to F,$$

where $F \in \text{Coh}(X)$ is a pure one dimensional sheaf, and $s$ is a morphism satisfying the condition

$$\dim \text{Coker}(s) = 0.$$

The *Pandharipande-Thomas (PT) invariants* are defined by the integration over the virtual class of the moduli space of stable pairs,

$$P(X) = \{(F, s) \mid s: \mathcal{O}_X \to F \text{ is a stable pair }\},$$

and the *DT-PT correspondence* \cite{27} is a conjectural relationship between generating functions of reduced DT theory, PT theory respectively.

On the DT-side, any ideal sheaf $I_C \subset \mathcal{O}_X$ is a Gieseker-stable sheaf, hence DT-invariants count stable objects in $\text{Coh}(X)$. On the other hand, the space $P(X)$ can be viewed as the moduli space of the two term complexes,

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X),$$

where $D^b(X)$ is the bounded derived category of coherent sheaves on $X$. Furthermore the obstruction theory which admits the virtual class on $P(X)$ is obtained from the deformation theory of objects in $D^b(X)$, not from that of stable pairs. From this observation, we guess that $I^\bullet$ might be stable objects with respect to a certain stability condition on $D^b(X)$, and PT-invariants count stable objects.

Now we are led to consider stability conditions on $D^b(X)$, and enumerative problem of stable objects in $D^b(X)$. In the next paragraph, we discuss stability conditions on derived categories.

### 1.2 Stability conditions on triangulated categories

Let $\mathcal{D}$ be a triangulated category, e.g. $\mathcal{D} = D^b(X)$ for a smooth projective variety $X$. The notion of stability conditions on $\mathcal{D}$ is introduced by Bridgeland \cite{6}. Roughly speaking a stability condition on $\mathcal{D}$ consists of data $\sigma = (Z, A)$,

$$Z: K(\mathcal{D}) \to \mathbb{C}, \quad A \subset \mathcal{D},$$

where $Z$ is a group homomorphism called a stability function, and $A$ is the heart of a bounded t-structure on $\mathcal{D}$, which satisfy some axiom. When $\mathcal{D} = D^b(X)$, the set of locally finite numerical stability conditions $\text{Stab}(X)$ is shown to have the complex structure by Bridgeland \cite{6}, and the quotient space

$$\text{Auteq}(\mathcal{D}) \setminus \text{Stab}(X)/\mathbb{C}$$

2
is a mathematical candidate of the stringy Kähler moduli space. The space $\text{Stab}(X)$ have been studied in several examples. For instance see [7], [8], [9], [17], [24], [30], [31], [32].

Although the notion of stability conditions on triangulated categories has drawn much interest recently, we are not able to study the most important case, $D = D^b(X)$ for a projective Calabi-Yau 3-fold $X$ at this time. In this case, there are some technical difficulties to construct examples of stability conditions, so we do not know whether $\text{Stab}(X)$ is non-empty or not. From the ideas in physical articles [11], [12], there should exist stability conditions corresponding to the neighborhood of the large volume limits, whose stability functions are given by,

$$Z_\sigma(E) = - \int e^{-(B+i\omega)} \text{ch}(E) \sqrt{\text{td}_X},$$

(1)

where $\sigma = B + i\omega \in H^2(X, \mathbb{C})$ with $\omega$ an ample class. Such stability conditions should be parameterized by elements of the complexified ample cone,

$$\sigma \in A(X)_C = \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is an ample class} \}.$$

Instead of working with Bridgeland’s stability conditions, we introduce and study a generalized notion of stability conditions which we call limit stability. The corresponding heart of a t-structure is the category of perverse coherent sheaves,

$$\mathcal{A}^p \subset D^b(X),$$

in the sense of Bezrukavnikov [5] and Kashiwara [21]. We will see that for $\sigma \in A(X)_C$, the stability function (1) together with taking $\omega \to \infty$ determines the set of (semi)stable objects in $\mathcal{A}^p$, which we call $\sigma$-limit (semi)stable objects. The notation “limit” is used to emphasize that our stability conditions should correspond to the limit point $\omega = \infty$. Some fundamental properties of limit stability (e.g. existence of Harder-Narasimhan filtrations, Jordan-Hölder filtrations,) will be studied in Section 2.

1.3 Main results

We shall study the enumerative problem of $\sigma$-limit stable objects $E \in \mathcal{A}^p$. Let us take $\beta \in H^4(X, \mathbb{Q})$ and $n \in \mathbb{Q}$. We first show the existence of the moduli space of limit stable objects. The following theorem will be shown in Section 3.

**Theorem 1.1.** There is a separated algebraic space of finite type $\mathcal{L}^\sigma_n(X, \beta)$, which parameterizes $\sigma$-limit stable objects $E \in \mathcal{A}^p$, satisfying $\det E = O_X$ and the following numerical condition,

$$(\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E), \text{ch}_3(E)) = (-1, 0, \beta, n) \in H^4(X, \mathbb{Q}).$$

It will turn out that the moduli space $\mathcal{L}^\sigma_n(X, \beta)$ could be non-empty only if $\beta$ is the Poincaré dual of the homology class of an effective one cycle on $X$, and $n \in \mathbb{Z}$. (cf. Remark 3.3) By Theorem 1.1 and using Behrend’s constructible function [4], $\nu_L: \mathcal{L}^\sigma_n(X, \beta) \to \mathbb{Z}$, we are able to define the counting invariant of limit stable objects,

$$L_{n,\beta}(\sigma) := \sum_{n \in \mathbb{Z}} ne(\nu_L^{-1}(n)) \in \mathbb{Z}.$$

We next show the relationship between the integers $L_{n,\beta}(\sigma)$ and $P_{n,\beta}$, where $P_{n,\beta}$ is the PT-invariant counting stable pairs $(F, s)$ with

$$\text{ch}_2(F) = \beta, \quad \text{ch}_3(F) = n.$$

See Definition 4.3 for the detail. We show the following theorem in Section 4.
Theorem 1.2. Let $\sigma = k\omega + i\omega$ for $k \in \mathbb{R}$. We have,
\[ L_{n,\beta}(\sigma) = P_{n,\beta}, \quad (k \ll 0), \quad L_{n,\beta}(\sigma) = P_{-n,\beta}, \quad (k \gg 0). \]

It seems that Theorem 1.2 is related to the rationality conjecture of the generating function of the PT-invariants,
\[ Z_{\beta}^{\text{PT}}(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n \in \mathbb{Q}(q). \]

It is proposed by Pandharipande and Thomas in [27, Conjecture 3.2] and they conjecture that $Z_{\beta}^{\text{PT}}(q)$ is a rational function of $q$, invariant under $q \mapsto 1/q$. This conjecture is solved when $\beta$ is an irreducible curve class in [29] by comparing $P_{n,\beta}$ and $P_{-n,\beta}$. In the following, we propose a conjectural wall-crossing formula of our invariants $L_{n,\beta}(\sigma)$, which combined with Theorem 1.2 provides a relationship between $P_{n,\beta}$ and $P_{-n,\beta}$ in a general situation.

Conjecture 1.3. There is a virtual counting of one dimensional $\omega$-Gieseker semistable sheaves $F$ with $(\text{ch}^2(F), \text{ch}^3(F)) = (\beta', n')$, denoted by $N_{n',\beta'} \in \mathbb{Q}$, such that
\[ L_{n,\beta}(\sigma_-) - L_{n,\beta}(\sigma_+) = \sum (-1)^{n'-1} n' N_{n',\beta'} L_{n'',\beta''}(\sigma_0). \]

Here in the above sum, $(\beta', n')$, $(\beta'', n'')$ must satisfy $\beta' + \beta'' = \beta$, $n' + n'' = n$ and $n'/\omega \beta' = \mu$.

See Paragraph 4.3 for the explanation of the above conjecture. In Section 5, we investigate the wall-crossing phenomena of limit stable objects and study Conjecture 1.3 in some examples.

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1.5 Notation and convention

We work over varieties over $\mathbb{C}$. For a variety $X$, we denote by $D^b(X)$, $K(X)$ the bounded derived category of coherent sheaves on $X$, the Grothendieck group of coherent sheaves respectively. For a triangulated category $\mathcal{D}$ and a set of subobjects $S \subset \mathcal{D}$, we denote by $\langle S \rangle \subset \mathcal{D}$ the smallest extension closed subcategory which contains $S$. If $S$ is a set of subobjects in an abelian category $\mathcal{A}$, we also use the same notation $\langle S \rangle \subset \mathcal{A}$.

2 Limit stability

2.1 Bridgeland’s stability conditions

Here we briefly review the definition of Bridgeland’s stability conditions [6]. Let us begin with the stability conditions on abelian categories.
**Definition 2.1.** [6] Let \( \mathcal{A} \) be an abelian category. A *stability function* on \( \mathcal{A} \) is a group homomorphism,

\[
Z : K(\mathcal{A}) \to \mathbb{C},
\]

such that for any non-zero \( E \in \mathcal{A} \), we have

\[
Z(E) \in \mathbb{H} := \{ r \exp(i\pi \phi) \mid r > 0, 0 < \phi \leq 1 \}.
\]

Given a non-zero object \( E \in \mathcal{A} \) and a stability function \( Z : K(\mathcal{A}) \to \mathbb{C} \), we can uniquely determine the *phase* of \( E \) by

\[
\phi(E) = \frac{1}{\pi} \text{Im} \log Z(E) \in (0, 1].
\]

We say \( E \in \mathcal{A} \) is *Z-semistable* if for any non-zero subobject \( F \subset E \) in \( \mathcal{A} \), we have

\[
\phi(F) \leq \phi(E).
\]

**Definition 2.2.** [6] A stability function \( Z : K(\mathcal{A}) \to \mathbb{C} \) is called a *stability condition* on \( \mathcal{A} \) if for any \( E \in \mathcal{A} \), there exists a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,
\]

such that each \( F_i = E_i/E_{i-1} \) is Z-semistable with

\[
\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).
\]

The above filtration is called a *Harder-Narasimhan filtration*.

It is easy to construct examples of stability conditions if \( \mathcal{A} \) has finite number of simple objects \( S_1, \cdots, S_N \in \mathcal{A} \) such that

\[
\mathcal{A} = \langle S_1, \cdots, S_N \rangle.
\]

e.g. \( \mathcal{A} = \text{mod} \ A \) for a finite dimensional \( k \)-algebra \( A \). In this case \( K(\mathcal{A}) \) is generated by \( [S_i] \in K(\mathcal{A}) \), and \( Z : K(\mathcal{A}) \to \mathbb{C} \) is a stability condition if and only if \( Z(S_i) \in \mathbb{H} \) for all \( i \).

In general, a sufficient condition for a stability function to be a stability condition is provided in [6, Proposition 2.4].

**Proposition 2.3.** [6, Proposition 2.4] Let \( Z : K(\mathcal{A}) \to \mathbb{C} \) be a stability function. Assume that

- there is no infinite sequence of inclusions in \( \mathcal{A} \),

\[
\cdots \hookrightarrow E_n \hookrightarrow \cdots \hookrightarrow E_1 \hookrightarrow E_0,
\]

with \( \phi(E_{i+1}) > \phi(E_i) \) for all \( i \).

- there is no infinite sequence of surjections in \( \mathcal{A} \),

\[
E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_n \twoheadrightarrow \cdots,
\]

with \( \phi(E_i) > \phi(E_{i+1}) \) for all \( i \).

Then \( Z \) is a stability condition.
Next let $\mathcal{D}$ be a triangulated category, e.g. $\mathcal{D} = D^b(X)$ for a variety $X$. The following is the definition of Bridgeland’s stability conditions.

**Definition 2.4.** [6] A stability condition on $\mathcal{D}$ consists of data $(Z, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z$ is a stability condition on $\mathcal{A}$.

**Remark 2.5.** A stability condition on $\mathcal{D}$ in [6] is originally given by data $(Z, \mathcal{P})$, where $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ is a group homomorphism, and $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ are full subcategories, satisfying some axioms. However as shown in [6, Proposition 4.2], this is equivalent to giving data $(Z, \mathcal{A})$ as in Definition 2.4.

**Remark 2.6.** Let $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ be a stability condition on an abelian category $\mathcal{A}$. Then the pair $(Z, \mathcal{A})$ is a stability condition on the triangulated category $D^b(\mathcal{A})$.

Let $\mathcal{D} = D^b(X)$ for a smooth projective variety. A stability condition $(Z, \mathcal{A})$ on $\mathcal{D}$ is called *numerical* if $Z: K(X) \rightarrow \mathbb{C}$ factors through the Chern character map,

$$
\begin{align*}
K(X) &\xrightarrow{Z} \mathbb{C} \\
ch &\downarrow \\
H^*(X, \mathbb{Q}) &
\end{align*}
$$

The set of numerical stability conditions on $\mathcal{D} = D^b(X)$ which satisfy the *local finiteness* (cf. [6, Definition 5.7]) is denoted by $\text{Stab}(X)$. In [6, Theorem 1.2], Bridgeland shows that $\text{Stab}(X)$ has a structure of a complex manifold.

The space $\text{Stab}(X)$ is studied when $\dim X = 1, 2$ in the articles [6], [7]. Unfortunately, we do not know how to construct examples of stability conditions for higher dimensional varieties. It seems that the following lemma is well-known, but we put it to emphasize that the construction problem is non-trivial.

**Lemma 2.7.** Let $X$ be a smooth projective variety with $d = \dim X \geq 2$. Then there is no numerical stability condition $(Z, \mathcal{A})$ on $D^b(X)$ with $\mathcal{A} = \text{Coh}(X)$.

**Proof.** It is enough to show that there is no stability function $Z: K(X) \rightarrow \mathbb{C}$ on $\text{Coh}(X)$ of the following form,

$$
Z(E) = \sum_{j=0}^d (u_j + iv_j) \text{ch}_j(E),
$$

for $u_j + iv_j \in H^{2d-2j}(X, \mathbb{C})$. Suppose that such a stability function $Z$ exists. Since $d \geq 2$, there is a smooth subvariety $S \hookrightarrow X$ with $\dim S = 2$. Then the composition

$$
K(S) \xrightarrow{i^*} K(X) \xrightarrow{Z} \mathbb{C},
$$

is a stability function on $\text{Coh}(S)$, hence we may assume $d = 2$. Let $C \subset X$ be a smooth curve and take a divisor $D$ on $C$. Since $\text{Im} Z(E) \geq 0$ for any $E \in \text{Coh}(X)$, we have

$$
\text{Im} Z(\mathcal{O}_C(D)) = v_2(\deg D + \text{ch}_2(\mathcal{O}_C)) + v_1 \cdot [C] \geq 0.
$$

Since we can take $D$ with an arbitrary degree, we must have $v_2 = 0$. Similarly we have

$$
\text{Im} Z(\mathcal{O}_X(mC)) = mv_1 \cdot [C] + v_0 \geq 0,
$$

6
for any $m \in \mathbb{Z}$, hence $v_1 \cdot |C| = 0$. Therefore $\text{Im} \ Z(O_C(D)) = 0$, and this implies

$$\text{Re} \ Z(O_C(D)) = u_2(\deg D + \text{ch}_2(O_C)) + u_1 \cdot |C| \leq 0,$$

since $Z(O_C(D)) \in \mathbb{H}$. Then the same argument shows that $u_2 = 0$, and this implies

$$Z(O_x) = u_2 + iv_2 = 0,$$

for any closed point $x \in X$. This contradicts that $Z(O_x) \in \mathbb{H}$.

**Remark 2.8.** In the case of $\dim X = 2$, the examples of stability conditions $(Z, A)$ are constructed in [7], [2] by setting $A$ to be the tilting of $\text{Coh}(X)$ with respect to certain torsion pairs. When $X$ is a K3 surface, the stability function $Z$ is given by

$$Z(B, \omega)(E) = -\int e^{-(B+i\omega)} \text{ch}(E) \sqrt{\text{td}_X},$$

for $B + i\omega \in H^2(X, \mathbb{C})$ with $\omega$ an ample class. When $\dim X \geq 3$ we expect that for $\omega \gg 0$, there are hearts of bounded t-structures $\mathcal{A}_{(B,\omega)}$ such that the pairs $(Z_{(B,\omega)}, \mathcal{A}_{(B,\omega)})$ determine stability conditions, giving the neighborhood of the large volume limits. However at this time, we are not able to find such $\mathcal{A}_{(B,\omega)}$.

### 2.2 Perverse coherent sheaves on Calabi-Yau 3-folds

From this paragraph, we focus on the case that $X$ is a Calabi-Yau 3-fold, i.e. $X$ is a smooth projective 3-fold with a trivial canonical class. Here we study the heart of a bounded t-structure $\mathcal{A}^p$, constructed as one of the perverse t-structures introduced by Bezrukavnikov [5] and Kashiwara [21]. In the notation of Remark 2.8, the desired category $\mathcal{A}_{(B,\omega)}$ should be constructed as an approximation of our category $\mathcal{A}^p$, so we hope that studying $\mathcal{A}^p$ in detail will solve the construction problem in a future. Let us recall the notion of torsion pairs and their tilting for the construction of $\mathcal{A}^p$.

**Definition 2.9.** Let $\mathcal{A}$ be an abelian category. A **torsion pair** on $\mathcal{A}$ is a pair of full subcategories $(T, F)$ such that

- For $T \in T$ and $F \in F$, we have $\text{Hom}(T, F) = 0$.
- For any $E \in \mathcal{A}$, there is an exact sequence $0 \to T \to E \to F \to 0$ in $\mathcal{A}$ such that $T \in T$, $F \in F$.

Given a torsion pair $(T, F)$ on $\mathcal{A}$, the following subcategory of $D^b(\mathcal{A})$,

$$\mathcal{A}^t = \langle F[1], T \rangle = \{ E \in D^b(\mathcal{A}) \mid \mathcal{H}^{-1}(E) \in F, \mathcal{H}^0(E) \in T, \mathcal{H}^i(E) = 0 \text{ for } i \neq -1, 0 \},$$

is known to be the heart of a bounded t-structure on $D^b(\mathcal{A})$, and it is called a **tilting** with respect to the torsion pair $(T, F)$. (cf. [13]) For a Calabi-Yau 3-fold $X$, we have the following torsion pair.

**Lemma 2.10.** The pair $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$,

$$\text{Coh}_{\leq 1}(X) := \{ E \in \text{Coh}(X) \mid \dim \text{Supp}(E) \leq 1 \},$$

$$\text{Coh}_{\geq 2}(X) := \{ E \in \text{Coh}(X) \mid \text{Hom}(F, E) = 0 \text{ for any } F \in \text{Coh}_{\leq 1}(X) \},$$

is a torsion pair of $\text{Coh}(X)$.
Proof. For an object $E \in \text{Coh}(X)$ there is an exact sequence,

$$0 \to T \to E \to F \to 0,$$

such that $T \in \text{Coh}_{\leq 1}(X)$ and $\dim \text{Supp}(F) \geq 2$. Since $\text{Coh}(X)$ is a noetherian abelian category, we can take $T$ to be maximum, i.e. there is no $T' \in \text{Coh}_{\leq 1}(X)$ with $T \subset T' \subset E$. Then it is easy to see that $F \in \text{Coh}_{\geq 2}(X)$. \qed

Our abelian category $\mathcal{A}^p$ is constructed as a tilting.

**Definition 2.11.** We define the heart of a perverse t-structure $\mathcal{A}^p \subset D^b(X)$ to be the tilting with respect to the torsion pair $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$, i.e.

$$\mathcal{A}^p = \langle \text{Coh}_{\geq 2}(X)[1], \text{Coh}_{\leq 1}(X) \rangle.$$

**Remark 2.12.** In general, a perverse t-structure introduced in [5], [21] is determined by choosing a perversity function, which is a map $p: X^{\text{top}} \to \mathbb{Z}$ satisfying a certain condition. One can easily check that our category $\mathcal{A}^p$ corresponds to the following perversity function,

$$p(x) = \begin{cases} -1 & \text{dim } \mathcal{O}_{X,x} \leq 1, \\ 0 & \text{dim } \mathcal{O}_{X,x} \geq 2. \end{cases}$$

**Remark 2.13.** The subcategory $\text{Coh}_{\leq 1}(X) \subset \mathcal{A}^p$ is easily seen to be closed under quotients and subobjects, hence it is an abelian subcategory. Since $\text{Coh}_{\leq 1}(X)$ is not artinian, the abelian category $\mathcal{A}^p$ is also not artinian.

**Remark 2.14.** The abelian category $\mathcal{A}^p$ is also not noetherian. In fact let us take a divisor $H \subset X$ and a curve $C \subset H$. Then there exists an infinite chain of surjections in $\mathcal{A}^p$,

$$\mathcal{O}_H[1] \to \mathcal{O}_H(C)[1] \to \mathcal{O}_H(2C)[1] \to \cdots.$$ 

**2.3 Torsion pair on $\mathcal{A}^p$ and the dualizing functor**

As we have seen in Remark 2.14, the abelian category $\mathcal{A}^p$ is worse than $\text{Coh}(X)$, and this fact sometimes causes difficulty to handle $\mathcal{A}^p$. In this paragraph, we introduce a certain torsion pair on $\mathcal{A}^p$ which makes $\mathcal{A}^p$ much more amenable. Let us set $\mathcal{A}^p_1$, $\mathcal{A}^p_{1/2}$ to be the subcategories of $\mathcal{A}^p$,

$$\mathcal{A}^p_1 := \langle F[1], \mathcal{O}_x \mid F \text{ is a pure two dimensional sheaf and } x \in X \rangle,$$

$$\mathcal{A}^p_{1/2} := \{ E \in \mathcal{A}^p \mid \text{Hom}(F, E) = 0 \text{ for any } F \in \mathcal{A}^p_1 \}.$$ 

The meaning of the subscript will be clear in Lemma 2.26.

**Remark 2.15.** For $E \in \mathcal{A}^p$, it is obvious that $E \in \mathcal{A}^p_1$ if and only if $\mathcal{H}^0(E)$ is zero dimensional and $\mathcal{H}^{-1}(E)$ is a torsion sheaf. Also $E \in \mathcal{A}^p_{1/2}$ if and only if $\mathcal{H}^{-1}(E)$ is torsion free and $\text{Hom}(\mathcal{O}_x, E) = 0$ for any $x \in X$. In particular, we have

$$\mathcal{A}^p_{1/2} \cap \text{Coh}_{\leq 1}(X) = \{ \text{pure one dimensional sheaves} \}.$$ 

We show the following lemma.

**Lemma 2.16.** The pair $(\mathcal{A}^p_1, \mathcal{A}^p_{1/2})$ is a torsion pair of $\mathcal{A}^p$. 

8
Proof. It is enough to show that for any \( E \in \mathcal{A}^p \), there is an exact sequence in \( \mathcal{A}^p \)

\[ 0 \rightarrow E_1 \rightarrow E \rightarrow E_{1/2} \rightarrow 0, \]

with \( E_i \in \mathcal{A}_i^p \) for \( i = 1, 1/2 \). Let \( F \subset \mathcal{H}^{-1}(E) \) be the maximum torsion subsheaf. We have the exact sequence in \( \mathcal{A}^p \):

\[ 0 \rightarrow F[1] \rightarrow E \rightarrow E' \rightarrow 0. \]

Note that \( F[1] \in \mathcal{A}_1^p \) and \( \mathcal{H}^{-1}(E') = \mathcal{H}^{-1}(E)/F \) is torsion free. Hence for any pure two dimensional sheaf \( F' \), we have

\[ \text{Hom}(F'[1], E') = \text{Hom}(F', \mathcal{H}^{-1}(E')) = 0. \]

Therefore if \( E' \) is not contained in \( \mathcal{A}^p_{1/2} \), there is a zero dimensional sheaf \( U \) such that \( \text{Hom}(U, E') \neq 0 \). By Remark 2.13, this means that there is a subobject \( U' \subset E' \) in \( \mathcal{A}^p \) such that \( U' \) is a zero dimensional sheaf. Moreover we can take \( U' \subset E \) to be maximum, i.e. there is no zero dimensional sheaf \( U'' \) with \( U' \subset U'' \subset E' \) in \( \mathcal{A}^p \). To show this, it is enough to check that any sequence of subobjects,

\[ U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots \subset E', \quad (5) \]

where \( U_i \) are zero dimensional sheaves, terminates. Let \( G_i = E'/U_i \in \mathcal{A}^p \). We have the exact sequence in \( \mathcal{A}^p \),

\[ 0 \rightarrow U_i/U_{i+1} \rightarrow G_i \rightarrow G_{i+1} \rightarrow 0. \]

Taking cohomology and noting that \( \mathcal{H}^{-1}(E') \) is torsion free, we see that

\[ \mathcal{H}^{-1}(E') \subset \mathcal{H}^{-1}(G_1) \subset \mathcal{H}^{-1}(G_2) \subset \cdots \subset \mathcal{H}^{-1}(G_n) \subset \cdots \subset \mathcal{H}^{-1}(E')^{\vee \vee}, \quad (6) \]

in \( \text{Coh}(X) \). The sequence (5) must terminate, say \( \mathcal{H}^{-1}(G_j) = \mathcal{H}^{-1}(G_{j+1}) = \cdots \). Replacing \( E' \) by \( G_j \), we may assume that \( \mathcal{H}^{-1}(E') = \mathcal{H}^{-1}(G_i) \) for any \( i \). Then each \( U_i \) are subsheaves of \( \mathcal{H}^0(E') \), thus (5) must terminate. Therefore there is a maximum zero dimensional sheaf \( U' \subset E' \).

Now let \( E'' = E'/U' \), and consider the exact sequences in \( \mathcal{A}^p \),

\[ 0 \rightarrow F' \rightarrow E \rightarrow E'' \rightarrow 0, \]
\[ 0 \rightarrow F[1] \rightarrow F' \rightarrow U' \rightarrow 0. \]

Here \( E \rightarrow E'' \) is obtained as the composition of the quotients in \( \mathcal{A}^p \), \( E \rightarrow E' \rightarrow E'' \). The bottom sequence shows \( F' \in \mathcal{A}_1^p \). By the construction, we also have \( E'' \in \mathcal{A}^p \).

Let \( \mathbb{D} : D^b(X) \rightarrow D^b(X)^{\text{op}} \) be the dualizing functor,

\[ \mathbb{D}(E) = \mathbb{R}\text{Hom}(E, \mathcal{O}_X[2]). \]

In the following lemma, we see the compatibility of the torsion pair \((\mathcal{A}_1^p, \mathcal{A}^p_{1/2})\) with the dualizing functor \( \mathbb{D} \).

**Lemma 2.17.** We have

\[ E \in \mathcal{A}^p_1 \Rightarrow \mathbb{D}(E) \in \mathcal{A}^p_1[-1], \]
\[ E \in \mathcal{A}^p_{1/2} \Rightarrow \mathbb{D}(E) \in \mathcal{A}^p_{1/2}. \]
Proof. First we show that $\mathcal{D}(E) \in \mathcal{A}_{1}^{p}[-1]$ for $E \in \mathcal{A}_{1}^{p}$. It is enough to check this for $E = G[1]$ and $E = \mathcal{O}_x$, where $G$ is a pure two dimensional sheaf and $x \in X$ is a closed point. Since $G$ is pure, we have

$$\text{Ext}^{i}_X(G, \mathcal{O}_X) = 0, \quad \text{for } i \neq 1,$$

and $\text{Ext}^{1}_X(G, \mathcal{O}_X)$ is a pure two dimensional sheaf. (cf. [14, Section 1.1].) Therefore $\mathcal{D}(G[1]) \in \mathcal{A}_{1}^{p}[-1]$. Also we have $\mathcal{D}(\mathcal{O}_x) = \mathcal{O}_x[-1] \in \mathcal{A}_{1}^{p}[-1]$.

Next let us take $E \in \mathcal{A}_{1/2}^{p}$ and check $\mathcal{D}(E) \in \mathcal{A}_{1/2}^{p}$. Since $\mathcal{H}^0(E)$ is a torsion sheaf and $E$ is concentrated on $[-1, 0]$, we can easily see $\mathcal{H}^i(\mathcal{D}(E)) = 0$ for $i \leq -2$. Suppose that $\mathcal{H}^k(\mathcal{D}(E)) \neq 0$ and $\mathcal{H}^i(\mathcal{D}(E)) = 0$ for any $i < k$. Then there is a closed point $x \in X$ such that

$$0 \neq \text{Hom}(\mathcal{D}(E), \mathcal{O}_x[-k]) = \text{Hom}(\mathcal{O}_x[k-1], E).$$

Therefore we have $k \leq 0$, and $\mathcal{D}(E)$ is concentrated on $[-1, 0]$. Let us take $F \in \text{Coh}_{\leq 1}(X)$. Since we have

$$\mathcal{H}^i(\mathcal{D}(F[1])) = 0 \quad \text{for } i \leq 0,$$

it follows that

$$\text{Hom}(F, \mathcal{H}^{-1}(\mathcal{D}(E))) = \text{Hom}(F[1], \mathcal{D}(E)) = \text{Hom}(E, \mathcal{D}(F[1])) = 0.$$

Hence $\mathcal{H}^{-1}(\mathcal{D}(E)) \in \text{Coh}_{\geq 2}(X)$. Let us take a codimension one point $p \in X$. Since $\mathcal{H}^{-1}(E)$ is torsion free, we have $E_p \cong \mathcal{O}_{X_p}^{\text{dir}}[1]$ for some $r$. Therefore $\mathcal{D}(E)_p \cong \mathcal{O}_{X_p}^{\text{dir}}[1]$, and this implies $\mathcal{H}^0(\mathcal{D}(E)) \in \text{Coh}_{\leq 1}(X)$, i.e. $\mathcal{D}(E) \in \mathcal{A}^p$. Moreover for any object $E' \in \mathcal{A}_{1}^{p}$, we have

$$\text{Hom}(E', E) \cong \text{Hom}(\mathcal{D}(E), \mathcal{D}(E')) = 0,$$

since $\mathcal{D}(E') \in \mathcal{A}_{1}^{p}[-1]$. Therefore we can conclude $\mathcal{D}(E) \in \mathcal{A}_{1/2}^{p}$.

Remark 2.18. According to [21], the abelian category $\mathcal{D}(\mathcal{A}^p)$ corresponds to the heart of a perverse t-structure $\mathcal{A}^{p^*}$ (up to shift) with the dual perversity function $p^* : X^{\text{top}} \to \mathbb{Z}$. Lemma 2.17 implies that $\mathcal{A}^{p^*}$ is obtained as a tilting with respect to the torsion pair $(\mathcal{A}_{1}^{p}, \mathcal{A}_{1/2}^{p})$.

Let us take $E, F \in \mathcal{A}_{1}^{p}$ and a morphism $f : E \to F$. The morphism $f$ is called a strict monomorphism if $f$ is injective in $\mathcal{A}^p$ and $\text{Coker}(f) \in \mathcal{A}_{1}^{p}$. Similarly $f$ is called a strict epimorphism if $f$ is surjective in $\mathcal{A}^p$ and $\text{Ker}(f) \in \mathcal{A}_{1}^{p}$. Although the category $\mathcal{A}^p$ is not artinian nor noetherian, each subcategories $\mathcal{A}_{i}^{p}$ have such properties.

Lemma 2.19. For $i = 1, 1/2$, the category $\mathcal{A}_{i}^{p}$ is of finite length with respect to strict monomorphisms, and strict epimorphisms, i.e. any infinite chains of strict monomorphisms, strict epimorphisms in $\mathcal{A}_{i}^{p}$,

$$\cdots \hookrightarrow E_n \hookrightarrow \cdots \hookrightarrow E_1 \hookrightarrow E_0,$$

must terminate.
Proof. By applying the dualizing functor $\mathbb{D}$ and using Lemma 2.4, it is enough to show that a chain $\{7\}$ terminates. Let us take an infinite chain $\{7\}$ in $\mathcal{A}_p^0$ with each $E_i \in \mathcal{A}_1$. Let $\omega$ be an ample divisor on $X$. Since $-\text{ch}_1(E_i) \cdot \omega^2 \geq 0$ for $E \in \mathcal{A}_1$, we have

$$-\text{ch}_1(E_i) \cdot \omega^2 \geq -\text{ch}_1(E_{i+1}) \cdot \omega^2 \geq 0.$$  

Hence we may assume that $\text{ch}_1(E_i) \cdot \omega^2 = \text{ch}_1(E_{i+1}) \cdot \omega^2$ for any $i$, and this implies that the induced morphism

$$\mathcal{H}^{-1}(E_i) \longrightarrow \mathcal{H}^{-1}(E_{i+1}),$$

is an isomorphism in codimension one. Let us take the exact sequence in $\mathcal{A}_p$,

$$0 \longrightarrow E_i \longrightarrow E_{i+1} \longrightarrow G_i \longrightarrow 0. \quad (9)$$

Then $\mathcal{H}^{-1}(G_i) = 0$ since otherwise $\mathcal{H}^{-1}(G_i)$ is one or zero dimensional, and contradicts that $\mathcal{H}^{-1}(G_i) \in \text{Coh}_{\geq 2}(X)$. Taking the cohomology of (9), we have the chain of inclusions of sheaves,

$$\cdots \subset \mathcal{H}^0(E_n) \subset \cdots \subset \mathcal{H}^0(E_1) \subset \mathcal{H}^0(E_0). \quad (10)$$

The sequence (10) must terminate since each $\mathcal{H}^0(E_j)$ is a zero dimensional sheaf by the definition of $\mathcal{A}_1$. Hence the chain $\{7\}$ also terminates.

Similarly let us take a chain $\{7\}$ with each $E_i \in \mathcal{A}_{1/2}$. Then we have

$$-\text{ch}_0(E_i) \geq -\text{ch}_0(E_{i+1}) \geq 0,$$

hence we may assume $-\text{ch}_0(E_i) = -\text{ch}_0(E_{i+1})$ for any $i$. Let us consider the exact sequence as in (9). Again $\mathcal{H}^{-1}(G_i) = 0$ since otherwise it is a two dimensional sheaf, which contradicts that $G_i \in \mathcal{A}_{1/2}$. Taking the cohomology of (9), we obtain the sequence (10). In this case, we have

$$\text{ch}_2(\mathcal{H}^0(E_i)) \cdot \omega \geq \text{ch}_2(\mathcal{H}^0(E_{i+1})) \cdot \omega \geq 0,$$

hence we may assume $\text{ch}_2(\mathcal{H}^0(E_i)) \cdot \omega = \text{ch}_2(\mathcal{H}^0(E_{i+1})) \cdot \omega$. Then $G_i = \mathcal{H}^0(G_i)$ is zero dimensional, thus $G_i = 0$ by the definition of $\mathcal{A}_{1/2}$. Therefore $\{7\}$ must terminate.  

\Box

2.4 Limit stability on $\mathcal{A}_p$

Here we introduce the notion of limit stability on $\mathcal{A}_p$. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{ B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is an ample class } \}. $$

For $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$, we consider the group homomorphism $Z_{\sigma} : K(X) \to \mathbb{C}$,

$$Z_{\sigma}(E) = -\int e^{-(B+i\omega)} \text{ch}(E) \sqrt{\text{td}X}. \quad (11)$$

The above function does not give a stability function on $\mathcal{A}_p$. However if we replace $\sigma$ by

$$\sigma_m = B + m i \omega, \quad \text{for } m \gg 0,$$

then we can define the well-defined argument of $Z_{\sigma_m}(E)$ for any non-zero $E \in \mathcal{A}_p$, which defines the set of (semi)-stable objects in $\mathcal{A}_p$. To see this in more detail, let us introduce the (twisted) Mukai vector,

$$v^B : K(X) \ni E \mapsto e^{-B} \text{ch}(E) \sqrt{\text{td}X} \in H^*(X, \mathbb{R}).$$
Let $v^B_i(E) \in H^{2i}(X, \mathbb{R})$ be the $H^{2i}$-component of $v^B_i(E)$. Then one can expand (11) and give the following formula,

$$Z_{\sigma m}(E) = -\int e^{-m\omega} v^B_i(E)$$

$$= \left(-v_3^B(E) + \frac{1}{2} \kappa^2 \omega^2 v_1^B(E)\right) + \left(\frac{1}{6} \kappa^3 \omega^3 v_0^B(E)\right) i.$$  \(13\)

We have the following lemma.

**Lemma 2.20.** For a non-zero object $E \in \mathcal{A}^p$, we have

$$Z_{\sigma m}(E) \in \left\{ r \exp(i\pi \phi) : r > 0, \frac{1}{4} < \phi < \frac{5}{4} \right\},$$

(14)

for $m \gg 0$.

**Proof.** Let us take $E \in \text{Coh}(X)$ with $\dim \text{Supp}(E) = 3 - i$ for $0 \leq i \leq 3$. It is easy to see that $v_j^B(E) = 0$ for $j < i$ and $v_i^B(E) \cdot \omega^{3-i} = c_i(E) \cdot \omega^{3-i} > 0$.

Therefore by the formula (13), the argument of $Z_{\sigma m}(E)$ for $m \to \infty$ goes, (modulo $2\pi$, )

$$\arg Z_{\sigma m}(E) \longrightarrow \begin{cases} 
\pi & \text{dim Supp}(E) = 0, \\
\frac{\pi}{2} & \text{dim Supp}(E) = 1, \\
0 & \text{dim Supp}(E) = 2, \\
-\frac{\pi}{2} & \text{dim Supp}(E) = 3.
\end{cases}$$  \(15\)

Since the category $\mathcal{A}^p$ is generated by $\text{Coh}_{\leq 1}(X)$ and $\text{Coh}_{\geq 2}(X)$[1], the above asymptotic behavior of $Z_{\sigma m}(E)$ shows the result. \hfill \Box

Given $\sigma \in A(X)_C$ and a non-zero object $E \in \mathcal{A}^p$, we can uniquely determine the phase of $Z_{\sigma m}(E)$ by

$$\phi_{\sigma m}(E) = \frac{1}{\pi} \text{Im log } Z_{\sigma m}(E) \in \left(\frac{1}{4}, \frac{5}{4}\right),$$

for $m \gg 0$. For non-zero $F, E \in \mathcal{A}^p$, we simply write

$$\phi_{\sigma m}(F) < \phi_{\sigma m}(E), \quad \phi_{\sigma m}(F) \leq \phi_{\sigma m}(E),$$

if $\phi_{\sigma m}(F) < \phi_{\sigma m}(E)$, $\phi_{\sigma m}(F) \leq \phi_{\sigma m}(E)$ for $m \gg 0$ respectively. Below we introduce the notion of limit (semi)stable objects.

**Definition 2.21.** For $\sigma \in A(X)_C$, a non-zero object $E \in \mathcal{A}^p$ is called $\sigma$-limit stable (resp. $\sigma$-limit semistable) if for any non-zero subobject $F \subset E$, one has

$$\phi_{\sigma m}(F) < \phi_{\sigma m}(E), \quad \phi_{\sigma m}(F) \leq \phi_{\sigma m}(E),$$

if $\phi_{\sigma m}(F) < \phi_{\sigma m}(E)$, $\phi_{\sigma m}(F) \leq \phi_{\sigma m}(E)$ for $m \gg 0$ respectively.

**Remark 2.22.** In Lemma 2.20, the smallest $m > 0$ for which (14) holds depends on $E$, the function $Z_{\sigma m}$ does not give stability functions on $\mathcal{A}^p$ for any $m$. On the other hand, the function $Z_{\sigma}$ induces the stability condition on the subcategory $\text{Coh}_{\leq 1}(X) \subset \mathcal{A}^p$ by the composition,

$$K(\text{Coh}_{\leq 1}(X)) \longrightarrow K(X) \xrightarrow{Z_{\sigma}} \mathbb{C}.$$

The induced stability condition is the same one constructed in [34, Lemma 3.4].
Remark 2.23. Our notion of limit stability is included in the notion of polynomial stability introduced by A. Bayer [3] independently. Some of the results in this section, especially Theorem 2.29 (i), are proved in [3] in more general setting, although the proofs are different.

Remark 2.24. It is easy to see some standard stability properties for limit stability. For example, let $E, F \in \mathcal{A}^p$ be $\sigma$-limit semistable with $\phi_\sigma(E) > \phi_\sigma(F)$. Then $\text{Hom}(E, F) = 0$. Also for $\sigma$-limit stable object $E \in \mathcal{A}^p$, one has $\text{Hom}(E, E) = \mathbb{C}$.

In the following, we give some examples of limit (semi)stable objects. The proofs are straightforward and we leave them to the readers.

Example 2.25. (i) Let $F$ be a $\mu$-stable vector bundle on $X$. Then $F[1] \in \mathcal{A}^p$ and it is $\sigma$-limit stable for any $\sigma \in A(X)_\mathbb{C}$.

(ii) Let us take $\sigma = B + i\omega \in A(X)_\mathbb{C}$ and $F \in \text{Coh}_{\leq 1}(X) \subset \mathcal{A}^p$. Then noting Remark 2.13 and Remark 2.22, we can easily see that $F$ is a $\sigma$-limit semistable if and only if $F$ is $(B, \omega)$-twisted semistable sheaf, i.e. for any non-zero subsheaf $F' \subset F$, one has $\mu_\sigma(F') \leq \mu_\sigma(F)$, where

$$\mu_\sigma(F) = \frac{\text{ch}_3(F) - B \text{ch}_2(F)}{\omega \text{ch}_2(F)} \in \mathbb{R}.$$ 

(iii) Let $x \in X$ be a closed point and $L_x \subset \mathcal{O}_X$ the ideal sheaf. Then $L_x$ is a Gieseker stable sheaf, but $L_x[1] \in \mathcal{A}^p$ is not $\sigma$-limit semistable. In fact we have the exact sequence in $\mathcal{A}^p$,

$$0 \rightarrow \mathcal{O}_x \rightarrow L_x[1] \rightarrow \mathcal{O}_X[1] \rightarrow 0,$$

with $\phi_\sigma(\mathcal{O}_x) = \phi_\sigma(L_x[1])$, which destabilizes $L_x[1]$.

For objects in $\mathcal{A}^p$, we have the following lemma.

Lemma 2.26. For a non-zero object $E \in \mathcal{A}^p_i$ ($i=1, 1/2,$) we have

$$\phi_{\sigma_m}(E) \rightarrow i, \quad \text{for } m \rightarrow \infty.$$ 

Proof. For $E = F[1]$ or $E = \mathcal{O}_x$, where $F$ is a pure two dimensional sheaf and $x \in X$ is a closed point, the result follows by the formula [13]. By the definition of $\mathcal{A}^p_1$, the result follows for any $E \in \mathcal{A}^p_1$. Next let us take a non-zero object $E \in \mathcal{A}^p_{1/2}$. Then by the definition of $\mathcal{A}^p_{1/2}$, we have either $\mathcal{H}^{-1}(E)$ is a torsion free sheaf, or $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a pure one dimensional sheaf. In both cases, the result follows by the formula [13].

We have the following characterization of limit stable objects.

Lemma 2.27. An object $E \in \mathcal{A}^p$ is $\sigma$-limit (semi)stable with $\phi_{\sigma_m}(E) \rightarrow i$ for $m \rightarrow \infty$ if and only if $E \in \mathcal{A}^p_i$ and for any strict monomorphism $0 \neq F \hookrightarrow E$ in $\mathcal{A}^p_i$, one has $\phi_\sigma(F) < \phi_\sigma(E)$. (resp. $\phi_\sigma(F) \leq \phi_\sigma(E)$.)

Proof. Suppose first that $E$ is $\sigma$-limit semistable. By Lemma 2.16 there is an exact sequence,

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_{1/2} \rightarrow 0,$$

in $\mathcal{A}^p$ with $E_i \in \mathcal{A}_i$. By Lemma 2.26, the limit semistability of $E$ implies $E_1 = 0$ or $E_{1/2} = 0$. Hence if $\phi_{\sigma_m}(E)$ goes to $i$, we have $E \in \mathcal{A}^p_i$. Next assume that $E \in \mathcal{A}^p_{1/2}$ and consider an exact sequence in $\mathcal{A}^p$,

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0.$$
By Lemma 2.16 there is an exact sequence $0 \to F_1 \to F \to F_{1/2} \to 0$ with $F_i \in \mathcal{A}^p$. Lemma 2.26 yields,

$$\phi_\sigma(F_1) \geq \phi_\sigma(F) \geq \phi_\sigma(F_{1/2}).$$

Composing the injections $F_1 \hookrightarrow F \hookrightarrow E$, we obtain the exact sequence in $\mathcal{A}^p$,

$$0 \to F_1 \to E \to G' \to 0.$$

For any $F' \in \mathcal{A}^p_{1/2}$, we have $\text{Hom}(G', F') \subset \text{Hom}(E, F') = 0$. Therefore $G' \in \mathcal{A}^p$, i.e. $F_1 \hookrightarrow E$ is a strict monomorphism. Hence if $\phi_\sigma(F_1) \leq \phi_\sigma(E)$ holds, then $\phi_\sigma(F) \leq \phi_\sigma(E)$, hence $E$ is $\sigma$-limit semistable. The proofs for limit stable objects and the case of $i = 1/2$ are similar and we leave them to the reader.

For $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$, let $\sigma^\vee = -B + i\omega$. Combining Lemma 2.17 with Lemma 2.27, we have the following compatibility of limit stability with the dualizing functor.

**Lemma 2.28.** We have the following.

$E \in \mathcal{A}^p_1$ is $\sigma$-limit (semi)stable $\iff \mathbb{D}(E)[1] \in \mathcal{A}^p$ is $\sigma^\vee$-limit (semi)stable,

$E \in \mathcal{A}^p_{1/2}$ is $\sigma$-limit (semi)stable $\iff \mathbb{D}(E) \in \mathcal{A}^p_{1/2}$ is $\sigma^\vee$-limit (semi)stable.

**Proof.** For $v \in H^{\text{even}}(X, \mathbb{R})$, let $v^\vee$ be the dual operator,

$$v = (v_0, v_1, v_2, v_3) \mapsto v^\vee = (v_0, -v_1, v_2, -v_3).$$

Here $v_i$ is the $H^{2i}$-component of $v$. Then we have $v^{-B}(\mathbb{D}(E)) = v^B(E)^\vee$, hence

$$Z_{\sigma^\vee}(\mathbb{D}(E)) = -\int e^{-m\omega} v^B(E)^\vee,$$

$$= -Z_\sigma(E).$$

Therefore the result follows from Lemma 2.17 with Lemma 2.27 directly. □

Finally in this section, we prove the following theorem.

**Theorem 2.29.** For $\sigma \in A(X)_{\mathbb{C}}$, we have the following.

(i) For a non-zero $E \in \mathcal{A}^p$, there exists a filtration in $\mathcal{A}^p$,

$$E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that each $F_i = E_i/E_{i+1}$ is $\sigma$-limit semistable with $\phi_\sigma(F_i) > \phi_\sigma(F_{i+1})$. i.e. (16) is a Harder-Narasimhan filtration.

(ii) For a $\sigma$-limit semistable object $E \in \mathcal{A}^p$, there exists a filtration in $\mathcal{A}^p$,

$$E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that each $F_i = E_i/E_{i+1}$ is $\sigma$-limit stable with $\phi_\sigma(F_i) = \phi_\sigma(F_{i+1})$. i.e. (17) is a Jordan-Hölder filtration.
Proof. (i) In Proposition 2.3, let us replace $A$, $\phi$, “inclusions”, “surjections”, by $A^p$, $\phi_\sigma$, “strict monomorphisms”, “strict epimorphisms”, respectively. As Lemma 2.19 provides the corresponding sufficient condition, we can follow the same proof of Proposition 2.3 in [6, Proposition 2.4], and show the following. For any $E \in A^p_1$, there is a finite sequence of strict monomorphisms in $A^p_1$,
\begin{equation}
E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = E,
\end{equation}
such that for any strict monomorphism $F \hookrightarrow F_i = E_i/E_{i+1} \in A^p_i$, one has $\phi_\sigma(F) \leq \phi_\sigma(F_i)$, and $\phi_\sigma(F_i) \succ \phi_\sigma(F_{i+1})$. By Lemma 2.27, $F_i$ is a $\sigma$-limit semistable object, hence the filtration (18) gives the Harder-Narasimhan filtration.

Let us take an object $E \in A^p$. We have an exact sequence,

$$
0 \to E_1 \to E \to E_{1/2} \to 0,
$$

with $E_i \in A^p_i$. Composing the Harder-Narasimhan filtrations of $E_1$, $E_{1/2}$, we obtain the Harder-Narasimhan filtration of $E$.

(ii) Since any $\sigma$-limit semistable object is contained in $A^p_1$ or $A^p_{1/2}$ by Lemma 2.27, the result follows from Lemma 2.19.

Remark 2.30. The existence of a Harder-Narasimhan filtration is guaranteed once we show that there are no infinite sequences such as (3), (4) for limit stability. Unfortunately this is not true. In fact in the notation of Remark 2.14, we have the following infinite sequence,

$$
O_X \oplus O_H[1] \mapsto O_X \oplus O_H(C)[1] \mapsto O_X \oplus O_H(2C)[1] \mapsto \cdots
$$

which satisfies that for $\sigma = i\omega$,

$$
\phi_\sigma(O_X \oplus O_H[1]) \succ \phi_\sigma(O_X \oplus O_H(C)[1]) \succ \phi_\sigma(O_X \oplus O_H(2C)[1]) \succ \cdots.
$$

3 Moduli spaces of limit stable objects

For a Calabi-Yau 3-fold $X$, let us take elements,

$$
\beta \in H^4(X, \mathbb{Q}), \quad n \in H^6(X, \mathbb{Q}) \cong \mathbb{Q}.
$$

This section is devoted to study the moduli problem of limit stable objects $E \in A^p$, satisfying $\det E = O_X$ and the following numerical condition,

$$
(ch_0(E), ch_1(E), ch_2(E), ch_3(E)) = (-1, 0, \beta, n).
$$

Note that if $E \in A^p$ is limit stable satisfying (19), then $E \in A^p_{1/2}$ by Lemma 2.27. For $\sigma \in A(X)_{\mathbb{C}}$, let $L_n(X, \beta)$, $L^p_n(X, \beta)$ be the sets of objects,

$$
L_n(X, \beta) := \{ E \in A^p_{1/2} \mid \det E = O_X \text{ and } ch(E) \text{ satisfies (19) } \},
$$

$$
L^p_n(X, \beta) := \{ E \in L_n(X, \beta) \mid E \text{ is } \sigma\text{-limit stable } \}.
$$

Remark 3.1. If $E$ is quasi-isomorphic to a two term complex $(O_X \xrightarrow{\delta} F)$, where $F$ is a pure one dimensional sheaf located in degree zero, and

$$
ch_2(F) = \beta, \quad ch_3(F) = n,
$$

then $E \in L_n(X, \beta)$.
From this section, we use the following notation. For a relatively perfect object (cf. [23, Definition 2.1.1]) $\mathcal{E} \in D^b(X \times S)$ and a morphism $T \rightarrow S$, we denote by $\mathcal{E}_T \in D^b(X \times T)$ the derived pull-back of $\mathcal{E}$. The moduli problem of objects in the derived category has been studied in some articles, see [16, 23, 33]. In this paper, we use the algebraic space constructed by Inaba [16], which provides a “mother space” of our moduli problem. Let $\mathcal{M}$ be the functor,

$$\mathcal{M}: (\text{Sch}/\mathbb{C}) \rightarrow \text{(Set)},$$

which sends a $\mathbb{C}$-scheme $S$ to a family of simple complexes $\mathcal{E} \in D^b(X \times S)$, (up to isomorphism,) where an object $E \in D^b(X)$ is called a simple complex if

$$\text{Hom}(E, E) = \mathbb{C}, \quad \text{Ext}^{-1}(E, E) = 0.$$  

(22)

Then Inaba [16] shows that the étale sheafication of $\mathcal{M}$, denoted by $\mathcal{M}^{\text{et}}$, is an algebraic space of locally finite type. Let $\mathcal{M}^{\text{et}}_0$ be the closed fiber at $[\mathcal{O}_X] \in \text{Pic}(X)$ with respect to the following morphism,

$$\det: \mathcal{M}^{\text{et}} \ni E \rightarrow \det E \in \text{Pic}(X).$$

Since any object $E \in L^g_n(X, \beta)$ satisfies (22), there is a subfunctor

$$\mathcal{L}^g_n(X, \beta) \subset \mathcal{M}^{\text{et}}_0,$$  

(23)

whose $S$-valued point consists of $\mathcal{E} \in \mathcal{M}^{\text{et}}_0(S)$ with $E_s \in L^g_n(X, \beta)$ for any $s \in S$. Our purpose in this section is to show that $\mathcal{L}^g_n(X, \beta)$ is an algebraic subspace of $\mathcal{M}^{\text{et}}_0$. We use the same strategy as in [33], namely we show that (23) is an open immersion and $L^g_n(X, \beta)$ is bounded.

3.1 Characterizations of limit stable objects

In this paragraph, we give some characterizations for objects in $L^g_n(X, \beta)$ to be limit stable. First we show the following.

**Lemma 3.2.** For an object $E \in L^g_n(X, \beta)$, there is a subscheme $C \subset X$ with $\mathcal{O}_C$ a pure one dimensional sheaf (or zero) such that $\mathcal{H}^{-1}(E)$ is isomorphic to the ideal sheaf $I_C \subset \mathcal{O}_X$.

**Proof.** Since $E \in \mathcal{A}_{1/2}^p$, $\mathcal{H}^{-1}(E)$ is a torsion free sheaf of rank one with trivial determinant. We have the injection,

$$\mathcal{H}^{-1}(E) \hookrightarrow \mathcal{H}^{-1}(E)^{\vee \vee} \cong \mathcal{O}_X,$$

which shows $\mathcal{H}^{-1}(E) \cong I_C$ for a subscheme $C \subset X$. By the condition [19], we have $\dim C \leq 1$.

Also if $\mathcal{O}_C$ contains a zero dimensional subsheaf, there is $x \in X$ with injections in $\mathcal{A}^p$,

$$\mathcal{O}_x \hookrightarrow \mathcal{O}_C \hookrightarrow I_C[1] \hookrightarrow E.$$

Here $\mathcal{O}_C \hookrightarrow I_C[1]$ corresponds to the extension $0 \rightarrow I_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$. Since $E \in \mathcal{A}_{1/2}^p$, this is a contradiction.

**Remark 3.3.** For $E \in L^g_n(X, \beta)$, Lemma 3.2 yields,

$$\beta = \text{ch}_2(\mathcal{O}_C) + \text{ch}_2(\mathcal{H}^0(E)) \in H^4(X, \mathbb{Z}),$$

$$n = \text{ch}_3(\mathcal{O}_C) + \text{ch}_3(\mathcal{H}^0(E)) \in H^6(X, \mathbb{Z}) \cong \mathbb{Z}.$$  

(24)

Hence below we always assume $\beta \in H^4(X, \mathbb{Z})$, $n \in \mathbb{Z}$, and $\beta$ is an effective class, i.e. the Poincaré dual of the homology class of an effective one cycle on $X$. 

16
Next we show the following.

**Lemma 3.4.** For $\sigma \in A(X)_C$, an object $E \in L_n(X, \beta)$ is $\sigma$-limit stable if and only if the following conditions hold.

(a) For any pure one dimensional sheaf $G \neq 0$ which admits a strict epimorphism $E \rightarrow G$ in $A^p_{1/2}$, one has $\phi_\sigma(E) \prec \phi_\sigma(G)$.

(b) For any pure one dimensional sheaf $F \neq 0$ which admits a strict monomorphism $F \hookrightarrow E$ in $A^p_{1/2}$, one has $\phi_\sigma(F) \prec \phi_\sigma(E)$.

**Proof.** For a $\sigma$-limit stable object $E \in L_n(X, \beta)$, the conditions (a), (b) follow from the definition of limit stability.

Next suppose that $E \in L_n(X, \beta)$ satisfies (a) and (b). Applying Lemma 2.27, it is enough to show that for any non-trivial exact sequence in $A^p_{1/2}$,

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0,$$

we have $\phi_\sigma(F) \prec \phi_\sigma(E)$. If $H^{-1}(F) = 0$, then $F$ is a pure one dimensional sheaf and $\phi_\sigma(F) \prec \phi_\sigma(E)$ follows from (b). If $H^{-1}(F) \neq 0$, then it is a torsion free sheaf of rank one by Lemma 3.2. It follows that $H^{-1}(G)$ is a torsion sheaf, hence zero because of $G \in A^p_{1/2}$. So $G$ is a pure one dimensional sheaf, and we obtain $\phi_\sigma(E) \prec \phi_\sigma(G)$ by (a).

\[\square\]

**Remark 3.5.** By Lemma 2.17 and the same argument as in Lemma 2.28, the condition (b) of Lemma 3.4 can be replaced by the following. For any pure one dimensional sheaf $G' \neq 0$ which admits a strict epimorphism $D(E) \rightarrow G'$ in $A^p_{1/2}$, one has $\phi_\sigma(D(E)) \prec \phi_\sigma(G')$.

**Remark 3.6.** Since $E \in A^p_{1/2}$ is concentrated on $[-1,0]$, giving a strict epimorphism $E \rightarrow G$ as in (a) of Lemma 3.4 is equivalent to giving a surjection of sheaves $H^0(E) \rightarrow G$.

As for strict monomorphism $F \hookrightarrow E$ in (b) of Lemma 3.4 we have the following.

**Lemma 3.7.** Let $F \hookrightarrow E$ be as in (b) of Lemma 3.4 and $C \subset X$ as in Lemma 3.2. Then there are subsheaves,

$$F_1 \subset O_C, \quad F_2 \subset H^0(E),$$

such that $F$ is written as an extension,

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0.$$

**Proof.** Let

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0,$$

be the exact sequence in $A^p$. Taking cohomology, we obtain the exact sequences of sheaves,

$$0 \rightarrow H^{-1}(E) \rightarrow H^{-1}(G) \rightarrow F_1 \rightarrow 0,$$

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0,$$

$$0 \rightarrow F_2 \rightarrow H^0(E) \rightarrow H^0(G) \rightarrow 0.$$

Since $H^{-1}(E)$ is torsion free and $F_1 \in \text{Coh}_{\leq 1}(X)$, we have

$$F_1 \subset H^{-1}(E)^{\vee}/H^{-1}(E) \cong O_C,$$

from the sequence (25). Therefore the sequence (26) gives the desired extension. \[\square\]
In Lemma 3.8 let us write the condition \( \phi_\sigma(F) \prec \phi_\sigma(E) \) in a simpler way. For \( F \in \text{Coh}_{\leq 1}(X) \), let \( \mu_\sigma(F) \in \mathbb{R} \) be as in Example 2.25 (ii).

**Lemma 3.8.** For \( \sigma = B + i \omega \in \mathcal{A}(X) \), \( E \in L_\omega(X, \beta) \) and \( F \in \text{Coh}_{\leq 1}(X) \), we have \( \phi_\sigma(F) \prec \phi_\sigma(E) \), (resp. \( \phi_\sigma(F) \succ \phi_\sigma(E) \),) if and only if one of the following conditions hold.

- We have the following inequality,
  \[
  \mu_\sigma(F) < -\frac{3B\omega^2}{\omega^3}, \quad \text{(resp. } \mu_\sigma(F) > -\frac{3B\omega^2}{\omega^3}) \tag{28}
  \]
- We have \( \mu_\sigma(F) = -3B\omega^2/\omega^3 \) and
  \[
  \omega v_2^B(E)\mu_\sigma(F) < v_3^B(E), \quad \text{(resp. } \omega v_2^B(E)\mu_\sigma(F) > v_3^B(E) \tag{29})
  \]

**Proof.** The condition \( \phi_\sigma(F) \prec \phi_\sigma(E) \) is equivalent to
\[
\frac{\text{Re } Z_{\sigma_m}(F)}{\text{Im } Z_{\sigma_m}(F)} > \frac{\text{Re } Z_{\sigma_m}(E)}{\text{Im } Z_{\sigma_m}(E)},
\]
for \( m \gg 0 \). Since we have
\[
v^B(F) = (0, 0, ch_2(F), ch_3(F) - B \cdot ch_2(F)),
\]
\[
v^B(E) = (-1, B, v_2^B(F), v_3^B(F)),
\]
the inequality (30) is equivalent to
\[
- \frac{\mu_\sigma(F)}{m} = \frac{-3ch_2(F) + B \cdot ch_2(F)}{m\omega \cdot ch_2(F)} \geq \frac{m^2\omega^2B/2 - v_3^B(E)}{m^3\omega^3/6 + m\omega v_2^B(E)},
\]
for \( m \gg 0 \) by the formula (12). The above inequality is equivalent to
\[
\frac{1}{6}m^2\omega^3 \left( \mu_\sigma(F) + \frac{3\omega^2B}{\omega^3} \right) < -v_2^B(E)\mu_\sigma(F) + v_3^B(E),
\]
for \( m \gg 0 \). Therefore (28) or (29) must be satisfied. \( \square \)

### 3.2 Evaluations of numerical classes

In this paragraph, we evaluate the numerical classes of \( \mathcal{H}^{-1}(E), \mathcal{H}^0(E) \) for \( E \in L_\omega(X, \beta) \). Below we fix an ample divisor \( H \) on \( X \), and set
\[
\mathcal{N}(\beta) := \{ \beta' \in H^4(X, \mathbb{Z}) \mid \beta' \text{ is an effective class with } 0 \leq \beta' \cdot H \leq \beta \cdot H \}. \tag{31}
\]

The following Lemma 3.9 and Lemma 3.10 seem well-known, but we give the proof for the reader’s convenience.

**Lemma 3.9.** The set \( \mathcal{N}(\beta) \) is a finite set.

**Proof.** For any ample divisor \( H' \) on \( X \), we have
\[
\mathcal{N}(\beta) \subset \{ \beta' \in H^4(X, \mathbb{R}) \mid \beta' \cdot H' \geq 0 \}.
\]

Since the ample cone is an open cone, one can find a compact convex polytope in \( H^4(X, \mathbb{R}) \) which contains \( \mathcal{N}(\beta) \). Therefore \( \mathcal{N}(\beta) \) is a finite set. \( \square \)
Next we set \( m(\beta) \in [-\infty, \infty) \) as follows,
\[
m(\beta) := \inf \{ \text{ch}_3(\mathcal{O}_C) \mid C \subset X \text{ satisfies } \dim C = 1, [C] \in \mathcal{N}(\beta) \}.
\]

**Lemma 3.10.** We have \( m(\beta) > -\infty \).

**Proof.** Let \( \text{Hilb}_n(X, \beta) \) be the Hilbert scheme of one dimensional subschemes \( C \subset X \) with \( \beta = [C], \ n = \text{ch}_3(\mathcal{O}_C) \).

If \( \text{Hilb}_{n-k}(X, \beta) \) is non-empty for \( k > 0 \), then we have
\[
\dim \text{Hilb}_n(X, \beta) \geq 3k,
\]
by adding \( k \)-floating points to a subscheme \( C' \subset X \) with \( [C'] = \beta, \ n-k = \text{ch}_3(\mathcal{O}_{C'}) \). Then the boundedness of \( \text{Hilb}_n(X, \beta) \) implies that \( \text{Hilb}_{n-k}(X, \beta) = \emptyset \) for \( k \gg 0 \).

Finally we show the following lemma.

**Lemma 3.11.** (i) The image of the map,
\[
L_n(X, \beta) \ni E \mapsto (\text{ch}_2(\mathcal{H}^{-1}(E)), \text{ch}_2(\mathcal{H}^0(E))) \in H^4(X, \mathbb{Z})^\oplus 2,
\]
is a finite set.

(ii) For \( E \in L_n(X, \beta) \), let \( \beta' = -\text{ch}_2(\mathcal{H}^{-1}(E)) \). Then we have,
\[
\text{ch}_3(\mathcal{H}^0(E)) \leq n - m(\beta') \leq n - m(\beta).
\]

Moreover for \( \sigma = B + i\omega \in A(X)_C \), the image of the map,
\[
L_n^\sigma(X, \beta) \ni E \mapsto (\text{ch}_3(\mathcal{H}^{-1}(E)), \text{ch}_3(\mathcal{H}^0(E))) \in \mathbb{Z}^\oplus 2,
\]
is a finite set.

**Proof.** (i) For \( E \in L_n^\sigma(X, \beta) \), the equality \((24)\) implies,
\[
(\text{ch}_2(\mathcal{H}^{-1}(E)), \text{ch}_2(\mathcal{H}^0(E))) \in \mathcal{N}(\beta) \times \mathcal{N}(\beta).
\]

Hence Lemma 3.9 yields the result.

(ii) For \( E \in L_n(X, \beta) \), we have \( \mathcal{H}^{-1}(E) = I_C \) where \( C \subset X \) is as in Lemma 3.2. Since \( [C] = \beta' \), we have
\[
\text{ch}_3(\mathcal{H}^0(E)) = n - \text{ch}_3(\mathcal{O}_C)
\leq n - m(\beta').
\]

Also since \( \beta' \in \mathcal{N}(\beta) \), we have \( n - m(\beta') \leq n - m(\beta) \). Suppose that \( E \) is \( \sigma \)-limit stable. If \( \mathcal{H}^0(E) \) is non-zero, Lemma 3.4 Remark 3.6 and Lemma 3.8 show,
\[
-\frac{3B\omega^2}{\omega^3} \leq \mu_\sigma(\mathcal{H}^0(E)) = \frac{\text{ch}_3(\mathcal{H}^0(E)) - B \text{ch}_2(\mathcal{H}^0(E))}{\omega \text{ch}_2(\mathcal{H}^0(E))}.
\]

Since \( \text{ch}_2(\mathcal{H}^0(E)) \in \mathcal{N}(\beta) \), the value \( \text{ch}_3(\mathcal{H}^0(E)) \in \mathbb{Z} \) is also bounded below by the above inequality and Lemma 3.9. \( \square \)
3.3 Boundedness of limit stable objects

In this paragraph, we show the boundedness of limit stable objects, which is relevant to the existence of the moduli space. Recall that a set of objects $S \subset D^b(X)$ is bounded if there is a finite type $\mathbb{C}$-scheme $Q$ and an object $E \in D^b(X \times Q)$ such that any object $E \in S$ is isomorphic to $E_q$ for some $q \in Q$. We first show the boundedness of some subsets of objects in $\text{Coh}_{\leq 1}(X)$.

For $\beta \in H^4(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we set

$$S_n(X, \beta) := \{ E \in \text{Coh}_{\leq 1}(X) \mid \text{ch}_2(E) = \beta, \text{ch}_3(E) = n \}.$$ 

Also let us fix $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$ and $\mu \in \mathbb{R}$.

Lemma 3.12. (i) The following set of objects is bounded.

$$S_n(X, \beta, \sigma, \mu) = \left\{ E \in S_n(X, \beta) : \mu_{\sigma}(G) \geq \mu \text{ for any surjection } E \twoheadrightarrow G \text{ in } \text{Coh}_{\leq 1}(X) \right\}.$$ 

(ii) The following set of objects is bounded.

$$S'_n(X, \beta, \sigma, \mu) = \left\{ G \in \text{Coh}_{\leq 1}(X) \cap \mathcal{A}_{1/2}^p : \text{there is } E \in S_n(X, \beta) \text{ and a surjection } E \twoheadrightarrow G \text{ in } \text{Coh}_{\leq 1}(X) \text{ and } \mu_{\sigma}(G) \leq \mu \right\}.$$ 

Proof. (i) For $E \in S_n(X, \beta, \sigma, \mu)$, let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_i = E$$

be the Harder-Narasimhan filtration with respect to $(B, \omega)$-twisted semistability. (Or equivalently Harder-Narasimhan filtration with respect to the induced stability condition $(\text{Coh}_{\leq 1}(X), Z_{\sigma})$.) Note that for $F_i = E_i/E_{i-1}$, we have $\text{ch}_2(F_i) \in N(\beta)$, and the function which sends $E \in S_n(X, \beta, \sigma, \mu)$ to the number of $(B, \omega)$-twisted semistable factors $l$ is bounded. By the definition of $S_n(X, \beta, \sigma, \mu)$, we see

$$\mu_{\sigma}(E) \geq \mu_{\sigma}(E/E_i) \geq \mu,$$

for any $i$. Therefore for each $i$, we have only finite number of possibilities for the pair,

$$(\text{ch}(E_i), \text{ch}(E/E_i)) \in H^*(X, \mathbb{Q})^{\oplus 2}.$$ 

Hence the possibilities for $\text{ch}(F_i) \in H^*(X, \mathbb{Q})$ is also bounded. On the other hand, the set of $(B, \omega)$-twisted semistable sheaves $F \in \text{Coh}_{\leq 1}(X)$ with a fixed numerical class is bounded. (See for instance [34 Lemma 3.8].) Therefore $S_n(X, \beta, \sigma, \mu)$ is bounded.

(ii) For $G \in S'_n(X, \beta, \sigma, \mu)$, note that $\text{ch}_2(G) \in N(\beta)$. Hence $\mu_{\sigma}(G) \leq \mu$ implies that there is $\mu' \in \mathbb{R}$, which depends only on $B$ and $\beta$, such that

$$\mu_{\omega}(G) = \frac{\text{ch}_2(G)}{\omega \text{ch}_2(G)} \leq \mu',$$

for any $G \in S'_n(X, \beta, \sigma, \mu)$. Since $G$ is a pure sheaf, one can apply [14 Lemma 1.7.9], and conclude that $S'_n(X, \beta, \sigma, \mu)$ is bounded.

In the following we show the boundedness of $L'_n(X, \beta)$.

Proposition 3.13. For $\sigma = B + i\omega \in A(X)_{\mathbb{C}}$, the set of objects $L'_n(X, \beta)$ is bounded.
Proof. It is enough to show the boundedness of the following sets of sheaves,

$$\{ \mathcal{H}^{-1}(E) \mid E \in L^0_n(X, \beta) \},$$  \hspace{1cm} (32) \\
$$\{ \mathcal{H}^0(E) \mid E \in L^0_n(X, \beta) \}. $$  \hspace{1cm} (33) \\

Also by Lemma 3.11 it is enough to show the boundedness of the above sets of objects satisfying

$$(ch_2(\mathcal{H}^{-1}(E)), ch_3(\mathcal{H}^{-1}(E))) = (\beta', -n'),$$ \\
$$(ch_2(\mathcal{H}^0(E)), ch_3(\mathcal{H}^0(E))) = (\beta'', n''),$$

for fixed numerical classes $(\beta', n')$, $(\beta'', n'')$. For $E \in L^0_n(X, \beta)$, we have $\mathcal{H}^{-1}(E) \in I_{n'}(X, \beta')$ by Lemma 3.2, in particular the set of sheaves (32) is bounded. Here $I_{n'}(X, \beta')$ is the Hilbert scheme as in the proof of Lemma 3.10. Also Lemma 3.4, Remark 3.6 and Lemma 3.8 show

$$\mathcal{H}^0(E) \in S_{n''}(X, \beta'', \sigma, \mu),$$

where $\mu = -3B\omega^2/\omega^3$. Therefore by Lemma 3.12(i), the set of sheaves (33) is also bounded. \qed

### 3.4 Openness of limit stability

The purpose of this paragraph is to show that the embedding $\mathcal{L}_n^0(X, \beta) \subset \mathcal{M}_0^0$ given in (23) is an open immersion, and complete the proof that $\mathcal{L}_n^0(X, \beta)$ is an algebraic space of finite type. First we see the openness of objects in $\mathcal{A}^p$ and $\mathcal{A}_1^p$.

**Lemma 3.14.** For a variety $S$ and an object $\mathcal{E} \in D^b(\mathcal{X} \times S)$, the sets

$$S^o = \{ s \in S \mid \mathcal{E}_s \in \mathcal{A}^p \}, \quad S_i^o = \{ s \in S \mid \mathcal{E}_s \in \mathcal{A}_i^p \},$$

are open subsets in $S$.

**Proof.** As in [2] Appendix A, Example 1], the torsion theory $(\text{Coh}_{\leq 1}(X), \text{Coh}_{\geq 2}(X))$ defines an open stack of torsion theories. Then [2] Theorem A.3] shows that $S^o$ is open. Let us show that $S_i^o$ is open in $S$. We set $\mathbb{D}(\mathcal{E})$ to be

$$\mathbb{D}(\mathcal{E}) = \mathbb{R} \mathcal{H}om(\mathcal{E}, \mathcal{O}_{\mathcal{X} \times S}[2]).$$

Then we have $\mathbb{D}(\mathcal{E}) \simeq \mathbb{D}(\mathcal{E}_s)$. By Lemma 2.17 $S_i^o$ are written as

$$S_i^o = S^o \cap \{ s \in S \mid \mathbb{D}(\mathcal{E})_s \in \mathcal{A}^p[-1] \},$$ \\
$$S_{i/2}^o = S^o \cap \{ s \in S \mid \mathbb{D}(\mathcal{E})_s \in \mathcal{A}^p \}.$$

Therefore the openness of $S_i^o$ follows from the openness of $S^o$. \qed

Next we show the boundedness of destabilizing objects for a family of objects in $\mathcal{A}^p_{1/2}$. Suppose that $\mathcal{E} \in D^b(\mathcal{X} \times S)$ satisfies $\mathcal{E}_s \in L_n(X, \beta)$ for any $s \in S$. We set the sets of objects $\mathcal{D}_e, \mathcal{D}_m$ as follows.

$$\mathcal{D}_e = \left\{ G \in \text{Coh}_{\leq 1}(X) \cap \mathcal{A}^p_{1/2} : \begin{array}{l} \text{there is } s \in S \text{ and a strict epimorphism } \\
\mathcal{E}_s \twoheadrightarrow G \text{ in } \mathcal{A}^p_{1/2} \text{ with } \phi_\sigma(G) \preceq \phi_\sigma(\mathcal{E}_s) \end{array} \right\}. $$

$$\mathcal{D}_m = \left\{ F \in \text{Coh}_{\leq 1}(X) \cap \mathcal{A}^p_{1/2} : \begin{array}{l} \text{there is } s \in S \text{ and a strict monomorphism } \\
\mathcal{E}_s \hookrightarrow F \text{ in } \mathcal{A}^p_{1/2} \text{ with } \phi_\sigma(F) \succeq \phi_\sigma(\mathcal{E}_s) \end{array} \right\}. $$

We have the following.
Lemma 3.15. The sets of objects $\mathcal{D}_e, \mathcal{D}_m$ are bounded.

Proof. Applying the dualizing functor $\mathcal{D}$, it suffices to show the boundedness of $\mathcal{D}_e$ as in Remark 3.15. By taking a flattening stratification, we may assume that $\mathcal{H}^i(\mathcal{E})$ is flat over $S$ for any $i$. As in Remark 3.10, any object $G \in \mathcal{D}_e$ is obtained as a surjection,

$$\mathcal{H}^0(\mathcal{E}_s) = \mathcal{H}^0(\mathcal{E})_s \to G.$$  \hspace{1cm} (34)

Let $(\text{ch}_2(\mathcal{H}^0(\mathcal{E})_s), \text{ch}_3(\mathcal{H}^0(\mathcal{E})_s)) = (\beta', n')$. By (34) and Lemma 3.8 we have

$$G \in S'_m(X, \beta', \sigma, \mu),$$

where $\mu = -3B\omega^2/\omega^3$. By Lemma 3.12 (ii), the set of objects $\mathcal{D}_e$ is bounded. \hfill \square

Based on the above lemma, we show the following proposition.

Proposition 3.16. (i) There exist a finite type $S$-scheme $\pi_e: Q_e \to S$, $\mathcal{G}_e \subset \text{Coh}(X \times Q_e)$, and a morphism $u_e: \mathcal{E}_{Q_e} \to \mathcal{G}_e$ such that

- For $q \in Q_e$, the morphism $u_{e,q}: \mathcal{E}_q \to \mathcal{G}_{e,q}$ is a strict epimorphism in $\mathcal{A}^p_{1/2}$.
- Any strict epimorphism $\mathcal{E}_s \to G$ in $\mathcal{A}^p_{1/2}$ for $G \in \mathcal{D}_e$ is isomorphic to $u_{e,q}$ for some $q \in \pi_e^{-1}(s)$.

(ii) There exist a finite type $S$-scheme $\pi_m: Q_m \to S$, $\mathcal{F}_m \subset \text{Coh}(X \times Q_m)$, and a morphism $u_m: \mathcal{E}_{Q_m} \to \mathcal{E}_m$ such that

- For $q \in Q_m$, the morphism $u_{m,q}: \mathcal{F}_{m,q} \to \mathcal{E}_q$ is a strict monomorphism in $\mathcal{A}^p_{1/2}$.
- Any strict monomorphism $F \hookrightarrow \mathcal{E}_s$ in $\mathcal{A}^p_{1/2}$ for $F \in \mathcal{D}_m$ is isomorphic to $u_{m,q}$ for some $q \in \pi_m^{-1}(s)$.

Proof. The proof is essentially same as in [33, Proposition 3.17], so we only give the outline of the construction of $Q_m$. Since $\mathcal{D}_m$ is bounded, there is a $\mathcal{C}$-scheme of finite type $Q$ and $\mathcal{F} \subset \text{Coh}(X \times Q)$, flat over $Q$, such that any $F \in \mathcal{D}_m$ is isomorphic to $\mathcal{F}_q$ for some $q \in Q$. We may assume that $\phi_\sigma(\mathcal{F}_q) \geq \phi_\sigma(\mathcal{E}_s)$ and $\mathcal{F}_q$ is a pure one dimensional sheaf for any $q \in Q$. Arguing as in [33, Proposition 3.17], there is an affine scheme of finite type $Q'$ and a morphism $Q' \to Q \times S$ such that

- $Q' \to Q \times S$ is bijective on closed points.
- There exists a locally free sheaf $\mathcal{U}$ on $Q'$ such that the functor

$$(T \to Q') \mapsto \mathcal{H}^0(\mathcal{R}_{qT*}\mathcal{R}\text{Hom}(\mathcal{F}_T, \mathcal{E}_T)) \in \text{Coh}(T),$$

is represented by the affine bundle $\mathcal{V}(\mathcal{U}) \to Q'$, where $q_T: X \times T \to T$ is the projection.

Here $\mathcal{F}_T, \mathcal{E}_T$ are obtained by the base changes of $\mathcal{F}, \mathcal{E}$ for the following morphisms respectively,

$$T \to Q' \to Q \times S \xrightarrow{p_1} Q, \quad T \to Q' \to Q \times S \xrightarrow{p_2} S,$$

and $p_1, p_2$ are projections. Let $u: \mathcal{F}_{\mathcal{V}(\mathcal{U})} \to \mathcal{E}_{\mathcal{V}(\mathcal{U})}$ be the universal morphism and take the distinguished triangle,

$$\mathcal{F}_{\mathcal{V}(\mathcal{U})} \xrightarrow{u} \mathcal{E}_{\mathcal{V}(\mathcal{U})} \to \mathcal{G}.$$
Note that \( u_q : \mathcal{F}_{\mathbb{V}(U), q} \to \mathcal{E}_{\mathbb{V}(U), q} \) is a strict monomorphism in \( \mathcal{A}^p_{1/2} \) if and only if \( G_q \in \mathcal{A}^p_{1/2} \). We construct \( Q_m \) as

\[
Q_m := \{ q \in \mathbb{V}(U) \mid G_q \in \mathcal{A}^p_{1/2} \}.
\]

Then \( Q_m \) is an open subscheme of \( \mathbb{V}(U) \) by Lemma 3.14, in particular it is of finite type. By the construction,

\[
\pi_m : Q_m \to S, \quad \mathcal{F}_m := \mathcal{F}_{\mathbb{V}(U)}|Q_m \in \text{Coh}(X \times Q_m), \quad u_m := u|Q_m,
\]
satisfy the desired property. \( \Box \)

**Remark 3.17.** By the construction and Lemma 3.2, the object \( \mathcal{E}_s \in \mathcal{L}_m(X, \beta) \) is \( \sigma \)-limit stable if and only if

\[
s \notin \pi_e(Q_e) \cup \pi_m(Q_m).
\]

Here we collect some well-known lemmas on a family of objects in \( D^b(X) \). For the lack of reference, we also put the proofs. Let \( R \) be a discrete valuation ring and \( K \) a quotient field of \( R \). We denote by \( t \in R \) the uniformizing parameter, and \( o \in \text{Spec} R \) the closed point. We set \( X_R = X \times \text{Spec} R, \ X_K = X \times \text{Spec} K \).

**Lemma 3.18.** Take \( \mathcal{F}, \mathcal{E} \in D^b(X \times \text{Spec} R) \) and a non-zero morphism \( f : \mathcal{F}_K \to \mathcal{E}_K \) in \( D^b(X_K) \). Then there is \( m \in \mathbb{Z} \) such that

1. The morphism \( t^m f : \mathcal{F}_K \to \mathcal{E}_K \) extends to a morphism \( t^m f : \mathcal{F} \to \mathcal{E} \).
2. The induced morphism \( t^m f|_{X \times \{ o \}} : \mathcal{F}_o \to \mathcal{E}_o \) is non-zero.

**Proof.** Since \( \text{Hom}(\mathcal{F}, \mathcal{E}) \) is a finitely generated \( R \)-module and

\[
\text{Hom}_{X_R}(\mathcal{F}, \mathcal{E}) \otimes_R K \cong \text{Hom}_{X_K}(\mathcal{F}_K, \mathcal{E}_K),
\]

there is \( m \in \mathbb{Z} \) such that \( t^m f \) extends to \( \mathcal{F} \to \mathcal{E} \) and \( t^{m-1} f \) does not extend to \( \mathcal{F} \to \mathcal{G} \). We have the exact sequence in \( \text{Coh}(X_R) \),

\[
0 \to \mathcal{E} \xrightarrow{x} \mathcal{E} \to \mathcal{E}_o \to 0.
\]

The above sequence shows that if \( t^m f|_{X \times \{ o \}} \) is zero, then \( t^m f \) factors though \( \mathcal{F} \to \mathcal{E} \xrightarrow{x} \mathcal{E}_o \), which gives an extension of \( t^{m-1} f \). Therefore \( t^m f|_{X \times \{ o \}} \) is non-zero. \( \Box \)

Let \( T \) be a (not necessary projective) smooth curve with a closed point \( o \in T \). We set \( T^o = T \setminus \{ o \} \).

**Lemma 3.19.** Suppose that \( \mathcal{F} \in \text{Coh}(X \times T^o) \) is flat over \( T^o \) and \( \mathcal{F}_t \in \text{Coh}_{\leq 1}(X) \) for any \( t \in T^o \).

1. Assume moreover that \( \mathcal{F}_t \) is \( (B, \omega) \)-twisted semistable for any \( t \in T^o \). Then there is \( \tilde{\mathcal{F}} \in \text{Coh}(X \times T) \), which is flat over \( T \), such that \( \tilde{\mathcal{F}}|_{X \times T^o} \cong \mathcal{F} \) and \( \tilde{\mathcal{F}}|_{X \times \{ o \}} \) is also \( (B, \omega) \)-twisted semistable.
2. There is an open subset \( T'^o \subset T^o \) and a filtration of flat sheaves over \( T'^o \),

\[
0 = \mathcal{F}_{T'^o, 0} \subset \mathcal{F}_{T'^o, 1} \subset \cdots \subset \mathcal{F}_{T'^o, n} = \mathcal{F}_{T'^o},
\]

such that for any \( t \in T'^o \), the induced filtration of \( \mathcal{F}_t \) is a Harder-Narasimhan filtration with respect to \( (B, \omega) \)-twisted semistability.
Proof. (i) For $\sigma = B + i\omega$, let us consider the induced stability condition $(Z_\sigma, \text{Coh}_{\leq 1}(X))$. Using [33] Proposition 2.8, we can assume that $B$ and $\omega$ are defined over $\mathbb{Q}$. After applying some element of $\mathbb{C}$ to $\text{Stab}(D^b(\text{Coh}_{\leq 1}(X)))$, we obtain a stability condition $(Z', \mathcal{A}')$ on $D^b(\text{Coh}_{\leq 1}(X))$ such that $\mathcal{A}'$ is a noetherian abelian category and $Z'(\mathcal{F}_s) \in \mathbb{R}_{<0}$ for any $s \in U$. (See [33] Remark 2.7.) Then we can apply [1] Theorem 4.1.1 and conclude the result.

(ii) If $B = 0$, this is shown in [14] Theorem 2.3.2. The twisted case is similarly discussed and we leave it to the reader. 

Now we are ready to show the following theorem.

**Theorem 3.20.** The embedding $\mathcal{L}_n^\sigma(X, \beta) \subset \mathcal{M}_0^{\text{st}}$ is an open immersion, and $\mathcal{L}_n^\sigma(X, \beta)$ is a separated algebraic space of finite type over $\mathbb{C}$.

**Proof.** First we show that $\mathcal{L}_n^\sigma(X, \beta) \subset \mathcal{M}_0^{\text{st}}$ is an open immersion. For a variety $S$, let us take a $S$-valued point, $\mathcal{E} \in \mathcal{M}_0^{\text{st}}(S)$. Suppose that $\mathcal{E}_s \in L_n^\sigma(X, \beta)$ for some point $s \in S$. We want to show that there exists a Zariski open subset $s \in U \subset S$ such that $\mathcal{E}_s' \in L_n^\sigma(X, \beta)$ for any $s' \in U$. Applying Lemma 3.14 we may assume $\mathcal{E}_s' \in L_n(X, \beta)$ for any $s' \in S$. Let us construct

$$
\pi_e : Q_e \to S, \quad G_e \in \text{Coh}(X \times Q_e), \quad u_e : \mathcal{E}_{Q_e} \to G_e,
$$

$$
\pi_m : Q_m \to S, \quad \mathcal{F}_m \in \text{Coh}(X \times Q_m), \quad u_m : \mathcal{F}_m \to \mathcal{E}_{Q_m},
$$

as in Proposition 3.16. Noting Remark 3.17 it is enough to show

$$
s \notin \overline{\pi_e(Q_e) \cup \pi_m(Q_m)}.
$$

Let us show $s \notin \overline{\pi_m(Q_m)}$. The proof of $s \notin \overline{\pi_e(Q_e)}$ is similar. Suppose by a contradiction that $s \in \pi_m(Q_m)$. Then we can find a smooth curve $T$ with a closed point $o \in T$ and a morphism $p : T \to S$ such that $p(o) = s$ and there is a commutative diagram,

$$
\begin{array}{ccc}
T^o & \longrightarrow & Q_m \\
\uparrow & & \downarrow \pi_m \\
T & \underset{p}{\longrightarrow} & S,
\end{array}
$$

where $T^o = T \setminus \{o\}$. By pulling back $\mathcal{F}_m \in \text{Coh}(X \times Q_m)$ to $X \times T^o$, we obtain the object $\mathcal{F}_{m,T^o} \in \text{Coh}(X \times T^o)$ and a morphism,

$$
u_{m,T^o} : \mathcal{F}_{m,T^o} \longrightarrow \mathcal{E}_{T^o},
$$

such that $\phi_{\sigma}(\mathcal{F}_{m,t}) \geq \phi_{\sigma}(\mathcal{E}_t)$ for any $t \in T^o$. Applying Lemma 3.19 (ii), we may assume that $\mathcal{F}_{m,t}$ is $(B, \omega)$-twisted semistable for any $t \in T^o$. Then Lemma 3.19 (i) shows that there is a flat family of $(B, \omega)$-twisted semistable sheaves,

$$
\mathcal{F}_m \in \text{Coh}(X \times T),
$$

which extends $\mathcal{F}_{m,T^o}$. Applying Lemma 3.18 (i) for $R = \mathcal{O}_{T,o}$, we obtain a non-zero morphism,

$$
\mathcal{F}_{m,o} \longrightarrow \mathcal{E}_s.
$$

Note that $\mathcal{F}_{m,o}$ is $\sigma$-limit semistable, $\mathcal{E}_s$ is $\sigma$-limit stable, and $\phi_{\sigma}(\mathcal{F}_{m,o}) \geq \phi_{\sigma}(\mathcal{E}_s)$. This implies that $\phi_{\sigma}(\mathcal{F}_{m,o}) = \phi_{\sigma}(\mathcal{E}_s)$, the object $\mathcal{E}_s$ is one of the Jordan-Hölder factors of $\mathcal{F}_{m,o}$, and the
morphism \([35]\) is surjective in \(A^p\). However in this case \(E_s\) must be an object in Coh\(_{<1}\)(\(X\)) as we remarked in Remark 2.13 which contradicts that \(E_s \in L_n(X, \beta)\). Therefore \(s \notin \pi_c(Q_c)\) holds.

Now we have proved \(L^\sigma_n(X, \beta) \subset M^\sigma_0\) is an open immersion, hence \(L^\sigma_n(X, \beta)\) is an algebraic space of locally finite type. Moreover \(L^\sigma_n(X, \beta)\) is bounded by Lemma 3.13 which implies that \(L^\sigma_n(X, \beta)\) is in fact of finite type.

Finally let us show that \(L^\sigma_n(X, \beta)\) is separated using valuative criterion. Let \(R, K, t \in R\) and \(o \in \text{Spec } R\) be as in Lemma 3.13. Take two \(R\)-valued points of \(L^\sigma_n(X, \beta)\), \(E_1, E_2 \in D^b(X_R)\). Suppose that there is an isomorphism in \(D^b(X_K)\),

\[
f: E_{1,K} \xrightarrow{\cong} E_{2,K}.
\]

 quotient field of \(R\). By the valuative criterion, it is enough to show that \(t^m f\) extends to an isomorphism \(E_1 \rightarrow E_2\) for some \(m \in \mathbb{Z}\). By Lemma 3.18 there is \(m \in \mathbb{Z}\) and a morphism

\[
f: E_1 \rightarrow E_2
\]

which extends \(t^m f\) and the induced morphism \(f_0: E_{1,o} \rightarrow E_{2,o}\) is non-zero. Since \(E_{1,o}\) and \(E_{2,o}\) are both \(\sigma\)-limit stable objects with the same numerical classes, the morphism \(f_0\) is an isomorphism. Hence \(f\) is also an isomorphism.

**Remark 3.21.** In [33], the author used the result of [1] Proposition 3.5.3 to show the openness of stability for the case of K3 surfaces. In the situation of our paper, the relevant abelian category \(A^n\) is not noetherian which prevents us to use the result of [1]. Instead we have used Lemma 3.18, Lemma 3.19 to show the openness.

**Remark 3.22.** It seems likely that \(L^\sigma_n(X, \beta)\) is a projective variety for a generic choice of \(\sigma\), which we are unable to prove at this time. We do not how to construct the moduli space as a GIT quotient. Also the main technical difficulty to show the properness is that we are unable to extension results of a family of objects as in [1] Theorem 4.1.1, since \(A^n\) is not noetherian again.

### 4 Counting invariants of limit stable objects

In this section, we again assume that \(X\) is a projective Calabi-Yau 3-fold. The purpose of this section is to construct virtual counting of \(\sigma\)-limit stable objects, and study their properties.

#### 4.1 Definitions of counting invariants

For \(\sigma \in A(X)_\mathbb{C}\), \(\beta \in H^4(X, \mathbb{Z})\) and \(n \in \mathbb{Z}\), let \(L^\sigma_n(X, \beta)\) be the algebraic space constructed in Theorem 3.20. In this paragraph, we give the definition of the counting invariant of \(\sigma\)-limit stable objects \(L_{n, \beta}(\sigma) \in \mathbb{Z}\) using \(L^\sigma_n(X, \beta)\). Since we are unable to conclude that \(L^\sigma_n(X, \beta)\) is proper, the integration of virtual classes does not make sense. Instead we use K. Behrend’s constructible function [4] to define counting invariants. Recall that Behrend [4] constructs on any scheme \(M\), (more generally \(M\) is a Deligne Mumford stack,) a canonical constructible function,

\[
\nu_M: M \rightarrow \mathbb{Z},
\]

which depends only on the scheme structure of \(M\). If \(M\) is smooth, \(\nu_M\) is given by \(\nu_M(p) = (-1)^{\dim M}\). Moreover if \(M\) is proper and carries a symmetric perfect obstruction theory, one has

\[
\sharp_{vir}(M) = \sum_{n \in \mathbb{Z}} ne(\nu^{-1}_M(n)),
\]

25
where \( \nu_{\text{vir}}^\sharp(M) \) is the integration over the virtual cycle, and \( e(\ast) \) is the euler number. In our situation, let

\[
\nu_L: \mathcal{L}^\sigma_n(X, \beta) \rightarrow \mathbb{Z},
\]

be Behrend’s constructible function.

**Definition 4.1.** We define the invariant \( L_{n,\beta}(\sigma) \in \mathbb{Z} \) by the formula,

\[
L_{n,\beta}(\sigma) = \sum_{n \in \mathbb{Z}} ne(\nu_L^{-1}(n)).
\]

Note that since an algebraic space of finite type is stratified by affine schemes of finite type, its euler number makes sense.

**Remark 4.2.** Suppose that \( \mathcal{L}^\sigma_n(X, \beta) \) is a proper algebraic space. Then by the same argument as in \[27, \text{Lemma 2.10}\] and \[15\], there is a virtual fundamental class,

\[
[\mathcal{L}^\sigma_n(X, \beta)]_{\text{vir}}^{\ast} \in A_0(\mathcal{L}^\sigma_n(X, \beta)).
\]

By the above argument, our invariant \( L_{n,\beta}(\sigma) \) coincides with the integration over the virtual class,

\[
L_{n,\beta}(\sigma) = \int_{[\mathcal{L}^\sigma_n(X, \beta)]_{\text{vir}}} 1.
\]

The purpose of this section is to relate the invariants \( L_{n,\beta}(\sigma) \) to the invariants of stable pairs on a Calabi-Yau 3-fold introduced by Pandharipande and Thomas \[27\]. Let us recall the notion of stable pairs.

**Definition 4.3.** \[27\] A stable pair consists of data \((F, s)\),

\[
s: \mathcal{O}_X \rightarrow F,
\]

where \( F \) is a pure one dimensional sheaf and \( s \) is a morphism satisfying

\[
\dim \text{Coker}(s) = 0.
\]

Given a stable pair \((F, s)\), we can associate the two term complex,

\[
I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \in D^b(X),
\]

where \( F \) is located in degree zero. As we mentioned in Remark 3.1, if \( F \) satisfies

\[
\text{ch}_2(F) = \beta, \quad \text{ch}_3(F) = n,
\]

we have \( I^\bullet \in L_n(X, \beta) \). By abuse of notation, we also call the two term complexes \[36\] as stable pairs. In \[27\], the moduli space of stable pairs \((F, s)\) satisfying the condition \[37\] is constructed as a projective variety, and denoted by \( P_n(X, \beta) \). The obstruction theory on \( P_n(X, \beta) \) is obtained from the deformation theory of the two term complexes \( I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \).

**Definition 4.4.** \[27\] A PT-invariant \( P_{n,\beta} \in \mathbb{Z} \) is defined by

\[
P_{n,\beta} = \int_{[P_n(X, \beta)]_{\text{vir}}} 1 = \sum_{n \in \mathbb{Z}} ne(\nu_F^{-1}(n)) \in \mathbb{Z}.
\]

Here \( \nu_F: P_n(X, \beta) \rightarrow \mathbb{Z} \) is Behrend’s constructible function.
4.2 Limit stable objects and stable pairs

The purpose of this paragraph is to investigate the relationship between \( L_{n,\beta}(\sigma) \) and \( P_{n,\beta} \). First we show the following lemma.

**Lemma 4.5.** An object \( E \in L_n(X, \beta) \) (cf. (20)) is isomorphic to a stable pair (36) if and only if \( \mathcal{H}^0(E) \) is zero dimensional.

**Proof.** Only if part is obvious, so we show the if part. For an object \( E \in L_n(X, \beta) \), suppose that \( \mathcal{H}^0(E) \) is zero dimensional. Applying \( \text{Hom}(\ast, \mathcal{O}_X[1]) \) for the triangle \( \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \), we obtain the exact sequence,

\[
\text{Hom}(\mathcal{H}^0(E), \mathcal{O}_X[1]) \to \text{Hom}(E, \mathcal{O}_X[1]) \to \text{Hom}(\mathcal{H}^{-1}(E), \mathcal{O}_X) \to \text{Hom}(\mathcal{H}^0(E), \mathcal{O}_X[2]).
\]

Since \( \mathcal{H}^0(E) \) is zero dimensional, the Serre duality implies

\[
\text{Hom}(\mathcal{H}^0(E), \mathcal{O}_X[j]) = H^{3-j}(X, \mathcal{H}^0(E)) = 0,
\]

for \( j = 1, 2 \). Hence we have the isomorphism

\[
\text{Hom}(E, \mathcal{O}_X[1]) \cong \text{Hom}(\mathcal{H}^{-1}(E), \mathcal{O}_X).
\]

By Lemma 3.2, we have \( \mathcal{H}^{-1}(E) = I_C \) for a one dimensional subscheme \( C \subset X \). Therefore there is a morphism \( u: E \to \mathcal{O}_X[1] \) corresponding to the inclusion \( I_C \subset \mathcal{O}_X \). Let us take the distinguished triangle,

\[
\mathcal{O}_X \to F \to E \xrightarrow{u} \mathcal{O}_X[1].
\]

It is enough to show that \( F \) is a pure one dimensional sheaf. Since \( E \in L_n(X, \beta) \), it is obvious that \( F \in \text{Coh}_{\leq 1}(X) \). For a closed point \( x \in X \), we have the exact sequence,

\[
0 = \text{Hom}(\mathcal{O}_x, \mathcal{O}_X) \to \text{Hom}(\mathcal{O}_x, F) \to \text{Hom}(\mathcal{O}_x, E).
\]

Since \( E \in \mathcal{A}_{1/2}^p \), we have \( \text{Hom}(\mathcal{O}_x, E) = 0 \), hence \( \text{Hom}(\mathcal{O}_x, F) = 0 \). Therefore \( F \) is a pure sheaf.

In the following, we focus on \( \sigma \in A(X) \mathbb{C} \) of the form,

\[
\sigma = k\omega + i\omega, \quad k \in \mathbb{R},
\]

and see how \( \mathcal{L}_n^\sigma(X, \beta) \) and \( L_{n,\beta}(\sigma) \) vary under change of \( k \in \mathbb{R} \). The advantage of setting \( \sigma \) as (38) is as follows.

**Lemma 4.6.** For \( F \in \text{Coh}_{\leq 1}(X) \) and \( E \in L_n(X, \beta) \), the condition \( \phi_\sigma(F) \preceq \phi_\sigma(E) \) (resp. \( \phi_\sigma(F) \succeq \phi_\sigma(E) \)) implies

\[
k \leq -\frac{\mu_{i\omega}(F)}{2}, \quad \text{(resp. } k \geq -\frac{\mu_{i\omega}(F)}{2} \text{)}.
\]

**Proof.** If \( \sigma = k\omega + i\omega \), we have

\[
\mu_\sigma(F) = \mu_{i\omega}(F) - k, \quad \frac{3B\omega^2}{\omega^3} = -3k,
\]

for \( B = k\omega \). Hence the inequality (39) follows from the same argument as in Lemma 3.8.
We also note that \((k\omega, \omega)\)-twisted (semi)stable sheaves coincide with \(\omega\)-Gieseker (semi)stable sheaves.

We set \(\mu_{n,\beta} \in \mathbb{Q}\) as follows,

\[
\mu_{n,\beta} = \max \left\{ \frac{n - m(\beta'')}{\omega \beta'} : 0 \neq \beta', \beta'' \in \mathcal{N}(\beta) \text{ and } \beta = \beta' + \beta'' \right\}.
\]

Since \(\mathcal{N}(\beta)\) is a finite set, we have \(\mu_{n,\beta} < \infty\). The following is the main result of this section.

**Theorem 4.7.** Let \(\sigma = k\omega + i\omega\) for \(k \in \mathbb{R}\). We have,

\[
\mathcal{L}_n^\sigma(X, \beta) = P_n(X, \beta), \quad L_{n,\beta}(\sigma) = P_{n,\beta}, \quad \text{if } k < -\mu_{n,\beta}/2,
\]

\[
\mathcal{L}_n^\sigma(X, \beta) = P_{-n}(X, \beta), \quad L_{n,\beta}(\sigma) = P_{-n,\beta}, \quad \text{if } k > \mu_{n,\beta}/2.
\]

**Proof.** First assume \(k < -\mu_{n,\beta}/2\) and take \(E \in L_n^\sigma(X, \beta)\). (cf. (21).) In order to show \(E\) is isomorphic to a stable pair \((36)\), it suffices to check that \(\mathcal{H}^0(E)\) is zero dimensional by Lemma 4.5. Suppose by a contradiction that \(\mathcal{H}^0(E)\) is one dimensional, and set

\[
\beta' = \text{ch}_2(\mathcal{H}^0(E)) \neq 0, \quad \beta'' = -\text{ch}_2(\mathcal{H}^{-1}(E)).
\]

By Lemma 3.11 we have

\[
\text{ch}_3(\mathcal{H}^0(E)) \leq n - m(\beta'').
\]

Since \(E\) is \(\sigma\)-limit stable, we must have \(\phi_\sigma(E) < \phi_\sigma(\mathcal{H}^0(E))\). Lemma 4.6 implies that

\[
k \geq -\frac{\mu_{\omega}(\mathcal{H}^0(E))}{2} = -\frac{\text{ch}_3(\mathcal{H}^0(E))}{2\omega \beta'} \geq \frac{n - m(\beta'')}{2\omega \beta'} \geq -\frac{\mu_{n,\beta}}{2}.
\]

This contradicts that \(k < -\mu_{n,\beta}/2\), hence \(E\) is isomorphic to a stable pair.

Conversely take a stable pair \(E \cong (\mathcal{O}_X \xrightarrow{s} F) \in P_n(X, \beta)\). We have to check the conditions \((a), (b)\) in Lemma 3.4. Let \(E \twoheadrightarrow G\) be a strict epimorphism as in \((a)\) in Lemma 3.4. By Remark 3.6 \(G\) is obtained as a surjection of sheaves \(\mathcal{H}^0(E) \twoheadrightarrow G\). Since \(\mathcal{H}^0(E)\) is zero dimensional, \(G\) is also a zero dimensional sheaf, hence \(G \notin \mathcal{A}_1^0\) provided \(G \neq 0\). This means that \((a)\) does not occur, so it is enough to check \((b)\) in Lemma 3.4.

Let \(F' \hookrightarrow E\) be a strict monomorphism as in \((b)\) in Lemma 3.4 and set \(\beta' = \text{ch}_2(F')\). Let \(C \subset X\) be a one dimensional subscheme with \(\mathcal{H}^{-1}(E) = I_C\). We can take subsheaves \(F_1 \subset \mathcal{O}_C, F_2 \subset \mathcal{H}^0(E)\) as in Lemma 3.7 such that \(F'\) is written as an extension

\[
0 \longrightarrow F_1 \longrightarrow F' \longrightarrow F_2 \longrightarrow 0.
\]

Since \(\mathcal{O}_C/F_1 \cong \mathcal{O}_{C'}\) for a subscheme \(C' \subset C\), we have

\[
\text{ch}_3(F_1) = \text{ch}_3(\mathcal{O}_C) - \text{ch}_3(\mathcal{O}_{C'}) \\
\leq \text{ch}_3(\mathcal{O}_C) - m(\beta''),
\]

where \(\beta'' = \text{ch}_2(\mathcal{O}_{C'})\). Hence we have

\[
\text{ch}_3(F') = \text{ch}_3(F_1) + \text{ch}_3(F_2) \\
\leq \text{ch}_3(\mathcal{O}_C) + \text{ch}_3(\mathcal{H}^0(E)) - m(\beta'') \\
= n - m(\beta'').
\]
We obtain the inequality,
\[ k < -\frac{\mu_{n,\beta}}{2} \leq -\frac{n-m(\beta')}{2\beta \omega} \leq -\frac{\mu_i \omega(F')}{2}, \]
which implies \( \phi_{\sigma}(F') < \phi_{\sigma}(E) \) by Lemma 4.6. Therefore (b) in Lemma 3.4 holds, hence \( E \in L_n^{\sigma}(X, \beta) \).

By the above arguments, the set of \( \mathbb{C} \)-valued points of \( L_n^{\sigma}(X, \beta) \) and \( P_n(X, \beta) \) are identified. Therefore we have the isomorphism of the moduli spaces,
\[ L_n^{\sigma}(X, \beta) \cong P_n(X, \beta), \]
since both are open algebraic subspaces of \( \mathcal{M}^{et}_0 \). In particular \( L_{n,\beta}(\sigma) = P_{n,\beta} \) follows. When \( k > \mu - n,\beta / 2 \), we have
\[ L_{n,\beta}(\sigma) = L_{-n,\beta}(\sigma^\vee) = P_{-n,\beta}, \]
by applying the dualizing functor \( D \) and using Lemma 2.28.

\[ \text{Remark 4.8.} \] The wall-crossing phenomena for stable pairs is also studied in Bayer’s polynomial stability conditions [3, Paragraph 6.2]. However our wall-crossing is crucially different from Bayer’s wall-crossing. In fact the complexified ample cone \( A(X) \), the stability parameter in our stability conditions, behaves itself as a wall in the wall-crossing of Bayer [3, Paragraph 6.2].

### 4.3 Wall-crossing phenomena of limit stable objects

Let \( \sigma = k\omega + i\omega \) be as in the previous paragraph. In this paragraph, we investigate how \( \sigma \)-limit stable objects vary under change of \( k \in \mathbb{R} \). As we have seen in Theorem 4.7, \( \sigma \)-limit stable objects coincide with stable pairs for \( k \ll 0 \), and the dual of stable pairs for \( k \gg 0 \). We look at the wall-crossing phenomena more closely, which hopefully might be helpful for the rationality conjecture of the generating functions of PT-invariants, proposed in [27].

For an effective class \( \beta \in H^4(X, \mathbb{Z}) \), we set \( S(\beta) \subset \mathbb{R} \) as
\[ S(\beta) := \left\{ \frac{m}{2\omega \gamma} : 0 \neq \gamma \in \mathcal{N}(\beta), m \in \mathbb{Z} \right\} \subset \mathbb{R}, \]
where \( \mathcal{N}(\beta) \) is introduced in (31). Note that \( S(\beta) \) is a discrete subset in \( \mathbb{R} \) because \( \mathcal{N}(\beta) \) is a finite set. In the following, we see that \( S(\beta) \) behaves as the set of walls.

\[ \text{Proposition 4.9.} \] Let \( \mathcal{C} \subset \mathbb{R} \setminus S(\beta) \) be one of the connected components. For \( k, k' \in \mathcal{C} \), we have
\[ \mathcal{L}_n^\sigma(X, \beta) = \mathcal{L}_n^{\sigma'}(X, \beta), \]
where \( \sigma = k\omega + i\omega, \sigma' = k'\omega + i\omega \). In particular the function,
\[ \mathbb{R} \ni k \mapsto L_{n,\beta}(k\omega + i\omega) \in \mathbb{Z}, \]
is constant on \( \mathcal{C} \).

\[ \text{Proof.} \] For \( E \in L_n^{\sigma}(X, \beta) \), assume that \( E \notin L_n^{\sigma'}(X, \beta) \). Then at least one of the conditions (a) or (b) in Lemma 3.2 does not hold. Suppose that (a) does not hold and let \( E \twoheadrightarrow G \) be a strict epimorphism with \( G \in \text{Coh}_{\leq 1}(X) \), which destablizes \( E \). Then Lemma 3.8 and Lemma 4.6 show,
\[ k' \leq -\frac{\mu_i \omega(G)}{2}. \]
Since \( \text{ch}_2(G) \in \mathcal{N}(\beta) \) by Remark 3.6, the right hand side of (11) is an element of \( \mathcal{S}(\beta) \). Therefore \( k \) satisfies the inequality \( k < -\mu_i \omega(G)/2 \), which implies \( \phi_\sigma(E) \succ \phi_\sigma(G) \). This contradicts that \( E \in L_n^\sigma(X, \beta) \).

The case that the condition (b) in Lemma 3.2 does not hold is similarly discussed, noting Lemma 3.7 which shows that \( \text{ch}_2(F) \in \mathcal{N}(\beta) \) for destabilizing monomorphism \( F \hookrightarrow E \). 

Next we investigate the wall-crossing phenomena at some point \( -\mu/2 \in \mathcal{S}(\beta) \). Let \( C_- \), \( C_+ \) be connected components in \( \mathbb{R} \setminus \mathcal{S}(\beta) \) such that

\[
C_- \subset \mathbb{R}_{< -\mu/2}, \quad C_+ \subset \mathbb{R}_{> -\mu/2}, \quad \overline{C}_- \cap \overline{C}_+ = \left\{ \frac{-\mu}{2} \right\}.
\]

Let us take \( k_- \in C_- \), \( k_+ \in C_+ \) and \( k_0 = -\mu/2 \). We set \( \sigma_+ = k_+ \omega + i \omega \) for \( * = \pm, 0 \). We have the following proposition.

**Proposition 4.10.** (i) Assume that \( E \in L_n^\sigma(X, \beta) \) is not \( \sigma_\pm \)-limit stable. Then there is an exact sequence in \( \mathcal{A}_{1/2}^\sigma \),

\[
0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,
\]

such that \( F \) is \( \omega \)-Gieseker semistable sheaf with \( \mu_i \omega = \mu \) and \( G \) is \( \sigma_\pm \)-limit stable with \( \phi_{\sigma_\pm}(F) \succ \phi_{\sigma_\pm}(G) \). i.e. (41) is a Harder-Narasimhan filtration in \( \sigma_+ \). The object \( G \) is also \( \sigma_0 \)-limit stable.

(ii) Assume that \( E \in L_n^\sigma(X, \beta) \) is not \( \sigma_- \)-limit stable. Then there is an exact sequence in \( \mathcal{A}_{1/2}^\sigma \),

\[
0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0,
\]

such that \( F \) is \( \omega \)-Gieseker semistable sheaf with \( \mu_i \omega = \mu \) and \( G \) is \( \sigma_- \)-limit stable with \( \phi_{\sigma_-}(G) \succ \phi_{\sigma_-}(F) \). i.e. (42) is a Harder-Narasimhan filtration in \( \sigma_- \). The object \( G \) is also \( \sigma_0 \)-limit stable.

**Proof.** The proof of (ii) is identical to (i), so we only show (i). Assume that \( E \in L_n^\sigma(X, \beta) \) is not \( \sigma_\pm \)-limit stable. By Lemma 3.4 we have one of the two possibilities.

(a') There is a non-zero pure one dimensional sheaf \( G \) which admits a strict epimorphism \( E \twoheadrightarrow G \) in \( \mathcal{A}_{1/2}^\sigma \) with \( \phi_{\sigma_+}(E) \succ \phi_{\sigma_+}(G) \).

(b') There is a non-zero pure one dimensional sheaf \( F \) which admits a strict monomorphism \( F \hookrightarrow E \) in \( \mathcal{A}_{1/2}^\sigma \) with \( \phi_{\sigma_+}(F) \succ \phi_{\sigma_+}(E) \).

Suppose that (a') occurs. By Lemma 4.6 we see that

\[
k_- < k_+ < -\frac{\mu_i \omega(G)}{2}.
\]

Therefore we also have \( \phi_{\sigma_-}(E) \succ \phi_{\sigma_-}(G) \), which contradicts that \( E \in L_n^\sigma(X, \beta) \). Hence (b') occurs. Let \( F \hookrightarrow E \) be as in (b'). Since \( \mathcal{A}_{1/2}^\sigma \) is of finite length, one can take such \( F \) to be maximal, i.e. there is no non-trivial strict monomorphism \( F \hookrightarrow \tilde{F} \hookrightarrow E \) such that \( \tilde{F} \) is also pure one dimensional sheaf with \( \phi_{\sigma_+}(\tilde{F}) \geq \phi_{\sigma_+}(E) \). Since \( \phi_{\sigma_+}(F) \geq \phi_{\sigma_+}(E) \) and \( \phi_{\sigma_-}(F) \prec \phi_{\sigma_-}(E) \), we have

\[
k_- < -\frac{\mu_i \omega(F)}{2} \leq k_+.
\]

Since \( \text{ch}_2(F) \in \mathcal{N}(\beta) \), we have \( -\mu_i \omega(F)/2 \in \mathcal{S}(\beta) \). Therefore we have

\[
\mu_i \omega(F) = \mu, \quad k_+ > -\frac{\mu_i \omega(F)}{2},
\]

30
which implies \( \phi_{\sigma_+}(F) \succeq \phi_{\sigma_+}(E) \). In order to show \( F \) is a \( \omega \)-Gieseker semistable sheaf, it is enough to show that \( F \) is \( \sigma_+ \)-limit semistable. Let \( F' \subset F \) be a strict monomorphism in \( A^0_{1/2} \).

Note that \( F' \) is also pure one dimensional sheaf. If \( \phi_{\sigma_+}(F') \succeq \phi_{\sigma_+}(F) \), then \( \phi_{\sigma_+}(F') \succeq \phi_{\sigma_+}(E) \), hence \( \mu_{\omega}(F') = \mu \) by the same argument as above. It follows that \( F \) is \( \sigma_+ \)-limit semistable.

Let us take the exact sequence in \( A^0_{1/2} \),

\[
0 \to F \to E \to G \to 0.
\]

We want to show that \( G \) is \( \sigma_+ \)-limit stable for \( * = +, 0 \). We show the case of \( * = 0 \), as the proof for the other case is similar. Suppose the contrary. By the same argument as above, there is a strict monomorphism \( F'' \to G \) in \( A^0_{1/2} \) such that \( \phi_{\sigma_0}(F'') \succeq \phi_{\sigma_0}(G) \). We have

\[
k_+ > k_0 \geq -\frac{\mu_{\omega}(F'')}{2},
\]

by Lemma 4.6 which in turn implies \( \phi_{\sigma_+}(F'') \succeq \phi_{\sigma_+}(E) \). Let \( F''' \) be the kernel of the composition of the strict epimorphisms

\[
E \to G \to G/F''.
\]

Then we have the non-trivial strict monomorphism, \( F \mapsto F'' \mapsto E \) with \( \phi_{\sigma_+}(F'') \succeq \phi_{\sigma_+}(E) \), which is a contradiction since \( F \mapsto E \) is maximal.

Let \( Z^\text{PT}_\beta \) be the generating series,

\[
Z^\text{PT}_\beta(q) = \sum_{n \in \mathbb{Z}} P_{n, \beta} q^n \in \mathbb{Q}(q).
\]

In [27, Conjecture 3.2], Pandharipande and Thomas conjecture that the generating series (43) is a rational function of \( q \), invariant under \( q \mapsto 1/q \). This conjecture (rationality conjecture) is solved when \( \beta \) is an irreducible curve class case in [29], and the crucial point is to find a relationship between \( P_{n, \beta} \) and \( P_{-n, \beta} \). By Theorem 4.7, it is possible to obtain such a relationship in a general situation by establishing a wall-crossing formula of our invariants \( L_{n, \beta} \). Suppose for instance that any \( F \in \text{Coh}_{\leq 1}(X) \) which appears in the sequence \( (41), (42) \) is in fact \( \omega \)-stable, and satisfies

\[
(ch_2(F), ch_3(F)) = (\beta', n').
\]

Let \( L^- \subset L^\sigma_-(X, \beta) \) be the unstable locus in \( \sigma_+ \)-limit stability, \( L^+ \subset L^\sigma_+(X, \beta) \) the similar locus, and \( M_{n'}(X, \beta') \) the moduli space of \( \omega \)-Gieseker stable sheaf \( F \in \text{Coh}_{\leq 1}(X) \) satisfying (44). The destabilizing sequences (41), (42) yield the following diagram,

\[
\begin{array}{ccc}
L^- & \xrightarrow{\pi_-} & M_{n'}(X, \beta') \\
\downarrow & & \downarrow \pi_+ \\
& & L^+ \\
\end{array}
\]

Here \( \beta' + \beta'' = \beta \) and \( n' + n'' = n \). From the above diagram, one might expect the formula something like

\[
L_{n, \beta}(\sigma_-) - L_{n, \beta}(\sigma_+) = (\sharp \text{Ext}^1(G, F) - \sharp \text{Ext}^1(F, G)) N_{n', \beta'} L_{n', \beta''}(\sigma_0),
\]

where \( N_{n', \beta'} \) is the virtual counting,

\[
N_{n', \beta'} = \int_{[M_{n'}(X, \beta')]} 1 \in \mathbb{Z}.
\]
We expect that the contribution of the term $(\sharp Ext^1(G,F) - \sharp Ext^1(F,G))$ is given by

$$( -1)^{n'-1} \chi(F,G) = (-1)^{n'-1} n',$$

doing the right hand side of (45) should be $( -1)^{n'-1} n' N_{n',\beta'} L_{n',\beta'}(\sigma_0)$.

In case a destabilizing sheaf $F \in \text{Coh}_{\leq 1}(X)$ which appears in (41) or (42) is strictly semistable, the construction of the invariant $N_{n',\beta}$ is problematic. It seems that Joyce’s motivic invariants of moduli stacks [18] Definition 3.18 are relevant for this problem, although Joyce’s invariants are not deformation invariant as they do not involve virtual classes. Hopefully it is possible to involve virtual classes (probably using Behrend’s constructible function [4]) and the following wall-crossing formula should hold.

**Conjecture 4.11.** There is a virtual counting of $\omega$-Gieseker semistable sheaves $F \in \text{Coh}_{\leq 1}(X)$ with $(\text{ch}_2(F), \text{ch}_3(F)) = (\beta', n')$, denoted by $N_{n',\beta'} \in \mathbb{Q}$, such that

$$L_{n,\beta}(\sigma_-) - L_{n,\beta}(\sigma_+) = \sum (-1)^{n'-1} n' N_{n',\beta'} L_{n',\beta'}(\sigma_0).$$

(46)

Here in the above sum, $(\beta', n'), (\beta'', n'')$ must satisfy $\beta' + \beta'' = \beta$, $n' + n'' = n$ and $n'/\omega \beta' = \mu$.

Note that a term in the right hand side of (46) is non-zero only if $\beta', \beta'' \in \mathcal{N}(\beta)$, so there are only finite number of non-zero terms. Also for a non-zero term in (46), the class $\beta''$ is smaller than $\beta$, i.e. $0 \leq \beta'' \cdot H < \beta \cdot H$ for an ample divisor $H$. Hence if Conjecture 4.11 is true, we can describe how the invariants $L_{n,\beta}(\sigma)$ vary under change of $\sigma$ inductively on $\beta$, and eventually provides a relationship between $P_{n,\beta}$ and $P_{-n,\beta}$. We expect this relationship will show the rationality conjecture of the generating series (43).

In the next paper [35], we will proceed this idea further using D. Joyce’s work [18] on the wall-crossing formula of counting invariants of semistable objects in abelian categories. It will turn out in [35] that the similar rationality property holds for the generating functions of euler numbers of the moduli spaces of stable pairs, using the results in this paper. We remark that Joyce’s work is applied for the invariants without virtual fundamental cycles. However by the recent progress in this field [22], [19], [20], we guess that the similar wall-crossing formula should hold after involving virtual classes. At this moment, the works [22], [20] are not enough to conclude the rationality conjecture. (The work [22] assumes [22] Conjecture 4] to show the main result [22], Theorem 8], and the result of [20] is only applied for counting invariants of coherent sheaves, not for those of objects in the derived category.) Finally we mention that T. Bridgeland [10] proved the rationality conjecture assuming the main result of Kontsevich-Soibelman [22], Theorem 8], without using any notion of stability conditions.

### 5 Examples

In this section, we see the wall-crossing phenomena of limit stable objects in several examples.

#### 5.1 $\beta$ is an irreducible curve class

Suppose that $\beta \in H^4(X,\mathbb{Z})$ is an irreducible class, i.e. $\beta = [C]$ for an irreducible and reduced curve $C \subset X$. For $\sigma = k\omega + i\omega$, we have

$$L_n^\sigma(X,\beta) = \begin{cases}
P_n(X,\beta) & \text{if } k < -\frac{n}{2\omega \beta}, \\
P_{-n}(X,\beta) & \text{if } k > -\frac{n}{2\omega \beta}
\end{cases}.$$
Note that \( \mu_{n, \beta} = n/\omega \beta \) in this case, so Theorem 4.7 yields the above result. The formula 4.6 becomes

\[
P_{n, \beta} - P_{-n, \beta} = (-1)^{n-1} n N_{n, \beta},
\]

which is proved in [29, Proposition 2.2]. Hence Conjecture 4.11 is true in this case.

5.2 \( \beta \) is a reducible curve class

Suppose that there are smooth rational curves \( C_1, C_2 \) on \( X \) such that

\[
N_{C_i/X} \cong O_{C_i}(-1)^{\oplus 2}, \quad \beta = [C_1] + [C_2], \quad C_1 \cap C_2 = \{p\},
\]

where \( C_1 \cap C_2 \) is the scheme theoretic intersection. Let \( C = C_1 \cup C_2, d_i = \omega \cdot C_i \) and assume that \( d_1 > d_2 \geq 0 \). This is possible if \( C_1 \) and \( C_2 \) determine linearly independent homology classes in \( H^2(X, \mathbb{R}) \). As for \( L^q(X, \beta) \), we have the following,

\[
L_1^q(X, \beta) = \begin{cases} 
P_1(X, \beta) \cong \text{Spec } \mathbb{C} & \text{if } k < -\frac{1}{2(d_1 + d_2)}, \\
P_{-1}(X, \beta) = \emptyset & \text{if } k > -\frac{1}{2(d_1 + d_2)}.
\end{cases}
\]

In fact \( \mu_{1, \beta} = 1/(d_1 + d_2) \) and \( \mu_{-1, \beta} = -1/(d_1 + d_2) \) in this case, so we can apply Theorem 4.7. The set of stable pairs \( E \in \mathcal{A}^p \) with \( (\text{ch}_2(E), \text{ch}_3(E)) = (1, \beta) \) consists of one element \( \{I_C[1]\} \), and we can easily compute

\[
\text{Ext}^1_X(I_C[1], I_C[1]) = 0.
\]

Hence scheme theoretically \( P_1(X, \beta) \) is isomorphic to \( \text{Spec } \mathbb{C} \). If \( k > -1/2(d_1 + d_2) \), then the exact sequence in \( \mathcal{A}^p \),

\[
0 \longrightarrow O_C \longrightarrow I_C[1] \longrightarrow O_X[1] \longrightarrow 0,
\]

destabilizes \( I_C[1] \). According to Proposition 4.10, we might obtain stable objects as an extension,

\[
0 \longrightarrow O_X[1] \longrightarrow E \longrightarrow O_C \longrightarrow 0.
\]

However since \( \text{Ext}^1_X(O_C, O_X[1]) = H^1(O_C) = 0 \), the above sequence splits, so \( E \) is not \( \sigma \)-limit stable. In fact we can check that \( P_{-1}(X, \beta) \) is empty in this case. The counting invariants are as follows,

\[
L_{1, \beta}(\sigma) = \begin{cases} 
1 & \text{if } k < -\frac{1}{2(d_1 + d_2)}, \\
0 & \text{if } k > -\frac{1}{2(d_1 + d_2)}.
\end{cases}
\]

The formula 4.6 is easily checked to hold in this case.

Next let us investigate \( L^q_2(X, \beta) \). The result is as follows.

\[
L_2^q(X, \beta) = \begin{cases} 
P_2(X, \beta) \cong C & \text{if } k < -\frac{1}{d_2}, \\
P_{-2}(X, \beta) = \emptyset & \text{if } -\frac{1}{2d_2} < k < -\frac{1}{d_2}, \\
P_{-2}(X, \beta) = \emptyset & \text{if } k > -\frac{1}{d_1 + d_2}.
\end{cases}
\]

In this case, we have \( \mu_{2, \beta} = 1/d_2 \) and \( \mu_{-2, \beta} = -2/(d_1 + d_2) \). Also giving a point of \( P_2(X, \beta) \) is equivalent to choosing a closed point of \( C \). By Theorem 4.7 together with some more arguments, we see

\[
L_2^q(X, \beta) = P_2(X, \beta) \cong C,
\]

for \( k < -1/2d_2 \).
Suppose that 

\[- \frac{1}{2d_2} < k < - \frac{1}{d_1 + d_2}\].

Any stable pair \(E \in P_2(X, \beta)\) admits an exact sequence in \(\mathcal{A}^p\),

\[
0 \rightarrow IC[1] \rightarrow E \rightarrow O_x \rightarrow 0,
\]

for \(x \in C\). If \(x \in C_2\), then we have the exact sequence in \(\mathcal{A}^p\),

\[
0 \rightarrow OC_2 \rightarrow E \rightarrow IC_1[1] \rightarrow 0,
\]

which destabilizes \(E\). In fact \(E \in \mathcal{A}^p\) given in (47) is \(\sigma\)-limit stable if and only if \(x \not\in C_2\). Hence \(C_1 \setminus \{p\} = C\) is embedded into \(L^0_2(X, \beta)\). It is compactified by adding a point corresponding to the (unique) extension,

\[
0 \rightarrow IC_1[1] \rightarrow E' \rightarrow OC_2 \rightarrow 0.
\]

One can check that \(L^\sigma_n(X, \beta) = (C_1 \setminus \{p\}) \cup \{E'\}\).

These objects are also obtained as two term complexes \(O_X \rightarrow F\), where \(F \in \text{Coh}_{\leq 1}(X)\) is a unique non-trivial extension,

\[
0 \rightarrow OC_1 \rightarrow F \rightarrow OC_2 \rightarrow 0.
\]

From these observations, we see that \(L^\sigma_2(X, \beta) = \mathbb{P}(H^0(F)) \cong \mathbb{P}^1\).

Finally suppose that \(k > -1/(d_1 + d_2)\). Then for a two term complex \(E = (O_X \rightarrow F)\) where \(F \in \text{Coh}_{\leq 1}(X)\) is a unique non-trivial extension,

\[
0 \rightarrow F \rightarrow E \rightarrow O_X[1] \rightarrow 0,
\]

destabilizes \(E\). Also there are no non-trivial extensions,

\[
0 \rightarrow OC_1[1] \rightarrow E' \rightarrow F \rightarrow 0,
\]

because \(\text{Hom}(F, O_X[2]) = H^1(F) = 0\). In this case, one can check that \(L^\sigma_2(X, \beta) = P_{-2}(X, \beta) = \emptyset\). For the counting invariants, we obtain

\[
L_{2, \beta}(\sigma) = \begin{cases} 
-1 & \text{if } k < -\frac{1}{2d_2}, \\
-2 & \text{if } -\frac{1}{2d_2} < k < -\frac{1}{d_1 + d_2}, \\
0 & \text{if } k > -\frac{1}{d_1 + d_2}.
\end{cases}
\]

The formula (46) also holds in this case. For instance, take

\[
k_- < -\frac{1}{2d_2}, \quad k_0 = -\frac{1}{2d_2}, \quad -\frac{1}{2d_2} < k_+ < -\frac{1}{d_1 + d_2}.
\]

A term in the sum of (46) is non-zero only if \(\beta' = [C_2], n' = 1, \beta'' = [C_1] \text{ and } n'' = 1\). We have \(N_{n', \beta'} = 1\), and \(L_{n', \beta''}(\sigma_0) = 1\). We can check (46) as follows,

\[
L_{2, \beta}(\sigma_-) - L_{2, \beta}(\sigma_+) = -1 - (-2) = 1,
\]

\[
(-1)^{1-n'}N_{1, [C_2]}L_{1, [C_1]}(\sigma_0) = (-1)^{1-1}1 \cdot 1 = 1.
\]
5.3 $\beta$ is a multiple curve class

Let $C \subset X$ be a smooth rational curve with

$$N_{C/X} \cong \mathcal{O}_C(-1)^{\otimes 2}, \quad \beta = 2[C].$$

Let $\sigma = k\omega + i\omega$ and set $d = \omega \cdot C$. For $\mathcal{L}_3^s(X, \beta)$, we have the following,

$$\mathcal{L}_3^s(X, \beta) = \begin{cases} P_3(X, \beta) \cong \mathbb{P}^1 & \text{if } k < -\frac{1}{d}, \\ P_{-3}(X, \beta) = \emptyset & \text{if } k > -\frac{1}{d}. \end{cases}$$

In this case, we have $\mu_{3,\beta} = 2/d$ and $\mu_{-3,\beta} = -2/d$ so Theorem [4.7] is applied. We have $P_3(X, \beta) \cong \mathbb{P}^1$ for $k < -1/d$ by [27, Section 4]. If $k > -1/d$, we have the exact sequence in $\mathcal{A}$ for $E \in P_3(X, \beta)$,

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow E \longrightarrow I_C[1] \longrightarrow 0,$$

which destabilizes $E$. Since Hom$(\mathcal{O}_C(1), I_C[2]) = 0$, there is no non-trivial extension,

$$0 \longrightarrow I_C[1] \longrightarrow E' \longrightarrow \mathcal{O}_C(1) \longrightarrow 0,$$

and in fact $\mathcal{L}_3^s(X, \beta) = P_{-3}(X, \beta) = \emptyset$ in this case. The counting invariants are

$$L_{3,\beta}(\sigma) = \begin{cases} -2 & \text{if } k < -\frac{1}{d}, \\ 0 & \text{if } k > -\frac{1}{d}. \end{cases}$$

The formula [4.6] also holds in this case.

In the same way, $\mathcal{L}_4^s(X, \beta)$ is as follows,

$$\mathcal{L}_4^s(X, \beta) = \begin{cases} P_4(X, \beta) & \text{if } k < -\frac{3}{2d}, \\ \text{Spec } \mathbb{C} & \text{if } -\frac{3}{2d} < k < -\frac{1}{d}, \\ P_{-4}(X, \beta) = \emptyset & \text{if } k > -\frac{1}{d}. \end{cases}$$

Note that $\mu_{4,\beta} = 3/d$ and $\mu_{-4,\beta} = -2/d$ in this case. If $-3/2d < k < -1/d$, the sequence

$$0 \longrightarrow \mathcal{O}_C(2) \longrightarrow E \longrightarrow I_C[1] \longrightarrow 0,$$

destabilizes $E \in P_4(X, \beta)$. Instead the unique non-trivial extension

$$0 \longrightarrow I_C[1] \longrightarrow E' \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,$$

becomes $\sigma$-limit stable. The object $E'$ is isomorphic to a two term complex $\mathcal{O}_X \xrightarrow{\delta} \mathcal{O}_C(1)^{\otimes 2}$, and the sequence

$$0 \longrightarrow \mathcal{O}_C(1)^{\otimes 2} \longrightarrow E' \longrightarrow \mathcal{O}_X[1] \longrightarrow 0,$$

destabilizes $E'$ if $k > -1/d$. We have Hom$(\mathcal{O}_C(1)^{\otimes 2}, \mathcal{O}_X[2]) = 0$, and in fact $\mathcal{L}_4^s(X, \beta) = P_{-4}(X, \beta) = \emptyset$ for $k > -1/d$. The counting invariants are as follows,

$$L_{4,\beta}(\sigma) = \begin{cases} 4 & \text{if } k < -\frac{3}{2d}, \\ 1 & \text{if } -\frac{3}{2d} < k < -\frac{1}{d}, \\ 0 & \text{if } k > -\frac{1}{d}. \end{cases}$$

For $P_{4,\beta} = 4$, see [27, Section 4]. If $k_- < -3/2d$, $k_0 = -3/2d$ and $-3/2d < k_+ < -1/d$, one can check that [4.6] holds. On the other hand, the formula [4.6] is problematic if

$$-\frac{3}{2d} < k_- < -\frac{1}{d}, \quad k_0 = -\frac{1}{d}, \quad k_+ > -\frac{1}{d},$$

35
Since $O_C(1)^{\otimes 2}$ is not $\omega$-Gieseker stable, we do not how to define $N_{4,2[C]}$. In this case, $N_{4,2[C]}$ should be defined by Joyce’s invariant [18, Definition 3.18] after involving virtual classes. For instance let us ignore virtual classes. By definition, Joyce’s invariant $N_{4,2[C]}$ is the “euler number” of the following “virtual” stack,

$$[\text{Spec } \mathbb{C}/\text{GL}(2, \mathbb{C})] - \frac{1}{2} [\text{Spec } \mathbb{C}/(\mathbb{A}^1 \times \mathbb{G}_m^2)],$$

which results $N_{4,2[C]} = -1/4$. We have

$$L_{4,\beta}(\sigma_-) - L_{4,\beta}(\sigma_+) = 1 - 0 = 1,$$

$$(-1)^{1-1}4N_{4,2[C]}L_{0,0}(\sigma_0) = -1 \cdot 4 \cdot (-1/4) \cdot 1 = 1,$$

as desired.

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