The Kobayashi-Hitchin correspondence of generalized holomorphic vector bundles over generalized Kähler manifolds of symplectic type

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Abstract

In the previous paper [16], the notion of an Einstein-Hermitian metric of a generalized holomorphic vector bundle over a generalized Kähler manifold of symplectic type was introduced from the moment map framework. In this paper we establish the Kobayashi-Hitchin correspondence, that is, the equivalence of the existence of an Einstein-Hermitian metric and $\psi$-polystability of a generalized holomorphic vector bundle over a compact generalized Kähler manifold of symplectic type. Poisson modules provide intriguing generalized holomorphic vector bundles and we obtain $\psi$-stable Poisson modules over complex surfaces which are not stable in the ordinary sense.

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1 Introduction

S. Kobayashi [27] and Lübke [29] showed that an Einstein-Hermitian vector bundle over a smooth compact Kähler manifold is polystable. Conversely, Donaldson [5], [6], [7] and Uhlenbeck-Yau [38], [39] proved that a polystable holomorphic vector bundle has an Einstein-Hermitian metric. The equivalence of the existence of an Einstein-Hermitian metric and polystability of a holomorphic vector bundle is called the Kobayashi-Hitchin correspondence (See also [24] for more details).

Generalized complex geometry and generalized Kähler geometry are introduced by Hitchin [19] and Gualtieri which are successful generalizations of the ordinary complex geometry and Kähler geometry. Holomorphic Poisson geometry is closely related to generalized Kähler geometry. In fact, it turns out that holomorphic Poisson structures on a Kähler manifold produce remarkable generalized Kähler structures of symplectic type [21], [23], [11], [12], [13], [4], [17]. Poisson modules also provide typical and prominent generalized holomorphic vector bundles [22], [23], [17].

Then an inevitable and important problem is to extend the Kobayashi-Hitchin correspondence to the cases of generalized holomorphic vector bundles over a generalized Kähler manifold. In the paper, from a view point of moment map framework, we focus on the class of generalized Kähler manifolds of symplectic...
A generalized Kähler manifold \((M, J_1, J_2)\) is of symplectic type if one of generalized complex structures \(J_2\) is induced from a symplectic form \(\omega\) and a real \(d\)-closed 2-form \(b\) on \(M\), which is denoted by \(J_\psi\), where \(\psi = e^{b-\sqrt{-1}\omega}\), is a \(d\)-closed, nondegenerate, pure spinor of type 0. Let \(E\) be a generalized holomorphic vector bundle and \(h\) an Hermitian metric over a generalized Kähler manifold of symplectic type \((M, J, J_\psi)\). The Einstein-Hermitian condition in [16] is defined by the following equation:

\[
K_h(\psi) = \lambda \psi \text{id}_E,
\]

where \(K_h(\psi)\) denotes the mean curvature of the canonical generalized connection (see Definition 3.3 and Definition 4.1). The degree of \(E\) is defined in terms of the first Chern class of a vector bundle \(E\) together with the class \([\psi]\) and then the slope is also given by \(\text{deg} E/\text{rank} E\). By using the slope inequality, we have the notions of \(\psi\)-stability and \(\psi\)-polystability of a generalized holomorphic vector bundle. Then the following is our main theorem which is proved in Subsection 9.8.

**Theorem 1.1.** [Kobayashi-Hitchin correspondence] There exists an Einstein-Hermitian metric on a \(\psi\)-polystable generalized holomorphic vector bundle. Conversely, a generalized holomorphic vector bundle admitting an Einstein-Hermitian metric is \(\psi\)-polystable.

It is known that a generalized Kähler structure \((J_1, J_2)\) gives rise to a bihermitian structure \((I_+, I_-, g, b)\), where \(I_+\) are two ordinary complex structures and a single \(g\) is an Hermitian metric with respect to both \(I_+\) and \(I_-\), and \(b\) is a real 2-form. Then a generalized holomorphic vector bundle provides a holomorphic vector bundle with respect to both \(I_+\) and \(I_-\). From the view point of bihermitian geometry, Hitchin [22] expected in general the following equation describes a stability condition of a generalized holomorphic vector bundle:

\[
\frac{1}{2} (F_+ \wedge \omega_+^{n-1} + F_- \wedge \omega_-^{n-1}) = \lambda \text{id}_E \text{vol}_M
\]

In [33], Hu, Moraru and Reza showed that the equation (1.2) is related with \(\alpha\)-stability which is introduced by using Hermitian forms \(\omega_\pm\) of the bihermitian structure if both \(\omega_\pm\) are Gauduchon metrics. However, a generalized Kähler manifold \((M, J_1, J_2)\) does not give Gauduchon metrics \(\omega_\pm\) if the dimension of \(M\) is greater than 4. The \(\psi\)-stability in this paper is topologically defined as in the ordinary Kähler cases, which is different from the \(\alpha\)-stability. Our Einstein-Hermitian equation (1.1) is introduced from the moment map framework, which is also different from the equation (1.2) in general

This paper is organized as follows. In Section 2 we shall give a brief review of generalized complex structures and generalized Kähler structures focusing on nondegenerate, pure spinors. In Subsection 2.3 we recall the stability theorem of generalized Kähler structures which is applied to construct nontrivial examples of generalized Kähler manifolds from holomorphic Poisson structures. In Section 3 and 4 we recall definitions of Einstein-Hermitian metrics and generalized holomorphic vector bundles and the canonical generalized connections.

In the cases of ordinary holomorphic vector bundles over a Kähler manifold, a weak holomorphic subbundle plays a crucial role to construct an Einstein-Hermitian metric. In Section 4.2 we also introduce the notion of a weak generalized holomorphic subbundle. Then the notions of \(\psi\)-stability and \(\psi\)-polystability are introduced in Section 5. In Section 6 we obtain the formula of the second fundamental form which

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\(^{1}\) Our results can be applied to more general cases that \(\psi\) is a \(d\)-closed, nondegenerate, pure spinor. In fact, the existence result as in Section 9 does hold. We also extend our result to the cases \(H\)-twisted generalized Kähler structures.

\(^{2}\) We also remark that in the cases of co-Higgs bundles over an ordinary Kähler manifold, our equation (1.1) coincides with the one as (1.2).
measures the difference between the first Chern form of $E$ and the one of a subbundle of $E$. The formula is used to show $\psi$-polystability of a generalized holomorphic vector bundle admitting an Einstein-Hermitian metric in Section 7. In Section 8 we give variation formulas of the curvature and the mean curvature of canonical generalized Hermitian connections under deformations of Hermitian metrics on a generalized holomorphic vector bundle. In Section 9 we construct an Einstein-Hermitian metric on a stable generalized holomorphic vector bundle. We use the continuity method. In Section 10 we construct Poisson modules by using the Serre construction over complex surfaces and discuss the $\psi$-stability of them. On $\mathbb{C}P^2$, by using a certain configuration of points on a line, we obtain a $\psi$-stable Poisson module which is not stable in the ordinary sense. Thus such a Poisson module does have an Einstein-Hermitian metric as a generalized holomorphic vector bundle, however which does not have any ordinary Einstein-Hermitian metric. In Section 11 we obtain vanishing theorems of a generalized holomorphic vector bundle with an Einstein-Hermitian metric over a generalized Kähler manifold of symplectic type, which give another proof of $\psi$-polystability of an Einstein-Hermitian generalized holomorphic vector bundle.

2 Generalized complex structures and generalized Kähler structures

2.1 Generalized complex structures and nondegenerate, pure spinors

Let $M$ be a differentiable manifold of real dimension $2n$. The bilinear form $\langle , \rangle_{T\oplus T^*}$ on the direct sum $T_M \oplus T_M^*$ over a differentiable manifold $M$ of dim $= 2n$ is defined by

$$\langle v + \xi, u + \eta \rangle_{T\oplus T^*} = \frac{1}{2} (\xi(u) + \eta(v)), \quad u, v \in T_M, \xi, \eta \in T_M^*.$$ 

Let $SO(TM \oplus T^*M)$ be the fibre bundle over $M$ with fibre $SO(2n, 2n)$ which is a subbundle of $\text{End}(TM \oplus T^*M)$ preserving the bilinear form $\langle , \rangle_{T\oplus T^*}$. An almost generalized complex structure $J$ is a section of $SO(TM \oplus T^*M)$ satisfying $J^2 = -\text{id}$. Then as in the case of almost complex structures, an almost generalized complex structure $J$ yields the eigenspace decomposition:

$$(T_M \oplus T_M^*)^c = \mathcal{L}_J \oplus \overline{\mathcal{L}_J},$$

(2.1)

where $\mathcal{L}_J$ is $-\sqrt{-1}$-eigenspace and $\overline{\mathcal{L}_J}$ is the complex conjugate of $\mathcal{L}_J$. The Courant bracket of $T_M \oplus T^*M$ is defined by

$$[u + \xi, v + \eta]_{\text{co}} = [u, v] + \mathcal{L}_u \eta - \mathcal{L}_v \xi - \frac{1}{2} (\mathcal{L}_u \xi - \mathcal{L}_v \eta),$$

where $u, v \in TM$ and $\xi, \eta$ is $T^*M$. If $\mathcal{L}_J$ is involutive with respect to the Courant bracket, then $J$ is a generalized complex structure, that is, $[e_1, e_2]_{\text{co}} \in \Gamma(\mathcal{L}_J)$ for any two elements $e_1 = u + \xi, e_2 = v + \eta \in \Gamma(\mathcal{L}_J)$. Let $\text{Cl}(T_M \oplus T^*_M)$ be the Clifford algebra bundle which is a fibre bundle with fibre the Clifford algebra $\text{Cl}(2n, 2n)$ with respect to $\langle , \rangle_{T\oplus T^*}$ on $M$. Then a vector $v$ acts on the space of differential forms $\oplus_{p=0}^{2n} \wedge^p T^*M$ by the interior product $i_v$ and a 1-form $\theta$ acts on $\oplus_{p=0}^{2n} \wedge^p T^*M$ by the exterior product $\theta \wedge$, respectively. Then the space of differential forms gives a representation of the Clifford algebra $\text{Cl}(TM \oplus T^*M)$ which is the spin representation of $\text{Cl}(TM \oplus T^*M)$. Thus the spin representation of the Clifford algebra arises as the space of differential forms

$$\wedge^* T_M = \bigoplus_p \wedge^p T^*_M = \wedge^{\text{even}} T^*_M \oplus \wedge^{\text{odd}} T^*_M.$$
The inner product $\langle \cdot , \cdot \rangle_s$ of the spin representation is given by

$$\langle \alpha , \beta \rangle_s := (\alpha \wedge \sigma \beta)_{2n},$$

where $(\alpha \wedge \sigma \beta)_{2n}$ is the component of degree $2n$ of $\alpha \wedge \sigma \beta \in \mathop{\bigoplus}_{p} \Lambda^p T^* M$ and $\sigma$ denotes the Clifford involution which is given by

$$\sigma \beta = \begin{cases} +\beta & \text{deg } \beta \equiv 0, 1 \mod 4 \\ -\beta & \text{deg } \beta \equiv 2, 3 \mod 4 \end{cases}$$

We define $\ker \Phi := \{ e \in (T_M \oplus T^*_M)^C | e \cdot \Phi = 0 \}$ for a differential form $\Phi \in \Lambda^{\text{even/odd}} T^*_M$. If $\ker \Phi$ is maximal isotropic, i.e., $\dim_{\mathbb{C}} \ker \Phi = 2n$, then $\Phi$ is called a pure spinor of even/odd type. A pure spinor $\Phi \in \Lambda^* T^*_M$ gives an almost generalized complex structure $J_\Phi$ which satisfies

$$J_\Phi e = \begin{cases} \sqrt{-1} e, & e \in \ker \Phi \\ \sqrt{1} e, & e \in \ker \Phi \end{cases}$$

Conversely, an almost generalized complex structure $J$ locally arises as $J_\Phi$ for a nondegenerate, pure spinor $\Phi$ which is unique up to multiplication by non-zero functions. Thus an almost generalized complex structure yields the canonical line bundle $K_J := \mathbb{C} \langle \Phi \rangle$ which is a complex line bundle locally generated by a nondegenerate, pure spinor $\Phi$ satisfying $J = J_\Phi$. An generalized complex structure $J_\Phi$ is integrable if and only if $d\Phi = \eta \cdot \Phi$ for a section $\eta \in T_M \oplus T^*_M$. The type number of $J = J_\Phi$ is defined as the minimal degree of the differential form $\Phi$. Note that type number $\text{Type } J$ is a function on a manifold which is not a constant in general.

**Example 2.1.** Let $J$ be a complex structure on a manifold $M$ and $J^*$ the complex structure on the dual bundle $T^* M$ which is given by $J^* \xi(v) = \xi(Jv)$ for $v \in TM$ and $\xi \in T^* M$. Then a generalized complex structure $J_J$ is given by the following matrix

$$J_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.$$ 

Then the canonical line bundle is the ordinary one which is generated by complex forms of type $(n,0)$. Thus we have $\text{Type } J_J = n$.

**Example 2.2.** Let $\omega$ be a symplectic structure on $M$ and $\tilde{\omega}$ the isomorphism from $TM$ to $T^* M$ given by $\tilde{\omega}(v) := i_v \omega$. We denote by $\tilde{\omega}^{-1}$ the inverse map from $T^* M$ to $TM$. Then a generalized complex structure $J_\psi$ is given by the following

$$J_\psi = \begin{pmatrix} 0 & -\tilde{\omega}^{-1} \\ \tilde{\omega} & 0 \end{pmatrix}, \quad \text{Type } J_\psi = 0$$

Then the canonical line bundle is given by the differential form $\psi = e^{-\sqrt{-1} \omega}$. Thus $\text{Type } J_\psi = 0$.

**Example 2.3 (b-field action).** A real $d$-closed 2-form $b$ acts on a generalized complex structure by the adjoint action of Spin group $e^b$ which provides a generalized complex structure $\text{Ad}_{e^b} J = e^b \circ J \circ e^{-b}$. 
EXAMPLE 2.4 (Poisson deformations). Let $\beta$ be a holomorphic Poisson structure on a complex manifold. Then the adjoint action of Spin group $e^{i\beta}$ gives deformations of new generalized complex structures by $J_{\beta t} := \text{Ad}_{e^{i\beta t}}J$. Then Type $J_{\beta x} = n - 2$ (rank of $\beta_x$) at $x \in M$, which is called the Jumping phenomena of type number.

Let $(M, J)$ be a generalized complex manifold and $\mathcal{L}_J$ the eigenspace of eigenvalue $\sqrt{-1}$. Then we have the Lie algebroid complex $\wedge^* \mathcal{L}_J$:

$$0 \rightarrow \wedge^0 \mathcal{L}_J \xrightarrow{\bar{\partial}_J} \wedge^1 \mathcal{L}_J \xrightarrow{\bar{\partial}_J} \wedge^2 \mathcal{L}_J \xrightarrow{\bar{\partial}_J} \wedge^3 \mathcal{L}_J \rightarrow \cdots$$

The Lie algebroid complex is the deformation complex of generalized complex structures. In fact, $\varepsilon \in \wedge^2 \mathcal{L}_J$ gives deformed isotropic subbundle $E_\varepsilon := \{e + [\varepsilon, e] | e \in \mathcal{L}_J\}$. Then $E_\varepsilon$ yields deformations of generalized complex structures if and only if $\varepsilon$ satisfies Generalized Mauer-Cartan equation

$$\bar{\partial}_J \varepsilon + \frac{1}{2} [\varepsilon, \varepsilon]_{\text{Sch}} = 0,$$

where $[\varepsilon, \varepsilon]_{\text{Sch}}$ denotes the Schouten bracket. The Kuranishi space of generalized complex structures is constructed. Then the second cohomology group $H^2(\wedge^* \mathcal{L}_J)$ of the Lie algebraic complex gives the infinitesimal deformations of generalized complex structures and the third one $H^3(\wedge^* \mathcal{L}_J)$ is the obstruction space to deformations of generalized complex structures. Let $\{e_i\}_{i=1}^n$ be a local basis of $\mathcal{L}_J$ for an almost generalized complex structure $J$, where $\langle e_i, e_j \rangle_{\mathcal{T}\otimes T^*} = \delta_{i,j}$. The almost generalized complex structure $J$ is written as an element of Clifford algebra,

$$J = \frac{\sqrt{-1}}{2} \sum_i e_i \cdot \bar{e}_i,$$

where $J$ acts on $TM \oplus T^*M$ by the adjoint action $[J, \cdot]$. Thus we have $[J, e_i] = -\sqrt{-1}e_i$ and $[J, \bar{e}_i] = \sqrt{-1}e_i$. An almost generalized complex structure $J$ acts on differential forms by the Spin representation which gives the decomposition into eigenspaces:

$$\wedge^* T^*_M = U^{-n} \oplus U^{-n+1} \oplus \cdots U^n,$$

where $U^i (= U^i_J)$ denotes the $i$-eigenspace.

2.2 Generalized Kähler structures

DEFINITION 2.5. A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ consisting of two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $\tilde{G} := -\mathcal{J}_1 \circ \mathcal{J}_2 = -\mathcal{J}_2 \circ \mathcal{J}_1$ gives a positive definite symmetric form $G := \langle \tilde{G}, \cdot \rangle$ on $TM \oplus T^*_M$. We call $G$ a generalized metric. A generalized Kähler structure of symplectic type is a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{J}_2$ is the generalized complex structure $\mathcal{J}_\psi$ which is induced from a $d$-closed, nondegenerate, pure spinor $\psi := e^{b\sqrt{-1}\omega}$.

Each $\mathcal{J}_i$ gives the decomposition $(TM \oplus T^*M)^C = \mathcal{L}_{\mathcal{J}_i} \oplus \mathcal{L}_{\mathcal{J}_i}$, for $i = 1, 2$. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ are commutative, we have the simultaneous eigenspace decomposition

$$(TM \oplus T^*M)^C = (\mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}) \oplus (\mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}) \oplus (\mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}) \oplus (\mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_2}).$$
Since $\hat{G}^2 = + \text{id}$, the generalized metric $\hat{G}$ also gives the eigenspace decomposition: $TM \oplus T^*M = C_+ \oplus C_-$, where $C_\pm$ denote the eigenspaces of $\hat{G}$ of eigenvalues $\pm 1$. We denote by $L^{\pm}_{\mathcal{J}_1}$ the intersection $L_{\mathcal{J}_1} \cap C^{\pm}_\mathbb{C}$. Then it follows

$$L_{\mathcal{J}_1} \cap L_{\mathcal{J}_2} = L_{\mathcal{J}_1}^{+} \cap L_{\mathcal{J}_2}^{+}, \quad \overline{L_{\mathcal{J}_1} \cap L_{\mathcal{J}_2}} = \overline{L_{\mathcal{J}_1}^{+}}$$

**Example 2.6.** Let $X = (M, J, \omega)$ be a Kähler manifold. Then the pair $(\mathcal{J}_1, \psi)$ is a generalized Kähler where $\psi = \exp(\sqrt{1-\omega})$.

**Example 2.7.** Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure. Then the action of $b$-fields gives a generalized Kähler structure $(\text{Ad}_{b_{\mathcal{J}_1}}, \text{Ad}_{b_{\mathcal{J}_2}})$ for a real $d$-closed 2-form $b$.

### 2.3 The deformation-stability theorem of generalized Kähler manifolds

It is known that the deformation-stability theorem of ordinary Kähler manifolds holds

**Theorem 2.8 (Kodaira-Spencer).** Let $X = (M, J)$ be a compact Kähler manifold and $X_t$ small deformations of $X = X_0$ as complex manifolds. Then $X_t$ inherits a Kähler structure.

The following deformation-stability theorem of generalized Kähler structures provides many interesting examples of generalized Kähler manifolds of symplectic type.

**Theorem 2.9.** [12] Let $X = (M, J, \omega)$ be a compact Kähler manifold and $(\mathcal{J}_1, \mathcal{J}_\psi)$ the induced generalized Kähler structure, where $\psi = e^{-\sqrt{-1}\omega}$. If there are analytic deformations $\{\mathcal{J}_1\}$ of $\mathcal{J}_0 = \mathcal{J}_1$ as generalized complex structures, then there are deformations of $d$-closed nondegenerate, pure spinors $\{\psi_t\}$ such that pairs $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ are generalized Kähler structures, where $\psi_0 = \psi$.

Then we have the following:

**Corollary 2.10.** Let $X = (M, J, \omega)$ be a compact Kähler manifold with a nontrivial holomorphic Poisson structure $\beta$. Then there exists a nontrivial deformations of generalized Kähler structures $(\mathcal{J}_{\beta t}, \mathcal{J}_{\psi_t})$ such that $\{\mathcal{J}_{\beta t}\}$ is the Poisson deformations given by Example 2.4 and $\{\psi_t\}$ is a family of $d$-closed nondegenerate, pure spinors and $\psi_0 = e^{-\sqrt{-1}\omega}$.

### 3 Einstein-Hermitian generalized connections

#### 3.1 Generalized connections over symplectic manifolds

Let $M$ be a compact real manifold of dimension $2n$ and $\omega$ a real symplectic structure on $M$. We denote by $\psi$ the exponential of $b - \sqrt{-1}\omega$, that is,

$$\psi := e^{b - \sqrt{-1}\omega} = 1 + (b - \sqrt{-1}\omega) + \frac{1}{2!}(b - \sqrt{-1}\omega)^2 + \cdots + \frac{1}{n!}(b - \sqrt{-1}\omega)^n,$$

where $b$ denotes a real $d$-closed 2-form on $M$. Then $\psi$ induces a generalized complex structure $\mathcal{J}_\psi$ which gives a decomposition

$$(TM \oplus T^*M)^{\mathbb{C}} = L_{\mathcal{J}_\psi} \oplus \overline{L_{\mathcal{J}_\psi}}$$
$TM \oplus T^*M$ acts on differential forms by the spin representation which is given by the interior product and the exterior product of $TM \oplus T^*M$ on differential forms. Then as in (2.2), we have a decomposition of differential forms on $M$,
$$\oplus_{i=0}^{2n} \wedge^i T^*_M = \oplus_{j=0}^{2n} U^{-n+j}_{\mathcal{J}_\psi},$$
where $U^n_{\mathcal{J}_\psi}$ is a complex line bundle generated by $\psi$ and $U^{-n+1}_{\mathcal{J}_\psi}$ is constructed by the spin action of $\wedge^i \mathcal{J}_\psi$ on $U^n_{\mathcal{J}_\psi}$. Let $E \to M$ be a complex vector bundle of rank $r$ over $M$ and $\Gamma(E)$ a set of smooth sections of $E$. We denote by $\Gamma(E \otimes (TM \oplus T^*M)^\n)$ the set of smooth sections of $E \otimes (TM \oplus T^*M)^\n$. A generalized connection $D^A$ is a map from $\Gamma(E)$ to $\Gamma(E \otimes (TM \oplus T^*M)^\n$ such that
$$D^A(fs) = s \otimes df + fD^A(s), \quad \text{for } s \in \Gamma(E), \ f \in C^\infty(M).$$

Let $h$ be an Hermitian metric of $E$. A generalized Hermitian connection is a generalized connection $D^A$ satisfying
$$dh(s, s') = h(D^A s, s') + h(s, D^A s'), \quad \text{for } s, s' \in \Gamma(E).$$

We denote by $u(E, h)(:= u(E))$ the set of skew-symmetric endomorphisms of $E$ with respect to $h$. Then $\text{End}(E)$ is decomposed as
$$\text{End}(E) = u(E, h) \oplus \text{Herm}(E, h), \quad (3.1)$$
where $\text{Herm}(E, h)$ denotes the set of Hermitian endomorphisms of $E$. Let $\{U_\alpha\}$ be an open covering of $M$ which gives local trivializations of $E$. We take $s_\alpha := (s_{\alpha,1}, \cdots, s_{\alpha,r})$ as a local unitary frame of $E$ over $U_\alpha$. The set of transition functions is denoted by $\{g_{\alpha\beta}\}$. Then an Hermitian generalized connection $D^A$ is written as
$$D^A(s_{\alpha,p}) = \sum_{q=1}^{r} s_{\alpha,q} A^q_{p,\alpha},$$
where $A_{\alpha} := (A^q_{p,\alpha})$ is called a connection form of a generalized connection which is a section of $u(E) \otimes_R \text{(TM } \oplus T^*M)_{U_\alpha}$. Then $A_{\alpha}$ is denoted as
$$A_{\alpha} = \sum_i A_{\alpha,i} e_i,$$
where $e_i \in (TM \oplus T^*M)$ and $A^i_{\alpha} \in u(E)$. Note that each $e_i$ is a real element of $TM \oplus T^*M$. Then $A_{\alpha}$ is also decomposed into
$$A_{\alpha} = A_{\alpha} + V_{\alpha}.$$ 

Then it turns out that $A_{\alpha} := (A^q_{p,\alpha})$ is an ordinary connection form and $V_{\alpha} := (V^q_{p,\alpha})$ gives a section of $u(E) \otimes T_M$ In fact, by using local trivializations $s_\alpha$, given a generalized connection $D^A$ is written as
$$D^A = d_{\alpha} + A_{\alpha},$$
Since $D^A : \Gamma(E) \to \Gamma(E \otimes (TM \oplus T^*M))$ is globally defined as a differential operator, we have
$$A_{\alpha} = - (dg_{\alpha\beta})^{-1} g_{\alpha\beta} A_{\beta}^{-1} g_{\alpha\beta}^{-1} \quad (3.2)$$
$$V_{\alpha} = g_{\alpha\beta} V_{\beta} g_{\alpha\beta}^{-1} \quad (3.3)$$
Thus it follows that $A_{\alpha}$ is a connection form of a connection $D^A$ and $V_{\alpha}$ is a $u(E)$-valued vector field. As shown in the ordinary connections, a generalized connection $D^A$ is extended to be an operator $\Gamma(\text{End}(E)) \to \Gamma(\text{End}(E) \otimes (TM \oplus T^*M))$ by the following:
$$(D^A\xi)s := D^A(\xi s) - \xi(\xi^A s), \quad \text{for } \xi \in \Gamma(\text{End}(E)), \ s \in \Gamma(E).$$
Then the extended operator $D^A$ is also written as follows in terms of $A_{\alpha} = \sum_{i} A_{\alpha,i} e_i$

$$D^A \xi = d\xi + \sum_{i} [A_{\alpha,i}, \xi] e_i. \quad (3.4)$$

Each $e_i \in TM \oplus T^* M$ acts on $\psi$ which is denoted by $e_i \cdot \psi \in U_{J_0}^{-n+1}$ and then each element $\xi \otimes e_i$ of $\text{End} (E) \otimes (TM \oplus T^* M)$ acts on $\psi$ which is denoted by $\xi \otimes (e_i \cdot \psi)$. Then the action on $\psi$ gives the map $\text{End} (E) \otimes (TM \oplus T^* M) \to \text{End} (E) \otimes U_{J_0}^{-n+1}$. Thus we obtain $D^A \xi \cdot \psi \in \text{End} (E) \otimes U_{J_0}^{-n+1}$ for $\xi \in \text{End} (E)$ and a generalized connection $D^A$. Thus an operator $d^A$ is defined by

$$d^A : \Gamma(\text{End} (E)) \to \Gamma(\text{End} (E) \otimes U_{J_0}^{-n+1})$$

$$\xi \mapsto D^A \xi \cdot \psi \quad (3.5)$$

We also extend $d^A$ to be an operator

$$d^A : \Gamma(\text{End} (E) \otimes U_{J_0}^{-n+1}) \to \Gamma(\text{End} (E) \otimes (U_{J_0}^{-n} \oplus U_{J_0}^{-n+2}))$$

by setting :

$$d^A(\xi \otimes e_i \cdot \psi) = (D^A \xi) \cdot e_i \cdot \psi + \xi \otimes d(e_i \cdot \psi)$$

Let $a = \sum a_i e_i$ be a section of $\text{End} (E) \otimes (TM \oplus T^* M)$, where $a_i \in \text{End} (E)$ and $e_i \in TM \oplus T^* M$. Then the extended operator $d^A$ is written in the following form:

$$d^A (a \cdot \psi) = \sum_{i} d^A (a_i e_i \cdot \psi) = \sum_{i} (d^A a_i) e_i \cdot \psi + a_i d(e_i \cdot \psi)$$

$$= \sum_{i,j} a_i a_j e_i \cdot \psi + a_i d(e_i \cdot \psi) + [A_{\alpha,j}, a_i] e_j \cdot e_i \cdot \psi$$

$$= d(a \cdot \psi) + [A \cdot a] \cdot \psi \quad (3.7)$$

where we are using the following notation: $[A \cdot a] := \sum_{i,j} [A_{\alpha,j}, a_i] e_j \cdot e_i$. By using the local trivialization and the decomposition $A_\alpha = A_\alpha + V_\alpha$, the operator $d^A$ is described as the following:

$$d^A (a \cdot \psi) = (d_\alpha + A_\alpha + V_\alpha) a \cdot \psi = d(a \cdot \psi) + [V_\alpha \cdot a] \cdot \psi + [A_\alpha \cdot a] \cdot \psi,$$

where $[\cdot \cdot]$ is the product of $\text{End} (E) \otimes \text{CL}$ which is defined as

$$[V_{\alpha} \cdot a] = \sum_{i,j} [V_{\alpha,i}, a_j] e_i \cdot e_j$$

$$[A_{\alpha} \cdot a] = \sum_{i,j} [A_{\alpha,i}, a_j] e_i \cdot e_j$$

Note that our new product $[\cdot \cdot]$ is the combinations of the bracket of Lie algebra $\text{End} (E)$ and the Clifford multiplications of Clifford algebra $\text{CL}$, that is,

$$[(A \otimes s) \cdot (A' \otimes s')] = [A, A'] \otimes s \cdot s' \in \text{End} (E) \otimes \text{CL}$$

for $A \otimes e, A' \otimes e'$, where $A, A' \in \text{End} (E), s, s' \in \text{CL}$. 

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3.2 Curvature of generalized connections

Let $D^A$ be a generalized Hermitian connection of an Hermitian vector bundle $(E,h)$ over a manifold $M$ which consists an ordinary Hermitian connection $D^A$ and a section $V \in \Gamma(u(E) \otimes T_M)$. Then the ordinary curvature $F_A$ is a section $\text{End}(E)$ valued 2-from which acts on $\psi$ by the spin representation to obtain $F_A \cdot \psi \in \text{End}(E) \otimes_{\mathbb{R}} (U^{-n} \oplus U^{-n+2})$. The ordinary connection is extended to be an operator as before

$$d^A : \text{End}(E) \otimes_{\mathbb{R}} U^{-n+1} \rightarrow \text{End}(E) \otimes_{\mathbb{R}} (U^{-n} \oplus U^{-n+2}).$$

(For simplicity, we denote by $U^p$ the eigenspace $U^p_{\mathcal{F}}$.) By applying $d^A$ to $V \cdot \psi \in \text{End}(E) \otimes_{\mathbb{R}} U^{-n+1}$, we have $d^A(V \cdot \psi) \in \text{End}(E) \otimes_{\mathbb{R}} (U^{-n} \oplus U^{-n+2})$. By the spin representation, $[V \cdot V] \in u(E) \otimes_{\mathbb{R}} \Lambda^2 T_M$ acts on $\psi$ to obtain $[V \cdot V] \cdot \psi \in \text{End}(E) \otimes_{\mathbb{R}} (U^{-n} \oplus U^{-n+2})$. Then we have the definition of curvature of generalized connection $D^A$:

**Definition 3.1.** [curvature of generalized connections] $\mathcal{F}_A(\psi)$ of a generalized connection $D^A$ is defined by

$$\mathcal{F}_A(\psi) := F_A \cdot \psi + d^A(V \cdot \psi) + \frac{1}{2}[V \cdot V] \cdot \psi \quad (3.8)$$

$\mathcal{F}(\psi)$ is a globally defined section of $\text{End}(E) \otimes (U^{-n} \oplus U^{-n+2})$ which is called the curvature of $D^A$.

**Remark 3.2.** The ordinary curvature $F_A$ of a connection $D^A$ is defined to be the composition $d^A \circ d^A$. However, $\mathcal{F}_A(\psi)$ is different from the composition $d^A \circ d^A$ which is not a tensor but a differential operator. In fact, we have

$$d^A \circ d^A = (d + A) \circ (d + A) = (d^A + V) \circ (d^A + V) \quad (3.9)$$

$$= F_A + d^A \circ V + V \circ d^A + V \circ V \quad (3.10)$$

Then we have

$$(V \circ V) \cdot \psi = \frac{1}{2}[V \cdot V] \cdot \psi$$

However, for $f \in C^\infty(M)$, $s \in \Gamma(E)$, we have

$$(d^A \circ d^A)(fs \otimes \psi) = f(d^A \circ d^A)(s \otimes \psi) - 2(df,Vs)_{T \otimes T} \cdot \psi$$

Thus $(d^A \circ d^A)$ is not a tensor but a differential operator.

$\text{End}(E) \otimes (U^{-n} \oplus U^{-n+2})$ is decomposed as

$$\text{End}(E) \otimes (U^{-n} \oplus U^{-n+2}) = (\text{End}(E) \otimes U^{-n}) \oplus (\text{End}(E) \otimes U^{-n+2})$$

We denote by $\pi_{U^{-n}}$ the projection from $\text{End}(E) \otimes (U^{-n} \oplus U^{-n+2})$ to the component $\text{End}(E) \otimes U^{-n}$. The line bundle $U^{-n}$ becomes the trivial complex line bundle by using the basis $\psi \in U^{-n}$. Then $\text{End}(E) \otimes_{\mathbb{C}} U^{-n}$ is identified with $\text{End}(E)$. Then it follows from (3.1) that we have

$$\text{End}(E) \otimes U^{-n} = u(E) \oplus \text{Herm}(E,h) \quad (3.11)$$

We define $\pi_{\text{Herm}}$ to be the projection to the component $\text{Herm}(E,h)$ and we denote by $\pi_{U^{-n}}$ the composition $\pi_{\text{Herm}} \circ \pi_{U^{-n}}$. Then we define $\mathcal{K}_A(\psi)$ by

$$\mathcal{K}_A(\psi) := \pi_{U^{-n}} \mathcal{F}_A(\psi) \in \text{Herm}(E,h)$$
**Definition 3.3.** [Einstein-Hermitian condition] A generalized Hermitian connection $D^A$ is **Einstein-Hermitian** if $D^A$ satisfies the following:

$$\mathcal{K}_A(\psi) = \lambda \text{id}_E,$$

for a real constant $\lambda$.

**Remark 3.4.** If $D^A$ is an ordinary Hermitian connection $D^A$ over a Kähler manifold with a Kähler form $\omega$, then the Einstein-Hermitian condition in Definition 3.3 coincides with the ordinary Einstein-Hermitian condition for $\psi = e^{-\sqrt{-1} \omega}$. In fact, the ordinary connection $D^A$ is Einstein-Hermitian connection if its curvature $F_A$ satisfies

$$\sqrt{-1} \Lambda \omega F_A = + \lambda \text{id}_E$$

or equivalently

$$\frac{\sqrt{-1} n F_A \wedge \omega^{n-1}}{\omega^n} = + \lambda \text{id}_E,$$

where $\lambda$ is a real constant. The projection $\pi_{U^{-n}}$ of $F_A$ is given by

$$\pi_{U^{-n}} F_A \cdot \psi = \frac{\langle F_A \cdot \psi, \overline{\psi} \rangle_s}{\langle \psi, \overline{\psi} \rangle_s} \psi$$

since $\psi = e^{-\sqrt{-1} \omega}$, we have

$$\frac{\langle F_A \cdot \psi, \overline{\psi} \rangle_s}{\langle \psi, \overline{\psi} \rangle_s} = \frac{F_A \wedge (-\sqrt{-1} \omega)^{n-1}}{(n-1)!} \frac{n!}{\sqrt{-1} n} \frac{F_A \wedge \omega^{n-1}}{\omega^n}$$

Since $F_A$ is in $u(E)$, it follows that

$$\frac{\sqrt{-1} n F_A \wedge \omega^{n-1}}{\omega^n} \in \text{Herm}(E, h).$$

Thus under the identification $U^{-n} \cong \mathbb{C} \psi$, we have

$$\pi^\text{Herm}_{U^{-n}} F_A \cdot \psi = \sqrt{-1} n \frac{F_A \wedge \omega^{n-1}}{\omega^n} \psi$$

Hence $\pi^\text{Herm}_{U^{-n}} F_A \cdot \psi = \lambda \text{id}_E$ is equivalent to $\sqrt{-1} \Lambda \omega F_A = + \lambda \text{id}_E$.

The unitary gauge group $U(E, h)$ acts on $\mathcal{F}_A(\psi)$ by the adjoint action. Thus the Einstein-Hermitian condition is invariant under the action of the unitary gauge group. Further our Einstein-Hermitian condition behaves nicely for the action of $b$-fields. A real $d$-closed 2-form $b$ acts on $\psi$ by $e^b \cdot \psi$ which is also a $d$-closed, nondegenerate, pure, spinor.

**Definition 3.5 (b-field action of generalized connections).** Let $D^A = d + A$ be a generalized connection of vector bundle $E$ over $(M, \psi)$. A $d$-closed 2-form $b$ acts on a generalized connection $D^A$ by $e^b D^A e^{-b} = d + e^b A e^{-b}$. Then $e^b A e^{-b}$ is given by

$$e^b A e^{-b} = A + e^b V e^{-b} = A + ad_b V + V,$$

where $A + ad_b V \in \text{End} (E) \otimes T_M$ and $d + A + ad_b V$ is an ordinary connection of $E$ and $V \in \text{End} (E) \otimes T_M$.

Note that $ad_b V$ is given by

$$ad_b V = \sum_i V_i \otimes [b, v_i],$$

for $V = \sum_i V_i \otimes v_i$, where $V_i \in u(E)$ and $v_i \in T_M$. 

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Proposition 3.6. Let $\text{Ad}_{A}D = d + \text{Ad}_{A}A$ be a generalized connection which is given by the $b$-field action on $D$. Then the curvature $F_{\text{Ad}_{A}A}$ is given by

$$F_{\text{Ad}_{A}A}(\psi) = e^{b}F_{A}(e^{-b}\psi)$$

Proof. The $b$-field action is given by $\text{Ad}_{b}A = d + A + \text{ad}_{b}V + V$, where $d + A + \text{ad}_{b}V$ is a connection and its curvature is given by

$$F_A + dA(\text{ad}_bV) + \frac{1}{2}[\text{ad}_bV, \text{ad}_bV]$$

Then we have

$$F_{\text{Ad}_{A}A}(\psi) = (F_A + dA(\text{ad}_bV) + \frac{1}{2}[\text{ad}_bV, \text{ad}_bV]) \cdot \psi$$

$$+ dA(V \cdot \psi) + [\text{ad}_bV \cdot V] \cdot \psi + \frac{1}{2}[V \cdot V] \cdot \psi$$

Applying $\text{Ad}_{b}V = V + \text{ad}_{b}V$ and $e^{b}F_{A}e^{-b} = F_{A}$ and $[e^{b}Ve^{-b} \cdot e^{b}Ve^{-b}]= e^{b}[V \cdot V]e^{-b}$, we have

$$F_{\text{Ad}_{A}A}(\psi) = F_{A} \cdot \psi + dA(e^{b}Ve^{-b} \cdot \psi) + \frac{1}{2}[e^{b}Ve^{-b} \cdot e^{b}Ve^{-b}] \cdot \psi$$

$$= e^{b}F_{A} \cdot e^{-b} \cdot \psi + e^{b}dA(V \cdot e^{-b} \psi) + \frac{1}{2}e^{b}[V \cdot V] \cdot e^{-b} \cdot \psi$$

$$= e^{b}F_{A}(e^{-b} \psi)$$

Thus our Einstein-Hermitian condition is equivalent under the action of $b$-field.

Proposition 3.7. Let $D$ be a generalized Hermitian connection of $E$ over $(M, \psi)$. Then $D$ is Einstein-Hermitian over $(M, \psi)$ if and only if $\text{Ad}_{A}D$ is an Einstein-Hermitian generalized connection of $E$ over $(M, e^{b}\psi)$.

Proof. We denote by $\pi_{U_{e^{b}A}^{-n}}$ the projection to the component $U_{e^{b}A}^{-n} = e^{b} \cdot U_{e^{b}A}^{-n}$. From Proposition 3.6, we have

$$\pi_{\text{Herm}}F_{\text{Ad}_{A}A}(e^{b}\psi) = \pi_{\text{Herm}}\langle F_{\text{Ad}_{A}A}(e^{b}\psi), e^{b} \cdot \overline{\psi} \rangle_{s}$$

$$= \pi_{\text{Herm}}\langle e^{b}F_{A}(\psi), e^{b} \cdot \overline{\psi} \rangle_{s}$$

Since $(e^{b}\psi, e^{b} \overline{\psi})_{s} = \langle \psi, \overline{\psi} \rangle_{s}$, we have

$$\pi_{\text{Herm}}F_{\text{Ad}_{A}A}(e^{b}\psi) = \pi_{\text{Herm}}\frac{\langle F_{A}(\psi), \cdot \overline{\psi} \rangle_{s}}{\langle \psi, \overline{\psi} \rangle_{s}} = \pi_{\text{Herm}}\frac{\langle F_{A}(\psi), \cdot \overline{\psi} \rangle_{s}}{\langle \psi, \overline{\psi} \rangle_{s}} = \pi_{U_{e^{b}A}^{-n}}F_{A}(\psi).$$

Hence $\mathcal{K}_{A}(\psi)$ is invariant under the action of $b$-fields. Thus $\pi_{U_{e^{b}A}^{-n}}F_{\text{Ad}_{A}A}(e^{b}\psi) = \lambda \text{id}_{E}$ if and only if $\pi_{U_{e^{b}A}^{-n}}F_{A}(\psi) = \lambda \text{id}_{E}$. Thus the result follows.
3.3 The first Chern class of $E$ and $\text{tr}\mathcal{F}_A(\psi)$

**Theorem 3.8.** $\frac{-1}{2\pi\sqrt{-1}} \text{tr}\mathcal{F}_A(\psi)$ is a $d$-closed differential form on $M$ which is a representative of the class $[c_1(\det E)] \cup [\psi] \in H^*(M)$.

**Proof.** As in Definition 3.1, $\text{tr}\mathcal{F}_A(\psi)$ is given by

$$\text{tr}\mathcal{F}_A(\psi) := \text{tr}\mathcal{F}_A \cdot \psi + \text{tr}dA (\cdot \psi) + \text{tr}1_{\mathbb{C}} \{ \cdot \psi \}.$$

We have $\text{tr}V \cdot \psi$ is a globally defined form on $M$, so

$$\frac{-1}{2\pi\sqrt{-1}} [\text{tr}\mathcal{F}_A(\psi)] = \frac{-1}{2\pi\sqrt{-1}} [\text{tr}\mathcal{F}_A \cdot \psi] = [c_1(\det E)] \cup [\psi].$$

**Proposition 3.9.** Let $D^A$ be a generalized Einstein-Hermitian connection which satisfies $D^A H = \lambda \text{id}_E$. Then $\lambda$ is given in terms of the first Chern class $c_1(E)$ of $E$ and the class $[\psi]$ by

$$\frac{-\lambda r}{2\pi\sqrt{-1}} \int_M \langle \psi, \overline{\psi} \rangle_s = \int_M \langle c_1(E) \land \psi, \overline{\psi} \rangle_s.$$

**Proof.** It follows from Theorem 3.8 that we have

$$\frac{-1}{2\pi\sqrt{-1}} \int_M \text{tr}\mathcal{F}_A(\psi), \overline{\psi} \rangle_s = \int_M \langle c_1(E) \land \psi, \overline{\psi} \rangle_s.$$

The $U^{-n}$-component of $\mathcal{F}_A(\psi)$ is written as

$$\mathcal{F}_A(\psi) = +\lambda \text{id}_E \psi + \xi \psi,$$

where $\lambda \in \mathbb{R}$ and $\xi \in u(E)$. Then we have

$$\frac{-1}{2\pi\sqrt{-1}} \text{tr}\mathcal{F}_A(\psi), \overline{\psi} \rangle_s = \frac{-1}{2\pi\sqrt{-1}} \text{tr}\langle +\lambda \psi, \overline{\psi} \rangle_s + \frac{-1}{2\pi\sqrt{-1}} \text{tr}\langle \xi \psi, \overline{\psi} \rangle_s$$

$$= \frac{-\lambda r}{2\pi\sqrt{-1}} \langle \psi, \overline{\psi} \rangle_s + \frac{-1}{2\pi\sqrt{-1}} \langle \xi \psi, \overline{\psi} \rangle_s.$$

We have

$$\frac{\langle c_1(E) \land \psi, \overline{\psi} \rangle_s}{\langle \psi, \overline{\psi} \rangle_s} = \frac{c_1(E) \land (\sqrt{-1} \omega)^{n-1}}{(n-1)!} = \frac{n!}{(\sqrt{-1} \omega)^n} \frac{c_1(E) \land \omega^{n-1}}{\omega^n} \frac{n}{\sqrt{-1} \omega} \in \sqrt{-1} \mathbb{R}.$$

Since $\xi \in u(E)$, $\text{tr}\xi$ is pure imaginary. Thus we have the result.

4 Generalized holomorphic vector bundles

Let $E$ be a complex vector bundle over a generalized complex manifold $(M, \mathcal{J})$. A generalized holomorphic structure of $E$ is a differential operator

$$\overline{\partial}_E^E : E \to E \otimes \overline{\mathcal{J}},$$

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which satisfies

\[ \overline{\partial}_J^E (fs) = s \otimes (\overline{\partial}_J f) + f(\overline{\partial}_J^E s), \quad \overline{\partial}_J^E \circ \overline{\partial}_J^E = 0, \quad \text{for } f \in C^\infty(M), s \in E. \]

A complex vector bundle equipped with a generalized holomorphic structure is called as a *generalized holomorphic vector bundle* [17]. Then a generalized holomorphic structure \( \overline{\partial}_J^E \) is extended to be an operator \( \overline{\partial}_J^E : E \otimes \wedge^i \mathcal{L}_J \to E \otimes \wedge^{i+1} \mathcal{L}_J \) by

\[ \overline{\partial}_J^E(s\alpha) = (\overline{\partial}_J^E s) \wedge \alpha + s(\overline{\partial}_J \alpha), \quad \text{for } s \in E, \alpha \in \wedge^i \mathcal{L}_J. \]

Then we obtain an elliptic complex which is the Lie algebroid complex.

\[ 0 \to E \to E \otimes \mathcal{L}_J \to E \otimes \wedge^2 \mathcal{L}_J \to \cdots \to E \otimes \wedge^i \mathcal{L}_J \to E \otimes \wedge^{i+1} \mathcal{L}_J \to \cdots \to 0 \]

We denote by \( C^0 \cdot = \{ C^0, i \} \) the Lie algebroid complex \((E \otimes \wedge^i \mathcal{L}_J, \overline{\partial}_J^E)\).

### 4.1 The canonical generalized connection

Let \((E, h)\) be an Hermitian vector bundle over a generalized complex manifold \((M, J)\). We assume that \(E\) admits a generalized holomorphic structure \( \overline{\partial}_J^E \). Then there is a unique generalized Hermitian connection \( D_h \) such that

\[ D_h = D^{1,0} + \overline{\partial}_J^E, \]

where \( D^{1,0} : E \to E \otimes \mathcal{L}_J \) denotes the \((1,0)\)-component of \( D_h \) with respect to \( J \). We call \( D_h \) as the canonical generalized connection of Hermitian vector bundle over a generalized complex manifold. This is an analogue of the canonical connection of Hermitian vector bundle over a complex manifold. In fact, \( D_h \) is determined by

\[ \overline{\partial}_J h(s_1, s_2) = h(D^{1,0}s_1, s_2) + h(s_1, \overline{\partial}_J^E s_2), \quad s_1, s_2 \in E. \]

We denote by \( \mathcal{K}_h(\psi) \) the mean curvature of the generalized connection \( D_h \).

**Definition 4.1.** [Einstein-Hermitian metric] If the canonical generalized connection \( D_h \) satisfies the Einstein-Hermitian condition,

\[ \mathcal{K}_h(\psi) = \lambda \text{id}_E \psi, \]

then \( h \) is called an *Einstein-Hermitian metric*.

### 4.2 Weak generalized holomorphic subbundles

Let \((E, \overline{\partial}_J^E)\) be a generalized holomorphic vector bundle over a generalized complex manifold \((M, J)\). Let \( F \) be a subbundle of \( E \). If \( F \) admits a generalized holomorphic structure \( \overline{\partial}_J^F \) such that \( j \circ \overline{\partial}_J^F = \overline{\partial}_J^E \circ j \), where \( j : F \to E \) denotes the inclusion of \( F \) into \( E \), then \( F \) is called a generalized holomorphic subbundle, that is,

\[ F \xrightarrow{\overline{\partial}_J^F} F \otimes \mathcal{L}_J \]

\[ E \xrightarrow{\overline{\partial}_J^E} E \otimes \mathcal{L}_J \]
In order to define the notion of stability of $E$, we need "a subbundle with singularities" which is already introduced in Uhlenbeck-Yau’s paper as "a weak holomorphic subbundle". We shall define a weak generalized holomorphic subbundle:

**Definition 4.2.** Let $E$ be a complex vector bundle with an Hermitian metric $h$. An element $\pi$ of $L^2_1(\text{End } (E))$ is called a weak generalized holomorphic subbundle of $E$ if the followings hold in $L^1(\text{End } (E))$:

$$\pi^* = \pi = \pi^2, \quad (\text{id}_E - \pi) \circ \overline{\partial}^E_\mathcal{J} \pi = 0 \quad (4.1)$$

We assume that there is another generalized complex structure $\mathcal{J}_2$ such that $(M, \mathcal{J}, \mathcal{J}_2)$ is a generalized Kähler manifold. From a view point of bihermitian structure $I_\pm$, a generalized holomorphic vector bundle $(E, \mathcal{O}_\mathcal{J})$ gives a locally free sheaf with respect to both complex structure $I_+$ and $I_-$. We shall define a weak holomorphic subbundle of $E$.

**Proposition 4.3.** A weak generalized holomorphic subbundle $\pi$ is a weak holomorphic subbundle with respect to both $I_+$ and $I_-$. 

**Proof.** Let $\pi_T : TM \oplus T^* M \to T_M$ be the projection from $TM \oplus T^* M$ to $T_M$. Then $\pi_T$ gives the identification $\mathcal{L}_\mathcal{J}^\pm \cong T_{I_\pm}^{1,0}$ and $\mathcal{L}_{\mathcal{J}}^- \cong T_{I_-}^{1,0}$, where $T_{I_\pm}^{1,0}$ denotes the $(1,0)$-component of $T^C_{I_\pm}$ with respect to $I_\pm$, respectively.

Thus we have the lift $\tilde{v}_\pm$ of $v \in T_M$ defined by $\pi_T(\tilde{v}_\pm) = v$, which is explicitly given by

$$\tilde{v}_\pm = v_\pm^{1,0} \pm g(v^{1,0}_\pm, ) + b(v^{0,1}_\pm, ) \quad (4.2)$$

where $(I_\pm, g, b)$ denotes the bihermitian structure corresponding to the generalized Kähler structure and $v^{1,0}_\pm \in T_{I_\pm}^{1,0}$ and $v^{0,1}_\pm \in T_{I_\pm}^{0,1}$ and

$$v_\pm^{1,0} \pm g(v^{1,0}_\pm, ) + b(v^{0,1}_\pm, ) \in \mathcal{L}_\mathcal{J}^\pm \quad (4.4)$$

$$v_\pm^{0,1} \pm g(v^{0,1}_\pm, ) + b(v^{0,1}_\pm, ) \in \mathcal{L}_{\mathcal{J}}^- \quad (4.5)$$

We denote by $\overline{\partial}^\pm$ the ordinary $\overline{\partial}$-operators with respect to $I_\pm$. Since $df = \partial_\mathcal{J} f + \overline{\partial}_\mathcal{J} f$ for a function $f$, we have

$$\langle \overline{\partial}_\mathcal{J} f, \tilde{v}_\pm \rangle_{T \oplus T^*} = \langle \overline{\partial}_\mathcal{J} f, v_\pm^{0,1} \pm g(v^{0,1}_\pm, ) + b(v^{0,1}_\pm, ) \rangle_{T \oplus T^*} \quad (4.6)$$

$$= \langle df, v_\pm^{0,1} \pm g(v^{0,1}_\pm, ) + b(v^{0,1}_\pm, ) \rangle_{T \oplus T^*} \quad (4.7)$$

$$= \langle df, v_\pm^{0,1} \rangle_{T \oplus T^*} \quad (4.8)$$

$$= \overline{\partial}^\pm f(v) \quad (4.9)$$

Then we have

$$\overline{\partial}^\pm_\nu s = \langle \overline{\partial}^E_\mathcal{J} s, \tilde{v}_\pm \rangle_{T \oplus T^*}$$

where $\tilde{v}_\pm \in C^\pm$ is the lift of $v$. Thus $\pi$ satisfies

$$(\text{id}_E - \pi) \circ \overline{\partial}^\pm \circ \pi = 0$$

Hence $\pi$ is a weak holomorphic subbundle with respect to both $I_\pm$. 

The following result ensures that a weak holomorphic subbundle gives rise to a subsheaf.
Theorem 4.4 (Uhlenbeck-Yau). A weak holomorphic subbundle $\pi$ of $E$ represents a coherent subsheaf $F$ of $E$. More precisely, there is a coherent subsheaf $F$ of $E$ and an analytic subset $S \subset M$ such that

1. $\text{codim } S \geq 2$
2. $\pi|_{M \setminus S}$ is $C^\infty$ and satisfies (4.1).
3. $F|_{M \setminus S} = \pi(E|_{M \setminus S}) \subset E|_{M \setminus S}$ is locally free, i.e., a holomorphic subbundle.

Then from Theorem 4.4 we have

Proposition 4.5. We denote by $E_{\pm}$ locally free sheaves which are given by a generalized holomorphic vector bundle $(E, \overline{\partial}_J^E)$ with respect to $I_{\pm}$, respectively. Let $\pi$ be a weak generalized holomorphic subbundle of $(E, \overline{\partial}_J^E)$. Then there are coherent subsheaves $F_{\pm}$ of $E_{\pm}$ which satisfy

1. $\text{codim } S \geq 2$
2. $\pi|_{M \setminus S}$ is $C^\infty$ and satisfies (4.1).
3. $F_{\pm}|_{M \setminus S} = \pi(E_{\pm}|_{M \setminus S}) \subset E_{\pm}$ is locally free, i.e., a holomorphic subbundle.

5 Definition of $\psi$-stability and $\psi$-polystability

Let $(E, \overline{\partial}_J^E)$ be a generalized holomorphic vector bundle. Then the degree of $E$ is given by

$$\deg(E) := \int_M i^n \langle c_1(E) \cdot \psi, \overline{\psi} \rangle_s,$$

where $c_1(E)$ denotes a d-closed 2-form representing the first Chern class of $E$ which acts on $\psi$. Then the slope $\mu(E)$ of $E$ is the ratio

$$\mu(E) := \frac{\deg E}{\text{rank } E}.$$

Let $h$ be an Hermitian metric on $E$. Then we shall define the stability of $E$ by using weak generalized holomorphic subbundles. A weak generalized holomorphic subbundle $F$ on a complement $M \setminus S$ of codim 2 subset $S$ from Proposition 4.4 Then we define $\text{rank}(\pi)$ to be the rank of the generalized subbundle $F$ on the complement. Since $F$ is a generalized holomorphic subbundle with the induced Hermitian metric $h|_F$, the canonical generalized connection $\mathcal{D}^E$ of $(F, \overline{\partial}_J^E, h|_F)$ gives the first Chern form $c_1(\pi, h|_F)$ on the complement $M \setminus S$. Then we define

$$\deg(\pi) := \int_{M \setminus S} i^n \langle c_1(\pi, h|_F) \cdot \psi, \overline{\psi} \rangle,$$

and then

$$\mu(\pi) := \frac{\deg(\pi)}{\text{rank}(\pi)}.$$

Remark 5.1. The degree of $F$ is well-defined which is given in terms of the second fundamental form (c.f. Section 6). It turns out that $\deg(\pi)$ is finite which coincides with the one given by the first Chern class of the coherent sheaves $F_{\pm}$. 

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Definition 5.2 (Stability). A generalized holomorphic vector bundle \((E, \overline{\partial}^E_J)\) is stable if and only if for every weak generalized holomorphic subbundle \(\pi\) with \(0 < \text{rank} \pi < \text{rank} E\), we have
\[
\mu(\pi) < \mu(E).
\]
If we have
\[
\mu(\pi) \leq \mu(E),
\]
then \(E\) is semistable. If \(E\) is decomposed into the direct sum \(\oplus_i E_i\) of generalized holomorphic subbundles with the same slope \(\mu(E)\), then \(E\) is polystable.

6 The second fundamental forms

Let \(E\) be a generalized holomorphic vector bundle with an Hermitian metric \(h\) over a generalized complex manifold \((M, J)\). We denote by \(S\) a generalized holomorphic subbundle of \(E\). Then we have the short exact sequence
\[
0 \to S \to E \to Q \to 0.
\]
The quotient \(Q\) is identified with the orthogonal complement \(S^\perp\) by using \(h\) and we have the decomposition \(E = S \oplus S^\perp\). Let \(\mathcal{D}^E\) be the canonical generalized Hermitian connection of \(E\) with respect to \(h\). Then we have the decomposition \(\mathcal{D}^E_s = \mathcal{D}^S_s + H^S(s)\) for \(s \in \Gamma(S)\) where \(\mathcal{D}^S(s) \in S\) and \(H^S(s) \in S^\perp\). We also denote by \(\partial_0\) the operator \(\mathcal{D}^{1,0}\).

Proposition 6.1. Let \(h_S\) be the Hermitian metric of \(S\) which is the restriction of \(h\) to \(S\). We denote by \(\pi_s\) the orthogonal projection from \(E\) to \(S\). Then we have
(1) \(\mathcal{D}^S\) is the canonical connection of the generalized holomorphic vector bundle \(S\) with respect to \(h_S\).
(2) \(H^S\) is a section \(\partial_0 \pi_s\) of \(L_J \otimes \text{Hom}(S, S^\perp)\), where \(\partial_0 = \mathcal{D}^{1,0}\) acts on \(\text{End}(E)\).

Proof. Since \(S\) is a generalized holomorphic subbundle, \(\mathcal{D}^{0,1}s = \overline{\partial}^E J = \Gamma(L_J \otimes S)\) for \(s \in \Gamma(S)\). Then (1) follows. Since \(\mathcal{D}^E_s = \mathcal{D}^S_s + H^S(s)\), it follows that \((H^S)^{0,1} = 0\). Hence we obtain \(H^S \in L_J \otimes \text{Hom}(S, S^\perp)\). Then \(H^S\) is given by
\[
H^S = (1 - \pi_s) \circ \mathcal{D}^E \circ \pi_s = (1 - \pi_s) \circ \partial_0 \circ \pi_s
\]
(6.1)
Since \((\partial_0 \pi_s)s = \partial_0 s - \pi_s(\partial_0 s)\) for \(s \in \Gamma(S)\), we see that
\[
H^S s - \partial_0 \pi_s s = (1 - \pi_s) \circ \partial_0 \circ \pi_s s - (\partial_0 \pi_s)s = \partial_0 s - \pi_s(\partial_0 s) - \partial_0 s + \pi_s(\partial_0 s) = 0
\]
Thus \(H^S = \partial_0 \pi_s\).

We also define \(\mathcal{D}^{S^\perp}\) and \(H^{S^\perp}\) by
\[
\mathcal{D}^E s_Q = H^{S^\perp} s_Q + \mathcal{D}^{S^\perp} s_Q,
\]
for \(s_Q \in \Gamma(S^\perp)\), where \(H^{S^\perp} s_Q \in S, \mathcal{D}^{S^\perp} s_Q \in S^\perp\). Then we also have

Proposition 6.2. (1) \(H^{S^\perp}\) is a section \(-\overline{\partial} J \pi \in \overline{L}_J \otimes \text{Hom}(S^\perp, S)\).
(2) \(H^S\) and \(H^{S^\perp}\) satisfy the following
\[
h(H^S, s') + h(s, H^{S^\perp} s') = 0
\]

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Proof. The results follows from the similar way as before.

Let \( \{ e_i \} \) be a local basis of \( L_J \). Then \( H^S \) and \( H^{S \perp} \) are written as follows

\[
H^S = \sum_i H^S_i e_i, \quad H^{S \perp} = -\sum_i (H^S)^* e_i, \tag{6.2}
\]

where \((H^S)^*\) denotes the adjoint of \( H^S \) with respect to \( h \).

7 From Einstein-Hermitian metrics to \( \psi \)-stability

At first, we assume that \( \psi = e^{-\sqrt{-1} \omega} \) and then we shall show that Einstein-Hermitian generalized holomorphic vector bundle is \( \psi \)-polystable. Then we shall show the result in general cases of \( \psi = e^{b - \sqrt{-1} \omega} \) by using the invariance under the action of \( b \)-field. Let \((E, \overline{\partial}^E_J)\) be a generalized holomorphic bundle with an Hermitian metric \( h \) and \( S \) a generalized holomorphic subbundle of \((E, \overline{\partial}^E_J)\). Let \( D^E \) be the canonical generalized connection of \((E, \overline{\partial}^E_J, h)\) over a generalized Kähler manifold \((M, J, J_\psi)\). A generalized connection \( D^E \) is written as

\[
D^E = d + A = d + A + V,
\]

where \( A \in T_M \otimes \text{End} (E) \) is an ordinary connection form and \( V \in T_M \otimes \text{End} (E) \). We identify the quotient bundle \( Q \) with the orthogonal complement \( S^\perp \). Then the decomposition \( E = S \oplus S^\perp \) gives the following decomposition of \( A \):

\[
A = \begin{pmatrix} A_{ss} & A_{sq} \\ A_{qs} & A_{qq} \end{pmatrix}
\]

where \( A_{ss} \in \text{Hom}(S, S) \), \( A_{sq} \in \text{Hom}(Q, S) \), \( A_{qs} \in \text{Hom}(S, Q) \) and \( A_{qq} \in \text{Hom}(Q, Q) \). The connection form \( A \) and the section \( V \) are also decomposed

\[
A = \begin{pmatrix} A_{ss} & A_{sq} \\ A_{qs} & A_{qq} \end{pmatrix}, \quad V = \begin{pmatrix} V_{ss} & V_{sq} \\ V_{qs} & V_{qq} \end{pmatrix}, \tag{7.1}
\]

Then the second fundamental forms are given by

\[
H^S = A_{qs} = A_{qs} + V_{qs}, \quad H^{S \perp} = A_{sq} = A_{sq} + V_{sq}. \tag{7.2}
\]

Let \( F^A(\psi) = F_A \cdot \psi + d^A (V \cdot \psi) + V \cdot V \cdot \psi \) be the generalized curvature of \( D^E \) of \((E, h)\). If we take \( \psi = e^{\sqrt{-1} \omega} \), then we have

\[
K^A(\psi) = \pi_{U-n}^\text{Herm} F^A(\psi) = \pi_{U-n}^\text{Herm} \left( F_A \cdot \psi + \frac{1}{2} [V \cdot V] \cdot \psi \right).
\]

(Note that \( \pi_{U-n}^\text{Herm} d^A (V \cdot \psi) = 0 \).)

Let \( \{ e_i \} \) be a local basis of \( \mathcal{L}_J = \mathcal{L}^+_J \oplus \mathcal{L}^-_J \). When we take the decomposition \( e_i = e^+_i + e^-_i \), then we assume that \( \{ e^+_i \} \) satisfies

\[
\langle e^+_i, e^+_j \rangle_s = \pm \delta_{i,j} \tag{7.3}
\]

Then we have
Lemma 7.1.

\[ \langle e_i \cdot e_j \cdot \psi, \bar{\psi} \rangle_s = -2 \delta_{ij} \langle \psi, \bar{\psi} \rangle_s \]  
(7.4)

\[ \langle e_i \cdot e_j \cdot \psi, \bar{\psi} \rangle_s = +2 \delta_{ij} \langle \psi, \bar{\psi} \rangle_s \]  
(7.5)

**Proof.** Since \( e_i \cdot \psi = e_i^\perp \cdot \psi \) and \( e_i^\perp \cdot \psi = 0 \) and \((e_i^\perp \cdot e_j^\perp + e_i^\parallel \cdot e_j^\parallel) = 2(e_i^\perp \cdot e_j^\perp)_{T \oplus T^*}\), we have

\[ \langle e_i \cdot e_j \cdot \psi, \bar{\psi} \rangle_s = (e_i^\perp \cdot e_j^\perp \cdot \psi, \bar{\psi})_s \]
(7.6)

\[ = ((e_i^\perp \cdot e_j^\perp + e_i^\parallel \cdot e_j^\parallel) \cdot \psi, \bar{\psi})_s \]
(7.7)

\[ = +2(e_i^\perp \cdot e_j^\perp)_{T \oplus T^*} \langle \psi, \bar{\psi} \rangle_s 
(7.8)

\[ = -2(e_i \cdot e_j \cdot \bar{\psi})_s \]  
(7.9)

Lemma 7.2. Let \( \alpha \) be a differential form which acts on \( \psi \). Then we have

\[ \pi_{U-n}(\alpha \cdot \psi) = \frac{\langle \alpha \cdot \psi, \bar{\psi} \rangle_s}{\langle \psi, \bar{\psi} \rangle_s} \psi \]

**Proof.** Let \( \pi_{U^p} \) denotes the projection to the component \( U^p \). Then we have

\[ \langle \pi_{U^p}(\alpha \cdot \psi), \bar{\psi} \rangle_s = 0 \]
for \( p \neq -n \). We denote by \( f \psi \) the component \( \pi_{U-n}(\alpha \cdot \psi) \), where \( f \) is a function. Then we have

\[ \frac{\langle \alpha \cdot \psi, \bar{\psi} \rangle_s}{\langle \psi, \bar{\psi} \rangle_s} \psi = \frac{(f \psi, \bar{\psi})_s}{\langle \psi, \bar{\psi} \rangle_s} \psi = f \psi = \pi_{U-n}(\alpha \cdot \psi). \]

The composition \( H^S \cdot H^{S^\perp} \) between \( H^S \in \mathcal{L}_J \otimes \text{Hom}(S,S^\perp) \) and \( H^{S^\perp} \in \mathcal{L}_J \otimes \text{Hom}(S^\perp,S) \) is a section of \( \mathcal{L}_J \wedge \mathcal{L}_J \otimes \text{End}(S) \). Then \((H^S \cdot H^{S^\perp}) \cdot \psi \) is a section of \((U^{-n} \oplus U^{-n+1}) \otimes \text{End}(S) \). Then the trace of the projection \( \pi_{U-n}^{\text{Herm}}(H^S \cdot H^{S^\perp}) \cdot \psi \) is given by the following:

Lemma 7.3.

\[ \text{tr} \pi_{U-n}^{\text{Herm}}(H^S \cdot H^{S^\perp}) \cdot \psi = + \|H^S\|^2 \psi \]  
(7.14)

\[ \text{tr} \pi_{U-n}^{\text{Herm}}(H^S \cdot H^{S^\perp}) \cdot \psi = - \|H^S\|^2 \psi \]  
(7.15)
Proof. By using (6.2) and (7.2), we have
\[ H_{S}^{\perp} \cdot H_{S} = A_{sq} \cdot A_{qs} = - \sum_{i,j} (H_{S}^{i})^{*}(H_{S}^{j})e_{i}e_{j} \quad (7.16) \]
\[ H_{S} \cdot H_{S}^{\perp} = A_{qs} \cdot A_{sq} = - \sum_{i,j} (H_{S}^{i})(H_{S}^{j})^{*}e_{i}e_{j} \quad (7.17) \]

From Lemma 7.1, we have
\[ \text{tr} \langle (H_{S}^{\perp} \cdot H_{S}) \cdot \psi, \overline{\psi} \rangle_{s} = - \sum_{i,j} \text{tr}((H_{S}^{i})^{*}(H_{S}^{j})^{*})\langle e_{i} \cdot e_{j} \cdot \psi, \overline{\psi} \rangle_{s} \quad (7.18) \]
\[ = + 2 \sum_{i} \| H_{S}^{i} \|^{2} \langle \psi, \overline{\psi} \rangle_{s} \quad (7.19) \]
\[ = \| H_{S} \|^{2} \langle \psi, \overline{\psi} \rangle_{s}, \quad (7.20) \]

Note that it follows from (7.3) that $e_{i} = e_{i}^{+} + e_{i}^{-}$ and $\| e_{i} \| = 2$ and $\| H_{S} \|^{2} = +2 \sum_{i} \| H_{S}^{i} \|^{2}$. From Lemma 7.2 we have
\[ \text{tr} \pi_{U^{-n}}^{\text{Herm}}(H_{S}^{\perp} \cdot H_{S}) \cdot \psi = \| H_{S} \|^{2} \psi. \quad (7.21) \]

We also have
\[ \text{tr} \pi_{U^{-n}}^{\text{Herm}}(H_{S} \cdot H_{S}^{\perp}) \cdot \psi, \overline{\psi} \rangle_{s} = - \sum_{i,j} \text{tr}((H_{S}^{i})(H_{S}^{j})^{*})\langle e_{i} \cdot e_{j} \cdot \psi, \overline{\psi} \rangle_{s} \quad (7.22) \]
\[ = - \| H_{S} \|^{2} \psi. \quad (7.23) \]

Thus we have
\[ \text{tr} \pi_{U^{-n}}^{\text{Herm}}(H_{S} \cdot H_{S}^{\perp}) \cdot \psi = - \| H_{S} \|^{2} \psi. \quad (7.24) \]

\[ \square \]

**Lemma 7.4.** Let $P$ be a skew Hermitian endmorphsim. Then we have
\[ \pi_{U^{-n}}^{\text{Herm}}(\theta \wedge v) \cdot \psi = 0, \]
where $v \in T_{M}$ and $\theta \in T_{M}^{*}$ are real.

Proof. This is a point-wise calculation. Since $\psi = e^{-\sqrt{-1}\omega}$, we have $v - \sqrt{-1}i_{v}\omega \in \mathcal{L}_{\mathcal{J}_{\psi}}$ and $\theta - \sqrt{-1}\omega^{-1}(\theta) \in \mathcal{L}_{\mathcal{J}_{\psi}}$. Then we have
\[ 4\pi_{U^{-n}}(\theta \wedge v) \cdot \psi = \pi_{U^{-n}}(\theta - \sqrt{-1}i_{v}\omega) \cdot (v - \sqrt{-1}i_{v}\omega) \cdot \psi \quad (7.25) \]
\[ = 2(\theta - \sqrt{-1}i_{v}\omega) \cdot (v - \sqrt{-1}i_{v}\omega) \cdot \psi \quad (7.26) \]
\[ = 2\theta(v) \psi, \quad (7.27) \]
where $\theta(v)$ denotes the coupling $\theta \in T_{M}^{*}$ and $v \in T_{M}$. Since $\theta(v)$ is real, $P\theta(v)$ is skew-Hermitian. Thus we have $\pi_{U^{-n}}^{\text{Herm}}(\theta \wedge v) \cdot \psi = 0. \quad \square$
Let $F^E_A(\psi)$ be the curvature of $E$ and $K^E_A(\psi)$ the mean curvature of $E$. Since \( \frac{1}{2}[V \cdot V] \cdot \psi = V \cdot V \cdot \psi \) and \( \frac{1}{2}[A \cdot A] \cdot \psi = A \cdot A \cdot \psi \) for \( V \in T_M \otimes \text{End} (E) \) and \( A \in T_M \otimes \text{End} (E) \), then the component of $F^E_A(\psi)$ of $\text{End} (S)$ is given by

\[
\pi_\varepsilon \circ F^E_A(\psi) \circ \pi_\varepsilon = \pi_\varepsilon \circ (F_A + V \cdot V) \cdot \psi \circ \pi_\varepsilon
\]

\[
= F^S \cdot \psi + V_{ss} \cdot V_{ss} \cdot \psi + A_{sq} \cdot A_{qs} \cdot \psi + V_{sq} \cdot V_{qs} \cdot \psi, \quad (7.29)
\]

where $\pi_\varepsilon$ is the orthogonal projection to $S$ and $i_S : S \to E$ denotes the inclusion and $F^S$ denotes the ordinary curvature of Hermitian connection $A_{ss}$ of the subbundle $S$. It must be noted that the curvature of $D^S$ of the subbundle $S$ is given by

\[
F^S_{A_{ss}}(\psi) = F^S \cdot \psi + V_{ss} \cdot V_{ss} \cdot \psi.
\]

Hence we obtain

\[
\pi_\varepsilon \circ K^E_A(\psi) \circ \pi_\varepsilon = K^S_{A_{ss}}(\psi) + \pi_\varepsilon \circ F^H_{U-n}(A_{sq} \cdot A_{qs} \cdot \psi + V_{sq} \cdot V_{qs} \cdot \psi)
\]

**Lemma 7.5.**

\[
2 \text{tr} \pi_{U-n} H^{S_+} (A_{sq} \cdot A_{qs} \cdot \psi + V_{sq} \cdot V_{qs} \cdot \psi) = \text{tr} \pi_{U-n} (H^{S_+} \cdot H^S - H^S \cdot H^{S_+} \cdot \psi)
\]

**Proof.** As in (7.2), we have $H^{S_+} = A_{sq}$, $H^S = A_{qs}$, where $A_{sq} = A_{sq} + V_{sq}$, $A_{qs} = A_{qs} + V_{qs}$. Taking a real basis \( \{v_i\} \) of $T_M$ and the real dual basis \( \{\theta^i\} \) of $T_M^*$, we obtain the following

\[
A_{sq} = \sum_i A_{sq,i} \theta^i, \quad A_{qs} = \sum_i A_{qs,i} \theta^i
\]

\[
V_{sq} = \sum_i V_{sq,i} v_i, \quad V_{qs} = \sum_i V_{qs,i} v_i, \quad (7.32)
\]

where $A_{sq,i} = -A_{qs,i}$, $V_{sq,i} = -V_{qs,i}$. Thus we have

\[
H^{S_+} \cdot H^S = A_{sq} \cdot A_{qs} = (A_{sq} + V_{sq}) \cdot (A_{qs} + V_{qs})
\]

\[
= (A_{sq} \cdot A_{qs} + V_{sq} \cdot V_{qs}) + (A_{sq} \cdot V_{qs} + V_{sq} \cdot A_{qs})
\]

Then we have

\[
(A_{sq} \cdot A_{qs} + V_{sq} \cdot V_{qs}) = - \sum_{i,j} (A^*_{qs,i} A_{qs,j} \theta^i \wedge \theta^j + V_{qs,i}^* V_{qs,j} v_i \wedge v_j)
\]

\[
(7.36)
\]

From $\theta^i \cdot v_j + v_j \cdot \theta^i = \delta_{i,j}$, we obtain

\[
(A_{sq} \cdot V_{qs} + V_{sq} \cdot A_{qs}) = - \sum_{i,j} (A^*_{qs,i} V_{qs,j} \theta^i v_j + V_{qs,i}^* A_{qs,j} v_i \theta^j)
\]

\[
= - \sum_{i,j} (A^*_{qs,i} V_{qs,j} \theta^i v_j + V_{qs,i}^* A_{qs,j} v_i \theta^j)
\]

\[
= - \sum_{i,j} (A^*_{qs,i} V_{qs,j} - V_{qs,j}^* A_{qs,i}) \theta^i v_j - \sum_i V_{qs,i}^* A_{qs,i}
\]

\[
= - \sum_{i,j} P_{i,j} \theta^i v_j - \sum_i V_{qs,i}^* A_{qs,i},
\]

\[
(7.40)
\]
where $P_{i,j}$ denotes the skew-Hermitian endomorphism $A^*_{qs,i} V_{qs,j} - V^*_{qs,j} A_{qs,i}$. We also have

$$H^S \cdot H^{S^+} = A_{qs} \cdot A_{sq} = (A_{qs} + V_{qs}) \cdot (A_{sq} + V_{sq}) = (A_{qs} \cdot A_{sq} + V_{qs} \cdot V_{sq}) + (A_{qs} \cdot V_{sq} + V_{qs} \cdot A_{sq})$$ (7.41)

Then we have

$$(A_{qs} \cdot A_{sq} + V_{qs} \cdot V_{sq}) = - \sum_{i,j} (A_{qs,i} A^*_{qs,j} \theta^i \wedge \theta^j + V_{qs,i} V^*_{qs,j} v_i \wedge v_j) = \sum_{i,j} (A_{qs,j} A^*_{qs,i} \theta^i \wedge \theta^j + V_{qs,j} V^*_{qs,i} v_i \wedge v_j)$$ (7.44)

We also calculate $(A_{qs} \cdot V_{sq} + V_{qs} \cdot A_{sq})$.

$$(A_{qs} \cdot V_{sq} + V_{qs} \cdot A_{sq}) = - \sum_{i,j} A_{qs,i} V^*_{qs,j} \theta^i v_j + V_{qs,i} A^*_{qs,j} v_j \theta^i = - \sum_{i,j} A_{qs,i} V^*_{qs,j} \theta^i v_j + V_{qs,j} A^*_{qs,i} v_j \theta^i = - \sum_{i,j} (A_{qs,i} V^*_{qs,j} - V_{qs,j} A^*_{qs,i}) \theta^i v_j - \sum_i V_{qs,i} A^*_{qs,i}$$ (7.46)

where $P_{i,j}'$ is a skew-Hermitian endomorphism $(A_{qs,i} V^*_{qs,j} - V_{qs,j} A^*_{qs,i})$. From (7.36) and (7.44), we have

$$\text{tr}(A_{sq} \cdot A_{qs} + V_{sq} \cdot V_{qs}) = - \text{tr}(A_{qs} \cdot A_{sq} + V_{qs} \cdot V_{sq})$$ (7.50)

From Lemma 7.4, we have $\pi^{\text{Herm}}_{U-n} P_{i,j}' \theta^i v_j = 0$ and $\pi^{\text{Herm}}_{U-n} P_{i,j}' \theta^i v_j = 0$. Thus we obtain

$$\text{tr}_{\pi^{\text{Herm}}_{U-n}} (A_{qs} \cdot V_{qs} + V_{qs} \cdot A_{qs}) = - \sum_i \text{tr}_{\pi^{\text{Herm}}_{U-n}} (V^*_{qs,i} A_{qs,i} - V_{qs,i} A^*_{qs,i}) \psi = - \sum_i \text{tr}_{\pi^{\text{Herm}}_{U-n}} (A_{qs,i} V^*_{qs,i} - V_{qs,i} A^*_{qs,i}) \psi = 0$$ (7.51)

Note that $(A_{qs,i} V^*_{qs,i} - V_{qs,i} A^*_{qs,i})$ is a skew-Hermitian endomorphism. Hence we have

$$\text{tr}_{\pi^{\text{Herm}}_{U-n}} (H^{S^+} \cdot H^S - H^S \cdot H^{S^+}) \cdot \psi = 2 \text{tr}_{\pi^{\text{Herm}}_{U-n}} (A_{qs} \cdot A_{qs} \cdot \psi + V_{sq} \cdot V_{qs} \cdot \psi).$$ (7.54)

□

**Proposition 7.6.**

$$\text{tr} \pi_S \circ K^E_{A} (\psi) \circ \pi_S = \text{tr} K^E_{A_{qs} \cdot \psi} + \frac{1}{2} \text{tr} \pi^{\text{Herm}}_{U-n} (H^{S^+} \cdot H^S - H^S \cdot H^{S^+}) \cdot \psi$$

**Proof.** The result follows from Lemma 7.4. □

**Proposition 7.7.**

$$\text{tr} (\pi_s \circ K^E (\psi) \circ \pi_s) = \text{tr} K^S_{A_{qs} (\psi) + \|H^S\|^2 \psi}$$
The slope is given by 
\[ \mu = \frac{\text{deg}(E)}{p} = \frac{1}{p} \int_M \frac{i^n}{2\pi} \text{tr} (\mathcal{K}^S_{\mathcal{A}}(\psi), \overline{\psi})_s \]

Then we have 
\[ \mu(S) = \frac{\text{deg}(S)}{p} = \frac{1}{p} \int_M \frac{i^n}{2\pi} \text{tr} (\mathcal{K}^S_{\mathcal{A}}(\psi), \overline{\psi})_s \]

\[ \leq \mu(E) - \int_M \frac{1}{2\pi} \|H^S\|^2 \text{vol}_M \leq \mu(E) \]  

(PROPOSITION 7.8. Let \( \psi = e^{-\sqrt{-1}\omega} \). If a generalized holomorphic vector bundle \( E \) over a generalized \( \mathbb{C}^n \) manifold \((M, \mathcal{J}, \mathcal{J}_b)\) admits an Einstein-Hermitian metric, then for a generalized holomorphic subbundle \( S \) of \( E \), we have 
\[ \mu(S) \leq \mu(E) \]

Proof. The inequality follows from (7.57). □

We shall extend our result to the general cases of \( \psi = e^{b-\sqrt{-1}\omega} \).

(PROPOSITION 7.9. Let \( \psi = e^{b-\sqrt{-1}\omega} \). If a generalized holomorphic vector bundle \( E \) over a generalized \( \mathbb{C}^n \) manifold \((M, \mathcal{J}, \mathcal{J}_b)\) admits an Einstein-Hermitian metric, then for a generalized holomorphic subbundle \( S \) of \( E \), we have 
\[ \mu(S) \leq \mu(E) \]

Proof. Let \( \mathcal{D}^E \) be the canonical connection of \((E, \overline{\mathcal{D}}^E, h)\) over \((M, \mathcal{J}, \mathcal{J}_b)\). Since \( b \) is a \( d \)-closed form, the action of \( b \)-field gives a generalized \( \mathbb{C}^n \) manifold \((M, \mathcal{J}_b, \mathcal{J}_b)\), where \( \mathcal{J}_b := \text{Ad}_{-\omega} \mathcal{J} \) and \( \psi_b = e^{b}\psi = e^{-\sqrt{-1}\omega} \). There is also an action of \( b \)-field on \( \overline{\mathcal{D}}^E \) by 
\[ \overline{\mathcal{D}}_{\mathcal{J}_b} := e^{-b} \circ \overline{\mathcal{D}}^E \circ e^b. \]

Then \( \overline{\mathcal{D}}_{\mathcal{J}_b} \) is a generalized holomorphic structure with respect to \( \mathcal{J}_b \). As in Proposition 3.6, the action of \( b \)-field gives a generalized connection \( \mathcal{D}^E_b \) by \( \mathcal{D}^E_b = d^* + \text{Ad}_{-\omega} \mathcal{A} = d + A - b(V) + V \), where \( b(V) \in T^*_M \otimes \text{End} (E) \) denotes the contraction of \( b \) and \( V \). Then \( \mathcal{D}^E_b \) is the canonical connection of a generalized holomorphic
vector bundle \((E, \bar{\partial}_{\mathcal{J}_0}, h)\). From Proposition \[\text{3.6}\] we have \(\mathcal{F}_A^E(\psi) = \text{Ad}_{-\psi} \mathcal{F}_A^E\). Thus \(\deg(E)\) is invariant under the action of \(b\)-field. Let \(S\) be a generalized holomorphic subbundle. Then \(b\)-field also acts on \((S, \bar{\partial}_{\mathcal{J}_0}^S)\) to give a generalized holomorphic subbundle \((S, \bar{\partial}_{\mathcal{J}_0}^S)\) of \((E, \bar{\partial}_{\mathcal{J}_0}^E)\). Since \(\deg(S)\) is also invariant under the action of \(b\)-field, we obtain \(\deg(S) \leq \deg(E)\) from Proposition \[\text{7.8}\].

**Proposition 7.10.** Let \(\psi = e^h - \sqrt{-1} \omega\). If a generalized holomorphic vector bundle \(E\) over a generalized Kähler manifold \((M, \mathcal{J}, \mathcal{J}_0)\) admits an Einstein-Hermitian metric, then for a weak generalized holomorphic subbundle \(\pi\) of \(E\), we have

\[\mu(\pi) \leq \mu(E)\]

**Proof.** Let \(F\) be a generalized holomorphic subbundle which is given by \(\pi\) on the complement \(M\setminus S\), where \(S\) is a subset of \(M\) of codim 2. From Proposition \[\text{6.1}\] and Proposition \[\text{7.7}\] on \(M\setminus S\), we have

\[\text{tr} \left( \pi \circ K_A^E(\psi) \circ \pi \right) = \text{tr} K_A^E(\psi) + \|\partial_0 \pi\|^2 \psi.
\]

Since \(\pi \in L^2_1 \text{End} (E)\), \(\partial_0 \pi\) is square-integrable. Thus if \((E, h)\) has an Einstein-Hermitian metric, then we also have

\[\mu(\pi) = \frac{\deg(S)}{p} = \frac{1}{p} \int_M \frac{i^n}{2\pi} \text{tr} \left( K_A^S(\psi) \circ \bar{\psi} \right)_S = \mu(E) = \int_M \frac{1}{2\pi} \|\partial_0 \pi\|^2 \text{vol}_M \leq \mu(E)\]

(7.58)

Thus we have \(\mu(\pi) \leq \mu(E)\). Since \(\mu(\pi)\) is also invariant under the action of \(b\)-field, we have the result. 

**Proposition 7.11.** If a generalized holomorphic vector bundle \(E\) admits an Einstein-Hermitian metric, then \(E\) is polystable.

**Proof.** Let \(E\) be an irreducible holomorphic vector bundle with an Einstein-Hermitian metric. Then it follows from \[\text{7.57}\] that \(\mu(S) \leq \mu(E)\) for a generalized holomorphic subbundle \(S\). If the second fundamental form \(H^S\) vanishes, \(E\) is decomposed into \(S \oplus S'\). Since \(E\) is irreducible, \(H^S\) does not vanish. Thus we have the strict inequality \(\mu(S) < \mu(E)\). Hence \(E\) is stable. We assume that \(E\) is decomposed into \(E_1 \oplus \cdots \oplus E_m\), where each \(E_i\) is irreducible. Then each \(E_i\) is stable with the same Einstein factor \(\lambda\).

**Remark 7.12.** From Proposition \[\text{7.7}\] and Proposition \[\text{6.2}\] we have

\[\text{tr} \left( \pi_S \circ K_A^E(\psi) \circ \pi_s \circ \pi_s \right) = \text{tr} K_A^S(\psi) + \|\bar{\partial} \pi\|^2 \psi.
\]

Since the first Chern form of the generalized subbundle is given by \(\text{tr} K_A^S(\psi)\) and \(\pi \in L^2_1(\text{End} (E))\), it turns out that the first Chern form is integrable which coincides with the one given by the first Chern class of the corresponding sheaves.

### 8 Variation formula of mean curvature

Let \((E, h_0)\) be a complex Hermitian vector bundle over a generalized Kähler manifold \((M, \mathcal{J}, \mathcal{J}_0)\). Let \(\bar{\partial}_{\mathcal{J}}^E\) be a generalized holomorphic structure on \(E\) and \(\mathcal{D}_0\) the canonical connection of \((E, h_0, \bar{\partial}_{\mathcal{J}}^E)\). We denote by \(\text{End} (E, h_0)\) Hermitian endmorphisms of \(E\) with respect to \(h_0\) and also denoted by \(\text{Herm}^+(E, h_0)\)
positive-definite Hermitian endmorphisms with respect to \( h_0 \). For \( f \in \text{Herm}^+(E, h_0) \), we define an Hermitian metric \( h_f \) by
\[
h_f(e_1, e_2) := h_0(f e_1, e_2),
\]
for \( e_1, e_2 \in E \). We denote by \( D_f \) the canonical connection of \((E, h_f, \bar{\nabla}_f)\) with the connection form \( A_f \).

**Lemma 8.1.** \( D_f^{1,0} = f^{-1} \circ D_0^{1,0} \circ f \)

**Proof.** From the definition of \( D_f \), we have
\[
\partial_J h_f(e_1, e_2) = h_f(D_f^{1,0} e_1, e_2) + h_f(e_1, \bar{\nabla}_f e_2) \tag{8.1}
\]
We also have
\[
\partial_J h_0(f e_1, e_2) = h_0(D_0^{1,0} (f e_1), e_2) + h_0(f e_1, \bar{\nabla}_f e_2)
\]
Since \( h_0(e_1, e_2) = h_f(f^{-1} e_1, e_2) \), we have
\[
\partial_J h_f(e_1, e_2) = h_f(f^{-1} \circ D_0^{1,0} (f e_1), e_2) + h_f(e_1, \bar{\nabla}_f e_2) \tag{8.2}
\]
Thus from (8.1) and (8.2), we obtain \( D_f^{1,0} = f^{-1} \circ D_0^{1,0} \circ f \). Hence the result follows.

Then the connection form \( A_f^{1,0} \) is given by
\[
A_f^{1,0} = f^{-1} \partial_J f + f^{-1} A_0^{1,0} f, \tag{8.3}
\]
where \( A_0^{1,0} \) denotes the \((1,0)\)-component of the connection form of the connection \( D_0 \). Let \( \{f_t\} \) be a smooth family of \( \text{Herm}^+(E, h_0) \) with \( f_0 = \text{id}_E \). We denote by \( K_{f_t}(\psi) \) the mean curvature of the generalized connection \( D_{f_t} \). Then the derivative of variation of the mean curvature \( K_{f_t}(\psi) \) is given by

**Proposition 8.2.** For all \( t_0 \), we have
\[
\pi_{\text{Herm}_{f_t}} \left( \frac{d}{dt} K_{f_t}(\psi) \right) \bigg|_{t=t_0} = \pi_{U^{\infty}} \left( \frac{d}{dt} A_{f_t} \right) \left( \frac{d}{dt} \dot{A}_{f_t} \cdot \psi \right) \bigg|_{t=t_0},
\]
where \( \dot{A}_{f_t} = \frac{d}{dt} A_{f_t} \) and \( \pi_{\text{Herm}_{f_t}} \) denotes the projection from \( \text{End}(E) \) to the Hermitian endmorphisms with respect to \( h_{f_t} \).

We need the following Proposition to prove Proposition 8.2 which implies that the mean curvature is regarded as the moment map in [16].

**Proposition 8.3.** Let \( h_0 \) be a fixed Hermitian metric on \( E \) and \( \{A_t\} \) a family of Hermitian generalized connection of \((E, h)\) with the respect to \( h_0 \). Then the derivative of variation of the mean curvature is given by
\[
\frac{d}{dt} K_{A_t}(\psi) = \pi_{U^{\infty}} \left( \frac{d}{dt} \dot{A}_t \cdot \psi \right), \tag{8.4}
\]
**Proof.** By using an action of \( b \)-field, our Proposition is reduced to the cases of \( \psi = e^{-\sqrt{-1} \omega} \). Thus it suffices to show our Proposition in the cases of \( \psi = e^{-\sqrt{-1} \omega} \). Then the mean curvature \( K_{A_t}(\psi) \) is given by
\[
K_{A_t}(\psi) = \pi_{U^{\infty}} \left( F_{A_t} \cdot \psi + \frac{1}{2} [V_t \cdot V_t] \cdot \psi \right), \tag{8.5}
\]

where \( A_t = A_t + V_t \) and \( d^A := d + A_t \) denotes the ordinary connections and \( V_t \in T_M \otimes u(E) \). In fact, from Lemma 7.4, we see that \( \pi_{U-n}^{Herm} d^A(V_t \cdot \psi) = 0 \) in the cases of \( \psi = e^{-\sqrt{-1} t \omega} \). Then we have

\[
\frac{d}{dt} \mathcal{K}_{A_t}(\psi) = \pi_{U-n}^{Herm} \left( d^A(A_t \cdot \psi) + [V_t \cdot V_t] \cdot \psi \right) \tag{8.6}
\]

Applying Lemma 7.4 again, we obtain \( \pi_{U-n}^{Herm} \left( d^A(V_t \cdot \psi) + [V_t \cdot A_t] \cdot \psi \right) = 0 \). Then the right hand side of (8.4) is given by

\[
\pi_{U-n}^{Herm} d^A(A_t \cdot \psi) = \pi_{U-n}^{Herm} \left( d^A(A_t \cdot \psi) + [V_t \cdot V_t] \cdot \psi \right). \tag{8.7}
\]

Thus we have the result. \( \square \)

**Proof of Proposition 8.2.** There is a square \( f^\frac{1}{2} \in \text{Herm}^+(E, h_0) \) such that \( h_f(e_1, e_2) = h_0(f^\frac{1}{2} e_1, f^\frac{1}{2} e_2) \). Since the relation between \( \pi^{Herm_f} \) and \( \pi^{Herm_h} \) is given by

\[
\text{Ad}_f f^\frac{1}{2} \circ \pi^{Herm_f} = \pi^{Herm_h} \circ \text{Ad}_f f^\frac{1}{2}, \tag{8.8}
\]

then \( f^\frac{1}{2} \circ D_f \circ f^{-\frac{1}{2}} \) is an Hermitian generalized connection with respect to \( h_0 \). In fact, \( dh_f(e_1, e_2) = h_f(D_f e_1, e_2) + h_f(e_1, D_f e_2) \) yields the following:

\[
dh_0(e_1, e_2) = dh_f(f^{-\frac{1}{2}} e_1, f^{-\frac{1}{2}} e_2) \tag{8.9}
\]

\[
= h_f(D_f \circ f^{-\frac{1}{2}} e_1, f^{-\frac{1}{2}} e_2) + h_f(f^{-\frac{1}{2}} e_1, D_f \circ f^{-\frac{1}{2}} e_2) \tag{8.10}
\]

\[
= h_0(f^\frac{1}{2} \circ D_f \circ f^{-\frac{1}{2}} e_1, e_2) + h_0(e_1, f^\frac{1}{2} \circ D_f \circ f^{-\frac{1}{2}} e_2) \tag{8.11}
\]

We denote by \( \tilde{D}_f \) the generalized connection \( f^\frac{1}{2} \circ D_f \circ f^{-\frac{1}{2}} \). Since \( f^\frac{1}{2} \) acts on the curvature by the Adjoint action, we also have

\[
F_{\tilde{D}_f}(\psi) = f^\frac{1}{2} \circ F_{D_f}(\psi) \circ f^{-\frac{1}{2}} = \text{Ad}_f f^\frac{1}{2} \left( F_{D_f}(\psi) \right) \tag{8.12}
\]

We shall go back to our proof of Proposition 8.2. We shall reduce the cases of Proposition 8.2 to the one of Proposition 8.3 by using the gauge transformation \( \text{Ad}_f f^\frac{1}{2} \).

When we replace \( h_0 \) by \( h_{f_{t_0}} \) and use \( f_t = f_{t_0} \circ f_{t_0} \), Proposition 8.3 reduces to the case \( t_0 = 0 \). Thus it suffices to show Proposition 8.3 in the case \( t_0 = 0 \). From (8.8) and (8.12), we obtain

\[
\text{Ad}_{f_t} f^\frac{1}{2} \circ \pi_{U-n}^{Herm_{f_t}} \left( F_{\tilde{D}_{f_t}}(\psi) \right) = \pi_{U-n}^{Herm_{f_0}} \circ \text{Ad}_{f_t} f^\frac{1}{2} \left( F_{\tilde{D}_{f_0}}(\psi) \right) \tag{8.13}
\]

\[
= \pi_{U-n}^{Herm_{f_0}} \left( F_{\tilde{D}_{f_t}}(\psi) \right) \tag{8.14}
\]

Since \( \pi_{U-n}^{Herm_{f_t}} F_{\tilde{D}_{f_t}}(\psi) = \mathcal{K}_{f_t}(\psi) \), applying \( \text{Ad}_{f_t} f^{-\frac{1}{2}} \) to the both sides and taking the differential at \( t = 0 \), we have

\[
\frac{d}{dt} \mathcal{K}_{\tilde{D}_{f_t}}(\psi)|_{t=0} = \text{Ad}_{f_{t_0}} f^{-\frac{1}{2}} \left( \frac{d}{dt} \pi_{U-n}^{Herm_{f_0}} F_{\tilde{D}_{f_t}}(\psi)|_{t=0} \right) + \frac{d}{dt} \text{Ad}_{f_t} f^{-\frac{1}{2}} \left( \mathcal{K}_{\tilde{D}_{f_0}}(\psi) \right)|_{t=0} \tag{8.15}
\]

Since \( f_0 = \text{id}_E \), we have \( F_{\tilde{D}_{f_0}}(\psi) = F_{\tilde{D}_{f_0}}(\psi) \). Thus we have

\[
\frac{d}{dt} \text{Ad}_{f_t} f^{-\frac{1}{2}} \left( \mathcal{K}_{\tilde{D}_{f_0}}(\psi) \right)|_{t=0} = \left[ \frac{d}{dt} f^{-\frac{1}{2}}, f_0 \right]. \tag{8.16}
\]
Since \( f_t \) and \((\mathcal{K}_D f_0 (\psi))\) are Hermitian with respect to \( h_{f_0} = h_0 \), the bracket of (8.16) is Skew-Hermitian. Thus
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_0 (\psi)) \bigg|_{t=t_0} = 0
\]
From (8.15) and \( f_0 = \text{id}_E \), we have
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_t (\psi)) \bigg|_{t=t_0} = \left( \frac{d}{dt} \pi_{U^{-n}} \mathcal{T}_{f_t}(\psi) \right|_{t=t_0}
\]
(8.17)
Since \( \mathcal{T}_{f_t} \) is an Hermitian generalized connection with respect to the metric \( h_0 \), we can apply our formula (8.4) in Proposition 8.3 to obtain
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_t (\psi)) \bigg|_{t=t_0} = \pi_{Herm_f} \left( \mathcal{K}_{\mathcal{D}_{f_t}} (\psi) \right) = \pi_{Herm_f} \left( \mathcal{D}_{f_t} \mathcal{A}_f \cdot \psi \right)
\]
(8.18)
From \( \mathcal{D}_{f_t} \mathcal{A}_f = f_t \cdot (\mathcal{D}_{f_t} \mathcal{A}_f) \cdot f_t^{-1} \), it follows from \( f_0 = \text{id}_E \) that we have \( \mathcal{D}_{f_0} \mathcal{A}_f = \mathcal{D}_{f_0} \mathcal{A}_f \). From (8.18) we obtain
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_t (\psi)) \bigg|_{t=t_0} = \pi_{\text{Herm}} (\mathcal{D}_{f_t} \mathcal{A}_f \cdot \psi) \bigg|_{t=t_0}
\]
Hence we obtain the result. \( \square \)

**Proposition 8.4.** For all \( t = t_0 \), we have
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_t (\psi)) \bigg|_{t=t_0} = \pi_{\text{Herm}} \left( \mathcal{D}^{1,0}_{\mathcal{J}} \mathcal{A}_f \cdot \psi \right) \bigg|_{t=t_0}
\]

**Proof.** Since \( \mathcal{D}_{f_t} \) is a family of canonical connections, i.e., \( \mathcal{D}_{f_t}^{0,1} = \mathcal{D}^{1,0}_{\mathcal{J}} \). Thus \( \mathcal{A}_f = \mathcal{A}_f^{1,0} \in \mathcal{L}_{\mathcal{J}} \otimes \text{End} (E) \). Since the \( \pi_{U^{-n}} \)-component is given by \( (1,1) \)-component, i.e., \( (\mathcal{L}_{\mathcal{J}} \cdot \mathcal{L}_{\mathcal{J}}) \cdot \psi \), we have
\[
\pi_{\text{Herm}_{f_t}} (\mathcal{D}_{f_t}^{1,0} \mathcal{A}_f) \bigg|_{t=t_0} = \pi_{\text{Herm}} (\mathcal{D}_{f_t}^{1,0} \mathcal{A}_f) \bigg|_{t=t_0}
\]
\( \square \)

**Proposition 8.5.**
\[
\pi_{\text{Herm}} (\mathcal{K}_D f_t (\psi)) = \pi_{\text{Herm}} \left( \mathcal{D}^{1,0}_{\mathcal{J}} \circ \mathcal{D}_{f_t}^{1,0} (f_t^{-1} \mathcal{A}_f) \cdot \psi \right)
\]
**Proof.** The generalized connection \( \mathcal{D}_{f_t}^{1,0} \) is given by the gauge transformation from Lemma 8.1
\[
\mathcal{D}_{f_t}^{1,0} = f_t^{-1} \circ \mathcal{D}_{f_t}^{1,0} \circ f_t
\]
Thus we have
\[
\mathcal{A}_f = \mathcal{D}_{f_t}^{1,0} (f_t^{-1} \mathcal{A}_f)
\]
Then the result follows from Proposition 8.2 \( \square \)

**Remark 8.6.** We also have the following:
\[
(f_t^{-1} \mathcal{A}_f) = h_t^{-1} \frac{\partial}{\partial h_t} h_t
\]
Then we have

\[ f \] and \[ \text{have} \]

Since \[ \text{Ad} \]

\[ f \in \text{Herm} \]

\[ h \]

\[ S \in \text{Herm} \]

**Proof.** The result follows from Proposition 8.2 and 8.5, since we have

\[ \text{Proposition 8.9} \]

8.8 \[ \text{Lemma} \]

Proposition 8.7.

Taking the trace of the both side of Proposition 8.5, we obtain

**Proposition 8.7.**

\[
\left( \frac{d}{dt} \text{tr} K_{f_t}(\psi) \right) = \pi_{U_{-n}}^{\text{Herm}_f} \left( \overline{\partial}_J^E \circ D_{f_t}^{1,0} \cdot \text{tr} (f_t^{-1} f_t) \cdot \psi \right)
\]

**Proof.** The result follows from Proposition 8.2 and 8.5, since we have

\[
\text{tr} \frac{d}{dt} \text{Ad}_{f_t^{-1}} \left( K_{D_{f_{t_0}}}(\psi) \right) \bigg|_{t=t_0} = \text{tr} \left[ \frac{d}{dt} f_t^{-1} f_t, (\mathcal{F}_{D_{f_{t_0}}}(\psi)) \right] = 0
\]

\[ \square \]

**Lemma 8.8.** Let \( h_0 \) be an Hermitian metric of \( E \) and \( h_f \) be an Hermitian metric given by \( f \in \text{Herm}^+(E, h_0) \) as before. Let \( T_f \) be a section of \( \text{Herm}(E, h_0) \) satisfying \([f,T_f] = 0\). Then for any section \( S \in \text{End}(E) \), we have

\[ h_0 \left( \pi_{\text{Herm}_f}^E(S), T_f \right) = h_0(S, T_f). \]

**Proof.** From 8.8, we have

\[ \text{Ad}_{f_t^{-1}} \circ \pi_{\text{Herm}_f} = \pi_{\text{Herm}_0} \circ \text{Ad}_{f_t^{-1}} \]

Since \( h_0(\text{Ad}_{f_t^{-1}} A, B) = h_0(A, \text{Ad}_{f_t} B) \) for \( A, B \in \text{End}(E) \) and \([f,T_f] = 0, T_f \in \text{Herm}(E, h_0) \), we have

\[ h_0 \left( \pi_{\text{Herm}_f}^E(S), T_f \right) = h_0(\pi_{\text{Herm}_0} \text{Ad}_{f_t^{-1}} S, \text{Ad}_{f_t^{-1}} T_f) \]

\[ = h_0(\text{Ad}_{f_t^{-1}} S, T_f) \]

\[ = h_0(S, T_f) \]

\[ \square \]

**Proposition 8.9.** Let \( h_0 \) be an Hermitian metric and \( h_f := h_0 f \) an Hermitian metric given by \( f \in \text{Herm}^+(E, h_0) \). Let \( T_f \in \text{Herm}(E, h_0) \) be an Hermitian endomorphism satisfying \([T_f, f] = 0\). Then we have

\[ h_0(\mathcal{K}_f(\psi) - \mathcal{K}_0(\psi), T_f) = h_0(\overline{\partial}_J^E (A_1 - A_0) \cdot \psi, T_f \overline{\psi}) \]

\[ = h_0(\overline{\partial}_J^E (f^{-1} \partial_0 f \cdot \psi), T_f \cdot \overline{\psi}) \]

**Proof.** Let \( \xi = \log(h_0^{-1} h_f) \). Then we have a family \( \{ f_t = e^{\xi t} \} \) of \( \text{Herm}^+(E, h_0) \) satisfying \( f_1 = f \) and \( f_0 = \text{id}_E \). Since \( f_t = e^{\xi t} \) commutes with \( T_f \), we have

\[ h_0(\mathcal{K}_f(\psi), T_f \cdot \overline{\psi}) = h_0(\text{Ad}_{f_t^{-1}}(\mathcal{K}_f(\psi)), T_f \cdot \overline{\psi}) \]

Since \( \text{Ad}_{f_t^{-1}}(\mathcal{K}_f(\psi)) = \mathcal{K}_{D_t} \), We have

\[ \text{Ad}_{f_t^{-1}}(\mathcal{K}_f(\psi)) = \mathcal{K}_{D_t} - \mathcal{K}_{D_0}. \]

Then we have

\[ \mathcal{K}_{D_t}(\psi) - \mathcal{K}_{D_0}(\psi) = \int_0^1 \frac{d}{dt} \mathcal{K}_{D_{t'}}(\psi) dt \]

(8.24)
Since $\check{D}_f$ is a family of generalized Hermitian connections with respect to the fixed $h_0$, we apply (8.4) to obtain

$$\int_0^1 \pi_{U-n}^{\text{Herm}} \frac{d}{dt} K_{\check{D}_f}(\psi) ds = \int_0^1 \pi_{U-n}^{\text{Herm}} \frac{d}{dt} \check{A}_{f_t} \cdot \psi$$

(8.25)

$$= \int_0^1 \pi_{U-n}^{\text{Herm}} \text{Ad}_{\check{A}_{f_t}} \left( d\check{A}_{f_t} \cdot \psi \right)$$

(8.26)

Since $T_f \in \text{Herm}(E, h_0)$, we have

$$\langle h_0(\mathcal{K}_f(\psi) - \mathcal{K}_0(\psi), T_f \cdot \overline{\psi}) = h_0^{\text{top}} \left( \int_0^1 \frac{d}{dt} K_{\check{D}_f}(\psi) dt, T_f \cdot \overline{\psi} \right)$$

(8.28)

$$= h_0^{\text{top}} \left( \int_0^1 \pi_{U-n}^{\text{Herm}} \frac{d}{dt} K_{\check{D}_f}(\psi) dt, T_f \cdot \overline{\psi} \right)$$

(8.29)

$$= \int_0^1 h_0^{\text{top}} (\text{Ad}_{\check{A}_{f_t}} \frac{d}{dt} \check{A}_{f_t} \cdot \psi), T_f \cdot \overline{\psi}) dt$$

(8.30)

$$= \int_0^1 h_0^{\text{top}} (\partial_{\check{A}_{f_t}} \frac{d}{dt} \check{A}_{f_t} \cdot \psi), T_f \cdot \overline{\psi}) dt$$

(8.31)

Thus we obtain the result.

**Remark 8.10.** For instance, we will apply Proposition 8.9 to $T_f = \log f$ or $f^\sigma$, for $0 \leq \sigma \leq 1$.

9 Construction of Einstein-Hermitian metrics on stable generalized holomorphic bundles

9.1 The continuity method

We shall use the continuity method to obtain Einstein-Hermitian metrics on polystable generalized holomorphic bundles. We will use the same notations as before. Given an Hermitian metric $h_0$ on a generalized holomorphic vector bundle $E$, we denote the canonical generalized connection by $D_0 = \partial_0 + \overline{\partial}^E D_0$ and the mean curvature by $K_0(\psi)$. An Hermitian metric $h_f$ is given by $h_f(s_1, s_2) = h_0(f s_1, s_2)$, where $f \in \text{Herm}^+(E, h_0)$ and $s_1, s_2 \in \Gamma(E)$. Then we have the canonical generalized connection $D_f$ which is associated with $h_f$ and $K_f(\psi)$ is the mean curvature of $D_f$. Then the Einstein-Hermitian condition is given by

$$K_f(\psi) = \lambda \text{id}_E \psi$$
where $\lambda$ is the Einstein constant. By using $\psi$, $\text{Herm}(E) \otimes U^{-n}$ is identified with $\text{Herm}(E)$. For abuse of notation, we often consider $K_f(\psi)$ as the Hermitian endmorphism under the identification. For $\varepsilon \in [0, 1]$, we introduce the following equation on which the continuity method is applied

$$L_\varepsilon(f) := K_f(\psi) - \lambda \text{id}_E + \varepsilon(\log f) = 0$$  \hfill (9.1)

Note that the solution of the equation gives an Einstein-Hermitian metric if $\varepsilon = 0$. We define the subset $S \subset [0, 1]$ by

$$S := \{ \varepsilon \in [0, 1] | \text{the equation [9.1] has a solution} \}$$

Let $\hat{K}_f(\psi) := K_f(\psi) - \lambda \text{id}_E$. Our construction of the solution is divided into four steps

**Step 0.** The subset $S$ contains 1.

**Step 1.** $S \subset [0, 1]$ is a nonempty, open set.

**Step 2.** $(0, 1] \subset S$

**Step 3.** $S = [0, 1]$

We shall show these steps one by one in the following subsections.

### 9.2 Preliminary results for Step 0 and Step 1

According to the decomposition $(TM \oplus T^*M)^\mathbb{C} = \mathcal{L}_\mathcal{J} \oplus \overline{\mathcal{L}_\mathcal{J}}$, the exterior derivative $d$ is decomposed into $d = \partial + \overline{\partial}$. In a generalized Kähler manifold, we also use the decomposition $\mathcal{L}_\mathcal{J} = \mathcal{L}_\mathcal{J}^+ \oplus \mathcal{L}_\mathcal{J}^-$ to obtain $\partial = \delta_+ + \delta_-$ and $\overline{\partial} = \overline{\delta}_+ + \overline{\delta}_-$. These four differential operators act on differential forms and we have the adjoint operators $\delta_\pm$ and $\overline{\delta}_\pm$. We denote by $\Delta_d$ the Laplacian of $d$. We also have the Laplacians $\Delta_{\mathcal{J}_d}$ and $\Delta_{\overline{\mathcal{J}}_d}$, respectively. Let $\Delta_{\delta_\pm}$ be the Laplacians $\delta_+ \delta_\pm^* + \delta_- \delta_\pm^*$ which acts on differential forms on $M$. Then the following generalized Kähler identity holds $\delta_+^* = -\delta_-$ and $\delta_-^* = \delta_+$. By the generalized Kähler identity, we have

$$\Delta_d = 2\Delta_{\delta_\pm} = 2\Delta_{\overline{\mathcal{J}}_d} = 2\Delta_\mathcal{J} = 4\Delta_{\delta_+} = 4\Delta_{\delta_-} = 4\Delta_{\overline{\delta}_+} = 4\Delta_{\overline{\delta}_-}$$

Let $h$ be an Hermitian metric on $E$ and $\xi \in \text{Herm}(E, h)$. We denote by $h_s(e_1, e_2) := h(e^{\xi_s}e_1, e_2)$, where $s$ is a parameter and $h_0 = h$. Let $(E, \theta^{\mathbb{C}})$ be a generalized holomorphic vector bundle and $\mathcal{D}_s = d + \mathcal{A}_s$ the 1-parameter family of canonical generalized Hermitian connections with respect to $h_s$. Then we have

**Lemma 9.1.**

$$\dot{\mathcal{A}} = \frac{d}{ds} \mathcal{A}_s|_{t=0} = \mathcal{D}_h^{1,0} \xi$$

**Proof.** From Lemma 8.1, the connection form $\mathcal{A}_s$ is given by

$$\mathcal{A}_s = (e^{-\xi_s\partial_\mathcal{J}}e^{\xi_s})s + e^{-\xi_s} \mathcal{A}_h^{1,0} e^{\xi_s},$$

where $\mathcal{A}_h = \mathcal{A}_0$ and $\mathcal{D}_h = d + \mathcal{A}_h$. Thus we have

$$\frac{d}{ds} \mathcal{A}_s|_{t=0} = \partial_\mathcal{J}\xi + [\mathcal{A}_h^{1,0}, \xi] = \mathcal{D}_h^{1,0} \xi.$$

\[\square\]

**Proposition 9.2.**

$$\pi^{\text{Herm}_h} \left( \frac{d}{ds} \hat{K}_f(\psi)|_{t=0} \right) = \Delta_h(\xi \cdot \psi),$$

where $\Delta_h$ is the Laplacian $\mathcal{D}_h^{1,0}(\mathcal{D}_h^{1,0})^* + (\mathcal{D}_h^{1,0})^* \mathcal{D}_h^{1,0}$ actin on $\text{End}(E)$.
Proof. The key point is Proposition 8.5:

\[ \pi_{Herm} \left( \frac{d}{ds} K_{h_s}(\psi) \big|_{s=0} \right) = \pi_{U-n} d^{D_+} (\hat{A} \cdot \psi), \]

where \( \hat{A} = \frac{d}{ds} A_{h_s} \big|_{s=0} \in \text{End}(E) \otimes (TM \oplus T^* M) \). From Lemma 9.1, we have

\[ \pi_{Herm} \left( \frac{d}{ds} K_{h_s}(\psi) \big|_{s=0} \right) = \pi_{Herm} U^{D_+} \]

Then we also have

\[ \pi_{U-n} d^{D_+} (\hat{D}^{1,0}_h (\xi \cdot \psi) \big|_{s=0}) = \pi_{U-n} (\hat{D}^{1,0}_h (\xi \cdot \psi) \big|_{s=0}) = 0. \]

Thus we have

\[ \pi_{Herm} \left( \frac{d}{ds} K_{h_s}(\psi) \big|_{s=0} \right) = \pi_{Herm} U^{D_+} \]

since \( \xi \cdot \psi \in U^{n} \), it follows \( (\hat{D}^{1,0}_h \xi \cdot \psi) = 0 \). Thus we have

\[ \pi_{Herm} \left( \frac{d}{ds} K_{h_s}(\psi) \big|_{t=0} \right) = \pi_{U-n} (\hat{D}^{1,0}_h (\xi \cdot \psi) \big|_{t=0}) \]

(9.2)

Applying the following generalized Kähler identity

\[ (\hat{D}^{1,0}_h - \hat{D}^{1,0}_h) = (\hat{D}^{1,0}_h - \hat{D}^{1,0}_h)^*, \]

We obtain

\[ \pi_{Herm} \left( \frac{d}{ds} K_{h_s}(\psi) \big|_{t=0} \right) = (\hat{D}^{1,0}_h \xi \cdot \psi) \]

(9.4)

(9.5)

since \( (\hat{D}^{1,0}_h)^*(\xi \psi) = 0 \) and \( \Delta_h \) is a real self adjoint operator.

9.3 Step 0

We use the same notations as before. Let \( Q := \hat{K}_f(\psi) - \hat{K}_{f_0}(\psi) \). Then we have

Proposition 9.3.

\[ \text{tr } Q = \Delta \text{tr } \log f \]

Proof. From Proposition 8.7, the trace of \( Q \) is given by

\[ \text{tr } Q = \int_0^1 \pi_{U-n} \left( \overline{\partial} \text{tr } \hat{A}_{f_t} \cdot \psi \right) dt \]

Then we see

\[ \text{tr } \hat{A}_{f_t} = \text{tr } \frac{d}{dt} f_t^{-1} (\partial_t f_t) \psi = \partial (\text{tr } f_t)^{\cdot} \]

Then we have

\[ \text{tr } Q = \int_0^1 \pi_{U-n} \left( \overline{\partial} (\text{tr } f_t) \cdot \psi \right) dt \]

By using the generalized Kähler identity, we have

\[ \pi_{U-n} \overline{\partial} (\text{tr } f_t)^{\cdot} \psi = \Delta \text{tr } f_t \psi, \]

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where $4\Delta$ is the ordinary Laplacian, which does not depend on $h$. Since $\text{tr} \log f_t$ is a real function, $\pi_{\text{Herm}}(t) \Delta \text{tr} \log f_t \psi = \Delta \text{tr} \log f_t \psi$. Thus we have

$$\text{tr} \, Q = \int_0^1 (\Delta \text{tr} \log f_t) dt = \Delta \text{tr} \log f$$

Hence we obtain the result.

**Lemma 9.4.** For every Hermitian metric $h_0$, there exists an Hermitian metric $h_f$ such that

$$\text{tr} \hat{K}_f(\psi) = 0.$$ 

**Proof.** Since

$$\int_M \langle \text{tr} \hat{K}_{f_0}(\psi), \overline{\psi} \rangle_s = 0,$$

we have a solution $f$ such that

$$\Delta \text{tr} \log f + \text{tr} K_f(\psi) = 0$$

Thus we have $\text{tr} \hat{K}_f(\psi) = 0$.

Hence it follows from Lemma 9.4 that we can choose $h_0$ which satisfies $\text{tr} \hat{K}_o(\psi) = 0$.

Then the trace of the equation (9.1) is written as

$$\text{tr} (K_f(\psi) - \lambda \text{id}_E) + \Delta \text{tr} \log f + \varepsilon \text{tr} \log f = 0$$

Since $\hat{K}_f(\psi) := (K_f(\psi) - \lambda \text{id}_E)$, then we have the following equation:

$$\text{tr} \hat{K}_f(\psi) + \Delta \text{tr} \log f + \varepsilon \text{tr} \log f = 0 \quad (9.7)$$

**Proposition 9.5.** If $f$ satisfies (9.1) for $\varepsilon > 0$, then $\text{tr} \log f = 0$, that is, $\det f = 1$.

**Proof.** Since $f$ satisfies (9.7) and $\text{tr} \hat{K}_o(\psi) = 0$, we have

$$\Delta \text{tr} \log f + \varepsilon \text{tr} \log f = 0$$

Since $\Delta + \varepsilon$ is invertible for $\varepsilon > 0$, it follows $\text{tr} \log f = 0$. Then $\det f = 1$.

**Proposition 9.6.** There exists an Hermitian metric $h_0$ and $f_1 \in \text{Herm}^+(E, h_0)$ such that $f_1$ satisfies (9.1) for $\varepsilon = 1$ and $\text{tr} \hat{K}_o(\psi) = 0$

**Proof.** From Lemma 9.4, we have an Hermitian metric $h$ such that $\text{tr} \hat{K}_h(\psi) = 0$. Let $\hat{K}_h(\psi) = \hat{K}_h \psi$, where $\hat{K}_h \in \text{Herm}(E, h_0)$. Then we define an Hermitian metric $h_0$ by

$$h_0 = h(e^{-\hat{K}_h}, \cdot)$$

Let $f_1 := e^{-\hat{K}_h}$. Then we have

$$\hat{K}_h + \log f_1 = 0$$

Since $h = h_0(e^{-\hat{K}_h}, \cdot)$ and $\hat{K}_h(\psi) = \hat{K}_{f_1}(\psi)$, we have a $h_0$ and $f$ satisfying (9.71) for $\varepsilon = 1$. Since $\text{tr} \log f_1 = -\text{tr} \hat{K}_h = 0$, we have

$$\text{tr} \hat{K}_{f_1}(\psi) - \text{tr} \hat{K}_{h_0}(\psi) = \Delta \text{tr} \log f_1 = 0$$

Thus we see $\text{tr} \hat{K}_{h_0}(\psi) = 0$. 

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9.4 Step 1

For a smooth section \( f \in \text{Herm}(E, h_0) \) and an integer \( s \geq 0 \), we define the norm \( \|f\|_{L^2_s}^2 \) by

\[
\|f\|_{L^2_s}^2 := \sum_{i=0}^{s} \int_M |\nabla^i f|^2 \text{vol}_M,
\]

where \( |\nabla^i f| \) denotes the point-wise norm of the \( i \)-th covariant derivative. We denote by \( H_s \) the Sobolev space \( L^2_s\text{Herm}(E, h_0) \) which is the completion of \( \text{Herm}(E, h_0) \) with respect to the Sobolev norm \( \|\|_{L^2_s} \).

Let \( L(f, \varepsilon) \) be the equation \( 9.1 \).

\[
L(f, \varepsilon) := \mathcal{K}_f(\psi) - \lambda(\text{id}_E) + (\varepsilon \log f) = 0
\]

Since \( L(f, \varepsilon) \in \text{Herm}(E, h_f) \), it follows from \( 8.8 \) that

\[
\text{Ad}_{\frac{d}{ds}} L(f, \varepsilon) \in \text{Herm}(E, h_0).
\]

Then we define an operator \( F : H_{s+2} \times [0, 1] \rightarrow H_s \) by

\[
F(f, \varepsilon) := \text{Ad}_{\frac{d}{ds}} L(f, \varepsilon),
\]

where we identify \( H_s \cdot \psi \) with the Sobolev space \( H_s \). Let \( f \) be a solution of \( L(f, \varepsilon) = 0 \). Then the differential \( DF \) of \( F \) along \( f \)-direction at \( (f, \varepsilon) \) is given by

\[
DF(f, \varepsilon)(\xi) = \frac{d}{ds} F (f \epsilon \xi_s, \varepsilon)|_{s=0},
\]

where \( \xi \in H_{s+2} \). Since \( L(f, \varepsilon) = 0 \) and \( L(f, \varepsilon) \in \text{Herm}(E, h_0) \), applying \( 8.8 \) and Proposition \( 9.2 \), we have

\[
DF(f, \varepsilon)(\xi) = \pi_{\text{Herm}h_0} \text{Ad}_{\frac{d}{ds}} \left( \frac{d}{ds} \mathcal{K}_f(\psi) + \varepsilon D(\log f)(\xi) \right) + \left[ \frac{d}{ds}, L(f, \varepsilon) \right] (9.8)
\]

\[
= \text{Ad}_{\frac{d}{ds}} \pi_{\text{Herm}h_f} \left( \frac{d}{ds} \mathcal{K}_f(\psi) + \varepsilon D(\log f)(\xi) \right) (9.9)
\]

\[
= \text{Ad}_{\frac{d}{ds}} \left( \triangle_f(\xi \cdot \psi) + \varepsilon D(\log f)(\xi) \right) (9.10)
\]

where \( \triangle_f := \triangle_{h_f} \) is the Laplacian as in Proposition \( 9.2 \).

In order to show that \( S \) is an open set, we shall show that the operator \( \triangle_f + \varepsilon D(\log f) : H_{s+2} \rightarrow H_s \) is an isomorphism. At first we need the following Proposition in order to show that the derivative \( D(\log f) \) gives an injective map.

**Proposition 9.7.** If \( h_f(D(\log f))(\xi, \xi) = 0 \), then \( \xi = 0 \).

**Proof.** Let \( \{f_t\} \) be a smooth family of \( \text{Herm}(E, h_0) \). We shall diagonalize \( f_t \) by using a unitary basis of \( E \) on an open dense subset \( W \). It is already know that there exits an open dense set \( W \) in \( M \) such that there is a unitary basis of \( \{s_i(t)\}_{t=1}^n \) of \( E|_W \) with respect to \( h_t := h_{f_t} \) satisfy the followings: \( s_i(t) \) are smoothly depending on \( t \), and \( f_t \) is written as

\[
f_t = \sum_i e^{\lambda_i(t)} s_i(t) \otimes s_i(t)^*,
\]

where \( \{s_i(t)^*\}_{t=1}^n \) denotes the dual basis of \( \{s_i(t)\}_{t=1}^n \). (This local diagonalization method is well-explained in Appendix 7.4 \[30\].) (see Proposition \( 9.12 \).)
Then \( \log f_t \) is given by
\[
\log f_t = \sum_i \lambda_i(t) s_i(t) \otimes s_i(t)^*.
\]

Then \( D(\log f)(\xi) \) is
\[
D(\log f)(\xi) = \frac{d}{dt} \log f_t|_{t=0},
\]
where \( \xi = f_t^{-1} f_t \). The derivative of \( \{s_i(t)\} \) is given by \( \frac{d}{dt} s_i(t) = \sum_{i,j} P_{i,j} s_j(t) \) for \( P = (P_{i,j}) \). Then we have
\[
D(\log f)(\xi) = \sum_i \lambda_i'(0) s_i(0) \otimes s_i(0)^* + [P, \log f]
\]
\[
= \sum_i \lambda_i'(0) s_i(0) \otimes s_i(0)^* + \sum_{i \neq j} P_{i,j}(\lambda_j(0) - \lambda_i(0)) s_i(0) \otimes s_j(0)^* \tag{9.11}
\]
\[
\sum_i \lambda_i'(0) s_i(0) \otimes s_i(0)^* + \sum_{i \neq j} P_{i,j}(\lambda_j(0) - \lambda_i(0)) s_i(0) \otimes s_j(0)^* \tag{9.12}
\]

Since \( \xi = f_t^{-1} f_t \), \( \xi \) is given by
\[
\xi = \sum_i \lambda_i'(0) s_i(0) \otimes s_i(0)^* + \sum_{i \neq j} P_{i,j}(e^{\lambda_i(0)} - \lambda_i(0)) s_i(0) \otimes s_j(0)^*
\]

We denote by \( x_{i,j} \) the real number \((\lambda_j(0) - \lambda_i(0))\). Then we have
\[
h_f(D(\log f))(\xi, \xi) = \sum_i |\lambda_i'(0)|^2 + \sum_{i \neq j} |P_{i,j}|^2 x_{i,j}(e^{x_{i,j}} - 1) \tag{9.13}
\]

Since \( x_{i,j}(e^{x_{i,j}} - 1) \geq 0 \), \( h_f(D(\log f))(\xi, \xi) = 0 \) implies that \( \xi = 0 \).

Then we have

**Proposition 9.8.** The operator \( \triangle_f + \varepsilon D(\log f) : H_{s+2} \to H_s \) is an isomorphism.

**Proof.** The operator \( \triangle_f + \varepsilon D(\log f) \) is an elliptic operator of index 0. Thus it suffices to show that the operator \( \triangle_f + \varepsilon D(\log f) \) is injective. We have
\[
\int_M h_f((\triangle_f + \varepsilon D(\log f))\xi \psi, \xi \overline{\psi}) = \int_M h_f(\triangle_f \xi \psi, \xi \overline{\psi}) + \varepsilon h_f(D(\log f)(\xi) \psi, \xi \overline{\psi}) \tag{9.14}
\]
\[
= \int_M h_f((D_{1,0}^{h_f}) \xi \psi, (D_{1,0}^{h_f}) \xi \overline{\psi}) + \varepsilon h_f(D(\log f)\xi \psi, \xi \overline{\psi}) \tag{9.15}
\]
\[
= \|D_{1,0}^{h_f}\xi\|_{L^2}^2 + \int_M h_f(D(\log f)\xi \psi, \xi \overline{\psi}). \tag{9.16}
\]

Thus \( (\triangle_f + \varepsilon D(\log f))\xi = 0 \) implies that \( h_f(D(\log f)\xi, \xi) = 0 \). From Proposition 9.7 we have \( \xi = 0 \). Hence \( \triangle_f + \varepsilon D(\log f) : H_{s+2} \to H_s \) is injective. Since \( \triangle_f + \varepsilon D(\log f) : H_{s+2} \to H_s \) is an elliptic operator whose index is 0. Then the result follows.

**Proposition 9.9.** \( S \subset (0, 1] \) is an open set.

**Proof.** From the implicit function theorem, we obtain that the region \( S \) is open set in \((0, 1] \).
9.5 Preliminary results for the Step 2 and Step 3

An Hermitian metric $h$ on $E$ yields a bilinear form $h$ of $\text{End}(E) \otimes \bigwedge^* T^*_M$ by

$$h(A \otimes \alpha, B \otimes \beta) := i^r \text{Re}\{(\alpha \wedge \beta) \langle A B^* \rangle \} \in \bigwedge^* T^*_M,$$

where $A, B \in \text{End}(E)$, $B^*$ is the adjoint of $B$ with respect to $h$ and $\alpha, \beta \in \bigwedge^* T^*_M$ and $\sigma$ is the Clifford involution. We denote by $h^{\text{top}}$ the $2n$-component of $h$, i.e., the form of top degree. Since $(\log f)\psi \in \text{Herm}_f \cap U^{-n}$, we have

**Lemma 9.10.**

$$h^{\text{top}}(\pi^{\text{Herm}}_{U^{-n}} E^c, (\log f)\psi) = h^{\text{top}}(E, (\log f)\psi)$$

**Proof.** The result follows from Lemma 8.8, applying $T_f = \log f$.

The following three Propositions are important for Step 2 and Step 3. Let $\alpha = f \text{vol}_M$ and $\beta = g \text{vol}_M$ be two differential forms of top degree. Then if $f > g$, we denote it by $\alpha > \beta$.

**Proposition 9.11.** Let $h_t = h_{f_t}$ be a family of Hermitian metrics with $f := f_1$.

We denote by $Q$ the difference $K_f(\psi) - K_{f_0}(\psi)$. Then we have

$$h^{\text{top}}_0(Q, (\log f)\psi) \geq \frac{1}{2} |\Delta| \log f|^2 \text{vol}_M,$$

**Proof.** From Proposition 8.9 and $f_1 = f$, we have

$$h^{\text{top}}(Q, (\log f)\psi) = \int_0^1 h^{\text{top}}(\bar{\partial} E, (\partial f)(\psi))dt, (\log f)\psi)dt$$

$$= h^{\text{top}}(\bar{\partial} E, (\partial f)(\psi)) - h^{\text{top}}(\bar{\partial} E, (\partial f)(\psi))$$

Then we have

$$h^{\text{top}}(Q, (\log f)\psi) = \pi^{\text{top}} \bar{\partial} E h((f^{-1}(\partial f)(\psi))), (\log f)\psi) + h^{\text{top}}((f^{-1}(\partial f)(\psi))), \bar{\partial} E (\log f)\psi)$$

We shall use the method of "local diagonalization" 30 again.

**Proposition 9.12.** Let $(E, h)$ be an Hermitian vector bundle and $f$ an Hermitian endmorphism with respect to $h$. Then there exists an open dense subset $W \subset M$ such that for every $x \in W$ the following holds: There exists an open neighborhood $U$ of $x$ a unitary basis $\{s_1\}$ for $E$ defined over $U$ and functions $\lambda_i \in C^\infty (U)$ such that

$$f(y) = \sum_{i=1}^r e^{\lambda_i(y)} s_i(y) \otimes s^i(y),$$

where $\{s^i(y)\}$ denotes the dual basis and $y \in U$, that is, $f$ is given in the following form of the diagonal matrix:

$$f = \begin{pmatrix}
e^\lambda_1 & \cdots & 0 \\
0 & e^{\lambda_2} & \cdots \\
& \vdots \\
0 & \cdots & e^{\lambda_r}
\end{pmatrix}$$

(9.21)
Then we have
\[ \partial_0(f\psi) = \partial(f\psi) + [A_0, f\psi] \] 
(9.22)
\[ = \sum_i \partial(\lambda_i\psi)e^{\lambda_i}s_i \otimes s_i^* + \sum_{i,j} A_{i,j}\psi(e^{\lambda_j} - e^{\lambda_i})s_i \otimes s_j^* \] 
(9.23)
\[ f^{-1}\partial_0(f\psi) = \sum_i \partial(\lambda_i\psi)s_i \otimes s_i^* + \sum_{i,j} A_{i,j}\psi(e^{\lambda_j} - \lambda - 1)s_i \otimes s_j^* \] 
(9.24)
\[ \partial_0(\log f\psi) = \partial\lambda_i s_i \otimes s_i^* + [A, \log f]\psi \] 
(9.25)
\[ \text{Thus we have} \]
\[ h\left(f^{-1}\partial_0(f\psi), \log f\psi\right) = \sum_i \lambda_i(\partial\lambda_i)\psi \wedge \sigma \psi + \pi_{\text{top}}\frac{\partial h(f^{-1}(\partial_0(f\psi))), \overline{\partial}^E(\log f)\psi)}{\sqrt{\text{vol} M}} \] 
(9.26)
Since \( \text{tr}(f^{-1}[A^{1,0}, f] \log f) = 0 \), the first term of the right hand side of (9.26) is given by
\[ \sum_i \lambda_i(\partial\lambda_i)\psi \wedge \sigma \psi + \pi_{\text{top}}\frac{\partial h(f^{-1}(\partial_0(f\psi))), \overline{\partial}^E(\log f)\psi)}{\sqrt{\text{vol} M}} \] 
We shall show that the second term of the R.H.S of (9.26) \( h((f^{-1}(\partial_0(f\psi))), \overline{\partial}^E(\log f)\psi) \) is greater than or equal to 0. Applying the local diagonalization, we have
\[ h((f^{-1}(\partial_0(f\psi))), \overline{\partial}^E(\log f)\psi) \geq 0 \] 
(9.29)
\[ \text{Thus Proposition 9.11 follows.} \]

**Lemma 9.13.**
\[ h(f^{-1}[A^{1,0}, f], [A^{1,0}, \log f]) \geq 0 \]
**Proof.** The result follows from the direct calculation by using the method of local diagonalization.

Then the second term of the R.H.S of (9.26) \( h((f^{-1}(\partial_0(f\psi))), \overline{\partial}^E(\log f)\psi) \geq 0 \). Thus Proposition 9.11 follows.

**Proposition 9.14.** For \( \sigma > 0 \), we have
\[ \pi_{\text{top}}^* dh(f^{-1}(\partial_0(f\psi)), f^\sigma\psi) = \frac{1}{\sigma} \langle \Delta (\text{tr} f^\sigma\psi), \overline{\psi}^* \rangle, \]
where \( \pi_{\text{top}}^* \) denotes the projection to \( \wedge^{2n} T_M^* \). Note that \( \Delta := \Delta_{\partial}^\sigma \) of \( (M, J, J_\psi) \) does not depend on \( h_f \).

**Proof.** By using the generalized Kähler identity, we obtain
\[ \frac{1}{\sigma} \langle \Delta (\text{tr} f^\sigma\psi), \overline{\psi}^* \rangle = \frac{1}{\sigma} \langle \overline{\partial}_J \partial_J (\text{tr} f^\sigma\psi), \overline{\psi}^* \rangle. \]
We see that
\[ \text{tr} f^{-1}[A^{1,0}, f] = \text{tr} [A^{1,0}, f^\sigma] = 0 \]
we have
\[ \operatorname{tr}(f^\sigma \partial_0 f) = \operatorname{tr}(f^\sigma \partial f) + \operatorname{tr} f^\sigma - [A^{1,0}, f] = \operatorname{tr}(f^\sigma \partial f) = \frac{1}{\sigma} \operatorname{tr}(\partial_0 f^\sigma) \]

The left hand side is given by
\[ \pi_{\text{top}} d h(f^{-1}(\partial_0 f \psi), f^\sigma \psi) = \pi_{\text{top}} d h(f^\sigma - (\partial_0 f \psi)), \psi) \]
\[ = \pi_{\text{top}} h(\overline{\partial}(f^\sigma - (\partial_0 f \psi)), \psi) \]
\[ = h(\pi_U - (\overline{\partial} f^\sigma - (\partial_0 f \psi)), \psi) \]
\[ = \operatorname{tr}(\overline{\partial}(f^\sigma - (\partial_0 f \psi)), \psi) \]
\[ = \frac{1}{\sigma} \operatorname{tr}(\overline{\partial} \partial_0 f^\sigma \psi, \psi) \]
\[ = \frac{1}{\sigma} (\overline{\partial}_f \partial f \psi, \psi) \]

Thus we obtain
\[ \pi_{\text{top}} d h(f^{-1}(\partial_0 f \psi), f^\sigma \psi) = \frac{1}{\sigma} \langle \Delta (\operatorname{tr} f^\sigma \psi), \psi \rangle, \]
\[ \square \]

**PROPOSITION 9.15.** Let \( f \in \text{Herm}(E, h_0) \) and \( 0 < \sigma \leq 1 \). Then we have the followings:

1. \[ h_0^{\text{top}}(f^{-1}(\partial_0 f \psi), \partial_0(f^\sigma \psi)) \geq |f^{-\sigma}(\partial_0 f^\sigma \psi)|^2_{h_0} \text{vol}_M \]

2. If \( f \) satisfies the equation (9.4) for some \( \varepsilon > 0 \), then we have
\[ \frac{1}{\sigma} \Delta (\operatorname{tr} f^\sigma \psi) \text{vol}_M + \varepsilon h_0^{\text{top}}((\log f) \psi, f^\sigma \psi) + |f^{-\sigma}(\partial_0 f^\sigma \psi)|^2_{h_0} \text{vol}_M \leq -h_0^{\text{top}}(\tilde{K}_0(\psi), f^\sigma \psi) \]

**PROOF.** (1) We use the same notations as before.
\[ \partial_0(f^\sigma \psi) = \partial(f^\sigma \psi) + [A^{1,0}, f^\sigma] \psi \]

Then the left hand side of (1) is given by
\[ h_0^{\text{top}}(f^{-1}(\partial_0 f \psi), \partial_0(f^\sigma \psi)) = h_0^{\text{top}}(f^{-1}(\partial f \psi) + f^{-1}[A^{1,0}, f] \psi, \partial(f^\sigma \psi) + [A^{1,0}, f^\sigma] \psi) \]
\[ = h_0^{\text{top}}(f^{-1}(\partial f \psi), \partial(f^\sigma \psi)) + h_0^{\text{top}}(f^{-1}(\partial f \psi), [A^{1,0}, f^\sigma] \psi) \]
\[ + h_0^{\text{top}}(f^{-1}[A^{1,0}, f] \psi, \partial(f^\sigma \psi)) + h_0^{\text{top}}(f^{-1}[A^{1,0}, f] \psi, [A^{1,0}, f^\sigma] \psi) \]

Then we see
\[ h_0^{\text{top}}(f^{-1}(\partial f \psi), [A^{1,0}, f^\sigma] \psi) = 0 \]
\[ h_0^{\text{top}}(f^{-1}[A^{1,0}, f] \psi, \partial(f^\sigma \psi)) = 0 \]

The right hand side of (1) is given by
\[ |f^{-\sigma}(\partial_0 f^\sigma \psi)|^2_{h_0} \text{vol}_M = h_0^{\text{top}}(f^{-\sigma}\partial_0(f^\sigma \psi), \partial_0(f^\sigma \psi)) \]
\[ = h_0^{\text{top}}(f^{-\sigma}\partial(f^\sigma \psi), \partial(f^\sigma \psi)) \]
\[ + h_0^{\text{top}}(f^{-\sigma}[A^{1,0}, f^\sigma] \psi, [A^{1,0}, f^\sigma] \psi) \]

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We also have\[ e \in (9.37) \text{ and } (9.42) \text{ are written as}\]
\[
h^0_0(f^{-1}(\partial f \psi), \partial (f^\sigma \psi)) = \sigma h^0_0(f^{\sigma-2} \partial (f \psi), \partial (f \psi)) \tag{9.44}
\]
\[
h^0_0(f^{-\sigma} \partial_0 (f^\sigma \psi), \partial_0 (f^\sigma \psi)) = \sigma^2 h_0(f^{\sigma-2} \partial (f \psi), \partial (f \psi)) \tag{9.45}
\]
since $f \in \text{Herm}(E)$. Since $0 \leq \sigma \leq 1$, we have
\[
h^0_0(f^{-1}(\partial f \psi), \partial (f^\sigma \psi)) \geq h^0_0(f^{\sigma-2} \partial (f \psi), \partial (f \psi)).
\]

In order to estimate the second term of $\text{(9.38)}$ and $\text{(9.43)}$, we diagonalize $f$ at each point of $M$ as in $\text{(9.21)}$. Let $A^{1,0} = \sum_k A^{1,0}_{(k)} e_k$, where $\{e_k\} \in \mathcal{L}_\mathcal{J}$ denotes the orthonormal basis. Then $(i,j)$-components of the matrix $[A^{1,0}_{(k)}, f], f^{-1}[A^{1,0}_{(k)}, f]$ and $[A^{1,0}_{(k)}, f^\sigma]$ are given by

\[
\begin{align*}
[A^{1,0}_{(k)}, f]_{i,j} &= a_{i,j}(e^{\lambda_j} - e^{\lambda_i}) \tag{9.46} \\
f^{-1}[A^{1,0}_{(k)}, f]_{i,j} &= a_{i,j}(e^{\lambda_j} - e^{\lambda_i}) \tag{9.47} \\
[A^{1,0}_{(k)}, f^\sigma]_{i,j} &= a_{i,j}(e^{\sigma \lambda_j} - e^{\sigma \lambda_i}) \tag{9.48}
\end{align*}
\]

Since $[A^{1,0}_{(l)}, f^\sigma]$ is Hermitian,
\[
h^0_0(f^{-1}[A^{1,0}_{(l)}, f] \psi, [A^{1,0}_{(l)}, f^\sigma] \psi) = \sum_{k,l} \text{tr}(f^{-1}[A^{1,0}_{(k)}, f][A^{1,0}_{(l)}, f^\sigma]) \langle e_k \cdot \psi, \bar{e}_l \cdot \bar{\psi}\rangle_s \tag{9.49}
\]
\[
= \sum_k 2\text{tr}(f^{-1}[A^{1,0}_{(k)}, f][A^{1,0}_{(k)}, f^\sigma])\text{vol}_M, \tag{9.50}
\]
since $i^n \langle e_k \cdot \psi, \bar{e}_l \cdot \bar{\psi}\rangle_s = 2\delta_{k,l} \text{vol}_M$. Then we have
\[
h^0_0(f^{-1}[A^{1,0}_{(k)}, f] \psi, [A^{1,0}_{(k)}, f^\sigma] \psi) = \sum_{i,j} (e^{\lambda_j} - e^{\lambda_i})(e^{\sigma \lambda_j} - e^{\sigma \lambda_i}) |a_{i,j}|^2 \text{vol}_M \tag{9.51}
\]
since $a_{i,j} = -a_{j,i}$. $\text{(9.43)}$ is also given by
\[
h^0_0(f^{-\sigma}[A^{1,0}_{(k)}, f^\sigma] \psi, [A^{1,0}_{(k)}, f^\sigma] \psi) = \sum_k \text{tr}(f^{-\sigma}[A^{1,0}_{(k)}, f^\sigma][A^{1,0}_{(k)}, f^\sigma])\text{vol}_M \tag{9.52}
\]
We also have
\[
\sum_k \text{tr}(f^{-\sigma}[A^{1,0}_{(k)}, f^\sigma][A^{1,0}_{(k)}, f^\sigma])\text{vol}_M = \sum_{i,j} (e^{\sigma \lambda_j} - e^{\sigma \lambda_i}) |a_{i,j}|^2 \text{vol}_M \tag{9.53}
\]
Let $\lambda = \lambda_j - \lambda_i$. Since $(e^{\lambda_1} - e^{\lambda_2})(e^{\sigma \lambda_1} - e^{\sigma \lambda_2}) \geq (e^{\lambda_1} - e^{\lambda_2})(e^{\sigma \lambda_1} - e^{\sigma \lambda_2})$ for $0 \leq \sigma \leq 1$, it follows from $\text{(9.51)}$ and $\text{(9.53)}$ that,
\[
h^0_0(f^{-1}[A^{1,0}_{(k)}, f] \psi, [A^{1,0}_{(k)}, f^\sigma] \psi) \geq h^0_0(f^{-\sigma}[A^{1,0}_{(k)}, f^\sigma] \psi, [A^{1,0}_{(k)}, f^\sigma] \psi)
\]
Hence we have
\[
h^0_0(f^{-1}(\partial_0 f \psi), \partial_0 (f^\sigma \psi)) \geq |f |\tilde{\mathbf{v}}^2 (\partial_0 f^\sigma \psi)|^2 h_0^0 \text{vol}_M
\]

Proof of (2) : Since $f$ satisfies the equation $\text{(9.71)}$ for some $\varepsilon > 0$, we have $\hat{K}_f(\psi) - \hat{K}_0(\psi) + \hat{K}_0(\psi) + \varepsilon (\log f) \psi = 0$
Then we have
\[
\hat{h}_0^{\text{top}}(\tilde{\mathcal{K}}_f(\psi) - \tilde{\mathcal{K}}_0(\psi), f^\sigma \psi) + \varepsilon \hat{h}_0^{\text{top}}((\log f)\psi, f^\sigma \psi) = -\hat{h}_0^{\text{top}}(\tilde{\mathcal{K}}_0(\psi), f^\sigma \psi)
\] (9.54)

From Proposition 8.9, we obtain
\[
\hat{h}_0^{\text{top}}(\tilde{\mathcal{K}}_f(\psi) - \tilde{\mathcal{K}}_0(\psi), f^\sigma \psi) = \hat{h}_0^{\text{top}}(\mathcal{D}(f^{-1}(\partial_0(f^\psi))), f^\sigma \psi)
\]

From Proposition 9.14 and Proposition 9.15 (1), we obtain
\[
\hat{h}_0^{\text{top}}(\mathcal{D}(f^{-1}(\partial_0(f^\psi))), f^\sigma \psi) = \pi^{\text{top}} dh_0(\varepsilon \mathcal{D}(f^{-1}(\partial_0(f^\psi))), f^\sigma \psi)
\] (9.55)

Hence we obtain
\[
\frac{1}{\sigma} \langle \triangle(\text{tr} f^\sigma \psi), \psi \rangle_{s} + |f^\sigma (\partial_0 f^\sigma \psi)|^2_{h_0 \text{vol}_M} + \varepsilon \hat{h}_0^{\text{top}}((\log f)\psi, f^\sigma \psi) \leq -\hat{h}_0^{\text{top}}(\tilde{\mathcal{K}}_0(\psi), f^\sigma \psi)
\] (9.56)

\begin{proof}

9.6 Step 2

Step 2 is divided into the following three steps. In Step 2-1, we shall show a $C^0$ estimate of $\log f_\varepsilon$, i.e.,
\[\varepsilon \| \log f_\varepsilon \|_{C^0} < C.\]
In Step 2-2, we shall show a $C^1$-estimate of $f_\varepsilon$ in terms of $m_\varepsilon := \| \log f_\varepsilon \|_{C^0}$. In Step 2-3, we shall obtain an $L^p$-estimate of $f_\varepsilon$ for all $p > 0$.

9.6.1 Step 2-1

We need the following well-known result through this section (c.f.[30]).

**Lemma 9.16.** Let $f \in C^2(M, \mathbb{R})$ be a function satisfying $f \geq 0$ and
\[
\triangle f \leq \lambda f + \mu,
\]
for $0 \leq \lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Then there is a positive constant $C$ depending only on $(M, \mathcal{J}, \mathcal{J}_\psi)$ and $\lambda$ such that
\[
\| f \|_{C^0} \leq C(\| f \|_{L^1} + \mu)
\]

**Proposition 9.17.** Let $f_\varepsilon \in \text{Herm}^+(E, h_0)$ be a solution of $L_\varepsilon(f)\psi = 0$ for some $\varepsilon > 0$. Then we have the followings:

(1) \[\frac{1}{2} \triangle(|\log f_\varepsilon|^2) + \varepsilon |\log f_\varepsilon|^2 \leq |\tilde{\mathcal{K}}_0| |\log f_\varepsilon|\]

(2) Let $m_\varepsilon$ be the $C^0$-norm of $\log f_\varepsilon$. Then
\[m_\varepsilon \leq \frac{1}{\varepsilon} |\tilde{\mathcal{K}}_0|_{C^0}
\]

(3) \[\| f_\varepsilon \|_{C^0} \leq C(\| \log f_\varepsilon \|_{L^2} + |\tilde{\mathcal{K}}_0|_{C^0})
\]

\end{proof}
Proof. Since \( f_\varepsilon \) is a solution of \( L_\varepsilon(f_\varepsilon) = 0 \), we have

\[
\mathcal{K}_f(\psi) - \mathcal{K}_0(\psi) + \hat{\mathcal{K}}_0(\psi) + \varepsilon(\log f_\varepsilon) = 0
\]

Then by using \( \log f_\varepsilon \) and \( h_0 \), we have

\[
h_0(\mathcal{K}_f(\psi) - \mathcal{K}_0(\psi), \log f_\varepsilon) + \varepsilon|\log f_\varepsilon|^2 \leq |h_0(-\hat{\mathcal{K}}_0, \log f_\varepsilon)| \leq |\hat{\mathcal{K}}_0| |\log f_\varepsilon|
\]

From Proposition \ref{prop9.11} we have

\[
\frac{1}{2} \Delta(|\log f_\varepsilon|^2) \leq h_0(\mathcal{K}_f(\psi) - \mathcal{K}_0(\psi), \log f_\varepsilon)
\]

Then we obtain (1)

\[
\frac{1}{2} \Delta(|\log f_\varepsilon|^2) + \varepsilon|\log f_\varepsilon|^2 \leq |\hat{\mathcal{K}}_0| |\log f_\varepsilon|
\]

(2) follows from (1) by using the Maximum principle. In fact, Let \( x \in M \) be a point such that \( |\log f_\varepsilon|^2 \) attains its maximum. Then \( \Delta(|\log f_\varepsilon(x)|^2) \geq 0 \). It follows from (1) that

\[
\varepsilon|\log f_\varepsilon(x)|^2 \leq |\hat{\mathcal{K}}_0| |\log f_\varepsilon(x)|
\]

Thus we have

\[
m_\varepsilon = \|\log f_\varepsilon\|_{C^0} \leq \varepsilon^{-1}\|\hat{\mathcal{K}}_0\|_{C^0}
\]

From (1), we have

\[
\Delta(|\log f_\varepsilon|^2) \leq 2|\hat{\mathcal{K}}_0| |\log f_\varepsilon| \leq |\log f_\varepsilon|^2 + |\hat{\mathcal{K}}_0|^2 \leq |\log f_\varepsilon|^2 + |\hat{\mathcal{K}}_0|^2_{C^0}
\]

where \( |\hat{\mathcal{K}}_0|_{C^0} \) is a constant. Applying Lemma \ref{lem9.16} we obtain

\[
\|\log f_\varepsilon\|_{C^0} \leq C(\|\log f_\varepsilon\|_{L^2} + |\hat{\mathcal{K}}_0|_{C^0}^2)
\]

Thus we have (3)

\[
\|\log f_\varepsilon\|_{C^0} \leq C(\|\log f_\varepsilon\|_{L^2} + |\hat{\mathcal{K}}_0|_{C^0})
\]

\[\square\]

9.6.2 Step 2-2

We use the following notations:

\[
m_\varepsilon := \|\log f_\varepsilon\|_{C^0} \tag{9.57}
\]

\[
\zeta_\varepsilon := \frac{d}{d\varepsilon} f_\varepsilon \tag{9.58}
\]

\[
\eta_\varepsilon := f_\varepsilon^{-\frac{1}{2}} \circ \zeta_\varepsilon \circ f_\varepsilon^{-\frac{1}{2}} \tag{9.59}
\]

\[
\hat{\eta} := f_\varepsilon^{-1} \frac{d}{d\varepsilon} f_\varepsilon \tag{9.60}
\]

Since \( \det f_\varepsilon = 1 \), we have \( C \leq |\log f_\varepsilon| \), where \( C \) are constant depending only on \( m_\varepsilon \). In this section, \( C(m_\varepsilon) \) means a positive constant which depends only on \( m_\varepsilon \). Note that \( C(m_\varepsilon) \) may be changed as we proceed. In order to obtain results of Step 2-2, we need the following inequality:
Proposition 9.18.
\[ \Delta(|\eta_\varepsilon|^2) + 2\varepsilon|\eta_\varepsilon|^2 + |df_\eta_\varepsilon|^2 \leq -2h_0(\log f_\varepsilon, \eta_\varepsilon) \]

Proof. Since \( L(f_\varepsilon, \varepsilon) = 0 \), we have \( \frac{d}{d\varepsilon}L(f_\varepsilon, \varepsilon) = 0 \).

Then as in Section 1 and subsection 2-1, the result follows from calculation.

Lemma 9.19.
\[ \text{tr} \eta_\varepsilon = 0. \]

Proof. The result follows from \( \text{det} f_\varepsilon = 1 \) for all \( \varepsilon \) since \( \text{tr} \eta_\varepsilon = \text{tr} f^{-1}_\varepsilon \frac{d}{d\varepsilon}f_\varepsilon \).

Proposition 9.20. For all \( \varepsilon \), it holds
\[ \|\zeta_\varepsilon\|_{C^0} \leq \|\frac{d}{d\varepsilon}f_\varepsilon\|_{C^0} \leq C(m_\varepsilon)\varepsilon^{-1} \]

Proof. From Proposition 9.18, we have
\[ \Delta(|\eta_\varepsilon|^2) + 2\varepsilon|\eta_\varepsilon|^2 + |df_\eta_\varepsilon|^2 \leq -2h_0(\log f_\varepsilon, \eta_\varepsilon) \leq 2|\log f_\varepsilon||\eta_\varepsilon| \]

Thus we have
\[ \Delta(|\eta_\varepsilon|^2) + 2\varepsilon|\eta_\varepsilon|^2 \leq -2h_0(\log f_\varepsilon, \eta_\varepsilon) \leq 2|\log f_\varepsilon||\eta_\varepsilon| \]

Integration over \( M \) yields
\[ \varepsilon \int_M |\eta_\varepsilon|^2 \leq m_\varepsilon \int_M |\eta_\varepsilon| \]

Thus we have
\[ \varepsilon\|\eta_\varepsilon\|_{L^2}^2 \leq m_\varepsilon\|\eta_\varepsilon\|_{L^1} \leq m_\varepsilon\text{Vol}_M^\frac{1}{2}\|\eta_\varepsilon\|_{L^2} \]

Hence we obtain
\[ \|\eta_\varepsilon\|_{L^2} \leq C(m_\varepsilon)\varepsilon^{-1} \]

From Proposition 9.18 we also have
\[ \Delta(|\eta_\varepsilon|^2) \leq -2h_0(\log f_\varepsilon, \eta_\varepsilon) \leq 2|\log f_\varepsilon||\eta_\varepsilon| \leq 2m_\varepsilon|\eta_\varepsilon| \leq m_\varepsilon|\eta_\varepsilon|^2 + m_\varepsilon \]

Applying Lemma 9.16 we have
\[ \|\eta_\varepsilon\|_{C^0} \leq C(m_\varepsilon)(\|\eta_\varepsilon\|_{L^1} + m_\varepsilon) \leq C(m_\varepsilon)(\|\eta\|_{L^2}^2 + m_\varepsilon) \]

Since \( \|\eta_\varepsilon\|_{L^2} < C(m_\varepsilon)\varepsilon^{-1} \), we have \( \|\eta_\varepsilon\|_{C^0} < C(m_\varepsilon)\varepsilon^{-1} \). Since \( \eta_\varepsilon = f_\varepsilon^{\frac{1}{2}} \circ \zeta_\varepsilon \circ f_\varepsilon^{\frac{1}{2}} \), we have the result.

9.6.3 Step 2-3

Let \( \varepsilon_0 \) be a constant in \((0, 1)\). We set \( m := \frac{\|K_0\|_{C^0}}{\varepsilon_0} \). Then \( m_\varepsilon := \|\log f_\varepsilon\|_{C^0} \leq m \) for all \( \varepsilon \in (\varepsilon_0, 1] \). We also have \( C(m_\varepsilon)\varepsilon^{-1} \leq C(m) \), where \( C(m) \) is a constant depending only on \( m \). We shall show that the following Proposition in the section.
Proposition 9.21. Let \( f_\varepsilon \) and \( \zeta_\varepsilon := \frac{d}{dx} f_\varepsilon \) be as before. Then for all \( p > 0 \) and \( \varepsilon \in (\varepsilon_0, 1] \), we have

\[
(1) \quad \|\zeta_\varepsilon\|_{L^p_2} \leq C(m) \left( 1 + \|f_\varepsilon\|_{L^p} \right)
\]

\[
(2) \quad \|f_\varepsilon\|_{L^p_2} \leq c^{C(m)(1-\varepsilon)} \left( 1 + \|f_1\|_{L^p_2} \right)
\]

Proof. (1) From Proposition 9.2, the differential of \( \hat{K}_{f_\varepsilon}(\psi) \) is given by

\[
\pi^{\text{Herm}} \frac{d}{dx} \hat{K}_{f_\varepsilon}(\psi) = \Delta_{f_\varepsilon}(\xi_\psi),
\]

where \( \xi = f_\varepsilon^{-1} \zeta_\varepsilon \). Since \( \pi_{U-n} \overline{\partial}^E \partial f_\varepsilon = \Delta_{f_\varepsilon} \) for \( U^{-n} \) and \( \partial f_\varepsilon = f_\varepsilon^{-1} \circ \partial_0 \circ f_\varepsilon \), from (8.3), we have

\[
\Delta_{f_\varepsilon}(\xi_\psi) = \pi_{U-n} \overline{\partial}^E \partial f_\varepsilon(\xi_\psi) = \pi_{U-n} \overline{\partial}^E \circ f_\varepsilon^{-1} \circ \partial_0 \circ f_\varepsilon(\xi_\psi)
\]

\[
= \pi_{U-n} \overline{\partial}^E f_\varepsilon^{-1} \circ \partial_0(\zeta_\psi) = \pi_{U-n} \overline{\partial}^E f_\varepsilon^{-1}(\partial_0 \zeta_\psi),
\]

where \( \Delta_0 = \pi_{U-n} \overline{\partial}^E \partial_0 \) denotes the Laplacian with respect to \( h_0 \). Since \( L(f_\varepsilon, \varepsilon) = 0 \), we have

\[
0 = \frac{d}{dx} L(f_\varepsilon, \varepsilon) = \pi^{\text{Herm}} \frac{d}{dx} \hat{K}_{f_\varepsilon}(\psi) + \varepsilon \frac{d}{dx} (\log f_\varepsilon)(\psi) + (\log f_\varepsilon)(\psi)
\]

\[
= f_\varepsilon^{-1} \Delta_0(\zeta_\psi) + \pi_{U-n} \overline{\partial}^E f_\varepsilon^{-1}(\partial_0 \zeta_\psi)
\]

\[
+ \varepsilon \frac{d}{dx} (\log f_\varepsilon)(\psi) + (\log f_\varepsilon)(\psi)
\]

Since the operator \( f^{-1} \Delta_0 + \varepsilon D(\log f) \) is invertible, we have the estimate

\[
\|\zeta_\varepsilon\|_{L^p_2} \leq C \|f_\varepsilon^{-1} \Delta_0(\zeta_\psi) + \varepsilon D(\log f_\varepsilon)(\zeta_\psi)\|_{L^p} \]

\[
(9.63)
\]

\[
(9.64)
\]

Since \( \varepsilon D(\log f_\varepsilon)(\zeta_\psi) = \varepsilon \frac{d}{dx} (\log f_\varepsilon)(\psi) \), it follows from (9.62) that we have

\[
\|\zeta_\varepsilon\|_{L^p} \leq C \|\pi_{U-n} \overline{\partial}^E f_\varepsilon^{-1}(\partial_0 \zeta_\psi) + (\log f_\varepsilon)(\psi)\|_{L^p} \]

\[
(9.65)
\]

\[
\leq C \|\pi_{U-n} \overline{\partial}^E f_\varepsilon^{-1}(\partial_0 \zeta_\psi)\|_{L^2} + \|(\log f_\varepsilon)(\psi)\|_{L^p} \]

\[
(9.66)
\]

\[
(9.67)
\]

Since \( \|f_\varepsilon^{-1}\|_{C^0} \) is bounded, using the Hölder inequality, we have

\[
\|\zeta_\varepsilon\|_{L^p_2} \leq C \|f_\varepsilon\|_{L^p_2} \|\zeta_\varepsilon\|_{L_{2p}} \]

\[
(9.68)
\]

Then using the general formula as in the book by Aubin, we have

\[
\|\zeta_\varepsilon\|_{L^p_2} \leq C \|f_\varepsilon\|_{L^{2p}} \|\zeta_\varepsilon\|_{L^{2p}} \]

\[
+ C \leq C \|f_\varepsilon\|_{L^{2p}} \|\zeta_\varepsilon\|_{L^{2p}} + C \]

Since \( Cx^\frac{1}{2} y^\frac{1}{2} \leq \frac{1}{2} (C^2 x + y) \), we have
Thus we obtain
\[ \| \zeta \|_{L^p_2} \leq C(m) \left( 1 + \| f_\epsilon \|_{L^p_2} \right). \]

(2) If \( x(\epsilon) \) is a smooth function with values in a normed vector space, then we have
\[ \frac{d}{d\epsilon} \| x(\epsilon) \| \geq -\| \frac{d}{d\epsilon} x(\epsilon) \|. \]
This follows from the triangle inequality. Applying this to \( x(\epsilon) := f_\epsilon \) with the norm \( \| \cdot \|_{L^p_2} \), we have
\[ \frac{d}{d\epsilon} \| f_\epsilon \|_{L^p_2} \geq -\| \zeta \|_{L^p_2} \geq -C(m) \left( 1 + \| f_\epsilon \|_{L^p_2} \right) \]
from (1). Then we have
\[
\log(1 + x(1)) - \log(1 + x(\epsilon)) = \int_1^\epsilon \frac{1}{1 + x(s)} \frac{d}{ds} x(s) ds \geq - C(m)(1 - \epsilon) \tag{9.69}
\]
Hence
\[ \frac{1 + x(1)}{1 + x(\epsilon)} \geq e^{-C(m)(1+\epsilon)} \tag{9.70} \]
Thus \( \| f_\epsilon \|_{L^p_2} \leq 1 + \| f_\epsilon \|_{L^p_2} \leq e^{C(m)(1+\epsilon)}(1 + \| f_1 \|_{L^p_2}). \)

**Proposition 9.22.** We have the followings

1. The set of solutions \( S \) contains \((0, 1]\).

2. If we have a uniform \( C^0 \)-upper bound \( \| f_\epsilon \|_{C^0} < C \) for all \( \epsilon \in (0, 1] \), then there is a solution \( f_0 \) of the equation \( L(f_0, 0) = 0 \), that is, \( h_{f_0} \) is an Einstein-Hermitian metric.

**Proof.** (1) From Proposition 9.21, there is a sequence \( \{ \epsilon_j \} \) such that \( f_{\epsilon_j} \) converges strongly \( f_{\epsilon_0} \) in \( L^p \) for \( p > 2n \), when \( \epsilon_j \to \epsilon_0 \). Then by using the Stokes theorem, we obtain that
\[ (L(f_{\epsilon_0}, \alpha))_{L^2} = 0 \]
for any smooth section \( \alpha \in \text{Herm}(E) \). Then by using the elliptic regularity, it turns out that \( f_{\epsilon_0} \) is smooth which gives rise to an Einstein-Hermitian metric \( h_{f_{\epsilon_0}} \). (2) is also shown by the same way as in (1).

**9.7 Step 3**

Let \( f_\epsilon \) be a solution of the equation for \( \epsilon > 0 \).
\[
\mathcal{K}_{f_\epsilon}(\psi) - \lambda \text{id}_E + \epsilon \log f_\epsilon = 0 \tag{9.71}
\]
We assume that \( \| f_\epsilon \|_{C^0} \to \infty \) (\( \epsilon \to 0 \)). Let \( \lambda_\epsilon(x) \) be the largest eigenvalue of \( \log f_\epsilon(x) \) at \( x \in M \). We denote by \( M_\epsilon \) maximum of \( \lambda_\epsilon \), i.e., \( M_\epsilon := \max \{ \lambda_\epsilon(x) \mid x \in M \} \). Set \( \rho_\epsilon = e^{-M_\epsilon} \). Since \( \text{tr} \log f_\epsilon = 0 \), we see that \( M_\epsilon = O(\epsilon^{-1}) \) from Lemma 9.17. We can assume \( \rho_\epsilon \leq 1 \).
Proposition 9.23. There is a sequence \( \{\varepsilon_i\} \) which goes to 0 (\( i \to \infty \)) and \( \rho(\varepsilon_i) \) also goes to 0 such that \( f_i := \rho(\varepsilon_i) f_{\varepsilon_i} \), satisfies the following properties:

1. When \( i \) goes to infinity, the \( f_i \) converges weekly in \( L^2_1 \) to a \( f_\infty \neq 0 \).

2. There is a sequence of numbers \( \{\sigma_j\} \) which goes to 0 (\( j \to \infty \)) such that the sequence \( \{f^{\sigma_j}_\infty\} \) converges weekly in \( L^2_1 \) to a \( f^{\sigma}_\infty \), where 0 < \( \sigma_j \leq 1 \).

3. \( \pi := \text{id}_E - f^{\sigma}_\infty \) is a weekly generalized holomorphic subbundle of \( E \).

Proof. (1) Since every eigenvalue of \( f_i \) is less than or equal to 1, the \( L^2 \)-norm of \( f_i \) is bounded by the volume of \( M \). Thus it suffices to show that the \( L^2 \)-norm of \( d_0 f_i \) is bounded from above by a constant \( C \) which does not depend on \( \varepsilon \). From Proposition [9.15](2), we obtain

\[
\frac{1}{\sigma} \Delta(\text{tr} f^\sigma) \text{vol}_M + \varepsilon h_0^\top((\log f)^{\psi}, f^\sigma \psi) + |f^\sigma(\partial_0 f^\sigma \psi)|^2_{h_0} \text{vol}_M \leq -h_0^\top(\tilde{\mathcal{K}}_0(\psi), f^\sigma \psi)
\]

Let \( \sigma = 1 \). Then we have

\[
\Delta(\text{tr} f_{\varepsilon}) \text{vol}_M \leq -h_0^\top(\tilde{\mathcal{K}}_0(\psi), f_{\varepsilon} \psi) - \varepsilon h_0^\top((\log f_{\varepsilon}) \psi, f_{\varepsilon} \psi)
\]

Since \( \varepsilon \| f_{\varepsilon} \| < C \), \( h_0(\tilde{\mathcal{K}}_0(\psi), f_{\varepsilon} \psi) < C \| f_{\varepsilon} \| \text{vol}_M \), we have

\[
\Delta(\text{tr} f_{\varepsilon}) \text{vol}_M < C \| f_{\varepsilon} \| \text{vol}_M < C \text{tr} f_{\varepsilon} \text{vol}_M
\]

This means \( \max(f_{\varepsilon}) \leq \| f_{\varepsilon} \|_{L^1} \). Since \( \rho_{\varepsilon} f_{\varepsilon} \) has at least one eigenvalue 1, we have

\[
1 \leq \max_{M}(\rho_{\varepsilon} \| f_{\varepsilon} \|_{L^1} < C \text{Vol}(M) \frac{1}{4} \rho_{\varepsilon} \| f_{\varepsilon} \|_{L^2} < C \| \rho_{\varepsilon} f_{\varepsilon} \|_{L^2},
\]

since \( \rho_{\varepsilon} \) is a constant and we denote by \( C \) nonzero constants which does not depend on \( \varepsilon \). This implies that

\[
0 < \frac{1}{C} \leq \| \rho_{\varepsilon} f_{\varepsilon} \|_{L^2}
\]

If \( f_i := \rho_{\varepsilon_i} f_{\varepsilon_i} \) converges strongly \( f_\infty \) in \( L^2 \), then \( f_\infty \neq 0 \). Further we see that

\[
\| \rho_{\varepsilon} f_{\varepsilon} \|_{L^2} \leq \| \text{id}_E \|_{L^2} = C
\]

Next we shall estimate \( L^2_1 \)-norm of \( \rho_{\varepsilon} f_{\varepsilon} \). Since \( f_{\varepsilon} = f^*_\varepsilon \) and \( |\partial_0(\rho_{\varepsilon} f_{\varepsilon})| = |\partial(\rho_{\varepsilon} f_{\varepsilon})| \), we have

\[
|d_0(\rho_{\varepsilon} f_{\varepsilon})|^2 = 2|\partial_0(\rho_{\varepsilon} f_{\varepsilon})|^2 \leq 2|\rho_{\varepsilon} f_{\varepsilon}|^{\frac{2}{\sigma}} \partial_0(\rho_{\varepsilon} f_{\varepsilon})|^2,
\]

where \( \| (\rho_{\varepsilon} f_{\varepsilon})^{\frac{1}{\sigma}} \| \geq \| \text{id}_E \| \).

By multiplying \( \rho_{\varepsilon}^2 \) on the both sides of Proposition [9.15](2), we obtain

\[
\frac{1}{\sigma} \Delta(\text{tr} f^\sigma) \text{vol}_M + \varepsilon h_0^\top((\log f_{\varepsilon}) \psi, f^\sigma \psi) + |f^\sigma(\partial_0 f^\sigma \psi)|^2_{h_0} \text{vol}_M \leq -h_0^\top(\tilde{\mathcal{K}}_0(\psi), f^\sigma \psi),
\]

Note that \( \rho_{\varepsilon} \) is a constant and \( f_i := \rho_{\varepsilon_i} f_{\varepsilon_i} \). Let \( \sigma = 1 \). Applying \( \int_M \Delta(\text{tr} f^\sigma) \text{vol}_M = 0 \), we obtain

\[
\int_M |f_i^{\frac{1}{\sigma}}(\partial_0 f_i \psi)|^2_{h_0} \text{vol}_M \leq -\int_M \varepsilon h_0^\top((\log f_{\varepsilon}) \psi, f_i \psi) - \int_M h_0^\top(\tilde{\mathcal{K}}_0(\psi), f_i \psi)
\]

(9.72)

(9.73)
because $|\log f_e| < C$ and $\max |\hat{\mathcal{K}}_0| < C$. Thus we have
\[
\int_M |d_0(\rho_\varepsilon f_e)|^2 < C
\]
Proof of (2) We also apply Proposition 9.15 (2) to $f_i^\sigma$. Then we obtain
\[
\int_M |f_i^\sigma(\partial_0 f_i^\sigma \psi)|^2 \frac{2}{\text{vol}_M} < C
\]
where $C$ does not depend on $i$ and $\sigma$. Since $|(f_i^\sigma)^{\frac{1}{2}}| \geq |\text{id}_E|$, we have
\[
|d_0(f_i^\sigma)|^2 = 2|\partial_0(f_i^\sigma)|^2 \leq 2(|f_i^\sigma|^{\frac{1}{2}} \partial_0(f_i^\sigma))^2,
\]
Thus we have
\[
\|f_i^\sigma\|_{L^2_1} < C.
\]
Then we have $\|f_\infty^\sigma\|_{L^2_1} < C$. $f_\infty^\sigma$ converges to $f_\infty^0$ weak in $L^2_1$ when we take a subsequence $\{\sigma_j\}$
Proof of (3) Since $f_i^\sigma$ is bounded, $f_i^\sigma$ converges strongly to $f_\infty^0$ in $L^2$ (we take a subsequence if necessary). Since $M$ is compact, $f_i^\sigma$ converges strongly to $f_\infty^0$ in $L^1$ also. Thus there is a subset $W$ of $M$ such that the measure of $M \setminus W$ is zero and $f_i^\sigma$ converges $f_\infty^0$ on $W$. Note that $f_\infty$ and $f_\infty^0$ are defined on $W$. Since $(f_i^\sigma)^* = f_i^\sigma$, this means that $(f_\infty^0)^* = f_\infty^0$. Since $\pi := \text{id}_E - f_\infty^0$, we have $\pi^* = \pi$. Since $f_\infty \in \text{Herm}(E, h_0)$, we can diagonalize $f_\infty$. Then we see
\[
\lim_{j \to \infty} f_\infty^\sigma_j = \lim_{j \to \infty} f_\infty^{2\sigma_j} = f_\infty^0
\]
Thus we have in $L^1$
\[
\pi = \lim_{j \to \infty} (\text{id}_E - f_\infty^\sigma_j)^2 = \lim_{j \to \infty} (\text{id}_E - 2f_\infty^\sigma_j + f_\infty^{2\sigma_j}) = \text{id}_E - 2f_\infty^0 + f_\infty^0 = \pi
\]
We shall show that $(\text{id}_E - \pi) \circ \overline{\partial}_j^E = 0$. Since $(\text{id}_E - \pi) \circ \pi = 0$, we have
\[
\overline{\partial}_j^E((\text{id}_E - \pi) \circ \pi) = \left(\overline{\partial}_j^E(\text{id}_E - \pi)\right) \circ \pi + (\text{id}_E - \pi) \circ \overline{\partial}_j^E \pi = 0
\]
Thus we have
\[
|(\overline{\partial}_j^E(\text{id}_E - \pi)) \circ \pi| = |(\text{id}_E - \pi) \circ \overline{\partial}_j^E \pi|.
\]
Since $\pi^* = \pi$, we have
\[
|(\overline{\partial}_j^E(\text{id}_E - \pi)) \circ \pi| = \left|\left((\overline{\partial}_j^E(\text{id}_E - \pi)) \circ \pi\right)^*\right| = |\pi \circ \partial_0(\text{id}_E - \pi)|
\]
Thus it suffices to show that $\|\pi \circ \partial_0(\text{id}_E - \pi)\|_{L^2} = 0$. Note that $L^1$-norm is bounded from above by $L^2$-norm.

**Lemma 9.24.** For $0 \leq \lambda \leq 1$ and $0 < s \leq \sigma \leq 1$, we have the following inequality:
\[
0 \leq \frac{s + \sigma}{s} (1 - \lambda^s) \leq \lambda^{-\sigma}
\]
**Proof.** This follows from calculation (c.f. [30]).
Since $f_i$ can be diagonalize such that every eigenvalue is positive and less than or equal to 1, applying Lemma 9.24 for $0 < s \leq \frac{\sigma}{2} \leq 1$, we have

$$0 \leq (\text{id}_E - f_i^s) \leq \frac{s}{s + \frac{\sigma}{2}} f_i^{-\frac{\sigma}{2}}$$  \hspace{1cm} (9.74)

Applying (9.74) and Proposition 9.15, we obtain

$$\int_M |(\text{id}_E - f_i^s) \circ \partial_0 f_i^s|^2 \leq \frac{s}{s + \frac{\sigma}{2}} \int_M |f_i^{-\frac{\sigma}{2}} \circ \partial_0 f_i^s|^2 \leq \frac{s}{s + \frac{\sigma}{2}} C,$$  \hspace{1cm} (9.75)

where $C$ does not depend on $s$, $\sigma$ and $i$. When $i$ goes to $\infty$, we have

$$\int_M |(\text{id}_E - f_\infty^s) \circ \partial_0 f_\infty^s|^2 \leq \frac{s}{s + \frac{\sigma}{2}} C.$$  \hspace{1cm} (9.76)

When $s$ goes to 0 , we have

$$\int_M |\pi \circ \partial_0 f_\infty^s|^2 \leq \lim_{s \to 0} \frac{s}{s + \frac{\sigma}{2}} C = 0$$

Then when $\sigma$ goes to 0, we obtain

$$\int_M |\pi \circ \partial_0 f_\infty^0|^2 = \int_M |\pi \circ \partial_0 (\text{id}_E - \pi)|^2 = 0$$

Thus we obtain the result.

The $\pi$ is a weak generalized holomorphic subbundle of $E$. We also denote it by $E_{\pi}$. Then $\pi$ gives rise to a coherent subsheaf $E_{\pi}$ with respect to both $I_{\pm}$ of the bihermitian structure.

**Proposition 9.25.** The rank of $E_{\pi}$ satisfies

$$0 < \text{rk} E_{\pi} < r = \text{rk} E.$$  \hspace{1cm} (9.77)

**Proof.** Since $f_i$ converges to $f_\infty$ on almost everywhere, we see that $f_\infty^0 \neq 0$ and $f_\infty^0 \neq \text{id}_E$ on almost everywhere. Thus $\pi := \text{id}_E - f_\infty^0 \neq 0$ and $\pi \neq \text{id}_E$ on almost everywhere. Thus we have the result. \hfill $\square$

**Proposition 9.26.** Let $\mu(E)$ be the slope of $E$ and $\mu(E_{\pi})$ the slope of $E_{\pi}$. Then we have

$$\mu(E_{\pi}) \geq \mu(E).$$

**Proof.** As in before, $\pi$ gives a holomorphic subbundle $E_{\pi}|_W$ on an open dense subset $W$ of $M$. Applying Proposition 7.7 to $\pi$ on $W$ and $S := E_{\pi}|_W$, we have

$$\text{tr} \left( \pi \circ K^E_A(\psi) \circ \pi \right) = \text{tr} K^S_{A_S}(\psi) + \| H^S \|^2 \psi,$$

where $A$ is the canonical connection with respect to $h_0$ and $K^E_A$ is the mean curvature of $E$ and $K^S_{A_S}(\psi)$ denotes the mean curvature of $S := E_{\pi}|_W$ and $H^S$ is the second fundamental form of $S$ of $E|_W$ with respect to $h_0$. From Proposition 9.1 (2), we obtain

$$\text{tr} \left( \pi \circ K^E_A(\psi) \circ \pi \right) = \text{tr} K^S_{A_S}(\psi) + ||\partial_0 \pi||^2 \psi,$$
Recall that the degree of $E$ is defined by

$$\deg(E) := \frac{1}{\text{Vol}_M} \int_M i^n \frac{1}{2\pi} \text{tr} \langle \mathcal{K}_M^E(\psi), \overline{\psi} \rangle_s$$

Thus the degree of $E_\pi$ is given by

$$\text{Vol}_M \deg(E_\pi) = \frac{1}{2\pi} \int_M i^n \text{tr} \langle \mathcal{K}_M \psi, \overline{\psi} \rangle_s = \frac{1}{2\pi} \int_M i^n \text{tr} \langle \pi \circ \mathcal{K}_M^E(\psi) \circ \pi, \overline{\psi} \rangle_s - \frac{1}{2\pi} \int_M \|\partial_0\pi\|^2 \text{vol}_M$$

(9.77)

where $\text{vol}_M = i^n \langle \psi, \overline{\psi} \rangle_s$. Let $r = \text{rk}E$ and $r' := \text{rk}E_\pi$. Then the slope is given by

$$\mu(E) = \frac{1}{r} \deg(E).$$

Let $\hat{\mathcal{K}}_M^E(\psi) = \mathcal{K}_M^E(\psi) - \mu(E)\psi\text{id}_E$. Then we have

$$\mu(E_\pi) = \frac{1}{r} \deg(E_\pi) = \frac{1}{2\pi r' \text{Vol}_M} \int_M i^n \text{tr} \langle \pi \circ \hat{\mathcal{K}}_M^E(\psi) \circ \pi, \overline{\psi} \rangle_s - \frac{1}{2\pi r' \text{Vol}_M} \int_M \|\partial_0\pi\|^2 + \mu(E)$$

Thus the inequality $\mu(E_\pi) \geq \mu(E)$ is equivalent to the following inequality:

$$\int_M i^n \text{tr} \langle \hat{\mathcal{K}}_M^E(\psi) \circ \pi, \overline{\psi} \rangle_s \geq \int_M \|\partial_0\pi\|^2 \text{vol}_M$$

since $\text{tr} \langle \pi \circ \hat{\mathcal{K}}_M^E(\psi) \circ \pi, \overline{\psi} \rangle_s = \text{tr}(\hat{\mathcal{K}}_M^E(\psi) \circ \pi).$ Recall that $\text{id}_E - f_i^\sigma$ strongly converges in $L^2$,

$$\pi = \lim_{\sigma \to 0, i \to \infty} \lim (\text{id}_E - f_i^\sigma).$$

Since $\text{tr} \hat{\mathcal{K}}_M^E(\psi) = 0$, we have

$$\int_M i^n \text{tr} \langle \hat{\mathcal{K}}_M^E(\psi) \circ \pi, \overline{\psi} \rangle_s = -\lim_{\sigma \to 0} \lim_{i \to \infty} \int_M i^n \text{tr} \langle \hat{\mathcal{K}}_M^E(\psi) \circ f_i^\sigma, \overline{\psi} \rangle_s$$

Since $f_i$ satisfies the equation (9.71) for some $\varepsilon_i > 0$, then from Proposition 9.15 (2), we have

$$\frac{1}{\sigma} \Delta (\text{tr} f_i^\sigma) \text{vol}_M + \varepsilon h_0^\top((\log f_i^\sigma), \psi, f_i^\sigma \psi) + (f_i^\sigma \overline{\sigma}) (\partial_0 f_i^\sigma \psi)|^2 \text{vol}_M \leq -h_0^\top(\hat{\mathcal{K}}_0(\psi), f_i^\sigma \psi)$$

since $\hat{\mathcal{K}}_0(\psi) = \hat{\mathcal{K}}_A(\psi)$.

**Lemma 9.27.** For real numbers $\sigma \leq 0$ and $\lambda_i$ with $\sum_i \lambda_i = 0$, we have

$$\sum_i \lambda_i e^{\sigma \lambda_i} \geq 0$$

**Proof.** This follows directly from calculation.

We can assume that $f_{\varepsilon_i}$ can be diagonalize with eigenvalues $\{e^{\lambda_i}\}$. Since $\text{tr}(\log f_{\varepsilon_i}) = 0$, applying Lemma 9.27, we have

$$\varepsilon h_0^\top((\log f_{\varepsilon_i}), f_{\varepsilon_i}^\sigma \psi) \geq 0$$

Substituting $f = f_i$ into Proposition 9.15 (2), we have

$$\int_M |f_i^\sigma (\partial_0 f_i^\sigma \psi)|^2 \text{vol}_M \leq -\int_M h_0^\top(\hat{\mathcal{K}}_0(\psi), f_i^\sigma \psi)$$

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since $\int_M \triangle f^\sigma = 0$. Thus we have
\[
\int_M |f^\sigma_1 \circ \partial \omega (\psi)|^2 \omega \leq |\int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega \leq |\int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega \leq \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega
\]
Thus we obtain
\[
\int_M |\partial \omega (f^\sigma_1 \psi)|^2 \omega \leq \int_M |\partial \omega (\psi)|^2 \omega \leq \int_M |\partial \omega (\psi)|^2 \omega
\]
since $f^\sigma_1 \geq \text{id}$ for $0 \leq \sigma < 1$. Since $\partial \omega (\psi)$ converges to $\partial \omega$ weak in $L^2$, we obtain
\[
\|\partial \omega \|_{L^2}^2 \leq (\partial \omega, \partial \omega)_{L^2} \leq \lim \lim (\partial \omega (\psi), \partial \omega (\psi))_{L^2}
\]
\[
\leq \lim \lim \|\partial \omega \|_{L^2}^2 \|\partial \omega (\psi), \partial \omega (\psi))_{L^2}
\]
\[
\leq \lim \lim \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega\]
\[
\leq \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega\]
\[
\leq \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega\]
\[
\leq \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega
\]
since $\text{tr} \hat{K}^E_A (\psi) = 0$. Thus we obtain $\|\partial \omega \|_{L^2}^2 \leq \int_M \int_{\frac{1}{2}}^\sigma \partial \omega (\psi)|^2 \omega$. Hence we have
\[
\mu (E) \geq \mu (E)
\]
\[
\square
\]

9.8 Proof of main theorem

Theorem 1.1 [Kobayashi-Hitchin correspondence] There exists an Einstein-Hermitian metric on a $\psi$-polystable generalized holomorphic vector bundle. Conversely, a generalized holomorphic vector bundle admitting an Einstein-Hermitian metric is $\psi$-polystable.

Proof. Let $E$ be a $\psi$-polystable generalized holomorphic vector bundle. Then we start the continuity method by solving the equation $\Box$. From Step 2, we already have the set of solutions $\{f \}$ for $\epsilon \in (0, 1]$. If we assume that $\|f_\epsilon\|_{C^0}$ goes to infinity when $\epsilon \to 0$, then from Step 3, we have a weak generalized holomorphic subbundle $\pi$ such that $\mu (\pi) \geq \mu (E)$. This is a contradiction since $E$ is $\psi$-stable. Thus $\|f_\epsilon\|_{C^0}$ is bounded and then there exits $f_0 := \lim_{\epsilon \to 0} f_\epsilon$ such that $h_{f_0}$ gives an Einstein-Hermitian metric. Conversely, a generalized holomorphic vector bundle with an Einstein-Hermitian metric is $\psi$-stable from Section 4. $\Box$

10 Einstein-Hermitian metrics and stable Poisson modules

10.1 Stable Poisson modules

Let $X := (M, J)$ be a complex manifold with a holomorphic Poisson structure $\beta$. The Poisson bracket $\{,\}$ is given by the contraction by the 2-vector $\beta$ and $df \wedge dg$ for a holomorphic function $f, g$.

$\{f, g\} := \beta (df \wedge dg)$
Let $E$ be a holomorphic vector bundle over $X$. We denote by $\mathcal{E}$ the sheaf $\mathcal{O}(E)$ of germs of holomorphic sections of $E$. Then a Poisson module structure of $\mathcal{E}$ is a map from $\mathcal{O}_X \times \mathcal{E}$ to $\mathcal{E}$ given by $s \mapsto \{f, s\}$ for $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$ which satisfies

$$\{f, gs\} = \{f, g\}s + g\{f, s\} \quad (10.1)$$

$$\{fg, s\} = f\{g, s\} + g\{f, s\} \quad (10.2)$$

$$\{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\} \quad (10.3)$$

for $f, g \in \mathcal{O}_X$ and $s \in \mathcal{E}$. An $\mathcal{O}_X$-module $\mathcal{E}$ with a Poisson module structure is called a Poisson module.

We denote by $\Theta$ the sheaf of germs of holomorphic vector fields of $X$. Then $\beta$ gives a differential operator $\delta_\beta : \mathcal{O}_X \to \Theta_X$ by

$$\delta_\beta f = \beta(df), \quad f \in \mathcal{O}_X,$$

where $\beta(df) \in \Theta$ is the contraction of $\beta \in \wedge^2 \Theta$ and $df \in \Omega^1$. Let $\delta^E_\beta$ be a differential operator whose principal part is given by $\delta_\beta$, that is,

$$\delta^E_\beta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Theta,$$

which satisfies

$$\delta^E_\beta(fs) = f\delta^E_\beta(s) + (\delta_\beta f)s, \quad s \in \mathcal{E}, \ f \in \mathcal{O}_X \quad (10.4)$$

Then a Poisson module structure $\{,\}$ gives a differential operator $\delta^E_\beta$ by

$$\{f, s\} = \langle \delta^E_\beta(s), df \rangle,$$

where $\langle, \rangle$ denotes the coupling between $\Theta$ and $\Omega^1$. Then $\delta^E_\beta$ gives the following sequence:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\delta^E_\beta} \mathcal{E} \otimes \Theta \xrightarrow{\delta^E_\beta} \mathcal{E} \otimes \wedge^2 \Theta \xrightarrow{\delta^E_\beta} \cdots$$

Then the condition (10.3) is equivalent to

$$\delta^E_\beta \circ \delta^E_\beta = 0.$$

Thus it turns out that a Poisson module structure is equivalent to a differential operator $\delta^E_\beta : \mathcal{E} \to \mathcal{E} \otimes \Theta$ which satisfies (10.4) and $\delta^E_\beta \circ \delta^E_\beta = 0$. Let $(M, J_\beta)$ be a generalized complex manifold given by Poisson deformation of $\beta$ as in Example 2.4. Then it is known that the followings (1) and (2) are equivalent

(1) a Poisson module $(\mathcal{E}, \delta^E_\beta)$ on $X$

(2) a generalized holomorphic vector bundle $(E, \overline{\mathcal{J}}^E_{J_\beta})$ over $(M, J_\beta)$

In fact, the exterior derivative $d$ is decomposed into $d = \partial_{\mathcal{J}_\beta} + \overline{\partial}_{\mathcal{J}_\beta}$, where $\overline{\partial}_{\mathcal{J}_\beta} = e^E \circ \overline{\partial} \circ e^{-\overline{\beta}} - \delta_\beta$, where $\overline{\partial}$ is the ordinary $\overline{\partial}$-operator and $e^E \circ \overline{\partial} \circ e^{-\overline{\beta}}$ denotes the Adjoint action of $e^E$ on $\overline{\partial}$. Thus a generalized holomorphic structure $\overline{\partial}^E_{J_\beta}$ is uniquely decomposed into

$$\overline{\partial}^E_{J_\beta} = e^E \circ \overline{\partial} \circ e^{-\overline{\beta}} - \delta^E_\beta,$$

where $\overline{\partial}^E$ denotes the ordinary holomorphic structure on $E$. Hence a Poisson structure $\delta^E_\beta$ on a holomorphic vector bundle $E$ gives a generalized holomorphic structure, and vice versa.
Then the cohomology group of the Lie algebroid complex complex \((E \otimes \Lambda^\bullet \mathcal{L}_{J_\beta}, \bar{\partial}^E_{J_\beta})\) is given by the hyper-cohomology of the Poisson complex \((\mathcal{E} \otimes \Lambda^\bullet \Theta, \delta^E_\beta)\). The degree of a Poisson module \((\mathcal{E}, \delta^E_\beta)\) is the degree of \(\mathcal{E}\) and the slope is defined to be the slope of \(E\) also. A sub Poisson module \((\mathcal{F}, \delta^F_\beta)\) is a subsheaf \(\mathcal{F}\) of \(\mathcal{E}\) satisfying

\[
\begin{array}{c}
\mathcal{F} \xrightarrow{i} \mathcal{E} \\
\delta^F_\beta \downarrow \quad \downarrow \delta^E_\beta \\
\mathcal{F} \otimes \Theta \quad \mathcal{E} \otimes \Theta
\end{array}
\]

where \(i : \mathcal{F} \to \mathcal{E}\) denotes the inclusion. Thus we have the stability of Poisson modules which coincides with the notion of the stability of generalized holomorphic vector bundles.

**Definition 10.1.** A Poisson module \((\mathcal{E}, \delta^E_\beta)\) is **stable** if for every sub Poisson module \((\mathcal{F}, \delta^F_\beta)\) of \((\mathcal{E}, \delta^E_\beta)\), the slope strict inequality holds

\[
\mu(\mathcal{F}) < \mu(\mathcal{E})
\]

We also have the notion of semistability and polystability: If \(\mathcal{E}\) satisfies \(\mu(\mathcal{F}) \leq \mu(\mathcal{E})\), then \(\mathcal{E}\) is **semistable**. If \(\mathcal{E}\) is a direct sum of Poisson modules \(\oplus_i (\mathcal{E}_i, \delta^E_{\beta_i})\) with the same slope \(\mu(\mathcal{E})\), then \(\mathcal{E}\) is **polystable**.

Then we have the following:

**Theorem 10.2.** Let \((E, \bar{\partial}^E_{J_\beta})\) be a generalized holomorphic vector bundle which is given by a Poisson module \((\mathcal{E}, \delta^E_\beta)\). Then the following (1) and (2) are equivalent

1. \((E, \bar{\partial}^E_{J_\beta})\) admits an Einstein-Hermitian metric
2. \((\mathcal{E}, \delta^E_\beta)\) is polystable.

**Proof.** The result follows from our main theorem. \(\square\)

### 10.2 Einstein-Hermitian metrics of Poisson modules

Let \(X = (M, J)\) be a Poisson manifold with a Poisson structure \(\beta\) and \(J_\beta\) the induced generalized complex structure. Then as in (2.1), \(J_\beta\) gives the eigenspace decomposition \((TM \oplus T^*M)^E = \mathcal{L}_{J_\beta} \oplus \bar{\mathcal{L}}_{J_\beta}\) and \(\mathcal{L}_{J_\beta} = \Lambda^1_{\beta} \oplus T^1_{\beta}\) and \(\bar{\mathcal{L}}_{J_\beta} = \Lambda^0_{\beta} \oplus T^0_{\beta}\), where and \(\Lambda^1_{\beta}\) and \(\Lambda^0_{\beta}\) are respectively given by

\[
\begin{align*}
\Lambda^1_{\beta} &= \{ \theta + [\beta, \theta] | \theta \in \Lambda^1_J \} \\
\Lambda^0_{\beta} &= \{ \bar{\theta} + [\bar{\beta}, \bar{\theta}] | \bar{\theta} \in \Lambda^0_J \}
\end{align*}
\]

(10.5) (10.6)

where \(\Lambda^1_J\) denotes forms of type \((1, 0)\) with respect to the ordinary complex structure \(J\) and \(\Lambda^0_J\) is the complex conjugate of \(\Lambda^0_J\) and \(T^0_J\) denotes vectors of type \((1, 0)\) and \(T^1_J\) is the complex conjugate of \(T^1_J\). The generalized complex structure \(J_\beta\) gives the decomposition \(d = \partial_\beta + \bar{\partial}_\beta\). For a function \(f\), \(\partial_\beta f\) and \(\bar{\partial}_\beta f\) are explicitly given by

\[
\begin{align*}
\partial_\beta f &= e^\beta \partial f e^{-\beta} - [\bar{\beta}, \bar{\partial} f] \\
\bar{\partial}_\beta f &= \bar{\partial} f + [\beta, \bar{\partial} f] - [\bar{\beta}, \bar{\partial} f] \in \mathcal{L}_{J_\beta}
\end{align*}
\]

(10.7) (10.8)

where \(\Lambda^0_{\beta}\) and \(\Lambda^1_{\beta}\) are respectively given by

\[
\begin{align*}
\Lambda^0_{\beta} &= \{ \theta + [\beta, \theta] | \theta \in \Lambda^0_J \} \\
\Lambda^1_{\beta} &= \{ \bar{\theta} + [\bar{\beta}, \bar{\theta}] | \bar{\theta} \in \Lambda^1_J \} \end{align*}
\]

(10.9)
where \( df = \partial_h f + \overline{\partial}_h f \).

Let \((E, h)\) be an Hermitian vector bundle and \( \overline{\partial}^E = \overline{\partial} + A^{0,1} \) a (ordinary) holomorphic structure of \( E \). Then we denote by \( d^A = \partial^A + \overline{\partial}^E \) the canonical connection of \((E, h)\) with a connection form \( A \). Let \( \overline{\partial}^E_\beta \) be a generalized holomorphic structure of \( E \) with respect to the generalized complex structure \( J_\beta \). Then as in Subsection 10.1, there is a holomorphic structure \( \overline{\partial}^E \) and a Poisson module structure \( \delta^E_\beta \) such that

\[
\overline{\partial}^E_\beta = e^\beta \circ \overline{\partial}^E \circ e^{-\overline{\beta}} - \delta^E_\beta.
\]

Then \( \overline{\partial}^E \) and \( \delta^E_\beta \) are respectively written as

\[
\begin{align*}
\overline{\partial}^E &= \overline{\partial} + A^{0,1} \\
\delta^E_\beta &= [\beta, \partial^E] + V^{1,0},
\end{align*}
\]

where \( \partial^E \) is the \((1,0)\)-component of the ordinary canonical connection \( d^A \) and \( A^{0,1} \in \wedge^0_{\beta} \otimes \text{End} (E) \) denotes the connection form of \( \overline{\partial}^E \) and \( V^{1,0} \in T^{1,0} \otimes \text{End} (E) \). Note that \( \delta^E_\beta \) is a holomorphic operator but \( \partial^E \) is not holomorphic. Since \( \overline{\partial}^E_\beta \circ \overline{\partial}^E = 0 \) and \( [e^\beta, \beta] = 0 \), we have

\[
\begin{align*}
\overline{\partial}^E \circ \overline{\partial}^E &= 0 \\
\overline{\partial}^E \circ \delta^E_\beta + \delta^E_\beta \circ \overline{\partial}^E &= 0 \\
\delta^E_\beta \circ \delta^E_\beta &= 0
\end{align*}
\]

For given generalized holomorphic structure \( \overline{\partial}^E_\beta \), we have the canonical generalized connection \( D^E_\beta = \overline{\partial}^E_\beta + \overline{\partial}^E \) of \((E, h)\), where \( D^1_{\beta} \in \mathcal{L}_{J_\beta} \otimes \text{End} (E) \). Then we have

**Proposition 10.3.**

\[
D^E_\beta = d^A + V
\]

where \( V = V^{0,1} + V^{1,0} \) and \(-V^{1,0}\) is the adjoint \((V^{1,0})^*\) of \( V^{1,0} \) with respect to \( h \).

**Proof.** The generalized holomorphic structure \( \overline{\partial}^E_\beta \) is explicitly given by

\[
\overline{\partial}^E_\beta s = \overline{\partial}^E s + [\beta, \overline{\partial}^E s] - [\beta, \overline{\partial}^E s] + V^{1,0} s \in \mathcal{L}_{J_\beta} \otimes \text{End} (E)
\]

We define the following operator \( \partial^1_\beta \) by

\[
\partial^1_\beta s = \partial^E s + [\beta, \partial^E s] - [\beta, \partial^E s] + V^{0,1} s \in \mathcal{L}_{J_\beta} \otimes \text{End} (E)
\]

Since \( d^A = \partial^E + \overline{\partial}^E \) is the ordinary canonical connection of \((E, h)\), we have

\[
\begin{align*}
\partial h(s_1, s_2) &= h(\partial^E s_1, s_2) + h(s_1, \overline{\partial}^E s_2), \\
\overline{\partial} h(s_1, s_2) &= h(\overline{\partial}^E s_1, s_2) + h(s_1, \partial^E s_2),
\end{align*}
\]

for sections \( s_1, s_2 \in E \). Since \( h \) is Hermitian, we also have

\[
\begin{align*}
[\beta, \partial h(s_1, s_2)] &= h(\beta, \partial^E s_1, s_2) + h(s_1, [\beta, \overline{\partial}^E s_2]) \\
[\overline{\beta}, \overline{\partial} h(s_1, s_2)] &= h(\overline{\beta}, \overline{\partial}^E s_1, s_2) + h(s_1, [\beta, \partial s_2])
\end{align*}
\]
Since \( V^{1,0} = -(V^{0,1})^* \), we have \( h(V^{1,0}s_1, s_2) + h(s_1, V^{1,0}s_2) = 0 \). From (10.8), we obtain
\[
\partial_{\bar{\beta}} h(s_1, s_2) = \partial h(s_1, s_2) + [\beta, \partial h(s_1, s_2)] - [\bar{\beta}, \bar{\partial} h(s_1, s_2)]
\]
Thus we have
\[
\partial_{\bar{\beta}} h(s_1, s_2) = h(\partial_{\bar{\beta}} s_1, s_2) + h(s_1, \partial_{\bar{\beta}} s_2)
\]
We also have
\[
\bar{\partial}_{\beta} h(s_1, s_2) = h(\bar{\partial}_{\beta} s_1, s_2) + h(s_1, \bar{\partial}_{\beta} s_2)
\]
Thus \( \mathcal{D}^E_{\beta} = \partial_{\beta}' \). Hence the canonical connection \( \mathcal{D}^E = \partial_{\beta}' + \overline{\partial}^E_{\beta} \) is given by
\[
\mathcal{D}^E_{\beta} = dA + V
\]

**Proposition 10.4.** Let \((E, h)\) be an Hermitian vector bundle over a generalized Kähler manifold \((M, J_{\beta}, J_{\psi})\). We denote by \( \mathcal{D}^E_{\beta} \) the canonical connection of \((E, h)\) as in Proposition 10.3. Then the curvature and the mean curvature of the canonical connection \( \mathcal{D}^E_{\beta} \) are respectively given by
\[
\mathcal{F}_{\mathcal{D}^E_{\beta}}(\psi) = F_A \cdot \psi + d^A(V \cdot \psi) + \frac{1}{2} [V \cdot V] \cdot \psi \tag{10.21}
\]
\[
K_{\mathcal{D}^E_{\beta}} = \pi_{U^{n,1}}^{Herm} \mathcal{F}_{\mathcal{D}^E_{\beta}}(\psi) \tag{10.22}
\]

**Proof.** The result follows from Proposition 10.3.

### 10.3 D-modules and Poisson modules over complex surfaces

A log Poisson structure is a holomorphic Poisson structure which is the dual of a log symplectic structure. A log Poisson structure admits remarkable features, that is, the Poisson cohomology is calculated by using the singular cohomology of the complement of zero divisor of a log Poisson structure coupled with datas of the Jacobi rings of singularities of the zero divisor. In particular, log Poisson structures on complex surfaces provide important classes of Poisson structures.

Let \( X = (M, J) \) be a complex surface. Then every holomorphic section \( \beta \) of the anti-canonical line bundle is a holomorphic Poisson structure. If the zero divisor \( C \) of \( \beta \) is smooth, \( \beta \) is a log Poisson structure. Then \( C \) is an elliptic curve which is of particular interest.

**Proposition 10.5.** Let \( X \) be a Poisson surface with a simple normal crossing anticanonical divisor \( C \). A Poisson module on a complex surface \( X \) is given by a meromorphic flat connection which admits single pole along the anti-canonical divisor \( C \). Thus a Poisson module \((\mathcal{E}, \delta^E_{\beta})\) gives a \( \mathcal{D} \)-module over \( X \)

**Proof.** A Poisson structure \( \beta \) gives a map from the sheaf of germs of holomorphic 1-forms \( \Omega_X^1 \) to \( \Theta \) by the contraction of \( \beta \) and \( \theta \in \Omega^1 \). Then the map is extended to a map \( i_{\beta} \) from \( \hat{\Omega}^1 \) to \( \Theta \), where \( \hat{\Omega}^1 \) is the sheaf of meromorphic 1-forms with single pole along \( C \). Then it turns out that the map is isomorphism
\[
i_{\beta} : \hat{\Omega}^1 \cong \Theta.
\]
The isomorphism \( i_{\beta} \) is extended to an isomorphism \( \wedge^p i_{\beta} : \wedge^p \hat{\Omega}^1 \cong \wedge^p \Theta \) by
\[
\theta_1 \wedge \cdots \wedge \theta_p \mapsto i_{\beta} \theta_1 \wedge \cdots \wedge i_{\beta} \theta_p.
\]
Let $\hat{\Omega}^p = \Lambda^p \hat{\Omega}^1$. Then it turns out that we have the isomorphism between the complex $(\hat{\Omega}^\bullet, d)$ and the Poisson complex $(\Lambda^\bullet, \delta_\beta)$ since $\Lambda^{p+1}i_\beta \circ d = \pm \delta_\beta \circ \Lambda^p i_\beta$. Tensoring $\mathcal{E}$ with the both complexes, we obtain the isomorphism between the complex $(\mathcal{E} \otimes \Lambda^\bullet \hat{\Omega}^1, \delta \mathcal{E})$ the Poisson module complex $(\mathcal{E} \otimes \Lambda^\bullet \Theta, \delta \mathcal{E})$, that is,

$$
\begin{array}{cccccccc}
0 & \to & \mathcal{E} & \xrightarrow{d\mathcal{E}} & \mathcal{E} \otimes \hat{\Omega}^1 & \xrightarrow{d\mathcal{E}} & \mathcal{E} \otimes \hat{\Omega}^2 & \xrightarrow{d\mathcal{E}} & \cdots \\
0 & \to & \mathcal{E} & \xrightarrow{\delta \mathcal{E}} & \mathcal{E} \otimes \Theta & \xrightarrow{\delta \mathcal{E}} & \mathcal{E} \otimes \wedge^2 \Theta & \xrightarrow{\delta \mathcal{E}} & \cdots \\
\end{array}
$$

Since $\delta \mathcal{E} \circ \delta \mathcal{E} = 0$, we have $d\mathcal{E} \circ d\mathcal{E} = 0$. Thus $d\mathcal{E}$ give a flat connection whose connection form allows a single pole along the anti-canonical divisor $C$. Then by using a meromorphic flat connection $\mathcal{E}$ becomes a $\mathcal{D}$-modules.

\section{10.4 The Serre construction of stable and unstable Poisson modules}

The book [35] is a good reference of the Serre construction in this subsection. Let $X$ be a del Pezzo surface which is a projective complex surface whose anticanonical line bundle $K_X^{-1} \cong \mathcal{O}(N)$ for a positive integer $N$. We shall construct examples of stable Poisson modules over $X$. Let $J_Y$ be the ideal sheaf which is given by a set of points $Y = \{p_1, \ldots, p_m\} \subset X$. Then we have a holomorphic vector bundle $E$ as an extension:

$$
0 \to \mathcal{O}_X \to E \to \mathcal{O}_X(k) \otimes J_Y \to 0, \quad (10.23)
$$

where $c_1(E) = k, c_2(E) = m$. Let $\mathcal{O}_X(k) \otimes J_Y = J_Y(k)$. Then an extension is given by an element $e$ of $\text{Ext}^1(J_Y(k), \mathcal{O}_X)$. The spectral sequence whose $E_2$-term is given by $\{H^p(X, \mathcal{E}xt^q(J_Y(k), \mathcal{O}_X))\}$ degenerates to $\text{Ext}^\bullet(J_Y(k), \mathcal{O}_X)$. Since $\text{Hom}(J_Y(k), \mathcal{O}_X) = \text{Hom}(\mathcal{O}_X(k), \mathcal{O}_X) = \mathcal{O}_X(-k)$, we have the local and global exact sequence:

$$
0 \to H^1(X, \mathcal{O}_X(-k))) \to \text{Ext}^1(J_Y(k), \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(J_Y(k), \mathcal{O}_X)) \xrightarrow{\partial} H^2(X, \mathcal{O}(-k)) \to \cdots
$$

Since $\mathcal{E}xt^1(J_Y(k), \mathcal{O}_X)$ is the Skyscraper sheaf with support $Y = \{p_1, \ldots, p_m\}$, we see

$$
H^0(X, \mathcal{E}xt^1(J_Y(k), \mathcal{O}_X)) = \oplus_{p_i \in Y} \mathbb{C}_{p_i}.
$$

Let $\{e_{p_i}\} \in \oplus_{p_i \in Y} \mathbb{C}_{p_i}$ be the image of $e \in \text{Ext}^1(J_Y(k), \mathcal{O}_X)$. Then it is known that $e$ gives a holomorphic vector bundle if and only if each $e_{p_i}$ does not vanish. From the Serre duality, we have $H^2(X, \mathcal{O}_X(-k)) = H^0(X, K_X(k)) = H^0(X, \mathcal{O}(k-N))$. If $k < N$, then $H^2(X, \mathcal{O}(-k)) = 0$. Thus we have an extension $e$ such that $e_{p_i} \neq 0$ for all $p_i \in Y$. In the case of $k \geq N$, $\dim H^0(X, \mathcal{O}(k-N)) = 0$. If there is an element $\{e_{p_i}\} \in \oplus_{p_i \in Y} \mathbb{C}_{p_i}$ such that $e_{p_i} \neq 0$ for all $p_i \in Y$ and $d_2(\{e_{p_i}\}) = 0$, then we obtain an extension as a vector bundle (c.f. Global duality in Chapter 5 of [10]).

**Proposition 10.6.** Let $C$ be the anti-canonical divisor on $X$ which is the zero set of a Poisson structure $\beta$. We assume that $C$ is a simple normal crossing divisor. Let $E$ be a holomorphic vector bundle $E$ which is an extension in the cases of $k = N$,

$$
0 \to \mathcal{O}_X \to E \to \mathcal{O}_X(N) \otimes J_Y \to 0, \quad (10.24)
$$

Then $E$ has a Poisson module structure if $Y$ is included in the anti-canonical divisor $C$.  

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Remark 10.7. Hitchin already constructed Poisson modules in the cases of a smooth elliptic curve by using Dolbeault type formulation of the Serre construction [23].

In order to show Proposition 10.6, we need local results of meromorphic extension of flat holomorphic connections on $\mathbb{C}^2$. Let $I$ be the ideal sheaf with support at the origin which is generated by $z_1, z_2$ over $\mathcal{O}_{\mathbb{C}^2}$, where $(z_1, z_2)$ be coordinates of $\mathbb{C}^2$. Then every nontrivial extension of $I$ by $\mathcal{O}_{\mathbb{C}^2}$ is given by

$$0 \xrightarrow{} \mathcal{O}_{\mathbb{C}^2} \xrightarrow{i} \mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{O}_{\mathbb{C}^2} \xrightarrow{j} I \xrightarrow{} 0,$$

where $i(1) = (\lambda z_1, \lambda z_2)$ and $j(f, g) = -z_2 f + z_1 g$, for $\lambda \neq 0 \in \mathbb{C}$. For simplicity, we consider the case of $\lambda = 1$. We denote by $C$ the curve which is given by $\{z_1 = 0\}$. Let $\nabla^I$ be a logarithmic connection of the complement $I|_{\mathbb{C}^2 \setminus C}$

$$d - \frac{dz_1}{z_1}$$

Then we have $\nabla^I z_1 = 0$. We denote by $\nabla^{(0)}$ the trivial connection of $\mathcal{O}_{\mathbb{C}^2}$. By using the splitting map $s$ such that $j \circ s = \text{id}$ on the complement $\mathbb{C}^2 \setminus C$, we have an isomorphism $i \oplus s : \mathcal{O}_{\mathbb{C}^2 \setminus C} \oplus I|_{\mathbb{C}^2 \setminus C} \rightarrow \mathcal{O}_{\mathbb{C}^2 \setminus C} \oplus \mathcal{O}_{\mathbb{C}^2 \setminus C}$. Then $\nabla^{(0)} \oplus \nabla^I$ induces a holomorphic connection $\nabla$ on $\mathcal{O}_{\mathbb{C}^2 \setminus C} \oplus \mathcal{O}_{\mathbb{C}^2 \setminus C}$ by using the isomorphism $i \oplus s$. Then we have

Lemma 10.8. The holomorphic connection $\nabla$ can be extended to be a meromorphic connection of $\mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{O}_{\mathbb{C}^2}$ with a simple pole along $C = \{z_1 = 0\}$.

Proof. Let $\{e_1, e_2\}$ be a basis $\mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{O}_{\mathbb{C}^2}$ which is given by $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Since $i \circ \nabla^{(0)} = \nabla \circ i$, it follows from $\nabla^{(0)} 1 = 0$ that

$$\nabla(z_1 e_1 + z_2 e_2) = (dz_1) e_1 + (dz_2) e_2 + z_1 \nabla e_1 + z_2 \nabla e_2 = 0.$$  \hspace{1cm} (10.26)

A splitting map $s$ is written as $s(z_1) = e_2 + f(z_1 e_1 + z_2 e_2)$ for $f \in \mathcal{O}_{\mathbb{C}^2}$. Since $\nabla s(z_1) = \nabla^I z_1 = 0$, it follows from (10.26) that

$$\nabla s(z_1) = \nabla e_2 + df(z_1 e_1 + z_2 e_2) = 0.$$  \hspace{1cm} (10.27)

Then from (10.26) and (10.27), we obtain

$$\nabla e_1 = - \frac{dz_1}{z_1} e_1 - \frac{dz_2}{z_1} e_2 + df(z_2 e_1 + \frac{z_2}{z_1} e_2)\hspace{1cm} (10.28)$$

$$\nabla e_2 = -d f(z_1 e_1 + z_2 e_2).$$  \hspace{1cm} (10.29)

Thus the connection $\nabla$ is extended to be a meromorphic connection on $\mathbb{C}^2$ with simple pole along $\{z_1 = 0\}$.

Lemma 10.9. Let $C'$ be a curve defined by $\{z_1 z_2 = 0\}$ on $\mathbb{C}^2$ and $I$ the ideal sheaf generated by $z_1, z_2$ as before. We define a connection $\nabla^{I'}$ by

$$d - \frac{dz_1}{z_1} - \frac{dz_2}{z_2}.$$  \hspace{1cm} (10.25)

Then by using a splitting map $s$ as in (10.25), the connection $\nabla^{(0)} \oplus \nabla^{I'}$ yields a connection $\nabla'$ on $\mathcal{O}_{\mathbb{C}^2 \setminus C'} \oplus \mathcal{O}_{\mathbb{C}^2 \setminus C'}$. Then $\nabla$ is extended to be a meromorphic connection on $\mathbb{C}^2$ with a simple pole along $C' = \{z_1 z_2 = 0\}$.

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PROOF. Our proof is the same as before. For a splitting map \( s \), we have 
\[ s(z_1 z_2) = z_2 e_2 + f(z_1 e_1 + z_2 e_2). \]
Then \( \nabla' s(z_1 z_2) = \nabla'' z_1 z_2 = 0 \). Thus we have
\[ z_2 \nabla' e_2 + d z_2 e_1 + (d f)(z_1 e_1 + z_2 e_2) = 0 \]
Since \( \nabla'(z_1 e_1 + z_2 e_2) = 0 \), we also have
\begin{align*}
\nabla' e_1 &= - \frac{dz_1}{z_1} e_1 - \frac{dz_2}{z_2} e_2 - \frac{z_2}{z_1} \nabla' e_2 \\
&= - \frac{dz_1}{z_1} e_1 - \frac{dz_2}{z_2} e_2 - \frac{dz_2}{z_2} e_1 - \frac{z_2 e_2}{z_1} (df)(e_1 + \frac{z_2}{z_1} e_2) 
\end{align*}
(10.30)
Hence \( \nabla' \) is extended to be a meromorphic connection with a simple pole along \( C' = \{ z_1 z_2 = 0 \} \). \( \square \)

**Proof of Proposition 10.6** Let \( X = \cup_\alpha U_\alpha \) be an covering of affine open sets. The zero set of \( \beta \) is given by \( \{ f_\alpha = 0 \} \) on \( U_\alpha \), where \( \{ f_\alpha \} \) gives a section of the anti-canonical line bundle. Then it turns out that
\[ \{ d - \frac{df_\alpha}{f_\alpha} \} \]
yields a meromorphic flat connection of \( K_X^{-1} \cong \mathcal{O}(N) \), which has a logarithmic pole along \( C \). Thus it follows from Proposition 10.5 that \( K_X^{-1} \) is a Poisson module. Let \( A_\alpha := \frac{-df_\alpha}{f_\alpha} \) be a connection form of the meromorphic connection of \( K_X^{-1} \). Since the complement \( X \setminus C \) is Stein, the extension \( \{ \cdot \} \) restricted to the complement \( X \setminus C \) gives the trivial extension:
\[ \begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_X|_{X \setminus C} & \longrightarrow & E|_{X \setminus C} & \longrightarrow & \mathcal{O}_X(N)|_{X \setminus C} & \longrightarrow & 0,
\end{array} \]
(10.32)
Thus \( E|_{X \setminus C} \cong \mathcal{O}_X|_{X \setminus C} \oplus \mathcal{O}_X(N)|_{X \setminus C} \) has the holomorphic flat connection \( \nabla \) which is induced from the product of flat connections of \( \mathcal{O}_X|_{X \setminus C} \oplus \mathcal{O}_X(N)|_{X \setminus C} \). From Lemma 10.8 and Lemma 10.9, it turns out that every holomorphic flat connection on \( X \setminus C \) can be extended as a meromorphic connection along a single pole along \( C \). From Proposition 10.5, \( E \) becomes a Poisson module. \( \square \)

Let \( \psi = e^{-\sqrt{-1} \omega} \), where \( \omega \) is a Kähler structure. Then we have the notion of the ordinary stability with respect to \( \omega \) and the notion of \( \psi \)-stability of generalized holomorphic vector bundles. As in before, a Poisson module \( (\mathcal{E}, \delta^E) \) gives the generalized holomorphic vector bundle \( (E, \mathcal{D}^E_{\mathcal{J}^E}) \), where \( \mathcal{E} = \mathcal{O}(E) \).

**Proposition 10.10.** If \( E \) as in (10.23) is stable as a holomorphic vector bundle, then \( E \) is \( \psi \)-stable as Poisson module (a generalized holomorphic vector bundle).

**Proof.** Since we have the slope inequality \( \mu(\mathcal{F}) < \mu(\mathcal{E}) \) for all subsheaves \( \mathcal{F} \), we also have the slope inequality for all Poisson subsheaves. \( \square \)

We shall explain the \( \psi \)-stability and \( \psi \)-semistability in the case \( X = \mathbb{C}P^2 \). For \( k = 3 \), we have the following:
\[ \begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & \mathcal{O}_X(3) \otimes \mathcal{J}_Y & \longrightarrow & 0,
\end{array} \]
(10.33)
Then it is known that \( E \) is stable if and only if \( Y = \{ p_1, p_2, p_3 \} \) is not contained in a line \( l \subset \mathbb{C}P^2 \). If \( Y \subset l \), then \( E \) is not stable as a holomorphic vector bundle, that is, \( E \) does not admit any Einstein-Hermitian metric in the ordinary sense.
Thus it turns out that the ordinary stability and semistability of \( E \) depend on a configuration of points \( p_1, \cdots, p_m \). However the following shows that \( \psi \)-stability of Poisson modules is different from the ordinary stability:

**Proposition 10.11.** We assume that \( C \) is smooth. A Poisson module \( E \) as in (10.33) is always \( \psi \)-stable. Thus \( E \) admits an Einstein-Hermitian metric of a generalized holomorphic vector bundle over generalized Kähler manifold \( (M, J_{J_0}, J_\psi) \).

The following Figure 1 in the cases of three points, explains the difference between the ordinary stability and \( \psi \)-stability:

![Diagram](image)

**Proof of Proposition 10.11.** Tensoring \( \mathcal{O}(-l) \) with the exact sequence (10.33), we obtain

\[
0 \longrightarrow \mathcal{O}_X(-l) \longrightarrow E(-l) \longrightarrow J_Y(3-l) \longrightarrow 0, \quad (10.34)
\]

where \( E(-l) := E \otimes \mathcal{O}(-l) \) and \( J_Y(3-l) := \mathcal{O}_X(3-l) \otimes J_Y \). This sequence is an exact sequence of Poisson modules, that is, meromorphic flat connections are preserved. Let \( \nabla^{E(-l)} \) be the meromorphic flat connection of \( E(-l) \). We denote by \( H^0_\beta(X, E(-l)) \) the space of holomorphic parallel sections of \( E(-l) \), that is, \( \{ s \in H^0(X, E(-l)) | \nabla^{E(-l)}s = 0 \} \). We also denote by \( H^0_\beta(X, J_Y(3-l)) \) the space of holomorphic parallel sections of \( J_Y(3-l) \). We have \( H^1(X, \mathcal{O}(-l)) = 0 \) since \( X = \mathbb{C}P^2 \). If \( l \geq 1 \), we have \( H^0(X, \mathcal{O}_X(-l)) = 0 \).

Then the long exact sequence gives an exact sequence of holomorphic parallel sections:

\[
0 \longrightarrow 0 \longrightarrow H^0_\beta(X, E(-l)) \longrightarrow H^0_\beta(X, J_Y(3-l)) \longrightarrow 0, \quad (10.35)
\]

Thus we have the isomorphism \( H^0_\beta(X, E(-l)) \cong H^0_\beta(X, J_Y(3-l)) \).
If \( l \geq 3 \), then \( H^0(X, J_Y(3-l)) = 0 \). In the case of \( l = 2 \), every section of \( H^0(X, J_Y(1)) \) has zero along a line. Since \( C \) is smooth, every section of \( H^0(X, J_Y(1)) \) has zero on the complement \( X \setminus C \). Since the flat connection of \( J_Y(3-l) \) is holomorphic on the complement \( X \setminus C \), it implies that \( H^0_\beta(X, J_Y(3-l)) = 0 \).

We assume that there exists a destabilizing object, that is, a Poisson subsheaf \( i : \mathcal{O}(1) \to E \) with \( l \geq 2 \). Thus the inclusion \( i \) gives a nonzero parallel section \( s \) of \( H^0_\beta(X, E(-l)) \). However \( H^0_\beta(X, E(-l)) = 0 \) and then \( s = 0 \). This is a contradiction.

On the other hand, if the anti-canonical divisor admits singularities, then a different aspect of \( \psi \)-stability appears.

**Proposition 10.12.** Let \( \beta \) be a Poisson structure on \( X := \mathbb{C}P^2 \) whose zero set consists of three lines \( l_1, l_2 \) and \( l_3 \) in a general position (See Figure 2: three points and three lines). We denote by \( E \) a Poisson module which is given by the extension as in \( \text{(10.33)} \). If the support \( Y \) of the ideal \( J_Y \) is included in one of lines, then \( E \) is not \( \psi \)-stable.

**Proof.** We assume that \( Y \) is included in a line \( l_1 \). Let \( X = \bigcup \alpha U_\alpha \) be an open covering as before and \( f^{(1)}_\alpha \) a defining equation of the line \( l_1 \) on each \( U_\alpha \). Then as before,

\[
d - \frac{df^{(1)}_\alpha}{f^{(1)}_\alpha}
\]

yields a meromorphic flat connection \( \nabla^{(1)} \) of \( \mathcal{O}(1) \) and the section \( s^{(1)} := \{ f^{(1)}_\alpha \} \) is a flat section of \( \mathcal{O}(1) \), with respect to \( \nabla^{(1)} \). Since the zero set of \( s^{(1)} \) is \( l_1 \) and \( Y \subset l_1 \), we see that \( H^0_\beta(X, J_Y(1)) \neq 0 \). Since \( H^0_\beta(X, J_Y(1)) \neq 0 \), there is an inclusion \( i : \mathcal{O}(2) \to E \) which preserves flat connections. Thus there is a Poisson submodule \( \mathcal{O}(2) \) of \( E \). Hence \( E \) is not \( \psi \)-stable since \( \mu(E) = \frac{3}{2} \). \( E \) is irreducible, since \( c_2(E) = m \neq 0 \). Thus \( E \) is not \( \psi \)-polystable.

The following Figure 2 explains that a Poisson module is not \( \psi \)-stable if an anticanonical divisor consists of three lines and three points are on a line of them.

**Figure 2: three points and three lines**
11 Vanishing theorems of generalized holomorphic vector bundles

11.1 Vanishing theorem in the case of $\psi = e^{-\sqrt{-1}\omega}$

Let $(E, h)$ be an Hermitian vector bundle over a generalized Kähler manifold of symplectic type $(M, J, J_\psi)$. In this subsection, we assume that $\psi = e^{\sqrt{-1}\omega}$ for simplicity. Let $\mathcal{D} : \Gamma(E) \rightarrow \Gamma(E \otimes (TM \oplus T^*M))$ be a generalized connection:

$$\mathcal{D} = d + A + V = D^A + V,$$

where $D^A = d + A$ is an ordinary unitary connection and $V$ is a section of $\text{End}(E) \otimes T_M$. We have the decomposition of $(TM \oplus T^*M)$ with respect to $J$, that is,

$$(TM \oplus T^*M)^C = L_J \oplus \overline{L_J}$$

and we also have another decomposition with respect to $J_\psi$

$$(TM \oplus T^*M)^C = L_\psi \oplus \overline{L_\psi}.$$

Since $J$ and $J_\psi$ commute, we have the simultaneous decomposition:

$$(TM \oplus T^*M)^C = L_J^+ \oplus L_J^- \oplus \overline{L_J^+} \oplus \overline{L_J^-},$$

where $L_J = L_J^+ \oplus L_J^-$, $L_\psi = L_\psi^+ \oplus \overline{L_\psi}$. Note that $L_\psi = \{ e \in TM \oplus T^*M \mid e \cdot \psi = 0 \}$. The following table explains our decomposition:

|   | $L_J$ | $\overline{L_J}$ |
|---|---|---|
| $L_\psi$ | $L_\psi^+$ | $\overline{L_\psi}$ |
| $\overline{L_\psi}$ | $\overline{L_\psi}^+$ | $\overline{L_\psi}^-$ |

According to the decomposition, a generalized connection $\mathcal{D}$ is decomposed into

$$\mathcal{D} = \mathcal{D}^1 \oplus \mathcal{D}^0, \quad \mathcal{D}^1 = L_J \otimes \text{End}(E), \quad \mathcal{D}^0 = L_\psi \otimes \text{End}(E)$$

(11.1)

$$\mathcal{D} = \mathcal{D}' \oplus \mathcal{D}'', \quad \mathcal{D}' \in L_\psi \otimes \text{End}(E), \quad \mathcal{D}'' \in \overline{L_\psi} \otimes \text{End}(E)$$

(11.2)

$$\mathcal{D}^1 \oplus \mathcal{D}^0 = \mathcal{D}^1_+ \oplus \mathcal{D}^0_+ + \mathcal{D}^1_- \oplus \mathcal{D}^0_-,$$

(11.3)

where $\mathcal{D}^1 \in L_\psi \otimes \text{End}(E)$ and $\mathcal{D}^0 \in \overline{L_\psi} \otimes \text{End}(E)$. Then we have

$$\mathcal{D}' = \mathcal{D}^1_+ \oplus \mathcal{D}^0_+, \quad \mathcal{D}'' = \mathcal{D}^1_- \oplus \mathcal{D}^0_-.$$

We shall introduce the following ordinary connections $D'$ and $D''$:

$$D' = D^A + \sqrt{-1}\omega(V),$$

(11.4)

$$D'' = D^A - \sqrt{-1}\omega(V)$$

(11.5)

where $\omega(V) \in T^*_M \otimes \text{u}(E)$ is given by the contraction between the real 2-form $\omega$ and $V \in \text{u}(E) \otimes T_M$. Then we have
LEMMA 11.1.

\[ dh(s_1, s_2) = h(D's_1, s_2) + h(s_1, D''s_2) \]
\[ dh(s_1, s_2) = h(D''s_1, s_2) + h(s_1, D's_2) \]

PROOF. Since \( D^A \) is a unitary connection, we have

\[ dh(s_1, s_2) = h(D^A s_1, s_2) + h(s_1, D^A s_2) \]

Since \( V \) is a Skew-Hermitian matrix valued vector field, then the contraction \( \sqrt{-1}\omega(V) \) is an Hermitian matrix-valued 1-form. Thus we have

\[ 0 = h(\sqrt{-1}\omega(V)s_1, s_2) + h(s_1, -\sqrt{-1}\omega(V)) \]

Thus we have \( dh(s_1, s_2) = h(D's_1, s_2) + h(s_1, D''s_2) \).

As in before, \( D', D'' \) are extended as the operators \( d^{D'}, d^{D''} \) of \( E \otimes \wedge^* T^*_M \), respectively. Note that \( D' \) and \( D'' \) are connections which are different from generalized connections \( D' \) and \( D'' \). The relations between them are given by

LEMMA 11.2. For a section \( s \in E \), we have

\[ (D''s) \cdot \psi = d^{D''}(s \psi) \]
\[ (D's) \cdot \bar{\psi} = d^{D'}(s \bar{\psi}) \]

PROOF. Since \( D = d + A + V = D' + D'' \) and \( \psi = e^{-\sqrt{-1}\omega} \), we have

\[ D' = \frac{1}{2}(d - \sqrt{-1}\omega^{-1}d + A - \sqrt{-1}\omega^{-1}(A) + V + \sqrt{-1}\omega(V)) \quad (11.6) \]
\[ D'' = \frac{1}{2}(d + \sqrt{-1}\omega^{-1}d + A + \sqrt{-1}\omega^{-1}(A) + V - \sqrt{-1}\omega(V)) \quad (11.7) \]

Then we have

\[ (D's) \cdot \bar{\psi} = ds \cdot \bar{\psi} + (As) \cdot \bar{\psi} + \sqrt{-1}(\omega(V)s) \cdot \bar{\psi} = d^{D'}(s \bar{\psi}) \]
\[ (D''s) \cdot \psi = ds \cdot \psi + (As) \cdot \psi - \sqrt{-1}(\omega(V)s) \cdot \psi = d^{D''}(s \psi) \]

For \( s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \in E \otimes \wedge^* T^*_M \), we define

\[ h(s_1 \otimes \alpha_1, s_2 \otimes \alpha_2) := h(s_1, s_2)\alpha_1 \wedge \sigma(\bar{\alpha}_2) \in \wedge^* T^*_M. \quad (11.8) \]

Taking the projection to the top forms, i.e., \( (2n\text{-forms}) \), we define

\[ h(s_1 \otimes \alpha_1, s_2 \otimes \alpha_2)_s := h(s_1, s_2)(\alpha_1 \wedge \sigma(\bar{\alpha}_2))_{\text{top}} \in \wedge^{2n} T^*_M. \]

Then we obtain

LEMMA 11.3.

\[ dh(s_1 \otimes \alpha_1, s_2 \otimes \alpha_2) = h(d^{D'}(s_1 \otimes \alpha_1), s_2 \otimes \alpha_2) + (-1)^{|\alpha_1|+|\alpha_2|}h(s_1 \otimes \alpha_1, d^{D'}(s_2 \otimes \alpha_2)), \]
\[ dh(s_1 \otimes \alpha_1, s_2 \otimes \alpha_2) = h(d^{D''}(s_1 \otimes \alpha_1), s_2 \otimes \alpha_2) + (-1)^{|\alpha_1|+|\alpha_2|}h(s_1 \otimes \alpha_1, d^{D''}(s_2 \otimes \alpha_2)), \]

where \( |\alpha_i| \) denotes the degree of differential form \( \alpha_i \).
Proof. We have the following:

\[ d\sigma(\alpha) = \begin{cases} +\sigma(d\alpha), & (|\alpha| = \text{even}) \\ -\sigma(d\alpha), & (|\alpha| = \text{odd}) \end{cases} \]

Then the result follows from Lemma \[11.1\] and \[11.8\] \[\square\]

Let \( s \in \Gamma(E) \). By the action of the exterior derivative \( d \) on both differential forms

\[ h(dD'(s\psi), s\psi) \quad \text{and} \quad h(dD''(s\psi), s\psi), \]

we obtain the following equalities of top forms from Lemma \[11.3\]

\[
\left( \frac{d}{dh}(dD'(s\psi), s\psi) \right)_{top} = h(D''(dD'(s\psi), s\psi))_{s} - h(D'(dD'(s\psi), s\psi))_{s} \tag{11.9}
\]

\[
\left( \frac{d}{dh}(dD''(s\psi), s\psi) \right)_{top} = h(D''(dD''(s\psi), s\psi))_{s} - h(D''(dD'(s\psi), s\psi))_{s}
\]

Since \( D' = d^A + \sqrt{-1}\omega(V) \), \( D'' = d^A - \sqrt{-1}\omega(V) \), we have

\[
dD'(dD'(s\psi)) = (d^A + \sqrt{-1}\omega(V))(d^A - \sqrt{-1}\omega(V))(s\psi) \tag{11.10}
\]

\[
= (F_A + \omega(V) \cdot \omega(V))(s\psi) \tag{11.11}
\]

\[
- \sqrt{-1}d^A(\omega(V)(s\psi)) + \sqrt{-1}\omega(V)d^A(s\psi) \tag{11.12}
\]

\[
dD''(dD'(s\psi)) = (F_A + \omega(V) \cdot \omega(V))(s\psi) \tag{11.13}
\]

\[
+ \sqrt{-1}d^A(\omega(V)(s\psi)) - \sqrt{-1}\omega(V)d^A(s\psi) \tag{11.14}
\]

Hence we have

**Lemma 11.4.**

\[
(dD''(dD' + dD''dD'))(s\psi) = 2(F_A + \omega(V) \cdot \omega(V))(s\psi)
\]

**Lemma 11.5.** Let \( s \) be a nonzero section of \( E \) satisfying \( D^{0,1}s = 0 \). \((s \neq 0)\). We set \( i^{\pm n}\langle \psi, \bar{\psi} \rangle \) to be a volume form on \( M \). Then the following \( 2n \)-forms are positive with respect to the volume form,

\[
i^{\pm n}h(dD'(s\psi), dD'(s\psi))_{s} > 0, \quad i^{\pm n}h(dD''(s\psi), dD''(s\psi))_{s} > 0 \tag{11.16}
\]

In other words, we have

\[
\frac{h(dD'(s\psi), dD'(s\psi))_{s}}{\langle \psi, \bar{\psi} \rangle_{s}} > 0, \quad \frac{h(dD''(s\psi), dD''(s\psi))_{s}}{\langle \psi, \bar{\psi} \rangle_{s}} > 0 \tag{11.17}
\]

We need the following two lemmas for our proof of Lemma \[11.5\] Let \( \{e_{-i,j}^{1,0}\}_{i=1}^{n} \) be a unitary basis of \( \mathcal{L}_{\bar{T}} \) which are locally defined, that is,

\[
\langle e_{-i,j}^{1,0}, e_{-i,j}^{1,0} \rangle_{T \otimes T^{*}} = -\delta_{i,j}
\]

**Lemma 11.6.**

\[
\langle e_{-i,j}^{1,0} \cdot \psi, e_{-i,j}^{1,0} \cdot \bar{\psi} \rangle_{s} = 2\delta_{i,j} \langle \psi, \bar{\psi} \rangle_{s}
\]
Proof.

\[
\langle e_{-i}^{1,0}, \overline{e_{-j}^{1,0}} \rangle_s = -\langle e_{-j}^{1,0}, e_{-i}^{1,0} \rangle_s
\]

(11.18)

\[
-\langle (e_{-j}^{1,0} + e_{-j}^{1,0} \cdot e_{-j}^{0,0}) \cdot \psi, \overline{\psi} \rangle_s
\]

(11.19)

\[
= -2\langle e_{-j}^{1,0} \cdot e_{-j}^{0,0} \rangle_s
\]

(11.20)

\[
= 2\delta_{i,j} \langle \psi, \overline{\psi} \rangle_s
\]

(11.21)

\[
-\langle (e_{-j}^{1,0} + e_{-j}^{1,0} \cdot e_{-j}^{0,0}) \cdot \psi, \overline{\psi} \rangle_s
\]

(11.22)

Since

\[
D''_{-i} = D_{-i}^{1,0} + D_{-i}^{1,0} \cdot s
\]

and

\[
D''_{-i} = 0,
\]

we have

\[
D''_{-i} = 0.
\]

Thus we obtain

\[
\langle e_{-j}^{1,0} \cdot \psi, \overline{\psi} \rangle_s = -2\delta_{i,j} \langle \psi, \overline{\psi} \rangle_s
\]

(11.23)

Proof of Lemma 11.7. First we shall show

\[
\frac{h(\langle d''(s\psi), d''(s\psi) \rangle_s)}{\langle \psi, \overline{\psi} \rangle_s} > 0.
\]

From Lemma 11.2, we obtain

\[
h(\langle d''(s\psi), d''(s\psi) \rangle_s) = h(\langle (D''_s \cdot \psi), (D''_s \cdot \psi) \rangle_s)
\]

(11.22)

Since \(D'' = D''_{-i} + D''_{+i}\) and \(D_{+i}^{0,1} = 0\), we have \(D_{+i}^{0,1} = 0\). Thus we obtain

\[
\langle e_{-j}^{1,0} \cdot \psi, \overline{\psi} \rangle_s = -2\delta_{i,j} \langle \psi, \overline{\psi} \rangle_s
\]

(11.24)

By using our unitary basis, \(D_{-i}^{1,0}\) is written by

\[
(D_{-i}^{1,0}) = \sum_{i} s_i e_{-i}^{1,0}
\]

Applying Lemma 11.6, we obtain

\[
\frac{h(\langle d''(s\psi), d''(s\psi) \rangle_s)}{\langle \psi, \overline{\psi} \rangle_s} = \sum_{i,j} h(s_i e_{-i}^{1,0} \cdot \psi, s_j e_{-j}^{1,0} \cdot \psi)_s
\]

(11.25)

\[
= \sum_{i,j} h(s_i, s_j) \langle e_{-i}^{1,0} \cdot \psi, e_{-j}^{0,0} \cdot \overline{\psi} \rangle_s
\]

(11.26)

Thus we obtain

\[
\frac{h(\langle d''(s\psi), d''(s\psi) \rangle_s)}{\langle \psi, \overline{\psi} \rangle_s} = \sum_{i} h(s_i, s_i) = \|D''_s\|^2 \geq 0
\]

Then it turns out that the equality holds if and only if \(s = 0\).
Secondly, we shall show

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s \langle \psi, \overline{\psi}\rangle_s > 0. \]

Taking the complex conjugate, we have

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s \langle \psi, \overline{\psi}\rangle_s > 0 \]

**Lemma 11.8.**

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s = -h\langle d^{D'}(s\overline{\psi}), d^{D'}(s\overline{\psi})\rangle_s \]

**Proof.** Let \( \{\theta_i\} \) be a symplectic basis of \( T^*_M \) which satisfies \( \omega = \sum_i \theta_i \wedge \theta_i^* \). Then \( D' \) is written as \( D' = \sum_i s_i \theta_i \). Since \( \langle \theta_i \psi, \theta_i \psi \rangle_s = 0 \), we have

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s = \sum_{ij} h(s_i, s_j) \langle \theta_i^* \overline{\psi}, \theta_j \psi \rangle_s \]

\[ = \sum_{i \neq j} h(s_i, s_j) \langle \theta_i^* \overline{\psi}, \theta_j \psi \rangle_s \]

\[ = -\sum_{i \neq j} h(s_j, s_i) \langle \theta_i^* \overline{\psi}, \theta_j \psi \rangle_s \]

\[ = -\sum_{i \neq j} h(s_j, s_i) \theta_i^* \overline{\psi}, s_i \theta_j \psi \rangle_s \]

\[ = -h\langle d^{D'}(s\overline{\psi}), d^{D'}(s\overline{\psi})\rangle_s \]

Then we obtain

\[ h\langle d^{D'}(s\overline{\psi}), d^{D'}(s\overline{\psi})\rangle_s = h\langle (D')^* s \cdot \overline{\psi}, (D')^* s \cdot \overline{\psi} \rangle_s \]

\[ = h\langle (D_{+0}^1 s) \cdot \overline{\psi}, (D_{+0}^1 s) \cdot \overline{\psi} \rangle_s \]

By using our local basis, we obtain

\[ (D_{+0}^1 s) = \sum_i s_i e_{+,i}^{1,0} \]

Then applying Lemma 11.7, we have

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s = -\sum_{ij} h(s_i e_{+,i}^{1,0} \cdot \overline{\psi}, s_j e_{+,j}^{1,0} \cdot \overline{\psi})_s \]

\[ = -\sum_{ij} h(s_i, s_j) e_{+,i}^{1,0} \cdot \overline{\psi}, e_{+,j}^{1,0} \cdot \overline{\psi})_s \]

\[ = 2 \sum_i h(s_i, s_i) \langle \overline{\psi}, \psi \rangle_s \]

Thus we obtain

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s = \sum_i h(s_i, s_i) \|D'\|^2 > 0 \]

Hence we have the result

\[ h\langle d^{D'}(s\psi), d^{D'}(s\psi)\rangle_s > 0 \]

\[ \square \]
Applying Lemma 11.4 to (11.9), we obtain
\[ dh\langle d'(s\psi), s\psi \rangle + h(d''(s\psi), s\psi) \rangle_{2n-1} = h(F_A + \omega(V) \cdot \omega(V))(s\psi), s\psi) \]
(11.37)
(11.38)
(11.39)

Then we have

**Theorem 11.9.**

\[ 0 = \int_M i^n h((F_A + \omega(V) \cdot \omega(V))(s\psi), s\psi) \]

A generalized Hermitian connection \( D \) satisfies the Einstein-Hermitian condition if and only if

\[ F_A + \omega(V) \cdot \omega(V) = -i\lambda \] id,

where \( \lambda \) is Einstein constant. This is equivalent to

\[ \sqrt{-1} \lambda \omega (F_A + \omega(V) \wedge \omega(V)) = \lambda \text{id} \]

Thus we have

\[ i^n ((F_A + \omega(V) \cdot \omega(V))(s\psi), s\psi) \]

Since \( \psi = e^\tau \), we have

\[ i^n (\frac{\lambda \omega}{i} \wedge \psi, \overline{\psi}) = \lambda i^n \frac{\omega}{i} \wedge (\frac{\omega}{i})^{n-1} \frac{1}{(n-1)!} \]

(11.40)

and we have

\[ \langle \psi, \overline{\psi} \rangle_{\lambda} = \frac{1}{n!} (\frac{\omega}{i})^n \]

Thus we have the following vanishing theorem:

**Theorem 11.10.** Let \( D \) be a generalized Hermitian connection which satisfies the Einstein-Hermitian condition for negative Einstein constant \( \lambda < 0 \). Then a generalized holomorphic section \( s \) of \( E \) which satisfies \( D^0,1 s = 0 \) is zero. If \( \lambda = 0 \), then a generalized holomorphic section \( s \) is parallel.

**Proof.** We have the following:

\[ \int_M i^n h((F_A + \omega(V) \cdot \omega(V))(s\psi), s\psi) \]

If \( \lambda < 0 \), then \( s = 0 \). If \( \lambda = 0 \), then \( D's = D''s = 0 \). Then \( Ds = 0 \).

11.2 In the general cases of \( \psi = e^{b - \sqrt{-1} \omega} \)

Let \( (E, h) \) be an Hermitian vector bundle over \( (M, J, \overline{J}) \), where \( \psi = e^{b - \sqrt{-1} \omega} \). Then by the action of the \( d \)-closed form \( b \), we have a generalized Kähler structure \( (\overline{J} b, \overline{J} \psi) \), where \( \overline{J} b = e^b \overline{J} e^b \) and \( \overline{J} \psi = e^{-\sqrt{-1} \omega} \).

Let \( D^A = D^{1,0} + D^{0,1} \) be a generalized Hermitian connection satisfying the followings:

\[ D^{0,1} \circ D^{0,1} = 0 \]
Let \( E \) be a generalized holomorphic sub bundle of \( E \) of rank \( p \). We denote by \( j : E_1 \to E \) the inclusion. Then taking \( p \)th skew-Symmetric power of both sides, we have the map \( \wedge^p : \wedge^p E_1 \to \wedge^p E \). Thus we have a non zero section \( s \) of \( \wedge^p \otimes (\wedge^p E_1)^* \) which is generalized holomorphic. Every line bundle admits a generalized Einstein-Hermitian metric and \( \wedge^p E \) also satisfies the Einstein-Hermitian condition.

**Theorem 11.11.** Let \( (E, h) \) be an Hermitian vector bundle over a compact generalized Kähler manifold \((M, J, J_\psi)\) of symplectic type. We assume that a generalized Hermitian connection \( D^A := D^{1,0} + D^{0,1} \) satisfies \( D^{0,1} \circ D^{0,1} = 0 \) and the Einstein-Hermitian condition

\[
\kappa_A(\psi) = \lambda \text{id}_E
\]

If \( \lambda \) is negative, then every section \( s \in \Gamma(E) \) satisfying \( D^{0,1}s = 0 \) is a zero section. If \( \lambda = 0 \), then every generalized holomorphic section is parallel.

**Proposition 11.12.** Let \( (E, h) \) be an Hermitian vector bundle over a compact generalized Kähler manifold \((M, J, J_\psi)\) of symplectic type. We assume that a generalized Hermitian connection \( D^A := D^{1,0} + D^{0,1} \) satisfies \( D^{0,1} \circ D^{0,1} = 0 \) and the Einstein-Hermitian condition

\[
\kappa_A(\psi) = \lambda \text{id}_E
\]

Then the dual bundle \((\otimes^p E) \otimes (\otimes^q E^*)\) also satisfies the Einstein-Hermitian condition with factor \((p-q)\lambda\). In particular, \( \wedge^p E \) satisfies the Einstein-Hermitian condition with factor \( p\lambda \).

**Proof.** The result directly follows from the standard connection theory. \( \square \)

We define the slope of \( E \) by the ratio

\[
\mu(E) := \frac{\langle c_1(M) \cdot \psi, \overline{\psi} \rangle_s}{\langle \psi, \overline{\psi} \rangle_s}
\]

**Theorem 11.13.** Let \( (E, h) \) be an Hermitian vector bundle over a compact generalized Kähler manifold \((M, J, J_\psi)\) of symplectic type. We assume that a generalized Hermitian connection \( D^A := D^{1,0} + D^{0,1} \) satisfies \( D^{0,1} \circ D^{0,1} = 0 \) and the Einstein-Hermitian condition

\[
\kappa_A(\psi) = \lambda \text{id}_E
\]

Let \( E_1 \) be a generalized holomorphic sub bundle of \( E \). Then we have the following inequality

\[
\mu(E_1) \leq \mu(E)
\]

**Proof.** Let \( E_1 \) be a generalized holomorphic subbundle of \( E \) of rank \( p \). We denote by \( j \) the inclusion \( j : E_1 \to E \). Then taking \( p \)th skew-Symmetric power of both sides, we have the map \( \wedge^p : \wedge^p E_1 \to \wedge^p E \). Thus we have a non zero section \( s \) of \( \wedge^p \otimes (\wedge^p E_1)^* \) which is generalized holomorphic. Every line bundle admits a generalized Einstein-Hermitian metric and \( \wedge^p E \) also satisfies the Einstein-Hermitian condition.
Thus \( \wedge^p \otimes (\wedge^p E_1)^* \) also satisfies the Einstein-Hermitian condition with factor \( -p\mu(E_1) + p\mu(E) \). Hence it follows from Theorem 11.11 that we have

\[
0 \leq -p\mu(E_1) + p\mu(E)
\]

Thus we have the inequality

\[ \mu(E_1) \leq \mu(E). \]

\[ \square \]

**Theorem 11.14.** Let \((E, h)\) be an Einstein-Hermitian generalized holomorphic vector bundle over a compact generalized Kähler manifold \((M, J, J_\psi)\). Then \( E \) is \( \psi \)-poly-stable and \( E \) is a direct sum

\[
(E, h) = \bigoplus_i (E_i, h_i)
\]

where each \((E_i, h_i)\) is \( \psi \)-stable generalized holomorphic vector bundle with the same factor.

**Proof.** The result follows from Theorem 11.11. \[ \square \]

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