The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces

Luiz C. B. da Silva

Abstract In this work we are interested in the differential geometry of surfaces in simply isotropic $\mathbb{I}^3$ and pseudo-isotropic $\mathbb{I}^3_p$ spaces, which is basically the study of $\mathbb{R}^3$ equipped with a degenerate metric such as $ds^2 = dx^2 \pm dy^2$. Extending previous results concerning simply isotropic surfaces [B. Pavković, Glas. Mat. Ser. III 15, 149 (1980)], here we introduce a Gauss map in both $\mathbb{I}^3$ and $\mathbb{I}^3_p$ taking values on a unit sphere of parabolic type, define a shape operator from it, and show that its determinant and trace give the known relative Gaussian and Mean curvatures, respectively. We show that every (admissible) pseudo-isotropic surface is timelike and that, in analogy to what happens in Lorentzian geometry, the pseudo-isotropic shape operator may fail to be diagonalizable. We also prove that the only totally umbilical surfaces in $\mathbb{I}^3_p$ are spheres of parabolic type and that the curvature tensor associated with the induced Levi-Civita connection vanishes identically for any pseudo-isotropic surface, as happens in simply isotropic space. Later, based on our Gauss map, we introduce a new notion of connection, named relative connection (or r-connection, for short), whose curvature tensor does not vanish identically and which is directly related to the relative Gaussian curvature. Finally, we compute the Gauss and Codazzi-Mainardi equations for the r-connection and show that geodesics on a sphere of parabolic type are obtained by intersections with planes passing through its center (focus).

Keywords Simply isotropic geometry · pseudo-isotropic geometry · Gauss map · shape operator · geodesic · curvature

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1 Introduction

The three dimensional (3d) simply isotropic $I^3$ and pseudo-isotropic $I^3_p$ spaces are examples of 3d Cayley-Klein (CK) geometries [7,8,18]. Basically, a CK geometry is the study of the properties in projective space $\mathbb{P}^3$ invariant by the action of the subgroup of projectivities that preserves the so-called absolute figure. In our case of interest, the absolute figure is given by a plane at infinity and a degenerate quadric of index zero [20,23] or one [7]. From the differential viewpoint, we are essentially led to the study of $\mathbb{R}^3$ equipped with a degenerate metric of index 0 or 1: $ds^2 = dx^2 \pm dy^2$. Besides its mathematical interest, see e.g. [4,7,20], isotropic geometry also finds applications in economics [2,6], elasticity [16], and in image processing [10,17]. The geometry of curves and surfaces in $I^3$ was began by Strubecker in a series of papers [23,24,25], see also Ref. [20], while the respective theory in $I^3_p$ has been recently initiated in [1,7].

It is known that the (induced) Riemann curvature tensor vanishes for any surface in $I^3$ [20] and then every simply isotropic surface is locally the same. Despite that, the concept of a second fundamental form is not trivial and allows one to introduce an alternative notion of Gaussian curvature $K$ (also named relative Gaussian curvature) whose expression in local coordinates and interpretation via normal curvatures are analogous to the respective Euclidean notions. Amazingly, $K$ can be expressed as the ratio between the area of a region under the (parabolic) spherical image and the isotropic area on the surface [20]. Finally, in the 1980’s Pavković interpreted $K$ in terms of a shape operator defined with respect to a unit sphere of parabolic type [15]. These results suggest that the relative Gaussian curvature is a proper substitute for the intrinsic one in $I^3$. However, to the best of our knowledge, a possible relation with a curvature tensor has not been investigated yet.

In this work we push previous investigations further and extend them to $I^3_p$. After preliminaries in Sect. 2, we define in Sect. 3 a Gauss map in both $I^3$ and $I^3_p$ taking values on a unit sphere of parabolic type, slightly distinct from that of [15], from which one defines a shape operator and the relative Gaussian and Mean curvatures. As happens in Lorentzian geometry [13], we show that the pseudo-isotropic shape operator may also fail to be diagonalizable. In addition, we prove that totally umbilical surfaces in $I^3_p$ should be planes and spheres of parabolic type, as also happens in $I^3$. Based on the Gauss map, we introduce in Sect. 4 a new connection in $I^3$, the relative connection or $r$-connection, whose curvature tensor does not vanish identically and which is directly related to the relative Gaussian curvature. We define $r$-geodesics as autoparallel curves in the $r$-connection and show that $r$-geodesics of spheres of parabolic type are obtained by intersections with planes passing through their center (focus). In Sect. 4 we compute the Gauss and Codazzi-Mainardi equations. In Sect. 5 we devote our attention to the surface theory in $I^3_p$, where we prove that every (admissible) surface is timelike, i.e., the induced metric has index 1, and show that analogous results to the previous sections are also valid in $I^3_p$.

In the remaining of this work we shall use the Einstein convention of summing on repeated indexes, e.g., $\Gamma^k_{ij}x_k := \sum_{k=1}^2 \Gamma^k_{ij}x_k$. 

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2 Preliminaries

In the spirit of Klein’s Erlangen Program, the simply isotropic \( \mathbb{I}^3 \) and pseudo-isotropic \( \mathbb{I}^3_p \) geometries are the study of those properties in \( \mathbb{R}^3 \) invariant by the action of the 6-parameter group \( \mathcal{B}_6 \) [20, 23] and \( \mathcal{B}^0_6 \) given, respectively, by

\[
\begin{align*}
\mathcal{B}_6 &= \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R} \} \\
\mathcal{B}^0_6 &= \{ (\alpha, \beta) | \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \}
\end{align*}
\]

where, \( a, b, c, c_1, c_2, \phi \in \mathbb{R} \). In other words, \( \mathcal{B}_6 \) and \( \mathcal{B}^0_6 \) give our rigid motions.

Observe that on the \( xy \) plane these geometries look exactly like the Euclidean \( \mathbb{E}^2 \) and Lorentzian \( \mathbb{E}^2_1 \) plane geometries. The projection of a vector \( \mathbf{u} = (u^1, u^2, u^3) \) and \( \mathbf{v} = (v^1, v^2, v^3) \) on the \( xy \) plane is called the top view of \( \mathbf{u} \) and we shall denote it by \( \tilde{\mathbf{u}} = (u^1, u^2, 0) \). The top view concept plays a fundamental role in the simply and pseudo isotropic spaces, since the \( z \)-direction is preserved under the action of \( \mathcal{B}_6 \) or \( \mathcal{B}^0_6 \). A line with this direction is called an isotropic line and a plane containing an isotropic line is an isotropic plane.

In addition, one may introduce a simply isotropic and a pseudo-isotropic inner product between two vectors \( \mathbf{u} = (u^1, u^2, u^3) \) and \( \mathbf{v} = (v^1, v^2, v^3) \) as

\[
(\mathbf{u}, \mathbf{v})_z = u^1v^1 + u^2v^2 \quad \text{and} \quad (\mathbf{u}, \mathbf{v})_{pz} = u^1v^1 - u^2v^2,
\]

respectively\(^1\). These inner products induce a (semi) norm in a natural way:

\[
\|\mathbf{u}\|_z = \sqrt{(\mathbf{u}, \mathbf{u})_z} = \|\tilde{\mathbf{u}}\| \quad \text{and} \quad \|\mathbf{u}\|_{pz} = \sqrt{(\mathbf{u}, \mathbf{u})_{pz}} = \|\tilde{\mathbf{u}}\|_1,
\]

respectively. Here, \( \| \cdot \| \) and \( \| \cdot \|_1 \) are the Euclidean and Lorentzian norms induced, respectively, by

\[
(\mathbf{u}, \mathbf{v}) = u^1v^1 + u^2v^2 + u^3v^3 \quad \text{and} \quad (\mathbf{u}, \mathbf{v})_1 = u^1v^1 - u^2v^2 + u^3v^3,
\]

whose corresponding vector products are \( \times \) and \( \times_1 \); notice that

\[
\mathbf{u} \times_1 \mathbf{v} = (u^2v^3 - u^3v^2, u^1v^3 - u^3v^1, u^1v^2 - u^2v^1).
\]

Finally, since the (pseudo-) isotropic metric is degenerate, the distance from a point \( (u^1, u^2, u^3) \) to \( (u^1, u^2, v^3) \) is zero: \( \tilde{\mathbf{u}} = \tilde{\mathbf{v}} \Rightarrow \|\mathbf{u} - \mathbf{v}\|_z, \|\mathbf{u} - \mathbf{v}\|_{pz} = 0 \). In such cases, one may define a (pseudo) co-metric by using the codistance

\[
\text{cd}(A, B) = |b^3 - a^3|.
\]

**Remark 1** The codistance \( \text{cd}(\cdot, \cdot) \) is a secondary concept and it is invariant by \( \mathcal{B}_6 \), or \( \mathcal{B}^0_6 \) only when applied to isotropic vectors [7, 20]. One should not see it as part of the definition of a (pseudo) isotropic distance. Indeed, the function

\[
G(\mathbf{u}, \mathbf{v}) = \begin{cases} 
(\mathbf{u}, \mathbf{v})_z & \text{if } \tilde{\mathbf{u}} \neq 0 \text{ or } \tilde{\mathbf{v}} \neq 0 \\
(\mathbf{u}, \mathbf{v})_{pz} & \text{if } \tilde{\mathbf{u}} = \tilde{\mathbf{v}} = 0
\end{cases}
\]

is not bilinear. For example, we have \( G((1, 0, 1) + (0, 0, 1), (0, 0, 1)) = 0 \), but \( G((1, 0, 1), (0, 0, 1)) + G((0, 0, 1), (0, 0, 1)) = 1 \neq 0 \). Thus, \( G \) cannot be a metric.

\(^1\) The index \( z \) is here to emphasize that \( z \) is the isotropic (degenerate) direction.
Remark 2 If instead of \( x_0 = x_1 = 0 \), we choose the pair of lines \( x_0 = x_1 \pm x_2 = 0 \) and \( x_0 = x_2 = 0 \) for the pseudo-isotropic absolute figure, then we would obtain a different group of pseudo-isotropic rigid motions \([7]\), which coincides with the choice made in the classical literature (see the following remark). These groups however lead to the same geometry and can be related by a convenient coordinate change on the top view plane \([7]\).

Remark 3 Concerning notation and terminology, in \([21,22]\) Strubecker, the pioneer of the isotropic geometry, possibly did not realize that one must distinguish between the cases where the pair of lines in the absolute figure is either complex or real: the choice between these two cases leads to the simply and pseudo isotropic spaces, which are not the same geometry. He used the metric \( ds^2 = dx\,dy \), see e.g., p. 244 of \([22]\), and denoted the corresponding geometry by \( I_3 \). In \([23]\) however, he started to consider the two intersecting lines as being complex and then used the metric \( ds^2 = dx^2 + dy^2 \) (he continues to call the space isotropic, still denoted by \( I_3 \), despite the fact that the metric changed). Around the 1930’s, Lense seems to be the first to pay more attention on the “degree of isotropy” \([11]\); e.g., when introducing a name to the doubly isotropic space, Brauner based its terminology on Lense’s work \([5]\). In \([19]\) Sachs denoted the pseudo-isotropic space by \( I_3^{(1)} \); this same notation were employed by Husty and Röschel five years earlier \([9]\). In his book, however, Sachs denoted the pseudo-isotropic space by \( I_3^{(1)}P \) \([20]\). This same notation were used recently by Mészáros \([14]\). Here, for the ease of notation, we shall write \( I_3 \) for the simply isotropic space (in modern geometry texts one usually puts the dimension in the upper index) and \( I_3^p \) for the pseudo-isotropic space.

2.1 Spheres in (pseudo-) isotropic space

In the following sections we shall use spheres of parabolic type in order to define a Gauss map and a shape operator for surfaces in isotropic spaces.

We define simply isotropic and pseudo-isotropic spheres as connected and irreducible surfaces of degree 2 given by the 4-parameter families

\[
(x^2 + y^2) + 2c_1x + 2c_2y + 2c_3z + c_4 = 0, \quad c_i \in \mathbb{R},
\]

and

\[
(x^2 - y^2) + 2c_1x + 2c_2y + 2c_3z + c_4 = 0, \quad c_i \in \mathbb{R},
\]

respectively. In addition, up to a rigid motion, we can express an isotropic sphere in one of the two normal forms below \([7,20]\):

1. **Spheres of parabolic type:** in \( I_3 \)

\[
z = \frac{1}{2p}(x^2 + y^2) \quad \text{with} \quad p \neq 0;
\]

and in \( I_3^p \)

\[
z = \frac{1}{2p}(x^2 - y^2) \quad \text{with} \quad p \neq 0;
\]
2. **Spheres of cylindrical type**: in $\mathbb{I}^3$

\[ x^2 + y^2 = r^2 \quad \text{with} \quad r > 0; \quad (10) \]

and in $\mathbb{I}^3_p$

\[ x^2 - y^2 = \pm r^2 \quad \text{with} \quad r > 0. \quad (11) \]

**Remark 4** The quantities $p$ and $r$ are isotropic invariants. Moreover, spheres of cylindrical type are precisely the set of points equidistant from a given center. However, they do not constitute “good” surfaces in isotropic geometry, since their tangent planes are isotropic (in the terminology of the following section, they are not admissible surfaces). On the other hand, spheres of parabolic type are admissible and have constant Gaussian and Mean curvatures, $K = 1/p^2$ and $H = 1/p$, and are totally umbilical (Props. 3 and 4).

### 3 Surfaces in isotropic spaces

Now we discuss on the differential geometry of surfaces in the simply isotropic and pseudo-isotropic spaces. For further details concerning the geometry in $\mathbb{I}^3$ the reader may consult Ref. [20]. On the other hand, the study of surfaces in pseudo-isotropic geometry was initiated in Ref. [1].

**Definition 1** Let $\mathbb{M}^3$ be $\mathbb{I}^3$ or $\mathbb{I}^3_p$. Then, $\mathbf{x} : S \rightarrow \mathbb{M}^3$ is an admissible surface if its tangent planes $T_qS$ are all non-isotropic.

Let $g$ be the metric in $S \subset \mathbb{M}^3$ induced by the immersion $\mathbf{x}$, i.e.,

\[ g(u,v) = \langle d\mathbf{x}(u), d\mathbf{x}(v) \rangle_z \quad \text{or} \quad g(u,v) = \langle d\mathbf{x}(u), d\mathbf{x}(v) \rangle_{pz}. \quad (12) \]

We define the coefficients of the first fundamental form $I$ as

\[ g_{ij} = g\left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right). \quad (13) \]

In a local parameterization $(u^1, u^2) \in U \subseteq \mathbb{R}^2 \mapsto \mathbf{x}(u^1, u^2) \in S$ we have $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle_z$, or $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle_{pz}$, where $\mathbf{x}_i = \partial \mathbf{x}/\partial u^i$. (We shall see in Proposition 1 that for any admissible surface in $\mathbb{I}^3_p$ the induced metric is non-degenerate with index 1, i.e., every surface is timelike.)

Any admissible surface $\mathbf{x}(u^1, u^2) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$ gives

\[ \frac{\partial (x^1, x^2)}{\partial (u^1, u^2)} = (x^1_1x^2_2 - x^1_2x^2_1) \neq 0, \quad (14) \]

and, therefore, they can be parameterized as a graph over the $xy$-plane,

\[ \mathbf{x}(u^1, u^2) = (u^1, u^2, f(u^1, u^2)), \quad (15) \]

which we call the normal form. The first fundamental form is then

\[ I = (du^1)^2 + (du^2)^2 \quad \text{or} \quad I = (du^1)^2 - (du^2)^2. \quad (16) \]

---

2. Observe that the center $P$ is not uniquely defined since any other point $Q$ with the same top view as $P$, i.e., $\tilde{Q} = \tilde{P}$, is also a center.
3.1 Isotropic Gauss map and shape operator

Denoting \( \mathbf{x} = (x^1, x^2, x^3) \) and \( \mathbf{x}_i = (x^1_i, x^2_i, x^3_i) \), we introduce the notations

\[
X = \begin{pmatrix}
  x^1_1 & x^2_1 & x^3_1 \\
  x^1_2 & x^2_2 & x^3_2 \\
  x^1_3 & x^2_3 & x^3_3
\end{pmatrix},
\]

\( X_{ij} = \det \begin{pmatrix}
  x^1_i & x^3_i \\
  x^1_j & x^3_j
\end{pmatrix} \).

(17)

It follows that \( \det(g_{ij}) = (X_{12})^2 > 0 \) for \( M^3 = \mathbb{I}^3 \) and \( \det(g_{ij}) = -(X_{12})^2 < 0 \) for \( M^3 = \mathbb{I}^3_p \) (here, \( X_{12}(q) = 0 \) would mean an isotropic tangent plane at \( q \)). We can suppose that \( X_{12} > 0 \) by exchanging \( u^1 \leftrightarrow u^2 \) if necessary.

**Proposition 1** In \( \mathbb{I}^3_p \), every admissible surface is timelike. In addition, there exists no spacelike surface and the only lightlike ones are non-admissible.

**Proof** In \( \mathbb{I}^3_p \) any admissible surfaces satisfy \( \det g_{ij} = -(X_{12})^2 < 0 \), which shows that they should be timelike, i.e., \( g_{ij} \) is non-degenerated and of index 1. In particular, there is no spacelike surface. Finally, a non-admissible surface gives \( \det g_{ij} = 0 \). Then, a surface is lightlike if, and only if, it is non-admissible.

\( \Box \)

Let \( \Sigma^2 \) be the unit (parabolic) sphere in \( M^3 \) given by

\[
\Sigma^2 = \{(x, y, z) \in \mathbb{I}^3 : z = -\frac{1}{2}(x^2 + y^2) + \frac{1}{2}\}
\]

(18)

or

\[
\Sigma^2 = \{(x, y, z) \in \mathbb{I}^3_p : z = -\frac{1}{2}(x^2 - y^2) + \frac{1}{2}\}
\]

(19)

The sphere \( \Sigma^2 \) will play a role in isotropic geometry analogous to that of \( S^2 \) in Euclidean geometry \( \mathbb{E}^3 \) and of \( S^2_1 \) in Lorentz-Minkowski geometry \( \mathbb{E}^3_1 \) (there is no isotropic counterpart of \( \mathbb{H}^3_0 \), since any surface in \( \mathbb{I}^3_p \) is timelike, prop. 1).

**Definition 2** Denoting by \( \{e_i\}_{i=1}^3 \) the canonical basis of \( \mathbb{R}^3 \), where \( e_3 \) is isotropic. The **Gauss map** \( \xi : S \to \Sigma^2 \) is defined as

\[
\xi(u^1, u^2) = \frac{X_{23}}{X_{12}} e_1 + \frac{X_{31}}{X_{12}} e_2 + \frac{1}{2} \left( 1 - \left( \frac{X_{23}}{X_{12}} \right)^2 + \left( \frac{X_{31}}{X_{12}} \right)^2 \right) e_3,
\]

(20)

for \( M^3 = \mathbb{I}^3 \) and

\[
\xi^p(u^1, u^2) = \frac{X_{23}}{X_{12}} e_1 + \frac{X_{31}}{X_{12}} e_2 + \frac{1}{2} \left( 1 - \left( \frac{X_{23}}{X_{12}} \right)^2 - \left( \frac{X_{31}}{X_{12}} \right)^2 \right) e_3,
\]

(21)

for \( M^3 = \mathbb{I}^3_p \).

**Remark 5** The definition above is inspired by Pavković’s findings in \( \mathbb{I}^3 \) [15]. In our definition, however, we made a translation in \( z \) in order to guarantee \( 0 \notin \Sigma^2 \) and \( \|\xi\|^2 + \xi^3 > 0 \), see Eqs. (44) and (84). In addition, \( \Sigma^2 \) is centered at \((0, 0, 0)\) since its focus is located at the origin of the coordinate system.
In local coordinates, the coefficients of the quadratic form II can be written as
\[ N_h = \frac{x_1 \times x_2}{\|x_1 \times x_2\|} = \frac{X_{23}}{X_{12}} e_1 + \frac{X_{31}}{X_{12}} e_2 + e_3. \] (22)

The z-coordinate of ξ was adjusted to have ξ ◦ x ∈ Σ^2. The same is true in \( \mathbb{P}_p^3 \):
\[ N_h = \frac{x_1 \times x_2}{\|x_1 \times x_2\|_1} = \frac{X_{23}}{X_{12}} e_1 + \frac{X_{31}}{X_{12}} e_2 + e_3 \Rightarrow \tilde{N}_h = \tilde{\xi}^p. \] (23)

**Definition 3** The shape operator \( L_q \) (or Weingarten map) is defined as
\[ L_q(w_q) = \begin{cases} -D_{w_q} \xi, \forall w_q \in T_q S \subset T_q \mathbb{P}_p^3 & \text{if } \gamma \text{ is a curve with } \gamma(0) = q \text{ and } \gamma'(0) = w_q. \\ -D_{w_q} \xi^p, \forall w_q \in T_q S \subset T_q \mathbb{P}_p^3 & \text{if } \gamma \text{ is a curve with } \gamma(0) = q \text{ and } \gamma'(0) = w_q. \end{cases} \] (24)

where \( D \) denotes the usual directional derivative in \( \mathbb{R}^3 \), i.e., \( D_{w_q} \xi = (\xi \circ \gamma)'(0) \).

Following similar steps to that of Ref. [15], it can be shown that
\[ L_q(x_i) = -\frac{1}{X_{12}} \det \left( A_i x_2^j \right) x_1 - \frac{1}{X_{12}} \det \left( x_2^i A_i \right) x_2, \] (25)

where \( A = X_{23}/X_{12} \) and \( B = X_{31}/X_{12} \) for \( \mathbb{P}_p^3 \) or \( B = X_{13}/X_{12} \) for \( \mathbb{P}_p^3 \). The equation above means that \( L_q \) can be seen as a linear operator on \( T_q S \). The planes \( T_q S \) and \( T_q(\xi(q)) \Sigma^2 \) are then parallel and can be canonically identified.

In addition, define the isotropic second fundamental form II by
\[ \forall u_q, v_q \in T_q S, \ II(u_q, v_q) = I(L_q(u_q), v_q). \] (26)

In local coordinates, the coefficients of the quadratic form II can be written as
\[ h_{ij} = II(x_i, x_j), x : S \to \mathbb{M}^3. \] (27)

Finally, the normal curvature \( \kappa_n \) along a unit direction \( w_q \in T_q S \) is
\[ \kappa_n(q, w_q) = II(w_q, w_q) = I(L_q(w_q), w_q). \] (28)

Indeed, using that for any tangent vector \( w_q \in T_q S \) one has
\[ 0 = \langle N_h, w_q \rangle = \langle \xi + e_3, w_q \rangle = \langle \xi, w_q \rangle_z + w_q^3 \Rightarrow \langle \xi, w_q \rangle_z = -w_q^3, \] (29)

we conclude that
\[ \langle L_q(w_q), w_q \rangle_z = D_{w_q} w_q^3 + \langle \xi, D_{w_q} w_q \rangle_z = \langle \xi + e_3, D_{w_q} w_q \rangle = \langle N_h, D_{w_q} w_q \rangle = \kappa_n. \] (30)

If \( \gamma \subset S \) is a curve with \( \gamma'(0) = w_q \), then the equation above is just the component of the acceleration \( \gamma'' \) in the direction of \( N_h \), which is precisely the isotropic normal curvature [20], p. 155. The same reasoning applies to \( \mathbb{P}_p^3 \) [1].
As done in [20] for curves in $\mathbb{I}^3$ and in [1] for curves in $\mathbb{P}^3$, the second fundamental form can be alternatively defined by first studying the normal curvatures $\kappa_n$ on the surface and then observing that the normal curvature at a unit direction $w_q = a_1 x_1 + b_2 x_2 \in T_q S$ is

$$\kappa_n(q, w_q) = \ell_{11} a^2 + 2 \ell_{12} ab + \ell_{22} b^2,$$

(31)

where

$$\ell_{ij} = \frac{\det(x_1, x_2, x_{ij})}{\det(x_1, x_2)}.$$  

(32)

Notice, however, that we can write the coefficient $\ell_{ij}$ above as $\ell_{ij} = \langle N_{x_i}, x_{ij} \rangle$, see Eq. (22). On the other hand, using that $(\xi, x_j)_{z} = -x_j^{3}$, we have

$$h_{ij} = \langle -D_{x_i} \xi, x_j \rangle_{z} = \langle \xi, D_{x_i} x_j \rangle_{z} + x_j^{3}$$

$$= \langle \xi, x_{ij} \rangle_{z} + x_j^{3} = \langle \xi + e_3, x_{ij} \rangle_{z} + 1 \cdot x_j^{3}$$

$$= \langle N_{x_i}, x_{ij} \rangle = \ell_{ij}.$$  

(33)

A similar reasoning applies for the case of $\mathbb{P}^3$. In short, our definitions coincides with the usual ones, as already mentioned by Pavković for surfaces in $\mathbb{P}^3$ [15].

Definition 4 The (pseudo-) isotropic Gaussian and Mean curvatures of an admissible surface $S \subset \mathbb{M}^3$ are respectively defined as

$$K(q) = \det(L_q) \text{ and } H(q) = \frac{1}{2} \text{tr}(L_q).$$  

(34)

If we write $L_q(x_i) = -A^k_i x_k$ in local coordinates, then

$$h_{ij} = \langle L_q(x_i), x_j \rangle = -A^k_i \text{tr}(x_k, x_j) = -A^k_i g_{kj}.$$  

(35)

From this relation it follows that $-A^k_i = g^{kj}h_{ji}$ and, therefore, we can write

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \text{ and } H = \frac{1}{2} \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}.$$  

(36)

Example 1 Let $x = (u^1, u^2, f(u^1, u^2))$ be an admissible surface parameterized in its normal form. Its 1st fundamental form is given in Eq. (16). On the other hand, we have $x_i = e_i + f_i e_3$, $x_{ij} = f_{ij} e_3$, and then

(a) in $\mathbb{I}^3$, we find $x_1 \times x_2 = (-f_1, -f_2, 1)$ and $\hat{x}_1 \times \hat{x}_2 = (0, 0, 1)$. Consequently, the 2nd fundamental form is $II = f_{11}(du^1)^2 + 2f_{12} du^1 du^2 + f_{22}(du^2)^2$ and the Gaussian and Mean curvatures are respectively

$$K = f_{11} f_{22} - f_{12}^2 \text{ and } H = \frac{f_{11} + f_{22}}{2};$$  

(37)

(b) in $\mathbb{P}^3$, we find $x_1 \times x_2 = (-f_1, f_2, 1)$ and $\hat{x}_1 \times \hat{x}_2 = (0, 0, 1)$. Consequently, the 2nd fundamental form is $II = f_{11}(du^1)^2 + 2f_{12} du^1 du^2 + f_{22}(du^2)^2$ and the Gaussian and Mean curvatures are respectively

$$K = f_{12}^2 - f_{11} f_{22} \text{ and } H = \frac{f_{11} - f_{22}}{2}.$$  

(38)
Proposition 2 Every admissible pseudo-isotropic minimal surface \( S \subset I^3_p \), i.e., zero mean curvature surfaces, can be parameterized as

\[
x(u^1, u^2) = (u^1, u^2, f(u^1 + u^2) + g(u^1 - u^2)),
\]

where \( f \) and \( g \) are smooth real functions.

Proof When written in its normal form \( x = (u^1, u^2, z(u^1, u^2)) \), a minimal pseudo-isotropic surface is associated with the solution of the homogeneous wave equation \( z_{11} - z_{22} = 0 \), whose general solution is of the form \( z(u^1, u^2) = f(u^1 + u^2) + g(u^1 - u^2) \) for some smooth functions \( f \) and \( g \).

Remark 6 The minimal surfaces in \( I^3 \) are associated with the solution of the Laplace equation \( z_{11} + z_{22} = 0 \). Consequently, \( z \) should be the real or imaginary part of a holomorphic function, a fact that allows for a generic description of simply isotropic minimal surfaces [20]. In \( I^3_p \), we just showed that every minimal surface is a special kind of an affine translation surface [3,12].

Spheres of parabolic type are graphs of quadratic polynomials \( f = [(u^1)^2 \pm (u^2)^2]/p + b_1 u^1 + b_2 u^2 + a_0 \), from which easily follows the

Proposition 3 Every sphere of parabolic type has constant Gaussian and Mean curvatures equal to \( K = 1/p^2 \) and \( H = 1/p \).

Finally, it is worth mentioning that \( K \) above may be also named as the relative Gaussian curvature in opposition to the absolute Gaussian curvature \( K_a \), which is the intrinsic curvature of the 1st fundamental form. In the simply isotropic geometry \( K_a \) vanishes for every surface [20]:

\[
K_a = \frac{1}{g_{11}} (T^2_{11,1} - T^2_{12,1} + T^2_{11} - T^2_{12} - T^2_{11}) \equiv 0,
\]

where, denoting by \( N = (0, 0, 1) \) the isotropic surface normal, the coefficients \( T^k_{ij} \) are the isotropic Christoffel symbols defined through the relation

\[
x_{ij} = T^k_{ij} x_k + h_{ij} N.
\]

We shall see in the following that by using the isotropic Gauss maps it will be possible to introduce a new connection in isotropic space in such a way that the intrinsic curvature is no longer trivial and is directly related to the (relative) Gaussian curvature. Besides the interpretation of \( K \) as the determinant of a shape operator and its relation with a new notion of connection (to be obtained from this shape operator), let us mention that \( K(q) \) can be seen as the ratio between the area of a region \( \xi(U) \) in \( \Sigma^2 \) under the Gauss map and the area of \( U \subset S \) in the limit \( U \rightarrow \{q\} \) [20], p. 178, in complete analogy with Euclidean geometry.
3.2 Principal curvatures and totally umbilical surfaces

If we fix a point \( q \in S \), then we can see \( \kappa_n \) as a function on the set of unit velocity vectors, i.e., \( \kappa_n(q, \cdot) : S^1 \subset T_qS \to \mathbb{R} \). In \( \mathbb{E}^3 \) the unit sphere \( S^1 \) corresponding to the unit velocity vectors is compact and then \( \kappa_n \) has both a maximum \( \kappa_1 \) and a minimum \( \kappa_2 \). These are the principal curvatures and they are precisely the eigenvalues of the shape operator. Therefore, it is possible to write \( K = \kappa_1 \kappa_2 \) and \( H = (\kappa_1 + \kappa_2)/2 \). On the other hand, in \( \mathbb{E}^p \) the unit sphere \( S^1 \) is no longer compact (in coordinates, their equation is \( x^2 - y^2 = \pm 1 \)) and then \( \kappa_n \) may fail to have both a maximum and a minimum. As a consequence, the shape operator may fail to be diagonalizable (see Sect. 5).

It may happen that all the directions in \( T_qS \) are eigenvectors of the shape operator, which happens precisely when \( I \) and \( II \) are multiple. Then, we have

**Definition 5** A point \( q \) where the 1st and 2nd fundamental forms are proportional is said to be an umbilic point, i.e., \( q \) is umbilic when

\[
II = \lambda I.
\]

A surface whose every point is umbilic is said to be totally umbilical.

In in \( \mathbb{E}^3 \) the only totally umbilical surfaces are the spheres of parabolic type and non-isotropic planes [20], p. 171. Analogously, in \( \mathbb{E}^p \) it is valid.

**Proposition 4** The only totally umbilical surfaces in \( \mathbb{E}^p \) are spheres of parabolic type and non-isotropic planes.

**Proof** Assume that the surface is given in its normal form, see example 1. In order to be totally umbilical, we must have \( f_{12} = 0 \) and \( f_{11} = -f_{22} \). From the first equation we deduce that \( f(u^1, u^2) = F_1(u^1) - F_2(u^2) \). On other hand, \( f_{11} = -f_{22} \) implies \( F''_1(u^1) = F''_2(u^2) \) and, therefore, there exists a constant \( 2c_0 \) such that \( F''_1(u^1) = 2c_0 \). So, we have \( F_i(u^i) = c_0 (u^i)^2 + b_i u^i + a_i \), for some constants \( a_i, b_i \). In short, \( S \) can be parameterized by

\[
x(u^1, u^2) = (u^1, u^2, c_0 \left[(u^1)^2 - (u^2)^2\right] + b_1 u^1 - b_2 u^2 + a_1 - a_2).
\]

Thus, \( S \) is a sphere if \( 1/2p_0 = c_0 \neq 0 \) or a non-isotropic plane if \( c_0 = 0 \). \( \square \)

4 Surface theory in simply isotropic space

4.1 Relative differential geometry in simply isotropic space

Let us introduce a new connection on \( \mathbb{E}^3 \) whose coefficients \( \Xi^k_{ij} \) come from

\[
x_{ij} = \Xi^k_{ij} x_k + \rho_{ij} \xi.
\]

The coefficient \( \rho_{ij} \) is unequivocally defined since \( \{x_1, x_2, \xi\} \) is always a basis for \( \mathbb{R}^3 \). Indeed, using Eq. (17) and the definition of \( \xi \) in Eq. (20), one has

\[
\langle x_1 \times x_2, \xi \rangle = \langle [X_{23}, X_{31}, X_{12}], \xi \rangle = \frac{(X_{23})^2 + (X_{31})^2 + (X_{12})^2}{2X_{12}} > 0.
\]
In addition, the coefficient \( \rho_{ij} \) satisfies
\[
h_{ij} = \langle \mathbf{x}_{ij}, N_h \rangle = \rho_{ij}(\xi, N_h) = \rho_{ij}(\tilde{\mathbf{e}} + \mathbf{e}_3, \tilde{\mathbf{e}} + \mathbf{e}_3) = \rho_{ij}(\|\tilde{\mathbf{e}}\|^2 + \xi^3) \tag{46}
\]
and then
\[
\rho_{ij} = \frac{h_{ij}}{\|\xi\|^2 + \xi^3} \Rightarrow \rho_{ij} = \rho_{ji} \quad \text{and} \quad \Xi^k_{ij} = \Xi^k_{ji} . \tag{47}
\]
Observe that \(2(\|\tilde{\mathbf{e}}\|^2 + \xi^3) = [(X_{23})^2 + (X_{13})^2 + (X_{12})^2](X_{12})^{-2} > 0 \) and then there is no singularity in the expression for \( \rho_{ij} \).

**Proposition 5** The coefficients \( \Xi^k_{ij} \) relate with \( \Gamma^k_{ij} \) according to
\[
\Xi^k_{ij} = \Gamma^k_{ij} + g^{kl} x^3_l \rho_{ij} . \tag{48}
\]

**Proof** Assume the notation \( A_k := \partial \mathcal{A}/\partial u^k \). From \( g_{ab} = \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_b \rangle \), we have
\[
g_{ab,c} = \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_b \rangle + \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_b \rangle - \rho_{ac} \tilde{\mathbf{x}}_d + \rho_{bc} \tilde{\mathbf{x}}_d
\]
\[
= \Xi^d_{ac} \langle \tilde{\mathbf{x}}_d, \tilde{\mathbf{x}}_b \rangle + \Xi^d_{bc} \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_d \rangle + \rho_{ac} \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_d \rangle + \rho_{bc} \langle \tilde{\mathbf{x}}_a, \tilde{\mathbf{x}}_d \rangle . \tag{49}
\]
Now, using Eq. (29), one finds
\[
g_{ab,c} = \Xi^d_{ac} g_{bd} + \Xi^d_{bc} g_{ad} - \rho_{ac} x_b^3 - \rho_{bc} x_a^3 . \tag{50}
\]
Finally, computing \( g_{ij,j} + g_{ij,i} - g_{ii,j} \) and using the symmetry \( \Xi^c_{ab} = \Xi^c_{ba} \), we can deduce that
\[
\Xi^k_{ij} = \frac{g^{kl}}{2} (g_{ij,j} + g_{ij,i} - g_{ii,j}) + g^{kl} x^3_l \rho_{ij} = \Gamma^k_{ij} + g^{kl} x^3_l \rho_{ij} . \tag{51}
\]
\( \Box \)

**Definition 6** We say that a curve \( \gamma : I \to S \) is a relative geodesic (or r-geodesic) if the acceleration vector \( \gamma'' \) is parallel to \( \xi \).

The coefficients \( \Xi^k_{ij} \) define a covariant derivative \( \nabla^r \) through
\[
\nabla^r_{\mathbf{e}_k} \mathbf{x}_j = \Xi^k_{ij} \mathbf{x}_k \tag{52}
\]
and then, for any \( v_q = v^i \mathbf{x}_i, w_q = w^i \mathbf{x}_i \in T_q S \), one has
\[
\nabla^r_{v_q} w_q = [v_q(w^i)] + v^i w^j \Xi^k_{ij} \mathbf{x}_k . \tag{53}
\]
We may refer to \( \nabla^r \) as the relative connection or r-connection. Now, computing \( \gamma'' = \nabla^r_{\gamma'} \gamma' \) we may deduce the standard result below.

**Proposition 6** A curve \( \gamma : I \to S \) is an r-geodesic if and only if
\[
\nabla^r_{\gamma'} \gamma' = 0 \Leftrightarrow \frac{d^2 u^k}{dt^2} + \Xi^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0 , \quad k \in \{1, 2\} , \tag{54}
\]
where \( \gamma(t) = \mathbf{x}(u^1(t), u^2(t)) \) and \( \mathbf{x} \) is a local parameterization of \( S \).
Remark 7 The relative connection $\mathcal{E}^k_{ij}$ is not metric (with respect to the induced isotropic metric $I$). As a corollary, $r$-geodesics are not necessarily parametrized by arc-length, see example 3 below.

Example 2 ($r$-geodesics on a plane) Let $S$ be a non-isotropic plane. Clearly, any straight line $t \mapsto q + tu$, $u \in T_q S \cong S$, is an $r$-geodesic. By the existence and uniqueness theorem for ODE’s, these are the only $r$-geodesics in $S$. □

Geodesics according to the Levi-Civita connection are easy to find: they are the intersection with isotropic planes, since the length minimization property is defined with respect to $ds^2 = dx^2 \pm dy^2$ on the top view plane. On the other hand, the computation of $r$-geodesics is not so trivial.

Example 3 ($r$-geodesics on a sphere of parabolic type) Let $S$ be the sphere of parabolic type $\Sigma^2(p) = \{z = p/2 - 1/2p(x^2 + y^2)\}$ centered at the origin. In $\mathbb{R}^3$ it is known that the geodesics on a sphere can be obtained by intersecting it with planes passing through its center. Now we show the same for $S = \Sigma^2(p)$.

The intersection $S \cap \Pi_{a,b} = \{z = -ax - by\}$, $\Pi_{a,b}$ non-isotropic, is the curve

$$\gamma(t) = p \left( R \cos \theta(t) + a, R \sin \theta(t) + b, -a^2 - b^2 - R(a \cos \theta(t) + b \sin \theta(t)) \right),$$

where $R = \sqrt{1 + a^2 + b^2}$ (3). In order to have $\gamma'' \parallel \gamma/\rho \circ \gamma = \gamma/p$, it is enough to find a function $\theta(t)$ such that $\gamma \times \gamma'' = 0$. This leads to

$$[\theta''(R + a \cos \theta + b \sin \theta) - \theta' \gamma(2)(a \sin \theta - b \cos \theta)](R, a, b, R) = 0. \tag{56}$$

By writing the constants $a, b$ as $(a, b) = \rho(\cos \phi, \sin \phi)$, with $\rho = \sqrt{a^2 + b^2} < R$, we find $R + a \cos \theta + b \sin \theta = R + \rho \cos(\theta - \phi) > 0$ and then Eq. (56) gives

$$\frac{d^2\theta}{dt^2} = \frac{\rho \sin(\theta - \phi)}{R + \rho \cos(\theta - \phi)} \left( \frac{d\theta}{dt} \right)^2. \tag{57}$$

Now define $\Theta = \theta'$ and observe that $\theta'' = \Theta' = \dot{\Theta} \theta'$, where a prime and a dot denotes differentiation with respect to $t$ and $\theta$, respectively. Then, Eq. (57) can be alternatively written as a 1st order differential equation

$$\frac{d\Theta}{d\theta} = \frac{\rho \sin(\theta - \phi)}{R + \rho \cos(\theta - \phi)} \Theta, \tag{58}$$

whose solution does exist and it is unique for any given initial condition. Thus, once we know a solution $F(\theta)$ of Eq. (58), we can find $\theta(t)$ by solving $\theta' = F(\theta)$, for which it is also valid the existence and uniqueness theorem for ODE’s.

On the other hand, if $\Pi$ is an isotropic plane passing through the origin, then the intersection $\gamma = \Pi \cap S$ can be written as

$$\gamma(t) = \left( x(t), 0, \frac{p}{2} - \frac{x^2(t)}{2p} \right), \tag{59}$$

Indeed, if $(x^*, y^*, z^*) \in \Pi_{a,b} \cap S$, then from $-ax^* - bx^* = p/2 - [(x^*)^2 + (y^*)^2]/2p$ we find $(x^* - ap)^2 + (y^* - bp)^2 = p^2(1 + a^2 + b^2)$. 

---

$\rho$, $a$, $b$ are constants. 

$\rho(\cos \phi, \sin \phi)$ is a parametrization of the sphere $S$. 

$\Pi_{a,b}$ is a non-isotropic plane. 

$\Theta = \theta'$, $\dot{\Theta} \theta'$ are derivatives. 

$F(\theta)$ is a solution of the differential equation. 

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where we are assuming, without loss of generality, that \( \Pi \) is the \( xz \)-plane. In order to have \( \gamma'' \parallel \gamma \) it is enough to find \( x(t) \) with \( \gamma \times \gamma'' = 0 \), which leads to

\[
\frac{d^2x}{dt^2} = -\frac{2x}{p^2 + x^2} \left( \frac{dx}{dt} \right)^2 \implies \frac{dX}{dx} = -\frac{2x}{p^2 + x^2} X.
\] (60)

By the same reasoning as before, it is possible to find a solution \( x(t) \).

In short, the intersection of \( S \) with a plane passing through its center can describe all the parameterized \( r \)-geodesics on a sphere of parabolic type. \( \square \)

4.2 Gauss and Codazzi-Mainardi equations for the relative connection

Let us exploit the equality \( x_{ab,c} = x_{ac,b} \). We have

\[
x_{ab,c} = (\Xi_{ab,c} + \Xi_{ab} \varepsilon_{cd} \rho_{ab} A_{c}^d - \rho_{ab} A_{c}^d) x_c + (\rho_{ab,c} + \Xi_{ab} \rho_{cd}) \xi,
\] (61)

where \( \xi_c = -A_{c}^d x_c \). From the coefficients of \( x_c \) in \( x_{ab,c} - x_{ac,b} = 0 \) we deduce the Gauss equation

\[
\Xi_{ab,c} - \Xi_{ac,b} + \Xi_{ab} \rho_{cd} - \Xi_{ac} \rho_{bd} = 0,
\] (62)

\[
= (h_{ab} h_{cd} - h_{ac} h_{bd}) \frac{g^{ed}}{\|\varepsilon\|^2 + \xi^3},
\] (63)

\[
= (\|\varepsilon\|^2 + \xi^3) (\rho_{ab} \rho_{cd} - \rho_{ac} \rho_{bd}) g^{ed},
\] (64)

where we used Eq. (35), i.e.,

\[
D \xi = -\left( \begin{array}{cc} A_{1}^{1} & A_{1}^{2} \\ A_{2}^{1} & A_{2}^{2} \end{array} \right) = \left( \begin{array}{cc} g^{11} & g^{12} \\ g^{21} & g^{22} \end{array} \right) \left( \begin{array}{cc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array} \right).
\] (65)

On the other hand, from the coefficient of \( \xi \) in \( x_{ab,c} - x_{ac,b} = 0 \) we deduce the Codazzi-Mainardi equation

\[
\rho_{ab,c} - \rho_{ac,b} + \Xi_{ab} \rho_{cd} - \Xi_{ac} \rho_{bd} = 0.
\] (66)

Finally, let us introduce the \( r \)-curvature tensor as

\[
\mathcal{R}_{ij}^l = \Xi_{ij,k}^l - \Xi_{ik,j}^l + \Xi_{ij}^k \Xi_{k}^l - \Xi_{ik}^k \Xi_{j}^l.
\] (67)

Now, using that \( g_{fe} g^{ed} = \delta_f^d \) and making \( (a,b,c) = (1,1,2) \) in the right-hand side of the Gauss equation Eq. (64), we find

\[
g_{ef} \mathcal{R}_{abc} = \frac{g_{ef} g^{cd}}{\|\varepsilon\|^2 + \xi^3} (h_{ab} h_{cd} - h_{ac} h_{bd}) = \frac{\delta_f^d}{\|\varepsilon\|^2 + \xi^3} (h_{ab} h_{cd} - h_{ac} h_{bd}),
\] (68)

which implies

\[
\mathcal{R}_{abc} := g_{cd} \mathcal{R}_{abc}^e = \frac{h_{ab} h_{cd} - h_{ac} h_{bd}}{\|\varepsilon\|^2 + \xi^3}.
\] (69)
Then, we deduce that
\[ K = \frac{\det(h)}{\det(g)} = \left(\|\tilde{\xi}\|^2 + \xi^2\right) \times \frac{R_{1112}}{g_{11}g_{22} - (g_{12})^2}. \] (70)
which, from \( x_{12} = x_{21} \) and \( \rho_{11} A_2^2 - \rho_{12} A_1^2 = (\|\tilde{\xi}\|^2 + \xi^2)^{-1} g_{11} \det(L_q) \), can be alternatively rewritten as
\[ K = \frac{\|\tilde{\xi}\|^2 + \xi^3}{g_{11}} \left( \Xi_{11,2}^2 - \Xi_{12,1}^2 + \Xi_{11}^2 - \Xi_{12}^2 \Xi_{11}^2 \right). \] (71)
Therefore, we can conclude that the (relative) Gaussian curvature only depends on the Gauss map \( \xi \) and on the coefficients \( \Xi_{ij}^k \) of the \( r \)-connection. This equation represents the Theorema Egregium for the (relative) Gaussian curvature according to the \( r \)-connection.

5 Surface theory in pseudo-isotropic space

In pseudo-Euclidean geometry, depending on the properties of the induced metric, we may associated a causal character with a surface. In \( \mathbb{P}^3 \), any admissible pseudo-isotropic surface has a metric \( \langle \cdot, \cdot \rangle_{\mathbb{P}^2} \) which is non-degenerate and of index one. In other words, according to Prop. 1, every admissible surface is timelike. (Non-admissible surfaces are all lightlike).

Despite the fact that the shape operator \( L_q \) is symmetric with respect to the induced metric, \( L_q \) may fail to be diagonalizable in the pseudo-isotropic space (in \( \mathbb{P}^3 \) the shape operator is always diagonalizable since the induced metric is always Riemannian). In \( \mathbb{P}^3 \), the diagonalization of the shape operator depends on the existence of real roots of its characteristic polynomial:
\[ C_{L_q}(\lambda) = \lambda^2 - \text{tr}(L_q) \lambda + \det(L_q) = \lambda^2 - 2H \lambda + K. \] (72)
The shape operator is diagonalizable if \( C_{L_q} \) has two distinct real roots, i.e., if \( H^2 - K > 0 \). However, if \( H^2 - K < 0 \), \( L_q \) is not diagonalizable and if \( H^2 - K = 0 \), then \( L_q \) may be diagonalizable or not. Finally, when \( S \) is totally umbilical \( L_q \) is diagonalizable and \( H^2 - K = 0 \).

Example 4 (\( H^2 - K = 0 \), but \( S \) not totally umbilical: \( L_q \) not diagonalizable)

Let \( S \subset \mathbb{P}^3 \) be the surface parameterized by
\[ x(u^1, u^2) = (u^1, u^1 + b u^2, u^1 u^2), \quad b > 0. \] (73)
Here \( x_1 = (1, 1, u^2), x_2 = (0, b, u^1), x_1 \times x_2 = (u^1 - b u^2, u^1, b), \) and \( S \) is admissible with metric \( I = -b du^1 du^2 - b^2 (du^2)^2 \). In addition, \( x_{11} = x_{22} = (0, 0, 0), x_{12} = (0, 0, 1), \) and then \( II = du^1 du^2 \). The Gaussian and Mean curvatures are respectively
\[ K = \frac{1}{h^2} \quad \text{and} \quad H = -\frac{1}{h} \Rightarrow H^2 - K \equiv 0. \] (74)
Finally, note that the shape operator is not diagonalizable and, despite that \( H^2 - K = 0 \), the surface is not totally umbilical (if it were umbilical, then the condition \( H^2 - K = 0 \) would imply that \( L_q \) is diagonalizable). \( \square \)
Example 5 \((H^2 - K < 0)\): \(L_q\) non diagonalizable) Let \(S \subset \mathbb{I}^3_p\) be the helicoidal surface
\[
x(u^1, u^2) = (u^1 \cosh(u^2), u^1 \sinh(u^2), c \cdot u^2), \quad c, u^1 > 0.
\] (75)
We have \(x_1 = (\cosh(u^2), \sinh(u^2), 0),\) \(x_2 = (u^1 \sinh(u^2), u^1 \cosh(u^2), c),\) and \(x_1 \times x_2 = (c \sinh(u^2), c \cosh(u^2), u^1)\). Then, \(S\) is admissible with metric \(I = (du^1)^2 - (u^1)^2(du^2)^2\). In addition, \(x_{11} = (0, 0, 0), \) \(x_{12} = (\sinh(u^2), \cosh(u^2), 0),\) and \(x_{22} = (u^1 \cosh(u^2), u^1 \sinh(u^2), 0)\). Consequently, \(\Pi = -(c/u^1)du^1du^2\). The Gaussian and Mean curvatures are respectively
\[
K = \frac{c^2}{(u^1)^4} \quad \text{and} \quad H = 0.
\] (76)
Finally, the shape operator is not diagonalizable since \(H^2 - K < 0\). \(\Box\)

Example 6 \((H^2 - K \geq 0)\): \(L_q\) diagonalizable) Let \(S \subset \mathbb{I}^3_p\) be the (hyperbolic) revolution surface
\[
x(u^1, u^2) = (u^1 \cosh(u^2), u^1 \sinh(u^2), z(u^1)), \quad u^1 > 0.
\] (77)
We have \(x_1 = (\cosh(u^2), \sinh(u^2), z'),\) \(x_2 = (u^1 \sinh(u^2), u^1 \cosh(u^2), 0),\) and \(x_1 \times x_2 = (-u^1 z' \cosh(u^2), -u^1 z' \sinh(u^2), u^1)\). Then, \(S\) is admissible with metric \(I = (du^1)^2 - (u^1)^2(du^2)^2\). In addition, we have \(x_{11} = (0, 0, z''), \) \(x_{12} = (\sinh(u^2), \cosh(u^2), 0),\) and \(x_{22} = (u^1 \cosh(u^2), u^1 \sinh(u^2), 0)\). Consequently, \(\Pi = z''(u^1)(du^1)^2 - u^1 z'(u^1)(du^2)^2\). The Gaussian and Mean curvatures are respectively
\[
K = \frac{z'z''}{u^1} \quad \text{and} \quad H = \frac{z''}{2-u^1} + \frac{z'}{2u^1}.
\] (78)
Finally, note that \(H^2 - K = (z''/2 - z'/2u^2)^2 \geq 0\). The equality occurs only for \(z(u^1) = c_0(u^1)^2 + c_1,\) i.e., when \(S\) is a sphere of parabolic type (which is totally umbilical) and, therefore, \(L_q\) is diagonalizable. Otherwise, \(H^2 - K > 0\) and \(L_q\) is also diagonalizable. \(\Box\)

5.1 Relative differential geometry in pseudo-isotropic space

As in \(\mathbb{I}^3\), the Christoffel symbols \(\Gamma^k_{ij}\) coming from the induced connection are
\[
x_{ij} = \Gamma^k_{ij} x_k + h_{ij} \mathcal{N},
\] (79)
where \(\mathcal{N} = (0, 0, 1)\) is the normal to \(S\) according to \(\langle \cdot, \cdot \rangle_{p^2}\). From the equality \(x_{ij,k} - x_{ik,j} = 0\) in \(\mathbb{I}^3\), the pseudo-isotropic Gauss and Codazzi-Mainardi equations with the induced pseudo-isotropic metric are respectively
\[
\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^s_{ij} \Gamma^t_{ks} - \Gamma^s_{ik} \Gamma^t_{js} = 0 \quad \text{and} \quad h_{ij,k} - h_{ik,j} + \Gamma^t_{ij} h_{tk} - \Gamma^t_{ik} h_{tj} = 0.
\] (80)
In analogy to what happens in \(\mathbb{I}^3\), Eq. (80) implies that the intrinsic (or absolute) Gaussian curvature associated with \(\langle \cdot, \cdot \rangle_{p^2}\) vanishes for any \(S \subset \mathbb{I}^3_p\).
Now, let us introduce a new connection on $\mathbb{R}^3$ whose coefficients $\Xi^k_{ij}$ are
\[x_{ij} = \Xi^k_{ij} x_k + \rho_{ij} x^p.\] (81)
The coefficient $\rho_{ij}$ is unequivocally defined whenever $x_1 \times x_2$ is not lightlike in the background metric $(\cdot, \cdot)_1$. Indeed, using Eq. (17) and $\xi^p$ in Eq. (21),
\[\langle x_1 \times x_2, \xi^p \rangle_1 = \langle (X_{23}, X_{13}, X_{12}), \xi^p \rangle_1 = \frac{(X_{23})^2 - (X_{13})^2 + (X_{12})^2}{2X_{12}}.\] (82)

We shall call a point $q \in S$ where $\|x_1(q) \times x_2(q)\|_1 = 0$ a lightlike point of $S$ (this notion should not be confused with the one coming from the induced metric. Indeed, $g(\cdot, \cdot) := (\cdot, \cdot)_{\mathbb{R}^3}$ is always timelike as shown in Prop. 1).

In addition, the coefficient $\rho_{ij}$ satisfies

\[h_{ij} = \langle x_{ij}, N_h \rangle_1 = \rho_{ij} (\xi, N_h)_1 = \rho_{ij} (\xi + \xi^k e_3, \xi + \xi^l e_3)_1 = \rho_{ij} \|\xi\|^2 + \xi^3\] (83)
and then

\[\rho_{ij} = \frac{h_{ij}}{\|\xi\|^2 + \xi^3} \Rightarrow \rho_{ij} = \rho_{ji} \text{ and } \Xi^k_{ij} = \Xi^k_{ji}.\] (84)

Observe that $2(\|\xi\|^2 + \xi^3) = [(X_{23})^2 - (X_{13})^2 + (X_{12})^2]/(X_{12})^2 \neq 0$ outside the set of lightlike points. In this case, there is no singularity in the expression for $\rho_{ij}$. With a proof analogous to that of surfaces in $\mathbb{R}^3$ we can show that

**Proposition 7** The coefficients $\Xi^k_{ij}$ relate with $\Gamma^k_{ij}$ according to
\[\Xi^k_{ij} = \Gamma^k_{ij} + g^{kl} x^3_{i} \rho_{kj}.\] (85)

The Gauss-Codazzi-Mainardi equations associated with $\Xi^k_{ij}$ in $\mathbb{R}^3$ are analogous to the simply isotropic ones obtained in subsect. 4.2. They allow us to reinterpret the Gaussian curvature, Eq. (34), as an intrinsic curvature.

**Example 7** ($r$-geodesics on a sphere of parabolic type) Let $S$ be the sphere of parabolic type $\Sigma^3(p) = \{z = p/2 - (x^2 - y^2)/2p\}$ centered at the origin. Their $r$-geodesics can be obtained by intersections with a plane passing through the origin. Indeed, the intersection $S \cap H_{a,b} = \{z = -ax + by\}$ is the curve
\[\gamma(t) = p \left(R \cosh \theta + a, R \sinh \theta + b, -a^2 + b^2 - R[a \cosh \theta - b \sinh \theta]\right),\] (86)
where $\theta = \theta(t)$ and $R^2 = 1 + a^2 - b^2 > 0$ (4): if $R^2 = 1 + a^2 - b^2 < 0$, then
\[\gamma(t) = p \left(R \sinh \theta + a, R \cosh \theta + b, a^2 - b^2 + R[a \sinh \theta - b \cosh \theta]\right),\] (87)

When $R = 0$, the intersection is a pair of lines, which are $r$-geodesics. To have $\gamma^\prime \parallel \gamma/p$, $\xi^p \circ \gamma = \gamma/p$, it is enough to find a function $\theta(t)$ such that $\gamma \times_1 \gamma^\prime = 0$. The resulting equations can be managed in a similar fashion to that of $\mathbb{R}^3$, example 3, by using the hyperbolic trigonometric functions instead of the usual ones. If $H$ is isotropic and passes through the origin, we can also proceed as in example 3).

In short, the intersection of $S$ with a plane passing through the origin can describe all $r$-geodesics on a pseudo-isotropic sphere of parabolic type. □
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