FELL BUNDLES OVER A COUNTABLE DISCRETE GROUP 
AND STRONG MORITA EQUIVALENCE FOR INCLUSIONS OF 
C*-ALGEBRAS

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ABSTRACT. We consider two saturated Fell bundles over a countable discrete 
group, whose unit fibers are σ-unital C*-algebras. Then by taking the reduced 
cross-sectional C*-algebras, we get two inclusions of C*-algebras. We suppose 
that they are strongly Morita equivalent as inclusions of C*-algebras. Also, 
we suppose that one of the inclusions of C*-algebras is irreducible, that is, the 
relative commutant of one of the unit fiber algebras, which is a σ-unital C*- 
algebra, in the multiplier C*-algebra of the reduced cross-sectional C*-algebra 
is trivial. We show that the two saturated Fell bundles are then equivalent up 
to some automorphism of the group.

1. INTRODUCTION

Let G be a countable discrete group and let \( A = \{A_t\}_{t \in G} \) be a Fell bundle 
over G. Let \( C^*_e(A) \) be the reduced cross-sectional C*-algebra of \( A \) and \( A_e = A \), 
a C*-algebra, where \( e \) is the unit element in G. Then we obtain an inclusion of 
C*-algebras \( A \subseteq C^*_e(A) \) and we call it the inclusion of C*-algebras induced by \( A \). 
By Abadie and Ferraro [1, Sections 3, 4], it is easy to show that if Fell bundles 
\( A = \{A_t\}_{t \in G} \) and \( B = \{B_t\}_{t \in G} \) over G are equivalent with respect to an equivalence 
bundle \( X = \{X_t\}_{t \in G} \) over G such that

\[
\overline{A(x_t, x_s)} = A_{ts}^{-1}, \quad \overline{B(x_t, x_s)} = B_{ts}^{-1}
\]

for any \( t, s \in G \), then the inclusions of C*-algebras induced by \( A \) and \( B \) are strongly 
Morita equivalent, where \( \overline{A(x_t, x_s)} \) means the closure of linear span of the set 
\( \{A(x, y) | x \in X_t, y \in X_s\} \) and \( \overline{B(x_t, x_s)} \) means the closure of the same set as above.

In this paper, we shall show the inverse direction as follows: Let \( A = \{A_t\}_{t \in G} \) 
and \( B = \{B_t\}_{t \in G} \) be saturated Fell bundles over G. We suppose that \( A_e = A \) and 
\( B_e = B \) are σ-unital C*-algebras and that \( A' \cap C^*_e(A) = \mathcal{C}1 \). If the inclusions 
of C*-algebras induced by \( A \) and \( B \) are strongly Morita equivalent, then there is 
an automorphism \( f \) of G such that \( A \) and \( B^f \) are equivalent with respect to an equivalence 
bundle \( X = \{X_t\}_{t \in G} \) such that

\[
\overline{A(x_t, x_s)} = A_{ts}^{-1}, \quad \overline{B(x_t, x_s)} = B_{ts}^{-1}
\]

for any \( t, s \in G \), where \( B^f \) is a Fell bundle over G induced by \( B = \{B_t\}_{t \in G} \) and \( f \), 
that is, \( B^f = \{B_{f(t)}\}_{t \in G} \).

We prove this result in the following way: Let \( K \) be the C*-algebra of all compact 
operators on a countably infinite dimensional Hilbert space. Let \( A^S \) and \( B^S \) be the 
Fell bundles over G induced by \( A, K \) and \( B, K \), respectively. Then since \( A \) and \( B \) 
are σ-unital, \( A^S \) and \( B^S \) satisfy the assumptions of Exel [3, Theorem 7.3]. Since

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1
\( \mathcal{A}^S \) and \( \mathcal{B}^S \) are saturated, by \cite[Theorem 7.3]{3} there are twisted actions \((\alpha, w_{\alpha})\) and \((\beta, w_{\beta})\) of \( G \) on the \( C^* \)-algebras \( A \otimes K \) and \( B \otimes K \), which are the unit fibre algebras of \( \mathcal{A}^S \) and \( \mathcal{B}^S \), such that \( \mathcal{A}^S \) and \( \mathcal{B}^S \) are isomorphic to the semidirect product bundles of \( A \otimes K, G \) and \( B \otimes K, G \) constructed by \((\alpha, w_{\alpha})\) and \((\beta, w_{\beta})\) as Fell bundles over \( G \), respectively. Since the inclusions of \( C^* \)-algebras induced by \( A \) and \( B \) are strongly Morita equivalent, so are the inclusions of \( C^* \)-algebras induced by \( \mathcal{A}^S \) and \( \mathcal{B}^S \). Hence the inclusions of \( C^* \)-algebras induced by \((\alpha, w_{\alpha})\) and \((\beta, w_{\beta})\) are strongly Morita equivalent. Then since the inclusions are irreducible, by \cite[Theorem 5.5]{4} there is an automorphism \( f \) of \( G \) such that \((\alpha, w_{\alpha})\) and \((\beta^f, w_{\beta}^f)\) are strongly Morita equivalent, where \((\beta^f, w_{\beta}^f)\) is the twisted action of \( G \) on \( B \otimes K \) induced by \((\beta, w_{\beta})\) and \( f \), that is, 
\[
\beta^f_t = \beta_{f(t)} \quad \text{and} \quad w_{\beta}^f(t, s) = w_{\beta}(f(t), f(s))
\]
for any \( t, s \in G \). Using this, we can prove the result.

2. Preliminaries

Let \( A \) be a \( C^* \)-algebra and we denote by \( M(A) \) the multiplier \( C^* \)-algebra of \( A \). Let \( \alpha \) be an automorphism of \( A \). Then there is a unique strictly continuous automorphism of \( M(A) \) extending \( \alpha \) by Jensen and Thomsen \cite[Corollary 1.1.15]{1}. We denote it by \( \alpha/e \).

Let \( G \) be a countable discrete group and \( e \) the unit element in \( G \). Let \( A = \{ A_t \}_{t \in G} \) be a Fell bundle over \( G \) and let \( A_e = A \), a \( C^* \)-algebra. Also, let \( B = \{ B_t \}_{t \in G} \) be a Fell bundle over \( G \) and let \( B_e = B \), a \( C^* \)-algebra. Following Abadie and Ferraro \cite[Definitions 2.1 and 2.2]{2}, we give the definition of an equivalence bundle:

**Definition 2.1.** (1) A right Hilbert \( B \)-bundle is a complex Banach bundle over \( G \), \( \mathcal{X} = \{ X_t \}_{t \in G} \) with continuous maps
\[
\mathcal{X} \times B \to X, \quad (x, b) \mapsto xb \quad \text{and} \quad \langle -, - \rangle_B : \mathcal{X} \times \mathcal{X} \to B, \quad (x, y) \mapsto \langle x, y \rangle_B
\]
such that:

1. \((\text{R})\) \( X_r B_s \subset X_{r+s} \) and \( \{ X_r, X_s \}_B \subset B_{r-s} \) for all \( r, s \in G \).
2. \((\text{R})\) \( X_r B_s \subset X_{r+s} \) and \( \langle x, y \rangle_B \) is bilinear for all \( r, s \in G \).
3. \((\text{R})\) \( X_r B_s \subset X_{r+s} \) and \( \langle x, y \rangle_B \) is linear for all \( x, y \in X_r \) and \( s \in G \).
4. \((\text{R})\) \( \langle x, y \rangle_B \geq 0 \) for all \( x \in X \) and \( \langle x, x \rangle_B = 0 \) implies \( x = 0 \). Beside, each fiber \( X_t \) is complete with respect to the norm \( x \mapsto \| (\langle x, x \rangle_B) \|^{1/2} \).
5. \((\text{R})\) \( \| (\langle x, x \rangle_B) \| \leq 2 \) for all \( x \in X \).
6. \((\text{R})\) \( \{ \langle X_t, X_s \rangle_B \mid s \in G \} = B_e \).

(2) A left Hilbert \( A \)-bundle is a complex Banach bundle over \( G \), \( \mathcal{X} = \{ X_t \}_{t \in G} \) with continuous maps
\[
A \times \mathcal{X} \to \mathcal{X}, \quad (a, x) \mapsto ax \quad \text{and} \quad A(-, -) : \mathcal{X} \times \mathcal{X} \to A, \quad (x, y) \mapsto A\langle x, y \rangle
\]
such that:

1. \((\text{L})\) \( A_r X_s \subset X_{s+r} \) and \( A\langle x, y \rangle \) is bilinear for all \( r, s \in G \).
2. \((\text{L})\) \( A_r X_s \subset X_{s+r} \) and \( A\langle x, y \rangle \) is linear for all \( x, y \in A_r \) and \( s \in G \).
3. \((\text{L})\) \( A\langle x, y \rangle = a A\langle x, y \rangle \) for all \( a \in A \) and \( x, y \in X \) and \( \langle x, x \rangle = 0 \) implies \( x = 0 \). Beside, each fiber \( X_t \) is complete with respect to the norm \( x \mapsto \| A\langle x, x \rangle \|^{1/2} \).
4. \((\text{L})\) \( \| A\langle x, x \rangle \| \leq 2 \) for all \( x \in X \).
5. \((\text{L})\) \( \{ A\langle X_t, X_s \rangle \mid s \in G \} = A_e \).
(3) We say that $\mathcal{X}$ is an $\mathcal{A} - \mathcal{B}$-equivalence bundle if $\mathcal{X}$ is both a left Hilbert $\mathcal{A}$-bundle, a right Hilbert $\mathcal{B}$-bundle and $\mathcal{A}(x,y)z = x(y,z)\mathcal{B}$ for all $x,y,z \in \mathcal{X}$. Besides, we say $\mathcal{A}$ is equivalent to $\mathcal{B}$ if there exists an $\mathcal{A} - \mathcal{B}$-equivalence bundle.

**Definition 2.2.** Let $\mathcal{A} = \{A_t\}_{t \in \mathcal{G}}$ be a Fell bundle over $\mathcal{G}$. We say that $\mathcal{A} = \{A_t\}_{t \in \mathcal{G}}$ is saturated if $A_tA_t^{-1} = A_e$ for any $t \in \mathcal{G}$.

Let $\mathcal{A} = \{A_t\}_{t \in \mathcal{G}}$ be a saturated Fell bundle over $\mathcal{G}$ and let $A_e = A$, a $C^*$-algebra. We suppose that $\mathcal{A}$ is a $\sigma$-unital $C^*$-algebra.

Let $C^*_r(\mathcal{A})$ be the reduced cross-sectional $C^*$-algebra of $\mathcal{A}$. Then for any $t \in \mathcal{G}$, $A_t$ is regarded as a closed subspace of $C^*_r(\mathcal{A})$ since $\mathcal{G}$ is discrete.

Let $K$ be the $C^*$-algebra of all compact operators on a countably infinite dimensional Hilbert space. Let $\{e_{ij}\}_{i,j \in \mathbb{N}}$ be a system of matrix units of $K$.

Let $A_t \otimes K$ be the closure of linear span of the subset

$$\{x \otimes k \in C^*_r(\mathcal{A}) \otimes K \mid x \in A_t, k \in K\}.$$

Let $\mathcal{A}^S = \{A_t \otimes K\}_{t \in \mathcal{G}}$. Then $\mathcal{A}^S$ is a saturated Fell bundle over $\mathcal{G}$ and $A \otimes K$ is its unit fibre algebra. Clearly $C^*_r(\mathcal{A}^S) = C^*_r(\mathcal{A}) \otimes K$. Since $\mathcal{A}$ is saturated, we can regard $A_t$ as an $\mathcal{A} - \mathcal{A}$-equivalence bimodule for any $t \in \mathcal{G}$ and by the definition of the product in $C^*_r(\mathcal{A}^S)$ we can regard $A_t \otimes K$ as the tensor product of the $\mathcal{A} - \mathcal{A}$-equivalence bimodule $A_t$ and the trivial $K - K$-equivalence bimodule $K$, which is $A \otimes K - A \otimes K$-equivalence bimodule for any $t \in \mathcal{G}$. Thus $\mathcal{A}^S$ is a saturated Fell bundle over $\mathcal{G}$. Since $A \otimes K$ is $\sigma$-unital, by [3] Theorem 7.3, there is a twisted action $(\alpha, w_\alpha)$ of $G$ on $A \otimes K$ such that $\mathcal{A}^S$ is isomorphic to the semidirect product bundle over $G$ induced by $(\alpha, w_\alpha)$ as Fell bundles over $G$.

**Lemma 2.1.** With the above notation, if $A' \cap M(C^*_r(\mathcal{A})) = C1$, then $(A \otimes K)' \cap M(\mathcal{A} \rtimes_{\alpha, w_\alpha}, r, G) = C1$.

**Proof.** Since $A' \cap M(C^*_r(\mathcal{A})) = C1$, by [3] Lemma 3.1] $(A \otimes K)' \cap M(C^*_r(\mathcal{A}) \otimes K) = C1$. Since $\mathcal{A}^S$ is isomorphic to the semidirect product bundle over $G$ of $A \otimes K$ induced by $(\alpha, w_\alpha)$ as Fell bundles over $G$, the inclusions $A \otimes K \subset C^*_r(\mathcal{A}) \otimes K$ and $A \otimes K \subset (A \otimes K) \rtimes_{\alpha, w_\alpha, r} G$ are isomorphic as inclusions of $C^*$-algebras. Thus $(A \otimes K)' \cap M((A \otimes K) \rtimes_{\alpha, w_\alpha, r} G) = C1$. }

Let $(\alpha, w_\alpha)$ and $(\beta, w_\beta)$ be twisted actions of a countable discrete group $G$ on $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. We suppose that $(\alpha, w_\alpha)$ and $(\beta, w_\beta)$ are strongly Morita equivalent with respect to a twisted action $\lambda$ of $G$ on an $\mathcal{A} - \mathcal{B}$-equivalence bimodule $X$, that is, $\lambda$ is a map from $G$ to $\text{Aut}(X)$ satisfying the following:

1. $\alpha_t(\lambda_t(x)y) = \lambda_t(x)y_t$,
2. $\beta_t((x,y)_B) = (\lambda_t(x), \lambda_t(y))_B$,
3. $(\lambda_t \circ \lambda_s)(x) = w_\alpha(t,s) \lambda_t(x)w_\beta(t,s)^*$

for any $t, s \in G$, $x, y \in X$, where we regard $X$ as a Hilbert $M(\mathcal{A}) - M(\mathcal{B})$-bimodule as in [3] Preliminaries.

Let $u$ and $v$ be unitary representations of $G$ to $M(\mathcal{A} \rtimes_{\alpha, w_\alpha, r} G)$ and $M(\mathcal{B} \rtimes_{\beta, w_\beta, r} G)$ implementing $\alpha$ and $\beta$, respectively, that is, $\alpha_t = \text{Ad}(u_t)$ and $\beta_t = \text{Ad}(v_t)$ for any $t \in G$. Let $\mathcal{A} = \{Au_t\}_{t \in \mathcal{G}}$ and $\mathcal{B} = \{Bu_t\}_{t \in \mathcal{G}}$ be the semidirect product Fell bundles over $G$ induced by $(\alpha, w_\alpha)$ and $(\beta, w_\beta)$, respectively.

For any $t \in G$, let $X_t = Xv_t$ as Banach spaces. We regard $X' = \{X_t\}_{t \in \mathcal{G}}$ as an $\mathcal{A} - \mathcal{B}$-equivalence bundle in the following (See [3] and [5]): For any $au_t \in Au_t,$
bundle $Y$. Hence by [6, Theorem 5.5] we obtain the conclusion.

**Proof.** Since

$$\langle xv_t, yv_t \rangle = \langle x, (\lambda_s \circ \lambda_t^{-1})(y) \rangle w_{\alpha}(t, s^{-1}) \alpha_{t^{-1}-1}(w_{\alpha}(s, s^{-1}))^* u_{t^{-1}-1},$$

$$\langle xv_t, yv_t \rangle_B = \beta^{-1}_s((\langle x, y \rangle_B) w_{\beta}(s^{-1}, s)^* w_{\beta}(s^{-1}, t) v_{s^{-1}}).$$

**Lemma 2.2.** With the above notation, $X = \{X_t\}_{t \in G}$ is an $A - B$-equivalence bundle over $G$ such that

$$A(\langle x_t, x_s \rangle) = A_{t^{-1}-1}, \langle x_t, x_s \rangle_B = B_{t^{-1}s}$$

for any $t, s \in G$.

**Proof.** By the definition of $X = \{X_t\}_{t \in G}$, it is clear that $X$ has Conditions (1R)-(4R) and (1L)-(4L) in Definition 2.1. For any $xv_t \in Xv_t$,

$$||\langle xv_t, xv_t \rangle_B|| = ||\beta^{-1}_t((x, x)_B)|| = ||(x, x)_B|| = ||x||^2 = ||xv_t||^2,$$

$$||A(\langle xv_t, xv_t \rangle)|| = ||A(x, x)|| = ||xv_t||^2.$$

Hence we see that $X_{t^2} \otimes_{\alpha, w_{\alpha}} B_{t^2}$ for any $t, s \in G$.

**3. Strong Morita equivalence**

Let $A = \{A_t\}_{t \in G}$ and $B = \{B_t\}_{t \in G}$ be saturated Fell bundles over $G$. Let $A_e = A$ and $B_e = B$ be $C^*$-algebras. Let $A^S$ and $B^S$ be the saturated Fell bundles over $G$ induced by $A$, $K$ and $B$, respectively. Let $(\alpha, w_{\alpha})$ and $(\beta, w_{\beta})$ be the twisted actions of $A$ on $A \otimes K$ and $B \otimes K$ such that $A^S$ and $B^S$ are isomorphic to the semidirect product bundles of $A \otimes K$ and $B \otimes K$ induced by $(\alpha, w_{\alpha})$ and $(\beta, w_{\beta})$, which are defined in Section 2.1. We suppose that $A \subset C^*_r(A)$ and $B \subset C^*_r(B)$ are strongly Morita equivalent and that $A' \cap M(C^*_r(A)) = C_1$.

**Lemma 3.1.** With the above notation and assumptions, there is an automorphism $f$ of $G$ such that $(\alpha, w_{\alpha})$ is strongly Morita equivalent to $(\beta, w_{\beta})$, $(\beta, w_{\beta})$ is the twisted action of $G$ on $B \otimes K$ induced by $(\beta, w_{\beta})$ and $f$, which is defined by

$$\beta^f_t = \beta_{f(t)}(t, s) = w_{\beta}(f(t), f(s))$$

for any $t, s \in G$.

**Proof.** Since $A' \cap M(C^*_r(A)) = C_1$, by Lemma 2.1,

$$(A \otimes K)' \cap M((A \otimes K) \rtimes_{\alpha, w_{\alpha}} G) = C_1.$$

Hence by [6, Theorem 5.5] we obtain the conclusion.

Let $B^S \otimes f = \{B_{f(t)} \otimes K\}_{t \in G}$ be the semidirect product bundle of $B \otimes K$ induced by $(\beta^f, w_{\beta}^f)$. Then by Lemma 2.2 and Lemma 3.1, there is an $A^S - B^S \otimes f$-equivalence bundle $Y = \{Y_t\}_{t \in G}$ such that

$$A^S(\langle Y_t, Y_s \rangle) = A_{t^{-1}} \otimes K, \quad \langle Y_t, Y_s \rangle_B^{S, f} = B_{f(t^{-1}s)} \otimes K$$

for any $t, s \in G$.

Let $B^f = \{B_{f(t)}\}_{t \in G}$ be the Fell bundle over $G$ induced by $B = \{B_t\}_{t \in G}$ and the automorphism $f$ of $G$.  

4
Lemma 3.2. With the above notation, $A$ and $B^f$ are equivalent with respect to an $A - B^f$-equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ such that

$$\mathcal{A}(X_t, X_s) = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_{B^f} = B_{f(t^{-1}s)}$$

Proof. Let $X_t = (1 \otimes e_{11})Y_t(1 \otimes e_{11})$ for any $t \in G$. Let $t, s \in G$ and let $a_t \in A_t, b_t \in B_t, x_s \in Y_s$ and $x_t, y_t \in Y_t$. Then

$$a_t(1 \otimes e_{11})x_s(1 \otimes e_{11}) = (1 \otimes e_{11})(a_t \otimes e_{11})x_s(1 \otimes e_{11}) \in X_{ts},$$

$$(1 \otimes e_{11})x_s(1 \otimes e_{11})b_t = (1 \otimes e_{11})x_s(b_t \otimes e_{11})(1 \otimes e_{11}) \in X_{st},$$

where we identify $(1 \otimes e_{11})(A_{st^{-1}} \otimes K)(1 \otimes e_{11})$ with $A_{st^{-1}}$. Also,

$$\mathcal{A}^s((1 \otimes e_{11})x_s(1 \otimes e_{11}), (1 \otimes e_{11})y_t(1 \otimes e_{11}))$$

$$= (1 \otimes e_{11})\mathcal{A}^s(x_s(1 \otimes e_{11}), y_t(1 \otimes e_{11}))(1 \otimes e_{11}),$$

where we identify $(1 \otimes e_{11})(A_{st^{-1}} \otimes K)(1 \otimes e_{11})$ with $A_{st^{-1}}$. Hence $\mathcal{X}$ has Conditions (1R)-(4R) and (1L)-(4L) in Definition 2.1. Furthermore,

$$\mathcal{A}^s((1 \otimes e_{11})x_t(1 \otimes e_{11}), (1 \otimes e_{11})x_t(1 \otimes e_{11}))$$

$$= (1 \otimes e_{11})\mathcal{A}^s(x_t(1 \otimes e_{11}), x_t(1 \otimes e_{11}))(1 \otimes e_{11}),$$

$$(1 \otimes e_{11})x_t(1 \otimes e_{11}), (1 \otimes e_{11})x_t(1 \otimes e_{11}))_{B^{s,t}}$$

$$= (1 \otimes e_{11})(1 \otimes e_{11})x_t(1 \otimes e_{11}), (1 \otimes e_{11})x_t(1 \otimes e_{11})).$$

These equations implies that $\mathcal{X}$ has Conditions (5R) and (5L) in Definition 2.1. It is clear that $\mathcal{X}$ has Conditions (6R) and (6L) in Definition 2.1. Moreover, let $c, d \in B \otimes K$. Then

$$\mathcal{A}^s((1 \otimes e_{11})x_c(1 \otimes e_{11}), (1 \otimes e_{11})y_t(1 \otimes e_{11}))$$

$$= (1 \otimes e_{11})\mathcal{A}^s(x_c(1 \otimes e_{11}), y_t(1 \otimes e_{11}))(1 \otimes e_{11}).$$

Since $1 \otimes e_{11}$ is full in $B \otimes K$, that is, $(B \otimes K)(1 \otimes e_{11})(B \otimes K) = B \otimes K$ and $Y_t(B \otimes K) = Y_t$ by [2, Proposition 1.7(i)],

$$\mathcal{A}^s((1 \otimes e_{11})x_s(1 \otimes e_{11}), (1 \otimes e_{11})y_t(1 \otimes e_{11})) = A_{st^{-1}} \otimes e_{11}.$$  

In the same way, we can see that

$$\mathcal{A}(X_t, X_s) = A_{ts^{-1}}, \quad \langle X_t, X_s \rangle_{B^f} = B_{f(t^{-1}s)}$$

for any $t, s \in G$. Since for any $x, y, z \in \mathcal{Y}$, $\mathcal{A}^s(x, y)z = x(y, z)_{B^{s,t}}$, we can see that

$$\mathcal{A}((1 \otimes e_{11})x(1 \otimes e_{11}), (1 \otimes e_{11})y(1 \otimes e_{11}))(1 \otimes e_{11})$$

$$= (1 \otimes e_{11})x(1 \otimes e_{11})(1 \otimes e_{11}), (1 \otimes e_{11})z(1 \otimes e_{11})).$$

for any $x, y, z \in \mathcal{Y}$. It follows that $\mathcal{X}$ is an $A - B^f$-equivalence bundle. Therefore, we obtain the conclusion. □

Theorem 3.3. Let $G$ be a countable discrete group and let $A = \{A_t\}_{t \in G}$ and $B = \{B_t\}_{t \in G}$ be saturated Fell bundles over $G$. We suppose that $A_e = A$ and $B_e = B$ are $\sigma$-unital $C^*$-algebras and that $A' \cap C^*_r(A) = C_1$, where $e$ is the unit element in $G$ and $C^*_r(A)$ is the reduced cross-sectional $C^*$-algebra of $A$. If the inclusions of $C^*$-algebras induced by $A$ and $B$ are strongly Morita equivalent, then
there is an automorphism $f$ of $G$ such that $\mathcal{A}$ and $\mathcal{B}^f$ are equivalent with respect to an $\mathcal{A} - \mathcal{B}^f$-equivalence bundle $\mathcal{X} = \{X_t\}_{t \in G}$ over $G$ such that

$$\mathcal{A}(X_t, X_s) = A_{ts^{-1}}, \quad (X_t, X_s)_{\mathcal{B}^f} = B_{f(t^{-1}s)}$$

for any $t, s \in G$, where $\mathcal{B}^f$ is a Fell bundle over $G$ induced by $\mathcal{B} = \{B_t\}_{t \in G}$ and $f$, that is, $\mathcal{B}^f = \{B_{f(t)}\}_{t \in G}$.

**Proof.** This is immediate by Lemmas 3.1, 3.2 and the discussions before Lemma 3.2.

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