KADOMTSEV-PETVIASHVILI EQUATION: ONE-CONSTRAINT METHOD AND LUMP PATTERN

JIE-YANG DONG, LIMING LING, AND XIAOEN ZHANG

ABSTRACT. The Kadomtsev-Petviashvili reduction method is a crucial method to derive the solitonic solutions of (1 + 1)-dimensional integrable system from high dimensional system. In this work, we explore to use the solutions of lower dimensional system to construct the solutions in the high dimensional one with the Darboux transformation. Especially, we utilize this method to disclose the relationship between the rogue wave and lump solutions. Under one-constraint method, the asymptotic analysis to the lump pattern of Kadomtsev-Petviashvili equation is given.

Keywords: Kadomtsev-Petviashvili equation, Lump solution, Asymptotic analysis, Rogue wave solution, Darboux transformation

1 Introduction

As one of the most important integrable (2 + 1) dimensional equations, the Kadomtsev-Petviashvili (KP) equation has a wide applications, it can be used to describe the capillary gravitational waves on a liquid surface and magneto-acoustic waves in plasma [1-3]. In general, the KP equation is written in the form

\[(4u_t - 12uu_x - u_{xxx})_x + 3u_{yy} = 0,\]

where \(u = u(x, y, t)\) denotes a scalar function with respect to the variables \(x, y\) and \(t\). Similar to the (1 + 1) integrable equation, the KP equation also has Lax pair, bilinear form and the symmetry constraint [4].

Due to its integrability, there are lots of research on it, including the soliton solution [5,6], the lump solution [7-10] and the quasi-periodic solution [11], the corresponding methods refer to the Darboux transformation [12,13], the bilinear method [14], the inverse scattering method [6] and the algebraic geometric method [15]. Among these interesting solutions, the lump solution is a special one, which is a type of the localized traveling waves in both \(x\) and \(y\) directions [16-19].

In [20], the authors constructed the high order lump solutions for KP equation and high order rogue wave solutions for nonlinear Schrödinger (NLS) equation respectively, and the result indicates that the high order rogue waves of NLS equation is similar to the high order lump solutions of KP equation. This incredible phenomena can be explained by the theory of “k-constraint”. Actually, this theory is very important to understand the KP equation and it can derive the KP hierarchy to the nonlinear system for finite number of dynamical coordinates [21-27]. Especially, under the framework of Sato theory [28] with one-constraint, the AKNS hierarchy can be derived through the KP hierarchy. As a result, the solution \(u(x, y, t)\) of KP equation can be represented as a product of \(\psi(x, y, t)\psi^*(x, y, t)\), in which, the function \(\psi(x, y, t)\) satisfies the second and the third flow of AKNS system. In [24], the authors gave a detailed description about the relation between the KP equation and the AKNS flows, which presents a good confirmation about the property between the rogue wave and lump solution again. Moreover, this method had been applied to other aspects, such as in [29], the authors constructed the quasiperiodic solution to KP equation with the nonlinearized Zakharov-Shabat eigenvalue problem. Apart from the KP equation, this one-constraint method can also be used to other (2 + 1) dimensional integrable equations [30].

In [14], Kodama gave the interaction patterns of \(N\)-soliton with the Young diagrams and gave a classification to the \(N\) soliton solutions. When \(t\) is large, the high order soliton or multi-solitons will be split into single soliton. However, the rogue wave is different from the soliton on the asymptotic analysis. As to the rogue wave in (1 + 1) dimensional system, when \(t\) is large, the rogue wave will go back to the constant background, thus its asymptotic behavior with respect to the variable \(t\) is meaningless. Whereas, the
geometry and the location character about the rogue waves are more diverse than the soliton, which is found to be determined by some free parameters. Very recently, in [31, 32], the authors analyzed the asymptotics of rogue wave with respect to these parameters and gave a general conclusion about the location distribution. Inspired by this theory, we would like to analyze the lump pattern of KP equation as well as its asymptotics. With the one-constraint method, the solution of KP equation is connected with the AKNS system, thus we try to analyze this asymptotics with the aid of the Darboux transformation of high dimensional AKNS system. As we know that the Darboux transformation of the AKNS system has a perfect architecture and theoretical framework, which has its merit in contrast to the Darboux transformation of the Lax pair of KP equation. Under the framework of this kind of Darboux transformation, the maximum solution can be analyzed easily, which has not been realized by the original Darboux transformation of KP equation.

This paper is organized as follows. In Section 2, we give a brief introduction about the KP hierarchy under one-constraint condition and convert the Lax pair of KP equation to a high dimensional AKNS system. With the one-constraint method, we construct the Darboux transformation to derive the solution formula in Section 3. By choosing a special spectral parameter under the plane wave background, we obtain the high order lump solution and rewrite it to a r function obtained by Ohta and Yang [33]. In Section 4, we analyze the lump patterns from several aspects, one is the asymptotics with respect to the time variable t and the other one is the asymptotics with one parameter $a_{2m+1}$, additionally, by the dynamics of solution at $t = 0$, we display a classification for lump pattern. We also give an auxiliary material in the appendix.

## 2 The KP hierarchy under one-constraint

In this section, we intend to present some introduction about the KP equation and the one-constraint method. It is well known that the KP hierarchy can be constructed with the Sato theory and the corresponding introduction had been reported in many previous works. The Sato theory is originated from the Kyoto school, which can be regarded as one of the most famous theories in the integrable system. In view of this point, the KP hierarchy is the fundamental one, and many other integrable equations can be derived from the KP hierarchies. Such as, the KdV equation, the Boussinesq equation, the NLS equation and so on. To analyze the KP equation, we first give a brief review about how the Lax pair of KP equation can be derived from the microdifferential operator [21, 28, 34, 35]. Suppose the operator $L$ as

\[
L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + u_4 \partial^{-3} + \cdots, \quad \partial = \partial/\partial x,
\]

where $u_i$, $i = 2, 3, \cdots$ are functions with respect to the variables $t = (t_1, t_2, t_3, \cdots)$. Consider a system of linear equations about the eigenfunction $\psi$,

\[
\begin{align*}
L \psi &= \lambda \psi, \\
\psi_{t_a} &= B_n \psi, \quad B_n = (L^n)_{+},
\end{align*}
\]

where $\lambda$ is the spectral parameter, the subscript $+$ denotes the positive part of the microdifferential operator. Through the compatibility conditions for the system (3), we can derive the Lax equation as

\[
L_{t_a} = [B_n, L],
\]

where $B_n$ denotes the differential part of $L^n$ and can be uniquely determined by the coordinates $u_2, u_3, \cdots$ and their x derivatives. With a simple calculation, the first few operators $B_n$ can be given as

\[
B_1 = \partial, \quad B_2 = \partial^2 + 2u_2, \quad B_3 = \partial^3 + 3u_2 \partial + 3u_3 + 3u_{2,x}.
\]

Based on the Lax equation (4), the KP hierarchy can be determined recursively. For later analysis, we give a generalized Leibniz rule,

\[
\partial^n f(x) = \sum_{j=1}^{\infty} \frac{n(n-1)(n-2) \cdots (n-j+1)}{j!} \frac{d^j f}{dx^j} \partial^{n-j},
\]

where $n$ is an integer. By choosing $n = 2$ and $n = 3$ in Eq.(4), we can obtain the following two equations:

\[
\begin{align*}
\partial^n f(x) &= \frac{n(n-1)(n-2) \cdots (n-j+1)}{j!} \frac{d^j f}{dx^j} \partial^{n-j}, \\
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\partial^n f(x) &= \frac{n(n-1)(n-2) \cdots (n-j+1)}{j!} \frac{d^j f}{dx^j} \partial^{n-j},
\end{align*}
\]
and
\[
(8a) \quad u_{2,t_3} = u_{2,xxx} + 3u_{3,xx} + 3u_{4,x} + 6u_2u_2x,
\]
\[
(8b) \quad u_{3,t_3} = u_{3,xxx} + 3u_{4,xx} + 3u_{5,x} + 6 (u_2u_3)_x,
\]
where the subscript , represents the derivative with respect to \( x \). Obviously, the functions \( u_3 \) and \( u_4 \) can be expressed as a function of \( u_2 \) via the Eq.(7a) and Eq.(7b),
\[
(9) \quad u_3 = \frac{1}{2} \partial^{-1}(u_{2,t_2} - u_{2,xx}),
\]
\[
(9) \quad u_4 = \frac{1}{4} \partial^{-2} (u_{2,t_2} - u_{2,xx}) - \frac{1}{4} (u_{2,t_2} - u_{2,xx}) - \partial^{-1} (u_2u_2x).
\]
Next, substitute Eq.(9) to Eq.(8a) and set \( u_2 = u, t_1 = x, t_2 = -i\eta, t_3 = t \), the KP equation (1) can be derived.

The Lax pair and its adjoint Lax pair for the KP equation are given by:
\[
(10a) \quad (-i) \psi_y = \psi_{xx} + 2u\psi,
\]
\[
(10b) \quad \psi_t = \psi_{xxx} + 3u\psi_x + 3 (u_x - i\partial_x^{-1} u_y) \psi,
\]
and
\[
(11a) \quad (-i) \psi_y^* = -\psi_{xx}^* - 2u\psi^*,
\]
\[
(11b) \quad \psi_t^* = \psi_{xxx}^* + 3u\psi_x^* + 3 (u_x - i\partial_x^{-1} u_y) \psi^*.
\]
where the superscript * represents the complex conjugate. If we identify the conserved covariant \( u \) with the covariant generator \( \psi\psi^* \),
\[
(12) \quad u = \psi\psi^* = qr,
\]
where \( q \equiv \psi \) and \( r \equiv \psi^* \). Then Eq.(10a) and Eq.(11a) can be reduced into
\[
(13) \quad iq_y = -q_{xx} - 2q^2r,
\]
\[
ir_y = r_{xx} + 2q^2r.
\]
From Eq.(13), we can easily get the identity \( i(qr)_y = (qr_x - qx r)_x \). Together with Eq.(10b) and Eq.(11b), we have
\[
(14) \quad q_t = q_{xxx} + 6q_xqr,
\]
\[
r_t = r_{xxx} + 6qr_x r.
\]
It can be seen that Eq.(13) and Eq.(14) are the second and third flow of the AKNS hierarchy [21]. Thus if \( q \) and \( r \) satisfy Eq.(13) and Eq.(14), then the potential \( u \) will satisfy KP equation Eq.(1). Therefore, the solutions of KP equation can be derived from AKNS system [21, 24].

Now we give a simple introduction about the one-constraint of KP equation. In general, the operator of \( k \)-constraint is set as \( L^k = B_k + q\partial^{-1}r \) [21]. When \( k = 1 \), the so called one-constraint operator is changed into
\[
(15) \quad L = \partial + q\partial^{-1}r.
\]
With this operator, consider the following coupled eigenvalue problems:
\[
(16a) \quad L\varphi = \varphi_x + q\varphi = \lambda \varphi,
\]
\[
(16b) \quad \varphi_x = r\varphi,
\]
and
\[
(17) \quad \varphi_{1n} = B_n\varphi,
\]
\[
\varphi_{1n} = A_n\varphi,
\]
where \( \varphi \) is a new “eigenfunction” defined by (16b) and \( A_n \) is the polynomial in \( \partial \) satisfying
\[
(18) \quad \partial A_n = rB_n - (B_n^r).
\]
where and

more, we can get the second and third flow with the compatibility condition of Eq.(20) by setting $\Phi$ and $\Phi_t$:

\[
\Phi_x = U(\lambda; x, t) \Phi, \quad \Phi_{tt} = V_n(\lambda; x, t) \Phi,
\]

where

\[
U(\lambda; x, t) = \begin{pmatrix} \lambda \sigma_3 + Q \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}, \quad V_n(\lambda; x, t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

$A, B, C, D$ are the polynomials with $\lambda$. The system (20) is the AKNS hierarchy with $su(2)$ symmetry. Furthermore, we can get the second and third flow with the compatibility condition of Eq.(20) by setting $n = 2, 3$ and $t_2 = -iy, t_3 = t$, then the corresponding Lax pair of Eq.(13) and Eq.(14) can be given as

\[
\Phi_x = U \Phi, \quad \Phi_y = V \Phi, \quad \Phi_t = W \Phi,
\]

with

\[
V = 2 \left( i \lambda^2 \sigma_3 + i \lambda Q - \frac{i}{2} \sigma_3 Q^2 + \frac{\sigma_3}{2} Q_x \right),
\]

\[
W = -4i \lambda^3 \sigma_3 - 4i \lambda^2 Q + 2 \lambda \left( i \sigma_3 Q^2 - \sigma_3 Q_x \right) + iQ_{xx} + 2iQ^3 + Q_x Q - QQ_x.
\]

As we all know that the traditional Lax pair of KP equation is Eq.(10a), but in this paper, we will use the Lax pair of AKNS system. During the calculation, we first give a constraint condition

\[
Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \right) = 0,
\]

\[
Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \right) = 0,
\]

\[
Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \right) = 0,
\]

\[
Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \right) = 0.
\]

With a simple calculation, Eq.(24a) and (24b) can be written as

\[
\begin{cases}
Z_{12}^{[1]} = ir_y - r_{xx} - 2qr^2 = 0, \\
Z_{12}^{[2]} = r_r - r_{xxx} - 6rqr_x = 0,
\end{cases}
\]

\[
\begin{cases}
Z_{21}^{[1]} = iq_y + q_{xx} + 2q^2r = 0, \\
Z_{21}^{[2]} = q_r - q_{xxx} - 6rqq_x = 0.
\end{cases}
\]

Moreover, $Z^{[3]}$ can be given as

\[
Z^{[3]} = \begin{pmatrix} Z^{[3]}_{11} & Z^{[3]}_{12} \\ Z^{[3]}_{21} & Z^{[3]}_{22} \end{pmatrix}, \quad Z^{[3]}_{11} = -Z^{[3]}_{11},
\]

\[
Z^{[3]}_{12} = -2 \left( qZ_{12}^{[1]} + rZ_{21}^{[1]} \right) \lambda - iZ^{[1]}_{12} + irZ^{[1]}_{21} + iq \left( Z^{[1]}_{12} \right)_x - irZ^{[2]}_{12} - iqZ^{[2]}_{21} - rZ^{[2]}_{21},
\]

\[
Z^{[3]}_{21} = 4Z^{[1]}_{12} \lambda^2 - 2i \left( \left( Z^{[1]}_{12} \right)_x - Z^{[2]}_{12} - 4qrZ^{[1]}_{12} - 2r^2Z^{[1]}_{21} - \left( Z^{[1]}_{12} \right)_x + \left( Z^{[2]}_{12} \right)_x \right),
\]

\[
Z^{[3]}_{22} = 4Z^{[1]}_{21} \lambda^2 + 2i \left( \left( Z^{[1]}_{21} \right)_x + Z^{[2]}_{21} - 2q^2Z^{[1]}_{12} - 4qrZ^{[1]}_{21} - \left( Z^{[1]}_{21} \right)_x - \left( Z^{[2]}_{21} \right)_x \right).
\]

From Eq.(26), we can see that if $q, r$ satisfy Eq.(25), they must satisfy Eq.(24c). Consequently, we only need to consider Eq.(25), then the solutions of KP equation can be given.

### 3 The one-constraint method to the KP equation

Last section, under the framework of Sato theory, we give the one-constraint of KP equation in a high dimensional AKNS system. During the calculation, we first give a constraint condition $u = qr$, where $u$ is the solution of KP equation and $q, r$ can be derived from the AKNS system. Compared to the original Lax pair Eq.(10a), the study to the new Lax pair has much more advantages. Such as, to derive the lump solution, if we use the original Lax pair Eq.(10a), we should construct the binary Darboux transformation, but if we use the Lax pair of AKNS system, we only need to modify the well-known Darboux matrix.
of AKNS system, which seems more simple. More importantly, we can establish the Riemann-Hilbert problem with the Darboux matrix of AKNS system and continue to study the asymptotics, which is hard with the original Lax pair Eq.(10a) to our knowledge. In [36, 37], the authors studied the soliton solutions on high dimensional AKNS system through the Darboux transformation, including the N-wave equation, Davey–Stewartson(DS) equation. Inspired by this idea, in this section, we will construct the lump solution to KP equation with the Lax pair(22).

The Darboux transformation for the AKNS system with su(2) symmetry is given by

\begin{equation}
T_1(\lambda; x, y, t) = \mathbb{I} - \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*} \begin{pmatrix} c_1 \end{pmatrix}(x, y, t), \quad \begin{pmatrix} c_1 \end{pmatrix}(x, y, t) = \frac{\phi_1 \phi_1^*}{\phi_1^* \phi_1}
\end{equation}

where

\begin{equation}
\phi_1 := \begin{bmatrix} \phi_{1,1}, \phi_{1,2} \end{bmatrix}^T = c_1(x, y, t) \Phi(\lambda_1; x, y, t) \Phi^{-1}(\lambda_1; 0, 0, 0) (1, -i)^T = c_1(x, y, t) \Phi(\lambda_1; x, y, t) (\alpha_1, \beta_1)^T
\end{equation}

and \(c_1(x, y, t)\) is an arbitrary function on \(x, y\) and \(t\). It can be verified that the Darboux matrix admits the following symmetry

\begin{equation}
T_1(\lambda; x, y, t) T_1^*(\lambda^*; x, y, t) = \mathbb{I}
\end{equation}

according to the symmetry of the Lax pair (22): \(U^*(\lambda^*; x, y, t) = -U(\lambda^*; x, y, t), V^*(\lambda^*; x, y, t) = -V(\lambda^*; x, y, t)\) and \(W^*(\lambda^*; x, y, t) = -W(\lambda^*; x, y, t)\). Then the Bäcklund transformation about the potential functions can be given by

\begin{equation}
Q^{[1]} = Q - (\lambda_1 - \lambda_1^*) [P_1, \sigma_3],
\end{equation}

which indicates the following identity:

\begin{equation}
q^{[1]}(x, y, t) = q(x, y, t) - 2(\lambda_1 - \lambda_1^*) \frac{\phi_{1,2}(\lambda_1; x, y, t) \phi_{1,1}^*(\lambda_1, x, y, t)}{\phi_1^*(\lambda_1; x, y, t) \phi_1(\lambda_1, x, y, t)}
\end{equation}

From this expression, we can get the maximal peak by the mean inequality, which is shown in the following proposition.

**Proposition 1.** The maximal peak about the new solution \(q^{[1]}(x, y, t)\) is given by

\begin{equation}
\max_{x,y,t}(|q^{[1]}(x, y, t)|) = \max_{x,y,t}(|q(x, y, t)|) + 2 |\text{Im}(\lambda_1)|,
\end{equation}

in particular, by set the special condition \(\phi_1 = \Phi(\lambda_1; x, y, t) \Phi^{-1}(\lambda_1; 0, 0, 0) (1, -i)^T\), the maximum point is located at the origin \((x, y, t) = (0, 0, 0)\).

**Remark 1.** Similar to the properties of rogue waves, the maximum modulus of \(q^{[N]}\) is also \(1 + 2N |\text{Im}(\lambda_1)|\), which is given by Proposition 1. Meanwhile, this kind of Darboux transformation provides a way to construct the high order lump solution with the maximal peak.

To construct the lump solution, we set the plane wave seed solution of \(q\) and \(r\) in Eq.(22) as:

\begin{equation}
q = be^{i\theta}, \quad r = be^{-i\theta}, \quad \theta = cx + (2b^2 - c^2) y + (6b^2 c - c^3) t,
\end{equation}

where \(b, c\) are arbitrary real number. Substituting the above special seed solution (33) to the corresponding Lax equation Eq.(22), we then have the fundamental solution

\begin{equation}
\Phi(\lambda; x, y, t) = e^{-\frac{i}{2}\theta\sigma_3} e^{i\xi\omega c_3}, \quad \xi = \sqrt{\left(\frac{\lambda + \frac{c}{2}}{b} \right)^2 + b^2},
\end{equation}

\begin{equation}
\omega = x + (2\lambda - c) y + 2 \left(\frac{b^2 - \frac{1}{2}c^2 + c\lambda - 2\lambda^2}{2} \right) t + a(\lambda),
\end{equation}

where \(a(\lambda)\) is independent with \(x, y\) and \(t\). Without loss of generality, we can set \(b = 1, c = 0\), other types of choice are equivalent to each other because of the Galilean symmetry:

\begin{equation}
q(x, y, t) = e^{-\frac{i}{2}i(3xe^{-ye^2-\frac{1}{2}te^3})} q \left( x - \frac{2}{3}ye - \frac{1}{3}te^2, y + te, t \right).
\end{equation}
Moreover, by iterating the above Darboux transformation step by step, then the N-fold Darboux transformation can be represented as a compact form. And the multi-solitonic solution can be derived with the N-fold Darboux transformation. With the theory of Darboux transformation for the AKNS system, we give the N-fold Darboux transformation for the system (22) in Theorem 1.

**Theorem 1.** Suppose there is a bounded smooth solution \( q(x, y, t) \in L^\infty (\mathbb{R}^3) \cap C^\infty (\mathbb{R}^3) \). And the matrix solution \( \Phi(\lambda; x, y, t) \) is analytic in the whole complex plane \( \mathbb{C} \), then the N-fold Darboux transformation can be represented as

\[
T_N(\lambda; x, y, t) = I + Y_N M^{-1} D^{-1} Y_N^*,
\]

where

\[
D = \text{diag} (\lambda - \lambda_1^*, \lambda - \lambda_2^*, \cdots, \lambda - \lambda_N^*), \quad Y_N = [\phi_1, \phi_2, \cdots, \phi_N], \quad M = \left( \frac{\phi_i^* \phi_j}{\lambda_i^* - \lambda_j^*} \right)_{1 \leq i, j \leq N},
\]

\[
\phi_i = [\phi_{i1}, \phi_{i2}]^T = c_i(x, y, t) \Phi(\lambda_i; x, y, t) \Phi^{-1}(\lambda_i; 0, 0, 0) (1, -i)^T,
\]

then the new potential can be given with the following Bäcklund transformation

\[
q[N] = q + 2 Y_{N2} M^{-1} Y_{N1}^*,
\]

where the subscript \( Y_{N,i} \) denotes the i-th row vector of \( Y_N \), \( i = 1, 2 \).

**Remark 2.** In view of the Darboux transformation, by choosing different spectral parameter \( \lambda_i \), we can get different types of solutions. If \( \lambda_i = \alpha i \) with \( 0 < \alpha < 1 \), we can get the breathers which is localized in y-direction and periodic in x-direction. If \( \lambda_i = \alpha i \) with \( \alpha > 1 \), we can get the breathers which is localized in x-direction and periodic in y-direction. If \( \lambda_i + \lambda_i^* \neq 0 \), the other types of breathers can be obtained. High order ones and their mixtures can also be obtained by performing the above transformation.

Our idea is planning to analyze the asymptotics by using the \( \tau \) function given by Ohta and Yang. To realize it, we should derive the \( \tau \) function from the Darboux transformation. Through the Eq. (31), (33), the solution \( q[1] \) can be rewritten as

\[
q[1] = e^{2iy} + 2(\lambda_1^* - \lambda_1) \frac{\phi_{12}^* \phi_{21}^*}{\phi_{12}^* \phi_{21}} = \frac{\phi_{11}^* \phi_{22} e^{2iy}}{-2i(\lambda_1^* - \lambda_1)} + i \phi_{12}^* \phi_{21}^*.
\]

By the property of determinant, the solution of Eq.(38) can be represented in the following form:

\[
q[N] = \frac{\det \left( m^{(1)}_{ij} \right)}{\det \left( m^{(0)}_{ij} \right)} e^{2iy}, \quad m^{(1)}_{ij} = \frac{\phi_{11}^* \phi_{22} e^{2iy}}{-2i(\lambda_1^* - \lambda_1)} + i \phi_{12}^* \phi_{21}^*, \quad m^{(0)}_{ij} = \frac{\phi_{i1}^* \phi_{j1}}{-2i(\lambda_i^* - \lambda_j)}.
\]

Moreover, from the definition of \( \phi_i \) in Eq.(37), the component \( m^{(0)}_{ij} \) and \( m^{(1)}_{ij} \) can be given in a quadric form

\[
m^{(0)}_{ij} = c_i^* c_j \left( \alpha_i^* e^{-i \xi_{ij}^* \omega_i}, \beta_i^* e^{i \xi_{ij}^* \omega_i} \right) \begin{pmatrix} \frac{\lambda_i^* + \xi_{ij}^*}{\lambda_i^* + \xi_{ij}^*} & \frac{1}{\lambda_i^* + \xi_{ij}^*} \\ \frac{1}{\lambda_i^* + \xi_{ij}^*} & \frac{\lambda_i^* + \xi_{ij}^*}{\lambda_i^* + \xi_{ij}^*} \end{pmatrix} \begin{pmatrix} \alpha_j e^{i \xi_{ij} \omega_j} \\ \beta_j e^{-i \xi_{ij} \omega_j} \end{pmatrix},
\]

\[
m^{(1)}_{ij} = c_i^* c_j \left( \alpha_i^* e^{-i \xi_{ij}^* \omega_i}, \beta_i^* e^{i \xi_{ij}^* \omega_i} \right) \begin{pmatrix} \frac{\lambda_i^* + \xi_{ij}^*}{\lambda_i^* + \xi_{ij}^*} & \frac{1}{\lambda_i^* + \xi_{ij}^*} \\ \frac{1}{\lambda_i^* + \xi_{ij}^*} & \frac{\lambda_i^* + \xi_{ij}^*}{\lambda_i^* + \xi_{ij}^*} \end{pmatrix} \begin{pmatrix} \alpha_j e^{i \xi_{ij} \omega_j} \\ \beta_j e^{-i \xi_{ij} \omega_j} \end{pmatrix}.
\]

For simplicity, we introduce some new notations

\[
p_i^* = -i (\lambda_i^* + \xi_{ij}^*), \quad q_j = i (\lambda_j + \xi_{ij}) \quad (j \neq i).
\]
According to the definition of $\alpha_i, \beta_j$ in Eq. (28) and the relationship $\xi^2 = 1 + \lambda^2$, we have

$$a_i^* = i(p_i^* + 1), \quad b_i^* = -i\left(\frac{1}{q_i^*} + 1\right), \quad a_j = -i(q_j + 1), \quad \beta_j = i\left(\frac{1}{q_j^*} + 1\right). \quad (44)$$

By choosing the proper parameters $c_i$, we know that the exponent term $i\xi_j\omega_j - i\xi_i^*\omega_i^*$ can be changed into another equivalent form, that is

$$i\xi_j\omega_j - i\xi_i^*\omega_i^* \sim \left[(i\lambda_j + i\xi_j) \left(x + a (\lambda_j)\right) - (i (\lambda_j + \xi_j))^2 i y + \left(i (\lambda_j + \xi_j)^3 + 3i (\lambda_j + \xi_j)\right) t\right]$$

$$+ \left(-i (\lambda_j + \xi_j) (x + a^* (\lambda_j)) + \left(-i (\lambda_j + \xi_j)^2 i y + (-i (\lambda_j + \xi_j)^3 - 3i (\lambda_j + \xi_j)\right) t\right]$$

$$= \left[ q_j x - q_j^2 i y + \left(q_j^3 + 3q_j\right) t + \sum_{k=0}^{\infty} \hat{a}_k (\ln q_j)^k \right] + \left[p_i^* x + p_i^* i y + \left(p_i^* i + 3p_i^*\right) t + \sum_{k=0}^{\infty} \hat{a}_k^* (\ln p_i^*)^k \right],$$

where $a(\lambda_j) = q_j^{-1}\sum_{k=0}^{\infty} \hat{a}_k (\ln q_j)^k$.

Furthermore, set the factor involving the exponent term $e^{i\xi_j\omega_j - i\xi_i^*\omega_i^*}$ in Eq.(41) and Eq.(42) as $\hat{B}_{ij}^{(0)}$ and $\hat{B}_{ij}^{(1)}$ respectively. With this definition, we introduce some new notations $\xi_i^*, \eta_j$ as

$$\xi_i^* = p_i^* x + p_i^* i y + \left(p_i^* i + 3p_i^*\right) t, \quad \eta_j = q_j x - q_j^2 i y + \left(q_j^3 + 3q_j\right) t,$$

then we have

$$\hat{B}_{ij}^{(0)}(p_i^*, q_j) = \frac{(p_i^* + 1)(q_j + 1)}{p_i^* + q_j} \exp \left(\xi_i^* + \eta_j + \sum_{k=0}^{\infty} \hat{a}_k^* (\ln p_i^*)^k + \sum_{k=0}^{\infty} \hat{a}_k (\ln q_j)^k\right), \quad (46)$$

$$\hat{B}_{ij}^{(1)}(p_i^*, q_j) = \frac{(p_i^* + 1)(q_j + 1)}{p_i^* + q_j} \left(-\frac{p_i^*}{q_j}\right) \exp \left(\xi_i^* + \eta_j + \sum_{k=0}^{\infty} \hat{a}_k^* (\ln p_i^*)^k + \sum_{k=0}^{\infty} \hat{a}_k (\ln q_j)^k\right). \quad (47)$$

Next, we make a minor revision to $\hat{B}_{ij}^{(0)}$ and $\hat{B}_{ij}^{(1)}$ with the purpose of not altering the value of $q^{[N]}$ in Eq.(40), that is

$$\hat{B}_{ij}^{(n)}(p_i^*, q_j) = \frac{\hat{B}_{ij}^{(n)}(p_i^*, q_j)}{\exp \left(\sum_{k=0}^{\infty} A_{2k}^+(\ln p_i^*)^{2k} + \sum_{l=0}^{\infty} A_{2l}^-(\ln q_j)^{2l}\right)}, \quad n = 0, 1, \quad (48)$$

where $A_{2k}^+, A_{2l}^-$ are given by the following expansion:

$$\xi_i^* + \eta_j + \sum_{k=0}^{\infty} \hat{a}_k^* (\ln p_i^*)^k + \sum_{k=0}^{\infty} \hat{a}_k (\ln q_j)^k = \sum_{k=0}^{\infty} A_k^+ (\ln p_i^*)^k + \sum_{l=0}^{\infty} A_l^- (\ln q_j)^l,$$

i.e.

$$A_k^+ = x + 2^k i y + 3^k t + 3t + \hat{a}_k^*, \quad A_l^- = x - 2^l i y + 3^l t + 3t + \hat{a}_l^-.$$

In Eq. (47), the even power terms of $\ln q_j$ and $\ln p_i^*$ in exponential factor can be eliminated since the elements determinants are invariant under the transformation $\ln q_j \rightarrow -\ln q_j$ and $\ln p_i^* \rightarrow -\ln p_i^*$. We can rewrite $\hat{B}_{ij}^{(n)}(p_i^*, q_j)$ as

$$\frac{1}{1 - \frac{(p_i^* - 1)(q_j - 1)}{(p_i^* + 1)(q_j + 1)}} \left(-1\right)^n \exp \left(\sum_{k=0}^{\infty} A_{2k+1}^+ (\ln p_i^*)^{2k+1} + n(\ln p_i^*) + \sum_{l=0}^{\infty} A_{2l+1}^- (\ln q_j)^{2l+1} - n(\ln q_j)\right). \quad (51)$$
Under the condition $\lambda_i = \lambda_j = -i$, we have $p_i^* = q_j = 1$, $\xi_i = \bar{\zeta}_j = 0$. From the definition $m_{ij}^{(n)} (n = 0, 1)$ in Eq.(41) and Eq.(42), we get a relation between $m_{ij}^{(n)}$ and $\tau$ function

$$m_{ij}^{(n)} = \exp \left( \sum_{k=0}^{\infty} A_k^{(2)}(\ln p_i^*)^{2k} + \sum_{l=0}^{\infty} A_k^{(2)}(\ln q_j)^{2l} \right)$$

$$= B_{ij}^{(n)} \left( p_i^*, q_j \right) - B_{ij}^{(n)} \left( p_{i^*}, 1, q_j \right) + B_{ij}^{(n)} \left( 1, 1, q_j \right)$$

$$= 4 \sum_{k,j=0}^{\infty} \tau_{k,j}^{(n)} (\ln p_i^*)^{2k+1} (\ln q_j)^{2l+1},$$

where $B_{ij}^{(n)} = \sum_{k,j=0}^{\infty} \tau_{k,j}^{(n)} (\ln p_i^*)^k (\ln q_j)^l$ is expanded at $\lambda_i = \lambda_j = -i$, and

$$\tau_{k,l}^{(n)} = \frac{1}{k!} \left( \frac{\partial}{\partial \ln p_i^*} \right)^k \frac{1}{l!} \left( \frac{\partial}{\partial \ln q_j} \right)^l B_{ij}^{(n)} \bigg|_{p_i^* = q_j = 1}, \quad n = 0, 1.$$

Correspondingly, the solution $q^{[N]}$ in Eq. (40) can be converted into the following form:

$$q^{[N]} = e^{2i\nu \det (P \tau^{(1)} Q)} \det (P \tau^{(0)} Q), \quad \tau^{(n)} = \begin{bmatrix} \tau_{1,1}^{(n)} & \cdots & \tau_{1,2}^{(n)} & \cdots & \tau_{1,n}^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau_{n,1}^{(n)} & \cdots & \tau_{n,2}^{(n)} & \cdots & \tau_{n,n}^{(n)} \end{bmatrix}, \quad n = 0, 1$$

$$P = \begin{bmatrix} \ln p_1^* & (\ln p_1^*)^3 & \cdots & (\ln p_1^*)^{2N-1} \\ \ln p_2^* & (\ln p_2^*)^3 & \cdots & (\ln p_2^*)^{2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \ln p_N^* & (\ln p_N^*)^3 & \cdots & (\ln p_N^*)^{2N-1} \end{bmatrix}, \quad Q = \begin{bmatrix} \ln q_1 & (\ln q_1)^3 & \cdots & (\ln q_1)^{2N-1} \\ \ln q_2 & (\ln q_2)^3 & \cdots & (\ln q_2)^{2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \ln q_N & (\ln q_N)^3 & \cdots & (\ln q_N)^{2N-1} \end{bmatrix},$$

especially, when $p, q \rightarrow 1, \ln p^*, \ln q \rightarrow 0$, the solution $q^{[N]}$ will be simplified into the following formula.

$$q^{[N]} = e^{2i\nu \det (P_{N,N}(\tau^{(1)})_{N,N}(Q_{N,N}))} = e^{2i\nu \det (P_{N,N}(\tau^{(0)})_{N,N}(Q_{N,N}))} \cdot e^{2i\nu \det (\tau^{(1)})_{N,N} \cdot \det (\tau^{(0)})_{N,N}} = e^{2i\nu \sigma_1 \sigma_0}, \quad \sigma_n = \frac{\det \left( \tau_{i,j}^{(n)} \right)}{\det (\tau_{i,j}^{(2i-1,j-1)})},$$

where $\tau_{i,j}^{(2i-1,j-1)}$ is defined in Eq.(53), which is a function with respect to $x, y, t$ and $\hat{a}_{2k+1}, (k = 1, 2, \cdots, N - 1)$. The determinant formula (55) is consistent with the result derived by the Hirota bilinear method in [33]. To the best of our knowledge, the universality of lump solution between the Darboux transformation and $\tau$ function had not been discovered in the previous research.

In fact, there are several distinct formulas on the rogue waves in the previous literatures based on different expansions. We show that they are equivalent to each other. In the reference [38], Guo, one of the authors and Liu used the expansion in $\epsilon$ by $\lambda = -i(1 + \epsilon^2)$. Then we have

$$\ln p = \ln \left( 1 + e^2 - \epsilon \sqrt{2 + \epsilon^2} \right) = -\int_0^\epsilon \frac{2}{\sqrt{2 + s^2}} ds = \sum_{i=1}^{\infty} p_i^{[i]} e^{2i-1},$$

where $p_i^{[i]} = -\sqrt{2} \left( \frac{1}{i} \right)^{1/2(2i+1)}$. Similarly, the function $\ln q = \ln \left( 1 + \eta^2 - \eta \sqrt{2 + \eta^2} \right)$ has the same expansion as $\eta$. Then we can obtain the following determinant formula

$$q^{[N]} = e^{2i\nu \dot{\sigma}_1 / \dot{\sigma}_0}, \quad \dot{\sigma}_n = \det \left( \tau_{i,j}^{(n)} \right)_{1 \leq i,j \leq N},$$
Thus we have

\[ m^{(n)} = \sum_{i,j=1}^{+\infty,+\infty} \hat{m}_{2i-1,2j-1}^{(n)} e^{2i-1\eta} 2^{j-1}. \]

By the Faà di Bruno formula, the coefficients between different expansions method have the following relation:

\[ \hat{m}_{2i-1,2j-1}^{(n)} = \sum_{g=1}^{j} B_{2i-1,2g-1} B_{2j-1,2h-1} \hat{m}_{2g-1,2h-1}^{(n)} \]

where

\[ B_{2i-1,2g-1} = \sum_{i=1}^{j} \frac{(2g-1)!}{k_1!k_2! \cdots k_i!} (p[1])^{k_1} (p[2])^{k_2} \cdots (p[n])^{k_i} \]

which indicates that

\[ \begin{bmatrix} \hat{m}_{1,1}^{(n)} & \hat{m}_{1,3}^{(n)} & \cdots & \hat{m}_{1,2N-1}^{(n)} \\ \hat{m}_{3,1}^{(n)} & \hat{m}_{3,3}^{(n)} & \cdots & \hat{m}_{3,2N-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{m}_{2N-1,1}^{(n)} & \hat{m}_{2N-1,3}^{(n)} & \cdots & \hat{m}_{2N-1,2N-1}^{(n)} \end{bmatrix} = \begin{bmatrix} B_{1,1}^{(n)} & 0 & \cdots & 0 \\ B_{1,3}^{(n)} & B_{3,3}^{(n)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{2N-1,1}^{(n)} & B_{2N-1,3}^{(n)} & \cdots & B_{2N-1,2N-1}^{(n)} \end{bmatrix} \]

Thus we have \( q^{[N]} = e^{2iy\hat{\theta}_{10}} = e^{2iy_{10}} \).

Furthermore, when \( p_1 = p_2 = \cdots = p_N = 1 \), the \( N \)-th order Darboux matrix will be changed into the following form:

\[ T_N(\lambda;x,y,t) = I + Y_N M^{-1} D Y_N^\dagger, \quad M = X^\dagger S X \]

and

\[ Y_N = \begin{bmatrix} \Phi_1[0], \Phi_1[1], \cdots, \Phi_1[N-1] \end{bmatrix}, \]

\[ D = \begin{bmatrix} \frac{1}{(\lambda-1)^2} & 0 & \cdots & 0 \\ \frac{1}{(\lambda-1)^3} & \frac{1}{(\lambda-1)^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(\lambda-1)^N} & \frac{1}{(\lambda-1)^{N-1}} & \cdots & \frac{1}{(\lambda-1)} \end{bmatrix}, \quad X = \begin{bmatrix} \Phi_1[0], \Phi_1[1], \cdots, \Phi_1[N-1] \end{bmatrix}, \]

\[ S = \begin{bmatrix} \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \xi} & \cdots & \frac{\partial}{\partial \xi} \\ \frac{1}{(2i)^2} & \frac{1}{(2i)^2} & \cdots & \frac{1}{(2i)^2} \\ \frac{1}{(2i)^3} & \frac{1}{(2i)^3} & \cdots & \frac{1}{(2i)^3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(2i)^N} & \frac{1}{(2i)^N} & \cdots & \frac{1}{(2i)^N} \end{bmatrix}, \]

with \( \Phi_1[k] = \frac{1}{k!} \left( \frac{d}{d\lambda} \right)^k \Phi_1 |_{\lambda=1} \) and

\[ \Phi_1 = e^{-\frac{i}{2} \theta_{10} \xi} \left\{ \frac{(i\lambda + 1) \sin(\xi \omega)}{\xi} \begin{bmatrix} 1 \\ i \end{bmatrix} + \cos(\xi \omega) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}. \]
To construct the high order lump solution in a compact form, we need to modify the expansion as shown in the previous section. Unlike the previous binary Darboux transformation, we give the Darboux transformation via the Lax pair of high dimensional AKNS system, which can be used to construct the corresponding Riemann-Hilbert problem with the theory in [39, 40]. Through the N-th order Darboux transformation, we can define the following sectional analytic matrix

\[ M^{[N]}_+(\lambda; x, y, t) := \begin{cases} M^{[N]}_+(\lambda; x, y, t) = \left( \frac{\lambda + i}{\lambda - i} \right)^{-N/2} T_N(\lambda; x, y, t), \\ M^{[N]}_-(\lambda; x, y, t) = T_N(\lambda; x, y, t)e^{-\frac{i}{4} \partial_\lambda} E^{i \lambda \omega_3} E^{-1} T_N^{-1}(\lambda; 0, 0, 0) E^{-i \lambda \omega_3} E^{-1} e^{\frac{i}{4} \partial_\lambda}. \end{cases} \]

Especially, if we choose \( a(\lambda) = 0 \), then

\[ \left( \frac{\lambda + i}{\lambda - i} \right)^{-N/2} T_N(\lambda; 0, 0, 0) = Q_c \left( \frac{\lambda + i}{\lambda - i} \right)^{\frac{N}{2} \sigma_3} Q_c^{-1}, \]

where \( Q_c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -i & 1 \end{pmatrix} \). Under the above special case, the matrix function \( M^{[N]}(\lambda; x, y, t) \) satisfies the following Riemann-Hilbert problem.

**Riemann-Hilbert problem 1.** Let \((x, y, t) \in \mathbb{R}^3\) be arbitrary parameters, and \( N \in \mathbb{Z}_{>0} \). Then we can find a \( 2 \times 2 \) matrix function \( M^{[N]}(\lambda; x, y, t) \) satisfying the following properties:

- **Analyticity:** \( M^{[N]}(\lambda; x, y, t) \) is analytic for \( \lambda \in \mathbb{C} \setminus \partial D_0 \), where \( D_0 \) is a big circle involving the point \( \lambda = \pm i \). It takes the continuous boundary values from the interior and exterior of \( \partial D_0 \).
- **Jump condition:** The jump condition in the boundary of \( \partial D_0 \) are related by

\[ \begin{cases} M^{[N]}_+(\lambda; x, y, t) = M^{[N]}_-(\lambda; x, y, t) E^{i \lambda \omega_3} E^{-1} Q_c \left( \frac{\lambda + i}{\lambda - i} \right)^{\frac{N}{2} \sigma_3} Q_c^{-1} E^{-i \lambda \omega_3} E^{-1}. \end{cases} \]

- **Normalization:** \( M^{[N]}(\lambda; x, y, t) = I + O(\lambda^{-1}) \), as \( \lambda \to \infty \).

With the aid of Deift-Zhou nonlinear steepest method, the spatial-temporal pattern for the high order lumps for the large \( N \) can be carried out. For the case \( t = 0 \), the asymptotics for the large order rogue waves was derived by Bilman and Miller very recently [41]. The infinite order rogue waves were given in [42] by combing the Darboux transformation and Riemann-Hilbert method. As for the large \( t \) and \( N \), we would like to explore it in the future work. In the following, we will study the large \( t \) asymptotics for the fixed order \( N \).

From the solutions of KP equation (1) by the Darboux transformation, we know \( u \) satisfies the non-vanishing boundary condition \( u = 1 \). Through a simple symmetry, the non-vanishing background can be eliminated by the transformation \( u \to u - 1 \), \( x \to x - 3t \), \( t \to t \), \( y \to y \). Thus in the later analysis, all of the lump solutions are changed into the zero background.

## 4 The lump pattern of KP equation

In last section, we have constructed the solutions \( q^{[N]}(x, y, t) \) by the Darboux transformation, which is given in a determinant form in Eq.(55). In this section, we will utilize the determinant formula to analyze the lump pattern. Actually, for the other form of solutions, we can also analyze the asymptotics. To describe the structure of lump solution more clearly, we would like to use the method provided in the reference [31].

From the definition of \( \tau_{ij}^{(n)} \) in Eq.(53), the elements of \( \tau \) can be rewritten as

\[ \tau_{ij}^{(n)} = \frac{1}{n!} (\partial \ln p^*)^j (\partial \ln q^*)^i B^{(n)} \bigg|_{p^* = q^* = 1}, \]

\[ B^{(n)} = \frac{(-1)^n}{1 - (e^{\ln p^* - 1})(e^{\ln q^* + 1})} \exp \left( \sum_{k=0}^{\infty} A_{2k+1}^{2k+1} (\ln p^*)^{2k+1} + n(\ln p^*) + \sum_{l=0}^{\infty} A_{2l+1}^{2l+1} (\ln q^*)^{2l+1} - n(\ln q^*) \right), \]

\[ \mathrm{det} M^{[N]}_+(\lambda; x, y, t) = e^{-\frac{x^2}{2} - \frac{y^2}{2}} \prod_{t=1}^{N} (\lambda - (2t-1)i). \]
where $A_{2k+1}^\pm, A_{2j+1}^\mp$ are given in Eq.(50). Then we can rewrite the coefficient of $B^{(n)}$ as

\begin{equation}
(1)^n \sum_{v=0}^{\infty} \left( \frac{(\ln p^* - 1)(\ln q - 1)}{\ln p^* + 1}\right)^v = (1)^n \sum_{v=0}^{\infty} \left( \frac{\ln p^* \ln q}{4} \right)^v \exp \left( v \sum_{j=1}^{\infty} s_j \left( (\ln p^*)^j + (\ln q)^j \right) \right),
\end{equation}

where $s_j$ is given by the following expansion:

\begin{equation}
\sum_{j=1}^{\infty} s_j \lambda^j = \ln \left[ \frac{2}{\lambda} \tanh \left( \frac{\lambda}{2} \right) \right].
\end{equation}

It is clear that $s_{2j+1} = 0, j = 0, 1, \cdots$. Next, we introduce some new notations for simplification:

\begin{align*}
x_1^+(n) &= A_1^+ + n = (x + 3t) + 2iy + n + 3t + a_1, \\
x_1^-(n) &= A_1^- - n = (x + 3t) - 2iy - n + 3t + a_1^*, \\
x_{2k+1}^+(n) &= A_{2k+1}^+ = \frac{(x + 3t) + 2^{(k+1)}i}{(2k+1)!} + \frac{3^{(k+1)}}{(2k+1)!} + a_{2k+1}, \\
x_{2k+1}^-(n) &= A_{2k+1}^- = \frac{(x + 3t) - 2^{(k+1)}i}{(2k+1)!} + \frac{3^{(k+1)}}{(2k+1)!} + a_{2k+1}^*, \quad k \geq 1,
\end{align*}

where $a_{2k+1} = a_{2k+1}^*(k = 1, 2, \cdots, N - 1)$ are given in Eq.(55). Without loss of generality, we can assume $a_1 = 0$, then $B^{(n)}$ can be rewritten as

\begin{equation}
(1)^n \sum_{v=0}^{\infty} \left( \frac{\ln p^* \ln q}{4} \right)^v \exp \left( v \sum_{j=1}^{\infty} s_j (\ln p^*)^j + (\ln q)^j \right) + \sum_{k=0}^{\infty} x_{2k+1}^+(\ln p^*)^{2k+1} + \sum_{l=0}^{\infty} x_{2l+1}^-(\ln q)^{2l+1}).
\end{equation}

Considering the Taylor expansion of $B^{(n)}$ in Eq.(64), we find that the coefficient $\tau_{ij}^{(n)}$ of $(\ln p^*)^i(\ln q)^j$ is related to the Schur polynomial:

\begin{equation}
\tau_{ij}^{(n)} = (-1)^n \sum_{v=0}^{\infty} \frac{1}{4^v} S_{i-j}(x^+(n) + vs) S_{j-v}(x^-(n) + vs),
\end{equation}

where $x^\pm(n) = \left( x_1^+(n), 0, x_2^+(n), 0, \cdots, 0, x_{2k+1}^+(n), 0, \cdots \right)$ and $s = (0, s_2, 0, s_4, \cdots, 0, s_{2k}, 0, \cdots)$. The definition of Schur polynomial $S_j(x)$ with $x = (x_1, x_2, \cdots)$ is

\begin{equation}
\sum_{j=0}^{\infty} S_j(x) \lambda^j = \exp \left( \sum_{j=1}^{\infty} x_j \lambda^j \right).
\end{equation}

Afterwards, the solution of KP equation can be expressed by the $\sigma_1$ and $\sigma_0$ function, which is shown in theorem 2.

**Theorem 2.** The $N$-th order lump solutions for the KP equation are given by

\begin{equation}
\rho_N(x, y, t; A) = \left| \frac{\sigma_1(x - 3t, y, t; a_3, \cdots, a_{2N-1})}{\sigma_0(x - 3t, y, t; a_3, \cdots, a_{2N-1})} \right|^2 - 1,
\end{equation}

where $N$ is the order of solution and $\rho_n$ is given by Eq.(55)/(66)/(67)/(69). $A = (a_3, a_5, \cdots, a_{2N-1})$ and $a_3, \cdots, a_{2N-1}$ are some free parameters in Eq.(67).

**Remark 3.** From the definition $\tau$ function in Eq.(69), if all the parameters $a_{2k+1}, (k = 1, 2, \cdots, 2N - 1)$ are purely imaginary number, then we have $x^\pm(n)|_{x^\pm \rightarrow x^\mp} = -x^\pm(n)$. With a simple calculation, we get a symmetry relation $\tau_{2i-1, 2i-1}^{(n)}|_{x^\pm \rightarrow x^\mp} = -\tau_{2i-1, 2i-1}^{(n)}$. Spontaneously, the solution $u$ will satisfy $u_N(x, y, t; A) = u_N(-x, y, -t; A)$.

**Remark 4.** With the method in [43], we know that the high order lump solution can also be expressed into another equivalent formula $u_N(x, y, t; A) = \frac{\partial^2}{\partial x^2} \ln(\sigma_0(x - 3t, y, t; A))$. 

4.1 The classification of lump solution and lump pattern

In theorem 2, we have derived the high order lump solution for KP equation, next we begin to analyze the asymptotics or the spatial-temporal pattern for these high order lump solutions. It can be seen that the solution in theorem 2 contains some free parameters $a_{2k+1}, (1 \leq k \leq N-1)$. In recent literature [31], the authors discussed the asymptotics about these parameters in rogue wave pattern for NLS equation. Unlike to the rogue waves, the lump solution involves three variables $x, y$ and $t$, where the added variable $y$ is similar to the time variable $t$ in $(1+1)$ dimensional system, that is, we can regard the variable $t$ as a new one. Then the asymptotics to lump solutions will be diverse, we can not only study the asymptotics or the spatial-temporal pattern for these high order lump solutions. It can be seen that the asymptotics or the spatial-temporal pattern will be diverse, we can not only study the asymptotics or the spatial-temporal pattern for these high order lump solutions. It can be seen that the asymptotics or the spatial-temporal pattern for these high order lump solutions. It can be seen that the asymptotics or the spatial-temporal pattern for these high order lump solutions.

Proposition 2. When $t$ is large and $|t| \gg |a_{2k+1}|^{3/2}, (1 \leq k \leq N-1)$, if $(x, y)$ satisfies $\sqrt{(x+3t)^2 + 4y^2} = O(|t|^{3/2})$, then the $N$-th lump solution $u_N(x, y, t; A)$ in Eq.(1) decays to the zero background, except at or near the point $(x, y, t) = (x_0, y_0, t)$, where

$$x_0 + 3t + 2iy_0 = A^\frac{3}{2}z_0, \quad A = -3t,$$

and $z_0$ is the non-zero root of Yablonskii-Vorob’ev polynomial $Q_N^{[t]}(z)$.

The Yablonskii-Vorobév polynomial, which is originated from the rational solutions of the second Painlevé equation (P$_I$) [44, 45],

$$w'' = 2w^3 + zw + \alpha.$$

When $\alpha$ is an integer, this equation has the rational solutions as

$$w(z; N) = \frac{d}{dz} \ln \frac{Q_{N-1}^{[m]}(z)}{Q_N^{[m]}(z)}, \quad N \geq 1, \quad w(z, 0) = 0, \quad w(z; -N) = -w(z; N),$$

where $m$ is a non-negative integer and $Q_N^{[m]}(z)$ is called the Yablonskii-Vorob’ev polynomial. Moreover, it is equivalent to a determinant composed by the Schur polynomial $p_k^{[m]}(z)$ [46];

$$Q_N^{[m]}(z) = c_N \det_{1 \leq i, j \leq N} [p_{2j-i}^{[m]}(z)], \quad c_N = \prod_{j=1}^N (2j-1)!!,$$

$$\sum_{k=0}^\infty p_k^{[m]}(z)\lambda^k = \exp \left( z\lambda - \frac{2m}{2m+1} \lambda^{2m+1} \right), \quad p_k^{[m]}(z) = 0, \quad (k < 0).$$

In the later asymptotic analysis, we find that the locations of lump solution have an intimate relationship with the root structures of the Yablonskii-Vorob’ev polynomial $Q_N^{[m]}(z)$, which has been studied in [47–51]. With the result in [49], we know the order of $Q_N^{[m]}(z)$ is $N(N+1)/2$, among which the number of nonzero roots is $N^0_m$ and the multiplicity of zero root is $N^0_0$, where

$$N^0_0 = \begin{cases} N \mod (2m + 1), & 0 \leq N \mod (2m + 1) \leq m, \\ 2m - (N \mod (2m + 1)), & N \mod (2m + 1) > m, \end{cases}$$

and

$$N^0_m = \frac{1}{2} \left[ N(N+1) - N^0_0 \left( N^0_0 + 1 \right) \right].$$

In addition, in [48] [49], the authors have proved that the nonzero roots of the Yablonskii-Vorob’ev polynomial are all simple.

Furthermore, in the neighbourhood of the special point $(x, y) = (x_0, y_0)$, the asymptotics will be different, which is shown in proposition 3.
**Proposition 3.** When $t$ is large and $|t| \gg |a_{2k+1}|^{\frac{3}{2}}$, $(1 \leq k \leq N-1)$, in the neighbourhood of the point $(x,y) = (x_0, y_0)$ with $\sqrt{(x-x_0)^2 + 4(y-y_0)^2} = O(1)$, where $(x_0, y_0)$ is given in proposition 2, then the $N$-th order lump solution $u_N(x,y,t;A)$ in Eq.(71) approaches a first-order lump $u_1(x-(x_0+3t), y-y_0, t)$, where

$$u_1(x,y,t) = \frac{-32(x+3t)^2 + 128y^2 + 8}{(4(x+3t)^2 + 16y^2 + 1)} + O(|t|^{-\frac{3}{2}}).$$

From this expression, it is easy to see that the velocity about $u_1$ is $v = (v_x, v_y)$, where $v_x = -3t, v_y = 0$.

What is more, the asymptotic behavior in the near point $(x,y) = (-3t,0)$ must be different, whose asymptotics is given in proposition 4.

**Proposition 4.** When $t$ is large and $|t| \gg |a_{2k+1}|^{\frac{3}{2}}$, $(1 \leq k \leq N-1)$, in the neighborhood of $(x,y) = (-3t,0)$ with $\sqrt{(x+3t)^2 + 4y^2} = O(1)$, the $N$-th order lump solution $u_N(x,y,t;A)$ in Eq.(71) will asymptotically approach a lower $N^{[1]}_0$-th order lump $u_{N^{[1]}_0}(x,y,t)$. In this case, $N^{[1]}_0$ is either 1 or 0 with the definition in Eq.(76). Thus we have

$$u_{N^{[1]}_0}(x,y,t) = \begin{cases} 0, & N^{[1]}_0 = 0, \\ -\frac{32(x+3t)^2 + 128y^2 + 8}{(4(x+3t)^2 + 16y^2 + 1)}, & N^{[1]}_0 = 1. \end{cases}$$

With the result in proposition 3, 4, when $t$ is large, the whole asymptotics contains two different types—in the neighborhood of $(x,y) = (x_0, y_0)$ or $(x,y) = (-3t,0)$. In total, the asymptotic expression about the KP equation is given in theorem 3.

**Theorem 3.** According to proposition 2, 3, 4, we can get a conclusion that when $t$ is large and satisfies $|t| \gg |a_{2k+1}|^{\frac{3}{2}}$, $(1 \leq k \leq N-1)$, the asymptotic expression about the lump soliton of KP equation is

$$u_N(x,y,t;A) \rightarrow u_{N^{[1]}_0}(x,y,t) + \sum_{j=1}^{N^{[1]}_p} \left[ -\frac{32(x-x_0^{(j)})^2 + 128(y-y_0^{(j)})^2 + 8}{(4(x-x_0^{(j)})^2 + 16(y-y_0^{(j)})^2 + 1)} \right].$$

where $N^{[1]}_0, N^{[1]}_p, (x_0, y_0)$ and $u_{N^{[1]}_j}(x,y,t)$ are given in Eq.(76), Eq.(77), Eq.(72) and Eq.(79) respectively. The superscript $(j)$ means the $j$-th $(x_0, y_0)$.

In fact, the number of the first-order lumps is always $N(N+1)/2$ when $t$ is large no matter $N^{[1]}_0$ is 0 or 1.

In the following proposition 5, we will analyze the asymptotics with respect to the internal parameter $a_{2k+1}$. By the calculation, we find an approximate critical point $\frac{2k}{2^{k+1}}$ about the parameters $a_{2k+1}$, and the behaviors of lump solution will change essentially whether $a_{2k+1}$ is greater than $\frac{2k}{2^{k+1}}$ or not.

**Proposition 5.** Suppose only one parameter $a_{2m+1}$ large enough with $|a_{2m+1}| \gg |a_{2m+1}|^{\frac{3}{2}}$, $(1 \leq m \leq N-1)$ and other parameters satisfying $a_{2k+1} = o(|a_{2m+1}|^{\frac{3}{2}})$, $(k \neq m)$, then the asymptotics can be summarized in the following aspects.

- **When $|t| \gg |a_{2m+1}|^{\frac{3}{2}}$, the $N$-th lump solution is still decomposed into $N(N+1)/2$ first-order lumps, whose asymptotic expression can also be expressed by Eq.(80).**

- **When $|t| \ll |a_{2m+1}|^{\frac{3}{2}}$, as $(x,y)$ is far away from the point $(-3t,0)$, the $N$-th order lump solution $u_N(x,y,t;a_3, a_5, \cdots, a_{2N-1})$ will be separated into $N^{[m]}_p$ first-order lumps, where $N^{[m]}_p$ is given in Eq.(77). That is, in the neighborhood of $(x,y) = (x_0^{[m]}, y_0^{[m]})$ with $\sqrt{(x-x_0^{[m]})^2 + 4(y-y_0^{[m]})^2} = O(1)$, the lump
solution asymptotically approaches a first-order lump \( u_1 \left( x - (x_0^m + 3t), y - y_0^m, t \right) \), where

\[
x_0^m = {\text{Re}} \left( \left( -\frac{2m + 1}{2a_{2m+1}} \right)^{\frac{1}{m+1}} z_0^m - 3t \right), \quad y_0^m = \frac{1}{2} \text{Im} \left( \left( -\frac{2m + 1}{2a_{2m+1}} \right)^{\frac{1}{m+1}} z_0^m \right),
\]

and \( z_0^m \) is the nonzero root of \( Q_N^{[m]} (z) \); additionally, in the neighborhood of the point \((x, y) = (3t, 0)\) with \( \sqrt{(x + 3t)^2 + 4y^2} = O(1) \), the \( N \)-th order lump solution will asymptotically approaches a lower \( N_0^{[m]} \)-th order lump solution \( u_{N_0^{[m]}} (x, y, t; a_3, a_5, \cdots, a_{2N_0^{[m]}}) \), where \( N_0^{[m]} \) is given in Eq.(76).

4.2 The classification of high order lumps with purely imaginary parameters \( A \)

In the last subsection, we give a detailed asymptotic analysis for the lump solutions of KP equation in proposition 2-5, which can be utilized to give the whole scenery of lumps. In this subsection, we would like to use these properties to give the classification of these lump solutions. Compared to the rogue wave in NLS equation, the lump solution of KP equation is 1 + 2 dimensional and has a two-dimensional dynamic graph as the variation of time variable \( t \). If \( t \) is large enough, through the proposition 2-4, we know that the patterns of lumps have the similar structure. But when \( t \) is small, the lump patterns will be diverse. Based on the result in Remark 3, we can classify the lump solution if we give a constraint to these parameters \( a_{2k+1}, k = 1, 2, \cdots, N - 1 \) and set them all purely imaginary. Under this choice, the evolution process is symmetry on \( t = 0 \) when \( x \) is changed into \( -x \), then we can deem that the lumps have the strongest interaction at \( t = 0 \), which is the reason of classification. At this time, the behavior of lump solution is consistent with the rogue wave in [31, 52], thus we would like to classify the lumps into the following four categories by the dynamics of solution at \( t = 0 \):

- **Complete polymerization type:** \( u_N (x, y, 0; A) \) has the maximal peak at \((x, y) = (0, 0)\).
- **Partially polymerization type:** the asymptotic state of \( u_N (x, y, 0; A) \) has a lower order lumps with the maximal peak \( u_K (x, y, 0; A = 0), 2 \leq K \leq N \).
- **Completely separating type:** the asymptotic state of \( u_N (x, y, 0; A) \) is separately distributed with the first order lumps \( u_i (x - x_i, y - y_i, 0), i = 1, 2, \cdots, N(N + 1)/2 \), where \( (x_i, y_i) \) are the central points of the first lumps. The least distance \( d_C \) between any two central points \((x_i, y_i)\) of these first lumps is enough wide (in this work, we set the criterion \( d_C \geq 1 \)).
- **Hybrid type:** the other case are not involved in the above three categories.

By the above analysis and the determinant formula, we know that the spatial distribution of lumps at \( t = 0 \) is affected by the parameters \( a_{2k+1}, k = 1, 2, \cdots, N - 1 \). In general, the properties for \( u_N (x, y, 0; A) \) with \( A \) purely imaginary are not completely determined. But for one large internal parameter and multiple internal parameters, the distribution can be almost determined by the roots of \( Q_N^{[m]} (z) \) as given in the reference [31]. Now, we give a description about the classification.

**Case 1: Completely polymerization type**

By the proposition 1 and Darboux matrices (62), we can prove that the lump solutions in Eq.(71) will attain the maximal value \( u_N (0, 0, 0; 0) = (2N + 1)^2 - 1 \) at the point \((x, y, t) = (0, 0, 0)\) by choosing \( A = 0 \), which will yield the high order lumps with completely polymerization type. One of calculation way for \( u_N (0, 0, 0; 0) \) was given in [53]. On the other hand, as \( t \to \pm \infty \), by the proposition 3 and 4, we know that high order lumps can be decomposed into \( \frac{1}{N} N(N + 1) \) first-order lumps, whose locations are determined by the roots of \( Q_N^{[m]} (z) \). We find that \( u_N (0, 0, 0; 0) = (2N + 1)^2 - 1 = 8 \times \frac{1}{4} N(N + 1) \), which means that all the first order lumps will collide at the origin, where \( 8 \) is the height of first order lump.

Especially, we would like to exhibit the dynamic behavior by the third and fifth-order lumps (Fig. 1). Firstly, we analyze the dynamics of high order lumps with large \( t \). For \( N = 3 \), by the formula (75), we have

\[
Q_3^{[1]} (z) = z^6 + 20z^3 - 80,
\]

which implies that the corresponding locations of lumps \((x_0, y_0)\) are given by Eq.(72):

\[
x_0 = \text{Re}\left( (-3t)^{\frac{1}{3}} z_0 \right) - 3t, \quad y_0 = \text{Im}\left( (-3t)^{\frac{1}{3}} z_0 \right)/2.
\]
The case for \( N = 5 \) can be calculated similarly. Obviously, the locations of lumps are related to the time variable \( t \). It can be seen that when \( t \) is large enough, the third and fifth order lump solution will be separated into 6 and 15 first-order lumps respectively. When \( t = 0 \), all the first-order lumps collide at the origin \((x, y, t) = (0, 0, 0)\) and form a maximal peak. We can find that the peak’s values of the third and fifth order lumps are \( 6 \times 8 = 48 \) and \( 15 \times 8 = 120 \) respectively, which exhibits the linear superposition for the first order lumps. Under this special setting \( A = 0 \), the rule holds for all high order lumps.

**Case 2: Partially polymerization type**

The partially polymerization lump solutions are defined by the asymptotic state of \( u_N(x, y, 0; A) \) having a lower order lumps with maximal peak. Under this case, we give a sufficient condition with the parameters choice of \( A \) with one parameter \( a_{2m+1} \) large enough. With the aid of proposition 5, there will appear a lower order (less than \( N \)) lump solution at \((x, y, t) = (0, 0, 0)\). Specifically, we set 
\[
|a_{2m+1}| \gg 2^{2m+1}
\]
and
\[
\begin{cases} 
  a_{2k+1} = 0, & 1 \leq k < N_0^{[m]} \\
  |a_{2k+1}| \ll |a_{2m+1}| 2^{\frac{m+1}{2}} & k \geq N_0^{[m]},
\end{cases}
\]
where \( N_0^{[m]} \) is defined by Eq.(76) and \( N_0^{[m]} \geq 2 \). Then the lower order lump solution located at \((x, y) = (0, 0)\) can reach the corresponding maximum with its own order. By choosing some proper parameters, we exhibit the evolution dynamics in Fig.2.

In Fig.2, we set \( N = 4 \) and \( a_7 = 50i \). With a simple calculation, we know \( m = 3, N_0^{[3]} = 2 \) and \( N_p^{[3]} = 7 \). At \( t = 0 \), there appear seven first-order lumps embraced with one second-order lump \( u_2(x, y, 0; 0) \) located at \((x, y) = (0, 0)\). The locations of these seven first-order lumps can be obtained by the nonzero roots of \( Q_4^{[3]}(z) \) in Eq.(75) and the transformation Eq.(81). When \( t = \pm 1/6 \), an asymmetrical second-order lump \( u_2(x, y, \pm 1/6; 0) \) is approximately in the neighborhood of \((x, y) = (\mp 1/2, 0)\).

**Case 3: Completely separating type**

The completely separating lump solutions are defined by the asymptotic state of \( u_N(x, y, 0; A) \) having the separated first-order lumps. Compared to the sufficient condition given in case 2, we can also give a similar sufficient condition to the parameter \( A \) with one parameter \( a_{2m+1} \) large enough, but in this case,
Figure 2. The fourth order lump solution of KP equation by choosing $a_3 = a_5 = 0, a_7 = 50i$. The center of each picture is $(x, y) = (-3t, 0)$. These red circles are the predicted locations of these first-order and the yellow circle gives the predicted location of the second-order lump.

$N_0^{[m]}$ should be 1 or 0. Then there will only appear the first-order lumps at anytime. By choosing some proper parameters, we show one example of this type in Fig. 3.

Figure 3. The third order lump solution of KP equation by choosing $a_3 = 0, a_5 = 20i$. The center of yellow circle is $(x, y) = (-3t, 0)$ and these red circles are the predicted locations by Eq.(81) and Eq.(83).

In Fig.3, choosing $N = 3, a_3 = 0, a_5 = 20i$, then we get the third-order lump with $m = 2, N_p^{[2]} = 5, N_0^{[2]} = 1$. When $t = 0$, there appear six first-order lump solutions, and the locations for the five lumps can be obtained by the roots of $Q_3^{[2]}(z)$ in Eq.(75) and the transformation Eq.(81). The left first-order lump is still located in the neighborhood of $(x, y, t) = (0, 0, 0)$.

**Case4: Hybrid type**

Apart from above three cases, we call the other lump solutions as the hybrid type ones. In this case, there exist a lot of choices to the parameters $a_{2k+1}$. And we list some ones as follows:

1. $|a_{2k+1}| < \frac{2^k}{2^{k+1}}, (1 \leq k \leq N - 1)$, but $A \neq 0$;
2. $|a_{2m+1}| > \frac{2^{m}}{2^{m+1}}$, but there exists $k$ such that $a_{2k+1} \neq 0$ with $1 \leq k < N_0^{[m]}$;
3. there are more than two parameters $a_{2m+1}$ large enough.

If these parameters satisfy the first one, then there only appear the first-order lump solutions, but the least distance between two adjacent lumps is too close to distinguish; if these parameters satisfy the second one, there maybe appear a higher order lump solution and some first-order lumps, still, the least distance is too close; if the parameters satisfy the third one, there maybe appear more than two higher order lump solutions, which are similar to the rogue wave in [52]. We only show the evolutional process in this case by choosing the parameters as the first one, which are shown in Fig.4.

From Fig. 1 to Fig. 4, we show four types of lump solutions by choosing different parameters. When $t = 0$, different types of lumps have different behaviors. But when $t$ is large, all of them have a similar geometric distribution, which can be seen from Fig.1(a), Fig.3(a) and Fig.4(a). This incredible phenomenon
Figure 4. The lump solutions of KP equation when $N = 3, 5$, in this case, these parameters $a_{2k+1} < \frac{2k}{2k+1}$ but $A \neq 0$. The red circles are the predicted locations by Eq.(83). The pictures in the first row are the third order lump solution $u_3(x,y,t;i,i)$ and the pictures in the second row are the fifth order lump solution $u_5(x,y,t;\frac{1}{10},\frac{1}{5},\frac{1}{2},i)$.

Remark 5. In the references [13, 54, 55], the authors give the lump solution of KP equation via the binary Darboux transformation and the $\tau$ function method, and they give the asymptotics for the lump solutions as $t$ is large. Our method is different from theirs and has its merit. In [13, 55], as $t$ is large, the locations of the peak is related to the real root of an orthogonal polynomials, while in our results, the locations are determined by the root of Yablonskii-Vorob'ev polynomial. More importantly, the maximal peak of our higher order lump can be obtained by choosing proper parameters. Apart from the asymptotics with $t$, we also analyze the asymptotics with respect to the parameter $a_{2k+1}$.

5 Discussions and conclusions

In this paper, we give a detailed analysis for the lump pattern on both theoretically and numerically via the one-constraint method, to our best knowledge, there are seldom articles about the asymptotics to the high dimensional system. The lump solution in the KP equation is similar to the rogue wave in the $(1+1)$ dimensional system, which is a hot topic in recent years. Thus the dynamical analysis to the lump solution is extremely significant. With the one-constraint method, we classify the lump patterns into four different types according to the behavior at $t = 0$. Observing the $\tau$ function in Eq.(69), we find that this solution has a good symmetry if all parameters $a_{2k+1}, (k = 1, 2, \cdots, N-1)$ are purely imaginary. Based on this fact, we can give this kind of classification, while if these parameters are arbitrary complex number, the symmetry of this solution is ambiguous, we have not found a location that can be used as the criterion for classification, this interesting topic can be studied in future.

In this paper, we only use the one-constraint condition. Apart from this, the equations derived from the two-constraint or the three-constraint are also diverse, such as in [22], the authors gave a $3 \times 3$ matrix eigenvalue Yajima-Oikawa equation from the two-constraint. Moreover, for a general $k$-constraint, there
are many high dimensional equations, which can also be reduced to a simple (1+1) dimensional equation. Similar to the one-constraint equation, our idea can be extended to the high order Lax pair and analyze the properties effectively. In [43, 56], with the Darboux transformation, the authors analyzed the asymptotics for the multi-component system, whose calculation is more complicated. Our method can also be applied to this similar multi-component system and give the corresponding solutions with the Darboux transformation and the τ function, which can be used to analyze the asymptotics efficiently, if possible, the location of the multi-component system can be determined by the root of a polynomial.

With the theory of inverse scattering, from the Darboux matrix of high dimensional AKNS system, we construct the corresponding Riemann-Hilbert problem. More importantly, this Riemann-Hilbert problem can be used to study the asymptotics when $N$ is large. In the future study, we prepare to analyze the asymptotics of lump solution for KP equation when $N$ is large by using this Riemann-Hilbert problem.

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### Appendix

In the appendix, we give the proofs about the above propositions and theorems in subsection 4.1. Actually, the following proofs follow the calculations in the Ref. [31] with a minor revision on the variables.

**Proof of Proposition 2.** Firstly, we set $\epsilon = |t|^{-\frac{1}{2}}$. From the definition of $\sigma_n$, we can give the element $\tau_{2i-1,2j-1}^{(n)}$ of $\sigma_n(x - 3t, y, t; A)$ as

$$
\tau_{2i-1,2j-1}^{(n)} = (-1)^n \sum_{\nu=0}^{\min(2i-1,2j-1)} \frac{1}{4^\nu} S_{2j-1-\nu}(x^+(n) + \nu s - t) S_{2i-1-\nu}(x^-(n) + \nu s - t),
$$

where $t = \left(3t, 0, \frac{3t}{3!}, 0, \ldots, 0, \frac{3t}{(2k+1)!}, 0, \ldots \right)$. Under the condition $|a_{2k+1}| \ll |t|^\frac{n}{2}$, which indicates $a_{2k+1}e^{2k+1} = o(e)$, when $\sqrt{(x + 3t)^2 + 4y^2} = O\left(|t|^{\frac{1}{2}}\right) = O\left(e^{-1}\right)$, we have

$$
S_k(x^+(n) + \nu s - t) = S_k(x + 3t + 2iy + n, vs_2, \ldots)
$$

$$
= e^{-k}S_k \left( (x + 3t + 2iy + n)e, vs_2, \frac{x + 3t + 2iy}{3!} + a_3 + \frac{(3^3 - 3)t}{3!} e^3, \ldots \right)
$$

$$
= e^{-k}S_k \left( (x + 3t + 2iy)e, 0, \frac{(3^3 - 3)t}{3!} e^3, 0, 0, 0, \ldots \right) \left[1 + O(e)\right]
$$

$$
\sim S_k(x + 3t + 2iy, 0, 4t, 0, 0, 0, \ldots).
$$

Thus the Schur polynomial can be given as

$$
S_k(x^+(n) + \nu s - t) \sim S_k(v), \quad |t| \to \infty
$$

where

$$
v = (x + 3t + 2iy, 0, 4t, 0, 0, \ldots),
$$

$$
\sum_{k=0}^{\infty} S_k(v) \lambda^k = \exp \left([x + 2iy + 3t] \lambda + 4t \lambda^3 \right).
$$

From the definition of $p_k^{[1]}(z)$ in Eq.(75), if we set

$$
A = -3t, \quad z = A^{-\frac{1}{2}}(x + 3t + 2iy),
$$

$$
p_k^{[1]}(z) = \frac{1}{\sqrt{2\pi i}} \int_{C} \frac{1}{\lambda^{k+1}} \exp \left(\frac{z^2}{4\lambda} - 4t \lambda^3 \right) d\lambda.
$$
then the polynomial \( p_k^{(1)}(z) \), \( S_k(x^+(n) + vs - t) \) and \( S_k(v) \) satisfy
\[
S_k(x^+(n) + vs - t) \sim S_k(v) = A^k p_k^{(1)}(z).
\]

Based on the linear algebra, we can rewrite \( \sigma_n(x - 3t, y; t; A) \) as a \( 3N \times 3N \) determinant:
\[
\sum_{0 \leq v_1 < v_2 < \cdots < v_N \leq 2N - 1} \det_{1 \leq i, j \leq N} \left[ \frac{1}{2 \pi^2} S_{2i-1-v_j} (x^+(n) + v_j s - t) \right] \times \det_{1 \leq i, j \leq N} \left[ \frac{1}{2 \pi^2} S_{2i-1-v_j} (x^-(n) + v_j s - t) \right].
\]
By using the relationship (89), we can reduce the determinant involving \( x^+(n) \) in Eq.(90) as
\[
2^{-\delta} (-3t)^{\frac{1}{2}} (N^2 - \delta) \det_{1 \leq i, j \leq N} \left[ p_{2i-1-v_j}(z) \right],
\]
where \( \delta = v_1 + v_2 + \cdots + v_N \). It is obvious that the highest order term of \( t \) can reach by choosing \( v_j = j - 1 \), then we have
\[
\det_{1 \leq i, j \leq N} \left[ \frac{1}{2 \pi^2} S_{2i-1-v_j} (x^+(n) + v_j s - t) \right] = 2^{-N(N-1)/2} (-3t)^{\frac{1}{2}} (N^2 - \delta) \cdot c_N^{-1} Q_N^{(1)}(z).
\]
From the expansion in Eq.(90), when \( |t| \rightarrow \infty \), the leading order term of \( \sigma_n(x - 3t, y; t; A) \) is
\[
2^{-N(N-1)/2} (-3t)^{\frac{1}{2}} (N^2 - \delta) \cdot c_N^{-1} \left[ Q_N^{(1)}(z) \right] + O(t^{N(N-1)-2}), \quad |t| \rightarrow \infty.
\]
We can see that the leading terms of \( \sigma_n(x - 3t, y; t; A) \) are independent of \( n \), therefore \( u_N(x, y; t; A) = \left| r_1(x-3t, y; A) \right|^2 \left| r_0(x-3t, y; A) \right|^2 \rightarrow 0 \) as \( |t| \rightarrow \infty \). It completes the proof.

**Proof of Proposition 3.** When \( t \) is large and \( |t| \gg |a_{2k+1}| \frac{3}{2}, (1 \leq k \leq N - 1) \), in the neighborhood of \((x, y) = (x_0, y_0)\), the coefficients about the highest term \( t^{N(N-1)/2} \) of \( \sigma_n(x - 3t, y; t; A) \) will be almost zero. Therefore we have to consider the second highest order term of \( t \). Then we should make a more refined asymptotics for \( S_k(x^+(n) + vs - t) \):
\[
S_k(x^+(n) + vs - t) = e^{-k} S_k \left( (x + 3t + 2iy + n) e^{v_2 e^2}, \cdots \right)
\]
\[
= e^{-k} S_k \left( (x + 3t + 2iy + n) e^{v_2 e^2}, 0, 0, 0, 0, \cdots \right) \left[ 1 + O \left( e^2 \right) \right]
\]
\[
= S_k(\tilde{v}) \left[ 1 + O \left( e^2 \right) \right],
\]
where \( \tilde{v} = (x + 3t + 2iy + n, 0, 4t, 0, 0, 0, \ldots) \). By setting \( \tilde{v} = A^{-\frac{1}{2}} (x + 3t + 2iy + n) \), then the Schur polynomial \( S_k(x^+(n) + vs - t) \) can be given with a similar formula as Eq.(92). Especially, \( p_k^{(1)}(\tilde{z}) \) and \( S_k(\tilde{v}) \) have the following relation:
\[
S_k(x^+(n) + vs - t) = S_k(\tilde{v}) \left[ 1 + O \left( e^2 \right) \right] = A^k p_k^{(1)}(\tilde{z}) \left[ 1 + O \left( e^2 \right) \right].
\]
According to the equations Eq.(90), (91), we know that the second highest order term is composed by two index choices of \( v \): one is \( v = (0, 1, \cdots, N - 1) \) and the other one is \( v = (0, 1, \cdots, N - 2, N) \). Then the leading order terms should be the combination of these two factors. Firstly, if \( v = (0, 1, \cdots, N - 1) \), with a similar calculation as Eq.(91)~(93), we can derive the determinant involving the \( x^+(n) \) as
\[
\det_{1 \leq i, j \leq N} \left[ \frac{1}{2 \pi^2} S_{2i-1-v_j} (x^+(n) + v_j s - t) \right] = 2^{-N(N-1)/2} (-3t)^{\frac{1}{2}} \cdot c_N^{-1} \cdot \left[ Q_N^{(1)}(\tilde{z}) \right] \left[ 1 + O \left( e^2 \right) \right].
\]
As \( Q_N^{(1)}(z_0) = 0 \), we should expand \( Q_N^{(1)}(\tilde{z}) \) at \( \tilde{z} = z_0 \),
\[
Q_N^{(1)}(\tilde{z}) = Q_N^{(1)}(z_0) + [Q_N^{(1)}]'(z_0) (\tilde{z} - z_0) + O((\tilde{z} - z_0)^2)
\]
\[
= [Q_N^{(1)}]'(z_0) A^{-\frac{1}{2}} [N - x_0 + 2i(y - y_0) + n] \left[ 1 + O(e) \right].
\]
Substitute the above equation into Eq. (96), then the determinant involving \( x^+ (n) \) changes into
\[
2^{-N(N-1)/2} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} \left[ (x - x_0) + 2i (y - y_0) + n \right] t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{(\frac{1}{2})} \right]' (z_0) [1 + O(\epsilon)],
\]
Similarly, the determinant involving \( x^- (n) \) becomes
\[
2^{-N(N-1)/2} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} \left[ (x - x_0) - 2i (y - y_0) - n \right] t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{(\frac{1}{2})} \right]' (z_0) [1 + O(\epsilon)].
\]
Now we consider the second part with \( v = (0, 1, \cdots, N - 2, N) \), then the determinant involving \( x^+ (n) \) is
\[
\det_{1 \leq i, j \leq N} \left[ \frac{1}{2^N} S_{2i-1 - v_j} \left( x^+(n) + v_j s - t \right) \right]
\]
\[
= \det_{1 \leq i, j \leq N} \left[ \frac{1}{2^N} S_{2i-1} (x^+ - t), \cdots, \frac{1}{2^N} S_{2i-N+1} [x^+ + (N - 2) s - t], \frac{1}{2^N} S_{2i-N-1} (x^+ + N s - t) \right]
\]
\[
= 2^{-\frac{N(N-1)}{2}} \frac{1}{2^N} \det_{1 \leq i, j \leq N} \left[ p_{2i-1}^{[1]} (z), p_{2i-2}^{[1]} (z), \cdots, p_{2i-N+1}^{[1]} (z), p_{2i-N-1}^{[1]} (z) \right] [1 + O(\epsilon^2)].
\]
From the definition of \( p_{k}^{[1]}(z) \), we have the relation \( p_{2i-(N+1)}^{[1]}(z) = [p_{2i-N}^{[1]}]'(z) \), which implies that the determinant in Eq. (100) is \( c_N^{-1} \left[ Q_N^{[1]} \right]' (z) \). Thus, Eq. (100) will be simplified as
\[
2^{-\frac{N(N-1)+2}{2}} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{[1]} \right]' (z) [1 + O(\epsilon^2)].
\]
Furthermore, by expansion \( \left[ Q_N^{[1]} \right]' (z) \) at \( z = z_0 \), we have
\[
\left[ Q_N^{[1]} \right]' (z) = \left[ Q_N^{[1]} \right]' (z_0) [1 + O(\epsilon)],
\]
then the Eq. (101) can be reduced into
\[
2^{-\frac{N(N-1)+2}{2}} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{[1]} \right]' (z_0) [1 + O(\epsilon)].
\]
Similarly, for the vector \( v = (0, 1, \cdots, N - 2, N) \), the determinant involving \( x^- (n) \) is
\[
2^{-\frac{N(N-1)+2}{2}} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{[1]} \right]' (z_0) [1 + O(\epsilon)].
\]
Substitute the above Eq. (98) (99) (103) (104) to Eq. (90), we have
\[
\sigma_n (x - 3t, y, t; A) = 2^{-\frac{N(N-1)+2}{2}} (-3)^{\frac{N(N+1)-2}{6}} c_N^{-1} t^{\frac{N(N+1)-2}{6}} \left[ Q_N^{[1]} \right]' (z_0) [1 + O(\epsilon)].
\]
According to the properties of the root structure of \( Q_N^{[1]} (z) \), we know \( \left[ Q_N^{[1]} \right]' (z_0) \neq 0 \), which means that the coefficient of the second highest term \( t^{\frac{N(N+1)-2}{6}} \) in Eq. (105) never equals to zero. In this case, when \( |t| \) is large, the asymptotic expression is
\[
\left| u_N (x, y, t; A) \right| = \left| \left( x - x_0 \right)^2 + 4 \left( y - y_0 \right)^2 - n^2 - \frac{1}{4} \right| [1 + O(\epsilon)].
\]
Clearly, the \( N \)-th order lump solution approaches to the first-order lump solution \( u_1 (x - (\hat{x}_0 + 3t), y - \hat{y}_0, t) \) as \( t \) is large with the error \( O(|t|^{-3}) \). It completes the proof.

**Proof of Proposition 4.** In Ref. [31], the authors had studied the rogue wave pattern near the origin for NLS equation. Similarly, we can analyze the asymptotics for the high order lump solutions. Compared to
the analysis for rogue waves, we only need to make a modified definition to the vectors \( x^\pm \) and \( y^\pm \). In this paper, we define \( y^\pm \) as

\[
(107) \quad x^+ - t = y^+ + (0, 0, 4t, 0, \ldots), \quad x^- - t = y^- + (0, 0, 4t, 0, \ldots).
\]

Following the method in [33], \( \sigma_n \) can be written as a \( 3N \times 3N \) determinant form

\[
(108) \quad \begin{vmatrix} O_{N \times N} & \Omega_{N \times 2N}^{(n)} & O_{2N \times 2N}^{(n)} \\ -\Psi_{2N \times N}^{(n)} & I_{2N \times 2N} \end{vmatrix},
\]

where \( \Omega_{i,j} = \frac{1}{2\pi} S_{2i-j} |x^+(n) + (j-1)s - t| \) and \( \Psi_{i,j} = \frac{1}{2\pi} S_{2i-j} |x^-(n) + (j-1)s - t| \). By a simple calculation to the determinant \( \sigma_n \), we get the asymptotics as

\[
(109) \quad \sigma_n = \beta |t|^{2(2m+1)+k(2N_0+1)} \begin{vmatrix} O_{N \times N} & \Omega_{N \times 2N} & O_{2N \times 2N} \\ -\Psi_{2N \times N} & I_{2N \times 2N} \end{vmatrix} \left[ 1 + O \left( |t|^{-1} \right) \right],
\]

where \( \tilde{\Omega}, \Psi, \beta \) are defined as Ref. [31]. With a similar analysis, in the neighborhood of \((x, y) = (-3t, 0)\), we have the following asymptotics:

\[
(110) \quad u_N (x, y; t; A) = \left( \frac{\sigma_1 (x-3t, y; t; A)}{\sigma_0 (x-3t, y; t; A)} \right)^2 - 1 = u_{N_0^{(1)}} (x, y; t) \left[ 1 + O \left( |t|^{-1} \right) \right], \quad |t| \to \infty.
\]

According to Eq.(76), we know that \( N_{0}^{(1)} \) can only be 0 or 1. It completes the proof.

**Proof of Proposition 5.** When an internal parameter \( a_{2m+1} \) is large enough: \(|a_{2m+1}| \gg \frac{2m}{2m+1}, (1 < m \leq N - 1)\) and other parameters satisfy \( a_{2k+1} = o \left( |a_{2m+1}|^{\frac{2}{2m+1}} \right) \), we need to compare the effect of \( a_{2m+1} \) and \( t \). If \( t \) is the dominant term and satisfies the condition \(|t| \gg |a_{2m+1}|^{\frac{2}{2m+1}}\), that is \( a_{2m+1} = o \left( \frac{2m}{t^2} \right) = o \left( e^{-2m} \right) \), we have \( (x_{2m+1}^{\pm} - \frac{3t}{2m+1}) e^{2m+1} = o(e) \) for \( m \neq 1 \), then the proposition 2-4 are still valid. Therefore, the high order lump solutions can be separated into \( N_{0}^{(1)} \) first-order lumps and theorem 3 is still valid. When \(|t| \ll |a_{2m+1}|^{\frac{2}{2m+1}}\), that is \( t = o \left( |a_{2m+1}|^{\frac{2}{2m+1}} \right) \), we need to analyze the asymptotics of lump solutions with respect to \( a_{2m+1} \). This case is similar to the asymptotics of rogue waves for NLS equation.

In Ref. [31], the authors discussed the asymptotics of rogue waves about the multiple internal parameters for NLS equation. Similar to this result, when \((x, y)\) is far away from \((-3t, 0)\) with \( \sqrt{(x+3t)^2 + 4y^2} = O \left( |a_{2m+1}|^{\frac{2}{2m+1}} \right) \), we set \( \hat{e} = |a_{2m+1}|^{-1/(2m+1)} \), which indicates \( t = o(\hat{e}^{-2}) \), then the Schur polynomials in theorem 3 change into

\[
S_k \left( x^+(n) + vs - t \right) = e^{-kS_k} \left( x + 3t + 2iy + n) e, vs2e^2, \ldots \right)
\]

\[
(111) \quad = e^{-kS_k} \left( (x + 3t + 2iy + n) e, 0, \ldots, 0, 1, 0, \ldots \right) [1 + o(\hat{e})] = S_k (x + 3t + 2iy + n, 0, \ldots, 0, a_{2m+1}, 0, \ldots) [1 + o(\hat{e})].
\]

Similarly, when \( t \) is large, we can get the highest order about \( a_{2m+1} \) as

\[
(112) \quad 2^{-N(N-1)} e^{-2} \left( \frac{2m + 1}{2m} \right)^{\frac{N(N+1)}{2m+1}} |a_{2m+1}|^{\frac{N(N+1)}{2m+1}} \left| Q_N^{[m]} (z) \right|^2,
\]

where

\[
(113) \quad z = \hat{A}^{\frac{1}{2m+1}} (x + 3t + 2iy), \quad \hat{A} = -\frac{2m + 1}{2m} a_{2m+1},
\]

and \( e^{-2} \) and \( Q_N^{[m]} (z) \) are defined as Eq.(75). Thus when \( Q_N^{[m]} (z) \neq 0 \), we have \( u_N(x, y; t; A) \to 0 \). But when \( Q_N^{[m]} (z) = 0 \), the highest order term will be zero, we can consider the asymptotics in the neighbourhood of the special point \((x, y, t) = (x_0^{[m]}, y_0^{[m]}, t)\) with the definition

\[
(114) \quad x_0^{[m]} + 3t + 2iy_0^{[m]} / 21 = \hat{A}^{\frac{1}{2m+1}} z_0^{[m]},
\]
where \( z_0^{[m]} \) is the root of Yablonskii-Vorob’ev polynomials \( Q_N^{[m]}(z) \), which has \( N_p^{[m]} \) simple nonzero roots.

Similar to the proof of proposition 3, if \((x, y)\) satisfies \( \left( x - x_0^{[m]} \right)^2 + 4 \left( y - y_0^{[m]} \right)^2 + 1 \), we have
\[
(115) \quad u_N(x, y, t; \mathbf{A}) = \frac{\sigma_1(x - 3t, y, t; \mathbf{A})}{\sigma_0(x - 3t, y, t; \mathbf{A})}^2 - 1 = -32 \left( x - x_0^{[m]} \right)^2 + 128 \left( y - y_0^{[m]} \right)^2 + 8 \left( 4 \left( x - x_0^{[m]} \right)^2 + 16 \left( y - y_0^{[m]} \right)^2 + 1 \right)^2 + O(\varepsilon).
\]

Then we can get a conclusion that, when \( a_{2n+1} \) is large enough, in the neighborhood of \((x, y) = (-3t, 0)\), the corresponding calculation is similar to the solution of NLS equation near the origin with one parameter large. Bo Yang et al. had studied this case for NLS equation in theorem 4 in Ref [31]. Hence, in the neighborhood of \((x, y) = (-3t, 0)\), the lump solution will decay to a \( N_0^{[m]} \)-th order lump solution. It completes the proof.

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