Probability Bracket Notation, Wick-Matsubara Relation, Density Operators, and Microscopic Probability Modeling

Xing M. Wang 1
Sherman Visual Lab, Sunnyvale, CA, USA

Abstract

Following Dirac’s vector bracket notation (VBN), we proposed the probability bracket notation (PBN) in our previous paper. We mentioned that under the special Wick rotation (imaginary time), a stationary Schrodinger equation in the Hilbert space transforms into the master equation of a microscopic probabilistic process (MPP) in the probability space. In this article, we first study the MPP of the system of a single particle, we show that the energy expectation of the MPP eventually approaches the lowest energy level in its initial condition and its von Neumann entropy finally vanishes. Then we explore the MPP for the quantum system of identical particles in the Fock space, we recover the expected occupation number of particles and the grand partition function in quantum statistics by connecting time with temperature (the Wick-Matsubara relation). We also reproduce the internal energy of an ideal gas in thermodynamics by using the relation. To address the entropy issue and relate the PBN with research topics of statistics in the literature, we express the density operators and the von Neumann entropy in the probability space by using the PBN. The Wick-Matsubara relation plus the PBN might provide a new way of microscopic probability modeling.

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1 xmwang@shermanlab.com
1. Introduction

The Dirac vector probability notation (VBN, see [1], p.96) is a powerful tool to manipulate vectors in the Hilbert space. The main beauty of the VBN is that many formulas in QM can be presented in a symbolic abstract way, independent of the state expansion or the basis selection, which, when needed, is easily done by inserting an identity operator. Inspired by the great success of VBN, we proposed the Probability Bracket Notation (PBN) [2]. Here is a brief introduction to the two notations.

The V-bracket in the VBN is an inner product of two vectors in the Hilbert space.

\[ \langle \psi_A | \psi_B \rangle \Rightarrow \text{V-bra:} \langle \psi_A |, \text{V-ket:} | \psi_B \rangle \]

where: \[ \langle \psi_A | \psi_B \rangle = \langle \psi_B | \psi_A \rangle^*, \ \langle \psi_A | = | \psi_A \rangle^* \]

V-basis and V-identity:

A discrete basis: \[ \hat{H} | \epsilon_i \rangle = \epsilon_i | \epsilon_i \rangle, \ \langle \epsilon_i | \epsilon_j \rangle = \delta_{ij}, \ \hat{I}_H = \sum_i | \epsilon_i \rangle \langle \epsilon_i | \]

A continuous basis: \[ \hat{x} | x \rangle = x | x \rangle, \ \langle x | x' \rangle = \delta_{x,x'}, \ \hat{I}_x = \int dx | x \rangle \langle x | \]

The normalization of a system V-ket \[ | \Psi \rangle \] using a continuous V-basis:

\[ 1 = \langle \Psi | \Psi \rangle = \langle \Psi | \hat{I}_x | \Psi \rangle = \int dx \langle \Psi | x \rangle \langle x | \Psi \rangle = \int dx | \Psi(x) \rangle^2 \]

The expectation of an operator \[ H \] with a discrete V-basis:

\[ \langle H \rangle = \hat{H} \equiv \langle \Psi | \hat{H} | \Psi \rangle = \sum_i \langle \Psi | \hat{H} | \epsilon_i \rangle \langle \epsilon_i | \Psi \rangle = \sum_i \epsilon_i | \epsilon_i \rangle^2 \]

The P-bracket in the PBN [2] is a conditional probability in the probability space.

\[ P(\varphi_A | \varphi_B) \Rightarrow \text{P-bra:} \ P(\varphi_A |, \text{P-ket:} | \varphi_B) \]

where: \[ P(\varphi_A | \varphi_B) \neq P(\varphi_B | \varphi_A)^*, \ P(\varphi_A | \neq | \varphi_A \rangle^*, \ P(A | B) = 1, \text{if } B \subseteq A \]

P-basis and P-identity:

A discrete case: \[ \hat{H} | \epsilon_i \rangle = \epsilon_i | \epsilon_i \rangle, \ P(\epsilon_i | \epsilon_j \rangle = \delta_{ij}, \ \hat{I}_H = \sum_i | \epsilon_i \rangle P(\epsilon_i | \]

A continuous case: \[ X | x \rangle = x | x \rangle, \ P(x | x') = \delta_{x,x'}, \ \hat{I}_x = \int_{x \in \Delta} dx | x \rangle P(\epsilon | \]

The normalization of a system P-ket with a continuous P-basis:

\[ 1 = P(\Omega | \Omega) = P(\Omega | \hat{I}_x | \Omega) = \int_{x \in \Omega} dx P(\Omega | x) P(x | \Omega) = \int_{x \in \Omega} dx P(x | \Omega) = \int_{x \in \Omega} dx P(x) \]
The expectation of an operator $H$ with a discrete P-basis:

$$\langle H \rangle = \sum_i P(\Omega_i | \hat{H} | \Omega) \epsilon_i P(\epsilon_i | \Omega) = \sum_i \epsilon_i P(\epsilon_i | \Omega)$$  \hspace{1cm} (1.12)

According to the Born statistic interpretation ([1], p.53) of the wave functions, there exists a relation between the system V-ket and the system P-ket:

$$| \Psi_i \rangle = \sum_c t_i \langle \psi_i | c \rangle \implies | \Omega_i \rangle = \sum_c P(i,t) | i \rangle; \quad P(i,t) = | c_i(t) |^2$$ \hspace{1cm} (1.13)

Then, for example, the expectation value can be written in both ways:

$$\langle H \rangle = \langle \Psi_i | \hat{H} | \Psi_i \rangle = \sum_i \epsilon_i | c_i(t) |^2 = P(\Omega_i | \hat{H} | \Omega) = \sum_i \epsilon_i P(i,t)$$ \hspace{1cm} (1.14)

Therefore, the PBN is a very convenient tool for handling the microscopic probability models based on Eq. (1.13-14).

In Ref. [2], we discussed several correlations revealed by the special Wick rotation (SWR, $it \rightarrow t$). In particular: the stationary Schrodinger equation in the Hilbert space transform into the master equations of microscopic probabilistic processes (MPPs) in the probability space under the SWR.

In the next section, we investigate the time evolution of the MPP for a single particle with a discrete or a contiguous energy spectrum. We find an MPP behaves like a freezing process. Then we explore the time evolution of the MPP for the system of many identical particles in the Fock space. The expected occupation number of particles and the grand partition function in quantum statistics are reproduced by associating time with temperature through the Wick-Matsubara relation $it \rightarrow t \rightarrow \hbar / kT$. We also recover the internal energy of an ideal gas in thermodynamics by using the relation. To address the entropy issue in an MPP and to associate the PBN with other related research topics in statistics, we express the density operators and the von Neumann entropy in the probability space by using the PBN. The Wick-Matsubara relation plus the PBN might provide a new way of microscopic probability modeling.

2. The Microscopic Probabilistic Processes of a Single Particle

By studying the path integrals with PBN [2], we demonstrated that the stationary Schrodinger equation in the Hilbert space naturally changes to the master equation in the probability space under the special Wick rotation (SWR). The master equation describes the time evolution of an induced microscopic diffusion, which we call a macroscopic probabilistic process (MPP) here.
The **Special Wick Rotation** (SWR) is defined by [2]:

\[
\text{SWR: } i\hbar t \to t, \quad |\psi(t)\rangle \to |\Omega_t\rangle, \quad \langle x_{b,t_b} | x_{a,t_a} \rangle \to P(x_{b,t_b} | x_{a,t_a}) \tag{2.1}
\]

Under the SWR, a stationary Schrödinger equation becomes a master equation of an MPP:

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} |\Psi(t)\rangle = -\frac{1}{\hbar} \hat{H} |\psi(t)\rangle \quad \Rightarrow \quad \frac{\partial}{\partial t} |\Omega_t\rangle = \frac{1}{\hbar} \hat{H} |\Omega_t\rangle, \quad \hat{\partial}_t \hat{H} = 0 \tag{2.2}
\]

In Ref. [2], we discussed the MPP of a single particle and its possible application to certain systems with non-Hermitian Hamiltonians [3]. Here we want to discuss the behavior of the MPPs in detail.

### 2.1. Discrete Energy Spectrum

For the system of a particle with a discrete energy spectrum, the time evolution of the energy eigenstates has the following correlation under the SWR (Eq. (115), [2]):

\[
|\psi_k(t)\rangle = e^{-i\varepsilon_k t/\hbar} |\psi_k\rangle \quad \Rightarrow \quad |\psi_k(t)\rangle = e^{-i\varepsilon_k t/\hbar} |\psi_k\rangle \tag{2.3}
\]

Suppose that the MPP has a normalized initial system P-ket:

\[
|\Omega_0\rangle = \sum_{k \in B} \eta_k |\psi_k\rangle, \quad P(\Omega | \Omega_0\rangle) = \sum_{k \in B} \eta_k = 1 \tag{2.4}
\]

It leads to the following unnormalized time-dependent system P-ket,

\[
|\Omega_t\rangle = \sum_{k \in B} \eta_k e^{-i\varepsilon_k t/\hbar} |\psi_k\rangle, \quad P(\Omega | \Omega_t\rangle) = \sum_{k \in B} \eta_k e^{-i\varepsilon_k t/\hbar} \neq 1 \quad \text{for } t > 0 \tag{2.5}
\]

Because it is unnormalized, the expectation of energy has the expression:

\[
\langle \hat{H} \rangle_t = P(\Omega | \hat{H} | \Omega_t\rangle) / P(\Omega | \Omega_t\rangle) \tag{2.6}
\]

Due to the exponential time dependence in Eq. (2.5), the basis P-ket of the lowest energy level eventually dominates the system P-ket. Let $\varepsilon_s, s \in B$ be the lowest energy, we have

\[
|\Omega_t\rangle = \sum_{k \in B} \eta_k e^{-i\varepsilon_s t/\hbar} |\psi_k\rangle \quad \Rightarrow \quad \eta_s e^{-i\varepsilon_s t/\hbar} |\psi_s\rangle \tag{2.7}
\]

Therefore, the MPP behaves like a process that will ultimately freeze to its lowest energy level, included in its initial condition (2.5) (see the discussion after Eq. (117), [2]):

\[
\langle \hat{H} \rangle_t = P(\Omega | \hat{H} | \Omega_t\rangle) / P(\Omega | \Omega_t\rangle) = \sum_{k \in B} \eta_k e^{-\varepsilon_k t} / \sum_{k \in B} \eta_k e^{-\varepsilon_k t} \to \varepsilon_s \tag{2.8}
\]
The system becomes a pure state with zero von Neumann entropy [4] and the MPP is like a freezing process.

### 2.2. Continuous Spectrum

Now let us consider a free particle with a uniform probability distribution in momentum $p$. Based on Eq. (2.3), its system P-ket takes the form:

$$|\Omega_p\rangle = \int dp \ e^{-\varepsilon_p^2/\hbar} \ |p\rangle = \int dp \ e^{-\varepsilon_p \beta} \ |p\rangle, \quad \varepsilon_p \equiv p^2 / 2m, \quad \beta \equiv t / \hbar$$  \hspace{1cm} (2.9)

Because it is unnormalized, according to Eq. (2.6), the expected energy reads:

$$\langle \hat{H} \rangle = P(\Omega | \hat{H} | \Omega_p \rangle / P(\Omega | \Omega_p \rangle) = \int dp \varepsilon_p e^{-\varepsilon_p \beta} / \int dp \ e^{-\varepsilon_p \beta} = -\partial_\beta \ln z_\beta$$  \hspace{1cm} (2.10)

$$z_\beta \equiv \int dp \ e^{-\varepsilon_p \beta}$$  \hspace{1cm} (2.11)

It can be easily calculated (see [5], §10.4):

$$z_\beta = \int dp \ e^{-\beta^2} = \int dp \ e^{-p^2 / 2m} = \sqrt{\pi \beta} / 2m$$  \hspace{1cm} (2.12)

$$\langle \hat{H} \rangle = -\partial_\beta \ln z_\beta = \frac{1}{2\beta} = \frac{\hbar}{2t}$$  \hspace{1cm} (2.13)

The explicit time dependency tells us that, at $t = 0$, the system has infinite expected energy, as predicted by its evenly distributed momentum, and its energy approaches zero when time gets to infinity. This picture resembles a microscopic freezing process: its energy level eventually falls to the lowest ($\varepsilon_p = 0$ in the continuous $p$-spectrum).

It is natural to ask: is the time (originally an imaginary time) in Eq. (2.5) or (2.13) related to temperature? The answer seems to be YES.

### 3. Many-Particle Systems and the Wick-Matsubara Relation

#### 3.1. The MPP of a Many-Particle System

For a system of many identical and weakly interacting particles, we have the following occupation number P-basis in the Fock space (Eq. (28), [2]):

$$\hat{n}_i \ |n_1, n_2, \ldots\rangle = \hat{n}_i \prod_k \ |n_k\rangle \equiv n_i \ |\tilde{n}\rangle$$  \hspace{1cm} (3.1)

$$P(\tilde{n}^* \tilde{n}) = \prod_i P(n_i^* \ n_i) \equiv \delta_{\tilde{n}^*, \tilde{n}}, \quad \sum_{\tilde{n}} P(\tilde{n} \ | n) \ P(\tilde{n}) = \prod_i \sum_{n_i} P(n_i) = \hat{1}_\tilde{n}$$  \hspace{1cm} (3.2)

The system has the following total Hamiltonian $H$ (Eq. (1.3), [6]):
$$\hat{H} = \hat{H} - \mu \hat{N} = \sum_{i} \sum_{n_i} \hat{n}_i (\varepsilon_i - \mu)$$  \hspace{1cm} (3.3)$$

Here $\hat{H}$, $\varepsilon_i$ and $\mu$ are the conventional dynamic Hamiltonian, the dynamic energy per particle at the $i^{th}$ level, and the chemical potential per particle. Based on Eq. (2.3) and (3.3), the time evolution of the basis $P$-ket at the $i^{th}$ level is:

$$\partial_t |n_i, t\rangle = -[\hat{n}_i (\varepsilon_i - \mu) / \hbar] |n_i, t\rangle, \quad |n_i, t\rangle = e^{-n_i (\varepsilon_i - \mu)t/\hbar} |n_i\rangle$$  \hspace{1cm} (3.4)$$

We assume that at $t = 0$, the probability of the system is evenly distributed:

$$|\Omega_{i,0}\rangle \equiv |\Omega_i\rangle = \sum_{n_i} |n_i\rangle, \quad |\Omega_{i,i}\rangle = \sum_{n_i} e^{-n_i (\varepsilon_i - \mu)t/\hbar} |n_i\rangle$$  \hspace{1cm} (3.5)$$

It is unnormalized and the expected occupation number of particles at the $i^{th}$ level reads:

$$n(\varepsilon_i) \equiv \langle \hat{n}_i \rangle = \frac{\sum_{n_i} n_i e^{-(\varepsilon_i - \mu)t/\hbar}}{\sum_{n_i} e^{-(\varepsilon_i - \mu)t/\hbar}}$$  \hspace{1cm} (3.6)$$

The expected total number of particles can be obtained:

$$\langle \hat{N} \rangle = \sum_i \langle \hat{n}_i \rangle = \sum_i \left\{ \frac{\sum_{n_i} n_i e^{-(\varepsilon_i - \mu)t/\hbar}}{\sum_{n_i} e^{-(\varepsilon_i - \mu)t/\hbar}} \right\}$$  \hspace{1cm} (3.7)$$

It would have the same freezing behavior as the single particle with a discrete energy spectrum as in Eq. (2.8). However, Eq. (3.7) is quite similar to the distribution function in quantum statistics (11.2-6, [5]):

$$\langle \hat{N} \rangle = \sum_i \left\{ \frac{\sum_{n_i} n_i e^{-(\varepsilon_i - \mu)t/\hbar}}{\sum_{n_i} e^{-(\varepsilon_i - \mu)t/\hbar}} \right\} = \sum_i \sum_{n_i} n_i e^{-(\varepsilon_i - \mu)t/\hbar} / Z_i$$  \hspace{1cm} (3.8)$$

Here $Z_i$ is the partition function of the $i^{th}$ energy level.

3.2. The Wick-Matsubara Relation: Comparing Eq. (3.7) with (3.8), we realize that the imaginary time in the Wick rotation corresponds to the absolute temperature (the Matsubara formalism, see §1.4, [6]), which we call:

**The Wick-Matsubara Relation:**

$$it \rightarrow t \rightarrow \hbar / kT$$  \hspace{1cm} (3.9)$$

For fermions, $n_i \in \{0, 1\}$ (the Fermi-Dirac distribution), all lowest energy states ($\varepsilon \leq \varepsilon_F$) are occupied when $T \rightarrow 0$ (or $t \rightarrow \infty$). For bosons, $n_i \geq 0$ (the Bose-Einstein distribution), when $T \rightarrow 0$ (or $t \rightarrow \infty$), all particles go to the ground energy level (the Bose condensation). So the systems (either of bosons or fermions) are eventually frozen at zero temperature with the lowest total energy and zero entropy (no chaos).
Using Eq. (3.9), Eq. (2.13) becomes:

$$\langle \hat{H} \rangle_t = \frac{\hbar}{2t} \rightarrow \langle \hat{H} \rangle = \frac{kT}{2}$$

(3.10)

Eq. (3.10) represents the expected energy for each particle and each dimension of an ideal monoatomic gas (\(\langle \mathcal{E} \rangle = kT / 2\)). For a system of \(N\) non-interacting identical particles in 3D, we recover its internal energy in thermodynamics (§10.7, [5]):

$$U = 3N \langle \hat{H} \rangle = \frac{3NkT}{2}$$

(3.11)

The grand partition function can be expressed in the VBN as (§11.2-6, [5]; page 37, [6]):

$$Z_G = \sum_n \langle \vec{n} \rangle e^{-\langle (\hat{H} - \mu \hat{N}) / kT \rangle} = \prod_i^\infty \sum_n e^{-\langle (\epsilon_i - \mu) \rangle / kT} = \prod_i^\infty Z_i$$

(3.12)

We can write it concisely in the PBN and it is easy to expand:

$$Z_G = P(\Omega | e^{-(\hat{H} - \mu \hat{N}) / kT} | \Omega) = P(\Omega | e^{-(\hat{H} - \mu \hat{N}) / kT} \hat{1}_n | \Omega) = \prod_{i=1}^\infty \sum_n e^{-(\epsilon_i - \mu) / kT}$$

(3.13)

And Eq. (3.6) can also be rewritten in a compact form:

$$\langle \hat{n}_i \rangle = \frac{P(\Omega | \hat{n}_i e^{-\hat{H} / \hbar} | \Omega)}{P(\Omega | e^{-\hat{H} / \hbar} | \Omega)} = \frac{P(\Omega | \hat{n}_i e^{-\hat{H} / \hbar} \hat{1}_n | \Omega)}{P(\Omega | e^{-\hat{H} / \hbar} \hat{1}_n | \Omega)} = \frac{\sum_n n_i e^{-\langle (\epsilon_i - \mu) \rangle / \hbar}}{\sum_n e^{-(\epsilon_i - \mu) / \hbar}}$$

(3.14)

The freezing behavior of an MPP implies that the system seems to be in contact with a huge reservoir at an absolute zero temperature. In reality, the system will have the temperature \(T\) of its environment (or the reservoir in contact) and its clock of time (imaginary, not real!) will stop at \(t = \hbar/kT\).

Since the master equation of an MPP describes a probabilistic process, even though Eq. (2.5) and Eq. (2.13) are about a single particle, the Wick-Matsubara relation could be applied and the time should be associated with temperature.

4. Density Operators and the von Neumann Entropy in the PBN

So far, we have not used the density operators in the probability space, because the system P-ket already has all the info for expectations as shown in Eq. (2.6). However, we need these operators to discuss the von Neumann entropy in MPPs and to associate the PBN with research topics of statistics in the literature [6].
4.1. Density Operators in the Hilbert Space: The density operator [7] (or matrix) \( \rho \) is defined in the Hilbert space \( \mathcal{H}^N \), having the following properties:

\[
\text{Tr} \rho = 1, \quad \rho = \rho^\dagger, \quad \langle \varphi | \rho | \varphi \rangle \geq 0 \quad \forall \ | \varphi \rangle \in \mathcal{H}^N
\]  

(4.1)

A density matrix \( \rho \) describes a pure state [8] if and only if \( \rho^2 = \rho \), i.e. the state is idempotent; it describes a mixed state [9] if \( \rho^2 \neq \rho \). The purity of a state is defined as \( \text{Tr} \rho^2 \).

Let \( \eta_k, k \in \{1, 2, \ldots, k\} \), be the eigenvalues of the density matrix, then we have the following expressions:

\[
\rho | \eta_k \rangle = \eta_k | \eta_k \rangle, \quad \langle \eta_i | \eta_k \rangle = \delta_{i,k}, \quad \rho = \sum_k | \eta_k \rangle \eta_k \langle \eta_k |, \quad \text{Tr} \rho = \sum_k \eta_k = 1
\]  

(4.2)

4.2. Density Operators in the Probability Space for MPPs: Using the PBN, we can define the density operator using the system P-ket and its bases, similar to Eq. (4.2) in the Hilbert space. We define the time-dependent density operator as:

\[
\rho_t = \sum_k [\eta_k e^{-i\varepsilon_k t \hbar} | \varepsilon_k \rangle P(\varepsilon_k)]/ \sum_k \eta_k e^{-i\varepsilon_k \hbar}, \quad \langle \hat{H} \rangle_t = \text{Tr} [\rho_t \hat{H}]
\]  

(4.3)

For an MPP, if the Hamiltonian of the system has a discrete spectrum as in Eq. (2.5), i.e.:

\[
| \Omega_t \rangle = \sum_{k \in \mathcal{B}} \eta_k e^{-i\varepsilon_k t \hbar} | \psi_k \rangle, \quad P(\Omega_t | \Omega_t \rangle = \sum_{k \in \mathcal{B}} \eta_k e^{-i\varepsilon_k \hbar} \neq 1 \quad \text{for} \ t > 0
\]  

(4.4)

The time-dependent density operator can be defined as:

\[
\rho_t = \sum_k [\eta_k(t) | \psi_k \rangle P(\psi_k)]/ \sum_k \eta_k(t), \quad \eta_k(t) = \eta_k e^{-i\varepsilon_k t \hbar}
\]  

(4.5)

The expectation of the Hamiltonian \( H \) is given by:

\[
\langle \hat{H} \rangle_t = \text{Tr} [\hat{H} \rho_t] = \sum_k \varepsilon_k \eta_k(t)/ \sum_k \eta_k(t)
\]  

(4.6)

Similarly, for the system of a free particle in Eq. (2.9), the density operator is:

\[
\rho_t = \frac{\int dp \ e^{-\varepsilon p t \hbar / \hbar} | p \rangle P(p)}{\int dp \ e^{-\varepsilon p t \hbar / \hbar}} = \frac{e^{-i\hat{H} t \hbar} \hat{I}_p}{\text{Tr}[e^{-i\hat{H} t \hbar} \hat{I}_p]} = \frac{e^{\frac{-\beta t}{2m}}}{z_\beta}, \quad \varepsilon_p = p^2 / 2m, \ \beta = t / \hbar
\]  

(4.7)

\[
\langle \hat{H} \rangle_t = \text{Tr} [\hat{H} \rho_t] = \frac{\text{Tr}[\hat{H} e^{-\beta t \hbar}]}{\text{Tr}[e^{-\beta t \hbar}]} = \frac{\int dp \ e^{-\varepsilon p t \hbar / \beta} \hat{I}_p}{z_\beta} = -\beta \ln z_\beta
\]  

(4.8)
For the system of many particles in Section 2.2, the density operator reads:

\[
\rho_\beta = \sum_n e^{-\eta_n (\varepsilon_n - \mu) \beta / h} \left| \hat{n} \right\rangle P(\hat{n}) | \hat{n} \rangle = \frac{e^{-\hat{H}_\beta} \hat{\Pi}}{\text{Tr}[e^{-\hat{H}_\beta} \hat{\Pi}] } = \frac{e^{-\hat{H}_\beta}}{Z_G(\beta)}, \quad \beta \equiv \frac{t}{h} \tag{4.9}
\]

The expected occupation number of particles at the \(i\)th level in Eq. (3.14) becomes:

\[
\langle \hat{n}_i \rangle = \text{Tr}(\hat{n}_i \rho_\beta) = \frac{\text{Tr}(\hat{n}_i e^{-\hat{H}_\beta})}{Z_G(\beta)} = \frac{\sum_n n_i e^{-\eta_n (\varepsilon_n - \mu) \beta / h}}{\sum_n e^{-\eta_n (\varepsilon_n - \mu) \beta / h}} \tag{4.10}
\]

The grand partition function in Eq. (3.13) can be written as (here \( \beta = 1/kT\)):

\[
Z_G = \text{Tr}\rho_\beta = \text{Tr}[e^{-(\hat{H} - \mu \hat{N}) \beta / h}] = \prod_{i=1}^\infty \sum_n e^{-\eta_n (\varepsilon_n - \mu) \beta / h} | \bar{\bar{\eta}}_n \rangle = \prod_{i=1}^\infty Z_i \tag{4.11}
\]

### 4.3. Quantum Entropy (QE)

For a quantum system, the quantum entropy (QE) or von Neumann entropy [10] is given by:

\[
S(\rho) = -\text{Tr}(\rho \ln \rho) \tag{4.12}
\]

It can be shown that:

\[
S(\rho) \geq 0; \quad S(\rho) = 0 \text{ if only if } \rho \text{ describes a pure state.} \tag{4.13}
\]

The maximally mixed state (MMS) in the \(N\)-dimensional Hilbert space \(\mathcal{H}_N\) has a density operator proportional to the \(N\)-d identity and has the maximal entropy [10]:

\[
\rho_{\text{max}} = \frac{1}{N} \hat{I}_N, \quad S(\rho_{\text{max}}) = \ln N \tag{4.14}
\]

It corresponds to the uniform distribution. Therefore, the distributions in Eq. (2.9) and (3.5) produce the maximal entropy at the initial time.

In the orthonormal basis of \(H\) in Eq. (4.5), the QE is given by:

\[
S(\rho_i) = -\text{Tr}(\rho_i \ln \rho_i) = -\sum_k \bar{\bar{\eta}}_k(t) \ln \bar{\bar{\eta}}_k(t), \quad \bar{\bar{\eta}}_k(t) \equiv \eta_k(t) / \sum_k \eta_k(t) \tag{4.15}
\]

When time goes to infinity, as described in Eq. (2.7):

\[
|\Omega_i\rangle = \sum_{k \in B} \eta_k e^{-\varepsilon_k t / h} |\psi_k\rangle \xrightarrow{t \to \infty} \eta_k e^{-\varepsilon_k t / h} |\psi_i\rangle \tag{4.16}
\]
\[
\rho_t = \frac{\sum_{k \in B} \eta_k(t) |\psi_k\rangle P(\psi_k) \xrightarrow{t \to \infty} \eta_t(t) |\psi_s\rangle P(\psi_s)}{\sum_{k \in B} \eta_k(t)} = |\psi_s\rangle P(\psi_s)
\]  
(4.17)

It ends up in a pure state, and its QE approaches zero. That was why we said that an MPP system becomes frozen with the lowest energy and zero entropy if time goes to infinity. Now we understand that there is the Wick-Matsubara relation, the (imaginary) time of an MPP will be in sync with its environment temperature.

Our expressions are comparable with those in the Literature. For example, if we set \( t/\hbar = 1/kT = \beta \) and \( \eta_k = 1 \) (uniform distribution) in our expressions of \( \rho_t \) in Eq. (4.3), we obtained equations similar to Eq. (1.1) and (1.11) in Ref [6]:

\[
\rho_\beta = \frac{\sum_k e^{-\beta \hat{H}} |\varepsilon_k\rangle P(\varepsilon_k)}{\sum_k e^{-\beta \varepsilon_k}} = \frac{e^{-\beta \hat{H}} \mathcal{Z}_\beta}{\mathcal{Z}_\beta} = e^{-\beta \hat{H}} \mathcal{Z}_\beta
\]  
(4.18)

\[
\langle \hat{A} \rangle = \text{Tr}[\hat{A} \rho_t] = \frac{\text{Tr}[\hat{A} e^{-\beta \hat{H}}]}{\mathcal{Z}_\beta}
\]  
(4.19)

5. Summary and Discussion

After introducing the basic ideas of the probability bracket notation (PBN), we investigated the implication of one of the correlations revealed by the special Wick rotation (SWR): A stationary Schrödinger Equation in the Hilbert space naturally transforms into the master equation of a microscopic probabilistic process (MPP) in the probability space under the SWR.

We demonstrated that the expected energy of the MPPs of a single particle approaches their lowest level and their entropy eventually vanishes when time goes to infinity. It implies that the system of an MPP is in contact with a huge reservoir at an absolute zero temperature. We then explored the MPP for the system of identical particles in the Fock space, we recovered the basic formulas in thermodynamics and quantum statistics using the Wick-Matsubara relation \( it \to t \to \hbar/kT \). Therefore, in reality, the system will have the temperature \( T \) of the reservoir in contact with and its (imaginary) time clock will be in sync with the temperature \( (t = \hbar/kT) \). Finally, to address the entropy issue and make our presentation comparable with research topics of statistics in the literature, we express the density operators and the von Neumann entropy in the probability space by using the PBN.

The Wick-Matsubara relation, together with the PBN, probably provides a new way to handle microscopic probability modeling: at first, translate (using the SWR, \( it \to t \)) the stationary Schrodinger equation to the master equation of an MPP in the PBN; next, solve the master equation with an appropriate initial condition; then: replace time \( (t \to \hbar/kT) \).
Of course, more investigations are needed to verify the PBN's consistency (or correctness), usefulness, and limitations.

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