ON GENERATING RELATIVE AND ABSOLUTE INVARIANTS
OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. A general expression for a relative invariant of a linear ordinary differential equations is given in terms of the fundamental semi-invariant and an absolute invariant. This result is used to established a number of properties of relative invariants, and it is explicitly shown how to generate fundamental sets of relative and absolute invariants of all orders for the general linear equation. Explicit constructions are made for the linear ODE of order five. The approach used for the explicit determination of invariants is based on an infinitesimal method.

1. Introduction

The first invariants found during the early days of development of the theory of invariants of differential equations were all relative invariants [1, 2, 3], and one of the very first determination of absolute invariants for differential equations is perhaps due to Brioschi [3], who obtained them as a quotient of two relative invariants. It has subsequently been shown that in the case of linear ordinary differential equations, every fundamental absolute invariant can be expressed as a rational function.

These basic properties of relative invariants have led to significant progress in the study of invariants of differential equations and their applications, and such studies have been largely influenced by the work of Halphen [4, 5], and Forsyth [6]. However the methods used by Halphen, Forsyth, and earlier researchers on the subject were very intuitive and most often ad hoc methods requesting tedious calculations just for finding a couple of invariants.

Based on recent advances in Lie group techniques [7-9], infinitesimal methods haven been increasingly used for the investigation of invariants of differential equations [10-12]. But although these infinitesimal methods provide a more systematic route for the treatment of invariants of differential equations and transformation groups, they have been applied to the determination of relative invariants of differential equations only in some very rare cases [13-14], and even in those cases only some very specific relative invariants were obtained in the usual way as absolute invariants corresponding to partial structure-preserving transformations, in which some of the variables in the equation are kept constant. It therefore appears that a number of interesting properties of these relative invariants have not yet been uncovered.

Restricting our attention to linear ordinary differential equations, we determine in this paper some properties of relative invariants, considered as semi-invariants of the full structure-preserving transformations of these equations. In particular

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we show that every absolute invariant can be expressed as a quotient of a relative invariant and the fundamental semi-invariant, and we derive a general expression for these relative invariants. We show how these relative invariants can be used to obtain invariants of all orders via invariant differentiation. Our approach for any explicit determination of invariants is based on the infinitesimal method recently proposed in [10], and which contrary to the former well-known method of [8], does not require the knowledge of the structure-preserving transformations, but rather provides it.

2. Basic properties of relative invariants

Let \( G \) be a Lie group of point transformations of the form

\[
x = \phi(z, w; \tau), \quad y = \psi(z, w; \tau),
\]

where \( \tau \) denotes collectively some arbitrary parameters specifying the group element in \( G \). Consider on the other hand a family \( \mathcal{D} \) of differential equations of the general form

\[
\Omega(x, y^{(n)}; \rho) = 0,
\]

in which \( y^{(n)} \) represents the dependent variable \( y = y(x) \) and all its derivatives up to the order \( n \), and \( \rho \) represents collectively some arbitrary functions of \( x \), or arbitrary constants specifying the family element in \( \mathcal{D} \). We say that \( G \) is the equivalence group of (2.2) if it is the largest group of transformations that maps elements of \( \mathcal{D} \) into itself. In this case, the transformed equation takes the same form \( \Omega(z, w^{(n)}; \theta) = 0 \), in terms of the transformed parameter \( \theta \), and (2.1) is called the structure-preserving transformations of (2.2). By a well known result of Lie [18], the transformations (2.1) induces another group of transformation \( G_c \) acting on the parameter \( \rho \) of the differential equation, and we shall be interested in the invariants of this group action and its prolongations, and which are commonly referred to as the invariants (or differential invariants) of (2.2). As sets, \( G \) and \( G_c \) are equal, except that they act on different spaces, and both are often referred to as the equivalence group of (2.2). Thus both sets will often be denoted simply by \( G \).

It should be noted that in (2.2), \( x \) and \( y \) each denote collectively all independent and dependent variables, respectively, so that the equation also includes in particular all partial differential equations. However, we shall be interested in this paper in a linear ordinary differential equations of the general form

\[
y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_ny = 0,
\]

where \( a_j = a_j(x) \) are arbitrary functions of the independent variable \( x \). The structure-preserving transformations of (2.3) can be written in the form

\[
x = \xi(z), \quad y = \eta(z)w,
\]

where \( \xi \) and \( \eta \) are arbitrary functions. Under (2.4), the transformed equation of (2.3) takes the form

\[
w^{(n)} + A_1w^{(n-1)} + A_2w^{(n-2)} + \cdots + A_nw = 0,
\]

where the \( A_j = A_j(z) \) are the new coefficients. Let \( a \) denote collectively the coefficients \( a_j = a_j(x) \). A differential function \( F = F(a, a(\tau)) \), where \( a(\tau) \) represents
as usual the derivative of $a$ up to a certain order $r$, is called a relative invariant of (2.3) if

$$F(\tau \cdot (a, a(r))) = w(\tau) \cdot F(a, a(r)),$$

(2.6)

for all $a$ and $r$, and for all $\tau \in G$. In (2.0), the weight function $w = w(\tau)$ must be a character of the group $G$. When $w$ is identically equal to one, the function $F$ is called an absolute invariant of (2.3). For simplicity, an expression of the form $F(\tau \cdot (a, a(r)))$ like in (2.6) will often be represented by $F_\tau$, for a given function $F$.

Let’s assign to an expression of the form $d^k a_j /dx^k$ the weight $j + k$. We say that a polynomial function $F$ in the coefficients $a_j$ and their derivatives has weight $m$ if each of its terms has constant weight $m$. By combining a result of Forsyth [6] according to which all absolute invariants of (2.3) can be obtained as rational functions, and certain results obtained by Halphen [5], we readily obtain the following result.

**Theorem 1.** Equation (2.3) has a fundamental set of absolute invariants consisting of rational functions, in which every element $F = S_1/S_2$ is the quotient of two relative invariants $S_1$ and $S_2$ of the same weight $m$, each of which satisfies a relation of the form

$$S(\tau \cdot (a, a(r))) = \xi(z)^m S(a, a(r)), \quad \text{that is} \quad S_\tau = \xi(z)^m S,$$

(2.7)

for every $\tau \in G$, where $S$ denotes any of the invariants $S_1$ and $S_2$.

The invariant $F$ of Theorem 1 is also said to be of weight $m$. In general, a relative invariant $S$ satisfying $S_\tau = \xi^r S$ for some $r \in \mathbb{R}$ is said to be of index $r$. We shall make the result of this theorem more precise in the next section.

3. The fundamental relative invariant

For a general group of transformations $G$ acting on a manifold $M$, a semi-invariant usually refers to a function $F$ satisfying a relation similar to that specified by (2.0), and which has the simple form

$$F(g \cdot p) = w(g) \cdot F(p),$$

(3.1)

for all $g \in G$ and $p \in M$ such that $g \cdot p$ is defined, and where the function $w$ is a character of $G$. If we let $v$ be a generic element in the generating system for the Lie algebra of $G$ and denote by $X$ the corresponding infinitesimal generator of the group action, then (3.1) implies that

$$X \cdot F = -\lambda F,$$

(3.2a)

$$\lambda = dw(e)(v),$$

(3.2b)

and where $dw(e)$ denotes the differential of $w$ at the identity element $e$ of $G$. A function $F$ satisfying (3.2) is often called a $\lambda$-semi-invariant. By a nontrivial semi-invariant, we shall mean a semi-invariant which is not an absolute one, and which in particular is not a constant function. We now establish the following result that relates every semi-invariant to a fundamental set of absolute invariants.

**Theorem 2.** A function $F$ defined on $M$ is a $\lambda$-semi-invariant of $G$ if and only if it is of the form

$$F = F_0 \Phi,$$

(3.3)
where $F_0$ is an arbitrarily chosen nontrivial $\lambda$-semi-invariant, and $\Phi$ is an absolute invariant, that is $\Phi = \Phi(I_1, \ldots, I_s)$, where $\{I_1, \ldots, I_s\}$ is a fundamental set of absolute invariants of $G$.

Proof. Suppose that in a coordinates system $\{u_1, \ldots, u_q\} = u$ of $M$ the infinitesimal generator $X$ of $G$ defined as in (3.2) has the form

$$X = X_1 \partial_{u_1} + \cdots + X_q \partial_{u_q},$$

where $X_j = X_j(u)$. Then by a well-known result [19], the equivalent system of characteristic equations associated with (3.2) is given by the sequence of $q$ equalities

$$\frac{du_1}{X_1} = \frac{du_2}{X_2} = \cdots = \frac{du_q}{X_q} = \frac{dF}{-\lambda F}, \quad (3.4)$$

in which the first $q-1$ equalities are the determining equations for the absolute invariants. To find the general solution to this system, we can first find the integral resulting from a combination of the last fraction $dF/(-\lambda F)$ with any suitable ones from among the first $q$ fractions. We may assume without loss of generality that such a combination is given by an equation of the form

$$\frac{dF}{F} = -\lambda du_k, \quad \text{for some } k, \ 1 \leq k \leq q.$$ 

This equation clearly has separable variables, since $F$ is considered as a new independent variable added to the coordinates system $\{u_1, \ldots, u_q\}$. Its integral can be written in the form $F\nu = C_{q+1}$, for some function $\nu = \nu(u)$. If we denote by $I_j(u) = C_j$, for $j = 1, \ldots, q-1$, the other integrals corresponding to the first $q-1$ equalities in (3.4), and in which the $C_j$ are arbitrary constants, then the $I_j$ are the absolute invariants of $G$ and the general solution of (3.4) takes the form

$$\Phi_1(I_1, \ldots, I_q, F\nu) = 0, \quad \text{or equivalently } F\nu = \Phi(I_1, \ldots, I_q),$$

for some arbitrary functions $\Phi_1$ and $\Phi$. To obtain the function $F_0$ of the theorem we only need to take $F_0 = 1/\nu$, and it readily follows from the Leibnitz property of the derivation $X$ and the definition of an absolute invariant that $F_0$ is a $\lambda$-semi-invariant.

It should be noted that Eq. (3.3) also holds with the same $F_0$ for all prolongations of the group $G$. Since $F_0$ is nontrivial, it follows in particular that the set $\{I_1, \ldots, I_q, F_0\}$ is functionally independent. In contrast to the case of absolute invariants, not every function of semi-invariants is again a semi-invariant, and although the set of all semi-invariants of $G$ forms a group under functions multiplication, the sum or difference of two semi-invariants is not in general a semi-invariant.

For simplicity, we let $I$ denote collectively all elements in a fundamental set of absolute invariants of Eq. (2.3). We shall also often denote by $\omega$ an element of the form $(a, a_{i(s)})$, which can be viewed as an element in the $s$th-jet space determined by the independent variable $x$ and the dependent variable $a$ of Eq. (2.3).

**Corollary 1.** Suppose that $S_1$ is a relative invariant of Eq. (2.3) of index $k$. Then every relative invariant $S_2$ of order $m$ of the same equation can be put into the form

$$S_2 = S_1^{m/k} \Phi,$$ 

for a certain function $\Phi = \Phi(I)$. In particular $S_1^{m}/S_2^{k}$ is an absolute invariant.
Proof. According to Theorem 2 to prove the first part of the corollary, we only need to show that $S_{1}^{m/k}$ is a $\lambda$-semi-invariant for the same function $\lambda$ as $S_{2}$, and by (3.2), we only need to show that $S_{1}^{m/k}$ is a relative invariant of the same index $m$ as $S_{2}$. But since $(S_{1})_{r} = \xi^r S_{1}$ by assumption, it readily follows that $(S_{1}^{m/k})_{r} = \xi^m S_{1}^{m/k}$. For the second part we notice that since $S^p$ has index $rp$ for every relative invariant $S$ of index $r$, we must have $(S_{1}^{m} / S_{2}^{k})_{r} = S_{1}^{m} / S_{2}^{k}$. □

The corollary says that every relative invariant of (2.3) can be expressed as a product of an arbitrarily chosen, but fixed relative invariant and an arbitrary function of $I$. We call such a fixed relative invariant the fundamental relative invariant of (2.3), and denote it and its index by $S_{0}$ and $\sigma$, respectively. We can now give without any need of calculations a simple formula relating every absolute invariant with the fundamental relative invariant.

Corollary 2. Eq. (2.3) has a fundamental set of absolute invariants in which every element has the form $F = S_{1}/S_{0}^{m/\sigma}$ for some relative invariant $S_{1}$ of index $m$.

Proof. By Theorem 1, we may write every fundamental absolute invariant of index $m$ in the form $R_{1}/R_{2}$, where $R_{1}$ and $R_{2}$ are relative invariant of index $m$. It follows from Corollary 1 that $R_{1} = S_{0}^{m/\sigma} \Phi_{1}$ and $R_{2} = S_{0}^{m/\sigma} \Phi_{2}$, for some functions $\Phi_{1}$ and $\Phi_{2}$. The corollary is thus proved by taking $S_{1} = S_{0}^{m/\sigma} \Phi_{1}/\Phi_{2}$. □

Corollary 3. The weight and the index of every relative invariant coincide.

Proof. Suppose that the relative invariant $S$ has weight $\mu$ and index $m$. Then $S/S_{0}^{m/\sigma}$ is an absolute invariant whose weight is the index of $S_{0}^{m/\sigma}$, which is $m$ by Theorem 1. By the same theorem, this weight is the same as the weight $\mu$ of $S$, and this proves the corollary. □

Corollary 4.

(a) Suppose that $\{ I_{1}, \ldots, I_{p} \}$ is a fundamental set of absolute invariants of a certain group of equivalence transformations $G$, and that $F_{0}$ is a nontrivial $\lambda$-semi-invariant of $G$. Then, $\{ F_{0}, F_{0} I_{1}, \ldots, F_{0} I_{p} \}$ is a maximal set of functionally independent $\lambda$-semi-invariants of $G$.

(b) If $I_{j} = S_{j}/S_{0}^{m/\sigma}$ are the fundamental absolute invariants of Eq. (2.3) for $j = 1, \ldots, p$, where $m_{j}$ is the index of $S_{j}$, then

$$S = \left\{ S_{0}^{m/\sigma}, S_{1}^{m/m_{1}}, \ldots, S_{p}^{m/m_{p}} \right\}$$

is a fundamental set of relative invariants of index $m$ of Eq. (2.3).

Proof. Since $F_{0}$ is a nontrivial semi-invariant of $G$, the given set $\{ F_{0}, F_{0} I_{1}, \ldots, F_{0} I_{p} \}$ clearly forms a functionally independent set of semi-invariants of $G$, by Theorem 2. By the same theorem, any other semi-invariant of $G$ has the form $S = F_{0} \Phi$, where $\Phi = \Phi(I_{1}, \ldots, I_{p})$, and this readily proves part (a) of the corollary.

On the other hand, every element in the given set $S$ is clearly a relative invariant of index $m$, if we replace each $I_{j}$ by

$$I_{j}^{m/m_{j}} = S_{j}^{m/m_{j}} / S_{0}^{m/\sigma},$$

the resulting set formed by the $I_{j}^{m/m_{j}}$ is a fundamental set of absolute invariants of the same index $m$ of Eq. (2.3). Thus by (3.2) the elements of $S$ are $\lambda$-semi-invariants.
corresponding to the same function \( \lambda \). Therefore, part (b) of the corollary readily follows from part (a).

It also follows from part (b) of this corollary that \( \{ S_0, S_1, \ldots, S_p \} \) is a functionally independent set of relative invariants, but whose elements do not have the same index.

4. Application to the determination of invariants

It is well-known, by a result of Lie, that all higher order differential invariants of a transformation group can be obtained from a generating system of lower order ones by means of invariant differentiation. This often reduces the problem of determination of invariants to finding a generating system of invariants and the invariant differential operators. To begin with, suppose that we know two functionally independent absolute invariants \( I_0 \) and \( I_1 \) of equation (2.3), and that these are rational functions of the form

\[
I_0 = R_0^m/S_0^m, \quad I_1 = S_1^m/S_0^m
\]

where \( S_0 \) is the fundamental relative invariant of index \( \sigma \) already introduced, while \( R_0 \) and \( S_1 \) are relative invariants of indices \( k \) and \( m \), respectively. We also assume that \( S_1 \) is of order \( \mu \) as a function of the independent variable \( x \), and that all other relative invariants are of order at most \( \mu \) in \( x \). Then by means of the differential operator \( \zeta D_x \), where \( \zeta = 1/I_0^\prime \), \( I_0^\prime = dI_0/dx \) and \( D_x = d/dx \), we can generate a new absolute invariant \( I_1^\prime = \zeta D_x(I_1) \), whose order is \( \mu + 1 \) in general. In terms of the relative invariants in (4.1), we have

\[
I_1^\prime = \frac{I_1}{I_0} \left( \frac{mS_1S_0^\prime S_0 - rS_0S_1^\prime}{kR_0S_0^\prime S_0 - rS_0R_0^\prime} \right), \quad (4.2)
\]

where \( h^\prime = dh/dx \) for every function \( h = h(x) \). It follows from Corollary \( \text{[2]} \) that to find an absolute invariant of higher order \( \mu + 1 \), we only need to find a relative invariant of order \( \mu + 1 \). We now introduce the notation

\[
\varphi(R_1, R_2) = m_1R_1R_2^\prime - m_2R_2R_1^\prime, \quad \chi(R_1, R_2) = \frac{[\varphi(R_1, R_2)]^{m_2}}{R_2^{m_1 + m_2 + 1}} \quad (4.3a)
\]

\[
\varphi_0(R_1, R_2) = R_1, \quad \chi_0(R_1, R_2) = \frac{R_1^{m_2}}{R_2^{m_1}} \quad (4.3b)
\]

for any relative invariants \( R_1 \) and \( R_2 \) of respective index \( m_1 \) and \( m_2 \). For simplicity of notation, when the second argument \( R_2 \) is fixed and there is no possibility of confusion, we shall set \( \varphi(R_1) = \varphi(R_1, R_2) \) and \( \chi(R_1) = \chi(R_1, R_2) \). By multiplying \( I_1^\prime \) by the absolute invariant \( I_0/I_1 \) and the relative invariant \( S_1/I_0 \), we obtain a relative invariant of the form \( F = \varphi(S_1, S_0)/\varphi(R_0, S_0) \). By an application of Theorem \( \text{[2]} \) it is easy to see that both \( \varphi(S_1) \) and \( \varphi(R_0) \) are relative invariants, and they clearly have indices \( m + r + 1 \) and \( k + r + 1 \), respectively. Moreover, \( \varphi(S_1) \) has the required order \( \mu + 1 \), and it gives rise to the absolute invariant

\[
\chi(S_1, S_0) = \frac{[\varphi(S_1)]^\mu}{S_0^{m+r+1}}. \quad (4.4)
\]

Owing to the arbitrariness of the function \( S_1 \), Eqs. (4.3) and (4.4) show that starting with any relative invariant \( S_1 \) of order \( \mu \), we can construct an indefinite sequence \( \varphi_q(S_1) = \varphi^q(S_1) \) of relative invariants and \( \chi_q(S_1) = \chi^q(S_1) \) of absolute invariants, each having a general term of order \( \mu + q \). To write down the expression for
the general terms of these sequences, we only need to note that \( \varphi_q(S_1) \) has index \( \theta(q) = m + q(\sigma + 1) \). Consequently, we have

\[
\varphi_q(S_1) = \theta(q - 1)\varphi^{q-1}(S_1)S'_{0} - rS_{0}\varphi^{q-1}(S_1)', \quad \chi_q(S_1) = \frac{[\varphi_q(S_1)]^\sigma}{S_0^\theta(q)}, \quad (4.5)
\]

where

\[\varphi^{q-1}(S_1)' = d(\varphi^{q-1}(S_1))/dx.\]

A similar sequence of relative and absolute invariants can be obtained using the relative invariant \( \varphi(R_0, S_0) \).

In the case of ODEs only one differential operator of the form \( \zeta D_x \) is required and once this operator is known, finding fundamental sets of invariants only requires a suitable generating system of invariants, and these invariants must be explicitly calculated. As already mentioned, earlier methods for finding these invariants such as those used in [5] were very intuitive and quite informal, and they were designed only for linear equations. However, based on ideas suggested by Lie [20], and starting with some works undertaken by Ovsyannikov [7], the development of infinitesimal methods started to grow and a number of Lie groups techniques for invariant functions have been proposed in the recent scientific literature [8, 21, 9, 10]. These Lie groups methods provide a systematic means for finding invariant functions and some of them have far reaching immediate applications (see [21, 10]). For instance, the method of [10] that we shall use for the explicit determination of the invariants also provides, in infinitesimal form, the structure-preserving transformations of any differential equation. When it is applied to Eq. (2.3), the coefficients \( a_j(x) \) of the equation are considered as dependent variables on the same footing as \( y \) and the infinitesimal generators \( X^0 \) of the equivalence group \( G_e \) takes the form

\[
X^0 = f \partial_x + \sum_{i=1}^{n} \phi_i \partial a_i. \quad (4.6)
\]

The explicit expression of \( X^0 \) depends on various canonical forms adopted for the general linear ODE. By a change of variable of the form \( x = z \) and \( y = \exp(-\int a_1 dz)w \), Eq. (2.3) can be put in the form

\[
w^{(n)} + b_2 w^{(n-2)} + \cdots + b_n w = 0 \quad (4.7)
\]

which is deprived of the term of second highest order, and in which the \( b_j = b_j(z) \) are the new coefficients. Similarly, by a change of variables of the form

\[
\{z, x\} = \frac{12}{n(n-1)(n+1)} a_2, \quad y = \exp\left(-\int a_1 dz\right)w, \quad (4.8a)
\]

where

\[
\{z, x\} = \left(z'z^{(3)} - (3/2)z''^2\right) z'^{-2} \quad (4.8b)
\]

is the Schwarzian derivative, and where \( z' = dz/dx \), we can put Eq. (2.3) into a form in which the terms of orders \( n-1 \) and \( n-2 \) do not appear. After the renaming of variables and coefficients of the new equation using the same notation as for the original equation (2.3), the transformed equation takes the form

\[
y^{(n)} + a_3 y^{(n-3)} + \cdots + a_n y = 0. \quad (4.9)
\]
In fact, due to the prominence in size of invariants of differential equations and the rapid rate at which this size increases with the number of nonzero coefficients in the equation, it is customary to use the canonical form (4.9) of (2.3) for the investigation of invariants [6, 3].

A determination of invariants of linear ODEs by means of the infinitesimal generator of the form (4.6) was made in [14] for equations of order up to five in the various canonical forms (2.3), (4.7), and (4.9), but only for low orders of prolongation of the operator \( X^0 \) not exceeding three. We now undertake the explicit determination of a generating system of invariants of all orders by an application of formulas (4.3)-(4.5). For this purpose we shall adopt the canonical form (4.9) for which the invariants have a much simpler expression, and we restrict our attention to the case of equations of order \( n = 5 \) which have the form

\[
y^{(5)} + a_3 y'' + a_4 y' + a_5 y = 0. \tag{4.10}
\]

For every \( n > 3 \), and for any order \( p \) of prolongation of the operator \( X^0 \), the number \( \Gamma \) of absolute invariants of Eq. (4.9) can be shown [14] to be given by

\[
\Gamma = n + 4 - p(n - 2), \tag{4.11}
\]

and this indicates that for a fixed value of \( n \), as the order of prolongation of \( X^0 \) increases by one unit, the number of absolute invariants increases by \( n - 2 \), which is precisely the number of coefficients in the equation. This means that a generating system of absolute invariants should contain \( n - 2 \) elements, which by virtue of (4.11) can be taken to be functionally independent, provided that none of the coefficients \( a_j \) vanishes. For (4.10), \( X^0 \) depends on three arbitrary constants \( k_1, k_2 \) and \( k_3 \) and has the explicit form

\[
X^0 = [k_1 + x(k_2 + k_3x)] \partial_x - 3a_3(k_2 + 2k_3x) \partial_{a_3} - 2 [-3a_3k_3 + 2a_4(k_2 + 2k_3x)] \partial_{a_4} + [-4a_4k_3 - 5a_5(k_2 + 2k_3x)] \partial_{a_5} \tag{4.12}
\]

and its first prolongation has four invariants given by

\[
I_0 = \frac{(3a_5 a_3 - a_4^2)^3}{27a_3^8} \tag{4.13a}
\]
\[
I_1 = \frac{(-a_4 + a_3')^3}{a_3^2} \tag{4.13b}
\]
\[
I_2 = \frac{(6a_3a_4' - a_4^2 - 6a_4 a_3')}{{216 a_3^8}} \tag{4.13c}
\]
\[
I_3 = \frac{5a_3^3 + 9a_3^2 a_4' - 3a_4 a_3(5a_5 + 2a_4') + 3a_4^2 a_3' + 3a_3^3 a_4'}{9a_3^2} \tag{4.13d}
\]

The fundamental relative invariant of (4.9) is readily seen to be given by \( S_0 = a_3(x) \), and Eq. (4.13) shows that the four functions

\[
R_0 = 3a_5 a_3 - a_4^2
\]
\[
S_1 = -a_4 + a_3'
\]
\[
S_2 = 6a_3a_4' - a_4^2 - 6a_4 a_3'
\]
\[
S_3 = 5a_3^3 + 9a_3^2 a_4' - 3a_4 a_3(5a_5 + 2a_4') + 3a_4^2 a_3'
\]
are relative invariants of respective index 8, 4, 8, and 12. In terms of these relative
invariants, we have
\[ I_0 = \frac{R_0^3}{S_0}, \quad I_1 = \frac{S_1^3}{S_0}, \quad I_2 = \frac{S_2^3}{S_0}, \quad I_3 = \frac{S_3^3}{S_0} \]  
(4.14)
Moreover, each of the four functions \( R_0, S_1, S_2, \) and \( S_3 \) is of order at most one, and
together they form a functionally independent set. Consequently, on in accordance
with (4.5), \( B = \{ I_0, I_1, I_2, I_3 \} \) forms a functionally independent generating system
of absolute invariants which may be called a basis of absolute invariants of (4.9).

It also follows that a fundamental set of absolute invariants of order \( p \geq 1 \) is
given by the functions
\[ I_0, \quad \chi_k(S_1), \quad \chi_k(S_2), \quad \chi_k(S_3), \quad \text{for } k = 0, \ldots, p - 1 \]  
(4.15)
and their number is \( 3p + 1 \), which is in accordance with equation (4.11). It follows
from Eq. (4.5) that to find explicit expressions for the absolute invariant \( \chi_k(S_j) \), we
only need to compute the corresponding relative invariant \( \varphi_k(S_j) \) and to determine
its index. For \( S_1 \), the corresponding higher order relative invariants up to the order
three are
\[ \varphi(S_1) = 4(a_4 - a_3')(a_3')^2 + 3a_3(-a_4' + a_3'' + 9a_4a_5) \]
\[ \varphi_2(S_1) = 32(a_4 - a_3')(a_3')^2 - 3a_3(9a_4'a_3 + (4a_4 - 13a_4'a_3') + 9a_3'' + a_3'(3) \]
and their indices are clearly 8 and 12, respectively. The size of these two invari-
ants are much more smaller compared to those corresponding to \( S_2 \) and \( S_3 \), simply
because \( S_1 \) has a much smaller size compared to \( S_2 \) and \( S_3 \). For instance the expres-
sion for \( \varphi_k(S_1) \) is about five times larger in size than that for \( \varphi_k(S_j) \), for \( k = 1, 2 \).
It turns out that the expression for the invariants computed explicitly by means
of the infinitesimal generator \( X^0 \) are considerably much smaller in size than those
computed from the indefinite sequence (4.5). Indeed, a direct computation shows
that a fundamental set of differential invariants of order two is given by
\[ I_4 = \frac{(7a_4^3 - 14a_4a_5' + 6a_3a_5'')^3}{216a_3^3} \]
\[ I_5 = \frac{4a_4^3 + 24a_3^2a_4' + 9a_3^2a_5'' - 9a_4a_3(3a_4' + a_5'')}{9a_3^3} \]
\[ I_6 = \frac{(-18a_4^3 - 18a_3a_4' + 18a_3^3a_5'' - 6a_4a_3^2(11a_4' + 2a_5'') + a_2^2a_3(55a_5 + 40a_4' + 6a_5'')^3}{5832a_3^{16}} \]
and that for differential invariants of order three is given by
\[ I_7 = \frac{-14a_4^3 + 42a_3^2a_4' - 36a_4a_3a_5'' + 9a_3^2a_3(3)}{9a_3^3} \]
\[ I_8 = \frac{(-2a_4^3 - 12a_3a_4' + 3a_2^2a_3(5a_5' + 3a_5'') + 2a_3^2a_4(3) - 2a_4a_3^2(5a_5' + a_3(3))}{8a_3^5} \]
\[ I_9 = (S_0, 1 + S_0, 2)^3/(5832a_3^{20}) \]
where
\[
S_{9,1} = 35a_5^3 + 45a_4^2a_3 - 10a_3^2a_4(11a_5 + 11a_4 + 3a_3') + 18a_4^4a_5^{(3)}
\]
\[
S_{9,2} = -12a_4a_3^2(9a_5' + a_4^{(3)} + 6a_4^2a_3^2(33a_5' + 11a_4'' + a_3^{(3)})^3).
\]

As already mentioned these absolute invariants of (4.10) are given in [14], but only for the second prolongation of the operator \(X^0\), and not in a form in which the index can be readily read off. They indeed appear to be smaller in size than those derived from the recurrence relations (4.5), which indicates that a further simplification of the expression of the function \(\varphi\) might be possible. However, such a simplification might lead to more complicated recurrence equations for the indefinite sequence of invariants. It’s however naturally much easier to generate higher order invariants using the indefinite sequence (4.5), rather than finding first the right prolongation of \(X^0\) and then solving the corresponding system of equations to find the invariants.

Similarly, using Eqs. (4.14), (4.13), and (4.5), we can also determine, by invoking Corollary [3] a fundamental set of relative invariants of all orders. For instance, fundamental relative invariants of order up to two of (4.10) are given by
\[
\varphi_k(S_j, S_0), \quad j = 1, 2, 3, \quad k = 0, 1
\]
\[
\varphi_k(R_0, S_0), \varphi_k(S_0, R_0), \quad k = 0, 1, 2.
\]

Incidently, Halphen [5] constructed by a different method an indefinite sequence of relative invariants similar to that given in (4.10), but for linear ODEs of order 4 in the canonical form (2.3). In particular the approach of [5] is not based on infinitesimal nor Lie groups methods and it aimed only at deriving and indefinite sequence of relative invariants from known ones. Consequently, the sequence of relative invariants obtained in the said paper for equations of order 4 is not associated with the determination of a fundamental set of absolute invariants.

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