QUOTIENTS OF JACOBIANS

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Abstract. Let $C$ be a curve of genus $g$, and let $G$ be a finite group of automorphisms of $C$. The group $G$ acts on the Jacobian $J$ of $C$; we prove that for $g \geq 21$ the quotient $J/G$ has canonical singularities and Kodaira dimension 0. On the other hand we give examples with $g \leq 4$ for which $J/G$ is uniruled.

Pour Enrico, après 50 ans d’amitié

1. Introduction

Let $C$ be a (smooth, projective, complex) curve of genus $g$, and let $G$ be a finite group of automorphisms of $C$. The group $G$ acts on the Jacobian $J$ of $C$. There are two possibilities for the quotient $J/G$: either it has canonical singularities and Kodaira dimension 0, or it is uniruled. The aim of this note is to show that the latter case is rather exceptional – in fact, it does not occur for $g \geq 21$. This bound is rough, and can certainly be lowered. On the other hand we give examples of uniruled $J/G$ for $g \leq 4$. When $g = 3$ this is an essential ingredient in our proof that the cycle $[C] - [(-1)^*C]$ on $J$ is torsion modulo algebraic equivalence [B-S]; in fact this note grew out of the observation that already in genus 3, the case where $J/G$ is uniruled is quite rare.

2. $J/G$ Canonical

Let $C$ be a curve of genus $g$ and $J$ its Jacobian. Let $G$ be a subgroup of Aut($C$), hence also of Aut($J$). By [K-L, Theorem 2], there are two possibilities:

- Either $J/G$ has canonical singularities, and Kodaira dimension 0;
- or $J/G$ is uniruled.

\[\text{I am indebted to A. Höring for pointing out this reference}\]
Proposition 1. Assume \( g \geq 21 \). The quotient variety \( J/G \) has canonical singularities, and Kodaira dimension 0.

Proof: We first observe that the fixed locus \( \text{Fix}(\sigma) \subset J \) of any element \( \sigma \neq 1 \) of \( G \) has codimension \( \geq 2 \). Indeed the dimension of \( \text{Fix}(\sigma) \) is the multiplicity of the eigenvalue 1 for the action of \( \sigma \) on \( H^0(C, K_C) \), that is, the genus of \( C/\langle \sigma \rangle \), which is \( \leq g - 2 \) as soon as \( g \geq 4 \) by the Hurwitz formula.

Let \( \sigma \) be an element of order \( r \) in \( G \), and let \( p \in J \) be a fixed point of \( \sigma \). The action of \( \sigma \) on the tangent space \( T_p(J) \) is isomorphic to the action on \( T_0(J) = H^0(C, K_C) \). Let \( \zeta = e^{2\pi i/r} \); we write the eigenvalues of \( \sigma \) acting on \( H^1(C, C) \) in the form \( \zeta a_1, \ldots, \zeta a_g \), with \( 0 \leq a_i < r \). By Reid’s criterion \([R, \text{Theorem 3.1}]\), \( J/G \) has canonical singularities if and only if \( \sum a_i \geq r \) for all \( \sigma \) in \( G \).

The eigenvalues of \( \sigma \) acting on \( H^1(C, C) \) are \( \zeta a_1, \ldots, \zeta a_g; \zeta^{-a_1}, \ldots, \zeta^{-a_g} \). Thus \( \text{Tr} \sigma^*_{H^1(C, C)} = 2 \sum \cos \frac{2\pi a_i}{r} \). By the Lefschetz formula, this trace is equal to \( 2 - f \), where \( f \) is the number of fixed points of \( \sigma \); in particular, it is \( \leq 2 \). Using \( \cos x \geq 1 - \frac{x^2}{2} \), we find

\[
1 \geq \sum \cos \frac{2\pi a_i}{r} \geq g - \frac{2\pi^2}{r^2} \sum a_i^2,
\]

hence \( (\sum a_i)^2 \geq \sum a_i^2 \geq g - \frac{1}{2\pi^2} r^2 \). Thus if \( g \geq 1 + 2\pi^2 = 20.739 \ldots \), we get \( \sum a_i > r \). \( \square \)

Remark. - The proposition does not extend to the case of an arbitrary abelian variety: indeed we give below examples where \( J/G \) is uniruled; then if \( A \) is any abelian variety, the quotient of \( J \times A \) by \( G \) acting trivially on \( A \) is again uniruled.

3. \( J/G \) uniruled

We will now give examples of (low genus) curves \( C \) such that \( J/G \) is uniruled. The genus 2 case is quite particular (and probably well known). We put \( \rho := e^{2\pi i/3} \). We assume \( G \neq \{1\} \).

Proposition 2. If \( g(C) = 2 \), \( J/G \) has Kodaira dimension 0 if and only if \( G = \langle \sigma \rangle \) and \( \sigma \) is the hyperelliptic involution, or an automorphism of order 3 with eigenvalues \((\rho, \rho^2)\) on \( H^0(C, K_C) \), or an automorphism of order 6 with eigenvalues \((-\rho, -\rho^2)\) on \( H^0(C, K_C) \).
Proof: Let $\sigma$ be an element of order $r$ in $G$, and let $\zeta$ be a primitive $r$-th root of unity. The eigenvalues of $\sigma$ acting on $H^0(C, K_C)$ are $\zeta^a, \zeta^b$, with $0 \leq a, b < r$. Suppose that $J/G$ has canonical singularities. By Reid’s criterion we must have $a + b \geq r$; replacing $\zeta$ by $\zeta^{-1}$ gives $2r - a - b \geq r$, hence $a + b = r$. Since $\sigma$ has order $r$, $\sigma^k$ acts non-trivially on $H^0(C, K_C)$ for $0 < k < r$; therefore $a$ and $r$ are coprime.

Replacing $\zeta$ by $\zeta^a$ we may assume that the eigenvalues of $\sigma$ are $\zeta$ and $\zeta^{-1}$. Put $\zeta = e^{i\alpha}$; the trace of $\sigma$ acting on $H^1(C, \mathbb{Z})$ is $2(\zeta + \zeta^{-1}) = 4 \cos \alpha$, and it is an integer. This implies $\cos \alpha = 0$ or $\pm \frac{1}{2}$, hence $\alpha \in \mathbb{Z} \cdot \frac{2\pi}{6}$ and $r \in \{2, 3, 6\}$. If $r = 2$ we must have $\zeta = \zeta^{-1} = -1$, hence $\sigma$ is the hyperelliptic involution; if $r = 3$ or $6$ we have $\{\zeta, \zeta^{-1}\} = \{\rho, \rho^3\}$ or $\{-\rho, -\rho^3\}$.

Thus $G$ contains elements of order 3 and 6, and at most one element of order 2. Since the order of $\text{Aut}(C)$ is not divisible by 9 [B-L, 11.7], $G$ is isomorphic to either $\mathbb{Z}/3$, or a central extension of $\mathbb{Z}/3$ by $\mathbb{Z}/2$ – that is, $\mathbb{Z}/6$. This proves the Proposition.

The last two cases are realized by the curves $C$ of the form $y^2 = x^6 + kx^3 + 1$, with the automorphisms $\sigma_1 : (x, y) \mapsto (\rho x, y)$ and $\sigma_2 : (x, y) \mapsto (\rho x, -y)$. The surfaces $S_i := J/\langle \sigma_i \rangle$ are (singular) K3 surfaces, each depending on one parameter. The surface $S_1$ is studied in detail in [B-V]: it has 9 singular points of type $A_2$. The surface $S_2$ is the quotient of $S_1$ by the involution induced by $(-1)_J$; it has one singular point of type $A_5$ (the image of $0 \in J$), 4 points of type $A_2$ corresponding to pairs of singular points of $S_1$, and 5 points of type $A_1$ corresponding to triples of points of order 2 of $J$.

It is more subtle to give examples with $g \geq 3$. By Reid’s criterion, we must exhibit an automorphism $\sigma$ of $C$, of order $r$, such that the eigenvalues of $\sigma$ acting on $H^0(C, K_C)$ are of the form $\zeta^{a_1}, \ldots, \zeta^{a_g}$, with $\zeta$ a primitive $r$-th root of unity, $a_i \geq 0$ and $\sum a_i < r$. In the following tables we give the curve $C$ (in affine coordinates), the order $r$ of $\sigma$ and $\zeta$, the expression of $\sigma$, a basis of eigenvectors for $\sigma$ acting on $H^0(C, K_C)$, the exponents $a_1, \ldots, a_g$; in the last column we check the inequality $\sum a_i < g$, which implies that $J/\langle \sigma \rangle$ is uniruled.
\[ q = 3 \]

| curve         | \( r \) | \( \sigma(x,y) = \)   | basis of \( H^0(K_C) \) | \( a_1, \ldots, a_3 \) | \( \sum a_i \) |
|---------------|--------|---------------------|-----------------|------------------|----------------|
| \( y^2 = x^8 - 1 \) | 8      | \( (\zeta x,y) \) | \( \frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y} \) | 1, 2, 3           | 6 < 8          |
| \( y^2 = x(x^6 - 1) \) | 12     | \( (\zeta^2 x, \zeta y) \) | \( \frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y} \) | 1, 3, 5           | 9 < 12         |
| \( y^2 = x(x^7 - 1) \) | 14     | \( (\zeta^2 x, \zeta y) \) | \( \frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y} \) | 1, 3, 5           | 9 < 14         |
| \( y^3 = x(x^3 - 1) \) | 9      | \( (\zeta^3 x, \zeta y) \) | \( \frac{dx}{y^2}, \frac{x dx}{y^2}, \frac{dx}{y} \) | 1, 4, 2           | 7 < 9          |
| \( y^3 = x^4 - 1 \) | 12     | \( (\zeta^{-3} x, \zeta^{-1} y) \) | \( \frac{dx}{y^2}, \frac{x dx}{y^2}, \frac{dx}{y} \) | 5, 2, 1           | 8 < 12         |

The fact that \( J/\langle \sigma \rangle \) is uniruled for the fourth curve \( C \) is an important ingredient of the proof in [B-S] that the Ceresa cycle \([C] - [(-1)^*C] \) in \( J \) is torsion modulo algebraic equivalence. The same property for the last curve is observed in [L-S]. The second curve has been provided to me by D. Conti, A. Ghigi and R. Pignatelli, using the database [CGP].

\[ q = 4 \]

| curve         | \( r \) | \( \sigma(x,y) = \) | basis of \( H^0(K_C) \) | \( a_1, \ldots, a_4 \) | \( \sum a_i \) |
|---------------|--------|---------------------|-----------------|------------------|----------------|
| \( y^2 = x(x^9 - 1) \) | 18     | \( (\zeta^2 x, \zeta y) \) | \( \frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y}, \frac{x^3 dx}{y} \) | 1, 3, 5, 7       | 16 < 18        |
| \( y^3 = x^5 - 1 \) | 15     | \( (\zeta^{-3} x, \zeta^{-5} y) \) | \( \frac{dx}{y^2}, \frac{x dx}{y^2}, \frac{x^2 dx}{y^2}, \frac{dx}{y} \) | 7, 4, 1, 2       | 14 < 15        |

In genus 5 there is no example:

**Proposition 3.** Assume \( q = 5 \). The quotient variety \( J/G \) has canonical singularities, and Kodaira dimension 0.
Proof: The paper [K-K] contains the list of possible automorphisms of genus 5 curves, and gives for each automorphism of order $r$ its eigenvalues $\zeta^{a_1}, \ldots, \zeta^{a_5}$ on $H^0(K)$, with $\zeta = e^{\frac{2\pi i}{r}}$ and $0 \leq a_i < 5$. We want to prove $\sum a_i \geq r$.

We first observe that if an eigenvalue and its inverse appear in the list, we have $\sum a_i \geq r$; this eliminates all cases with $r \leq 10$. For $r \geq 11$ the numbers $a_1, \ldots, a_5$ are always distinct and nonzero; therefore $\sum a_i \geq 1 + 2 + \cdots + 5 = 15$. The remaining cases are $r = 20$ and $22$; one checks immediately that for all $k$ coprime to $r$ we have $\sum b_i \geq r$, where $b_i \equiv ka_i \pmod{r}$ and $0 < b_i < r$. This proves the Proposition.

Added in proof: R. Serova just proved that there are in fact no examples in genus $\geq 6$ (Quotients of Jacobians, preprint arXiv:2407.00860).

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