Proposal for reading out anyon qubits in non-abelian $\nu = 12\text{=}5$ quantum Hall state

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Abstract

To detect non-abelian statistics in the $\nu = 12\text{=}5$ quantum Hall state through interferometry, we apply an analysis similar to the ones proposed for the non-abelian $\nu = 5\text{=}2$ quantum Hall state. The result is that the amplitude of the Aharonov-Bohm oscillation of this interference is dependent on the internal states of quasiholes, but, in contrast to the $\nu = 5\text{=}2$ quantum Hall state, independent of the number of quasiholes. However, if the quasiholes are in a superposition state, it is necessary for the interferometer to have certain additional features to obtain the coefficients.

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I. INTRODUCTION

In two dimensions, particles are no longer constrained to obey Bose or Fermi statistics. It is possible for an exchange of identical particles to result in the multiplication of the wavefunction by an arbitrary phase factor $e^{i\theta}$, not just -1 or +1 [1]. Particles with such properties are called anyons [2].

There exists an even more exotic possibility. If there is a set of $g > 1$ degenerate states $\psi_a$, $a = 1, 2, \cdots, g$, for anyons with identical configurations, exchanging particles $i$ and $j$ can rotate one state into another in the space spanned by the $\psi_a$'s:

$$\psi_a \rightarrow T_{ab}^{ij} \psi_b : \quad (1)$$

There is no reason to expect the matrices $T_{ab}^{ij}$ to commute in general, and when they do not commute, the particles are said to obey non-abelian braiding statistics [3].

The transformation Eq.(1) is possible because, in two dimensions, particle exchange represents a topologically nontrivial manipulation. External potentials or impurities, if weak enough, cannot induce such a transformation, so the $g$-fold degeneracy is unaffected by such perturbations. These characteristics make non-abelian systems attractive candidates for fault-tolerant computation [4, 5]. There exist possible quantum Hall states where the braiding rules for a set of $N$ quasiparticles coincide with the braiding rules for $N$-point conformal blocks in the level-$k$ SU(2) Wess-Zumino-Witten model. In the case $k = 3$, these braiding rules can be used to construct a universal quantum computer [5]. In particular a universal set of quantum gates realized by anyon braiding in a $k = 3$ system has been found [6].

Although it is not adequate for quantum computation, the $k = 2$ case has received more attention. This is partly because it is simpler but also because there is experimental [7] and numerical evidence [8, 9] that such a system actually exists in the $\nu = 5=2$ quantum Hall system [10]. However, the $\nu = 12=5$ quantum Hall state is a possible candidate for the $k = 3$ case, although evidence is not as strong. Consequently it is of interest to construct a readout scheme for the $k = 3$ case similar to the $k = 2$ scheme exhibited in [11, 12].

In this paper, we will show that in the Aharonov-Bohm (AB) oscillation experiment similar to the one proposed for the $\nu = 5=2$ quantum Hall system [16, 17], the $\nu = 12=5$ quantum Hall system will show the effects of its non-abelian statistics. The paper is organized as follows. In Section II, we discuss the analysis of the AB oscillation experiment in the $\nu = 5=2$ quantum Hall system. In
FIG. 1: A two point-contact interferometer for measuring the quasiparticle statistics. The light-gray region contains an incompressible fractional quantum Hall liquid. The front gates (black rectangles) are used to bring the opposite edge currents (indicated by arrows) close to each other to form two tunneling junctions. Applying voltage to the central gate creates an antidot in the middle and adjusts the number of quasiholes contained there. In addition a side gate can be used in such a way that a voltage applied to it would keep the filling fraction constant even as the applied magnetic field is changed.

Section III, we show how to construct the non-abelian quantum Hall state at $\nu = 12/5$. In Section IV, we present our main result, the analysis of the AB oscillation in the $\nu = 12/5$ quantum Hall system. Section V is a conclusion and discussion.

II. DETECTING NON-ABELIAN STATISTICS IN $\nu = 5/2$ QUANTUM HALL SYSTEM

The basic scheme for the experiment in both filling fractions $\nu = 5/2$ and $\nu = 12/5$ is a two point-contact interferometer composed of a quantum Hall bar with two front gates that is shown in Fig.1 [11, 13–17]. By biasing the front gates, we can create constrictions in the Hall bar, and so adjust the tunneling amplitude of $t_L$ and $t_R$ across the constrictions. The tunneling between opposite-edge currents leads to deviations of $\sigma_{xy}$ from its quantized value, or equivalently, to the appearance of $\sigma_{xx}$. This $\sigma_{xx}$ can be measured by connecting one edge to a constant current source and measuring the voltage drop across the other edge. For this experiment, the interest is on the case where the tunneling amplitudes $t_L$ and $t_R$ are small. This must be so in order to ensure that the tunneling current is entirely due to lowest charge ($e=4$) quasiholes with no contribution from higher charge composites [16].

To the lowest order in $t_L$ and $t_R$, the tunneling current and, hence, longitudinal conductivity $\sigma_{xx}$ in this system will be proportional to the probability that current entering the bottom edge leaves
through the top edge [15, 16]:

$$\sigma_{xx} \propto (U_L + t_R U_R) \Psi j \Psi i j$$

$$= j L j^2 + j R j^2 + 2 \text{Re} f t t R j \Psi L j \Psi R^j i j$$

$$= j L j^2 + j R j^2 + 2 \text{Re} f t t R \Psi M_n \Psi i j; \tag{2}$$

In this expression, $U_L$ and $U_R$ are the unitary evolution operators for a quasihole taking the two respective paths, and $\Psi i j$ is the initial state of the system. In the third line, the operator $M_n$ is the transformation solely due to the braiding statistics of winding a single quasihole around $n$ quasiholes [15], and $e^{i \alpha}$ is the abelian phase factor that includes the AB phase.

In this Section, we will consider $\Psi j \Psi j M_n \Psi i j$ for the $n$ quasiholes in the interferometer in the case $\nu = 5 = 2$. The essential point is that this operator $M_n$ does not probe individually the local properties of the quasiholes, because in this experiment, these $n$ quasiholes are seen only as a single composite entity. Therefore one first needs to consider what is the fusion rule for these anyons. It is known, through the machinery of conformal field theory, that the Moore-Read state can be built from the $\mathbb{Z}_2$ Ising anyon model. This model has three particle types, conventionally denoted as: $I$ (vacuum), $\sigma$ (spin/vortex), and $\psi$ (Majorana fermion). Its non-trivial fusion rules are:

$$\sigma \psi = \sigma; \quad \sigma \sigma = I + \psi; \quad \psi \psi = I; \tag{3}$$

(Note that the magnetic flux of the quasiholes does not show up in this formalism. However, that is a less interesting issue, since for all cases, the flux can be accounted for by some abelian phase factors.)

Eq.(3) tells us that for the case $n$ even, the composite can turn out to be either $I$ or $\psi$, whereas for $n$ odd it can only be $\sigma$. But (3) also tells us that fusion of $\sigma$ and $\psi$ (not to mention $\sigma$ and $I$) has a unique result. Therefore the operator $M_n$, which is equivalent to encircling a $\sigma$, representing the tunneling quasihole, around the composite in the island, would only lead to wavefunction modification by some phase factor in this case. Diagrammatically this means that a diagram where one particle winds around another can be reduced to an unwound diagram. The phase factor has been worked out [16, 18], and $M_n$ in this case is effectively reduced to the diagrammatic braiding

4
rules worked out by Bonderson et al. [16]:

\[
\sigma I = \sigma I (4)
\]

\[
\sigma \psi = (1) \sigma \psi (5)
\]

(One way to obtain these phase factors is by examining the exponent of the operator product expansion of these two particles.) Therefore if one adjusts the magnetic field while maintaining the filling fraction (something one can do by varying the size of the island region using a side gate voltage), one will observe AB interference.

The situation is different for the case \( n \) odd. Eq.(3) tells us that the composite in this case will always be \( \sigma \). However, the second fusion rule of Eq.(3) indicates that there are two available states in this case. Although one can diagonalize \( M_n = (T_{te})^2 \) for this case (where ‘\( t \)' in the superscript represents the tunneling quasihole and ‘\( c \)' the composite particle in the island), there are two different eigenvalues in this case [15]. This means that the diagram with two \( \sigma \)'s winding around each other,

\[
(6)
\]

cannot be reduced to the diagram of two unwound \( \sigma \)'s multiplied by some phase factor. Hence, the effect of braiding cannot be reduced to some phase factor as it was for the diagrams (4) and (5). It had been found by Bonderson et al. that the diagram (6) is actually proportional to the diagram of two unwound \( \sigma \)'s exchanging a \( \psi \) particle [16]. Their result is consistent with Stern and Halperin’s observation that for two different tunneling quasiholes, the \( M_n \)'s anticommute [17, 19]. However, for the purpose of this paper, it is sufficient to use a less general method presented below.

The interference term \( \hbar \Psi M_n \Psi \) is the expectation value of \( M_n \) evaluated at the initial state \( \Psi \). Diagrammatically, this is represented by the standard closure, where each worldline is looped back onto itself in such a way as to introduce no further braiding. From Eq.(6), for \( n \) odd, \( \hbar \Psi M_n \Psi \)
would be equal to the following diagram [15, 16]:

\[ \sigma \bigcirc \sigma \]  

(7)

provided that we set loop propagators to unity. It is important to consider the meaning of the initial state $\Psi_i$. It includes not only the internal state of the quasiholes in the island but also the edge state as well. This is so because the operator $M_n$ should involve creation and annihilation of the chiral $\sigma$ field at the edge if it is to account for tunneling of a quasihole. In this sense, the diagrammatic calculation performed here is taking the expectation value over tunneling quasiholes as well [19].

The anyon wordlines of the diagram (7) are said to have formed a *Hopf link*. Provided that unlinked loops are normalized to the value of each anyon’s *quantum dimension*, the *topological S matrix* of the anyon model can be defined in terms of these Hopf links [20]. If one of the anyons forming the Hopf link is the vacuum, the value of this Hopf link is merely the quantum dimension of the other anyon. In addition, the S matrix is unitary [20, 21]. For the Ising model, these results, together with the results we have from the diagrams (4) and (5), enable us to obtain all the elements of the S matrix, and thus evaluate the diagram (7).

An alternative way to calculate the diagram (7), which we shall employ in our analysis of the $\nu = 12=5$ case, is to view it as involving not two, but four $\sigma$ particles. (This is possible because by Eq.(3), the $\sigma$ particle can be regarded as its own antiparticle.) Consider the situation where two pairs of $\sigma$ particles are created out of vacuum. Then, have one $\sigma$ particle from one pair wind counterclockwise around one $\sigma$ particle from the other pair. Finally the two pairs are fused. The diagram (7) is equal to the amplitude of obtaining vacuum for both fusion processes [19]. It is known that such amplitude is zero [11, 12, 15, 18, 22]; indeed the proposed NOT gate of Das Sarma *et al.* [11] depends on this result. As a consequence, there is no AB oscillation for the case $n$ odd.

III. CONSTRUCTING $\nu = 12=5$ QUANTUM HALL SYSTEM FROM $\mathbb{Z}_3$ PARAFERMION

Although the evidence is not quite as strong as that of the Pfaffian state at $\nu = 5=2$ [23], the $k = 3$ parafermion state is considered the most likely candidate for the $\nu = 12=5$ quantum Hall system. As conceived by Read and Rezayi [24], $k$-cluster quantum Hall states are generalizations
of the Pfaffian state, where the wavefunction vanishes when \(k+1\) electrons come together but, in the ground state, not vanishing when \(k\) or fewer electrons come together. In this scheme, the original Pfaffian state has \(k=2\). The fusion rules of a Majorana fermion, too, can be generalized into the fusion rule of \(Z_k\) parafermions:

\[
\psi_l \{z\}^k \overset{\text{k factors}}{\rightarrow} \psi_{l'}
\]

where \(l = 1; \ldots; k - 1\).

\(Z_k\) parafermions originally arose from the \(Z_k\) generalization of the two-dimensional Ising model [25]. (In case of \(k=3\), this would be equivalent to the three-state Potts model [25, 26].) In such a model, there would be a “spin” variable \(\sigma\) on each site, taking \(k\) values \(\omega^p (p = 0, 1, 2; \ldots; k - 1)\), where

\[
\omega = \exp \left(\frac{2\pi i}{k}\right):
\]

If the system is at a critical point, one can treat \(\sigma^l\) as a continuous conformal field \(\sigma_l(x), x \in \mathbb{R}^2\). The critical theory retains the \(Z_k\)-symmetry, so the correlations are invariant under the transformation

\[
\sigma_l(x) ! \omega^m \sigma_l(x)
\]

for arbitrary integer \(m\). From Eq.(10), one can say that the \(Z_k\) charge of \(\sigma_l(x)\) is \(l\). This model is self-dual - that is, there exist dual conformal fields \(\mu_n(x) (n = 1, 2; \ldots; k - 1)\) and all correlation functions are invariant under the interchange \(\sigma \leftrightarrow \mu\).

It is helpful to examine the \(k = 4\) cases. The Hamiltonian on a square lattice can be expressed as

\[
\mathcal{H} = J \sum_{\langle i,j \rangle} [\sigma_i(\sigma_j)] + c \mathcal{V}:
\]

Through a generalized Wannier-Kramer transformation, one can define the dual spin variables \(\mu\) on each dual lattice site. The lattice model of Eq.(11) is self-dual for these cases [27], which can be seen by generalizing the case of the Ising model treated in [28]. To obtain the correlation function \(\langle \mu_{j_1}^{n_1} \mu_{j_2}^{n_2} 1 \rangle\), we must first calculate a transformed partition function, in which the Hamiltonian Eq.(11) is modified by

\[
\sigma_i(\sigma_j) ! t_{ij} \sigma_i(\sigma_j):
\]

In Eq.(12), \(t_{ij} = 1\) if the link is not on a path \(\Gamma\) connecting \(j_1\) to \(j_2\). But if the link \(ij\) is on \(\Gamma\), either \(t_{ij} = \omega^n\) or \(t_{ij} = \omega^n\) (so that if \(\Gamma\) is closed, the modification results in the mapping to an
equivalent problem where all spins in sites inside gets multiplied by $\omega^n$), hence the term ‘disorder’ for $\mu$ variables. The correlation function $\h \mu_{j_1} \mu_{j_2} \cdots \mu_{j_n}$ is the ratio of this transformed partition function to the original partition function. Since $\mu^n$ can be treated as $\mu_n(\chi)$ when the system is at a critical point, Eq.(12) tells us that $\sigma_l(\chi)$ and $\mu_n(\chi)$ are mutually *semilocal* with the mutual locality exponent of $\gamma_{ln} = l \ln = k$. That is, one would pick up a phase of $\exp(2i\pi \gamma_{ln}) = \omega^{ln}$ when $\sigma_l(\chi)$ winds counterclockwise around $\mu_n(\chi)$ or vice versa.

This result can be generalized to any $k$ values [25]. A useful analogy in thinking about the mutual locality exponent is to regard $\sigma_l(\chi)$ as possessing charge $l = k$ and $\mu_n(\chi)$ as possessing vortex winding $n$; the negative of this vortex winding is termed the dual $\tilde{Z}_k$ charge. (A similar conclusion can be also obtained from the $Z_N$-Villain model treated in [29].) Given the formulation of $\mu_n(\chi)$ given here, it is natural that there exists dual $\tilde{Z}_k$ invariance

$$\mu_n(\chi) \cong \omega^{s_n} \mu_n(\chi)$$

for arbitrary integer $s$.

One can regard the $Z_k$ parafermion field $\psi_l$ as a holomorphic field originating from fusing $\sigma_l$ and $\mu_l$ [25]. Therefore the $Z_k$ $\tilde{Z}_k$ charge of $\psi_l$ is $\psi(l)$. One result of this formulation is that the $\psi_l$’s are not mutually local; between $\psi_l$ and $\psi_{l0}$ the mutual locality exponent is $\gamma_{l/l0} = 2l\gamma = k$. (Together with the conservation of $Z_k$ $\tilde{Z}_k$ charge, this sets both the conformal dimension and its spin at $\Delta_l = l(2l-k) = k$ [25, 26].) Let us now consider the problem of constructing the electron operator from $\psi_l$. One wants to obtain an anticommuting fermion operator with the least possible flux attached to it [24]. In order to do so, one should set $l = 1$ and obtain

$$V_{\text{para}}^{\text{el}} = \psi_1 : \exp \left[ \frac{\phi}{1 + 2l \Phi_0} \right] :$$

where $\phi_c$ is a chiral free boson uniquely defined by the two-point correlation function $\langle \phi_c(z) \phi_e(0) \rangle = \ln z$. (Note that from this point on, as we will be dealing with quantum Hall states, we are interested only in the holomorphic part of our conformal fields.) This sets the mutual locality exponent of the vertex operator part (and flux attached to one electron in units of $\Phi_0 = \hbar = e$) at $1 + 2l = k$, and therefore the electron operator of Eq.(14) is mutually local. (Note that for any value of $k$, this operator has the conformal dimension of $3/2$.) The wavefunction obtained by Read and Rezayi was constructed by taking the many-point correlation function of the electron operator of Eq.(14). For $k = 3$ it turns out to be [24]

$$\psi_{\text{para}}^{(M)}(\xi_1; \cdots; \xi_N) = \prod_{i} \psi_1(\xi_1) \prod_{i<j} (\xi_i - \xi_j)^{3/2} :$$

(15)
The next step is to determine the quasihole operator for $k = 3$. Before doing this we need to consider the complete fusion rules for the $\mathbb{Z}_3$ Potts model. These are [30]:

\[
\begin{align*}
\psi_i \psi_i &= \psi_3 \psi_i; \quad \psi_i \psi_3 \psi_i = \mathbb{I}; \\
\psi_i \sigma_i &= \mathbb{I}; \quad \psi_i \sigma_3 \psi_i = \mathbb{I}; \quad \psi_i \varepsilon = \sigma_3 \psi_i; \\
\sigma_i \sigma_i &= \psi_i + \sigma_3 \psi_i; \quad \sigma_i \sigma_3 \psi_i = \mathbb{I} + \varepsilon; \quad \sigma_i \varepsilon = \psi_3 \psi_i + \sigma_i
\end{align*}
\]

and

\[
\begin{align*}
\varepsilon \varepsilon &= \mathbb{I} + \varepsilon;
\end{align*}
\]

where $i = 1, 2$. The $\sigma_i$'s are the primary field of the parafermion algebra. (Note that for $\sigma_i$, the labeling scheme we are using here follows Read and Rezayi [24] which is different from what is used in [30].) Since we are only concerned with the holomorphic part of these fields here, the operator product expansions (OPEs) need to be modified from their original result in the three-state Potts model. This is exactly analogous to the situation in the $\mathbb{Z}_2$ Ising model. In this holomorphic setting, the conformal dimensions of fields are $\Delta_\psi = 2/3$, $\Delta_\sigma = 1/5$, and $\Delta_\varepsilon = 2/5$. As a result the least singular OPE that emerges between $\psi_1$ and $\sigma_i$ or $\varepsilon$ is [24]

\[
\psi_1(z) \sigma_1(\emptyset) \propto \prod_{i<j} (z_i - z_j)^{1/5} = \prod_{i<j} (w_i - w_j)^{1/5} \prod_{ij} (\xi_i \cdot w_j)^{1/5} \prod_{i<j} (\xi_i \cdot z_j)^{1/3}.
\]

There are two condition for constructing the quasihole operator. One is that it should have the least possible charge, or equivalently, least flux attached. The other is that it should be mutually local with the electron operator [31, 32]. Eq.(18) tells us that $\sigma_1$ needs to be included to satisfy the first condition. The vertex operator part then needs to be adjusted to satisfy the second condition. (In order to satisfy the second condition one needs $\Delta_{el} + \Delta_{qh} \Delta_{\lambda^0}$ to be an integer, where $\lambda^0$ is the fusion product of electron and quasihole operators [31, 32].) The answer we get is

\[
\mathcal{V}_{\text{para} \ qh} = \sigma_1 : \exp \left( \frac{1}{15} \varphi_c \right) :.
\]

Now with Eq.(19), the wavefunction with quasiholes can be constructed:

\[
\mathcal{\Psi}^{\text{(M)}}_{\text{para+qh}}(z_1; \ldots; z_N; w_1; \ldots; w_{3n}) = \mathcal{\Psi}_1(z_1) \cdot \mathcal{\Psi}(w_1) \cdot \mathcal{\Psi}_3(w_1) \cdot \mathcal{\Psi}(z_1) \cdot \mathcal{\Psi}_3(z_1) = \mathcal{\Psi}_1(z_1) \cdot \mathcal{\Psi}(w_1) \cdot \mathcal{\Psi}_3(w_1) \cdot \mathcal{\Psi}(z_1) \cdot \mathcal{\Psi}_3(z_1) \cdot \mathcal{\Psi}_1(z_1) \cdot \mathcal{\Psi}(w_1) \cdot \mathcal{\Psi}_3(w_1) \cdot \mathcal{\Psi}(z_1) \cdot \mathcal{\Psi}_3(z_1).
\]

We have seen that the electron operator is mutually local to both electrons and quasiholes. Consequently the many-electron wavefunction in (20) is analytic in the electron coordinate, the fractional
exponent of the Laughlin-like term being canceled out by the parafermion correlation function part. The filling fraction for this wavefunction is determined entirely by the Laughlin factor. This gives \( \nu = 3 \times 5 \) [24]. In order to obtain \( \nu = 2 + (1 \times 3 \times 5) \) from the wavefunction in (20), we need to fill the first Landau level with electrons of both spins and then apply a particle-hole transformation to the second Landau level. From the exponent of the \( z_i \ w_j \) term in Eq.(20), one can see that the electric charge of a quasihole is \( e = 5 \) in the parafermion \( \nu = 3 \times 5 \) quantum Hall state [24]. Since the \( \nu = 12 \times 5 \) state is obtained by applying particle-hole transformation on the \( \nu = 3 \times 5 \) state, the quasiholes of Eq.(20) becomes excitations with the charge \( e = 5 \) in the parafermion \( \nu = 12 \times 5 \) quantum Hall state. Conversely, there exists charge \( + e = 5 \) excitations, such as the ones on the antidot [19], which originate from quasielectron excitations of the \( \nu = 3 \times 5 \) state.

For most of the next section, the case of the \( \nu = 3 \times 5 \) parafermion state will be considered, the particle-hole inversion being applied at the end. The inversion will result in inverting the signs of both the quasiparticle charge and the statistical angle.

IV. QUBIT MEASUREMENT IN \( \nu = 12 \times 5 \) QUANTUM HALL SYSTEM

As stated at the end of the last section, this section will chiefly deal with the AB interference that would arise if Eq.(20) is the second Landau level electron wavefunction. However, for \( \nu = 12 \times 5 \), the wavefunction Eq.(20) should be that of holes, not electrons. One can consider an analogous situation in an abelian fractional quantum Hall state. There, the sign of the charge would be reversed while that of the flux remains the same. As a result, the sign of the nontrivial phase that gets accumulated when one quasihole encircles another is reversed. There should be the same reversal of the sign of the phase in this interference also. However it should be noted that in this case not all phase comes from the abelian U(1) sector; reversal of sign of the charge alone cannot explain this sign reversal. One can formulate this sign change precisely by applying Girvin’s particle-hole transformation in the lowest Landau level [33], which include taking the complex conjugates of the quasiparticle coordinates.

One thing to be noticed from Eq.(16) is that fusions involving \( \psi_i \) produce a single operator and not a sum of operators. This indicates that the braiding \( \psi_i \)’s only contribute abelian phase factors, and that in the case of braiding, replacing \( \sigma_1 \) or \( \sigma_2 \) with \( \epsilon \) will only result in changing the phase factor. With these consideration, the conclusion is that all non-abelian statistics in this model can be derived from the fusion rule Eq.(17) of the \( \epsilon \) particles. This is a very important point because
Eq.(17) is equivalent to the fusion rule for the Fibonacci anyons discussed by Preskill [20], with \(\varepsilon\) as the Fibonacci anyon. That braids of these Fibonacci anyons can yield universal quantum computation was explicitly shown by Bonesteel et al. [6] (Fibonacci anyons may also be realized in quantum spin systems [34, 35] and rotating Bose condensates [36].)

The question now is whether there is any way one can probe the internal state of these anyons using the method explained in Section II. In other words, we need to see if different internal states can lead to different results for Eq.(2). Let us first consider the system of Fibonacci anyons. Braiding in such a system would differ from that of the \(k = 3\) parafermion state only by some phase factors which we will calculate later. For this system, one can always have either \(I\) or \(\varepsilon\) when two or more anyons are fused. (This fusion result is termed *anyonic charge*. This charge is conserved in the braiding transformation.) In this probe, since anyons in the interferometer are seen only as a single entity, they form a two-state system; in this sense, they can be considered to have formed a qubit.

If the result of fusing all the quasiholes in the interferometer is \(I\), \(M_n\) should be the same as it is for the Ising model in the diagram (4). Therefore \(\mathcal{H} \mathcal{M}_n \mathcal{H} \dagger = 1\); except for some phase factors, which we will deal with later, the AB oscillation should be the same as the case with no quasiholes in the island.

The situation is quite different if the fusion result is \(\varepsilon\). Since in this interferometer, evaluating \(\mathcal{H} \mathcal{M}_n \mathcal{H} \dagger\) involves taking standard closure to the worldlines of particles [15], just as in the case of the \(Z_2\) Ising model, the diagram to be evaluated is:

\[
\varepsilon \bigcirc \varepsilon
\]

(For our purpose, all propagators are set to unity, as they were in Section II. Furthermore, all unlinked loops are normalized to unity.)

From Eq.(17), one can see that the \(\varepsilon\) particle can be regarded as its own antiparticle. Therefore, the diagram (21) can be evaluated in exactly the same way as the diagram (7). Again one can consider the situation where two pairs of \(\varepsilon\) particles are created out of vacuum, and one \(\varepsilon\) particle from one pair is wound counterclockwise around one \(\varepsilon\) particle from the other pair; the diagram (21) is equal to the amplitude of the fusion result of both pairs being vacuum. From Eq.(2), one can see that this amplitude would be equal to the amplitude of the AB oscillation up to a possible phase factor. This in turn means that for the \(v = 12 = 5\) quantum Hall state, the internal state of quasiholes
FIG. 2: Diagrammatic representation of two different bases for the Hilbert space of four Fibonacci anyons. The first can be labeled by $j( (( ; a); b); c)$ and the second $j( (( ; a); ( ; b)); c)$. Here, $i$ indicates one Fibonacci anyon and $a;b;c$ indicate the fusion result of the anyons inside the bracket.

in the island region determine the amplitude of oscillation. Unlike in the case of the $\nu = 5=2$ quantum Hall state, the amplitude of oscillation is not determined by the number of quasiholes in the island.

Now the task is to calculate the diagram (21). The elementary braid transformations for Fibonacci anyons were derived in [20, 22], and the corresponding transformation matrices in the three-anyon system in the basis $j( (( ; a); b); c)$ is given in [6]. In particular the matrix for interchanging the second and third anyons in this basis is

$$
\sigma_2 = \begin{pmatrix}
0 & \tau e^{i\pi/5} & e^{i\pi/10} & 0 \\
\tau e^{i\pi/5} & 0 & \tau & 0 \\
e^{i2\pi/5} & 0 & 0 & e^{i2\pi/5}
\end{pmatrix}
$$

(22)

where $\tau = 2 \cos 2\pi/5 = (\frac{\sqrt{5} + 1}{2}) = 2$. The natural way to generalize this basis to a four-anyon system would be to take the $j( (( ; a); b); c)$ basis shown in Fig.2. If we are to consider the transformation $\hat{U}$ in which the second anyon winds around the third anyon, the last fusion result $c$ is unaffected by this transformation, so the transformation matrix will come out in a block-diagonal form in this basis. Therefore one obtains the following relation:

$$
h((( ; a); b); c)\hat{U}j((( ; a); b); c)= \delta_{cz} h((( ; a); b)\hat{F}_2^2j((( ; a); ( ; b)); c)$$

(23)

From Eq.(22) and Eq.(23) one can obtain the following matrix for this winding transformation in the $j( (( ; a); b); c)$ basis.
basis:

\[
\begin{array}{cccc}
& 0 & (1 \tau) & i5^{1+4}\tau e^{i\pi=5} \\
i5^{1+4}\tau e^{i\pi=5} & (1 \tau)e^{i2\pi=5} & 0 & 0 \\
0 & 0 & (1 \tau) & i5^{1+4}\tau e^{i\pi=5} \\
0 & 0 & i5^{1+4}\tau e^{i\pi=5} & (1 \tau)e^{i2\pi=5} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i4\pi=5}
\end{array}
\]

\[U = \begin{pmatrix}
0 & (1 \tau) & i5^{1+4}\tau e^{i\pi=5} \\
i5^{1+4}\tau e^{i\pi=5} & (1 \tau)e^{i2\pi=5} & 0 & 0 \\
0 & 0 & (1 \tau) & i5^{1+4}\tau e^{i\pi=5} \\
0 & 0 & i5^{1+4}\tau e^{i\pi=5} & (1 \tau)e^{i2\pi=5} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i4\pi=5}
\end{pmatrix}
\]

The diagram (21), however, needs to be calculated in the other basis of Fig. 2, for it is equal to

\[\h(( \sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp) = 0 \]

unless \(a = I\) and \(c = I\). Since there is only one state in the \((\sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp\) basis with \(a = I\) and \(b = I\) for some real number \(\delta\),

\[\h(( \sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp = \exp(\psi \delta) \h(( \sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp) \]

The amplitude for the AB oscillation when the result of fusing all quasiholes in the interferometer is \(\epsilon\) can be given now:

\[\h\Psi M_n \Psi^i = \epsilon \left( \begin{array}{c}
\epsilon
\end{array} \right) = \h(( \sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp = \exp(\psi \delta) \h(( \sigma_1; \psi_1 \psi_1'; \psi_2 \psi_2')_\perp
\]

Note that there is also the phase factor of -1.

(From Preskill [20], one can easily obtain the S matrix for the Fibonacci anyon model. This is possible because the braiding involving the vacuum is trivial and the S matrix here is a 2 \times 2 unitary matrix. The S matrix obtained in this way agrees with the amplitude of AB oscillation in Eq.(27).)

Some phase factors were lost by identifying the \(\sigma_i\)'s with \(\epsilon\) and the \(\psi_i\)'s with \(I\). Two points need to be made in order to figure these out. First, from Eq.(16), \(\sigma_1\) can be regarded as resulting from fusing the Fibonacci anyon \(\epsilon\) and the parafermion \(\psi_2\); the quasihole tunneling can be regarded as
tunneling of the composite of the parafermion \( \psi_2 \) and the Fibonacci anyon \( \epsilon \). Second, from the operator product expansion

\[
\psi_i(z)\epsilon(0) \propto \frac{1}{z} \sigma_3 \psi_i(0);
\]

one can see that the \( \psi_i \)'s are relatively local to \( \epsilon \) - that is, there is no accumulation of nontrivial phase factor when \( \psi_i \) winds around \( \epsilon \). Therefore the phase factors that need to be calculated comes entirely from \( \psi_i \)'s. In other words, the phase factors to be considered now originate from the \( \mathbb{Z}_3 \mathbb{Z}_3 \) charge of the parafermion.

If the number of quasiholes in the island is \( 3n+1 \), the fusion result is

\[
\sigma_1 \{z\}^{3n+1}_i \sigma \psi_2 (I + \epsilon) ;
\]

(If \( n = 0 \) there cannot be a fusion result of \( \psi_2 \), of course.) So the effect that had been ignored is that of encircling \( \psi_2 \) counterclockwise around each other. From Section III. this phase factor can be found from the mutual locality exponent - the phase factor is \( \exp(2\pi i = 3) \).

Similarly,

\[
\sigma_1 \{z\}^{3n+2} \sigma \psi_1 (I + \epsilon) ;
\]

Here the phase factor is the same as the one that arises when \( \psi_1 \) circles counterclockwise around \( \psi_2 \), or vice versa, which turns out to be \( \exp(2\pi i = 3) \).

On the other hand,

\[
\sigma_1 \{z\}^{3n} \sigma I + \epsilon ;
\]

gives rise to no further phase factor.

This shows that whereas the number of quasiholes in the interferometer does not determine the amplitude of the oscillation, it does induce phase shift in the oscillation. The phase shift due to the electric charge and magnetic flux of quasiholes now needs to be accounted for. For the quantum Hall state of Eq.(20), the quasihole has charge of \( e = 5 \) and flux of \( \Phi_0 = 3 \), the flux phase factor that arises when one quasihole is added is \( 2\pi = 15 \). Combining these two phase shifts, we see that there is a \( 2\pi = 3 + 2\pi = 15 = 4\pi = 5 \) phase shift per one quasihole.

We now have the result of Eq.(2) for the \( \nu = 12 = 5 \) parafermion quantum Hall state, keeping in mind, however, that the particle-hole inversion gives a negative sign to these phase shifts due to the quasiholes. In this case the fusion result of \( n \) quasiholes in the island region would have the
three-state Potts label of $\psi_1$ or $\psi_2$:

$$\sigma_{xx} \propto \frac{4\pi}{5} n \cos \alpha + \arg (\tau_2 = \tau_1) : \quad (32)$$

Otherwise the three-state Potts label of the fusion result would be $\varepsilon$, $\sigma_2$ or $\sigma_1$ and one would have

$$\sigma_{xx} \propto \frac{4\pi}{5} n \cos \alpha + \arg (\tau_2 = \tau_1) : \quad (33)$$

The first two phase terms, $\alpha + \arg (\tau_2 = \tau_1)$, can be varied by changing $B$. Since the quasiparticle charge is $e=5$, the period of AB oscillation is $5\Phi_0$.

As far as the phase of the oscillation is concerned, Eqs. (32) and (33) give the same result as the hierarchial $\nu = 12 = 5$ quantum Hall state; the non-abelian nature of the parafermion quantum Hall state is manifest only through the changed amplitude of the oscillation in Eq. (33).

V. DISCUSSION

As analyzed above, comparing the AB oscillation in the $\nu = 5/2$ and $12/5$ quantum Hall states shows that different fusion rules leads to qualitatively different results. The biggest difference is that, unlike the $\nu = 5/2$ case, there is no instance in the $\nu = 12/5$ case where the interference vanishes due to the number of quasiholes in the island region. In fact, in the case of $\nu = 12/5$ for any number of quasiholes in the interferometer, the change in the amplitude of oscillation due to the internal state of the quasiholes occur in the same manner.

This paper does not present a complete readout scheme for the internal state of the quasiholes of the $\nu = 12 = 5$ quantum Hall state to the extent done in the $p_x + ip_y$ superconductor [12]. Given a set of quasiholes, their total anyonic charge is conserved in any topological process. This means that, in the case that the quasiholes inside the interferometer are not in a state with definite total anyonic charge, it is not possible to obtain some of the phase relations between coefficients in the superposition state. Even if the total anyonic charge is fixed, as we saw in Section IV, once there are three or more quasiholes, they can be in a linear superposition of more than one internal state. It is impossible to probe the internal Hilbert space of such a quasihole cluster unless we can move the quasiholes adiabatically out of the interferometer region. In addition, if we are to obtain any phase relation between the coefficients, we must be able to braid quasiholes adiabatically. Without introducing such additional features, we cannot extract the coefficients of the internal quasihole superposition state from this interference experiment.
Lastly it should be noted that for these “qubits”, the initialization process is not known. So far, it is not clear how one can prepare quantum mechanically pure states; more work needs to be done in this direction.

**Note Added**

While this paper was in preparation, the authors learned about a similar work by P. Bonderson, K. Shtengel, and J. K. Slingerland [37]. They demonstrated that the monodromy matrix element can be written in terms of the S matrix. By obtaining S matrix for the general $\mathbb{Z}_k$ parafermion theory, they obtained the same conclusion on the phase and amplitude of AB oscillation presented in this paper.

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