Quantization of the mean decay time for non-Hermitian quantum systems

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We consider a quantum system with a general Hermitian Hamiltonian $\hat{H}$ and one decay channel, through which probability dissipates to the environment with rate $\Gamma$. When the system is initially prepared exactly in the decay state, the mean decay time $\langle T \rangle$ is quantized and equal to $w/(2\Gamma)$. $w$ is the number of distinct energy levels, i.e., eigenvalues of $\hat{H}$, that have overlap with the decay state, and is also the winding number of a transform of the resolvent in the complex plane. Apart from the integer $w$, $\langle T \rangle$ is completely independent of the system’s dynamics. We compute the complete decay time distribution in the strong and weak dissipation limit and show how its decay modes are obtained from an effective charge theory in general. We discuss our findings in the two-level atom in a continuous-wave field and in a disordered tight-binding model.

Introduction and Model The Schrödinger equation with an Hermitian Hamiltonian $\hat{H}$ is at the core of conventional quantum physics. Its unitary dynamics conserve probability. Yet, Hermiticity is not sacred. In many applications, loss effects are modeled by adding a non-Hermitian term:

$$i\hbar \frac{d}{dt}\psi(t) = \hat{H}\psi(t) - i\hbar\Gamma|\psi_d\rangle\langle\psi_d|\psi(t)$$

(1)

Among these applications are models with a finite lifetime of certain states, dissipative optical media [1–3], and quantum search and transport [4,7]. Non-Hermitian physics is used to estimate the fidelity of quantum channels [8], and the efficiency of light-harvesting complexes [9]. Eq. (1) is used in quantum jump approaches to fluorescence resonance to determine the time of the next photon emission [10–14]. While $\hat{H}$ mediates coherent transport in the system, the other term models dissipation of the wave function through the channel $|\psi_d\rangle$ with a rate $\Gamma$. An example would be a single excitation on a molecule/graph that is emitted through a localized sink $|\psi_d\rangle = |\tau_d\rangle$ [5]. The normalization of $|\psi(t)\rangle$ is a measure of the excitation’s survival in the system and decays. One fundamental quantity of interest is the expected decay time. We show in this letter that when the system is initially prepared in $|\psi(t = 0)\rangle = |\psi_d\rangle$, and when $\hat{H}$ has a finite discrete spectrum, the mean time of decay $\langle T \rangle$ is quantized and completely independent of the dynamical details:

$$\langle T \rangle = \frac{w}{2\Gamma}.$$  

(2)

It only depends on the dissipation rate $\Gamma$ and an integer winding number $w$. $w$ is equal to the number of distinct energy levels $E_l$ (i.e., the eigenvalues of $\hat{H}$), that have overlap with $|\psi_d\rangle$. In other words, $w$ is the number of energy levels connected to the decay channel. It is also the number of stationary points of a related electrostatic potential (see below).

The quantity $w$ can only be altered by a qualitative change of the energy spectrum. It is the winding number of a certain complex transformation of the resolvent $\hat{R}(s) := |\psi_d\rangle[s + i\hat{H}/\hbar]^{-1}|\psi_d\rangle$, see Fig. 1. This contrasts with the usual discussion of topological effects in condensed matter, where the corresponding invariants are often defined from tracing the Hamiltonian’s eigenstates over adiabatic cycles [15,16]. Here, the quantization is a consequence of the discrete nature of the energy spectrum. Hence a “change of the topological class” is achieved by either certain modifications of the decay channel, or of the Hamiltonian $\hat{H}$, e.g., through lifting its degeneracies. Both are demonstrated later on.

A similar quantization rule was reported in repeatedly strongly measured systems by Grünbaum et al. [19], Eq. (1) was proposed as a perturbative description for this setup [20–22]. Inspired by [19], we provide a firm basis for topological effects in non-Hermitian systems without any reference to collapse theory.
Quantization  To understand the behavior of $\langle T \rangle$, we need to solve Eq. (1) with the initial condition $|\psi_0\rangle$. This can be achieved via a Laplace transform. Let the solution be $|\psi(t)\rangle$. Its squared norm is the system’s survival probability that steadily decays. The negative derivative of this norm $F(t) := -d/dt \langle \psi(t)|\psi(t)\rangle = 2\Gamma|\langle \psi_0|\psi(t)\rangle|^2$ describes the instantaneous decay rate or the probability density function of the decay time. In the supplementary material (SM), we obtain the Laplace transform of “the wave function” $\hat{\Psi}(t) := \langle \psi_0|\psi(t)\rangle$, namely $\hat{\Psi}(s) := \int_0^\infty dt e^{-st} \langle \psi_0|\psi(t)\rangle$. This is sufficient to obtain $\langle T \rangle$, which is the first moment of the distribution $F(t)$:

$$\langle T \rangle = \int_0^\infty dt \int s \frac{ds}{2\pi i} \hat{\Psi}^*(s)(-s)(-\frac{1}{\pi i})\hat{\Psi}(s). \quad (3)$$

Here $f^*(s) = [f(s^*)]^*$, $z^*$ is the complex conjugate of $z$, and we have used standard rules to write the time-domain integral as a Laplace-domain integral. The complex integral is taken along the Bromwich path $B = \{t^* + iw|\omega \in \mathbb{R}\}$ which lies immediately to the right of the imaginary axis and gathers all residues of poles in the complex left half plane. Then,

$$\hat{\Psi}(s) = \frac{R(s)}{1 + GR(s)}, \quad R(s) := \langle \psi_0|\frac{1}{s + i\hbar H}|\psi_0\rangle, \quad (4)$$

where $R(s)$ is the Hamiltonian’s resolvent. Using the fact that $R^*(-s) = -R(s)$ for $s = i\omega$, we can rewrite the integrand in Eq. (3) as

$$\langle T \rangle = 2\Gamma \int_B \frac{ds}{2\pi i} \frac{R(s)R^*(s)}{(1 - GR(s))(1 + GR(s))} = I(s). \quad (5)$$

We call the integrand $I(s)$. To complete our argument, we need to specify the Hamiltonian. We assume that $H = \sum_i E_i \sum_{m=1}^{N_i} |E_{i,m}\rangle\langle E_{i,m}|$, and thus:

$$R(s) = \sum_{i=1}^{w} \frac{p_i}{s + \frac{i}{\hbar}E_i}, \quad (6)$$

where $E_i$ are the $g_i$-fold degenerate energy levels with eigenstates $|E_{i,m}\rangle$. $p_i := \sum_{m=1}^{N_i} |\langle E_{i,m}|\psi_0\rangle|^2$ is the overlap of $|\psi_0\rangle$ with the $E_{i}$-eigenspace, such that $\sum_i p_i = 1$. We have finite-dimensional Hamiltonians in mind. However, only the energy levels with overlap in $|\psi_0\rangle$ are relevant. Therefore, our system could have a continuous spectrum or more energy levels than those appearing in Eq. (4). These do not matter. The only important fact is that there are finitely many, namely $w$, discrete energy levels that appear in $|\psi_0\rangle$.

$\langle T \rangle$ is now computed by brute force: We plug Eq. (6) into Eq. (5) and use the residue theorem. This involves finding the poles $s_{p,l}$ of the integrand $I(s)$ of Eq. (5), which lie in the complex left-half plane. At these second order poles, the resolvent assumes the value

$$R(s_{p,l}) = -1/\Gamma. \quad (7)$$

The residue of $I(s)$ at $s_{p,l}$ only depends on $R(s_{p,l})$ but not on any of $R(s)$’s derivatives; it equals $\text{Res}_{s_{p,l}} I(s) = 1/(4\Gamma^2)$ for all poles. Since all residues are equal, we find that $2\Gamma \langle T \rangle$ is an integer, namely the number of solutions of $R(s) = -1/\Gamma$. In light of Eq. (6), this number must be $w$, thus one has Eq. (2). Note that the singularities of $R(s)$ at $s = -\Omega E_i/\hbar$ play no role, because they cancel in $I(s)$.

An alternative path to Eq. (2) starts with the resolvent $R(s)$ and the realization that $I(s)$ can be written as a logarithmic derivative:

$$I(s) = -\frac{1}{4\Gamma^2} \frac{d}{ds} \ln C(\Gamma R(s)), \quad (7)$$

where $C(z) = M(z)e^{M(z)}$ and $M(z) = (z - 1)/(z + 1)$. According to the argument principle, $2\Gamma \langle T \rangle$ is given by the winding number $w$ of the path $C(\Gamma R(B))$ around the origin, which is depicted in Fig. 1. To determine this number, we note that $f(E) := -iR(iE/\hbar)$ is always real for real $E$ (due to the Hermiticity of $H$). This function maps each interval $[-E_{\text{ini}}, -E_{\text{fin}}]$ as well as the outer segment $[-\infty, -E_{\text{ini}}) \cup [-E_{\text{fin}}, \infty)$ to the complete real line. The figure in the SM demonstrates this nicely. Therefore, the resolvent $R(s)$ maps $B$ exactly $w$ times to itself. The Möbius transform $M(z)$ maps $B$ to the unit circle, which gets mapped to another curve circling the origin by $ze^z$. The direction is reversed, hence the winding number is $-w$, and $\langle T \rangle$ is related to a “topological invariant” of the curve $C(\Gamma R(s))$. Again we find Eq. (2).

Two-Level system  Let us now consider an elementary quantum optics model, an atom with an excited ($e$) and a ground state ($g$) in a continuous-wave (cw) field close to resonance. Under the rotating wave approximation,
The poles $s_{p,l}$ (+) are the stationary points of a 2D-electrostatic field $V(s)$, Eq. [9], heat map, stream lines equal gradient, which consists of a constant force together and point charges at $-E_l/\hbar$ on the imaginary axis of magnitude $p_l$ (blue circles). Since all charges are positive, we have $w$ stationary points from electrostatics. Here for the Hamiltonian [15] with $L = 6$ and $\epsilon = 0$, where $\Gamma = 4\gamma/\hbar$.

the Schrödinger equation for this system reads [23]:

\[
i\begin{pmatrix}
\dot{\psi}_e(t) \\
\dot{\psi}_g(t)
\end{pmatrix} = \begin{pmatrix}
\frac{i}{2} & \Omega \\
\Omega & -\frac{i}{2}
\end{pmatrix} \begin{pmatrix}
\psi_e(t) \\
\psi_g(t)
\end{pmatrix} - \begin{pmatrix}
0 & \Gamma \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\psi_e(t) \\
\psi_g(t)
\end{pmatrix}.
\]

(8)

Here, $\delta$ is the detuning, $\Omega$ is the Rabi frequency and $\Gamma$ is the inverse lifetime of the excited state. Our theory claims that the mean decay time depends only on $\Gamma$ and the number $w$ of energy levels that have overlap with $|\psi_d\rangle = |e\rangle = (1,0)^T$. Since the eigenstates of the Hermitian part of Eq. [5] are given by $|E_{\pm\nu}\rangle = N_{\nu}[|g\rangle + (\delta \pm \sqrt{\delta^2 + 4\Gamma^2})|e\rangle]/2$, with appropriate normalization $N_{\nu}$, we find $w = 2$ and the obvious result $(T) = \Gamma^{-1}$. The mean decay time is equal to the inverse life time. The exception occurs when the $\epsilon$ field is turned off and $\Omega$ vanishes. Then $|e\rangle = |\psi_d\rangle = |E_{+\nu}\rangle$ becomes an eigenstate of $\hat{H}$ and $w = 1$. $(T)$ is halved at this point, thus discontinuous. Although the change in $(T)$ is discontinuous, the fluctuations of the mean decay time diverge as $\Omega$ vanishes, as presented in Fig. 2, Var[T] is computed similarly to $\langle T \rangle$ in the SM. This divergence is due to the tiny portion of the wave function that escapes to the ground state before it decays. There it spends a long time before returning to $|\psi_d\rangle = |e\rangle$, where it can decay. To see that this picture is actually generic for large $\Gamma$, we set out to find the complete distribution of decay times, which can be computed from the poles $s_{p,l}$ of $\Psi(s)$.

**Electrostatic analogy** Finding these poles is generally a hard task. Separately considering the real and imaginary part of $R(s) + 1/\Gamma = 0$ reveals that the poles $s_{p,l} = x + iy$ are given by the stationary points $\nabla V = 0$ of a 2D-electrostatic potential (similarly to [19]):

\[
V(x, y) := \frac{x}{\Gamma} + \sum_{l=1}^{w} p_l \ln \frac{1}{\Gamma} \sqrt{x^2 + (y + \frac{E_l}{\hbar})^2}.
\]

(9)

$w$ 2D-point charges with magnitude equal to the overlap $p_l$ are placed at $-E_l/\hbar$ on the $y$-axis, see Fig. 3. Additionally there is a $\Gamma$-dependent constant force. The gradient of $V(x, y)$ yields $R(x + iy) + 1/\Gamma$. This picture enables one to draw many general conclusions about the position of the poles. For instance they all must have negative real part due to the constant force. In general, a small charge $p_l$ will have a pole close to $-iE_l/\hbar$. Furthermore, two very close charges with small $|E_l - E_{l+1}|$ will have a pole in their vicinity. Refs. [24] fully exploit this method, here we use it to find the poles $s_{p,l}$ in the weak and strong dissipation limit.

**Decay time distribution** The aforementioned $s_{p,l}$ appear as simple poles in $\Psi(s)$ and thus give the full decay time distribution $F(t) = 2\Gamma |\Psi(t)|^2$ via the inverse Laplace transform:

\[
\Psi(t) := \sum_{l} \text{Res}_{s_{p,l}} \Psi(s)e^{st} = \sum_{l} e^{s_{p,l}t} \frac{R(s_{p,l})}{\Gamma R(s_{p,l})}.
\]

(10)

In the weak dissipation limit $\Gamma \to 0$, we expand $-\Gamma = 1/R(s_{p,l})$ around $-iE_l/\hbar$. This reveals the poles $s_{p,l} = -p_l \Gamma - iE_l/\hbar$ to be close to the charges. Together with $R(s_{p,l}) = -1/\Gamma$ and $R'(s_{p,l}) = -1/(p_l \Gamma^2)$, we obtain

\[
\Psi(t) \sim \sum_{l=1}^{w} p_l e^{-t(p_l \Gamma + i \frac{E_l}{\hbar})}, \quad \Gamma \to 0.
\]

(11)

When $\Gamma$ is small, the coherent dynamics are only minimally disturbed by a slow drain from every energy level. In the strong dissipation limit $\Gamma \to \infty$, the poles are found from the Taylor expansion of $-1/\Gamma = R(s_{p,l})$ around the zeros $-i\omega_l$ of the resolvent $R(s)$. Considering real and imaginary part separately and omitting higher order terms in $1/\Gamma$, one finds $s_{p,l}^\dagger \sim -i\omega_l - \lambda_l/\Gamma$, where

\[
\lambda_l := \frac{1}{R(-i\omega_l)} = \left[ \langle \psi_d | \frac{1}{\omega_l - \frac{1}{\Gamma} \hat{H}} | \psi_d \rangle \right]^{-1}.
\]

(12)

This yields $w - 1$ poles $s_{p,l}^\dagger$, $1 \leq l < w$, one for each zero of $R(s)$, see the figure in the SM. The remaining pole is obtained from a similar expansion around $s = -\infty$:

\[
s_{p,0}^\dagger \sim -\Gamma - i\omega_0, \quad \omega_0 := \frac{1}{\Gamma} \langle \psi_d | \hat{H} | \psi_d \rangle,
\]

(13)

Plugging the poles in Eq. [10] together with $R(s_{p,0}) = -1/\Gamma$, $R'(s_{p,0}) \sim R'(-i\omega_0) = 1/\lambda_0$, and $R'(s_{p,l}) = -1/\Gamma^2$ we obtain for $\Gamma \to \infty$:

\[
\Psi(t) \sim e^{-t(\Gamma + i\omega_0)} = \frac{1}{\Gamma^2} \sum_{l=1}^{w-1} \lambda_l e^{-t(\frac{2}{\Gamma} + i\omega_l)}.
\]

(14)
Here $\gamma$ is the hopping energy, $\epsilon_x$ are random on-site energies, $|x\rangle$ are position eigenstates on a ring with $L$ sites and periodic boundary conditions $|x + L\rangle = |x\rangle$. $\epsilon_x$ is drawn uniformly from $[-\epsilon, \epsilon]$. We equip one site ($r_d$) with a sink and set $|\psi_d\rangle = |r_d\rangle$ in Eq. (14). This equation with Hamiltonian (15) is then solved numerically.

Let’s assume that $L$ is even. In the absence of disorder, $\epsilon = 0$, almost all energy levels are two-fold degenerate, and it is easy to see that $w = 1 + L/2$. This value changes abruptly when even the slightest bit of disorder is introduced ($\epsilon > 0$) and the degeneracy is broken. In that case the number of energy levels that have overlap with $|\psi_d\rangle = |r_d\rangle$ assumes the much higher value $w = L$. However, the drastic change in the mean is accompanied with seemingly moderate adjustments in the distribution $F(t)$ when disorder is turned on. After some large crossover time, disorder manifests in very slowly decaying tails, see Fig. 4. A similar crossover effect was reported in connection to a magnetic field [25]. Although the number of poles $w$ and thus the number of modes changes abruptly when disorder is introduced, the actual values of the decay rates, i.e. of the $\lambda$s, only change marginally, see the inset of Fig. 4. However, the new, very slow decay rates dominate the fluctuations $\text{Var}(T)$ of the decay time.

Furthermore, Anderson localization has no influence on our result Eq. (2). Although the overlaps $p_{l}$ with the localized eigenstates are exponentially small, they are non-zero. As a consequence all eigenstates participate in Eq. (6). The actual decay dynamics in $F(t)$ however, will be ultra-slow and one will find gigantic fluctuations of $T$ in disordered systems.

**Summary** We investigated the mean decay time $\langle T \rangle$ through a single channel $|\psi_d\rangle$ in an otherwise Hermitian system. We showed that $\langle T \rangle$ is quantized by the number of distinct energy levels that have overlap with the decay channel. It was shown how the poles, which determine the dissipation dynamics, can be obtained from the stationary points of an electrostatic charge field. These are used to find the complete distribution of decay times in the strong and weak dissipation limit. We demonstrated our findings in a two-level model and in a tight-binding model with and without disorder.

**Discussion** Nowhere in this letter did we discuss the total dissipation probability $P_d = \int_0^\infty dt F(t)$. However, since the system is prepared in the decay state $|\psi_d\rangle$ initially, the theory of invariant subspaces ensures $P_d = 1$, see [9]. This can be confirmed from a small or large $\Gamma$ expansion of Eqs. (11) and (14), respectively.

As mentioned, Refs. [19, 26] reported a quantization of the “recurrence time” $\langle T \rangle$ in stroboscopically measured systems. Here the system evolves unitarily, but is repeatedly projectively measured in state $|\psi_d\rangle$ in order to find its first detected return, see also [21-27, 29]. One finds $\langle T \rangle_{\text{strob}} = w \tau$, where $\tau = 2/\Gamma$ is the time between successive detection attempts. As Refs [20-22, 30-32] suggested, both setups are closely related in the strong dissipation limit $\Gamma \to \infty$, which will be explored in a longer publication [33].

Drawing further from the analogy, we can expect a rational quantization of $\langle T \rangle$ when multiple decay channels are considered (see [34] for the stroboscopic case). A detailed investigation of the non-Hermitian system with multiple decay channels is left to the future.

As our second example showed, our topological invariant does not ensure “protection from disorder”, but rather “protection of disorder”. $\langle T \rangle$ is very robust in systems void of degeneracy; a general feature that we also demonstrated in the two-level system.

Our most remarkable result is how $\langle T \rangle$ completely lacks any dependence on the system’s dynamical details. This is due to the special preparation state $|\psi(t = 0)\rangle = |\psi_d\rangle$. Other initial states lead to a complex behavior of $\langle T \rangle$ that depends on all details of the system, see Ref. [35]. The fundamental two-level system should prove a wonderful experimental playground for demonstration.
The excited state can easily be prepared by application of a π-pulse to a ground state atom. The decay time is obtained from registering the time between the pulse and the first photon emission and can be compared present and absent additional Rabi driving.

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Appendix A: Derivation of the formal solution

In this section, we derive the formal solution of Eq. (1).

This derivation closely follows Ref. [4]. Apply a Laplace transform to both sides of Eq. (1). We write $|\psi(s)| = L[|\psi(t)|; s] := \int_0^\infty dt e^{-st} |\psi(t)|$ and use the initial condition $|\psi(t=0)| = |\psi_0|$:

$$ih[s|\psi(s)| − |\psi_d|] = \hat{H} |\psi(s)| − i\hbar |\psi_d\rangle \langle \psi(s)| . \tag{A.1}$$

Rearranging the equation yields:

$$|\psi(s)| = \left[s + i\frac{\hbar}{\Gamma}\right]^{-1} \left[|\psi_0| − i\Gamma |\psi_d\rangle \langle \psi_d|\psi(s)|\right] . \tag{A.2}$$

When the equation is multiplied with $\langle \psi_d|\psi(s)|$ from the left and solved for $\langle \psi_d|\psi(s)| = \Psi(s)$, Eq. (4) is obtained.

To see how $F(t)$ and $\Psi(t)$ are related, one considers the survival probability, whose negative derivative is equal to $F(t)$. The survival probability is equal to the not yet decayed probability, i.e. equal to the norm of the state $|\psi(t)|$. We find:

$$F(t) = −\frac{d}{dt} \langle \psi(t)|\psi(t)\rangle \tag{A.3}$$

$$= \left(\frac{d}{dt} \langle \psi(t)|\psi(t)\rangle\right) − \langle \psi(t)|\frac{d}{dt} |\psi(t)\rangle

= −\langle \psi(t)| \left[i\hbar \hat{H} − i\frac{\hbar}{\Gamma} \hat{H} − 2\Gamma |\psi_d\rangle \langle \psi_d|\psi(t)\right].$$

Hence $F(t) = 2\Gamma|\Psi(t)|^2$ as stated in the main text. To obtain $⟨T⟩$ in Laplace domain, we write

$$⟨T⟩ = \lim_{σ→0} 2\Gamma \int_0^\infty dt e^{-σt} |\Psi(t)|^* [\Psi(t)] \tag{A.4}$$

in terms of a Laplace transform. We have $|\Psi(t)|^* → \Psi^*(s)$ and $[\Psi(t)] → (−d/ds)Ψ(s)$. Next we use the fact that products in time domain transform to convolutions in image domain:

$$⟨T⟩ = \lim_{σ→0} 2\Gamma \int_0^∞ s^2 \Psi^*(s) \left(−\frac{d}{ds}\right)^m Ψ(s), \tag{A.5}$$

where the integral only considers the poles of $Ψ(s)$. Taking the limit yields Eq. (3).

Appendix B: The two-level system

The Schrödinger equation is given by Eq. (8) of the main text. The energy levels (i.e. the eigenvalues of $\hat{H}$) are:

$$E_± = ±\hbar|Ω| \sqrt{1 + x^2} \quad \tag{B.1}$$

The corresponding eigenstates are:

$$|E_±⟩ = \frac{1}{\sqrt{2(1 + x^2 ± \sqrt{1 + x^2})}} \left(x ± \sqrt{1 + x^2}\right), \tag{B.2}$$

where $x := δ/(2Ω)$. From here we obtain with $|ψ_d⟩ = \langle \psi| = (1,0)^T$:

$$R(s) = \frac{|⟨ψ_d|E_+⟩|^2}{s^2 + \frac{4}{\hbar}E_+} + \frac{|⟨ψ_d|E_−⟩|^2}{s^2 + \frac{4}{\hbar}E_−} = \frac{s − iδ}{s^2 + \frac{4}{\hbar} + Ω^2}, \tag{B.3}$$

which yields two poles $R(s_{p±}) = −1/Γ$:

$$s_{p±} = ±\frac{1}{2} ± \sqrt{\frac{1}{4}(1 + iδ)^2} − Ω^2. \tag{B.4}$$

The wave function in Laplace domain is given by $Ψ(s) = R(s)/(1 + ΓR(s))$ and its conjugate by $Ψ^*(−s) = −R^*(s)/(1 − ΓR(s))$. The normalization $P_{det} = \int_0^∞ dt F(t)$ of the decay time distribution $F(t) = 2Γ|Ψ(t)|^2$ is given by the integral:

$$P_{det} = \frac{2Γ}{2πi} \int_B ds Ψ^*(−s)Ψ(s) \tag{B.5}$$

which can be evaluated using the residue theorem. The relevant poles are $s_{p±}$ in the left half complex plane. Higher moments $⟨T^m⟩ = \int_0^∞ dt t^m F(t)$ are given by similar integrals:

$$⟨T^m⟩ = \frac{2Γ}{2πi} \int_B ds Ψ^*(−s) \left(−\frac{d}{ds}\right)^m Ψ(s) \tag{B.6}$$

and similarly evaluated using the residue theorem. We find

$$P_{det} = 1, \quad ⟨T⟩ = \frac{1}{Γ}, \quad \text{Var}[T] = \frac{Γ^2 + δ^2 + 2Ω^2}{2Γ^2Ω^2}, \tag{B.7}$$

where of course $\text{Var}[T] = ⟨T^2⟩ − ⟨T⟩^2$. These are the quantities plotted in Fig. 2.

An exception occurs for vanishing Rabi frequency. Then the decay state is an eigenstate of $\hat{H}$ and $w$ is equal to one. The resolvent reads $R(s) = 1/(s + iδ/2)$ and there is only one pole $s_p = −1 − iδ/2$. In this case, we find

$$P_{det} = 1, \quad ⟨T⟩ = \frac{1}{2Γ}, \quad \text{Var}[T] = \frac{1}{4Γ^2}. \tag{B.8}$$
FIG. 5: The resolvent $R(s)$, defined by Eq. (6) maps the imaginary axis $w$ times to itself, where $w$ is given by the number of energy levels that have overlap with the decay state. Here we plot $R(s)$ for one realization of the disordered tight-binding model, Eq. (15), with $L = 6$ and disorder strength $\epsilon = \gamma$. The dash-dotted vertical lines are the negative energy levels $-E_l/\hbar$. Each of the intervals $[-E_{l+1}/\hbar, -E_l/\hbar)$ and the outer segments get mapped once to the complete imaginary axis (different colors). The inset shows the curve $C(\Gamma R(B))$ in the complex plane (i.e. Fig. 1 from the main text), whose winding number $w$ determines the mean decay time by Eq. (2).

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