On the Hawking-Page Transition and the Cardy-Verlinde formula

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The free energies of the conformal field theories dual to charged adS and rotating adS black holes show Hawking-Page phase transition. We study the transition by constructing boundary free energies in terms of order parameters. This is done by employing Landau’s phenomenological theory of first order phase transition. The Cardy-Verlinde formula is then showed to follow quite naturally. We further make some general observations on the Cardy-Verlinde formula and the first order phase transition.
In recent literature, many evidences suggest a correspondence between gravitational physics in anti-de Sitter (adS) space-time and a class of conformal field theories (CFTs) in one lower dimension \[1\]. This holographic duality, in turn, allows us to explore various space-time physics in terms of field theories which are non-gravitational in nature. In particular, the thermodynamics of various black holes, in adS space, is understood as the high-temperature phase of CFTs. The Hawking-Page (HP) phase transition \[2\] of these black holes is then found to correspond to a confining-deconfining phase transition of \(N = 4\) Yang-Mills gauge theory \[3\]. Furthermore, Verlinde observed \[4\] that for these CFTs, the entropy can be expressed as a Cardy like formula of two dimensional CFTs \[5\]. This is now commonly known as Cardy-Verlinde (CV) formula. Subsequently, the consequences of having such an entropy formula for boundary CFTs were explored in various directions in the literature \[6\] - \[17\]. In particular, in \[14\], the HP transition of the boundary CFT related to adS-Schwarzschild black hole was studied using Landau phenomenological theory of first order phase transition. The Cardy-Verlinde formula was shown to arise from such an analysis in a quite elegant manner. This motivates us to explore further the relation between such an entropy formula and the first order phase transition.

In this letter, we first construct the boundary free energies, in terms of the order parameter and the other relevant thermodynamic variables, for the charged and the rotating adS black holes. For each of these black holes, the free energy is then found to interpolate between two phases around the HP transition point. Next, we derive the CV formula from our expressions of free energies. This note ends with some general observations on the CFTs with first order phase transitions. We argue here that the Verlinde’s scaling argument, along with the knowledge of the temperature and the entropy of the boundary CFT at the phase transition point, is enough to completely fix the extensive and the sub-extensive part of the boundary energy.

We start by briefly reviewing the work of \[4\] and \[14\] as this will be useful for our later purpose. As emphasised in \[4\], for CFT, in a region with boundary or in a curved manifold, the energy is not an extensive quantity. Rather, it has an extensive part \((E_E)\) and a sub-extensive part \((E_c)\), often called the casimir energy. The total energy is then defined as

\[
E = E_E + \frac{1}{2} E_c \quad \text{with} \quad E_c = (n-1)(E + pV - TS) = (n-1)(F + pV),
\]

in \(n\) space-time dimensions. Here, \(F\) is the free energy. A rather general scaling argument then suggests that \(E_E\) and \(E_c\) depend on the entropy as

\[
RE_E = cS^{1+\frac{1}{n-1}}, \quad RE_c = dS^{1-\frac{1}{n-1}}.
\]

Here, \(c\) and \(d\) are two constants that depend on the detail of the CFT in question. \(R\) is typically the linear size of the system. By specifying the bulk, one can explicitly determine the constants through adS/CFT correspondence. Furthermore, by expressing the entropy (using \[14\]) in terms of \(E_E\) and \(E_c\), one gets the CV formula for the boundary entropy. We will show this following \[14\]. We start by considering a \(n+1\) dimensional adS black hole of the form

\[
ds^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2 d\Omega^2_{n-1},\]

with

\[
h(a) = 1 - \frac{m}{a^{n-2}} + \frac{a^2}{l^2}.
\]
The parameter $m$ is related to the ADM mass $M$ as

$$M = \frac{(n - 1)\omega_{n-1}m}{16\pi g}, \quad (5)$$

where $\omega_{n-1}$ is the volume of the unit $n-1$ dimensional sphere, and $g$ is the Newton’s constant in $n+1$ dimensions. The entropy and the temperature are given by

$$\tilde{S} = \frac{a_+^{n-1}\omega_{n-1}}{4g}, \quad \tilde{T} = \frac{n}{4\pi l^2}a_+ + \frac{(n - 2)}{4\pi a_+}, \quad (6)$$

where $a_+$ is the location of the horizon. For large $a$, on a hypersurface of constant $a$, the metric behaves as

$$ds^2 = \frac{a^2}{l^2}(-dt^2 + l^2d\Omega_{n-1}^2), \quad (7)$$

which is conformally equivalent to $S^{n-1} \times \mathbb{R}$ with $S^{n-1}$ being of radius $l$. More generally, by rescaling the time as $\tau = \frac{R}{l}t$, we can rewrite the metric as

$$ds^2 = \frac{a^2}{R^2}(-d\tau^2 + R^2d\Omega_{n-1}^2), \quad (8)$$

such that the $S^{n-1}$ is now having radius $R$. This scaling will enable us to vary the volume of $S^{n-1}$ as we will see later. The holographic principle then suggests that the theory living on the bulk which asymptotes to $\mathcal{S}$ is holographically dual to a CFT living on $\mathcal{S}$ with $R$ being the radius and $\tau$ being the time. This prescription, in turn, allows us to identify thermodynamic quantities of the CFT at high temperature with the corresponding thermodynamic quantities of the bulk black hole. Following the correspondence, we get the energy $E$ and temperature $T$ of the CFT on $S^{n-1}$ as

$$E = \frac{Ml}{R}, \quad T = \frac{\tilde{T}l}{R}, \quad S = \tilde{S}, \quad (9)$$

where $M$ and $\tilde{T}$ are given in $\mathcal{S}$ and $\mathcal{S}$ respectively. From $\mathcal{S}$, it is now easy to compute casimir energy $E_c$ and $E_E$. $E_c$, for example, is given by

$$E_c = \frac{(n - 1)\omega_{n-1}a_+^{n-2}}{8\pi g R}. \quad (10)$$

In deriving $E_c$, we have used the fact that the free energy is $F = E - TS$ and the pressure is $p = -(\partial F/\partial V)_\beta$ with $\beta$ being the inverse temperature and $V = \omega_{n-1}R^{n-1}$ is the volume of $S^{n-1}$. The equations $\mathcal{S}$, $\mathcal{S}$ and $\mathcal{S}$ then immediately show that the boundary entropy satisfies

$$S = \frac{2\pi R}{n-1}\sqrt{E_c(2E - E_c)} \quad (11)$$

This formula is indeed the Cardy’s entropy formula in two dimension once we identify $RE_c \sim c$ (the central charge) and $RE \sim L_0$ (the zero mode of the Virasoro generators) $\mathcal{S}$.

The above formula can be re-derived in an elegant manner by exploiting the first order nature of the HP transition for the adS-Schwarzschild black hole. Following $\mathcal{S}$, we can calculate
the boundary action for the bulk metric given in [8]. Without repeating the calculation, we give the result here:

\[ I = \frac{\kappa \omega_{n-1} \hat{a}^{n-1}(1 - \hat{a}^2)}{4n \hat{a}^2 + 4(n - 2)}. \]  

(12)

In the above expression, we have written the Newton’s constant as \( g = \frac{\mu^{n-1}}{\kappa} \) and introduced dimensionless quantity \( \hat{a} = \frac{a}{l} \). One then finds the free energy at the boundary \( F_{BH} = \beta^{-1} I \) as

\[ F_{BH} = \frac{\kappa \omega_{n-1} \hat{a}^{n-2}(1 - \hat{a}^2)}{16\pi R}, \]

(13)

The boundary field theory undergoes HP phase transition at a critical value \( \hat{a} = 1 \). This corresponds to a critical temperature \( T_c \) that can be obtained from (9). Below the critical temperature, the system prefers pure adS bulk geometry than adS black hole. While for temperature above \( T_c \), it is the black hole phase which lowers the free energy.

As usual for the first order phase transition [18], the entropy changes discontinuously around the transition point \( T = T_c \). Following [14], we write the free energy as a function of order parameter (\( \hat{a} \)) and temperature in the following form *

\[ F(\hat{a}, T) = \frac{1}{2} E_c(\hat{a})(1 - 2\beta_c T \hat{a} + \hat{a}^2), \quad \text{where} \quad \beta_c = \frac{1}{T_c} = \frac{2\pi R}{n - 1}. \]

(14)

Substituting \( E_c \) and \( T \) from (10) and (9) respectively, the above expression for the free-energy reduces to (13). We note that for any \( E_c \), which is a monotonously growing function of \( \hat{a} \) with \( E_c(0) = 0 \), the free energy (14) describes a first order phase transition from \( \hat{a} = 0, \ T < T_c \) to \( \hat{a} > 0, \ T > T_c \). Furthermore, from the expression of free energy, it follows that the entropy and the energy are given by

\[ S = \beta_c E_c(\hat{a})\hat{a}, \]

\[ E = \frac{1}{2} E_c(\hat{a})(1 + \hat{a}^2). \]

(15)

We then have

\[ S = \beta_c \sqrt{E_c(2E - E_c)}. \]

(16)

This is same as the entropy relation given in (11).

In the next part of the paper, we would like to carry out a similar exercise for various other adS black holes. The examples that we will study here are the charged adS black hole and the rotating adS black hole. All these black holes show HP transition within certain range of parameters. However, due to the presence of various other thermodynamic potentials, the

*This expression can be derived following the standard Landau’s phenomenological construction of free energy for theories with first order phase transition. We first make an ansatz,

\[ F(\hat{a}, T) = \frac{\kappa \omega_{n-1} \hat{a}^{n-2}}{R}(p \hat{a}^{n-2} - q T \hat{a}^{n-1} + r \hat{a}^n), \]

where \( p, q, r \) are three constants. \( \omega_{n-1} \) is the volume of unit \( (n - 1) \) sphere. The constants can be determined as follows. Condition of extremisation, \( \partial F/\partial \hat{a} = 0 \), must reproduce the temperature \( T \) given in (9). This determines two of the three constants in terms of the third. Furthermore, the condition that the substitution of \( T \) from (9) in \( F \) should give (13) determines the third constant.
construction of free energies as a function of order parameters in the Landau’s phenomenological frame work, is somewhat involved. We will elaborate on these constrictions in the next section of the paper. We will also see that for all these cases, CV formula appears in a quite natural manner.

**Charged adS black holes:** The metric of a charged black hole in \((n+1)\) adS(RNadS) space-time is given by

\[
ds^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2d\Omega^2_{n-1},
\]

with

\[
h(a) = 1 - \frac{m}{a^{n-2}} + \frac{q^2}{a^{2n-4}} + \frac{a^2}{l^2}.
\]

The parameters \(m\) and \(q\) are related to ADM mass \(M\) and charge \(\tilde{Q}\) as

\[
M = \frac{(n-1)\omega_{n-1}m}{16\pi g},
\]

\[
\tilde{Q} = \frac{\sqrt{2(n-1)(n-2)}\omega_{n-1}q}{8\pi g}.
\]

Here, \(\omega_{n-1}\) is again the volume of the unit \(n-1\) sphere. The gauge potential is given by

\[
A_t = -\frac{q}{\alpha a^{n-2}} + \tilde{\Phi}, \text{ with } \alpha = \sqrt{\frac{2(n-2)}{n-1}}.
\]

In the above equation, \(\tilde{\Phi}\) is a constant. We work with \(\tilde{\Phi}\) such that the gauge potential at the horizon \(a = a_+\) is zero. Hence,

\[
\tilde{\Phi} = \frac{q}{\alpha a^{n-2}}.
\]

At the horizon of the black hole \(h(a_+) = 0\). This condition can be re-written as

\[
m = a_+^{n-2} + \frac{q^2}{a_+^{2n-4}} + \frac{a_+^n}{l^2}.
\]

This expression will be of use later. The temperature and the entropy associated with the configuration are given by

\[
\tilde{T} = \frac{n}{4\pi l^2}a_+ + \frac{(n-2)(1-\alpha^2\tilde{\Phi}^2)}{4\pi a_+}, \quad \tilde{S} = \frac{\omega_{n-1}a_+^{n-1}}{4g}.
\]

The on-shell action at fixed gauge potential was calculated for example in [19] and is given by

\[
I = \frac{\omega_{n-1}\tilde{\beta}}{16\pi gl^2} \left(l^2a_+^{n-2}(1-\alpha^2\tilde{\Phi}^2) - a_+^n\right)
\]

where \(\tilde{\beta}\) is the inverse of the temperature given in (23). The grand-canonical potential \(\tilde{G}(\tilde{T}, \tilde{\Phi})\) is then given by

\[
\tilde{G}(\tilde{T}, \tilde{\Phi}) = \frac{I}{\tilde{\beta}} = \frac{\omega_{n-1}(1-\alpha^2\tilde{\Phi}^2)a_+^{n-2}}{16\pi g} - \frac{\omega_{n-1}a_+^n}{16\pi gl^2}.
\]
Following the adS/CFT prescription, $n-1$ dimensional CFT on the boundary has the temperature, chemical potential, entropy and energy

$$
T = \frac{l}{R} \tilde{T}, \quad Q = \tilde{Q}, \quad \Phi = \frac{l}{R} \tilde{\Phi}, \quad S = \tilde{S}, \quad E = \frac{l}{R} M.
$$

(26)

and the Gibbs potential is then

$$
G = \omega_{n-1} l (1 - \alpha^2 \tilde{\Phi}^2) \hat{a}^{n-2} - \frac{\omega_{n-1} a_n}{16\pi g R} \hat{a}^{n-1} + \frac{\kappa (n-1) \omega_{n-1} a_n}{16\pi R} \hat{a}^{n}.
$$

(27)

Let us now consider $G$ at a fixed $\tilde{\Phi} < 1/\alpha$. It is straightforward to check that $G$ is negative for

$$
a_+ > l \sqrt{1 - \alpha^2 \tilde{\Phi}^2}.
$$

(28)

Defining $a_c = l \sqrt{1 - \alpha^2 \tilde{\Phi}^2}$, we see that the black hole is stable for $a_+ > a_c$ and for $a_+ < a_c$ the adS is preferred with constant electric potential $\Phi$ all over. From (26), we see that this happens at a critical temperature

$$
T_c = \frac{(n-1) \sqrt{1 - \alpha^2 \tilde{\Phi}^2}}{2\pi R}.
$$

(29)

It is now straightforward to write the Gibbs potential for the boundary theory as an expansion in terms of order parameter ($\hat{a}$ as defined earlier) in the Landau-Ginzburg framework. It is given by\footnote{We skip here the systematic constructional detail. The prescription is similar to that of the last footnote and also can be found in [18].}

$$
G(\hat{a}, T) = \frac{\kappa \omega_{n-1} (n-1) (1 - \alpha^2 \tilde{\Phi}^2)}{16\pi R} \hat{a}^{n-2} - \frac{\kappa \omega_{n-1} T}{4} \hat{a}^{n-1} + \frac{\kappa (n-1) \omega_{n-1} a_n}{16\pi R} \hat{a}^{n}.
$$

(30)

We may mention here that the condition of extremum of the free energy with respect to $\hat{a}$, that is

$$
\frac{\partial G}{\partial \hat{a}} \bigg|_{\Phi} = 0,
$$

(31)

gives the temperature $T$ as in (26). For $T = T_c$, $G$ has degenerate minima at $\hat{a} = 0$ and $\hat{a} = \hat{a}_c$. When $T < T_c$, $G$ has an absolute minimum at $\hat{a} = 0$. For $T > T_c$, the minimum is determined by the larger solution of (31). The energy at equilibrium can easily be calculated from (30) as follows. We first find the Helmholtz free energy $F = G - Q\Phi$. Then the energy $E$ is given by:

$$
E(T, V, \Phi) = F(T, V, \Phi) - T \left( \frac{\partial F}{\partial T} \right)_{V, \Phi} = \frac{(n-1) \kappa \omega_{n-1} \hat{a}^{n-2}}{16\pi R} (1 + \alpha^2 \tilde{\Phi}^2 + \hat{a}^2).
$$

(32)

The energy can be seen to be equal to $\frac{lM}{R}$ where $M$ is given in (19) and (22). Using the generalisation of the formula for $E_c$ given in (11) as

$$
E_c = (n-1)(E + pV - TS - \Phi Q) = (n-1)(F + pV),
$$

(33)

we get

$$
E_c = \frac{\kappa (n-1) \hat{a}^{n-2} \omega_{n-1}}{8\pi R}.
$$

(34)
Here we have used $p = -(\partial G/\partial V)_{T,Q}$. We can therefore rewrite (30) as

$$G(\hat{a}, T) = \frac{1}{2}E_c\left((1 - \alpha^2\hat{\Phi}^2) - \frac{4\pi RT}{n - 1}\hat{a} + \hat{a}^2\right).$$  \hspace{1cm} (35)

From this expression, we get the CV formula for RNAdS black hole as

$$S = \frac{2\pi R}{n - 1}\sqrt{E_c\left(2(E - E_Q) - E_c\right)},$$  \hspace{1cm} (36)

where $E_Q = Q\Phi/2$ is the zero temperature energy of the CFT which makes contribution to the free energy.

**Kerr-adS black hole:** The rotating adS black hole metric in $n + 1$ dimensional space-time is given by

$$ds^2 = -\frac{\Delta_a}{\rho^2}\left[dt - \frac{b\sin^2 \theta}{1 - \frac{b^2}{l^2}}d\phi\right]^2 + \frac{\rho^2}{\Delta_a}da^2 + \frac{\rho^2d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta\sin^2 \theta}{\rho^2}\left[b dt - \frac{a^2 + b^2}{1 - \frac{b^2}{l^2}}d\phi\right]^2 + a^2 \cos^2 \theta d\Omega_{n-3}^2,$$ \hspace{1cm} (37)

where

$$\Delta_a = (a^2 + b^2)(1 + \frac{a^2}{l^2}) - 2ma^{4-n},$$

$$\Delta_\theta = 1 - \frac{b^2}{l^2}\cos^2 \theta,$$

$$\rho^2 = a^2 + b^2 \cos^2 \theta.$$ \hspace{1cm} (38)

The parameters $m$ and $b$ are related to the black hole energy and angular momentum as defined later. Note that the metric for large $a$, on a hypersurface of fixed $a$, is a rotating Einstein universe \[20\]. The temperature ($\tilde{T}$), free energy ($\tilde{F}$), entropy ($\tilde{S}$), energy ($\tilde{E}$), angular momentum ($\tilde{J}$) and angular velocity ($\tilde{\Omega}$) of the black hole, calculated with respect to the adS background, are given by \[21\]

$$\tilde{T} = \frac{(n - 2)(1 + \frac{b^2}{l^2})a_+ + n\frac{a^3}{l^3} + (n - 4)\frac{b^2}{a_+}}{4\pi(a_+^2 + b^2)},$$

$$\tilde{F} = \frac{\kappa\omega_{n-1}}{16\pi(1 - \frac{b^2}{l^2})^{n-1}}a_{+}^{n-4}(a_+^2 + b^2)(1 - \frac{a_+^2}{l^2}),$$

$$\tilde{S} = \frac{\kappa\omega_{n-1}}{4(1 - \frac{b^2}{l^2})^{n-1}}a_{+}^{n-3}(a_+ + b^2),$$

$$\tilde{E} = \frac{(n - 1)\kappa\omega_{n-1}}{16\pi(1 - \frac{b^2}{l^2})^{n-1}}a_{+}^{n-4}(a_+^2 + b^2)(1 + \frac{a_+^2}{l^2}),$$

$$\tilde{J} = \frac{\kappa\omega_{n-1}b}{8\pi(1 - \frac{b^2}{l^2})^{2n-1}}a_{+}^{n-4}(1 + \frac{a_+^2}{l^2})(a_+^2 + b^2),$$

$$\tilde{\Omega} = \frac{b(1 + \frac{a^2}{l^2})}{(a_+^2 + b^2)}.$$ \hspace{1cm} (39)
Here, $a_+$ corresponds to the location of the horizon of the black hole. This is found by setting $\Delta_0 = 0$ in (38). We first define the dimensionless parameters $\hat{b} = b/l$ and $\hat{a} = a_+/l$. Throughout this section, we will work with the rotation parameter $\hat{b} < 1$. Note that the free energy changes sign and becomes negative for $\hat{a} > 1$ signaling a Hawking-Page phase transition. This happens at a critical temperature $\tilde{T}_c$ given by

$$\tilde{T}_c = \frac{(n - 1) + (n - 3)\hat{b}^2}{2\pi(1 + \hat{b}^2)}.$$

(40)

As before, we can understand such a transition at the boundary in the Landau-Ginzburg framework. The boundary free energy, in the grand canonical ensemble, can be written in terms of order parameter $(\hat{a})$ as

$$F(\hat{a}, \Omega, T) = \frac{(n - 3)\kappa \omega_{n-1} \hat{b}^2}{16\pi(1 - \hat{b}^2)R} \hat{a}^{n-4} - \frac{\kappa \omega_{n-1} \hat{b}^2 T}{4(1 - \hat{b}^2)} \hat{a}^{n-3} + \frac{\kappa \omega_{n-1}}{16\pi(1 - \hat{b}^2)R} \left( (n - 1) + (n - 3)\hat{b}^2 \right) \hat{a}^{n-2} - \frac{\kappa \omega_{n-1} T}{4(1 - \hat{b}^2)} \hat{a}^{n-1} + \frac{(n - 1)\kappa \omega_{n-1}}{16\pi(1 - \hat{b}^2)R} \hat{a}^{n}.$$

(41)

Note that we could have eliminated $\hat{b}$ in favour of $\Omega$. However, the expression of the free energy becomes more involved. That is why we preferred to retain $\hat{b}$ in (41). Note also that, in the above, the boundary thermodynamic quantities are written without tildes. More precisely, boundary angular momentum ($J$), temperature ($T$) are related to $\tilde{J}$ and $\tilde{T}$ by a $1/R$ scaling whereas angular velocity ($\Omega$) remains same as $\tilde{\Omega}$. After somewhat long but straightforward calculation, it follows that the location of the extremum of the free energy as a function of order parameter $\hat{a}$

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure1.png}
\caption{Free energy of the rotating-adS black hole as a function of order parameter $\hat{a}$ for fixed $R$. Three curves correspond to three different choices of temperature $T$ as shown.}
\end{figure}

This expression of free energy can be systematically constructed following Landau’s prescription. But, we leave here the constructional detail.
\[ \partial F \bigg|_\Omega = 0 \]  
\[ (42) \]

This corresponds to the boundary temperature \((T)\)
\[ T = \frac{(n - 2)(1 + \hat{b}^2)\hat{a} + n\hat{a}^3 + (n - 4)\hat{b}^2}{4\pi(\hat{a}^2 + \hat{b}^2)R}. \]  
\[ (43) \]

This is nothing but the black hole temperature given in (39) scaled by appropriate power of \(R\). Furthermore, substituting (43) into (41), we get the free energy of (39), again up to the boundary scaling. It is now easy to check that \(F\) is negative for \(\hat{a} > 1\) and is identically zero at the boundary critical temperature \(T_c\), given by
\[ T_c = \frac{(n - 1) + (n - 3)\hat{b}^2}{2\pi(1 + \hat{b}^2)R}. \]  
\[ (44) \]

Typical boundary phase transition curves are shown in the figure. We may mention here that the other boundary thermodynamic quantities can be found from (41). For example,
\[ J = -\frac{\partial F}{\partial \Omega} \bigg|_T \]  
\[ (45) \]

can easily be evaluated. Consequently, we find that the casimir part of the boundary energy \((E_c)\) can be written as
\[ E_c = (n - 1)(F + pV) = \frac{(n - 1)\kappa \omega_{n-1}(n^2 + \hat{b}^2)}{8\pi(1 - b^2)}. \]  
\[ (46) \]

Here we have used the thermodynamic relation \(p = -\left(\frac{\partial E}{\partial V}\right)_{S,J}\) and \(V = \omega_{n-1}R^{n-1}/(1 - \hat{b}^2)\). The CV formula then follows immediately as
\[ S = \frac{2\pi R}{n - 1} \sqrt{E_c(2E - E_c)}. \]  
\[ (47) \]

**Scaling and phase transition:** We end this note with some general observations related to finite temperature CFT and first order phase transition. As we discussed in the beginning, Verlinde’s scaling argument determines the dependence of \(RE_c\) and \(RE_E\) on the entropy up to some proportionality constants as given in (2). These constants, \(c\) and \(d\), depend on the detail of the system. We will try to argue here that if the system further shows a first order phase transition, \(c\) and \(d\) can be determined uniquely in terms of critical entropy and temperature. To begin with, let us consider (2). These equations can be inverted to write
\[ S = \left(\frac{RE_c}{d}\right)^\frac{n}{n-1}. \]  
\[ (48) \]

Hence the free energy \(F = E - TS\) can be written as
\[ F = \frac{c}{R} S^{1 + \frac{1}{n-1}} + \frac{d}{2R} S^{1 - \frac{1}{n-1}} - TS \]
\[ = \frac{1}{2} E_c \left(1 + \frac{2c}{d} S^{\frac{2}{n-1}} - \frac{2R}{d} TS^{\frac{1}{n-1}}\right). \]  
\[ (49) \]
In writing the above set of equations, we have used (1). We will now assume that the system described by (49) shows a first order phase transition as we continue to change $T$, around which the order parameter changes discontinuously (we will continue to represent the order parameter by $\hat{a}$). We will further assume that the entropy of the system is a monotonic function of $\hat{a}$ with $S = 0$ at $\hat{a} = 0$. If we now insist that the free energy vanishes at a non-zero value of $\hat{a}$ where $\partial F/\partial \hat{a} = 0$, we can then uniquely determine $c$ and $d$. They are given by

$$c = \frac{1}{2} RT_c S_c^{-\frac{1}{n-1}}, \quad d = RT_c S_c^{-\frac{1}{n-1}}. \quad (50)$$

Here, $S_c$ and $T_c$ are respectively the entropy and the temperature at the critical point. We therefore have the following scenario. The system shows a first order phase transition at some critical value of order parameter at which point $F$ vanishes. However, the system has a finite non-zero entropy. Below the critical temperature, system prefers a state of zero order parameter which is chosen to have zero entropy. When the system has such a phase structure, the extensive and the casimir part of the energy can be completely determined in terms of entropy and critical temperature. We see this by substituting $c$ and $d$ from (50) to (2):

$$E_E = \frac{T_c}{2} \left( \frac{S}{S_c} \right)^{-\frac{1}{n-1}} S, \quad E_c = T_c \left( \frac{S_c}{S} \right)^{-\frac{1}{n-1}} S. \quad (51)$$

Note that the explicit dependence of the linear size of the system on energies has disappeared in these two expressions.

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