Operator theory induced by powers of the de Branges-Rovnyak kernel and its application

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Abstract

In this note, we give a new property of de Branges-Rovnyak kernels. As the main theorem, it is shown that the exponential of de Branges-Rovnyak kernel is strictly positive definite if the inner part of the corresponding Schur class function is nontrivial.

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H^\infty$ be the Banach algebra consisting of all bounded analytic functions on $\mathbb{D}$. Then, we set

$$\mathcal{S} = \{ \varphi \in H^\infty : |\varphi(\lambda)| \leq 1 \ (\lambda \in \mathbb{D}) \},$$

and which is called the Schur class. For any function $\varphi$ in $H^\infty$, it is well known that $\varphi$ belongs to $\mathcal{S}$ if and only if

$$\frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\lambda}z}$$

(1)

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is positive semi-definite. This equivalence relation based on the properties of the Szegö kernel is crucial in the operator theory on the Hardy space over $\mathbb{D}$, in particular, theories of Pick interpolation, de Branges-Rovnyak spaces and sub-Hardy Hilbert spaces (see Agler-McCarthy [2], Ball-Bolotnikov [4], Fricain-Mashreghi [6] and Sarason [15]). The kernel (1) is called the de Branges-Rovnyak kernel.

Before introducing our study, we should mention that not only the original de Branges-Rovnyak kernel but also its variants have been studied by a number of authors. For example, Zhu [17, 18] initiated the study on the kernel

$$\frac{1 - \varphi(\lambda)\varphi(z)}{(1 - \lambda z)^2}$$

in the Bergman space over $\mathbb{D}$. The reproducing kernel Hilbert space induced by the kernel (2) is called a sub-Bergman Hilbert space (see also Abkar-Jafarzadeh [1], Ball-Bolotnikov [3], Chu [5], Nowak-Rososzczuk [12] and Sultanic [16]). Further, powers of the de Branges-Rovnyak kernel

$$\left(\frac{1 - \varphi(\lambda)\varphi(z)}{1 - \lambda z}\right)^n \quad (n \in \mathbb{N})$$

are naturally obtained from the theory of hereditary functional calculus for weighted Bergman spaces on $\mathbb{D}$ (see Example 14.48 in [2] for the case where $n = 2$) and have appeared also in p. 3672 of Jury [9].

Now, the purpose of this paper is to study the structure of the kernel

$$\exp\left(\frac{1 - \varphi(\lambda)\varphi(z)}{1 - \lambda z}\right) \quad (t > 0).$$

(4)

Note that our kernel (4) is obtained by binding all kernels in (3) together. Thus, we expect that new properties of the de Branges-Rovnyak kernel (1) are drawn out from our kernel (4). In fact, as the main result, we will show that the exponential of the de Branges-Rovnyak kernel is strictly positive definite if the inner part of $\varphi$ is nontrivial.

Here, we shall give some remarks on strictly positive definite kernels. In general, it is not difficult to construct positive semi-definite kernels. On the other hand, for strictly positive definite kernels, nontrivial methods depending on each case are often needed (for example, see Micchelli [10]). Moreover, it might be worth while mentioning that strictly positive definite kernels have received attention in machine learning (see Rasmussen-Williams [14]).

This paper is organized as follows. In Section 2, basic properties of the reproducing kernel Hilbert space $\mathcal{H}_t(\varphi)$ constructed from our kernel (4) are given. In Section 3, unbounded multipliers on $\mathcal{H}_t(\varphi)$ are introduced and studied. In Section 4, main results are given.

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2 Preliminaries

For $t > 0$, let $\mathcal{H}_t(\varphi)$ denote the reproducing kernel Hilbert space with kernel

$$tk^\varphi(z, \lambda) = \frac{1 - \varphi(\lambda)\varphi(z)}{1 - \lambda z}, \quad (\varphi \in \mathcal{S}),$$

and we will use notations $tk^\varphi_\lambda(z) = tk^\varphi(z, \lambda)$ and $\mathcal{H}(\varphi) = \mathcal{H}_1(\varphi)$. Then, since

$$\langle tk^\varphi_\lambda, tk^\varphi_\mu \rangle_{\mathcal{H}_t(\varphi)} = tk^\varphi(z, \lambda) = t^{-1} \langle tk^\varphi_\lambda, tk^\varphi_\mu \rangle_{\mathcal{H}(\varphi)},$$

the trivial linear mapping $f \mapsto f$ from $\mathcal{H}(\varphi)$ onto $\mathcal{H}_t(\varphi)$ is bounded and invertible. Particularly, $\mathcal{H}_t(\varphi) = \mathcal{H}(\varphi)$ as vector spaces. In this section, we construct the exponential of $\mathcal{H}_t(\varphi)$ and give its basic properties. The contents of this section are well known to specialists. For example, see Exercise (k) in p. 320 of Nikolski [11] and Chapter 7 in Paulsen-Raghupathi [13]. However, we give the details for the sake of readers.

2.1 Construction of exp $\mathcal{H}_t(\varphi)$

Let $\mathcal{H}_t(\varphi)^n$ be the reproducing kernel Hilbert space obtained by the pull-back construction with the $n$-fold tensor product space

$$\mathcal{H}_t(\varphi)^{\otimes n} = \mathcal{H}_t(\varphi) \otimes \cdots \otimes \mathcal{H}_t(\varphi)$$

and the $n$-dimensional diagonal map

$$\Delta_n : \mathbb{D} \to \mathbb{D}^n, \lambda \mapsto (\lambda, \ldots, \lambda)$$

(for the pull-back construction, see Theorem 5.7 in [13]). We note that $(tk^\varphi_\lambda)^{\otimes n} \circ \Delta_n = (tk^\varphi)^n_\lambda$ is the reproducing kernel of $\mathcal{H}_t(\varphi)^n$. Let $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ denote the Hilbert space with the inner product

$$\langle (f_0, f_1, \ldots)^T, (g_0, g_1, \ldots)^T \rangle_{\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n} = \sum_{n=0}^\infty \frac{1}{n!} \langle f_n, g_n \rangle_{\mathcal{H}_t(\varphi)^n},$$

where we set $\mathcal{H}_t(\varphi)^0 = \mathbb{C}$. Moreover, we define linear map $\Gamma$ as follows:

$$\Gamma : \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \mapsto \sum_{n=0}^\infty \frac{1}{n!} f_n \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \in \oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n.$$

Proposition 2.1. The following statements hold:

(i) $\Gamma$ is a map from $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ to $\text{Hol}(\mathbb{D})$.

(ii) $\ker \Gamma$ is closed.
Proof. For any $F = (f_0, f_1, \ldots)^\top$ in $\oplus_{n=0}^{\infty} \mathcal{H}_t(\varphi)^n$, we have
\[
\left| \sum_{\ell=n+1}^{m} \frac{1}{\ell!} f_\ell(\lambda) \right| \leq \sum_{\ell=n+1}^{m} \left| \frac{1}{\ell!} f_\ell(\lambda) \right|
\leq \sum_{\ell=n+1}^{m} \frac{1}{\ell!} \| f_\ell \|_{\mathcal{H}_t(\varphi)^n} \| (tk_\lambda^{\varphi})^\ell \|_{\mathcal{H}_t(\varphi)^n}
\leq \left( \sum_{\ell=n+1}^{m} \frac{1}{\ell!} \| f_\ell \|_{\mathcal{H}_t(\varphi)^n}^2 \right)^{1/2} \left( \sum_{\ell=n+1}^{m} \frac{1}{\ell!} \| (tk_\lambda^{\varphi})^\ell \|_{\mathcal{H}_t(\varphi)^n}^2 \right)^{1/2}
= \left( \sum_{\ell=n+1}^{m} \frac{1}{\ell!} \| f_\ell \|_{\mathcal{H}_t(\varphi)^n}^2 \right)^{1/2} \left( \sum_{\ell=n+1}^{m} \frac{1}{\ell!} \| (tk_\lambda^{\varphi})^\ell \|_{\mathcal{H}_t(\varphi)^n}^2 \right)^{1/2}.
\]

Hence,
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f_n(\lambda)
\]
converges uniformly on any compact subset of $\mathbb{D}$. This concludes (1). Next, suppose that $F_\ell = (f_0^{(\ell)}, f_1^{(\ell)}, \ldots)^\top$ belongs to $\text{ker} \Gamma$ and $F_\ell$ converges to $F$ in $\oplus_{n=0}^{\infty} \mathcal{H}_t(\varphi)^n$. Then, for sufficiently large $L$, we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \| f_n^{(\ell)} - f_n \|_{\mathcal{H}_t(\varphi)^n}^2 = \| F_\ell - F \|_{\oplus_{n=0}^{\infty} \mathcal{H}_t(\varphi)^n}^2 < 1 \quad (\ell \geq L).
\]

Hence we have $\| f_n^{(\ell)} - f_n \|_{\mathcal{H}_t(\varphi)^n} < \sqrt{n!}$ for every $n \geq 0$ if $\ell \geq L$. It follows from this inequality that, for any $\ell \geq L$,
\[
\left| \frac{1}{n!} f_n^{(\ell)}(\lambda) \right| \leq \frac{1}{n!} \| f_n^{(\ell)} \|_{\mathcal{H}_t(\varphi)^n} \| (tk_\lambda^{\varphi})^n \|_{\mathcal{H}_t(\varphi)^n}
\leq \left( \| f_n \|_{\mathcal{H}_t(\varphi)^n}^2 + \sqrt{n!} \right) \| (tk_\lambda^{\varphi})^n \|_{\mathcal{H}_t(\varphi)^n}
\leq \frac{(M+1) \| (tk_\lambda^{\varphi})^n \|_{\mathcal{H}_t(\varphi)^n}}{\sqrt{n!}},
\]
where we set $M = \sup_{n \geq 0} (\| f_n \|_{\mathcal{H}_t(\varphi)^n}/n!)^{1/2}$. Then, by the ratio test,
\[
\sum_{n=0}^{\infty} \frac{(M+1) \| (tk_\lambda^{\varphi})^n \|_{\mathcal{H}_t(\varphi)^n}}{\sqrt{n!}}
\]
is finite. Hence, by the Lebesgue dominated convergence theorem and the assumption that $F_\ell = (f_0^{(\ell)}, f_1^{(\ell)}, \ldots)^\top$ belongs to $\text{ker} \Gamma$, we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f_n(\lambda) = \sum_{n=0}^{\infty} \lim_{\ell \to \infty} \frac{1}{n!} f_n^{(\ell)}(\lambda) = \lim_{\ell \to \infty} \sum_{n=0}^{\infty} \frac{1}{n!} f_n^{(\ell)}(\lambda) = 0.
\]
This concludes (2). □

By Proposition 2.1, the pull-back construction can be applied to \( \Gamma \).

**Definition.** We define \( \exp \mathcal{H}_t(\varphi) \) as the reproducing kernel Hilbert space obtained by the pull-back construction with the linear map

\[
\Gamma : \oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n \to \text{Hol}(\mathbb{D}).
\]

### 2.2 Basic properties of \( \exp \mathcal{H}_t(\varphi) \)

We summarize basic properties of \( \exp \mathcal{H}_t(\varphi) \).

**Proposition 2.2.** \( \exp \mathcal{H}_t(\varphi) \) is a reproducing kernel Hilbert space consisting of holomorphic functions on \( \mathbb{D} \). More precisely, for any \( f \) in \( \exp \mathcal{H}_t(\varphi) \), there exists a vector \( (f_0, f_1, \ldots,)^\top \) in \( \oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n \) such that

\[
f = \sum_{n=0}^\infty \frac{1}{n!} f_n
\]

converges uniformly on any compact subset of \( \mathbb{D} \). Moreover,

(i) the following norm estimate holds:

\[
\|f\|_{\exp \mathcal{H}_t(\varphi)}^2 \leq \sum_{n=0}^\infty \frac{1}{n!} \|f_n\|_{\mathcal{H}_t(\varphi)^n}^2,
\]

(ii) the reproducing kernel of \( \exp \mathcal{H}_t(\varphi) \) is

\[
\sum_{n=0}^\infty \frac{1}{n!} (tk^{\varphi}_X)^n = \exp tk^{\varphi}_X,
\]

that is,

\[
f(\lambda) = \langle f, \exp tk^{\varphi}_X \rangle_{\exp \mathcal{H}_t(\varphi)}
\]

for any \( \lambda \) in \( \mathbb{D} \),

(iii) the following growth condition holds:

\[
|f(\lambda)|^2 \leq \|f\|_{\exp \mathcal{H}_t(\varphi)}^2 \exp \left( \frac{1 - |\varphi(\lambda)|^2}{1 - |\lambda|^2} \right)
\]

for any \( \lambda \) in \( \mathbb{D} \).
Proof. By the definition of the norm and the inner product of $\exp \mathcal{H}_t(\varphi)$, we have (1) and (2). We shall show (3). By (2) and the Cauchy-Schwarz inequality, we have

$$|f(\lambda)|^2 = |(f, \exp tk^\varphi_\lambda)_{\exp \mathcal{H}_t(\varphi)}|^2 \leq \|f\|_{\exp \mathcal{H}_t(\varphi)} \cdot \|\exp tk^\varphi_\lambda\|_{\exp \mathcal{H}_t(\varphi)} = \|f\|_{\exp \mathcal{H}_t(\varphi)} \cdot \exp tk^\varphi_\lambda(\lambda) \leq \|f\|_{\exp \mathcal{H}_t(\varphi)} \cdot \exp t(1 - |\varphi(\lambda)|^2)$$

Thus we have (3).

3 Unbounded multipliers

We shall investigate into unbounded multipliers of $\exp \mathcal{H}_t(\varphi)$.

Lemma 3.1. Let $\psi$ be a function in $\mathcal{H}_t(\varphi)$. Then, for any function $f$ in $\mathcal{H}_t(\varphi)^n$, $\psi f$ belongs to $\mathcal{H}_t(\varphi)^{n+1}$.

Proof. We define bounded linear operator $\tau_\psi$ as follows:

$$\tau_\psi : \mathcal{H}_t(\varphi)^n \rightarrow \mathcal{H}_t(\varphi)^{n+1}, \quad F \mapsto \psi \otimes F.$$

Then, the following diagram commutes:

$${\mathcal{H}_t(\varphi)^n \rightarrow \mathcal{H}_t(\varphi)^{n+1}} \quad \tau_\psi \downarrow \quad \Delta_n$$

$${\mathcal{H}_t(\varphi) \rightarrow \mathcal{H}_t(\varphi)^{n+1}, \quad M_\psi|_{\mathcal{H}_t(\varphi)^n} \rightarrow \Delta_{n+1}}$$

where $M_\psi$ denotes the multiplication operator with symbol $\psi$. This concludes the proof.

Theorem 3.2. Let $\psi$ be a function in $\mathcal{H}_t(\varphi)$. Then, the multiplication operator $M_\psi$ is a densely defined closable linear operator in $\exp \mathcal{H}_t(\varphi)$.

Proof. Let $F = (f_0, f_1, \ldots, f_N, 0 \ldots)^\top$ be a vector having finite support in $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$. We set $\Gamma F = f$. Then,

$$\psi f = \psi \sum_{n=0}^N \frac{1}{n!} f_n = \sum_{n=0}^N \frac{1}{n!} \psi f_n = \sum_{n=0}^N \frac{1}{(n+1)!} (n+1) \psi f_n = \sum_{n=1}^{N+1} \frac{1}{n!} n \psi f_{n-1},$$

where we note that $n \psi f_{n-1}$ belongs to $\mathcal{H}_t(\varphi)^n$ by Lemma 3.1. Hence, setting

$$G = (0, \psi f_0, 2 \psi f_1, \ldots, (N+1) \psi f_N, 0, \ldots)^\top,$$

$G$ belongs to $\oplus_{n=0}^\infty \mathcal{H}_t(\varphi)^n$ and $\Gamma G = \psi f$, that is, $\psi f$ belongs to $\exp \mathcal{H}_t(\varphi)$. Therefore, $M_\psi$ is a densely defined linear operator in $\exp \mathcal{H}_t(\varphi)$. Moreover, it is easy to see that $M_\psi$ is closable.

\qed
Corollary 3.3. Let $\psi$ be a function in $\mathcal{H}_t(\varphi)$. Then the adjoint operator $M^*_\psi$ of $M_\psi$ is a densely defined closed linear operator in $\exp \mathcal{H}_t(\varphi)$, and every $\exp t k_\lambda^\varphi$ is an eigenfunction of $M^*_\psi$. More precisely,

$$M^*_\psi \exp t k_\lambda^\varphi = \psi(\lambda) \exp t k_\lambda^\varphi.$$ 

4 Main results

Let $X$ be a set. A function $k$ on $X \times X$ is called a strictly positive definite kernel on $X$ if $k(x, y) = k(y, x)$ for any $x$ and $y$ in $X$ and

$$\sum_{i,j=1}^n c_i \overline{c_j} k(x_j, x_i) > 0$$

for any $n$ in $\mathbb{N}$, any $(c_1, \ldots, c_n)^\top$ in $\mathbb{C}^n \setminus \{0\}$ and any $n$ distinct points $x_1, \ldots, x_n$ in $X$. For example, it is well known that $k(z, \lambda) = \exp(\overline{\lambda} z)$ is a strictly positive definite kernel on $\mathbb{C}$. In fact, this is the reproducing kernel of the Segal-Bargmann space. Now, we note that if $\varphi = z^2$ then

$$e^{-1} \exp \left( \frac{1 - \varphi(\lambda) \varphi(z)}{1 - \overline{\lambda} z} \right) = \exp(\overline{\lambda} z).$$

Motivated by this observation, we shall give new examples of strictly positive definite kernels.

Let $\varphi$ be a function in $\mathcal{S}$. For the canonical factorization $\varphi = \alpha z^N BSF$ of $\varphi$ (see Section 5 in Chapter II in Garnett [7]), where $\alpha$ is a unimodular constant, $B$ is a Blaschke product consisting of nonzero zero points, $S$ is a singular inner function and $F$ is an outer function, we consider three conditions (C1) $N \geq 2$, (C2) $B$ is nontrivial and (C3) $S$ is nontrivial. We need the following lemma.

Lemma 4.1. Let $\lambda_1, \ldots, \lambda_n$ be $n$ distinct points in $\mathbb{D}$. Suppose one of (C1), (C2) and (C3). Then there exists a function $\psi$ in $\mathcal{H}_t(\varphi)$ such that $\psi(\lambda_i) \neq \psi(\lambda_j)$ ($i \neq j$).

Proof. Since $\mathcal{H}_t(\varphi) = \mathcal{H}(\varphi)$ as vector spaces, it suffices to show the statement for $\mathcal{H}(\varphi)$. First, we assume (C1). Then, since $\varphi/z$ is in $\mathcal{S}$ by the Schwarz lemma and $(\varphi/z)(0) = 0$, we have

$$(I - T_\varphi T_\varphi^*) z = z - T_\varphi T_\varphi^* T_\varphi z = z - T_\varphi T_\varphi^* z = T_\varphi z = 1 = z,$$

where $T_\varphi$ denotes the Toeplitz operator with symbol $\varphi$ on the Hardy space $H^2$ over $\mathbb{D}$. Hence $z$ belongs to $\mathcal{H}(\varphi)$, and we may take $\psi = z$.

Secondly, we assume (C2). Let $\mu$ be a nonzero zero point of $\varphi$. Then, we have

$$k_\mu^\varphi = (I - T_\varphi T_\varphi^*) k_\mu = k_\mu.$$
Hence \((1 - \overline{\mu}z)^{-1}\) belongs to \(\mathcal{H}(\varphi)\), and we may take \(\psi = (1 - \overline{\mu}z)^{-1}\).

Thirdly, we assume (C3). Then, observe that there exists a point \(\theta \in [0, 2\pi)\) such that \(\varphi(re^{\sqrt{-1}\theta}) \to 0\) as \(r \uparrow 1\) (see Theorem 6.2 in [7]). Without loss of generality, we may assume that \(\theta = 0\). Setting

\[
\delta_{ij}(r) = |k_r^\varphi(\lambda_i) - k_r^\varphi(\lambda_j)| \quad \text{and} \quad \delta(r) = \min_{1 \leq i < j \leq n} \delta_{ij}(r) \quad (0 < r < 1),
\]

it suffices to show that \(\delta(r) > 0\) for any \(r\) sufficiently close to 1. For any distinct \(\lambda_i\) and \(\lambda_j\), we suppose that, for any \(m \in \mathbb{N}\), there exists \(r_m \in (1 - m^{-1}, 1)\) such that \(\delta_{ij}(r_m) = 0\). Then, we have

\[
\left| \frac{1}{1 - \lambda_i} - \frac{1}{1 - \lambda_j} \right| = \lim_{m \to \infty} \left| \frac{1 - \varphi(r_m)\varphi(\lambda_i)}{1 - r_m\lambda_i} - \frac{1 - \varphi(r_m)\varphi(\lambda_j)}{1 - r_m\lambda_j} \right| = \lim_{m \to \infty} \delta_{ij}(r_m) = 0.
\]

This concludes that \(\lambda_i = \lambda_j\), however, which contradicts the assumption \(\lambda_i \neq \lambda_j\). Hence there exists \(r_{ij} \in (0, 1)\) such that \(\delta_{ij}(r) > 0\) for any \(r \in (r_{ij}, 1)\). Setting \(R = \max_{1 \leq i < j \leq n} r_{ij}\), we have \(\delta(r) > 0\) for any \(r \in (R, 1)\). Then, we may take \(\psi = k_r^\varphi\).

**Theorem 4.2.** Let \(\varphi\) be a function in \(\mathcal{S}\). If \(\varphi\) satisfies one of (C1), (C2) and (C3), then the kernel

\[
k_t(z, \lambda) = \exp \left( t \frac{1 - \varphi(\lambda)\varphi(z)}{1 - \lambda z} \right) \quad (t > 0)
\]

is strictly positive definite.

**Proof.** It suffices to show that \(\{\exp tk_{\lambda_j}^\varphi\}_{j=1}^n\) is linearly independent for any \(n \in \mathbb{N}\) and any \(n\) distinct points \(\lambda_1, \ldots, \lambda_n\) in \(\mathbb{D}\). Suppose that

\[
\sum_{j=1}^n c_j \exp tk_{\lambda_j}^\varphi = 0
\]

for some \(n \in \mathbb{N}\), some \(n\) distinct points \(\lambda_1, \ldots, \lambda_n\) in \(\mathbb{D}\), and some \(c_1, \ldots, c_n\) in \(\mathbb{C}\). Then, for any function \(\psi\) in \(\mathcal{H}_t(\varphi)\), by Corollary 3.3 and the assumption, we have

\[
\begin{pmatrix}
\frac{1}{\psi(\lambda_1)} & \cdots & \frac{1}{\psi(\lambda_n)} \\
\vdots & \ddots & \vdots \\
\frac{1}{\psi(\lambda_1)^{n-1}} & \cdots & \frac{1}{\psi(\lambda_n)^{n-1}}
\end{pmatrix}
\begin{pmatrix}
c_1 \exp tk_{\lambda_1}^\varphi \\
c_2 \exp tk_{\lambda_2}^\varphi \\
\vdots \\
c_n \exp tk_{\lambda_n}^\varphi
\end{pmatrix}
= \begin{pmatrix}
\sum_{j=1}^n c_j \exp tk_{\lambda_j}^\varphi \\
\sum_{j=1}^n \frac{c_j \exp tk_{\lambda_j}^\varphi}{\psi(\lambda_j)} \\
\vdots \\
(M\psi)^{n-1} \sum_{j=1}^n c_j \exp tk_{\lambda_j}^\varphi
\end{pmatrix}
= 0.
\]
Further, by Lemma 4.1, there exists a function $\psi$ in $H_t(\varphi)$ such that
$$
\prod_{1 \leq i < j \leq n} (\psi(\lambda_i) - \psi(\lambda_j)) \neq 0.
$$

Then, the Vandermonde matrix
\[
\begin{pmatrix}
1 & \cdots & 1 \\
\psi(\lambda_1) & \cdots & \psi(\lambda_n) \\
\vdots & \vdots & \vdots \\
\psi(\lambda_1)^{n-1} & \cdots & \psi(\lambda_n)^{n-1}
\end{pmatrix}
\]
is nonsingular. Therefore, we have that
\[
\begin{pmatrix}
c_1 \exp tk_{\lambda_1}^\varphi \\
c_2 \exp tk_{\lambda_2}^\varphi \\
\vdots \\
c_n \exp tk_{\lambda_n}^\varphi
\end{pmatrix} = 0.
\]
This concludes that $c_1 = \cdots = c_n = 0$. \qed

The well-known fact mentioned at the beginning of this section is included in Theorem 4.2.

**Corollary 4.3.** The kernel function
\[
k(z, \lambda) = \exp(\lambda z)
\]
is strictly positive definite on $\mathbb{C}$.

**Proof.** For any $n$ distinct points $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, we set $R = \max_{1 \leq j \leq n} |\lambda_j| + 1$. Then $\lambda_1/R, \ldots, \lambda_n/R$ are in $\mathbb{D}$. Hence, by Theorem 4.2 in the case where $\varphi = z^2$ and $t = R^2$, we have
\[
\sum_{i,j=1}^{n} c_i \overline{c_j} \exp(\overline{\lambda_i} \lambda_j) = e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp(R^2 + \overline{\lambda_i} \lambda_j)
\]
\[
= e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp(R^2(1 + (\lambda_i/R) \overline{(\lambda_j/R)}))
\]
\[
= e^{-R^2} \sum_{i,j=1}^{n} c_i \overline{c_j} \exp \left( R^2 \frac{1 - \varphi(\lambda_i/R) \overline{\varphi(\lambda_j/R)}}{1 - (\lambda_i/R) \overline{(\lambda_j/R)}} \right) > 0
\]
for any $(c_1, \ldots, c_n)^\top$ in $\mathbb{C}^n \setminus \{0\}$. \qed
Although the next result is just a simple consequence of Theorem 4.2, from the viewpoint of the theory of model spaces (see Garcia-Mashreghi-Ross [8]), it will be worth while mentioning it as a theorem.

**Theorem 4.4.** Let \( \varphi \) be an inner function. If \( \varphi \) is neither a constant nor \( e^{i\theta} z \), then the kernel

\[
k_t(z, \lambda) = \exp \left( t \frac{1 - \overline{\varphi(\lambda)} \varphi(z)}{1 - \lambda z} \right) \quad (t > 0)
\]

is strictly positive definite.

Further, with help of the theory of sub-Hardy Hilbert spaces, we have

**Theorem 4.5.** Let \( \varphi \) be a function in \( S \). If \( \varphi \) is a nonextreme point of the closed unit ball in \( H^\infty \), then the kernel

\[
k_t(z, \lambda) = \exp \left( t \frac{1 - \overline{\varphi(\lambda)} \varphi(z)}{1 - \lambda z} \right) \quad (t > 0)
\]

is strictly positive definite.

**Proof.** By Theorem (IV-3) in [15] (or Theorem 23.13 in [6]), the polynomials are dense in \( \mathcal{H}(\varphi) \). Hence \( z \) belongs to \( \mathcal{H}_t(\varphi) \). Then, setting \( \psi = z \), the proof of Theorem 4.2 applies to this case.

\( \square \)

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