Abstract

We introduce a family of quasi-Banach spaces — which we call wave packet smoothness spaces — that includes those function spaces which can be characterised by the sparsity of their expansions in Gabor frames, wave atoms, and many other frame constructions. We construct Banach frames for and atomic decompositions of the wave packet smoothness spaces and study their embeddings in each other and in a few more classical function spaces such as Besov and Sobolev spaces.

Résumé

Nous introduisons une famille d’espaces affines quasi-normés complets – que nous appellerons espaces de paquets d’ondelettes réguliers – qui incluent de nombreux espaces de fonctions caractérisés par leurs transformées, comme celle de Gabor ou en ondelettes, clairsemées. Nous construisons des cadres de Banach et des décompositions atomiques pour ces espaces et étudions leurs inclusions l’un dans l’autre ainsi que dans quelques espaces de fonctions devenus classiques tels que ceux de Sobolev ou de Besov.

Keywords: Wave packets, Banach frames, atomic decompositions, embeddings, analysis and synthesis sparsity, decomposition spaces, α-modulation spaces, Sobolev and Besov spaces

2010 MSC: 42B35, 42C15, 46E15, 46E35, 42C40

1. Introduction

A large number of different frame constructions are used in harmonic analysis both in practical applications such as image denoising [9, 43, 5, 53], restoring truncated signals [51, 52], edge detection [50, 41] and compressed sensing [1] and in pure mathematics for characterising function spaces in terms of the frame coefficients [14, 64, 27, 20, 58] or for characterising the wave front set of distributions [17, 39, 8]. The most important of these frame constructions are wavelets [14], Gabor frames [27, 29], shearlets [38, 40], curvelets [7], ridgelets [6, 30] and wave atoms [15].

All of these frames are constructed by applying dilations, modulations and translations to a finite set of prototype functions. Since the seminal work of Cordoba and Fefferman [12] — who used such systems to study the mapping properties of pseudo-differential operators — it has become customary to refer to such systems as wave packet systems. While the first papers mainly considered wave packet systems with continuous index sets [12, 48], nowadays the focus lies on discrete wave packet systems, which are special generalised shift invariant systems [31, 43, 21, 59]. As particular highlights of the theory of wave packets, we mention the characterisation of the Parseval property [35, 42] for such systems and the use of Gaussian wave packets for approximating solutions of the homogeneous wave equation [2].

* Corresponding authors. E-mail addresses: dimitri.bytchenkoff@univ-lorraine.fr (D. Bytchenkoff), felix@voigtlaender.xyz (F. Voigtlaender).
In the present paper, we will concentrate on the case of functions on $\mathbb{R}^2$ and consider the class of $(\alpha, \beta)$ wave packet systems as introduced in [15]. Here, the parameter $\alpha \in [0, 1]$ describes the frequency bandwidth relationship of the system, while $\beta \in [0, 1]$ describes its directional selectivity.

More precisely, if $(\psi_i)_{i \in I}$ is a system of $(\alpha, \beta)$ wave packets and if $\psi_i$ is concentrated at frequency $\xi \in \mathbb{R}^2$, then the bandwidth of $\psi_i$ is approximately $(1 + |\xi|)^\alpha$. The parameter $\alpha$ describes how multi-scale the system is. For instance, for Gabor systems, the bandwidth of the frame elements is independent of the frequency ($\alpha = 0$), while for wavelets, the bandwidth is proportional to the frequency ($\alpha = 1$).

The parameter $\beta$ determines how many different directions the wave packet system can distinguish at each frequency scale; that is, on the dyadic frequency ring $\{\xi \in \mathbb{R}^2 : |\xi| \approx 2^j\}$, an $(\alpha, \beta)$-wave packet system distinguishes approximately $2^{(1-\beta)j}$ different directions. For instance, wavelets are directionally insensitive ($\beta = 1$), while Gabor frames have high directional sensitivity ($\beta = 0$). Figure 1 shows how wave packet systems, including their most important examples, relate to $\alpha$ and $\beta$.

![Figure 1: Parametrisation of $(\alpha, \beta)$ wave packet systems including their most important special cases. In this work we focus on the regime where $0 \leq \beta \leq \alpha \leq 1$ (hatched in the figure).](image)

Our contribution. In this work, we provide a rigorous mathematical framework for studying the properties of $(\alpha, \beta)$ wave packet systems. Specifically, for given $0 \leq \beta \leq \alpha \leq 1$, we define a family of wave packet smoothness spaces $W^{p,q}_s(\alpha, \beta)$, parametrised by the integrability exponents $p, q \in (0, \infty]$ and the smoothness parameter $s \in \mathbb{R}$, and investigate properties of these spaces.

One of our main results is that if $W(\alpha, \beta) = (\psi_i)_{i \in I}$ is a sufficiently regular frame of $(\alpha, \beta)$-wave packets, then $W(\alpha, \beta)$ constitutes a Banach frame and an atomic decomposition for a whole family of wave packet smoothness spaces. We would like to emphasise that the wave packet system is not required to be band-limited; on the contrary, we show that if the generators of the wave packet system are compactly supported and smooth enough, then the resulting wave packet system will constitute a Banach frame and an atomic decomposition for a family of wave packet smoothness spaces provided that the sampling density of the wave packet system is fine enough.

In a nutshell, this means that the wave packet smoothness space is characterised by the decay of the frame coefficients with respect to the wave packet system. More precisely, there is an explicitly given coefficient space $C^{p,q}_s$ such that

$$W^{p,q}_s(\alpha, \beta) = \left\{ f : ((f \mid \psi_i)_{L^2})_{i \in I} \in C^{p,q}_s \right\} = \left\{ f = \sum_{i \in I} c_i \psi_i : c = (c_i)_{i \in I} \in C^{p,q}_s \right\}. \quad (1.1)$$

Moreover, a function $f \in W^{p,q}_s(\alpha, \beta)$ can be continuously reconstructed from its analysis coefficients $((f \mid \psi_i)_{L^2})_{i \in I}$, and the synthesis coefficients $c(f) = (c_i)_{i \in I} \in C^{p,q}_s$ satisfying $f = \sum_{i \in I} c_i \psi_i$ can be chosen to depend linearly and continuously on $f$. 

In a less technical terminology, the identity \([1,1]\) means that analysis and synthesis sparsity are equivalent for sufficiently regular wave packet systems, where sparsity is quantified by the coefficient space \(C^{p,q}_s\). We note that \(C^{p,p}_s = L^p\) for a suitable choice of \(s = s(p, \alpha, \beta)\). For non-tight frames, this equivalence between analysis- and synthesis sparsity is nontrivial, but often useful. For instance, it is usually relatively simple to verify that a certain class of functions has sparse — or quickly decaying — analysis coefficients, which amounts to estimating the inner products \(\langle f, \psi_i \rangle_{L^2}\). In contrast, it can be quite difficult to construct coefficients \(c = (c_i)_{i \in I}\) such that \(f = \sum_{i \in I} c_i \psi_i\), even without requiring that the sequence \(c\) has good decay properties. For applications in approximation theory or for studying the boundedness properties of operators, however, it is usually much more useful to know that \(f = \sum_{i \in I} c_i \psi_i\) with sparse coefficients \(c\), rather than that the analysis coefficients of \(f\) are sparse.

The second of our main findings are several useful results concerning embeddings of the wave packet smoothness spaces. First, we study the existence of embeddings

\[
W^{p,q}_{s_1}(\alpha, \beta) \hookrightarrow W^{p',q'}_{s_2}(\alpha', \beta')
\]  

(1.2)

between wave packet spaces with different parameters. Given \([1,1]\), this amounts to asking whether sparsity of a function \(f\) with respect to an \((\alpha, \beta)\) wave packet system implies some, possibly worse, sparsity with respect to an \((\alpha', \beta')\) wave packet system. If \(\beta \leq \beta'\) and \(\alpha \leq \alpha'\) or if \(\beta' \leq \beta\) and \(\alpha' \leq \alpha\), we can completely characterise the existence of the embedding \([1,2]\). Furthermore, we show that distinct parameter choices yield distinct wave packet smoothness spaces; that is, \(W^{p,q}_{s_1}(\alpha, \beta) \neq W^{p',q'}_{s_2}(\alpha', \beta')\) unless \((p_1, q_1, s_1, \alpha, \beta) = (p_2, q_2, s_2, \alpha', \beta')\) or \((p_1, q_1) = (2, 2) = (p_2, q_2)\) and \(s_1 = s_2\).

Finally, we also consider embeddings between wave packet smoothness spaces on the one hand and Besov- or Sobolev spaces on the other hand. For the case of Besov spaces, we again obtain a complete characterisation of the existence of the embeddings \(W^{p,q}_{s_1}(\alpha, \beta) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^2)\) and of the reverse embedding; as a corollary, we show that \(B^{s}_{p,q}(\mathbb{R}^2) = W^{p,q}_{s}(1,1)\). For the case of Sobolev spaces, we can completely characterise the existence of the embedding \(W^{p,q}_{s}(\alpha, \beta) \hookrightarrow W^{k,l}_{r}(\mathbb{R}^2)\) for \(r \in [1, 2] \cup \{\infty\}\). For \(r \in (2, \infty)\) we establish certain necessary and certain sufficient conditions for the existence of the embedding, which are not equivalent.

In particular, we show that if \(s \geq k + c(p)\), then \(W^{p,q}_{s}(\alpha, \beta) \hookrightarrow C^{k}_{c}(\mathbb{R}^2)\), so that all functions in the wave packet smoothness space are \(k\)-times continuously differentiable. This is one of the reasons for calling the spaces \(W^{p,q}_{s}\) smoothness spaces.

One can in principle define wave packet systems for arbitrary \(\alpha, \beta \in [0, 1]\). In this work, however, we restrict ourselves to the case where \(0 \leq \beta \leq \alpha \leq 1\) for defining the wave packet smoothness spaces and to \(0 \leq \beta \leq \alpha < 1\) for constructing Banach frames and atomic decompositions for these spaces. This restriction \(\alpha < 1\) is mainly done for convenience, since the case \(\alpha = 1\) was already explored in \([3]\), which studies \(\alpha\)-shearlet systems and the associated smoothness spaces for \(\alpha \in (0, 1]\). In our terminology, \(\alpha\)-shearlets are \((1, \alpha)\) wave packets.

In contrast — at least with our construction of the wave packet smoothness spaces — the restriction \(\beta \leq \alpha\) seems to be unavoidable. Precisely, we define the wave packet spaces as decomposition spaces \([10]\) with respect to a certain covering of the frequency space, which we call the \((\alpha, \beta)\) wave packet covering. To show that this construction yields well-defined spaces, the wave packet covering needs to satisfy a bounded overlap property; for this, the assumption \(\beta \leq \alpha\) seems to be essential. Finally, we are unaware of any frame construction that results in \((\alpha, \beta)\) wave packets with \(\beta > \alpha\); as seen in Figure \([1]\) all commonly used frame constructions fall into the regime \(\beta \leq \alpha\).

Structure of the paper. To define the wave packet smoothness spaces \(W^{p,q}_{s}(\alpha, \beta)\), we shall use the formalism of decomposition spaces, originally introduced in \([10]\). In order to define such a decomposition space \(\mathcal{D}(\mathcal{Q}, L^p, L^c)\), one needs a covering \(\mathcal{Q} = (Q_i)_{i \in I}\) of the frequency domain which has to satisfy certain regularity criteria; namely, it has to be admissible and, preferably, almost-structured.

In Section \([2]\) we recall those parts of the existing theory of decomposition spaces that are essential for our work. In particular, we recapitulate the notions of admissible and almost-structured coverings, the existing theory concerning the existence of embeddings between different decomposition spaces, and the recent theory of structured Banach frame decompositions of decomposition spaces.

In Section \([3]\) we introduce the wave packet coverings \(\mathcal{Q}^{(\alpha, \beta)}\) that we shall use to define the wave packet smoothness spaces and verify that \(\mathcal{Q}^{(\alpha, \beta)}\) indeed covers the whole frequency plane. That the covering \(\mathcal{Q}^{(\alpha, \beta)}\) is admissible and almost-structured is shown in Sections \([4]\) and \([5]\) respectively.
The wave packet smoothness spaces will be defined in Section 6 where we also study many of their properties. First, we show that they are indeed well-defined quasi-Banach spaces. Second, we study the existence of embeddings $W_p^{α,β}(α, β) \hookrightarrow W_q^{α',β'}(α', β')$ between wave packet smoothness spaces with different parameters and show that distinct parameters yield distinct spaces. Third, we will completely characterise the existence of embeddings between inhomogeneous Besov spaces and wave packet smoothness spaces. Fourth, we study the conditions under which the wave packet smoothness spaces embed into the classical Sobolev spaces $W^{k,p}(\mathbb{R}^d)$. For the range $p \in [1, 2] \cup \{∞\}$ we characterise these conditions completely. Finally, we show that the $(α, α)$ wave packet smoothness spaces are identical to $α$-modulation spaces.

Since our construction of the covering $Q^{α,β}$ involves some non-canonical choice of parameters, the spaces $W_p^{α,β}(α, β)$ might appear rather esoteric. In Section 7 we show that this is not the case. Precisely, we introduce the natural class of $(α, β)$ coverings, and show that any two $(α, β)$ coverings give rise to the same family of decomposition spaces. We also verify that $Q^{α,β}$ is indeed an $(α, β)$ covering. This shows that the wave packet smoothness spaces are natural objects, and it allows us to show that the wave packet smoothness spaces are invariant under dilation with respect to arbitrary invertible matrices (see Section 8).

Finally, in Section 9 we formally define the notion of $(α, β)$ wave packet systems. We then show that the wave packet smoothness spaces can be described using the decay of the analysis or synthesis coefficients with respect to such a system. More formally, we show that if the generators of the wave packet system are sufficiently smooth and decay fast enough, then the associated wave packet system constitutes a Banach frame and an atomic decomposition for a whole range of wave packet smoothness spaces provided that the sampling density is fine enough.

The proofs of some particularly lengthy auxiliary statements were transferred to Appendices A – C. All general mathematical notions used in this manuscript are summarised in Appendix D.

2. Decomposition spaces and their relation to frames and sparsity

Decomposition spaces, originally introduced by Feichtinger and Gröbner [19], provide a unified generalisation of modulation and Besov spaces and were used to introduce the $α$-modulation spaces [25], which have recently received great attention [36, 55, 32, 19, 31, 33, 13, 4].

The essential element of the decomposition space $D(Q, L^p, ℓ^q_ω)$ is the covering $Q = (Q_1)_{i \in I}$ of the frequency domain $\mathbb{R}^d$. Given this covering, the Fourier transform $\hat{g}$ of a given function $g$ can be decomposed into the components $\varphi_i \cdot \hat{g}$, where $(\varphi_i)_{i \in I}$ is a suitable partition of unity subordinate to $Q$. The Fourier inverse $g_i := F^{-1}(\varphi_i \cdot \hat{g})$ of the components $\varphi_i \cdot \hat{g}$ are frequency-localised components of the function $g$. The contribution of each of these components $g_i$ to the decomposition space quasi-norm of $g$ is measured by the $L^p$-norm, in the time domain, and the total quasi-norm of $g$ is obtained by using the weighted $ℓ^q$ space $ℓ^q_ω$ as follows:

$$∥g∥_{D(Q, L^p, ℓ^q_ω)} = ∥∥g_i∥_{L^p}∥_{i \in I}∥_{ℓ^q_ω} = ∥∥w_i \cdot ∥F^{-1}(\varphi_i \cdot \hat{g})∥_{L^p}∥_{i \in I}∥_{ℓ^q_ω}. \tag{2.1}$$

To ensure that the decomposition space $D(Q, L^p, ℓ^q_ω)$ is indeed a well-defined quasi-Banach space, certain conditions must be imposed on the covering $Q$, the partition of unity $(\varphi_i)_{i \in I}$ and the weight $w = (w_i)_{i \in I}$. These conditions and elementary properties of the decomposition spaces $D(Q, L^p, ℓ^q_ω)$ will be reminded in Section 2.1.

An attractive feature of decomposition spaces is the recently developed theory of structured Banach frame decompositions of decomposition spaces [63], which shows that there is a close connexion between the existence of a sparse expansion of a given function $f$ in terms of a given frame, and the membership of $f$ in a certain decomposition space, which depends on the frame under consideration.

This theory will be formally introduced in Section 2.2 here, we outline the underlying intuition. We first note that most frame constructions used in harmonic analysis have two crucial properties:

- The frame is a generalised shift-invariant system (see [31, 59, 11, 24, 83] for more about these systems), i.e., it is of the form $Ψ = (L_x \psi_j)_{j \in I, x \in Γ_j}$ for suitable generators $(\psi_j)_{j \in I}$ and certain lattices $Γ_j = δB_j \mathbb{Z}^d$ where the matrices $B_j \in GL(\mathbb{R}^d)$ are determined by the frame construction and $δ > 0$ globally stands for the sampling density.
The matrices $B_j$ determine the relative step size of the translations that are applied to each of the generators $\psi_j$. For example, in a Gabor system, $B_j = \text{id}$ for all $j \in I$, while in a wavelet system, $B_j = 2^{-j} \cdot \text{id}$, so that the wavelets on higher scales have a much smaller step size than the wavelets on lower scales.

- The generators $\psi_j$ of the GSI system $\Psi$ have a characteristic frequency concentration. For instance, in a Gabor frame, $I = \mathbb{Z}^d$ and $\psi_j(x) = e^{2\pi i (j, x)} \cdot \psi(x)$ where $\psi$ is a given window function. Hence, if $\hat{\psi}$ is concentrated in a subset $Q$ of the frequency domain $\mathbb{R}^d$, then $\hat{\psi}_j$ is concentrated in $Q_j = Q + j$; the frequency tiling associated with a Gabor frame is thus uniform. Similarly, the frequency tiling associated with a wavelet system is dyadic.

For most frame constructions, the Fourier transforms $\hat{\psi}_j$ of the generators $\psi_j$ are concentrated in subsets of the frequency domain of the form $Q_j = T_j Q + b_j$. Here, $Q \subset \mathbb{R}^d$ is a fixed bounded set, $b_j \in \mathbb{R}^d$ and $T_j = B_j^{-1}$ where $\psi_j$ is, as mentioned above, translated along the lattice $\Gamma_j = \delta B_j \mathbb{Z}^d$.

Given such a system $\Psi = (L_x \psi_j)_{j \in I, x \in \Gamma_j}$, the theory of structured Banach frame decompositions provides verifiable conditions on the generators $\psi_j$ so that if these conditions are satisfied and if the sampling density $\delta > 0$ is fine enough, then $\Psi$ forms a Banach frame and an atomic decomposition for the decomposition spaces $D(Q, L^p, l^p_w)$, where $Q = (Q_j)_{j \in I}$.

In Subsection 2.2 we shall discuss in detail what these two notions mean and what kind of conditions the generators have to satisfy. Here, we merely note that for $p = q \in (0, 2]$ and a suitable choice of the weight $w = w(p)$, there is the following equivalence for any $f \in L^2(\mathbb{R}^d)$:

$$f \in D(Q, L^p, l^p_w) \iff ((f \mid L_x \psi_j)_{j \in I, x \in \Gamma_j}) \in l^p$$

$$\iff \exists (c_{j,x})_{j \in I, x \in \Gamma_j} \in l^p : f = \sum_{j \in I, x \in \Gamma_j} (c_{j,x} \cdot L_x \psi_j). \tag{2.2}$$

In other words, a function $f$ is sparse with respect to the frame $\Psi$ if and only if $f$ belongs to the decomposition space $D(Q, L^p, l^p_w)$.

We mention that the theory of structured Banach frames is related to the findings in [49]. In that paper, Nielsen and Rasmussen establish the existence of compactly supported Banach frames for certain decomposition spaces. The main difference between these results and those in [63] is that the theory of structured Banach frames does not just establish the existence of Banach frames; rather, it allows to verify whether a given set of prototype functions generates a Banach frame or an atomic decomposition. Moreover, the theory of structured Banach frames applies to more general coverings $Q$ than those considered in [49].

Finally, as we shall be interested in a whole family of wave packet systems, parametrised by $0 < \beta \leq \alpha < 1$, it is sensible to ask oneself whether sparsity of a function $f$ in a given wave packet system $W_{\alpha_1, \beta_1}^l(\psi_1, \delta_1)$ entails some, albeit worse, sparsity in another wave packet system $W_{\alpha_2, \beta_2}^l(\psi_2, \delta_2)$. Given (2.1), this is equivalent to the question of whether $D(Q^{(\alpha_1, \beta_1)}, L^p, l^p_w)$ is a subset of $D(Q^{(\alpha_2, \beta_2)}, L^p, l^p_w)$ where the frequency covering $Q^{(\alpha, \beta)}$ is the one associated with $(\alpha, \beta)$-wave packet systems. In many cases this question can be answered using the recently developed theory of embeddings for decomposition spaces [61] [60], which we shall briefly present in Subsection 2.3.

### 2.1. Definition of decomposition spaces

As explained after (2.1), one needs to impose certain conditions on the covering $Q$, the weight $w$, and the partition of unity $(\varphi_i)_{i \in I}$ in order to obtain well-defined decomposition spaces. Precisely, the covering $Q$ should be almost structured, the partition of unity $(\varphi_i)_{i \in I}$ should be regular, and the weight $(w_i)_{i \in I}$ should be $Q$-moderate. Let us now give the definitions of these notions:

**Definition 2.1.** (Definition 2.5 in [61]; inspired by [4])

A family $Q = (Q_i)_{i \in I}$ is called an almost structured covering of $\mathbb{R}^d$, if there is an associated family $(T_i \bullet + b_i)_{i \in I}$ of invertible affine-linear maps such that the following properties hold:

1. $Q$ is admissible; that is, the sets
   $$i^* := \{ \ell \in I : Q_\ell \cap Q_i \neq \emptyset \} \quad \text{for} \quad i \in I \tag{2.3}$$
   have uniformly bounded cardinality.
There is \( C > 0 \) such that \( \|T_i^{-1}T_\ell\| \leq C \) for all \( i, \ell \in I \) for which \( Q_i \cap Q_\ell \neq \emptyset \).

There are \( n \in \mathbb{N} \) and open, non-empty, bounded sets \( Q_1^{(0)}, \ldots, Q_n^{(0)}, P_1, \ldots, P_n \subset \mathbb{R}^d \) such that

- for each \( i \in I \) there is some \( k_i \in \{1, \ldots, n\} \) such that \( Q_i = T_i Q_k^{(0)} + b_i \);
- \( P_k \) is compactly contained in \( Q_k^{(0)} \); that is, \( \overline{T_k} \subset Q_k^{(0)} \) for all \( k \in \{1, \ldots, n\} \);
- \( \mathbb{R}^d = \bigcup_{i \in I} (T_i P_k + b_i) \).

If it is possible to choose \( n = 1 \), then the covering becomes **structured**, as it was defined in [4].

**Definition 2.2.** (Definition 2.4 in [22]; inspired by [3])

Let \( Q = (Q_i)_{i \in I} \) be an almost structured covering of \( \mathbb{R}^d \) with associated family \( (T_i \bullet + b_i)_{i \in I} \). A family of functions \( \Phi = (\varphi_i)_{i \in I} \) is called a regular partition of unity subordinate to \( Q \), if

1. \( \varphi_i \in C_c^\infty(\mathbb{R}^d) \) with \( \text{supp} \varphi_i \subset Q_i \) for all \( i \in I \);
2. \( \sum_{i \in I} \varphi_i \equiv 1 \) on \( \mathbb{R}^d \); and
3. \( \sup_{i \in I} \| D^\alpha \varphi_i^2 \|_{L^\infty} < \infty \) for all \( \alpha \in \mathbb{N}_0^d \), where \( \varphi_i^2 : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \varphi_i(T_i \xi + b_i) \).

**Remark.** The classical definition of decomposition spaces in [16] uses so-called BAPUs (bounded admissible partitions of unity) to define the decomposition spaces. The notion of regular partitions of unity is a modification of the concept of a BAPU and is necessary for handling the spaces \( L^p \) for \( p \in (0, 1) \), which are not considered in [16].

**Definition 2.3.** (Definition 3.1 in [16])

Let \( Q = (Q_i)_{i \in I} \) be an almost structured covering of \( \mathbb{R}^d \). A **weight** on \( I \) is a sequence \( w = (w_i)_{i \in I} \) where \( w_i \in (0, \infty) \) for all \( i \in I \). Such a weight is called \( Q \)-**moderate**, if there is \( C > 0 \) such that \( w_i \leq C \cdot w_\ell \) for all \( i, \ell \in I \) for which \( Q_i \cap Q_\ell \neq \emptyset \). We write \( C_{Q, w} \) for the smallest constant \( C \) for which this holds; that is, \( C_{Q, w} = \sup_{i \in I} \sup_{\ell \in I^I} w_i/w_\ell \).

The following theorem ensures that, given an almost structured covering, one can always find an associated regular partition of unity:

**Theorem 2.4.** (Theorem 2.8 in [22]; inspired by Proposition 1 in [4])

Let \( Q = (Q_i)_{i \in I} \) be an almost structured covering of \( \mathbb{R}^d \). Then the index set \( I \) is countably infinite and there exists a regular partition of unity subordinate to \( Q \).

In principle, the decomposition space \( D(Q, L^p, 1) \) could be defined as the set of all tempered distributions \( g \in S'(\mathbb{R}^d) \) for which \( \|g\|_{D(Q, L^p, 1)} < \infty \) with the quasi-norm \( \|g\|_{D(Q, L^p, 1)} \) as defined in [21]. However, the decomposition space defined in this way would not necessary be complete (see the example in Section 5 in [22]). To avoid this possible incompleteness, we shall use a slightly different set than the space of tempered distributions for defining the decomposition spaces:

**Definition 2.5.** (inspired by [56])

Let us define the set \( Z := \mathcal{F}(C_c^\infty(\mathbb{R}^d)) \subset S(\mathbb{R}^d) \) and equip it with the unique topology that makes the Fourier transform \( \mathcal{F} : C_c^\infty(\mathbb{R}^d) \to Z \) into a homeomorphism. The topological dual space \( Z' \) of \( Z \) will be called the **reservoir**. We write \( \langle \phi, g \rangle_{Z'} := \langle \phi, g \rangle := \phi(g) \) for the bilinear dual pairing between \( Z' \) and \( Z \).

As in the space of tempered distributions, the **Fourier transform in the reservoir** \( Z' \) can be defined by using its duality with the space \( Z \), i.e.

\[
\mathcal{F} : Z' \to D'(\mathbb{R}^d), \phi \mapsto \mathcal{F} \phi := \hat{\phi} := \phi \circ \mathcal{F} \quad \text{and therefore} \quad \langle \mathcal{F} \phi, g \rangle_{Z'} = \langle \hat{\phi}, g \rangle_{Z'} \text{ for } g \in C_c^\infty(\mathbb{R}^d).
\]

When \( Z' \) and \( D'(\mathbb{R}^d) \) are both equipped with their respective weak-*topologies, this Fourier transform is a homeomorphism with its inverse being given by \( \mathcal{F}^{-1} : D'(\mathbb{R}^d) \to Z', \phi \mapsto \phi \circ \mathcal{F}^{-1} \).

We can now define our decomposition spaces.
Definition 2.6. Let \( Q = (Q_i)_{i \in I}, \Phi = (\varphi_i)_{i \in I}, \) and \( w = (w_i)_{i \in I} \) be an almost structured covering of \( \mathbb{R}^d, \) a regular partition of unity subordinate to \( Q \) and a \( Q \)-moderate weight, respectively, and let \( p, q \in (0, \infty]. \)

The decomposition space with the covering \( Q, \) the weight \( w, \) and the integrability exponents \( p \) and \( q \) is defined as
\[
\mathcal{D}(Q, L^p, \ell^q_w) := \left\{ g \in \mathcal{Z}' : \|g\|_{\mathcal{D}(Q, L^p, \ell^q_w)} < \infty \right\},
\]
where the decomposition space quasi-norm \( \|g\|_{\mathcal{D}(Q, L^p, \ell^q_w)} \) of any \( g \in \mathcal{Z}' \) is defined as
\[
\|g\|_{\mathcal{D}(Q, L^p, \ell^q_w)} := \left\| (w_i \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \hat{g})\|_{L^p})_{i \in I} \right\|_{\ell^q_w} \in [0, \infty].
\]

Remark. At this point a few comments regarding the notions introduced in Definition 2.6 are appropriate.

First, we see from Definition 2.5 of the Fourier transform \( \mathcal{F} : Z' \rightarrow \mathcal{D}'(\mathbb{R}^d) \) that \( \hat{g} \in \mathcal{D}'(\mathbb{R}^d) \) for \( g \in \mathcal{Z}' \), whence \( \varphi_i \cdot \hat{g} \) is a tempered distribution with compact support. Furthermore the Paley-Wiener theorem (Theorem 7.23 in [54]) allows us to infer that \( \|\mathcal{F}^{-1}(\varphi_i \cdot \hat{g})\|_{L^p} \in [0, \infty] \) makes sense. Given the convention that \( \|c_i\|_{\ell^q} = \infty \) if \( c_i = \infty \) for some \( i \in I, \) we can now conclude that \( \|g\|_{\mathcal{D}(Q, L^p, \ell^q_w)} \in [0, \infty] \) is indeed well-defined.

Second, the combination of Corollary 2.7 in [62] and Corollary 3.18 in [61] allows us to conclude that any two regular partitions of unity will yield equivalent quasi-norms as in Equation (2.4). Therefore, the space \( \mathcal{D}(Q, L^p, \ell^q_w) \) is independent of the choice of the regular partition of unity.

Third, we chose to use the somewhat unusual reservoir \( \mathcal{Z}' \) to make sure that our decomposition space is complete. Indeed, in Theorem 3.12 in [61], which uses the same definition of decomposition spaces as we do here, it is shown that \( (\mathcal{D}(Q, L^p, \ell^q_w), \|\cdot\|_{\mathcal{D}(Q, L^p, \ell^q_w)}) \) is a quasi-Banach space; that is, a complete quasi-normed vector space.

2.2. Structured Banach frame decompositions for decomposition spaces

Let us select a particular almost structured covering \( Q = (Q_i)_{i \in I} \) of \( \mathbb{R}^d \) with associated family \( (T_i \bullet + b_i)_{i \in I}. \) By definition, there are non-empty, open, bounded sets \( Q_1^{(0)}, \ldots, Q_n^{(0)} \subset \mathbb{R}^d \) and for each \( i \in I \) some \( k_i \in \{1, \ldots, n\} \) such that \( Q_i = T_i Q_{k_i}^{(0)} + b_i. \) Let us select one particular such family \( (k_i)_{i \in I} \), which we shall use in the rest of this subsection. Furthermore, let us define \( Q_i' := Q_{k_i}^{(0)} \) for \( i \in I. \)

In this section, we consider generalised shift-invariant systems of the form
\[
\Gamma^{(i)} := \left( \gamma_{i, k}^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d} := \left( L_{\delta T_i^{-1} \bullet - k} \gamma_i^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d}
\]
where the generators \( \gamma_i^{[i]} \) are given by
\[
\gamma_i^{[i]} := |\det T_i|^{1/2} : M_{b_i} \gamma_i (T_i \bullet)
\]
with a suitable prototype functions \( \gamma_i \in L^2(\mathbb{R}^d). \) The suitable choice of these prototype functions \( \gamma_i \) will ensure that the system \( \Gamma^{(i)} \) is compatible with the frequency covering \( Q. \) Indeed, if the Fourier transform \( \hat{\gamma}_i \) of \( \gamma_i \) decays rapidly outside the set \( Q_i', \) then \( \hat{\gamma}_i^{[i]} = |\det T_i|^{-1/2} \cdot \hat{\gamma}_i (T_i^{-1} \bullet - b_i), \) so that the Fourier transform of \( \gamma_i^{[i]} \) decays rapidly outside the set \( Q_i = T_i Q_i' + b_i. \) Therefore, except for their normalisation, the \( \gamma_i^{[i]} \) are similar to a regular partition of unity \( \Phi = (\varphi_i)_{i \in I} \) subordinate to \( Q. \) Thus, with the generalised shift invariant system \( \Gamma^{(i)} \) defined in Equation (2.5), one would intuitively expect that the membership of a function \( g \) in the decomposition space \( \mathcal{D}(Q, L^p, \ell^q_w) \) could be characterised in terms of the decay of its coefficients \( \left( \langle g, \gamma_{i, k}^{[i]} \rangle \right)_{i \in I, k \in \mathbb{Z}^d}. \)

The theory of structured Banach frames [63], whose elements essential to this work we shall remind here, makes this intuition precise and provides criteria on the prototype functions \( \gamma_i \) that, if satisfied, will guarantee that the system \( \Gamma^{(i)} \) constitutes a Banach frame or an atomic decomposition for the decomposition space \( \mathcal{D}(Q, L^p, \ell^q_w) \) provided the sampling density \( \delta > 0 \) is sufficiently fine.

The concept of Banach frames and atomic decompositions [26] are generalisations of the notion of frames in Hilbert spaces. By definition, a frame \( (\psi_j)_{j \in J} \) in a Hilbert space \( \mathcal{H} \) satisfies
\[ \|x\|_\mathcal{H}^2 \approx \sum_{j \in J} \|x_j\|_{\mathcal{H}}^2 \] for all \( x \in \mathcal{H} \). In other words, the norm of an element \( x \) of the Hilbert space can be characterised in terms of its coefficients \( (x_j)_{j \in J} \). This, given the rich structure of Hilbert spaces, has far-reaching consequences. In particular, \( (\psi_j)_{j \in J} \) has a dual frame \( (\widetilde{\psi}_j)_{j \in J} \) (see Theorem 5.1.6 in [10]) which satisfies

\[ x = \sum_{j \in J} (x \mid \widetilde{\psi}_j)_{\mathcal{H}} \psi_j = \sum_{j \in J} (x \mid \psi_j)_{\mathcal{H}} \widetilde{\psi}_j \quad \forall x \in \mathcal{H}. \]

Thus, any \( x \in \mathcal{H} \) can be, on the one hand, stably recovered from its coefficients \( (x_j)_{j \in J} \in \ell^2(J) \) and, on the other hand, represented as a series \( x = \sum_{j \in J} c_j \psi_j \), where the coefficients \( (c_j)_{j \in J} \in \ell^2(J) \) depend linearly and continuously on \( x \). Each of these properties can be shown to be equivalent to \( (\psi_j)_{j \in J} \) being a frame for \( \mathcal{H} \) and thus are equivalent to each other. In Banach spaces, however, these properties are no longer equivalent, thus leading to the introduction of the concepts of Banach frames and atomic decompositions.

In the space \( \ell^2(J) \), if \( c = (c_j)_{j \in J} \in \ell^2(J) \) and \( e = (e_j)_{j \in J} \) are such that \( |e_j| \leq |c_j| \) for all \( j \in J \), then \( e \in \ell^2(J) \) and \( \|e\|_\ell^2 \leq \|c\|_\ell^2 \). More generally, a quasi-Banach space \( X \subset C^J \) — which consists of sequences with index set \( J \) — with the analogous property is called solid.

**Definition 2.7.** A family \( \Psi = (\psi_i)_{i \in I} \) in a quasi-Banach space \( Y \) is called an atomic decomposition of \( Y \) with coefficient space \( X \), if

1. \( X \subset \mathbb{C}^J \) is a solid quasi-Banach space;

2. the synthesis map \( S_\Psi : X \rightarrow Y, (c_j)_{j \in J} \mapsto \sum_{j \in J} c_j \psi_j \) is well-defined and bounded, with convergence of the series in a suitable topology; and

3. there is such a bounded linear coefficient map \( C_\Psi : Y \rightarrow X \) that \( S_\Psi \circ C_\Psi = \text{id}_Y \).

**Definition 2.8.** A family \( \Theta = (\theta_j)_{j \in J} \) in the dual space \( Y' \) of a quasi-Banach space \( Y \) is called a Banach frame for \( Y \) with coefficient space \( X \), if

- \( X \subset \mathbb{C}^J \) is a solid quasi-Banach space;

- the analysis map \( A_\Psi : Y \rightarrow X, f \mapsto ((f, \theta_j)_{Y \times Y'})_{j \in J} \) is well-defined and bounded; and

- there is such a bounded linear reconstruction map \( R_\Psi : X \rightarrow Y \) that \( R_\Psi \circ A_\Psi = \text{id}_Y \).

We now introduce the associated sequence spaces which we shall use in the theory of structured Banach frames for decomposition spaces.

**Definition 2.9.** (Definition 2.8 in [64])

For \( p, q \in (0, \infty) \) and \( w = (w_i)_{i \in I} \), the associated coefficient space \( C_w^{p,q} \subset C^I \times \mathbb{Z}^d \) is defined as

\[ C_w^{p,q} := \left\{ c = (c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} : \|c\|_{C_w^{p,q}} := \left\| \left( \left| \det T_i \right|^{\frac{1}{2} - \frac{1}{p}} \cdot w_i \cdot \|c_k^{(i)}\|_{\ell_q} \right)_{i \in I} \right\|_{\ell_q} < \infty \right\}. \]

The following theorem on structured atomic decompositions for decomposition spaces is a combination of Theorem 2.10 and Proposition 2.11 in [64], which provide simplified versions of the results obtained in [63].

**Theorem 2.10.** Let \( \epsilon, p_0, q_0 \in (0, 1], p, q \in (0, \infty] \) such that \( p \geq p_0 \) and \( q \geq q_0 \) and \( w = (w_i)_{i \in I} \) be \( \mathcal{Q} \)-moderate. Let \( \gamma_i^{(0)}, \ldots, \gamma_n^{(0)} \in L^1(\mathbb{R}^d) \) and \( \gamma_i := \gamma_i^{(0)} \) for \( i \in I \). Let us define

\[ \Lambda := 1 + \frac{d}{\min\{1, p\}} \quad \text{and} \quad N := \frac{d + \epsilon}{\min\{1, p\}} \]

and assume that, for each \( k \in \{1, \ldots, n\} \), there is a non-negative function \( q_k \in L^1(\mathbb{R}^d) \) such that the following hold:

1. \( \mathcal{F}\gamma_k^{(0)} \in C^\infty(\mathbb{R}^d) \) and all partial derivatives of \( \mathcal{F}\gamma_k^{(0)} \) are of polynomial growth at most;
Finally, let us define
\[
\tau := \min\{1, p, q\}, \quad \vartheta := (p^{-1} - 1)_+ \quad \text{and} \quad \sigma := \begin{cases} \tau \cdot (d + 1), & \text{if } p \in [1, \infty), \\
\tau \cdot (p^{-1} \cdot d + \lfloor p^{-1} \cdot (d + \epsilon) \rfloor), & \text{if } p \in (0, 1) \end{cases}
\]
and
\[
N_{i,j} := \left(\frac{w_i}{w_j} \cdot \left(\frac{\det T_j}{\det T_i}\right)^\vartheta \cdot (1 + \|T_j^{-1}T_i\|)\sigma \cdot \left(\frac{\det T_i}{\det T_j}\right)^{\tau}\int q_k(T^{-1}_j(\xi - b_j))\,d\xi\right)^\tau
\]
for \(i, j \in I\) and assume that
\[
K_1 := \sup_{i \in I} \sum_{j \in I} N_{i,j} < \infty \quad \text{and} \quad K_2 := \sup_{j \in I} \sum_{i \in I} N_{i,j} < \infty. \tag{2.7}
\]
Then there is a \(\delta_0 > 0\) such that for any \(\delta \in (0, \delta_0]\), the family \(\Gamma^{(\delta)}\) as defined by \(2.6\) and \(2.5\) constitutes an atomic decomposition for the decomposition space \(\mathcal{D}(Q, L^p, \ell^q_w)\) with associated coefficient space \(C_w^{p,q}\) as introduced in Definition 2.9.

More specifically,
\begin{enumerate}
  \item there is a constant \(C = C(p_0, q_0, \varepsilon, d, Q, \gamma_1^{(0)}, \ldots, \gamma_n^{(0)}) > 0\) that allows us to choose
  \[
  \delta_0 = \min \left\{ 1, \left[ C \cdot (K_1^{1/\tau} + K_2^{1/\tau}) \right]^{-1} \right\};
  \]
  \item the synthesis map
  \[
  S_{\Gamma^{(\delta)}} : C_{w}^{p,q} \to \mathcal{D}(Q, L^p, \ell^q_w), (c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} c_k^{(i)} \cdot L_{\delta T_i^{-1}k}^{\gamma[i]}
  \]
  is well-defined and bounded for all \(\delta \in (0, 1]\). Moreover, for each \(i \in I\) the inner series
  \[
  \sum_{k \in \mathbb{Z}^d} c_k^{(i)} \cdot L_{\delta T_i^{-1}k}^{\gamma[i]}
  \]
  converges absolutely to a function \(g_i \in L^1_{\text{loc}}(\mathbb{R}^d) \cap S'(\mathbb{R}^d)\) and the series \(S_{\Gamma^{(\delta)}}(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} = \sum_{i \in I} g_i\) converges unconditionally in the weak*-sense in \(Z'\); and
  \item for \(0 < \delta \leq \delta_0\), there is a coefficient operator \(C^{(\delta)} = C_{p,q,w}^{(\delta)} : \mathcal{D}(Q, L^p, \ell^q_w) \to C_w^{p,q}\) such that
  \[
  S_{\Gamma^{(\delta)}} \circ C^{(\delta)} = \text{id}_{\mathcal{D}(Q, L^p, \ell^q_w)}.
  \]
  Furthermore, the action of \(C^{(\delta)}\) on a given \(f \in \mathcal{D}(Q, L^p, \ell^q_w)\) is independent of \(p, q\) and \(w\), thus justifying the notation \(C^{(\delta)}\).
\end{enumerate}

Remark. 1) The description of the convergence of the series in \(2.8\) might appear quite technical. Luckily, for \(p, q < \infty\), this description can be simplified. Indeed, the finitely supported sequences are dense in \(C_w^{p,q}\) if \(p, q < \infty\). Combined with the boundedness of the synthesis map \(S_{\Gamma^{(\delta)}}\), this implies that the series \(\sum_{i,k \in \mathbb{Z}^d} c_k^{(i)} L_{\delta T_i^{-1}k}^{\gamma[i]}\) converges unconditionally in \(\mathcal{D}(Q, L^p, \ell^q_w)\).

2) We note that the conditions \([1]\) and \([3]\) are satisfied as long as all \(\gamma_k^{(0)}\) are bounded and have compact supports. In the case of the condition \([1]\), this is a consequence of the Paley-Wiener theorem.

For the next theorem — which is a combination of Theorem 2.9 and Lemma 5.12 in [64] — we shall need a GSI system \(\tilde{\Gamma}^{(\delta)}\) that differs slightly from the system \(\Gamma^{(\delta)}\) given by \(2.6\) and \(2.5\). Precisely, let us define
\[
\tilde{\Gamma}^{(\delta)} := \left( L_{\delta T_i^{-1}k}^{\gamma[i]} \right)_{i \in I, k \in \mathbb{Z}^d}, \quad \text{where} \quad \tilde{g}(x) = g(-x). \tag{2.9}
\]

**Theorem 2.11.** Let \(\varepsilon, p_0, q_0 \in (0, 1]\) and \(p, q \in (0, \infty]\) such that \(p \geq p_0\) and \(q \geq q_0\). Let \(\Phi = (\varphi_i)_{i \in I}\) and \(w = (w_i)_{i \in I}\) be a regular partition of unity for \(Q\) and \(Q\)-moderate weight, respectively. Let \(\gamma_1^{(0)}, \ldots, \gamma_n^{(0)} \in L^1(\mathbb{R}^d)\) and let us define \(\gamma_i := \gamma_k^{(0)}\) for \(i \in I\). Let us assume that \(\gamma_k^{(0)}\) for all \(k \in \{1, \ldots, n\}\),
(1) \( \mathcal{F} \gamma_k^{(0)} \in C^\infty(\mathbb{R}^d) \) and all partial derivatives of \( \mathcal{F} \gamma_k^{(0)} \) are of polynomial growth at most;

(2) \( \mathcal{F} \gamma_k^{(0)}(\xi) \neq 0 \) for all \( \xi \in \overline{Q}_k^{(0)} \);

(3) \( \gamma_k^{(0)} \in C^1(\mathbb{R}^d) \) and \( \nabla \gamma_k^{(0)} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \).

Finally, let us define

\[
N := \left\lfloor \frac{d + \varepsilon}{\min\{1, p\}} \right\rfloor, \quad \tau := \min\{1, p, q\}, \quad \theta := \tau \cdot \left( N + \frac{d}{\min\{1, p\}} \right) \]

and

\[
M_{j,i} := \left( \frac{w_i}{w_j} \right) \tau \cdot (1 + \|T_1^{-1}T_i\|) \cdot \max_{|\beta| \leq 1} \left( \left| \det T_i \right|^{-1} \int_{Q_i} \max_{|\alpha| \leq N} \left| \partial^\alpha \partial^j_{\beta} \gamma_j \right| \left| T^{-1}_j(\xi - b_j) \right| \, d\xi \right) ^\tau
\]

for \( i, j \in I \) and assume that

\[ C_1 := \sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty \quad \text{and} \quad C_2 := \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty. \]

Then there is a \( \delta_0 \in (0, 1) \) such that for any \( \delta \in (0, \delta_0) \), the family \( \tilde{\Gamma}(\delta) \) defined in Equations (2.6) and (2.9) constitutes a Banach frame for the decomposition space \( D(Q, L^p, \ell^q_w) \) with associated coefficient space \( C_{w_{\ell}}^{p,q} \) as introduced in Definition 2.9.

More specifically,

(1) There is a constant \( C = C(p_0, q_0, \varepsilon, d, Q, \gamma_1^{(0)}, \ldots, \gamma_n^{(0)}) > 0 \) that allows us to choose

\[
\delta_0 = \delta_0(p_0, q_0, w) = 1 \left/ \left[ 1 + C \cdot C_{Q, w} \cdot (C_1^{1/\tau} + C_2^{1/\tau}) \right] \right. \]

(2) The analysis map

\[
A_{\Gamma(\delta)} : D(Q, L^p, \ell^q_w) \to C_{w_{\ell}}^{p,q}, \quad f \mapsto \left( [\gamma[k] \ast f](\delta \cdot T_i^{-1}k) \right)_{i \in I, k \in \mathbb{Z}^d},
\]

where the convolution \( [\gamma[k] \ast f] \), i.e.

\[
([\gamma[k] \ast f])(x) = \sum_{\ell \in \ell} \mathcal{F}^{-1}(\gamma[k] \cdot \varphi_{\ell} \cdot \hat{f})(x),
\]

is well-defined and bounded for all \( \delta \in (0, 1) \) and the series in \((2.10)\) converges normally in \( L^\infty(\mathbb{R}^d) \).

Moreover, if \( f \in L^2(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d) \hookrightarrow Z' \), then the convolution defined by \((2.10)\) agrees with its usual definition and

\[
A_{\Gamma(\delta)} f = \left( (f, L_{\delta - T_i^{-1}k}(\gamma[k])) \right)_{i \in I, k \in \mathbb{Z}^d} \quad \forall f \in L^2(\mathbb{R}^d) \cap D(Q, L^p, \ell^q_w).
\]

(3) For \( 0 < \delta \leq \delta_0 \), there is such a bounded linear reconstruction map \( R_{\delta}^{(\delta)} : C_{w_{\ell}}^{p,q} \to D(Q, L^p, \ell^q_w) \) that \( R_{p,q,w}^{(\delta)} \circ A_{\Gamma(\delta)} = \text{id}_{D(Q, L^p, \ell^q_w)} \).

(4) If the assumptions of the current theorem are valid for \( (p, q, w) = (p_{\ell}, q_{\ell}, w_{\ell}) \) for \( \ell \in \{1, 2\} \) and \( 0 < \delta \leq \min\{\delta_0(p_0, q_0, w^{(1)}), \delta_0(p_0, q_0, w^{(2)})\} \), then

\[
\forall f \in D(Q, L^{p_2}, \ell^{q_2}_{w^{(2)}}) : f \in D(Q, L^{p_1}, \ell^{q_1}_{w^{(1)}}) \iff A_{\Gamma(\delta)} f \in C_{w_{\ell}}^{p_1,q_1}.
\]
2.3. Embeddings of decomposition spaces

In this subsection, we recall from [61] the results concerning the existence of embeddings between two decomposition spaces \( D(Q, L^p, \ell^q_w) \) and \( D(P, L^p, \ell^q_w) \) which we shall need in the following. Furthermore, we recall a few notions and results established by Feichtinger and Gröbner [16] on which we shall rely in this work.

**Definition 2.12.** Let \( Q \) and \( P \) be two almost-structured coverings of \( \mathbb{R}^d \) and \( w, u \) be a \( Q \)-moderate weight and a \( P \)-moderate weight, respectively and let \( p_1, p_2, q_1, q_2 \in (0, \infty] \).

We shall write \( D(Q, L^p, \ell^q_w) \hookrightarrow D(P, L^p, \ell^q_u) \) and say that \( D(Q, L^p, \ell^q_w) \) embeds in \( D(P, L^p, \ell^q_u) \), if \( D(Q, L^p, \ell^q_w) \subset D(P, L^p, \ell^q_u) \) and if the identity map \( D(Q, L^p, \ell^q_w) \to D(P, L^p, \ell^q_u), f \mapsto f \) is bounded.

**Remark.** From the closed graph theorem (see Theorem 2.15 in [54]), in combination with the embeddings \( D(Q, L^p, \ell^q_w) \hookrightarrow Z' \) and \( D(P, L^p, \ell^q_u) \hookrightarrow Z' \) (see Theorem 3.21 in [61]), we infer that, if \( D(Q, L^p, \ell^q_w) \subset D(P, L^p, \ell^q_u) \), then \( D(Q, L^p, \ell^q_w) \hookrightarrow D(P, L^p, \ell^q_u) \); that is, the identity map is always bounded if the decomposition spaces are included in each other.

To be able to provide meaningful criteria allowing to decide whether such an embedding holds, one needs a certain compatibility between the coverings \( Q \) and \( P \). The required type of compatibility is discussed in the following definition.

**Definition 2.13.** (Definition 3.3 in [16]) Let \( Q = (Q_i)_{i \in I} \) be an admissible covering of \( \mathbb{R}^d \). Using the notation \( i^* \) as introduced in (2.3), let us define \( L^* := \bigcup_{i \in I} L^* \subset I \) for any \( L \subset I \). Moreover, let us inductively define \( L^{0*} := L \), and \( L^{(n+1)*} := (L^{n*})^* \) for \( n \in \mathbb{N}_0 \). Finally, let us write \( i^{n*} := \{i\}^{n*} \) and \( Q_i^{n*} := \bigcup_{j \in i^{n*}} Q_j \) for \( i \in I \) and \( n \in \mathbb{N} \).

Now, let \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two admissible coverings of \( \mathbb{R}^d \). Let us define

\[
I_j := \{i \in I : Q_i \cap P_j \neq \emptyset\} \quad \text{for } j \in J \quad \text{and} \quad J_i := \{j \in J : P_j \cap Q_i \neq \emptyset\} \quad \text{for } i \in I. \tag{2.12}
\]

We shall say that

(1) \( Q \) is **weakly subordinate** to \( P \) if \( \sup_{i \in I} |J_i| \) is finite, that is, if the number of elements of the sets \( J_i \) is uniformly bounded;

(2) \( Q \) is **almost subordinate** to \( P \) if

\[
\exists N \in \mathbb{N}_0 \quad \forall i \in I \quad \exists j_i \in J : Q_i \subset P_{j_i}^{N*};
\]

(3) \( Q \) and \( P \) are **weakly equivalent** if \( Q \) is weakly subordinate to \( P \) and if also \( P \) is weakly subordinate to \( Q \); and

(4) \( Q \) and \( P \) are **equivalent**, if \( Q \) is almost subordinate to \( P \) and if also \( P \) is almost subordinate to \( Q \).

Most of the results in [61] concerning embeddings of decomposition spaces will require \( Q \) to be almost subordinate to \( P \), or vice versa. However, this almost subordinateness is often quite difficult to verify. Since it is often easier to verify that one covering is weakly subordinate to another, the following lemma will be useful.

**Lemma 2.14.** (slightly corrected version of Proposition 3.6 in [16]; see also Lemma 2.12 in [61]) Let \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two admissible coverings of \( \mathbb{R}^d \) such that each \( Q_i \) is path-connected and each \( P_j \) is open.

Then \( Q \) is weakly subordinate to \( P \) if and only if \( Q \) is almost subordinate to \( P \).

In addition to the different concepts of subordinateness, we shall also need the following two notions of relative moderateness.

**Definition 2.15.** Let \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two almost structured coverings of \( \mathbb{R}^d \) with associated families \((T_i \cdot + b_i)_{i \in I}\) and \((S_j \cdot + c_j)_{j \in J}\) and let \( w = (w_i)_{i \in I} \) be a weight. We shall say that
(1) \( w \) is **relatively \( \mathcal{P} \)-moderate** if there is a constant \( C > 0 \) such that
\[
  w_i \leq C \cdot w_\ell \quad \text{for all} \ i, \ell \in I \text{ and all} \ j \in J \text{ for which} \ Q_i \cap P_j \neq \emptyset \neq Q_\ell \cap P_\ell ;
\]

(2) \( Q \) is relatively \( \mathcal{P} \)-moderate if the weight \( (|\det T_i|)_{i \in I} \) is relatively \( \mathcal{P} \)-moderate.

We now state the two embedding results on which we shall rely. In the first, we assume \( \mathcal{P} \) to be almost subordinate to \( Q \), while in the second we will assume \( Q \) to be almost subordinate to \( \mathcal{P} \).

**Theorem 2.16.** *(special case of Theorem 7.2 in [61]*)

Let \( p_1, p_2, q_1, q_2 \in (0, \infty) \). Let \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two almost structured coverings of \( \mathbb{R}^d \) with associated families \( (T_i \bullet + b_i)_{i \in I} \) and \( (S_j \bullet + c_j)_{j \in J} \). Let \( w = (w_i)_{i \in I} \) and \( v = (v_j)_{j \in J} \) be \( Q \)-moderate and \( \mathcal{P} \)-moderate, respectively.

Assume that \( \mathcal{P} \) is almost subordinate to \( Q \) and that \( \mathcal{P} \) and \( v \) are relatively \( Q \)-moderate. Finally, for each \( i \in I \), let us choose an index \( j_i \in J \) that \( Q_i \cap P_{j_i} \neq \emptyset \). Then \( D(Q, L^{p_1}, \ell_{w_i}^1) \hookrightarrow D(\mathcal{P}, L^{p_2}, \ell_{v_j}^2) \) if and only if
\[
p_1 \leq p_2 \quad \text{and} \quad \left\| \frac{v_j}{w_i} \cdot |\det T_i|^\nu \cdot |\det S_{j_i}|^{p_1^{-1} - p_2^{-1} - \nu} \right\|_{\ell_{p_2}(q_1/q_2)^J} < \infty
\]
where
\[
  \nu := (q_2^{-1} - p_1^*)_+ \quad \text{with} \quad x_+ := \max\{0, x\} \quad \text{and} \quad p_1^* := \min\{p_1^{-1}, 1 - p_1^{-1}\},
\]
and where the exponent \( q_2 \cdot (q_1/q_2)^J \in (0, \infty) \) is defined by
\[
  \frac{1}{q_2 \cdot (q_1/q_2)^J} = (q_2^{-1} - q_1^{-1})_+ \cdot .
\]

In particular, \( q_2 \cdot (q_1/q_2)^J = \infty \) if and only if \( q_1 \leq q_2 \).

**Remark.** The definition \((2.13)\) results in the same value as when computing \( q_2 \cdot (q_1/q_2)^J \) as usual (with the conjugate exponent as defined in Appendix \( D \)) if the latter expression is defined; the advantage of \((2.13)\) is that it is defined in some cases where \( q_2 \cdot (q_1/q_2)^J \) is not — for instance if \( q_2 = \infty \).

**Theorem 2.17.** *(special case of Theorem 7.4 in [61]*)

Let \( p_1, p_2, q_1, q_2 \in (0, \infty) \), let \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two almost structured coverings of \( \mathbb{R}^d \) with associated families \( (T_i \bullet + b_i)_{i \in I} \) and \( (S_j \bullet + c_j)_{j \in J} \), and let \( w = (w_i)_{i \in I} \) and \( v = (v_j)_{j \in J} \) be \( Q \)-moderate and \( \mathcal{P} \)-moderate, respectively.

Let us assume that \( Q \) is almost subordinate to \( \mathcal{P} \) and that \( Q \) and \( w \) are relatively \( \mathcal{P} \)-moderate. Finally, for each \( j \in J \), let us choose \( i_j \in I \) such that \( Q_{i_j} \cap P_j \neq \emptyset \). Then \( D(Q, L^{p_1}, \ell_{w_i}^1) \hookrightarrow D(\mathcal{P}, L^{p_2}, \ell_{v_j}^2) \) if and only if
\[
p_1 \leq p_2 \quad \text{and} \quad \left\| \frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{p_1^{-1} - p_2^{-1} - \mu} \cdot |\det S_{j_i}|^{\mu} \right\|_{\ell_{p_2}(q_1/q_2)^J} < \infty
\]
where the exponent \( q_2 \cdot (q_1/q_2)^J \in (0, \infty) \) is as defined in \((2.13)\) and where
\[
  \mu := (p_2^{**} - q_1^{-1})_+ \quad \text{with} \quad p_2^{**} := (\min\{p_2, p_2^*\})^{-1}.
\]
Here, \( p_2^* \) is the conjugate exponent of \( p_2 \in (0, \infty) \), as defined in Appendix \( D \).

Finally, we shall also need the following rigidity result, which shows that if two decomposition spaces are identical, then the “ingredients” used to define the decomposition spaces are closely related.

**Theorem 2.18.** *(Theorem 6.9 in [61]*)

Let \( p_1, p_2, q_1, q_2 \in (0, \infty) \), \( Q = (Q_i)_{i \in I} \) and \( P := (P_j)_{j \in J} \) be two almost structured coverings of \( \mathbb{R}^d \) and \( w = (w_i)_{i \in I} \) and \( v = (v_j)_{j \in J} \) be \( Q \)-moderate and \( \mathcal{P} \)-moderate, respectively.

If \( D(Q, L^{p_1}, \ell_{w_i}^1) = D(\mathcal{P}, L^{p_2}, \ell_{v_j}^2) \), then \((p_1, q_1) = (p_2, q_2)\) and there is a constant \( C > 0 \) such that
\[
  C^{-1} \cdot w_i \leq v_j \leq C \cdot w_i \quad \forall i \in I \text{ and } j \in J \text{ for which } Q_i \cap P_j \neq \emptyset.
\]
If furthermore \((p_1, q_1) \neq (2, 2)\) then \( Q \) and \( \mathcal{P} \) are weakly equivalent.
3. Defining the wave packet covering $Q^{(\alpha,\beta)}$

In order to define the wave packet smoothness spaces, we shall need suitable coverings of the frequency plane $\mathbb{R}^2$, which we now introduce. We recall that $N = \{1, 2, 3, \ldots\}$, $N_0 = \{0\} \cup \mathbb{N}$ and $B_r(x)$ is the Euclidean ball of radius $r$ with its centre in $x \in \mathbb{R}^d$.

**Definition 3.1.** Let $0 \leq \beta \leq \alpha \leq 1$. First, let

\begin{align*}
N &:= 10, \\
m_{j}^{\max} &:= m_{j}^{\max,\alpha} := \left[2^{(1-\alpha)j-1}\right], \quad \text{and} \quad \ell_{j}^{\max,\beta} := \left[N \cdot 2^{(1-\beta)j}\right] 
\end{align*}

and furthermore

\begin{align*}
I_{0}^{(\alpha,\beta)} &:= \{0\} \cup I_{0}^{(\alpha,\beta)}, \quad \text{where} \quad I_{0}^{(\alpha,\beta)} := \{(j, m, \ell) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}_0 : m \leq m_{j}^{\max} \text{ and } \ell \leq \ell_{j}^{\max}\}. 
\end{align*}

Second, let us choose $\varepsilon \in (0, 1/32)$ and define

\begin{align*}
Q &:= (-\varepsilon, 1 + \varepsilon) \times (-1-\varepsilon, 1 + \varepsilon) \quad \text{and} \quad P := [0, 1] \times [-1, 1].
\end{align*}

Third, for all $j \in \mathbb{N}$ and all $m \in \mathbb{N}_0$ such that $m \leq m_{j}^{\max}$, let us define

\begin{align*}
A_{j} &:= \begin{pmatrix} 2^{\alpha j} & 0 \\ 0 & 2^{\beta j} \end{pmatrix} \quad \text{and} \quad c_{j,m} := \begin{pmatrix} 2^{j-1} + m \cdot 2^{\alpha j} \\ 0 \end{pmatrix}.
\end{align*}

Fourth, for all $\ell \in \mathbb{N}_0$ such that $\ell \leq \ell_{j}^{\max}$, let us define

\begin{align*}
R_{j,\ell} &:= \begin{pmatrix} \cos \Theta_{j,\ell} & -\sin \Theta_{j,\ell} \\ \sin \Theta_{j,\ell} & \cos \Theta_{j,\ell} \end{pmatrix} \quad \text{where} \quad \Theta_{j,\ell} := \Theta_{j,\ell}^{(\beta)} := 2\ell \cdot \phi_{j} \quad \text{and} \quad \phi_{j} := \phi_{j}^{(\beta)} := \frac{\pi}{N} \cdot 2^{(\beta-1)j}.
\end{align*}

Finally, for all $(j, m, \ell) \in I_{0}^{(\alpha,\beta)}$, let us define

\begin{align*}
Q_{j,m,\ell} &:= R_{j,\ell} (A_{j} Q + c_{j,m}) \quad \text{and} \quad P_{j,m,\ell} := R_{j,\ell} (A_{j} P + c_{j,m}),
\end{align*}

and set $Q_{0} := B_{4}(0)$ and $P_{0} := B_{3}(0)$. The family $Q^{(\alpha,\beta)} := \{Q_{i}\}_{i \in I}$ will be called the $(\alpha, \beta)$ wave packet covering of $\mathbb{R}^2$.

![Figure 2: An element $Q_{j,m,\ell}$ of the wave packet covering introduced in Definition 3.1. The distance between the rectangle $Q_{j,m,\ell}$ and the origin $O$ of the frequency plane is approximately $2^{j-1} + m \cdot 2^{\alpha j}$. The length of $Q_{j,m,\ell}$, in the radial direction, is approximately $2^{\alpha j}$ while its width, in the angular direction, is approximately $2^{\beta j}$. Its axis of symmetry intersecting the origin $O$ deviates from the $\xi_1$-axis by the angle $\Theta_{j,\ell} = \frac{\pi}{N} \cdot 2^{(\beta-1)j}$. For a given $j$, all rectangles $Q_{j,m,\ell}$ are contained in the dyadic ring $\{\xi \in \mathbb{R}^2 : |\xi| \asymp 2^{j}\}$. We also note that $m_{j}^{\max} \asymp 2^{(1-\alpha)j}$ and $\ell_{j}^{\max} \asymp 2^{(1-\beta)j}$.]
In other words, the elements $Q_{j,m,\ell}$ of the covering are generated from the rectangle $Q$ by scaling, shifting and rotating — the corresponding operators being represented by $A_j$, $c_{j,m}$ and $R_{j,\ell}$ — as schematically shown in Figure 2. For a given $j$, all rectangles $Q_{j,m,\ell}$ are contained in the dyadic ring $\{\xi \in \mathbb{R}^2 : |\xi| \cong 2^j\}$. Moreover, the length of the rectangle $Q_{j,m,\ell}$, in the radial direction, is approximately $2^{\alpha j}$ while its width, in the angular direction, is approximately $2^{\beta j}$.

We shall now prove that the family $Q^{(\alpha, \beta)}$ introduced in Definition 3.1 is indeed a covering of $\mathbb{R}^2$. Indeed, we shall prove the following stronger statement.

**Lemma 3.2.** Let $0 \leq \beta \leq \alpha \leq 1$. The sets $(P_i)_{i \in I^{(\alpha, \beta)}}$ and $(Q_i)_{i \in I^{(\alpha, \beta)}}$ introduced in Definition 3.1 satisfy

\[
\mathbb{R}^2 = P_0 \cup \bigcup_{j=1}^{\infty} \bigcup_{m=0}^{m_{max}} \bigcup_{\ell=0}^{\ell_{max}} P_{j,m,\ell} = Q_0 \cup \bigcup_{j=1}^{\infty} \bigcup_{m=0}^{m_{max}} \bigcup_{\ell=0}^{\ell_{max}} Q_{j,m,\ell}.
\]  

(3.7)

**Proof.** First of all we note that $P_0 \subset Q_0$ and $P \subset Q$, whence $P_{j,m,\ell} \subset Q_{j,m,\ell} \subset \mathbb{R}^2$ for all $(j, m, \ell) \in I^{(\alpha, \beta)}_0$. Therefore, the second equality in (3.7) indeed holds, provided that the first holds.

Second we note that

\[
S_{j,m,\ell} := R_{j,\ell} S_{j,m,0} \subset P_{j,m-1,\ell} \cup P_{j,m,\ell}
\]

for all $(j, m, \ell) \in I^{(\alpha, \beta)}_0$ for which $m \geq 1$,

(3.8)

where

\[
S_{j,m,0} := \left\{ \xi = r \cdot \left( \frac{\cos \phi}{\sin \phi} \right) : 2^{j-1} + m 2^{\alpha j} \leq r \leq 2^{j-1} + (m+1) 2^{\alpha j} \text{ and } |\phi| \leq \phi_j \right\}
\]

(3.9)

and $\phi_j = \frac{\pi}{N} \cdot 2^{(\beta-1)j}$ as defined in (3.5).

Indeed, from the definitions in (3.6) and (3.8) we see that $P_{j,m,\ell}$ and $S_{j,m,\ell}$ can be obtained by rotating $P_{j,m,0}$ and $S_{j,m,0}$ through the angle $\Theta_{j,\ell} = 2\cdot\phi_j$, respectively. Therefore, we would prove (3.8) in general, should we prove it for $\ell = 0$. To do so, we first note from (3.6) and (3.4) that

\[
P_{j,m,0} = [2^{j-1} + m 2^{\alpha j}, 2^{j-1} + (m+1) 2^{\alpha j}] \times [-2^{\beta j}, 2^{\beta j}] =: P^{(1)}_{j,m,0} \times P^{(2)}_{j,m,0},
\]

(3.10)

and therefore $\xi \in P_{j,m-1,0} \cap P_{j,m,0}$ if and only if $2^{j-1} + (m+1) 2^{\alpha j} \leq \xi_1 \leq 2^{j-1} + (m+1) 2^{\alpha j}$ and $|\xi_2| \leq 2^{\beta j}$. We now verify that these conditions hold for $\xi \in S_{j,m,0}$.

Indeed, from (3.9) we see that if $\xi = r \cdot (\cos \phi, \sin \phi) \in S_{j,m,0}$, then $\xi_1 \leq |\xi_1| \leq r \leq 2^{j-1} + (m+1) 2^{\alpha j}$ and

\[
\xi_1 = r \cos \phi = r \cos |\phi| \geq r \cos \phi_j \geq (2^{j-1} + m 2^{\alpha j}) \cdot \left(1 - \frac{2}{N} 2^{(\beta-1)j}\right),
\]

where we noticed that $|\phi| \leq \phi_j \leq \pi/2$ as $N = 10$ and $0 \leq \beta \leq \alpha \leq 1$ and that the cosine is a decreasing function on $[0, \pi/2]$ that satisfies

\[
1 - \frac{2}{\pi} \phi \leq \cos \phi \leq \frac{\pi}{2} \left(1 - \frac{2}{\pi} \phi\right) = \frac{\pi}{2} - \phi \quad \text{for} \quad 0 \leq \phi \leq \frac{\pi}{2};
\]

(3.11)

see Appendix B for a proof.

Furthermore, since $m \leq m_{max}^j \leq 1 + 2^{(1-\alpha)j-1}$, and noting that $\beta \leq \alpha \leq 1$ and hence $2^{(\beta-\alpha)j} \leq 1$ and $2^{(\beta-1)j+1} \leq 2$, we establish the following chain of implications:

\[
(2^{j-1} + m 2^{\alpha j}) \left(1 - \frac{2}{N} \cdot 2^{(\beta-1)j}\right) \geq 2^{j-1} + (m-1) 2^{\alpha j}
\]

\[
\iff 1 - \frac{2}{N} \cdot 2^{(\beta-1)j} \geq \frac{2^{j-1} + (m-1) 2^{\alpha j}}{2^{j-1} + m 2^{\alpha j}} = 1 - \left(2^{(1-\alpha)j-1} + m\right)^{-1}
\]

\[
\iff N \geq 2 \cdot 2^{(\beta-1)j} \cdot \left(2^{(1-\alpha)j-1} + m\right)
\]

\[
\iff N \geq 2 \cdot 2^{(\beta-\alpha)j-1} + \left(1 + 2^{(1-\alpha)j-1} \cdot 2^{(\beta-1)j}\right) = 2^{(\beta-\alpha)j} + 2^{(\beta-1)j+1} + 2^{(\beta-\alpha)j}
\]

\[
\iff N \geq 4.
\]
The last inequality does indeed hold, since \( N = 10 \) by Definition 3.1. Thus we have demonstrated that \( 2^{j-1} + (m-1)2^{\alpha j} \leq \xi_1 \leq 2^{j-1} + (m+1)2^{\alpha j} \) if \( \xi \in S_{j,m,0} \).

Now we estimate \( \xi_2 \) for \( \xi \in S_{j,m,0} \). Write \( \xi = r \cdot (\cos \phi, \sin \phi)^t \) with \( r, \phi \) as in Equation 3.9. Next, note as a consequence of the definition of \( m_{j}\max \) in Equation 3.1, that \( m+1 \leq 2 + 2(1-\alpha)j-1 \), and recall that \( \alpha - 1 \leq 0 \) and \( N = 10 \). In combination with the estimate \( |\sin \phi| \leq |\phi| \), this implies

\[
|\xi_2| = r \cdot |\sin \phi| \leq r \cdot |\phi| \leq (2^{j-1} + (m + 1)2^{\alpha j}) \cdot \frac{\pi}{N} \cdot 2^{(\beta-1)j} \leq \frac{2^{\beta j} \pi}{N} \cdot (1 + 2^{(\alpha - 1)j}) \leq \frac{3\pi}{N} \cdot 2^{\beta j} \leq 2^{\beta j}.
\]

Overall, we have thus shown \( \xi \in P_{j,m-1,0} \cup P_{j,m,0} \) for all \( \xi \in S_{j,m,0} \). As discussed above, we have thus proven Equation (3.8).

Third, we note that

\[
S_{j,0,\ell} := R_{j,\ell} S_{j,0} \subset P_{j,0,\ell} \quad \text{for} \quad j \in \mathbb{N} \quad \text{and} \quad \ell \in \mathbb{N}_0 \quad \text{with} \quad \ell \leq \ell_{\max}, \tag{3.12}
\]

where

\[
S_{j,0} := \left\{ r \cdot \left( \frac{\cos \phi}{\sin \phi} \right) : 2^{j-1} + 2^{\alpha(j-1)} \leq r \leq 2^{j-1} + 2^{\alpha j} \quad \text{and} \quad |\phi| \leq \phi_j \right\}. \tag{3.13}
\]

Indeed, since \( S_{j,0,\ell} \) and \( P_{j,0,\ell} \) can be obtained by rotating \( S_{j,0} = S_{j,0,0} \) and \( P_{j,0,0} \) using the matrix \( R_{j,\ell} \), we would prove (3.12) in general, for any \( \ell \), if we prove it for \( \ell = 0 \).

To do so, we infer from (3.4) and (3.6) that

\[
P_{j,0,0} = [2^{j-1}, 2^{j-1} + 2^{\alpha j}] \times [-2^{\beta j}, 2^{\beta j}] =: P_{j,0,0}^{(1)} \times P_{j,0,0}^{(2)}.
\]

Furthermore, from (3.13) we deduce that, if \( \xi = r \cdot (\cos \phi, \sin \phi)^t \in S_{j,0,0} \), then on the one hand \( \xi_1 \leq |\xi| \leq 2^{j-1} + 2^{\alpha j} \), but on the other hand, thanks to (3.11),

\[
\xi_1 = r \cos \phi = r \cos |\phi| \geq r \cos \phi_j \geq (2^{j-1} + 2^{\alpha(j-1)}) \left( 1 - \frac{2}{N} 2^{(\beta-1)j} \right) \geq 2^{j-1},
\]

where the last inequality is justified by the following chain of implications:

\[
\frac{2}{N} 2^{(\beta-1)j} \cdot (2^{j-1} + 2^{\alpha(j-1)}) \leq 2^{\alpha(j-1)} \iff N \geq 2^{1+\alpha_1} \cdot 2^{(\beta-\alpha)j-1 + 2^{-\alpha}2^{(\beta-1)j}} \iff 0 \leq \beta \leq \alpha \leq 1 \quad N \geq 8.
\]

The last inequality does indeed hold, since \( N = 10 \). Thus we have shown that \( \xi_1 \in P_{j,0,0}^{(1)} \) if \( \xi \in S_{j,0,0} \).

Furthermore, if \( \xi = r \cdot (\cos \phi, \sin \phi)^t \in S_{j,0,0} \), then

\[
|\xi_2| = r \cdot |\sin \phi| \leq r \cdot |\phi| \leq (2^{j-1} + 2^{\alpha j}) \cdot \frac{\pi}{N} 2^{(\beta-1)j} = \frac{\pi}{N} 2^{\beta j} \cdot (2^{\alpha j} - 1) \leq \frac{3\pi}{2N} \cdot 2^{\beta j} \leq 2^{\beta j}
\]

and hence \( \xi_2 \in P_{j,0,0}^{(2)} \) and \( \xi \in P_{j,0,0} \). This completes the proof of (3.12) for \( \ell = 0 \) and hence in general, for any \( \ell \).

Finally, from (3.8) we deduce that

\[
\bigcup_{j=1}^{\infty} \bigcup_{m=0}^{m_{j}\max} \bigcup_{\ell=0}^{\ell_{j}\max} P_{j,m,\ell} \supset \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{m_{j}\max} \bigcup_{\ell=0}^{\ell_{j}\max} S_{j,m,\ell} \supset \bigcup_{j=1}^{\infty} \{ \xi \in \mathbb{R}^2 : 2^{j-1} + 2^{\alpha j} \leq |\xi| \leq 2^{j} + 2^{\alpha j} \} \tag{3.14}
\]

Here, we noted that \( \ell_{j}\max \geq \pi/\phi_j \) and \( m_{j}\max = [2^{(1-\alpha)j-1}] \) thanks to (3.1) and (3.5) and therefore

\[
\bigcup_{\ell=0}^{\ell_{j}\max} [\phi_j(2\ell - 1), \phi_j(2\ell + 1)] = [-\phi_j, \phi_j(2 \cdot \ell_{j}\max + 1)] \supset [0, 2\pi]
\]

and

\[
\bigcup_{m=1}^{m_{j}\max} [2^{j-1} + m2^{\alpha j}, 2^{j-1} + (m+1)2^{\alpha j}] = [2^{j-1} + 2^{\alpha j}, 2^{j-1} + \left(m_{j}\max + 1\right)2^{\alpha j}]
\]

\[
\supset [2^{j-1} + 2^{\alpha j}, 2^{j-1} + (2^{(1-\alpha)j-1} + 1)2^{\alpha j}] = [2^{j-1} + 2^{\alpha j}, 2^{j} + 2^{\alpha j}].
\]
Similarly, from (3.12), we infer that
\[
\bigcup_{j=2}^{\infty} \bigcup_{\ell=0}^{\ell_{\max}} P_{j,0,\ell} \supset \bigcup_{j=2}^{\infty} \bigcup_{\ell=0}^{\ell_{\max}} S_{j,0,\ell} \supset \bigcup_{j=2}^{\infty} \left\{ \xi \in \mathbb{R}^2 : 2^{j-1} + 2^{\alpha(j-1)} \leq |\xi| \leq 2^{j-1} + 2^{\alpha(j)} \right\}
\]
\[
= \bigcup_{j=1}^{\infty} \left\{ \xi \in \mathbb{R}^2 : 2^j + 2^{\alpha(j)} \leq |\xi| \leq 2^j + 2^{\alpha(j+1)} \right\}.
\]
Combining (3.14) and (3.15) results in
\[
\bigcup_{j=1}^{\infty} \bigcup_{m=0}^{m_{\max}} \bigcup_{\ell=0}^{\ell_{\max}} P_{j,m,\ell} \supset \bigcup_{j=1}^{\infty} \left\{ \xi \in \mathbb{R}^2 : 2^{j-1} + 2^{\alpha(j-1)+1} \leq |\xi| \leq 2^j + 2^{\alpha(j+1)} \right\} = \left\{ \xi \in \mathbb{R}^2 : |\xi| \geq 1 + 2^\alpha \right\}.
\]
Since $1 + 2^\alpha \leq 3$, this implies $B_\beta(0) \cup \bigcup_{j=1}^{\infty} \bigcup_{m=0}^{m_{\max}} \bigcup_{\ell=0}^{\ell_{\max}} P_{j,m,\ell} = \mathbb{R}^2$. \qed

4. Proving admissibility of the wave packet covering

Our next lemma will clarify in more detail the geometric structure of the wave packet covering and will be useful in proving its admissibility. The lemma makes clear how the Euclidean length $|\xi|$ and the angle $\angle(\xi)$ of the vectors $\xi \in Q_{j,m,\ell}$ are influenced by the indices $j, m$ and $\ell$, respectively.

**Lemma 4.1.** Let $0 \leq \beta \leq \alpha \leq 1$. With notation as in Definition 3.1, let $(j, m, \ell) \in I_{0}^{(\alpha, \beta)}$ and $\xi \in Q_{j,m,\ell}$. Then
\[
2^{-2j} < 2^{-j} + 2^\alpha j (m - \varepsilon) \leq |\xi| \leq 2^{-j} + 2^\alpha j (m + 2 + 2\varepsilon) \leq 2^{-j} + 2^\alpha j (3 + 2\varepsilon) < 2^{j+3}
\]
and
\[
\exists \varphi \in \mathbb{R} : \quad \xi = |\xi| \cdot e^{i\varphi} \quad \text{and} \quad |\varphi - \Theta_{j,\ell}| \leq 4 (1 + \varepsilon) \cdot 2^{(\beta-1)j},
\]
where the vector $(\cos \varphi, \sin \varphi) \ell \in \mathbb{R}^2$ is identified with the complex number $e^{i\varphi}$.

**Proof.** Since $Q_{j,m,\ell} = R_{j,\ell} Q_{j,m,0}$ can be obtained from $Q_{j,m,0}$ by rotation through the angle $\Theta_{j,\ell}$ and since rotations preserve the Euclidean norm, we would prove (4.1), in general, for $\xi \in Q_{j,m,\ell}$, should we prove it for $\xi \in Q_{j,m,0}$. To do so, directly from Definition 3.1, we infer that
\[
Q_{j,m,0} = (2^{j-1} + 2^\alpha j (m - \varepsilon), 2^{j-1} + 2^\alpha j (m + 1 + \varepsilon)) \times \left( (1 + \varepsilon) 2^{bj}, (1 + \varepsilon) 2^{bj} \right) = \sqrt{2^{j-1} + 2^\alpha j (m + 2 + 2\varepsilon)} \times \left( (1 + \varepsilon) 2^{bj}, (1 + \varepsilon) 2^{bj} \right).
\]
As $\varepsilon \in (0, 1/32)$ and $\alpha \leq 1$, we conclude that
\[
|\xi| \geq |\xi_1| \geq 2^{-j} + 2^\alpha j (m - \varepsilon) \geq 2^{-j} - \varepsilon 2^\alpha j > 2^{j-2} > 0 \quad \forall \xi \in Q_{j,m,0}.
\]
This completes the proof of the lower bound in (4.1).

Similarly, since $\xi_1 \geq 0$ for $\xi \in Q_{j,m,0}$ and since $\beta \leq \alpha$, we infer from (4.3) that, for any $\xi \in Q_{j,m,0}$,
\[
|\xi| \leq |\xi_1| + |\xi_2| \leq 2^{-j} + 2^\alpha j (m + 1 + \varepsilon) + (1 + \varepsilon) 2^{bj} \leq 2^{-j} + 2^\alpha j (m_{\max} + 2 + 2\varepsilon)
\]
(Definition of $m_{\max}$, see Eq. (3.1)) \[
\leq 2^{j-1} + 2^\alpha j \cdot (2^{(1-\alpha)j-1} + 3 + 2\varepsilon) = 2^j + 2^\alpha j (3 + 2\varepsilon) < 2^{j+3}.
\]
This completes the proof of the upper bound in (4.1).

To prove (4.2), let us first consider the case where $\xi \in Q_{j,m,0}$ and choose $\varphi \in [-\pi, \pi]$ such that $\xi = |\xi| \cdot e^{i\varphi}$. Since $\xi_1 = |\xi| \cdot \cos(\varphi)$ and $\xi_1 > 0$ for $\xi \in Q_{j,m,0}$ (see Equation (4.4)), we conclude that $\varphi \in (-\pi/2, \pi/2)$. Since the derivative $\tan'(\varphi) = 1 + \tan^2(\varphi)$ of $\tan \varphi$ is not less than one for $\varphi \in (-\pi/2, \pi/2)$ and since $\tan(0) = 0$, we conclude that $\tan(\varphi) \geq \varphi \geq 0$ for $\varphi \in [0, \pi/2)$ and $|\tan(\varphi)| = \tan(|\varphi|) \geq |\varphi| \geq 0$ for $\varphi \in (-\pi/2, \pi/2)$. Therefore,
\[
|\varphi| \leq |\tan(\varphi)| \leq \frac{|\xi_2|}{|\xi_1|}
\]
\[
(\xi_1 \geq 2^{-j} \text{ and } |\xi_2| \leq 1 + 2^\alpha j (m_{\max} + 2 + 2\varepsilon) \text{ for } \xi \in Q_{j,m,0}) \leq \frac{1 + \varepsilon}{2^{j-2}} \leq 4 \cdot (1 + \varepsilon) \cdot 2^{(\beta-1)j}.
\]
This completes the proof of (4.2) for $\xi \in Q_{j,m,0}$.

In general, if $\xi \in Q_{j,m,\ell} = R_{j,\ell} Q_{j,m,0}$, there is $\xi' = |\xi'| \cdot e^{i\varphi_0} \in Q_{j,m,0}$ such that $|\varphi_0| \leq 4(1 + \varepsilon) \cdot 2^{(\beta-1)j}$ and $\xi = R_{j,\ell} \xi'$. Therefore, $\varphi := \varphi_0 + \Theta_{j,\ell}$ satisfies $\xi = |\xi| \cdot e^{i\varphi}$ and $|\varphi - \Theta_{j,\ell}| = |\varphi_0| \leq 4(1 + \varepsilon) \cdot 2^{(\beta-1)j}$. \qed
We now turn to the proof of the admissibility of the covering from Lemma 3.2.

**Lemma 4.2.** Let $0 \leq \beta \leq \alpha \leq 1$. Then the covering $Q := Q^{(\alpha, \beta)} := (Q_i)_{i \in I}$ from Definition 3.1 is admissible.

More specifically,

a) for any given $(j, m, \ell), (j', m', \ell') \in I_0^{(\alpha, \beta)},$

$$Q_{j,m,\ell} \cap Q_{j',m',\ell'} = \emptyset \text{ unless } |j - j'| \leq 3;$$

b) for any given $(j, m, \ell) \in I_0^{(\alpha, \beta)}$ and $j' \in \mathbb{N}$, there are at most five different values of $m' \in \mathbb{N}_0$ such that there is $\ell' \in \mathbb{N}_0$ with $(j', m', \ell') \in I_0^{(\alpha, \beta)}$ and $Q_{j,m,\ell} \cap Q_{j',m',\ell'} \neq \emptyset;$

c) for any given $(j, m, \ell), (j', m', \ell') \in I_0^{(\alpha, \beta)},$

$$Q_{j,m,\ell} \cap Q_{j',m',\ell'} = \emptyset \text{ unless } \min_{k \in \{-2, -1, 0, 1, 2\}} |\Theta_{j,\ell} - \Theta_{j',\ell'} - 2\pi k| \leq 4(1 + \varepsilon)(2^{(\beta - 1)j} + 2^{(\beta - 1)j'}) = 2^{j - 4}$$

d) for any given $(j, m, \ell) \in I_0^{(\alpha, \beta)}$ and $j' \in \mathbb{N}$, there are at most 65 different values of $\ell' \in \mathbb{N}_0$ such that there is $m' \in \mathbb{N}_0$ with $(j', m', \ell') \in I_0^{(\alpha, \beta)}$ and $Q_{j,m,\ell} \cap Q_{j',m',\ell'} \neq \emptyset;$ and
e) there are at most 135 different values of $(j', m', \ell') \in I_0^{(\alpha, \beta)}$ such that $Q_0 \cap Q_{j',m',\ell'} \neq \emptyset$.

**Remark.** The derived bounds concerning the number of intersections are quite pessimistic, but sufficient for our purposes. The reason for the unappealing bounds is that we provide uniform bounds that apply simultaneously for all values of $0 \leq \beta \leq \alpha \leq 1$.

**Proof.**

**Proof of a)** Assume there is some $\xi \in Q_{j,m,\ell} \cap Q_{j',m',\ell'}$. We claim that $|j - j'| \leq 3$. To show this, let us assume the contrary, i.e., $|j - j'| \geq 4$. By symmetry, we can assume that $j \geq j'$, whence $0 \leq j' \leq j - 4$ and $2^{\alpha j} \leq 2^{j} \leq 2^{j - 4}$. Thus, we infer from (4.1) that

$$2^{j - 1} - \varepsilon 2^{j} \leq 2^{j - 1} - \varepsilon 2^{\alpha j} \leq |\xi| \leq 2^{j'} + 2^{\alpha j'}(3 + 2\varepsilon) \leq 2^{j - 4} + 2^{2(3 + 2\varepsilon)} = 2^{j - 4}(4 + 2\varepsilon).$$

Multiplying this estimate by $2^{j - 4}$, we obtain $2^{j - 4}\varepsilon \leq 4 + 2\varepsilon$ and hence $\varepsilon \geq \frac{2}{9}$, which contradicts our choice of $\varepsilon \in (0, \frac{1}{9})$.

**Proof of b)** We assume that $\xi \in Q_{j,m,\ell} \cap Q_{j',m',\ell'}$ and derive restrictions for the possible values of $m'$. To do this, we distinguish three possible cases:

**Case 1:** $j' = j$. From Lemma 4.1, we infer that

$$2^{j - 1} + 2^{\alpha j}(m - \varepsilon) \leq |\xi| \leq 2^{j - 1} + 2^{\alpha j}(m' + 2 + 2\varepsilon) = 2^{j - 1} + 2^{\alpha j}(m' + 2 + 2\varepsilon)$$

and hence $m - m' \leq 2 + 3\varepsilon$. By symmetry (interchanging the indices $(j, m, \ell)$ and $(j', m', \ell')$), this yields $|m - m'| \leq 2 + 3\varepsilon < 4$, that is, $|m - m'| \leq 3$. Thus, in case $j' = j$, the index $m'$ can take five different values at most.

**Case 2:** $j' < j$. Thanks to (4.5), we can write $j = j' + \kappa$ where $\kappa \in \{1, \ldots, 3\}$. From Lemma 4.1, we infer that $2^{j - 1} - \varepsilon 2^{\alpha j} \leq |\xi| \leq 2^{j - 1} + 2^{\alpha j}(m' + 2 + 2\varepsilon)$ and hence $2^{j - 1} - 2^{j - 1} \leq 2^{\alpha j}(m' + 2 + 2\varepsilon)$. Taking into account the possible values of $\kappa$, we conclude that $2^{j - 1} \leq 2^{\alpha j}(m' + 2 + 2\varepsilon)$. Combining the last two estimates with $2^{\alpha j} = 2^{\alpha (j + \kappa)} \leq 2^{\alpha j'}$, results in

$$2^{j - 1} \leq 2^{\alpha j} + 2^{\alpha j'}(m' + 2 + 2\varepsilon) \leq 2^{\alpha j'}(m' + 3 + 2\varepsilon) < 2^{\alpha j'}(m' + 4),$$

whence $2^{(1 - \alpha)j' - 4} \leq m' \leq m_{\text{min}}^{j'} \leq 2^{(1 - \alpha)j' - 1} + 1$. Thus, in case $j' < j$, the index $m'$ can take five different values at most.

**Case 3:** $j' > j$ and thus $j' \geq j + 1$. From Lemma 4.1, we infer that

$$2^{j - 1} + 2^{\alpha j'}(m' - \varepsilon) \leq |\xi| \leq 2^{j'} + 2^{\alpha j'}(3 + 2\varepsilon) \leq 2^{j - 1} + 2^{\alpha j'}(3 + 2\varepsilon)$$

and hence $0 \leq m' \leq 3(1 + \varepsilon) < 4$. Thus, in case $j' > j$, the index $m'$ can take four different values at most.
Combining our conclusions of the three cases completes the proof of b).

Proof of [c] If \( \xi \in Q_{\beta,m,\ell} \cap Q_{j',m',\ell'}, \) then (4.2) implies that there are \( \varphi, \varphi' \in \mathbb{R} \) such that \( \xi = |\xi| \cdot e^{i\varphi} \) where \( |\varphi - \Theta_j,\ell| \leq 4(1 + \varepsilon) \cdot 2^{(3-1)j} \) and such that \( \xi = |\xi| \cdot e^{i\varphi'} \) where \( |\varphi' - \Theta_{j',\ell'}| \leq 4(1 + \varepsilon) \cdot 2^{(3-1)j'} \). Moreover, Equation (4.1) shows that \( |\xi| > 0 \). Therefore, \( e^{i\varphi} = e^{i\varphi'} \) so that there is \( k \in \mathbb{Z} \) such that \( \varphi - \varphi' = 2\pi k \).

Taking into account that \( \ell \leq \ell_j^{\text{max}} \leq 1 + N \cdot 2^{(1-\beta)} \), \( N = 10 \), that \( \beta \leq 1 \) and \( \varepsilon \leq \frac{1}{32} \), we conclude that

\[
0 \leq \Theta_{j,\ell} = \frac{2\pi}{N} \cdot 2^{(3-1)j} \cdot \ell \leq \frac{2\pi}{N} \cdot 2^{(3-1)j} + 2\pi \leq \frac{22}{10} \pi
\]  

(4.7)

and hence

\[
-\frac{14}{10} \pi < -4(1 + \varepsilon) \leq -|\varphi - \Theta_{j,\ell}| \leq \varphi - \Theta_{j,\ell} + \Theta_{j,\ell} = \varphi \leq |\varphi - \Theta_{j,\ell}| + \Theta_{j,\ell} \leq 4(1 + \varepsilon) + \frac{22}{10} \pi < \frac{36}{10} \pi.
\]

In the same way, we also see that \( -\frac{14}{10} \pi < \varphi' < \frac{36}{10} \pi \) and hence \( -5\pi < \varphi - \varphi' < 5\pi \), so that \( |k| = \frac{|\varphi - \varphi'|}{2\pi} < \frac{5}{2} < 3 \), or, in other words, \( k \in \{-2, -1, 0, 1, 2\} \). Finally, we conclude, as claimed, that

\[
|\Theta_{j,\ell} - \Theta_{j',\ell'} - 2\pi k| \leq |\Theta_{j,\ell} - \varphi| + |\varphi - \varphi' - 2\pi k| + |\varphi' - \Theta_{j',\ell'}| \leq 4(1 + \varepsilon) \cdot (2^{(3-1)j} + 2^{(3-1)j'})
\]  

(4.8)

Proof of [d] Given (4.8) and the definition of \( \Theta_{j',\ell'} \), we see that

\[
\left| \frac{2\pi}{N} \cdot 2^{(3-1)j'} \cdot (\ell' - \lambda_{j,\ell,k,j'}) \right| \leq 4(1 + \varepsilon) \cdot (2^{(3-1)j} + 2^{(3-1)j'})
\]

where

\[
\lambda_{j,\ell,k,j'} := \frac{N}{2\pi} \cdot 2^{(1-\beta)j'} \cdot (\Theta_{j,\ell} - 2\pi k) \in \mathbb{R}.
\]

Multiplying this estimate by \( \frac{N}{2\pi} \cdot 2^{(1-\beta)j'} \) and noting that \( 2^{(1-\beta)(j'-j)} \leq 2^{(3-1) \leq 8} \), we conclude that \( |\ell' - \lambda_{j,\ell,k,j'}| \leq 6N \). Since, for given \( (j, m, \ell) \) and \( j' \), the parameter \( k \in \{-2, \ldots, 2\} \) can only take up to five different values, the index \( \ell' \) can take at most \( 5 \cdot 13N = 65N \) different values, as claimed.

Proof of [e] For \( \xi \in Q_0 \cap Q_{j',m',\ell'}, \) the estimate (4.1) implies that \( 2^{j'-2} \leq |\xi| < 4 = 2^2 \) and hence \( j' \leq 3 \) if \( Q_0 \cap Q_{j',m',\ell'} \neq \emptyset \). Furthermore

\[
0 \leq \ell \leq \ell_j^{\text{max}} = \left\lceil N \cdot 2^{(1-\beta)j'} \right\rceil \leq 8N \quad \text{and} \quad 0 \leq m' \leq m_j^{\text{max}} = \left\lceil 2^{(1-\alpha)j' - 1} \right\rceil \leq 4,
\]

as \( j' \leq 3 \). Hence there can be at most \( 3 \cdot 5 \cdot 9N = 135N \) different triples \( (j', m', \ell') \in I_0^{(\alpha,\beta)} \) such that \( Q_0 \cap Q_{j',m',\ell'} \neq \emptyset \).

Finally, we can prove the admissibility of \( Q^{(\alpha,\beta)} \). Combining [a], [b] and [d], we conclude that, for any given \( i \in I_0^{(\alpha,\beta)} \), there are at most \( 7 \cdot 5 \cdot 65 \cdot N + 1 \) different values of \( i' \in I^{(\alpha,\beta)} \) such that \( Q_i \cap Q_{i'} \neq \emptyset \). Part [e] shows that this also holds for \( i = 0 \).

\[\square\]

5. Proving almost-structuredness of the wave packet covering

We now prove that the wave packet covering \( Q^{(\alpha,\beta)} \) is almost structured.

Lemma 5.1. Let \( 0 \leq \beta \leq \alpha \leq 1 \) and let us define, with notations as in Definition 3.1.

\[Q_1^{(0)} := Q, \quad T_i := T_{j,m,\ell} := R_{j,\ell} A_j \quad \text{and} \quad b_i := b_{j,m,\ell} := R_{j,\ell} c_{j,m} \quad \text{for} \quad i = (j, m, \ell) \in I_0^{(\alpha,\beta)} \]

(5.1)

and \( Q_2^{(0)} := B_3(0), \quad T_0 := \text{id} \) and \( b_0 := 0 \). Finally, set \( k_i := 1 \) for \( i \in I_0^{(\alpha,\beta)} \) and \( k_0 := 2 \) and \( Q_i := Q_0^{(0)} \) for \( i \in I^{(\alpha,\beta)} \).

Then the admissible covering \( Q^{(\alpha,\beta)} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I} \) with associated family \((T_i \bullet + b_i)_{i \in I}\) is almost structured.
Proof. First of all, note that the family \( Q^{(\alpha, \beta)} = (Q_i)_{i \in I} \) indeed satisfies \( Q_0 = T_0B_1(0) + b_0 = T_0Q_0' + b_0 \) and \( Q_{j,m,\ell} = T_{j,m,\ell}Q + b_{j,m,\ell} = T_{j,m,\ell}Q_{j,m,\ell}' + b_{j,m,\ell} \) and that \( Q_1^{(0)}, Q_2^{(0)} \subset \mathbb{R}^2 \) are nonempty, open, and bounded.

Moreover, \( Q_1^{(0)} \supset P_1 \) and \( Q_2^{(0)} \supset P_2 \) for the non-empty, open, bounded sets
\[
P_1 := (-\varepsilon/2, 1 + \varepsilon/2) \times (-1 - \varepsilon/2, 1 + \varepsilon/2) \quad \text{and} \quad P_2 := B_3(0).
\]

From Lemma 3.2, we infer that the family \( (T_i P_k + b_i)_{i \in I} \) covers the entire frequency plane \( \mathbb{R}^2 \), and Lemma 4.2 shows that \( Q^{(\alpha, \beta)} \) is admissible. Therefore, to prove that the covering \( Q^{(\alpha, \beta)} \) is almost structured, it is enough to show that there exists a constant \( 0 < C < \infty \) such that
\[
\|T_i^{-1}T'_{i'}\| \leq C \quad \forall i, i' \in I^{(\alpha, \beta)} \text{ for which } Q_i \cap Q_{i'} \neq \emptyset.
\] (5.2)

To do so we first consider the case where neither \( i \) nor \( i' \) are zero, i.e., \( i = (j, m, \ell) \) and \( i' = (j', m', \ell') \) belong to \( I_0^{(\alpha, \beta)} \). Note that \( T_i^{-1}T_{i'} = A_j^{-1}R_{j,k}^{-1}R_{j,k}A_j' \), so that a direct computation shows that
\[
T_{j,m,\ell}^{-1}T_{j',m',\ell'}' = \left( \frac{2^{\alpha(j'-j)}}{2^{2\alpha(j'-j)}} \cdot \cos(\Theta_{j',m',\ell} - \Theta_{j,m,\ell}) - \frac{2^{\beta j' - \alpha j}}{2^{2(j' - j)}} \cdot \sin(\Theta_{j',m',\ell} - \Theta_{j,m,\ell}) \right) =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\] (5.3)

From (4.5) we infer that \( |j - j'| \leq 3 \), since \( Q_i \cap Q_{i'} \neq \emptyset \). Recalling that \( 0 \leq \beta \leq \alpha \leq 1 \), we thus see that
\[
|a| \leq 2^{\alpha(j'-j)} \cdot 2^{2\alpha(j'-j)} \leq 2^3, \quad |b| \leq 2^{\beta j' \cdot \alpha j} \leq 2^{\alpha(j'-j)} \leq 2^3, \quad |c| \leq 2^{\alpha j' - \beta j} \cdot |\sin(\Theta_{j',m',\ell} - \Theta_{j,m,\ell} + 2\pi k)| \leq 2^{\alpha j' - \beta j} \cdot (1 + \varepsilon) \cdot (2^{(\beta - 1)j} + 2^{(\beta - 1)j'})
\]
\[
\leq 5 \cdot (2^{\alpha j' - \beta j} + 2^{(\alpha - 1)j' + \beta j'})) \leq 5 \cdot (2^3 + 2^{3\beta}) \leq 80.
\]

Thus, we have shown that \( \|T_{j,m,\ell}^{-1}T_{j',m',\ell'}\| \leq 2^3 + 2^3 + 2^3 + 80 = 104 \).

We now consider the case where \( i = 0 \) or \( i' = 0 \). If \( i = i' = 0 \), then \( \|T_0^{-1}T_{i'}\| = 1 \leq 104 \).

Furthermore, if \( \xi \in Q_0 \cap Q_{j,m,\ell} \neq \emptyset \), then Lemma 4.1 shows that \( 2^{j' - 2} \leq |\xi| < 4 \), and hence \( j \leq 3 \). Therefore, since \( |R_{j,k}| = |R_{j,k}'| = 1 \) and since \( |A_j| = 1 \) and \( |A_j'| = 2^{2\varepsilon} \), we finally deduce that \( \|T_{j,m,\ell}^{-1}T_0\| = \|A_j^{-1}\| \leq 1 \leq 104 \) and \( \|T_0^{-1}T_{j,m,\ell}\| \leq \|A_j\| \leq 2^3 \leq 104 \). This completes the proof of Equation (5.2) with \( C = 104 \).

6. Defining the wave packet smoothness spaces and investigating their properties

Having proved that \( Q^{(\alpha, \beta)} \) is an almost structured and admissible covering of \( \mathbb{R}^2 \), we shall now define the wave packet smoothness spaces \( W^{p,q}_s(\alpha, \beta) \) as decomposition spaces associated with \( Q^{(\alpha, \beta)} \) and investigate their basic properties. In particular, we shall demonstrate that the spaces \( W^{p,q}_s(\alpha, \beta) \) are embedded in the space of tempered distributions and, under certain restrictions on its parameters, in classical function spaces such as Besov and Sobolev spaces. We shall also investigate the conditions under which one wave packet smoothness space \( W^{p,q}_{s_1}(\alpha, \beta) \) is embedded in another wave packet smoothness space \( W^{p,q}_{s_2}(\alpha', \beta') \). Furthermore, we show that any two wave packet spaces \( W^{p,q}_{s_1}(\alpha, \beta) \) and \( W^{p,q}_{s_2}(\alpha', \beta') \) are distinct, unless their parameters satisfy \( (p_1, q_1, s_1, \alpha, \beta) = (p_2, q_2, s_2, \alpha', \beta') \) or \( (p_1, q_1, s_1) = (2, 2, s) = (p_2, q_2, s_2) \) for some \( s \in \mathbb{R} \). Finally, we show that if \( \alpha = \beta \), then \( W^{p,q}_s(\alpha, \alpha) \) coincides with the \( \alpha \)-modulation space \( M^{p,q}_{\alpha/2}(\mathbb{R}^2) \).

6.1. Defining the wave packet smoothness spaces

The \( (\alpha, \beta) \) wave packet covering \( Q^{(\alpha, \beta)} = (Q_i)_{i \in I} \) with \( I = I^{(\alpha, \beta)} = \{0\} \cup I^{(0, \beta)}_0 \) is an almost structured covering of \( \mathbb{R}^2 \) as we saw in Lemma 5.1. In Section 2.1, we explained that this guarantees that the associated decomposition spaces \( D(Q^{(\alpha, \beta)}) \), \( L^p \) are well-defined quasi-Banach spaces, as long as the weight \( w = (w_i)_{i \in I} \) is \( Q^{(\alpha, \beta)} \)-moderate. For the weights we are interested in, this is verified in the following lemma:
Lemma 6.1. For $0 \leq \beta \leq \alpha \leq 1$ and $s \in \mathbb{R}$, define

$$w_i^s := \begin{cases} 2^{js}, & \text{if } i = (j, m, \ell) \in I_0^{(\alpha, \beta)}, \\ 1, & \text{if } i = 0. \end{cases}$$

Then $w^s = (w_i^s)_{i \in I}$ is $Q^{(\alpha, \beta)}$-moderate.

Proof. Let $i, i' \in I$ with $\emptyset \neq Q_i \cap Q_{i'} \ni \xi$. Our goal is to show that $w_i^s / w_{i'}^s \leq 2^{|s|}$.

First, let us consider the case where $i = (j, m, \ell) \in I_0$ and $i' = (j', m', \ell') \in I_0$. Then Equation (4.5) shows that $|j - j'| \leq 3$, whence $w_i^s / w_{i'}^s = 2^{s(j-j')} \leq 2^{|s|} |j-j'| \leq 2^{|s|}$.

Second, we consider the case $i = (j, m, \ell) \in I_0$, but $i' = 0$. By virtue of Equation (4.1), this entails $2^{j-2} < |\xi|$. Since $Q_0 = B_4(0)$, this implies that $2^{j-2} < |\xi| < 2^2$ and hence $j \leq 3$. Therefore, $w_i^s / w_{i'}^s = 2^{js} \leq 2^{j|s|} \leq 2^{|s|}$.

Third, if $i = 0$ and $i' = (j', m', \ell') \in I_0$, then we see as in the preceding case that $j' \leq 3$, whence $w_i^s / w_{i'}^s = 2^{-s j'} \leq 2^{|s|} |j'| \leq 2^{|s|}$.

Finally, if $i = i' = 0$, then $w_i^s / w_{i'}^s = 1 \leq 2^{|s|}$ as well.

With the preceding lemma, we know that the spaces introduced below are well-defined quasi-Banach spaces.

Definition 6.2. Let $0 \leq \beta \leq \alpha \leq 1$. For $s \in \mathbb{R}$ and $p, q \in (0, \infty)$, the $(\alpha, \beta)$ wave packet smoothness space associated with the parameters $p, q, s$ is the decomposition space

$$W^p_q(\alpha, \beta) := D(Q^{(\alpha, \beta)}, L^p, L^q_w).$$

Remark. Recall from Lemma 4.1 that $1 + |\xi| \simeq 2^j$ for $\xi \in Q_{j,m,\ell}$. Therefore, the weight $w_i^s$ satisfies

$$w_i^s \asymp (1 + |\xi|)^s \quad \text{for} \quad \xi \in Q_i \quad \text{and} \quad i \in I^{(\alpha, \beta)}. \quad (6.1)$$

Therefore, the weight $w^s$ here is similar to that in Besov- and modulation spaces.

6.2. Investigating the conditions for inclusions between different wave packet smoothness spaces

In order to use the theory of embeddings for decomposition spaces to establish conditions under which the inclusion

$$W^{p_1,q_1}_{s_1}(\alpha, \beta) \subset W^{p_2,q_2}_{s_2}(\alpha', \beta')$$

holds, we first have to determine for which values of $\alpha, \beta$ and $\alpha', \beta'$ the covering $Q^{(\alpha, \beta)}$ is almost subordinate to the covering $Q^{(\alpha', \beta')}$. This will be done in the following lemma. In proving this lemma, we shall often use arguments similar to those in the proof of Lemma 4.2.

In what follows, we shall write $T_i^{(\alpha, \beta)}$ rather than $T_i$ and $Q_i^{(\alpha, \beta)}$ rather than $Q_i$. This will be done to avoid any confusion when we consider the two coverings $Q^{(\alpha, \beta)}$ and $Q^{(\alpha', \beta')}$ at the same time. We also remind the reader of the notations $n_{j}^{\max, \alpha}$ and $\ell_{j}^{\max, \alpha}$, $\Theta_{j, \ell}^{(\beta)}$ and $\phi_{j, \ell}^{(\beta)}$ introduced in Definition 3.1.

Proposition 6.3. Let $0 \leq \beta \leq \alpha \leq 1$ and $0 \leq \beta' \leq \alpha' \leq 1$ and let the coverings $Q^{(\alpha, \beta)}$ and $Q^{(\alpha', \beta')}$ be as introduced in Definition 3.1. Then

$$\forall (j, m, \ell) \in I_0^{(\alpha, \beta)} \quad \forall (j', m', \ell') \in I_0^{(\alpha', \beta')} : \text{if } Q_{j,m,\ell}^{(\alpha, \beta)} \cap Q_{j',m',\ell'}^{(\alpha', \beta')} \neq \emptyset \text{ then } |j - j'| \leq 4. \quad (6.2)$$

Moreover, $Q^{(\alpha, \beta)}$ is almost subordinate to $Q^{(\alpha', \beta')}$ if and only if $\beta \leq \beta'$.

Proof. First of all, if $\xi \in Q_{j,m,\ell}^{(\alpha, \beta)} \cap Q_{j',m',\ell'}^{(\alpha', \beta')}$, then (4.1) implies that both $2^{j-2} < |\xi| < 2^{j+3}$ and $2^{j'-2} < |\xi| < 2^{j'+3}$. Combining these results immediately in (6.2).

Part 1: In this part, we assume that $\alpha \leq \alpha'$ and $\beta \leq \beta'$ and prove that $Q^{(\alpha, \beta)}$ is almost subordinate to $Q^{(\alpha', \beta')}$. To do so, let us define

$$J_i := \{ i' \in I^{(\alpha', \beta')} : Q_i^{(\alpha, \beta)} \cap Q_{i'}^{(\alpha', \beta')} \neq \emptyset \} \quad \text{for} \quad i \in I^{(\alpha, \beta)}.$$ 

Since the coverings $Q^{(\alpha, \beta)}$ and $Q^{(\alpha', \beta')}$ consist of open path-connected, indeed convex, sets, Lemma 2.14 shows that $Q^{(\alpha, \beta)}$ is almost subordinate to $Q^{(\alpha', \beta')}$ if and only if there is $K > 0$ such that $|J_i| \leq K$ for all $i \in I^{(\alpha, \beta)}$. To verify this, it will be enough to prove the following claims:
a) For any given \( i = (j, m, \ell) \in I^\prime_0(\alpha, \beta) \) and \( j' \in \mathbb{N} \), there are at most five different values of \( m' \in \mathbb{N}_0 \) such that there is some \( \ell' \in \mathbb{N}_0 \) with \((j', m', \ell') \in I_0^\alpha(\alpha, \beta) \cap J_i;\)

b) For any given \( i = (j, m, \ell) \in I^\prime_0(\alpha, \beta) \) and \( j' \in \mathbb{N}, m' \in \mathbb{N}_0 \), there are at most 125N different values of \( \ell' \in \mathbb{N}_0 \) with \((j', m', \ell') \in I_0^\alpha(\alpha, \beta) \cap J_i;\) and

c) \( J_0 \cap I^\prime_0(\alpha, \beta) \) contains at most 135N elements.

Indeed, as \( I^\alpha(\alpha, \beta) = \{0\} \cup I_0^\alpha(\alpha, \beta) \), the statements a),b) and c) together with Equation (6.2) imply that

\[
|J_i| \leq \max\{1 + 135N, 1 + 9 \cdot 125N\} = 1 + 5625N \quad \text{for all} \quad i \in I^\alpha(\alpha, \beta).
\]

Proof of a) We suppose that \( Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \neq \emptyset \) and derive restrictions on the possible values of \( m' \). To do so, we distinguish three possible cases:

Case 1: \( j = j' \). Let \( m'_{\text{min}} \) and \( m'_{\text{max}} \) be respectively the minimal and the maximal values of \( m' \) such that \( Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \neq \emptyset \). Therefore, there exist \( \xi \in Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \) and \( \eta \in Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \).

Since \( j = j' \), Equation (4.1) implies that

\[
2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon) \leq |\xi| \leq 2^j - 2 \alpha' \cdot (m'_{\text{min}} + 2 + \epsilon) \cdot (m'_{\text{max}} + 2 + \epsilon)
\]

Combining these estimates results in

\[
2^j \cdot (m'_{\text{max}} - m'_{\text{min}}) = [2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon)] - [2^j - 2 \alpha' \cdot (m'_{\text{max}} + 2 + \epsilon)] + (2 + 3 \epsilon) \cdot 2^j \\
\leq [2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon)] - [2^j - 2 \alpha' \cdot (m'_{\text{max}} + 2 + \epsilon)] + (2 + 3 \epsilon) \cdot 2^j \\
\leq (2 + 3 \epsilon) \cdot (2^j + 2 \alpha' \cdot 2^j).
\]

Since \( \alpha \leq \alpha' \) and \( \epsilon < \frac{1}{127} \), this finally implies that \( m'_{\text{max}} - m'_{\text{min}} \leq (2 + 3 \epsilon) \cdot (2^{\alpha' \cdot \alpha' + 1}) + 2 \cdot (2 + 3 \epsilon) < 5 \).

Therefore, \( m' \) can take at most five different values if \( j = j' \).

Case 2: \( j' < j \) and hence \( j' \leq j - 1 \). Since \( Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \neq \emptyset \), there is some \( \xi \in Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \).

Therefore, from Equation (4.1) we infer that \( 2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon) \leq |\xi| \leq 2^j - 2 \alpha' \cdot (m'_{\text{max}} + 2 + \epsilon) \cdot (m'_{\text{max}} + 2 + \epsilon) \) and thus

\[
2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon) \leq |\xi| \leq 2^j - 2 \alpha' \cdot (m'_{\text{max}} + 2 + \epsilon) \cdot (m'_{\text{max}} + 2 + \epsilon) \]

(since \( \epsilon < 2^{-5} \) and \( j \leq j' + 4 \) (see (4.2)) and \( \alpha \leq \alpha' \)) \leq 2^j \cdot (m'_{\text{max}} + 2 + \epsilon) < 2^j \cdot (m'_{\text{max}} + 4),

since \( 2^j \leq 2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon) \). From this we infer that \( 2^{(1 - \alpha')j)} - 4 - 4 \leq m' \leq \min \frac{m'_{\text{max}} - \alpha'}{2^{(1 - \alpha')j} + 1} \). Thus, \( m' \) can take at most five different values if \( j' < j \).

Case 3: \( j' > j \) and thus \( j \leq j' - 1 \). Here there exists again \( \xi \in Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \) and from (4.1) we infer that

\[
2^j - 2 \alpha' \cdot (m'_{\text{max}} - \epsilon) \leq |\xi| \leq 2^j + 2 \alpha' \cdot (m'_{\text{max}} + 2 + \epsilon) \cdot (m'_{\text{max}} + 2 + \epsilon) \]

and hence \( 0 \leq m' \leq m'_{\text{max}} + 2 + \epsilon \cdot (m'_{\text{max}} + 2 + \epsilon) \leq 3 + 3 \leq 4 \), since \( \alpha \leq \alpha' \). Thus, \( m' \) can take at most four different values if \( j' > j \).

Having considered all three possible cases, we conclude that \( m' \) can take at most five different values, as claimed in a).

Proof of b) Here again there exists \( \xi \in Q^\alpha(\alpha, \beta) \cap Q_j^{\alpha, \beta} \) and from (4.2) we infer that there are \( \varphi, \varphi' \in \mathbb{R} \) such that \( |\xi| \cdot e^{\varphi \varphi} = \xi \cdot e^{\varphi \varphi'} \), \( |\varphi - \Theta_j,\ell^{(\beta)}| \leq 4(1 + \epsilon) \cdot 2^{(\beta - 1)(3 + 2 \epsilon)} \leq 4(1 + \epsilon) \cdot 2^{(\beta - 1)(3 + 2 \epsilon)} \) and furthermore \( |\varphi - \Theta_j,\ell^{(\beta)}| \leq 4(1 + \epsilon) \cdot 2^{(\beta - 1)(3 + 2 \epsilon)} \leq 4(1 + \epsilon) \).

Using essentially the same arguments as in the proof of Lemma (4.2), we conclude that there is some \( k \in \{-2, \ldots, 2\} \) such that \( \varphi - \varphi' = 2 \pi k \).

Finally, defining

\[
\lambda_{j,\ell,k,j'} := \frac{N}{2 \pi} \cdot 2^{(1 - \alpha')j'} \cdot (\Theta_j,\ell^{(\beta)} - 2 \pi k) = \frac{1}{2} \cdot (\Theta_j,\ell^{(\beta)} - 2 \pi k),
\]
we conclude that

\[ |\ell' - \lambda_{j,\ell,k,j'}| = \frac{1}{2} \left( \phi_{j'}^{(\beta')} \right)^{-1} \left[ 2 \phi_{j'}^{(\beta')} - (\Theta_{j,\ell}^{(\beta)} - 2 \pi k) \right] = \frac{1}{2} \left( \phi_{j'}^{(\beta')} \right)^{-1} \left[ \Theta_{j,\ell}^{(\beta')} - \Theta_{j,\ell}^{(\beta)} + \varphi - \varphi' \right] \]

\[ \leq \left( \phi_{j'}^{(\beta')} \right)^{-1} \cdot 2(1 + \varepsilon) \cdot (2(\beta-1)j + 2(\beta-1)j') = \frac{N}{\pi} \cdot 2(1 + \varepsilon) \cdot (2(\beta-1)j - (\beta-1)j' + 1) \]

(since \( \beta \leq \beta' \))

\[ \leq \frac{N}{\pi} \cdot 2(1 + \varepsilon) \cdot (2(\beta'(1-\beta')-1) + 1) \leq \frac{34N}{\pi} (1 + \varepsilon) \leq 12N , \]

since \(|j - j'| \leq 5\), according to Equation (6.2).

Because of \(|\ell' - \lambda_{j,\ell,k,j'}| \leq 12N\) and \(k \in \{-2, \ldots, 2\}\), the index \(\ell'\) can take at most \(5 \cdot 25N = 125N\) different values, for given \(j, \ell\) and \(j'\).

**Proof of c)** The proof of this part is identical to that of part c) of Lemma 4.2, since the set \(Q_0^{(\alpha,\beta)} = B_4(0)\) is independent of the choice of \(\alpha\) and \(\beta\).

**Part 2:** In this part, we prove that \(Q^{(\alpha,\beta)}\) is not almost subordinate to \(Q^{(\alpha',\beta')}\) if \(\alpha > \alpha'\) or \(\beta > \beta'\).

To do so, it will be enough to prove the following two properties:

d) If \(\alpha' < \alpha\), then

\[ \lim_{j \to \infty} |J_{(j,0,0)}| \geq \lim_{j \to \infty} \left| \left\{ m' \in \mathbb{N}_0 : (j, m', 0) \in I_0^{(\alpha',\beta')} \text{ and } Q_{j,m',0}^{(\alpha',\beta')} \cap Q_{j,0,0}^{(\alpha,\beta)} \neq \emptyset \right\} \right| = \infty . \]

e) If \(\alpha \leq \alpha'\) but \(\beta' < \beta\), then

\[ \lim_{j \to \infty} |J_{(j,0,0)}| \geq \lim_{j \to \infty} \left| \left\{ \ell' \in \mathbb{N}_0 : (j, 0, \ell') \in I_0^{(\alpha',\beta')} \text{ and } Q_{j,0,\ell'}^{(\alpha',\beta')} \cap Q_{j,0,0}^{(\alpha,\beta)} \neq \emptyset \right\} \right| = \infty . \]

Indeed, d) and e) show that \(Q^{(\alpha,\beta)}\) is not weakly subordinate to \(Q^{(\alpha',\beta')}\). Thanks to Lemma 2.14 this implies that \(Q^{(\alpha,\beta)}\) is not almost subordinate to \(Q^{(\alpha',\beta')}\).

**Proof of d)** From the definition of \(Q_{j,m',\ell}\), we infer that

\[ Q_{j,0,0}^{(\alpha,\beta)} \supset [2^{j-1}, 2^{j-1} + 2^{\alpha'}] \times \{ 0 \} \quad \text{and} \quad Q_{j,m',0}^{(\alpha',\beta')} \supset \left[ 2^{j-1} + m' \cdot 2^{\alpha'}, 2^{j-1} + (m' + 1) \cdot 2^{\alpha'} \right] \times \{ 0 \} . \]

The latter implies that \(\xi_{j,m'} := (2^{j-1} + m' \cdot 2^{\alpha'}, 0)^{\ell} \in Q_{j,m',0}^{(\alpha',\beta')}\).

Let us now choose \(m' \in \mathbb{N}_0\) with \(m' \leq 2^{(j-\alpha')}-1\). Then, on the one hand, \((j, m', 0) \in I_0^{(\alpha',\beta')}\) since \(m' \leq 2^{(1-\alpha')}-1 \leq m_{j,\max}^{\alpha'}\). On the other hand, \(\xi_{j,m'} \in Q_{j,0,0}^{(\alpha,\beta)}\) since \(m' \cdot 2^{\alpha'} \leq 2^{\alpha'-1} \leq 2^{\alpha'}\).

Put together, this implies, as \(\alpha > \alpha'\), that

\[ |J_{(j,0,0)}| \geq \left| \left\{ m' \in \mathbb{N}_0 : (j, m', 0) \in I_0^{(\alpha',\beta')} \text{ and } Q_{j,m',0}^{(\alpha',\beta')} \cap Q_{j,0,0}^{(\alpha,\beta)} \neq \emptyset \right\} \right| = 1 + \left| 2^{(j-\alpha')-1} \right| \rightarrow j \rightarrow \infty \infty . \]

**Proof of e)** Here we shall write \(R_{j,\ell}^{(\beta)}\) instead of \(R_{j,\ell}\) to clearly indicate the value of \(\beta\) that determines this matrix.

From the definition of \(Q_{j,m,\ell}^{(\alpha,\beta)}\) we infer that

\[ Q_{j,0,0}^{(\alpha,\beta)} \supset [2^{j-1}, 2^{j-1} + 2^{\alpha'}] \times \{ 0 \} \quad \text{and} \quad Q_{j,0,0}^{(\alpha,\beta)} \supset \left[ 2^{j-1} + 2^{\alpha'} \right] \times \{ 0 \} . \]

For \(j \in \mathbb{N}\) define \(\theta_j := \min \left\{ \frac{1}{2} 2^{(\beta'-1)j}, \frac{2^{\alpha'-1}j}{2^{\alpha'}} \right\} \). Below, we shall prove the following technical auxiliary claim:

\[ \forall j \in \mathbb{N} \forall \theta \in [-\theta_j, \theta_j] \exists z_{j,\theta} \in [2^{j-1}, 2^{j-1} + 2^\alpha'] : \left( z_{j,\theta, \cos \theta} \right) \in \left[ 2^{j-1} - 1, 2^{j-1} + 2^\alpha' \right] \times \left[-2^\beta, 2^\beta \right] . \]

(6.4)

Accepting this for the moment, we can combine Equations (6.4) and (6.3) to conclude that if \(j \in \mathbb{N}\) and \(\ell' \in \mathbb{N}_0\) with \(\ell' \leq \ell_{j,\max}^{\alpha,\beta}\) are such that \(\theta(j, \ell') := \Theta_{j,\ell'}^{(\beta')}\) satisfies \(|\theta(j, \ell')| \leq \theta_j\), then

\[ R_{j,\ell'}^{(\beta')} \left( z_{j,\theta(j,\ell'), 0} \right) = \left( z_{j,\theta(j,\ell'), \cos \theta(j,\ell')} \right) \in Q_{j,0,0}^{(\alpha,\beta)} \cap R_{j,\ell'}^{(\beta')} Q_{j,0,0}^{(\alpha',\beta')} = Q_{j,0,0}^{(\alpha,\beta)} \cap Q_{j,0,0}^{(\alpha',\beta')} . \]
and hence
\[ |J_{(j,0,0)}| \geq \left\{ \ell' \in \mathbb{N}_0 : \ell' \leq \ell_{j}^{\text{max},\beta'} \quad \text{and} \quad |\Theta_{j,\ell'}^{(\beta')}| \leq \theta_j \right\} \]
(Def. of \( \Theta_{j,\ell'}^{(\beta')}, \ell_{j}^{\text{max},\beta'} \), and \( \theta_j \))
\[ \geq \left\{ \ell' \in \mathbb{N}_0 : \ell' \leq N \cdot 2^{(1-\beta')j} \quad \text{and} \quad \ell' \leq \frac{N}{2^j} \cdot \min \left\{ \frac{1}{2^2} \Theta_{j,\ell'}^{(\beta-\beta')}, 2^{1-\frac{(1-\alpha')}{2}} \right\} \right\} \]
\[ \xrightarrow{j \to \infty} \infty. \]

Here we noted in the very last step that \( \beta' < \beta \leq 1 \) and that \( \beta' \leq \alpha' \), so that \( 2^{(1-\beta')j}, 2^{(\beta-\beta')j} \) and \( 2^{1-\frac{(1-\alpha')}{2}} \) all tend to \( \infty \) as \( j \to \infty \). Thus, we shall prove Claim e), if we prove \( (6.4) \).

To prove that \( (6.4) \) is indeed satisfied, let \( j \in \mathbb{N} \) and \( \theta \in [-\theta_j, \theta] \). We first show that we can choose \( z = z_{j,\theta} \in [2^{j-1}, 2^{j-1} + 2^{\alpha'}j] \) such that \( z \cdot \cos \theta \in [2^{j-1}, 2^{j-1} + 2^{\alpha'}j] \). Note that \( |\theta| \leq \theta_j \leq 1 - \frac{\pi}{2} \) and hence \( \cos \theta > 0 \). Thus, our goal is to show that we can choose
\[ z_{j,\theta} \in [2^{j-1}, 2^{j-1} + 2^{\alpha'}j] \cap \left[ \frac{2^{j-1}}{\cos \theta}, \frac{2^{j-1} + 2^{\alpha'}j}{\cos \theta} \right]. \] (6.5)

This is possible if and only if the first condition in the following chain of equivalences is satisfied:
\[ [2^{j-1}, 2^{j-1} + 2^{\alpha'}j] \cap \left[ \frac{2^{j-1}}{\cos \theta}, \frac{2^{j-1} + 2^{\alpha'}j}{\cos \theta} \right] \neq \emptyset \iff 2^{j-1} \leq \frac{2^{j-1} + 2^{\alpha'}j}{\cos \theta} \quad \text{and} \quad \frac{2^{j-1}}{\cos \theta} \leq 2^{j-1} + 2^{\alpha'}j \]
\[ \iff \frac{2^{j-1}}{\cos \theta} \leq \cos \theta \leq \frac{2^{j-1} + 2^{\alpha'}j}{2^{j-1}} \quad \iff \cos \theta \geq 1 - \frac{2^{\alpha'}j}{2^{j-1} + 2^{\alpha'}j}. \]

To prove that the latter condition is satisfied, we recall from Equation \( (B.4) \) that \( \cos \theta \geq 1 - \frac{\theta^2}{2} \) for all \( \theta \in \mathbb{R} \), and hence
\[ \cos \theta \geq 1 - \frac{\theta^2}{2} \geq 1 - \frac{\theta^2}{2} \geq 1 - \frac{2^{(\alpha'-1)j}}{2} = 1 - \frac{2^{\alpha'}j}{2j} \geq 1 - \frac{2^{\alpha'}j}{2^{j-1} + 2^{\alpha'}j}, \]
as desired. Here we noted in the last step that \( 2^{j-1} + 2^{\alpha'}j \leq 2^{j} + 2^{j} \) since \( \alpha' \leq 1 \). Overall, we have shown that one can indeed choose \( z_{j,\theta} \) as in Equation \( (6.5) \).

Thus, to prove Equation \( (6.4) \), it suffices to verify that \( z_{j,\theta} \cdot \sin \theta \leq 2^j \). But this is a consequence of the estimate \( |\sin \phi| \leq |\phi| \) combined with \( 0 \leq z_{j,\theta} \leq 2^{j-1} + 2^{\alpha'}j \leq 2 \cdot 2^j \) and \( |\theta| \leq \theta_j \leq 1 - \frac{\pi}{2} \); indeed, these estimates imply that \( z_{j,\theta} \cdot \sin \theta \leq 2 \cdot 2^j \cdot \frac{1}{2} (2^{(\beta-1)j}) = 2^j \). \( \square \)

In the next corollary, we verify the conditions concerning relative moderation of coverings and weights that we shall need to apply Theorems 2.16 and 2.17.

**Corollary 6.4.** Let \( 0 \leq \beta \leq \alpha \leq 1 \) and \( 0 \leq \beta' \leq \alpha' \leq 1 \). Then, for any fixed \( s \in \mathbb{R} \), the weight \( w^s \) — considered as a weight for \( Q^{(\alpha,\beta)} \) — is relatively \( Q^{(\alpha,\beta)} \)-moderate; more specifically,
\[ w^s \cong w^s, \quad \text{if} \quad Q^{(\alpha,\beta)} \cap Q^{(\alpha',\beta')} \neq \emptyset. \]
Furthermore, the covering \( Q^{(\alpha,\beta)} \) is relatively \( Q^{(\alpha',\beta')} \)-moderate, and
\[ |\det T^{(\alpha,\beta)}| \cong w^{|\alpha + \beta|}, \quad \text{if} \quad Q^{(\alpha,\beta)} \cap Q^{(\alpha',\beta')} \neq \emptyset. \]

**Proof.** If \( i = (j,m,\ell) \in I^{(\alpha,\beta)}_0 \) and \( i' = (j',m',\ell') \in I^{(\alpha',\beta')}_0 \) satisfy \( Q^{(\alpha,\beta)} \cap Q^{(\alpha',\beta')} \neq \emptyset \), then \( (6.2) \) implies that \( |j - j'| \leq 4 \). Therefore,
\[ \frac{w^s}{w^{s'}} = 2^{(j-j')\cdot s} \leq 2^{4|s|} \quad \text{and} \quad \frac{w^s}{w^{s'}} = 2^{(j-j')\cdot s} \geq 2^{4|s|}. \]
Moreover, if \( \emptyset \neq Q^{(\alpha,\beta)} \cap Q^{(\alpha',\beta')} \) then \( x \neq 2^j \), \( |\xi| \geq 2^{j'} \), and hence \( j' \leq 4 \). Therefore, \( w^s/w^{s'} = 2^{-s \cdot j'} \leq 2^{4|s|} \cdot j' \leq 2^{4|s|} \) and \( w^s/w^{s'} = 2^{-s \cdot j'} \geq 2^{-|s| \cdot j'} \geq 2^{-|s| \cdot 4} \).

23
where \( \nu \) is identical to the one introduced in Theorem 2.17. Finally, let us select, for each \( \mu \) Proposition 6.3 and Corollary 6.4. Furthermore, note that the constant \( \alpha, \beta \) of that theorem are indeed satisfied, as can be seen from Lemmas 5.1 and 6.1, considering the chain of embeddings \( Q^{(\alpha, \beta)} \cap Q_i^{(\alpha', \beta')} \neq \emptyset \) proving that \( w^s \) as considered as a weight for \( Q^{(\alpha, \beta)} \) — is relatively \( Q^{(\alpha', \beta')} \)-moderate.

To prove that \( Q^{(\alpha, \beta)} \) is relatively \( Q^{(\alpha', \beta')} \)-moderate, we note that

\[
\det T_i = \det(R_{j, \ell} A_j) = \det A_j = 2^{a_j} \cdot 2^{a_j + \nu} = w_i^{a_j + \nu} \quad \text{for} \quad i = (j, m, \ell) \in I^{(\alpha, \beta)}.
\]

Similarly, \( \det T_i = 1 = w_i^{a_j + \nu} \) for \( i = 0 \). Thus, we conclude that \( |\det T_i^{(\alpha, \beta)}| = w_i^{a_j + \nu} \) if \( Q_i^{(\alpha, \beta)} \cap Q_i^{(\alpha', \beta')} \neq \emptyset \).

We can now state and prove the main theorem of this subsection.

**Theorem 6.5.** Let \( 0 \leq \beta \leq \alpha \leq 1 \) and \( 0 \leq \beta' \leq \alpha' \leq 1 \) be such that \( \alpha \leq \alpha' \) and \( \beta \leq \beta' \). Let \( p_1, p_2, q_1, q_2 \in (0, \infty) \) and \( s_1, s_2 \in \mathbb{R} \).

Then

\[
W^{p_1, q_1}_{s_1}(\alpha, \beta) \hookrightarrow W^{p_2, q_2}_{s_2}(\alpha', \beta')
\]

if and only if \( p_1 \leq p_2 \) and

\[
\left\{
\begin{aligned}
& s_1 > s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) + \mu(\alpha' - \alpha + \beta' - \beta) + (2 - \alpha' - \beta')(q_2^{-1} - q_1^{-1}), & \text{if } q_1 > q_2, \\
& s_1 \geq s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) + \mu(\alpha' - \alpha + \beta' - \beta), & \text{if } q_1 \leq q_2,
\end{aligned}
\right.
\]

where \( \mu = (p_2^* - q_1^{-1})_+ \) and \( p_2^* = (\min\{p_2, p_2^*\})^{-1} \) and where the conjugate exponent \( p_2^* \in [1, \infty) \) of \( p_2 \in (0, \infty) \) is defined as in Appendix D.

Conversely,

\[
W^{p_1, q_1}_{s_1}(\alpha, \beta') \hookrightarrow W^{p_2, q_2}_{s_2}(\alpha, \beta)
\]

if and only if \( p_1 \leq p_2 \) and

\[
\left\{
\begin{aligned}
& s_1 > s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) + \nu(\alpha' - \alpha + \beta' - \beta) + (2 - \alpha' - \beta')(q_2^{-1} - q_1^{-1}), & \text{if } q_1 > q_2, \\
& s_1 \geq s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) + \nu(\alpha' - \alpha + \beta' - \beta), & \text{if } q_1 \leq q_2,
\end{aligned}
\right.
\]

where \( \nu = (q_2^{-1} - p_1^*)_+ \) and \( p_1^* = \min\{p_1^{-1}, 1 - p_2^{-1}\} \).

**Remark.** 1) Note that this theorem cannot be applied if one of the conditions \( \alpha \leq \alpha' \) or \( \beta \leq \beta' \) does not hold. Nevertheless, *sufficient* conditions for embeddings can still be derived, for instance by considering the chain of embeddings

\[
W^{p_1, q_1}_{s_1}(\alpha, \beta) \hookrightarrow W^{p_2, q_2}_{s_2}(\max\{\alpha, \alpha'\}, \max\{\beta, \beta'\}) \hookrightarrow W^{p_2, q_2}_{s_2}(\alpha', \beta'),
\]

for suitable parameters \( p, q, s \) under certain conditions on \( p_1, p_2, q_1, q_2 \) and \( s_1, s_2 \). Alternatively, one can use embedding criteria provided in [61] which are applicable to coverings that are not almost subordinate to each other. This, however, is outside the scope of the present paper.

2) In Subsection 6.5 we shall see that the wave packet smoothness spaces \( W^{p, q}_{s}(\alpha, \alpha) \) are identical to the \( \alpha \)-modulation spaces \( M^{p, q}_{\alpha}(\mathbb{R}^2) \) introduced in Gröbner’s PhD thesis [25] and studied further in [3, 19, 55, 33, 36, 32, 61]. Therefore, Theorem 6.5 can be seen as a generalisation of the characterisation of the embeddings between \( \alpha \)-modulation spaces, which were first studied in [25, 33] and fully understood in [60, 32, 61].

**Proof.** To characterise the embedding \( W^{p_1, q_1}_{s_1}(\alpha, \beta) \hookrightarrow W^{p_2, q_2}_{s_2}(\alpha', \beta'), \) we shall apply Theorem 2.17 to the coverings \( Q = Q^{(\alpha, \beta)} \) and \( P = Q^{(\alpha', \beta')} \) and the respective weights \( w = w^{s_1} \) and \( v = v^{s_2} \). All assumptions of that theorem are indeed satisfied, as can be seen from Lemmas 5.1 and 6.1, Proposition 6.3 and Corollary 6.4. Furthermore, note that the constant \( \mu \) defined in the present theorem is identical to the one introduced in Theorem 2.17. Finally, let us select, for each \( i' \in I^{(\alpha', \beta')} \),
such an index $i' \in I^{(\alpha, \beta)}$ that $Q_i^{(\alpha, \beta)} \cap Q_i^{(\alpha, \beta)} \neq \emptyset$. Then, Theorem 2.17 implies that the embedding $W^{p_1,q_1}_{s_1}(\alpha, \beta) \hookrightarrow W^{p_2,q_2}_{s_2}(\alpha', \beta')$ holds if and only if $p_1 \leq p_2$ and

$$\infty > \left\| \left( \frac{w_i^{s_2}}{w_i^{s_1}} \cdot | \det T_{i'}^{(\alpha, \beta)} |_{p_1 - p_2 - \mu} \cdot | \det T_{i'}^{(\alpha', \beta')} |_{p_1 - p_2 - \nu} \right)_{i' \in I^{(\alpha', \beta')}} \right\|_{p_2'/(q_1/q_2)},$$

(Corollary 6.4)

First, we note that the single term with index 0 $i' \in I^{(\alpha', \beta')}$ alone has no influence on whether the norm in (6.6) is finite or not. Therefore, it is enough to consider only the terms $i' \in I_0^{(\alpha', \beta')}$. Next, since the set

$$\Omega_{j'} := \{(m', \ell') \in \mathbb{N}_0 \times \mathbb{N}_0 : (j', m', \ell') \in I_0^{(\alpha', \beta')}\}$$

satisfies $|\Omega_{j'}| \asymp 2^{(1-\alpha+1-\beta')j'}$ and since the weight $w_i^{\gamma} = 2^{\gamma j'}$ is independent of $m', \ell'$ for $i' = (j', m', \ell')$, we conclude that

$$\left\| (w_i^{\gamma})_{i' \in I_0^{(\alpha', \beta')}} \right\|_{p_2'/(q_1/q_2)} < \left\| (2^{\gamma j'}(\gamma + (2-\alpha-\beta')q)/q)_{j' \in \mathbb{N}} \right\|_{p_2'} \quad \forall \gamma \in \mathbb{R} \text{ and } q \in (0, \infty).$$

The right-hand side of (6.7) is finite if and only if

$$\gamma + (2 - \alpha' - \beta')/q < 0, \quad \text{if } q < \infty, \quad \gamma + (2 - \alpha' - \beta')/q < 0, \quad \text{if } q < \infty, \quad \gamma + (2 - \alpha' - \beta')/q < 0, \quad \text{if } q < \infty, \quad \gamma < 0, \quad \text{if } q = \infty.$$

Therefore, by recalling the identity (2.13), we infer that (6.6) is satisfied if and only if

$$\begin{cases} s_2 - s_1 + (\alpha + \beta)(p_1 - p_2 - 1) - \mu + \mu(\alpha' + \beta') + (2 - \alpha - \beta') \cdot (q_2^{-1} - q_1^{-1}) < 0, & \text{if } q_1 > q_2, \\ s_2 - s_1 + (\alpha + \beta)(p_1 - p_2 - 1) - \mu + \mu(\alpha' + \beta') \leq 0, & \text{if } q_1 \leq q_2, \end{cases}$$

which is equivalent to the conditions stated in the theorem.

To characterise the converse embedding $W^{p_1,q_1}_{s_1}(\alpha, \beta) \hookrightarrow W^{p_2,q_2}_{s_2}(\alpha', \beta')$, we apply Theorem 2.16 to the coverings $Q = Q^{(\alpha', \beta')}$ and $P = Q^{(\alpha, \beta)}$ and the respective weights $w = w^{s_1}$ and $v = v^{s_2}$. As before, we see that all assumptions of that theorem are indeed satisfied. Furthermore, we note that the constant $\nu$ defined in the present is identical to the one introduced in Theorem 2.16. Therefore, we see as above that the desired embedding holds if and only if $p_1 \leq p_2$ and

$$\infty > \left\| \left( \frac{w_i^{s_2}}{w_i^{s_1}} \cdot | \det T_{i'}^{(\alpha, \beta)} |_{p_1 - p_2 - \mu} \cdot | \det T_{i'}^{(\alpha', \beta')} |_{p_1 - p_2 - \nu} \right)_{i' \in I^{(\alpha', \beta')}} \right\|_{p_2'/(q_1/q_2)},$$

(Corollary 6.4)

Precisely as before, we thus see that the embedding holds if and only if the conditions stated in the theorem are satisfied.

6.3. Characterising the coincidence of two wave packet smoothness spaces

In this short subsection, we show that two wave packet spaces $W^{p_1,q_1}_{s_1}(\alpha, \beta)$ and $W^{p_2,q_2}_{s_2}(\alpha', \beta')$ can coincide only if all their parameters are identical. This is almost true as stated; a small exception occurs for the case $p_1 = q_1 = p_2 = q_2 = 2$ in which the wave packet smoothness spaces are simply $L^2$-Sobolev spaces, independently of the parameters $\alpha, \beta$.

**Theorem 6.6.** Let $0 \leq \beta \leq \alpha \leq 1$, $0 \leq \beta' \leq \alpha' \leq 1$, $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$. If $W^{p_1,q_1}_{s_1}(\alpha, \beta) = W^{p_2,q_2}_{s_2}(\alpha', \beta')$, then $(p_1, q_1, s_1) = (p_2, q_2, s_2)$. If furthermore $(p_1, q_1) \neq (2, 2)$, then $(\alpha, \beta) = (\alpha', \beta')$.

Finally, for arbitrary $s \in \mathbb{R}$, $W^{2,2}_{s}(\alpha, \beta) = H^s(\mathbb{R}^2)$ with equivalent norms, where the $L^2$-Sobolev space $H^s(\mathbb{R}^2)$ is given by $H^s(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) : (1 + |x|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^2) \}$ (see for instance Section 9.3 in [18]).
Proof. Let us assume that \( \mathcal{W}_{s}^{p_{1},q_{1}}(\alpha, \beta) = \mathcal{W}_{s}^{p_{2},q_{2}}(\alpha', \beta') \). Since \( \mathcal{W}_{s}^{p,q}(\alpha, \beta) = \mathcal{D}(Q_{\alpha, \beta}, L^{p}, \ell^{q}_{s/\mu}) \), Theorem 2.18 implies that \((p_{1}, q_{1}) = (p_{2}, q_{2})\) and that there is \(C > 0\) such that \(C^{-1} \cdot |w_{s}^{i}| \leq w_{s}^{i} \leq C \cdot |w_{s}^{i}|\) for all \(i \in I^{(\alpha, \beta)}\) and \(i' \in I^{(\alpha', \beta')}\) for which \(Q_{s}^{i} \cap Q_{s}^{i'} \neq \emptyset\). Because of \((2^{i-1})^{j} \in Q_{j,0}^{(\alpha, \beta)} \cap Q_{j,0}^{(\alpha', \beta')}\) for arbitrary \(j \in \mathbb{N}\), this implies \(C^{-1} \cdot 2^{s_{1}} \leq 2^{s_{2}} \leq C \cdot 2^{s_{1}}\) for all \(j \in \mathbb{N}\), which implies that \(s_{1} = s_{2}\).

Furthermore, in case of \((p_{1}, q_{1}) \neq (2, 2)\), Theorem 2.18 shows that \(Q_{s}^{(\alpha, \beta)}\) and \(Q_{s}^{(\alpha', \beta')}\) are weakly equivalent. Since the coverings \(Q_{s}^{(\alpha, \beta)}\) and \(Q_{s}^{(\alpha', \beta')}\) consist of open, path-connected sets, Lemma 2.14 shows that \(Q_{s}^{(\alpha, \beta)}\) and \(Q_{s}^{(\alpha', \beta')}\) are in fact equivalent coverings. Therefore, Proposition 6.3 shows that \(\alpha, \beta = (\alpha', \beta')\).

Finally, since \(w_{s}^{i} \asymp |1 + |\xi||^{s} \asymp (1 + |\xi|^{2})^{s/2}\) for all \(\xi \in Q_{i}^{(\alpha, \beta)}\) and \(i \in I^{(\alpha, \beta)}\) (see Equation (6.1)), Lemma 6.10 in [61] implies that
\[
\mathcal{W}_{s}^{2,2}(\alpha, \beta) = \{ f \in Z': (1 + |\xi|^{2})^{s/2} \cdot \hat{f} \in L^{2}(\mathbb{R}^{2}) \} = \{ f \in S'(\mathbb{R}^{2}) : (1 + |\xi|^{2})^{s/2} \cdot \hat{f} \in L^{2}(\mathbb{R}^{2}) \} = \mathcal{H}^{s}(\mathbb{R}^{2}),
\]
where the penultimate equality is justified by the smoothness and growth properties of the weight \(\xi \mapsto (1 + |\xi|^{2})^{s/2}\), which imply that if \(g = \hat{f} \in \mathcal{D}'(\mathbb{R}^{2})\) satisfies \((1 + |\xi|^{2})^{s/2} \cdot g \in L^{2}(\mathbb{R}^{2})\), then \(g \in S'(\mathbb{R}^{2})\) and hence \(f \in S'(\mathbb{R}^{2})\).

\(\square\)

6.4. Establishing embeddings of wave packet smoothness spaces in classical spaces

In this subsection, we study the conditions on the parameters \(\alpha, \beta\) and \(p, q, s\) under which the wave packet smoothness space \(\mathcal{W}_{s}^{p,q}(\alpha, \beta)\) embeds in the Sobolev space \(W^{k, \ell}(\mathbb{R}^{2})\) or the inhomogeneous Besov space \(B_{s}^{p,q}(\mathbb{R}^{2})\). For the Besov spaces, we also study the converse question, that is, whether the Besov spaces embed in the wave packet smoothness spaces. As an application, we show that the Besov spaces arise as special cases of the wave packet smoothness spaces for the case \(\alpha = \beta = 1\).

We start by analysing the existence of embeddings between wave packet smoothness and Besov spaces.

Theorem 6.7. Let \(0 \leq \beta \leq \alpha \leq 1\), \(p_{1}, p_{2}, q_{1}, q_{2} \in (0, \infty)\) and \(s_{1}, s_{2} \in \mathbb{R}\). Let \(B_{s}^{p,q}(\mathbb{R}^{2})\) be the inhomogeneous Besov spaces as introduced for instance in Definition 2.2.1 in [24] or in Definition 2 of Section 2.3.1 in [57]. Let us define \(p_{1}^{1} := \min\{p_{1}^{-1}, 1 - p_{1}^{-1}\}\) and \(p_{2}^{1} := (\min\{p_{2}, p_{2}^{1}\})^{-1}\).

Then,
\[
\mathcal{W}_{s_{1}}^{p_{1},q_{1}}(\alpha, \beta) \hookrightarrow B_{s_{2}}^{p_{2},q_{2}}(\mathbb{R}^{2})
\]
if and only if \(p_{1} \leq p_{2}\) and
\[
\begin{cases}
 s_{1} \geq s_{2} + (\alpha + \beta)(p_{1}^{-1} - p_{2}^{-1} - \mu) + 2\mu, & \text{if } q_{1} \leq q_{2}, \\
 s_{1} > s_{2} + (\alpha + \beta)(p_{1}^{-1} - p_{2}^{-1} - \mu) + 2\mu, & \text{if } q_{1} > q_{2},
\end{cases}
\]
where \(\mu := (p_{2}^{1} - q_{1}^{1})_{+}\). \quad (6.9)

Conversely,
\[
B_{p_{1},q_{1}}(\mathbb{R}^{2}) \hookrightarrow \mathcal{W}_{s_{2}}^{p_{2},q_{2}}(\alpha, \beta)
\]
if and only if \(p_{1} \leq p_{2}\) and
\[
\begin{cases}
 s_{1} \geq s_{2} + (\alpha + \beta)(p_{1}^{-1} - p_{2}^{-1} - \nu) + 2\nu, & \text{if } q_{1} \leq q_{2}, \\
 s_{1} > s_{2} + (\alpha + \beta)(p_{1}^{-1} - p_{2}^{-1} - \nu) + 2\nu, & \text{if } q_{1} > q_{2},
\end{cases}
\]
where \(\nu := (q_{2}^{-1} - p_{1}^{1})_{+}\). \quad (6.10)

Remark. Let us somewhat clarify this statement. The Besov space \(B_{s}^{p,q}(\mathbb{R}^{2})\) is defined as a subspace of \(S'(\mathbb{R}^{2})\), while the wave packet smoothness space \(\mathcal{W}_{s}^{p,q}(\alpha, \beta)\) is a subspace of \(Z'\) (see Definition 2.5).

Therefore, validity of the embedding \(B_{p_{1},q_{1}}(\mathbb{R}^{2}) \hookrightarrow \mathcal{W}_{s_{2}}^{p_{2},q_{2}}(\alpha, \beta)\), strictly speaking, that the map \(B_{p_{1},q_{1}}(\mathbb{R}^{2}) \rightarrow \mathcal{W}_{s_{2}}^{p_{2},q_{2}}(\alpha, \beta), f \mapsto f|_{Z}\) is well-defined and bounded. Likewise, validity of the embedding \(\mathcal{W}_{s_{1}}^{p_{1},q_{1}}(\alpha, \beta) \hookrightarrow B_{p_{2},q_{2}}(\mathbb{R}^{2})\) means that each \(f \in \mathcal{W}_{s_{1}}^{p_{1},q_{1}}(\alpha, \beta) \subset Z'\) can be extended to a uniquely determined tempered distribution \(f|_{Z}\) and that the map \(\mathcal{W}_{s_{1}}^{p_{1},q_{1}}(\alpha, \beta) \rightarrow B_{p_{2},q_{2}}(\mathbb{R}^{2}), f \mapsto f|_{Z}\) is well-defined and bounded.

Proof. It was shown in Lemma 9.15 in [61] that the map
\[
B_{s}^{p,q}(\mathbb{R}^{2}) \rightarrow \mathcal{D}(\mathcal{B}, L^{p}, \ell^{q}_{s/\mu}), f \mapsto f|_{Z}
\]
(6.11)
is an isomorphism of quasi-Banach spaces. Here, the inhomogeneous Besov covering $\mathcal{B} = (B_n)_{n \in \mathbb{N}_0}$ is given by $B_0 = B_4(0)$ and $B_n = B_{2^{n+2}}(0) \setminus \overline{B}_{2^{n-2}}(0)$ for $n \in \mathbb{N}$ and the weight $\nu^{(s)}$ is given by $\nu^{(s)}_n = 2^{sn}$ for $n \in \mathbb{N}_0$. It was shown in Lemma 9.10 in [61] for

$$S_n := 2^n \text{id}, \quad e_n := 0, \quad B_0^{(1)} := B_4(0) \setminus \overline{B}_{1/4}(0), \quad B_2^{(0)} := B_4(0), \quad k_n := \begin{cases} 1, & \text{if } n \in \mathbb{N}, \\ 2, & \text{if } n = 0 \end{cases}$$

that $\mathcal{B} = (B_n)_{n \in \mathbb{N}_0} = (S_n B_{k_n}^{(0)} + e_n)_{n \in \mathbb{N}_0}$ is an almost structured covering of $\mathbb{R}^2$ with associated family $(S_n \bullet + e_n)_{n \in \mathbb{N}_0}$.

Given the isomorphism [61] and the remark we made after the theorem, we need to characterise the existence of the embeddings $\mathcal{D}(\mathcal{Q}^{(\alpha,\beta)}, L^p, \ell_{u^{(1)}}^{(q)}) = W_{i \ll \alpha, \beta}^{p_1, q_1}(\mathcal{Q}^{(\alpha,\beta)})$ and $\mathcal{D}(\mathcal{B}, L^p, \ell_{u^{(1)}}^{(q)}) \hookrightarrow W^{p_2, q_2}_i(\mathcal{Q}^{(\alpha,\beta)}) = \mathcal{D}(\mathcal{Q}^{(\alpha,\beta)}, L^p, \ell_{u^{(2)}}^{(q)})$. To do so, we shall rely on Theorems 2.17 and 2.16 respectively. The main prerequisite for applying these theorems is that $\mathcal{Q}^{(\alpha,\beta)}(Q_i)_{i \in I^{(\alpha,\beta)}}$ be almost subordinate to $\mathcal{B}$ and that $\mathcal{Q}^{(\alpha,\beta)}$ and $u^{(2)}$ be relatively $\mathcal{B}$-moderate.

Since $\mathcal{Q}^{(\alpha,\beta)}$ consists only of open and path-connected sets, and since $\mathcal{B}$ consists only of open sets, Lemma 2.14 implies that $\mathcal{Q}^{(\alpha,\beta)}$ is almost subordinate to $\mathcal{B}$, if it is weakly subordinate; that is, we need to show that sup$_{i \in I^{(\alpha,\beta)}} |J_i| < \infty$ where $J_i := \{ n \in \mathbb{N}_0 : B_n \cap Q_i \neq \emptyset \}$ for $i \in I^{(\alpha,\beta)}$. To see that this is the case, let $i = (j, m, \ell) \in I_0^{(\alpha,\beta)}$ be arbitrary. For any $n \in \mathbb{N}$ with $\emptyset \neq B_n \cap Q_i \ni \xi$, [61] implies that

$$2^{n-2} \leq |\xi| \leq 2^{j+3} \quad \text{and} \quad 2^{j-2} \leq |\xi| \leq 2^{n+2},$$

and hence $j - 3 \leq n \leq j + 4$. Thus, $J_i \subset \{0\} \cup \{ j - 3, \ldots, j + 4 \}$, which implies that $|J_i| \leq 9$ for all $i \in I_0^{(\alpha,\beta)}$. Finally, if $\emptyset \neq B_n \cap Q_0 \ni \xi$ for some $n \in \mathbb{N}$, then $2^{n-2} \leq |\xi| \leq 2^2$ and hence $n \leq 3$. Therefore, $J_0 \subset \{0, \ldots, 3\}$ and thus $|J_i| \leq 9$ for all $i \in I^{(\alpha,\beta)}$. We have thus shown that $\mathcal{Q}^{(\alpha,\beta)}$ is almost subordinate to $\mathcal{B}$.

To verify that, for arbitrary $\sigma \in \mathbb{R}$, the weight $u^{(\sigma)}$ is relatively $\mathcal{B}$-moderate, we recall from [6.1] that $u^{(\sigma)}_1 = 1 + |\sigma|$ for arbitrary $\xi \in Q_i$ and $i \in I^{(\alpha,\beta)}$. Since $2 \geq 1 + |\sigma| = 2^n$ for $\xi \in B_n$ and any $n \in \mathbb{N}_0$, this implies that

$$w^{(\sigma)}_1 \asymp_\sigma (1 + |\sigma|)^\sigma \asymp_\sigma 2^{2n} \quad \text{for any } \sigma \in \mathbb{R} \text{ and all } i \in I^{(\alpha,\beta)} \text{ and } n \in \mathbb{N}_0 \text{ with } Q_i \cap B_n \neq \emptyset. \quad (6.12)$$

In particular, $w^{(\sigma)}_n \asymp 2^{2n} \times w^{(\sigma)}_0$, if $Q_i \cap B_n \neq \emptyset \neq Q_j \cap B_n$. Hence, $u^{(\sigma)}$ is relatively $\mathcal{B}$-moderate.

From this, we conclude that the wave packet covering $\mathcal{Q}^{(\alpha,\beta)}$ is relatively $\P$-moderate. Indeed,

$$|\det T_{i'}| = 2^{(\alpha+\beta)} |S_{i'}^{(\alpha+\beta)}| = |\det T_i| \quad \forall i = (j, m, \ell) \in I_0^{(\alpha,\beta)}. \quad (6.13)$$

Likewise, $|\det T_0| = 1 = w_0^{(\alpha+\beta)}$, so that (6.13) is also true for $i = 0$. In particular, we see that

$$|\det T_{i'}| = w_i^{(\alpha+\beta)} \asymp w_{i'}^{(\alpha+\beta)} = |\det T_{i'}| \quad \text{for } i, i' \in I^{(\alpha,\beta)} \text{ such that } Q_i \cap B_n \neq \emptyset \neq Q_{i'} \cap B_n,$$

or, in other words, $\mathcal{Q}^{(\alpha,\beta)}$ is relatively $\mathcal{B}$-moderate.

Now, let us choose, for each $n \in \mathbb{N}_0$, such an index $i_n \in I^{(\alpha,\beta)}$ that $Q_{i_n} \cap B_n \neq \emptyset$. Then, for $\mu$ as defined in the present theorem, Theorem 2.17 shows that $\mathcal{D}(\mathcal{Q}^{(\alpha,\beta)}, L^p, \ell_{u^{(1)}}^{(q)}) \hookrightarrow \mathcal{D}(\mathcal{B}, L^p, \ell_{u^{(2)}}^{(q)})$ holds if and only if $p_1 \leq p_2$ and

$$\infty > \left\| \left( \frac{1}{w_{i_n}^{(s_2)}} \cdot |\det T_{i_n}|^{p_1-1-p_2^{-1}-\tau_1} \cdot |\det S_{i_n}|^{\tau_1} \right)_{n \in \mathbb{N}_0} \right\|_{\ell_{p_2}^{(q_1/q_2)}} \quad (6.12 \text{ and } 6.13) \times \left\| \left( \frac{Q_{i_n}^{(s_2-s_1)} n + n(n+\alpha+\beta)(p_1^{-1}-p_2^{-1}-\tau_1) + 2n\tau_1}{n \in \mathbb{N}_0} \right) \right\|_{\ell_{p_2}^{(q_1/q_2)}}.$$
As before, we see that this norm is finite if and only if Condition (6.10) holds.

As a direct application of the preceding theorem to the case $\alpha = \beta = 1$, we conclude that the inhomogeneous Besov spaces are special examples of the wave packet smoothness spaces.

**Corollary 6.8.**

$$\mathcal{W}_s^{p,q}(1,1) = B_{p,q}^s(\mathbb{R}^2)$$

for all $p, q \in (0, \infty]$ and $s \in \mathbb{R}$.

From Section 2.3.3 in [57], we know that $S(\mathbb{R}^2) \hookrightarrow B_{p,q}^s(\mathbb{R}^2) \hookrightarrow S'(\mathbb{R}^2)$. Combining this with the previous theorem, we conclude that, for arbitrary $p, q \in (0, \infty]$ and $s \in \mathbb{R}$,

$$S(\mathbb{R}^2) \hookrightarrow B_{p,q}^s(\mathbb{R}^2) \hookrightarrow \mathcal{W}_s^{p,q}(\alpha, \beta) \hookrightarrow B_{p,q}^\sigma(\mathbb{R}^2) \hookrightarrow S'(\mathbb{R}^2)$$

if $\sigma$ is sufficiently large and $\rho$ sufficiently small (negative). We have thus established the following corollary.

**Corollary 6.9.**

$$S(\mathbb{R}^2) \hookrightarrow \mathcal{W}_s^{p,q}(\alpha, \beta) \hookrightarrow S'(\mathbb{R}^2)$$

for arbitrary $0 \leq \beta \leq \alpha \leq 1$, $p, q \in (0, \infty]$, and $s \in \mathbb{R}$.

We now turn to studying conditions under which the wave packet smoothness space $\mathcal{W}_s^{p,q}(\alpha, \beta)$ are embedded in the Sobolev space $W^{k,r}(\mathbb{R}^2)$. The following theorem will also justify the name “smoothness spaces,” since it will show that, if the smoothness parameter $s$ is chosen so that $s > k + 2(1 + p^{-1})$, then the wave packet smoothness space $\mathcal{W}_s^{p,q}(\alpha, \beta)$ consists of $C^k$ functions.

**Theorem 6.10.** Let $0 \leq \beta \leq \alpha \leq 1$, $p, q \in (0, \infty]$, $k \in \mathbb{N}_0$, $s \in \mathbb{R}$ and $r \in [1, \infty]$, and let us define $r^\vee := \min\{r, r'\}$. If

$$p \leq r \quad \text{and} \quad \begin{cases} s \geq k + (\alpha + \beta)(p^{-1} - r^{-1}), & \text{if } q \leq r^\vee, \\ s > k + (\alpha + \beta)(p^{-1} - r^{-1}) + (2 - \alpha - \beta)\left(\frac{1}{r^\vee} - \frac{1}{q}\right), & \text{if } q > r^\vee, \end{cases}$$

then $\mathcal{W}_s^{p,q}(\alpha, \beta) \hookrightarrow W^{k,r}(\mathbb{R}^2)$; that is, there is an injective bounded linear map

$$\iota: \mathcal{W}_s^{p,q}(\alpha, \beta) \to W^{k,r}(\mathbb{R}^2) \quad \text{such that} \quad \iota f = f \quad \forall f \in S(\mathbb{R}^2) \text{ with } \hat{f} \in C^\infty_c(\mathbb{R}^2).$$

Moreover, if (6.14) is satisfied for $r = \infty$, then $\iota f \in C^k_b(\mathbb{R}^2)$ for all $f \in \mathcal{W}_s^{p,q}(\alpha, \beta)$ where

$$C^k_b(\mathbb{R}^2) = \{f \in C^k(\mathbb{R}^2; \mathbb{C}) : \partial^\alpha f \in L^\infty(\mathbb{R}^2) \quad \forall \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \leq k\}.$$

Conversely, assume that there is such a $C > 0$ that $\|f\|_{W^{k,r}} \leq C\|f\|_{\mathcal{W}_s^{p,q}(\alpha, \beta)}$ for all such $f \in S(\mathbb{R}^2)$ that $\hat{f} \in C^\infty_c(\mathbb{R}^2)$. Then

$$p \leq r \quad \text{and} \quad \begin{cases} s \geq k + (\alpha + \beta)(p^{-1} - r^{-1}), & \text{if } q \leq r, \\ s > k + (\alpha + \beta)(p^{-1} - r^{-1}) + (2 - \alpha - \beta)(r^{-1} - q^{-1}), & \text{if } q > r. \end{cases}$$

Furthermore, if $r = \infty$, then (6.14) is satisfied and if $r \in (2, \infty)$, then

$$\begin{cases} s \geq k + (\alpha + \beta)(p^{-1} - 2^{-1})_+, & \text{if } q \leq 2, \\ s > k + (\alpha + \beta)(p^{-1} - 2^{-1})_+ + (2 - \alpha - \beta)(2^{-1} - q^{-1}), & \text{if } q > 2. \end{cases}$$

**Remark.** Note that this theorem gives a complete characterisation of the existence of the embedding for $r \in [1, 2] \cup \{\infty\}$. Indeed, for $r = \infty$, this results from the theorem statement; moreover, $r^\vee = r$ for $r \in [1, 2]$, so that (6.14) and (6.15) are identical.

For $r \in (2, \infty)$, on the other hand, there is a gap between the necessary and the sufficient conditions.
Proof. Let us use the notations of Lemma 5.1 and additionally define
\[ v_i := | \det T_i |^{p - 1 - r - 1} \quad \text{and} \quad u_i := | \det T_i |^{p - 1 - r - 1} \cdot (\sum b_i^{k} + \| T_i \|^k) \] for \( i \in I := I^{(\alpha, \beta)} \).

Next, remember that \( W^p_q(\alpha, \beta) = D(\mathbb{Q}^{(\alpha, \beta)}, L^p, \ell^q_{\alpha, \beta}) \) where \( Q_i = (Q_i)_{i \in I} \) with \( Q_i = T_i Q_i' + b_i \) is an almost structured covering, and thus — according to Theorem 2.8 in \[62\] — a regular covering of \( \mathbb{R}^2 \). We can thus apply \[62\] Corollary 3.5 to conclude that the embedding \( W^p_q(\alpha, \beta) \hookrightarrow W^{k, r}(\mathbb{R}^2) \) — which is to be understood as in the statement of the theorem — holds as long as
\[ p \leq r \quad \text{and} \quad \ell_{w^*}^q(I^{(\alpha, \beta)}) \hookrightarrow \ell_{u^*}^p(I^{(\alpha, \beta)}), \quad \text{as well as} \quad \ell_{w^*}^q(I^{(\alpha, \beta)}) \hookrightarrow \ell_{u^*}^p(I^{(\alpha, \beta)}). \] (6.17)

To verify (6.17), we simplify the weights \( v = (v_i)_{i \in I} \) and \( u = (u_i)_{i \in I} \). First, for \( i = (j, m, \ell) \in I_0^{(\alpha, \beta)} \), we infer from the definition of \( b_i \) in Lemma 5.1 and that of \( c_{j,m} \) in Definition 5.1 that
\[ |b_i| = |c_{j,m}| = \left[ \left( \frac{2^{i+1} + m - 2^{\alpha j}}{2} \right) \right] \leq 2^j, \quad \text{since} \quad 0 \leq m \leq m_j^{\max} = 2(1-\alpha)j - 1 \leq 2(1-\alpha)j. \] Second, since \( T_i = R_{j, \ell} A_j \) and \( A_j = diag(2^\alpha j, 2^\beta j) \) and since \( \beta \leq \alpha \), we conclude that
\[ \| T_i \| = \| A_j \| = \max\{2^\alpha j, 2^\beta j\} = 2^\alpha j. \] Combining these two results leads to \( |b_i|^k + \| T_i \|^k \geq 2^j k + 2^\alpha j k \geq 2^{j+k} = w_i^k \) for \( i = (j, m, \ell) \in I_0^{(\alpha, \beta)} \). On the other hand, if \( i = 0 \), then \( b_i = 0 \) and \( T_i = id \) and thus \( |b_i|^k + \| T_i \|^k = 1 = w_i^k \) as well. Therefore,
\[ |b_i|^k + \| T_i \|^k \geq w_i^k \quad \forall i \in I^{(\alpha, \beta)}. \] (6.18)

Furthermore, we note that \( | \det T_i | = w_i^{\alpha + \beta} \). Therefore,
\[ v_i \leq w_i^{\alpha + \beta}, \quad \text{and} \quad u_i \leq w_i^{k+\alpha+\beta}(p-1-r-1). \] (6.19)

Finally, since \( w_i \geq 1 \) for all \( i \in I \) and since \( k \geq 0 \), we conclude that (6.17) holds if and only if
\[ p \leq r \quad \text{and} \quad \ell_{w^*}^q(I^{(\alpha, \beta)}) \hookrightarrow \ell_{u^*}^p(I^{(\alpha, \beta)}). \] This, according to Lemma 5.1 in \[62\] and the remark that follows it, holds if and only if
\[ p \leq r \quad \text{and} \quad \frac{w_i^{k+\alpha+\beta}(p-1-r-1)}{w^*} \in \ell^{r', q'(r')}((I^{(\alpha, \beta)}) \quad \text{where} \quad \frac{1}{r^q \cdot \langle q, r^q \rangle} = \left( \frac{1}{r^q} - \frac{1}{q} \right). \] (6.20)

In particular, this shows that \( r^q \cdot \langle q, r^q \rangle = \infty \) if and only if \( q \leq r^q \). Now, using the same arguments as in the proof of Theorem 6.5 — see especially (6.7) and (6.8) — we conclude that (6.20) holds if and only if \( (j, m, \ell) \in I_0^{(\alpha, \beta)} \) does.

It remains to prove the converse statement. To do so, we shall apply Theorem 4.7 in \[62\], which is fully applicable only if there is such an \( M > 0 \) that \( \| T_i \| \leq M \) for all \( i \in I \). This can be easily verified in our case, since \( T_0^{-1} = id \) and since \( \| T_i^{-1} \| = \| A_j^{-1} \| = \max\{2^{-\alpha j}, 2^{-\beta j}\} \leq 1 \) for all \( i = (j, m, \ell) \in I_0^{(\alpha, \beta)} \).

Now, let us define
\[ u_{i}^{(\sigma, \tau)} := | \det T_i |^{\sigma - 1 - r - 1} \cdot (\sum b_i^{k} + \| T_i \|^k) \quad \text{for} \quad i \in I \quad \text{and} \quad \sigma, \tau \in (0, \infty) \] and note that \( u_{i}^{(\sigma, \tau)} \geq w_i^{k+\alpha+\beta}(\sigma-1-r-1) \) as a consequence of \( | \det T_i | = w_i^{\alpha + \beta} \) and of (6.18). Since by assumption \( \| f \|_{W^{k, q}(\alpha, \beta)} \leq \| f \|_{W^{k, q}(\alpha, \beta)} \) for all \( f \in \mathbb{S}(\mathbb{R}^2) \) with \( \widehat{f} \in C_c^\infty(\mathbb{R}^2) \), we can combine Theorems 4.4 and 4.7 and Lemma 5.1 in \[62\] to conclude that \( p \leq r \), and that
\[ w^{k+\alpha+\beta}(\sigma-1-r-1) \geq \frac{u_{i}^{(\sigma, \tau)}}{w^*} \in \ell^{r', q'}((I^{(\alpha, \beta)}) \] holds for the following choices of \( \sigma, \tau, q \):
(1) \((\sigma, \tau, q) = (p, r, r)\);
(2) \((\sigma, \tau, q) = (p, r, 1) = (p, r, r^\gamma)\) if \(r = \infty\);
(3) \((\sigma, \tau, q) = (p, 2, 2)\) if \(r \in (2, \infty)\); and
(4) \((\sigma, \tau, q) = (p, p, 2)\) if \(r \in (2, \infty)\).

Using the same arguments as in the proof of Theorem 6.5, we conclude that (1) implies (6.15), while (2) implies (6.14). Similarly, (3) and (4) imply, respectively, that

\[
\begin{cases}
  s \geq k + (\alpha + \beta)(p^{-1} - 2^{-1}), & \text{if } q \leq 2, \\
  s > k + (\alpha + \beta)(p^{-1} - 2^{-1}) + (2 - \alpha - \beta)(2^{-1} - q^{-1}), & \text{if } q > 2
\end{cases}
\]

and

\[
\begin{cases}
  s \geq k, & \text{if } q \leq 2, \\
  s > k + (2 - \alpha - \beta)(2^{-1} - q^{-1}), & \text{if } q > 2.
\end{cases}
\]

Combining these shows that (6.16) is satisfied.

\[\square\]

6.5. Identifying \(\alpha\)-modulation spaces as wave-packet smoothness spaces

In this subsection, we show that, for arbitrary \(\alpha \in [0, 1]\), the \(\alpha\)-modulation spaces \(M^{s,\alpha}_{p,q}(\mathbb{R}^2)\) are identical — up to canonical identifications — to the wave packet smoothness spaces \(W^{s,\alpha}_{p,q}(\alpha, \alpha)\). In particular, we show that the wave packet smoothness spaces \(W^{s,\alpha}_{p,q}(0, 0)\) are identical to the modulation spaces \(M^{s}_{p,q}(\mathbb{R}^2)\), which play a crucial role in time-frequency analysis [27, 23]. Precisely, we prove the following theorem.

**Theorem 6.11.** Let \(\alpha \in [0, 1]\), \(p, q \in (0, \infty)\) and \(s \in \mathbb{R}\). Then, for the \(\alpha\)-modulation space \(M^{s,\alpha}_{p,q}(\mathbb{R}^2)\) as defined in Definition 2.4 in [23] and the space \(Z\) as introduced in Definition 2.5, the map

\[M^{s,\alpha}_{p,q}(\mathbb{R}^2) \rightarrow W^{s,\alpha}_{p,q}(\alpha, \alpha), f \mapsto f|_Z\]

is an isomorphism of quasi-Banach spaces. In other words, \(M^{s,\alpha}_{p,q}(\mathbb{R}^2) = W^{s,\alpha}_{p,q}(\alpha, \alpha)\), up to canonical identifications.

**Remark.** Usually \(\alpha\)-modulation spaces for \(\alpha = 1\) are understood as inhomogeneous Besov spaces. With this interpretation, Corollary 6.8 shows that the preceding theorem also remains valid for \(\alpha = 1\).

**Proof.** It was shown in Corollary 9.16 in [61] that the map

\[M^{s,\alpha}_{p,q}(\mathbb{R}^2) \rightarrow D(\mathcal{P}^{(\alpha)}, L^p, \ell^q_{(1/(1-\alpha))}), f \mapsto f|_Z\]

is an isomorphism of quasi-Banach spaces. Here, the covering \(\mathcal{P}^{(\alpha)}\) is given by

\[\mathcal{P}^{(\alpha)} = (P_k)_{k \in \mathbb{Z}^2 \setminus \{0\}}, \quad \text{where } P_k := B_{r|k|^a_0} (|k|^\alpha_0 k) \quad \text{and } \alpha_0 := \frac{\alpha}{1 - \alpha},
\]

and where \(r > 0\) is chosen large enough so that \(\mathcal{P}^{(\alpha)}\) is a structured admissible covering of \(\mathbb{R}^2\); this is possible due to Lemma 9.3 in [61]. Furthermore, for arbitrary \(\theta \in \mathbb{R}\), the weight \(w^{(\theta)} = (v^{(\theta)}_k)_{k \in \mathbb{Z}^2 \setminus \{0\}}\) is given by

\[v^{(\theta)}_k = (1 + |k|^2)^{\theta/2}.
\]

Given (6.22), it is enough to prove that \(D(\mathcal{P}^{(\alpha)}, L^p, \ell^q_{(1/(1-\alpha))}) = W^{s,\alpha}_{p,q}(\alpha, \alpha)\). To do so, Lemma 6.11 in [61] and the identity \(W^{s,\alpha}_{p,q}(\alpha, \alpha) = D(Q^{(s,\alpha), \alpha}, L^p, \ell^q_{(1-\alpha)+})\) show that it is enough to prove that the covering \(\mathcal{P}^{(\alpha)}\) is equivalent to the \((\alpha, \alpha)\)-wave packet covering \(Q^{(\alpha, \alpha)} = (Q_i)_{i \in I^{(\alpha, \alpha)}}\) and that

\[v^{(s/(1-\alpha))}_k \asymp w^*_i \quad \text{for all } i \in I^{(\alpha, \alpha)} \text{ and } k \in \mathbb{Z}^2 \setminus \{0\} \text{ with } P_k \cap Q_i \neq \emptyset.
\]

We start by proving the latter. From (4.1), we conclude that \(1 + |\xi| \asymp |\xi| \asymp 2^j = w_i\) for all \(\xi \in Q_i\) and \(i = (j, m, \ell) \in I^{(\alpha, \alpha)}_0\). On the other hand, \(Q_0 = B_4(0)\) and thus \(1 + |\xi| \asymp 1 = w_0\) for all \(\xi \in Q_0\) as well.
Furthermore, Lemma 9.2 in [61] shows that $1 + |\xi| \asymp (1 + |\xi|^2)^{1/2} \asymp v_k^{(1/(1-\alpha))}$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and $\xi \in P_k$. Therefore, if there is some $\xi \in P_k \cap Q_i \neq \emptyset$, then

$$v_k^{(1/(1-\alpha))} = (v_k^{(1/(1-\alpha))})^* \asymp (1 + |\xi|)^* \asymp w_\vartheta^*.$$

It remains to prove that the coverings $Q^{(\alpha,\alpha)}$ and $P^{(\alpha)}$ are equivalent. Since both coverings consist of open, path-connected sets, Lemma 2.14 shows that it suffices to prove that the two coverings are weakly equivalent. To prove this, we shall use Lemma B.2 in [3], which implies that any two $\alpha$-coverings of $\mathbb{R}^d$ are weakly equivalent. Therefore, it suffices to show that both $Q^{(\alpha,\alpha)}$ and $P^{(\alpha)}$ are $\alpha$-coverings of $\mathbb{R}^2$.

To present the notion of $\alpha$-coverings as introduced in Definition 2.1 in [3], we need yet another notation: For a bounded open set $\Omega \subset \mathbb{R}^d$, we shall write

$$R_\Omega := \inf\{R > 0 : \exists \xi \in \mathbb{R}^d : \Omega \subset \xi + B_R(0)\} \quad \text{and} \quad r_\Omega := \sup\{r > 0 : \exists \xi \in \mathbb{R}^d : \xi + B_r(0) \subset \Omega\}.$$

With this, a family $(\Omega_\ell)_{\ell \in \mathcal{L}}$ is called an $\alpha$-covering if it satisfies the following:

1. $(\Omega_\ell)_{\ell \in \mathcal{L}}$ is an admissible covering of $\mathbb{R}^d$ consisting of open bounded sets;
2. there is such a constant $K \geq 1$ that $R_{\Omega_\ell}/r_{\Omega_\ell} \leq K$ for all $\ell \in \mathcal{L}$; and
3. $\lambda(\Omega_\ell) \asymp (1 + |\xi|)^{da}$ for all $\ell \in \mathcal{L}$ and $\xi \in \Omega_\ell$ where the implied constant is independent of $\ell$ and $\xi$. Here, $\lambda$ denotes the Lebesgue measure.

The first of these three conditions is satisfied for $Q^{(\alpha,\alpha)}$ and $P^{(\alpha)}$, as shown by Lemmas 3.2 and 4.2 and by Theorem 2.6 in [3], respectively.

Now, since the covering $P^{(\alpha)} = (P_k)_{k \in \mathbb{Z}^2 \setminus \{0\}}$ consists of open balls, $R_{P_k} = r_{P_k}$ for all $k \in \mathbb{Z}^2 \setminus \{0\}$. Hence, $P^{(\alpha)}$ also satisfies the second condition from above. Finally, since we are dealing with coverings of $\mathbb{R}^2$ and hence $d = 2$,

$$\lambda(P_k) = \lambda(B_{|k|^{\alpha_0}}(|k|^{\alpha_0} k)) \asymp (r|k|^{\alpha_0})^2$$

for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and $\xi \in P_k$. Here, we noted that $1 + |\xi| \asymp v_k^{(1/(1-\alpha))}$ for $\xi \in P_k$, as seen above.

It remains to show that $Q^{(\alpha,\alpha)}$ satisfies conditions (2) and (3). To do so, we first note that if $\Omega = TQ + b$ with $T \in \text{GL}(\mathbb{R}^d)$ and $b \in \mathbb{R}^d$ and with an open bounded set $Q$ such that $Q \subset B_\vartheta(\xi)$, then $\Omega \subset b + T\Omega_1 + TB_\vartheta(0) \subset b + T\xi + B_\vartheta(T\xi)$ and hence $R_{\Omega_1} \leq |T| \vartheta$. Conversely, if $\Omega = TQ + b$ with $Q \supset B_\vartheta(\xi)$, then

$$b + T\xi + B_\vartheta(T\xi) \subset b + T\xi + T^{-1}B_\vartheta(T^{-1}(0)) \subset b + TB_\vartheta(0) \subset TQ + b = \Omega,$$

and hence $r_{\Omega_1} \geq \vartheta/||T^{-1}||$.

Moreover, we note that $Q_i = T_iQ + b_i$ for $i = (j, m, \ell) \in I_0^{(\alpha,\alpha)}$ with $T_i, b_i$ as in Lemma 5.1 and with $Q = (1 - \varepsilon, 1 + \varepsilon) \times (1 - \varepsilon, 1 + \varepsilon)$ for a certain $\varepsilon \in (0, 1)$. From the definition of $Q$, we conclude that $Q \subset B_4(0)$ and $Q \supset B_{1/2}(\xi_0)$ for $\xi_0 = (1/2, 0)^t \in \mathbb{R}^2$. Finally, recalling that $\beta = \alpha$, we see that

$$||T_i|| = ||A_j|| = \max\{2^{\alpha_j}, 2^{\beta_j}\} = 2^{\alpha_j} \quad \text{and} \quad ||T_i^{-1}|| = ||A_j^{-1}|| = \max\{2^{-\alpha_j}, 2^{-\beta_j}\} = 2^{-\alpha_j}.$$

All these considerations imply that

$$\frac{1}{2} \cdot 2^{\alpha_j} = \frac{1}{2} \cdot ||T_i^{-1}||^{-1} \leq r_{Q_i} \leq R_{Q_i} \leq 4 \cdot ||T_i|| \leq 4 \cdot 2^{\alpha_j},$$

and hence $R_{Q_i}/r_{Q_i} \leq 8$ for all $i = (j, m, \ell) \in I_0^{(\alpha,\alpha)}$. Furthermore, since $Q_0 = B_4(0)$, we also see that $R_{Q_0}/r_{Q_0} = 1 \leq 8$. Put together, this shows that the $(\alpha, \alpha)$-wave packet covering $Q^{(\alpha,\alpha)}$ satisfies Condition (2).

To verify Condition (3), we recall that $1 + |\xi| \asymp 2^j$ for all $\xi \in Q_i$ and $i = (j, m, \ell) \in I_0^{(\alpha,\alpha)}$. Noting that $Q_i = T_iQ + b_i$ and $\alpha = \beta$, this shows that, for $i = (j, m, \ell) \in I_0^{(\alpha,\alpha)}$,

$$\lambda(Q_i) \asymp |\det T_i| = 2^{\alpha_j} \cdot 2^{\beta_j} = 2^{\alpha_j} \cdot (1 + |\xi|)^{2\alpha} = (1 + |\xi|)^{da} \quad \forall \xi \in Q_i.$$

Finally, $1 + |\xi| \asymp 1$ and hence $\lambda(Q_0) \asymp 1 \asymp (1 + |\xi|)^{da}$ for all $\xi \in Q_0 = B_4(0)$. Therefore, $Q^{(\alpha,\alpha)}$ satisfies Condition (3) and thus is an $\alpha$-covering of $\mathbb{R}^2$. \qed
7. Universality of the wave packet coverings

Even though we showed in Section 6.5 that the \( \alpha \)-modulation spaces arise as special cases of the wave packet smoothness spaces, it might still be objected that the construction of the coverings \( Q^{(\alpha,\beta)} \) involves a lot of arbitrariness, so that these coverings and the decomposition spaces associated with them are rather esoteric.

In the present section, we show that this is not the case. Generalising the concept of \( \alpha \)-coverings \([25, 3]\), we introduce the natural class of \((\alpha, \beta)\) coverings of \(\mathbb{R}^2\). We then prove that any two \((\alpha, \beta)\) coverings determine the same class of decomposition spaces. Finally, we show that the wave packet coverings \(Q^{(\alpha,\beta)}\) are indeed \((\alpha, \beta)\) coverings. In summary, this shows that the wave packet coverings \(Q^{(\alpha,\beta)}\) and the wave packet smoothness spaces \(W^{p,q}_{\alpha,\beta}(\alpha, \beta)\) are natural objects universal among, respectively, all coverings and function spaces with a similar frequency concentration.

We begin by introducing the class of \((\alpha, \beta)\) coverings, drawing on the intuition that an element \(Q_i\) of an \((\alpha, \beta)\)-covering \(Q = (Q_i)_{i \in I}\) should essentially be a set or a union of two sets symmetric with respect to the origin of the frequency plane of length \(\approx (1 + |\xi|)^{\alpha}\) in the radial direction and of width \(\approx (1 + |\xi|)^{\beta}\) in the angular direction where \(\xi \in Q_i\) is chosen arbitrarily.

In the rest of this section, we shall identify vectors \((x, y) \in \mathbb{R}^2\) with corresponding complex numbers \(x + iy\).

**Definition 7.1.** Let \(\alpha, \beta \in [0, 1]\). A family \(Q = (Q_i)_{i \in I}\) of open bounded subsets of \(\mathbb{R}^2\) is called an \((\alpha, \beta)\)-covering of \(\mathbb{R}^2\), if it satisfies the following conditions:

1. \(Q\) is an admissible covering of \(\mathbb{R}^2\);
2. \(\lambda(Q_i) \approx (1 + |\xi|)^{\alpha + \beta}\) for all \(i \in I\) and \(\xi \in Q_i\) where \(\lambda\) denotes the Lebesgue measure;
3. (for each \(i \in I\), there is an interval \(L_i \subset [0, \infty)\) such that
   a. \(\lambda(L_i) \leq (1 + |\xi|)^{\alpha}\) for all \(\xi \in Q_i\) and
   b. \(|\xi| \in L_i\) for all \(\xi \in Q_i\); and
4. (for each \(i \in I\), there is an angle \(\phi_i \in \mathbb{R}\) such that
   \[\forall \xi \in Q_i \quad \exists \phi \in \mathbb{R}: \min_{\omega \in \{0, \pi\}} |\phi - (\phi_i + \omega)| \lesssim (1 + |\xi|)^{\beta-1} \quad \text{and} \quad \xi = |\xi| \cdot e^{i\phi}. \] (7.1)

Here, all the constants must be independent of the choice of \(i \in I\) and \(\xi \in Q_i\).

Our first goal is to prove that any two \((\alpha, \beta)\) coverings are weakly equivalent. To do so, the following lemma will be helpful.

**Lemma 7.2.** Let \(\alpha, \beta \in [0, 1]\) and let \(Q = (Q_i)_{i \in I}\) be an \((\alpha, \beta)\) covering of \(\mathbb{R}^2\). Then \(1 + |\xi| \approx 1 + |\eta|\) for arbitrary \(i \in I\) and \(\xi, \eta \in Q_i\).

**Proof.** Let us first consider the case where \(\alpha = 0\). Then, the intervals \(L_i\) introduced in Condition (3) of Definition 7.1 satisfy \(\lambda(L_i) \leq C \cdot (1 + |\xi|)^{\alpha} = C\) for all \(i \in I\). Since, on the other hand, the Lebesgue measure of an interval is simply its length and since Definition 7.1 implies that \(|\xi|, |\eta| \in L_i\) if \(\xi, \eta \in Q_i\), we conclude that \(|\xi| - |\eta| | \leq C\). Therefore,

\[1 + |\xi| \leq 1 + |\eta| + |\xi| - |\eta| \leq 1 + C + |\eta| \leq (1 + C) \cdot (1 + |\eta|).\]

By symmetry, we also infer that \(1 + |\eta| \leq (1 + C) \cdot (1 + |\xi|),\) thereby proving the claim for the case \(\alpha = 0\).

If \(\alpha > 0\) then \(\alpha + \beta > 0\), and so \(1 + |\xi| \approx [\lambda(Q_i)]^{1/(\alpha+\beta)} \approx 1 + |\eta|\) for arbitrary \(\xi, \eta \in Q_i\), according to Condition (2) in Definition 7.1.

We can now prove that any two \((\alpha, \beta)\) coverings are weakly equivalent.

**Theorem 7.3.** Let \(\alpha, \beta \in [0, 1]\) and let \(Q = (Q_i)_{i \in I}\) and \(P = (P_j)_{j \in J}\) be two \((\alpha, \beta)\) coverings of \(\mathbb{R}^2\).

Then \(Q\) and \(P\) are weakly equivalent. Furthermore, if \(\beta \leq \alpha\), then

\[\lambda(Q_i - P_j) \lesssim (1 + |\xi|)^{\alpha+\beta} \times \min\{\lambda(Q_i), \lambda(P_j)\}\]

for all \(i \in I\) and \(j \in J\) such that \(\emptyset \neq Q_i \cap P_j \ni \xi\) where we write \(A - B = \{a - b : a \in A, b \in B\}\) for \(A, B \subset \mathbb{R}^2\).
Proof. Let us fix, for each \( j \in J \), some \( \zeta_j \in P_j \). By symmetry, it suffices to prove that \( P \) is weakly subordinate to \( Q \). Therefore, defining

\[
I_j := \{ i \in I : Q_i \cap P_j \neq \emptyset \} \quad \text{for } j \in J,
\]
we have to find such an \( N \in \mathbb{N} \) that \(|I_j| \leq N\) for all \( j \in J \).

**Step 1:** Our first goal is to show that there are \( C_1, C_2 > 0 \) such that, for each \( j \in J \), there is an interval \( \Lambda_j \subset [0, \infty) \) of length \( \lambda(\Lambda_j) \leq C_1 \cdot (1 + |\zeta_j|)\alpha \) and such that

\[
Q_i \subset \{ r \cdot e^{i\phi} : r \in \Lambda_j \ \text{and} \ \min_{\omega \in [0, \pi]} |\phi - (\phi_j + \omega)| \leq C_2 \cdot (1 + r)^{\beta - 1} \} =: \Omega_j \quad \forall i \in I_j
\]

(7.2)

where \( \phi_j \in \mathbb{R} \) is the angle associated with \( P_j \) according to Condition (4) in Definition 7.1.

To prove (7.2), we recall from Lemma 7.2 that there is such a \( C_3 \geq 1 \) that \( 1 + |\xi| \leq C_3 \cdot (1 + |\eta|) \) for all \( \xi, \eta \in Q_i \) and for all \( \xi, \eta \in P_j \). Furthermore, according to Condition (3) in Definition 7.1, there is such a \( C_4 > 0 \) that

\[
\lambda(L_i) \leq C_4 \cdot (1 + |\xi|)\alpha \quad \forall \xi \in Q_i \quad \text{and} \quad \lambda(L_j) \leq C_4 \cdot (1 + |\xi|)\alpha \quad \forall \xi \in P_j.
\]

Now, let \( j \in J \) and \( i \in I_j \) be arbitrary and let us fix some \( \xi_i \in Q_i \cap P_j \). Then, \( |\xi_i|, |\xi_j| \in I_j \) for arbitrary \( \xi \in Q_i \). Since the Lebesgue measure \( \lambda(L_i) \) is the length of the interval \( L_i \) and since \( \xi_i, \xi_j \in P_j \), this implies that

\[
|\xi - \xi_i| \leq \lambda(L_i) \leq C_4 \cdot (1 + |\xi_i|)\alpha \leq C_3^2C_4 \cdot (1 + |\zeta_j|)\alpha.
\]

Likewise, \( |\xi_j|, |\xi_i| \in L_j \) and hence \( |\xi_i - \xi_j| \leq \lambda(L_j) \leq C_4 \cdot (1 + |\xi_j|)\alpha \). Combining these estimates results in \(|\xi - \xi_j| \leq \lambda(L_i) \leq C_3^2C_4 + C_4 \cdot (1 + |\zeta_j|)\alpha \). Therefore, defining \( C_1 := 2C_4 \cdot (1 + C_3^2) \), we see that

\[
\Lambda_j := [0, \infty) \cap \left[ |\zeta_j| - \frac{C_1}{2} \cdot (1 + |\zeta_j|)\alpha, |\zeta_j| + \frac{C_1}{2} \cdot (1 + |\zeta_j|)\alpha \right]
\]

satisfies \( \lambda(\Lambda_j) \leq C_1 \cdot (1 + |\zeta_j|)\alpha \) and \( |\xi| \in \Lambda_j \ \forall \xi \in Q_i \) and \( j \in I_j \).

Having estimated the Euclidean norm of \( \xi \in Q_i \) for \( i \in I_j \), we now estimate the angle of \( \xi \). To do so, let us choose a \( C_5 > 0 \) larger than the constant in (7.1) for both coverings \( Q \) and \( P \). Furthermore, for \( \phi, \psi \in \mathbb{R} \), let us define

\[
d(\phi, \psi) := \min_{\omega \in [0, \pi]} |\phi - (\psi + \omega)| \in [0, \infty).
\]

It is not hard to see that the minimum is indeed attained and that

\[
d(\phi, \psi) = d(\psi, \phi) \quad \text{and} \quad d(\phi, \theta) \leq d(\phi, \psi) + d(\psi, \theta) \quad \forall \phi, \psi, \theta \in \mathbb{R}.
\]

Now, let \( j \in J \) and \( i \in I_j \) be arbitrary. Then \( Q_i \cap P_j \) is a non-empty open set, so that we can find a non-zero \( \xi_i \in Q_i \cap P_j \). According to (7.1), we can find \( \phi, \psi \in \mathbb{R} \) and \( \omega_1, \omega_2 \in [0, \pi] \) such that

\[
|\xi_i| \cdot e^{i\psi} = \xi_i = |\xi_i| \cdot e^{i\phi} \quad \text{and} \quad \max \{|\phi - (\phi_1 + \omega_1)|, |\psi - (\phi_j + \omega_2)|\} \leq C_5 \cdot (1 + |\xi_i|)\beta - 1.
\]

On the one hand, since \( |\xi_i| \neq 0 \), this entails \( e^{i\psi} = e^{i\phi} \) and hence \( \phi = \psi + 2\pi k \) for some \( k \in \mathbb{Z} \); therefore, \( d(\phi, \psi) = 0 \). On the other hand, the preceding estimate implies that \( d(\phi, \phi_1) \leq C_5 \cdot (1 + |\xi_i|)\beta - 1 \) and \( d(\psi, \phi_j) \leq C_5 \cdot (1 + |\xi_j|)\beta - 1 \).

Finally, from (7.1) we conclude that, for arbitrary \( \xi \in Q_i \), there are \( \theta \in \mathbb{R} \) and \( \omega \in [0, \pi] \) such that

\[
\xi = |\xi| \cdot e^{i\theta} \quad \text{and} \quad |\theta - (\phi_i + \omega)| \leq C_5 \cdot (1 + |\xi|)\beta - 1 \leq C_3^{1-\beta}C_5 \cdot (1 + |\xi|)\beta - 1.
\]

In particular, \( d(\theta, \phi_i) \leq C_3^{1-\beta}C_5 \cdot (1 + |\xi_i|)\beta - 1 \). By combining our observations, we see that

\[
d(\theta, \phi_j) \leq d(\theta, \phi_i) + d(\phi_i, \phi) + d(\phi, \psi) + d(\psi, \phi_j) \leq C_5 \cdot (C_3^{1-\beta} + 1 + 0 + 1) \cdot (1 + |\xi|)\beta - 1 \leq C_3^{1-\beta}C_5 \cdot (2 + C_3^{1-\beta}) \cdot (1 + |\xi|)\beta - 1 := C_2 \cdot (1 + |\xi|)\beta - 1.
\]

(7.4)
Here, in the penultimate step, we noted that $\xi, \xi_i \in Q_i$ and hence $1 + |\xi| \leq C_3 \cdot (1 + |\xi_i|)$. The estimate (7.4) shows that there is some $k \in \mathbb{Z}$ such that $|\tau - k\pi| - \delta_j \leq C_2 \cdot (1 + |\xi|)^{\beta - 1}$. Writing $k = 2 \ell + m$ with $m \in \{0, 1\}$ and $\ell \in \mathbb{Z}$, we thus see that $|\tau - 2\ell\pi| - (\delta_j + m\pi)| \leq C_2 \cdot (1 + |\xi|)^{\beta - 1}$.

All in all, defining $r := |\xi|$ and $\varphi := \tau - 2\ell\pi$, we have shown that $\xi = r \cdot e^{i\varphi} = r \cdot e^{i\tau}$ where $\min_{\omega \in \{0, \pi\}} |\varphi - (\delta_j + \omega)| \leq C_2 \cdot (1 + r)^{\beta - 1}$ and $r \in \Lambda_j$, according to (7.3). Therefore, (7.2) holds.

**Step 2:** We now estimate the measure of the set $\Omega_j$ on the right-hand side of Equation (7.2). To do so, let $W_j := \{r \cdot e^{i\varphi} : r \in \Lambda_j \text{ and } |\varphi| \leq C_2 \cdot (1 + r)^{\beta - 1}\}$.

Then, since $-1 = e^{-i\pi}$, we see that

$$-\{r \cdot e^{i\varphi} : r \in \Lambda_j \text{ and } |\varphi - \pi| \leq C_2 \cdot (1 + r)^{\beta - 1}\} = \{r \cdot e^{i(\varphi - \pi)} : r \in \Lambda_j \text{ and } |\varphi - \pi| \leq C_2 \cdot (1 + r)^{\beta - 1}\} = W_j.$$  

Therefore,

$$e^{-i\varphi} \Omega_j = \{r \cdot e^{i(\varphi - \pi)} : r \in \Lambda_j \text{ and } \min_{\omega \in \{0, \pi\}} |\varphi - (\delta_j + \omega)| \leq C_2 \cdot (1 + r)^{\beta - 1}\}$$

$$= \bigcup_{\omega \in \{0, \pi\}} \{r \cdot e^{i\psi} : r \in \Lambda_j \text{ and } |\psi - \omega| \leq C_2 \cdot (1 + r)^{\beta - 1}\} = W_j \cup (-W_j) \quad (7.5)$$

And hence

$$\lambda(\Omega_j) = \lambda(e^{-i\varphi} \Omega_j) = \lambda(W_j \cup (-W_j)) \leq 2\lambda(W_j). \quad (7.6)$$

Now, let us define

$$a_j := \max\left\{0, |\zeta_j| - \frac{C_1}{2} (1 + |\zeta_j|)^{\alpha}\right\} \quad \text{and} \quad b_j := |\zeta_j| + \frac{C_1}{2} (1 + |\zeta_j|)^{\alpha}, \quad (7.7)$$

so that $\Lambda_j = [a_j, b_j]$ according to (7.3). Introducing polar coordinates allows us to write

$$\lambda(W_j) = \int_{(a_j, b_j)} r \cdot \int_0^{2\pi} I_{W_j}(r \cdot e^{i\varphi}) \, d\varphi \, dr,$$

where $I_{W_j}$ denotes the indicator function of the set $W_j$. On the other hand, if $I_{W_j}(r \cdot e^{i\varphi}) = 1$ for some $r \in (a_j, b_j) \subset (0, \infty)$ and $\varphi \in [-\pi, \pi]$, then $r \cdot e^{i\varphi} = s \cdot e^{i\psi}$ for some $s \in \Lambda_j$ and $\psi \in \mathbb{R}$ satisfies $|\psi| \leq C_2 \cdot (1 + s)^{\beta - 1}$. Since $r \cdot e^{i\varphi} = s \cdot e^{i\psi}$ where $r > 0$, we conclude that $s = r$ and $\varphi - \psi \in 2\pi \mathbb{Z}$. We claim that this implies $|\varphi| \leq C_2 \cdot (1 + r)^{\beta - 1}$. As $C_2 \cdot (1 + r)^{\beta - 1} \geq \pi$ this is trivial. Therefore we shall assume that $C_2 \cdot (1 + r)^{\beta - 1} < \pi$. This implies that $|\psi| < \pi$ and hence $|\varphi - \psi| \leq |\varphi| + |\psi| < 2\pi$, since $\varphi \in [-\pi, \pi]$. Since $\varphi - \psi \in 2\pi \mathbb{Z}$, this entails $\varphi = \psi$ and thus $|\varphi| = |\psi| \leq C_2 \cdot (1 + r)^{\beta - 1}$ also in this case. In combination with the estimate $1 + b_j = 1 + |\zeta_j| + \frac{C_1}{2} (1 + |\zeta_j|)^{\alpha} \leq (1 + C_1)(1 + |\zeta_j|)$, these considerations show that

$$\lambda(W_j) \leq \int_{(a_j, b_j)} r \cdot 2C_2 \cdot (1 + r)^{\beta - 1} \, dr \leq 2C_2 \int_{(a_j, b_j)} (1 + r)^{\beta} \, dr$$

$$\leq 2C_2 \cdot (1 + b_j)^{\beta} \cdot (b_j - a_j) \leq 2C_2 \cdot (1 + C_1)^{\beta} \cdot (1 + |\zeta_j|)^{\beta} \cdot C_1 \cdot (1 + |\zeta_j|)^{\alpha} \quad (7.8)$$

**Step 3:** We now show that there is such an $N > 0$ that $|I_j| \leq N$ for all $j \in J$, or, in other words, $\mathcal{P}$ is weakly subordinate to $\mathcal{Q}$. Since $\mathcal{Q}$ is an admissible covering, there is such an $N_0 \in \mathbb{N}$ that $\sum_{i \in I} I_{Q_i} \leq N_0$. Combined with (7.2), this implies $\sum_{i \in I_j} I_{Q_i} \leq N_0 \cdot I_{\Omega_j}$. Now, from Condition (2) in Definition 7.1 we infer that there is such an $C_6 > 0$ that

$$\lambda(Q_i) \geq C_6 \cdot (1 + |\xi_i|)^{\alpha + \beta} \geq C_6 C_3^{-(\alpha + \beta)} \cdot (1 + |\zeta_j|)^{\alpha + \beta} \quad \forall i \in I_j$$

where $\xi_i \in Q_i \cap P_j$ can be chosen arbitrarily.

Therefore, if $\Gamma \subset I_j$ is an arbitrary finite subset, then

$$C_6 C_3^{-(\alpha + \beta)} \cdot (1 + |\zeta_j|)^{\alpha + \beta} \cdot |\Gamma| \leq \sum_{i \in \Gamma} \lambda(Q_i) = \int_{\mathbb{R}^2} \sum_{i \in \Gamma} I_{Q_i}(\xi) \, d\xi \leq \int_{\mathbb{R}^2} N_0 \cdot I_{\Omega_j}(\xi) \, d\xi$$

(Eqs. (7.6) and (7.8)) $\leq 4C_1 C_2 (1 + C_1) N_0 \cdot (1 + |\zeta_j|)^{\alpha + \beta}$

34
and hence $|\Gamma| \leq 4C_1C_2(1 + C_1)C_3^{\alpha + \beta}N_0/C_6 =: N$. Since $\Gamma \subset I_j$ was an arbitrary finite subset and the right-hand side of the last estimate is independent of $j \in J$, this completes the proof of the statement of this step.

**Step 4:** We assume that $\beta \leq \alpha$ and estimate $\lambda(\overline{Q_i} - \overline{P_j})$. To do so, we first estimate the set $W_j$ introduced in Step 2. Let us recall the definition (7.7) of $a_j$ and $b_j$, where we saw that $\Lambda_j = [a_j, b_j]$. Any $\xi \in W_j$ is given by $\xi = r \cdot e^{i\phi}$ for $r \in [a_j, b_j]$ and $|\phi| \leq C_2 \cdot (1 + r)^{\beta - 1}$.

Now, we note that $1 + r \leq 1 + b_j \leq (1 + C_1) \cdot (1 + |\xi_j|)$ and so

$$r \cdot (1 + r)^{\beta - 1} \leq (1 + r)^\beta \leq (1 + C_1)^\beta \cdot (1 + |\xi_j|) \leq (1 + C_1) \cdot (1 + |\xi_j|)^\alpha,$$

where our assumption $\beta \leq \alpha$ was used in the last step. On the other hand, $\cos \phi \geq 1 - |\phi|$, since the cosine is 1-Lipschitz with $\cos(0) = 1$. Therefore, defining $C_7 := (1 + C_1)(1 + C_2)$ and recalling the definition of $a_j$, we see that

$$\xi_1 = r \cdot \cos \phi \geq r \cdot (1 - |\phi|) \geq r - C_2 \cdot r \cdot (1 + r)^{\beta - 1} \geq a_j - (1 + C_1)C_2 \cdot (1 + |\xi_j|)^\alpha \geq |\xi_j| - C_7 (1 + |\xi_j|)^\alpha.$$

Conversely,

$$\xi_1 \leq r \leq b_j \leq |\xi_j| + C_7 \cdot (1 + |\xi_j|)^\alpha.$$

Finally, noting that $|\sin \phi| \leq |\phi| \leq C_2 \cdot (1 + r)^{\beta - 1}$ and $1 + r \leq 1 + b_j \leq (1 + C_1) (1 + |\xi_j|)$, we conclude that

$$|\xi_2| \leq r \cdot C_2 \cdot (1 + r)^{\beta - 1} \leq C_2 \cdot (1 + r)^\beta \leq (1 + C_1)^\beta C_2 \cdot (1 + |\xi_j|)^\beta.$$

Defining $C_8 := (1 + C_1)^\beta C_2$ and $\gamma_j := (|\xi_j|, 0) \in \mathbb{R}^2$, we thus see that

$$\overline{W_j} \subset \gamma_j + R_j \quad \text{where} \quad R_j = \left( C_7 \cdot (1 + |\xi_j|)^\alpha \cdot [-1, 1] \right) \times \left( C_8 \cdot (1 + |\xi_j|)^\beta \cdot [-1, 1] \right). \quad (7.9)$$

Using the symmetry of the rectangle $R_j$, we see that $\varepsilon_1 \overline{W_j} + \varepsilon_2 \overline{W_j} \subset \varepsilon_1 \gamma_j + \varepsilon_2 \gamma_j + (R_j + R_j)$ for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, from where we deduce that $\lambda(R_j + R_j) \leq C_9 \cdot (1 + |\xi_j|)^{\alpha + \beta}$. Combining this with Equation (7.5) results in

$$\lambda(\overline{Q_j} - \overline{P_j}) = \lambda\left( e^{-i\phi_j} \overline{Q_j} - e^{-i\phi_j} \overline{P_j} \right) \leq \lambda\left( |\overline{W_j} \cup (-W_j)| - |W_j \cup (-W_j)| \right) \leq \sum_{\varepsilon_1, \varepsilon_2 \in \{\pm 1\}} \lambda(\varepsilon_1 \overline{W_j} + \varepsilon_2 \overline{W_j}) \leq 4C_9 \cdot (1 + |\xi_j|)^{\alpha + \beta}.$$

Finally, since $Q$ covers the whole $\mathbb{R}^2$, we see that $P_j \subset \bigcup_{i \in I_j} Q_i$, and hence $P_j \subset \Omega_j$, according to (7.2). Therefore, $\lambda(\overline{Q_i} - P_j) \leq \lambda(\overline{Q_j} - \overline{P_j}) \leq 4C_9 \cdot (1 + |\xi_j|)^{\alpha + \beta}$ for all $j \in J$ and $i \in I_j$. Since $\lambda(\Omega_j) \simeq (1 + |\xi_j|)^{\alpha + \beta} \leq \lambda(P_j)$ for all $\xi \in Q_i \cap P_j$, according to Condition (2) in Definition 7.1 and since $1 + |\xi_j| \simeq 1 + |\xi_j|$ for such $\xi$, the proof is complete. \hfill \Box

We now show that the wave packet covering $Q^{(\alpha, \beta)}$ fits the framework of $(\alpha, \beta)$ coverings.

**Lemma 7.4.** For any $0 \leq \beta \leq \alpha \leq 1$, the wave packet covering $Q^{(\alpha, \beta)}$ is an $(\alpha, \beta)$ covering of $\mathbb{R}^2$.

**Remark.** This shows in particular that almost structured $(\alpha, \beta)$ coverings exist for $0 \leq \beta \leq \alpha \leq 1$.

By considering coverings that consist of sectors of rings, one can show that $(\alpha, \beta)$ coverings also exist for $\beta > \alpha$. It seems to be an open questions, however, whether such coverings can be chosen to be almost structured. To the best of our knowledge, no coverings that would satisfy this condition have been reported.

**Proof.** As shown by Lemmas 3.2 and 4.2, $Q^{(\alpha, \beta)} = (Q_i)_{i \in I^{(\alpha, \beta)}}$ is an admissible covering of $\mathbb{R}^2$. Thus, it remains to verify conditions (2)-(4) of Definition 7.1. First, we shall do so only for $i = (j, m, \ell) \in I_0^{(\alpha, \beta)}$; the case $i = 0$ will be considered afterwards.

Recall from Equation (4.1) that $|\xi_j| \simeq 1 + |\xi_j| \simeq 2^j$ for all $\xi \in Q_i = T_i Q + b_i$. Therefore, we see that Condition (2) in Definition 7.1 is satisfied; indeed, we see that

$$\lambda(Q_i) \simeq |\det T_i| = |\det A_j| = 2^{a_j} 2^{\beta_j} \simeq (1 + |\xi_j|)^{\alpha + \beta} \quad \forall i = (j, m, \ell) \in I_0^{(\alpha, \beta)} \text{ and } \xi \in Q_i.$$
Now, let us define
\[ L_i := [0, \infty) \cap \left[ 2^{i-1} + 2^{i+1}(m - \varepsilon), 2^{i-1} + 2^{i+1}(m + 2 + \varepsilon) \right] \quad \text{for} \quad (j, m, \ell) \in \mathcal{I}_0^{(\alpha, \beta)}, \]

note that \( \lambda(L_i) \leq (2 + 3\varepsilon) \cdot 2^{i+1} \leq (1 + |\xi|)^\alpha \) for all \( \xi \in Q_i \) and recall from \( (4.1) \) that \( |\xi| \in L_i \) for all \( \xi \in Q_i \). Hence, Condition (3) in Definition 7.1 is satisfied.

Finally, let \( \phi_i := \Theta_{j, \ell} \) with \( \Theta_{j, \ell} \) as in \( (3.5) \). With this, Equation \( (4.2) \) shows that — for an arbitrary \( \xi \in Q_i \) — there is \( \phi \in \mathbb{R} \) such that \( \xi = |\xi| \cdot e^{i\phi} \) and
\[
\min_{\omega \in \{0, \pi\}} |\phi - (\phi_j + \omega)| \leq |\phi - \Theta_{j, \ell}| \leq 5 \cdot 2^{(\beta - 1)j} \leq (1 + |\xi|)^{\beta - 1},
\]
as required in \( (7.1) \).

Since \( Q_0 = B_4(0) \), we have \( |\xi| \in L_0 := [0, 4] \) and furthermore \( 1 + |\xi| \approx 1 \), \( \lambda(Q_0) \approx 1 \approx (1 + |\xi|)^{\alpha + \beta} \), and \( \lambda(L_0) \leq 1 \leq (1 + |\xi|)^\alpha \) for all \( \xi \in Q_0 \). Thus, it is not hard to see that Conditions (2)-(4) of Definition 7.1 remain valid for \( i = 0 \) as well, after adjusting the implicit constants.

As our final result in this section, we shall now show that any two \((\alpha, \beta)\) coverings induce the same family of decomposition spaces. To make this result as general as possible, we first introduce slightly broader classes of coverings than the almost-structured coverings that we introduced in Section 2.1.

**Definition 7.5.** (Definition 2.5 in \[61\]; inspired by \[4\])
A family \( Q = (Q_i)_{i \in I} \) is called a semi-structured covering of \( \mathbb{R}^d \), if there is an associated family \((T_i \bullet + b_i)_{i \in I}\) of invertible affine-linear maps such that the following properties hold:

1. \( Q \) covers \( \mathbb{R}^d \), that is, \( \mathbb{R}^d = \bigcup_{i \in I} Q_i \);
2. \( Q \) is admissible, that is, the sets \( i^* \) introduced in \( (2.3) \) have uniformly bounded cardinality;
3. there is a family \((Q'_i)_{i \in I}\) of non-empty open sets \( Q'_i \subset \mathbb{R}^d \) such that \( Q_i = T_iQ'_i + b_i \) for all \( i \in I \), and such that the \( Q'_i \) are uniformly bounded, that is, \( \sup_{i \in I} \sup_{\xi \in Q'_i} |\xi| < \infty \); and
4. there is such a \( C > 0 \) that \( \|T_i^{-1}T_j\| \leq C \) for all such \( i, \ell \in I \) that \( Q_i \cap Q_\ell \neq \emptyset \).

We also need to impose less restrictive conditions on the partitions of unity than those imposed on regular partitions of unity.

**Definition 7.6.** (Definitions 3.5 and 3.6 in \[61\]; inspired by Definition 2.2 in \[15\] and by Definition 2 in \[4\])
Let \( Q = (Q_i)_{i \in I} \) be an admissible covering of \( \mathbb{R}^d \). A family of functions \( \Phi = (\varphi_i)_{i \in I} \) is called an \( L^p\)-bounded admissible partition of unity \((L^p\text{-BAPU})\) subordinate to \( Q \) for all \( 1 \leq p \leq \infty \), if

1. \( \varphi_i \in C_c^\infty(\mathbb{R}^d) \) with \( \varphi_i(\xi) = 0 \) for all \( \xi \in \mathbb{R}^d \setminus Q_i \) and any \( i \in I \);
2. \( \sum_{i \in I} \varphi_i \equiv 1 \) on \( \mathbb{R}^d \); and
3. \( \sup_{i \in I} \|F^{-1}\varphi_i \|_{L^1} < \infty. \)

If there is such an \( L^p\text{-BAPU} \), the covering \( Q \) is called an \( L^p\text{-decomposition covering} \) of \( \mathbb{R}^d \), for all \( 1 \leq p \leq \infty \).

Now, let \( p \in (0, 1) \) and let us assume that \( Q \) is semi-structured with associated family \((T_i \bullet + b_i)_{i \in I}\). A family \( \Phi = (\varphi_i)_{i \in I} \) is called an \( L^p\text{-BAPU} \) subordinate to \( Q \), if it is an \( L^p\text{-BAPU} \) for all \( 1 \leq q \leq \infty \), and \( \sup_{i \in I} |\det T_i|^{p^* - 1} \cdot \|F^{-1}\varphi_i \|_{L^p} < \infty. \) If there is such an \( L^p\text{-BAPU} \), we shall say that \( Q \) is an \( L^p\text{-decomposition covering} \) of \( \mathbb{R}^d \).

Replacing the regular partition of unity by an \( L^p\text{-BAPU} \), one can define decomposition spaces associated with \( L^p\text{-decomposition coverings}, \) proceeding exactly as in Definition 2.6.

Now, we state the main result of this section, whose proof is slightly deferred.

**Theorem 7.7.** Let \( p, q \in (0, \infty], \) \( \alpha, \beta \in [0, 1], \) \( Q = (Q_i)_{i \in I} \) and \( P = (P_j)_{j \in J} \) be two \((\alpha, \beta)\) coverings of \( \mathbb{R}^2 \), and \( w = (w_i)_{i \in I} \) and \( v = (v_j)_{j \in J} \) be \( Q\)-moderate and \( P\)-moderate, respectively. Let us assume that
\[ w_i \preceq v_j \quad \text{for any } i \in I \text{ and } j \in J \text{ satisfying } Q_i \cap P_j \neq \emptyset. \]
Then the following holds:
(1) If $p \in [1, \infty]$ and if $Q, P$ are $L^p$-decomposition coverings, then
\[ D(Q, L^p, \ell_w^q) = D(P, L^p, \ell_v^q) \] with equivalent quasi-norms.

(2) If $p \in (0, 1)$, if $\beta \leq \alpha$, and if $Q, P$ are semi-structured $L^p$-decomposition coverings, then
\[ D(Q, L^p, \ell_w^q) = D(P, L^p, \ell_v^q) \] with equivalent quasi-norms.

The wave packet covering $Q^{(\alpha, \beta)} = (Q_i)_{i \in I(\alpha, \beta)}$ is an $(\alpha, \beta)$ covering and $Q^{(\alpha, \beta)}$ is almost-structured, so that it is a semi-structured $L^p$-decomposition covering for all $p \in (0, \infty]$ and therefore satisfies the assumptions of the preceding theorem. Moreover, Equation (6.1) and Lemma 7.2 imply that $L_w$ so that it is a semi-structured smoothness spaces are invariant under dilation with arbitrary invertible matrices.

8. Dilation invariance of the wave packet smoothness spaces

In this section, we shall use the universality of the wave packet coverings to show that the wave packet smoothness spaces are invariant under dilation with arbitrary invertible matrices $B \in \text{GL}(\mathbb{R}^2)$.

To do so, we shall first show that it suffices to establish an embedding between certain decomposition spaces to derive the dilation invariance. To show this, let us fix $B \in \text{GL}(\mathbb{R}^2)$ and define the dilation $f \circ B$ of an element $f$ of the reservoir $Z'$ as usual\footnote{This definition is natural, since $\langle f \circ B, g \rangle = |\det B|^{-1} \langle f, g \circ B^{-1} \rangle$ if $f : \mathbb{R}^2 \to \mathbb{C}$ is of moderate growth and $g \in \mathcal{S}(\mathbb{R}^2)$}.

\[
f \circ B : Z \to \mathbb{C}, \varphi \mapsto |\det B|^{-1} : \langle f, \varphi \circ B^{-1} \rangle_{Z'}.
\] (8.1)
We now show that this indeed defines a well-defined element \( f \circ B \in Z' \) and, at the same time, compute the Fourier transform \( \mathcal{F}[f \circ B] \). Namely, \( f \circ B \in Z' \) if and only if \( \mathcal{F}[f \circ B] = (f \circ B) \circ \mathcal{F} \in \mathcal{D}'(\mathbb{R}^2) \). This is indeed the case, since for \( \psi \in C_c^\infty(\mathbb{R}^2) \)

\[
(\mathcal{F}[f \circ B], \psi)_{\mathcal{D}'} = (f \circ B, \hat{\psi})_{Z'} = |\det B|^{-1} (\hat{f}, \hat{\psi} \circ B^{-1})_{Z'} = |\det B|^{-1} (\hat{f}, \mathcal{F}^{-1}[(\hat{\psi} \circ B^{-1})]_{\mathcal{D}'}) = (\hat{f}, \psi \circ B^t)_{\mathcal{D}'},
\]

from where it is easy to see that the map \( \psi \mapsto (\hat{f}, \psi \circ B^t)_{\mathcal{D}'} \) is a well-defined distribution on \( \mathbb{R}^2 \).

Now, let us fix \( 0 \leq \beta \leq \alpha \leq 1 \) and a regular partition of unity \((\varphi_i)_{i \in I}\) subordinate to the \((\alpha, \beta)\) wave packet covering \(Q^{(\alpha, \beta)} = (Q_i)_{i \in I}\). By using the computation of \(\mathcal{F}[f \circ B]\) and recalling how the Fourier transform is computed for compactly supported distributions (see Theorem 7.23 in [54]), we conclude that

\[
\mathcal{F}^{-1}((\varphi_i \cdot \mathcal{F}[f \circ B])(x)) = (\mathcal{F}[f \circ B], e^{2\pi i(x, \cdot)} \cdot \varphi_i)_{\mathcal{D}'} = (\hat{f}, e^{2\pi i(x, B^t \cdot \cdot)} \cdot (\varphi_i \circ B^t))_{\mathcal{D}'}
\]

for all \( i \in I \) and \( x \in \mathbb{R}^2 \). Therefore, for all \( i \in I \),

\[
\|\mathcal{F}^{-1}((\varphi_i \cdot \mathcal{F}[f \circ B])\|_{L^p} = \|\mathcal{F}^{-1}((\varphi_i \circ B^t) \cdot \hat{f}) \circ B\|_{L^p} = |\det B|^{-1/p} \cdot \|\mathcal{F}^{-1}((\varphi_i \circ B^t) \cdot \hat{f})\|_{L^p}. \tag{8.2}
\]

It is straightforward to verify that the family \(B^{-t}Q^{(\alpha, \beta)} := (B^{-t}Q_i)_{i \in I}\) is an almost structured admissible covering of \( \mathbb{R}^2 \) with associated family \((B^{-t}T_i + B^{-t}b_i)_{i \in I}\) where \( T_i \) and \( b_i \) are as defined in Lemma 5.1. Likewise, it follows directly from the definitions that \((\varphi_i \circ B^t)_{i \in I}\) is a regular partition of unity subordinate to \(B^{-t}Q^{(\alpha, \beta)}\). Furthermore, \(B^{-t}Q_i \cap B^{-t}Q_j \neq \emptyset\) if and only if \(Q_i \cap Q_j \neq \emptyset\), so that the weight \(w^s\) introduced in Lemma 6.1 is \(B^{-t}Q^{(\alpha, \beta)}\)-moderate. Thus, the decomposition spaces \(\mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s})\) are well-defined and Equation (8.2) implies that, for \( f \in Z' \),

\[
\|f \circ B\|_{\mathcal{W}^p_{\ell^q}(\alpha, \beta)} \lessapprox \left(\left(\|\mathcal{F}^{-1}(\varphi_i \circ B^t)\|_{L^p}\right)_{i \in I}\right)_{\ell^q_{w^s}} \leq \left(\left(\|\mathcal{F}^{-1}((\varphi_i \circ B^t) \cdot \hat{f})\|_{L^p}\right)_{i \in I}\right)_{\ell^q_{w^s}} \leq \|f\|_{\mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s})}.
\]

If we knew that \(\mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s}) = \mathcal{W}^p_{\ell^q}(\alpha, \beta)\) with equivalent quasi-norms, then the preceding equation would show that every \( f \in \mathcal{W}^p_{\ell^q}(\alpha, \beta) = \mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s}) \) satisfies \( f \circ B \in \mathcal{W}^p_{\ell^q}(\alpha, \beta) \) and \(\|f \circ B\|_{\mathcal{W}^p_{\ell^q}(\alpha, \beta)} \lessapprox \|f\|_{\mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s})} \lessapprox \|f\|_{\mathcal{W}^p_{\ell^q}(\alpha, \beta)}\). Thus, if we establish the identity \(\mathcal{D}(B^{-t}Q^{(\alpha, \beta)}, \ell^q_{w^s}) = \mathcal{W}^p_{\ell^q}(\alpha, \beta)\), we shall prove the following theorem concerning the dilation invariance of the wave packet spaces \(\mathcal{W}^p_{\ell^q}(\alpha, \beta)\):

**Theorem 8.1.** Let \( 0 \leq \beta \leq \alpha \leq 1, p, q \in (0, \infty], s \in \mathbb{R}, \) and \(B \in \text{GL}(\mathbb{R}^2)\). With \( f \circ B \) as defined in Equation (8.1), the linear map

\[\mathcal{W}^p_{\ell^q}(\alpha, \beta) \rightarrow \mathcal{W}^p_{\ell^q}(\alpha, \beta), f \mapsto f \circ B\]

is well-defined and bounded.

**Remark.** Roughly speaking, the lemma shows that if \( x = \pm r e^{i\varphi} \) with \( \varphi \approx \varphi_0 \), then also \( Bx = \pm s e^{i\varphi'} \) with \( \varphi' \approx \psi_0 \). Here, the angle \( \psi_0 \) is the one satisfying \(B(e^{i\psi_0}) = r_B e^{i\psi_0}\) for some \( r_B > 0 \).
Proof. Step 1: In this step, we prove that
\[
\frac{2}{\pi} \cdot \min_{\ell \in \mathbb{Z}} |x - \pi \ell| \leq |\sin x| \leq \min_{\ell \in \mathbb{Z}} |x - \pi \ell| \quad \forall x \in \mathbb{R}.
\]
To prove the upper bound, simply note that \(x \mapsto |\sin x|\) is a 1-Lipschitz and \(\pi\)-periodic function and that \(\sin(0) = 0\), whence \(|\sin x| = |\sin(x - \pi \ell) - \sin(0)| \leq |x - \pi \ell|\) for all \(\ell \in \mathbb{Z}\).

To prove the lower bound, note that the functions \(x \mapsto |\sin x|\) and \(x \mapsto \min_{\ell \in \mathbb{Z}} |x - \pi \ell|\) are both \(\pi\)-periodic, so that it is enough to prove the claim for \(x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\). On this interval, \(|\sin x| = |\sin(x)|\) and \(\min_{\ell \in \mathbb{Z}} |x - \pi \ell| = |x|\), so that it is enough to show that \(\frac{2}{\pi} x \leq |\sin x|\) for \(x \in [0, \frac{\pi}{2}]\). To see this, note that the sine is concave on \([0, \frac{\pi}{2}]\), since \(\sin'' = -\sin \leq 0\) on this interval. Now, if \(x \in [0, \frac{\pi}{2}]\), then \(t := \frac{2}{\pi} x \in [0, 1]\) and \(x = (1-t) \cdot 0 + t \cdot \frac{\pi}{2}\), so that \(|\sin(x)| \geq (1-t) \sin(0) + t \sin(\pi/2) = t = \frac{2}{\pi} x\), completing the proof of Equation (8.3).

Step 2: In this step, we show that if \(r, s, r_B, s_B > 0\) and \(\varphi, \psi, \varphi_B, \psi_B \in \mathbb{R}\) are arbitrary with \(B(re^{i\varphi}) = r_B e^{i\varphi_B}\) and \(B(se^{i\psi}) = s_B e^{i\psi_B}\), then
\[
|\sin(\psi_B - \varphi_B)| \leq \|B^{-1}\| \cdot |\det B| \cdot |\sin(\psi - \varphi)|.
\]
To see this, let \(\rho, \sigma > 0\) and \(\theta, \omega \in \mathbb{R}\) be arbitrary and define
\[
\Delta_{\theta, \omega}^r := A_{\rho, \sigma}^r(\Delta) \quad \text{where} \quad A_{\rho, \sigma}^r := \begin{pmatrix} \rho \cos \theta & \sigma \cos \omega \\ \rho \sin \theta & \sigma \sin \omega \end{pmatrix} \in \mathbb{R}^{2 \times 2}
\]
and \(\Delta := \{ (\mu, \nu) \in [0, 1]^2 : \mu + \nu \leq 1 \}\). Since the Lebesgue measure of \(\Delta\) is given by \(\lambda(\Delta) = \frac{1}{2}\), we see that
\[
\lambda(\Delta_{\theta, \omega}^r) = \frac{1}{2} \cdot |\det A_{\rho, \sigma}| \cdot |\cos \theta \sin \omega - \sin \theta \cos \omega| = \frac{\rho \sigma}{2} \cdot |\sin(\omega - \theta)|.
\]
Now, note that
\[
A_{r_B, s_B}^{\varphi_B, \psi_B} = (r_B e^{i\varphi_B}) s_B e^{i\psi_B} = B(r e^{i\varphi}) B(s e^{i\psi}) = B A_{r, s}^{\varphi, \psi}
\]
and hence \(\lambda(\Delta_{r_B, s_B}^{\varphi_B, \psi_B}) = \lambda(B(\Delta_{r, s}^{\varphi, \psi})) = |\det B| \cdot \lambda(\Delta_{r, s}^{\varphi, \psi}).\) Therefore, by applying Equation (8.5) twice, we conclude that
\[
\frac{r_B \cdot s_B}{2} \cdot |\sin(\psi_B - \varphi_B)| = \lambda(\Delta_{r_B, s_B}^{\varphi_B, \psi_B}) = |\det B| \cdot \lambda(\Delta_{r, s}^{\varphi, \psi}) = |\det B| \cdot \|B^{-1}\| \cdot |\sin(\psi - \varphi)|.
\]
Now, note that \(r = |B^{-1}| \cdot |x_B| = \|B^{-1}\| \cdot r_B\) since of \(x_B := B(r e^{i\varphi}) = r_B e^{i\varphi_B}\) and similarly \(s \leq \|B^{-1}\| \cdot s_B\). Introducing these estimates into Equation (8.6), resuls in (8.4).

Step 3: In this step, we complete the proof. With \(\varphi_0\) as in the statement of the lemma, let \(x_0 := e^{i\varphi_0}\) and write \(Bx_0 = r_0 e^{i\psi_0}\) for suitable \(r_0 > 0\) and \(\psi_0 \in \mathbb{R}\).

Let \(r \geq 0\) and \(\varphi \in \mathbb{R}\) be arbitrary. If \(r = 0\), we can simply choose \(\varphi' = \psi_0\), so that the claimed estimate trivially holds. Thus, we shall assume that \(r > 0\), define \(x := r e^{i\varphi}\) and write \(Bx = r_B e^{i\varphi_B}\) for suitable \(r_B > 0\) and \(\varphi_B \in \mathbb{R}\). Let us choose \(k \in \mathbb{Z}\) such that \(|\varphi_B - (\psi_0 + \pi k)| = \min_{\ell \in \mathbb{Z}} |\varphi_B - (\psi_0 + \pi \ell)|\). Finally, we write \(k = 2n + \kappa\) with \(n \in \mathbb{Z}\) and \(\kappa \in \{0, 1\}\) and define \(\varphi' := \varphi_B - 2\pi n\). Then we have \(B(r e^{i\varphi}) = r_B e^{i\varphi_B} = r B e^{i\varphi} = r B e^{i\varphi_B}\) and
\[
\min_{\omega \in \{0, \pi\}} |\psi' - (\psi_0 + \omega)| \leq |(\varphi_B - 2\pi n) - (\psi_0 + \pi \kappa)| = |\varphi_B - (\psi_0 + \pi \kappa)| = \min_{\ell \in \mathbb{Z}} |\varphi_B - (\psi_0 + \pi \ell)|
\]
(Step 1 and then Step 2) \(
\leq \frac{\pi}{2} \cdot |\sin(\psi_B - \psi_0)| \leq \frac{\pi}{2} \cdot \|B^{-1}\| \cdot |\det B| \cdot |\sin(\varphi - \varphi_0)|
\]
(Step 1) \(
\leq \frac{\pi}{2} \cdot \|B^{-1}\| \cdot |\det B| \cdot \min_{\theta \in [0, \pi]} |\varphi - (\varphi_0 + \theta)|.
\)

Using Lemma 8.2 we can now show that \(B^{-1}Q^{(\alpha, \beta)}\) is an \((\alpha, \beta)\)-covering. As explained above, this will also prove Theorem 8.1.

Lemma 8.3. Let \(0 \leq \beta \leq \alpha \leq 1\) and \(B \in \text{GL}(\mathbb{R}^2)\). Then \(B^{-1}Q^{(\alpha, \beta)}\) is an \((\alpha, \beta)\)-covering of \(\mathbb{R}^2\).

Proof. It is straightforward to verify that \(B^{-1}Q^{(\alpha, \beta)} = (B^{-1}Q_t)_{t \in \mathbb{I}}\) is an almost structured admissible covering of \(\mathbb{R}^2\) with associated family \((B^{-1}T_i \cdot + B^{-1}b_i)_{i \in \mathbb{I}}\). We now verify the remaining three properties in Definition 7.1.
(2) Since \( Q^{(\alpha,\beta)} \) is an \((\alpha, \beta)\)-covering (see Lemma 7.4), \( \lambda(Q_i) \asymp (1+|\eta|)^{\alpha+\beta} \) for all \( i \in I \) and \( \eta \in Q_i \). Since \( B \) is invertible and since \( \eta = B^t \xi \in Q_i \) for \( \xi \in B^{-t}Q_i \), we conclude that, for all \( i \in I \) and \( \xi \in B^{-t}Q_i \), \( \lambda(B^{-t}Q_i) = |\det B|^{-1} \lambda(Q_i) \asymp (1+|B^t\xi|)^{\alpha+\beta} \asymp (1+|\xi|)^{\alpha+\beta} \).

(3) Let us define \( L_1 := [0, \infty) \cap [\gamma_i, \lambda_i] := [0, \infty) \cap [B^{-t}b_i - 4B^{-t}b_i \cdot 2^{\alpha j}, B^{-t}b_i + 4B^{-t}b_i \cdot 2^{\alpha j}] \) for \( i = (j, m, \ell) \in I_0^{(\alpha,\beta)} \). It is not hard to see that \( |\gamma| \leq 4 \) for all \( \eta \in Q = (-\varepsilon, 1 + \varepsilon) \times (-\varepsilon, 1 + \varepsilon) \) and \( ||T_i|| = \|A_j\| = 2^{\alpha j} \). Therefore, we see for \( \xi \in B^{-t}Q_i = B^{-t}(T_iQ + b_i) \) that

\[
\gamma_i = |B^{-t}b_i - 4B^{-t}b_i \cdot ||T_i|| \leq |\xi| \leq |B^{-t}b_i + 4B^{-t}b_i \cdot ||T_i|| = \lambda_i
\]

and hence \( |\xi| \in L_1 \) for all \( \xi \in B^{-t}Q_i \) and \( \lambda_i \in L_0^{(\alpha,\beta)} \). Finally, Equation (4.1) indicates that, for \( \xi \in B^{-t}Q_i, 2 \varepsilon^2 \leq |B^t\xi| \leq \|B\| \cdot |\xi| \) and hence \( \lambda(L_1) \leq 8\|B^{-t}b_i \cdot 2^{\alpha j} \leq (\|B\|^{-1} \cdot 2^{\alpha j})^\alpha \leq (1+|\xi|)^\alpha \) for all \( \xi \in Q_i \) and \( \lambda_i \in L_0^{(\alpha,\beta)} \).

For the remaining case \( i = 0 \), define \( L_0 := [0, 4 \|B^{-1}b_i\|) \) and note that, if \( \xi \in B^{-t}Q_0 = B^{-t}B_4(0) \) is arbitrary, then \( |\xi| \in L_0 \) and \( \lambda(L_0) \leq 1 \leq (1+|\xi|)^\alpha \).

(4) Lemma 7.4 shows that \( Q^{(\alpha,\beta)} \) is an \((\alpha, \beta)\)-covering. Thus, there is such a constant \( C > 0 \) and, for each \( i \in I \), such an angle \( \phi_i \in \mathbb{R} \) that, for each \( \eta \in Q_i \), there is another angle \( \phi \in \mathbb{R} \) that satisfies \( \eta = |\eta| \cdot e^{i\phi} \) and \( \min_{\omega \in (0, \pi)} |\phi - (\phi_i + \omega)| \leq C \cdot (1 + |\eta|)^{\beta-1} \). Therefore, Lemma 8.2 (applied to \( B^{-t} \) instead of \( B \)) yields a constant \( C' = C'(B) > 0 \) and for each \( i \in I \) an angle \( \psi_i \in \mathbb{R} \) such that for each \( \xi = B^{-t} \eta \in B^{-t}Q_i \) there is another angle \( \psi \in \mathbb{R} \) satisfying \( \xi = B^{-t} \eta = |\xi| \cdot e^{i\psi} \) and

\[
\min_{\omega \in (0, \pi)} |\psi - (\psi_i + \omega)| \leq C' \cdot \min_{\theta \in (0, \pi)} |\phi - (\phi_i + \theta)| \leq C' \cdot (1 + |\eta|)^{\beta-1} \leq C' \cdot (1 + \|B^{-1}b_i\|)^{1-\beta}(1+|\xi|)^{\beta-1},
\]

since \( 1 + |\xi| \leq (1 + \|B^{-1}b_i\|) \cdot (1 + |\eta|) \).

9. Constructing Banach frame decompositions of the wave packet smoothness spaces

In this section, we introduce so-called wave packet systems (as informally introduced in [15]). Furthermore, we prove that if the generators of such a system are nice enough, then the system constitutes an atomic decomposition and a Banach frame for certain wave packet smoothness spaces. To do so, we shall concentrate on the case where \( \alpha < 1 \); results for the case \( \alpha = 1 \) can be found in [64].

**Definition 9.1.** Let \( 0 \leq \beta < \alpha < 1 \) \( \delta > 0 \) and \( \varphi, \gamma \in L^1(\mathbb{R}^2) \). Moreover let \( I_0 = I_0^{(\alpha,\beta)} \) and \( I = I^{(\alpha,\beta)} \) be as defined in Equation (4.2). Finally, let \( T_i := R_{j,\ell} A_j b_i \) and \( c_{j,m} \) for \( i = (j, m, \ell) \in I_0 \) and \( T_0 := 0 \) be 0, \( A_j, R_{j,\ell}, \) and \( c_{j,m} \) as introduced in Definition 3.1.

The family \( \{L_{\delta, T_{-t}, j, \ell} \psi \}_{i \in I} \) with

\[
\gamma[i] := \begin{cases} 
\det A_j / |A_j| \cdot M_{R_{j,\ell}}(2^{j-1+2^{\alpha j} m}) \cdot \gamma \circ A_j \circ R_{j,\ell} & \text{if } (j, m, \ell) \in I_0, \\
\varphi & \text{if } (j, m, \ell) = 0
\end{cases}
\]

is called the \((\alpha, \beta)\)-wave packet system with generators \( \varphi, \gamma \) and sampling density \( \delta > 0 \).

**Remark.** If \( \gamma \in C^\infty_c(\mathbb{R}^2) \), then \( \supp \gamma[i] = R_{j,\ell}^{-1} A_j^{-1} \supp \gamma \) is essentially a rectangle that can be obtained by rotating the axis-aligned rectangle of the dimensions \( 2^{-\alpha j} \times 2^{-\beta j} \) with its centre at the origin through the angle \( \Theta_{j,\ell} \approx 2 \pi / 2^{\beta j+\alpha j} \).

Furthermore, \( \gamma \circ A_j \circ R_{j,\ell} \) oscillates roughly at frequencies \( \xi \) such that \( |\xi| \lesssim |A_j| \approx 2^{\alpha j} < 2^j \).

Since \( |2^{j-1} + 2^{\alpha j} m| \geq 2^{j-1} \sim 2^j \), the behaviour of \( \gamma[i] \) will be largely determined by the oscillations at frequency \( \approx 2^j \) in direction \( R_{j,\ell}(1/2) \) that are caused by applying the modulation \( M_{R_{j,\ell}}(2^{j-1+2^{\alpha j} m}) \) to the function \( \gamma \circ A_j \circ R_{j,\ell} \).

Therefore, our \((\alpha, \beta)\)-wave packet systems are similar to those introduced in [15] Section 1.1, with one notable exception. Namely, the frequency support of the elements of our wave packet systems are not symmetric with respect to the origin, while those in [15] are.

To formulate our discretisation results for the wave packet smoothness spaces, we need the following definition.
Definition 9.2. Let \( 0 \leq \beta \leq \alpha < 1 \), \( p, q \in (0, \infty) \) and \( s \in \mathbb{R} \). The set of sequences of complex numbers

\[
C^p,q_s(\alpha, \beta) := \left\{ c = (c_k^{(i)})_{i \in I^{(\alpha, \beta)}, k \in \mathbb{Z}^2} \in C^{I^{(\alpha, \beta)} \times \mathbb{Z}^2} : \| c \|_{C^p,q_s} < \infty \right\}
\]

where

\[
\| (c_k^{(i)})_{i \in I^{(\alpha, \beta)}, k \in \mathbb{Z}^2} \|_{C^p,q_s} := \left\| \left( w_k^{s_0 + (\alpha + \beta) \cdot (\frac{1}{2} + \frac{1}{4})} \cdot \| (c_k^{(i)})_{k \in \mathbb{Z}^2} \|_{\ell^p} \right)_{i \in I^{(\alpha, \beta)}} \right\|_q \in [0, \infty]
\]

is called the **coefficient space** associated with the wave packet smoothness space \( W^p,q_s(\alpha, \beta) \).

The main goal of the present section is to prove the following two theorems that provide condition on the functions \( \varphi, \gamma \) that — if satisfied — guarantee that the \( (\alpha, \beta) \)-wave packet system generated by \( \varphi, \gamma \) forms an atomic decomposition or a Banach frame for the wave packet smoothness spaces \( W^p,q_s(\alpha, \beta) \).

Theorem 9.3. Let \( 0 \leq \beta \leq \alpha < 1 \), \( s_0 \geq 0 \) and \( \omega, p_0, q_0 \in (0, 1] \). Moreover let \( \varphi, \gamma \in L^1(\mathbb{R}^2) \) be such that:

1. \( \hat{\varphi}, \hat{\gamma} \in C^\infty(\mathbb{R}^2) \) and all partial derivatives of \( \hat{\varphi} \) and \( \hat{\gamma} \) are of polynomial growth at most;
2. \( \hat{\varphi}(\xi) \neq 0 \) for all \( \xi \in \mathbb{B}_4(0) \) and \( \hat{\gamma}(\xi) \neq 0 \) for all \( \xi \in [-\varepsilon, 1 + \varepsilon] \times [-1 - \varepsilon, 1 + \varepsilon] \); and
3. \( \sup_{x \in \mathbb{R}^2} (1 + |x|)^{1 + 2p_0^{-1}} |\varphi(x)| < \infty \) and \( \sup_{x \in \mathbb{R}^2} (1 + |x|)^{1 + 2p_0^{-1}} |\gamma(x)| < \infty \).

Finally, let us assume that there is such a constant \( C > 0 \) that

\[
| (\partial^\theta \hat{\varphi})(\xi) | \leq C \cdot (1 + |\xi|)^{-(\frac{\alpha}{2} + s_0)} \cdot (1 + |\xi_1|)^{-(\alpha + \beta - \frac{\alpha}{2})} \cdot (1 + |\xi_2|)^{-(\alpha + \beta - \frac{\alpha}{2})}
\]

and

\[
| (\partial^\theta \hat{\gamma})(\xi) | \leq C \cdot (1 + |\xi|)^{-(\frac{\alpha}{2} + s_0)} \cdot (1 + |\xi_1|)^{-(\alpha + \beta - \frac{\alpha}{2})} \cdot (1 + |\xi_2|)^{-(\alpha + \beta - \frac{\alpha}{2})}
\]

for all \( \xi \in \mathbb{R}^2 \) and all such \( \theta \in \mathbb{N}_0^2 \) that \( |\theta| \leq N_0 \) where

\[
N_0 := \lfloor p_0^{-1}(2 + \omega) \rfloor , \quad \kappa_1 := \frac{2}{\min\{p_0, q_0\}} , \quad \kappa_2 := 3 + \frac{2}{(1 - \beta) \min\{p_0, q_0\}} + \frac{5}{p_0} , \quad \kappa_0 := (1 - \alpha)^{-1} \cdot \left( 3 + s_0 + \frac{3 + \alpha}{\min\{p_0, q_0\}} + \frac{6\alpha + 9\beta}{p_0} + \frac{2\beta}{(1 - \beta) \min\{p_0, q_0\}} \right) .
\]

Then there is such a \( \delta_0 = \delta_0(\alpha, \beta, \omega, p_0, q_0, s_0, \varphi, \gamma, C) > 0 \), that, for each \( \delta \in (0, \delta_0] \), \( p \in [p_0, \infty) \), \( q \in [q_0, \infty) \) and \( s \in [-s_0, s_0] \), the \( (\alpha, \beta) \)-wave packet system with generators \( \varphi, \gamma \) and sampling density \( \delta \) is an atomic decomposition for \( W^p,q_s(\alpha, \beta) \) with coefficient space \( C^p,q_s(\alpha, \beta) \).

Remark. Note that Conditions (1) and (3) in the theorem are satisfied as long as \( \varphi, \gamma \in C^c(\mathbb{R}^2) \).

Furthermore, Condition (9.1) is satisfied if \( \varphi, \gamma \in C^k_c(\mathbb{R}^2) \) where \( k \geq 4 + \kappa_0 + \kappa_1 + \kappa_2 \). This last observation is due to the fact that \( \partial^\theta \hat{\varphi}(\xi) = | \mathcal{F}( (-2\pi i)^{\alpha} \cdot f ) |(\xi) \) where \( (-2\pi i)^{\alpha} \cdot f \in C^k_c(\mathbb{R}^2) \) if \( f \in C^k_c(\mathbb{R}^2) \).

Theorem 9.4. Let \( 0 \leq \beta \leq \alpha < 1 \), \( s_0 \geq 0 \) and \( \omega, p_0, q_0 \in (0, 1] \). Moreover let \( \varphi, \gamma \in L^1(\mathbb{R}^2) \) be such that Properties (1)–(2) from Theorem 9.3 are satisfied and that

\( \begin{align*}
(3') & \quad \varphi, \gamma \in C^1(\mathbb{R}^2) \quad \text{and} \quad \partial^\mu \varphi, \partial^\mu \gamma \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \quad \text{for all} \quad \mu \in \mathbb{N}_0^2 \quad \text{with} \quad |\mu| \leq 1 .
\end{align*} \)

Finally, let us assume that there is such a constant \( C > 0 \) that

\[
| (\partial^\nu \partial^\mu \varphi)(\xi) | \leq C \cdot (1 + |\xi|)^{-\kappa_0} \cdot (1 + |\xi_1|)^{-\kappa_1} \cdot (1 + |\xi_2|)^{-\kappa_2} \quad \text{and} \quad | (\partial^\nu \partial^\mu \gamma)(\xi) | \leq C \cdot (1 + |\xi|)^{-\kappa_0} \cdot (1 + |\xi_1|)^{-\kappa_1} \cdot (1 + |\xi_2|)^{-\kappa_2}
\]

for all \( \xi \in \mathbb{R}^2 \) and all such \( \mu, \nu \in \mathbb{N}_0^2 \) that \( |\nu| \leq N_0 \) and \( |\mu| \leq 1 \) where \( N_0, \kappa_1, \kappa_2 \) are as in Theorem 9.3

and where

\[
\kappa' := (1 - \alpha)^{-1} \cdot \left( 3 + s_0 + \frac{1 + \alpha}{\min\{p_0, q_0\}} + \frac{5\alpha + 10\beta}{p_0} + \frac{2}{(1 - \beta) \min\{p_0, q_0\}} \right) .
\]
Then there exists $\delta_0 = \delta_0(\alpha, \beta, \omega, p_0, q_0, s_0, \varphi, \gamma, C) > 0$ such that, for any $\delta \in (0, \delta_0]$, $p \in [p_0, \infty]$, $q \in [q_0, \infty]$ and $s \in [-s_0, s_0]$, the $(\alpha, \beta)$-wave packet system with generators $\varphi, \gamma$ and sampling density $\delta > 0$ is a Banach frame for $\mathcal{W}_s^{p,q}(\alpha, \beta)$ with coefficient space $C_0^{\alpha,\beta}(\alpha, \beta)$.

More precisely, there is such a bounded analysis map $A^{(\delta)} : \mathcal{W}_s^{p,q}(\alpha, \beta) \to C_0^p$ that

$$A^{(\delta)} f = \left( f \mid L_{\delta,T_i^{-1}}, (\gamma^{[i]} L_2) \right)_{i \in I, k \in \mathbb{Z}^2} \quad \forall f \in L^2(\mathbb{R}^2) \cap \mathcal{W}_s^{p,q}(\alpha, \beta),$$

(9.3)

and $A^{(\delta)}$ has a bounded linear left inverse whose action is independent of the choice of $p \in [p_0, \infty]$, $q \in [q_0, \infty]$ and $s \in [-s_0, s_0]$.

**Remark.** Conditions (1) and (3') are satisfied as long as $\varphi, \gamma \in C_0^1(\mathbb{R}^2)$. Furthermore, Condition (9.2) is satisfied if $\varphi, \gamma \in C_0^1(\mathbb{R}^2)$ where $k \geq 1 + \kappa_0' + \kappa_1 + \kappa_2$.

To prove Theorems 9.3 and 9.4 we shall use Theorems 2.10 and 2.11 respectively. To do so, we need to show that the constants $K_1, K_2$ and $C_1, C_2$, as they were introduced in those theorems, are finite. Given that these constants differ only marginally from each other, we shall slightly reformulate this problem and by doing so prove that $K_1, K_2, C_1, C_2$ are all finite at once. Specifically, let us define

$$\psi : \mathbb{R}^2 \to (0, \infty), \xi \mapsto (1 + |\xi|)^{-\kappa_0} \cdot (1 + |\xi_1|)^{-\kappa_1} \cdot (1 + |\xi_2|)^{-\kappa_2},$$

(9.4)

where $\kappa_0, \kappa_1, \kappa_2 \geq 0$ are fixed, but arbitrary. In the remainder of this section, we shall establish conditions on $\kappa_0, \kappa_1, \kappa_2$ and $B$ so that

$$\sup_{i' \in I} \sum_{i \in I} M_{i,i'}^{(1)} \leq B < \infty \quad \text{and} \quad \sup_{i' \in I} \sum_{i \in I} M_{i,i'}^{(1)} \leq B < \infty,$$

(9.5)

where, for $i = (j, m, \ell) \in I_0$ and $i' = (j', m', \ell') \in I_0$, the quantity $M_{i,i'}^{(1)}$ is given by

$$M_{i,i'}^{(1)} := 2^{(j-j')s} \cdot (1 + \|T_i^{-1} T_{i'}\|)^{\sigma} \cdot \left( \int_{Q_{i,i'}} \psi(T_i^{-1}(\xi - b_i)) \, d\xi \right)^{\tau},$$

(9.6)

with $s \in \mathbb{R}$ and $\sigma, \tau \in (0, \infty)$ fixed, but arbitrary and with $T_i, b_i$ as in Lemma 5.1. Here, we used the notation $\int_M f(x) \, dx := \frac{1}{\lambda(M)} \int_M f(x) \, dx$ where $\lambda$ denotes the Lebesgue measure.

Similarly, we define

$$M_{0,i'}^{(1)} := 2^{-j's} \cdot (1 + \|T_0^{-1} T_{i'}\|)^{\sigma} \cdot \left( \int_{Q_{0,i'}} \psi(T_0^{-1}(\xi - b_0)) \, d\xi \right)^{\tau},$$

$$M_{i,0}^{(1)} := 2^{j's} \cdot (1 + \|T_i^{-1} T_0\|)^{\sigma} \cdot \left( \int_{Q_{0,i}} \psi(T_i^{-1}(\xi - b_i)) \, d\xi \right)^{\tau},$$

and

$$M_{0,0}^{(1)} := (1 + \|T_0^{-1} T_0\|)^{\sigma} \cdot \left( \int_{Q_{0,0}} \psi(T_0^{-1}(\xi - b_0)) \, d\xi \right)^{\tau},$$

(9.7)

where, again $i = (j, m, \ell) \in I_0$ and $i' = (j', m', \ell') \in I_0$. Precisely, we shall prove the following theorem, from which we shall then deduce Theorems 9.3 and 9.4.

**Theorem 9.5.** If $\sigma \geq 0$, $\tau > 0$,

$$\kappa_1 \geq \max \left\{ 2, \frac{2}{\tau} \right\} \quad \text{and} \quad \kappa_2 \geq \max \left\{ 1 + \frac{\sigma + 2}{\tau}, 2 + \kappa_2^{(0)} \right\} \quad \text{with} \quad \kappa_2^{(0)} := \frac{2 - \alpha + \beta + \sigma(\alpha + \beta)}{(1 - \beta)^{\tau}}$$

(9.8)

and if

$$\kappa_0 \geq \max \left\{ \frac{3 + |s + \tau + \alpha + (\alpha + \beta)|}{(1 - \alpha)^{\tau}}, \frac{2 + |s + \tau + \beta \kappa_2^{(0)} + \max\{\tau, \sigma\}(\alpha + \beta)}{(1 - \alpha)^{\tau}} \right\}$$

(9.9)

and

$$B := N \cdot 2^{3\tau + 8\sigma + \tau(10 + 5\kappa_0 + 6\kappa_2^{(0)} + \kappa_2)},$$

(9.10)

then (9.5) holds.
Structure of the section. To prove Theorem 9.5, we first estimate the different terms occurring in (9.5); this will be done in Subsection 9.1. In Subsections 9.2 and 9.3 we estimate respectively the former and the latter series in (9.5) for \( i, i' \in I_0 \). In Subsection 9.4 we estimate these series for \( i' = 0 \) or \( i = 0 \), respectively. Finally, in Subsection 9.5 we prove Theorems 9.3 and 9.4 by using Theorem 9.5.

For \( i \in I_0 \) or \( i' \in I_0 \), we shall use the convention \( i = (j, m, \ell) \) and \( i' = (j', m', \ell') \) throughout this section, without mentioning it explicitly.

9.1. Estimating various terms occurring in \( M_{i,i'}^{(1)} \)

Let \( i = (j, m, \ell) \in I_0 \) and \( i' = (j', m', \ell') \in I_0 \) and let us define

\[
i_0 := (j, \ell), \quad i_0' := (j', \ell'), \quad i_* := (j, m), \quad i_*' := (j', m'), \quad \text{and} \quad \varepsilon_{i_0, i_0'} := \Theta_{j, \ell} - \Theta_{j, \ell'}.
\]

(9.11)

Since \( 0 \leq \Theta_{j, \ell} < 3\pi \) according to (4.7), \( \varepsilon_{i_0, i_0'} \in (-3\pi, 3\pi) \). Thus there exists \( k = k_{i_0, i_0'} \in \{-1, 0, 1, 2\} \) such that

\[
\varepsilon_{i_0, i_0'} := \varepsilon_{i_0, i_0'} + 2\pi k \in [0, 2\pi).
\]

(9.12)

With the change of variables \( \eta = T_i^{-1}(\xi - b_i) \), we obtain

\[
M_{i,i'}^{(1)} = M_{i,i'}^{(2)} := 2^{(j-j')^a} \left( 1 + ||T_i^{-1}T_{i'}||^\beta \right)^\gamma \left( \int_{\Omega_{i,i'}} \psi(\eta) \, d\eta \right)^\gamma \quad \text{where} \quad \Omega_{i,i'} := T_i^{-1}(Q_\nu - b_i).
\]

(9.13)

To estimate the integral in (9.13), we first estimate the Euclidean norm \( |\xi| \) of \( \xi \in \Omega_{i,i'} \).

Lemma 9.6. Let \( i, i' \in I_0 \) and \( \xi \in \Omega_{i,i'} \), then:

a) \( 1 + |\xi| \geq 2^{-5} \cdot 2^{(1-\alpha)|j-j'|}; \) and

b) if \( |j - j'| \geq 5 \), then \( |\xi| \geq 2^{-5} \cdot 2^{\max\{j,j'\}-\alpha} \geq 2^{-5} \cdot 2^{(1-\alpha)\max\{j,j'\}} \).

Proof. Since \( \xi \in \Omega_{i,i'} \), we see that \( \eta = T_i \xi + b_i \in Q_\nu \). Thus, Equation (4.4) implies \( 2^{j'-2} < |\eta| < 2^{j'+3} \).

Furthermore, for \( c_{j,m} \) as defined in (3.3),

\[
2^{j-1} \leq |c_{j,m}| = |b_i| = |c_{j,m}| \leq 2^{j-1} + (1 + 2^{-(1-\alpha)j-1}) \cdot 2^{aj} = 2^{j-1} + 2^{aj} + 2^{j-1} \leq 2^{j+1}.
\]

Finally, we see that \( ||T_i|| = ||A_j|| = \max\{2^{aj}, 2^{bj}\} = 2^{aj} \) since \( \beta \leq \alpha \) and \( A_j = \text{diag}(2^{aj}, 2^{bj}) \).

After this preparation, we first prove Part b) so that we are working under the assumption \( |j - j'| \geq 5 \). Thus, there are two cases:

Case 1: \( j' \geq j + 5 \). Then \( 2^{j'-2} < |\eta| = |T_i \xi + b_i| \leq 2^{aj} |\xi| + 2^{j+1} \) and hence

\[
|\xi| \geq 2^{-\alpha j} \cdot (2^{j'-2} - 2^{j+1}) \geq 2^{-\alpha j} \cdot (2^{j'-2} - 2^{j'-3}) = 2^{-3} \cdot 2^{j'-\alpha j} \geq 2^{-5} \cdot 2^{\max\{j,j'\}-\alpha j}.
\]

Case 2: \( j \geq j' + 5 \). Then

\[
2^{aj} \cdot |\xi| \geq |T_i \xi| = |\eta - b_i| \geq |b_i| - |\eta| \geq 2^{j-1} - 2^{j'+3} \geq 2^{j-1} - 2^{j'-2} = 2^{j-2},
\]

and therefore \( |\xi| \geq 2^{-2} \cdot 2^{-\alpha j} \geq 2^{-5} \cdot 2^{\max\{j,j'\}-\alpha j} \).

Combining these two cases proves the first estimate in Part b). To prove the second, we note that

\[
\max\{j, j'\} - \alpha j \geq \max\{j, j'\} - \alpha \max\{j, j'\} = (1 - \alpha) \max\{j, j'\}.
\]

Finally, to prove Part a) we note that \( 2^{-5} 2^{(1-\alpha)|j-j'|} \leq 1 \leq 1 + |\xi| \) if \( |j - j'| \leq 5 \). If otherwise \( |j - j'| \geq 5 \), then Part b) implies that

\[
1 + |\xi| \geq 2^{-5} \cdot 2^{(1-\alpha)\max\{j,j'\}} \geq 2^{-5} \cdot 2^{(1-\alpha)(\max\{j,j'\} - \min\{j,j'\})} = 2^{-5} \cdot 2^{(1-\alpha)|j-j'|}.
\]

To prove (9.5), we shall rely on the following two lemmata.

Lemma 9.7. (see Lemma C.1 and ensuing remark in [64])

Let \( N \in (0, \infty) \), \( \tau, \beta_0, \ell, L \in (0, \infty) \) and \( M \in \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{C} \) be measurable and such that

\[
|f(x)| \leq C \cdot (1 + |x|)^{-N+2} \quad \forall x \in \mathbb{R}.
\]

Then,

\[
\sum_{k \in \mathbb{Z}} (1 + \beta_0 k + M)^N \left( \int_{[\beta_0 k + M]} |f(x)| \, dx \right)^\tau \leq 2^{3+\tau+N} \cdot 10^{N+3} \cdot C^\tau \cdot L^\tau \cdot (1 + L^N) \cdot \left( 1 + \frac{L + 1}{\beta_0} \right).
\]

43
Lemma 9.8. Let $N, \gamma \in [0, \infty)$, $L, \tau \in (0, \infty)$, $M \in \mathbb{R}$ and $0 < \beta_1 \leq \beta_2$. Furthermore, let $f : \mathbb{R} \to \mathbb{C}$ be measurable and assume that there is

$$q \geq 1 + \frac{N + 2}{\tau}$$

such that

$$|f(x)| \leq C_0 \cdot (1 + |x|)^{-q} \quad \text{for all } x \in \mathbb{R}, \quad (9.14)$$

and let $C := 2^{4+N+\tau+\tau q}$. Then

$$\sum_{k \in \mathbb{Z}, \text{ with } k \neq 0} \left| \gamma \cdot (k + M) \right|^N \cdot \left( \int_0^{\beta_2 \cdot (k+M)} \left| f(x) \right| \, dx \right)^\tau \leq C \cdot \left( \frac{\beta_2}{\beta_1} \right)^\tau \left( \frac{\gamma}{\beta_2} \right)^N \cdot C_0^\tau \cdot (1 + L^\tau + N) \cdot \left( 1 + \frac{L + 1}{\beta_1} \right).$$

Proof. See Appendix A. \hfill \Box

We shall also need a slightly reformulated form of this lemma.

Corollary 9.9. Let $N, \gamma \in [0, \infty)$, $L, \tau \in (0, \infty)$, $M \in \mathbb{R}$, and $0 < \beta_2 \leq \beta_1$. Furthermore, let $f : \mathbb{R} \to \mathbb{C}$ be measurable and such that (9.14) is satisfied.

Then, with the same constant $C$ as in Lemma 9.8,

$$\sum_{k \in \mathbb{Z}, \text{ with } k \neq 0} \left| \gamma \cdot (k + M) \right|^N \cdot \left( \int_0^{\beta_2 \cdot (k+M)+L} \left| f(x) \right| \, dx \right)^\tau \leq C \cdot \left( \frac{\beta_1}{\beta_2} \right)^\tau \left( \frac{\gamma}{\beta_2} \right)^N \cdot C_0^\tau \cdot (1 + L^\tau + N) \cdot \left( 1 + \frac{L + 1}{\beta_2} \right).$$

Proof. See Appendix A. \hfill \Box

To use the preceding results for proving (9.5), we have to verify that the domain of integration $\Omega_{i',\ell'}$ in (9.13) is contained in a Cartesian product of intervals that comply with the requirements of the lemmata. To this end, let us define

$$R_{i_0,i_0'} := \begin{pmatrix} \cos \vartheta_{i_0,i_0'} & -\sin \vartheta_{i_0,i_0'} \\ \sin \vartheta_{i_0,i_0'} & \cos \vartheta_{i_0,i_0'} \end{pmatrix} = \begin{pmatrix} \cos \vartheta_{i_0,i_0'} & -\sin \vartheta_{i_0,i_0'} \\ \sin \vartheta_{i_0,i_0'} & \cos \vartheta_{i_0,i_0'} \end{pmatrix}, \quad (9.15)$$

and recall that $T_i = R_{j,i}A_{ji}$, to conclude that

$$\Omega_{i',j'} = T_i^{-1}(Q_{i',j'} - b_i) = A_j^{-1}R_{j,i}^{-1}(R_{j,i}A_{j'i'}Q_{j'i'} + c_{j'i'}) - R_{j,i}c_{j,i}$$

$$= A_j^{-1}(R_{i_0,i_0'}Q_{j'i'}m',0 - c_{j,i}). \quad (9.16)$$

Next, we investigate the set $Q_{j'i'm',0}$ and, in doing so, introduce a convenient notation:

Lemma 9.10. For $i' = (j',m',\ell') \in I_0$, let us define

$$x_{i'}^- := 2^{j'-1} + (m' - \varepsilon) \cdot 2^{\alpha j'} \quad \text{and} \quad x_{i'}^+ := 2^{j'-1} + (m' + 1 + \varepsilon) \cdot 2^{\alpha j'}, \quad \text{and} \quad y_{j'} := 2^{\beta j'}+1.$$

Then

$$Q_{j,i'm',0} \subseteq [x_{i'}^-, x_{i'}^+] \times [-y_{j'}, y_{j'}] \quad \text{and} \quad \frac{1}{4} \cdot 2^{j'} \leq x_{i'}^- \leq x_{i'}^+ \leq 4 \cdot 2^{j'}.$$ 

Proof. Since $Q = (-\varepsilon, 1 + \varepsilon) \times (-1 - \varepsilon, 1 + \varepsilon) \subseteq (-\varepsilon, 1 + \varepsilon) \times (-2, 2)$ and $A_{j'i'} = \text{diag}(2^{\alpha j'}, 2^{\beta j'})$, we conclude that

$$Q_{j,i'm',0} = A_{j'i'}Q + c_{j,i'm'} = \begin{pmatrix} 2^{\alpha j'} & (0,1) \\ 0 & 2^{\beta j'} \end{pmatrix} + \begin{pmatrix} 2^{j'-1} + m' \cdot 2^{\alpha j'} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} 2^{j'-1} + 2^{\alpha j'}(m' - \varepsilon), 2^{j'-1} + 2^{\alpha j'}(m' + 1 + \varepsilon) \end{pmatrix} \subseteq [-2^{\beta j'}+1, 2^{\beta j'}+1],$$

which completes the proof of the first claim of the lemma.

To prove the second claim, let us recall that $m' \leq m_j^{\text{max}} \leq 1 + 2^{(1-\alpha)j'-1}$, whence

$$x_{i'}^+ \leq 2^{j'-1} + (2^{(1-\alpha)j'-1} + 2 + \varepsilon) \cdot 2^{\alpha j'} = 2^{j'} + (2 + \varepsilon) \cdot 2^{\alpha j'} \leq 4 \cdot 2^{j'}.$$

Clearly, $x_{i'}^- \leq x_{i'}^+$. Finally, since $m' \geq 0$ and $\varepsilon \leq 1/32 \leq 1/4$, we see that

$$x_{i'}^- \geq 2^{j'-1} - \varepsilon \cdot 2^{\alpha j'} \geq 2^{j'-1} - (2 + \varepsilon) \cdot 2^{\alpha j'} = 2^{j'} \cdot \left( \frac{1}{2} - \varepsilon \right) \geq \frac{1}{4} \cdot 2^{j'}.$$

\hfill \Box
We now investigate the set $\Omega_{t,t'}$ defined in (9.13).

**Lemma 9.11.** Recall that $\vartheta_{t_0,t_0'} \in [0,2\pi)$ and define

$$\theta_{t_0,t_0'} := \vartheta_{t_0,t_0'} - \frac{\pi}{2} \in \left(0, \frac{\pi}{2}\right) \quad \text{if} \quad \vartheta_{t_0,t_0'} \in \left(0, \frac{\pi}{2}\right) \quad \text{for some} \quad \ell \in \{0,1,2,3\}. \quad (9.17)$$

Let us also define

$$\psi_{t_0,t_0'} := \begin{cases} 
\frac{\pi}{2} \cdot \cos \theta_{t_0,t_0'} \pm y \cdot \sin \theta_{t_0,t_0'}, & \text{if} \ \vartheta_{t_0,t_0'} \in \left(0, \frac{\pi}{2}\right), \\
-\frac{\pi}{2} \cdot \sin \theta_{t_0,t_0'} \pm y \cdot \cos \theta_{t_0,t_0'}, & \text{if} \ \vartheta_{t_0,t_0'} \in \left[\frac{\pi}{2}, \pi\right), \\
\frac{\pi}{2} \cdot \cos \theta_{t_0,t_0'} \pm y \cdot \sin \theta_{t_0,t_0'}, & \text{if} \ \vartheta_{t_0,t_0'} \in \left[\pi, \frac{3\pi}{2}\right), \\
-\frac{\pi}{2} \cdot \sin \theta_{t_0,t_0'} \pm y \cdot \cos \theta_{t_0,t_0'}, & \text{if} \ \vartheta_{t_0,t_0'} \in \left[\frac{3\pi}{2}, 2\pi\right). 
\end{cases} \quad (9.18)$$

Then

$$R_{t_0,t_0'} Q_{t',m',0} \subset [u_{t_0,t_0'}, v_{t_0,t_0'}] \times [v_{t_0,t_0'}, v_{t_0,t_0'}] \quad (9.20)$$

and

$$\Omega_{t,t'} \subset I_{1}^{(i,i')} \times I_{2}^{(i,i')} \quad (9.21)$$

where

$$I_{1}^{(i,i')} := I_1 := \left[2^{-\alpha j} \cdot (u_{t_0,t_0'} - 2^{-j-1} - m \cdot 2^{\alpha j}) + 2^{-\alpha j} \cdot (v_{t_0,t_0'} - 2^{-j-1} - m \cdot 2^{\alpha j})\right]$$

and

$$I_{2}^{(i,i')} := I_2 := \left[2^{-\beta j} \cdot v_{t_0,t_0'}, 2^{-\beta j} \cdot v_{t_0,t_0'}\right].$$

**Remark.** Note that the angle $\theta_{t_0,t_0'}$ was introduced to ensure that $\cos \theta_{t_0,t_0'} \geq 0$ and $\sin \theta_{t_0,t_0'} \geq 0$, which will prove convenient.

**Proof of Lemma 9.11.** See Appendix C

Having estimated the domain of integration $\Omega_{t,t'}$ in (9.13), we still need to estimate the quantities $(1 + ||T_{t,t'}||)^{\sigma}$ and $\psi(\eta)$, for $\eta \in \Omega_{t,t'}$, in such a way that Lemmas 9.7 and 9.8 can be readily applied.

First, from the definition of $\psi$ and from Lemma 9.6 we infer that

$$\psi(\eta) \leq 2^{5\alpha_{0}} \cdot 2^{-(1-\alpha)|\alpha_{0}|} \cdot (1 + |\eta_3|)^{-\kappa_1} \cdot (1 + |\eta_2|)^{-\kappa_2} \quad (9.23)$$

Second, by recalling Equation (5.3) from the proof of Lemma 5.1 and by recalling the definitions of $\theta_{t_0,t_0}'$ and $\vartheta_{t_0,t_0}'$ (see Equations (9.11) and (9.12)), we conclude that

$$T_{t,t'}^{-1} T_{t'} = E_{i,i'} := \begin{pmatrix} 
2^{\alpha j' - j} \cdot \cos \theta_{t_0,t_0'} & -2^{\beta j' - j} \cdot \sin \theta_{t_0,t_0'} \\
2^{\alpha j' - j} \cdot \sin \theta_{t_0,t_0'} & 2^{\beta j' - j} \cdot \cos \theta_{t_0,t_0'}
\end{pmatrix}.$$ 

To estimate the matrix elements of $E_{i,i'}$, we recall that $\beta \leq \alpha$, whence

$$|E_{i,i'}^{(1)}| \leq 2^{\alpha j' - j} \leq 2^{\alpha (j' - j)}, \quad |E_{i,i'}^{(2)}| \leq 2^{\beta j' - j} \leq 2^{\beta (j' - j)} \leq 2^{\alpha (j' - j)} + 2^{\alpha j' - j} \sin \theta_{t_0,t_0'}$$

and

$$|E_{i,i'}^{(4)}| \leq 2^{\beta (j' - j)} \leq 2^{\beta (j' - j)} + 2^{\alpha j' - j} \sin \theta_{t_0,t_0'}.$$ 

Therefore,

$$1 + ||T_{t,t'}^{-1} T_{t'}|| = 1 + \|E_{i,i'}\| = 4 \cdot 2^{\alpha j' - j} + 2^{\alpha j' - j} \sin \theta_{t_0,t_0'} \leq 4 \cdot 2^{\alpha j' - j} \cdot (1 + 2^{\alpha j' - j} \sin \theta_{t_0,t_0'})$$

and hence

$$(1 + ||T_{t,t'}^{-1} T_{t'}||)^{\sigma} \leq 4^{\sigma} \cdot 2^{(\alpha - \beta) j'} \cdot (1 + 2^{(\alpha - \beta) j} \sin \theta_{t_0,t_0'})^{\sigma}. \quad (9.24)$$
Finally, since we need to convert the mean integral in (9.13) to an ordinary integral, we need to establish a lower bound for \( \Omega_{i,j'} \). Given (9.16) and recalling that \( A_j = \text{diag}(2^{\alpha_j}, 2^{\beta_j}) \), we conclude that

\[
\lambda(\Omega_{i,j'}) = |\det A_j^{-1}| \cdot \lambda(Q_{j', m', 0}) = |\det A_j|^{-1} \cdot |\det A_{j'}| \cdot \lambda(Q) \geq 2^{(\alpha + \beta)(j' - j)},
\]

and hence

\[
[\lambda(\Omega_{i,j'})]^{-1} \leq 2^{(\alpha + \beta)(j' - j)}.
\]

(9.25)

Combining the estimates (9.23)–(9.25) and recalling that \( \Omega_{i,j'} \subset I_1^{(i,j')} \times I_2^{(i,j')} \), we conclude that

\[
M_{i,i'}^{(2)} \leq 4^\sigma \cdot 2^{5\sigma \kappa_0} \cdot 2^{\omega_{j,j'}} \cdot M_{i,i'}^{(3)}
\]

(9.26)

where

\[
\omega_{j,j'} := (s + \tau(\alpha + \beta)) \cdot (j' - j) + \alpha \sigma \cdot (j' - j) + (1 - \alpha) \kappa_0 \tau \cdot |j - j'|
\]

(9.27)

and

\[
M_{i,i'}^{(3)} := (1 + 2^{(\alpha + \beta)j}) |\sin \theta_{i_{0,i'}}|^{\sigma} \cdot \left( \int_{I_1^{(i,j')}} (1 + |\eta_1|)^{-\sigma_1} \, d\eta_1 \cdot \int_{I_2^{(i,j')}} (1 + |\eta_2|)^{-\sigma_2} \, d\eta_2 \right)^{\frac{1}{\sigma}}
\]

(9.28)

The estimate (9.26) and the inclusion \( \Omega_{i,j'} \subset I_1^{(i,j')} \times I_2^{(i,j')} \) from Lemma 9.11 are the main ingredients for applying Lemmas 9.7 and 9.8. This will be done in the next two subsections.

9.2. Estimating the sum over \( i \in I_0 \)

We fix \( i' = (j', m', \ell') \in I_0 \) for this whole subsection. To be able to apply Lemmas 9.7 and 9.8 we investigate the intervals \( I_1^{(i,j')} \) and \( I_2^{(i,j')} \) a little further.

Lemma 9.12. Let \( j \in \mathbb{N} \) and \( \ell \in \mathbb{N}_0 \) such that \( \ell \leq \ell_{j}^{\max} \). Then there is a number \( S_{j, \ell} \in \mathbb{R} \) such that

\[
I_1^{(i,j')} \subset \left[ -m + S_{j, \ell} - 2^{\alpha(j' - j)} \right] \times \left[ -m + S_{j, \ell} + 2^{\alpha(j' - j)} \right]
\]

for all \( m \in \mathbb{N}_0 \) for which \( i = (j, m, \ell) \in I_0 \).

Proof. Let us define \( x := 2^{j-1} + m \cdot 2^{\alpha j'} \). From the definition of \( x_{i_{0,i'}}^{+} \) in Lemma 9.10 we infer that

\[
|x_{i_{0,i'}}^{+} - x| \leq (1 + \varepsilon) \cdot 2^{\alpha j'} \leq 2^{1 + \alpha j'}.
\]

Let \( m \in \mathbb{N}_0 \) be arbitrary with \( i = (j, m, \ell) \in I_0 \), and let us define

\[
S_{j, \ell}^{(0)} := \begin{cases} x \cdot \cos \theta_{i_{0,i'}} & \text{if } \theta_{i_{0,i'}} \in \left[ 0, \frac{\pi}{2} \right], \\ -x \cdot \sin \theta_{i_{0,i'}} & \text{if } \theta_{i_{0,i'}} \in \left[ \frac{\pi}{2}, \pi \right], \\ -x \cdot \cos \theta_{i_{0,i'}} & \text{if } \theta_{i_{0,i'}} \in \left[ \pi, \frac{3\pi}{2} \right], \\ x \cdot \sin \theta_{i_{0,i'}} & \text{if } \theta_{i_{0,i'}} \in \left[ \frac{3\pi}{2}, \pi \right] \end{cases}
\]

and

\[
S_{j, \ell} := 2^{-\alpha j} \cdot (S_{j, \ell}^{(0)} - 2^{-j-1}),
\]

and compare the definition of \( S_{j, \ell}^{(0)} \) with the definition of \( u_{i_{0,i'}}^{+} \) in (9.18). Recalling that \( \beta \leq \alpha \) and \( y_{j'} = 2^{\beta j'} + 1 \), and that \( |\sin \theta_{i_{0,i'}}|, |\cos \theta_{i_{0,i'}}| \leq 1 \), we conclude that

\[
|u_{i_{0,i'}}^{+} - S_{j, \ell}^{(0)}| \leq |y_{j'}| + |x - x_{i_{0,i'}}^{+}| \leq 2^{1+\beta j'} + 2^{1+\alpha j'} \leq 2^{\alpha j'}
\]

and

\[
2^{-\alpha j} \cdot (u_{i_{0,i'}}^{+} - 2^{-j-1} - m \cdot 2^{\alpha j}) - (S_{j, \ell} - m) = 2^{-\alpha j} \cdot (u_{i_{0,i'}}^{+} - S_{j, \ell}^{(0)}) \leq 2^{2+\alpha(j' - j)}.
\]

Combining this and the definition of \( I_1^{(i,j')} \) in (9.22) results in

\[
I_1^{(i,j')} \subset \left[ -m + S_{j, \ell} - 2^{2+\alpha(j' - j)} \right] \times \left[ -m + S_{j, \ell} + 2^{2+\alpha(j' - j)} \right].
\]
Lemma 9.13. Let $\iota \in \{0, 1, 2, 3\}$, $k \in \{-1, 0, 1, 2\}$, and $j \in \mathbb{N}$. Define

$$J^{i,k}_j := \left\{ \ell \in \mathbb{N}_0 : \ell \leq \ell^{\text{max}}_j \text{ and } i_0,i_0' \in \iota \cdot \frac{\pi}{2} + \left[0, \frac{\pi}{2}\right) \text{ and } k_{i_0,i_0'} = k \right\},$$

with $k_{i_0,i_0'}$ as defined in (9.12). Furthermore, let

$$\beta_1 := N^{-1} \cdot 2^{j'-j}, \quad \beta_2 := \frac{8\pi}{N} \cdot 2^{j'-j}, \quad \text{and } L_j := 2 \cdot 2^{\beta(j'-j)}.$$

Then there are $S_{k,i,j} \in \mathbb{R}$ and $\nu_i \in \{\pm 1\}$ such that:

1. for any $m \in \mathbb{N}_0$ and $\ell \in J^{i,k}_j$ satisfying $i = (j, m, \ell) \in I_0$, we have

$$\begin{cases} I_2^{(i,j)} \subset [\beta_1 \cdot (S_{k,i,j} + \nu_i \ell) - L_j, \beta_2 \cdot (S_{k,i,j} + \nu_i \ell) + L_j], & \text{if } \iota \in \{0, 1\}, \vspace{3pt} \\ I_2^{(i,j)} \subset [\beta_2 \cdot (S_{k,i,j} + \nu_i \ell) - L_j, \beta_1 \cdot (S_{k,i,j} + \nu_i \ell) + L_j], & \text{if } \iota \in \{2, 3\}; \end{cases}$$

2. for any $\ell \in J^{i,k}_j$, we have

$$\begin{cases} S_{k,i,j} + \nu_i \ell \geq 0, & \text{if } \iota \in \{0, 1\}, \vspace{3pt} \\ S_{k,i,j} + \nu_i \ell \leq 0, & \text{if } \iota \in \{2, 3\}; \end{cases}$$

3. $2^{\alpha_j'-\beta_j} \cdot |\sin \theta_{i_0,i_0'}| \leq 2\pi \cdot \beta_1 \cdot |S_{k,i,j} + \nu_i \ell|$ for all $\ell \in J^{i,k}_j$.

**Proof.** See Appendix C. \hfill \qed

Given Lemmas 9.12 and 9.13, we can finally show that the first supremum in (9.5) is finite, provided that (9.8) and (9.9) are satisfied.

To show this, let us define $(\langle x \rangle := 1 + |x|)$ for $x \in \mathbb{R}$ and fix $j \in \mathbb{N}$ for the moment. Since

$$\sum_{\ell=0}^{\ell^{\text{max}}_j} \sum_{m=0}^{m^{\text{max}}_j} M^{(3)}_{i,i'} \leq \sum_{\ell=0}^{3} \sum_{k=-1}^{2} \sum_{\ell \in J^{i,k}_j} \sum_{m=0}^{m^{\text{max}}_j} M^{(3)}_{i,i'},$$

it suffices to estimate the inner double sum for fixed $i \in \{0, \ldots, 3\}$ and $k \in \{-1, \ldots, 2\}$. Let $\beta_1, \beta_2, S_{k,i,j}, \nu_i, L_j$ as in Lemma 9.13. If $\iota \in \{0, 1\}$, let us define $\beta_1 := \beta_1$ and $\beta_2 := \beta_2$; otherwise, if $\iota \in \{2, 3\}$, let us define $\beta_1 := \beta_2$ and $\beta_2 := \beta_1$ instead. Now, by definition of $M^{(3)}_{i,i'}$ and by Lemmas 9.12 and 9.13, we see that, for $\ell \in J^{i,k}_j$ and $0 \leq m \leq m^{\text{max}}_j$ with $S_{j,k}$, as in Lemma 9.12

$$M^{(3)}_{i,i'} \leq (1 + 2\pi \beta_1 \cdot |S_{k,i,j} + \nu_i \ell|)^{\sigma} \cdot \left( \int_{S_{j,k}-m+2^{j'+\alpha}(j'-j)}^{S_{j,k}-m+2^{j'+\alpha}(j'-j)} (\eta_2)^{-\kappa_2} d\eta_2 \right)^{\tau} \cdot \left( \int_{S_{j,k}-m-2^{j'+\alpha}(j'-j)}^{S_{j,k}-m-2^{j'+\alpha}(j'-j)} (\eta_1)^{-\kappa_1} d\eta_1 \right)^{\tau}.$$

On the other hand, since $\kappa_1 \geq \frac{\sigma}{2}$, Lemma 9.7 (applied with $N = 0$, $\beta_0 = 1$, $L = 2^{2+\alpha}(j'-j)$ and $M = -S_{j,k}$) yields the estimate

$$\sum_{m \in \mathbb{Z}} \left( \int_{S_{j,k}-m-2^{j'+\alpha}(j'-j)}^{S_{j,k}-m+2^{j'+\alpha}(j'-j)} (1 + |\eta_1|)^{\tau - \kappa_1} d\eta_1 \right)^{\tau} \leq \sum_{m \in \mathbb{Z}} \left( \int_{m-S_{j,k}-2^{j'+\alpha}(j'-j)}^{m-S_{j,k}+2^{j'+\alpha}(j'-j)} (1 + |\xi_1|)^{-\kappa_1} d\xi_1 \right)^{\tau} \leq 2^{3+\tau} \cdot 10^3 \cdot 2^{2\tau+\tau}(j'-j) \cdot 2 \cdot (2 + 2^{2+\alpha}(j'-j)) \leq 2^{17+3\tau} \cdot 2^{\alpha(j'-j)} \cdot 2^{\alpha(j'-j)} + \Gamma_j^{(1)}.$$

Next, we note that

$$(1 + 2\pi \beta_1 \cdot |S_{k,i,j} + \nu_i \ell|)^{\sigma} \leq 2^{\sigma} \cdot \sum_{E \in \{0, \sigma\}} |2\pi \beta_1 \cdot (S_{k,i,j} + \nu_i \ell)|^{E}.$$

Let us fix $E \in \{0, \sigma\}$ for the moment.
Set \( \varepsilon := 1 \) if \( i \in \{0, 1\} \) and \( \varepsilon := -1 \) otherwise. Then Lemma \ref{lem:9.13} shows that \( \varepsilon \cdot (S_{k,i,j} + \nu_i \ell) \geq 0 \) for \( \ell \in J_i^{j,k} \). Furthermore, the change of variable \( \lambda = \nu_i \ell \) combined with an application of Lemma \ref{lem:9.8} or Corollary \ref{cor:9.9} (depending on whether \( \varepsilon = 1 \) or \( \varepsilon = -1 \)) shows that, for \( \Gamma_j^{(2)} := 2^{4 + \sigma + \tau + \kappa_2} \),
\[
\sum_{\ell \in \mathbb{Z}, \varepsilon(S_{k,i,j} + \nu_i \ell) \geq 0} |2\pi \beta_1(S_{k,i,j} + \nu_i \ell)|^E \left( \int_{\beta_1^{-1}(S_{k,i,j} + \nu_i \ell) - L_j}^{\beta_1^{-1}(S_{k,i,j} + \nu_i \ell) + L_j} \eta_2^{-\kappa_2} \, d\eta_2 \right)^\tau
\leq \Gamma_j^{(2)} \cdot (8\pi)^\tau \cdot (2\pi)^\nu \cdot (1 + \max\{1, L_j\})^\tau \cdot (1 + 2N \cdot 2^{(1-\beta)(j-j')} + N \cdot 2^{j-j'})
\leq \Gamma_j^{(2)} \cdot 2 \cdot 2^{\beta(j-j') + 4\beta \cdot (j-j')} \cdot 4N \cdot 2^{j-j'}+
\leq N \cdot \Gamma_j^{(2)} \cdot 2^{3 + \beta \cdot (2 \sigma - \sigma)} \cdot 2^{j-j'} + \beta(\tau + \sigma)(j-j') =: \Gamma_j^{(3)}.
\]
Here, we noted in the penultimate estimate that \( L_j = 2 \cdot 2^{\beta(j-j')} \leq 2^{\beta(j-j')} \). Furthermore, we noted that \( \kappa_2 \geq 1 + \frac{\sigma + 2}{2} \) (see Equation \ref{eq:9.8}), so that Lemma \ref{lem:9.8} and Corollary \ref{cor:9.9} are indeed applicable.

Summarising all these estimates, we finally conclude that
\[
\sum_{\ell=0}^{m_{\text{max}}} \sum_{m=0}^{m_{\text{max}}} M_{i,i'}^{(3)} \leq 4 \cdot 4 \cdot 2^\sigma \cdot 2 \cdot \Gamma_j^{(1)} \cdot \Gamma_j^{(3)} \leq \Gamma_j^{(4)},
\]
where a straightforward but tedious calculation shows that one can choose
\[
\Gamma_j^{(4)} := 2^{30 + 6\sigma + (10 + \kappa_2)\tau} \cdot N \cdot 2^{(\alpha + \beta(\tau + \sigma))(j-j') + j-j'} + \alpha(j-j').
\]
Finally, by recalling Equations \ref{eq:9.13} and \ref{eq:9.26}, we see that
\[
\sum_{i=(j,m,l) \in I_0} M_{i,i'}^{(1)} \leq 4^\sigma \cdot 2^{5\tau \kappa_0} \cdot \sum_{j=1}^{\infty} \omega_{j,j'} \cdot \Gamma_j^{(4)} \leq \Gamma^{(5)} \cdot \sum_{j=1}^{\infty} 2\tilde{\omega}_{j,j'}
\]
with
\[
\tilde{\omega}_{j,j'} = \tau \alpha(j-j') + j-j' + (\alpha + \beta(\tau + \sigma))(j-j') + \omega_{j,j'} \quad \text{and} \quad \Gamma^{(5)} := 2^{30 + 5\kappa_0 + 8\sigma + (10 + \kappa_2)\tau} \cdot N \cdot
\]
A direct calculation shows that
\[
\tilde{\omega}_{j,j'} = \begin{cases} \frac{|j' - j|}{(1 - \alpha)\tau} \cdot \frac{(\alpha + \beta(\alpha + \beta) - \beta)}{(1 - \alpha)\tau} - \kappa_0, & \text{if } j \leq j', \\
\frac{|j' - j|}{(1 - \alpha)\tau} \cdot \frac{1 + \tau \beta}{(1 - \alpha)\tau} - \kappa_0, & \text{otherwise}. \end{cases}
\]
Given our choice of \( \kappa_0 \) (see Equation \ref{eq:9.9}), we therefore conclude that \( \tilde{\omega}_{j,j'} \leq -|j' - j| \) and
\[
\sum_{i=(j,m,l) \in I_0} M_{i,i'}^{(1)} \leq \Gamma^{(5)} \cdot \sum_{j=1}^{\infty} 2^{-|j' - j|} \leq 3 \cdot \Gamma^{(5)} < \infty \quad \forall i' \in I_0.
\]
Since \( \Gamma^{(5)} \) is independent of the choice of \( i' = (j', m', l') \in I_0 \), we have thus shown that the first supremum in \ref{eq:9.5} is finite, as long as both \( i \) and \( i' \) are restricted to \( I_0 \) instead of to \( I = \{0\} \cup I_0 \).

\subsection*{9.3. Estimating the sum over \( i' \in I_0 \)}

For this whole subsection, we fix \( i = (j, m, \ell) \in I_0 \).

In the preceding subsection, we used the inclusion \( I_1^{(i,i')} \subset [S_{j,l} - m - 2^{2 + \alpha(j-j')}, S_{j,l} - m + 2^{2 + \alpha(j-j')} \] to then apply Lemma \ref{lem:9.7}. Observe that the parameter \( m \) appears on the right-hand side of this inclusion without a factor in front. In contrast, we shall prove in Lemma \ref{lem:9.15} that the inclusion \( I_1^{(i,i')} \subset [S_{j,k} \cdot \beta_{j',k'} \cdot m' + S_{j,k} \cdot L_{j'}', s_{j,k} \cdot \beta_{j',k'} \cdot m' + S_{j,k} + L_{j'}'] \) holds where \( s_{j,k} \in \{\pm 1\} \) and the factor \( \beta_{j',k'} \) depends on \( \cos \theta_{i_0,i_0} \) or \( \sin \theta_{i_0,i_0} \), so that possibly \( \beta_{j',k'} \approx 0 \). If this happens, the bound
provided by Lemma 9.7 will be ineffective. Therefore, we first deal with this special case using another method. Precisely, let us define

\[ t_{j',\ell'} := \begin{cases} 
\cos\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [0, \pi/2), \\
\sin\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [\pi/2, \pi), \\
\cos\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [\pi, 3\pi/2), \\
\sin\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [3\pi/2, 2\pi).
\end{cases} \]

Note that this is well-defined, i.e. the right-hand side indeed only depends on \( j', \ell' \), since \( i = (j, m, \ell) \) is fixed and \( i_0' = (j', \ell') \). Furthermore, we note that \( t_{j',\ell'} \) is, up to a sign, the coefficient of \( x_{i_0}^\pm \) in the definition of \( u_{i_0,i'}^\pm \); see Equation (9.18). Finally, for fixed \( j' \in \mathbb{N} \), let us define

\[ J^{(j')}_{\text{special}} := \left\{ \ell' \in \mathbb{N}_0 : \ell' \leq t_{j',\ell'}^\max \text{ and } t_{j',\ell'} \leq \frac{1}{10} \right\} \]

and \( J^{(j')}_{\text{normal}} := \left\{ \ell' \in \mathbb{N}_0 : \ell' \leq t_{j',\ell'}^\max \text{ and } t_{j',\ell'} > \frac{1}{10} \right\} \).

The following lemma provides the crucial ingredient for estimating the contribution of the terms with problematic indices \( \ell' \in J^{(j')}_{\text{special}} \).

**Lemma 9.14.** For \( j' \in \mathbb{N} \), \( \ell' \in J^{(j')}_{\text{special}} \) and \( m' \in \mathbb{N}_0 \) with \( i' = (j', m', \ell') \in I_0 \), any \( \eta_2 \in I_2^{(i,\ell')} \) (see Equation (9.22)) satisfies

\[ 1 + |\eta_2| \geq \frac{2^{j' - \beta_j}}{40}. \]

**Proof.** Let us define

\[ \tilde{t}_{j',\ell'} := \begin{cases} 
\sin\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [0, \pi/2), \\
\cos\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [\pi/2, \pi), \\
\sin\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [\pi, 3\pi/2), \\
\cos\theta_{i_0,i_0'} & \text{if } \theta_{i_0,i_0'} \in [3\pi/2, 2\pi),
\end{cases} \]

and note that

\[ \tilde{t}_{j',\ell'} \geq 1 - t_{j',\ell'} \tag{9.30} \]

as a consequence of Equation (3.3) and since \( t_{j',\ell'}, \tilde{t}_{j',\ell'} \geq 0 \).

According to Equation (9.22), \( I_2^{(i,\ell')} = 2^{-\beta_j} \cdot [v_{i_0,i'}^-, v_{i_0,i'}^+] \). To derive from this the desired estimate, we distinguish two cases, depending on \( \theta_{i_0,i_0'} \).

**Case 1:** \( \theta_{i_0,i_0'} \in [0, \pi] \). In this case, \( v_{i_0,i'} = x_{i_0,i'} \cdot \tilde{t}_{j',\ell'} - y_{j'} \cdot t_{j',\ell'} \) where \( y_{j'} = 2^{j'+1} \leq 2^{j'+1} \) and \( x_{i'} \geq 2^{j'/2} \); see Lemma 9.10. This and \( t_{j',\ell'} \leq 1/10 \) for \( \ell' \in J^{(j')}_{\text{special}} \) allows us to conclude that

\[ v_{i_0,i'}^- \geq 2^{j'-2} \cdot \tilde{t}_{j',\ell'} - 2^{j'+1} \cdot t_{j',\ell'} \geq 2^{j'-2} \cdot \left(1 - t_{j',\ell'} \cdot 2^{-\beta_j} \cdot t_{j',\ell'}\right) \geq 2^{j'-2} \cdot \frac{1}{10} > 0. \]

Therefore, we see that \( 1 + |\eta_2| \geq \eta_2 \geq 2^{-\beta_j} \cdot v_{i_0,i'}^- \geq 2^{j'-\beta_j}/40 \), as desired.

**Case 2:** \( \theta_{i_0,i_0'} \in [\pi, 2\pi] \). In this case, \( -v_{i_0,i'}^+ = x_{i_0,i'} \cdot \tilde{t}_{j',\ell'} - y_{j'} \cdot t_{j',\ell'} \). Precisely as in the previous case, we note that \( -v_{i_0,i'}^+ \geq 2^{j'-2}/10 > 0 \). This and \( \eta_2 \leq 2^{-\beta_j} \cdot v_{i_0,i'}^+ \), leads to

\[ 1 + |\eta_2| \geq -\eta_2 \geq 2^{-\beta_j} \cdot (-v_{i_0,i'}^+) \geq 2^{j'-\beta_j}/40. \]

Since \( \kappa_2 \geq 2 + \kappa_2^{(0)} \) (see Equation (9.8)), Lemma 9.14 shows that, for \( \ell' \in J^{(j')}_{\text{special}} \) and \( m' \in \mathbb{N}_0 \) with \( m' \leq m_{j'}^\max \),

\[ \int_{I_2^{(i,\ell')}} (1 + |\eta_2|)^{-\kappa_2} d\eta_2 \leq \left( \frac{2^{j' - \beta_j}}{40} \right)^{-\kappa_2^{(0)}} \cdot \int_{\mathbb{R}} (1 + |\eta_2|)^{-2} d\eta_2 \leq 2 \cdot \left( \frac{2^{j' - \beta_j}}{40} \right)^{-\kappa_2^{(0)}}. \]
Likewise, since $\kappa_1 \geq 2$, \( \int_{I_1(i',i)} \left(1 + |\eta_i| \right)^{-\kappa_1} \, d\eta_i \leq \int_{\mathbb{R}} \left(1 + |\eta_i| \right)^{-2} \, d\eta_i \leq 2 \). Furthermore,

\[
1 + 2^{\alpha j' - \beta j} |\sin \vartheta_{i_0,i_0'}| \leq 1 + 2^{(\alpha - \beta) j'} \cdot 2^{\beta (j' - j)} \leq 2 \cdot 2^{\beta (j' - j)} \cdot 2^{(\alpha - \beta) j'} .
\]

Finally,

\[
2^{j' - \beta j} = 2^{(1 - \beta) j'} \cdot 2^{\beta (j' - j)} \geq 2^{(1 - \beta) j'} \cdot 2^{-\beta (j' - j)} .
\]

Combined with the definition of $M_{i,i'}^{(j)}$ (see Equation (9.28)), these estimates imply that, for any index $i' = (j', m', \ell') \in I_0$ with $\ell' \in J_{\text{special}}^{(j')}$,

\[
M_{i,i'}^{(j)} \leq 2^{2 + 2 \tau} \cdot 40 \tau \kappa_0 \cdot 2 \beta (j' - j) +, 2 \beta \kappa_2 (0) (j' - j) +, 2 \beta \tau \kappa_2 (1) - \beta j' .
\]

Since $1 + m_{j'}^{\text{max}} \leq 3 \cdot 2^{(1 - \beta) j'}$ and $1 + m_{j'}^{\text{max}} \leq 3 \cdot 2^{(1 - \alpha) j'}$ as well, combining this with Equations (9.13) and (9.26) results in

\[
\sum_{\ell' \in J_{\text{special}}^{(j')}} \sum_{m' = 0}^{m_{j'}^{\text{max}}} M_{i,i'}^{(j)} \leq N \cdot 2^{4 + 3 \sigma + 2 \tau} + 5 \tau \kappa_0 + 6 \tau \kappa_0 \cdot 2 \sigma \kappa_2 (0) (j' - j) +, 2 \tau \kappa_2 (1) - \beta j' .
\]

for each fixed $j' \in \mathbb{N}$. But by definition of $\kappa_2 (0)$ and by our choice of $\kappa_0$ (see Equations (9.8) and (9.9)), we see that $\omega \beta \sigma (j' - j) +, 2 \beta \kappa_2 (0) (j' - j) +, 2 \beta \tau \kappa_2 (0) (j' - j) +, 2 \beta \tau \kappa_2 (1) - \beta j'$ is so small that the first step for estimating the remaining series is to further estimate the intervals $I_1^{(i', i')}$ and $I_2^{(i', i')}$, which we shall do in the following two lemmata.

**Lemma 9.15.** Let $j' \in \mathbb{N}$ and $\ell' \in J_{\text{special}}^{(j')}$ be arbitrary and define

\[
\beta_{j', \ell'} := \beta_{j', \ell'} \cdot 2^{\alpha (j' - j)} \quad \text{and} \quad L_{j'}^* := 4 \cdot 2^{\alpha (j' - j)} .
\]

Then there are such $s_{j', \ell'} \in \{ \pm 1 \}$ and $S_{j', \ell'} \in \mathbb{R}$ that

\[
I_1^{(i', i')} \subseteq \left\{ s_{j', \ell'} \cdot \beta_{j', \ell'} \cdot m' + S_{j', \ell'} - L_{j'}^* \right\}, \quad s_{j', \ell'} \cdot \beta_{j', \ell'} \cdot m' + S_{j', \ell'} + L_{j'}^*
\]

for all $m' \in \mathbb{N}_0$ with $i' = (j', m', \ell') \in I_0$.

**Proof.** Directly from the definition of $\bar{x}_{i'}$, we see that $|x_{i'}^+ - (2^{j' - 1} + m' \cdot 2^{\alpha j'})| \leq 2 \cdot 2^{\alpha j'}$. Therefore, defining

\[
s_{j', \ell'} := \begin{cases} -1 & \text{if } \vartheta_{i_0,i_0'} \in [\frac{3}{2}, \frac{5}{2} \pi), \\ 1 & \text{otherwise} \end{cases}
\]

and recalling the definitions of $u_{i_0,i_0'}^{\pm}$ and $t_{j', \ell'}^{\pm}$, we conclude that

\[
|s_{j', \ell'} \cdot \beta_{j', \ell'} \cdot m' + S_{j', \ell'} - L_{j'}^*| \leq 2 \cdot 2^{\alpha j'} + |y_{j'}| \leq 4 \cdot 2^{\alpha j'},
\]

where we noted in the last step that $\beta \leq \alpha$ and that $y_{j'} = 2^{\beta j' + 1}$; see Lemma 9.16.

By the definition of $I_1^{(i', i')}$, we thus see that, for

\[
S_{j', \ell'} := -2^{(1 - \alpha) j' - 1} - m + s_{j', \ell'} \cdot t_{j', \ell'} \cdot 2^{j' - \alpha j' - 1},
\]

\[
I_1^{(i', i')} = 2^{\alpha j'} \cdot [u_{i_0,i_0'}^+, 2^{j' - 1} + m \cdot 2^{\alpha j'} - u_{i_0,i_0'}^-, - 2^{j' - 1} - m \cdot 2^{\alpha j'}]
\]

\[
= 2^{\alpha j'} \cdot \left( -2^{j' - 1} - m \cdot 2^{\alpha j'} + s_{j', \ell'} \cdot t_{j', \ell'} \cdot 2^{j' - 1} + m' \cdot 2^{\alpha j'} + [4 \cdot 2^{\alpha j'}, 4 \cdot 2^{\alpha j'}] \right)
\]

\[
= [s_{j', \ell'} \cdot \beta_{j', \ell'} \cdot m' + S_{j', \ell'} - L_{j'}^* , s_{j', \ell'} \cdot \beta_{j', \ell'} \cdot m' + S_{j', \ell'} + L_{j'}^*].
\]

\[\square\]
Lemma 9.16. Let us fix $\iota \in \{0, 1, 2, 3\}$, $k \in \{-1, 0, 1, 2\}$ and $j' \in \mathbb{N}$ and define
\[
J_{j'}^{\iota, k} := \left\{ \ell' \in \mathbb{N}_0 : \ell' \leq \ell_{j'}^{\max} \text{ and } \vartheta_{i_0, i_0'k} \in \iota \cdot \frac{\pi}{2} + \left[ 0, \frac{\pi}{2} \right] \text{ and } k_{i_0, i_0'} = k \right\},
\]
with $k_{i_0, i_0'}$ as in Equation (9.12).

Furthermore, let us define
\[
\beta_1 := N^{-1} \cdot 2^{\beta(j'-j)} \quad \text{ and } \quad \beta_2 := \frac{8\pi}{N} \cdot 2^{\beta(j'-j)} \quad \text{ and } \quad L_{j'} := 2 \cdot 2^{\beta(j'-j)}.
\]

Then there are such $S_{k, i, j'} \in \mathbb{R}$ and $\nu_i \in \{\pm 1\}$ that:

1. For any $m' \in \mathbb{N}_0$ and $\ell' \in J_{j'}^{\iota, k}$ with $\ell' = (j', m', \ell') \in I_0$,
\[
\begin{cases}
\{i_2^{(t, r)} \in [\beta_1 \cdot (S_{k, i, j'} + \nu_i \ell') - L_{j'}, \beta_2 \cdot (S_{k, i, j'} + \nu_i \ell') + L_{j'}] \}, \text{ if } \iota \in \{0, 1\}, \\
\{i_2^{(t, r)} \in [\beta_2 \cdot (S_{k, i, j'} + \nu_i \ell') - L_{j'}, \beta_1 \cdot (S_{k, i, j'} + \nu_i \ell') + L_{j'}] \}, \text{ if } \iota \in \{2, 3\}.
\end{cases}
\]

2. For any $\ell' \in J_{j'}^{\iota, k}$,
\[
\begin{cases}
S_{k, i, j'} + \nu_i \ell' \geq 0, \text{ if } \iota \in \{0, 1\}, \\
S_{k, i, j'} + \nu_i \ell' \leq 0, \text{ if } \iota \in \{2, 3\}.
\end{cases}
\]

3. Finally $2^{\alpha j' - \beta j} \cdot |\sin \vartheta_{i_0, i_0'}| \leq 2\pi \cdot \beta_1 \cdot |S_{k, i, j'} + \nu_i \ell'|$ for all $\ell' \in J_{j'}^{\iota, k}$.

Proof. See Appendix C.

Given Lemmas 9.15 and 9.16, we can finally show that the second supremum in (9.5) is finite provided that Conditions (9.8) and (9.9) are satisfied.

For the moment, we fix $j' \in \mathbb{N}$ and define $\langle x \rangle := 1 + |x|$ for $x \in \mathbb{R}$. Recall that we already estimated the series as $\ell' \in J_{j'}^{\iota, k}$ special; see Equation (9.31). Therefore, it suffices to consider
\[
\sum_{\ell' \in J_{j'}^{\iota, k} \cap J_{j'}^{\iota, k}} \sum_{m' = 0}^{m_{j'}^{\max}} \sum_{m' = 0}^{m_{j'}^{\max}} M_{i, i'}^{(3)} = 3 \sum_{\iota = 0}^{2} \sum_{k = -1}^{2} \sum_{\ell' \in J_{j'}^{\iota, k} \cap J_{j'}^{\iota, k}} \sum_{m' = 0}^{m_{j'}^{\max}} M_{i, i'}^{(3)}.
\]

Therefore, let us fix $\iota \in \{0, \ldots, 3\}$ and $k \in \{-1, \ldots, 2\}$ for the moment. Let $\beta_1, \beta_2, S_{k, i, j'}, \nu_i, L_{j'}$ as in Lemma 9.16. If $\iota \in \{0, 1\}$, let us define $\beta_1 := \beta_1$ and $\beta_2 := \beta_2$; otherwise, if $\iota \in \{2, 3\}$, let us define $\beta_1 := \beta_2$ and $\beta_2 := \beta_1$ instead. By definition of $M_{i, i'}^{(3)}$ and by Lemmas 9.15 and 9.16 we see that, for $\ell' \in J_{j'}^{\iota, k}$ and $0 \leq m' \leq m_{j'}^{\max}$ with $\beta_{j', \iota', L_{j'}^*}$ and $s_{j', \iota', S_{j', \iota'}}$ as in Lemma 9.15
\[
M_{i, i'}^{(3)} \leq \left( 1 + 2\pi \beta_1 \cdot |S_{k, i, j'} + \nu_i \ell'| \right)^{\sigma} \left( \int_{\beta_2 \cdot (S_{k, i, j'} + \nu_i \ell') - L_{j'}}^{\beta_1 \cdot (S_{k, i, j'} + \nu_i \ell') + L_{j'}} (\eta_2)^{-\kappa_2} d\eta_2 \right)^{\tau} \left( \int_{s_{j', \iota', \nu_i \ell'} - L_{j'}}^{s_{j', \iota', \nu_i \ell'} + L_{j'}^*} (\eta_1)^{-\kappa_1} d\eta_1 \right).
\]

But with $S_{j', \iota'} := s_{j', \iota', \nu_i \ell'}$, the change of variables $\xi = s_{j', \iota', \nu_i \ell'}$ results in
\[
\int_{s_{j', \iota', \nu_i \ell'} - L_{j'}}^{s_{j', \iota', \nu_i \ell'} + L_{j'}^*} (\eta_1)^{-\kappa_1} d\eta_1 = \int_{s_{j', \iota', \nu_i \ell'} - L_{j'}}^{s_{j', \iota', \nu_i \ell'} + L_{j'}^*} (\xi)^{-\kappa_1} d\xi.
\]

Since $\kappa_1 \geq \frac{2}{3}$, Lemma 9.7 — applied with $N = 0$, $\beta_0 = \beta_{j', \iota'}$, and $L = L_{j'}^* = 4 \cdot 2^{\alpha (j'-j)} —$ gives the estimate
\[
\sum_{m' \in \mathbb{Z}} \left( \int_{s_{j', \iota', \nu_i \ell'} - L_{j'}}^{s_{j', \iota', \nu_i \ell'} + L_{j'}^*} (\xi)^{-\kappa_1} d\xi \right)^{\tau} \leq 2^{3+\tau} \cdot 10^3 \cdot (L_{j'}^*)^\tau \cdot 2 \cdot (1 + \beta_{j', \iota'} - 1 \cdot (1 + L_{j'}^*)) \leq 2^{20+3\tau} \cdot 2^{2\alpha (j'-j)} \cdot 2^{\alpha (j'-j)+} := \Gamma_{j'}^{(1)}.
\]
Here, we noted that $\beta_{j',\ell'} = t_{j',\ell'} \cdot 2^\alpha(j'-j)$ where $t_{j',\ell'} \geq 1/10$ since $\ell' \in J_{normal}^{(j')}$.

Next, we note that

$$(1 + 2\pi \cdot \beta_1 \cdot |S_{k,i,j'} + \nu_i \ell'|)^2 \leq 2^\rho \cdot \sum_{E \in \{0,\sigma\}} |2\pi \beta_1 (S_{k,i,j'} + \nu_i \ell'|)^E.$$ 

Let us fix $E \in \{0,\sigma\}$ for the moment and define $\varepsilon := 1$ if $\ell \in \{0,1\}$ and $\varepsilon := -1$ otherwise. Then Lemma 9.16 shows that $\varepsilon \cdot (S_{k,i,j'} + \nu_i \ell') \geq 0$ for $\ell' \in J_{normal}^{(j')}$. Furthermore, the change of variable $\lambda = \nu_i \ell'$ combines with an application of Lemma 9.8 or Corollary 9.9 — depending on whether Lemma 9.16 shows that $\varepsilon = 1$ or $\varepsilon = -1$ — with $N = E \leq \sigma$, $\gamma = 2\pi \beta_1$, and $L = L_{j'}$ shows that, for $\Gamma_j^{(3)} := 2^{3+\sigma+\tau + \tau} \varepsilon$,

$$\sum_{\ell' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} |2\pi \beta_1 (S_{k,i,j'} + \nu_i \ell'|)^E \left( \frac{1}{\beta_1 (S_{k,i,j'} + \nu_i \ell') - L_{j'}} \right)^{\tau} \eta \leq \Gamma_j^{(2)} \cdot 2^5 + 3 \cdot 2^{\beta(j'-j)_+ + \langle \tau + \sigma \rangle(j'-j)_+}.$$ 

Summarising all these estimates, we finally see that

$$\sum_{\ell' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} \sum_{m'=0}^{m_{\max}} M_{i,\ell'}^{(3)} \leq 4 \cdot 4 \cdot 2^\sigma \cdot 2 \cdot \Gamma_j^{(1)} \cdot \Gamma_j^{(3)} \leq \Gamma_j^{(4)},$$

from where a straightforward but tedious calculation shows that one can choose

$$\Gamma_j^{(4)} := 2^{33+6\sigma+(10+\kappa_0)\tau} \cdot N \cdot 2^{\alpha(j-j')_+ + (\tau + \sigma)(j-j')_+ + \omega_{j'} \cdot \Gamma_j^{(4)}}.$$ 

Finally, by recalling Equations (9.13) and (9.26), we see that

$$\sum_{j'=1}^\infty \sum_{\ell' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} \sum_{m'=0}^{m_{\max}} M_{i,\ell'}^{(1)} \leq 2^\sigma \cdot 2^{\tau} \cdot 2 \cdot \Gamma_j^{(4)} \cdot \Gamma_j^{(4)} \leq \Gamma_j^{(6)} \cdot \sum_{j'=1}^\infty \sum_{\ell' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} \sum_{m'=0}^{m_{\max}} \omega_{j',\ell'} \cdot \Gamma_j^{(4)}$$

with

$$\tilde{\omega}_{j',\ell'} = \alpha[(j-j')_+ + \tau(j'-j)] + \beta[(j-j')_+ + (\tau + \sigma)(j'-j)_+] + \omega_{j',\ell'} \quad \text{and} \quad \Gamma_j^{(6)} := 2^{33+8\sigma+(10+\kappa_0+2\kappa_2)\tau} N.$$ 

A direct calculation shows that

$$\tilde{\omega}_{j',\ell'} = \begin{cases} (j'-j) \cdot (1 - \alpha) \tau \cdot \frac{(1)_{\ell} - \kappa_0}{(1 - \alpha)\tau}, & \text{if } j \leq j' \\
(j'-j) \cdot (1 - \alpha) \tau \cdot \frac{(1)_{\ell} + \kappa_0}{(1 - \alpha)\tau}, & \text{otherwise}. \end{cases}$$

Given our choice of $\kappa_0$ (see Equation 9.9) and recalling that $\alpha, \beta \leq 1$, we therefore see that $\tilde{\omega}_{j',\ell'} \leq -[j'-j]$, and thus

$$\sum_{j'=1}^\infty \sum_{\ell' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} \sum_{m'=0}^{m_{\max}} M_{i,\ell'}^{(1)} \leq \Gamma_j^{(6)} \cdot \sum_{j'=1}^\infty \sum_{j' \in \ell', \varepsilon(S_{k,i,j'} + \nu_i \ell') \geq 0} 2^{-[j'-j]} \leq 3 \cdot \Gamma_j^{(6)} < \infty.$$ 

Combining this with Equation (9.31) finally results in

$$\sum_{i \in I_0} M_{i,i}^{(1)} \leq 3 \cdot \Gamma_j^{(6)} + N \cdot 2^{6+3\sigma+2\tau+5\tau} \cdot 6 \cdot \Gamma_j^{(6)} := \Gamma_j^{(7)} \quad \forall i \in I_0.$$
9.4. Estimating the contribution of the low-pass part

In this subsection, we estimate the series $\sum_{i \in I} M_{i,0}^{(1)}$ and $\sum_{i' \in I} M_{0,i'}^{(1)}$ where $M_{i,0}^{(1)}$ and $M_{0,i'}^{(1)}$ are as defined in Equation (9.7).

We first estimate $\sum_{i' \in I_0} M_{0,i'}^{(1)}$. To this end, we recall that, for $i' = (j', m', \ell') \in I_0$, $T_{i'} = R_{j', \ell'} A_{j'}$. Since the rotation matrix $R_{j', \ell'}$ does not change the norm and since $A_{j'} = \text{diag}(2^{\alpha j'}, 2^{\beta j'})$ we conclude that $\|T_0^{-1} T_{i'}\| = \|A_{j'}\| = 2^{\alpha j'}$, since $\beta \leq \alpha$. Furthermore, Lemma 4.1 shows that $1 + |\xi| \geq |\xi| \geq 2^{j'-2}$ for all $\xi \in Q_{i'}$ and hence

$$\psi(T_0^{-1}(\xi - b_{0})) = \psi(\xi) \leq (2^{j'-2} - \kappa_0) \leq 2^{2\kappa_0} \cdot 2^{-\kappa_0} \leq 2^{2\kappa_0} \cdot 2^{-((1-\alpha)\kappa_0)} \quad \forall \xi \in Q_{i'}.$$ 

Overall, these considerations imply

$$M_{0,i'}^{(1)} \leq 2^{2+2\kappa_0} \cdot 2^{j'((\alpha\sigma-s)-(1-\alpha)\kappa_0)}.$$ 

Finally, we note that $1 + m_j^{\max} \geq 3 \cdot 2^{(1-\alpha)j'} \leq 3 \cdot 2^{j'}$ and $1 + \ell_j^{\max} \leq 3N \cdot 2^{(1-\beta)j'} \leq 3N \cdot 2^{j'}$ and thus

$$\sum_{i' \in I_0} M_{0,i'}^{(1)} \leq 9N \cdot 2^{2+2\kappa_0} \cdot \sum_{j'=1}^{\infty} 2^{j'(2+\alpha\sigma-s-(1-\alpha)\kappa_0)} \leq N \cdot 2^{4+\sigma+2\kappa_0}.$$ 

Here, in the last step we noted that $2 + \alpha\sigma - s - (1-\alpha)\kappa_0 \leq -1$ thanks to our assumptions regarding $\kappa_0$; see Equation (9.9).

Next, we estimate $\sum_{i \in I_0} M_{i,0}^{(1)}$. As above, we see that, for $i = (j, m, \ell) \in I_0$,

$$\|T_i^{-1} T_0\| = \|T_i^{-1}\| = \|A_j^{-1}\| = \max\{2^{-\alpha j}, 2^{-\beta j}\} \leq 1.$$ 

In particular, this implies that, for $\xi \in Q_0 = B_4(0), |T_i^{-1} \xi| \leq \|T_i^{-1}\| \cdot |\xi| \leq 4$. Furthermore,

$$T_i^{-1} b_i = A_j^{-1} R_{j, \ell}^{-1} R_{j, \ell} c_{j,m} = A_j^{-1} c_{j,m} = \text{diag}(2^{-\alpha j}, 2^{-\beta j}) \cdot (2^{j-1} + m \cdot 2^{\alpha j}, 0)^t = (2^{(1-\alpha)j-1} + m, 0)^t.$$ 

Combining this results in

$$2^{(1-\alpha)j-1} \leq 1 + |T_i^{-1} b_i| \leq 1 + |T_i^{-1} (b_i - \xi)| + |T_i^{-1} \xi| \leq 5 \cdot (1 + |T_i^{-1}(\xi - b_i)|)$$

and thus

$$\psi(T_i^{-1}(\xi - b_i)) \leq (1 + |T_i^{-1}(\xi - b_i)|)^{-\kappa_0} \leq 2^{4\kappa_0} \cdot 2^{-(1-\alpha)\kappa_0 j}.$$ 

Overall, we have thus shown that

$$M_{i,0}^{(1)} \leq 2^{2+4\kappa_0} \cdot 2^{(s-\tau\kappa_0(1-\alpha))}.$$ 

Noting once again that $1 + m_j^{\max} \geq 3 \cdot 2^{(1-\alpha)j} \leq 3 \cdot 2^{j}$ and $1 + \ell_j^{\max} \leq 3N \cdot 2^{(1-\beta)j} \leq 3N \cdot 2^{j}$, we finally see that

$$\sum_{i \in I_0} M_{i,0}^{(1)} \leq 9N \cdot 2^{2+4\kappa_0} \cdot \sum_{j=1}^{\infty} 2^{j(2+s-\tau\kappa_0(1-\alpha))} \leq N \cdot 2^{4+\sigma+4\kappa_0}$$

since our assumptions regarding $\kappa_0$ imply that $2 + s - \tau\kappa_0(1-\alpha) \leq -1$; see Equation (9.9).

Finally, since $\psi \leq 1$, we see $M_{0,0}^{(1)} \leq 2^\sigma$. Combining this with the preceding estimates from this subsection, we conclude that

$$\sum_{i \in I} M_{i,0}^{(1)} \leq N \cdot 2^{5+\sigma+4\kappa_0} \quad \text{and} \quad \sum_{i' \in I} M_{0,i'}^{(1)} \leq N \cdot 2^{5+\sigma+2\kappa_0}.$$ 

(9.33)

Concluding the proof of Equation (9.5). Combing the estimates (9.29), (9.32) and (9.33), we finally see that Equation (9.5) is satisfied for $B$ as in (9.10).
9.5. Proving Theorems 9.3 and 9.4

Proof of Theorem 9.3. We shall derive the claims by applying Theorem 2.10 (with \( \omega \) instead of \( \varepsilon \)). To this end, let us choose \( Q_1^{(0)} := Q = (-\varepsilon, 1 + \varepsilon) \times (-1 - \varepsilon, 1 + \varepsilon) \) and \( Q_2^{(0)} := B_2(0) \). Furthermore, let \( k_i := 1 \) for \( i \in I_0^{(\alpha, \beta)} \) and \( k_0 := 2 \). With \( T_i, b_i \) as in Lemma 5.1, \( Q_i = T_i Q_k^{(0)} + b_i \) for all \( i \in I^{(\alpha, \beta)} \), as required at the beginning of Section 2.2.

Furthermore, in the notation of Theorem 2.10, let us define \( \gamma_1^{(0)} := \gamma \) and \( \gamma_2^{(0)} := \varphi \). Still in the notation of Theorem 2.10, let us define

\[
\varrho_k : \mathbb{R}^2 \to (0, \infty), \xi \mapsto C \cdot (1 + |\xi|)^{-\kappa_0} \cdot (1 + |\xi_1|)^{-\kappa_1} \cdot (1 + |\xi_2|)^{-\kappa_2}
\]

for \( k \in \{1, 2\} \), noting that \( \varrho_k = C \cdot \psi \) with \( \psi \) as in Equation (9.4). Finally, we choose \( \tau, \vartheta \), and \( \sigma \) as in Theorem 2.10 and note that \( N_0 \) as defined in Theorem 9.3 satisfies \( N_0 \geq N \) for \( N \) as in Theorem 2.10.

It is then not hard to see that the wave packet system \( (L_{\xi k} \gamma_k^{(0)})_{\xi,k \in \mathbb{Z}^2} \) introduced in Definition 9.1 coincides with the system \( \Gamma^{(\delta)} \) from Equation (2.5). Furthermore, the assumptions of Theorem 9.3 imply that the first three assumptions of Theorem 2.10 are satisfied. In addition, since we are working in dimension \( d = 2 \), so that \( d + 1 + \omega \leq 4 \) and given our choice of \( \varrho_1, \varrho_2 \), Equation (9.1) shows that the fourth assumption of Theorem 2.10 is fulfilled.

Therefore, it remains to verify the last condition in Theorem 2.10, namely that the constants \( K_1, K_2 \) introduced in Equation (2.7) are finite. To this end, we first show that the entries \( N_{i,j} \) of the infinite matrix \((N_{i,j})_{i,j \in \mathbb{N}}\) can be estimated in terms of the numbers \( M_{i,j}^{(1)} \) from Equations (9.6) and (9.7). To see this, first recall that

\[
|\det T_i| = |\det \text{diag}(2^{a_j}, 2^{b_j})| = 2^{(a_j + b_j)} \quad \forall (j, m, \ell) \in I_0^{(\alpha, \beta)}
\]

and hence

\[
|\det T_i| = 2^{(a_j + b_j)(j' - j)} \quad \forall i = (j, m, \ell), i' = (j', m', \ell') \in I_0^{(\alpha, \beta)}.
\]

It should be noted that Equation (9.34) also implies that the coefficient space introduced in Definition 9.2 coincides with the one in Definition 2.9 with identical quasi-norms.

Furthermore, since \( Q = (-\varepsilon, 1 + \varepsilon) \times (-1 - \varepsilon, 1 + \varepsilon) \subset (-2, 2)^2 \), we see that

\[
\lambda(Q_i) = \lambda(T_i Q + b_i) = |\det T_i| \cdot \lambda(Q) \leq 2^6 \cdot |\det T_i| \quad \forall i \in I_0^{(\alpha, \beta)}.
\]

Therefore, \( |\det T_i|^{-1} \int_{Q_i} f d\xi = \frac{\lambda(Q_i)}{|\det T_i|} \int_{Q_i} f d\xi \leq 2^6 \cdot \int_{Q_i} f d\xi \) for every non-negative measurable function \( f : \mathbb{R}^2 \to [0, \infty) \) and each \( i \in I_0^{(\alpha, \beta)} \).

By comparing the definition of \( N_{i,j} \) (where \( w = w^* \)) in Theorem 2.10 with that of \( M_{i,j}^{(1)} \), and by using the observations from the two preceding paragraphs, it is not hard to see that

\[
N_{i,j} \leq C^{\tau} \cdot 2^{6\sigma} \cdot M_{i,j}^{(1)} \quad \forall i, j \in I_0^{(\alpha, \beta)} \quad \text{where } M_{i,j}^{(1)} \text{ is defined using } \tau[\vartheta(\alpha + \beta) - s] \text{ instead of } s.
\]

Since \( \lambda(B_2(0)) = \pi \cdot 4^2 \leq 2^6 \), it is not hard to see that this estimate remains valid for all \( i, j \in I^{(\alpha, \beta)} \). Let us define \( \bar{s} := \tau[\vartheta(\alpha + \beta) - s] \).

Therefore, if we prove that \( \kappa_0, \kappa_1, \kappa_2 \) in Theorem 9.3 satisfy the conditions of Theorem 9.5, we shall also prove that, for any \( \ell \in \{1, 2\} \), the constant \( K_\ell \) defined in Equation (2.7) satisfies

\[
K_\ell^{1/\tau} \leq 2^6 C \cdot \left( \sup_{i \in I^{(\alpha, \beta)}} \sum_{j \in I^{(\alpha, \beta)}} M_{i,j}^{(1)} \right)^{1/\tau} \leq 2^6 C \cdot \left( N \cdot 2^{37 + 8\sigma + \tau(10 + 5\kappa_0 + 6\kappa_2 + 9)} \right)^{1/\tau} \leq C \cdot 2^{24 + 11/\tau_0 + 5\kappa_0 + 6\kappa_2 + 12/((1 - \beta)\tau_0) + 9}/p_0,
\]

where we noted that

\[
\tau = \min\{1, p, q\} \geq \tau_0 := \min\{p_0, q_0\} \quad \text{and} \quad \vartheta = (p^{-1} - 1) + \leq p_0^{-1} \leq \tau_0^{-1},
\]

so that \( 1/\tau \leq 1/\tau_0 \). Furthermore, we noted that

\[
\frac{\sigma}{\tau} \leq \frac{2}{p_0} + [p_0^{-1}(2 + \omega)] =: \sigma_0 \leq 1 + \frac{5}{p_0},
\]

(9.36)
which follows directly from the definition of \( \sigma \) in Theorem \( 2.10 \) by recalling that we use \( \omega \) instead of \( \varepsilon \) and that \( d = 2, p \geq p_0 \) and \( \omega \leq 1 \). Finally, we also invoked the estimate

\[
\kappa_2^{(0)} \leq \frac{2}{(1 - \beta)\tau_0} + \sigma_0,
\]

(9.37)

which can be obtained directly from (9.8), given (9.36) and recalling that \( \alpha - \beta \leq 1 - \beta \). Since the right-hand side of the estimate above only depends on \( \alpha, \beta, p_0, q_0, s, C \), Theorem \( 2.10 \) finally yields the claim.

Overall, it remains to verify that the choice of \( \kappa_0, \kappa_1, \kappa_2 \) in Theorem \( 9.3 \) satisfy the assumptions of Theorem \( 9.5 \) where \( \tilde{s} \) is used instead of \( s \). To see this, we first of all note that indeed \( \kappa_1 = \frac{2}{\tau_0} \geq \max \{2, \frac{2}{\tau} \} \). Second, from Equations (9.37) and (9.36) we infer that

\[
2 + \kappa_2^{(0)} = 2 + \frac{2}{(1 - \beta)\tau_0} + \sigma_0 \leq 3 + \frac{5}{p_0} + \frac{2}{(1 - \beta)\tau_0} = \kappa_2
\]

and furthermore \( 1 + \frac{\sigma + 2}{\tau} \leq 1 + \frac{2}{\tau_0} + \sigma_0 \leq 2 + \frac{2}{(1 - \beta)\tau_0} + \frac{5}{p_0} \leq \kappa_2 \), as required in Theorem \( 9.5 \).

Regarding \( \kappa_0 \), we note \( \left[ \frac{2}{\tau} \right] = |\vartheta(\alpha + \beta) - s| \leq s_0 + p_0 - 1(\alpha + \beta) \), which implies

\[
\frac{3 + \tilde{s}}{(1 - \alpha)\tau} + \frac{\tau + \alpha + (\alpha + \beta)\sigma}{(1 - \alpha)\tau} \leq (1 - \alpha)^{-1} \left( \frac{3}{\tau_0} + s_0 + \frac{\alpha + \beta}{p_0} + 1 + \frac{\alpha}{\tau_0} + (\alpha + \beta)\sigma_0 \right)
\]

\[
\leq (1 - \alpha)^{-1} \left( \frac{3 + \frac{3 + \alpha}{\tau_0} + s_0 + \frac{6\alpha + 6\beta}{p_0}}{(1 - \beta)\tau_0} \right) \leq \kappa_0.
\]

Finally, we see that \( \max \{\tau, \sigma\} \leq \max \{1, \sigma_0\} \leq 1 + \frac{5}{p_0} \) and hence

\[
\frac{2 + \tilde{s}}{(1 - \alpha)\tau} + \frac{\tau + \alpha + (\alpha + \beta)\sigma}{(1 - \alpha)\tau} \leq (1 - \alpha)^{-1} \left( \frac{2}{\tau_0} + s_0 + \frac{\alpha + \beta}{p_0} + \beta \left( \frac{2}{1 - \beta} + \frac{5}{p_0} \right) + (\alpha + \beta) \cdot \left( 1 + \frac{5}{p_0} \right) \right)
\]

\[
\leq (1 - \alpha)^{-1} \left( \frac{3 + \frac{2}{\tau_0} + s_0 + \frac{6\alpha + 11\beta}{p_0} + \frac{2\beta}{(1 - \beta)\tau_0}}{(1 - \beta)\tau_0} \right) \leq \kappa_0,
\]

as required in (9.9).

\( \square \)

**Proof of Theorem 9.4.** The proof is very similar to the one of Theorem 9.3 and therefore only sketched here.

Instead of Theorem 2.10, we use Theorem 2.11 but with \( \gamma_1^{(0)} := \tilde{\gamma} \) and \( \gamma_2^{(0)} := \tilde{\varphi} \) instead of \( \gamma_1^{(0)} = \gamma \) and \( \gamma_2^{(0)} = \varphi \) in the preceding proof; here we recall the notation \( \tilde{f}(x) = f(-x) \). To justify this choice of \( \gamma_1^{(0)}, \gamma_2^{(0)} \), we recall the elementary identity \( \mathcal{F} \tilde{f} = \tilde{f} \). With this, it is not hard to see that \( \tilde{\gamma} \) and \( \tilde{\varphi} \) also satisfy the assumptions (1)-(2) of Theorem 9.3 and (3') of Theorem 9.4 and thus assumptions (1)-(3) of Theorem 2.11.

Given our assumptions, it is not hard to verify — as in the proof of Theorem 9.3 — that the matrix elements \( M_{j,i} \) introduced in Theorem 2.11 satisfy

\[
M_{j,i} \leq C^\tau \cdot 2^{\theta\tau} \cdot M_{j,i}^{(1)} \quad \text{where } M_{j,i}^{(1)} \text{ is defined using } s\tau \text{ instead of } s \text{ and } \theta \text{ instead of } \sigma
\]

where \( \theta \) is as defined in Theorem 2.11. The remainder of the proof is then almost identical to that of Theorem 9.3 with one exception Namely we still need to verify Equation (9.3).

For this, to avoid confusion in notations of the family \( \gamma^{(i)} \) defined in Equation (2.6) with those introduced in Definition 9.1, let us write \( \gamma^{(i)} := |\det T_i|^{1/2} \cdot M_{b_i}(\gamma_i(T_i\bullet)) \) for the family defined in Equation (2.6). Now, we recall that \( \gamma_i = \gamma_{k_i} = \gamma_1^{(0)} = \tilde{\gamma} \) for \( i \in \mathcal{I}_0^{(\alpha, \beta)} \) and \( \gamma_0 = \gamma_{k_0} = \tilde{\varphi} \). Finally, we note that

\[
\overline{M_{j,g}} = M_{j,g} \tilde{g} \quad \text{for any measurable } g : \mathbb{R}^2 \to \mathbb{C} \text{ and any } \xi \in \mathbb{R}^2,
\]

which shows that

\[
\langle f, L_{\delta T_i^{-1}k} \overline{\gamma^{(i)}} \rangle = \langle f, |L_{\delta T_i^{-1}k} \overline{\gamma^{(i)}}| \rangle_{L^2} = \langle f, |\det T_i|^{1/2}. L_{\delta T_i^{-1}k} \left[ M_{b_i}(\overline{\gamma_i(T_i)}) \right] \rangle_{L^2} = \langle f, |L_{\delta T_i^{-1}k} \gamma^{(i)}| \rangle_{L^2}
\]
for all \( f \in L^2(\mathbb{R}^2), i \in I^{(\alpha, \beta)} \) and \( k \in \mathbb{Z}^2 \).

Therefore, Equation (2.11) finally shows that, for \( f \in L^2(\mathbb{R}^2) \cap \mathcal{W}_p^q(\alpha, \beta) \), the analysis map \( A^{(\delta)} := A_{\Gamma(\delta)} \) constructed in Theorem 2.11 satisfies

\[
A_{\Gamma(\delta)} f = \left( \langle f, L_{\delta T_i^{-k} \gamma^{(i)}} \rangle \right)_{i \in I, k \in \mathbb{Z}^2} = \left( \langle f | L_{\delta T_i^{-k} \gamma^{(i)}} \rangle_{L^2} \right)_{i \in I, k \in \mathbb{Z}^2},
\]
as desired.

\[\square\]

A. Proof of Lemma 9.8 and Corollary 9.9

For proving Lemma 9.8 we shall use the following auxiliary result.

Lemma A.1. For any \( \beta > 0 \) and \( x \in \mathbb{R} \),

\[
\sum_{k \in \mathbb{Z}} (1 + |\beta(k + x)|)^{-2} \leq 2 + \frac{10}{\beta} \leq 2^4 (1 + \beta^{-1}).
\]

Proof. First of all, we note that the function

\[ g : \mathbb{R} \rightarrow [0, \infty], x \mapsto \sum_{k \in \mathbb{Z}} (1 + |\beta(k + x)|)^{-2} \]

is periodic with period one. Therefore, to prove that \( g(x) \leq 2 + \frac{10}{\beta} \), it is enough to consider only the case when \( x \in [0, 1] \). We now distinguish three cases determined by the value of \( k \in \mathbb{Z} \).

Case 1: \( k \geq \frac{1}{\beta} \). This implies that \( \beta(k + x) \geq \beta k \geq 1 \), whence

\[
\sum_{k \geq \frac{1}{\beta}} (1 + |\beta(k + x)|)^{-2} \leq \sum_{k \geq \frac{1}{\beta}} (\beta k)^{-2} = \beta^{-2} \cdot \sum_{k \geq \frac{1}{\beta}} k^{-2}.
\]

Furthermore, if \( y > 0 \) and \( n \in \mathbb{Z}_{\geq y} \), then \( n \geq y > 0 \), whence \( n \geq 1 \) and \( n + 1 \leq 2n \). Therefore, if \( z \in [n, n + 1] \), then \( z^{-2} \geq (n + 1)^{-2} \geq (2n)^{-2} = n^{-2}/4 \) and thus

\[
\sum_{n \in \mathbb{Z}_{\geq y}} n^{-2} = \sum_{n \geq y} \sum_{n \geq y} n^{-2} dz \leq 4 \sum_{n \geq y} n^{-2} dz \leq 4 \cdot \int_{y}^{\infty} z^{-2} dz = \frac{4}{z-1} \bigg|_{z=y} = \frac{4}{y} \tag{A.1}
\]

and

\[ \sum_{k \geq \frac{1}{\beta}} (1 + |\beta(k + x)|)^{-2} \leq \beta^{-2} \cdot \sum_{k \geq \frac{1}{\beta}} k^{-2} \leq \beta^{-2} \cdot \frac{4}{1/\beta} = \frac{4}{\beta}. \]

Case 2: \( k \leq -\frac{1}{\beta} - 1 \), and hence \(- (k + 1) \geq \frac{1}{\beta} \). For any \( x \in [0, 1] \), this implies that

\[ \beta(k + x) \leq \beta(k + 1) \leq \beta \cdot \left( -\frac{1}{\beta} \right) = -1 < 0 \quad \text{and hence} \quad |\beta(k + x)| = -\beta(k + x) \geq -\beta(k + 1) > 0. \]

Therefore, we can again apply Equation (A.1) to obtain

\[
\sum_{k \in \mathbb{Z}_{\leq -\frac{1}{\beta} - 1}} (1 + |\beta(k + x)|)^{-2} \leq \sum_{k \in \mathbb{Z}_{\leq -\frac{1}{\beta} - 1}} (-\beta(k + 1))^{-2}
\]

\[
\text{(with } \ell = -(k + 1)) = \sum_{\ell \in \mathbb{Z}_{\geq \frac{1}{\beta}}} (\beta \ell)^{-2} \leq \beta^{-2} \cdot \frac{4}{1/\beta} = \frac{4}{\beta}.
\]

Case 3: \( -\frac{1}{\beta} - 1 \leq k \leq \frac{1}{\beta} \). This implies that \( k \in \mathbb{Z} \cap [-\frac{1}{\beta} - 1, \frac{1}{\beta}] \) and hence \( k \) can take at most \( 2 + \frac{2}{\beta} \) different values. Therefore,

\[
\sum_{-\frac{1}{\beta} - 1 \leq k \leq \frac{1}{\beta}} (1 + |\beta(k + x)|)^{-2} \leq 2 \left( 1 + \frac{1}{\beta} \right).
\]

Combining the three cases results in \( g(x) \leq 2 + \frac{4}{\beta} + 2 \left( 1 + \frac{1}{\beta} \right) = 2 + \frac{10}{\beta} \) for all \( x \in [0, 1] \) and, indeed, for all \( x \in \mathbb{R} \), since \( g \) is periodic with period 1. \( \square \)
We can now turn to the proof of Lemma 9.8.

**Proof of Lemma 9.8** Let

\[ I_k := [\beta_1 \cdot (k + M) - L, \beta_2 \cdot (k + M) + L] \]

for \( k \in \mathbb{Z} \) such that \( k + M \geq 0 \). The Lebesgue measure \( \lambda(I_k) \) of \( I_k \) is given by

\[ \lambda(I_k) = 2L + \frac{\beta_2 - \beta_1}{\beta_1} \cdot \beta_1 \cdot (k + M). \]  \hspace{1cm} (A.2)

We now distinguish two cases determined by the value of \( k \in \mathbb{Z} \).

*Case 1:* \( k \) is such that \( \beta_1 \cdot (k + M) \geq 2L > 0 \). Given (A.2), this implies that

\[ \lambda(I_k) \leq \beta_1 \cdot (k + M) \cdot \left[ 1 + \frac{\beta_2 - \beta_1}{\beta_1} \right] = \frac{\beta_2}{\beta_1} \cdot \beta_1 \cdot (k + M) \leq \frac{\beta_2}{\beta_1} \cdot (1 + |\beta_1 \cdot (k + M)|). \]

Furthermore, we note that each \( x \in I_k \) satisfies

\[ x \geq \beta_1 \cdot (k + M) - L \geq \frac{1}{2} \cdot \beta_1 \cdot (k + M) = \frac{1}{2} \cdot |\beta_1 \cdot (k + M)|. \]

Therefore,

\[ |f(x)| \leq C_0 \cdot (1 + |x|)^{-q} \leq C_0 \cdot \left( 1 + \frac{1}{2} |\beta_1 \cdot (k + M)| \right)^{-q} \leq C_0 \cdot 2^q \cdot (1 + |\beta_1 \cdot (k + M)|)^{-q}, \]

and

\[ |\gamma \cdot (k + M)|^N = \left( \frac{\gamma}{\beta_1} \right)^N \cdot |\beta_1 \cdot (k + M)|^N \leq \left( \frac{\gamma}{\beta_1} \right)^N \cdot (1 + |\beta_1 \cdot (k + M)|)^N. \]

Put together, this results in

\[ |\gamma \cdot (k + M)|^N \cdot \left( \int_{I_k} |f(x)| \, dx \right)^\tau \leq \left( \frac{\gamma}{\beta_1} \right)^N \left( \frac{\beta_2}{\beta_1} \right)^\tau \cdot C_0^\tau \cdot 2^q \cdot (1 + |\beta_1 \cdot (k + M)|)^{N+\tau(1-q)} \]

\[ \leq \left( \frac{\gamma}{\beta_1} \right)^N \left( \frac{\beta_2}{\beta_1} \right)^\tau \cdot C_0^\tau \cdot 2^q \cdot (1 + |\beta_1 \cdot (k + M)|)^{-2}, \]

where, in the last step, we recalled that \( q \geq 1 + \tau^{-1}(N + 2) \), whence \( N + \tau(1-q) \leq -2 \).

Finally, applying Lemma [A.1], we conclude that

\[ \sum_{k \in \mathbb{Z} \text{ with } \beta_1 \cdot (k + M) \geq 2L} \left[ |\gamma \cdot (k + M)|^N \cdot \left( \int_{I_k} |f(x)| \, dx \right)^\tau \right] \leq \left( \frac{\gamma}{\beta_1} \right)^N \left( \frac{\beta_2}{\beta_1} \right)^\tau \cdot C_0^\tau \cdot 2^q \cdot \sum_{k \in \mathbb{Z}} (1 + |\beta_1 \cdot (k + M)|)^{-2} \]

\[ \leq \left( \frac{\gamma}{\beta_1} \right)^N \left( \frac{\beta_2}{\beta_1} \right)^\tau \cdot C_0^\tau \cdot 2^q \cdot \left( 1 + \frac{1}{\beta_1} \right). \]

*Case 2:* \( k \) is such that \( 0 \leq \beta_1 \cdot (k + M) \leq 2L \), or equivalently \( 0 \leq k + M \leq 2 \cdot \frac{L}{\beta_1} \). Therefore, \( k \) can take at most \( 1 + 2 \cdot \frac{L}{\beta_1} \) different values.

Furthermore, from Equation (A.2) we infer that

\[ \lambda(I_k) = 2L + \frac{\beta_2 - \beta_1}{\beta_1} \cdot \beta_1 \cdot (k + M) \leq 2L + \frac{\beta_2 - \beta_1}{\beta_1} \cdot 2L = 2 \cdot \frac{\beta_2}{\beta_1} \cdot L. \]

Moreover, \( |f(x)| \leq C_0 \) and

\[ |\gamma \cdot (k + M)|^N \leq \left( \frac{\gamma}{\beta_1} \cdot \beta_1 \cdot (k + M) \right)^N \leq \left( \frac{\gamma}{\beta_1} \right)^N \cdot (2L)^N. \]
Put together, this results in
\[
\sum_{k \in \mathbb{Z}, \text{ with } k + M \geq 0} \left[ |\gamma \cdot (k + M)|^N \cdot \left( \int_{I_k} |f(x)| \, dx \right)^T \right] \leq \left( 1 + 2 \frac{L}{\beta_1} \right)^N \cdot \left( \frac{\gamma}{\beta_1} \right)^N \cdot (2L)^N \cdot C_0^\tau \cdot \left( 2 \cdot \frac{\beta_2}{\beta_1} \cdot L \right)^T.
\]

Finally, combining the estimates obtained in Cases 1 and 2 results in the claimed estimate
\[
\sum_{k \in \mathbb{Z}, \text{ with } k + M \geq 0} \left[ |\gamma \cdot (k + M)|^N \cdot \left( \int_{I_k} |f(x)| \, dx \right)^T \right] \leq 2^{4 + \tau + \tau q} \cdot \left( \frac{\beta_2}{\beta_1} \right)^N \cdot \left( \frac{\gamma}{\beta_1} \right)^N \cdot C_0^\tau \cdot (1 + L^{\tau + N}) \left( 1 + \frac{L + 1}{\beta_1} \right) .
\]

\[\square\]

**Proof of Corollary 9.9.** Let \( \tilde{\beta}_2 := \beta_1, \tilde{\beta}_1 := \beta_2, \tilde{\hat{M}} := -M \) and \( \tilde{f} : \mathbb{R} \to \mathbb{C}, x \mapsto f(-x) \). Together with \( f, \tilde{f} \) also satisfies (9.14). Furthermore, \( 0 < \tilde{\beta}_1 \leq \tilde{\beta}_2 \). Thus, after the substitution \( y = -x \) and the change of summation index \( \ell = -k \), we can apply Lemma 9.8 (with \( \tilde{\beta}_1, \tilde{\beta}_2 \) instead of \( \beta_1, \beta_2 \) and with \( \tilde{\hat{M}} \) instead of \( M \)) to obtain
\[
\sum_{k \in \mathbb{Z}, k + M \leq 0} \left[ |\gamma \cdot (k + M)|^N \cdot \left( \int_{\beta_1 \cdot (k + M) - L}^{\beta_2 \cdot (k + M) + L} |f(x)| \, dx \right)^T \right] = \sum_{k \in \mathbb{Z}, -\{(k + M) \leq 0} \left[ |\gamma \cdot ((-k + \tilde{\hat{M}})|^N \cdot \left( \int_{\beta_1 \cdot ((-k + \tilde{\hat{M}}) - L}^{\beta_2 \cdot ((-k + \tilde{\hat{M}}) + L} |\tilde{f}(y)| \, dy \right)^T \right]
\]
\[
= \sum_{\ell \in \mathbb{Z}, \ell + \tilde{\hat{M}} \geq 0} \left[ |\gamma \cdot (\ell + \tilde{\hat{M}})|^N \cdot \left( \int_{\beta_1 \cdot (\ell + \tilde{\hat{M}}) - L}^{\beta_2 \cdot (\ell + \tilde{\hat{M}}) + L} |\tilde{f}(y)| \, dy \right)^T \right]
\]
\[
\leq C \cdot \left( \frac{\tilde{\beta}_2}{\tilde{\beta}_1} \right)^T \cdot \left( \frac{\gamma}{\tilde{\beta}_1} \right)^N \cdot C_0^\tau \cdot (1 + L^{\tau + N}) \cdot \left( 1 + \frac{L + 1}{\beta_1} \right) ,
\]
which easily yields the claim. \[\square\]

**B. Estimates for Sine and Cosine**

In this appendix, we first state and prove linear bounds of the sine and cosine on the interval \([0, \pi]\). Second, we prove an elementary relation between the absolute values of sine and cosine. Finally, we prove a quadratic lower bound for the cosine. Even though these bounds are probably well-known, we prefer to provide a proof, since they play an important role in the proofs of Proposition 6.3 and of Lemmas 9.13 and 9.16 and thus in our proof of the existence of Banach frames and atomic decompositions for the wave packet smoothness spaces.

First, we show that
\[
\frac{2}{\pi} \cdot \phi \leq \sin \phi \leq \phi \quad \forall \phi \in [0, \pi].
\]

Indeed, the upper bound follows from the stronger estimate \(| \sin \phi | \leq | \phi | \) for all \( \phi \in \mathbb{R} \), which results from \(| \frac{d}{d\phi} \sin \phi | = | \cos \phi | \leq 1 \) combined with \( \sin 0 = 0 \).
To estimate the lower bound, we note that \( \frac{d}{d\phi} \sin \phi = -\sin \phi \leq 0 \) on \([0, \frac{\pi}{2}]\), so that the sine is concave on this interval. This together with \( \lambda := \frac{2}{\pi} \cdot \phi \in [0, 1] \) for \( \phi \in [0, \frac{\pi}{2}] \), implies, as claimed, that

\[
\sin \phi = \sin \left( (1 - \lambda) \cdot 0 + \lambda \cdot \frac{\pi}{2} \right) \geq (1 - \lambda) \cdot \sin 0 + \lambda \cdot \sin \frac{\pi}{2} = \lambda = \frac{2}{\pi} \cdot \phi.
\]

Next, we show that

\[
1 - \frac{2}{\pi} \cdot \phi \leq \cos \phi \leq \frac{\pi}{2} \cdot \left(1 - \frac{2}{\pi} \cdot \phi\right) \quad \forall \phi \in [0, \frac{\pi}{2}].
\]  

To see this, we recall that \( \cos \phi = \cos(-\phi) = \sin(\frac{\pi}{2} - \phi) \) and then apply Equation (B.1), noting that \( \frac{\pi}{2} - \phi \in [0, \frac{\pi}{2}] \) since \( \phi \in [0, \frac{\pi}{2}] \).

Next, we show that

\[
|\sin \phi| \geq 1 - |\cos \phi| \quad \text{and} \quad |\cos \phi| \geq 1 - |\sin \phi| \quad \forall \phi \in \mathbb{R}.
\]  

To see this, we recall that \( \sin^2 \phi + \cos^2 \phi = 1 \). Thus, it suffices to prove that

\[
|x| \geq 1 - |y| \quad \forall x, y \in \mathbb{R} \text{ with } x^2 + y^2 = 1.
\]

We note that \(|x|, |y| \leq 1 \) because \( x^2 + y^2 = 1 \). Thus, both sides of the desired inequality are non-negative, so that the inequality is equivalent to

\[
|x|^2 \geq (1 - |y|)^2 \iff x^2 \geq 1 - 2|y| + y^2 \iff 0 \geq 1 - x^2 - 2|y| + y^2 = 2(y^2 - |y|) = 2|y| \cdot (|y| - 1).
\]

This last estimate is satisfied since \(|y| \leq 1\).

Finally, we establish the quadratic lower bound

\[
\cos \theta \geq 1 - \frac{\theta^2}{2} \quad \forall \theta \in \mathbb{R}.
\]  

To prove this, we first of all note that both sides of the inequality are even functions, so that it is enough to consider the case \( \theta \geq 0 \). Next, we also note that \( \frac{d}{d\theta} \left( \frac{\theta^2}{2} + \cos \theta - 1 \right) = -\sin \theta \geq \theta - |\sin \theta| \geq \theta - |\theta| = 0 \) for all \( \theta \geq 0 \). Hence, we see that, as claimed, \( \frac{\theta^2}{2} + \cos \theta - 1 \geq \frac{\theta^2}{2} + \cos 0 - 1 = 0 \) for all \( \theta \geq 0 \). 

C. Proof of Lemmas 9.11, 9.13, and 9.16

Proof of Lemma 9.11. First of all, elementary properties of the sine and cosine imply that

\[
\cos \vartheta_{i_0,i_0'} = \begin{cases} 
\cos \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [0, \frac{\pi}{2}), \\
-\sin \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\frac{\pi}{2}, \pi), \\
\cos \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\pi, \frac{3\pi}{2}), \\
\sin \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\frac{3\pi}{2}, 2\pi).
\end{cases}
\]

and

\[
\sin \vartheta_{i_0,i_0'} = \begin{cases} 
\sin \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [0, \frac{\pi}{2}), \\
\cos \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\frac{\pi}{2}, \pi), \\
-\sin \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\pi, \frac{3\pi}{2}), \\
-\cos \theta_{i_0,i_0'}, & \text{if } \theta_{i_0,i_0'} \in [\frac{3\pi}{2}, 2\pi).
\end{cases}
\]

Next, Lemma 9.10 shows that, for \( \xi \in Q_{j',m',0}, x_{i_0}^+ \leq \xi_1 \leq x_{i_0'}^+ \) and \(|\xi_2| \leq y_{j'}\). Furthermore, by definition of \( R_{i_0,i_0'} \),

\[
(R_{i_0,i_0'} \xi)_1 = \xi_1 \cdot \cos \vartheta_{i_0,i_0'} - \xi_2 \cdot \sin \vartheta_{i_0,i_0'} \quad \text{and} \quad (R_{i_0,i_0'} \xi)_2 = \xi_1 \cdot \vartheta_{i_0,i_0'} + \xi_2 \cdot \cos \vartheta_{i_0,i_0'}.
\]

Finally, since \( \theta_{i_0,i_0'} \in [0, \pi/2) \), \( \sin \theta_{i_0,i_0'} \geq 0 \) and \( \cos \theta_{i_0,i_0'} \geq 0 \). By combining these observations, we see that (9.20) is true. Indeed, we distinguish four cases:

Case 1: \( \vartheta_{i_0,i_0'} \in [0, \frac{\pi}{2}) \). In this case, \( \vartheta_{i_0,i_0'} = \theta_{i_0,i_0'} \) and hence

\[
u_{i_0,i_0'}^{+} = x_{i_0'}^{+} \cdot \vartheta_{i_0,i_0'} - y_{j'} \cdot \sin \vartheta_{i_0,i_0'} \leq (R_{i_0,i_0'} \xi)_1 \leq x_{i_0'}^{+} \cdot \cos \theta_{i_0,i_0'} + y_{j'} \cdot \sin \theta_{i_0,i_0'} = u_{i_0,i_0'}^{+}
\]

and

\[
u_{i_0,i_0'}^{-} = x_{i_0}^{+} \cdot \sin \theta_{i_0,i_0'} - y_{j'} \cdot \cos \theta_{i_0,i_0'} \leq (R_{i_0,i_0'} \xi)_2 \leq x_{i_0}^{+} \cdot \sin \theta_{i_0,i_0'} + y_{j'} \cdot \cos \theta_{i_0,i_0'} = v_{i_0,i_0'}^{+}.
\]
Case 2: $\theta_{t_0,i_0} \in \left[\frac{3\pi}{2}, \pi\right)$. In this case, $(R_{t_0,i_0} \xi)_{1} = -\xi_1 \cdot \sin \theta_{t_0,i_0} - \xi_2 \cdot \cos \theta_{t_0,i_0}$ and thus
\[
u_{t_0,i_0} = -\frac{x_0^+}{t_0^+} \cdot \sin \theta_{t_0,i_0} - y_0^+ \cdot \cos \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{1} \leq -\frac{x_0^+}{t_0^+} \cdot \sin \theta_{t_0,i_0} + y_0^+ \cdot \cos \theta_{t_0,i_0} = u_{t_0,i_0}^+.
\]
Likewise, $(R_{t_0,i_0} \xi)_{2} = \xi_1 \cdot \cos \theta_{t_0,i_0} - \xi_2 \cdot \sin \theta_{t_0,i_0}$ and thus
\[
u_{t_0,i_0} = \frac{x_0^-}{t_0^-} \cdot \cos \theta_{t_0,i_0} - y_0^- \cdot \sin \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{2} \leq \frac{x_0^-}{t_0^-} \cdot \cos \theta_{t_0,i_0} + y_0^- \cdot \sin \theta_{t_0,i_0} = v_{t_0,i_0}^+.
\]

Case 3: $\theta_{t_0,i_0} \in \left[\pi, \frac{3\pi}{2}\right)$. In this case, $(R_{t_0,i_0} \xi)_{1} = -\xi_1 \cdot \cos \theta_{t_0,i_0} + \xi_2 \cdot \sin \theta_{t_0,i_0}$ and thus
\[
u_{t_0,i_0} = -\frac{x_0^+}{t_0^+} \cdot \cos \theta_{t_0,i_0} - y_0^+ \cdot \sin \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{1} \leq -\frac{x_0^+}{t_0^+} \cdot \cos \theta_{t_0,i_0} + y_0^+ \cdot \sin \theta_{t_0,i_0} = u_{t_0,i_0}^+.
\]
Likewise, $(R_{t_0,i_0} \xi)_{2} = -\xi_1 \cdot \sin \theta_{t_0,i_0} - \xi_2 \cdot \cos \theta_{t_0,i_0}$ and thus
\[
u_{t_0,i_0} = \frac{x_0^-}{t_0^-} \cdot \sin \theta_{t_0,i_0} - y_0^- \cdot \cos \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{2} \leq \frac{x_0^-}{t_0^-} \cdot \sin \theta_{t_0,i_0} + y_0^- \cdot \cos \theta_{t_0,i_0} = v_{t_0,i_0}^+.
\]

Case 4: $\theta_{t_0,i_0} \in \left[\frac{3\pi}{2}, 2\pi\right)$. In this case, $(R_{t_0,i_0} \xi)_{1} = \xi_1 \cdot \sin \theta_{t_0,i_0} + \xi_2 \cdot \cos \theta_{t_0,i_0}$, and thus
\[
u_{t_0,i_0} = \frac{x_0^+}{t_0^+} \cdot \sin \theta_{t_0,i_0} - y_0^+ \cdot \cos \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{1} \leq \frac{x_0^+}{t_0^+} \cdot \sin \theta_{t_0,i_0} + y_0^+ \cdot \cos \theta_{t_0,i_0} = u_{t_0,i_0}^+. 
\]
Likewise, $(R_{t_0,i_0} \xi)_{2} = -\xi_1 \cdot \cos \theta_{t_0,i_0} + \xi_2 \cdot \sin \theta_{t_0,i_0}$ and thus
\[
u_{t_0,i_0} = \frac{x_0^-}{t_0^-} \cdot \cos \theta_{t_0,i_0} - y_0^- \cdot \sin \theta_{t_0,i_0} \leq (R_{t_0,i_0} \xi)_{2} \leq \frac{x_0^-}{t_0^-} \cdot \cos \theta_{t_0,i_0} + y_0^- \cdot \sin \theta_{t_0,i_0} = v_{t_0,i_0}^+.
\]

Finally, (9.21) results from combination of (9.20) and (9.16), since $A_j = \text{diag}(2^{\alpha_j}, 2^{\beta_j})$. \qed

**Proof of Lemma 9.13** First we recall from Lemma 9.10 that
\[
\frac{1}{4} \cdot 2^{j'} \leq \nu_{t_0,i_0}^+ \leq 4 \cdot 2^{j'} \text{ and } y_{t_0,i_0} = 2^{\beta j'+1}. \tag{C.1}
\]

Next, we recall Equations (9.11), (9.12), (9.17) and (3.5) to see that, for $\ell \in J_j^{k}$,
\[
\theta_{t_0,i_0} = \theta_{t_0,i_0} - \ell \cdot \frac{\pi}{2} = 2^0 k + \Theta_{j',\ell} - \ell \cdot \frac{\pi}{2} - \Theta_{j,\ell} = 2^{\pi k} + \Theta_{j',\ell} - \ell \cdot \frac{\pi}{2} - \Theta_{j,\ell} \tag{C.2}
\]
and hence
\[
1 - \frac{2}{\pi} \cdot \theta_{t_0,i_0} = \frac{4}{N} \cdot 2^{(\beta-1) j} \cdot (S_{k^{(1)}_{j,\ell}} + \ell) \tag{C.3}
\]
where $S_{k^{(1)}_{j,\ell}} := \frac{N}{2^\ell} \cdot (2^{1-\beta_j} - (2^{\pi k} + \Theta_{j',\ell} - \ell \cdot \frac{\pi}{2})$ and $S_{k^{(1)}_{j,\ell}} := \frac{N}{4} \cdot 2^{(1-\beta_j)} - S_{k^{(1)}_{j,\ell}}$. In particular, since $0 \leq \theta_{t_0,i_0} < \frac{\pi}{2}$, we see that $S_{k^{(1)}_{j,\ell}} + \ell \geq 0$ for all $\ell \in J_j^{k}$.

As a further preparation, we recall from Appendix B the estimates
\[
\frac{2}{\pi} \cdot \phi \leq \sin \phi \leq \phi \text{ and } 1 - \frac{2}{\pi} \cdot \phi \leq \cos \phi \leq \frac{1}{2} \cdot \left(1 - \frac{2}{\pi} \cdot \phi\right) \quad \forall \phi \in \left[0, \frac{\pi}{2}\right]. \tag{C.4}
\]

Finally, to actually prove the claim, we distinguish the four possible values of $\ell$.

**Case 1**: $\ell = 0$. Let $S_{k^{(1)}_{j,\ell}} := S_{k^{(1)}_{j,i_0}}$ and $\nu_{t_0,i_0} := -1$. With this definition, Equation (C.2) shows that $\theta_{t_0,i_0} = \frac{2}{N} \cdot 2^{(\beta-1) j} \cdot (S_{k^{(1)}_{j,i_0}} + \ell) \text{ and } S_{k^{(1)}_{j,i_0}} + \nu_{t_0,i_0} \geq 0 \text{ for all } \ell \in J_j^{k}.$

Next, recalling the definition of $v_{t_0,i_0}^+$ (see Equation (9.19)) and combining Equations (C.1), (C.2) and the identity $\theta_{t_0,i_0} = \frac{2}{N} \cdot 2^{(\beta-1) j} \cdot (S_{k^{(1)}_{j,i_0}} + \nu_{t_0,i_0})$, we see that, for any $\ell \in J_j^{k}$,
\[
v_{t_0,i_0}^+ \leq 2^{\beta j'+1} + 4 \cdot 2^{j'} \cdot \sin \theta_{t_0,i_0} \leq 2^{\beta j'+1} + 4 \cdot 2^{j'} \cdot \theta_{t_0,i_0} = 2^{\beta j'+1} + \frac{8\pi}{N} \cdot 2^{j'+(\beta-1) j} \cdot (S_{k^{(1)}_{j,i_0}} + \nu_{t_0,i_0}) \leq 2^{\beta j'+1} + \frac{8\pi}{N} \cdot 2^{j'+(\beta-1) j} \cdot (S_{k^{(1)}_{j,i_0}} + \nu_{t_0,i_0}) \tag{C.5}
\]
and
\[
v_{t_0,i_0}^- \leq -2^{\beta j'+1} + \frac{1}{4} \cdot 2^{j'} \cdot \sin \theta_{t_0,i_0} \leq -2^{\beta j'+1} + 1 \cdot 2^{j'} \cdot \theta_{t_0,i_0} = -2^{\beta j'+1} + \frac{8\pi}{N} \cdot 2^{j'+(\beta-1) j} \cdot (S_{k^{(1)}_{j,i_0}} + \nu_{t_0,i_0}) \tag{C.6}
\]
Since $I_2^{(i,i',\ell)} = [2^{-\beta j} \cdot v_{i_0,i'}^{-} + 2^{-\beta j} \cdot v_{i_0,i'}^{+}]$, this proves the desired estimate for $I_2^{(i,i',\ell)}$. Here we noted again that $S_{k,i,j} + \nu_{\ell, \ell} \leq 0$.

Finally, (C.2), $\theta_{i_0,i'} = \theta_{i_0,i'}$ implies that, for $\ell \in J_{j}^{r,k}$,

$$2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| \leq 2^\alpha \eta_{j}^{i',-j} \cdot \theta_{i_0,i'} = \frac{2\pi}{N} \cdot 2^{(a-1)j'} \cdot 2^{j'-j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right) \leq 2\pi \cdot \beta_1 \cdot \left( S_{k,i,j} + \nu_{\ell} \right).$$

**Case 2:** $\ell = 1$. Let $S_{k,i,j} := S_{k,i,j}^{(1)}$ and $\nu_{\ell} := 1$. On the one hand, as seen after Equation (C.3), this ensures that $S_{k,i,j} + \nu_{\ell} \leq 0$ for all $\ell \in J_{j}^{r,k}$.

On the other hand, by combining Equations (C.1), (C.3), and (C.4), and by recalling the definition of $v_{i_0,i'}^\pm$, we see for any $\ell \in J_{j}^{r,k}$ that

$$v_{i_0,i'}^+ \leq 2^{\beta j} + 4 \cdot 2 \cdot \cos \theta_{i_0,i'} \leq 2^{\beta j} + 2 \cdot 2 \cdot \left( 1 - \frac{2\pi}{\theta_{i_0,i'}} \right) = 2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right)$$

and

$$v_{i_0,i'}^- \geq -2^{\beta j} + 4 \cdot 2 \cdot \cos \theta_{i_0,i'} \geq -2^{\beta j} + \frac{2\pi}{\theta_{i_0,i'}} \left( 1 - \frac{2\pi}{\theta_{i_0,i'}} \right) = -2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right).$$

Just as in Case 1, this yields the desired estimate for $I_2^{(i,i',\ell)}$.

Finally, since $\theta_{i_0,i'} = \theta_{i_0,i'} + \pi/2$ for $\ell \in J_{j}^{r,k}$, we see that $\sin \theta_{i_0,i'} = \cos \theta_{i_0,i'}$. This combined with (C.4) and (C.3) results in

$$2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| \leq 2\pi \cdot \beta_1 \cdot |S_{k,i,j} + \nu_{\ell}| \text{ for } \ell \in J_{j}^{r,k}.$$  

**Case 3:** $\ell = 2$. Let $S_{k,i,j} := -S_{k,i,j}^{(0)}$ and $\nu_{\ell} := 1$. On the one hand, as can be seen from Equation (C.3), this ensures that $S_{k,i,j} + \nu_{\ell} \leq 0$ for all $\ell \in J_{j}^{r,k}$.

On the other hand, recalling the definition of $v_{i_0,i'}^\pm$ and combining Equations (C.1), (C.4), and (C.2), we see that, for any $\ell \in J_{j}^{r,k}$,

$$v_{i_0,i'}^+ \leq 2^{\beta j} + \frac{2\pi}{\theta_{i_0,i'}} \leq 2^{\beta j} + 1 \cdot \frac{2\pi}{\theta_{i_0,i'}} = 2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right)$$

and

$$v_{i_0,i'}^- \geq -2^{\beta j} + 4 \cdot 2 \cdot \cos \theta_{i_0,i'} \geq -2^{\beta j} + 4 \cdot 2 \cdot \left( 1 - \frac{2\pi}{\theta_{i_0,i'}} \right) = -2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right).$$

Given the previous two estimates and bearing in mind that $S_{k,i,j} + \nu_{\ell} \leq 0$, we get the inclusion $I_{2}^{(i,i')} \subset \left[ \beta_2(S_{k,i,j} + \nu_{\ell}) - L_j, \beta_1(S_{k,i,j} + \nu_{\ell}) + L_j \right]$.

Finally, we have $\theta_{i_0,i'} = \theta_{i_0,i'} + \pi$ for $\ell \in J_{j}^{r,k}$ and thus $\sin \theta_{i_0,i'} = -\sin \theta_{i_0,i'}$, whence

$$2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| \leq 2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| = 2^{(a-1)j'} \cdot 2^{j'-j} \cdot \frac{2\pi}{\theta_{i_0,i'}} \cdot |S_{k,i,j} + \nu_{\ell}|.$$  

As in the previous cases, this shows that $2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| \leq 2\pi \cdot \beta_1 \cdot |S_{k,i,j} + \nu_{\ell}| \text{ for } \ell \in J_{j}^{r,k}.$

**Case 4:** $\ell = 3$. Let $S_{k,i,j} := -S_{k,i,j}^{(1)}$ and $\nu_{\ell} := -1$. On the one hand, this, as can be seen from Equation (C.3), ensures that $S_{k,i,j} + \nu_{\ell} \leq 0$ for all $\ell \in J_{j}^{r,k}$.

On the other hand, by recalling the definition of $v_{i_0,i'}^\pm$, and by combining Equations (C.1), (C.2), and (C.4), we see for any $\ell \in J_{j}^{r,k}$ that

$$v_{i_0,i'}^+ \leq 2^{\beta j} + \frac{2\pi}{\theta_{i_0,i'}} \leq 2^{\beta j} + \frac{2\pi}{\theta_{i_0,i'}} \leq 2^{\beta j} + \frac{2\pi}{\theta_{i_0,i'}} = 2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right)$$

and

$$v_{i_0,i'}^- \geq -2^{\beta j} + 4 \cdot 2 \cdot \cos \theta_{i_0,i'} \geq -2^{\beta j} + 4 \cdot 2 \cdot \left( 1 - \frac{2\pi}{\theta_{i_0,i'}} \right) = -2^{\beta j} + 8\pi \cdot 2^{(a-1)j} \cdot \left( S_{k,i,j} + \nu_{\ell} \right).$$

As in the previous cases, this shows that $2^\alpha \eta_{j}^{i',-j} \cdot |\sin \theta_{i_0,i'}| \leq 2\pi \cdot \beta_1 \cdot |S_{k,i,j} + \nu_{\ell}| \text{ for } \ell \in J_{j}^{r,k}.$
and
\[ \nu_{i_0,i'} \geq -2^{\beta_j+1} - 4 \cdot 2^{j'} \cdot \cos \theta_{i_0,i'} \geq -2^{\beta_j+1} - 2\pi \cdot 2^{j'} \left( 1 - \frac{2}{\pi} \theta_{i_0,i'} \right) = -2^{\beta_j+1} + \frac{8\pi}{N} \cdot 2^{j'+(\beta-1)j} \cdot (S_{k,i,j} + \nu_i \ell). \]

As in Case 3, this yields the desired estimate for \( I_2^{(i,i')} \).

Finally, since \( \vartheta_{i_0,i'} = \theta_{i_0,i'_0} + \frac{2}{\pi} \) for \( \ell' \in J_{j_k}^k \), we see that \( \sin \vartheta_{i_0,i'_0} = -\cos \theta_{i_0,i'_0} \leq 0 \). Therefore, Equations (C.4) and (C.3) imply that
\[ 2^{\alpha j'-\beta_j} \cdot |\sin \vartheta_{i_0,i'_0}| = 2^{\alpha j'-\beta_j} \cdot \cos \theta_{i_0,i'_0} \leq 2^{\alpha j'-\beta_j} \cdot \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \theta_{i_0,i'_0} \right) \leq -\frac{2}{\pi} \cdot 2^{(\alpha-1)j'} \cdot 2^{j'-j} \cdot (S_{k,i,j} + \nu_i \ell). \]

As in the previous cases, since \( S_{k,i,j} + \nu_i \ell \leq 0 \), this proves that \( 2^{\alpha j'-\beta_j} \cdot |\sin \vartheta_{i_0,i'_0}| \leq 2^{\alpha j'-\beta_j} \cdot |S_{k,i,j} + \nu_i \ell| \) for \( \ell' \in J_{j_k}^k \).

Proof of Lemma 9.14 This proof is quite similar to that of Lemma 9.13. Therefore we shall only outline it briefly. First, combining Equations (9.12), (9.17) and (3.5) results in
\[ \theta_{i_0,i'_0} = 2\pi k + \Theta_{j',\ell} - \frac{\pi}{2} - \Theta_{j,\ell} = 2\pi k - \Theta_{j,\ell} - \frac{\pi}{2} + \frac{2\pi}{N} \cdot 2^{(\beta-1)j'} \ell' = \frac{2\pi}{N} \cdot 2^{(\beta-1)j'} \cdot (S_{k,i,0}^{(0)} + \ell'). \]

and hence
\[ 1 - \frac{2}{\pi} \cdot \theta_{i_0,i'_0} = \frac{4}{N} \cdot 2^{(\beta-1)j'} \cdot (S_{k,i,j'}^{(1)} - \ell') \]
where \( S_{k,i,j'}^{(0)} := \frac{N}{2\pi} \cdot 2^{(1-\beta)j'} \cdot (2\pi k - \Theta_{j,\ell} + \ell - \frac{\pi}{2}) \) and \( S_{k,i,j'}^{(1)} := \frac{N}{2\pi} \cdot 2^{(1-\beta)j'} - S_{k,i,j'}^{(0)} \).

Since \( 0 \leq \theta_{i_0,i'_0} < \frac{\pi}{2} \), Equations (C.5) and (C.6) show that \( S_{k,i,j'}^{(0)} + \ell' \geq 0 \) and \( S_{k,i,j'}^{(1)} - \ell' \geq 0 \) for all \( \ell' \in J_{j_k}^k \).

To prove the lemma, we now distinguish the four possible values of \( i \).

Case 1: \( i = 0 \). Let \( S_{k,i,j'} := S_{k,i,j'}^{(0)} \) and \( \nu_i := 1 \). On the one hand, this, as can be seen from (C.6), ensures that \( S_{k,i,j'} + \nu_i \ell' \geq 0 \) for all \( \ell' \in J_{j_k}^k \).

On the other hand, precisely as in Case 1 in the proof of Lemma 9.13, we see that, for \( \ell' \in J_{j_k}^k \),
\[ v_{i_0,i'}^+ \leq 2^{\beta_j+1} + 4 \cdot 2^{j'} \cdot \theta_{i_0,i'_0} \quad \text{and} \quad v_{i_0,i'}^- \geq -2^{\beta_j+1} + \frac{2^{j'}}{2\pi} \cdot \theta_{i_0,i'_0}. \]

Given (C.5), our choice of \( S_{k,i,j'} \), and \( \nu_i \), this implies that
\[ v_{i_0,i'}^- \geq 2^{\beta_j+1} + \frac{8\pi}{N} \cdot 2^{\beta(j'-j)} \cdot (S_{k,i,j'} + \nu_i \ell') \quad \text{and} \quad v_{i_0,i'}^- \geq -2^{\beta_j+1} + N^{-1} \cdot 2^{\beta_j} \cdot (S_{k,i,j'} + \nu_i \ell'). \]

Combining these estimates results in the stated inclusion for \( I_2^{(i,i')} \).

Finally, since \( \theta_{i_0,i'_0} = \vartheta_{i_0,i'_0} \) for \( \ell' \in J_{j_k}^k \) and \( \alpha \leq 1 \), we see that
\[ 2^{\alpha j'-\beta_j} \cdot |\sin \vartheta_{i_0,i'_0}| \leq 2^{\alpha j'-\beta_j} \cdot \theta_{i_0,i'_0} = \frac{2\pi}{N} \cdot 2^{(\alpha-1)j'} \cdot 2^{j'-j} \cdot (S_{k,i,j} + \nu_i \ell'). \]

Case 2: \( i = 1 \). Let \( S_{k,i,j'} := S_{k,i,j'}^{(1)} \) and \( \nu_i := -1 \). On the one hand, this, as can be seen from (C.6), ensures that \( S_{k,i,j'} + \nu_i \ell' \geq 0 \) for all \( \ell' \in J_{j_k}^k \).

On the other hand, precisely as in Case 2 of the proof of Lemma 9.13, we see that, for \( \ell' \in J_{j_k}^k \),
\[ v_{i_0,i'}^+ \leq 2^{\beta_j+1} + 2\pi \cdot 2^{j'} \cdot (1 - \frac{2}{\pi} \theta_{i_0,i'_0}) \quad \text{and} \quad v_{i_0,i'}^- \geq -2^{\beta_j+1} + \frac{2^{j'}}{\pi} \cdot (1 - \frac{2}{\pi} \theta_{i_0,i'_0}). \]

Given (C.6) and our choice of \( S_{k,i,j'} \) and \( \nu_i \), this implies that
\[ v_{i_0,i'}^- \geq 2^{\beta_j+1} + \frac{8\pi}{N} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell') \quad \text{and} \quad v_{i_0,i'}^- \geq -2^{\beta_j+1} + N^{-1} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell'). \]

This yields the stated inclusion for \( I_2^{(i,i')} \).

Finally, we see that, exactly as in Case 2 of the proof of Lemma 9.13
\[ 2^{\alpha j'-\beta_j} \cdot |\sin \vartheta_{i_0,i'_0}| \leq \frac{\pi}{2} \cdot 2^{\alpha j'-\beta_j} \cdot \left( 1 - \frac{2}{\pi} \theta_{i_0,i'_0} \right), \]

62
which, given (C.6) and \( \alpha \leq 1 \), implies that \( 2^{\alpha j' - \beta j} \cdot |\sin \theta_{i_0,i_0'}| \leq 2 \pi \cdot \beta_1 \cdot (S_{k,i,j'} + \nu_i \ell') \) as in the previous case.

**Case 3:** \( \ell = 2 \). Let \( S_{k,i,j'} := -S_{k,i,j'}^{(0)} \) and \( \nu_i := -1 \). On the one hand, this, as can be seen from (C.6), ensures that \( S_{k,i,j'} + \nu_i \ell' \leq 0 \) for all \( \ell' \in J_{j'}^k \).

On the other hand, precisely as in Case 3 of the proof of Lemma 9.13, we see that, for \( \ell' \in J_{j'}^k \),
\[
v^{+}_{i_0,i'} \leq 2^{\beta j' + 1} + N^{-1} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell') \quad \text{and} \quad v^{-}_{i_0,i'} \geq -2^{\beta j' + 1} - 4 \cdot 2^{\beta j'} \cdot \theta_{i_0,i_0'}.
\]
Given (C.5) and our choice of \( S_{k,i,j'} \) and \( \nu_i \), this implies that
\[
v^{+}_{i_0,i'} \leq 2^{\beta j' + 1} + N^{-1} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell') \quad \text{and} \quad v^{-}_{i_0,i'} \geq -2^{\beta j' + 1} + \frac{8 \pi}{N} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell')
\]
These estimates together imply the stated inclusion for \( I_2^{(i,i')}. \)

Finally, as in Case 3 of the proof of Lemma 9.13, we see that, for \( \ell' \in J_{j'}^k \),
\[
v^{+}_{i_0,i'} \leq 2^{\beta j' + 1} + N^{-1} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell') \quad \text{and} \quad v^{-}_{i_0,i'} \geq -2^{\beta j' + 1} + \frac{8 \pi}{N} \cdot 2^{\beta j'} \cdot (S_{k,i,j'} + \nu_i \ell')
\]
These estimates together imply the stated inclusion for \( I_2^{(i,i')}. \)

Finally, as in Case 4 of the proof of Lemma 9.13, we see that
\[
2^{\alpha j' - \beta j} \cdot |\sin \theta_{i_0,i_0'}| \leq 2^{\alpha j' - \beta j} \cdot \frac{\pi}{2} \cdot (1 - \frac{2}{\pi} \cdot \theta_{i_0,i_0'}).
\]
Given (C.6) and our choice of \( S_{k,i,j'} \) and \( \nu_i \), this implies \( 2^{\alpha j' - \beta j} \cdot |\sin \theta_{i_0,i_0'}| \leq 2 \pi \cdot \beta_1 \cdot |S_{k,i,j'} + \nu_i \ell'| \) since \( \alpha \leq 1 \).

**D. Notation**

We recall a few elements of the theory of distributions and introduce the notations that we use throughout this work:

For an open non-empty set \( U \subset \mathbb{R}^d \), we define \( C^\infty_c(U) := \{ g \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{C}) : \supp g \subset U \text{ compact} \} \). There is a canonical topology on \( C^\infty_c(U) \) that makes this space into a topological vector space; see Sections 6.3-6.6 in [54] for the definition of this topology. The topological dual space \( \mathcal{D}'(U) := [C^\infty_c(U)]' \) of \( C^\infty_c(U) \) is called the space of **distributions** on \( U \). The bilinear pairing between \( \mathcal{D}'(U) \) and \( C^\infty_c(U) \) is denoted by \( \langle \phi, g \rangle_{\mathcal{D}'(U)} := \langle \phi, g \rangle := \phi(g) \). We use the characterisation of \( \mathcal{D}'(U) \) given in Theorem 6.8 in [54]: a linear functional \( \phi : C^\infty_c(U) \to \mathbb{C} \) belongs to \( \mathcal{D}'(U) \) if and only if, for every compact set \( K \subset U \), there are numbers \( C = C(K, \phi) > 0 \) and \( N = N(K, \phi) \in \mathbb{N} \) such that \( |\phi(g)| \leq C \cdot \max_{|\alpha| \leq N} \| \partial^\alpha g \|_{L^\infty} \) for all \( g \in C^\infty_c(U) \) with supp \( g \subset K \). Here, we used **multi-index** notation: Any \( \alpha \in \mathbb{N}^d_0 \) is called a multi-index. We write \( \partial^\alpha f = \frac{\partial^{\alpha_d} f}{\partial x_{d}^{\alpha_d}} \cdots \frac{\partial^{\alpha_1} f}{\partial x_{1}^{\alpha_1}} \) if \( f : U \subset \mathbb{R}^d \to \mathbb{C} \) is sufficiently smooth for the derivative to be defined. Similarly, we write \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \) for \( x \in \mathbb{R}^d \). Finally, we write \( |\alpha| = \alpha_1 + \ldots + \alpha_d \). Even though we use the same notation \( |\xi| \) for the Euclidean norm of a vector \( \xi \in \mathbb{R}^d \), this, we believe, should not lead to any confusion.

Any locally integrable function \( f \in L^1_{\text{loc}}(U) \) induces a distribution \( T_f \in \mathcal{D}'(U) \) that is given by \( \langle T_f, g \rangle := \int_U f(x) g(x) \, dx \). A distribution \( \phi \in \mathcal{D}'(U) \) is called **regular** if \( \phi = T_f \) for some \( f \in L^1_{\text{loc}}(U) \). For the sake of simplicity, we shall often write \( f \) instead of \( T_f \), i.e., \( \langle \phi, f \rangle := \langle T_f, g \rangle \).

Various operations in the space of distributions \( \mathcal{D}'(U) \) can be defined using its duality with the space \( C^\infty_c(U) \), i.e., for any \( h \in C^\infty_c(U) \), \( \phi \in \mathcal{D}'(U) \) and \( \alpha \in \mathbb{N}^d_0 \), the distributions \( \partial^\alpha \phi \in \mathcal{D}'(U) \) and \( h \cdot \phi \in \mathcal{D}'(U) \) are defined by \( \langle \partial^\alpha \phi, f \rangle = (-1)^{|\alpha|} \cdot \langle \phi, \partial^\alpha f \rangle \), and \( \langle h \cdot \phi, f \rangle = \langle \phi, h \cdot f \rangle \), respectively.
With the topology induced by the family of norms $(\| \cdot \|_N)_{N \in \mathbb{N}_0}$, the \textbf{Schwartz function space} $S(\mathbb{R}^d)$, defined as

$$S(\mathbb{R}^d) := \left\{ g \in C^\infty(\mathbb{R}^d; \mathbb{C}) : \forall N \in \mathbb{N}_0 : \|g\|_N := \max_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} \sup_{x \in \mathbb{R}^d} (1 + |x|)^N : |\partial^\alpha g(x)| < \infty \right\},$$

becomes a topological vector space and its topological dual space $S'(\mathbb{R}^d)$, called the space of \textbf{tempered distributions}, becomes a topological vector space when equipped with the weak-$*$-topology. We write $\langle \phi, g \rangle_{S'} := \langle \phi, g \rangle := \phi(g)$ for the dual pairing between $S'(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$. The reader is referred to Sections 8.1 and 9.2 in [13] for more details on these spaces.

In contrast to the \textit{bilinear} dual pairings for distributions and tempered distributions, we write $\langle f, g \rangle_{L^2} := \int f \cdot \overline{g} \, dx$, which is sesquilinear.

The \textbf{Fourier transform} of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i (x, \xi)} \, dx \quad \text{for} \quad \xi \in \mathbb{R}^d.$$
[2] M. Berra, M.V. de Hoop, and J.L. Romero, *A multi-scale Gaussian beam parametrix for the wave equation: the Dirichlet boundary value problem*, arXiv preprints (2017), https://arxiv.org/abs/1705.00337.

[3] L. Borup and M. Nielsen, *Banach frames for multivariate α-modulation spaces*, J. Math. Anal. Appl. 321 (2006), no. 2, 880–895.

[4] ____, *Frame decomposition of decomposition spaces*, J. Fourier Anal. Appl. 13 (2007), no. 1, 39–70.

[5] D. Bytchenkoff, S. Rodts, P. Moucheront, and T. Fen-Chong, *Cardinal series to sort out defective samples in magnetic resonance data sets*, J. Magn. Reson. 202 (2010), 147–154.

[6] E.J. Candès and D.L. Donoho, *Ridgelets: a key to higher-dimensional intermittency?*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 357 (1999), no. 1760, 2495–2509.

[7] ____, *New tight frames of curvelets and optimal representations of objects with piecewise $C^2$ singularities*, Comm. Pure Appl. Math. 57 (2004), no. 2, 219–266.

[8] ____, *Continuous curvelet transform. I. Resolution of the wavefront set*, Appl. Comput. Harmon. Anal. 19 (2005), no. 2, 162–197.

[9] A. Cordoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations 3 (1978), no. 11, 979–1005.

[10] H.G. Feichtinger and P. Gröbner, *Banach spaces of distributions defined by decomposition methods, I*, Math. Nachr. 123 (1985), no. 1, 97–120.

[11] J. Fell, H. Führ, and F. Voigtländer, *Resolution of the wavefront set using general continuous wavelet transforms*, J. Fourier Anal. Appl. 22 (2016), no. 5, 997–1058.

[12] G.B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., Pure and Applied Mathematics, Wiley, New York, 1999.

[13] M. Fornasier, *Banach frames for α-modulation spaces*, Appl. Comput. Harmon. Anal. 22 (2007), no. 2, 157–175.

[14] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), no. 1, 34–170.

[15] H. Führ and J. Lemvig, *System bandwidth and the existence of generalized shift-invariant frames*, J. Funct. Anal. 276 (2019), no. 2, 563–601.

[16] H. Führ and F. Voigtländer, *Wavelet coorbit spaces viewed as decomposition spaces*, J. Funct. Anal. 269 (2015), 80–154.
A pedestrian’s approach to pseudodifferential operators, Harmonic analysis and applications, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2006, pp. 139–169.

K. Gröchenig and S. Koppensteiner, Gabor frames: Characterizations and coarse structure, arXiv preprints (2018), https://arxiv.org/abs/1803.05271

P. Grohs, Ridgelet-type frame decompositions for Sobolev spaces related to linear transport, J. Fourier Anal. Appl. 18 (2012), no. 2, 309–325.

W. Guo, D. Fan, H. Wu, and G. Zhao, Sharpness of complex interpolation on α-modulation spaces, J. Fourier Anal. Appl. 25 (2015), no. 2, 427–461.

W. Guo, D. Fan, and G. Zhao, Full characterization of the embedding relations between α-modulation spaces, Sci. China Math. 61 (2018), no. 7, 1243–1272.

J. Han and B. Wang, α-modulation spaces (I) scaling, embedding and algebraic properties, J. Math. Soc. Japan 66 (2014), no. 4, 1315–1373.

E. Hernández, D. Labate, and G. Weiss, A unified characterization of reproducing systems generated by a finite family. II, J. Geom. Anal. 12 (2002), no. 4, 615–662.

E. Hernández, D. Labate, G. Weiss, and E. Wilson, Oversampling, quasi-affine frames, and wave packets, Appl. Comput. Harmon. Anal. 16 (2004), no. 2, 111–147.

T. Kato, The inclusion relations between α-modulation spaces and $L^p$-Sobolev spaces or local Hardy spaces, J. Funct. Anal. 272 (2017), no. 4, 1340–1405.

T. Kato and N. Tomita, Pseudodifferential operators with symbols in the Hörmander class $S^0_{0,0}$ on α-modulation spaces, Monatsh. Math. 188 (2019), no. 4, 667–687.

P. Kittipoom, G. Kutyniok, and W.-Q. Lim, Construction of compactly supported shearlet frames, Constr. Approx. 35 (2012), no. 1, 21–72.

G. Kutyniok and D. Labate, Resolution of the wavefront set using continuous shearlets, Trans. Amer. Math. Soc. 361 (2009), no. 5, 2719–2754.

G. Kutyniok and W.-Q. Lim, Compactly supported shearlets are optimally sparse, J. Approx. Theory 163 (2011), no. 11, 1564–1589.

G. Kutyniok and P. Petersen, Classification of edges using compactly supported shearlets, Appl. Comput. Harmon. Anal. 42 (2017), no. 2, 245–293.

D. Labate, G. Weiss, and E. Wilson, An approach to the study of wave packet systems, Wavelets, frames and operator theory, Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 215–235.

J. Lemvig and J.T. van Velthoven, Criteria for generalized translation-invariant frames, Stud. Math. (To Appear. arXiv:1704.06510).

Y. Liang, Y. Sawano, T. Ullrich, D. Yang, and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, J. Fourier Anal. Appl. 18 (2012), no. 5, 1067–1111.

F. Luisier, T. Blu, and M. Unser, A new SURE approach to image denoising: interscale orthonormal wavelet thresholding, IEEE Trans. Image Process. 16 (2007), no. 3, 593–606.

Y. Meyer, Ondelettes et opérateurs. II, Actualités Mathématiques. [Current Mathematical Topics], Hermann, Paris, 1990, Opérateurs de Calderón-Zygmund. [Calderón-Zygmund operators].

Y. Meyer, Wavelets and operators, Cambridge Studies in Advanced Mathematics, vol. 37, Cambridge University Press, Cambridge, 1992.

B. Nazaret and M. Holschneider, An interpolation family between Gabor and wavelet transformations: application to differential calculus and construction of anisotropic Banach spaces, Nonlinear hyperbolic equations, spectral theory, and wavelet transformations, Oper. Theory Adv. Appl., vol. 145, Birkhäuser, Basel, 2003, pp. 363–394.

M. Nielsen and K.N. Rasmussen, Compactly supported frames for decomposition spaces, J. Fourier Anal. Appl. 18 (2012), no. 1, 87–117.

R. Reisenhofer, J. Kiefer, and E. J. King, Shearlet-based detection of flame fronts, Exp. Fluids 57:41 (2016).

S. Rodts and D. Bytchenkoff, Cardinal series to restore NMR-signals dominated by strong inhomogeneous broadening, J. Magn. Reson. 212 (2011), no. 1, 26–39.

S. Rodts and D. Bytchenkoff, Extrapolation and phase correction of non-uniformly broadened signals, J. Magn. Reson. 233 (2013), 64–73.
[53] S. Rodts, D. Bytchenkoff, and T. Fen-Chong, *Cardinal series to filter oversampled truncated magnetic resonance signals*, J. Magn. Reson. **204** (2010), 64–75.

[54] W. Rudin, *Functional Analysis*, International series in pure and applied mathematics, McGraw-Hill, 1991.

[55] M. Speckbacher, D. Bayer, S. Dahlke, and P. Balazs, *The \( \alpha \)-modulation transform: admissibility, coorbit theory and frames of compactly supported functions*, Monatsh. Math. **184** (2017), no. 1, 133–169.

[56] H. Triebel, *Fourier Analysis and Function Spaces (selected topics)*, Teubner Verlagsgesellschaft, Leipzig, 1977.

[57] ______, *Theory of Function Spaces I*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

[58] ______, *Theory of function spaces. III*, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.

[59] J.T. van Velthoven, *On the local integrability condition for generalised translation-invariant systems*, Collect. Math. (2019), [https://doi.org/10.1007/s13348-019-00238-5](https://doi.org/10.1007/s13348-019-00238-5).

[60] F. Voigtlaender, *Embedding Theorems for Decomposition Spaces with Applications to Wavelet Coorbit Spaces*, Ph.D. thesis, RWTH Aachen University, 2015, [http://publications.rwth-aachen.de/record/564979](http://publications.rwth-aachen.de/record/564979).

[61] ______, *Embeddings of decomposition spaces*, arXiv preprints (2016), [https://arxiv.org/abs/1605.09705](https://arxiv.org/abs/1605.09705).

[62] ______, *Embeddings of Decomposition Spaces into Sobolev and BV Spaces*, arXiv preprints (2016), [http://arxiv.org/abs/1601.02201](http://arxiv.org/abs/1601.02201).

[63] ______, *Structured, compactly supported Banach frame decompositions of decomposition spaces*, arXiv preprints (2016), [http://arxiv.org/abs/1612.08772](http://arxiv.org/abs/1612.08772).

[64] F. Voigtlaender and A. Pein, *Analysis vs. synthesis sparsity for \( \alpha \)-shearlets*, arXiv preprints (2017), [http://arxiv.org/abs/1702.03559](http://arxiv.org/abs/1702.03559).