Irreducibility of Discrete Painlevé Equation of Type $D_7^{(1)}$

By

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Abstract. In this paper, we will study the irreducibility of the discrete Painlevé equation of type $D_7^{(1)}$ in the sense of decomposable extensions. The irreducibility here particularly implies that the transcendental function solution cannot be built from rational functions by reiterating algebraic operations, the taking of a solution of a linear difference equation and the taking of a solution of a first-order algebraic difference equation. We also study non-existence of algebraic function solution. A modification to the definition of the decomposable extension is mentioned.

Key Words and Phrases. Irreducibility, Decomposable extension, Discrete Painlevé equation, Transcendence.

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1. Introduction

In about 1900, P. Painlevé introduced certain second-order ordinary differential equations called Painlevé equations. For example, Painlevé equation of type I is

$$y'' = 6y^2 + x.$$ 

He asserted that they define new functions (cf. [15]). However there was controversy on his assertion (see articles in Comptes rendus hebdomadaires des séances de l’Académie des sciences, around 1903). In his paper [7] published in 1988, K. Nishioka defined a decomposable extension of differential fields, which clarifies the problem through a concept of reducibility of differential equations, and proved irreducibility of Painlevé equation of type I in his sense. Since any chain of strongly normal extensions and algebraic extensions is decomposable, the irreducibility particularly implies that the transcendental function solution cannot be built from rational functions by reiterating algebraic operations and the taking of solutions of linear differential equations. It is considered that the problem for Painlevé equation of type I was solved by his work and H. Umemura’s work [20] using his notion of the classical functions. Also, for all the other types, the problem has been solved by several authors in the same sense.
In the 1990s, B. Grammaticos, A. Ramani and other authors introduced certain difference/discrete equations called discrete Painlevé equations, which are counterparts of Painlevé differential equations (see [4, 16, 17]). In his paper [18], H. Sakai classified them into 21 types—1 elliptic-difference equation, 11 \( q \)-difference equations and 9 (usual) difference equations (cf. [19]). The author defined decomposable extensions of difference fields, and for several \( q \)-difference Painlevé equations, he proved that almost all transcendental function solutions never belong to any decomposable extension of \( C(x) \) (see [8, 9, 11, 12] and [6] with N. Nakazono). This kind of result is also called the irreducibility of \( q \)-Painlevé equations. However, there has not been any result on the irreducibility of (usual) difference equations of Painlevé type. It has not been clear whether the problem could be solved in the same way as for \( q \)-Painlevé equations. For that reason, the object of study here is one of discrete Painlevé equations,

\[
d-P(D_7): \quad y(x + \delta) + y(x - \delta) = \frac{s x}{y(x)} - \frac{s^2}{y(x)^2}, \quad \delta, s \in C^*,
\]

which is called discrete Painlevé equation of type \( D_7^{(1)} \). Unlike \( q \)-Painlevé equations, the form of this equation is not new. In fact, it is also seen in a Bäcklund transformation of the third Painlevé differential equation of type \( D_7^{(1)} \) (cf. §2.3 of the paper [14]),

\[
P(D_7): \quad \frac{d^2 q}{dt^2} = \frac{1}{q} \left( \frac{dq}{dt} \right)^2 - \frac{1}{t} \frac{dq}{dt} - \frac{2q^2}{t^2} + \beta - \frac{1}{4t} - \frac{1}{q}.
\]

In the above two equations, \( q, t, \beta \) correspond to \( y, s, x \) respectively. Although it is known that \( P(D_7) \) is irreducible because of the absence of invariant divisors, we are not certain whether the notion of invariant divisors could be formalized for difference equations.

In this paper, we will modify the definition of the decomposable difference field extensions, and study the irreducibility of \( d-P(D_7) \) in the sense of the new decomposable extensions. For example, a strongly normal extension of difference fields introduced by A. Bialynicki-Birula [1], a difference field extension generated by a solution of a linear difference equation and one generated by a solution of a first-order algebraic difference equation are all decomposable by taking algebraic closures. Moreover, a chain of those extensions is also decomposable. Hence the irreducibility here particularly implies that the transcendental function solution cannot be built from rational functions by reiterating algebraic operations, the taking of a solution of a linear difference equation and the taking of a solution of a first-order algebraic difference equation. The minor change of the decomposable extensions does
not affect the above examples. It makes the definition simpler and more consistent with the differential counterpart, and makes the proof of irreducibility only a little more complicated. We also study non-existence of algebraic function solutions.

We introduce the terminology of difference algebra before more exact results.

**Notation.** Throughout the paper every field is of characteristic zero. When $K$ is a field and $\tau$ is an isomorphism of $K$ into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. We call $\tau$ the (transforming) operator and $K$ the underlying field. For a difference field $\mathcal{K}$, $K$ often denotes its underlying field. For a difference field $K$, $a \in K$, the element $\tau^n a \in K \ (n \in \mathbb{Z})$, if it exists, is called the $n$-th transform of $a$ and is sometimes denoted by $a_n$. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. If $K/\tau K$ is algebraic, we say that $\mathcal{K}$ is almost inversive. For an algebraic closure $\bar{K}$ of $K$, the transforming operator $\tau$ is extended to an isomorphism $\bar{\tau}$ of $\bar{K}$ into itself, not necessarily in a unique way (cf. the book [21], Ch. II, § 14, Theorem 33). We call the difference field $\bar{\mathcal{K}} = (\bar{K}, \bar{\tau})$ an algebraic closure of $\mathcal{K}$. For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if $K'/K$ is a field extension and $\tau'|_K = \tau$. In this case, we say that $\mathcal{K}'$ is a difference overfield of $\mathcal{K}$ and that $\mathcal{K}$ is a difference subfield of $\mathcal{K}'$. For brevity we sometimes use $(K, \tau')$ instead of $(K, \tau'|_K)$. We define a difference intermediate field in the proper way. Let $\mathcal{K}$ be a difference field, $L = (L, \tau)$ a difference overfield of $\mathcal{K}$ and $B$ a subset of $L$. The difference subfield $\mathcal{K}\langle B \rangle_{\mathcal{K}}$ of $L$ is defined to be the difference field $(K(B, \tau B, \tau^2 B, \ldots), \tau)$ and is denoted by $\mathcal{K}\langle B \rangle$ for brevity. A solution of a difference equation over $\mathcal{K}$ is defined to be an element of some difference overfield of $\mathcal{K}$ which satisfies the equation (cf. the books [3, 5]).

In Section 2, we modify the definition of the decomposable extensions, and review their properties. In Section 3, we study the following difference equation over an arbitrary difference field $\mathcal{K}$,

$$(y_2 + y)y_1^2 = \alpha y_1 + \beta, \quad \alpha, \beta \in K, \beta \neq 0,$$

where $\alpha_1 \neq \alpha$ or $\beta_2 \neq \beta$. In the typical case, $y_1$, $y_2$, $\alpha_1$, $\beta_2$ correspond to $y(x + \delta)$, $y(x + 2\delta)$, $\alpha(x + \delta)$, $\beta(x + 2\delta)$ respectively. We prove that there is no transcendental solution in any decomposable extension. In Section 4, we prove that the discrete Painlevé equation of type $D_7^{(1)}$ has no algebraic function solution when $\delta$ is transcendental over $Q(s)$.

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2. Decomposable extensions

The decomposable extension was defined by the author to study irreducibility of second-order algebraic difference equations including the discrete Painlevé equations and Poincaré's multiplication formulae (cf. [11, 13]). Since we found that a certain condition in the definition is unnecessary, we will modify the definition of the \((\mathcal{U})\)-decomposable extensions.

**Definition 1.** Let \(\mathcal{U}\) be a difference field and \(L/\mathcal{K}\) a difference field extension in \(\mathcal{U}\) of finite transcendence degree. We define \(\mathcal{U}\)-decomposable extensions by induction on \(\text{tr.deg} \ L/\mathcal{K}\).

(i) If \(\text{tr.deg} \ L/\mathcal{K} \leq 1\), then \(L/\mathcal{K}\) is \(\mathcal{U}\)-decomposable.

(ii) In the case \(\text{tr.deg} \ L/\mathcal{K} \geq 2\), \(L/\mathcal{K}\) is \(\mathcal{U}\)-decomposable if there are difference subfields \(\mathcal{E}, \mathcal{M}\) of \(\mathcal{U}\) such that

(a) \(\mathcal{E}\) is a difference overfield of \(\mathcal{K}\),

(b) \(E\) and \(L\) are free over \(K\),

(c) \(\mathcal{M}\) is a difference intermediate field of \(LE/\mathcal{E}\),

(d) \(\text{tr.deg} \ LE/\mathcal{M} \geq 1\), \(\text{tr.deg} \ \mathcal{M}/\mathcal{E} \geq 1\) and

(e) \(LE/\mathcal{M}\) and \(\mathcal{M}/\mathcal{E}\) are \(\mathcal{U}\)-decomposable.

**Definition 2.** Let \(L/\mathcal{K}\) be a difference field extension of finite transcendence degree such that \(L\) is algebraically closed. We define decomposable extensions by induction on \(\text{tr.deg} \ L/\mathcal{K}\).

(i) If \(\text{tr.deg} \ L/\mathcal{K} \leq 1\), then \(L/\mathcal{K}\) is decomposable.

(ii) In the case \(\text{tr.deg} \ L/\mathcal{K} \geq 2\), \(L/\mathcal{K}\) is decomposable if there are difference fields \(\mathcal{U}, \mathcal{E}, \mathcal{M}\) such that

(a) \(\mathcal{U}\) is an algebraically closed difference overfield of \(L\),

(b) \(\mathcal{E}\) is a difference overfield of \(\mathcal{K}\) in \(\mathcal{U}\),

(c) \(E\) and \(L\) are free over \(K\),

(d) \(\mathcal{M}\) is a difference intermediate field of \(LE/\mathcal{E}\),

(e) \(\text{tr.deg} \ LE/\mathcal{M} \geq 1\), \(\text{tr.deg} \ \mathcal{M}/\mathcal{E} \geq 1\) and

(f) \(LE/\mathcal{M}\) and \(\mathcal{M}/\mathcal{E}\) are decomposable, where \(LE\) and \(\mathcal{M}\) are the algebraic closure of \(LE\) and \(\mathcal{M}\) in \(\mathcal{U}\) respectively.

**Remark.** In the above definitions, the condition \(\text{tr.deg} \ E/\mathcal{K} < \infty\) is eliminated. Since it is the only change, we easily find that Bialynicki-Birula’s strongly normal extension of difference fields and a difference field extension \(\mathcal{K}\langle f \rangle/\mathcal{K}\) generated by a solution \(f\) of a linear difference equation over \(\mathcal{K}\) are all \(\mathcal{U}\)-decomposable in the new sense. See the papers [1, 8, 9] for details of Bialynicki-Birula’s strongly normal extension, and the paper [11] for details of a linear difference equation. Moreover, the following Proposition 3 and Proposition 4 hold also for the generalized decomposable extensions. They
make it possible to treat a chain of those extensions as a decomposable extension by taking algebraic closures. The proves of them are the same as the proves of the corresponding propositions in the paper [11].

**Proposition 3.** Let \( \mathcal{K} \subset \mathcal{L} \subset \mathcal{N} \) be a chain of decomposable extensions. Then \( \mathcal{N} / \mathcal{K} \) is decomposable.

**Proposition 4.** Let \( \mathcal{U} \) be a difference field, \( \mathcal{L}/\mathcal{K} \) a \( \mathcal{U} \)-decomposable extension, \( \mathcal{U} \) an algebraic closure of \( \mathcal{U} \) and \( \mathcal{D} \) the algebraic closure of \( \mathcal{L} \) in \( \mathcal{U} \). Then \( \mathcal{D}/\mathcal{K} \) is decomposable.

The following lemma is used in the next section.

**Lemma 5.** Let \( \mathcal{D}/\mathcal{K} \) be a decomposable extension and \( B \) a subset of \( D \). Suppose that for any difference overfield \( \mathcal{L} \) of \( \mathcal{K} \) and any difference overfield \( \mathcal{U} \) of \( \mathcal{L} \) and \( \mathcal{K}(B)/\mathcal{K} \), the following holds:

\[
\text{tr. deg } \mathcal{L}(B)_{\mathcal{U}}/\mathcal{L} \leq 1 \Rightarrow \text{ any } f \in B \text{ is algebraic over } \mathcal{L}.
\]

Then any \( f \in B \) is algebraic over \( K \).

**Proof.** The proof is almost the same as the proof of Lemma 9 in [11]. \( \square \)

3. **Proof of irreducibility**

In this section, We study irreducibility of difference equations of the form

\[
(y_2 + y)y_1^2 = \alpha y_1 + \beta.
\]

We need the following two lemmas. The first one is needed because of the refinement of the decomposable extensions.

**Lemma 6.** Let \( \mathcal{L} \) be a difference field and \( \mathcal{N} = (\mathcal{N}, \tau) \) a difference overfield of \( \mathcal{L} \). Let \( f, g \in \mathcal{N} \) be transcendental over \( L \) but algebraically dependent over \( L \). If all the transforms \( f_i \) and \( g_i \) \((i = 1, 2, 3, \ldots)\) are transcendental over \( L \), then there exist \( k \in \mathbb{Z}_{\geq 0} \) and an irreducible polynomial \( F \in \mathcal{L}[Y, Z] \{0\} \) such that

(i) \( F(f_k, g_k) = 0 \) and

(ii) \( F^\tau \) is irreducible over \( L \), where \( F^\tau \) is the polynomial whose coefficients are the transforms of the corresponding coefficients of \( F \) by \( \tau \).
Proof. Every pair of transforms $f_k$ and $g_k$ ($k = 0, 1, 2, \ldots$) are algebraically dependent over $L$, for the pair $f$ and $g$ are algebraically dependent over $L$. Hence for each $k \geq 0$, there exists an irreducible polynomial $F^{(k)} \in L[Y, Z] \setminus \{0\}$ such that $F^{(k)}(f_k, g_k) = 0$. Since the numbers $\deg F^{(k)}$ have the minimum, we choose such $k$.

From $F^{(k)}(f_k, g_k) = 0$, we obtain $F^{(k)\tau}(f_{k+1}, g_{k+1}) = 0$. Let $F^{(k)\tau} = AB$ ($A, B \in L[Y, Z] \setminus \{0\}$), where $A$ is an irreducible polynomial satisfying $A(f_{k+1}, g_{k+1}) = 0$. Since $F^{(k+1)} \in L[Y, Z]$ is irreducible and satisfy $F^{(k+1)}(f_{k+1}, g_{k+1}) = 0$, $A$ is divisible by $F^{(k+1)}$ (cf. the book [21], Ch. II, § 13, Lemma 2). This implies

$$\deg F^{(k+1)} \leq \deg A \leq \deg F^{(k)\tau} = \deg F^{(k)}.$$ 

We also have $\deg F^{(k)} \leq \deg F^{(k+1)}$ by the assumption about $k$. Hence we find

$$\deg A = \deg F^{(k)\tau} = \deg A + \deg B,$$

which yields $\deg B = 0$. Since $A \in L[Y, Z]$ is irreducible, we conclude that $F^{(k)\tau} = AB \in L[Y, Z]$ is also irreducible. \hfill \Box

Lemma 7. For $n, l \in \mathbb{Z}_{\geq 0}$ with $l \leq n/2$, we consider the matrix

$$M_{nl} = \begin{pmatrix} \binom{n}{0} & \binom{n-1}{0} & \ldots & \binom{n-l-1}{0} \\ \binom{n}{1} & \binom{n-1}{1} & \ldots & \binom{n-l}{1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{l} & \binom{n-1}{l} & \ldots & \binom{n-l+1}{l} \end{pmatrix}.$$ 

This satisfies

$$\det M_{nl} = 1.$$ 

Proof. We prove this by induction on $l$. In the case $l = 0$, we find

$$M_{nl} = \begin{pmatrix} n \\ 0 \end{pmatrix} = (1),$$

and thus $\det M_{nl} = 1$. We suppose $l \geq 1$ and the result is true for $l - 1$. Using the fact

$$\binom{i}{j} - \binom{i-1}{j} = \binom{i-1}{j-1} \quad (i > j \geq 1),$$

we are able to calculate the value of $\det M_{nl}$ in the following manner,
\[
\text{det } M_{nl} = \\
\begin{vmatrix}
(n-1) & (n-l+1) & \ldots & (n) \\
(n-1) & (n-l+1) & \ldots & (n) \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & (n-l+1) & \ldots & (n) \\
\end{vmatrix}
\]
\[
= \\
\begin{vmatrix}
(n-1) & (n-l+1) - (n-1) & \ldots & (n-1) - (n-2) & (n) - (n-1) \\
(n-1) & (n-l+1) - (n-1) & \ldots & (n-1) - (n-2) & (n) - (n-1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(n-1) & (n-l+1) - (n-1) & \ldots & (n-1) - (n-2) & (n) - (n-1) \\
\end{vmatrix}
\]
\[
= \\
\begin{vmatrix}
1 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \ldots & 0 & 0 \\
\end{vmatrix}
\]
\[
= \text{det } M_{n-1,l-1} = 1. \quad \square
\]

**Theorem 8.** Let \( \mathcal{L} \) be a difference field and \( f \) a solution of the equation over \( \mathcal{L} \),
\[(y + y')^2 = ax + \beta,\]
where \( a, \beta \in L, \beta \neq 0 \). If \( \text{tr. deg } \mathcal{L} \langle f \rangle / \mathcal{L} = 1 \), then \( a = x \) and \( \beta = y \).

**Proof.** Let \( \tau \) be the operator of the difference field \( \mathcal{L} \langle f \rangle \). We find that all the \( f_i, i = 0, 1, 2, \ldots \) are transcendental over \( L \). In fact, let \( i \geq 1 \) and suppose that this is true for \( i - 1 \). Since we have
\[(f_{i+1} + f_{i-1})f_i^2 = \alpha_{i-1}f_i + \beta_{i-1} \]
and \( f_i \neq 0 \), which is obtained from \( f_{i-1} \neq 0 \), we find
\[f_{i-1} = \frac{\alpha_{i-1}f_i + \beta_{i-1}}{f_i^2} - f_{i+1} \in L(f_i, f_{i+1}).\]
If \( f_i \) is algebraic over \( L \), then this implies that \( f_{i-1} \) is also algebraic over \( L \), which contradicts the induction hypothesis. Hence \( f_i \) is transcendental over \( L \).

Applying Lemma 6 to the pair \( f \) and \( f_1 \), we see that there exist \( k \in \mathbb{Z}_{\geq 0} \) and an irreducible polynomial \( F \in L[Y, Z] \setminus \{0\} \) such that \( F(f_k, f_{k+1}) = 0 \) and \( F^\tau \) is irreducible over \( L \). We fix such \( k \) and \( F \). We may assume
\[F = \sum_{i=0}^{m_0} \sum_{j=0}^{m_1} a_{ij}Y^iZ^j, \quad a_{ij} \in L,\]
where \( n_0 = \deg_Y F, \ n_1 = \deg_Z F \) and \( a_{n_0 n_1} \in \{0, 1\} \). From \( F(f_k, f_{k+1}) = 0 \), we obtain two equations,

\[
0 = F^\tau(f_{k+1}, f_{k+2}) = F \left( f_{k+1}, \frac{z_k f_{k+1} + \beta_k}{f^2_{k+1}} - f_k \right),
\]

\[
0 = F(f_k, f_{k+1}) = F \left( \frac{z_k f_{k+1} + \beta_k}{f^2_{k+1}} - f_{k+2}, f_{k+1} \right).
\]

Define two polynomials \( F_1 \) and \( F_0 \) as follows,

\[
F_1 = Z^{2n} F^\tau \left( Z, \frac{z_k Z + \beta_k}{Z^2} - Y \right) \in L[Y, Z \setminus \{0\}],
\]

\[
F_0 = Y^{2m} F \left( \frac{z_k Y + \beta_k}{Y^2} - Z, Y \right) \in L[Y, Z \setminus \{0\}].
\]

We easily see \( F_1(f_k, f_{k+1}) = F_0(f_{k+1}, f_{k+2}) = 0 \). Since the polynomials \( F \) and \( F^\tau \) are irreducible over \( L \), these imply \( F|F_1 \) and \( F^\tau|F_0 \), respectively. Hence the following is obtained,

\[
n_0 = \deg_Y F \leq \deg_Y F_1 \leq n_1 = \deg_Z F = \deg_Z F^\tau \leq \deg_Z F_0 \leq n_0,
\]

which yields \( n_0 = n_1 \). Let \( n = n_0 = n_1( \geq 1 ) \) for brevity.

Let \( F_1 = PF \) and \( F_0 = QF^\tau \), where \( P, Q \in L[Y, Z \setminus \{0\}] \). We find \( P \in L[Z] \subset L[Y, Z] \) and \( Q \in L[Y] \subset L[Y, Z] \), for the above equality implies

\[
\deg_Y P = \deg_Y F_1 - \deg_Y F = 0,
\]

\[
\deg_Z Q = \deg_Z F_0 - \deg_Z F^\tau = 0.
\]

To compare the coefficients of both sides of the equation \( F_1 = PF \), we begin by calculating each side separately,

\[
F_1 = Z^{2n} F^\tau \left( Z, \frac{z_k Z + \beta_k}{Z^2} - Y \right)
\]

\[
= Z^{2n} \sum_{i=0}^{n} \sum_{j=0}^{n} \tau(a_{ij}) Z^i \left( \frac{z_k Z + \beta_k}{Z^2} - Y \right)^j
\]

\[
= Z^{2n} \sum_{i=0}^{n} \sum_{j=0}^{n} \tau(a_{ij}) Z^i \sum_{h=0}^{j} \binom{j}{h} \left( \frac{z_k Z + \beta_k}{Z^2} \right)^{j-h} (-1)^h Y^h
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{h=0}^{j} \binom{j}{h} \tau(a_{ij}) Z^i Z^{2n} \left( \frac{z_k Z + \beta_k}{Z^2} \right)^{j-h} (-1)^h Y^h
\]
\[
\begin{align*}
&= \sum_{i=0}^{n} \sum_{h=0}^{n} \sum_{j=h}^{n} \left( \frac{j}{h} \right) \tau(a_{ij}) Z^i Z^{2(n-j+h)} (\alpha_k Z + \beta_k)^{j-h} (-1)^h Y^h \\
&= -\sum_{h=0}^{n} (-1)^h \left\{ \sum_{j=h}^{n} \left( \frac{j}{h} \right) Z^{2(n-j+h)} (\alpha_k Z + \beta_k)^{j-h} \sum_{i=0}^{n} \tau(a_{ij}) Z^i \right\} Y^h, \\
PF &= P \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} Y^i Z^j = \sum_{i=0}^{n} \left\{ P \sum_{j=0}^{n} a_{ij} Z^j \right\} Y^i = \sum_{h=0}^{n} \left\{ P \sum_{j=0}^{n} a_{ij} Z^j \right\} Y^h.
\end{align*}
\]

Comparing the coefficients of \( Y^h \), we obtain

\[
(1) \quad (-1)^h \sum_{j=h}^{n} \left( \frac{j}{h} \right) Z^{2(n-j+h)} (\alpha_k Z + \beta_k)^{j-h} \sum_{i=0}^{n} \tau(a_{ij}) Z^i = P \sum_{j=0}^{n} a_{ij} Z^j
\]

for \( 0 \leq h \leq n \). We use the symbol eq.(1)(h) to refer to the above equation with specific value of \( h \). For example, eq.(1)(n) and eq.(1)(0) denote the following respectively,

\[
(-1)^n Z^{2n} \sum_{i=0}^{n} \tau(a_{in}) Z^i = P \sum_{j=0}^{n} a_{ij} Z^j,
\]

\[
\sum_{j=0}^{n} Z^{2(n-j)} (\alpha_k Z + \beta_k)^{j} \sum_{i=0}^{n} \tau(a_{ij}) Z^i = P \sum_{j=0}^{n} a_{ij} Z^j.
\]

In the same way, it follows from \( F_0 = QF^T \) that

\[
(2) \quad (-1)^h \sum_{i=h}^{n} \left( \frac{i}{h} \right) Y^{2(n-i+h)} (\alpha_k Y + \beta_k)^{i-h} \sum_{j=0}^{n} a_{ij} Y^j = Q \sum_{i=0}^{n} \tau(a_{ih}) Y^i
\]

for \( 0 \leq h \leq n \). Eq.(2)(n) and eq.(2)(0) are the following respectively,

\[
(-1)^n Y^{2n} \sum_{j=0}^{n} a_{nj} Y^j = Q \sum_{i=0}^{n} \tau(a_{in}) Y^i,
\]

\[
\sum_{i=0}^{n} Y^{2(n-i)} (\alpha_k Y + \beta_k)^{i} \sum_{j=0}^{n} a_{ij} Y^j = Q \sum_{i=0}^{n} \tau(a_{0i}) Y^i.
\]

From eq.(1)(n) and eq.(2)(n), we obtain two equations,

\[
(-1)^n X^{2n} \sum_{i=0}^{n} \tau(a_{in}) X^i = P(Y, X) \sum_{j=0}^{n} a_{nj} X^j,
\]

\[
(-1)^n X^{2n} \sum_{j=0}^{n} a_{nj} X^j = Q(X, Z) \sum_{i=0}^{n} \tau(a_{in}) X^i.
\]
where \( P(Y, X), Q(X, Z) \in L[X] \). Multiplying them together, we obtain

\[
X^{4n} \left( \sum_{i=0}^{n} \tau(a_{i})X^{i} \right) \left( \sum_{j=0}^{n} a_{nj}X^{j} \right)
= P(Y, X)Q(X, Z) \left( \sum_{j=0}^{n} a_{nj}X^{j} \right) \left( \sum_{i=0}^{n} \tau(a_{i})X^{i} \right).
\]

By \( \deg_Y F = \deg_Z F = n \), we find that \( \sum_{j=0}^{n} a_{nj}X^{j} \) and \( \sum_{i=0}^{n} \tau(a_{i})X^{i} \) are non-zero. Hence the above equation yields

\[
X^{4n} = P(Y, X)Q(X, Z).
\]

We also find \( \deg P \leq 2n \) and \( \deg Q \leq 2n \). In fact, there exists some \( 0 \leq h' \leq n \) such that \( a_{h'n} \neq 0 \). Comparing the degrees of both sides of eq.(1)(\( h' \)), we obtain

\[
\deg P + n \leq \max_{0 \leq h' \leq n} \{ 2(n - j + h') + j - h' + n \}
= \max_{0 \leq h' \leq n} \{ 3(n - j + h') \} \leq 3n,
\]

and thus \( \deg P \leq 2n \). The other result can be seen in the same way. Hence the equation (3) implies that \( P = cZ^{2n} \) and \( Q = dY^{2n} \), where \( cd = 1 \).

Let \( m = \max \{ l \in Z \mid 0 \leq l < n/2 \} \). We will show that the following holds for all \( l = 0, 1, \ldots, m \):

\[
\begin{align*}
& a_{0n} = a_{1n} = \cdots = a_{2l+1,n} = 0, \\
& a_{0,n-1} = a_{1,n-1} = \cdots = a_{2l-1,n-1} = 0, \\
& \vdots \\
& a_{0,n-l} = a_{1,n-l} = 0.
\end{align*}
\]

We prove this by induction. In the case \( l = 0 \), we obtain \( \beta_{k}^{n} \tau(a_{0n}) = 0 \) by comparing the coefficients of \( Z^{0} \) of eq.(1)(0), which yields \( a_{0n} = 0 \) by the assumption \( \beta \neq 0 \). Comparing the coefficients of \( Z^{1} \), we obtain \( \beta_{k}^{n} \tau(a_{1n}) = 0 \), which yields \( a_{1n} = 0 \). We suppose \( 1 \leq l \leq m \) and the result is true for \( l - 1 \). For each \( h = 0, 1, \ldots, l \), we compare the coefficients of \( Z^{2l+2h} \) of eq.(1)(\( h \)), where \( 2l + 2h \leq 4m < 2n \):

\[
(-1)^{h} \left\{ \binom{n}{h} \beta_{k}^{n-h} \tau(a_{2l,n}) + \binom{n-1}{h} \beta_{k}^{n-1-h} \tau(a_{2l-2,n-1}) + \cdots \\
+ \binom{n-p}{h} \beta_{k}^{n-p-h} \tau(a_{2l-2p,n-p}) + \cdots + \binom{n-l}{h} \beta_{k}^{n-l-h} \tau(a_{0,n-l}) \right\} = 0.
\]
This implies
\[ \sum_{p=0}^{l} \binom{n-p}{h} \beta_k^{n-p} \tau(a_{2l-2p,n-p}) = 0, \]
and thus
\[ M_{nl} \begin{pmatrix} \beta_k^{n-l} \tau(a_{0,n-l}) \\ \beta_k^{n-l+1} \tau(a_{2,n-l+1}) \\ \vdots \\ \beta_k^n \tau(a_{2l,n}) \end{pmatrix} = 0, \]
where \( M_{nl} \) is the matrix defined in Lemma 7. Since \( \det M_{nl} \) is non-zero, we find \( a_{0,n-l} = a_{2,n-l+1} = \cdots = a_{2l,n} = 0. \)

For each \( h = 0, 1, \ldots, l \), we also compare the coefficients of \( Z^{2l+1+2h} \) of eq.(1)(h). Since the degree satisfies
\[ 2l + 1 + 2h \leq 4m + 1 \leq 2(n-1) + 1 < 2n, \]
we find
\[ (-1)^h \binom{n}{h} \beta_k^{n-h} \tau(a_{2l+1,n}) + \binom{n-1}{h} \beta_k^{n-1-h} \tau(a_{2l-1,n-1}) + \cdots + \binom{n-p}{h} \beta_k^{n-p-h} \tau(a_{2l+1-2p,n-p}) + \cdots + \binom{n-l}{h} \beta_k^{n-l-h} \tau(a_{1,n-l}) = 0, \]
which implies
\[ \sum_{p=0}^{l} \binom{n-p}{h} \beta_k^{n-p} \tau(a_{2l+1-2p,n-p}) = 0. \]
Hence we obtain
\[ M_{nl} \begin{pmatrix} \beta_k^{n-l} \tau(a_{1,n-l}) \\ \beta_k^{n-l+1} \tau(a_{3,n-l+1}) \\ \vdots \\ \beta_k^n \tau(a_{2l+1,n}) \end{pmatrix} = 0, \]
and thus \( a_{1,n-l} = a_{3,n-l+1} = \cdots = a_{2l+1,n} = 0. \) As a result, we obtained
\[
\begin{aligned}
a_{0n} &= a_{1n} = \cdots = a_{2m+1,n} = 0, \\
a_{0,n-1} &= a_{1,n-1} = \cdots = a_{2m-1,n-1} = 0, \\
&\vdots \\
a_{0,n-m} &= a_{1,n-m} = 0.
\end{aligned}
\]
In the same way, it follows from the equation (2) that

\[
\begin{align*}
 a_{n0} &= a_{n1} = \cdots = a_{n, 2m+1} = 0, \\
 a_{n-1, 0} &= a_{n-1, 1} = \cdots = a_{n-1, 2m-1} = 0, \\
 &\vdots \\
 a_{n-m, 0} &= a_{n-m, 1} = 0.
\end{align*}
\]

If \( n \) was odd, then \( m \) would be \( (n - 1)/2 \) by definition. In this case, we would see \( 2m + 1 = n \) and thus \( a_{0n} = a_{1n} = \cdots = a_{nn} = 0 \). Hence \( n \) is even. By definition, we find \( m = n/2 - 1 \), and thus \( 2m + 1 = n \). This implies \( a_{0n} = a_{1n} = \cdots = a_{n-1, n} = 0 \). Since we supposed \( a_{nn} \in \{0, 1\} \), we obtain \( a_{nn} = 1 \). Comparing the coefficients of \( Z^{3n} \) of eq.(1), we obtain \((-1)^n \tau(a_{nn}) = c \alpha_m \), which yields \( c = d = 1 \).

We continue comparing the coefficients of the equations (1) and (2). For each \( h = 0, 1, \ldots, m \), we compare the coefficients of \( Z^{n+2h} \) of eq.(1)(\( h \)), where \( n + 2h \leq n + 2m = 2n - 2 < 2n \):

\[
(-1)^h \left\{ \binom{n}{h} \beta_k^{n-h} \tau(a_{nn}) + \cdots + \binom{n-p}{h} \beta_k^{n-p-h} \tau(a_{n-2p, n-p}) \right. \\
+ \cdots + \left. \binom{n-(m+1)}{h} \beta_k^{n-(m+1)-h} \tau(a_{0, n-(m+1)}) \right\} = 0.
\]

This implies

\[
\binom{n}{h} \beta_k^n + \sum_{p=1}^{m+1} \binom{n-p}{h} \beta_k^{n-p} \tau(a_{n-2p, n-p}) = 0,
\]

and thus

\[
(4) \quad M_{n-1, m} \begin{pmatrix} 
\beta_k^{n-(m+1)} \tau(a_{0, n-(m+1)}) \\
\beta_k^{n-m} \tau(a_{2, n-m}) \\
\vdots \\
\beta_k^{n-1} \tau(a_{n-2, n-1}) 
\end{pmatrix} = -\beta_k^n \begin{pmatrix} 
\binom{n}{0} \\
\binom{n}{1} \\
\vdots \\
\binom{n}{m} 
\end{pmatrix}.
\]

In the same way, we obtain the following from the equation (2),

\[
(5) \quad M_{n-1, m} \begin{pmatrix} 
\beta_k^{n-(m+1)} a_{n-(m+1), 0} \\
\beta_k^{n-m} a_{n-m, 2} \\
\vdots \\
\beta_k^{n-1} a_{n-1, n-2} 
\end{pmatrix} = -\beta_k^n \begin{pmatrix} 
\binom{n}{0} \\
\binom{n}{1} \\
\vdots \\
\binom{n}{m} 
\end{pmatrix}.
\]
Hence it follows that

\[ \beta_k^{-p} a_{n-p,n-2p} = \beta_k^{-p} \tau(a_{n-2p,n-p}) \quad (p = 1, 2, \ldots, m + 1), \]

which yields

\[ \tau(a_{n-2p,n-p}) = a_{n-p,n-2p} \quad (p = 1, 2, \ldots, m + 1), \]

especially \( \tau(a_{n-2,n-1}) = a_{n-1,n-2} \).

Next, we compare the coefficients of \( Z^{n+1+2h} \) of eq.(1)(h) for each \( h = 0, 1, \ldots, m \), where \( n + 1 + 2h \leq n + 1 + 2m = 2n - 1 < 2n \):

\[
(-1)^h \left\{ \binom{n}{h} (n-h) \sum_{k} \beta_k^{-h-1} \tau(a_{mh}) + \cdots \\
+ \binom{n-p}{h} ((n-p-h) \sum_{k} \beta_k^{-p-h-1} \tau(a_{n-2p,n-p}) + \beta_k^{-p-h} \tau(a_{n-2p+1,n-p})) \\
+ \cdots + \binom{n-(m+1)}{h} ((n-(m+1)-h) \sum_{k} \beta_k^{-n-(m+1)-h-1} \tau(a_{0,n-(m+1)}) \\
+ \beta_k^{-n-(m+1)-h} \tau(a_{1,n-(m+1)})) \right\} \\
= 0.
\]

This implies

\[
\binom{n}{h} (n-h) \sum_{k} \beta_k^{-n-1} \\
+ \sum_{p=1}^{m+1} \binom{n-p}{h} ((n-p-h) \sum_{k} \beta_k^{-p-n-1} \tau(a_{n-2p,n-p}) + \beta_k^{-p} \tau(a_{n-2p+1,n-p})) \\
= 0,
\]

and thus

\[
\binom{n-1}{h} \sum_{k} \beta_k^{-n-1} + \sum_{p=1}^{m+1} \binom{n-p-1}{h} (n-p) \sum_{k} \beta_k^{-p-n-1} \tau(a_{n-2p,n-p}) \\
+ \sum_{p=1}^{m+1} \binom{n-p}{h} \beta_k^{-p-n} \tau(a_{n-2p+1,n-p}) = 0.
\]

Hence we obtain
Finally, we obtain the following from the equation (2) in the same way,

\[
\begin{pmatrix}
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\
0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & (n-1) & \cdots & (n-1)
\end{pmatrix}
+ M_{n-1,m}
\begin{pmatrix}
\beta_k^n a_{n-m,1} \\
\beta_k^{n-1} a_{n-m,2} \\
\vdots \\
\beta_k^{n-(m+1)} a_{n-1,1}
\end{pmatrix} = 0.
\]

By the equations (6), (7) and (8), we find

\[
M_{n-1,m}
\begin{pmatrix}
\beta_k^n a_{n-m,1} \\
\beta_k^{n-1} a_{n-m,2} \\
\vdots \\
\beta_k^{n-(m+1)} a_{n-1,1}
\end{pmatrix} = M_{n-1,m}
\begin{pmatrix}
\beta_k^{n-m} a_{n-m,1} \\
\beta_k^{n-1} a_{n-m,2} \\
\vdots \\
\beta_k^{n-(m+1)} a_{n-1,1}
\end{pmatrix},
\]

which yields

\[
\tau(a_{n-2p+1,n-p}) = a_{n-p,n-2p+1} \quad (p = 1, 2, \ldots, m + 1),
\]

especially \(\tau(a_{n-1,n-1}) = a_{n-1,n-1}\).

We derive \(\tau(x) = x\) from eq.(1)(n - 1),

\[
\begin{align*}
\left\{ nZ^{2n-2}(x_kZ + \beta_k) \sum_{i=0}^{n} \tau(a_{m})Z^i + Z^{2n} \sum_{i=0}^{n} \tau(a_{n-1})Z^i \right\} \\
= Z^{2n} \sum_{j=0}^{n} a_{n-1,j}Z^j.
\end{align*}
\]
Comparing the coefficients of $Z^{3n-1}$, we obtain
\[-(nz_k \tau(a_{nn}) + \tau(a_{n-1,n-1})) = a_{n-1,n-1},\]
and thus $z_k = -(2/n)a_{n-1,n-1}$ by $\tau(a_{n-1,n-1}) = a_{n-1,n-1}$. Hence it follows that
\[\tau(z_k) = -\frac{2}{n} \tau(a_{n-1,n-1}) = -\frac{2}{n} a_{n-1,n-1} = z_k,\]
which implies $\tau(z) = z$.

We will prove $\tau^2(\beta) = \beta$, the remaining. Comparing the coefficients of $Z^{3n-2}$ of the above equation, we obtain
\[-(n\beta_k \tau(a_{nn}) + \tau(a_{n-2,n-1})) = a_{n-1,n-2},\]
and thus $\beta_k = -(2/n)a_{n-1,n-2}$ by $\tau(a_{n-2,n-1}) = a_{n-1,n-2}$. We also compare the coefficients of $Z^{3n-1}$ of eq.(1)(n-2),
\[\sum_{j=n-2}^{n} \binom{n}{j} Z^{2(n-j+n-2)}(z_k Z + \beta_k)_{j-(n-2)} \sum_{i=0}^{n} \tau(a_i) Z^j = Z^{2n} \sum_{j=0}^{n} a_{n-2,j} Z^j.\]

Then we obtain
\[\binom{n-1}{n-2} z_k \tau(a_{n,n-1}) + \tau(a_{n-1,n-2}) = a_{n-2,n-1},\]
and thus $\tau(a_{n-1,n-2}) = a_{n-2,n-1}$ by $\tau(a_{n,n-1}) = 0$. Hence we find
\[\tau(\beta_k) = -\frac{2}{n} \tau(a_{n-1,n-2}) = -\frac{2}{n} a_{n-2,n-1},\]

\[\tau^2(\beta_k) = -\frac{2}{n} \tau(a_{n-2,n-1}) = -\frac{2}{n} a_{n-1,n-2} = \beta_k,\]
which implies $\tau^2(\beta) = \beta$. 

Corollary 9. Let $K$ be a difference field and $f$ a solution of the equation over $K$ of
\[(y_2 + y) y_1^2 = z y_1 + \beta, \quad z, \beta \in K, \beta \neq 0,\]
where $z_1 \neq z$ or $\beta_2 \neq \beta$. If there exists a decomposable extension $D/K$ such that $K \langle f \rangle \subset D$, then $f$ is algebraic over $K$. 

Proof. For any difference overfield $\mathcal{L}$ of $\mathcal{K}$ and any difference overfield $\mathcal{U}$ of $\mathcal{L}$, if we suppose $\text{tr.deg} \mathcal{L} \langle f \rangle_{\mathcal{L}} / \mathcal{L} \leq 1$, then $f \in \mathcal{U}$ is a solution of the equation over $\mathcal{L}$,

$$(y_2 + y)\gamma_1^2 = \alpha \gamma_1 + \beta,$$

for we have

$$\mathcal{K} \langle f \rangle = \mathcal{K} \langle f \rangle_{\mathcal{L}} = \mathcal{K} \langle f \rangle_{\mathcal{U}} \subset \mathcal{U}.$$ 

In this case, we find $\text{tr.deg} \mathcal{L} \langle f \rangle_{\mathcal{U}} / \mathcal{L} = 0$ by Theorem 8 and the assumption, $\alpha_1 \neq \alpha$ or $\beta_2 \neq \beta$. Hence $f$ is algebraic over $\mathcal{L}$.

By Lemma 5, we conclude that $f$ is algebraic over $\mathcal{K}$. \hfill \qed

4. Discrete Painlevé equation of type $D_7^{(1)}$

Let $\mathcal{K}$ be an algebraically closed field and $\mathcal{K}(x)$ a rational function field. Let $\delta \in \mathcal{K}^\times$ and $\mathcal{K} = (\mathcal{K}(x), x \mapsto x + \delta)$. In this section, the discrete Painlevé equation of type $D_7^{(1)}$ is the difference equation over $\mathcal{K}$,

$$d-\text{P}(D_7): (y_2 + y)\gamma_1^2 = s(x + \delta) \gamma_1 - s^2, \quad s \in \mathcal{K}^\times.$$

By Corollary 9, it is seen that $d-\text{P}(D_7)$ is irreducible in the sense of decomposable extensions. We will study non-existence of algebraic function solutions by using the following lemma.

Lemma 10 (Lemma 2 in S. Nishioka [10]). Let $t$ be transcendental over $\mathcal{K}$, $\mathcal{F} = \mathcal{K}(t)$ a finite algebraic field extension, and $\tau \in \text{Aut} (\mathcal{F}/\mathcal{K})$ satisfy $\tau t = t + 1$. Then $\mathcal{F} = \mathcal{K}(t)$.

Theorem 11. Suppose that $\delta$ is transcendental over $\mathcal{K}(x)$. Then any solution $f$ of $d-\text{P}(D_7)$ is transcendental over $\mathcal{K}(x)$.

Proof. Let $f$ be a solution and assume that $f$ is algebraic over $\mathcal{K}(x)$. We will derive a contradiction. By Lemma 10 with $t = x/\delta$, we find that $f$ is a rational function, namely $f \in \mathcal{K}(x)$. Let $f = A/B$, where $A, B \in \mathcal{K}[x] \setminus \{0\}$ are relatively prime. Then we obtain

$$\left(\frac{A_2}{B_2} + \frac{A}{B} \right) A_1^2 \frac{A_1}{B_1}^2 = s(x + \delta) \frac{A_1}{B_1} - s^2,$$

and thus

$$(A_2 B + AB_2) A_1^2 = B_2 B_1 B (s(x + \delta) A_1 - s^2 B_1).$$
Comparing the degrees, we find

$$3 \deg A = 2 \deg B + \deg(s(x + \delta)A_1 - s^2B_1).$$

In the above calculation, we note $\deg(A_2B + AB_2) = \deg A + \deg B$. From this equation, we will obtain $\deg A = \deg B - 1$. If $\deg A \geq \deg B$, then the right side is $2 \deg B + \deg A + 1$, which implies $2 \deg A = 2 \deg B + 1$. The number of left side is even and the other is odd, a contradiction. Hence $\deg A \leq \deg B - 1$. If $\deg A < \deg B - 1$, then the right side is $3 \deg B$, which implies $\deg A = \deg B - 1$.

Since we obtained $\deg B - \deg A = 1$, the solution $f = A/B$ can be represented by the following power series,

$$f = \sum_{i=1}^{\infty} \frac{a_i}{x^i}, \quad a_i \in C, \ a_1 \neq 0.$$

Let

$$\bar{f} = f_1 = \sum_{i=1}^{\infty} \frac{a_i}{x^i} \left(1 - \frac{\delta}{x} + \frac{\delta^2}{x^2} - \cdots \right)^i,$$

$$\underline{f} = f_{-1} = \sum_{i=1}^{\infty} \frac{a_i}{x^i} \left(1 + \frac{\delta}{x} + \frac{\delta^2}{x^2} + \cdots \right)^i.$$

Then

$$(\bar{f} + \underline{f}) f^2 = sxf - s^2,$$

where the right side is

$$sa_1 - s^2 + s \sum_{i=1}^{\infty} \frac{a_{i+1}}{x^i}.$$

Comparing the coefficients of $1/x^0$ of the above equation, we find $sa_1 - s^2 = 0$, and thus $a_1 = s$. Comparing the coefficients of $1/x^1$ and $1/x^2$, we also find $a_2 = 0$ and $a_3 = 0$ respectively. In the same way, we see that for all $k \geq 3$,

$$sa_{k+1} \in \mathbb{Z}[\delta, a_1, \ldots, a_{k-2}].$$

In fact, the coefficient of $1/x^k$ of $(\bar{f} + \underline{f}) f^2$ equals that of

$$\left(\sum_{i=1}^{k-2} \frac{a_i}{x^i} \left(1 - \frac{\delta}{x} + \cdots \right)^i + \sum_{i=1}^{k-2} \frac{a_i}{x^i} \left(1 + \frac{\delta}{x} + \cdots \right)^i\right) \left(\sum_{i=1}^{k-2} \frac{a_i}{x^i}\right)^2,$$

which is in $\mathbb{Z}[\delta, a_1 \ldots, a_{k-2}]$. Hence we find $a_i \in \mathbb{Z}[\delta, s, s^{-1}]$ by induction.
Let $f$ be the homomorphism of $\mathbb{Q}(s)[\delta]$ to $\mathbb{Q}(s)$ such that $\phi(\delta) = 0$ and $\phi|_{\mathbb{Q}(s)} = \text{id}$, and let $\varphi$ be the homomorphism of $\mathbb{Q}(s)[[1/x]]$ to $\mathbb{Q}(s)[[1/x]]$ such that $\varphi(d) = 0$ and $\varphi|_{\mathbb{Q}(s)} = \text{id}$. Let $g = \varphi(f)$ for brevity.

We will prove $g \in C(x)$. Since $f \in C(x) = C(1/x)$, there exist $n \in \mathbb{Z}_{\geq 0}$ and $m_0 \in \mathbb{Z}_{\geq 0}$ such that $m \geq m_0 \Rightarrow F_f(m, n) = 0$,

where $F_f(m, n)$ is the Hankel determinant of $f$, namely

$$F_f(m, n) = \det(a_{m+i+j})_{0 \leq i, j \leq n}.$$

Refer to the book [2] by J. W. S. Cassels for the Hankel determinant. For all $m \geq m_0$,

$$F_g(m, n) = \det(\phi(a_{m+i+j}))_{0 \leq i, j \leq n} = \phi(\det(a_{m+i+j})_{0 \leq i, j \leq n}) = \phi(F_f(m, n)) = \phi(0) = 0,$$

which implies $g \in C(1/x) = C(x)$.

Let $g = P/Q$, where $P, Q \in C[x] \setminus \{0\}$ are relatively prime and $Q$ is monic. Then we obtain

$$2P^3/Q^3 = sxP - s^2Q,$$

and thus

$$2P^3 = Q^2(sxP - s^2Q).$$
It follows that $Q|P^3$, which implies $Q = 1$. Hence the above equation is rewritten as follows,

$$2P^3 = sxP - s^2.$$  

Comparing the degrees, we find $3 \deg P = 1 + \deg P$, and thus $2 \deg P = 1$. This is impossible.

**Corollary 12.** Suppose that $\delta$ is transcendental over $Q(s)$. Then there is no solution of $\delta$-$P(D_D)$ in any decomposable extension $\mathcal{D}/\mathcal{H}$.

**Proof.** This is straightforwardly proved by Corollary 9 and Theorem 11. 

**References**

[1] Bialynicki-Birula, A., On Galois theory of fields with operators, Amer. J. Math., 84 (1962), 89–109.

[2] Cassels, J. W. S., *Local Fields*, Cambridge University Press, 1986.

[3] Cohn, R. M., *Difference Algebra*, Interscience Publishers, New York · London · Sydney, 1965.

[4] Grammaticos, B., Ramani, A. and Papageorgiou, V. G., Do integrable mappings have the Painlevé property?, Phys. Rev. Lett., 67 (1991), 1825–1828.

[5] Levin, A., *Difference Algebra*, Springer Science+Business Media B.V., 2008.

[6] Nakazono, N. and Nishioka, S., Solutions to a $q$-analog of Painlevé III equation of type $D_9^{(1)}$, Funkcialaj Ekvacioj, 56 (2013), 415–439.

[7] Nishioka, K., A note on the transcendency of Painlevé’s first transcendent, Nagoya Math. J., 109 (1988), 63–67.

[8] Nishioka, S., Difference algebra associated to the $q$-Painlevé equation of type $A_1^{(1)}$, Differential equations and exact WKB analysis, 167–176, RIMS Kōkyūroku Bessatsu, B10, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[9] Nishioka, S., On Solutions of $q$-Painlevé Equation of Type $A_7^{(1)}$, Funkcial. Ekvac., 52 (2009), 41–51.

[10] Nishioka, S., Transcendence of solutions of $q$-Painlevé equation of type $A_7^{(1)}$, Aequat. Math., 79 (2010), 1–12.

[11] Nishioka, S., Decomposable extensions of difference fields, Funkcial. Ekvac., 53 (2010), 489–501.

[12] Nishioka, S., Irreducibility of $q$-Painlevé equation of type $A_6^{(1)}$ in the sense of order, J. Differ. Equ. Appl., 18 (2012), 313–333.

[13] Nishioka, S., Functions satisfying Poincaré’s multiplication formula, Osaka J. Math., 51 (2014), 141–160.

[14] Ohyama, Y., Kawamuko, H., Sakai, H. and Okamoto, K., Studies on the Painlevé equations. V. Third Painlevé equations of special type $P_{III}(D_7)$ and $P_{III}(D_8)$, J. Math. Sci. Univ. Tokyo, 13 (2006), 145–204.

[15] Painlevé, P., Leçon de Stockholm, Oeuvres de P. Painlevé I, pp. 199–818, Éditions du C.N.R.S., Paris, 1972.

[16] Ramani, A., Grammaticos, B. and Hietarinta, J., Discrete versions of the Painlevé equations, Phys. Rev. Lett., 67 (1991), 1829–1832.
[17] Ramani, A. and Grammaticos, B., Discrete Painlevé equations: coalescences, limits and degeneracies, Physica A, 228 (1996), 160–171.
[18] Sakai, H., Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys., 220 (2001), 165–229.
[19] Sakai, H., Problem: discrete Painlevé equations and their Lax forms, Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies, 195–208, RIMS Kôkyûroku Bessatsu, B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.
[20] Umemura, H., On the Irreducibility of the First Differential Equation of Painlevé, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, pp. 771–789, Kinokuniya, Tokyo, 1987.
[21] Zariski, O. and Samuel, P., Commutative Algebra Volume I, Springer-Verlag, New York, NY, 1958.

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