THE PROBABILITY MEASURE CORRESPONDING TO 2-PLANE TREES

BY

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Abstract. We study the probability measure \( \mu_0 \) for which the moment sequence is \( \binom{3n}{n} \frac{1}{n+1} \). We prove that \( \mu_0 \) is absolutely continuous, find the density function and prove that \( \mu_0 \) is infinitely divisible with respect to the additive free convolution.

2000 AMS Mathematics Subject Classification: Primary: 44A60; Secondary: 46L54.

Key words and phrases: Beta distribution, additive and multiplicative free convolution, Meijer G-function.

1. INTRODUCTION

A 2-plane tree is a planted plane tree such that each vertex is colored black or white and for each edge at least one of its ends is white. Gu and Prodinger [3] proved that the number of 2-plane trees on \( n + 1 \) vertices with black (white) root is \( \binom{3n+1}{n} \frac{1}{3n+1} \) (Fuss–Catalan number of order three, sequence A001764 in OEIS [10]) and \( \binom{3n+2}{n} \frac{2}{3n+2} \) (sequence A006013 in OEIS) respectively (see also [4]). We will study the sequence

\[
\binom{3n}{n} \frac{2}{n+1} = \binom{3n+1}{n} \frac{1}{3n+1} + \binom{3n+2}{n} \frac{2}{3n+2},
\]

which begins with

2, 3, 10, 42, 198, 1001, 5304, 29070, 163438, \ldots,

of total numbers of such trees (A007226 in OEIS).

* W. M. is supported by the Polish National Science Center grant No. 2012/05/B/ST1/00626.
** K. A. P. acknowledges support from Agence Nationale de la Recherche (Paris, France) under Program PHYSCOMB No. ANR-08-BLAN-0243-2.
Both the sequences on the right-hand side of (1.1) are positive definite (see [5] and [6]), therefore so is the sequence \( \binom{3n}{n} \frac{2}{n+1} \) itself. In this paper we study the corresponding probability measure \( \mu_0 \), i.e. such that the numbers \( \binom{3n}{n} \frac{1}{n+1} \) are moments of \( \mu_0 \). First we prove that \( \mu_0 \) is Mellin convolution of two beta distributions, in particular \( \mu_0 \) is absolutely continuous. Then we find the density function of \( \mu_0 \). In the last section we prove that \( \mu_0 \) can be decomposed as additive free convolution \( \mu_1 \boxplus \mu_2 \) of two measures \( \mu_1 \) and \( \mu_2 \), which are both infinitely divisible with respect to \( \boxplus \) and are related to the Marchenko–Pastur distribution. In particular, the measure \( \mu_0 \) itself is \( \boxplus \)-infinitely divisible.

2. THE GENERATING FUNCTION

Let us consider the generating function

\[
G(z) = \sum_{n=0}^{\infty} \binom{3n}{n} \frac{2z^n}{n+1}.
\]

According to (1.1), \( G \) can be represented as a sum of two generating functions. The former is usually denoted by \( B_3 \),

\[
B_3(z) = \sum_{n=0}^{\infty} \binom{3n+1}{n} \frac{z^n}{3n+1},
\]

and satisfies the equation

\[
B_3(z) = 1 + z \cdot B_3(z)^3. \tag{2.1}
\]

Lambert’s formula (see (5.60) in [2]) implies that the latter is just square of \( B_3 \),

\[
B_3(z)^2 = \sum_{n=0}^{\infty} \binom{3n+2}{n} \frac{2z^n}{3n+2},
\]

so we have

\[
G(z) = B_3(z) + B_3(z)^2. \tag{2.2}
\]

Combining (2.1) and (2.2), we obtain the following equation for \( G \):

\[
2 - z - (1 + 2z)G(z) + 2zG(z)^2 - z^2G(z)^3 = 0, \tag{2.3}
\]

which will be applied later on.

Now we will give a formula for \( G(z) \).

**Proposition 2.1.** For the generating function of the sequence (1.1) we have

\[
G(z) = \frac{12 \cos^2 \alpha + 6}{(4 \cos^2 \alpha - 1)^2}, \tag{2.4}
\]

where \( \alpha = \frac{1}{3} \arcsin \left( \sqrt{\frac{27z}{4}} \right) \).
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Proof. Defining \((a)_n := a(a + 1) \ldots (a + n - 1)\) we have

\[
\frac{2(3n)!}{(n + 1)! (2n)!} = \frac{-2 \left( \frac{-2}{3} \right)_{n+1} \left( \frac{-1}{2} \right)_{n+1}}{3(n + 1)! \left( \frac{-1}{2} \right)_{n+1}} 2^{7n+1}.
\]

Therefore

\[
G(z) = \frac{2 - 2 \cdot 2 \cdot F_1 \left( \frac{-2}{3}, \frac{-1}{3}; \frac{1}{2} \mid \frac{27z}{4} \right)}{3z}.
\]

Now we apply the formula

\[
2F1 \left( \frac{-2}{3}, \frac{-1}{3}, \frac{-1}{2} \mid u \right) = \frac{1}{3} \sqrt{u} \sin \left( \frac{1}{3} \arcsin \left( \sqrt{u} \right) \right) + \sqrt{1 - u} \cos \left( \frac{1}{3} \arcsin \left( \sqrt{u} \right) \right),
\]

which can be checked by verifying the hypergeometric equation (note that both the functions \(w \mapsto w \sin \left( \frac{1}{3} \arcsin (w) \right)\) and \(w \mapsto \cos \left( \frac{1}{3} \arcsin (w) \right)\) are even, so the right-hand side is well defined for \(|u| < 1\)). Putting \(\alpha = \frac{1}{3} \arcsin \left( \sqrt{u} \right), u = 27z/4\), we have \(\sqrt{u} = \sin 3\alpha, \sqrt{1 - u} = \cos 3\alpha,\) which leads to (2.4).

3. The Measure

Now we want to study the (unique) measure \(\mu_0\) for which \(\{ (\binom{3n}{n})_{n=0} \infty \) is the moment sequence. We will show that \(\mu_0\) can be expressed as the Mellin convolution of two beta distributions. Then we will provide an explicit formula for the density function \(V(x)\) of \(\mu_0\).

Recall (see [1]) that for \(\alpha, \beta > 0\), the beta distribution \(\text{Beta}(\alpha, \beta)\) is the absolutely continuous probability measure defined by the density function

\[
f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1} (1 - x)^{\beta-1}
\]

for \(x \in (0, 1)\). The moments of \(\text{Beta}(\alpha, \beta)\) are

\[
\int_0^1 x^n f_{\alpha,\beta}(x) \, dx = \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(\alpha + \beta + n)} = \prod_{i=0}^{n-1} \frac{\alpha + i}{\alpha + \beta + i}.
\]

For probability measures \(\nu_1, \nu_2\) on the positive half-line \([0, \infty)\) the Mellin convolution is defined by

\[
(\nu_1 \circ \nu_2)(A) := \int_0^\infty \int_0^\infty \chi_A(xy) d\nu_1(x) d\nu_2(y)
\]
for every Borel set \( A \subseteq [0, \infty) \) \( (\chi_A \) denotes the indicator function of the set \( A \)).
This is the distribution of the product \( X_1 \cdot X_2 \) of two independent nonnegative random variables with \( X_i \sim \nu_i \). In particular, if \( c > 0 \) then \( \nu \circ \delta_c \) is the dilation of the measure \( \nu \):
\[
(\nu \circ \delta_c) (A) = D_c \nu (A) := \nu \left( \frac{1}{c} A \right),
\]
where \( \delta_c \) denotes the Dirac delta measure at \( c \).
If both the measures \( \nu_1, \nu_2 \) have all moments \( s_n(\nu_i) := \int_0^\infty x^n d\nu_i(x) \) finite, then so has \( \nu_1 \circ \nu_2 \) and
\[
s_n \left( \nu_1 \circ \nu_2 \right) = s_n(\nu_1) \cdot s_n(\nu_2)
\]
for all \( n \). The method of Mellin convolution has been recently applied to a number of related problems, see for example [6] and [8].
From now on we will study the probability measure corresponding to the sequence \( \left( \frac{3^n}{n} \right)_{n+1} \).

**Proposition 3.1.** Define \( \mu_0 \) as the Mellin convolution:
\[
\mu_0 = \text{Beta}(1/3, 1/6) \circ \text{Beta}(2/3, 4/3) \circ \delta_{27/4}.
\]
Then the numbers \( \left( \frac{3^n}{n} \right)_{n+1} \) are moments of \( \mu_0 \):
\[
\int_0^{27/4} x^n d\mu_0(x) = \left( \frac{3n}{n} \right) \frac{1}{n+1}.
\]

**Proof.** It is sufficient to check that
\[
\frac{(3n)!}{(n+1)! (2n)!} = \prod_{i=0}^{n-1} \frac{1}{i+1/2} \cdot \prod_{i=0}^{n-1} \frac{2/3+i}{2+i} \cdot \left( \frac{27}{4} \right)^n.
\]
In view of formula (3.2), the measure \( \mu_0 \) is absolutely continuous and its support is the interval \([0, 27/4]\). Now we want to find the density function \( V(x) \) of the measure \( \mu_0 \).

**Theorem 3.1.** Let
\[
V(x) = \frac{\sqrt{3}}{2^{10/3} \pi x^{2/3}} \left( 3 \sqrt{1 - 4x/27} - 1 \right) \left( 1 + \sqrt{1 - 4x/27} \right)^{1/3}
+ \frac{1}{2^{8/3} \pi x^{1/3} \sqrt{3}} \left( 3 \sqrt{1 - 4x/27} + 1 \right) \left( 1 + \sqrt{1 - 4x/27} \right)^{-1/3},
\]
x \in (0, 27/4). Then \( V \) is the density function of \( \mu_0 \), i.e.
\[
\int_0^{27/4} x^n V(x) \, dx = \left( \frac{3n}{n} \right) \frac{1}{n+1}
\]
for \( n = 0, 1, 2, \ldots \).
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(a) The densities of $\mu_1$, $\mu_2$

(b) The density of $\mu_0 = \mu_1 \boxplus \mu_2$

Figure 1. The densities of $\mu_1$, $\mu_2$ and $\mu_0 = \mu_1 \boxplus \mu_2$

The density $V(x)$ of $\mu_0$ is represented in Figure 1 (b).

Proof. Putting $n = s - 1$ and applying the Gauss–Legendre multiplication formula

$$\Gamma(mz) = (2\pi)^{(1-m)/2}m^{mz-1/2}\Gamma(z)\Gamma(z + \frac{1}{m})\Gamma(z + \frac{2}{m})\cdots\Gamma(z + \frac{m-1}{m})$$

we obtain

$$\binom{3n}{n} = \frac{\Gamma(3n+1)}{\Gamma(n+2)\Gamma(2n+1)} = \frac{\Gamma(3s-2)}{\Gamma(s+1)\Gamma(2s-1)}$$

$$= \frac{2}{27}\sqrt{\frac{3}{\pi}} \left( \frac{27}{4} \right)^s \frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)} := \psi(s).$$

Then $\psi$ can be extended to an analytic function on the complex plane, except for the points $1/3 - n$, $2/3 - n$, $n = 0, 1, 2, \ldots$

Now we want to apply a particular type of the Meijer $G$-function, see [9] for details. Let $\tilde{V}$ denote the inverse Mellin transform of $\psi$. Then we have

$$\tilde{V}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \psi(s) \, ds$$

$$= \frac{2}{27}\sqrt{\frac{3}{\pi}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)} \left( \frac{4x}{27} \right)^{-s} \, ds$$

$$= \frac{2}{27}\sqrt{\frac{3}{\pi}} G_{2,2}^{1,1} \left( \frac{4x}{27}, -1/2, 1 \mid -2/3, -1/3 \right).$$
where \( x \in (0, 27/4) \) (see [11] for the role of \( c \) in the integrals). On the other hand, for the parameters of the \( G \)-function we have

\[
(-2/3 - 1/3) - (-1/2 + 1) = -3/2 < 0,
\]

and hence the assumptions of formula 2.24.2.1 in [9] are satisfied. Therefore we can apply the Mellin transform on \( \tilde{V}(x) \):

\[
\int_0^{27/4} x^{s-1} \tilde{V}(x) \, dx = 2 \sqrt{\frac{3}{\pi}} \frac{27}{4} \frac{4x}{27} \begin{pmatrix} 4x \mid -1/2, 1 \end{pmatrix} \begin{pmatrix} -2/3, -1/3 \end{pmatrix}^{-2/3} \begin{pmatrix} 2 \begin{pmatrix} -2\frac{2}{3}, 5\frac{2}{3}; \frac{4x}{27} \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 2 \begin{pmatrix} -1\frac{1}{3}, 7\frac{4}{3}; \frac{4x}{27} \end{pmatrix} \end{pmatrix} = \psi(s)
\]

whenever \( \Re s > 2/3 \). Consequently, \( \tilde{V} = V \).

Now we use Slater’s formula (see [9], formula 8.2.2.3) and express \( V \) in terms of the hypergeometric functions:

\[
V(x) = 2 \sqrt{\frac{3}{\pi}} \frac{27}{4} \frac{4x}{27} \begin{pmatrix} 4x \mid -1/2, 1 \end{pmatrix} \begin{pmatrix} -2/3, -1/3 \end{pmatrix}^{-2/3} \begin{pmatrix} 2 \begin{pmatrix} -2\frac{2}{3}, 5\frac{2}{3}; \frac{4x}{27} \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 2 \begin{pmatrix} -1\frac{1}{3}, 7\frac{4}{3}; \frac{4x}{27} \end{pmatrix} \end{pmatrix} = \sqrt{\frac{3}{4\pi x^{2/3}}} 2F1\left(\frac{-2\frac{2}{3}, 5\frac{2}{3}; \frac{4x}{27}}{3\frac{6}{3}}\right) + \frac{1}{2\pi \sqrt{3}x^{1/3}} 2F1\left(\frac{-1\frac{1}{3}, 7\frac{4}{3}; 4x}{3\frac{6}{3}}\right).
\]

Applying the formula

\[
2F1\left(\frac{t - 2}{2}, \frac{t + 1}{2}; t \mid z\right) = \frac{2t}{2t} \left( t - 1 + \sqrt{1 - z} \right) \left( 1 + \sqrt{1 - z} \right)^{1-t}
\]

(see [6]) for \( t = 2/3 \) and \( t = 4/3 \) we complete the proof.

### 4. RELATIONS WITH FREE PROBABILITY

In this part we describe relations of \( \mu_0 \) with free probability. In particular, we will show that \( \mu_0 \) is infinitely divisible with respect to the additive free convolution. Let us briefly describe the additive and multiplicative free convolutions. For details we refer to [12] and [7].

Denote by \( \mathcal{M}_c \) the class of probability measures on \( \mathbb{R} \) with compact support. For \( \mu \in \mathcal{M}_c \), with moments

\[
s_m(\mu) := \int_{\mathbb{R}} t^m \, d\mu(t)
\]
and the moment generating function

\[ M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m = \int_\mathbb{R} \frac{d\mu(t)}{1 - tz}. \]

we define its R-transform \( R_\mu(z) \) by the equation

\[ (4.1) \quad R_\mu(z M_\mu(z)) + 1 = M_\mu(z). \]

Then the additive free convolution of \( \mu', \mu'' \in \mathcal{M}^c \) is defined as the unique measure \( \mu' \boxplus \mu'' \in \mathcal{M}^c \) which satisfies

\[ R_{\mu' \boxplus \mu''}(z) = R_{\mu'}(z) + R_{\mu''}(z). \]

If the support of \( \mu \in \mathcal{M}^c \) is contained in the positive half-line \([0, +\infty)\) then we define its S-transform \( S_\mu(z) \) by

\[ (4.2) \quad M_\mu\left(\frac{z}{1 + z} S_\mu(z)\right) = 1 + z \quad \text{or} \quad R_\mu(z S_\mu(z)) = z \]

on a neighborhood of zero. If \( \mu', \mu'' \) are such measures then their multiplicative free convolution \( \mu' \boxtimes \mu'' \) is defined by

\[ S_{\mu' \boxtimes \mu''}(z) = S_{\mu'}(z) \cdot S_{\mu''}(z). \]

Recall that for dilated measure \( D_{c,\mu} \) we have

\[ M_{D_{c,\mu}}(z) = M_\mu(c z), \quad R_{D_{c,\mu}}(z) = R_\mu(c z), \quad \text{and} \quad S_{D_{c,\mu}}(z) = S_\mu(z) / c. \]

The operations \( \boxplus \) and \( \boxtimes \) can be regarded as free analogs of the classical and Mellin convolution.

For \( t > 0 \) let \( \varpi_t \) denote the Marchenko–Pastur distribution with parameter \( t \),

\[ (4.3) \quad \varpi_t = \max\{1 - t, 0\} \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} \, dx, \]

with the absolutely continuous part supported on \([ (1 - \sqrt{t})^2, (1 + \sqrt{t})^2 \]). Then

\[ (4.4) \quad M_{\varpi_t}(z) = \frac{2}{1 + z - tz + \sqrt{(1 - z - tz)^2 - 4tz^2}} \]

\[ = 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{n}{k} \left( \frac{n}{k - 1} \right) \frac{t^k}{n}, \]

\[ (4.5) \quad R_{\varpi_t}(z) = \frac{tz}{1 - z}, \quad S_{\varpi_t}(z) = \frac{1}{t + z}. \]

In free probability the measures \( \varpi_t \) play the role of the Poisson distributions. Note that by (4.5) the family \( \{ \varpi_t \}_{t > 0} \) constitutes a semigroup with respect to \( \boxplus \), i.e. we have \( \varpi_s \boxplus \varpi_t = \varpi_{s+t} \) for \( s, t > 0 \).
THEOREM 4.1. The measure \( \mu_0 \) can be decomposed as the additive free convolution \( \mu_0 = \mu_1 \boxplus \mu_2 \), where \( \mu_1 = D_2 \varpi_1/2 \), so that

\[
\mu_1 = \frac{1}{2} \delta_0 + \frac{\sqrt{8 - (x - 3)^2}}{4 \pi x} \chi_{(3 - \sqrt{8}, 3 + \sqrt{8})}(x) \, dx,
\]

and \( \mu_2 = \frac{1}{2} \delta_0 + \frac{1}{2} \varpi_1 \), i.e.

\[
\mu_2 = \frac{1}{2} \delta_0 + \frac{\sqrt{4x - x^2}}{4 \pi x} \chi_{(0, 4)}(x) \, dx.
\]

The measures \( \mu_1, \mu_2 \) are infinitely divisible with respect to the additive free convolution \( \boxplus \), and, consequently, so is \( \mu_0 \).

The absolutely continuous parts of the measures \( \mu_1 \) and \( \mu_2 \) are represented in Figure 1 (a).

Proof. The moment generating function of \( \mu_0 \) is \( M_{\mu_0}(z) = G(z)/2 \). Then we have \( M_{\mu_0}(0) = 1 \) and, by (2.3),

\[
2 - z - 2(1 + 2z) M_{\mu_0}(z) + 8z M_{\mu_0}(z)^2 - 8z^2 M_{\mu_0}(z)^3 = 0.
\]

Let \( T(z) \) be the inverse function for \( M_{\mu_0}(z) - 1 \), so that we have \( T(0) = 0 \) and \( M_{\mu_0}(T(z)) = 1 + z \). Then

\[
2 - T(z) + (-1 - 2T(z))2(1 + z) + 8T(z)(1 + z)^2 - 8T(z)^2(1 + z)^3 = 0,
\]

which gives

\[
8(1 + z)^3 T(z)^2 - (8z^2 + 12z + 3) T(z) + 2z = 0,
\]

and finally

\[
T(z) = \frac{8z^2 + 12z + 3 - \sqrt{9 + 8z}}{16(1 + z)^3} = \frac{4z}{8z^2 + 12z + 3 + \sqrt{9 + 8z}}.
\]

Therefore we can find the \( S \)-transform of \( \mu_0 \):

\[
S_{\mu_0}(z) = \frac{1 + z}{z} T(z) = \frac{8z^2 + 12z + 3 - \sqrt{9 + 8z}}{16z(1 + z)^2} = \frac{4(1 + z)}{8z^2 + 12z + 3 + \sqrt{9 + 8z}};
\]

consequently, from (4.2) we get the \( R \)-transform

\[
R_{\mu_0}(z) = \frac{4z - 1 + \sqrt{1 - 2z}}{2(1 - 2z)}.
\]
Now we observe that $R_{\mu_0}(z)$ can be decomposed as follows:

$$R_{\mu_0}(z) = \frac{z}{1 - 2z} + \frac{1 - \sqrt{1 - 2z}}{2\sqrt{1 - 2z}} = R_1(z) + R_2(z).$$

Comparing this formula with (4.5) we observe that $R_1(z)$ is the $R$-transform of $\mu_1 = D_2 \varpi_{1/2}$, which implies that $\mu_1$ is $\boxplus$-infinitely divisible.

Consider the Taylor expansion of $R_2(z)$:

$$R_2(z) = \sum_{n=1}^{\infty} \left(\frac{2n}{n}\right) z^{n-1} \frac{z^n}{2 + z^2} \sum_{n=0}^{\infty} \left(\frac{2(n+2)}{n+2}\right) z^n = R_1(z) + R_2(z).$$

Since the numbers $\left(\frac{2n}{n}\right)$ are moments of the arcsine distribution

$$\frac{1}{\pi \sqrt{x(4-x)}} \chi_{(0,4)}(x) \, dx,$$

the coefficients of the last sum constitute a positive definite sequence. So $R_2(z)$ is the $R$-transform of a probability measure $\mu_2$, which is $\boxplus$-infinitely divisible (see Theorem 13.16 in [7]). Now using (4.1) we obtain

$$M_{\mu_2}(z) = 1 + 2z - \sqrt{1 - 4z} = 1 + 2z - 2\sqrt{1 - 4z} = \frac{1}{2} + \frac{1}{1 + \sqrt{1 - 4z}}.$$

Comparing this formula with (4.4) for $t = 1$ we see that $\mu_2 = \frac{1}{2} \delta_0 + \frac{1}{2} \varpi_1$. 

Let us now consider the measures $\mu_1, \mu_2$ separately. For $\mu_1 = D_2 \varpi_{1/2}$ the moment generating function is

$$M_{\mu_1}(z) = \frac{2}{1 + z + \sqrt{1 - 6z + z^2}} = 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{n}{k-1}\right) \frac{2^{n-k}}{n},$$

so the moments are

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots$$

This is the A001003 sequence in OEIS (little Schroeder numbers). $s_n(\mu_1)$ is the number of ways to insert parentheses in product of $n + 1$ symbols. There is no restriction on the number of pairs of parentheses. The number of objects inside a pair of parentheses must be at least two.

On the subject of $\mu_2$, applying (4.2) we can find the $S$-transform:

$$S_{\mu_2}(z) = \frac{2(1 + z)}{(1 + 2z)^2} = \frac{1 + z}{1/2 + z} \cdot \frac{1}{1 + 2z}.$$ 

One can check that $(1 + z)/(1/2 + z)$ is the $S$-transform of $\frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$, which yields

$$\mu_2 = \left(\frac{1}{2} \delta_0 + \frac{1}{2} \delta_1\right) \boxplus \mu_1.$$ 

(4.8)
Acknowledgments. We would like to thank G. Aubrun, C. Banderier, K. Górska, and H. Prodinger for fruitful interactions.

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