Computational Complexity of Synchronization under Sparse Regular Constraints

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Abstract. The constrained synchronization problem (CSP) asks for a synchronizing word of a given input automaton contained in a regular set of constraints. It could be viewed as a special case of synchronization of a discrete event system under supervisory control. Here, we study the computational complexity of this problem for the class of sparse regular constraint languages. We give a new characterization of sparse regular sets, which equal the bounded regular sets, and derive a full classification of the computational complexity of CSP for letter-bounded regular constraint languages, which properly contain the strictly bounded regular languages. Then, we introduce strongly self-synchronizing codes and investigate CSP for bounded languages induced by these codes. With our previous result, we deduce a full classification for these languages as well. In both cases, depending on the constraint language, our problem becomes NP-complete or polynomial time solvable.

Keywords: automata theory · constrained synchronization · computational complexity · sparse languages · bounded languages · strongly self-synchronizing codes

1 Introduction

A deterministic semi-automaton is synchronizing if it admits a reset word, i.e., a word which leads to a definite state, regardless of the starting state. This notion has a wide range of applications, from software testing, circuit synthesis, communication engineering and the like, see [13,45,47]. The famous Černý conjecture [11] states that a minimal synchronizing word, for an $n$ state automaton, has length at most $(n - 1)^2$. We refer to the mentioned survey articles for details [15,47].

Due to its importance, the notion of synchronization has undergone a range of generalizations and variations for other automata models. The paper [19] introduced the constrained synchronization problem (CSP). In this problem, we search for a synchronizing word coming from a specific subset of allowed input sequences. To sketch a few applications:

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1 In computer science the acronym CSP is usually used for the constraint satisfaction problem [55]. However, as here we are not concerned with constrained satisfaction problems at all, no confusion should arise.
Reset State. In [19] one motivating example was the demand that a system, or automaton thereof, to synchronize has to first enter a “directing” mode, perform a sequence of operations, and then has to leave this operating mode and enter the “normal operating mode” again. In the most simple case, this constraint could be modelled by $ab^*a$, which, as it turns out [19], yields an NP-complete CSP. Even more generally, it might be possible that a system — a remotely controlled rover on a distant planet, a satellite in orbit, or a lost autonomous vehicle — is not allowed to execute all commands in every possible order, but certain commands are only allowed in certain order or after other commands have been executed. All of this imposes constraints on the possible reset sequences.

Part Orienters. Suppose parts arrive at a manufacturing site and they need to be sorted and oriented before assembly. Practical considerations favor methods which require little or no sensing, employ simple devices, and are as robust as possible. This could be achieved as follows. We put parts to be oriented on a conveyor belt which takes them to the assembly point and let the stream of the parts encounter a series of passive obstacles placed along the belt. Much research on synchronizing automata was motivated by this application [12,17,25,37,38,47] and I refer to [47] for an illustrative example. Now, furthermore, assume the passive components could not be placed at random along the belt, but have to obey some restrictions, or restrictions in what order they are allowed to happen. These could be due to the availability of components, requirements how to lay things out or physical restrictions.

Supervisory Control. The CSP could also be viewed of as supervisory control of a discrete event system (DES) that is given by an automaton and whose event sequence is modelled by a formal language [10,12,50]. In this framework, a DES has a set of controllable and uncontrollable events. Dependent on the event sequence that occurred so far, the supervisor is able to restrict the set of events that are possible in the next step, where, however, he can only limit the use of controllable events. So, if we want to (globally) reset a finite state DES [2] under supervisory control, this is equivalent to CSP.

Biocomputing. In [4,5] DNA molecules have been used as both hardware and software for finite automata of nanoscaling size, see also [47]. For instance, Benenson et al [4] produced “a ‘soup of automata’, that is, a solution containing $3 \times 10^{12}$ identical automata per µl. All these molecular automata can work in parallel on different inputs, thus ending up in different and unpredictable states. In contrast to an electronic computer, one cannot reset such a system by just pressing a button; instead, in order to synchronously bring each automaton to its start state, one should splice the soup with (sufficiently many copies of) a DNA molecule whose nucleotide sequences encodes a reset word” [47]. Now, it might be possible that certain sequences, or subsequences, are not possible as they might have unwanted biological side-effects, or might destroy the molecules at all.

Reduction Procedure. This example is more formal and comes from attempts to solve the Černý conjecture [47]. In [27] a special rank factorization [41]
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for automata was introduced from which smaller automata could be derived. Then, it was shown that the original automaton is synchronizing if and only if the reduced automaton admits a synchronizing word in a certain regular constraint language, and the reset threshold, i.e. the lengths of the shortest synchronizing word, of the original automaton could be bounded by that of the shortest one in the constraint language for the reduced automaton.

In [19], a complete analysis of the complexity landscape when the constraint language is given by small partial automata was done. It is natural to extend this result to other language classes.

In general there exist constraint languages yielding \text{PSPACE}-complete constrained problems [19]. A language is polycyclic [31], if it is recognizable by an automaton such that every strongly connected component forms a single cycle, and a language is sparse [51] if only polynomially many words of a specific length are in the language. As shown in [31] for polycyclic languages, which, as we show, equal the sparse regular languages, the problem is always in \text{NP}. This motivates investigating this class further. Also, as written in more detail in Remark 7, a subclass of these languages has a close relation to the commutative languages, and as for commutative constraint languages a trichotomy result has been established [30], tackling the sparse languages seems to be the next logical step. In fact, we show a dichotomy result for a subclass that contains the class corresponding to the commutative languages. Additionally, as has been noted in [19], the constraint language $ab^*a$ is the smallest language, in terms of a recognizing automaton, giving an \text{NP}-complete CSP. The class of languages for which our dichotomy holds true contains this language.

Let us mention that restricting the solution space by a regular language has also been applied in other areas, for example to topological sorting [3], solving word equations [15,16], constraint programming [39], or shortest path problems [43]. The famous road coloring theorem [1,46] states that every finite strongly connected and directed aperiodic graph of uniform out-degree admits a labelling of its edges such that a synchronizing automaton results. A related problem to our problem of constrained synchronization is to restrict the possible labelling(s), and this problem was investigated in [49].

Outline and Contribution. Here, we look at the complexity landscape for sparse regular constraint languages. In Section 3 we introduce the sparse languages and show that the regular sparse languages are characterized by polycyclic automata introduced in [31]. A similar characterization in terms of non-deterministic automata was already given in [21, Lemma 2]. In this sense, we extend this characterization to the deterministic case. As for polycyclic constraint automata the constrained problem is always in \text{NP}, see [31, Theorem 2], we can deduce the same for sparse regular constraint languages, which equal the bounded regular languages [34].

In Section 4 we introduce the letter-bounded languages, a proper subset of the sparse languages, and show that for letter-bounded constraint languages, the constrained synchronization problem is either in \text{P} or \text{NP}-complete.
The difficulty why we cannot handle the general case yet lies in the fact that in the reductions, in the general case, we need auxiliary states and it is not clear how to handle them properly, i.e., how to synchronize them properly while staying inside the constraint language.

In Section 5 we introduce the class of strongly self-synchronizing codes. The strongly self-synchronizing codes allow us to handle these auxiliary states mentioned before. We show that for homomorphisms given by such codes, the constrained problem for the homomorphic image of a language has the same computational complexity as for the original language. This result holds in general, and hence is of independent interest. Here we apply it to the special case of bounded, or sparse, regular languages given by such codes.

Lastly, we present a bounded language giving an NP-complete constrained problem that could not be handled by our methods so far.

2 Preliminaries and Definitions

We assume the reader to have some basic knowledge in computational complexity theory and formal language theory, as contained, e.g., in [32]. For instance, we make use of regular expressions to describe languages. By theory and formal language theory, as contained, e.g., in [32]. For instance, we make use of regular expressions to describe languages. By theory and formal language theory, as contained, e.g., in [32]. For instance, we make use of regular expressions to describe languages.
state \( q \in Q \) with \( \delta(Q, w) = \{ q \} \) for some \( w \in \Sigma^* \) a synchronizing state. For a semi-automaton (or PDFA) with state set \( Q \) and transition function \( \delta : Q \times \Sigma \to Q \), a state \( q \) is called a sink state, if for all \( x \in \Sigma \) we have \( \delta(q, x) = q \). Note that, if a synchronizing automaton has a sink state, then the synchronizing state is unique and must equal the sink state.

In [19] the constrained synchronization problem (CSP) was defined for a fixed PDFA \( B = (\Sigma, P, \mu, p_0, F) \).

**Decision Problem 1:** [19] \( L(B)\)-Constr-Sync

**Input:** DCSA \( A = (\Sigma, Q, \delta) \).

**Question:** Is there a synchronizing word \( w \in \Sigma^* \) for \( A \) with \( w \in L(B) \)?

The automaton \( B \) will be called the constraint automaton. If an automaton \( A \) is a yes-instance of \( L(B)\)-Constr-Sync we call \( A \) synchronizing with respect to \( B \). Occasionally, we do not specify \( B \) and rather talk about \( L\)-Constr-Sync. For example, for the unconstrained case, we have \( \Sigma^*\)-Constr-Sync \( \in \mathbb{P} \) [14].

In our \( \text{NP} \)-hardness reduction, we will need the following problem from [31].

**Decision Problem 2:** \text{DisjointSetTransporter}

**Input:** DCSA \( A = (\Sigma, Q, \delta) \) and disjoint \( S, T \subseteq Q \).

**Question:** Is there a word \( w \in \Sigma^* \) such that \( \delta(p, S \cup w) \subseteq T \)?

**Theorem 1.** For unary deterministic and complete input semi-automata the problem \text{DisjointSetTransporter} is \( \text{NP}\)-complete.

A PDFA \( A = (\Sigma, Q, \delta, q_0, F) \) is called polycyclic, if for each \( q \in Q \) there exists \( u \in \Sigma^* \) such that \( \{ w \in \Sigma^* \mid \delta(q, w) = q \} \subseteq u^* \). A PDFA is polycyclic if and only if every strongly connected component consists of a single cycle [31]. Proposition 3, where each transition in the cycle is labelled by precisely one letter. Formally, for each strongly connected component \( S \subseteq Q \) and \( q \in S \), we have \( \{ x : x \in \Sigma \text{ and } \delta(q, x) \text{ is defined and in } S \} \leq 1 \) (note that in the special case \( |S| = 1 \), the aforementioned set might be empty if the single state in \( S \) has no self-loops). A precursor of this characterization of polycyclic automata in a special case was given in [20] under the term linear cycle automata.

The following slight generalization of [19] Theorem 27 will be needed.

**Proposition 2.** Let \( \varphi : \Sigma^* \to F^* \) be a homomorphism. Then, for each regular \( L \subseteq \Sigma^* \), we have \( \varphi(L)\)-Constr-Sync \( \preceq^\text{log} L\)-Constr-Sync.

### 3 Sparse and Bounded Regular Languages

Here, in Theorem 5 we establish that for constraint languages from the class of sparse regular languages, which equals the class of the bounded regular languages [34], the constrained problem is always in \( \text{NP} \).

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\(^2\) In [31], I made an error in my formalization by writing \( |\{ \delta(q, x) : x \in \Sigma, \delta(q, x) \text{ is defined} \} \cap S| \leq 1 \).
A language $L \subseteq \Sigma^*$ is sparse, if there exists $c \geq 0$ such that, for every $n \geq 0$, we have $L \cap \Sigma^n \in O(n^c)$. Sparse languages were introduced into computational complexity theory by Berman & Hartmanis [6]. Later, it was established by Mahaney that if there exists a sparse NP-complete set (under polynomial-time many-one reductions), then $P = NP$ [30]. For a survey on the relevance of sparse sets in computational complexity theory, see [28].

A language $L \subseteq \Sigma^*$ is called bounded, if there exist $w_1, \ldots, w_k \in \Sigma^*$ such that $L \subseteq w_1^* \cdots w_k^*$. Bounded languages were introduced by Ginsburg & Spanier [23].

We will need the following representation of the bounded regular languages.

**Theorem 3 ([24]).** A language $L \subseteq w_1^* \cdots w_k^*$ is regular if and only if it is a finite union of languages of the form $L_1 \cdots L_k$, where each $L_i \subseteq w_i^*$ is regular.

It is known that the class of sparse regular languages equals the class of bounded regular languages [34], or see [40,51], where the bounded languages are not mentioned but the equivalence is implied by their results and Theorem 3.

The next results links this class to the polycyclic PDFAs.

**Proposition 4.** Let $L \subseteq \Sigma^*$ be regular. Then, $L$ is sparse if and only if it is recognizable by a polycyclic DFA.

In [31] Theorem 2] it was shown that for polycyclic constraint languages, the constrained problem is always in NP. So, we can deduce the next result.

**Theorem 5.** If $L \subseteq \Sigma^*$ is sparse and regular, then $L$-CONSTR-SYNC $\in$ NP.

We will need the following closure property stated in [51] Theorem 3.8] of the sparse regular languages.

**Proposition 6.** The class of sparse regular languages is closed under homomorphisms.

Note that the connection of the polycyclic languages to the sparse or bounded languages was not noted in [31]. However, a condition characterizing the sparse regular languages in terms of forbidden patterns was given in [40], and it was remarked that “a minimal deterministic automaton recognises a sparse language if and only if it does not contain two cycles reachable from one another”. This is quite close to our characterization.

### 4 Letter-Bounded Constraint Languages

Fix a constraint automaton $B = (\Sigma, P, \mu, p_0, F)$. Let $a_1, \ldots, a_k \in \Sigma$ be a sequence of (not necessarily distinct) letters. In this section, we assume $L(B) \subseteq a_1^* \cdots a_k^*$. A language which fulfills the above condition is called letter-bounded. Note that the language $ab^*a$ given in the introduction as an example is letter-bounded. In fact, it is the language with the smallest recognizing automaton yielding an NP-complete constrained problem [19].
A language such that the $a_i$ are pairwise distinct, i.e., $a_i \neq a_j$ for $i \neq j$, is called strictly bounded. The class of strictly bounded languages has been extensively studied \cite{[14],[22],[23],[24],[29]}, where in \cite{[22],[23],[24]} no name was introduced for them and in \cite{[29]} they were called strongly bounded. The class of letter-bounded languages properly contains the strictly bounded languages.

Remark 7. Let $\Sigma = \{b_1, \ldots, b_r\}$ be an alphabet of size $r$. Then, the mappings

$$\Phi(L) = L \cap b_1^* \cdots b_r^*$$

and

$$\text{perm}(L) = \{w \in \Sigma^* \mid \exists u \in L \forall a \in \Sigma : |u|_a = |w|_a\}$$

for $L \subseteq \Sigma^*$ are mutually inverse and inclusion preserving between the languages in $b_1^* \cdots b_r^*$ and the commutative languages in $\Sigma^*$, where a language $L \subseteq \Sigma^*$ is commutative if $\text{perm}(L) = L$. Furthermore, for strictly bounded languages of the form $B_1 \cdots B_r \subseteq b_1^* \cdots b_r^*$ with $B_j \subseteq \{b_j\}^*$, $j \in \{1, \ldots, r\}$, we have $\text{perm}(B_1 \cdots B_r) = B_1 \sqcup \cdots \sqcup B_r$, where $U \sqcup V = \{u_1v_1 \cdots u_nv_n \mid u_i, v_i \in \Sigma^*, u_1 \cdots u_n \in U, v_1 \cdots v_n \in V\}$ for $U, V \subseteq \Sigma^*$. Hence, $\text{perm}(L)$ is regularity-preserving for strictly bounded languages. More specifically, the above correspondence between the two language classes is regularity-preserving in both directions. For commutative constraint languages, a classification of the complexity landscape has been achieved \cite{[30]}. By the close relationship between commutative and certain strictly bounded languages, it is natural to tackle this language class next. However, as shown in \cite{[30]}, for commutative constraint languages, we can realize PSPACE-complete problems, but, by Theorem \cite{[3]} for strictly bounded languages, the constrained problem is always in NP. However, by the above relations, Theorem \cite{[3]} for languages in $b_1^* \cdots b_r^*$ is equivalent to \cite{[30]} Theorem 5), a representation result for commutative regular languages.

Our first result says, intuitively, that if in $A_1 \cdots A_k$ with $A_j$ unary and regular, if no infinite unary language $A_j$ over $\{a_j\}$ lies between non-empty unary languages over a distinct letter\footnote{Hence different from $\{\varepsilon\}$, as $\{\varepsilon\} \subseteq \{a\}^*$ for $a \in \Sigma$.} than $a_j$, then $(A_1 \cdots A_k)\text{-CONSTR-SYNC}$ is in P.

Proposition 8. Let $A_j \subseteq \{a_j\}^*$ be unary regular languages for $j \in \{1, \ldots, k\}$. Set $L = A_1 \cdots A_k$. If for all $j \in \{1, \ldots, k\}$, $A_j$ infinite implies that $A_i \subseteq \{a_j\}^*$ for all $i < j$ or $A_i \subseteq \{a_j\}^*$ for all $i > j$ (or both), then $L\text{-CONSTR-SYNC} \in \mathcal{P}$.

Now, we state a sufficient condition for NP-hardness over binary alphabets. This condition, together with Proposition \cite{[2]} allows us to handle the general case in Theorem \cite{[10]} Its application together with Proposition \cite{[2]} shows, in some respect, that the language $ab^*a$ is the prototypical language giving NP-hardness. We give a proof sketch of Lemma \cite{[6]} at the end of this section.

Lemma 9. Suppose $\Sigma = \{a, b\}$. Let $L(\mathcal{B}) \subseteq \Sigma^*$ be letter-bounded. Then, $L(\mathcal{B})\text{-CONSTR-SYNC}$ is NP-hard if $L(\mathcal{B}) \cap \Sigma^*ab^*b^*a \Sigma^* \neq \emptyset$.

So, finally, we can state our main theorem of this section. Recall that by Theorem \cite{[3]} and as the class of bounded regular languages equals the class of sparse regular languages \cite{[34]}, for bounded regular constraint languages, the constrained problem is, in our case, in NP.
Theorem 10 (Dichotomy Theorem). Let \( a_1, \ldots, a_k \in \Sigma \) be a sequence of letters and \( L \subseteq a_1^* \cdots a_k^* \) be regular. The problem \( L\text{-}\text{Constr-Sync} \) is \( \text{NP} \)-complete if

\[
L \cap \left( \bigcup_{1 \leq j_1 < j_2 < j_3 \leq k} L_{j_1,j_2,j_3} \right) \neq \emptyset
\]

with \( L_{j_1,j_2,j_3} = \Sigma^* a_{j_1} \Sigma^* a_{j_2}^{|P|} \Sigma^* a_{j_3} \Sigma^* \) for \( 1 \leq j_1 < j_2 < j_3 \leq k \) and solvable in polynomial time otherwise.

As the languages \( L_{j_1,j_2,j_3} \) are regular, we can devise a polynomial-time algorithm which checks the condition mentioned in Theorem 10.

Corollary 11. Given a PDFA \( B \) and a sequence of letters \( a_1, \ldots, a_k \) as input such that \( L(B) \subseteq a_1^* \cdots a_k^* \), the complexity of \( L(B)\text{-}\text{Constr-Sync} \) is decidable in polynomial-time.

Proof. An automaton for each \( L_{j_1,j_2,j_3} \) has size linear in \( |P| \). So, by the product automaton construction\footnote{[32]}, non-emptiness of \( L(B) \) with each \( L_{j_1,j_2,j_3} \) could be checked in time \( O(|P|^2) \). Doing this for every \( L_{j_1,j_2,j_3} \) gives a polynomial-time algorithm to check non-emptiness of the language written in Theorem 10.

Example 12. For the following constraint languages CSP is \( \text{NP} \)-complete: \( ab*a, \; aa(aaa)^*bbb^*d \cup a^*b \cup d^*, \; bbcc^*d^* \cup a \).

For the following constraint languages CSP is \( \text{P} \): \( a^5bd \cup cd^4, \; a^5bd \cup cd^8, \; aa^*bbbcd^* \cup bbdd^*d \).

Proof (Proof Sketch for Lemma 7). We construct a reduction from an instance of \textsc{DisjointSetTransporter}\footnote{Note that the problem \textsc{DisjointSetTransporter} is over a unary alphabet, but for \( L\text{-}\text{Constr-Sync} \) we have \( |\Sigma| > 1 \). Indeed, we need the additional letters in \( \Sigma \).} for unary input automata.

To demonstrate the basic idea, we only do the proof in the case \( L \subseteq a^*b^*a^* \).

By assumption we can deduce \( a^{r_1}b^{r_2}a^{r_3} \in L(B) \) with \( p_2 \geq |P| \) and \( r_1, r_3 \geq 1 \). By the pigeonhole principle, in \( B \), when reading the factor \( b^{r_2} \), at least one state has to be traversed twice. Hence, we find \( 0 < p_2 \leq |P| \) such that \( a^{r_1}b^{r_2+i}a^{r_3} \subseteq L(B) \) for each \( i \geq 0 \).

Let \( A = (\{e\}, Q, \delta) \) and \((A, S, T)\) be an instance of \textsc{DisjointSetTransporter}. We can assume \( S \) and \( T \) are non-empty, as for \( S = \emptyset \) it is solvable, and if \( T = \emptyset \) we have no solution. Construct \( A' = (\Sigma, Q', \delta') \) by setting \( Q' = S_{r_2} \cup \ldots \cup S_1 \cup Q \cup Q_1 \cup \ldots \cup Q_{p_2-1} \cup \{t\} \), where \( t \) is a new state, \( S_i = \{s_i \mid s \in S\} \) for \( i \in \{1, \ldots, r_2\} \) are pairwise disjoint copies of \( S \) and \( Q_i = \{q^i \mid q \in Q\} \) are also pairwise disjoint copies of \( Q \). Note that also
$S_i \cap Q_j = \emptyset$ for $i \in \{1, \ldots, r_2\}$ and $j \in \{1, \ldots, p_2 - 1\}$. Set $S_0 = S$ as a shorthand. Choose any $s \in S_{r_2}$, then, for $q \in Q$ and $x \in \Sigma$, the transition function is given by

$$
\delta'(q, x) = \begin{cases} 
  s_{i-1} & \text{if } x = b \text{ and } q = s_i \in S_i \text{ for some } i \in \{1, \ldots, r_2\}; \\
  \hat{s} & \text{if } x = a \text{ and } q \in (Q \cup Q_1 \cup \ldots \cup Q_{p_2 - 1}) \setminus S; \\
  s_{r_2} & \text{if } x = a \text{ and } q = s_i \in S_i \text{ for some } i \in \{0, \ldots, r_2\}; \\
  t & \text{if } x = a \text{ and } q \in T; \\
  q^{p_2 - 1} & \text{if } x = b \text{ and } q \in Q; \\
  q^{i-1} & \text{if } x = b \text{ and } q = q_i \in Q_i \text{ for some } i \in \{2, \ldots, p_2 - 1\}; \\
  \delta(q, c) & \text{if } x = b \text{ and } q = q^1 \in Q_1; \\
  q & \text{otherwise.} 
\end{cases}
$$

![Figure 1. The reduction from the proof sketch sketch of Lemma 9. The letter $a$ transfers everything surjectively onto $S_{r_2}$, indicated by four large arrows at the top and bottom and labelled by $a$. The auxiliary states $Q_1, \ldots, Q_{p_2 - 1}$, which are meant to interpret a sequence $b^{p_2}$ like a single symbol in the original automaton, are also only indicated inside of $A$, but not fully written out.](image)

Please see Figure 1 for a sketch of the reduction. For the constructed automaton $A'$, the following could be shown: $\exists m \geq 0 : \delta(S, c^m) \in T$ if and only if $A'$ has a synchronizing word in $ab^{p_2}(b^{p_2})^* a$ if and only if $A'$ has a synchronizing word in $ab^* a$ if and only if $A'$ has a synchronizing word in $a^* b^* a^*$.

Now, suppose $\delta(s, c^m) \subseteq T$ for some $m \geq 0$. By the above, $A'$ has a synchronizing word $u$ in $ab^{p_2}(b^{p_2})^* a$. Then, $a^{r_1 - 1} u a^{r_3 - 1} \in L(B)$ also synchronizes $A'$.

Conversely, suppose we have a synchronizing word $w \in L$ for $A'$. As $L \subseteq a^* b^* a^*$ by the above equivalences, $\delta(S, c^m) \subseteq T$ for some $m \geq 0$. $\square$
5 Constraints from Strongly Self-Synchronizing Codes

Here, we introduce strongly self-synchronizing codes and investigate \( L \)-Constr-Sync for bounded constraint languages \( L \subseteq w_1^{*} \cdots w_k^{*} \) where \( \{w_1, \ldots, w_k\} \) is such a code.

Let \( C \subseteq \Sigma^{+} \) be non-empty. Then, \( C \) is called a self-synchronizing code \cite{7,8,33}, if \( C^2 \cap \Sigma^{+} C \Sigma^{+} = \emptyset \). If, additionally, \( C \subseteq \Sigma^{n} \) for some \( n > 0 \), then it is called a comma-free code \cite{26}. Every self-synchronizing code is an infix code, i.e., no proper factor of a word from \( C \) is in \( C \) \cite{33}. A strongly self-synchronizing code is a self-synchronizing code \( C \subseteq \Sigma^{+} \) such that, additionally, \( \text{Pref}(C) \subseteq \Sigma^{*} C \Sigma^{+} = \emptyset \).

To give some intuition for the strongly self-synchronizing codes, we also present an alternative characterization, a few examples and a way to construct such codes.

Proposition 13. A non-empty \( C \subseteq \Sigma^{+} \) is a strongly self-synchronizing code if and only if, for all \( u \in \text{Pref}(C) \) and \( v \in C \), if we write \( uv = x_1 \cdots x_n \) with \( x_i \in \Sigma \) for \( i \in \{1, \ldots, n\} \), then, for all \( j \in \{1, \ldots, n\} \) and \( k \geq 1 \) where \( j + k - 1 \leq n \), we have that \( x_j x_{j+1} \cdots x_{j+k-1} \in C \) implies \( j = |u| + 1 \) and \( k = |v| \) or \( j = 1 \) and \( k = |u| \). Intuitively, in \( uv \) only the last \( |v| \) symbols form a factor in \( C \) and possibly the first \( |u| \) symbols.

When passing from letters to words by applying a homomorphism, in the reductions, we have to introduce additional states. The definition of the strongly synchronizing codes was motivated by the demand that these states also have to be synchronized, which turns out to be difficult in general.

Example 14. The code \{aacc, bbc, bac\} is strongly self-synchronizing. The code \{aab, bccc, abc\} is self-synchronizing, but not strongly self-synchronizing as, for example, \( (a)(bc) \) contains abc.

Remark 15 (Construction). Take any non-empty finite language \( X \subseteq \Sigma^{n}, n > 0 \), and a symbol \( c \in \Sigma \) such that \( \{c\} \Sigma^{*} \cap X = \emptyset \). Let \( k = \max \{ \ell \geq 0 \mid \exists u, v \in \Sigma^{*} : uc^{\ell} v \in X \} \). Then, \( Y = c^{k+1} X \) is a strongly self-synchronizing code.

Example 16. Let \( \Sigma = \{a, b, c\} \) and \( C = \{ab, ba, aa\} \). Then, \( \{cab, cba, caa\} \) or \( \{bbab, bbaa\} \) are strongly self-synchronizing codes by Remark 15.

Our next result, which holds in general, states conditions on a homomorphism such that we not only have a reduction from the problem for the homomorphic image to our original problem, as stated in Proposition 2, but also a reduction in the other direction.

Theorem 17. Let \( \varphi : \Sigma^{*} \rightarrow \Gamma^{*} \) be a homomorphism such that \( \varphi(\Sigma) \) is a strongly self-synchronizing code and \( |\varphi(\Sigma)| = |\Sigma| \). Then, for each regular \( L \subseteq \Sigma^{*} \) we have \( L \text{-Constr-Sync} \equiv_{m}^{\log} \varphi(L) \text{-Constr-Sync} \).

\(^6\) In \cite{33} this distinction is not made and self-synchronizing codes are also called comma-free codes.
Finally, we apply Theorem 17 to bounded languages such that \( \{w_1, \ldots, w_k\} \) forms a strongly self-synchronizing code.

Theorem 18. Let \( L \subseteq w_1 \cdots w_k^* \) be regular such that \( \{w_1, \ldots, w_k\} \) is a strongly self-synchronizing code. Then, L-CONSTR-SYNC is either \( \text{NP-complete} \) or in \( \text{P} \).

Example 19. (1) \((aa)(bb)(ba)^*\)-CONSTR-SYNC is \( \text{NP-complete}\).
(2) \((bb)(aa)(ba)^* \cup (bb)^*\)-CONSTR-SYNC is in \( \text{P} \).

6 Conclusion and Discussion

We have looked at the constrained synchronization problem (Problem 1) – CSP for short – for letter-bounded regular constraint languages and bounded languages induced by strongly self-synchronizing codes, thereby continuing the investigation started in [19]. The complexity landscape in these cases is completely understood. Only the complexity classes \( \text{P} \) and \( \text{NP-complete} \) arise. In [31] the question was raised if we can find sparse constraint languages that give constrained problems complete for some candidate \( \text{NP-intermediate complexity class}. \) At least for the language classes investigated here this is not the case. For general sparse regular languages, it is still open if a corresponding dichotomy theorem holds, or candidate \( \text{NP-intermediate} \) problems arise. By the results obtained so far and the methods of proofs, we conjecture that in fact a dichotomy result holds true.

Let us relate our results to the previous work [31], where partial results for \( \text{NP-hardness} \) and containment in \( \text{P} \) were given. Namely, by setting Fact\((L) = \{v \in \Sigma^* : \exists u, w \in \Sigma^* : uvw \in L\} \) and \( B_{p,E} = (\Sigma, P, \mu, q_0, F) \) for \( B = (\Sigma, P, \mu, p_0, F) \) with \( E \subseteq P \) and \( q \in P \), the following was stated.

Proposition 20 ([31]). Suppose we find \( u, v \in \Sigma^* \) such that we can write \( L = uw^*U \) for some non-empty language \( U \subseteq \Sigma^* \) with \( u \notin \text{Fact}(v^*) \), \( v \notin \text{Fact}(U) \) and \( \text{Pref}(v^*) \cap U = \emptyset \). Then L-CONSTR-SYNC is \( \text{NP-hard} \).

Proposition 21 ([31]). Let \( B = (\Sigma, P, \mu, p_0, F) \) be a polycyclic PDFA. If for every reachable \( p \in P \) with \( L(B_{p,E}) \neq \varepsilon \) we have \( L(B_{p_0,E}) \subseteq \text{Suff}(L(B_{p,E})) \), then the problem \( L(B)\)-CONSTR-SYNC is solvable in polynomial time.

Note that Proposition 21 implies that \( ab^*a \) gives an \( \text{NP-complete} \) CSP. However, in the letter-bounded case there exist constraint languages giving \( \text{NP-complete} \) problems for which this is not implied by Proposition 21 for example: \( ab^*ba, ab^*ab, aa^*abb^*a \) or \( ba^*b \cup a \). Also, Proposition 21 is weaker than our Proposition 8 in the case of letter-bounded constraints. For example, it does not apply to \( ab^*b \), every PDFA for this languages has a loop exclusively labelled by the letter \( b \) and reachable after reading the letter \( a \) from the start state, and so words along this loop cannot have a word starting with \( a \) as a suffix.

For general bounded languages, let us note the following implication of Propositions 2 and 8.
Proposition 22. Let $u, v \in \Sigma^*$. If $L \subseteq u^*v^*$ is regular, then $L$-CONSTR-SYNC is solvable in polynomial time.

Next, in Proposition 23, we give an example of a bounded regular language yielding an NP-complete synchronization problem, but for which this is not directly implied by the results we have so far.

Proposition 23. The problem $((ab)(ba)^*(ab))$-CONSTR-SYNC is NP-complete.

By Proposition 23 for the homomorphism $\varphi : \{a, b\}^* \rightarrow \{a, b\}^*$ given by $\varphi(a) = ab$ and $\varphi(b) = ba$ both problems $ab^*a$ and $\varphi(ab^*a) = ab(ba)^*ab$ are NP-complete. So, this is a homomorphisms which preserves, in this concrete instance, the computational complexity. But its image $\{ab, ba\}$ is not even a self-synchronizing code. However, I do not know if this homomorphism always preserves the complexity. Similarly, I do not know if the condition from Theorem 17 characterizes those homomorphisms which preserve the complexity.

In the reduction used in Lemma 9 the resulting automaton has a sink state. However, in general, for questions of synchronizability it makes a difference if we have a sink state or not, at least with respect to the Černý conjecture [11], as for automata with a sink state this conjecture holds true, even with the better bound $\frac{n^2}{2}$ [44,48]. However, even in [19] certain reductions establishing PSPACE-completeness use only automata with a sink state. Hence, for hardness these automata are sufficient at least in certain instances. So, it might be interesting to know if in terms of computational complexity of the CSP, we can, without loss of generality, limit ourselves to input automata with a sink state. The methods of proof for the letter-bounded constraints show that in this case, we can actually do this, as these input automata are sufficient to establish all cases of intractability.

Lastly, let us mention the following related problem one could come up with. Fix a deterministic and complete semi-automaton $A$. Then, for input PDFAs $B$, what is the computational complexity to determine if $A = (\Sigma, Q, \delta)$ has a synchronizing word in $L(B)$? As the set of synchronizing words $\{w \in \Sigma^* : |\delta(Q, w)| = 1\} = \bigcup_{q \in Q} \bigcap_{q' \in Q} L((\Sigma, Q, \delta, q', \{q\}))$ is a regular language, we have to test for non-emptiness of intersection of this fixed regular language with $L(B)$. This could be done in $\text{NL}$, hence in $\text{P}$.

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7 In [11] erroneously the bound $n(n+1)/2$ was reported as being sharp, but the overall argument in fact works to yield the sharp bound $n(n - 1)/2$.
8 This was actually suggested by a reviewer of a previous version.
References

1. Adler, R., Weiss, B.: Similarity of Automorphisms of the Torus. American Mathematical Society: Memoirs of the American Mathematical Society, American Mathematical Society (1970)

2. Alves, L.V., Pena, P.N.: Synchronism recovery of discrete event systems. IFAC-PapersOnLine 53(2), 10474–10479 (2020), 21th IFAC World Congress

3. Amarilli, A., Paperman, C.: Topological sorting with regular constraints. In: Chatzigiannakis, I., Kaklamanis, C., Marx, D., Sannella, D. (eds.) 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic. LIPIcs, vol. 107, pp. 115:1–115:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2018)

4. Benenson, Y., Adar, R., Paz-Eliyahu, T., Livneh, Z., Shapiro, E.: DNA molecule provides a computing machine with both data and fuel. Proceedings of the National Academy of Sciences of the United States of America 100, 2191–2196 (2003)

5. Benenson, Y., Paz-Eliyahu, T., Adar, R., Keinan, E., Livneh, Z., Shapiro, E.: Programmable and autonomous computing machine made of biomolecules. Nature 414, 430–434 (2001)

6. Berman, L., Hartmanis, J.: On isomorphisms and density of NP and other complete sets. SIAM J. Comput. 6(2), 305–322 (1977)

7. Berstel, J., Perrin, D.: Theory of codes. Pure and Applied Mathematics, 117. Orlando etc.: Academic Press, Inc. XIV, 433 (1985)

8. Berstel, J., Perrin, D., Reutenauer, C.: Codes and Automata, Encyclopedia of mathematics and its applications, vol. 129. Cambridge University Press (2010)

9. Blättner, M., Cremers, A.B.: Observations about bounded languages and developmental systems. Math. Syst. Theory 10, 253–258 (1977)

10. Cassandras, C.G., Lafortune, S.: Introduction to Discrete Event Systems, Second Edition. Springer (2019)

11. Černý, J.: Poznámka k homogénnym experimentom s konečnými automatmi. Matematicko-fyzikálny časopis 14(3), 208–216 (1964)

12. Chen, Y., Ierardi, D.: The complexity of oblivious plans for orienting and distinguishing polygonal parts. Algorithmica 14(5), 367–397 (1995)

13. Cho, H., Jeong, S., Somencz, F., Pixley, C.: Synchronizing sequences and symbolic traversal techniques in test generation. J. Electron. Test. 4(1), 19–31 (1993)

14. Dassow, J., Paun, G.: On the regularity of languages generated by context-free evolutionary grammars. Discret. Appl. Math. 92(2-3), 205–209 (1999)

15. Diekert, V.: Makanin’s algorithm for solving word equations with regular constraints. Report, Fakultät Informatik, Universität Stuttgart (03 1998)

16. Diekert, V., Gutiérrez, C., Hagenah, C.: The existential theory of equations with rational constraints in free groups is PSPACE-complete. Inf. Comput. 202(2), 105–140 (2005)

17. Eppstein, D.: Reset sequences for monotonic automata. SIAM J. Comput. 19(3), 500–510 (1990)

18. Erdmann, M.A., Mason, M.T.: An exploration of sensorless manipulation. IEEE J. Robotics Autom. 4(4), 369–379 (1988)

19. Fernau, H., Gusev, V.V., Hoffmann, S., Holzer, M., Volkov, M.V., Wolf, P.: Computational complexity of synchronization under regular constraints. In: Rossmanith, P., Hegernes, P., Katoen, J. (eds.) 44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26-30, 2019, Aachen, Germany. LIPIcs, vol. 138, pp. 63:1–63:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2019)
20. Ganardi, M., Hucke, D., König, D., Lohrey, M., Mamouras, K.: Automata theory on sliding windows. In: Niedermeier, R., Vallée, B. (eds.) 35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France. LIPIcs, vol. 96, pp. 31:1–31:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2018)
21. Gawrychowski, P., Krieger, D., Rampersad, N., Shallit, J.O.: Finding the growth rate of a regular or context-free language in polynomial time. Int. J. Found. Comput. Sci. 21(4), 597–618 (2010)
22. Ginsburg, S.: The Mathematical Theory of Context-free Languages. McGraw-Hill (1966)
23. Ginsburg, S., Spanier, E.H.: Bounded ALGOL-like languages. Transactions of the American Mathematical Society 113(2), 333–368 (1964)
24. Ginsburg, S., Spanier, E.H.: Bounded regular sets. Proceedings of the American Mathematical Society 17(5), 1043–1049 (1966)
25. Goldberg, K.Y.: Orienting polygonal parts without sensors. Algorithmica 10(2-4), 210–225 (1993)
26. Golomb, S.W., Gordon, B., Welch, L.R.: Comma-free codes. Canadian Journal of Mathematics 10, 202–209 (1958)
27. Gusev, V.V.: Synchronizing automata of bounded rank. In: Moreira, N., Reis, R. (eds.) Implementation and Application of Automata - 17th International Conference, CIAA. LNCS, vol. 7381, pp. 171–179. Springer (2012)
28. Hartmanis, J., Mahaney, S.R.: An essay about research on sparse NP complete sets. In: Dembinski, P. (ed.) Mathematical Foundations of Computer Science 1980 (MFCS'80), Proceedings of the 9th Symposium, Rydzyna, Poland, September 1-5, 1980. Lecture Notes in Computer Science, vol. 88, pp. 40–57. Springer (1980)
29. Herrmann, A., Kutrib, M., Malcher, A., Wendlandt, M.: Descriptional complexity of bounded regular languages. Journal of Automata, Languages and Combinatorics 22(1-3), 93–121 (2017)
30. Hoffmann, S.: Computational complexity of synchronization under regular commutative constraints. In: Kim, D., Uma, R.N., Cai, Z., Lee, D.H. (eds.) Computing and Combinatorics - 26th International Conference, COCOON 2020, Atlanta, GA, USA, August 29-31, 2020, Proceedings. Lecture Notes in Computer Science, vol. 12273, pp. 460–471. Springer (2020)
31. Hoffmann, S.: On a class of constrained synchronization problems in NP. In: Cordasco, G., Gargano, L., Rescigno, A. (eds.) Proceedings of the 21th Italian Conference on Theoretical Computer Science, ICTCS 2020, Ischia, Italy. CEUR Workshop Proceedings. CEUR-WS.org (2020)
32. Hopcroft, J.E., Ullman, J.D.: Introduction to Automata Theory, Languages, and Computation. Addison-Wesley Publishing Company (1979)
33. Hsieh, C., Hsu, S., Shyr, H.J.: Some algebraic properties of comma-free codes. Tech. rep., Kyoto University Research Information Repository (KURENAI) (1989)
34. Latteux, M., Thierrin, G.: On bounded context-free languages. Elektronische Informationsverarbeitung und Kybernetik (Journal of Information Processing and Cybernetics) 20(1), 3–8 (1984)
35. Lecoutre, C.: Constraint Networks: Techniques and Algorithms. John Wiley & Sons, Ltd (2009)
36. Mahaney, S.R.: Sparse complete sets of NP: solution of a conjecture of Berman and Hartmanis. J. Comput. Syst. Sci. 25(2), 130–143 (1982)
37. Natarajan, B.K.: An algorithmic approach to the automated design of parts orienters. In: 27th Annual Symposium on Foundations of Computer Science, Toronto, Canada, 27-29 October 1986. pp. 132–142. IEEE Computer Society (1986)
38. Natarajan, B.K.: Some paradigms for the automated design of parts feeders. Int. J. Robotics Res. 8(6), 98–109 (1989)
39. Pesant, G.: A regular language membership constraint for finite sequences of variables. In: Wallace, M. (ed.) Principles and Practice of Constraint Programming - CP 2004, 10th International Conference, CP 2004, Toronto, Canada, September 27 - October 1, 2004, Proceedings. Lecture Notes in Computer Science, vol. 3258, pp. 482–495. Springer (2004)
40. Pin, J.: Mathematical Foundations of Automata Theory (2020), https://www.irif.fr/~jep/PDF/MPRI/MPRI.pdf
41. Piziak, R., Odell, P.L.: Full rank factorization of matrices. Mathematics Magazine 72(3), 193–201 (1999)
42. Ramadge, P.J., Wonham, W.M.: Supervisory control of a class of discrete event processes. SIAM Journal on Control and Optimization 25, 206–230 (1987)
43. Romeuf, J.: Shortest path under rational constraint. Inf. Process. Lett. 28(5), 245–248 (1988)
44. Rystsov, I.: Reset words for commutative and solvable automata. Theor. Comput. Sci. 172(1-2), 273–279 (1997)
45. Sandberg, S.: Homing and synchronizing sequences. In: Broy, M., Jonsson, B., Katoen, J.P., Leucker, M., Pretschner, A. (eds.) Model-Based Testing of Reactive Systems. LNCS, vol. 3472, pp. 5–33. Springer (2005)
46. Trahtman, A.N.: The road coloring problem. Israel Journal of Mathematics 172, 51–60 (2009)
47. Volkov, M.V.: Synchronizing automata and the Černý conjecture. In: Martín-Vide, C., Otto, F., Fernau, H. (eds.) Language and Automata Theory and Applications, Second International Conference, LATA. LNCS, vol. 5196, pp. 11–27. Springer (2008)
48. Volkov, M.V.: Synchronizing automata preserving a chain of partial orders. Theor. Comput. Sci. 410(37), 3513–3519 (2009)
49. Vorel, V., Roman, A.: Complexity of road coloring with prescribed reset words. J. Comput. Syst. Sci. 104, 342–358 (2019)
50. Wonham, W.M., Cai, K.: Supervisory Control of Discrete-Event Systems. Springer (2019)
51. Yu, S.: Regular languages. In: Rozenberg, G., Salomaa, A. (eds.) Handbook of Formal Languages, Volume I: Word, Language, Grammar, pp. 41–110. Springer (1997)
A Proofs for Section 2 (Preliminaries and Definitions)

The following obvious remark, stating that the set of synchronizing words is a two-sided ideal, will be used frequently without further mentioning.

**Lemma 24.** Let $\mathcal{A} = (\Sigma, Q, \delta)$ be a deterministic and complete semi-automaton and $w \in \Sigma^*$ be a synchronizing word for $\mathcal{A}$. Then for every $u, v \in \Sigma^*$, the word $uwv$ is also synchronizing.

**Example 25.** A few examples and non-examples of polycyclic automata (start and final states not indicated, as they are irrelevant for these examples).

Note that a complete DFA over a non-unary alphabet is never polycyclic. This could be seen with Theorem 1 as the complement of a language with polynomial growth cannot have polynomial growth itself, or by using the property that every strongly connected component is a single cycle. Then, for two distinct letters and a strongly connected component, if one maps a state in the cycle into the cycle, the other one must leave the cycle. However, if we topologically sort the strongly connected components, the component at the end must be closed under any letter.

We will need the following result from [Hof20a].

**Lemma 26 ([Hof20a]).** Let $\mathcal{X}$ denote any of the complexity classes $P$, $NP$ or $PSPACE$. If $L(B)$ is a finite union of languages $L(B_1), L(B_2), \ldots, L(B_n)$ such that for each $1 \leq i \leq n$ the problem $L(B_i)$-Constr-Sync $\in \mathcal{X}$, then we have $L(B)$-Constr-Sync $\in \mathcal{X}$.

**Proposition 2.** Let $\varphi : \Sigma^* \rightarrow \Gamma^*$ be a homomorphism. Then, for each regular $L \subseteq \Sigma^*$, we have $\varphi(L)$-Constr-Sync $\leq_{\text{log}} L$-Constr-Sync.

**Proof.** Let $\mathcal{A} = (\Gamma, Q, \delta)$ be a DCSA. We want to know if it is synchronizing with respect to $\varphi(L)$. Build the automaton $\mathcal{A}' = (\Sigma, Q, \delta')$ according to the rule

$$\delta'(p, x) = q \text{ if and only if } \delta(p, \varphi(x)) = q,$$

for $x \in \Sigma^*$. As $\varphi$ is a mapping, $\mathcal{A}'$ is indeed deterministic and complete, as $\mathcal{A}$ is a DCSA. As the homomorphism $\varphi$ is independent of $\mathcal{A}$, automaton $\mathcal{A}'$ can be constructed from $\mathcal{A}$ in logarithmic space. Next we prove that the translation is indeed a reduction.
If \( u \in \varphi(L) \) is some synchronizing word for \( A \), then there is some \( s \in Q \) such that \( \delta(r, u) = s \), for all \( r \in Q \). By choice of \( u \), we find \( w \in L \) such that \( u = \varphi(w) \). As with \( \delta(r, \varphi(w)) = s \), it follows \( \delta'(r, w) = s \), hence \( w \) is a synchronizing word for \( A \). Conversely, if \( w \in L \) is a synchronizing word for \( A' \), then there is some \( s \in Q \) such that \( \delta'(r, w) = s \), for all \( r \in Q \). Further, \( \varphi(w) \) is a synchronizing word for \( A \), as by definition for all \( r \in Q \), we have \( \delta(r, \varphi(w)) = s \).

In the proofs of this appendix, we will need the following results and constructions.

If \(|L(B)| = 1\), then \( L(B)\text{-}\text{Constr-Sync} \) is obviously in \( P \). Simply feed this single word into the input semi-automaton for every state and check if a unique state results. Hence by Lemma [26] the next is implied.

**Lemma 27.** Let \( B = (\Sigma, P_1, \mu, p_0, F) \) be a constraint automaton such that \( L(B) \) is finite, then \( L(B)\text{-}\text{Constr-Sync} \in P \).

**Lemma 28.** Every regular language could be written as a finite union of regular languages recognizable by automata with a single final state.

**Proof.** Let \( A = (\Sigma, Q, \delta, q_0, F) \) be a DFA. Then, \( L(A) = \bigcup_{q \in F} \{ w \in \Sigma^* | \delta(q_0, w) = q \} \). In the proofs of this appendix, we will need the following results and constructions.

Let \( \Sigma = \{a\} \) be a unary alphabet. Suppose \( L \subseteq \Sigma^* \) is regular with a recognizing complete deterministic automaton \( A = (\Sigma, Q, \delta, q_0, F) \). Then, by considering the sequence of states \( \delta(q_0, a^i), \delta(q_0, a^2), \delta(q_0, a^3), \ldots \), we find numbers \( i \geq 0, p > 0 \) with \( i + p \) minimal such that \( \delta(q_0, a^i) = \delta(q_0, a^{i+p}) \). We call these numbers the index \( i \) and the period \( p \) of the automaton \( A \). If \( Q = \{ \delta(q_0, a^m) \mid m \geq 0 \} \), i.e., every state is reachable from the start state, then \( i + p = |Q| \). In our discussion, unary languages that are recognized by automata with a single final state appear.

**Lemma 29 ([Hof19]).** Let \( L \subseteq \{a\}^* \) be a unary language that is recognized by an automaton with a single final state, index \( i \) and period \( p \). Then either \( L = \{u\} \) with \( |u| < i \) (and if the automaton is minimal we would have \( p = 1 \)), or \( L \) is infinite with \( L = a^{i+m}(a^p)^* \) and \( 0 \leq m < p \). Hence two words \( u, v \) with \( \min(|u|, |v|) \geq i \) are both in \( L \) or not if and only if \( |u| \equiv |v| \) (mod \( p \)).

We need an additional construction, which we call **inflating** a automaton \( A = (\Sigma, Q, \delta, q_0, F) \)

**Definition 30 (Aut. Inflation by a factor \( N > 0 \)).** Let \( A = (\Sigma, Q, \delta, q_0, F) \) be a given automaton and \( N > 0 \). Then, the inflated automaton (of \( A \)) by \( N \) is \( A' = (\Sigma, Q', \delta', q_0, F) \), where

\[
Q' = Q \cup \bigcup_{x \in \Sigma} (Q_{1,x} \cup \ldots \cup Q_{N-1,x})
\]

\(^9\) Note that these numbers are independent of the language recognized by the automaton.
and the $Q_{1,x}$ are disjoint copies of $Q$. The states $Q_{1,x}, \ldots, Q_{N-1,x}$ are called auxiliary states in this context. Then, for $q \in Q'$ and $x \in \Sigma$, set

$$
\delta'(q, x) = \begin{cases} 
q_{i+1, x} & \text{if } q = q_{i, x} \land i \in \{1, \ldots, N - 2\}; \\
\delta(q, x) & \text{if } q = q_{N-1, x}; \\
q_{1, x} & \text{if } q \in Q; \\
q & \text{otherwise.}
\end{cases}
$$

Intuitively, a transition labelled by $x$ is replaced by a path labelled by $x^N$. Note that, for $q, q' \in Q$

$$
\delta(q, x) = q' \text{ in } A \iff \delta'(q, x^N) = q' \text{ in } A'
$$

and $\delta'(q, w) \in Q$ implies that $|w|$ is divisible by $N$.

### B Proofs for Section 3 (Sparse and Bounded Regular Languages)

**Proposition 4.** Let $L \subseteq \Sigma^*$ be regular. Then, $L$ is sparse if and only if it is recognizable by a polycyclic PDFA.

**Proof.** In [LT84] was shown that the context-free sparse languages are precisely the context-free bounded languages, which gives our first two equivalences. A result from [GKRS10, Lemma 2] readily implies that if a language is recognized by a polycyclic PDFA, then it must be sparse. Lastly, we show that every bounded regular language is recognizable by a polycyclic automaton, which finishes the proof.

**Claim:** For $w \in \Sigma^*$. Then, any regular $L \subseteq w^*$ is recognizable by a polycyclic PDFA.

**Proof of the Claim.** Let $w \in \Sigma^*$ and $L \subseteq w^*$ be a regular language. If $w = \varepsilon$, then $L = \{\varepsilon\}$, which is obviously recognizable by a polycyclic automaton. So, suppose $|w| > 0$. Let $a$ be an arbitrary symbol and define a homomorphism $\varphi : \{a\}^* \rightarrow \Sigma^*$ by $\varphi(a^i) = w^i$, which is injective as $|w| > 0$ by assumption. Then, the unary language $\varphi^{-1}(L) = \{a^i \mid w^i \in L\}$ is regular, as inverse homomorphisms preserve regularity. Hence, we can write it as a union of languages recognizable by automata with a single final state, which, by Lemma [29], have the form $\{a^i\}$ for some $i \geq 0$ or $\{a^{i+jp} \mid j \geq 0\}$ for some $i \geq 0, p > 0$. As the application of functions preserves union, and $L = \varphi(\varphi^{-1}(L))$ here, the language $L$ is the union of the images of these languages. We have $\varphi(\{a^i\}) = \{w^i\},$ and this singleton language is obviously recognizable by a polycyclic automaton, and we have $\varphi(\{a^{i+jp} \mid j \geq 0\}) = \{w^{i+jp} \mid j \geq 0\},$ and this language is also recognizable by an automaton that has an initial tail labelled by $w^i$ and a cycle labelled by $w^p$. So, as the polycyclic languages are closed under union [Hof20b, Proposition 6], we have shown that the language $L$ is recognizable by some polycyclic automaton. [End, Proof of the Claim]
Finally, as the languages recognizable by polycyclic automata are closed under concatenation and union [Hof20b, Proposition 5 and Proposition 6], by Theorem 3 every bounded regular language is recognizable by a polycyclic automaton.

Note that sparse languages in general are not closed under homomorphic mappings [Pin20]. As it is easy to see that the bounded languages are closed under homomorphic mappings, this also implies that, in general, the bounded languages do not equal the sparse languages.

C Proofs for Section 4 (Letter-Bounded Constraint Languages)

Note that the work [HKMW17] seems to deviate from the standard terminology, for example by calling bounded languages as introduced here word-bounded and refers to letter-bounded simply as bounded languages.

Proposition 8. Let $A_j \subseteq \{a_j\}^*$ be unary regular languages for $j \in \{1, \ldots, k\}$. Set $L = A_1 \cdots A_k$. If for all $j \in \{1, \ldots, k\}$, $A_j$ infinite implies that $A_i \subseteq \{a_j\}^*$ for all $i < j$ or $A_i \subseteq \{a_j\}^*$ for all $i > j$ (or both), then $L\text{-Constr-Sync} \in \mathbb{P}$.

Proof. Let $L = A_1 \cdots A_k$ with $A_j \subseteq \{a_j\}^*$ fulfill the assumption. If $A_j$ is infinite and for all $i < j$ we have $A_i \subseteq \{a_j\}^*$, then $A_1 \cdots A_j \subseteq \{a_j\}^*$, and similarly if for all $i > j$ we have $A_i \subseteq \{a_j\}^*$. So, by considering $(A_1 \cdots A_j)A_{j+1} \cdots A_k$ or $A_1 \cdots A_{j-1}(A_j \cdots A_k)$, with $j$ maximal in the former case and minimal in the latter, without loss of generality, we can assume $j = 1$ or $j = k$, i.e., we only have the cases $A_1$ is infinite, $A_k$ is infinite or both are infinite or none is infinite, and, by maximality or minimality of $j$, in all these cases the languages $A_2, \ldots, A_{k-1}$ are all finite.

Then, by Lemma 28, we can write $A_1$ and $A_k$ as a finite union of unary languages recognizable by automata with a single final state. As concatenation distributes over union, if we do this for $A_1$ and $A_k$ and rewrite the language using the mentioned distributivity, we get a finite union of languages of the form

$$A_1' A_2 \cdots A_i A_i' A_k'$$

where $A_i'$ and $A_i''$ are recognizable by unary automata with a single final state and are either finite or infinite. Hence, by Lemma 26 if we can show that the problem is in $\mathbb{P}$ for each such language, the result follows. So, without loss of generality, we assume from the start that $A_1$ or $A_k$ are recognizable by automata with a single final state.

If all $A_j$, $j \in \{1, \ldots, k\}$ are finite, then $L$ is finite, and $L\text{-Constr-Sync} \in \mathbb{P}$ by Lemma 27. We handle the remaining cases separately.

(i) Only $A_1$ is infinite.

By assumption, every $A_j \subseteq \{a_j\}^*$, $j \in \{1, \ldots, k\}$, is recognizable by a single state automaton. Hence, by Lemma 29, we can write, as $A_1$ is infinite,
$A_i = a_i^{(a_i^p)^*}$ with $i \geq 0$ and $p > 0$. Let $A = (\Sigma, Q, \delta)$ be an input semi-automaton for $L$-Constr-Sync. As $\delta(Q, a) \subseteq Q$, we have, for any $n \geq 0$, $\delta(Q, a_1^{n+1}) \subseteq \delta(Q, a_1^n)$. So, as $Q$ is finite and the sequence of subsets cannot get arbitrarily small, for some $0 \leq n < |Q|$ we have $|\delta(Q, a_1^{n+1})| = |\delta(Q, a_1^n)|$. But $|\delta(Q, a_1^{n+1})| = |\delta(Q, a_1^n)|$, as $\delta(Q, a_1^n) \subseteq \delta(Q, a_1^n)$, implies $\delta(Q, a_1^{n+1}) = \delta(Q, a_1^n)$. Then, the symbol $a_1$ permutes the set $\delta(Q, a_1^n)$. Hence, $\delta(Q, a_1^{n+m}) = \delta(Q, a_1^n)$ for any $m \geq 0$. So, combining these observations,

$$\{\delta(Q, a_1^n) \mid n \geq 0\} = \{\delta(Q, a_1^n) \mid n \in \{0, \ldots, |Q| - 1\}\}$$

and $\delta(Q, a_1^{(|Q|-1+m)}) = \delta(Q, a_1^{(|Q|-1)})$ for any $m \geq 0$. Now, note that the language $A_2 \cdots A_k$ is finite. So, to find out if we have any $a_i^{j+l}u$ with $u \in A_2 \cdots A_k$ that synchronizes the input semi-automaton, we only have to test the finite number of words $a_i^{j+l}u$, with $u \in A_2 \cdots A_k$ and $l$ such that $i + lp \leq \max\{|Q| - 1 + p, i\}$, synchronizes $A$. The number (and the length) of these words is linear bounded in $|Q|$ and each could be checked in polynomial time by feeding it into the input semi-automaton for each state and checking if a unique state results. Hence the problem is solvable in polynomial time.

(ii) Only $A_k$ is infinite.

Let $u \in A_1 \cdots A_{k-1}$. By assumption, there are only finitely many such words $u$. Set $S = \delta(Q, u)$ and $T = \delta(Q, a_1^{(|Q|-1)}u)$. As in case (i), $a_k$ permutes the states in $T$ and as $S \subseteq Q$, we have $\delta(S, a_k^{(|Q|-1)}) \subseteq T$. So, as $a_k$ permutes $T$, it acts injective on the subset $\delta(S, a_k^{(|Q|-1)})$. This gives $|\delta(S, a_k^{(|Q|-1)+n})| = |\delta(S, a_k^{(|Q|-1)})|$ for any $n \geq 0$. Together with $|\delta(S, a_k^n)| \leq |\delta(S, a_k^n)|$, we have

$$\exists n \in \mathbb{N}_{0} : |\delta(S, a_k^n)| = 1 \iff |\delta(S, a_k^{(|Q|-1)})| = 1.$$  

Choose any fixed $N \geq |Q| - 1$ with $a_k^N \in A_k$. Then, with the above considerations, we only have to test the finite number of words

$$u \cdot a_k^N, \quad u \in A_1 \cdots A_{k-1}.$$  

The length of these words is linear bounded in $|Q|$ and as each test, i.e., feeding the word into the input semi-automaton for each state and testing if a unique state results, could be performed in polynomial time, the problem is solvable in polynomial time.

(iii) Both $A_1$ and $A_k$ are infinite.

This is essentially a combination of the arguments of case (i) and (ii). Let $A = (\Sigma, Q, \delta)$ be an input semi-automaton and $B = (\Sigma, P, \mu, p_0, F)$ be a constraint automaton with $L = L(B)$. First, consider only the language $A_1 \cdots A_{k-1}$. Then, as in case (i), see Equation (1),

$$\{\delta(Q, a_1^n) \mid a_1^n \in A_1\} = \{\delta(Q, a_1^n) \mid 0 \leq n < |Q| - 1 + |P| \text{ and } a_1^n \in A_1\}.$$  

Note that we have written $0 \leq n < |Q| - 1 + |P|$ and not merely $\leq |Q| - 1$ as an upper bound. The reason is that otherwise, if $a_1^{(|Q|-1)} \notin A_1$, we might miss
the set \( \delta(Q, a_1^{Q|-1}) \), but as \( \delta(Q, a_1^{Q|-1+m}) = \delta(Q, a_1^{Q|-1}) \) for any \( m \geq 0 \) and \( A_1 \) is infinite, \( \delta(Q, a_1^{Q|-1}) \in \{ \delta(Q, a_1^n) \mid a_1^n \in A_1 \} \). However, if \( a_1^n \in A_1 \) for some \( n \geq |Q|-1 + |P| \), then, with \( s = \mu(p_0, a_1^{Q|-1}) \), by finiteness of \( P \), among the states \( s, \mu(s, a_1), \ldots, \mu(s, a_1^{n-|Q|+1}) \) we find \( 0 \leq m \leq |P| - 1 \) and \( 0 < r \leq |P| \) with \( m + r \leq |P| \) such that \( \mu(s, a_1^{m+r}) = \mu(s, a_1^n) \).

Then we have found a cycle and we can skip it, i.e.,

\[
\mu(p_0, a_1^n) = \mu(s, a_1^{n-|Q|+1}) = \mu(s, a_1^{m+r}), a_1^{n-|Q|+1-(m+r)} = \mu(s, a_1^{m+n-|Q|+1-m-r}) = \mu(s, a_1^{n-r}).
\]

But, as then \( \mu(s, a_1^{n-r}) = \mu(s, a_1^n) \in F \) we find \( a_1^{n-r} \in A_1 \). Repeating this procedure, if \( n-r \geq |Q|-1+|P| \), we ultimately find \( |Q|-1 \leq m < |Q|-1+|P| \) such that \( a_1^n \in A_1 \) and \( \delta(Q, a_1^{n-|Q|}) = \delta(Q, a_1^n) \). Note that the language \( A_2 \cdots A_{k-1} \) is finite. Then, as in case (i), we only have to consider the words, whose length and number is linear bounded in \( |Q| \),

\[
a_1^n \cdot u, \quad 0 \leq n < |Q| - 1 + |P|, a_1^n \in A_1, u \in A_2 \cdots A_{k-1}
\]

and the corresponding sets

\[
S = \delta(Q, a_1^n \cdot u),
\]

and these are all possible sets in \( \{ \delta(Q, a_1^n u) \mid a_1^n \in A_1, u \in A_2 \cdots A_{k-1} \} \).

Fix any such subset \( S \). Then, as in case (ii) and Equation (2), choose any \( N \geq |Q|-1 \) with \( a_1^N \in A_k \) and we only have to compute \( \delta(S, a_1^N) \) and test if it is a singleton set. So, in total, we only have to test the words

\[
a_1^n \cdot u a_k^N, 0 \leq n < |Q| - 1 + |P|, a_1^n \in A_1, u \in A_2 \cdots A_{k-1}.
\]

Their length and number is linear bounded in \( |Q| \) and computing the reachable state from each state of the input automaton, and testing if a unique state results, could be performed in polynomial time. Hence, the overall procedure could be performed in polynomial time.

So, we have handled every case and the proof is complete. \( \square \)

In the proof of Lemma 31, we will need the following two lemmata. For \( n > 0 \), set

\[
L_n = (\Sigma^* a \Sigma^* b^{|P|} \Sigma^*)^n.
\]

Recall that \( \mathcal{B} = (\Sigma, P, \mu, p_0, F) \).

**Lemma 31.** Let \( \Sigma = \{a, b\} \) and \( L(\mathcal{B}) \subseteq a_1^* \cdots a_k^* \) with \( a_i \in \Sigma \) and \( n > 0 \). Then, the following are equivalent:
1. \( L(B) \cap L_n \neq \emptyset \),
2. there exist \( u_0, \ldots, u_n \in \Sigma^* a \Sigma^* \) and \( p_1, \ldots, p_n \geq |P| \) such that
\[
u_0 b^{p_n} u_1 \cdots u_{n-1} b^{p_n} u_n \in L(B),
\]
3. there exist \( u_0, \ldots, u_n \in \Sigma^* a \Sigma^* \) and \( p_1, \ldots, p_n > 0 \) such that
\[
u_0 (b^{p_i})^* u_1 \cdots u_{n-1} (b^{p_i})^* u_n \subseteq L(B).
\]

**Proof.** That (1) implies (2) is obvious. As \( \mu_i \geq |P| \), when reading these factors they have to induce a loop in \( B \), which implies (3). Lastly, if (3) holds true, as
\[
u_0 b^{[P]} p_1 u_1 \cdots u_{n-1} b^{[P]} p_n u_n \in L(B)
\]
and \( u_i \in \Sigma^* a \Sigma^* \), we also find \( u_0 b^{[P]} p_1 u_1 \cdots u_{n-1} b^{[P]} p_n u_n \in L_n \) and (1) follows.

**Lemma 32.** Let \( \Sigma = \{a, b\} \) and \( L(B) \subseteq a_1^* \cdots a_n^* \) with \( a_i \in \Sigma \). Then, there exists a maximal \( n \) such that \( L(B) \cap L_n \neq \emptyset \) and for this maximal \( n \), we can assume that \( u_i \notin \Sigma^* b[P] \Sigma^* \) for the \( u_i, i \in \{0, \ldots, n\} \), as in the previous lemma and \( n \leq |P| \).

**Proof.** Recall \( B = (\Sigma, P, \mu, p_0, F) \) Note that \( B \) must necessarily be polycyclic (this is a slightly stronger claim than Theorem 4 as this theorem only asserts existence of some polycyclic automaton) after removing all states that are not coaccessible, i.e., states from which no final state is reachable, which could obviously be done without altering \( L(B) \). For if \( B \) is then not polycyclic, then some strongly connected component does not consists of a single cycle only and we find two distinct words \( u, v \) and a state \( p \in P \) such that \( \mu(p, u) = \mu(p, v) = p \) (see also the forbidden pattern in [Pin20, Theorem 4.29]). But then, if we choose \( x, y \in \Sigma^* \) such that \( \mu(p_0, x) = p \) and \( \mu(p, y) \in F \), we find \( x(u + v)^* y \subseteq L(B) \).

Set \( m = \max\{|u|, |v|\} \) Then, for \( i > 0 \),
\[
\{ w \in x(u + v)^* y : |w| \leq |x| + i \cdot m \}
\]
contains \( x(u + v)^i \), and \( |x(u + v)^i| = 2^i \). So, \( L(B) \cap \{ w \in \Sigma^* : |w| \leq n \} \)
contains at least \( 2^{n-(|x|+|y|)/m} \) many words, i.e., it is not sparse. Furthermore, as \( L \cap \Sigma^n \in O(n^{c'}) \) as a function of \( n \) and only if \( L \cap \{ w \in \Sigma^* : |w| \leq n \} \in O(n^{c'}) \)
as a function of \( n \) for some \( c, c' \geq 0 \), the claim follows.

So, we can assume \( B \) is polycyclic and every state is coaccessible. Now, note that this implies that every loop in \( B \) (or strongly connected component in this case) must be labelled by a single letter, for if we have \( \mu(p, u) = p \) with \( |u|_a > 0 \) and \( |u|_b > 0 \) and choose again \( x, y \) such that \( \mu(p_0, x) = p \) and \( \mu(p, y) \in F \), we find \( x u^k y \in L(B) \), which contradict \( L(B) \subseteq a_1^* \cdots a_n^* \).

But then, note that if, for example, \( aba \in L(B) \), we must have \( |P| \geq 2 \), as \( \mu(p_0, ab) \notin \{p_0, \mu(p_0, a)\} \). Similarly, if we have a word that switches letters, every time a letter-switch occurs the state we end up in \( B \) must be a new state not visited before, for otherwise we would have a loop whose transition are not exclusively labelled by a single letter.
So, this implies that if we have a word as written in Lemma 31 in $L(B)$, then $n \leq |P|$ which implies that we can find a maximal $n$. That $u_i \notin \Sigma^* b|^P| \Sigma^*$ is also implied by Lemma 31 and the maximality of $n$. \hfill \square

**Lemma 32** Suppose $\Sigma = \{a, b\}$. Let $L(B) \subseteq \Sigma^*$ be letter-bounded. Then, $L(B)$-\textsc{Constr-Sync} is \textsc{NP}-hard if $L(B) \cap \Sigma^* a b|^P| b^* a \Sigma^* \neq \emptyset$.

**Proof (Proof of Lemma 32)** First, using Lemma 32 choose $J > 0$ maximal such that

$$L(B) \cap (\Sigma^* a \Sigma^* b|^P| \Sigma^*)^J \neq \emptyset.$$ 

As stated in the lemma, we have $J \leq |P|$ (which implies the construction to follow could be carried out in polynomial time). Then, by Lemma 31 there exist $u_0, \ldots, u_J \in \Sigma^* a \Sigma^*$ and $p_1, \ldots, p_J > 0$ ($J > 0$) such that

$$u_0(b^{p_1})^* u_1 \cdots u_{J-1}(b^{p_J})^* u_J \subseteq L(B).$$

Let $N = |P|$ times the least common multiple of the numbers $p_1, \ldots, p_J$. We give a reduction from the \textsc{DFA-Intersection} for unary input automata, which is \textsc{NP}-complete in this case [SM73, FK17]. Let $A_i = (\{b\}, Q_i, \delta_i, q_i, F_i)$ for $i \in \{1, \ldots, k\}$ be unary input automata, and we want to know if they all accept a common word. The problem remains \textsc{NP}-complete if we assume for no input automaton, a start state is also a final state. This is easily seen but could also be shown similar to [Hof20b, Proposition 1]. Also, we can assume $F_i \neq \emptyset$ for all $i \in \{1, \ldots, k\}$.

We are going to construct a semi-automaton $C = (\{a, b\}, Q, \delta)$.

Write $u_i = u_{i,1} \cdots u_{i,|u_i|}$ with $u_j \in \Sigma$. For each $i \in \{0, \ldots, J\}$, we construct a path labelled with $u_i$. Formally, let $P_i = \{q_{i,0}, \ldots, q_{i,|u_i|}\} \subseteq Q$ be new states and set

$$\delta(q_{i,j-1}, u_j) = q_{i,j}.$$ 

Then, for each $A_i$ we construct $J$ (disjoint) copies of $A_i$ and inflate them according to Definition 30 by $N$. Call the results $A_{i,1}, A_{i,2}, \ldots, A_{i,J}$ with $A_{i,j} = (\{b\}, Q_{i,j}, \delta_{i,j}, s_{i,j}, F_{i,j})$. Note these are unary automata over the letter $b$. Also, let $t \in Q$ be a new state, which will be a (global) sink state in $C$, i.e., we set $\delta(t, a) = \delta(t, b) = t$. Next, we describe how we interconnect these automata with the paths and with $t$. See also Figure 2 for a sketch of the reduction in the special case $J = 3$ and two input automata.

1. Let $j \in \{1, \ldots, J\}$. For each final state $q \in F_{i,j}$ let $P_{i,q}$ be a disjoint copy of the path $P_i$ constructed above, except for one final state $q$ were we simply retain the path $P_i$, but also name it by $P_{i,q}$. By identifying states, we mean states that we have previously constructed are now merged to a single state in $Q$. We have to pay attention that this procedure does not introduces any non-determinism. We identify the state $q_{i,0}$ with $q$ and continue to identify the states $q_{i,j}$ and $q' \in Q_{i,j}$ if $q_{i,j-1}$ and $q'' \in Q_{i,j}$ were identified and $u_{i,j} = b$ and $q' = \delta_{i,j}(q'', b)$. As $u_i \in \Sigma^* a \Sigma^*$, this process has to come to a halt before
we have identified < J states. Note that the first state such that \( q_{i,j-1} \) and \( q'' \in Q \) were identified but not \( q_{i,j} \) and \( \delta_{i,j}(q'', b) \), i.e., were \( u_{i,j} = a \), we have added an \( a \)-transition to \( q_{i,j} \) from \( q'' = q_{i,j-1} \) in \( A_{i,j} \), i.e., this is the first \( a \)-transition we have added to \( A_{i,j} \) and it branches out of \( A_{i,j} \). Then, if \( j < J - 1 \), identify the state \( q_{i,j} \) with the start state \( s_{i,j+1} \) of \( A_{i,j+1} \), i.e., the path \( P_{i,q} \) ends at this state. And if \( j = J \), we identify the state \( q_{i,j} \) with \( t \).

2. For the path \( P_0 \) identify its end state \( q_{i,|u_0|} \) with the start state \( s_{i,1} \) of \( A_{i,1} \).

3. Up to now, we still have missing transitions. In all the paths created, every missing \( b \)-transition, i.e., were we have a state with an \( a \)-transition leading out but no \( b \)-transition, we add a self-loop labelled with \( b \) to that state. For each path \( P \) (including the copies constructed in the first step) let \( p \in P \) be that state closest to the end state, but that does not equal the end state (by the identifications above, some end state might already have an \( a \)-transition that goes out of some automaton \( A_{i,j} \) and has an outgoing \( a \)-transition. Such a state exists as the \( u_i \in \Sigma^* a \Sigma^* \). Then, for every state in \( P \) that does not have an \( a \)-transition we add an \( a \)-transition going to \( p \). Consider \( A_{i,j} \) and let \( P \) some path (the specific choice does not matter) ending at the start state of \( A_{i,j} \). For each state \( q \in Q_{i,j} \) that does not has an outgoing \( a \)-transition up to now, add an \( a \)-transition going to the state \( p \in P \) described above in that path. This ensures later that, by reading an \( a \), we end up in a well-defined situation.

Then, put all the states created so far, i.e., those of the \( A_{i,j} \) and those of the paths constructed, into \( Q \) (note for each \( i \in \{1, \ldots, k\} \) we have constructed paths and automata, intuitively we have copied each \( A_i \), inflated the copies and interconnected them with the paths given by the \( u_j \) and let \( \delta \) be the transition as defined above or as given by \( A_{i,j} \) on the state of these automata.

We need the following property of \( C \). Suppose \( i \in \{1, \ldots, k\}, j \in \{1, \ldots, |J|\} \) and \( w \in \{a, b\}^* \).

Claim: Let \( q \in Q_{i,j} \) \( F_{i,j} \) with \( \delta(q, w) = t \). Then, there exist

\[
u_1, u_2, \ldots, u_{|J| - i + 1} \subseteq \{b\}^*
\]

and \( u, v \in \{a, b\}^* \) such that \( |u_i| \geq N \) and \( |u_i| \) is divisible by \( N \) for all \( i \in \{1, \ldots, |J| - i + 1\} \) and \( v_1, \ldots, v_{|J| - i + 1} \in \{a, b\}^* a \{a, b\}^* \) so that

\[
w = uv_1u_2v_2 \cdots u_{|J| - i + 1} v_{|J| - i + 1} u
\]

and \( v \notin \Sigma^* b^N \Sigma^* \).

Proof of the Claim. First, the state \( q \in Q_{i,j} \) has to be mapped to a final state, which could only be done by a word containing at least \( N \) times the letter \( b \), as in the inflated construction we can only go from non-auxiliary states to non-auxiliary states by reading at least that number of letters. However, before that we might read some word \( v \in \{a, b\}^* \) that moves states around, does not has a consecutive sequence of more
than $N$ b’s and hence, every a goes back to the start state. But at some point, this has to come to an end and we have to read a sequence of more than $N$ consecutive b’s. Additionally, by the construction of the inflation, the word that moves from a non-auxiliary state to another non-auxiliary state must have a number of b’s that is divisible by $N$. Also, observe that such a word must consists entirely of b, because for non-final states in $A_{i,j}$ every a maps back to the start state. Then, by construction (recall $u_i \in \Sigma^*a\Sigma^*$ for the labels of the paths constructed above) to move between the automata $A_{i,j}$ inside of $C$ we have to traverse a path where, on some part, we can only move forward by reading the letter a. After this, when we are at the start state of $A_{i,j}$, as by assumption the start state is not final, we again have to read at least $N$ times the letter b and so on, until we have reached a final state in $A_{i,j}$. Then, we have to read at least one a to map the final state to $t$, from which on, as $t$ is a sink state, we can read any word. [End, Proof of the Claim]

The automaton $C$ has a synchronizing word in $L$ if and only if all the $A_i$, $i \in \{1, \ldots, k\}$, accept a common word.

1. Assume we have a word $b^n$ accepted by all $A_i$ for $i \in \{1, \ldots, k\}$. Then, for

$$w = u_0 b^{N-n} u_1 \cdots u_{j-1} b^{N-n} u_j$$

we have $w \in L$ and $w$ synchronizes $A$. Note that, after reading $u_{j-1}$, the automaton $A_{i,j}$ is either in its start state, or the final a in $u_{j-1}$ has mapped some state in $A_{i,j}$ to a state outside of $Q_{i,j}$. So, when reading $b^{N-n}$, as $A_{i,j}$ equals the inflation of $A_i$ by $N$, we end up in a final state $F_{i,j}$. Then, we read $u_j$ to map those final states to the start state of the next automaton $A_{i,j+1}$ or to $t$ if $j = J$. Note that all states in-between are either mapped to a start state of some $A_{i,j}$, moved inside of some $A_{i,j}$, or, when an a is read and they are not mapped back to a state that ultimately ends in a start state of some $A_{i,j}$ are moved toward the state $t$. As we always read enough a to always make a step towards the sink state $t$ the result follows.

2. Assume $A$ has a synchronizing word $w \in L$. Then, as $t$ is a sink state, the word $w$ must map every state to $t$. Consider the start state of some $A_{i,1} = (\{b\}, Q_{i,1}, \delta_{i,1}, q_{i,1}, F_{i,1})$. By the above claim, there exist $u_1, u_2, \ldots, u_J \subseteq \{b\}^*$ such that $|u_i| \geq N$ and $|u_i|$ is divisible by $N$ for all $i \in \{1, \ldots, |J|\}$ and $v_1, \ldots, v_J \in \{a, b\}^*a\{a, b\}^*$ and $v, u \in \{a, b\}^*$ so that

$$w = vu_1v_1u_2v_2 \cdots u_Jv_Ju.$$  

By the above claim, Lemma 32 and the maximal choice of $J$, we have

$$\{v, v_1, \ldots, v_J\} \cap \Sigma^* b_{|P|}\Sigma^* = \emptyset,$$

i.e., these words does not contains a sequence of more than $|P|$, and so in particular not more than $N$, consecutive b’s.
Now, let $b^n$ be a maximal non-empty factor whose length $n$ is divisible by $N$ of $vu_1v_1$ and using only the letter $b$. Note that, by construction of $A_{i,1}$, if we write $vu_1v_1 = xb^n y$, we have $\delta_{i,1}(q_{i,1}, x) = q_{i,1}$. Then, we claim that $b^{n/N}$ is accepted by every automaton $A_i$. Fix an index $i \in \{1, \ldots, k\}$. By the construction of the inflation, this is equivalent with the condition that $b^n$ drives every automaton $A_{i,j}$ for $j \in \{1, \ldots, |J|\}$ from the start state to some final state. Suppose this is not the case. As the automata $A_{i,j}$ are isomorphic, i.e., they are copies of each other, we can assume this is not the case for $A_{i,1}$, i.e., we have $\delta_{i,1}(q_{i,1}, b^n) \notin F_{i,1}$. Then, consider the following suffix of $w$ (recall $xb^n y = vu_1v_1$, and $y$ has to start with an $a$)

$$yu_2v_2 \cdots u_{|J|}v_{|J|}u.$$ 

Note that if we have in $u$ a consecutive sequence of $b$’s of length more than $N$, the rest of $u$ also must consist of $b$’s only, i.e., we cannot read an $a$ anymore. For suppose this is not the case and $u \in \Sigma^* b^N \Sigma^* a \Sigma^*$. We have $\delta(q_{i,1}, xb^n) = \delta(q_{i,1}, b^n) \in Q_{i,1} \backslash F_{i,1}$. By assumption, $\delta(q_{i,1}, w) = t$, and so we must have $\delta(q_{i,1}, yu_2v_2 \cdots u_{|J|}v_{|J|}u) = t$. Applying the above claim again, yields that we can factorize $yu_2v_2 \cdots u_{|J|}v_{|J|}u$ such that we have at least $|J|$ blocks of consecutive $b$’s broken up by at least one occurrence of the letter $a$ between each such block. However, then then

$$w = xb^n yu_2v_2 \cdots u_{|J|}v_{|J|}u,$$

as $y$ starts with an $a$, we would get a factorization of $w$ with $|J| + 1$ blocks of consecutive $b$’s separated by words with at least one $a$, which is not possible by the maximal choice of $J$ and Lemma [31].

So, this shows that this is a valid reduction. \(\Box\)

**Theorem 19 (Dichotomy Theorem).** Let $a_1, \ldots, a_k \in \Sigma$ be a sequence of letters and $L \subseteq a_1^* \cdots a_k^*$ be regular. The problem $L$-CONSTR-SYNC is $\text{NP}$-complete if

$$L \cap \left( \bigcup_{1 \leq j_1 < j_2 < j_3 \leq k} L_{j_1, j_2, j_3} \right) \neq \emptyset$$

with $L_{j_1, j_2, j_3} = \Sigma^* a_{j_1} \Sigma^* a_{j_2}^{[P]} \Sigma^* a_{j_3} \Sigma^*$ for $1 \leq j_1 < j_2 < j_3 \leq k$ and solvable in polynomial time otherwise.

**Proof.** Set $L = L(\mathcal{B})$. Let $j_1, j_2, j_3 \in \{1, \ldots, k\}$ be such that $a_{j_2} \notin \{a_{j_1}, a_{j_3}\}$, $j_1 < j_2 < j_3$ and

$$L(\mathcal{B}) \cap L_{j_1, j_2, j_3} \neq \emptyset.$$

Then, there exists a word $u_1a_{j_1}u_2a_{j_2}^{[P]}u_3a_{j_3}u_4 \in L(\mathcal{B})$ with $u_1, u_2, u_3, u_4 \in \Sigma^*$. By the pigeonhole principle, when reading the factor $b^{[P]}$, at least one state has
to be traversed twice and we find \( p > 0 \) such that \( u_1au_2b^{\lceil p \rceil + 1}Pt \) for any \( i \geq 0 \).

Define a homomorphism \( \varphi : \Sigma^* \rightarrow \{a, b\}^* \) by \( \varphi(a_{j_1}) = \varphi(a_{j_2}) = a, \varphi(a_{j_3}) = b \) and, for the remaining letters, \( \varphi(a) = \varepsilon \), if \( a \in \Sigma \setminus \{a_{j_1}, a_{j_2}, a_{j_3}\} \). Then, \( \varphi(L) \subseteq \varphi(a_{j_1}) \cdots \varphi(a_{j_k}) \) is letter-bounded. Set \( \Gamma = \{a, b\} \) and let \( B' = (\Gamma, P', \mu', p_0, F') \) be a recognizing PDFA for \( \varphi(L) \). We have

\[
\varphi(u_1)a\varphi(u_2)b^{\lceil p \rceil + 1}\varphi(u_3)a\varphi(u_4) \in \varphi(L)
\]

for any \( i \geq 0 \). So, \( \varphi(L) \cap \Gamma^*a^+\Gamma^*b^+\Gamma^*a^+\Gamma^* \neq \emptyset \). By Lemma 9, \( \varphi(L)\) is \( \text{NP-hard} \), and so, with Theorem 5, also \( L\) is \( \text{NP-hard} \), and so, with Theorem 5, \( \text{NP-complete} \).

Now, suppose \( L(B) \cap \left( \bigcup_{1 \leq j_1 < j_2 < j_3 \leq k} L_{j_1,j_2,j_3} \right) = \emptyset \). By Theorem 3, we can write \( L(B) = \bigcup_{j=1}^k A_1^{(j)} \cdots A_k^{(j)} \) with unary regular languages \( A_j^{(i)} \subseteq \{a_j\}^* \) for \( j \in \{1, \ldots, k\} \). Then,

\[
(A_1^{(i)} \cdots A_k^{(i)}) \cap \left( \bigcup_{1 \leq j_1 < j_2 < j_3 \leq k} L_{j_1,j_2,j_3} \right) = \emptyset
\]

for any \( i \in \{1, \ldots, n\} \). However, this implies that for any \( i \in \{1, \ldots, n\} \), if there exists \( j \in \{1, \ldots, k\} \) such that \( A_j^{(i)} \) is infinite, then for all \( j' < j \), or for all \( j' > j \), we have \( A_{j'} \subseteq \{a_j\}^* \) (recall that if \( A_{j'} = \{\varepsilon\} \), then this is also fulfilled).

Hence, by Proposition 8, we have \( (A_1^{(i)} \cdots A_k^{(i)}) \)-\( \text{CONSTR-SYNC} \in \text{P} \) and then, by Lemma 9, \( L(B)\)-\( \text{CONSTR-SYNC} \in \text{P} \).

Next, we supply the proof of the claim made in the proof sketch of Lemma 9 from the main text.

Claim: For the constructed automaton \( \mathcal{A}' \) from the proof sketch of Lemma 9 in the main text, we have:

\[
\exists m \geq 0 : \delta(S, c^m) \subseteq T \iff \mathcal{A}' \text{ has a synchronizing word in } ab^{r_2}b^{2m}a.
\]

\[
\iff \mathcal{A}' \text{ has a synchronizing word in } ab^*a.
\]

\[
\iff \mathcal{A}' \text{ has a synchronizing word in } a^*b^*a^*.
\]

Proof of the Claim. First, suppose \( \delta(S, c^m) \subseteq T \). By construction of \( \mathcal{A}' \), for any \( q, q' \in Q \),

\[
\delta(q, c) = q' \text{ in } \mathcal{A} \iff \delta'(q, b^{2^m}) = q' \text{ in } \mathcal{A}'.
\]

Also, \( \delta'(Q' \setminus \{t\}, a) = S_{r_2} \) and \( \delta'(S_{r_2}, b^{2^m}) = S \). Combining these facts, we find

\[
\delta'(Q', ab^{r_2}b^{2^m}) \subseteq T \cup \{t\}.
\]

A final application of \( a \) then maps all states in \( T \) to the single sink state \( t \).
Clearly, as \( ab^r z(b^p)^* a \subseteq ab^*a \) and \( ab^*a \subseteq a^*b^*a^* \), the next two implications are shown. Finally, to complete the argument, let \( u = a^p b^q a^r \) be a synchronizing word, \( p, q, r \geq 0 \). Then, as \( t \) is a sink state, \( \delta'(Q', u) = \{t\} \). The only way to enter \( t \) from the outside is to read \( a \) at least once, and as \( t \) is a sink state, we have \( \delta'(Q', a^p b^q a^r) = \{t\} \). Also, as for \( q \neq T \), we have \( \delta'(q, a) \notin T \), we must have \( \delta'(Q', a^p b^q) \subseteq T \cup \{t\} \), or more specifically, \( \delta'(Q' \setminus \{t\}, a^p b^q) \subseteq T \). We distinguish two cases for \( p \).

1. If \( p = 0 \), then, in particular, \( \delta'(Q, b^q) \subseteq T \). By construction of \( A' \), for any \( q \in Q \),
   \[ \delta'(q, b^n) \in Q \]
   if and only if \( n \equiv 0 \pmod{p_2} \). So, \( q = p_2 m \) for some \( m \geq 0 \). Hence, by Equation 3 above from the first case, in \( A \), we find \( \delta(S, c^m) \subseteq T \).

2. If \( p > 0 \), then \( \delta'(Q' \setminus \{t\}, a^p) = S_{r_2} \). The only way to leave any state in \( S_{r_2} \)
   is to read \( b \), which transfers \( S_{r_2} \) to \( S_{r_2-1} \). Reasoning similarly, we find that we have to read in \( b \) at least \( r_2 \) many times, which finally maps \( S_{r_2} \) onto
   \( S_0 = S \). So, \( q \geq r_2 \). By construction of \( A' \), for any \( q \in Q \),
   \[ \delta'(q, b^n) \in Q \]
   if and only if \( n \equiv 0 \pmod{p_2} \). So, as \( \delta'(S, b^{r_2}) \subseteq T \), \( q - r_2 = p_2 m \) for some \( m \geq 0 \). Hence, by Equation 3 above, in \( A \), we find \( \delta(S, c^m) \subseteq T \).

This ends the proof of the claim. [End, proof of the Claim.]

D Proofs for Section 5 (Constraints from Strongly Self-Synchronizing Codes)

Proposition 13 A non-empty \( C \subseteq \Sigma^+ \) is a strongly self-synchronizing code if and only if, for all \( u \in \text{Pref}(C) \) and \( v \in C \), if we write \( uv = x_1 \cdots x_n \) with \( x_i \in \Sigma \)
for \( i \in \{1, \ldots, n\} \), then, for all \( j \in \{1, \ldots, n\} \) and \( k \geq 1 \) where \( j + k - 1 \leq n \), we have that \( x_j \cdots x_{j+k-1} \in C \) implies \( j = |u| + 1 \) and \( k = |v| \) or \( j = 1 \) and \( k = |u| \). Intuitively, in \( uv \) only the last \( |v| \) symbols form a factor in \( C \) and
possibly the first \( |u| \) symbols.

Proof. Let \( C \subseteq \Sigma^+ \) be a strongly self-synchronizing code. Suppose \( u \in \text{Pref}(C) \)
and \( v \in C \). If \( u \notin C \), then we must have \( uv \notin \Sigma^*C^+ \), so that, if \( uv = x_1 \cdots x_n \) as
in the statement, we have \( x_j \cdots x_{j+k-1} \) if and only if \( j = |u| + 1 \) and \( k = |u| + |v| \).
If \( u \in C \), then, as \( uv \notin \Sigma^+C^+ \), we find that we have only the possibilities \( j = 1 \)
and \( k = |u| \) or \( j = |u| + 1 \) and \( k = |u| + |v| \).

Conversely, suppose \( C \subseteq \Sigma^+ \) is non-empty and fulfills the condition mentioned
in the statement. If \( u, v \in C \) and \( uv \in \Sigma^+C^+ \), then we can write \( uv = x_1 \cdots x_n \) with \( x_i \in \Sigma \)
for \( i \in \{1, \ldots, n\} \) and find \( 2 \leq i \leq j \leq n - 1 \) such that \( x_i \cdots x_j \in C \), which
contradicts the condition in the statement. Similarly, if \( u \in \text{Pref}(C) \setminus C \) and \( v \in C \) with \( uv \in \Sigma^*C^+ \), then we can write \( uv = x_1 \cdots x_n \) with \( x_i \in \Sigma \)
for \( i \in \{1, \ldots, n\} \) and find \( 1 \leq i \leq j \leq n - 1 \) such that \( x_i \cdots x_j \in C \),
which would contradict the condition too. So, we must have \( C^2 \cap \Sigma^+C^+ = \emptyset \)
and \( (\text{Pref}(C) \setminus C) \cap \Sigma^*C^+ = \emptyset \). □
To give a proof of the claim made in Example 14.

**Proposition 33.** The code \{aacc, bbc, bac\} is strongly self-synchronizing.

**Proof.** By checking all cases to combine prefixes with code words:

Non-empty prefixes of aacc: (a)aacc  (a) bbc  (a) bac
(a)aacc  (a) bbc  (a) bac
(aac)aacc  (aac) bbc  (aac) bac
(aacc)aacc  (aac) bbc  (aac) bac

Non-empty prefixes of bbc: (b)aacc  (b) bbc  (b) bac
(b)bacc  (b) bbc  (b) bac
(bcc)aacc  (ccc) bbc  (ccc) bac

Non-empty prefixes of bac: (b)aacc  (b) bbc  (b) bac
(ba)aacc  (ba) bbc  (ba) bac
(bac)aacc  (bac) bbc  (bac) bac.

So, we see that the defining conditions are satisfied. Note that \(C \cap \Sigma^* \Sigma^+ = \emptyset\) is always satisfied for self-synchronizing codes, as they are infix codes.

**Theorem 17.** Let \(\varphi : \Sigma^* \rightarrow \Gamma^*\) be a homomorphism such that \(\varphi(\Sigma)\) is a strongly self-synchronizing code and \(|\varphi(\Sigma)| = |\Sigma|\). Then, for each regular \(L \subseteq \Sigma^*\) we have \(L\text{-Constr-Sync} \equiv_{\log} \varphi(L)\text{-Constr-Sync}\).

**Proof.** By Proposition 2 we have \(\varphi(L)\text{-Constr-Sync} \leq_{m} L\text{-Constr-Sync}\). Next, we give a reduction from \(L\text{-Constr-Sync}\) to \(\varphi(L)\text{-Constr-Sync}\).

Write \(\Sigma = \{a_1, \ldots, a_n\}\) with \(n = |\Sigma|\) and \(u_i = \varphi(a_i)\) for \(i \in \{1, \ldots, n\}\). Let \(A = (\Sigma, Q, \delta)\) be an input semi-automaton for \(L\text{-Constr-Sync}\).

We construct a semi-automaton \(A' = (\Gamma, Q', \delta')\). The state set will be

\[Q' = \{ q_z \mid q \in Q, x \in \text{Pref}(\{u_1, \ldots, u_n\}) \setminus \{u_1, \ldots, u_n\} \} .\]

By identifying \(q_z\) with the state \(q \in Q\), we can assume \(Q \subseteq Q'\). Then, for \(q_z \in Q'\) and \(y \in \Sigma\), let \(z\) be the longest suffix of \(xy\) such that \(z \in \text{Pref}(\{u_1, \ldots, u_n\})\) and set\(^{10}\)

\[\delta'(q_z, y) = \begin{cases} q_z & \text{if } z \in \text{Pref}(\{u_1, \ldots, u_n\}) \setminus \{u_1, \ldots, u_n\}; \\ \delta(q, a_i) & \text{if } \exists i \in \{1, \ldots, n\} : z = u_i. \end{cases} \tag{4}\]

As \(|\varphi(\Sigma)| = |\Sigma|\) and \(\{u_1, \ldots, u_n\}\) is a prefix code\(^{11}\), the transition function is well-defined. By construction, for any \(u \in \Sigma^*\) and \(q \in Q\), we have

\[\delta(q, u) = \delta'(q, \varphi(u)). \tag{5}\]

\(^{10}\) Note the implicit correspondence between the states \(q\) and \(q_z\) for \(z \in \text{Pref}(\varphi(\Sigma)) \setminus \varphi(\Sigma)\).

\(^{11}\) A code is a prefix code, if no code word is the proper prefix of another code word.
Let \( x \in \text{Pref}(\varphi(\Sigma)) \setminus \varphi(\Sigma) \) and \( u_i \in \varphi(\Sigma), i \in \{1, \ldots, n\} \). Then, as \( \varphi(\Sigma) \) is a strongly self-synchronizing code, the word \( xu_i \) does not contain a word from \( \varphi(\Sigma) \), except the suffix \( u_i \), as a factor. Next, we will argue that, for the unique \( a_i \in \Sigma \) with \( \varphi(a_i) = u_i \), the following equations holds true:
\[
\delta'(q_x, u_i) = \delta'(q, u_i) = \delta(q, a_i).
\] (6)

For if \( v \in \text{Pref}\{\{u_i\}\} \cap \Sigma \), then the longest suffix of \( xv \) in \( \text{Pref}(\varphi(\Sigma)) \) must be \( v \). First, it is a suffix from this set. Second, if there exists longer one, say \( w \), then write \( ww' \in \varphi(\Sigma) \) for some \( w' \in \Sigma^* \). In that case, with \( xw = x'w' \) (\( |x'| < |x| \)), we have \( x'w' \in xu_i \Sigma^* \) or \( xu_i \in x'w' \Sigma^+ \). In the first case, \( wu' \) contains the proper factor \( u_i \in \varphi(\Sigma) \), which is not possible as \( \varphi(\Sigma) \) is, in particular, an infix code. In the second case, \( \{x'\} \cap \Sigma^* \varphi(\Sigma) \Sigma^+ \neq \emptyset \), which is excluded by the property of \( \varphi(\Sigma) \) being strongly self-synchronizing. So, by the defining equation of \( \delta' \), Equation (4), if \( v \notin \varphi(\Sigma) \), we have
\[
\delta'(q_x, v) = q_v,
\]
and if \( v \in \varphi(\Sigma) \), then \( v = u_i \), as \( \varphi(\Sigma) \) is a prefix code, and
\[
\delta'(q_x, v) = \delta'(q_x, u_i) = \delta'(q, u_i) = \delta(q, a_i)
\]
with the unique \( a_i \in \Sigma \) as above. So, in the latter case Equation (4) was established. In the former case, if \( u_i = vv'v'' \), then \( \delta'(q_x, v') = q_vu' \) which is easily seen as we always read in a word giving a prefix from \( \varphi(\Sigma) \), hence this word itself is the longest suffix from \( \varphi(\Sigma) \). So, after reading the entire word \( u_i \), by Equation (4), Equation (6) is implied.

Lastly, we show that this gives a valid reduction.

Claim: The automaton \( \mathcal{A} = (\Sigma, Q, \delta) \) has a synchronizing word in \( L \) if and only if \( \mathcal{A}' = (\Gamma, Q', \delta') \) has a synchronizing word in \( \varphi(L) \).

**Proof of the Claim:** First, suppose there exists \( u \in L \) such that \( |\delta(Q, u)| = 1 \). If \( |Q| = 1 \) every word is synchronizing and the statement is obviously true. So, we can assume \( |Q| > 1 \), which implies \( |u| > 0 \). Write \( u = av \) with \( a \in \Sigma \). By Equation (6), then, for any \( x \in \text{Pref}(\varphi(\Sigma)) \setminus \varphi(\Sigma) \),
\[
\delta'(q_x, \varphi(a)) = \delta(q, a).
\]
Hence, \( \delta'(Q', \varphi(a)) = \delta(Q, a) \). As \( \delta'(Q', \varphi(a)) \subseteq Q \), by Equation (4), or its formulation for the special case of states in \( Q \), Equation (5), we find
\[
\delta'(\delta(Q, a), \varphi(v)) = \delta(Q, a), v = \delta(Q, u).
\]
The last set is, by assumption, a singleton set. Hence, the word \( \varphi(u) \) synchronizes \( \mathcal{A}' \).

Now, suppose there exists \( u \in \varphi(L) \) such that \( |\delta'(Q', u)| = 1 \). Let \( v \in \Sigma^* \) be such that \( \varphi(v) = u \). By Equation (5) (or Equation (4)), we have
\[
\delta(Q, v) = \delta'(Q, \varphi(v)).
\]
By assumption, the set on the right side is a singleton set. Hence, \( v \) synchronizes \( \mathcal{A} \). [End, Proof of the Claim]
So, we find \( L\text{-CONSTR-SYNC} \leq_{\text{log}} \varphi(L)\text{-CONSTR-SYNC} \) and the proof is done.

**Theorem 18.** Let \( L \subseteq w_1^* \cdots w_k^* \) be regular such that \( \{w_1, \ldots, w_k\} \) is a strongly self-synchronizing code. Then, \( L\text{-CONSTR-SYNC} \) is either \( \text{NP} \)-complete or in \( \text{P} \).

**Proof.** Let \( \Gamma = \{a_1, \ldots, a_n\} \) be a new alphabet and let \( \varphi : \Gamma^* \rightarrow \Sigma^* \) be the homomorphism given by \( \varphi(a_i) = w_i \) for \( i \in \{1, \ldots, n\} \). Let \( U = \varphi^{-1}(L) \). As every word in \( L \) is a concatenation of words from \( \{w_1, \ldots, w_n\} \), we have \( L \subseteq \varphi(\Gamma^*) \).

By Theorem 17, the languages \( U \) and \( L \) have the same computational complexity. Also, as is easy to check, we have \( U \subseteq a_1^* \cdots a_n^* \) and \( U \) is regular. So, by Theorem 10 the constrained synchronizing problem for \( L \) is either \( \text{NP} \)-complete or in \( \text{P} \).

## E Proofs for Section 6 (Conclusion and Discussion)

**Proposition 22.** Let \( u, v \in \Sigma^* \). If \( L \subseteq u^*v^* \) is regular, then \( L\text{-CONSTR-SYNC} \) is solvable in polynomial time.

**Proof.** Let \( \Gamma = \{a, b\} \) and \( \varphi : \Gamma^* \rightarrow \Sigma^* \) be the homomorphism given by \( \varphi(a) = u \) and \( \varphi(b) = v \). Define \( N = \{(i, j) \mid u^i v^j \in L\} \) and set \( L' = \{a^ib^j \mid (i, j) \in N\} \subseteq a^*b^* \). Then, \( \varphi(L') = L \) and by Proposition 8 we have \( L'\text{-CONSTR-SYNC} \in \text{P} \). So, with Proposition 2 also \( L\text{-CONSTR-SYNC} \in \text{P} \).

**Proposition 23.** The problem \( (ab)(ba)^*(ab))\text{-CONSTR-SYNC} \) is \( \text{NP} \)-complete.

**Proof.** We give a reduction from \( \text{DISJOINTSETTRANSPORTER} \) for unary input semi-automata, which is \( \text{NP} \)-complete by Theorem 1. Let \( A = (\{c\}, Q, \delta) \) with \( S, T \subseteq Q \) being disjoint. Construct the automaton \( A' = (\{a, b\}, Q', \delta') \) with

\[
Q' = Q \cup \{q_a \mid q \in Q\} \cup \{q_b \mid q \in Q\} \cup \{t\}.
\]

Fix some \( \delta \). Then, for \( q \in Q \), set

\[
\delta'(t, x) = t \quad \text{for } x \in \Sigma \quad \text{and} \quad \delta'(q, x) = q_x \quad \text{for } x \in \Sigma;
\]

\[
\delta'(q_a, b) = t \quad \text{if } q_a \in Q' \quad \text{and} \quad \delta'(q_b, a) = \delta(q, c);
\]

\[
\delta'(q_a, a) = q_a \quad \text{for } q_a \in Q';
\]

\[
\delta'(q, b) = \begin{cases} 
\delta(q, c) & \text{if } q \in Q \setminus (T \cup S); \\
q & \text{if } q \in S; \\
t & \text{if } q \in T.
\end{cases}
\]

Then, there exists \( n \geq 0 \) with \( \delta(S, c^n) \subseteq T \) if and only if \( A' \) has a synchronizing word in \( L \).

First, suppose there exists \( n \geq 0 \) such that \( \delta(S, c^n) \subseteq T \). By construction, \( S \subseteq \delta'(Q', ab) \subseteq S \cup \{t\} \cup Q_a \), or more precisely \( \delta'(Q', ab) = S \cup \{t\} \cup \{q_b \mid q \in \delta(Q, c)\} \). Note that, as \( S \) and \( T \) are disjoint, we must have \( n > 0 \). As, for any \( q \in Q \), \( \delta'_{\delta}(q, ab) = \delta(q, c) \) and \( \delta(q_b, b) = t \), we find \( \delta'(\delta'(Q', ab), (ba)^n) \subseteq T \cup \{t\} \).
where we needed $n > 0$ to map those states in \( \{ q_b \mid q \in \delta(Q, c) \} \) to \( T \). Finally, \( \delta(T \cup \{ t \}, ab) = \{ t \} \) and so \( \delta'(Q', ab(ab)^n ab) = \{ t \} \).

Conversely, suppose there exists \( n \geq 0 \) such that \( \delta'(Q', ab(ab)^n ab) \) is a singleton set. So, as \( t \) is a sink state, \( \delta'(Q', ab(ab)^n ab) = \{ t \} \). By construction, a state in \( Q' \) is mapped to \( t \) by \( ab \) if and only if it is contained in \( T \cup \{ t \} \). Hence, \( \delta'(Q', ab(ab)^n ab) \subseteq T \cup \{ t \} \). As before, \( \delta'(Q', ab) = S \cup \{ q_b : q \in \delta(Q, c) \} \). In particular, we must have \( \delta'(S, (ba)^{\infty}) \subseteq T \cup \{ t \} \). As, for any \( q \in Q \), \( \delta'(q, ba) = \delta(q, c) \), this implies that \( \delta'(S, (ba)^{\infty}) \subseteq T \) and that for \( u = c^a \) we have \( \delta(S, c^a) \subseteq T \).

By Theorem \( L \)-Constr-Sync \( \in \text{NP} \) and by the above reduction the problem is \( \text{NP} \)-complete.

References for the Appendix

FK17. Henning Fernau and Andreas Krebs. Problems on finite automata and the exponential time hypothesis. *Algorithms*, 10(1):24, 2017.

GKRS10. Pawel Gawrychowski, Dalia Krieger, Narad Rampersad, and Jeffrey O. Shallit. Finding the growth rate of a regular or context-free language in polynomial time. *Int. J. Found. Comput. Sci.*, 21(4):597–618, 2010.

HKMW17. A. Herrmann, M. Kutrib, A. Malcher, and M. Wendlandt. Descriptive complexity of bounded regular languages. *Journal of Automata, Languages and Combinatorics*, 22(1-3):93–121, 2017.

Hof19. Stefan Hoffmann. Commutative regular languages - properties and state complexity. In Miroslav Cirić, Manfred Droste, and Jean-Éric Pin, editors, *Algebraic Informatics - 8th International Conference, CAI 2019, Niš, Serbia, June 30 - July 4, 2019, Proceedings*, volume 11545 of *Lecture Notes in Computer Science*, pages 151–163. Springer, 2019.

Hof20a. Stefan Hoffmann. Computational complexity of synchronization under regular commutative constraints. In Donghyun Kim, R. N. Uma, Zhipeng Cai, and Dong Hoon Lee, editors, *Computing and Combinatorics - 26th International Conference, COCOON 2020, Atlanta, GA, USA, August 29-31, 2020, Proceedings*, volume 12273 of *Lecture Notes in Computer Science*, pages 460–471. Springer, 2020.

Hof20b. Stefan Hoffmann. On a class of constrained synchronization problems in \( \text{NP} \). In Gennaro Cordasco, Luisa Gargano, and Adele Rescigno, editors, *Proceedings of the 21th Italian Conference on Theoretical Computer Science, ICTCS 2020, Ischia, Italy*, CEUR Workshop Proceedings. CEUR-WS.org, 2020.

LT84. Michel Latteux and Gabriel Thierrin. On bounded context-free languages. *Elektronische Informationsverarbeitung und Kybernetik (Journal of Information Processing and Cybernetics)*, 20(1):3–8, 1984.

Pin20. Jean-Éric Pin. *Mathematical Foundations of Automata Theory*. 2020.

SM73. Larry J. Stockmeyer and Albert R. Meyer. Word problems requiring exponential time (preliminary report). In *Proceedings of the fifth annual ACM Symposium on Theory of Computing, STOC*, pages 1–9. ACM, 1973.
Fig. 2. The reduction from the proof of Lemma 9 in the special case $J = 3$ (see the proof for the definition of $J$) and two input automata $A_1, A_2$ over $\{b\}$. The automata $A_{i,j}$ are inflated, according to Definition 30, copies of $A_i$ for $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. The letter $a$ maps every state not associated with a path inside each $A_{i,j}$ to the last innermost state that is hit by an $a$ along the path leading into this automaton. This is only drawn for $A_{1,1}$ but left out for the other automata, also, to give a more “high-level” drawing, the $b$-transitions are not drawn. On the right end is the sink state $t$. The paths stay inside the automata but leave as soon as an $a$ is read.