On Non-minimal Coupling of Magnetic Field with Gravity in the Schwarzschild Spacetime

Kumar Ravi
Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math 711202, West Bengal, India

Petar Pavlović
Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math 711202, West Bengal, India, Institute for Cosmology and Philosophy of Nature, Trg svetog Florijana 16, Križevci, Croatia

(Dated: October 1, 2021)

In this work we study the effects of non-minimal coupling between electromagnetism and gravity, which are motivated by quantum effects such as vacuum polarization. We investigate the modification of both asymptotically dipole and uniform magnetic fields around the Schwarzschild black hole that come as the result of the non-minimal coupling. The case of the asymptotically dipole magnetic field is considered first, while improving and critically analysing the results of recent earlier investigations of this topic. Furthermore, the asymptotically homogeneous field configuration is also considered, while providing the effective potential analysis for both configurations. While in both cases (the dipole and the uniform) the magnetic fields get enhanced or suppressed with respect to the case of minimal coupling, depending on the sign of non-minimal coupling parameter, in the case of a background uniform magnetic field the direction of magnetic field also alters in the vicinity of the black hole horizon. We have discussed the possible astrophysical and cosmological sources where the vacuum polarization may be at play, while also discussing the observational effects. We conclude that such observational signatures could be used to constrain the value of the non-minimal coupling parameter.

I. INTRODUCTION

The description of electromagnetism on macroscopic scales in terms of Maxwell equations represents one of the most successful, well established and also the oldest field theory. The standard description of electromagnetic fields in the presence of gravitational fields, which is of interest in cosmology and astrophysics, therefore assumes a direct mathematical generalization of Maxwell’s equations on curved spacetime[1]. On the other hand, the effect of electromagnetic fields on spacetime is in the standard approach described by the Einstein’s equation, where the stress-energy tensor of electromagnetic fields curves the spacetime in the same fashion as any other source of energy density. This type of dependence between gravity and electromagnetism is called minimal, since the field equations for this case are derived by a simple addition of gravitational and electromagnetic part of the Lagrangian, without any cross terms. There are however strong reasons to expect that this picture should be changed for very strong gravitational fields, where the non-minimal coupling - the presence of Lagrangian terms containing direct contractions between gravitational and electromagnetic tensors - should arise. First of all, we could expect that on sufficiently high energies gravity and electromagnetism will get united and described by a single field, similar to what was already found for electricity and magnetism and electromagnetism and weak nuclear interaction. Since in this regime gravitational and electromagnetic sector could change into each other and would manifest a deep interconnection, the investigation of non-minimal coupling between electromagnetism and gravity can serve as an effective model for such effects. There is also a more concrete reason for consideration of non-minimal coupling in regimes of strong gravitational fields. In [2] the effect of QED vacuum polarization on the curved spacetime was studied for the case of a photon propagating in vacuum and it was found that it leads to a non-minimal coupling between gravity and electromagnetism. This can be understood as coming from the transition of a photon into an electron/positron pair and the consequent tidal influences of the spacetime geometry along the characteristic Compton wavelength [2, 3]. In the one-loop approach it was demonstrated [2] that the vacuum polarization effects thus leads to the effective Lagrangian density of the following form

\[ \mathcal{L} = \frac{R}{\kappa} + \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \mathcal{L}_{\text{matter}}, \]  

(1)

where \( \kappa = 8\pi G/c^4 \) (from now on we will set \( c = G = 1 \)), \( R \) is the Ricci scalar, \( F^{\mu\nu} \) is the Maxwell tensor obeying \( F^{\mu\nu} = \nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu} \) where \( \nabla_{\mu} \) is the covariant derivative and \( \mathcal{L}_{\text{matter}} \) is the Lagrangian of neutral matter. The effects of the vacuum polarization are contained in the tensor \( R^{\mu\nu\rho\sigma} \) (not to be confused with the Riemann tensor \( R^{\mu\nu\rho\sigma} \)), which is defined as follows

\[ R^{\mu\nu\rho\sigma} = \frac{q_1}{2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) R + \frac{q_2}{2} (R^{\mu\rho} g^{\nu\sigma} - R^{\mu\sigma} g^{\nu\rho} + R^{\nu\sigma} g^{\mu\rho} - R^{\nu\rho} g^{\mu\sigma}) + q_3 R^{\mu\nu\rho\sigma}, \]  

(2)
where \( q_1 \), \( q_2 \) and \( q_3 \) are the the coupling constants, and as usual \( R^{\mu \nu} \) is the Ricci tensor and \( R^{\mu \nu \rho \sigma} \) is the Riemann tensor.

In [3] it was proposed that the signatures of non-minimal coupling between electromagnetism and gravity could be in principle observed, or at least constrained, by studying the magnetic fields around the event horizons of black holes and that the same effect could be used for constraining the sizes of primordial black holes. With this aim, the problem of modification of magnetic fields due to the vacuum polarization effect around the Schwarzschild black hole was for the first time studied in [3] and the consequences for the orbits of charged particles around such black holes were for the first time studied in [4]. In this work we try to further develop the investigation of this topic. We study the uniform magnetic field configuration, which was not analysed before in this context, and also improve and critically discuss the previously studied dipole magnetic field case. We then investigate the motion in equatorial plane by using the effective potential and demonstrate that most of the conclusions that can be reached regarding the effects of the non-minimal coupling on the trajectories and scattering of the charged particles can be simply understood with the help of the effective potential study.

This paper is organised as follows: In Sec. II we have briefly reviewed the theory of non-minimal coupling. In Sec. III we have presented the analytical solutions(minimal coupling) and numerical solutions(non-minimal coupling) of the Maxwell equations in Schwarzschild spacetime for the cases (i) when a static and asymptotically dipole magnetic field is placed at origin of the Schwarzschild black hole and (ii) when a Schwarzschild black hole is placed in a static and asymptotically uniform magnetic field. In Sec. IV we study the motion of a test charged particle in the equatorial plane of the Schwarzschild spacetime in the effective potential formalism for both of these magnetic field configurations. In Sec. V we briefly explore the energetics, collision and the possibility of acceleration of test charged particles near the event horizon when a Schwarzschild black hole is placed in a static and asymptotically uniform magnetic field. In Sec. VI we discussed the possible astrophysical and cosmological scenarios where the considerations of vacuum polarization becomes important and hence the application non-minimal coupling. Also in this section we have discussed the possible observational signatures. In Sec. VII we have concluded this work with the discussion for future scopes.

II. NON-MINIMAL COUPLING

There are of course numerous ways in which the non-minimal coupling between gravitational and electromagnetic sector can be achieved, such as coupling between curvature tensors of different rank and order with vector potentials, Maxwell tensors and their contractions etc. [5–12]. Many of those options are, however, completely arbitrary and lacking an additional motivation and some of them lead to the violation of important physical principles, such as gauge invariance. We will focus on the type of non-minimal coupling discussed in Sec. I. and considered in [2] since: i) it is motivated by the quantum-electrodynamical one-loop corrections in curved spacetime, ii) it involves the contraction of all fundamental electromagnetic and curvature tensors to the leading order and iii) it satisfies the gauge invariance. When the variational procedure is performed on the Lagrangian (1) we obtain the following equations of motion:

\[
\nabla_\mu (F^{\mu \nu} + R^{\mu \nu \rho \sigma} F_{\rho \sigma}) = 0. \tag{3}
\]

III. NON-MINIMAL COUPLING IN A SCHWARZSCHILD SPACETIME

At this point we need to stress the most important physical approximations which we take in our study, which were also assumed in earlier works [3, 4]. We first assume that magnetic fields are weak enough so that the their effect on the spacetime stays negligible, and can therefore still be described by the Schwarzschild metric. Finding the complete solutions to the both non-minimally coupled Maxwell and Einstein equation is, however, of significant interest, and should be considered in future research. Such full solutions of the non-minimal problem, describing both electromagnetic and gravitational sector, could be of physical importance in the very early Universe, where the non-minimal coupling effects and the resulting magnetic fields could become very strong (see discussion in Sec. VI). The weak field and negligible feedback on the spacetime approximation we use here should, however, be justified for all known astrophysical systems (see detailed discussion in [3]). The next crucial assumption we made is to ignore the electric fields and consider only the magnetic part of the electromagnetic tensor. This is justified because of the very high value of the conductivity of the Universe, causing the electric fields to be completely insignificant – and this assumption therefore represents one of the standard elements in studies of astrophysical electromagnetism. On the other hand, magnetic fields are existing on all scales of the observable Universe [13–19]. This fact makes magnetic fields an excellent candidate for investigation of the effect of non-minimal coupling between electromagnetism and gravity. We therefore do not need to consider a special mechanism for creation of magnetic field around black holes since every black hole will naturally be immersed at least within the galactic magnetic field. The physical picture we approach here is therefore as follows: we assume the existence of a magnetic field, going to its asymptotical dependence far away from the black hole – where the
spacetime is approximately flat, while being influenced by the black hole spacetime near the black hole horizon. The magnetic field we treat here is thus not the magnetic field of the black hole itself, but the external magnetic field, which can be galactic or belonging to some other source, which we immerse in the Schwarzschild spacetime.

The Schwarzschild metric in a coordinate system adapted to spherical symmetry \((t, r, \theta, \phi)\) is

\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\tag{4}\]

where the mass \(M\) is the source of gravity. Schwarzschild solution of Einstein field equation is a unique solution and obtained with the assumptions of spherical symmetry, asymptotic flatness, and vacuum outside the spherical object of mass \(M\) and the solution turns out to be static also.

We assume a magnetic field, expressed in a local Lorentz frame(LLF)\([20, 21]\)

\[
F_{\theta\phi}^{LLF} = B_r = \frac{2\mu \cos \theta}{r^3} \xi(r),
F_{\phi r}^{LLF} = B_\theta = \frac{\mu \sin \theta}{r^3} \psi(r),
F_{r\theta}^{LLF} = B_\phi = 0,\tag{5}
\]

where \(\mu\) is magnetic moment and the functions \(\xi(r)\) and \(\psi(r)\) take into account the effects of curved spacetime on magnetic filed. We will later see that \(\xi(r)\) and \(\psi(r)\) are to be obtained from a second order linear differential equations, and so we get two linearly independent equations: one corresponding to a dipole magnetic field while another corresponding to a uniform magnetic field (we will also see that the uniform magnetic field solution is directed along the negative \(z\) direction whereas the dipole solution has its dipole moment directed along the positive \(z\) direction).

After doing the transformation from local Lorentz frame, the components of Maxwell tensor in Schwarzschild spacetime are

\[
F_{\theta\phi} = \frac{2\mu \sin \theta \cos \theta}{r} \xi(r),
F_{\phi r} = \frac{\mu \sin^2 \theta}{r^2 \sqrt{1 - \frac{2M}{r}}} \psi(r),
F_{r\theta} = 0.\tag{6}
\]

As for any vacuum solution of the Einstein field equation, the Ricci scalar\((R)\) and Ricci tensor\((R_{\mu\nu})\) are zero, so the Eq. \(3\) reduces to

\[
\nabla_\mu (F^{\mu\nu} + q_3 R^{\mu\nu\rho\sigma} F_{\rho\sigma}) = 0.\tag{7}
\]

The Maxwell tensor components are also subject to another Maxwell equation

\[
\nabla_\lambda F^{\lambda\mu\nu} = 0.\tag{8}
\]

After substituting Maxwell tensor components from Eq. 6 in Eq. 7 and in Eq. 8, we get

\[
\frac{d}{dr} \left[ \left(1 - \frac{2M}{r^3}\right)^{1/2} \psi(r) \right] + \frac{1}{1 + 4Mq_3} \frac{2\xi(r)}{r^3} = 0,\tag{9}
\]

and

\[
\frac{d}{dr} \left[ \frac{\xi(r)}{r^2 \sqrt{1 - \frac{2M}{r}}} \right] + \frac{\psi(r)}{r^2}\sqrt{1 - \frac{2M}{r}} = 0,\tag{10}
\]

respectively. These equations can easily be decoupled to get

\[
\frac{d}{dr} \left[ \left(1 - \frac{2M}{r^3}\right) \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \left(\frac{\xi(r)}{r}\right) \right] - \frac{1}{1 + 4Mq_3} \frac{2\xi(r)}{r^3} = 0.\tag{11}
\]

Here we do not write the decoupled equation for \(\psi(r)\), not just for the reason of its complicated form, but also for the fact that it will not be used in our computations. Namely, it is much more easier to use the solutions of Eq. 11 and insert them into Eq. 10, thus obtaining the solutions for \(\psi(r)\), rather than solving another differential equation from beginning.

Before getting into the solutions of Eq. 9 and Eq. 10 i.e. for the case of non-minimal coupling, we briefly review the available solutions for the case of minimal coupling; the utility of doing so will become clear later.

### A. Minimal Coupling: A Brief Review of Analytical Solutions

As expected with \(q_3 = 0\) the Eqs. 9 and 10 reduce to corresponding equations of \([20]\) and \([21]\). Though for the case of minimal coupling(i.e. \(q_3 = 0\)), getting analytical solutions is straight forward with the application of Frobenius series solution technique for a second order ordinary linear homogeneous equation, we could not obtain the same for \(q_3 \neq 0\). The solutions for the case of minimal coupling are \([20–22]\)

\[
\xi^d(r) = -\frac{3r^3}{8M^3} \ln \left(1 - \frac{2M}{r}\right) + \frac{2M}{r} \left(1 + \frac{M}{r}\right),
\]

\[
\psi^d(r) = \frac{3r^2}{4M^2} \left[1 + \left(1 - \frac{2M}{r}\right)^{-1}\right] + \frac{r}{M} \ln \left(1 - \frac{2M}{r}\right) \sqrt{1 - \frac{2M}{r}},\tag{12}
\]

\[
\xi^d(r) = -\frac{3}{8M^3} r^3 \ln \left(1 - \frac{2M}{r}\right) - \frac{2M}{r^2} \left(1 + \frac{M}{r}\right),
\]

\[
\psi^d(r) = \frac{3}{4M^2} r^2 \left[1 + \left(1 - \frac{2M}{r}\right)^{-1}\right] + \frac{r}{M} \ln \left(1 - \frac{2M}{r}\right) \sqrt{1 - \frac{2M}{r}},\tag{12}
\]

\[
\xi^d(r) = -\frac{3}{8M^3} r^3 \ln \left(1 - \frac{2M}{r}\right) - \frac{2M}{r^2} \left(1 + \frac{M}{r}\right),
\]

\[
\psi^d(r) = \frac{3}{4M^2} r^2 \left[1 + \left(1 - \frac{2M}{r}\right)^{-1}\right] + \frac{r}{M} \ln \left(1 - \frac{2M}{r}\right) \sqrt{1 - \frac{2M}{r}},\tag{12}
\]
and,
\[ \xi^u(r) = \frac{B_0}{2\mu} r^3, \]
\[ \psi^u(r) = -\frac{B_0}{\mu} r^3 \sqrt{1 - \frac{2M}{r}}, \] (13)

where \( B_0 \) is the constant of integration and can be interpreted as magnitude of the asymptotically uniform background magnetic field. We have used superscripts ‘d’ and ‘u’ for dipole solution and for uniform solution, respectively. It can easily be checked that for the dipole solution (Eq. 12), as \( r \to \infty \), i.e., asymptotically, \( \xi(r) \to 1 \) and \( \psi(r) \to 1 \) and so Eq. 5 reduces to the familiar flat spacetime solutions. So finally in LLF, the expressions for the asymptotically dipole magnetic field solution is

\[ B_r^u = -\frac{3\mu}{4M^3} \left[ \ln \left( 1 - \frac{2M}{r} \right) + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right], \]
\[ B_\theta^u = \frac{3\mu \sin \theta}{4M^2 r} \left[ 1 + \left( 1 - \frac{2M}{r} \right)^{-1} + \frac{r}{M} \ln \left( 1 - \frac{2M}{r} \right) \right], \] (14)

and for the asymptotically uniform magnetic field solution is

\[ B_r^u = B_0 \cos \theta, \]
\[ B_\theta^u = -B_0 \sin \theta \sqrt{1 - \frac{2M}{r}}. \] (15)

We should note that \( \xi(r) \) and \( \psi(r) \) basically represent the ratios of magnetic field components in a curved (here Schwarzschild’s) spacetime to those in the flat spacetime (e.g. the quantities on vertical axis in the Fig. 1 of [3]) and these ratios are precisely what we are interested in. Now we will proceed with the investigations for the non-minimal coupling scenario.

**B. Non-minimal Coupling: Numerical Solutions**

In order to numerically solve the coupled system of differential equations Eq. 9 and Eq. 10 we need two initial/boundary conditions, namely \( \xi(r_0) \) and \( \psi(r_0) \) at some \( r_0 \). For a chosen value of \( q_3 \), we can choose large enough \( r_0 \) such that \( 4Mq_3/r_0^3 \ll 1 \) and then we can safely assume that the solutions there at \( r_0 \), to an excellent approximation, are equal to those of minimal coupling scenario. So, for numerically obtaining the asymptotically dipole solution we can choose the initial conditions compatible with Eq. 12 and for the asymptotically uniform solution compatible with Eq. 13. For instance, \( r_0 \approx 100(2M) \), \( \xi(r_0) \approx 1 \) and \( \psi(r_0) \approx 1 \) are valid initial conditions for seeking the dipole solution whereas for the uniform solution \( \xi(r_0) \approx \text{const.} \times r_0^3/2 \) and \( \psi(r_0) \approx -\text{const.} \times r_0^3 \sqrt{1 - 2M/r_0} \) are valid initial conditions. We define a dimensionless form of the coupling parameter as \( \tilde{q} = q_3/(2M)^2 \).

Following the above described numerical recipe, we have obtained the magnetic field solutions for the non-minimal coupling case. In the Fig. 1(2) we have plotted the ratio of radial (azimuthal) component of asymptotically dipole magnetic field modified by gravity to that of a dipole magnetic field in the flat spacetime, both for the case of minimal and non-minimal coupling scenarios, i.e., \( \xi(r) / \psi(r) \). We can see from these figures that for a positive (negative) coupling constant \( \tilde{q} \) there is enhancement (suppression) of magnetic field near the horizon.

We should compare the results of [3, 4] with the present
work. Although the main conclusions and the qualitative behaviour is the same in these previous works to the one we found in this study, there are still some quantitative differences present. Though far from the event horizon, in the asymptotically flat region, both this presentation and that of [3, 4] don’t differ significantly, and both have the same typical dipole form ($B_r(r, \theta) = 2\text{const.} \times \cos \theta / r^3$ and $B_\theta(r, \theta) = \text{const.} \times \sin \theta / r^3$), but near the event horizon they differ. The radial dependence of both the components $B_r(r, \theta)$ and $B_\theta(r, \theta)$ in [3, 4], namely $B_{\text{rad}}(r)$, is identical but here we have different radial dependence for these two components, namely, $\xi(r)$ and $\psi(r)$. These differences come from the different mathematical treatment. In [3, 4] the geometry of the magnetic field configuration was prescribed \textit{a priori}, assuming that $B_\theta(r, \theta) = \tan \theta B_r(r, \theta) / 2$. This type of the assumption regarding the $B_\theta$ and $B_r$ is based on the configuration satisfying the second Maxwell equation on the flat spacetime, viz. $\nabla \times B = 0$. The modifications of magnetic field were then inspected using Eq. (7), while no further reference was made with respect to the other Maxwell equation (Eq. 8) since it is not modified by the presence of the non-minimal coupling. In this work, however, no \textit{a priori} assumption on the field configuration, and thus the relationship between $B_\theta$ and $B_r$ is made, and both Maxwell’s equations are solved simultaneously, making the treatment complete and self-consistent and thus improving the mathematical analysis. The predicted enhancement/suppression for a given $\tilde{q}$ in [3, 4] is more than what the present work predicts about the component $B_r$ whereas that for $B_\theta$ component of present work is more than that in [3, 4].

Modification of an asymptotically uniform magnetic field near a Schwarzschild black hole in a minimal coupling scenario has been studied by many researchers (see [25] and references therein). As expected, depending on sign of the coupling constant $\tilde{q}$, the components of asymptotically uniform magnetic field get enhanced/suppressed near the event horizon (see Figs. 3 and 4). There are two points to note here: (i) in minimal coupling scenario the radial component, $B_r$, remains unaffected whereas in case of non-minimal coupling both the components, $B_r$ and

![Fig. 3](image1.png)

**FIG. 3.** The radial component of the uniform magnetic field in the Schwarzschild spacetime scaled by the same in the flat spacetime as a function of $r$ i.e., $\xi(r)$ for different values of $\tilde{q}$. The enhancement of the magnetic field component near the horizon for $\tilde{q} = 0.5$ with respect to the case of minimal coupling ($\tilde{q} = 0$) can be seen. For $\tilde{q} = -0.5$ the change in sign and magnitude of the field component near the horizon can be seen.

![Fig. 4](image2.png)

**FIG. 4.** The azimuthal component of the uniform magnetic field in the Schwarzschild spacetime scaled by the same in the flat spacetime as a function of $r$ i.e., $\psi(r)$ for different values of $\tilde{q}$. The enhancement of the magnetic field component near the horizon for $\tilde{q} = -0.5$ with respect to the case of minimal coupling ($\tilde{q} = 0$) can be seen. For $\tilde{q} = 0.5$ the change in sign and magnitude of the field component near the horizon can be seen.

![Fig. 5](image3.png)

**FIG. 5.** Asymptotically uniform magnetic field near the Schwarzschild horizon for the minimal coupling scenario.
magnetic field gets modified qualitatively also, apart from common quantitative enhancement/suppression in both the cases. The manifestations of this qualitative change on the motion of a charged particle would be worth further exploration. But since we will study the motion in the equatorial plane so the effect of direction change near poles(for $\tilde{q} > 0$) will not show up there.

IV. MOTION IN EQUATORIAL PLANE: EFFECTIVE POTENTIAL

From the symmetry of the problem we investigating, the 4-potential can be assumed to have following form [23]

$$A_\mu = (0, 0, 0, A_\phi(r, \theta)),$$

and the Lagrangian for the motion of a test charged particle of charge $Q$ and mass $m$ can be written as [24]

$$\mathcal{L} = \frac{1}{2} m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + Q A_\mu \dot{x}^\mu,$$

where $g_{\mu\nu}$ denotes metric components and $\dot{x}^\mu$ is 4-velocity. From this Lagrangian apart from the equation of motion

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{Q}{m} F^\mu_{\nu\text{tot}} \frac{dx^\nu}{d\tau},$$

following two conserved quantities can also be obtained

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -m \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \equiv -E,$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \frac{d\phi}{d\tau} + Q A_\phi \equiv l.$$

Here we stress that $F^\mu_{\nu\text{tot}}$ represents the total electromagnetic tensor, which includes the corrections to magnetic fields coming from the non-minimal coupling effect (and not only the “background” part of the field), $F^\mu_{\nu\text{tot}} = F^\mu_{\nu} + R^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The potential, $A_\mu$, discussed here is defined with respect to this total field in the usual manner, i.e. $F^\mu_{\nu\text{tot}} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $E$ and $l$ can be interpreted as (effective) energy and angular momentum, respectively. We define $\tilde{E} \equiv E/m$, $\tilde{l} \equiv l/m$ and $q \equiv Q/m$. From the equation of motion 18 and the conservation equations 19 and 20, for motion in the equatorial plane we get

$$\tilde{E}^2 = \left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}^2$$

where

$$V_{\text{eff}}^2 = \left(1 - \frac{2M}{r}\right) \left[1 + \frac{1}{r^2} \left(\tilde{l} - q A_\phi(r)\right)^2\right],$$

where $A_\phi(r)$ is subject to the following ordinary differential equation

$$\frac{dA_\phi}{dr} = -\mu \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{\psi}{r^2}.$$
A. Dipole Magnetic Field

We should make note of available analytical expression for $A_\phi(r, \theta)$ for the case of minimal coupling scenario[21]

$$A_\phi = -\frac{3\mu \sin^2 \theta}{8M^3} r^2 \left[ \ln \left( 1 - \frac{2M}{r} \right) + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right].$$

For the non-minimal coupling scenario $V_{\text{eff}}^2$ can be obtained numerically by solving the system of Eqs. 9, 10 and 23. The proper initial/boundary conditions to be used is $A_\phi(r_0) \approx 0$ when $r_0 \to \infty$ (which is compatible with Eq. 24) and again those for $\xi(r)$ and $\psi(r)$ are as discussed earlier. We introduce two dimensionless parameters $\lambda \equiv \mu q/M^2$ and $H \equiv l/M$ as an aid to study the effective potential.

1. Case: Angular Momentum Parallel to Magnetic Moment

Let us first review some basic features of this setting in the minimal coupling case. In general the most rich and feature-wise type of the effective potential has two maxima and two minima[21]. Between these two maxima a potential well(minima) is present. The maxima nearest to the horizon are shown in the respective figure insets while the minima farthest from the horizon are not shown as they are not of significant physical interest. For a given value of $H$, increasing $\lambda$ amounts to flattening-out of the potential well, raising the level of the first maximum, lowering the level of the outer maximum and shifting the first minimum outwards - in the case if it exists. For a given $\lambda$, increasing $H$ amounts to narrowing the potential well, raising the level of outer maximum, lowering the level of inner maximum and shifting the first minimum inwards(see Figs. 8 or 11). So, if we focus on the case for which no outer maximum (and thus no potential well) initially exists for a given $\lambda$ and $H$, then with increasing $H$ at one point the outer maximum appears. Furthermore, its level increases with $H$, and thus the potential well becomes narrower and deeper. Finally, it reaches a maximum depth when both the maxima attain equal height. If $H$ is then further increased, the depth of potential well starts decreasing as the inner maximum level decreases. Hence, increasing the $H$ further, the potential well will at some point again vanish as the inner maximum is vanishing. (e.g., $\lambda = 50$, $H \sim 20 - 150$). All this was stated in the context of minimal coupling. In case of non-minimal coupling, a positive value of coupling constant $\tilde{q}$ shifts the potential well outwards, doesn’t affect the depth of potential well significantly (reducing it slightly) and raises the level of the first maximum significantly(see the inset of Fig. 8). A negative value of $\tilde{q}$ shifts the potential well inwards and lowers the first minimum significantly (so it can affect the potential well, see Fig. 11). Depending on the particle’s energy $E$ it will be subjected to: scattering, stable/unstable circular orbit, bound orbit with Larmor motion, or it may plunge into the black hole.

Scattering: A particle coming from infinity and having energy lesser than either of the maxima will be scattered away to infinity again. When inner maximum is the absolute maximum, depending on the sign of $\tilde{q}$, the spectrum(i.e., energy dependence) of scattered particles will differ. For a positive $\tilde{q}$ particles with higher energies will be scattered away, while for a negative $\tilde{q}$ the scattered particles will have lesser energies, in comparison to the case of minimal coupling(see Figs. 8 and 11). When the outer maximum is the absolute maximum (e.g. $\lambda = 30$, $H = 70.7816$ [21]) nonzero $\tilde{q}$ will not affect scattering significantly as the outer maximum is not significantly influenced by the effect of vacuum polarization.

Plunging into the black hole A particle coming from infinity with energy more than both of the maxima will plunge into the black hole. The effects of nonzero $\tilde{q}$ on the spectrum of particles plunging into the black hole will be the opposite to that discussed related to scattering.

Circular motion: A particle coming from infinity with energy equal to either of the maxima will have an unstable circular orbit. So when $\tilde{q}$ is positive(negative), particles of respectively higher(lower) energies, in comparison with the case of minimal coupling, will be able to have unstable circular orbit. On the other hand, a particle having the energy which is equal to the first minima and of local origin(i.e. not coming from infinity) will have a stable circular orbit. This kind of circular orbit goes by a popular term “innermost stable circular orbit(isco)”.

In the absence of any magnetic field a charged particle can not have any stable/unstable circular orbit below $6M$ radius, while a magnetic field allows circular orbits well below this radius. Furthermore, in the non-minimal coupling case, with nonzero $\tilde{q}$, this radius gets additionally affected (it increases/decreases for positive/negative values of $\tilde{q}$, see Figs. 8 and 11). This will have implications in modelling accretion disks, as will be discussed later.

Particle trapping: A particle of local origin and having energy lesser than both of the maxima will see two turning points and will be trapped in a gyrating(Larmor motion) bound orbit. Depending on the values of $\lambda$ and $H$ the two maxima can have same height and so the depth of the corresponding potential well will be maximum(e.g. $\lambda = 50$, $H = 68.5249$). A positive value of $\tilde{q}$ apart from shifting the potential well outwards, hardly makes any difference in comparison with minimal coupling scenario(see Fig. 10). Whereas a negative $\tilde{q}$ by affecting the first maximum significantly affects potential well and so the trapping of particles(see Fig. 9). A more negative value of $\tilde{q}$ can bring down the first maximum even lower with respect to the outer maximum and can hence make the potential well shallower( or can even wash it out completely(see Fig. 10)). On the other hand, in the case when a negative $\tilde{q}$ doesn’t make the potential well shallower(for certain combination of values of $\lambda$, $H$, e.g., Fig. 9), then - as the potential well is in this case
positioned nearer to the event horizon - this well is capable to hold more energetic particle trapped in itself, in comparison to the minimal coupling scenario.

In all the settings discussed above, the study of spectrum of scattered particles stands out as the most distinguished case for manifesting the difference between minimal and non-minimal coupling scenarios. With respect to the spectrum of scattered particles, all other aspects of motion represent a less significant kind of probe that could serve as a signature for the existence of the effect of non-minimal coupling.

2. Case: Angular Momentum Anti-Parallel to Magnetic Moment

For a particle having its angular momentum anti-parallel to magnetic dipole, the effective potential has only one maximum (very near to the horizon) and a minimum far away from the horizon. So effectively such particle will not see any potential well. If energy $E$ of the particle is greater than the maximum, it will plunge into the black hole, if lesser then it will be scattered away. There also exists a possibility of unstable circular orbit very near the event horizon - if the particle’s energy is equal to the maximum, and a stable circular orbit far away from the event horizon - for less energetic particles.

The effects of nonzero $\tilde{q}$ on the scattering of particles are similar to what discussed in case of $H > 0$(see Figs. 11 and 12).

B. Uniform Magnetic Field

Again we make note of available analytical expression for $A_\phi(r, \theta)$ for the case of minimal coupling scenario [25]

$$A_\phi = \frac{B_0}{2} r^2 \sin^2 \theta$$ (25)
where $B_0$ is the strength of magnetic field at infinity. To obtain the effective potential numerically for the case of non-minimal coupling, we follow the same set of procedures as described earlier for the case of dipole field. We define one more dimensionless quantity $\beta \equiv qB_0M/m$ and will make use of $H$ as defined earlier. For radial distance we will use a dimensionless parameter $\rho = r/M$ and $\rho_+$ and $\rho_-$ will be used for denoting the radius of isocho when the angular momentum is parallel and anti-parallel, respectively, to the magnetic field. Unlike in the case of dipole magnetic field, the uniform magnetic field at far away radii from the horizon plays an important role as a barrier for the particles coming from infinity (see Figs. 13 or, 14).

1. Case: Angular Momentum Parallel to Magnetic Field

For our investigation of non-minimal coupling of an asymptotically uniform magnetic field with Schwarzschild spacetime, we have used the values of parameters $\beta$ and $H$ from [25] (from the caption of Fig. 3). We also note that the quoted $l \approx 3.22$ in that figure caption is a typographical error and we find it to be $l \approx 4.6$. The parameter $\rho$, $\beta$ and $H$ defined here corresponds to $2\rho$, $b$ and $2l$ in [25], respectively. We will focus our discussion mainly for $\tilde{q} \neq 0$ case as the rest is nicely done in [25].

In the case of minimally coupled magnetic field, for lower values of $H$ there at first exist no extrema of the effective potential, then with increase in $H$ we see one extremum (inflection point) and with further increase in $H$ we see one maximum near the horizon and one minimum away from the horizon (as can be seen in the Fig. 13 for $H \approx 2.36, 4.14, 9.2$). With further increase in $H$ both of these extrema shift outwards from the horizon with the increased values of effective potential. A positive $\tilde{q}$ introduces none, one or two more extrema near the horizon, depending on $H$. So, for those values of $\beta$ and $H$ for which there is no extrema at all for the minimal coupling case, a positive $\tilde{q}$ can introduce an inflection point

![Figure 11](image1.png)

**FIG. 11.** Same as Fig. 8 but here for correspondingly negative values of $\tilde{q} = 0.5$ are shown in the inset. Increase in the heights of the peaks for $\tilde{q} = 0.5$ with respect to the case of minimal coupling can be seen.

![Figure 12](image2.png)

**FIG. 12.** Same as Fig. 11 but here for $\tilde{q} = -0.5$. The lowering of the heights of the peaks for negative coupling with respect to the case of minimal coupling can be seen.

![Figure 13](image3.png)

**FIG. 13.** The effective potential for the motion of a charged particle in the equatorial plane of the Schwarzschild spacetime in asymptotically uniform magnetic field, for the magnetic field parameter $\beta = 0.5$, when magnetic field and angular momentum are parallel. The colors blue, red and black corresponds to the angular momentum parameter($H$) values 2.36, 4.14 and 9.2, respectively. The solid, dashed, dashdot curves are for $\tilde{q} = 0.0$ and dotted, dashdotdotted, densely dashdotted curves are for $\tilde{q} = 0.5$, respectively. Because of lack of space in the current figure window we have presented these linestyle in the **figure legends** of Fig.14 (only difference is the sign $\tilde{q}$ but not the magnitude). To highlight the two interesting features (one extra maxima and minima near the horizon) in the effective potential due to positive coupling, the inset plot is included. Only the right side labeling of the vertical axis corresponds to the inset plots.
or a minimum and hence lead to the existence of isco or potential well. For those values of $\beta$ and $H$ for which there is an inflection point, a positive $\tilde{q}$ can introduce a minimum and hence a potential well. For those higher values of $\beta$ and $H$ for which there is one maximum and a minimum, a positive $\tilde{q}$ can introduce one more maximum and minimum and hence one more potential well. Effectively a positive $\tilde{q}$ introduces bound trajectories (and stable/unstable circular orbits) very near the horizon, separated from such kind of trajectories away from the horizon. This observation can have its implications in modeling of accretion disks and while studying any emission (possibly synchrotron emission) from the particles gyrating in the bounded orbits.

A negative $\tilde{q}$ introduces at most one inflection point or minima near the horizon when no such features are there in case of minimal coupling. To an already existing potential well, a negative $\tilde{q}$ increases the depth (see Fig. 14). Unlike the case of $\tilde{q} > 0$, here we do not see two sets of bounded orbits but with increased depth of the potential well, here the particles trapped will have more energies in comparison with the case of minimal coupling.

In the Fig. 15 we have plotted the dependence of radius of isco on magnetic field parameter $\beta$. Clearly in the figure we can see the isco radius at 6$M$ when there is no magnetic field (i.e., when $\beta=0$). For the case of minimal coupling with increase in the strength of magnetic field parameter $\beta$, the isco radius saturates as $r_{isco} \rightarrow 2M$ as $\beta \rightarrow \infty$.

or a minimum and hence lead to the existence of isco or potential well. For those values of $\beta$ and $H$ for which there is an inflection point, a positive $\tilde{q}$ can introduce a minimum and hence a potential well. For those higher values of $\beta$ and $H$ for which there is one maximum and a minimum, a positive $\tilde{q}$ can introduce one more maximum and minimum and hence one more potential well. Effectively a positive $\tilde{q}$ introduces bound trajectories (and stable/unstable circular orbits) very near the horizon, separated from such kind of trajectories away from the horizon. This observation can have its implications in modeling of accretion disks and while studying any emission (possibly synchrotron emission) from the particles gyrating in the bounded orbits.

A negative $\tilde{q}$ introduces at most one inflection point or minima near the horizon when no such features are there in case of minimal coupling. To an already existing potential well, a negative $\tilde{q}$ increases the depth (see Fig. 14). Unlike the case of $\tilde{q} > 0$, here we do not see two sets of bounded orbits but with increased depth of the potential well, here the particles trapped will have more energies in comparison with the case of minimal coupling.

In the Fig. 15 we have plotted the dependence of radius of isco on magnetic field parameter $\beta$. Clearly in the figure we can see the isco radius at 6$M$ when there is no magnetic field (i.e., when $\beta=0$). For the case of minimal coupling with increase in the strength of magnetic field parameter $\beta$, the isco radius saturates as $r_{isco} \rightarrow 2M$ as $\beta \rightarrow \infty$.

or a minimum and hence lead to the existence of isco or potential well. For those values of $\beta$ and $H$ for which there is an inflection point, a positive $\tilde{q}$ can introduce a minimum and hence a potential well. For those higher values of $\beta$ and $H$ for which there is one maximum and a minimum, a positive $\tilde{q}$ can introduce one more maximum and minimum and hence one more potential well. Effectively a positive $\tilde{q}$ introduces bound trajectories (and stable/unstable circular orbits) very near the horizon, separated from such kind of trajectories away from the horizon. This observation can have its implications in modeling of accretion disks and while studying any emission (possibly synchrotron emission) from the particles gyrating in the bounded orbits.

A negative $\tilde{q}$ introduces at most one inflection point or minima near the horizon when no such features are there in case of minimal coupling. To an already existing potential well, a negative $\tilde{q}$ increases the depth (see Fig. 14). Unlike the case of $\tilde{q} > 0$, here we do not see two sets of bounded orbits but with increased depth of the potential well, here the particles trapped will have more energies in comparison with the case of minimal coupling.

In the Fig. 15 we have plotted the dependence of radius of isco on magnetic field parameter $\beta$. Clearly in the figure we can see the isco radius at 6$M$ when there is no magnetic field (i.e., when $\beta=0$). For the case of minimal coupling with increase in the strength of magnetic field parameter $\beta$, the isco radius saturates as $r_{isco} \rightarrow 2M$ as $\beta \rightarrow \infty$.

or a minimum and hence lead to the existence of isco or potential well. For those values of $\beta$ and $H$ for which there is an inflection point, a positive $\tilde{q}$ can introduce a minimum and hence a potential well. For those higher values of $\beta$ and $H$ for which there is one maximum and a minimum, a positive $\tilde{q}$ can introduce one more maximum and minimum and hence one more potential well. Effectively a positive $\tilde{q}$ introduces bound trajectories (and stable/unstable circular orbits) very near the horizon, separated from such kind of trajectories away from the horizon. This observation can have its implications in modeling of accretion disks and while studying any emission (possibly synchrotron emission) from the particles gyrating in the bounded orbits.

A negative $\tilde{q}$ introduces at most one inflection point or minima near the horizon when no such features are there in case of minimal coupling. To an already existing potential well, a negative $\tilde{q}$ increases the depth (see Fig. 14). Unlike the case of $\tilde{q} > 0$, here we do not see two sets of bounded orbits but with increased depth of the potential well, here the particles trapped will have more energies in comparison with the case of minimal coupling.

In the Fig. 15 we have plotted the dependence of radius of isco on magnetic field parameter $\beta$. Clearly in the figure we can see the isco radius at 6$M$ when there is no magnetic field (i.e., when $\beta=0$). For the case of minimal coupling with increase in the strength of magnetic field parameter $\beta$, the isco radius saturates as $r_{isco} \rightarrow 2M$ as $\beta \rightarrow \infty$. We choose $|\tilde{q}| = 0.125$ instead of 0.5 to show this transitory feature clearly, since for $\tilde{q} = 0.5$ this feature shows near $\beta \sim 0.1$. Therefore, we see that beyond certain values of $\beta$ there are two sets of stable circular orbits (well within 6$M$ radius) separated from each other. The termination of plot for negative $\tilde{q}$ should be interpreted as isco grazing the horizon and being occupied with angular momentum of the particles $H \rightarrow 0$. Because of limited resolution ($\Delta H$) used in our code this curve is terminating there. This feature is not surprising as we carefully observe effective potential plot in the Fig. 14 for $H = 2.36$. Like the case of minimal coupling for $\beta \rightarrow \infty$ [25] we expect the radius of isco to saturate to some value (depending on $\tilde{q}$) near
$2M$ (the Schwarzschild radius).

In the Fig. 16 we have plotted the angular momentum for isco as a function of magnetic field parameter $\beta$. We can see that, starting from the angular momentum of $\sqrt{12}M$ for $\beta = 0$, the particles with lower angular momentum occupy isco initially, then eventually – with the increase in the strength of magnetic field – particles of higher angular momentum orbit in the isco. Again, the transitory feature near $\beta \approx 1.3$ for $q = 0.125$ reflects the introduction of inflection point or shallow potential well near the horizon because of non-minimal coupling (see the inset of Fig. 13). The termination of the curve for $q = -0.125$ near $\beta \approx 2.2$ (with $H \approx 0.4$) reflects the fact that for the continuation of this curve forward one needs progressively finer and finer resolution ($\Delta H$) requiring heavy computation. As the gained information is not justifying such heavy computations, we have terminated our calculations beyond these limits.

2. Case: Angular Momentum Anti-Parallel to Magnetic Field

When the angular momentum of a particle is anti-parallel to the magnetic field we hardly see any interesting features in the effective potential in both the coupling scenarios. Results in case of non-minimal coupling are just slightly enhanced/suppressed quantitatively. We can speculate the possible reason for such situation to be the saturation of the isco radius ($r_{isco}$) comparatively (with the case of parallel angular momentum) far from the horizon, $r_{isco} \to 4.3M$ as $\beta \to \infty$ [25].

In the Figs. 17 and 18 we have presented few illustrative plots for the effective potential. Depending on $\beta$ and $H$, there exist a maxima (and hence unstable circular orbit) near the horizon followed by a minima (and hence stable circular orbit and/or potential well for particle trapping) away from the horizon. A positive/negative $q$ raises/lowers the height of the maxima and shifts inward/outwards. The effects of positive/negative $q$ on the minima is to shift it inwards/outwards and hardly makes any change in its height.

In the Fig. 19 we have plotted the radii ($r_{isco}$) as a function of $\beta$ when angular momentum of the particle is anti-parallel to the uniform magnetic field (asymptotically). The colors black, blue and red are for the values of $q = 0.5, 0.0$ and $-0.5$, respectively. As expected in the absence of any magnetic field radius should be $6M$. For the case of minimal coupling with increase in the strength of magnetic field parameter $\beta$, the isco radius saturates as $r_{isco} \to 4.3M$ as $\beta \to \infty$ [25].
corresponding angular momentum $-\sqrt{12}M$.

3. Energy of a Charged Particle in a Marginally Stable Circular Orbit

As a prequel to the study of collision of charged particles, in this section we briefly discuss the energetics of charged particles moving in a marginally stable circular orbit. In the case of angular momentum being parallel to the asymptotically uniform magnetic field, a non-minimal coupling makes it possible to have isco nearer to the horizon, when compared with minimal coupling (of course beyond a certain value of magnetic field strength this feature is common to both +ve/-ve $\tilde{q}$, see Fig. 15, Fig. 21). The break/gap in the energy plot for $\tilde{q} = 0.125$ is understandable if we observe Fig. 15, since for $\beta < 5$ there are no isco radii between $\sim 2.12 - 2.47$, while with the increase in $\beta$ this gap will fill. When the angular momentum is anti-parallel to the magnetic field, similar to the results in the case of minimal coupling, the non-minimal coupling also enables trapping of the particles with higher energies ($\tilde{E} > 1$, see Fig. 22). For a positive/negative value of coupling constant $\tilde{q}$, for any given $\beta$, these high energies particles can move in more near/far isco in comparison to the case of case of minimal coupling.

V. PARTICLE COLLISION/ACCELERATION

Taking the case of an asymptotically uniform magnetic field, we consider the collision of two charged particles of equal mass ($m_0$), opposite charge and moving in opposite direction on an isco(hence they have the same angular momentum). For this we calculate the centre of mass energy of the two colliding particle, given by

$$E_{cm}^2 = 2m_0^2 \left(1 - g_{\mu\nu}u_1^\mu u_2^\nu\right)$$

where $u_1^\mu$ and $u_2^\mu$ are their 4-velocities. For this configuration of collision the 4-velocities of the particles take the following form

$$u_{\pm}^\alpha = \left(-\frac{dt}{d\tau}, 0, 0, \pm \frac{d\phi}{d\tau}\right)$$

and are subject to normalization $u^\alpha u_\alpha = -1$. Using Eqs. 19, 20 and 22, we get

$$\frac{E_{cm}}{m_0} = \sqrt{2} \left(1 - \frac{2M}{r}\right)^{-1/2} \sqrt{\tilde{E}^2 + V_{\text{eff}}^2}.$$  

In the Fig. 23 we have plotted the centre of mass energy over rest energy of a charged particle as function of
is $E_{cm}/m_0$ near the $isco$ radius is not to be interpreted as gain in energy due to black hole, but coming from the fact that these particles are already energetic (see Fig. 22). To put in other words, they have high energies when measured by an observer at infinity, i.e., the black hole is not functioning as an accelerator here [26].

VI. POSSIBLE APPLICATION IN ASTROPHYSICAL AND COSMOLOGICAL SCENARIOS

Due to the required amount of work and space, it is not possible to present any detailed numerical and quantitative predictions regarding the systems in which the non-minimal coupling could have its astrophysical application. While leaving such more detailed and extensive investigations for the future, in this section we will try to explore many possible implications of the non-minimal coupling qualitatively. Applications of this effect for objects like neutron star, sun, earth are discussed in [3] and so we will not discuss them here. In [3] the application to primordial black holes is also discussed, but nevertheless here we would like to present it in a different light.

One of the central issues which appears when discussing the potential observational effects of the non-minimal coupling is the value of the coupling parameter. Broadly speaking, one approach to this question is to consider the coupling constant $\tilde{q}$ to be a free parameter that has to be obtained or constrained phenomenologically i.e. from observation of astrophysical sources, as have been considered by many other researchers [5, 6]. On the other hand, Drummond & Hathrell [2] have presented a derivation of the non-minimal coupling from the first principles, demonstrating it to be the consequence of the QED vacuum polarization on the curved spacetime. In their derivation they have found that the coupling constant is determined to be $q_3 = -\alpha \lambda_e^2/90\pi \sim -10^{-28}$ m$^2$, where $\alpha$ is fine structure constant and $\lambda_e$ is the Compton wavelength of electron. It is, however, important to take into account that this value is determined for a very specific and elementary setting of a photon propagating in vacuum on the curved spacetime. It is not at all simple to determine what is the connection between this very simple case of one photon and a very complex setting of photons leading to macroscopic magnetic field distributions, such as the dipole field. Therefore, it is not possible to simply claim that the coupling parameter for macroscopic configurations of magnetic fields i.e. the ones encountered in astrophysics - is related to the value calculated in [2]. In principle, the value of the coupling parameter could
even vary for different field strengths and configurations. It therefore seems that the value of the coupling parameter for the macroscopic magnetic field configurations of interest is basically unknown and should be constrained from the phenomenological considerations. However, the significance of the value determined in [2] comes from the fact that it can serve at least as a conservative lower limit for the value of $\tilde{q}$. Therefore, when discussing the potential observational implications we can take it as a reasonable assumption that the value of $\tilde{q}$ should be at least of the order of magnitude predicted in [2] or higher. In the following paragraph we will discuss the cosmological consequences of non-minimal coupling focusing on the case of this limit. Taking this conservative lower limit, $\tilde{q} = q_3/r_0^2$, where $r_0 \sim 2M$, to be of order 1, the mass $M$ has to be $\sim 10^{13}$ kg and so for most of the astrophysical sources of mass comparable to or higher than solar mass $\tilde{q} \ll 1$. Therefore, it follows that in such a case the observable effects will be negligible. But an interesting candidate relevant even for this conservative limit is given by hypothetical astrophysical objects called Primordial Black Holes (PBHs) [37, 38], which have mass range conjectured (depending on the model, i.e., on the time when they formed after the Big Bang) to be between $10^{-8}$ kg to many solar masses. Another point to note is that Hawking’s theory of black holes evaporation [27] predicts that any PBH of mass less than $\sim 10^{11}$ kg would have evaporated by now. Needles to say, it is of course very questionable to expect that this approximation – which assumes weak field and one-loop approximation – can be applicable for a PBH of mass $\sim 10^{-8}$ kg ($\tilde{q} \sim 10^{41}$). But, on the other hand, if we take the Drummond & Hathrell theory to be applicable for $\tilde{q} \sim 1 - 10$ (i.e., mass $\sim 10^{13} - 10^{11}$ kg, since $\tilde{q} \propto 1/\sqrt{\text{mass}}$) then it can be utilized for both kinds of PBHs — the ones which are already evaporated and the ones which are still there. With regard to evaporated PBHs it is in principle quite likely that the local distribution of magnetic fields previously associated with them would still be present at their location. Thus, an astrophysical observation of local suppression of magnetic field can be further analysed as possible candidate for the detection of PBH. The investigation of PBH’s in relation to non-minimal coupling can in general take two directions. In the case where a localised significant suppression of magnetic field is observed (while previously not being recognised as a PHB candidate), we can take non-minimal coupling considerations into account and claim this localised suppression to be a signature of the PBH. On the other hand, if some spot is already claimed to be a PBH candidate by some other model or consideration, then – if a significant suppression of magnetic field around this point is established – it can serve as a means to establish a constraint on the value of $\tilde{q}$. For the so far survived PBHs ($10^{11}$ kg $\lesssim$ mass $\lesssim 10^{13}$ kg) some arguments could be made here as for the usual black holes in [3, 4]. We note that the discussed question of PBH’s is also interesting because it connects the question of non minimal coupling with the question of evolution of cosmological magnetic fields, which could also be connected with the question of the origin of the Universe [28].

In any realistic modeling of an accretion disk around a black hole, the study of factors which affect isco is important. The implications of correct modeling of an accretion disk is also important for analysis the black hole images [43–45] and for such future observations. In the present work we saw that a non-minimal coupling affects isco not only quantitatively but also qualitatively. One of the means to distinguish between a black hole and hypothetical/exotic objects (like naked singularity, non-singular black hole etc.) has been the investigation of the properties of isco in absence of and in presence of magnetic field (in the minimal coupling scenario). Including a non-minimal coupling considerations in these studies can answer the questions like — can a black hole, when its gravity is non-minimally coupled with a magnetic field, have same/distinguishable observational signatures as those of exotic objects? For instance $\gamma$-metric — one of the candidates for naked singularity — shows two well separated regions of isco for $\gamma \in (1/\sqrt{5}, 1/2)$ (in absence of [29] and in presence of magnetic fields [30]). We can see from Fig. 15 that this particular feature is present for a black hole too when the non-minimal coupling is considered. So it would be interesting to further investigate how a black hole and exotic objects differ in their observational signature when non-minimal coupling of electromagnetic field with strong gravity is considered. Very often the estimation of spin of a black hole candidate has been derived from the theoretical relationship between spin and isco radius [31, 32]. Though in the present work we have not considered a rotating black hole (a Kerr black hole) but we can expect that there too the properties of isco would be different if non-minimal coupling is taken into consideration from so far studies in minimal coupling.

From observational astronomy it has been well established that many astrophysical objects (active galactic nuclei, radio galaxies, both stellar mass and super-massive black holes, etc.) often have jets associated with them. Many proposed mechanisms for formation of jets assumes the presence of magnetic field in the system comprising of a central object (often a neutron star or black hole), accretion disk and jet. The transfer of energy and material from accretion disk to jet, in many models, is facilitated by magnetic fields [33]. So far only the minimal coupling of magnetic field with gravity have been considered in the models of jet formation. A study of non-minimal coupling of magnetic field with gravity can possibly shed some new light.

For the possibility of synchrotron radiation from the vicinity of a black hole one needs trapping of high energy particles near the horizon [24]. In absence of any magnetic field the possibility particle trappings within 6M in unbound and unstable orbits [34–36] has limitations. In presence of magnetic field, even in a minimal coupling scenario, highly energetic particles can be trapped in sta-
ble circular orbits within $6M$ and near the horizon. We saw here that non-minimal coupling makes further facilitates the trapping of high energy particles near the horizon and so in any study of synchrotron radiation from the vicinity of a black hole the inclusion of non-minimal considerations can be useful.

VII. CONCLUDING REMARKS

New astrophysical observations, such as the results obtained by the Event Horizon Telescope and the study of super-massive black holes, make it possible to investigate more precisely the structure of magnetic fields near the event horizon [39–42]. This topic is also of fundamental theoretical interest since black hole horizons are associated with strongest gravitational fields currently accessible to observations. It is precisely in this strong gravity setting that one could expect to find some signatures of new effects going beyond the standard description of relationship between gravity and electromagnetism. As we have discussed, one of such effects – that can be motivated from the fundamental field-theory considerations – is the non-minimal coupling of gravity and electromagnetism. For instance, this type of non-minimal coupling naturally comes as a consequence of the effect of QED vacuum polarization on the curved spacetime. With an aim to study this effect in a setting which is relevant for the astrophysical application, we have considered the case of non-minimal coupling between magnetic fields and gravity on the Schwarzschild spacetime, in the weak magnetic field limit. We have thus followed, improved and considered some new applications of the ideas recently proposed in [3, 4], with the primary motivation to elaborate the physical consequences of the non-minimal coupling between gravity and electromagnetism and to help bring them closer to possibility of observational verification.

We have first reviewed the already known solutions for the case of minimally coupled magnetic fields on Schwarzschild spacetime and then considered the non-minimal coupling effects for both the asymptotically dipole and uniform magnetic fields in detail, while discussing the changes induced by the non-minimal coupling with respect to the minimal coupling case. We have also provided a detailed discussion of the changes that come as a result of non-minimal coupling in both of the mentioned magnetic field configurations. In all of the discussed settings we have found that even the modest values of the coupling parameter, $\tilde{q}$ - which do not seem to appear to be in the contradiction with other observations - can cause a significant change in the magnetic fields near the event horizon. These changes can have the character of amplification or suppression depending on the sign of the non-minimal coupling parameter. Apart from this quantitative change, which appears for both asymptotically dipole and uniform case, in the case of asymptotically uniform field a further qualitative change also appears: the direction of magnetic fields changes near the horizon - for the positive value of $\tilde{q}$ the direction alters near the equatorial plane, while for the positive values the direction alters near the two poles. We have also studied in detail how does the non-minimal coupling change the features of the effective potential in equatorial plane. The non-minimal coupling can thus increase/decrease the effective potential, change the position of the potential well, raise or decreases the level of the maxima and minima. For these reasons the non-minimal coupling can under proper conditions significantly change the spectrum of the scattered particles, affect the innermost stable circular orbit (isco), and influence the energy spectrum of trapped particles and change the center of mass energy during the acceleration of charged particles. All of these processes have been extensively qualitatively and quantitatively discussed in the text. The most promising effect for manifesting the difference between minimal and non-minimal coupling seems to be the study of spectrum of scattered particles for the case of dipole field configuration.

We have also discussed various physical scenarios with observational interest in which the non-minimal coupling effects could play a significant role. Such scenarios and settings include: primordial magnetic fields and primordial black holes, consequences of the studies of isco – for instance, of interest for discrimination between black holes and potential exotic objects with similar properties (like non-singular black holes, naked singularities etc.), potential effects of the non-minimal coupling in the jet formation and the synchrotron radiation.

We believe that a future theoretical research done in this topics - combined with the further improvement in the experimental techniques for the study of magnetic fields near the black hole horizons - can lead to the potential observation of of the non-minimal coupling, or at least to strongly constraining the value of the coupling parameter, in the near future. Such progress could be a significant step further in our understanding of strong gravitational fields and quantum effects on them, as well as the connection between electromagnetism and gravity. In fact, if observed, the non-minimal coupling between electromagnetism and gravity could be the first detection of a curved spacetime quantum process.

[1] C. Tsagas, Classical Quantum Gravity 22, 393 (2005). [2] I. T. Drummond and S. J. Hathrell, Phys. Rev. D 22, 343
[16] A. R. Prasanna, Phys. Lett. 37A, 331 (1971).
[17] A. R. Prasanna, Lett. Nuovo Cim. 6, 420 (1973).
[18] A. B. Balakin, J. P. S. Lemos, Class. Quant. Grav. 22, 1867 (2005)
[19] K. Bamba, S. D. Odintsov, JCAP 0804, 024 (2008)
[20] T. Dereli, Ö. Sert, Phys. Rev. D 83, 065005 (2011)
[21] T. Dereli, Ö. Sert, Eur. Phys. J. C71, 1589 (2011)
[22] Ö. Sert, J. Math. Phys. (N.Y.) 57, 032501 (2016).
[23] Ö. Sert, Eur. Phys. J. C 78, 241 (2018).
[24] R. Beck, Space Sci. Rev. 99, 243 (2001).
[25] K. Subramanian, Rep. Prog. Phys. 79, 076901 (2016).
[26] M. S. Turner and L. M. Widrow, Phys. Rev. D 37, 2743 (1988).
[27] A. B. Balakin and W. Zimdahl, Phys. Rev. D 71, 124014 (2005).
[28] M. L. Bernet, F. Miniati, S. J. Lilly, P. P. Kronberg, and M. Dessauges-Zavadsky, Nature (London) 454, 302 (2008).
[29] J. L. Han, IAU Symp. 259, 455 (2009).
[30] M. Giovannini, Int. J. Mod. Phys. D 13, 391 (2004).
[31] V. L. Ginzburg and L. M. Ozernoi, Sov. Phys. JETP 20, 689 (1965).
[32] A. R. Prasanna and R. K. Varma, Pramana 8, 229 (1977).
[33] I. Wasserman and S. L. Shapiro, ApJ 265, 1036 (1983).
[34] J. A. Petterson, Phys. Rev. D12, 2218 (1975).
[35] Preti, Giovanni, Classical and Quantum Gravity, 21(14), 3433, (2004).
[36] V. P. Frolov and A. A. Shoom, Phys. Rev. D82, 084034 (2010).
[37] V. P. Frolov, Phys. Rev. D 85, 024020 (2012).
[38] S.W. Hawking, Comm. Math. Phys. 43, 199 (1975).
[39] N. Leite, P. Pavlović, arXiv:1805.06036 [gr-qc]
[40] A. N. Chowdhury, M. Patil, D. Malafarina, and P. S. Joshi, Phys. Rev. D 85, 104031 (2012).
[41] C. A. Benavides-Gallego, A. Abdujabbarov, D. Malafarina, B. Ahmedov, and C. Bambi, Phys. Rev. D99, 044012 (2019).
[42] R. Shafee, J. E. McClintock, R. Narayan, S. W. Davis, L.-X. Li, and R. A. Remillard, Astrophys. J. Lett. 636, L113 (2006).
[43] R. Shafee, R. Narayan, and J. E. McClintock, Astrophys. J. Lett. 676, 549 (2008).
[44] R. D. Blandford and R. L. Znajek, Mon. Not. R. Astron. Soc. 179, 433 (1977).
[45] M. Davis, R. Ruffini, J. Tiomno and F. Zerilli, Phys. Rev. Lett. 28 1352 (1972).
[46] J. M. Bardeen, W. H. Press and S. T. Teukolsky, Astrophys. J. 178, 347 (1972).
[47] C. W. Misner et. al., Phys. Rev. Lett. 28, 998 (1972).
[48] Ya. B. Zeldovich and I.D. Novikov, Sov. Phys. Astron. J. 10 , 602 (1967).
[49] B. J. Carr, S. W. Hawking, Mon. Not. Roy. Astron. Soc. 168, 399 (1974).
[50] Event Horizon Telescope Collaboration, ApJL 910(1), L13, (2021).
[51] A. Ricarte, R. Qiu, R. Narayan, MNRAS 505(1) , 523, (2021).
[52] R.P. Eatough et.al, Nature 501, 391, (2013).
[53] M.Yu. Piotrovich, A.G. Mikhailov, S.D. Buliga, T.M. Natvlishvili, MNRAS 495 (1), 614, (2020)
[54] K. Akiyama et al. (Event Horizon Telescope Collaboration), Astrophys. J. Lett. 875, L1 (2019).
[55] K. Akiyama et al. (Event Horizon Telescope Collaboration), Astrophys. J. Lett. 875, L4 (2019).
[56] K. Akiyama et al. (Event Horizon Telescope Collaboration), Astrophys. J. Lett. 875, L6 (2019).