Initial Successive Coefficients for Certain Classes of Univalent Functions

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Abstract—We consider functions of the type \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) from a family of all analytic and univalent functions in the unit disk. The aim of this article is to investigate the bounds of the difference of moduli of initial successive coefficients, i.e., \(|a_{n+1}| - |a_n|\) for \( n = 1, 2 \) and for some subclasses of analytic univalent functions. In addition, we found that all the estimations are sharp in nature by constructing some extremal functions.

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1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions \( f \) of the form
\[
 f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots ,
\]
with \( a_1 = 1 \), which are analytic in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{S} \) be the set all functions \( f \in \mathcal{A} \) that are univalent in \( \mathbb{D} \). For a general theory of univalent functions, we refer to the classical books [3, 4, 22].

A function \( f \in \mathcal{A} \) is called starlike if \( f(\mathbb{D}) \) is a starlike domain with respect to origin. The class of univalent starlike functions is denoted by \( \mathcal{S}^* \). There is one natural generalization of starlike functions, namely the concept of \( \gamma \)-spirallike functions of order \( \alpha \) which leads to a useful criterion for univalency. The family \( \mathcal{S}_\gamma(\alpha) \) of \( \gamma \)-spirallike functions of order \( \alpha \) is defined by
\[
 \mathcal{S}_\gamma(\alpha) = \left\{ f \in \mathcal{A} : \text{Re} \left( e^{-i\gamma} \frac{zf'(z)}{f(z)} \right) > \alpha \cos \gamma \right\},
\]
where \( 0 \leq \alpha < 1 \) and \( -\pi/2 < \gamma < \pi/2 \). Each function in \( \mathcal{S}_\gamma(\alpha) \) is univalent in \( \mathbb{D} \) (see [13]). Functions in \( \mathcal{S}_\gamma(0) \) are called \( \gamma \)-spirallike, but they do not necessarily belong to the starlike family \( \mathcal{S}^* \). For example, the function \( f(z) = z(1 - iz)^{\alpha-1} \) is \( \pi/4 \)-spirallike but \( f \notin \mathcal{S}^* \). The class \( \mathcal{S}_\gamma(0) \) was introduced by \( \check{S}p\acute{a}cek \) [32] (see also [3]). Moreover, \( \mathcal{S}_0(\alpha) =: \mathcal{S}^*(\alpha) \) is the usual class of starlike functions of order \( \alpha \), and \( \mathcal{S}^*(0) = \mathcal{S}^* \). Recall that the class \( \mathcal{S}_\gamma(\alpha) \), for \( 0 \leq \alpha < 1 \), is studied by several authors in different perspective (see, for instance [8, 13, 27]).

We consider another family of functions that includes the class of convex functions as a proper subfamily. For \( -\pi/2 < \gamma < \pi/2 \) and \( 0 \leq \alpha < 1 \), we say that \( f \) belongs to the family \( \mathcal{C}_\gamma(\alpha) \) of \( \gamma \)-convex functions of order \( \alpha \) provided \( f \in \mathcal{A} \) is locally univalent in \( \mathbb{D} \) and \( z f'(z) \) belongs to \( \mathcal{S}_\gamma(\alpha) \), i.e.
\[
 \mathcal{C}_\gamma(\alpha) = \left\{ f \in \mathcal{A} : \text{Re} \left( e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > \alpha \cos \gamma \right\}.
\]

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We may set $C_\gamma(0) =: C_\gamma$ and observe that the class $C_0(\alpha) =: C(\alpha)$ consists of the normalized convex functions of order $\alpha$. The set $C(0) =: C$ is the usual class of convex functions. For general values of $\gamma$ ($|\gamma| < \pi/2$), a function in $C_\gamma$ needs not be univalent in $\mathbb{D}$. For example, the function $f(z) = i(1 - z)^{-i} - i$ is known to belong to $C_{\pi/4}\setminus S$. In [20], Pflatzgraff has shown that $f \in C_\gamma$ is univalent whenever $0 < \cos \gamma \leq 1/2$. On the other hand, in [31] it was also shown that a function $f$ in $C_\gamma$ which satisfies $f'''(0) = 0$ is univalent for all real values of $\gamma$ with $|\gamma| < \pi/2$. For a general reference about these special classes we refer to [4].

Let $\mathcal{LU}$ denote the subclass of $A$ consisting of all locally univalent functions; namely, $\mathcal{LU} = \{ f \in A : f'(z) \neq 0, z \in \mathbb{D} \}$. A family $G(\lambda)$, $\lambda > 0$, of functions $f \in \mathcal{LU}$ is defined by

$$G(\lambda) = \left\{ f \in \mathcal{LU} : \Re \left(1 + \frac{z f'''(z)}{f'(z)}\right) < 1 + \frac{\lambda}{2}\right\}.$$ 

The class $G := G(1)$ was first introduced by Ozaki [18] and he proved the inclusion relation $G \subseteq S$. Later in [34] Umezawa discussed a general version of this class. Moreover, functions in $G$ are proved to be starlike in $\mathbb{D}$, see for eg. [23, 26]. The Taylor coefficient problem for the class $G(\lambda)$, $0 < \lambda \leq 1$, is discussed by Obradović et al. [16]. In 2018, Obradović et al. [17] obtained the sharp logarithmic coefficient bounds for functions in the class $G(\lambda)$. The radius of convexity of the functions in the class $G(\lambda)$, $\lambda > 0$, is obtained in [9]. In 2020, Ponnusamy et al. [24] computed the sharp estimates for the initial three logarithmic coefficients for the class $G(\lambda)$, $0 < \lambda \leq 1$. This class, with special choices of the parameter $\lambda$, has also been considered by many researchers in the literature for different purposes; see for instance [2, 25].

The estimation of the difference of moduli of successive coefficients $||a_{n+1}|-|a_n||$ is an important problem in the study of univalent functions. It is well-known that the difference $|a_{n+1}|-|a_n|$ is bounded for $f \in S$. Indeed, Hayman [7] proved $||a_{n+1}|-|a_n|| \leq A$ for $f \in S$, where $A \geq 1$ is an absolute constant. Pommerenke [21] conjectured that $||a_{n+1}|-|a_n|| \leq 1$ for the class $S^*$ which was proved by Leung [10]. On the other hand, a sharp bound is known only for $n = 2$ (see [3, Theorem 3.11]), namely

$$-1 \leq |a_3| - |a_2| \leq 1.029 \ldots.$$ 

For convex functions, Li and Sugawa [12] investigated the sharp upper bound of $|a_{n+1}| - |a_n|$ for $n \geq 2$, and sharp lower bounds for $n = 2, 3$. Several results are known in this direction. These observations are also addressed in the recent paper [1], where a bound for $||a_{n+1}|-|a_n||$, $n \geq 2$, for the class $S_1(\alpha)$ is obtained. However, the bound contains an unknown constant.

The successive coefficient problem was first studied for the class of univalent functions to prove the classical Bieberbach conjecture. Although, the Bieberbach conjecture has been proved after sixty-nine years by De Branges in 1985, the successive coefficient problem is still an open problem for several important classes of functions including the whole class of analytic univalent functions. Thus, the successive coefficient problem is still under consideration for the class of univalent functions and its subclasses even for some particular values of $n$. In the present paper, we study the sharp bounds for $|a_2| - |a_1|$ and $|a_3| - |a_2|$ for functions belonging to the classes $S_1(\alpha)$, $C_1(\alpha)$, and $G(\lambda)$. Note that, in 2021, Sim and Thomas [30] estimated the sharp bound for $|a_3| - |a_2|$ when functions $f$ are in the class $S_\gamma(\alpha)$. However this paper provides an alternate proof for the above problem. Related work in the direction of the present investigation can also be found in [6, 11, 19, 28–30].

Let $P$ denote the class of all analytic functions $p$ having positive real part in $\mathbb{D}$, with the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

A member of $P$ is called a Carathéodory function. It is known that $|c_n| \leq 2$ for a function $p \in P$ and for all $n \geq 1$ (see [3]). Parametric representations of the coefficients are often more useful. Libera and Zlotkiewicz [14, 15] derived the following parameterizations of possible values of $c_2$ and $c_3$.

**Lemma 1.** Let $-2 \leq c_1 \leq 2$ and $c_2, c_3 \in \mathbb{C}$. Then there exists a function $p \in P$ with

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

if and only if $2c_2 = c_1^2 + (4 - c_1^2)x$ and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x^2 - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)y$$
for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

For simplification, we assume $\mu = e^{i\gamma} \cos \gamma$, $-\pi/2 < \gamma < \pi/2$ and $a_1 = 1$ throughout the paper. Next, recall the well known characterization of $\gamma$-spirallike functions of order $\alpha$.

**Lemma 2.** For $-\pi/2 < \gamma < \pi/2$ and $0 \leq \alpha < 1$, a function $f \in S_\gamma(\alpha)$ if and only if

$$f(z) = z \exp \left\{ (1 - \alpha) \mu \int_0^z \frac{p(t) - 1}{t} \, dt \right\},$$

where

$$p(z) = \frac{1}{1 - \alpha} \left\{ \frac{1}{\cos \gamma} \left( e^{-i\gamma} z f'(z) + i \sin \gamma \right) - \alpha \right\} \in \mathcal{P}. \quad (2)$$

The family $\mathcal{G}(\lambda)$ can be characterized in terms of the Carathéodory function as follows.

**Lemma 3.** For $0 < \lambda \leq 1$, a function $f \in \mathcal{G}(\lambda)$ if and only if

$$f'(z) = \exp \left\{ \frac{-\lambda}{2} \int_0^z \frac{p(t) - 1}{t} \, dt \right\},$$

where

$$p(z) = \frac{1}{\lambda} \left( \lambda - \frac{2zf''(z)}{f'(z)} \right) \in \mathcal{P}. \quad (3)$$

Our main objective in this paper is to estimate the sharp bounds of $|a_2| - |a_1|$ and $|a_3| - |a_2|$ for function $f$ belong to $S_\gamma(\alpha)$, $C_\gamma(\alpha)$, and $\mathcal{G}(\lambda)$. The rest of the structure of this paper is as follows. Section 2 is devoted to the statements of main results. Finally, the proofs of main results are given in Section 3.

## 2. MAIN RESULTS

We now state our first main results which provide sharp bounds for $|a_2| - |a_1|$ when the functions $f$ are $\gamma$-spirallike functions of order $\alpha$.

**Theorem 1.** Let $-\pi/2 < \gamma < \pi/2$ and $0 \leq \alpha < 1$. For every $f \in S_\gamma(\alpha)$ of the form (1), we have

$$-1 \leq |a_2| - |a_1| \leq 2(1 - \alpha) \cos \gamma - 1.$$  

Equality holds on right-hand side for the rotations of

$$k_{\gamma, \alpha}(z) = \frac{z}{(1 - z)^2(1 - \alpha)\mu}, \quad z \in \mathbb{D}, \quad (4)$$

and on the left-hand side equality holds for the rotations of

$$h_{\gamma, \alpha}(z) = \frac{z}{(1 - z^2)^2(1 - \alpha)\mu}, \quad z \in \mathbb{D}. \quad (5)$$

If we choose $\alpha = 0$ in Theorem 1, it produces the following result for the class of spirallike functions.

**Corollary 1.** Let $f \in S_\gamma$, $-\pi/2 < \gamma < \pi/2$, given by (1). Then we have $-1 \leq |a_2| - |a_1| \leq 2 \cos \gamma - 1$. Both inequalities are sharp.

In the next theorem, we estimate the sharp bound of $|a_3| - |a_2|$ when the functions $f$ are $\gamma$-spirallike functions of order $\alpha$.

**Theorem 2.** Let $-\pi/2 < \gamma < \pi/2$ and $0 \leq \alpha < 1$. For every $f \in S_\gamma(\alpha)$ of the form (1), we have

$$\frac{-2(1 - \alpha) \cos \gamma}{\sqrt{1 + T(\alpha, \gamma)}} \leq |a_3| - |a_2| \leq (1 - \alpha) \cos \gamma,$$

where $T(\alpha, \gamma) := \sqrt{1 + 4(1 - \alpha)(2 - \alpha) \cos^2 \gamma}$.  

Equality holds on the right-hand side for the rotations of $h_{\gamma, \alpha}$ given by (5) and on the left-hand side for the rotations of

$$f_{\gamma, \alpha}(z) = \frac{z}{((1 - \epsilon_1 z)^{\gamma_1} (1 - \epsilon_2 z)^{\gamma_2})^{2(1 - \alpha)\mu}, \quad z \in \mathbb{D}, \quad (7)$$
where
\[ |\epsilon_1| = |\epsilon_2| = 1, \quad \epsilon_1 \neq \epsilon_2, \quad \gamma_1, \gamma_2 > 0, \quad \gamma_1 + \gamma_2 = 1, \]
\[ \gamma_1 \epsilon_1 + \gamma_2 \epsilon_2 = \frac{c}{2}, \quad \text{and} \quad \gamma_1 \epsilon_1^2 + \gamma_2 \epsilon_2^2 = \frac{(c^2 + (4 - c^2)x)}{4} \]
with
\[ c = c_0 := \frac{2}{\sqrt{T(\alpha, \gamma) + 1}} \quad \text{and} \quad x = -\frac{(1 + 2(1 - \alpha) \cos^2 \gamma + i(1 - \alpha) \sin(2\gamma))}{T(\alpha, \gamma)}. \]

If we put \( \alpha = 0 \) in Theorem 2, then we obtain the following result.

**Corollary 2.** Let \( f \in S_{\gamma}, \pi/2 < \gamma < \pi/2, \) given by (1). Then we have
\[ -\frac{2 \cos \gamma}{\sqrt{1 + T(0, \gamma)}} \leq |a_3| - |a_2| \leq \cos \gamma. \]

Both the inequalities are sharp.

It is pertinent to mention that Corollary 2 coincides with [11, Theorem 1.4]. Also note that for \( \alpha = 0 \) and \( \gamma = 0 \), Theorems 1 and 2 extend the result of Leung [10] from starlike to \( \gamma \)-spirallike functions of order \( \alpha \).

Now let us recall the classical Alexander theorem which gives the close analytic connection between the convex and starlike functions. Analogues to this, it can be verified that \( f \) belongs to \( C_\gamma(\alpha) \) if and only if \( z f' \) belongs to \( S_{\gamma}(\alpha) \). This will be used in the sequel to prove the next theorem.

In the next two theorems, we will discuss about the sharp bounds for \( |a_2| - |a_1| \) and \( |a_3| - |a_2| \) when the functions \( f \) run over the class \( C_\gamma(\alpha) \).

**Theorem 3.** Let \( -\pi/2 < \gamma < \pi/2 \) and \( 0 \leq \alpha < 1 \). For every \( f \in C_\gamma(\alpha) \) be of the form (1), we have
\[ -1 \leq |a_2| - |a_1| \leq (1 - \alpha) \cos \gamma - 1. \]

The right-hand side inequality becomes equality for the rotations of
\[ l_{\gamma, \alpha}(z) = \frac{1}{2(1 - \alpha)\mu - 1} \left( \frac{1}{(1 - z)^{2(1 - \alpha)\mu - 1}} - 1 \right), \quad z \in D. \]

The left-hand side inequality becomes equality for the rotations of
\[ q_{\gamma, \alpha}(z) = \int_0^z \frac{1}{1 - (t^2)(1 - \alpha)\mu} dt, \quad z \in D. \]

Note that \( k_{\gamma, \alpha}(z) = z l'_{\gamma, \alpha}(z) \) and \( h_{\gamma, \alpha}(z) = z q'_{\gamma, \alpha}(z) \). For the special case \( \gamma = 0 \), we get the following result.

**Corollary 3.** Let \( 0 \leq \alpha < 1 \). For every \( f \in C(\alpha) \) of the form (1), we have \(-1 \leq |a_2| - |a_1| \leq -\alpha. \)
Both the inequalities are sharp.

**Theorem 4.** Let \( -\pi/2 < \gamma < \pi/2 \) and \( 0 \leq \alpha < 1 \). For every \( f \in C_\gamma(\alpha) \) of the form (1), we have
\[ -\frac{(1 - \alpha) \cos \gamma}{\sqrt{1 + T(\alpha, \gamma)}} \leq |a_3| - |a_2| \leq \frac{(1 - \alpha) \cos \gamma}{3}. \]

Equality holds on the right-hand side for the rotations of the functions \( q_{\gamma, \alpha} \) given by (12) and on the left-hand side for the rotations of
\[ g_{\gamma, \alpha}(z) = \int_0^z \frac{1}{((1 - \epsilon_1 t)^{\gamma_1}(1 - \epsilon_2 t)^{\gamma_2})^{2(1 - \alpha)\mu}} dt, \quad z \in D, \]
where \( T(\alpha, \gamma) \) is given by (6) and \( \gamma_1, \gamma_2, \epsilon_1, \epsilon_2, x \) satisfy (8) and (9) with \( c_0 \) and \( x \) given by (10).
The following corollary immediately follows, by choosing $\gamma = 0$, from Theorem 4 for the class of convex functions of order $\alpha$.

**Corollary 4.** Let $0 \leq \alpha < 1$. For every $f \in C(\alpha)$ of the form (1), we have
\[
\frac{-(1-\alpha)}{\sqrt{1+T(\alpha,0)}} \leq |a_3| - |a_2| \leq \frac{(1-\alpha)}{3}.
\]
Both inequalities are sharp.

We remark that Theorem 4, for $\gamma = 0$ and $\alpha = 0$, is obtained by Li and Sugawa [12] for the class of convex functions.

Our next two results are related to the sharp bounds for $|a_2| - |a_1|$ and $|a_3| - |a_2|$ when the functions $f$ are belonging to the family $G(\lambda)$.

**Theorem 5.** Let $0 < \lambda \leq 1$. If $f \in G(\lambda)$ is given by (1), then $-1 \leq |a_2| - |a_1| \leq \lambda/2 - 1$. Equality holds on the right-hand side for the rotations of $G(\lambda)$.

Equality holds on the left-hand side for the rotations of
\[
G_{\lambda}(z) = \frac{(1+z)^{1+\lambda} - 1}{\lambda + 1}, \quad z \in \mathbb{D}.
\]

**Theorem 6.** For $0 < \lambda \leq 1$, let every function $f \in G(\lambda)$ be defined by (1). Then we have $|a_3| - |a_2| \leq \lambda/6$. The inequality becomes equality for the rotations of $H_{\lambda}$ defined by (15). Furthermore,
\[
|a_3| - |a_2| \geq \begin{cases}
\frac{\lambda(4\lambda-1)}{24(2-\lambda)}, & \text{for } 0 < \lambda \leq 1/2, \\
\frac{\lambda(\lambda+2)}{6}, & \text{for } 1/2 \leq \lambda \leq 1.
\end{cases}
\]

The inequality becomes equality for the rotations of
\[
F_{\lambda}(z) = \int_{0}^{z} \left((1-\epsilon_1 t)\gamma_1 (1-\epsilon_2 t)\gamma_2 \right)^{\lambda}, \quad z \in \mathbb{D},
\]
where $\gamma_1, \gamma_2, \epsilon_1, \epsilon_2$ satisfy (8) and (9) with $x = -1$ and $c = \overline{c}$ given by (35).

3. PROOF OF THE MAIN RESULTS

This section is devoted to the detailed discussion on our proof of our main results.

3.1. Proof of Theorem 1

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_{\gamma}(\alpha)$. Then there exist $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P$ of the form (2) which is equivalent to writing
\[
((1-\alpha)p(z) + \alpha) \cos \gamma - i \sin \gamma = e^{-i\gamma} \frac{z f'(z)}{f(z)}.
\]

By using the Taylor representations of the functions $f$ and $p$, and comparing the coefficients of $z^n$ ($n = 1, 2$) on both the sides, we get
\[
a_2 = (1-\alpha)\mu c_1 \quad \text{and} \quad 2a_3 = (1-\alpha)^2 \mu^2 c_1^2 + (1-\alpha)\mu c_2.
\]

So,
\[
|a_2| - |a_1| = |a_2| - 1 = (1-\alpha) \cos \gamma |c_1| - 1 \leq 2(1-\alpha) \cos \gamma - 1,
\]
where the last inequality comes by using $|c_n| \leq 2$ for $n \geq 1$. For the right-hand side equality, let us consider the function $f = k_{\gamma, \alpha}$ given by (4) for which $a_2 = 2(1 - \alpha)\mu$. Then it is a simple exercise to see that

$$\text{Re}\left(e^{-i\gamma} \frac{zh_{\gamma, \alpha}(z)}{k_{\gamma, \alpha}(z)}\right) = \cos \gamma \left(1 + 2(1 - \alpha)\text{Re}\left(\frac{z}{1 - z}\right)\right) > \alpha \cos \gamma,$$

from which we can easily conclude that $k_{\gamma, \alpha} \in S_\gamma(\alpha)$ and $|a_2| = |a_1| = 2(1 - \alpha) \cos \gamma - 1$.

On the other hand, $|a_1| - |a_2| = 1 - (1 - \alpha) \cos \gamma |c_1| \leq 1$. Consider the function $f = h_{\gamma, \alpha}$ given by (5). In this case $a_1 = 1$ and $a_2 = 0$. A simple calculation shows that

$$\text{Re}\left(e^{-i\gamma} \frac{zh_{\alpha, \gamma}(z)}{h_{\gamma, \alpha}(z)}\right) = \cos \gamma \left(1 + 2(1 - \alpha)\text{Re}\left(\frac{z^2}{1 - z^2}\right)\right) > \alpha \cos \gamma.$$

Thus, the left-hand side equality holds for the function $f = h_{\gamma, \alpha} \in S_\gamma(\alpha)$. This completes the proof. □

### 3.2. Proof of Theorem 2

Let $f \in S_\gamma(\alpha)$. Then from equation (16) we obtain

$$|a_3| - |a_2| = \sqrt{(1 - \alpha)^2 \mu^2 c_1^2 + (1 - \alpha)\mu c_2} - |(1 - \alpha)\mu c_1| = \frac{(1 - \alpha)\mu}{2} \left|(1 - \alpha)\mu c_1^2 + c_2 - 2|c_1|\right|.$$ 

As $|a_3| - |a_2|$ is invariant under rotation, to simplify the calculation we assume that $c_1 = c \in [0, 2]$. Therefore, by Lemma 1, for some $x \in \mathbb{D}$ we have

$$|a_3| - |a_2| = \frac{(1 - \alpha)\mu}{4} \left|c^2 + (4 - c^2)x + 2(1 - \alpha)\mu c^2 - 4c\right| = \frac{(1 - \alpha)\mu}{4} \left(\psi(x, c) - 4c\right) \quad \text{(17)}$$

with

$$\psi(x, c) := |c^2 + (4 - c^2)x + 2(1 - \alpha)\mu c^2|.$$ \hspace{1cm} \text{(18)}$$

By letting $x = re^{i\theta}$, we compute

$$\psi(x, c) = |c^2 + (4 - c^2)r(\cos \theta + i \sin \theta) + 2(1 - \alpha) \cos \gamma (\cos \gamma + i \sin \gamma)c^2|$$

$$= \left|\sqrt{P \cos \theta + Q \sin \theta + R}\right|, \quad \text{where}$$

$$P = 2r(4 - c^2)c^2(1 + 2(1 - \alpha) \cos \gamma), \quad Q = 2r(1 - \alpha)(4 - c^2)c^2 \sin 2\gamma,$$

$$R = c^4 + (4 - c^2)^2r^2 + 4c^4(1 - \alpha)^2 \cos^2 \gamma + 4c^4(1 - \alpha) \cos^2 \gamma.$$ 

Clearly, $\sqrt{P^2 + Q^2} = 2rc^2(4 - c^2)T(\alpha, \gamma)$ with $T(\alpha, \gamma) = \sqrt{1 + 4(1 - \alpha)(2 - \alpha) \cos^2 \gamma}$. Next, we use the following well-known inequality

$$-\sqrt{P^2 + Q^2} \leq P \cos \theta + Q \sin \theta \leq \sqrt{P^2 + Q^2} \quad \text{(20)}$$

to obtain

$$\psi(x, c) \leq \sqrt{\sqrt{P^2 + Q^2}^2 + R} = \sqrt{2rc^2(4 - c^2)\sqrt{T(\alpha, \gamma)} + (4 - c^2)^2r^2 + c^4T^2(\alpha, \gamma)}$$

$$= c^2T(\alpha, \gamma) + (4 - c^2)r \leq c^2T(\alpha, \gamma) + 4 - c^2.$$ 

Since $\gamma \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$, it is easy to check that $T(\alpha, \gamma) < 3$. Hence,

$$\psi(x, c) \leq 2c^2 + 4. \quad \text{(21)}$$

By substituting (21) into (17), we get

$$|a_3| - |a_2| \leq \frac{(1 - \alpha)\mu}{4}(2c^2 + 4 - 4c) \leq \cos \gamma(1 - \alpha),$$
as required. Recall from the proof of Theorem 1, $h_{\gamma,\alpha}$ defined by (5) belongs to $S_{\gamma}(\alpha)$. It is evident that the equality holds for the function $h_{\gamma,\alpha}$ in which the coefficient of $z^2$ is 0 and $z^3$ is $(1 - \alpha)\mu$. Thus, the right-hand equality of the theorem has been proved.

On the other hand, by (19) and (20) we have
\begin{equation}
\psi(x, c) \geq \sqrt{-\sqrt{P^2 + Q^2} + R}
= \sqrt{-2rc^2(4 - c^2)T(\alpha, \gamma) + (4 - c^2)^2r^2 + c^4T^2(\alpha, \gamma)} = |c^2T(\alpha, \gamma) - (4 - c^2)r|.
\end{equation}
Here, the equality occurs if $\cos \theta = -(1 + 2(1 - \alpha)\cos^2 \gamma)/T(\alpha, \gamma)$ and $\sin \theta = (1 - \alpha)\sin(2\gamma)/T(\alpha, \gamma)$. The equation (17) and the inequality (22) together lead to
\begin{equation}
|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{4} \left(4c - |c^2T(\alpha, \gamma) - (4 - c^2)r|\right).
\end{equation}
To simplify the above inequality we have to consider following two cases.

**Case 1:** Let $c \in [0, c_0]$, where $c_0$ is defined by (10). Then
\begin{align*}
|a_2| - |a_3| &\leq \frac{(1 - \alpha)|\mu|}{4} \left(4c - |c^2T(\alpha, \gamma) - (4 - c^2)r|\right) \\
&\leq (1 - \alpha)|\mu|c \leq (1 - \alpha)|\mu|c_0 = \frac{2(1 - \alpha)|\mu|}{\sqrt{1 + T(\alpha, \gamma)}}.
\end{align*}

**Case 2:** Let $c \in [c_0, 2]$. Note that
\begin{equation}
\begin{aligned}
c^2T(\alpha, \gamma) - (4 - c^2)r &\geq c^2T(\alpha, \gamma) - (4 - c^2) \geq 0 \\
\end{aligned}
\end{equation}
for $c_0 \leq c \leq 2$. Then (23) becomes
\begin{equation}
|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{4} \left(4c - c^2T(\alpha, \gamma) + (4 - c^2)r\right) \leq \frac{(1 - \alpha)|\mu|}{4} \left(4(c_0 + 1) - c^2(T(\alpha, \gamma) + 1)\right).
\end{equation}
It is easy to check that $4(c_0 + 1) - c^2(T(\alpha, \gamma) + 1)$ is an decreasing function of $c$ in the interval $[c_0, 2]$. Hence, the maximum is attained in the above inequality for $c = c_0$. It follows that
\begin{equation}
|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{4} \left(4(c_0 + 1) - c_0^2(T(\alpha, \gamma) + 1)\right) = \frac{2(1 - \alpha)\cos \gamma}{\sqrt{1 + T(\alpha, \gamma)}}.
\end{equation}
Thus, by combining both the above cases, we find the desired inequality.

We now proceed to prove the left-hand side equality part. Choose $x = -1$ and $c = c_0$ given by (10). Note that $4 - c_0^2 \geq 0$ and $|x| = 1$. Thus, by making use of the Cauchy–Schwarz theorem (see [5, 33]), we obtain that $p$ is of the following form
\begin{equation}
p(z) = \gamma_1 \frac{1 + \epsilon_1 z}{1 - \epsilon_1 z} + \gamma_2 \frac{1 + \epsilon_2 z}{1 - \epsilon_2 z} \in \mathcal{P},
\end{equation}
where $\gamma_1, \gamma_2, \epsilon_1, \epsilon_2$ satisfy (8) and (9). Now Lemma 2 gives
\begin{equation}
f_{\gamma,\alpha}(z) = \frac{z}{(1 - \epsilon_1 z)^{2(1-\alpha)|\mu|} (1 - \epsilon_2 z)^{2(1-\alpha)|\mu|}},
\end{equation}
which belongs to $S_{\gamma}(\alpha)$. This can indeed be verified by using the series expansion of $f_{\gamma,\alpha}$ to get
\begin{equation}
a_2 = \frac{2(1 - \alpha)|\mu|}{\sqrt{1 + T(\alpha, \gamma)}} \quad \text{and} \quad a_3 = \frac{(1 - \alpha)|\mu|}{1 + T(\alpha, \gamma)} \left(2(1 - \alpha)\mu + 1 + xT(\alpha, \gamma)\right).
\end{equation}
This gives that
\begin{equation}
|a_3| - |a_2| = \frac{-2(1 - \alpha)\cos \gamma}{\sqrt{1 + T(\alpha, \gamma)}},
\end{equation}
which yields the desired result. \hfill \Box
3.3. Proof of Theorem 3

Suppose \( f \in C_\gamma(\alpha) \). Then there exist a function \( p \in \mathcal{P} \) such that

\[
p(z) = \frac{1}{1 - \alpha} \left\{ \frac{1}{\cos \gamma} \left( e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right) + i \sin \gamma \right) - \alpha \right\},
\]

or equivalently

\[
((1 - \alpha)p(z) + \alpha) \cos \gamma - i \sin \gamma = e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right).
\]

(25)

We write \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \). Equating the coefficients of \( z^n \) on both the sides of the equation (25) for \( n = 1, 2 \), we obtain

\[
a_1 = 1, \quad 2a_2 = (1 - \alpha)\mu c_1 \quad \text{and} \quad 6a_3 = (1 - \alpha)^2 \mu^2 c_1^2 + (1 - \alpha)\mu c_2.
\]

(26)

Now we compute and estimate

\[
|a_2| - |a_1| = \left| \frac{(1 - \alpha)\mu c_1}{2} \right| - 1 \leq (1 - \alpha) \cos \gamma - 1.
\]

The last inequality holds since \( |c_1| \leq 2 \). It is easy to check that the equality holds for the function \( l_{\gamma,\alpha} \) is given by (11) and which satisfies \( zl'_{\gamma,\alpha} = k_{\gamma,\alpha} \), where \( k_{\gamma,\alpha} \in \mathcal{S}_\gamma(\alpha) \) is given by (4). Thus, \( l_{\gamma,\alpha} \in C_\gamma(\alpha) \) and the coefficient of \( z^2 \) in \( l_{\gamma,\alpha} \) is \( (1 - \alpha)\mu \).

Secondly, we estimate the lower bound for \( |a_1| - |a_2| = (1 - \alpha)|\mu c_1|/2 \leq 1 \). For the sharpness, let us consider the function \( q_{\gamma,\alpha} \) satisfying \( zq'_{\gamma,\alpha} = h_{\gamma,\alpha} \), where \( h_{\gamma,\alpha} \) is defined by (5). Since \( h_{\gamma,\alpha} \in \mathcal{S}_\gamma(\alpha) \), it concludes that \( q_{\gamma,\alpha} \in C_\gamma(\alpha) \). Also,

\[
q_{\gamma,\alpha}(z) = \frac{1}{\int_0^z (1 - t^2)^{(1 - \alpha)/\mu} dt} = \frac{z}{1 + (1 - \alpha)\mu t^2 + \frac{(1 - \alpha)\mu((1 - \alpha)\mu + 1)}{2} t^4 + \ldots}
\]

\[
= z + \frac{(1 - \alpha)\mu}{3} z^3 + \frac{(1 - \alpha)\mu((1 - \alpha)\mu + 1)}{10} z^5 + \ldots.
\]

For this function, we have

\[
a_1 = 1, \quad a_2 = 0, \quad \text{and} \quad a_3 = \frac{(1 - \alpha)\mu}{3},
\]

(27)

and hence \( |a_2| - |a_1| = -1 \). This completes the proof. \( \square \)

3.4. Proof of Theorem 4

To prove this theorem, we use the similar technique that is adopted in Theorem 2. Let \( f \in C_\gamma(\alpha) \). Then, by means of equation (26), we see that

\[
|a_3| - |a_2| = \left| \frac{(1 - \alpha)^2 \mu^2 c_1^2 + (1 - \alpha)\mu c_2}{6} \right| - \left| \frac{(1 - \alpha)\mu c_1}{2} \right| = \frac{(1 - \alpha)|\mu|}{6} \left| \mu(1 - \alpha) c_1^2 + c_2 \right| - 3|c_1|.
\]

As \( |a_3| - |a_2| \) is invariant under rotations, to simplify the calculation we assume that \( c_1 = c \in [0, 2] \). Thus, Lemma 1 yields

\[
|a_3| - |a_2| = \frac{(1 - \alpha)|\mu|}{12} (\psi(x, c) - 6c),
\]

(28)

where \( \psi(x, c) \) is given by (18), for some \( x \in \mathbb{D} \). By using equation (21) we derive the desired inequality

\[
|a_3| - |a_2| \leq \frac{(1 - \alpha)|\mu|}{12} \left( 2c^2 + 4 - 6c \right) \leq \frac{(1 - \alpha) \cos \gamma}{3}.
\]

It is clear from (27) that the equality holds for the function \( q_{\gamma,\alpha} \in C_\gamma(\alpha) \) given by (12).
We next find the lower bound of $|a_3| - |a_2|$. The inequality (22) and equation (28) together lead to

$$|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{12} \left(6c - |c^2T(\alpha, \gamma) - (4 - c^2)r|\right).$$

(29)

Next we consider the following two cases in order to complete the proof.

**Case 1:** If $0 \leq c \leq c_0$, where $c_0$ is defined by (10). Then from (29)

$$|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{12} \left(6c - |c^2T(\alpha, \gamma) - (4 - c^2)r|\right) \leq \frac{(1 - \alpha)|\mu|c}{2} \leq \frac{(1 - \alpha)|\mu|c_0}{2}.$$

**Case 2:** If $c_0 \leq c \leq 2$. Then

$$c^2T(\alpha, \gamma) - (4 - c^2)r \geq c^2T(\alpha, \gamma) - (4 - c^2) \geq 0.$$

Thus, from (29), we have

$$|a_2| - |a_3| \leq \frac{(1 - \alpha)|\mu|}{12} \left(6c0 - c_0^2T(\alpha, \gamma) + 4 - c_0^2\right) = \frac{(1 - \alpha)\cos \gamma}{\sqrt{1 + T(\alpha, \gamma)}}.$$

The last inequality holds since $6c + 4 - c^2(T(\alpha, \gamma) + 1)$ is an decreasing function of $c$ in the interval $[c_0, 2]$. Therefore, the maximum is attained at $c = c_0$. Summarizing parts from Case 1 and 2, it follows the desired inequality.

For the equality, we consider the function $g_{\gamma, \alpha}$ defined as $zg_{\gamma, \alpha} = f_{\gamma, \alpha}$, where $f_{\gamma, \alpha} \in S_\gamma(\alpha)$ is given by (7). Thus, $g_{\gamma, \alpha} \in C_\gamma(\alpha)$ with the representation (13). The series expansion of $g_{\gamma, \alpha}$ has the form

$$g_{\gamma, \alpha}(z) = z + \frac{\mu(1 - \alpha)}{\sqrt{1 + T(\alpha, \gamma)}} z^2 + \frac{(1 - \alpha)\mu(2 - (1 - \alpha)\mu + 1 + T(\alpha, \gamma)\mu)}{3(1 + T(\alpha, \gamma))} z^3 + \ldots.$$

This completes the proof.

### 3.5. Proof of Theorem 5

Let $f \in G(\lambda)$. Then there exists a function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P$ satisfying (3). Hence $\lambda f'(z)p(z) = \lambda f'(z) - 2z f''(z)$. After writing $f$ and $p$ in the series form and by comparing the coefficients of $z$ and $z^2$ in the above equation, we obtain the relations

$$a_2 = -\frac{\lambda c_1}{4} \quad \text{and} \quad a_3 = \frac{\lambda^2 c_2^2 - 2\lambda c_2}{24}.$$

(30)

Consider $|a_2| - |a_1| = \lambda |c_1|/4 - 1 \leq \lambda/2 - 1$. To prove the equality part, consider the function $G_\lambda$ provided by (14) for which $a_2 = \lambda/2$. An easy computation yields

$$\text{Re} \left(1 + \frac{zG_\lambda''(z)}{G_\lambda'(z)}\right) = 1 + \lambda \text{Re} \left(\frac{z}{1 + z}\right) < 1 + \frac{\lambda}{2},$$

which shows that $G_\lambda \in G(\lambda)$.

Next, $|a_1| - |a_2| = 1 - \lambda |c_1|/4 \leq 1$. As the function $H_\lambda$, defined by (15), satisfies

$$\text{Re} \left(1 + \frac{zH_\lambda''(z)}{H_\lambda'(z)}\right) = 1 - \lambda \text{Re} \left(\frac{z^2}{1 - z^2}\right) < 1 + \frac{\lambda}{2},$$

and

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = \lambda/6,$$

(31)

hence $|a_1| - |a_2| = 1$. Thus the proof of the theorem is now complete. 

□
3.6. Proof of Theorem 6

We use the equation (30) to compute

\[ |a_3| - |a_2| = \left| \frac{\lambda^2 c_1^2 - 2\lambda c_2}{24} \right| - \left| \frac{\lambda c_1}{4} \right| = \frac{\lambda}{24} \left( |\lambda c_1^2 - 2c_2| - 6|c_1| \right). \]

We can check that the functional \(|a_3| - |a_2|\) is rotationally invariant, so we assume \(c_1 = c \in [0, 1]\). Also, we note that

\[ |a_3| - |a_2| = \frac{\lambda}{24} \left( |c^2(1 - \lambda) + x(4 - c^2)| - 6c \right) \]

follows from Lemma 1 for some \(x \in \mathbb{R}\) and therefore by substituting \(x = re^{i\theta}\) we deduce that

\[
|a_3| - |a_2| = \frac{\lambda}{24} \left( \sqrt{(1 - \lambda)^2 c^4 + r^2(4 - c^2)^2 + 2rc^2(1 - \lambda)(4 - c^2) \cos \theta} - 6c \right) \\
\leq \frac{\lambda}{24} \left( ((1 - \lambda)c^2 + r(4 - c^2)) - 6c \right) \leq \frac{\lambda}{24} \left( (1 - \lambda)c^2 + (4 - c^2) - 6c \right) \\
= \frac{\lambda}{24} (4 - \lambda c^2 - 6c) \leq \frac{\lambda}{6}. \tag{32}
\]

The desired inequality thus follows. As we noted in the previous theorem, the function \(H_\lambda\) given by (15) belongs to the class \(\mathcal{G}(\lambda)\). Therefore, from (31) we conclude now that the equality occurs for \(H_\lambda\).

We now proceed to prove the left-hand side inequality. By using the equality (32) and the inequality \(\cos \theta \geq -1\) we have that

\[ |a_2| - |a_3| \leq \frac{\lambda}{24} (6c - |(1 - \lambda)c^2 - r(4 - c^2)|). \tag{33} \]

Set \(c^* := \frac{2}{\sqrt{2} - \lambda}\). Next we consider the following two cases in order to complete the proof.

**Case 1:** Let \(0 \leq c \leq c^*\). Then from (33)

\[ |a_2| - |a_3| \leq \frac{\lambda}{24} (6c - |(1 - \lambda)c^2 - r(4 - c^2)|) \leq \frac{\lambda c}{4} \leq \frac{\lambda c^*}{4}. \]

**Case 2:** Let \(c^* \leq c \leq 2\). Then

\((1 - \lambda)c^2 - r(4 - c^2) \geq (1 - \lambda)c^2 - (4 - c^2) \geq 0.\)

Therefore, from (33), it follows that

\[ |a_2| - |a_3| \leq \frac{\lambda}{24} (6c - (1 - \lambda)c^2 + (4 - c^2)r) \leq \frac{\lambda}{24} (6c + 4 - (2 - \lambda)c^2) = \frac{\lambda}{24} \phi(c). \tag{34} \]

It is easy to check that \(d := 3/2 - \lambda\) is the critical point of \(\phi\) and gives the maximum value. But \(d \notin [c^*, 2]\) for \(\lambda > 1/2\). Thus,

**a)** If \(0 < \lambda \leq 1/2\), then \(\phi(c) \leq \phi(d)\) for \(c \in [c^*, 2]\). Therefore, from (34) we obtain

\[ |a_2| - |a_3| \leq \frac{\lambda}{24} \phi(c) \leq \frac{\lambda}{24} \phi(d) = \frac{\lambda(17 - 4\lambda)}{24(2 - \lambda)}. \]

**b)** If \(1/2 < \lambda \leq 1\), then \(\phi\) is increasing in the interval \([c^*, 2]\). Hence \(\phi(c) \leq \phi(2)\). Therefore from (34) we obtain

\[ |a_2| - |a_3| \leq \frac{\lambda}{24} \phi(c) \leq \phi(2) = \frac{\lambda(\lambda + 2)}{6}. \]

Note here that equalities hold simultaneously above for a suitable choice of \(x\) and \(c\). Choose \(x = -1\) and

\[
\tau = \begin{cases} 
\frac{3}{2-x}, & \text{for } 0 < \lambda \leq 1/2, \\
2, & \text{for } 1/2 < \lambda \leq 1.
\end{cases} \tag{35}
\]
Then, by the Carathéodory–Toeplitz theorem, it follows that $p$ has the form (24) satisfying (8). Now Lemma 3 together with $p$ gives the required form of the extremal function

$$F_\lambda(z) = \int_0^z ((1 - \epsilon_1 t)^{\gamma_1} (1 - \epsilon_2 t)^{\gamma_2})^\lambda dt.$$ 

This completes the proof of Theorem 5. □

**Remark.** In this paper, we found the sharp bounds of $|a_{n+1}| - |a_n|$ for the class $S_c(\alpha), C_c(\alpha),$ and $G(\lambda)$ only for $n = 1, 2$. For the remaining positive values of $n$ (i.e., for $n \geq 3$) this problem is still open. Investigation for a complete solution to this problem may lead to new techniques in this development.

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