Abstract.

We derive bilateral estimates for the constants appearing in the Fourier transform restricted theorems on the Euclidean sphere for the ordinary and especially radial functions belonging to the Lebesgue - Riesz spaces as well as belonging to the Grand Lebesgue Spaces.

We obtain an exact estimate for the norm of the restriction Fourier transform operator acting on the radial functions.

Key words and phrases. Ordinary and restricted Fourier transform, norm of vector and operator, inner (scalar) product, Ordinary and Grand Lebesgue Spaces (GLS) and norms, Gaussian function, radii and radial function, Bessel’s functions, surface measure, upper and lower estimates, Tomas - Stein inequality, unitary dilation operator, extremal function, examples.

1 Definitions. Statement of problem. Previous results.
Define as ordinary the Fourier transform $\hat{f}(k) = \hat{f}[f](k)$ for the integrable function on the whole Euclidean space $\mathbb{R}^d$, $d = 2, 3, \ldots$ as follows

$$\hat{f}(k) = \hat{f}[f](k) \overset{def}{=} \int_{\mathbb{R}^d} e^{-i(k,x)} f(x) \, dx.$$ \hspace{1cm} (1)

Henceforth $(k,x)$ denotes as usually the inner (scalar) product $kx = \sum_{j=1}^{d} k_j x_j$, $r = |x| := \sqrt{x \cdot x}$, $s = |k| := \sqrt{k \cdot k}$.

Denote as ordinary by $||f||_p = ||f||_{L^p}$ the classical Lebesgue - Riesz norm for the measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ or $f : \mathbb{R}^d \rightarrow \mathbb{C}$, $\mathbb{C}$ is complex plane, over the whole space

$$||f||_p = ||f||_{L^p(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}.$$ 

Denote also the usually surface Euclidean measure defined on the measurable subsets of the unit sphere $S(d-1) := \{ x, \; x \in \mathbb{R}^d, \; |x| = 1 \}$, by $\gamma$. For instance,

$$A(d) := \gamma(S(d-1)) = \frac{d \; \pi^{d/2}}{\Gamma(1+d/2)} = \frac{2 \; \pi^{d/2}}{\Gamma(d/2)}$$

is the area of sphere $S(d-1)$. As usually, $\Gamma(\cdot)$ denotes the Euler’s Gamma function.

The ordinary Lebesgue - Riesz surface norm $||g||_{L^q}(S(d-1))$, $q \geq 1$ for the measurable function $g : S(d-1) \rightarrow \mathbb{C}$ is defined alike one on the whole space

$$||g||_{L^q(S(d-1))} \overset{def}{=} \left( \int_{S(d-1)} |g(y)|^q \; \gamma(dy) \right)^{1/q}, \; q \geq 1.$$ \hspace{1cm} (2)

**Definition 1.1.** The inequality of the form

$$||\hat{f}||_{L^q(S(d-1))} \leq K_d(p,q) \; ||f||_{L^p(\mathbb{R}^d)},$$ \hspace{1cm} (3)

with finite coefficient $K_d(p,q)$ which holds true for arbitrary $f \in L_p(\mathbb{R}^d)$ and for some non - trivial domain of values $(p,q) \in B$, is named Fourier restriction inequality.

Here the non - trivial domain $B = B(d)$ consists on the whole set of the values of the parameters $(p,q)$, $p,q \geq 1$, for which $K_d(p,q) < \infty$.

Another name: Tomas - Stein inequality, see an pioneer work [30], 1975; in this article was obtained in particular the following important necessary conditions for this inequality

$$1 \leq p \leq \frac{2d+2}{d+3}, \; q \leq \frac{d-1}{d+1} \cdot \frac{p}{p-1}.$$ \hspace{1cm} (4)
There are huge works devoted to the calculation, or at last evaluation, of the domain $B$, as well as the upper estimation the coefficient $K_{d}(p,q)$; on the other words, the norm of correspondent restriction Fourier operator on some regular surfaces, in particular on the sphere, see e.g. in the works [4], [8], [15] - [16], [17], [21], [24] - [25], [30], [31] - [33], [34] etc.

We intent in this short report to offer the simple lower bounds for this coefficients, calculate its exact value only for the radial functions, find in this case the extremal functions.

We extent also these results on the Grand Lebesgue Spaces instead ordinary Lebesgue - Riesz ones.

We can and will to take as the value $K_{d}(p,q)$ its minimal value:

$$K_{d}(p,q) \overset{def}{=} \sup_{f: \|f\|_{p} = 1} Z_{d}(||\hat{f}||_{L_{q}(S(d-1))}, ||f||_{p}),$$

where

$$Z_{d}(||\hat{f}||_{L_{q}(S(d-1))}, ||f||_{p}) = Z_{d}(f) \overset{def}{=} \frac{||\hat{f}||_{L_{q}(S(d-1))}}{||f||_{p}}, \ (p,q) \in B,$$

and separately only for radial functions

$$K_{d}^{\text{Rad}}(p,q) \overset{def}{=} \sup_{f: f \in \text{Rad}, \|f\|_{p} = 1} Z_{d}(||\hat{f}||_{L_{q}(S(d-1))}, ||f||_{p}),$$

Introduce also an extremal function, which is not uniquely determined, e.g. up to change of sign and up to permutation of arguments

$$g(x) = g_{d}[p,q](x) \overset{def}{=} \arg\max_{g: \|g\|_{p} \leq 1} Z_{d}(||\hat{g}||_{L_{q}(S(d-1))}, ||g||_{p}),$$

so that evidently $||g||_{p} = ||g_{d}[p,q]||_{p} = 1$ and

$$Z_{d}(||\hat{g}||_{L_{q}(S(d-1))}, ||g||_{p}) = K_{d}(p,q), \ (p,q) \in B,$$

and analogously for the radial functions:

$$g^{\text{Rad}}(x) = g^{\text{Rad}}_{d}[p,q](x) \overset{def}{=} \arg\max_{g: g \in \text{Rad}, \|g\|_{p} \leq 1} Z_{d}(||\hat{g}||_{L_{q}(S(d-1))}, ||g||_{p}),$$

so that evidently $||g^{\text{Rad}}||_{p} = ||g^{\text{Rad}}_{d}[p,q]||_{p} = 1$ and of course

$$Z^{\text{Rad}}_{d}(||g^{\text{Rad}}||_{L_{q}(S(d-1))}, ||g^{\text{Rad}}||_{p}) = K^{\text{Rad}}_{d}(p,q), \ (p,q) \in B.$$
About radial functions.

Recall that the function $f = f(x), \ x \in \mathbb{R}^d$ is said to be radial, or equally spherical symmetry, iff it dependent only on the polar radii (Euclidean norm) of the argument vector $x: r = |x|:$

$$\exists F = F(r), \ f(x) = F(|x|), \ x \in \mathbb{R}^d; \ r \in (0, \infty).$$  \hspace{1cm} (10)

Write: $f(\cdot) \in \text{Rad}$. If this function $f(\cdot)$ is radial, then $\hat{f}(\cdot)$ is one, as well.

More detail, introduce the following kernel function

$$V_d(s,r) \overset{\text{def}}{=} (2\pi)^{d/2} J_{(d-2)/2}(s\ r) s^{(2-d)/2} r^{d/2},$$  \hspace{1cm} (11)

where $J_l(\cdot)$ denotes as usually the Bessel's function of order $l; \ l \geq 0$.

Then the function $\hat{f}(k)$ has a form $\hat{f}(k) = G(s) = G(|k|),$ where

$$G(s) = \int_0^\infty V_d(s,r) F(r) \ dr.$$  \hspace{1cm} (12)

In particular, in this radial case

$$\int_{\mathbb{R}^d} f(k) \ dk = \int_{\mathbb{R}^d} F(|x|) \ dx = G(0+) = \frac{2}{\Gamma(d/2)} \int_0^\infty r^{d-1} \ F(r) \ dr.$$  \hspace{1cm} (13)

and

$$\int_{\mathbb{R}^d} |F(|x|)|^p \ dx = \frac{2}{\Gamma(d/2)} \int_0^\infty r^{d-1} |F(r)|^p \ dr.$$  \hspace{1cm} (14)

2 Main result: lower estimate.

Let us consider the following example (Gaussian radial density function)

$$h(x) = h_{\sigma}(x) \overset{\text{def}}{=} (2\pi)^{-d/2} \sigma^{-d} \exp \left\{ -0.5 \ \sigma^{-2} \ ||x||^2 \right\}, \ \sigma = \text{const} \in (0, \infty).$$  \hspace{1cm} (15)

The Fourier transform of this function has a form

$$\hat{h}(k) = \exp \left\{ -0.5 \ \sigma^2 \ ||k||^2 \right\}, \ k \in \mathbb{R}^d.$$  \hspace{1cm} (16)

Note that if $k \in S(d-1),$ then $\hat{h}(k) = \exp \left\{ -0.5 \ \sigma^2 \right\},$ so that this function is constant on the surface of the unit sphere $S(d-1)$ and following

$$||\hat{h}||_{q,S(d-1)} = \exp(-\sigma^2/2) \ A^{1/q}(d-1), \ q \geq 1.$$  \hspace{1cm} (17)
Further,

$$||h||_p = (2\pi)^{-0.5d(1-1/p)} \sigma^{-d(1-1/p)} \, p^{-d/(2p)}, \ p \geq 1.$$ 

Therefore, for all the positive values $\sigma$

$$K_d(p,q) \geq e^{-\sigma^2/2} A^{1/q} (d - 1) (2\pi)^{0.5d(1-1/p)} \, p^{d/(2p)} \, \sigma^{d(1-1/p)},$$

and we conclude after maximization over $\sigma$

**Proposition 2.1.** We deduce that for all the values of parameters for which

$$d \geq 2, \ p, q \geq 1 \Rightarrow K_d(p,q) \geq$$

$$A^{1/q} (d - 1) (2\pi)^{0.5d(1-1/p)} \, p^{d/(2p)} \times [d(1 - 1/p)]^{0.5d(1-1/p)}. \quad (16)$$

### 3 Main result: sharp radial estimate.

Note first of all that there ara some reasons to consider the case of radial functions $f = f(|x|)$. Indeed, denote by $U$ an arbitrary unitary linear operator acting from the space $\mathbb{R}^d$ into oneself. It follows immediately from the expression for this function (26) after changes of variables that $g(Ux) = g(x)$.

In detail, introduce the so-called unitary dilation operator $T_U[g](x)$ for the function $g : \mathbb{R}^d \to \mathbb{R}$ and for arbitrary unitary linear operator $U : \mathbb{R}^d \to \mathbb{R}^d$

$$T_U[g](x) \overset{df}{=} g(Ux), \ x \in \mathbb{R}^d. \quad (17)$$

We have

$$||T_U[g]||_p = \int_{\mathbb{R}^d} |g(Ux)|^p \, dx = \int_{\mathbb{R}^d} |g(x)|^p \, dx = ||g||_p^p;$$

and quite alike

$$||\hat{T}_Ug||_{L(q,S(d - 1))} = ||\hat{g}||_{L(q,S(d - 1))}.$$

Following, if the function $g(\cdot), ||g||_p \leq 1$ is extremal and if she was the only one, its unitary delation $T_U[g]$ is also. Therefore, under this hypothesis (which is false!)

$$T_U[g] = \pm g, \ L. g(\cdot) \in \text{Rad}.$$  

We can and will assume without loss of generality that $T_U[g] = g, \Rightarrow g(x) = g_o(|x|)$ for some (measurable) numerical valued radial function $g_o = g_o(r), \ r > 0.$


Thus, the function \( g(x) \) is radial, and one can find the extremal in (5), (6) only among the set of all radial functions: \( g(x) = F(|x|) = F(r), \; r > 0 \).

Note that here

\[
||F||_p = \sqrt[p]{\frac{2}{\Gamma(d/2)}} \left[ \int_0^\infty r^{d-1} |F(r)|^p \, dr \right], \; p > 0.
\]  

(18)

Further, it follows from (12)

\[
G(s) = \int_0^\infty V_d(r) \, F(r) \, dr, \; |s| = 1,
\]  

(19)

i.e. this function is constant on the surface of unit sphere \( S(d-1) \). Obviously,

\[
V_d(r) = (2\pi)^{d/2} J_{(d-2)/2}(r) \, r^{d/2}, \; r > 0.
\]

Therefore

\[
||F||_{q,S(d-1)} = A^{1/q}(d-1) \int_0^\infty V_d(r) \, F(r) \, dr.
\]

(21)

This problem may be transformed as follows. Introduce the following measure \( d\nu = \nu_d(dr) := r^{d-1} \, dr \); then we get to the following problem

\[
\Phi[F] = \Phi_{d,p,q}[F] \overset{\text{def}}{=} \left[ 2^{-1/p} \; \Gamma^{1/p}(d/2) \; A^{1/q}(d-1) \right] \times
\]

\[
\frac{\int_0^\infty V_d(r) \, F(r) \, dr}{\left[ \int_0^\infty r^{d-1}|F(r)|^p \, dr \right]^{1/p}} \rightarrow \sup \{ F \in L(p, R_+, r^{d-1}dr) \}.
\]  

(22)

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\]

\[
\frac{\int_0^\infty V_d(r) \, F(r) \, \nu_d(dr)}{||F||_{p,\nu}} \rightarrow \sup \{ F \in L(p, \nu) \}.
\]  

(23)

(24)

To summarize.

**Theorem 3.1.**
\[ K^{\text{Rad}}(d; p, q) = P(d; p, q) \cdot \left\{ \int_0^\infty r^{(2+d(p-2))/(2(p-1))} J_{(d-2)/2}^p(r) \, dr \right\}^{1/p'} . \]  

(25)

Herewith the arbitrary extremal function \( F_0(r) \) has a form

\[ F_0(r) = C \cdot [V_d(r)]^{1/(p-1)}, \quad p > 1, \quad C = \text{const}. \]  

(26)

Notice that the integral in (25) convergent under our assumptions iff

\[ 1 < p < \frac{2d}{d+1}. \]  

(27)

As regards for the properties of Bessel’s functions see, e.g. [35]. Namely, as \( z \to 0^+ \)

\[ J_\alpha(z) \asymp C_0(\alpha) z^\alpha, \]

and correspondingly as \( z \to \infty \)

\[ J_\alpha(z) \asymp C_\infty(\alpha) z^{-1/2}, \quad \alpha = \text{const} \geq 0. \]

4 Generalization on the Grand Lebesgue Spaces.

Let \( (a, b) = \text{const}, \; 1 \leq a < b \leq \infty, \) and let \( \psi = \psi(p), \; p \in (a, b) \) be bounded from below: \( \inf_{p\in(a,b)} \psi(p) > 0 \) measurable function. The set of all such a functions will be denoted by \( \Psi(a, b) \); put also

\[ \Psi := \cup_{(a,b): 1 < a < b < \infty} \Psi(a, b). \]  

Definition 4.1. The Grand Lebesgue Space \( G\psi, \; \psi \in \Psi(a, b) \) builded over the set \( \mathbb{R}^d \), or equally over the sphere \( S(d-1) \), consists by definition on all the integrable functions having a finite norm

\[ \|f\|_{G\psi}(\mathbb{R}^d) \overset{\text{df}}{=} \sup_{p \in (a, b)} \left\{ \frac{\|f\|_p}{\psi(p)} \right\}. \]  

(28)

These space was investigated in many works, see e.g. [5], [7], [9], [10], [11], [12], [13], [14], [18], [19], [20], [26] - [29]. In particular, the belonging of the function to certain Grand Lebesgue Space \( G\psi \) is closely related with its tail behavior and is related with its moment generating function \( \int \exp(\lambda f(x)) \, dx \).

Suppose now that the function \( f = f(x) \) belongs to some GLS \( G\psi(a, b), \; \exists \psi \in \Psi : \)
\[ \|f\| G\psi \leq \psi(p), \quad p \in (a, b). \]

Define the (cut) set
\[ D = \{ \ p, \ \exists q \Rightarrow (p, q) \in B \ \}, \]
and assume its non-triviality: \( D \neq \emptyset \). We have
\[ \|\hat{f}\|_{q, S(d-1)} \leq \|f\| G\psi \cdot \psi(p) \cdot K(p, q), \quad p \in (a, b), \ q \in D. \]

Therefore
\[ \|\hat{f}\|_{q, S(d-1)} \leq \|f\| G\psi \cdot \psi(p) \cdot \zeta(q), \quad q \in D, \]
where
\[ \zeta(q) := \inf_{p \in D} \left[ \psi(p) \cdot K(p, q) \right]. \]

To summarize:

**Theorem 4.1.** We deduce under formulated restrictions
\[ \|\hat{f}\|_{G\zeta, S(d-1)} \leq 1 \times \|f\| G\psi, \]
where the constant ”1” is the best possible.

The non-improvability for this constant is grounded in particular in [27].

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