Spreading for the generalized nonlinear Schrödinger equation with disorder

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Abstract

The dynamics of an initially localized wavepacket is studied for the generalized nonlinear Schrödinger Equation with a random potential, where the nonlinearity term is $|\psi|^p \psi$ and $p$ is arbitrary. Mainly short times for which the numerical calculations can be performed accurately are considered. Long time calculations are presented as well. In particular the subdiffusive behavior where the average second moment of the wavepacket is of the form $\langle m_2 \rangle \approx t^a$ is computed. Contrary to former heuristic arguments, no evidence for any critical behavior as function of $p$ is found. The properties of $\alpha(t)$ are explored.
We consider the discrete nonlinear Schrödinger Equation (NLSE) with a random potential in one dimension:

\[ i \frac{\partial \psi_n}{\partial t} = -\psi_{n+1} - \psi_{n-1} + \varepsilon_n \psi_n + \beta |\psi_n|^p \psi_n \]  

(1)

where \( \varepsilon_n \) are i.i.d. random variables uniformly distributed in the interval \([-\frac{W}{2}, \frac{W}{2}]\), \( p \) is the degree of nonlinearity and \( \beta \) is its strength. For \( \beta = 0 \) this equation reduces to the Anderson model where all the states are exponentially localized [7]. Consequently, for \( \beta = 0 \) if one starts with a localized wavepacket it will not spread indefinitely. In the absence of a random potential spreading takes place for all \( p \) [14]. In fact the continuous version of (1) for \( p = 2 \) and without the disorder is integrable [14]. The case of \( p = 2 \) is of experimental relevance in classical optics [11] and in the field of Bose-Einstein condensates, where the NLSE is known as the Gross-Pitaevskii equation [2, 10]. This equation was studied extensively in the recent years, mainly for \( p = 2 \). In particular, the growth of the second moment was explored and it was found (numerically) to grow subdiffusively [4, 9, 13], namely, for a particle initially at \( n = 0 \),

\[ \langle m_2(t) \rangle = Dt^\alpha \]  

(2)

where \( m_2 = \sum_n n^2 |\psi_n|^2 \) and \( \alpha \) was found to be \( \alpha \approx 0.33 \) (for \( p = 2 \)). The average \( \langle ... \rangle \) denotes an average over the realizations of the random potential.

The analytical and intuitive understanding of (1) is quite poor. The simplest intuitive argument is that if a wavepacket spreads for long enough time, the amplitude of each state becomes negligible (since the norm, \( \sum_n |\psi_n|^2 = 1 \), is conserved) and as a result the nonlinear term weakens and becomes irrelevant, consequently, localization takes place. The difficulty with this argument (in addition with the fact that it disagrees with numerical results [4, 9, 13]) is that although absolute value of the nonlinear term becomes smaller it should be compared to an energy scale that may decrease as well. For \( p = 2 \) such an argument was developed by Pikovsky and Shepelyansky [9] (It is very similar to an argument that was found to work remarkably well for another system [12]). We generalize this argument to an arbitrary value of \( p \). Assuming that after some time the packet \( \psi \) is spread over \( \Delta n \) states while its norm is preserved, than typically \( |\psi_n|^2 \approx \frac{1}{\Delta n} \) and therefore the nonlinear term produces an energy shift of the order \( \delta E = \beta |\psi_n|^p \) that is of the order of \( \delta E \approx \Delta n^{-\frac{p-2}{2}} \).

Comparing this term with the typical distance between the energies of the linear problem, \( \Delta E \approx \frac{1}{\Delta n} \), gives \( \frac{\delta E}{\Delta E} \approx \beta \Delta n^{-\frac{p-2}{2}} \). Based on this argument, Pikovsky and Shepelyansky that
were interested in the case $p = 2$, where $\frac{\delta E}{\Delta E} \approx \beta$ concluded that there is a critical value denoted by $\beta_c$ such that for $\beta < \beta_c$ the nonlinear term is negligible compared to the level spacings of the linear problem and therefore Anderson localization holds. For $\beta > \beta_c$ the levels of the linear problem are mixed and presumably Anderson localization breaks down and spreading takes place. From this argument it turns out that $p = 2$ is a critical degree of nonlinearity and for $p > 2$, $\frac{\delta E}{\Delta E} \to 0$ as $\Delta n$ grows and localization holds. Existence of a critical value of $p$ was not considered in [9] since only the case $p = 2$ was studied. Also for the nonlinear Schrödinger equation without disorder, $p = 2$ has a critical meaning [1, 6]. In the present paper no evidence for the criticality at $p = 2$ was found. This leads one to question the validity of the arguments implying the criticality of $p = 2$ for spreading. Recently, Flach, Krimer and Skokos presented arguments that $\alpha = \frac{1}{p+1}$ and there is no critical value of $\beta$ or $p$ [4, 5, 13]. Their arguments are supported by some numerical calculations. It is unclear how $\alpha$ should behave when the limit $p \to 0$ is taken since in this limit localization takes place and one expects $\alpha = 0$. This is another motivation for the present work. Some arguments presented in [4, 9, 13] involve assumptions on chaoticity of various modes. The present work does not test these assumptions.

There are conjectures based on perturbation theory [3] and rigorous results [15] claiming that asymptotically the second moment of the wave packet cannot grow faster than logarithmically as a function of time. Nevertheless, numerical data predicts a power law growth of the second moment. If we trust the conjectures (their violation will be very surprising and of great interest) it is reasonable that the available numerical data is either not asymptotic (the time scale of this problem is unknown and therefore also the time when the system enters the asymptotic regime) or not reliable due to computational errors. Considering this, we concentrate on the short time behavior of a wave packet. Our results for the long time behavior are also presented for completeness.

In order to follow the dynamics of a wave packet, we use the SABA algorithm, which belongs to the family of split step algorithms and evaluates the wave packet in small steps, changing from coordinate space to momentum space. We apply the disorder and nonlinear interaction in the coordinates space, transform the wave to momentum space and apply there the kinetic energy term, transform it back to the coordinate space and so on. Nearly all numerical calculations for this problem use such methods. Additional details on the SABA algorithm, can be found in reference [13]. Like any numerical algorithm, the SABA algorithm
accumulates errors during the calculation which grow with the time of the integration. We use two criteria to determine whether our results are reliable or not: (t1) time reversal and (t2) comparison with data which is obtained using smaller time steps. Time reversal means integrating \[ I \] from time 0 to some later time and then integrating back to time 0. At the end of this process (if there are no errors) we should get the initial wave packet. To measure the accumulated errors, we define \[ \delta_{tr} = \sum_n |\psi_{initial} - \psi_{reversed}| \] and demand \( \delta_{tr} < 0.1 \). The comparison with smaller time step is done as follows: we calculate the second moment \( m_2(t) \) for representative realizations and then recalculate it using smaller time step (half of the original one). We define

\[
\delta_{m_2} = \frac{1}{T} \int \left| \frac{m_{2,dt} - m_{2,\frac{1}{2}dt}}{m_{2,\frac{1}{2}dt}} \right| dt
\]

and demand \( \delta_{m_2} < 0.01 \). Nearly all published numerical calculations used a more relaxed test: (t3) where in (t2) \( m_2 \) is replaced by the average over realizations.

We calculate \( m_2 \) for various values of \( \beta \) and \( p \) and average over 5,000 realizations until time 1000. We verified that (t1) and (t2) are satisfied for representative realizations. We use time steps of 0.1, 0.02, 0.01 and 0.00025 for \( \beta = 0.25, 0.5, 0.75 \) and 1, respectively, that were chosen to satisfy (t1) and (t2). In addition, data is presented for \( \beta = 2 \) and 4 using time steps of 0.1 where (t1) and (t2) are not satisfied. The results are shown in Fig. 1b where \( \alpha \) is obtained from fits similar to the one presented in Fig. 1a. Only the data in the interval \( 500 \leq t \leq 1000 \), that does not involve the initial spread was used in the fit of \( \alpha \).

If we choose the time interval to be \( 300 \leq t \leq 1000 \) or \( 800 \leq t \leq 1000 \), our results do not change in a significant way. As we could expect, \( \alpha (p \to 0) \to 0 \) and when \( p \) is large, \( \alpha \) is very small. The maximal \( \alpha \) is obtained for \( p \approx \frac{1}{2} \) and nothing special happens for \( p = 2 \). We do not see any discontinuity for \( p \to 0 \). All the lines in Fig. 1b have similar shape and after forming linear transformations \( \alpha = c_1 \alpha + c_2 \) where \( c_1 \) and \( c_2 \) are independent of \( \alpha \) and \( p \) and depend only on \( \beta \), all the lines approximately coincide as shown in Fig. 2, leading us to the conclusion that there might be some scaling property.

In our short time runs \( (t \leq 1000) \) the wavepacket didn’t spread over many sites. In the case of maximal spreading \( (\beta = 4, p = \frac{1}{2}) \) the second moment reached to a maximal value of 150 and for the parameters \( \beta = 1, p = 2 \) the second moment was smaller then 55, while the localization length is about 6. When we follow the dynamics for longer times (for which (t1) and (t2) are not satisfied but (t3) is satisfied) the results support our previous conclusion.
Figure 1: (a) $\langle m_2 \rangle$ for $\beta = 1$ and $p = 2$ as a function of time. The blue solid curve is the second moment calculated numerically and the green dashed line is the fit which we use in order to find $\alpha$. (b) $\alpha(p)$ for different values of $\beta$. From top to the bottom: $\beta = 4, 2, 1, 0.75, 0.5, 0.25$ (yellow stars, purple triangles, turquoise asterisks, red circles, green squares and blue diamonds). Only the data for $\beta = 0.25, 0.5, 0.75$ and 1 where points are connected by lines satisfy (t1) and (t2). For all realizations, $W = 4$ and maximal localization length is of 6 lattice sites. At the initial time the wavepacket populates one site ($n = 0$).

that nothing critical happens for $p = 2$. We see that the wave packet spreads for all powers of nonlinearity $p$ in a similar way, as shown in Fig. 3 for $p = 0, 1.5, 2, 2.5, 4$ and 8. Similar results are found in detailed studies of Mulansky [8].

In conclusion, we have found that for short times, there is no evidence of any critical phenomena neither for $p = 0$ nor any $p = 2$. This conclusion is supported by long time calculations. In addition, we found that $\alpha$ has a maximum for $p = \frac{1}{2}$ and there is evidence for scaling (Fig. 2). Understanding the physics of the $\alpha(p)$ plots, explaining why is the maximal spreading obtained for $p = \frac{1}{2}$, explaining of the origin of the scaling and finding the asymptotic behavior of $\alpha(p)$ are left for future research.
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Figure 3: $m^2(t)$ for a representative realization as a function of time for $\beta = 1$ and $W = 4$. From top to bottom: $p = 1.5, 2, 2.5, 4, 8, 0$ (green, red, turquoise, purple, yellow and blue).

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