Kerr black hole as a quantum rotator

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Abstract

It has been proposed by Bekenstein and others that the horizon area of a
black hole conforms, upon quantization, to a discrete and uniformly spaced
spectrum. In this paper, we consider the area spectrum for the highly non-
trivial case of a rotating (Kerr) black hole solution. Following a prior work
by Barvinsky, Das and Kunstatter, we are able to express the area spectrum
in terms of an integer-valued quantum number and an angular-momentum
operator. Moreover, by using an analogy between the Kerr black hole and a
quantum rotator, we are able to quantize the angular-momentum sector. We
find the area spectrum to be \( A_{n,J_{cl}} = 8\pi\hbar(n + J_{cl} + 1/2) \), where \( n \) and \( J_{cl} \)
are both integers. The quantum number \( J_{cl} \) is related to but distinct from
the eigenvalue \( j \) of the angular momentum of the black hole. Actually, it
represents the “classical” angular momentum and, for \( J_{cl} \gg 1 \), \( J_{cl} \approx j \).

I. INTRODUCTION

As is well known since the early seventies, black holes behave dynamically as thermodynamic systems. In particular, the surface area (\( A \)) of the horizon plays the role of the entropy (\( S \)) and the surface gravity (\( \kappa \)) at the horizon serves as the temperature (\( T \)); that is:

\[
S = \frac{A}{4l_p^2} \quad \text{and} \quad T = \frac{\kappa}{2\pi},
\]

(Here and throughout, the spacetime dimensionality is four, \( l_p^2 \sim \hbar \) is the Planck constant
and the fundamental constants \( c, G, k_B \) have been set equal to unity.) Thanks to Hawking’s
discovery that quantum black holes radiate at precisely the above value of temperature, this thermodynamic analogy has since been elevated to the status of a physical theory.

One of the outstanding open questions in gravitational theory is the microscopic origin
of this thermodynamic behavior. In all likelihood, such a question can only be truly resolved

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in the context of a quantum theory of gravity; a theory for which our understanding is conspicuously incomplete. Nonetheless, there are still fundamental issues that can be addressed even in the absence of the full-fledged quantum theory. One such question is what is the quantum spectrum of the black hole observables?

That the black hole horizon area, in particular, should be quantized was first argued for by Bekenstein [4] (also see [3,2]). The support for this argument comes from the observation that $A$ behaves, for a slowly changing black hole, as an adiabatic invariant [7]. It is significant that, as Bekenstein pointed out, a classical adiabatic invariant corresponds to a quantum observable with a discrete spectrum, by virtue of Ehrenfest’s principle.

On quite general grounds, Bekenstein has suggested the following explicit form for the area spectrum [4,5]:

$$A = \epsilon l_p^2 n, \quad n = 0, 1, 2, ...$$

where $\epsilon$ is a numerical factor of the order unity. (Note that a non-vanishing but positive zero-point term may also be considered.) The crucial point in this formulation is the equal spacing between the levels. This can be viewed as a consequence of the uncertainty principle, as a quantum point particle cannot be localized better than one Compton length, and this naturally leads to a minimal increase in the horizon area of $(\Delta A)_{\text{min}} = \epsilon l_p^2 [4,6]$.

Since the original heuristic arguments of Bekenstein, there has been a substantial amount of work in trying to derive the spectrum (2) by more rigorous means (see [9] for a list of relevant references). An example of a more rigorous proof of the equally spaced area spectrum, as well as the degeneracy of the area levels, can be found in the algebraic approach to black hole quantization [8,9,10].

Of particular relevance to the upcoming analysis is a program that was initiated by Barvinsky and Kunstatter [10]. Their methodology is based on expressing the black hole dynamics in terms of a reduced phase space [1] and then applying an appropriate process of quantization. For a static, uncharged black hole, this phase space consists of only the black hole mass observable and its canonical conjugate [12,13]. (This simplicity can be viewed as a manifestation of either Birkhoff’s theorem [14] or the “no-hair” principles of black holes [15].) One vital assumption was required in this analysis; namely, the authors assumed that the conjugate to the mass is periodic over an interval of $2\pi/\kappa$. They did, however, justify this input by way of Euclidean considerations. (We elaborate on the logistics of this point later on in the paper.) Ultimately, the area spectrum (2) was indeed reproduced with the particular value of $\epsilon = 8\pi$ (and a zero-point contribution of $4\pi l_p^2$).

The general procedure of [11] was later extended by Barvinsky, Das and Kunstatter to the case of a charged but still static black hole [16] (also see [17]). In this case, the reduced phase space now consists of the two relevant observables (the mass and the charge, $Q$) and their respective conjugates [18]. Assuming the same periodicity condition as before, the authors found the following for the area spectrum:

1To achieve the desired form of phase space, one requires a midisuperspace type of approximation - for instance, by imposing spherical symmetry - so as to sufficiently reduce the number of black hole degrees of freedom.
where \( A_{\text{ext}}(Q) \) is the extremal value of the horizon area (expressed as a function of the charge). Significantly, this extremal value represents, for a given value of \( Q \), a lower bound on the horizon area of a classical black hole. Note, however, that because of the zero-point term in Eq. (3), the quantum black hole can not approach this extremal value. (The authors of [16] attributed this censoring feature to the effects of quantum fluctuations.)

Barvinsky et al went on to quantize the charge sector of the theory and ultimately found that [16]

\[
A = 8\pi l_p^2 \left(n + \frac{p}{2} + \frac{1}{2}\right), \quad n, p = 0, 1, 2, ..., \quad (4)
\]

where the “new” quantum number \( p \) is related to the black hole charge according to \( Q^2 = \hbar p \).

The objective of the current paper is to further extend the above program to the case of a rotating black hole. (We will be assuming, for sake of simplicity, an uncharged black hole and always a four-dimensional spacetime.) This seems an *a priori* difficult task, given that there is no rigorous evidence that a rotating black hole can be described by an analogously simple form of reduced phase space. Nonetheless, we argue that, on the basis of the “no-hair” principles [15], that this should indeed be the case, with the relevant observables in the phase space now being the mass and an angular-momentum vector. The latter inclusion necessitates six additional degrees of freedom; for instance, the three Cartesian components of the angular momentum and their respective conjugates. (However, it will be shown later that the choice of Cartesian components is inappropriate and we will work, instead, with the Euler components as the initial basis.) Let us emphasize that this conjectural form of reduced phase space and the periodicity constraint on the conjugate to the mass [11] are the *only* assumptions used in the following analysis. (Also note that, later on, we will provide additional, independent support for this periodicity constraint.)

Before discussing the contents of this paper, let us point out that the area spectrum of a rotating black hole has recently been considered by Makela et al [19] (also see [20] for earlier studies on static black holes). Their approach, which differs substantially from that of Barvinsky et al, is based on formulating a Schrödinger-like equation for the black hole observables and quantizing this equation via a WKB analysis. Even without bringing rotation into the discussion, the results of [19] are somewhat different than those discussed above. For instance, the spacing between levels was found to be \( \epsilon = 32\pi \) (translated to our notation), and the quantity being quantized is not \( A - A_{\text{ext}} \) but rather \( A + A_- \) (where \( A_- \) represents the area of the inner black hole horizon). This latter distinction makes a direct comparison between the two approaches rather non-trivial.

The remainder of the paper is organized as follows. In the next section, we consider some relevant properties, at the classical level, of a rotating (Kerr) black hole. We then propose a reduced phase space and transform it into a form that is suitable for the subsequent analysis.

Note that a charged or rotating black hole typically has a pair of distinct horizons, with their coincidence determining the point of extremality. Further note that, throughout this paper, an unqualified \( A \) always signifies the area of the outermost horizon.
quantum analysis. In Section 3, following the general methodology of Barvinsky et al. [16], we are able to quantize the reduced phase space. This eventually yields an expression for the area spectrum in a form which is analogous to that of Eq.(3). In Section 4, we focus on the angular-momentum sector, and demonstrate that the spin eigenvalues are necessarily restricted to taking on integer values. In this way, we are able to derive an explicit, unambiguous form of the area spectrum, which is clearly evenly spaced and behaves as intuitively expected in the limiting cases of interest. The final section contains a summary.

II. CLASSICAL ANALYSIS

Let us begin here by considering the physically relevant model of interest; namely, a four-dimensional spacetime containing a rotating black hole. In this analysis, we will focus on the Kerr black hole, which may be regarded as the most general solution of the vacuum Einstein equations with vanishing electrostatic charge. In this particular section, considerations will be restricted to the classical level.

Thanks to the “no-hair” principles of black holes [15], we are safe in assuming that an external observer can describe the system strictly in terms of a few macroscopic parameters; in particular, the black hole mass, $M$, and an angular momentum, $J_{cl}$. Moreover, the well-known first law of black hole mechanics [1,2] relates these quantities in the following manner:

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ_{cl}. \quad (5)$$

Here, $A$ is the (outermost) horizon area, $\kappa$ is the surface gravity at this horizon, $\Omega$ is the angular velocity of the black hole, and $J_{cl} = |\vec{J}_{cl}|$ is the magnitude of the angular-momentum vector.

For the black hole of interest, the above quantities are explicitly known [14]:

$$A = 8\pi M \left[ M + \sqrt{M^2 - \frac{J_{cl}^2}{M^2}} \right] \quad (6)$$

or equivalently:

$$M^2 = \frac{A}{16\pi} + 4\pi \frac{J_{cl}^2}{A}, \quad (7)$$

and:

$$\kappa = 8\pi \left. \frac{\partial M}{\partial A} \right|_{J_{cl}} = \frac{1}{4M} - 16\pi^2 \frac{J_{cl}^2}{MA^2}, \quad (8)$$

3We include a subscript on this classical form of the angular momentum so as to avoid confusion in the later analysis.
\[ \Omega = \left. \frac{\partial M}{\partial J} \right|_A = 4\pi \frac{J_{cl}}{AM}. \]  

Extrapolating the well-understood dynamics of static black holes [12,13,18], we will assume that any classical black hole can be described (by an external observer) in terms of a reduced phase space consisting of the physical observables and their respective canonical conjugates. (For a relevant discussion in the context of rotating black holes, see [19].) Focusing on the current scenario, one might be inclined to describe the reduced phase space in terms of \( M, J_x, J_y \) and \( J_z \) (where \( J_x, \text{etc.} \) are the usual angular-momentum components in Cartesian coordinates). However, these variables are actually a poor choice because of their failure to commute (in terms of Poisson brackets). Therefore, the set \( M, J_x, J_y \) and \( J_z \) cannot be considered as a set of generalized coordinates. We can, however, rectify this situation by alternatively considering the Euler components [21] of the angular momentum:

\[ J_\alpha, J_\beta, J_\gamma, \]  

along with their respective conjugates, the three Euler angles, \( \alpha, \beta \) and \( \gamma \). The Cartesian components of the angular momentum can be written in terms of the Euler components [21]:

\[ J_x = - \cos \alpha \cot \beta J_\alpha - \sin \alpha J_\beta + \frac{\cos \alpha}{\sin \beta} J_\gamma \]
\[ J_y = - \sin \alpha \cot \beta J_\alpha + \cos \alpha J_\beta + \frac{\sin \alpha}{\sin \beta} J_\gamma \]
\[ J_z = J_\alpha \]  

If we adopt the common-sense assumption that the horizon area is invariant under rotation, it is clear that

\[ A, J_\alpha, J_\beta, J_\gamma, \]  
\[ P_A, \alpha, \beta, \gamma, \]

forms the desired set of generalized (commuting) coordinates (12) and their canonical conjugates (13). However, we would like to work with a set that includes \( M \) because, later on, the periodicity of its conjugate, \( P_M \), will be exploited in order to obtain the area spectrum.

The set

\[ M, J_\alpha, J_\beta, J_\gamma, \]

on the other hand, is a poor choice because

\[ \{ M, J_\beta \} \neq 0, \]  

where \( \{, \} \) denotes a commutator (Poisson) bracket in the Dirac sense [22]. To prove Eq.(14), it is enough to show that \( J_{cl} \) does not commute with \( J_\beta \) (cf, Eq.(7)). This can, in fact, be seen from the explicit expression for \( J_{cl} \):

\[ \text{The derivatives are taken with respect to the generalized coordinates in (12) and their canonical conjugates (13).} \]
\[ J_{cl}^2 = J_x^2 + J_y^2 + J_z^2 \]
\[ = \frac{1}{\sin^2 \beta} \left[ J_\alpha^2 + J_\gamma^2 - 2 \cos \beta J_\alpha J_\gamma \right] + J_\beta^2, \tag{15} \]

where, in this section, we treat \( J_\alpha, J_\beta, J_\gamma \) as classical (i.e., non-operating) quantities. Note the presence of \( \beta \) in the above relation, as this clearly demonstrates that \( \{J_{cl}, J_\beta\} \neq 0 \).

Eq. (13) also shows that \( J_{cl} \) commutes with both \( J_\alpha \) and \( J_\gamma \). This prompts us to introduce a new set of variables:

\[ M = M(A, J_{cl}, J_\alpha, J_\gamma), \tag{16} \]

along with their \textit{hypothetical} conjugates:

\[ \Pi_A, \Pi_{J_{cl}}, \Pi_\alpha, \Pi_\gamma. \tag{17} \]

At this point, we use the qualifier “hypothetical”, as it is not \textit{a priori} clear that there exists a transformation from Eqs. (12, 13) to Eqs. (16, 17) that is truly canonical. To be explicit, such a transformation requires that

\[ \{M, P_M\} = \{J_{cl}, P_{cl}\} = \{J_\alpha, P_\alpha\} = \{J_\gamma, P_\gamma\} = 1, \tag{18} \]
\[ \{\text{all other combinations}\} = 0, \tag{19} \]

where for arbitrary \( \mu \) and \( \nu \):

\[ \{\mu, \nu\} = \frac{\partial \mu}{\partial A} \frac{\partial \nu}{\partial P_A} - \frac{\partial \mu}{\partial P_A} \frac{\partial \nu}{\partial A} + \frac{\partial \mu}{\partial J_\alpha} \frac{\partial \nu}{\partial J_\alpha} - \frac{\partial \mu}{\partial J_\alpha} \frac{\partial \nu}{\partial J_\alpha} + \frac{\partial \mu}{\partial J_\beta} \frac{\partial \nu}{\partial J_\beta} - \frac{\partial \mu}{\partial J_\beta} \frac{\partial \nu}{\partial J_\beta} + \frac{\partial \mu}{\partial J_\gamma} \frac{\partial \nu}{\partial J_\gamma} - \frac{\partial \mu}{\partial J_\gamma} \frac{\partial \nu}{\partial J_\gamma}. \tag{20} \]

As it so happens, the canonical transformation in question does indeed exist, as can be shown in two steps. First, we make a \textit{canonical} transformation from Eqs. (12, 13) to the set:

\[ A, J_{cl}, J_\alpha, J_\gamma, \tag{21} \]
\[ P_A, P_{cl}, P_\alpha, P_\gamma, \tag{22} \]

where we have exchanged \( J_\beta \) with \( J_{cl} \). Then, after some lengthy but straightforward calculations, one can verify that Eqs. (18, 19) are consistently satisfied with the following conjugates:

\[ \Pi_A = \frac{8\pi}{\kappa} P_A, \tag{23} \]
\[ \Pi_{cl} = -\frac{8\pi}{\kappa} \Omega P_A + P_{cl}, \tag{24} \]
\[ \Pi_\alpha = P_\alpha, \tag{25} \]
\[ \Pi_\gamma = P_\gamma. \tag{26} \]
III. QUANTIZING THE AREA

With the black hole mass ($M$) and its conjugate ($\Pi_M$) contained within the reduced phase space, we are now well positioned to begin a process of quantization in the manner of Barvinsky et al.\[16\]. In following the prescribed methodology, we must necessarily invoke the following condition of periodicity:

$$\Pi_M = \Pi_M + \frac{2\pi}{\kappa}.$$  \hspace{1cm} (27)

Although an assumption, this condition follows quite naturally from a pair of observations. (i) The conjugate to the mass, $\Pi_M$, can be identified with the time separation at infinity \[13\]; that is, $\Pi_M$ directly measures the difference in Schwarzschild-like time between the ends of a spacelike slice that extends across the relevant Kruskal diagram. (ii) In the Euclidean (or imaginary time) sector of a black hole spacetime, the Schwarzschild-like time is periodic \[23\], with the period given precisely by $2\pi/\kappa$.

At least naively, these two observations, when take together, suggest that $\Pi_M$ should be constrained with the specified periodicity. On the other hand, the first observation follows from a purely Lorentzian perspective (Kruskal coordinates extend over the entire Lorentzian spacetime, whereas Euclidean coordinates reduce the black hole interior to a single point), and so it is unclear if (i) can be translated into the Euclidean framework of (ii). For this reason, the above condition should, at this point, be regarded as a well-motivated but conjectural input. For further justification and related discussion, see \[16\] (especially, pages 15-16 in the archival version). We also provide, in the next section, an independent argument that further substantiates the validity of Eq. (27).

Again following \[16\], let us now introduce a new pair of variables that directly incorporate the periodic nature of $\Pi_M$:

$$X = \sqrt{\frac{B(M, J_d, J_\alpha, J_\gamma)}{\pi}} \cos(\Pi_M \kappa),$$  \hspace{1cm} (28)

$$P_X = \sqrt{\frac{B(M, J_d, J_\alpha, J_\gamma)}{\pi}} \sin(\Pi_M \kappa).$$  \hspace{1cm} (29)

Here, we have included a yet-to-be-determined function, $B$, of the phase-space observables.\[5\] The underlying premise is that $B$ can be (at least partially) fixed with the constraint that Eqs. (16,17) transform canonically into the set of observables:

$$X, J_d, J_\alpha, J_\gamma$$  \hspace{1cm} (30)

and their conjugates:

$$P_X, P_d, P_\alpha, P_\gamma.$$  \hspace{1cm} (31)

Note that, as written above, $B$ has units of area; that is, $B \sim h$.\[5\]
With the above in mind, let us consider the following necessary and sufficient condition for a canonical transformation:

\[ \mathcal{P}_X \delta X + \mathcal{P}_{cl} \delta J_{cl} + \mathcal{P}_\alpha \delta J_\alpha + \mathcal{P}_\gamma \delta J_\gamma = \Pi_M \delta M + \Pi_{cl} \delta J_{cl} + \Pi_\alpha \delta J_\alpha + \Pi_\gamma \delta J_\gamma. \]  

(32)

Up to a total variation, it can be shown that

\[ \mathcal{P}_X \delta X = \frac{\kappa \Pi_M}{2\pi} \left[ \frac{\partial B}{\partial M} \delta M + \frac{\partial B}{\partial J_{cl}} \delta J_{cl} + \frac{\partial B}{\partial J_\alpha} \delta J_\alpha + \frac{\partial B}{\partial J_\gamma} \delta J_\gamma \right]. \]  

(33)

Substituting Eq.(33) into Eq.(32), we are then able to deduce the following:

\[ \frac{\partial B}{\partial M} = \frac{2\pi}{\kappa}, \]  

(34)

\[ \frac{\partial B}{\partial J_{cl}} = \frac{2\pi}{\kappa \Pi_M} (\Pi_{cl} - \mathcal{P}_{cl}), \]  

(35)

\[ \frac{\partial B}{\partial J_\alpha} = \frac{2\pi}{\kappa \Pi_M} (\Pi_\alpha - \mathcal{P}_\alpha), \]  

(36)

\[ \frac{\partial B}{\partial J_\gamma} = \frac{2\pi}{\kappa \Pi_M} (\Pi_\gamma - \mathcal{P}_\gamma). \]  

(37)

It is informative to compare Eq.(34) with Eq.(8), which immediately indicates that \( \partial A/\partial M = 4 \partial B/\partial M \). Hence, we can write

\[ B(M, J_{cl}, J_\alpha, J_\gamma) = \frac{1}{4} A(M, J_{cl}) + F(J_{cl}, J_\alpha, J_\gamma), \]  

(38)

where \( F \) is an essentially arbitrary function of the angular momentum. That is to say, for any well-behaved choice of \( F \), one will always be able to find expressions for \( \mathcal{P}_{cl}, \mathcal{P}_\alpha \) and \( \mathcal{P}_\gamma \) that satisfy Eqs.(35-37).

In spite of this freedom in choosing \( F \), there is only one particular form that will be useful for the quantization of the area \( A \). First, it is relevant that, regardless of the choice of \( F \), the function \( B \) is bounded from below. This follows from the lower bound that exists on the area, \( A \). To be precise, for a rotating black hole, \( A \) can not, classically, fall below its extremal value.\(^6\) This occurs when \( M^2 = J_{cl} \) (cf. Eq.(6)), and so:

\[ A \geq A_{ext} = 8\pi J_{cl}. \]  

(39)

As elaborated on below, it turns out to be convenient if \( F \) is chosen so that Eq.(38) translates into \( B \geq 0 \). Following this prescription, we can unambiguously set \( F = -8\pi J_{cl}/4 \) and thus obtain

\[^{6}\text{This realization follows from the censorship of naked singularities, which is usually assumed to be the case \cite{[14]}.}\]
\[ B = \frac{1}{4} [A(M, J_{cl}) - 8\pi J_{cl}] . \]  

Let us now recall Eqs.\,(28,29), which can be squared and summed to yield \( B = \pi (X^2 + P_X^2) \). Hence, Eq.\,(40) can be suggestively re-expressed as follows:

\[ X^2 + P_X^2 = \frac{1}{4\pi} [A(M, J_{cl}) - 8\pi J_{cl}] \geq 0. \]  

In this way, we have mapped the mass and its conjugate, \( M \) and \( \Pi_M \), into a complete two-dimensional plane, \( X \) and \( P_X \). Any other choice of \( F \) would have left a “hole” in this plane and complicated the prospective quantization with the need for non-trivial boundary conditions.

Next, let us elevate any classically defined quantity in Eq.\,(41) to the status of a quantum operator. Adopting the conventional “hat” notation, we then have

\[ \frac{1}{2\pi} \hat{B} = \frac{1}{8\pi} [\hat{A} - 8\pi \hat{J}_{cl}] = \frac{\hat{X}^2}{2} + \frac{\hat{P}_X^2}{2}. \]  

Since the domain of \( \hat{X} \) and \( \hat{P}_X \) is an entire two-dimensional plane, the quantization of the right-hand side becomes trivial. Indeed, the spectrum is readily identifiable with that of a harmonic oscillator, and so:

\[ \frac{1}{2\pi} B_n = 8\pi \hbar \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots, \]  

where \( B_n \) are the eigenstates of the operator \( \hat{B} \).

Our task is not, of course, complete until the spectra for \( \hat{A} \) and \( \hat{J}_{cl} \) have been explicitly separated. There is, however, an interesting observation that can be made without any further analysis. Namely, we can see from Eq.\,(43) that quantum fluctuations will always prevent the rotating black hole from ever reaching a precise state of extremality (since the right-hand side can never quite vanish). This result can best be viewed as a quantum black hole version of the third law of thermodynamics. Note that a similar observation was also made for charged (non-rotating) black holes in the prior work of Barvinsky et al \cite{16}.

\section*{IV. QUANTIZING THE ANGULAR MOMENTUM}

Since our principle objective is to find the area spectrum for a rotating black hole, the preceding outcome \cite{13} emphasizes the importance in knowing the spectrum of \( \hat{J}_{cl} \). Fortunately, it turns out that the spectrum of \( \hat{J}_{cl} \) can be obtained by way of some simple calculations.

To proceed in the stated direction, let us first take note of the operator form of this angular momentum (cf, Eq.\,(15)):

\[ \hat{J}_{cl}^2 = \frac{1}{\sin^2 \beta} \left[ \hat{J}_\alpha^2 + \hat{J}_\gamma^2 - 2 \cos \beta \hat{J}_\alpha \hat{J}_\gamma \right] + \hat{J}_\beta^2. \]  

In the above, the order of the operators in each element is not important because \( \beta, \hat{J}_\alpha \) and \( \hat{J}_\gamma \) all commute with each other. Therefore, the transition from the classical \( J_{cl} \) to the
quantum \( \hat{J}_d \) is well defined. Furthermore, it should be kept in mind that this expression is obtained by first summing the squares of the individual components \((J_x, J_y, J_z)\) and then quantizing. This makes \( \hat{J}_d \) distinct from the “traditional” quantum operator, \( \hat{J} \), which is obtained by first quantizing the components and then summing the squares. The importance of this distinction will become evident below.

In order to work with the quantum Euler components of the angular momentum, we will employ the usual identification of the operators, \( \hat{J}_\eta = -i\hbar \frac{\partial}{\partial \eta} \) (for any component \( \eta = \alpha, \beta \) or \( \gamma \)). Usefully, it can be shown [21] that Eq.(11) remains valid when the classical components of the angular momentum are replaced by these quantum operators. Therefore, the square of the conventional angular momentum is given by

\[
\hat{J}_2^d = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \frac{1}{\sin^2 \beta} \left[ \hat{J}_\alpha^2 + \hat{J}_\gamma^2 - 2 \cos \beta \hat{J}_\alpha \hat{J}_\gamma \right] + \hat{J}_\beta^2 - i\hbar \cot \beta \hat{J}_\beta. \tag{45}
\]

That is (cf, Eq.(44)):

\[
\hat{J}_2^d - \hat{J}_2^d = \hbar^2 \cot \beta \frac{\partial}{\partial \beta}. \tag{46}
\]

Hence, the spectrum of \( \hat{J}_2^d \) must be different than \( \hbar^2 j(j + 1) \).

Since both \( \hat{J} \) and \( \hat{J}_d \) commute with both of \( \hat{J}_\alpha \) and \( \hat{J}_\gamma \), there are two natural sets of angular-momentum eigenstates: the conventional set \(|j, m_\alpha, m_\gamma\rangle\) (where \( j = 0, 1/2, 1, ... \) and \( m_\alpha, m_\gamma = -j, -j + 1, ..., j \)) and \(|J_d, m_\alpha, m_\gamma\rangle\) (with \( \{J_d\} \) being the eigenvalues of \( \hat{J}_d \)). The first basis is able to diagonalize simultaneously \( \hat{J}, \hat{J}_\alpha \) and \( \hat{J}_\gamma \) and the second basis does likewise for \( \hat{J}_d, \hat{J}_\alpha \) and \( \hat{J}_\gamma \). Hence, we can write the eigenstate \(|J_d, m_\alpha, m_\gamma\rangle\) in terms of the eigenstates \(|j, m_\alpha, m_\gamma\rangle\):

\[
|J_d, m_\alpha, m_\gamma\rangle = \sum_j C_{j,J_d} |j, m_\alpha, m_\gamma\rangle, \tag{47}
\]

where \( C_{j,J_d} \) are complex coefficients that depend only on \( j \) and \( J_d \). To put it another way, any \(|J_d, m_\alpha, m_\gamma\rangle\) is a superposition of states \(|j, m_\alpha, m_\gamma\rangle\) with the same \( m_\alpha \) and \( m_\gamma \) but different \( j \).

Since we are only interested in the eigenvalues \( \{J_d\} \), let us restrict ourselves to the normalized eigenfunctions

\[
\Psi_{J_d,0,0}(\alpha, \beta, \gamma) \equiv \langle \alpha, \beta, \gamma | J_d, 0, 0 \rangle, \tag{48}
\]

where \( m_\alpha \) and \( m_\gamma \) have been set to zero for convenience. This enables us to write

\[
\hat{J}_d^2 \Psi_{J_d,0,0} = \hat{J}_\beta^2 \Psi_{J_d,0,0} = -\hbar^2 \frac{\partial^2}{\partial \beta^2} \Psi_{J_d,0,0}, \tag{49}
\]

\footnote{Note that the degeneracy of the angular momentum is \((2j + 1)^2\), just as it appears in quantum rotators [21].}
where Eq.(44) has also been incorporated. Moreover, since $\Psi_{J_d,0,0}$ is an eigenfunction of $\hat{J}_d$, it follows that

$$-\frac{\partial^2}{\partial \beta^2} \Psi_{J_d,0,0} = J_d^2 \Psi_{J_d,0,0}. \quad (50)$$

Inspecting the above equation, we are able to deduce the following:

$$\Psi_{J_d,0,0} \sim \cos(J_d \beta) \quad \text{with} \quad J_d = 0, 1, 2, \ldots$$
$$\Psi_{J_d,0,0} \sim \sin(J_d \beta) \quad \text{with} \quad J_d = 1/2, 3/2, 5/2, \ldots \quad (51)$$

where the identification $\beta + \pi = \pi - \beta \quad [21]$ has been employed. It is, essentially, this identification of the Euler angel $\beta$ that constrains $J_d$ in the above manner. However, this is not yet the full story because, as stressed above, any state $|J_d\rangle$ can be written as a superposition of states $|j\rangle$ (with the other, redundant labels having been suppressed). It just so happens that $\Psi_{j,0,0}$ is a symmetric function of $\beta \quad [21]$ and, therefore, $\Psi_{J_d,0,0}$ must be as well. On this basis, we can discard the lower line in Eq.(51); thus restricting $J_d$ to strictly integer values. Moreover, we will find further support for this restriction below. (Also, one might intuitively argue that such an intrinsically classical form of angular momentum should be constrained in precisely this way.)

Combining the above outcome with Eqs.(42,43), we finally have an explicit expression for the area spectrum of a rotating black hole:

$$A_{n,J_d} = 8\pi \hbar \left(n + J_d + \frac{1}{2}\right), \quad n, J_d = 0, 1, 2, \ldots \quad (52)$$

This formulation for the area spectrum is the main result of the paper. Significantly, we have found the spectrum to be evenly spaced, with the importance of this feature having been stressed in the introductory section.

That the quantum number $J_d$ should be restricted to taking on integer values can also be seen, independently of the above considerations, by way of the following discussion. Before elaborating on the logistics, let us point out that the same argument will provide some further motivation for the periodicity conjecture of Eq.(27).

Firstly, it is useful to consider, in the coordinate representation with $\hat{J}_d = -i\hbar \partial/\partial P_d$, the wavefunctions for the angular-momentum eigenstates. That is:

$$\Psi_{J_d}(P_d) \sim \exp \left[iJ_d P_d\right], \quad (53)$$

where $J_d$ is, as before, the eigenvalue of $\hat{J}_d/\hbar$; however, for the moment, we are assuming no knowledge with regard to this spectrum. In view of this formulation, we can make the following identification:

$$J_d P_d \sim J_d P_d + 2\pi p, \quad (54)$$

where $p$ is an arbitrary integer.

Next, let us recall Eq.(41). Also employing the explicit form of $B \quad [10]$ and the first law of black hole mechanics \cite{11}, we can elegantly re-express this relation as follows:

$$P_d = \chi + \theta, \quad (55)$$
where we have defined $\chi \equiv \Pi_{cl} + \Omega \Pi_M$ and $\theta \equiv \kappa \Pi_M$. When $\chi$ is held constant, then Eqs. (54, 55) tell us that $\theta$ should be constrained according to:

$$J_{cl} \theta \sim J_{cl} \theta + 2\pi p; \quad (56)$$

that is, $J_{cl} \theta$ must be an angle. However, $\theta$ is, itself, an angle by hypothesis (cf., Eq. (27)); and so Eq. (56) really says that $J_{cl}$ must be strictly an integer, thus reconfirming our prior finding. Alternatively, we could have used the spectrum of $\hat{J}_{cl}$ and Eq. (56) to argue that $\theta$ should be an angle, thus supporting the periodicity constraint (27) via independent means.

Although our work here is essentially done, one important question remains: how does the “classical” spin eigenvalue, $J_{cl}$, relate to the more conventional spin eigenvalue, $j$? As will be shown below, $J_{cl} \approx j$ for $j \gg 1$.

To establish our claim, we begin by using Eq. (46) to evaluate $\langle j, m_\alpha, m_\gamma | \hat{J}_{cl}^2 - \hat{J}_{cl}^2 | j, m_\alpha, m_\gamma \rangle$. It can be seen from the inverted form of Eq. (47) that this expectation value is independent of both $m_\alpha$ and $m_\gamma$. Hence, we denote it by $\langle \hat{J}_{cl}^2 - \hat{J}_{cl}^2 \rangle_j$ and, without loss of generality, make the convenient choice of $m_\alpha = m_\gamma = j$, for which the wavefunction is known [21]:

$$\Psi_{j,j,j} = \frac{(i)^j}{2\pi} \sqrt{\frac{2j + 1}{2}} \cos^2j \left(\frac{\beta}{2}\right) \exp \left[-ij(\alpha + \gamma)\right] \quad (57)$$

and is normalized as follows:

$$\int_0^{2\pi} d\alpha \int_0^\pi d\beta \int_0^{2\pi} d\gamma \Psi^* \sin\beta \Psi = 1. \quad (58)$$

Directly applying the above formalism and Eq. (46), we obtain the following:

$$\langle \hat{J}_{cl}^2 - \hat{J}_{cl}^2 \rangle_j = \hbar^2 \int_0^{2\pi} d\alpha \int_0^\pi d\beta \int_0^{2\pi} d\gamma \Psi^* \cos\beta \frac{\partial}{\partial \beta} \Psi_{j,j,j}$$

$$= -\frac{\hbar^2}{2} \left(j - \frac{1}{2}\right). \quad (59)$$

Moreover, since $\langle \hat{J}_{cl}^2 \rangle_j = \hbar^2 j(j + 1)$, it follows that

$$\langle \hat{J}_{cl}^2 \rangle_j = \hbar^2 \left(j^2 + \frac{j}{2} + \frac{1}{4}\right) \sim j^2 + O[j]. \quad (60)$$

This means that, for the physically interesting case of $J_{cl} \gg 1$, we have $J_{cl} \sim j$ and the area spectrum (52) simplifies to

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8One might be concerned that we are treating $\chi$ and $\theta$ as independent variables, whereas both depend on the conjugate $\Pi_M$. However, $\chi$ also depends on a variable, $\Pi_{cl}$, which is clearly independent of $\Pi_M$. Hence, we can, without loss of generality, restrict ourselves to the case in which $\chi$ is held constant.
\[ A_{n,j} \sim 8\pi (n + j) . \] (61)

A related point of interest is the mass spectrum of the rotating black hole. In principle, this spectrum is obtainable by way of Eqs.(60). Here, we focus on the regime of large \( J_{cl} \sim J \) and take note of the following cases:

\[ < \hat{M} > \sim \frac{\sqrt{n}}{2} \text{ if } n >> J >> 1 , \] (62)

\[ < \hat{M} > \sim \sqrt{J} \text{ if } J >> n >> 1 , \] (63)

\[ < \hat{M} > \sim \frac{1}{2} \sqrt{5n} \text{ if } J \sim n >> 1 . \] (64)

Finally, let us consider the “inverse” of the calculation in Eq.(60); that is,

\[ \langle J_{cl}, m_\alpha, m_\gamma | \hat{J}^2 | J_{cl}, m_\alpha, m_\gamma \rangle \equiv < \hat{J}^2 >_{J_{cl}} . \]

It follows from prior considerations that this expectation value should indeed be independent of \( m_\alpha \) and \( m_\gamma \). Hence, we can make this evaluation for the particularly simple case of \( m_\alpha = m_\gamma = 0 \). Incorporating \( \Psi_{J_{cl},0,0} \sim \cos(J_{cl} \beta) \) (cf, Eq.(51)), into the same general framework as depicted in Eq.(59), we find that \(< \hat{J}^2_{cl} - \hat{J}^2 >_{J_{cl}} \) is identically vanishing. In view of this outcome, it directly follows that

\[ < \hat{J}^2 >_{J_{cl}} = < \hat{J}^2_{cl} >_{J_{cl}} = J^2_{cl} . \] (65)

Therefore, when the system is expressed in terms of the unorthodox (but completely legitimate) set of eigenstates \( |J_{cl}, m_\alpha, m_\gamma \rangle \), the operators \( \hat{J} \) and \( \hat{J}_{cl} \) are effectively indistinguishable.

V. CONCLUSION

In summary, we have studied the area spectrum of a rotating (Kerr) black hole in four dimensions of spacetime. Extending a treatment by Barvinsky et al. [16], we have demonstrated that the area spectrum is evenly spaced, as it depends exclusively on a pair of integer-valued quantum numbers. To quantize the spin sector, we have applied a novel approach that utilizes the Euler components of the classical angular momentum. We have shown that the operator form of this classical angular momentum - which is closely related to but nevertheless distinct from the “conventional” quantum spin operator - has a spectrum of eigenvalues that is restricted to integer values. We have shown that, when the angular momentum is large (as expected to be the case for a physically realistic black hole), this spectrum is in asymptotic agreement with the “usual” quantum spin number, \( j \). We have also demonstrated that quantum fluctuations prevent extremal black holes from appearing in the physical spectrum. Notably, an analogous censoring mechanism has already been found for the case of charged (but static) black holes [16].
Let us again point out that our approach incorporates a pair of conjectural inputs. Firstly, we have assumed that a rotating black hole can be described in terms of a reduced phase space, consisting of a “handful” of physical observables and their respective conjugates. In view of prior works on static black holes [12,13,18], this appears to be a reasonable assumption, but one that should still be formally addressed. Secondly, we have followed [16] in assuming that the canonical conjugate to the mass is periodic, with the period fixed in accordance with purely Euclidean considerations. This seems difficult to establish on a rigorous level, but appears intuitively correct when one considers that the Euclidean sector plays a fundamental role in the very notion of black hole thermodynamics [23]. We have also provided support for this periodicity condition by way of an independent argument.

It would be interesting to compare our outcomes with that of a prior, related work by Makela et al [19]. However, because of a discrepancy with regard to precisely what quantity is being quantized - $A - A_{ext}$ for us versus $A + A_-$ for them\footnote{Here, $A_-$ represents the area of the inner black hole horizon, whereas $A_{ext}$ describes the area of either horizon at extremality. See Section 1 for further details.} - a direct comparison would be highly non-trivial. Nonetheless, one might expect that the qualitative features of the spectrum persevere for the case of large angular momentum, and this does indeed seem to be the case.

Finally, let us comment on the possibility of future directions. One might naively expect that extending the analysis to include charge would be trivial; however, this is not quite correct, as we will fully elaborate on in an upcoming paper [24]. Meanwhile, a change in the spacetime dimensionality would involve technical difficulties (one would require the higher-dimensional analogues of the Euler components), but should be straightforward in principle. Another interesting problem would be to relate our findings to those of other studies, such as the surface quantization approach of Khriplovich [25] or the hyperspin formalism advocated by one of the authors [10]. We defer such intrigue until a future time.

VI. ACKNOWLEDGMENTS

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