CLASSICAL AND SOBOLEV ORTHOGONALITY OF THE NONCLASSICAL JACOBI POLYNOMIALS WITH PARAMETERS $\alpha = \beta = -1$

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We dedicate this paper to the memory of our teacher, mentor, and friend Professor W. N. Everitt (1924-2011)

Abstract. In this paper, we consider the second-order differential expression

$$\ell[y](x) = (1 - x^2)(-y'(x)') + k(1 - x^2)^{-1}y(x) \quad (x \in (-1, 1)).$$

This is the Jacobi differential expression with non-classical parameters $\alpha = \beta = -1$ in contrast to the classical case when $\alpha, \beta > -1$. For fixed $k \geq 0$ and appropriate values of the spectral parameter $\lambda$, the equation $\ell[y] = \lambda y$ has, as in the classical case, a sequence of (Jacobi) polynomial solutions $\{P_n^{(-1, -1)}\}_{n=0}^{\infty}$. These Jacobi polynomial solutions of degree $\geq 2$ form a complete orthogonal set in the Hilbert space $L^2((-1, 1); (1 - x^2)^{-1})$. Unlike the classical situation, every polynomial of degree one is a solution of this eigenvalue equation. Kwon and Littlejohn showed that, by careful selection of this first degree solution, the set of polynomial solutions of degree $\geq 0$ are orthogonal with respect to a Sobolev inner product. Our main result in this paper is to construct a self-adjoint operator $T$, generated by $\ell[\cdot]$, in this Sobolev space that has these Jacobi polynomials as a complete orthogonal set of eigenfunctions. The classical Glazman-Krein-Naimark theory is essential in helping to construct $T$ in this Sobolev space as is the left-definite theory developed by Littlejohn and Wellman.

1. Introduction

For $\alpha, \beta > -1$, the spectral properties of the classical Jacobi differential expression

$$\ell_{\alpha, \beta}[y](x) := \frac{1}{\omega_{\alpha, \beta}(x)} \left[ \left((1-x)^{\alpha+1}(1+x)^{\beta+1}\right)y'(x)' + k(1-x)^{\alpha}(1+x)^{\beta}y(x) \right]$$

where $k \geq 0$ is fixed, $x \in (-1, 1)$ and

$$\omega_{\alpha, \beta}(x) := (1-x)^{\alpha}(1+x)^{\beta},$$

are well understood. In this case, the $n^{th}$ degree Jacobi polynomial $y = P_n^{(\alpha, \beta)}(x)$ is a solution of the equation

$$\ell_{\alpha, \beta}[y](x) = (n(n+\alpha+\beta+1) + k)y(x) \quad (n \in \mathbb{N}_0);$$

details of the properties of these polynomials can be found in the classic texts [14] and [30]. The right-definite spectral analysis has been studied in [1] and [17]. Through the Glazman-Krein-Naimark (GKN) theory [27], it has been known that there exists a self-adjoint operator $A^{(\alpha, \beta)},$
generated from the Jacobi differential expression (1.1), in the Hilbert space $L^2((-1,1); w_{\alpha,\beta})$ of Lebesgue measurable functions $f: (-1,1) \to \mathbb{C}$ satisfying
\[
\|f\|_{\alpha,\beta}^2 := \int_{-1}^{1} |f(x)|^2 w_{\alpha,\beta}(x) \, dx < \infty
\]
which has the Jacobi polynomials as a complete set of eigenfunctions.

For $\alpha, \beta \geq -1$, let
\[
L^2_{\alpha,\beta}(-1,1) := L^2((-1,1); w_{\alpha,\beta})
\]
be the weighted Hilbert space with usual inner product
\[
(f,g)_{\alpha,\beta} = \int_{-1}^{1} f(x)\overline{g(x)} w_{\alpha,\beta}(x) \, dx
\]
and related norm $\| \cdot \|_{\alpha,\beta} = (\cdot, \cdot)^{1/2}_{\alpha,\beta}$. In this paper, we will study the spectral theory of the Jacobi expression (1.1) in $L^2_{-1,-1}(-1,1)$ (that is when $\alpha = \beta = -1$) as well as in several other Hilbert spaces in which the associated Jacobi polynomials are orthogonal.

For $\alpha, \beta > -1$ and $n \in \mathbb{N}_0$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree $n$ is defined (see [14, 28, 30, Chapter IV]) to be any non-zero multiple of
\[
P_n^{(\alpha,\beta)}(x) := \sum_{j=0}^{n} \binom{n + \alpha}{j} \binom{n + \beta}{n - j} \left( \frac{x - 1}{2} \right)^j \left( \frac{x + 1}{2} \right)^{n-j};
\]
it is well known, in this case, that the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ form a complete orthogonal set in $L^2_{\alpha,\beta}(-1,1)$. From (1.6), notice that
\[
P_0^{(\alpha,\beta)}(x) = 1 \quad \text{and} \quad P_1^{(\alpha,\beta)}(x) = \alpha + 1 + (\alpha + \beta + 2) \left( \frac{1-x}{2} \right)
\]
so, in particular from this definition, we see that $P_1^{(-1,1)}(x) \equiv 0$. Furthermore,
\[
P_0^{(-1,1)}(x) \notin L^2_{-1,-1}(-1,1)
\]
because of the singularities in the weight function $w_{-1,-1}(x) = (1-x^2)^{-1}$. However, as we will see, it is the case that $\{P_n^{(-1,1)}\}_{n=2}^\infty$ does form a complete orthogonal set in $L^2_{-1,-1}(-1,1)$. In this weighted Hilbert space, we will apply the Glazman-Krein-Naimark (GKN) theory [27, Chapter IV] to construct the (unique) self-adjoint operator $A = A^{(-1,-1)}$, generated by $\ell_{-1,-1}[]$, having $\{P_n^{(-1,1)}\}_{n=2}^\infty$ as eigenfunctions. As the reader will see, this operator $A$ will be key to subsequent analysis that we develop.

When $\alpha = \beta = -1$, every first degree polynomial is a solution of (1.3). Kwon and Littlejohn [28] showed that, by careful choice of this first degree polynomial, the entire sequence $\{P_n^{(-1,-1)}\}_{n=0}^\infty$ forms an orthogonal set in a certain Sobolev space $W$ generated by the Sobolev inner product
\[
\phi(f,g) := \frac{1}{2} f(-1)\overline{g(-1)} + \frac{1}{2} f(1)\overline{g(1)} + \int_{-1}^{1} f'(t)\overline{g'(t)} \, dt.
\]
Moreover, in fact, these polynomials form a complete orthogonal set in $W$. We note that, by Favard’s Theorem, the entire set $\{P_n^{(-1,-1)}\}_{n=0}^\infty$, for any choice of the first degree polynomial $P_1^{(-1,-1)}(x)$, cannot be orthogonal on the real line with respect to a measure, signed or otherwise.

The main part of this paper is, however, to construct a self-adjoint operator $T$, generated by $\ell[]$, in $W$ that has the Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n=0}^\infty$ as eigenfunctions. The GKN theory, as
well as the general left-definite operator theory developed by Littlejohn and Wellman [24], is of paramount importance in the construction of this self-adjoint operator.

We note that, for \( m \in \mathbb{N} \), the Jacobi polynomials \( \{ P_n^{(\alpha,-m)} \}_{n=0}^\infty \) are orthogonal with respect to inner products of the form (1.7) but whose integrand involves the \( m \)th derivative of the functions. In this respect, we refer the reader to [3], [4], [5], [22], and [23] where general results on the Sobolev orthogonality of the Jacobi or Gegenbauer polynomials, when one or both parameters \( \alpha \) and \( \beta \) are negative integers, are obtained. Bruder and Littlejohn [11] developed the spectral theory when \( \alpha > -1 \) and \( \beta = -1 \). The analysis in [11] is similar in some respects to some of the results of this paper but, overall, quite different; whenever possible, we omit proofs which are similar to those given in [11].

The contents of this paper are as follows. In Section 2, we discuss several well known properties of the Jacobi polynomials that will be useful for subsequent analysis. Section 3 deals with an important operator inequality [13] that is essential for much of the hard analytic results that we develop. The classical GKN theory is used in Section 4 to construct the self-adjoint operator \( A \) in \( L^2((-1,1);(1-x^2)^{-1}) \); in this section, we also prove several properties of functions in the domain \( \mathcal{D}(A) \) that are necessary later in the paper. A short review of the general left-definite theory developed by Littlejohn and Wellman is given in Section 5. It is remarkable that this theory is important in developing the spectral theory of the Jacobi expression \( \ell[\cdot] \) in the Sobolev space \( W \), whose properties we develop in Section 7. In Section 6, the left-definite theory of \( \ell[\cdot] \) is developed. Results from this section are then used to construct the self-adjoint operator \( T \) in \( W \) in Section 7.

2. Preliminaries: Classical Properties of the Jacobi Polynomials

For \( \alpha, \beta > -1 \), it is convenient for later purposes to define the Jacobi polynomials \( \{ P_n^{(\alpha,\beta)} \}_{n=0}^\infty \) as

\[
P_n^{(\alpha,\beta)}(x) := k_n^{\alpha,\beta} \sum_{j=0}^{n} \frac{(1 + \alpha)_n(1 + \alpha + \beta)_n}{j!(n-j)!(1+\alpha)_j(1+\alpha+\beta)_n} \left( \frac{1-x}{2} \right)^j,
\]

where

\[
k_n^{\alpha,\beta} := \frac{(n!)^{1/2}(1 + \alpha + \beta + 2n)^{1/2}}{2^{\alpha+\beta+1}/(\Gamma(\alpha+n+1))^{1/2}}.
\]

With this normalization, these Jacobi polynomials form a complete orthonormal set in \( L^2_{\alpha,\beta}((-1,1)) \).

Their derivatives satisfy the identity

\[
\frac{d^j}{dx^j} P_n^{(\alpha,\beta)}(x) = a^{(\alpha,\beta)}(n,j) P_{n-j}^{(\alpha+\gamma,\beta+\gamma)}(x) \quad (n,j \in \mathbb{N}_0),
\]

where

\[
a^{(\alpha,\beta)}(n,j) = \frac{(n!)^{1/2}(\Gamma(\alpha+\beta+n+1+j))^{1/2}}{(n-j)!(\Gamma(\alpha+\beta+n+1))^{1/2}} \quad (j = 0, 1, ..., n),
\]

and \( a^{(\alpha,\beta)}(n,j) = 0 \) if \( j > n \). Furthermore, for \( n,r,j \in \mathbb{N}_0 \), we have the orthogonality relation

\[
\int_{-1}^{1} \frac{d^j}{dx^j} P_n^{(\alpha,\beta)}(x) \frac{d^l}{dx^l} P_r^{(\alpha,\beta)}(x) w_{\alpha+\gamma,\beta+\gamma}(x) dx = \frac{n!\Gamma(\alpha+\beta+n+1+j)}{(n-j)!(\Gamma(\alpha+\beta+n+1))^{1/2}} \delta_{n,r}.
\]

For \( \alpha = \beta = -1 \), the ‘natural’ setting for an analytical study of the Jacobi polynomials is the Hilbert space \( L^2_{-1,-1}((-1,1)) \). However, because of the singularities in the associated orthogonalizing weight function \( w_{-1,-1}(x) = (1-x^2)^{-1} \), only the Jacobi polynomials \( P_n^{(-1,-1)}(x) \) of degree \( n \geq 2 \) belong to this space; see Section 7.
We note that there is a similar situation with the Laguerre polynomials \( Muckenhoupt \) [26] in 1972.

The following result is well known and can be found, for example, in [30], Chapter III, Theorem 3.1.5.

**Lemma 2.1.** The sequence \( \{P_n^{(1,1)}(x)\}_{n=0}^{\infty} \) forms a complete orthogonal set in the Hilbert space \( L^2_{1,1}(-1,1) \).

From this lemma, we prove the following result.

**Lemma 2.2.** The sequence \( \{P_n^{(-1,-1)}(x)\}_{n=2}^{\infty} \) forms a complete orthogonal set in the Hilbert space \( L^2_{-1,-1}(-1,1) \). Equivalently, the set of all polynomials \( \mathcal{P}_{-1}[-1,1] \) of degree \( \geq 2 \) satisfying \( p(\pm 1) = 0 \) is dense in \( L^2_{-1,-1}(-1,1) \). In particular,

\[
P_n^{(-1,-1)}(\pm 1) = 0 \quad (n \geq 2).
\]

**Proof.** Note that

\[
\int_{-1}^{1} |f(x)|^2 (1-x^2)^{-1} dx = \int_{-1}^{1} |(1-x^2)^{-1} f(x)|^2 (1-x^2) dx,
\]

i.e., \( f \in L^2((-1,1); (1-x^2)^{-1}) \iff (1-x^2)^{-1} f \in L^2((-1,1); (1-x^2)) \), and in this case,

\[
\|f\|_{-1,-1} = \|(1-x^2)^{-1} f\|_{1,1}.
\]

Let \( f \in L^2((-1,1); (1-x^2)^{-1}) \), and let \( \epsilon > 0 \). Hence

\[
(1-x^2)^{-1} f \in L^2((-1,1); (1-x^2)),
\]

so by Lemma 2.1, there exists a polynomial \( q(x) \) with

\[
\|(1-x^2)^{-1} f - q\|_{1,1} < \epsilon.
\]

Let \( p(x) = (1-x^2)q(x) \) so \( q(x) = (1-x^2)^{-1}p(x) \). Then \( p \) is a polynomial of degree \( \geq 2 \), with \( p(\pm 1) = 0 \), and

\[
\epsilon > \|(1-x^2)^{-1} f - (1-x^2)^{-1} p\|_{1,1} = \|(1-x^2)^{-1} (f - p)\|_{1,1} = \|f - p\|_{-1,-1} \text{ by (2.5)}.
\]

Lastly, to establish (2.4), we note that it is straightforward to see, from the definition in (1.6), that \( P_n^{(-1,-1)}(x) \) is a non-zero multiple of \( (1-x^2)P_{n-2}^{(1,1)}(x) \); details can be found in [10], Lemma 5.4.

We note that there is a similar situation with the Laguerre polynomials \( \{L_n^{(a)}\} \) at \( x = 0 \) when \( a \) is a negative integer; see [30], p. 102.

\[ \square \]

3. An Operator Inequality

The following result, due to Chisholm and Everitt [12], is important in establishing the main analytic results in this paper; it has been a remarkably useful tool in obtaining general properties of functions in certain operator domains. Theorem 3.1 was extended to the general case of conjugate indices \( p \) and \( q \) \((p, q > 1)\) in [13] in 1999. Several years after publication, the authors of [13] learned that this result was first established by Talenti [31] and Tomaselli [32], both in 1969, and later by Muckenhoupt [26] in 1972.
Define \( K \) \((4.2) \) \( (1 - x^2) = 0 \). Then if \( x \in (a, b) \), then \( \phi \in L^2((a, b); w) \) and \( \psi \in L^2([c, b]; w) \);\n
(iii) for all \( [\alpha, \beta] \subset (a, b) \),
\[
\int_a^\alpha |\phi(t)|^2w(t)dt > 0 \quad \text{and} \quad \int_\beta^b |\psi(t)|^2w(t)dt > 0.
\]

Define the linear operators \( A \) and \( B \) on \( L^2((a, b); w) \) and \( L^2((a, b); w) \), respectively, by
\[
(Ag)(x) := \int_a^b \phi(t)g(t)w(t)dt \quad (x \in (a, b) \text{ and } g \in L^2((a, b); w))
\]
\[
(Bg)(x) := \int_a^b \psi(t)g(t)w(t)dt \quad (x \in (a, b) \text{ and } g \in L^2((a, b); w));
\]

then
\[
A : L^2((a, b); w) \to L^2_{\text{loc}}((a, b) : w)
\]
\[
B : L^2((a, b); w) \to L^2_{\text{loc}}((a, b); w).
\]

Define \( K(\cdot) : (a, b) \to (0, \infty) \) by
\[
K(x) := \left\{ \int_a^x |\phi(t)|^2w(t)dt \right\}^{1/2} \left\{ \int_x^b |\psi(t)|^2w(t)dt \right\}^{1/2} \quad (x \in (a, b))
\]
and the number \( K \in (0, \infty] \)
\[
K := \sup \{ K(x) \mid x \in (a, b) \}.
\]

Then a necessary and sufficient condition that \( A \) and \( B \) are bounded linear operators on \( L^2((a, b); w) \)
is that the number \( K \) is finite, i.e.
\[
K \in (0, \infty).
\]

Furthermore, the following operator inequalities are valid:
\[
\|Af\|_2 \leq 2K \|f\|_2 \quad (f \in L^2((a, b); w))
\]
\[
\|Bg\|_2 \leq 2K \|g\|_2 \quad (g \in L^2((a, b); w));
\]
the number \( 2K \) given in the above inequalities is best possible for these inequalities to hold.

4. Right-Definite Spectral Analysis of the Jacobi Expression when \( \alpha = \beta = -1 \)

In the special case when \( \alpha = \beta = -1 \), the Jacobi differential expression \( (1.1) \) simplifies to be
\[
\ell[y](x) := \ell_{-1, -1}[y](x) = (1 - x^2) \left( ( -y' \right)' + k(1 - x^2)^{-1}y(x) \right)
\]
\[
= -(1 - x^2)y''(x) + ky(x) \quad (x \in (-1, 1));
\]
here \( k \) is a fixed, non-negative constant. The maximal domain associated with \( \ell[\cdot] \) in \( L^2_{-1, -1}(-1, 1) \) is
\[
\Delta := \{ f : (-1, 1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1, 1); f, \ell[f] \in L^2_{-1, -1}(-1, 1) \}.
\]
Observe that if \( f \in \Delta \), then \( (1 - x^2)f'' \in L^2_{-1, -1}(-1, 1) \) or, equivalently
\[
(1 - x^2)^{1/2}f'' \in L^2(-1, 1) \quad (f \in \Delta);
moreover, \( f \in L^2_{-1,-1}(-1,1) \) is equivalent to
\[
(1 - x^2)^{-1/2} f \in L^2(-1,1) \quad (f \in \Delta).
\]

For \( f, g \in \Delta \) and \([a,b] \subset (-1,1)\), we have Dirichlet’s formula
\[
\int_a^b \ell[f](x)\overline{g}(x)(1-x^2)^{-1} \, dx = -f'(x)\overline{g}(x) \bigg|^b_a + \int_a^b \left[ f''(x)\overline{g}(x) + k(1-x^2)^{-1}f(x)\overline{g}(x) \right] \, dx
\]
and Green’s formula
\[
\int_a^b \ell[f](x)\overline{g}(x)(1-x^2)^{-1} \, dx = [f(x)\overline{g}(x) - f'(x)\overline{g}(x)] \bigg|^b_a + \int_a^b f(x)\overline{g}'(x)(1-x^2)^{-1} \, dx.
\]

**Theorem 4.1.** The Jacobi differential expression \((4.1)\) is strong limit-point (SLP) and Dirichlet at \( x = \pm 1 \). That is to say, for \( f, g \in \Delta \)

(i) (Dirichlet) \( \int_0^1 |f'(t)|^2 \, dt < \infty \) and \( \int_{-1}^0 |f'(t)|^2 \, dt < \infty \),

(ii) (SLP) \( \lim_{x \to \pm 1} f'(x)\overline{g}(x) = 0 \).

The proof of this theorem will follow immediately after the following three lemmas are established.

**Lemma 4.1 (Dirichlet Condition).** For \( f \in \Delta \), \( f' \in L^2(-1,1) \). In particular, we may assume that \( f \in AC[-1,1] \) for all \( f \in \Delta \).

**Proof.** We prove that \( f' \in L^2[0,1] \); a similar proof establishes \( f' \in L^2(-1,0) \). Since \( f' \in AC_{loc}[0,1] \), we see that
\[
f'(x) = f(0) + \int_0^x f''(t)\sqrt{1-t^2} \, dt \quad (x \in [0,1]).
\]
We now apply Theorem 3.1 with \( \varphi(x) = 1 \) and \( \psi(x) = 1/\sqrt{1-x^2} \). Since
\[
\int_0^1 \psi^2(t) \, dt \cdot \int_x^1 \varphi^2(t) \, dt = \frac{1}{2} (1-x) \ln \left( \frac{1+x}{1-x} \right)
\]
is bounded on \([0,1]\), we see that \( \int_0^x f''(t)\sqrt{1-t^2} \, dt \in L^2[0,1] \). Hence, from (4.6), \( f' \in L^2[0,1] \). \( \square \)

**Lemma 4.2.** For all \( f \in \Delta \), \( f(\pm 1) = 0 \).

**Proof.** Note that from the previous lemma, \( f \in AC[-1,1] \) and the limits
\[ f(\pm 1) := \lim_{x \to \pm 1} f(x) \]
exist and are finite. Suppose that \( f(1) \neq 0 \); we can, without loss of generality, assume that \( f(1) > 0 \).

By continuity, there exists \( x^* \in (0,1) \) such that
\[ f(x) > \frac{f(1)}{2} \quad \text{for } x \in [x^*,1). \]
Then
\[ \infty > \int_0^1 |f(t)|^2 (1-t^2)^{-1} \, dt \geq \int_{x^*}^1 |f(t)|^2 (1-t^2)^{-1} \, dt \geq \frac{(f(1))^2}{4} \int_{x^*}^1 (1-t^2)^{-1} \, dt = \infty, \]
a contradiction. A similar argument shows that \( f(-1) = 0 \). \( \square \)

**Lemma 4.3 (Strong Limit-Point Condition).** For all \( f, g \in \Delta \), \( \lim_{x \to \pm 1} f(x)g'(x) = 0 \).
Proof. Let \( f, g \in \Delta \). It suffices to prove that
\[
\lim_{x \to 1^-} f(x)g'(x) = 0;
\]
a similar argument establishes the other limit. We assume that \( f \) and \( g \) are both real-valued. Note, by Hölder’s inequality, \((4.2)\), and \((4.3)\) that \( fg'' \in L^1(-1, 1) \) so that \( \lim_{x \to 1^-} \int_0^x f(t)g''(t)dt \) exists and is finite. Now, by integration by parts,
\[
\int_0^x f(t)g''(t)dt = f(t)g'(t) |^0_x - \int_0^x f'(t)g'(t)dt.
\]
By Lemma \(4.1\), \( \lim_{x \to 1^-} \int_0^x f'(t)g'(t)dt \) exists and is finite. Hence, we see that \( \lim_{x \to 1^-} f(x)g'(x) \) exists and is finite. Suppose that \( \lim_{x \to 1^-} f(x)g'(x) = c \neq 0 \); we may assume that \( c > 0 \). For \( x \) close to 1, we may also assume that
\[
f(x) > 0 \text{ and } g'(x) > 0.
\]
Hence, there exists \( x^* \in [0, 1) \) such that \( g'(x) \geq \frac{c}{2f(x)} \) for \( x \in [x^*, 1) \). Therefore,
\[
|f'(x)g'(x)| \geq \frac{c|f'(x)|}{2f(x)} \quad (x \in [x^*, 1)).
\]
Integrate to obtain
\[
\int_{x^*}^x |f'(t)g'(t)| dt \geq \frac{c}{2} \int_{x^*}^x \frac{|f'(t)|}{f(t)} dt \geq \frac{c}{2} \left| \int_{x^*}^x \frac{f'(t)}{f(t)} dt \right| = \frac{c}{2} |\ln f(x)| + \gamma,
\]
where \( \gamma \) is some constant of integration. Now let \( x \to 1^- \); we see from Lemma \(4.2\) that
\[
\infty > \int_{x^*}^1 |f'(t)g'(t)| dt \geq \frac{c}{2} \lim_{x \to 1^-} |\ln f(x)| + k = \infty.
\]
This contradiction shows that \( c = 0 \) and this establishes the lemma. \(\square\)

We now define the operator
\[
A : \mathcal{D}(A) \subset L^2_{-1,-1}(-1, 1) \to L^2_{-1,-1}(-1, 1)
\]
by
\[
Af = \ell[f]
\]
\( f \in \mathcal{D}(A) := \Delta. \)

Since \( x = \pm 1 \) are SLP, we see from the Glazman-Krein-Naimark theory [27, Chapter V] that \( A \) is an unbounded, self-adjoint operator in \( L^2_{-1,-1}(-1, 1) \) with spectrum
\[
\sigma(A) = \{n(n-1) + k \mid n = 2, 3, \ldots\}
\]
Moreover, from \((4.4)\), \((4.5)\), and Theorem \(4.1\) we have the classic Green’s formula,
\[
(Af, g)_{-1,-1} = \int_{-1}^1 \ell[f](x)\overline{g}(x)(1-x^2)^{-1}dx = (f, Ag)_{L^2_{-1,-1}(-1, 1)},
\]
and Dirichlet’s formula,
\[
(Af, g)_{-1,-1} = \int_{-1}^1 \left[f'(x)\overline{g}'(x) + k(1-x^2)^{-1}f(x)\overline{g}(x)\right]dx.
\]
In particular, observe that

\[(4.8) \quad (Af, f)_{-1,-1} = \int_{-1}^{1} \left[ |f'(x)|^2 + k(1-x^2)^{-1} |f(x)|^2 \right] dx \geq k (f, f)_{-1,-1}; \]

in other words, the self-adjoint operator \( A \) is bounded below in \( L^2_{-1,-1}(-1, 1) \) by \( kI \) where \( I \) is the identity operator. This observation will be important from the viewpoint of left-definite theory, which we now briefly discuss.

5. General Left-Definite Theory

In [24], Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint operator \( T \) that is bounded below in a Hilbert space \( (H, \langle \cdot, \cdot \rangle) \). The results contained here are important for our construction of the Jacobi self-adjoint operator \( T \) in the Sobolev space \( W \), see Section 7 having the Jacobi polynomials \( \{P_m^{(-1,-1)}\}_{m=0}^\infty \) as a complete set of eigenfunctions.

There is a strong connection between left-definite theory and the theory of Hilbert scales; indeed, our left-definite spaces are Hilbert scales. We refer the reader to the recent texts of Albeverio and Kurasov [2] Chapter 1.2.2 and Simon [29] Chapter 12 as well as the classic texts of Berezanskiï [9] and Maz’ya [25].

Let \( V \) be a vector space over \( \mathbb{C} \) with inner product \( \langle \cdot, \cdot \rangle \) such that \( H = (V, \langle \cdot, \cdot \rangle) \) is a Hilbert space. Let \( s > 0 \) and suppose that \( V_s \) is a vector subspace of \( V \) with inner product \( \langle \cdot, \cdot \rangle_s \); let \( H_s = (V_s, \langle \cdot, \cdot \rangle_s) \) denote this inner product space.

Throughout this section, we assume that \( T : D(T) \subset H \rightarrow H \) is a self-adjoint operator that is bounded below by \( rI \) for some \( r > 0 \), where \( I \) is the identity operator on \( H \); that is to say

\[ (Tx, x) \geq r(x, x) \quad (x \in D(T)). \]

It is well known that, for \( s > 0 \), the operator \( T^s \) is self-adjoint and bounded below in \( H \) by \( r^s I \).

**Definition 5.1.** We say that \( H_s = (V_s, \langle \cdot, \cdot \rangle_s) \) is an \( s^{th} \) left-definite space associated with the pair \( (H, A) \) if

(i) \( H_s \) is a Hilbert space;
(ii) \( D(T^s) \), the domain of \( T^s \), is a vector subspace of \( V_s \);
(iii) \( D(T^s) \) is dense in \( H_s \);
(iv) \( (x, x)_s \geq r^s(x, x) \) for all \( x \in V_s \);
(v) \( (x, y)_s = (T^s x, y) \) for all \( x \in D(T^s), y \in V_s \).

Littlejohn and Wellman in [24] Theorem 3.1 prove the following existence/uniqueness result.

**Theorem 5.1.** Let \( T : D(T) \subset H \rightarrow H \) be a self-adjoint operator that is bounded below by \( rI \) for some \( r > 0 \). Let \( s > 0 \) and define \( H_s = (V_s, \langle \cdot, \cdot \rangle_s) \) by

\[ V_s = D(T^{s/2}) , \]

and

\[ (x, y)_s = (T^{s/2} x, T^{s/2} y) \quad (x, y \in V_s). \]

Then \( H_s \) is the unique left-definite space associated with the pair \( (H, T) \).

**Definition 5.2.** For \( s > 0 \), let \( H_s = (V_s, \langle \cdot, \cdot \rangle_s) \) be the \( s^{th} \) left-definite space associated with \( (H, T) \). If there exists a self-adjoint operator \( T_s : D(T_s) \subset H_s \rightarrow H_s \) satisfying

\[ T_s x = Tx \quad (x \in D(T_s) \subset D(T)), \]
then we call such an operator an $s^{th}$ left-definite operator associated with the pair $(H,T)$.

In [24, Theorem 3.2], the authors establish the following existence/uniqueness result.

**Theorem 5.2.** For any $s > 0$, let $H_s = (V_s, (\cdot, \cdot)_s)$ denote the $s^{th}$ left-definite space associated with $(H,T)$. Then there exists a unique left-definite operator $T_s$ in $H_s$ associated with $(H,T)$. Furthermore, $T_s x = T x$ for all $s > 0$ and $x \in \mathcal{D}(T_s)$, where

$$\mathcal{D}(T_s) = V_{s+2} \subset \mathcal{D}(T).$$

The last theorem that we list in this section shows that there is a non-trivial left-definite theory only in the unbounded case; see [24, Section 8].

**Theorem 5.3.** Let $H = (V, (\cdot, \cdot))$ be a Hilbert space. Suppose $T : \mathcal{D}(T) \subset H \rightarrow H$ is a self-adjoint operator that is bounded below by $r I$ for some $r > 0$. For each $s > 0$, let $H_s = (V_s, (\cdot, \cdot)_s)$ and $A_s$ denote the $s^{th}$ left-definite space and $s^{th}$ left-definite operator, respectively, associated with $(H,T)$.

1. Suppose $T$ is bounded. Then, for each $s > 0$,
   (i) $V = V_s$;
   (ii) the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_s$ are equivalent;
   (iii) $T = T_s$.

2. Suppose $T$ is unbounded. Then
   (i) $V_s$ is a proper subspace of $V$;
   (ii) $V_s$ is a proper subspace of $V_t$ whenever $0 < t < s$;
   (iii) the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_s$ are not equivalent for any $s > 0$;
   (iv) the inner products $(\cdot, \cdot)_s$ and $(\cdot, \cdot)_t$ are not equivalent for any $s,t > 0, s \neq t$;
   (v) $\mathcal{D}(T_s)$ is a proper subspace of $\mathcal{D}(T)$ for each $s > 0$;
   (vi) $\mathcal{D}(T_s)$ is a proper subspace of $\mathcal{D}(T_t)$ whenever $0 < t < s$;
   (vii) the point spectrum of each $T_s$ and $T$ are equal; that is, $\sigma_p(T_s) = \sigma_p(T)$ for each $s > 0$;
   (viii) the continuous spectrum of each $T_s$ and $T$ are equal; that is, $\sigma_c(T_s) = \sigma_c(T)$ for each $s > 0$.

From [17,8], we see that there is a non-trivial left-definite theory associated with the Jacobi self-adjoint operator $A$ defined in [4,7]. It is natural to ask: what are the left-definite spaces $\{H_s\}_{s>0}$ and left-definite operators $\{T_s\}_{s>0}$ associated with $(L^2_{1,-1}(-1,1), A)$? To answer this, we observe from Definition [5,11](v) that the $s^{th}$ left-definite inner product $(\cdot, \cdot)_s$ is generated by the $s^{th}$ power of $A$, or in our case, the $s^{th}$ composite power of the Lagrangian symmetric form of $\ell[\cdot]$. Practically speaking, we can only effectively determine these powers when $s$ is a positive integer.

6. Left-Definite Theory of the Jacobi Differential Expression when $\alpha = \beta = -1$

In [17], the authors show that, for each $n \in \mathbb{N}$, the $n^{th}$ composite power of the Jacobi differential expression [4,1] is given by

$$\ell^n[y](x) = \sum_{j=0}^{n} (-1)^j c_j(n,k) \left( (1 - x^2)^j y^{(j)}(x) \right)^{(j)},$$

where the coefficients $c_j(n,k)$ ($j = 0, 1, \ldots n$) are non-negative and defined as

\[
c_0(n,k) := \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0 \end{cases} \quad \text{and} \quad c_j(n,k) := \begin{cases} \binom{n}{j} & \text{if } k = 0 \\ \sum_{r=0}^{j} \binom{n}{r} \binom{n-r}{j} k^r & \text{if } k > 0. \end{cases}
\]
In (6.1), the numbers \( \{n\} \) are called Jacobi-Stirling numbers, defined by
\[
\binom{n}{j} := \delta_{n,j} \quad (j = 0, 1; \ n \in \mathbb{N}_0),
\]
and, when \( j \geq 2 \) and \( n \in \mathbb{N}_0 \), by
\[
\binom{n}{j} := \sum_{r=2}^{j} (-1)^{r+j} \frac{(2r-1)(r-2)! [r(r-1)]^n}{r!(j-r)!(j+r-1)!}.
\]

The coefficients \( c_j(n, k) \) were originally defined in [17] through the identity
\[
\sum_{j=0}^{n} c_j(n, k) \frac{m!(m+j-2)!}{(m-j)!(m-2)!} = (m(m-1) + k)^n.
\]

For a discussion of the combinatorial properties of \( \{n\} \), and the more general Jacobi-Stirling numbers \( \{n\}_1 \), see the recent papers [7], [8], and [19]. The following table lists some of these Jacobi-Stirling numbers \( \{n\}_1 \) for small values of \( n \) and \( j \).

| \( j/n \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( n = 5 \) | \( n = 6 \) | \( n = 7 \) | \( n = 8 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( j = 0 \) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( j = 1 \) | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( j = 2 \) | 0 | 0 | 1 | 2 | 4 | 8 | 16 | 32 | 64 |
| \( j = 3 \) | 0 | 0 | 0 | 1 | 8 | 52 | 320 | 1936 | 11648 |
| \( j = 4 \) | 0 | 0 | 0 | 0 | 1 | 20 | 292 | 3824 | 47824 |
| \( j = 5 \) | 0 | 0 | 0 | 0 | 0 | 1 | 40 | 1092 | 25664 |
| \( j = 6 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 70 | 3192 |
| \( j = 7 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 112 | |
| \( j = 8 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |

Jacobi-Stirling numbers (e.g. \( \{5\}_1 = 1092 \))

For example, if \( k = 0 \), we see from this table that
\[
\ell^5[y](x) = -\left(1 - x^2\right)^5 y^{(5)}(x) + \left(20(1 - x^2)^4 y^{(4)}(x)\right) + 52(1 - x^2)^3 y''(x)'' + \left(8(1 - x^2)^2 y''(x)''\right).
\]

It is interesting to note that \( \{n\}_1 = \{n-1\}_1 \); the numbers \( \{n\}_1 \) are called the Legendre-Stirling numbers which are the subject of several recent papers (see, for example, [6], [7], [8], and [16]).

Let \( k > 0 \). For each \( n \in \mathbb{N} \), define the inner product space
\[
H_n := (V_n; (\cdot, \cdot)_n),
\]
where
\[
V_n := \{ f : (-1, 1) \to \mathbb{C} \mid f \in AC_{loc}^{n-1}(1-1, 1); f^{(j)} \in L^2((-1, 1); (1 - x^2)^{j-1}), j = 0, 1, \ldots, n \}
\]
and
\[
(f, g)_n := \sum_{j=0}^{n} c_j(n, k) \int_{-1}^{1} f^{(j)}(x) g^{(j)}(x)(1 - x^2)^{j-1} dx.
\]
For later purposes, we note that

\[(f,g)_1 = \int_{-1}^{1} (f'(x)g'(x) + kf(x)g(x)(1 - x^2)^{-1})dx.\]

We shall show that \(H_n\) is the \(n^{th}\) left-definite space associated with the pair \((L_{-1,-1}^2(-1,1); A)\), where \(A\) is the self-adjoint Jacobi operator defined in (4.7). Mimicking the results from [17], let 

\[\text{mutatis mutandis, Theorem 6.1 follows; for a specific proof see the thesis [10] of Bruder.}\]

**Theorem 6.1.** Let \(k > 0\). For each \(n \in \mathbb{N}\), \(H_n\) is a Hilbert space.

It is clear, from (6.4) and the non-negativity of each \(c_j(n,k)\), that

\[(f,f)_n = \sum_{j=0}^{n} c_j(n,k) ||f^{(j)}||^2_{j-1,j-1} \geq c_0(n,k) ||f||^2_{-1,-1} = k^n (f,f)_{-1,-1} \quad (f \in H_n).\]

Since the proofs of the next two results are similar to, respectively, the proofs given in Theorem 7 and Lemma 8 in [11], we omit them.

**Theorem 6.2.** The Jacobi polynomials \(\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}\) form a complete orthogonal set in \(H_n\) for each \(n \in \mathbb{N}\). Equivalently, the vector space \(P_{-1}[-1,1]\), defined in Lemma 6.2, is dense in \(H_n\).

**Lemma 6.1.** For \(n \geq 2\) and \(p,q \in P_{-1}[-1,1],\)

\[(p,q)_n = (A^n p,q)_{-1,-1}.\]

We are now in position to prove the main result of this section.

**Theorem 6.3.** For \(k > 0\), let

\[A : \mathcal{D}(A) \subset L^2((-1,1); (1 - x^2)^{-1}) \to L^2((-1,1); (1 - x^2)^{-1})\]

be the Jacobi self-adjoint operator defined in (4.7) having the Jacobi polynomials \(\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}\) as eigenfunctions. For each \(n \in \mathbb{N}\), let \(H_n, V_n, \text{ and } (\cdot,\cdot)_n\) be as given in (6.4), (6.5), and (6.6), respectively. Then \(H_n\) is the \(n^{th}\)-left-definite space associated with \((L^2((-1,1); (1 - x^2)^{-1}), A)\). Moreover, the Jacobi polynomials \(\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}\) form a complete orthogonal set in each \(H_n\); specifically, they satisfy the orthogonality relation

\[\left(P_m^{(-1,-1)}, P_l^{(-1,-1)}\right)_n = (m(m-1)+k)^n\delta_{m,l}.\]

Furthermore, define

\[B_n := \mathcal{D}(B_n) \subset H_n \to H_n\]

by

\[B_nf := \ell [f] \quad (f \in \mathcal{D}(B_n) := V_{n+2}).\]

Then \(B_n\) is the \(n^{th}\) left-definite (self-adjoint) operator associated with the pair \((L_{-1,-1}^2(-1,1), A)\). Lastly, the spectrum of \(B_n\) is given by

\[\sigma(B_n) = \{m(m-1)+k \mid m = 2, 3, 4, \ldots\} = \sigma(A).\]
Proof. Fix \( n \in \mathbb{N} \). We need to show that \( H_n \) satisfies the five properties given in Definition 5.1.

(i) \( H_n \) is a Hilbert space; this is the statement given in Theorem 6.1.

(ii) We need to show that \( D(A^n) \subset V_n \). Let \( f \in D(A^n) \). Since the Jacobi polynomials \( \left\{ P_m^{(-1,-1)} \right\}_{m=2}^{\infty} \) form a complete orthonormal set in \( L_{-1,-1}^2(-1,1) \), we see that

\[

p_j \to f \quad \text{in} \quad L_{-1,-1}^2(-1,1) \quad \text{as} \quad j \to \infty,

\]

where

\[
p_j(t) = \sum_{m=2}^{j} c_m P_m^{(-1,-1)}(t) \quad (t \in (-1,1)),
\]

and

\[
c_m := \left( f, P_m^{(-1,-1)} \right)_{-1,-1} = \int_{-1}^{1} f(t) P_m^{(-1,-1)}(t) \left( 1 - t^2 \right)^{-1} \, dt \quad (m \geq 2).
\]

Since \( A^n f \in L_{-1,-1}^2(-1,1) \), we see that, as \( j \to \infty \),

\[

\sum_{m=2}^{j} \widetilde{c}_m P_m^{(-1,-1)} \to A^n f \quad \text{in} \quad L_{-1,-1}^2(-1,1)
\]

However, from the self-adjointness of \( A^n \), we find that

\[
\widetilde{c}_m := \left( A^n f, P_m^{(-1,-1)} \right)_{-1,-1} = \left( f, A^n P_m^{(-1,-1)} \right)_{-1,-1}
\]

\[
= (m(m - 1) + k)^n \left( f, P_m^{(-1,-1)} \right)_{-1,-1}
\]

Consequently,

\[
A^n p_j \to A^n f \quad \text{in} \quad L_{-1,-1}^2(-1,1) \quad \text{as} \quad j \to \infty.
\]

Moreover, by Lemma 6.1,

\[
\left( \| p_j - p_r \|_n \right)^2 = (A^n [p_j - p_r], p_j - p_r)_{-1,-1}
\]

\[
\to 0 \quad \text{as} \quad j, r \to \infty
\]

i.e. \( \{p_j\}_{j=0}^{\infty} \) is Cauchy in \( H_n \). Since \( H_n \) is a Hilbert space, there exists \( g \in H_n \subset L_{-1,-1}^2(-1,1) \) such that

\[
p_j \to g \quad \text{in} \quad H_n \quad \text{as} \quad j \to \infty.
\]

Furthermore, from (6.8), we see that

\[
\| p_j - g \|_{-1,-1} \leq k^{-n/2} \| p_j - g \|_n,
\]

and hence,

\[
p_j \to g \quad \text{in} \quad L_{-1,-1}^2(-1,1).
\]

Comparing (6.9) and (6.10),

\[
f = g \in H_n.
\]

(iii) By Theorem 6.2, \( \mathcal{P}_{-1} \subset \mathbb{C} \) is dense in \( H_n \). Since \( \mathcal{P}_{-1} \subset D(A^n) \), we see that \( D(A^n) \) is dense in \( H_n \).

(iv) We already showed, in (6.8), that \( (f,f)_n \geq k^n (f,f)_{-1,-1} \) for all \( f \in V_n \).
(v) We need to show that \((f, g)_n = (A^n f, g)_{-1, -1}\) for \(f \in D(A^n)\) and \(g \in V_n\). This is true for any \(f, g \in \mathcal{P}_{-1}[-1, 1]\) by Lemma 6.1. Let \(f \in D(A^n) \subset H_n, g \in H_n\). Since \(\mathcal{P}_{-1}[-1, 1]\) is dense in both \(H_n\) and \(L^2_{-1, -1}(-1, 1)\) (see Lemma 2.2 and Theorem 6.2), and (by (iv)), convergence in \(H_n\) implies convergence in \(L^2_{-1, -1}(-1, 1)\), there exist sequences \(\{p_j\}_{j=0}^\infty, \{q_j\}_{j=0}^\infty \subset \mathcal{P}_{-1}[-1, 1]\) such that

\[
p_j \to f \quad \text{in } H_n \text{ as } j \to \infty
\]

\[
A^n p_j \to A^n f \quad \text{in } L^2_{-1, -1}(-1, 1) \text{ as } j \to \infty \quad \text{(from the proof of part (ii))}
\]

and

\[
q_j \to g \quad \text{in } H_n \text{ and } L^2_{-1, -1}(-1, 1) \text{ as } j \to \infty.
\]

Hence,

\[
(A^n f, g)_{-1, -1} = \lim_{j \to \infty} (A^n p_j, q_j)_{-1, -1}
\]

\[
= \lim_{j \to \infty} (p_j, q_j)_{n} \quad \text{by Lemma 6.1}
\]

\[
= (f, g)_n.
\]

The results listed in the theorem on the left-definite operator \(B_n\) and the spectrum of \(B_n\) follow immediately from the general left-definite theory discussed in Section 5. □

7. Sobolev Orthogonality and Spectral Theory of the Jacobi Expression

As discussed in Section 4, any polynomial \(p(x)\) of degree one is a solution of the Jacobi differential equation

\[
\ell[y](x) := -(1 - x^2)y''(x) + ky(x) = (n(n - 1) + k)y(x);
\]

moreover, it is important to note that the Jacobi polynomial \(P_n^{(-1, -1)}(x)\) as defined, say, in [30], is identically zero. If we define

\[
P_n^{(-1, -1)}(x) := x/\sqrt{3}
\]

and renormalize the Jacobi polynomials (2.1), for \(\alpha = \beta = -1\), of degree \(n \geq 2\), by

\[
P_n^{(-1, -1)}(x) := \frac{\sqrt{4n - 2}}{(n - 1)} \sum_{j=0}^{n} \binom{n-1}{n-j} \binom{n-1}{j} \left( \frac{x-1}{2} \right)^{j} \left( \frac{x+1}{2} \right)^{n-j},
\]

then Kwon and Littlejohn prove the following theorem in [23].

**Theorem 7.1.** The Jacobi polynomials \(\left\{\tilde{P}_n^{(-1, -1)}(x)\right\}_{n=0}^\infty\), as given in (7.1) and (7.2), are orthonormal with respect to the Sobolev inner product

\[
\phi(f, g) := \frac{1}{2} f(-1)\overline{g}(-1) + \frac{1}{2} f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx.
\]

A key step in establishing this orthogonality is the fact that \(\tilde{P}_n^{(-1, -1)}(\pm 1) = 0\) for \(n \geq 2\); see [23].

**Definition 7.1.** Define

\[
W := \left\{ f : [-1, 1] \to \mathbb{C} \mid f \in AC[-1, 1]; f' \in L^2(-1, 1) \right\}
\]

and, with \(\phi(\cdot, \cdot)\) being the inner product defined in (7.3), let \(\|f\|_{\phi} := \phi(f, f)^{1/2}\) \((f \in W)\) be the associated norm.
Theorem 7.2. \((W, \phi(\cdot, \cdot))\) is a Hilbert space.

Proof. Let \(\{f_n\} \subset W\) be a Cauchy sequence. Hence
\[
\|f_n - f_m\|_\phi^2 = \frac{1}{2} |f_n(-1) - f_m(-1)|^2 + \frac{1}{2} |f_n(1) - f_m(1)|^2 + \int_{-1}^{1} |f'_n(x) - f'_m(x)|^2 \, dx
\]
\[
\rightarrow 0 \quad \text{as } n, m \to \infty.
\]
In particular, since
\[
\int_{-1}^{1} |f'_n(x) - f'_m(x)|^2 \, dx \leq \|f_n - f_m\|_\phi^2,
\]
we see that \(\{f'_n\}\) is Cauchy in \(L^2(-1, 1)\). Since \(L^2(-1, 1)\) is complete, there exists \(g \in L^2(-1, 1)\) such that
\[
f'_n \to g \quad \text{as } n \to \infty \quad \text{in } L^2(-1, 1).
\]
Also, since
\[
\frac{1}{2} |f_n(-1) - f_m(-1)|^2 \leq \|f_n - f_m\|_\phi^2 \quad \text{and} \quad \frac{1}{2} |f_n(1) - f_m(1)|^2 \leq \|f_n - f_m\|_\phi^2,
\]
we see that the sequences \(\{f_n(\pm 1)\}\) are both Cauchy in \(\mathbb{C}\) and, hence, there exists \(A_{\pm 1} \in \mathbb{C}\) such that
\[
f_n(1) \to A_1 \text{ in } \mathbb{C}
\]
\[
f_n(-1) \to A_{-1} \text{ in } \mathbb{C}.
\]
Furthermore, since \(f_n \in AC [-1, 1] \quad (n \in \mathbb{N})\), we see that
\[
\int_{-1}^{1} g(t) dt \leftarrow \int_{-1}^{1} f'_n(t) dt = f_n(1) - f_n(-1) \to A_1 - A_{-1};
\]
that is,
\[
A_1 = A_{-1} + \int_{-1}^{1} g(t) dt.
\]
Define \(f : [-1, 1] \to \mathbb{C}\) by
\[
f(x) = A_{-1} + \int_{-1}^{x} g(t) dt.
\]
It is clear that \(f \in AC [-1, 1]\) and \(f'(x) = g(x) \in L^2(-1, 1)\) for a.e. \(x \in [-1, 1]\), so \(f \in W\). Furthermore, \(f(-1) = A_{-1}\) and \(f(1) = A_{-1} + \int_{-1}^{1} g(t) dt = A_1\) by (7.7). Now
\[
\|f_n - f\|_\phi^2 = \frac{1}{2} |f_n(-1) - f(-1)|^2 + \frac{1}{2} |f_n(1) - f(1)|^2 + \int_{-1}^{1} |f'_n(t) - f'(t)|^2 \, dt
\]
\[
= \frac{1}{2} |f_n(-1) - A_{-1}|^2 + \frac{1}{2} |f_n(1) - A_1|^2 + \int_{-1}^{1} |f'_n(t) - g(t)|^2 \, dt
\]
\[
\rightarrow 0 \quad \text{as } n \to \infty.
\]
Thus, \((W, \phi(\cdot, \cdot))\) is complete. \(\square\)

With \(W\) and \(\phi(\cdot, \cdot)\) as given above, define
\[
W_1 := \{ f \in W \mid f(\pm 1) = 0 \}
\]
\[
W_2 := \{ f \in W \mid f'(x) = c \text{ for some constant } c = c(f) \}.
\]
Remark 7.1. It is clear, from the definition, that \( W_2 \) is two-dimensional and, in fact, \( W_2 = \text{span}\{\widetilde{P}_0^{(-1,-1)}, \widetilde{P}_1^{(-1,-1)}\} \).

Theorem 7.3. The spaces \( W_1 \) and \( W_2 \) are closed, orthogonal subspaces of \((W, \phi(\cdot, \cdot))\) and
\[
W = W_1 \oplus W_2.
\]

Proof. Since \( W_2 \) is two-dimensional, it is a closed subspace of \( W \). By definition, the orthogonal complement of \( W_2 \) is given by
\[
W_2^\perp = \{f \in W \mid \phi(f, g) = 0 \text{ for all } g \in W_2\}.
\]
To see that \( W_1 \subset W_2^\perp \), let \( f \in W_1 \), \( g \in W_2 \) and consider
\[
\phi(f, g) = \frac{1}{2} f(-1)\overline{g}(-1) + \frac{1}{2} f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx.
\]
The first two terms on the right hand side vanish since \( f \in W_1 \); furthermore, \( \overline{g}'(x) = c \) for some constant \( c \in \mathbb{C} \) since \( g \in W_2 \). Moreover,
\[
\phi(f, g) = \int_{-1}^{1} f'(x)\overline{g}'(x)dx = c \int_{-1}^{1} f'(x)dx = c(f(1) - f(-1)) = 0,
\]
so \( f \in W_2^\perp \). Conversely, let \( f \in W_2^\perp \). Then, for any choice of constants \( A, B \in \mathbb{C} \), it is the case that
\[
0 = \phi(f(x), Ax + B)
= \frac{1}{2} f(-1)(-\overline{A} + \overline{B}) + \frac{1}{2} f(1)(\overline{A} + \overline{B}) + \overline{A} \int_{-1}^{1} f'(x)dx
= -\frac{3}{2}Af(-1) + \frac{3}{2} Af(1) + \frac{7}{2}(f(-1) + f(1)).
\]
By choosing \( A = 0, B \neq 0 \) and then \( A \neq 0 \) and \( B = 0 \), we find that \( f(\pm 1) = 0 \) so \( f \in W_1 \).

We note that, given \( f \in W \), we can (uniquely) write
\[
f = f_1 + f_2 \quad (f_i \in W_i \ (i = 1, 2)),
\]
where
\[
f_1(x) := f(x) - f_2(x) \quad \text{and} \quad f_2(x) := \frac{f(1) - f(-1)}{2} x + \frac{f(1) + f(-1)}{2} \quad (x \in [-1, 1]).
\]

We now turn our attention to the construction of the self-adjoint operator \( T \) in \((W, \phi(\cdot, \cdot))\), generated by the Jacobi differential expression \( \ell(\cdot) \) given in [4.1], that has the entire sequence of Jacobi polynomials \( \{\widetilde{P}_n^{(-1,-1)}(x)\}_{n=0}^{\infty} \) as eigenfunctions. The main idea is to use the decomposition in Theorem 7.3 to construct self-adjoint operators \( T_1 \) in \( W_1 \) and \( T_2 \) in \( W_2 \), both generated by \( \ell(\cdot) \), that have, respectively, the Jacobi polynomials \( \{\widetilde{P}_n^{(-1,-1)}(x)\}_{n=2}^{\infty} \) and \( \{\widetilde{P}_0^{(-1,-1)}(x), \widetilde{P}_1^{(-1,-1)}(x)\} \) as eigenfunctions. The operator \( T \) is then specifically defined to be the direct sum of \( T_1 \) and \( T_2 \). The construction of \( T_2 \) is straightforward, but constructing \( T_1 \) needs special attention. Indeed, it requires the first left-definite operator \( B_1 \), defined in Theorem 6.3, associated with the pair \((A, L^2((-1, 1); 1 - x^2)^{-1}) \), where \( A \) is the self-adjoint operator defined in [4.7]. The construction of \( T_1 \) begins with the following remarkable, and surprising, identification of the function spaces \( W_1 \) and \( V_1 \).
Theorem 7.4. \( W_1 = V_1 \), where \( V_1 \) is defined as in (6.5).

Proof. For the sake of completeness, we note that
\[
V_1 = \{ f : (-1, 1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1, 1); (1 - x^2)^{-1/2} f, f' \in L^2(-1, 1) \}.
\]
and observe that the condition \((1 - x^2)^{-1/2} f \in L^2(-1, 1)\) is equivalent to \( f \in L^2_{-1,-1}(-1, 1) \).

1. We first show that \((1) = 0\). Let \( f \in V_1 \). In particular, \( f \in AC[-1, 1] \). For \( 0 \leq x < 1 \),
\[
\int_0^x f'(t) dt = f(x) - f(0);
\]
consequently, since \( f' \in L^2(-1, 1) \subset L^1(-1, 1) \), we see that \( \lim_{x \to 1^-} f(x) \) exists and is finite. Similarly, \( \lim_{x \to 1^+} f(x) \) exists and is finite. Define
\[
f(\pm 1) := \lim_{x \to \pm 1^\pm} f(x),
\]
so \( f \in AC[-1, 1] \). It suffices to show that \( f(\pm 1) = 0 \). Suppose that \( f(1) \neq 0 \). Hence, for some \( c > 0 \), there exists \( 0 < \delta < 1 \) such that
\[
|f(x)| \geq c > 0
\]
for all \( x \in [\delta, 1] \). Since \( f \in L^2_{-1,-1}(-1, 1) \), we see that
\[
\infty > \int_1^\delta |f(x)|^2 (1 - x^2)^{-1} dx
\]
\[
\geq c^2 \int_0^\delta (1 - x^2)^{-1} dx = \infty,
\]
a contradiction. Hence, \( f(1) = 0 \); similarly, \( f(-1) = 0 \), so \( f \in W_1 \).

2. Let \( f \notin W_1 \). It suffices to show that \( f \in L^2((-1, 1); (1 - x^2)^{-1}) \). For \(-1 < x < 0\),
\[
(1 - x^2)^{-1/2} \int_{-1}^x f'(t) dt = (1 - x^2)^{-1/2} f(x)
\]
since \( f(-1) = 0 \). We use Theorem 3.1 on \((-1, 0)\) with
\[
\psi(x) = (1 - x^2)^{-1/2}, \ \varphi(x) = 1.
\]
Clearly, \( \psi \) is square integrable near 0 and \( \varphi \) is square integrable near \(-1\). Moreover,
\[
\int_{-1}^x dt \int_x^0 \frac{dt}{1 - t^2} \leq \int_{-1}^x dt \int_x^0 \frac{dt}{1 + t} = -(x + 1) \ln(1 + x),
\]
which is a bounded function on \((-1, 0)\). By Theorem 3.1 it follows that
\[
f \in L^2((-1, 0); (1 - x^2)^{-1}) ;
\]
a similar argument shows \( f \in L^2([0, 1); (1 - x^2)^{-1}) \). Hence \( W_1 \subset V_1 \). \( \square \)

Theorem 7.5. The inner products \( \phi(\cdot, \cdot) \) and \( (\cdot, \cdot)_1 \), where \( (\cdot, \cdot)_1 \) is defined in (6.7), are equivalent on \( W_1 = V_1 \).
Proof. First of all, we note that both \((W_1, \phi(\cdot, \cdot))\) and \((V_1, (\cdot, \cdot)_1)\) are Hilbert spaces. Let \(f \in W_1 = V_1.\) Since
\[
\|f\|_\phi^2 = \int_{-1}^{1} |f'(x)|^2\,dx \leq \int_{-1}^{1} \left[ |f'(x)|^2 + k |f(x)|^2 (1 - x^2)^{-1} \right]\,dx = \|f\|_1^2,
\]
we see, by the Open Mapping Theorem (see [21, Theorem 4.12-2 and Problem 9, p. 291]), that these inner products are equivalent. \(\square\)

Remark 7.2. Since, by Theorem 6.7, the Jacobi polynomials \(\left\{ \tilde{P}_n^{(-1,-1)} \right\}_{n=2}^\infty \) form a complete orthogonal set in the first left-definite space \(H_1 = (V_1, (\cdot, \cdot)_1),\) it follows from Theorem 7.3 that they are also a complete orthogonal set in \((W_1, \phi(\cdot, \cdot)).\) Together with Remark 7.4, we see that the full sequence of Jacobi polynomials \(\left\{ \tilde{P}_n^{(-1,-1)} \right\}_{n=0}^\infty \) form a complete orthogonal set in \(W = W_1 \oplus W_2.\)

We now construct a self-adjoint operator \(T_1\) in the space \(W_1,\) generated by the Jacobi expression \(\ell[],\) defined in (4.1), having the sequence of Jacobi polynomials \(\left\{ \tilde{P}_n^{(-1,-1)} \right\}_{n=2}^\infty\) as eigenfunctions. Recall that the first left-definite operator
\[
B_1 : D(B_1) := V_3 \subset H_1 \rightarrow H_1,
\]
associated with \((A, L^2_{-1, -1}(-1, 1),\) is self-adjoint in the first left-definite space \(H_1\) (see (6.4)) and given specifically by
\[
B_1[f](x) := \ell[f](x) = -(1 - x^2)f''(x) + kf(x),
\]
where \(f \in D(B_1) := V_3\)
\[
= \left\{ f : (-1, 1) \rightarrow \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1, 1); \right\}
\]
\[
(1 - x^2)f'', (1 - x^2)^{1/2}f', (1 - x^2)^{-1/2}f \in L^2(-1, 1) \right\}.
\]

More specifically, \(B_1\) is self-adjoint with respect to the first left-definite inner product \((\cdot, \cdot)_1.\) We now set out to prove that the operator \(T_1 : D(T_1) \subset W_1 \rightarrow W_1\) given by
\[
T_1 f = B_1 f = \ell[f]
\]
\[
f \in D(T_1) := V_3
\]
is self-adjoint in \((W_1, \phi(\cdot, \cdot)).\)

Theorem 7.6. Let \(f, g \in V_3.\) Then
\[
\lim_{x \rightarrow \pm 1} \frac{1}{2} (1 - x^2)f''(x)g'(x) = 0.
\]

Proof. It suffices to prove this result for \(x \rightarrow 1^-\). Let \(f, g \in V_3.\) Without loss of generality, assume that \(f, g\) are both real-valued. Since \(V_3 \subset V_1\) and \(T_1 f \in V_1\), we see that
\[
f', (T_1 f)', g' \in L^2(-1, 1).
\]
Hence \((T_1 f)'g', f'g' \in L^1(-1, 1).\) For \(0 \leq x < 1,\)
\[
\int_0^x (T_1 f)'(t)g'(t)\,dt = -\int_0^x ((1 - t^2)f''(t))'g'(t)\,dt + k \int_0^x f'(t)g'(t)\,dt.
\]
It follows that
\[
\lim_{x \rightarrow 1} \int_0^x ((1 - t^2)f''(t))'g'(t)\,dt
\]
exists and is finite. Integration by parts shows that
\[
\int_0^x ((1 - t^2)f''(t))'g'(t)\,dt = (1 - t^2)f''(t)g'(t) \bigg|_0^x - \int_0^x (1 - t^2)f''(t)g''(t)\,dt.
\]
Since \((1 - x^2)^{1/2}f''(x)\) and \((1 - x^2)^{1/2}g''(x)\) belong to \(L^2(-1, 1)\), we see that
\[(7.10) \quad (1 - x^2)f''(x)g''(x) \in L^1(-1, 1),
\]
so
\[
\lim_{x \to 1^-} x \int_0^x (1 - t^2)f''(t)g''(t)\,dt
\]
exists and is finite. It follows that
\[
\lim_{x \to 1^-} (1 - x^2)f''(x)g'(x)
\]
exists and is finite. Suppose
\[
\lim_{x \to 1^-} (1 - x^2)f''(x)g'(x) =: 2c
\]
where we assume that \(c \neq 0\). Without loss of generality, assume \(c > 0\). Then there exists \(x_0 \in [0, 1)\) such that
\[(7.11) \quad (1 - x^2)f''(x)g'(x) \geq c \quad \text{and} \quad f''(x) > 0, g'(x) > 0 \quad (x \in [x_0, 1)),
\]
implying
\[
(1 - x^2)f''(x) |g''(x)| \geq c |g''(x)| g'(x) \quad (x \in [x_0, 1)).
\]
Hence,
\[
\int_{x_0}^x (1 - t^2)f''(t) |g''(t)| \,dt \geq c \int_{x_0}^x \frac{|g''(t)|}{g'(t)} \,dt
\]
\[(7.12) \quad \geq c \int_{x_0}^x \frac{g''(t)}{g'(t)} \,dt = c \ln \left( g'(x) \right) - c_1 \quad (x \in [x_0, 1)).
\]
Therefore,
\[
\lim_{x \to 1^-} \sup_{x \to 1^-} |\ln \left( g'(x) \right)| < \infty.
\]
\textbf{Claim:} There exist positive constants \(M_1, M_2\) such that
\[(7.13) \quad M_1 < g'(x) < M_2 \quad (x \in [x_0, 1)).
\]
Otherwise, if \(g'(x)\) is unbounded above, there exists a sequence \(\{x_n\}_{n \geq 1} \subset [x_0, 1)\) such that
\[g'(x_n) \to \infty \text{ as } n \to \infty.
\]
It follows from \((7.12)\) that
\[(1 - x^2)f''(x)g''(x) \notin L^1(x_0, 1),
\]
contradicting \((7.10)\); hence, \(M_2 > 0\) exists as claimed. If an \(M_1\), satisfying \((7.13)\), does not exist, then there exists a sequence \(\{y_n\} \subset [x_0, 1)\) such that
\[g'(y_n) \to 0 \text{ as } n \to \infty.
\]
Again, it follows from \((7.12)\) that
\[(1 - x^2)f''(x)g''(x) \notin L^1(-1, 1),
\]
Hence, Remark 7.2), it suffices to show that
\[ T \]

Theorem 7.7. \( T \)

Since
\[ a \]

Proof. \[ \]

Integrating over \([x_0, 1]\) and using the fact that \((1 - x^2)^{1/2}f''(x) \in L^2(-1, 1)\), we see that
\[ \infty > \int_{x_0}^1 (1 - t^2)(f''(t))^2 dt > \bar{c}^2 \int_{x_0}^1 \frac{dt}{1 - t^2} = \infty, \]

a contradiction unless \( \bar{c} = c = 0 \). This completes the proof. \( \square \)

**Theorem 7.7.** \( T_1 \) is symmetric in \((W_1, \phi(\cdot, \cdot))\).

**Proof.** Since \( T_1 \) has the Jacobi polynomials \( \left\{ P_n^{(-1, -1)} \right\}_{n=2}^{\infty} \) as a complete set of eigenfunctions (see Remark 7.2), it suffices to show that \( T_1 \) is Hermitian. Let \( f, g \in D(T_1) = V_3 \). Since \( V_3 \subset V_1 \) and \( T_1 f, T_1 g \in V_1 \), we see that
\[ f(\pm 1) = g(\pm 1) = 0 = T_1 f(\pm 1) = T_1 g(\pm 1). \]

Hence,
\[
\phi(T_1 f, g) = \int_{-1}^{1} (T_1 f)'(x) \overline{g}(x) dx \\
= \int_{-1}^{1} \left[-((1 - x^2)f''(x))' + kf'(x) \right] \overline{g}(x) dx \\
= -(1 - x^2)f''(x) \overline{g}(x) \big|_{-1}^{1} + \int_{-1}^{1} \left[(1 - x^2)f''(x) \overline{g}'(x) + kf'(x) \overline{g}(x) \right] dx \\
= \int_{-1}^{1} \left[(1 - x^2)f''(x) \overline{g}'(x) + kf'(x) \overline{g}(x) \right] dx
\]

since \(-(1 - x^2)f''(x) \overline{g}(x) \big|_{-1}^{1} = 0 \) by Theorem 7.6. A similar calculation shows that
\[
\phi(f, T_1 g) = \int_{-1}^{1} \left[-((1 - x^2)\overline{g}'(x))' + k\overline{g}'(x) \right] f'(x) dx \\
= \int_{-1}^{1} \left[(1 - x^2)f''(x) \overline{g}'(x) + kf'(x) \overline{g}(x) \right] dx.
\]

Hence \( \phi(f, T_1 g) = \phi(T_1 f, g) \); that is, \( T_1 \) is symmetric in \((W_1, \phi(\cdot, \cdot))\). \( \square \)

**Theorem 7.8.** The operator \( T_1 \) has the following properties:

(i) \( T_1 \) is self-adjoint in \((W_1, \phi(\cdot, \cdot))\);
(ii) \( \sigma(T_1) = \{ n(n - 1) + k \mid n \geq 2 \} \);
(iii) \( \left\{ P_n^{(-1, -1)} \right\}_{n=2}^{\infty} \) is a complete orthonormal set of eigenfunctions of \( T_1 \) in \((W_1, \phi(\cdot, \cdot))\).

**Proof.** Part (iii) is established in Remark 7.2. Since it is well known (for example, see [20] Theorem 3, p. 373 and Theorem 6, p. 184) that a closed, symmetric operator with a complete set of
eigenfunctions is self-adjoint, it suffices, in order to establish (i), to show \(T_1\) is closed. To this end, let \(\{f_n\} \subseteq \mathcal{D}(T_1) = V_3\) such that 
\[
\begin{align*}
  f_n & \to f & \text{in } (W_1, \phi(\cdot, \cdot)) \\
  T_1 f_n & \to g & \text{in } (W_1, \phi(\cdot, \cdot)).
\end{align*}
\]
We show that \(f \in \mathcal{D}(T_1)\) and \(T_1 f = g\). Since, by Theorem 7.5, \(\phi(\cdot, \cdot)\) and \((\cdot, \cdot)_1\) are equivalent, there exist positive constants \(c_1\) and \(c_2\) such that
\[
c_1 \|f\|_\phi \leq \|f\|_1 \leq c_2 \|f\|_\phi \quad (f \in W_1 = V_1).
\]
Hence,
\[
\|f_n - f\|_1 \leq c_2 \|f_n - f\|_\phi \to 0;
\]
in particular,
\[
f_n \to f \quad \text{in } (W_1, (\cdot, \cdot)_1).
\]
Similarly,
\[
\|T_1 f_n - g\|_1 \leq c_2 \|T_1 f_n - g\|_\phi \to 0
\]
so
\[
T_1 f_n \to g \quad \text{in } (W_1, (\cdot, \cdot)_1).
\]
Since \(T_1\) is self-adjoint in \((W_1, (\cdot, \cdot)_1)\), it is closed implying that \(f \in \mathcal{D}(T_1)\) and \(T_1 f = g\). Also, we know that, for \(n \geq 2\),
\[
T_1 P_n^{(-1,-1)} = \ell[P_n^{(-1,-1)}] = (n(n-1) + k)P_n^{(-1,-1)}.
\]
This implies
\[
\{n(n-1) + k \mid n \geq 2\} \subseteq \sigma(T_1).
\]
However, from the completeness of \(\left\{P_n^{(-1,-1)}\right\}_{n=2}^{\infty}\) and since \(\lambda_n := n(n-1) + k \to \infty\), it follows from well-known results that
\[
\sigma(T_1) = \{n(n-1) + k \mid n \geq 2\},
\]
which proves (ii). \(\square\)

Next, we define the operator \(T_2 : \mathcal{D}(T_2) \subset W_2 \to W_2\) by
\[
(T_2 f)(x) = \ell[f](x)
\]
\[
\mathcal{D}(T_2) := W_2.
\]
It is straightforward to check that \(T_2\) is symmetric in \(W_2\) and, since \(\mathcal{D}(T_2) = W_2\), it follows that \(T_2\) is self-adjoint.

We now construct the self-adjoint operator \(T\) in \((W, \phi(\cdot, \cdot))\), generated by the Jacobi differential expression \(\ell[\cdot]\), which has the entire set of Jacobi polynomials \(\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}\) as eigenfunctions and spectrum \(\sigma(T) = \{n(n-1) + k \mid n \in \mathbb{N}_0\}\).

We define the domain of this operator \(T\) to be
\[
\mathcal{D}(T) := \mathcal{D}(T_1) \oplus \mathcal{D}(T_2) = V_3 \oplus W_2.
\]
Then each \(f \in \mathcal{D}(T)\) can be written as \(f = f_1 + f_2\), where \(f_i \in \mathcal{D}(T_i)\) (\(i = 1, 2\)). Define \(T : \mathcal{D}(T) \subset W \to W\) by
\[
Tf := T_1 f_1 + T_2 f_2 = \ell[f_1] + \ell[f_2] = \ell[f].
\]
A proof that operators of this form are self-adjoint can be found in [18 Theorem 11.1]. Furthermore, since we know explicitly the domains of $T_1$ and $T_2$, we can specifically determine the domain $D(T)$ of $T$.

**Theorem 7.9.** $T$ is self-adjoint in $(W, \phi(\cdot, \cdot))$ and has domain

$$D(T) = \{ f : [-1, 1] \to \mathbb{C} \mid f \in AC[-1, 1]; f', f'' \in AC_{\text{loc}}(-1, 1); (1 - x^2)f''' + (1 - x^2)^{1/2}f'', f' \in L^2(-1, 1) \}.$$  

Furthermore, $\sigma(T) = \{ n(n-1) + k \mid n \in \mathbb{N}_0 \}$ and has the Jacobi polynomials \( \{ P_n^{(-1,-1)} \}_{n=0}^\infty \) as a complete set of eigenfunctions.

**Proof.** Define

$$D := \{ f : [-1, 1] \to \mathbb{C} \mid f \in AC[-1, 1]; f', f'' \in AC_{\text{loc}}(-1, 1); (1 - x^2)f''' + (1 - x^2)^{1/2}f'', f' \in L^2(-1, 1) \}.$$  

Since $D(T_i) \subset D$ for $i = 1, 2$, it is clear that $D(T) = D(T_1) \oplus D(T_2) \subset D$. Conversely, let $f \in D$. Writing $f = f_1 + f_2$ where each $f_i$ $(i = 1, 2)$ is given as in (7.8), we see that $f \in D(T)$. The proof of the last statement in the theorem is clear. \[ \square \]

Using Theorem 7.1 we can further refine the domain of $T$; we leave the details to the reader.

**Corollary 7.1.** The domain of $T$ is given by

$$D(T) = \{ f : [-1, 1] \to \mathbb{C} \mid f \in AC[-1, 1]; f', f'' \in AC_{\text{loc}}(-1, 1); (1 - x^2)f''' \in L^2(-1, 1) \}.$$  



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