CONVERGENCE ABCISSAS FOR DIRICHLET SERIES WITH MULTIPLICATIVE COEFFICIENTS

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ABSTRACT. This note deals with the relationship between the abscissas of simple, uniform and absolute convergence for the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, when the coefficients $a_n$ are either multiplicative or completely multiplicative.

Consider the ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + it.$$

A basic fact is that Dirichlet series converge in half-planes, just as power series converge in discs. However, Dirichlet series can have different types of convergence in distinct half-planes. It was H. Bohr [4, 6] who first studied the relationship between the following three convergence abscissas:

$$\sigma_c(f) = \inf \left\{ \sigma : \sum_{n=1}^{\infty} a_n n^{-\sigma} \text{ converges} \right\} \quad \text{(Simple)},$$

$$\sigma_b(f) = \inf \left\{ \sigma : \sum_{n=1}^{\infty} a_n n^{-\sigma-it} \text{ converges uniformly for } t \in \mathbb{R} \right\} \quad \text{(Uniform)},$$

$$\sigma_a(f) = \inf \left\{ \sigma : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \text{ converges} \right\} \quad \text{(Absolute)}.$$

Clearly $\sigma_c \leq \sigma_b \leq \sigma_a$, and it is easy to deduce that $\sigma_a(f) - \sigma_c(f) \leq 1$. Under the assumption that the Dirichlet series $f$ does not converge at $s = 0$, the Cauchy–Hadamard type formulas for these abscissas are:

$$\sigma_c(f) = \limsup_{x \to \infty} \frac{1}{\log x} \log \left( \sum_{n \leq x} |a_n| \right),$$

$$\sigma_b(f) = \limsup_{x \to \infty} \frac{1}{\log x} \log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n \leq x} a_n n^{-it} \right| \right),$$

$$\sigma_a(f) = \limsup_{x \to \infty} \frac{1}{\log x} \log \left( \sum_{n \leq x} |a_n| \right).$$

By choosing $a_n = \pm 1$ in a suitable manner, it is now easy to construct a Dirichlet series with $\sigma_a - \sigma_c = \alpha$, for any $\alpha \in [0, 1]$. Moreover, the Cauchy–Schwarz inequality can be applied to show that $\sigma_a - \sigma_b \leq 1/2$. The fact that there are Dirichlet series with $\sigma_a - \sigma_b = \beta$ for any $\beta \in [0, 1/2]$. The authors are supported by Grant 227768 of the Research Council of Norway.
is a result due to Bohnenblust–Hille [3]. See [2] for an excellent exposition of these results, containing clear proofs using modern techniques.

The inequality used in [3] to obtain this result was recently substantially improved [1] [11], and the improved version can be used to get a precise qualitative version of the optimality of \( \beta = 1/2 \) in view of the Cauchy–Hadamard formulas given above (see [10]).

It is interesting to consider the difference between these abscissas when the coefficients have some added multiplicative structure (recall that \( a_n \) is multiplicative if \( a_{mn} = a_m a_n \) whenever \( \gcd(m, n) = 1 \) and is completely multiplicative if this relationship persists for any choice of \( m \) and \( n \)). For example, the Riemann hypothesis is equivalent to \( \sigma_a - \sigma_c = 1/2 \) for the series

\[
1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s} = \prod_p (1 - p^{-s}),
\]

where \( \mu(n) \) is the Möbius function, which of course is multiplicative.

Lévy [16] argued that any random model of the Möbius function should take into account the multiplicative nature of \( \mu(n) \), and, following this, Wintner [17] showed that the Dirichlet series represented by the Euler product

\[
\prod_p (1 + \varepsilon_p p^{-s})
\]

has \( \sigma_c = 1/2 \) almost always, and concluded that “the Riemann hypothesis is almost always true”. Here \( \varepsilon_p \) denotes the Rademacher random variables which assumes the values \( \pm 1 \) with equal probability.

Motivated by this result regarding “typical” behavior, we will investigate the possible values for \( \sigma_a(f) - \sigma_c(f) \) and \( \sigma_a(f) - \sigma_b(f) \), when the coefficients of the Dirichlet series \( f \) are either multiplicative or completely multiplicative. For the first quantity, we have the following.

**Theorem 1.** There exists a Dirichlet series \( f \) with completely multiplicative coefficients such that \( \sigma_a(f) - \sigma_c(f) = \alpha \) for any \( \alpha \in [0, 1] \).

**Proof.** The cases \( \alpha = 0 \) and \( \alpha = 1 \) follow from considering the Riemann zeta function and the Dirichlet \( L \)-function of a non-principal character, respectively.

For \( 0 < \alpha < 1 \), consider

\[
g_a(s) = (1 - 3^{1-a-s})^{-1} = \sum_{k=0}^{\infty} 3^{(1-a)k} 3^{-k s}.
\]

We now let \( \chi \) denote the non-principal character of modulus 3 and we consider the Dirichlet series given by the product

\[ f(s) = g_a(s)L(s, \chi). \]

Clearly, \( f(s) \) has completely multiplicative coefficients, since \( \chi(3) = 0 \) and since \( g_a(s) \) is a geometric series. The latter fact also implies that \( \sigma_c(g_a) = \sigma_a(g_a) = 1 - \alpha \), and for the \( L \)-function of a non-principal character we have \( \sigma_c = 0 \) and \( \sigma_a = 1 \). Now, the product of a conditionally convergent series and an absolutely convergent series is conditionally convergent, so we have \( \sigma_c(f) \leq 1 - \alpha \). This cannot be improved, since \( f(1-\alpha) \) does not converge (an infinite number of the terms have modulus 1), so \( \sigma_c(f) = 1 - \alpha \).

The product of two absolutely convergent series is absolutely convergent, so \( \sigma_a(f) \leq 1 \). We let \( |f|(s) \) denote the Dirichlet series where we have replaced the coefficients by their absolute values. We see that \( |f|(1) \) diverges since \( L(1, |\chi|) \) diverges, the coefficients of \( g_a \) are positive, and \( g_a(1) \neq 0 \). In conclusion, we have \( \sigma_a(f) - \sigma_c(f) = 1 - (1 - \alpha) = \alpha \). \( \square \)
Of course, \( g_a(s) \) can be replaced by any power series in \( 3^{-s} \) with non-negative coefficients and \( \sigma_a = 1 - \alpha \) to obtain an example which is multiplicative, but not completely multiplicative.

Our next result can be considered as an example of the following scheme: A contractive function theoretic result concerning power series, can possibly be applied multiplicatively to obtain a similar result for ordinary Dirichlet series. A recent example of this type of result is [9, Thm. 2]. See also the proof of the main theorem in [14].

**Theorem 2.** Suppose that the Dirichlet series \( f \) has multiplicative coefficients. Then \( \sigma_a = \sigma_b \).

It was H. Bohr who realized the connection between Dirichlet series and function theory in polydiscs [5], through the correspondence \( p_j^{-s} \leftrightarrow z_j \). Inspecting the prime factorization \( n = \prod_j p_j^{a_j} \), we associate to the integer \( n \) the multi-index \( a(n) = (a_1, a_2, \ldots) \). The Bohr lift of the Dirichlet series \( f(s) = \sum_{n \geq 1} a_n n^{-s} \) is the power series

\[
\mathcal{B} f(z) = \sum_{n=1}^{\infty} a_n z^{a(n)}.
\]

Using Kronecker's theorem [12, Ch. 13] (see also [13, Sec. 2.2]), we may conclude that

\[
\| f \|_\infty := \sup_{\sigma > 0} |f(s)| = \sup_{z \in \mathbb{D}^\infty \cap \mathbb{C}_0} |\mathcal{B} f(z)|.
\]

Now, let us suppose that \( f \) has multiplicative coefficients. We may then factor

\[
f(s) = \prod_j \left(1 + \sum_{k=1}^{\infty} a_{p_j^k} p_j^{-ks}\right) = \prod_j f_j(s),
\]

at least for \( \sigma > \sigma_a \). In particular, since each prime only appears in one factor, we also obtain

\[
\| f \|_\infty = \sup_{z \in \mathbb{D}^\infty \cap \mathbb{C}_0} |\mathcal{B} f(z)| = \prod_j \sup_{z_j \in \mathbb{D}} |\mathcal{B} f_j(z_j)| = \prod_j \| f_j \|_\infty.
\]

To complete the proof of Theorem 2 we will require the following.

**Lemma.** Let \( F(z) = \sum_{m \geq 0} b_m z^m \) and suppose that \( \sup_{z \in \mathbb{D}} |F(z)| < \infty \). Let \( 0 \leq r < 1 \). Then

\[
\sum_{m=0}^{\infty} |b_m| r^m \leq C(r) \sup_{z \in \mathbb{D}} |F(z)|,
\]

where

\[
C(r) = \begin{cases} 
1, & 0 \leq r \leq 1/3, \\
1/\sqrt{1 - r^2}, & 1/3 < r < 1.
\end{cases}
\]

**Proof.** The first estimate is Bohr's inequality [7], the second follows from the Cauchy–Schwarz inequality, Parseval's formula and the maximum modulus principle. \( \square \)

The contractive function theoretic result for power series mentioned earlier is that \( C(r) = 1 \) when \( 0 \leq r \leq 1/3 \). It should also be pointed out that the values \( C(r) \) prescribed above are not optimal when \( r > 1/3 \), and that precise estimates in this range can be found in [8].

**Proof of Theorem 2.** Let the coefficients of \( f(s) = \sum_{n \geq 1} a_n n^{-s} \) be multiplicative, and fix \( \epsilon > 0 \). Since uniform convergence implies boundedness, we may (after a horizontal translation) assume that \( \sigma_b(f) = -\epsilon \) so that \( \| f \|_\infty < \infty \). We then want to prove that under this assumption we have

\[
\sum_{n=1}^{\infty} |a_n| n^{-\epsilon} < \infty,
\]
so that \( \sigma_a(f) \leq \epsilon \), and hence \( \sigma_a(f) - \sigma_b(f) \leq 2\epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( \sigma_a(f) = \sigma_b(f) \). By the discussion preceding it and the lemma, we obtain

\[
\sum_{n=1}^{\infty} |a_n| n^{-\epsilon} = \prod_{p} \left( 1 + \sum_{k=1}^{\infty} |a_{p^k}| p^{-k\epsilon} \right) \leq \left( \prod_{p^\epsilon < 3} \| f_p \|_{\infty} \right) \left( \prod_{3 \leq p^\epsilon < \infty} 1 \cdot \| f_p \|_{\infty} \right)
\]

\[
= \left( \prod_{p^\epsilon < 3} \frac{1}{\sqrt{1 - p^{-2\epsilon}}} \right) \left( \prod_{p} \| f_p \|_{\infty} \right) = \left( \prod_{p^\epsilon < 3} \frac{1}{\sqrt{1 - p^{-2\epsilon}}} \right) \| f \|_{\infty} < \infty.
\]

Theorem 2 allows us to provide a strengthening of a result of Bohr in the case of Dirichlet series with multiplicative coefficients.

**Corollary.** Let \( f(s) = \sum_{n=1} a_n n^{-s} \) have multiplicative coefficients and suppose that \( f \) is somewhere convergent. If \( f \) has a bounded analytic continuation to \( \sigma \geq \sigma_0 + \epsilon \), for every \( \epsilon > 0 \), then \( \sigma_a(f) = \sigma_0 \).

**Proof.** Bohr's theorem states that \( \sigma_b(f) = \sigma_0 \) without any assumptions on the coefficients of \( f \). By Theorem 2 we have \( \sigma_a(f) = \sigma_b(f) = \sigma_0 \). \( \square \)

**Note added in proof**

In a recent paper [15], J. Kaczorowski and A. Perelli have independently proven Theorem 2 under the additional assumption that the Dirichlet series belongs to the Selberg class. Their methods are slightly different and do not involve analysis on the polydisc.

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