SMOOTHNESS OF THE LAW OF THE SUPREMUM OF THE FRACTIONAL BROWNIAN MOTION

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Submitted 5 May 2003, accepted in final form 25 July 2003

AMS 2000 Subject classification: 60H07, 60G18
Keywords: Malliavin calculus, fractional Brownian motion, fractional calculus

Abstract
This note is devoted to prove that the supremum of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ has an infinitely differentiable density on $(0, \infty)$. The proof of this result is based on the techniques of the Malliavin calculus.

1 Introduction

A fractional Brownian motion (fBm for short) of Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B = \{B_t, t \in [0, 1]\}$ with the covariance function

$$ R_H(t, s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right). \tag{1} $$

Notice that if $H = \frac{1}{2}$, the process $B$ is a standard Brownian motion. From (1) it follows that

$$ E |B_t - B_s|^2 = |t - s|^{2H}, $$

and, as consequence, $B$ has $\alpha$-Hölder continuous paths for any $\alpha < H$.

The Malliavin calculus is a suitable tool for the study of the regularity of the densities of functionals of a Gaussian process. We refer to [7] and [8] for a detailed presentation of this theory. This approach is particularly useful when analytical methods are not available. In [5] the Malliavin calculus has been applied to derive the smoothness of the law of the supremum.

1SUPPORTED BY THE DGES GRANT BFM2000-0598
of the Brownian sheet. In order to obtain this result, the authors establish a general criterion for the smoothness of the density, assuming that the random variable is locally in $D^1$. The aim of this paper is to study the smoothness of the law of the supremum of a fBm using the general criterion obtained in [5].

The organization of this note is as follows. In Section 2 we present some preliminaries on the fBm and we review the basic facts on the Malliavin calculus and on the fractional calculus that will be used in the sequel. In Section 3 we state the general criterion for the smoothness of densities and we apply it to the supremum of the fBm.

2 Preliminaries

2.1 Fractional Brownian motion

Fix $H \in (0, 1)$ and let $B = \{B_t, t \in [0, 1]\}$ be a fBm with Hurst parameter $H$. That is, $B$ is a zero mean Gaussian process with covariance function given by (1). Let $\{\mathcal{F}_t, t \in [0, 1]\}$ be the family of sub-$\sigma$-fields of $\mathcal{F}$ generated by $B$ and the $P$-null sets of $\mathcal{F}$. We denote by $\mathcal{E} \subset \mathcal{H}$ the class of step functions on $[0, 1]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(s, t).$$

The mapping $1_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with $B$. The covariance kernel $R_H(t, s)$ can be written as

$$R_H(t, s) = \int_0^{\wedge t} K_H(t, r)K_H(s, r)dr,$$

where $K_H$ is a square integrable kernel given by (see [4]):

$$K_H(t, s) = \Gamma(H + 1)\frac{1}{2}^{-1}(t - s)^{H - \frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - \frac{t}{s}),$$

$F(a, b, c, z)$ being the Gauss hypergeometric function. Consider the linear operator $K_H^*$ from $\mathcal{E}$ to $L^2([0, 1])$ defined by

$$(K_H^*\varphi)(s) = K_H(1, s)\varphi(s) + \int_s^1 (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s)dr. \quad (2)$$

For any pair of step functions $\varphi$ and $\psi$ in $\mathcal{E}$ we have (see [3])

$$\langle K_H^*\varphi, K_H^*\psi \rangle_{L^2([0, 1])} = \langle \varphi, \psi \rangle_{\mathcal{H}}. \quad (3)$$

As a consequence, the operator $K_H^*$ provides an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0, 1])$. Hence, the process $W = \{W_t, t \in [0, T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1}(1_{[0,t]})) \quad (4)$$

is a Wiener process, and the process $B^H$ has an integral representation of the form

$$B_t^H = \int_0^t K_H(t, s)dW_s, \quad (5)$$

because $(K_H^*1_{[0,t]}) (s) = K_H(t, s)$.
2.2 Fractional calculus

We refer to [9] for a complete survey of the fractional calculus. Let us introduce here the main definitions. If \( f \in L^1([0,1]) \) and \( \alpha > 0 \), the right and left-sided fractional Riemann-Liouville integrals of \( f \) of order \( \alpha \) on \([0,1] \) are given almost surely for all \( t \in [0,1] \) by

\[
I^\alpha_0 f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds
\]

and

\[
I^\alpha_1 f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s) \, ds
\]

respectively, where \( \Gamma \) denotes the Gamma function.

Fractional differentiation can be introduced as an inverse operation. For any \( p > 1 \) and \( \alpha > 0 \), \( I^\alpha_p, (L^p) \) (resp. \( I^\alpha_1 (L^p) \)) will denote the class of functions \( f \in L^p([0,1]) \) which may be represented as an \( I^\alpha_p \) (resp. \( I^\alpha_1 \))- integral of some function \( \Phi \) in \( L^p([0,1]) \). If \( f \in I^\alpha_p (L^p) \) (resp. \( I^\alpha_1 (L^p) \)), the function \( \Phi \) such that \( f = I^\alpha_p \Phi \) (resp. \( I^\alpha_1 \Phi \)) is unique in \( L^p([0,1]) \) and is given by

\[
D^\alpha_0 f(t) = \frac{(-1)^{\alpha+1}}{\Gamma(1-\alpha)} \left( \frac{f(s)}{s^\alpha} - \alpha \int_0^t \frac{f(t) - f(s)}{(t-s)^{1+\alpha}} \, ds \right)
\]

\[
D^\alpha_1 f(t) = \frac{(-1)^{\alpha+1}}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(1-s)^\alpha} - \alpha \int_t^1 \frac{f(s) - f(t)}{(s-t)^{1+\alpha}} \, ds \right)
\]

where the convergence of the integrals at the singularity \( t = s \) holds in the \( L^p \)-sense.

When \( \alpha p > 1 \) any function in \( I^\alpha_p (L^p) \) is \((\alpha - \frac{1}{p})\)- Hölder continuous. On the other hand, any Hölder continuous function of order \( \beta > \alpha \) has fractional derivative of order \( \alpha \). That is, \( C^\beta([a,b]) \subset I^\alpha_p (L^p) \) for all \( p > 1 \).

Recall that by construction for \( f \in I^\alpha_1 (L^p) \),

\[
I^\alpha_1 (D^\alpha_p f) = f
\]

and for general \( f \in L^1([a,b]) \) we have

\[
D^\alpha_p (I^\alpha_1 f) = f.
\]

The operator \( K^*_H \) can be expressed in terms of fractional integrals or derivatives. In fact, if \( H > \frac{1}{2} \), we have

\[
(K^*_H \varphi)(s) = c_H \Gamma(H - \frac{1}{2}) s^{\frac{1}{2} - H} (I^{H - \frac{1}{2}}_1 u^{H - \frac{1}{2}} \varphi(u))(s), \quad (10)
\]

where \( c_H = \left[ \frac{H(H-1)}{2} \right]^{1/2} \), and if \( H < \frac{1}{2} \), we have

\[
(K^*_H \varphi)(s) = d_H s^{\frac{1}{2} - H} (D^{\frac{1}{2} - H}_1 u^{H - \frac{1}{2}} \varphi(u))(s), \quad (11)
\]

where \( d_H = c_H \Gamma(H + \frac{1}{2}) \).
2.3 Malliavin calculus

We briefly recall some basic elements of the stochastic calculus of variations with respect to the fBm $B$. For more complete presentation on the subject, see [7] and [8].

The process $B = \{B_t, t \in [0, 1]\}$ is Gaussian and, hence, we can develop a stochastic calculus of variations (or Malliavin calculus) with respect to it. Let $C_b^\infty(\mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables $F$ of the form

$$F = f(B(h_1), \ldots, B(h_n)), \quad (12)$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $h_1, \ldots, h_n \in \mathcal{H}$.

The derivative operator $D$ of a smooth and cylindrical random variable $F$ of the form (12) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n)) h_i.$$ 

In this way the derivative $DF$ is an element of $L^2(\Omega; \mathcal{H})$. The iterated derivative operator of $D$ is denoted by $D^k$. It is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}^\otimes k)$ for each $k \geq 1$, and each $p \geq 1$. We denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\| F \|^p_{k,p} = E(|F|^p) + \sum_{j=1}^k E\left(\| D^j F \|^p_{\mathcal{H}^\otimes j} \right).$$

We set $\mathbb{D}^\infty = \cap_{k,p} \mathbb{D}^{k,p}$.

For any given Hilbert space $V$, the corresponding Sobolev space of $V$-valued random variables can also be introduced. More precisely, let $\mathcal{S}_V$ denote the family of $V$-valued smooth random variables of the form

$$F = \sum_{j=1}^n F_j v_j, \quad (v_j, F_j) \in V \times \mathcal{S}.$$

We define

$$D^k F = \sum_{j=1}^n D^k F_j \otimes v_j, \quad k \geq 1.$$ 

Then $D^k$ is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; \mathcal{H}^\otimes k \otimes V)$ for any $p \geq 1$. For any integer $k \geq 1$ and for any real number $p \geq 1$, a norm is defined on $\mathcal{S}_V$ by

$$\| F \|^p_{k,p,V} = E(|F|^p_V) + \sum_{j=1}^k E\left(\| D^j F \|^p_{\mathcal{H}^\otimes j \otimes V} \right).$$ 

We denote by $\mathbb{D}^{k,p}(V)$ the completion of $\mathcal{S}_V$ with respect to the norm $\| \cdot \|_{k,p,V}$. We set $\mathbb{D}^\infty(V) = \cap_{k,p} \mathbb{D}^{k,p}(V)$.

Our main result will be based on the application of the following general criterion for smoothness of densities for one-dimensional random variable established in [5].

**Theorem 1** Let $F$ be a random variable in $\mathbb{D}^{1,2}$. Let $A$ be an open subset of $\mathbb{R}$. Suppose that there exist an $\mathcal{H}$-valued random variable $u_A$ and a random variable $G_A$ such that

$$F = f(B(h_1), \ldots, B(h_n)),$$ 

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $h_1, \ldots, h_n \in \mathcal{H}$.

The process $B = \{B_t, t \in [0, 1]\}$ is Gaussian and, hence, we can develop a stochastic calculus of variations (or Malliavin calculus) with respect to it. Let $C_b^\infty(\mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables $F$ of the form

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In this way the derivative $DF$ is an element of $L^2(\Omega; \mathcal{H})$. The iterated derivative operator of $D$ is denoted by $D^k$. It is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}^\otimes k)$ for each $k \geq 1$, and each $p \geq 1$. We denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\| F \|^p_{k,p} = E(|F|^p) + \sum_{j=1}^k E\left(\| D^j F \|^p_{\mathcal{H}^\otimes j} \right).$$

We set $\mathbb{D}^\infty = \cap_{k,p} \mathbb{D}^{k,p}$.

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Then $D^k$ is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; \mathcal{H}^\otimes k \otimes V)$ for any $p \geq 1$. For any integer $k \geq 1$ and for any real number $p \geq 1$, a norm is defined on $\mathcal{S}_V$ by

$$\| F \|^p_{k,p,V} = E(|F|^p_V) + \sum_{j=1}^k E\left(\| D^j F \|^p_{\mathcal{H}^\otimes j \otimes V} \right).$$ 

We denote by $\mathbb{D}^{k,p}(V)$ the completion of $\mathcal{S}_V$ with respect to the norm $\| \cdot \|_{k,p,V}$. We set $\mathbb{D}^\infty(V) = \cap_{k,p} \mathbb{D}^{k,p}(V)$.

Our main result will be based on the application of the following general criterion for smoothness of densities for one-dimensional random variable established in [5].

**Theorem 1** Let $F$ be a random variable in $\mathbb{D}^{1,2}$. Let $A$ be an open subset of $\mathbb{R}$. Suppose that there exist an $\mathcal{H}$-valued random variable $u_A$ and a random variable $G_A$ such that
(i) \( u_A \in \mathcal{D}^\infty (\mathcal{H}) \),
(ii) \( G_A \in \mathcal{D}^\infty \) and \( G_A^{-1} \in L^p (\Omega) \) for any \( p \geq 2 \) and,
(iii) \( \langle DF, u_A \rangle_{\mathcal{H}} = G_A \) on \( \{ F \in A \} \).
Then the random variable \( F \) possesses an infinitely differentiable density on the set \( A \).

3 Supremum of the fractional Brownian motion

The process \( B \) has a version with continuous paths as result of being \( \alpha \)-Hölder continuous for any \( \alpha < H \). Set
\[
M = \sup_{0 \leq s \leq 1} B_s.
\]
From results of [10] we know that \( M \) possesses an absolutely continuous density on \((0, \infty)\). In order to apply Theorem 1, we will first recall some results on this supremum.

Lemma 2 The process \( B \) attains its maximum on a unique random point \( T \).

Proof. The proof of this lemma would follow by the same arguments as the proof of Lemma 3.1 of [5], applying the criterion for absolute continuity of the supremum of a Gaussian process established in [10].

The following lemma will ensure the weak differentiability of the supremum of the fBm and give the value of its derivative.

Lemma 3 The random variable \( M \) belongs to \( \mathcal{D}^{1,2} \) and it holds \( D_tM = 1_{[0,T]}(t) \), for any \( t \in [0,1] \), where \( T \) is the point where the supremum is attained.

Proof. Similar to the proof of Lemma 3.2. in [5].

With the above results in hands, we are in position to prove our main result.

Proposition 4 The random variable \( M = \sup_{0 \leq s \leq 1} B_s \) possesses an infinitely differentiable density on \((0, \infty)\).

Proof. Fix \( a > 0 \) and set \( A = (a, \infty) \). Define the following random variable
\[
T_a = \inf \left\{ t \in [0, 1] \mid \sup_{0 \leq s \leq t} B_s > a \right\}.
\]
Recall that \( T_a \) is a stopping time with respect to the filtration \( \{ \mathcal{F}_t, t \in [0, 1] \} \) and notice that \( T_a \leq T \) on the set \( \{ M > a \} \). Hence, by Lemma 3, it holds that
\[
\{ M > a, t \leq T_a \} \subset \{ D_tM = 1 \}.
\]
Set
\[
\Delta = \left\{ (p, \gamma) \in \mathbb{N}^* \times (0, \infty) \mid \frac{1}{2p} < \gamma < H \right\}.
\]
For any \( (p, \gamma) \in \Delta \), we define the process \( Y \) on \([0, 1]\) by setting, for any \( t \in [0, 1] \)
\[
Y_t = \int_0^t \int_0^t \frac{|B_s - B_r|^{2p}}{|s - r|^{2p+1}} ds dr.
\]
We will need the following property: There exists a constant $R$ depending on $a, \gamma$ and $p$ such that
\[ Y_t < R \text{ implies that } \sup_{0 \leq s \leq t} B_s \leq a. \tag{14} \]

To prove this fact we use the Garsia, Rodemich and Rumsey Lemma in [6]. This lemma applied to the function $s \in [0, t] \to B_s$, with the hypothesis that $Y_t < R$, implies
\[ |B_s - B_r| \leq C_{p, \gamma} R^{\frac{p}{p}} |s - r|^{\gamma - \frac{1}{p}} \text{ for all } s, r \in [0, t]. \]

This implies that $\sup_{0 \leq s \leq t} |B_s| \leq C_{p, \gamma} R^{\frac{p}{p}}$. It suffices to choose $R$ in such a way that $C_{p, \gamma} R^{\frac{p}{p}} < a$.

Let $\psi : \mathbb{R}^+ \to [0, 1]$ be an infinitely differentiable function such that
\[ \psi(x) = \begin{cases} 0 & \text{if } x > R, \\ \psi(x) & \text{if } x \in \left[ \frac{R}{2}, R \right], \\ 1 & \text{if } x \leq \frac{R}{2}. \end{cases} \]

Consider the $\mathcal{H}$-valued random variable given by
\[ u_A = \left( K_H^* \right)^{-1} \left( K_H^{*, \text{adj}} \right)^{-1} (\psi(Y_t)), \tag{15} \]
where $K_H^*$ is the operator defined in (2) and $K_H^{*, \text{adj}}$ denotes its adjoint in $L^2([0, 1])$. We claim that the random element $u_A$ introduced in (15) and the random variable $G_A = \int_0^1 \psi(Y_t) \, dt$ satisfy the conditions of Theorem 1.

Let us first show that $u_A$ belongs to $D^\infty(\mathcal{H})$. Fix an integer $j \geq 0$. It suffices to show that for any $q \geq 1$,
\[ E \| D^j u_A \|_{\mathcal{H}^\otimes (j+1)}^q < \infty. \tag{16} \]

The $j$-th order derivative $D^j$ of the function $\psi(Y_t)$ is evaluated with the help of the Faà di Bruno formula, see formula [24.1.2] in [1], as follows
\[ D^j \psi(Y_t) = \sum_{n=1}^{j} \psi^{(n)}(Y_t) \sum_{i, l, \sum_{i=1}^j l_i = n, \sum_{i=1}^j i l_i = j} \prod_{i=1}^j \frac{1}{i!} \left( D^i Y_t \right)^{l_i}. \]

Hence, in order to show (16) it suffices to check that
\[ E \left\| \left( K_H^* \right)^{-1} \left( K_H^{*, \text{adj}} \right)^{-1} \left[ \psi^{(n)}(Y_t) \prod_{i=1}^j \left( D^i Y_t \right)^{l_i} \right] \right\|_{\mathcal{H}^\otimes (j+1)}^q < \infty. \tag{17} \]

for all $1 \leq n \leq j, \sum_{i=1}^j l_i = n, \sum_{i=1}^j i l_i = j$. Set
\[ \Lambda_t = \psi^{(n)}(Y_t) \prod_{i=1}^j \left( D^i Y_t \right)^{l_i}. \]

By (3)
\[ \left\| \left( K_H^* \right)^{-1} \left( K_H^{*, \text{adj}} \right)^{-1} \Lambda_t \right\|_{\mathcal{H}^\otimes (j+1)} = \left\| \left( K_H^{*, \text{adj}} \right)^{-1} \Lambda_t \right\|_{\mathcal{H}^\otimes (j+1) \otimes L^2([0, 1])}. \tag{18} \]
From (10), if $H > \frac{1}{2}$, we obtain
\[
(K_{H}^{*, \text{adj}})^{-1} \Lambda_t = d_H t^{H-\frac{1}{2}} I_{0+} t^{\frac{1}{2}-H} \Lambda_t
\]
\[
= \frac{d_H}{\Gamma \left( \frac{3}{2} - H \right)} \left( t^{\frac{3}{2}-H} \Lambda_t - \left( H - \frac{1}{2} \right) t^{H-\frac{1}{2}} \int_{0}^{t} \frac{t^{\frac{3}{2}-H} \Lambda_t - s^{\frac{3}{2}-H} \Lambda_s}{(t-s)^{H+\frac{1}{2}}} ds \right)
\]
where $d_H = (c_H \Gamma(H - \frac{1}{2}))^{-1}$. After some computations we get
\[
(K_{H}^{*, \text{adj}})^{-1} \Lambda_t = \beta(t) \Lambda_t + \int_{0}^{t} R(t, \theta) \Lambda_{\theta} d\theta,
\] (19)
where
\[
\beta(t) = \frac{d_H}{\Gamma \left( \frac{3}{2} - H \right)} \left( t^{\frac{3}{2}-H} - \left( H - \frac{1}{2} \right) t^{H-\frac{1}{2}} \int_{0}^{t} \frac{t^{\frac{3}{2}-H} - s^{\frac{3}{2}-H}}{(t-s)^{H+\frac{1}{2}}} ds \right)
\]
and
\[
R(t, \theta) = -\frac{d_H \left( H - \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - H \right)} \int_{0}^{\theta} s^{\frac{3}{2}-H} (t-s)^{-H-\frac{1}{2}} ds.
\]
On the other hand, if $H < \frac{1}{2}$, from (11) we obtain
\[
(K_{H}^{*, \text{adj}})^{-1} \Lambda_t = e_H t^{H-\frac{1}{2}} I_{0+} t^{\frac{1}{2}-H} \Lambda_t,
\] (20)
where $e_H = (c_H \Gamma(H + \frac{1}{2}))^{-1}$.

In the sequel $C_H$ will denote a generic constant depending on $H$. If $H > \frac{1}{2}$, (19) yields
\[
\left\| (K_{H}^{*, \text{adj}})^{-1} \Lambda_t \right\|_{\mathcal{H}^{(\oplus)} \otimes L^2([0,1])}^2 = \left\| \beta(t) \Lambda_t + \int_{0}^{t} R(t, \theta) \Lambda_{\theta} d\theta \right\|_{\mathcal{H}^{(\oplus)} \otimes L^2([0,1])}^2
\]
\[
\leq 2 \int_{0}^{1} \beta(t)^2 \left\| \Lambda_t \right\|_{\mathcal{H}^{(\oplus)}}^2 dt
\]
\[
+ C_H \int_{0}^{1} \left\| \Lambda_t \right\|_{\mathcal{H}^{(\oplus)}}^2 dt,
\] (21)
and for $H < \frac{1}{2}$, (20) yields
\[
\left\| (K_{H}^{*, \text{adj}})^{-1} \Lambda_t \right\|_{\mathcal{H}^{(\oplus)} \otimes L^2([0,1])}^2 \leq C_H \int_{0}^{1} \left\| \Lambda_t \right\|_{\mathcal{H}^{(\oplus)}}^2 dt.
\] (22)

We have
\[
\left\| \Lambda_t \right\|_{\mathcal{H}^{(\oplus)}} \leq \prod_{i=1}^{j} \left\| D^i Y_t \right\|_{\mathcal{H}^{(\oplus)}}^{1/2},
\] (23)

Taking into account that
\[
D^i Y_t = \int_{[0,t]^2} \frac{(B_r - B_s)^{2p-i}}{|r-s|^{2p+i+1}} 1_{[r,s]} dr ds,
\]
we obtain
\[ \| D^i Y_t \|_{\mathcal{H}^q} \leq \int_{[0,t]^2} \frac{|B_r - B_s|^{2p-i}}{|t-s|^{2p\gamma+1-H}} dr ds, \]
and this implies that
\[ \sup_{0 \leq t \leq 1} E \left[ \| D^i Y_t \|_{\mathcal{H}^q}^q \right] < \infty, \tag{24} \]
for any \( q \geq 1. \)

On the other hand, from
\[ \frac{d}{dt} \left( \psi^{(n)}(Y_t) \prod_{i=1}^j (D^i Y_t)^l_i \right) \]
we get
\[ \| \Lambda'_t \|_{\mathcal{H}^q} \leq \sum_{m=1}^j l_m \| D^m Y_t \|_{\mathcal{H}^q}^{l_m-1} \| D^m Y_t \|_{\mathcal{H}^q} \prod_{i=1}^j \| D^i Y_t \|_{\mathcal{H}^q}^{l_i}, \]
(25)

From
\[ D^i Y_t' = \int_0^t \frac{(B_t - B_s)^{2p-i}}{|t-s|^{2p\gamma+1-H}} ds, \]
we obtain
\[ \| D^i Y_t' \|_{\mathcal{H}^q} \leq \int_0^t \| B_t - B_s \|^{2p-i} ds, \]
and this implies that
\[ \sup_{0 \leq t \leq 1} E \left[ \| D^i Y_t' \|_{\mathcal{H}^q}^q \right] < \infty, \tag{26} \]
for any \( q \geq 1. \)

Finally, (24), (23), (21), (22), (18), (26) and (25) imply (17). This shows condition (i) of Theorem 1.

In order to show condition (iii) notice that
\[ \langle D M, u A \rangle_{\mathcal{H}} = \langle 1_{[0,T]}, u A \rangle_{\mathcal{H}} = \langle K_H^* 1_{[0,T]}, K_H^* u A \rangle_{L^2((0,1))} \]
\[ = \langle 1_{[0,T]}, K_H^{*,adj} K_H^* u A \rangle_{L^2((0,1))} \]
\[ = \int_0^T \psi(Y_t) dt. \]
On the other hand, on the set \( \{ M > a \} \), taking into account (13) and (14), it holds that
\[
\psi(Y_t) > 0 \Rightarrow t \leq T,
\]
and, as a consequence, \( \int_0^T \psi(Y_t) \, dt = G_A \).

Finally, it remains to show condition (ii), that is, \( G_A^{-1} \in L^q(\Omega) \) for any \( q \geq 2 \). We have
\[
G_A \geq \int_0^1 \psi(Y_t) 1\{Y_t < \frac{R}{2}\} \, dt
\]
\[
= \int_0^1 1\{Y_t < \frac{R}{2}\} \, dt
\]
\[
= \lambda \left\{ t \in [0, 1] : Y_t < \frac{R}{2} \right\}
\]
\[
= Y_t^{-1} \left( \frac{R}{2} \right),
\]
because \( Y \) is non-decreasing and is continuous. For any \( \epsilon > 0 \) we get
\[
P \left( Y_t^{-1} \left( \frac{R}{2} \right) < \epsilon \right) = P \left( \frac{R}{2} < Y_\epsilon \right)
\]
\[
\leq \left( \frac{2}{R} \right)^p E |Y_\epsilon|^p
\]
\[
\leq \left( \frac{2}{R} \right)^p \left[ \int_{[0, \epsilon]^2} \frac{\|B_r - B_s\|_{L^p(\Omega)}}{r-s} 2p \, dr \, ds \right]^p
\]
\[
\leq R^{-p} c_p \left[ \int_{[0, \epsilon]^2} |r-s|^{2pH-2p\gamma-1} \, dr \, ds \right]^{\frac{p}{p}},
\]
\[
= R^{-p} c_p (2p(H-\gamma+1)).
\]
This completes the proof of the proposition. \( \blacksquare \)

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