Averaging method and coherence applied to Rabi oscillations in a two-level system

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Abstract

We study Rabi oscillations in a two-level system within the semiclassical approximation as an archetypical test field of the Averaging Method (AM). The population transfer between the two levels is approached within the first and the second order AM. We systematically compare AM predictions with the rotating wave approximation (RWA) and with the complete numerical solution utilizing standard algorithms (NRWA). We study both the resonance (∆ = 0) and out-of-resonance (∆ ≠ 0) cases, where ∆ = ω − Ω, and hΩ = E₂ − E₁ is the two-level energetic separation, while ω is the (cyclic) frequency of the electromagnetic field. We introduce three types of dimensionless factors ε, i.e., ΩR/∆, ΩR/Σ, and ΩR/ω, where ΩR is the Rabi (cyclic) frequency and Σ = ω + Ω and explore the range of ε where the AM results are equivalent to NRWA. Finally, by allowing for a phase difference in the initial electron wave functions, we explore the prospects coherence can offer. We illustrate that even with equal initial probabilities at the two levels, but with phase difference, strong oscillations can be generated and manipulated.

1. Introduction

The Averaging Method (AM) in nonlinear dynamical systems belongs to asymptotic methods [1]. The simplest form of averaging is periodic averaging, which deals with solving a perturbation problem of the standard form

\[ \dot{x} = εf(x, t) + ε^2g(x, t) + \ldots \]  

(1)

We write the periodic (with period T) function f as

\[ f(x, t) = \overline{f}(x) + \tilde{f}(x, t). \]  

(2)

\( \overline{f}(x) \) is an idiotypic temporal average of f in the regime [0, T]: We average over t, but assuming x(t) constant, hypothesizing that x(t) is a slowly varying function, i.e.,

\[ \overline{f}(x) = \frac{1}{T} \int_{0}^{T} f(x, t) dt. \]  

(3)

Similarly with f(x, t), we treat g(x, t), \ldots Below we use similar separation of functions f, g, h \ldots to \( \overline{f}, \overline{g}, \overline{h}, \ldots \). Characteristic problems solved with AM can be found in Refs. [1, 2]. In recent years, among other applications, the AM has been used in Robotics [3], Engineering [4–8], Biology [9] and Physics [10–14]. In this work, we introduce the AM as an approximative solution of the archetypical semiclassical approximation of Rabi oscillations in a two-level system. The innovative thinking here is to present the AM as a method, which can solve successfully important problems in physics. We introduce the AM in great detail and hope that the readers will appreciate its importance.

Oscillations of electron probabilities between (usually two) energy levels due to the presence of an oscillating perturbing electric field are usually termed Rabi oscillations, although originally Rabi studied magnetic moment...
in the presence of a magnetic field [15]. Rabi oscillations can be treated fully quantum-mechanically [16, 17], but here we use the semiclassical approach and the two-level system (2LS) as an archetype: Our aim is merely to examine the first and mainly the second order AM and compare it with standard Rotating Wave Approximation (RWA) as well as with a full numerical treatment without any compromise, NRWA. Such oscillations are of archetypal importance, but also timely today. Recently, dissipative 2LS under strong off-resonant driving [18] and multiphoton Raman transitions and Rabi oscillations in driven spin systems [19] have been studied.

We obtained the idea of using the 2LS in the semiclassical approach as a benchmark for the AM by coming across [20]. However, we develop AM in a much different way: We introduce three types of small quantities, i.e., $\frac{\Omega}{\Delta}$, $\frac{\Sigma}{\Delta}$ and $\frac{\Omega}{\Sigma}$ for non-resonance ($\Delta \neq 0$), but only one type of $\epsilon$, i.e., $\frac{\Delta}{\Sigma}$, for resonance ($\Delta = 0$). $\Delta = \omega - \Omega$, $\Sigma = \omega + \Omega$, $\Omega = E_2 - E_1$ is the two-level energetic separation, while $\omega$ is the (cyclic) frequency of the electromagnetic (EM) field. $\Omega_R = \mathcal{P} \Omega_0 / h$ is the Rabi (cyclic) frequency, where $\mathcal{P}$ is the non-diagonal matrix element of the dipole moment along the electric field direction and $\mathcal{E}_0$ is the electric field amplitude.

We use first as well as second order AM, cf. section 3. We systematically compare the AM results with NRWA numerical results as well as with RWA. We explore the range of $\epsilon$ parameters for AM to work successfully, i.e., so that the second order AM results are equivalent with the NRWA numerical results. We also explore the prospects coherence can offer.

The article is organized as follows: In section 2 we define RWA and NRWA within the semiclassical approach. In section 3 we introduce the AM. In section 4 we compare our results for AM, RWA and NRWA. Finally, in section 5 we state our conclusions.

2. Rabi oscillations with or without Rotating Wave Approximation

Rabi oscillations of electron probabilities $|C_1(t)|^2$ and $|C_2(t)|^2$, as functions of time $t$, in a 2LS interacting with an electromagnetic field, are described by [17]

$$
C_1(t) = C_2(t) \frac{i \Omega_R}{2} \left[ e^{i(\omega-\Omega)t} + e^{-i(\omega+\Omega)t} \right]
$$

$$
\hat{\mathcal{C}}_2(t) = C_2(t) \frac{i \Omega_R}{2} \left[ e^{i(\omega+\Omega)t} + e^{-i(\omega-\Omega)t} \right].
$$

Using $\Delta = \omega - \Omega$ and $\Sigma = \omega + \Omega$, we obtain

$$
\begin{bmatrix}
    C_1(t) \\
    \hat{\mathcal{C}}_2(t)
\end{bmatrix} = 
\frac{i \Omega_R}{2} \begin{bmatrix}
    0 & e^{i\Delta t} + e^{-i\Delta t} \\
    e^{-i\Delta t} + e^{i\Delta t} & 0
\end{bmatrix} \begin{bmatrix}
    C_1(t) \\
    \hat{\mathcal{C}}_2(t)
\end{bmatrix}
$$

or

$$
\mathbf{x}(t) = i \mathbf{A}(t) \mathbf{x}(t).
$$

To appreciate the complexity of equation (6), we notice that trying the eigenvector-eigenvalue method, i.e., solutions of the form $\mathbf{x}(t) = \mathbf{v} e^{\lambda t}$, we obtain $\mathbf{A}(t) \mathbf{v} = \lambda \mathbf{v}$, i.e., a different eigenvector-eigenvalue problem for each $t$.

Equation (5) describes a separation into counter-rotating terms containing $\Delta$ or $\Sigma$. Taking into account only terms containing $\Delta$ is termed Rotating Wave Approximation (RWA). We shall call the full problem containing both the $\Delta$ and the $\Sigma$ terms NRWA (no RWA). To solve equation (5) numerically, within NRWA, we utilize matlab, using trapezoid and Runge-Kutta (4,5) algorithms. The probability to find the electron at the lower (higher) level is $P_1(t) = |C_1(t)|^2$ ($P_2(t) = |C_2(t)|^2$).

RWA stems from the assumption that if $\omega$ is close to $\Omega$, the $\Delta$ terms are slow and the $\Sigma$ terms are fast. Hence, in any remarkable time scale, the fast terms are somehow expected to have negligible effect. RWA is the claim that we can ignore the fast terms, i.e.,

$$
\begin{bmatrix}
    \hat{C}_1(t) \\
    \hat{C}_2(t)
\end{bmatrix} = 
\frac{i \Omega_R}{2} \begin{bmatrix}
    0 & e^{i\Delta t} + e^{-i\Delta t} \\
    e^{-i\Delta t} + e^{i\Delta t} & 0
\end{bmatrix} \begin{bmatrix}
    C_1(t) \\
    \hat{C}_2(t)
\end{bmatrix}
$$

The quantitative difference between NRWA and RWA is explored in the work, rather systematically. The analytical solution within RWA is known [17].

For example, with initial conditions $C_1(0) = 1$, $C_2(0) = 0$, we obtain
\[ P_{1,RWA}(t) = 1 - \frac{\Omega_{R}^{2}}{\Omega_{R}^{2} + \Delta^{2}} \sin^{2}(\lambda t), \]
\[ P_{2,RWA}(t) = \frac{\Omega_{R}^{2}}{\Omega_{R}^{2} + \Delta^{2}} \sin^{2}(\lambda t), \]

\[ 2\lambda = \sqrt{\Omega_{R}^{2} + \Delta^{2}}. \]

Hence, the period of the oscillations,
\[ T_{RWA} = \frac{2\pi}{\sqrt{\Omega_{R}^{2} + \Delta^{2}}}, \]

and the maximum transfer percentage,
\[ A_{RWA} = \frac{\Omega_{R}^{2}}{\Omega_{R}^{2} + \Delta^{2}}. \]

The period at resonance (\( \Delta = 0 \)), \( T_{RWA,0} = 2\pi / \Omega_{R} \), offers a convenient time scale, which will be exploited later, in the presentation of our results. However, we notice that when \( \Omega_{R} \) is significant, even at resonance, the frequency of the oscillations predicted by RWA, \( f_{RWA} = 1 / T_{RWA} \), does not coincide with the main frequency of NRWA, which, additionally, has richer frequency content.

If we assume the initial condition \( C(t) = \frac{1}{\sqrt{2}} e^{i\phi} \) and \( C_{2}(0) = \frac{1}{\sqrt{2}} e^{i\phi} \), i.e., we place the electron, at time zero with equal probability at both levels, \( |C_{0}(0)|^{2} = |C_{2}(0)|^{2} = \frac{1}{2} \), but we allow phase to vary, we obtain
\[ P_{1}(t) = \frac{1}{2} - \frac{\Omega_{R} \Delta}{2(\Omega_{R}^{2} + \Delta^{2})} \cos(\theta - \phi)(1 - \cos(2\lambda t)) + \frac{\Omega_{R}}{2\sqrt{\Omega_{R}^{2} + \Delta^{2}}} \sin(2\lambda t) \sin(\theta - \phi) \]
\[ P_{2}(t) = \frac{1}{2} + \frac{\Omega_{R} \Delta}{2(\Omega_{R}^{2} + \Delta^{2})} \cos(\theta - \phi)(1 - \cos(2\lambda t)) - \frac{\Omega_{R}}{2\sqrt{\Omega_{R}^{2} + \Delta^{2}}} \sin(2\lambda t) \sin(\theta - \phi) \]

\[ 2\lambda = \sqrt{\Omega_{R}^{2} + \Delta^{2}}. \]

In case of resonance (\( \Delta = 0 \Rightarrow \omega = \Omega \)), we have
\[ P_{1}(t) = \frac{1}{2} + \frac{1}{2} \sin(\Omega_{R} t) \sin(\theta - \phi) \]
\[ P_{2}(t) = \frac{1}{2} - \frac{1}{2} \sin(\Omega_{R} t) \sin(\theta - \phi) \]

Hence, if \( \theta - \phi = 2k\pi \) with \( k = 0, 1, 2, \ldots \), then in case of resonance, Rabi oscillations disappear, i.e., the probabilities are \( P_{1}(t) = P_{2}(t) = \frac{1}{2} \), constantly.

Finally, we notice that if \( \Omega_{R} \gg \Delta \) and \( \Omega_{R} \gg \Sigma \), so that in equation (i) each exponential term within the square brackets can be approximated by 1, then,
\[ P_{1,a}(t) = \cos^{2}(\Omega_{R} t), \]
\[ P_{2,a}(t) = \sin^{2}(\Omega_{R} t), \]

Hence, the period of the oscillations,
\[ T_{a} = \frac{\pi}{\Omega_{R}}, \]

and the maximum transfer percentage,
\[ A_{a} = 1. \]

3. Averaging method

For non-resonance (section 3.1), we employ three types of small quantities \( \epsilon \), i.e., \( \frac{\Omega_{R}}{\Delta} \), \( \frac{\Omega_{R}}{\Sigma} \) and \( \frac{\Omega_{R}}{\omega} \). Unavoidably, when \( \Delta \) becomes smaller, at some point, \( \frac{\Omega_{R}}{\Delta} \) gets so large that non-resonant AM is not successful anymore.

Hence, resonance must be treated via a different path, using only one type of \( \epsilon \), i.e., \( \frac{\Omega_{R}}{\omega} \) (section 3.2). This will become evident also in the Results, section 4.
3.1. Non-resonance

3.1.1. First order AM

In case of non-resonance ($\Delta = 0 \Rightarrow \omega = \Omega$), we can write equation (5) in the AM form of equation (1),

$$\dot{x}(t) = \frac{i\Omega_0}{2} \begin{bmatrix} 0 & e^{i\Delta t} + e^{-i\Delta t} \\ e^{-i\Delta t} + e^{i\Delta t} & 0 \end{bmatrix} x(t),$$

(20)

i.e., we call $f(x, t)$ the right-hand side of equation (5) and

$$g(x, t) = 0.$$  

(21)

Equation (20) includes two periods, $T_\Delta = \frac{2\pi}{\Delta}$ and $T_\Omega = \frac{2\pi}{\Omega}$. If $\frac{T_\Delta}{T_\Omega}$ is a rational number, the system is periodic with common period, $T$, the least common multiple of $T_\Delta$ and $T_\Omega$. (In numerical calculations, since $T_\Delta$ and $T_\Omega$ are represented as floats $\frac{T_\Delta}{T_\Omega}$ is always a rational number.)

In our case, combining equation (3) and equation (20), we obtain

$$\epsilon f(x, t) = \frac{i\Omega_0}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

(22)

Combining Eqs. (2), (20) and (22), we obtain

$$\epsilon f(x, t) = \frac{i\Omega_0}{2} \begin{bmatrix} 0 & e^{i\Delta t} + e^{-i\Delta t} \\ e^{-i\Delta t} + e^{i\Delta t} & 0 \end{bmatrix} x(t).$$  

(23)

In first order AM, we define

$$x(t) = y(t) + \epsilon w[y(t), t],$$

(24)

where $w$, a function of $y(t)$ and $t$, is defined so that

$$\bar{f}(y, t) = \frac{\partial w}{\partial t}.$$ 

(25)

The reason behind this definition becomes clear when we perform detailed calculations, cf. equation (A.10). After the detailed calculations shown in appendix A, we obtain

$$y = \epsilon \bar{f}(y) + \epsilon^2 \left( \frac{\partial \bar{f}(y)}{\partial y} w + \frac{\partial \bar{f}(y, t)}{\partial y} w + g(y, t) - \frac{\partial w}{\partial y} \bar{f}(y) \right) + \mathcal{O}(\epsilon^3)$$

(26)

In equation (26), if we ignore terms of order $\epsilon^2$ and above, we obtain the first order AM of equation (1), i.e.,

$$\dot{y} = \epsilon \bar{f}(y).$$

(27)

In our case, $\bar{f}(y) = 0$, therefore,

$$y = 0.$$  

(28)

Therefore, $y$ is a constant, i.e.,

$$y = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}.$$ 

(29)

We determine $y_{10}, y_{20}$ by applying the initial conditions.

Finally, from Eqs. (23), (25), (A.7), we obtain

$$\epsilon w(y, t) = \frac{i\Omega_0}{2} \begin{bmatrix} 0 & e^{i\Delta t} - e^{-i\Delta t} \\ e^{-i\Delta t} + e^{i\Delta t} & 0 \end{bmatrix} y(t).$$ 

(30)

3.1.2. Second order AM

In second order AM, we start from equation (26), and call $h(y, t)$ the $\mathcal{O}(\epsilon^2)$ function, i.e.,

$$y = \epsilon \bar{f}(y) + \epsilon^2 \left( \frac{\partial \bar{f}(y)}{\partial y} w + \frac{\partial \bar{f}(y, t)}{\partial y} w + g(y, t) - \frac{\partial w}{\partial y} \bar{f}(y) \right) + \mathcal{O}(\epsilon^3).$$  

(31)
Then, as with equation (2), we define

\[ h(y, t) = \tilde{h}(y) + \tilde{h}(y, t), \]  

(32)

and similarly to equation (24), we define

\[ y(t) = z(t) + \epsilon^2 u[z(t), t], \]

(33)

where \( u \) is a function of \( z(t) \) and \( t \), defined so that

\[ \tilde{h}(z, t) = \frac{\partial u}{\partial t} \]  

(34)

Function \( u \) in equation (34), in second order AM, has the same role as function \( w \) in equation (25), in first order AM. After detailed calculations shown in appendix B, we obtain

\[ \dot{z} = \epsilon \tilde{f}(z) + \epsilon^2 \tilde{h}(z) + \mathcal{O}(\epsilon^3) \]  

(35)

In equation (35), if we ignore terms of order \( \epsilon^3 \) and above, we obtain the second order AM of equation (1), i.e.,

\[ \dot{z} = \epsilon \tilde{f}(z) + \epsilon^2 \tilde{h}(z), \]

(36)

where

\[ h(z, t) = \frac{\partial \tilde{f}(z)}{\partial z} w(z, t) + \frac{\partial \tilde{f}(z, t)}{\partial z} w(z, t) + g(z, t) - \frac{\partial w(z, t)}{\partial z} \tilde{f}(z). \]  

(37)

In our case, \( \tilde{f}(z) = 0 \) and \( g(z, t) = 0 \), cf. equations (22) and (21), respectively. Hence,

\[ h(z(t), t) = \frac{\partial \tilde{f}(z, t)}{\partial z} w(z, t). \]  

(38)

Therefore, via equations (22) and (38), equation (36) reads

\[ \dot{z} = \epsilon^2 \frac{\partial \tilde{f}(z, t)}{\partial z} w(z, t). \]  

(39)

Combining equations (23), (30) and (38),

\[ \epsilon^2 h(z(t), t) = i \left( \frac{\Omega_0}{2} \right)^2 \left( \frac{1}{\Sigma} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] + \frac{1}{\Delta} \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] \right) z(t) \]

\[ + i \left( \frac{\Omega_0}{2} \right)^2 \left[ \begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array} \right] e^{i(\Delta + \Sigma)t} z(t) \]

\[ + i \left( \frac{\Omega_0}{2} \right)^2 \left[ \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] e^{i(\Delta + \Sigma)t} z(t) \]  

(40)

where

\[ \epsilon^2 \tilde{h}(z) = \epsilon^2 \left( \frac{\partial \tilde{f}(z, t)}{\partial z} w(z, t) \right) = i \left( \frac{\Omega_0}{2} \right)^2 \left( \frac{1}{\Sigma} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] + \frac{1}{\Delta} \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] \right) z(t). \]  

(41)

From equations (32), (40), (41) and \( \Delta + \Sigma = 2 \omega \) we have

\[ \epsilon^2 \tilde{h}(z, t) = \left( \frac{\Omega_0}{2} \right)^2 \left( \frac{1}{\Delta} \left[ \begin{array}{cc} -e^{-i2\omega t} & 0 \\ 0 & e^{i2\omega t} \end{array} \right] + \frac{1}{\Sigma} \left[ \begin{array}{cc} e^{i2\omega t} & 0 \\ 0 & -e^{-i2\omega t} \end{array} \right] \right) z(t) \]  

(42)

Therefore, equation (39) becomes

\[ \dot{z} = iA \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] z(t), \]  

(43)

\[ A = \left( \frac{\Omega_0}{2} \right)^2 \left( \frac{2\Omega}{\omega^* - \Omega^*} \right). \]  

(44)

The solution of equation (43) is

\[ z(t) = \left[ \begin{array}{c} z_1(t) \\ z_2(t) \end{array} \right] = \left[ \begin{array}{c} z_{10} e^{-iAt} \\ z_{20} e^{iAt} \end{array} \right] \]  

(45)
Also, \( u \) is calculated from the Eqs. (34), (42)
\[
\epsilon^2 u(z, t) = i \left( \frac{\Omega_R}{2} \right)^2 \begin{bmatrix}
\frac{e^{-i2\omega t}}{i2\omega\Delta} + \frac{e^{i2\omega t}}{i2\omega\Sigma} & 0 \\
0 & \frac{e^{i2\omega t}}{i2\omega\Delta} + \frac{e^{-i2\omega t}}{i2\omega\Sigma}
\end{bmatrix} z(t)
\]
(46)

Finally, we epitomize our AM results. For \textit{first order}:
\[
x = y + \epsilon \, w(y, t),
\]
(47)
\[
y = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix},
\]
(48)
\[
\epsilon \, w(y, t) = i \frac{\Omega_R}{2} \begin{bmatrix}
0 & \frac{e^{i\Delta t}}{i\Delta} - \frac{e^{-i\Delta t}}{i\Sigma} \\
-\frac{e^{-i\Delta t}}{i\Delta} + \frac{e^{i\Delta t}}{i\Sigma} & 0
\end{bmatrix} y.
\]
(49)

For \textit{second order}:
\[
x(t) = z(t) + \epsilon \, w(z, t) + \epsilon^2 u(z, t),
\]
(50)
\[
z(t) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_{10}e^{-i\Delta t} \\ z_{20}e^{i\Delta t} \end{bmatrix},
\]
(51)
\[
\epsilon \, w(z, t) = i \frac{\Omega_R}{2} \begin{bmatrix}
0 & \frac{e^{i\Delta t}}{i\Delta} - \frac{e^{-i\Delta t}}{i\Sigma} \\
-\frac{e^{-i\Delta t}}{i\Delta} + \frac{e^{i\Delta t}}{i\Sigma} & 0
\end{bmatrix} z,
\]
(52)
\[
\epsilon^2 u(z, t) = i \left( \frac{\Omega_R}{2} \right)^2 \begin{bmatrix}
\frac{e^{-i2\omega t}}{i2\omega\Delta} + \frac{e^{i2\omega t}}{i2\omega\Sigma} & 0 \\
0 & \frac{e^{i2\omega t}}{i2\omega\Delta} + \frac{e^{-i2\omega t}}{i2\omega\Sigma}
\end{bmatrix} z.
\]
(53)

Three types of \( \epsilon \) occur in our first and second order equations: \( \frac{\Omega_R}{\Delta} \), \( \frac{\Omega_R}{\Sigma} \), and \( \frac{\Omega_R}{\omega} \). The constants \( y_{10}, y_{20}, z_{10}, z_{20} \) are calculated from the initial conditions.

### 3.2. Resonance

Similarly, we treat the resonant case \( \Delta = 0 \Rightarrow \omega = \Omega \). Details can be found in [21]. Below we summarize our AM results. For \textit{first order}:
\[
x = y + \epsilon \, w(y, t),
\]
(54)
\[
y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_{11} \cos \left( \frac{\Omega_R}{2} t \right) + B_{11} \sin \left( \frac{\Omega_R}{2} t \right) \\ A_{21} \cos \left( \frac{\Omega_R}{2} t \right) + B_{21} \sin \left( \frac{\Omega_R}{2} t \right) \end{bmatrix},
\]
(55)
\[
\epsilon \, w(y, t) = \frac{\Omega_R}{4\omega} \begin{bmatrix}
0 & -e^{-i2\omega t} \\
e^{i2\omega t} & 0
\end{bmatrix} y.
\]
(56)

For \textit{second order}:
\[
x(t) = z(t) + \epsilon \, w(z, t) + \epsilon^2 u(z, t),
\]
(57)
\[
z(t) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A_{12} \cos(\omega t) + B_{12} \sin(\omega t) \\ A_{22} \cos(\omega t) + B_{22} \sin(\omega t) \end{bmatrix},
\]
(58)

where
\[
B = \sqrt{\left( \frac{\Omega_R}{2} \right)^2 + \left( \frac{\Omega_R}{2} \right)^2}
\]
(59)
\[
\epsilon \, w(z, t) = \frac{\Omega_R}{4\omega} \begin{bmatrix}
0 & -e^{-i2\omega t} \\
e^{i2\omega t} & 0
\end{bmatrix} z,
\]
(60)
\[
\epsilon^2 u(z, t) = i \left( \frac{\Omega k}{2} \right)^2 \frac{1}{2 \omega^2} \sin 2\omega \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z.
\] (61)

In the resonant case \( \epsilon \) is introduced in the first and second order equations with just one form, \( \frac{\Omega k}{\omega} \). The constants \( A_{11}, A_{21}, B_{11}, B_{21}, A_{12}, A_{22}, B_{12}, B_{22} \) are calculated from the initial conditions.

### 4. Results with NRWA, RWA, first and second order AM

We compare our results of NRWA, RWA, first and second order AM. In the figures, in the horizontal axes we use the dimensionless quantity \( \frac{\Omega k}{2\epsilon} \), i.e., time \( t \) divided by \( T_{RWA,0} \) and in the vertical axes we present the probability at the lower level, \( P_1 \). For non-resonance we employ three types of small quantities \( \epsilon \), i.e., \( \frac{\Omega k}{\Delta}, \frac{\Omega k}{\omega} \) and \( \frac{\Omega k}{|\Delta|} \).

Unavoidably, when \( \Delta \) becomes smaller, at some point, \( \frac{\Omega k}{\Delta} \) gets so large that non-resonant AM is not successful anymore and resonance must be treated via a different path, using just one type of \( \epsilon \), i.e., \( \frac{\Omega k}{\omega} \). In general, the accuracy of second order AM has a range between the outcomes, \( \frac{\Omega k}{\Delta} \), \( \frac{\Omega k}{\omega} \) and \( \frac{\Omega k}{|\Delta|} \), due to the fact that the terms above appear in the final equations (cf. equations (50)-(53)) and they are the last terms that we don’t ignore. First order AM is frequently away from the numerical solution. We include it in the figures below just for comparison.

The values of \( \epsilon \) were chosen with the purpose of introducing cases where AM approaches success. Hence, the values are down to the order of magnitude \( 0.01 \). For smaller values, AM is successful. Furthermore, very small values of \( \epsilon \) means very small perturbation, i.e., these are trivial cases. As far as the values of \( \theta \) and \( \phi \) are concerned, we explored the whole phase space.

#### 4.1. Initial condition (1,0)

We assume initial condition \( C_1(0) = 1, C_2(0) = 0 \), placing the electron, at \( t = 0 \), at the lower level. Sections 4.1.1 (4.1.2) refers to non-resonance (resonance).

##### 4.1.1. Non-resonance

In figure 1 we modify \( \epsilon_1 = \frac{\Omega k}{\omega} \) keeping \( \epsilon_1 = \frac{\Omega k}{\Delta} = -0.5 \) (\( \epsilon_1 = 0.5 \)) on the left (right) column. For \( \epsilon_1 > 0 \), as \( \epsilon_2 \) gets smaller, RWA is identified with NRWA. Second order AM is very close to NRWA in all cases. For \( \epsilon_1 < 0 \), as \( \epsilon_2 \) gets smaller, AM and RWA are identified with NRWA. The different behavior of AM for negative and positive \( \epsilon_1 \) stems from \( \epsilon_3 = \frac{\Omega k}{\omega} \) being different: for \( \epsilon_1 > 0 \), \( \epsilon_3 \) is smaller than for \( \epsilon_1 < 0 \). In figure 2 we modify \( \epsilon_1 = \frac{\Omega k}{\Delta} \) and keep \( \epsilon_2 = \frac{\Omega k}{\omega} = 0.01 \). On left (right) column \( \epsilon_1 < 0 \) (\( \epsilon_1 > 0 \)). RWA is identified with NRWA, but not with second order AM. As \( \epsilon_1 \) gets smaller, AM gradually approaches NRWA. Oscillations diminish as \( \epsilon_1 \) becomes smaller. Oscillations at the same row but in different columns are a little different due to the different value of \( \epsilon_3 \). In figures 1, 2, the two panels of the same line seem similar, because \( \epsilon_2 = \frac{\Omega k}{\omega} \) are almost identical, except for the first line in figure 1. For example, in figure 1, the two panels of the last line have \( \epsilon_1 = -0.5 \), \( \epsilon_2 = 0.4 \), \( \epsilon_3 = \frac{1}{49} \) and \( \epsilon_1 = 0.5 \), \( \epsilon_2 = 0.01 \), \( \epsilon_3 = \frac{1}{27} \), respectively, while the two panels of the first line have \( \epsilon_1 = -0.5 \), \( \epsilon_2 = 0.4 \), \( \epsilon_3 = \frac{4}{9} \), and \( \epsilon_1 = 0.5 \), \( \epsilon_2 = 0.4 \), \( \epsilon_3 = \frac{4}{9} \), respectively. Hence, second order AM is identified with NRWA when \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) are sufficiently small.

##### 4.1.2. Resonance

In figure 3 we illustrate \( P_1 \) vs. \( \frac{\Omega k}{2\epsilon} \), modifying \( \epsilon = \frac{\Omega k}{\omega} \). As \( \epsilon \) gets smaller, AM is identified with NRWA.

#### 4.2. Initial condition (1/2,1/2)

Here we explore the prospects coherence can offer, by defining a phase difference in the initial electron wave functions. Specifically, we assume initial condition \( C_1(0) = \frac{1}{\sqrt{2}} e^{i\theta}, C_2(0) = \frac{1}{\sqrt{2}} e^{i\phi}, \) i.e., at time zero, equal probability at both levels, \( |C_1(0)|^2 = |C_2(0)|^2 = \frac{1}{2} \), but with a phase difference \( \theta - \phi \). Sections 4.2.1 (4.2.2) is devoted to non-resonance (resonance).

##### 4.2.1. Non-resonance

In figure 4 we vary \( \epsilon_1 = \frac{\Omega k}{\Delta} \), keeping \( \epsilon_1 = \frac{\Omega k}{\omega} = -0.5 \) (\( \epsilon_1 = 0.5 \)) on the left (right) column, with \( \theta - \phi = \frac{\pi}{2} \). As \( \epsilon_1 \) gets smaller, second order AM approaches NRWA. Although the initial probabilities at the two levels are equal, phase difference of the initial wave functions leads to strong oscillations, a clear coherent phenomenon. Decreasing \( \epsilon_2 \), second order AM approaches NRWA.

In figure 5 we modify \( \epsilon_1 = \frac{\Omega k}{\Delta} \), keeping \( \epsilon_2 = \frac{\Omega k}{\omega} = 0.01 \) with \( \theta - \phi = \frac{\pi}{2} \). We observe strong oscillations, depending on the magnitude of \( \epsilon_1 \), although the initial probabilities are equal, a pure coherent
phenomenon, due to the initial phase difference of the wave functions. Decreasing $|\epsilon_1|$, second order AM approaches NRWA.

The discussion on the effect of the relative magnitude of $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ (section 4.1.1), applies here, too. In figure 6 we keep $\epsilon_1 = 0.5$ and $\epsilon_2 = 0.01$, varying the initial phase difference of the wave functions, $\theta - \phi$. We observe another aspect of coherence, a vertical and horizontal displacement of the oscillations.

4.2.2. Resonance
In figure 7 we modify $\epsilon = \frac{\Omega_0}{2\pi}$, for initial phase difference, $\theta - \phi = \frac{\pi}{3}$. Now, since we are in resonance, oscillations are particularly strong, of the order of one. As $\epsilon$ gets smaller, AM is identified with NRWA.

In figure 8 we keep $\epsilon = 0.1$, varying the initial phase difference, $\theta - \phi$. We observe that the amplitude of the oscillations can be readily manipulated this way.

4.3. Non-resonant AM vs. resonant AM
The reader might wonder why we have introduced two different versions of the AM, one for non-resonance (section 3.1) and another for resonance (section 3.2). In the discussion at the beginning of section 3, we have already explained the reason: When $\Delta$ becomes very small, $\Omega_0$ gets very large so that non-resonant AM is not successful anymore. Therefore, in resonance, the AM has to be manipulated in a different way.

Here we give a few examples. In figure 9, we vary $\epsilon_1 = \frac{\Omega_0}{\Delta}$ and keep $\epsilon_2 = \frac{\Omega_0}{2\pi} = 0.01$. We observe that for $\epsilon_1 < 1$, the second order AM for non-resonance is closer to the numerical solution (NRWA) than the second
order AM for resonance. However, for $\epsilon_1 > 1$, $\Delta$ is so small that the second order AM for resonance comes closer to NRWA than the second order AM for non-resonance. We have already mentioned that the first order AM is usually far from NRWA, and we include it in the figures just to underline this fact.

5. Conclusion

To explore the potential of the Averaging Method (AM) to handle coupled differential equations, we chose an archetypical system, i.e., Rabi oscillations in a two-level system (2LS), in the semiclassical approximation.

We illustrated the need to manipulate in a different manner non-resonant AM from resonant AM. We introduced three dimensionless ratios $\epsilon_i$, namely, $\frac{\Omega_i}{2\Delta}$, $\frac{\Omega_i}{\Sigma}$, and $\frac{\Omega_i}{\omega}$, to use properly the AM at non-resonance ($\Delta \neq 0$).

In manipulating resonance ($\Delta = 0$), we had to consider that unavoidably, when $\Delta$ becomes smaller, at some point, $\frac{\Omega_i}{\Delta}$ gets so large that non-resonant AM is not successful anymore. Therefore, resonance had to be treated in a different way, using just one type of $\epsilon_i$, i.e., $\frac{\Omega_i}{\Delta}$.

We compared first and second order AM with the full numerical solution (NRWA) as well as with Rotating Wave Approximation (RWA).

First order AM is usually away from the numerical solution. However, second order AM can closely approach NRWA, provided the $\epsilon$ ratios are small enough. We explored the range where second order AM is successful. We also explored the range where RWA is successful.

![Figure 2. $\Delta = 0. P_1$ vs. $\frac{\Omega_i}{2\Delta}$ for $\epsilon_1 = \frac{\Omega_i}{\Delta} = 0.01$, varying $\epsilon_1 = \frac{\Omega_i}{\Sigma}$. (a) $\epsilon_1 = -0.9$. (b) $\epsilon_1 = 0.9$. (c) $\epsilon_1 = -0.6$. (d) $\epsilon_1 = 0.6$. (e) $\epsilon_1 = -0.2$. (f) $\epsilon_1 = 0.2$. (g) $\epsilon_1 = -0.1$. (h) $\epsilon_1 = 0.1$. Lines correspond to NRWA (continuous --), RWA (dashed --), second order AM (dotted --), first order AM (dash-dotted --).](image-url)
Finally, we investigated various coherent phenomena, at resonance and at non-resonance, putting a phase difference in the initial wave functions. Even with equal initial probabilities at the two levels, but with phase difference, strong oscillations can be generated and manipulated.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. More equations for first order AM

We proceed with detailed calculations.

\[ dx(t) = dy(t) + \epsilon \, dw[y(t), t], \]  
(A.1)

\[ dw[y(t), t] = \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial t} dt. \]  
(A.2)

By \( \frac{\partial w}{\partial t} \) we denote the derivative of \( w \) relative to \( t \), keeping \( y(t) \) constant, hypothesizing that \( y(t) \) is a slowly varying function. Hence,

\[ \dot{x} = \dot{y} + \epsilon \frac{\partial w}{\partial y} \dot{y} + \epsilon \frac{\partial w}{\partial t}. \]  
(A.3)

Therefore, equation (1) becomes

\[ \dot{y} + \epsilon \frac{\partial w}{\partial y} \dot{y} + \epsilon \frac{\partial w}{\partial t} = \epsilon f(y + \epsilon w, t) + \epsilon^2 g(y + \epsilon w, t). \]  
(A.4)

Using equation (2), we obtain

\[ f(y + \epsilon w, t) = \tilde{f}(y + \epsilon w) + \epsilon \tilde{f}(y, t). \]  
(A.5)

Using a Taylor expansion vs. \( y \) we obtain

\[ \tilde{f}(y + \epsilon w) = \tilde{f}(y) + \epsilon \frac{\partial \tilde{f}(y)}{\partial y} w + \mathcal{O}(\epsilon^2 w^2), \]  
(A.6)

\[ \tilde{f}(y + \epsilon w, t) = \tilde{f}(y, t) + \epsilon \frac{\partial \tilde{f}(y, t)}{\partial y} w + \mathcal{O}(\epsilon^2 w^2), \]  
(A.7)
Equation (A.4), using equations (2), (A.6), (A.7), (A.8), becomes

\[\dot{y} + \epsilon \frac{\partial w}{\partial y} + \epsilon \frac{\partial w}{\partial t} = \epsilon \left( \tilde{f}(y) + \tilde{f}(y, t) \right) + \epsilon^2 \left( \frac{\partial \tilde{f}(y)}{\partial y}w + \frac{\partial \tilde{f}(y, t)}{\partial y}w + g(y, t) \right) + \mathcal{O}(\epsilon^3)\]  

(A.9)

Rearranging,

\[\left( I + \epsilon \frac{\partial w}{\partial y} \right) \dot{y} = \epsilon \left( \tilde{f}(y) + \tilde{f}(y, t) - \frac{\partial w}{\partial t} \right) + \epsilon^2 \left( \frac{\partial \tilde{f}(y)}{\partial y}w + \frac{\partial \tilde{f}(y, t)}{\partial y}w + g(y, t) \right) + \mathcal{O}(\epsilon^3)\]  

(A.10)

\(I\) is the unit relevant to the nature of \(y\). If \(y\) is a simple function of \(t\), \(I = 1\). If \(y\) is a column matrix, as in our case, \(I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). Now, the use of equation (25) to simplify equation (A.10) is obvious.

\[\dot{y} = \left( I + \epsilon \frac{\partial w}{\partial y} \right)^{-1} \left( \epsilon \tilde{f}(y) + \epsilon^2 \left( \frac{\partial \tilde{f}(y)}{\partial y}w + \frac{\partial \tilde{f}(y, t)}{\partial y}w + g(y, t) \right) + \mathcal{O}(\epsilon^3) \right),\]  

(A.11)
Appendix B. More equations for second order AM

We proceed with detailed calculations.

\[ \frac{dy(t)}{dt} = \frac{dz(t)}{dt} + \epsilon^2 \frac{du[z(t), t]}{dt}, \]
\[ du[z(t), t] = \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt. \]

By \( \frac{\partial u}{\partial t} \) we denote the derivative of \( u \) relative to \( t \), keeping \( z(t) \) constant, hypothesizing that \( z(t) \) is a slowly varying function. Hence,

\[ \dot{y} = \dot{z} + \epsilon^2 \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial u}{\partial t}. \]

Therefore, equation (31) becomes

\[ \dot{z} + \epsilon^2 \frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial u}{\partial t} = \epsilon f(z + \epsilon^2 u) + \epsilon^2 (\tilde{h}(z + \epsilon^2 u, t) + \tilde{h}(z + \epsilon^2 u, t)) + \mathcal{O}(\epsilon^3) \]
\[ \Delta = 0. \] $P_1$ vs. $p_w$, for $\epsilon_1 = \frac{\Omega_m}{2\pi}$ = 0.5 and $\epsilon_2 = \frac{\Omega_m}{2\pi}$ = 0.01, varying $\theta - \phi$. (a) $\theta - \phi = 0$, (b) $\theta - \phi = \frac{\pi}{4}$, (c) $\theta - \phi = \frac{\pi}{2}$, (d) $\theta - \phi = \pi$, (e) $\theta - \phi = \frac{3\pi}{2}$, (f) $\theta - \phi = \frac{5\pi}{4}$. Lines correspond to NRWA (continuous ---), RWA (dashed --), second order AM (dotted - - -), first order AM (dash-dotted - -). 

\[ \Omega = 0. \] $P_1$ vs. $p_w$, varying $\epsilon = \frac{\Omega_m}{2\pi}$ with $\theta - \phi = \frac{\pi}{2}$. (a) $\epsilon = 0.9$, (b) $\epsilon = 0.5$, (c) $\epsilon = 0.1$, (d) $\epsilon = 0.05$. Lines correspond to NRWA (continuous ---), RWA (dashed --), second order AM (dotted - - -), first order AM (dash-dotted - -).
Using a Taylor expansion vs. \( z \) we obtain

\[
f(z + \epsilon^2 u) = f(z) + \epsilon^2 \frac{\partial f}{\partial z} u + \mathcal{O}(\epsilon^4 u^2),
\]

(B.5)
\[ h(z + \epsilon^2 u) = \tilde{h}(z) + \epsilon^2 \frac{\partial \tilde{h}(z)}{\partial z} u + \mathcal{O}(\epsilon^4 u^2), \]  
(B.6)  
\[ \tilde{h}(z + \epsilon^2 u, t) = \tilde{h}(z, t) + \epsilon^2 \frac{\partial \tilde{h}(z, t)}{\partial z} u + \mathcal{O}(\epsilon^4 u^2). \]  
(B.7)

Equation (B.4) using equations (32), (B.5), (B.6), (B.7), becomes

\[ \dot{z} + \epsilon^2 \frac{\partial u}{\partial z} \dot{z} + \epsilon^2 \frac{\partial u}{\partial t} = \epsilon^2 \tilde{f}(z) + \epsilon^2 (\tilde{h}(z) + \dot{\tilde{h}}(z, t)) + \mathcal{O}(\epsilon^3) \]  
(B.8)

Rearranging

\[ \left( I + \epsilon^2 \frac{\partial u}{\partial z} \right) \dot{z} = \epsilon^2 \tilde{f}(z) + \epsilon^2 (\tilde{h}(z) + \dot{\tilde{h}}(z, t) - \frac{\partial u}{\partial t}) + \mathcal{O}(\epsilon^3) \]  
(B.9)

\( I \) is the unit relevant to the nature of \( z \). If \( z \) is a simple function of \( t \), \( I = 1 \). If \( z \) is a column matrix, as in our case, \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Now, the use of equation (34) to simplify equation (B.9) is obvious.

\[ \dot{z} = \left( I + \epsilon^2 \frac{\partial u}{\partial z} \right)^{-1} \epsilon^2 \tilde{f}(z) + \epsilon^2 (\tilde{h}(z) + \mathcal{O}(\epsilon^3)) \]  
(B.10)

\[ \left( I + \epsilon^2 \frac{\partial u}{\partial z} \right)^{-1} = I - \epsilon^2 \frac{\partial u}{\partial z} + \mathcal{O}(\epsilon^4). \]  
(B.11)

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**References**

[1] Sanders J A, Verhulst F and Murdock I 2007 *Averaging Methods in Nonlinear Dynamical Systems* 2nd edn (New York: Springer) 978-0-387-48918-6  
[2] Verhulst F 1975 *Celest. Mech.* 11 95–129  
[3] Taha H E, Kiani M, Hedrick T L and Greeter I S M 2020 *Science Robotics* 5 eabb1302  
[4] Ogundele A D, Agboola O A and Sinha S C 2021 *Commun. Nonlinear Sci. Numer. Simul.* 95 105668  
[5] Mani A and Narayanan M 2020 *Nonlinear Dyn.* 100 999  
[6] Wilson D 2020 *Sci. Rep.* 10 5922  
[7] Rega G 2020 *Nonlinear Dyn.* 99 11–34
[8] Du K, Ma Q, Kang Y and Fu W 2020 IEEE Transactions on Systems, Man, and Cybernetics: Systems 1–11
[9] Wilson D 2020 J. Math. Biol. 81 25–64
[10] Wilson D 2020 Phys. Rev. E 101 022220
[11] Fajman D, Heißel G and Jang JW 2021 Classical Quantum Gravity 38 085005
[12] Fajman D, Heißel G and Maliborski M 2020 Classical Quantum Gravity 37 135009
[13] Podvigina O and Krasilnikov P 2020 Icarus 335 113371
[14] Gokler C 2021 Found. Phys. 51 10
[15] Rabi II 1937 Phys. Rev. 51 652–4
[16] Loudon R 2000 The Quantum Theory of Light 3rd edn (Oxford: Oxford University Press)
[17] Simserides C 2016 Quantum Optics and Lasers 1st Edn (Athens: Kallipos, Hellenic Academic Libraries Link) (http://hdl.handle.net/11419/2108)
[18] Saiko A P, Markevich S A and Fedaruk R 2016 Phys. Rev. A 93 063834
[19] Saiko A P, Markevich S A and Fedaruk R 2018 Phys. Rev. A 98 043814
[20] Batista A A 2015 arXiv:1507.05124v1 [quant-ph]
[21] Chalkopiadis L 2021 Rabi oscillations in a two-level and in a multi-level system with and without rotating wave approximation
   National and Kapodistrian University of Athens, Department of Physics Diploma Thesis (in Greek) Athens, Greece (https://pergamos.lib.uoa.gr/uoa/dl/object/2946058)