Cancellation of quantum mechanical higher loop contributions to the gravitational chiral anomaly.

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ABSTRACT
We give an explicit demonstration, using the rigorous Feynman rules developed in [1], that the regularized trace \( \text{Tr} \gamma_5 e^{-\beta \hat{R}} \) for the gravitational chiral anomaly expressed as an appropriate quantum mechanical path integral is \( \beta \)-independent up to two-loop level. Identities and diagrammatic notations are developed to facilitate rapid evaluation of graphs given by these rules.

It is an old observation of Alvarez-Gaumé and Witten [2] that anomalies of quantum field theories, expressed in the Fujikawa [3] approach as the regulated trace of a jacobian \( \mathcal{J} \)

\[
\mathcal{A} \text{(nomaly)} = \lim_{\beta \to 0} \text{Tr} \mathcal{J} e^{-\frac{\beta}{2} \hat{R}} \tag{1}
\]

may be represented by quantum mechanical path integrals. However to fully understand such path integrals one must carefully address issues such as: the precise definition of the measure, which action corresponds to the particular operator ordering of \( \hat{R} \) and the correct Feynman rules for the perturbative expansion of such path integrals. In their original exposition, Alvarez-Gaumé and Witten consider chiral anomalies for which, due to their topological nature, the expression (1) is \( \beta \)-independent and calculable without such subtleties via a semiclassical expansion.

Recently, de Boer et.al. [4] have shown explicitly how to define the measure, action and Feynman rules for quantum mechanical path integrals for both bosons and fermions in curved space. The exact rules they obtain, although novel, follow directly from a rigorous treatment of the measure constructed from insertions of complete sets of coherent states. In this note we begin with their results and return to the gravitational chiral anomaly to verify through two loop order that their Feynman rules give the correct \( \beta \)-independent result for (1).

The gravitational chiral anomaly is given by the index of the Dirac operator

\[
\mathcal{A} = \text{Tr} \gamma_5 e^{-\frac{\beta}{2} \hat{D}} = n_+ - n_- \tag{2}
\]
where $n_{\pm}$ are the number of positive/negative parity zero modes of $\Psi = e_{\mu}^{\gamma}(\partial_\mu + \frac{1}{4}\omega_{\mu}^{\alpha\beta}\gamma_\alpha\gamma_\beta)$. $\mathcal{A}$ may be represented by a quantum mechanical path integral

$$\mathcal{A} = \int_M \frac{d^n y \sqrt{g(y)}}{(2\pi i)^2} d^n\Psi \ Z[y^\mu, \Psi^a],$$

(3)

where the path integral $Z$ depends on constant background fields $y^\mu$ and real fermionic (Majorana) $\Psi^a$ ($a = 1 \ldots n = \text{dim} M$ for some $n$-manifold $M$). Schematically

$$Z[y^\mu, \Psi^a] = \int [dq^\mu(t) dq^\alpha(t) da^\mu(t) da^\nu(t) db^\mu(t) db^\nu(t)] \exp\{ -\frac{1}{\hbar} (S_{\text{kin}} + S_{\text{int}}) \},$$

(4)

where

$$S_{\text{kin}} = \int_0^1 dt [\bar{\chi}_A \dot{\chi}^A + \frac{1}{2} g_{\mu\nu}(y)(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + 2b^\mu b^\nu)]$$

$$S_{\text{int}} = \int_0^1 dt \left[ (g_{\mu\nu}(q+y) - g_{\mu\nu}(y))(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + 2b^\mu b^\nu) \right.$$

$$\left. + \omega_{\mu\nu}(q+y)\dot{q}^\mu(\Psi + \bar{\psi})^a(\Psi + \bar{\psi})^b + \frac{\hbar^2}{4} \left( \Gamma_{\beta\mu}^\alpha \Gamma_{\alpha\nu}^\beta + \frac{1}{2} \omega_{\mu\nu}\omega^{\mu\nu} \right)(q+y)g^{\mu\nu}(q+y) \right].$$

(5)

(6)

As emphasized in [1], the above expression is the continuum limit of a rigorous discrete result for $Z[y^\mu, \Psi^a]$. Propagators must be derived from the discrete expressions and in this way ambiguities arising from products of distributions are resolved. Vertices may, however, be read directly from the continuum $S_{\text{int}}$ given above. In the kinetic action we have written $n/2$ complexified spinors $\chi^A$ as reminder that the correct Majorana propagators are obtained using the complexified $\chi^A$ as an intermediate step in order to construct fermionic coherent states. One could, of course, work completely in the complex basis but the interactions are then more complicated and depend on the arbitrary choice of complexification. The terms proportional to $\hbar^2$ in $S_{\text{int}}$ occur since the path integral was derived by Weyl ordering the regulator $\mathcal{D}\bar{\mathcal{D}}$ so that matrix elements could be calculated using the midpoint rule. However they contribute as extra vertices at three and higher loop level only so can be disregarded here. Of course, in higher loops we do expect these extra vertices to conspire with the new Feynman rules to give a $\beta$ independent result.

The number of graphs we must consider is greatly reduced if we choose Riemann normal coordinates $y^\mu$ such that geodesics through $O$ (say) are straight lines. I.e. any geodesic through $O$ with arc length $s$ has coordinates $y^\mu(s) = \xi^\mu s$. Taylor expanding in $s$ gives $y^\mu(s) = y^\mu(O) + s\dot{y}^\mu(O) + (1/2!) s^2\ddot{y}^\mu(O) + \cdots$ from which we see that the second and higher derivatives of $y^\mu(s)$ at $O$ vanish. But the geodesic equation $\ddot{y}^\mu(s) + \Gamma_{\alpha\beta}^\mu \dot{y}^\alpha(s) \dot{y}^\beta(s) = 0$ evaluated at $O$ gives $\Gamma_{\alpha\beta}^\mu(O) = 0$ and taking higher derivatives w.r.t. $s$ about $O$ of the geodesic equation shows that all symmetrized derivatives of the connection coefficients vanish $\partial_a[\ldots \Gamma_{\alpha\beta}^\mu(O) = 0$. It is now easy to find a

\footnote{We make two minor deviations from the notation of [1], firstly the worldline parameter $t \in [0,1]$ not $[-1,0]$ and the anticommuting ghosts appear as $g_{\alpha\beta} b^\alpha c^\beta$ rather than $(1/2) g_{\mu\nu} b^\mu c^\nu$.}
covariant expression for derivatives on any tensor at \( O \), in particular for the metric one finds

\[
\partial_\alpha g_{\mu\nu}(O) = 0 \tag{7}
\]

\[
\partial^2_{\alpha_1\alpha_2} g_{\mu\nu}(O) = \frac{1}{3} R_{\mu(\alpha_1 \alpha_2)\nu}. \tag{8}
\]

We also need Riemann normal coordinates for spinors to handle derivatives on the spin connection. Let us choose frames \( e_\mu^a \) such that the components of any vector with a flattened index \( v^a(s) \) parallely transported along a geodesic through \( O \) are constant \( \dot{v}^a(s) = 0 \). Taylor expanding \( v^a(s) = v^a(O) + s \dot{v}^a(O) + (1/2!) s^2 \ddot{v}^a(O) + \cdots \) shows that all derivatives of \( v^a(s) \) vanish at \( O \) so that the parallel transport equation \( Dv^a(s)/Ds = \dot{v}^a(s) + \dot{y}^\mu \omega_{\mu b} v^b(s) = 0 \) and derivatives thereof give that all symmetrized derivatives of the spin connection vanish \( \partial^n_{(\alpha_1 \cdots \alpha_n)\mu} a^b(O) = 0 \) for \( n = 0, 1, \ldots \). Hence we readily obtain

\[
\omega_{\mu b}^a(O) = 0 \tag{9}
\]

\[
\partial_\alpha \omega_{\mu b}^a(O) = \frac{1}{2} R_{\alpha\mu}^a b \tag{10}
\]

\[
\partial^2_{\alpha_1\alpha_2} \omega_{\mu b}^a(O) = \frac{1}{3} D_{(\alpha_1 R_{\beta})\mu}^a b. \tag{11}
\]

Let us now give all vertices relevant to our calculations.

\[
\begin{align*}
-\frac{1}{4} R_{\mu\nu ab} \Psi^a \Psi^b & \int_0^1 q^\mu \dot{q}^\nu & = \frac{1}{2} R_{\mu\nu ab} \Psi^b \int_0^1 q^\mu \dot{q}^\nu \\
-\frac{1}{3} R_{\mu(\alpha\beta)\nu} \frac{1}{2!} & \int_0^1 q^\mu \dot{q}^\nu \dot{q}^\alpha \dot{q}^\beta & = -\frac{1}{2} \partial^n_{(\alpha_1 \cdots \alpha_n)\mu} a_{\mu b} \Psi^a \Psi^b \frac{1}{n!} \int_0^1 q^{\alpha_1} \cdots q^{\alpha_n} \dot{q}^\mu 
\end{align*}
\]

Wiggly lines denote bosons and straight lines Majorana fermions. A dot on a boson line indicates a \( \dot{q}^\mu \) at the vertex and the on the end of fermion lines denote external fermion background fields \( \Psi^a \). It is expedient to leave the factor \( \partial^n_{(\alpha_1 \cdots \alpha_n)\mu} \omega_{\mu b} \) as it stands rather than grinding out directly its covariant Riemann normal coordinate expression since in graphs only certain antisymmetrized combinations will appear which are then readily covariantized. Note that we ignore ghost vertices. Strictly speaking we should replace \( \frac{1}{2!} \int_0^1 q^\mu \dot{q}^\nu \dot{q}^\alpha \dot{q}^\beta \rightarrow \frac{1}{2!} \int_0^1 q^\mu \dot{q}^\nu (\frac{1}{2!} \dot{q}^\alpha \dot{q}^\beta + \frac{1}{2!} \alpha^\alpha a^b + b^\alpha c^\beta) \), however at two loop level the ghosts only arise in self energy loops where they exactly cancel the delta function divergence of the accompanying \( \dot{q} \dot{q} \) self energy contraction.

We now give the propagators of the theory,

\[
\begin{align*}
\langle q^\mu(s) q^\nu(t) \rangle &= g^{\mu\nu}(y) \Delta(s, t) \quad (13) \\
\langle \dot{q}^\mu(s) \dot{q}^\nu(t) \rangle &= g^{\mu\nu}(y) \Delta(s, t) \quad (14)
\end{align*}
\]
\[
\begin{align*}
\langle \dot{q}^\mu(s) \dot{q}^\nu(t) \rangle &= g^{\mu\nu}(y) \Delta(s, t) \\
\langle a^\mu(s) a^\nu(t) \rangle &= g^{\mu\nu}(y) (\Delta(s, t) + 1) \\
\langle b^\mu(s) c^\nu(t) \rangle &= -g^{\mu\nu}(y) (\Delta(s, t) + 1) \\
\langle \psi^a(s) \psi^b(t) \rangle &= \delta^{ab} (1/2) \epsilon(s - t) + K^{ab}.
\end{align*}
\]

Where we have denoted
\[
\Delta(s, t) \equiv s \cdot \Delta_t \cdot t = t(1 - s) \theta(s - t) + s(1 - t) \theta(t - s)
\]
\[
\tilde{\Delta}(s, t) \equiv s \cdot \tilde{\Delta}_t \cdot t = (1 - s) \theta(s - t) - s \theta(t - s) = (d/dt) \Delta(s, t)
\]
\[
\Delta(s, t) \equiv s \cdot \Delta_t \cdot t = \delta(s - t) - 1 = (d^2/dsdt) \Delta(s, t).
\]

Further the theta function is defined as
\[
\theta(s - t) \equiv \begin{cases} 
1 & s > t \\
\frac{1}{2} & s = t \\
0 & s < t
\end{cases}
\]

and \( \epsilon(s - t) \equiv \theta(s - t) - \theta(t - s) \). When handling products of distributions the delta function is treated as a Kronecker delta, for example \( \int_0^1 ds \theta(s - t) \delta(s - t) = 1/2 \). The term \( K^{ab} \) in the fermion propagator is a relic of the complexification (see [2]) of the original real spinors, here we need only that \( K^{ab} = -K^{ab} \). Of course final (physical) results should be \( K^{ab} \) independent.

Now all graphs are of the factorized form \( (\text{general rel.}) \times (\text{integrals over } \Delta, \tilde{\Delta}, \Delta \text{ and } \epsilon) \), which are, in principle, elementary to perform. Of course in practice there is a large amount of trivial algebra to perform which can be greatly simplified if one first derives certain identities for the propagators (19)-(21). Firstly observe that propagators of the form \( \mathcal{D}(s, t) = d_1(s, t) \theta(s - t) + d_2(s, t) \theta(t - s) \) close under multiplication
\[
[\mathcal{D} \tilde{\mathcal{D}}](s, t) \equiv \int_0^1 dr \mathcal{D}(s, r) \tilde{\mathcal{D}}(r, t)
\]
\[
= \theta(s - t) \left( \int_0^t d_1(s, r) \tilde{d}_2(r, t) + \int_t^s d_1(s, r) \tilde{d}_1(r, t) + \int_s^1 d_2(s, r) \tilde{d}_1(r, t) \right) + \theta(t - s) \left( \int_0^s d_1(s, r) \tilde{d}_2(r, t) + \int_s^t d_2(s, r) \tilde{d}_2(r, t) + \int_t^1 d_2(s, r) \tilde{d}_1(r, t) \right).
\]

Let us now adopt a diagrammatic notation in which propagators are depicted as in (19)-(21) where a dot at the end of a line or vertex denotes a point yet to be integrated over (see the ends of the propagators above, we usually also attach a variable \( s, t, \ldots \) for clarity) and a cross denotes a point where an integration \( \int_0^1 \) has been performed. For example, applying (23) in this notation
\[
\begin{align*}
\begin{array}{c}
\text{s} \\
\text{••••••••t}
\end{array}
& \equiv \int_0^1 dr \Delta(s, r) \Delta(r, t) \\
& = (1/2)(s - t)(t(1 - s) \theta(s - t) + s(1 - t) \theta(t - s)) \\
& = \frac{1}{2}(s - t) \Delta(s, t) \\
& = -\left(\begin{array}{c}
\text{s} \\
\text{••••••••t}
\end{array}\right)
\end{align*}
\]
The last line of (24) is an example of an allowed (and very useful) integration by parts. In it is stressed that ad hoc integrations by parts are not compatible with the Kronecker delta prescription for the delta function, however (24) is correct since it is an example of the more general result

$$(d^n/dt^n)\Delta(0, t) = 0 = (d^n/dt^n)\Delta(1, t).$$

(25)

It is highly expedient to make integrations by parts at vertices with a single dotted line, in diagrammatic notation

$$1 \rightarrow \cdot \cdot \cdot 2 = -\sum_{i=2}^{m} 1 \rightarrow \cdot \cdot \cdot i.$$ 

(26)

This relation holds even if the outgoing lines form self-energy loops since $\Delta(s, s) = (1/2)(d/ds)\Delta(s, s)$.

The propagator $\Delta(s, t)$ seems to be an odd bunny but may be easily handled by noticing that its effect in any graph is to produce the difference of two graphs, the first in which the points $s$ and $t$ are pinched together (doing the delta function) and the second in which the $\Delta$ is simply absent (see the example in figure 1.). The only exception (at two loops, although a similar statement holds at higher loops) are self energy loops where, due to the aforementioned ghosts, $\Delta(s, s) = \delta(0) - 1 \rightarrow -1$.

In table 1 we list the results for various products of propagators, plus the results of integrating over the ends of these concatenated propagators and forming loops from them, included also are the results for the graphs with fermion propagators in which we denote $(1/2)\epsilon(s-t) \equiv s \rightarrow t$. Such graphs may also easily be evaluated using the techniques discussed above if one makes use of the identity

$$s \rightarrow t = \frac{1}{2} d/dt \epsilon(s-t) = -\delta(s-t),$$

(27)

for convenience however, we give the explicit results for these graphs. To avoid confusion, note that table 1 is really just a table of integrals if one decodes the graphical notation used here.

Before considering the graphs required to calculate the anomaly $\mathcal{A}$, let us discuss one more property of the propagators (19)-(21). Suppose we preferred the convention of integrating $f_{-1}^0$ to $f_0^1$, then we could convert our results via the variable change $s' = -s$ or $s'' = s - 1$ under both of which $f_0^1 ds \rightarrow f_{-1}^0 ds$. However, although the propagators $\Delta(s, t)$ and $\Delta(s, t)$ transform identically under both of these relabellings, the $q\dot{q}$ propagator transforms with a relative sign between the two variable changes. Hence in any graph $\mathcal{G} = f_0^1(\prod_i ds_i)(\prod \Delta)(\prod^N \Delta)(\prod \Delta)$, by subsequently changing variables $s' = s - 1$ and $s'' = -s'$ we have $\mathcal{G} = (-)^N \mathcal{G}$ so that only graphs with an even number of $q\dot{q}$ propagators are non-vanishing. Indeed study of possible bosonic graphs (graphs without fermion propagators) shows that the vertex

$$-\frac{1}{2} \partial_{a_1 \ldots a_n} \omega_{\mu ab} \Psi^a \Psi^b \frac{1}{n!} f_0^1 q^{a_1} \ldots q^{a_n} \dot{q}^\mu$$

Here we assume that the graph is expressed only in terms of propagators $\Delta$, $\Delta$ and $\Delta$. 


|\begin{align*}
\bullet & \xrightarrow{s} = \frac{1}{2} s(1 - s) & \xrightarrow{\frac{1}{12}} = \frac{1}{6} & \xrightarrow{\frac{1}{6}} = s(1 - s) \\
\bullet & \xrightarrow{s} = 0 & \xrightarrow{\frac{1}{2} - t} = 0 & \xrightarrow{\frac{1}{2} - s} = 0 \\
\bullet \xrightarrow{s} = \frac{1}{6} t(1 - s)(2s - s^2 - t^2) & \theta(s - t) + \frac{1}{6} s(1 - t)(2t - t^2 - s^2) & \theta(t - s) \\
\bullet \xrightarrow{s} = \frac{1}{24} s(1 - s)(1 + s - s^2) & \xrightarrow{\frac{1}{120}} = \frac{1}{90} & \xrightarrow{\frac{1}{90}} = \frac{1}{3}s^2(1 - s)^2 \\
\bullet \xrightarrow{s} = \frac{1}{2}(s - t)(t(1 - s)\theta(s-t) + s(1-t)\theta(t-s)) = \frac{1}{2} (s-t)\Delta(s,t) = -\left(s\xrightarrow{t}\right) \\
\bullet \xrightarrow{s} = \frac{1}{12} s(1 - s)(2s - 1) & \xrightarrow{0} = 0 & \xrightarrow{0} = 0 \\
\bullet \xrightarrow{s} = \frac{1}{6} t(t^2 + 3s^2 + 2 - 6s) & \theta(s - t) + \frac{1}{6} (1 - t)(2t - t^2 - 3s^2) & \theta(t - s) \\
\bullet \xrightarrow{s} = \frac{1}{24}(2s - 1)(2s^2 - 2s - 1) & \xrightarrow{\frac{1}{3}s(1 - s)(1 - 2s) \\
\bigcirc & \xrightarrow{\frac{1}{80}} = 0 & \bigcirc \xrightarrow{\frac{1}{240}} = 0 & \bigcirc \xrightarrow{\frac{1}{240}} = 0 & \bigcirc \xrightarrow{\frac{-1}{240}} = 0 \\
\end{align*}|

Table 1. “Table of Integrals”. 
must appear an even number of times\footnote{This is shows that bosonic graphs must have $4k + 4, \ k = 0, 1, \ldots$ external $\Psi$'s which is consistent with the chiral anomaly existing in $4k + 4$ dimensions only.} so that any bosonic graph has an even number of “dots” and therefore an even number of $qq$ propagators. Furthermore notice that $\epsilon(s - t)$ changes sign under $s' = -s$ but is invariant under $s' = s - 1$ so that we can now argue that the total number of $qq$ propagators plus the number of $\epsilon(s - t)$'s in any graph must be even. Therefore any non-vanishing graph containing $\epsilon(s - t)$ vanishes when a single $\epsilon(s - t)$ is replaced by $K^{ab}$. In fact one finds via this argument (or directly) that all two loop graphs involving $K^{ab}$ vanish separately. At higher loops only an even number of $K^{ab}$'s can appear (since the total number of $qq$ and fermion propagators must be even for any graph with $4k + 4$ external $\Psi$'s) and must cancel amongst themselves or as contractions on symmetric invariants.

Using the results in table 1 and the vertices and propagators given above one may now write down all relevant graphs and their results by inspection (at worst one may need to perform a single integral \textbf{\textit{f}} after appropriate integrations by parts and manipulations as explained above, see also figure 1 for examples). We work in $n = 4$ dimensions so all graphs must have four external fermions to saturate the Grassmann integral $\int d^4 \Psi$. To calculate $Z[q^\mu, \Psi^a]$ we must include all graphs, disconnected, connected and one-particle irreducible. At one loop order we find

$$
\begin{pmatrix}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{pmatrix}
= \frac{1}{192} R_{\mu\nu ab} R^{\mu\nu cd} \Psi^a \Psi^b \Psi^c \Psi^d.
$$

For brevity we leave the dots off the graphs, to reinstate them one must write all independent combinations of dots allowed by the vertices \cite{12}. Here there are two such independent graphs,
however using the integration by parts given in (24) and anti-symmetry of $R_{\mu
u}ab$ in $\mu$ and $\nu$ one needs only calculate two times one of them. With this result we can already calculate the gravitational chiral anomaly using (3) and find the correct result

$$\mathcal{A} = -\frac{1}{384\pi^2} \int d^4y \sqrt{g} R_{\mu
u}ab \left( \frac{1}{2} \epsilon^{abcd} R^\mu_{\nu cd} \right).$$

At two loops the results are

$$\left( \begin{array}{c} \text{graph 1} \\ \text{graph 2} \end{array} \right) = -\frac{1}{4608} R R_{\mu
u}ab R^\mu_{\nu cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (30)$$

$$= \frac{1}{5760} D_\alpha R^\alpha_{\mu ab} D_\beta R^\beta_{\mu cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (31)$$

$$= -\frac{1}{2880} R^{(\nu\rho)} R_{\mu
u}ab R_{\rho cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (32)$$

$$= \frac{1}{2880} R_{\mu\nu}ab R^\nu_{\alpha cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (33)$$

$$= \frac{1}{1440} D_\alpha R_{\mu
u}ab D^\alpha R^\mu_{\nu cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (34)$$

$$= \frac{1}{960} \Box \delta_{[\mu} \omega_{\nu]ab} R^\mu_{\nu cd} \Psi^a \Psi^b \Psi^c \Psi^d \quad (35)$$

$$= 0 \quad (36)$$

One may check that the fermion graphs (36) vanish, for the $K_{ab}$ pieces this cancellation is graph by graph as predicted above but for the pieces with an $\epsilon(s-t)$ the four independent graphs (after considering all combinations of dots) conspire to cancel and the relevant integrals are given in table 1. Also we still need to use Riemann normal coordinates to give a covariant expression for (35). Using $\Gamma^\mu_{\alpha\beta}(O) = 0 = \omega_{\mu ab}(O)$, $\partial_\mu \Gamma^\nu_{\alpha\beta}(O) = -(1/3) R^\nu_{(\alpha\beta)\mu}$ and $\partial_\mu \omega_{\nu ab}(O) = (1/2) R_{\mu
u}ab$ we find

$$\Box \delta_{[\mu} \omega_{\nu]ab} = D^2 R_{\mu
u}ab + \frac{1}{3} R^\gamma_{[\mu} R_{\nu]ab} + \frac{1}{2} R_\alpha \epsilon^{\gamma\alpha\mu\nu} [\mu R_{\nu]ab}. \quad (37)$$

Hence we have for the two loop $(O(h))$ contribution to $\mathcal{A}$

$$\mathcal{A}_1 = \int \mu^a_{ab} \left\{ \begin{array}{c} 8D^2 R_{\mu
u}ab R^\mu_{\nu cd} - 8 R_{\mu\nu}ab R^\nu_{\alpha cd} \\ -24 R^\alpha_{\mu ab} R_{\alpha cd} R^\nu_{\mu cd} - 8 R^{[\nu\rho]} R_{\mu
u}ab R_{\rho cd} \end{array} \right\}. \quad (38)$$
where \( \mu \equiv d^4 y \sqrt{g(y)}/( (2\pi i)^2 60 \cdot 384) \). It remains now only to show that the set of invariants built from three Riemann tensors \( R_{\mu\nu\rho\sigma} \) in (38) vanishes. To this end one needs only the usual symmetries and Bianchi identities for the Riemann tensor

\[
R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\rho\sigma\nu} = R_{\rho\sigma\nu\mu}; \quad R_{\mu(\nu\rho\sigma)} = 0 = D_\alpha R_{\mu\nu\rho\sigma}
\]  

and the identity

\[
\delta_{\mu}^\nu \epsilon^{abcd} = \delta_{\nu}^\alpha \epsilon^{abcd} + \delta_{\nu}^\beta \epsilon^{a\mu cd} + \delta_{\nu}^\gamma \epsilon^{ab\mu d} + \delta_{\nu}^\delta \epsilon^{abcd}.
\]  

(40)

Let us give some details. Rewrite the fourth term in (38) using \( R_{\mu(\nu\sigma)}^\rho R_{\rho\mu ab} = -{(3/2)} R_{\mu\nu\rho\sigma} R_{\rho\mu ab} \) and apply (40) twice to \( \epsilon^{abcd} R_{\mu\nu} = R_{\rho\lambda \sigma\eta} \delta_{\sigma}^\rho \delta_{\eta}^\lambda \epsilon^{abcd} \) so that the sixth term in (38) becomes

\[
\int \mu \epsilon^{abcd} R_{\mu\nu}^\rho R_{\rho\mu ab} R_{\mu\nu cd} = \int \mu \epsilon^{abcd} (2R_{\mu\nu\rho\sigma} R_{\rho\mu ab} R_{\mu\nu cd} - 8 R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd}).
\]  

(41)

In a similar fashion one can rewrite the second term in (38) as

\[
\int \mu \epsilon^{abcd} R_{\mu\nu}^\rho R_{\rho\mu ab} R_{\mu\nu cd} = \int \mu \epsilon^{abcd} (-2 R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd} + 2 R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd}).
\]  

(42)

For the fifth term in (38), use the Bianchi identity on the indices \( \alpha ab \) and \( \beta cd \), so that integrating by parts and using the antisymmetry of \( \epsilon^{abcd} \) one gets a commutator \([D_a, D_b] \) which may be expressed as curvatures whereby

\[
\int \mu \epsilon^{abcd} D_\alpha R_{\mu\nu ab} D_\beta R_{\mu\nu cd} = 2 \int \mu \epsilon^{abcd} R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd}.
\]  

(43)

where the last line was obtained by using (40). In a similar fashion the first term of (38) may be expressed in terms of curvatures as

\[
\int \mu \epsilon^{abcd} D^2 R_{\mu ab} R_{\mu\nu cd} = \int \mu \epsilon^{abcd} (4 R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd} - 4 R_{\mu a}^\alpha e R_{\alpha eb} R_{\mu\nu cd}).
\]  

(44)

Orchestrating the above manipulations, one finds

\[
A_1 = 8 \int \mu \epsilon^{abcd} R_{\mu a}^\alpha e (R_{\alpha eb} + R_{\alpha eb} + R_{\alpha eb}) R_{\mu\nu cd},
\]  

(45)

which clearly vanishes. This concludes our two loop demonstration of the \( \beta \)-independence of the anomaly \( A \).

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