On the inhomogeneous T-Q relation for quantum integrable models

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Abstract

The off-diagonal Bethe Ansatz method [1] is used to revisit the periodic XXX Heisenberg spin-$\frac{1}{2}$ chain. It is found that the spectrum of the transfer matrix can be characterized by an inhomogeneous T-Q relation, a natural but nontrivial extension of Baxter’s T-Q relation [2].
One of Baxter’s important discoveries is the T-Q relation \[2\], which provides a convenient and universal parametrization for eigenvalues of transfer matrices of most quantum integrable models. Nevertheless, the Q operator does not allow polynomial solutions for some integrable models such as the XYZ quantum spin chain with an odd number of sites and generic coupling constants, the chiral Potts model and the quantum spin chains with non-diagonal boundaries, despite the fact that the transfer matrix is a polynomial operator. It is obvious that the T-Q parametrization for transfer matrix is not the unique one because there are many ways to characterize a polynomial function, e.g., with its roots or with its coefficients. In a recent series of works (see ref.\[1\] and the references therein), a generalization of the T-Q relation with an extra inhomogeneous term, i.e., the inhomogeneous T-Q relation, was proposed and used in solving some integrable models without \(U(1)\) symmetry. This generalization seems to be a universal solution of the Hirota type equations (recursive inversion identities) and can account for the boundary conditions self-consistently without losing a polynomial Q operator.

In this note, we show that the inhomogeneous T-Q relation can also characterize the spectrum of the ordinary integrable models that can be characterized by Baxter’s T-Q relation and can be solved with the ordinary Bethe Ansatz methods.

Let us consider the periodic XXX spin-\(\frac{1}{2}\) chain. The corresponding \(R\)-matrix reads

\[
R_{0,j}(u) = u + \eta P_{0,j} = u + \frac{1}{2}\eta(1 + \vec{\sigma}_0 \cdot \vec{\sigma}_j),
\]

where \(\eta\) is the crossing parameter (we put \(\eta = 1\) in this case), \(\vec{\sigma}_j = (\sigma^x_j, \sigma^y_j, \sigma^z_j)\) are the Pauli matrices, and \(P_{i,j}\) is the permutation operator possessing the properties:

\[
P_{i,j}O_j = O_i P_{i,j}, \quad P_{i,j}^2 = \text{id}, \quad tr_j P_{i,j} = tr_i P_{i,j} = \text{id},
\]

for arbitrary operator \(O\) defined in the corresponding tensor space. This \(R\)-matrix satisfies the Yang-Baxter equation

\[
R_{1,2}(u - v)R_{1,3}(u)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u)R_{1,2}(u - v).
\]

It is easy to show that the \(R\)-matrix \([1]\) also satisfies the following relations:

Initial condition : \(R_{1,2}(0) = P_{1,2},\)

Unitary relation : \(R_{1,2}(u)R_{2,1}(-u) = -u(u - 1) \times \text{id},\)

Crossing relation : \(R_{1,2}(u) = -\sigma^y_1 R_{1,2}^{p_1}(-u - 1)\sigma^y_1.\)
The monodromy matrix and the corresponding transfer matrix of the periodic XXX spin-$\frac{1}{2}$ chain are respectively defined as

\[
T_0(u) = R_{0,N}(u - \theta_N) \cdots R_{0,1}(u - \theta_1) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix},
\]

\[
t(u) = tr_0 T_0(u) = A(u) + D(u),
\]

with \( \{\theta_j| j = 1, \cdots, N\} \) being some generic site-dependent inhomogeneity parameters. With the Yang-Baxter equation we can show that \([t(u), t(v)] = 0\). The Hamiltonian of the XXX spin-$\frac{1}{2}$ chain is thus expressed as

\[
H = \frac{1}{2} \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = \frac{\partial \ln t(u)}{\partial u} \bigg|_{u=\theta_j=0} - \frac{1}{2} N,
\]

with the periodic boundary condition \( \vec{\sigma}_N+1 \equiv \vec{\sigma}_1 \).

In order to get some functional relations of the transfer matrix, we evaluate the transfer matrix \( t(u) \) at the particular points \( u = \theta_j \) and \( u = \theta_j - 1 \). Let us apply the initial condition of the \( R \)-matrix to express the transfer matrix \( t(\theta_j) \) as

\[
t(\theta_j) = tr_0 \{ R_{0,N}(\theta_j - \theta_N) \cdots R_{0,j+1}(\theta_j - \theta_{j+1}) \\
\times P_{0,j} R_{0,j-1}(\theta_j - \theta_{j-1}) \cdots R_{0,1}(\theta_j - \theta_1) \} \\
= R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) \\
\times tr_0 \{ R_{0,N}(\theta_j - \theta_N) \cdots R_{0,j+1}(\theta_j - \theta_{j+1}) P_{0,j} \} \\
= R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) \\
\times R_{j,N}(\theta_j - \theta_N) \cdots R_{j,j+1}(\theta_j - \theta_{j+1}).
\]

In deriving the above equation, the initial condition (11) of the \( R \)-matrix plays a key role, which allows us to rewrite the transfer matrix as a product of \( R \)-matrices at the special spectral parameter points \( \theta_j \). The transfer matrix \( t(\theta_j) \) is a reduced monodromy matrix if the \( j \)-th quantum space is treated as the auxiliary space.

The crossing relation (6) makes it possible to express the transfer matrix \( t(\theta_j - 1) \) as

\[
t(\theta_j - 1) = tr_0 \{ R_{0,N}(\theta_j - \theta_N - 1) \cdots R_{0,1}(\theta_j - \theta_1 - 1) \} \\
= (-1)^N tr_0 \{ \sigma_{0}^y R_{0,N}^0(-\theta_j + \theta_N) \cdots R_{0,1}^0(-\theta_j + \theta_1) \} \\
= (-1)^N tr_0 \{ R_{0,1}(-\theta_j + \theta_1) \cdots R_{0,N}(-\theta_j + \theta_N) \} \\
= (-1)^N R_{j,j+1}(-\theta_j + \theta_{j+1}) \cdots R_{j,N}(-\theta_j + \theta_N) \\
\times R_{j,1}(-\theta_j + \theta_1) \cdots R_{j,j-1}(-\theta_j + \theta_{j-1}).
\]
Using the unitary relation (5), we have

\[ t(\theta_j)t(\theta_j - 1) = a(\theta_j)d(\theta_j - 1), \quad j = 1, \ldots, N, \]

(12)

\[ a(u) = \prod_{j=1}^{N} (u - \theta_j + 1), \quad d(u) = \prod_{j=1}^{N} (u - \theta_j). \]

(13)

The homogeneous analogue of (12) reads

\[ \frac{\partial^l}{\partial u^l} \left\{ t(u)t(u - 1) - a(u)d(u - 1) \right\} |_{u=0, \theta_j=0} = 0, \quad l = 0, \ldots, N - 1. \]

(14)

Applying (12) to an eigenstate of \( t(u) \), the corresponding eigenvalue \( \Lambda(u) \) thus satisfies

\[ \Lambda(\theta_j)\Lambda(\theta_j - 1) = a(\theta_j)d(\theta_j - 1), \quad j = 1, \ldots, N. \]

(15)

In addition, from the definition of the transfer matrix (8) it is easy to show that \( \Lambda(u) \) is a degree \( N \) polynomial of \( u \),

\[ \text{(16)} \]

with the asymptotic behavior

\[ \Lambda(u) = 2u^N + \cdots. \]

(17)

The relations (15)-(17) determine the spectrum of the model completely.

We can easily demonstrate that for any given parameter \( \phi \), the following inhomogeneous T-Q relation satisfies (15)-(17) and therefore characterizes the spectrum of the transfer matrix \( t(u) \) of the periodic XXX spin-\( \frac{1}{2} \) chain completely

\[ \Lambda(u) = e^{i\phi}a(u)\frac{Q(u - 1)}{Q(u)} + e^{-i\phi}d(u)\frac{Q(u + 1)}{Q(u)} + 2(1 - \cos \phi)\frac{a(u)d(u)}{Q(u)}, \]

(18)

\[ Q(u) = \prod_{j=1}^{N} (u - \mu_j), \]

(19)

provided that the Bethe roots \( \{\mu_j|j = 1, \ldots, N\} \) satisfy the Bethe Ansatz equations (BAEs)

\[ e^{i\phi}a(\mu_j)Q(\mu_j - 1) + e^{-i\phi}d(\mu_j)Q(\mu_j + 1) = 2(\cos \phi - 1)a(\mu_j)d(\mu_j), \]

(20)

and the selection rules \( \mu_j \neq \mu_l, \mu_j \neq \theta_l, \theta_l - 1 \).

**Proposition 1:** Each solution of (15)-(17) can be parameterized in terms of the inhomogeneous T-Q relation (18) with a polynomial Q-function (19).
Proof: Given a degree $N$ polynomial $\Lambda(u)$ satisfying (15)-(17), we seek the degree $N$ polynomial solution of $Q$-function satisfying the equation

$$Q(u)\Lambda(u) = e^{i\phi}a(u)Q(u-1) + e^{-i\phi}d(u)Q(u+1) + 2(1 - \cos \phi)a(u)d(u). \quad (21)$$

We note that the above equation is a polynomial one of degree $2N$. If the equation holds at $2N + 1$ independent points of $u$, the equation must also hold for arbitrary $u$. Obviously, the above equation holds for $u \rightarrow \infty$. In addition, as $d(\theta_j) = a(\theta_j - 1) = 0$, we readily obtain that

$$Q(\theta_j)\Lambda(\theta_j) = e^{i\phi}a(\theta_j)Q(\theta_j - 1), \quad (22)$$

$$Q(\theta_j - 1)\Lambda(\theta_j - 1) = e^{-i\phi}d(\theta_j - 1)Q(\theta_j). \quad (23)$$

From (15) we deduce that only one of (22) and (23) is independent. Obviously, (22) (or equivalently (23)) allows a degree $N$ polynomial solution of $Q(u)$

$$Q(u) = u^N + \sum_{n=0}^{N-1} \tilde{I}_n u^n = \prod_{j=1}^{N}(u - \mu_j). \quad (24)$$

Substituting the above Ansatz into (22) we have $N$ linear equations for the $N$ coefficients $\{\tilde{I}_n | n = 0, \ldots, N - 1\}$ which have a unique solution for a given $\Lambda(u)$. Taking $u = \mu_j$ in (21), we readily have the BAEs (20). □

**Proposition 2:** The functional relations (15)-(17) are the sufficient and necessary conditions to completely characterize the spectrum of the transfer matrix (8) with the $R$-matrix (1).

Proof: Let us introduce the rotated monodromy matrix

$$T_\phi(u) = \begin{pmatrix} A_\phi(u) & B_\phi(u) \\ C_\phi(u) & D_\phi(u) \end{pmatrix} = \begin{pmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}. \quad (25)$$

For each solution $\Lambda(u)$ of the functional relations (15)-(17), in terms of the inhomogeneous T-Q relation (18), we can construct the following eigenstate of the transfer matrix:

$$|\mu_1, \ldots, \mu_N\rangle = \prod_{j=1}^{N} B_\phi(\mu_j) |0\rangle, \quad (26)$$
The eigenvalues \( E_n \) calculated from (27) are the same as those from the exact diagonalization of the Hamiltonian (9). The symbol \( n \) denotes the number of the energy levels and \( d \) indicates the number of degeneracy.

| \( \mu_1 \) | \( \mu_2 \) | \( \mu_3 \) | \( \mu_4 \) | \( E_n \) | \( d \) |
|---|---|---|---|---|---|
| \(-2.97259 + 1.15909i\) | \(-2.51751 - 1.42184i\) | \(-0.50990 + 0.26274i\) | \(-1.50000\) | 2 |
| \(-2.97259 - 1.15909i\) | \(-2.51751 + 1.42184i\) | \(-0.50990 - 0.26274i\) | \(-1.50000\) | 2 |
| \(-2.88462 + 0.00000i\) | \(-1.55769 - 2.56650i\) | \(-1.55769 + 2.56650i\) | \(1.50000\) | 4 |

Table 2: The numerical solutions of the BAEs (20) for \( N = 4 \), \( \phi = -0.69315i \) and \( \{\theta_j = 0\} \). The eigenvalues \( E_n \) calculated from (27) are the same as those from the exact diagonalization of the Hamiltonian (9). The symbol \( n \) denotes the number of the energy levels and \( d \) indicates the number of degeneracy.

| \( \mu_1 \) | \( \mu_2 \) | \( \mu_3 \) | \( \mu_4 \) | \( E_n \) | \( d \) |
|---|---|---|---|---|---|
| \(-3.46085 - 2.04638i\) | \(-3.46085 + 2.04638i\) | \(-0.53915 + 0.28370i\) | \(-0.53915 + 0.28370i\) | \(-4.00000\) | 1 |
| \(-3.49754 - 0.00000i\) | \(-2.00000 + 2.49853i\) | \(-2.00000 - 2.49853i\) | \(-2.00000 - 0.00000i\) | \(-2.00000\) | 3 |
| \(-3.41695 - 0.01463i\) | \(-2.20702 + 2.20734i\) | \(-1.88461 + 2.68745i\) | \(-0.49142 + 0.49474i\) | \(-0.00000\) | 3 |
| \(-3.41695 + 0.01463i\) | \(-2.20702 - 2.20734i\) | \(-1.88461 - 2.68745i\) | \(-0.49142 - 0.49474i\) | \(-0.00000\) | 3 |
| \(-3.38446 - 2.02080i\) | \(-3.38446 + 2.02080i\) | \(-1.11571 + 0.00000i\) | \(-0.11537 + 0.00000i\) | \(0.00000\) | 1 |
| \(-3.07558 + 1.25638i\) | \(-3.07558 - 1.25638i\) | \(-0.92442 + 3.56865i\) | \(-0.92442 - 3.56865i\) | \(2.00000\) | 5 |

while \(|0\rangle\) is the all spin-up state. Therefore, each solution of the functional relations (15)-(17) corresponds to an eigenvalue of the transfer matrix. \( \Box \)

The corresponding eigenvalues of the Hamiltonian (9) read

\[
E = \left. \frac{\partial \ln \Lambda(u)}{\partial u} \right|_{u=0,\{\theta_j = 0\}} - \frac{1}{2} N. \tag{27}
\]

The numerical solutions of the BAEs (20) and the corresponding eigenvalues of the Hamiltonian (9) for \( N = 3 \) and \( N = 4 \) with an arbitrarily chosen \( \phi \) are shown in Table 1.1 and Table 1.2 respectively. Those numerical simulations imply that the inhomogeneous T-Q relation (18) and the BAEs (20) indeed give the correct and complete spectrum of the periodic XXX spin-\(\frac{1}{2}\) chain model.

In conclusion, we showed that the spectrum of the periodic Heisenberg spin-chain model can also be characterized by an inhomogeneous T-Q relation. This conclusion can be generalized to other quantum integrable models. We remark that in the present case, the inhomogeneous T-Q relation can be reduced to Baxter’s homogeneous T-Q relation by taking \( \phi = 0 \), and the degree of the Q polynomial can be reduced to \( M \) with \( 0 \leq M \leq N \) by
taking some of the Bethe roots to be infinity. However, for most of the quantum integrable models without $U(1)$ symmetry, the inhomogeneous term is indeed irreducible.

References

[1] Y. Wang, W.-L. Yang, J. Cao and K. Shi, *Off-Diagonal Bethe Ansatz for Exactly Solvable Models* (Springer Berlin Heidelberg, 2015).

[2] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).