THE CATEGORY OF MAXIMAL COHEN–MACAULAY MODULES AS A
RING WITH SEVERAL OBJECTS

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ABSTRACT. Over a commutative local Cohen–Macaulay ring, we view and study the category of maximal Cohen–Macaulay modules as a ring with several objects. We compute the global dimension of this category and thereby extend a result of Leuschke to the case where the ring has arbitrary (as opposed to finite) CM-type.

1. INTRODUCTION

Let \( R \) be a commutative local Cohen–Macaulay ring with Krull dimension \( d \). Suppose that \( R \) has finite CM-type; this means that, up to isomorphism, \( R \) admits only finitely many indecomposable maximal Cohen–Macaulay modules \( X_1, \ldots, X_n \). In this case, the category \( \text{MCM} \) of maximal Cohen–Macaulay \( R \)-modules has a representation generator, i.e. a module \( X \in \text{MCM} \) that contains as direct summands all indecomposable maximal Cohen–Macaulay \( R \)-modules (for example, \( X = X_1 \oplus \cdots \oplus X_n \) would be such a module). A result by Leuschke \([11, \text{Thm. 6}]\) shows that the endomorphism ring \( E = \text{End}_R(X) \) has finite global dimension \( \leq \max\{2, d\} \), and that equality holds if \( d \geq 2 \). This short paper is motivated by Leuschke’s result.

If \( R \) does not have finite CM-type, then \( \text{MCM} \) has no representation generator and there is a priori no endomorphism ring \( E \) to consider. However, regardless of CM-type, one can always view the entire category \( \text{MCM} \) as a “ring with several objects” and then study its (finitely presented) left/right “modules”, i.e. covariant/contravariant additive functors from \( \text{MCM} \) to abelian groups. The category \( \text{MCM-\text{mod}} \) of finitely presented left modules over the “several object ring” \( \text{MCM} \) is the natural object to study in the general case. Indeed, if \( R \) has finite CM-type, then this category is equivalent to the category \( \text{E-\text{mod}} \) of finitely generated left \( E \)-modules, where \( E \) is the endomorphism ring introduced above.

It turns out that \( \text{MCM-\text{mod}} \) and \( \text{mod-\text{MCM}} \), i.e. the categories of finitely presented left and right modules over \( \text{MCM} \), are abelian with enough projectives. Thus one can naturally speak of the global dimensions of these categories; they are called the left and right global dimensions of \( \text{MCM} \), and they are denoted \( l_{\text{gldim}}(\text{MCM}) \) and \( r_{\text{gldim}}(\text{MCM}) \). We show that there is an equality \( l_{\text{gldim}}(\text{MCM}) = r_{\text{gldim}}(\text{MCM}) \); this number is simply called the global dimension of \( \text{MCM} \), and it is denoted by \( \text{gldim}(\text{MCM}) \). Our first main result, Theorem \([4.10]\), shows that there are inequalities,

\[
(\ast)\quad d \leq \text{gldim}(\text{MCM}) \leq \max\{2, d\},
\]

and thus it extends Leuschke’s theorem to the case of arbitrary CM-type. We prove the left inequality in \((\ast)\) by showing that \( \text{MCM} \) always admits a finitely presented module with projective dimension \( d \). Actually, we show that if \( M \) is any Cohen–Macaulay \( R \)-module of dimension \( t \), then \( \text{Ext}_R^{d-t}(M, -) \) is a finitely presented left \( \text{MCM} \)-module and \( \text{Hom}_R(-, M) \)

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is a finitely presented right MCM-module both with projective dimension equal to \(d - t\). Our second main result, Theorem 4.15, shows that if \(d = 0, 1\), then the left inequality in (*) is an equality if and only if \(R\) is regular, that is, there are equivalences:

\[
\text{gldim}(\text{MCM}) = 0 \quad \iff \quad R\text{ is a field.}
\]
\[
\text{gldim}(\text{MCM}) = 1 \quad \iff \quad R\text{ is a discrete valuation ring.}
\]

2. Preliminaries

2.1 Setup. Throughout, \((R, m, k)\) is a commutative noetherian local Cohen–Macaulay ring with Krull dimension \(d\). It is assumed that \(R\) has a dualizing (or canonical) module \(\Omega\).

The category of finitely generated projective \(R\)-modules is denoted \(\text{proj}\); the category of maximal Cohen–Macaulay \(R\)-modules (defined below) is denoted \(\text{MCM}\); and the category of all finitely generated \(R\)-modules is denoted \(\text{mod}\).

The depth of a finitely generated \(R\)-module \(M \neq 0\), denoted \(\text{depth}_R M\), is the supremum of the lengths of all \(M\)-regular sequences \(x_1, \ldots, x_n \in m\). This numerical invariant can be computed homologically as follows:

\[
\text{depth}_R M = \inf \{ i \in \mathbb{Z} \mid \text{Ext}_R^i(k, M) \neq 0 \}.
\]

By definition, \(\text{depth}_R 0 = \inf \emptyset = +\infty\). For a finitely generated \(R\)-module \(M \neq 0\) one always has \(\text{depth}_R M \leq d\), and \(M\) is called maximal Cohen–Macaulay if equality holds. The zero module is also considered to be maximal Cohen–Macaulay; thus an arbitrary finitely generated \(R\)-module \(M\) is maximal Cohen–Macaulay if and only if \(\text{depth}_R M \geq d\).

2.2. It is well-known that the dualizing module \(\Omega\) gives rise to a duality on the category of maximal Cohen–Macaulay modules; more precisely, there is an equivalence of categories:

\[
\begin{array}{ccc}
\text{MCM} & \cong & \text{MCM}^{\text{op}} \\
\text{Hom}_R(-, \Omega) & \cong & \text{Hom}_R(-, \Omega)
\end{array}
\]

We use the shorthand notation \((-)^\dagger\) for the functor \(\text{Hom}_R(-, \Omega)\). For any finitely generated \(R\)-module \(M \neq 0\) there is a canonical homomorphism \(\delta_M : M \to M^\dagger\dagger\), which is natural in \(M\), and because of the equivalence above, \(\delta_M\) is an isomorphism if \(M\) belongs to \(\text{MCM}\).

We will need the following results about depth; they are folklore and easily proved.

2.3 Lemma. Let \(n \geq 0\) be an integer and let \(0 \to X_n \to \cdots \to X_0 \to M \to 0\) be an exact sequence of finitely generated \(R\)-modules. If \(X_0, \ldots, X_n\) are maximal Cohen–Macaulay, then one has \(\text{depth}_R M \geq d - n\). \(\square\)

2.4 Lemma. Let \(m \geq 0\) be an integer and let \(0 \to K_m \to X_{m-1} \to \cdots \to X_0 \to M \to 0\) be an exact sequence of finitely generated \(R\)-modules. If \(X_0, \ldots, X_{m-1}\) are maximal Cohen–Macaulay, then one has \(\text{depth}_R K_m \geq \min\{d, \text{depth}_R M + m\}\). In particular, if \(m \geq d\) then the \(R\)-module \(K_m\) is maximal Cohen–Macaulay. \(\square\)

We will also need a few notions from relative homological algebra.

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1 One way to prove Lemmas 2.3 and 2.4 is by induction on \(n\) and \(m\), using the behaviour of depth on short exact sequences recorded in Bruns and Herzog [4] Prop. 1.2.9].
2.5 Definition. Let \( \mathcal{A} \) be a full subcategory of a category \( \mathcal{M} \). Following Enochs and Jenda [1, def. 5.1.1] we say that \( \mathcal{A} \) is precovering (or contravariantly finite) in \( \mathcal{M} \) if every \( M \in \mathcal{M} \) has an \( \mathcal{A} \)-precover (or a right \( \mathcal{A} \)-approximation); that is, a morphism \( \pi : A \to M \) with \( A \in \mathcal{A} \) such that every other morphism \( \pi' : A' \to M \) with \( A' \in \mathcal{A} \) factors through \( \pi \), as illustrated by the following diagram:

\[
\begin{array}{ccc}
A' & \xrightarrow{\pi'} & M \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi} & M
\end{array}
\]

The notion of \( \mathcal{A} \)-preenvelopes (or left \( \mathcal{A} \)-approximations) is categorically dual to the notion defined above. The subcategory \( \mathcal{A} \) is said to be preenveloping (or covariantly finite) in \( \mathcal{M} \) if every \( M \in \mathcal{M} \) has an \( \mathcal{A} \)-preenvelope.

The following result is a consequence of Auslander and Buchweitz’s maximal Cohen–Macaulay approximations.

2.6 Theorem. Every finitely generated \( R \)-module has an MCM-precover.

Proof. By [2] Thm. A any finitely generated \( R \)-module \( M \) has a maximal Cohen–Macaulay approximation, that is, a short exact sequence,

\[
0 \longrightarrow I \longrightarrow X \xrightarrow{\pi} M \longrightarrow 0,
\]

where \( X \) is maximal Cohen–Macaulay and \( I \) has finite injective dimension. A classic result of Ischebeck [10] (see also [4, Exerc. 3.1.24]) shows that \( \text{Ext}^1_R(X', I) = 0 \) for every \( X' \) in MCM, and hence \( \text{Hom}_R(X', \pi) : \text{Hom}_R(X', X) \to \text{Hom}_R(X', M) \) is surjective. ☐

3. Rings with several objects

The classic references for the theory of rings with several objects are Freyd [8,9] and Mitchell [12]. Below we recapitulate a few definitions and results that we need.

A ring \( A \) can be viewed as a preadditive category \( \bar{A} \) with a single object \( * \) whose endo hom-set \( \text{Hom}_{\bar{A}}(*,*) \) is \( A \), and where composition is given by ring multiplication. The category \( \bar{A} \) of additive covariant functors from \( \bar{A} \) to the category of abelian groups is naturally equivalent to the category \( A\text{-Mod} \) of left \( A \)-modules. Indeed, an additive functor \( F : \bar{A} \to \bar{Ab} \) yields a left \( A \)-module whose underlying abelian group is \( M = F(*) \) and where \( A \)-multiplication is given by \( am = F(a)(m) \) for \( a \in A = \text{Hom}_{\bar{A}}(*,*) \) and \( m \in M = F(*) \). Note that the preadditive category associated to the opposite ring \( A^{\text{op}} \) of \( A \) is the opposite (or dual) category of \( \bar{A} \); in symbols: \( \bar{A}^{\text{op}} = \bar{A}^{\text{op}} \). It follows that the category \( \bar{A}^{\text{op}} \) of additive covariant functors \( \bar{A}^{\text{op}} \to \bar{Ab} \) (which correspond to additive contravariant functors \( \bar{A} \to \bar{Ab} \)) is naturally equivalent to the category \( A\text{-Mod} \) of right \( A \)-modules.

These considerations justify the well-known viewpoint that any skeletally small preadditive category \( \mathcal{A} \) may be thought of as a ring with several objects. A left \( \mathcal{A} \)-module is an additive covariant functor \( \mathcal{A} \to \bar{Ab} \), and the category of all such is denoted by \( \mathcal{A}\text{-Mod} \). Similarly, a right \( \mathcal{A} \)-module is an additive covariant functor \( \mathcal{A}^{\text{op}} \to \bar{Ab} \) (which corresponds to an additive contravariant functor \( \mathcal{A} \to \bar{Ab} \)), and the category of all such is denoted \( \text{Mod-}\mathcal{A} \).

From this point, we assume for simplicity that \( \mathcal{A} \) is a skeletally small additive category which is closed under direct summands (i.e. every idempotent splits). The category \( \mathcal{A}\text{-Mod} \) is a Grothendieck category, see [8] prop. 5.21, with enough projectives. In fact, it follows from Yoneda’s lemma that the representable functors \( \mathcal{A}(A, -) \), where \( A \) is in \( \mathcal{A} \), constitute
a generating set of projective objects in \( \mathcal{A}\text{-Mod} \). A left \( \mathcal{A} \)-module \( F \) is called \textit{finitely generated}, respectively, \textit{finitely presented} (or \textit{coherent}), if there exists an exact sequence \( \mathcal{A}(A, -) \rightarrow F \rightarrow 0 \), respectively, \( \mathcal{A}(B, -) \rightarrow \mathcal{A}(A, -) \rightarrow F \rightarrow 0 \), for some \( A, B \in \mathcal{A} \). The category of finitely presented left \( \mathcal{A} \)-modules is denoted by \( \mathcal{A}\text{-mod} \). The Yoneda functor,

\[
\mathcal{A}^{\text{op}} \rightarrow \mathcal{A}\text{-Mod} \quad \text{given by} \quad A \mapsto \mathcal{A}(A, -),
\]

is fully faithful, see [8, thm. 5.36]. Moreover, this functor identifies the objects in \( \mathcal{A} \) with the finitely generated projective left \( \mathcal{A} \)-modules, that is, a finitely generated left \( \mathcal{A} \)-module is projective if and only if it is isomorphic to \( \mathcal{A}(A, -) \) for some \( A \in \mathcal{A} \); cf. [8, exerc. 5-G].

Here is a well-known, but important, example:

3.1 Example. Let \( A \) be any ring and let \( \mathcal{A} = \text{A-proj} \) be the category of all finitely generated projective left \( \mathcal{A} \)-modules. In this case, the category \( \mathcal{A}\text{-mod} = (\text{A-proj})\text{-mod} \) is equivalent to the category \( \text{mod}\text{-A} \) of finitely presented right \( \mathcal{A} \)-modules. Let us explain why:

Let \( F \) be a left \( (\text{A-proj})\)-module, that is, an additive covariant functor \( F: \text{A-proj} \rightarrow \text{Ab} \).

For \( a \in A \) the homothety map \( \chi_a: A \rightarrow A \) given by \( b \mapsto ba \) is left \( A \)-linear and so induces an endomorphism \( F(\chi_a) \) of the abelian group \( F(A) \). Thus \( F(A) \) has a natural structure of a right \( \mathcal{A} \)-module given by \( xa = F(\chi_a)(x) \) for \( a \in A \) and \( x \in F(A) \). This right \( \mathcal{A} \)-module is denoted \( e(F) \), and we get a functor \( e \), called \textit{evaluation}, displayed in the diagram below.

The other functor \( f \) in the diagram, called \textit{functorification}, is given by \( f(M) = M \otimes_A - \) (restricted to \( \text{A-proj} \)) for a right \( \mathcal{A} \)-module \( M \).

\[
\begin{array}{ccc}
(A\text{-proj})\text{-Mod} & \xrightarrow{e} & \text{Mod}\text{-A} \\
\downarrow f & & \\
\end{array}
\]

The functors \( e \) and \( f \) yield an equivalence of categories: For every right \( \mathcal{A} \)-module \( M \) there is obviously an isomorphism \((e \circ f)(M) = M \otimes_A A \). We must also show that every left \( (\text{A-proj})\)-module \( F \) is isomorphic to \((f \circ e)(F) = F(A) \otimes_A -\). For every \( P \in \text{A-proj} \) and \( y \in P \) the left \( \mathcal{A} \)-linear map \( \mu^P_y: A \rightarrow P \) given by \( a \mapsto ay \) induces a group homomorphism \( F(\mu^P_y): F(A) \rightarrow F(P) \), and thus one has a group homomorphism \( \tau_P: F(A) \otimes_A P \rightarrow F(P) \) given by \( x \otimes y \mapsto F(\mu^P_y)(x) \). It is straightforward to verify that \( \tau \) is a natural transformation. To prove that \( \tau_P \) is an isomorphism for every \( P \in \text{A-proj} \) it suffices, since the functors \( F(A) \otimes_A - \) and \( F \) are both additive, to check that \( \tau_A: F(A) \otimes_A A \rightarrow F(A) \) is an isomorphism. However, this is evident.

It is not hard to verify that the functors \( e \) and \( f \) restrict to an equivalence between finitely presented objects, as claimed.

3.2 Observation. Example 3.1 shows that for any ring \( A \), the category \( (\text{A-proj})\text{-mod} \) is equivalent to \( \text{mod}\text{-A} \). Since there is an equivalence of categories,

\[
\begin{array}{ccc}
\text{A-proj} & \xrightarrow{\text{Hom}_A(\cdot, A)} & (\text{proj}\text{-A})^{\text{op}} \\
\downarrow \text{Hom}_A(\cdot, A) & & \\
\end{array}
\]

it follows\(^2\) that \( (\text{A-proj})\text{-mod} \) is further equivalent to \( ((\text{proj}\text{-A})^{\text{op}})\text{-mod} \), which is the same as \( \text{mod}\text{-}(\text{proj}\text{-A}) \). In conclusion, there are equivalences of categories:

\[
(\text{A-proj})\text{-mod} \simeq \text{mod}\text{-A} \simeq \text{mod}\text{-}(\text{proj}\text{-A}) .
\]

Of course, by applying this to the opposite ring \( A^\circ \) one obtains equivalences:

\[
(\text{proj}\text{-A})\text{-mod} \simeq \text{A-mod} \simeq \text{mod}\text{-}(\text{A-proj}) .
\]

\(^2\) If the category \( \mathcal{A} \) is only assumed to be preadditive, then one would have to modify the definitions of finitely generated/presented accordingly. For example, in this case, a left \( \mathcal{A} \)-module \( F \) is called finitely generated if there is an exact sequence of the form \( \bigoplus_{i=1}^n A(A_i, -) \rightarrow F \rightarrow 0 \) for some \( A_1, \ldots, A_n \in \mathcal{A} \).

\(^3\) Cf. the proof of Proposition 3.4.
In general, the category \( \mathcal{A}\text{-mod} \) of finitely presented left \( \mathcal{A} \)-modules is an additive category with cokernels, but it is not necessarily an abelian subcategory of \( \mathcal{A}\text{-Mod} \). A classic result of Freyd describes the categories \( \mathcal{A} \) for which \( \mathcal{A}\text{-mod} \) is abelian. This result is stated in Theorem 3.4 below, but first we explain some terminology.

A pseudo-kernel (also called a weak kernel) of a morphism \( \beta: B \to C \) in \( \mathcal{A} \) is a morphism \( \alpha: A \to B \) such that the sequence
\[
\mathcal{A}(-,A) \xrightarrow{\mathcal{A}(-,\alpha)} \mathcal{A}(-,B) \xrightarrow{\mathcal{A}(-,\beta)} \mathcal{A}(-,C)
\]
is exact in \( \text{Mod-}\mathcal{A} \). Equivalently, one has \( \beta \alpha = 0 \) and for every morphism \( \alpha': A' \to B \) with \( \beta \alpha' = 0 \) there is a (not necessarily unique!) morphism \( \theta: A' \to A \) with \( \alpha \theta = \alpha' \).

We say that \( \mathcal{A} \) has pseudo-kernels if every morphism in \( \mathcal{A} \) has a pseudo-kernel. Pseudo-cokernels (also called weak cokernels) are defined dually.

### 3.3 Observation
Suppose that \( \mathcal{A} \) is a full subcategory of an abelian category \( \mathcal{M} \).

If \( \mathcal{A} \) is precovering in \( \mathcal{M} \), see Definition 2.5 then \( \mathcal{A} \) has pseudo-kernels. Indeed, given a morphism \( \beta: B \to C \) in \( \mathcal{A} \) it has a kernel \( \iota: M \to B \) in the abelian category \( \mathcal{M} \); and it is easily verified that if \( \pi: A \to M \) is any \( \mathcal{A} \)-precover of \( M \), then \( \alpha = \iota \pi: A \to B \) is a pseudo-kernel in \( \mathcal{A} \) of \( \beta \).

A similar argument shows that if \( \mathcal{A} \) is preenveloping in \( \mathcal{M} \), then \( \mathcal{A} \) has pseudo-cokernels.

### 3.4 Theorem
The category \( \text{mod-}\mathcal{A} \) (respectively, \( \mathcal{A}\text{-mod} \)) of finitely presented right (respectively, left) \( \mathcal{A} \)-modules is an abelian subcategory of \( \text{Mod-}\mathcal{A} \) (respectively, \( \mathcal{A}\text{-Mod} \)) if and only if \( \mathcal{A} \) has pseudo-kernels (respectively, has pseudo-cokernels).

**Proof.** See Freyd [9, thm. 1.4] or Auslander and Reiten [3, prop. 1.3].

### 3.5 Example
Let \( A \) be a left and right noetherian ring. As \( A \) is left noetherian, the category \( \mathcal{M} = \mathcal{A}\text{-mod} \) of finitely presented left \( \mathcal{A} \)-modules is abelian, and evidently \( \mathcal{A} = \mathcal{A}\text{-proj} \) is precovering herein. As \( A \) is right noetherian, \( \mathcal{A}\text{-proj} \) is also preenveloping in \( \mathcal{A}\text{-mod} \); cf. [7, Exa. 8.3.10]. It follows from Observation 3.3 that \( \mathcal{A}\text{-proj} \) has both pseudo-kernels and pseudo-cokernels, and therefore the categories \( \text{mod-}(\mathcal{A}\text{-proj}) \) and \( (\mathcal{A}\text{-proj})\text{-mod} \) are abelian by Theorem 3.4. Of course, this also follows directly from Observation 3.2 which shows that \( \text{mod-}(\mathcal{A}\text{-proj}) \) and \( (\mathcal{A}\text{-proj})\text{-mod} \) are equivalent to \( \mathcal{A}\text{-mod} \) and \( \mathcal{A}\text{-mod} \), respectively.

Note that if \( \mathcal{A}\text{-mod} \) is abelian, i.e. if \( \mathcal{A} \) has pseudo-cokernels, then every finitely presented left \( \mathcal{A} \)-module \( F \) admits a projective resolution in \( \mathcal{A}\text{-mod} \), that is, an exact sequence
\[
\cdots \longrightarrow \mathcal{A}(A_1,-) \longrightarrow \mathcal{A}(A_0,-) \longrightarrow F \longrightarrow 0
\]
where \( A_0, A_1, \ldots \) belong to \( \mathcal{A} \). Thus one can naturally speak of the projective dimension of \( F \) (i.e. the length, possibly infinite, of the shortest projective resolution of \( F \) in \( \mathcal{A}\text{-mod} \)) and of the global dimension of the category \( \mathcal{A}\text{-mod} \) (i.e. the supremum of projective dimensions of all objects in \( \mathcal{A}\text{-mod} \)).
3.6 Definition. In the case where the category \( \mathcal{A}-\text{mod} \) (respectively, \( \text{mod-}\mathcal{A} \)) is abelian, then its global dimension is called the left (respectively, right) global dimension of \( \mathcal{A} \), and it is denoted \( l.\text{gldim} \mathcal{A} \) (respectively, \( r.\text{gldim} \mathcal{A} \)).

Note that \( l.\text{gldim} (\mathcal{A}^{op}) \) is the same as \( r.\text{gldim} \mathcal{A} \) (when these numbers make sense).

3.7 Example. Let \( A \) be a left and right noetherian ring whose global dimension\(^4\) we denote \( \text{gldim} A \). Recall that \( \text{gldim} A \) can be computed as the supremum of projective dimensions of all \( \text{finitely generated} \) (left or right) \( A \)-modules. It follows from Observation 3.2 that
\[
\text{l.\text{gldim}} (A-\text{proj}) = \text{gldim} A = r.\text{gldim} (A-\text{proj}) .
\]

4. The global dimension of the category MCM

We are now in a position to prove the results announced in the Introduction.

4.1 Example. Suppose that \( R \) has finite CM-type and let \( X \) be any representation generator of the category \( \text{MCM} \), cf. Section 1. This means that \( \text{MCM} = \text{add}_R X \) where \( \text{add}_R X \) denotes the category of direct summands of finite direct sums of copies of \( X \). Write \( E = \text{End}_R (X) \) for the endomorphism ring of \( X \); this \( R \)-algebra is often referred to as the Auslander algebra. Note that \( X \) has a canonical structure as a left-\( R \)-left-\( E \)–bimodule \( R.E \). It is easily verified that there is an equivalence, known as Auslander’s projectivization, given by:
\[
\text{MCM} = \text{add}_R X \xrightarrow{\text{Hom}_R (X,-)} \text{proj}-E .
\]

It now follows from Observation 3.2 that there are equivalences of categories:
\[
\text{MCM-mod} \simeq (\text{proj}-E)\text{-mod} \simeq E\text{-mod} .
\]

Similarly, there is an equivalence of categories: \( \text{mod-}\text{MCM} \simeq \text{mod-}E \).

4.2 Proposition. The category \( \text{MCM} \) has pseudo-kernels and pseudo-cokernels.

Proof. As \( \text{MCM} \) is precovering in the abelian category \( \text{mod} \), see Theorem 2.6, we get from Observation 3.3 that \( \text{MCM} \) has pseudo-kernels. To prove that \( \text{MCM} \) has pseudo-cokernels, let \( \alpha : X \rightarrow Y \) be any homomorphism between maximal Cohen–Macaulay \( R \)-modules. With the notation from \( \text{2.2} \) we let \( \iota : Z \rightarrow Y^{\dagger} \) be a pseudo-kernel in \( \text{MCM} \) of \( \alpha^{\dagger}: Y^{\dagger} \rightarrow X^{\dagger} \). We claim that \( \iota^{\dagger} \delta_Y : Y \rightarrow Z^{\dagger} \) is a pseudo-cokernel of \( \alpha \), i.e. that the sequence

\[
\text{Hom}_R (Z^{\dagger}, U) \xrightarrow{\text{Hom}_R (\iota^{\dagger} \delta_Y, U)} \text{Hom}_R (Y, U) \xrightarrow{\text{Hom}_R (\alpha, U)} \text{Hom}_R (X, U)
\]

is exact for every \( U \in \text{MCM} \). From the commutative diagram
\[
\begin{array}{cccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\iota^{\dagger} \delta_Y} & Z^{\dagger} \\
\cong \downarrow \phi_X & & \cong \downarrow \phi_Y & & \cong \downarrow \phi_Y \\
X^{\dagger \dagger} & \xrightarrow{\alpha^{\dagger \dagger}} & Y^{\dagger \dagger} & \xrightarrow{\iota^{\dagger}} & Z^{\dagger}
\end{array}
\]

it follows that the sequence (1) is isomorphic to
\[
\text{Hom}_R (Z^{\dagger}, U) \xrightarrow{\text{Hom}_R (\iota^{\dagger}, U)} \text{Hom}_R (Y^{\dagger \dagger}, U) \xrightarrow{\text{Hom}_R (\alpha^{\dagger \dagger}, U)} \text{Hom}_R (X^{\dagger \dagger}, U) .
\]

\(^4\) Recall that for a ring which is both left and right noetherian, the left and right global dimensions are equal; indeed, they both coincide with the weak global dimension.
Recall from [2.2] that there is an isomorphism $U \cong U^{††}$. From this fact and from the “swap” isomorphism [6] (A.2.9), it follows that the sequence (2) is isomorphic to

$$\text{Hom}_R(U^{†}, Z^{†}) \xrightarrow{\text{Hom}_R(U^{†}, Y^{†})} \text{Hom}_R(U^{†}, Y^{††}) \xrightarrow{\text{Hom}_R(U^{†}, X^{††})} \text{Hom}_R(U^{†}, X^{†††}).$$

Finally, the commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\iota} & Y^{†} \\
\downarrow{\delta_Z} & & \downarrow{\delta_{Y^{†}}} \\
Z^{††} & \xrightarrow{\alpha} & X^{†} \\
\downarrow{\delta_{Z^{††}}} & & \downarrow{\delta_{X^{†}}} \\
Y^{††} & \xrightarrow{\alpha^{†}} & X^{††} \\
\downarrow{\delta_{Y^{††}}} & & \downarrow{\delta_{X^{††}}} \\
X^{†††}
\end{array}$$

shows that the sequence (3) is isomorphic to

$$\text{Hom}_R(U^{†}, Z) \xrightarrow{\text{Hom}_R(U^{†}, Y^{†})} \text{Hom}_R(U^{†}, Y^{††}) \xrightarrow{\text{Hom}_R(U^{†}, X^{†})} \text{Hom}_R(U^{†}, X^{††}),$$

which is exact since $\iota: Z \rightarrow Y^{†}$ is a pseudo-kernel of $\alpha^{†}: Y^{†} \rightarrow X^{†}$.

We shall find the following notation useful.

4.3 Definition. For an $R$-module $M$, we use the notation $(M, -)$ for the left MCM-module $\text{Hom}_R(M, -)|_{\text{MCM}}$, and $(-, M)$ for the right MCM-module $\text{Hom}_R(-, M)|_{\text{MCM}}$.

Theorem [3.4] and Proposition [4.2] shows that MCM-mod and mod-MCM are abelian, and hence the left and right global dimensions of the category MCM are both well-defined; see Definition [3.6]. In fact, they are equal:

4.4 Proposition. The left and right global dimensions of MCM coincide, that is,

$$1. \text{gldim}(\text{MCM}) = r. \text{gldim}(\text{MCM}).$$

This number is called the global dimension of MCM, and it is denoted $\text{gldim}(\text{MCM})$. □

Proof. The equivalence in [2.2] induces an equivalence between the abelian categories of (all) left and right MCM-modules given by:

$$\begin{array}{ccc}
\text{MCM-Mod} & \xrightarrow{F \mapsto F \circ (-)^{†}} & \text{MCM} \\
G \circ (-)^{†} \xleftarrow{G} & & \text{Mod-MCM}
\end{array}$$

These functors preserve finitely generated projective modules. Indeed, if $P = (X, -)$ with $X \in \text{MCM}$ is a finitely generated projective left MCM-module, then the right MCM-module $P \circ (-)^{†} = (X, (-)^{†})$ is isomorphic to $(-, X^{†})$, which is finitely generated projective. Similarly, if $Q = (-, Y)$ with $Y \in \text{MCM}$ is a finitely generated projective right MCM-module, then $Q \circ (-)^{†} = ((-)^{†}, Y)$ is isomorphic to $(Y^{†}, -)$, which is finitely generated projective.

Since the functors in (4) are exact and preserve finitely generated projective modules, they restrict to an equivalence between finitely presented objects, that is, MCM-mod and mod-MCM are equivalent. It follows that MCM-mod and mod-MCM have the same global dimension, i.e. the left and right global dimensions of MCM coincide. □

We begin our study of $\text{gldim}(\text{MCM})$ with a couple of easy examples.

4.5 Example. If $R$ is regular, in which case the global dimension of $R$ is equal to $d$, then one has $\text{MCM} = \text{proj}$, and it follows from Example [3.7] that $\text{gldim}(\text{MCM}) = d$. 

4.6 Example. Assume that \( R \) has finite CM-type and denote the Auslander algebra by \( E \). It follows from Example \[4.1\] that \( \text{gldim}(\text{MCM}) = \text{gldim} E \).

We turn our attention to projective dimensions of representable right MCM-modules.

4.7 Proposition. Let \( M \) be a finitely generated \( R \)-module. Then \( (\cdot, M) \) is a finitely presented right \( M \)-module with projective dimension equal to \( d - \text{depth}_R M \).

**Proof.** First we argue that \( (\cdot, M) \) is finitely presented. By Theorem \[2.6\] there is an MCM-precover \( \pi : X \rightarrow M \), which by definition yields an epimorphism \( (\cdot, \pi) : (\cdot, X) \rightarrow (\cdot, M) \). Hence \( (\cdot, M) \) is finitely generated. As the \( \text{Hom} \) functor is left exact, the kernel of \( (\cdot, \pi) \) is the functor \( (\cdot, \text{Ker} \pi) \). Since \( \text{Ker} \pi \) is a finitely generated \( R \)-module, the argument above shows that \( (\cdot, \text{Ker} \pi) \) is finitely generated, and therefore \( (\cdot, M) \) is finitely presented.

If \( M = 0 \), then \( (\cdot, M) \) is the zero functor which has projective dimension \( d - \text{depth}_R M = -\infty \). Thus we can assume that \( M \) is non-zero such that \( m := d - \text{depth}_R M \) is an integer. By successively taking MCM-precovers, whose existence is guaranteed by Theorem \[2.6\], we construct an exact sequence of \( R \)-modules
\[
0 \rightarrow K_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0,
\]
where \( X_0, \ldots, X_{m-1} \) are maximal Cohen–Macaulay and \( K_m = \text{Ker}(X_{m-1} \rightarrow X_{m-2}) \), such that the sequence
\[
0 \rightarrow (\cdot, K_m) \rightarrow (\cdot, X_{m-1}) \rightarrow \cdots \rightarrow (\cdot, X_1) \rightarrow (\cdot, X_0) \rightarrow (\cdot, M) \rightarrow 0
\]
in mod-MCM is exact. Lemma \[2.3\] show that \( \text{depth}_R K_m \geq \min\{d, \text{depth}_R M + m\} = d \), and hence \( K_m \) is maximal Cohen–Macaulay. Thus exactness of the sequence displayed above shows that the projective dimension of \( (\cdot, M) \) is \( m \).

To prove that the projective dimension of \( (\cdot, M) \) is \( m \), we must show that if
\[
\begin{array}{c}
0 \rightarrow (-, Y_0) \rightarrow \cdots \rightarrow (-, Y_1) \rightarrow (-, Y_0) \rightarrow (-, M) \rightarrow 0
\end{array}
\]
is any exact sequence in mod-MCM, where \( Y_0, \ldots, Y_n \) are maximal Cohen–Macaulay, then \( n \geq m \). By Yoneda’s lemma, each \( \tau_i \) has the form \( \tau_i = (\cdot, \beta_i) \) for some homomorphism \( \beta_i : Y_i \rightarrow Y_{i-1} \) when \( 1 \leq i \leq n \) and \( \beta_0 : Y_0 \rightarrow M \). By evaluating the sequence on the maximal Cohen–Macaulay module \( R \), it follows that the sequence of \( R \)-modules,
\[
\begin{array}{c}
0 \rightarrow Y_n \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 \rightarrow M \rightarrow 0,
\end{array}
\]
is exact. Thus Lemma \[2.3\] yields \( \text{depth}_R M \geq d - n \), that is, \( n \geq m \). \( \square \)

In contrast to what is the case for representable right MCM-modules, representable left MCM-modules are “often” zero. For example, if \( d > 0 \) then \( \text{Hom}_R(k, X) = 0 \) for every maximal Cohen–Macaulay \( R \)-module \( X \), and hence \( (k, \cdot) \) is the zero functor. In particular, the projective dimension of a representable left MCM-module is typically not very interesting. Proposition \[4.9\] below gives concrete examples of finitely presented left MCM-modules that do have interesting projective dimension.

4.8 Lemma. For every Cohen–Macaulay \( R \)-module \( M \) of dimension \( t \) there is the following natural isomorphism of functors \( \text{MCM} \rightarrow \text{Ab} \),
\[
\text{Hom}_R((\cdot)^t, \text{Ext}_R^{d-t}(M, \Omega)) \cong \text{Ext}_R^{d-t}(M, (-)).
\]

**Proof.** Since \( M \) is Cohen–Macaulay of dimension \( t \) one has \( \text{Ext}_R^i(M, \Omega) = 0 \) for \( i \neq d - t \); see [4, Thm. 3.3.10]. Thus there is an isomorphism in the derived category of \( R \),
\[
\text{Ext}_R^{d-t}(M, \Omega) \cong \Sigma^{d-t} \text{RHom}_R(M, \Omega).
\]
In particular, there is an isomorphism $X^\dagger = \text{Hom}_R(X, \Omega) \cong \text{RHom}_R(X, \Omega)$ for $X \in \text{MCM}$. This explains the first isomorphism below. The second isomorphism is trivial, the third one is by “swap” [6, (A.4.22)], and the fourth one follows as $\Omega$ is a dualizing $R$-module.

$$\text{RHom}_R(X^\dagger, \text{Ext}^{d-t}_R(M, \Omega)) \cong \text{RHom}_R(\text{RHom}_R(X, \Omega), \Sigma^{d-t} \text{RHom}_R(M, \Omega))$$
$$\cong \Sigma^{d-t} \text{RHom}_R(\text{RHom}_R(X, \Omega), \text{RHom}_R(M, \Omega))$$
$$\cong \Sigma^{d-t} \text{RHom}_R(M, \text{RHom}_R(\text{RHom}_R(X, \Omega), \Omega))$$
$$\cong \Sigma^{d-t} \text{RHom}_R(M, X).$$

The assertion now follows by taking the zero'th homology group $H_0$. □

4.9 Proposition. If $M$ is any Cohen–Macaulay $R$-module of dimension $t$, then the functor $\text{Ext}^{d-t}_R(M, -)|_{\text{MCM}}$ is a finitely presented left $\text{MCM}$-module with projective dimension equal to $d - t$.

Proof. As $M$ is Cohen–Macaulay of dimension $t$, so is $\text{Ext}^{d-t}_R(M, \Omega)$; see [4, Thm. 3.3.10]. Proposition 4.7 shows that $\text{Hom}_R(-, \text{Ext}^{d-t}_R(M, \Omega))|_{\text{MCM}}$ is a finitely presented right $\text{MCM}$-module with projective dimension equal to $d - t$. The proof of Prop. 4.4 now gives that

$$\text{Hom}_R((-)^\dagger, \text{Ext}^{d-t}_R(M, \Omega))|_{\text{MCM}}$$

is a finitely presented left $\text{MCM}$-module with projective dimension $d - t$, and Lemma 4.8 finishes the proof. □

4.10 Theorem. The category $\text{MCM}$ has finite global dimension. In fact, one has

$$d \leq \text{gldim}(\text{MCM}) \leq \max\{2, d\}.$$

In particular, if $d \geq 2$ then there is an equality $\text{gldim}(\text{MCM}) = d$.

Proof. The residue field $k$ of $R$ is a finitely generated $R$-module with depth 0. Thus Proposition 4.7 shows that $(-, k)$ is finitely presented right $\text{MCM}$-module with projective dimension $d$. Consequently, we must have $d \leq \text{gldim}(\text{MCM})$.

To prove the other inequality, set $m = \max\{2, d\}$ and let $G$ be any finitely presented right $\text{MCM}$-module. Take any exact sequence in $\text{mod-MCM}$,

$$(5) \quad (-, X_{m-1}) \xrightarrow{\tau_{m-1}} \cdots \xrightarrow{\tau_1} (-, X_1) \xrightarrow{\epsilon} (-, X_0) \xrightarrow{\epsilon} G \xrightarrow{0},$$

where $X_0, X_1, \ldots, X_{m-1}$ are in $\text{MCM}$. Note that since $m \geq 2$ there is at least one “$\tau$” in this sequence. By Yoneda’s lemma, every $\tau_i$ has the form $\tau_i = (-, \alpha_i)$ for some homomorphism $\alpha_i: X_i \to X_{i-1}$. By evaluating (5) on $R$, we get an exact sequence of $R$-modules:

$$X_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_0} X_0.$$

Since $m \geq d$ it follows from Lemma 4.4 that the module $X_m = \text{Ker} \alpha_{m-1}$ is maximal Cohen–Macaulay. As the $\text{Hom}$ functor is left exact, we see that $0 \to (-, X_m) \to (-, X_{m-1})$ is exact. This sequence, together with (5), shows that $G$ has projective dimension $\leq m$. □

In view of Example 4.5 and Theorem 4.10 we immediately get the following result due to Leuschke [11, Thm. 6].

4.11 Corollary. Assume that $R$ has finite CM-type and let $X$ be any representation generator of $\text{MCM}$ with Auslander algebra $E = \text{End}_R(X)$. There are inequalities,

$$d \leq \text{gldim} E \leq \max\{2, d\}.$$

In particular, if $d \geq 2$ then there is an equality $\text{gldim} E = d$. □
4.12 Example. If \( d = 0 \) then \( \text{MCM} = \text{mod} \) and hence \( \text{gldim} \,(\text{MCM}) = \text{gldim} \,(\text{mod}) \). Since \( \text{mod} \) is abelian, it is a well-known result of Auslander \([1]\) that the latter number must be either 0 or 2 (surprisingly, it can not be 1). Thus one of the inequalities in Theorem 4.10 is actually an equality. If, for example, \( R = k[x]/(x^2) \) where \( k \) is a field, then \( \text{gldim} \,(\text{mod}) = 2 \).

4.13 Example. If \( d = 1 \) then Theorem 4.10 shows that \( \text{gldim} \,(\text{MCM}) = 1, 2 \). The 1-dimensional Cohen–Macaulay ring \( R = k[x,y]/(x^2) \) does not have finite CM-type \([2]\) and since it is not regular, it follows from Theorem 4.15 below that \( \text{gldim} \,(\text{MCM}) = 2 \).

Recall that in any abelian category with enough projectives (such as \( \text{mod}-\mathcal{A} \) in the case where \( \mathcal{A} \) has pseudo-kernels) one can well-define and compute \( \text{Ext} \) in the usual way.

4.14 Lemma. Assume that \( \mathcal{A} \) is precovering in an abelian category \( \mathcal{M} \) (in which case the category \( \text{mod}-\mathcal{A} \) is abelian by Observation 3.3 and Theorem 3.4). Let

\[
0 \rightarrow A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A''
\]

be an exact sequence in \( \mathcal{M} \) where \( A, A', A'' \) belong to \( \mathcal{A} \). Consider the finitely presented right \( \mathcal{A} \)-module \( G = \text{Coker} \, A(-, \alpha) \), that is, \( G \) is defined by exactness of the sequence

\[
\mathcal{A}(-, A) \xrightarrow{A(-, \alpha)} \mathcal{A}(-, A'') \xrightarrow{\alpha} G \xrightarrow{\beta} 0 .
\]

For any finitely presented right \( \mathcal{A} \)-module \( H \) there is an isomorphism of abelian groups,

\[
\text{Ext}^2_{\text{mod}-\mathcal{A}}(G, H) \cong \text{Coker} \, H(\alpha') .
\]

Proof. By the definition of \( G \) and left exactness of the Hom functor, the chain complex

\[
(6) \quad 0 \rightarrow \mathcal{A}(-, A') \xrightarrow{\mathcal{A}(-, \alpha')} \mathcal{A}(-, A) \xrightarrow{\mathcal{A}(-, \alpha)} \mathcal{A}(-, A'') \xrightarrow{\alpha} 0 ,
\]

is a non-augmented projective resolution in \( \text{mod}-\mathcal{A} \) of \( G \). To compute \( \text{Ext}^2_{\text{mod}-\mathcal{A}}(G, H) \) we must first apply the functor \( (\text{mod}-\mathcal{A})(?, H) \) to (6) and then take the second cohomology group of the resulting cochain complex. By Yoneda’s lemma there is a natural isomorphism

\[
(\text{mod}-\mathcal{A})(\mathcal{A}(-, B), H) \cong H(B)
\]

for any \( B \in \mathcal{A} \); hence application of \( (\text{mod}-\mathcal{A})(?, H) \) to (6) yields the cochain complex

\[
0 \rightarrow H(A'') \xrightarrow{H(\alpha')} H(A) \xrightarrow{H(\alpha)} H(A') \xrightarrow{\beta} 0 .
\]

The second cohomology group of this cochain complex is \( \text{Coker} \, H(\alpha') \).

Recall that a commutative ring is called a discrete valuation ring (DVR) if it is a principal ideal domain with exactly one non-zero maximal ideal. There are of course many other equivalent characterizations of such rings.

4.15 Theorem. If \( \text{gldim} \,(\text{MCM}) \leq 1 \), then \( R \) is regular. In particular, one has

- \( \text{gldim} \,(\text{MCM}) = 0 \iff R \) is a field.
- \( \text{gldim} \,(\text{MCM}) = 1 \iff R \) is a discrete valuation ring.

\[5\] See Buchweitz, Greuel, and Schreyer \([5\, \text{Prop. 4.1}]\) for a complete list of the indecomposable maximal Cohen–Macaulay modules over this ring.
Proof. Assume that \( \text{gldim}(\text{MCM}) \leq 1 \). Let \( X \) be any maximal Cohen–Macaulay \( R \)-module and let \( \pi: L \to X \) be an epimorphism where \( L \) is finitely generated and free. Note that \( Y = \text{Ker} \pi \) is also maximal Cohen–Macaulay by Lemma 2.4, so we have an exact sequence,

\[
0 \to Y \xrightarrow{\iota} L \xrightarrow{\pi} X \to 0,
\]

of maximal Cohen–Macaulay \( R \)-modules. With \( G = \text{Coker}(-, \pi) \) and \( H = (-, Y) \) we have

\[
\text{Coker}(\iota, Y) \cong \text{Ext}^2_{\text{mod-MCM}}(G, H) \cong 0;
\]

here the first isomorphism comes from Lemma 4.14, and the second isomorphism follows from the assumption that \( \text{gldim}(\text{MCM}) \leq 1 \). Hence the homomorphism

\[
\text{Hom}_R(L, Y) \xrightarrow{\iota^*} \text{Hom}_R(Y, Y)
\]

is surjective. Thus \( \iota \) has a left inverse and \( X \) becomes a direct summand of the free module \( L \). Therefore every maximal Cohen–Macaulay \( R \)-module is projective, so \( R \) is regular.

The displayed equivalences now follows in view of Example 4.5 and the fact that a regular local ring has Krull dimension 0, respectively, 1, if and only if it is a field, respectively, a discrete valuation ring. □

As a corollary, we get the following addendum to Leuschke’s theorem (see 4.11).

4.16 Corollary. Assume that \( R \) has finite CM-type and let \( X \) be any representation generator of \( \text{MCM} \) with Auslander algebra \( E = \text{End}_R(X) \). If \( \text{gldim} E \leq 1 \), then \( R \) is regular. □

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