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A numerical study of fractional rheological models and fractional Newell-Whitehead-Segel equation with non-local and non-singular kernel

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\section*{ABSTRACT}

In the recent years, few type of fractional derivatives which have non-local and non-singular kernel are introduced. In this work, we present fractional rheological models and Newell-Whitehead-Segel equations with non-local and non-singular kernel. For solving these equations, we present a spectral collocation method based on the shifted Legendre polynomials. To do this, we extend the unknown functions and its derivatives using the shifted Legendre basis. These expansions and the properties of the shifted Legendre polynomials along with the spectral collocation method will help us to reduce the main problem to a set of nonlinear algebraic equations. Finally, The accuracy and efficiency of the proposed method are reported by some illustrative examples.

\section*{1. Introduction}

The study of fractional calculus started at the end of the seventeenth century. It is a branch of mathematical analysis in which integer order derivatives and integrals extend to a real or complex number [1,2]. In the end of nineteenth century basic theory of fractional calculus was developed with the studies of Liouville, Grünwald, Letnikov, and Riemann. It has been shown that fractional derivative operators are useful in describing dynamical processes with memory or hereditary properties such as creep or relaxation processes in viscoelasticplastic materials [3,4], impact problem [5], plasma physics [1], diffusion process models [6–9], chaotic systems [10], control problems [11,12], dynamics modeling of coronavirus (2019-nCov) [13], etc.

Since in definition of the most important fractional operators such as Riemann-Liouville (RL) and Caputo exists a kernel of type local and singular, it is difficult or impossible to describe many non-local dynamics systems. Hence novel definitions for fractional integral and derivative operators have been introduced such as Caputo–Fabrizio (CF) [14] and Atangana–Baleanu (AB) operators [15]. The most important advantage of these operators is the existence of the non-local and non-singular kernel which introduced to describe complex physical problems [16–23].

The AB and CF derivatives show crossover properties for the meansquare displacement, while the RL derivative is scale invariant. Their probability distributions are also a Gaussian to non-Gaussian crossover, with the difference that the CF kernel has a steady state...
between the transition. Only the AB kernel is a crossover for the waiting time distribution from stretched exponential to power law. The CF derivative is less noisy while the fractional AB derivative provides an excellent description, due to its Mittag-Leffler memory, able to distinguish between dynamical systems taking place at different scales without steady state [24,25].

Orthogonal basis functions have been generally used to achieve approximate solution for many problems in various fields of science. Approximation of the solution using these functions is known as a useful tool in solving many classes of equations, numerically, e.g., differential equations [26,27], integro-differential equations [28,29] and partial differential equations [30] of various orders (fixed, fractional or variable order).

2. Basic concepts

In this section, many definitions of new fractional operators together with their important properties are recall which will be used further.

Definition 2.1. (See Yang [31]) Let \( 0 < \omega \). The RL–integral is defined as

\[
\mathcal{R}L_t^\omega \xi(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \xi(s) \, ds.
\]

The RL–integral of order \( \omega \) satisfies the following property

\[
\mathcal{R}L_t^\omega \xi = \frac{1}{\Gamma(\nu + 1)} \frac{\Gamma(\nu + \omega)}{\Gamma(\omega)} \xi^{\nu + \omega}, \quad \nu \geq 0.
\]

Definition 2.2. (See Atangana and Baleanu [15]) Let \( 0 < \omega \leq 1 \), \( \epsilon \in H^1(0, 1) \) and \( AB(\omega) \) be a normalization function suchthat \( AB(0) = AB(1) = 1 \) and \( AB(\omega) = 1 - \omega + \frac{\omega}{1!} \).

i. The Caputo AB–derivative is defined as

\[
\mathcal{A}^{\omega}_C \xi(t) = \frac{AB(\omega)}{1 - \omega} \int_0^t E_{\omega}(\frac{\omega}{1 - \omega}(t-s)\xi'(s)) \, ds,
\]

where \( E_{\omega}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\omega + 1)} \) is the Mittag-Leffler function.

ii. The AB–integral is given as

\[
\mathcal{A}^{\omega}_t \xi(t) = \frac{1 - \omega}{AB(\omega)} \xi(t) + \frac{\omega}{AB(\omega) \Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \xi(s) \, ds.
\]

Let \( \alpha_\omega = \frac{1 - \omega}{AB(\omega)} \) and \( \beta_\omega = \frac{1}{AB(\omega) \Gamma(\omega)} \), then we can rewrite (1) as

\[
\mathcal{A}^{\omega}_t \xi(t) = \alpha_\omega \xi(t) + \beta_\omega \Gamma(\omega + 1) \mathcal{R}L_t^{\omega} \xi(t).
\]

It is easy to report that the AB–integral satisfies the following properties [32]

\[
\mathcal{A}^{\omega}_t C = C(\alpha_\omega + \beta_\omega t^\omega), \quad C \in \mathbb{R},
\]
\[
\mathcal{A}^{\omega}_t t^\nu = t^\nu (\alpha_\omega + \beta_\omega (\nu + \omega + 1) B(\nu + 1, \omega + 1) t^\omega),
\]
\[
\mathcal{A}^{\omega}_t (\mathcal{A}^{\nu}_C D^\omega t^\nu) = \xi(t) - \xi(0),
\]

where \( B(\cdot, \cdot) \) is the Beta function.

Theorem 2.1. Let \( C[0, 1] \) be the space of all continuous functions defined on \([0, 1]\) and \( f, g \in C[0, 1] \). Then the following inequality can be established [15]

\[
\| \mathcal{A}^{\omega}_t f(t) - \mathcal{A}^{\omega}_t g(t) \|_\infty \leq \delta \| f(t) - g(t) \|_\infty,
\]

where \( \delta \) is a constant number.

Theorem 2.2. Suppose that \( f \) and \( g \) satisfy the assumptions of Theorem 2.1, then we have

\[
\| \mathcal{A}^{\omega}_t f(t) - \mathcal{A}^{\omega}_t g(t) \|_\infty \leq \epsilon \| f(t) - g(t) \|_\infty,
\]

where \( \epsilon = \alpha_\omega + \beta_\omega \).

Proof. According to definition of the AB–integral, we have
\[ ||^{\text{h}} g_t f (t) - ^{\text{h}} g_t (t) ||_{\infty} = ||^{\text{h}} g_t (f (t) - g (t)) ||_{\infty} \\
= ||^{\text{h}} g_t (f (t) - g (t)) ||_{\infty} \\
= \alpha \| f (t) - g (t) \|_{\infty} + \beta \| \Gamma (\omega + 1)^{\text{RL}} g_t (f (t) - g (t)) \|_{\infty} \\
\leq \alpha \| f (t) - g (t) \|_{\infty} + \beta \| \Gamma (\omega + 1)^{\text{RL}} g_t (f (t) - g (t)) \|_{\infty} \\
\leq (\alpha + \beta) \| f (t) - g (t) \|_{\infty}. \]  
(2)

Taking \( \varepsilon = \alpha + \beta \), the proof is complete. \( \square \)

**Definition 2.3.** (See Caputo and Fabrizio [14]) Let \( \omega \in (0, 1], \varepsilon \in H^1(0, 1) \) and \( CF(\omega) \) be a normalization function such that \( CF(0) = CF(1) = 1 \) and \( CF(\omega) = \frac{2}{\pi - \omega} \). Then

i. The CF–derivative is defined as

\[ ^{CF} D^\varepsilon_t e = \frac{(2 - \omega)CF(\omega)}{2(1 - \omega)} \int_0^t e^{-\frac{\omega}{\pi-\omega}(t-s)\varepsilon(s)} ds. \]

ii. The CF–integral is given as

\[ ^{CF} I^\varepsilon_t e = \frac{2(1 - \omega)}{(2 - \omega)CF(\omega)}\varepsilon(t) + \frac{2\omega}{(2 - \omega)CF(\omega)} \int_0^t \varepsilon(s) ds. \]  
(3)

Let \( \alpha = \frac{2(1 - \omega)}{(2 - \omega)CF(\omega)} \) and \( \beta = \frac{2\omega}{(2 - \omega)CF(\omega)} \), then we can rewrite (3) as

\[ ^{CF} I^\varepsilon_t e = \alpha \varepsilon(t) + \beta \varepsilon. \]

The CF–integral satisfies the following properties

\[ ^{CF} I^\varepsilon_t C = C(\alpha + \beta \varepsilon), \quad C \in \mathbb{R}, \]
\[ ^{CF} I^\varepsilon_t e^t = e^t\left[ \alpha \varepsilon(t) + \beta \varepsilon \right], \]
\[ ^{CF} D^\varepsilon_t (^{CF} D^\varepsilon_t e) = \varepsilon(t) - \varepsilon(0). \]

**Theorem 2.3.** Let \( C[0, 1] \) be the space of all continuous functions defined on \( [0, 1] \) and \( f, g \in C[0, 1] \). Then the following inequality can be established

\[ ||^{CF} I^\varepsilon_t f (t) - ^{CF} I^\varepsilon_t g (t) ||_{\infty} \leq \varepsilon || f (t) - g (t) ||_{\infty}, \]

where \( \varepsilon = \alpha + \beta \varepsilon. \)

**Proof.** According to definition of the CF-integral, we have

\[ ||^{CF} I^\varepsilon_t f (t) - ^{CF} I^\varepsilon_t g (t) ||_{\infty} = ||^{CF} I^\varepsilon_t (f (t) - g (t)) ||_{\infty} \\
= ||\alpha || f (t) - g (t) ||_{\infty} + \beta \| \Gamma (\omega + 1)^{\text{RL}} I^\varepsilon_t (f (t) - g (t)) \|_{\infty} \\
\leq \alpha || f (t) - g (t) ||_{\infty} + \beta \| \Gamma (\omega + 1)^{\text{RL}} I^\varepsilon_t (f (t) - g (t)) \|_{\infty} \\
\leq (\alpha + \beta) || f (t) - g (t) ||_{\infty}. \]  
(4)

Taking \( \varepsilon = \alpha + \beta \varepsilon \), the proof completes. \( \square \)

3. The shifted Legendre polynomials and their properties

The shifted Legendre polynomials (SLPs) on the interval \([0, 1]\) are defined by

\[ L_n(t) = L_n(2t - 1), \quad n = 0, 1, 2, \ldots, \]  
(5)

where \( L_n(t) \) is the well-known Legendre polynomial (LP) of degree \( n \). The recursive formula of LP on \([-1, 1]\) is given by

\[ L_0(t) = 1, \]
\[ L_1(t) = t, \]
\[ L_{n+1}(t) = \frac{2n+1}{n+1}L_n(t) - \frac{n}{n+1}L_{n-1}(t), \quad n = 1, 2, 3, \ldots, \]

The given SLPs \((L_n(t))\) in the Eq. (5), could be written the following analytic form
\[
\mathcal{L}_n(t) = \sum_{k=0}^{n} \xi_{n,k} t^k,
\]
\[
where \quad \xi_{n,k} = \frac{(-1)^{n+k}(n+k)!}{(n-k)!k!}.
\]

For two arbitrary functions \( h, p \in L^2[0, 1] \) the inner product and norm in this space are defined, respectively, by
\[
\langle h(t), p(t) \rangle = \int_0^1 h(t)p(t) \, dt,
\]
\[
\|h(t)\|_2 = \left( \int_0^1 |h(t)|^2 \, dt \right)^{1/2}.
\]

For the SLPs, the orthogonality condition is as follows
\[
\langle \mathcal{L}_m(t), \mathcal{L}_n(t) \rangle = \begin{cases} 
\frac{1}{2m+1}, & m = n, \\
0, & m \neq n.
\end{cases}
\]

Suppose that \( \varepsilon(t) \in L^2[0, 1] \). Then, the function \( \varepsilon(t) \) can be expanded in terms of the SLPs by
\[
\varepsilon(t) = \sum_{i=0}^{m} \xi_i \mathcal{L}_i(t),
\]
where
\[
\xi_i = \frac{\langle \varepsilon(t), \mathcal{L}_i(t) \rangle}{\langle \mathcal{L}_i(t), \mathcal{L}_i(t) \rangle} = (2i+1) \int_0^1 \varepsilon(t) \mathcal{L}_i(t) \, dt.
\]

By taking only the first \( M + 1 \) terms in (8), \( \varepsilon(t) \) can be approximated as
\[
\varepsilon(t) \approx \varepsilon_M(t) = \sum_{i=0}^{M} \xi_i \mathcal{L}_i(t) = C^T \mathcal{L}(t),
\]
where \( C = [\xi_0, \xi_1, ..., \xi_M]^T \) and
\[
\mathcal{L}(t) = [\mathcal{L}_0(t), \mathcal{L}_1(t), ..., \mathcal{L}_M(t)]^T.
\]

**Theorem 3.1.** Suppose that \( \varepsilon \in C^{M+1}[0, 1] \) and \( H = \text{span} \{\mathcal{L}_0(t), \mathcal{L}_1(t), ..., \mathcal{L}_M(t)\} \subset L^2[0, 1] \). Assume \( \varepsilon_M \) is the best approximation of \( \varepsilon \) into \( H \), then the error bound is as follows
\[
\|\varepsilon(t) - \varepsilon_M(t)\|_2 \leq \frac{\rho}{(M+1)!2^{2M+1}},
\]
where \( \rho = \sup_{\varepsilon \in [0, 1]} |\varepsilon^{(M+1)}(\theta)| \).

**Proof.** Suppose that \( P_M \) is the interpolating polynomials to \( \varepsilon \) at points \( t_i \), where \( t_i, i = 0, 1, ..., M \) are the roots of \( (M+1) \)-degree shifted Chebychev polynomials on \([0, 1]\). Then
\[
\varepsilon(t) - P_M(t) = \frac{\varepsilon^{(M+1)}(\theta)}{(M+1)!} \prod_{i=0}^{M} (t - t_i), \quad \theta \in [0, 1].
\]
So,
\[
|\varepsilon(t) - P_M(t)| \leq \frac{\rho}{(M+1)!2^{2M+1}},
\]
where \( \rho = \sup_{\varepsilon \in [0, 1]} |\varepsilon^{(M+1)}(\theta)| \).

Since \( \varepsilon_M \) is the best approximation of \( \varepsilon \) in \( H \), we get
\[
\|\varepsilon(t) - \varepsilon_M(t)\|_2 \leq \|\varepsilon(t) - P_M(t)\|_2 \leq \int_0^1 |\varepsilon(t) - P_M(t)|^2 \, dt = \int_0^1 \left( \frac{\rho}{(M+1)!2^{2M+1}} \right)^2 \, dt = \left( \frac{\rho}{(M+1)!2^{2M+1}} \right)^2.
\]

By taking the squared root from both sides (12), the proof completes. \( \square \)

Similarly, any function \( \varepsilon(x, t) \in H^* = L^2([0, 1] \times [0, 1]) \) can be approximated in terms of the SLPs as
\[
\varepsilon(x, t) \approx \mathcal{L}^T(x) C \mathcal{L}(t),
\]
where $\mathbf{C} = [c_{ij}]$ is an $(M + 1) \times (M + 1)$ matrix whose elements are given by

$$c_{ij} = \frac{\langle \varepsilon(t, s), \mathcal{L}_i(s) \rangle}{\| \mathcal{L}_i(s) \|^2} \| \mathcal{L}_j(s) \|^2, \quad i, j = 0, 1, \ldots, M.$$ 

**Theorem 3.2.** Let $H^* = \text{span} \{ \mathcal{L}_0(x), \mathcal{L}_0(t), \ldots, \mathcal{L}_0(x), \mathcal{L}_0(t), \ldots, \mathcal{L}_M(x), \mathcal{L}_0(t), \ldots, \mathcal{L}_M(x), \mathcal{L}_0(t) \}$ and $\varepsilon \in H^*$ be a smooth function defined on $I = [0, 1] \times [0, 1]$ with bounded derivatives as follows

$$\max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial x^{M+1}} \right| \leq \theta_1, \quad \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial t^{M+1}} \right| \leq \theta_2, \quad \max_{(x, t) \in I} \left| \frac{\partial^{3M+2}\varepsilon(x, t)}{\partial x^{M+1}\partial t^{M+1}} \right| \leq \theta_3,$$

where $\theta_1$, $\theta_2$, and $\theta_3$ are positive constants. If $\varepsilon = \mathcal{L}^\varepsilon(x)\mathcal{L}^\varepsilon(t)$ be the best approximation of $\varepsilon$ into $H^*$, then

$$\| \varepsilon(x, t) - \varepsilon(x, t) \|_2 \leq \frac{1}{(M + 1)^2 2^{M+1}} \left( \theta_1 + \theta_2 + \frac{\theta_3}{(M + 1)^2 2^{M+1}} \right).$$

**Proof.** Let that $P_M$ is the interpolating polynomials to $\varepsilon$ at points $(x_i, t_j)$, where $x_i, i = 0, 1, \ldots, M$ and $t_j, j = 0, 1, \ldots, M$ are the roots of $(M + 1)$-degree shifted Chebyshev polynomials on $[0, 1]$. Then

$$\varepsilon(x, t) - P_M(x, t) = \frac{\partial^{M+1}\varepsilon(x, t)}{\partial x^{M+1}} \prod_{i=0}^{M} (x - x_i) + \frac{\partial^{M+1}\varepsilon(x, t)}{\partial t^{M+1}} \prod_{j=0}^{M} (t - t_j) - \frac{\partial^{3M+2}\varepsilon(x, t)}{\partial x^{M+1}\partial t^{M+1}} \prod_{i=0}^{M} (x - x_i) \prod_{j=0}^{M} (t - t_j),$$

where $\xi, \xi', \eta, \eta' \in [0, 1]$. Then we obtain

$$\| \varepsilon(x, t) - P_M(x, t) \|_2 \leq \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial x^{M+1}} \right| \prod_{i=0}^{M} |x - x_i| + \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial t^{M+1}} \right| \prod_{j=0}^{M} |t - t_j| + \max_{(x, t) \in I} \left| \frac{\partial^{3M+2}\varepsilon(x, t)}{\partial x^{M+1}\partial t^{M+1}} \right| \prod_{i=0}^{M} |x - x_i| \prod_{j=0}^{M} |t - t_j|,$$

where $\xi, \xi', \eta, \eta' \in [0, 1]$. Then we obtain

$$\| \varepsilon(x, t) - P_M(x, t) \|_2 \leq \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial x^{M+1}} \right| \prod_{i=0}^{M} |x - x_i| + \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial t^{M+1}} \right| \prod_{j=0}^{M} |t - t_j| + \max_{(x, t) \in I} \left| \frac{\partial^{3M+2}\varepsilon(x, t)}{\partial x^{M+1}\partial t^{M+1}} \right| \prod_{i=0}^{M} |x - x_i| \prod_{j=0}^{M} |t - t_j|.$$

Since $\varepsilon(x, t)$ is a smooth function on $I$, then there exist constants $\theta_1, \theta_2, \theta_3$ such that

$$\max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial x^{M+1}} \right| \leq \theta_1, \quad \max_{(x, t) \in I} \left| \frac{\partial^{M+1}\varepsilon(x, t)}{\partial t^{M+1}} \right| \leq \theta_2, \quad \max_{(x, t) \in I} \left| \frac{\partial^{3M+2}\varepsilon(x, t)}{\partial x^{M+1}\partial t^{M+1}} \right| \leq \theta_3.$$  

(15)

By substituting (15) into (14) and employing the estimates for Chebyshev interpolation nodes, we have

$$\| \varepsilon(x, t) - P_M(x, t) \|_2 \leq \frac{1}{(M + 1)^2 2^{M+1}} \left( \theta_1 + \theta_2 + \frac{\theta_3}{(M + 1)^2 2^{M+1}} \right).$$

(16)

Since $\varepsilon_M$ is the best approximation of $\varepsilon$ in $H^*$, that is

$$\| \varepsilon(x, t) - \varepsilon_M(x, t) \|_2 \leq \| \varepsilon(x, t) - \varepsilon^*(x, t) \|_2,$$

where $\varepsilon^*$ is any arbitrary polynomial in $H^*$. Then, using (16) we obtain

$$\| \varepsilon(x, t) - \varepsilon_M(x, t) \|_2^2 = \int_0^1 \int_0^1 |\varepsilon(x, t) - \varepsilon_M(x, t)|^2 \, dx \, dt \leq \int_0^1 \int_0^1 |\varepsilon(x, t)|^2 \, dx \, dt = \int_0^1 \int_0^1 \left( \frac{1}{(M + 1)^2 2^{M+1}} \left( \theta_1 + \theta_2 + \frac{\theta_3}{(M + 1)^2 2^{M+1}} \right) \right)^2 \, dx \, dt.$$

(17)

Finally, taking the square root of both sides of (17) completes the proof. □

### 3.1. Operational matrices of the SLPs

This subsection is devoted to introducing some operational matrices (OMs) of the SLPs basis vector which will be used further.

(1) The OM of the integration of the vector $\mathcal{L}(t)$ given by (11) can be approximated as

$$\int_0^1 \mathcal{L}(s) \, ds \approx P \mathcal{L}(t),$$

(18)
where $P$ is given as [29]

$$
P = \frac{1}{2} \begin{bmatrix}
1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
2 & 0 & \frac{1}{2} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \frac{1}{2M+1} & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{2M+1} \\
\end{bmatrix},
$$

(2) The OM of AB–integral of order $\omega$ of the vector $\mathcal{L}(t)$ is obtained as

$$A^B_{\omega} \mathcal{L}(t) = \alpha_\omega \mathcal{L}(t) + \mathcal{P}_\omega \Gamma(\omega + 1) A^B_{\omega} \mathcal{L}(t).$$

(19)

Now, we must obtain the OM of RL–integral of order $\omega$. To do this, we apply the LR–integral operator, $R^L_{\omega}$, on $\mathcal{L}(t)$, $i = 0, 1, ..., M$ as

$$R^L_{\omega} \mathcal{L}(t) = R^L_{\omega} \left( \sum_{i=0}^{i} s_i \mathcal{L}(t) \right) = \sum_{i=0}^{i} s_i R^L_{\omega} \mathcal{L}(t) = \sum_{i=0}^{i} \frac{\Gamma(r + 1)}{\Gamma(r + \omega + 1)} s_i \mathcal{L}(t)^{r+\omega}.$$  

By approximating the function $t^{r+\omega}$ in terms of the SLPs, we have

$$t^{r+\omega} \approx \sum_{k=0}^{K} e_{r,k} \mathcal{L}(t).$$

(20)

In view of (20) and for $i = 0, 1, ..., M$, we get

$$R^L_{\omega} \mathcal{L}(t) \approx \sum_{k=0}^{K} \frac{\Gamma(r + 1)}{\Gamma(r + \omega + 1)} \left( \sum_{k=0}^{K} e_{r,k} \mathcal{L}(t) \right) = \sum_{k=0}^{K} \left( \sum_{r=0}^{R} \frac{\Gamma(r + 1)}{\Gamma(r + \omega + 1)} e_{r,k} \mathcal{L}(t) \right) \mathcal{L}(t) = \sum_{k=0}^{K} \left( \sum_{r=0}^{R} e_{r,k} \right) \mathcal{L}(t).$$

Therefore, for $i = 0, 1, ..., M$, we can write

$$R^L_{\omega} \mathcal{L}(t) = \mathcal{F}^{\omega} \mathcal{L}(t),$$

(21)

where

$$\mathcal{F}^{\omega} = \begin{bmatrix}
\rho_{0,0,0} & \rho_{0,1,0} & \ldots & \rho_{0,M,0} \\
\sum_{r=0}^{R} \rho_{1,0,r} & \sum_{r=0}^{R} \rho_{1,1,r} & \ldots & \sum_{r=0}^{R} \rho_{1,M,r} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{r=0}^{R} \rho_{M,0,r} & \sum_{r=0}^{R} \rho_{M,1,r} & \ldots & \sum_{r=0}^{R} \rho_{M,M,r} \\
\end{bmatrix},$$

with

$$\rho_{r,k} = \frac{\Gamma(r + 1) s_{r,k} e_{r,k}}{\Gamma(r + \omega + 1)}.$$

By substituting (21) into (19), the proof completes.

(3) The OM of CF–integral of order $\omega$ of the vector $\mathcal{L}(t)$ is obtained as

$$C^F_{\omega} \mathcal{L}(t) = \mathcal{F}^{\omega} \mathcal{L}(t) + \bar{\mathcal{P}}_{\omega} \mathcal{L}(t).$$

In view of (18), we have

$$C^F_{\omega} \mathcal{L}(t) = \mathcal{F}^{\omega} \mathcal{L}(t) + \bar{\mathcal{P}}_{\omega} \mathcal{L}(t) = (\mathcal{F}^{\omega} + \bar{\mathcal{P}}_{\omega} \mathcal{L}(t) = I^\omega \mathcal{L}(t),$$

(22)

where $I$ is an $(M + 1) \times (M + 1)$ identity matrix and $I^\omega = \mathcal{F}^{\omega} + \bar{\mathcal{P}}_{\omega}$. The matrix $I^\omega$ is called the OM of CF–integral based on the SLPs.

4. Applications

4.1. Rheological models

i. Classic approach The behavior of linear viscoelastic materials can be described by linear differential equations. In general, a constitutive equation for a linear viscoelastic material is given by

$$\mathcal{A} \varepsilon = \mathcal{B} \varepsilon.$$
where $\mathcal{A} = \sum_{i=0}^{a} a_i D_i^j, \mathcal{B} = \sum_{i=0}^{b} b_i D_i^j, a_i, b_i \in \mathbb{R}, i = 0, 1, \ldots, n, j = 0, 1, \ldots, m$, and $n \geq m$. Thus, models including connected common mechanical elements such as springs and dampers can be used to visualize the constitutive equation in a convenient way. These descriptions are known as rheological models constructed by combining linear springs and dampers in series and parallel. Three well-known rheological models called Kelvin-Voigt, Maxwell, and Zener models. More complex rheological models with more realistic responses can be constructed by including additional elements [2].

A Kelvin-Voigt element is composed of a linear spring and damper connected in parallel, and its constitutive equation is given as

$$\eta D_t \varepsilon(t) + E \varepsilon(t) = \sigma(t),$$

where $E$ is the elasticity modulus and $\eta$ is the viscosity. Under a creep test with $\sigma(t) = \sigma_0$ and $\varepsilon(0) = 0$, the response is obtained as

$$\varepsilon(t) = \left(1 - \exp \left[ - \frac{E}{\eta} t \right]\right) \sigma_0.$$

A Maxwell element is composed of a linear spring and damper connected in series, and its constitutive equation is given as

$$D_t \varepsilon(t) = \frac{D_t \sigma(t)}{E} + \frac{\sigma(t)}{\eta}.$$

Under a creep test with $\sigma(t) = \sigma_0$ and $\varepsilon(0) = 0$, the response is

$$\varepsilon(t) = \left(\frac{1}{E} + \frac{1}{\eta} t\right) \sigma_0.$$

A Zener element is composed of a linear spring and a Maxwell element connected in parallel, and its constitutive equation is given as

$$D_t \varepsilon(t) + \frac{E_1 E_2}{E_1 + E_2} \varepsilon(t) = \frac{1}{E_1 + E_2} D_t \sigma(t) + \frac{E_2}{\eta(E_1 + E_2)} \sigma(t).$$

Under a creep test with $\sigma(t) = \sigma_0$ and $\varepsilon(0) = 0$, the response of the model is

$$\varepsilon(t) = \left(1 - \exp \left[ - \frac{E_1 E_2}{E_1 + E_2} t\right]\right) \sigma_0.$$

### ii. Fractional approach

In general, consider the following FDE:

$$D_t^\omega \varepsilon(t) + \lambda_1 \varepsilon(t) + \lambda_2 D_t^\omega \varepsilon(t) + \lambda_3 \sigma(t),$$

(23)

where $0 < \omega \leq 1$ and $\lambda_i \in \mathbb{R}^+, i = 1, 2, 3$. $D_t^\omega$ is denoted either the AB ($ABC_t^\omega$) derivative or CF ($CDF_t^\omega$) derivative. The constitutive equation of the proposed fractional rheological models can be obtained by adjusting the parameters $\lambda_i$ in the equation (23).

a. The constitutive equation of the fractional Kelvin-Voigt model is obtained when

$$\lambda_1 = \frac{E}{\eta}, \lambda_2 = 0, \lambda_3 = \frac{1}{\eta}, \omega = v.$$

b. The constitutive equation of the fractional Maxwell model is obtained when

$$\lambda_1 = 0, \lambda_2 = \frac{1}{E}, \lambda_3 = \frac{1}{\eta}, \omega = v.$$

c. The constitutive equation of the fractional Zener model is obtained when

$$\lambda_1 = \frac{E_1 E_2}{E_1 + E_2}, \lambda_2 = \frac{1}{E_1 + E_2}, \lambda_3 = \frac{E_2}{\eta(E_1 + E_2)}, \omega = v.$$

All the previous settings for $\lambda_i$ yield the classic rheological models when $\omega = v = 1$.

### iii. The method

In here, we introduce a numerical method for the solution of the form Eq. (23). Let in the Eq. (23), the derivative is described in the AB (or CF) sense. For solving the Eq. (23), first we approximate $ABC_t^\omega \varepsilon(t)$ and $ABC_t^\omega \sigma(t)$ as

$$ABC_t^\omega \varepsilon(t) \approx C_1^T \varepsilon(t),$$

$$ABC_t^\omega \sigma(t) \approx C_2^T \varepsilon(t).$$

(24)

By taking the AB–integral of (24) and using initial conditions ($\varepsilon(0) = 0, \sigma(0) = \sigma_0$), we have
By approximating \( \sigma(0) \approx C_t^2 J^0 \mathcal{L}(t) \), \( \sigma(t) \) can be rewritten as
\[
\sigma(t) \approx C_t^2 J^0 \mathcal{L}(t) + \sigma(0).
\] 
By putting (24)-(26) in (23), we have
\[
C_t^2 + \lambda_1 C_t^4 J^m - \lambda_2 C_t^2 - \lambda_3 C_t = 0.
\]
By solving the system (27), the unknown parameters are obtained. Finally, the approximate solution can be computed by (25).

iv. Test examples
The creep behavior of the fractional Kelvin-Voigt model for different values of \( E, \omega \) and \( M = 5 \) is shown in Figs. 1 and 2, when \( \eta = 0.5 \) and \( a_0 = 1 \) for the AB and CF derivatives, respectively. We used Mathematica for computation.

4.2. The Newell-Whitehead-Segel equation

i. Classic approach
Consider the Newell-Whitehead-Segel equation
\[
D_t \varepsilon(x, t) = \lambda_1 D_x \varepsilon(x, t) + \lambda_2 \varepsilon(x, t) + \lambda_3 \varepsilon(x, t)^3,
\]
with initial condition
\[
\varepsilon(x, 0) = f_0(x),
\]
and boundary conditions
\[
\varepsilon(0, t) = f_1(t), \quad \varepsilon(1, t) = f_2(t),
\]
where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are real numbers with \( \lambda_1 > 0 \), and \( \lambda_4 \) is a positive integer number. When \( \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1 \) and \( \lambda_4 = 2 \), the Eq. (28) is called the Fishers equation.

ii. Fractional approach
Consider the Newell-Whitehead-Segel equation by
ii. The method

Here, we introduce a numerical method for the solution of the form Eq. (31). Let in the Eq. (31), the derivative is described in the AB (or CF) sense. For solving the Eq. (31), first we approximate $D_{xx} \varepsilon(x, t)$ as

$$L L D_{xx} \varepsilon(x, t) \approx \Sigma^T(x)C_1 \Sigma(t),$$

(32)

where $C_1$ is an $(M + 1) \times (M + 1)$ unknown vector. By integrating from (32) respect $x$ twice, we get

$$\varepsilon(x, t) \approx \Sigma^T(x)(P^2)^T C_1 \Sigma(t) + x\varepsilon(0, t) + \varepsilon(0, t).$$

(33)

By putting $x = 1$ into (33), we have

$$\varepsilon(0, t) = \varepsilon(1, t) - \varepsilon(0, t) - \Sigma^T(1)(P^2)^T C_1 \Sigma(t).$$

(34)

Let

$$1 \approx C_2^T \Sigma(t) (or \ C_2^T \Sigma(x)), \quad x \approx C_2^T \Sigma(x), \quad \varepsilon(0, t) \approx C_3^T \Sigma(t), \quad \varepsilon(1, t) \approx C_4^T \Sigma(t),$$

(35)

where the elements of $C_2$, $C_3$, $C_4$ and $C_5$ vectors can be calculated by (9). With helping (34) and (35), $\varepsilon(x, t)$ can be rewritten as

$$\varepsilon(x, t) \approx \Sigma^T(x)C_6 \Sigma(t),$$

(36)

where $C_6 = (P^2)^T C_1 + C_2 C_7 - C_1 \Sigma^T(1)(P^2)^T C_1 - C_4 C_7 + C_2 C_3^T$. Let $F(x, t)$, $\varepsilon(x, t) = \lambda_2 \varepsilon(x, t) - \lambda_3 \varepsilon(x, t)^{\lambda_4}$. We approximate $F$ and $\varepsilon(x, 0)$ using the shifted Legendre basis by

$$F(x, t, \varepsilon(x, t)) = \Sigma^T(x)C_7 \Sigma(t),$$

(37)

$$\varepsilon(x, 0) = C_8^T \Phi(t),$$

(38)

where $C_7$ is an $(M + 1) \times (M + 1)$ unknown vector and the elements of $C_8$ vector can be obtained by (9). By taking the AB-integral of both sides of the Eq. (31) and using (32), (36)–(38), we get

$$C_6 - C_2 C_7^T - \lambda_1 C_1 J^w - C_8 J^w = 0.$$  

(39)

Now, by putting (36) into (37) and using the collocation points $x_i = \frac{i}{M+1}$, $i = 1, 2, ..., M + 1$ and $t_j = \frac{j}{M+2}$, $j = 1, 2, ..., M + 1$, gives

Fig. 2. The creep behavior of the Kelvin-Voigt model for different values of $E$, $\omega$ and $M = 5$, by considering the CF derivative.
Eqs. (39) and (40) form a system of $2(M + 1)(M + 1)$ nonlinear equations of the vectors of $C_1$ and $C_7$. By solving this system, the unknown parameters of the vectors of $C_1$ and $C_7$ are obtained. Finally the approximate solution can be computed by (36).

iv. Test examples
Consider the Newell-Whitehead-Segel equation in the following cases

Case 1. By considering $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ and $\lambda_4 = 2$, the Newell-Whitehead-Segel equation with the exact solution $\varepsilon(x, t) = \frac{2}{\omega^2 + t^2}$ is as follows

$$D_\omega^\omega \varepsilon(x, t) = D_\omega x_\omega \varepsilon(x, t) + \varepsilon(x, t) - \varepsilon(x, t)^2.$$  

Case 2. By considering $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ and $\lambda_4 = 3$, the Newell-Whitehead-Segel equation with the exact solution $\varepsilon(x, t) = \frac{1}{2} \left( 1 + \tanh \left( \frac{x + y}{4} \right) \right)$ is as follows

$$D_\omega^\omega \varepsilon(x, t) = D_\omega x_\omega \varepsilon(x, t) + \varepsilon(x, t) - \varepsilon(x, t)^3.$$  

Case 3. By considering $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4$ and $\lambda_4 = 3$, the Newell-Whitehead-Segel equation with the exact solution $\varepsilon(x, t) = \frac{2}{\sqrt{2}} \frac{e^{\omega x} - e^{\omega y}}{e^{\omega x} + e^{\omega y}}$ is as follows.

Fig. 3. The numerical results for different values of $\omega$ and $M = 5$, by considering the AB derivative.

$$F(x_i, t_j, \delta^T(x_i)C_2\delta(t_j)) - \delta^T(x_i)C_1\delta(t_j) = 0.$$  

(40)

$D_\omega^\omega \varepsilon(x, t) = D_\omega x_\omega \varepsilon(x, t) + \varepsilon(x, t) - \varepsilon(x, t)^2.$
By solving the Newell-Whitehead-Segel equation in above cases using the proposed method, the numerical results for different values of $\omega$ are reported in Tables 1, 2 and 3 and Figs. 3, 4 and 5.

### Table 1
Comparison of the absolute error at some selected points ($M = 5, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 2$).

| $(x, t)$ | $\omega = 0.5$ | $\omega = 0.7$ | $\omega = 0.99$ |
|---|---|---|---|
| AB | CF | AB | CF | AB | CF |
| $(0.1, 0.1)$ | $7.525e - 2$ | $6.619e - 3$ | $6.486e - 3$ | $5.450e - 3$ | $4.428e - 4$ | $2.744e - 4$ |
| $(0.3, 0.3)$ | $1.494e - 3$ | $1.362e - 2$ | $1.361e - 2$ | $6.958e - 3$ | $3.825e - 4$ | $7.925e - 6$ |
| $(0.5, 0.5)$ | $1.472e - 2$ | $1.761e - 3$ | $1.642e - 2$ | $3.408e - 3$ | $1.690e - 4$ | $1.527e - 4$ |
| $(0.7, 0.7)$ | $9.731e - 3$ | $2.522e - 3$ | $1.684e - 3$ | $3.422e - 4$ | $7.833e - 6$ | $1.905e - 4$ |
| $(0.9, 0.9)$ | $9.743e - 2$ | $5.600e - 4$ | $1.574e - 3$ | $1.288e - 3$ | $2.798e - 5$ | $7.392e - 5$ |

$$D_\omega^\omega \varepsilon(x, t) = D_{\omega^2} \varepsilon(x, t) + 3\varepsilon(x, t) - 4\varepsilon(x, t)^3.$$  

By solving the Newell-Whitehead-Segel equation in above cases using the proposed method, the numerical results for different values of $\omega$ are reported in Tables 1, 2 and 3 and Figs. 3, 4 and 5.

### 5. Conclusion

In this work, we have presented a numerical method for solving fractional rheological models and Newell-Whitehead-Segel
Table 2
Comparison of the absolute error at some selected points (M = 5, \( \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 3 \)).

| (x, t) | \( \omega = 0.5 \) | \( \omega = 0.7 \) | \( \omega = 0.99 \) |
|--------|----------------|----------------|----------------|
|        | AB             | CF             | AB             | CF             | AB             | CF             |
| (0.1,0.1) | 1.264 \( \times 10^{-2} \) | 1.112 \( \times 10^{-2} \) | 1.093 \( \times 10^{-2} \) | 9.147 \( \times 10^{-3} \) | 7.272 \( \times 10^{-4} \) | 4.29809 \( \times 10^{-4} \) |
| (0.3,0.3) | 1.977 \( \times 10^{-2} \) | 1.325 \( \times 10^{-2} \) | 1.495 \( \times 10^{-2} \) | 7.890 \( \times 10^{-3} \) | 2.671 \( \times 10^{-4} \) | 2.690225 \( \times 10^{-4} \) |
| (0.5,0.5) | 1.316 \( \times 10^{-2} \) | 3.731 \( \times 10^{-3} \) | 8.061 \( \times 10^{-3} \) | 1.091 \( \times 10^{-3} \) | 2.576 \( \times 10^{-4} \) | 5.29674 \( \times 10^{-4} \) |
| (0.7,0.7) | 3.730 \( \times 10^{-3} \) | 4.513 \( \times 10^{-3} \) | 5.362 \( \times 10^{-3} \) | 6.466 \( \times 10^{-3} \) | 4.170 \( \times 10^{-3} \) | 4.66574 \( \times 10^{-3} \) |
| (0.9,0.9) | 9.471 \( \times 10^{-4} \) | 4.303 \( \times 10^{-3} \) | 1.770 \( \times 10^{-3} \) | 4.135 \( \times 10^{-3} \) | 1.778 \( \times 10^{-3} \) | 1.23433 \( \times 10^{-3} \) |

Table 3
Comparison of the absolute error at some selected points (M = 5, \( \lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 3 \)).

| (x, t) | \( \omega = 0.5 \) | \( \omega = 0.7 \) | \( \omega = 0.99 \) |
|--------|----------------|----------------|----------------|
|        | AB             | CF             | AB             | CF             | AB             | CF             |
| (0.1,0.1) | 2.488 \( \times 10^{-2} \) | 2.203 \( \times 10^{-2} \) | 2.165 \( \times 10^{-2} \) | 1.823 \( \times 10^{-2} \) | 1.253 \( \times 10^{-3} \) | 5.482 \( \times 10^{-4} \) |
| (0.3,0.3) | 1.728 \( \times 10^{-2} \) | 8.122 \( \times 10^{-3} \) | 1.119 \( \times 10^{-2} \) | 1.191 \( \times 10^{-3} \) | 1.064 \( \times 10^{-4} \) | 1.610 \( \times 10^{-5} \) |
| (0.5,0.5) | 2.379 \( \times 10^{-3} \) | 1.179 \( \times 10^{-2} \) | 6.092 \( \times 10^{-3} \) | 1.488 \( \times 10^{-3} \) | 1.106 \( \times 10^{-4} \) | 8.904 \( \times 10^{-4} \) |
| (0.7,0.7) | 8.982 \( \times 10^{-3} \) | 1.514 \( \times 10^{-2} \) | 9.800 \( \times 10^{-3} \) | 1.391 \( \times 10^{-3} \) | 8.813 \( \times 10^{-4} \) | 6.349 \( \times 10^{-4} \) |
| (0.9,0.9) | 4.675 \( \times 10^{-3} \) | 6.516 \( \times 10^{-3} \) | 4.367 \( \times 10^{-3} \) | 4.932 \( \times 10^{-3} \) | 4.051 \( \times 10^{-3} \) | 1.552 \( \times 10^{-3} \) |

equations.
The derivative is considered in the Caputo–Fabrizio and Atangana–Baleanu sense. Our numerical method is based on the operational matrices of the shifted Legendre polynomials. By this way, the main problem is reduced to a system of nonlinear algebraic equations which greatly simplifies the problem. An error estimation is proved for the approximate solution. Finally, some examples have been presented to demonstrate the accuracy and efficiency of the proposed method.

Author contributions
All authors discussed the results and contributed to the final manuscript.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References
[1] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, 2000.
[2] D. Baleanu, A. Mendes, Lopes, Handbook of Fractional Calculus with Applications, Applications in Engineering, Life and Social Sciences, Part A 7 Southampton: Comput Mech Publicat, Berlin, Boston: De Gruyter. Retrieved 28 August, 2019.
[3] N. Heymans, J.C. Bauwens, Fractal rheological models and fractional differential equations for viscoelastic behavior, Rheol Acta 33 (3) (1994) 210–219.
[4] F. Mainardi, G. Spada, Creep, relaxation and viscosity properties for basic fractional models in rheology, Eur. Phys. J. Spec. Top. 193 (1) (2011) 133–160.
[5] A. Dabiri, E.A. Butcher, M. Nazari, Coefficient of restitution in fractional viscoelastic compliant impacts using fractional Chebyshev collocation, J. Sound Vib. 388 (2017) 230–244.
[6] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, Comput. Math. Appl. 57 (3) (2009) 483–487.
[7] P. Pandey, S. Kumar, S. Das, Approximate analytical solution of coupled fractional order reaction-advection-diffusion equations, Eur. Phys. J. Plus 134 (2019) 364. 
[8] K.D. Dwivedi, S. Das, D. Baleanu, Numerical solution of nonlinear space time fractional-order advection reaction diffusion equation, ASME. J. Comput. Nonlinear Dyn 15 (6) (June 2020) 061005.
[9] S. Das, R. Kumar, Approximate analytical solutions of fractional gas dynamic equations, Appl. Math. Comput. 217 (24) (2011) 9905–9915.
[10] V.K. Yadav, R. Kumar, A. Y. T. Leungb, S. Das, Dual phase and dual anti-phase synchronization of fractional order chaotic systems in real and complex variables with uncertainties, Chin. J. Phys. 57 (2019) 282–308.
[11] A. Jajarmi, D. Baleanu, Optimal control of nonlinear dynamical systems based on a new parallel eigenvalue decomposition approach, Optim. Control Appl. Methods 39 (2) (2018) 1071–1083.
[12] J.A.T. Machado, A.M. Lopesb, A fractional perspective on the trajectory control of redundant and hyper-redundant robot manipulators, Appl. Math. Model. 46 (2017) 716–726.
[13] M.A. Khan, A. Atangana, Modeling the dynamics of novel coronavirus (2019-nCoV) with fractional derivative, Alexandria Eng. J. 59 (4) (2020) 2379–2389, https://doi.org/10.1016/j.aej.2020.02.033.
