The Boundary of the Moduli Space of Stable Cubic Fivefolds

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Abstract
According to Mumford’s geometric invariant theory, the moduli space of stable cubic fivefolds is compactified by adding nonstable semi-stable (i.e., strictly semi-stable) locus. In this paper, we prove that this locus consists of 21 irreducible components. Moreover, we give a description of equation and singularity of the cubic fivefold corresponding to the generic point in each component.

0 Introduction
There exists an analogy between the geometry of cubic threefolds (i.e., hypersurfaces of degree 3 in the projective space $\mathbb{P}^4$) with their intermediate Jacobians (principally polarized Abelian varieties of dimension 5) and the geometry of algebraic curves with their Jacobians[3]. For example, the Abel-Jacobi map from the Albanese variety of the Fano surface of lines on a smooth cubic threefold to its intermediate Jacobian is an isomorphism, and this is in agreement with the Torelli theorem for cubic threefolds, which states that two cubic threefolds are isomorphic if their intermediate Jacobians are isomorphic.

The intermediate Jacobian of a cubic fivefold (i.e., a hypersurface of degree 3 in $\mathbb{P}^6$) is principally a polarized Abelian variety of dimension 21. The Abel-Jacobi map from the Albanese variety of the Fano surface of 2-planes on a generic cubic fivefold to its intermediate Jacobian is an isomorphism[4]. The above-mentioned examples indicate an analogy between the geometry of cubic fivefolds (or the geometry of cubic threefolds) and the geometry of algebraic curves. Availability of limited results about the geometry of cubic fivefolds necessitates the need to establish foundations. As it is important to consider compactifications of the moduli space of algebraic curves and the moduli space of cubic hypersurfaces of lower dimensions, we use the geometric invariant theory (GIT) compactification of the moduli space of cubic fivefolds in this study.

The GIT compactification of the moduli space of cubic hypersurfaces in $\mathbb{P}^n$ presents the categorical quotient

$$\mathbb{P}(\text{Sym}_{n+1}^3)^{ss} / \text{SL}(n + 1),$$

where $\text{Sym}_{n+1}^3$ is the vector space of homogeneous polynomials of degree 3 comprising $n + 1$ variables. The categorical quotient is projective, hence complete.
Let us look at the results of the GIT compactifications for $n = 3, 4, 5$.

For $n = 3$, the result is as follows\cite{7, 10}.

**Theorem 0.1.** Let $S$ be a cubic surface in $\mathbb{P}^3$.

1. $S$ is stable if and only if it has only rational double points of type $A_1$.
2. $S$ is semi-stable if and only if it has only rational double points of type $A_1$ or $A_2$.
3. The moduli space of stable cubic surfaces is compactified by adding one point corresponding to the semi-stable cubic surface $x_0x_1x_2 + x_3^3 = 0$ with three $A_2$ singularities.

For $n = 4$, the main result is as follows\cite{1, 11}.

**Theorem 0.2.** Let $X$ be a cubic threefold.

1. $X$ is stable if and only if it has double points of type $A_k$ with $k \leq 4$.
2. $X$ is semi-stable if and only if it has only double points of type $A_k$ with $k \leq 5$, $D_4$, or $A_\infty$.
3. The moduli space of stable cubic threefolds is compactified by adding two components. One is isomorphic to $\mathbb{P}^1$, and the other is an isolated point corresponding to the semi-stable cubic threefold $x_0^3 + x_1^3 + x_2x_3x_4 = 0$ with three $D_4$ singularities.

For $n = 5$, the following result is a portion of what is known\cite{12}. Refer to \cite{8} for further results.

**Theorem 0.3.** A cubic fourfold $X$ is not stable if and only if it satisfies either

1. $\text{Sing}X$ contains a conic,
2. $\text{Sing}X$ contains a line,
3. $\text{Sing}X$ contains the intersection of two hyperquadrics in a space,
4. $X$ has a double point of rank $\leq 2$,
5. there exist a double point $p$ of rank 3 and a hyperplane section $Y$ through $p$ with a line $L$ as a singular locus such that the point $p$ on $L$ is of rank 1 and any points on $L$ are of rank $\leq 2$, or
6. there exist a double point $p$ of rank 3 such that the singular locus of the tangent cone at $p$ of $X$ is a 2-plane in $X$.

We study the case $n = 6$ in this study. We give a list of polynomials of cubic fivefolds that are strictly semi-stable (see Theorem3.2 and Theorem3.6). They form the boundary of the moduli space of stable cubic fivefolds and consist of 21 irreducible components. Further, we investigate the singular loci of these strictly
semi-stable cubic fivefolds \((\text{see Theorem} \, \ref{thm:semi-stable})\). For isolated singular points, we give Milnor numbers, Tjurina numbers, and coranks but not Arnold’s symbols because its calculation is complex \([2]\).

In section 1, we specify the numerical criterion for cubic fivefolds using convex geometry. In section 2, we fix a maximal torus \(T\) in \(\text{SL}(7)\) and list all strictly semi-stable cubic fivefolds with respect to \(T\) using an algorithm that terminates in finite steps. We obtain 22 strictly semi-stable cubic fivefolds with respect to \(T\). In section 3, we consider all maximal tori in \(\text{SL}(7)\) and the inclusion relations of these 22 cubic fivefolds modulo \(\text{SL}(7)\) actions. We obtain 21 strictly semi-stable cubic fivefolds. In section 4, we investigate the singular loci of these 21 cubic fivefolds and use Groebner basis and Hilbert polynomials to calculate these singularities.

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## 1 Numerical criterion for cubic fivefolds

In this section, we review the numerical criterion for stability or semi-stability of cubic fivefolds. We use the following notations.

- Let \(\mathbb{C}[x_0, \cdots, x_6]_3\) be the set of homogeneous polynomials of degree 3.
- For a vector \(x \in \mathbb{Q}^7\), \(\text{wt}(x) = \sum_{k=0}^{6} x_k\) is called the weight of \(x\).
- We define \(Z_{\geq 0} = \{x = (x_0, x_1, \cdots, x_6) \in \mathbb{Z}^7|x_k \geq 0(k = 0, 1, \cdots, 6)\}\), \(Z_{d} = \{x \in \mathbb{Z}^7|\text{wt}(x) = d\}\), and \(\mathbb{I} = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{\geq 0}\), and it is simply called the simplex.
- For \(r \in \mathbb{Q}^7\), we define \(I(r) = \{i \in \mathbb{I}|r \cdot i \geq 0\}\) and \(I(r) > 0 = \{i \in \mathbb{I}|r \cdot i > 0\}\); here, \(\cdot\) denotes the standard inner product of vectors.
- For a polynomial \(f = \sum_{\text{wt}(i)=3} a_i x^i \in \mathbb{C}[x_0, \cdots, x_6]_3\), we define the support of \(f\) by \(\text{Supp}(f) = \{i \in \mathbb{I}|a_i \neq 0\}\)
- We set \(\eta = (3/7, 3/7, 3/7, 3/7/3, 3/7, 3/7/3, 3/7/3) \in \mathbb{Q}^7\) and it is called the barycenter of the simplex \(\mathbb{I}\).
- A vector \(r \in \mathbb{Z}^7\) is said to be reduced when there is no integer \(\alpha\) such that \(|\alpha| \geq 2\) and \(|\alpha| r \in \mathbb{Z}^7\).

We fix a maximal torus \(T\) of \(\text{SL}(7)\). Consider a one parameter subgroup \((1-\text{PS for short}) \lambda : \mathbb{G}_m \to \text{SL}(7)\) whose image is contained in \(T\). For suitable basis of \(\mathbb{C}^7\), \(\lambda\) can be expressed as a diagonal matrix \(\text{diag}(t^{r_0}, t^{r_1}, \cdots, t^{r_6})\), where \(t \neq 0\)
is a parameter of $G_m$. Let us choose and fix this basis. Then, $\lambda$ corresponds to an element $r = (r_0, r_1, \cdots, r_6)$ in $\mathbb{Z}_7^6$. We can regard an element of $\mathbb{Z}_7^6$ as a 1-PS of $T$.

**Definition 1.1.** Let $s$ be a subset of $I$. We say that $s$ is not stable (resp. unstable) with respect to $T$ when $s \subseteq I(r)_{>0}$ (resp. $s \subseteq I(r)_{>0}$) for some 1-PS $r$. For $0 \neq f \in \mathbb{C}[x_0, \cdots, x_6]_3$, we say that $f$ is not stable (resp. unstable) with respect to $T$ when $\text{Supp}(f) \subseteq I$ is not stable (resp. unstable) with respect to $T$.

See Figure 1 for more details.

The following theorem is the numerical criterion for stability via the language of convex geometry.

**Theorem 1.2.** The cubic fivefold defined by $f \in \mathbb{C}[x_0, \cdots, x_6]_3$ is not stable (resp. unstable) if and only if there exists an element $\sigma \in \text{SL}(7)$ such that $f^\sigma$ is not stable (resp. unstable) with respect to $T$.

In particular, $f$ is strictly semi-stable if and only if

1. There exist $\sigma \in \text{SL}(7)$ such that $f^\sigma$ is not stable with respect to $T$, and
2. For any $\sigma \in \text{SL}(7)$, $f^\sigma$ is semi-stable with respect to $T$.

**Proof.** See Theorem 9.3 of [6].

**2 Maximal cubic fivefolds that is strictly semi-stable with respect to the maximal torus $T$**

In this section, we list the irreducible components corresponding to strictly semi-stable cubic fivefolds. For this purpose, we list all strictly semi-stable cubic fivefolds with respect to the maximal torus $T$. To solve this problem, we will consider the set of maximal strictly semi-stable subset of $I$. The order in the set of subsets of $I$ is given by inclusion. For this purpose, we list the set of all maximal elements of $S = \{I(r)_{>0} | r \in \mathbb{Z}_7^6 \}$. 

![Figure 1: Various concepts of stability](image-url)
We solve this problem using a computer. We need an algorithm to obtain them in finite steps. Before giving this algorithm, we remark that \( I(r) \geq 0 \) and \( I(r') \geq 0 \) might be the same for two different vectors \( r, r' \in Z^7_{(0)} \).

**Lemma 2.1.** Let \( \mathbb{I}(r) \geq 0 \) be a maximal element of \( S \), where \( r \in Z^7_{(0)} \). Then, there exist 5 elements \( x_1, x_2, \ldots, x_5 \in \mathbb{I} \) and a vector \( r' \in Z^7_{(0)} \) such that they satisfy the following three conditions:

1. The vector subspace \( W \) of \( Q^7 \) spanned by \( x_1, \ldots, x_5, \eta \) over \( Q \) has dimension 6.
2. The vector \( r' \) is orthogonal to the subspace \( W \) of \( Q^7 \).
3. \( \mathbb{I}(r) \geq 0 = \mathbb{I}(r') \geq 0 \)

**Proof.** Let us put \( C = \mathbb{I}(r) \cup \eta \). We consider the convex hull \( \hat{C} \) of \( C \) in \( Q^7 \). Let \( F \) be a face of \( \hat{C} \) containing the point \( \eta \). There is a normal vector \( r' \) of \( F \) in \( Z^7_{(0)} \) such that \( \hat{C} \subseteq \{ x \in Q^7 | r' \cdot x \geq 0 \} \). We have \( \text{wt}(r') = 0 \) because the hyperplane defined by \( \{ x \in Q^7 | r' \cdot x = 0 \} \) passes through the point \( \eta \). By the definition of the faces of a convex set in \( Q^7 \), we can take 5 points \( x_1, x_2, \ldots, x_5 \) from the set \( \mathbb{I} \cap F \) such that \( x_1, x_2, \ldots, x_5, \eta \) are linearly independent over \( Q \). In general we have \( \mathbb{I}(r) \geq 0 \subseteq \mathbb{I}(r') \geq 0 \), and by the assumption that \( \mathbb{I}(r) \geq 0 \) is maximal in \( S \), we conclude that \( \mathbb{I}(r) \geq 0 = \mathbb{I}(r') \geq 0 \).

By this lemma, we can determine the set of maximal elements of \( S \) up to permutations of coordinates in finite steps using the following algorithm.

**Algorithm 2.2.** Let \( F \) be the set of five different points of \( \mathbb{I} \). We fix a total order on \( F \). As an initial data, we set \( S' = \emptyset \) and \( x = (x_0, \ldots, x_5) \) as the minimum element of \( F \). We will modify \( S' \) using the following algorithm.

- **Step 1.** If the subspace \( W \) spanned by \( x_0, \ldots, x_5, \eta \) of \( Q^7 \) has dimension 6, then take a reduced normal vector \( r = (r_0, \ldots, r_6) \in Z^7_{(0)} \) of \( W \) and go to step 2, else go step 5.
- **Step 2.** If \( r = (r_0, \ldots, r_6) \) satisfy the condition \( r_0 \geq \cdots \geq r_6 \) or \( r_0 \leq \cdots \leq r_6 \), then go to step 3, else go to step 5.
- **Step 3.** If \( r_0 \geq \cdots \geq r_6 \) (resp. \( r_0 \leq \cdots \leq r_6 \)), add \( \mathbb{I}(r) \) (resp. \( \mathbb{I}(-r) \)) to \( S' \) and go to step 4.
- **Step 4.** Delete all elements of \( S' \), which is not maximal in \( S' \), and go to step 5.
- **Step 5.** We replace the element \( x \) by the next element if \( x \) is not the maximum element and go to step 1. Otherwise, we stop the algorithm.

We remark that step 2 kills the symmetry \( S_7 \) action on the variables \( x_0, \ldots, x_6 \). We also remark that step 4 is not essential but technical to save the memory of a computer. After running this algorithm on a computer, we find 23 elements.
\[ \mathbb{I}(r_1)_{\geq 0}, \cdots, \mathbb{I}(r_{23})_{\geq 0} \] in \( S' \), where \( r_k = (r_0, \cdots, r_6) \in \mathbb{Z}_{(0)}^7 \) is a reduced vector with \( r_0 \geq \cdots \geq r_6 \). When we compute the convex hulls of \( \mathbb{I}(r_1)_{\geq 0}, \cdots, \mathbb{I}(r_{23})_{\geq 0} \) in \( \mathbb{Q}^7 \), only one of the convex hulls of \( \mathbb{I}(r_k)_{\geq 0} \) does not contain \( \eta \). Let us call it \( \mathbb{I}(r_{23})_{\geq 0} \). Because only \( \mathbb{I}(r_{23})_{\geq 0} \) is unstable with respect to \( T \), we do not treat it if not required. Thus, we can conclude that there are 22 maximal strictly semi-stable cubic fivefolds for the fixed maximal torus \( T \). As a consequence of this algorithm, we have the following proposition.

**Proposition 2.3.** The set \( \mathcal{M} = \{ \mathbb{I}(r_1)_{\geq 0}, \cdots, \mathbb{I}(r_{22})_{\geq 0} \} \) is given as follows.

| \( r_1 \)       | \( r_2 \)       | \( r_3 \)       | \( r_4 \)       | \( r_5 \)       | \( r_6 \)       | \( r_7 \)       | \( r_8 \)       | \( r_9 \)       | \( r_{10} \)   |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( (8,3,2,-1,-2,-4,-6) \) | \( (6,4,1,-1,-2,-3,-5) \) | \( (4,2,1,-1,-2,-3) \) | \( (2,2,0,-1,-1,-1,-1) \) | \( (3,2,1,0,-1,-2,3) \) | \( (4,2,1,0,-1,-2,-3) \) | \( (6,4,2,1,-2,-3,-8) \) | \( (4,1,1,0,-2,-2,2) \) | \( (2,2,0,0,-1,-1,-2) \) | \( (3,2,1,0,-1,-2,-4) \) |
| \( (2,1,0,0,-1,-1,-1) \) | \( (2,1,0,0,-1,-1,-2) \) | \( (2,0,0,0,0,-1,-1) \) | \( (2,1,1,0,0,-2,-2) \) | \( (1,1,0,0,-1,-1,-2) \) | \( (1,1,0,0,-1,-2,2) \) | \( (1,0,0,0,0,0,-1) \) |

For example, \( \mathbb{I}(r_1)_{\geq 0} \) is
\[
\mathbb{I}(r_1)_{\geq 0} = \{ x_0^3, x_0^2 x_1, x_0^2 x_2, x_0^2 x_3, x_0^2 x_4, x_0^2 x_5, x_0^2 x_6, x_0 x_1 x_2, x_0 x_1 x_3, x_0 x_1 x_4, \\
x_0 x_1 x_5, x_0 x_1 x_6, x_0 x_2 x_3, x_0 x_2 x_4, x_0 x_2 x_5, x_0 x_2 x_6, x_0 x_3 x_4, x_0 x_3 x_5, x_0 x_3 x_6, \\
x_0 x_4 x_5, x_0 x_4 x_6, x_0 x_5 x_6, x_0 x_6 x_1, x_1^2 x_2, x_1^2 x_3, x_1^2 x_4, x_1^2 x_5, x_1^2 x_6, x_1 x_2 x_3, x_1 x_2 x_4, \\
x_1 x_2 x_5, x_1 x_2 x_6, x_1 x_3 x_4, x_1 x_3 x_5, x_1 x_3 x_6, x_2 x_3 x_4, x_2 x_3 x_5, x_2 x_3 x_6 \}.
\]

Here we use the notation \( x_0^{i_0} x_1^{i_1} \cdots x_6^{i_6} \) for an element \( (i_0, i_1, \cdots, i_6) \in \mathbb{Z}_{(3)}^7 \) to save space.

**Remark 2.4.** There are several vectors that give the set \( \mathbb{I}(r_{23})_{\geq 0} \). For example, we can take \( r_{23} = (8,5,3,2,-4,-4,-10) \).

### 3 21 maximal strictly semi-stable cubic fivefolds under the action of \( \text{SL}(7) \)

An element \( \mathbb{I}(r_k)_{\geq 0} \) of \( \mathcal{M} \) represents a family of cubic fivefold whose defining polynomial’s support is contained in \( \mathbb{I}(r_k)_{\geq 0} \). In this section, we will investigate inclusion relations among \( \mathbb{I}(r_k)_{\geq 0} \) under the action of \( \text{SL}(7) \). Let \( f_k \) be a generic polynomial whose support is \( \mathbb{I}(r_k)_{\geq 0} \). If we express \( f_k \) directly it becomes too long, so we prepare a notation.

**Definition 3.1.** The symbols \( c, q, l, \alpha \) stand for a cubic form, quadratic form, linear form, and constant term, respectively. Similarly, the symbols \( q_i, l_i, \alpha_i \) stand for the \( i \)-th quadratic form, linear form, constant term respectively.

The following theorem is a direct consequence of the list in proposition 2.3.
Theorem 3.2. Using the above notations, the generic polynomials of \( f_1, \ldots, f_{22} \) are the following forms.

- \( f_1 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2) + \{q_3(x_0, x_1, x_2) + x_0l_3(x_3, x_4)\}x_5 + \alpha_2x_0x_5^2 + \{q_4(x_0, x_1) + x_0l_4(x_2, x_3)\}x_6 \)

- \( f_2 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3 + l_3(x_0, x_1)x_4 + q_3(x_0, x_1) + l_4(x_0, x_1)x_3 + \alpha_1x_0x_4 + \alpha_2x_0x_5^2 + \{q_4(x_0, x_1) + l_6(x_0, x_1)x_2 + \alpha_3x_0x_3\}x_6 \)

- \( f_3 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3 + l_3(x_0, x_1)x_4 + q_3(x_0, x_1, x_2) + x_0l_4(x_3, x_4)x_5 + \alpha_1x_0x_5^2 + \{q_4(x_0, x_1) + l_5(x_0, x_1)x_2 + x_0l_6(x_3, x_4)\}x_6 \)

- \( f_4 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2) + l_3(x_0, x_1)x_3 + q_3(x_0, x_1, x_2)x_4 + \alpha_1x_0x_4 + \alpha_2x_0x_5^2 + \{q_4(x_0, x_1) + l_5(x_0, x_1)x_2 + l_6(x_0, x_1)x_3 + \alpha_3x_0x_3\}x_6 \)

- \( f_5 = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + l_2(x_0, x_1)x_2 + \{q_3(x_0, x_1, x_2) + l_5(x_0, x_1)x_3 + \alpha_1x_0x_4\}x_5 + \{q_4(x_0, x_1) + l_4(x_0, x_1)x_2 + \alpha_2x_0x_3\}x_6 \)

- \( f_6 = c(x_0, x_1, x_2, x_3) + \{q_1(x_0, x_1, x_2, x_3) + l_1(x_0, x_1, x_2)x_3\}x_4 + \{q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3\}x_4 + \{q_3(x_0, x_1, x_2) + l_3(x_0, x_1)x_3\}x_4 + \alpha_1x_0x_4 + \alpha_2x_0x_5^2 + \{q_4(x_0, x_1) + x_0l_4(x_2, x_3)\}x_6 \)

- \( f_7 = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + l_1(x_0, x_1, x_2)x_3 + \alpha_1x_0x_4 + \alpha_2x_0x_5^2 + \{q_3(x_0, x_1) + x_0l_3(x_2, x_3)\}x_6 \)

- \( f_8 = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + l_1(x_0, x_1, x_2)x_3 + \{q_2(x_0, x_1, x_2) + l_2(x_0, x_1, x_2)x_3\}x_4 + \alpha_1x_0x_4 + \alpha_2x_0x_5^2 + \{q_3(x_0, x_1) + x_0l_3(x_2, x_3)\}x_6 + \alpha_3x_0x_2 \)

- \( f_9 = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + \alpha_1x_0x_3 + \alpha_2x_0x_4 + \{q_3(x_0, x_1, x_2) + x_0l_3(x_3, x_4)\}x_5 + \alpha_3x_0x_2 + \alpha_4x_0x_3 \)

- \( f_{10} = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3 + \{q_2(x_0, x_1, x_2) + l_3(x_0, x_1)x_2 + l_4(x_0, x_1)x_3 + l_5(x_0, x_1)x_4\}x_5 + \{q_3(x_0, x_1) + x_0l_3(x_2, x_3)\}x_6 \)

- \( f_{11} = c(x_0, x_1, x_2, x_3) + \{q_1(x_0, x_1, x_2, x_3) + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3\}x_4 + \alpha_1x_0x_2 + \alpha_2x_0x_3 + \{q_2(x_0, x_1, x_2) + l_3(x_0, x_1)x_2 + l_4(x_0, x_1)x_3 + l_5(x_0, x_1)x_4\}x_5 + \alpha_3x_0x_2 + \alpha_4x_0x_3 \)

- \( f_{12} = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + l_1(x_0, x_1)x_2 + \{q_2(x_0, x_1, x_2, x_3) + l_3(x_0, x_1)x_2 + \alpha_1x_0x_3\}x_5 + \{q_3(x_0, x_1) + x_0l_3(x_2, x_3)\}x_6 \)

- \( f_{13} = c(x_0, x_1, x_2, x_3) + q_1(x_0, x_1, x_2, x_3)x_4 + \alpha_1x_0x_2 + \alpha_2x_0x_3 + \{q_2(x_0, x_1, x_2, x_3) + \alpha_1x_0x_3\}x_5 + \alpha_2x_0x_4 + \alpha_3x_0x_5 + \{q_3(x_0, x_1, x_2) + \alpha_4x_0x_3\}x_6 \)
Proposition 3.4. There are two relations $f$ in through all polynomials with $\text{Supp}(\cdot\cdot\cdot)\equiv f\cdot f\cdot f\cdot f\cdot f\cdot f$ such that

\[ f_{14} = c(x_0, x_1, x_2, x_3) + \{q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3\}x_4 + l_1(x_0, x_1, x_2)x_5^2 + q_2(x_0, x_1, x_2) + l_2(x_0, x_1, x_2)x_3 + l_3(x_0, x_1, x_2)x_4 \cdot l_4(x_0, x_1, x_2)x_5 + l_4(x_0, x_1, x_2)x_6^2 + q_3(x_0, x_1, x_2)x_6 \]

\[ f_{15} = c(x_0, x_1, x_2, x_3, x_4) + x_0l_1(x_0, x_1, x_2, x_3, x_4)x_5 + \alpha_1 x_0 x_5^2 + x_0 l_2(x_0, x_1, x_2, x_3, x_4) x_6 + \alpha_2 x_0 x_6^2 \]

\[ f_{16} = c(x_0, x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1, x_2) + x_0 l_1(x_3, x_4)\}x_5 + q_2(x_0, x_1, x_2) + x_0 l_2(x_3, x_4) x_6 \]

\[ f_{17} = c(x_0, x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1) + l_1(x_0, x_1)x_5 + l_1(x_0, x_1)x_6 + \alpha_1 x_0 x_5^2 + q_2(x_0, x_1) + x_0 l_4(x_2, x_3, x_4) x_6 \]

\[ f_{18} = c(x_0, x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3 + l_2(x_0, x_1, x_2)x_4\}x_5 + q_2(x_0, x_1, x_2)x_6 \]

\[ f_{19} = c(x_0 x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1) + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3 + l_3(x_0, x_1)x_4\}x_5 + q_2(x_0, x_1) + l_4(x_0, x_1)x_2 + l_5(x_0, x_1)x_3 + l_6(x_0, x_1)x_4 x_6 \]

\[ f_{20} = c(x_0, x_1, x_2, x_3, x_4) + q_1(x_0, x_1, x_2, x_3)x_5 + q_2(x_0, x_1, x_2, x_3)x_6 \]

\[ f_{21} = c(x_0, x_1, x_2, x_3, x_4, x_5) + q(x_0, x_1)x_6 \]

\[ f_{22} = c(x_0, x_1, x_2, x_3, x_4, x_5) + x_0 l(x_0, x_1, x_2, x_3, x_4, x_5)x_6 \]

For an element $\sigma$ in $\text{SL}(7)$ and $J \subseteq I$, we set $J^\sigma = \cup_f \text{Supp}(f^\sigma)$, where $f$ runs through all polynomials with $\text{Supp}(f) \subseteq J$.

Definition 3.3. We denote $I(\mathbf{r}_k)_{\geq 0} \subseteq I(\mathbf{r}_l)_{\geq 0} \mod \text{SL}(7)$ when there exists $\sigma \in \text{SL}(7)$ such that $I(\mathbf{r}_k)_{\geq 0} \subseteq I(\mathbf{r}_l)_{\geq 0}$ and say that $I(\mathbf{r}_k)_{\geq 0}$ is included in $I(\mathbf{r}_l)_{\geq 0}$ modulo $\text{SL}(7)$.

We construct a smaller subset $M'$ of $M$ such that (1) any element $I(\mathbf{r}_k)_{\geq 0}$ in $M$ is included in some element $I(\mathbf{r}_l)_{\geq 0}$ in $M' \mod \text{SL}(7)$, (2) any element $I(\mathbf{r}_k)_{\geq 0}$ in $M'$ is not included in other $I(\mathbf{r}_l)_{\geq 0}$ in $M' \mod \text{SL}(7)$ ($1 \leq l \leq 22$).

Proposition 3.4. There are two relations

\[ I(\mathbf{r}_{21})_{\geq 0} \subseteq I(\mathbf{r}_{22})_{\geq 0} \mod \text{SL}(7) \]

\[ I(\mathbf{r}_{22})_{\geq 0} \subseteq I(\mathbf{r}_{21})_{\geq 0} \mod \text{SL}(7) \]

Proof. $f_{22} = c(x_0, \cdots, x_5) + x_0 l(x_0, \cdots, x_5)x_6$

\[ \equiv c(x_0, \cdots, x_5) + x_0 l(x_0, x_1)x_6 \]

\[ \equiv c(x_0, \cdots, x_5) + q(x_0, x_1)x_6 \]

$= f_{21}$

Here, $\equiv$ means linear transformation by $\text{SL}(7)$ \hfill \Box
From this proposition, we can delete $f_{22}$ from the list. Thus, we obtain a list of 21 types of cubic fivefolds.

**Proposition 3.5.** If $1 \leq k, l \leq 21 (k \neq l)$, there is no inclusion

$$\mathbb{I}(r_k) \geq 0 \subseteq \mathbb{I}(r_l) \geq 0 \mod \text{SL}(7).$$

**Proof.** We have to check all combinations of $k, l$; therefore, the proof becomes computationally expensive. As it is not difficult, we omit it. \qed

Hence, we obtain following theorem.

**Theorem 3.6.** The moduli space of strictly semi-stable cubic fivefolds has 21 irreducible components that are represented by $f_1, f_2, \ldots, f_{21}$.

### 4 Singular loci of the 21 cubic fivefolds

In this section, we investigate the singular locus of each cubic fivefold defined by $f_k = 0, (k = 0, \ldots, 21)$. Furthermore, we give Milnor numbers, Tjurina numbers, and coranks for isolated singular points. We denote generic cubic fivefold defined by $f_k = 0$ as $X_k$. We give a theorem that describes the information of the singular loci of $X_k$. The theorem is proved by calculating the Groebner basis and Hilbert polynomials of the ideals $\{f_k, \frac{\partial f_k}{\partial x_0}, \ldots, \frac{\partial f_k}{\partial x_6}\}$. For Groebner basis, see [5].

**Theorem 4.1.** The singular loci of $X_k$ are the following:

1. The cubic fivefold $X_1$ has a conic curve with multiplicity 1 as the singular locus in $P = \{x_0 = x_1 = x_2 = x_3 = 0\}$. That is, $P$ is the projective plane whose homogeneous coordinates are $(x_4 : x_5 : x_6)$;
2. The cubic fivefold $X_2$ has a non-degenerate space curve of degree 4 with multiplicity 1 as the singular locus in the projective space $S = \{x_0 = x_1 = x_2 = 0\}$;
3. The cubic fivefold $X_3$ has a non-degenerate space curve of degree 4 with multiplicity 1 as the singular locus in the projective space $S = \{x_0 = x_1 = x_2 = 0\}$;
4. The cubic fivefold $X_4$ has a non-degenerate space curve of degree 4 with multiplicity 2 as the singular locus in the projective space $S = \{x_0 = x_1 = x_2 = 0\}$;
5. The cubic fivefold $X_5$ has a line with multiplicity 1 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;
6. The cubic fivefold $X_6$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 21, Tjurina number is 19, and corank is 3;
(7) The cubic fivefold $X_7$ has a line with multiplicity 1 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(8) The cubic fivefold $X_8$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 19, Tjurina number is 17, and corank is 3;

(9) The cubic fivefold $X_9$ has a conic curve with multiplicity 2 as the singular locus in $P = \{x_0 = x_1 = x_2 = x_3 = 0\}$;

(10) The cubic fivefold $X_{10}$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 25, Tjurina number is 23, and corank is 2;

(11) The cubic fivefold $X_{11}$ has a conic curve with multiplicity 2 as the singular locus in $P = \{x_0 = x_1 = x_2 = x_3 = 0\}$;

(12) The cubic fivefold $X_{12}$ has a line with multiplicity 1 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(13) The cubic fivefold $X_{13}$ has two distinct lines with multiplicities 1 as the singular locus in $L = \{x_0 = x_1 = x_2 = x_3 = 0\}$;

(14) The cubic fivefold $X_{14}$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 18, Tjurina number is 16, and corank is 3;

(15) The cubic fivefold $X_{15}$ has two isolated singular points of the same modulus as the singular locus. Each point has Milnor number 16, Tjurina number 16, and corank 4;

(16) The cubic fivefold $X_{16}$ has a line with multiplicity 2 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(17) The cubic fivefold $X_{17}$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 20, Tjurina number is 18, and corank is 3;

(18) The cubic fivefold $X_{18}$ has a line with multiplicity 1 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(19) The cubic fivefold $X_{19}$ has a line with multiplicity 2 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(20) The cubic fivefold $X_{20}$ has a line with multiplicity 2 as the singular locus that is given by $L = \{x_0 = x_1 = x_2 = x_3 = x_4 = 0\}$;

(21) The cubic fivefold $X_{21}$ has an isolated singular point as the singular locus at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$. Its Milnor number is 16, Tjurina number is 15, and corank is 4.
Remark 4.2. *It is not possible to give Arnold’s symbols for isolated singular points whose coranks are 2 to 3 due to a shortage of available memory. Calculating using the program Singular, 768GB of memory space is not sufficient.*

References

[1] D. Allcock, The moduli space of cubic threefolds, J. Algebraic Geom. 12, 2 (2003) 2, 201-223.

[2] V.I.Arnold, S.M.Gusein-Zade, and Varchenko, Singularities of differentiable maps. Vol. I, Monographs in Mathematics, vol. 82, Birkhäuser, Boston, MA, 1985.

[3] C.H. Clemens and P.A.Griffiths, The intermediate Jacobian of the cubic threefold, Ann. Math., Second Series, Vol. 95, No. 2, 281-356, 1972.

[4] Collino A., The Abel-Jacobi isomorphism for the cubic fivefold, pacific J. Math., 122, 1 (1986), 43-55.

[5] D. Cox, J. Littele and D. O’shea, Ideals, varieties, and algorithms, Springer, New York. 2007.

[6] I. Dolgachev, Lectures on invariant theory, London Mathematical Society Lecture Note Series 296. Cambridge University Press, Cambridge, 2003.

[7] D. Hilbert, Uber die vollen invariantensysteme, Mathem. Annalen 42 (1893), 313-373.

[8] R. Laza, the moduli space of cubic fourfolds, J. Algebraic Geom. 18 (2009), 511-545.

[9] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, Third edition. Springer-Verlag, Berlin, 1982.

[10] D. Mumford, Stability of projective varieties, Extrait de L’Enseignement Mathématique T. 23 (1977), 39–110.

[11] M. Yokoyama, Stability of cubic 3-fold, Tokyo J. Math. 25. (2002),85-105.

[12] M. Yokoyama, Stability of cubic hypersurfaces of dimension 4, RIMS Kôkyûroku Bessatsu B9, 2008, 189-204.