Phenomenological theory of the underdoped phase of a high-$T_c$ superconductor.

A. M. Tsvelik$^1$ and A. V. Chubukov$^2$

$^1$Department of Condensed Matter Physics and Materials Science, Brookhaven National Laboratory, Upton, NY 11973-5000, USA and
$^2$Department of University of Wisconsin, Madison, WI 53706, USA

(Dated: March 23, 2022)

We model the Fermi surface of the cuprates by one-dimensional nested parts near $(0, \pi)$ and $(\pi, 0)$ and unnested parts near the zone diagonals. Fermions in the nested regions form 1D spin liquids, and develop spectral gaps below some $\sim T^*$, but superconducting order is prevented by 1D phase fluctuations. We show that the Josephson coupling between order parameters at $(0, \pi)$ and $(\pi, 0)$ locks their relative phase at a crossover scale $T^{**} < T^*$. Below $T^{**}$, the system response becomes two-dimensional, and the system displays Nernst effect. The remaining total phase gets locked at $T_c < T^{**}$, at which the system develops a (quasi-) long-range superconducting order.

PACS numbers: 71.10.Pm, 72.80.Sk

The copper oxide materials in their underdoped phase combine features typical for systems of different dimensionality. The existence of stripes [1, 2], the absence of quasiparticle peaks above $T_c$, the presence of the spectral gap near $(\pi, 0)$ and $(0, \pi)$ at $T > T_c$, and the filling of the gap by thermal fluctuations (instead of closing of the gap) [3, 4], are hallmarks of 1D strongly correlated systems. On the other hand, the $d$-wave form of the pairing gap and existence of curved portions of the Fermi surface (FS) near zone diagonals are essentially two-dimensional features responsible for the physics of superconducting cuprates at the lowest energies (see [2]). An intriguing aspect of the cuprate physics is the existence of striped phases near $T^*$, and has a different doping dependence.

In this letter we propose a semi-phenomenological model which proposes an interpretation of the origin of the difference between $T_c$, $T^{**}$ and $T^*$. We assume that fermions in antinodal regions near $(0, \pi)$ and $(\pi, 0)$ interact attractively and form quasi-one-dimensional spin liquids (SL) with a large spin gap. These electrons then generate low energy collective modes whose dispersion is highly anisotropic in space. This assumption is consistent with with ARPES experiments [5] showing incoherent spectral functions near the antinodes. The assumption of strong anisotropy of collective modes is also consistent with the ‘checkerboard’-like density modulation seen in the STM experiments [6], and with the recently observed softening of the optical phonons at wave-vectors $(4k_F, 0)$ and $(0, 4k_F)$ (Ref. [7]).

The effective interaction in spin liquids is attractive and scales to strong coupling. At the other hand, it is reasonable to suggest that the coupling at the unnested parts and between the unnested and nested parts remains weak and repulsive. The increase in doping leads to a decrease in the anisotropy which explains opposite trends in doping dependence of $T^*$ and $T^{**}$. Without discussing any microscopic details of the 1D SL, we just exploit the known fact that interactions there open up quasiparticle gaps (pseudogaps) below some crossover temperature $T^*$, which is the largest energy scale in our consideration. In 3D, the gap opening would imply spin or charge ordering, in 1D the ordering is prevented by strong quantum phase fluctuations.

We consider the Josephson-type coupling between the two fluctuating superconducting order parameters formed near $(0, \pi)$ and $(\pi, 0)$ and demonstrate that below a crossover temperature $T^{**} < T^*$, fluctuations of the relative phase of the two order parameters at $(0, \pi)$ and $(\pi, 0)$ acquire a finite mass $m \sim T^{**}$, i.e., the relative phase gets locked. This implies that at $T^{**}$ the system crosses over into a 2D fluctuational regime with enhanced diamagnetic response, and begins displaying Nernst effect. We further show that due to topological excitations (vortices) in the 2D regime, the (quasi)-long-range superconducting order develops only at a smaller $T_c < T^{**}$, where the system undergoes XY-model type phase transition. Below $T_c$ the $d$-wave-like spectral gap opens on the entire FS due to the proximity effect.

Our approach bears some similarities to earlier theories, but there are important differences. The “stripe"
scenario [2] also departs from 1D physics, but it assumes the existence of two different 1D subsystems, separated in real space, one with a spin gap, and one without. In this situation, the dominant interaction is within each subsystem. In our scenario, the 1D SLs near \((0, \pi)\) and \((\pi, 0)\) are identical, both having a spin gap, and the dominant interaction is between these two SLs. Besides, in stripe scenario, the nodal quasiparticles are produced by the interaction between the stripes, while we obtain nodal quasiparticles due to the interaction between 1D and curved parts of the FS. In the RVB scenario [10, 11], SL is supposed to be two-dimensional, and strong phase fluctuations at \(T = 0\) emerge from the fact that the (Hubbard) interaction is assumed to be larger than the fermionic bandwidth unlike in our model where strong phase fluctuations are due to quasi-1D behavior. In “quantum-critical” scenarios [12, 13], the strongest interaction, responsible for \(T^*\), is also assumed to be within the antinodal region, and the \(d\)-wave superconducting gap emerges due to the residual interaction between the antinodal regions and the rest of the FS. However, in these scenarios, the FS is assumed to be almost circular, and the quasi-1D behavior is absent.

In our model, the FS, shown in Fig. 1, has nested regions, and the system can be separated into two quasi-1D systems represented by the electronic states close to the antinodal regions, and a 2D system represented by the states close to the unnested part of the Fermi surface. The low energy physics is dominated by two gapless collective modes originating from critical fluctuations of the charge in 1D regions, and by nodal quasiparticles. Each critical mode is very anisotropic dispersing either in the \(x\) or \(y\) direction. As is always the case in 1D, such modes describe simultaneously fluctuations of the charge density wave (CDW) \(\Delta_{\text{CDW}}\) and the superconducting (SC) \(\Delta_{\text{SC}} = \Delta\) order parameter. The corresponding phases \(\Theta_x, \Theta_y\) are related to each other by duality \(\partial_x X = e^{i\nu} \partial_y X\), \((\mu, \nu = x, y)\), and the 1D effective action can be equally well expressed in terms of \(X\) or \(\Theta_x\). Below we use \(\Delta_{x,y}\) as the basic variables (subindex refers to fluctuations near \((0, \pi)\) and \((\pi, 0)\), respectively), and also switch from \(\Delta_{x,y}\) to the phases \(X, Y\) using \(\Delta_x = |\Delta| e^{iX}\), \(\Delta_y = |\Delta| e^{iY}\). At energies \(E \ll |\Delta| \sim T^*\), the order parameter amplitudes are frozen, but the phases fluctuate. We assume that the most relevant interaction between fermions near \((0, \pi)\) and \((\pi, 0)\) is the Josephson coupling between the phases of the two SC order parameters \(\Delta_x\) and \(\Delta_y\) (note that the CDW phases \(\Theta_{x,y}\) cannot be coupled due to the momentum conservation).

The dynamics of the order parameters \(X, Y\) is described by the effective action in the form of coupled 1D XY models: \(S = \int d\tau d\xi \frac{d\delta}{\lambda} \mathcal{L}_1\), where \(\mathcal{L}_1\) is

\[
\frac{\rho}{2} \left[ (\dot{X})^2 + (\dot{Y})^2 \right] - \frac{t}{\lambda^2} \cos[X(x, y) - X(x + \lambda, y)] - \frac{t}{\lambda^2} \cos[Y(x, y) - Y(x, y + \lambda)] + J \cos[X(x, y) - Y(x, y)]
\]

Here \(2\pi/\lambda\) is the size of the nested area of the Fermi surface along a particular direction (experimentally \(\lambda \approx 4a\), where \(a\) is the lattice constant). For later convenience we discretized the gradient term in the action replacing it by the cosine. The Coulomb interaction can be absorbed into the model parameters as it is screened by the ungauged nodal quasiparticles. The last term in \(\mathcal{L}_1\) describes the Josephson coupling between the two SC order parameters. We assume that this coupling is repulsive, i.e., \(J > 0\), in which case it favors \(X - Y \approx \pi\).

At \(J = 0\) the action \(\mathcal{L}_1\) describes critical and purely 1D fluctuations of the phases \(X, Y\). Parameters of the model can be expressed in terms of collective mode velocity \(v\) and the scaling dimension \(d\) of the SC order parameter

\[
\rho = (4\pi dv)^{-1}, \quad t = v/4\pi d
\]

which can be extracted from the order parameter correlation function, e.g., \(\chi_{xx} = \langle \langle e^{iX(\tau, x, y)} e^{-iX(0, 0, y)} \rangle \rangle\).

\[
\chi_{xx} = \left[ \frac{(\pi T)^2}{\sin^2(\pi T) + \sin^2(\pi T/v)} \right] d
\]

(\(\chi_{yy}\) is obtained by interchanging of \(x\) and \(y\)). Deviations from one-dimensionality near \((0, \pi)\) or \((\pi, 0)\) are taken into account by the terms of the form

\[
\mathcal{L}_2 \sim b_{\phi}(\cos[X(x, y) - X(x, y + \lambda)] + \cos[Y(x, y) - Y(x + \lambda, y)] + (X, Y) \rightarrow 4\pi \Theta_{x,y} d)(4)
\]

These extra terms by itself lead to a finite temperature phase transition to a superconducting or a CDW state (depending on what type of coupling is more relevant), even at \(J = 0\) [2]. We assume, however, that the corresponding transition temperature is the smallest energy scale in the problem, and neglect it, focusing instead on the Josephson interaction.

The coupling of the SC order parameters to the unnested part of the Fermi surface is given by

\[
\gamma(k_x) \Delta_{x}(q) + \gamma(k_y) \Delta_{y}(q)|\psi^\dagger_{k+q/2} \psi^\dagger_{k+q/2} + H.c. \quad (5)
\]

where \(\gamma(\ldots)\) are phenomenological parameters. This interaction opens the SC gap on the curved parts of FS through the proximity effect when the phases of \(\Delta_x\) and \(\Delta_y\) lock. At \(k_x = k_y\), the couplings \(\gamma\) are equal, and for \(X - Y = \pi\) \((i.e., \Delta_x = -\Delta_y)\) the interaction vanishes. Setting \(\gamma(k_x) = \gamma \cos k_x\), we then reproduce the \(d\)-wave order parameter along the whole FS. When the phases \(X, Y\) are not locked, the interaction [5] is unable to generate a gap at the unnested part of FS, but it can destroy the quasiparticle coherence along the whole Fermi surface. Indeed, evaluating the self energy of these electrons from [5] in the leading order in \(\gamma\) we obtained that it is local and behaves as

\[
Im \Sigma \sim \gamma^2 \Delta^2 \int d\tau \frac{\sin(\omega \tau)}{\tau} |x|^{-2d} \sim \omega^{2d}
\]
It can be shown that the higher order terms are less singular in \( \omega \). If \( d < 1 \) this self-energy exceeds \( \omega^2 \) coming from other, non-singular interactions. Experimentally, \( \Sigma(\omega) \) scales nearly linearly with \( \omega \) above \( T_\lambda \). In our phenomenological theory, this implies that \( d \approx 1/2 \).

We now return to effective action (11). Assuming that \( X, Y(x, y) \) are slowly varying functions of \( x \) and \( y \) respectively and replacing \( t/\lambda^2 \) terms in (11) by quadratic functions, we obtain the action quadratic in \( \Phi^{(+)} = (X + Y)/\sqrt{2} \). After integrating out \( \Phi^{(+)} \) we obtain the action for \( \Phi^{(-)} = (X - Y)/\sqrt{2} \) in the form

\[
S_{\text{eff}} = \int \frac{dωdkdk_y}{(2π)^3} \mathcal{L}_{\text{eff}}(k, \omega)
\]

where

\[
\mathcal{L}_{\text{eff}}(k, \omega) = \frac{1}{2} \left[ \Phi^{(-)} \right] G_0^{-1}(ω, k_x) + G_0^{-1}(ω, k_y) \Phi^{(-)} + J(2π)^3 δ(k) δ(ω) \left[ \cos(\sqrt{2}Φ^{(-)}) \right]_{k, ω},
\]

(7)

where \( G_0(ω, k) = 2\lambda d[(ω^2/v + vk^2)]^{-1} \) and \( [.]_{k, ω} \) means Fourier-transform.

Models of this kind have been discussed before in the context of sliding Luttinger liquid phases [14]. It was assumed there that the theory is renormalizable. By inspecting the perturbation theory series in \( g \) we have found that this is not the case. In this situation, arguments based on a supposed relevance of various operators, used in [14] are questionable. In our analysis, we restrict ourselves to the summation of the most diverging RPA diagrams. The RPA series for the pairing susceptibility are given by

\[
χ_{xx}(ω; k) = \left\{ [χ_0(ω, k_x)]^{-1} - J^2 χ_0(ω, k_y) \right\}^{-1},
\]

(8)

where \( χ_0(ω, k) \) is the Fourier transform of \( \mathcal{L} \). Evaluating \( χ_0(k, ω) \) (see Ref. [13]) and substituting into (5) we find that \( χ_{xx}(0, 0) \) diverges at the temperature

\[
T^{**} = \frac{\Lambda}{2π} \left( \frac{J\lambda}{\Lambda} \sin πd \int \frac{Γ^2(d/2)Γ^2(1 - d)}{Γ^2(1 - d/2)} \right)^{-1/2π},
\]

(9)

where \( Λ = 2πd/\lambda \). The divergence implies, that within RPA, \( T^{**} \) is the true superconducting ordering temperature [10]. Below \( T^{**} \), phase fluctuations acquire a finite mass \( m(T) \). The spectrum can be obtained by extending RPA analysis to the ordered state, i.e., by replacing cosine Josephson potential by a quadratic function of \( \Phi^{(-)} = \sqrt{2}Φ^{(-)} - π \) as

\[
J \cos[\sqrt{2}Φ^{(-)}] ≈ -\frac{m^2}{2} \left( \Phi^{(-)} \right)^2
\]

\[
m^2 = J[\cos[\sqrt{2}Φ^{(-)}]] = J \exp \left[ -\frac{1}{2} \left( \Phi^{(-)} \right)^2 \right].
\]

(10)

Plugging this back into (7), we get

\[
\langle [Φ_i(-ω, -k) Φ_j(ω, k)] \rangle = \frac{1}{D(ω, k)} \left( \frac{ρω^2 + tk_y^2 + m^2}{m^2} \right) \frac{m^2}{ρω^2 + tk_x^2 + m^2}
\]

where \( i, j = x, y \), \( D = (ρω^2 + tk_y^2)/(ρω^2 + tk_x^2 + m^2) \) its poles at \( iω = E_± \) yield the excitation spectrum:

\[
E_±^2 = \frac{k_x^2}{2} + (m/2)^2 ± \sqrt{(m/2)^4 + \frac{1}{4}k^4 \cos^2(2θ)}
\]

(12)

where \( k_x = k \cos θ, k_y = k \sin θ \). The mode \( E_- \) is gapless, as it should be in the ordered state. A substitution of (11) into (10) yields the self-consistent equation for \( m \) solving which we reproduce (12).

As we shall see, the vortices transform \( T^{**} \) into a crossover scale at which fluctuations of \( Φ^{(-)} \) develop a gap, and the system response becomes 2D. The static diamagnetic susceptibility, however, still remains finite, i.e., the system does not become a superconductor. To demonstrate this, we use the Villain approximation replacing all cosine potentials in the effective action (1) by periodic quadratic functions, e.g.,

\[
-\cos[X(x, y) - X(x + λ, y)] →
\]

\[
\frac{1}{2} [X(x, y) - X(x + λ, y) - 2πN_1(x, x + λ; y)]^2
\]

(13)

We further represent \( N_1(x, x + λ; y) = \tilde{N}_1(x, y) - \tilde{N}_1(x + N_1, x + λ; y) \). The integers \( N_1, y \) live on links of the lattice, while \( \tilde{N}_1 \) live on its sites. One can easily verify that the actions for \( \tilde{N}_1 \) and \( N_1 \) decouple from each other. Because of the space limitations, we derive only the action for \( \tilde{N}_1 \). These fields enter \( S_{\text{eff}} \) in combination \( \tilde{N}_1 - \bar{N}_1 ≡ n \). The Lagrangian density is

\[
\mathcal{L} = ρ[(\dot{X})^2 + (\dot{Y})^2] + t(∂_x X)^2 + t(∂_y Y)^2 + m^2[X(x, y) - Y(x, y) - 2πn(x, y)]^2
\]

(14)

Integrating over \( X, Y \) fields we obtain the effective action for \( n \):

\[
S[n] = 2π^2 m^2 T \sum_{ω, k} n(-ω, -k) K(ω, k)n(ω, k)
\]

(15)

\[
K(ω, k) = \frac{(ρω^2 + tk_y^2)(ρω^2 + tk_x^2)}{(ρω^2 + tk_y^2)(ρω^2 + tk_x^2) + m^2(2ρω^2 + tk^2)}
\]

At finite \( T \) the most important contribution comes from zero frequency where the kernel is

\[
K(0, k) = tk_x^2 k_y^2/(tk_x^2 k_y^2 + m^2 k^2)
\]

(16)

This form of the kernel suggests that the relevant integer variable is \( q(x, y) = ∂_x ∂_y n(x, y) \) so that

\[
n(x, y) = \sum_j e_j \theta(x - x_j) \sum_k e'_k \theta(y - y_k), \text{where } e_j, e'_k \text{ are integers.}
\]

Thus from \( S[n] \) we arrive to the classical action for the Coulomb gas of “charges” \( q(j, k) = e_j e'_k \):

\[
S[q] = \frac{2πt}{T} \sum_r |q_r| \ln ||r - r'|/b| q_r, \ b ≈ m^{-1}
\]

(17)
This form implies that the actual ordering transition belongs to the universality class of XY model (2D one for a single layer or anisotropic 3D one for a system of weakly coupled layers). The transition temperature then can be estimated as

\[ T^c = \frac{\pi T}{2\lambda} - \frac{\pi v}{8d\lambda} = \frac{\Lambda}{16d} \] (18)

Below \( T^c \), the vortices are bound into pairs, both the relative and total phase of the two order parameters are locked, and the system displays Meissner effect. The coupling to the unexploited part of the FS then gives rise to the emergence of the \( d^-\) wave parameter all along the FS. By construction, the phases X, Y have to be locked first, hence \( T^c \) must be smaller than \( T^*\). Parametrically, this requires \( J \) to be quite large: \( J\lambda > \Lambda = 2\pi v/\lambda \sim (\pi/2)v/a \). However, numberwise, for \( d = 1/2 \), we obtain from (19) and (13) \( T^*/T^c \approx 70(J\lambda/\Lambda) \), hence \( T^* > T^c \) even if \( J \lambda \leq \Lambda \).

The analysis of the effective action for integer fields \( N_{1,2} \) proceeds in a similar fashion. After the integration over \( X, Y \) we obtain

\[ S = 2\pi^2 t \sum (N_1, N_2) K \left( \begin{array}{c} N_1 \\ N_2 \end{array} \right) \]

The kernel matrix \( K \) is such that at zero frequency we have the same Coulomb gas action (17) (with the same \( T^c \)) for the charges \( q = \partial_y N_1 - \partial_x N_2 \). In the absence of a magnetic field the link variables are modified to \( N_1 \to 2e\lambda A_x/c + N_1 \) and \( N_2 \to 2e\lambda A_y/c + N_2 \), and the charges acquire an extra piece \( q \to q + \frac{2e\lambda^2 B}{c} \), where \( B \) is the magnetic field. This implies that these charges are directly coupled to a magnetic field, i.e., between \( T^* \) and \( T^c \) the system develops a strong diamagnetic response and displays Nernst effect.

The existence of the diamagnetic response in a finite range above \( T^c \) is consistent with the measurements of the Nernst effect in the cuprates [1]. In agreement with our theory the temperature up to which Nernst effect has been observed is definitely much smaller than \( T^* \). Experimentally, the onset temperature for the Nernst effect follows the same trend as \( T^c \) and goes down in strongly underdoped cuprates implying that the ratio \( J\lambda/\Lambda \) is independent on doping. Our theory also predicts that at \( T > T^c \), the interaction between nested and curved parts of the FS destroys fermionic coherence all along the Fermi surface.

To summarize, in this paper we proposed a phenomenological model for the low energy physics of underdoped cuprates. We assumed that at some large energy scale \( T^* \), fermions within the nested regions (see Fig.) form 1D spin liquids. Strong quantum 1D fluctuations prevent the superconductivity to develop at \( T^* \); these fluctuations are suppressed only at much lower temperature \( T^{**} \) where the relative phase of the two order parameters made of electronic states near \((0, \pi)\) and \((\pi, 0)\) gets locked, and the system crosses over into a 2D fluctuational regime with enhanced diamagnetic response. Because of the topological vortex fluctuations the total phase of the two order parameters becomes locked only at even smaller \( T_c < T^{**} \), below which the system develops a (quasi-) long-range superconducting order.

AVC acknowledges the support from Theory Institute for Strongly Correlated and Complex Systems at BNL and NSF DMR 0240238. AMT is grateful to Abdus Salam ICTP for hospitality and acknowledges the support from US DOE under contract number DE-AC02 -98 CH 10886. We are grateful to A. A. Nersesyan for valuable ideas, and to A. G. Abanov, D. Basov, G. Blumberg, E. Fradkin, A. Parmekanti, T. M. Rice, O. Starykh and J. Tranquada for discussions and interest to the work.

[1] J. M. Tranquada et al., Nature 429, 534 (2004).
[2] M. Granath et al., Phys. Rev. Lett. 87, 167011 (2001). See also S. A. Kivelson, V. J. Emery and O. Zachar, Phys. Rev. B 56, 6120 (1997).
[3] T. Timusk and B. Statt, Rev. Prog. Phys. 62, 61 (1999).
[4] A. Kanigel et al, Nature Physics 2, 447 (2006) and references therein.
[5] S. Damascelli et al., Rev. Mod. Phys. 75, 473 (2003); J. C. Campuzano, M. R. Norman, M. Randeria in "Physics of Superconductors", Vol. II, ed. K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2004), p. 167-273; T. Valla et al, cond-mat/0512685.
[6] see e.g., T. Dahm et al, Phys. Rev. B 72, 214512 (2005); I. Eremin et al, Phys. Rev. Lett 94, 147001 (2005); E. Bascones and T. M. Rice, cond-mat/0511661; J. Paaske and D.V. Khveshchenko, Physica C, 341, 265 (2000); A. V. Chubukov and A. M. Tsvelik, Phys. Rev. B 73, 220503 (2006).
[7] Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B 73, 024510 (2006).
[8] T. Hanaguri et al, Nature 430, 1001 (2004).
[9] D. Reznik et al., Nature 440, 1170 (2006). We emphasize that the existence of critical fluctuations at the wave vector \( 4k_F \) is the characteristic feature of a doped 1D SL.
[10] P. W. Anderson et al., J.of Phys. Cond. Mat. 16, R755 (2004); Parmekanti et al., Phys. Rev. B 70, 054504 (2004).
[11] A. Lauchli, C. Honerkamp and T. M. Rice, Phys. Rev. Lett. 92, 037006 (2004).
[12] Ar. Abanov, A. Chubukov, and J. Schmalian, Adv. Phys. 52, 119 (2003).
[13] A. Perali et al., Phys. Rev. B 54, 16216-16225 (1996).
[14] R. Mukhopadhyay, C. L. Kane and T. C. Lubensky, Phys. Rev. B 64, 045120 (2001).
[15] K. B. Efetov and A. I. Larkin, Soviet Physics JETP 42, 390 (1975). See also S. T. Carr and A. M. Tsvelik, Phys. Rev. B 65, 195121 (2002).
[16] A similar RPA analysis for the spin susceptibility has been performed in O. A. Starykh, R. R. P. Singh and G. C. Levine, Phys. Rev. Lett. 88, 167203 (2002). They, however, considered the susceptibility at \( k = (\pi, \pi) \), for which RPA does not lead to an instability.