General Gauss-Bonnet brane cosmology

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Abstract

We consider 5-dimensional spacetimes of constant 3-dimensional spatial curvature in the presence of a bulk cosmological constant. We find the general solution of such a configuration in the presence of a Gauss-Bonnet term. Two classes of non-trivial bulk solutions are found. The first class is valid only under a fine tuning relation between the Gauss-Bonnet coupling constant and the cosmological constant of the bulk spacetime. The second class of solutions are static and are the extensions of the AdS-Schwarzchild black holes. Hence in the absence of a cosmological constant or if the fine tuning relation is not true, the generalised Birkhoff’s staticity theorem holds even in the presence of Gauss-Bonnet curvature terms. We examine the consequences in brane world cosmology obtaining the generalised Friedmann equations for a perfect fluid 3-brane and discuss how this modifies the usual scenario.

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I. INTRODUCTION

The intriguing possibility that our Universe is only part of a higher dimensional spacetime \[1, 2\] has raised a lot of interest in the physics community recently \[3, 4, 5, 6, 7, 8\]. In particular 5 dimensional brane Universe models and their cosmology have been extensively studied (see for example \[9, 10, 11\]). The Universe in this case is a gravitating homogeneous and isotropic brane or domain wall evolving in a constant negative curvature spacetime.

Amongst the interesting features of such a toy-model configuration is that it verifies a generalised version of Birkhoff’s staticity theorem \[10, 11\]: a constant curvature spacetime of constant 3-space curvature is locally static; more specifically an ADS-black hole solution \[12\],

\[
ds^2 = -h(r) dt^2 + h^{-1}(r) dr^2 + r^2 \left( \frac{d\chi^2}{1 - \kappa \chi^2} + \chi^2 d\Omega^2_{II} \right)
\]

where \( h(r) = \kappa - \frac{\mu}{r^2} + \kappa^2 r^2 \), with \( \kappa = 0, \pm 1 \) and \( \mu, k^2 \) are related to the black hole mass and bulk cosmological constant respectively. To ensure the validity of this theorem it is essential firstly that the brane Universe is of co-dimension 1 i.e. a domain wall type defect and secondly that the brane Universe is homogeneous and isotropic. The theorem in turn implies a certain number of physical properties for the configuration, in particular that the only dynamical degree of freedom is the wall’s trajectory or equivalently, for the 4-dimensional observer stuck on the brane, the expansion rate of the Universe. Hence although we have introduced an extra dimension, the number of dynamical degrees of freedom does not alter with respect to standard 4 dimensional FLRW cosmology. Just like in 4 dimensional cosmology, once given an equation of state relating energy density and pressure one obtains the expansion rate or equivalently the brane Universe trajectory. It follows rather elegantly, \[11\], that it is totally equivalent to study a brane Universe evolving in a static background in the manner of \[10\] to a fixed brane Universe in a time-dependant background \[9\].

When treating higher dimensional gravity theories we should keep in mind that 4 dimensional gravity is quite special for a numerous number of reasons. For example \( D = 4 \) gives
the minimal number of dimensions where the graviton is non trivial and has exactly two polarisation degrees of freedom whereas at the same time gauge interactions of the Standard Model are renormalisable. Another special property of 4 dimensional gravity is the uniqueness of the Einstein-Hilbert action.

In $D > 4$, however, the situation is quite different. In 5 dimensions, in order to obtain the most general unique action, i.e. giving rise to a second order symmetric and divergence-free tensor, and to field equations that are of second order in the metric components, we have to add the Gauss-Bonnet term to the usual Einstein-Hilbert plus cosmological constant action. This is part of Lovelock’s theorem \cite{14}. Furthermore in an effective action approach of string theory, the Gauss-Bonnet term corresponds to the leading order quantum correction to gravity in particular in the case of the heterotic string \cite{15}. The Gauss-Bonnet coupling constant is related to the Regge slope parameter or string scale. Furthermore one of the important properties of string theories is that they contain no ghosts. Interestingly as was demonstrated in \cite{16} the only curvature squared terms to give ghost-free self-interactions for the graviton (around flat spacetime) is precisely the Gauss-Bonnet combination.

The reason for all these nice properties shared by the Einstein-Hilbert and the Gauss-Bonnet terms can be understood from a purely geometrical point of view. The Gauss-Bonnet term is the generalised Euler characteristic of a 4 dimensional spacetime. It yields in $D = 4$ a boundary term hence a topological and not dynamical contribution. This is a quite general and elegant fact. Indeed we remind the reader that in a similar fashion the Einstein-Hilbert action in 2 dimensions is related to the usual Euler characteristic of a 2-dimensional manifold. For example in string field theory the Euler characteristic $\chi$ is related to the string coupling constant $g_s$, governing “surface diagrams” in the perturbative regime. In general every spacetime of even dimension $2n$ is accompanied by its generalised Euler characteristic; which we have to add to the gravitational action of a $2n + 1$ manifold in order to preserve uniqueness. Thus for example in 10 dimensions one has the Euler characteristics of 0 (cosmological constant), 2 (Einstein-Hilbert), 4 (Gauss-Bonnet), 6, and 8 dimensional manifolds \cite{14}. So from this discussion it would seem natural to include the
Gauss-Bonnet term in a 5 dimensional spacetime, all the more since we are interested in toy models merging string theory with standard cosmology.

Madore and collaborators have considered this term in order to stabilise the 5th dimension in Kaluza-Klein theories [17] whereas there was a lot of effort in the 80’s to obtain exact solutions in Gauss-Bonnet theories in view to their relevance to quantum gravity corrections of string theory (see for instance [18], [19], [20], [21], [22]). More recently in the context of brane universe it has been shown that the localised graviton zero mode persists in the RS model in the presence of a Gauss-Bonnet term [23], [24], [25]. Cosmological consequences have also been studied in [26]. However, only particular solutions in the bulk have been considered. Here we shall attack the problem in its full generality. We shall first of all find and discuss the full bulk solutions, and then we shall investigate the brane cosmology they induce. Not surprisingly Birkhoff’s theorem will be in the centre of our analysis and its physical consequences.

In the next section we set up the basic ingredients of the problem. In Section III we solve by brute force the field equations and find the general solution for the bulk spacetime. In Section IV we discuss the relevance of the bulk solutions to brane Universe cosmology in 5 dimensions. We conclude in section V.

II. GENERAL SETTING

Consider the following 5-dimensional action,

$$S = \frac{M^3}{2} \int d^5x \sqrt{-g} \left[ R + 12k^2 + \alpha(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \right], \quad (2)$$

where $M$ is the fundamental mass scale of the 5-dimensional theory, $\Lambda = -6k^2$ is the negative bulk cosmological constant and the Gauss-Bonnet coupling constant $\alpha$ of dimension $(\text{length})^2$, which we leave free, is the additional physical parameter of the problem. Setting $\alpha = 0$ we obviously get the usual Einstein-Hilbert action with cosmological constant in 5 dimensions. As we discussed in the introduction, just like the Einstein-Hilbert action with
cosmological constant is unique in 4 dimensions, the gravitational action \((2)\) is unique in 5 dimensions. To put it in a nutshell, \((2)\) is the most general action that will yield second order partial differential equations with respect to the metric components in 5 dimensions. For this reason and for clarity we shall restrict ourselves to 5 dimensions.

Let us now consider a spacetime with constant three-dimensional spatial curvature. A general metric can then be written,

\[
ds^2 = e^{2\nu(t,z)}B(t, z)^{-2/3}(-dt^2 + dz^2) + B(t, z)^{2/3}\left(\frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega^2_{II}\right) \tag{3}
\]

where \(B(t, z)\) and \(\nu(t, z)\) are the unknown component fields of the metric and \(\kappa = 0, \pm 1\) is the normalised curvature of the 3-dimensional homogeneous and isotropic surfaces. We choose to use the conformal gauge in order to take full advantage of the 2-dimensional conformal transformations in the \(t - z\) plane. This is the setup for a cosmological wall or brane-Universe of co-dimension 1. We note on passing that a co-dimension 2 or higher set-up would have lost the two dimensional conformal freedom. We will see in what follows that this freedom is essential for the integrability of the system whether or not we include the Gauss-Bonnet combination.

The field equations we are seeking to solve are found by varying the above action \((2)\) with respect to the background metric and read

\[
e_{\mu\nu} = G_{\mu\nu} - 6k^2 g_{\mu\nu} - \alpha \left[\frac{g_{\mu\nu}}{2}(R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2)
- 2RR_{\mu\nu} + 4R_{\mu\gamma}R_{\nu}^{\gamma} + 4R_{\gamma\delta}R_{\mu\nu}^{\gamma\delta} - 2R_{\mu\gamma\delta\lambda}R_{\nu}^{\gamma\delta\lambda}\right] = 0 \tag{4}
\]

where now the symmetric tensor \(e_{\mu\nu}\) has replaced the usual Einstein tensor \(G_{\mu\nu}\) and is also divergence free. Taking the trace of \((4)\) one can show that for a solution, the Ricci scalar is a multiple of the Lagrangian in \((2)\) (see for instance \([21]\)). Thus the behaviour (in particular singularities) of the scalar curvature \(R\) is shared by the Gauss-Bonnet scalar in \((2)\). Hence not surprisingly we can deduce that although the field equations change radically, spacetime curvature still plays the same physical role for the critical points of the action \((2)\).
III. THE GENERAL SOLUTION FOR THE BULK SPACETIME

Before plunging into the field equations\footnote{For the full field equations see Appendix.} it is rather useful to review the generalisation of Birkhoff’s theorem in the presence of a cosmological constant as it appeared recently in \cite{11}. Furthermore we shall use exactly the same method to derive the general solution.

In this subcase the field equations are obtained setting $\alpha = 0$ in (2) and read

\[ R_{\mu\nu} = -\frac{2\Lambda}{3} g_{\mu\nu}. \]

There are two key ingredients in this method. First of all in order to make use of the $t - z$ conformal symmetries of (3) it is important to pass to light-cone coordinates,

\[ u = \frac{t - z}{2}, \quad v = \frac{t + z}{2}. \tag{5} \]

Secondly taking the combination $R_{tt} + R_{zz} \pm 2R_{tz} = 0$, one obtains the integrability conditions which read\footnote{From now on $B_{,u}$ represents the partial derivative of $B$ with respect to $u$ etc.}

\[ B_{,uu} - 2\nu_{,u} B_{,u} = 0, \tag{6} \]

\[ B_{,vv} - 2\nu_{,v} B_{,v} = 0. \tag{7} \]

Note then that these are ordinary differential equations with respect to $u$ and $v$ respectively and are independent of the physical parameter of the problem, namely, the cosmological constant $\Lambda$. As their name indicates they are directly integrable giving

\[ B = B(U + V) \quad e^{2\nu} = B'U'V'. \tag{8} \]

where $U = U(u)$ and $V = V(v)$ are arbitrary functions of $u$ and $v$, and a prime stands for the total derivative of the function with respect to its unique variable. Using a conformal transformation,

\[ U = \frac{\tilde{z} - \tilde{t}}{2}, \quad V = \frac{\tilde{z} + \tilde{t}}{2} \]
gives that the solution is locally static \( B = B(\tilde{z}) \) and Birkhoff’s theorem is therefore true. Starting from a general time and space dependant metric, spacetime has been shown to be locally static or equivalently that there exists a timelike Killing vector field (here \( \frac{\partial}{\partial \tilde{t}} \)). Note that we did not have to find the precise form of the solution for \( B \). The integrability conditions actually suffice to prove staticity and thus Birkhoff’s theorem. By use of the remaining field equations we can then find the form of \( B \), leading after coordinate transformation to the topological black hole solution \( \boxed{} \). Note that the solution becomes \( \tilde{t} \)-dependent as we cross the event horizon of the black hole. For more details the reader can consult \[11\].

Let us now turn to our case of interest with \( \alpha \neq 0 \). In the presence of the Gauss-Bonnet term we can expect that if the system is indeed integrable then some integrability equation should be reproduced. Putting away technicalities this is the essence of what we shall do here. In analogy to the previous case let us take the combination, \( E_{tt} + E_{zz} + 2E_{t \tilde{z}} = 0 \). On passing to light cone coordinates \( \boxed{} \) we get after some manipulations

\[
\begin{align*}
\left( 9B^{4/3} e^{2\nu} + 36\alpha \kappa B^{2/3} e^{2\nu} + 4\alpha B_{,u}B_{,v} \right) (B_{,uu} - 2\nu_{,u}B_{,u}) &= 0 \\
\left( 9B^{4/3} e^{2\nu} + 36\alpha \kappa B^{2/3} e^{2\nu} + 4\alpha B_{,u}B_{,v} \right) (B_{,vv} - 2\nu_{,v}B_{,v}) &= 0
\end{align*}
\]

Note how the Gauss-Bonnet terms factorise nicely leaving the integrability equations \( \boxed{} \) we had in the absence of \( \alpha \).

Let us neglect for the moment the degenerate case where either \( B_{,u} = 0 \) or \( B_{,v} = 0 \) corresponding to flat solutions \[27\] (see Appendix). For \( B_{,u} \neq 0 \) and \( B_{,v} \neq 0 \) the situation is clear: either we have static solutions and Birkhoff’s theorem holds as in the case above or we will have

\[
e^{2\nu} = \frac{4\alpha (B_{zz}^2 - B_{,\tilde{z}}^2)}{9B^{2/3} (B^{2/3} + 4\alpha \kappa)}
\]

Let us first examine the latter case, that we will call Class I solution. The two remaining field equations \( E_{\chi\chi} = 0 \) and \( E_{tt} - E_{zz} = 0 \) give after some algebra the simple relation,

\[
8\alpha k^2 = 1
\]
This is quite remarkable: if the coupling constants obey this simple relation (11) then the $B$ field is an arbitrary function of space and time. Note in passing that Class I solutions exist in arbitrary dimension $d$ if the fine tuning relation \[ \frac{96\alpha k^2}{(d-1)(d-2)} = 1 \] is satisfied. We can already deduce that Birkhoff’s theorem does not hold for non zero cosmological constant. However in the absence of a cosmological constant it is always trivially true since (11) is impossible. Also we can note from (11) that a positive Gauss-Bonnet constant $\alpha > 0$, as in heterotic string theory, demands a negative cosmological constant and vice-versa. The Class I metric reads,

\[ ds^2 = \frac{4\alpha (B_t^2 - B_z^2)}{9B^{4/3}(B^{2/3} + 4\alpha \kappa)} (-dt^2 + dz^2) + B^{2/3} \left( \frac{d\chi^2}{1 - \kappa \chi^2} + \chi^2 d\Omega_{II}^2 \right) \]  

(12)

under the constraint (11) where we emphasize that $B(t, z)$ is an arbitrary function of $t$ and $z$. To simplify somewhat set $B = R^3$ to get,

\[ ds^2 = \frac{R^2}{\kappa + \frac{4\alpha}{R^2}} (-dt^2 + dz^2) + R^2 \left( \frac{d\chi^2}{1 - \kappa \chi^2} + \chi^2 d\Omega_{II}^2 \right) \]  

(13)

This solution has generically a curvature singularity for $R, z = \pm R, t$. The parameter $\alpha$ is related here to the 5-dimensional cosmological constant via (11). The Class I static solutions are given by,

\[ ds^2 = -\frac{A(R)^2}{\kappa + \frac{4\alpha}{R^2}} dt^2 + \frac{dR^2}{\kappa + \frac{4\alpha}{R^2}} + R^2 \left( \frac{d\chi^2}{1 - \kappa \chi^2} + \chi^2 d\Omega_{II}^2 \right) \]  

(14)

with $A = A(R)$ now an arbitrary function of $R$. Time-dependent solutions for $\alpha > 0$ are only possible for $R^2 < 4\alpha$ and $\kappa = -1$.

In order to obtain $t$ and $z$ dependent solutions it suffices to take the functional $R$ to be a non-harmonic function. Take for instance $R = \exp(f(t) + g(z))$, with $f$ and $g$ arbitrary functions of a timelike and spacelike coordinate respectively. Let us also assume $\kappa = 0$ for simplicity, the Class I metric in proper time reads,

\[ ds^2 = -\frac{A(R)^2}{\kappa + \frac{4\alpha}{R^2}} dt^2 + \frac{dR^2}{\kappa + \frac{4\alpha}{R^2}} + R^2 \left( \frac{d\chi^2}{1 - \kappa \chi^2} + \chi^2 d\Omega_{II}^2 \right) \]  

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(14)

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3 Note however that for a non-zero charge $Q$ and spherical symmetry ($\kappa = 1$) Birkhoff’s theorem is always true as was shown by Wiltshire [19] (see also [20]).

4 For $\alpha < 0$ the situation is interchanged with static solutions possible only for $\kappa = 1$ and $R^2 < -4\alpha$. 

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\[ ds^2 = -d\tau^2 + \frac{4\alpha dg^2}{1 + 4\alpha f^2_{\tau}} + e^{2(f+g)} \left( \frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega^2_{II} \right). \] (15)

Note here again that \( f \) is an arbitrary function of time.

On the other hand if (11) does not hold then Birkhoff’s theorem remains true in the presence of the Gauss-Bonnet terms i.e. \textit{the general solution assuming the presence of a cosmological constant in the bulk and 3 dimensional constant curvature surfaces is static if and only if (11) is not satisfied.} In this case the remaining two equations give the same ordinary differential equation for \( B(U + V) \) which after one integration reads,

\[ B' + 9B^{2/3}(k^2 B^{2/3} + \kappa) + 18\alpha \left( \frac{B'}{9B^{2/3}} + \kappa \right)^2 = 9\mu \] (16)

where \( \mu \) is an arbitrary integration constant. Then by making \( B \) the spatial coordinate and setting \( B^{1/3} = r \) we get the solution discovered and discussed in detail by Boulware-Deser \[18\] (\( \kappa = 1 \)) and Cai \[28\] (\( \kappa = 0, -1 \)).

\[ ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 \left( \frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega^2_{II} \right), \] (17)

where \( V(r) = \kappa + \frac{r^2}{4\alpha} \left[ 1 \pm \sqrt{1 - 8\alpha k^2 + 8\frac{\mu}{r^4}} \right] \), and \( \mu \) plays the role of the gravitational mass. The maximally symmetric solutions are obtained by setting \( \mu = 0 \). There are two AdS branches permitted by the solution for \( \alpha > 0 \) (\[28\], \[18\]). We can have both de-Sitter and Anti-de-Sitter for \( \alpha < 0 \). Generically, as shown by Boulware and Deser \[18\], only one of the branches is physical, the \('+\) branch being classically unstable to small perturbations and yielding a graviton ghost. For \( \alpha > 0 \) and the \('-\) branch there is a black hole singularity at \( r = 0 \), a unique event horizon and asymptotically one approaches the 5-d topological black hole solutions \[12\], \[13\] (see also \[30\]). For a general and thorough analysis of the Gauss Bonnet black hole solutions, and their thermodynamics we refer the reader to \[28\], \[18\]. It is interesting to point out that only the planar black hole \( \kappa = 0 \) obeys the entropy-area formula.

\( ^5\text{We have kept the same label as in (3) for the rescaled time coordinate.} \)

\( ^6\text{Perturbative AdS black holes with } R^2 \text{ curvature terms and their thermodynamics have been discussed in [31])} \)
Indeed it turns out, [28] that the planar black hole shares exactly the same thermodynamic properties as the planar topological black hole ($\alpha = 0$) although the two solutions differ considerably. We will come back to this point in the next section. Furthermore for small $\alpha$ we have, \( V(r) = \kappa + k^2 r^2 (1 + 2\alpha k^2) - \frac{\mu}{r^2} [1 + 2\alpha (2k^2 - \frac{\mu}{4r^2})] + O(\alpha^2) \) and indeed for $\alpha = 0$ we get the usual Kottler solution [12].

Now notice how (11) is a particular ‘end’ point for (17) since the maximally symmetric solution is defined only for $1 \geq 8\alpha k^2$ (for $\alpha < 0$ there is no such restriction). We can deduce in all generality that for $1 \geq 8\alpha k^2$ there is a unique static solution (17). When (11) is satisfied and $\mu = 0$, the two branches coincide and $V = \kappa + \frac{r^2}{4\alpha}$, which is then a particular Class I solution (12) for the value $A(r) = V(r)$. For $1 \leq 8\alpha k^2$ no solutions exist.

### IV. BRANE WORLD COSMOLOGY

Having evaluated the general solution in the bulk we now consider a 4-dimensional 3-brane where matter is confined. We furthermore suppose following the symmetries of our metric (3) that matter on the brane is modelled by a perfect fluid of energy density $\rho$ and pressure $p$. The brane is fixed at $z = 0$, and the energy-momentum tensor associated with the brane takes the form,

\[
T^{(b)}_{\mu\nu} = \frac{\delta(z)}{\sqrt{g_{zz}}} \text{diag}(-\rho(t), p(t), p(t), p(t), 0).
\]

and the field equations read,

\[
e_{\mu\nu} = M^{-3}T^{(b)}_{\mu\nu} \delta(z).
\]

We will assume $Z_2$ symmetry across the location of the brane at $z = 0$, and set $M^3 = 1$ for the time being.

Now before proceeding there are three important points to take into account. First of all the Israel junction conditions are no longer valid since we have included the Gauss-Bonnet

\[\text{[29]}\]
term in the gravitational action. Although the Gauss-Codazzi integrability conditions are universal for any spacelike or timelike hypersurface (see for instance \[32\]), the Israel junction conditions have to be generalised in order to take into account the addition of the Gauss-Bonnet term \[33\] in the gravitational action \(2\). So in order to evaluate the brane junction conditions we choose to bifurcate the Israel junction conditions, integrating the field equations \(18\) on an infinitely small interval across the brane location at \(z = 0\).

The second important point is that since the field equations are of second order we will encounter at most second derivatives of \(z\) and therefore the metric component fields have to be continuous. Indeed first order derivatives contain a jump in the metric given by means of the Heaviside distribution whereas second order derivatives contain a Dirac distribution at \(z = 0\), to be matched with the brane energy-momentum tensor \(18\). Note that had we considered any other combination of quadratic curvature terms in the action, the situation would have been different. The good behavior of the Gauss-Bonnet combination is coherent with the fact that \(2\) is unique in 5 dimensions, just as ordinary Einstein Hilbert gravity plus cosmological constant is unique in 4 dimensions. Hence we can expect a regular gravity theory and hence regular boundary conditions.

The final remark turns out to be crucial for the correct evaluation of the junction conditions and has been a source of confusion in the related literature. Indeed note that, in \(18\), first order derivatives with respect to \(z\), multiplying second order derivatives of the metric functions, are always encountered as squares. Thus, the first order part involving Heaviside distributions turns out to be equal to +1 everywhere except at \(z = 0\). Although this is a removable discontinuity it occurs in the location of the Dirac distribution and hence the junction conditions are not obtained by simply matching the Dirac distributions in the field equations \(18\). It is imperative that we integrate over an infinitely small interval across the brane location \(8\) and then take the limit. In doing so the \(t – t\) and \(\chi – \chi\) components of \(18\) give respectively the energy density and pressure on the brane:

\[8\]We thank Stephen Davis for discussions on this point
\[ \rho(t) = -\frac{2}{9} e^{-3\nu} B^{-2} B_z (I_1 + \frac{8\alpha}{3} B_{z}^2) \]  

(19)

\[ p(t) = \frac{2}{9} e^{-3\nu} B^{-2} \left[ I_1 (B_{\nu z} - \frac{1}{2} B_z) + B_z [6e^{2\nu} B^{4/3} + 8\alpha B(B_{tt} - \nu t B_t) - \frac{8\alpha}{3} B_{t}^2] \right] \]  

(20)

with \( I_1 = 9B^{4/3} e^{2\nu} + 36\alpha k B^{2/3} e^{2\nu} - 4\alpha (B_z^2 - B_{z}^2) \) (see appendix). All the functions in the RHS of (19) and (20) are evaluated at \( z = 0^+ \) since we have assumed \( Z_2 \)-symmetry. The domain wall case (\( \rho = -p \)) corresponding to a Poincaré invariant brane has been treated in [24], [25].

We will first focus on the Class II solution in the bulk. In this case the junction conditions remain invariant under the conformal boost \( u \to f(u), \nu \to f(\nu) \) just as for \( \alpha = 0 \). Therefore for a fixed brane (or boundary) at \( z = 0 \) there is a single degree of freedom \( U' \) or \( V' \) for the bulk spacetime which is evolving in time. By virtue of Birkhoff’s theorem this is equivalent to taking a moving brane (or boundary) in the static black hole background. We make use of this fact now to pass on to the static bulk configuration (for a detailed discussion see (17)).

Consider a brane Universe observer. The expansion parameter (or wall’s trajectory) reads, \( R(\tau) = B^{1/3}(t, 0) \) whereas proper time \( \tau \) is given by \( d\tau = e^{\nu(t,0)} B^{-1/3}(t,0) dt \). For the solution (17) we have relations (8) and for example the Hubble expansion rate is given by,

\[ H = \frac{1}{R} \frac{dR}{d\tau} = \frac{(U' + V')B'}{6e^{\nu} B^{2/3}} \]

First, using (8), (16), (19) and (20), we may obtain the standard conservation equation on the brane:

\[ \frac{d\rho}{d\tau} + 3H(p + \rho) = 0 \]  

(21)

which is a consequence of the Bianchi and Bach-Lanczos identities for the Einstein tensor and Gauss-Bonnet terms respectively in (4). Then, using (8), (16) and (19) we get the generalised Friedmann equation:

\[ \left( \frac{\rho}{16\alpha} \right)^2 = \left( H^2 + \frac{V(R)}{R^2} \right)^3 + C \left( H^2 + \frac{V(R)}{R^2} \right)^2 + \frac{1}{4} C^2 \left( H^2 + \frac{V(R)}{R^2} \right) \]  

(22)
where we have defined

\[ C = \frac{3}{4\alpha} \left(1 - 8\alpha k^2 + \frac{8\alpha \mu}{R^4}\right)^{1/2} \]

and \( V(R) \) is the black hole potential (17). This equation relates the brane trajectory \( R(\tau) \) with the energy density of the brane. Equations (22) and (21) fix the unique degree of freedom in the bulk \( R = R(\tau) \). In Einstein-Hilbert brane cosmology where \( \alpha = 0 \), \( \rho^2 \) depends linearly on \( H^2 + \frac{h(R)}{R^2} \), where \( h(R) \) is the Kottler potential (1). Indeed taking heuristically the limit of small Gauss-Bonnet coupling \( (\alpha \to 0) \) in (22) yields the usual Friedmann equation for a 3-brane embedded into a (Einstein-Hilbert) five-dimensional bulk (see for instance [10]):

\[ H^2 = \frac{\rho^2}{36} - \frac{\kappa}{R^2} - k^2 + \frac{\mu}{R^4} + O(\alpha) \]  

(23)

for the lower ‘+’ sign in (22). This sign corresponds to the stable branch for the black hole solution (17) as demonstrated by Boulware and Deser [18]. At early times, (23) leads to unconventional cosmology \( H^2 \propto \rho^2 \) [4], in contrast with standard four-dimensional cosmology where \( H^2 \propto \rho \). Now, note from (22) how Gauss-Bonnet gravity yields \( \rho^2 \) depending also on \( (H^2 + \frac{V(R)}{R^2})^2 \) and \( (H^2 + \frac{V(R)}{R^2})^3 \). In general the three powers in the RHS of (22) are expected to dominate successively during the cosmological evolution of the universe. This may have interesting consequences for early as for late time brane cosmology. Generically, the cosmological evolution resulting from (22) strongly depends on the epoch under consideration and on the order of magnitude of the bulk lagragian parameters \( \alpha \) and \( k \). A detailed study lies beyond the scope of a simple application of Birkhoff’s theorem to higher dimensional theories and will be undertaken in future work.

As an illustrative example we consider here the late time cosmology of a spatially flat \( (\kappa = 0) \) expanding universe, for the particular relation between the bulk parameters

\[ 8\alpha k^2 = -3 \]  

(24)

Note that for \( \alpha \to 0 \) the upper ‘+’ branch of (17) yields a singular negative term in the RHS of (22), which ties in nicely with the results of [18] showing the instability of this branch.
We will consider the ‘−’ (upper) sign in (22) and expand up to \(O(\frac{1}{R^4})\) (large scale factor).

In this particular case, both the effective cosmological constant on the brane and the black hole term \(\mu\) in \(\frac{d}{dt}\) vanish. Equation (22) then reduces to

\[
\frac{4}{3} \alpha H^6 + H^4 - \frac{\mu}{2 R^4} H^2 = \frac{\rho^2}{192 \alpha} + O\left(\frac{1}{R^8}\right)
\]

At sufficiently late times, the leading contribution for the brane energy density is \(H^2 \propto \rho\), as in standard four-dimensional cosmology, without the need to introduce any brane tension.

For \(\alpha > 0\) we have a positive cosmological constant in the five-dimensional bulk action, but however the bulk space-time is AdS, as may be seen through the expression of the black-hole potential \(V(R)\) (17). Domain wall solutions (\(\rho = -p\)) without brane tension and in the presence of higher curvature terms, have been studied in [25]. It has been proved that they may allow for a massless normalisable four-dimensional graviton. We can regard (24) as a special relation involving the bulk parameters in contrast to the Randall-Sundrum relation relating brane and bulk parameters. The catch however is that we used the ‘+’ (upper) branch of the solution in (17), which is unstable according to [18]. It would be interesting to study the cosmology of similar cases in higher dimensions with higher order Euler densities, which may be stable [25].

Let us now consider the case \(8 \alpha k^2 = 1\) and \(\mu = 0\). Then, (22) becomes:

\[
H^2 = \left(\frac{k^2}{2} \rho\right)^{2/3} - 2k^2 - \frac{\kappa}{R^2}
\]

This is the result we obtain for a Class I solution in the bulk, by directly using (10) and (15) (extended for \(\kappa \neq 0\)) in (19). In this case, with (20), one also obtains (21). Again for Class I solutions, there is only one dynamical degree of freedom in brane cosmology, namely the function \(f(\tau)\) appearing in (13).

\[\text{This term is usually referred to as the 'dark radiation term' due its 'radiation' like behavior. This is however misleading since the bulk solution does not radiate (Birkhoff’s theorem) and there are furthermore no 'radiation' like particles in the bulk.}\]
Now it is interesting to study the late time cosmology of an expanding brane-universe resulting from the generalised Friedmann equation \([22]\). In doing so, we take into account the tension (vacuum energy) \(T\) of the brane \((\rho \rightarrow T + \rho)\), and keep only linear terms in \(\rho\) and \(\frac{\mu}{R}\) (large scale factor). We concentrate on the physical Class II solutions (‘-’ branch in \((17)\)), with a negative cosmological constant in the bulk action and with \(\alpha > 0\) (as required by string theory), so that \(0 \leq 8\alpha k^2 << 1\). For a zero effective cosmological constant on the brane, one has to impose the modified Randall-Sundrum fine-tuning condition:

\[
T = \left[\frac{1 - \sqrt{1 - 8\alpha k^2}}{\alpha}\right]^{\frac{1}{2}} (2 + \sqrt{1 - 8\alpha k^2})
\] (25)

which indeed allows for a Kaluza-Klein zero-mode localized on the brane and a finite volume element, \(m_{Pl} < \infty\) \([25]\). Then, up to \(O(\rho)\) and \(O(\frac{\mu}{R})\), \((22)\) gives

\[
H^2 = \frac{m^2_{Pl}}{3} \rho - \frac{\kappa}{R^2} + (2 + \sqrt{1 - 8\alpha k^2})^{-1} \frac{\mu}{R^4}
\] (26)

where the reduced four-dimensional Planck mass is given by

\[
m^2_{Pl} = M^{-3} \frac{2}{2\sqrt{\alpha}} \left(\frac{1 - \sqrt{1 - 8\alpha k^2}}{2 - \sqrt{1 - 8\alpha k^2}}\right)^{1/2}\]
(27)

and we have restored the fundamental mass scale \(M\) of the 5-dimensional theory. Note that the “cosmological” Planck mass, as defined above, agrees with the four-dimensional Newton constant obtained through estimation of the static gravitational potential at long distances along the brane \([24]\).

Repeating the same procedure as above in the special case \(8\alpha k^2 = 1\), one finds that \((25)\) and \((27)\) still hold, while the \(\mu\)-dependent term in the Friedmann equation \((22)\) turns out to be in \(\frac{\mu^2}{R^6}\). Thus for \(0 < 8\alpha k^2 \leq 1\), one sees from \((27)\) that:

\[
M^3 < m^2_{Pl} k
\]

whereas strict equality holds in the absence of the Gauss-Bonnet term (which is the leading quantum gravity correction term). Hence for fixed 4-dimensional Planck mass and cosmological constant in the bulk, “quantum corrections” for gravity in the bulk tend to decrease the fundamental mass scale \(M\) of the 5-dimensional theory.
V. CONCLUSIONS

In this paper we have studied Gauss-Bonnet brane cosmology in a 5 dimensional space-time. Our main motivation for including the Gauss-Bonnet term is that the usual 5 dimensional gravitational action (2) is then unique [14] as we noted in the Introduction. Furthermore the Gauss-Bonnet coupling constant $\alpha$ provides a window to the leading quantum gravity correction coming from string theory. Throughout our analysis the technical difficulties induced by the inclusion of the higher order curvature term where seen to be overcome quite elegantly for the Gauss-Bonnet combination.

Indeed starting from a homogeneous and isotropic 3-space in constant bulk curvature we found the general spacetime solutions to the field equations. Under a particular relation between the bulk cosmological constant and the Gauss-Bonnet coupling, a space and time dependant solution (Class I) of the field equations was found. If however this special relation is not satisfied then the unique solution is the black hole solution discovered and discussed in [18], [21], [19] and [28]. Therefore quite elegantly Birkhoff’s staticity theorem holds and all its interesting properties go through just like in the ordinary $\alpha = 0$ case [11].

As a concrete application to the generalised Birkhoff’s theorem we studied brane cosmology in 5 dimensions for a 4-dimensional perfect-fluid brane. As it turns out special care has to be taken when deriving the generalised Friedmann equations for the brane. Although there are no ill defined distributional products (unlike any other higher order curvature theory) a limiting procedure has to be undertaken in order to obtain the junction conditions. On doing so it is found that the generalised Friedmann equation involves a third order polynomial in $H^2$ which yields drastic changes to conventional ($\alpha = 0$) brane cosmology. Also generically Gauss-Bonnet gravity tends to decrease the 5 dimensional fundamental mass scale, which is interesting if we interpret the Gauss Bonnet term as the leading string quantum gravity correction. It is now important to investigate whether the Gauss-Bonnet term can give ordinary late time FLRW cosmology without the usual fine tuning conditions needed in Einstein-Hilbert brane cosmology. Work in this direction is under way.
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Note added: [34] has been brought to our attention, which appeared quasi-simultaneously to our paper and discusses some related issues.

APPENDIX: FIELD EQUATIONS

The field equations obtained from (4) read,

\[ \varepsilon_{\chi \chi} = -\frac{1}{9e^{4\nu}}(B_{tt} - B_{zz})[12e^{2\nu}B^{2/3}(2\alpha\kappa + B^{2/3}) - I_1] \]

\[ + \frac{8\alpha}{9e^{4\nu}}[B_{zz}B_{tt} + (\nu_t^2 - \nu_z^2)(B^2_t - B^2_z) + B_{tz}(2\nu_t B_z + 2\nu_z B_t - B_{tt})] \]

\[ - (B_{tt} + B_{zz})(\nu_t B_t + \nu_z B_z)] \]

\[ - \frac{20\alpha}{81e^{4\nu}B^2}(B^2_t - B^2_z)^2 - \frac{I_1}{9e^{4\nu}}(\nu_{tt} - \nu_{zz}) \]

\[ - 6k^2B^{2/3} - \frac{\kappa}{3e^{2\nu}B^{4/3}}[I_1 - 6e^{2\nu}B^{2/3}(6\alpha\kappa + B^{2/3})] = 0 \quad (A1) \]

\[ \varepsilon_{tt} - \varepsilon_{zz} = \frac{I_1}{9e^{2\nu}B^{7/3}}(B_{tt} - B_{zz}) + \frac{12e^{2\nu}k^2}{B^{2/3}} \]

\[ + \frac{6\kappa e^{2\nu}}{B^{4/3}} - \frac{8\alpha}{27B^{10/3}e^{2\nu}}(B^2_t - B^2_z)^2 \]

\[ - \frac{4\alpha\kappa}{3B^{8/3}}(B^2_t - B^2_z) = 0 \quad (A2) \]

The integrability conditions \( \varepsilon_{tt} + \varepsilon_{zz} \pm 2\varepsilon_{tz} \) are:

\[ I_1(B_{tt} + B_{zz} + 2B_{tz} - 2\nu_t B_t - 2\nu_z B_z + 2\nu_t B_z + 2\nu_z B_t + B_{tt}\nu_z) = 0 \quad (A3) \]

\[ I_1(B_{tt} + B_{zz} - 2B_{tz} - 2\nu_t B_t - 2\nu_z B_z - 2\nu_t B_z - 2\nu_z B_t + B_{tt}\nu_z) = 0 \quad (A4) \]

where \( I_1 = 9B^{4/3}e^{2\nu} + 36\alpha\kappa B^{2/3}e^{2\nu} - 4\alpha(B^2_{t-z} - B^2_{z-t}) \).

17
In the degenerate case where either $B_{,u} = 0$ or $B_{,v} = 0$ (Class I solutions according to Taub) there are now two subcases. Either we obtain the Class I solution of Taub [27] which is simply flat Minkowski spacetime or we obtain,

$$ds^2 = e^{2\nu}(-4\alpha\kappa)^{-1}(-dt^2 + dz^2) + (-4\alpha\kappa)\frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega_{II}^2$$  \hspace{1cm} (A5)

under once again (11). Note once more that $\nu(t, z)$ is an arbitrary function of $t$ and $z$ and planar symmetry is not permitted.
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