On High Moments of Strongly Diluted Large Wigner Random Matrices *

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Abstract

We consider a dilute version of the Wigner ensemble of \( n \times n \) random real symmetric matrices \( H^{(n,\rho)} \), where \( \rho \) denotes an average number of non-zero elements per row. We study the asymptotic properties of the moments \( M_{2s}^{(n,\rho)} = \mathbb{E} \text{Tr}(H^{(n,\rho)})^{2s} \) in the limit of infinite \( n, s \) and \( \rho \).

Our main result is that in the limit of infinite \( n \) and \( \rho \), the sequence \( M_{2s}^{(n,\rho)} \) with \( s_n = \chi \rho_n, \rho_n \to \infty \) and \( \rho_n = o(n^{1/4}) \) is asymptotically close to a sequence of numbers \( \hat{m}_{s}^{(\rho)} \), where \( \hat{m}_{s}^{(\rho)} \) are determined by an explicit recurrence that involves the second and the fourth moments of the random variables \( (H^{(n,\rho)})_{ij} \). This recurrent relation generalizes the one that determines the moments of the Wigner’s semicircle law given by \( \hat{m}_{s} = \lim_{\rho \to \infty} m_{s}^{(\rho)}, s \in \mathbb{N} \).

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1 Introduction and main result

Spectral properties of high dimensional random matrices attract considerable and constantly increasing interest. This interest is motivated by deep and fruitful relations of random matrices with theoretical physics as well as by their strong connection with various branches of modern mathematics (see monographs [1, 20] and [22]).

The spectral theory of random matrices of infinitely increasing dimensions was started by the works of E. Wigner, where the ensemble of real symmetric matrices of the form

\[
(A^{(n)})_{ij} = \frac{1}{\sqrt{n}}a_{ij}
\]

(1.1)

was studied and the semi-circle law for the limiting eigenvalue distribution of (1.1) was proved in the limit \( n \to \infty \) [30]. The random matrix entries of \( A^{(n)} \) are given by jointly independent

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random variables \( \{a_{ij}, i \leq j\} \) that have all moments finite and the odd moments zero. This ensemble is known at present as the Wigner ensemble of random matrices. The convergence of the eigenvalue distribution of \( A^{(n)} \) to a non-random limit with the density of the semi-circle form is referred at present as the semicircle (or Wigner) law for random matrix ensembles.

The semicircle law was generalized in many aspects. One group of generalizations concern the classes of probability distributions of the elements \( a_{ij} \), another one is related with the studies of the spectral norm of \( A^{(n)} \) and other local properties of the eigenvalue distribution at the border of the limiting spectrum or inside of it (see [1, 20, 22] and references therein).

A large groups of works is related with various generalizations of the form of random matrices under consideration. In the present paper we study one of such generalizations of the Wigner ensemble given by the ensemble of dilute random matrices. We consider a family of real symmetric random matrices \( \{H^{(n,\rho)}\} \) whose elements are determined by equality

\[
\left( H^{(n,\rho)} \right)_{ij} = a_{ij} b_{ij}^{(n,\rho)}, \quad 1 \leq i \leq j \leq n, \tag{1.2}
\]

where \( A = \{a_{ij}, 1 \leq i \leq j\} \) is an infinite family of jointly independent identically distributed random variables and \( B_n = \{b_{ij}^{(n,\rho)}, 1 \leq i \leq j \leq n\} \) is a family of jointly independent between themselves random variables that are also independent from \( A \). We denote by \( E = E_n \) the mathematical expectation with respect to the measure \( \mathbf{P} = \mathbf{P}_n \) generated by random variables \( \{A, B_n\} \). We assume that the probability distribution of random variables \( a_{ij} \) is symmetric and denote their even moments by

\[
V_{2l} = E(a_{ij})^{2l}, \quad l \geq 1.
\]

Random variables \( b_{ij}^{(n,\rho)} \) are proportional to the Bernoulli ones,

\[
b_{ij}^{(n,\rho)} = \frac{1}{\sqrt{\rho}} \begin{cases} 1 - \delta_{ij}, & \text{with probability } \rho/n, \\ 0, & \text{with probability } 1 - \rho/n, \end{cases} \tag{1.3}
\]

where \( \delta_{ij} = \delta_{i,j} \) is the Kronecker \( \delta \)-symbol. In the case when the dilution parameter \( \rho \) is equal to \( n \), one gets the Wigner ensemble of real symmetric random matrices \( A_n \) (1.1). Let us note that the random matrix \( B^{(n,\rho)} \) with the entries \( \sqrt{\rho}b_{ij} (1.3) \) can be regarded as the adjacency matrix of the Erdős-Rényi random graph [4]. In this interpretation, the dilution parameter \( \rho \) represents the average degree of a given vertex of this graph.

The interest to this kind of generalizations of Wigner ensemble was originated by theoretical physics works (see, for instance, [21] and [24] and also [14] for more references), where the statistical mechanical properties of large systems with broken interaction were considered. This kind of random matrices is also important in the studies of various mathematical models, such as random graphs [5, 7, 19], Laplace operators on random graphs [15] and matrix models related with Laplace operator on graphs [10] and others.

In the present paper we study the asymptotic behavior of the moments of \( H^{(n,\rho)} \) given by expression

\[
M_{2s}^{(n,\rho)} = E \left( \text{Tr} \left( H^{(n,\rho)} \right)^{2s} \right). \tag{1.4}
\]

The study of the moments of random matrices is important in many applications and was initiated by the pioneering works of E. Wigner [30]. In particular, the semicircle law was proved initially by the studies of the moments \( M_{2s}^{(n,n)} \) in the limit of infinite \( n \) and given \( s \).
The next in turn step was related with the studies of the moments \( M^{(n,n)}_{2s} \) in the limit \( n \to \infty \) with infinitely increasing \( s = s_n \). The asymptotic upper bound of \( M^{(n,n)}_{2s} \) allows one to estimate the spectral norm of \( A^{(n)} \) \([2, 7, 8]\) as well as to consider the local spectral properties of these matrices at the border of the limiting spectrum \([25, 26, 28]\). In particular, in these studies, the Tracy-Widom law for random matrices \( A^{(n)} \) with \( a_{ij} \) normally distributed is shown to be true in the general case of arbitrary distributed \( a_{ij} \) \([28]\). The latter result partly confirmed the universality conjecture for local spectral properties of random matrices at the border of the limiting spectrum \([20, 23]\) for the history and discussion.

The high moments of large dilute random matrices were studied in \([13]\) in the asymptotic regime when \( \rho = \rho_n = O(n^\alpha) \) with \( 2/3 < \alpha < n \). It was proved that the upper bound of the moments \( M^{(n,\rho_n)}_{2s_n} \) with \( s_n = \chi n^{2/3} \), \( \chi > 0 \) is the same as the one of the moments of the Wigner random matrices and that the limiting expressions for them coincide. This fact can be regarded as an evidence of the universal behavior of the local eigenvalue statistics for weakly dilute random matrices, i.e. when the dilution parameter \( \rho \) is sufficiently large. In the present paper we study the complementary asymptotic regime given by the case of strongly dilute random matrices, i.e. when the dilution parameter \( \rho_n \) tends to infinity as \( n \to \infty \) but with much lower range \( \alpha, \alpha < 1/4 \).

Our main result is given by the following statement.

**Theorem 1.1.** Assume that random variables \( a_{ij} \) are bounded with probability 1,
\[
|a_{ij}| \leq U. \tag{1.5}
\]
There exists \( \chi_0 > 0 \) such that for any given \( 0 < \chi < \chi_0 \), in the limit
\[
n, \rho_n \to \infty, \quad \rho_n = o(n^{1/4}), \quad s_n = \chi \rho_n, \tag{1.6}
\]
the following asymptotic relation holds,
\[
M^{(n,\rho_n)}_{2s_n} = nm^{(\rho)}_{s_n}(1 + o(1)), \quad n \to \infty, \tag{1.7}
\]
where the sequence \( \{m^{(\rho)}_s, s = 0, 1, 2, \ldots\} \) is given by equality
\[
m^{(\rho)}_s = \sum_{r=0}^{s} S(s,r),
\]
and the numbers \( \{S(k,r), k \geq 1, r \geq 1\} \) are determined by recurrent relations
\[
S^{(\rho)}(k,r) = V_2 \sum_{u=0}^{k-r} \sum_{v=0}^{u} S^{(\rho)}(u,v) S^{(\rho)}(k-u-1, r-1)
+ \frac{V_1}{\rho} \sum_{u=0}^{k-r} (r-1) \sum_{v=0}^{u} (v+1) S^{(\rho)}(u,v) S^{(\rho)}(k-u-2, r-2), \quad k \geq r \geq 2 \tag{1.8a}
\]
and
\[
S(k,1) = V_2 \sum_{v=0}^{k-1} S(k-1,v) \tag{1.8b}
\]
with initial condition \( S(k,0) = \delta_{k,0} \).
Remark 1. Let us note that relation (1.7) does not mean that the sequence $nm(\rho_n)$ has a limit when $n, \rho_n \to \infty$ and $s_n = \chi \rho_n$. This assertion can be considered as a conjecture that we put forward (see more discussion at the end of Section 5).

Remark 2. We consider the asymptotic regime (1.6) not to overload the paper with cumbersome computations. With some more work performed, relation (1.7) can be proved in the limit when $\rho_n \to \infty$ and $\rho_n = o(n^{1/2})$. Moreover, one can expect that Theorem 1.1 remains valid in the asymptotic regime when $\rho_n = n^\alpha$ with $0 < \alpha < 2/3$ that is complementary to the one studied in [13].

Also let us point out that our main aim of this paper is to prove relations (1.7) and (1.8) and therefore we choose the simplest class of random variables $a_{ij}$ determined by the condition (1.5). Theorem 1.1 could be valid under less restrictive conditions than (1.5). We postpone this question to subsequent publications.

Remark 3. Regarding (1.8) in the limit $\rho \to \infty$, it is easy to see that the numbers $\hat{m}_s = \lim_{\rho \to \infty} m_s^{(\rho)}$ exist and verify the following recurrent relation,

$$\hat{m}_s = V_2 \sum_{j=0}^{s-1} \hat{m}_{s-j-1} \hat{m}_j, \quad s \geq 1, \quad \hat{m}_0 = 1. \tag{1.9}$$

These numbers determine the moments of the semicircle (or Wigner) distribution [30].

Let us describe the main ingredients of the approach we use to prove Theorem 1.1. It is based on the method of the study of high moments of Wigner random matrices developed by Ya. Sinai and A. Soshnikov in papers [25, 26]. This method was completed in paper [18] and further modified and improved in papers [11, 13]. Following the original E. Wigner’s idea, it is proposed to consider the right-hand side of (1.3) as the weighted sum over paths of $2s$ steps. In the case of dilute random matrices, we can write that

$$M_{2s}^{(n,\rho)} = \sum_{i=1}^{n} \mathbb{E} \left( \hat{H}^{(n,\rho)} \right)_{ii}^{2s} \sum_{\mathcal{I}_{2s} \in \mathbb{I}_{2s}(n)} \Pi_{n,b}(\mathcal{I}_{2s}) = \sum_{\mathcal{I}_{2s} \in \mathbb{I}_{2s}(n)} \Pi_{a}(\mathcal{I}_{2s}) \Pi_{b}(\mathcal{I}_{2s}), \tag{1.10}$$

where the sequence $\mathcal{I}_{2s} = (i_0, i_1, \ldots, i_{2s-1}, i_0), i_k = \{1, 2, \ldots, n\}$ is regarded as a closed path of $2s$ steps $(i_{t-1}, i_t)$ with the discrete time $t \in [0, 2s]$. We will also say that $\mathcal{I}_{2s}$ is a trajectory of $2s$ steps. The set of all possible trajectories of $2s$ steps over $\{1, \ldots, n\}$ is denoted by $\mathbb{I}_{2s}(n)$. The weights $\Pi_{a}(\mathcal{I}_{2s})$ and $\Pi_{b}(\mathcal{I}_{2s})$ are determined as the mathematical expectations of the products of corresponding random variables,

$$\Pi_{a}(\mathcal{I}_{2s}) = \mathbb{E} \left( a_{i_0i_1} \cdots a_{i_{2s-1}i_0} \right), \quad \Pi_{b}(\mathcal{I}_{2s}) = \mathbb{E} \left( b_{i_0i_1} \cdots b_{i_{2s-1}i_0} \right). \tag{1.11}$$

Here and below, we omit the superscripts in $b_{ij}^{(n,\rho)}$ when no confusion can arise.

The proof of Theorem 1.1 consists of two parts. To give an immediate image of the matter, let us say that one can represent a trajectory $\mathcal{I}_{2s}$ as a multi-graph with $2s$ edges. Postponing rigorous definitions to the next section, we can say that the set of all trajectories can be separated into two classes, the family of trajectories $\mathcal{I}_{2s}$ of the tree-type structure, i. e. those whose graphs are trees and the remaining part $\hat{\mathcal{I}}_{2s}$ of trajectories that are not of the tree-structure.
On the first stage of the proof of Theorem 1.1, we show that the contribution of the trajectories $I_{2s}$ is vanishing with respect to the contribution of the tree-type trajectories $I_{2s}$. On the second stage, we prove that the leading contribution to $(1.10)$ is given by a certain sub-class of tree-type trajectories; these trajectories are such that in their graphs each edge is passed two or four times when counting them in both directions. The total weight of this class of trajectories is determined by the sequence $m_{s}^{(p)}$ (1.8).

Finally, let us say that Theorem 1.1 proves the conjecture put forward in [13]. Its relations with the universality conjecture are discussed in Section 6.

2 Trajectories, walks and graphical representations

In the present section we characterize the classes of trajectories with the help of notions of [25, 26] and [13, 16] and their modifications as well. Each trajectory $I_{2s}$ generates in a natural way a walk

$$W_{2s} = W_{2s}^{(I_{2s})} = \{W(t), t \in [0,2s]\}, \quad \text{where} \quad [0,2s] = \{0,1,2,\ldots,2s\},$$

that we determine as a sequence of $2s + 1$ symbols (or equivalently, letters) from an ordered alphabet, say $A = \{\alpha_1, \alpha_2, \ldots\}$. The walk $W_{2s}^{(I_{2s})}$ is constructed with the help of the following recurrence rules [16]. Given a trajectory $I_{2s}$, we write that $I_{2s}(t) = i_t, t \in [0,2s]$ and consider a subset $U(I_{2s}; t) = \{I_{2s}(t'), 0 \leq t' \leq t\} \subseteq \{1,2,\ldots,n\}$. We denote by $|U(I_{2s}; t)|$ its cardinality. Then

1) $W_{2s}(0) = \alpha_1$;
2) if $I_{2s}(t + 1) \notin U(I_{2s}; t)$, then $W_{2s}(t + 1) = \alpha_{|U(I_{2s}; t)| + 1}$;

if there exists $t' \leq t$ such that $I_{2s}(t + 1) = I_{2s}(t')$, then $W_{2s}(t + 1) = W_{2s}(t')$.

For example, $I_{16} = (5,2,7,9,7,1,5,2,7,9,7,2,7,2,7,1,5)$ produces the walk

$$W_{16} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_2, \alpha_3, \alpha_2, \alpha_3, \alpha_5, \alpha_1).$$

(2.1)

We say that the pair $(W_{2s}(t - 1), W_{2s}(t))$ represents the $t$-th step of the walk $W_{2s}$, and that $\alpha_1$ represents the root of the walk $W_{2s}$.

Two trajectories $I_{2s}$ and $I'_{2s}$ are equivalent if $W_{2s}^{(I_{2s})} = W_{2s}^{(I'_{2s})} = W_{2s}$. We denote by $C_{W_{2s}}$, the corresponding class of equivalence. It is clear that

$$|C_{W_{2s}}| = n(n-1)\cdots(n-|U(I_{2s}; 2s)|+1).$$

(2.2)

Given $W_{2s}$, one can introduce a graphical representation $g(W_{2s}) = (V_g, E_g)$ that can be considered as a kind of multigraph with the set of vertices $V_g = \{\alpha_1, \ldots, \alpha_{|U(I_{2s}; 2s)|}\}$ and the set $E_g$ of $2s$ oriented edges (or equivalently, arcs) labelled by $t \in \{1,\ldots,2s\}$. To describe the properties of $g(W_{2s})$ in general situations, we will use greek letters $\alpha, \beta, \gamma, \ldots$ instead of the symbols from $A$. In this case, the root of the walk will be denoted by $\varnothing$. By a slight abuse of terminology, we refer to $g(W_{2s})$ as to the graph of the walk $W_{2s}$.

Let us define the current multiplicity of the couple of vertices $\{\beta, \gamma\}, \beta, \gamma \in V_g$ up to the instant $t$ by the following variable

$$m_{W}^{(\beta, \gamma)}(t) = \#\{t' \in [1,t] : (W(t' - 1), W(t')) = (\beta, \gamma) \quad \text{or} \quad (W(t' - 1), W(t')) = (\gamma, \beta)\}$$
and say that \(m_W^{(\beta, \gamma)}(2s)\) represents the total multiplicity of the couple \(\{\beta, \gamma\}\).

The probability law of \(a_{ij}\) being symmetric, the weight of \(I_{2s}\) \((1.11)\) is non-zero only in the case when \(I_{2s}\) is such that in the corresponding graph of the walk \(W_{2s}^{(I_{2s})}\) each couple \(\{\alpha, \beta\}\) has an even multiplicity \(m_W^{(\alpha, \beta)}(2s) = 0(\mod 2)\). We refer to the walks of such trajectories as to the even closed walks \([25]\) and denote by \(W_{2s}\) the set of all possible even closed walks of \(2s\) steps. In what follows, we consider the even closed walks only and refer to them simply as to the walks.

It is natural to say that the pair \((W_{2s}(t-1), W_{2s}(t)) = s_t\) represents the step of the walk number \(t\). Given \(W_{2s} \in \mathbb{W}_{2s}\), we say that the instant of time \(t\) is marked \([25]\) if the couple \(\{\alpha, \beta\} = \{W_{2s}(t-1), W_{2s}(t)\}\) has an odd current multiplicity \(m_W^{(\alpha, \beta)}(t) = 1(\mod 2)\). We also say that the corresponding step \(s_t\) and the edge \(e_t\) of \(g(W_{2s})\) are marked. All other steps and edges are called the non-marked ones. Regarding a collection of the marked edges \(E_g\) of \(g(W_{2s})\), we consider a multigraph \(\bar{g}_s = (\bar{V}_g, \bar{E}_g)\). Clearly, \(\bar{V}_g = V_g\) and \(|\bar{E}_g| = s\). It is useful to keep the time labels of the edges \(\bar{E}_g\) as they are in \(E_g\). Given two edges \(e' = e_t\) and \(e'' = e_{t''}\) such that \(t' < t''\), we write that \(e' < e''\). Sometimes we denote \(t' = t(e')\).

Any even closed walk \(W_{2s} \in \mathbb{W}_{2s}\) generates a sequence \(\theta_{2s}\) of \(s\) marked and \(s\) non-marked instants. Corresponding sequence of \(2s\) signs \(+\) and \(-\) is known to encode a Dyck path of \(2s\) steps. We denote by \(\theta_{2s} = \theta(W_{2s})\) the Dyck path of \(W_{2s}\) and say that \(\theta(W_{2s})\) represents the Dyck structure of \(W_{2s}\).

Let us denote by \(\Theta_{2s}\) the set of all Dyck paths of \(2s\) steps. It is known that \(\Theta_{2s}\) is in one-by-one correspondence with the set of all half-plane rooted trees \(T_s \in T_s\) constructed with the help of \(s\) edges. The correspondence between \(\Theta_{2s}\) and \(T_s\) can be established with the help of the chronological run \(\mathcal{R}\) over the edges of \(T_s\). It is known that the cardinality of \(T_s\) are given by the Catalan numbers (cf. \((1.9)\))

\[
|T_s| = t_s = \frac{(2s)!}{s!(s+1)!}, \quad s = 0, 1, 2, \ldots
\]

We refer to the elements of \(T_s\) as to the Catalan trees. We consider the edges of the tree \(T_s\) as the oriented ones in the direction away from the root of \(T_s\).

Given a Catalan tree \(T_s \in T_s\), one can label its vertices with the help of letters of \(\mathcal{A}\) according to \(\mathcal{R}_T\). The root vertex gets the label \(\alpha_1\) and each new vertex that has no label is labelled by the next in turn letter. We denote the walk obtained by \(\mathcal{W}_{2s}[T_s]\) and the corresponding Dyck path \(\theta_{2s} = \theta(\mathcal{W}_{2s})\) will be denoted also as \(\theta_{2s} = \theta(T_s)\).

Any Dyck path \(\theta_{2s}\) generates a sequence \((\xi_1, \xi_2, \ldots, \xi_s), \xi_i \in \{1, 2, \ldots, 2s - 1\}\) such that each step \(s_{\xi_i}, 1 \leq i \leq s\) of \(\mathcal{W}_{2s}[\theta_{2s}]\) is marked. We denote this sequence by \(\Xi_s = \Xi(\theta_{2s})\). Given \(\Xi_s\) and \(\tau \in [1, s]\), one can uniquely reconstruct \(\theta_{2s}\) and find corresponding instant of time \(\xi_\tau \in \{1, \ldots, 2s - 1\}\). We will say that the interval \([1, s]\) represents the \(\tau\)-marked instants or instants of marked time that varies from 1 to \(s\); sometimes we will simply say that \(\tau \in [1, s]\) is the marked instant when no confusion with the term ”marked instant of time” can arise.

Given a walk \(W_{2s}\) and a letter \(\beta\) such that \(\beta \in \mathcal{V}_g(W_{2s})\), we say that the instant of time \(t'\) such that \(W_{2s}(t') = \beta\) represents an arrival \(a\) at \(\beta\). If \(t'\) is marked, we will say that \(a\) is the marked arrival at \(\beta\). In \(W_{2s}\), there can be several marked arrival instants of time at \(\beta\) that we denote by \(1 \leq t^{(\beta)}_1 < \cdots < t^{(\beta)}_N\). For any non-root vertex \(\beta\), we have \(N = N_{\beta} \geq 1\). The
first arrival instant of time $\beta$ is always the marked one. We can say that $\beta$ is created at this instant of time. To unify the description, we assume that the root vertex $\varrho$ is created at the zero instant of time $t_1^{(\rho)} = 0$ and add the corresponding zero marked instant to the list of the marked arrival instants at $\varrho$.

If $N_{\beta} \geq 2$, then we say that the $N$-plet $(t_1^{(\beta)}, \ldots, t_N^{(\beta)})$ of marked arrival instants of time represents the self-intersection of $W_{2s}$. $\beta$ is the vertex of self-intersection, and this self-intersection is of the degree $N$ [25]. We say that the self-intersection degree $\kappa(\beta)$ is equal to $N$ and denote this by $\kappa(\beta) = N_{\beta}$. If $N_{\beta} = 1$, then we will say that $\kappa(\beta) = 1$.

Finally, let us consider a vertex $\beta$ and a collection of the marked edges of the form $(\beta, \alpha_i)$. We say that this collection is the exit cluster of $\beta$ and denote it by $\Delta(\beta)$,

$$\Delta(\beta) = \Delta_{W}(\beta) = \{e \in \mathbb{V}(W_{2s}) : e = (\beta, \alpha_i)\}. \tag{2.4}$$

Sometimes we will say that $\Delta(\beta)$ is given by the collection of corresponding vertices $\alpha_i$.

Finally, let us note that we can determine the tree-type and non-tree type trajectories mentioned at the end of Section 1 in terms of corresponding walks and the graphs of the walks. Namely, given $W_{2s}$, we can say that it is of the tree-type form if $\tilde{g}(W_{2s})$ has no cycles, i.e. is the tree when the multiple edges are regarded as the single ones. If $\tilde{g}(W_{2s})$ has at least one cycle, we can say that corresponding $W_{2s}$ is not of the tree-type structure. In the next section, we present a rigorous definition of tree-type walks and non-tree type walks based on the classification of the vertices of their graphs.

### 3 Walks of non-tree type

Let us consider a walk $W_{2s}$ and its graph $\tilde{g}(W_{2s})$ that is not necessarily a tree. Given a vertex $\beta \in \mathbb{V}(\tilde{g})$, it can happen that there exists a number of distinct vertices, $\alpha_1, \ldots, \alpha_m$ such that the marked edges of the from $(\alpha_i, \beta), i = 1, \ldots, m$ exist in $E(\tilde{g})$. Let us denote by $E_{\alpha_i}(\beta) = E_i(\beta)$ the collection of all marked edges of the form $(\alpha_i, \beta)$ of $W_{2s}$. Also, we denote by $\bar{e}(E_i(\beta)) = \min\{e : e \in E_i(\beta)\}$ the first passage of the multi-edge $E_i(\beta)$. Clearly, the multi-edges $E_i(\beta), i = 1, \ldots, m$ can be ordered between themselves according the instant of their first passages $t(\bar{e}_i)$. Then we can say that $E_i(\beta)$ represent the $i$-th distinct arrival at $\beta$.

#### 3.1 Classification of edges and vertices

Given a vertex $\beta$ with $m \geq 2$ distinct arrivals, let us consider a multiple edge $E_{\alpha_j}(\beta)$ with $j \geq 2$. We say that $E_{\alpha_j}(\beta)$ belongs to a set $\mathcal{E}(\beta)$ if the minimal edge $\bar{e}_{\alpha_j}(\beta)$ is such that there is either no marked edges of the form $(\beta, \alpha_j)$ performed by $W_{[0, t_j - 1]}$ or the multi-edge $E_{\beta}(\alpha_j)$ does not represent the first or the second distinct arrival at $\alpha_j$.

If $\mathcal{E}(\beta)$ is non-empty, then we say that $\beta$ is the $r$-vertex and color it in blue. We consider the minimal element $\bar{E}(\beta)$ of $\mathcal{E}(\beta)$ and say that the first passage of it given by $\bar{e}(\bar{E}(\beta))$ is the blue $r$-edge. Regarding $E_1(\beta)$, we say that $\bar{e}(E_1(\beta))$ is also the blue $r$-edge. All other edges of the form $(\alpha_i, \beta)$ that are not the blue ones, are referred to as the $u$-edges and are colored in black.

In what follows, we attribute to the blue $r$-edges the weight $V_2/n$. The black $u$-edges will be attributed by the weight $U^2/\rho$. 

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Regarding the vertices that are not blue $r$-vertices, we classify them according to the properties of the second arrivals $a_2$ to them. If $\beta$ is such that $e(a_2(\beta)) \in E_1(\beta)$, then we say that $\beta$ is the green $p$-vertex. We color the edge of the first arrival $e(a_1(\beta))$ in green and attribute to it the weight $v_2/n$. We say that the edge $e(a_2(\beta))$ is the $p$-edge and color it in black color. All other edges of the form $(\gamma, \beta)$ and $u \in E$ multi-edge $\beta$ are attributed by the weight $U^2/\rho$. Let us consider $e(a_2(\beta)) = (\alpha_2, \beta) \in E_2(\beta)$. Remembering that $\beta$ is not the blue $r$-vertex, we conclude that there exists an edge $e' = (\beta, \alpha_2)$ such that $e' < e(a_2)$ and the multi-edge $E_\beta(\alpha_2)$ represents either the first distinct arrival at $\alpha_2$ or the second one. Then we say that $\beta$ is the $q$-vertex and color it in red. The edge of the first arrival $e(a_1(\beta))$ is also colored in red. We attribute to it the weight $V_2/\rho$.

We say that $e(a_2(\beta))$ is the $q$-edge and color it in black color. All other edges of the form $(\gamma, \beta)$ are referred to as the black $u$-edges. The $q$-edge and $u$-edges are attributed by the weight $U^2/\rho$.

The notion of blue $r$-vertices can be used to give a rigorous definition of the classes of tree-type walks and the walks of non-tree type mentioned at the end of Section 2.

**Definition 3.1.** Given a walk $W_{2s}$, we say that is it a tree-type walk if it graph $\tilde{g}(W_{2s})$ does not contain any blue $r$-vertex. We denote by $\mathcal{W}_{2s}$ a collection of tree-type walks. If $W_{2s}$ is such that its graph $\tilde{g}(W_{2s})$ contains at least one blue $r$-vertex, then we say that $W_{2s}$ is of non-tree-type. We denote a collection of all non-tree-type walks by $\mathcal{W}_{2s}$.

Regarding the example walk $\mathcal{W}_{16}$ (2.1), we see that its graphical presentation contains two vertices of the self-intersection degree 2 (these are $\alpha_1$ and $\alpha_4$) and one vertex $\alpha_2$ of the self-intersection degree 3. Among vertices of $\tilde{g}(\mathcal{W}_{16})$, there is one $p$-vertex $\alpha_4$ and one $q$-vertex $\alpha_2$. The root vertex $\alpha_1$ has one blue edge of the (mute) first arrival and one blue edge $e(6)$ of the second distinct arrival. So, the root vertex $\alpha_1$ is the blue $r$-vertex and $\mathcal{W}_{16}$ is of non-tree type.

The following simple statement plays an important role in the studies of walks.

**Lemma 3.1.** If $W_{2s}$ is such that its graphical representation $g(W_{2s})$ has at least one green $q$-vertex, then $g(W_{2s})$ contains at least one blue $r$-vertex.

We prove Lemma 3.1 in Section 5.

**Lemma 3.2.** Let $\mathcal{I}_{2s}$ be such that the graph of its walk $W_{2s} = W(\mathcal{I}_{2s})$ contains $r$ blue $r$-vertices, $p$ green $p$-vertices, $q$ red $q$-vertices. Also we assume that $\tilde{g}(W_{2s})$ has $u$ black $u$-edges. Then

$$
\Pi_a(\mathcal{I}_{2s}) \Pi_b(\mathcal{I}_{2s}) = \Pi_a(W_{2s}) \Pi_b(W_{2s}) \leq \left( \frac{V_2^2}{n^2} \right)^r \left( \frac{V_2 U^2}{n \rho} \right)^{p+q} \left( \frac{U^2}{\rho} \right) u \left( \frac{V_2}{n} \right)^{s-u-2(r+p+q)}. \tag{3.1}
$$

**Proof of Lemma 3.2.** The proof of (3.1) follows from the definition of $r$-vertices, $p$-vertices and $q$-vertices and the description of weights attributed to corresponding edges. □

Now we can formulate the main technical statement of this section.
Theorem 3.1. Let us denote
\[ \tilde{Z}_{2s}(n, \rho) = \sum_{I_{2s}} \Pi_a(I_{2s}) \Pi_b(I_{2s}), \]
Then under conditions of Theorem 1.1, the following relation holds (cf. (1.7)),
\[ \tilde{Z}_{2s}(n, \rho) = o(n t_s V^{*n}_2), \quad \text{as} \quad n \to \infty, \]
where \( t_s \) are given by (2.3).

Remark. Let us denote
\[ \hat{Z}_{2s}(n, \rho) = \sum_{I_{2s}} \Pi_a(I_{2s}) \Pi_b(I_{2s}). \]
Observing that all terms of the right-hand side of this equality are non-negative, we conclude that \( \hat{Z}_{2s}(n, \rho) \geq n t_s V^{*n}_2 \) and that relation (3.3) implies the asymptotic estimate
\[ \tilde{Z}_{2s}(n, \rho) = o(\hat{Z}_{2s}(n, \rho)), \quad n \to \infty \]
valid under conditions of Theorem 2.1.

3.2 Diagrams \( \mathcal{G}^{(c)}(\bar{\nu}) \) and their realizations

Each walk \( W_{2s} \) generates a set of numerical data, \( \bar{\nu} = (\nu_2, \nu_3, \ldots, \nu_s) \), where \( \nu_k \) is the number of vertices \( \beta_i \) of \( \bar{g}(W_{2s}) \) such that their self-intersection degree is equal to \( k \), \( \zeta(\beta_i) = k \). To estimate the number of elements of the set \( \tilde{W}_{2s} \), we construct a kind of diagrams \( \mathcal{G}(\bar{\nu}) \).

To explain general principles of the estimates, let us start with the construction of non-colored diagram \( \mathcal{G}(\bar{\nu}) \). This diagram consists of \( |\bar{\nu}| = \sum_{k=2}^s \nu_k \) vertices. We arrange these vertices in \( s-1 \) levels, the \( k \)-th level contains obviously \( \nu_k \) vertices. Each vertex \( v \) of \( k \)-th level is attributed by \( k \) half-edges joined to \( v \) by their heads. At the tails of these edges we put square boxes (or in other words, windows). Then any vertex \( v \) of this \( k \)-th level has \( k \) edge-boxes (or edge-windows) attached to it.

Given \( \mathcal{G}(\bar{\nu}) \), one can attribute to its edge-windows the values from the set \( \{1, 2, \ldots, s\} \) such that there is no pair of windows with the same value. Then we get a realization of \( \mathcal{G}(\bar{\nu}) \) that we denote by \( \langle \mathcal{G}(\bar{\nu}) \rangle \).

The main principle of the estimates given by the Sinai-Soshnikov technique is based on the observation that any walk is determined by its values in the marked instant of time added by a family of rules that determine its values in the non-marked instant of time. The values in the marked instants of time are determined by given Dyck path \( \theta_s \) and a realization \( \langle \mathcal{G}(\bar{\nu}) \rangle \).

The values in the non-marked instant of time are determined by a family of rules \( \mathbb{Y}(\bar{\nu}) \) that indicate the way to leave a vertex \( \beta \) of self-intersection with the help of the non-marked step out. It is shown in [25, 26] that if \( \zeta(\beta) = k \), then the number of the exit rules at this vertex is bounded as follows, \( |\mathbb{Y}(\beta)| \leq (2k)^k \). The rigorous proof of this upper bound is given in [18] (see also [11]). No such rule as \( \mathbb{Y} \) is needed for the non-marked instants of time when the walk leaves a vertex of the self-intersection degree 1 because in this case this continuation
is uniquely determined. The estimate of the total number of rules is given by the following inequality,

$$|\mathcal{Y}(\hat{\nu})| \leq \prod_{k=2}^{s} (2k)^{k\nu_k}. \quad (3.5)$$

The number of all possible realizations of $\mathcal{G}(\bar{\nu})$ is given by the following expression

$$\sum_{\langle \mathcal{G}(\bar{\nu}) \rangle_s} 1 = \frac{s!}{\nu_2!(2!)^{\nu_2}\nu_3!(3!)^{\nu_3}\cdots \nu_s!(s!)^{\nu_s} \cdot (s-\|\bar{\nu}\|)!},$$

where $\|\bar{\nu}\| = \sum_{k=2}^{s} k\nu_k$. It is easy to see that the following upper bound is true,

$$\sum_{\langle \mathcal{G}(\bar{\nu}) \rangle_s} 1 \leq \prod_{k=2}^{s} \frac{1}{\nu_k!} \left( \frac{s^k}{k!} \right)^{\nu_k}. \quad (3.6)$$

Combining this inequality with (2.3) and (3.5), we conclude that the number of elements in $\mathcal{W}_{2s}(\bar{\nu})$ can be estimated as follows,

$$|\mathcal{W}_{2s}| \leq t_s \prod_{k=2}^{s} \frac{1}{\nu_k!} \left( \frac{(2k)^k s^k}{k!} \right)^{\nu_k} \leq t_s \prod_{k=2}^{s} \frac{(C_1 s)^{k\nu_k}}{\nu_k!}, \quad (3.7)$$

where $C_1 = \sup_{k \geq 2} \frac{(2k)^k}{(k!)^{1/k}}$.

The upper bound (3.7) clearly explains the role of the diagrams $\mathcal{G}(\bar{\nu})$ in the estimates of the number of walks. However, it is rather rough and does not give inequalities needed in the majority of cases of interest. In particular, it is not compatible with the weight of walks (3.1) in the case of dilute random matrices.

To adapt the diagram technique to our model, we introduce the color diagrams and formulate the filtering principle of estimates of the number of walks. The first use of this principle is due to the works by Ya. Sinai and A. Soshnikov. The rigorous formulation of the filtration technique is given in [11]. In paper [13], the filtration procedure is formulated in more precise form adapted to the study of the moments of dilute random matrices.

The color diagram $\mathcal{G}^{(c)}(\bar{\nu}, \bar{p}, \bar{q})$ is determined by parameters $\bar{\nu} = (\nu_2, \ldots, \nu_s)$, $\bar{p} = (p_2, \ldots, p_s)$ and $\bar{q} = (q_2, \ldots, q_s)$. Regarding $\nu_k$ vertices of the $k$-th level of $\mathcal{G}(\bar{\nu})$, we fill the second edge-box attached to each vertex by the values from the set $\{1, \ldots, s\}$. This can be done by

$$\frac{s!}{\nu_k!(s-\nu_k)!} = \frac{s^{\nu_k}}{\nu_k!} \leq \frac{s^k}{k!},$$

ways. Then we color the $\nu_k$ vertices in blue, red and green colors by one of $\frac{r_k}{\nu_k p_k q_k}$ ways, where $r_k = \nu_k - p_k - q_k$. Taking the empty $k-2$ edge-boxes attached to green or red vertex, we fill them with the values from the set $\{1, \ldots, s\}$. This can be done by not more than $s^{k-2}/(k-2)!$ ways.

Let us consider $k-2$ empty edge-boxes attached to a blue vertex. We have to choose one of $k-1$ places between them to settle the position of already filled edge-box. Then it remains
to fill \( k - 2 \) edge-boxes with the values from \( \{1, \ldots, s\} \) in the way that takes into account the position of the already filled edge-box. Ignoring this restriction, we estimate the number of ways to do this again by \( s^{k-2}/(k-2)! \). Finally, we attribute values from \( \{1, \ldots, s\} \) to the first edge at blue vertices.

This procedure being performed at each level independently, we get the following estimate from above of the number of different realizations of color diagrams,

\[
\sum_{\langle G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \rangle_{s, t}} 1 \leq \prod_{k=2}^{s} \frac{1}{r_k!} \left( \frac{s^k}{(k-2)!} \right)^{r_k} \cdot \frac{1}{p_k!} \left( \frac{s^k}{(k-2)!} \right)^{p_k} \cdot \frac{1}{q_k!} \left( \frac{s^k}{(k-2)!} \right)^{q_k}. \tag{3.8}
\]

Let us note that the color diagrams we consider are adapted to the asymptotic regime (1.6) and are different from those used in [13].

The **filtration principle** is follows: we consider a realization of the color diagram \( \langle G^{(c)} \rangle_{s, t}^{(b)} \) such that all blue and black boxes of edge-windows of \( G^{(c)} \) are filled with different values of \( \{1, \ldots, s\} \). Then we choose a Dyck path \( \theta \) and a rule \( \Upsilon \in \Upsilon (\tilde{\nu}) \) and start the run of the corresponding walk \( W \) till the marked instant of the first \( p \)-edge or \( q \)-edge appear. Let us denote this instant by \( \tau' \) with \( t' = \xi_{\tau'} \) and assume that the sub-walk \( W_{[0, t'-1]} \) get its end value \( \beta = W(t' - 1) \). Then at the instant of time \( t' \) the walk has to choose one of the vertices of the exit cluster \( \Delta(\beta) \) in the case of the \( p \)-edge or one of the two possible vertices \( \alpha_1, \alpha_2 \) of the first two distinct arrival edges at \( \beta \), in the case of \( q \)-edge. When this is done, we fill the green (or red) box attached to the corresponding vertex \( v \) of \( G^{(c)} \) by the marked instant that characterizes the vertex \( W(t') \) and proceed till the box-window of the next green or red vertex. When all the walk is constructed, if it exists, we denote by \( \langle \langle G^{(c)} \rangle_{s, t}^{(b)} \rangle_{W} \) the set of values obtained during this run of \( W \). We refer to this procedure as to the filtration principle because the set of all possible values to be put into the green and red boxes are filtered according to one or another condition.

**Lemma 3.3.** Given a color diagram \( G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \), let us consider a realization of the blue and black edge-windows \( \langle G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \rangle_{s, t}^{(b)} \). We denote by \( \mathcal{W}_{2s}(D, \langle G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \rangle_{s, t}^{(b)}, \Upsilon) \) the set of walks \( \mathcal{W}_{2s} \) that have this realization of \( G^{(c)} \), follow the rule \( \Upsilon \) and such that the maximal exit degree

\[
D(\mathcal{W}_{2s}) = \max_{\beta \in \mathcal{V}(\mathcal{W}_{2s})} |\Delta(\beta)|
\]

is equal to \( D, D(\mathcal{W}_{2s}) = D \). Then the number of possible realizations of the values in red and green boxes can be estimated by

\[
|\langle \langle G^{(c)} \rangle_{s, t}^{(b)} \rangle_{W}| \leq 2^{|\bar{q}|} D^{|\bar{p}|}, \tag{3.9}
\]

where \( |\bar{q}| = \sum_{k=2}^{s} q_k \) and \( |\bar{p}| = \sum_{k=2}^{s} p_k \) and therefore

\[
|\mathcal{W}_{2s}(D, \langle G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \rangle_{s, t}^{(b)}, \Upsilon)| \leq 2^{|\bar{q}|} D^{|\bar{p}|} t_s.
\]

**Proof of Lemma 3.3.** The proof of (3.9) follows immediately from the filtration principle for the color diagrams formulated above. Namely, we take a Dyck path \( \theta_s \) and perform the run of the walk according to the data given by the self-intersection of \( \langle G^{(c)}(\tilde{\nu}, \tilde{\mu}, \tilde{\rho} \tilde{\theta}) \rangle_{s, t}^{(b)} \) and \( \Upsilon \) till the value \( \xi_{\tau'} \) appear, where \( \tau' \) is attributed to the second arrival edge at a red or green
vertex \( v' \) of \( G^{(c)} \). We estimate the number of possibilities to fill the first arrival edge at \( v' \) by 2 or \( D \) respectively. When one of appropriate values is chosen, we proceed till the instant of time \( \xi_{v''} \), where \( v'' \) is attribute to the second edge of the second red or green vertex \( v'' \) of \( G^{(c)} \). Finally, we get the estimate (3.9). □

As a corollary of Lemma 3.3, we can estimate the number of walks that have a color diagram \( G^{(c)}(\vec{v}, \bar{p}, \bar{q}) \) and the maximal exit degree \( D \),

\[
|\mathbb{W}_{2s}(D, G^{(c)}(\vec{v}, \bar{p}, \bar{q}))| \leq t_s \prod_{k=2}^{s} \frac{(4k^3(C_1s)^{k-2})^r_k}{r_k!} \cdot \frac{(k^2Ds(C_1s)^{k-2})^p_k}{p_k!} \cdot \frac{(4k^2(C_1s)^{k-2})^q_k}{q_k!}.
\]

(3.10)

This relation follows from inequalities (3.5), (3.8) and (3.9).

We will use Lemma 3.3 and a version of relation (3.10) in the proof of Theorem 2.1 below. However, to get the estimates we need, we have to show that the number of Catalan trees \( T_s \) generated by the elements of \( \mathbb{W}_{2s}(D, G^{(c)}) \) is exponentially small with respect to the total number \( t_s \) of all \( T_s \) [26, 28]. To do this, we need to study the vertex of maximal exit degree of walks \( \mathbb{W}_{2s} \) in more details.

### 3.3 Vertex of maximal exit degree, arrival cells and BTS-instants

Let us consider a walk \( \mathbb{W}_{2s} \) and find the first letter that we denote by \( \tilde{\beta} \) such that (cf. (3.8))

\[
|\Delta(\tilde{\beta})| = D(\mathbb{W}_{2s}).
\]

(3.11)

We will refer to \( \tilde{\beta} \) as to the vertex of maximal exit degree and denote for simplicity \( D = D(\mathbb{W}_{2s}) \).

To classify the arrival instants at \( \tilde{\beta} \), we need to determine reduction procedures similar to those considered in [18] and further modified in [13]. Certain elements of the reduction procedure of [18] were independently introduced in paper [6].

#### 3.3.1 Reduction procedures and reduced sub-walks

Given \( \mathbb{W}_{2s} \), let \( t' \) be the minimal instant of time such that

1. the step \( s_{t'} \) is the marked step of \( \mathbb{W}_{2s} \);
2. the consecutive to \( s_{t'} \) step \( s_{t'+1} \) is non-marked;
3. \( \mathbb{W}_{2s}(t' - 1) = \mathbb{W}_{2s}(t' + 1) \).

If such \( t' \) exists, we apply to the ensemble of steps \( \bar{S} = \{s_t, 1 \leq t \leq 2s, s_t \in \mathbb{W}_{2s}\} \) a reduction \( \bar{R} \) that removes from \( \bar{S} \) two consecutive elements \( s_{t'} \) and \( s_{t'+1} \); we denote \( \bar{R}(\bar{S}) = \bar{S}' \). The ordering time labels of elements of \( \bar{S}' \) are inherited from those of \( \bar{S} \).

The new sequence \( \bar{S}' \) can be regarded again as an even closed walk. We can apply to this new walk the reduction procedure \( \bar{R} \). Repeating this operation maximally possible number of times \( m \), we get the walk

\[
\bar{W}_{2s} = (\bar{R})^m(\mathbb{W}_{2s}), \quad s = s - m,
\]

that we refer to as the strongly reduced walk. We denote \( \bar{S} = (\bar{R})^m(\bar{S}) \) and say that \( \bar{R} \) is the strong reduction procedure.

We introduce a weak reduction procedure \( \bar{R} \) of \( \bar{S} \) that removes from \( \bar{S}_{2s} \) the pair \( (s_{t'}, s_{t'+1}) \) in the case when the conditions (i)-(iii) are verified and
iv) $W_2s(t') \neq  \tilde{\beta}$.

We denote by

$$W_\ddot{s} = (\ddot{R})'(W_{2s}), \quad \ddot{s} = s - l \quad (3.12)$$

the result of the action of maximally possible number of consecutive weak reductions $\ddot{R}$ and denote $\ddot{S} = (\ddot{R})'(\ddot{S})$. In what follows, we sometimes omit the subscripts $2\ddot{s}$ and $2\ddot{S}$. Regarding the example walk $W_{16}$ (2.1), we observe that $\ddot{\beta} = \alpha_3$ and that the strongly and weakly reduced walks coincide and are as follows,

$$W_8 = \ddot{W}_8 = (\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_1).$$

Taking the difference $\ddot{S} \setminus \ddot{\ddot{S}} = \ddot{\ddot{S}}$, we see that it represents a collection of sub-walks, $\ddot{W} = \bigcup_j \ddot{W}^{(j)}$. Each sub-walk $\ddot{W}^{(j)}$ can be reduced by a sequence of the strong reduction procedures $\ddot{R}$ to an empty walk. We say that $\ddot{W}^{(j)}$ is of the Dyck-type structure. It is easy to see that any $\ddot{W}^{(j)}$ starts by a marked step and ends by a non-marked steps and there is no steps of $\ddot{W}$ between these two steps of $\ddot{W}^{(j)}$. We say that $\ddot{W}^{(j)}$ is non-split sub-walk.

It is not hard to see that the collection of steps $\ddot{S} = \ddot{S} \setminus \ddot{\ddot{S}}$ is given by a collection of subsets $\ddot{\ddot{S}} = \cup_k \ddot{\ddot{S}}^{(k)}$, each of $\ddot{\ddot{S}}^{(k)}$ represents a non-split Dyck-type sub-walk $\ddot{W}^{(k)}$,

$$\ddot{W} = \cup_k \ddot{W}^{(k)}. \quad (3.13)$$

In this definition, we assume that each sub-walk $\ddot{W}^{(k)}$ is maximal by its length.

### 3.3.2 Arrival instants and Dyck-type sub-walks attached to $\ddot{\beta}$

Given $W_{2s}$, let us consider the instants of time $0 \leq t_1 < t_2 < \ldots < t_R \leq 2s$ such that for all $i = 1, \ldots, R$ the walk arrives at $\ddot{\beta}$ by the steps of $W_{2\ddot{s}}$,

$$W_{2s}(t_i) = \ddot{\beta} \quad \text{and} \quad s_{t_i} \in \ddot{W}_{2s}, \quad i = 1, 2, \ldots, R. \quad (3.14)$$

We say that $t_i$ are the $\ddot{t}$-arrival instants of time of $W_{2s}$. Let us consider a sub-walk that corresponds to the subset $\ddot{S}_{[t_i + 1, t_{i+1}]} = \{s_{t_i}, t_i + 1 \leq t \leq t_{i+1}\} \subseteq \ddot{S}$; we denote this sub-walk by $W_{\ddot{t}_i, t_{i+1}}$. In general situation, we also denote by $W_{\ddot{t}', t''}$ a sub-walk that is not necessary even and/or closed.

Let us consider the interval of time $[t_{i+1}, t_{i+1} - 1]$ between two consecutive $\ddot{t}$-arrivals at $\ddot{\beta}$. It can happen that $W_{2s}$ arrives at $\ddot{\beta}$ at some instants of time $t' \in [t_{i+1}, t_{i+1} - 1]$, $W_{2s}(t') = \ddot{\beta}$. We denote by $\ddot{t}_{(i)}$ the maximal value of such $t'$.

**Lemma 3.4.** The sub-walk $W_{\ddot{t}_{i}, t_{(i)}}$ coincides with one of the maximal Dyck-type sub-walks $\ddot{W}^{(k)}$ of (3.13).

Lemma 3.4 is proved in [13].
we get equality \( D = \sum_{j=1}^{R} d_j, d_j \geq 0 \). Clearly, any exit sub-cluster is attributed to a uniquely determined \( l \)-arrival instant at \( \hat{\beta} \).

Regarding the reduced walk \( \tilde{\mathcal{W}}_{2s} \) of \( \mathcal{W}_{2s} \) (3.12), we can determine corresponding Dyck path \( \theta_{2s} = \theta(\tilde{\mathcal{W}}_{2s}) \) and the tree \( \tilde{T}_s = \mathcal{T}(\tilde{\mathcal{W}}_{2s}) \). It is easy to show that \( \tilde{T}_s \) is a sub-tree of the original tree \( T_s = \mathcal{T}(\theta(\mathcal{W}_{2s})) \). One can introduce the difference \( \mathcal{T}^s = T_s \setminus \tilde{T}_s \) and say that it is represented by a collection of sub-trees \( \mathcal{T}^{(i)} \).

Returning to the Catalan tree \( \mathcal{T}(\theta_{2s}) \), let us consider the chronological run over it \( \mathcal{R}_\tau \). Then the \( l \)-arrival instant \( t \) (3.14) determines the step \( \varpi_l \) of \( \mathcal{R}_\tau \). Also the corresponding vertex \( \tilde{\varpi}_l \) of the tree \( \tilde{T}_s \) are determined. It is clear that \( \tilde{\varpi}_l \) are not necessarily different for different \( l \).

The sub-trees \( \mathcal{T}^{(i)} \) are attached to \( \tilde{\varpi}_l \) and the chronological run over \( \mathcal{T}^{(i)} \) starts immediately after the step \( \varpi_l \) is performed. We will say that these steps \( \varpi_l, 1 \leq l \leq R \) represent the nest cells from where the sub-trees \( \mathcal{T}^{(i)} \), \( 1 \leq l \leq L \) grow. It is clear that the sub-tree \( \mathcal{T}_l \) has \( d_l \geq 0 \) edges attached to its root \( q_l \) and this root coincides with the vertex \( \tilde{\varpi}_l \). Returning to \( \mathcal{W}_{2s} \), we will say that the arrival instants of time \( \bar{t}_l \) represent the arrival cells at \( \hat{\beta} \). In the next sub-section, we describe a classification of the arrival cells at \( \hat{\beta} \) that represents a natural improvement of the approach proposed in [18].

### 3.3.3 Classification of arrival cells at \( \hat{\beta} \)

Let us consider a walk \( \mathcal{W}_{2s} \) together with its reduces counterparts \( \mathcal{W} = \mathcal{W}_{2s} \) and \( \tilde{\mathcal{W}}_{2s} = \tilde{\mathcal{W}} \). Let \( t_i \) denote a \( l \)-arrival cell (3.8). If the step \( s_{t_i} \) of \( \mathcal{W}_{2s} \) is marked, then we say that \( t_i \) represents a proper cell at \( \hat{\beta} \). If the step \( s_{t_i} \) is non-marked and \( s_{t_i} \in W = W \setminus \tilde{W} \), then we say that \( t_i \) represents a mirror cell at \( \hat{\beta} \). If the step \( s_{t_i} \in \tilde{W} \) is non-marked, then we say that \( t_i \) represents an imported cell at \( \hat{\beta} \).

Let us consider \( I \) proper cells \( l_i \) such that \( s_{l_i} \) belongs to \( \tilde{\mathcal{S}} \). We denote by \( x_i \) the corresponding marked instants, \( x_i = \xi_{i,l}, 1 \leq i \leq I \) and write that \( \bar{x}_I = (x_1, \ldots, x_I) \). It is easy to see that each proper cell \( x_i \) can be attributed by a number 1 or 0 in dependence of whether it produces a corresponding mirror cell at \( \hat{\beta} \) or not. We denote this number by \( m_i \in \{0,1\} \) and write that \( M = \sum_{i=1}^I m_i \) and \( \bar{m}_I = (m_1, \ldots, m_I) \). Clearly, \( M \leq I \).

Regarding the strongly reduced walk \( \mathcal{W}_{2s} \), we denote by \( t_k \) the proper cells such that the steps \( s_{t_k} \in \tilde{\mathcal{S}} \). Corresponding to \( t_k \) marked instants will be denoted by \( z_k, 1 \leq k \leq K \). Then \( \bar{z}_K = (z_1, \ldots, z_K) \) and clearly \( \mathcal{R}(\tilde{\beta}) = I + K \).

Given \( \mathcal{W}_{2s} \) with non-empty set \( \tilde{\mathcal{S}} \), there exists at least one pair of elements of \( \tilde{\mathcal{S}} \) denoted by \( (s', s'') \) such that \( s' \) is a marked step of \( \mathcal{W}_{2s} \), \( s'' \) is the non-marked one and \( s'' \) follows immediately after \( s' \) in \( \tilde{\mathcal{S}} \). We refer to each pair of this kind as to the pair of broken tree structure steps of \( \mathcal{W}_{2s} \), or in abbreviated form, the BTS-pair of \( \mathcal{W}_{2s} \). If \( t' \) is the marked instant that corresponds to \( s' \), we will simply say that \( t' \) is the BTS-instant of \( \mathcal{W}_{2s} \) [18].

Regarding the strongly reduced walk \( \tilde{\mathcal{W}} \), let us consider a non-marked arrival step at \( \hat{\beta} \) that we denote by \( \bar{s} = s_l \). Then one can find the uniquely determined marked instant \( t' \) such that all steps \( s_{t'} \in \tilde{\mathcal{S}} \) with \( \xi_{t'} + 1 \leq t' \leq \bar{t} \) are the non-marked ones. Let us denote by \( t'' \) the instant of time of the first non-marked step \( s_{t''} \in \tilde{\mathcal{S}} \) of this series of non-marked steps. Then \( (s_{t'}, s_{t''}) \) with \( t' = \xi_{t'} \) is the BTS-pair of \( \mathcal{W}_{2s} \), that corresponds to \( \bar{t} \). We will say that \( \bar{t} \) is
attributed to the corresponding BTS-instant $\tau'$. It can happen that several arrival instants $\bar{t}_i$ are attributed to the same BTS-instant $\tau'$. We will also say that the BTS-instant $\tau'$ generates the imported cells that are attributed to it.

Let us consider a BTS-instant $\tau$ such that $\mathcal{W}_{2s}(\xi_\tau) = \tilde{\beta}$. As it is said above, we denote such marked instants by $z_k$, $1 \leq k \leq K$. Assuming that a marked BTS-instant $z_k$ generates $f_k' \geq 0$ imported cells, we denote by $\varphi_1^{(k)}, \ldots, \varphi_{f'_k}^{(k)}$ the positive numbers such that

$$\mathcal{W}_{2s}(\xi_{z_k} + \sum_{j=1}^{l} \varphi_j^{(k)}) = \tilde{\beta} \quad \text{for all} \quad 1 \leq l \leq f'_k. \quad (3.15)$$

If for some $\bar{k}$ we have $f'_\bar{k} = 0$, then we will say that $z_\bar{k}$ does not generate any imported cell at $\tilde{\beta}$. We denote $\varphi^{(k)} = (\varphi_1^{(k)}, \ldots, \varphi_{f'_k}^{(k)})$.

Let us consider a BTS-instant $\tau$ that generates imported cells at $\tilde{\beta}$ and such that $\mathcal{W}_{2s}(\xi_\tau) \neq \tilde{\beta}$. We denote such BTS-instants by $y_j$, $1 \leq j \leq J$. Assuming that a marked BTS-instant $y_j$ generates $f''_j + 1$ imported cells, $f''_j \geq 0$, we denote by $\Lambda_j, \psi_1^{(j)}, \ldots, \psi_{f''_j}^{(j)}$ the positive numbers such that $\mathcal{W}_{2s}(\xi_{y_j} + \Lambda_j) = \tilde{\beta}$ and

$$\mathcal{W}_{2s} \left( \xi_{y_j} + \Lambda_j + \sum_{l=1}^{k} \psi_l^{(j)} \right) = \tilde{\beta} \quad \text{for all} \quad 1 \leq k \leq f''_j. \quad (3.16)$$

In this case we will say that the first arrival at $\tilde{\beta}$ given by the instant of time $\xi_{y_j} + \Lambda_j$ represents the principal imported cell at $\tilde{\beta}$. All subsequent arrivals at $\tilde{\beta}$ given by (3.13) represent the secondary imported cells at $\tilde{\beta}$. We will say that $y_j$ is the remote BTS-instant with respect to $\tilde{\beta}$ and will use denotations $\bar{y}_j = (y_1, \ldots, y_J)$ and $\bar{\Lambda}_J = (\Lambda_1, \ldots, \Lambda_J)$. We also denote $\bar{\psi}^{(j)} = (\psi_1^{(j)}, \ldots, \psi_{f''_j}^{(j)})$.

We see that for a given walk $\mathcal{W}_{2s}$, the proper, mirror and imported cells at its vertex of maximal exit degree are characterized by the set of parameters, $(\bar{x}, \bar{m})_I$, $(\bar{z}, \bar{F}, \bar{f}'_K)$, where $\Phi_K = (\varphi_1^{(1)}, \ldots, \varphi_{f'_K}^{(K)})$, $\bar{f}'_K = (f'_1, \ldots, f'_K)$ and $(\bar{y}, \bar{\Lambda}, \bar{\Psi}, \bar{f}'')_J$, where $\Psi_J = (\psi_1^{(1)}, \ldots, \psi^{(J)})$, $\bar{f}''_J = (f''_1, \ldots, f''_J)$. We also denote

$$F' = \sum_{k=1}^{K} f'_k \quad \text{and} \quad F'' = \sum_{j=1}^{J} f''_j.$$

Summing up, we observe that the vertex $\tilde{\beta}$ with $\alpha(\tilde{\beta}) = I + K$ has the total number of cells given by $R = I + M + K + J + F$, where $F = F' + F''$. In what follows, we will denote the parameters described above as

$$\mathcal{P}_R = \{ (\bar{x}, \bar{m})_I, (\bar{y}, \bar{\Lambda}, \bar{\Psi}, \bar{f}'')_J, (\bar{z}, \bar{F}, \bar{f}')_K \}. \quad (3.17)$$
3.4 Proof of Theorem 3.1

We are going to estimate the number of walks in a family of walks $\tilde{W}_{2s}(\bar{D})$ that have a vertex of maximal exit degree $D$. We rewrite (3.2) in the following from

$$\tilde{Z}_{2s}(n, \rho) = \sum_{D=1}^{\bar{s}} \sum_{\bar{W}_{2s} \in \tilde{W}_{2s}(D)} \Pi_a(\bar{W}_{2s}) \Pi_b(\bar{W}_{2s}) \cdot |C_{\bar{W}_{2s}}|,$$

where $C_{\bar{W}_{2s}}$ is given by (2.2). To estimate the number of elements in $\tilde{W}_{2s}(D)$, we have to consider a kind of color diagrams that have a separate vertex $\bar{v}$ attributed by the parameters from the family $\mathcal{P}_R$, namely by $\bar{x}_I$ and $\bar{z}_K$. Also we have to include into the color diagrams the parameters $\bar{y}_J$. Let us describe this new type of color diagrams.

3.4.1 Color diagrams with a vertex of maximal exit degree

Let us consider a vertex $\bar{v}$ and attach to it $I + K$ edge-boxes. We denote by $\langle \bar{v}, I, K \rangle_s$ a realization of the values of marked instants that fill these boxes. Given $\bar{v}, \bar{p}$ and $\bar{q}$, we consider a realization of the corresponding color diagram $\langle \mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q}) \rangle_s$ and point out $J$ edge-boxes that will provide the marked instants $\bar{y}$. Joining such a realization with chosen $J$ edge-boxes $\langle \mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q}) \rangle_{s(b)}$ with $\langle \bar{v}, I, K \rangle_s$, we get a realizations of the diagram we need,

$$\langle \mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q}) \rangle_{s(b)} = \langle \bar{v}, I, K \rangle_s \cup \langle \mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q}) \rangle_{s(b)} = \langle \bar{y} \cup \mathcal{G}^{(c)} \rangle_{s(b)}.$$

The last equality of the formula presented above introduces a denotation for a realization of the diagram we consider.

The number of different realizations of the color diagram $\mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q})$ is estimated by the right-hand side of (3.8). Regarding realizations $\langle \bar{v}, I, K \rangle_s$, we can write that

$$|\langle \bar{v}, I, K \rangle_s| \leq \frac{s^{I+K}}{(I+K)!} 2^{I+K},$$

where the last factor gives the upper bound for the choice of $K$ elements among $I + K$ ones to be marked as the values of $\bar{z}_K$. The vertex $\bar{\beta}$ of the walk can be attributed by the weight

$$\Pi_a(\bar{\beta}) \Pi_b(\bar{\beta}) = \begin{cases} \frac{V_2}{n^2}, & \text{if } \kappa(\bar{\beta}) = 1, \\ \frac{1}{n^2 \rho^{I+K-2}} V_2 U^{2(I+K)-4}, & \text{if } \bar{\beta} \text{ is } r \text{-vertex}, \\ \frac{1}{n^2 \rho^{I+K-1}} V_2 U^{2(I+K)-2}, & \text{if } \bar{\beta} \text{ is } p \text{-vertex or } q \text{-vertex}. \end{cases}$$

In the first and in the third cases of (3.19), at least one blue $r$-vertex is necessarily present in $\mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q})$.

Regarding $\langle \mathcal{G}^{(c)}(\bar{v}, \bar{p}, \bar{q}) \rangle_s$, one can choose $J$ edge-boxes to be labeled as the values of the realization $\langle \bar{y} \rangle$ among $\sum_{k=2}^{n} (k-1)u_k = \|\bar{v}\|_1$ edges only. This is because the first arrival to a vertex cannot be the marked BTS-instant. The number of ways to choose $J$ ordered places among $\|\bar{v}\|_1$ ordered edges can be estimated as follows,

$$\binom{\|\bar{v}\|_1}{J} \leq \left( \frac{\|\bar{v}\|_1}{J!} \right)^J \leq \frac{1}{h_o} \exp \{h_o \|\bar{v}\|_1\},$$

where $h_0 > 1$ is a constant.
3.4.2 Exit sub-clusters and cells at $\bar{\beta}$

The maximal exit degree of a walk $W_{2s} \in \mathcal{W}_{2s}(D)$ can be represented as follows, $D = D + \tilde{D} + \hat{D}$, where $\tilde{D}$ is the number of marked edges of the form $(\bar{\beta}, \gamma)$ that belong to the strongly reduced walk $W$ (3.12). $\tilde{D}$ represents the exit edges that belong to $W = W \setminus W$. It is known that $\tilde{D} = F + J$ and that $F \leq K$ [18] (see also Lemma 5.1 of [11]). Also we observe that $\hat{D} = M$. Taking into account that $M \leq I$, we can write that

$$\hat{D} = D - M - F - J \geq D - I - K - J. \quad (3.21)$$

The remaining $\hat{D}$ edges of $\bar{\mathcal{E}}(W_{2s})$ belong to the exit sub-clusters of the Dyck-type sub-walks $W_{2s}^{(k)}$ (3.13) attached to $\bar{\beta}$. They are distributed among $R$ arrival cells at $\bar{\beta}$. We denote by $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_R)$ a particular distribution such that $\sum_{i=1}^{R} \bar{d}_i = \hat{D}$.

The number of cells $R$ depends on $\langle G^{(c)} \rangle^{(b)}$, $\theta_s$ and $\Upsilon$. However, the inequalities used to get (3.21) show that

$$R = I + K + M + F + J \leq 2I + 2K + J = R^* \quad (3.22)$$

Then

$$\sum_{\bar{d}_R:|\bar{d}_R| = \hat{D}} 1 = \binom{\hat{D} + R - 1}{R - 1} \leq \binom{D + R^* - 1}{R^* - 1}.$$ 

Elementary analysis shows that if $D \geq 2$, then

$$\binom{D + R^* - 1}{R^* - 1} \leq h_0^{R^*} \sup_{R^* \geq 1} \frac{1}{h_0^{R^* - 1}} \binom{D + R^* - 1}{R^* - 1} \leq h_0^{2I + 2K + J} \exp \left\{ \frac{eD}{h_0^2} \right\}, \quad h_0 > 1. \quad (3.23)$$

Now we are ready to perform the estimates that prove Theorem 3.1.

3.4.3 Exponential estimates and $\tilde{Z}_{2s}$

The following statement improves the estimate (3.9) of Lemma 3.3.

**Lemma 3.4.** Given $D$ and a realization $\langle \bar{\nu} \oplus G^{(c)} \rangle^{(b)}$ (3.18), let us consider a family of walks $W_{2s}(D, \langle \bar{\nu} \oplus G^{(c)} \rangle^{(b)}, \Upsilon)$ such that the vertex of maximal exit degree $\bar{\beta}$ determined by $\bar{\nu}$ has $D$ exit edges of the form $(\bar{\beta}, \gamma_i)$ and the rule $\Upsilon$ is given. Then

$$|W_{2s}(D, \langle \bar{\nu} \oplus G^{(c)} \rangle^{(b)}, \Upsilon)| \leq 2|\bar{\nu}|D|\bar{\nu}| (e^{\eta}h_0^{2})^{I+K+J} e^{-\eta D + eD/h_0 t_s}, \quad (3.24)$$

where $\eta = \ln(4/3)$.

We prove Lemma 3.5 in Section 5.

Let us write that

$$\tilde{Z}_{2s}(n, \rho) = \tilde{Z}_{2s}^{(1)} + \tilde{Z}_{2s}^{(2)}, \quad (3.25)$$

where

$$\tilde{Z}_{2s}^{(1)} = \sum_{D=1}^{s} \sum_{J=0}^{s} \left( \prod_{k=2}^{s} \sum_{c_k} \langle J \rangle^{(1)} \sum_{k: \bar{c}_k \in \omega_k} \sum_{\langle \bar{c}_s \rangle^{(b)} \in W_{2s} \in W_{2s} (D, \langle \bar{c}_s \rangle^{(b)})} \Pi_{a,b}^{(1)}(W_{2s}) |C_{W_{2s}}| \right).$$

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and

\[ \hat{Z}_{2s}^{(2)} = \sum_{D=1}^{s} \sum_{J=0}^{s} \prod_{k=2}^{J} r_k \sum_{q_k} \left( \sum_{l=I,K}^{s} \sum_{(\psi_k^q,)}^{(2)} \Pi_{r_k}^{(2)}(W_{2s})|C_{W_{2s}}| \right) . \]

In these relations, the superscripts 1 and 2 are introduced to indicate the situations when a blue r-vertex is attributed to the diagram \( G^{(c)} \) or to the vertex \( \hat{r} \), respectively. The weights \( \Pi_{r_k}^{(2)}(W_{2s}), i = 1, 2 \) are adapted to these situations according to relation (3.20). The superscript \((J)\) means that the sum over \( r_k, p_k \) and \( q_k \) is such that

\[ \sum_{k=2}^{s} (k - 1)\nu_k \geq J. \] (3.26)

Let us consider the first term from the right-hand side of (3.25). Using relations presented above, we can write that

\[ \hat{Z}_{2s}^{(1)} \leq \sum_{D=1}^{s} \sum_{J=0}^{s} \prod_{k=2}^{J} r_k \sum_{q_k} \left( \sum_{I+K \geq 1}^{(2s)(I+K)} \frac{(2s)(I+K)}{(I+K)!} \right) \]

\[ \times \left( \frac{(k^3(C_1 s)^k)}{r_k!} \right) \left( \frac{(k(C_1 s)^k-1)^{p_k}}{p_k!} \right) \left( \frac{(2k(C_1 s)^{k-1})^{q_k}}{q_k!} \right) \]

\[ \times \frac{V_2 U^2(I+K-1)}{n \rho^{I+K-1}} \left( \frac{V_2}{n} \right)^{r_k} \left( \frac{V_2 U^2}{n \rho} \right)^{p_k+q_k} \left( \frac{U^2}{\rho} \right)^{(k-2)\nu_k} \left( \frac{V_2}{n} \right)^{s-\|\hat{r}\|_2-(I+K-1)} . \] (3.27)

where we denoted \( \|\hat{r}\|_2 = \sum_{k=2}^{s} (k - 2)\nu_k \).

Taking into account conditions (3.26) \( \sum_k r_k \geq 1 \) and using definition \( s_n = \chi_{\rho_n} \), we deduce from (3.27) the following inequality,

\[ \hat{Z}_{2s}^{(1)} \leq \sum_{D=1}^{s} \sum_{J=0}^{s} \prod_{k=2}^{J} \left( \frac{2sC_1 e^{\eta h_0} (2\chi C_1 U^2 e^{\eta h_0})^{I+K-1}}{s^2 h_0 e^{\eta h_0} n} \right) \exp \left\{ \frac{(D + 2)\chi}{\rho} + \frac{s^2 h_0 e^{\eta h_0}}{n} + \frac{eD}{h_0} - \eta D \right\} , \] (3.28)

where we denoted \( U^2 = U^2/V_2 \). Remembering that \( \eta = \ln(4/3) \), we assume that

\[ \frac{8}{3} \chi C_1 U^2 e^{2h_0} = \delta < 1. \] (3.29)

Denoting \( A = \sum_{i=1}^{s} (l + 1)\delta^l \) and \( B = \sum_{k=2}^{s} (2 + k^3)\delta^{k-2} \), we get from (3.28) the following upper bound,

\[ \hat{Z}_{2s}^{(1)} \leq n s V_2^s A e^{B + 2\chi + 1} \left( \frac{C_2 s}{n} \right) \sum_{D=1}^{s} \exp \left\{ - (\eta - \chi - e h_0^{-1}) D \right\} , \] (3.30)
where \( C_2 = 2C_1e^{2\eta+h_0}h_0^3 \). The value of the technical parameter \( h_0 > 1 \) is at our disposal. Taking \( h_0 = \frac{4e}{(3\eta)} \), we observe that if
\[
0 < \chi \leq \chi_0 = \min \left\{ \frac{\eta}{4}, \frac{3V_2}{8C_1U_2e^{2h_0}} \right\}, \quad \eta = \ln(4/3),
\]
then (3.29) is verified and the last series of (3.30) converges. Thus
\[
\tilde{Z}_{2s}^{(1)} = o(nT_{V_2}^s), \quad n \to \infty
\]
(3.32)
under conditions of Theorem 2.1.

Let us consider the second term of (3.25). It is easy to see that \( \tilde{Z}_{2s}^{(2)} \) can be estimated from above by the expression given by the right-hand side of (3.27), where the sum over \( I, K \) is performed over the range \( I + K \geq 2 \) and the weight factor \( V_2U_2^{2(I+K-1)}/(n^2\rho^{I+K-2}) \) is replaced by \( V_2U_2^{2(I+K-2)}/(n^2\rho^{I+K-2}) \) (see relation (3.20)). Then we get the following upper bound,
\[
\tilde{Z}_{2s}^{(2)} \leq nt_s V_2^s \sum_{D=1}^{s} \sum_{J=0}^{s} \sum_{I+K \geq 2} \frac{2s^2C_1e^{\eta h_0^2}}{n} (2\chi C_1U_2^2e^{\eta h_0^3})^{I+K-2} \times \prod_{k=2}^{s} \exp \left\{ (2 + k^3) \left( \chi C_1U_2^2e^{\eta h_0^3} \right)^{k-2} \right\} \cdot \exp \left\{ (D + 2)\chi + \frac{\eta^2}{n} + \frac{D}{h_0 - \eta} \right\} ,
\]
(3.33)
Assuming that (3.31) is verified and using the same denotations as above, we deduce from (3.33) that (cf. (3.30))
\[
\tilde{Z}_{2s}^{(2)} \leq nt_s V_2^s A e^{B + 2\chi + 1} \frac{C_3 s^3}{n} \sum_{D=1}^{s} \exp \left\{ - (\eta - \chi - h_0^{-1}) D \right\} ,
\]
where \( C_3 = 2C_1e^{\eta h_0^2} \). Then obviously
\[
\tilde{Z}_{2s}^{(2)} = o(nT_{V_2}^s), \quad n \to \infty.
\]
(3.34)
Relations (3.32) and (3.34) imply (3.3). Theorem 3.1 is proved. □

4 Tree-type walks

Let us consider the family \( \mathcal{W}_{2s} \) of tree-type walks and separate it in two part as follows,
\[
\mathcal{W}_{2s} = \mathcal{X}_{2s}^{(2,4)} \cup \mathcal{X}_{2s}^{(2,6)},
\]
where \( \mathcal{X}_{2s}^{(2,4)} \) contains the walks \( W_{2s} \) such that their weights have the factors \( V_2 \) and \( V_4 \) only, \( \Pi_a(W_{2s}) = V_2^{u_1}V_4^{u_2}, \quad u_1 + 2u_2 = s \). We denote
\[
\mathcal{X}_{2s}^{(2,4)} = \sum_{W_{2s} \in \mathcal{X}_{2s}^{(2,4)}} \Pi_a(W_{2s}) \Pi_b(W_{2s}) \quad \text{and} \quad \mathcal{X}_{2s}^{(2,6)} = \sum_{W_{2s} \in \mathcal{X}_{2s}^{(2,6)}} \Pi_a(W_{2s}) \Pi_b(W_{2s}).
\]

Theorem 4.1. Under conditions of Theorem 2.1, the following asymptotic relations are true,

\[
\limsup_{n, \rho_n \to \infty} \frac{1}{nt_{s_n} V^2_{2s_n}} \lambda^{(2,4)}_{2s_n}(n, \rho_n) < +\infty,
\]

and

\[
\lambda^{(2,4)}_{2s_n}(n, \rho_n) = o \left( \lambda^{(2,4)}_{2s_n}(n, \rho_n) \right), \quad n, \rho_n \to \infty,
\]

where \( s_n = \chi \rho_n \) with \( 0 < \chi \leq \chi_0 \) (see (1.6)).

Remark. In fact, we are going to show that

\[
\lambda^{(2,6)}_{2s_n}(n, \rho_n) = o(n t_{s_n} V^2_{s_n}), \quad n, \rho_n \to \infty.
\]

Taking into account the obvious observation \( \lambda^{(2,4)}_{2s_n} \geq n t_{s_n} V^2_{s_n} \), we deduce from (4.3) relation (4.2). We will refer to the elements of \( X^{(2,4)}_{2s_n} \) as to the tree-type (2,4)-walks or simply as to the (2,4)-walks. Everywhere below we omit the subscripts \( n \) in \( s_n \) and \( \rho_n \) when no confusion can arise.

Let us describe the scheme of the proof of Theorem 4.1. The strategy is similar to that used to prove Theorem 3.1. Namely, we estimate the contribution of trajectories such that their walks \( W_{2s} \in \mathcal{W}_{2s}(D, \bar{\nu}) \) have a vertex of maximal exit degree \( \hat{\beta} \), \( |\Delta(\hat{\beta})| = D \) (see (3.11)) and generate a set of vertices of self-intersection characterized by \( \bar{\nu} = (\nu_2, \ldots, \nu_4) \). In the proof of Theorem 3.1, the contribution of the vertex \( \hat{\beta} \) to the weight of trajectories is estimated by \( V^2_{s} U^{2k-2} s^k / \rho^{k-1} = s V^2_{s} (\chi U^2)^{k-1} \), where \( k = \varkappa(\hat{\beta}) \). The "extra" factor \( s \) is compensated by the presence of the factor \( s^2 / n \) given by a blue \( r \)-vertex that necessarily exists in the non-tree type walks. In the case of tree-type walks \( \bar{W}_{2s} \), this description of the walks and the vertex \( \hat{\beta} \) of maximal exit degree does not lead to the estimates we need. So, we have to introduce a new description of \( \hat{\beta} \) and the properties of the corresponding trees \( T_s = T(W_{2s}) \).

4.1 The vertex of maximal exit degree \( \hat{\beta} \) of tree-type walks

Let us consider a walk \( W_{2s} \in \mathcal{W}_{2s}(D) \) with its graph \( \bar{g}(W_{2s}) = (\bar{E}, V) \) and find the first vertex \( \hat{\beta} \) such that

\[
|\Delta(\hat{\beta})| = \max_{\hat{\beta} \in \bar{V}(\bar{g})} |\Delta(\hat{\beta})|.
\]

Regarding a collection of all multi-edges of the form \( \bar{E}_{\hat{\beta}} = \{E_{\hat{\beta}}(\gamma_i), i = 1, \ldots, t, |E_{\hat{\beta}}(\gamma_i)| \geq 2\} \), we consider the maximal edge \( \hat{e}_i \) of each multiple edge and order the ensemble \( \bar{E}_{\hat{\beta}} \) in natural way according the order of \( \hat{e}_i, 1 \leq i \leq t \). Let us denote \( \hat{e} = (\hat{\beta}, \hat{\gamma}) = \max\{\hat{e}_1, \ldots, \hat{e}_t\} \).

The vertex \( \hat{\beta} \) is joined with the root vertex \( \varrho \) with the help of a chain of \( m \) oriented multi-edges of the form \( E_{\alpha_j}(\alpha_{j-1}), 1 \leq j \leq m, \alpha_0 = \hat{\beta}, \alpha_m = \varrho \). Let us consider the chain of minimal edges

\[
\hat{B}^{(1)} = \left\{ e_j^{(1)} = \min E_{\alpha_j}(\alpha_{j-1}), j = 1, \ldots, m \right\}
\]

and say that \( \hat{B}^{(1)} = \hat{B}^{(1)}_{m_1} = \{(\hat{\beta}, \alpha_1), (\alpha_2, \alpha_1), \ldots, (\varrho, \alpha_{m_1-1})\} \) is the first inverse branch of the length \( m_1 = m \). The sequence of the instants if time \( t(e_1^{(1)}), \ldots, t(e_m^{(1)}) \) is monotone decreasing, so we say that the branch \( \hat{B}^{(1)} \) is the monotone branch.
Taking the ensemble of the second minimal edges \( e^{(2)}_j = \min\{e \in E_{a_j}(\alpha_{j-1}), t(e) > t(e^{(1)}_j)\} \), we construct the corresponding second minimal monotone branch \( \mathcal{B}^{(2)}_{m_2} \) with \( m_2 \leq m_1 \). The third monotone branch \( \mathcal{B}^{(3)}_{m_3} \) with \( m_3 \leq m_1 \), if it exists, is determined by the similar procedure, and so on.

We denote the family of \( L \) monotone branches obtained by \( \mathcal{L}_\bar{\beta}(\bar{\ell}_m) = \{\mathcal{B}^{(1)}_{m_1}, \ldots, \mathcal{B}^{(L)}_{m_L}\} \) and say that \( \mathcal{L}_\bar{\beta}(\bar{\ell}_m) \) is the support (or pillar) structure of \( \bar{\beta} \), where \( \bar{\ell}_m = (\ell_1, \ell_2, \ldots, \ell_m + 1) \) and \( \ell_j \geq 0 \) denotes the number of additional monotone branches of the length \( j \). Let us note that the base minimal branch \( \mathcal{B}^{(1)}_{m_1} \) always exists and we do not count it in \( \ell_j \).

We represent the exit cluster as \( \Delta(\bar{\beta}) = \Delta'(\bar{\beta}) \cup \Delta''(\bar{\beta}) \), where \( \Delta'(\bar{\beta}) = \{(\bar{\beta}, \gamma_i) \leq \bar{e}\} \) and denote by \( \mathcal{V}(\Delta'(\bar{\beta})) = \{\delta_i : (\bar{\beta}, \delta_i) \in \Delta'(\bar{\beta})\} \) the corresponding collection of vertices. The elements of \( \mathcal{V}(\Delta') \) are characterized by \( \bar{\sigma} = (\zeta; \sigma_2, \ldots, \sigma_{D'}) \), where \( \zeta = \mathcal{N}(\bar{\gamma}) \), \( (\bar{\beta}, \bar{\gamma}) = \bar{e} \) and \( \sigma_i \) is the number of vertices \( \delta_j \neq \bar{\gamma} \) such that \( \mathcal{N}(\delta_j) = i \). We denote

\[
\|\bar{\sigma}\| = \sum_{i=2}^{D'} i\sigma_i
\]

and write that \( \sigma_1 = D' - \|\bar{\sigma}\| \). Regarding the remaining vertices \( \mathcal{V}(\bar{\gamma}) \setminus \left( \mathcal{V}(\Delta') \cup \mathcal{V}(\mathcal{L}_\bar{\beta}) \right) \), where \( \mathcal{V}(\mathcal{L}_\bar{\beta}) = \{\bar{\beta}, \alpha_1, \ldots, \alpha_{m_1-1}, \varrho\} \), we classify them by their self-intersection degrees and denote the data obtained by \( \bar{\nu} = (\nu_2, \ldots, \nu_\alpha) \).

We are going to estimate the contribution of trajectories such that their walks \( \mathcal{W}_{2s}(D', \bar{\ell}_m) \) have the vertex of maximal exit degree \( \bar{\beta} \) characterized by the set of parameters \( D' = |\Delta'(\bar{\beta})| \) and \( \bar{\ell}_m \). To do this, we consider the family of the corresponding underlying trees \( \mathcal{T}_s = \mathcal{T}(\mathcal{W}_{2s}) \). In the next subsection, we will show that the family of such trees \( \mathcal{T}(D', \bar{\ell}_m) \) admits an exponential estimate with respect to \( D' \) (cf. (3.24)). This will allow us to prove Theorem 4.1.

### 4.2 The set of underlying trees \( \mathcal{T}_s(D', \bar{\ell}) \) and exponential estimates

Given \( D' \) and \( \bar{\ell}_m \), let us describe a construction of the set of underlying trees \( \mathcal{T}(D', \bar{\ell}_m) \). We start with the root vertex \( \varrho \) and attach to it \( \ell_m + 1 \) branches of the length \( m \). Regarding the collection of edges \( (\varrho, v_{(1)}^i), 1 \leq i \leq m + 1 \), we say that the vertices \( \mathcal{F}^{(1)} = \{v_{(1)}^1, \ldots, v_{(m+1)}^m\} \) represent the first level \( \mathcal{F}^{(1)} \) of the pillar structure \( \mathcal{L}(\bar{\ell}_m) \) we construct. Let us note that \( \mathcal{L} \) of the present subsection differs from the pillar structure of the previous subsection and should be denoted by another letter. Since it is obvious, we ignore this difference.

On the next step, we attach to the elements of \( \mathcal{F}^{(1)} \) the branches of the length \( m - 1 \). The number of these branches is given by \( \ell_m - 1 \) and therefore the number of ways to attach them to \( \ell_m + 1 \) vertices is given by

\[
\binom{\ell_m + \ell_m - 1}{\ell_m} \leq 2^{\ell_m + \ell_m - 1}. \tag{4.5}
\]

Then we get \( \ell_m + \ell_{m-1} + 1 \) vertices of the second level \( v_{(2)}^j \) that we denote by \( \mathcal{F}^{(2)} \). The branches of the length \( m - 2 \) are attached to the elements of \( \mathcal{F}^{(2)} \) and the number of ways to
do this is bounded by

\[
\left( \ell_m + \ell_{m-1} + \ell_{m-2} \right) \leq 2^{\ell_m + \ell_{m-1} + \ell_{m-2}}. \tag{4.6}
\]

We continue this process till the last group of \( \ell \) branches of length 1 is attached and denote by \( \mathcal{F}^{(m)} \) the set of vertices of the \( m \)-th level obtained. It follows from the upper bounds (4.5) and (4.6) that the total number of pillar structures \( \mathcal{L}(\bar{\ell}_m) \) is bounded as follows,

\[
|\{ \mathcal{L}(\bar{\ell}_m) \}| = |\mathcal{L}(\bar{\ell}_m)| \leq 2^{(m-1)\ell_m + (m-1)\ell_{m-1} + \cdots + 2\ell_2 + \ell_1} = 2^{\|\bar{\ell}\|},
\]

where \( \|\bar{\ell}\| = \sum_{j=1}^{m} j \ell_j \).

Regarding the set \( \mathcal{F}^{(m)} \), we denote by \( L \) its cardinality, where \( L = \sum_{i=1}^{m} \ell_i = |\bar{\ell}_m| \) and choose a distribution \( \bar{d} = (d_1, \ldots, d_L) \) of \( D' \) edges over \( L \) vertices. We estimate the number of these distributions as follows (cf. (3.23)),

\[
|\{ \bar{d}_L \}| = \left( \frac{D' + L - 1}{L - 1} \right) \leq h_0 L^e e^{D'/h_0}, \quad h_0 > 1. \tag{4.8}
\]

Each vertex \( v_j^{(m)} \), \( 1 \leq j \leq L \) of \( \mathcal{F}^{(m)} \) is attached by a corresponding exit sub-cluster \( \Delta_j' \) such that \( |\Delta_j'| = d_j \). We denote the tree obtained by \( \mathcal{T} = \mathcal{L}(\bar{\ell}_m) \cup \{ \Delta_1', \ldots, \Delta_L' \} \). Regarding all vertices of \( \mathcal{T} \) excepting \( v_j^{(m)}, \ldots, v_{L_1}^{(m)} \), we attach to them the sub-trees that are constructed with the help of \( s - (|\bar{\ell}_m| + m + D') \) edges. The last vertex \( v_{L_1}^{(m)} \) is attached by \( D'' = D - D' \) edges to the left of the sub-cluster \( \Delta_L' \). This completes the construction of a tree from the set \( \mathcal{T}(D', D'', \bar{\ell}_m) \).

**Lemma 4.1.** The set \( \mathcal{T}(D', D'', \bar{\ell}_m) \) admits the following upper bound (cf. (3.24)),

\[
|\mathcal{T}(D', D'', \bar{\ell}_m)| \leq 2^{|\bar{\ell}_m|} h_0 L^e e^{-\eta(D'+D'') + eD'/h_0} t_s \tag{4.9}
\]

and therefore

\[
\sum_{D'+D''=D} |\mathcal{T}(D', D'', \bar{\ell}_m)| \leq 2^{|\bar{\ell}_m|} h_0 L^e e^{-\eta D + eD'/h_0} t_s, \tag{4.10}
\]

where \( \eta = \ln(4/3) > 0 \).

We prove lemma 4.1 in Section 6.

### 4.3 Proof of Theorem 4.1

Let us write that

\[
\hat{Z}_{2s} = \sum_{D'=2}^{D} \sum_{m=1}^{s} \sum_{\ell_m} \sum_{\bar{\sigma}} \sum_{\bar{\nu}} \sum_{W_{2s} \in \mathcal{W}_{2s}(D', \bar{\ell}_m, \bar{\sigma}, \bar{\nu})} |\Pi_{a,b}(\mathcal{W}_{2s})| \cdot |\mathcal{L}_{W_{2s}}|, \tag{4.11}
\]

where \( \mathcal{W}_{2s}(D', \bar{\ell}_m, \bar{\sigma}, \bar{\nu}) \) denotes the family of walks determined by parameters \( D', \bar{\ell}_m, \bar{\sigma} \) and \( \bar{\nu} \). The weight of the walk can be represented as follows,

\[
\Pi_{a,b}(\mathcal{W}_{2s}) = \Pi_{a,b}(\mathcal{L}(\bar{\ell})) \cdot \Pi_{a,b}(\Delta(\bar{\sigma})) \cdot \Pi_{a,b}(g(\bar{\nu})), \tag{4.12}
\]

where...
Assuming that the vertex \( \langle G \rangle \) number of (\( k \)) Clearly, the total number of such realizations is bounded as follows,

\[
\Pi_{a,b}(\Delta^{'}(\bar{\sigma})) = \frac{V_{2k}}{n^{\rho k-1}} \prod_{i=1}^{D'} \left( \frac{V_{2i}}{n^{\rho^{i-1}}} \right)^{\sigma_{i}} \leq \frac{V_{2k}U^{2\zeta-2}}{n^{\rho \zeta^{i-1}}} \prod_{i=1}^{D'} \left( \frac{V_{2k}U^{2i-2}}{n^{\rho^{i-1}}} \right)^{\sigma_{i}}, \quad \zeta \geq 2,
\]

and

\[
\Pi_{a,b}(g(\bar{\nu})) = \prod_{k=1}^{s} \left( \frac{V_{2k}}{n^{\rho k-1}} \right)^{\nu_{k}} \leq \prod_{k=1}^{s} \left( \frac{V_{2k}U^{2k-2}}{n^{\rho k-1}} \right)^{\nu_{k}},
\]

where \( \sigma_{1} \) and \( \nu_{1} \) are determined in the obvious manner. Equalities of (4.13), (4.14) and (4.15) can be regarded as the definitions of the factors of the right-hand side of (4.12).

Let us estimate the number of walks in the family \( \mathcal{W}_{2s}(T(\bar{d}, \mathcal{L}(\ell_{m})), \bar{\sigma}, \bar{\nu}) \) with given tree \( T \). To do this, we construct a diagram \( \mathcal{G}(\bar{\nu}) \) as it is described in sub-section 3.2. The next step is to consider a realization \( \langle \mathcal{G}(\bar{\nu}) \rangle_{s}^{(l)} \) of the values at the last edge-boxes at each vertex. Clearly, the total number of such realizations is bounded as follows,

\[
\sum_{\langle \mathcal{G}(\bar{\nu}) \rangle_{s}^{(l)}} 1 \leq \prod_{k=2}^{s} \frac{s^{\nu_{k}}}{\nu_{k}!}.
\]

We rearrange the vertices of \( \langle \mathcal{G}(\bar{\nu}) \rangle_{s}^{(l)} \) according to the values in their last edge-boxes given by \( \tau_{1} < \tau_{2} < \cdots < \tau_{|\bar{\nu}|} \) and denote the diagram obtained by \( \langle \mathcal{G}(\bar{\nu}) \rangle_{s}^{(l,s)} = \langle \mathcal{G} \rangle^{s} \).

Let us consider the self-intersections that can be performed by the walk at remaining \( m + \|\ell_{m}\| + \Delta' \) marked instants of time. These instants are completely determined by the particular tree \( T(\bar{d}, \mathcal{L}(\ell_{m})) \). Regarding the edge-boxes of \( \mathcal{L} \), we observe that the marked instants are uniquely attributed to them and therefore the corresponding self-intersections of \( \mathcal{W}_{2s} \) are uniquely determined. We choose a repartition of the edges of \( D' \) into groups according to \( \bar{\sigma} \). The number of such repartitions is estimated by the following expression,

\[
\frac{d!}{(\zeta - 1)! \cdot (\sigma_{1}! \cdot 2^{\sigma_{2}} \cdot \cdots \cdot \sigma_{d}! \cdot (d - \zeta + 1 - \|\bar{\sigma}\|)!))} \leq \frac{d^{\zeta - 2}}{(\zeta - 1)!} \prod_{i=2}^{d} \left( \frac{d^{\sigma_{i}}}{\sigma_{i}!} \right),
\]

where we denoted \( d = D' \).

Given a tree \( T \in \mathcal{T}(\bar{d}_{L}, D'', \ell_{k}) \), \( D'' = D - D' \) and assuming that the edges of \( \Delta' \) are split into groups according to \( \bar{\sigma} \), we start the run of the walk till the value \( \xi_{\tau_{1}} - 1 \). The vertex \( \gamma_{1} = \mathcal{W}(\xi_{\tau_{1}} - 1) \) is completely determined as well as its exit cluster \( \Delta(\gamma_{1}) \), \( |\Delta(\gamma_{1})| \leq D \). Assuming that the vertex \( v_{1} \) of \( G^{s} \) is of the self-intersection degree \( k_{1} \), we can estimate the number of \( (k_{1} - 1) \)-plets by

\[
\frac{D^{k_{1} - 1}}{(k_{1} - 1)!}.
\]

Choosing one of such \( (k_{1} - 1) \)-plets, we obtain a sub-walk \( \mathcal{W}_{[0, \xi_{\tau_{1}}]} \), if it exists, and continue its run till the value \( \xi_{\tau_{2}} - 1 \). Assuming that the vertex \( v_{2} \) of \( G^{k} \) is of the self-intersection degree \( k_{2} \), we choose of the \( (k_{2} - 1) \)-plets with the values from \( \Delta(\gamma_{2}) \), where \( \gamma_{2} = \mathcal{W}(\xi_{\tau_{2}} - 1) \). Then
we get the estimate of the form (4.18) with $k_1$ replaced by $k_2$. Let us stress that the value $\tau_1$
cannot be seen in the edge-boxes attached to $v_2$ by the agreement that $\tau_1$ is the last arrival
instant at the vertex of the self-intersection of the degree $k_1$.

Then we get the following upper bound,

$$
|\mathcal{W}(T(d_L, D'', \mathcal{L}(\ell_m)), (\mathcal{G})^*)| \leq \frac{d^{k-1}}{(\zeta - 1)!} \prod_{i=2}^d \frac{1}{\sigma_i!} \left( \frac{d^i}{i!} \right) \prod_{k=2}^s \frac{1}{\nu_k!} \left( \frac{s(d + D'')^{k-1}}{(k-1)!} \right)^{\nu_k},
$$

(4.19)

where $d = D' = |\tilde{d}|$.

Taking into account that $|C_{V_{2\ell}}| = n^{1+m+D'+\sum_{i=1}^s \nu_k (1+o(1))}$, $n \to \infty$ and using relations
(4.7), (4.8), (4.13), (4.14) and (4.15), we can deduce from (4.11) the following inequality,

$$
\hat{Z}_{2s} \leq nV_2^s \sum_{d=2}^s \sum_{D''=0}^s \sum_{m=1}^s \sum_{\ell_m} \mathcal{Q}_c(\tilde{\ell}_m) \sum_{\sigma} \mathcal{Q}_\Delta(D, \tilde{\sigma}) \sum_{\tilde{\nu}} \mathcal{Q}_g(D, \tilde{\nu}) \cdot |\mathbb{P}(d, D'', \mathcal{L}(\ell_m))|,
$$

(4.20)

where

$$
\mathcal{Q}_c(\tilde{\ell}_m) = \prod_{j=1}^m \left( \frac{U_{2j}^2}{V} \right)^{(j)}, \quad U_{2j}^2 = \frac{U_2^2}{V},
$$

$$
\mathcal{Q}_\Delta(d, \sigma) = \frac{d^{k-1}U_{2\sigma - 2}^{k-2}}{(\zeta - 1)! \rho^k - 1} \prod_{i=2}^d \frac{1}{\sigma_i!} \left( \frac{d^{i}U_{2i - 2}}{i! \rho^i - 1} \right)^{\sigma_i},
$$

$$
\mathcal{Q}_g(D, \tilde{\nu}) = \frac{1}{\nu_k!} \left( \frac{sD^{k-1}U_{2\nu_k - 2}}{(k-1)! \rho^{k-1}} \right)^{\nu_k}.
$$

Substituting these expressions into (4.20) and using (4.10), we obtain the following inequality,

$$
\hat{Z}_{2s} \leq nV_2^s \sum_{d=2}^s \sum_{m=1}^s \frac{dU_{2j}^2}{\rho} \cdot \exp \left[ \frac{dU_{2j}^2}{\rho} \right] \cdot \exp \left\{ \sum_{i=2}^d \frac{d^{i}U_{2i - 2}}{i! \rho^i - 1} + \sum_{k=2}^s \frac{s d^{k-1}U_{2\nu_k - 2}}{(k-1)! \rho^{k-1}} + \frac{cd}{h_0} - \eta d \right\},
$$

(4.21)

where we replaced $d + D''$ of (4.20) by $d$. Taking into account relation $s_n = \chi \rho_n$ and denoting
$\varepsilon = \chi U_{2j}^2$, we deduce from (4.21) that

$$
\hat{Z}_{2s} \leq nV_2^s \sum_{d=2}^s \frac{2\varepsilon \varepsilon^d}{d} \exp \left\{ \frac{2\varepsilon \varepsilon^d}{d} + \frac{cd}{h_0} - \eta d \right\},
$$

(4.22)

for sufficiently large $\rho$ such that $(1 - 2h_0 U_{2j}^2 / \rho) > 1/2$. It is clear that the last series of (4.22) converges if

$$
\frac{2e}{\eta} < h_0 \quad \text{and} \quad 8\varepsilon \varepsilon^\varepsilon < \eta.
$$

(4.23)

Therefore

$$
\limsup_{n \to \infty} \frac{1}{n \tau_s V_2^s} \hat{Z}_{2s} \leq C(V_2, U, \chi) = \frac{\eta}{4} \sum_{d=2}^\infty \frac{d e^{-\eta d/4}},
$$

and relation (4.1) follows.
Looking at the right-hand side of (4.21), it is easy to understand why that relation (4.2) is true. To explain this, let us write down the following version of the upper bound (4.20),

\[
\lambda_{2s, s}^{(26)} \leq nV_2 \sum_{d=2}^{s} \sum_{m=1}^{s} \sum_{\ell_m} \sum_{\sigma} \sum_{\nu} \left[ Q_L(\ell_m) Q_{\Delta}(d, \sigma) Q_{g}(d, \nu) \right]^* \cdot |\mathbb{T}(d, L(\ell_m))|, \tag{4.24}
\]

where the brackets \([\cdot]^*\) mean that at least one factor \(V_{2q}, q \geq 2\) is present in the weight of (4.13) or (4.14) or (4.15). We consider each of these three situations separately.

Let us consider the sum related with \(Q_{\Delta}\). Denoting \(\eta' = \eta - e/h_0\), we can write that

\[
\sum_{d \geq 2} e^{-\eta'd} \sum_{\sigma: (\zeta - 2) + \sum_{i=2}^{d} \sigma_i \geq 1} \frac{sd^{K-1}U_{V_2}^{2\zeta - 2}}{(\zeta - 1)! \rho^{K-1}} \cdot \prod_{i=2}^{d} \frac{1}{\sigma_i!} \left( \frac{d!U_{V_2}^{2i-2}}{i! \rho^{i-1}} \right) \leq \sum_{d \geq 2} e^{-\eta'd} \sum_{\zeta \geq 3} \frac{sd^{K-1}U_{V_2}^{2\zeta - 2}}{(\zeta - 1)! \rho^{K-1}} \exp\{\varepsilon \varepsilon^c\} + \sum_{d \geq 2} e^{-\eta'd} \varepsilon \varepsilon^c \exp\{\varepsilon \varepsilon^c\} \sum_{i=2}^{s} \frac{d!U_{V_2}^{2i}}{i! \rho^{i-1}}. \tag{4.25}
\]

The series of the right-hand side of (4.25) converge and we can change the order of summation there. Accepting that

\[
\sum_{d \geq 2} \frac{d!}{\eta'u^l} e^{-\eta'd} \leq \left( \frac{2}{\eta'} \right)^l, \quad l = 1, 2, \ldots, \tag{4.26}
\]

we conclude that the right-hand side of (4.25) is bounded from above by the following expression

\[
\varepsilon \varepsilon^c \sum_{\zeta \geq 3} \frac{s(2U_{V_2})^{\zeta - 1}}{(\eta' \rho)^{\zeta - 1}} + \varepsilon \varepsilon^c \exp\{\varepsilon \varepsilon^c\} \sum_{l=2}^{s} \frac{(2U_{V_2})^l}{\rho^{l-1} (\eta')^l} = O(\rho^{-1}). \tag{4.27}
\]

Let us consider the terms of (4.24) related with \(Q_{L}\). It is sufficient to estimate the following sum,

\[
\sum_{\ell_m: \sum_{j=1}^{\ell_m} l_j \geq 2} \prod_{j=1}^{\ell_m} \left( \frac{2U_{V_2}}{\rho} \right)^{l_j} \leq m \sum_{l=1}^{m} \left( \frac{U_{V_2}}{\rho^l} \right) \sum_{\ell_m} \prod_{j=1}^{\ell_m} \left( \frac{2U_{V_2}}{\rho} \right)^{l_j} = O(\rho^{-1}), \quad n \to \infty. \tag{4.28}
\]

In the last relation, we have taken into account that \(m \leq s\).

Let us consider the last situation, when the multiple edge with the weight \(V_{2k'}, k' \geq 3\) is present in the terms related with \(Q_{g}(d, \nu)\). We need to estimate the sum

\[
\sum_{d \geq 2} e^{-\eta'd} \sum_{\nu: \sum_{k=3}^{\nu} \nu_k \geq 1} \frac{1}{\nu_k!} \left( \frac{sd^{k-1}U_{V_2}^{2k-2}}{(k-1)! \rho^{k-1}} \right)^{\nu_k} \leq \sum_{d \geq 2} e^{-\eta'd} \exp\{\varepsilon d\} \sum_{l=3}^{s} \frac{sd^{l-1}U_{V_2}^{2l-2}}{(l-1)! \rho^{l-1}} \exp \left\{ \sum_{k=3}^{s} \frac{sd^{k-1}U_{V_2}^{2k-2}}{(k-1)! \rho^{k-1}} \right\} \leq \sum_{l=3}^{s} \frac{sU_{V_2}^{2l-2}}{\rho^{l-1}} \sum_{d \geq 2} e^{-\eta'd + \varepsilon de^c} \frac{d!}{(l-1)!}. \tag{4.29}
\]
Now it is clear that the right-hand side of (4.29) is bounded from above by the following expression,

\[
\frac{s}{\rho^2} \sum_{k=0}^{\infty} \left( \frac{f(\eta', \varepsilon) U^2_{\nu}}{\rho} \right)^k = O(\rho^{-1}), \quad n \to \infty,
\]

where \(f(\eta', \varepsilon)\) is expressed in terms of the last sum of (4.29).

It is easy to see that relations (4.27), (4.28) and (4.30) imply the following asymptotic estimate

\[
X_{2s}^{(\geq 6)} = O(nt_s V_2^s \rho^{-1}), \quad n \to \infty.
\]

This gives (4.2). Theorem 4.1 is proved. \(\square\)

4.4 Proof of Theorem 1.1

Let us consider a tree-type walk \(W_{2s} \in \tilde{\mathcal{W}}_{2s}\) and attribute to it the weight

\[
\Pi_{V, \rho}(W_{2s}) = \prod_{E \in \tilde{\mathcal{G}}(W_{2s})} \frac{V_{2s}^{|E|}}{\rho^{|E|-1}},
\]

where \(E\) denotes an oriented multi-edge of the multigraph \(\tilde{\mathcal{G}}(W_{2s})\) and \(|E|\) is the multiplicity of \(E\). In papers [16, 17] (see also [3]), it is shown that the sequence of numbers

\[
M_{s}^{(\nu, \rho)} = \sum_{W_{2s} \in \tilde{\mathcal{W}}_{2s}} \Pi_{V, \rho}(W_{2s}), \quad s = 1, 2, \ldots
\]

can be found with the help of a system of recurrent relations. This system is rather cumbersome and we do not present it here.

According to the results of Theorem 4.1, in the limit \(n, \rho_n \to \infty, \sigma_n = \chi \rho_n\), the leading contribution to the moments \(M_{2s_n}^{(n, \rho_n)}\) (1.4) is determined by the trajectories whose walks \(W_{2s}\) are the (2,4)-walks, \(W_{2s} \in \mathcal{X}_{2s}^{(2,4)}\). Theorems 3.1 and 4.1 imply the following asymptotic relation,

\[
M_{2s_n}^{(n, \rho_n)} = X_{2s_n}^{(2,4)}(n, \rho_n)(1 + o(1)) = n m_{s_n}^{(\rho_n)}(1 + o(1)), \quad n, \rho_n \to \infty, \sigma_n = \chi \rho_n.
\]

(4.32)

Slightly modifying the reasoning of [16] and [17], we can prove the following statement.

Lemma 4.2. Denote

\[
m_{s}^{(\nu, \rho)} = \sum_{W_{2s} \in \mathcal{X}_{2s}^{(2,4)}} \Pi_{V, \rho}(W_{2s}), \quad s = 1, 2, \ldots
\]

(4.32)

Then numbers \(m_{s}^{(\rho)}\) are determined by relations \(m_{s}^{(\rho)} = \sum_{r=0}^{s} S(s, r) \text{ and (1.8)}\).

We prove Lemma 4.2 in Section 6. One can interpret the result of Theorem 4.1 as the following asymptotic relation,

\[
M_{2s}^{(\nu, \rho)} = m_{s}^{(\rho)}(1 + o(1)), \quad s, \rho \to \infty, \quad s = \chi \rho.
\]

(4.36)
Recurrent relations (1.8) that determine \(m_s^{(\rho)}\) represent a simplified version of those that determine the sequence \(M_{2s}^{(V,\rho)}\). However, they are still cumbersome and difficult to study the limiting transition \(s, \rho \to \infty, s = \chi \rho\) and thus to prove equality (4.36) directly.

Regarding the expansion

\[ m_s^{(\rho)} = V_s^2 t_s + \sum_{l=1}^{[s/2]} V_s^{s-2l} \left( \frac{V_s}{\rho} \right)^l R_s^{(l)}, \tag{4.37} \]

we deduce from (5.4) that

\[ R_s^{(1)} = N_s^{(2)} = \left( s - \frac{3s}{s + 2} \right) t_s. \tag{4.38} \]

Therefore equality (4.37) can be transformed into the formal expansion of \(m_s^{(\rho)}\) as \(\rho \to \infty\) with respect to the powers of \(\chi\),

\[ m_s^{(\rho)} = V_s^s t_s \left( 1 + \frac{V_s}{V_s^2} \chi + \ldots \right), s = \chi \rho, \rho \to \infty. \]

The next terms of this asymptotic expansion are related with the limiting expressions

\[ P_l = \lim_{s \to \infty} \frac{1}{s^l} R_s^{(l)}. \tag{4.39} \]

Unfortunately, it seems to be difficult to get an explicit expression for \(R_s^{(l)}\), even in the simplest case of \(l = 2\). By analogy with (4.38) we expect \(R_s^{(2)}\) to be proportional to \(s^2 t_s\).

5 Auxiliary statements

In this section we collect the auxiliary statements and prove lemmas needed for the proof of Theorems 3.1 and 4.1. We also prove Lemma 4.2.

5.1 Proof of Lemma 3.1

Let us consider the \(q\)-vertex \(\beta\) such that the edge of the second arrival \(e_2 = e(a_2) = (\beta, \alpha_2)\) is the minimal \(q\)-edge over the whole walk \(W_{2s}\). We denote by \(t_2\) the instant of time such that \(e_2 = e(t_2)\) and consider the sub-walk \(W_{[0,t_2-1]} = W^*\) only.

If the edge \([\beta, \alpha_2]\) represents the second distinct arrival at \(\alpha_2\) by \(W^*\), then \(\alpha_2\) is the blue \(r\)-vertex and we are done. Let us consider the case when \([\beta, \alpha_2] = E'_1\) is the first distinct arrival at \(\alpha_2\) by \(W^*\) and denote by \(e'_{\text{max}} = \max\{e, e \in E'_1(\alpha_2)\}\). This edge \(e'_{\text{max}}\) is closed in \(W^*\) by a non-marked edge \(f\). We consider two possible orientations of \(f\) separately.

Let us consider first the cases when \(f = (\alpha_2, \beta)\). Then \(W^*\) has to go from \(\beta\) to \(\alpha_2\) after \(t(f)\) to create the \(q\)-edge \(e(t_2)\). It can arrive at \(\alpha_2\) only with by a non-marked step \(h = (\gamma, \alpha_2), \gamma \neq \beta\) that closes the marked edge \((\alpha_2, \gamma) = \hat{e}\). Thus, the sub-walk \(W^*\) has to go from \(\beta\) to \(\gamma\) to perform \(h\). If \(W^*\) arrives at \(\gamma\) by a marked edge \((\delta, \gamma)\), then \(\gamma\) is the blue \(r\)-vertex because \(\delta \neq \alpha_2\). If \(W^*\) arrives at \(\gamma\) by a non-marked step \((\delta, \gamma)\), then this step closes a marked
edge \{\delta, \gamma\}. If \{d, \gamma\} = (\delta, \gamma), then \gamma is the blue r-vertex. If \{\delta, \gamma\} = (\gamma, \delta), then we get the recurrence, where the couple \alpha_2, \gamma is replaced by \delta, \gamma. Since \kappa_{W^*}(\beta) = 1 by \mathcal{E}_1, then this recurrence will be terminated before we come to \beta and the r-vertex will be specified.

Let us consider the case when \( f = (\beta, \alpha_2) \). To perform this step, the sub-walk \( W^* \) has to go from \( \alpha_2 \) to \( \beta \) before \( t(f) \). Assume that it arrives at \( \beta \) by the step \( h = (\gamma, \beta), \gamma \neq \alpha_2 \) that has to be the non-marked one.

Let us consider first the case when \( \gamma \neq \alpha_1 \). The sub-walk has to go from \( \alpha_2 \) to \( \gamma \) and arrive at \( \gamma \) by the step \( g = (\delta, \gamma) \). If this step is marked, then \( \gamma \) is the blue r-vertex and we are done. If \( g \) is non-marked, then it closes the marked edge \{\gamma, \delta\}. If \{\gamma, \delta\} = (\delta, \gamma), then \( \gamma \) is the blue r-vertex. If \{\gamma, \delta\} = (\gamma, \delta), then we get a recurrence. Since \( \kappa_{W^*}(\alpha_2) = 1 \) by \( \mathcal{E}_1 \), then this recurrence will be terminated by a blue r-vertex.

Finally, let us consider the case when \( \gamma = \alpha_1 \) and \( h = (\alpha_1, \beta) \). Then the sub-walk has to go from \( \alpha_2 \) to \( \alpha_1 \) and arrive it by the step \( g = (\gamma, \alpha_1) \). If this step is marked, then \( \alpha_1 \) is the blue r-vertex. If \( g \) is non-marked, then either \( \gamma = \epsilon \) or \( \gamma \neq \epsilon \), where the edge \( (\epsilon, \alpha_1) \in \mathcal{E}_1(\alpha_1) \).

If \( g = \epsilon \), then we get a recurrence with the couple \( \alpha_1, \beta \) replaced by \( \epsilon, \alpha_1 \). Please note that the fact that \( (\epsilon, \alpha_1) \) generally is not the first arrival at \( \alpha_1 \) does not alter this recurrence. Then we terminate with the blue r-edge.

If \( \gamma \neq \epsilon \), then \( g \) closes a marked edge \{\gamma, \alpha_1\}. If \{\gamma, \alpha_1\} = (\gamma, \alpha_1), then \( \alpha_1 \) is the blue r-vertex. If \{\gamma, \alpha_1\} = (\alpha_1, \gamma), then we get a recurrence that will terminate before \( \alpha_2 \) and the blue r-vertex will be specified. Lemma 3.1 is proved. \( \square \)

### 5.2 Catalan trees and exponential estimates

The Catalan numbers \( \{t_s, s = 0, 1, 2, \ldots\} \) are determined by the following well-known recurrent relation (cf. (1.9)),

\[
t_s = \sum_{j=0}^{s-1} t_{s-1-j} t_j, \quad s \geq 1, \quad t_0 = 1.
\]

This relation can be used in the proof of the following statement proved in [13].

**Lemma 5.1.** Consider the family of Catalan trees constructed with the help of \( s \) edges and such that the root vertex \( \varrho \) has \( d \) edges attached to it and denote by \( t_s^{(d)} \) its cardinality,

\[
t_s^{(d)} = \sum_{u_1 + \cdots + u_{d-1} + u_d = s-d} t_{u_1} t_{u_2} \cdots t_{u_{d-1}} t_{u_d},
\]

where the sum runs over all possible \( u_i \geq 0 \). Then the upper bound

\[
t_s^{(d)} \leq e^{-\eta d} t_s, \quad \eta = \ln(4/3)
\]

is true for any given integers \( d \) and \( s \) such that \( 1 \leq d \leq s \).

**Remark.** According to the definitions of subsection 3.3, we can say that \( t_s^{(d)} \) represents the number of Catalan trees such that their root vertex \( \varrho \) has the exit sub-cluster of cardinality \( d \). The numbers \( \{t_s^{(d)}, 1 \leq d \leq s\} \) verify the following recurrent relation,

\[
t_s^{(d)} = t_s^{(d-1)} - t_{s-1}^{(d-2)}, \quad 3 \leq d \leq s
\]
with the initial values \( t_s^{(1)} = t_{s-1}, \ s \geq 1 \) and \( t_s^{(2)} = t_{s-1}, \ s \geq 2 \) (see papers [9, 13] for the proof). Using this relation, one can show that

\[
t_s^{(d)} = \begin{cases} 
(2m-1)t_{s-m} \prod_{i=1}^{m-1} \frac{s+1-m-i}{s+1-i}, & \text{if } d = 2m-1, \ m \geq 1, \\
mt_{s-m} \prod_{i=1}^{m-1} \frac{s-m-i}{s+1-i}, & \text{if } d = 2m, \ m \geq 0,
\end{cases}
\]  
(5.3)

Let us denote by \( \mathcal{N}^{(m)}_s \) the number of even closed walks \( \mathcal{W}_{2s} \) such that their graphs contain one edge of total multiplicity \( 2m \) and all other edges of multiplicity \( 2 \). We assume also that \( \mathcal{W}_{2s} \) has not other self-intersections and therefore is of the Dyck-type structure. It is argued in [13] that

\[
\mathcal{N}^{(m)}_s = \frac{(2s)!}{(s-m)!(s+m)!}, \ 2 \leq m \leq s.
\]  
(5.4)

In paper [13], this equality is proved for \( m = 2 \) and \( m = 3 \).

### 5.3 D-lemma

In the present subsection we prove Lemma 3.5. Let us introduce a collection of variables

\[
\mathcal{H} = (\tilde{m}_I, (\bar{A}, \Psi, \bar{f}^m)_J, (\Phi, \bar{f}^I)_K)
\]  
(5.5)

that represent a subset of parameters \( \mathcal{P}_R \) (3.17) and consider its numerical realization \( \langle \mathcal{H} \rangle \). Then relation (3.24) can be rewritten in the following form,

\[
|\sqcup_{\langle \mathcal{H} \rangle} \mathcal{W}_{2s}(D, (\bar{v} \uplus \mathcal{G}^{(c)}), (\mathcal{H}), \mathcal{Y})| \leq 2^{[i]D} D^{[\bar{v}]} (e^{\eta h^2_0})^{I+J+K} e^{-\eta D_0 D h_0} t_s,
\]  
(5.6)

where the disjoint union is taken over the set of all possible realizations \( \mathcal{H} = \{\langle \mathcal{H} \rangle\} \). The main observation related with (5.6) is that given a realization \( \langle \mathcal{P}_R \rangle \), the nest cells of the exit sub-clusters \( \Delta_1, \ldots, \Delta_R \) of the underlying trees \( T(\mathcal{W}_{2s}) \) are uniquely determined (see subsection 3.3.2 for rigorous definitions). Then we can apply inequalities of the form (5.3) to get the estimate of the set of underlying trees. This estimate is exponential with respect to the sum \( \sum_{i=1}^{R} d_i, \ d_i = |\Delta_i| \).

We prove (5.6) by recurrence with respect to \( N = I + J + K \).

#### 5.3.1 The case of \( N = 1, R = 1 \)

If \( R = 1 \), then either \( \mathcal{P}_1 = x_1 \) or \( \mathcal{P}_1 = z_1 \) and the set of variables \( \mathcal{H} \) is empty. We consider for simplicity the former case and denote \( x_1 = \tau_1 \). Regarding a walk \( \mathcal{W}_{2s} \) from the left-hand side of (5.6) and the corresponding tree \( T_s = T(\mathcal{W}_{2s}) \), we observe that its vertex \( \bar{v} \) such that \( \bar{v} = \mathcal{R}(\xi_{\tau_1}) \) is attached by a sub-cluster \( \Delta_1 \) of \( d_1 \) edges. We construct the set \( T_s(\tau_1, d_1) \) with the help of the following procedure.

Let us consider a root vertex \( \varrho \) and attach to it a linear branch \( \mathcal{B} \) of \( l \) edges. The first \( l \) vertices ordered by the chronological run over \( \mathcal{B} \) (including \( \varrho \)) are considered as the sub-roots for the subtrees \( T_{a_1}, \ldots, T_{a_l} \) with given \( \bar{a} = (a_1, \ldots, a_l) \) such that \( |\bar{a}| = a_1 + \cdots + a_l = \tau_1 - l \). We denote the tree obtained by \( T_s(\tau_1, d_1)(\bar{a}) \).

The extreme vertex of \( \mathcal{B} \) being \( \bar{v} \), we attach to it the sub-cluster \( \Delta_1 \) of \( D = d \) edges. Using \( d + l \) vertices obtained, we construct \( d + l \) sub-trees \( T_{b_1}, \ldots, T_{b_d}, T_{c_1}, \ldots, T_{c_l} \).
Then we can write that
\[ T_s(\tau_1, d) = |T_s(\tau_1, d)| = \sum_{l=1}^{\tau_1} \sum_{\tilde{a}, \tilde{b}, \tilde{c}} t_{a_1} \cdots t_{a_l} \times t_{b_1} \cdots t_{b_d} \times t_{c_1} \cdots t_{c_l}, \]  
(5.7)
where the second sum is such that $|\tilde{a}| = \tau_1 - l$ and $|\tilde{b}| + |\tilde{c}| = s - \tau_1 - d$. It follows from (5.3) that
\[ \sum_{\tilde{b}: |\tilde{b}| = m} t_{b_1} \cdots t_{b_d} \leq e^{-\eta d} t_{m+d}. \]
(5.8)
Then we can deduce from (5.7) the following inequality,
\[ T_s(\tau_1, d) \leq e^{-\eta d} \sum_{l=1}^{\tau_1} \sum_{\tilde{a}, \tilde{b}, \tilde{c}} t_{a_1} \cdots t_{a_l} \times t_{b} \times t_{c_1} \cdots t_{c_l} \leq e^{-\eta d} t_s. \]
(5.9)
Using the filtration estimate (3.9), we get the needed upper bound,
\[ |W_{2s}(D, \tilde{v}(x_1) \uplus \langle G^{(c)}(\tilde{p}, \tilde{\rho}, \tilde{q})^{(b)}_s, \tau) \rangle| \leq 2^{|d|} D^{|\tilde{p}|} e^{-\eta D} t_s. \]
(5.10)

5.3.2 The cases of $N = 2, R = 2$

a) Let $\mathcal{P}_2$ be such that $x_1 = \tau_1, x_2 = \tau_2$. Then the collection of parameters $\mathcal{H}$ (5.5) is empty and the situation is very close to the one of the case when $N = 1$ and $R = 1$. The mirror cells are not present at $\tilde{\beta}$ and therefore the tree $T_s = T(W_{2s})$ is such that the vertices $\tilde{v}_1$ and $\tilde{v}_2$ lie on two different branches. Thus we can apply (5.8) two times with respect to the sub-trees with the exit sub-clusters $\Delta_1$ and $\Delta_2$ and get the estimate
\[ \sum_{\hat{b}^{(1)}: |\hat{b}^{(1)}| = m_1} t_{b_1^{(1)}} \cdots t_{b_d^{(1)}} \times \sum_{\hat{b}^{(2)}: |\hat{b}^{(2)}| = m_2} t_{b_1^{(2)}} \cdots t_{b_d^{(2)}} \leq e^{-\eta(d_1 + d_2)} t_{m_1 + d_1} t_{m_2 + d_2} \]
(5.11)
Then one can write down an inequality of the form (5.9) that gives the exponential bound with respect to $D = d_1 + d_2$. The number of ways to distribute $D$ balls over two boxes is given by $D + 1$ and we conclude that (5.6) is verified in the case of $I = 2$.

b) Let us consider the case of two cells $\mathcal{P}_2 = (x_1, (y_1, \Lambda))$ such that $x_1 = \tau_1$ and $y_1 = \tau_2$ are given as well as a particular value $\langle \Lambda \rangle$. Then $\tilde{\beta}$ is attributed by one proper cell and one imported cell and $\langle \mathcal{H} \rangle = \langle \Lambda \rangle$.

We first study the case when $y_1$ does not fill the edge-box attached to a red or to a green vertex of $G^{(c)}$. There is no mirror cells at $\tilde{\beta}$ and therefore the vertices $\tilde{v}_1 = \mathfrak{R}(\xi_{\tau_1})$ and $\tilde{v}_2 = \mathfrak{R}(\xi_{\tau_2})$ are situated on different branches of $T_s$. The the form of $T_s$ is as follows:

there is a branch $\mathcal{B}_1$ of the length $l_1$ with $l_1$ sub-trees to the left of the vertex $\tilde{v}_1$; this vertex is attached by the sub-cluster $\Delta_1$ of $d_1$ edges; the second branch $\mathcal{B}_2$ of the length $l_2$ is attached to one of the vertices of the descending part of $\mathcal{B}_1$, let us denote this vertex by $v'$.
Then $\phi$ steps down are performed along the descending part of $\mathcal{B}_2$ and the remaining part of $\mathcal{B}_1$ from $v'$ to $\rho$ and the vertex $\tilde{v}_2$ is determined to be attached by the sub-cluster $\Delta_2$ of $d_2$ edges.

The remaining $s - l_1 - l_2 - d_1 - d_2$ edges are used to construct the sub-trees on $2l_1 + 2l_2 + d_1 + d_2 + 1$ vertices. It is clear that we get the expression similar to the right-hand side
of (5.7), but with two sub-clusters. Therefore we can use again inequality (5.11) to get the exponential estimate of the number of trees.

Since the edge-box of $y_1$ is attached to a blue vertex $v'$ of $\mathcal{G}^{(2)}$, then the corresponding vertex of self-intersection $\gamma'$ of $g(\mathcal{W}_{[0,\xi_{\tau_2}]}^{(2)})$ is uniquely determined. Given $\Upsilon$, the path from $\gamma'$ to $\bar{\beta}$ as well as its length $\Lambda'$ are uniquely determined too. Therefore the union with respect to $\langle \mathcal{H} \rangle = \langle \Lambda \rangle$ in the left-hand side of (5.6) represents only one element such that $\langle \Lambda \rangle = \Lambda'$. Now it is clear that the upper bound (5.11) can be also used in the case under consideration and relation (5.6) is true.

Let us point out that the observation that the union $\biguplus_{\phi}$ disappears from the estimates obtained is true in general situation. In fact, the auxiliary variables $\mathcal{H}$ (5.5) are introduced to ease the construction of the sub-trees with given nest cells.

c) Let us consider the case of $\langle \mathcal{P}_2 \rangle = (\tau_1, (\tau_2, \Lambda))$ such that $y_1 = \tau_2$ fills the edge-box than belongs to a red vertex or to a green vertex of $\mathcal{G}^{(c)}$ and this edge-box represents the edge of the second arrival at the corresponding vertex.

We first construct a tree $\tilde{T} = \tilde{T}_{\tau_2}$ with the help of $l_1 + d_1 + j_1 + l_2 - 1$ sub-trees on $l_1$ ascending vertices of $B_1$, $d_1$ vertices of $\Delta_1$, $j_1$ of descending vertices of $B_1$ till the vertex $v'$, and $l_2 - 1$ ascending vertices of the second branch $B_2$.

Then we perform the run of the sub-walk $\mathcal{W} = \mathcal{W}_{[0,1_{l_2-1}]}^{(\tilde{T})}$, $t_2 = \xi_{\tau_2}^{(\tilde{T})}$ following the prescriptions of $\langle \tilde{v} \cup \mathcal{G}^{(c)} \rangle_{s}^{(b)}$ and the rule $\Upsilon$. The vertex $\alpha = \mathcal{W}$ being determined, the exit cluster $\Delta(\alpha) = \{\gamma_1, \ldots, \gamma_m\}$ is also uniquely determined.

At the instant of time $\bar{\xi}_{\tau_2}$ the walk has to choose a vertex from $\Delta(\alpha)$ such that is situated on the distance of $\langle \Lambda \rangle = \phi$ non-marked steps from $\bar{\beta}$.

We denote this choice by $\langle \mathcal{W}(\bar{\xi}_{\tau_2}) \rangle_{\phi}$. We see that the union with respect to all possible values of $\phi$ permits to include into consideration all vertices of $\Delta(\alpha)$ possible, and the number of the vertices admitted is still not greater than $m$. Then we construct the remaining part of the walk $\mathcal{W}_{[\bar{\xi}_{\tau_2}+1,2s]}^{(\tilde{T})}$.

Denoting $\mathcal{W}_{x_2s} = \mathcal{W}_{x_2s}(d_1, d_2; \langle \tilde{v} \cup \mathcal{G}^{(c)} \rangle_{s}, \langle \mathcal{H} \rangle, \Upsilon)$, we can write the following equality

$$\mathcal{W}_{x_2s}^{\ast} = \biguplus_{\tilde{\gamma}} \left\{ \mathcal{W}_{[0,\tilde{\xi}_{\tau_2-1}]} \right\} \otimes \left\{ \mathcal{W}(\bar{\xi}_{\tau_2})_{\phi} \right\} \otimes \biguplus_{\tilde{T}} \left\{ \mathcal{W}_{[\bar{\xi}_{\tau_2}+1,2s]}^{(\tilde{T})} \right\},$$

where the curly brackets denote the families of realizations of corresponding sub-walks and the values of $\langle \mathcal{W}(\bar{\xi}_{\tau_2}) \rangle_{\phi}$. Then

$$| \biguplus_{\phi} \mathcal{W}_{x_2s}^{\ast} | \leq \prod_{\tilde{\gamma}} \# \left\{ \mathcal{W}_{[0,\tilde{\xi}_{\tau_2-1}]} \right\} \times \sum_{\phi} \# \left\{ \mathcal{W}(\bar{\xi}_{\tau_2})_{\phi} \right\} \times e^{-\eta d_2} 2^{q''} D^{p''-1} \# \left\{ \tilde{T} \right\},$$

where denotations $q''$ and $p''$ have an obvious meaning. We have shown that

$$\sum_{\phi} \# \left\{ \mathcal{W}(\bar{\xi}_{\tau_2})_{\phi} \right\} \leq m.$$

Using this inequality and obvious estimates

$$\# \left\{ \mathcal{W}_{[0,\tilde{\xi}_{\tau_2-1}]} \right\} \leq 2^{q'} D^{p'},$$

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and
\[ \prod_{\delta} 1 \cdot \# \{ \tilde{T} \} \leq t_s, \quad (5.14) \]
we deduce from (5.13) the following upper bound
\[ |\bigcup_{\delta} \mathcal{W}^s_{2s}| \leq 2^{|\bar{\delta}|} D^{\bar{|\delta|}} e^{-\eta(d_1+d_2)} t_s. \]

Now it is easy to show that (5.6) is true in the case under consideration.

### 5.3.3 The cases of \( N = 2, R \geq 3 \)

Let us consider \( \mathcal{P}_R \) such that three cells at \( \tilde{\beta} \) are given by \((x_1, x_2)\) with the third one represented by the mirror cell. The presence of one mirror cell, \( m_2 = 1 \) means that in the Dyck-type part of the walk, and in the corresponding tree the vertices \( v_1 = \mathcal{R}(\xi_{x_1}) \) and \( v_2 = \mathcal{R}(\xi_{x_2}) \) lie on the same branch of edges that starts at the root vertex \( \varrho \). Therefore the tree \( T \) is of the following structure: we choose a length \( l_1 \) and construct the branch \( \mathcal{B}_1 \) of \( l_1 \) edges that starts at \( \varrho \) and ends by \( v_1 \). Then we attach to \( v_1 \) another linear branch \( \mathcal{B}_2 \) of \( l_2 \) edges that ends by \( v_2 \). We attach the exit sub-cluster \( \Delta_1 \) to \( v_1 \) at the instant \( l_1 \) of the chronological run \( \mathcal{R}(\mathcal{B}_1 \cup \mathcal{B}_2) \) and the sub-cluster \( \Delta_3 \) to the vertex \( v_1 \) at the instant \( l_1 + 2l_2 + 1 \) of \( \mathcal{R}(\mathcal{B}_1 \cup \mathcal{B}_2) \).

Then we attach the sub-cluster \( \Delta_2 \) at the vertex \( v_2 \). Regarding \( 2(l_1 + l_2) + 1 + d_1 + d_2 + d_3 \) vertices of the obtained construction, we attach to them the sub-trees of the total number of edges \( s = (l_1 + l_2 + d_1 + d_2 + d_3) \). Then we obviously get the exponential estimate we need for the number of trees obtained.

To complete the study of the initial step of the proof of Lemma 5.1, we consider the case of numerous imported cells of the form \( \mathcal{P}_R = (z_1, (y_1, \Lambda, \psi_1, \ldots, \psi_f)) \), where \( f = f_1^R \) and \( R = 3 + f \). We assume for simplicity that \((z_1, y_1) = (\tau_1, \tau_2)\) and that \( \tau_1 < \tau_2 \). The reasoning presented below can be applied without any changes to the case of imported cells generated by the local BTS instants \((z_1, z_2) = (\tau_1, \tau_2)\). Let us point out that in this situation the number \( f \) can take one of two values, either \( f = 0 \) or \( f = 1 \) (see inequality (3.22)). However, we include into considerations the general case of greater values of \( f \). Another remark is that we can ignore the presence of the proper cell \( z_1 = \tau_1 \) with the exit sub-cluster \( \Delta_1 \) at \( \tilde{\beta} \) and consider the imported cells and corresponding exit sub-clusters only. We also assume for simplicity that \( y_2 \) is attributed to a blue \( r \)-vertex \( \nu' \) of \( G^{(c)} \).

To get a realization of \( (\mathcal{H}) \), we take an integer \( f \) and then attribute numerical values to the variables \( \Lambda, \psi_1, \ldots, \psi_f \). Let us take a tree \( \tilde{T} = \tilde{T}_{\tau_2} \) and consider a part of the chronological run over it \( \mathcal{R}[0,t'] \) with \( t' = \xi_{\tau_2} \). Following this run, we construct a sub-walk \( \mathcal{W}[0,t'-1] \) according to the rules prescribed by \( (\tilde{v} \cup G^{(c)})_s \) and \( \Upsilon \). At the instant of time \( t'_1 \), the walk has to join a vertex \( \gamma \) of \( g(\mathcal{W}[0,t'-1]) \) prescribed by the values of marked instants of the edge-boxes attached to \( \nu' \). This vertex \( \gamma \) is uniquely determined and therefore we are able to conclude whether the set of numerical data \( f, (\langle \Lambda, \psi_1, \ldots, \psi_f \rangle) \) is compatible with \( \mathcal{W}[0,t'-1] \) or not. We mean that is becomes clear whether there exists a path from \( \gamma \) to \( \tilde{\beta} \) of \( \Lambda \) non-marked steps that the walk can perform according to the rules \( \Upsilon \) or not. The same concern \( f \) consecutive returns to \( \tilde{\beta} \) with the help of \( \psi_1 \) non-marked steps.

The \( f + 1 \) nest cells are uniquely determined in \( \tilde{T}_{\tau_2} \) and the exit sub-clusters of the total cardinality \( D = D - (f + 1) \) are to be distributed to these nest cells. Let us denote by \( d_{f+1} \)
this distribution. We also denote by $T_s(\hat{T}_2 \cup \{\hat{\Delta}_1, \ldots, \hat{\Delta}_{f+1}\})$ a collection of Catalan trees constructed over the base tree $\hat{T}$ with the exit sub-clusters $\hat{\Delta}_i$ attached.

Using (5.8) and (5.11) several times, one can easily prove the exponential estimate for the number of trees

$$|T_s(\hat{T}_2 \cup \{\hat{\Delta}_1, \ldots, \hat{\Delta}_{f+1}\})| \leq e^{-\eta D}|T_s(\hat{T}_2)|.$$  

(5.15)

By changing somehow the point of view, we can say that given $\langle \hat{\varepsilon} \cup \mathcal{G}^{(c)} \rangle_s^{(b)}$ and $\mathcal{Y}$, the set of all possible values of $\varepsilon$ and $\left\langle (\Lambda, \psi, \ldots, \psi_f) \right\rangle$ is filtered by the run of the walk $\mathcal{W}_{[0, \nu'-1]}$. The values $f$ and $\hat{D} = D - (f + 1)$ depend on the realization of $\mathcal{W}_{[0, \nu'-1]}$. With the help of the filtration principle and relation (5.15), we get the following inequality,

$$|\bigcup_{\mathcal{Y}} \mathcal{W}^{(\hat{T})}_s(D, \langle \hat{\varepsilon} \cup \mathcal{G}^{(c)} \rangle_s^{(b)}, \langle \mathcal{H} \rangle, \mathcal{Y})| \leq 2^{\bar{\eta}h_0}f^{\bar{\eta}h_0} \sup_f \left\{ e^{\eta(f+1)}\left(\frac{\hat{D} + f}{f}\right) \right\} e^{-\eta D}|T_s(\hat{T}_2)|,$$  

(5.16)

where the superscript $\hat{T}$ means that the walks have this tree as the first part of the underlying trees. Taking into account the upper bound $f \leq K = 1$ (see (3.22)), we can apply to the right-hand side of (5.16) relations (3.21) and (3.23) and write that

$$\sup_f \left\{ e^{\eta(f+1)}\left(\frac{\hat{D} + f}{f}\right) \right\} \leq e^{2\eta h_0^2}e^{D/h_0}.$$  

(5.17)

Using (5.14), we get from (5.16) and (5.17) the upper bound (5.6).

### 5.3.4 The general step of recurrence

The general step of recurrent estimate of (5.6) is to show that if this estimate is true for $N = I + J + K$, then it is true in the case of $N' = N + 1$, where $N' = I' + J' + K'$. Let us consider the case when $K' = K + 1$ and $I' = I$, $J' = J$. This means that if the set $(\bar{x}_I, \bar{y}_J, \bar{z}_k)$ is represented by $N$ marked instants of time $\tau_1 < \tau_2 < \cdots < \tau_N$, then $\tau_{N+1} > \tau_n$ and $\tau_{K+1} = \tau_{N+1}$. Obviously, the numbers $f'_n, f'_K = f$ and $\bar{\varphi}(K+1) = (\varphi_1(K+1), \ldots, \varphi_f(K+1))$ are also joined to the set of parameters $\langle \mathcal{P}_R \rangle$ (3.17).

Let us briefly describe the steps that we perform to get the estimate needed. Regarding the vertices and the edge-boxes of realization of the color diagram $\langle \mathcal{G}^{(c)}(\bar{p}, \bar{q}, \bar{n}) \rangle_s^{(b)}$, we separate the edge-boxes of each vertex into two groups in dependence of whether the values in the boxes are less than $\tau_{N+1}$ or greater than $\tau_{N+1}$. Clearly, the vertex attached by the edge-box with $\tau_{N+1}$ plays a special role here. By this procedure, we obtain realizations of two sub-diagrams $\langle \hat{\mathcal{G}} \rangle$ and $\langle \bar{\mathcal{G}} \rangle$ determined in obvious way.

The underlying trees $T_s = T(\mathcal{W}_{2s})$ of the walks are to be of the following structure: there exists a branch $\mathcal{B}_{N+1}$ from the vertex $\bar{q}$ to $\hat{u}_{N+1}$ of the total length $\lambda_{N+1}$ not less than $\bar{\varphi}(K+1) = \sum_{i=1}^f \varphi_i(K+1)$. At the vertex $\hat{u}_{N+1}$ and corresponding $f$ vertices of the descending part of $\mathcal{B}_{N+1}$, the sub-clusters of the total number of $D_{N+1}$ edges are attached. Then the remaining edges are used to construct sub-trees attached to $l_{N+1} + D_{N+1} - f$ vertices. We denote this part of $T$ by $\hat{T}$.

It is clear that the set of the walks under consideration can be represented in the form of the right-hand side of (5.12) with $\tau_2$ replaced by $\tau_{N+1}$ and that the exponential estimate with the factor $e^{-\eta D_{N+1}}$ can be obtained for the family of trees $\{\hat{T}\}$ (see also inequality (5.13),

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where \( q'' \) and \( p'' \) are determined with the help of sub-diagram \( G \). Using (5.17), it is not hard to complete the proof of (5.6) in the case of \( N' = N + 1 \). We omit the detailed computations here because they repeat in major part those performed earlier in this sub-section (see also [13] for more discussion of the general step of recurrent estimates).

5.4 \( D \)-lemma for tree-type walks

Let us prove Lemma 4.1. We consider a family of trees \( T_s \in \mathcal{T}(d_1, \ldots, d_L, L(\ell)) \) such that the vertices \( \nu^{(m)}_j, 1 \leq j \leq L \) of the set \( \mathcal{F}(m) \) of \( L(\ell) \) are attached by the sub-trees constructed with the help of \( d_j + a_j \) edges, respectively, where \( a_j \geq 0 \) and \( \sum_{j=1}^L d_j = D' \). Relation (5.3) implies the following inequality,

\[
t^{(d_j)}_{d_j + a_j} \leq e^{-\eta d_j} t_{d_j + a_j}.
\]

Then

\[
|\mathcal{T}(d_1, \ldots, d_L, L(\ell))| \leq e^{-\eta D'} t_s.
\]

(5.18)

Using (4.7) and (4.8), it is easy to deduce from (5.18) the upper bound (4.9).

It is easy to see that the presence of \( D'' = D - D' \) additional edges implies the estimate of the form (5.18),

\[
|\mathcal{T}(d_1, \ldots, d_L, D'', L(\ell))| \leq e^{-\eta D'' t_s}.
\]

This inequality together with the upper bound (4.8) rewritten appropriately implies (4.10). Lemma 4.1 is proved.

5.5 Total weight of tree-type (2,4)-walks

We prove Lemma 4.2 with the help of reasoning of [16] (see also [17]). Let us denote by \( S(k, r) \) the sum of weights (4.31) of the walks \( W_{2s} \in X^{(2,4; r)}_{2s} \) such that \( W_{2s} \) performs \( r \) steps out of the root vertex \( \varrho \). We represent \( X^{(2,4; r)}_{2s} \) as a disjoint union of the walks of two types,

\[
X^{(2,4; r)}_{2s} = \bar{X}^{(2,4; r)}_{2s} \sqcup \bar{X}^{(2,4; r)}_{2s},
\]

where \( \bar{X}^{(2,4; r)}_{2s} \) contains the walks such that their first step is \((\varrho, \alpha_1)\) and that the corresponding edge \((\varrho, \alpha_1)\) is passed two times, there and back. Then it is clear that

\[
\mathcal{X}^{(2,4; r)}_{2s} = \sum_{W_{2s} \in \bar{X}^{(2,4; r)}_{2s}} \Pi_{\nu, \rho}(W_{2s}) = V_2 \sum_{u=0}^{k-r} \sum_{v=0}^{u} S^{(\rho)}(u, v) S^{(\rho)}(k - u - 1, r - 1).
\]

(5.19)

Let us consider the family of the walks \( \bar{X}^{(2,4; r)}_{2s} \) such that their first step is \((\varrho, \alpha_1)\) and that the edge \((\varrho, \alpha_1)\) is passed four times there and back. Let us denote by \( a_2 \) the second arrival at \( \alpha_1 \) by the edge \((\varrho, \alpha_1)\). We get the following relation,

\[
\mathcal{X}^{(2,4; r)}_{2s} = \sum_{W_{2s} \in \bar{X}^{(2,4; r)}_{2s}} \Pi_{\nu, \rho}(W_{2s})
\]
\[ = \frac{V_s}{\rho} \sum_{s=0}^{k-r} (r-1) \sum_{v=0}^{u} (v+1) S^{(\rho)}(u,v) S^{(\rho)}(k-u-2, r-2), \ k \geq r \geq 2. \quad (5.20) \]

The factor \( v+1 \) is due to the fact that the walk \( W_{2s} \in \mathcal{X}_{2s}^{(2,4;r)} \) that performs \( v \) marked steps out of the vertex \( \alpha \) can perform some of them either before or after the second arrival \( \alpha_2 = (\rho, \alpha_1) \). The same concerns \( r-2 \) steps out of \( \rho \) by marked edges other than \((\rho, \alpha_1)\). This gives the factor \( r-1 \).

Taking into account that

\[ \lambda_2^{(2,4)} = \sum_{r=1}^{s} \lambda_2^{(2,4;r)} + \sum_{r=1}^{s} \lambda_2^{(2,4;r)}, \]

we conclude that \( m_s^{(\rho)} = \sum_{r=1}^{s} S(s, r) \), where \( S(s, r) \) verifies relation (1.8a). Relation (1.8b) follows directly from definitions. The initial value \( S(s, 0) = \delta_{s,0} \) makes relations (1.8) true for all values of \( r \leq s \).

### 6 Summary and discussion

We have studied asymptotic behavior of the moments of large dilute random matrices

\[ M_{2s}^{(n,\rho)} = \mathbb{E} \left( \text{Tr} \left( H^{(n,\rho)} \right)^{2s} \right) = \mathbb{E} \left( \sum_{j=1}^{n} \lambda_j^{(n,\rho)} \right)^{2s}, \quad (6.1) \]

where \( \lambda_j^{(n,\rho)} = \lambda_j(H^{(n,\rho)}) \) are the eigenvalues of \( H^{(n,\rho)} \), in the limit of the strong dilution given by (1.6). We have shown that the leading contribution to \( M_{2s}^{(n,\rho_n)} \) with \( s_n = \chi \rho_n, \chi > 0 \) is related with the total weight of \( (2,4) \)-type walks that depends on the second and the fourth moments of the matrix entries of \( H^{(n,\rho)}, V_2 \) and \( V_4 \), respectively and does not depend on the moments \( V_{2k} \) with \( k \geq 3 \).

This situation is opposite to the case of weakly diluted random matrices \( H^{(n,\rho_n)} \) when \( \rho_n = O(n^{2/3+\epsilon}), n \to \infty \) with \( 0 < \epsilon < 1/3 \). In this case the limiting expressions for \( M_{2s}^{(n,\rho_n)} \) with \( s_n = \chi n^{2/3}, \chi > 0 \) are related with \( V_2 \) and do not depend on the higher moments \( V_4, V_6, \ldots \) of the matrix entries of \( H^{(n,\rho_n)} \) [13]. The situation here is similar to the case of the Wigner ensemble of random matrices \( A^{(n)} \) (1.1), where the fact of asymptotic independence of the moments (6.1) of \( V_{2k} \), \( k \geq 2 \) is regarded as an evidence of the universal behavior of the spectral statistics at the edge of the limiting spectra [28] (see also [6]). In particular, one can conclude that the probability distribution of the maximal in absolute value eigenvalue \( \lambda_{\text{max}}(A^{(n)}) = \max_j |\lambda_j(A^{(n)})| \) follows the famous Tracy-Widom law (see e.g. monograph [1] and references therein). It should be noted that a part of reasoning of [28] should be supplemented with the studies of the moment analog of the inverse participation ratio of random matrices (see, for example, [12]).

Regarding the result of Theorem 1.1, we deduce that the Tracy-Widom law cannot be valid for the ensemble of strongly dilute random matrices \( H^{(n,\rho)} \) (1.2) while the results of paper [13] support its validity in the case of weakly diluted random matrices. It is known [9] that the maximal eigenvalue of strongly diluted random matrices \( \lambda_{\text{max}}^{(n,\rho_n)} \) converges to \( 2\sqrt{\nu} \).
provided $\rho_n = (\log n)^{1+\varepsilon}$, $\varepsilon > 0$. However, instead of the Tracy-Widom distribution of the rescaled $\lambda_{\text{max}}^{(n,\rho_n)}$, another distribution could appear that would depend on both parameters, $V_2$ and $V_4$. Therefore one can put forward a conjecture about a new type of universality, say $(2,4)$-universality at the border of the spectrum of strongly diluted random matrices.

It should be noted that this universality, however, can be ensemble-sensitive in the sense that for certain types of the dilution different from that determined by $H^{(n,\rho)}$ (1.2) the Tracy-Widom law is still valid [27].

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