Rigorous Results for the Periodic Oscillation of an Adiabatic Piston

by

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Abstract

We study a heavy piston of mass $M$ that moves in one dimension. The piston separates two gas chambers, each of which contains finitely many ideal, unit mass gas particles moving in $d$ dimensions, where $d \geq 1$. Using averaging techniques, we prove that the actual motions of the piston converge in probability to the predicted averaged behavior on the time scale $M^{1/2}$ when $M$ tends to infinity while the total energy of the system is bounded and the number of gas particles is fixed. Neishtadt and Sinai previously pointed out that an averaging theorem due to Anosov should extend to this situation.

When $d = 1$, the gas particles move in just one dimension, and we prove that the rate of convergence of the actual motions of the piston to its averaged behavior is $O(M^{-1/2})$ on the time scale $M^{1/2}$. The convergence is uniform over all initial conditions in a compact set. We also investigate the piston system when the particle interactions have been smoothed. The convergence to the averaged behavior again takes place uniformly, both over initial conditions and over the amount of smoothing.

In addition, we prove generalizations of our results to $N$ pistons separating $N + 1$ gas chambers. We also provide a general discussion of averaging theory and the proofs of a number of previously known averaging results. In particular, we include a new proof of Anosov’s averaging theorem for smooth systems that is primarily due to Dolgopyat.
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Chapter 1

Introduction

What can be rigorously understood about the nonequilibrium dynamics of chaotic, many particle systems? Although much progress has been made in understanding the infinite time behavior of such systems, our understanding on finite time scales is still far from complete. Systems of many particles contain a large number of degrees of freedom, and it is often impractical or impossible to keep track of their full dynamics. However, if one is only interested in the evolution of macroscopic quantities, then these variables form a small subset of all of the variables. The evolution of these quantities does not itself form a closed dynamical system, because it depends on events happening in all of the (very large) phase space. We must therefore develop techniques for describing the evolution of just a few variables in phase space. Such descriptions are valid on limited time scales because a large amount of information about the dynamics of the full system is lost. However, the time scales of validity can often be long enough to enable a good prediction of the observable dynamics.

Averaging techniques help to describe the evolution of certain variables in some physical systems, especially when the system has components that move on different time scales. The primary results of this thesis involve applying averaging techniques to chaotic microscopic models of gas particles separated by an adiabatic piston for the purposes of justifying and understanding macroscopic laws.

This thesis is organized as follows. In Section 1.1 we briefly introduce the the adiabatic piston problem and our results. In Section 1.2 we review the physical motivations for our results. The following three chapters may each be read independently. Chapter 2 presents an introduction to averaging theory and the proofs of a number of averaging theorems for smooth systems that motivate our later proofs for the piston problem. Chapter 3 contains our results for piston systems in one dimension, and Chapter 4 contains our results for the piston system in dimensions two and three.
1.1 The adiabatic piston

Consider the following simple model of an adiabatic piston separating two gas containers: A massive piston of mass $M \gg 1$ divides a container in $\mathbb{R}^d$, $d = 1, 2, \text{ or } 3$, into two halves. The piston has no internal degrees of freedom and can only move along one axis of the container. On either side of the piston there are a finite number of ideal, unit mass, point gas particles that interact with the walls of the container and with the piston via elastic collisions. When $M = \infty$, the piston remains fixed in place, and each gas particle performs billiard motion at a constant energy in its sub-container. We make an ergodicity assumption on the behavior of the gas particles when the piston is fixed. Then we study the motions of the piston when the number of gas particles is fixed, the total energy of the system is bounded, but $M$ is very large.

Heuristically, after some time, one expects the system to approach a steady state, where the energy of the system is equidistributed amongst the particles and the piston. However, even if we could show that the full system is ergodic, an abstract ergodic theorem says nothing about the time scale required to reach such a steady state. Because the piston will move much slower than a typical gas particle, it is natural to try to determine the intermediate behavior of the piston by averaging techniques. By averaging over the motion of the gas particles on a time scale chosen short enough that the piston is nearly fixed, but long enough that the ergodic behavior of individual gas particles is observable, we will show that the system does not approach the expected steady state on the time scale $M^{1/2}$. Instead, the piston oscillates periodically, and there is no net energy transfer between the gas particles.

The results of this thesis follow earlier work by Neishtadt and Sinai [Sin99, NS04]. They determined that for a wide variety of Hamiltonians for the gas particles, the averaged behavior of the piston is periodic oscillation, with the piston moving inside an effective potential well whose shape depends on the initial position of the piston and the gas particles’ Hamiltonians. They pointed out that an averaging theorem due to Anosov [Ano60, LM88], proved for smooth systems, should extend to this case. The main result of the present work, Theorem 4.1.1, is that Anosov’s theorem does extend to the particular gas particle Hamiltonian described above. Thus, if we examine the actual motions of the piston with respect to the slow time $\tau = t/M^{1/2}$, then, as $M \to \infty$, in probability (with respect to Liouville measure) most initial conditions give rise to orbits whose actual motion is accurately described by the averaged behavior for $0 \leq \tau \leq 1$, i.e. for $0 \leq t \leq M^{1/2}$.

A recent study involving some similar ideas by Chernov and Dolgopyat [CD06a] considered the motion inside a two-dimensional domain of a single heavy, large gas particle (a disk) of mass $M \gg 1$ and a single unit mass point particle. They assumed that for each fixed location of the heavy particle, the light particle moves
inside a dispersing (Sinai) billiard domain. By averaging over the strongly hyperbolic motions of the light particle, they showed that under an appropriate scaling of space and time the limiting process of the heavy particle’s velocity is a (time-inhomogeneous) Brownian motion on a time scale $O(M^{1/2})$. It is not clear whether a similar result holds for the piston problem, even for gas containers with good hyperbolic properties, such as the Bunimovich stadium. In such a container the motion of a gas particle when the piston is fixed is only nonuniformly hyperbolic because it can experience many collisions with the flat walls of the container immediately preceding and following a collision with the piston.

The present work provides a weak law of large numbers, and it is an open problem to describe the sizes of the deviations for the piston problem [CD06b]. Although our result does not yield concrete information on the sizes of the deviations, it is general in that it imposes very few conditions on the shape of the gas container. Most studies of billiard systems impose strict conditions on the shape of the boundary, generally involving the sign of the curvature and how the corners are put together. The proofs in this work require no such restrictions. In particular, the gas container can have cusps as corners and need satisfy no hyperbolicity conditions.

If the piston divides a container in $\mathbb{R}^2$ or $\mathbb{R}^3$ with axial symmetry, such as a rectangle or a cylinder, then our ergodicity assumption on the behavior of the gas particles when the piston is fixed does not hold. In this case, the interactions of the gas particles with the piston and the ends of the container are completely specified by their motions along the normal axis of the container. Thus, this system projects onto a system inside an interval consisting of a massive point particle, the piston, which interacts with the gas particles on either side of it. These gas particles make elastic collisions with the walls at the ends of the container and with the piston, but they do not interact with each other. For such one-dimensional containers, the effects of the gas particles are quasi-periodic and can be essentially decoupled, and we recover a strong law of large numbers with a uniform rate, reminiscent of classical averaging over just one fast variable in $S^1$: The convergence of the actual motions to the averaged behavior is uniform over all initial conditions, with the size of the deviations being no larger than $O(M^{-1/2})$ on the time scale $M^{+1/2}$. See Theorem 3.1.1 Gorelyshev and Neishtadt [GN06] independently obtained this result.

For systems in $d = 1$ dimension, we also investigate the behavior of the system when the interactions of the gas particles with the walls and the piston have been smoothed, so that Anosov’s theorem applies directly. Let $\delta \geq 0$ be a parameter of smoothing, so that $\delta = 0$ corresponds to the hard core setting above. Then the averaged behavior of the piston is still a periodic oscillation, which depends smoothly on $\delta$. We show that the deviations of the actual motions of the piston from the averaged behavior are again not more than $O(M^{-1/2})$ on the time scale.
$M^{1/2}$. The size of the deviations is bounded uniformly, both over initial conditions and over the amount of smoothing, Theorem 3.1.2.

Our results for a single heavy piston separating two gas containers generalize to the case of $N$ heavy pistons separating $N+1$ gas containers. Here the averaged behavior of the pistons has them moving like an $N$-dimensional particle inside an effective potential well. Compare Section 3.1.3.

The systems under consideration in this work are simple models of an adiabatic piston. The general adiabatic piston problem [Cal63], well-known from physics, consists of the following: An insulating piston separates two gas containers, and initially the piston is fixed in place, and the gas in each container is in a separate thermal equilibrium. At some time, the piston is no longer externally constrained and is free to move. One hopes to show that eventually the system will come to a full thermal equilibrium, where each gas has the same pressure and temperature. Whether the system will evolve to thermal equilibrium and the interim behavior of the piston are mechanical problems, not adequately described by thermodynamics [Gru99], that have recently generated much interest within the physics and mathematics communities following Lieb’s address [Lie99]. One expects that the system will evolve in at least two stages. First, the system relaxes deterministically toward a mechanical equilibrium, where the pressures on either side of the piston are equal. In the second, much longer, stage, the piston drifts stochastically in the direction of the hotter gas, and the temperatures of the gases equilibrate. See for example [GPL03, CL02, Che04] and the references therein. Previously, rigorous results have been limited mainly to models where the effects of gas particles recolliding with the piston can be neglected, either by restricting to extremely short time scales [LSC02, CLS02] or to infinite gas containers [Che04].

1.2 Physical motivation for the results

In this section, we briefly review the physical motivations for our results on the adiabatic piston.

Consider a massive, insulating piston of mass $M$ that separates a gas container $D$ in $\mathbb{R}^d$, $d = 1, 2,$ or $3$. See Figure 1.1. Denote the location of the piston by $Q$ and its velocity by $dQ/dt = V$. If $Q$ is fixed, then the piston divides $D$ into two subdomains, $D_1(Q) = D_1$ on the left and $D_2(Q) = D_2$ on the right. By $|D_i|$ we denote the area (when $d = 2$, or length, when $d = 1$, or volume, when $d = 3$) of $D_i$. Define

$$\ell := \frac{\partial |D_1(Q)|}{\partial Q} = -\frac{\partial |D_2(Q)|}{\partial Q},$$

so that $\ell$ is the piston’s cross-sectional length (when $d = 2$, or area, when $d = 3$). If $d = 1$, then $\ell = 1$. By $E_i$ we denote the total energy of the gas inside $D_i$. 


We are interested in the dynamics of the piston when the system’s total energy is bounded and $M \to \infty$. When $M = \infty$, the piston remains fixed in place, and each energy $E_i$ remains constant. When $M$ is large but finite, $MV^2/2$ is bounded, and so $V = O(M^{-1/2})$. It is natural to define

$$\varepsilon = M^{-1/2},$$

$$W = \frac{V}{\varepsilon},$$

so that $W$ is of order 1 as $\varepsilon \to 0$. This is equivalent to scaling time by $\varepsilon$, and so we introduce the slow time

$$\tau = \varepsilon t.$$

If we let $P_i$ denote the pressure of the gas inside $\mathcal{D}_i$, then heuristically the dynamics of the piston should be governed by the following differential equation:

$$\frac{dQ}{dt} = V, \quad M \frac{dV}{dt} = P_1 \ell - P_2 \ell,$$

i.e.

$$\frac{dQ}{d\tau} = W, \quad \frac{dW}{d\tau} = P_1 \ell - P_2 \ell.$$  \hspace{1cm} (1.1)

To find differential equations for the energies of the gases, note that in a short amount of time $dt$, the change in energy should come entirely from the work done on a gas, i.e. the force applied to the gas times the distance the piston has moved,
because the piston is adiabatic. Thus, one expects that
\[
\frac{dE_1}{dt} = -VP_1 \ell, \quad \frac{dE_2}{dt} = +VP_2 \ell,
\]
i.e.
\[
\frac{dE_1}{d\tau} = -WP_1 \ell, \quad \frac{dE_2}{d\tau} = +WP_2 \ell.
\]

To obtain a closed system of differential equations, it is necessary to insert an expression for the pressures. \(P_i \ell\) should be the average force from the gas particles in \(D_i\) experienced by the piston when it is held fixed in place. Whether such an expression, depending only on \(E_i\) and \(D_i(Q)\), exists and is the same for (almost) every initial condition of the gas particles depends strongly on the microscopic model of the gas particle dynamics. Sinai and Neishtadt [Sin99, NS04] pointed out that for many microscopic models where the pressures are well defined, the solutions of Equations (1.1) and (1.2) have the piston moving according to a model-dependent effective Hamiltonian.

Because the pressure of an ideal gas in \(d\) dimensions is proportional to the energy density, with the constant of proportionality \(2/d\), we choose to insert
\[
P_i = \frac{2E_i}{d|D_i|}.
\]
Later, we will make assumptions on the microscopic gas particle dynamics to justify this substitution. However, if we accept this definition of the pressure, we obtain the following ordinary differential equations for the four macroscopic variables of the system:
\[
\frac{d}{d\tau} \begin{bmatrix} Q \\ W \\ E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} W \\ \frac{2E_1 \ell}{d|D_1(Q)|} - \frac{2E_2 \ell}{2WE_1 \ell} \\ -\frac{d|D_1(Q)|}{2WE_2 \ell} \\ +\frac{d|D_2(Q)|}{|D_2(Q)|} \end{bmatrix}.
\]
For these equations, one can see the effective Hamiltonian as follows. Since
\[
\frac{d \ln(E_i)}{d\tau} = -\frac{2}{d} \frac{d \ln(|D_i(Q)|)}{d\tau},
\]
\[
E_i(\tau) = E_i(0) \left( \frac{|D_i(Q(0))|}{|D_i(Q(\tau))|} \right)^{2/d}.
\]
Hence
\[
\frac{d^2 Q(\tau)}{d\tau^2} = \frac{2\ell E_1(0) |D_1(Q(0))|^{2/d}}{d |D_1(Q(\tau))|^{1+2/d}} - \frac{2\ell E_2(0) |D_2(Q(0))|^{2/d}}{d |D_2(Q(\tau))|^{1+2/d}},
\]
and so \((Q, W)\) behave as if they were the coordinates of a Hamiltonian system describing a particle undergoing motion inside a potential well. The effective Hamiltonian may be expressed as

\[
\frac{1}{2} W^2 + \frac{E_1(0) |D_1(Q(0))|^{2/d}}{|D_1(Q)|^{2/d}} + \frac{E_2(0) |D_2(Q(0))|^{2/d}}{|D_2(Q)|^{2/d}}. \tag{1.4}
\]

The question is, do the solutions of Equation (1.3) give an accurate description of the actual motions of the macroscopic variables when \(M\) tends to infinity? The main result of this thesis, Theorem 4.1.1, is that, for an appropriately defined system, the answer to this question is affirmative for \(0 \leq t \leq M^{1/2}\), at least for most initial conditions of the microscopic variables. Observe that one should not expect the description to be accurate on time scales much longer than \(O(M^{1/2}) = O(\varepsilon^{-1})\). The reason for this is that, presumably, there are corrections of size \(O(\varepsilon)\) in Equation (1.3) that we are neglecting. For \(\tau = \varepsilon t > O(1)\), these corrections should become significant. Such higher order corrections for the adiabatic piston were studied by Crosignani et al. [CDPS96].
Chapter 2

Background Averaging Material

In this chapter, we present a number of well-known classical averaging results for smooth systems, as well as a proof of Anosov’s averaging theorem, which is the first general multi-phase averaging result. All of these theorems are at least 45 years old. However, we present them here because our proofs of the classical results are at least slightly novel, and the ideas in them lend themselves well to certain higher-dimensional generalizations. In particular, they are fairly close to the ideas in the proof we give for our piston results in one dimension. The proof of Anosov’s theorem is a new and unpublished proof due mainly to Dolgopyat, with some further simplifications made. The ideas in this proof underly the ideas we will use to prove the weak law of large numbers for our piston system in dimensions two and three.

We begin by giving a discussion of a framework for general averaging theory and some averaging results. A number of classical averaging theorems are then proved, followed by the proof of Anosov’s theorem.

2.1 The averaging framework

In this section, consider a family of ordinary differential equations

\[ \frac{dz}{dt} = Z(z, \varepsilon) \]  

(2.1)

on a smooth, finite-dimensional Riemannian manifold \( \mathcal{M} \), which is indexed by the real parameter \( \varepsilon \in [0, \varepsilon_0] \). Assume

- **Regularity:** the functions \( Z \) and \( \partial Z / \partial \varepsilon \) are both \( C^1 \) on \( \mathcal{M} \times [0, \varepsilon_0] \).

We denote the flow generated by \( Z(\cdot, \varepsilon) \) by \( z_\varepsilon(t, z) = z_\varepsilon(t) \). We will usually suppress the dependence on the initial condition \( z = z_\varepsilon(0, z) \). Think of \( z_\varepsilon(\cdot) \) as being a random variable whose domain is the space of initial conditions for
the differential equation (2.1) and whose range is the space of continuous paths (depending on the parameter $t$) in $\mathcal{M}$.

- **Existence of smooth integrals:** $z_0(t)$ has $m$ independent $C^2$ first integrals $h = (h_1, \ldots, h_m) : \mathcal{M} \to \mathbb{R}^m$.

Then $h$ is conserved by $z_0(t)$, and at every point the linear operator $\partial h/\partial z$ has full rank. It follows from the implicit function theorem that each level set $\mathcal{M}_c := \{h = c\}$ is a smooth submanifold of co-dimension $m$, which is invariant under $z_0(t)$. Further, assume that there exists an open ball $U \subset \mathbb{R}^m$ satisfying:

- **Compactness:** $\forall c \in U$, $\mathcal{M}_c$ is compact.

- **Preservation of smooth measures:** $\forall c \in U$, $z_0(t)|_{\mathcal{M}_c}$ preserves a smooth measure $\mu_c$ that varies smoothly with $c$, i.e. there exists a $C^1$ function $g : \mathcal{M} \to \mathbb{R}_{>0}$ such that $g|_{\mathcal{M}_c}$ is the density of $\mu_c$ with respect to the restriction of Riemannian volume.

Set

$$h_\varepsilon(t, z) = h_\varepsilon(t) := h(z_\varepsilon(t)).$$

Again, think of $h_\varepsilon(\cdot)$ as being a random variable that takes initial conditions $z \in \mathcal{M}$ to continuous paths (depending on the parameter $t$) in $U$. Since $dh_0/dt \equiv 0$, Hadamard’s Lemma allows us to write

$$\frac{dh_\varepsilon}{dt} = \varepsilon H(z_\varepsilon, \varepsilon)$$

for some $C^1$ function $H : \mathcal{M} \times [0, \varepsilon_0] \to U$. Observe that

$$\frac{dh_\varepsilon}{dt}(t) = Dh(z_\varepsilon(t))Z(z_\varepsilon(t), \varepsilon) = Dh(z_\varepsilon(t))(Z(z_\varepsilon(t), \varepsilon) - Z(z_\varepsilon(t), 0)),$$

so that

$$H(z, 0) = \mathcal{L}_{\frac{dZ}{d\varepsilon}}|_{\varepsilon=0}^t h.$$

Here $\mathcal{L}$ denotes the Lie derivative.

Define the averaged vector field $\bar{H}$ by

$$\bar{H}(h) = \int_{\mathcal{M}_h} H(z, 0) d\mu_h(z).$$

(2.2)

Then $\bar{H}$ is $C^1$. Fix a compact set $\mathcal{V} \subset U$, and introduce the slow time

$$\tau = \varepsilon t.$$
Let $\bar{h}(\tau, z) = \bar{h}(\tau)$ be the random variable that is the solution of
\[
\frac{d\bar{h}}{d\tau} = \bar{H}(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0).
\]

We only consider the dynamics in a compact subset of phase space, so for initial conditions $z \in h^{-1}U$, define the stopping time
\[
T_\varepsilon(z) = T_\varepsilon = \inf\{\tau \geq 0 : \bar{h}(\tau) \notin \mathcal{V} \text{ or } h_\varepsilon(\tau/\varepsilon) \notin \mathcal{V}\}.
\]

Heuristically, think of the phase space $M$ as being a fiber bundle whose base is the open set $U$ and whose fibers are the compact sets $M_h$. See Figure 2.1. Then the vector field $Z(\cdot, 0)$ is perpendicular to the base, so its orbits $z_0(t)$ flow only along the fibers. Now when $0 < \varepsilon \ll 1$, the vector field $Z(\cdot, \varepsilon)$ acquires a component of size $O(\varepsilon)$ along the base, and so its orbits $z_\varepsilon(t)$ have a small drift along the base, which we can follow by observing the evolution of $h_\varepsilon(t)$. Because of this, we refer to $h$ as consisting of the slow variables. Other variables, used to complete $h$ to a parameterization of (a piece of) phase space, are called fast variables. Note that $h_\varepsilon(t)$ depends on all the dimensions of phase space, and so it is not the flow of a vector field on the $m$-dimensional space $U$. However, because the motion along each fiber is relatively fast compared to the motion across fibers, we hope to be able to average over the fast motions and obtain a vector field on $U$ that gives a good description of $h_\varepsilon(t)$ over a relatively long time interval, independent of where the solution $z_\varepsilon(t)$ started on $M_{h_\varepsilon(0)}$. Because our averaged vector field, as defined by Equation (2.2), only accounts for deviations of size $O(\varepsilon)$, we cannot expect this time interval to be longer than size $O(1/\varepsilon)$. In terms of the slow time $\tau = \varepsilon t$, this length becomes $O(1)$. In other words, the goal of the first-order averaging method described above should be to show that, in some sense, $\sup_{0 \leq \tau \leq 1 \wedge T_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \to 0$ as $\varepsilon \to 0$. This is often referred to as the averaging principle.

Note that the assumptions of regularity, existence of smooth integrals, compactness, and preservation of smooth measures above are not sufficient for the averaging principle to hold in any form. As an example of just one possible obstruction, the level sets $\mathcal{M}_c$ could separate into two completely disjoint sets, $\mathcal{M}_c = \mathcal{M}_c^+ \sqcup \mathcal{M}_c^-$. If this were the case, then it would be implausible that the solutions of the averaged vector field defined by averaging over all of $\mathcal{M}_c$ would accurately describe $h_\varepsilon(t, z)$, independent of whether $z \in \mathcal{M}_c^+$ or $z \in \mathcal{M}_c^-$. 

**Some averaging results**

So far, we are in a general averaging setting. Frequently, one also assumes that the invariant submanifolds, $\mathcal{M}_h$, are tori, and that there exists a choice of coordinates
\[
z = (h, \varphi)
\]
\[ M = \{(h, \varphi)\} \quad \varphi = \text{“fast variables”} \]

Figure 2.1: A schematic of the phase space \( M \). Note that although the level set \( M_c = \{h = c\} \) is depicted as a torus, it need not be a torus. It could be any compact, co-dimension \( m \) submanifold.

on \( M \) in which the differential equation (2.1) takes the form

\[
\frac{dh}{dt} = \varepsilon H(h, \varphi, \varepsilon), \quad \frac{d\varphi}{dt} = \Phi(h, \varphi, \varepsilon).
\]

Then if \( \varphi \in S^1 \) and the differential equation for the fast variable is regular, i.e. \( \Phi(h, \varphi, 0) \) is bounded away from zero for \( h \in U \),

\[
\sup_{\text{initial conditions}} \sup_{0 \leq \tau \leq 1} \left| h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau) \right| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0.
\]

See for example Chapter 5 in [SV85], Chapter 3 in [LM88], or Theorem 2.2.3 in the following section.

When the differential equation for the fast variable is not regular, or when there is more than one fast variable, the typical averaging result becomes much weaker than the uniform convergence above. For example, consider the case when \( \varphi \in \mathbb{T}^n \), \( n > 1 \), and the unperturbed motion is quasi-periodic, i.e. \( \Phi(h, \varphi, 0) = \Omega(h) \). Also assume that \( H \in C^{n+2} \) and that \( \Omega \) is nonvanishing and satisfies a nondegeneracy condition on \( U \) (for example, \( \Omega : U \to \mathbb{T}^n \) is a submersion). Let \( P \) denote Riemannian volume on \( M \). Neishtadt [LM88, Nei76] showed that in this
situation, for each fixed $\delta > 0$,

$$P \left( \sup_{0 \leq \tau \leq 1 / T \varepsilon} \left| h_\varepsilon(\tau / \varepsilon) - \bar{h}(\tau) \right| \geq \delta \right) = O(\sqrt{\varepsilon / \delta}),$$

and that this result is optimal. Thus, the averaged equation only describes the actual motions of the slow variables in probability on the time scale $1 / \varepsilon$ as $\varepsilon \to 0$.

Neishtadt’s result was motivated by a general averaging theorem for smooth systems due to Anosov. This theorem requires none of the additional assumptions in the averaging results above. Under the conditions of regularity, existence of smooth integrals, compactness, and preservation of smooth measures, as well as

- **Ergodicity:** for Lebesgue almost every $c \in U$, $(z_0(\cdot), \mu_c)$ is ergodic,

Anosov showed that $\sup_{0 \leq \tau \leq 1 / T \varepsilon} \left| h_\varepsilon(\tau / \varepsilon) - \bar{h}(\tau) \right| \to 0$ in probability (w.r.t. Riemannian volume on initial conditions) as $\varepsilon \to 0$, i.e.

**Theorem 2.1.1 (Anosov’s averaging theorem [Ano60]).** For each $T > 0$ and for each fixed $\delta > 0$,

$$P \left( \sup_{0 \leq \tau \leq T / T \varepsilon} \left| h_\varepsilon(\tau / \varepsilon) - \bar{h}(\tau) \right| \geq \delta \right) \to 0$$

as $\varepsilon \to 0$.

We present a recent proof of this theorem in Section 2.3 below.

If we consider $h_\varepsilon(\cdot)$ and $\bar{h}(\cdot)$ to be random variables, Anosov’s theorem is a version of the weak law of large numbers. In general, we can do no better: There is no general strong law in this setting. There exists a simple example due to Neishtadt (which comes from the equations for the motion of a pendulum with linear drag being driven by a constant torque) where for no initial condition in a positive measure set do we have convergence of $h_\varepsilon(t)$ to $\bar{h}(\varepsilon t)$ on the time scale $1 / \varepsilon$ as $\varepsilon \to 0$ [Kif04b]. Here, the phase space is $\mathbb{R} \times S^1$, and the unperturbed motion is (uniquely) ergodic on all but one fiber.

### 2.2 Some classical averaging results

In this section we present some simple, well-known averaging results. See for example Chapter 5 in [SV85] or Chapter 3 in [LM88].
2.2.1 Averaging for time-periodic vector fields

Consider a family of time dependent ordinary differential equations

\[ \frac{dh}{dt} = \varepsilon H(h, t, \varepsilon), \]  

indexed by the real parameter \( \varepsilon \geq 0 \), where \( h \in \mathbb{R}^m \). Fix \( \mathcal{V} \subset \subset \mathcal{U} \subset \mathbb{R}^m \), and suppose

- **Regularity:** \( H \in \mathcal{C}^1(\mathcal{U} \times \mathbb{R} \times [0, \infty)) \).

- **Periodicity:** There exists \( T > 0 \) such that for each \( h \in \mathcal{U} \), \( H(h, t, 0) \) is \( T \)-periodic in time.

Then

\[ \frac{dh}{dt} = \varepsilon H(h, t, 0) + \mathcal{O}(\varepsilon^2). \]

Let \( h_\varepsilon(t) \) denote the solution of Equation (2.3). We seek a time independent vector field whose solutions approximate \( h_\varepsilon(t) \), at least for a long length of time. It is natural to define the averaged vector field \( \bar{H} \) by

\[ \bar{H}(h) = \frac{1}{T} \int_{0}^{T} H(h, s, 0) ds. \]

Then \( \bar{H} \in \mathcal{C}^1(\mathcal{U}) \). Let \( \bar{h}(\tau) \) be the solution of

\[ \frac{d\bar{h}}{d\tau} = \bar{H}(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0). \]

It is reasonable to hope that \( \bar{h}(\varepsilon t) \) and \( h_\varepsilon(t) \) are close together for \( 0 \leq t \leq \varepsilon^{-1} \).

We only consider the dynamics in a compact subset of phase space, so for initial conditions in \( \mathcal{U} \), we define the stopping time

\[ T_\varepsilon = \inf\{\tau \geq 0 : h(\tau) \notin \mathcal{V} \text{ or } h_\varepsilon(\tau/\varepsilon) \notin \mathcal{V}\}. \]

**Theorem 2.2.1** (Time-periodic averaging). For each \( T > 0 \),

\[ \sup_{h_\varepsilon(0) \in \mathcal{V}} \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau) \right| = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0. \]

**Proof.** We divide our proof into three essential steps.
Step 1: Reduction using Gronwall's Inequality. Now, $\bar{h}(\tau)$ satisfies the integral equation
\[
\bar{h}(\tau) - \bar{h}(0) = \int_{0}^{\tau} \bar{H}(\bar{h}(\sigma))d\sigma,
\]
while $h_\varepsilon(\tau/\varepsilon)$ satisfies
\[
h_\varepsilon(\tau/\varepsilon) - h_\varepsilon(0) = \varepsilon \int_{0}^{\tau/\varepsilon} H(h_\varepsilon(s), s, \varepsilon)ds
= \mathcal{O}(\varepsilon) + \varepsilon \int_{0}^{\tau/\varepsilon} H(h_\varepsilon(s), s, 0)ds
= \mathcal{O}(\varepsilon) + \varepsilon \int_{0}^{\tau/\varepsilon} H(h_\varepsilon(s), s, 0) - \bar{H}(h_\varepsilon(s))ds + \int_{0}^{\tau} \bar{H}(h_\varepsilon(\sigma/\varepsilon))d\sigma
\]
for $0 \leq \tau \leq T \wedge T_\varepsilon$.

Define
\[
e_\varepsilon(\tau) = \varepsilon \int_{0}^{\tau/\varepsilon} H(h_\varepsilon(s), s, 0) - \bar{H}(h_\varepsilon(s))ds.
\]
It follows from Gronwall’s Inequality that
\[
\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |\bar{h}(\tau) - h_\varepsilon(\tau/\varepsilon)| \leq \left( \mathcal{O}(\varepsilon) + \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_\varepsilon(\tau)| \right) e^{\text{Lip}(\bar{H}|V)T}.
\]

Step 2: A sequence of times adapted for ergodization. Ergodization refers to the convergence along an orbit of a function’s time average to its space average. We define a sequence of times $t_k$ for $k \geq 0$ by $t_k = kT$. This sequence of times is motivated by the fact that
\[
\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} H(h_0(s), s, 0)ds = \bar{H}(h_0).
\]
Note that $h_0(t)$ is independent of time. Thus,
\[
\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_\varepsilon(\tau)| \leq \mathcal{O}(\varepsilon) + \varepsilon \sum_{t_k \leq \tau \leq T \wedge T_\varepsilon} \left| \int_{t_k}^{t_{k+1}} H(h_\varepsilon(s), s, 0) - \bar{H}(h_\varepsilon(s))ds \right|. \quad (2.4)
\]

Step 3: Control of individual terms by comparison with solutions of the $\varepsilon = 0$ equation. The sum in Equation (2.4) has no more than $\mathcal{O}(1/\varepsilon)$ terms, and so it suffices to show that each term $\int_{t_k}^{t_{k+1}} H(h_\varepsilon(s), s, 0) - \bar{H}(h_\varepsilon(s))ds$ is no larger than $\mathcal{O}(\varepsilon)$. We can accomplish this by comparing the motions of $h_\varepsilon(t)$ for $t_k \leq t \leq t_{k+1}$ with $h_{k,\varepsilon}(t)$, which is defined to be the solution of the $\varepsilon = 0$ ordinary differential equation satisfying $h_{k,\varepsilon}(t_k) = h_\varepsilon(t_k)$, i.e. $h_{k,\varepsilon}(t) \equiv h_\varepsilon(t_k)$. 

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Lemma 2.2.2. If \( t_{k+1} \leq \frac{T_k T_{k+1}}{T} \), then \( \sup_{t_k \leq t \leq t_{k+1}} |h_{k,\varepsilon}(t) - h_{\varepsilon}(t)| = O(\varepsilon) \).  

Proof. \( dh_{\varepsilon}/dt = O(\varepsilon) \). □

Using that \( H \) and \( \bar{H} \) are Lipschitz continuous, we conclude that 
\[
\int_{t_k}^{t_{k+1}} H(h_{\varepsilon}(s), s, 0) - \bar{H}(h_{\varepsilon}(s)) ds = O(\varepsilon) + 0 + O(\varepsilon) = O(\varepsilon).
\]

Thus we see that \( \sup_{0 \leq \tau \leq T \wedge T_{k+1}} |h_{\varepsilon}(\tau/\varepsilon) - \bar{h}(\tau)| \leq O(\varepsilon) \), independent of the initial condition \( h_{\varepsilon}(0) \in \mathcal{V} \). □

Remark 2.2.1. Note that the \( O(\varepsilon) \) control in Theorem 2.2.1 on a time scale \( t = O(\varepsilon^{-1}) \) is generally optimal. For example, take \( H(h, t, \varepsilon) = \cos(t) + \varepsilon \).

2.2.2 Averaging for vector fields with one regular fast variable

For \( h \in \mathbb{R}^m \) and \( \varphi \in S^1 = [0, 1]/0 \sim 1 \), consider the family of ordinary differential equations
\[
\frac{dh}{dt} = \varepsilon H(h, \varphi, \varepsilon), \quad \frac{d\varphi}{dt} = \Phi(h, \varphi, \varepsilon),
\]  
indexed by the real parameter \( \varepsilon \geq 0 \). With \( z = (h, \varphi) \), we write this family of differential equations as \( dz/dt = Z(z, \varepsilon) \).

Fix \( \mathcal{V} \subset \subset \mathcal{U} \subset \mathbb{R}^m \), and suppose
- Regularity: \( Z \in C^1(\mathcal{U} \times S^1 \times [0, \infty)) \).
- Regular fast variable: \( \Phi(h, \varphi, 0) \) is bounded away from 0 for \( h \in \mathcal{U} \), i.e. \( \inf_{(h, \varphi) \in \mathcal{U} \times S^1} |\Phi(h, \varphi, 0)| > 0 \).

Without loss of generality, we assume that \( \Phi(h, \varphi, 0) > 0 \).
Let $z_\varepsilon(t) = (h_\varepsilon(t), \varphi_\varepsilon(t))$ denote the solution of Equation (2.5). Then $z_0(t)$ leaves invariant the circles $M_c = \{h = c\}$ in phase space. In fact, $z_0(t)$ preserves an uniquely ergodic invariant probability measure on $M_c$, whose density is given by

$$d\mu_c = \frac{1}{K_c} \frac{d\varphi}{\Phi(c, \varphi, 0)},$$

where $K_c = \int_0^1 \frac{d\varphi}{\Phi(c, \varphi, 0)}$ is a normalization constant.

The averaged vector field $\bar{H}$ is defined by averaging $H(h, \varphi, 0)$ over $\varphi$:

$$\bar{H}(h) = \int_0^1 H(h, \varphi, 0) d\mu_h(\varphi) = \frac{1}{K_h} \int_0^1 H(h, \varphi, 0) \frac{d\varphi}{\Phi(h, \varphi, 0)}.$$

Then $\bar{H} \in C^1(U)$. Let $\bar{h}(\tau)$ be the solution of

$$\frac{d\bar{h}}{d\tau} = \bar{H}(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0).$$

For initial conditions in $U \times S^1$, we have the usual stopping time $T_\varepsilon = \inf\{\tau \geq 0 : \bar{h}(\tau) \notin V \text{ or } h_\varepsilon(\tau/\varepsilon) \notin V\}$.

**Theorem 2.2.3** (Averaging over one regular fast variable). For each $T > 0$,

$$\sup_{\text{initial conditions}} \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau) \right| = O(\varepsilon) \text{ as } \varepsilon \to 0.$$

**Remark 2.2.2.** This result encompasses Theorem 2.2.1 for time-periodic averaging. For example, if $T = 1$, simply take $\varphi = t \mod 1$ and $\Phi(h, \varphi, \varepsilon) = 1$.

**Remark 2.2.3.** Many of the proofs of the above theorem of which we are aware hinge on considering $\varphi$ as a time-like variable. For example, one could write

$$\frac{dh}{d\varphi} = \frac{dh}{dt} \frac{dt}{d\varphi} = \varepsilon \frac{H(h, \varphi, 0)}{\Phi(h, \varphi, 0)} + O(\varepsilon^2),$$

and this looks very similar to the time-periodic situation considered previously. However, it does take some work to justify such arguments rigorously, and the traditional proofs do not easily generalize to averaging over multiple fast variables. Our proof essentially uses $\varphi$ to mark off time, and it will immediately generalize to a specific instance of multiphase averaging.

**Proof.** Again, we have three steps.
Step 1: Reduction using Gronwall’s Inequality. Now

\[ \bar{h}(\tau) - \bar{h}(0) = \int_0^\tau \bar{H}(\bar{h}(\sigma))d\sigma, \]

and

\[ h_\varepsilon(\tau/\varepsilon) - h_\varepsilon(0) = \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s), \varepsilon)ds = O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s), 0)ds \]

\[ = O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds + \int_0^{\tau} \bar{H}(h_\varepsilon(\sigma/\varepsilon))d\sigma \]

for \( 0 \leq \tau \leq T \wedge T_\varepsilon \).

Define

\[ e_\varepsilon(\tau) = \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds. \]

It follows from Gronwall’s Inequality that

\[ \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |\bar{h}(\tau) - h_\varepsilon(\tau/\varepsilon)| \leq \left( O(\varepsilon) + \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_\varepsilon(\tau)| \right) e^{\text{Lip}(\bar{H}|_\nu)T}. \]

Step 2: A sequence of times adapted for ergodization. Now for each initial condition in our phase space and for each fixed \( \varepsilon \), we define a sequence of times \( t_k, \varepsilon \) and a sequence of solutions \( z_k, \varepsilon \) inductively as follows:

\( t_0, \varepsilon = 0 \) and \( z_0, \varepsilon(t) = z_0(t) \). For \( k > 0 \),

\( t_{k+1, \varepsilon} = \inf \{ t > t_k, \varepsilon : \varphi_{k-1, \varepsilon}(t) = \varphi_{\varepsilon}(0) \} \),

and \( z_k, \varepsilon(t) \) is defined as the solution of

\[ \frac{dz_{k, \varepsilon}}{dt} = Z(z_{k, \varepsilon}, 0) = (0, \Phi(z_{k, \varepsilon}, 0)), \quad z_{k, \varepsilon}(t_{k, \varepsilon}) = z_\varepsilon(t_{k, \varepsilon}). \]

This sequence of times is motivated by the fact that

\[ \frac{1}{t_{k+1, \varepsilon} - t_{k, \varepsilon}} \int_{t_{k, \varepsilon}}^{t_{k+1, \varepsilon}} H(z_{k, \varepsilon}(s), 0)ds = \bar{H}(h_{k, \varepsilon}). \]

Recall that \( h_{k, \varepsilon}(t) \) is independent of time. The elements of this sequence of times are approximately uniformly spaced, i.e. if we fix \( \omega > 0 \) such that \( \omega > 0 \) such that \( z \in \mathcal{V} \times S^1 \Rightarrow 1/\omega < \Phi(z, 0) < \omega \), then if \( t_{k+1, \varepsilon} \leq (T \wedge T_\varepsilon)/\varepsilon, 1/\omega < t_{k+1, \varepsilon} - t_{k, \varepsilon} < \omega \).

Thus,

\[ \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_\varepsilon(\tau)| \leq O(\varepsilon) + \varepsilon \sum_{t_{k+1, \varepsilon} \leq T \wedge T_\varepsilon} \left| \int_{t_{k, \varepsilon}}^{t_{k+1, \varepsilon}} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds \right|, \]

where the sum in in this equation has no more than \( O(1/\varepsilon) \) terms.
Step 3: Control of individual terms by comparison with solutions along fibers. It suffices to show that each term \( \int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds \) is no larger than \( O(\varepsilon) \). We can accomplish this by comparing the motions of \( z_\varepsilon(t) \) for \( t_k,\varepsilon \leq t \leq t_{k+1,\varepsilon} \) with \( z_{k,\varepsilon}(t) \).

Lemma 2.2.4. If \( t_{k+1,\varepsilon} \leq \frac{T^{\land}T_\varepsilon}{\varepsilon} \), then \( \sup_{t_k,\varepsilon \leq t \leq t_{k+1,\varepsilon}} |z_{k,\varepsilon}(t) - z_\varepsilon(t)| = O(\varepsilon) \).

Proof. Without loss of generality, we take \( k = 0 \), so that \( z_{k,\varepsilon}(t) = z_0(t) \). Since \( h_0(t) = h_\varepsilon(0) \) and \( dh_\varepsilon/dt = O(\varepsilon) \), \( \sup_{t_k,\varepsilon \leq t \leq t_{k+1,\varepsilon}} |h_0(t) - h_\varepsilon(t)| = O(\varepsilon) \).

Now \( \varphi_\varepsilon(t) - \varphi_\varepsilon(0) = \int_0^t \Phi(h_\varepsilon(s), \varphi_\varepsilon(s), \varepsilon)ds \), and because \( \Phi \) is Lipschitz, we find that
\[
|\varphi_\varepsilon(t) - \varphi_\varepsilon(0)| \leq O(\varepsilon) + \text{Lip}(\Phi) \int_0^t |\varphi_\varepsilon(s) - \varphi_\varepsilon(0)|ds
\]
for \( 0 \leq t \leq \omega \). The result follows from Gronwall’s Inequality.

Using that \( H \) and \( \bar{H} \) are Lipschitz continuous, we conclude that
\[
\int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds
= \int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} H(z_\varepsilon(s), 0) - H(z_{k,\varepsilon}(s), 0)ds
+ \int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} H(z_{k,\varepsilon}(s), 0) - \bar{H}(h_{k,\varepsilon}(s))ds
+ \int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} \bar{H}(h_{k,\varepsilon}(s)) - \bar{H}(h_\varepsilon(s))ds
= O(\varepsilon) + 0 + O(\varepsilon)
= O(\varepsilon).
\]

Thus we see that \( \sup_{0 \leq \tau \leq T^{\land}T_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| = O(\varepsilon) \), independent of the initial condition \( (h_\varepsilon(0), \varphi_\varepsilon(0)) \in V \times S^1 \).

2.2.3 Multiphase averaging for vector fields with separable, regular fast variables

As explained in Section 2.1 when the differential equation for the fast variable is not regular, or when there is more than one fast variable, the typical averaging result becomes much weaker than the uniform convergence in Theorems 2.2.1 and 2.2.3 above. Nonetheless, if the differential equations under consideration
satisfy some very specific hypotheses, the proof in the previous section immediately
generalizes to yield uniform convergence.

For $h \in \mathbb{R}^m$ and $\varphi = (\varphi^1, \ldots, \varphi^n) \in \mathbb{T}^n = ([0, 1)/0 \sim 1)^n$, consider the family of ordinary differential equations

$$
\frac{dh}{dt} = \varepsilon H(h, \varphi, \varepsilon), \quad \frac{d\varphi}{dt} = \Phi(h, \varphi, \varepsilon),
$$

indexed by the real parameter $\varepsilon \geq 0$. We also write $z = (h, \varphi)$ and $dz/dt = Z(z, \varepsilon)$.

Fix $V \subset \subset U \subset \mathbb{R}^m$, and suppose

- **Regularity:** $Z \in C^1(U \times \mathbb{T}^n \times [0, \infty))$.

- **Separable fast variables:** $H(h, \varphi, 0)$ and $\Phi(h, \varphi, 0)$ have the following specific forms:
  
  - There exist $C^1$ functions $H_j(h, \varphi^j)$ such that $H(h, \varphi, 0) = \sum_{j=1}^n H_j(h, \varphi^j)$. This can be thought of as saying that, to first order in $\varepsilon$, each fast variable affects the slow variables independently of the other fast variables.
  
  - The components $\Phi^j$ of $\Phi$ satisfy $\Phi^j(h, \varphi, 0) = \Phi^j(h, \varphi^j, 0)$, i.e. the unperturbed motion has each fast variable moving independently of the other fast variables. Note that this assumption is satisfied if the unperturbed motion is quasi-periodic, i.e. $\Phi(h, \varphi, 0) = \Omega(h)$.

- **Regular fast variables:** For each $j$,

$$
\inf_{(h, \varphi^j) \in U \times S^1} |\Phi^j(h, \varphi^j, 0)| > 0.
$$

Let $z_\varepsilon(t) = (h_\varepsilon(t), \varphi_\varepsilon(t))$ denote the solution of Equation (2.6). Then $z_0(t)$ leaves invariant the tori $\mathcal{M}_c = \{h = c\}$ in phase space. In fact, $z_0(t)$ preserves a (not necessarily ergodic) invariant probability measure on $\mathcal{M}_c$, whose density is given by

$$
d\mu_c = \prod_{j=1}^n \frac{1}{K_j^c} \frac{d\varphi^j}{|\Phi^j(c, \varphi^j, 0)|},
$$

where $K_j^c = \int_0^1 \frac{d\varphi^j}{|\Phi^j(c, \varphi^j, 0)|}$.

The averaged vector field $\bar{H}$ is defined by

$$
\bar{H}(h) = \int_{\mathcal{M}_c} H(h, \varphi, 0) d\mu_h(\varphi) = \sum_{j=1}^n \int_{\mathcal{M}_c} H_j(h, \varphi^j) d\mu_h(\varphi)
$$

$$
= \sum_{j=1}^n \frac{1}{K_j^c} \int_0^1 \frac{H_j(h, \varphi^j)}{|\Phi^j(h, \varphi^j, 0)|} d\varphi^j := \sum_{j=1}^n \bar{H}_j(h).
$$
Let \( \bar{h}(\tau) \) be the solution of
\[
\frac{d\bar{h}}{d\tau} = \bar{H}(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0),
\]
and the stopping time \( T_\varepsilon = \inf\{ \tau \geq 0 : \bar{h}(\tau) \notin \mathcal{V} \text{ or } h_\varepsilon(\tau/\varepsilon) \notin \mathcal{V} \} \).

**Theorem 2.2.5** (Averaging over multiple separable, regular fast variables). For each \( T > 0 \),
\[
\sup_{\text{initial conditions } h_\varepsilon(0) \in \mathcal{V}} \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| = O(\varepsilon) \text{ as } \varepsilon \to 0.
\]

**Proof.** The proof is essentially the same as the proof of Theorem 2.2.3. As before, we need only show that \( \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_\varepsilon(\tau)| = O(\varepsilon) \), where
\[
e_\varepsilon(\tau) = \varepsilon \int_{0}^{\tau/\varepsilon} H(z_\varepsilon(s), 0) - \bar{H}(h_\varepsilon(s))ds.
\]
But by our separability assumptions, it suffices to show that for each \( j \),
\[
\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |e_{j,\varepsilon}(\tau)| = O(\varepsilon),
\]
where \( e_{j,\varepsilon}(\tau) \) is defined by
\[
e_{j,\varepsilon}(\tau) = \varepsilon \int_{0}^{\tau/\varepsilon} H_j(h_\varepsilon(s), \varphi_j^\varepsilon(s)) - \bar{H}_j(h_\varepsilon(s))ds.
\]
Thus, we have effectively separated the effects of each fast variable, and now the proof can be completed by essentially following steps 2 and 3 in the proof of Theorem 2.2.3. \( \square \)

### 2.3 A proof of Anosov’s theorem

Anosov’s original proof of Theorem 2.1.1 from 1960 may be found in [Ano60]. An exposition of the theorem and Anosov’s proof in English may be found in [LM88]. Recently, Kifer [Kif04a] proved necessary and sufficient conditions for the averaging principle to hold in an averaged with respect to initial conditions sense. He also showed explicitly that his conditions are met in the setting of Anosov’s theorem. The proof of Anosov’s theorem given here is mainly due to Dolgopyat [Dol05], although some further simplifications have been made.
Proof of Anosov’s theorem. We begin by showing that without loss of generality we may take $T_{\varepsilon} = \infty$. This is just for convenience, and not an essential part of the proof. To accomplish this, let $\psi(h)$ be a smooth bump function satisfying

- $\psi(h) = 1$ if $h \in V$,
- $\psi(h) > 0$ if $h \in \text{interior}(\tilde{V})$,
- $\psi(h) = 0$ if $h \notin \tilde{V},$

where $\tilde{V}$ is a compact set chosen such that $V \subset \subset \text{interior}(\tilde{V}) \subset \subset U$. Next, set $\tilde{Z}(z,\varepsilon) = \psi(h(z))Z(z,\varepsilon)$. Because the bump function was chosen to depend only on the slow variables, our assumption about preservation of measures is still satisfied; on each fiber, $\tilde{Z}(z,0)$ is a scalar multiple of $Z(z,0)$. Furthermore, the flow of $\tilde{Z}(\cdot,0)|_{M_h}$ is ergodic for almost every $h \in \tilde{V}$. Then it would suffice to prove our theorem for the vector fields $\tilde{Z}(z,\varepsilon)$ with the set $\tilde{V}$ replacing $V$. We assume that this reduction has been made, although we will not use it until Step 5 below.

Step 1: Reduction using Gronwall’s Inequality. Observe that $\bar{h}(\tau)$ satisfies the integral equation

$$\bar{h}(\tau) - \bar{h}(0) = \int_0^\tau \bar{H}(\bar{h}(\sigma))d\sigma,$$

while $\bar{h}_\varepsilon(\tau/\varepsilon)$ satisfies

$$\bar{h}_\varepsilon(\tau/\varepsilon) - \bar{h}_\varepsilon(0) = \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s),\varepsilon)ds$$

$$= \mathcal{O}(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s),0)ds$$

$$= \mathcal{O}(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s),0) - \bar{H}(h_\varepsilon(s))ds + \int_0^\tau \bar{H}(h_\varepsilon(\sigma/\varepsilon))d\sigma$$

for $0 \leq \tau \leq T \wedge T_{\varepsilon}$. Here we have used the fact that $h^{-1}V \times [0,\varepsilon_0]$ is compact to achieve uniformity over all initial conditions in the size of the $\mathcal{O}(\varepsilon)$ term above. We use this fact repeatedly in what follows. In particular, $H$, $\bar{H}$, and $Z$ are uniformly bounded and have uniform Lipschitz constants on the domains of interest.

Define

$$e_\varepsilon(\tau) = \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s),0) - \bar{H}(h_\varepsilon(s))ds.$$

It follows from Gronwall’s Inequality that

$$\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} |\bar{h}(\tau) - h_\varepsilon(\tau/\varepsilon)| \leq \left( \mathcal{O}(\varepsilon) + \sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} |e_\varepsilon(\tau)| \right) e^{\text{Lip}(\bar{H}V)T}. \quad (2.7)$$
Step 2: Introduction of a time scale for ergodization. Choose a real-valued function $L(\epsilon)$ such that $L(\epsilon) \to \infty$, $L(\epsilon) = o(\log \epsilon^{-1})$ as $\epsilon \to 0$. Think of $L(\epsilon)$ as being a time scale which grows as $\epsilon \to 0$ so that ergodization, i.e. the convergence along an orbit of a function’s time average to a space average, can take place. However, $L(\epsilon)$ doesn’t grow too fast, so that on this time scale $z_\epsilon(t)$ essentially stays on one fiber, where we have our ergodicity assumption. Set $t_{k,\epsilon} = kL(\epsilon)$, so that

$$\sup_{0 \leq \tau \leq T \wedge T_\epsilon} |e_\epsilon(\tau)| \leq \mathcal{O}(\epsilon L(\epsilon)) + \epsilon \sum_{k=0}^{T \wedge T_\epsilon - 1} \left| \int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} H(z_\epsilon(s), 0) - \bar{H}(h_\epsilon(s))ds \right|. \quad (2.8)$$

Step 3: A splitting for using the triangle inequality. Now we let $z_{k,\epsilon}(s)$ be the solution of

$$\frac{dz_{k,\epsilon}}{dt} = Z(z_{k,\epsilon}, 0), \quad z_{k,\epsilon}(t_{k,\epsilon}) = z_\epsilon(t_{k,\epsilon}).$$

Set $h_{k,\epsilon}(t) = h(z_{k,\epsilon}(t))$. Observe that $h_{k,\epsilon}(t)$ is independent of $t$. We break up the integral $\int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} H(z_\epsilon(s), 0) - \bar{H}(h_\epsilon(s))ds$ into three parts:

$$\int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} H(z_\epsilon(s), 0) - \bar{H}(h_\epsilon(s))ds$$

$$= \int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} H(z_\epsilon(s), 0) - H(z_{k,\epsilon}(s), 0)ds$$

$$+ \int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} H(z_{k,\epsilon}(s), 0) - \bar{H}(h_{k,\epsilon}(s))ds$$

$$+ \int_{t_{k,\epsilon}}^{t_{k+1,\epsilon}} \bar{H}(h_{k,\epsilon}(s))ds - \bar{H}(h_\epsilon(s))ds$$

$$:= I_{k,\epsilon} + II_{k,\epsilon} + III_{k,\epsilon}.$$ 

The term $II_{k,\epsilon}$ represents an “ergodicity term” that can be controlled by our assumptions on the ergodicity of the flow $z_0(t)$, while the terms $I_{k,\epsilon}$ and $III_{k,\epsilon}$ represent “continuity terms” that can be controlled using the following control on the drift from solutions along fibers.

Step 4: Control of drift from solutions along fibers. Lemma 2.3.1. If $0 < t_{k+1,\epsilon} \leq \frac{T \wedge T_\epsilon}{\epsilon}$,

$$\sup_{t_{k,\epsilon} \leq t \leq t_{k+1,\epsilon}} |z_{k,\epsilon}(t) - z_\epsilon(t)| \leq \mathcal{O}(\epsilon L(\epsilon)L^{\text{lip}}(Z)L(\epsilon))$$
Proof. Without loss of generality we may set \( k = 0 \), so that \( z_{k,\varepsilon}(t) = z_0(t) \). Then for \( 0 \leq t \leq L(\varepsilon) \),

\[
|z_0(t) - z_\varepsilon(t)| = \left| \int_0^t Z(z_0(s), 0) - Z(z_\varepsilon(s), \varepsilon) \, ds \right|
\leq \text{Lip}(Z) \int_0^t |\varepsilon| + |z_0(s) - z_\varepsilon(s)| \, ds
= O(\varepsilon L(\varepsilon)) + \text{Lip}(Z) \int_0^t |z_0(s) - z_\varepsilon(s)| \, ds.
\]

The result follows from Gronwall’s Inequality.

From Lemma 2.3.1 we find that \( I_{k,\varepsilon}, III_{k,\varepsilon} = O(\varepsilon L(\varepsilon)^2 e^{\text{Lip}(Z)L(\varepsilon)}) \).

Step 5: Use of ergodicity along fibers to control \( II_{k,\varepsilon} \). From Equations (2.7) and (2.8) and the triangle inequality, we already know that

\[
\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| \bar{h}(\tau) - h_\varepsilon(\tau/\varepsilon) \right|
\leq O(\varepsilon) + O(\varepsilon L(\varepsilon)) + \varepsilon \frac{T}{\varepsilon L(\varepsilon)} O(\varepsilon L(\varepsilon)^2 e^{\text{Lip}(Z)L(\varepsilon)}) + O \left( \varepsilon \sum_{k=0}^{T \wedge T_\varepsilon - 1} |II_{k,\varepsilon}| \right)
= O(\varepsilon L(\varepsilon) e^{\text{Lip}(Z)L(\varepsilon)}) + O \left( \varepsilon \sum_{k=0}^{T \wedge T_\varepsilon - 1} |II_{k,\varepsilon}| \right).
\] (2.9)

Fix \( \delta > 0 \). Recalling that \( T_\varepsilon = \infty \), it suffices to show that

\[
P \left( \varepsilon \sum_{k=0}^{T \wedge T_\varepsilon - 1} |II_{k,\varepsilon}| \geq \delta \right) \to 0
\]
as \( \varepsilon \to 0 \).

For initial conditions \( z \in M \) and for \( 0 \leq k \leq \frac{T}{\varepsilon L(\varepsilon)} \) define

\[
B_{k,\varepsilon} = \left\{ z : \frac{1}{L(\varepsilon)} |II_{k,\varepsilon}| > \frac{\delta}{2T} \right\},
B_z,\varepsilon = \{ k : z \in B_{k,\varepsilon} \}.
\]

Think of these sets as describing “bad ergodization.” For example, roughly speaking, \( z \in B_{k,\varepsilon} \) if the orbit \( z_\varepsilon(t) \) starting at \( z \) spends the time between \( t_{k,\varepsilon} \) and \( t_{k+1,\varepsilon} \)
in a region of phase space where the function \( H(\cdot, 0) \) is “poorly ergodized” on the time scale \( L(\varepsilon) \) by the flow \( z_0(t) \) (as measured by the parameter \( \delta/2T \)). As \( II_{k,\varepsilon} \) is clearly never larger than \( O(L(\varepsilon)) \), it follows that
\[
\sum_{k=0}^{T_{\varepsilon L(\varepsilon)}^{-1}} |II_{k,\varepsilon}| \leq \frac{\delta}{2} + O(\varepsilon L(\varepsilon) \#(B_{z,\varepsilon}))
\]
Therefore it suffices to show that
\[
P\left( \#(B_{z,\varepsilon}) \geq \frac{\delta}{\text{const} \varepsilon L(\varepsilon)} \right) \to 0 \quad \text{as} \quad \varepsilon \to 0
\]
By Chebyshev’s Inequality, we need only show that
\[
E(\varepsilon L(\varepsilon) \#(B_{z,\varepsilon})) = \varepsilon L(\varepsilon) \sum_{k=0}^{T_{\varepsilon L(\varepsilon)}^{-1}} P(B_{k,\varepsilon})
\]
tends to 0 with \( \varepsilon \).
In order to estimate the size of \( P(B_{k,\varepsilon}) \), it is convenient to introduce a new measure \( P^f \) that is uniformly equivalent to the restriction of Riemannian volume \( P \) to \( h^{-1} \mathcal{V} \). Here the \( f \) stands for “factor,” and \( P^f \) is defined by
\[
dP^f = dh \cdot d\mu_h,
\]
where \( dh \) represents integration with respect to the uniform measure on \( \mathcal{V} \).
Observe that \( B_{0,\varepsilon} = z_\varepsilon(t_{k,\varepsilon})B_{k,\varepsilon} \). In words, the initial conditions giving rise to orbits that are “bad” on the time interval \([t_{k,\varepsilon}, t_{k+1,\varepsilon}]\), moved forward by time \( t_{k,\varepsilon} \), are precisely the initial conditions giving rise to orbits that are “bad” on the time interval \([t_0,\varepsilon, t_1,\varepsilon]\). Because the flow \( z_0(\cdot) \) preserves the measure \( P^f \), we expect \( P^f(B_{0,\varepsilon}) \) and \( P^f(B_{k,\varepsilon}) \) to have roughly the same size. This is made precise by the following lemma.

**Lemma 2.3.2.** There exists a constant \( K \) such that for each Borel set \( B \subset \mathcal{M} \) and each \( t \in [-T/\varepsilon, T/\varepsilon] \),
\[
P^f(z_\varepsilon(t)B) \leq e^{KT} P^f(B).
\]
Proof. Assume that \( P^f(B) > 0 \), and set \( \gamma(t) = \ln(P^f(z_\varepsilon(t)B)/P^f(B)) \). Then \( \gamma(0) = 0 \), and
\[
\frac{d\gamma}{dt}(t) = \frac{d}{dt} \frac{\int_{z_\varepsilon(t)B} \tilde{f}(z)dz}{\int_{z_\varepsilon(t)B} \tilde{f}(z)dz} = \frac{\int_{z_\varepsilon(t)B} \text{div}_{P^f} Z(z,z,\varepsilon)dz}{\int_{z_\varepsilon(t)B} \tilde{f}(z)dz},
\]
where \( \tilde{f} > 0 \) is the \( C^1 \) density of \( P^f \) with respect to Riemannian volume on \( h^{-1} \mathcal{V} \), \( dz \) represents integration with respect to that volume, and \( \text{div}_{P^f} Z(z,z,\varepsilon) = \text{div}_{z} \tilde{f}(z)Z(z,z,\varepsilon) \). Because \( z_0(\cdot) \) preserves \( P^f \), \( \text{div}_{P^f} Z(z,0) \equiv 0 \). By Hadamard’s Lemma, it follows that \( \text{div}_{P^f} Z(z,z,\varepsilon) = O(\varepsilon) \) on the compact set \( h^{-1} \mathcal{V} \). Hence \( d\gamma(t)/dt = O(\varepsilon) \), and the result follows. \( \square \)
Returning to our proof of Anosov’s theorem, it suffices to show that
\[ P_f (B_0, \varepsilon) = \int_V dh \cdot \mu_h \left\{ z : \frac{1}{L(\varepsilon)} \left| \int_0^{L(\varepsilon)} H(z_0(s), 0) - \bar{H}(h_0(0))ds \right| \geq \frac{\delta}{2T} \} \]
tends to 0 with \( \varepsilon \). By our ergodicity assumption, for almost every \( h \),
\[ \mu_h \left\{ z : \frac{1}{L(\varepsilon)} \left| \int_0^{L(\varepsilon)} H(z_0(s), 0) - \bar{H}(h_0(0))ds \right| \geq \frac{\delta}{2T} \} \to 0 \text{ as } \varepsilon \to 0. \]
Finally, an application of the Bounded Convergence Theorem finishes the proof.

\[ \square \]

2.4 Moral

From the proofs of the theorems in this chapter, it should be apparent that there are at least two key steps necessary for proving a version of the averaging principle in the setting presented in Section 2.1.

The first step is estimating the continuity between the \( \varepsilon = 0 \) and the \( \varepsilon > 0 \) solutions of
\[ \frac{dz}{dt} = Z(z, \varepsilon). \]
In particular, on some relatively long timescale \( L = L(\varepsilon) \ll \varepsilon^{-1} \), we need to show that
\[ \sup_{0 \leq t \leq L} |z_0(t) - z_\varepsilon(t)| \to 0 \]
as \( \varepsilon \to 0 \). As long as \( L \) is sub-logarithmic in \( \varepsilon^{-1} \), such estimates for smooth systems can be made using Gronwall’s Inequality.

The second step is estimating the rate of ergodization of \( H(\cdot, 0) \) by \( z_0(t) \), i.e. estimating how fast
\[ \frac{1}{L} \int_0^L H(z_0(s), 0) ds \to \bar{H}(h_0) \]
(generally as \( L \to \infty \)). Note that the estimates in this step compete with those in the first step in that, if \( L \) is small we obtain better continuity, but if \( L \) is large we usually obtain better ergodization. Also, we do not need the full force of the assumption of ergodicity of \( (z_0(t), \mu_h) \) on the fibers \( \mathcal{M}_h \). We only need \( z_0(t) \) to ergodize the specific function \( H(\cdot, 0) \). Compare the proof of Theorem 2.2.5.

Note that in the setting of Anosov’s theorem, uniform ergodization leads to uniform convergence in the averaging principle. Returning to the proof of Theorem 2.1.1 above, suppose that
\[ \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} H(z_0(s), 0) ds \to \bar{H}(h_0) \]
uniformly over all initial conditions as $L(\varepsilon) \to \infty$. Then for all $\varepsilon$ sufficiently small and each $k$, $\mathcal{B}_{k,\varepsilon} = \emptyset$, and hence for all $\varepsilon$ sufficiently small and each $z$, $\#(\mathcal{B}_{z,\varepsilon}) = 0$. From Equation (2.9), it follows that $\sup_{0 \leq \tau \leq T \wedge T^\varepsilon} |\bar{h}(\tau) - h(\tau/\varepsilon)| \to 0$ as $\varepsilon \to 0$, uniformly over all initial conditions $z \in \bar{h}^{-1} \mathcal{V}$. However, uniform convergence in Birkhoff’s Ergodic Theorem is extremely rare and usually comes about because of unique ergodicity, so it is unreasonable to expect this sort of uniform convergence in most situations where Anosov’s theorem applies.
Chapter 3

Results for piston systems in one dimension

In this chapter, we present our results for piston systems in one dimension. These results may also be found in [Wri06].

3.1 Statement of results

3.1.1 The hard core piston problem

Consider the system of $n_1 + n_2 + 1$ point particles moving inside the unit interval indicated in Figure 3.1. One distinguished particle, the piston, has position $Q$ and mass $M$. To the left of the piston there are $n_1 > 0$ particles with positions $q_{1,j}$ and masses $m_{1,j}$, $1 \leq j \leq n_1$, and to the right there are $n_2 > 0$ particles with positions $q_{2,j}$ and masses $m_{2,j}$, $1 \leq j \leq n_2$. These gas particles do not interact with each other, but they interact with the piston and with walls located at the end points of the unit interval via elastic collisions. We denote the velocities by $dQ/dt = V$ and $dx_{i,j}/dt = v_{i,j}$. There is a standard method for transforming this system into a billiard system consisting of a point particle moving inside an $(n_1 + n_2 + 1)$-dimensional polytope [CM06a], but we will not use this in what follows.

We are interested in the dynamics of this system when the numbers and masses of the gas particles are fixed, the total energy is bounded, and the mass of the piston tends to infinity. When $M = \infty$, the piston remains at rest, and each gas particle performs periodic motion. More interesting are the motions of the system when $M$ is very large but finite. Because the total energy of the system is bounded, $MV^2/2 \leq \text{const}$, and so $V = \mathcal{O}(M^{-1/2})$. Set

$$\varepsilon = M^{-1/2},$$

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Figure 3.1: The piston system with $n_1 = 3$ and $n_2 = 4$. Note that the gas particles do not interact with each other, but only with the piston and the walls.

and let

$$W = \frac{V}{\varepsilon},$$

so that

$$\frac{dQ}{dt} = \varepsilon W$$

with $W = O(1)$.

When $\varepsilon = 0$, the system has $n_1 + n_2 + 2$ independent first integrals (conserved quantities), which we take to be $Q$, $W$, and $s_{i,j} = |v_{i,j}|$, the speeds of the gas particles. We refer to these variables as the slow variables because they should change slowly with time when $\varepsilon$ is small, and we denote them by

$$h = (Q, W, s_{1,1}, s_{1,2}, \cdots, s_{1,n_1}, s_{2,1}, s_{2,2}, \cdots, s_{2,n_2}) \in \mathbb{R}^{n_1 + n_2 + 2}.$$  

We will often abbreviate by writing $h = (Q, W, s_{1,j}, s_{2,j})$. Let $h_\varepsilon(t, z) = h_\varepsilon(t)$ denote the dynamics of these variables in time for a fixed value of $\varepsilon$, where $z$ represents the dependence on the initial condition in phase space. We usually suppress the initial condition in our notation. Think of $h_\varepsilon(\cdot)$ as a random variable which, given an initial condition in the $2(n_1 + n_2 + 1)$-dimensional phase space, produces a piecewise continuous path in $\mathbb{R}^{n_1 + n_2 + 2}$. These paths are the projection of the actual motions in our phase space onto a lower dimensional space. The goal of averaging is to find a vector field on $\mathbb{R}^{n_1 + n_2 + 2}$ whose orbits approximate $h_\varepsilon(t)$.  

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The averaged equation

Sinai [Sin99] derived

\[
\frac{d}{d\tau} \begin{bmatrix} Q \\ W \\ s_{1,j} \\ s_{2,j} \end{bmatrix} = \bar{H}(\bar{h}) := \begin{bmatrix} \frac{\sum_{j=1}^{n_1} m_{1,j} s_{1,j}^2}{Q} - \frac{\sum_{j=1}^{n_2} m_{2,j} s_{2,j}^2}{1-Q} \\ -\frac{s_{1,j} W}{Q} \\ -\frac{s_{1,j} W}{1-Q} \\ \frac{s_{2,j} W}{Q} + \frac{s_{2,j} W}{1-Q} \end{bmatrix}
\]  

(3.1)

as the averaged equation (with respect to the slow time \( \tau = \varepsilon t \)) for the slow variables. We provide a heuristic derivation in Section 3.2. Sinai solved this equation as follows: From

\[
\frac{d\ln(s_{1,j})}{d\tau} = -\frac{d\ln(Q)}{d\tau},
\]

\(s_{1,j}(\tau) = s_{1,j}(0)Q(0)/Q(\tau)\). Similarly, \(s_{2,j}(\tau) = s_{2,j}(0)(1-Q(0))/(1-Q(\tau))\). Hence

\[
\frac{d^2Q}{d\tau^2} = \frac{\sum_{j=1}^{n_1} m_{1,j} s_{1,j}(0)^2 Q(0)^2}{Q^3} - \frac{\sum_{j=1}^{n_2} m_{2,j} s_{2,j}(0)^2 (1-Q(0))^2}{(1-Q)^3},
\]

and so \((Q,W)\) behave as if they were the coordinates of a Hamiltonian system describing a particle undergoing periodic motion inside a potential well. If we let

\[
E_i = \sum_{j=1}^{n_i} \frac{m_{i,j} s_{i,j}^2}{2}
\]

be the kinetic energy of the gas particles on one side of the piston, the effective Hamiltonian may be expressed as

\[
\frac{1}{2} W^2 + \frac{E_1(0)Q(0)^2}{Q^2} + \frac{E_2(0)(1-Q(0))^2}{(1-Q)^2}.
\]  

(3.2)

Hence, the solutions to the averaged equation are periodic for all initial conditions under consideration.

Main result in the hard core setting

The solutions of the averaged equation approximate the motions of the slow variables, \(h_\varepsilon(t)\), on a time scale \(O(1/\varepsilon)\) as \(\varepsilon \to 0\). Precisely, let \(\bar{h}(\tau, z) = \bar{h}(\tau)\) be the solution of

\[
\frac{d\bar{h}}{d\tau} = H(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0).
\]
Again, think of \( h(\cdot) \) as being a random variable that takes an initial condition in our phase space and produces a path in \( \mathbb{R}^{n_1+n_2+2} \).

Next, fix a compact set \( V \subset \mathbb{R}^{n_1+n_2+2} \) such that \( h \in V \Rightarrow Q \subset (0,1), W \subset \mathbb{R} \), and \( s_{i,j} \subset (0,\infty) \) for each \( i \) and \( j \). For the remainder of this discussion we will restrict our attention to the dynamics of the system while the slow variables remain in the set \( V \). To this end, we define the stopping time

\[
T_{\varepsilon}(z) = T_{\varepsilon} := \inf \{ \tau \geq 0 : \bar{h}(\tau) \notin V \text{ or } h_{\varepsilon}(\tau/\varepsilon) \notin V \}.
\]

**Theorem 3.1.1.** For each \( T > 0 \),

\[
\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} \sup_{\text{s.t. } h_{\varepsilon}(0) \in V} \left| h_{\varepsilon}(\tau/\varepsilon) - \bar{h}(\tau) \right| = O(\varepsilon) \text{ as } \varepsilon = M^{-1/2} \to 0.
\]

This result was independently obtained by Gorelyshev and Neishtadt \[GN06\].

Note that the stopping time does not unduly restrict the result. Given any \( c \) such that \( h = c \Rightarrow Q \in (0,1), s_{i,j} \in (0,\infty) \), then by an appropriate choice of the compact set \( V \) we may ensure that, for all \( \varepsilon \) sufficiently small and all initial conditions in our phase space with \( h_{\varepsilon}(0) = c, T_{\varepsilon} \geq T \). We do this by choosing \( V \ni c \) such that the distance between \( \partial V \) and the periodic orbit \( \bar{h}(\tau) \) with \( \bar{h}(0) = c \) is positive. Call this distance \( d \). Then \( T_{\varepsilon} \) can only occur before \( T \) if \( h_{\varepsilon}(\tau/\varepsilon) \) has deviated by at least \( d \) from \( \bar{h}(\tau) \) for some \( \tau \in [0,T) \). Since the size of the deviations tends to zero uniformly with \( \varepsilon \), this is impossible for all small \( \varepsilon \).

### 3.1.2 The soft core piston problem

In this section, we consider the same system of one piston and gas particles inside the unit interval considered in Section 3.1.1 but now the interactions of the gas particles with the walls and with the piston are smooth. Let \( \kappa : \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) function satisfying

- \( \kappa(x) = 0 \) if \( x \geq 1 \),
- \( \kappa'(x) < 0 \) if \( x < 1 \).

Let \( \delta > 0 \) be a parameter of smoothing, and set

\[
\kappa_{\delta}(x) = \kappa(x/\delta).
\]

\footnote{We have introduced this notation for convenience. For example, \( h \in V \Rightarrow Q \subset (0,1) \) means that there exists a compact set \( A \subset (0,1) \) such that \( h \in V \Rightarrow Q \in A \), and similarly for the other variables.}
Then consider the Hamiltonian system obtained by having the gas particles interact with the piston and the walls via the potential

$$\sum_{j=1}^{n_1} \kappa \delta(q_{1,j}) + \kappa \delta(Q - q_{1,j}) + \sum_{j=1}^{n_2} \kappa \delta(q_{2,j} - Q) + \kappa \delta(1 - q_{2,j}).$$

As before, we set $\varepsilon = M^{-1/2}$ and $W = V/\varepsilon$. If we let

$$E_{1,j} = \frac{1}{2} m_{1,j} v_{1,j}^2 + \kappa \delta(q_{1,j}) + \kappa \delta(Q - q_{1,j}), \quad 1 \leq j \leq n_1,$$

$$E_{2,j} = \frac{1}{2} m_{2,j} v_{2,j}^2 + \kappa \delta(q_{2,j} - Q) + \kappa \delta(1 - q_{2,j}), \quad 1 \leq j \leq n_2,$$

then $E_{i,j}$ may be thought of as the energy associated with a gas particle, and $W^2/2 + \sum_{j=1}^{n_1} E_{1,j} + \sum_{j=1}^{n_2} E_{2,j}$ is the conserved energy.

When $\varepsilon = 0$, the Hamiltonian system admits $n_1 + n_2 + 2$ independent first integrals, which we choose this time as $h = (Q, W, E_{1,j}, E_{2,j})$. While discussing the soft core dynamics we use the energies $E_{i,j}$ rather than the variables $s_{i,j} = \sqrt{2E_{i,j}/m_{i,j}}$, which we used for the hard core dynamics, for convenience.

For comparison with the hard core results, we formally consider the dynamics described by setting $\delta = 0$ to be the hard core dynamics described in Section 3.1.1. This is reasonable because we will only consider gas particle energies below the barrier height $\kappa(0)$. Then for any $\varepsilon, \delta \geq 0$, $h^\delta(\tau)$ denotes the actual time evolution of the slow variables. While discussing the soft core dynamics we often use $\delta$ as a superscript to specify the dynamics for a certain value of $\delta$. We usually suppress the dependence on $\delta$, unless it is needed for clarity.

Main result in the soft core setting

We have already seen that when $\delta = 0$, there is an appropriate averaged vector field $\bar{H}^0$ whose solutions approximate the actual motions of the slow variables, $h^0(\tau)$. We will show that when $\delta > 0$, there is also an appropriate averaged vector field $\bar{H}^\delta$ whose solutions still approximate the actual motions of the slow variables, $h^\delta(\tau)$. We delay the derivation of $\bar{H}^\delta$ until Section 3.4.1.

Fix a compact set $\mathcal{V} \subset \mathbb{R}^{n_1 + n_2 + 2}$ such that $h \in \mathcal{V} \Rightarrow Q \subset (0, 1), W \subset \mathbb{R}$, and $E_{i,j} \subset (0, \kappa(0))$ for each $i$ and $j$. For each $\varepsilon, \delta \geq 0$ we define the functions $\bar{h}^\delta(\cdot)$ and $T^\delta_{\varepsilon}$ on our phase space by letting $\bar{h}^\delta(\tau)$ be the solution of

$$\frac{d\bar{h}^\delta}{d\tau} = \bar{H}^\delta(\bar{h}^\delta), \quad \bar{h}^\delta(0) = h^\delta(0),$$

and

$$T^\delta_{\varepsilon} = \inf\{\tau \geq 0 : \bar{h}^\delta(\tau) \notin \mathcal{V} \text{ or } h^\delta(\tau/\varepsilon) \notin \mathcal{V}\}.$$
Theorem 3.1.2. There exists $\delta_0 > 0$ such that the averaged vector field $\vec{H}^\delta(h)$ is $C^1$ on the domain \( \{(\delta,h) : 0 \leq \delta \leq \delta_0, h \in \mathcal{V}\} \). Furthermore, for each $T > 0$,

\[
\sup_{0 \leq \delta \leq \delta_0} \sup_{\text{initial conditions}} \sup_{0 \leq \tau \leq T \land T^\delta_\varepsilon} \left| h^\delta_\varepsilon(\tau/\varepsilon) - \bar{h}^\delta(\tau) \right| = O(\varepsilon) \quad \text{as} \quad \varepsilon = M^{-1/2} \to 0.
\]

As in Section 3.1.1, for any fixed $c$ there exists a suitable choice of the compact set $\mathcal{V}$ such that for all sufficiently small $\varepsilon$ and $\delta$, $T^\delta_\varepsilon \geq T$ whenever $h^\delta_\varepsilon(0) = c$.

As we will see, for each fixed $\delta > 0$, Anosov’s theorem 2.1.1 applies to the soft core system and yields a weak law of large numbers, and Theorem 2.2.5 applies and yields a strong law of large numbers with a uniform rate of convergence. However, neither of these theorems yields the uniformity over $\delta$ in the result above.

3.1.3 Applications and generalizations

Relationship between the hard core and the soft core piston

It is not a priori clear that we can compare the motions of the slow variables on the time scale $1/\varepsilon$ for $\delta > 0$ versus $\delta = 0$, i.e. compare the motions of the soft core piston with the motions of the hard core piston on a relatively long time scale. It is impossible to compare the motions of the fast-moving gas particles on this time scale as $\varepsilon \to 0$. As we see in Section 3.4, the frequency with which a gas particle hits the piston changes by an amount $O(\delta)$ when we smooth the interaction. Thus, on the time scale $1/\varepsilon$, the number of collisions is altered by roughly $O(\delta/\varepsilon)$, and this number diverges if $\delta$ is held fixed while $\varepsilon \to 0$.

Similarly, one might expect that it is impossible to compare the motions of the soft and hard core pistons as $\varepsilon \to 0$ without letting $\delta \to 0$ with $\varepsilon$. However, from Gronwall’s Inequality it follows that if $\bar{h}^\delta(0) = \bar{h}^0(0)$, then

\[
\sup_{0 \leq \tau \leq T \land T^\delta_\varepsilon \land T^\delta_0} \left| \bar{h}^\delta(\tau) - \bar{h}^0(\tau) \right| = O(\delta).
\]

From the triangle inequality and Theorems 3.1.1 and 3.1.2 we obtain the following corollary, which allows us to compare the motions of the hard core and the soft core piston.

Corollary 3.1.3. As $\varepsilon = M^{-1/2}, \delta \to 0$,

\[
\sup_{c \in \mathcal{V}} \sup_{\text{initial conditions}} \sup_{0 \leq t \leq (T \land T^\delta_\varepsilon \land T^\delta_0)/\varepsilon} \left| h^\delta_\varepsilon(t) - h^0_\varepsilon(t) \right| = O(\varepsilon) + O(\delta).
\]

This shows that, provided the slow variables have the same initial conditions,

\[
\sup_{0 \leq t \leq 1/\varepsilon} \left| h^\delta_\varepsilon(t) - h^0_\varepsilon(t) \right| = O(\varepsilon) + O(\delta).
\]

Thus the motions of the slow variables converge on the time scale $1/\varepsilon$ as $\varepsilon, \delta \to 0$, and it is immaterial in which order we let these parameters tend to zero.

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The adiabatic piston problem

We comment on what Theorem 3.1.1 says about the adiabatic piston problem. The
initial conditions of the adiabatic piston problem require that
\[ W(0) = 0. \]
Although our system is so simple that a proper thermodynamical pressure is not defined,
we can define the pressure of a gas to be the average force received from the gas
particles by the piston when it is held fixed, i.e.
\[ P_1 = \sum_{j=1}^{n_1} 2m_{1,j} s_{1,j} \frac{v_{1,j}}{2m_{1,j}} = 2E_1/Q \]
and
\[ P_2 = 2E_2/(1 - Q). \]
Then if \( P_1(0) > P_2(0) \), the initial condition for our
averaged equation (3.1) has the motion of the piston starting at the left turning
point of a periodic orbit determined by the effective potential well. Up to errors not
much bigger than \( M^{-1/2} \), we see the piston oscillate periodically on the time scale
\( M^{1/2} \). If \( P_1(0) < P_2(0) \), the motion of the piston starts at a right turning point.
However, if \( P_1(0) = P_2(0) \), then the motion of the piston starts at the bottom of
the effective potential well. In this case of mechanical equilibrium, \( \bar{h}(\tau) = \bar{h}(0) \),
and we conclude that, up to errors not much bigger than \( M^{-1/2} \), we see no motion
of the piston on the time scale \( M^{1/2} \). A much longer time scale is required to see
if the temperatures equilibrate.

Generalizations

A simple generalization of Theorem 3.1.1 proved by similar techniques, follows.
The system consists of \( N - 1 \) pistons, that is, heavy point particles, located inside
the unit interval at positions \( Q_1 < Q_2 < \ldots < Q_{N-1} \). Walls are located at \( Q_0 \equiv 0 \)
and \( Q_N \equiv 1 \), and the piston at position \( Q_i \) has mass \( M_i \). Then the pistons divide
the unit interval into \( N \) chambers. Inside the \( i^{th} \) chamber, there are \( n_i \geq 1 \) gas
particles whose locations and masses will be denoted by \( x_{i,j} \) and \( m_{i,j} \), respectively,
where \( 1 \leq j \leq n_i \). All of the particles are point particles, and the gas particles
interact with the pistons and with the walls via elastic collisions. However, the
gas particles do not directly interact with each other. We scale the piston masses
as \( \tilde{M}_i = \tilde{M}_i/\varepsilon^2 \) with \( \tilde{M}_i \) constant, define \( \tilde{W}_i \) by \( dQ_i/dt = \varepsilon \tilde{W}_i \), and let \( E_i \) be
the kinetic energy of the gas particles in the \( i^{th} \) chamber. Then we can find an
appropriate averaged equation whose solutions have the pistons moving like an
\((N-1)\)-dimensional particle inside a potential well with an effective Hamiltonian

\[ \frac{1}{2} \sum_{i=1}^{N-1} \tilde{M}_i \tilde{W}_i^2 + \sum_{i=1}^{N} \frac{E_i(0)(Q_i(0) - Q_{i-1}(0))^2}{(Q_i - Q_{i-1})^2}. \]

If we write the slow variables as \( h = (Q_i, W_i, |v_{i,j}|) \) and fix a compact set \( \mathcal{V} \) such
that \( h \in \mathcal{V} \Rightarrow Q_{i+1} - Q_i \subset \subset (0,1), W_i \subset \subset \mathbb{R}, \) and \( |v_{i,j}| \subset \subset (0,\infty) \), then the
convergence of the actual motions of the slow variables to the averaged solutions
is exactly the same as the convergence given in Theorem 3.1.1.
Remark 3.1.1. The inverse quadratic potential between adjacent pistons in the effective Hamiltonian above is also referred to as the Calogero-Moser-Sutherland potential. It has also been observed as the effective potential created between two adjacent tagged particles in a one-dimensional Rayleigh gas by the insertion of one very light particle inbetween the tagged particles [BTT07].

3.2 Heuristic derivation of the averaged equation for the hard core piston

We present here a heuristic derivation of Sinai’s averaged equation (3.1) that is found in [Dol05].

First, we examine interparticle collisions when $\varepsilon > 0$. When a particle on the left, say the one at position $q_{1,j}$, collides with the piston, $s_{1,j}$ and $W$ instantaneously change according to the laws of elastic collisions:

$$
\begin{align*}
\left[ s_{1,j}^+ \right] &= \frac{1}{2m_{1,j} + M} \left[ \frac{m_{1,j} - M}{2m_{1,j}} - 2M \right] \left[ s_{1,j}^{-} \right], \\
\left[ W^+ \right] &= \frac{1}{2m_{1,j} + M} \left[ \frac{m_{1,j} - M}{2m_{1,j}} - 2M \right] \left[ W^{-} \right].
\end{align*}
$$

(3.5)

If the speed of the left gas particle is bounded away from zero, and $W = M^{1/2}V$ is also bounded, it follows that for all $\varepsilon$ sufficiently small, any collision will have $v_{1,j}^+ > 0$ and $v_{1,j}^- < 0$. In this case, when we translate Equation (3.5) into our new coordinates, we find that

$$
\Delta s_{1,j} = s_{1,j}^+ - s_{1,j}^- = -2\varepsilon W^- + O(\varepsilon^2),
\Delta W = W^+ - W^- = +2\varepsilon m_{1,j} s_{1,j}^- + O(\varepsilon^2).
$$

The situation is analogous when particles on the right collide with the piston. For all $\varepsilon$ sufficiently small, $s_{2,j}$ and $W$ instantaneously change by

$$
\Delta W = W^+ - W^- = -2\varepsilon m_{2,j} s_{2,j}^- + O(\varepsilon^2),
\Delta s_{2,j} = s_{2,j}^+ - s_{2,j}^- = +2\varepsilon W^- + O(\varepsilon^2).
$$

We defer discussing the rare events in which multiple gas particles collide with the piston simultaneously, although we will see that they can be handled appropriately.

Let $\Delta t$ be a length of time long enough such that the piston experiences many collisions with the gas particles, but short enough such that the slow variables change very little, in this time interval. From each collision with the particle at
position \( q_{1,j} \), \( W \) changes by an amount \(+2\varepsilon m_{1,j}s_{1,j} + \mathcal{O}(\varepsilon^2)\), and the frequency of these collisions is approximately \( \frac{n_{1,j}}{2Q} \). Arguing similarly for collisions with the other particles, we guess that

\[
\frac{\Delta W}{\Delta t} = \varepsilon \sum_{j=1}^{n_{1}} 2m_{1,j}s_{1,j} \frac{s_{1,j}}{2Q} - \varepsilon \sum_{j=1}^{n_{2}} 2m_{2,j}s_{2,j} \frac{s_{2,j}}{2(1-Q)} + \mathcal{O}(\varepsilon^2).
\]

Note that not only does the position of the piston change slowly in time, but its velocity also changes slowly, i.e. the piston has inertia. With \( \tau = \varepsilon t \) as the slow time, a reasonable guess for the averaged equation for \( W \) is

\[
\frac{dW}{d\tau} = \frac{\sum_{j=1}^{n_{1}} m_{1,j}s_{1,j}^2}{Q} - \frac{\sum_{j=1}^{n_{2}} m_{2,j}s_{2,j}^2}{1-Q}.
\]

Similar arguments for the other slow variables lead to the averaged equation (3.1).

### 3.3 Proof of the main result for the hard core piston

#### 3.3.1 Proof of Theorem 3.1.1 with only one gas particle on each side

We specialize to the case when there is only one gas particle on either side of the piston, i.e. we assume that \( n_1 = n_2 = 1 \). We then denote \( x_{1,1} \) by \( q_1 \), \( m_{2,2} \) by \( m_2 \), etc. This allows the proof’s major ideas to be clearly expressed, without substantially limiting their applicability. At the end of this section, we outline the simple generalizations needed to make the proof apply in the general case.

**A choice of coordinates on the phase space for a three particle system**

As part of our proof, we choose a set of coordinates on our six-dimensional phase space such that, in these coordinates, the \( \varepsilon = 0 \) dynamics are smooth. Complete the slow variables \( h = (Q, W, s_1, s_2) \) to a full set of coordinates by adding the coordinates \( \varphi_i \in [0, 1]/0 \sim 1 = S^1 \), \( i = 1, 2 \), defined as follows:

\[
\varphi_1 = \varphi_1(q_1, v_1, Q) = \begin{cases} \frac{q_1}{2Q} & \text{if } v_1 > 0 \\ 1 - \frac{q_1}{2Q} & \text{if } v_1 < 0 \end{cases}
\]

\[
\varphi_2 = \varphi_2(q_2, v_2, Q) = \begin{cases} \frac{1-q_2}{2(1-Q)} & \text{if } v_2 < 0 \\ 1 - \frac{1-q_2}{2(1-Q)} & \text{if } v_2 > 0 \end{cases}
\]
When $\varepsilon = 0$, these coordinates are simply the angle variable portion of action-angle coordinates for an integrable Hamiltonian system. They are defined such that collisions occur between the piston and the gas particles precisely when $\varphi_1$ or $\varphi_2 = 1/2$. Then $z = (h, \varphi_1, \varphi_2)$ represents a choice of coordinates on our phase space, which is homeomorphic to a subset of $\mathbb{R}^4 \times T^2$. We abuse notation and also let $h(z)$ represent the projection onto the first four coordinates of $z$.

Now we describe the dynamics of our system in these coordinates. When $\varphi_1, \varphi_2 \neq 1/2$,

$$
\frac{d\varphi_1}{dt} = \begin{cases} \frac{s_1}{2Q} - \frac{\varepsilon W}{Q} \varphi_1 & \text{if } 0 \leq \varphi_1 < 1/2 \\ \frac{s_1}{2Q} + \frac{\varepsilon W}{Q} (1 - \varphi_1) & \text{if } 1/2 < \varphi_1 \leq 1 \end{cases}
$$

$$
\frac{d\varphi_2}{dt} = \begin{cases} \frac{s_2}{2(1-Q)} + \frac{\varepsilon W}{1-Q} \varphi_2 & \text{if } 0 \leq \varphi_2 < 1/2 \\ \frac{s_2}{2(1-Q)} - \frac{\varepsilon W}{1-Q} (1 - \varphi_2) & \text{if } 1/2 < \varphi_2 \leq 1 \end{cases}
$$

Hence between interparticle collisions, the dynamics are smooth and are described by

$$
\begin{align*}
\frac{dQ}{dt} &= \varepsilon W, \\
\frac{dW}{dt} &= 0, \\
\frac{ds_1}{dt} &= 0, \\
\frac{ds_2}{dt} &= 0, \\
\frac{d\varphi_1}{dt} &= \frac{s_1}{2Q} + \mathcal{O}(\varepsilon), \\
\frac{d\varphi_2}{dt} &= \frac{s_2}{2(1-Q)} + \mathcal{O}(\varepsilon).
\end{align*}
$$

(3.7)

When $\varphi_1$ reaches $1/2$, while $\varphi_2 \neq 1/2$, the coordinates $Q, s_2, \varphi_1$, and $\varphi_2$ are instantaneously unchanged, while $s_1$ and $W$ instantaneously jump, as described by Equation (3.6). As an aside, it is curious that $s_1^+ + \varepsilon W^+ = s_1^- - \varepsilon W^-$, so that $d\varphi_1/dt$ is continuous as $\varphi_1$ crosses $1/2$. However, the collision induces discontinuous jumps of size $\mathcal{O}(\varepsilon^2)$ in $dQ/dt$ and $d\varphi_2/dt$. Denote the linear transformation in Equation (3.6) with $j = 1$ by $A_{1,\varepsilon}$. Then

$$
A_{1,\varepsilon} = \begin{bmatrix} 1 & -2\varepsilon \\ 2\varepsilon m_1 & 1 \end{bmatrix} + \mathcal{O}(\varepsilon^2).
$$

The situation is analogous when $\varphi_2$ reaches $1/2$, while $\varphi_1 \neq 1/2$. Then $W$ and $s_2$ are instantaneously transformed by a linear transformation

$$
A_{2,\varepsilon} = \begin{bmatrix} 1 & -2\varepsilon m_2 \\ 2\varepsilon & 1 \end{bmatrix} + \mathcal{O}(\varepsilon^2).
$$
We also account for the possibility of all three particles colliding simultaneously. There is no completely satisfactory way to do this, as the dynamics have an essential singularity near \( \{\varphi_1 = \varphi_2 = 1/2\} \). Furthermore, such three particle collisions occur with probability zero with respect to the invariant measure discussed below. However, the two \( 3 \times 3 \) matrices

\[
\begin{bmatrix}
A_{1,\varepsilon} & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & A_{2,\varepsilon}
\end{bmatrix}
\]

have a commutator of size \( \mathcal{O}(\varepsilon^2) \). We will see that this small of an error will make no difference to us as \( \varepsilon \to 0 \), and so when \( \varphi_1 = \varphi_2 = 1/2 \), we pretend that the left particle collides with the piston instantaneously before the right particle does. Precisely, we transform the variables \( s_1, W, \) and \( s_2 \) by

\[
\begin{bmatrix}
s_1^+ \\
W^+ \\
s_2^+
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & A_{2,\varepsilon}
\end{bmatrix}
\begin{bmatrix}
s_1^- \\
W^- \\
s_2^-
\end{bmatrix}.
\]

We find that

\[
\Delta s_1 = s_1^+ - s_1^- = -2\varepsilon W^- + \mathcal{O}(\varepsilon^2),
\]
\[
\Delta W = W^+ - W^- = +2\varepsilon m_1 s_1^- - 2\varepsilon m_2 s_2^- + \mathcal{O}(\varepsilon^2),
\]
\[
\Delta s_2 = s_2^+ - s_2^- = +2\varepsilon W^- + \mathcal{O}(\varepsilon^2).
\]

The above rules define a flow on the phase space, which we denote by \( z_\varepsilon(t) \). We denote its components by \( Q_\varepsilon(t), W_\varepsilon(t), s_{1,\varepsilon}(t), \) etc. When \( \varepsilon > 0 \), the flow is not continuous, and for definiteness we take \( z_\varepsilon(t) \) to be left continuous in \( t \).

Because our system comes from a Hamiltonian system, it preserves Liouville measure. In our coordinates, this measure has a density proportional to \( Q(1 - Q) \). That this measure is preserved also follows from the fact that the ordinary differential equation (3.7) preserves this measure, and the matrices \( A_{1,\varepsilon}, A_{2,\varepsilon} \) have determinant 1. Also note that the set \( \{\varphi_1 = \varphi_2 = 1/2\} \) has co-dimension two, and so \( \bigcup_t z_\varepsilon(t) \{\varphi_1 = \varphi_2 = 1/2\} \) has co-dimension one, which shows that only a measure zero set of initial conditions will give rise to three particle collisions.

**Argument for uniform convergence**

**Step 1: Reduction using Gronwall’s Inequality.** Define \( H(z) \) by

\[
H(z) = 
\begin{bmatrix}
W \\
2m_1 s_1 \delta_{\varphi_1 = 1/2} - 2m_2 s_2 \delta_{\varphi_2 = 1/2} \\
-2W \delta_{\varphi_1 = 1/2} \\
2W \delta_{\varphi_2 = 1/2}
\end{bmatrix}.
\]
Here we make use of Dirac delta functions. All integrals involving these delta functions may be replaced by sums. We explicitly deal with any ambiguities arising from collisions occurring at the limits of integration.

**Lemma 3.3.1.** For \(0 \leq t \leq \frac{T\sqrt{\varepsilon}}{\varepsilon},\)

\[
h_{\varepsilon}(t) - h_{\varepsilon}(0) = \varepsilon \int_0^t H(z_{\varepsilon}(s))ds + O(\varepsilon),
\]

where any ambiguity about changes due to collisions occurring precisely at times 0 and \(t\) is absorbed in the \(O(\varepsilon)\) term.

**Proof.** There are four components to verify. The first component requires that \(Q_{\varepsilon}(t) - Q_{\varepsilon}(0) = \varepsilon \int_0^t W_{\varepsilon}(s)ds + O(\varepsilon)\). This is trivially true because \(Q_{\varepsilon}(t) - Q_{\varepsilon}(0) = \varepsilon \int_0^t W_{\varepsilon}(s)ds\).

The second component states that \(W_{\varepsilon}(t) - W_{\varepsilon}(0) = \varepsilon \int_0^t 2m_1 s_{1,\varepsilon}(s)\delta_{\varphi_{1,\varepsilon}(s)=1/2} - 2m_2 s_{2,\varepsilon}(s)\delta_{\varphi_{2,\varepsilon}(s)=1/2}ds + O(\varepsilon)\). (3.8)

Let \(r_k\) and \(q_j\) be the times in \((0, t)\) such that \(\varphi_{1,\varepsilon}(r_k) = 1/2\) and \(\varphi_{2,\varepsilon}(q_j) = 1/2\), respectively. Then

\[
W_{\varepsilon}(t) - W_{\varepsilon}(0) = \sum_{r_k} \Delta W_{\varepsilon}(r_k) + \sum_{q_j} \Delta W_{\varepsilon}(q_j) + O(\varepsilon).
\]

Observe that there exists \(\omega > 0\) such that for all sufficiently small \(\varepsilon\) and all \(h \in V, 1/\omega < \frac{d\varphi_i}{dt} < \omega\). Thus the number of collisions in a time interval grows no faster than linearly in the length of that time interval. Because \(t \leq T/\varepsilon\), it follows that

\[
W_{\varepsilon}(t) - W_{\varepsilon}(0) = \varepsilon \sum_{r_k} 2m_1 s_{1,\varepsilon}(r_k) - \varepsilon \sum_{q_j} 2m_2 s_{2,\varepsilon}(q_j) + O(\varepsilon),
\]

and Equation (3.8) is verified. Note that because \(V\) is compact, there is uniformity over all initial conditions in the size of the \(O(\varepsilon)\) terms above. The third and fourth components are handled similarly. \(\square\)

Next, \(\bar{h}(\tau)\) satisfies the integral equation

\[
\bar{h}(\tau) - \bar{h}(0) = \int_0^\tau \bar{H}(\bar{h}(\sigma))d\sigma,
\]

while \(h_{\varepsilon}(\tau/\varepsilon)\) satisfies

\[
h_{\varepsilon}(\tau/\varepsilon) - h_{\varepsilon}(0) = O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_{\varepsilon}(s))ds
\]

\[
= O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_{\varepsilon}(s)) - \bar{H}(h_{\varepsilon}(s))ds + \int_0^\tau \bar{H}(h_{\varepsilon}(\sigma/\varepsilon))d\sigma
\]
for \(0 \leq \tau \leq T \wedge T_{\varepsilon}\).

Define

\[ e_{\varepsilon}(\tau) = \varepsilon \int_0^{\tau/\varepsilon} H(z_{\varepsilon}(s)) - \bar{H}(h_{\varepsilon}(s)) ds. \]

It follows from Gronwall’s Inequality that

\[
\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} \left| \bar{h}(\tau) - h_{\varepsilon}(\tau/\varepsilon) \right| \leq \left( O(\varepsilon) + \sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} |e_{\varepsilon}(\tau)| \right) e^{\text{Lip}(H)\nu T}. \tag{3.9}
\]

Gronwall’s Inequality is usually stated for continuous paths, but the standard proof (found in [SV85]) still works for paths that are merely integrable, and \(\bar{h}(\tau) - h_{\varepsilon}(\tau/\varepsilon)\) is piecewise smooth.

**Step 2: A splitting according to particles.** Now

\[
H(z) - \bar{H}(h) = \begin{bmatrix} 0 & \varepsilon \int_0^{\tau/\varepsilon} s_{1,\varepsilon}(s) \delta_{\varphi_{1,\varepsilon}}(s) ds - \frac{s_{1,\varepsilon}(s)^2}{2Q_{\varepsilon}(s)} ds \\ -2Wm_2s_2\delta_{\varphi_{1,\varepsilon}}(s) + s_1W/Q & 0 \\ 2W\delta_{\varphi_{2,\varepsilon}}(s) - s_2W/(1 - Q) & 0 \end{bmatrix},
\]

and so, in order to show that \(\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} |e_{\varepsilon}(\tau)| = O(\varepsilon)\), it suffices to show that

\[
\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} \left| \int_0^{\tau/\varepsilon} s_{1,\varepsilon}(s) \delta_{\varphi_{1,\varepsilon}}(s) ds - \frac{s_{1,\varepsilon}(s)^2}{2Q_{\varepsilon}(s)} ds \right| = O(1),
\]

as well as two analogous claims about terms involving \(\varphi_{2,\varepsilon}\). Thus we have effectively separated the effects of the different gas particles, so that we can deal with each particle separately. We will only show that

\[
\sup_{0 \leq \tau \leq T \wedge T_{\varepsilon}} \left| \int_0^{\tau/\varepsilon} s_{1,\varepsilon}(s) \delta_{\varphi_{1,\varepsilon}}(s) ds - \frac{s_{1,\varepsilon}(s)^2}{2Q_{\varepsilon}(s)} ds \right| = O(1).
\]

The other three terms can be handled similarly.

**Step 3: A sequence of times adapted for ergodization.** Ergodization refers to the convergence along an orbit of a function’s time average to its space average. For example, because of the splitting according to particles above, one can easily check that

\[
\frac{1}{t} \int_0^t H(z_0(s)) ds = \bar{H}(h_0) + O(1/t),
\]

even when \(z_0(\cdot)\) restricted to the invariant tori \(\mathcal{M}_{h_0}\) is not ergodic. In this step, for each initial condition \(z_\varepsilon(0)\) in
Lemma 3.3.2. If \( t \) in the sum has the same form, without loss of generality we will only examine the \( t \)
comparing the motions of \( O \) suffices to show that each term is no larger than our phase space, we define a sequence of times \( t_{k,\varepsilon} \) inductively as follows: \( t_{0,\varepsilon} = \inf \{ t \geq 0 : \varphi_{1,\varepsilon}(t) = 0 \} \), \( t_{k+1,\varepsilon} = \inf \{ t > t_{k,\varepsilon} : \varphi_{1,\varepsilon}(t) = 0 \} \). This sequence is chosen because \( \delta_{\varphi_{1,\varepsilon}(s)} = 1/2 \) is “ergodizd” as time passes from \( t_{k,\varepsilon} \) to \( t_{k+1,\varepsilon} \). If \( \varepsilon \) is sufficiently small and \( t_{k+1,\varepsilon} \leq (T \wedge T_\varepsilon)/\varepsilon \), then the spacings between these times are uniformly of order 1, i.e. \( 1/\omega < t_{k+1,\varepsilon} - t_{k,\varepsilon} < \omega \). Thus,

\[
\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| \int_0^{\tau/\varepsilon} s_{1,\varepsilon}(s)\delta_{\varphi_{1,\varepsilon}(s)} = 1/2 - \frac{s_{1,\varepsilon}(s)^2}{2Q_\varepsilon(s)} ds \right| \leq O(1) + \sum_{t_{k+1,\varepsilon} \leq \tau \leq T \wedge T_\varepsilon} \left| \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} s_{1,\varepsilon}(s)\delta_{\varphi_{1,\varepsilon}(s)} = 1/2 - \frac{s_{1,\varepsilon}(s)^2}{2Q_\varepsilon(s)} ds \right| .
\] (3.10)

**Step 4: Control of individual terms by comparison with solutions along fibers.** The sum in Equation (3.10) has no more than \( O(1/\varepsilon) \) terms, and so it suffices to show that each term is no larger than \( O(\varepsilon) \). We can accomplish this by comparing the motions of \( z_\varepsilon(t) \) for \( t_{k,\varepsilon} \leq t \leq t_{k+1,\varepsilon} \) with the solution of the \( \varepsilon = 0 \) version of Equation (3.7) that, at time \( t_{k,\varepsilon} \), is located at \( z_\varepsilon(t_{k,\varepsilon}) \). Since each term in the sum has the same form, without loss of generality we will only examine the first term and suppose that \( t_{0,\varepsilon} = 0 \), i.e. that \( \varphi_{1,\varepsilon}(0) = 0 \).

**Lemma 3.3.2.** If \( t_{1,\varepsilon} \leq \frac{T \wedge T_\varepsilon}{\varepsilon} \), then \( \sup_{0 \leq t \leq t_{1,\varepsilon}} |z_\varepsilon(t) - z_0(t)| = O(\varepsilon) \).

**Proof.** To check that \( \sup_{0 \leq t \leq t_{1,\varepsilon}} |h_0(t) - h_\varepsilon(t)| = O(\varepsilon) \), first note that \( h_0(t) = h_0(0) = h_\varepsilon(0) \). Then \( dQ_\varepsilon/dt = O(\varepsilon) \), so that \( Q_0(t) - Q_\varepsilon(t) = O(\varepsilon t) \). Furthermore, the other slow variables change by \( O(\varepsilon) \) at collisions, while the number of collisions in the time interval \([0, t_{1,\varepsilon}]\) is \( O(1) \).

It remains to show that \( \sup_{0 \leq t \leq t_{1,\varepsilon}} |\varphi_{i,0}(t) - \varphi_{i,\varepsilon}(t)| = O(\varepsilon) \). Using what we know about the divergence of the slow variables,

\[
\varphi_{1,0}(t) - \varphi_{1,\varepsilon}(t) = \int_0^t \frac{s_{1,0}(s)}{2Q_0(s)} \varepsilon - \frac{s_{1,\varepsilon}(s)}{2Q_\varepsilon(s)} + O(\varepsilon) ds = \int_0^t O(\varepsilon) ds = O(\varepsilon)
\]

for \( 0 \leq t \leq t_{1,\varepsilon} \). Showing that \( \sup_{0 \leq t \leq t_{1,\varepsilon}} |\varphi_{2,0}(t) - \varphi_{2,\varepsilon}(t)| = O(\varepsilon) \) is similar. \( \square \)

From Lemma 3.3.2, \( t_{1,\varepsilon} = t_{1,0} + O(\varepsilon) = 2Q_0/s_{1,0} + O(\varepsilon) \). We conclude that

\[
\int_0^{t_{1,\varepsilon}} s_{1,\varepsilon}(s)\delta_{\varphi_{1,\varepsilon}(s)} = 1/2 - \frac{s_{1,\varepsilon}(s)^2}{2Q_\varepsilon(s)} ds = O(\varepsilon) + \int_0^{t_{1,\varepsilon}} s_{1,0}(s)\delta_{\varphi_{1,\varepsilon}(s)} = 1/2 - \frac{s_{1,0}(s)^2}{2Q_0(s)} ds
\]

\[
= O(\varepsilon) + s_{1,0} - t_{1,\varepsilon} \frac{s_{1,0}^2}{2Q_0} = O(\varepsilon).
\]

It follows that \( \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| h_\varepsilon(\tau/\varepsilon) - h(\tau) \right| = O(\varepsilon) \), independent of the initial condition in \( h^{-1}V \).
3.3.2 Extension to multiple gas particles

When \( n_1, n_2 > 1 \), only minor modifications are necessary to generalize the proof above. We start by extending the slow variables \( h \) to a full set of coordinates on phase space by defining the angle variables \( \varphi_{i,j} \in [0, 1] / 0 \sim 1 = S^1 \) for \( 1 \leq i \leq 2 \), \( 1 \leq j \leq n_i \):

\[
\varphi_{1,j} = \varphi_{1,j}(q_{1,j}, v_{1,j}, Q) = \begin{cases} 
\frac{q_{1,j}}{2Q} & \text{if } v_{1,j} > 0 \\
1 - \frac{q_{1,j}}{2Q} & \text{if } v_{1,j} < 0
\end{cases}
\]

\[
\varphi_{2,j} = \varphi_{2,j}(q_{2,j}, v_{2,j}, Q) = \begin{cases} 
\frac{1 - q_{2,j}}{2(1 - Q)} & \text{if } v_{2,j} < 0 \\
1 - \frac{1 - q_{2,j}}{2(1 - Q)} & \text{if } v_{2,j} > 0
\end{cases}
\]

Then \( d\varphi_{1,j}/dt = s_{1,j}(2Q)^{-1} + O(\varepsilon) \), \( d\varphi_{2,j}/dt = s_{2,j}(2(1 - Q))^{-1} + O(\varepsilon) \), and \( z = (h, \varphi_{1,j}, \varphi_{2,j}) \) represents a choice of coordinates on our phase space, which is homeomorphic to \( (\mathbb{R}^{n_1 + n_2 + 2}) \times \mathbb{T}^{n_1 + n_2} \). In these coordinates, the dynamical system yields a discontinuous flow \( z_\varepsilon(t) \) on phase space. The flow preserves Liouville measure, which in our coordinates has a density proportional to \( Q^{n_1}(1 - Q)^{n_2} \). As is Section 3.3.1, one can show that the measure of initial conditions leading to multiple particle collisions is zero.

Next, define \( H(z) \) by

\[
H(z) = \left[ \sum_{j=1}^{n_1} 2m_{1,j}s_{1,j}\delta_{\varphi_{1,j}=1/2} - \sum_{j=1}^{n_2} 2m_{2,j}s_{2,j}\delta_{\varphi_{2,j}=1/2} - 2W\delta_{\varphi_{1,j}=1/2} - 2W\delta_{\varphi_{2,j}=1/2} \right].
\]

For \( 0 \leq t \leq \frac{T_{\varepsilon}}{\varepsilon} \), \( h_\varepsilon(t) - h_\varepsilon(0) = \varepsilon \int_0^t H(z_\varepsilon(s))ds + O(\varepsilon) \). From here, the rest of the proof follows the same arguments made in Section 3.3.1.

3.4 Proof of the main result for the soft core piston

For the remainder of this chapter, we consider the family of Hamiltonian systems introduced in Section 3.1.2 which are parameterized by \( \varepsilon, \delta \geq 0 \). For simplicity, we specialize to \( n_1 = n_2 = 1 \). As in Section 3.3, the generalization to \( n_1, n_2 > 1 \) is not difficult. The Hamiltonian dynamics are given by the following ordinary
differential equation:
\[
\begin{align*}
\frac{dQ}{dt} &= \varepsilon W, \\
\frac{dW}{dt} &= \varepsilon (-\kappa'_\delta(Q - x_1) + \kappa'_\delta(x_2 - Q)), \\
\frac{dx_1}{dt} &= v_1, \\
\frac{dv_1}{dt} &= \frac{1}{m_1} (-\kappa'_\delta(x_1) + \kappa'_\delta(Q - x_1)), \\
\frac{dx_2}{dt} &= v_2, \\
\frac{dv_2}{dt} &= \frac{1}{m_2} (-\kappa'_\delta(x_2 - Q) + \kappa'_\delta(1 - x_2)).
\end{align*}
\] (3.11)

Recalling the particle energies defined by Equation (3.3), we find that
\[
\frac{dE_1}{dt} = \varepsilon W \kappa'_\delta(Q - x_1), \quad \frac{dE_2}{dt} = -\varepsilon W \kappa'_\delta(x_2 - Q).
\]

For the compact set \( \mathcal{V} \) introduced in Section 3.1.2, fix a small positive number \( \varepsilon \) and an open set \( \mathcal{U} \subset \mathbb{R}^4 \) such that \( \mathcal{V} \subset \mathcal{U} \) and \( h \in \mathcal{U} \Rightarrow Q \in (\mathcal{E}, 1 - \mathcal{E}), W \subset \subset \mathbb{R}, \) and \( \mathcal{E} < E_1, E_2 < \kappa(0) - \mathcal{E}. \) We only consider the dynamics for \( 0 < \delta < \mathcal{E}/2 \) and \( h \in \mathcal{U}. \)

Define
\[
U_1(q_1) = U_1(q_1, Q, \delta) := \kappa_\delta(q_1) + \kappa_\delta(Q - q_1), \\
U_2(q_2) = U_2(q_2, Q, \delta) := \kappa_\delta(q_2 - Q) + \kappa_\delta(1 - q_2).
\]

Then the energies \( E_i \) satisfy \( E_i = m_i v_i^2/2 + U_i(x_i). \)

Let \( T_1 = T_1(Q, E_1, \delta) \) and \( T_2 = T_2(Q, E_2, \delta) \) denote the periods of the motions of the left and right gas particles, respectively, when \( \varepsilon = 0. \)

**Lemma 3.4.1.** For \( i = 1, 2, \)
\[
T_i \in \mathcal{C}^1 \{(Q, E_i, \delta) : Q \in (\mathcal{E}, 1 - \mathcal{E}), E_i \in (\mathcal{E}, \kappa(0) - \mathcal{E}), 0 \leq \delta < \mathcal{E}/2 \}.
\]
Furthermore,
\[
T_1(Q, E_1, \delta) = \sqrt{\frac{2m_1}{E_1}} Q + \mathcal{O}(\delta),
\]
\[
T_2(Q, E_2, \delta) = \sqrt{\frac{2m_2}{E_2}} (1 - Q) + \mathcal{O}(\delta).
\]
The proof of this lemma is mostly computational, and so we delay it until Section 3.5. Note especially that the periods can be suitably defined such that their regularity extends to $\delta = 0$.

In this section, and in Section 3.5 below, we adopt the following convention on the use of the $O$ notation. All use of the $O$ notation will explicitly contain the dependence on $\varepsilon$ and $\delta$ as $\varepsilon, \delta \to 0$. For example, if a function $f(h, \varepsilon, \delta) = O(\varepsilon)$, then there exists $\delta', \varepsilon' > 0$ such that $\sup_{0 < \varepsilon \leq \varepsilon', 0 < \delta \leq \delta', h \in V} |f(h, \varepsilon, \delta) / \varepsilon| < \infty$.

When $\varepsilon = 0$, \[ \frac{dx_i}{dt} = \pm \sqrt{\frac{2}{m_i} (E_i - U_i(x_i))}. \]

Define $a = a(E_i, \delta)$ by \[ \kappa(a) = \kappa(a / \delta) = E_i, \]
so that $a(E_i, \delta)$ is a turning point for the left gas particle. Then $a = \delta \kappa^{-1}(E_i)$, where $\kappa^{-1}$ is defined as follows: $\kappa : [0, 1] \to [0, \kappa(0)]$ takes 0 to $\kappa(0)$ and 1 to 0. Furthermore, $\kappa \in C^2([0, 1])$, $\kappa' \leq 0$, and $\kappa'(x) < 0$ if $x < 1$. By monotonicity, $\kappa^{-1} : [0, \kappa(0)] \to [0, 1]$ exists and takes 0 to 1 and $\kappa(0)$ to 0. Also, by the Implicit Function Theorem, $\kappa^{-1} \in C^2((0, \kappa(0))]$, $(\kappa^{-1})'(y) < 0$ for $y > 0$, and $(\kappa^{-1})'(y) \to -\infty$ as $y \to 0$.

Because we only consider energies $E_i \in (\mathcal{E}, \kappa(0) - \mathcal{E})$, it follows that $a(E_i, \delta)$ is a $C^2$ function for the domains of interest.

### 3.4.1 Derivation of the averaged equation

As we previously pointed out, for each fixed $\delta > 0$, Anosov’s theorem 2.1.1 and Theorem 2.2.5 apply directly to the family of ordinary differential equations in Equation (3.11), provided that $\delta$ is sufficiently small. The invariant fibers $\mathcal{M}_h$ of the $\varepsilon = 0$ flow are tori described by a fixed value of the four slow variables and \[ \{(Q, W, q_v, v_1, q_2, v_2) : E_1 = m_1 v_1^2 / 2 + U_1(q_1, Q, \delta), E_2 = m_2 v_2^2 / 2 + U_2(q_2, Q, \delta)\}. \]

If we use $(q_1, q_2)$ as local coordinates on $\mathcal{M}_h$, which is valid except when $v_1$ or $v_2 = 0$, the invariant measure $\mu_h$ of the unperturbed flow has the density \[ dq_1 dq_2 \quad \frac{1}{T_1 \sqrt{\frac{2}{m_1} (E_1 - U_1(q_1))} \cdot T_2 \sqrt{\frac{2}{m_2} (E_2 - U_2(q_2))}}. \]

The restricted flow is ergodic for almost every $h$. See Corollary 3.5.1 in Section 3.5.

Now \[ \frac{dh_i^\delta}{dt} = \varepsilon \begin{bmatrix} W \\ -\kappa_i'(Q - q_1) + \kappa_i'(q_2 - Q) \\ W \kappa_i'(Q - q_1) \\ -W \kappa_i'(q_2 - Q) \end{bmatrix}, \]
and
\[
\int_{\mathcal{M}_h} \kappa'_\delta(Q - q_1) d\mu_h = \frac{2}{T_1} \int_{\mathcal{Q}}^{Q - \delta} dq_1 \kappa'_\delta(Q - q_1) \sqrt{\frac{2}{m_1} (E_1 - U_1(q_1))}
\]
\[
= \sqrt{\frac{2m_1}{T_1}} \int_{Q - \delta}^{Q - a} dq_1 \kappa'_\delta(Q - q_1) \sqrt{E_1 - \kappa_\delta(Q - q_1)}
\]
\[
= -\sqrt{\frac{2m_1}{T_1}} \int_{E_1}^{E_1} du \sqrt{E_1 - u}
\]
\[
= -\sqrt{8m_1E_1} T_1.
\]

Similarly,
\[
\int_{\mathcal{M}_h} \kappa'_\delta(q_2 - Q) d\mu_h = -\frac{\sqrt{8m_2E_2}}{T_2}.
\]

It follows that the averaged vector field is
\[
\bar{H}^\delta(h) = \begin{bmatrix}
W \\
-\sqrt{8m_1E_1} T_1 \\
-\sqrt{8m_2E_2} T_2 \\
+\sqrt{8m_1E_1} T_1 \\
\end{bmatrix},
\]

where from Lemma 3.4.1 we see that $\bar{H}(\cdot) \in \mathcal{C}^1((\delta, h) : 0 \leq \delta < \mathcal{E}/2, h \in \mathcal{V})$. $\bar{H}^0(h)$ agrees with the averaged vector field for the hard core system from Equation (3.1), once we account for the change of coordinates $E_i = m_is_i^2/2$.

**Remark 3.4.1.** An argument due to Neishtadt and Sinai [NS04] shows that the solutions to the averaged equation (3.4) are periodic. This argument also shows that, as in the case $\delta = 0$, the limiting dynamics of $(Q, W)$ are effectively Hamiltonian, with the shape of the Hamiltonian depending on $\delta$, $Q(0)$, and the initial energies of the gas particles. The argument depends heavily on the observation that the phase integrals
\[
I_i(Q, E_i, \delta) = \int_{u = \frac{1}{2}m_1v^2 + U_i(x, Q, \delta) \leq E_i} dxdv
\]
are adiabatic invariants, i.e. they are integrals of the solutions to the averaged equation. Thus the four-dimensional phase space of the averaged equation is foliated by invariant two-dimensional submanifolds, and one can think of the effective Hamiltonians for the piston as living on these submanifolds.
3.4.2 Proof of Theorem 3.1.2

The following arguments are motivated by our proof in Section 3.3, although the details are more involved as we show that the rate of convergence is independent of all small $\delta$.

A choice of coordinates on phase space

We wish to describe the dynamics in a coordinate system inspired by the one used in Section 3.3.1. For each fixed $\delta \in (0, \delta_0]$, this change of coordinates will be $C^1$ in all variables on the domain of interest. However, it is an exercise in analysis to show this, and so we delay the proofs of the following two lemmas until Section 3.5.

We introduce the angular coordinates $\varphi_i \in [0, 1]$, $i = 1, 2$, defined by

$$\varphi_1 = \varphi_1(q_1, v_1, Q) = \begin{cases} 
0 & \text{if } q_1 = a \\
\frac{1}{T_1} \int_a^q \sqrt{\frac{m_1/2}{E_1 - U_1(s)}} \, ds & \text{if } v_1 > 0 \\
\frac{1}{2} & \text{if } q_1 = Q - a \\
1 - \frac{1}{T_1} \int_a^q \sqrt{\frac{m_1/2}{E_1 - U_1(s)}} \, ds & \text{if } v_1 < 0
\end{cases} \quad (3.12)$$

$$\varphi_2 = \varphi_2(q_2, v_2, Q) = \begin{cases} 
0 & \text{if } q_2 = 1 - a \\
\frac{1}{T_2} \int_{q_2}^{1-a} \sqrt{\frac{m_2/2}{E_2 - U_2(s)}} \, ds & \text{if } v_2 < 0 \\
\frac{1}{2} & \text{if } q_2 = Q + a \\
1 - \frac{1}{T_2} \int_{q_2}^{1-a} \sqrt{\frac{m_2/2}{E_2 - U_2(s)}} \, ds & \text{if } v_2 > 0
\end{cases}$$

Then $z = (h, \varphi_1, \varphi_2)$ is a choice of coordinates on $h^{-1}\mathcal{U}$. As before, we will abuse notation and let $h(z)$ denote the projection onto the first four coordinates of $z$.

There is a fixed value of $\delta_0$ in the statement of Theorem 3.1.2. However, for the purposes of our proof, it will be convenient to progressively choose $\delta_0$ smaller when needed. At the end of the proof, we will have only shrunk $\delta_0$ a finite number of times, and this final value will satisfy the requirements of the theorem. Our first requirement on $\delta_0$ is that it is smaller than $\mathcal{E}/2$.

Lemma 3.4.2. If $\delta_0 > 0$ is sufficiently small, then for each $\delta \in (0, \delta_0]$ the ordinary differential equation (3.11) in the coordinates $z$ takes the form

$$\frac{dz}{dt} = Z^\delta(z, \varepsilon),$$

(3.13)

where $Z^\delta \in C^1(h^{-1}\mathcal{U} \times [0, \infty))$. When $z \in h^{-1}\mathcal{U}$,
We start by proving the following lemma, which essentially says that an orbitArgумент for uniform convergenceLemma 3.4.3. If \( \delta > 0 \) is chosen sufficiently small, there exists a constant \( K \) such that for all \( \delta \in (0, \delta_0] \), \( \kappa'(|Q - x_i(z)|) = 0 \) unless \( \varphi_i \in [1/2 - K\delta, 1/2 + K\delta] \).

**Argument for uniform convergence**

We start by proving the following lemma, which essentially says that an orbit \( z^\delta(t) \) only spends a fraction \( O(\delta) \) of its time in a region of phase space where \( |H^\delta(z^\delta(t), \varepsilon)| = |H^\delta(z^\delta(t), 0)| \) is of size \( O(\delta^{-1}) \).

**Lemma 3.4.4.** For \( 0 \leq T' \leq T \leq \frac{T + T^\delta}{\varepsilon} \),

\[
\int_{T'}^{T} |H^\delta(z^\delta(s), 0)| \, ds = O(1 \vee (T - T')).
\]

**Proof.** Without loss of generality, \( T' = 0 \). From Lemmas 3.4.1 and 3.4.2 it follows that if we choose \( \delta_0 \) sufficiently small, then there exists \( \omega > 0 \) such that for all sufficiently small \( \varepsilon \) and all \( \delta \in (0, \delta_0] \), \( h \in V \Rightarrow 1/\omega < \frac{d\varphi^\delta_1}{dt} < \omega \). Define the set \( B = [1/2 - K\delta, 1/2 + K\delta] \), where \( K \) comes from Lemma 3.4.3. Then we find a crude bound on \( \int_0^T |\kappa'_\delta(Q^\delta_2(s) - q_1(z^\delta(s)))| \, ds \) using that

\[
\frac{d\varphi^\delta_1}{dt} \text{ is } \begin{cases} 
\geq 1/\omega & \text{if } \varphi^\delta_1 \in B \\
\leq \omega & \text{if } \varphi^\delta_1 \in B^c.
\end{cases}
\]
This yields
\[
\int_0^T \left| \kappa_\delta'(Q_\varepsilon^\delta(s) - q_1(z_\varepsilon^\delta(s))) \right| \, ds \leq \frac{\text{const}}{\delta} \int_0^T \left| \nu_{1,\varepsilon}^\delta(s) \right| \, ds \\
\leq \frac{\text{const}}{\delta} \left( \frac{2K\omega\delta}{2K\omega\delta + \frac{1-2K\delta}{\omega}} T + 2K\omega\delta \right) \\
= \mathcal{O}(1 \vee T).
\]

Similarly, \( \int_0^T \left| \kappa_\delta'(q_2(z_\varepsilon^\delta(s)) - Q_\varepsilon^\delta(s)) \right| \, ds = \mathcal{O}(1 \vee T) \), and so \( \int_0^T \left| H_\varepsilon^\delta(z_\varepsilon^\delta(s), 0) \right| \, ds = \mathcal{O}(1 \vee T) \).

We now follow steps one through four from Section 3.3.1 making modifications where necessary.

**Step 1: Reduction using Gronwall's Inequality.** Now \( h_\varepsilon^\delta(\tau/\varepsilon) \) satisfies
\[
h_\varepsilon^\delta(\tau/\varepsilon) - h_\varepsilon^\delta(0) = \varepsilon \int_0^{\tau/\varepsilon} H_\varepsilon^\delta(z_\varepsilon^\delta(s), 0) \, ds.
\]
Define
\[
e_\varepsilon^\delta(\tau) = \varepsilon \int_0^{\tau/\varepsilon} \left( H_\varepsilon^\delta(z_\varepsilon^\delta(s), 0) - \bar{H}^\delta(h_\varepsilon^\delta) \right) \, ds.
\]
It follows from Gronwall's Inequality and the fact that \( \bar{H}^\delta(\cdot) \in \mathcal{C}^1(\{(\delta, h) : 0 \leq \delta \leq \delta_0, h \in \mathcal{V}\}) \) that
\[
\sup_{0 \leq \tau \leq T \vee T_\varepsilon^\delta} \left| h_\varepsilon^\delta(\tau/\varepsilon) - \bar{H}^\delta(\tau) \right| \leq \left( \sup_{0 \leq \tau \leq T \vee T_\varepsilon^\delta} \left| e_\varepsilon^\delta(\tau) \right| \right) e^{\text{Lip}(\bar{H}^\delta)T} \\
= \mathcal{O} \left( \sup_{0 \leq \tau \leq T \vee T_\varepsilon^\delta} \left| e_\varepsilon^\delta(\tau) \right| \right).
\]

**Step 2: A splitting according to particles.** Next,
\[
H_\varepsilon^\delta(z, 0) - \bar{H}^\delta(h) = \\
\begin{bmatrix}
0 \\
\kappa_\delta'(Q - q_1(z)) - \frac{\sqrt{8m_1E_1}}{T_1} \\
W\kappa_\delta'(Q - q_1(z)) + W\frac{\sqrt{8m_1E_1}}{T_1} \\
0
\end{bmatrix} + \\
\begin{bmatrix}
0 \\
\kappa_\delta'(q_2(z) - Q) + \frac{\sqrt{8m_2E_2}}{T_2} \\
-W\kappa_\delta'(q_2(z) - Q) - W\frac{\sqrt{8m_2E_2}}{T_2}
\end{bmatrix},
\]

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and so, in order to show that \( \sup_{0 \leq \tau \leq T \wedge T^\delta_{\epsilon}} |c^\delta_{\epsilon}(\tau)| = \mathcal{O}(\epsilon) \), it suffices to show that for \( i = 1, 2 \),

\[
\sup_{0 \leq \tau \leq T \wedge T^\delta_{\epsilon}} \int_0^{\tau/\epsilon} \kappa'_{\delta}\left(|Q^\delta_{\epsilon}(s) - x_i(z^\delta_{\epsilon}(s))|\right) + \frac{\sqrt{8m_1E^\delta_{1,\epsilon}(s)}}{T_1(Q^\delta_{\epsilon}(s), E^\delta_{1,\epsilon}(s), \delta)} \, ds = \mathcal{O}(1),
\]

\[
\sup_{0 \leq \tau \leq T \wedge T^\delta_{\epsilon}} \int_0^{\tau/\epsilon} W_\epsilon(s)\kappa'_{\delta}\left(|Q^\delta_{\epsilon}(s) - x_i(z^\delta_{\epsilon}(s))|\right) + W_\epsilon(s) \frac{\sqrt{8m_1E^\delta_{1,\epsilon}(s)}}{T_1(Q^\delta_{\epsilon}(s), E^\delta_{1,\epsilon}(s), \delta)} \, ds = \mathcal{O}(1).
\]

We only demonstrate that

\[
\sup_{0 \leq \tau \leq T \wedge T^\delta_{\epsilon}} \int_0^{\tau/\epsilon} \kappa'_{\delta}\left(Q^\delta_{\epsilon}(s) - q_1(z^\delta_{\epsilon}(s))\right) + \frac{\sqrt{8m_1E^\delta_{1,\epsilon}(s)}}{T_1(Q^\delta_{\epsilon}(s), E^\delta_{1,\epsilon}(s), \delta)} \, ds = \mathcal{O}(1).
\]

The other three terms are handled similarly.

**Step 3: A sequence of times adapted for ergodization.** Define the sequence of times \( t^\delta_{k,\epsilon} \) inductively by \( t^\delta_{0,\epsilon} = \inf\{t \geq 0 : \varphi^\delta_{1,\epsilon}(t) = 0\} \), \( t^\delta_{k+1,\epsilon} = \inf\{t > t^\delta_{k,\epsilon} : \varphi^\delta_{1,\epsilon}(t) = 0\} \). If \( \epsilon \) and \( \delta \) are sufficiently small and \( t^\delta_{k+1,\epsilon} \leq (T \wedge T^\delta_{\epsilon})/\epsilon \), then it follows from Lemma 3.4.2 and the discussion in the proof of Lemma 3.4.3 that \( 1/\omega < t^\delta_{k+1,\epsilon} - t^\delta_{k,\epsilon} < \omega \). From Lemmas 3.4.2 and 3.4.3 it follows that

\[
\sup_{0 \leq \tau \leq T \wedge T^\delta_{\epsilon}} \int_0^{\tau/\epsilon} \kappa'_{\delta}\left(Q^\delta_{\epsilon}(s) - q_1(z^\delta_{\epsilon}(s))\right) + \frac{\sqrt{8m_1E^\delta_{1,\epsilon}(s)}}{T_1(Q^\delta_{\epsilon}(s), E^\delta_{1,\epsilon}(s), \delta)} \, ds \leq \mathcal{O}(1) + \sum_{t^\delta_{k+1,\epsilon} - T^{\delta_{\epsilon}}/\epsilon}^{T \wedge T^\delta_{\epsilon}} \int_{t^\delta_{k,\epsilon}}^{t^\delta_{k+1,\epsilon}} \kappa'_{\delta}\left(Q^\delta_{\epsilon}(s) - q_1(z^\delta_{\epsilon}(s))\right) + \frac{\sqrt{8m_1E^\delta_{1,\epsilon}(s)}}{T_1(Q^\delta_{\epsilon}(s), E^\delta_{1,\epsilon}(s), \delta)} \, ds \, ds.
\]

(3.17)

**Step 4: Control of individual terms by comparison with solutions along fibers.** As before, it suffices to show that each term in the sum in Equation (3.17) is no larger than \( \mathcal{O}(\epsilon) \). Without loss of generality we will only examine the first term and suppose that \( t^\delta_{0,\epsilon} = 0 \), i.e. that \( \varphi^\delta_{1,\epsilon}(0) = 0 \).

**Lemma 3.4.5.** If \( t^\delta_{1,\epsilon} \leq T^{\delta_{\epsilon}}/\epsilon \), then \( \sup_{0 \leq t \leq t^\delta_{1,\epsilon}} |z^\delta_{\epsilon}(t) - z^\delta_{\epsilon}(t)| = \mathcal{O}(\epsilon) \).
Proof. By Lemma 3.4.4
\[ h_0^\delta(t) - h_{\epsilon}^\delta(t) = h_0^\delta(0) - h_{\epsilon}^\delta(0) - \varepsilon \int_0^t H^\delta(z_{\epsilon}^\delta(s), 0)ds = O(\varepsilon(1 + t)) \] for \( t \geq 0 \).

Using what we know about the divergence of the slow variables, we find that
\[
\varphi_1^\delta(0) - \varphi_1^{\delta, \epsilon}(0) = \int_0^t \frac{1}{T_1(Q_0^\delta(s), E_0^\delta(s), \delta)} - \frac{1}{T_1(Q_{\epsilon}^\delta(s), E_{\epsilon}^\delta(s), \delta)} + O(\varepsilon)ds
\]
for \( 0 \leq t \leq t_{1, \epsilon}^\delta \). Lemmas 3.4.1 and 3.4.2 ensure the desired uniformity in the sizes of the orders of magnitudes. Showing that \( \sup_{0 \leq t \leq t_{1, \epsilon}^\delta} \abs{\varphi_2^\delta(0) - \varphi_2^{\delta, \epsilon}(0)} = O(\varepsilon) \) is similar.

From Lemma 3.4.5 we find that \( t_{1, \epsilon} = t_{1, 0} + O(\varepsilon) = T_1(Q_0^\delta, E_0^\delta, \delta) + O(\varepsilon) \). Hence
\[
\int_0^{t_{1, \epsilon}} \frac{\sqrt{8m_1 E_{1, \epsilon}^\delta(s)}}{T_1(Q_{\epsilon}^\delta(s), E_{1, \epsilon}^\delta(s), \delta)} ds = O(\varepsilon) + \int_0^{t_{1, 0}} \frac{\sqrt{8m_1 E_{1, 0}^\delta(s)}}{T_1(Q_{\epsilon}^\delta(s), E_{1, 0}^\delta(s), \delta)} ds
\]
\[
= O(\varepsilon) + \sqrt{8m_1 E_{1, 0}^\delta}
\]
But when \( q_1(z_{\epsilon}^\delta) < Q_\epsilon^\delta - a \),
\[
\frac{d}{ds} \sqrt{E_{1, \epsilon}^\delta(s) - \kappa_\delta(Q_\epsilon^\delta(s) - q_1(z_{\epsilon}^\delta(s)))} = \frac{\text{sign}(v_1(z_{\epsilon}^\delta(s))))\kappa'_\delta(Q_\epsilon^\delta(s) - q_1(z_{\epsilon}^\delta(s)))}{\sqrt{2m_1}}
\]
and so
\[
\int_0^{t_{1, \epsilon}} \kappa'_\delta(Q_\epsilon^\delta(s) - q_1(z_{\epsilon}^\delta(s))) ds = - \sqrt{2m_1 E_{1, \epsilon}^\delta(0)} - \sqrt{2m_1 E_{1, \epsilon}^\delta(t_{1, \epsilon}^\delta)}
\]
\[
= O(\varepsilon) - \sqrt{8m_1 E_{1, 0}^\delta}
\]
Hence,
\[
\int_0^{t_{1, \epsilon}} \kappa'_\delta(Q_\epsilon^\delta(s) - q_1(z_{\epsilon}^\delta(s))) ds + \frac{\sqrt{8m_1 E_{1, \epsilon}^\delta(s)}}{T_1(Q_\epsilon^\delta(s), E_{1, \epsilon}^\delta(s), \delta)} ds = O(\varepsilon),
\]
as desired.
3.5 Appendix to Section 3.4

Proof of Lemma 3.4.1

Proof. For \(0 < \delta < \mathcal{E}/2\),

\[
T_1 = T_1(Q, E_1, \delta) = 2 \int_a^{Q-a} \sqrt{\frac{m_1/2}{E_1 - U_1(s)}} ds, \\
T_2 = T_2(Q, E_2, \delta) = 2 \int_{Q+a}^{1-a} \sqrt{\frac{m_2/2}{E_2 - U_2(s)}} ds.
\]

We only consider the claims about \(T_1\), and for convenience we take \(m_1 = 2\). Then

\[
T_1(Q, E_1, \delta) = 2 \int_a^{Q-a} ds \sqrt{E_1 - U_1(s)} = 4 \int_a^{Q/2} ds \sqrt{E_1 - \kappa \delta(s)}
\]

\[
= 4 \left( \frac{Q/2 - \delta}{\sqrt{E_1}} + \int_{a}^{\delta} \frac{ds}{\sqrt{E_1 - \kappa \delta(s)}} \right)
\]

\[
= \frac{2Q - 4\delta}{\sqrt{E_1}} + 4\delta \int_{\kappa^{-1}(E_1)}^{1} \frac{ds}{\sqrt{E_1 - \kappa(s)}}.
\]

Define

\[
F(E) := \int_{\kappa^{-1}(E)}^{1} \frac{ds}{\sqrt{E - \kappa(s)}} = \int_{0}^{E} \frac{-(\kappa^{-1})'(u)}{\sqrt{E - u}} du.
\]

Notice that \((\kappa^{-1})'(u)\) diverges as \(u \to 0^{+}\), while \((E - u)^{-1/2}\) diverges as \(u \to E^{-}\), but both functions are still integrable on \([0, E]\). It follows that \(F(E)\) is well defined. Then it suffices to show that \(F : [\mathcal{E}, \kappa(0) - \mathcal{E}] \to \mathbb{R}\) is \(C^1\).

Write

\[
F(E) = \int_{0}^{\mathcal{E}/2} \frac{-(\kappa^{-1})'(u)}{\sqrt{E - u}} du + \int_{\mathcal{E}/2}^{E} \frac{-(\kappa^{-1})'(u)}{\sqrt{E - u}} du
\]

\[
:= F_1(E) + F_2(E).
\]

A standard application of the Dominated Convergence Theorem allows us to differentiate inside the integral and conclude that \(F_1 \in C^\infty([\mathcal{E}, \kappa(0) - \mathcal{E}])\), with

\[
F_1'(E) = \int_{0}^{\mathcal{E}/2} \frac{(\kappa^{-1})'(u)}{2(E - u)^{3/2}} du.
\]

To examine \(F_2\), we make the substitution \(v = E - u\) to find that

\[
F_2(E) = \int_{0}^{E-\mathcal{E}/2} \frac{-(\kappa^{-1})'(E - v)}{\sqrt{v}} dv.
\]
Using the fact that \((\kappa^{-1})' \in C^1([\mathcal{E}/2, \kappa(0)])\) and the Dominated Convergence Theorem, we find that \(F_2\) is differentiable, with
\[
F'_2(E) = -\frac{(\kappa^{-1})'(\mathcal{E}/2)}{\sqrt{E - \mathcal{E}/2}} + \int_0^{E - \mathcal{E}/2} -\frac{(\kappa^{-1})''(E - v)}{\sqrt{v}} \, dv.
\]

Another application of the Dominated Convergence Theorem shows that \(F'_2\) is continuous, and so \(F_2 \in C^1([\mathcal{E}, \kappa(0) - \mathcal{E}])\).

Thus
\[
T_1(Q, E_1, \delta) = \frac{2Q}{\sqrt{E_1}} + 4\delta \left[ -E_1^{-1/2} + F_1(E_1) + F_2(E_1) \right]
\]
has the desired regularity. For future reference, we note that
\[
\frac{\partial T_1}{\partial Q} = \frac{2}{\sqrt{E_1}}, \quad \frac{\partial T_1}{\partial E_1} = -\frac{Q}{E_1^{3/2}} + \mathcal{O}(\delta). \tag{3.18}
\]

**Corollary 3.5.1.** For all \(\delta\) sufficiently small, the flow \(z_0^\delta(t)\) restricted to the invariant tori \(\mathcal{M}_c = \{h = c\}\) is ergodic (with respect to the invariant Lebesgue measure) for almost every \(c \in \mathcal{U}\).

**Proof.** The flow is ergodic whenever the periods \(T_1\) and \(T_2\) are irrationally related. Fix \(\delta\) sufficiently small such that \(\frac{\partial T_1}{\partial E_1} = -\frac{Q}{E_1^{3/2}} + \mathcal{O}(\delta) < 0\). Next, consider \(Q, W,\) and \(E_2\) fixed, so that \(T_2\) is constant. Because \(T_1 \in C^1\), it follows that, as we let \(E_1\) vary, \(\frac{T_1}{T_2} \notin \mathbb{Q}\) for almost every \(E_1\). The result follows from Fubini’s Theorem. \(\square\)

**Proof of Lemma [3.4.2]**

**Proof.** For the duration of this proof, we consider the dynamics for a small, fixed value of \(\delta > 0\), which we generally suppress in our notation. For convenience, we take \(m_1 = 2\).

Let \(\psi\) denote the map taking \((Q, W, q_1, v_1, q_2, v_2)\) to \((Q, W, E_1, E_2, \varphi_1, \varphi_2)\). We claim that \(\psi\) is a \(C^1\) change of coordinates on the domain of interest. Since \(E_1 = v_1^2 + \kappa\delta(q_1) + \kappa\delta(Q - q_1), E_1\) is a \(C^2\) function of \(q_1, v_1,\) and \(Q\). A similar statement holds for \(E_2\).

The angular coordinates \(\varphi_i(x_i, v_i, Q)\) are defined by Equation (3.12). We only consider \(\varphi_1\), as the statements for \(\varphi_2\) are similar. Then \(\varphi_1(q_1, v_1, Q)\) is clearly \(C^1\) whenever \(q_1 \neq a, Q - a\). The apparent difficulties in regularity at the turning points are only a result of how the definition of \(\varphi_1\) is presented in Equation (3.12). Recall that the angle variables are actually defined by integrating the elapsed time along
orbits, and our previous definition expressed $\varphi_1$ in a manner which emphasized the dependence on $q_1$. In fact, whenever $|v_1| < \sqrt{E_1}$,

$$
\varphi_1(q_1, v_1, Q) = \begin{cases} 
-\frac{2}{T_1} \int_0^{v_1} (\kappa_\delta^{-1})' (E_1 - v^2) dv & \text{if } q_1 < \delta \\
\frac{1}{2} + \frac{2}{T_1} \int_0^{v_1} (\kappa_\delta^{-1})' (E_1 - v^2) dv & \text{if } q_1 > Q - \delta.
\end{cases}
$$  \hspace{1cm} (3.19)

Here $E_1$ is implicitly considered to be a function of $q_1, v_1$, and $Q$. One can verify that $D\psi$ is non-degenerate on the domain of interest, and so $\psi$ is indeed a $C^1$ change of coordinates.

Next observe that $d\varphi_1,0/dt = 1/T_1$, so Hadamard’s Lemma implies that

$$
d\varphi_1,\varepsilon/dt = \frac{1}{T_1} + O(\varepsilon f(\delta)).
$$  

It remains to show that, in fact, we may take $f(\delta) = 1$. It is easy to verify this whenever $q_1 \leq Q - \delta$ because $dE_1/dt = 0$ there. We only perform the more difficult verification when $q_1 > Q - \delta$.

When $q_1 > Q - \delta, |v_1| < \sqrt{E_1}$ and $E_1 = v_1^2 + \kappa_\delta (Q - q_1)$. From Equation (3.19) we find that

$$
\varphi_1 = \frac{1}{2} + \frac{2\delta}{T_1(Q, E_1, \delta)} \int_0^{v_1} (\kappa^{-1})' (E_1 - v^2) dv.
$$  \hspace{1cm} (3.20)

To find $d\varphi_1/dt$, we consider $\varphi_1$ as a function of $v_1, Q$, and $E_1$, so that

$$
\frac{d\varphi_1}{dt} = \frac{\partial \varphi_1}{\partial v_1} \frac{dv_1}{dt} + \frac{\partial \varphi_1}{\partial Q} \frac{dQ}{dt} + \frac{\partial \varphi_1}{\partial E_1} \frac{dE_1}{dt}.
$$

Then, using Equations \(3.18\) and \(3.20\), we compute

$$
\frac{\partial \varphi_1}{\partial v_1} \frac{dv_1}{dt} = \frac{2}{T_1} \left(\kappa_\delta^{-1}' (E_1 - v_1^2) \frac{\kappa_\delta' (Q - q_1)}{2} \right) = \frac{1}{T_1},
$$

$$
\frac{\partial \varphi_1}{\partial Q} \frac{dQ}{dt} = \frac{1/2 - \varphi_1}{T_1} \frac{\partial T_1}{\partial Q} (\varepsilon W) = \varepsilon W 1/2 - \varphi_1 \frac{2}{\sqrt{E_1}},
$$

$$
\frac{\partial \varphi_1}{\partial E_1} \frac{dE_1}{dt} = \left( \frac{1/2 - \varphi_1}{T_1} \frac{\partial T_1}{\partial E_1} + \frac{2\delta}{T_1} \int_0^{v_1} (\kappa^{-1})'' (E_1 - v^2) dv \right) (\varepsilon W \kappa_\delta' (Q - q_1)).
$$

Using that $\kappa_\delta' (Q - q_1) = \kappa' (\kappa^{-1} (E_1 - v_1^2)) / \delta = (\delta (\kappa^{-1})' (E_1 - v_1^2))^{-1}$, we find that

$$
\frac{\partial \varphi_1}{\partial E_1} \frac{dE_1}{dt} = \varepsilon O \left( \frac{1/2 - \varphi_1}{\delta} \right) + \varepsilon O \left( \frac{1}{(\kappa^{-1})' (E_1 - v_1^2)} \int_0^{v_1} (\kappa^{-1})'' (E_1 - v^2) dv \right).
$$

But here $1/2 - \varphi_1$ is $O(\delta)$. See the proof of Lemma 3.4.3 below. Thus the claims about $d\varphi_1/dt$ will be proven, provided we can uniformly bound

$$
\frac{1}{(\kappa^{-1})' (E_1 - v_1^2)} \int_0^{v_1} (\kappa^{-1})'' (E_1 - v^2) dv.
$$
Note that the apparent divergence of the integral as \(|v_1| \to \sqrt{E_1}\) is entirely due to the fact that our expression for \(\varphi_1\) from Equation (3.20) requires \(|v_1| < \sqrt{E_1}\). If we make the substitution \(u = E_1 - v^2\) and let \(e = E_1 - v_2^2\), then it suffices to show that

\[
\sup_{\varepsilon \leq E_1 \leq \kappa(0) - \varepsilon} \sup_{0 < e \leq E_1} \left| \frac{1}{(\kappa^{-1})'(e)} \int_e^{E_1} \frac{\kappa^{-1}(u)''(u)}{\sqrt{E_1 - u}} \, du \right| < +\infty.
\]

The only difficulties occur when \(e\) is close to 0. Thus it suffices to show that

\[
\sup_{\varepsilon \leq E_1 \leq \kappa(0) - \varepsilon} \sup_{0 < e \leq E/2} \left| \frac{1}{(\kappa^{-1})'(e)} \int_e^{E/2} \frac{\kappa^{-1}(u)''(u)}{\sqrt{E_1 - u}} \, du \right|
\]

is finite. But this is bounded by

\[
\sup_{0 < e \leq E/2} \left| \frac{1}{(\kappa^{-1})'(e)} \int_e^{E/2} \frac{\kappa^{-1}(u)''(u)}{\sqrt{E/2}} \, du \right| = \sup_{0 < e \leq E/2} \left| \frac{\sqrt{2/\varepsilon}}{(\kappa^{-1})'(e)} \left((\kappa^{-1})'(E/2) - (\kappa^{-1})'(e)\right) \right|
\]

which is finite because \((\kappa^{-1})'(e) \to -\infty\) as \(e \to 0^+\). The claims about \(d\varphi_2/dt\) can be proven similarly.

\[\square\]

**Proof of Lemma 3.4.3.**

Proof. We continue in the notation of the proofs of Lemmas 3.4.1 and 3.4.2 above, and we set \(m_1 = 2\). Then from Equation (3.20), we see that \(\kappa'(Q - q_1) = 0\) unless \(|\varphi_1 - 1/2| \leq \left| \frac{2\delta}{T_1} \int_0^{\sqrt{E_1}} (\kappa^{-1})'(E_1 - v^2) \, dv \right| = \delta F(E_1)/T_1 = O(\delta)\). Dealing with \(\varphi_2\) is similar.

\[\square\]
Chapter 4

The periodic oscillation of an adiabatic piston in two or three dimensions

In this chapter, we present our results for the piston system in two or three dimensions. These results may also be found in [Wri07].

4.1 Statement of the main result

4.1.1 Description of the model

Consider a massive, insulating piston of mass $M$ that separates a gas container $\mathcal{D}$ in $\mathbb{R}^d$, $d = 2$ or 3. See Figure 4.1. Denote the location of the piston by $Q$, its velocity by $dQ/dt = V$, and its cross-sectional length (when $d = 2$, or area, when $d = 3$) by $\ell$. If $Q$ is fixed, then the piston divides $\mathcal{D}$ into two subdomains, $\mathcal{D}_1(Q) = \mathcal{D}_1$ on the left and $\mathcal{D}_2(Q) = \mathcal{D}_2$ on the right. By $E_i$ we denote the total energy of the gas inside $\mathcal{D}_i$, and by $|\mathcal{D}_i|$ we denote the area (when $d = 2$, or volume, when $d = 3$) of $\mathcal{D}_i$.

We are interested in the dynamics of the piston when the system’s total energy is bounded and $M \to \infty$. When $M = \infty$, the piston remains fixed in place, and each energy $E_i$ remains constant. When $M$ is large but finite, $MV^2/2$ is bounded, and so $V = O(M^{-1/2})$. It is natural to define

$$
\varepsilon = M^{-1/2}, \\
W = \frac{V}{\varepsilon},
$$

so that $W$ is of order 1 as $\varepsilon \to 0$. This is equivalent to scaling time by $\varepsilon$. 

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Next we precisely describe the gas container. It is a compact, connected billiard domain $\mathcal{D} \subset \mathbb{R}^d$ with a piecewise $C^3$ boundary, i.e. $\partial \mathcal{D}$ consists of a finite number of $C^3$ embedded hypersurfaces, possibly with boundary and a finite number of corner points. The container consists of a “tube,” whose perpendicular cross-section $\mathcal{P}$ is the shape of the piston, connecting two disjoint regions. $\mathcal{P} \subset \mathbb{R}^{d-1}$ is a compact, connected domain whose boundary is piecewise $C^3$. Then the “tube” is the region $[0, 1] \times \mathcal{P} \subset \mathcal{D}$ swept out by the piston for $0 \leq Q \leq 1$, and $[0, 1] \times \partial \mathcal{P} \subset \partial \mathcal{D}$. If $d = 2$, $\mathcal{P}$ is just a closed line segment, and the “tube” is a rectangle. If $d = 3$, $\mathcal{P}$ could be a circle, a square, a pentagon, etc.

Our fundamental assumption is as follows:

**Main Assumption.** For almost every $Q \in [0, 1]$ the billiard flow of a single particle on an energy surface in either of the two subdomains $\mathcal{D}_i(Q)$ is ergodic (with respect to the invariant Liouville measure).

If $d = 2$, the domain could be the Bunimovich stadium [Bun79]. Another possible domain is indicated in Figure 4.1. The ergodicity of billiards in such domains, which produce hyperbolic flows, goes back to the pioneering work of Sinai [Sin70], although a number of individuals have contributed to the theory. A full accounting of this history can be found in [CM06a]. Polygonal domains satisfying our assumptions can also be constructed [Vor97]. Suitable domains in $d = 3$ dimensions can be constructed using a rectangular box with shallow spherical caps adjoined [BR98]. Note that we make no assumptions regarding the hyperbolicity of the billiard flow in the domain.

The Hamiltonian system we consider consists of the massive piston of mass $M$ located at position $Q$, as well as $n_1 + n_2$ gas particles, $n_1$ in $\mathcal{D}_1$ and $n_2$ in $\mathcal{D}_2$. Here $n_1$
and $n_2$ are fixed positive integers. For convenience, the gas particles all have unit mass, though all that is important is that each gas particle has a fixed mass. We denote the positions of the gas particles in $D_i$ by $q_{i,j}$, $1 \leq j \leq n_i$. The gas particles are ideal point particles that interact with $\partial D$ and the piston by hard core, elastic collisions. Although it has no effect on the dynamics we consider, for convenience we complete our description of the Hamiltonian dynamics by specifying that the piston makes elastic collisions with walls located at $Q = 0$, 1 that are only visible to the piston. We denote velocities by $dQ/dt = V = \varepsilon W$ and $dq_{i,j}/dt = v_{i,j}$, and we set

$$E_{i,j} = v_{i,j}^2/2, \quad E_i = \sum_{j=1}^{n_i} E_{i,j}.$$  

Our system has $d(n_1 + n_2) + 1$ degrees of freedom, and so its phase space is $(2d(n_1 + n_2) + 2)$-dimensional.

We let $h(z) = h = (Q, W, E_{1,1}, E_{1,2}, \cdots, E_{1,n_1}, E_{2,1}, E_{2,2}, \cdots, E_{2,n_2})$, so that $h$ is a function from our phase space to $\mathbb{R}^{n_1+n_2+2}$. We often abbreviate $h = (Q, W, E_{1,j}, E_{2,j})$, and we refer to $h$ as consisting of the slow variables because these quantities are conserved when $\varepsilon = 0$. We let $h_\varepsilon(t, z) = h_\varepsilon(t)$ denote the actual motions of these variables in time for a fixed value of $\varepsilon$. Here $z$ represents the initial condition in phase space, which we usually suppress in our notation. One should think of $h_\varepsilon(\cdot)$ as being a random variable that takes initial conditions in phase space to paths (depending on the parameter $t$) in $\mathbb{R}^{n_1+n_2+2}$.

### 4.1.2 The averaged equation

From the work of Neishtadt and Sinai [NS04], one can derive

$$\frac{d}{d\tau} \begin{bmatrix} Q \\ W \\ E_{1,j} \\ E_{2,j} \end{bmatrix} = \bar{H}(h) := \begin{bmatrix} W \\ \frac{2E_1 \ell}{d|D_1(Q)|} - \frac{2E_2 \ell}{d|D_2(Q)|} \\ -\frac{2WE_{1,j} \ell}{2WE_1 \ell} \\ + \frac{2WE_{2,j} \ell}{2WE_2 \ell} \end{bmatrix}$$  

(4.1)

as the averaged equation (with respect to the slow time $\tau = \varepsilon t$) for the slow variables. Later, in Section 4.2.3, we will give another heuristic derivation of the averaged equation that is more suggestive of our proof.

Neishtadt and Sinai [Sin99, NS04] pointed out that the solutions of Equation (1.3) have $(Q, W)$ behaving as if they were the coordinates of a Hamiltonian
system describing a particle undergoing motion inside a potential well. As in Section 1.2, the effective Hamiltonian is given by
\[
\frac{1}{2} W^2 + \frac{E_1(0) \left| D_1(Q(0)) \right|^{2/d}}{|D_1(Q)|^{2/d}} + \frac{E_2(0) \left| D_2(Q(0)) \right|^{2/d}}{|D_2(Q)|^{2/d}}.
\]
This can be seen as follows. Since
\[
\frac{\partial \left| D_1(Q) \right|}{\partial Q} = \ell = - \frac{\partial \left| D_2(Q) \right|}{\partial Q},
\]
\[
d \ln(E_{i,j})/d\tau = -(2/d)d \ln(|D_i(Q)|)/d\tau,
\]
and so
\[
E_{i,j}(\tau) = E_{i,j}(0) \left( \frac{|D_i(Q(0))|}{|D_i(Q(\tau))|} \right)^{2/d}.
\]
By summing over \( j \), we find that
\[
E_i(\tau) = E_i(0) \left( \frac{|D_i(Q(0))|}{|D_i(Q(\tau))|} \right)^{2/d}
\]
and so
\[
\frac{d^2 Q(\tau)}{d\tau^2} = \frac{2\ell E_1(0) \left| D_1(Q(0)) \right|^{2/d}}{d} \left| D_1(Q(\tau)) \right|^{1+2/d} - \frac{2\ell E_2(0) \left| D_2(Q(0)) \right|^{2/d}}{d} \left| D_2(Q(\tau)) \right|^{1+2/d}.
\]

Let \( \bar{h}(\tau, z) = \bar{h}(\tau) \) be the solution of
\[
\frac{d\bar{h}}{d\tau} = H(\bar{h}), \quad \bar{h}(0) = h_\varepsilon(0).
\]
Again, think of \( \bar{h}(\cdot) \) as being a random variable.

4.1.3 The main result

The solutions of the averaged equation approximate the motions of the slow variables, \( h_\varepsilon(t) \), on a time scale \( O(1/\varepsilon) \) as \( \varepsilon \to 0 \). Precisely, fix a compact set \( \mathcal{V} \subset \mathbb{R}^{n_1+n_2+2} \) such that \( h \in \mathcal{V} \Rightarrow Q \subset (0,1), W \subset \mathbb{R} \), and \( E_{i,j} \subset (0,\infty) \) for each \( i \) and \( j \). We will be mostly concerned with the dynamics when \( h \in \mathcal{V} \). Define
\[
Q_{\min} = \inf_{h \in \mathcal{V}} Q, \quad Q_{\max} = \sup_{h \in \mathcal{V}} Q,
\]
\[
E_{\min} = \inf_{h \in \mathcal{V}} \frac{1}{2} W^2 + E_1 + E_2, \quad E_{\max} = \sup_{h \in \mathcal{V}} \frac{1}{2} W^2 + E_1 + E_2.
\]

\[1\] We have introduced this notation for convenience. For example, \( h \in \mathcal{V} \Rightarrow Q \subset (0,1) \) means that there exists a compact set \( A \subset (0,1) \) such that \( h \in \mathcal{V} \Rightarrow Q \in A \), and similarly for the other variables.
For a fixed value of $\varepsilon > 0$, we only consider the dynamics on the invariant subset of phase space defined by
\[ \mathcal{M}_\varepsilon = \{(Q, V, q_{i,j}, v_{i,j}) \in \mathbb{R}^{2(d(n_1+n_2)+2)} : Q \in [0,1], \ q_{i,j} \in D_i(Q), \ E_{\text{min}} \leq \frac{M}{2} V^2 + E_1 + E_2 \leq E_{\text{max}} \}. \]

Let $P_\varepsilon$ denote the probability measure obtained by restricting the invariant Liouville measure to $\mathcal{M}_\varepsilon$. Define the stopping time
\[ T_\varepsilon(z) = T_\varepsilon = \inf\{\tau \geq 0 : \tilde{h}(\tau) \notin \mathcal{V} \text{ or } h_\varepsilon(\tau/\varepsilon) \notin \mathcal{V} \}. \]

**Theorem 4.1.1.** If $\mathcal{D}$ is a gas container in $d = 2$ or $3$ dimensions satisfying the assumptions in Subsection 4.1.1 above, then for each $T > 0$,
\[ \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| h_\varepsilon(\tau/\varepsilon) - \tilde{h}(\tau) \right| \to 0 \text{ in probability as } \varepsilon = M^{-1/2} \to 0, \]
i.e. for each fixed $\delta > 0$,
\[ P_\varepsilon \left( \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} \left| h_\varepsilon(\tau/\varepsilon) - \tilde{h}(\tau) \right| \geq \delta \right) \to 0 \text{ as } \varepsilon = M^{-1/2} \to 0. \]

**Remark 4.1.1.** It should be noted that the stopping time in the above result is not unduly restrictive. If the initial pressures of the two gasses are not too mismatched, then the solution to the averaged equation is a periodic orbit, with the effective potential well keeping the piston away from the walls. Thus, if the actual motions follow the averaged solution closely for $0 \leq \tau \leq T \wedge T_\varepsilon$, and the averaged solution stays in $\mathcal{V}$, it follows that $T_\varepsilon > T$.

**Remark 4.1.2.** The techniques of this work should immediately generalize to prove the analogue of Theorem 4.1.1 above in the nonphysical dimensions $d > 3$, although we do not pursue this here.

**Remark 4.1.3.** As in Subsection 3.1.3, Theorem 4.1.1 can be easily generalized to cover a system of $N - 1$ pistons that divide $N$ gas containers, so long as, for almost every fixed location of the pistons, the billiard flow of a single gas particle on an energy surface in any of the $N$ subcontainers is ergodic (with respect to the invariant Liouville measure). The effective Hamiltonian for the pistons has them moving like an $(N-1)$-dimensional particle inside a potential well.

### 4.2 Preparatory material concerning a two-dimensional gas container with only one gas particle on each side

Our results and techniques of proof are essentially independent of the dimension and the fixed number of gas particles on either side of the piston. Thus, we focus
on the case when $d = 2$ and there is only one gas particle on either side. Later, in Section 4.4, we will indicate the simple modifications that generalize our proof to the general situation. For clarity, in this section and next, we denote $q_{1,1}$ by $q_1$, $v_{2,1}$ by $v_2$, etc. We decompose the gas particle coordinates according to whether they are perpendicular to or parallel to the piston’s face, for example $q_1 = (q_1^\perp, q_1^\parallel)$.

See Figure 4.2.

The Hamiltonian dynamics define a flow on our phase space. We denote this flow by $z\varepsilon(t, z) = z\varepsilon(t)$, where $z = z\varepsilon(0, z)$. One should think of $z\varepsilon(\cdot)$ as being a random variable that takes initial conditions in phase space to paths in phase space. Then $h\varepsilon(t) = h(z\varepsilon(t))$. By the change of coordinates $W = V/\varepsilon$, we may identify all of the $M\varepsilon$ defined in Section 4.4 with the space

$$M = \{(Q, W, q_1, v_1, q_2, v_2) \in \mathbb{R}^{10} : Q \in [0, 1], q_1 \in D_1(Q), q_2 \in D_2(Q), E_{\min} \leq \frac{1}{2} W^2 + E_1 + E_2 \leq E_{\max}\}.$$ 

and all of the $P\varepsilon$ with the probability measure $P$ on $M$, which has the density

$$dP = \text{const} dQ dW dq_1^\perp dq_1^\parallel dv_1^\perp dv_1^\parallel dq_2^\perp dq_2^\parallel dv_2^\perp dv_2^\parallel.$$ 

(Throughout this work we will use const to represent generic constants that are independent of $\varepsilon$.) We will assume that these identifications have been made, so that we may consider $z\varepsilon(\cdot)$ as a family of measure preserving flows on the same space that all preserve the same probability measure. We denote the components of $z\varepsilon(t)$ by $Q\varepsilon(t), q_1^\perp(t), t$, etc.

The set $\{z \in M : q_1 = Q = q_2\}$ has co-dimension two, and so $\bigcup_t z\varepsilon(t)\{q_1 = Q = q_2\}$ has co-dimension one, which shows that only a measure zero set of initial conditions will give rise to three particle collisions. We ignore this and other measures zero events, such as gas particles hitting singularities of the billiard flow, in what follows.

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Now we present some background material, as well as some lemmas that will assist us in our proof of Theorem 4.1.1. We begin by studying the billiard flow of a gas particle when the piston is infinitely massive. Next we examine collisions between the gas particles and the piston when the piston has a large, but finite, mass. Then we present a heuristic derivation of the averaged equation that is suggestive of our proof. Finally we prove a lemma that allows us to disregard the possibility that a gas particle will move nearly parallel to the piston’s face – a situation that is clearly bad for having the motions of the piston follow the solutions of the averaged equation.

4.2.1 Billiard flows and maps in two dimensions

In this section, we study the billiard flows of the gas particles when $M = \infty$ and the slow variables are held fixed at a specific value $h \in V$. We will only study the motions of the left gas particle, as similar definitions and results hold for the motions of the right gas particle. Thus we wish to study the billiard flow of a point particle moving inside the domain $D_1$ at a constant speed $\sqrt{2E_1}$. The results of this section that are stated without proof can be found in [CM06a].

Let $T D_1$ denote the tangent bundle to $D_1$. The billiard flow takes place in the three-dimensional space $M^1_1 = M^1 = \{(q_1, v_1) \in T D_1 : q_1 \in D_1, |v_1| = \sqrt{2E_1}\}/\sim$. Here the quotient means that when $q_1 \in \partial D_1$, we identify velocity vectors pointing outside of $D_1$ with those pointing inside $D_1$ by reflecting through the tangent line to $\partial D_1$ at $q_1$, so that the angle of incidence with the unit normal vector to $\partial D_1$ equals the angle of reflection. Note that most of the quantities defined in this subsection depend on the fixed value of $h$. We will usually suppress this dependence, although, when necessary, we will indicate it by a subscript $h$.

We denote the resulting flow by $y(t, y) = y(t)$, where $y(0, y) = y$. As the billiard flow comes from a Hamiltonian system, it preserves Liouville measure restricted to the energy surface. We denote the resulting probability measure by $\mu$. This measure has the density $d\mu = dq_1 dv_1/(2\pi \sqrt{2E_1} |D_1|)$. Here $dq_1$ represents area on $\mathbb{R}^2$, and $dv_1$ represents length on $S^1_1 = \{v_1 \in \mathbb{R}^2 : |v_1| = \sqrt{2E_1}\}$.

There is a standard cross-section to the billiard flow, the collision cross-section $\Omega = \{(q_1, v_1) \in T D_1 : q_1 \in \partial D_1, |v_1| = \sqrt{2E_1}\}/\sim$. It is customary to parameterize $\Omega$ by $\{x = (r, \varphi) : r \in \partial D_1, \varphi \in [-\pi/2, +\pi/2]\}$, where $r$ is arc length and $\varphi$ represents the angle between the outgoing velocity vector and the inward pointing normal vector to $\partial D_1$. It follows that $\Omega$ may be realized as the disjoint union of a finite number of rectangles and cylinders. The cylinders correspond to fixed scatterers with smooth boundary placed inside the gas container. If $F : \Omega \circledast$ is the collision map, i.e. the return map to the collision cross-section, then $F$ preserves the projected probability measure $\nu$, which has the density $d\nu = \cos \varphi d\varphi dr/(2 |\partial D_1|)$. Here $|\partial D_1|$ is the length of $\partial D_1$. 

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We suppose that the flow is ergodic, and so $F$ is an invertible, ergodic measure preserving transformation. Because $\partial D_1$ is piecewise $C^3$, $F$ is piecewise $C^2$, although it does have discontinuities and unbounded derivatives near discontinuities corresponding to grazing collisions. Because of our assumptions on $D_1$, the free flight times and the curvature of $\partial D_1$ are uniformly bounded. It follows that if $x \notin \partial \Omega \cup F^{-1}(\partial \Omega)$, then $F$ is differentiable at $x$, and

$$\|DF(x)\| \leq \frac{\text{const} \cos \varphi(Fx)}{\cos \varphi(Fx)}, \quad (4.2)$$

where $\varphi(Fx)$ is the value of the $\varphi$ coordinate at the image of $x$.

Following the ideas in Section 4.5, we induce $F$ on the subspace $\hat{\Omega}$ of $\Omega$ corresponding to collisions with the (immobile) piston. We denote the induced map by $\hat{F}$ and the induced measure by $\hat{\nu}$. We parameterize $\hat{\Omega}$ by $\{(r,\varphi) : 0 \leq r \leq \ell, \varphi \in [-\pi/2, +\pi/2]\}$. As $\nu \hat{\Omega} = \ell/|\partial D_1|$, it follows that $\hat{\nu}$ has the density $d\hat{\nu} = \cos \varphi d\varphi dr/(2\ell)$.

For $x \in \Omega$, define $\zeta_x$ to be the free flight time, i.e. the time it takes the billiard particle traveling at speed $\sqrt{2E_1}$ to travel from $x$ to $Fx$. If $x \notin \partial \Omega \cup F^{-1}(\partial \Omega)$,

$$\|D\zeta(x)\| \leq \frac{\text{const} \cos \varphi(Fx)}{\cos \varphi(Fx)}, \quad (4.3)$$

Santaló’s formula \cite{San76, Che97} tells us that

$$E_{\nu}\zeta = \frac{\pi |D_1|}{|v_1| |\partial D_1|}. \quad (4.4)$$

If $\hat{\zeta} : \hat{\Omega} \to \mathbb{R}$ is the free flight time between collisions with the piston, then it follows from Proposition 4.5.1 that

$$E_{\hat{\nu}}\hat{\zeta} = \frac{\pi |D_1|}{|v_1| \ell}. \quad (4.5)$$

The expected value of $|v_1^+|$ when the left gas particle collides with the (immobile) piston is given by

$$E_{\hat{\nu}}|v_1^+| = E_{\hat{\nu}} \sqrt{2E_1} \cos \varphi = \frac{\sqrt{2E_1}}{2} \int_{-\pi/2}^{+\pi/2} \cos^2 \varphi d\varphi = \sqrt{2E_1} \frac{\pi}{4}. \quad (4.6)$$

We wish to compute $\lim_{t \to \infty} t^{-1} \int_0^t |v_1^+(s)| \delta_{q_1}(s)=Q ds$, the time average of the change in momentum of the left gas particle when it collides with the piston. If this limit exists and is equal for almost every initial condition of the left gas particle, then it makes sense to define the pressure inside $D_1$ to be this quantity divided by $\ell$. Because the collisions are hard-core, we cannot directly apply Birkhoff’s Ergodic Theorem to compute this limit. However, we can compute this limit by using the map $\hat{F}$. 

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Lemma 4.2.1. If the billiard flow $y(t)$ is ergodic, then for $\mu - a.e. \, y \in \mathcal{M}^1$,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |v_1^+(s)| \delta_{q_1^+(s)=Q} ds = \frac{E_1 \ell}{2|\mathcal{D}_1(Q)|}.
\]

Proof. Because the billiard flow may be viewed as a suspension flow over the collision cross-section with $\zeta$ as the height function, it suffices to show that the convergence takes place for $\hat{\nu} - a.e. \, x \in \hat{\Omega}$. For an initial condition $x \in \hat{\Omega}$, define
\[
\hat{N}_t = \# \left\{ s \in (0, t] : y(s, x) \in \Omega \right\}.
\]
By the Poincaré Recurrence Theorem, $\hat{N}_t \to \infty$ as $t \to \infty$, $\hat{\nu} - a.e.$

But
\[
\frac{\hat{N}_t}{\sum_{n=0}^{\hat{N}_t} \hat{\zeta}(\hat{F}^n x)} \frac{1}{N_t} \sum_{n=1}^{\hat{N}_t} |v_1^+| (\hat{F}^n x) \leq \frac{1}{t} \int_0^t |v_1^+(s)| \delta_{q_1^+(s)=Q} ds \leq \frac{\hat{N}_t}{\sum_{n=0}^{\hat{N}_t-1} \hat{\zeta}(\hat{F}^n x)} \frac{1}{N_t} \sum_{n=0}^{\hat{N}_t} |v_1^+| (\hat{F}^n x),
\]
and so the result follows from Birkhoff’s Ergodic Theorem and Equations (4.5) and (4.6).

Corollary 4.2.2. If the billiard flow $y(t)$ is ergodic, then for each $\delta > 0$,
\[
\mu \left\{ y \in \mathcal{M}^1 : \left| \frac{1}{t} \int_0^t |v_1^+(s)| \delta_{q_1^+(s)=Q} ds - \frac{E_1 \ell}{2|\mathcal{D}_1(Q)|} \right| \geq \delta \right\} \to 0 \text{ as } t \to \infty.
\]

4.2.2 Analysis of collisions

In this section, we return to studying our piston system when $\varepsilon > 0$. We will examine what happens when a particle collides with the piston. For convenience, we will only examine in detail collisions between the piston and the left gas particle. Collisions with the right gas particle can be handled similarly.

When the left gas particle collides with the piston, $v_1^+$ and $V$ instantaneously change according to the laws of elastic collisions:
\[
\begin{bmatrix} v_1^{++} \\ V^+ \end{bmatrix} = \frac{1}{1 + M} \begin{bmatrix} 1 - M & 2M \\ 2 & M - 1 \end{bmatrix} \begin{bmatrix} v_1^{+-} \\ V^- \end{bmatrix}.
\]

In our coordinates, this becomes
\[
\begin{bmatrix} v_1^{++} \\ W^+ \end{bmatrix} = \frac{1}{1 + \varepsilon^2} \begin{bmatrix} \varepsilon^2 - 1 & 2\varepsilon \\ 2\varepsilon & 1 - \varepsilon^2 \end{bmatrix} \begin{bmatrix} v_1^{+-} \\ W^- \end{bmatrix}.
\]
Recalling that \(v_1, W = \mathcal{O}(1)\), we find that to first order in \(\varepsilon\),
\[
v_1^+ = -v_1^- + \mathcal{O}(\varepsilon), \quad W^+ = W^- + \mathcal{O}(\varepsilon).
\] (4.8)

Observe that a collision can only take place if \(v_1^+ > \varepsilon W^-\). In particular, \(v_1^+ > -\varepsilon \sqrt{2E_{\text{max}}}\). Thus, either \(v_1^+ > 0\) or \(v_1^+ = \mathcal{O}(\varepsilon)\). By expanding Equation (4.7) to second order in \(\varepsilon\), it follows that
\[
E_1^+ - E_1^- = -2\varepsilon W |v_1^+| + \mathcal{O}(\varepsilon^2),
\]
\[
W^+ - W^- = +2\varepsilon |v_1^+| + \mathcal{O}(\varepsilon^2).
\] (4.9)

Note that it is immaterial whether we use the pre-collision or post-collision values of \(W\) and \(|v_1^+|\) on the right hand side of Equation (4.9), because any ambiguity can be absorbed into the \(\mathcal{O}(\varepsilon^2)\) term.

It is convenient for us to define a “clean collision” between the piston and the left gas particle:

**Definition 4.2.1.** The left gas particle experiences a **clean collision** with the piston if and only if \(v_1^+ > 0\) and \(v_1^- < -\varepsilon \sqrt{2E_{\text{max}}}\).

In particular, after a clean collision, the left gas particle will escape from the piston, i.e. the left gas particle will have to move into the region \(q_1^+ \leq 0\) before it can experience another collision with the piston. It follows that there exists a constant \(C_1 > 0\), which depends on the set \(\mathcal{V}\), such that for all \(\varepsilon\) sufficiently small, so long as \(Q \geq Q_{\text{min}}\) and \(|v_1^+| > \varepsilon C_1\) when \(q_1^+ \in [Q_{\text{min}}, Q]\), then the left gas particle will experience only clean collisions with the piston, and the time between these collisions will be greater than \(2Q_{\text{min}}/(\sqrt{2E_{\text{max}}})\). (Note that when we write expressions such as \(q_1^+ \in [Q_{\text{min}}, Q]\), we implicitly mean that \(q_1\) is positioned inside the “tube” discussed at the beginning of Section 4.1.) One can verify that \(C_1 = 5\sqrt{2E_{\text{max}}}\) would work.

Similarly, we can define clean collisions between the right gas particle and the piston. We assume that \(C_1\) was chosen sufficiently large such that for all \(\varepsilon\) sufficiently small, so long as \(Q \leq Q_{\text{max}}\) and \(|v_2^+| > \varepsilon C_1\) when \(q_2^+ \in [Q, Q_{\text{max}}]\), then the right gas particle will experience only clean collisions with the piston.

Now we define three more stopping times, which are functions of the initial conditions in phase space.

\[
T_\epsilon' = \inf\{\tau \geq 0 : Q_{\text{min}} \leq q_{1,\epsilon}(\tau/\varepsilon) \leq Q_{\text{max}} \text{ and } |v_{1,\epsilon}(\tau/\varepsilon)| \leq C_1 \varepsilon\},
\]
\[
T_\epsilon'' = \inf\{\tau \geq 0 : Q_{\epsilon}(\tau/\varepsilon) \leq q_{2,\epsilon}(\tau/\varepsilon) \leq Q_{\text{max}} \text{ and } |v_{2,\epsilon}(\tau/\varepsilon)| \leq C_1 \varepsilon\},
\]
\[
\hat{T}_\epsilon = T \land T_\epsilon' \land T_\epsilon''
\]
Define $H(z)$ by

$$H(z) = \begin{bmatrix}
W + 2 |v_1^\perp| \delta_{q^1_1 = 0} - 2 |v_2^\perp| \delta_{q^2_1 = 0} \\
-2W |v_1^\perp| \delta_{q^1_2 = 0} \\
+2W |v_2^\perp| \delta_{q^2_2 = 0}
\end{bmatrix}.$$ 

Here we make use of Dirac delta functions. All integrals involvng these delta functions may be replaced by sums.

The following lemma is an immediate consequence of Equation (4.9) and the above discussion:

**Lemma 4.2.3.** If $0 \leq t_1 \leq t_2 \leq \tilde{T}_\varepsilon/\varepsilon$, the piston experiences $O((t_2 - t_1) \vee 1)$ collisions with gas particles in the time interval $[t_1, t_2]$, all of which are clean collisions. Furthermore,

$$h_\varepsilon(t_2) - h_\varepsilon(t_1) = \mathcal{O}(\varepsilon) + \varepsilon \int_{t_1}^{t_2} H(z_\varepsilon(s)) ds.$$

Here any ambiguities arising from collisions occurring at the limits of integration can be absorbed into the $\mathcal{O}(\varepsilon)$ term.

### 4.2.3 Another heuristic derivation of the averaged equation

The following heuristic derivation of Equation (4.1) when $d = 2$ was suggested in [Dol05]. Let $\Delta t$ be a length of time long enough such that the piston experiences many collisions with the gas particles, but short enough such that the slow variables change very little, in this time interval. From each collision with the left gas particle, Equation (4.9) states that $W$ changes by an amount $+2\varepsilon |v_1^\perp| + \mathcal{O}(\varepsilon^2)$, and from Equation (4.6) the average change in $W$ at these collisions should be approximately $\varepsilon \pi \sqrt{2E_1/2 + \mathcal{O}(\varepsilon^2)}$. From Equation (4.5) the frequency of these collisions is approximately $\sqrt{2E_1 \ell/(\pi |D_1|)}$. Arguing similarly for collisions with the other particle, we guess that

$$\frac{\Delta W}{\Delta t} = \varepsilon \frac{E_1 \ell}{|D_1(Q)|} - \varepsilon \frac{E_2 \ell}{|D_2(Q)|} + \mathcal{O}(\varepsilon^2).$$

With $\tau = \varepsilon t$ as the slow time, a reasonable guess for the averaged equation for $W$ is

$$\frac{dW}{d\tau} = \frac{E_1 \ell}{|D_1(Q)|} - \frac{E_2 \ell}{|D_2(Q)|}.$$
Similar arguments for the other slow variables lead to the averaged equation (4.1), and this explains why we used \( P_i = E_i/|D_i| \) for the pressure of a 2-dimensional gas in Section 1.2.

There is a similar heuristic derivation of the averaged equation in \( d > 2 \) dimensions. Compare the analogues of Equations (4.5) and (4.6) in Subsection 4.4.2.

4.2.4 A priori estimate on the size of a set of bad initial conditions

In this section, we give an a priori estimate on the size of a set of initial conditions that should not give rise to orbits for which \( \sup_{0 \leq \tau \leq T} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \) is small. In particular, when proving Theorem 4.1.1, it is convenient to focus on orbits that only contain clean collisions with the piston. Thus, we show that \( P\{\tilde{T}_\varepsilon < T \wedge T_{\varepsilon}\} \) vanishes as \( \varepsilon \to 0 \). At first, this result may seem surprising, since \( P\{T'_{\varepsilon} \wedge T''_{\varepsilon} = 0\} = O(\varepsilon) \), and one would expect \( \bigcup_{t=0}^{T/\varepsilon} z_\varepsilon(-t)\{T'_{\varepsilon} \wedge T''_{\varepsilon} = 0\} \) to have a size of order 1. However, the rate at which orbits escape from \( \{T'_{\varepsilon} \wedge T''_{\varepsilon} = 0\} \) is very small, and so we can prove the following:

**Lemma 4.2.4.**

\[
P\{\tilde{T}_\varepsilon < T \wedge T_{\varepsilon}\} = O(\varepsilon).
\]

In some sense, this lemma states that the probability of having a gas particle move nearly parallel to the piston’s face within the time interval \([0, T/\varepsilon]\), when one would expect the other gas particle to force the piston to move on a macroscopic scale, vanishes as \( \varepsilon \to 0 \). Thus, one can hope to control the occurrence of the “nondiffusive fluctuations” of the piston described in [CD06a] on a time scale \( O(\varepsilon^{-1}) \).

**Proof.** As the left and the right gas particles can be handled similarly, it suffices to show that \( P\{T'_{\varepsilon} < T\} = O(\varepsilon) \). Define

\[
\mathcal{B}_\varepsilon = \{ z \in \mathcal{M} : Q_{\min} \leq q^1 \leq Q \leq Q_{\max} \text{ and } |v^1_1| \leq C_1 \varepsilon \}.
\]

Then \( \{T'_\varepsilon < T\} \subset \bigcup_{t=0}^{T/\varepsilon} z_\varepsilon(-t)\mathcal{B}_\varepsilon \), and if \( \gamma = Q_{\min}/\sqrt{8E_{\max}} \),

\[
P\left( \bigcup_{t=0}^{T/\varepsilon} z_\varepsilon(-t)\mathcal{B}_\varepsilon \right) = P\left( \bigcup_{t=0}^{T/\varepsilon} z_\varepsilon(t)\mathcal{B}_\varepsilon \right) = P\left( \mathcal{B}_\varepsilon \cup \bigcup_{t=0}^{T/\varepsilon} (z_\varepsilon(t)\mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \right)
\]

\[
\leq P\mathcal{B}_\varepsilon + P\left( \bigcup_{k=0}^{T/(\varepsilon\gamma)} z_\varepsilon(k\gamma) \bigcup_{t=0}^{\gamma} (z_\varepsilon(t)\mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \right)
\]

\[
\leq P\mathcal{B}_\varepsilon + \left( \frac{T}{\varepsilon\gamma} + 1 \right) P\left( \bigcup_{t=0}^{\gamma} (z_\varepsilon(t)\mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \right).
\]
Now \( P\mathcal{B}_\varepsilon = O(\varepsilon) \), so if we can show that \( P(\bigcup_{t=0}^{\tau} (z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon) = O(\varepsilon^2) \), then it will follow that \( P\{T_\varepsilon^* < T\} = O(\varepsilon) \).

If \( z \in \bigcup_{t=0}^{\tau} (z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \), it is still true that \( |v^\perp_t| = O(\varepsilon) \). This is because \( |v^\perp_t| \) changes by at most \( O(\varepsilon) \) at the collisions, and if a collision forces \( |v^\perp_t| > C_1\varepsilon \), then the gas particle must escape to the region \( q^\perp_t \leq 0 \) before \( v^\perp_1 \) can change again, and this will take time greater than \( \gamma \). Furthermore, if \( z \in \bigcup_{t=0}^{\tau} (z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \), then at least one of the following four possibilities must hold:

- \( |q^\perp_t - Q_{\text{min}}| \leq O(\varepsilon) \),
- \( |Q - Q_{\text{min}}| \leq O(\varepsilon) \),
- \( |Q - Q_{\text{max}}| \leq O(\varepsilon) \),
- \( |Q - q^\perp_t| \leq O(\varepsilon) \).

It follows that \( P(\bigcup_{t=0}^{\tau} (z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon) = O(\varepsilon^2) \). For example,

\[
\int_M 1_{\{|v^\perp_1| \leq O(\varepsilon), |q^\perp_t - Q_{\text{min}}| \leq O(\varepsilon)\}} dP = \text{const} \int \left\{ E_{\min} \leq W^2/2 + v_1^2/2 + v_2^2/2 \leq E_{\max} \right\} 1_{\{|v^\perp_1| \leq O(\varepsilon)\}} dW d\tau^1 dv_1^\parallel dv_2^\parallel \times \int \{Q\in[0,1], q_1\in\mathcal{D}_1, q_2\in\mathcal{D}_2\} 1_{\{|q^\perp_t - Q_{\text{min}}| \leq O(\varepsilon)\}} dQ dq_1^\parallel dq_2 \equiv O(\varepsilon^2).
\]

\( \square \)

4.3 Proof of the main result for two-dimensional gas containers with only one gas particle on each side

As in Section 4.2, we continue with the case when \( d = 2 \) and there is only one gas particle on either side of the piston.

4.3.1 Main steps in the proof of convergence in probability

By Lemma 4.2.4, it suffices to show that \( \sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \tilde{h}(\tau)| \to 0 \) in probability as \( \varepsilon = M^{-1/2} \to 0 \). Several of the ideas in the steps below were inspired by a recent proof of Anosov’s averaging theorem for smooth systems that is due to Dolgopyat [Dol05].
Step 1: Reduction using Gronwall’s Inequality. Observe that $\bar{h}(\tau)$ satisfies the integral equation

$$\bar{h}(\tau) - \bar{h}(0) = \int_0^\tau \bar{H}(\bar{h}(\sigma))d\sigma,$$

while from Lemma 4.2.3,

$$h_\varepsilon(\tau/\varepsilon) - h_\varepsilon(0) = \mathcal{O}(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} \bar{H}(z_\varepsilon(s))ds$$

$$= \mathcal{O}(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds + \int_0^\tau \bar{H}(h_\varepsilon(\sigma/\varepsilon))d\sigma$$

for $0 \leq \tau \leq \tilde{T}_\varepsilon$. Define

$$e_\varepsilon(\tau) = \varepsilon \int_0^{\tau/\varepsilon} H(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds.$$

It follows from Gronwall’s Inequality that

$$\sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \leq \left( \mathcal{O}(\varepsilon) + \sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |e_\varepsilon(\tau)| \right) e^{\text{Lip}(\bar{H}|V)T}. \quad (4.10)$$

Gronwall’s Inequality is usually stated for continuous paths, but the standard proof (found in [SV85]) still works for paths that are merely integrable, and $|h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)|$ is piecewise smooth.

Step 2: Introduction of a time scale for ergodization. Let $L(\varepsilon)$ be a real valued function such that $L(\varepsilon) \to \infty$, but $L(\varepsilon) \ll \log \varepsilon^{-1}$, as $\varepsilon \to 0$. In Section 4.3.2 we will place precise restrictions on the growth rate of $L(\varepsilon)$. Think of $L(\varepsilon)$ as being a time scale that grows as $\varepsilon \to 0$ so that ergodization, i.e. the convergence along an orbit of a function’s time average to a space average, can take place. However, $L(\varepsilon)$ doesn’t grow too fast, so that on this time scale $z_\varepsilon(t)$ essentially stays on the submanifold $\{ h = h_\varepsilon(0) \}$, where we have our ergodicity assumption. Set $t_{k,\varepsilon} = kL(\varepsilon)$, so that

$$\sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |e_\varepsilon(\tau)| \leq \mathcal{O}(\varepsilon L(\varepsilon)) + \varepsilon \sum_{k=0}^{\tilde{T}_\varepsilon/L(\varepsilon) - 1} \left| \int_{t_k,\varepsilon}^{t_{k+1,\varepsilon}} H(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds \right|. \quad (4.11)$$

Step 3: A splitting according to particles. Now $H(z) - \bar{H}(h(z))$ divides into two pieces, each of which depends on only one gas particle when the piston
is held fixed:

\[
H(z) - \bar{H}(h(z)) = \begin{bmatrix}
2 |v_1^+| \delta_{q_1^+}=Q - \frac{E_1\ell}{|D_1(Q)|} & 0 & 0 \\
-2W |v_1^+| \delta_{q_1^+}=Q + \frac{\bar{E}_2\ell}{|D_2(Q)|} & 0 & 0 \\
0 & -\frac{W\bar{E}_2\ell}{|D_2(Q)|} + 2W |v_2^+| \delta_{q_2^+}=Q & 0
\end{bmatrix}.
\]

We will only deal with the piece depending on the left gas particle, as the right particle can be handled similarly. Define

\[
G(z) = |v_1^+| \delta_{q_1^+}=Q, \quad \bar{G}(h) = \frac{E_1\ell}{2|D_1(Q)|}.
\]

Returning to Equation (4.11), we see that in order to prove Theorem 4.1.1, it suffices to show that both

\[
\varepsilon \sum_{k=0}^{\mathcal{E}(\varepsilon)} \left| \int_{t_{k+1,\varepsilon}}^{t_{k,\varepsilon}} G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s))ds \right|
\]

and

\[
\varepsilon \sum_{k=0}^{\mathcal{E}(\varepsilon)} \left| \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} W_{\varepsilon}(s)(G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)))ds \right|
\]

converge to 0 in probability as \( \varepsilon \to 0 \).

**Step 4: A splitting for using the triangle inequality.** Now we let \( z_{k,\varepsilon}(s) \) be the orbit of the \( \varepsilon = 0 \) Hamiltonian vector field satisfying \( z_{k,\varepsilon}(t_{k,\varepsilon}) = z_{\varepsilon}(t_{k,\varepsilon}) \). Set \( h_{k,\varepsilon}(t) = h(z_{k,\varepsilon}(t)) \). Observe that \( h_{k,\varepsilon}(t) \) is independent of \( t \).

We emphasize that so long as \( 0 \leq t \leq \bar{T}_{\varepsilon}/\varepsilon \), the times between collisions of a specific gas particle and piston are uniformly bounded greater than 0, as explained before Lemma 4.2.3. It follows that, so long as \( t_{k+1,\varepsilon} \leq \bar{T}_{\varepsilon}/\varepsilon \),

\[
\sup_{t_{k,\varepsilon} \leq t \leq t_{k+1,\varepsilon}} |h_{k,\varepsilon}(t) - h_{\varepsilon}(t)| = \mathcal{O}(\varepsilon L(\varepsilon)).
\]

This is because the slow variables change by at most \( \mathcal{O}(\varepsilon) \) at collisions, and \( dQ_{\varepsilon}/dt = \mathcal{O}(\varepsilon) \).

Also,

\[
\int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} W_{\varepsilon}(s)(G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)))ds
\]

\[
= \mathcal{O}(\varepsilon L(\varepsilon)^2) + W_{k,\varepsilon}(t_{k,\varepsilon}) \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s))ds,
\]
and so
\[
\sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |\int_{t_k}^{t_{k+1}} W_{\varepsilon}(s) G(z_{\varepsilon}(s)) ds| \leq O(\varepsilon L(z)) + \varepsilon \text{ const } \sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |\int_{t_k}^{t_{k+1}} G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)) ds|.
\]

Thus, in order to prove Theorem 4.1.1, it suffices to show that
\[
\sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |\int_{t_k}^{t_{k+1}} G(z_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)) ds| \leq \varepsilon \sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |I_{k,\varepsilon}| + |II_{k,\varepsilon}| + |III_{k,\varepsilon}|
\]
converges to 0 in probability as \( \varepsilon \to 0 \), where
\[
I_{k,\varepsilon} = \int_{t_k}^{t_{k+1}} G(z_{\varepsilon}(s)) - G(z_{k,\varepsilon}(s)) ds,
\]
\[
II_{k,\varepsilon} = \int_{t_k}^{t_{k+1}} G(z_{k,\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)) ds,
\]
\[
III_{k,\varepsilon} = \int_{t_k}^{t_{k+1}} \bar{G}(h_{\varepsilon}(s)) - \bar{G}(h_{\varepsilon}(s)) ds.
\]

The term \( II_{k,\varepsilon} \) represents an “ergodicity term” that can be controlled by our assumptions on the ergodicity of the flow \( z_0(t) \), while the terms \( I_{k,\varepsilon} \) and \( III_{k,\varepsilon} \) represent “continuity terms” that can be controlled by controlling the drift of \( z_\varepsilon(t) \) from \( z_{k,\varepsilon}(t) \) for \( t_k \leq t \leq t_{k+1} \).

**Step 5: Control of drift from the \( \varepsilon = 0 \) orbits.** Now \( \bar{G} \) is uniformly Lipschitz on the compact set \( \mathcal{Y} \), and so it follows from Equation (4.13) that
\[
III_{k,\varepsilon} = O(\varepsilon L(z)^2).
\]

Thus, \( \varepsilon \sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |III_{k,\varepsilon}| = O(\varepsilon L(z)) \to 0 \) as \( \varepsilon \to 0 \).

Next, we show that for fixed \( \delta > 0 \), \( P\left(\varepsilon \sum_{k=0}^{\frac{T}{\varepsilon L(z)}} |I_{k,\varepsilon}| \geq \delta \right) \to 0 \) as \( \varepsilon \to 0 \).

For initial conditions \( z \in \mathcal{M} \) and for integers \( k \in [0, T/(\varepsilon L(z)) - 1] \) define
\[
\mathcal{A}_{k,\varepsilon} = \left\{ z : \frac{1}{L(z)} |I_{k,\varepsilon}| > \frac{\delta}{2T} \text{ and } k \leq \frac{T}{\varepsilon L(z)} - 1 \right\},
\]
\[
\mathcal{A}_{z,\varepsilon} = \{ k : z \in \mathcal{A}_{k,\varepsilon} \}.
\]

Think of these sets as describing “poor continuity” between solutions of the \( \varepsilon = 0 \) and the \( \varepsilon > 0 \) Hamiltonian vector fields. For example, roughly speaking, \( z \in \mathcal{A}_{k,\varepsilon} \) if the orbit \( z_\varepsilon(t) \) starting at \( z \) does not closely follow \( z_{k,\varepsilon}(t) \) for \( t_k \leq t \leq t_{k+1} \).
One can easily check that $|I_{k, \varepsilon}| \leq O(L(\varepsilon))$ for $k \leq \tilde{T}_\varepsilon/(\varepsilon L(\varepsilon)) - 1$, and so it follows that

$$\frac{\varepsilon}{L(\varepsilon)} \sum_{k=0}^{\tilde{T}_\varepsilon - 1} |I_{k, \varepsilon}| \leq \frac{\delta}{2} + O(\varepsilon L(\varepsilon) \#(A_{z, \varepsilon})).$$

Therefore it suffices to show that $P(\#(A_{z, \varepsilon}) \geq \delta \text{const} \ v L(\varepsilon)) \to 0$ as $\varepsilon \to 0$. By Chebyshev’s Inequality, we need only show that

$$E_P(\varepsilon L(\varepsilon) \#(A_{z, \varepsilon})) = \varepsilon L(\varepsilon) \sum_{0 \leq k \leq \tilde{T}_\varepsilon - 1} P(A_{k, \varepsilon}) \leq \text{const} \ P(A_{0, \varepsilon}).$$

tends to 0 with $\varepsilon$.

Observe that $z_\varepsilon(t_{k, \varepsilon}) A_{k, \varepsilon} \subset A_{0, \varepsilon}$. In words, the initial conditions giving rise to orbits that are “bad” on the time interval $[t_{k, \varepsilon}, t_{k+1, \varepsilon}]$, moved forward by time $t_{k, \varepsilon}$, are initial conditions giving rise to orbits which are “bad” on the time interval $[t_{0, \varepsilon}, t_{1, \varepsilon}]$. Because the flow $z_\varepsilon(\cdot)$ preserves the measure, we find that

$$\varepsilon L(\varepsilon) \sum_{0 \leq k \leq \tilde{T}_\varepsilon - 1} P(A_{k, \varepsilon}) \leq \text{const} \ P(A_{0, \varepsilon}).$$

To estimate $P(A_{0, \varepsilon})$, it is convenient to use a different probability measure, which is uniformly equivalent to $P$ on the set $\{z \in \mathcal{M} : h(z) \in \mathcal{V} \supset \{T_\varepsilon \geq \varepsilon L(\varepsilon)\}$. We denote this new probability measure by $P^f$, where the $f$ stands for “factor.” If we choose coordinates on $\mathcal{M}$ by using $h$ and the billiard coordinates on the two gas particles, then $P^f$ is defined on $\mathcal{M}$ by $dP^f = dh \, d\mu_h^1 \, d\mu_h^2$, where $dh$ represents the uniform measure on $\mathcal{V} \subset \mathbb{R}^4$, and the factor measure $d\mu_h^i$ represents the invariant billiard measure of the $i^{th}$ gas particle coordinates for a fixed value of the slow variables. One can verify that $1_{ \{h(z) \in \mathcal{V}\}} \, dP \leq \text{const} \, dP^f$, but that $P^f$ is not invariant under the flow $z_\varepsilon(\cdot)$ when $\varepsilon > 0$.

We abuse notation, and consider $\mu_h^1$ to be a measure on the left particle’s initial billiard coordinates once $h$ and the initial coordinates of the right gas particle are fixed. In this context, $\mu_h^1$ is simply the measure $\mu$ from Subsection 4.2.1. Then

$$P^f(A_{0, \varepsilon}) \leq \int dh \, d\mu_h^2 \cdot \mu_h^1 \left\{ \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) \, ds \geq \frac{\delta}{2T} \text{ and } \varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon \right\},$$

and we must show that the last term tends to 0 with $\varepsilon$. By the Bounded Convergence Theorem, it suffices to show that for almost every $h \in \mathcal{V}$ and initial
condition for the right gas particle,
\[
\mu^1_h \left\{ z : \left| \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) \, ds \right| \geq \frac{\delta}{2T} \text{ and } \varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon \right\} \to 0 \text{ as } \varepsilon \to 0. \tag{4.14}
\]

Note that if \( G \) were a smooth function and \( z_\varepsilon(\cdot) \) were the flow of a smooth family of vector fields \( Z(z,\varepsilon) \) that depended smoothly on \( \varepsilon \), then from Gronwall’s Inequality, it would follow that \( \sup_{0 \leq t \leq L(\varepsilon)} |z_\varepsilon(t) - z_0(t)| \leq \mathcal{O}(\varepsilon L(\varepsilon) e^{\text{Lip}(Z)L(\varepsilon)}) \). If this were the case, then
\[
\left| L(\varepsilon)^{-1} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) \, ds \right| = \mathcal{O}(\varepsilon L(\varepsilon) e^{\text{Lip}(Z)L(\varepsilon)}),
\]
which would tend to 0 with \( \varepsilon \). Thus, we need a Gronwall-type inequality for billiard flows. We obtain the appropriate estimates in Section 4.3.2.

Step 6: Use of ergodicity along fibers to control \( II_{k,\varepsilon} \). All that remains to be shown is that for fixed \( \delta > 0 \), \( P \left( \varepsilon \sum_{k=0}^{t_k(\varepsilon)-1} |II_{k,\varepsilon}| \geq \delta \right) \to 0 \text{ as } \varepsilon \to 0. \)

For initial conditions \( z \in \mathcal{M} \) and for integers \( k \in [0, T/(\varepsilon L(\varepsilon)) - 1] \) define
\[
B_{k,\varepsilon} = \left\{ z : \frac{1}{L(\varepsilon)} |II_{k,\varepsilon}| > \frac{\delta}{2T} \text{ and } k \leq \frac{\tilde{T}_\varepsilon}{\varepsilon L(\varepsilon)} - 1 \right\},
\]
\[
B_{z,\varepsilon} = \{ k : z \in B_{k,\varepsilon} \}.
\]

Think of these sets as describing “bad ergodization.” For example, roughly speaking, \( z \in B_{k,\varepsilon} \) if the orbit \( z_\varepsilon(t) \) starting at \( z \) spends the time between \( t_k(\varepsilon) \) and \( t_{k+1,\varepsilon} \) in a region of phase space where the function \( G(\cdot) \) is “poorly ergodized” on the time scale \( L(\varepsilon) \) by the flow \( z_0(t) \) (as measured by the parameter \( \delta/2T \)). Note that \( G(z) = |v_1^z| \delta_{q_1^z=Q} \) is not really a function, but that we may still speak of the convergence of \( t^{-1} \int_0^t G(z_0(s)) \, ds \) as \( t \to \infty \). As we showed in Lemma 4.2.1, the limit is \( \bar{G}(h_0) \) for almost every initial condition.

Proceeding as in Step 5 above, we find that it suffices to show that for almost every \( h \in \mathcal{V} \),
\[
\mu^1_h \left\{ z : \left| \frac{1}{t} \int_0^t G(z_0(s)) \, ds - \bar{G}(h_0(0)) \right| \geq \frac{\delta}{2T} \right\} \to 0 \text{ as } t \to \infty.
\]
But this is simply a question of examining billiard flows, and it follows immediately from Corollary 4.2.2 and our Main Assumption.

4.3.2 A Gronwall-type inequality for billiards

We begin by presenting a general version of Gronwall’s Inequality for billiard maps. Then we will show how these results imply the convergence required in Equation (4.14).
Some inequalities for the collision map

In this section, we consider the value of the slow variables to be fixed at $h_0 \in V$. We will use the notation and results presented in Section 4.2.1, but because the value of the slow variables is fixed, we will omit it in our notation.

Let $\rho$, $\gamma$, and $\lambda$ satisfy $0 < \rho \ll \gamma \ll 1 \ll \lambda < \infty$. Eventually, these quantities will be chosen to depend explicitly on $\varepsilon$, but for now they are fixed.

Recall that the phase space $\Omega$ for the collision map $F$ is a finite union of disjoint rectangles and cylinders. Let $d(\cdot, \cdot)$ be the Euclidean metric on connected components of $\Omega$. If $x$ and $x'$ belong to different components, then we set $d(x, x') = \infty$. The invariant measure $\mu$ satisfies $\mu < \text{const} \cdot (\text{Lebesgue measure})$. For $A \subset \Omega$ and $a > 0$, let $N_a(A) = \{x \in \Omega : d(x, A) < a\}$ be the $a$-neighborhood of $A$.

For $x \in \Omega$ let $x_k(x) = x_k = F^k x$, $k \geq 0$, be its forward orbit. Suppose $x \notin C_{\gamma, \lambda}$, where

$$C_{\gamma, \lambda} = (\bigcup_{k=0}^{\lambda} F^{-k} N_\gamma(\partial \Omega)) \bigcup \left( \bigcup_{k=0}^{\lambda} F^{-k} N_\gamma(F^{-1} N_\gamma(\partial \Omega)) \right).$$

Thus for $0 \leq k \leq \lambda$, $x_k$ is well defined, and from Equation (4.2) it satisfies

$$d(x', x_k) \leq \gamma \Rightarrow d(Fx', x_{k+1}) \leq \frac{\text{const}}{\gamma} d(x', x_k).$$

(4.15)

Next, we consider any $\rho$-pseudo-orbit $x'_k$ obtained from $x$ by adding on an error of size $\leq \rho$ at each application of the map, i.e. $d(x'_0, x_0) \leq \rho$, and for $k \geq 1$, $d(x'_k, Fx'_{k-1}) \leq \rho$. Provided $d(x_j, x'_j) < \gamma$ for each $j < k$, it follows that

$$d(x_k, x'_k) \leq \rho \sum_{j=0}^{k} \left( \frac{\text{const}}{\gamma} \right)^j \leq \text{const} \rho \left( \frac{\text{const}}{\gamma} \right)^k.$$  \hspace{1cm} (4.16)

In particular, if $\rho$, $\gamma$, and $\lambda$ were chosen such that

$$\text{const} \rho \left( \frac{\text{const}}{\gamma} \right)^\lambda \leq \gamma,$$

(4.17)

then Equation (4.16) will hold for each $k \leq \lambda$. We assume that Equation (4.17) is true. Then we can also control the differences in elapsed flight times using Equation (4.3):

$$|\zeta x_k - \zeta x'_k| \leq \frac{\text{const} \rho}{\gamma} \left( \frac{\text{const}}{\gamma} \right)^k.$$  \hspace{1cm} (4.18)

It remains to estimate the size $\nu C_{\gamma, \lambda}$ of the set of $x$ for which the above estimates do not hold. Using Lemma 4.3.1 below,

$$\nu C_{\gamma, \lambda} \leq (\lambda + 1) \left( \nu N_\gamma(\partial \Omega) + \nu N_\gamma(F^{-1} N_\gamma(\partial \Omega)) \right) \leq O(\lambda(\gamma + \gamma^{1/3})) = O(\lambda \gamma^{1/3}).$$

(4.19)
Lemma 4.3.1. As \( \gamma \to 0 \),

\[
\nu \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial\Omega)) = \mathcal{O}(\gamma^{1/3}).
\]

This estimate is not necessarily the best possible. For example, for dispersing billiard tables, where the curvature of the boundary is positive, one can show that 
\( \nu \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial\Omega)) = \mathcal{O}(\gamma) \). However, the estimate in Lemma 4.3.1 is general and sufficient for our needs.

Proof. First, we note that it is equivalent to estimate \( \nu \mathcal{N}_\gamma(F\mathcal{N}_\gamma(\partial\Omega)) \), as \( F \) has the measure-preserving involution \( \mathcal{I}(r, \varphi) = (r, -\varphi) \), i.e. \( F^{-1} = \mathcal{I} \circ F \circ \mathcal{I} \) [CM06b].

Fix \( \alpha \in (0, 1/2) \), and cover \( \mathcal{N}_\gamma(\partial\Omega) \) with \( \mathcal{O}(\gamma^{-1}) \) starlike sets, each of diameter no greater than \( \mathcal{O}(\gamma) \). For example, these sets could be squares of side length \( \gamma \). Enumerate the sets as \( \{A_i\} \). Set \( \mathcal{G} = \{i : FA_i \cap \mathcal{N}_\gamma(\partial\Omega) = \emptyset\} \).

If \( i \in \mathcal{G} \), \( F|_{A_i} \) is a diffeomorphism satisfying \( \|DF\| \leq \mathcal{O}(\gamma^{-\alpha}) \). See Equation (4.2). Thus diameter \( (FA_i) \leq \mathcal{O}(\gamma^{1-\alpha}) \), and so

\[
\text{diameter} \ (\mathcal{N}_\gamma(FA_i)) \leq \mathcal{O}(\gamma^{1-\alpha}).
\]

Hence \( \nu \mathcal{N}_\gamma(FA_i) \leq \mathcal{O}(\gamma^{2(1-\alpha)}) \), and \( \nu \mathcal{N}_\gamma(\cup_{i \in \mathcal{G}} FA_i) \leq \mathcal{O}(\gamma^{-2\alpha}) \).

If \( i \notin \mathcal{G} \), \( A_i \cap F^{-1}(\mathcal{N}_\gamma(\partial\Omega)) \neq \emptyset \). Thus \( A_i \) might be cut into many pieces by \( F^{-1}(\partial\Omega) \), but each of these pieces must be mapped near \( \partial\Omega \). In fact, \( FA_i \subset \mathcal{N}_\mathcal{O}(\gamma^\alpha)(\partial\Omega) \). This is because outside \( F^{-1}(\mathcal{N}_\gamma(\partial\Omega)) \), \( \|DF\| \leq \mathcal{O}(\gamma^{-\alpha}) \), and so points in \( FA_i \) are no more than a distance \( \mathcal{O}(\gamma/\gamma^\alpha) \) away from \( \mathcal{N}_\gamma(\partial\Omega) \), and \( \gamma < \gamma^{1-\alpha} < \gamma^\alpha \). It follows that \( \mathcal{N}_\gamma(FA_i) \subset \mathcal{N}_\mathcal{O}(\gamma^\alpha)(\partial\Omega) \), and \( \nu \mathcal{N}_\mathcal{O}(\gamma^\alpha)(\partial\Omega) = \mathcal{O}(\gamma^\alpha) \).

Thus \( \nu \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial\Omega)) = \mathcal{O}(\gamma^{-2\alpha} + \gamma^\alpha) \), and we obtain the lemma by taking \( \alpha = 1/3 \).

Application to a perturbed billiard flow

Returning to the end of Step 5 in Section 4.3.1, let the initial conditions of the slow variables be fixed at \( h_0 = (Q_0, W_0, E_{1,0}, E_{2,0}) \in \mathcal{V} \) throughout the remainder of this section. We can assume that the billiard dynamics of the left gas particle in \( \mathcal{D}_1(Q_0) \) are ergodic. Also, fix a particular value of the initial conditions for the right gas particle for the remainder of this section. Then \( z_\epsilon(t) \) and \( T_\epsilon \) may be thought of as random variables depending on the left gas particle’s initial conditions \( y \in \mathcal{M}^1 \). Now if \( h_\epsilon(t) = (Q_\epsilon(t), W_\epsilon(t), E_{1,\epsilon}(t), E_{2,\epsilon}(t)) \) denotes the actual motions of the slow variables when \( \epsilon > 0 \), it follows from Equation (4.13) that, provided \( \epsilon L(\epsilon) \leq T_\epsilon \),

\[
\sup_{0 \leq t \leq L(\epsilon)} |h_0 - h_\epsilon(t)| = \mathcal{O}(\epsilon L(\epsilon)). \tag{4.20}
\]
Furthermore, we only need to show that
\[
\mu \left\{ y \in \mathcal{M}^1 : \left| \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) ds \right| \geq \frac{\delta}{2T} \text{ and } \varepsilon L(\varepsilon) \leq \tilde{T} \varepsilon \right\} \to 0
\]
(4.21)
as \varepsilon \to 0, where \( G \) is defined in Equation (4.12).

For definiteness, we take the following quantities from Subsection 4.3.2 to depend on \( \varepsilon \) as follows:
\[
L(\varepsilon) = L = \log \log \frac{1}{\varepsilon}, \quad \gamma(\varepsilon) = \gamma = e^{-L}, \quad \lambda(\varepsilon) = \lambda = \frac{2}{E_\nu \zeta} L, \quad \rho(\varepsilon) = \rho = \text{const} \varepsilon L.
\]
(4.22)
The constant in the choice of \( \rho \) and \( \rho \)'s dependence on \( \varepsilon \) will be explained in the proof of Lemma 4.3.3, which is at the end of this subsection. The other choices may be explained as follows. We wish to use continuity estimates for the billiard map to produce continuity estimates for the flow on the time scale \( L \). As the divergence of orbits should be exponentially fast, we choose \( L \) to grow sublogarithmically in \( \varepsilon^{-1} \). Since from Equation (4.13) the expected flight time between collisions with \( \partial D_1(Q_0) \) when \( \varepsilon = 0 \) is \( E_\nu \zeta = \pi |D_1(Q_0)| / (\sqrt{2E_{1,0}} |\partial D_1(Q_0)|) \), we expect to see roughly \( \lambda/2 \) collisions on this time scale. Considering \( \lambda \) collisions gives us some margin for error. Furthermore, we will want orbits to keep a certain distance, \( \gamma \), away from the billiard discontinuities. \( \gamma \to 0 \) as \( \varepsilon \to 0 \), but \( \gamma \) is very large compared to the possible drift \( \mathcal{O}(\varepsilon L) \) of the slow variables on the time scale \( L \). In fact, for each \( C, m, n > 0 \),
\[
\frac{\varepsilon L^m}{\gamma^n} \left( \frac{C}{\gamma} \right)^\lambda = \mathcal{O}(\varepsilon e^{\text{const} L^2}) \to 0 \text{ as } \varepsilon \to 0.
\]
(4.23)

Let \( X : \mathcal{M}^1 \to \Omega \) be the map taking \( y \in \mathcal{M}^1 \) to \( x = X(y) \in \Omega \), the location of the billiard orbit of \( y \) in the collision cross-section that corresponds to the most recent time in the past that the orbit was in the collision cross-section. We consider the set of initial conditions
\[
\mathcal{E}_\varepsilon = X^{-1}(\Omega \setminus C_{\gamma,\lambda}) \bigcap \left\{ x \in \Omega : \sum_{k=0}^{\lambda} \zeta(F^k x) > L \right\}.
\]
Now from Equations (4.19) and (4.22), \( \nu C_{\gamma,\lambda} \to 0 \) as \( \varepsilon \to 0 \). Furthermore, by the
ergodicity of $F$,

$$
\nu \left\{ x \in \Omega : \sum_{k=0}^{\lambda} \zeta(F^k x) \leq L \right\} = \nu \left\{ x \in \Omega : \lambda^{-1} \sum_{k=0}^{\lambda} \zeta(F^k x) \leq E_x \zeta/2 \right\} \to 0
$$
as $\varepsilon \to 0$. But because the free flight time is bounded above, $\mu X^{-1} \leq \text{const}\cdot \nu$, and so $\mu \mathcal{E}_\varepsilon \to 1$ as $\varepsilon \to 0$. Hence, the convergence in Equation (4.21) and the conclusion of the proof in Section 4.3.1 follow from the lemma below and Equation (4.23).

**Lemma 4.3.2** (Analysis of deviations along good orbits). As $\varepsilon \to 0$,

$$
\sup_{y \in \mathcal{E}_\varepsilon \cap \{\varepsilon L \leq \tilde{T}_\varepsilon\}} \left| \frac{1}{L} \int_0^L G(z_\varepsilon(s)) - G(z_0(s))ds \right| = \mathcal{O} \left( \rho \left( \frac{\text{const}^{-\lambda}}{\gamma} \right) \right) + \mathcal{O}(L^{-1}) \to 0.
$$

**Proof.** Fix a particular value of $y \in \mathcal{E}_\varepsilon \cap \{\varepsilon L \leq \tilde{T}_\varepsilon\}$. For convenience, suppose that $y = X(y) = x \in \Omega$. Let $y_0(t)$ denote the time evolution of the billiard coordinates for the left gas particle when $\varepsilon = 0$. Then there is some $N \leq \lambda$ such that the orbit $x_k = F^k x = (r_k, \varphi_k)$ for $0 \leq k \leq N$ corresponds to all of the instances (in order) when $y_0(t)$ enters the collision cross-section $\Omega = \Omega_{h_0}$ corresponding to collisions with $\partial \mathcal{D}_1(Q_0)$ for $0 \leq t \leq L$. We write $\Omega_{h_0}$ to emphasize that in this subsection we are only considering the collision cross-section corresponding to the billiard dynamics in the domain $\mathcal{D}_1(Q_0)$ at the energy level $E_{1,0}$. In particular, $F$ will always refer to the return map on $\Omega_{h_0}$.

Also, define an increasing sequence of times $t_k$ corresponding to the actual times $y_0(t)$ enters the collision cross-section, i.e.

$$
t_0 = 0,

\quad t_k = t_{k-1} + \zeta x_{k-1} \text{ for } k > 0.
$$

Then $x_k = y_0(t_k)$. Furthermore, define inductively

$$
N_1 = \inf \left\{ k > 0 : t_k \text{ corresponds to a collision with the piston} \right\},

N_j = \inf \left\{ k > N_{j-1} : t_k \text{ corresponds to a collision with the piston} \right\}.
$$

Next, let $y_\varepsilon(t)$ denote the time evolution of the billiard coordinates for the left gas particle when $\varepsilon > 0$. We will construct a pseudo-orbit $x_{k,\varepsilon}' = (r_{k,\varepsilon}', \varphi_{k,\varepsilon}')$ of points in $\Omega_{h_0}$ that essentially track the collisions (in order) of the left gas particle with the boundary under the dynamics of $y_\varepsilon(t)$ for $0 \leq t \leq L$.

First, define an increasing sequence of times $t'_{k,\varepsilon}$ corresponding to the actual times $y_\varepsilon(t)$ experiences a collision with the boundary of the gas container or the moving piston. Define

$$
N'_\varepsilon = \sup \left\{ k \geq 0 : t'_{k,\varepsilon} \leq L \right\},

N'_{1,\varepsilon} = \inf \left\{ k > 0 : t'_{k,\varepsilon} \text{ corresponds to a collision with the piston} \right\},

N'_{j,\varepsilon} = \inf \left\{ k > N'_{j-1,\varepsilon} : t'_{k,\varepsilon} \text{ corresponds to a collision with the piston} \right\}.
$$
Because \( L \leq \tilde{T}_\varepsilon(y)/\varepsilon \), we know that as long as \( N_{j+1,\varepsilon} \leq N'_j \), then \( N'_{j+1,\varepsilon} - N'_{j,\varepsilon} \geq 2 \). See the discussion in Subsection 4.2.2. Then we define \( x'_{k,\varepsilon} \in \Omega_0 \) by

\[
x'_{k,\varepsilon} = \begin{cases} 
  y_\varepsilon(t'_{k,\varepsilon}) & \text{if } k \notin \{N'_{j,\varepsilon}\}, \\
  F^{-1}x'_{k+1,\varepsilon} & \text{if } k \in \{N'_{j,\varepsilon}\}.
\end{cases}
\]

**Lemma 4.3.3.** Provided \( \varepsilon \) is sufficiently small, the following hold for each \( k \in [0, N \land N'_{\varepsilon}) \). Furthermore, the requisite smallness of \( \varepsilon \) and the sizes of the constants in these estimates may be chosen independent of the initial condition \( y \in \mathcal{E}_\varepsilon \cap \{\varepsilon L \leq \tilde{T}_\varepsilon\} \) and of \( k \):

(a) \( x'_{k,\varepsilon} \) is well defined. In particular, if \( k \notin \{N'_{j,\varepsilon}\}, y_\varepsilon(t'_{k,\varepsilon}) \) corresponds to a collision point on \( \partial D_1(Q_0) \), and not to a collision point on a piece of \( \partial D \) to the right of \( Q_0 \).

(b) If \( k > 0 \) and \( k \notin \{N'_{j,\varepsilon}\} \), then \( x'_{k,\varepsilon} = Fx'_{k-1,\varepsilon} \).

(c) If \( k > 0 \) and \( k \in \{N'_{j,\varepsilon}\} \), then \( d(x'_k, Fx'_{k-1,\varepsilon}) \leq \rho \) and the \( \varphi \) coordinate of \( y_\varepsilon(t'_{k,\varepsilon}) \) satisfies \( \varphi(y_\varepsilon(t'_{k,\varepsilon})) = \varphi'_{k,\varepsilon} + \mathcal{O}(\varepsilon) \).

(d) \( d(x_k, x'_{k,\varepsilon}) \leq \text{const} \rho(\text{const}/\gamma)^k \).

(e) \( k = N'_{j,\varepsilon} \) if and only if \( k = N_j \).

(f) If \( k > 0 \), \( t'_{k,\varepsilon} - t'_{k-1,\varepsilon} = t_k - t_{k-1} + \mathcal{O}(\rho(\text{const}/\gamma)^k) \).

We defer the proof of Lemma 4.3.3 until the end of this subsection. Assuming that \( \varepsilon \) is sufficiently small for the conclusions of Lemma 4.3.3 to be valid, we continue with the proof of Lemma 4.3.2.

Set \( M = N \land N'_{\varepsilon} - 1 \). Note that \( M \leq \lambda \sim L \). From (f) in Lemma 4.3.3 and Equations (4.22) and (4.23), we see that

\[
|t_M - t'_{M,\varepsilon}| \leq \sum_{k=1}^{M} |t'_{k,\varepsilon} - t'_{k-1,\varepsilon} - (t_k - t_{k-1})| = \mathcal{O} \left( \rho \frac{\text{const}^\lambda}{\gamma^\lambda} \right) \to 0 \text{ as } \varepsilon \to 0.
\]

Because the flight times \( t'_{k,\varepsilon} - t'_{k-1,\varepsilon} \) and \( t_k - t_{k-1} \) are uniformly bounded above, it follows from the definitions of \( N \) and \( N'_{\varepsilon} \) that \( t_M, t'_{M,\varepsilon} \geq L - \text{const} \). But from Subsection 4.2.2 the time between the collisions of the left gas particle with the piston are uniformly bounded away from zero. Using (c) and Equation (4.20), it
Equation (4.17) is satisfied. This is possible by Equation (4.23).

Proof of Lemma 4.3.3. The proof is by induction. We take follows that
\[ k < l \]
(a)-(f) have been verified for all \( k \in \{N_j: N_j \leq M\} \).

But using (d),
\[
\sum_{k \in \{N_j: N_j \leq M\}} |\cos \varphi_k - \cos \varphi'_{k,\varepsilon}| \leq \sum_{k=0}^{M} O(\rho(\text{const}/\gamma)^k) = O(\rho(\text{const}/\gamma)^\lambda).
\]

Since \( \varepsilon L^2 = O(\rho(\text{const}/\gamma)^\lambda) \), this finishes the proof of Lemma 4.3.2.

**Proof of Lemma 4.3.3.** The proof is by induction. We take \( \varepsilon \) to be so small that Equation (4.17) is satisfied. This is possible by Equation (4.23).

It is trivial to verify (a)-(f) for \( k = 0 \). So let \( 0 < l < N \cap N' \), and suppose that (a)-(f) have been verified for all \( k < l \). We have three cases to consider:

**Case 1: \( l - 1 \) and \( l \notin \{ N'_{j,\varepsilon} \} \):**

In this case, verifying (a)-(f) for \( k = l \) is a relatively straightforward application of the machinery developed in Subsection 4.3.2 because for \( t'_{l-1,\varepsilon} \leq t \leq t'_{l,\varepsilon} \), \( y_{\varepsilon}(t) \) traces out the billiard orbit between \( x'_{l-1,\varepsilon} \) and \( x'_{l,\varepsilon} \) corresponding to free flight in the domain \( D_1(Q_0) \). We make only two remarks.

First, as long as \( \varepsilon \) is sufficiently small, it really is true that \( x'_{l,\varepsilon} = y_{\varepsilon}(t'_{l,\varepsilon}) \) corresponds to a true collision point on \( \partial D_1(Q_0) \). Indeed, if this were not the case, then it must be that \( Q_{\varepsilon}(t'_l) > Q_0 \), and \( y_{\varepsilon}(t'_{l,\varepsilon}) \) would have to correspond to a collision with the side of the “tube” to the right of \( Q_0 \). But then \( x''_{l,\varepsilon} = Fx'_{l-1,\varepsilon} \in \Omega_{\varepsilon} \) would correspond to a collision with an immobile piston at \( Q_0 \) and would satisfy \( d(x_k, x''_{k,\varepsilon}) \leq \text{const} \rho(\text{const}/\gamma)^k \leq \text{const} \rho(\text{const}/\gamma)^\lambda = o(\gamma) \), using Equations (4.16) and (4.23). But \( x_k \notin \mathcal{N}_\gamma(\partial \Omega_{\varepsilon}) \), and so it follows that when the trajectory of \( y_{\varepsilon}(t) \) crosses the plane \( \{ Q = Q_0 \} \), it is at least a distance \( \sim \gamma \) away from the boundary of the face of the piston, and its velocity vector is pointed no closer than \( \sim \gamma \) to being parallel to the piston’s face. As \( Q_{\varepsilon}(t'_{l,\varepsilon}) - Q_0 = O(\varepsilon L) = O(\gamma) \), and it is geometrically impossible (for small \( \varepsilon \)) to construct a right
triangle whose sides $s_1$, $s_2$ satisfy $|s_1| \geq \sim \gamma$, $|s_2| \leq \mathcal{O}(\varepsilon L)$, with the measure of the acute angle adjacent to $s_1$ being greater than $\sim \gamma$, we have a contradiction. After crossing the plane $\{Q = Q_0\}$, $y_\varepsilon(t)$ must experience its next collision with the face of the piston, which violates the fact that $l \notin \{N_{j,\varepsilon}^t\}$. From Equation (4.20), the face of the piston, which violates the fact that $l \notin \{N_{j,\varepsilon}^t\}$, because $v_{1,\varepsilon} = v_{1,0} + \mathcal{O}(\varepsilon L)$. See Equation (4.20). From Equation 4.18, $|\zeta x_{t-1,\varepsilon} - \zeta x^t_{t-1,\varepsilon}| \leq \mathcal{O}((\rho/\gamma)(\text{const/}\gamma)^{l-1})$. As $t_1 - t_{l-1} = \zeta x_{l-1}$ and $\varepsilon L = \mathcal{O}((\rho/\gamma)(\text{const/}\gamma)^{l-1})$, we obtain (f).

**Case 2: There exists $i$ such that $l = N_{i,\varepsilon}^t$:**

For definiteness, we suppose that $Q_{\varepsilon}(t_{l,\varepsilon}^t) \geq Q_0$, so that the left gas particle collides with the piston to the right of $Q_0$. The case when $Q_{\varepsilon}(t_{l,\varepsilon}^t) \leq Q_0$ can be handled similarly.

We know that $x_{l-1,\varepsilon}, x_1, x_{l+1,\varepsilon} \notin N_{\varepsilon}(\partial \Omega_{h_0}) \cup N_{\varepsilon}(F^{-1}N_{\varepsilon}(\partial \Omega_{h_0}))$. Using the inductive hypothesis and Equation (4.10), we can define

$$x_{l,\varepsilon}'' = F x_{l-1,\varepsilon}, \quad x_{l+1,\varepsilon}'' = F^2 x_{l-1,\varepsilon},$$

and $d(x_{l,\varepsilon}, x_{l,\varepsilon}'') \leq \text{const } \rho(\text{const/}\gamma)^{l}$, $d(x_{l,\varepsilon}', x_{l+1,\varepsilon}'') \leq \text{const } \rho(\text{const/}\gamma)^{l+1}$. In particular, $x_{l,\varepsilon}'$ and $x_{l+1,\varepsilon}'$ are both a distance $\sim \gamma$ away from $\partial \Omega_{h_0}$. Furthermore, when the left gas particle collides with the moving piston, it follows from Equation (4.3) that the difference between its angle of incidence and its angle of reflection is $\mathcal{O}(\varepsilon)$. Referring to Figure 4.3 this means that $\varphi_{l,\varepsilon} = \varphi_{l,\varepsilon}'' + \mathcal{O}(\varepsilon)$. Geometric arguments similar to the one given in Case 1 above show that the $y_{\varepsilon}$-trajectory of the left gas particle has precisely one collision with the piston and no other collisions with the sides of the gas container when the gas particle traverses the region $Q_0 \leq Q \leq Q_{\varepsilon}(t_{l,\varepsilon}^t)$. Note that $x_{l,\varepsilon}'$ was defined to be the point in the collision cross-section $\Omega_{h_0}$ corresponding to the return of the $y_{\varepsilon}$-trajectory into the region $Q \leq Q_0$. See Figure 4.3. From this figure, it is also evident that $d(r_{l,\varepsilon}', x_{l,\varepsilon}'') \leq \mathcal{O}(\varepsilon L/\gamma)$. Thus $d(x_{l,\varepsilon}'', x_{l,\varepsilon}') = \mathcal{O}(\varepsilon L/\gamma)$, and this explains the choice of $\rho(\varepsilon)$ in Equation (1.22).

From the above discussion and the machinery of Subsection 4.3.2 (a)-(e) now follow readily for both $k = l$ and $k = l + 1$. Furthermore, property (f) follows in much the same manner as it did in Case 1 above. However, one should note that $t_{l,\varepsilon}' - t_{l-1,\varepsilon}' = \zeta x_{l-1,\varepsilon}' + \mathcal{O}(\varepsilon L) + \mathcal{O}(\varepsilon L/\gamma)$ and $t_{l+1,\varepsilon}' - t_{l,\varepsilon}' = \zeta x_{l,\varepsilon}' + \mathcal{O}(\varepsilon L) + \mathcal{O}(\varepsilon L/\gamma)$, because of the extra distance $\mathcal{O}(\varepsilon L/\gamma)$ that the gas particle travels to the right of $Q_0$. But $\varepsilon L/\gamma = \mathcal{O}((\rho/\gamma)(\text{const/}\gamma)^{l-1})$, and so property (f) follows.

**Case 3: There exists $i$ such that $l = N_{i,\varepsilon}^t$:**

As mentioned above, the inductive step in this case follows immediately from our analysis in Case 2.

\[ \Box \]
Figure 4.3: An analysis of the divergences of orbits when \( \varepsilon > 0 \) and the left gas particle collides with the moving piston to the right of \( Q_0 \). Note that the dimensions are distorted for visual clarity, but that \( \varepsilon L \) and \( \varepsilon L/\gamma \) are both \( o(\gamma) \) as \( \varepsilon \to 0 \).

Furthermore, \( \phi''_{l,\varepsilon} \in (-\pi/2 + \gamma/2, \pi/2 - \gamma/2) \) and \( \phi'_{l,\varepsilon} = \phi''_{l,\varepsilon} + O(\varepsilon) \), and so \( r'_{l,\varepsilon} = r''_{l,\varepsilon} + O(\varepsilon L/\gamma) \). In particular, the \( y_\varepsilon \)-trajectory of the left gas particle has precisely one collision with the piston and no other collisions with the sides of the gas container when the gas particle traverses the region \( Q_0 \leq Q \leq Q_\varepsilon(t'_{l,\varepsilon}) \).
4.4 Generalization to a full proof of Theorem 4.1.1

It remains to generalize the proof in Sections 4.2 and 4.3 to the cases when \( n_1, n_2 \geq 1 \) and \( d = 3 \).

4.4.1 Multiple gas particles on each side of the piston

When \( d = 2 \), but \( n_1, n_2 \geq 1 \), only minor modifications are necessary to generalize the proof above. As in Subsection 4.2.2, one defines a stopping time \( \tilde{T}_\varepsilon \) satisfying
\[
P\{\tilde{T}_\varepsilon < T \wedge T_\varepsilon\} = O(\varepsilon)
\]
such that for \( 0 \leq t \leq \tilde{T}_\varepsilon/\varepsilon \), gas particles will only experience clean collisions with the piston.

Next, define \( H(z) \) by
\[
H(z) = \begin{bmatrix}
+2 \sum_{j=1}^{n_1} \left| v_{1,j} \right| \delta_{q_{1,j}=Q} - 2 \sum_{j=1}^{n_2} \left| v_{2,j} \right| \delta_{q_{2,j}=Q} \\
-2W \left| v_{1,j} \right| \delta_{q_{1,j}=Q} \\
+2W \left| v_{2,j} \right| \delta_{q_{2,j}=Q}
\end{bmatrix}
\]

It follows that for \( 0 \leq t \leq \tilde{T}_\varepsilon/\varepsilon \), \( h_\varepsilon(t) - h_\varepsilon(0) = O(\varepsilon) + \varepsilon \int_0^t H(z_\varepsilon(s))ds \). From here, the rest of the proof follows the same steps made in Subsection 4.3.1. We note that at Step 3, we find that \( H(z) - \bar{H}(h(z)) \) divides into \( n_1 + n_2 \) pieces, each of which depends on only one gas particle when the piston is held fixed.

4.4.2 Three dimensions

The proof of Theorem 4.1.1 in \( d = 3 \) dimensions is essentially the same as the proof in two dimensions given above. The principal differences are due to differences in the geometry of billiards. We indicate the necessary modifications.

In analogy with Section 4.2.1, we briefly summarize the necessary facts for the billiard flows of the gas particles when \( M = \infty \) and the slow variables are held fixed at a specific value \( h \in V \). As before, we will only consider the motions of one gas particle moving in \( D_1 \). Thus we consider the billiard flow of a point particle moving inside the domain \( D_1 \) at a constant speed \( \sqrt{2E_1} \). Unless otherwise noted, we use the notation from Section 4.2.1.

The billiard flow takes place in the five-dimensional space \( M^1 = \{(q_1, v_1) \in TD_1 : q_1 \in D_1, \ |v_1| = \sqrt{2E_1}/\sim \} \). Here the quotient means that when \( q_1 \in \partial D_1 \), we identify velocity vectors pointing outside of \( D_1 \) with those pointing inside \( D_1 \) by reflecting orthogonally through the tangent plane to \( \partial D_1 \) at \( q_1 \). The billiard flow preserves Liouville measure restricted to the energy surface. This measure
has the density \[ d\mu = dq_1dv_1/(8\pi E_1 |D_1|) \]. Here \( dq_1 \) represents volume on \( \mathbb{R}^3 \), and \( dv_1 \) represents area on \( S^2_{\sqrt{2|E_1|}} = \{v_1 \in \mathbb{R}^3 : |v_1| = \sqrt{2|E_1|}\} \).

The collision cross-section \( \Omega = \{(q_1,v_1) \in TD_1 : q_1 \in \partial D_1, |v_1| = \sqrt{2|E_1|}\}/\sim \) is properly thought of as a fiber bundle, whose base consists of the smooth pieces of \( \partial D_1 \) and whose fibers are the set of outgoing velocity vectors at \( q_1 \in \partial D_1 \). This and other facts about higher-dimensional billiards, with emphasis on the dispersing case, can be found in [BCST03]. For our purposes, \( \Omega \) can be parameterized as follows. We decompose \( \partial D_1 \) into a finite union \( \bigcup_j \Gamma_j \) of pieces, each of which is diffeomorphic via coordinates \( r \) to a compact, connected subset of \( \mathbb{R}^2 \), with \( \partial \mathbb{R}^2 \) boundary. The \( \Gamma_j \) are nonoverlapping, except possibly on their boundaries. Next, if \( (q_1,v_1) \in \Omega \) and \( v_1 \) is the outward going velocity vector, let \( \hat{v} = v_1/|v_1| \). Then \( \Omega \) can be parameterized by \( \{x = (r, \hat{v})\} \). It follows that \( \Omega \) it is diffeomorphic to \( \bigcup_j \Gamma_j \times S^2^{+} \), where \( S^2^{+} \) is the upper unit hemisphere, and by \( \partial \Omega \) we mean the subset diffeomorphic to \( (\bigcup_j \partial \Gamma_j \times S^2^{+}) \cup (\bigcup_j \Gamma_j \times \partial S^2^{+}) \). If \( x \in \Omega \), we let \( \varphi \in [0, \pi/2] \) represent the angle between the outgoing velocity vector and the inward pointing normal vector \( n \) to \( \partial D_1 \), i.e. \( \cos \varphi = \langle \hat{v}, n \rangle \). Note that we no longer allow \( \varphi \) to take on negative values. The return map \( F : \Omega \rightharpoonup \mathbb{R} \) preserves the projected probability measure \( \nu \), which has the density \( d\nu = \cos \varphi d\hat{v}dr/(\pi |\partial D_1|) \).

Here \( |\partial D_1| \) is the area of \( \partial D_1 \).

\( F \) is an invertible, measure preserving transformation that is piecewise \( C^2 \). Because of our assumptions on \( D_1 \), the free flight times and the curvature of \( \partial D_1 \) are uniformly bounded. The bound on \( \|DF(x)\| \) given in Equation (4.2) is still true. A proof of this fact for general three-dimensional billiard tables with finite horizon does not seem to have made it into the literature, although see [BCST03] for the case of dispersing billiards. For completeness, we provide a sketch of a proof for general billiard tables in Section 4.6.

We suppose that the billiard flow is ergodic, so that \( F \) is ergodic. Again, we induce \( F \) on the subspace \( \hat{\Omega} \) of \( \Omega \) corresponding to collisions with the (immobile) piston to obtain the induced map \( \hat{F} : \hat{\Omega} \rightharpoonup \mathbb{R} \) that preserves the induced measure \( \hat{\nu} \).

The free flight time \( \zeta : \Omega \rightarrow \mathbb{R} \) again satisfies the derivative bound given in Equation (4.3). The generalized Santaló's formula [Che97] yields

\[ E_{\nu} \zeta = \frac{4 |D_1|}{|v_1| |\partial D_1|}. \]

If \( \hat{\zeta} : \hat{\Omega} \rightarrow \mathbb{R} \) is the free flight time between collisions with the piston, then it follows from Proposition 4.5.1 that

\[ E_{\hat{\nu}} \hat{\zeta} = \frac{4 |D_1|}{|v_1| \ell}. \]

The expected value of \( |v_1^+| \) when the left gas particle collides with the (immo-
bile) piston is given by

\[ E_\hat{\nu} \left| v_1^+ \right| = E_\hat{\nu} \sqrt{2E_1} \cos \varphi = \frac{\sqrt{2E_1}}{\pi} \int_{S^{2+}} \cos^2 \varphi \, d\hat{v}_1 = \sqrt{2E_1^2} \frac{2}{3}. \]

As a consequence, we obtain

**Lemma 4.4.1.** For \( \mu - a.e. \, y \in \mathcal{M}^1 \),

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \left| v_1^+(s) \right| \delta_{q_1^+(s)=Q} ds = \frac{E_1 \ell}{3 |D_1(Q)|}. \]

Compare the proof of Lemma 4.2.1.

With these differences in mind, the rest of the proof of Theorem 4.1.1 when \( d = 3 \) proceeds in the same manner as indicated in Sections 4.2, 4.3 and 4.4.1 above. The only notable difference occurs in the proof of the Gronwall-type inequality for billiards. Due to dimensional considerations, if one follows the proof of Lemma 4.3.1 for a three-dimensional billiard table, one finds that

\[ \nu_{\gamma'}(F^{-1}_{\gamma'}(\partial \Omega)) = O(\gamma^{1-4\alpha} + \gamma^{\alpha}). \]

The optimal value of \( \alpha \) is 1/5, and so \( \nu_{\gamma'}(F^{-1}_{\gamma'}(\partial \Omega)) = O(\gamma^{1/5}) \) as \( \gamma \to 0 \). Hence \( \nu_{\gamma,\lambda} = O(\lambda \gamma^{1/5}) \), which is a slightly worse estimate than the one in Equation (4.19). However, it is still sufficient for all of the arguments in Section 4.3.2 and this finishes the proof.

### 4.5 Inducing maps on subspaces

Here we present some well-known facts on inducing measure preserving transformations on subspaces. Let \( F : (\Omega, \mathfrak{B}, \nu) \to (\hat{\Omega}, \hat{\mathfrak{B}}, \hat{\nu}) \) be an invertible, ergodic, measure preserving transformation of the probability space \( \Omega \) endowed with the \( \sigma \)-algebra \( \mathfrak{B} \) and the probability measure \( \nu \). Let \( \hat{\Omega} \in \hat{\mathfrak{B}} \) satisfy \( 0 < \nu \hat{\Omega} < 1 \). Define \( R : \hat{\Omega} \to \mathbb{N} \) to be the first return time to \( \hat{\Omega} \), i.e.

\[ R_\omega = \inf \{ n \in \mathbb{N} : F^n \omega \in \hat{\Omega} \}. \]

Then if \( \hat{\nu} := \nu(\cdot \cap \hat{\Omega})/\nu \hat{\Omega} \) and \( \hat{\mathfrak{B}} := \{ B \cap \hat{\Omega} : B \in \mathfrak{B} \} \), \( \hat{F} : (\hat{\Omega}, \hat{\mathfrak{B}}, \hat{\nu}) \to (\hat{\Omega}, \hat{\mathfrak{B}}, \hat{\nu}) \) defined by \( \hat{F} \omega = F^{R_\omega} \omega \) is also an invertible, ergodic, measure preserving transformation [Pet83]. Furthermore \( E_{\nu} R = \int_{\hat{\Omega}} R d\hat{\nu} = (\nu \hat{\Omega})^{-1} \).

This last fact is a consequence of the following proposition:

**Proposition 4.5.1.** If \( \zeta : \Omega \to \mathbb{R}_{\geq 0} \) is in \( L^1(\nu) \), then \( \hat{\zeta} = \sum_{n=0}^{R-1} \zeta \circ F^n \) is in \( L^1(\hat{\nu}) \), and

\[ E_{\hat{\nu}} \hat{\zeta} = \frac{1}{\nu \hat{\Omega}} E_{\nu} \zeta. \]
Proof.

\[ \nu\hat{\Omega} \int_{\hat{\Omega}} \sum_{n=0}^{R-1} \zeta \circ F^n \, d\hat{\nu} = \int_{\hat{\Omega}} \sum_{n=0}^{R-1} \zeta \circ F^n \, d\nu = \sum_{k=1}^{\infty} \int_{\hat{\Omega} \cap \{ R = k \}} \sum_{n=0}^{k-1} \zeta \circ F^n \, d\nu = \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \int_{F^n(\hat{\Omega} \cap \{ R = k \})} \zeta \, d\nu = \int_{\Omega} \zeta \, d\nu, \]

because \( \{ F^n(\hat{\Omega} \cap \{ R = k \}) : 0 \leq n < k < \infty \} \) is a partition of \( \Omega \).

\[ \square \]

4.6 Derivative bounds for the billiard map in three dimensions

Returning to Section 4.4.2, we need to show that for a billiard table \( \mathcal{D}_1 \subset \mathbb{R}^3 \) with a piecewise \( C^3 \) boundary and the free flight time uniformly bounded above, the billiard map \( F \) satisfies the following: If \( x_0 \notin \partial\Omega \cup F^{-1}(\partial\Omega) \), then

\[ \| DF(x_0) \| \leq \frac{\text{const}}{\cos \varphi(Fx_0)}. \]

Fix \( x_0 = (r_0, \hat{v}_0) \in \Omega \), and let \( x_1 = (r_1, \hat{v}_1) = Fx_0 \). Let \( \Sigma \) be the plane that perpendicularly bisects the straight line between \( r_0 \) and \( r_1 \), and let \( r_{1/2} \) denote the point of intersection. We consider \( \Sigma \) as a “transparent” wall, so that in a neighborhood of \( x_0 \), we can write \( F = F_2 \circ F_1 \). Here, \( F_1 \) is like a billiard map in that it takes points (i.e. directed velocity vectors with a base) near \( x_0 \) to points with a base on \( \Sigma \) and a direction pointing near \( r_1 \). \( (F_1 \) would be a billiard map if we reflected the image velocity vectors orthogonally through \( \Sigma \).) \( F_2 \) is a billiard map that takes points in the image of \( F_1 \) and maps them near \( x_1 \). Let \( x_{1/2} = F_1 x_0 = F_2^{-1} x_1 \). Then \( \| DF(x_0) \| \leq \| DF_1(x_0) \| \| DF_2(x_{1/2}) \| \).

It is easy to verify that \( \| DF_1(x_0) \| \leq \text{const} \), with the constant depending only on the curvature of \( \partial \mathcal{D}_1 \) at \( r_0 \). In other words, the constant may be chosen independent of \( x_0 \). Similarly, \( \| DF_2^{-1}(x_1) \| \leq \text{const} \). Because billiard maps preserve a probability measure with a density proportional to \( \cos \varphi \), \( \det DF_2^{-1}(x_1) = \cos \varphi_1 / \cos \varphi_{1/2} = \cos \varphi_1 \). As \( \Omega \) is 4-dimensional, it follows from Cramer’s Rule for the inversion of linear transformations that

\[ \| DF_2(x_{1/2}) \| \leq \frac{\text{const} \| DF_2^{-1}(x_1) \|^3}{\det DF_2^{-1}(x_1)} \leq \frac{\text{const}}{\cos \varphi_1}, \]

and we are done.
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