Twisted symmetries and integrable systems

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Summary. Symmetry properties are at the basis of integrability. In recent years, it appeared that so called twisted symmetries are as effective as standard symmetries in many respects (integrating ODEs, finding special solutions to PDEs). Here we discuss how twisted symmetries can be used to detect integrability of Lagrangian systems which are not integrable via standard symmetries.

Keywords: Symmetry of differential equations; integrable systems; conservation laws.

Introduction

Integrable systems are characterized by a high degree of symmetry – and a favourable structure of the underlying symmetry algebra.

In recent years, the standard concept of symmetry for differential equations [1, 2, 9, 13, 20, 36, 37, 41, 43] has been generalized in several directions (see e.g. [19] for an overview). Here we are interested in a special case among these generalizations; actually this is special not only in the sense it is a specific one but also in that it differs from all other ones in a substantial way. That is, in dealing with symmetry of differential equations we always consider a vector field acting in the basic (independent and dependent) variables, and then prolong it to derivatives of suitable order – the order of the differential equation to be considered, or maybe to infinite order. While “usual” generalizations amount to
generalize the admitted vector fields acting on basic variables, **twisted symmetries** modify the prolongation operation itself. In this note we will focus specifically on twisted symmetries, and their role in analyzing integrability.

Our main result will be that systems which are characterized by a high degree of twisted symmetry – and a favorable structure of the underlying symmetry algebra – are integrable.

Twisted symmetries are not a new concept, but a collective name to include different types of symmetries (\(C^\infty\)-symmetries, also known as \(\lambda\)-symmetries, \(\mu\)-symmetries, and the recently introduced \(\rho\)-symmetries), all of them involving a deformation of the prolongation operation. They were first introduced in the context of scalar ODEs as \(C^\infty\)-symmetries or \(\lambda\)-symmetries in a seminal paper by Muriel and Romero [23], who also extended their ideas to more general settings [24, 25, 26, 27, 28, 29, 30, 31, 32, 33]; see also [18, 38]. The name “\(\lambda\)-symmetries” refers to the key role played by a \(C^\infty\) function \(\lambda(x,u,u_x)\). The case dealing with PDEs was then christened under the name of \(\mu\)-symmetries [11, 17] both for alphabetic continuity and because in this context the key role is played by a semi-basic matrix-valued one-form \(\mu = \Lambda_i dx^i\). Still a different name in use is that of “\(\rho\)-symmetries” for a specific class of \(\mu\)-symmetries which allow reduction of system of ODEs [7, 8]. The basic ideas behind twisted symmetries as well as these different special types of twisted symmetries will be briefly reviewed in section 2 below; a more substantial review is provided in [16].

## 1 Notation

We will start by fixing some general (standard) notation, to be freely used in the following.

We will consider problems defined on a phase bundle \((M, \pi, B)\) with fiber \(\pi^{-1}(x) = U\); here \(B\) and \(U\) are smooth real manifolds of dimensions \(p\) and \(q\) respectively, and we will use local coordinates \(\{x^1, ..., x^p\}\) in \(B\) and \(\{u^1, ..., u^q\}\) in \(U\). As usual, when dealing with differential equations we will think of the \(x\) as independent variables and the \(u\) as dependent ones (fields). Associated to the bundle \(M\) are the \(k\)-th order jet bundles \(J^kM\); there are natural coordinates in these, provided by \(x, u\) and by partial derivatives of the \(u\) with respect to the \(x\). In dealing with these, we will freely use the multi-index notation, see e.g. [36] for details.\(^1\)

\(^1\)In this note we will actually mainly focus on systems with only one independent variable. However we prefer to deal with the general case as this makes the geometry behind twisted symmetries – and their properties – more transparent: the special case of ODEs is indeed degenerate in several respects, and it is highly remarkable that Muriel and Romero were able to deal with it at first.
Vector fields

Consider now a vector field (VF) on $M$; this will be written in coordinates as

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \varphi^a(x, u) \frac{\partial}{\partial u^a}.$$  \hfill (1)

We will routinely omit to write the dependencies of the functions (such as $\xi$ and $\varphi$) to avoid an exceedingly heavy notation.

As well known, in many cases it is convenient to consider the evolutionary representative (or vertical representative) of a VF; this describes the action of the VF on a section of the bundle $(M, \pi, B)$ and is written in coordinates as

$$X_v = (\varphi^a - \xi^i u^a_i) \frac{\partial}{\partial u^a} = Q^a \frac{\partial}{\partial u^a}.$$  \hfill (2)

A vector field acting in $M$ also acts naturally in $J^k M$: once the action on independent and dependent variables is given, the action on derivatives of any order can be readily computed. The lift of the $X$ action from $M$ to $J^k M$ is also known as the prolongation operation \cite{1, 9, 13, 20, 36, 37, 41, 43}. In coordinates, the prolonged vector field $X^*$ (or $X^n$ if we consider prolongation only up to order $n$) is given by

$$X^* = \xi^i \frac{\partial}{\partial x^i} + \psi^a_j \frac{\partial}{\partial u^a_j},$$

where $J$ are multi-indices and

$$\psi^a_0 = \varphi^a.$$  \hfill (3)

The coefficients $\psi^a_j$ are then determined by the prolongation formula

$$\psi^a_j \iota = D_i \psi^a_j - u^a_{j,k} D_i \xi^k.$$  \hfill (4)

Here $D_i$ is the total derivative with respect to $x^i$, i.e.

$$D_i = \frac{\partial}{\partial x^i} + u^a_{j,i} \frac{\partial}{\partial u^a_j}.$$  \hfill (5)

These relations are specially simple for vertical vector fields: in this case we have

$$X_v^* = Q^a_j \frac{\partial}{\partial u^a_j}, \quad \text{with} \quad Q^a_{j,i} = D_i Q^a_j.$$  \hfill (5)

\footnote{Here and everywhere below we understand summation over repeated indices unless otherwise stated.}

\footnote{In fact, a simple computation shows that if a section $\sigma$ is given in coordinates by $\sigma = \{(x, u) / u = f(x)\}$, then under the infinitesimal action of $X$ it is mapped to a new section $\tilde{\sigma} = \{(x, u) / u = f(x)\}$ with $\tilde{f}(x) = f(x) + \varepsilon \left[\varphi^a - \xi^a u^a\right]_{\sigma}$, where the functions within the square bracket should be computed on the section $\sigma$.}
It will be convenient to have a more intrinsic characterization of the prolongation operation.

The jet bundles \( J^n M \) are naturally equipped with a **contact structure**, i.e. with a set of contact forms \( \theta^a_J \), given in coordinates by

\[
\theta^a_J := du^a_J - u^a_J, \quad (|J| = 0, ..., n - 1).
\]

We will denote by \( \mathcal{E} \) the ideal generated by these forms (with coefficients in \( C^\infty(J^n M) \)).

Then the prolonged vector field \( X^* = Y \) (we use this notation for graphical ease) is the only vector field which coincides with \( X \) on \( M \) and which **preserves** the contact ideal \( \mathcal{E} \), i.e. such that

\[
L_Y(\mathcal{E}) \in \mathcal{E} ;
\]

where \( L \) is the Lie derivative. This means that for any \( \vartheta \in \mathcal{E} \), \( L_Y(\vartheta) \in \mathcal{E} \).

The condition (7) can be expressed equivalently in terms of conditions involving the commutator of \( Y \) with the total derivative operators \( D_i \); in particular, it is equivalent to either one of

\[
[D_i, Y] \vartheta = 0 \quad \forall \vartheta \in \mathcal{E} ,
\]

\[
[D_i, Y] = h^m_i D_m + V ,
\]

with \( h^m_i \in C^\infty(J^n M) \) and \( V \) a vertical vector field in \( J^n M \) seen as a bundle over \( J^{n-1} M \) (i.e. it has components only along derivatives of maximal order \( n \)).

It is appropriate, for further reference and since it has just been mentioned, to recall that the jet bundles have several fibered structures; in particular, \( J^k M \) can be seen both as a bundle \((J^k M, \pi_k, B)\) over \( B \) and as a bundle \((J^k, \sigma_k, M)\) over \( M \):

\[
\begin{array}{ccc}
J^k M & \xrightarrow{\sigma_k} & J^k \\
\downarrow \pi_k & & \\
B & \xleftarrow{\pi} & M
\end{array}
\]

**Differential equations and their symmetries**

A differential equation – or system of differential equations – of order \( n \), which we will denote by \( \Delta \), identifies a submanifold \( S_\Delta \subset J^n M \), its solution manifold. That is, \( S_\Delta \) is the set of points of \( J^n M \) in which the relations \( \Delta \) are satisfied. If the equations involve only smooth coefficients, then \( S_\Delta \) is smooth, and we will assume that the equation is non-degenerate\(^4\), so that they correspondence between \( \Delta \) and \( S_\Delta \) is one-to-one [36].

\(^4\)That is, if \( \Delta \) is given by \( E^a(x, u^{(\alpha)}) = 0 \), we assume that the derivatives of \( E^a \) are nonzero in directions transversal to \( S_\Delta \).
The vector field $X$ is a symmetry (or more precisely, being a VF, a symmetry generator) for $\Delta$ if its $n$-th prolongation $X^{(n)}$ leaves $S_{\Delta}$ invariant, i.e.

$$X^{(n)} : S_{\Delta} \to T S_{\Delta}.$$ 

In this case, the one-parameter group generated by $X$ maps solutions to $\Delta$ into solutions.

We assume the reader is familiar with the use of symmetries for the analysis of differential equations, referring to [1, 13, 20, 36, 37, 41] for details and applications; here we just wanted to stress that the very concept of symmetry of differential equations is based on the prolonged vector fields and hence on the prolongation operation.

## 2 Twisted prolongations and symmetries

We will now introduce and discuss the twisted prolongation $Y$ of a vector field $X$; we anticipate that if $Y$ satisfies the symmetry condition

$$Y : S_{\Delta} \to T S_{\Delta},$$

then $X$ is a twisted symmetry of the differential equation $\Delta$.

Here we will deal directly with the most general setting ($p$ independent and $q$ dependent variables for a system of PDEs of order $n$); we refer to [16] for a review and discussion of relevant special cases: e.g. scalar equations, or ODEs; the latter will also be discussed below.

The relevant contact structure in this case is spanned by the contact forms $(6)$; it is convenient to see them as the components of a vector-valued contact form $\partial_J$ [42]. We will denote by $\Theta$ the module over $q$-dimensional smooth matrix functions generated by the $\partial_J$, i.e. the set of vector-valued forms which can be written as $\eta = (R_J)^d_b \partial^b_J$ with $R_J : J^n M \to \text{Mat}(q)$ smooth matrix functions.

The manifold of dependent variables – that is, the fiber of $(M, \pi, B)$ – has tangent space $U \simeq R^q$, on which is defined an action of $G = GL(q, R)$; the corresponding Lie algebra is $\mathcal{G} = g\ell(q)$ (we omit from now on the indication that all our manifolds, spaces and actions are real).\footnote{We will think the $G$ and hence $\mathcal{G}$ action is fixed once for all, and hence – for the sake of notation – do not distinguish notationally between the group (or algebra) and its representation.}

Consider a $\mathcal{G}$-valued semi-basic one-form on $J^1 M$,

$$\mu := \Lambda_i dx^i;$$

the $\Lambda_i = \Lambda_i(x, u, u_x)$ are smooth matrix functions (with values in $\mathcal{G}$) satisfying some additional compatibility conditions discussed below, see (15).

We will say that the vector field $Y$ on $J^n M \mu$-preserves the vector contact structure $\Theta$ if, for all $\vartheta \in \Theta$,

$$\mathcal{L}_Y(\vartheta) + (Y \lrcorner \Lambda_i)^a_b \partial^b_J dx^i \in \Theta; \quad (12)$$

the $\Lambda_i = \Lambda_i(x, u, u_x)$ are smooth matrix functions (with values in $\mathcal{G}$) satisfying some additional compatibility conditions discussed below, see (15). We will say that the vector field $Y$ on $J^n M \mu$-preserves the vector contact structure $\Theta$ if, for all $\vartheta \in \Theta$,
this should be compared to standard preservation of the contract structure in the form (7).

In terms of the coefficients of $Y$, see (3), this is equivalent to the requirement that the $\Psi^a_J$ obey the vector $\mu$-prolongation formula

$$\Psi^a_{J,i} = (\nabla_i)^a_b \Psi^b_J - u^b_{J,m} [(\nabla_i)^m_b \xi]$$

(13)

where we have introduced the (matrix) differential operators

$$\nabla_i := ID_i + \Lambda_i$$

$I$ is of course the $q \times q$ identity matrix. Needless to say, for $\mu = 0$ (i.e. $\Lambda_i = 0$ for all $i$), this reduces to the standard prolongation formula.

Note for later reference that in the case of vertical vector fields $X = Q^a(\partial/\partial u^a)$, (13) yields for the coefficients of the first prolongation $Y = X + \psi^a_i(\partial/\partial u^a)$, simply

$$\psi^a_i = (\nabla_i)^a_b Q^b = D_i Q^a + (R_i)^a_b Q^b.$$

(14)

As mentioned above, the functions $\Lambda_i$ defining the form $\mu$ in (11) are not arbitrary: they must satisfy a compatibility condition (this guarantees the $\Psi^a_J$ defined by (13) are uniquely determined), expressed in coordinates by

$$[\nabla_i, \nabla_k] \equiv D_i \Lambda_k - D_k \Lambda_i + [\Lambda_i, \Lambda_k] = 0.$$

(15)

This is nothing else than the coordinate expression of the horizontal Maurer-Cartan equation

$$D\mu + \frac{1}{2} [\mu, \mu] = 0.$$

(16)

Based on this condition – and on classical results of differential geometry [12, 6, 39] – it follows easily that in any contractible neighborhood $A \subseteq J^n M$, there exists $\gamma_A : A \rightarrow GL(q)$ such that (locally in $A$) $\mu$ is the Darboux derivative of $\gamma_A$.

In other words, any $\mu$-prolonged vector field is locally gauge-equivalent to a standard prolonged vector field [11, 22], the gauge group being $G = GL(q)$.

The result stated above means that if $Y$ is the $\mu$-prolongation of a vector field $X$, then there are vector fields $W$ and $Z$, gauge-equivalent via the same gauge transformation (acting respectively as $\gamma^{(k)}$ in $T(J^k)M$ and as $\gamma$ in $T(M)$) to $Y$ and $X$, and such that $W$ is the standard prolongation of $Z$. This is schematically summarized in the following diagram:

$$X \xrightarrow{\gamma^{(k)}} Z$$

$$\mu-\text{prol} \quad \text{prol}$$

$$Y \xrightarrow{\gamma^{(k)}} W$$

__6__This expresses the requirement that the standard Maurer-Cartan equation is satisfied modulo contact forms, i.e. $d\mu + (1/2)[\mu, \mu] \in \mathcal{E}$.

__7__When $J^n M$ is topologically nontrivial, or $\mu$ presents singular points, one can have $\mu$-prolonged vector fields which are not globally gauge equivalent to standardly prolonged ones (and in this sense non-trivial $\mu$-symmetries); see [11] for concrete examples.
For these considerations, it is convenient to deal with evolutionary representatives of vector fields \[36\], which we will implicitly do from now on.

It should also be stressed that the gauge group \(\Gamma\) (modelled over a Lie group \(G\)) acts in the same way on the vector \(\{\varphi^1, \ldots, \varphi^q\}\) of the components of the vector field \(X\) in \(M\), and on the vectors \(\{\psi^1, \ldots, \psi^q\}\) of components (relative to a given multi-index \(J\), i.e. to partial derivatives with respect to the same array of independent variables) of the vector field \(Y\) in \(J^kM\). One also says that \(\Gamma\) acts via a \textit{Jet representation}.

Summarizing, one finds out that the twisted prolongation operation amounts locally to standard prolongation seen in a different reference frame, i.e. under a gauge transformation.\(^8\)

It appears now fully naturally that – for what concerns properties which are both \textit{local} and \textit{frame-independent} – twisted symmetries are as good as standard ones.

The existence of conserved quantities in Mechanics – or conserved currents in Field Theory – satisfies these criteria, and it should thus be no surprise that one can analyze these with the help of twisted symmetries as well as of standard ones.

### 3 Lagrangians, twisted symmetries and conservation laws

The reader who attempts some very simple computations could find very strange the above statement, that twisted symmetries of the Lagrangian are related to conserved quantities. In fact, it is easy to check – e.g. in the case of first order Lagrangians \(L(q, \dot{q}; t)\) which we will consider for the sake of simplicity – that in general a twisted symmetry of the Lagrangian \(L\) does not correspond to a symmetry, either regular or twisted, of the corresponding Euler-Lagrange equations

\[
\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0 ;
\]

and even less to a conserved quantity.\(^9\)

The point is that if we change reference frame acting on the variables \(q^i\) via a gauge transformation \(\Lambda^i_j\), the variational equations corresponding to the Lagrangian \(L(q, \dot{q}; t)\) are not the standard Euler-Lagrange equations (17), but rather the “twisted Euler-Lagrange equations”

\[
\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^j} \Lambda^j_i = 0 ;
\]

for a derivation and discussion of these equations, see \[10\].

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\(^8\)This point of view was first presented in \[11, 17\] and is further discussed in \[14, 15, 16\].

\(^9\)Needless to say, the same is true \textit{a fortiori} in the case of a field Lagrangian \(L(u, u_x; x^1, \ldots, x^d)\) and the corresponding Euler-Lagrange equations \((\partial L/\partial u^a) - (d/dx^i)(\partial L/\partial u_x^a) = 0\): no conserved quantity is in general associated to a twisted symmetry of the Lagrangian. See \[10\] for a detailed discussion.
Remark. Albeit we decided to restrict in general to the Mechanics case, we would now like to mention that in the general field theoretical case, with fields $u^a$ and space-time variables $x^i$, we have several matrix functions $\Lambda_i$ (one for each space-time variable), subject to the compatibility conditions discussed in section 2 (which we assume are satisfied); introducing the notations $\pi^i_a := \left(\partial L/\partial u^a_i\right)$ for the momenta and $D_i$ for the total derivative with respect to the variable $x^i$, the resulting equations read

$$\frac{\partial L}{\partial u^a} - D_i \pi^i_a = - \left(\Lambda^T\right)_a^b \pi^i_b . \quad (19)$$

The reader is again referred to [10] for a derivation.

It is then a simple matter to check that the following statement (which is Theorem 9 from [10]) holds true:

**Proposition 1.** Let $\mathcal{L}$ be a first order field Lagrangian, admitting the vector field $X = \varphi^a(\partial/\partial u^a)$ as a $\mu$-symmetry for a certain form $\mu = \Lambda dt$. Then the vector $P$ of components $P^i = \varphi^i \pi^i_a$ defines a standard conservation law, $D_i P^i = 0$, for the flow of the associated $\mu$-Euler-Lagrange equations (19).

In the case of a Mechanical Lagrangian $\mathcal{L}(q, \dot{q}; t)$, the above Proposition 1 reduces to a statement about the existence of first integrals:

**Proposition 2.** Let $\mathcal{L}(q, \dot{q}; t)$ be a first order mechanical Lagrangian, admitting the evolutionary vector field $X = \varphi^i(\partial/\partial q^i)$ as a $\mu$-symmetry for a certain form $\mu = \Lambda dt$. Then the function $J = \varphi^i(\partial L/\partial \dot{q}^i)$ is a first integral for the flow of the associated $\mu$-Euler-Lagrange equations (18), i.e. $dJ/dt = 0$.

The reader is once again referred to [10] for applications and examples. See also the earlier paper [33], which started application of twisted symmetries to the study of variational problems.

4 Multiple twisted symmetries and reduction

As we have discussed in the previous Section 3, each twisted symmetry of the Lagrangian yields a first integral and hence allow for a reduction of the variational problem. If we have several symmetries, we can try to use them one after the other to reduce by stages the variational problem [21]; however, this will be effective only if the Lie algebra of vector fields generating the Lagrangian symmetries have a convenient structure. This corresponds to what happens in the

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10 And also for a discussion of how twisted symmetries induce “$\mu$-conservation laws” for the standard Euler-Lagrange equations, and correspondingly how standard symmetries induce the same kind of “$\mu$-conservation laws” for the twisted Euler-Lagrange equations.
usual application of Lie symmetries to differential equations, where in general only a solvable algebra can be fully used for reduction.\footnote{It should be mentioned that after reduction one could have extra symmetries beyond those inherited from the unreduced problem, see e.g. the discussion in \cite{36}; this remark \cite{3} was one of the motivations for the introduction of twisted symmetries, and on the other hand leads to considering solvable structures \cite{4, 40}. See \cite{5} for a recent discussion blending (twisted) symmetries and solvable structures in the reduction of ODEs.}

Needless to say, the same will hold in this case. However, the situation here is slightly more complex, and could appear much more complex if one is not aware of the basic mechanism at work, i.e. that twisted symmetries correspond – locally – to standard symmetries in a different reference frame. Note that as the Lie algebraic structure of sets of vector fields depends only on local properties, we can make full use of this feature in the present context.

We stress that the twisting should be \textit{the same for all vector fields}, i.e. we operate with the same matrix $\Lambda$ for the prolongation of different vector fields; this correspond to the fact that the associated gauge transformation is the same.

Let us denote the vector fields generating twisted symmetries as

$$X_a = \varphi_a (\partial/\partial q^i),$$

and their first prolongation as $Y_a$. We have

$$Y_a = X_a + \psi_a (\partial/\partial \dot{q}^i);$$

we recall that

$$\psi_a = D_t \varphi_a + \Lambda^i_j \varphi_a^j.$$

Let us determine the reference frame in which prolongations are just standard ones. Acting on component of vector fields by an invertible matrix function\footnote{We recall that in general $R$ can depend not only on the independent variable $t$, but in the dependent variables $q^i$ as well.} $R$, and writing $\varphi = R\xi$, $\psi = R\eta$, eq.(22) reads

$$R\eta = D_t (R\xi) + \Lambda R\xi,$$

which also reads

$$\eta = D_t \xi + \left[(R^{-1}D_t R) + (R^{-1}\Lambda R)\right] \xi.$$ 

Thus the components of the prolonged vector field satisfy the standard prolongation formula (for vertical vector fields) $\eta = D_t \xi$ if and only if $R$ and $\Lambda$ are related by

$$(R^{-1} D_t R) + (R^{-1} \Lambda R) = 0.$$ \hspace{1cm} (23)

Needless to say, this just expresses the request that the gauge transformation $R$ maps $\Lambda$ into the identically null matrix function, as the left hand side of (23) is the standard expression for a gauge transformation, see e.g. \cite{6, 12, 34, 35}. We thus have

$$R = \exp \left[- \int \Lambda \, dt \right].$$ \hspace{1cm} (24)
The commutator of vector fields $X_a, X_b$ as above is given by

$$[X_a, X_b] = \{\varphi_a, \varphi_b\}^i (\partial/\partial q^i),$$

where we have defined

$$\{\varphi_a, \varphi_b\}^i := \varphi_a^j \frac{\partial \varphi_b^i}{\partial q^j} - \varphi_b^j \frac{\partial \varphi_a^i}{\partial q^j}. \tag{25}$$

When we see this as twisted vector fields, i.e., vector fields on which the gauge transformation $R$ acted, we rewrite $\varphi = R\xi$ and hence

$$X_a = (R^i_j \xi^j) (\partial/\partial q^i).$$

In this way we get immediately

$$[X_a, X_b] = \{R\xi_a, R\xi_b\}^i (\partial/\partial q^i) := \{\xi_a, \xi_b\}_{(R)} (\partial/\partial q^i).$$

We can summarize this relation, with the notations introduced above, as

$$\{\varphi_a, \varphi_b\} = \{\xi_a, \xi_b\}_{(R)};$$

this also reads

$$\{\varphi_a, \varphi_b\}_{(R^{-1})} = \{\xi_a, \xi_b\}.$$

Now we note that the $\xi$ correspond to the reference frame in which prolongations are not twisted, i.e., $\Lambda = 0$, and the Euler-Lagrange equations are the standard ones. In this frame a solvable Lie algebra of vector fields allows for reduction by stages, and the structure of the Lie algebra is recovered by considering $\{\xi_a, \xi_b\}$.

Thus, if the twisted symmetries $X_a$ of $\mathcal{L}$, written in the form (20), are obtained by twisting the prolongation by $\Lambda$, one determines the corresponding $R$ and should then check that the vector of vector fields component $\varphi$ form a solvable Lie algebra with the bracket

$$\{\ldots\}_{(R^{-1})}. \tag{26}$$

If the maximal solvable algebra (under this bracket) of twisted symmetries has dimension $n$, then the system described by the $n$-dimensional Lagrangian $\mathcal{L}$ is integrable.

Finally, we would like to stress that we conducted our discussion locally; integrability is however usually of interest when is a global property, so that one should be able to patch together the analysis in different local charts to extend it over the whole manifold. In this sense the analysis in terms of the bracket (26) is more convenient – at least notationally – than simply passing to the gauge transformed frame (in which the twisted prolongation is mapped to a regular one), in that different gauge transformations and hence different $R$ would be used in different local charts.

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13By performing the corresponding transformation $q = R\chi$ on the dependent variables we would get rid of $R$, but here we want to briefly discuss how the algebraic relations between the $X_a$ are affected by this transformation. This is easily done by using the bracket $\{\ldots\}$ defined above, and a generalization to be defined in a moment.
5 Discussion and outlook

In this final section we collect some remarks on the subject discussed here as well as comments about desirable further developments.

(1) The point of view adopted here is to use as far as possible the result that locally twisted symmetries are standard ones described in a different (twisted) frame of reference, i.e. deformed by a gauge transformation. In this way, many of the results obtained for twisted symmetries appear more and less obvious, and focus should be shifted to global properties. Needless to say, this also applies to Integrability; we have focused on the local aspects in that the transition from local to global ones does not present any special feature in the case of twisted symmetries, except in some respect for what concerns the analysis of the Lie algebraic properties of sets of different twisted symmetries; for these we have suggested in Section 4 a way to analyze the situation in terms of a deformed bracket which takes into account the twisting in different local charts, i.e. under the different local gauge transformations mapping twisted prolongations into standard ones.

(2) The relation between twisted symmetries and gauge-transformed standard ones also suggests that one could proceed in a different way. That is, the problem here arises from the fact that in considering standard (partial or ordinary) derivatives, as common in Applied Mathematics, they transform in a different way when one changes the reference frame. The cure for this is well known in Physics, and consists in using covariant derivatives. One could thus consider equations in terms of covariant derivatives, and prolong vector fields acting in the phase manifold $M$ by considering their action on covariant (rather than standard) derivatives, i.e. by employing “covariant jet spaces”. We hope this approach can be implemented in future works.

(3) Working instead with standard derivatives, it is quite clear that the approach based on gauge transformations allows to easily produce integrable systems simply by starting from a known one and applying gauge transformations on it. The transformed systems will in general not be integrable in usual sense (that is, not pass the usual integrability tests), in particular not possess a suitable algebra of standard symmetries, but will – just by construction! – possess a suitable algebra of twisted symmetries and be integrable in the sense considered in the present paper. It goes without saying that albeit here we considered Lagrangian systems only, this remark applies to any kind of Integrable System.

(4) It is also rather clear that albeit we only discussed first order mechanical Lagrangians, the geometrical framework and hence the validity of the present approach are quite more general. In particular they also apply both to higher order Lagrangians and to the framework of Field Theory.

(5) Finally we recall that, as already mentioned above, the Hamiltonian aspects of this approach have received only limited attention, and only some preliminary results are at present available.
Appendix. A concrete example

In order to illustrate our discussion, we discuss a fully explicit (but slightly artificial) example, considering a mechanical Lagrangian in three degrees of freedom; we denote by $t$ the independent variable and by $(x, y, z)$ the dependent ones.

A.1 The Lagrangian and its twisted symmetries

We choose the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left( \dot{z} + (x^2 + y^2) z f ((x^2 + y^2)z) \right) \left( \dot{x}^2 + \dot{y}^2 \right) - g ((x^2 + y^2)z).$$  \hfill (A.1)

Here $f$ and $g$ are arbitrary smooth functions of their argument $\beta = [(x^2 + y^2)z]$, and we will assume $f \neq 0$ (in order to avoid discussing trivial cases). The case $g = 0$ is simple but not trivial, and the reader willing to consider a specially simple example at first can set $g = 0$ in all the following formulas.

Let us now consider the two vector fields

$$X = x \partial_y - y \partial_x, \quad Y = x \partial_x + y \partial_y - 2z \partial_z.$$  \hfill (A.2)

Any smooth function of $\beta$ is invariant under both of these, so in particular the potential part $g(\beta)$ of the Lagrangian $\mathcal{L}$, as well as the coefficient $[\beta f(\beta)]$, are surely invariant. Note moreover that $X$ and $Y$ commute, $[X, Y] = 0$.

As for the prolongations of $X$ and $Y$, we start by considering their standard prolongations, which are easily seen to be

$$X^{(1)} = X - \dot{y} \partial_z + \dot{x} \partial_y, \quad Y^{(1)} = Y + \dot{x} \partial_x + \dot{y} \partial_y - 2 \dot{z} \partial_z.$$ 

Applying these on the Lagrangian (A.1) we see easily that

$$X^{(1)} \cdot \mathcal{L} = 0, \quad Y^{(1)} \cdot \mathcal{L} = [(x^2 + y^2)(\dot{x}^2 + \dot{y}^2)] z f ((x^2 + y^2)z) \neq 0$$

(the last inequality depends on the assumption $f \neq 0$). It is easily checked that – as also guaranteed by a general theorem [36] given that the vector fields themselves commute – these standard prolongations commute, $[X^{(1)}, Y^{(1)}] = 0$.

Let us now consider twisted prolongation, with the twisting matrix

$$\Lambda = (x^2 + y^2) f((x^2 + y^2)z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

In this case we get, applying the general formulas,

$$X_A^{(1)} = X - \dot{y} \partial_z + \dot{x} \partial_y,$$

$$Y_A^{(1)} = Y + \dot{x} \partial_x + \dot{y} \partial_y - 2 (\dot{z} + (x^2 + y^2)\dot{z}) f((x^2 + y^2)z) \partial_z.$$ 

These remarks suggest directions for further developments. We hope some of the readers the present paper will contribute to these.
In this case, the twisted prolongations do still commute, \([X^{(1)}_\Lambda, Y^{(1)}_\Lambda] = 0\).

By applying these twisted prolongations to \(\mathcal{L}\), we get

\[ X^{(1)}_\Lambda \cdot \mathcal{L} = 0, \quad Y^{(1)}_\Lambda \cdot \mathcal{L} = 0; \]

we thus have a two-dimensional (abelian) algebra of twisted symmetries for the three-degrees-of-freedom Lagrangian \(\mathcal{L}\), and our results apply.

### A.2 Conservation laws

We want to check in particular that the conservation of quantities \(J_X\) and \(J_Y\) (associated to the vector fields \(X\) and \(Y\) respectively) under the twisted Euler-Lagrange equations of motion, granted by Proposition 2, holds in this case.

As easy to foresee in view of our choice for \(\Lambda\), the twisted Euler-Lagrange equations (18) are only slightly more complex that the standard ones (17), and actually differ from these only for the equation related to \(z\). It will be convenient to introduce a simplified notation, with

\[
F = f(x^2 + y^2)z, \quad G = g((x^2 + y^2)z).
\]

In this way the twisted Euler-Lagrange equations for \(\mathcal{L}\) turn out to be

\[
-2G_{\beta}xz - 2F\dot{x}y\dot{y}z - Fx(\dot{x}^2 - \dot{y}^2)z + \dot{x}(-Fx^2z - Fy^2z - \dot{z}) - Fx^2\dot{x}z \\
-\dot{F}y^2\dot{z} - F_{\beta}\rho^2 z(x(\dot{x}^2 - \dot{y}^2)z + x^2\dot{x}z + \dot{x}y(2\dot{y}z + y\dot{z})) - \dot{x}z = 0 \\
-2G_{\beta}yz + F\dot{x}\dot{y}y - 2F\dot{x}\dot{y}y - Fy^2z + \dot{y}(-Fx^2z - Fy^2z - \dot{z}) - Fx^2\dot{y}z \\
-\dot{F}y^2\dot{y}z - F_{\beta}\rho^2 z(-\dot{x}^2yz + 2x\dot{x}y\dot{z} + \dot{y}(y\dot{y}z + x^2\dot{z} + \dot{y}^2z)) - \dot{y}z = 0, \\
-\dot{x}\dot{x} - \dot{y}\dot{y} + (1/2)(2G_{\beta}\rho^2  + Fr^2\rho^2 + r^2\rho^2(F + F_{\beta}r^2z)) = 0.
\]

After some rearrangement, these provide

\[
\dot{x} = \frac{[\rho^2(Fr^2z + \dot{z})]^{-1}}{\left[(1/2)(2F^2r^4\rho^2 \dot{x}z + (-2G_{\beta} + Fr^2\rho^2z)(2xy^2z + \dot{x}z + y^2\dot{z})) + F(F_{\beta}r^6\rho^2 \dot{x}^2z + 2(2G_{\beta} + Fr^2\rho^2z) + \dot{x}(-2y\dot{y}z + x^2\dot{z} + y^2\dot{z}))\right]} \\
\dot{y} = \frac{[\rho^2(Fr^2z + \dot{z})]^{-1}}{\left[-\dot{x}(xy - y\dot{y}z)(-2G_{\beta} + Fr^2\rho^2 + F_{\beta}r^2\rho^2z) + r^2(1/2)\dot{y}(-2G_{\beta} + 2Fr^2\rho^2 + F_{\beta}r^2\rho^2z)(-Fr^2z - \dot{z})\right]} \\
\dot{z} = \frac{[\rho^2]^{-1}}{\left[(2F^2r^4\rho^2z - 2G_{\beta}(-2\dot{x}\dot{z} + x^2\dot{z} + y(-2\dot{y}z + y\dot{z})) + F_{\beta}r^6\rho^2z(2x\dot{x}z + 3x^2\dot{z} + y(2\dot{y}z + 3y\dot{z})) + F(F_{\beta}r^6\rho^2z^2 + 2(-G_{\beta}\rho^4z + \rho^2(x\dot{x}z + 2x^2\dot{z} + y(\dot{y}z + 2y\dot{z})))\right]}.
\]

As for the conserved quantities associated to \(X\) and \(Y\) according to Proposition 2, these are respectively

\[
J_X = x\dot{y}(Fr^2z + \dot{z}) - y\dot{x}(Fr^2z + \dot{z}), \\
J_Y = x\dot{x}(Fr^2z + \dot{z}) + y\dot{y}(Fr^2z + \dot{z}) - \rho^2z.
\]
Their time derivatives are respectively
\[ D_t(J_X) = -2F \dot{x} \dot{y} \dot{z} + 2F x^2 \dot{x} \dot{y} \dot{z} - 2F x y \dot{y} \dot{z} + 2F x y \dot{z} - F x^2 \dot{y} \dot{z} \]
\[ + \ddot{x}(-F x^2 \dot{y} \dot{z} - F y^3 \dot{z} - y \dot{z}) + F x^2 (2 \dot{x} \dot{z} + x^2 \dot{z} + y(2 \dot{y} \dot{z} + y \dot{z})) + (x \dot{x} + y \dot{y}) \dot{z}; \]
\[ D_t(J_Y) = 3F x^2 \dot{x}^2 z + F \dot{x}^2 y^2 z + 4F x \dot{x} \dot{y} \dot{z} + F x^2 \dot{y}^2 z + 3F y^2 \dot{y}^2 z \]
\[ + F x^3 \dot{z} + F x \dot{y} \dot{z} + F x \dot{y} \dot{z} + F y \dot{y} \dot{z} \]
\[ + F \dot{z}(F x^3 \dot{z} - 2 \dot{x} \dot{z} + F x y^2 z + x \dot{z}) + F \dot{y}(F x^2 \dot{y} + F y^3 \dot{z} - 2 \dot{y} \dot{z} + y \dot{z}) \]
\[ + F \dot{y}^2(2 \dot{x} \dot{z} + x^2 \dot{z} + y(2 \dot{y} \dot{z} + y \dot{z}))) + (x \dot{x} + y \dot{y}) \dot{z}. \]

Inserting (A.3) into these, one checks that indeed
\[ D_t(J_X) = 0 = D_t(J_Y). \]

We also note that while \( J_X \) is also conserved under the standard Euler-Lagrange equations, for \( J_Y \) we would get a non-zero time derivative, given explicitly by \( Fr^2 \rho^2 z \).

**A.3 Gauge transformation**

Finally, we should check that a gauge transformation transforms our problem with twisted symmetries into one with standard symmetries.

The form of the Lagrangian \( L \), which we rewrite using the simplified notation \( \beta = [(x^2 + y^2)z] \) as
\[ L = (1/2) \left( \dot{z} + \beta f(\beta) \right)(\dot{x}^2 + \dot{y}^2) - g(\beta), \]
suggests that it can be written as
\[ L = (1/2) \left[ \nabla_t z \right] \left[ (\nabla_t x)^2 + (\nabla_t y)^2 \right] - g(\beta), \]
in terms of covariant time derivatives, defined as
\[ \nabla_t x = dx/dt, \quad \nabla_t y = dy/dt, \quad \nabla_t z = (dz/dt) + \beta f(\beta). \]

In other words, the operator \( \nabla_t \) acts on the vector \((x, y, z)^T\) by
\[ \nabla_t = \frac{d}{dt} + \Lambda \]
with \( \Lambda \) as given above.

The covariant \( t \)-derivative of \( z \) can be mapped into a standard \( t \) derivative by a change of variables; as we need
\[ \nabla_t z = z_t + \beta f(\beta) = \zeta_t, \]
this yields
\[ z = \sigma \zeta \text{ with } \sigma = \exp \left[ - \int \beta f(\beta) dt \right]. \]
Needless to say, we do not need to change variables for $x$ and $y$; we will introduce new variables $\xi = x$ and $\eta = y$ just for the sake of stressing the different set of variables. In these variables, we have

$$\mathcal{L} = \frac{1}{2} \dot{\xi}^2 + \dot{\eta}^2 - g(\beta); \quad (A.4)$$

Note that writing $\beta$ in terms of the new variables is in general a nontrivial task, as the change of variables depends itself on $\beta$.

As the gauge transformation we considered – and hence the change of variables needed to set the Lagrangian in the form (A.4) – do not act on the $(x, y)$ variables, we can forget about the vector field $X$ and concentrate on $Y$.

In the new variables the basic differential operators are written as $\partial_x = (\partial\xi/\partial x)\partial_\xi + (\partial\eta/\partial x)\partial_\eta + (\partial\zeta/\partial x)\partial_\zeta$, and so on. This yields explicitly

$$\begin{align*}
\partial_x &= \partial_\xi - \frac{z}{\sigma^2} \sigma z \partial_\zeta, \\
\partial_y &= \partial_\eta - \frac{z}{\sigma^2} \sigma z \partial_\zeta, \\
\partial_z &= \left(\frac{1}{\sigma} - \frac{z}{\sigma^2} \sigma z\right) \partial_\zeta.
\end{align*}$$

Note however that $\sigma$ depends on $(x, y, z)$ only through the function $\beta$; thus we have $\sigma_x = \sigma \beta_x$, and so on. It follows from this that

$$X = -\eta \partial_\xi + \xi \partial_\eta; \quad Y = \xi \partial_\xi + \eta \partial_\eta - 2\lambda(\beta) \zeta \partial_\zeta;$$

Here the exact expression of $\lambda(\beta)$ is inessential, but for the sake of completeness we mention that, writing $A = \beta f(\beta)$, it is given by

$$\lambda(\beta) = \frac{1}{\sigma} \left(1 + \beta \int \frac{dA/d\beta}{dt} \right).$$

According to our general discussion, there should be a vector field $W$, obtained via a gauge transformation from $Y$, which leaves the Lagrangian $\mathcal{L}$ invariant. This is immediately seen in the new variables, i.e. with the representation (A.4) for $\mathcal{L}$. The gauge transformation is simply $\Gamma = [1/\lambda(\beta)] M$, so that

$$W = \xi \partial_\xi + \eta \partial_\eta - 2\zeta \partial_\zeta;$$

Needless to say, this is a rescaling in the new variables, and the invariance of $\mathcal{L}$ under this, or more precisely its standard prolongation, is immediate.

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