All face 2-colorable d-angulations are Grünbaum colorable

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Abstract

A d-angulation of a surface is an embedding of a 3-connected graph on that surface that divides it into d-gonal faces. A d-angulation is said to be Grünbaum colorable if its edges can be d-colored so that every face uses all d colors. Up to now, the concept of Grünbaum coloring has been related only to triangulations (d = 3), but in this note, this concept is generalized for an arbitrary face size d ≥ 3. It is shown that the face 2-colorability of a d-angulation P implies the Grünbaum colorability of P. Some wide classes of triangulations have turned out to be face 2-colorable.

Keywords: coloring; d-angulation of surface.

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1 Terminology and notation

In this note we consider only simple graphs—that is, graphs without loops or parallel edges, and we embed the graphs only on closed surfaces—that is, surfaces without boundaries, such as a sphere. We mainly follow the standard terminology and notation of graph theory (10).

A d-angulation P of a surface means a d-gonal embedding of a 3-connected graph G = G(P) on that surface—that is, an embedding each face of which is bounded by a simple cycle of G with fixed length d ≥ 3. Combinatorially, P is defined by the triple of sets V(P), E(P), and F(P) of vertices, edges, and faces, respectively. The dual graph G*(P) is defined to be the graph the vertex sets of which corresponds to F(P) and in which two vertices are adjacent if and only if the corresponding faces of P are adjacent. Notice that G*(P) is d-regular—that is, the degree of each vertex is equal to d.
In this note we only consider $d$-angulations $P$ whose dual graphs are simple graphs, and therefore we will suppose that $G(P)$ is 3-connected, which ensures the simplicity of $G^*(P)$ in the following two important cases:

(i) $d = 3$,
(ii) the carrier surface is a sphere.

In case ii, by Steinitz’s Theorem, every 3-connected planar graph $G$ is the 1-skeleton of a convex polytope (in 3-space) with boundary complex $P$, and the dual graph $G^*(P)$ appears to be the 1-skeleton of the dual polytope with boundary complex $P^*$.

A vertex, edge, or face $k$-coloring of a $d$-angulation $P$ is a surjection of the set $V(P)$, $E(P)$, or $F(P)$ onto a set of $k$ distinct colors such that the images of adjacent vertices, edges, or faces are different, respectively. Especially, any edge 3-coloring of a 3-regular graph is called a Tait coloring. The vertex, edge, and face chromatic numbers of $P$ are defined to be the smallest values of $k$ possible to obtain corresponding $k$-colorings, and are denoted by $\chi(P)$, $\chi'(P)$, and $\chi''(P)$, respectively. The numbers $\chi(P)$ and $\chi'(P)$ are also called the vertex and edge chromatic numbers of the graph $G(P)$ itself and denoted as $\chi(G(P))$ and $\chi'(G(P))$, respectively.

Clearly, any face $k$-coloring of an arbitrary $d$-angulation $P$ corresponds to some vertex $k$-coloring of the dual graph $G^*(P)$, and conversely, whence

$$\chi''(P) = \chi(G^*(P)).$$

Interestingly, since $G^*(P)$ is $d$-regular, it follows that

$$\chi(G^*(P)) \in \{2, 3, \ldots, d, d+1\},$$

and that

$$\chi'(G^*(P)) \in \{3, \ldots, d, d+1\}$$

by Vizing’s Theorem [18].

2 Grünbaum colorings

A Grünbaum coloring is a coloring of the edges of a $d$-angulation $P$ with $d$ colors such that for each face $f$ all $d$ colors occur at the edges incident to $f$. Up to now, the concept of Grünbaum coloring has been related only to triangulations—that is, the case $d = 3$, but in this note we will generalize this concept for an arbitrary face size $d \geq 3$. If $T$ is a triangulation, then $\chi'(G^*(T)) \in \{3, 4\}$ by Vizing’s Theorem. The equality $\chi'(G^*(T)) = 3$ means that $G^*(T)$ is Tait colorable, which, in the dual form, means that $T$ is Grünbaum colorable.
Conjecture 1 (Grünbaum [9], 1969). Every triangulation $T$ of an orientable surface is Grünbaum colorable—that is, $\chi'(G^*(T)) = 3$.

Conjecture 1 stood for 40 years, until Kochol [11] constructed infinite families of counterexamples on orientable surfaces with genus $g$ for all $g \geq 5$. Here we put forward another conjecture about triangulations by strengthening the vacuous restriction $\chi''(T) \leq 4$ (which obviously holds for any $T$) to the restriction $\chi''(T) \leq 3$, but without restricting the surface’s orientability class:

Conjecture 2. Every triangulation $T$ of (a) an orientable or (b) nonorientable surface with $\chi''(T) \leq 3$ is Grünbaum colorable.

In Section 3 we establish (Theorem 2) that for Grünbaum colorability of a $d$-angulation $P$ (that is, for the equality $\chi'(G^*(P)) = d$ to hold), it suffices that $\chi''(P) = 2$, without the orientability restriction. In Sections 4 and 5 we establish the face 2-colorability in some known, and quite wide, classes of triangulations.

3 Key Theorem

Let $P$ be a $d$-angulation of an orientable or nonorientable surface (whose dual graph is a simple graph). Since the dual graph $G^*(P)$ is $d$-regular, the following lemma is obvious.

Lemma 1. In order for the equality $\chi'(G^*(P)) = d$ to hold, it is necessary and sufficient that the graph $G^*(P)$ be 1-factorable—that is, be the sum of $d$ one-factors.

In a classical article, König [12] (also see [14]) proved that each bipartite $d$-regular graph expands to the sum of $d$ one-factors. Since a graph is bipartite if and only if it is vertex 2-colorable, we get the following reformulation of König’s Theorem:

Theorem 1 (König). If $\chi(G^*(P)) = 2$, then $G^*(P)$ is 1-factorable.

By a combination of Lemma 1 and Theorem 1, we obtain our key theorem which states that each face 2-colorable $d$-angulation of an orientable or nonorientable surface is Grünbaum colorable:

Theorem 2 (Key Theorem). If $\chi(G^*(P)) = 2$, then $\chi'(G^*(P)) = d$.

Dual formulation: If $\chi''(P) = 2$, then $P$ is Grünbaum colorable.

As a particular case of Theorem 2, when $d = 3$, we can state that Conjecture 1 certainly holds for all face 2-colorable triangulations of orientable and nonorientable surfaces. Notice that in Theorem 2 the face chromaticity condition is only minimally strengthened in comparison to that in Conjecture 2.

Conjecture 1 in full generality is obviously false if extended to the nonorientable case. The best known counterexample is provided by the minimal triangulation $T_{min}$...
of the projective plane by the complete 6-graph $G = K_6$. In this case, $G^*(T_{\text{min}})$ turns out to be the Petersen Graph \cite{15} which cannot be decomposed into the sum of three 1-factors (see \cite{10}, \cite{14}, \cite{15}) and by Lemma 1 has edge chromatic number at least 4. (This number is in fact equal to 4, which, by the way, easily implies the non-Hamiltonicity of the Petersen Graph—these are excellent creative exercises for a college course on Discrete Mathematics!)

4 Triangulations by complete graphs

In this section, we establish the existence of Grünbaum colorable triangulations on orientable and nonorientable surfaces by complete graphs $K_n$ for at least half of the residue classes in the spectrum of possible values of $n$.

We begin with the orientable case, in which there exists a triangulation by $K_n$ if and only if $n \equiv 0, 3, 4$ or 7 (mod 12); see \cite{16}. Grannell, Griggs, and Širáň \cite{6} noticed that, when $n \equiv 0$ or 4 (mod 12), such triangulations are not face 2-colorable because for face 2-colorability it is necessary that all vertex degrees should be even, that is, $n$ should be odd. Furthermore, they established that the orientable triangulations constructed by Ringel \cite{16} for all $n \equiv 3$ (mod 12) are face 2-colorable; however, since $K_3$ is not 3-connected, we have to enforce $n \neq 3$ (see Section 1). Finally, they established that there exists a face 2-colorable triangulation for each $n \equiv 7$ (mod 12) among the orientable triangulations constructed by Youngs \cite{19}. We summarize these results in the following theorem.

**Theorem 3 (Ringel \cite{16}; Youngs \cite{19}; Grannell, Griggs, Širáň \cite{6}).** There exists a face 2-colorable triangulation of an orientable surface by the complete graph $K_n$ if and only if $n \equiv 3$ or 7 (mod 12), $n \neq 3$.

If one triangulation is face 2-colorable and the other is not, the two triangulations are certainly nonisomorphic—that is, there is no bijection between their vertex sets that extends to a homeomorphism between the surfaces carrying the triangulations.

Historically, the first examples of pairs of nonisomorphic orientable triangulations with the same complete graph were constructed \cite{19} in 1970. In those examples, the non-isomorphism follows immediately from the fact that one of the triangulations is face 2-colorable while the other is not; see review \cite{3}. After a quarter of a century, in \cite{13}, there was constructed an example of more than two nonisomorphic orientable triangulations with the same complete graph, namely: there were constructed three such triangulations, only one of which is face 2-colorable. In 2000, it was shown \cite{2} (also see \cite{3}) that the number of nonisomorphic orientable triangulations with graph $K_n$ actually grows very rapidly as $n \to \infty$ even within the class of face 2-colorable triangulations; for instance, when $n \equiv 7$ or 19 (mod 36), that number is at least $2n^2/54 + o(n^2)$.

The following corollary can be proved by a combination of Theorems 3 and 2.
Corollary 1. For each \( n \equiv 3 \) or \( 7 \) (mod 12), \( n \neq 3 \), there exists a Grünbaum colorable orientable triangulation by the complete graph \( K_n \).

To turn to the nonorientable case, recall [16] that \( K_n \) triangulates a nonorientable surface if and only if \( n \equiv 0 \) or \( 1 \) (mod 3), \( n \geq 6 \) and \( n \neq 7 \). Thus, under these conditions, \( n \) is odd if and only if \( n \equiv 1 \) or \( 3 \) (mod 6), \( n \geq 9 \). For all these values of \( n \), face 2-colorable triangulations of the corresponding nonorientable surface by the graph \( K_n \) are constructed in [16] and [8].

Theorem 4 (Ringel [16]; Grannell, Korzhik [8]). There exists a face 2-colorable triangulation of a nonorientable surface by the complete graph \( K_n \) if and only if \( n \equiv 1 \) or \( 3 \) (mod 6), \( n \geq 9 \).

The following corollary can be proved by a combination of Theorems 4 and 2.

Corollary 2. For each \( n \equiv 1 \) or \( 3 \) (mod 6), \( n \geq 9 \), there exists a Grünbaum colorable nonorientable triangulation by the complete graph \( K_n \).

Theorems 3 and 4 guarantee that we have not missed any face 2-colorable triangulations when applying Theorem 2 for obtaining Corollaries 1 and 2 (respectively).

5 Triangulations by tripartite graphs

Firstly, we notice that the existence of an orientable triangulation by the complete tripartite graph \( K_{n,n,n} \) was established by Ringel and Youngs [17] for each \( n \). Secondly, the face 2-colorability of each such triangulation was established by Grannell, Griggs, and Knor [4] (also see [3]). A combination of these two results with Theorem 2 leads to the following statement: For each \( n \geq 2 \), all triangulations of the corresponding orientable surface by the complete tripartite graph \( K_{n,n,n} \) are Grünbaum colorable. However, as observed by Archdeacon [1], it is very easy to prove this fact directly, even without using the completeness or orientability conditions: if the vertex parts are \( A, B, C \), then color the edges between \( A \) and \( B \) red, those between \( B \) and \( C \) blue, and those between \( A \) and \( C \) green, and we are done!

At first sight, the statement of the preceding paragraph may seem to be subjectless; however, as shown in [5] (see also [3]) in the case \( n \) is prime, there exist at least \((n - 2)!/(6n)\) nonisomorphic orientable triangulations by \( K_{n,n,n} \). Furthermore, [7] provides improved bounds on the number of such triangulations; for instance, when \( n \equiv 6 \) or \( 30 \) (mod 36), there exist at least \( n^{2^2}/144-o(n^2) \) nonisomorphic orientable triangulations by \( K_{n,n,n} \).

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