Invariant Surfaces for Toric Type Foliations in Dimension Three

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Abstract

In this paper we show that every toric type foliation on \( (\mathbb{C}^3, 0) \), without saddle-nodes, has invariant surface. The difficulty concentrates at the compact dicritical components of the exceptional divisor of the (combinatorial) reduction of singularities. These components are naturally endowed with a structure of projective toric surface. This allows us to enlarge the concept of partial separatrix in a consistent way and, finally, to detect the existence of invariant surface. The result of Ortiz-Rosales-Voronin, about the distribution of invariant curves in dimension two, is a key argument in our proof.

1 Introduction

The problem of existence of invariant hypersurfaces for a holomorphic codimension one foliation is a leitmotif in the theory of holomorphic singular foliations, coming from an initial question of René Thom. The main result in this paper is a contribution to this problem, that we state as follows:

“Every torifiable complex hyperbolic foliation on \( (\mathbb{C}^3, 0) \) has an invariant surface”.

A foliation is torifiable (of toric type) when it admits a combinatorial reduction of singularities, with respect to some coordinate system. The expression “complex hyperbolic” means that we can not extract saddle-nodes from the foliation. Let us remark that this type of foliations may be dicritical, in the sense that there are some generically transversal irreducible components of the exceptional divisor after reduction of singularities.

The existence of invariant hypersurface has a positive answer in the non-dicritical situation. The prove is due to Camacho-Sad in the bidimensional case [2], to Cano-Cerveau in the three-dimensional case [6] and to Cano-Mattei in general ambient dimension [8]. In contrast to what it happens in dimension two, there are dicritical examples of codimension one foliations in dimension three without invariant surface; the first family of such examples was given by Jouanolou [13].

In order to prove the existence of invariant surface for a dicritical foliation on \( (\mathbb{C}^3, 0) \), it is essential to have “good properties” for the restriction of the foliation to compact dicritical components after reduction of singularities. In the context of toric type foliations, we see that the compact components of the exceptional divisor are projective toric surfaces, in the sense of Toric Geometry, endowed in a natural way of a normal crossings divisor, compatible with the ambient divisor. In a previous work [15], we have proved that a toric type foliation \( \mathcal{G} \) on a projective toric surface \( S \), with the associated divisor \( D_S \), satisfies that:

“Every isolated invariant branch \( (\Gamma, p) \) extends to a global curve \( Y \subset S \); moreover, all the branches of \( Y \) at the points of \( Y \cap D_S \) are isolated.”

In a general way, if we have this “prolongation property for isolated invariant branches”, we can extend the argument of Cano-Cerveau in [6] to prove the existence of invariant surface, provided we have at least one non-corner type (trace type) simple singular point, after reduction of singularities. The details of this argument may be found in Subsection 3.2.
Now, it would be enough to find a trace type singular point after reduction of singularities, in the toric type context. Such a point appears if and only if there is at least one invariant component in the exceptional divisor, as we show in Section 4. In the proof of this result, we invoke a refined version of Camacho-Sad’s theorem proved by Ortiz-Rosales-Voronin in [10]. The remaining case corresponds to toric type foliations admitting reduction of singularities without invariant components. In this situation only blowing-ups centered in curves are allowed, in an étale way over an initial one, and the existence of invariant surface follows straightforward.

2 Generalities on Foliated Spaces

We introduce basic definitions and results concerning the theory of holomorphic singular foliations. These contents can be essentially found at [3].

2.1 Foliations

Let us recall that a nonsingular complex analytic space $M$ of dimension $n$ is a $\mathbb{C}$-ringed space $M = (\mathcal{M}, \mathcal{O}_M)$ in local $\mathbb{C}$-algebras of functions, covered by open subsets isomorphic to open subsets of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. Let $\Omega^1_M$ be the sheaf of germs of holomorphic one-forms on $M$. A codimension one holomorphic singular foliation $\mathcal{F}$ on $M$ (short, a foliation on $M$) is an integrable and invertible coherent $\mathcal{O}_M$-submodule $\mathcal{F} \subset \Omega^1_M$ such that the quotient $\Omega^1_M/\mathcal{F}$ is a torsion-free $\mathcal{O}_M$-module. The foliation $\mathcal{F}$ is locally generated at each point $p \in M$ by a holomorphic one-form $\omega \in \Omega^1_{M,p}$ satisfying $\omega \wedge d\omega = 0$, that we write in local coordinates as

$$\omega = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n, \quad f_i \in \mathcal{O}_{M,p}$$

where the coefficients $f_i$ have no common factors. The order $\nu_p(\mathcal{F})$ of $\mathcal{F}$ at $p$ is defined by

$$\nu_p(\mathcal{F}) = \nu_p(f_1, f_2, \ldots, f_n) = \min\{\nu_p(f_i) \mid i = 1, 2, \ldots, n\},$$

where $\nu_p(f_i)$ is the order at $p$ of the coefficient $f_i$. The singular locus $\text{Sing}(\mathcal{F})$ is the closed analytic subset of $M$ locally defined by

$$\text{Sing}(\mathcal{F}) = (f_1 = f_2 = \cdots = f_n = 0).$$

Note that $p \in \text{Sing}(\mathcal{F})$ if and only if $\nu_p(\mathcal{F}) > 0$. Since the coefficients of $\omega$ have no common factors, the codimension of $\text{Sing}(\mathcal{F})$ in $M$ is at least two.

Denote by $\Theta_M$ the sheaf of germs of holomorphic vector fields on $M$. Given a point $p \in M$, a germ $\xi \in \Theta_{M,p}$ is called tangent to $\mathcal{F}$ at $p$ when $\omega(\xi) = 0$, where $\omega$ is a local generator of $\mathcal{F}$. The dimensional type $\tau_p(\mathcal{F})$ of $\mathcal{F}$ at $p$ is given by

$$\tau_p(\mathcal{F}) = n - k \geq 1,$$

where $k$ is the dimension of the $\mathbb{C}$-vector space spanned by the vectors $\xi(p)$, with $\xi$ a tangent germ of vector field. As a result of the rectification theorem of vector fields, there are local coordinates $x = (x_1, x_2, \ldots, x_n)$ such that $\mathcal{F}$ is locally generated by a one-form $\omega$ of the type

$$\omega = \sum_{i=1}^\tau f_i(x_1, x_2, \ldots, x_r)dx_r, \quad \tau = \tau_p(\mathcal{F}).$$

Such a coordinate system is called minimal. The foliation is locally an analytic cylinder over a codimension one foliation on a space of dimension $\tau$. Note that $\tau = 1$ if and only if $p \notin \text{Sing}(\mathcal{F})$.

Let us consider a holomorphic morphism $\phi : N \to M$, where $N$ is a nonsingular connected complex analytic space. The morphism $\phi$ is called invariant for $\mathcal{F}$ when $\phi^*\omega$ is identically zero for the local generators $\omega$ of $\mathcal{F}$. Otherwise, we say that $\phi$ is generically transversal to $\mathcal{F}$; in this case, there is an induced foliation $\phi^{-1}\mathcal{F}$ on $N$, locally defined by the pull-backs $\phi^*\omega$, after dividing by the common factors of the coefficients.

A closed analytic subspace $Y \subset M$ is called invariant for $\mathcal{F}$ at $p \in Y$ if each morphism $\phi : (\mathbb{C}, 0) \to (M, p)$ factoring through $(Y, p)$ is invariant. We say that $Y$ is invariant for $\mathcal{F}$ when the property holds at each point $p \in Y$. Being invariant at a point is an open and closed property on $Y$. Hence, an irreducible subspace $Y$ invariant at a point is invariant.
Remark 2.1. Every subspace $Y \subset \text{Sing}(\mathcal{F})$ is invariant for $\mathcal{F}$.

Recall that a hypersurface $H$ of $M$ is locally given at $p \in H$ by a reduced equation $f = 0$, where $f \in \mathcal{O}_{\mathcal{H},p}$. An analytic subspace $Y \subset M$ of dimension $k$ is a complete intersection if it is given by intersection of $n-k$ hypersurfaces $H_i \subset M$. We know that a complete intersection $Y$ is invariant for $\mathcal{F}$ at a point $p \in Y$ if and only if there is a local generator $\omega$ of $\mathcal{F}$ such that $\omega \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_k \mid_Y = 0$, where $f_i = 0$ are reduced local equations of $H_i$, for $i = 1, 2, \ldots, n-k$.

Remark 2.2. Frobenius’ theorem implies that there is a unique germ of invariant hypersurface of $\mathcal{F}$ through each nonsingular point.

Let us recall the classical definitions of presimple and simple singularities in the bidimensional case studied by Seidenberg in [17] (for more details, see [2]). Let $\mathcal{G}$ be a foliation on $(\mathbb{C}^2,0)$ generated by a holomorphic one-form $\omega = f_1 dx_1 + f_2 dx_2$ and let us assume that the origin $0 \in \text{Sing}(\mathcal{G})$. Consider the tangent vector field

$$\xi = f_2 \partial/\partial x_1 - f_1 \partial/\partial x_2,$$

and denote by $L_\xi$ its linear part. The origin is a presimple singularity for $\mathcal{G}$ if $(\lambda, \mu) \neq (0,0)$, where $\lambda$ and $\mu$ are the eigenvalues of $L_\xi$. We say that it is a simple singularity for $\mathcal{G}$ if we also have $\lambda/\mu \notin \mathbb{Q}_{>0}$. A simple singularity such that $\lambda \mu = 0$ is called a saddle-node. In section 2.3 we precise and extend these notions to arbitrary dimension.

Next definitions can be found in [2].

**Definition 2.1.** A foliation $\mathcal{F}$ is complex hyperbolic at $p \in M$ if there is no generically transversal morphism $\phi : (\mathbb{C}^2,0) \to (M,p)$ with 0 being a saddle-node for $\phi^{-1}\mathcal{F}$. The foliation $\mathcal{F}$ is complex hyperbolic (for short, CH) if the property holds at each $p \in M$.

**Definition 2.2.** A foliation $\mathcal{F}$ is dicritical at $p \in M$ when there is a generically transversal morphism $\phi : (\mathbb{C}^2,0) \to (M,p)$, such that $\phi^{-1}\mathcal{F} = (dx = 0)$ and $\phi(y = 0)$ is invariant for $\mathcal{F}$. The foliation $\mathcal{F}$ is non-dicritical if it is not dicritical at each $p \in M$.

Remark 2.3. Note that $y = 0$ is a curve, hence $\phi(y = 0)$ is either the point $p$ or a germ at $p$ of analytic curve. Moreover, we can prove that $\phi(y = 0) \subset \text{Sing}(\mathcal{F})$.

### 2.2 Foliated Spaces

Let $M$ be an $n$-dimensional nonsingular complex analytic space and let $\{E_i\}_{i \in I}$ be a finite family of connected nonsingular hypersurfaces. The union $E = \cup_{i \in I} E_i$ is called a normal crossings divisor of $M$ if for each point $p \in M$, there is a local coordinate system $x = (x_1, x_2, \ldots, x_n)$ such that

$$E \subset (x_1 x_2 \cdots x_n = 0).$$

Such coordinate systems are called adapted to $E$.

The non-invariant irreducible components of $E$ are also called dicritical components. We write the index set as $I = I_{\text{inv}} \cup I_{\text{dic}}$, where $I_{\text{inv}}$ corresponds to the invariant components and $I_{\text{dic}}$ corresponds to the dicritical ones. We also denote $E_{\text{inv}} = \cup_{i \in I_{\text{inv}}} E_i$ and $E_{\text{dic}} = \cup_{i \in I_{\text{dic}}} E_i$. Given an irreducible analytic subspace $Y \subset M$, we denote by $e_Y(E)$ the number of irreducible components of $E$ containing $Y$.

**Remark 2.4.** Note that $e_p(E_{\text{inv}}) \leq \tau_p(\mathcal{F})$ and $e_p(E_{\text{inv}}) \leq \nu_p(\mathcal{F}) + 1$, for every $p \in M$.

Let $\mathcal{F}$ be a foliation on $M$. We say that $\mathcal{F}$ and $E$ have normal crossings at $p \in M$ when $p \notin \text{Sing}(\mathcal{F})$ and $E \cup H$ is a local normal crossings divisor, where $(H,p)$ is the only germ of invariant hypersurface through $p$. The adapted singular locus $\text{Sing}(\mathcal{F},E)$ is defined by

$$\text{Sing}(\mathcal{F},E) = \{p \in M; \mathcal{F} \text{ and } E \text{ have not normal crossings at } p\}.$$
Proof. We work locally at a point \( p \in M \). If \( \nu_p(E_{\text{inv}}) = 0 \), we have \( \text{Sing}(\mathcal{F}, E) = \text{Sing}(\mathcal{F}) \) and we are done. In a general way, we consider local coordinates such that \( E_{\text{dic}} = \bigcup_{j \in A_{\text{dic}}} (x_j = 0) \), where \( A_{\text{dic}} \subseteq \{1, 2, \ldots, n\} \). Let \( \omega = f_1dx_1 + f_2dx_2 + \cdots + f_n dx_n \) be a local generator of \( \mathcal{F} \). Define the closed analytic subsets

\[
Z_j = (x_j = 0; \ j \in J) \cap (f_j = 0; \ j \in \{1, 2, \ldots, n\} \setminus J), \quad J \subseteq A_{\text{dic}}.
\]

It is easy to check that \( \text{Sing}(\mathcal{F}, E) = \bigcup Z_j \), locally at \( p \).

Note that \( Z_0 = \text{Sing}(\mathcal{F}) \), hence \( \text{codim}(Z_0) \geq 2 \). We also have \( \text{codim}(Z_j) \geq 2 \) when \( \# J \geq 2 \). Then, it is enough to show that \( Z_j \) has codimension at least two, for every \( j \in A_{\text{dic}} \). Let us assume that there is \( j \in A_{\text{dic}} \) such that \( Z_j = (x_j = 0) \). Then we have

\[
f_k = x_j \tilde{f}_k, \quad \tilde{f}_k \in \mathcal{O}_{M, p}, \ k \in \{1, 2, \ldots, n\} \setminus \{j\}
\]

and hence \( (x_j = 0) \) is invariant. This is a contradiction.

We are interested in considering not only a foliation \( \mathcal{F} \) but a pair \((\mathcal{F}, E)\), where \( E \subseteq M \) is a normal crossings divisor. In order to do this, it is useful to introduce the logarithmic concept of \( E \)-foliation.

Let us consider the sheaf of germs of logarithmic one-forms along \( E \), which is denoted by \( \Omega^1_{\mathcal{F}}(\log E) \). An \( E \)-foliation \( \mathcal{L} \) on \( M \) is an integrable and invertible coherent \( \mathcal{O}_M \)-submodule \( \mathcal{L} \subseteq \Omega^1_{\mathcal{F}}(\log E) \), such that the quotient \( \Omega^1_{\mathcal{F}}(\log E)/\mathcal{L} \) is torsion-free. The \( E \)-foliation \( \mathcal{L} \) is locally generated at each point \( p \in M \) by an integrable logarithmic one-form \( \eta \in \Omega^1_{\mathcal{F}, p}(\log E) \), that we write in local coordinates adapted to \( E \) as

\[
\eta = \sum_{j \in A} a_j \frac{dx_j}{x_j} + \sum_{j \in B} a_j dx_j, \quad E = \bigcup_{j \in A} (x_j = 0); \quad A \cup B = \{1, 2, \ldots, n\},
\]

where the coefficients \( a_j \in \mathcal{O}_{M, p} \) have no common factors. The order \( \nu_p(\mathcal{L}, E) \) of \( \mathcal{L} \) at \( p \) is \( \nu_p(\mathcal{L}, E) = \nu_p(a_1, a_2, \ldots, a_n) \). Let \( \text{Fol}(M, E) \) and \( \text{Fol}(M) \) be respectively the sets of \( E \)-foliations and foliations on \( M \). There is a bijection

\[
\text{Fol}(M) = \text{Fol}(M, \emptyset) \rightarrow \text{Fol}(M, E), \quad \mathcal{F} \mapsto \mathcal{L}_\mathcal{F},
\]

defined in terms of local generators at \( p \in M \) by the relation \( \omega = (\prod_{j \in A_{\text{inv}}} x_j)^{-1} \eta_j \), where \( E_{\text{inv}} = \bigcup_{j \in A_{\text{inv}}} (x_j = 0) \) is the invariant part of \( E \) with respect to the foliation \( \mathcal{F} \). A local generator \( \eta \) of \( \mathcal{L}_\mathcal{F} \) is also called a local generator of \( \mathcal{F} \) adapted to \( E \). We define the order \( \nu_p(\mathcal{F}, E) \) of \( \mathcal{F} \) adapted to \( E \) by the equality

\[
\nu_p(\mathcal{F}, E) = \nu_p(\mathcal{L}_\mathcal{F}, E).
\]

Observe that \( \nu_p(\mathcal{F}) = \nu_p(\mathcal{F}, E) + \nu_p(E_{\text{inv}}) - 1 \).

**Remark 2.6.** An irreducible component \( x_j = 0 \) of \( E \) is a dicritical component if and only if there is \( \tilde{a}_j \in \mathcal{O}_{M, p} \) such that \( a_j = x_j \tilde{a}_j \).

**Definition 2.3.** An ambient space \( \mathcal{M} = (M, E) \) is a pair consisting of a nonsingular complex analytic space \( M \) and a normal crossings divisor \( E \subseteq M \). A foliated space \( (\mathcal{M}, \mathcal{F}) \) is the data of an ambient space and a foliation \( \mathcal{F} \) on \( M \). A foliated space is complex hyperbolic when the corresponding foliation is complex hyperbolic.

**Remark 2.7.** We consider also ambient spaces where \( M \) is a germ of nonsingular complex analytic space around a compact analytic subset \( K \). We can transfer to this case all the definitions of this text, just by taking a small enough open set around \( K \).

**Remark 2.8.** The name “foliated space” is inspired in the terminology “foliated manifold” introduced by A. Belotto (see [1]).

### 2.3 Presimple Points

We slightly modify the definitions of simple and presimple singularities given for dimension two in Subsection [2.2](#2.2) taking into account not only the foliation, but also the normal crossings divisor. We also extend the definitions to higher dimension.
that is, we can write \( \omega \) be a local generator of \( \mathcal{F} \) adapted to \( E \) as in Equation 2. Write \( E_{\text{inv}} = \bigcup_{j \in A_{\text{inv}}} (x_j = 0) \) and \( E_{\text{dic}} = \bigcup_{j \in A_{\text{dic}}} (x_j = 0) \), where we have \( A = A_{\text{inv}} \cup A_{\text{dic}} \).

**Definition 2.4** (See [5, 11]). The point \( p \) is presimple for \( (\mathcal{M}, \mathcal{F}) \) if either \( \nu_p(\mathcal{F}, E) = 0 \) or there is \((j, k) \in A \times B \) such that \( \partial a_j / \partial x_k(p) \neq 0 \).

**Proposition 2.2.** If \( p \) is presimple for \( (\mathcal{M}, \mathcal{F}) \), then \( \nu_p(E_{\text{inv}}) \leq \tau_p(\mathcal{F}) \leq \nu_p(E_{\text{inv}}) + 1 \).

**Sketch of the proof.** (For more details, see [5, Lemma 5] and [11]). Recall that \( a_j = x_j \bar{a}_j \) for every \( j \in A_{\text{dic}} \), hence \( \nu_p(a_j) > 0 \) and \( \partial a_j / \partial x_k(p) = 0 \) for every \( k \in B \). When \( \nu_p(\mathcal{F}, E) = 0 \), we distinguish two cases:

- There is a \( j \in A_{\text{inv}} \) such that \( a_j \) is a unit. The germs of vector fields
  \[ \xi_k = \partial / \partial x_k - (a_k / a_j) x_j \partial / \partial x_k, \quad k \in B; \quad \xi_\ell = \partial / \partial x_\ell - (\bar{a}_\ell / a_j) x_j \partial / \partial x_\ell, \quad \ell \in A_{\text{dic}}. \]
  are tangent to \( \mathcal{F} \) at \( p \). The dimensional type is \( \tau_p(\mathcal{F}) = \nu_p(E_{\text{inv}}) \), since there are no more independent trivializing vector fields.

- For every \( \ell \in A \), we have \( \nu_p(a_\ell) > 0 \) and there is a \( j \in B \) such that \( a_j \) is a unit. The germs of vector fields
  \[ \xi_k = \partial / \partial x_k - (a_k / a_j) \partial / \partial x_k, \quad k \in B \setminus \{j\}; \quad \xi_\ell = \partial / \partial x_\ell - (\bar{a}_\ell / a_j) \partial / \partial x_\ell, \quad \ell \in A_{\text{dic}}. \]
  are tangent to \( \mathcal{F} \) at \( p \). The dimensional type is \( \tau_p(\mathcal{F}) = \nu_p(E_{\text{inv}}) + 1 \).

When \( \nu_p(\mathcal{F}, E) > 0 \) but \( \partial a_j / \partial x_k(p) \neq 0 \) for some \( j \in A, k \in B \), trivializing vector fields may be obtained thanks to the integrability condition. We obtain also that \( \tau_p(\mathcal{F}) = \nu_p(E_{\text{inv}}) + 1 \).

**Definition 2.5.** Let \( p \in M \) be a presimple point for \( (\mathcal{M}, \mathcal{F}) \). We say that \( p \) is of corner type when \( \tau_p(\mathcal{F}) = \nu_p(E_{\text{inv}}) \) and we say that \( p \) is of trace type when \( \tau_p(\mathcal{F}) = \nu_p(E_{\text{inv}}) + 1 \).

**Remark 2.9.** In view of proof of Proposition 2.2 the point \( p \) is presimple of corner type if and only if \( \nu_p(a_j) = 0 \) for every \( j \in A_{\text{inv}} \).

**Proposition 2.3.** Let us suppose that \( \mathcal{F} \) is a complex hyperbolic foliation at \( p \). We have that \( p \) is a presimple point for \( (\mathcal{M}, \mathcal{F}) \) if and only if \( \nu_p(\mathcal{F}, E) = 0 \). Moreover, if \( \nu_p(E_{\text{inv}}) \geq 1 \), we have that \( p \) is presimple of corner type for \( (\mathcal{M}, \mathcal{F}) \) if and only if \( \nu_p(a_j) = 0 \), for every \( j \in A_{\text{inv}} \).

**Proof.** Let us prove the first assertion. Assume that \( \nu_p(\mathcal{F}, E) > 0 \) and there is \((j, k) \in A \times B \) such that \( \partial a_j / \partial x_k(p) \neq 0 \). We obtain that \( \bar{\omega} = 0 \) gives a saddle-node, where
\[ \bar{\omega} = \omega_{(x_\ell = 0, \xi(\ell, k))} = \bar{a}_\ell dx_\ell + x_\ell \bar{a}_k dx_k, \quad \bar{a}_\ell = a_\ell(x_\ell = 0, \xi(\ell, k)). \]

Let us prove the second statement. By Remark 2.9 the point \( p \) is presimple of corner type if and only if there is \( j \in A_{\text{inv}} \) such that \( \nu_p(a_j) = 0 \). If there is an \( \ell \in A_{\text{inv}} \) with \( \nu_p(a_\ell) > 0 \), we have a saddle-node given by the restriction \( \omega_{(x_\ell = 0, \xi(\ell, \ell))} \).

### 2.4 Residual Vectors and Simple Points

Let us consider a foliated space \((\mathcal{M}, \mathcal{F})\) and a point \( p \in M \). Denote \( \tau = \tau_p(\mathcal{F}) \) the dimensional type. We know that there is a minimal local coordinate system \( x = (x_1, x_2, \ldots, x_n) \) and a local generator \( \omega \) of \( \mathcal{F} \) with
\[ \omega(\partial / \partial x_j) = 0, \quad \partial / \partial x_j(\omega(\partial / \partial x_k)) = 0, \quad j > \tau \text{ and } \ell \leq \tau. \]

That is, we can write \( \omega \) as in Equation 1. We say that \( x \) is a minimal coordinate system adapted to \( E \) when \( E_{\text{inv}} = \bigcup_{j=1}^n (x_j = 0) \) and \( E_{\text{dic}} \subseteq \bigcup_{j=\tau+1}^{n+1} (x_j = 0) \). Note that \( e \leq \tau \). The generator of \( \mathcal{F} \) adapted to \( E \) defined by \( \eta = (1 / \prod_{j=1}^n x_j) \omega \) can be written as
\[ \eta = \sum_{j=1}^e a_j(x_1, x_2, \ldots, x_e) dx_j \frac{x_j}{x_j} + \sum_{j=e+1}^r a_j(x_1, x_2, \ldots, x_r) dx_j \quad (3) \]

The above expression of the vector fields \( \xi_k \) and \( \xi_\ell \) in the proof of Proposition 2.2 allows to prove the following statement:
Proposition 2.4. There is a minimal coordinate system adapted to $E$ at a given presimple point $p \in M$.

Assume that $p$ is a presimple point for $(M, F)$. Recall that $\epsilon \leq \tau \leq \epsilon + 1$ and take a minimal coordinate system $x$ and an adapted generator $\eta$ as in Equation [18]. The residual vector $\lambda_{p, x} \in \mathbb{C}^*$ is defined by

$$
\lambda_{p, x} = \left\{ \begin{array}{ll}
(\lambda_1, \lambda_2, \ldots, \lambda_r) & \text{with } \lambda_i = a_i(p) \quad \text{if } \tau = c \\
(\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \mu) & \text{with } \lambda_i = \frac{\partial \phi}{\partial x_i}(p), \mu = a_r(p) \quad \text{if } \tau = c + 1.
\end{array} \right.
$$

Remark 2.10. Since $p$ is presimple point, we have that $\lambda_{p, x} \neq 0$.

Lemma 2.1. Let $p$ be a presimple point for $(M, F)$. Consider minimal coordinate systems $x$ and $x'$ adapted to $E$, such that $(x_j = 0) = (x'_j = 0)$ for every $j \in \{1, 2, \ldots, q\}$. Given $\eta$ and $\eta'$ generators of $F$ adapted to $E$ as in Equation [18], there is a constant $c \in \mathbb{C}^*$ such that $\lambda_{p, x} = c\lambda_{p, x'}$.

Proof. We have that there are units $u, u_j \in \mathcal{O}_{M, p}$ with $\partial/\partial x_k(u) = \partial/\partial x_k(u_j) = 0$ for $k > \tau$, such that $\eta' = u\eta$ and $x'_j = u_j x_j$, for every $j \in \{1, 2, \ldots, q\}$. Moreover, when $\tau = c + 1$, we have $x'_j = \alpha x_j + \phi(x_1, x_2, \ldots, x_{r-1}) + x_r \psi(x_1, x_2, \ldots, x_r)$, where $\alpha \in \mathbb{C}^*$ and $\phi(p) = \psi(p) = 0$. If $x = x'$, we have that $\lambda_{p, x}' = c\lambda_{p, x}$, with $c = u(p)$. Assume now that $\eta = \eta'$ and let us write

$$
\eta = \sum_{i=1}^{r-1} a_i dx_i/x_i + x'_r a'_r dx_r/x_r = \sum_{i=1}^{r-1} a'_i dx'_i/x'_i + x'_r a'_r dx'_r/x'_r, \quad \epsilon = \tau - c.
$$

If $(x_1', x_2', \ldots, x_r') = (x_1, x_2, \ldots, x_r)$, we are done. Suppose that there is $j \in \{1, 2, \ldots, q\}$ such that $u_j \neq 1$ and $x'_j = x_j$ for every $i \neq j$. We have that

$$
a'_j = a_j(1 + x_j h_j), \quad a'_i = a_i + x_i h_i a_j, \quad i \in \{1, 2, \ldots, \tau\} \setminus \{j\},
$$

where $du_j/u_j = \sum_{i=1}^{r-1} x_i h_i dx_i/x_i$. As a consequence $\lambda_{p, x} = \lambda_{p, x'}$. In case $\tau = c + 1$ and $x'_j = x_j$ for every $i \in \{1, 2, \ldots, q\}$, we obtain

$$
a'_i = a_i - c x_i a_r \partial \phi/\partial x_i + x_r \partial \psi/\partial x_i, \quad a'_r = c a_r (1 - \psi - x_r \partial \psi/\partial x_r),
$$

with $c = 1/\alpha$. Hence $\lambda_{p, x} = c\lambda_{p, x'}$.

Lemma 2.2. If $F$ is a complex hyperbolic foliation at a point $p$ presimple for $(M, F)$, then the residual vector $\lambda_{p, x}$ belongs to $(\mathbb{C}^*)^q$.

Proof. If $p$ is of corner type, in view of Proposition 2.3 we are done. Assume that $p$ is a presimple point of trace type. By Proposition 2.3 and Remark 2.9 we have that $\mu \neq 0$. Then, if $\lambda_j = 0$ for some $j \in \{1, 2, \ldots, q\}$, the restriction

$$
\omega|_{(x_j = 0, i \phi(j, r))} = a_j dx_j + x_j a_r dx_r, \quad \bar{a}_\ell = a_{\bar{\ell}}|_{(x_j = 0, i \phi(j, r))},
$$

would give a saddle-node.

Given $s \in \mathbb{Z}_{>0}$, let us consider a vector $\beta = (\beta_1, \beta_2, \ldots, \beta_s) \in \mathbb{C}^*$. A resonance $r$ for $\beta$ is an $s$-tuple of non-negative integers $r = (r_1, r_2, \ldots, r_s) \in \mathbb{Z}_{>0}^s$ such that $\beta_1 r_1 + \beta_2 r_2 + \cdots + r_s \beta_s = 0$. We say that $\beta$ is non-resonant if it does not have resonances different from $r = 0$.

Remark 2.11. In this work we consider just non-negative resonances. Observe that Martinet-Ramis resonances in $\mathbb{Z}^3$ allow negative entries in $r$.

Let us give a definition of simple point in the complex hyperbolic frame. This definition coincides with the general one introduced in [5, 6].

Definition 2.6. Assume that $F$ is a complex hyperbolic foliation at $p$. We say that $p$ is a simple point for $(M, F)$ if it is presimple and the residual vectors $\lambda_{p, x}$ are non-resonant (this property does not depend on the particular choice of $x$ and $\eta$).

Remark 2.12. Being simple is an open property on $M$. That is, if $p$ is a simple point, there is a small enough open set $U \subset M$ containing $p$ such that each $q \in U$ is a simple point.

Definition 2.7. A foliated space $(M, F)$ is desingularized if each $p \in M$ is a simple point.
Remark 2.13. We have that \( \text{Sing}(\mathcal{F}, E) = \text{Sing}(\mathcal{F}) \) when \((\mathcal{M}, \mathcal{F})\) is desingularized.

Remark 2.14. Let \( p \) be a simple point for \((\mathcal{M}, \mathcal{F})\), where \( \mathcal{F} \) is a complex hyperbolic foliation at \( p \). If \( p \) is of corner type, the only invariant hypersurfaces for \( \mathcal{F} \) through it are contained in \( E \). If \( p \) is of trace type, there is exactly one germ of invariant hypersurface \((H, p)\) not contained in \( E \) (for the general case, see [6]). The singular locus, locally at \( p \), is given by

\[
\text{Sing}(\mathcal{F}) = \bigcup_{i \leq i < j \leq \tau} E_i \cap E_j, \quad i, j \in \text{inv},
\]

taking \( E_r = H \) when \( p \) is of trace type. Denote by \( T_{\mathcal{F}, E} \) the set of trace type simple singularities for \((\mathcal{M}, \mathcal{F})\). If \( p \) is of corner type, we have that \( T_{\mathcal{F}, E} = \emptyset \), locally at \( p \). When \( p \) is a trace type simple singularity, we have

\[
T_{\mathcal{F}, E} = (E_1 \cap H) \cup (E_2 \cap H) \cup \cdots \cup (E_{r-1} \cap H).
\]

2.5 Reduction of Singularities and Toric Type Foliated Spaces

The concept of toric type foliated space was introduced in [4, 15] for the bidimensional case. Here we generalize it to higher dimension.

Let us consider an ambient space \( \mathcal{M} = (M, E) \) and a connected nonsingular analytic subspace \( Y \subseteq M \). We say that \( Y \) and \( E \) have normal crossings if for each point \( p \in M \) there is a local coordinate system \( x = (x_1, x_2, \ldots, x_n) \) adapted to \( E \) and a subset \( B \subseteq \{1, 2, \ldots, n\} \) such that \( Y = \cap_{b \in B} (x_i = 0) \), locally at \( p \). In this case, the blowing-up \( \pi : M' \to M \) centered at \( Y \) gives a new normal crossings divisor \( E' = \pi^{-1}(E \cup Y) \) and we get a new ambient space \( M' = (M', E') \). Given a foliation \( \mathcal{F} \) on \( M \), we say that \( Y \) is an admissible center for \((\mathcal{M}, \mathcal{F})\) if, in addition, the subspace \( Y \) is invariant for \( \mathcal{F} \). We write, for short

\[
\pi : (\mathcal{M}', \mathcal{F}') \to (\mathcal{M}, \mathcal{F}),
\]

where \( \mathcal{F}' \) is the transform of \( \mathcal{F} \) by \( \pi \). We also say that \( \pi \) is an admissible blowing-up of foliated spaces. The blowing-up \( \pi \) is called dicritical if the exceptional divisor \( \pi^{-1}(Y) \) is a dicritical component of \( E' \).

Remark 2.15. Note that given an admissible blowing-up \( \pi : (\mathcal{M}', \mathcal{F}') \to (\mathcal{M}, \mathcal{F}) \), the following properties hold:

1. The blowing-up \( \pi \) is dicritical only if \( \mathcal{F} \) is dicritical.
2. If the foliation \( \mathcal{F} \) is non-dicritical, then \( \mathcal{F}' \) is non-dicritical.
3. If the foliation \( \mathcal{F} \) is complex hyperbolic, then \( \mathcal{F}' \) is also complex hyperbolic.

Let \( \sigma : (\mathcal{M}', \mathcal{F}') \to (\mathcal{M}, \mathcal{F}) \) be a morphism obtained, up to isomorphism, by composition of a finite family of admissible blowing-ups. That is,

\[
\sigma : (\mathcal{M}', \mathcal{F}') = (\mathcal{M}', \mathcal{F}') \xrightarrow{\pi_{r}} (\mathcal{M}'^{r-1}, \mathcal{F}'^{r-1}) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_1} (\mathcal{M}', \mathcal{F}') = (\mathcal{M}, \mathcal{F}),
\]

where each \( \pi_i \) is an admissible blowing-up of foliated spaces. If \((\mathcal{M}', \mathcal{F}')\) is desingularized, we say that \( \sigma \) is a reduction of singularities of \((\mathcal{M}, \mathcal{F})\).

Remark 2.16. In general, the existence of reduction of singularities of a foliated space \((\mathcal{M}, \mathcal{F})\) is only known when \( \dim M \leq 3 \) (see [3, 17]). Nevertheless, it always exists if \( M = ((\mathbb{C}^n, 0) \setminus \emptyset) \) and \( \mathcal{F} \) is a complex hyperbolic non-dicritical foliation. Indeed, it is given by a reduction of singularities of the invariant hypersurfaces of \( \mathcal{F} \) (see [3, 12]). For this reason, these foliations are also called “generalized hypersurfaces”. For \( n \geq 4 \), the existence of reduction of singularities is an open problem, even in the dicritical complex hyperbolic case.

An admissible blowing-up \( \pi : (\mathcal{M}', \mathcal{F}') \to (\mathcal{M}, \mathcal{F}) \) centered at \( Y \subseteq M \) is called combinatorial when \( Y \) is a connected component of an \( E_J = \cap_{j \in J} E_j \), with \( J \subseteq I \).

Definition 2.8. A foliated space \((\mathcal{M}, \mathcal{F})\) is of toric type if it has a combinatorial reduction of singularities. A foliation \( \mathcal{F} \) on a complex analytic space \( M \) is called torifiable if there is a normal crossings divisor \( E \) on \( M \) such that the foliated space \(((M, E), \mathcal{F})\) is of toric type.
Remark 2.17. A complex hyperbolic foliated space \((M, F)\), where \(M = ((\mathbb{C}^n, 0), E)\) and the origin is a presimple corner is of toric type (see [11]). Moreover, if the origin is not simple, the foliation \(F\) is dicritical. In the two-dimensional being of toric type is very close to be Newton non-degenerate (see [15]).

3 Invariant Surfaces and Extended Partial Separatrices

Partial separatrices have been introduced in [9] to formalize the arguments in [6] for the construction of invariant surfaces of non-dicritical foliations in ambient dimension three. We extend this concept to the dicritical case and we give properties that assure the existence of invariant surfaces supported by them.

3.1 The Prolongation Property

Along this subsection we consider a foliated surface \((S, G)\), where \(S = (S, D)\) and \(S\) is a surface. In dimension two we only blow-up with center at points, hence every blowing-up is automatically admissible.

Let \((\Gamma, p) \not\subset (D, p)\) be an invariant branch of \(G\). We say that \((\Gamma, p)\) is isolated for \((S, G)\) when for each composition \(\sigma : (S', G') \to (S, G)\) of a finite sequence of blowing-ups, the point \(p'\) belongs to the adapted singular locus \(\text{Sing}(G', D')\), where \((\Gamma', p')\) is the strict transform of \((\Gamma, p)\) by \(\sigma\). Note that \(p \in \text{Sing}(G, D)\) when \((\Gamma, p)\) is an isolated invariant branch, just by taking \(\sigma\) to be the identity. Besides, in order to verify if \((\Gamma, p)\) is isolated or not, it is enough to consider blowing-ups centered at the infinitely near points of \((\Gamma, p)\).

From now on, in order to simplify the exposition, each time we say “a curve \(Y\) in \(S\)”, we make reference to a “closed irreducible analytic curve \(Y\) in \(S\)”.

Definition 3.1. A foliated surface \((S, G)\) has the prolongation property for isolated branches if for every isolated invariant branch \((\Gamma, p)\) the next conditions hold:

1. There is a curve \(Y \subset S\) extending \((\Gamma, p)\), that is, such that \((\Gamma, p) \subset (Y, p)\). (Note that \(Y\) is unique).
2. If \(Y \subset S\) is the curve extending \((\Gamma, p)\) and \(q \in Y \cap D\), each branch \((Y, q) \subset (Y, q)\) is isolated.

Remark 3.1. The second condition implies that \(Y \cap D \subset \text{Sing}(G, D)\).

Lemma 3.1. A desingularized foliated surface \((S, G)\) has the prolongation property for isolated branches if and only if for each trace type singularity \(p \in \text{Sing}(G)\), the following conditions hold for the only invariant branch \((\Gamma, p) \not\subset (D, p)\) through it:

1. There is a curve \(Y \subset S\) extending \((\Gamma, p)\).
2. If \(Y \subset S\) is the curve extending \((\Gamma, p)\), then \(Y \cap D_{\text{dic}} = \emptyset\).

Proof. Since \((S, G)\) is desingularized, we have that an invariant branch \((\Gamma, p) \not\subset (D, p)\) is isolated if and only if \(p\) is a trace type simple singularity. Assume that there is a curve \(Y \subset S\) extending \((\Gamma, p)\). Observe that \((Y, q) \not\subset (D, q)\) is analytically irreducible for every \(q \in Y\), because there is at most one invariant branch different from the divisor at \(q\). As a consequence, it is enough to see that given \(q \in Y \cap D\), we have that \((Y, q)\) is isolated if and only \(e_q(D_{\text{dic}}) = 0\).

If \((Y, q)\) is isolated, then \(q\) is a simple singularity; hence \(e_q(D_{\text{dic}}) \leq 2 - \tau_q(G) = 0\). Conversely, when \(e_q(D_{\text{dic}}) = 0\), we have that \((Y, q)\) and \((D, q)\) are two invariant branches; then \(q\) is a simple singularity and \((Y, q)\) is an isolated invariant branch.

Recall that a toric surface is an irreducible complex surface \(S\) containing a two-dimensional complex torus \(T \simeq (\mathbb{C}^*)^2\) as a Zariski open subset, such that the action of \(T\) on itself extends to an algebraic action on \(S\) (see, for instance [10]). The union of the non-dense orbits of the torus action on \(S\) is a normal crossings divisor \(D_S\). Hence \(S\) gives in a natural way a toric ambient surface \((S, D_S)\). In the work [15], we have provided a proof of the following statement:
Proposition 3.1. The prolongation property for isolated branches holds for every toric type foliated surface \(((S,DS),\mathcal{F})\) over a nonsingular projective toric ambient surface.

3.2 Extended Partial Separatrices

We consider a desingularized foliated space \((\mathcal{M},\mathcal{F})\) in dimension three, where \(\mathcal{M} = (M,E)\) and \(E\) is a germ of complex analytic set around a compact analytic subset. By Remark 3.1, the set of trace type simple singularities \(T_{F,E}\) is a closed analytic subspace of \(M\); it is a union of nonsingular curves \(Y\) with \(c_Y(E) = c_Y(E_{\text{inv}}) = 1\). A connected component \(C\) of \(T_{F,E}\) is called a partial separatrix. Given \(q \in C\), there is an open set \(U \subset M\) containing \(q\) and an irreducible surface \(S_q\) on \(U\), such that the germ of \(S_q\) at each \(q' \in C \cap U\) is the only invariant surface through \(q'\) different from \(E\). Moreover \(S_q \cap E_{\text{inv}} = T_{F,E}\) in \(U\). When \(C \cap E_{\text{dic}} = \emptyset\), the surface \(S_q\) extends to an irreducible closed surface \(S_C \subset M\) invariant for \(\mathcal{F}\).

We are interested in connecting partial separatrices of \((\mathcal{M},\mathcal{F})\) through invariant curves contained in the dicritical components of \(E\). Denote by \(\Sigma\) the set whose elements are the curves \(Z \subset E_{\text{dic}}\) with \(c_Z(E) = 1\), invariant for \(\mathcal{F}\) and satisfying \(Z \cap \text{Sing}(\mathcal{F}) \neq \emptyset\). Note that \(\Sigma\) is a finite set. Indeed, there are finitely many dicritical components \(E_k \subset E_{\text{dic}}\). Moreover \(\text{Sing}(\mathcal{F}) \cap E_k\) is finite and there is at most one invariant branch contained in \(E_k\) but not in \(E_{\text{inv}}\) through each \(p \in \text{Sing}(\mathcal{F}) \cap E_k\). Let us denote \(U_{F,E}\) the closed analytic subspace of \(M\) given by

\[U_{F,E} = T_{F,E} \cup (\bigcup_{Z \in \Sigma} Z).\]

Note that every \(q \in U_{F,E}\) is a trace type simple point, in particular \(c_q(E) \leq 2\).

Definition 3.2. The extended partial separatrices are the connected components of \(U_{F,E}\).

Lemma 3.2. Let us consider an extended partial separatrix \(E \subset U_{F,E}\). For each point \(q \in E\), there is an open set \(U_q \subset M\) containing \(q\) and an irreducible surface \(S_q\) defined in \(U_q\), such that \(E \cap U_q \subset S_q\). Moreover, the germ of \(S_q\) at each \(q' \in E \cap U_q\) is the only invariant surface through \(q'\) not contained in \(E\).

Proof. It follows from the local structure of the singular locus, described in Subsection 1.3, and from the fact that there is a unique invariant surface through a regular point.

Definition 3.3. An extended partial separatrix \(E \subset U_{F,E}\) is complete when \((S_q,q) \cap E = (E,q)\) for every \(q \in E\), where \(S_q\) is the only germ of surface invariant for \(\mathcal{F}\) through \(q\) that is not contained in \(E\).

Corollary 3.1. If \(E \subset U_{F,E}\) is a complete extended partial separatrix, there is a unique irreducible closed surface \(S_E \subset M\) invariant for \(\mathcal{F}\), such that \(S_E \cap E = E\).

Proof. It follows from Lemma 3.2 and Definition 3.3 in a similar way as Cano-Cerveau’s argument in [9] (see also [10]).

Let \(E_k\) be a dicritical component of \(E\). We know that the foliation \(\mathcal{F}\) induces by restriction a foliation \(\mathcal{F}|_{E_k}\) on \(E_k\). We define the restriction \(\mathcal{M}|_{E_k}\) of the ambient space \(\mathcal{M}\) to \(E_k\) by

\[\mathcal{M}|_{E_k} = (E_k,\mathcal{E}|_{E_k}) , \quad \mathcal{E}|_{E_k} = E \cap E_k \cup E_k.\]

We obtain in this way a foliated surface \((\mathcal{M},\mathcal{F})|_{E_k}\) given by \((\mathcal{M},\mathcal{F})|_{E_k} = (\mathcal{M}|_{E_k},\mathcal{F}|_{E_k})\). Since \((\mathcal{M},\mathcal{F})\) is desingularized, the restriction \((\mathcal{M},\mathcal{F})|_{E_k}\) is also desingularized. Moreover, we have that \((\mathcal{E}|_{E_k})_{\text{inv}} = E_{\text{inv}}|_{E_k}\) and \((\mathcal{E}|_{E_k})_{\text{dic}} = E_{\text{dic}}|_{E_k}\).

Proposition 3.2. Let \((\mathcal{M},\mathcal{F})\) be a desingularized foliated space. Assume that the foliated surface \((\mathcal{M},\mathcal{F})|_{E_k}\) has the prolongation property for isolated branches, for each dicritical component \(E_k\) of \(E\). Then every extended partial separatrix is complete.

Proof. Let us consider an extended partial separatrix \(E\). Note that \(1 \leq c_q(E) \leq 2\) for every point \(q \in E\). When \(c_q(E) = 1\), there is a unique curve \(Y \subset E\) containing \(q\). The intersection of \((S_q,q)\) and \(E\) is a branch trough \(q\) containing the germ \((Y,q)\); as a consequence, we have that \((S_q,q) \cap E = (Y,q) = (E,q)\). Let us assume now \(E = E_1 \cup E_2\), locally at \(q\). We distinguish three cases:
• $E_{\text{inv}} = E_1 \cup E_2$. There is a partial separatrix $C \subset \mathcal{E}$, such that $(C, q) = (\mathcal{E}, q)$. Then, we conclude by the study of the partial separatrices in [6] and [9].
• $E_{\text{inv}} = E_1$. There is a curve $Y \subset E_1$ with $q \in Y$ such that $(Y, q) = (\mathcal{F}|_Y, q)$. The foliated surface $(\mathcal{M}, \mathcal{F})|_{E_1}$ is desingularized and $q \in \text{Sing}(\mathcal{F}|_{E_1})$ is of trace type. The unique branch $(\Gamma, q) \not\subset (\mathcal{E}|_{E_1}; q)$ invariant for $\mathcal{F}|_{E_1}$ extends to a curve $Z \subset E_2$, by the prolongation property for isolated branches. Note that $Z \subset \Sigma$, hence $Z \subset \mathcal{E}$. We conclude that $(S_q, q) \cap E = (Y \cup Z, q) = (\mathcal{E}, q)$.
• $E_{\text{inv}} = \emptyset$. In this case $E_{\text{disc}} = E_1 \cup E_2$. Let us see that this situation does not hold. Note that $q$ is a regular point. There is a curve $Z \subset \Sigma$ with $q \in Z$. We can assume $Z \subset E_1$. By definition of $\Sigma$, there is $p \in \text{Sing}(\mathcal{F}) \cap Z$. Note that $e_{p}(E_{\text{inv}}) = 1$ and $(Z, p)$ is the unique invariant branch for $\mathcal{F}|_{E_1}$ not contained in $E|_{E_1}$. In view of Lemma 3.3, we have that $Z \cap (E|_{E_1})_{\text{disc}} = \emptyset$. We find a contradiction, since $q \in E|_{E_1} = E|_{E_1} = (E|_{E_1})_{\text{disc}}$.

3.3 Invariant Surfaces for Torifiable Foliations

The main result in this paper is the following:

**Theorem 3.1.** Every torifiable complex hyperbolic foliation on $(\mathbb{C}^3, 0)$ has an invariant surface.

We present now the structure of the proof of Theorem 3.1. Let us consider a germ of complex hyperbolic foliation $\mathcal{F}$ and a strong normal crossings divisor $E^0$ on $(\mathbb{C}^3, 0)$. Denote $\mathcal{M}_0 = ((\mathbb{C}^3, 0), E^0)$ and assume that $(\mathcal{M}_0, \mathcal{F}_0)$ is of toric type. Let us fix a combinatorial reduction of singularities $\sigma : (\mathcal{M}, \mathcal{F}) \to (\mathcal{M}_0, \mathcal{F}_0)$. The proof of Theorem 3.1 follows from the next statements:

**Proposition 3.3.** If $E_{\text{inv}} = \emptyset$, there is a closed surface $S$ of $M$, invariant for $\mathcal{F}$, with $S \not\subset \mathcal{E}$.

**Proposition 3.4.** If $E_{\text{inv}} \neq \emptyset$ and $E^0 = E_{\text{disc}}^0$, there is an extended partial separatrix of $(\mathcal{M}, \mathcal{F})$.

**Proposition 3.5.** Every extended partial separatrix of $(\mathcal{M}, \mathcal{F})$ is complete.

Let us see how we deduce Theorem 3.1 from these propositions. If $E_{\text{inv}}^0 \neq \emptyset$, there is an invariant surface for $\mathcal{F}_0$ contained in $E^0$ and we are done. Hence, we can assume $E^0 = E_{\text{disc}}^0$. If $E_{\text{inv}} = \emptyset$, Proposition 3.3 assures the existence of a closed surface $S \subset M$ invariant for $\mathcal{F}$, with $S \not\subset \mathcal{E}$. If $E_{\text{inv}} \neq \emptyset$, there is an extended partial separatrix $E$ by Proposition 3.5 that is complete by Proposition 3.5 as a consequence of Corollary 3.4. There is a closed surface $S = S_{\mathcal{F}} \subset M$ invariant for $\mathcal{F}$, with $S \not\subset \mathcal{E}$. In both cases, by Grauert’s Proper Mapping Theorem, we obtain a surface $S_0 = \sigma(S)$ of $(\mathbb{C}^3, 0)$ invariant for $\mathcal{F}_0$, with $S_0 \not\subset E^0$.

Proposition 3.3 and Proposition 3.4 are proved in Section 3.4 and Proposition 3.5 is proved in Section 3.1. For the proof of Proposition 3.3 we use the next version of the “refined Camacho-Sad’s Theorem” in [16].

**Proposition 3.6.** Let us consider a complex hyperbolic foliated surface $(S_0, \mathcal{G}_0)$, where the ambient surface is $S_0 = ((\mathbb{C}^2, 0), D^0)$ and let $\sigma : (S, \mathcal{G}) \to (S_0, \mathcal{G}_0)$ be the composition of a finite sequence of blowing-ups. Assume that there is a connected component $K$ of $D_{\text{inv}}$ with the property that every point $p \in K$ is simple for $(S, \mathcal{G})$. Then, we have $T_{S, \mathcal{G}} \cap K \neq \emptyset$.

**Proof.** After completing the reduction of singularities of $(S, \mathcal{G})$, we apply similar arguments to the ones in the proof of [16].

4 The Hunt of Trace Singularities

The aim of this section is to prove Proposition 3.3 and Proposition 3.4. Recall that we have a complex hyperbolic foliation $\mathcal{F}$ such that the foliated space $(\mathcal{M}_0, \mathcal{F}_0)$ is of toric type, where $\mathcal{M}_0 = ((\mathbb{C}^3, 0), E^0)$, with $E^0 = E_{\text{disc}}^0$. Moreover, we have a fixed combinatorial reduction of singularities $\sigma : (\mathcal{M}, \mathcal{F}) \to (\mathcal{M}_0, \mathcal{F}_0)$. 

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Remark 4.1. If \( E_{\text{inv}} = \emptyset \), then \( e_0(E^0) \leq 2 \). Indeed, if \( e_0(E^0) = 3 \), we always find points \( p \in M \) with \( e_p(E) = 3 \); since \( p \) is a simple point, we have \( E_{\text{inv}} \neq \emptyset \). If \( E_{\text{inv}} \neq \emptyset \), then \( e_0(E^0) \geq 2 \). Indeed, if \( e_0(E^0) \in \{0, 1\} \) we have that \( \sigma \) is the identity morphism, since it is combinatorial. Hence, we have that \( E = E^0 \) and \( E_{\text{inv}} = \emptyset \).

Proof of Proposition 3.3. If \( \sigma \) is the identity morphism, the foliation \( F_0 \) is regular and it has normal crossings with \( E^0 \). By the Frobenius theorem, there is a unique germ of surface \( S_0 \) in \( (C^3, 0) \) invariant for \( F_0 \), with \( S_0 \not\subset E^0 \) and we are done. When \( \sigma \) is not the identity morphism, we have that \( e_0(E^0) = 2 \), in view of Remark 4.1 and it is a composition of blowings-ups with one-dimensional combinatorial centers. Note that \( K = \sigma^{-1}(0) \) is a finite union of curves and \( M \) is a germ around it. Since every point \( p \in K \) is simple, we have that

\[
0 = e_p(E_{\text{inv}}) \leq \tau_p(F) \leq e_p(E_{\text{inv}}) + 1 = 1.
\]

As a consequence \( \tau_p(F) = 1 \) and hence \( p \notin \text{Sing}(F) \). Moreover \( F \) and \( E \) have normal crossings at \( p \). Then, there is a unique germ of invariant surface \( (S_p, p) \), with \( (S_p, p) \not\subset (E, p) \). We distinguish two situations:

- If \( K \) is a closed analytic subspase invariant for \( F \), there is a regular point \( p \in K \) such that \( S_p \cap K = \{p\} \). Hence \( S = (S_p, p) \) defines a closed surface in \( M \), since \( M \) is a germ around \( K \).
- If \( K \) is invariant for \( F \), we have that \( (K, p) \subset (S_p, p) \) for every \( p \in K \), as a consequence, the germ of surface \( (S_p, p) \) extends to a closed surface \( S \) defined in \( \sigma \) invariant for \( F \), with \( K \subset S \). The argument is similar to the one in the proof of Corollary 3.1.

Proof of Proposition 3.4. It is enough to prove that the set of trace type singularities \( T_{F, E} \) is not empty. Indeed, in this case we find at least one partial separatrix for \( (M, F) \) that is naturally contained in an extended partial separatrix. We divide the proof in two steps:

1. Let \( \Gamma^0 = E^0_0 \cap E^0_2 \), where \( E^0_0 \) and \( E^0_2 \) are two irreducible components of \( E^0 \). Let us assume that \( \Gamma^0 \) has been used as a center of blowing-up in \( \sigma \) and hence \( D = \sigma^{-1}(\Gamma \setminus \{0\}) \) is a normal crossings divisor. If \( D_{\text{inv}} \neq \emptyset \), then \( T_{F, E} \neq \emptyset \).
2. If \( e_0(E^0) = 3 \), then \( T_{F, E} \neq \emptyset \).

Note that Step 1 gives the complete proof when \( e_0(E^0) = 2 \), since in this case we have \( E = D \cup E_1 \cup E_2 \), where \( E_i \) is the strict transform of \( E_i^0 \) by \( \sigma \), for \( i = 1, 2 \).

Step 1: Let us write \( D = \bigcup_{i=1}^{n+1} D_i \), \( D_0 = E_1 \) and \( D_{n+1} = E_2 \), in such a way that \( D_i \cap D_{i+1} \neq \emptyset \), for every \( i \in \{0, 1, \ldots, n\} \). Assume \( T_{F, E} = \emptyset \) and let us find a contradiction.

We consider a local coordinate system \( (x_1, x_2, y) \) at the origin of \( C^3 \) with \( E^0_0 = (x_1, x_2 = 0) \) and satisfying that \( (y = 0) \) is not invariant for \( F_0 \) (this is always possible). In this situation, there is \( \varepsilon > 0 \) such that \( \Delta_0^0 = (y = \varepsilon) \) is generically transversal to \( F_0 \) through the point \( p_c = (0, 0, c) \), for every \( 0 < |c| < \varepsilon \). Consider the foliated surface

\[
(M_0, F_0)|_{\Delta_0^0} = (\Delta_0^0, E^0_0|_{\Delta_0^0}, F_0)|_{\Delta_0^0}, \quad E^0_0|_{\Delta_0^0} = E^0 \cap \Delta_0^0.
\]

Let \( \Delta_c \) be the strict transform of \( \Delta_0^0 \) by \( \sigma \). Note that \( (M, F)|_{\Delta_c} \) is a foliated surface obtained from \( (M_0, F_0)|_{\Delta_0^0} \) by a sequence of blowing-ups induced by \( \sigma \). Recalling that \( D_{\text{inv}} \neq \emptyset \) and that \( D_0, D_{n+1} \) are dicritical components, there are indices \( j, k \in \{1, 2, \ldots, n\} \) with \( j \leq k \) such that \( D_i \) is invariant for every \( j \leq i \leq k \) and \( D_{j-1}, D_{k+1} \) are dicritical components. We write \( Y_{c,i} = D_i \cap \Delta_c \), for every \( i \in \{0, 1, \ldots, n+1\} \). The fiber \( \sigma^{-1}(p_c) \) is given by

\[
\sigma^{-1}(p_c) = D \cap \Delta_c = \bigcup_{i=1}^{n+1} Y_{c,i}.
\]

Observe that \( Y_{c,i} \) is an invariant component of \( E|_{\Delta_c} \), for every \( j \leq i \leq k \). Since \( T_{F, E} = \emptyset \), we have that each \( p \in Y_{c,i} \), with \( j \leq i \leq k \), is a simple point of corner type for \( (M, F) \) and as a consequence, also for \( (M, F)|_{\Delta_c} \). In particular

\[
p_{c,j-1} = Y_{c,j-1} \cap Y_{c,j}, \quad p_{c,k} = Y_{c,k} \cap Y_{c,k+1},
\]

are regular points for \( (M, F)|_{\Delta_c} \). We conclude that \( Y_{c,j-1} \) and \( Y_{c,k+1} \) are dicritical components of \( E|_{\Delta_c} \). In this way, we find a contradiction with Proposition 3.6.
are the strata defined by $H$.

Each compact irreducible component $E$ that provides in a natural way a toric ambient variety $(\mathcal{M}, \mathcal{F})$ is a desingularized foliated surface, obtained from $(\mathcal{M}_0, \mathcal{F}_0)|_{E^0}$ by a sequence of blowing-ups induced by $\sigma$. Note that $E|_{E_1} = (\tilde{E} \cap E_1) \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_j = D_j \cap E_1$ for $j = 2, 3$. An irreducible component of $E|_{E_1}$ is invariant for $\mathcal{F}|_{E_1}$ if and only if it is the intersection of $E_1$ with an invariant component of $E$, since $(\mathcal{M}, \mathcal{F})$ is desingularized and $E_1$ is a dicritical component of $E$. In particular, if the branch $\Gamma_3$ is invariant, we have that $\Gamma_3$ has been used as a center of blowing-up, hence $D_3$ is a normal crossings divisor; moreover, we obtain that $D_1^{\text{inv}} \neq \emptyset$. In this case, we conclude by Step 1. We argue in the same way when $\Gamma_2$ is invariant.

Let us suppose now that $\Gamma_2$ and $\Gamma_3$ are not invariant for $\mathcal{F}|_{E_1}$. There are points $p \in E_1$, with $c_p(E|_{E_1}) = 2$. Since $(\mathcal{M}, \mathcal{F})|_{E_1}$ is a desingularized foliated surface, we conclude that $(E|_{E_1})^{\text{inv}} = \tilde{E}^{\text{inv}} \cap E_1 \neq \emptyset$. By Proposition 3.6 there is a point $p \in \tilde{E}^{\text{inv}} \cap E_1$ that is a singularity of trace type for $(\mathcal{M}, \mathcal{F})|_{E_1}$. We have that $p$ is also a singularity of trace type for $(\mathcal{M}, \mathcal{F})$ and we are done.

5 Prolongation Property in Toric Type Foliated Surfaces

In this section we prove Proposition 3.5. Recall that the ambient space is $\mathcal{M}_0 = ((\mathbb{C}^3, 0), E)$, the foliation $\mathcal{F}_0$ is complex hyperbolic and the foliated space $(\mathcal{M}_0, \mathcal{F}_0)$ is of toric type. Moreover, we have a fixed combinatorial reduction of singularities $\sigma : (\mathcal{M}, \mathcal{F}) \to (\mathcal{M}_0, \mathcal{F}_0)$.

In view of Proposition 3.2 we have that Proposition 3.5 comes from the following statement:

**Proposition 5.1.** For each dicritical component $E_k$ of $E$, the foliated surface $(\mathcal{M}, \mathcal{F})|_{E_k}$ has the prolongation property for isolated branches.

The foliated surface $(\mathcal{M}, \mathcal{F})|_{E_k}$ has automatically the prolongation property for isolated branches, when $E_k$ is a non-compact dicritical component of $E$. Otherwise, in view of Proposition 3.3 the proof of Proposition 5.1 follows from the next statement:

**Lemma 5.1.** Every compact component $E_k$ of $E$ has a structure of toric surface, where $E|_{E_k}$ is the natural divisor given by the torus action. That is $(E_k, E|_{E_k})$ is a toric ambient surface.

**Proof.** An irreducible component $E_k$ of $E$ is compact if and only if $\sigma(E_k) = \{0\}$. Since we are in a combinatorial situation, we are allowed to blow-up the origin only if $e_0(E^0) = 3$. Hence, let us assume $e_0(E^0) = 3$. We fix local coordinates $(x_1, x_2, x_3)$ at the origin of $\mathbb{C}^3$ such that $E^0 = \{x_1 x_2 x_3 = 0\}$. This allows us to give an immersion of $(\mathbb{C}^3, 0) \subset \mathbb{P}^3_{\mathbb{C}}$ by

$$(a_1, a_2, a_3) \mapsto [1, a_1, a_2, a_3].$$

Let $H = H_0 \cup H_1 \cup H_2 \cup H_3$ be the union of the coordinate planes of $\mathbb{P}^3_{\mathbb{C}}$, in such a way that $H_i \cap (\mathbb{C}^3, 0) = \{x_i = 0\}$, for $i = 1, 2, 3$. The projective space $\mathbb{P}^3_{\mathbb{C}}$ has a structure of toric variety that provides in a natural way a toric ambient variety $(\mathbb{P}^3_{\mathbb{C}}, \hat{H})$; that is, the orbits are the strata defined by $H$. In this situation, the combinatorial sequence of blowing-ups $\sigma : \mathcal{M} \to \mathcal{M}_0$ lifts to a combinatorial (equivariant) sequence of blowing-ups

$$\sigma : (\mathbb{P}^3_{\mathbb{C}}, \hat{H}) \to (\mathbb{P}^3_{\mathbb{C}}, H).$$

Each compact irreducible component $E_k$ of $E$ is also an irreducible component of $\hat{H}$. Moreover, we have that $E|_{E_k} = \hat{H}|_{E_k}$. Hence $(E_k, E|_{E_k})$ is a toric ambient surface.

The proof of Theorem 3.1 is ended.
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References

[1] A. Belotto da Silva, *Local resolution of ideals subordinated to a foliation*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 110 (2016), pp. 841-862.

[2] C. Camacho; P. Sad, *Invariant varieties through singularities of vector fields*. Ann. of Math. 115 (1982), pp. 579-595.

[3] C. Camacho; A. Lins Neto; P. Sad, *Topological invariants and equidesingularization for holomorphic vector fields*. J. Differential Geometry Vol. 20 No. 1 (1984), pp. 143-174.

[4] M.I.T. Camacho; F. Cano, *Singular foliations of toric type*. Annales de la faculté des sciences de Toulouse, 6e série, tome 8, No. 1, (1999) pp. 45-52.

[5] F. Cano, *Reduction of singularities of codimension one foliations in dimension three*. Ann. of Math (2) 160 (3) (2004), pp. 907-1011.

[6] F. Cano; D. Cerveau, *Desingularization of non-dicritical holomorphic foliations and existence of separatrices*. Acta Math. 169 (1992), pp. 1-103.

[7] F. Cano; D. Cerveau; J. Déserti, *Théorie élémentaire des feuilletages holomorphes singuliers*. Berlin Education Editions. ISBN-13: 978-2701174846 (2013).

[8] F. Cano; J-F. Mattei, *Hypersurfaces intégrales des feuilletages holomorphes*. Annales de l’institut Fourier 42.1-2 (1992), pp. 49-72.

[9] F. Cano; M. Ravara-Vago, *Local Brunella’s alternative II. Partial separatrices*. Int. Math. Res. Not. IMRN, no. 23 (2015), pp. 12840–12876.

[10] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*. Springer-Verlag New York Berlin Heidelberg, ISBN-10: 0-387-94755-8 (1996).

[11] M. Fernández-Duque, *Elimination of resonances in codimension one foliations*. Publicacions Matematiques, Vol. 59, No. 1 (2015), pp. 75-97.

[12] P. Fernández; J. Mozo-Fernández, *On generalized surfaces in (C^3, 0)*. Astérisque 323 (2009), pp. 261-268.

[13] J. P. Jouanolou, *Équations de Pfaff algébriques*. Lecture Notes in Mathematics, 708. Springer-Verlag ISBN-10: 3-540-09239-0 (1979)

[14] J. Martinet; J-P. Ramis, *Classification analytique des équations différentielles non linéaires résonnantes du premier ordre*. Annales scientifiques de l’É.N.S. 4e sÃľrie, tome 16, No. 4 (1983), pp. 571-621

[15] B. Molina-Samper, *Global Invariant Branches of Non-degenerate Foliations on Projective Toric Surfaces*. ArXiv: 1902.04875 (Submitted on 13 Feb 2019).

[16] L Ortiz; E. Rosales; S. Voronin, *On Camacho-Sad’s theorem about the existence of a separatrix*. International Journal of Mathematics, Vol. 21, No. 11 (2010), pp. 1413-1420.

[17] A. Seidenberg, *Reduction of singularities of the differential equation Ady=Bdx*. Amer. J. Math., 90 (1968), pp. 248-269.