Anisotropic spacetimes in $f(T, B)$ theory III: LRS Bianchi III Universe

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Abstract We study the asymptotic dynamics of $f(T, B)$-theory in an anisotropic Bianchi III background geometry. We show that an attractor always exists for the field equations, which depends on a free parameter provided by the specific $f(T, B)$ functional form. The attractor is an accelerated spatially flat FLRW or non-accelerated LRS Bianchi III geometry. Consequently, the $f(T, B)$-theory provides a spatially flat and isotropic accelerated Universe.

1 Introduction

The family of spatially homogeneous Bianchi cosmologies includes an important gravitational model, such as the Mixmaster Universe or the isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes [1–4]. Nine anisotropic Bianchi models exist based on the three-dimensional real Lie algebra classification. They act as isometries. In these spacetimes, three-dimensional hypersurfaces are defined by the orbits of three isometries. An important characteristic of the Bianchi models is that the physical variables depend only on the time variable. The latter means that the field equations are a system of ordinary differential equations [5, 6].

The FLRW spacetimes follow as the limit for some Bianchi models where the anisotropy vanishes. Indeed, the flat, the open and the closed FLRW geometries are related to the Bianchi I, the Bianchi III and the Bianchi IX spacetimes, respectively [7]. In general, the Bianchi spacetimes are defined by three scale factors [1] however, the locally rotational spacetimes (LRS) admit an extra fourth isometry, and the LRS Bianchi-line elements admit two independent scale factors. It is interesting to mention that the LRS Bianchi IX spacetime is related to the Kantowski–Sachs geometry [8].

We have devoted a series of papers to obtain conditions under which the $f(T, B)$-model anisotropic model tends to the homogeneous and isotropic FRW model. A related question is how the parameters and initial conditions of the model influence the isotropization process. For example, it is well-known that inflation is the most successful candidate to explain why the observable Universe is currently homogeneous and isotropic with great precision. However, the problem is not completely solved in the literature. That is, one usually assumes from the beginning that the Universe is homogeneous and isotropic, as given by the FLRW metrics, and then examines the evolution of the perturbations, rather than starting with an arbitrary metric, and showing that inflation does occur and that the Universe evolves toward homogeneity and isotropy. The complete analysis is complex, even using numerical tools [9]. Thus, one should impose another assumption to extract analytical information: consider anisotropic but homogeneous cosmologies. This class of geometries [10] exhibits very interesting cosmological features, both in inflationary and postinflationary epochs [11]. Along these lines, isotropization is a crucial question. Finally, the class of anisotropic geometries has recently gained much interest due to anisotropic anomalies in the Cosmic Microwave Background (CMB) and large-scale structure data, with strong evidence of a violation of the Cosmological Principle in its isotropic aspect [12, 13]. The Bianchi I spacetime reduces to the spatially flat FLRW geometry, while the Bianchi III and the Kantowski–Sachs geometries reduce to the open and closed FLRW geometries. Furthermore, Kantowski–Sachs geometry can be naturally separated from Bianchi I and III since it gives a closed model, Bianchi I is flat, and Bianchi III is open. The different geometries provide different topologies.

This third work analysis the dynamics of modified teleparallel $f(T, B)$-theory in anisotropic spacetimes. We determine selection rules in which initial conditions with anisotropy and curvature can lead to an isotropic and accelerated spatially flat FLRW geometry within the $f(T, B)$-theory. In [14] we performed a detailed analysis on the dynamics for the Bianchi I Universe, while in [15] we focused on the Kantowski–Sachs geometry. Paper [14] is the first part of a series of studies on analyzing the $f(T, B)$-theory considering some anisotropic spacetimes. The previous analysis of $f(T, B)$-gravity was reviewed, and the global dynamics of a locally rotational Bianchi I background geometry were investigated. A criterion for solving the homogeneity problem in the $f(T, B)$-theory was

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deduced. Finally, the integrability properties for the field equations were investigated by applying the Painlevé analysis, obtaining an analytic solution in terms of a right Painlevé expansion. On the other hand, in [15], we construct a family of exact anisotropic solutions while investigating the evolution of the field equations using dynamical system analysis in \( f(T, B) = T + F(B) \) theory of gravity. From the analysis, it follows that in this theory, with initial conditions of Kantowski–Sachs geometry, the future attractor of the Universe, can be a spatially flat spacetime which describes acceleration. The de Sitter spacetime exists for a specific value of the free parameter.

In these two analyses, we found that future attractors exist where the Universe is spatially flat and isotropic. We perform a similar analysis for the LRS Bianchi III Universe in the following. For previous studies of Bianchi III Universes in various gravitational theories we refer the reader to [16–27] and references therein. The application of teleparallelism in anisotropic geometries is presented in detail in [14], and [15]. Thus we continue with the plan of the present paper.

In Sect. 2 we briefly present the gravitational field equations of \( f(T, B) \)-theory. In Sect. 3 we focus on the LRS Bianchi III Universe, and we derive the field equations. Some exact solutions of special interest are studied in Sect. 4. Section 5 includes the main results of this analysis, where we present a detailed study of the dynamics of the field equations. Finally, in Sect. 6 we discuss our results.

2 \( f(T, B) \) gravity

In this section, we briefly discuss the field equations in the modified teleparallel theory of our consideration; more details can be found in [28] or in the previous article of this series of studies [14].

In \( f(T, B) \)-theory the Gravitational Action Integral is defined [29]

\[
S_{f(T, B)} = \frac{1}{16\pi G} \int d^4x f(T, B)
\]

(1)

where \( T \) is the torsion scalar for the Weitzenböck connection [30] and \( B = 2e^{-1} \partial_\nu(e^\mu T_\mu^{\nu}) \). The Ricciscalar \( R \) and the torsion scalar \( T \) are related as \( B = T + R \).

Variation of the Action Integral (1) for the vierbein fields lead to the field equations

\[
0 = ef_T G^a_{\mu} + \left[ \frac{1}{4}(T f_T - f)eh^a_{\mu} + e(f_{,\mu})_{,\lambda}S^a_{\lambda} \right]
+ \left[ e(f_{,\mu})_{,\lambda}S^a_{\lambda} - \frac{1}{2}e\left( h^a_{\mu}(f_{,\mu})_{,\lambda} - h^a_{\mu}(f_{,\lambda})_{,\nu}g_{\mu\nu} \right) + \frac{1}{4}eBh^a_{\mu}f_{,\mu} \right].
\]

(2)

For the special case of \( f(T, B) = T + F(B) \) theory which we will study in this work, we can define the scalar field \( \phi \) and the potential function as \( \phi = F_B \) and \( V(\phi) = F - BF_B \) such that the field equations can be written in the equivalent form

\[
eG^a_{\mu} + \left( e\phi_{,\mu}S^a_{\lambda} - \frac{1}{2}e\left( h^a_{\mu}\phi_{,\lambda} - h^a_{\lambda}\phi_{,\mu}g_{\mu\nu} \right) + \frac{1}{4}eh^a_{\mu}V(\phi) + \frac{1}{4}eh^a_{\mu}f \right) = 0
\]

(3)

We consider the background geometry to be that of LRS Bianchi III spacetime.

3 LRS Bianchi III Universe

The line element for the LRS Bianchi III Universe is

\[
ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left( e^{2\beta(t)}dx^2 + e^{-\beta(t)}dy^2 + \sinh^2(y)dz^2 \right)
\]

(4)

where \( N(t) \) is the lapse function, \( \alpha(t) \) is the scale factor for the three-dimensional hypersurface and \( \beta(t) \) is the anisotropic parameter. For \( \beta(t) \to 0 \), the line element (4) reduces to the closed FLRW geometry.

We assume the vierbein fields [28]

\[
e^1 = Ndt
\]

\[
e^2 = ie^{\alpha+\beta}\cos z \sinh y \ dx + e^{\alpha-\beta} (\cosh y \cos z \ dy - \sinh y \sin z \ dz)
\]

\[
e^3 = ie^{\alpha+\beta} \sin z \ \sinh y \ dx + e^{\alpha-\beta} (\cosh y \sin z \ dy - \sinh y \cos z \ dz)
\]

\[
e^4 = -e^{\alpha+\beta} \cosh y \ dx - i e^{\alpha-\beta} \sinh y \ dy
\]

which provides...
\[ T = \frac{1}{N^2} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) + 2e^{-2\alpha+\beta}, \]  
(5)  
such that TEGR is recovered.

Moreover, the boundary term is calculated

\[ B = \frac{6}{N^2} \left( \ddot{\alpha} - \ddot{\alpha} \frac{\dot{N}}{N} + 3\ddot{\alpha}^2 \right). \]  
(6)  
Hence, the modified gravitational field equations are

\[ 0 = 6H^2 - \frac{3}{2} \dot{\beta}^2 - 6H\dot{\phi} + V(\phi) - 2e^{-2\alpha+\beta}, \]  
(7)  
\[ 0 = \dot{H} + 3H^2 + \frac{1}{6} V_{,\phi} , \]  
(8)  
\[ 0 = \ddot{\beta} + 3H\dot{\beta} + \frac{2}{3} e^{-2\alpha+\beta} , \]  
(9)  
and

\[ 0 = \ddot{\phi} + 3H^2 + \frac{1}{2} V(\phi) + \frac{1}{3} V_{,\phi} + \frac{3}{4} \dot{\beta}^2 + \frac{1}{3} e^{-2\alpha+\beta} . \]  
(10)  

Where without loss of generality we have selected \( N = 1 \) and \( H = \dot{\alpha} \).

An important characteristic is the existence of a minisuperspace description for the theory. Indeed, there exists the point-like Lagrangian

\[ \mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, \phi, \dot{\phi}) = \frac{1}{N} \left( e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} \dot{\alpha} \dot{\phi} \right) + Ne^{3\alpha} V(\phi) + 2Ne^{\alpha+\beta} , \]  
(11)  

which generates the field equations. The existence of this point-like Lagrangian is essential because techniques from Analytic Mechanics can be applied for the study of the field equations, such is the Noether symmetry analysis for the determination of conservation laws or the quantization process, such approaches are discussed in [31].

4 Exact solutions

Let us now investigate the existence of a power-law solution for the field equations (7)–(10). We assume that \( \alpha = \alpha_0 \ln t \) then, Eq (9) ends with

\[ 0 = \ddot{\beta} + 3\alpha_0 t^{-1} \dot{\beta} + \frac{2}{3} e^{-\beta} e^{-2\alpha_0 t} . \]  
(12)  
Hence, a closed-form solution of the latter equation is

\[ \beta(t) = 2(\alpha_0 - 1) \ln t , \alpha_0 = \frac{2}{3} . \]  
(13)  
Moreover, by replacing in the rest of the field equations we end with the system

\[ V(\phi) = -4t^{-1} \dot{\phi} , \]  
(14)  
and

\[ t\ddot{\phi} - 2\dot{\phi} = 0. \]  
(15)  
For the scalar field we find \( \phi(t) = \frac{\phi_1}{3} t^3 + \phi_1 \), while for the scalar field potential it follows

\[ V(\phi) = -4 \phi_1^2 \frac{2}{3} (3(\phi - \phi_0))\frac{3}{4} . \]  
(16)  
We proceed with the analysis of the dynamics.
5 Asymptotic dynamics

We define the dimensionless variables

\[ \Sigma = \frac{\beta}{2H}, \quad x = \frac{\phi}{H}, \quad y = \frac{V(\phi)}{H^2}, \quad \Omega_R = \frac{e^{-2a+\beta}}{3H^2}, \quad \lambda = -\frac{V_0}{V} \]  

(17)

where as in the previous studies we assume \( V(\phi) = V_0 e^{-\lambda \phi} \), such that \( \lambda = \text{const} \). The selection of the exponential potential function is two-fold. In terms of dynamics, for such potential function, the dimension of the dynamical system is reduced by one; however this can provide the stationary points and for other potential functions in the limit where \( \lambda = \text{const} \), see for instance the discussion in [32]. Additionally, the exponential potential is of special interest in terms of an isotropic universe. In previous studies, [33, 34]; it was found that such potential is cosmological viable and can explain the main epochs of the cosmological evolution. Finally, the exponential potential has been found to provide integrable cosmological equations in the case of isotropic and spatially flat universe [32].

Thus, in the new variables \((\Sigma, x, y, \eta)\) the field equations are written as the following system of algebraic-differential equations

\[ \frac{d\Sigma}{d\tau} = -\lambda y \Sigma - \Omega_R, \]  

(18)

\[ \frac{dx}{d\tau} = 3(\Sigma^2 - 1) + (2\lambda - 3)y + x(3 - \lambda y) - \Omega_R, \]  

(19)

\[ \frac{dy}{d\tau} = -y(\lambda(x + 2y) - 6), \]  

(20)

and

\[ \frac{d\Omega_R}{d\tau} = 2(2 - \lambda y + \Sigma)\Omega_R, \]  

(21)

with algebraic equation

\[ 1 - x - y - \Sigma^2 - \Omega_R = 0, \]  

(22)

and \( d\tau = H dt \).

Furthermore, the deceleration parameter \( q = -1 - \frac{\dot{H}}{H^2} \) is calculated

\[ q(\Sigma, x, y, \eta) = 2 - \lambda y. \]  

(23)

We determine the stationary points for the dynamical system (18)–(22). Because of the constraint equation (22), the dimension of the dynamical system can be reduced by one, and without loss of generality, we select to replace \( y = 1 - x - \Sigma^2 - \Omega_R \) in the field equations and we end with the system (18), (19) and (21). For the asymptotic solution to describe a real solution, it follows that \( \Omega_R \geq 0 \). The rest of the variables are not a constraint.

5.1 Analysis at the finite regime

We replace \( y \) from (22) in Eqs. (18), (19) and (21) and we end with the system

\[ \frac{d\Sigma}{d\tau} = -\lambda y \Sigma - \Omega_R, \]  

(24)

\[ \frac{dx}{d\tau} = 3(\Sigma^2 - 1) + (2\lambda - 3)(1 - x - \Sigma^2 - \Omega_R) + x(3 - \lambda(1 - x - \Sigma^2 - \Omega_R) - \Omega_R), \]  

(25)

\[ \frac{d\Omega_R}{d\tau} = 2(2 - \lambda(1 - x - \Sigma^2 - \Omega_R) + \Sigma)\Omega_R, \]  

(26)

where now the deceleration parameter becomes

\[ q(\Sigma, x, \eta) = 2 - \lambda(1 - x - \Sigma^2 - \Omega_R). \]  

(27)

The stationary points \( A = (\Sigma(A), x(A), \Omega_R(A)) \) and the corresponding eigenvalues for the dynamical system are

\[ A_1 = (\Sigma, (1 - \Sigma^2), 0), \]  

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with eigenvalues

\[ e_1(A_1) = 0, \quad e_2(A_1) = 2(\Sigma + 2), \quad e_3(A_1) = (6 + \lambda(\Sigma^2 - 1)). \]  

(28)

The asymptotic solutions described by the family of points \( A_1 \) are that of anisotropic Bianchi I spacetimes. Because for \( \Omega_R = 0 \), the dynamical system is reduced to that of Bianchi I, the analysis presented in [14] is valid. From there, we infer that the anisotropic solutions of \( A_1 \) are always unstable.
Fig. 1 Phase-space portrait for the dynamical system on the-dimensional surface \((\Sigma, x)\) for \(\Omega_R = \frac{3(\lambda - 4)(2 + \lambda)}{(1 + 2\lambda)^2}\) from where we observe that the Kantoski-Sacks solution described by \(A_3\) is always a saddle point.

Table 1 Stationary points at the finite regime

| Point   | Existence | Spacetime      | \(q < 0\) | Stable? |
|---------|-----------|----------------|-----------------|---------|
| \(A_1\) | Always    | Bianchi I      | No              | No      |
| \(A_2\) | \(\lambda \neq 0\) | FLRW (Flat) | \(\lambda < 4\) | \(\lambda < 4\) |
| \(A_3\) | \(\lambda \geq 4, \lambda \leq -2\) | Bianchi III | No              | \(\lambda > 4\) |

\[
A_2 = \left(0, 2 - \frac{6}{\lambda}, 0\right),
\]

with eigenvalues
\[
e_1(A_2) = (\lambda - 6), \quad e_2(A_2) = (\lambda - 6), \quad e_3(A_2) = 2(\lambda - 4).
\]

The stationary point \(A_2\) describes a spatially flat FLRW geometry with acceleration when \(\lambda < 4\), while in the particular case in which \(\lambda = 3\) the de Sitter spacetime is recovered. We note that at this point, the kinetic part of the scalar field and the potential term contributes to the cosmological fluid. Moreover, the point is a sink for \(\lambda < 4\).

\[
A_3 = \left(\frac{4 - \lambda}{1 + 2\lambda}, \frac{6(\lambda - 1)}{\lambda(1 + 2\lambda)}, \frac{3(\lambda - 4)(\lambda + 2)}{(1 + 2\lambda)^2}\right)
\]

with corresponding eigenvalues
\[
e_1(A_3) = -\frac{3(2 + \lambda)}{1 + 2\lambda}, \quad e_{2,3}(A_3) = \frac{-3\lambda - 6 \pm i \sqrt{3(2 + \lambda)(\lambda(16\lambda - 59) - 38)}}{2(1 + 2\lambda)}.
\]

The point is physically accepted when \(\lambda \geq 4\) and \(\lambda \leq -2\). The deceleration parameter is \(q(A_3) = \frac{\lambda - 4}{2 + \lambda}\) which means that \(q(A_3) \geq 0\) when the point exists. Thus, there is no acceleration. Finally from the eigenvalues we infer that for \(\lambda > 4\) the asymptotic solution is stable, that is, \(A_3\) is a sink, while for \(\lambda < -2\), point \(A_3\) is a saddle point.

In Fig. 1 we present phase-space portraits for the dynamical system on the surface \(\Omega_R = \frac{3(\lambda - 4)(2 + \lambda)}{(1 + 2\lambda)^2}\), where it is clear that for \(\lambda > 4\) the stationary point \(A_3\) is an attractor.

Table 1 summarizes the present analysis of the finite regime.

5.2 Analysis at the infinity

We define the Poincaré variables
\[
x = \frac{\rho}{\sqrt{1 - \rho^2}} \cos \Theta, \quad \Sigma = \frac{\rho}{\sqrt{1 - \rho^2}} \sin \Theta \cos \Psi,
\]
\[
\Omega_R = \frac{\rho^2}{1 - \rho^2} \sin^2 \Theta \sin^2 \Psi, \quad d\sigma = \sqrt{1 - \rho^2} d\tau,
\]

with \(\rho \in [0, 1], \Theta \in [0, \pi]\) and \(\Psi \in [0, \pi]\).
Therefore, the field equations read
\[
\frac{4 \, d \rho}{d \sigma} = -4 \rho^4 \cos(\Theta)((\lambda - 2) \cos(2\Theta) + 2 \sin^2(\Theta) \cos(2\Psi) - 2\lambda + 8) \\
+ 4 \rho^2 \cos(\Theta)((\lambda - 2) \cos(2\Theta) + 2 \sin^2(\Theta) \cos(2\Psi) - 4\lambda + 14) \\
- 2 \sqrt{1 - \rho^2} \rho ((2\lambda - 5) \cos(2\Theta) + 2 \sin^2(\Theta) \cos(2\Psi) + 4\lambda - 7) \\
- 4 \sin(\Theta)\left( \rho \left( \sqrt{1 - \rho^2} \cos(\Theta)(-2\lambda + \cos(2\Psi) + 5) + (\lambda - 2) \rho \cos(2\Theta) + 2\rho \sin^2(\Theta) \cos(2\Psi) \right) \right) \\
- \frac{\rho}{4 \sin(\Theta)}(\lambda - 3) \cos(\Theta) \\
- 2 \sqrt{1 - \rho^2} \rho^3 ((\lambda - 5) \cos(2\Theta) + 2 \sin^2(\Theta) \cos(2\Psi) + 5\lambda - 7),
\]

(33)

\[
\frac{d \Theta}{d\sigma} = \rho^2 \left( (-2(\lambda - 2) \cos(2\Theta) + \cos(2(\Theta - \Psi)) + \cos(2(\Theta + \Psi)) \\
+ 6\lambda - 2 \cos(2\Psi) - 16) \right) + 2 \sqrt{1 - \rho^2} \rho \cos(\Theta)(-2\lambda + \cos(2\Psi) + 5) + 4(\lambda - 3),
\]

(34)

\[
\frac{d \Psi}{d\sigma} = \sin(\Psi) \left( \rho \sin(\Theta) + 2 \sqrt{1 - \rho^2} \cos(\Psi) \right).
\]

(35)

Infinity is reached when \( \rho = 1 \), thus the stationary points at the infinity are defined on the two-dimensional surface \((\Theta, \Psi)\). The points are

\[
B^1 = (0, \Psi), \quad B^2 = (\pi, \Psi)
\]

for arbitrary \( \lambda \), and

\[
D^1 = (\Theta, 0), \quad D^2 = (\Theta, \pi)
\]

when \( \lambda = 3 \). Similarly to the study presented in [15] for the Kantowski–Sachs Universe, the stationary points at the infinity, describe isotropic spatially flat FLRW universes for arbitrary \( \lambda \), and Bianchi I geometry for \( \lambda = 3 \). The eigenvalues for the stationary points are

\[
e_1(B_1) = 0, \quad e_2(B_1) = 0, \quad e_3(B_1) = -2\lambda,
\]

(36)

\[
e_1(B_2) = 0, \quad e_2(B_2) = 0, \quad e_3(B_2) = 2\lambda.
\]

(37)

We investigate the stability properties of the stationary points by using numerical results. In Fig. 2 we present two-dimensional phase-space portraits for the dynamical system at the Poincare variables for various values of parameters \( \lambda \). From the evolution of the trajectories, it is straightforward to conclude that the stationary points at the infinity always describe unstable solutions.

6 Conclusions

We performed a detailed analysis of the global dynamics of \( f(T, B) \)-theory for an LRS Bianchi III spacetime. Specifically, we consider the \( f(T, B) = T + F(B) \) theory where \( F(B) \) function can be seen that introduces small deviations from the TEGR. The field equations can be written in the equivalent of a second-order theory with a scalar field for this specific theory. This work is part of our analysis of modified teleparallelism in anisotropic background geometries.

We determined the stationary points corresponding to asymptotic solutions for the field equations. The stability properties were investigated such that to construct the complete cosmological history. For the \( F(B) = -\frac{1}{2} B \) in \( B \), we found that for \( \lambda < 4 \), the final attractor of the field equations is a spatially flat and accelerated FLRW Universe, where the de Sitter Universe is recovered for \( \lambda = 3 \). However, for \( \lambda > 4 \) the future attractor is an anisotropic Universe described by the Bianchi III geometry. These are the two unique attractors for the field equations. The value \( \lambda < 4 \) agrees with the previous studies on the Bianchi I and Kantowski–Sachs geometries. For \( \lambda < 4 \), the \( f(T, B) \) can solve the flatness and isotropic problems by leading to an accelerated Universe. Hence, the isotropization process of the Universe depends only on the parameter \( \lambda \). We conclude that in \( f(T, B) \) theory, the initial conditions which describe a homogeneous and anisotropic open Universe can provide a spatially flat isotropic universe and explain the inflationary epoch.
Fig. 2 Phase-space portrait for the three-dimensionless dynamical system on the Poincaré variables. The plot clearly shows that the stationary points at the infinity always describe unstable solutions.

Last but not least, we mention that there are no attractors for the dynamical system at the infinity regime, and the trajectories of the field equations have the origin at the finite and infinity regimes; thus, the attractors are in the finite regime.

In [31] we continue our study by applying the Noether symmetries for the construction of conservation laws.

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