Program Verification via Predicate Constraint Satisfiability Modulo Theories

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This paper presents a verification framework based on a new class of predicate Constraint Satisfaction Problems called $\text{pCSP}^{\lambda}$, where constraints are represented as clauses modulo first-order theories over function variables and predicate variables that may represent well-founded predicates. The verification framework generalizes an existing one based on Constrained Horn Clauses (CHCs) to arbitrary clauses, function variables, and well-foundedness constraints. While it is known that the satisfiability of CHCs and the validity of queries for Constrained Logic Programs (CLP) are inter-reducible, we show that, thanks to the added expressiveness, $\text{pCSP}^{\lambda}$ is expressive enough to express $\mu$CLP queries. $\mu$CLP itself is a new extension of CLP that we propose in this paper. It extends CLP with arbitrarily nested inductive and co-inductive predicates and is equi-expressive as first-order fixpoint logic. We show that $\mu$CLP can naturally encode a wide variety of verification problems including but not limited to termination/non-termination verification and even full modal mu-calculus model checking of programs written in various languages. To establish our verification framework, we present (1) a sound and complete reduction algorithm from $\mu$CLP to $\text{pCSP}^{\lambda}$ and (2) a constraint solving method for $\text{pCSP}^{\lambda}$ based on stratified CounterExample-Guided Inductive Synthesis (CEGIS) of (co-)inductive invariants, ranking functions, and Skolem functions witnessing existential quantifiers. Stratified CEGIS combines CEGIS with stratified families of templates to achieve relative completeness and faster and stable convergence of CEGIS by avoiding the overfitting problem. We have implemented the proposed framework and obtained promising results on diverse verification problems that are beyond the scope of the previous verification frameworks based on CHCs.

1 INTRODUCTION

In the formal verification community, a class of predicate constraints called Constrained Horn Clauses (CHCs) [Bjørner et al. 2015] has been widely adopted as a "common intermediate language" for uniformly expressing verification problems for various programming paradigms, such as functional and object-oriented languages. Example uses of the CHCs framework include safety property verification [Grebenshchikov et al. 2012; Gurfinkel et al. 2015; Kahsai et al. 2016] and refinement type inference [Jhala et al. 2011; Kobayashi et al. 2011; Terauchi 2010; Unno and Kobayashi 2009; Zhu et al. 2015]. The wide applicability of CHCs is due in no small part to its expressiveness: it is known that the satisfiability of CHCs and the validity of queries for Constrained Logic Programs (CLP) [Jaffar and Maher 1994] are inter-reducible. The separation of constraint generation and solving has facilitated the rapid development of constraint generation tools such as RCAML [Unno and Kobayashi 2009], SeaHorn [Gurfinkel et al. 2015], and JayHorn [Kahsai et al. 2016] as well as efficient constraint solving tools such as SPACER [Komuravelli et al. 2014], ELDARICA [Hojjat and Rümmer 2018], and HoIce [Champion et al. 2018].

In this paper we show that the same phenomenon—separating constraint generation from solving—can empower a wider class of verification problems. To this end, we generalize CHCs and introduce a new class of predicate Constraint Satisfaction Problems called $\text{pCSP}^{\lambda}$ where constraints are arbitrary (i.e., possibly non-Horn) clauses modulo first-order theories over function...
variables and (possibly well-founded) predicate variables. We then show that $\text{pCSP}^{\lambda}$ can encode a wider range of verification problems including but not limited to termination/non-termination verification and even linear-time & branching-time temporal verification (full modal mu-calculus model checking) of programs written in various languages. All these become possible due to the increased expressiveness: we show that $\text{pCSP}^{\lambda}$ can express queries for a new extension of CLP, denoted $\mu\text{CLP}$, that has arbitrarily nested inductive and co-inductive predicates. $\mu\text{CLP}$ can naturally encode the above classes of verification problems and subsumes first-order fixpoint logics (which have recently been applied to temporal verification of imperative and functional programs [Kobayashi et al. 2019; Nanjo et al. 2018]).

The first part of this paper is a sound and complete reduction algorithm from $\mu\text{CLP}$ to $\text{pCSP}^{\lambda}$. The algorithm generalizes the recently proposed deductive system for the validity of first-order fixpoint logic [Nanjo et al. 2018] to $\mu\text{CLP}$. It obtains a collection $C$ of clause constraints that have placeholder function variables $T_\lambda, U_\lambda, \ldots$ and predicate variables $P, Q, \ldots$, including some for well-founded relations $R_\parallel, S_\parallel, \ldots$ (see Section 4 for the definition).

Next, we give a constraint solving method for $\text{pCSP}^{\lambda}$ based on stratified CounterExample-Guided Inductive Synthesis (CEGIS) of (co-)inductive invariants, ranking functions, and Skolem functions witnessing existential quantifiers. Stratified CEGIS combines CEGIS [Solar-Lezama et al. 2006] with stratified families of templates [Jhala and McMillan 2006; Terauchi and Unno 2015] to achieve relative completeness, a theoretical guarantee of convergence, and a faster and stable convergence by avoiding the overfitting problem of expressive templates to counterexamples [Padhi et al. 2019]. The constraint solving method naturally generalizes a number of previous techniques developed for CHCs solving and invariant/ranking function synthesis to the new class $\text{pCSP}^{\lambda}$. It proceeds with an iterative algorithm that attempts to discover appropriate functions/predicates or counterexamples to the given $\text{pCSP}^{\lambda} C$. Each iteration consists of a synthesis phase that attempts to guess the function/predicate variables (represented as a function/predicate substitution over the stratified families of templates) and a validation phase that determines whether the guess was valid. The validation is done by substituting the guess to $C$ and using an SMT solver to determine whether $C$ is satisfiable. When the substitution yields a satisfiable $C$ we conclude the $\mu\text{CLP}$ queries to be valid. Meanwhile, iterations maintain example instances of $C$ from failed attempts by previous candidates and if these examples become unsatisfiable, we conclude the queries to be invalid.

We have implemented the above framework. The implementation supports various widely used background theories: Booleans, linear integer and rational arithmetic. We have applied our tool to a diverse collection of verification problems (modal mu-calculus, CTL*, CTL, LTL, termination, and safety) and obtained promising results. The benchmark problems used for experiments go beyond the capabilities of the existing related tools (such as CHCs solvers and program verification tools).

The rest of the paper is organized as follows. Section 2 gives a brief overview with some examples. Section 3 defines $\mu\text{CLP}$ and discusses its expressiveness and applications. Section 4 defines $\text{pCSP}^{\lambda}$ and Section 5 formalizes the reduction from $\mu\text{CLP}$. We present our constraint solving method for $\text{pCSP}^{\lambda}$ based on stratified CEGIS in Section 6. Section 7 reports on the implementation and experimental evaluation of the presented framework. We discuss related work in Section 8 and conclude with a remark on future work in Section 9.

2 OVERVIEW

We now highlight the contributions of our work through a series of representative examples that we will return to later in the paper.
2.1 Modeling language $\mu$CLP: Generalizing CLP.

Our first contribution is $\mu$CLP. It generalizes CLP to allow describing a wider range of verification problems. Let us consider the termination verification problem of the following program obtained from the benchmark set of the FUNCtion tool [Urban 2013; Urban and Miné 2014], which is available from its web interface\(^1\):

while ($x_1 >= 0$ & & $x_2 >= 0$) {
    if (nondet()) {
        while ($x_2 <= 10$ & & nondet()) {
            $x_2 = x_2 + 1$
        }
        $x_1 = x_1 - 1$
    }
    $x_2 = x_2 - 1$
}

where nondet() returns a non-deterministic Boolean value. This program is always terminating for any external integer inputs $x_1, x_2$ and any internal Boolean non-deterministic choices.\(^2\)

The termination verification problem for the program can be modularly encoded as the following $\mu$CLP $P_{term}$ using both least and greatest fixpoints in the style of extended refinement type systems [Nanjo et al. 2018; Unno et al. 2017a]:

$$
\text{Query : } \forall x_1, x_2: \text{int. } I(x_1, x_2)
$$

$$
\text{Program : } \begin{cases}
I(x_1, x_2) =_{\mu} & \neg(x_1 \geq 0 \land x_2 \geq 0) \lor \\
& \left( J(x_2) \land \\
& \left( \forall x'_2: \text{int. } NP(x_2, x'_2) \lor I(x_1 - 1, x_2' - 1) \land \\
& I(x_1, x_2 - 1) \right) \right) \\
J(x_2) =_{\mu} & \neg(x_2 \leq 10) \lor J(x_2 + 1) \\
NP(x_2, x'_2) =_{\nu} & \neg(x_2 \leq 10) \land x'_2 = x_2 \lor \\
& x_2 \leq 10 \land (\neg NP(x_2 + 1, x'_2) \lor x'_2 = x_2)
\end{cases}
$$

Here, $J$ is an inductive predicate defined as the least fixpoint of the function $\mathcal{F}(J) = \lambda x_2. \neg(x_2 \leq 10) \lor J(x_2 + 1)$ over predicates (indicated by $\mu$ in $=_{\mu}$). Likewise, $I$ is also an inductive predicate and is defined as a least fixpoint. By contrast, $NP$ is a co-inductive predicate defined as the greatest fixpoint of the function $G(NP) = \lambda (x_2, x'_2). \neg(\neg(x_2 \leq 10) \land x'_2 = x_2 \lor x_2 \leq 10 \land (\neg NP(x_2 + 1, x'_2) \lor x'_2 = x_2))$. Intuitively, $I(x_1, x_2)$ and $J(x_2)$ characterize the weakest pre-conditions for the termination of the outer and the inner loops, respectively; Note that the inner loop always terminates (regardless of non-deterministic choices) if and only if $\neg(x_2 \leq 10)$ eventually holds after a finite number of iterations of incrementing $x_2$, which is here enforced by the least-fixpoint definition of $J$. $NP(x_2, x'_2)$ denotes the complement of the following inductive predicate $P(x_2, x'_2)$ which characterizes the strongest post-condition of the inner loop:

$$
P(x_2, x'_2) =_{\mu} \neg(x_2 \leq 10) \land x'_2 = x_2 \lor x_2 \leq 10 \land (P(x_2 + 1, x'_2) \lor x'_2 = x_2).
$$

In the definition of $I$, $NP(x_2, x'_2)$ is used to bind $x'_2$ to a possible value of the program variable $x_2$ upon the termination of the inner loop, encapsulating the internal behavior of the inner loop. Thus, the query formula is valid if and only if the program is always terminating for all initial integer valuations of $x_1$ and $x_2$ and for all internal non-deterministic choices. Though we could encode the termination verification problem using only least fixpoints by regarding the given program as a single monolithic transition system, this example demonstrates an advantage of the use of $\mu$CLP for modularly and naturally encoding verification problems.

To demonstrate another advantage of $\mu$CLP, let us now consider verifying non-termination for the same program. Thanks to the expressiveness of $\mu$CLP, this can be encoded as the following

\(^1\)https://www.di.ens.fr/~urban/FuncTion.html

\(^2\)Note that the termination is witnessed by, for example, the lexicographic order of $x_1, x_2$. 


\( \mu \text{CLP} \) \( \mathcal{P}_{\text{nterm}} \), which is simply the De Morgan dual of \( \mathcal{P}_{\text{term}} \):

**Query:** \( \exists x_1, x_2 : \text{int.} \ N\mathcal{I}(x_1, x_2) \)

**Program:**

\[
\begin{align*}
N\mathcal{I}(x_1, x_2) &= \nu \left( x_1 \geq 0 \land x_2 \geq 0 \land \right. \\
&\left. (\exists x'_2 : \text{int.} \ P(x_2, x'_2) \land N\mathcal{I}(x_1 - 1, x'_2 - 1)) \lor \\ 
&\left. N\mathcal{I}(x_1, x_2 - 1) \right) \\
N\mathcal{J}(x_2) &= \nu \ x_2 \leq 10 \land N\mathcal{J}(x_2 + 1) \\
P(x_2, x'_2) &= \mu \neg (x_2 \leq 10) \land x'_2 = x_2 \lor x_2 \leq 10 \land (P(x_2 + 1, x'_2) \lor x'_2 = x_2)
\end{align*}
\]

Intuitively, \( N\mathcal{I}(x_1, x_2) \) and \( N\mathcal{J}(x_2) \) respectively characterize the weakest pre-conditions for the non-termination of the outer and the inner loops, which generalize the recurrent sets [Gupta et al. 2008] whose inhabitant witnesses the non-termination of the given program, for the purpose of modular encoding. Recall that \( P(x_2, x'_2) \) characterizes the strongest post-condition of the inner loop.

Our validity checker \( \text{MuVal} \) for \( \mu \text{CLP} \) tries to solve the primary \( \mathcal{P}_{\text{term}} \) and the dual \( \mathcal{P}_{\text{nterm}} \) in parallel. The primal-dual approach turns out to be particularly useful for branching-time temporal verification where we found either the primary or the dual is often easier to solve than the other (cf. Section 7).

As we show in Section 3.3 and Appendix A, \( \mu \text{CLP} \) is expressive enough to naturally encode a diverse class of program verification problems.

- Linear-time temporal verification of labeled transition systems. Section 3.3 explains a reduction from \( \omega \)-regular model checking where the specifications are given as Büchi word automata (which strictly subsume LTL).
- Bisimulation and bisimilarity verification between labeled transition systems (Appendix A.1).
- Infinite state, infinite duration games. Safety games, reachability games and so-called LTL games (Appendix A.2).

Also, it immediately follows from existing results that \( \mu \text{CLP} \) can encode:

- Linear-time temporal verification of functional programs [Kobayashi et al. 2019; Nanjo et al. 2018].
- Branching-time temporal verification. A reduction algorithm from modal mu-calculus model checking of imperative programs is shown in [Kobayashi et al. 2019].

### 2.2 Intermediate representation \( \text{pCSP}^{\mu\lambda} \)

\( \mu \text{CLP} \) is a very expressive language, and existing verification intermediate representations such as Constrained Horn Clauses (CHCs) [Bjørner et al. 2015] are not powerful enough to capture the full class of \( \mu \text{CLP} \). We therefore introduce a new verification intermediate representation: a class of *predicate* Constraint Satisfaction Problems denoted \( \text{pCSP}^{\mu\lambda} \). \( \text{pCSP}^{\mu\lambda} \) is a generalization of CHCs to arbitrary clauses, function variables, and well-foundedness constraints over predicate variables.

We will present a sound and complete reduction from \( \mu \text{CLP} \) validity to \( \text{pCSP}^{\mu\lambda} \) satisfiability in Section 5. It is inspired by a recently proposed deductive system for first-order fixpoint logic [Nanjo et al. 2018] that eliminates least and greatest fixpoints by over- and under-approximations via (co-)inductive invariants and well-founded relations, and eliminates quantifiers by Skolemization. For the termination verification problem \( \mathcal{P}_{\text{term}} \), our reduction gives the following set of clauses
whose term variables are implicitly universally quantified:

\[
\begin{align*}
I(x_1, x_2) & \equiv (x_1 \geq 0 \land x_2 \geq 0) \\
J(x_2) & \equiv (x_2 \leq 10) \lor J(x_2 + 1) \land J_I(x_2, x_2 + 2), \\
NP(x_2, x_2') & \equiv \neg(x_2 \leq 10) \land x_2' = x_2 \lor x_2 \leq 10 \land (\neg NP(x_2 + 1, x_2') \lor x_2' \neq x_2)
\end{align*}
\]

Here \(I, J,\) and \(NP\) are predicate variables that represent an under-approximation of the (co-)inductive predicates \(I, J,\) and \(NP,\) respectively. \(I_I\) and \(J_I\) are well-founded predicate variables that are required to represent a well-founded relation and used here to enforce a bounded unfolding of the inductive predicates \(I\) and \(J,\) respectively. Note here that, in the third clause, \(J_I(x_2, x_2 + 1)\) requires that the formal argument \(x_2\) of \(J\) and the actual argument \(x_2 + 1\) of the recursive call to \(J\) are related by a well-founded relation. Similarly, \(I_I(x_1, x_2, x_1 - 1, x_2' - 1)\) and \(J_I(x_1, x_2, x_1, x_2 - 1)\) in the second clause require that the pair \((x_1, x_2)\) of the formal arguments and the pairs of the actual arguments of the two recursive calls are respectively related by a well-founded relation. This transformation of inductive predicates generalizes binary reachability analysis that has been studied for termination verification of imperative [Cook et al. 2006; Podelski and Rybalchenko 2004b] and functional programs [Kuwahara et al. 2014]. The obtained pCSP\(^{\downarrow\lambda}\) is satisfiable: our satisfiability checker PCSat for pCSP\(^{\downarrow\lambda}\) reports the following satisfying predicate assignment:

\[
I(x_1, x_2) \mapsto \top, \quad J(x_2) \mapsto x_2 \geq 0, \quad NP(x_1, x_2) \mapsto \bot
\]

\[
I_I(x_1, x_2, x_1', x_2') \mapsto x_1 \geq 0 \land x_1 + x_2 \geq 0 \land (x_1 > x_1' \lor x_1 \geq x_1' \land x_1 + x_2 > x_1' + x_2')
\]

\[
J_I(x_2, x_2') \mapsto x_2 \geq 0 \land x_2' \geq 0 \land \max(22 - x_2, x_2') \geq 0 \land \max(22 - x_2, x_2') > \max(22 - x_2', x_2')
\]

Here, \(\max(t_1, t_2)\) represents the maximum of integer terms \(t_1\) and \(t_2,\) \(I_I\) and \(J_I\) represent the well-founded relations respectively induced by the lexicographic ranking function \(\lambda x_2 \geq 0.\ max(22 - x_2, x_2)\) and the piecewise-defined ranking function \(\lambda x_2 \geq 0.\ max(22 - x_2, x_2).\)

For the non-termination verification problem \(P_{\text{term}},\) our reduction gives the following pCSP\(^{\downarrow\lambda}\):

\[
\begin{align*}
NI(S_\lambda, T_\lambda) & \equiv x_1 \geq 0 \land x_2 \geq 0 \land \\
NI(x_1, x_2) & \equiv \left(\frac{NI(x_2)}{NI(x_1, x_2 - 1)} \lor \frac{NI(x_1 - 1, U_\lambda(x_1, x_2) - 1)) \lor}{P(x_2, x_2') \equiv \left(\frac{\neg(x_2 \leq 10) \land x_2' = x_2 \lor}{x_2 \leq 10 \land (P(x_2 + 1, x_2') \land P_I(x_2, x_2', x_2 + 1, x_2') \lor x_2' = x_2)\right)}
\end{align*}
\]

Here \(NI, NJ,\) and \(P\) are predicate variables that represent an under-approximation of the (co-)inductive predicates \(NI, NJ,\) and \(P,\) respectively. \(P_I\) is a well-founded predicate variable used here to enforce a bounded unfolding of \(P.\) \(S_\lambda, T_\lambda,\) and \(U_\lambda\) are function variables that represent total functions to be synthesized and used here to Skolemize the existential quantification of the term variables \(x_1, x_2\) in the query and \(x_2'\) in the body of \(NJ,\) respectively.\(^3\) Not surprisingly, this pCSP\(^{\downarrow\lambda}\) is unsatisfiable.

\(^3\)Note that we regard \(S_\lambda, T_\lambda\) as integer variables that represent integer functions of the arity 0.
2.3 CounterExample-Guided Inductive Synthesis

Our reduction from µCLP to pCSP[λ] may generate constraints that go beyond the class of CHCs. We thus present a new constraint solving method that can handle the general class of constraints, which is formally defined in Section 4 and here explained informally using the following example pCSP[λ]:

\[
C \triangleq \begin{cases}
    n \geq 0 \Rightarrow X(n), & (1) \\
    X(x) \Rightarrow (Y(x) \land X(x + 1)), & (2) \\
    Y(y) \Rightarrow (y = 0 \lor Y(y - 1) \land Y_\parallel(y, y - 1)) & (3)
\end{cases}
\]

Our constraint solving method is based on a CounterExample Guided Inductive Synthesis (CEGIS) approach to finding a solution of the given constraint set \(C\), i.e., a predicate substitution for \(X, Y, Y_\parallel\) that satisfies all the three formulas in \(C\) and the well-foundedness condition of \(Y_\parallel\). Our method is designed as a general constraint solving schema and this paper presents an instantiation of the schema based on template-based synthesis [Garg et al. 2014; Sharma et al. 2013b]. This section is mostly dedicated to informally reviewing well-known CEGIS with template-based synthesis in order to make the paper self-contained. Detailed exposition, in particular, our extensions with stratified families of function/predicate templates and unsat-core-based template refinement, are given in Section 6).

Our method first prepares predicate templates \(T_X, T_Y, T_\parallel\), with unknown parameters to be inferred, respectively for the predicate variables \(X, Y, Y_\parallel\) to restrict the solution space to be explored. For example, let us here use the templates:

\[
T_X \triangleq \lambda x. a \cdot x + b \geq 0, \quad T_Y \triangleq \lambda y. c \cdot y + d \geq 0, \\
T_\parallel \triangleq \lambda (z, z'). d \cdot z + e \geq 0 \land d \cdot z + e > d \cdot z' + e.
\]

Here, \(a, b, c, d, e\) are unknown parameters of the predicate templates. Note that the form of the predicate template \(T_\parallel\) for \(Y_\parallel\) guarantees that \(Y_\parallel\) is a well-founded relation for any valuation of \(d, e\): the function \(\lambda z. d \cdot z + e\) represents an affine ranking function whose return value strictly decreases, when the input changes from \(z\) to \(z'\). These templates are geared to the background theory, but other templates could be used for other theories.

Our constraint solving schema then iteratively accumulates examples \(E\) of the constraints in \(C\) by instantiating term variables to concrete values in a counterexample guided manner and enumerates candidate solutions using \(E\) until a genuine solution for \(C\) is obtained. More specifically, at each iteration \(i\), our schema consists of two phases: Synthesis Phase asks a synthesizer to obtain a candidate solution \(\sigma^{(i)}\) that satisfies the set of examples \(E^{(i)}\) and Validation Phase checks whether \(\sigma^{(i)}\) is a genuine solution of \(C\). If it is the case, our schema returns \(\sigma^{(i)}\) as a solution and otherwise repeats with \(E^{(i+1)}\) obtained from \(E^{(i)}\) by adding new examples that are not satisfied by \(\sigma^{(i)}\).

We now illustrate this procedure, using the running example. There will be four iterations, each with two phases.

First iteration. In the first iteration, we have no examples of the constraints yet, so we start with examples \(E^{(1)} = \emptyset\). The template-based synthesizer, using e.g. an SMT solver, may generate any candidate solution such as:

\[
\sigma^{(1)} \triangleq \vartheta^{(1)}(\{X \mapsto T_X, Y \mapsto T_Y, Y_\parallel \mapsto T_\parallel\}), \\
\vartheta^{(1)} \triangleq \{a \mapsto 0, b \mapsto 0, c \mapsto 0, d \mapsto 0, e \mapsto 0\}
\]

Our stratified template families further support templates of more general shapes: disjunctions of conjunctions of atomic formulas for ordinary predicate variables, well-founded relation templates induced by lexicographic piecewise-defined affine ranking for well-founded predicate variables, and piecewise-defined affine function templates for function variables.
where \( \theta^{(1)} \) is a substitution of values for template parameters. We next use this parameter assignment, substituting it back into the templates. In this example, substituting \( a \mapsto 0 \) and \( b \mapsto 0 \) into \( T_{\bar{X}} \), yields \( \lambda x.0 + 0 \geq 0, \) or \( \lambda x.\top \). We similarly obtain \( T_Y = \lambda y.\top \) but obtain \( T_{\bar{y}} = \lambda (z, z').\bot \). Together, we have

\[
\sigma^{(1)} \equiv \{ \bar{X} \mapsto \lambda x.\top, \bar{Y} \mapsto \lambda y.\top, \bar{y} \mapsto \lambda (z, z').\bot \}.
\]

We now enter the second phase of the first iteration: we need to check whether \( \sigma^{(1)} \) is a genuine solution of \( C \). We substitute \( \sigma^{(1)} \) back into \( C \) and for Eqn. 3, to obtain

\[
\sigma^{(1)}(Y)(y) \Rightarrow (y = 0 \lor \sigma^{(1)}(Y)(y - 1) \land \sigma^{(1)}(Y_{\bar{y}})(y, y - 1)).
\]

Using an SMT solver, we can find that this is not valid and, thus, the constraint is not satisfied. An SMT solver may, for example, generate a model \( y \mapsto 1 \), which gives us an example \( Y(1) \Rightarrow Y(0) \land Y_{\bar{y}}(1, 0) \) of \( C \) that is not satisfied by \( \sigma^{(1)} \). This provides us with a new example \( \mathcal{E}^{(2)} \) of \( C \) that is not satisfied by \( \sigma^{(1)} \):

\[
\mathcal{E}^{(2)} \equiv \{ Y(1) \Rightarrow Y(0) \land Y_{\bar{y}}(1, 0) \}.
\]

Remaining iterations 2, 3, and 4. The next iterations proceed similarly, and yield the following solutions and examples:

\[
\sigma^{(2)} \equiv \{ \bar{X} \mapsto \lambda x.\top, \bar{Y} \mapsto \lambda y.\top, \bar{y} \mapsto \lambda (z, z').z \geq 0 \land z > z' \}
\]

\[
\mathcal{E}^{(3)} \equiv \mathcal{E}^{(2)} \cup \{ Y(-1) \Rightarrow Y(-2) \land Y_{\bar{y}}(-1, -2) \}
\]

\[
\sigma^{(3)} \equiv \{ \bar{X} \mapsto \lambda x.\top, \bar{Y} \mapsto \lambda y.y \geq 0, \bar{y} \mapsto \lambda (z, z').z \geq 0 \land z > z' \}
\]

\[
\mathcal{E}^{(4)} \equiv \mathcal{E}^{(3)} \cup \{ X(-1) \Rightarrow (Y(-1) \land X(0)) \}
\]

\[
\sigma^{(4)} \equiv \{ \bar{X} \mapsto \lambda x.x \geq 0, \bar{Y} \mapsto \lambda y.y \geq 0, \bar{y} \mapsto \lambda (z, z').z \geq 0 \land z > z' \}
\]

From iteration 3, we have refined \( X \), requiring that it constrains \( y \) to be positive. This constraint eliminates the issue that arose from parameters to \( Y_{\bar{y}} \) being negative. Iteration 4 similarly teaches us that \( x \) must be positive. In the final iteration’s second phase, we find that \( \sigma^{(4)} \) is a genuine solution of \( C \) and exit the procedure.

## 3 Extended Constraint Logic Programs \( \mu \text{CLP} \)

This section defines the syntax and the semantics of the extension \( \mu \text{CLP} \) of constraint logic programs CLP [Jaffar and Maher 1994] with arbitrarily nested inductive and co-inductive predicates. We also discuss its application to temporal verification.

### 3.1 Syntax

Let \( \mathcal{T} \) be a (possibly many-sorted) first-order theory with the signature \( \Sigma \). The syntax of \( \mathcal{T} \)-formulas and \( \mathcal{T} \)-terms is:

- (formulas) \( \phi ::= X(t_1, \ldots, t_{\text{ar}(X)}) \mid p(t_1, \ldots, t_{\text{ar}(p)}) \mid \neg \phi \mid \phi_1 \lor \phi_2 \mid \phi_1 \land \phi_2 \mid \exists x: s.\phi \mid \forall x: s.\phi \)
- (terms) \( t ::= x \mid F(t_1, \ldots, t_{\text{ar}(F)}) \mid f(t_1, \ldots, t_{\text{ar}(f)}) \)

Here, the meta-variables \( x, X, \) and \( F \) respectively range over term, predicate, and function variables. The meta-variables \( p \) and \( f \) respectively denote predicate and function symbols of the signature \( \Sigma \). We use \( s \) as a meta-variable ranging over sorts of the signature \( \Sigma \). We write \( \bullet \) for the sort of propositions and \( s_1 \rightarrow s_2 \) for the sort of functions from \( s_1 \) to \( s_2 \). We henceforth regard a predicate variable as a function variable whose return sort is \( \bullet \). We write \( \text{ar}(o) \) and \( \text{sort}(o) \) respectively for the arity and the sort of a syntactic element \( o \). A function \( f \) represents a constant if \( \text{ar}(f) = 0 \). We write
$(\mu v)(\phi), \upsilon v(\phi),$ and $\nu v(\phi)$ respectively for the set of free term, predicate, and function variables that occur in $\phi$. Note that $\upsilon v(\phi) \subseteq \mu v(\phi)$ always holds. We write $\bar{x}$ for a sequence of term variables, $[\bar{x}]$ for the length of $\bar{x}$, and $e$ for the empty sequence. We often abbreviate $\neg \phi_1 \lor \phi_2$ as $\phi_1 \Rightarrow \phi_2$. We henceforth consider only well-sorted formulas and terms.

A $\mu$CLP $P$ over the theory $T$ is a sequence of mutually (co-)recursive equations of the form:\footnote{If we fix $T$ to integer arithmetic, $\mu$CLP coincides Mu-Arithmetic, a fixpoint logic with integer arithmetic studied in [Bradfield 1999; Lubarsky 1993] and reformalized as hierarchical equation systems (HES) in [Kobayashi et al. 2019].}

$$\forall i, j \in \{1, \ldots , m\}, X_i = a_i \phi_i; \ldots ; (X_m(\bar{x}_m) = a_m \phi_m)$$

Here, $a_i \in \{\mu, \nu\}$ and for any $i, j \in \{1, \ldots , m\}, X_i$ may occur only positively in $\phi_j$. An equation $X(\bar{x}) = \mu \phi$ that satisfies $\mu v(\phi) \subseteq \{X\}$ represents the inductive predicate $\mu X(\bar{x})$. $\phi$ defined as the least fixpoint of the function $\mathcal{F}(X) = \lambda \bar{x} \phi$ over predicates. Similarly, $X(\bar{x}) = \nu \phi$ that satisfies $\nu v(\phi) \subseteq \{X\}$ represents the co-inductive predicate $\nu X(\bar{x})$. $\phi$ defined as the greatest fixpoint of $\mathcal{F}$. Note here that $\mathcal{F}$ is monotonic because the bound predicate variable $X$ occurs only positively in the body $\phi$. In cases where $\mu v(\phi) \subseteq \{X\}$ does not hold, more sophisticated semantic treatment is required. We formalize this point later in Section 3.2. We define dom($P$) = $\{X_1, \ldots , X_m\}$. A query for a $\mu$CLP $P$ is defined as a $T$-formula $\phi$. The De Morgan dual $\neg \mathcal{P}$ of a $\mu$CLP $P$ = $(X_i(\bar{x}_i) = a_i \phi_i)_{i=1}^{m}$ is defined by $(\neg \phi_i)_{i=1}^{m}$ where $\sigma = \{(X_1 \mapsto \neg X_1', \ldots , X_m \mapsto \neg X_m')\}, -\mu \triangleq \nu$, and $-\nu \triangleq \mu$.

Remark 1. Note that quantifiers over recursively enumerable (r.e.) domains (e.g., integers) can be eliminated in $\mu$CLP. We can encode $\exists x: \text{int} \phi$ and $\forall x: \text{int} \phi$ with the bound integer variable $x$ respectively as $E(0)$ and $A(0)$ using the following inductive and co-inductive predicates $E$ and $A$:

$$E(x) = \mu \phi \lor [-x/x] \phi \lor E(x+1) \quad A(x) = \nu \phi \land [-x/x] \phi \land A(x+1)$$

Intuitively, $E$ and $A$ are required to hold for some and for all integer $x$, respectively. This encoding strategy, however, cannot apply to non r.e. domains like real numbers and is not useful in practice even for r.e. domains like rational numbers that have no simple way to enumerate all its elements. This is the reason why we apply Skolemization via function variables instead in our reduction algorithm to pCSpI$\lambda$ (see Section 5 for details).

3.2 Semantics

This section formalizes the denotational semantics of $\mu$CLP. Let $\mathcal{A} = (D, \Sigma, I)$ be the structure of the background first-order theory $T$. Here, $D$ is the universe, $\Sigma$ is the signature, and $I$ is the interpretation function for the predicate and function symbols in $\Sigma$. We write $D_s$ for the set of values in $D$ of the sort $s$. In particular, We define $D_s = \{\top, \bot\}$ for the sort $\bullet$ of propositions. For a sequence $\bar{s} = s_1, \ldots , s_m$ of sorts with $m \geq 0$, we write $D_{\bar{s}}$ for the sequence $D_{s_1}, \ldots , D_{s_m}$. We define $D_{\bar{s}} = D_{s_1} \to D_{s_2} \to \cdots \to D_s$. We assume that $I(p) \in D_{\bar{s}} \to D_s$ if sort($p$) = $\bar{s} \to \bullet$, and $I(f) \in D_{\bar{s}} \to D_s$ if sort($f$) = $\bar{s} \to s$. We introduce the partially ordered sets $(D_{\bar{s} \to \bullet}, \sqsubseteq_{\bar{s} \to \bullet})$ by defining

$$\sqsubseteq_{\bullet} = \{(\top, \top), (\bot, \bot), (\bot, \top), \} \quad \sqsubseteq_{\bar{s} \to \bullet} = \{(f, g) | \forall \bar{v} \in D_{\bar{s}}. f(\bar{v}) \sqsubseteq_{\bullet} G(\bar{v})\}.$$ 

The least upper bound $\sqcup_{\bar{s} \to \bullet}$ and the greatest lower bound $\sqcap_{\bar{s} \to \bullet}$ operators with respect to $\sqsubseteq_{\bar{s} \to \bullet}$ are then defined as follows:

$$\sqcap_{\bullet} = \top \quad \sqcup_{\bullet} = \bot \quad \sqcap_{\bar{s} \to \bullet} = \bot \quad \sqcup_{\bar{s} \to \bullet} = \top$$

Note that $(D_{\bar{s} \to \bullet}, \sqsubseteq_{\bar{s} \to \bullet})$ forms a complete lattice. The least and greatest elements of $D_{\bar{s} \to \bullet}$ are $\lambda \bar{x} \cdot \bot$ and $\lambda \bar{x} \cdot \top$ respectively.
Given a \( \mathcal{T} \)-formula \( \phi \) and an interpretation \( \rho \) of free term and function/predicate variables in \( \phi \), we write \([\phi](\rho)\) for the truth value of \( \phi \) which is defined as follows:

\[
\begin{align*}
[\forall x:\ s\ .\ \phi](\rho) & \triangleq \bigwedge\{ [\phi](\rho[x \mapsto v]) \mid v \in \mathcal{D}_s \} \\
[\exists x:\ s\ .\ \phi](\rho) & \triangleq \bigvee\{ [\phi](\rho[x \mapsto v]) \mid v \in \mathcal{D}_s \} \\
[-\phi](\rho) & \triangleq \begin{cases} 
\top & ([\phi](\rho) = \bot) \\
\bot & ([\phi](\rho) = \top) 
\end{cases} \\
[x](\rho) & \triangleq \rho(x) \\
[F(\overline{t})](\rho) & \triangleq \rho(F)(\overline{t})(\rho) \\
[I](\rho) & \triangleq I(\overline{f})(\overline{t})(\rho)
\end{align*}
\]

Here, we assume that \( \rho(x) \in \mathcal{D}_{\text{sort}(x)}, \rho(X) \in \mathcal{D}_s \rightarrow \mathcal{D}_s \) if \( \text{sort}(X) = \overline{s} \rightarrow \bullet \), and \( \rho(F) \in \mathcal{D}_s \rightarrow \mathcal{D}_s \) if \( \text{sort}(X) = \overline{s} \rightarrow s \). We write \( \rho \models \phi \) if and only if \([\phi](\rho) = \top \) holds for any extension \( \rho' \) of \( \rho \) for the term and function/predicate variables in \( (\text{ftv}(\phi) \cup \text{ftv}(\phi)) \setminus \text{dom}(\rho) \), where \( \text{dom}(\rho) \) represents the domain of \( \rho \). We say the given formula \( \phi \) is valid and write \( \models \phi \) if and only if \( \emptyset \models \phi \) holds.

Given a \( \mu \text{CLP} \ P \) and an interpretation \( \rho \) of free term and function/predicate variables in \( P \), we write \([P](\rho)\) for the predicate interpretation for \( \text{dom}(P) \) induced by \( P \) which is defined by:

\[
[X(\overline{x}) = _\alpha \phi]_{P}(\rho) \triangleq [X(\overline{x}) = _\alpha \phi]_{P}(\rho) \cup [P](\rho \cup [X(\overline{x}) = _\alpha \phi]_{P}(\rho))
\]

where \( \text{dom}(\rho) \cap \text{dom}(P) = \emptyset \) and the fixpoint operator \( \text{FP}^\overline{s} \rightarrow \bullet (\bullet) \) is defined by:

\[
\begin{align*}
\text{FP}^\overline{s} \rightarrow \bullet (\mu)(F) & \triangleq \bigcap\{ q \in \mathcal{D}_s \rightarrow \bullet \mid F(q) \sqsubseteq (\overline{q} \rightarrow \bullet) \} \\
\text{FP}^\overline{s} \rightarrow \bullet (\nu)(F) & \triangleq \bigcup\{ q \in \mathcal{D}_s \rightarrow \bullet \mid q \sqsupseteq (\overline{q} \rightarrow \bullet) \}
\end{align*}
\]

We write \( P \models \phi \) if and only if \([P](\emptyset) \models \phi \) holds.

**Example 3.1.** Let us consider \( \mu \text{CLP} P_{\mu} \triangleq (X = _\nu X \land Y); \ (Y = _\mu X \lor Y) \) and \( \mu \text{CLP} P_{\mu \nu} \triangleq (Y = _\mu X \lor Y); \ (X = _\nu X \land Y) \). Note that the semantics of \( P_{\mu} \) and \( P_{\mu \nu} \) are different as shown below, though the definition of \( P_{\mu \nu} \) and \( P_{\mu \nu} \) only differ in the order of the equations:

\[
\begin{align*}
[P_{\nu \mu}](\emptyset) & = [X = _\nu X \land Y](Y = _\mu X \lor Y)(\emptyset) \cup [Y = _\mu X \lor Y]_e\{[X = _\nu X \land Y]_{(Y = _\mu X \lor Y)}(\emptyset)\} \\
& = \emptyset \cup [Y = _\mu X \lor Y]_e([X \mapsto \top]) \\
& = \{X \mapsto \top, Y \mapsto \top\}
\end{align*}
\]
We now demonstrate the expressiveness of \( \mu \)-CLP verification. In recent years, a wide variety of techniques and tools have emerged for verifying temporal properties of programs. Here are some examples. In the setting of infinite-state imperative programs, there have been works that prove CTL properties \([\text{Beyene et al. } 2013; \text{Cook et al. } 2011, 2013]\), LTL properties \([\text{Cook and Koskinen } 2011; \text{Dietsch et al. } 2015]\), and others such as CTL* properties \([\text{Cook et al. } 2015]\). For infinite-state higher-order programs, \([\text{Murase et al. } 2016]\) and \([\text{Koskinen and Terauchi } 2014]\) respectively present automata-theoretic and type-based approaches to verification of \( \omega \)-regular properties (that subsume LTL). As already mentioned, there are recent proposals of reductions from temporal program verification to validity checking in fixpoint logic \([\text{Kobayashi et al. } 2019, 2018; \text{Nanjo et al. } 2018; \text{Watanabe et al. } 2019]\). Our validity checking method for \( \mu \)-CLP can be combined with their reductions to yield an automated temporal verification method for infinite-state imperative and functional programs that can solve classes of verification problems beyond the reach of the existing verification tools.

As an exemplary instance of such a reduction, we next formalize the reduction from linear temporal property verification of infinite-state systems to \( \mu \)-CLP. First, we review the notion of labeled transition system (LTS). A LTS is a triple \( M = (S, T, L) \) where \( S \subseteq D^n \) is the set of states, \( L \) is the finite set of labels, \( T \subseteq S \times L \times S \) is the transition relation. (Note that \( S \) may be infinite and therefore we allow infinite-state systems.) For \( M = (S, T, L) \), we often write \( S_M \) for \( S \), \( T_M \) for \( T \), and

\[
\rho^\nu\mu = \{ X \mapsto FP^*_\mu (\lambda q. [X \land Y] (\{ X \mapsto q \} \cup [Y =_\mu X \lor Y] (\{ X \mapsto q \}))) \} = \{ X \mapsto \bigcup_i \{ q \in D_i \mid q \models [X \land Y] (\{ X \mapsto q \} \cup [Y =_\mu X \lor Y] (\{ X \mapsto q \})) \} = \{ X \mapsto \bigcup_i \{ q \in D_i \mid q \models [X \land Y] (\{ X \mapsto q \} \cup \rho^\nu_i (Y)) \} = \{ X \mapsto \bigcup_i \{ q \in D_i \mid q \models [X \land Y] (\{ X \mapsto q \} \lor \rho^\nu_i (Y)) \} = \{ X \mapsto T \}
\]

\[
\rho^\nu_i = \{ Y \mapsto FP^*_\mu (\lambda q'. [X \lor Y] (\{ X \mapsto q, Y \mapsto q' \})) \} = \{ Y \mapsto q \}
\]

\[
[\mathcal{P}_{\mu \nu}](\emptyset) = [Y =_\mu X \lor Y]_{(X=\nu,X \land Y)}(\emptyset) = \{ Y \mapsto \bot \} \cup [X =_\nu X \land Y]_{\epsilon}(\{ Y \mapsto \bot \}) = \{ X \mapsto \bot, Y \mapsto \bot \}
\]

**Definition 3.2.** A validity checking problem \((\phi, \mathcal{P})\) of a query \( \phi \) for a \( \mu \)-CLP \( \mathcal{P} \) is that of deciding \( \mathcal{P} \models \phi \), which we will also write \( \models (\phi, \mathcal{P}) \).

**Remark 2.** The validity of \( \mu \)-CLP \((\phi_0, ((X_1(\overline{x}_1) =_{\alpha_1} \phi_1); \cdots; (X_m(\overline{x}_m) =_{\alpha_m} \phi_m))) \) has an equivalent \( \mu \)-CLP validity if all the following conditions are met: (1) \( \alpha_i = v \) for all \( i = 1, \ldots, m \), (2) \( X_1, \ldots, X_m \) occur only positively in \( \phi_0 \), and (3) universal (resp. existential) quantifiers occur only positively (resp. negatively) in \( \phi_i \)’s. We henceforth call this fragment of \( \mu \)-CLP, validity-reducible. Similarly, the invalidity of \( \mu \)-CLP has an equivalent \( \mu \)-CLP validity if: (1) \( \alpha_i = v \) for all \( i = 1, \ldots, m \), (2) \( X_1, \ldots, X_m \) occur only positively in \( \phi_0 \), (3a) \( \phi_i \)'s only free variables are \( \overline{x}_i \), and (3b) universal (resp. existential) quantifiers occur only negatively (resp. positively) in \( \phi_i \)’s. We call this \( \mu \)-CLP fragment, invalidity-reducible.

### 3.3 Application to Temporal Property Verification

We now demonstrate the expressiveness of \( \mu \)-CLP by showing that it can encode temporal property verification. In recent years, a wide variety of techniques and tools have emerged for verifying temporal properties of programs. Here are some examples. In the setting of infinite-state imperative programs, there have been works that prove CTL properties \([\text{Beyene et al. } 2013; \text{Cook et al. } 2011, 2013]\), LTL properties \([\text{Cook and Koskinen } 2011; \text{Dietsch et al. } 2015]\), and others such as CTL* properties \([\text{Cook et al. } 2015]\). For infinite-state higher-order programs, \([\text{Murase et al. } 2016]\) and \([\text{Koskinen and Terauchi } 2014]\) respectively present automata-theoretic and type-based approaches to verification of \( \omega \)-regular properties (that subsume LTL). As already mentioned, there are recent proposals of reductions from temporal program verification to validity checking in fixpoint logic \([\text{Kobayashi et al. } 2019, 2018; \text{Nanjo et al. } 2018; \text{Watanabe et al. } 2019]\). Our validity checking method for \( \mu \)-CLP can be combined with their reductions to yield an automated temporal verification method for infinite-state imperative and functional programs that can solve classes of verification problems beyond the reach of the existing verification tools.

As an exemplary instance of such a reduction, we next formalize the reduction from linear temporal property verification of infinite-state systems to \( \mu \)-CLP. First, we review the notion of labeled transition system (LTS). A LTS is a triple \( M = (S, T, L) \) where \( S \subseteq D^n \) is the set of states, \( L \) is the finite set of labels, \( T \subseteq S \times L \times S \) is the transition relation. (Note that \( S \) may be infinite and therefore we allow infinite-state systems.) For \( M = (S, T, L) \), we often write \( S_M \) for \( S \), \( T_M \) for \( T \), and
We regard the variables in \( \text{ftv} \) for \( L_M \) for \( L \). We write \( s \xrightarrow{\ell} M s' \) when \((s, \ell, s') \in T_M \). We omit the subscript \( M \) when it is clear from the context.

We now review the notion of a Büchi automaton. A (non-deterministic) Büchi automaton \( A \) is a tuple \((Q, L, \delta, q_{\text{init}}, F)\) where \( Q \) is the finite set of states (unrelated to the states of the LTS), \( L \) is the finite set of labels, \( \delta \subseteq Q \times L \times Q \) is the transition relation, \( q_{\text{init}} \in Q \) is the starting state, and \( F \subseteq Q \) is the set of final states. For \( q \in Q \) and \( \ell \in L \), we write \( \delta(q, \ell) \) for the set \( \{q' \in Q \mid (q, \ell, q') \in \delta\} \). An infinite word \( \ell_0\ell_1 \cdots \in L^\omega \) is accepted by \( A \) if and only if there exists an infinite sequence of states \( q_0, q_1, \ldots \) such that \( q_0 = q_{\text{init}}, q_{i+1} \in \delta(q_i, \ell) \) for all \( i \geq 0 \), and some state in \( F \) occurs infinitely often.

We consider the temporal property verification problem in which we are given a LTS \( M \), a predicate \( \phi_{\text{init}}(\bar{x}) \) on states of the LTS, and a Büchi automaton \( A \) such that the label set of \( A \) is \( L_M \). Recall that we allow a LTS to be infinite-state.) The goal of the verification is to decide if for any (infinite) execution of \( M \) from a state satisfying \( \phi_{\text{init}}(\bar{x}) \), the infinite sequence of labels of the execution is accepted by \( A \). That is, the goal is to verify whether the given LTS satisfies the linear temporal property specified by the given Büchi automaton. The problem can be expressed in \( \mu \text{CLP} \) by defining the mutually-recursive least-and-greatest fixpoint predicates \( \text{LV}_{q, \alpha}(\bar{x}) \) for each \( q \in Q \) and \( \alpha \in \{\mu, \nu\} \):

\[
\text{LV}_{q, \alpha}(\bar{x}) = \alpha \bigwedge_{\ell \in L} \forall \bar{y}. (\overline{\alpha}) \xrightarrow{\ell} (\overline{\bar{y}}) \Rightarrow \bigvee_{q' \in \delta(q, \ell)} \text{LV}_{q', \alpha(q')}(\bar{y}).
\]

Here, \( \alpha(q) = \nu \) if \( q \in F \) and \( \alpha(q) = \mu \) otherwise. Then, the LTS satisfies the temporal property if and only if \( \phi_{\text{init}}(\bar{x}) \Rightarrow \text{LV}_{q_{\text{init}}, \nu}(\bar{x}) \) is valid. The correctness of the construction follows from the fact that \( \text{LV}_{q, \alpha}(\bar{x}) \) represents the set of states from which the labels along the execution of the LTS is accepted by \( A \) when \( A \) is run from the state \( q \). Note that the occurrence of a predicate in the body of the recursive definition becomes the \( \text{LV}_{\cdot, \mu} \) variant when no state in \( F \) is visited in the corresponding execution step. This ensures that there must be a path in which a state from \( F \) is visited infinitely often.

4 PREDICATE CONSTRAINT SATISFACTION PROBLEMS \( \text{pCSP}_{\ll}^{\ll} \)

We now describe a new verification intermediate representation, that generalizes CHCs, and serves as an intermediary for automating \( \mu \text{CLP} \) validity queries. Specifically, we formalize the class \( \text{pCSP}_{\ll}^{\ll} \) of predicate constraint satisfaction problems. We use \( \varphi \) as a meta-variable ranging over \( T \)-formulas (cf. Section 3) without quantifiers and predicate variables (but possibly with non-predicate function variables whose return sort is not \( \bullet \)). First, we define a \( \text{pCSP} \) (without function variables and well-founded predicate variables) to be a finite set of clauses of the form

\[
\varphi \lor \left( \bigvee_{i=1}^{\ell} X_i(t_i) \right) \lor \left( \bigvee_{i=\ell+1}^{m} \neg X_i(t_i) \right)
\]

where \( 0 \leq \ell \leq m \) and \( \text{ffv}(\varphi) = \emptyset \).

We write \( \text{ftv}(c) \) and \( \text{ftv}(C) \) for the set of free term variables that occur in \( c \) and \( C \), respectively. We regard the variables in \( \text{ftv}(c) \) as implicitly universally quantified. We write \( \text{fpv}(C) \) (resp. \( \text{ffv}(C) \)) for the set of free predicate (resp. function) variables that occur in \( C \). A \( \text{pCSP} \) \( C \) is called \( \text{CHCs} \) if \( \ell \leq 1 \) for all clauses \( c \in C \), and co-\( \text{CHCs} \) if \( m \leq \ell + 1 \) for all \( c \in C \). A \( \text{pCSP} \) \( C \) is called linear \( \text{CHCs} \) (or linear co-\( \text{CHCs} \)) if \( C \) is both \( \text{CHCs} \) and co-\( \text{CHCs} \). A function/predicate substitution \( \sigma \) is a finite map from non-predicate function variables \( F \) to closed functions of the form \( \lambda x_1, \ldots, x_{\text{arf}(F)}. t \) and predicate variables \( X \) to closed predicates of the form \( \lambda x_1, \ldots, x_{\text{arf}(X)} . \varphi \). We write \( \sigma(C) \) for the application of \( \sigma \) to \( C \) and \( \text{dom}(\sigma) \) for the domain of \( \sigma \). We call \( \sigma \) a syntactic solution for \( C \) if
is the formula obtained from that represents an under-approximation of $\rho$ with the predicate variable $X$ where $\varphi$ with co-inductive predicates defined by $\mu$. This section defines our reduction algorithm from the given variables. Formally, the reduction algorithm is:

- a finite set $C$ of pCSP-clauses over function/predicate variables without necessarily satisfying the pCSP-restriction $ffv(\varphi) = \emptyset$ of the $\varphi$-part of each clause and
- a set $R$ of well-founded predicate variables that are required to represent well-founded relations.

We write $\rho \models WF(X)$ if the interpretation $\rho(X)$ of the predicate variable $X$ is well-founded, that is, $\text{sort}(X) = (\bar{s}, \bar{s}) \rightarrow \bullet$ for some sequence $\bar{s}$ of sorts and there is no infinite sequence $\bar{v}_1, \bar{v}_2, \ldots$ of sequences $\bar{v}_i$ of values of the sorts $\bar{s}$ such that $(\bar{v}_i, \bar{v}_{i+1}) \in \rho(X)$ for all $i \geq 1$. We call a function/predicate interpretation $\rho$ a semantic solution for $(C, R)$ if $\rho$ is a semantic solution of $C$ and $\rho \models WF(X)$ for all $X \in R$. The notion of syntactic solution can be similarly generalized to pCSP$^{\|\lambda}$.

**Definition 4.1 (Satisfiability of pCSP$^{\|\lambda}$).** The predicate satisfiability problem of a pCSP$^{\|\lambda}$ $(C, R)$ is that of deciding whether it has a semantic solution.

It is well known that the satisfiability of CHCs and the validity of CLP are inter-reducible. In Section 5, we will show a sound and complete reduction from the validity of $\mu$CLP to the satisfiability of pCSP$^{\|\lambda}$. The reduction is of practical importance because the latter problem is often easier to address: we may find a certificate of the satisfiability instead of exhaustively checking all possible cases.

# 5 REDUCTION ALGORITHM FROM $\mu$CLP TO pCSP$^{\|\lambda}$

This section defines our reduction algorithm from the given $\mu$CLP validity problem $(\varphi, P)$ (cf. Definition 3.2) to a pCSP$^{\|\lambda}$ satisfiability problem $(C, R)$ (cf. Definition 4.1). We assume without loss of generality that the (co-)inductive predicates $X \in \text{dom}(P)$ occur only positively in the query $\varphi$: we can always transform the given query into this form by replacing each negative occurrence of $X$ in $\varphi$ with $\neg X$ where the predicate $X$ is defined by the De Morgan dual $\neg P$.

Our reduction consists of three steps: The first step, $\text{elim}_3$, Skolemizes positive occurrences of existential quantifiers and negative occurrences of universal quantifiers by introducing fresh function variables. The second step, $\text{elim}_\mu$, replaces inductive predicates defined by $\mu$-equations with co-inductive predicates defined by $v$-equations with guards (i.e., well-foundedness constraints) for co-recursion added to preserve the semantics. The third step, $\text{elim}_v$, further eliminates co-inductive predicates by replacing them with uninterpreted predicates represented as fresh predicate variables. Formally, the reduction algorithm is:

$$\text{reduct}(\varphi, P) \triangleq \begin{cases} \text{let } (\varphi_\mu, P_\mu) = \text{elim}_3(\varphi, P) \text{ in } \\
\text{let } (\varphi_v, P_v, R) = \text{elim}_\mu(\varphi_\mu, P_\mu, \emptyset) \text{ in } \text{elim}_v(\varphi_v, P_v, R) \end{cases}$$

Here, the $\mu$CLP $(\varphi_\mu, P_\mu)$ is obtained from $(\varphi, P)$ by eliminating existential quantifiers with fresh function variables as stated above. The definition of $\text{elim}_v(\varphi, P)$ is given as:

$$\text{elim}_v(\varphi, e) \triangleq \text{cnf}(\bar{\varphi})$$

$$\text{elim}_v(\varphi, P; (X(\bar{x}) = v \varphi')) \triangleq \text{elim}_v(\varphi, P) \cup \text{cnf}(X(\bar{x}) \Rightarrow \varphi')$$

where $\bar{\varphi}$ is the formula obtained from $\varphi$ by replacing each occurrence of a predicate $X \in \text{dom}(P)$ with the predicate variable $X$ that represents an under-approximation of $X$. $\text{cnf}(\varphi)$ converts $\varphi$ into its prenex and conjunctive normal form $\forall \bar{x}. \land C$ and returns the set $C$ of clauses. The most tricky
part of the algorithm, namely, \( \text{elim}_\mu(\phi, \mathcal{P}, \mathcal{R}) \), is defined by:

\[
\text{elim}_\mu(\phi, (X_i(x_i) =_\nu \phi_i)_{i=1}^m, \mathcal{R}) \triangleq (\phi, (X_i(x_i) =_\nu \phi_i)_{i=1}^m, \mathcal{R}) \quad \text{(base)}
\]

\[
\text{elim}_\mu(\phi, \mathcal{P}; (X(x) =_\mu \phi'), (X_i(x_i) =_\nu \phi_i)_{i=1}^m, \mathcal{R}) \triangleq \text{elim}_\mu(\phi_0, \mathcal{P}; (X(x) =_\sigma \sigma(X)(\phi')), (X_i(x_i) =_\nu \sigma_i(\phi_i))_{i=1}^m, \mathcal{R} \cup \{X_i\})
\quad \text{(recursive)}
\]

\[
\sigma_0 \triangleq \{ X_i \mapsto \lambda \bar{y}.X_i(\bot, \bar{v}, \bar{y}) \mid i = 1, \ldots, m \}
\]

\[
\sigma_X \triangleq \{ X \mapsto \lambda \bar{y}.X(\bar{y}) \wedge X_i(\bar{x}, \bar{y}) \mid i = 1, \ldots, m \}
\]

\[
\sigma_i \triangleq \{ X \mapsto \lambda \bar{y}.X(\bar{y}) \wedge (b_i \implies X_i(\bar{x}, \bar{y})) \} \cup \{ X_j \mapsto \lambda \bar{y}.X_j(b_j, \bar{x}, \bar{y}) \mid j = 1, \ldots, m \}
\]

The third argument of \( \text{elim}_\mu \) accumulates generated fresh well-founded predicate variables. The base case of \( \text{elim}_\mu(\phi, \mathcal{P}, \mathcal{R}) \) just returns the converted \( \nu \)-only \( \mu \text{CLP} (\phi, \mathcal{P}) \) that contains well-founded predicate variables in \( \mathcal{R} \). In the recursive step, for the definition \( X(\bar{x}) =_\mu \phi' \) of the right-most inductive (i.e., \( \mu \)) predicate \( X \) in the input \( \mu \text{CLP} \), we generate a fresh well-founded predicate variable \( X_{\bar{y}} \) and use it as the guard for each co-recursion in the converted co-inductive definition \( X(\bar{x}) =_\nu \sigma_X(\phi') \): we use the substitution \( \sigma_X \) to replace each call \( X(\bar{t}) \) in the body \( \phi' \) of \( X \) with \( X(\bar{t}) \wedge X_{\bar{y}}(\bar{x}, \bar{t}) \) that checks that the formal arguments \( \bar{x} \) of \( X \) and the actual arguments \( \bar{t} \) of the co-recursion are related by the well-founded relation represented by \( X_{\bar{y}} \). At the same time, we extend the formal arguments of each co-inductive (i.e., \( \nu \)) predicate \( X_i \) in the right-hand side of the equation for \( X \) with arguments \( \bar{x} \) of the same sort as the formal arguments of \( X \) and a Boolean argument \( b_i \), where we assume that the formal arguments \( \bar{x}_i \) of \( X_i \) are \( \alpha \)-renamed to avoid a name conflict between \( \bar{x}_i \) and \( \bar{x}, b_i \). The extended formal arguments \( \bar{x}_i \) of \( X_i \) are used to receive the actual arguments previously passed to a call to the inductive predicate \( X \) and are related by \( X_{\bar{y}} \), in the converted definition of \( X_i \), with the actual arguments passed to each indirect recursive call to \( X \) in \( X \).\(^6\) Dummy values are passed as \( \bar{x} \) when no such previous call to \( X \) exists and the extended Boolean formal argument \( b_i \) of \( X_i \) indicates whether there indeed is such a call to \( X \) and its actual arguments are passed as \( \bar{x} \) to \( X_i \) or the dummy values are passed as \( \bar{x} \) to \( X_i \). In fact, we use the substitution \( \sigma_0 \) to replace each call \( X_i(\bar{t}) \) in the query \( \bar{t} \) and the definition of the predicates \( \mathcal{P} \) in the left-hand side of the equation for \( X \) with \( X_i(\bot, \bar{v}, \bar{t}) \) for some sequence \( \bar{v} \) of dummy values of the same sorts as the formal arguments \( \bar{x} \) of \( X \). For the definition \( X(\bar{x}) =_\mu \phi' \), we use the substitution \( \sigma_X \) to replace each call \( X_i(\bar{t}) \) in \( X \) with \( X_i(\bar{t}) \wedge X_{\bar{y}}(\bar{x}, \bar{t}) \). For the definition \( X_j(\bar{x}_j) =_\nu \phi_j \) of each co-inductive predicate \( X_j \) in the right-hand side of the equation for \( X \), we use \( \sigma_j \) to replace each call \( X_i(\bar{t}) \) in \( X_j \) with \( X_i(\bar{t}) \wedge (b_i \implies X_{\bar{y}}(\bar{x}, \bar{t})) \) that checks that if \( \bar{x} \) are not dummy (i.e., \( b_i = \top \)), the actual arguments of a previous call to \( X \) passed around to \( X_j \) as its extended formal arguments \( \bar{x} \) are related by \( X_{\bar{y}} \) with the actual arguments \( \bar{t} \) of the indirect recursive call to \( X \). In the resulting \( \mu \text{CLP} \), the generated well-founded predicate variables occur only positively.

**Example 5.1.** Let us consider the \( \mu \text{CLP} (\phi, \mathcal{P}) \) where \( \phi \equiv \forall x.X(x) \wedge Y(x) \) and \( \mathcal{P} \equiv (X(x) =_\mu Y(x - 1)) \wedge (Y(y) =_\nu y \leq 0 \lor X(y - 1)) \). We obtain \( \text{elim}_\exists(\phi, \mathcal{P}, \emptyset) = (\phi, \mathcal{P}) \) and

\[
\text{elim}_\mu(\phi, \mathcal{P}, \emptyset) = \text{elim}_\mu(\phi, (X(x) =_\mu Y(x - 1)) \wedge (Y(y) =_\nu y \leq 0 \lor X(y - 1)), \emptyset)
\]

\[
= \left( \forall x.X(x) \wedge Y(\bot, \emptyset, x), \right.
\]

\[
\left. (X(x) =_\nu Y(\top, x, x - 1)), \right)
\]

\[
(Y(b, x, y) =_\nu y \leq 0 \lor X(y - 1) \wedge (b \implies X_{\bar{y}}(x, y - 1))), \right\}
\]

\[
\{X_{\bar{y}}\}
\]

\(^6\)This transformation is similar in spirit to binary reachability analysis [Cook et al. 2006; Kuwahara et al. 2014; Podelski and Rybalchenko 2004b] for termination verification.
Here, in the first step of the transformation, the inductive definition of \( Y \) is simply replaced by the co-inductive definition because the body of \( Y \) has no recursive call to \( Y \). The indirect recursive call to \( X \) in \( Y \) is properly handled in the second step by adding the formal arguments \( b \) and \( x \) to \( Y \). Note also that in the call \( Y(\bot, 0, x) \) in the query, 0 is used as a dummy value for the extended formal argument \( x \) of \( Y \).

We thus get pCSP \( \text{reduct}(\phi, P) = (C, \{X_0\}) \) where

\[
C \triangleq \left\{ \begin{array}{l}
X(x), Y(\bot, 0, x), \neg X(x) \lor Y(\top, x, x - 1), \\
\neg Y(b, x, y) \lor y \leq 0 \lor X(y - 1), \neg Y(b, x, y) \lor y \leq 0 \lor \neg b \lor X_0(y, y - 1)
\end{array} \right\}
\]

**Remark 3.** In the implementation of \( \text{reduct}(\phi, P) \) in our \( \mu\text{CLP} \) validity checker MuVAL, unnecessary arguments addition is suppressed. For example, from the \( \mu\text{CLP} \) \( (\phi, P) \) where

\[
\phi \triangleq \forall x. X(x) \\
P \triangleq (X(x) =_\mu Y(x - 1)); (Y(y) =_\mu Z(y - 1)); (Z(z) =_\mu z \leq 0 \lor X(z - 1))
\]

\( \text{reduct}(\phi, P) \) generates the pCSP \( (C, \{X_0\}) \) where

\[
C \triangleq \left\{ \begin{array}{l}
X(x), X(x) \Rightarrow Y(\top, x, x - 1), Y(b_1, y, y) \Rightarrow Z(b_1, x, y, y - 1), \\
\neg Y(b, x, y) \Rightarrow y \leq 0 \lor X(y - 1), \neg Y(b, x, y) \Rightarrow z \leq 0 \lor X(z - 1)
\end{array} \right\}
\]

By contrast, MuVAL gets a simpler but equi-satisfiable pCSP \( (C', \{X_0\}) \) where

\[
C' \triangleq \left\{ \begin{array}{l}
X(x), X(x) \Rightarrow Y(x, x - 1), Y(x, y) \Rightarrow Z(x, x - 1), \\
\neg Y(x, z) \Rightarrow z \leq 0 \lor X(z - 1), \neg Z(x, z) \Rightarrow z \leq 0 \lor X_0(x, z - 1)
\end{array} \right\}
\]

Note that the query \( \phi \) calls \( X \), \( X \) calls \( Y \), \( Y \) calls \( Z \), and \( Z \) recursively calls \( X \). Thus, if we start from the query, \( Z \) is always the actual argument of the previous call to \( X \) and, therefore, \( b_1 \) is always \( \top \) and so unneeded. Likewise, \( b_2 \) is also always \( \top \) and unneeded.

We now show the following soundness and the completeness of the reduction algorithm.

**Theorem 5.2.** \( \text{reduct}(\phi, P) \) has a semantic solution if and only if \( \models (\phi, P) \).

This follows from the lemmas for each steps of \( \text{reduct}(\phi, P) \): The following lemma for the first-step elim\(_3\)(\( \phi, P \)) follows immediately from the well-known soundness and completeness of Skolemization for first-order logic.

**Lemma 5.3.** \( \models (\phi, P) \) if and only if there is an interpretation \( \rho \) for the function variables introduced by the Skolemization such that \( \rho \models \text{elim}_3(\phi, P) \).

The following lemma for the third-step elim\(_3\)(\( \phi, P \)) follows from the maximality of the greatest fixpoints (i.e., co-induction principle) (see e.g., Corollary 1 in [Unno et al. 2017b] for a formal related discussion of least fixpoints that occur in negative positions).

**Lemma 5.4.** Let \( \rho \) be any interpretation of ffv(\( \phi, P \)). \( \text{elim}_3(\phi, P) \) has a semantic solution that extends \( \rho \) if and only if \( \rho \models (\phi, P) \).

We finally show the soundness and completeness of the second-step elim\(_\mu\)(\( \phi, P \)).

**Lemma 5.5.** Suppose that elim\(_\mu\)(\( \phi, P \)) = (\( \phi', P', R \)). We then have \( \models (\phi, P) \) if and only if there is an interpretation \( \rho \) of \( R \) such that \( \rho \models \text{WF}(X) \) for all \( X \in R \) and \( \rho \models (\phi', P') \).

This can be shown as a corollary of the following lemma.
LEMMA 5.6. \( \rho \models (\phi, \mathcal{P}; (X(x) = \mu \phi); (X_i(x_i) = \nu \phi_i)_{i=1}^{m}) \) if and only if there is an interpretation \( \rho' \) of \( X_{\|} \) such that \( \rho' \models \text{WF}(X_{\|}) \) and \( \rho \cup \rho' \models (\sigma_0(\phi), \sigma_0(\mathcal{P}); (X(x) = \nu \sigma_X(\phi')); (X_i(b_i, x_i, x_i) = \nu \sigma_i(\phi_i))_{i=1}^{m} \) where

\[
\begin{align*}
\sigma_0 & \triangleq \{ X_i \mapsto \lambda y_i X_i(\bot, \overline{v}, \overline{y}) \mid i = 1, \ldots, m \} \\
\sigma_X & \triangleq \{ X \mapsto \lambda y. X(\overline{y}) \land X_{\|}(\overline{x}, \overline{y}) \} \cup \{ X_i \mapsto \lambda y_i X_i(\top, \overline{x}, \overline{y}) \mid i = 1, \ldots, m \} \\
\sigma_i & \triangleq \{ X \mapsto \lambda y_i X_i(\overline{y}) \land (b_i \Rightarrow X_i(\overline{x}, \overline{y})) \} \cup \{ X_j \mapsto \lambda y_i X_j(b_i, \overline{x}, \overline{y}) \mid j = 1, \ldots, m \}
\end{align*}
\]

Remark 4. Let \( (\phi, \mathcal{P}) \) be a \( \mu \text{CLP} \). If \( (\phi, \mathcal{P}) \) is validity-reducible (recall Remark 2), \( \text{reduct}(\phi, \mathcal{P}) \) always generates co-CHCs. Similarly, if \( (\phi, \mathcal{P}) \) is invalidity-reducible, \( \text{reduct}(\phi, \neg(\mathcal{P})) \) always generates co-CHCs. Note also that the satisfiability of the co-CHCs \( C \) can be further reduced to that of the CHCs \( C^\land \), obtained from \( C \) by replacing, each literal of the form \( X(\overline{t}) \) with \( \neg X^\land(\overline{t}) \) and \( \neg X(\overline{t}) \) with \( X^\land(\overline{t}) \) for \( X \in \text{fpv}(C) \), where \( X^\land \) is a fresh predicate variable that represents the negation of \( X \). Thus we can use off-the-shelf CHC solvers to discharge the validity-reducible and invalidity-reducible fragments of \( \mu \text{CLP} \). Our constraint solving method described in the next section can handle the full classes of \( \text{pCSP}^{\beta/\lambda} \) and \( \mu \text{CLP} \) (via the reduction).

6 CONSTRAINT SOLVING METHOD FOR \( \text{pCSP}^{\beta/\lambda} \)

This section describes our CEGIS-based method for finding a (syntactic) solution—in other words, (co-)inductive invariants, ranking functions, and witnesses for existential quantifiers—of the given \( \text{pCSP}^{\beta/\lambda} \) \((C, \mathcal{R})\). Our method iteratively accumulates example instances of \( C \), which are defined to be \( \text{pCSP}^{\beta/\lambda} \)-clauses without term variables obtained from \( C \) by instantiating \( \text{ftv}(C) \), from which a sequence of candidate solutions for \( C \) is generated by using a synthesizer \( S \) (whose details are deferred to Section 6.2), until a genuine solution or a counterexample (i.e., unsatisfiable example instances) is found. We write \( \mathcal{E}^{(i)} \) for the set of example instances accumulated before the iteration \( i \). Starting from \( \mathcal{E}^{(1)} = \emptyset \), for each iteration \( i \geq 1 \), our method performs the following:

1. **Synthesis Phase**: We check whether the set of instances \( \mathcal{E}^{(i)} \cup \mathcal{R} \) is unsatisfiable. If so, we return \( \mathcal{E}^{(i)} \) as a counterexample to the input \( \text{pCSP}^{\beta/\lambda} \) \((C, \mathcal{R})\). Otherwise, we let the synthesizer \( S \) find a syntactic solution \( \sigma^{(i)} \) (with \( \text{dom}(\sigma^{(i)}) = \text{ftv}(C) \)) of the instances \( \mathcal{E}^{(i)} \cup \mathcal{R} \), which will be used as a candidate solution for \((C, \mathcal{R})\).

2. **Validation Phase**: We check whether \( \sigma^{(i)} \) is a genuine solution to \((C, \mathcal{R})\) by using an off-the-shelf SMT solver. If so, we return \( \sigma^{(i)} \) as the solution. Otherwise, for each clause \( c \in C \) unsatisfied by \( \sigma^{(i)} \), we obtain a counterexample, that is, a term substitution \( \theta_c \) such that \( \text{dom}(\theta_c) = \text{ftv}(c) \) and \( \not\models \theta_c(\sigma^{(i)}(c)) \). We then update the example set by adding a new example instance for each unsatisfied clause (i.e., \( \mathcal{E}^{(i+1)} = \mathcal{E}^{(i)} \cup \{ \theta_c(c) \mid c \in C \land \not\models \sigma^{(i)}(c) \} \)), and proceed to the next iteration with \( \mathcal{E}^{(i+1)} \).

Remark 5. In our \( \text{pCSP}^{\beta/\lambda} \) satisfiability checker \( \text{PCSAT} \) (Section 7), we implemented a third phase that we call the resolution phase for accelerating the convergence of the CEGIS loop. There, we first apply unit propagation repeatedly to the example instances \( \mathcal{E} \) to obtain the set \( \mathcal{E}^+ \) of positive examples of the form \( X(\overline{v}) \) and the set \( \mathcal{E}^- \) of negative examples of the form \( \neg X(\overline{v}) \). We then repeatedly apply resolution principle to the clauses in the original \( \text{pCSP}^{\beta/\lambda} \) \( C \) and the clauses in \( \mathcal{E}^+ \cup \mathcal{E}^- \) to obtain new positive/negative examples (without containing term variables), which are then added to \( \mathcal{E} \).

In general, the above CEGIS procedure may diverge, which is inevitable due to the undecidability of \( \text{pCSP}^{\beta/\lambda} \). But it satisfies the so-called progress property: any counterexample and candidate solution found in an iteration are never generated again in succeeding iterations. Furthermore, if we carefully design a synthesizer \( S \) as discussed in Section 6.2 by incorporating our idea of stratified CEGIS, we can show the relative completeness in the sense of [Jhala and McMillan 2006; Terauchi and Unno...
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2015: if the given pCSP\(^{\lambda}\) \((C, \mathcal{R})\) has a syntactic solution expressible in the stratified families of templates, a solution of the pCSP\(^{\lambda}\) is eventually found by the procedure.

The rest of this section discusses the details of the synthesis phase. Section 6.1 discusses how to check the unsatisfiability of example instances \((E, \mathcal{R})\). Section 6.2 discusses the synthesis based on stratified template families and unsat-core-based template refinement. For simplicity, we focus on the theory of quantifier-free linear integer arithmetic (QFLIA) in the description of the synthesis phase. Designing stratified template families for richer theories such as arrays, algebraic data types, and heaps is actually non-trivial and will be discussed as a future work in Section 9.

6.1 Unsatisfiability Checking of Example Instances

If \(\mathcal{R} = \emptyset\), the unsatisfiability of the given example instances \((E, \mathcal{R})\) can be decided by an off-the-shelf SAT solver (if \(\text{ffv}(E) \setminus \text{fpv}(E) = \emptyset\)) or SMT solver (otherwise) because \(E\) is a finite set of clauses not containing term variables. Otherwise, we use the following (CDCL-like) iterative algorithm staring from \(E_0 = E\): For each iteration \(i \geq 0\), we first check whether \((E_i, \emptyset)\) is unsatisfiable. If so, then we conclude that \((E, \mathcal{R})\) is unsatisfiable. Otherwise, we obtain a satisfying assignment \(\sigma\) for \(E_i\). Then, for each \(X \in \mathcal{R}\), we consider the graph comprising the edges \(\{(\overline{v}_1, \overline{v}_2) | \sigma(X(\overline{v}_1, \overline{v}_2))\}\) and enumerate its simple cycles (e.g., by using the algorithm of [Johnson 1975]). Note that such cycles would be counterexamples to the well-foundedness constraint \(X\). If no such cycles exist, we conclude that \((E, \mathcal{R})\) is satisfiable. Otherwise, we let \(E_{i+1}\) be \(E_i\) but with the following new learnt clauses added:

- \(\neg X(\overline{v}_1, \overline{v}_2) \lor \ldots \lor \neg X(\overline{v}_{m-1}, \overline{v}_m)\) for each simple cycle \(\overline{v}_1, \ldots, \overline{v}_m = \overline{v}_1\) of each \(X \in \mathcal{R}\).

We then proceed to the next iteration with \(E_{i+1}\).

It is worth mentioning here that if \(\mathcal{R} = \emptyset\) and the original pCSP \((C, \mathcal{R})\) is unsatisfiable, there always exists an unsatisfiable finite set \(E\) of example instances of \(C\). However, there is, in general, no such finite witness of the unsatisfiability if \(\mathcal{R} \neq \emptyset\). This fact also supports an advantage of our primal-dual approach to verification based on \(\mu\)-CLP.

6.2 Function/Predicate Synthesis with Stratified Families of Templates

We do a template-based search for a solution of the given example instances to be returned as a candidate solution of the input pCSP\(^{\lambda}\) \((C, \mathcal{R})\). Templates can effectively restrict the solution space to explore and be made to satisfy the well-foundedness constraints at the same time. There however is a trade-off between expressiveness and generalizability. With less expressive templates like intervals, we may miss actual solutions. By contrast, with very expressive templates like polyhedra, there could be many solutions, and a solution thus returned is liable to overfitting and therefore is of low generalizability. That is, the solution is likely too specific to be a solution of \((C, \mathcal{R})\). [Padhi et al. 2019] discusses a similar overfitting problem in the context of grammar-based synthesis.

Our remedy to the problem is to use stratified families of predicate templates that have been used in prior work to guarantee the convergence of counterexample-guided refinement iterations [Jhala and McMillan 2006; Terauchi and Unno 2015]. Initially, we assign each predicate variable a less expressive template and gradually refine it in a counterexample-guided manner: we try to find a solution expressible in the current templates and if no solution is found, we generate and analyze an unsat core of the constraint over the unknown parameters of the templates to identify the parameters of the families of templates that are necessary to be updated.

6.2.1 Stratified Families of Templates. We have designed three stratified families of templates respectively for (1) ordinary predicates, (2) (non-predicate) functions, and (3) well-founded predicates.
(1) For ordinary predicates \( X \in (fpv(\mathcal{C}) \setminus \mathcal{R}) \), the stratified family of templates \( T_X^\star (nd, nc, ac, ad) \) and its accompanying constraint \( \phi_X^\star (nd, nc, ac, ad) \) are defined as:

\[
T_X^\star (nd, nc, ac, ad) \triangleq \lambda(x_1, \ldots, x_{\text{ar}(X)}). \bigvee_{i=1}^{nd} \bigwedge_{j=1}^{nc} c_{i,j,0} + \sum_{k=1}^{\text{ar}(X)} c_{i,j,k} \cdot x_k \geq 0
\]

\[
\phi_X^\star (nd, nc, ac, ad) \triangleq \bigwedge_{i=1}^{nd} \bigwedge_{j=1}^{nc} \sum_{k=1}^{\text{ar}(X)} |c_{i,j,k}| \leq ac \land |c_{i,j,0}| \leq ad
\]

Here, \( c_{i,j,k} \)'s are fresh unknown parameters to be inferred. Note that the parameter \( nd \) (resp. \( nc \)) is the number of disjuncts (resp. conjuncts) and the parameter \( ac \) is the upper bound of the sum of the absolute value of coefficients \( c_{i,j,k} \) (\( k > 0 \)) of the variables. The parameter \( ad \) is the upper bound of the absolute value of constant term \( c_{i,j,0} \).

(2) For (non-predicate) functions \( F \in (fpv(\mathcal{C}) \setminus fpv(\mathcal{C})) \), we define the stratified family of templates \( T_F^\lambda (nd, nc, dc, dd, ec, ed) \) and its accompanying constraint \( \phi_F^\lambda (nd, nc, dc, dd, ec, ed) \) as:

\[
T_F^\lambda (nd, nc, dc, dd, ec, ed) \triangleq \lambda(x_1). t_1(x)
\]

\[
\phi_F^\lambda (nd, nc, ec, ed, dc, dd) \triangleq \bigwedge_{i=1}^{nd} \sum_{j=1}^{\text{ar}(X)} |c_{i,j}| \leq ec \land |c_{i,0}| \leq ed \land
\]

\[
\bigwedge_{i=1}^{nd-1} \bigwedge_{j=1}^{nc} \sum_{k=1}^{\text{ar}(X)-1} |c'_{i,j,k}| \leq dc \land |c'_{i,j,0}| \leq dd
\]

where

\[
t_{nd}(x) \triangleq e_{nd}(x), \quad t_1(x) \triangleq \text{ITE}(D_1(x), e_1(x), t_{i+1}(x)) \quad (\text{for } 1 \leq i < nd),
\]

\[
e_{i}(x) \triangleq c_{i,0} + \sum_{j=1}^{\text{ar}(X)-1} c_{i,j} \cdot x_j, \quad D_1(x) \triangleq \bigwedge_{j=1}^{nc} c'_{i,j,0} + \sum_{k=1}^{\text{ar}(X)-1} c'_{i,j,k} \cdot x_k \geq 0.
\]

Here, \( c_{i,j} \)'s and \( c'_{i,j,k} \)'s are fresh unknown parameters to be inferred. \( T_F^\lambda \) characterizes a piecewise-defined affine function with discriminators \( D_1, \ldots, D_{nd-1} \) and branch expressions \( e_1, \ldots, e_{nd} \). The parameter \( nc \) is the number of conjuncts in each discriminator. The parameters \( dc, dd, ec, ed \) are the upper bounds similar to \( ac, ad \) for \( T_X^\star \). Note that for any substitution \( \theta \) for the unknown parameters in \( T_F^\lambda, \theta(T_F^\lambda) \) represents a total function.

(3) For well-founded predicates \( X \in \mathcal{R} \), the stratified family of templates \( T_X^= (nl, np, nc, rc, rd, dc, dd) \) and its accompanying constraint \( \phi_X^= (nl, np, nc, rc, rd, dc, dd) \) are defined as:

\[
T_X^= (nl, np, nc, rc, rd, dc, dd) \triangleq \lambda(x, y). \left( \bigwedge_{i=1}^{nl} \bigwedge_{j=1}^{np} r_{i,j}(x, y) \geq 0 \right) \land \left( \bigwedge_{i=1}^{nl} \bigwedge_{j=1}^{np} \sum_{k=1}^{\text{ar}(X)} |c_{i,j,k}| \leq rc \land |c_{i,j,0}| \leq rd \land \bigwedge_{i=1}^{nl} \bigwedge_{j=1}^{np} \sum_{k=1}^{\text{ar}(X)} |c'_{i,j,k}| \leq dc \land |c'_{i,j,k,0}| \leq dd \right)
\]

where

\[
GT_1(x, y) \triangleq \bigvee_{j=1}^{np} D_{i,j}(x) \geq 0 \land \bigwedge_{k=1}^{\text{ar}(X)} (D_{i,k}(y) \geq 0 \Rightarrow r_{i,j}(x, y) > r_{i,k}(y))
\]

\[
\text{GEQ}_1(x, y) \triangleq \bigvee_{j=1}^{np} D_{i,j}(x) \geq 0 \land \bigwedge_{k=1}^{\text{ar}(X)} (D_{i,k}(y) \geq 0 \Rightarrow r_{i,j}(x) \geq r_{i,k}(y))
\]

\[
D_{1,j}(x) \triangleq \bigwedge_{k=1}^{nc} c'_{i,j,k,0} + \sum_{l=1}^{\text{ar}(X)/2} c'_{i,j,k,l} \cdot x_l \geq 0
\]

\[
r_{i,j}(x) \triangleq c_{i,j,0} + \sum_{k=1}^{\text{ar}(X)/2} c_{i,j,k} \cdot x_k
\]

Here, \( c_{i,j,k} \)'s and \( c'_{i,j,k,l} \)'s are fresh unknown parameters to be inferred. \( T_X^= \) represents the well-founded relation induced by the \( nl \)-lexicographic \( np \)-piecewise-defined affine ranking function where \( r_{i,j} \) are the affine ranking function template for the \( j \)-th region specified by the discriminator \( D_{i,j} \) of the \( i \)-th lexicographic component. The parameters \( rc, rd, dc, dd \) are the upper bounds similar to \( ac, ad \) for \( T_X^\star \). The first conjunct of \( T_X^= \) asserts that the return value of all the affine ranking functions is non-negative and the second conjunct asserts
To evaluate the presented verification framework, we have implemented:

- PCSAT, a satisfiability checking tool for pCSP$^{\mathbb{N}}$ based on stratified CEGIS.
- MuVal, a validity checking tool for μCLP based on the reduction algorithm presented in Section 5 and the satisfiability checker PCSAT.

PCSAT supports the theory of Booleans and the quantifier-free theory of linear inequalities over integers/rationals. The tools are implemented in OCaml, using Z3 [de Moura and Bjørner 2008] and MiniSat [Eén and Sörensson 2004] as the backend SMT and SAT solvers, respectively.
We compare PCSAT with the state-of-the-art SyGuS (syntax-guided synthesis) solver LoopInvGen [Padhi et al. 2019] which is the winner of the Inv Track of SyGuS-Comp 2018. We also compare with the state-of-the-art CHCs solvers HoIce [Champion et al. 2018] and SPACER [Gurfinkel et al. 2015]. We run the tools on the following benchmark sets:

(a) SyGuS-Comp 2018 (Invariant Synthesis Track).
(b) CHC-COMP 2019 (LIA-nonlin Track) for CHCs over the theory of QFLIA.

We remark that the SyGuS benchmarks only contain linear CHCs with each constraint set containing only a single predicate variable. To compare, we have selected non-linear instances from CHC-COMP.

We have also tested MuVAL on the benchmark sets below encoded as µCLP and compared the results with Mu2CHC [Kobayashi et al. 2019], which is a recently proposed tool for solving fixpoint logic constraints:

(c1) The standard benchmark set for CTL verification (small) [Cook and Koskinen 2013].
(c2) The standard benchmark set for CTL verification (industrial) [Cook and Koskinen 2013].
(d) The benchmark set of Mu-Arithmetic (i.e., µCLP restricted to integer arithmetic) [Kobayashi et al. 2019] which consists of some properties of integer arithmetic encoded in Mu-Arithmetic (Problems 1–6), linear-time temporal properties of first-order functional programs encoded by a translation in [Kobayashi et al. 2019] (Problems 7–22), branching-time temporal properties (some are only expressible in CTL* or modal-\(\mu\)) of imperative programs encoded by a translation similar to one in [Watanabe et al. 2019] (Problems 23–28).
(e) The termination verification benchmark set for FuncTion.\(^7\)

All experiments have been conducted on 3.1GHz Intel Xeon Platinum 8000 CPU and 32 GiB RAM with the time limit of 300 seconds.

The experimental results except (b) are summarized in Figure 1. The cactus plot (left) compares the results of PCSAT on (a) with those of HoIce, SPACER, and LoopInvGen. For the number of solved instances, PCSAT obtained comparable results with LoopInvGen [Padhi et al. 2019]: PCSAT (denoted “Stratified”) solved 113 SAT and 8 UNSAT instances while LoopInvGen solved 116 SAT and 5 UNSAT instances. PCSAT obtained better results than the highly-tuned CHCs solvers HoIce (109 SAT, 9 UNSAT, and 2 wrong answers) and SPACER (101 SAT and 9 UNSAT). PCSAT however is often slower compared to the other mature tools. This is partly because PCSAT does not use incremental SMT solving across CEGIS iterations and therefore becomes significantly slower as the number of example instances grows. This inefficiency caused PCSAT to obtain suboptimal results on (b) the CHC-COMP benchmarks: PCSAT solved 97 SAT and 55 UNSAT instances while HoIce solved 123 SAT and 79 UNSAT, and SPACER solved 147 SAT and 117 UNSAT instances. From our analysis of the failed runs, we found that PCSAT often failed to solve CHCs containing multiple Boolean variables. This is because the current version of PCSAT naively generates \(2^n\)-copies of templates over integer variables for each Boolean valuation where \(n\) is the number of Boolean variables in CHCs. We plan to design improved families of templates for Boolean variables. Though it is rather out of the scope of this paper, we believe this dramatically improves the experiment results because most benchmarks from (b) have multiple Boolean variables. Also, we found that PCSAT is general but not well-tuned for proving the unsatisfiability when applied to the subclass CHCs of pCSP\(\lambda\). We could exploit the restricted (i.e. Horn) form of constraints for efficiently finding a resolution derivation of the contradiction via SLD-resolution.

The cactus plot (left) also shows the trade-off between expressiveness and generalizability of templates. Interval, Octagon, Octahedron, and Polyhedron are PCSAT restricted to use respective

\(^7\)https://www.di.ens.fr/~urban/FuncTion.html
fixed predicate templates, and the plot shows that they obtained significantly worse results compared to PCSat with stratified families of templates. Also note that the results with the Polyhedron and the Interval templates are even worse than those of the Octahedron and the Octagon templates. We believe that these results show that the Polyhedron templates suffer from the overfitting problem [Padhi et al. 2019] due to their high expressiveness, while the Interval templates suffer from their low expressiveness.

The scatter plot (right) in Figure 1 compares the results of MuVal on (c1), (c2), (d), and (e) with those of Mu2CHC: MuVal solved 76 VALID and 72 INVALID instances (out of 159 instances) and Mu2CHC solved 74 VALID and 74 INVALID instances. MuVal failed to solve 5 temporal verification benchmarks from (c2) and (d) that were solved by Mu2CHC. We believe that this is because the highly-tuned invariant synthesis engine (i.e., SPACER and HoIce) used in Mu2CHC worked better for the benchmarks. By contrast Mu2CHC failed to solve 5 termination verification benchmarks that were solved by MuVal, which require synthesis of piecewise-defined and/or lexicographic affine ranking functions. We believe that this shows a limitation of the Mu2CHC approach that separately synthesize termination arguments and inductive invariants, and cannot quickly feedback a failure of invariant synthesis to ranking function synthesis.

8 RELATED WORK

The class of problems $\text{pCSP}_{\lambda}^\exists$ that we have introduced in this paper is closely related to existentially-quantified Horn clauses (E-CHCs) introduced in [Beyene et al. 2013]. We conjecture that $\text{pCSP}_{\lambda}^\exists$ and E-CHCs are inter-reducible, though it is not trivial to fill the gap, without changing the background theory, between our well-foundedness and their disjunctive well-foundedness constraints and our function variables and their existentially-quantified heads. We believe inter-reducibility is often a desirable feature: even though DFAs and regular expressions are inter-reducible, each format has its own benefits. In our case, for instance, having the direct support for general disjunctions in $\text{pCSP}$ can be advantageous compared to encoding them indirectly by existentials in E-CHCs. In particular, general disjunctions can be handled by PCSat without any additional twist, and can be used to completely encode branching-time temporal properties verification problems of imperative programs with finitely-bounded non-determinism, for which existential quantifications in E-CHCs
and function variables in pCSP\(\lambda\) are probably overkill. Also, the class of pCSP without function variables and well-founded predicates is closed under negation like in \(\mu\)CLP (cf. Remark 4). Besides the logical beauty, the property is also useful in practice: we can mechanically compute the De Morgal dual of the given CHCs and check the satisfiability of the primary and dual CHCs in parallel or cooperatively. Also, it is well known that well-founded relations used in pCSP\(\lambda\) and disjunctively well-founded relations used in E-CHCs are both complete for termination (see [Podelski and Rybalchenko 2004b]) but have different benefits. To solve E-CHCs, [Beyene et al. 2013] proposes a method called E-HSF which reduces the given E-CHCs to (ordinary) CHCs by synthesizing candidate witnesses for existentially quantified variables iteratively in a counterexample-guided manner. The generated CHCs are then solved (possibly itself via a counterexample-guided iteration) by an off-the-shelf CHCs solver. By contrast, our method, while also based on counterexample-guided iteration, reduces the problem to quantifier-free SMT solving by simultaneously synthesizing candidate invariants, well-founded relations, and quantifier witnesses. We believe that there are two advantages to our approach. One is that the simultaneous synthesis facilitates finding candidates that depend amongst each other, for instance, well-founded relations that depend on quantifier witnesses, by sharing useful information via faster feedbacks from synthesis failures. Another advantage is that pCSP\(\lambda\) can directly express non-Horn clauses whereas handling such clauses in E-CHCs would incur introducing additional existential quantifiers.

An extension of CLP called co-Constraint Logic Programs (co-CLP) with mixed inductive and co-inductive predicates has been proposed in [Saeedloei and Gupta 2012]. Unlike our \(\mu\)CLP, co-CLP does not support mutually recursive inductive and co-inductive predicates which are necessary to directly express modal-\(\mu\) temporal verification problems. Also related to our \(\mu\)CLP is Mu-Arithmetic [Bradfield 1999; Lubarsky 1993] which is a first-order fixpoint logic of integer arithmetic. It has recently been applied to temporal property verification in [Kobayashi et al. 2019] where they present a method called Mu2CHC for checking the validity of formulas expressed in the logic.\(^8\) Mu2CHC works by reducing the problem to (ordinary) CHCs. This is done by conservatively approximating fixpoints by asserting some (symbolic) bound on their unfolding depths. The resulting CHCs are then solved by an off-the-shelf CHCs solver. By contrast, our MuVal reduces the problem to pCSP\(\lambda\) and therefore has the advantages of simultaneous synthesis remarked above.\(^9\) In fact, this difference resulted in the better results of MuVal on the termination verification benchmark set that requires synthesis of lexicographic and/or piecewise-defined ranking functions (recall discussion in Section 7). And, the completeness of the reduction to pCSP\(\lambda\) allows MuVal to conclude the invalidity of the original \(\mu\)CLP from the unsatisfiability of the reduced pCSP\(\lambda\) unlike Mu2CHC. Also, Mu2CHC is specialized to integer arithmetic, for example, relying on that particular domain to encode existential quantifiers as fixpoints, as explained in Remark 1. Generalizing their method to other theories (such as the theory of reals) may require non-trivial extensions. By contrast, MuVal is designed for the full class of \(\mu\)CLP which can be seen as a generalization of Mu-Arithmetic to arbitrary first-order theories. However, we remark that both Mu2CHC and E-HSF have an advantage over our approach in that they can utilize highly-tuned off-the-shelf CHCs solvers. Indeed, for this reason, we have noticed that our approach is often less efficient than theirs on (ordinary) CHCs instances.

Our pCSP\(\lambda\) solving technique generalizes a number of previous techniques developed for CHCs solving and invariant/ranking function discovery. Most closely related to our work are the data-driven approaches to solving subclasses of CHCs based on CEGIS [Solar-Lezama et al. 2006] combined with template-based synthesis via SMT solver [Garg et al. 2014; Sharma et al. 2013b],

\(^8\)Technically, their method works on hierarchical equation systems (HES) which is a reformulation of Mu-Arithmetic.

\(^9\)In fact, Mu2CHC has no feedback from CHCs solving to fixpoints approximation.
greedy set covering with logic minimization [Padhi et al. 2016; Sharma et al. 2013a], decision tree learning [Champion et al. 2018; Ezudheen et al. 2018; Garg et al. 2016; Krishna et al. 2015; Zhu et al. 2018], and grammar-based synthesis [Fedyukovich et al. 2018; Padhi et al. 2019]. Our stratified CEGIS adopts the idea of stratified families of templates [Jhala and McMillan 2006; Terauchi and Unno 2015]. Our approach is similar in spirit to [Padhi et al. 2019] but they use a stratified family of grammars instead and also do not use unsat cores for updating grammars. The idea presented in [Fedyukovich et al. 2018] of extracting grammars for enumerating ranking functions and recurrent sets and our idea of stratifying templates are orthogonal and could be better together. Besides the data-driven approach, various CHCs solving approaches have been proposed: counterexample-guided abstraction refinement and Craig interpolation [Hojjat and Rümmer 2018; Unno and Kobayashi 2009], generalized property directed reachability [Hoder and Bjørner 2012; Komuravelli et al. 2014], constraint specialization [Angelis et al. 2014; Kafle et al. 2016], and inductive theorem proving [Unno et al. 2017b]. A number of existing techniques for program verification can be applied straightforwardly to invariant synthesis for linear CHCs and ranking function synthesis. Some use templates for invariants [Colón et al. 2003; Sankaranarayanan et al. 2004] and ranking functions [Leike and Heizmann 2014] but many of them involve costly non-linear constraint solving. RankFinder [Podolski and Rybalchenko 2004a] synthesizes linear ranking functions via linear constraint solving. However, none of the above methods can be used to solve the full class of pCSP$^{\text{HL}}$.

9 CONCLUSION

We have introduced the class $\mu$CLP of constraint logic programs with arbitrarily nested inductive and co-inductive predicates and the class pCSP$^{\text{HL}}$ of predicate constraint satisfaction problems that generalizes CHCs with arbitrary clauses, function variables, and well-foundedness constraints. We have then established a program verification framework based on $\mu$CLP by showing that (1) $\mu$CLP can naturally encode various classes of verification problems, (2) the validity of $\mu$CLP can be reduced to the satisfiability of pCSP$^{\text{HL}}$, and (3) existing CHCs solving and invariants/ranking function synthesis techniques can be adopted to pCSP$^{\text{HL}}$ solving and further improved with the idea of stratified CEGIS for simultaneously achieving relative completeness (Theorem 6.1) and practical effectiveness (Figure 1, left).

Though we presented a sound and complete reduction from $\mu$CLP to pCSP$^{\text{HL}}$ and the classes of CHCs and co-CHCs correspond to fragments of $\mu$CLP as discussed in Remark 4, any $\mu$CLP is reduced to the satisfiability of a co-CHCs$^{\text{HL}}$ that is a strict syntactic fragment of pCSP$^{\text{HL}}$ and recent semantic results based on the recursion theory [Tsukada 2020] imply that the full class of pCSP$^{\text{HL}}$ is strictly more expressive than $\mu$CLP, meaning that the full class of pCSP$^{\text{HL}}$ is not necessary for the validity of $\mu$CLP. It would thus be interesting to investigate the potential of the full class of pCSP$^{\text{HL}}$ in practice and to find some (non-syntactic) restriction that would capture the full class of $\mu$CLP.

To further widen the applicability of our framework, we plan to extend our tools MuVal and PCSat to support other first-order theories beyond LIA/LRA such as arrays, algebraic data types (ADTs), and heaps [Duck et al. 2013]. As far as the semantics of $\mu$CLP is concerned, there is no issue with the background theory being incomplete (i.e., undecidable). However, the constraint solving method may require non-trivial extensions to support the above theories because it involves designing appropriate stratified families of templates. For example, certificates (i.e., invariants, ranking functions, witnesses for quantifiers) over arrays often require quantifiers, and those over heaps and ADTs often require inductive predicates. Future work also includes extensions of the framework to higher-order predicates and probabilities. The former extension is useful for precisely analyzing higher-order recursive functions (cf. HoCHC [Burn et al. 2018] and HFL($\mathbb{Z}$) [Kobayashi]...
et al. 2018; Watanabe et al. 2019]). The latter extension is for reasoning about programs and systems that exhibit uncertain or probabilistic behaviors (cf. PCHC [Albarghouthi 2017]).

REFERENCES

Aws Albarghouthi. 2017. Probabilistic Horn Clause Verification. In SAS ’17. Springer, 1–22.
Emanuele De Angelis, Fabio Fioravanti, Alberto Pettorossi, and Maurizio Proietti. 2014. VeriMAP: A tool for verifying programs through transformations. In TACAS ’14. Springer, 568–574.
Tewodros Beyene, Swarat Chaudhuri, Corneliu Popeea, and Andrey Rybalchenko. 2014. A Constraint-based Approach to Solving Games on Infinite Graphs. In POPL ’14 (San Diego, California, USA). ACM, 221–233.
Tewodros A. Beyene, Corneliu Popeea, and Andrey Rybalchenko. 2013. Solving Existentially Quantified Horn Clauses. In CAV ’13 (LNCS), Vol. 8044. Springer, 869–882.
Nikolaj Bjørner, Arie Gurfinkel, Kenneth L. McMillan, and Andrey Rybalchenko. 2015. Horn Clause Solvers for Program Verification. In Fields of Logic and Computation II: Essays Dedicated to Yuri Gurevich on the Occasion of His 75th Birthday (LNCS), Vol. 9300. Springer, 24–51.
Marijke H. L. Bodlaender, Cor A. J. Hurkens, Vincent J. J. Kusters, Frank Staals, Gerhard J. Woeginger, and Hans Zantema. 2012. Cinderella versus the Wicked Stepmother. In IFIP TCS. 57–71.
J. Richard Buchi and Lawrence H. Landweber. 1969. Solving Sequential Conditions by Finite-State Strategies. Trans. Amer. Math. Soc. 138 (1969), 295–311.
Byron Cook, Heidy Khlafa, and Nir Piterman. 2015. On Automation of CTL* Verification for Infinite-State Systems. In CAV ’15. Springer, 13–29.
Byron Cook and Eric Koskinen. 2011. Making Prophecies with Decision Predicates. In POPL ’11 (Austin, Texas, USA). ACM, 399–410.
Byron Cook and Eric Koskinen. 2013. Reasoning About Nondeterminism in Programs. In PLDI ’13 (Seattle, Washington, USA). ACM, 219–230.
Byron Cook, Eric Koskinen, and Moshe Vardi. 2011. Temporal Property Verification As a Program Analysis Task. In CAV ’11 (Snowbird, UT). Springer, 333–348.
Byron Cook, Andreas Podelski, and Andrey Rybalchenko. 2006. Termination proofs for systems code. In PLDI ’06. ACM, 415–426.
Byron Cook, Abigail See, and Florian Zuleger. 2013. Ramsey vs. Lexicographic Termination Proving. In TACAS ’13 (LNCS), Vol. 7795. Springer, 47–61.
Leonardo de Moura and Nikolaj Bjørner. 2008. Z3: An Efficient SMT Solver. In TACAS ’08 (Budapest, Hungary, March 29 – April 6) (LNCS), Vol. 4963. Springer, 337–340.
Daniel Dietsch, Matthias Heizmann, Vincent Langenfeld, and Andreas Podelski. 2015. Fairness Modulo Theory: A New Approach to LTL Software Model Checking. In CAV ’15. Springer, 49–66.
Gregory J. Duck, Joxan Jaffar, and Nicolas C. H. Koh. 2013. Constraint-Based Program Reasoning with Heaps and Separation. In CP ’13. Springer, 282–298.
Erich Grädel, Wolfgang Thomas, and Thomas Wilke (Eds.). 2002. Automata, Logics, and Infinite Games: A Guide to Current Research. LNCS, Vol. 2500. Springer.
A EXPRESSING VERIFICATION PROBLEMS IN $\mu$CLP

We now consider a variety of verification problems, showing that each of them can be expressed as validity in $\mu$CLP. Specifically, we discuss applications of $\mu$CLP to bisimulation and bisimilarity verification in Appendix A.1 and infinite state and infinite duration two player game solving in Appendix A.2.

A.1 Bisimulation and Bisimilarity Verification

Bisimulation and bisimilarity are a prototypical application of greatest fixpoint and co-induction in computer science [Sangiorgi 2011]. We show that the notions and various problems thereof can be naturally expressed in our framework.

Given two LTS $M_1$ and $M_2$ with $L = L_{M_1} = L_{M_2}$, the bisimilarity relation $\text{Bisim}_{M_1, M_2}$ between $M_1$ and $M_2$ can be defined in $\mu$CLP as follows:

\[
\text{Bisim}_{M_1, M_2}(x_1, x_2) = \nu \left( \bigwedge_{e \in L} \forall y_1. \langle x_1 \rangle \xrightarrow{e} M_1 \langle y_1 \rangle \implies \exists y_2. \langle x_2 \rangle \xrightarrow{e} M_2 \langle y_2 \rangle \land \text{Bisim}_{M_1, M_2}(y_1, y_2) \right)
\land \exists y_1. \langle x_1 \rangle \xrightarrow{e} M_1 \langle y_1 \rangle \land \text{Bisim}_{M_1, M_2}(y_1, y_2) \land \forall y_2. \langle x_2 \rangle \xrightarrow{e} M_2 \langle y_2 \rangle \implies \exists y_1. \langle x_1 \rangle \xrightarrow{e} M_1 \langle y_1 \rangle \land \text{Bisim}_{M_1, M_2}(y_1, y_2)
\]

Note that the equation defines $\text{Bisim}_{M_1, M_2}(x_1, x_2)$ as a greatest fixpoint. A basic problem of interest in bisimulation is deciding whether two (concrete) states, say $n_1 \in S_{M_1}$ and $n_2 \in S_{M_2}$, are bisimilar. This is expressed in our logic by the formula $\text{Bisim}_{M_1, M_2}(n_1, n_2)$. More generally, we may be interested in knowing if every pair of states $x_1 \in S_{M_1}$ and $x_2 \in S_{M_2}$ satisfying $\phi(x_1, x_2)$ are bisimilar, where $\phi$ is some property on pairs of states. This can be expressed by the formula $\phi(x_1, x_2) \implies \text{Bisim}_{M_1, M_2}(x_1, x_2)$.

Such queries are instances of checking if a formula is a lower-bound of a greatest fixpoint formula, and can be solved by our constraint solving method described in Sections 5 and 6. As we shall show
there, our technique for solving such a constraint corresponds to the well-known technique of proof by co-induction.

While co-induction can be used to prove lower-bounds of greatest fixpoints, a different, new technique is required to prove their upper-bounds. For instance, suppose that we wish to check if all bisimilar pairs of states satisfy a certain property, say $\psi$. The query can be expressed in our logic by: $\text{Bisim}_{M_1, M_2}(\bar{x}_1, \bar{x}_2) \Rightarrow \psi(\bar{x}_1, \bar{x}_2)$. Solving such greatest-fixpoint upper-bound queries are beyond the scope of previous methods. Nonetheless, our method is able to solve them by use of well-founded relations as we show in Sections 5 and 6.

Next, we instantiate the above with a concrete instance. Let us consider a concrete LTS $M$ with labels $L_M = \{+, -, \}$, states $S_M = \mathbb{Z}^2$, and the following transition relation:

\[ \langle x, y \rangle \xrightarrow{+} \langle x + 1, y \rangle \quad \text{if} \quad x + 1 \leq y \]
\[ \langle x, y \rangle \xrightarrow{-} \langle x - 1, y \rangle \quad \text{if} \quad x - 1 \geq y \]

Let us consider $\text{Bisim}_{M, M}$, that is, we consider the bisimulation relation between two states of the same system $M$. We may then check if two states, for instance $(0, 1) \in S_M$ and $(1, 2) \in S_M$, are bisimilar by proving if $\text{Bisim}_{M, M}(0, 1, 1, 2)$ is true. In this case, our method is able to do the proof by synthesizing the co-inductive invariant $\text{Bisim}_{M, M}(x_1, y_1, x_2, y_2) \triangleq x_1 = 0 \land x_2 = 1 \land x_3 = 1 \land x_4 = 2 \lor x_1 = 1 \land x_2 = 1 \land x_3 = 2 \land x_4 = 2$. Our method can also prove a more general property that any states of $M$ such that $y - x$ is the same are bisimilar, by synthesizing the co-inductive invariant $\text{Bisim}_{M, M}(x_1, y_1, x_2, y_2) \triangleq y_1 - x_1 = y_2 - x_2$.

Next, suppose that we wish to prove that every pair of bisimilar states $(x_1, y_1) \in S_M$ and $(x_2, y_2) \in S_M$ satisfies $y_1 - x_1 = y_2 - x_2$. That is, every bisimilar states of $M$ have equal direction and distance from $x$ to $y$. The query can be expressed in our logic by the following formula:

$$\text{Bisim}_{M, M}(x_1, y_1, x_2, y_2) \Rightarrow y_1 - x_1 = y_2 - x_2$$

which is equivalent to

$$y_1 - x_1 \neq y_2 - x_2 \Rightarrow \text{Bisim}_{M, M}^\neg(x_1, y_1, x_2, y_2)$$

where $\text{Bisim}_{M, M}^\neg$ is the de Morgan dual of $\text{Bisim}_{M, M}$ defined by:

$$\text{Bisim}_{M_1, M_2}^\neg(\bar{x}_1, \bar{x}_2) = \mu \ell \left( \exists \bar{y}_1. \langle \bar{x}_1 \rangle \xrightarrow{\ell} M_1 \langle \bar{y}_1 \rangle \land \langle \forall \bar{y}_2. \langle \bar{x}_2 \rangle \xrightarrow{\ell} M_2 \langle \bar{y}_2 \rangle \Rightarrow \text{Bisim}_{M_1, M_2}^\neg(\bar{y}_1, \bar{y}_2) \right)$$

As remarked above, such a “property checking” query on greatest fixpoints can be, as a result, handled by our method by using well-founded relations. Here, our method synthesizes the inductive invariant $\text{Bisim}_{M, M}^\neg(x_1, y_1, x_2, y_2) \triangleq y_1 - x_1 \neq y_2 - x_2$ and the well-founded relation $\text{Bisim}_{M, M}^\neg(y_1, y_1, x_2, y_2) \triangleq r(x_1, y_1, x_2, y_2) > r(x_1', y_1', x_2', y_2')$ where $r(x_1, y_1, x_2, y_2) = (y_1 - x_1) + (y_2 - x_2)$.

### A.2 Infinite State and Infinite Duration Games Solving

Two-player turn-based infinite-duration games are games in which two players take turns in moving a token along the edges of a graph. A player wins if the (infinite) sequence of nodes visited by the token satisfies a certain condition. Classically, the games are played over a finite graph representing the state transition diagram of a finite-state transition system, and there is a rich body of work relating such games to the verification and synthesis of finite state systems [Grädel et al. 2002]. For instance, in the synthesis of reactive systems [Buchi and Landweber 1969; Pnueli and Rosner 1989; Thomas 1995], a game with two players, Sys and Env, is considered over a graph with edges from
one player’s node to the other player’s node. The edges from Sys’s nodes describe the possible (one-step) execution choices of the system to be synthesized and those from Env’s nodes describe the possible external inputs to the system. The goal of Sys is to satisfy the given specification (given, e.g., by a temporal logic formula) whereas the goal of Env is to violate it. The desired system is realizable if and only if Sys has a winning strategy.

Recently, the line of work has been extended to infinite-state systems with which one can express the verification and synthesis problems for infinite-state systems [Beyene et al. 2014; Farzan and Kincaid 2017]. We show that our framework is expressive enough to express such infinite-state infinite-duration games. Following the literature [Beyene et al. 2014; Farzan and Kincaid 2017], we consider three classes of games: Safety games, Reachability games, and LTL games.\(^\text{10}\)

Each game is played over a graph formed by a LTS. Specifically, we consider a LTS of the form \(M = (S, T, L_A \cup L_E)\) where \(L_A \cap L_E = \emptyset\), that is, the labels are partitioned into \(L_A\) and \(L_E\). For each \(\ell \in L_E\) (resp. \(\ell \in L_A\)), a transition \(s \xrightarrow{\ell} s'\) denotes E player’s (resp. A player’s) move from node \(s\) to node \(s'\). We write \(s \xrightarrow{E} s'\) (resp. \(s \xrightarrow{A} s'\)) when there exists \(\ell \in L_E\) (resp. \(\ell \in L_A\)) such that \(s \xrightarrow{\ell} s'\). Below, we describe each class of games and our encoding of them in \(\mu\)CLP. For simplicity, we assume that each game starts with A’s turn.

A.2.1 Safety games. In a safety game, we are given predicates \(\phi_{\text{init}}(x)\) and \(\phi_{\text{safe}}(x)\). The E player wins the game if only states satisfying \(\phi_{\text{safe}}(x)\) are visited along any sequence of plays starting from any state satisfying \(\phi_{\text{init}}(x)\). The game can be expressed in \(\mu\)CLP by defining the greatest fixpoint predicate \(SG(x)\) as follows:

\[
SG(x) = \forall y. \langle x \rangle \xrightarrow{A} \langle y \rangle \Rightarrow \phi_{\text{safe}}(y) \wedge \exists z. \langle y \rangle \xrightarrow{E} \langle z \rangle \wedge SG(z).
\]

Then, E has a winning strategy if and only if \(\phi_{\text{init}}(x) \Rightarrow SG(x)\) is valid. The correctness of the encoding can be readily seen by observing that \(SG(x)\) describes exactly the set of states from which E can force the plays to stay in the states satisfying \(\phi_{\text{safe}}(x)\).

A.2.2 Reachability games. In a reachability game, we are given predicates \(\phi_{\text{init}}(x)\) and \(\phi_{\text{reach}}(x)\). The E player wins the game if for any play starting from a state satisfying \(\phi_{\text{init}}(x)\), a state satisfying \(\phi_{\text{reach}}(x)\) is eventually visited. As clear from the definition, reachability games are the dual of safety games. That is, a safety game with the objective \(\phi_{\text{safe}}(x)\) is won by E if and only if A wins the reachability game with the objective \(\phi_{\text{reach}}(x) = \neg \phi_{\text{safe}}(x)\) on the same graph but with the players’ edge sets swapped. The game can be expressed in \(\mu\)CLP by defining the least fixpoint predicate \(RG(x)\) as follows:

\[
RG(x) = \mu y. \phi_{\text{reach}}(x) \vee \forall y. \langle x \rangle \xrightarrow{A} \langle y \rangle \Rightarrow \phi_{\text{reach}}(y) \vee \exists z. \langle y \rangle \xrightarrow{E} \langle z \rangle \wedge RG(z).
\]

Then, E has a winning strategy if and only if \(\phi_{\text{init}}(x) \Rightarrow RG(x)\) is valid. The correctness of the encoding can be readily seen by observing that \(RG(x)\) describes exactly the set of states from which E can force the plays to eventually reach a state satisfying \(\phi_{\text{reach}}(x)\).

Our \(\mu\)CLP formulations of safety games and reachability games show a striking resemblance, reflecting the inherent duality of the two classes of games. This is in contrast to their formulations in existentially-quantified Horn clauses [Beyene et al. 2014, 2013] that used rather different encodings for the two classes.

\(^{10}\)“LTL games” is a misnomer as the games actually go beyond LTL properties. [Farzan and Kincaid 2017] also considers another class of games called Satisfiability games which is only finite duration and therefore is omitted from our discussion.
A.2.3 LTL games. In a LTL game, we are given a predicate $\phi_{\text{init}}(x)$ and a Büchi automaton $A$ such that the label set of $A$ is $L_E \cup L_A$ (cf. Section 3.3 for the definition of Büchi automaton). The E player wins the game if for any play starting from a state satisfying $\phi_{\text{init}}(x)$, the infinite sequence of labels of the play is accepted by $A$. The game can be expressed in $\mu$CLP by defining the mutually-recursive least-and-greatest fixpoint predicates $LG_{q,\alpha}(x)$ for each $q \in Q$ and $\alpha \in \{\mu, \nu\}$:

$$LG_{q,\alpha}(x) = \alpha \bigwedge_{\ell \in L_A} \forall y.\langle x \rangle \xrightarrow{\ell} \langle y \rangle \Rightarrow \bigvee_{q' \in \delta(q,\ell)} \bigvee_{q'' \in \delta(q',\ell')} \exists z.\langle y \rangle \xrightarrow{\ell'} \langle z \rangle \land LG_{q'',\alpha(q',q'')}\langle z \rangle.$$  

Here, $\alpha(q_1, q_2) = \nu$ if $q_1 \in F$ or $q_2 \in F$, and $\alpha(q_1, q_2) = \mu$ otherwise. Then, E has a winning strategy if and only if $\phi_{\text{init}}(x) \Rightarrow LG_{q_{\text{init}},\nu}(x)$ is valid. The correctness of the construction follows from an argument similar to that of the linear temporal property verification reduction shown in Section 3.3.

Here is an example of a simple LTL game for the property $GF(\text{restore})$. The game consists of a single integer variable $x$ whose value is initially 0. The LTS is defined with the label sets $L_E = \{\text{restore}, \text{incr}, \text{decr}\}$ and $L_A = \{\text{break}, \text{skip}\}$, and the transition relation is shown in the left column below:

- $\langle x \rangle \xrightarrow{\text{restore}} \langle x' \rangle$ if $x = x' = 0$
- $\langle x \rangle \xrightarrow{\text{incr}} \langle x' \rangle$ if $x' = x + 1$
- $\langle x \rangle \xrightarrow{\text{decr}} \langle x' \rangle$ if $x' = x - 1$
- $\langle x \rangle \xrightarrow{\text{break}} \langle x' \rangle$ if $x = 0$ and $x' \neq 0$
- $\langle x \rangle \xrightarrow{\text{skip}} \langle x' \rangle$ if $x = x'$.

A Büchi automaton expressing the property $GF(\text{restore})$ is shown in the right column.

The winning strategy is obvious: whenever player A breaks away from $q_0$ by randomly assigning to $x$, E must either incr or decr to get $x$ back to 0 and then restore. Following the encoding of $LG_{q,\alpha}$ above, it is straightforward to define this game in $\mu$CLP. The encoding will be as follows:

$$LG_{q,\alpha}(x) = \alpha (\forall x'. (x = 0 \land x' \neq 0) \Rightarrow \phi(x')) \land (\forall x'. x' = x \Rightarrow \phi(x'))$$

$$\phi(x') = \bigvee \begin{cases} \exists x''. x'' = x' - 1 \land LG_{q_{\text{init}},\mu}(x'') \\ \exists x''. x'' = x' + 1 \land LG_{q_{\text{init}},\mu}(x'') \\ \exists x''. x'' = x' = 0 \land LG_{q_{\text{init}},\nu}(x'') \end{cases}$$

where $q \in \{q_0, q_1\}$ and $\alpha \in \{\mu, \nu\}$. The universally quantified actions pertain to the A player performing a break or skip action. The existentially quantified actions pertain to the E player performing a decr, incr, or restore action.

A.2.4 Cinderella-Stepmother game. As a concrete example of the three classes of games, let us consider the Cinderella-Stepmother game [Bodlaender et al. 2012; Hurkens et al. 2011], which is also used as examples in [Beyene et al. 2014; Farzan and Kincaid 2017]. The game comprises five buckets of water arranged in a circle. Each bucket can hold some constant $c$ amount of water. The two players, Cinderella and Stepmother, take turns emptying and filling the buckets. In each of her turns, Stepmother brings 1 unit of additional water and distributes it among the five buckets. In turn, Cinderella chooses two adjacent buckets and empties them. Cinderella wins if none of the buckets ever overflow. It is known that Cinderella has a winning strategy exactly when $c > 2$. For instance, when $c < 1$, it is easy to see that Stepmother wins in one round by pouring the entire additional water to a single bucket. Also, when $c \geq 3$, it is easy to see that Cinderella can win by adopting the round-robin strategy whereby she goes around the circle and in each round empties two buckets that are adjacent to the buckets that were emptied in the previous round. However, as
 remarked in [Beyene et al. 2014; Farzan and Kincaid 2017], synthesizing a winning strategy (for Cinderella or Stepmother) when \(1 \leq c < 3\) is non-trivial.

We formalize the game in our framework. The state space of the game is \(S = \mathbb{Q}_{\geq 0}^5\) where \(\mathbb{Q}_{\geq 0}\) is the set of non-negative rational numbers. Each \((b_0, b_1, b_2, b_3, b_4) \in S\) represents the state of the five buckets with \(b_i\) being the amount of water in the \(i\)-th bucket. The set of labels for Stepmother is \(L_{SM} = \{\text{ov}, \text{sm}\}\) where \(\text{ov}\) indicates that a bucket is overflowing after Stepmother has made the move. The set of labels for Cinderella is the singleton set \(L_{CD} = \{\text{cd}\}\). Let us write \([0, 4]\) for the set \([0, 1, 2, 3, 4]\). The transition relation is defined by:

- \((b_0, b_1, b_2, b_3, b_4) \xrightarrow{\text{sm}} (b'_0, b'_1, b'_2, b'_3, b'_4)\) if \(1 + \sum_{i \in [0, 4]} b_i = \sum_{i \in [0, 4]} b'_i\) and \(b_i \leq b'_i \leq c\) for each \(i \in [0, 4]\).
- \((b_0, b_1, b_2, b_3, b_4) \xrightarrow{\text{ov}} (b'_0, b'_1, b'_2, b'_3, b'_4)\) if \(1 + \sum_{i \in [0, 4]} b_i = \sum_{i \in [0, 4]} b'_i\), \(b_i \leq b'_i\) for each \(i \in [0, 4]\), and there exists \(i \in [0, 4]\) such that \(b'_i > c\).
- \((b_0, b_1, b_2, b_3, b_4) \xrightarrow{\text{cd}} (b'_0, b'_1, b'_2, b'_3, b'_4)\) if there exists \(i \in [0, 4]\) such that \(b'_i = b'_{(i+1)\%5} = 0\) and \(b_j = b'_j\) for each \(j \in [0, 4] \setminus \{i, (i + 1)\%5\}\).

The set of initial states is described by \(\varphi_{\text{init}}(b_0, b_1, b_2, b_3, b_4) \triangleq \bigwedge_{i \in [0, 4]} b_i = 0\), that is, the buckets are initially all empty.

For Cinderella, the game can be formalized as a safety game where \(L_E = L_{CD}, L_A = L_{SM}\), and the safety objective is \(\varphi_{\text{safe}}(b_0, b_1, b_2, b_3, b_4) \triangleq \bigwedge_{i \in [0, 4]} b_i \leq c\). Dually, for Stepmother, the game can be formalized as a reachability game where \(L_E = L_{SM}, L_A = L_{CD}\), and the reachability objective is \(\varphi_{\text{reach}}(b_0, b_1, b_2, b_3, b_4) \triangleq \neg \varphi_{\text{safe}}(b_0, b_1, b_2, b_3, b_4) = \bigvee_{i \in [0, 4]} b_i > c\).

As in [Beyene et al. 2014], let us also consider variants of the game with LTL objectives. For instance, consider the Büchi automata \(A_1\) and \(A_2\) shown in Fig. 2. \(A_1\) corresponds to the LTL formula \(\text{FG}(\neg \text{ov})\). That is, it accepts exactly the plays in which overflows happen only finitely often. By contrast, \(A_2\) corresponds to the LTL formula \(\text{GF}(\neg \text{ov})\) and it accepts exactly the plays in which buckets are in a non-overflowing state infinitely often. As also remarked in [Beyene et al. 2014], the automata are examples of weakened objectives for Cinderella which may allow her to win the game more often. Using \(A_1\) or \(A_2\) as the objective Büchi automaton and letting \(L_E = L_{CD}\) and \(L_A = L_{SM}\), our framework is able to model the weakened variants as LTL games.

Fig. 2. Büchi automata for Cinderella-Stepmother LTL games.