On finite simple and nonsolvable groups acting on closed 4-manifolds

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Abstract. We show that the only finite nonabelian simple groups which admit a locally linear, homologically trivial action on a closed simply connected 4-manifold $M$ (or on a 4-manifold with trivial first homology) are the alternating groups $A_5$, $A_6$ and the linear fractional group $PSL(2,7)$ (we note that for homologically nontrivial actions all finite groups occur). The situation depends strongly on the second Betti number $b_2(M)$ of $M$ and has been known before if $b_2(M)$ is different from two, so the main new result of the paper concerns the case $b_2(M) = 2$. We prove that the only simple group that occurs in this case is $A_5$, and then give a short list of finite nonsolvable groups which contains all candidates for actions of such groups.

1. Introduction

We are interested in actions of finite groups on closed orientable 4-manifolds. All actions in the present paper will be locally linear, faithful and orientation-preserving. An action is locally linear if the isotropy group of each point leaves invariant a neighbourhood of the point which is equivariantly homeomorphic to an invariant neighbourhood of the origin in a linear action on some Euclidean space $\mathbb{R}^n$ (e.g. smooth actions).

It has been shown in [MZ1] that the only finite nonabelian simple groups acting on a homology 4-sphere are the alternating groups $A_5$ and $A_6$, and from this a short list of finite nonsolvable groups is deduced containing all candidates for an action on a homology 4-sphere (and in particular on the 4-sphere; the corresponding situation in dimension three is considered in [MZ2,3] and [Z2]). On the other hand, since each finitely presented group is the fundamental group of a closed 4-manifold, each finite group $G$ admits a free action on a simply connected closed 4-manifold (the universal covering of a closed 4-manifold with fundamental group $G$); as a consequence of the Lefschetz fixed point Theorem, such a free action has to act nontrivially on homology. For homologically trivial actions on simply connected 4-manifolds, and more generally on 4-manifolds $M$ with trivial first homology $H_1(M)$, there are again strong restrictions. Building on previous work of various authors and concentrating on the basic case of nonabelian simple groups, the following holds.

1
Theorem. Let $G$ be a finite nonabelian simple group which admits a homologically trivial action on a closed 4-manifold $M$ with trivial first homology. Then $G$ is isomorphic to $A_5$, $A_6$ or $PSL(2,7)$.

For various cases the Theorem has been known previously (and also in greater generality); in fact, the situation depends strongly on the second Betti number $b_2(M)$ of $M$, and we shall discuss the different cases in the following; $M$ will always denote a closed 4-manifold with trivial first homology $H_1(M)$.

I. The case $b_2(M) \geq 3$

In this case the possible finite groups which admit an action are very restricted; in particular, no nonabelian simple groups occur. The following is the main result of [Mc1].

**Theorem 1.** ([Mc1]) Let $G$ be a finite group with a homologically trivial action on a closed 4-manifold $M$ with trivial first homology.

i) If $b_2(M) \geq 3$ then $G$ is abelian of rank at most two (cyclic or a product of two cyclic groups), and $G$ has a global fixed point.

ii) If $b_2(M) \geq 2$ and $G$ has a global fixed point then again $G$ is abelian of rank at most two.

Also in the next case a complete classification is still known.

II. The case $b_2(M) = 1$

Now $M$ is a homology complex projective plane $\mathbb{C}P^2$, so a reference model here is the group of projectivities $PGL(3,\mathbb{C})$ of $\mathbb{C}P^2$. The case has been considered in [W1] and [HL] for homologically trivial actions of arbitrary finite groups (see also [W2]), and the following holds.

**Theorem 2.** ([W1],[HL]) Let $G$ be a finite group which admits a homologically trivial action on a closed 4-manifold $M$ with $b_2(M) = 1$ and trivial first homology (e.g. the complex projective plane $\mathbb{C}P^2$). Then $G$ is isomorphic to a subgroup of $PGL(3,\mathbb{C})$ and in particular, if $G$ is nonabelian simple, to $A_5$, $A_6$ or $PSL(2,7)$.

These are exactly the finite nonabelian simple subgroups of $PGL(3,\mathbb{C})$. A main ingredient of the proof of Theorem 2 is the classification of the finite simple groups of 2-rank at most two (i.e. without subgroups $(\mathbb{Z}_2)^3$).

In the two remaining cases, a complete classification seems still far, so we concentrate on the basic case of finite simple groups in the following.
III. The case $b_2(M) = 0$ of a homology 4-sphere

Now the following holds.

**Theorem 3.** ([MZ1]) A finite nonabelian simple group acting on a homology 4-sphere, and in particular on the 4-sphere $S^4$, is isomorphic to an alternating group $A_5$ or $A_6$.

This result is used in [MZ1] to obtain a short list of finite nonsolvable groups which contains all candidates for actions on a homology 4-sphere of such groups. A reference model here is the orthogonal group $SO(5)$ acting on $S^4$. We note that now subgroups $(\mathbb{Z}_2)^4$ may occur, so the 2-rank of a finite simple group acting on a homology 4-sphere might, in principle, be equal to four; in fact, the main ingredient of the proof is the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four (i.e. each 2-subgroup is generated by at most four elements).

Finally, we come to the last and main case of the present paper.

IV. The case $b_2(M) = 2$

The main results of the present paper are the following two theorems.

**Theorem 4.** Let $G$ be a finite nonabelian simple group which admits an action on a closed 4-manifold $M$ with $b_2(M) = 2$ and trivial first homology (e.g. $S^2 \times S^2$). Then $G$ is isomorphic to the alternating group $A_5$, and $M$ has the intersection form \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] of $S^2 \times S^2$.

In particular, if $M$ is simply connected then by [F] it is homeomorphic to $S^2 \times S^2$, so a reference model here is $S^2 \times S^2$ and the group $SO(3) \times SO(3)$ that consists of isometries of $S^2 \times S^2$ that act trivially on homology (whose only finite simple subgroup is $A_5$). Again subgroups $(\mathbb{Z}_2)^4$ might in principle occur and, as for Theorem 3, a main tool of the proof will be the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four (as well as part ii) of Theorem 1).

On the basis of Theorem 4 we obtain a short list of finite nonsolvable groups which contains all candidates for actions on this class of manifolds; the result is summarized in the following theorem.

**Theorem 5.** Let $G$ be a finite nonsolvable group that admits a homologically trivial action on a closed 4-manifold $M$ with $b_2(M) = 2$ and trivial first homology. Then $G$ contains, of index at most two, a normal subgroup isomorphic to one of the following groups:

$A_5 \times C$, $A_5^* \times \mathbb{Z}_2$, $A_5 \times A_5$, $A_5 \times A_4$

where $C$ is a cyclic group.
This is close to the list of the finite nonsolvable subgroups of $\text{SO}(3) \times \text{SO}(3)$, except that we are not able to exclude the binary dihedral group $A_5^*$ at moment (we suppose that it does not act). Some information about the possible 2-extensions can be deduced from the proof of Theorem 5.

2. Proof of Theorem 4

In the following, $M$ will always denote a closed 4-manifold with $b_2(M) = 2$ and trivial first homology $H_1(M)$, and $G$ a finite nonabelian simple group acting faithfully and locally linearly on $M$. Since the finite subgroups of $\text{GL}(2,\mathbb{Z})$ are cyclic or dihedral and $G$ is nonabelian simple, the action of $G$ is homologically trivial. We start with some preliminary results.

Lemma 1. Let $g$ be an orientation-preserving periodic map of $M$ which is not the identity and acts trivially on the homology of $M$. Then the fixed point set of $g$ is of one of the following types:

i) four isolated points;

ii) a 2-sphere $S^2$ and two isolated points;

iii) two 2-spheres $S^2$.

Proof. By a version of the Lefschetz fixed point Theorem (see e.g. [TD]), the Euler characteristic of the fixed point set of $g$ equals the alternating sum of the traces of the maps induced by $g$ on the rational homology $H_*(M;\mathbb{Q})$ of $M$; since $g$ acts trivially on homology and $H_1(M) = 0$ this is equal to four. By [E, Prop.2.4], the fixed point set of $g$ has no 1-dimensional components and consists of isolated points and 2-spheres; this leaves the three possibilities of the Lemma, thus finishing its proof.

Lemma 2. Let $S$ be a finite 2-group which admits a faithful, homologically trivial action on $M$. Then $S$ is generated by at most four elements. In particular, $G$ has sectional 2-rank at most four (each 2-subgroup is generated by at most four elements).

Proof. Let $g$ be a central involution in $S$. The possible fixed point sets of $g$ are listed in Lemma 1. If the fixed point set $\text{Fix}(g)$ consists of a 2-sphere $S^2$ and two isolated points then the 2-sphere $S^2$ is invariant under $S$. By a result of Edmonds [E], see also [Mc1, Theorem 2], the action on $S^2$ is orientation preserving. The subgroup of $S$ acting trivially on $S^2$ is cyclic (since $S$ acts homologically trivial and hence orientation-preserving on $M$), its factor group acts faithfully on $S^2$ and, being a 2-group, is a subgroup of a dihedral group. Clearly $S$ is generated by at most three elements.

Suppose that $\text{Fix}(g)$ consists of two 2-spheres. Then a subgroup of index at most two of $S$ leaves invariant both 2-spheres and, considering the first case, $S$ is generated by at most four elements.

Finally, suppose that $\text{Fix}(g)$ consists of four isolated points, invariant under $S$. Let $S_0$ be the subgroup of $S$ fixing one of these four points, of index at most four. By
Theorem 1 ii), $S_0$ is abelian of rank at most two and hence $S$ is generated by at most four elements.

This finishes the proof of Lemma 2.

We apply the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four (see [G1,p.6], [Su2,chapter 6, Theorem 8.12]). By Lemma 2, $G$ has sectional 2-rank at most four and hence is one of the groups in the Gorenstein-Harada list; the groups are the following ($q$ denotes an odd prime power):

$$
\begin{align*}
&\text{PSL}(m,q), \; \text{PSU}(m,q), \; m \leq 5, \\
&G_2(q), \; 3\text{D}_4(q), \; \text{PSp}(4,q), \; 2G_2(3^{2m+1}) \; (m \geq 1), \\
&\text{PSL}(2,8), \; \text{PSL}(2,16), \; \text{PSL}(3,4), \; \text{PSU}(3,4), \; \text{Sz}(8), \\
&A_m \; (7 \leq m \leq 11), \; M_i \; (i \leq 23), \; J_i \; (i \leq 3), \; \text{McL}, \; \text{Ly}.
\end{align*}
$$

In the following, we will exclude all of these groups except $A_5$. We consider first the linear fractional groups $\text{PSL}(2,p)$, for a prime $p \geq 5$. The group $\text{PSL}(2,p)$ has a meta-cyclic subgroup (semidirect product) $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$ (represented by all upper triangular matrices), with an effective action of $\mathbb{Z}_{(p-1)/2}$ (the diagonal matrices) on the normal subgroup $\mathbb{Z}_p$ (the matrices having both entries one on the diagonal).

**Lemma 3.** For an odd prime $p$ and an integer $q \geq 2$, let $U = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ be a metacyclic group, with an effective action of $\mathbb{Z}_q$ on the normal subgroup $\mathbb{Z}_p$.

i) If $U$ admits a faithful, orientation-preserving action on a homology 3-sphere then $q = 2$.

ii) If $U$ admits a faithful, orientation-preserving action on a closed 4-manifold $M$ as in Theorem 4 then $q = 2$ or $q = 4$.

**Proof.** i) See [Z1, Proof of Prop.1].

ii) We denote by $g$ a generator of the normal subgroup $\mathbb{Z}_p$ of $U$. If we are in case ii) of Lemma 1 then $\mathbb{Z}_p$ fixes pointwise a 2-sphere $S^2$ which is invariant under $\mathbb{Z}_q$ and $U$. If $q > 2$ then $\mathbb{Z}_q$ and hence $U$ have a global fixed point on $S^2$; now a $U$-invariant regular neighbourhood in $M$ of this fixed point is a 3-sphere with a faithful action of $U$, contradicting part i) of the Lemma.

If the fixed point set of $g$ consists of two 2-spheres then a subgroup of index two of $U$ fixes pointwise both 2-spheres. As before, this is possible only for $q = 2$ or $q = 4$.

Finally, suppose that the fixed point set of $\mathbb{Z}_p$ consists of four isolated points. Then a subgroup of index at most four of $U$ has a fixed point and acts faithfully on a 3-sphere. Again by part i) of the Lemma, this is possible only for $q = 2$, 4 or 8. If $q = 8$ then a dihedral subgroup $\mathbb{Z}_p \rtimes \mathbb{Z}_2$ of $U$ has a fixed point in $M$. The situation for actions of
dihedral groups has been analyzed in [Mc1], in particular it follows from [Mc1, Prop.13] that in the case $b_2(M) = 2$ a dihedral group has to act without fixed points on $M$. This contradiction excludes $q = 8$ and finishes the proof of Lemma 3.

**Lemma 4.** i) If $G = \text{PSL}(2, p)$, for a prime $p \geq 5$, then $p = 5$ and $G$ is isomorphic to $A_5 \cong \text{PSL}(2, 5)$.

ii) If $G = \text{PSL}(2, q)$, for a prime power $q = p^n$ with $n > 1$, then $q = 4$ so again $G$ is isomorphic to $A_5 \cong \text{PSL}(2, 4)$.

iii) Let $\tilde{G}$ be a finite central extension, with nontrivial center, of a nonabelian simple group $G$ (e.g., the central extension $\text{SL}(2, q)$ of $\text{PSL}(2, q)$). If $\tilde{G}$ acts faithfully on $M$ then $G$ is isomorphic to the dodecahedral group $A_5$.

**Proof.** i) Since $\text{PSL}(2, p)$ has a metacyclic subgroup $U = \mathbb{Z}_p \ltimes \mathbb{Z}_{(p-1)/2}$, with an effective action of $\mathbb{Z}_{(p-1)/2}$ on the normal subgroup $\mathbb{Z}_p$, part i) follows from Lemma 3.

ii) The group $G = \text{PSL}(2, p^n)$ has a subgroup $U = (\mathbb{Z}_p)^n \ltimes \mathbb{Z}_{(q-1)/2}$ if $p$ is odd, resp. $U = (\mathbb{Z}_p)^n \times \mathbb{Z}_{q-1}$ if $p = 2$, with an effective action of $\mathbb{Z}_{(q-1)/2}$ resp. $\mathbb{Z}_{q-1}$ on $(\mathbb{Z}_p)^n$. Since $\text{PSL}(2, p)$ is a subgroup of $\text{PSL}(2, p^n)$, part i) of the Lemma implies that $p = 2, 3$ or 5.

Suppose that $p = 3$ or 5. We consider the subgroup $(\mathbb{Z}_p)^n$ of $U$ and a nontrivial element $g$ in $(\mathbb{Z}_p)^n$. If $g$ fixes pointwise a 2-sphere (cases ii) and iii) of Lemma 1) then this 2-sphere is invariant under $(\mathbb{Z}_p)^n$, hence there is a faithful action of $(\mathbb{Z}_p)^{n-1}$ on $S^2$ which is possible only if $n \leq 2$. If $g$ fixes four isolated points then $(\mathbb{Z}_p)^n$ has a global fixed point, and by Theorem 1 ii) again we have $n \leq 2$.

Suppose that $G = \text{PSL}(2, 25)$, with a subgroup $U = (\mathbb{Z}_5)^2 \ltimes \mathbb{Z}_{12}$. If the element $g$ in $(\mathbb{Z}_5)^2$ fixes pointwise one or two 2-spheres then $(\mathbb{Z}_5)^2$ has two or four fixed points and some nonabelian subgroup of $U$ has a global fixed point contradicting Theorem 1. If $g$ fixes four isolated points then also $(\mathbb{Z}_5)^2$ fixes these points, and again a nonabelian subgroup of $U$ has a global fixed point. So $\text{PSL}(2, 25)$ does not occur.

Next we consider $\text{PSL}(2, 9)$, with a subgroup $(\mathbb{Z}_3)^2 \ltimes \mathbb{Z}_4$; let $g$ be a nontrivial element in $(\mathbb{Z}_3)^2$. If $g$ fixes four isolated points then all nontrivial elements in $(\mathbb{Z}_3)^2$ have exactly four fixed points (since all subgroups $\mathbb{Z}_4$ are conjugate). The whole group $(\mathbb{Z}_3)^2$ fixes at least one of the four fixed points of $g$ and hence admits a free action on $S^3$; but $(\mathbb{Z}_3)^2$ does not admit a free action on $S^3$ (see [B]) so this case does not occur. A similar argument applies if $g$ fixes two isolated points and a 2-sphere. Finally, if $g$ fixes pointwise two 2-spheres then each of these 2-sphere is invariant under $(\mathbb{Z}_3)^2$ and $(\mathbb{Z}_3)^2$ has two global fixed points on it. By [Mc1, Prop.14], the singular set of $(\mathbb{Z}_3)^2$ consists of exactly four 2-spheres intersecting pairwise at their poles; this contradicts the fact that all subgroups $\mathbb{Z}_3$ of $(\mathbb{Z}_3)^2$ are conjugate and hence fix pointwise two 2-spheres. So $\text{PSL}(2, 9)$ does not occur.
This leaves us with the groups PSL(2, 2^n), for n ≥ 3, with a subgroup U = (Z_2)^n ⋊ Z_{q-1} such that all involutions in (Z_2)^n are conjugate. Suppose that n ≥ 4. Let g be an involution in (Z_2)^n. If g has four isolated fixed points or fixes two points and a 2-sphere, then a subgroup (Z_2)^2 of (Z_2)^n has a global fixed point and acts freely on S^3 which is a contradiction. Suppose that the fixed point set of g consists of two 2-spheres; then another involution in (Z_2)^n leaves each of these 2-spheres invariant and acts orientation-preservingly on it (by [Mc1, Theorem 2]), any involution has to act orientation-preservingly on such a 2-sphere since the action is homologically trivial). Now again a subgroup (Z_2)^2 has a global fixed point, and a contradiction to [Mc1, Prop.14] is obtained as in the previous case of the subgroup (Z_3)^2 of PSL(2, 9).

Finally, we exclude the group PSL(2, 8) which has a subgroup (Z_2)^3 ⋊ Z_7 such that all involutions are conjugate; let g be an involution in (Z_2)^3. If g fixes two 2-spheres then a subgroup (Z_2)^2 has four fixed points. By [Mc1, Prop.14] the singular set of (Z_2)^2 is a union of four 2-spheres which is a contradiction since each involution in (Z_2)^2 fixes pointwise two 2-spheres. If g fixes a 2-sphere and two isolated points then again a subgroup (Z_2)^2 has four fixed points which is a contradiction to [Mc1, Prop.14] since each involution fixes exactly one 2-sphere.

Suppose that g has four isolated fixed points. None of these points is fixed by a subgroup (Z_2)^2 since otherwise (Z_2)^2 would act freely on a 3-sphere S^3. So each orbit under (Z_2)^3 of a fixed point of an involution has exactly four elements, and there are exactly seven such orbits. This is exactly the situation excluded for a group (Z_2)^2 in the proof of [Mc2, Lemma 4.5], so PSL(2, 8) does not act.

iii) Let g be a nontrivial central element of ˜G. If g fixes pointwise one or two 2-spheres then such a 2-sphere is invariant under the factor group ˜G/ < g > or under a subgroup of index two, and hence G ≅ A_5. If g has four isolated fixed points then a subgroup of index at most four of ˜G fixes each of these four points; by Theorem 1, such a group has to be abelian so this case does not occur.

This finishes the proof of Lemma 4.

Continuing with the proof of Theorem 4 we consider next the groups G = PSL(m, q) in the Gorenstein-Harada list, where q = p^n is odd or m = 3 and q = 4. We note that PSL(m, p) is a subgroup of PSL(m, q); also, for r < m, the linear group SL(r, q) is a subgroup of the linear fractional group PSL(m, q) (see [Su2, chapter 6.5]). Applying Lemma 3 and Lemma 4, it suffices then to exclude the groups PSL(3, 4), PSL(3, 3) and PSL(3, 5); but PSL(3, 4) has a subgroup PSL(3, 2) ≅ PSL(2, 7), the group PSL(3, 3) has a metacyclic subgroup Z_13 ⋊ Z_3 and PSL(3, 5) a metacyclic subgroup Z_31 ⋊ Z_3 which are all excluded by Lemma 3 or Lemma 4 (see [C] for information about the subgroup structure of the finite simple groups). Thus among the linear fractional groups PSL(m, q) there remains only the group PSL(2, 5) ≅ A_5.
The proof for the unitary groups $\text{PSU}(m, q)$ and the symplectic groups $\text{PSp}(4, q)$ is similar, noting that $\text{PSU}(2, q) \cong \text{PSL}(2, q) \cong \text{PSp}(2, q)$. The unitary groups $\text{PSU}(3, 3)$ and $\text{PSU}(3, 5)$ are excluded since both have a subgroup $\text{PSL}(2, 7)$, and $\text{PSU}(3, 4)$ because it has a subgroup $\mathbb{Z}_{13} \ltimes \mathbb{Z}_3$. Noting that $\text{SU}(r, q)$ is a subgroup of $\text{PSU}(m, q)$, for $r < m$, by Lemma 4 iii) this excludes all unitary groups $\text{PSU}(m, q)$ except $\text{PSU}(2, 5) \cong \text{PSL}(2, 5)$.

Concerning the symplectic groups $\text{PSp}(4, q)$, $q$ odd, we note that $\text{PSp}(4, 3) \cong \text{PSU}(4, 2)$ has a subgroup $(\mathbb{Z}_3)^3$ and $\text{PSp}(4, 5)$ a subgroup $(\mathbb{Z}_5)^3$. Choosing an element $g$ in this subgroup $(\mathbb{Z}_3)^3$ resp. $(\mathbb{Z}_5)^3$ and applying Lemma 1, it is easy to see that either such a subgroup must have a global fixed point contradicting Theorem 1 ii), or there is a subgroup $(\mathbb{Z}_3)^2$ resp. $(\mathbb{Z}_5)^2$ acting faithfully on a 2-sphere which gives again a contradiction; hence these groups do not act on $M$. Since $\text{Sp}(2, q)$ is a subgroup of $\text{PSp}(4, q)$ this excludes also the symplectic groups $\text{PSp}(4, q)$.

Considering the remaining groups in the Gorenstein-Harada list, up to central extension there are inclusions $3D_4(q) \supset G_2(q) \supset \text{PSL}(3, q)$ which excludes these groups (see [St, Table 0A8], [GL, Table 4-1]). The Ree groups $2G_2(3^{2m+1})$ have one conjugacy class of involutions, the centralizer of an involution is $\mathbb{Z}_2^2 \times \text{PSL}(2, 3^{2m+1})$ ([G2,p.164]) so for $m \geq 1$ they do not act (the group $2G_2(3)$ is not simple).

The Sylow 2-subgroup $S_2$ of the Suzuki group $\text{Sz}(8)$ has order 64 and a normal subgroup $(\mathbb{Z}_2)^3$, and all involutions are conjugate. Let $g$ be a central involution in $S_2$. If $g$ has four isolated fixed points, or fixes two isolated points and a 2-sphere, then a subgroup of order at least 16 fixes one of these isolated fixed points and hence is abelian by Theorem 1. Since $\text{Sz}(8)$ has no elements of order eight, a subgroup $(\mathbb{Z}_2)^2$ fixes a point and hence admits a free action on $S^3$ which is a contradiction. If the fixed point set of $g$ consists of two 2-spheres then a subgroup of order at least 16 of $S_2$ acts faithfully and orientation-preservingly on a 2-sphere; since there are no elements of order eight this gives again a contradiction, so $\text{Sz}(8)$ does not occur.

Finally, $A_7$ has a subgroup $\text{PSL}(2, 7)$, and also the Mathieu groups $M_i$, the Janko groups $J_i$, the McLaughlin group $\text{McL}$ and the Lyons group $\text{Ly}$ have metacyclic or linear fractional subgroups excluded by Lemma 3 or Lemma 4 (see [C]).

Hence we have excluded all finite simple groups from the Gorenstein-Harada list except the alternating group $A_5$, and for the proof of Theorem 4 it remains to show that the 4-manifold $M$ has intersection form of $S^2 \times S^2$.

Now $G = A_5$ has a subgroup $A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_3$, and we consider the normal subgroup $(\mathbb{Z}_2)^3$ of $A_4$. By [Mc1, Prop.14] either $M$ has the right intersection form or $(\mathbb{Z}_2)^3$ has a global fixed point, so we can assume the latter. Let $g$ be an involution in $(\mathbb{Z}_2)^3$. According to Lemma 1 we consider three cases.
If \( g \) has four isolated fixed points then \((\mathbb{Z}_2)^2\) has two or four global fixed points. But then also \( A_4 \) has a global fixed point which contradicts Theorem 1.

If \( g \) fixes pointwise two 2-spheres then each of these 2-spheres is invariant under the action of \((\mathbb{Z}_2)^2\); moreover by [Mc1, Theorem 2], since \((\mathbb{Z}_2)^2\) acts homologically trivial it acts orientation-preservingly on each of these 2-spheres. Then \((\mathbb{Z}_2)^2\) has again exactly four global fixed points and we get a contradiction as in the first case.

Finally, if the fixed point set of \( g \) consists of a 2-sphere and two isolated points then \((\mathbb{Z}_2)^2\) has two or four global fixed points, so \( A_4 \) has a global fixed point contradicting Theorem 1.

3. Proof of Theorem 5

Recall that a finite \( Q \) group is quasisimple if it is perfect (the abelianized group is trivial) and the factor group of \( Q \) by its center is a nonabelian simple group. A finite group \( E \) is semisimple if it is perfect and the factor group of \( E \) by its center is a direct product of nonabelian simple groups (see [Su2, chapter 6.6]). A semisimple group is a central product of quasisimple groups that are uniquely determined. Any finite group \( G \) contains a unique maximal semisimple normal group \( E(G) \) (the subgroup \( E(G) \) may be trivial); the subgroup \( E(G) \) is characteristic in \( G \) and the quasisimple factors of \( E(G) \) are called the components of \( G \). To prove the Theorem 5 we consider first the case of trivial maximal normal semisimple subgroup and prove that in this case the groups are solvable.

**Lemma 5.** Let \( G \) be a finite group with trivial maximal normal semisimple subgroup \( E(G) \). If \( G \) admits a homologically trivial action on a closed 4-manifold \( M \) with \( b_2(M) = 2 \) and trivial first homology, then \( G \) is solvable.

**Proof.** We consider first the case of \( G \) containing a normal nontrivial cyclic subgroup \( H \) and we prove that in this case if \( E(G) \) is trivial then \( G \) is solvable. We can suppose that \( H \) has prime order \( p \) so each nontrivial element of \( H \) has the same fixed point set; since \( G \) normalizes \( H \) then \( G \) fixes setwise the fixed point set of \( H \).

If the fixed point set of \( H \) consists of two isolated points and a 2-sphere there exists a subgroup \( G_0 \) of index at most two in \( G \) such that \( G_0 \) fixes both points; the subgroup \( G_0 \) is abelian by Theorem 1 and consequently \( G \) is solvable.

If the fixed point set of \( H \) consists of four isolated points there exists a normal subgroup \( G_0 \) of \( G \) that fixes each point; \( G_0 \) is abelian by Theorem 1. The quotient group \( G/G_0 \) is isomorphic to a subgroup of \( S_4 \), the symmetry group over four elements that is a solvable group and we can conclude that \( G \) is solvable. Finally suppose that the fixed point set of \( H \) consists of two 2-spheres; there exists a subgroup \( G_0 \) of index at most two in \( G \) such that \( G_0 \) leaves invariant both 2-spheres. We consider in \( G_0 \) the normal subgroup \( K \) of elements fixing pointwise \( S^2_+ \), one of the two 2-spheres; the subgroup \( K \) contains
and since $K$ acts locally by rotations around $S^2_+$ then $K$ is cyclic. The factor group $G_0/K$ acts faithfully on $S^2_+$. If $G_0/K$ is solvable, we get the thesis; otherwise we can suppose that $G_0/K$ is isomorphic to $A_5$ because it is the only nonsolvable finite group acting orientation-preservingly on the 2-sphere (the action is orientation-preserving by a result of Edmonds [E], see also [Mc1, Theorem 2]). The action of $A_5$ by conjugation on $K$ is trivial because $K$ is cyclic and its automorphism group is abelian; then $G_0$ is a central extension of $A_5$. The derived group $G'_0$ is a quasisimple normal subgroup of $G_0$ (see [Su1, Theorem 9.18, pag.257]); this fact implies that $E(G_0)$, and consequently $E(G)$, are not trivial in contradiction with our hypothesis. The proof of this particular case is now complete and in the following we can use this fact.

**Fact:** if a subgroup $N$ of $G$ contains a nontrivial cyclic normal subgroup then either $N$ is solvable or $E(N)$ is not trivial.

We consider now the general case. We denote by $F$ the Fitting subgroup of $G$ (the maximal nilpotent normal subgroup of $G$). Since $E(G)$ is trivial, the Fitting subgroup $F$ coincides with the generalized Fitting subgroup that is the product of the Fitting subgroup with the maximal semisimple normal subgroup. The generalized Fitting subgroup $F$ contains its centralizer in $G$ and $F$ is not trivial ([Su2, Theorem 6.11, pag.452]).

Since $F$ is nilpotent it is the direct product of its Sylow $p$-subgroups. In particular any Sylow subgroup of $F$ is normal in $G$; since $F$ is not trivial we have a nontrivial $p$-subgroup $P$ which is normal in $G$. We consider the maximal elementary abelian $p$-subgroup $Z$ contained in the center of $P$; this subgroup is not trivial and it is normal in $G$.

Suppose first that we can chose $p$ odd (the order of $F$ is not a power of two). If $Z$ contains an element with fixed point set consisting of four points or of two points and one 2-sphere, the group $Z$ has global fixed point set and has rank at most two by Theorem 1. If $Z$ contains an element with fixed point set consisting of two 2-spheres each element of $Z$ leaves invariant both 2-spheres; a quotient group of $Z$ by a cyclic group acts faithfully on the 2-spheres. This quotient group must be cyclic and it acts on the 2-spheres by rotations then the group $Z$ has also in this case global fixed point set and it has rank at most two. If $Z$ is cyclic, by the first part of the proof, $G$ is solvable. If $Z$ has rank two, since it has global fixed point set it is described by [Mc1, Prop.14]. The fixed point set of $Z$ consists of four points. The whole group $G$ leaves invariant the fixed point set of $Z$ and there exists a normal subgroup $G_0$ that fixes each point. The quotient group $G/G_0$ is isomorphic to a subgroup of $S_4$ that is solvable. By Theorem 1 $G_0$ is abelian and consequently $G$ is solvable.

Suppose now that the order of $F$ is a power of two; in this case $F = P$ is a 2-group and $Z$ is an elementary abelian 2-group of rank at most four (by Lemma 2). If $Z$ has rank one by the first part of the proof $G$ is solvable. If $Z$ has rank two we consider $C_G(Z)$ the centralizer of $Z$ in $G$ that is normal because $Z$ is normal; $C_G(Z)$ contains a
nontrivial normal cyclic subgroup and it is solvable. The factor $G/C_G(Z)$ is isomorphic to a subgroup of $GL(2, \mathbb{Z}_2)$, the automorphism group of an elementary abelian 2-group of rank two; since $GL(2, \mathbb{Z}_2)$ is a solvable group we can conclude that $G$ is solvable. Suppose that $Z$ has rank three. In this case the factor group $G/C_G(Z)$ is isomorphic to a subgroup of $GL(3, \mathbb{Z}_2)$, the automorphism group of an elementary abelian 2-group of rank three; $GL(3, \mathbb{Z}_2)$ has order $2^3 \cdot 3 \cdot 7$ and any element of order seven permutes cyclically all the involutions of $(\mathbb{Z}_2)^3$. The group $G/C_G(Z)$ can not contain any element of order 7 otherwise all involutions in $Z$ are conjugated and this can be excluded by the same argument used to exclude case PSL(2, 8) in the proof of Lemma 4; so the group $G/C_G(Z)$ has order $2^\alpha 3^\beta$ and it is solvable. This fact implies that $G$ is solvable.

It remains the case $Z$ of rank four; the factor group $G/C_G(Z)$ is isomorphic to a subgroup of $GL(3, \mathbb{Z}_2)$, the automorphism group of an elementary abelian 2-group of rank four.

We analyze the fixed point set of the elements in $Z$. The group $Z$ cannot contain any element with fixed point set consisting of two points and one 2-sphere. In this case a subgroup of index at most two of $Z$ has global fixed point set and this impossible by Theorem 1.

Suppose first that $h$ is an involution such that its fixed point set $Fix(h)$ consists of two 2-spheres. If an element of $Z$ leaves invariant both components of $Fix(h)$, it acts on both 2-spheres orientation preservingly (by [Mc1, Theorem 2]). Since there exists only one involution acting trivially on $Fix(h)$ and the maximal elementary 2-group acting faithfully on a 2-sphere has rank two, the group $Z$ contains with index two a subgroup $Z_0$ that leaves invariant both 2-spheres in $Fix(h)$. Any involution of $Z_0$ different from $h$ acts nontrivially and orientation preservingly (again by [Mc1, Theorem 2]) on the 2-spheres, thus it fixes pointwise on each 2-sphere two points, this implies that the subgroup of rank two generated by $h$ and by the other involution has global fixed point set; this 2-rank subgroup is described by [Mc1, Prop.14] and it contains two involutions different from $h$, one with 0-dimensional fixed point set and one with 2-dimensional fixed point set. We obtain that if the fixed point set of $h$ consists of two 2-spheres there exist exactly three involutions with 0-dimensional fixed point set and with fixed point set contained in $Fix(h)$. We consider now $h'$ an involution such that its fixed point set $Fix(h')$ consists of four isolated points. We consider $Z_1$ the subgroup of $Z$ that fixes pointwise $Fix(h')$; since a maximal elementary 2-group in the symmetric group $S_4$ has rank two, the subgroup $Z_1$ has index at most four but by Theorem 1 the subgroup $Z_1$ has rank at most two; we can conclude that the rank of $Z_1$ is exactly two. In this case $Z_1$ is completely described by [Mc1, Prop. 14] and there exist exactly two involutions in $Z$ with 2-dimensional fixed point set that contain $Fix(h')$. If $n$ is the number of involutions in $Z$ with 0-dimensional fixed point set and $m$ is the number of involutions with 2-dimensional fixed point set we have that $n + m = 15$ and for the previous computations $3m = 2n$; we obtain that $n = 9$ and $m = 6$. We recall that the factor group $G/C_G(Z)$ is isomorphic to a subgroup of $GL(3, \mathbb{Z}_2)$; the group $GL(3, \mathbb{Z}_2)$
has order $2^6 \cdot 3^2 \cdot 5 \cdot 7$; an automorphism of order five does not centralize any involution of $(\mathbb{Z}_2)^3$ (we have three orbits with five elements) and an automorphism of order seven centralizes exactly one involution (two orbits with seven elements and one orbit with only one element). Since we have nine elements with 0-dimensional fixed point set and six elements with 2-dimensional fixed point set, the group $G/C_G(Z)$ can not contain elements of order five and seven and it has order $2^3 \cdot 3^2$; we have that $G/C_G(Z)$ is solvable and consequently $G$ is solvable. This fact concludes the proof.

The following lemma considers the case of semisimple groups.

**Lemma 6.** Let $G$ be a finite semisimple group that admits a homologically trivial action on a closed 4-manifold $M$ with $b_2(M) = 2$ and trivial first homology; then $G$ is isomorphic to one of the groups $A_5$, $A_5^*$ or $A_5 \times A_5$.

**Proof.** By Lemma 4(iii), if $G$ is quasisimple, then $G$ is isomorphic either to $A_5$ or to $A_5^* \cong \text{SL}(2,5)$ that is the unique perfect central extension of $A_5$.

We consider now the case of $G$ with two quasisimple components; since in our list of quasisimple groups $A_5^*$ is the unique group with nontrivial center, then either $G \cong A_5^* \times \mathbb{Z}_2$ or $G$ is the direct product of two quasisimple subgroups.

We prove that no involution can be contained in the center of $G$ and the only possibility with two components remains $A_5 \times A_5$. Suppose that $h$ is an involution contained in the center of $G$. If the fixed point set $\text{Fix}(h)$ of $h$ consists of two points and one 2-sphere then $G$ has a subgroup of index at most two that fixes both points and by Theorem 1 this group should be abelian of rank two, that is impossible.

If the fixed point set of $h$ consists of four isolated points there exists a normal subgroup $G_0$ of $G$ that fixes each point of $\text{Fix}(h)$; by Theorem 1 $G_0$ is abelian of rank two. The quotient $G/G_0$ acts faithfully on the four points of $\text{Fix}(h)$ and it is isomorphic to a subgroup of $S_4$. The group $G$ contains a normal subgroup isomorphic either to $A_5^*$ or to $A_5$; no quotient of these groups by an abelian normal subgroup is isomorphic to a subgroup of $S_4$. Finally we consider when $\text{Fix}(h)$ consists of two 2-spheres, in this case $G$ leaves invariant both 2-spheres because it does not contain any subgroup of index two ($G$ is perfect). The subgroup acting trivially on the 2-spheres is cyclic and normal in $G$; the quotient of $G$ by it acts faithfully on the 2-spheres and it is again the product of two quasisimple groups. This fact can not occur. By the previous part if $G$ has three or more components the quasisimple factors are all isomorphic to $A_5$ but these groups cannot occur because the sectional 2-rank of $G$ is smaller then four.

**Proof of Theorem 5.**

If the maximal semisimple normal subgroup $E(G)$ is trivial then $G$ is solvable by Lemma 5. We can suppose that $E(G)$ is not trivial and it is isomorphic by Lemma 6 to $A_5$, $A_5^*$
of $A_5 \times A_5$. We denote by $C$ the centralizer of $E(G)$ in $G$; since $E(G)$ is normal its centralizer consist of two 2-spheres. Let $h$ be a nontrivial element in $C$. If the fixed point set of $h$ consists of two points and one 2-sphere then $G$ has a subgroup of index at most two that fixes both points and by Theorem 1 this group should be abelian of rank two, that is impossible. If the fixed point set of $h$ consists of four isolated points then there exists a normal subgroup $G_0$ of $G$ that fixes each point of $\text{Fix}(h)$; by Theorem 1 the group $G_0$ is abelian of rank two. The quotient $G/G_0$ acts faithfully on the four points of $\text{Fix}(h)$ and it is isomorphic to a subgroup of $S_4$. The group $G$ contains a normal subgroup isomorphic either to $A_5^+$ or to $A_5$; no quotient of these groups by an abelian normal subgroup is isomorphic to a subgroup of $S_4$.

Case 1: $E(G)$ is isomorphic to $A_5 \times A_5$

If $E(G)$ is isomorphic to $A_5 \times A_5$ the subgroup $C$ is trivial. In fact if $C$ contains a non-trivial element $f$ all the group $A_5 \times A_5$ fixes setwise the fixed point set of $f$. The group $E(G)$ does not contain any subgroup of index two (it is perfect), so $A_5 \times A_5$ fixes setwise each 2-sphere; the group $E(G)$ does not contain any nontrivial normal cyclic subgroup so it acts faithfully on both 2-spheres and this is impossible. This fact implies that $G$ is isomorphic to a subgroup of $\text{Aut}(E(G))$, the automorphism group of $E(G)$, that contains with index two the subgroup $\text{Aut}(A_5) \times \text{Aut}(A_5) \cong S_5 \times S_5$ and any element not in $S_5 \times S_5$ exchange the two quasisimple components of $E(G)$ (see [GLS, Theorem 3.23, p.13]). We recall that the quasisimple components of a group are permuted by any automorphism of the group (see [GLS, Theorem 3.5, p.7]). We consider $S$ the Sylow 2-subgroup of $E(G)$ that is an elementary abelian 2-group of rank two; in the proof of Lemma 5 we have proved that $S$ contains six elements with fixed point set consisting of two 2-spheres and nine elements with fixed point set consisting of four points. Let $h$ be an element with 2-dimensional fixed point set; $h$ is contained in a subgroup $A$ of rank two with global fixed point set. By [Mc1, Prop. 14] each of the four 2-spheres in the singular set of $A$ represents a primitive class in $H_2(M)$ and together these classes generate $H_2(M)$. Since the 2-spheres of the fixed point of $h$ are exchanged by some elements of $S$ and the action is homologically trivial we can conclude that the two 2-spheres of the fixed point of $h$ represent the same class. Each element with 2-dimensional fixed point set gives one primitive class in homology and the 2-spheres in the singular set of $S$ generate $H_2(M)$ (that has rank two). The conjugacy classes of involutions of $S$ in $E(G) \cong A_5 \times A_5$ are three. The six involutions contained in the quasisimple components form two conjugacy classes, one class for each component; the third class contains the remaining nine involutions. Since the action is homologically trivial, we have that the six involutions in the quasisimple components have 2-dimensional fixed point set and the two conjugacy classes in the quasisimple components represent different elements in $H_2(M)$. This fact implies immediately that $G$ cannot contain any element that by
conjugation exchanges the two quasisimple components and $G$ is isomorphic to a subgroup of $\text{Aut}(A_5) \times \text{Aut}(A_5) \cong S_5 \times S_5$. Moreover we consider $t$ an involution contained in a quasisimple component. The fixed point set of $t$, $\text{Fix}(t)$, consists of two 2-spheres. The other involutions contained in $S$ that are in the same quasisimple component of $t$ act on $\text{Fix}(t)$ exchanging the two 2-spheres otherwise the Sylow 2-subgroup of the quasisimple component has global fixed point set and it contains an involution with 0-dimensional fixed point set that is impossible. Suppose that a subgroup isomorphic to $S_5 \times A_5$ is contained in $G$. We consider $t$ an involution contained in the second factor (that is a quasisimple component of the group), its centralizer in $G$ contains a subgroup isomorphic to $S_5 \times Z_2 \times Z_2$. Since the involutions in the same quasisimple component exchange the two 2-spheres of $\text{Fix}(t)$ there exists a group isomorphic to $S_5 \times Z_2$ that leaves invariant the two 2-spheres. We have that $S_5$ acts faithfully on the two 2-spheres and this is impossible. We obtain that either $G \cong A_5 \times A_5$ or $G$ contains with index two $E(G)$ and $G$ is isomorphic to the extension of $A_5 \times A_5$ by the outer automorphism that acts nontrivially on both quasisimple components.

**Case 2: $E(G)$ is isomorphic to $A_5$**

We consider $F(G)$ the Fitting subgroup of $G$; the Fitting subgroup centralizes $E(G)$ [Su2, pag.452], then all the elements in the Fitting have two 2-spheres as fixed point set. Suppose first that $F(G)$ contains a nontrivial cyclic subgroup normal in $G$; in this case all the group fixes setwise the two 2-spheres that are the fixed point set of this group. We consider $K$ the normal subgroup consisting of the elements acting trivially on the two 2-spheres; the subgroup $K$ is cyclic. The subgroup $E(G) \cong A_5$ intersects trivially $K$ and since $E(G)$ does not contain subgroup of index two it leaves invariant both 2-spheres. We recall that by [Mc1, Theorem 2] the action of $G$ on the two 2-spheres has to be orientation preserving and $A_5$ is maximal between the finite groups acting orientation preserving on a 2-sphere. Moreover since the automorphism group of $K$ is abelian, the group $E(G)$ acts trivially by conjugation on $K$. We can conclude in this case that the subgroup of $G$ of index at most two that leaves invariant both the 2-spheres in the fixed point of $K$ is exactly $E(G) \times K$. We suppose now that the order of the Fitting subgroup is not a power of two; the Fitting subgroup contains a characteristic $p$-subgroup with $p$ odd. The center of this subgroup is normal in $G$ and it has to be cyclic, otherwise the centralizer of any element in the center contains a subgroup isomorphic to $A_5 \times Z_p \times Z_p$ that is impossible because $A_5$ is maximal between the finite groups acting orientation preservingly on a 2-sphere. We are in the case considered previously and we get the thesis. We can suppose finally that the order of $F(G)$ is a power of two. We consider $Z$ the center of $F(G)$ that is a normal subgroup of $G$. Since the sectional 2-rank of the group $G$ is at most four by Lemma 2 and $E(G)$ contains an elementary 2-subgroup of rank two, we obtain that $Z$ has rank at most
two. If it is cyclic we have the thesis, so we can suppose that $Z$ has rank two. We consider the fixed point set of an involution in $Z$ that consists of two 2-spheres; by the same argument used previously we can obtain that the subgroup of index at most two of $E(G) \times F(G)$ that leaves invariant both the 2-spheres is isomorphic to $A_5 \times Z_{2m}$. Since $Z$ has rank two we obtain that $F(G) \cong \mathbb{Z}_{2m} \times \mathbb{Z}_2$. If $m > 1$ there exists a central involution in $G$ (in $F(G)$ the exist a unique involution that is not primitive) and we get the thesis. It remains the case when $F(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case $F(G)$ is the center of the generalized Fitting subgroup $F^*(G)$ [Su2, Theorem 6.10., pag.452], $G/F(G)$ is isomorphic to a subgroup of $\text{Aut}(F^*(G)) \cong S_5 \times \text{GL}(2, \mathbb{Z}_2) \cong S_5 \times S_3$. We can suppose in $G/F(G)$ the existence of an element of order three permuting the three involutions in $F(G)$, otherwise one involution in $F(G)$ is centralized by the whole group and we can conclude that $G$ contains a subgroup of index two isomorphic to $A_5 \times Z_m$. We denote by $\tilde{f} \in G/F(G)$ such an element that we can suppose of order three. We denote by $f \in G$ a preimage of $\tilde{f}$ with respect to the projection of $G$ onto $G/F(G)$; we can chose $f$ of order three. Since the automorphisms of $A_5$ of order three are inner, we can compose $f$ with the appropriate element of $A_5$ and we can suppose that the action of $f$ on $E(G) \cong A_5$ is trivial. Now we want to prove that the whole group $\text{Aut}(F^*(G))$ cannot occur as the quotient $G/F(G)$; in particular we exclude the presence in $G/F(G)$ of an involution acting on $A_5$ as a non-inner automorphism, acting trivially on $F(G)$ and acting trivially by conjugation on $\text{Aut}(F(G))$. Let $\tilde{t}$ be an involution of this type; we denote by $t \in G$ a preimage of $\tilde{t}$ with respect to the projection of $G$ onto $G/F(G)$. The element $t$ is an involution otherwise its square is an involution in $F(G)$ and this is in contradiction with the existence of $f$ (we should have that $ft^2f^{-1}$ is an involution different from $t^2$ and on the other hand $ftf^{-1} = t$). The element $t$ is an involution that acts trivially by conjugation on $F(G)$, thus we have a subgroup of $G$ isomorphic to $S_5 \times F(G) \cong S_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where the fixed point set of each involution in $F(G)$ consists of two 2-spheres. In the Case 1 we proved that this group cannot occur. We have that $E(G) \times F(G)$ is contained in $G$ with index at most six. The split extension of $E(G) \times F(G)$ by $f$ is isomorphic to $A_5 \times A_4$ and it is contained in $G$ with index at most two.

Case 3: $E(G)$ is isomorphic to $A_5^2$

We denote by $h$ the involution in the center of $E(G)$ and by $\text{Fix}(h)$ the fixed point set of $h$ that consists consists of two 2-spheres. The center of $E(G)$ is normal in $G$ and $G$ fixes setwise $\text{Fix}(h)$. We consider $G_0$ the subgroup of $G$ of the elements that fixes setwise each 2-sphere in $\text{Fix}(h)$. The subgroup $G_0$ has index at most two in $G$ and it contains $E(G)$. In $G_0$ we consider $K$ the cyclic and normal subgroup of the elements that act trivially on one of the two 2-spheres; the quotient $G_0/K$ acts faithfully on the 2-sphere and it contains a subgroup isomorphic to $A_5$ (the quotient of $E(G)$ by its center). We recall that by [Mc1, Theorem 2] the action of $G_0$ on the two 2-spheres has to be orientation preserving and $A_5$ is maximal between the finite groups acting orientation preserving
on a 2-sphere. We obtain that $G_0/K \cong A_5$. The group $G_0$ is generated by $E(G)$ and $K$. The action by conjugation of $E(G)$ on $K$ has to be trivial because the automorphism group of $K$ is abelian. This implies that $G_0$ is the central product of $F(G)$ and $K$.

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