COHOMOLOGY OF SEMISIMPLE LOCAL SYSTEMS AND THE DECOMPOSITION THEOREM

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Abstract. In this paper, we study the cohomology of semisimple local systems in the spirit of classical Hodge theory. On the one hand, we construct a generalized Weil operator from the complex conjugate of the cohomology of a semisimple local system to the cohomology of its dual local system, which is functorial with respect to smooth restrictions. This operator allows us to study the Poincaré pairing, usually not positive definite, in terms of a positive definite Hermitian pairing. On the other hand, we prove a global invariant cycle theorem for semisimple local systems.

As an application, we give a new proof of Sabbah’s Decomposition Theorem for the direct images of semisimple local systems under proper algebraic maps, by adapting the method of de Cataldo-Migliorini, without using the category of polarizable twistor D-modules. This answers a question of Sabbah.

Introduction

The classical Hodge theory provides two fundamental tools for studying the cohomology of complex algebraic varieties: polarizations and weights. The polarizations allow one to study the Poincaré pairing using a positive definitive pairing; the theory of weights puts strong restrictions on morphisms between cohomology groups of algebraic varieties. On the other hand, the study of nontrivial local systems leads to deeper understanding of topology of algebraic varieties. For example it has applications to Kähler groups [31], higher dimensional uniformization [30] and Shafarevich conjecture [17]. Therefore it is natural to seek generalization of results on polarizations and weights to nontrivial local systems. The purpose of this paper is to study the cohomology of semisimple local systems in this spirit and provide some new tools. In particular, we construct a (functorial) generalized Weil operator, which induces polarizations on the pure twistor structures associated to semisimple local systems; we also prove a global invariant cycle theorem. As an application, we give a new proof of Sabbah’s Decomposition Theorem [27], which might offer some new viewpoint on polarizations.

Our guiding principle is Simpson’s “meta-theorem” [33], which leads to the solution of Kashiwara’s conjecture [26]. But even for the cohomology of semisimple local systems, which is the simplest case, we discover something new: there are some hidden structures related to the polarization, e.g. a pre-Weil operator, which seems invisible in Hodge theory.

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Turning to details, let $X$ be a complex smooth projective variety and let $V$ be a local system on $X$. Denote the dual local system by $V^*$. Let $\eta$ be an ample line bundle on $X$ and fix an integer $k \leq \dim X$. The twisted Poincaré pairing is defined by

$$S : H^k(X, V) \otimes H^k(X, V^*) \to \mathbb{C},$$

$$[\alpha] \otimes [\beta] \mapsto (-1)^{k(k-1)/2} \cdot \int_X c_1(\eta)^{\dim X - k} \wedge \alpha \wedge \beta,$$

where $\alpha$ is a $k$-form on $X$ with coefficient in $V$, same for $\beta$. Among arbitrary local systems, semisimple local systems are special as they are more manageable by analysis. From now on, we assume $V$ is semisimple. Corlette [5] showed that the $C^\infty$-bundle $H = V \otimes_\mathbb{C} C_X^k$ admits a unique harmonic metric up to a linear transformation. Unlike the constant local system $\mathbb{C}$, the cohomology groups of $V$ do not necessarily underlie pure Hodge structures. However, using the harmonic metric, Simpson [33] proved that there is a natural semistable holomorphic bundle $E$ of slope $k$ over $P^1$ whose fiber at 1 satisfies

$$E|_{z=1} \cong H^k(X, V).$$

This can be viewed as a generalized pure Hodge structure of weight $k$ on $H^k(X, V)$. Indeed, Simpson defined a pure twistor structure of weight $k$ to be a semistable holomorphic bundle of slope $k$ over $P^1$. Our first result is the construction of a pre-Weil operator

$$\phi : E|_{z=-1} \xrightarrow{\sim} H^k(X, V^*),$$

which is an isomorphism and is functorial with respect to restriction to smooth subvarieties. Precomposing an identification map $E|_{z=1} \to E|_{z=-1}$ associated to the pure twistor structure $E$, this gives an isomorphism

$$C : \overline{H^k(X, V)} \xrightarrow{\sim} H^k(X, V^*).$$

In particular, it recovers the Weil operator on $H^k(X, \mathbb{C})$. In the appendix, we reinterpret $C$ using Hodge star operators. Using Simpson’s Lefschetz decomposition [31, Lemma 2.6] on $H^k(X, V)$, we can use $C$ to prove the following generalization of Hodge-Riemann bilinear relation.

**Theorem A** (Theorem 1.4.10). The bilinear pairing

$$S(\bullet, C(\bullet)) : H^k(X, V) \otimes \overline{H^k(X, V)} \to \mathbb{C}$$

is Hermitian, and positive definite on each primitive space $\eta^m H^{k-2m}(X, V)_{prim}$, up to a non-zero constant, where $H^k(X, V)_{prim}$ denotes the primitive space with respect to $\eta$. In particular, the bilinear pairing

$$S(\bullet, \phi(\bullet)) : E|_{z=1} \otimes \overline{E|_{z=-1}} \to \mathbb{C}$$

polarizes the pure twistor structure on $H^k(X, V)_{prim}$, up to a non-zero constant.

This theorem means that the pre-Weil operator $\phi$ should be an intrinsic structure associated to the polarization of the pure twistor structure on $H^k(X, V)_{prim}$. As in Hodge theory, the existence of polarizations enables one to keep track of non-degeneracy of topological pairings under restrictions. To illustrate the idea, let us consider a smooth subvariety $T \subseteq X$ and the restriction maps

$$R : H^k(X, V)_{prim} \to H^k(T, V|_T), \quad \bar{R} : H^k(X, V^*)_{prim} \to H^k(T, V^*|_T).$$
Then the twisted Poincaré pairing $S$ restricts to a non-degenerate pairing:

$$S : \text{Ker } R \otimes \text{Ker } \tilde{R} \to \mathbb{C}.$$ 

Even though this is a purely topological statement, it seems difficult to find a topological proof.

Turning to weights, let $X$ be a complex quasi-projective variety and let $V$ be a local system on $X$ coming from the restriction of a semisimple local system on a smooth projective compactification $\overline{X} \supseteq X$. In [33], Simpson showed that $H^k(X, V)$ underlies a natural mixed twistor structure, which is functorial in $X$ and generalizes Deligne's mixed Hodge structures. This means that there is a holomorphic vector bundle $E$ over $\mathbb{P}^1$ with an increasing filtration $W^\bullet E$ by strict subbundles such that $E|_{z=1} \cong H^k(X, V)$ and $W^\ell E / W^{\ell-1} E$ is a pure twistor structure of weight $\ell$. In [33], after stating this result, Simpson remarked that the same yoga of weights also holds, which is stated as follows.

**Simpson’s theory of weights** (Theorem 1.3.3). Let $W^\bullet$ denote the weight filtration on $H^k(X, V)$ induced by the natural mixed twistor structure. We have

- if $X$ is proper, then $\text{Gr}_i^W H^k(X, V) = 0$ for $i > k$,
- if $X$ is smooth, then $\text{Gr}_i^W H^k(X, V) = 0$ for $i < k$,
- if $X$ is smooth and proper, then $\text{Gr}_i^W H^k(X, V) = 0$ for $i \neq k$.

Using this, we prove a version of global invariant cycle theorem for semisimple local systems, generalizing Deligne’s classical result [13].

**Theorem B.** Consider the following chain of inclusion maps:

$$Z \xrightarrow{\alpha} U \xrightarrow{j} X,$$

where $X$ is a smooth projective variety, $U$ is a Zariski open subset of $X$, and $Z$ is a proper subvariety of $X$ contained in $U$. Let $V$ be a semisimple local system on $X$. Then, for any integer $k$, the following two restriction maps have the same image:

$$(j \circ \alpha)^* : H^k(X, V) \to H^k(Z, (j \circ \alpha)^* V),$$

$$\alpha^* : H^k(U, j^* V) \to H^k(Z, (j \circ \alpha)^* V).$$

Using these new results on the cohomology of semisimple local systems, we give a relatively short and geometric proof of Sabbah’s Decomposition Theorem for semisimple local systems under proper algebraic maps, by adapting the method of de Cataldo and Migliorini in [8]. Sabbah’s proof relies on the complicated category of polarizable twistor $\mathcal{D}$-modules, we only need work in the category of constructible complexes. This answers a question of Sabbah [27, Page 8]. We would like to point out that the adaption of the de Cataldo-Migliorini method is nontrivial, which will be explained at the end of introduction. The main point is the compatibility of the pre-Weil operator $\phi$ with the perverse cohomology functor and smooth restrictions: on the one hand it allows us to keep track of non-degeneracy of various Poincaré pairings after restrictions to hyperplanes (as illustrated after Theorem A); on the other hand it helps us translate the positivity coming from the polarization to the non-degeneracy of certain adjunction morphisms (see Proposition 2.8.3), which is needed for the splitting criterion.

Now let us state the result. Let $f : X \to Y$ be a morphism between projective varieties, where $X$ is smooth. Let $V$ be a semisimple local system on $X$ and $\eta$ be an $f$-ample line.
bundle on $X$. Denote by $K := \mathcal{V}[\dim X]$ the associated perverse sheaf. Let $\mathcal{P}H$ denote the perverse cohomology functor and $f_*$ denote the total direct image functor.

**Theorem C.** We have

(i) **(Relative Hard Lefschetz Theorem)** the following cup product map is an isomorphism:

$$\eta^0 : \mathcal{P}H^{-\ell}(f_*, K) \xrightarrow{\sim} \mathcal{P}H^\ell(\eta^0, K).$$

(ii) **(Decomposition Theorem)** There exists an isomorphism in $\mathcal{D}^b(Y)$:

$$f_*K \cong \bigoplus_{\ell} \mathcal{P}H^\ell(\eta^0, K)[-\ell],$$

where $\mathcal{D}^b(Y)$ denotes the derived category of constructible sheaves on $Y$.

(iii) **(Semisimplicity Theorem)** For each $\ell$, $\mathcal{P}H^\ell(\eta^0, K)$ is a semisimple perverse sheaf. More precisely, given any stratification for $f$ so that $Y = \coprod_{d=\dim S_d} S_d$, there is a canonical isomorphism in $\text{Perv}(Y)$:

$$\mathcal{P}H^\ell(\eta^0, K) \cong \bigoplus_{d=0}^{\dim Y} \text{IC}_{S_d}^\mathcal{P}(L_{\ell,d}),$$

where the local systems $L_{\ell,d} := \mathcal{H}^{-\ell}(\mathcal{P}H^\ell(\eta^0, K))|_{S_d}$ on $S_d$ are semisimple. Here $\text{Perv}(Y)$ denotes the category of perverse sheaves on $Y$ and $\text{IC}_{S_d}^\mathcal{P}(L_{\ell,d})$ denotes the intersection complex associated to the local system $L_{\ell,d}$.

Sabbah’s original theorem holds for a proper map from $U \to Y$, where $U$ is a Zariski open subset of $X$ and $Y$ is a complex manifold. We will discuss in Theorem 2.10.1 how to obtain the Decomposition Theorem for a proper map $U \to Y$, where $Y$ is an algebraic variety. It is not clear to us whether or not our method can recover [27, Main Theorem 2], which is about semisimplicity of nearby and vanishing cycles.

**Theorem D.** Let $\ell, j \in \mathbb{Z}$. Then the following cup product maps are isomorphisms:

$$\eta^\ell : H^j_{-\ell}(X, K) \xrightarrow{\sim} H^j_{\ell}(X, K), \quad L^j : H^{\ell-j}_\ell(X, K) \xrightarrow{\sim} H^{\ell-j}_\ell(X, K).$$

As a corollary, there is a double Lefschetz decomposition

$$H_{-\ell}^{-j}(X, K) = \bigoplus_{i, m \in \mathbb{Z}} \eta^{\ell-i}L^{-j+m}P_{\ell-2i}^j,$$

where $P_{-\ell}^j := \text{Ker} \eta^{\ell+1} \cap \text{Ker} L^j \subseteq H_{-\ell}^{-j}(X, K)$ for $\ell, j \geq 0$ and $P_{-\ell}^j = 0$ if $\ell < 0$ or $j < 0$. 

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Set $K^* = \mathcal{V}[\dim X]$ and we denote by $\tilde{P}^{-j}_{-\ell} \subseteq H^{-\ell-j}_{-\ell}(X, K^*)$ the corresponding primitive pieces. For $\ell, j \geq 0$, there is a similar bilinear pairing as in [8, Theorem 2.1.8],

\begin{equation}
S_{\ell j}^{nL} : H^{-\ell-j}_{-\ell}(X, K) \otimes \mathbb{C} H^{-\ell-j}_{-\ell}(X, K^*) \to \mathbb{C},
\end{equation}

\[ [\alpha \otimes e] \otimes [\beta \otimes \lambda] \mapsto C(n - \ell - j) \cdot \int_X \lambda(e) \cdot c_1(\eta)^\ell \wedge c_1(L)^j \wedge \alpha \wedge \beta. \]

Here $C(k)$ denotes the constant $(-1)^{k(k-1)/2}$. In the next theorem, we prove that the pairing $S_{\ell j}^{nL}$ and the pre-Weil operator $\phi$ can be used to polarize the pure twistor structures on primitive pieces.

**Theorem E.** The pairing $S_{\ell j}^{nL}$ is well-defined and non-degenerate. The double Lefschetz decompositions of $H^{-\ell-j}_{-\ell}(X, K)$ and $H^{-\ell-j}_{-\ell}(X, K^*)$ are orthogonal with respect to $S_{\ell j}^{nL}$. Furthermore, each direct summand $\eta^{-\ell+i} L^{-j+m} \tilde{P}^{-2m}_{-2i}$ underlies a natural pure twistor structure $F$ induced by the pure twistor structure on $H^{n-\ell-j}(X, \mathcal{V})$. The pre-Weil operator $\phi$ induces an isomorphism

\[ \phi : \overline{F|_{z=-1}} \sim \eta^{-\ell+i} L^{-j+m} \tilde{P}^{-2m}_{-2i}, \]

so that $F$ is polarized by the bilinear pairing $S_{\ell j}^{nL} (\bullet, \phi(\bullet))$, up to multiplication by certain power of $\sqrt{-1}$ (see Remark 1.6.8).

To conclude the introduction, let us give a very brief historical discussion of the Decomposition Theorems (see [27] for a more detailed account). The Decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber [1], says that Theorem C holds for the constant local system $\mathbb{C}$ and any proper map between algebraic varieties. Inspired by the Theorem of BBDG and M. Saito’s Decomposition Theorem for Hodge modules [28], Kashiwara [21] conjectured that the Decomposition Theorem should hold for arbitrary semisimple holonomic $\mathcal{D}$-modules.

Theorem C, originally due to Sabbah [27], establishes Kashiwara’s conjecture for semisimple local systems. Sabbah’s proof drew on his deep theory of polarizable twistor $\mathcal{D}$-modules. The main idea is that semisimple local systems on smooth projective varieties underlie polarizable twistor $\mathcal{D}$-modules, where Sabbah proved a Decomposition Theorem in this category by extending Saito’s method in [28]. This allows him to translate the results back to get the Decomposition Theorem for semisimple local systems. Building on Sabbah’s work, T. Mochizuki established the full Kashiwara’s conjecture using his seminal work of harmonic bundles on quasi-projective varieties [21, 25, 26]. Meanwhile, Kashiwara’s conjecture for semisimple perverse sheaves was also proved by Drinfeld [14], Gaitsgory [18], Böckle-Khare [3] using arithmetic methods.

There are other approaches to Decomposition Theorems. Recently, Bhatt-Lurie [2] construct a $p$-adic Riemann-Hilbert functor from certain constructible sheaves to filtered $\mathcal{D}$-modules, and then deduce the Decomposition Theorem for filtered $\mathcal{D}$-modules of geometric origin from BBDG. Budur-Wang [4] used absolute sets to deduce the Decomposition Theorem for rank one perverse sheaves from the Theorem of BBDG. El Zein-Lê-Ye [16] gave another proof for the case of intermediate extensions of polarizable variations of Hodge structures using a local purity theorem.

Finally, let us highlight a couple of differences from the ideas of Sabbah and those of de Cataldo-Migliorini. In addition, we also want to mention how our results differ.
Remark 0.0.1. (I) Comparing with Sabbah’s proof, instead of putting polarizable twistor $\mathcal{D}$-modules structures on semisimple local systems, we simply put polarizable twistor structures on their cohomology groups with an extra pre-Weil operator $\phi$. It turns out that to run the de Cataldo-Migliorini method, one only needs to check the compatibility of the pre-Weil operator $\phi$ with perverse cohomology functors and adjunction morphisms. This allows us to run inductive arguments in the category of constructible complexes, which is technically simpler. Furthermore, this topological method reduces the Decomposition Theorem for arbitrary maps to the constant map (i.e. Simpson’s Hard Lefschetz theorem and Theorem $[A]$), with the help of Simpson’s results for smooth projective maps and Theorem $[B]$. It is also easier than Zucker’s theorem for tame harmonic bundles $[27, \S5]$.

(II) Comparing with de Cataldo and Migliorini’s method, the main difficulty we encounter is that there is no off-the-shelf Hodge theory. The idea is to use Simpson’s theory of mixed twistor structures as a replacement. However, Simpson’s result is not enough, especially there is no obvious relation between his polarization and the twisted Poincaré pairing, which is one of the main discovery of this paper. On the other hand, when the local system $\mathcal{V}$ is $\mathbb{C}$, our argument gives a Hodge-decomposition-free proof of de Cataldo-Migliorini’s results. In other words, we realize that it is polarization, not Hodge decomposition, that is essential for the Decomposition Theorem. For example, the proof of Theorem $[D]$ is different from that of $[8, \text{Proposition } 5.2.3]$. The proof of Theorem $[E]$ is also different, where we need a new Corollary $2.7.10$.

(III) One key property used in the proof of de Cataldo-Migliorini is that the constant local system $\mathbb{C}$ is self-dual, while we need to run the inductive arguments for the local system $\mathcal{V}$ and its dual $\mathcal{V}^*$ together. This is the main reason for introducing notions of weight filtrations for two companion vector spaces in §1.6. Moreover, quite some efforts are needed to check that the pre-Weil operator $\phi$ is functorial and compatible with various geometric operations (see Lemma 1.4.8 and Corollary 1.5.6) and show non-degeneracy is preserved under restrictions. This enables us to prove non-degeneracy of certain adjunction morphisms using the positivity of polarizations, which is in turn needed to split the perverse cohomologies (see Lemma 2.8.3 and Proposition 2.9.1).

(IV) To apply the splitting criterion for the proof of Theorem $[C](iii)$ in the case of $\ell = 0$, one needs to relate adjunction morphisms with Poincare pairings. This is probably well-known to experts, we give a detailed discussion in §2.4 for the lack of appropriate references.

Structure of the paper. This paper consists of two parts. In the first part §1, we study the cohomology of semisimple local systems. In the second part §2, we give a new proof of Sabbah’s Decomposition Theorem.

In §1.2, we introduce an equivalent definition of polarization on pure twistor structures. In §1.3, we discuss Simpson’s theory of weights and prove Theorem $[B]$. In §1.4, we construct the generalized Weil operator and prove Theorem $[A]$. In §1.5, we show that perverse filtrations underlie natural pure twistor structures. In §1.6, we set up various definitions concerning weight filtrations for nilpotent operators and polarizations on the associated graded spaces on two vector spaces related by a non-degenerate pairing.
In §2, building on results in previous sections, we give a new proof of Sabbah’s Decomposition Theorem. We collect and prove some topological results about constructible complexes in §2.2 - §2.4. Then we give the proof in the rest of subsections.

In Appendix A, we sketch the construction of Hodge star operators for differential forms with coefficients in harmonic bundles.

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1. Cohomology of semisimple local systems

In this section, we study the cohomology of semisimple local systems using polarizations and weights, and prove Theorem A and Theorem B. The main set-up of this section is

Set-up 1.0.1. Let $X$ be a smooth projective variety of dimension $n$ and $V$ be a semisimple local system on $X$ with a flat connection $\nabla$. Let $\eta$ be an ample line bundle on $X$. Denote $V^\ast$ to be the dual local system and $H := V \otimes_\mathbb{C} \mathcal{O}_X^{\infty}$ to be the associated $\mathcal{C}^{\infty}$ flat bundle. Let $A^k_X$ denote the sheaf of $\mathcal{C}^{\infty}$ $k$-forms on $X$, similarly for $A^{p,q}_X$, and let $(\mathcal{A}^\ast_X(H), \nabla)$ denote the de Rham complex associated to $H$ and $\nabla$

$$(\mathcal{A}^\ast_X(H), \nabla) : = \{ H \xrightarrow{\nabla} A^1_X \otimes_{\mathcal{C}^{\infty}_X} H \to \cdots \to A^{2 \dim X}_X \otimes_{\mathcal{C}^{\infty}_X} H \},$$

with cohomology degree 0 at the term $H$.

Notation 1.0.2. Denote $\Omega_0$ and $\Omega_\infty$ to be the standard $\mathbb{A}^1$-neighborhoods for 0 and $\infty$ in $\mathbb{P}^1$ and set $G_m : = \Omega_0 \cap \Omega_\infty$.

1.1. Classical results.

Theorem 1.1.1 (Hodge-Simpson). We have the following results.

- (Hard Lefschetz Theorem) For each $j \geq 1$, the cup product map with $\eta$ is an isomorphism

$$\eta^j : H^{n-j}(X, V) \simto H^{n+j}(X, V).$$

- (Lefschetz decomposition) For each $k \geq 1$, $H^k(X, V)$ underlies a natural pure twistor structure of weight $k$. Moreover, for $k \leq n$, there is a direct sum decomposition

$$H^k(X, V) \cong \bigoplus_{m \geq 0} \eta^m H^{k-2m}(X, V)_{\text{prim}},$$

where $H^k(X, V)_{\text{prim}} := \text{Ker } \eta^{\dim X-k+1} \subseteq H^k(X, V)$ and each primitive space $\eta^m H^{k-2m}(X, V)_{\text{prim}}$ underlies a natural pure sub-twistor structure.
Proof. The first statement is [31, Lemma 2.6]. The second statement comes from [33, Theorem 4.1] (see also [27, Theorem 2.2.4]).

**Theorem 1.1.2** (Semisimplicity Theorem for smooth projective maps). Let \( f : X \to Y \) be a projective morphism between smooth projective varieties. Let \( \mathcal{V} \) be a semisimple local system on \( X \). Let \( Y_0 \subseteq Y \) be the open subset over which \( f \) is smooth and let \( X_0 = f^{-1}(Y_0) \). Then \( R^q f_*(\mathcal{V}|_{X_0}) \) is a semisimple local system on \( Y_0 \) for every \( q \geq 0 \).

Proof. This is proved by Simpson in [32, Corollary 4.5].

**Corollary 1.1.3.** Let \( T \subseteq Y_0 \) be a smooth subvariety and let \( X_T = f^{-1}(Y_T) \subseteq X_0 \). Then \( R^q f_*(\mathcal{V}|_{X_T}) \) is a semisimple local system on \( T \) for every \( q \geq 0 \).

Proof. Eliminating the indeterminacies of maps, we can find the following commutative diagram:

\[
\begin{array}{ccc}
X_T & \longrightarrow & X \\
\downarrow f & & \downarrow f \\
T & \longrightarrow & Y,
\end{array}
\]

where \( T \) is a smooth projective variety containing \( T \) as a Zariski open subset and \( X_T \) is smooth projective. Using the metric characterization of semisimple local systems on smooth projective varieties (cf. Corollary 1.2.12), we know the local system \( h^*\mathcal{V} \) is semisimple. Then by Theorem 1.1.2 and Remark 1.1.4, it follows that

\[ R^q f_*(\mathcal{V}|_{X_T}) = R^q f_*(h^*\mathcal{V}|_{X_T}) \]

is semisimple on \( T \).

**Remark 1.1.4.** Let \( Y \) be a smooth variety and let \( \mathcal{V} \) be a local system on \( Y \). Let \( Y' \subseteq Y \) be a Zariski dense open subset, then \( \mathcal{V} \) is semisimple if and only if \( \mathcal{V}|_{Y'} \) is semisimple. This follows from the surjectivity of the natural map \( \pi_1(Y') \to \pi_1(Y) \). (For example, see [19, X, Théorème 2.3], and notice that \( Y \setminus Y' \) has real codimension at least 2.)

1.2. **Cohomology of smooth projective varieties and pure twistor structures.** In this subsection, we review Simpson’s notion of pure twistor structures following [33]. For our purposes, we introduce an equivalent definition of polarization. This allows us to relate the polarization with the twisted Poincaré pairing, via the pre-Weil operator \( \phi \) constructed in §1.4.

**Definition 1.2.1.** A twistor structure is a holomorphic vector bundle \( E \) on \( \mathbb{P}^1 \). Morphisms of twistor structures are morphisms between holomorphic vector bundle over \( \mathbb{P}^1 \). We say a complex vector space \( V \) underlies a twistor structure \( E \) if \( V \cong E|_{z=1} \) (the fiber over the point \( 1 \in \mathbb{P}^1 \)). We also say that \( E \) is a twistor structure on \( V \). A twistor structure \( E \) is pure of weight \( w \) if \( E \) is a direct sum of copies of \( \mathcal{O}_{\mathbb{P}^1}(w) \).

**Remark 1.2.2.** It is clear that the category of pure twistor structures with a fixed weight is equivalent to the category of complex vector spaces.

For the study of polarizations, we define a functorial Identification map associated to any pure twistor structure.
**Definition 1.2.3.** Let $E$ be a pure twistor structure. The Identification map $\text{Iden} : E|_{z=1} \to E|_{z=-1}$ is defined as follows. If the weight of $E$ is 0, define

$$\text{Iden} : E|_{z=1} \xrightarrow{(\text{ev}_{z=1})^{-1}} H^0(\mathbf{P}^1, E) \xrightarrow{\text{ev}_{z=-1}} E|_{z=-1},$$

where $\text{ev}_{z=z_0}$ is the isomorphic evaluation map for global sections.

If the weight of $E$ is $w \neq 0$, we do a Tate twist to weight zero: choose $\mu \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ to be the unique section up to scaling so that $\mu|_{z_0}$ is nowhere zero. The evaluation of $\mu^{\otimes w}$ at $z = z_0 \in \Omega_0$ identifies $\mathcal{O}_{\mathbf{P}^1}(w)|_{z=z_0}$ with $\mathbb{C}$. We use $\text{ev}^w_{z=z_0}$ to denote the composition of following maps:

$$\text{ev}^w_{z=z_0} : H^0(\mathbf{P}^1, E(-w)) \xrightarrow{\text{ev}^w_{z=z_0}} E(-w)|_{z=z_0} \xrightarrow{\mu^{\otimes w}|_{z=z_0}} E|_{z=z_0}.$$  

Then we define

$$\text{Iden} : E|_{z=1} \xrightarrow{(\text{ev}^w_{z=1})^{-1}} H^0(\mathbf{P}^1, E(-w)) \xrightarrow{\text{ev}^w_{z=-1}} E|_{z=-1}.$$ 

It is direct to see that it does not depend on the choice of $\mu$.

**Remark 1.2.4.** In fact, one can actually define Identification map between any two fibers of $E$. Here 1 and $-1$ are special because they are related to Weil operators in Hodge theory (see Corollary [2.4.7]).

**Remark 1.2.5.** It is direct to see that the Identification map is functorial with respect to pure sub-twistor structures.

Now let us turn to polarizations. Let $E$ be a pure twistor structure of weight $w$. Let $\sigma$ denote the antipodal involution of $\mathbf{P}^1$, where $\sigma(z) = -1/z$. As in [33, §2], $\sigma$ induces another pure twistor structure $\sigma^*(E)$ with an induced isomorphism:

$$\sigma : H^0(\mathbf{P}^1, E) \xrightarrow{\sim} H^0(\mathbf{P}^1, \sigma^*E).$$

**Definition 1.2.6** (Simpson’s Polarization). Consider a morphism of pure twistor structures

$$P : E \otimes_{\mathcal{O}_{\mathbf{P}^1}} \sigma^*E \to \mathcal{O}_{\mathbf{P}^1}(2w),$$

which is equivalent to $P(-2w) : E(-w) \otimes_{\mathcal{O}_{\mathbf{P}^1}} \sigma^*(E(-w)) \to \mathcal{O}_{\mathbf{P}^1}$. We say $P$ is a polarization on $E$ if the induced morphism on global sections

$$H^0(\mathbf{P}^1, E(-w)) \otimes_{\mathbb{C}} H^0(\mathbf{P}^1, E(-w)) \xrightarrow{\text{Id} \otimes \sigma} H^0(\mathbf{P}^1, E(-w)) \otimes_{\mathbb{C}} H^0(\mathbf{P}^1, \sigma^*(E(-w))) \to \mathbb{C}$$

is a positive definite Hermitian pairing.

In fact, any pure twistor structure is polarizable. The polarization on $H^k(X, V)$ we use in this paper is the canonical one that comes from twisted Poincaré pairings. For this purpose, we would like to have the following lemma.

**Lemma 1.2.7.** The pure twistor structure $E$ is polarized by a morphism $P$ if and only if the bilinear pairing $S := P|_{z=1}$ as

$$S : E|_{z=1} \otimes_{\mathbb{C}} E|_{z=-1} \to \mathbb{C},$$

induces a positive definite Hermitian pairing

$$S(\bullet, \text{Iden}(\bullet)) : E|_{z=1} \otimes_{\mathbb{C}} E|_{z=1} \to \mathbb{C}.$$  

Here $\text{Iden}$ comes from Definition [1.2.3].
Proof. The proof is straightforward via the definition of Identification map. For details, see [35, Lemma 2.3.20].

As a result, we can use bilinear pairings to polarize pure twistor structures.

**Definition 1.2.8** (Polarization by a bilinear pairing). Given a bilinear pairing \( S : E|_{z=1} \otimes C \rightarrow C \), we say \( E \) is polarized by \( S \), if the pairing
\[
S(\bullet, \text{Iden}(\bullet)) : E|_{z=1} \otimes C E|_{z=1} \rightarrow E|_{z=1} \otimes C E|_{z=1} \rightarrow C
\]
is positive definite and Hermitian, where Iden is the Identification map in Definition 1.2.3.

The following two statements are direct to check.

**Lemma 1.2.9.** If \( S \) is a bilinear pairing that polarizes \( E \), then \( S \) is non-degenerate.

**Lemma 1.2.10.** Let \( G \subseteq E \) be a pure sub-twistor structure. If \( E \) is polarized by a bilinear pairing \( S_E : E|_{z=1} \otimes C E|_{z=1} \rightarrow C \), then its restriction
\[
S_G : G|_{z=1} \otimes C G|_{z=1} \rightarrow C
\]
polarizes the pure twistor structure \( G \).

1.2.1. **Cohomology of smooth projective varieties with semisimple coefficients.** We use Set-up 1.0.1. The following result is proved by Corlette [5, Theorem 3.3] and Simpson [31, Theorem 1]. We follow [31, Page 13, paragraph 2] with slightly different notations, where Simpson’s \( D, \partial, \bar{\partial}, \theta, \bar{\theta}, d', d'' \) are our \( \nabla, \partial', \partial'', \theta', \theta'' \), \( D', D'' \) respectively.

**Theorem 1.2.11.** The bundle \( H \) admits a harmonic metric \( h \) inducing the decomposition
\[
\nabla = \partial' + \theta' + \partial'' + \theta'',
\]
with
\[
\partial' : H \rightarrow \mathcal{A}^{1,0}_X \otimes C_X H, \quad \theta' : H \rightarrow \mathcal{A}^{1,0}_X \otimes C_X H,
\]
\[
\partial'' : H \rightarrow \mathcal{A}^{0,1}_X \otimes C_X H, \quad \theta'' : H \rightarrow \mathcal{A}^{0,1}_X \otimes C_X H,
\]
so that \( \partial' \) is a \((1,0)\)-connection, \( \partial'' \) is a \((0,1)\)-connection and \( \partial' + \partial'' \) is a metric connection of \( h \). The operator \( \theta' \) is \( C_X \)-linear and \( \theta'' \) is the adjoint of \( \theta' \) with respect to \( h \). Moreover, if we set \( D' := \partial' + \theta' \) and \( D'' := \partial'' + \theta'' \), then
\[
(D')^2 = (D'')^2 = 0.
\]
Conversely, if a local system admits a harmonic metric, then it is semisimple.

**Corollary 1.2.12.** Let \( f : Z \rightarrow X \) be a morphism from a smooth quasi-projective variety \( Z \). Then \( f^* V \) is semisimple.

**Proof.** Use Theorem 1.2.11, Remark 1.1.3 and the fact that pullback preserves harmonic metrics (see [30, Page 18]).

The following result is proved by Simpson, see [33, Page 25, line 8 - line 13].

**Theorem 1.2.13** (Simpson). There is an isomorphism
\[
H^k(X, V) \cong \text{Harm}^k(X, H) := \{ \alpha \in C^\infty(\mathcal{A}^k_X \otimes C_X H) : \Delta_V(\alpha) = 0 \},
\]
where \( \Delta_V \) is the Laplacian of \( V \). Moreover, for any \((a, b) \neq (0, 0)\), there is an isomorphism
\[
H^k(X, \mathcal{A}^a_X(H); aD' + bD'') \cong \{ \alpha \in C^\infty(\mathcal{A}^k_X \otimes C_X H) : \Delta_{aD' + bD''}(\alpha) = 0 \} = \text{Harm}^k(X, H).
\]
If we use the coordinate \(1.2.14\) can be calculated in the following way. Let \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be a harmonic representative of \(\psi\) be 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In the remark under [33] Theorem 5.2, Simpson said that the same theory of weights hold as in [13] Théorème 8.2.4. In particular, we have

**Theorem 1.3.3 (Simpson’s theory of weights).** With the assumptions in Theorem 1.3.2. Let $W_\cdot$ denote the weight filtration on $H^k(U, \mathcal{V})$ induced by the natural mixed twistor structure. Then we have

- if $U$ is proper, then $\text{Gr}_i^W H^k(U, \mathcal{V}) = 0$ for $i > k$,
- if $U$ is smooth, then $\text{Gr}_i^W H^k(U, \mathcal{V}) = 0$ for $i < k$,
- if $U$ is smooth and proper, then $\text{Gr}_i^W H^k(U, \mathcal{V}) = 0$ for $i \neq k$.

In [33], Simpson only proved the last statement of Theorem 1.3.3. For some proof details, see [35] Corollary 2.3.87 and Corollary 2.3.97.

1.3.1. **Mixed Twistor Complexes.** To construct the mixed twistor structures, Simpson used the notion of mixed twistor complexes, patching construction and an analytic construction for the pure twistor structures.

**Definition 1.3.4.** A mixed twistor complex is a filtered complex $(M^\cdot, W^\cdot_{\text{pre}})$ of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules on $\mathbb{P}^1$ such that $\mathcal{H}^i(\text{Gr}_i^W(M^\cdot))$ is a pure twistor structure of weight $\ell + i$. $W^\cdot_{\text{pre}}$ is called the pre-weight filtration.

**Lemma 1.3.5.** Let $(M^\cdot, W^\cdot_{\text{pre}})$ be a mixed twistor complex. Suppose $\ell$ is the smallest integer so that $W^\cdot_{\ell, \text{pre}} M^\cdot \neq 0$, then

$$W^\cdot_{\ell, \text{pre}} H^k(M^\cdot) = \text{Im} \{ \mathcal{H}^k(W^\cdot_{\ell, \text{pre}} M^\cdot) \to \mathcal{H}^k(M^\cdot) \},$$

where the map is induced by $W^\cdot_{\ell, \text{pre}} M^\cdot \hookrightarrow M^\cdot$.

**Proof.** It follows from [33] Lemma 5.3 and [34] Lemma 8.24. For details, see [35] Corollary 2.3.74.

Simpson introduced a patching construction to glue several filtered complexes over subsets of $\mathbb{P}^1$ to a filtered complex over $\mathbb{P}^1$ [33 Page 30-32]. For clarity, we briefly recall his construction. Let $M^\cdot, N^\cdot$ and $P^\cdot$ be filtered complexes of sheaves of $\mathcal{O}$-modules, respectively over $\Omega_0, \Omega_\infty$ and $G_m$. Assume there are filtered quasi-isomorphisms $M^\cdot|_{G_m} \xrightarrow{\ell} P^\cdot \xrightarrow{g} N^\cdot|_{G_m}$, then we define by

$$\text{Patch}(M \xleftarrow{P} \to N) := \text{Cone}(P^\cdot \to M^\cdot \oplus N^\cdot),$$

where $M^\cdot_{\text{ex}}, N^\cdot_{\text{ex}}, P^\cdot_{\text{ex}}$ are filtered complexes over $\mathbb{P}^1$ induced by a fixed functorial choice of right derived functor of some extension functor for the inclusion $\Omega_0 \hookrightarrow \mathbb{P}^1$ (as well as for $\Omega_\infty$ and $G_m$). If there are filtered quasi-isomorphisms $M^\cdot|_{G_m} \xrightarrow{f} P^\cdot \xrightarrow{g} N^\cdot|_{G_m}$, then we define

$$\text{Patch}(M \to P \xleftarrow{N}) := \text{Cone}(M^\cdot_{\text{ex}} \oplus N^\cdot_{\text{ex}} \to P^\cdot_{\text{ex}})[−1].$$

There is also a version for gluing 5 complexes. Suppose $P^\cdot, Q^\cdot, R^\cdot$ are filtered complexes of $\mathcal{O}$-modules over $G_m$ and $M^\cdot, N^\cdot$ are filtered complexes of $\mathcal{O}$-modules over $\Omega_0$ and $\Omega_\infty$ respectively. Assume we have filtered quasi-isomorphisms

$$M^\cdot|_{G_m} \xrightarrow{f} P^\cdot \xleftarrow{g} Q^\cdot \to R^\cdot \to N^\cdot|_{G_m},$$

then we define

$$\text{Patch}(M, P, Q, R, N) := \text{Patch}(M^\cdot \xleftarrow{P^\cdot} \to Q^\cdot \xrightarrow{R^\cdot} \to N^\cdot)[−1].$$
1.3.2. **Mixed twistor structures on open varieties.** Let $\Omega$ be an open subset with a simple normal crossing boundary divisor $\Delta$. Then there is a natural filtered quasi-isomorphism

$$H^k_{\text{ind}}(\Omega) \cong H^k(\Omega, \mathcal{V}),$$

where the latter is induced by $\tau$. This gives filtered quasi-isomorphisms on $\mathcal{G}$.

**Proof.** It is straightforward by definition of cones. For details, see [33, Lemma 2.3.80].

It turns out that the Construction 1.2.14 can be recovered using the patching construction. Now we work with Set-up 1.0.1 and follow the notations in [33, Page 23-24]. Simpson constructed a triple $(\mathcal{F}, \mathcal{L}, \mathcal{F}')$ associated to $\mathcal{V}$ and showed that the bundle $R^k\mathcal{P}_{2,*}(\xi\Omega_X^{\bullet}(\mathcal{F}))$ over $\Omega_0$ and the bundle $R^k\mathcal{P}_{2,*}(\xi\Omega_X^{\bullet}(\mathcal{F}'))$ over $\Omega_\infty$ can be identified over $\mathcal{G}_m$ with $R\mathcal{P}_{2,*}(\mathcal{L})$, and therefore glued to a holomorphic bundle $\oplus\mathcal{O}_{\mathcal{P}^1}(k)$, which is canonically isomorphic to the twistor structure in Construction 1.2.14 (see [33, Page 25-26]). It can be summarized as follows.

**Lemma 1.3.7.** Consider the complex of sheaves of $\mathcal{O}_{\mathcal{P}^1}$-modules with trivial filtrations:

$$\text{Patch}(Rp_{2,*}(\xi\Omega_X^{\bullet}(\mathcal{F}))) \to R\mathcal{P}_{2,*}(\mathcal{L}) \to R\mathcal{P}_{2,*}(\xi\Omega_X^{\bullet}(\mathcal{F}')),$$

where the morphisms are induced by Dolbeault resolutions. Then the $k$-th cohomology of this complex is the natural pure twistor structure $E^k$ on $H^k(X, \mathcal{V})$.

### 1.3.2. Mixed twistor structures on open varieties.

We work with Set-up 1.0.1. Let $j : U \to X$ be an open subset with a simple normal crossing boundary divisor $D = X \setminus U$. In [33, Page 34], the functorial mixed twistor structure on $H^k(U, \mathcal{V}|_U) := H^k(U, j^*\mathcal{V})$ is constructed as follows. Let $p$ denote the second projection map to the $\mathcal{P}^1$-direction for subsets of $X^{\text{top}} \times \mathcal{P}^1$.

Using the triple $(\mathcal{F}, \mathcal{L}, \mathcal{F}')$ associated to $\mathcal{V}$ on $X$, he defined 5 filtered complexes:

$$M^\bullet := (\xi\Omega_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{F}, W^p_{\mathcal{O}^\mathcal{P}^1}), \quad N^\bullet := (\xi\Omega_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{F}', W^p_{\mathcal{O}^\mathcal{P}^1}),$$

$$P^\bullet := (\xi\Omega_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{F}, \tau), \quad Q^\bullet := (j_*\mathcal{A}_{\mathcal{U}}^\mathcal{P} \otimes_{\mathcal{O}_{\mathcal{G}_m}} \mathcal{L}, \tau),$$

$$R^\bullet := (\xi\Omega_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{F}', \tau),$$

where $\tau$ is the filtration induced by the truncation functor and $W^p_{\mathcal{O}^\mathcal{P}^1}$ is the filtration induced by the weight filtration on $\Omega_X^\bullet(\log D)$. Then the isomorphisms $\Omega_X^\bullet(\log D) \cong j_*\mathcal{O}_{\mathcal{U}} \cong j_*\mathcal{A}_{\mathcal{U}}^\mathcal{P}$ induces filtered isomorphisms

$$M^\bullet|_{\mathcal{X} \times \mathcal{G}_m} \leftarrow P^\bullet \to Q^\bullet \leftarrow R^\bullet \to N^\bullet|_{\mathcal{X} \times \mathcal{G}_m}.$$

This gives filtered quasi-isomorphisms on $\mathcal{G}_m$:

$$R_{p*}M^\bullet|_{\mathcal{G}_m} \leftarrow R_{p*}P^\bullet \to R_{p*}Q^\bullet \leftarrow R_{p*}R^\bullet \to R_{p*}N^\bullet|_{\mathcal{G}_m}.$$

Define the patching complex

$$\text{MTC}(\mathcal{V}|_U) := \text{Patch}(R_{p*}M^\bullet, R_{p*}P^\bullet, R_{p*}Q^\bullet, R_{p*}R^\bullet, R_{p*}N^\bullet).$$

**Theorem 1.3.8.** [33, Page 34] The complex $\text{MTC}(\mathcal{V}|_U)$ is a mixed twistor complex.
As a corollary, for any integer $k$, $H^k(U, \mathcal{V}|_U)$ underlies a natural mixed twistor structure, which is functorial with respect to $U \hookrightarrow X$.

**Remark 1.3.9.** To construct the mixed twistor structure on $H^k(U, \mathcal{V}|_U)$, it is actually enough to use three complexes $P^\bullet, Q^\bullet, R^\bullet$ and $R^\bullet$ naturally extend to $\Omega_0$ and $\Omega_\infty$ respectively) and consider $\text{Patch}(Rp_*P^\bullet \to Rp_*Q^\bullet \leftarrow Rp_*R^\bullet)$. The advantage of using the extra two complexes is that the explicit weight filtration on $\Omega_X^\bullet(\log D)$ is useful for the proof of Lemma 1.3.11.

**Corollary 1.3.10.** With Set-up 1.0.1. Let $j : U \hookrightarrow X$ be an arbitrary Zariski open subset. Then $H^k(U, \mathcal{V}|_U)$ underlies a natural and functorial mixed twistor structure $(E^k_U, W^*_UE^k_U)$. Moreover, for $[z_0, 1] \in \mathbf{P}^1 \setminus \{0, \infty\}$, there is an isomorphism

$$E^k_U|_{z=z_0} \cong H^k(U, \mathcal{A}^*_U(H|_U); (z_0 \partial'+ \partial''+ \theta'+ z_0 \theta'')|_U),$$

where $\partial', \partial'', \theta', \theta''$ are the operators associated to $\mathcal{V}$ and $H$ in Theorem 1.2.11.

**Proof.** First we assume $X \setminus U$ is a simple normal crossing divisor and use notation above. The first statement is obtained in Theorem 1.3.8 Lemma 1.3.7 and (2) imply that $H^k(X, \mathcal{L}|_{X \times \{z_0\}}) \cong H^k(X, \mathcal{A}^*_X(H); z_0 \partial' + \partial'' + \theta' + z_0 \theta'')$.

Therefore, we have

$$E^k_U|_{z=z_0} \cong (R^k p_*Q^\bullet)|_{z_0} = H^k(X, (\xi j_*\mathcal{A}^*_U \otimes \mathcal{L})|_{X \times \{z_0\}}) \cong H^k(U, \mathcal{A}^*_U(H|_U); (z_0 \partial'+ \partial''+ \theta'+ z_0 \theta'')|_U),$$

where $p : X^\text{top} \otimes \mathbf{G}_m \to \mathbf{G}_m$ is the second projection. This proves (3) for $X \setminus U$ normal crossing.

For arbitrary $U \subseteq X$, consider the following diagram

$$\begin{array}{ccc}
U & \xrightarrow{j} & \tilde{X} \\
\text{id} & \downarrow & \downarrow \pi \\
U & \xrightarrow{j} & X,
\end{array}$$

where $\pi : \tilde{X} \to X$ is a log resolution of $(X, X \setminus U)$, so that $\tilde{X}$ is smooth projective, $\pi$ is an isomorphism over $U$, and $\tilde{X} \setminus \pi^{-1}(U)$ is a simple normal crossing divisor. In particular, we can view $U$ as an open subset of $\tilde{X}$. Let $\tilde{\mathcal{V}} = \pi^*\mathcal{V}$ be the pullback local system. By Corollary 1.2.12 $\tilde{\mathcal{V}}$ is semisimple and the corresponding bundle $\tilde{H} = \tilde{\mathcal{V}} \otimes \mathcal{C}_X^\infty$ admits the harmonic metric $\tilde{h} = \pi^*h$. Let $\tilde{\partial}', \tilde{\partial}'', \tilde{\theta}', \tilde{\theta}''$ be the operators associated to $(\tilde{H}, \tilde{h})$. Since $\pi$ is an isomorphism over $U$, we have $\tilde{\mathcal{V}}|_U = \mathcal{V}|_U$ and $(\tilde{H}, \tilde{h})|_U = (H, h)|_U$, and $(\tilde{\partial}', \tilde{\partial}'', \tilde{\theta}', \tilde{\theta}'')|_U = (\partial', \partial'', \theta', \theta'')|_U$.

Now we can apply the previous case to conclude that $H^k(U, \mathcal{V}|_U) = H^k(U, \tilde{\mathcal{V}}|_U)$ admits a natural mixed twistor structure $(\tilde{E}^k_U, W_*\tilde{E}^k_U)$. Moreover,

$$\tilde{E}^k_U|_{z=z_0} \cong H^k(U, \mathcal{A}^*_U(\tilde{H}|_U); (z_0 \tilde{\partial}' + \tilde{\partial}'' + \tilde{\theta}' + z_0 \tilde{\theta}'')|_U) = H^k(U, \mathcal{A}^*_U(H|_U); (z_0 \partial' + \partial'' + \theta' + z_0 \theta'')|_U).$$

This proves (3) for arbitrary $U \subseteq X$ and finishes the proof of the corollary. \hfill \Box

**Lemma 1.3.11.** With the notations in Corollary 1.3.10, then

$$W_kH^k(U, \mathcal{V}|_U) = j^*H^k(X, \mathcal{V}).$$
Proof. First, consider the following commutative diagram
\[
\begin{array}{ccc}
U & \xrightarrow{j} & \tilde{X} \\
\downarrow{\text{id}} & & \downarrow{\pi} \\
U & \xrightarrow{j} & X,
\end{array}
\]
where \(\pi\) is a log resolution so that \(\tilde{X} \setminus U\) is a normal crossing divisor. It is easy to see that \(j^*H^k(\tilde{X}, \pi^*\mathcal{V}) = j^*H^k(X, \mathcal{V})\) by looking at the Leray spectral sequence for the map \(\pi : \tilde{X} \to X\) and \(\pi^*\mathcal{V}\). Since semisimplicity is preserved under pull-backs (see Corollary 1.2.12), we can assume \(X \setminus U\) is normal crossing. Let \(E^k\) be the natural twistor structure on \(H^k(X, \mathcal{V})\). By Lemma 1.3.7 and Lemma 1.3.6,
\[
E^k \cong H^k\text{Patch} \left( Rp_*(\xi\Omega^*_X(\mathcal{F})) \to Rp_*(\mathcal{L}) \leftarrow Rp_*(\xi\Omega^*_X(\mathcal{F}')) \right)
\]
\[
\cong H^k\text{Patch} \left( Rp_*(\xi\Omega^*_X(\mathcal{F})), Rp_*(\xi\Omega^*_X(\mathcal{F}))|_{G_m}, Rp_*(\xi\Omega^*_X(\mathcal{F}'))|_{G_m}, Rp_*(\xi\Omega^*_X(\mathcal{F}')) \right)
\]
By Theorem 1.3.8, the map \(j^* : H^k(X, \mathcal{V}) \to H^k(U, \mathcal{V}|_U)\) can be lifted to a morphism of mixed twistor structures, which is induced by the following morphisms of complexes:
\[
\xi\Omega^*_X \otimes \mathcal{F} \to \xi\Omega^*_X(\log D) \otimes \mathcal{F}, \quad \mathcal{L} \to \xi j_*\Omega^*_U \otimes \mathcal{L}, \quad \xi\Omega^*_X \otimes \mathcal{F}' \to \xi\Omega^*_X(\log D) \otimes \mathcal{F}'.
\]
By Lemma 1.3.3, \(W_kH^k(U, \mathcal{V}|_U)\) underlies the gluing of the images of the inclusion maps over \(\Omega_0, G_m, \Omega_\infty\):
- \(R^kp_*(W^0\xi\Omega^*_X(\log D) \otimes \mathcal{F}) \to R^kp_*(\xi\Omega^*_X(\log D) \otimes \mathcal{F})\),
- \(R^kp_*(\tau_0\xi\Omega^*_X(\log D) \otimes \mathcal{F}) \to R^kp_*(\xi\Omega^*_X(\log D) \otimes \mathcal{F})\),
- \(R^kp_*(\tau_0\xi j_*\Omega^*_U \otimes \mathcal{L}) \to R^kp_*(\xi j_*\Omega^*_U \otimes \mathcal{L})\).
By the residue map in [33] Page 36, we have
\[
W^0\xi\Omega^*_X(\log D) \otimes \mathcal{F} \cong \xi\Omega^*_X \otimes \mathcal{F}.
\]
Topological calculations from [12] 3.1.8.1 show that
\[
\tau_0\xi\Omega^*_X(\log D) \otimes \mathcal{F} = H^0(\xi\Omega^*_X(\log D) \otimes \mathcal{F}) \cong \xi\Omega^*_X \otimes \mathcal{F},
\]
and by definition of the truncation functor
\[
\tau_0\xi j_*\Omega^*_U \otimes \mathcal{L} = H^0(\xi j_*\Omega^*_U \otimes \mathcal{L}) \cong \mathcal{L}.
\]
In particular, on each corresponding subset of \(\mathbb{P}^1\), the lowest weight filtration coincides with the image of restriction from \(X\). Since the patching construction preserves the quasi-isomorphism, the lemma follows. \(\square\)

Now we can prove the global invariant cycle theorem.

Proof of Theorem [12] Theorem 1.3.3 implies that
- if \(i > k\), then \(Gr^W H^k(Z, (j \circ \alpha)^*\mathcal{V}) = 0\) [\(Z\) is proper],
- if \(i < k\), then \(Gr^W H^k(U, j^*\mathcal{V}) = 0\) [\(U\) is smooth].
By the strictness of morphism between mixed twistor structures and Lemma 1.3.11, we have
\[
\text{Im } \alpha^* = \text{Im } \alpha^* \cap W_kH^k(Z, (j \circ \alpha)^*\mathcal{V})
\]
\[
= \alpha^*(W_kH^k(U, j^*\mathcal{V})) = \alpha^*j^*H^k(X, \mathcal{V}) = \text{Im } (j \circ \alpha)^*.
\]
\(\square\)
1.4. Twisted Poincaré pairings and polarizations. In this subsection, we construct the pre-Weil operator \( \phi \) using harmonic metrics and Sabbah’s rescaling map. We show that the twisted Poincaré pairing can be used to polarize the pure twistor structures on the cohomology groups semisimple local systems.

Let \( U \) be a smooth quasi-projective variety and \( V \) be a semisimple local system on \( U \) with a flat connection \( \nabla \), which comes from the restriction of a semisimple local system on a smooth projective compactification of \( U \). By Theorem 1.2.11, there is a harmonic bundle \( (H, h) \) associated to \( V \) with the decomposition \( \nabla = \partial' + \theta' + \partial'' + \theta'' \), where \( h \) is unique up to a linear transformation. We view the harmonic metric \( h \) as a \( C^\infty_U \)-linear morphism \( h : H \otimes_{C^\infty_U} \bar{H} \to C^\infty_U \).

Let \( V^* \) be the dual local system of \( V \) and \( H^* \) be the dual \( C^\infty \)-bundle of \( H \). In [31, Page 14], Simpson showed that \( H^* \) is equipped with the following harmonic structure so that it is the harmonic bundle associated to \( V^* \). Following Simpson, we will abuse notation and still use \( D'' \), \( \nabla \) etc to represent the corresponding operator on \( H^* \). Note that Simpson’s dual construction in [31] actually works without the compactness assumption. This is crucial for us, because in the proof of Decomposition theorem one needs to deal with non-compact varieties.

Construction 1.4.1 (Dual harmonic bundle). The dual metric \( h^* : H^* \otimes \bar{H^*} \to C^\infty_U \) is defined as follows: for two sections \( \lambda, \mu \in C^\infty(H^*) \), set

\[
h^*(\lambda, \mu) := \lambda(e),
\]

where \( e \) is the unique section in \( C^\infty(H) \) satisfying \( \mu(\cdot) = h(\cdot, \bar{e}) \). The dual Higgs operator \( D'' = \partial'' + \theta'' \) is defined by

\[
(\theta' \lambda)(e) + \lambda(\theta' e) = 0, \quad (\partial'' \lambda)(e) + \lambda(\partial'' e) = \overline{(\partial' \lambda(e))},
\]

where \( \lambda \in C^\infty(H^*), e \in C^\infty(H) \). The dual flat connection \( \nabla \) on \( V^* \) is defined by

\[
\nabla := \partial' + \theta' + \partial'' + \theta'',
\]

where \( \partial' + \partial'' \) is a metric connection for \( h^* \) and \( \theta'' \) is the adjoint of \( \theta' \) with respect to \( h^* \).

Remark 1.4.2. We want the reader to be aware that the dual connection \( \nabla \) on \( V^* \) is not the dual of \( \nabla \) on \( V \) with respect to the harmonic metric \( h \).

Lemma 1.4.3. With the notations above, the harmonic metric \( h \) induces a \( \mathbb{C} \)-linear isomorphism between \( C^\infty_U \)-bundles:

\[
h : \overline{H^*} \to H^*, \quad \overline{\mu} \mapsto h(\cdot, \overline{\mu}), \quad \forall \mu \in C^\infty(H),
\]

so that it induces a quasi-isomorphism

\[
h : (A^*_U(H), \partial' - \theta' + \partial'' - \theta'') \xrightarrow{\sim} V^*,
\]

and an isomorphism

\[
h : H^k(U, A^*_U(H); \partial' - \theta' + \partial'' - \theta'') \xrightarrow{\sim} H^k(U, V^*).
Proof. There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{h} & H^* \\
\downarrow h^*(\nabla) & & \downarrow \nabla \\
\mathcal{H} \otimes A^1_U & \xrightarrow{h \otimes \text{Id}} & H^* \otimes A^1_U.
\end{array}
\]

We claim that

\[h^*(\nabla) = \overline{\nabla'} - \overline{\theta'} + \overline{\nabla} - \overline{\theta},\]

and each operator in the equation is the conjugated operator on \(H\) with respect to \(\nabla\) on \(H^*\). Granting this for now, since \(H^*\) is the harmonic bundle associated to \(\nabla^*\) and the de Rham complex \(A^1_U(H^*)\) is determined by the first order differential operator \(\nabla\), we see that \(h\) induces a quasi-isomorphism \(\mathcal{H} \cong (A^*_X(H), \partial' - \partial'' + \partial - \theta')\).

To prove the claim, using the isomorphism \(h : \mathcal{H} \rightarrow H^*\), we define a map so that any operator \(A : H^* \rightarrow H^* \otimes A^1_U\) is sent to the unique operator \(B : \mathcal{H} \rightarrow \mathcal{H} \otimes A^1_U\) satisfying

\[A \circ h = (h \otimes \text{Id}) \circ B.\]

Let us see how each operator in \(\nabla = \partial' + \theta' + \partial'' + \theta''\) changes under the map defined above. We fix \(e_1, e_2 \in C^\infty(H)\) and \(\lambda = h(e_1, e_2) \in C^\infty(H^*)\).

(A) \(\theta' \mapsto -\overline{\theta'}\). We have

\[(\theta' \lambda)(e_1) = -h(\theta' e_1, e_2) = h(e_1, -\overline{\theta'' e_2}).\]

The last equality uses that \(\theta'\) is the adjoint of \(\theta''\). Hence

\[\theta' h(e_1, e_2) = h(e_1, -\overline{\theta'' e_2}),\]

which means that the image of \(\theta'\) is \(-\overline{\theta''}\).

(B) \(\partial'' \mapsto \overline{\partial'}\). We have

\[(\partial'' \lambda)(e_1) = \overline{\partial} h(e_1, e_2) - h(\partial'' e_1, e_2) = h(e_1, \overline{\partial e_2}).\]

The last equality uses that \(\partial' + \partial''\) is the metric connection with respect to \(h\). Hence

\[\partial'' h(e_1, e_2) = h(e_1, \overline{\partial e_2}),\]

which means that \(\partial''\) is mapped to \(\overline{\partial'}\).

(C) \(\theta'' \mapsto -\overline{\theta'}\) and \(\partial' \mapsto \overline{\partial'}\) can be verified similarly using their definition via the dual harmonic metric \(h^*\).

\[\square\]

To define the pre-Weil operator \(\phi\), we recall Sabbah’s rescaling map in [27, equation (2.2.6)].

Definition 1.4.4. Define a map of complexes

\[\iota : C[z] \otimes_C A^{p,q} \otimes_{C^\infty} H \rightarrow z^{-p}C[z] \otimes_C A^{p,q} \otimes_{C^\infty} H\]

\[\alpha^{p,q} \otimes m \mapsto z^{-p}\alpha^{p,q} \otimes m.\]

Under \(\iota\), the differential \(z\partial' + \theta' + \partial'' + z\theta''\) is changed into \(\partial' + z^{-1}\theta' + \partial'' + z\theta''\). For \(z_0 \in G_m\), the evaluation at \(z = z_0\) gives

\[\iota_{z_0} : A^{p,q} \otimes_{C^\infty} H \rightarrow A^{p,q} \otimes_{C^\infty} H, \quad \alpha^{p,q} \otimes m \mapsto z_0^{-p}\alpha^{p,q} \otimes m.\]
Abusing notations, we denote the induced map on the cohomology by
\[ \iota_{z_0} : H^k(U, A^\bullet_U(H); z_0 \partial' + \theta' + \partial'' + z_0 \theta'') \rightarrow H^k(U, A^\bullet_U(H); \partial' + z_0^{-1} \theta' + \partial'' + z_0 \theta''). \]

Let \( U \) be a smooth quasi-projective variety and \( \mathcal{V} \) be a local system on \( U \) coming from the restriction of a semisimple local system on a smooth projective compactification of \( U \). By Corollary \[1.3.10\] there exists a natural twistor structure \( E^k \) on \( H^k(U, V) \) so that
\[ E^k|_{z=-1} \cong H^k(U, A^\bullet_U(H); -\partial' + \theta' + \partial'' - \theta''). \]

**Definition 1.4.5.** With the notation above. Let us fix the choice of a harmonic metric on \( H \). The pre-Weil operator \( \phi \) associated to \((U, \mathcal{V})\), at the level of de Rham complexes, is defined by
\[ \phi : (\mathcal{A}^\bullet_U(H); -\partial' + \theta' + \partial'' - \theta'') \xrightarrow{\iota_{-1}} (\mathcal{A}^\bullet_U(H); \partial' - \theta' + \partial'' - \theta'') \rightarrow \mathcal{V}^\ast. \]

The pre-Weil operator at the level of cohomology is the induced map
\[ \phi : E^k|_{z=-1} \xrightarrow{\iota_{-1}} H^k(U, A^\bullet_U(H); \partial' - \theta' + \partial'' - \theta'') \rightarrow H^k(U, \mathcal{V}^\ast), \]
where \( h \) is the map in Lemma \[1.4.3\] and \( \iota_{-1} \) is Sabbah’s rescaling map.

**Remark 1.4.6.** In the rest of paper, we also refer pre-Weil operators to these isomorphisms induced by \( \phi \) through perverse cohomology functors and smooth restrictions.

**Proposition 1.4.7.** Let \( X \) be a smooth projective variety and \( \mathcal{V} \) be a semisimple local system. Then the map
\[ C : \overline{H^k(X, \mathcal{V})} \cong \overline{E}|_{z=1} \xrightarrow{\text{Iden}} \overline{E}|_{z=-1} \xrightarrow{\phi} H^k(X, \mathcal{V}^\ast), \]
recovers the Weil operator for the pure Hodge structure on \( H^k(X, \mathbb{C}) \), sending \( \alpha^{p,q} \) to \((-1)^p \alpha^{p,q} \) for \( \alpha^{p,q} \in H^{p,q}(X, \mathbb{C}) \), where Iden is the Identification map in Definition \[1.2.3\].

**Proof.** When \( \mathcal{V} = \mathbb{C} \), the associated harmonic metric is the constant Hermitian metric. Therefore the map \( h \) in Lemma \[1.4.3\] is an identity under the identification \( \mathbb{C} \cong \mathbb{C}^\ast \). Then the corollary follows from Remark \[1.2.15\] and the construction of the rescaling map in Definition \[1.4.4\]. \( \square \)

**Lemma 1.4.8.** With the notation in Definition \[1.4.5\]. The pre-Weil operator \( \phi \) is functorial with respect to smooth restrictions and is compatible with cup products:
- if \( N \) is any line bundle on \( U \), there is a commutative diagram:
  \[
  \begin{array}{c}
  E^k|_{z=-1} \\
  \phi \downarrow \ \
  \overline{H^k(U, \mathcal{V}^\ast)} \\
  \downarrow \phi \\
  E^{k+2\ell}|_{z=-1} \\
  \end{array}
  \xrightarrow{\wedge c_1(N)^\ell} \overline{H^{k+2\ell}(U, \mathcal{V}^\ast)}
  \]
where \( F : E^k \rightarrow E^{k+2\ell} \) is the morphism of twistor structures induced by the cup product with \( c_1(N)^\ell \),
• for any smooth subvariety $T \subseteq U$, denote $E^k_T$ to be the associated twistor structure, then there is a commutative diagram:

$$
\begin{array}{ccc}
E^k|_{z=-1} & \xrightarrow{G|_{z=-1}} & E^k_T|_{z=-1} \\
\downarrow \phi & & \downarrow \phi_T \\
H^k(U, \mathcal{V}^*) & \xrightarrow{\bar{R}} & H^k(T, \mathcal{V}^*|_T)
\end{array}
$$

where $G : E^k \to E^k_T$ and $\bar{R}$ are restriction maps.

**Proof.** The first statement follows from the compatibility of $\phi$ with the cup product with $c_1(N)^t$ on the level of de Rham complex $\mathcal{A}^*_U(H)$.

For the second statement, denote by $(H, h)$ the harmonic bundle associated to the semisimple local system $\mathcal{V}$ on $U$. Consider the restriction $\mathcal{V}_T = \mathcal{V}|_T$ and $(H_T, h_T) = (H, h)|_T$. Because harmonic metrics are preserved under pullback (see [30, Page 18]), we know that $h_T$ is a harmonic metric on $H_T$. This induces the complex $(\mathcal{A}^*_T(H_T); \lambda \partial_T + \theta_T + \partial_T^* + \lambda \theta_T^*)$, with the differential operators $\partial_T$, etc., which are associated with the harmonic metric $h_T$. It is clear that they are equal to the restriction of $\partial'$, etc. to $T$. This implies that the pre-Weil operator

$$
\frac{H^k(U, \mathcal{A}^*_U(H); -\partial' + \theta' + \partial'' - \theta'')}{h_{\mathcal{V}}} \xrightarrow{\bar{R}} H^k(U, \mathcal{V}^*)
$$

is compatible with restriction to $T$.

It remains to prove that the isomorphism $E^k|_{z=-1} \cong H^k(U, \mathcal{A}^*_U(H); -\partial' + \theta' + \partial'' - \theta'')$ from [3] in Corollary 1.3.10 is also compatible with $T \hookrightarrow U$. To do this, we need to recall the construction of this isomorphism. Let $X$ be the smooth projective compactification of $U$ so that $\mathcal{V}$ is the restriction of a semisimple local system on $X$. As in the proof of Corollary 1.3.10, using sufficiently many blow-ups of $X$ which do not change $T$ and $U$, we can assume that we have the following diagram

$$
\begin{array}{ccc}
T & \xleftarrow{i_T} & U \\
\downarrow j_U & & \downarrow j_U \\
\overline{T} & \xleftarrow{i} & X
\end{array}
$$

where $X \setminus U$ and $\overline{T} \setminus T$ are simple normal crossing divisors in $X$ and $\overline{T}$, respectively and $\overline{T} \subseteq X$ is a smooth projective variety containing $T$.

Let $\mathcal{L}_X$ be the holomorphic family of local systems on $X^{\text{top}} \times G_m$ constructed by Simpson [33, Page 23-24], as a part of the triple $(\mathcal{F}_X, \mathcal{L}_X, \mathcal{F}_X')$ associated to the local system $\mathcal{V}_X$. Denote by $H_X$ the associated harmonic bundle on $X$ and $E^k_X$ the pure twistor structure on $H^k(X, \mathcal{V}_X)$. We denote by $(\_)|_\overline{T}$ for the corresponding objects of $\overline{T}$. Then $\mathcal{L}_\overline{T} = (i \times \text{id})^* \mathcal{L}_X$ and $H_{\overline{T}} = i^* H_X$. Lemma 1.3.7 and (2) imply that

$$
E^k_X|_{z=-1} = H^k(X, \mathcal{L}_X|_{X \times \{z\}^-}) \cong H^k(X, \mathcal{A}^*_X(H_X); -\partial_X + \partial'_X + \theta'_X - \theta''_X).
$$

The same holds for $\overline{T}$. The natural map $\mathcal{L}_X \to (i \times \text{id})_* \mathcal{L}_\overline{T}$ and $\mathcal{A}^*_X(H_X) \to i_* i^* \mathcal{A}^*_X(H_X) \cong_{\text{qiso}} i_* \mathcal{A}^*_\overline{T}(H_{\overline{T}})$ induce the restriction map $E^k_X \to E^k_{\overline{T}}$ and

$$
H^k(X, \mathcal{A}^*_X(H_X); -\partial_X + \partial'_X + \theta'_X - \theta''_X) \to H^k(\overline{T}, \mathcal{A}^*_\overline{T}(H_{\overline{T}}); -\partial_T + \partial''_T + \theta'_T - \theta''_T)
$$

so that they are compatible with the isomorphism (4) and the one for $\overline{T}$. 
Now note that the twistor structures $E^k_T$ and $E^k$ are induced by $L_T$ and $L_X$ so that

$$E^k_{|z=0} \cong H^k(X, (\xi_j U_* A^*_\nu) \otimes L_X)|_{X \times \{z_0\}}, \quad E^k_T_{|z=0} \cong H^k(T, (\xi_j T_* A^*_\nu) \otimes L_T|_{T \times \{z_0\}}).$$

This gives the desired compatibility with $T \Rightarrow U$, because all of them are induced by $L_X$ and all harmonic bundles $H_-$ (for $\_ = U, T, T$) are restrictions of $H_X$. This finishes the proof of the Lemma.

In the rest of this subsection, we work with Set-up 1.0.1 and assume $k \leq \dim X$.

**Definition 1.4.9.** We define the twisted Poincaré pairing $S$ to be the bilinear pairing

$$S : H^k(X, \mathcal{V}) \otimes \mathbb{C} H^k(X, \mathcal{V}^*) \to \mathbb{C}$$

$$[\alpha \otimes e] \otimes [\beta \otimes \lambda] \mapsto (-1)^{k(k-1)/2} \int_X \lambda(e) \cdot c_1(\eta)^{\dim X-k} \wedge \alpha \wedge \beta.$$

Here $\alpha, \beta$ are $k$-forms on $X$ and $e, \lambda$ are global sections of $H$ and $H^*$ respectively.

Let $E^k_{\text{prim}}$ be the natural pure sub-twistor structure on $H^k(X, \mathcal{V})_{\text{prim}}$ induced by the one on $H^k(X, \mathcal{V})$. We show that one can use $S$ and $\phi$ to polarize $E^k_{\text{prim}}$.

**Theorem 1.4.10.** The pre-Weil operator $\phi$ restricts to

$$\phi : E^k_{\text{prim}}_{|z=-1} \tilde{\to} H^k(X, \mathcal{V}^*)_{\text{prim}} := \text{Ker} \eta^{\dim X-k+1} \subseteq H^k(X, \mathcal{V}^*).$$

Moreover, $E^k_{\text{prim}}$ is polarized by the bilinear form

$$i^{-k} : S(\bullet, \phi(\bullet)) : E^k_{\text{prim}}_{|z=1} \otimes E^k_{\text{prim}}_{|z=-1} \frac{\text{Id} \otimes \phi}{1} H^k(X, \mathcal{V})_{\text{prim}} \otimes H^k(X, \mathcal{V}^*)_{\text{prim}} \overset{S}{\to} \mathbb{C}.$$ in the sense of Definition 1.2.8. Similarly, $i^{-(k-2m)}(-1)^{m(2m-k+1)} \cdot S(\bullet, \phi(\bullet))$ polarizes the natural pure twistor structure $F$ on the primitive space $\eta^m H^{k-2m}(X, \mathcal{V})_{\text{prim}}$, with the induced isomorphism $\phi : F_{|z=-1} \tilde{\to} \eta^m H^{k-2m}(X, \mathcal{V}^*)_{\text{prim}}$.

**Proof.** We will only deal with the case of $H^k(X, \mathcal{V})_{\text{prim}}$ and leave other cases to the reader. The statement on restriction of $\phi$ follows from Lemma 1.4.8. For any element in $H^k(X, \mathcal{V})$, consider its harmonic representative $\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}$, where $\alpha^{p,q}$ are $(p, q)$-forms and $m_{p,q}$ are sections of the associated harmonic bundle $H$. By Remark 1.2.15, we see that

$$\phi(\text{Iden} \left[ \sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q} \right]) = \phi(\left[ \sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q} \right]) = \sum_{p+q=k} (-1)^p \left[ \alpha^{p,q} \otimes m_{\nu}^\mathbb{V} \right]$$

where $m_{\nu}^\mathbb{V} \in C^\infty(H^*)$ is a section satisfying $m_{\nu}^\mathbb{V}(\bullet) = h(\bullet, \overline{m}_{p,q})$. Therefore,

$$i^{-k} : S \left( \left[ \sum \alpha^{p,q} \otimes m_{p,q} \right], \phi \circ \text{Iden} \left[ \sum \alpha^{p,q} \otimes m_{p,q} \right] \right)$$

$$= \sum_{p+q=k} i^{-k} : S \left( [\alpha^{p,q} \otimes m_{p,q}], ([(-1)^p \alpha^{p,q} \otimes m_{\nu}^\mathbb{V}] \right)$$

$$= \sum_{p+q=k} (-1)^p \cdot i^{-k}(-1)^{k(k-1)/2} \int_X h(m_{p,q}, \overline{m}_{p,q}) \cdot c_1(\eta)^{n-k} \wedge \alpha^{p,q} \wedge \overline{\alpha^{p,q}}$$

$$= \sum_{p+q=k} i^{p-q}(-1)^{k(k-1)/2} \int_X h(m_{p,q}, \overline{m}_{p,q}) \cdot c_1(\eta)^{n-k} \wedge \alpha^{p,q} \wedge \overline{\alpha^{p,q}} > 0.$$
The positivity follows from the classical calculation of the Hodge star operators for primitive forms [34, Theorem 6.29].

**Corollary 1.4.11.** Let \( T \subseteq X \) be a smooth subvariety. Consider the restrictions maps

\[
R : H^k(X, \mathcal{V})_{\text{prim}} \to H^k(T, \mathcal{V}|_T), \quad \tilde{R} : H^k(X, \mathcal{V}^*)_{\text{prim}} \to H^k(T, \mathcal{V}^*|_T).
\]

Then the twisted Poincaré pairing restricts to a non-degenerate pairing

\[
S : \ker R \otimes \ker \tilde{R} \to \mathbb{C}.
\]

**Proof.** It follows from Theorem 1.4.10, the compatibility Lemma 1.4.8 and Lemma 1.2.10. □

We conclude this subsection by reinterpreting the pre-Weil operator \( \phi \) via the Hodge star operator.

**Lemma 1.4.12.** Assume that \( \eta \) is an ample line bundle associated to a Kähler metric \( g \) on \( X \). Consider

\[
(\eta^{n-k})^{-1} \circ * : H^k(X, \mathcal{V})_{\text{prim}} \to H^{2n-k}(X, \mathcal{V}^*)_{\text{prim}} \to H^k(X, \mathcal{V}^*_{\text{prim}}),
\]

where \( * \) is the Hodge star operator in Definition [A.0.4]. Then this map can be identified up to a scaling constant with the composition map

\[
\phi \circ \text{Iden} : H^k(X, \mathcal{V})_{\text{prim}} = \mathcal{E}^k_{\text{prim}}|_{z=1} \to \mathcal{E}^k_{\text{prim}}|_{z=-1} \to H^k(X, \mathcal{V}^*_{\text{prim}}).
\]

**Proof.** Let \( \sum \alpha^{p,q} \otimes m_{p,q} \) be a primitive harmonic form, where \( \alpha^{p,q} \) is a form and \( m_{p,q} \) is a global section of \( H \). By Lemma [A.0.5] we have

\[
(\eta^{n-k})^{-1} \circ *(\alpha^{p,q} \otimes m_{p,q}) = \frac{(-1)^{k(k+1)/2} p-q}{(n-k)!} \alpha^{p,q} \otimes m^\vee_{p,q} = C \cdot (-1)^p \alpha^{p,q} \otimes m^\vee_{p,q}.
\]

Here \( C = \frac{(-1)^{k(k+1)/2} p-q}{(n-k)!} \) and \( m^\vee_{p,q} \) is the section of \( H^* \) so that \( m^\vee_{p,q}(\bullet) = h(\bullet, m_{p,q}) \). On the other hand, the proof of Theorem 1.4.10 implies that

\[
\phi(\text{Iden} \big[ \sum \alpha^{p,q} \otimes m_{p,q} \big]) = \sum (-1)^{k(k+1)/2} \alpha^{p,q} \otimes m^\vee_{p,q}.
\]

□

### 1.5. Compatibility with perverse filtrations.

In this subsection, we show that the pre-Weil operator \( \phi \) is compatible with perverse cohomology functors, using the geometric description found by de Cataldo and Migliorini [10]. Let \( f : V \to W \) be an algebraic morphism between quasi-projective varieties. Let \( K \) be a bounded complex with constructible cohomology sheaves on \( V \). Denote \( \mathcal{F}_\tau \leq \ell \) to be the perverse truncation functor.

**Notation 1.5.1.** We denote the perverse Leray filtration on \( H^b(V, K) \) by

\[
H^b_{\leq \ell}(V, K) := \text{Im} \{ H^b(W, \mathcal{F}_\tau \leq \ell f_* K) \to H^b(W, f_* K) \} \subseteq H^b(V, K).
\]

We also set

\[
H^b_{\ell}(V, K) := H^b_{\leq \ell}(V, K)/H^b_{\leq \ell-1}(V, K).
\]

The positivity follows from the classical calculation of the Hodge star operators for primitive forms [34, Theorem 6.29].
1.5.1. **Perverse filtrations via flag filtrations.** Let us recall the main results of [10]. Suppose that \( \dim \ W = k \) and let \( \ W \subseteq P^N \) be a fixed affine embedding.

**Definition 1.5.2.** A linear \( k \)-flag \( \mathfrak{F} \) on \( P^N \) is defined to be

\[
\mathfrak{F} := \{ P^N = \Lambda_0 \supseteq \Lambda_{-1} \supseteq \cdots \supseteq \Lambda_{-k} \},
\]

where \( \Lambda_{-\ell} \) is a codimension \( \ell \) linear subspace. A linear \( k \)-flag \( \mathfrak{F} \) is general, if it belongs to a suitable Zariski dense open subset of the corresponding flag variety parametrizing all such \( k \)-flags.

Let \( \mathfrak{F}^1 = \{ \Lambda^1 \}, \mathfrak{F}^2 = \{ \Lambda^2 \} \) be two, possibly identical, linear \( k \)-flags on \( P^N \). They induce two pre-image flags \( V_* , Z_* \) on \( V \):

\[
V = V_0 \supseteq V_{-1} \supseteq \cdots \supseteq V_{-k} \supseteq V_{-k-1} = \emptyset , \quad \text{with } V_{\ell} := f^{-1}(\Lambda^1_{\ell} \cap \ W).
\]

\[
V = Z_0 \supseteq Z_{-1} \supseteq \cdots \supseteq Z_{-k} \supseteq Z_{-k-1} = \emptyset , \quad \text{with } Z_{\ell} := f^{-1}(\Lambda^2_{\ell} \cap \ W).
\]

**Notation 1.5.3.** Let \( i : T \hookrightarrow V \) be a locally closed embedding. We use \( (-)_T := i_! i^*(-) \)

to denote the complex compactly supported on \( T \). We use \( H^*_T(-) \) to denote the local cohomology group with support in \( T \).

**Theorem 1.5.4** (Theorem 4.2.1 [10]). Given a pair of general flags \( \mathfrak{F}^1 , \mathfrak{F}^2 \), there is an isomorphism

\[
H^b_{\leq \ell}(V , K) \cong \text{Im} \left\{ \bigoplus_{j+i=b+\ell} H^b_{Z_{-j}}(V , K_{V \setminus V_i}) \to H^b(V , K) \right\}.
\]

In [10], the authors use the decreasing filtrations. We transform their results to increasing filtrations, which is consistent with the notations in this paper.

1.5.2. **Perverse filtrations are twistor-theoretic.** From now on, we work with Set-up [1.0.1] and a map \( f : X \to Y \). Set \( K = \mathcal{V}[\dim X] , K^* = \mathcal{V}^*[\dim X] \).

**Lemma 1.5.5.** For any integers \( \ell \) and \( b \), the subspace \( H^b_{\leq \ell}(X , K) \subseteq H^b(X , K) \) underlies a pure sub-twistor structure of the natural pure twistor structure on \( H^b(X , K) \) in Theorem [1.1.1]. Therefore the quotient space

\[
H^b_{\leq \ell}(X , K) := H^b_{\leq \ell}(X , K) / H^b_{\leq \ell-1}(X , K)
\]

inherits a pure twistor structure \( F \) of weight \( (b + \dim X) \). Moreover, the pre-Weil operator \( \phi \) in Definition [1.4.3] induces an isomorphism \( \phi : F|_{z=-1} \cong H^b_{\ell}(X , K^*) \). In other words, the pre-Weil operator \( \phi \) is compatible with perverse filtrations.

**Proof.** We use Theorem [1.5.4] to study \( H^b_{\leq \ell}(X , K) \). We use the notation from [1.5.1] with \( V = X \). For each \( j \), the local cohomology group \( H^b_{Z_{-j}}(X , K) \) fits into the long exact sequence associated to the closed and open embeddings \( Z_{-j} \overset{\Delta}{\hookrightarrow} X \overset{\pi}{\longrightarrow} U := X \setminus Z_{-j} :\)

\[
H^b_{Z_{-j}}(X , K) = H^b(X , A^1 \overset{\Delta^1}{\longrightarrow} K) \to H^b(X , K) \to H^b(U , B^* K).
\]
Claim: $H^b_{Z_{-j}}(X,K)$ underlies a mixed twistor structure $E_1$ with an isomorphism $\phi_{\text{loc}}$ so that the exact sequence (3) underlies an exact sequence of mixed twistor structures with the following commutative diagram:

$$
\begin{array}{c}
E_1|_{z=-1} \rightarrowtail E_2|_{z=-1} \twoheadrightarrow E_3|_{z=-1} \\
\downarrow_{\phi_{\text{loc}}} \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow_{\phi} \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow_{\phi_U} \\
H^b_{Z_{-j}}(X,K^*) \rightarrow H^b(X,K^*) \rightarrow H^b(U,B^*K^*)
\end{array}
$$

where $E_2$ and $E_3$ are natural twistor structures on $H^b(X,K)$ and $H^b(X,B_*B^*K)$, $\phi$ and $\phi_{X\setminus Z_{-j}}$ are the corresponding isomorphisms.

**Proof of claim.** Since $U = X \setminus Z_{-j}$ is smooth, the square involving $E_2$ and $E_3$ is commutative by Lemma 1.4.8. Recall that $H$ is the harmonic bundle associated to $V$. Define a complex

$$A^k_X(H)_{Z_{-j}} := \text{Cone}(A^k_X(H) \to A^k_U(H|_U))[-1].$$

We have

$$H^b_{Z_{-j}}(X,K) \cong H^b(X,A^k_X(H)_{Z_{-j}}).$$

Adapting the proof of Lemma 1.3.11, one can then put a natural mixed twistor structure on $H^b_{Z_{-j}}(X,K)$. Adapting the proof of Lemma 1.4.3 and applying the rescaling map $\iota_1$ in Definition 1.3.7, we obtain the isomorphism $\phi_{\text{loc}}$ with the desired commutativity. One can also use [15, Proposition III.2.1] for the construction of mixed twistor structures on local cohomology.

Observe that $K_{X \setminus X_i}$ fits into the distinguished triangle in $D^b_c(X)$ associated to the closed and open embeddings $X_i \hookrightarrow X \leftarrow X \setminus X_i$:

$$K_{X \setminus X_i} = \beta_1 \beta^* K = \beta_1 \alpha^* K \to K \to \alpha_* \alpha^* K \xrightarrow{[1]}.$$ 

By a similar argument as above, we can show that $H^b_{Z_{-j}}(X,K_{X \setminus X_i})$ underlies a mixed twistor structure $E_0$ so that there is an isomorphism $\phi_0 : E_0|_{z=-1} \cong H^b_{Z_{-j}}(X,K_{X \setminus X_i})$. Since $H^b(X,K)$ underlies a pure twistor structure, using Theorem 1.5.4 and [33, Lemma 1.3] (which says that the category of mixed twistor structures is abelian and any morphism is strict with respect to the weight filtrations), we conclude that the space $H^b_{Z_{-j}}(X,K)$ admits a pure sub-twistor structure $E_0$ so that the pre-Weil operator $\phi$ restricts to

$$\phi : E_0|_{z=-1} \cong \text{Im} \left\{ H^b_{Z_{-j}}(X,K^*_{X \setminus X_i}) \to H^b(X,K^*) \right\}.$$ 

The statement for $H^b_{Z_{-j}}(X,K)$ follows immediately.

We show that the pre-Weil operator $\phi$ on perverse filtrations is compatible with smooth restrictions.

**Corollary 1.5.6.** With the notation above and let $T$ be a smooth subvariety of $X$. Set $V_T = V|_T$, $K_T = V|_T[\dim T]$, and $K_T^* = V^*_T[\dim T]$. Let $H^b_{T}(T,K_T)$ denote the space associated to $f|_T : T \to Y$ as in Notation 1.5.4. Then we have

- $H^b_{T}(T,K_T)$ underlies a natural mixed twistor structure $F_T$, with an isomorphism $\phi_T : F_T|_{z=-1} \cong H^b_{T}(T,K_T^*)$. 
The restriction map

\[ R : H^b_\ell(X, K) \to H^b_\ell(T, K_T) \]

underlies a morphism of twistor structures \( G : F \to F_T \) and there is a commutative diagram:

\[
\begin{array}{ccc}
F|_{z=-1} & \xrightarrow{\partial|_{z=-1}} & F_T|_{z=-1} \\
\downarrow \phi & & \downarrow \phi_T \\
H^b_\ell(X, K^*) & \xrightarrow{\tilde{R}} & H^b_\ell(T, K^{*}_T)
\end{array}
\]

Proof. Apply Theorem 1.3.8, Lemma 1.4.8 and Lemma 1.5.5. \( \square \)

1.6. Weight filtrations. In this subsection, we discuss weight filtrations in the category of polarized pure twistor structures following [8, §4.5]. It is important for us to set everything up so that they are compatible with the pre-Weil operator \( \phi \). We only give proofs when necessary. The equation (9) in Corollary 1.6.5 is a new input and will play an important role in the proof of the Decomposition Theorem (via Theorem E, or more precisely Corollary 2.7.10).

1.6.1. Bilinear pairings. Let \( H \) and \( \tilde{H} \) be two finite dimensional vector spaces and let \( S : H \otimes \tilde{H} \to \mathbb{C} \) be a bilinear pairing. As in the case \( H = \tilde{H} \), one can also talk about notions of orthogonal complement, decompositions orthogonal with respect to \( S \) and non-degenerate pairings etc. For reader’s benefit, we record the following statement, which is used to prove Corollary 2.7.10 and is an important piece of the proof of Hodge-Riemann bilinear relation Theorem E.

**Lemma 1.6.1.** Let \( S : H \otimes \tilde{H} \to \mathbb{C} \) be a bilinear pairing. Assume there are direct sum decompositions orthogonal with respect to \( S \): \( H = H_1 \oplus H_2, \tilde{H} = \tilde{H}_1 \oplus \tilde{H}_2 \). Then the induced direct sum decomposition

\[
H/\tilde{H}^+ = H_1/(H_1 \cap \tilde{H}^+) \oplus H_2/(H_2 \cap \tilde{H}^+)
\]

is orthogonal with respect to the descent pairing \( \hat{S} : H/\tilde{H}^+ \otimes \tilde{H}/H^+ \to \mathbb{C} \). Moreover, the pairing

\[
\hat{S} : H_1/(H_1 \cap \tilde{H}^+) \otimes \tilde{H}_1/(\tilde{H}_1 \cap H^+) \to \mathbb{C}
\]

is non-degenerate. We have the same statement if one switches \( H \) with \( \tilde{H} \). We also have

\[
(\tilde{H}_2/(\tilde{H}_2 \cap H^+))^\perp = H_1/(H_1 \cap \tilde{H}^+),
\]

where the first orthogonal complement is taken with respect to \( \hat{S} \).

1.6.2. One weight filtration. Let \( (H, N) \) be a finite dimensional vector space \( H \) equipped with a nilpotent endomorphism \( N \). By [29, Lemma 6.4], there is a unique increasing filtration \( W \) so that we have \( NW_i \subseteq W_{i-2} \), hard Lefschetz isomorphism \( N^i : \text{Gr}^W_i H \cong \text{Gr}^W_{i-2} H \), and a Lefschetz decomposition

\[
\text{Gr}^W_i H = \bigoplus_{\ell \in \mathbb{Z}} N^{-i+\ell} P^{i-2\ell}, \quad i \in \mathbb{Z},
\]
where $P^{-i} := \text{Ker} N^{i+1} \subseteq \text{Gr}_i^W H$ for $i \geq 0$ and $P^{-i} := 0$ for $i < 0$. We use $W^N$ to denote the unique filtration and call it the weight filtration of $N$. For ease of notation, we set $\text{Gr}_i^N H := \text{Gr}_i^{W^N} H$.

**Definition 1.6.2** (Infinitesimal automorphisms). Consider two pairs $(H, N)$ and $(\tilde{H}, \tilde{N})$ so that $\dim H = \dim \tilde{H}$. We say that $(N, \tilde{N})$ are infinitesimal automorphisms of $(H, \tilde{H}, S)$ if there is a non-degenerate bilinear pairing

$$S : H \otimes \tilde{H} \to \mathbb{C},$$

which is either symmetric or skew-symmetric and satisfies

$$S(N a, \tilde{b}) + S(a, \tilde{N} \tilde{b}) = 0, \quad \forall a \in H, \tilde{b} \in \tilde{H}.$$

**Lemma 1.6.3.** One has

$$(W_i^N)^\perp = W_{-i}^{\tilde{N}}, \quad \forall i \in \mathbb{Z}.$$  

**Proof.** It is similar to the original case. For details, see [35, Lemma 2.3.107]. \qed

**Corollary 1.6.4.** The pairing (8) descends to non-degenerate pairings for $\ell \geq 0$:

$$S_\ell : \text{Gr}_\ell^N H \otimes \text{Gr}_\ell^{\tilde{N}} \tilde{H} \to \mathbb{C},$$

where $S_\ell([a], [\tilde{b}]) := S(a, \tilde{N}^\ell \tilde{b})$. Moreover, the Lefschetz decompositions (7) for $\text{Gr}_\ell^N H$ and $\text{Gr}_\ell^{\tilde{N}} \tilde{H}$ are orthogonal with respect to $S_\ell$.

**Corollary 1.6.5.** We have

$$(9) \quad \text{Ker} N \cap (\text{Ker} \tilde{N})^\perp = \text{Ker} N \cap W_{-1}^N,$$

and the pairing (8) descends to a non-degenerate pairing

$$S : \text{Ker} N / (\text{Ker} N \cap W_{-1}^N) \otimes \text{Ker} \tilde{N} / (\text{Ker} \tilde{N} \cap W_{-1}^{\tilde{N}}) \to \mathbb{C}.$$  

**Proof.** It suffices to show (9). On the one hand, the isomorphism $\tilde{N} : \text{Gr}_1^{\tilde{N}} H \cong \text{Gr}_{-1}^{\tilde{N}} H$ gives $\text{Ker} \tilde{N} \subseteq W_0^{\tilde{N}}$. Combining with Lemma 1.6.3, we see that

$$W_{-1}^N = (W_0^\tilde{N})^\perp \subseteq (\text{Ker} \tilde{N})^\perp.$$  

On the other hand, because $S$ is non-degenerate, we have $(\text{Ker} \tilde{N})^\perp = \text{Im} N$. Therefore

$$\text{Ker} N \cap (\text{Ker} \tilde{N})^\perp = \text{Ker} N \cap \text{Im} N \subseteq \text{Ker} N \cap W_{-1}^N.$$  

The last inclusion comes from the convolution formula for $W_{-1}^N$ (see [8, equation (23)]). \qed

1.6.3. Two weight filtrations. Let $H$ and $\tilde{H}$ be two vector spaces with a non-degenerate bilinear pairing $S : H \otimes \tilde{H} \to \mathbb{C}$. Let $N, M$ be two commuting nilpotent operators on $H$ and $\tilde{N}, \tilde{M}$ be two commuting nilpotent operators on $\tilde{H}$ such that $(N, \tilde{N})$ and $(M, \tilde{M})$ are infinitesimal automorphisms of $(H, \tilde{H}, S)$. Assume the shifted weight filtration $W^M[j]$ (with $W^M[j], = W^M_{j+i}$) induces the monodromy weight filtration of $M$ on $\text{gr}_j^N H$ for every $j \in \mathbb{Z}$ (see [8, §4.5]). Make the same assumption for $\tilde{M}$ and $\tilde{N}$. It means that

$$M^\ell : \text{Gr}_j^{M+\ell} \text{Gr}_j^N H \cong \text{Gr}_j^{M-\ell} \text{Gr}_j^N H,$$  

whenever $\ell \geq 0$.

For $\ell, j \geq 0$, define

$$P_{-\ell}^j := \text{Ker} M^{\ell+1} \cap \text{Ker} N^{j+1} \subseteq \text{Gr}_j^M \text{Gr}_j^N H.$$
Otherwise, set $P_{-i}^{-j} = 0$. Then we have the double Lefschetz decomposition:

$$\text{Gr}_{j+\ell}^M \text{Gr}_{j}^N H \cong \bigoplus_{i,m \in \mathbb{Z}} M^{-\ell+i} N^{-j+m} P_{-2i}, \quad \ell, j \in \mathbb{Z}. \tag{10}$$

We also have similar statements for $\tilde{M}$, $\tilde{N}$ and define $\tilde{P}_{-i}^{-j}$ to be the corresponding spaces for $\tilde{M}$ and $\tilde{N}$.

**Lemma 1.6.6.** The non-degenerate pairing $S_{\ell}$ in Corollary 1.6.4 descends to a non-degenerate pairing for $\ell, j \geq 0$:

$$S_{\ell,j} : \text{Gr}_{j+\ell}^M \text{Gr}_{j}^N H \otimes \text{Gr}_{j+\ell}^{\tilde{M}} \text{Gr}_{j}^{\tilde{N}} \tilde{H} \to \mathbb{C}, \tag{11}$$

where $S_{\ell,j}([a], [b]) = [S(a, \tilde{M}^\ell \tilde{N}^j b)]$ for any $\ell, j \geq 0$. Moreover, the double Lefschetz decompositions (10) for $\text{Gr}_{j+\ell}^M \text{Gr}_{j}^N H$ and $\text{Gr}_{j+\ell}^{\tilde{M}} \text{Gr}_{j}^{\tilde{N}} \tilde{H}$ are orthogonal with respect to $S_{\ell,j}$.

The next corollary will be used in Lemma 2.7.7, as a part of Theorem E.

**Corollary 1.6.7.** For each $\ell, j \geq 0$, consider the vector spaces

$$\text{Ker } N^{j+1} \subseteq \text{Gr}_{j+\ell}^M \text{Gr}_{j}^N H, \quad \text{Ker } \tilde{N}^{j+1} \subseteq \text{Gr}_{j+\ell}^{\tilde{M}} \text{Gr}_{j}^{\tilde{N}} \tilde{H}.$$

Then the pairing (11) restricts to a non-degenerate pairing:

$$S_{\ell,j} : \text{Ker } N^{j+1} \otimes \text{Ker } \tilde{N}^{j+1} \to \mathbb{C}.$$

**Proof.** We can use (10) and the commutability between $M$ and $N$ to obtain decompositions

$$\text{Ker } N^{j+1} = \bigoplus_{i \in \mathbb{Z}} M^{-\ell+i} P_{-2i}, \quad \text{Ker } \tilde{N}^{j+1} = \bigoplus_{i \in \mathbb{Z}} \tilde{M}^{-\ell+i} \tilde{P}_{-2i},$$

which are orthogonal with respect to $S_{\ell,j}$ by Lemma 1.6.6. Then the non-degeneracy follows. □

1.6.4. **Filtrations on $H^*(X, \mathcal{V})$.** Now we apply the previous discussion to the cohomology of local systems. We work with Set-up 1.0.1. Let $f : X \to Y$ be a map between smooth projective varieties, let $A$ be an ample line bundle on $Y$, and set $L := f^* A$. Denote $K := \mathcal{V}[\dim X]$. Consider the two vector spaces

$$H = H^*(X, \mathcal{V}) := \bigoplus_{j \in \mathbb{Z}} H^j(X, \mathcal{V}), \quad \tilde{H} = H^*(X, \mathcal{V}^*) := \bigoplus_{j \in \mathbb{Z}} H^j(X, \mathcal{V}^*).$$

Since $X$ is compact, there is a non-degenerate Poincaré pairing

$$S : H \otimes \tilde{H} \to \mathbb{C}, \tag{12}$$

$$S(\sum e_k \otimes \alpha_k, \sum \lambda_k \otimes \beta_k) := \sum_k C(k) \int_X \lambda_{2n-k}(e_k) \cdot \alpha_k \land \beta_{2n-k},$$

where $e_k$ and $\lambda_k$ are global sections of the harmonic bundle $H$ and dual harmonic bundle $H^*$ respectively, and $\alpha_k, \beta_k$ are $k$-forms on $X$, $C(k) = (-1)^{k(k-1)/2}$.

Let $\eta$ and $L$ denote the commuting nilpotent operators on $H$, induced by the cup product maps. To keep track the filtrations on $\tilde{H}$, we use $\tilde{\eta}$ and $\tilde{L}$ to denote the corresponding
operators on $\tilde{H}$. It is direct to check that $(\eta, \tilde{\eta})$ and $(L, \tilde{L})$ are infinitesimal automorphisms for $(H, \tilde{H}, S)$:

$$S(\eta a, b) + S(a, \tilde{\eta} b) = 0, \quad S(L a, b) + S(a, \tilde{L} b) = 0.$$  

Theorem 1.1.1 implies that the weight filtrations of $W^\eta$ on $H$ is

$$W^\eta_i = \bigoplus_{\ell \geq n-i} H^\ell(X, V) = \bigoplus_{b \geq -i} H^b(X, K).$$

Consider the total filtration on $H$ (see Notation 1.5.1):

$$W^\text{tot}_j := \bigoplus_{b \in \mathbb{Z}} H^b(X, K)_{b \leq b+j}.$$  

Let $W^\tilde{\eta}$ and $W^\text{tot}$ denote the corresponding filtrations on $\tilde{H}$. Then we have for all $\ell, j \in \mathbb{Z}$:

$$H_{-\ell}^{-\ell-j}(X, K) = \text{Gr}_{j+\ell}^W H, \quad H_{-\ell}^{-\ell-j}(X, K^*) = \text{Gr}_{j+\ell}^{\tilde{\eta}} \text{Gr}_{j}^{\text{tot}} \tilde{H}.$$  

As in [8], we will show $W^L = W^\text{tot}$ in Corollary 2.7.3.

Remark 1.6.8. By Theorem 1.4.10 we see that $i^{-(n-\ell-j)} S_{\ell j}(\bullet, \phi(\bullet))$ polarizes the pure twistor structure on $P_{-\ell}^{-\ell-j} = \text{Ker} \eta^{j+1} \cap \text{Ker} L^{j+1}$. Therefore, we can get the appropriate signs for other primitive pieces in the double Lefschetz decomposition, as in [8, Remark 4.5.2].

2. An application to the Decomposition Theorem

In this section, building on the results developed in the previous sections, we give a new proof of Sabbah’s Decomposition Theorem C. Here is the main set-up of this section.

Set-up 2.0.1. -

- Let $f: X \to Y$ be a morphism between projective varieties, where $X$ is smooth of dimension $n$ and $\dim f(X) = m$. Let $\eta$ be an ample line bundle on $X$, let $A$ be an ample line bundle on $Y$, and set $L := f^* A$.
- Let $\mathcal{V}$ be a semisimple local system on $X$ and denote $\mathcal{V}^*$ to be the dual local system. Set $K := \mathcal{V}[^{\dim X}], K^* := \mathcal{V}[^{\dim X}]$ to be the associated perverse sheaves on $X$.

2.1. Outline of the proof. Since the general strategy is quite close to [8 §2.6], we may skip some details and leave them to the reader. But we will elaborate on extra difficulties and give more details. We prove Theorem C, Theorem D and Theorem E by double induction on the defect of semismallness $r = r(f)$ and $m = \dim f(X)$. The basic reason for double induction is that in the proof of Theorem D and Theorem E there are two different ways of cutting hyperplanes. Cutting on $X$ gives $r' < r$ and cutting on $Y$ gives $r' \leq r$ and $m' < m$.

Step 1 is similar to the original strategy and Step 2-4 consist of additional difficulties, this is where the pre-Weil operator $\phi$ comes in to overcome the difficulty.

Step 1 By Deligne’s Lefschetz splitting criterion and the construction of universal hyperplanes, it suffices to prove Theorem C(i) and Theorem C(iii) and the key step is to prove Theorem C(iii) for $\ell = 0$. To achieve this, we first prove Theorem D and Theorem E by induction and we elaborate this in Step 2 and 3.

Step 2 Theorem C(i) and the inductive Theorem E imply Theorem D This needs the new input about the compatibility of the pre-Weil operator $\phi$ and perverse filtration (Lemma 1.5.5, Lemma 1.4.8 and Corollary 1.5.6).
Step 3 To prove Theorem E, we need the setup of weight filtrations on two companion vector spaces from §1.6. This is the most difficult part and our argument is not the same as the original proof. Then by induction, one can reduce it to the case of the constant map (Theorem A).

Step 4 We can complete the proof the Semisimplicity Theorem C(iii) using the splitting criterion of de Cataldo-Migliorini. To apply the criterion, we need Lemma 2.4.3 to relate the adjunction morphism with the twisted Poincaré pairing and then apply the positivity coming from the polarization induced by the twisted Poincaré pairing and the pre-Weil operator $\phi$. The semisimplicity of local systems over strata are deduced from Simpson’s theorem in the case of smooth projective maps (see Corollary 1.1.3).

2.2. The cup product with a line bundle. We need a duality result on cup products. Let $X$ be a projective variety and let $\eta$ be a line bundle on $X$. Let $K$ be a constructible complex of $\mathbb{C}$-vector spaces on $X$. Let $D_X$ denote the Verdier dual functor and set $K^* := D_X(K)$. The first Chern class of $\eta$ corresponds to an element in $H^2(X, \mathbb{C}) \cong \text{Hom}_{D^b(X)}(\mathbb{C}_X, \mathbb{C}_X[2])$.

Definition 2.2.1. The cup product map $\eta : K \to K[2]$ is defined by

$$K \cong K \otimes \mathbb{C}_X \xrightarrow{\text{Id} \otimes \eta} K \otimes \mathbb{C}_X[2] \cong K[2].$$

Lemma 2.2.2. The dual $D_X(\eta)$ of the morphism $\eta : K \to K[2]$ is isomorphic to the morphism $\eta : (K^* \to K^*[2])[-2]$.

Proof. It follows from [8, Remark 4.4.1] and the compatibility of functors involved in the description of $K \to K[2]$ with the Verdier dual functor. For details, see [35, Lemma 2.5.2].

Corollary 2.2.3. Suppose we are in the Set-up 2.0.1 then the dual $D_Y(\eta^f)$ of the morphism $\eta^f : \mathcal{H}^{-\ell} f_\ast K \to \mathcal{H}^\ell f_\ast K$ is isomorphic to the morphism $\eta^f : \mathcal{H}^{-\ell} f_\ast K^* \to \mathcal{H}^\ell f_\ast K^*$.

2.3. Weak-Lefschetz-type results. To run the inductive proof of Theorem C via cutting with hyperplanes, let us recall results from [8, §4.7, §5.2, §5.3] enriched by pure twistor structures and the pre-Weil operator $\phi$.

Proposition 2.3.1. There exists $m_0$ so that for any $m \geq m_0$, the following statements hold: if $i : X^1 \to X$ is a general hyperplane section of $|\eta^{\otimes m}|$ and $f^1 : X^1 \to Y$ is the restricted map, set $K^1 := i^\ast K[-1]$, then

- $r(f^1) \leq \max\{r(f) - 1, 0\}$.
- The restriction map $\mathcal{H}^{-\ell} f_\ast K \to \mathcal{H}^{-\ell+1}(f^1_\ast K)$ is iso for $\ell \geq 2$ and mono for $\ell = 1$.
- Assume Theorem C(ii) holds for $f$, then there is an injective morphism

$$i^* : P_{-\ell}^j(X, K) \to P_{-\ell+1}^j(X^1, K^1), \quad \ell \geq 1, j \geq 0,$$

which underlies a morphism of pure twistor structures and are compatible with the pre-Weil operator $\phi$.

Proof. For the second statement, we need to apply [8, Lemma 3.5.4(b)] to get $i^* K[-1] = i^! K[1]$. The statement on twistor structures follows from Corollary 1.5.6 (functoriality with restriction) and Lemma 1.4.8 (compatibility with cup products).
Since \( f : X \to Y \) is an algebraic map between algebraic varieties, the algebraic version of Thom isotopy lemmas (for example see [8] Theorem 3.2.3) imply that there exist finite algebraic Whitney stratifications \( \mathcal{X} \) of \( X \) and \( \mathcal{Y} \) of \( Y \) such that 1) given any connected component \( S \) of a \( \mathcal{Y} \) stratum \( S_t \) on \( Y \), then \( f^{-1}(S) \) is an union of connected components of strata of \( \mathcal{X} \), each of which is mapping submersively to \( S \), 2) \( \forall y \in S \), there exists an euclidean open neighborhood \( U \) of \( y \) in \( S \) and a stratum-preserving homeomorphism \( h : U \times f^{-1}(y) \to f^{-1}(U) \) such that \( f \circ h \) is the projection to \( U \). For the next statement, let us fix a choice of such stratifications on \( X \) and \( Y \).

Applying Bertini Theorem to linear systems \( |A| \) and \( f^*|A| \) simultaneously, we can choose a general section \( Y_1 \in |A| \) such that 1) \( Y_1 \) is transversal to all positive-dimensional strata of \( Y \) and avoids the 0-dimensional strata, 2) \( X_1 = f^{-1}(Y_1) \) is smooth.

**Proposition 2.3.2.** With the notation above. Let \( f_1 : X_1 \to Y_1 \) denote the restriction of \( f \), let \( i : X_1 \to X \) denote the inclusion map and set \( K_1 := i^*K[-1] \). Then

- \( r(f_1) \leq r(f) \).
- the restriction map \( i^* : H^{-j}(Y, \mathcal{H}^0(f_1K_1)) \to H^{-j+1}(Y_1, \mathcal{H}^0(f_1K_1)) \) is an isomorphism for \( j \geq 2 \) and an injection for \( j = 1 \),
- the Gysin pushforward \( i_! : H^j(Y_1, \mathcal{H}^0(f_1K_1)) \to H^{-j}(Y, \mathcal{H}^0(f_1K)) \) is an isomorphism for \( j \geq 2 \) and a surjection for \( j = 1 \),
- assuming Theorem C(iii) holds for \( f \), there is an injective morphism

\[
i^*(P_0^{-j}(X, K)) \subseteq P_0^{-j+1}(X_1, K_1), \quad j \geq 1,
\]

which is an equality for \( j \geq 2 \).

Moreover, all morphisms underlie morphisms of pure twistor structures and are compatible with the pre-Weil operator \( \phi \).

**Proof.** The proposition can be proved using [8] Lemma 4.7.6 and Proposition 4.7.7 as in the proof of [8] Lemma 5.3.1. The compatibility with \( \phi \) follows from Corollary 1.5.6 and Lemma 1.4.8. \( \square \)

### 2.4. Polarization and splitting criterions.

In this section, we relate adjunction morphisms with Poincaré-type pairings. Let \( T \) be a closed subset of \( Y \). We use \( T \to Y \leftarrow Y \setminus T \) to denote the closed and open embedding. For any perverse sheaf \( P \) on \( Y \), we have two distinguished triangles

\[
\alpha_1 \alpha^!P \to P \to \beta_\alpha \beta^* P \overset{[1]}{\to}, \quad \beta_1 \beta^!P \to P \to \alpha_\beta \alpha^* P \overset{[1]}{\to}.
\]

They induce a map of complexes \( \alpha_1 \alpha^!P \to P \to \alpha_\beta \alpha^* P \), whose cohomology gives a map

\[
H^k(Y, \alpha_1 \alpha^!P) \to H^k(Y, \alpha_\beta \alpha^* P), \quad \forall k \in \mathbb{Z}.
\]

This map induces various maps in the splitting criterions of de Cataldo-Migliorini [8] Lemma 4.1.3 and of MacPherson-Vilonen [23] (see [9] Remark 5.7.5). We interpret the map (16) and other related maps via various Poincaré-type pairings. This should be known to experts, but we would like to include the discussion here because we cannot find sufficient references in the literature.
First, recall the Poincaré pairing from (12):

\begin{equation}
S: H^k(X, \mathcal{V}) \otimes H^{2n-k}(X, \mathcal{V}^*) \to \mathbb{C},
\end{equation}

where \( A, B \) are forms with coefficients in \( \mathcal{V} \) and \( \mathcal{V}^* \). For \( k = n \), the pairing induces

\begin{equation}
S: H^0(X, K) \to H^0(X, K^*)^\vee, \quad A \mapsto (B \mapsto S(A, B)),
\end{equation}

where \( K = \mathcal{V}[n] \). Let \( D_Z \) denote the Verdier dual functor on a space \( Z \) and let \( p_X : X \to \text{pt} \) denote the constant map. Since \( X \) is proper, there is a canonical equivalence of functors \( (p_X)_* \cong D_{\text{pt}} \circ (p_X)_* \circ D_X \). Applying to \( K \), we get the following map

\begin{equation}
V: H^0(X, K) \to H^0(X, K^*)^\vee.
\end{equation}

**Lemma 2.4.1.** The map \( V \) coincides with the map \( S \), up to a sign \( C(n) = (-1)^{n(n-1)/2} \).

**Proof.** When \( \mathcal{V} \) is the constant local system \( \mathbb{C}_X \), this is mentioned in [22, §3.1]. To simplify the notation, we denote by \( p \) the constant map \( p_X : X \to \text{pt} \). Let us denote by \( \omega_X^* = p^!\mathbb{C}_{\text{pt}} \) the dualizing complex on \( X \) so that \( D_X(-) = R\mathcal{H}om(-, \omega_X^*) \).

First let us recall that \( p_* \circ D_X \sim D_{\text{pt}} \circ p_* \circ D_X \) is the existence of an equivalence of functors

\begin{equation}
p_* R\mathcal{H}om(-, p^!\mathbb{C}_{\text{pt}}) \sim R\mathcal{H}om(p_*(-), \mathbb{C}_{\text{pt}}),
\end{equation}

which induces the isomorphism \( p_* \cong D_{\text{pt}} \circ p_* \circ D_X \) and thus the map \( V \). By the adjunction map \( p_* p^! \to \text{Id} \), the equivalence above is defined by the composition of morphisms

\begin{equation}
p_* R\mathcal{H}om(-, \omega_X^*) \xrightarrow{p_*} R\mathcal{H}om(p_*(-), p_* \omega_X^*) \to R\mathcal{H}om(p_*(-), \mathbb{C}_{\text{pt}}),
\end{equation}

where first arrow is induced by \( p_* \).

Now, let us take a closer look at the first arrow \( p_* \) in (19). Let \( K \) be any constructible complex of \( \mathbb{C} \)-vector spaces on \( X \). The tensor-hom adjunction gives a natural isomorphism

\begin{equation}
\text{Hom}(p_* R\mathcal{H}om(K, \omega_X^*), R\mathcal{H}om(p_* K, p_* \omega_X^*)) \sim \text{Hom}(p_* R\mathcal{H}om(K, \omega_X^*) \otimes p_* K, p_* \omega_X^*).
\end{equation}

By adjunction, the first arrow \( p_* \) in (19) is transformed to the composition of morphisms

\begin{equation}
p_* R\mathcal{H}om(K, \omega_X^*) \otimes p_* K \xrightarrow{p_* \otimes \text{Id}} R\mathcal{H}om(p_* K, p_* \omega_X^*) \otimes p_* K \to p_* \omega_X^*,
\end{equation}

and the second arrow is induced by the natural morphism \( R\mathcal{H}om(A, B) \xrightarrow{L} A \to B \) for complexes \( A, B \). Using the projection formula, one can show that the morphism (20) can also be factorized as

\begin{equation}
p_* R\mathcal{H}om(K, \omega_X^*) \otimes p_* K \to p_*(R\mathcal{H}om(K, \omega_X^* \otimes K)) \to p_* \omega_X^*,
\end{equation}

where the first map is induced by the natural transformation \( p_*(-) \otimes p_*(-) \to p_*(- \otimes -) \).

Note that the second arrow in (19) is induced by \( p_* \omega_X^* \to \mathbb{C}_{\text{pt}} \). In our case, it takes a special form and relates to integration on \( X \). Since \( X \) is an orientable manifold, there exists an isomorphism

\[ \mathbb{C}_X[2n] \sim \omega_X^* = p^!\mathbb{C}_{\text{pt}}. \]

By the adjunction map \( p_* p^! \to \text{Id} \), this induces a map

\[ p_* \mathbb{C}_X[2n] \to p_* \omega_X^* \to \mathbb{C}_{\text{pt}}. \]
Since the first term above sits in degrees \([-2n, 0]\), there is a factorization

\[(21) \quad p_* C_X[2n] \to R^{2n} p_* C_X \to p_* C_{pt},\]

where the second map is induced by the isomorphism \(H^{2n}(X, \mathbb{C}) \cong \mathbb{C}\) coming from the orientation of \(X\). In particular, we can understand \(p_* \omega_X^* \to \mathbb{C}_{pt}\) using (21).

Summarizing the discussion above, if we plug \(K = \mathcal{V}[n]\) into (19), then it induces the following map

\[(22) \quad p_* R\text{Hom}(K, \omega_X^*) \otimes p_*(K) \to p_* (R\text{Hom}(K, \omega_X^*) \otimes K) \to p_* \omega_X^* \to p_* C_X[2n] \to \mathbb{C}_{pt}.\]

The last two maps come from (21) and \(\omega_X^* \cong \mathcal{C}[2n]\). Moreover, by taking the 0-th cohomology, the fact that the map \(V\) being isomorphism is equivalent to the statement that the induced pairing

\[(23) \quad H^0(X, K^*) \otimes H^0(X, K) \to H^0(X, K^* \otimes K) \to H^0(X, \omega_X^*) \cong H^{2n}(X, \mathbb{C}) \to \mathbb{C}\]

is a perfect pairing, so that the map \(V\) is induced by (23). Here because \(X\) is an orientable manifold, we can choose the last isomorphism to be induced by the integration \(\omega \mapsto \int_X \omega\), for any top degree form \(\omega\).

Note that we have \(K = \mathcal{V}[n], K^* \cong \mathcal{V}^*[n]\), and thus \(K \otimes K^* \cong \mathcal{V} \otimes \mathcal{V}^*[2n]\). Then the map (23) coincides with the pairing

\[(24) \quad H^n(X, \mathcal{V}) \otimes H^n(X, \mathcal{V}^*) \to H^{2n}(X, \mathcal{V} \otimes \mathcal{V}^*) \to H^{2n}(X, \mathbb{C}) \to \mathbb{C},\]

where the first map is the cup product map, the second map is induced by the natural map \(\mathcal{V} \otimes R\text{Hom}(\mathcal{V}, C_X) \to C_X\). Since we can represent each cohomology class in \(H^n(X, \mathcal{V})\) by forms with coefficients in the vector bundle \(H = \mathcal{V} \otimes C_X^\infty\) (same for \(\mathcal{V}^*\), it is then clear that the pairing (24) is equal to the Poincaré pairing (17), up to a sign \(C(n) = (-1)^{n(n-1)/2}\).

Therefore we conclude that the map \(V\) coincides with the map \(S\), up to a sign \((-1)^{n(n-1)/2}\). \(

\[
\square
\]

Now, we extend Lemma 2.4.1 to perverse filtrations on \(H^0(X, K)\). Let \(\alpha : \{y\} \to Y\) be a point in \(Y\). The adjunction map \(\alpha_! \alpha^! f_* K \to f_* K\) induces a cycle map

\[\text{cl} : H^0(Y, \alpha_! \alpha^! f_* K) \to H^0(Y, f_* K) = H^0(X, K).\]

We denote the corresponding cycle map for \(K^*\) by \(\tilde{\text{cl}}\) (instead of \(\text{cl}^*\), which may get confused with the pullback of \(\text{cl}\)). They induce the following pairing

\[(25) \quad S_y : H^0(Y, \alpha_! \alpha^! f_* K) \otimes H^0(Y, \alpha_! \alpha^! f_* K^*) \to \mathbb{C}, \quad A \otimes B \mapsto S(\text{cl}(A), \tilde{\text{cl}}(B)).\]

Let \(S_y\) also denote the induced map between vector spaces. Using the canonical isomorphisms

\[(26) \quad D_Y \circ \alpha_! \cong \alpha_* \circ D_Y, \quad D_Y \circ \alpha^! \cong \alpha^* \circ D_Y, \quad D_Y \circ f_* \cong f_* \circ D_X,\]

we have the following isomorphism

\[(27) \quad \text{V.D.} : H^0(Y, \alpha_! \alpha^! f_* K^*) \to H^0(Y, \alpha_! \alpha^! f_* K^*) \to H^0(Y, \alpha_* \alpha^* f_* K).\]

**Lemma 2.4.2.** The map

\[H^0(Y, \alpha_! \alpha^! f_* K) \xrightarrow{S_y} H^0(Y, \alpha_! \alpha^! f_* K^*) \xrightarrow{\text{V.D.}} H^0(Y, \alpha_* \alpha^* f_* K)\]
coincides with the map from \[16\]
\[ H^0(Y, \alpha \alpha^1 f_s K) \rightarrow H^0(Y, \alpha_s \alpha^* f_s K), \]
up to a sign.

**Proof.** The map \( \alpha \alpha^1 f_s K \rightarrow \alpha_s \alpha^* f_s K \) can be decomposed as
\[ \alpha \alpha^1 f_s K \rightarrow f_s K \sim \rightarrow D_Y \circ D_Y (f_s K) \sim \rightarrow D_Y (f_s K^*) \rightarrow D_Y (\alpha \alpha^1 f_s K^*) \sim \rightarrow \alpha_s \alpha^* f_s K. \]
The corresponding map on hypercohomology can be decomposed as
\[ H^0(\alpha \alpha^1 f_s K) \xrightarrow{\cl} H^0(Y, f_s K) = H^0(X, K) \xrightarrow{V} H^0(X, K^*)^\vee \]
\[ = H^0(Y, f_s K^*)^\vee \xrightarrow{(\tilde{\cl})^\vee} H^0(Y, \alpha \alpha^1 f_s K^*)^\vee \xrightarrow{\V.D.} H^0(Y, \alpha_s \alpha^* f_s K). \]
Using Lemma 2.4.1 for any \( A \in H^0(Y, \alpha \alpha^1 f_s K) \) and \( B \in H^0(Y, \alpha \alpha^1 f_s K^*) \), we have
\[ ( (\tilde{\cl})^\vee \circ V \circ \cl ) (A) : B \mapsto C(n) \cdot S(\cl(A), \tilde{\cl}(B)), \]
which is equal to the map \( S_\eta \) induced by \[25\], up to a sign. \( \square \)

Lastly, let us explain the adaption of Lemma 2.4.2 to \( \wp H^0 f_s K \). Assume Theorem \[ C \](ii) holds for \( f_s K \), and we have the induced cycle map
\[ \cl : H^0(Y, \alpha \alpha^p \wp H^0 f_s K) \rightarrow H^0(Y, \wp H^0 f_s K). \]
Similarly we use \( \tilde{\cl} \) to denote the cycle map of \( K^* \). The pairing \[25\] induces a pairing
\[ S_{00}^{\eta L} : H^0(Y, \alpha \alpha^p \wp H^0 f_s K) \otimes H^0(Y, \alpha \alpha^p \wp H^0 f_s K^*) \rightarrow C, \quad A \otimes B \mapsto S(\cl(A), \tilde{\cl}(B)). \]
Using the canonical isomorphisms from \[26\] and
\[ D_Y \circ \wp \tau_{\leq 0} \cong \wp \tau_{\geq 0} \circ D_Y, \quad D_Y \circ \wp \tau_{\geq 0} \cong \wp \tau_{\leq 0} \circ D_Y, \quad D_Y \circ \wp \mathcal{H}^k \cong \wp \mathcal{H}^{-k} \circ D_Y, \]
we have the following map as in \[27\]:
\[ (28) \quad \V.D. : H^0(Y, \alpha \alpha^p \wp H^0 f_s K^*)^\vee \rightarrow H^0(Y, \alpha_s \alpha^* \wp H^0 f_s K). \]

**Lemma 2.4.3.** Assume Theorem \[ C \](ii) holds for \( f_s K \), then the map
\[ H^0(Y, \alpha \alpha^p \wp H^0 f_s K) \xrightarrow{S_{00}^{\eta L}} H^0(Y, \alpha \alpha^p \wp H^0 f_s K^*)^\vee \xrightarrow{\V.D.} H^0(Y, \alpha_s \alpha^* \wp H^0 f_s K) \]
coinsides with the map from \[16\]
\[ H^0(Y, \alpha \alpha^p \wp H^0 f_s K) \rightarrow H^0(Y, \alpha_s \alpha^* \wp H^0 f_s K), \]
where \( k = 0 \) and \( P = \wp H^0 f_s K \), up to a sign.

**Proof.** Note that the map \[16\] \( \alpha \alpha^p \tau_{\leq 0} f_s K \rightarrow \alpha_s \alpha^* \wp \tau_{\leq 0} f_s K \) can be decomposed as
\[ \alpha \alpha^p \tau_{\leq 0} f_s K \rightarrow \wp \tau_{\leq 0} f_s K \sim \rightarrow D_Y \circ D_Y (\wp \tau_{\leq 0} f_s K) \sim \]
\[ D_Y (\wp \tau_{\geq 0} f_s K^*) \rightarrow D_Y (\alpha \alpha^p \tau_{\geq 0} f_s K^*) \sim \rightarrow \alpha_s \alpha^* \wp \tau_{\leq 0} f_s K. \]
In addition, the map from \[16\] for \( f_s (K) \) and \( \wp H^0 f_s K \) are compatible in the sense that one can represent an element in \( H^0(Y, \alpha \alpha^p \wp H^0 f_s K) \) using its lifting to \( H^0(Y, \alpha \alpha^1 f_s K) \). Since the bilinear pairing \( S_{00}^{\eta L} \) is defined in the same way, one proceeds similarly as in the proof of Lemma 2.4.2. Again, it is important to notice that these two maps coincide up to a sign. \( \square \)
2.5. **Set up of the proof.** Let us fix two finite algebraic Whitney stratifications $\mathfrak{X}$ on $X$ and $\mathfrak{Y}$ on $Y$ adapted to the map $f$, so that all perverse sheaves we work with are constructible with respect to these stratifications. For precise definitions, the reader can consult [8 §6.1].

For the base case, we start from the case $m = 0$ and arbitrary $r$. Then all the statements follow from Hodge-Simpson Theorem [14.4.1] and Theorem A.

**Assumption 2.5.1.** Let $r = r(f) \geq 0$ and $m > 0$. Assume that the results of Theorem [C] and Theorem [D] hold for every projective map $g : X \to Y$ and $\mathcal{V}$, where either $r(g) < r$, or $\dim g(X) < m$ and $r(g) \leq r$.

We will prove that if we are in Assumption 2.5.1 then all three Theorems hold for $f : X \to Y$ and $\mathcal{V}$ with $r(f) = r$ and $\dim f(X) = m$.

2.6. **Proof of Theorem [C], except the Semisimplicity Theorem for $\ell = 0$.** There are two cases. If $r(f) = 0$, then $f$ is semismall. Since the proof of [20] Prop 8.2.30 also works for any local system, we conclude that $f_*K = {}^pH^0(f_*K)$ is a perverse sheaf. In particular, Theorem [C] except for Theorem [C](iii) $\ell = 0$ automatically hold.

If $r(f) > 0$, choose an integer $k$ large enough so that $\eta^{\otimes k}$ is very ample, and fix the projective embedding $X \subseteq P$ induced by the linear system $|\eta^{\otimes k}|$. Consider the following commutative diagram from [8 §4.7]:

\[
\begin{array}{ccc}
X & \xleftarrow{p_X} & \mathcal{X} \\
\downarrow & & \downarrow g \\
Y & \xleftarrow{p_Y} & \mathcal{Y}
\end{array}
\]

Here $\mathcal{Y} = Y \times P^\vee$, $\mathcal{X} = \{(x,h) : h(x) = 0\} \subseteq X \times P^\vee$, the map $g$ is defined by $g(x,h) = (f(x), h)$ and the two horizontal maps $p_X$ and $p_Y$ are projections to the first factors. Let $d = \dim P$ and $M = p_X^*\mathcal{V}[\dim \mathcal{X}]$. By Corollary 1.2.12, $p_X^*\mathcal{V}$ is a semisimple local system on $\mathcal{X}$. Since we know that $r(g) < r(f)$ by [8 Lemma 4.7.4], we can apply the inductive assumption on $g$ to proceed as follows. For Theorem [C](i), since the functor $p_Y^*[d]$ is fully-faithful, it suffices to show that

$$p_Y^*(\eta)[d] : p_Y^*({}^pH^{\ell}(f_*K))[d] \to p_Y^*({}^pH^\ell(f_*K))[d]$$

is an isomorphism. There are two cases.

**Case I:** $\ell \geq 2$. It follows by induction on $g$ and [8 Proposition 4.7.8(i)]. Note that we actually need Theorem [C](i) to hold for $f$-ample line bundles, but this can be deduced from the ample line bundle case as in [8 Remark 5.1.2].

**Case II:** $\ell = 1$. We need some additional care. The cup product can be factored as

$$p_Y^*(\eta)[d] : p_Y^*({}^pH^{-1}(f_*K))[d] \to {}^pH^0(g_*M) \to p_Y^*({}^pH^1(f_*K))[d].$$

Since $^pH^0(g_*M)$ is semisimple, the proof of [8 Lemma 5.1.1] implies that

(Cup)

$$\eta : {}^pH^{-1}(f_*K) \to {}^pH^1(f_*K)$$

is a monomorphism. Since $K^* = \mathcal{V}^*[\dim X]$ and $\mathcal{V}^*$ is semisimple, we also know that

(Cup*)

$$\eta : {}^pH^{-1}(f_*K^*) \to {}^pH^1(f_*K^*)$$

is a monomorphism. By Corollary 2.2.3 the Verdier dual of (Cup*) can be identified with the morphism (Cup). Hence the morphism (Cup) is also an epimorphism. This finishes the inductive proof of Theorem [C](i).
By Deligne’s Lefschetz splitting criterion [11, Theorem 1.5], Theorem [C(i)] implies Theorem [C(ii)]. Regarding Theorem [C(iii)] \( \ell \neq 0 \), the semisimplicity of \( \nu^* \mathcal{H}(f, K) \) follows from [8, Proposition 4.7.8] and the inductive semisimplicity of \( \nu^* \mathcal{H}(f, K) \) and \( \nu^* \mathcal{H}(g, M) \).

2.7. **Proof of Theorem [D] and Theorem [E]**. In order to prove Theorem [C(iii)] in the case of \( \ell = 0 \), we first need to prove Theorem [D] and Theorem [E].

**Proposition 2.7.1.** With the assumption [2.5.1], then Theorem [D] holds for \( f \).

**Proof.** At this point, Theorem [C(ii)] (the Decomposition Theorem) holds for \( f, K \), hence the complex \( f_* K \) is \( p \)-split in the sense of [8, Definition 4.3.1] and let us fix a choice of the splitting. We obtain an isomorphism

\[
\nu : H^b(X, K) \cong H^{b-\ell}(Y, \nu^* \mathcal{H}(f, K)).
\]

By [8, Lemma 4.4.2 and Remark 4.4.3], the cup product maps with the first Chern classes of \( \eta \) and \( A \) are compatible with the isomorphism \( \nu \), respectively. Therefore Theorem [D] is equivalent to the following two statements:

\[
\eta^j : H^j(Y, \nu^* \mathcal{H}(f, K)) \cong H^j(Y, \nu^* \mathcal{H}(f, K)), \text{ whenever } \ell \geq 0, j \in \mathbb{Z},
\]

\[
A^j : H^{-j}(Y, \nu^* \mathcal{H}(f, K)) \cong H^{-j}(Y, \nu^* \mathcal{H}(f, K)), \text{ whenever } j \geq 0, \ell \in \mathbb{Z}.
\]

The statement for \( \eta \) follows from Theorem [C(i)] for \( f \). For \( A \), the plan is to cut \( X \) by hyperplane sections in \( |\eta| \) or \(|L|\) and use the corresponding weak Lefschetz theorem.

**Case I:** \( \ell \neq 0 \). By Theorem [C(ii)], we can assume \( \ell < 0 \). Choose a general hyperplane section \( X_1 \in |\eta| \) and set \( f^1 : X^1 \rightarrow Y \). For proving that \( A^\ell \) is an isomorphism, using Proposition [2.3.1] we can replace \( \eta \) by \( \eta^{\otimes m} \) for some integer \( m \), so that \( r(f^1) \leq r(f) - 1 \) or \( r(f^1) = r(f) = 0 \) and \( \dim f^1(X^1) < \dim f(X) \). The injectivity of \( A^\ell \) follows from inductive assumption on \( f^1 \) and Proposition [2.3.1]. The surjectivity follows from a dual argument as in the Case II of the proof of Theorem [C(i)].

**Case II:** \( \ell = 0 \) and \( j \geq 2 \). Using Bertini Theorem, we can choose \( Y_1 \) to be a sufficiently general hyperplane section in \( |A| \) so that \( f^{-1}(Y_1) \) is nonsingular and \( Y_1 \) is transversal to all strata of \( Y \). Set \( f_1 : X_1 := f^{-1}(Y_1) \rightarrow Y_1 \). Proposition [2.3.2] implies that \( r(f_1) \leq r(f) \) and \( \dim f_1(X_1) < \dim f(X) \). The bijectivity of \( A^\ell \) follows from the inductive assumption on \( (A|_{Y_1})^{\ell-1} \) and Proposition [2.3.2].

**Case III:** \( \ell = 0 \) and \( j = 1 \). This step requires a different argument since there is no Hodge decomposition to use, compared with [8, Proposition 5.2.3]. Instead, we use the pre-Weil operator \( \phi \) to keep track of non-degeneracy of Poincaré pairings. We use the same choice as in Case II. By Theorem [C(i)] we have \( \eta^\ell : \nu^* \mathcal{H}(f, K) \cong \nu^* \mathcal{H}(f, K) \), which gives a Lefschetz decomposition

\[
(29) \quad \nu^* \mathcal{H}(f, K) = \bigoplus_{m \geq 0} \eta^m \mathcal{P}^{2m}, \quad \mathcal{P}^{2m} := \ker \eta^{2m+1} \subseteq \nu^* \mathcal{H}^{-2m}(f, K).
\]

If \( m \geq 1 \), then \( A : H^{-1}(Y, \eta^m \mathcal{P}^{2m}) \rightarrow H^1(Y, \eta^m \mathcal{P}^{2m}) \) is an isomorphism by Case I. Therefore we just need to deal with \( m = 0 \), i.e.

\[
(30) \quad A : H^{-1}(Y, \mathcal{P}^0) \rightarrow H^1(Y, \mathcal{P}^0)
\]

is an isomorphism. The idea is to decompose it into restriction and Gysin maps, which come from the corresponding maps on \( X \).
Let us denote by $i_X : X_1 \hookrightarrow X$ the inclusion map. Since $X_1 \in |L|$, the cup product with $L$ on $X$ can be decomposed into

$$H^{n-1}(X, \mathcal{V}) \xrightarrow{R = i_X} H^{n-1}(X_1, \mathcal{V}|_{X_1}) \xrightarrow{G = i_X^*} H^{n+1}(X, \mathcal{V})$$

(31)

$$H^{-1}(X, K) \xrightarrow{R} H^0(X_1, K[-1]|_{X_1}) \xrightarrow{G} H^1(X, K)$$

where $R$ and $G$ denote the restriction map and the Gysin map respectively for $i_X$. Similarly, for $\mathcal{V}^*$ we denote

$$\tilde{L} : H^{n-1}(X, \mathcal{V}^*) \xrightarrow{\tilde{R}} H^{n-1}(X_1, \mathcal{V}^*|_{X_1}) \xrightarrow{\tilde{G}} H^{n+1}(X, \mathcal{V}^*)$$

Note that $G$ can be identified with the dual of $\tilde{R}$ using the Poincaré duality.

**Claim 2.7.2.** The maps $R$ and $G$ in (31) give a decomposition of the cup product map (30) into

$$A : H^{-1}(Y, \mathcal{P}^0) \xrightarrow{R} H^0(Y_1, \mathcal{P}^0_1) \xrightarrow{G} H^1(Y, \mathcal{P}^0),$$

(32)

where $\mathcal{P}^0_1 := \text{Ker} \eta|_{X_1} \subseteq \mathcal{P}^0(f_*K)[-1]|_{Y_1}$. Similarly, we have

$$A : H^{-1}(Y, \tilde{\mathcal{P}}^0) \xrightarrow{\tilde{R}} H^0(Y_1, \tilde{\mathcal{P}}^0_1) \xrightarrow{\tilde{G}} H^1(Y, \tilde{\mathcal{P}}^0)$$

for corresponding objects associated with $K^*$. Moreover, $G$ is the dual of $\tilde{R}$ under the Poincaré duality.

**Proof of claim.** Step 1: we first show that $R$ and $G$ restrict to the cup product with $A$ on $\mathcal{P}^0(f_*K)$. Since we have proven that $f_*K \cong \oplus \mathcal{P}^\ell(f_*K)[-\ell]$, by [8, Remark 4.4.3] again, the cup product map $L : H^{-1}(X, K) \to H^1(X, K)$ restricts to the cup product map with $A$ on the component $\mathcal{P}^0(f_*K)$:

$$A : H^{-1}(Y, \mathcal{P}^0(f_*K)) \to H^1(Y, \mathcal{P}^0(f_*K)).$$

Denote by $i_Y : Y_1 \hookrightarrow Y$ the inclusion. Since $Y_1 \in |A|$, the cup product with $A$ can also be decomposed as

$$A : H^{-1}(Y, \mathcal{P}^0(f_*K)) \xrightarrow{R = i_Y} H^0(Y_1, \mathcal{P}^0(f_*K)[-1]|_{Y_1}) \xrightarrow{G = i_Y^*} H^1(Y, \mathcal{P}^0(f_*K)).$$

(33)

We claim that the maps $R$ and $G$ in (33) are the restriction of $R$ and $G$ in (31). Since $L = f^*A$, one has a commutative diagram

$$\begin{array}{ccc}
X_1 & \xrightarrow{i_X} & X \\
\downarrow f_1 & \downarrow f & \\
Y_1 & \xrightarrow{i_Y} & Y
\end{array}$$

Since $Y_1$ is chosen to be transverse to the strata of $Y$ (adapted to $f$), the embedding $Y_1 \hookrightarrow Y$ is a normally nonsingular inclusion (see [8, Page 714] for the definition and discussion). Furthermore, by [8, Remark 3.5.1] (or see the proof of [8, Lemma 4.3.8]) $i_Y^*[-1]$ is $t$-exact and one has

$$i_Y^*\mathcal{P}^0(f_*K)[-1] \cong \mathcal{P}^0(f_1^*(i_X^*K)[-1]))$$
This implies that the map \( R \) in (31) induces the map \( R \) in (33). On the other hand, since the closed embedding \( i_X \) is affine and quasi-finite, one has \( i_{X,*} \) is \( t \)-exact and thus \( p\mathcal{H}^0f_* \circ i_{X,*} \cong i_{Y,*} \circ p\mathcal{H}^0f_{1,*} \). This implies that
\[
p\mathcal{H}^0f_* (i_{X,*}(i_X^*K[-1])) = i_{Y,*} p\mathcal{H}^0(f_{1,*}(i_X^*K[-1])) = i_{Y,*} i_Y^* p\mathcal{H}^0(f_*K)[-1],
\]
and hence the map \( G \) in (31) restricts to the map \( G \) in (33).

Step 2: Now it suffices to show that \( R \) and \( G \) in (33) restrict to \( R \) and \( G \) in (32). This requires additional two substeps.

Step 2.1: we show that \( R \) and \( G \) in (31) are compatible with the cup product map \( \eta : K \to K[2] \). Explicitly, we claim that the map \( i_X^*(K \to K[2]) \) is the cup product map \( \eta|_{X_1} : i_X^*K \to i_X^*K[2] \). The proof is similar to the proof of Lemma 2.2.2 for reader’s convenience, we give some details here. By [8, Remark 4.4.1], one can choose a section \( s \in \Gamma(X, \eta) \) whose zero locus defines a normally nonsingular inclusion \( \alpha : \{ s = 0 \} \hookrightarrow X \), so that the map \( \eta : K \to K[2] \) can be described as
\[
K \to \alpha_* \alpha^*K \sim \alpha_1 \alpha^1K[2] \to K[2].
\]
On the other hand, consider the following diagram
\[
\begin{array}{ccc}
\{ s = 0 \} \cap X_1 & \xrightarrow{\alpha_1} & X_1 \\
\downarrow i_{X,1} & & \downarrow i_X \\
\{ s = 0 \} & \xrightarrow{\alpha} & X
\end{array}
\]
We can choose \( s \) such that \( \alpha_1 \) is also a normally nonsingular inclusion and so the cup product with \( \eta|_{X_1} \) can be described using \( \alpha_1 \). Since \( \alpha \) is proper, we have
\[
i_X^* \alpha_* \alpha^* = \alpha_1^* i_{X,1}^*.
\]
Then
\[
i_X^*(K \to K[2]) = i_X^*K \to i_X^* \alpha_* \alpha^*K \sim i_X^* \alpha_1 \alpha^1K[2] \to i_X^*K[2]
\]
\[
= i_X^*K \to i_1 \alpha_1^* i_X^*K \sim \alpha_1 \alpha_1^1(i_X^*K)[2] \to i_X^*[2]
\]
\[
= i_X^*K \xrightarrow{\eta|_{X_1}} i_X^*K[2].
\]
Here we use the additional fact that \( \alpha^1 = \alpha^*[-2] \) [8 Lemma 3.5.4]. Using similar arguments, we can also show that the map \( i_{X,*}(i_X^*K \to i_X^*K[2]) \) is the map \( i_{X,*}i_X^*K \sim i_{X,*}i_X^*K[2] \).

We conclude that the Lefschetz decomposition with respect to \( \eta \) on \( X \) is compatible with \( R \) and \( G \) in (31).

Step 2.2: we need to argue that this compatibility descends to \( Y \), i.e. the maps \( R \) and \( G \) in (33) are compatible with the cup product map \( p\mathcal{H}^0f_*(\eta) : p\mathcal{H}^0f_*K \to p\mathcal{H}^2f_*K \), which would imply that they are compatible with the Lefschetz decomposition (29). For \( R \), we need to show that the map
\[
i_Y^* \left( p\mathcal{H}^0(f_*K) \xrightarrow{p\mathcal{H}^0f_*(\eta)} p\mathcal{H}^2(f_*K) \right)
\]
is the cup product map
\[
i_Y^* p\mathcal{H}^0(f_*K) \xrightarrow{p\mathcal{H}^0f_*(\eta|_{X_1})} i_Y^* p\mathcal{H}^2(f_*K).
\]
This follows from the $t$-exactness of $\iota_Y^*$ and the compatibility between the cup product map $\eta : K \to K[2]$ and $R$ in Step 2.1. For the compatibility of $G$, the argument is similar and we leave the details to the reader.

We conclude that the restriction map $R$ and Gysin map $G$ with respect to $L$ in (31) reduce to the corresponding ones with respect to $A$ on $\mathcal{P}^0$ in (32) and we finish the proof of claim.

Now it suffices to show that the cup product map $A$ in (32) is bijective. To do this, we will use the pre-Weil operator $\phi$ to produce a natural non-degenerate pairing $S : \text{Ker} G \otimes \text{Ker} \tilde{G} \to \mathbb{C}$, where $G$ and $\tilde{G}$ are Gysin maps for $Y_1 \to Y$ and $\mathcal{P}^0$.

**Step 1:** Consider the Poincaré pairing in (12):

$$S : H^*(X_1, \mathcal{V}|_{X_1}) \otimes H^*(X_1, \mathcal{V}^*|_{X_1}) \to \mathbb{C}.$$  

By definition, $R$ and $\tilde{G}$ are adjoint to each other with respect to $S$:

$$(34) \quad S(R(\alpha), \tilde{\beta}) = S(\alpha, \tilde{G}(\tilde{\beta})),$$

for any $\alpha \in H^{n-1}(X, \mathcal{V})$ and $\tilde{\beta} \in H^{n-1}(X, \mathcal{V}^*|_{X_1})$.

**Step 2.** Consider the vector spaces

$$\text{Ker} A|_{Y_1} \subseteq H^0(Y_1, \mathcal{P}^0_1), \quad \text{Ker} \tilde{A}|_{Y_1} \subseteq H^0(Y_1, \tilde{\mathcal{P}}^0_1).$$

Denote by $K_1 = K[-1]|_{X_1}$ and $f_1 : X_1 \to Y_1$ the induced map. By [8, Theorem 4.4.4(c)], the isomorphism $H^0(Y_1, \mathcal{P}^0_1) \cong H^0_0(X_1, K_1)$ identifies

$$H^0(Y_1, \mathcal{P}^0_1) \cong \text{Ker} \eta|_{X_1} \subseteq H^0_0(X_1, K_1).$$

Using [8, Remark 4.4.3], one can further identify

$$(35) \quad \text{Ker} A|_{Y_1} \cong P^0_0(X_1) := \text{Ker} \eta|_{X_1} \cap \text{Ker} L|_{X_1} \subseteq H^0_0(X_1, K_1),$$

where $P^0_0(X_1)$ is the primitive piece with respect to the double Lefschetz decomposition in Theorem [D]. The same applies for $\text{Ker} \tilde{A}|_{Y_1}$. Moreover, by Lemma [1.6.6] $S$ induces a pairing

$$S^0_{0L}(X_1) : H^0_0(X_1, K_1) \otimes H^0_0(X_1, K_1^*) \to \mathbb{C}.$$  

The inductive Theorem [E] for $f_1 : X_1 \to Y_1$ and $K_1$ implies that $S^0_{0L}(X_1)(\bullet, \phi(\bullet))$ polarizes the natural pure twistor structure $F$ on $P^0_0(X_1)$, where $\phi : \overline{F}_{|_{z=-1}} \to \tilde{P}^0_0(X_1)$ is induced by the pre-Weil operator. Therefore the pairing

$$S^0_{0L}(X_1) : P^0_0(X_1) \otimes \tilde{P}^0_0(X_1) \to \mathbb{C}$$

is non-degenerate by Lemma [1.2.9]. To simplify the notation, we still denote by $S : \text{Ker} A|_{Y_1} \otimes \text{Ker} \tilde{A}|_{Y_1} \to \mathbb{C}$ the pairing corresponding to $S^0_{0L}(X_1)$, under the identification (35).

Summarizing the discussion above, we know that $S$ induces a non-degenerate pairing

$$S : \text{Ker} A|_{Y_1} \otimes \text{Ker} \tilde{A}|_{Y_1} \to \mathbb{C},$$

and $\text{Ker} A|_{Y_1}$ underlies a pure twistor structure $F$ polarized by $S(\bullet, \phi(\bullet))$, where $\phi : \overline{F}_{|_{z=-1}} \simeq \text{Ker} \tilde{A}|_{Y_1}$ is induced by the pre-Weil operator $\phi$. By Corollary [1.5.6] and the fact that $G$ is the dual of $\tilde{R}$, the Gysin map $G : H^0(Y_1, \mathcal{P}^0_1) \to H^1(Y, \mathcal{P}^0)$ underlies a morphism of pure twistor structures. Moreover since $A|_{Y_1} = R \circ G$, we have an inclusion map

$$\text{Ker} G \subseteq \text{Ker} A|_{Y_1},$$
which underlies a morphism of pure twistor structure. Then by Corollary \[1.5.6\] and Lemma \[1.2.10\] Ker \(G\) underlies a pure twistor structure \(E\) with an induced isomorphism \(\phi_E : E_{|z=-1} \sim \text{Ker} \tilde{G}\) so that \(E\) is polarized by

\[
S(\bullet, \phi_E(\bullet)) : \text{Ker} G \otimes E_{|z=-1} \rightarrow \text{Ker} G \otimes \text{Ker} \tilde{G} \rightarrow \mathbb{C}.
\]

By Lemma \[1.2.9\] and the fact \(\phi_E\) is an isomorphism, we conclude that the restricted pairing

\[
(36) \quad S : \text{Ker} G \otimes \text{Ker} \tilde{G} \rightarrow \mathbb{C}
\]

is non-degenerate.

Now we prove the injectivity of \(A\). Suppose by contradiction that there is a nonzero \(\alpha \in \text{Ker} A = \text{Ker} G \circ R\), then \(R(\alpha) \in \text{Ker} G\). Since the pairing \[(36)\] is non-degenerate, we can find an element \(\beta \in \text{Ker} \tilde{G}\) so that

\[
0 \neq S(R(\alpha), \beta) = S(\alpha, \tilde{G}(\beta)) = 0,
\]

where the first equality comes from \[(34)\]. But this is a contradiction! Therefore \(A\) is injective. The surjectivity of \(A\) follows from the injectivity of \(A\) for \(\mathbb{V}^*\). \(\square\)

**Corollary 2.7.3.** Let \(W^L\) and \(W^n\) be the weight filtrations on \(H^*(X, \mathbb{V})\) in \[(1.6.4)\]. Then

- \(W^L = \bigoplus_{b \geq 0} H^b(X, \mathbb{V})\), \(\text{Gr}^L_i = H^i(X, \mathbb{V})\).
- \(W^n_i = \bigoplus_{b \geq 0} H^b(X, \mathbb{V})\), \(\text{Gr}^n_i = H^i(X, \mathbb{V})\).
- The filtration \(W^n_j[i]\) induces the monodromy weight filtration of \(\eta\) on \(\text{Gr}^L_i\). Therefore, for \(\ell, j \in \mathbb{Z}\), we have a double Lefschetz decomposition:

\[
H_{-\ell}^{-j}(X, \mathbb{V}) = \bigoplus_{m, i \in \mathbb{Z}} \eta^{-\ell+i}L^{-j+m}P^{-2m}_{\ell-2i},
\]

where \(P^{-j}_{-\ell}\) are the primitive subspaces in Theorem \[(1)\].

- \(\text{Gr}^n_j \otimes \text{Gr}^L_i \rightarrow H^*(X, \mathbb{V}) = H_{-\ell}^{-j}(X, \mathbb{V})\) so that the bilinear pairing \(S^{\eta L}_{ij}\) in \[(1)\] is well-defined and non-degenerate.

**Proof.** Since at this stage we have Theorem \[(1)\], we can just proceed as in the proof of \[(8)\] Proposition 5.2.4], using Lemma \[1.6.6\] and Theorem \[1.1.1\]. \(\square\)

**Proposition 2.7.4.** With the assumption \[2.3.1\], Theorem \[2\] holds for \(f\), i.e. we have

- The double Lefschetz decomposition \[(37)\] satisfies

\[
S^{\eta L}_{ij}(\eta^{-\ell+i}L^{-j+m}P^{-2m}_{\ell-2i}, \eta^{-\ell+i}'L^{-j+m'}P^{-2m'}_{\ell-2i'}) = 0, \quad \forall (i, m) \neq (i', m').
\]

- Each direct summand \(\eta^{-\ell+i}L^{-j+m}P^{-2m}_{\ell-2i}\) underlies a natural pure sub-twistor structure \(F\) so that the pre-Weil operator \(\tilde{\phi}\) restricts to \(\tilde{\phi} : F_{|z=-1} \sim \eta^{-\ell+i}L^{-j+m}P^{-2m}_{\ell-2i}\).

- The pure twistor structure \(F\) is polarized by \(S^{\eta L}_{ij}(\bullet, \tilde{\phi}(\bullet))\) up to a non-zero constant and \(S^{\eta L}_{ij}\) is non-degenerate.

**Proof.** The first two statements follow from Lemma \[1.6.6\] and Lemma \[1.5.5\]. We focus on the last statement, whose proof is divided into two cases.

**Case I:** Consider all \(\eta^{-\ell+i}L^{-j+m}P^{-2m}_{\ell-2i} \neq P^0_0\). It suffices to deal with the case \(i \geq \ell, m \geq j\) by definition and Remark \[1.6.8\]. We can also assume \(i = \ell, m = j\). Then one uses Proposition \[2.3.1\] and Proposition \[2.3.2\] and the compatibility of the bilinear paring \(S^{\eta L}_{ij}\) between \(X\) and its hyperplanes, similar to the proof of \[8\] Proposition 5.3.2].
Case II: $P_0^0$. This is the most difficult part of the entire proof, as in [8, §5.4], where we will show that $i^{-n} \cdot \mathcal{S}_{00}^L(\bullet, \phi(\bullet))$ polarizes the twistor structure on $P_0^0$. For any $\epsilon > 0$, define

$$\Lambda_\epsilon := \ker(L + \epsilon \eta) \subseteq H^0(X, K), \quad \tilde{\Lambda}_\epsilon := \ker(L + \epsilon \eta) \subseteq H^0(X, K^*)$$

Let $E$ be the natural pure twistor structure on $H^0(X, K)$, with the pre-Weil operator $\phi: E|_{z=-1} \xrightarrow{\sim} H^0(X, K^*)$. By Theorem 1.1.1, each $\Lambda_\epsilon$ has the same dimension $b := \dim H^0(X, K) - \dim H^2(X, K)$ and underlies a pure sub-twistor structure $F_\epsilon$. By Lemma 1.4.8 the pre-Weil operator restricts to $\phi_\epsilon: F_\epsilon|_{z=-1} \xrightarrow{\sim} \tilde{\Lambda}_\epsilon$.

Recall from (12) there is a Poincaré pairing:

$$S: H^0(X, K) \otimes H^0(X, K^*) \rightarrow \mathbb{C}.$$ 

Lemma 2.7.5. Consider the limiting spaces in the Grassmannians:

$$\Lambda := \lim_{\epsilon \rightarrow 0} \Lambda_\epsilon \in G(b, H^0(X, K)), \quad \tilde{\Lambda} := \lim_{\epsilon \rightarrow 0} \tilde{\Lambda}_\epsilon \in G(b, H^0(X, K^*)�)$$

Then $\Lambda$ underlies a pure sub-twistor structure $F \subseteq E$, with the isomorphism:

$$\phi_0: F|_{z=-1} \xrightarrow{\sim} \tilde{\Lambda},$$

as the limit of $\phi_\epsilon$. Moreover, the bilinear pairing

$$i^{-n} \cdot S(\bullet, \phi_0 \circ \text{Iden}(\bullet)) : \Lambda \otimes \tilde{\Lambda} \rightarrow \Lambda \otimes \tilde{\Lambda}|_{z=-1} \rightarrow \Lambda \otimes \tilde{\Lambda} \xrightarrow{S} \mathbb{C},$$

is positive semi-definite, where Iden is the Identification map from Definition 1.2.3.

Proof. Remark 1.2.2 implies that $\Lambda$ underlies a twistor structure $F$ and the pairing is positive semi-definite, as it is the limit of positive definite pairings. The only thing we need to argue is that $\phi_0 = \lim_{\epsilon \rightarrow 0} \phi_\epsilon$ is still an isomorphism. By Lemma 1.4.8 and the definition of $\phi_0$, we have a commutative diagram

$$\begin{array}{ccc}
F|_{z=-1} & \xrightarrow{\phi_0} & E|_{z=-1} \\
\downarrow & & \downarrow \phi \\
\tilde{\Lambda} & \xrightarrow{\sim} & H^0(X, K^*)
\end{array}$$

where both horizontal maps are inclusions. Since $\phi$ is injective ($\phi$ is an isomorphism) and $\phi_0$ is the restriction of $\phi$, we know that $\phi_0$ is also injective. On the other hand, we know

$$\dim F|_{z=-1} = \dim F|_{z=1} = \dim \Lambda = b = \dim \tilde{\Lambda}.$$ 

Therefore $\phi_0$ must be an isomorphism.

\[\square\]

Notation 2.7.6. Denote the cup product operators with $L$ by

$$L^k_r: H^{-r}(X, K) \rightarrow H^{-r+2k}(X, K), \quad \tilde{L}^k_r: H^{-r}(X, K^*) \rightarrow H^{-r+2k}(X, K^*)$$

Similarly, the cup products with $\eta$ are denoted by

$$\eta: H^i(X, K) \rightarrow H^{i+2}(X, K), \quad \tilde{\eta}: H^i(X, K^*) \rightarrow H^{i+2}(X, K^*).$$
For any subspace $V \subseteq H^0(X, K^*)$, recall that the orthogonal complement of $V$ in $H^0(X, K)$ with respect to $S$ is defined to be
\[ V^\perp := \{ w \in H^0(X, K) | S(w, v) = 0, \forall v \in V \}. \]
The following two results are direct adaptations of [8, Lemma 5.4.1, Lemma 5.4.2]. For details, see [35, Lemma 2.7.8 and Lemma 2.7.9].

**Lemma 2.7.7.** With the notations above, we have
\[ \eta \operatorname{Ker} L_2^i \cap (\eta^i \operatorname{Ker} \tilde{L}_2^i)^\perp \cap \cdots \cap (\eta^i \operatorname{Ker} \tilde{L}_2^{2i})^\perp = \{ 0 \} \subset H^0(X, K), \quad i \gg 0. \]

Since $\operatorname{Ker} L_0^1 \subseteq H^0(X, K)$, we have a restricted pairing
\[ (38) \quad S : \operatorname{Ker} L_0^1 \otimes \operatorname{Ker} \tilde{L}_0^1 \rightarrow \mathbb{C}. \]

**Lemma 2.7.8.** We have
\[ \Lambda = \operatorname{Ker} L_0^1 \cap \left( \bigcap_{i \geq 1} (\eta^i \operatorname{Ker} \tilde{L}_2^i)^\perp \right), \quad \tilde{\Lambda} = \operatorname{Ker} \tilde{L}_0^1 \cap \left( \bigcap_{i \geq 1} (\eta^i \operatorname{Ker} L_2^i)^\perp \right). \]

We also have direct sum decompositions
\[ \operatorname{Ker} L_0^1 = \Lambda \oplus \eta \operatorname{Ker} L_2^1, \quad \operatorname{Ker} \tilde{L}_0^1 = \tilde{\Lambda} \oplus \tilde{\eta} \operatorname{Ker} \tilde{L}_2^1 \]
which are orthogonal with respect to (38).

**Remark 2.7.9.** Starting from this point, our argument is different from [8, §5.4]. It seems to us more appropriate to prove the counterpart of [8, Lemma 5.4.4] first and then prove the counterpart of [8, Lemma 5.4.3].

Corollary 2.7.3 implies that
\[ \frac{\operatorname{Ker} L_0^1}{\operatorname{Ker} L_0^1 \cap H^0_{\leq -1}(X, K)} \subseteq \frac{W_0^L \cap H^0(X, K)}{W_0^L \cap H^0_{\leq -1}(X, K)} = \frac{H^0_{\leq 0}(X, K)}{H^0_{\leq -1}(X, K)} = H^0_0(X, K). \]
Define
\[ \Lambda_0 := \Lambda / (\Lambda \cap H^0_{\leq -1}(X, K)) \]
and similarly for $\tilde{\Lambda}_0$.

**Corollary 2.7.10.** The pairing (38) induces a non-degenerate pairing
\[ (39) \quad S_0^{nl} : \operatorname{Ker} L_0^1 / (\operatorname{Ker} L_0^1 \cap H^0_{\leq -1}(X, K)) \otimes \operatorname{Ker} \tilde{L}_0^1 / (\operatorname{Ker} \tilde{L}_0^1 \cap H^0_{\leq -1}(X, K^*)) \rightarrow \mathbb{C}, \]
which is the restriction of the pairing $S_0^{nl}$ from (1). There are direct sum decompositions
\[ \operatorname{Ker} L_0^1 / (\operatorname{Ker} L_0^1 \cap H^0_{\leq -1}(X, K)) = \Lambda_0 \oplus (\eta \operatorname{Ker} L_2^1 / \eta \operatorname{Ker} L_2^1 \cap H^0_{\leq -1}(X, K)) \]
\[ \operatorname{Ker} \tilde{L}_0^1 / (\operatorname{Ker} \tilde{L}_0^1 \cap H^0_{\leq -1}(X, K^*)) = \tilde{\Lambda}_0 \oplus (\tilde{\eta} \operatorname{Ker} \tilde{L}_2^1 / \tilde{\eta} \operatorname{Ker} \tilde{L}_2^1 \cap H^0_{\leq -1}(X, K^*)) \]
which are orthogonal with respect to (39). Moreover, the restricted bilinear pairing
\[ S_0^{nl} : \Lambda_0 \otimes \tilde{\Lambda}_0 \rightarrow \mathbb{C} \]
is non-degenerate and we have
\[ (40) \quad \Lambda_0 = \frac{(\tilde{\eta} \operatorname{Ker} \tilde{L}_2^1)^\perp \cap \operatorname{Ker} L_0^1}{(\eta \operatorname{Ker} L_2^1)^\perp \cap \operatorname{Ker} L_0^1 \cap H^0_{\leq -1}(X, K)}. \]
Remark 2.7.11. We decide to give a careful proof here because there are some subtleties about taking orthogonal complement with respect to a general (not necessarily positive definite) pairing. This is where (39) is used.

Proof. By Lemma 2.7.8, we can apply Lemma 1.6.1 to $H = \text{Ker } L_0^1$, $H_1 = \Lambda$, $H_2 = \eta \text{Ker } L_2^1$ and the pairing (38).

To deduce the first two statements, we need to calculate the orthogonal complement of $\text{Ker } \tilde{L}_0^1$ inside $\text{Ker } L_0^1$ with respect to (38), which is

$$\text{Ker } L_0^1 \cap (\text{Ker } \tilde{L}_0^1) = \text{Ker } L_0^1 \cap (W_{-1}^L \cap H^0(X, K)) = \text{Ker } L_0^1 \cap H^0_{\leq -1}(X, K).$$

Here the equalities follows from (9) and $W_{-1}^L = W_{-1}^\text{tot}$ in Corollary 2.7.3 Similarly

$$\Lambda \cap (\text{Ker } \tilde{L}_0^1) = \Lambda \cap H^0_{\leq -1}(X, K).$$

Therefore the non-degeneracy of $S^{\text{nl}}_{00} : \Lambda_0 \otimes \tilde{\Lambda}_0 \to \mathbb{C}$ follows from Lemma 1.6.1.

To obtain (40), one needs to calculate the orthogonal complement of

$$\tilde{\eta} \text{Ker } \tilde{L}_2^1 / \tilde{\eta} \text{Ker } \tilde{L}_2^1 \cap H^0_{\leq -1}(X, K^*)$$

with respect to the pairing (39) inside $\text{Ker } L_0^1 / (\text{Ker } L_0^1 \cap H^0_{\leq -1}(X, K))$, which is

$$(\tilde{\eta} \text{Ker } \tilde{L}_2^1) \cap \text{Ker } L_0^1 / (\tilde{\eta} \text{Ker } \tilde{L}_2^1 \cap \text{Ker } L_0^1 \cap H^0_{\leq -1}(X, K)).$$

This is because the pairing (39) is induced from $S : H^0(X, K) \otimes H^0(X, K^*) \to \mathbb{C}$. \[\square\]

Lemma 2.7.12. With the notations above, $\Lambda_0$ underlies a pure twistor structure $F_0$, with a descent isomorphism from Lemma 2.7.3 to $\phi : F_0|_{z=-1} \to \tilde{\Lambda}_0$ so that $F_0$ is polarized by the form $i^{-n} \cdot S^{\text{nl}}_{00}(\bullet, \phi(\bullet))$.

Proof. Since $H^0_{\leq -1}(X, K)$ underlies a natural pure twistor structure by Lemma 1.5.3, $\Lambda_0$ underlies a pure twistor structure $F_0$ with the pre-Weil operator $\phi : F_0|_{z=-1} \to \tilde{\Lambda}_0$. Lemma 2.7.3 implies that the pairing

$$i^{-n} \cdot S(\bullet, \phi \circ \text{Id}(\bullet)) : \Lambda \otimes \tilde{\Lambda} \to \Lambda \otimes \tilde{\Lambda} \to \mathbb{C}$$

is positive semi-definite. On the other hand, the pairing $S^{\text{nl}}_{00} : \Lambda_0 \otimes \tilde{\Lambda}_0 \to \mathbb{C}$ is non-degenerate by Corollary 2.7.10 Therefore the pairing on the quotients

$$i^{-n} \cdot S^{\text{nl}}_{00}(\bullet, \phi \circ \text{Id}(\bullet)) : \Lambda_0 \otimes \tilde{\Lambda}_0 \to \Lambda_0 \otimes \tilde{\Lambda}_0 \to \mathbb{C}$$

is positive definite, since it is positive semi-definite and non-degenerate at the same time. In particular, $\Lambda_0$ is polarized by $i^{-n} \cdot S^{\text{nl}}_{00}(\bullet, \phi(\bullet))$. \[\square\]

Now we can finish the proof of Theorem E for $P^0_0$. By definition

$$P^0_0 = \text{Ker } \eta \cap \text{Ker } L \subseteq \text{Gr}^n_0 \text{Gr}^L_0 H^*(X, \mathcal{V}).$$

It is direct to check that

$$P^0_0 \subseteq \text{Ker } L_0^1 / (\text{Ker } L_0^1 \cap W_{-1}^L).$$

The next claim is mentioned without proof in [3] Proof of Theorem 2.1.8], again we would like to add some details here.

Claim 2.7.13. $P^0_0 \subseteq (\tilde{\eta} \text{Ker } \tilde{L}_2^1)^\perp / (\tilde{\eta} \text{Ker } \tilde{L}_2^1)^\perp \cap W_{-1}^L$. 

Proof of claim. Let \( a \in H^0(X, K) = \Gr_0^L H^*(X, V) \) be an element so that \([a] \in P^0_0\). Then \( \eta[a] = 0 \in \Gr_{L_0}^{L_0} \Gr_0^L \), which means that \( \eta a \in W^L_{-1} H^*(X, V) \). Using the hard Lefschetz \( L^i : \Gr_i^L \cong \Gr_{i-1}^L \), we can find an element \( b \in W^L_{-1} \) so that \( \eta a = Lb \). Now let \( \tilde{c} = \tilde{\eta} \tilde{d} \) be an element so that \( \tilde{d} \in \ker L^1_{\tilde{\beta}} \). Then

\[
S(a, c) = S(a, \tilde{\eta} \tilde{d}) = -S(\eta a, \tilde{d}) = -S(Lb, \tilde{d}) = S(a, \tilde{L} \tilde{d}) = 0.
\]

We conclude that \( a \in (\tilde{\eta} \ker \tilde{L}^1_{\tilde{\beta}})^\perp \).

Now combining with (40), we have

\[
P^0_0 \subseteq \frac{(\tilde{\eta} \ker \tilde{L}^1_{\tilde{\beta}})^\perp \cap \ker L^1_{\tilde{\beta}}}{(\tilde{\eta} \ker \tilde{L}^1_{\tilde{\beta}})^\perp \cap \ker L^1_{\tilde{\beta}} + W^L_{-1}} = \frac{(\tilde{\eta} \ker \tilde{L}^1_{\tilde{\beta}})^\perp \cap \ker L^1_{\tilde{\beta}}}{(\tilde{\eta} \ker \tilde{L}^1_{\tilde{\beta}})^\perp \cap \ker L^1_{\tilde{\beta}} + H^0_{\leq -1}(X, K)} = \Lambda_0.
\]

Moreover, the inclusion \( P^0_0 \subseteq \Lambda_0 \) underlies a morphism of pure twistor structures. Since \( \Lambda_0 \) is polarized by \( i^{-n} \cdot S^{0L}_{i0}(\cdot, \rho(\cdot)) \) by Lemma 2.7.12 we conclude by Lemma 1.2.10 that \( P^0_0 \) is polarized by the restriction of \( i^{-n} \cdot S^{0L}_{i0}(\cdot, \rho(\cdot)) \).

2.8. Splitting criterion of perverse sheaves. After the proof of Theorem E now we can prove some auxiliary facts needed for the proof of Theorem C(iii) for \( \ell = 0 \), where the functoriality of the pre-Weil operator \( \rho \) is again crucial.

Let us first recall the splitting criterion of perverse sheaves due to de Cataldo-Migliorini [S]. Let \( Y \) be an algebraic variety and \( T \) is a closed subset of \( Y \). We use \( T \subset Y \prec Y \setminus T \) to denote the closed and open embeddings. For any perverse sheaf \( P \) on \( Y \), as in 2.4 the distinguished triangles associated to \( \alpha \) and \( \beta \) induces the following map

\[
H^k(Y, \alpha \alpha^1 P) \to H^k(Y, \alpha_\ast \alpha^* P), \quad \forall k \in \mathbb{Z}.
\]

Furthermore, suppose there is a stratification \( \mathcal{X} \) of \( Y \) with

\[
Y = U \cup T, \quad U = \bigcup_{d' > d} S_{d'}, \quad T = S_d,
\]

where \( S_{d'} \) denotes a \( d' \)-dimensional stratum.

**Lemma 2.8.1.** [S, Lemma 4.1.3] Let \( P \) be a perverse sheaf on \( Y \), constructible with respect to \( \mathcal{X} \). Assume

\[
\dim \mathcal{H}^{-d}(\alpha \alpha^1 P)_y = \dim \mathcal{H}^{-d}(\alpha_\ast \alpha^* P)_y
\]

for any \( y \in T \). Then the following statements are equivalent:

- \( \mathcal{H}^{-d}(\alpha \alpha^1 P) \to \mathcal{H}^{-d}(P) \) is an isomorphism.
- \( P \cong \beta_{\ast} \beta^* P \oplus \mathcal{H}^{-d}(P)[d] \).

Here \( \beta_{\ast} \) is the intermediate extension functor.

Now we prove that for \( T = \{y\} \), \( P = p\mathcal{H}^0(f_\ast K) \) and \( k = 0 \), the map in (41) is an isomorphism. This implies that (42) is satisfied in this case.

**Lemma 2.8.2.** For each point \( \alpha : \{y\} \hookrightarrow Y \) in the support of the sheaf \( \mathcal{H}^0(p\mathcal{H}^0(f_\ast K)) \) on \( Y \), the restriction map

\[
H^0(Y, p\mathcal{H}^0(f_\ast K)) \to H^0(Y, \alpha_\ast \alpha^* p\mathcal{H}^0(f_\ast K))
\]

is surjective. Moreover, the following cycle map

\[
cl : H^0(Y, \alpha_\ast \alpha^* p\mathcal{H}^0(f_\ast K)) \to H^0(Y, p\mathcal{H}^0(f_\ast K))
\]
is injective.

**Proof.** Since we have Theorem [C(ii)] for \( f_*K \) and Theorem [E] the proof of the first statement follows the same line as in [S Proposition 6.2.2]. For the second statement, apply the surjectivity statement to \( K^* \).

**Proposition 2.8.3.** Let \( \alpha : \{ y \} \hookrightarrow Y \) be a point in the support of \( \mathcal{H}^0(\mathcal{H}^0(f_*K)) \). Then the map (11) for \( k = 0 \) and \( P = \mathcal{H}^0(f_*K) \):

\[
H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \to H^0(Y, \alpha!*p\mathcal{H}^0(f_*K))
\]

is an isomorphism.

**Proof.** By construction in [24], this map decomposes as

\[
H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \xrightarrow{\text{cl}} H^0(Y, \beta!*p\mathcal{H}^0(f_*K)) \to H^0(Y, \beta!*p\mathcal{H}^0(f_*K)).
\]

where \( \text{cl} \) is the injective cycle map in Lemma 2.8.2. Let \( \beta : U = Y \setminus \{ y \} \hookrightarrow Y \) be the open embedding. The distinguished triangle associated to \( \{ y \} \hookrightarrow Y \beta \) gives rise to the following short exact sequence:

\[
0 \to H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \to H^0(Y, \beta!*p\mathcal{H}^0(f_*K)) \to H^0(Y, \beta!*p\mathcal{H}^0(f_*K)).
\]

Corollary [1.5.6] implies that \( H^0(Y, \beta!*p\mathcal{H}^0(f_*K)) \) underlies a mixed twistor structure \( E_U \) with an isomorphism \( E_U|_{z=-1} \cong H^0(Y, \beta!*p\mathcal{H}^0(f_*K^*)) \). Therefore, the kernel of the restriction map, underlies a pure sub-twistor structure \( F \) so that the isomorphism \( \phi \) in Corollary [2.7.3] restricts to

\[
\phi : F|_{z=-1} \cong H^0(Y, \alpha!*p\mathcal{H}^0(f_*K^*)).
\]

Now we want to use \( \phi \) to polarize different pieces of \( H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \). Note that

\[
\text{Im}(\text{cl}) \subseteq \text{Ker} L \subseteq H^0(X, K).
\]

By the additivity of \( \alpha!\alpha^! \), this inclusion is compatible with the double Lefschetz decomposition with respect to \( \eta \) and \( L \). Therefore, each direct summand of \( \text{Im}(\text{cl}) \) is polarized by \( \eta \). Then the pairing \( S_{00}^L \) in (11) restricts to a non-degenerate pairing

\[
S_{00}^L : H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \otimes H^0(Y, \alpha!*p\mathcal{H}^0(f_*K^*)) \to \mathbb{C}.
\]

Using Verdier duality map (28), this pairing gives rise to the map

\[
H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)) \xrightarrow{S_{00}^L} H^0(Y, \alpha!*p\mathcal{H}^0(f_*K^*)) \xrightarrow{\text{V.D.}} H^0(Y, \alpha!*p\mathcal{H}^0(f_*K)).
\]

which coincides with the map (43) by Lemma [2.4.3]. Therefore, the non-degeneracy of \( S_{00}^L \) implies the desired isomorphism.

**2.9. Semisimplicity Theorem for \( \ell = 0 \).** In this subsection, we finish the proof of Theorem [C] by showing that \( p\mathcal{H}^0(f_*K) \) is a semisimple perverse sheaf. To do this, we first show that \( p\mathcal{H}^0(f_*K) \) decomposes into the direct sum of intersection complexes associated to local systems over the strata of \( Y = \bigsqcup_d S_d \), where \( S_d \) is a smooth locally closed subset of pure dimension \( d \). An extra care is also needed here compared to [3 Proposition 6.3.2 and Lemma 6.1.3], because the assumption in [S Lemma 4.1.3] is not automatically satisfied.
Proposition 2.9.1. There is a canonical isomorphism in $\text{Perv}(Y)$:
\[ \mathcal{H}^0(f_*K) \cong \bigoplus_{d=\dim S_d} \text{IC}_{S_d}(\mathcal{H}^{-d}(\mathcal{H}^0(f_*K)|_{S_d})). \]

Proof. For any $d$, consider a $d$-dimensional stratum $S_d$ and set $U_d := \bigcup_{d' \geq d} S_{d'}$. We denote two embeddings by $S_d \hookrightarrow U_d \overset{\beta}{\leftarrow} U_{d+1}$. By Deligne’s formula, one has $\beta_* \cong \tau_{\leq d-1} \beta_*$. Then it suffices to show that
\[ \mathcal{H}^0(f_*K)|_{U_d} \cong \beta_*(\mathcal{H}^0(f_*K)|_{U_{d+1}}) \oplus \mathcal{H}^{-d}(\mathcal{H}^0(f_*K)|_{S_d})[d]. \]

To achieve this, let us prove the following two claims.

(A) $\dim \mathcal{H}^{-d}(\alpha_! \alpha^! \mathcal{H}^0(f_*K))_y = \dim \mathcal{H}^{-d}(\alpha_* \alpha^* \mathcal{H}^0(f_*K))_y$,
(B) the morphism $\mathcal{H}^{-d}(\alpha_! \alpha^! \mathcal{H}^0(f_*K)) \to \mathcal{H}^{-d}(\mathcal{H}^0(f_*K))$ is an isomorphism at $y$.

For both statements, we use the following induction method. If $d \geq 1$ and $y \in S_d$, choose a generic $d$-dimensional complete intersection $Y_d \subseteq Y$ which contains $y$ and is transversal to all strata of $Y$ (adapted to $f$) and $X_d = X \times_Y Y_d$ is smooth. It induces the following Cartesian diagram
\[
\begin{array}{ccc}
X_d & \xrightarrow{i_d} & X \\
\downarrow f_d & & \downarrow f \\
Y_d & \longrightarrow & Y
\end{array}
\]
By the repeated use of Proposition 2.3.2 we have $\dim Y_d < \dim Y$ and $r(f_d) \leq \max\{r(f) - d, 0\}$. Set $K_d := i_y^* K[-d]$ and denote the inclusion by $i : \{y\} \hookrightarrow Y_d$. We have the following commutative diagram
\[
\begin{array}{cccc}
\mathcal{H}^0(i_! i^! \mathcal{H}^0(f_*K_d))_y & \longrightarrow & \mathcal{H}^0(i_! \mathcal{H}^0(f_*K_d))_y & \longrightarrow & \mathcal{H}^0(i_* i^* \mathcal{H}^0(f_*K_d))_y \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathcal{H}^{-d}(\alpha_! \alpha^! \mathcal{H}^0(f_*K))_y & \longrightarrow & \mathcal{H}^{-d}(\mathcal{H}^0(f_*K))_y & \longrightarrow & \mathcal{H}^{-d}(\alpha_* \alpha^* \mathcal{H}^0(f_*K))_y
\end{array}
\]

The isomorphisms in the vertical direction follow from [8 Lemma 4.3.8]. By the inductive Theorem (iii) for the morphism $f_d$ (recall that we are under Assumption 2.5.1 and $X_d$ is smooth), $\mathcal{H}^0(f_*K_d)$ is a direct sum of intersection cohomology complexes. Therefore by [8 Remark 4.1.2] we have
\[ \dim \mathcal{H}^0(i_! i^! \mathcal{H}^0(f_*K_d))_y = \dim \mathcal{H}^0(i_* i^* \mathcal{H}^0(f_*K_d))_y. \]
Moreover, by Lemma 2.8.1 the morphism
\[ \mathcal{H}^0(i_! i^! \mathcal{H}^0(f_*K_d))_y \to \mathcal{H}^0(i_! \mathcal{H}^0(f_*K_d))_y \]
is an isomorphism.

We first prove (A) for any $d$. The case $d = 0$ is obtained in Proposition 2.9.1. For $d \geq 1$ and $y \in S_d$, we apply the left and right vertical isomorphisms in the diagram (45) to reduce (A) to the equality (46), which holds by induction.

At this point (A) is proved. Then by Lemma 2.8.1 (the splitting criterion for perverse sheaves), to prove (44), it suffices to prove that (B) holds at any point of any stratum $S_d$ for any $d \geq 0$. For convenience, we denote this statement by (B)$_d$. For $d = 0$, (B)$_0$ follows from
Proposition 2.9.1. When \( d \geq 1 \), we apply the left vertical isomorphism in the diagram (45) to reduce (B) to (17) being an isomorphism, which also holds by induction. Therefore (44) holds and we finish the proof of this proposition. 

\[ \square \]

Proposition 2.9.2. \( p^{\mathcal{H}^0}(f_*K) \) is semisimple.

Proof. By Proposition 2.9.1, it suffices to show the local system

\[ \mathcal{H}^{-d}(p^{\mathcal{H}^0}(f_*K)|_{S_d}) \]

is semisimple over \( S_d \) for each \( d = \dim S_d \). Let \( i : y \hookrightarrow Y \) be a point lying in \( S_d \), the stalk of the local system at \( y \) is

\[ H^{-d}(Y, i_* i^* p^{\mathcal{H}^0}(f_*K)) \subseteq H^{-d}(f^{-1}(y), K|_{f^{-1}(y)}) = H^{n-d}(f^{-1}(y), V|_{f^{-1}(y)}). \]

Case I. \( d = 0 \) or \( d = m = \dim f(X) \). If \( d = 0 \), the semisimplicity is trivial. Suppose \( d = m \). By the construction of Whitney stratification, \( S_d \) is a smooth Zariski dense open subset of the open subset of \( Y \) over which \( f \) is smooth. By passing to a resolution of singularity of \( Y \) which does not modify \( S_d \), we can assume \( Y \) is smooth projective. It follows from Corollary 1.1.3 that \( R^{n-d} f_* V|_{S_d} \) is semisimple, so is the sub-local system \( \mathcal{H}^{-d}(p^{\mathcal{H}^0}(f_*K)|_{S_d}) \).

Case II. \( 1 \leq d = \dim S_d \leq m - 1 \). We use a geometric construction from [8, §6.4]. Extra care is needed to keep track of semisimplicity. Choose \( A \) to be a very ample line bundle on \( Y \) and let \( \mathbb{P}^V = |A| \) be the associated linear system. Set \( \Pi := (\mathbb{P}^V)^d \). Consider the universal \( d \)-fold complete intersection families

\[ \mathcal{Y} := \{(y, (H_1, \ldots, H_d)) : y \in \bigcap_{j=1}^d H_j, \quad H_j \text{ is a hyperplane of } Y \} \subseteq Y \times \Pi \]

and set \( \mathcal{X} := \mathcal{Y} \times_Y \Pi \times \Pi \). For any subset \( W \subseteq Y \), we write \( \mathcal{Y}_W := \mathcal{Y} \times_Y W \), \( \mathcal{X}_W := \mathcal{X} \times_Y \mathcal{Y}_W \).

In [8, §6.4, Page 743], de Cataldo and Migliorini showed that there is a commutative diagram with Cartesian squares:

\[
\begin{array}{ccc}
\mathcal{X}_T & \xrightarrow{p'} & \mathcal{X} & \xrightarrow{p''} & X \\
\downarrow \phi & & \downarrow f & & \\
\mathcal{Y}_T & \xrightarrow{q'} & \mathcal{Y} & \xrightarrow{q''} & Y \\
\downarrow \pi & & \downarrow & & \\
T & \longrightarrow & Y
\end{array}
\]

satisfying the following conditions:

- \( T \subseteq S_d \) is a Zariski open subset.
- \( \mathcal{Y}_T \rightarrow T \) admits a section \( \theta \) and \( \Phi \) inherits a stratification from the Whitney stratification on \( f \) chosen in [2.5]. Moreover all strata on \( \mathcal{Y}_T \) map surjectively and smoothly onto \( T \).
- The morphism \( \pi \circ \Phi \) is surjective and smooth projective of relative dimension \( (\dim \mathcal{X}_T - \dim T) \) and \( \dim \mathcal{X}_T = \dim X = n \).
Set
\[ p_X = p'' \circ p', \quad p_Y = q'' \circ q'. \]
Since \( \mathcal{V} \) is semisimple on \( X \), by Corollary \[1.2.12\] we have \( p^*_X \mathcal{V} \) is semisimple on \( \mathcal{X}_T \). At this point, we already know that the Decomposition Theorem holds for \( f_* (\mathcal{V} [\dim X]) \), by the base-change property, the Decomposition holds for \( \Phi_* (p^*_X \mathcal{V} [\dim \mathcal{X}_T]) \) as well. Therefore, we can apply Lemma \[2.9.3\] to \( p^*_X \mathcal{V} \) and the diagram
\[ \mathcal{X}_T \xrightarrow{\Phi} \mathcal{Y}_T \xrightarrow{\pi} T \xrightarrow{\theta} \mathcal{Y}_T, \]
so that we obtain a surjection map of local systems over \( T \):
\[ R^{\dim \mathcal{X}_T - d} (\pi \circ \Phi)_* (p^*_X \mathcal{V}) \to \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (\Phi_* p^*_X \mathcal{V} [\dim \mathcal{X}_T])) = \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (\Phi_* p^*_X K)) \]
As in Case I, by passing to a suitable resolution of \( Y \), we can assume \( Y \) is smooth projective. Since \( \mathcal{X} \) is smooth projective, \( T \) is a smooth subvariety contained in the open subset over which the map \( \mathcal{X} \to \mathcal{Y} \to Y \) is smooth, we can apply Corollary \[1.1.3\] to conclude that
\[ R^{\dim \mathcal{X}_T - d} (\pi \circ \Phi)_* (p^*_X \mathcal{V}) = R^{\dim \mathcal{X}_T - d} (\pi \circ \Phi)_* ((p'')^* \mathcal{V} |_{\mathcal{X}_T}) \]
is semisimple on \( T \), where \( (p'')^* \mathcal{V} \) is semisimple on \( \mathcal{X} \) because of Corollary \[1.2.12\]. Therefore the quotient local system \( \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (\Phi_* p^*_X K)) \) is also semisimple. Now the standard base change formula implies that
\[ \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (\Phi_* p^*_X K)) \cong \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (p^*_Y f_* K)) \cong \mathcal{H}^{-d} (\theta^* p^*_Y \mathcal{H}^0 (f_* K)) [\text{\( p^*_Y \) is t-exact}] \]
Since \( T \) is Zariski-dense in \( S_d \), by Remark \[1.1.4\] we conclude that \( \mathcal{H}^{-d} (\mathcal{H}^0 (f_* K) |_{S_d}) \) is semisimple on \( S_d \). In particular, \( \mathcal{H}^0 (f_* K) \) is semisimple and this finishes the proof of Theorem \[C\].

**Lemma 2.9.3.** Consider the following diagram
\[ \xymatrix{ \mathcal{X} \ar[r]^\Phi \ar[d]^F & \mathcal{Y} \ar[d]^\pi \\ T \ar[r]^=} & T } \]
and \( \theta : T \to \mathcal{Y} \) is a section of \( \pi \). Suppose
- \( \mathcal{X} \) is smooth of dimension \( n \), \( T \) is smooth of dimension \( d \), and \( F \) is surjective and smooth projective of relative dimension \( n - d \).
- The map \( \Phi \) is stratified and the strata of \( \mathcal{Y} \) map smoothly and surjectively onto \( T \). \( \theta (T) \) is a stratum of \( \mathcal{Y} \).
- The decomposition theorem holds for \( \Phi_* \mathcal{V} [\dim \mathcal{X}] \), where \( \mathcal{V} \) is a semisimple local system on \( \mathcal{X} \).

Then, there is a surjective map of local systems on \( T \):
\[ R^{\dim \mathcal{X} - d} F_* \mathcal{V} \to \mathcal{H}^{-d} (\theta^* \mathcal{H}^0 (\Phi_* \mathcal{V} [\dim \mathcal{X}])). \]

**Proof.** The proof is basically identical to \[8\] Lemma 6.4.1: by working stalkwise, one can reduce to Lemma \[2.8.2\] which is the semisimple local system version of \[8\] Proposition 6.2.2. \[\square\]
2.10. **Sabbah’s Theorem.** Using the standard reductions in [7, Page 71-74] and [6], we obtain the following version of Sabbah’s Decomposition Theorem in [27].

**Theorem 2.10.1.** Let \( f : U \to Y \) be a proper map between algebraic varieties, where \( U \) is a Zariski open subset of a smooth projective variety \( X \). Let \( \mathcal{V} \) be a semisimple local system on \( X \). Then Theorem \( C(ii) \) and Theorem \( C(iii) \) hold for \( f \) and \( \mathcal{V}|_U \). If in addition, \( f \) is projective and \( \eta \) is an \( f \)-ample line bundle, then Theorem \( C(i) \) holds as well.

**Proof.** One needs to keep track of the semisimplicity of \( \mathcal{V} \). For details, see [35, §2.9]. □

### Appendix A. Hodge star operators

In this appendix, for the lack of appropriate references in the literature, we state the construction and basic properties of Hodge star operators for differential forms with Harmonic bundle coefficients, adapting [34, §5]. We only give formulas, and the proof can be found in [35, §2.4]. It is used to understand the natural pure twistor structures on \( H^k(X, \mathcal{V}) \) in Theorem 1.1.1 and the pre-Weil operator in Definition 1.4.5.

Let \( X \) be a compact Kähler manifold of complex dimension \( n \) with a Kähler metric \( g \). Let \( \mathcal{A}_X^k \) denote the sheaf of \( \mathcal{C}^\infty \) \( k \)-forms on \( X \) and let \( \mathcal{A}_X^{p,q} \) denote the sheaf of \( (p,q) \)-forms. Following [34, §5.1.1], there is an induced Hermitian metric on \( \mathcal{A}_X^k \): if \( e_1, \ldots, e_n \) is an orthonormal basis for \( (T_{x,X}, g_x) \) and \( e^*_i \) is the dual basis, then \( e^*_i \wedge \cdots \wedge e^*_k \) form an orthonormal basis for the Hermitian metric on \( \mathcal{A}_X^k \). Let \( H \) be a harmonic bundle on \( X \) with harmonic metric \( h \). Together, \( h \) and the metric on \( \mathcal{A}_X^k \) induce a Hermitian metric \( (\cdot, \cdot)_x \) on \( H_x \otimes_\mathbb{C} \mathcal{A}_X^k \). Denote \( H^* \) to be the dual harmonic bundle from Construction 1.4.1.

**Definition A.0.1** \((L^2 \text{ metric})\). For each integer \( k \), there is a \( L^2 \) Hermitian metric on the space of \( k \)-forms with coefficients in \( H \) defined by

\[
(A, B)_{L^2} := \int_X (A, B) \text{Vol},
\]

where \( \text{Vol} \) is the volume form of \( X \) relative to \( g \), \( A, B \in \mathcal{C}^\infty(H \otimes \mathcal{A}_X^k) \), and \( (A, B) \) is the function \( x \mapsto (A_x, B_x)_x \).

**Definition A.0.2** \((\text{Hodge star operator})\). The Hodge star operator for harmonic bundles is defined to be the composition of the following \( \mathbb{C} \)-linear map

\[
*: H_x \otimes \mathcal{A}_X^k \to \text{Hom}(H_x \otimes \mathcal{A}_X^k, \mathbb{C}) \to H_x^* \otimes \mathcal{A}_X^{2n-k},
\]

where the first map is induced by \( (\cdot)_x \) and the second map is induced by the exterior product

\[
(H_x \otimes \mathcal{A}_X^k) \otimes (H_x^* \otimes \mathcal{A}_X^{2n-k}) \to \mathcal{A}_X^{2n} \cong \mathbb{C},
\]

so that we have

\[
(A_x, B_x)_x \text{Vol}_x = A_x \wedge B_x^*,
\]

where \( A, B \in \mathcal{C}^\infty(H \otimes \mathcal{A}_X^k) \). In particular,

\[
(A, B)_{L^2} = \int_X A_x \wedge B_x^*.
\]

**Lemma A.0.3.** The Hodge star operator restricts to

\[
*: H \otimes \mathcal{A}_X^{p,q} \to H^* \otimes \mathcal{A}_X^{n-p,n-q} = H^* \otimes \mathcal{A}_X^{n-q,n+p}.
\]
Let $e$ be a $C^\infty$ section of $H$. One constructs a section $\overline{e^\vee} \in C^\infty(H^*)$ via
$$\overline{e^\vee}(\bullet) = h(e, \bullet).$$
where $h$ is the harmonic metric.

**Lemma A.0.4.** The Hodge star operator for $H$ satisfies
$$*(e \otimes \alpha) = \overline{e^\vee} \otimes (*)\alpha,$$
where $e \otimes \alpha \in C^\infty(H \otimes \mathcal{A}_X^k)$ and $*\alpha$ is the Hodge star operator for $k$-forms so that
$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

**Lemma A.0.5.** Let $L$ be the ample line bundle on $X$ associated to the Kähler metric $g$. Let $e \otimes \alpha$ be a primitive $(p,q)$-form with coefficient in $H$ so that $p + q = k$. Then for $k \leq n$, we have
$$*(e \otimes \alpha) = C\overline{e^\vee} \otimes L^{n-k} \wedge \alpha,$$
where $C = (-1)^{(k+1)/2}p^q/n-k!$ is a constant. For $k \geq n$ we have
$$*(e \otimes \alpha) = C\overline{e^\vee} \otimes (L^{k-n})^{-1}(\alpha),$$
here $(L^{k-n})^{-1}$ represents the inverse of the cup product map
$$L^{k-n} : \mathcal{A}_X^{2n-k} \to \mathcal{A}_X^k.$$

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