Mixed schemes of finite element method for non-standard boundary value problems of the nonlinear theory of thin elastic shells

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Abstract. Variational statements of equilibrium problems for a thin elastic shell within a geometrically nonlinear theory of mean bending for different types of principal boundary conditions, including non-classical, are given. Two classes of shell models are considered: (1) a geometrically nonlinear equilibrium model for an anisotropic shell of a material obeying generalized Hooke's law; (2) a geometrically and physically nonlinear equilibria model for a shallow shell. Sufficient conditions for their generalized solvability in the corresponding Sobolev spaces are derived. In the case of the non-shallow shell the implicit function theorem is used. In the case of the shallow shell the variation problem is investigated using the generalized Weierstrass principle. Mixed finite element methods based on the use of the second derivatives of the deflection as auxiliary variables are constructed for approximate solutions of these problems. Sufficient conditions for solvability of the corresponding discrete problems are obtained. The convergence of approximate solutions is investigated. Accuracy estimates in the case of sufficiently smooth solutions of the original problems are given. Additional conditions are proposed for the problem on the shallow shell to ensure of implementation of the inequalities of the type of strong monotonicity and Lipschitz-continuity of the differential operator.

1. Introduction
The proposed report presents the results of the study of mixed schemes of the finite element method for solving boundary value problems of nonlinear theory of thin elastic shells. Applied by us mixed schemes, apparently, were for the first time considered by G. P. Astrakhantsev [1]. These schemes based on the direct use of second derivatives of the deflection as a subsidiary unknown and therefore characterized by great versatility. In addition, they use the simplest conformal Lagrangian finite elements and thus give an optimal coordination of estimation of accuracy of the approximate solution and the requirement of smoothness of the desired solution (in sense of belonging to the corresponding Sobolev space). In this paper, we consider two classes of shell theory problems: the equilibrium problems of blobs of arbitrary geometry from a physically linear material obeying the generalized for stake Hooke; the problem of equilibrium of a shallow shell, the geometry of the middle surface, which is identify with the geometry of the plane, from a physically nonlinear material. Everywhere you are assumed fulfillment of the conditions of applicability of the hypotheses of the Kirchhoff—Love theory and the middle of the bend. Consideration of the approximate method we always precede the formulation of a conditions on the data to ensure the solvability of the original differential problems.
At the same time for shells of arbitrary geometry, we limit ourselves to considering only Dirichlet boundary conditions, and for shallow of shells are considered and more complex conditions such as rigid contact of the boundary of the shell and normal boundary load.

2. Equilibrium equations of a thin elastic shell

The problem of equilibrium of a thin elastic shell in the framework of applicability of Kirchhoff — Love hypotheses, it can be formulated in the form of the problem of finding the critical points of the functional

$$F(u) = \int_S (a(\varepsilon, \varepsilon) + b(\kappa, \kappa)) dS - \int_S f \cdot udS.$$  

Here $S$ is the middle surface of the shell, which we consider to be related to the lines of the main curvatures. The corresponding curvilinear coordinates on $S$ will be denoted by $x_1$, $x_2$, and $u$ is displacement vector of the middle surface points, $u_1$ and $u_2$ are tangential components of the vector $u$, $u_3$ is normal component of the vector $u$, $f$ is the density of the external loads, $\Omega \subset \mathbb{R}^2$ is the domain of variation for $x = (x_1, x_2)$. So $S = \varphi(\Omega)$, and $\varphi$ is a one-to-one mapping of class $C^\infty(\Omega); \Gamma$ is the boundary of the domain $\Omega$, $\{e_{ij}\}_{i,j=1}^2 = \{\kappa_{ij}\}_{i,j=1}^2$ are matrices of membrane and flexural deformation components, respectively; $a$, $b$ are symmetric uniformly positive definite and uniformly bounded bilinear forms. Matrices $\varepsilon$ and $\kappa$ are determined by the following relations:

$$\partial_1 = A_{11}^{-1}u_{1,n} + k_1u_1, \quad \partial_2 = A_{22}^{-1}u_{2,n} + k_2u_2,$$

$$\sigma_1 = A_{11}^{-1}u_{2,n} - A_{22}^{-1}A_{11}^{-1}A_{11}u_1, \quad \sigma_2 = A_{22}^{-1}u_{2,n} - A_{22}^{-1}A_{22}^{-1}A_{22}u_2,$$

$$\varepsilon_1 = A_{11}^{-1}u_{1,n} - A_{11}^{-1}A_{11}^{-1}A_{11}^{-1}A_{11}u_1, \quad \varepsilon_2 = A_{22}^{-1}u_{2,n} - A_{22}^{-1}A_{22}^{-1}A_{22}^{-1}A_{22}u_2,$$

$$\tau_1 = A_{11}^{-1}\partial_1 + A_{11}^{-1}A_{11}^{-1}A_{11}^{-1}A_{11}\partial_1, \quad \tau_2 = A_{22}^{-1}\partial_2 + A_{22}^{-1}A_{22}^{-1}A_{22}^{-1}A_{22}\partial_2,$$

$$\kappa_{11} = A_{11}^{-1}\partial_1 + A_{11}^{-1}A_{11}^{-1}A_{11}^{-1}A_{11}\partial_1, \quad \kappa_{22} = A_{22}^{-1}\partial_2 + A_{22}^{-1}A_{22}^{-1}A_{22}^{-1}A_{22}\partial_2,$$

$$\kappa_{12} = (\tau_1 + \tau_2 + k_1\omega_1 + k_2\omega_2)/2 \quad \omega = \omega_1 + \omega_2,$$

$$\epsilon_{ij} = \epsilon_i + \dot{\xi}^i / 2, \quad e_{ij} = e_j + \dot{\xi}^j / 2, \quad e_{ij} = \omega + \partial_1\partial_2.$$

Here $A_1$, $A_2$ are Lame parameters, $k_1$, $k_2$ are the main curvatures of the surface $S$. It is assumed that $A_1$, $A_2 \in C^\infty(\Omega)$, $A_1, A_2 > 0$, $k_1, k_2, a_{ijkl}, b_{ijkl} \in C^\infty(\Omega), \Gamma \in C^\infty$.

3. The study of the solvability of the equilibrium of the shell

We write the matrix components of membrane deformations in the form $e(u) = e(u) + d(u, u)$, where $e(u)$ are linear components of membrane deformations, $d(u, v)$ are bilinear symmetric matrix functions, and, as it is easy to see, the estimate

$$|d(u, v)| \leq c(|u| + |\nabla u_1| + |v| + |\nabla v_1|), \quad c = \text{const} > 0,$$

holds.

The problem of finding a stationary point of the functional $F$ can be formulated as a problem of solving the equation

$$Au = f, \quad x \in \Omega,$$

with main boundary conditions, where $A$ is the differential operator that can be defined with the help of identity
The derivative of the operator $A$ is formally determined by the relationship
\[
\int_{\Omega} A'(u)v - \eta d\Omega = \int_{\Omega} (a(e(v), e(\eta)) + b(\kappa(v), \kappa(\eta)))d\Omega \quad \forall u, \eta \in C^0_0(\Omega), \quad d\Omega = A_1 A_2 dx
\]

It is important to note that $A'(0)$ is the the operator of linear shell theory. In this regard, for the investigation of the solvability of equation (1) is useful.

**Theorem 1 (Kantorovich).** Let $P: E_1 \rightarrow E_2$, and $E_1, E_2$ be a Banach spaces. Operator $P$ has the Frechet derivative, and
\[
\|P'(u) - P'(v)\| \leq L\|u - v\| \quad \forall u, v \in B(R) = \{u \in E_1, \|u\| \leq R\},
\]
\[
\|P'(0)^{-1}\| \leq h_0, \quad \|P'(0)^{-1}P(0)\| \leq \eta_0, \quad h_0 = h_0 L \eta_0 \leq 1/2,
\]
\[
r_0 = \eta_0 (1 - \sqrt{1 - 2h_0}) / h_0 \leq R.
\]

Then in the ball $B(r_0)$ there is a solution of the equation $Pu = 0$, and it can be found at help of Newton-Kantorovich iterative process
\[
P'(u^n)(u^{n+1} - u^n) = -P(u^n), \quad n = 0, 1, \ldots, u^0 = 0.
\]

If $h_0 < 1/2$, then the solution of the equation $Pu = 0$ in the ball $B(r_0)$ is unique.

The estimates necessary for the application of Theorem 1 are based on the coercivity inequality for the operator $A'(0)$ (or on so called Korn type inequality)
\[
\|u_1, u_2\|_{W^1_2(\Omega)} + \|u_3\|_{W^1_2(\Omega)} \leq c\left(\|e(u_1)\|_{W^1_2(\Omega)} + \|\kappa(u_1)\|_{W^1_2(\Omega)} + \|u_1, u_2\|_{W^1_2(\Omega)} + \|u_3\|_{W^1_2(\Omega)}\right)
\]

And on the description of the subspace of rigid displacements:
\[
R = \{u: e(u) = 0, \kappa(u) = 0, x \in \Omega\} = \{u(x) = a + W\phi(x), W = -W^T \quad \forall x \in \overline{\Omega}\},
\]

where vector $a$ and matrix $W$ are independent of $x$, see [3]–[6], and [7].

We denote by $V_0$ the subspace of the functions of $W^1_2(\Omega) \times W^2_2(\Omega) \times W^2_2(\Omega)$ satisfying the main (kinematic) boundary conditions. Analogously, [8], we set

**Theorem 2.** If
\[
\int_{\Omega} f \cdot v d\Omega = 0 \quad \forall v \in V_0 \cap R,
\]
then there is such a constant $c_1 > 0$ defined only by the coefficients of bilinear forms $a$, $b$ and geometry of the surface $S$ that the functional $F$ has at least one critical point on the space $V_0$, if additionally $\|f\|_{L^2(\Omega)}^2 \leq c_1$. 

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For example, if \( V_0 = \{ u : u(x) = 0, \partial u(x) / \partial x = 0, x \in \Gamma_0 \} \), where \( \Gamma_0 \subset \Gamma \), mes(\( \Gamma_0 \)) > 0, \( \nu \) is normal to \( \Gamma \), i.e., the conditions of rigid fixing are fulfilled on \( \Gamma_0 \), then (see [6] as well as [7]) and the functional \( F \) has at least one critical point on the space \( V_0 \) for any sufficiently small external load \( f \).

4. Mixed finite-element method

Let \( T_h \) be a regular conformal triangulation of the domain \( \Omega \), and \( h \) be the maximum of the diameters of the triangles of \( T_h \), \( H_i (H_i) \) is the subspace of \( W^1_p(\Omega) (W^1_p(\Omega)) \) of the functions, which are polynomials of degree \( l \geq 1 \) by the set of variables on each of the triangles of \( T_h \). Let also \( V_h = H_{l+1} \times H_{l+1} \times H_l, l \geq 2 \).

We will say that \( y \in V_h \) is an approximate solution the shell equilibrium problem if \( y \) is the critical point of the functional

\[
F_h(u) = \int (a(e, \kappa) + b(\kappa, \kappa))dS - \int f \cdot udS,
\]

on the space \( V_h \). In this case, instead of the second derivatives of the function \( y \) by their approximations we use the functions \( w^\delta (y, \kappa) \in H_i, i = 1,2 \), defined from equations

\[
\int w^\delta (y, \kappa) \eta d\Omega - \frac{1}{2} \int_\Omega \left( \frac{\partial^2 y}{\partial x_i \partial \eta} + \frac{\partial^2 y}{\partial x_j \partial \eta} \right) d\Omega \quad \forall \eta \in H_i.
\]

This method is not conformal, i.e., \( V_h \nsubseteq V \). Solvability of the discrete problem and a priori estimates (estimates of accuracy) can be established only by sufficiently small \( f \) and \( h \). In particular, if an addition

\[
u_i, u_2 \in W^1_2(\Omega), u_3 \in W^{l+1}_2(\Omega),
\]

then

\[
\sum_{i=1}^3 \| v_i - y_i \|_{W^1_2(\Omega)} + \sum_{i,j=1}^3 \| u_{ij} - w^\delta (y, \kappa) \|_{W^1_2(\Omega)} \leq ch^{l+1}, \quad c = \text{const} > 0.
\]

5. Shallow shells. Problem statement. The investigation of solvability

The problem of equilibrium of a shallow shell is formulated as a problem of minimization of the functional (potential energy)

\[
\Phi(u) = \int_\Omega \varphi(\kappa, \epsilon) dx - \int f \cdot u dx - \int g \cdot u dx.
\]

Here \( \Omega \subset \mathbb{R}^2 \) is a bounded domain identified with the middle surface of the shell, \( \Gamma \) is the boundary of the domain \( \Omega \). \( \varphi : \mathbb{R}^6 \rightarrow \mathbb{R} \) is the density of the potential energy of deformation of the middle surface of the shell. The middle surface of the shell will be referred to the Cartesian coordinate system \( x_1, x_2, x_3 \), the axis \( x_3 \) is directed along the normal to the middle surface, \( f, g \), are given functions, characterizing the density of external loads. The tangential deformation \( \kappa \) and bending deformation of the shell is determined by the following values

\[
\kappa_{ij} (u) = \kappa_{ij} (u) + \kappa_{ij} u_{ij} + \frac{1}{2} \frac{\partial u_{ij}}{\partial x_i} \frac{\partial u_{ij}}{\partial x_j}, \quad \kappa_{ij} (u) = 1 \left( \frac{\partial u_{ij}}{\partial x_i} + \frac{\partial u_{ij}}{\partial x_j} \right), \quad i, j = 1,2,
\]

\[
e_{ij} (u) = e_{ij} (u) + \epsilon_{ij} u_{ij} + \frac{1}{2} \frac{\partial u_{ij}}{\partial x_i} \frac{\partial u_{ij}}{\partial x_j}, \quad \epsilon_{ij} (u) = 1 \left( \frac{\partial u_{ij}}{\partial x_i} + \frac{\partial u_{ij}}{\partial x_j} \right), \quad i, j = 1,2,
\]
where \(k_{ij}, i, j = 1,2\), are initial curvatures of the shell. In the simplest case, when the shell material obeys Hooke's law,

\[
\varphi(\varepsilon, \kappa) = \sum_{i,j,l=1}^{2} a_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \sum_{i,j,l=1}^{2} b_{ijkl} \kappa_{ij} \kappa_{kl},
\]

where quadratic forms with coefficients \(a_{ijkl}, b_{ijkl}\) are positively defined. It is assumed, in general, that the function \(\varphi\) is continuous, convex, and there are constants \(c_0, c_i > 0, p \in (1, \infty)\) such that

\[
c_{0} | \xi |^{p} \leq \varphi(\xi) \leq c_{1} | \xi |^{p} \quad \forall \xi \in \mathbb{R}^{6}.
\]

Let \(V = W_{p}^{1}(\Omega) \times W_{p}^{1}(\Omega) \times W_{p}^{2}(\Omega), p > 1\). We denote through \(V_0\) the subspace of the space \(V\) obtained by the closure of the set \(C_{0}^{\infty}(\Omega) \times C_{0}^{\infty}(\Omega) \times C_{0}^{\infty}(\Omega)\). We denote through \(V_{1}\) the subspace of the space \(V\) obtained by the closure of the set of the functions from \(C^{\infty}(\Omega) \times C^{\infty}(\Omega) \times C^{\infty}(\Omega)\). satisfying the boundary conditions \(\bar{u}(x) \cdot n(x) = 0, x \in \Gamma\), where \(\bar{u}(x) = (u_{i}(x), u_{j}(x))\) is the vector of the tangential displacements; \(V_{2}\) is the subspace of functions from \(V\) obtained by the closure of the set of functions \(C^{\infty}(\Omega) \times C^{\infty}(\Omega) \times C_{0}^{\infty}(\Omega)\) with boundary conditions \(\bar{u}(x) \cdot \tau(x) = 0, x \in \Gamma\), where \(\tau\) is the tangent vector to \(\Gamma\). We believe further that \(f \in [L_{q}(\Omega)]^{3}, g \in [L_{q}(\Gamma)]^{3}, 1/p + 1/q = 1, k_{ij} \in L_{p_{i}}(\Omega), p_{i} > p\).

We will consider the following problems.

I. The problem of minimizing of the functional \(\Phi\) on the space\(V_{0}\). It is the Dirichlet problem.

II. The problem of minimization of the functional \(\Phi\) on the space \(V_{1}\). It is the problem of the rigid contact of the boundary of the shell.

III. The problem of minimizing the functional \(\Phi\) on the space \(V_{2}\). It is the problem of the normal load.

Applied to the systems of the linear differential equations, in particular, to the systems of equations of the linear elasticity theory similar problems were considered in [9]–[15].

Let \(R_{2}\) be the linear space of vector functions \(\bar{u}(x) = (u_{i}(x), u_{j}(x))\) defined on \(\Omega\) and such that \(u_{i}(x) = a_{i} + \delta x_{i}, u_{j}(x) = a_{j} - \delta x_{j}\), where \(a_{i}, a_{j}, \delta\) are constants. Let \(V_{2}, V_{1,2}, V_{2,2}\) be the linear spaces of two-dimensional vector functions defined on \(\Omega\) and such that the their components coincide with the first two components of the spaces \(V_{2}, V_{1}, V_{2}\), respectively.

**Lemma 1.** If the domain \(\Omega\) is a circle, then \(R_{2} \cap V_{1,2}\) is the set of all vector functions \(\bar{u}(x) = (u_{i}(x), u_{j}(x))\) of the form \(u_{i}(x) = x_{i}, u_{j}(x) = -\delta x_{j}, x \in \Omega, \delta = \text{const}\). Otherwise \(R_{2} \cap V_{1,2} = \{0\}\).

**Lemma 2.** For any region \(\Omega\) with boundary of class \(C^{1}\) we have \(R_{2} \cap V_{1,2} = \{0\}\).

**Lemma 3.** For the solvability of problem II in the case where \(\Omega\) is a circle (centered at the beginning coordinates), the following condition must be fulfilled

\[
\int_{\Omega} (f_{1}x_{2} - f_{2}x_{1}) dx + \int_{\Gamma} (g_{3}x_{2} - g_{2}x_{3}) dx = 0.
\]

In accordance with Lemma 3, the solution of problem II will be further understood as the solution of the functional \(\Phi\) minimization problem on the space \(V_{1,2}\). In this case, for the loads \(f, g\) can be considered completion condition (2).

**Theorem 3.** Let the above conditions be satisfied for the function \(\varphi: \mathbb{R}^{6} \rightarrow \mathbb{R}\).

Then each of the problems I III has a solution under one of the following conditions:

1) \(p > 2\);
2) $1 < p < 2$, $f_1, f_2, g_1, g_2 = 0$;
3) $p = 2$ and $f_1, f_2, g_1, g_2$ are sufficiently small in the sense of the norms $L_p(\Omega), L_2(\Gamma)$, respectively.

6. Shallow shell. Mixed finite element method. Construction and investigation of solvability

Let us limit ourselves in further of the consideration of the problem II. The problems I and III are studied similarly. It is assumed that the region $\Omega$ is a polygon. The regularity condition of the Dirichlet problem for the Poisson equation is fulfilled

$$\int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \eta dx = 0, \quad \forall \eta \in H_1,$$

which is the solution of the minimization problem

$$\min_{y \in V_0} \Phi_h(y),$$

where

$$\Phi_h(y) = \int_\Omega \varphi(y, \kappa^h(y)) dx - \int_\Omega f \cdot u dx - \int_\Omega g \cdot y dx,$$

and functions $\kappa^h(y) = \{\kappa_{ij}^h(y)\}_{i,j=1}^2$ are determined by means of relations

$$\int_\Omega \kappa_{ij}^h(y) \eta dx = \frac{1}{2} \int_\Omega \left( \frac{\partial y_i}{\partial x_j} \frac{\partial \eta}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \frac{\partial \eta}{\partial x_i} \right) dx \quad \forall \eta \in H_1,$$

which are, in fact, the systems of linear equations with respect to the nodal parameters of the functions $\kappa_{ij}^h(y), i, j = 1, 2$.

The study of the solvability of the problem (3) is similar to the study of the solvability of the original problem (problem II) and is based on the generalized Weierstrass theorem (see, for example, [17]). The differences and additional difficulties are caused by the fact that problem (3) is not an internal (conformal) approximation of the problem II. The main role is played the discrete analogue of the embedding theorem

**Lemma 4.** For each function $y \in \mathcal{H}_1$

$$\|y\|_{W^{1,2}_p(\Omega)} \leq c \sum_{i,j=1}^2 \left\|w_{i,j=1}^h(y)\right\|_{L_2(\Omega)},$$

where $w_{i,j}^{h,j} = -\kappa_{i,j}^h$.

7. Convergence of the method, and accuracy estimates

The convergence of the proposed method establishes

**Theorem 4.** There is a sequence $y = (y_1, y_2, y_3, y_4)$ of solutions of the problem (3) such that $y_1 \to u_1, i = 1, 2, 3,$ $W^1_p(\Omega), \ k_{ij}^h(y_3) \to -u_{3,j}, e_{ij}^h(y) \to e_{ij}^h(u), i, j = 1, 2,$ in $L_p(\Omega)$ weakly at $h \to 0$, where $u = (u_1, u_2, u_3)$ is a solution of the problem II.
We assume by the obtaining the accuracy estimates, that the function $\varphi$ is differentiable on $R^6$, and, if $p \geq 2$, then
\[
|\nabla \varphi(\xi) - \nabla \varphi(\eta)| \leq c_1 |\xi - \eta| (|\xi| + |\eta|)^{p-1} \quad \forall \xi, \eta \in R^6,
\]
\[
(\nabla \varphi(\xi) - \nabla \varphi(\eta)) \cdot (\xi - \eta) \geq c_2 |\xi - \eta|^p \quad \forall \xi, \eta \in R^6.
\]
If $1 < p < 2$, then
\[
|\nabla \varphi(\xi) - \nabla \varphi(\eta)| \leq c_1 |\xi - \eta|^{p-1} \quad \forall \xi, \eta \in R^6,
\]
\[
(\nabla \varphi(\xi) - \nabla \varphi(\eta)) \cdot (|\xi| + |\eta|)^{2-p} \cdot (\xi - \eta) \geq c_4 |\xi - \eta|^2 \quad \forall \xi, \eta \in R^6,
\]
where $c_1, \ldots, c_4$ are positive constants.

**Theorem 5.** Let $l > 2$, $u_1, u_2 \in W_p^l(\Omega)$, $u_3 \in W_p^{l+1}(\Omega)$. Then if $p > 2$, we have
\[
\sum_{i,j=1}^{2} \left\| \frac{\partial^2 u_3}{\partial x_i \partial x_j} - w^p(y_3) \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left\| t_i - y_i \right\|_{W_p^l(\Omega)}^2 \leq c h^{l-4}(p-1),
\]
if $1 < p < 2$,
\[
\sum_{i,j=1}^{2} \left\| \frac{\partial^2 u_3}{\partial x_i \partial x_j} - w^p(y_3) \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left\| t_i - y_i \right\|_{W_p^l(\Omega)}^2 \leq c h^{l-4}(p-1).
\]
For more details of the results of the sections 5–7 see [18].

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