Extrinsic Black Hole Uniqueness in Pure Lovelock Gravity

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Received: 12 July 2021 / Accepted: 21 September 2021 / Published online: 5 October 2021
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Abstract
We define a notion of extrinsic black hole in pure Lovelock gravity of degree $k$ which captures the essential features of the so-called Lovelock-Schwarzschild solutions, viewed as rotationally invariant hypersurfaces with null $2k$-mean curvature in Euclidean space $\mathbb{R}^{n+1}$, $2 \leq 2k \leq n - 1$. We then combine a regularity argument with a rigidity result by Araújo and Leite (Indiana University Mathematics Journal pp. 1667–1693, 2012) to prove, under a natural ellipticity condition, a global uniqueness theorem for this class of black holes. As a consequence we obtain, in the context of the corresponding Penrose inequality for graphs established by Ge et al. (Advances in Mathematics 266: 84–119, 2014), a local rigidity result for the Lovelock-Schwarzschild solutions.

Keywords Lovelock gravity · Black hole solution · $p$th mean curvature · Penrose inequality

Mathematics Subject Classification 53C21 · 53C80 · 83C57 · 83D05

L.L. de Lima has been partially supported by CNPq/Brazil grant 312485/2018-2, F. Girão by CNPq/Brazil grant 307239/2020-9 and J. Natário by FCT/Portugal grant UIDP/MAT/04459/2020 The authors have also benefited from support coming from FUNCAP/CNPq/PRONEX grant 00068.01.00/15.

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1 Introduction

Since the seminal work by Israel (1967), the study of uniqueness properties of black hole solutions of Einstein field equations remains an active field of research; see (Heusler 1996; Robinson 2009; Chruściel et al. 2012) for recent reviews on this subject. From a physical viewpoint, this body of knowledge captures the accepted perception that “black holes have no hair”, which roughly means that stationary black hole solutions satisfying suitable boundary conditions (both at the horizon and at spatial infinity) should be completely characterized by a few externally observable conserved charges (mass, angular momentum, etc.) For instance, in the static, asymptotically flat case, a celebrated result by Bunting and Masood-ul Alam (1987) guarantees that the black hole is completely determined by its ADM mass (and hence coincides with a Schwarzschild solution). A remarkable feature of this latter contribution is that its proof uses in an essential way the rigidity statement of the positive mass theorem due to Schoen and Yau (1981).

In recent years, there has been a renewed interest in studying gravity theories in higher dimension $N = n + 1$, $n \geq 4$, possibly coupled to certain kinds of matter fields (scalar, gauge, etc.), as these abstractions potentially qualify as low energy limits of certain versions of superstring theory. It turns out, however, that some of these theories, even if restricted to the Einsteinian setting, display exotic black hole solutions which can not be fully characterized by their conserved charges. In particular, black hole uniqueness, at least as classically envisaged, fails to hold; see (Emparan and Reall 2008; Horowitz 2012; Hollands and Ishibashi 2012) for discussions of these phenomena.

The purpose of this note is to indicate that, at least for pure Lovelock gravity, a natural and elegant extension of Einstein’s general relativity in higher dimensions, an appropriate version of black hole uniqueness may be restored under suitable assumptions. As explained in Sect. 2, we consider asymptotically flat, time-symmetric vacuum solutions in pure Lovelock gravity whose initial data set carries a compact (but not necessarily connected) inner horizon. These objects appear as extremal configurations in a conjectured Penrose inequality which so far has been checked in rather special cases (Ge et al. 2014a, b; Li et al. 2014; de Sousa and Girão 2019). Notice that these competing solutions are not necessarily stationary, so our result somehow departs from the classical uniqueness theorems discussed above. On the other hand, our method of proof demands that the “degree $k$” Lovelock black holes we consider display the essential features of the so-called Lovelock-Schwarzschild solutions, viewed as embedded, rotationally invariant hypersurfaces with null $2k$-mean curvature in Euclidean space $\mathbb{R}^{n+1}$, $2 \leq 2k \leq n - 1$ (this explains the qualification “extrinsic” in the title). If we further assume that a certain ellipticity condition is satisfied everywhere, then our main result (Theorem 1 below) confirms that the black hole solution is actually congruent to a Lovelock-Schwarzschild solution. In particular, it is completely determined by its Gauss–Bonnet–Chern mass, a feature which clearly makes contact with the standard formulation of black hole uniqueness. As a consequence we obtain, in the context of the corresponding Penrose inequality for graphs established by Ge et al. (2014b), a local rigidity result for the Lovelock–Schwarzschild solutions.
This note is organized as follows. In Sect. 2 we review the pertinent aspects of pure Lovelock gravity of degree $k$, including the explicit description of the corresponding Lovelock–Schwarzschild black holes as embedded, rotationally invariant hypersurfaces with null $2k$-mean curvature in Euclidean space $\mathbb{R}^{n+1}$, $2 \leq 2k \leq n - 1$. This section also contains the precise statement of our main result (Theorem 1), whose proof is presented in Sect. 4, with a preliminary discussion of the approach occurring in Sect. 3. The main technical ingredient in the proof of Theorem 1 is a remarkable rigidity result by (Araújo and Leite 2012, Theorem 3.1), which extends previous contributions in de Lima and Sousa (2011), Hounie and Leite (1999), Schoen (1983). This result completely classifies two-ended extrinsic Lovelock black hole solutions and our reasoning essentially boils down to doubling the given one-ended Lovelock black hole solution across the horizon, which is assumed to lie in a fixed hyperplane $\Pi \hookrightarrow \mathbb{R}^{n+1}$. It turns out, however, that the so constructed two-ended black hole is in principle only of class $C^{1,1}$ along the horizon. It is precisely at this point that the ellipticity assumption comes into play, as it allows us to explicitly carry out a regularity argument in order to restore $C^2$-smoothness. Finally, Theorem 4 describes the Penrose-type local rigidity result mentioned above.

2 Pure Lovelock Gravity and Statement of the Main Result

Starting with Einstein’s relativity, it has been highly desirable to characterize feasible gravity theories which have a Lorentzian metric $\mathcal{g}$ as its gravitational dynamical field by a few natural properties. According to this axiomatic approach, general covariance should lead to field equations of the type

$$G(\mathcal{g})_{ij} = T_{ij},$$  \hspace{1cm} (1)

where the left-hand side is a twice covariant symmetric tensor whose (local) dependence on $\mathcal{g}$ must involve, in the simplest case we will be interested in, derivatives up to second order, whereas the right-hand side $T$ collects the non-gravitational fields in the theory. Moreover, the usual (local) conservation law imposes a divergence-free condition on $G$:

$$\nabla^i G(\mathcal{g})_{ij} = 0,$$  \hspace{1cm} (2)

where $\nabla$ is the Levi-Civita connection associated to $\mathcal{g}$. A celebrated result by Lovelock (1972) confirms that in the physical dimension $3 + 1$, these conditions uniquely determine $G(\mathcal{g})$ as the well-known Einstein tensor plus a constant multiple of the metric:

$$G(\mathcal{g}) = \text{Ric}_{\mathcal{g}} - \frac{S_{\mathcal{g}}}{2} \mathcal{g} + \Lambda \mathcal{g},$$  \hspace{1cm} (3)

where $\text{Ric}_{\mathcal{g}}$ is the Ricci tensor of $\mathcal{g}$ and $S_{\mathcal{g}} = \text{tr}_{\mathcal{g}} \text{Ric}_{\mathcal{g}}$ is the scalar curvature. Here and in the following we set the cosmological constant $\Lambda$ to vanish.
However, in higher dimension \( N = n + 1 \), \( n \geq 4 \), new phenomena emerge and this characterization of the Einstein tensor only remains true if we further assume that \( G \) depends linearly on the second derivatives \( \partial^2 \bar{g} \) of \( \bar{g} \). More generally, symmetric tensors satisfying (2) and with an arbitrary dependence on \( \partial^2 \bar{g} \) have been classified in Lovelock (1971); see (Navarro and Navarro 2011) for a modern account of this foundational result. It then follows that if we further assume that the dependence on \( \partial^2 \bar{g} \) is homogeneous of degree \( k \) then \( G(\bar{g}) \approx L_{2k}(\bar{g}) \), where the pure Lovelock tensor \( L_{2k}(\bar{g}) \) is given by

\[
L_{2k}(\bar{g})_{ij} = \bar{g}_{ij} \delta^{i_1 j_1 ... i_{2k-1} j_{2k}} R^{j_1 j_2}_{i_1 i_2} ... R^{j_{2k-1} j_{2k}}_{i_{2k-1} i_{2k}}, \quad 1 \leq k \leq \frac{n}{2}.
\]  

Here, \( \bar{R}^{ij}_{kl} \) are the coefficients of the Riemann curvature tensor of \( \bar{g} \) with respect to a local orthonormal basis of tangent vectors and the symbol \( \approx \) relates quantities that possibly differ by a universal multiplicative constant.

### 2.1 Pure Lovelock Gravity

The field equations for pure Lovelock gravity in vacuum, namely,

\[
L_{2k}(\bar{g}) = 0,
\]

arise from a variational principle involving a geometric Lagrangian density, the so-called Gauss-Bonnet curvature

\[
S_{2k}(\bar{g}) \approx \text{tr}_\bar{g} L_{2k}(\bar{g}) \approx \delta^{i_1 j_1 ... i_{2k-1} j_{2k}}_{j_1 j_2 ... j_{2k-1} j_{2k}} \bar{R}^{j_1 j_2}_{i_1 i_2} ... \bar{R}^{j_{2k-1} j_{2k}}_{i_{2k-1} i_{2k}}.
\]

Thus, a metric \( \bar{g} \) extremizes the Lovelock action

\[
\bar{g} \mapsto \int S_{2k}(\bar{g})
\]

if and only if (5) is satisfied (Lovelock and Rund 1989; Labbi 2008). Notice that \( S_{2}(\bar{g}) \) is the full contraction of \( \bar{R} \), hence a multiple of \( S_\bar{g} \). Also, similarly to the Einstein tensor (\( k = 1 \)), the Lovelock tensor decomposes as

\[
L_{2k}(\bar{g}) \approx \text{Ric}_{2k}(\bar{g}) - \frac{H_{2k}(\bar{g})}{2} \bar{g},
\]

where \( \text{Ric}_{2k}(\bar{g}) \) is the so-called 2\( k \)-Ricci tensor. In the language of double forms, one has \( \text{Ric}_{2k}(\bar{g}) = c^{2k-1} \bar{R} \) and \( H_{2k}(\bar{g}) = c^{2k} \bar{R} \approx S_{2k}(\bar{g}) \), where \( \bar{R} \) is the Riemann tensor of \( \bar{g} \) and \( c_{\bar{g}} \) is the contraction induced by \( \bar{g} \) (Definition 2.2 Labbi 2008). In particular, \( \text{Ric}_2(\bar{g}) = \text{Ric}_{\bar{g}} \). In this way we obtain a full-fledged gravitational theory (pure Lovelock gravity of degree \( k \)) which happens to be a natural generalization of Einstein’s relativity in vacuum (this corresponds to \( k = 1 \)).
Remark 1 Another compelling indication that pure Lovelock is the right setup for gravity in higher dimensions comes from the writings of N. Dadhich and collaborators, where it is argued that this theory shares many of the congenial features of Einstein’s relativity, including kinematicity (Dadhich 2016), the existence of bound orbits around static subjects (Dadhich et al. 2013) and thermodynamical universality (Dadhich et al. 2012); see (Dadhich 2016) and the references therein for more on these issues.

Remark 2 It is useful to provide a coordinate free formulation of Lovelock theory. If \((\bar{M}, \bar{g})\) is the underlying Lorentzian manifold, \(N = n + 1\), recall that the metric \(\bar{g}\) induces a natural bundle isomorphism \(T\bar{M} \cong T^*\bar{M}\). Given a vector field \(Z \in \Gamma(T\bar{M})\) and a (local) volume element \(\Omega\) one has, by Cartan’s magic formula,

\[
di_Z \Omega = di_Z \Omega + i_Z d\Omega = \mathcal{L}_Z \Omega = (\text{div}_\bar{g} Z) \Omega,
\]

where \(i_Z\) is contraction with \(Z\), \(\mathcal{L}_Z\) is Lie derivative and \(\text{div}_\bar{g}\) is the divergence. Thus, the correspondence \(Z \leftrightarrow \omega = i_Z \Omega\) gives rise to an isomorphism between \(\Gamma(T^*\bar{M}) \cong \Gamma(T\bar{M})\) and \(\Gamma(\Lambda^n T^*\bar{M})\) such that \(\text{div}_\bar{g} Z = 0\) if and only if \(d\omega = 0\). Similarly, the correspondence \(Z_1 \otimes Z_2 \leftrightarrow i_{Z_1} \Omega \otimes i_{Z_2} \Omega\) defines an isomorphism between \(\Gamma(\otimes^2 T^*\bar{M})\), the space of twice covariant tensors, and \(\Gamma(\Lambda^n T^*\bar{M} \otimes \Lambda^n T^*\bar{M})\), the space of \(n\)-forms with values in \(n\)-forms, which is well defined globally even if \(\bar{M}\) is not orientable. Also, restriction to \(S^2(T^*\bar{M}) \subset \Gamma(\otimes^2 T^*\bar{M})\), the space of symmetric \((2, 0)\)-tensors, defines an isomorphism between \(S^2(T^*\bar{M})\) and \(S^2(\Lambda^n T^*\bar{M})\), where, by definition, \(S^2(\Lambda^p T^*\bar{M})\) is formed by those \(\eta \in \Gamma(\Lambda^p T^*\bar{M} \otimes \Lambda^p T^*\bar{M})\) which are symmetric in the sense that

\[
\eta(v_1 \wedge \ldots \wedge v_p \otimes w_1 \wedge \ldots \wedge v_p) = \eta(w_1 \wedge \ldots \wedge w_p \otimes v_1 \wedge \ldots \wedge v_p). \quad (7)
\]

Also, a computation shows that \(T \in S^2(T^*\bar{M})\) satisfies \(\text{div}_\bar{g} T = 0\) if and only the corresponding \(\eta \in S^2(\Lambda^n T^*\bar{M})\) satisfies \(d\text{div}_\bar{g} \eta = 0\), where, as an operator acting on \((7)\), \(d\text{div}_\bar{g}\) is the covariant exterior derivative defined by using the standard exterior derivative in the first factor and the covariant derivative \(\nabla_\bar{g}\) induced by the Levi-Civita connection in the second factor. Now, \(\bar{g} \in S^2(\Lambda^1 T^*\bar{M})\) and \(\bar{R} \in S^2(\Lambda^2 T^*\bar{M})\), so we may define

\[
\tilde{L}_{2k}(\bar{g}) = \underbrace{\bar{R} \wedge \ldots \wedge \bar{R} \wedge \bar{g} \wedge \ldots \wedge \bar{g}}_{k} \in S^2(\Lambda^n T^*\bar{M}), \quad 1 \leq k \leq \frac{n}{2}. \quad (8)
\]

Since \(d\text{div}_\bar{g} \bar{g} = 0\) (metric compatibility) and \(d\text{div}_\bar{g} \bar{R} = 0\) (Bianchi identity), we get \(d\text{div}_\bar{g} \tilde{L}_{2k}(\bar{g}) = 0\). Thus, the corresponding \((2, 0)\)-tensor \(\tilde{L}_{2k}(\bar{g}) \in S^2(T^*\bar{M})\) satisfies \(\text{div}_\bar{g} \tilde{L}_{2k}(\bar{g}) = 0\). By Lovelock’s theorem mentioned earlier, \(\tilde{L}_{2k}(\bar{g})\) is a (universal) multiple of the Lovelock tensor \(L_{2k}(\bar{g})\).
2.2 Lovelock–Schwarzschild black holes

It turns out that the analogy of (pure) Lovelock gravity with Einstein relativity goes one step further, the reason being that the field Eq. (5) admit a one-parameter family of static, spherically symmetric black hole solutions (Crisostomo et al. 2000; Cai and Ohta 2006; Kastikainen 2019). More precisely, in standard coordinates \((t, r, \theta)\), \(\theta \in \mathbb{S}^{n-1}\), the solution is given by

\[
\overline{g}_{k,m} = -V_{k,m}(r) dt^2 + g_{k,m}, \quad g_{k,m} = \frac{dr^2}{V_{k,m}(r)} + r^2 d\theta^2, \quad V_{k,m}(r) = 1 - \frac{2m}{r^{2k-2}},
\]

(9)

where \(m > 0\) is a real parameter whose physical interpretation we discuss later on; see the discussion surrounding (27). Since the space-like slice \(t = 0\) is totally geodesic, it follows that the Riemannian metric \(g_{k,m}\) satisfies the curvature condition

\[
S_{2k}(g_{k,m}) = 0.
\]

Conversely, at least in the time-symmetric case, this curvature condition may be viewed as the constraint equation a space-like initial data set should satisfy in order to have its Cauchy development yielding a solution of (5) (Choquet-Bruhat 1988; Teitelboim and Zanelli 1987).

The black hole character of the metric \(\overline{g}_{k,m}\) in (9) manifests itself in the fact that it displays an event horizon located at \(r_{k,m} = (2m)^{k/(n-2k)}\). As in the classical situation, this suggests that the solution may be continued somehow beyond the horizon. Clearly, it suffices to perform this continuation for the space-like metric \(g_{k,m}\) and this may be accomplished in two rather distinct, but related, ways. First, we may work intrinsically and verify that

\[
g_{k,m} = \left(1 + \frac{m}{2\rho^{2k-2}}\right)^{\frac{2k}{n-2k}} \left(d\rho^2 + \rho^2 d\theta^2\right), \quad \rho > 0,
\]

(11)

which provides a conformal representation of the black hole metric with the horizon now located at \(\rho_{k,m} = (m/2)^{k/(n-2k)}\). The corresponding lapse is

\[
l(\rho) = \left(1 - \frac{m}{2\rho^{2k-2}}\right)^{-1} \left(1 + \frac{m}{2\rho^{2k-2}}\right)^{2},
\]

which confirms the null character of the event horizon \(\rho = \rho_{k,m}\). We refer to the pair \((M_{k,m}, g_{k,m})\), where \(M_{k,m} := (r_{k,m}, +\infty) \times \mathbb{S}^{n-1}\), as a one-ended Lovelock-Schwarzschild black hole (associated to the pair \((k, m)\)). When \(k = 1\) we recover the Schwarzschild solution of Einstein gravity.
2.3 Extrinsic Lovelock–Schwarzschild black holes

For our purposes, it is convenient to work extrinsically and realize the Lovelock–Schwarzschild black holes mentioned earlier by means of an isometric embedding $(M_{k,m}, g_{k,m}) \hookrightarrow (\mathbb{R}^{n+1}, \delta)$, where $\delta$ is the flat metric. The key observation is that the passage from (4) to (6), when adapted to a general Riemannian $n$-manifold $(M, g)$, implies that $S_{2k}(g) = \text{tr}_g L_{2k}(g)$. If we further assume that $(M, g) \hookrightarrow (\mathbb{R}^{n+1}, \delta)$ isometrically then by Gauss equation the curvature tensor $R$ of $g$ satisfies

$$R = \frac{1}{2} A \wedge A,$$

where $A \in S^2(\Lambda^1 T^* M)$ is the shape operator of the embedding. In this Riemannian setting, (8) becomes

$$\tilde{L}_{2k}(g) = \frac{1}{2^k} \frac{A \wedge \ldots \wedge A \wedge g \wedge \ldots \wedge g}{n-1-2k}, \quad 1 \leq k \leq \frac{n-1}{2},$$

and a computation shows that $S_{2k}(g) \approx \sigma_{2k}(A)$, so that the time-symmetric constraint (10) becomes

$$\sigma_{2k}(A) = 0. \quad (12)$$

Here, $\sigma_p(A) = \sigma_p(\kappa_1, \ldots, \kappa_n)$, the $p$-mean curvature of $M$, is the elementary symmetric function of degree $p$ in the eigenvalues $\{\kappa_i\}$ of $A$ (the principal curvatures). This discussion suggests that each $g_{k,m}$ should be realized as the induced metric of some rotationally invariant isometric embedding of $M_{k,m}$ satisfying (12). We now proceed to indicate how these invariant embeddings may be classified; compare with (Leite 1990).

Consider the orthogonal group $O_n$ acting isometrically on $(\mathbb{R}^{n+1}, \delta)$ and leaving invariant the axis $x_{n+1} = t$. If $x_1 = s$ is another coordinate axis orthogonal to $x_{n+1}$ at the origin then the half-plane $P = \{(s, t); s \geq 0\}$ is the orbit space of the $O_n$-action. If $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ is rotational, i.e. $O_n$-invariant, then we denote by $\alpha \subset P$ its profile curve. We assume that $\alpha = \alpha(\tau)$ is parametrized by arc length:

$$s^2 + t^2 = 1, \quad (13)$$

where the dot stands for derivation with respect to $\tau$. If $\theta = (\theta_1, \ldots, \theta_{n-1})$ are standard coordinates in the spherical orbit then the induced metric on $\Sigma$ is

$$g = d\tau^2 + s(\tau)^2 d\theta^2.$$

From this we easily compute the sectional curvatures of $g$:

$$K(\partial_{\theta_i}, \partial_{\theta_j}) = \frac{1 - \dot{s}^2}{s^2}, \quad K(\partial_{\theta_i}, \partial_{\tau}) = -\frac{\ddot{s}}{s},$$

where $\dot{s} = \frac{ds}{d\tau}$ and $\ddot{s} = \frac{d^2s}{d\tau^2}$.
which by Gauss equation give the principal curvatures of $\Sigma$:

$$\kappa_1 = \cdots = \kappa_{n-1} = \frac{\sqrt{1 - \dot{s}^2}}{s}, \quad \kappa_n = -\frac{\ddot{s}}{\sqrt{1 - \dot{s}^2}}.$$ 

Thus,

$$\sigma_p(A) = \left(\frac{n - 1}{p}\right) \left(\frac{\sqrt{1 - \dot{s}^2}}{s}\right)^p - \left(\frac{n - 1}{p - 1}\right) \frac{\dot{s}}{\sqrt{1 - \dot{s}^2}} \left(\frac{\sqrt{1 - \dot{s}^2}}{s}\right)^{p-1},$$

so that $\sigma_{2k}(A) = 0$, $2 \leq 2k \leq n - 1$, if and only if

$$(n - 2k)(1 - \dot{s}^2) - 2ks\dot{s} = 0. \quad (14)$$

We proceed by observing that the quantity

$$C(s, \dot{s}) = s^{\frac{n}{k} - 2}(1 - \dot{s}^2)$$

satisfies $\dot{C} = 0$, so $C$ is a first integral for (14). Thus, we are left with the task of classifying profile curves $\alpha = (s, t)$ satisfying

$$s^{\frac{n}{k} - 2}(1 - \dot{s}^2) = 2c, \quad c > 0. \quad (15)$$

By eliminating $t$ from (15) and (13) we see that $t = t(s)$ satisfies

$$\left(\frac{dt}{ds}\right)^2 = \frac{2c}{s^{\frac{n}{k} - 2} - 2c}, \quad s > (2c)^{\frac{k}{n-2k}}. \quad (16)$$

This shows that $\Sigma$ is the union of two vertical graphs $\Sigma^{\pm}$ associated to functions $t = t^{\pm}$ satisfying (16). Since $|dt^{\pm}/ds|(2c)^{k/(n-2k)} = +\infty$, these graphs meet together along the hyperplane $\mathbb{R}^n = \{x_{n+1} = 0\}$ and they both intersect the hyperplane orthogonally. Notice that $\Sigma$ is smooth along the common horizon because (16) easily implies that

$$\frac{d^r s}{d(t^{\pm})^r} \approx s^{\frac{r}{k} - 1 - r}, \quad r \geq 2, \quad (17)$$

which remains finite and converges to a common value as $t^{\pm} \to 0$. In these coordinates, the induced metric on $\Sigma = \Sigma^{+} \cup \Sigma^{-}$ is

$$g_{k,c} = \frac{ds^2}{V_{k,c}(s)} + s^2 d\theta^2, \quad V_{k,c}(s) = 1 - \frac{2c}{s^{\frac{n}{k} - 2}}. \quad (18)$$

Thus, if we set $s = r$ and $c = m$ we see that $(\Sigma^{+}, g_{k,m})$ recovers the one-ended Lovelock–Schwarzschild black hole $(M_{k,m}, g_{k,m})$ as the induced geometry on the
corresponding $O_n$-invariant graph satisfying (12). Also, $(\Sigma, g_{k,m})$ reproduces the two-ended conformal representation in (11) with the horizon being located at the intersection $\Sigma \cap \mathbb{R}^n$. Notice that from (18) we easily check that, as $s \to +\infty$,

$$(g_{k,m})_{ij} = \left(1 + 2ms \frac{n}{2} \right) \delta_{ij} + O(s^{-\frac{n}{2} + 1}),$$

(19)

where $\delta$ is the flat metric in $\mathbb{R}^n$.

### 2.4 Extrinsic Lovelock black holes and the main theorem

After this preliminary discussion, we finally describe the class of extrinsic black holes we are interested in, which appear prominently in our main result (Theorem 1 below). The idea is to consider a one-ended abstract black hole $(M, g)$ which can be isometrically embedded in $(\mathbb{R}^{n+1}, \delta)$ in such a way that it shares the essential features of the extrinsic model $(\Sigma^+, g_{k,m})$ described above. More precisely, we require the following from $(M, g)$:

I. from the abstract viewpoint, the $n$-manifold $M$ carries a (a not necessarily connected) compact inner boundary $\Gamma$ and a unique end, say $E$, which is diffeomorphic to the complement of a ball in $\mathbb{R}^n$;

II. under the isometric embedding $(M, g) \hookrightarrow (\mathbb{R}^{n+1}, \delta)$, $M$ lies entirely inside the half-space $x_{n+1} \geq 0$, $\Gamma$ lies in $\mathbb{R}^n$, and $M \cap \mathbb{R}^n = \Gamma$, with the intersection being orthogonal along $\Gamma$;

III. besides $\sigma_2(A) = 0$, we also assume that $\sigma_{2k+1}(A) \neq 0$ everywhere along $M$, where $A$ is the shape operator of the embedding;

IV. the end $E$ may be written as the graph of a smooth function $u : \mathbb{R}^n \setminus \{|x| \leq R\} \to \mathbb{R}$, $R$ large. Moreover, as $|x| \to +\infty$, $u$ displays the same asymptotic behavior as $t^+$ defined by (16) under the identification $s = |x|$.

Notice that (II) implies that $\Gamma \hookrightarrow M$ is totally geodesic and hence qualifies to be a horizon. In (III), besides the constraint equation $\sigma_2(A) = 0$ (compare with (10) and (12)) we also require the “ellipticity condition” $\sigma_{2k+1}(A) \neq 0$; see Sect. 3 below for more on this point. We emphasize that these conditions are met by our model $(M_{k,m}, g_{k,m})$ viewed as an $O_n$-invariant hypersurface in $\mathbb{R}^{n+1}$.

We now elaborate on the prescribed asymptotic behavior of $u$ as indicated in (IV). For this it is convenient to set

$$q = q_{k,n} = \frac{n}{2k} - 1 \implies \frac{1}{2k} \leq q \leq \frac{n}{2} - 1.$$  

We then consider the following asymptotic regimes mentioned in (IV), which can be read off from the corresponding asymptotic expansion of $t$ in (16); compare with Definitions 3.1 and 4.2 in Araújo and Leite (2012).

- if $q > 1$ then

$$u(x) = a_1 - \frac{a}{q - 1} |x|^{1-q} + \langle c, x \rangle |x|^{-q-1} + O(|x|^{-q-1});$$

(20)
– if $q = 1$ then
\[ u(x) = a_1 + a \log |x| + \langle c, x \rangle |x|^{-2} + O(|x|^{-2}); \]  

– if $(2k)^{-1} \leq q < 1$ we decompose
\[
\left[ \frac{1}{2k}, 1 \right) = \left\{ \frac{1}{2k} \right\} \cup \left( \frac{1}{2k}, \frac{1}{2k-1} \right) \cup \left\{ \frac{1}{2k-1} \right\} \cup \left( \frac{1}{2k-1}, \frac{1}{2k-2} \right] \cup \ldots \cup \left\{ \frac{1}{2} \right\} \\
\ldots \cup \left( \frac{1}{3}, \frac{1}{2} \right] \cup \left( \frac{1}{2}, 1 \right].
\]

To match the notation in (Araújo and Leite 2012, Section 4), we label the intervals in this decomposition as
\[
I_m^1 = \left( \frac{1}{2m+2}, \frac{1}{2m+1} \right), \quad I_m^2 = \left( \frac{1}{2m+3}, \frac{1}{2m+2} \right], \quad 0 \leq m \leq k - 1,
\]

whereas the points are labeled by $q_m = (2m+3)^{-1}$, $0 \leq m \leq k - 3/2$. With this notation, the asymptotics of $u$ in this range of $q$ may be prescribed as follows.

1. if $q \in I_m^1$ then
\[
u(x) = a_1 + P_m(a, |x|) + \langle c, x \rangle |x|^{-1-q} + O(|x|^{1-(2m+3)q});
\]

2. if $q \in I_m^2$ then
\[
u(x) = a_1 + P_m(a, |x|) + a_2 |x|^{1-(2m+3)q} \]
\[+ \langle c, x \rangle |x|^{-1-q} + O(|x|^{1-(2m+5)q});
\]

3. if $q = q_m$ then
\[
u(x) = a_1 + P_m(a, |x|) + B_{m+1} a^{1/q} \log |x| \]
\[+ \langle c, x \rangle |x|^{-1-q} + O(|x|^{-2q}).
\]

Here, for $(2j+1)q < 1$ we set
\[
P_j(a, |x|) = C_0 a |x|^{1-q} + C_1 a^3 |x|^{1-3q} + \cdots + C_j a^{2j+1} |x|^{1-(2j+1)q},
\]

where
\[
C_j = \frac{B_j}{1 - (2j + 1)q}, \quad B_j = \frac{1 \times 3 \times \cdots \times (2j - 1)}{2^j \times j!}.
\]

Let us agree that, in these expansions, $a > 0$, $a_1, a_2 \in \mathbb{R}$ and $c \in \mathbb{R}^n$. Notice also that $C_0 = (1 - q)^{-1}$. 

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An important point in these expansions is that an examination of the (non-constant) leading term, which equals \( a(1 - q)^{-1}|x|^{1-q} \) for \( q \neq 1 \) and \( a \log |x| \) for \( q = 1 \), shows that the induced metric is given by

\[
g_{ij} = \delta_{ij} + \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
\]

\[
= \delta_{ij} + a^2 |x|^{-2q-2}x_i x_j + O(|x|^{-2q-1})
\]

\[
= \delta_{ij} + a^2 O(|x|^{-2q}),
\]

(25)

which matches with (19). In fact, a comparison of these expressions shows that

\[
a^2 = 2m.
\]

(26)

**Definition 1** We say that a pair \((M, g)\) is an extrinsic Lovelock black hole if there exists an isometric embedding \((M, g) \hookrightarrow (\mathbb{R}^{n+1}, \delta)\) satisfying conditions (I)-(IV) above (for some \(k\) such that \(2 \leq 2k \leq n - 1\)).

Thus, by (26) an extrinsic Lovelock black hole has the same asymptotic expansion as the model Lovelock-Schwarzschild black hole (at least up to first order). In any case, with this definition at hand we may finally state our extrinsic black hole uniqueness theorem in pure Lovelock gravity.

**Theorem 1** Let \((M, g) \hookrightarrow (\mathbb{R}^{n+1}, \delta)\) be an extrinsic Lovelock black hole as in Definition 1 above (for some \(k\)). Then \((M, g)\) is congruent to the Lovelock-Schwarzschild black hole \((M_{k,m}, g_{k,m})\) viewed as an \(O_n\)-invariant hypersurface in \(\mathbb{R}^{n+1}\) (for some \(m > 0\)).

**3 Preliminary Remarks on the Proof of Theorem 1**

The key technical ingredient in the proof of Theorem 1 is a remarkable result by (Theorem 3.1 Araújo and Leite (2012)), which extends previous contributions in de Lima and Sousa (2011), Hounie and Leite (1999), Schoen (1983). More precisely, they classify complete, embedded two-ended hypersurfaces \(M' \hookrightarrow \mathbb{R}^{n+1}\) such that:

a. its 2\(k\)-mean curvature vanishes \(\sigma_{2k}(A) = 0\);

b. it is elliptic in the sense that \(\sigma_{2k+1}(A) \neq 0\) everywhere;

c. each of its ends is asymptotically rotationally symmetric (they can be be written as a graph over the complement of a ball in a hyperplane \(\Pi \hookrightarrow \mathbb{R}^{n+1}\) associated to a function, say \(u\), which behaves precisely as in (20)–(24), depending on the value of \(q = q_{k,n}\)). Ends with this property are termed regular in Araújo and Leite (2012).

We now elaborate a bit on these assumptions. We first note that the null 2\(k\)-mean curvature condition in (a) is the Euler–Lagrange equation associated to a natural variational problem, namely, that related to the functional \(\int \sigma_{2k-1}(A)\) (Reilly 1973). In
this variational setting, the corresponding Jacobi operator is given by

$$J_k = \text{div}_g (N_{2k-1}(A) \nabla \cdot) - (2k + 1)\sigma_{2k+1}(A),$$

where

$$N_p(A) = \sigma_p(A) I - \sigma_{p-1}(A) A + \cdots + (-1)^p A^p$$

is the so-called Newton tensor. It turns out that the ellipticity condition $\sigma_{2k+1}(A) \neq 0$ in (b) is equivalent to $N_{2k-1}(A)$ being positive or negative definite, which means of course that $J_k$ is elliptic as a differential operator. Ellipticity and embededness are key assumptions in Araújo and Leite (2012) as their argument relies heavily on Aleksandrov Reflection Principle. As for (c), we first note that the asymptotics (25) guarantees that each regular end as above has a so-called Gauss-Bonnet-Chern mass $m_k(g)$ attached to it (Ge et al. 2014b, a). A direct computation shows that

$$m_k(g) \approx a^{2k} \approx a^k.$$  \hspace{1cm} (27)

From our extrinsic viewpoint, this invariant may be accessed as follows. For each end $E$ and for each smooth cycle $C \subset E$ generating $H_{n-1}(E)$ we may consider the flux quantity

$$\text{Flux}(E) = \int_C \langle N_{2k-1}(A) \xi, \nu \rangle d\text{vol}_C,$$

where $\nu$ is the outward unit normal vector field to $C \hookrightarrow E$ and $\xi$ is the unit normal vector field to the hyperplane $\Pi$ over which $E$ is graphically represented as in (c). We now recall that any coordinate function $x_i$, when restricted to $M'$, satisfies

$$\text{div}_g (N_{2k-1}(A) \nabla x_i) = 0;$$

see (Reilly 1973, Theorem C). Thus, an application of the divergence theorem shows that $\text{Flux}(E)$ is a homological invariant of $C$ and hence may be computed in the asymptotic limit by means of the expansion for $u$ in (20)–(24). According to (Theorem 2.4 Araújo and Leite (2012)), the final result is

$$\text{Flux}(E) \approx a^{2k},$$  \hspace{1cm} (28)

which shows by (27) that $E$ has the same flux (or Gauss–Bonnet–Chern mass) as the one-ended Lovelock–Schwarzschild black hole to which it is asymptotic. This flux formula is the very first step in the proof of the main result in Araújo and Leite (2012). Indeed, when the given hypersurface has exactly two ends, then (28) and the balance of fluxes imply that these ends are parallel to each other with opposite orientations and the same mass $m_k$ as in (27). The Reflection Method is then used twice: first to guarantee that the hypersurface is symmetric across some “horizontal” hyperplane (this involves adjusting the constant terms in the expansions) and then, after locating
the “vertical” axis of symmetry (which involves choosing coordinates so that \( c = 0 \)),
to make sure that the hypersurface is symmetric across hyperplanes containing this
axis. This argument leads to the next remarkable rigidity assertion, which follows from
the main results in Araújo and Leite (2012).

**Theorem 2** (Araújo and Leite 2012, Theorems 3.5 and 4.5) Any complete, embedded
hypersurface \( M' \hookrightarrow \mathbb{R}^{n+1} \) with two regular ends and satisfying \( \sigma_{2k}(A) = 0 \) and
\( \sigma_{2k+1}(A) \neq 0 \) everywhere (for some \( k \) such that \( 2 \leq 2k \leq n - 1 \)) is congruent to
some two-ended extrinsic Lovelock–Schwarzschild black hole \((\Sigma, g_k, m)\).

We now briefly sketch the proof of Theorem 1, which is detailed in Sect. 4. We first
reflect a given one-ended extrinsic Lovelock black hole \((M, g) \hookrightarrow (\mathbb{R}^{n+1}, \delta)\)
across the hyperplane containing the horizon \( \Gamma \) so as to obtain a two-ended hypersurface
\( M \) satisfying all the requirements in Theorem 2, except that in principle it is only \( C^{1,1} \)
along \( \Gamma \); compare this to the discussion surrounding (17), which guarantees that the
Lovelock–Schwarzschild solution is smooth under reflection across the horizon. We
then use the ellipticity condition to prove a regularity result (Proposition 1) showing
that actually \( M' \) is \( C^2 \) (in fact, smooth) along the horizon. Thus, Theorem 2 now
applies and we conclude that the original Lovelock black hole is congruent to some
one-ended Lovelock–Schwarzschild black hole \((M_{k,m}, g_{k,m})\), which completes the
argument.

### 4 A Regularity Result and the Proof of Theorem 1

As explained above, the proof of Theorem 1 involves the consideration of the embedded
\( C^{1,1} \) hypersurface \( M' \) obtained from our one-ended Lovelock black hole after reflection
across the hyperplane containing the horizon \( \Gamma \). More precisely, Theorem 1 follows
immediately from the rigidity result in Theorem 2 if we are able to show that \( M' \)
actually is of class \( C^2 \) along \( \Gamma \). Since the argument is local, we fix \( q \in \Gamma \) and write
locally \( M' \) around \( q \) as the graph of a \( C^{1,1} \) function \( v \) defined in a small neighborhood \( U \)
of the origin \( 0 \in T_M \Gamma \). Choose rectangular coordinates \((y_1, \ldots, y_n)\) in \( U \) so that the
hypersurface \( \Gamma_0 \subset U \) defined by \( y_n = 0 \) is such that \( v|_{\Gamma_0} \) is the graph representation of
\( \Gamma \). Notice that \( \Gamma_0 \) determines a decomposition \( U = U^+ \cup U^- \), where \( U^+ \) (respectively,
\( U^- \)) is given by \( y_n \geq 0 \) (respectively, \( y_n \leq 0 \)). Clearly, \( U^+ \cap U^- = \Gamma_0 \). We also
set \( v^\pm = v|_{U^\pm} \). Moreover, let us agree on the index ranges \( 1 \leq i, j, \ldots \leq n \) and
\( 1 \leq \alpha, \beta, \ldots \leq n - 1 \).

We now observe that the following properties hold:
- The partial derivatives \( v_i^\pm \) are \( C^1 \) along \( \Gamma_0 \) with \( v_i^+ = v_i^- \) there;
- The function \( v^\pm \) is \( C^2 \) on \( U^\pm \) and \( v_{ij}^+ = v_{ij}^- \) along \( \Gamma_0 \).

Let us agree that, when decorating \( v \), subscripts corresponding to partial differentiation
with respect to \( y \).

These properties entail the following facts. First, the second property implies that,
as we approach \( \Gamma_0 \) by interior points of \( U^\pm \), all second order derivatives \( v_{ij}^\pm \) exist in the
limit and are continuous on \( U^\pm \). The point here is to check whether these derivatives
agree along \( \Gamma_0 \) for each \((i, j)\), so that \( v \) is indeed \( C^2 \) on \( U \), which implies that \( M' \) is
by the fact that \( q \) has been arbitrarily chosen. We already know that \( v_{\alpha\beta}^+ = v_{\alpha\beta}^- \) and, moreover, by the content of the first property applied to \( v_n \), we see that \( v_{\alpha n}^+ = v_{\alpha n}^- \) along \( \Gamma_0 \) as well. Thus, we are led with the task of checking whether \( v_{nn}^+ = v_{nn}^- \) along \( \Gamma_0 \).

We notice that \( M'_\pm = v(U_\pm) \) both have a well-defined shape operator, say \( A_\pm \), with the usual properties (symmetry, etc.) holding up to \( \Gamma_0 \). We note for further reference that, in nonparametric coordinates,

\[
A_{ij}^\pm = B_{ij}^\pm + C_{ij}^\pm, \tag{29}
\]

where

\[
B_{ij}^\pm = \frac{v_{ij}^\pm}{W}, \quad C_{ij}^\pm = -\frac{1}{W^3} \sum_k v_{i k}^\pm v_k^\pm v_{jk}^\pm, \quad W = \sqrt{1 + |\nabla v^\pm|^2}. \tag{30}
\]

As usual, given a symmetric matrix \( A \), we denote by \( \sigma_p(A) \) the elementary symmetric function of degree \( p \) in the eigenvalues of \( A \). In particular, we set \( \sigma_p(u^\pm) := \sigma_p(A^\pm) \), so that the following property follows from the assumptions of Theorem 1 and the way \( M' \) was constructed from \( M' \):

- \( v^\pm \) is an elliptic solution of \( \sigma_{2k}(v^\pm) = 0 \) in the sense that \( \sigma_{2k+1}(v^\pm) \neq 0 \).

The following proposition provides the regularity result we are looking for.

**Proposition 1** Under the conditions above, \( v_{nn}^+ = v_{nn}^- \) along \( \Gamma_0 \). In particular, \( M' \) is of class \( C^2 \) (in fact, smooth).

We start the proof by observing that, in general, \( \sigma_p(A) \) is the sum of the principal minors of order \( p \) of the symmetric matrix \( A \), so that

\[
\sigma_{2k}(A) = \frac{1}{(2k)!} \sum \delta_{i_1 \ldots i_{2k}}^j \prod_{p=1}^{2k} A_{i_p j_p}, \tag{31}
\]

It then follows from (29) that

\[
\sigma_{2k}(v^\pm) = \sigma_{2k}(B^\pm) + \sigma_{2k, \geq 1}(B^\pm, C^\pm), \tag{32}
\]

where

\[
\sigma_{2k, \geq 1}(B^\pm, C^\pm) = \sum_{r=1}^{2k} \sigma_{2k, r}(B^\pm, C^\pm) \tag{33}
\]

and

\[
\sigma_{2k, r}(B^\pm, C^\pm) = \frac{1}{(2k)!} \sum \delta_{i_1 \ldots i_{2k}}^j \prod_{p=1}^{2k} C_{i_p j_p}^\pm B_{i_p j_p}^\pm. \tag{34}
\]
We now observe that, due to $v_i^\pm(0) = 0$, the ellipticity condition implies that the matrix

$$\frac{\partial \sigma_{2k}(u^\pm)}{\partial v_{ij}^\pm}(0) = \frac{\partial \sigma_{2k}(B^\pm)}{\partial v_{ij}^\pm}(0)$$

is positive or negative definite (see (Sect. 1 Hounie and Leite (1999)) for a clarification of this point).

**Lemma 1** Define

$$D^\pm = \frac{1}{(2k)!} \sum_{\alpha_1 \ldots \alpha_{2k-1}} \frac{\partial^{\alpha_1 \ldots \alpha_{2k-1}}}{\partial v_{nn}} \sigma_{2k,r \geq 1}(B^\pm, C^\pm)(0),$$

where the index $n$ is not allowed on the right-hand side. Then $D^\pm(0) \neq 0$.

**Proof** Note that $\sigma_{2k,r \geq 1}(B^\pm, C^\pm) \text{ always carries at least a first derivative of } v^\pm$. Thus, $v_i^\pm(0) = 0$ implies that

$$\frac{\partial}{\partial v_{nn}} \sigma_{2k,r \geq 1}(B^\pm, C^\pm)(0) = 0.$$ 

From (32) we then get

$$\frac{\partial}{\partial v_{nn}} \sigma_{2k}(v^\pm)(0) = \frac{\partial}{\partial v_{nn}} \sigma_{2k}(B^\pm)(0) = D^\pm(0),$$

where in the last step we used that

$$\sigma_{2k}(B^\pm) = D^\pm v_{nn} + \sigma_{2k}^\dagger(B^\pm),$$

where $\sigma_{2k}^\dagger(B^\pm)$ collects the terms not depending on $v_{nn}$. Now, observe that the left-hand side in (36) is precisely the $(n, n)$-entry of (35).

We now use that $\sigma_{2k}(v^\pm) = 0$ in (32), which together with (37) leads to

$$D^\pm v_{nn}^\pm + \sigma_{2k}^\dagger(B^\pm) + \sigma_{2k,r \geq 1}(B^\pm, C^\pm) = 0.$$ 

It follows from (36) that $D^\pm$ does not vanish and remains bounded in a neighborhood of the origin, so that

$$v_{nn}^\pm = -\frac{\sigma_{2k}^\dagger(B^\pm)}{D^\pm} - \frac{\sigma_{2k,r \geq 1}(B^\pm, C^\pm)}{D^\pm}$$

in this neighborhood. Due to the fact that $\sigma_{2k,r \geq 1}(B^\pm, C^\pm)$ depends at least quadratically on the first order derivatives, the last term on the right-hand side vanishes as we approach the origin, since all second order derivatives, including $v_{nn}^\pm$, remain bounded.
there. On the other hand, the first term on the right-hand side only depends on $v_{i}^{\pm}$ and $v_{m}^{\pm}$, and, since they coincide at the origin, we conclude that $v_{m}^{+}(0) = v_{m}^{-}(0)$, as desired. This completes the proof of Proposition 1 and hence, by the comments in the end of the previous section, of Theorem 1.

The ellipticity of the Lovelock–Schwarzschild black holes $(M_{k,m}, g_{k,m})$ leads to a nice application of Theorem 1, which we now describe. We first observe that for $(M_{k,m}, g_{k,m})$ the precise location $r_{k,m} = (2m)^{k/(n-2k)}$ of the horizon allows us to solve for the mass $m_{k}(g_{k,m})$ in terms of the area $A$. More precisely, there holds

$$m_{k}(g_{n,k}) = c_{k,n}A_{\frac{n-1}{n-2k}}, \quad A = \text{Area}_{n-1}(M_{k,m}, g_{k,m}),$$

where $c_{k,n} > 0$ is a certain universal constant. This motivates the following version of the Penrose inequality in Lovelock gravity, which has been proved in Ge et al. (2014b); see also (Li et al. 2014; de Sousa and Girão 2019) for the higher co-dimensional case. This extends to the Lovelock setting previous contributions in Lam (2010), de Lima and Girão (2015), which handled the classical case $k = 1$.

**Theorem 3** Let $(M, g)$ satisfy (I) and (II) above and assume further that

1. $(M, g)$ is globally a graph over a “horizontal” hyperplane $\Pi$ containing the horizon $\Gamma$, which is assumed to be convex;
2. the graphing function in the previous item behaves as in item (IV) of Subsection 2.4, so that the induced metric satisfies the asymptotics (19);
3. the energy condition $\sigma_{2k}(A) \geq 0$ is satisfied everywhere.

Then the following Penrose-type inequality holds:

$$m_{k}(g, A) \geq c_{k,n}A_{\frac{n-1}{n-2k}}.$$  \hspace{1cm} (40)

Given this lower bound for the Gauss–Bonnet–Chern mass in terms of the area, a fundamental question is to determine which configurations attain the equality. It follows from the arguments leading to (40) that this necessarily implies that $\sigma_{2k}(A) = 0$. In general, this extra piece of information does not suffice to say much about the corresponding $(M, g)$, except in case $k = 1$, where this question has been treated in Huang and Wu (2015), de Lima and Girão (2012). However, at least in case $(M, g)$ is a small perturbation of $(M_{k,m}, g_{k,m})$ with the same asymptotics, Theorem 1 applies to yield the expected characterization.

**Theorem 4** Let $(M, g)$ be as in Theorem 3 and assume further that $\sigma_{2k}(A) = 0$ everywhere. Also, assume that $M$ is a sufficiently small $C^{2}$ perturbation of an extrinsic Lovelock–Schwarzschild black hole $(M_{k,m}, g_{k,m})$ with the same asymptotics. Then $(M, g)$ is congruent to $(M_{k,m}, g_{k,m})$.

**Proof** Given that $(M_{k,m}, g_{k,m})$ is elliptic, the result follows from Theorem 1 and the obvious fact that the assumptions $M \cap \Pi = \Gamma$ and $\sigma_{2k+1}(A) \neq 0$ are both preserved under such small $C^{2}$ perturbations. \hfill $\Box$
5 Further Comments on the Case $\Lambda < 0$

We may also consider a version of pure Lovelock gravity in the presence of a negative cosmological constant, so that (5) gets replaced by

$$L_{2k}(\bar{g}) + \Lambda \bar{g} = 0, \quad \Lambda < 0.$$  

After a proper normalization of $\Lambda$, this field equation admits a vacuum black hole solution, the so-called *Lovelock-adS metric* given by

$$\bar{g}_{k,m,-} = -V_{k,m,-}(\tau)dt^2 + g_{k,m,-}, \quad m > 0,$$

where the space-like slice is

$$g_{k,m,-} = \frac{dt^2}{V_{k,m,-}(\tau)} + \tau^2d\theta^2, \quad V_{k,m,-}(\tau) = 1 + \tau^2 - \frac{2m}{\tau^2 - 2};$$

compare with (9). Notice that $g_{k,m,-}$ is defined on the manifold $M_{k,m,-} = (r_{k,m,-}, +\infty) \times S^{n-1}$, where $r_{k,m,-}$ is the positive zero of $V_{k,m,-}(\tau) = 0$. Hence, as expected, this solution displays a horizon located at $\tau = r_{k,m,-}$. Moreover, as in the $\Lambda = 0$ case, the constant $m^k$ may be identified to the Gauss-Bonnet-Chern mass of the asymptotically hyperbolic manifold $(M_{k,m,-}, g_{k,m,-})$ as defined in Ge et al. (2015).

We observe that the space-like slice $(M_{k,m,-}, \bar{g}_{k,m,-})$ above may be embedded as an asymptotically hyperbolic graph in hyperbolic space $(H^{n+1}, \bar{g}_{0,-})$, where

$$\bar{g}_{0,-} = \left(1 + \tau^2\right)dt^2 + g_{0,-}, \quad g_{0,-} = \frac{d\tau^2}{1 + \tau^2} + \tau^2d\theta^2.$$  

Similarly to (16), we now have

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{2m}{(1 + \tau^2)^2(\tau^{-2} + \tau^{-n} - 2m)},$$

so that solving for $t = t(\tau)$ defines the (two-sheeted) graph. A direct computation, quite similar to the one appearing in Sect. 2.3, confirms that this graphical realization of the Lovelock-adS black holes exhaust, up to an isometry, the family of rotationally invariant hypersurfaces in $\mathbb{H}^{n+1}$ satisfying the curvature condition $\sigma_{2k}(\Lambda) = 0$. This clearly suggests that a notion of *extrinsic* Lovelock black hole should be available in this setting and that a corresponding black hole uniqueness result in the line of Theorem 1 should hold true under a suitable ellipticity condition. As in the case discussed in the bulk of this paper, in order to carry out this program we should be able to establish a version of Theorem 2 in this hyperbolic context. We also mention that a Penrose-type inequality for asymptotically hyperbolic graphs suitably embedded in $\mathbb{H}^{n+1}$ has been proved in Ge et al. (2015); the corresponding result in the Einsteinian setting ($k = 1$) has been established in de Lima and Girão (2016), with a previous (non-sharp) contribution appearing in Dahl et al. (2013). Thus, the appropriate version
of Theorem 4 is also expected to hold true in this setting. We hope to address the issues raised in this section elsewhere.

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