A soothing invisible hand: moderation potentials in optimal control

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Abstract

A moderation incentive is a continuously differentiable control-dependent cost term that is identically zero on the boundary of the admissible control region, and is subtracted from the ‘do or die’ cost function to reward sub-maximal control utilization in optimal control systems. A moderation potential is a function on the cotangent bundle of the state space such that the solutions of Hamilton’s equations satisfying appropriate boundary conditions are solutions of the synthesis problem—the control-parametrized Hamiltonian system central to Pontryagin’s Maximum Principle. A multi-parameter family of moderation incentives for affinely controlled systems with quadratic control constraints possesses simple, readily calculated moderation potentials. One member of this family is a shifted version of the kinetic energy-style control cost term frequently used in geometric optimal control. The controls determined by this family approach those determined by a logarithmic penalty function as one of the parameters approaches zero, while the cost term itself is bounded.

1 Introduction

When modeling a conscious agent, the constant cost function of a traditional time minimization problem can be interpreted as representing a uniform stress or risk throughout the task, while a generalized time minimization cost function models varying stresses and risks that depend on the current state of the system. Implementation of agent limitations via cost terms may be more natural—particularly for biological systems—than a possible/impossible dichotomy, in which constraints are explicitly incorporated in the state space. For example, consider the classic ‘falling cat’ problem, in which a cat is suspended upside down and then released. (See, e.g., [12, 7, 14].) Marey [12] gave a qualitative description of the self-righting maneuver, supported by high-speed photographs: the cat rotates the front and back halves of its body, altering the positions of its head and limbs to adjust its moments of inertia, causing the narrowed half to rotate significantly faster than the thickened half, with zero net angular velocity. Kane and Scher [7] introduced a simple mathematical model of a cat, consisting of a pair of coupled rigid

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bodies and showed that self-righting with zero angular momentum is possible without alteration of the moments of inertia of the two halves of the body. To rule out the mechanically efficient but fatal solution in which the front and back halves simply counter-rotate, resulting in a 360° twist in the ‘cat’, Kane and Scher imposed a no-twist condition in their model. However, actual cats can and do significantly twist their bodies, and splay or tuck their limbs; the images in [7] generated using the mathematical model and superposed on photographs of an actual cat significantly underestimate the relative motion between the front and back halves of the body. Replacing the no-twist condition with a deformation-dependent term in the cost function that discourages excessively large relative motions allows more realistic motions.

The optimal control values for purely state-dependent cost function lie on the boundary, if any, of the admissible controls set. In some situations, geometric optimization and integration methods can be used (see, e.g., [11, 8, 19, 10]) to solve the restriction of the optimization problem to the boundary of the admissible region. If geometric methods are not available or desirable, penalty functions can be used to construct algorithms on an ambient vector space that respect the boundary due to the prohibitive (possibly infinite) expense of crossing the boundary; see, e.g., [5, 3], and references therein. For some state- and control-dependent cost functions, trajectories approaching the boundary of the admissible control region are so extravagant that the boundary can safely be left out of the mathematical model. However, a close approach to the boundary of the admissible region may be appropriate when making the best of a bad situation. Consider again the situation of a cat that is suspended in mid-air facing upward, then released: the cat is presumably eager to change its orientation before striking the ground—a typical cat can right itself when dropped from heights of approximately one meter. Selection of an appropriate cost function is essential; a very high price for near-maximal control values may yield overly conservative solutions, while excessively low costs may result in near-crisis responses in almost all situations. Modeling a system using a family of cost functions parametrized by moderation or urgency can reveal qualitative features of optimal solutions that are not readily seen using a single cost function.

In time minimization problems, the time required to complete the maneuver is obviously not specified a priori. In more general situations, in which the cost function is a non constant function of state and/or control values, the duration is still allowed to vary unless specified as fixed. One of Pontryagin’s necessary conditions for optimality when the duration is free to vary is the requirement that a trajectory lie within the zero level set of the Hamiltonian [16]. Shifting the Hamiltonian by a constant leaves the evolution equations unchanged, but can significantly influence the optimal trajectories via the initial conditions. Hence formulation

\[ x(t_0) = x_0 \text{ and } x(t_1) = x_1. \]

appears on page 13; the treatment of fixed time problems is deferred to page 66.

\[ x(t_0) = x_0 \text{ and } x(t_1) = x_1. \]
of intuitive, geometrically meaningful criteria for the specification of the constant is central to the analysis of modified time minimization problems. In [9] we considered generalizations of traditional time minimization problems, allowing both a purely dependent term modeling a do-or-die, ‘whatever it takes’ approach, and a control-dependent term that equals zero on the boundary of the admissible control region; the decreased cost on the interior can be interpreted as an incentive rewarding sub-maximal control efforts. We introduced two families of control cost term; one was modeled on a quadratic control cost (see, e.g., [2], [4], [13], and references therein), shifted so as to equal zero on the boundaries of the admissible regions, while the other, the elliptical moderation incentives, was modeled on the $L_2$ norm with respect to state-dependent inner products.

Optimal control on nonlinear manifolds has received significant attention in recent years, particularly situations in which the controls can be modeled as elements of a distribution within the tangent bundle of the state manifold, corresponding to (partially) controlled velocities. See, e.g. [15, 17, 18, 4], and references therein. Pontryagin’s Maximum Principle on manifolds involves Hamiltonian dynamics with respect to the canonical symplectic structure on the cotangent bundle of the state space. We extend the notion of a moderation incentive to control systems on manifolds and develop general conditions under which moderation incentives determine a unique optimal control value for each point in the cotangent bundle of the state space.

One of the advantages of the ‘kinetic energy’ control cost term for systems with boundaryless admissible control regions is that the optimal control value is straightforward to compute—it can be computed directly as a non-parametrized Hamiltonian system on $T^*S$. However, if the admissible control regions are bounded, a ‘kinetic energy’ cost term can lead to non-differentiable controls. We construct a multi-parameter family of moderation incentives for affine nonlinear control systems with admissible control regions determined by quadratic forms, and determine the optimal control values for the associated cost functions. For all but one value of one of the parameters, this family yields differentiable optimal controls on the interior of the admissible control region. The upper limit of one of the parameter ranges determines generalizations of the shifted kinetic energy’ cost, with continuous controls that fail to be differentiable on the boundary of the admissible control regions. The lower limit, which is not attained, yields controls equal to those determined by a traditional logarithmic penalty function.

Using the optimal control functions, we can construct functions, which we call moderation potentials, on the cotangent bundle of the state space such that the associated Hamiltonian dynamics are solutions of Pontryagin’s synthesis problem. Thus, rather than work with a family of parametrized Hamiltonians, we can find solutions of a traditional canonical Hamiltonian system. The moderation potentials for the multi-parameter family of moderation incentives constructed here have a relatively simple form. Some members of the family have particularly simple, geometrically meaningful, expressions. These members provide a promising alternative to ‘kinetic energy’ cost terms for systems with bounded control regions, allowing use of the full range of Hamiltonian machinery.

We illustrate some of the key features of the moderated control problems using a simple two-dimensional controlled velocity problem: a vertically launched projectile is guided towards a fixed target; the speed is bounded by a function $\rho$ of the horizontal component of the position.
The optimal velocity and launch point are to be determined. A cost term depending only on the horizontal component of the projectile’s position models risk from ground-based defense of the target. The cost function is a combination of a term depending only on the horizontal component of the projectile’s position models risk from ground-based defense of the target and a ‘moderating’ function of the control. We explore the behavior of the solutions of the synthesis problem as the parameters in the cost function are varied.

2 Constants matter: moderation incentives

We first establish notation and context: We assume that the set $S$ of possible states is a smooth manifold, and consider control problems with state variable $z \in S$ and control $u$ in the state-dependent admissible control region $A_z$ for $z$. We assume that both the state space $S$ and the set $A := \{(z, u) : z \in S \text{ and } u \in A_z\}$ of admissible state/control pairs are smooth manifolds. The evolution of the state variable is determined by a controlled vector field $X$. Specifically, $\dot{z} = X(z, u)$ for some continuous map $X : A \to T S$ satisfying $X(z, u) \in T_z S$ for all $(z, u) \in A$. (Here $T S$ denotes the tangent bundle of the state space $S$, and $T_z S$ denotes the fiber of over $z$ in $T S$. See, e.g. [1, 4].) The control problem is to find a duration $t_f$ and piecewise continuous curve $(z, u) : [0, t_f] \to A$ satisfying the boundary conditions $z(0) = z_0$ and $z(t_f) = z_f$, with piecewise continuously differentiable state component $z$. The optimal control problem with instantaneous cost function $C : A \to \mathbb{R}$ is to find the solution that minimizes the total cost

$$\int_0^{t_f} C(z(t), u(t)) dt$$

over the set of all solutions of the control problem. The fixed time problem is defined analogously, but $t_f$ is specified.

Given a purely state-dependent cost function $\hat{C} : S \to \mathbb{R}$, we construct the cost function $C : A \to \mathbb{R}$ by subtracting a control-dependent term $\tilde{C}(z, u)$ from the unmoderated cost $\hat{C}(z)$. Shifting the cost function by a constant doesn’t change the evolution equations of the corresponding Hamiltonian system. However, when seeking solutions of the optimization problem, shifting the Hamiltonian by a constant can have a significant effect via the condition that the Hamiltonian equal zero along solutions of the synthesis problem. To guide the selection of the control-dependent function $\tilde{C}$, we regard that term as an incentive for sub-maximal control investment, rather a penalty. This motivates the condition that $\tilde{C}(z, u) = 0$ if $u \in \partial A_z$.

**Definition 1** Given an admissible space $A$, $\tilde{C} \in C^1(A, [0, \infty))$ is a moderation incentive for $A$ if for all $z \in S u \in \partial A_z$ implies $\tilde{C}(z, u) = 0$.

If there are continuous functions $q : A \to [0, 1]$ and $\Phi : S \times [0, 1] \to \mathbb{R}$ with $\Phi^{-1}(0) = S \times \{1\}$ such that for every $z \in S$, $\partial A_z = \{u \in A_z : q(z, u) = 1\}$ and $s \mapsto \Phi(z, s)$ is a decreasing function, then $\tilde{C}(z, u) := \Phi(z, q(z, u))$ is a monotonic moderation incentive for $A$ and $q$. 
The following example illustrates the influence of shifting the cost function of an optimal control system for which the time is not fixed. We consider a two dimensional system with controlled velocities. Starting from the horizontal axis, with vertical initial velocity, the goal is to hit a target \((x_f, y_f)\). We assume that the projectile starts to the right of the target and consider a non-increasing unmoderated position-dependent cost term \(\hat{C} : [x_f, \infty) \to \mathbb{R}^+\) depending only on the horizontal component of the position, modeling risk due to ground-based defense of the target, combined with a control- (and possibly position-) dependent moderation term. Given the final height \(y_f\), we seek smooth trajectories \((x, y) : [0, t_f] \to \mathbb{R}^2\) satisfying \(y(0) = \dot{x}(0) = 0\), \(x(t_f) = x_f\), and \(y(t_f) = y_f\). Neither the launch point \((x_0, 0)\) nor final time \(t_f\) are specified in advance. We have direct control over the velocity, with the constraint that the speed of the projectile never exceeds one.

We consider a pair of two-parameter families of cost functions, differing only by a constant. One parameter, \(c\), scales a purely position-dependent term; the second, \(\mu\), scales a control-dependent term. One family yields inflexible solutions—the solution path is entirely determined by the boundary conditions, while the speed is simply rescaled by the ratio of the two parameters. The other family, in which the parameter \(\mu\) scales a moderation incentive, has solutions for which the optimal path and speed both depend nontrivially on the parameters \(c\) and \(\mu\). Here we summarize some of the key features of this system—our intent is only to remind the reader that analogous choices can have a profound influence on the solutions of optimal control systems if the total time is not specified a priori, and hence should be systematically selected. A generalization of this system is analyzed in §5.

The cost functions

\[
C_{ke}(x, y, \dot{x}, \dot{y}; \mu) := \frac{\mu}{2} \| (\dot{x}, \dot{y}) \|^2 + \frac{c}{2} x^2
\]

and

\[
C_{mi}(x, y, \dot{x}, \dot{y}; \mu) := C_{ke}(x, y, \dot{x}, \dot{y}; \mu) + 1 - \frac{\mu}{2}.
\]

differ only by a constant, but there are important differences in the behavior of the solutions. The Hamiltonians associated to \(C_{ke}\) and \(C_{mi}\) (see §4) equal that of a point mass with mass \(\mu\) and potential energy \(-\frac{c}{2\mu} + \text{constant}\). The solutions trace out segment of ellipses with principal axes \(\sqrt{1 - (x_f/x_0)^2}\) and \(y_f/x_0\). The constant influences the solutions of the synthesis problem through the condition that the Hamiltonian equal zero on an optimal trajectory.

- An optimal solution for \(C_{ke}\), traces an arc of a circle centered at the origin. The parameters \(c\) and \(\mu\) influence the solution only through the rescaling of the speed by \(\sqrt{c/\mu}\).

- The optimal starting point \(x_0\) for a solution minimizing \(C_{mi}\) depends nontrivially on both \(c\) and \(\mu\); specifically,

\[
x_0^2 = b + \sqrt{\left(\frac{b}{2}\right)^2 + a d}
\]

where \(a = x_f^2 + y_f^2\), \(d = \frac{c}{2 - \mu}\), \(b = x_f^2 - d\). \(x_0\) and \(t_f\) are increasing functions of \(\mu\). Since smooth solutions satisfying the control constraint \(1 \geq \dot{x}^2 + \dot{y}^2\) exist only if \(1 + \frac{c}{2x_f^2} = \mu_{\min} \leq \mu_{\max} = 2\), we must have \(x_f^2 \geq \frac{c}{2}\). The projectile follows a circular arc when \(\mu = \mu_{\max} = 2\).
Figure 1: Solutions of the vertical take-off targeting problem problem for sample target positions \((x_f, 2)\), cost function \(C_{mi}\), and defense strengths \(c\). Blue: \(c = \frac{1}{2}\), red: \(c = \frac{3}{2}\); left: \(x_f = 1\), right: \(x_f = 3\). Dots indicate projectile position at times \(t = \frac{j}{2}, j \in \mathbb{N}\); colored lines indicate the traces of solutions for \(\mu = \mu_{\text{min}}\); solid grey lines indicate those for \(\mu = \mu_{\text{max}} = 2\). All solutions for the cost function \(C_{ke}\) lie on the grey curves.

3 Affine nonlinear control systems and ellipsoidal admissible control regions

In [9] we introduced the notion of a moderation incentive for control systems in which both the state space and control regions were subsets of \(\mathbb{R}^n\) and \(\mathbb{R}^k\), and focused on the situation \(n = k\). Some of the incentives considered yielded simple explicit expressions for the optimal control values as rescalings of the auxiliary variable (the Lagrange multiplier in the Pontryagin formulation of the control problem). We now extend that strategy to nonlinear manifolds and introduce a family of moderation incentives for systems in which the admissible control regions \(A_z\) are the unit balls with respect to state-dependent norms. The optimal controls for these incentives are rescalings of the image of the auxiliary variable (now an element of the cotangent bundle of the state manifold) under a mapping determined by the norms.

Definition 2 Given a family of positive-definite quadratic forms \(Q_z\) on \(\mathbb{R}^k\), such that \(z \mapsto Q_z\) is \(C^1\), we will say that a control problem with admissible region

\[
A := \left\{ (z, u) \in S \times R^k : Q_z(u) \leq 1 \right\}
\]

has ellipsoidal control regions.

If there are continuous vector fields \(f\) and \(g_j, j = 0, \ldots, k\), on \(S\) such that

\[
\dot{z} = X(z, u) = f(z) + \sum_{j=1}^{k} u_j g_j(z),
\]
the system is said to be affine nonlinearly controlled; \( f \) is the drift vector field. (See, e.g., [4][18].)

Given a control system with ellipsoidal control regions and affine controls, for each \( z \in \mathcal{S} \), let \( L_z \) and \( \langle \cdot , \cdot \rangle_z \) denote respectively the invertible symmetric linear map and inner product on \( \mathbb{R}^k \) satisfying
\[
Q_z(u) = \langle u, L_z^{-1}u \rangle \quad \text{and} \quad \langle u, v \rangle_z = \langle u, L_z^{-1}v \rangle
\]
for all \( u \in \mathbb{R}^k \), and define the maps \( M_z : \mathbb{R}^k \rightarrow TS, \lambda : T^*\mathcal{S} \rightarrow \mathbb{R}^k \), and \( \ell : T^*\mathcal{S} \rightarrow [0, \infty) \) by
\[
M_z u := \sum_{j=1}^k u_j g_j(z), \quad \lambda(\psi_z) := L_z(M_z^* \psi_z),
\]
and
\[
\ell(\psi_z)^2 := Q_z(\lambda(\psi_z)) = \psi_z \cdot (X(z, \lambda(\psi_z)) - X_0(z)).
\]
(Here \( T^*\mathcal{S} \) denotes the cotangent bundle of \( \mathcal{S} \); see, e.g., [4][1].) Finally, define the map \( \nu : \ell^{-1}(\mathbb{R}^+) \rightarrow \mathbb{R}^k \) by
\[
\nu(\psi_z) := \frac{1}{\ell(\psi_z)} \lambda(\psi_z) \in \partial A_z.
\]

**Proposition 1** Consider a control system with ellipsoidal control regions and affinely controlled evolution. Let \( F \in \mathcal{C}^0(\mathcal{S} \times [0, 1], [0, \infty)) \) be a function satisfying \( F^{-1}(0) = \mathcal{S} \times \{0\} \) and such that for every \( z \in \mathcal{S} \), \( F_z(x) := F(z, x) \) is increasing and differentiable on \((0, 1)\), with \( \lim_{z \rightarrow 1} F_z(x) < \infty \).

Given \( p \geq 1 \), define \( x : \mathcal{S} \times \mathbb{R}^+ \times [0, 1] \rightarrow \mathbb{R} \) by
\[
x(z, \ell, s) := \ell s + F(z, 1 - s^p).
\]
If there is a function \( \sigma \) on \( \mathcal{S} \times \mathbb{R}^+ \) such that for every \((z, \ell) \in \mathcal{S} \times \mathbb{R}^+ \) \( s \mapsto x(z, \ell, s) \) achieves its maximum exactly at \( \sigma(z, \ell) \), then the moderation incentive
\[
\tilde{C}(z, u) := F \left( z, 1 - Q_z(u)^{\frac{p}{p}} \right)
\]
has optimal control value
\[
v(\psi_z) := \begin{cases} 
\sigma(\psi_z) \nu(\psi_z) & \ell(\psi_z) \neq 0 \\
0 & \ell(\psi_z) = 0
\end{cases}
\]
at \( \psi_z \in T^*\mathcal{S} \).

**Proof:** Fix \( \psi_z \in T^*\mathcal{S} \). Define \( F_z : [0, 1] \rightarrow [0, \infty) \) by \( F_z(x) := F(z, x) \), \( \tilde{F}_{\psi_z} : A_z \rightarrow \mathbb{R} \) by
\[
\tilde{F}_{\psi_z}(u) := \psi_z \cdot (X(z, u) - f(z)) + F_z \left( 1 - Q_z(u)^{\frac{p}{p}} \right) = \langle \lambda(\psi_z), u \rangle_z + F_z \left( 1 - Q_z(u)^{\frac{p}{p}} \right),
\]
and \( x_{\psi_z} : [0, 1] \rightarrow \mathbb{R} \) by \( x_{\psi_z}(s) := x(z, \ell(\psi_z), s). \)

If \( \lambda(\psi_z) = 0 \), \( \tilde{F}_{\psi_z}(u) = F_z \left( 1 - Q_z(u)^{\frac{p}{p}} \right) \), which achieves its maximum at \( u = 0 \).

We now show that if \( \lambda(\psi_z) \neq 0 \), then \( \tilde{F}_{\psi_z} \) takes its maximum on the line segment
\[
\{ sv(\psi_z) : 0 \leq s \leq 1 \},
\]
and hence, since $\tilde{F}_{\psi}(s \nu(\psi_z)) = x_{\psi_z}(s)$, the maximum of $\tilde{F}_{\psi}$ coincides with the maximum of $x_{\psi_z}$. The restriction of $\tilde{F}_{\psi}$ to the interior of $A$ is differentiable, with gradient

$$\nabla \tilde{F}_{\psi}(u) = \lambda(\psi_z) - p \, F_z'(1 - Q_z(u)\frac{p}{2}) Q_z(u)\frac{p}{2} - 1.$$ 

Hence if $\lambda(\psi_z) \neq 0$, any critical point of $\tilde{F}_{\psi}$ in the interior of $A$ has the form $u = s \nu(\psi_z)$ for $s \in (0, 1)$ satisfying

$$\frac{\ell(\psi_z)}{p} = s^{p-1} F_z'(1 - Q_z(s \nu(\psi_z)) \frac{p}{2}) = s^{p-1} F_z'(1 - s^p).$$ (9)

Note that if $\lambda(\psi_z) \neq 0$, and hence $\ell(\psi_z) \neq 0$, then $s$ satisfies (9) iff $s$ is a critical point of $x_{\psi_z}$.

Since $F(z, 0) = 0$ implies that $F_z'(1 - Q_z(u)\frac{p}{2}) = 0$ for $u \in \partial A_z$,

$$\max_{u \in \partial A_z} \tilde{F}_{\psi}(u) = \max_{Q_z(u) = 1} \langle \lambda(\psi_z), u \rangle.$$ 

Hence a standard Lagrange multiplier argument shows that the restriction of $\tilde{F}_{\psi}$ to $\partial A_z$ achieves its maximum, $\ell(\psi_z) = x_{\psi_z}(1)$, at $\nu(\psi_z)$.

Finally, $\tilde{F}_{\psi}(0) = F_z(1) = x_{\psi_z}(0)$. ■

Remark: If (8) is the optimal control for a moderation incentive of the form (7), with scaling factor $\sigma$, then given $\mu \in C^0(S, \mathbb{R}^+)$,

$$\tilde{C}_{\mu}(z, u) := \mu(z) F_z\left(z, 1 - Q_z(u)\frac{p}{2}\right)$$

is a moderation incentive with scaling factor obtained by replacing $\ell(\psi_z)$ with $\ell_\mu(\psi_z) := \frac{\ell(\psi_z)}{\mu(z)}$ in (8). ■

A moderation incentive is required to take the value zero on the boundary of the admissible control regions, but is not required to have a finite derivative there. If the limit of the derivative of the incentive as $\partial A_z$ is approached is infinite, the optimal control must lie in the interior of $A_z$.

**Lemma 1** If $F : S \times [0, 1] \to [0, \infty)$ satisfies $F_z^{-1}(0) = S \times \{0\}$, and for every $z \in S$ the function $F_z(x) := F(z, x)$ is $C^2$ on $(0, 1)$, with decreasing positive derivative satisfying

$$\lim_{x \to 0} F_z'(x) = \infty,$$ (10)

then $s \mapsto x(z, \ell, s)$ given by (6) achieves its maximum at a unique point $s_*(z, \ell) \in (0, 1)$ if $p > 1$, or if $p = 1$ and $F_z$ is strictly decreasing.

The associated map $\sigma : \ell^{-1}(\mathbb{R}^+) \to (0, 1)$ given by $\sigma(\psi_z) := s_*(z, \ell(\psi_z))$ is $C^1$. 

Figure 2: Plots of $\rho_{\alpha,p}^{-1}(\ell_{\mu})$ for different values of the dogleg parameters $\alpha$ and $p$. Purple: $\alpha = \frac{99}{100}$; blue: $\alpha = \frac{3}{4}$; red: $\alpha = \frac{1}{2}$; orange: $\alpha = \frac{1}{4}$; gray: limiting case $\alpha \to 0$. Left: $p = 2$. Right: $p = 1.01, 1.5, 2, 2.5, 5$; convexity for small values of $s$ increases with $p$. (The approximate step function in the right hand graph is associated to $\alpha = \frac{99}{100}, p = 1.01$.)

**Proof:** Setting $y = 1 - s^p$ and $c = \frac{\ell(\psi_z)}{p} > 0$, (9) takes the form $c(1-y)^{\frac{1}{p}-1} = F'_z(y)$, with unique solution $y(c) \in (0, 1)$. The map $s \mapsto s^{p-1} F'_z(1 - s^p)$ has a $C^1$ strictly positive derivative on $(0, 1)$. Hence the Implicit Function Theorem implies that the map $\sigma$ determined by (9) is $C^1$ on $\ell^{-1}(R^+)$. □

We now define a family of monotonic moderation incentives for affinely controlled systems with quadratic control costs. These generalize the moderation incentives introduced in [19]. The ‘dogleg parameters’ $\alpha \in (0, 1]$ and $p \geq 1$ can be interpreted as tuning the overall shape of the control response curve, while the state-dependent moderation strength function $\mu \in C^0(S, R^+)$ scales the instantaneous control cost. (Use of the term ‘dogleg’ is motivated by the shape of the response curve for values of $\alpha$ near 1; varying these parameters alters the abruptness of the dogleg bend.)

**Theorem 1** Given $0 < \alpha \leq 1 \leq p$, excluding $\alpha = 1 = p$, and $\mu \in C^0(S, R^+)$,

$$\tilde{C}_{\alpha,p}(z, u; \mu) = \frac{\mu(z)}{p\alpha} \left(1 - Q_z(u)^{\frac{1}{2}}\right)^\alpha$$

(11)

is a monotonic moderation incentive for $A$.

If $0 < \alpha < 1$, the unique optimal control parameter (8) for $\tilde{C}_{\alpha,p}$ has the scaling

$$\sigma_{\alpha,p}(\psi_z; \mu) = \rho_{\alpha,p}^{-1}(\ell_{\mu}(\psi_z)),$$

(12)

where $\rho_{\alpha,p} : [0, 1] \to [0, \infty)$ and $\ell_{\mu} : T^*S \to [0, \infty)$ are given by

$$\rho_{\alpha,p}(s) := s^{p-1} (1 - s^p)^{\alpha-1} \quad \text{and} \quad \ell_{\mu}(\psi_z) := \frac{\ell(\psi_z)}{\mu(z)}.$$
If $\alpha = 1 < p$, then
\[
\sigma_{1,p}(\psi; \mu) := \min \left\{ \frac{1}{p} \frac{1}{1 - s^{p-1}}, 1 \right\}
\]
is the optimal scaling.

Proof: For $0 < \alpha \leq 1$, $F_\alpha(x) := \frac{1}{\alpha} x^\alpha$ is differentiable, with decreasing positive derivative, on $(0, 1]$. For $0 < \alpha < 1$, $\lim_{x \to 0} F_\alpha'(x) = \infty$, so the rescaling of $F_\alpha$ by $\frac{\mu(z)}{p}$ satisfies the conditions of Lemma 1.

The case $\alpha = 1 < p$ requires a direct application of Proposition 1 since $F'_1 \equiv 1$. In this case,
\[
\frac{x_{\psi}(s)}{\mu(z)} = \ell_\mu(\psi) s + \frac{1}{p} (1 - s^p)
\]
is the restriction of a polynomial to $[0, 1]$. If $\ell_\mu(\psi) \leq 1$, the maximum of $x_{\psi}$ coincides with that of the polynomial, which occurs at $s = \ell_\mu(\psi)$. If $\ell_\mu(\psi) \geq 1$, the maximum occurs at one of the endpoints; since
\[
x_{\psi}(0) = \frac{\mu(z)}{p} < \ell(\psi) = x_{\psi}(1),
\]
in this case $x_{\psi}$ achieves its maximum of $\ell(\psi)$ at 1.

3.1 Special cases: $\alpha = 1$, $\alpha \to 0$, and $\frac{1}{\alpha} = p$

For some special values of the parameters $\alpha$ and $p$, simple closed form expressions for the optimal scaling $\sigma_{\alpha, p}$ exist.
The dogleg parameter values \( \alpha = 1, p = 2 \) correspond to the widely used ‘kinetic energy’ style control cost, which is used in the targeted attack problem in \([1]\) and \([5]\). Note that when \( \alpha = 1 \), the optimal scaling is not differentiable at \( \partial A_z \).

The limit \( \lim_{\alpha \to 0} \tilde{C}_{\alpha,p} \) is not well-defined, but

\[
\rho_{0,p}(s) := \lim_{\alpha \to 0} \rho_{\alpha,p}(s) = \frac{sp^{p-1}}{1 - sp^{p}} = -\frac{d}{ds} \ln \left( \frac{1 - s^{p}}{s^{p}} \right)
\]

is well-defined and invertible on \([0,1)\). In particular, the optimal scaling associated to the logarithmic control cost

\[
\tilde{C}(z,u) := -\frac{1}{2} \ln(1 - Q(z)(u))
\]

is \( \rho_{0,2}^{-1}(\ell_\mu(\psi_z)) \). Thus the controls determined by the family \( \tilde{C}_{\alpha,2} \) determine a homotopy between optimal controls determined by a ‘kinetic energy’ control cost and a logarithmic ‘penalty function’ cost. (Logarithmic penalty functions are widely used in the engineering literature to enforce inequality constraints.)

In the case \( \alpha = \frac{1}{p} < 1 \), we can explicitly invert \( \rho_{\frac{1}{p},p} \).

**Corollary 1**

\[
\sigma_{\frac{1}{p},p}(\psi_z;\mu) = \left( 1 + \ell_\mu(\psi_z)^{-q} \right)^{-\frac{1}{p}} \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

and hence

\[
u_{\frac{1}{p},p}(\psi_z;\mu) = \frac{\ell(\psi_z)^{q-2}}{(\mu(z)^q + \ell(\psi_z)^q)^{\frac{1}{p}}} \lambda(\psi_z)
\]

if \( \lambda(\psi_z) \neq 0 \).

**Remark:** When the drift field is trivial, the optimal rescaling has the following geometric interpretation: the optimal control \( \nu_{\frac{1}{2},2} \) satisfies

\[
u_{\frac{1}{2},2}(\psi_z) = \frac{\lambda(\psi_z)}{\| (\lambda(\psi_z), \mu(z)) \|_Q},
\]

where \( \|(u,t)\|_Q^2 = Q(u) + t^2 \) is the norm on \( \mathbb{R}^{k+1} \) induced by a quadratic form \( Q \) on \( \mathbb{R}^k \). Thus \( \nu_{\frac{1}{2},2}(\psi_z) \) is the control component of the projection of \( (\lambda(\psi_z), \mu(z)) \) onto the \( \| \|_Q \) unit ball in \( \mathbb{R}^{k+1} \). (See Figure 3.1.) We will further investigate the moderated controls for \( \alpha p = 1 \), particularly that for \( p = 2 \), in future work.
Figure 4: The optimal control for $\frac{1}{n} = p = 2$ implemented as lift into control–moderation space, followed by projection onto the unit sphere, then projection back into control space. Black solid arrows: $\lambda(\psi_z)$; black dashed: $(\lambda(\psi_z), \mu(z))$; red dashed: $\frac{(\lambda(\psi_z), \mu(z))}{\|\lambda(\psi_z), \mu(z)\|}Q_z$; red solid: $\lambda(\psi_z)\|\lambda(\psi_z), \mu(z)\|Q_z$.

4 Moderation potentials and the synthesis problem

Pontryagin’s Maximum Principle relates optimal control to Hamiltonian dynamics: If the state space $S$ is an $n$-dimensional subset of $\mathbb{R}^n$, then associated to a solution $(z,u) : [0,t_f] \rightarrow A \subseteq \mathbb{R}^n \times \mathbb{R}^k$ of the control problem minimizing the total cost there is a curve $\lambda : [0,t_f] \rightarrow \mathbb{R}^n$ such that the curve $(z,\lambda)$ satisfies Hamilton’s equations for the time-dependent Hamiltonian

$$H_t(\tilde{z}, \lambda) := \lambda^T X(\tilde{z}, u(t)) - C(\tilde{z}, u(t)),$$

and

$$H_t(z(t), \lambda(t)) = \max_{u \in A_{z(t)}} \left( \lambda(t)^T X(z(t), u) - C(z(t), u) \right).$$

(See [16] for the precise statement and proof of the Maximum Principle.) Pontryagin’s optimality conditions are necessary, but not sufficient. Their appeal lies in their constructive nature—well-known results and techniques for boundary value problems and Hamiltonian dynamics can be used to construct the pool of possibly optimal trajectories. This construction is referred to as the synthesis problem in [16]; we will make use of that terminology here.)

The generalization of Hamilton’s equations to a nonlinear state manifold $S$ utilizes the canonical symplectic structure on the cotangent bundle $T^*S$ of the state manifold. See, e.g., [4] for additional background and discussion. We now introduce the formulation of the synthesis problem that will be used here. Given our focus on systems with state-dependent admissible control regions, we introduce the condition that the parametrized Hamiltonians be extendable to neighborhoods of possibly optimal values.

Given that the admissible control regions can vary with the state variable, we explicitly require that the vector field $X$ and cost term $C$ with fixed control value be extensible to neighborhoods of the points of interest. Let $\pi : T^*S \rightarrow S$ denote the canonical projection, with $\pi^{-1}(z) = T^*_zS$,

$$P := \{ (\psi_z, u) : (\pi(\psi_z), u) \in A \},$$

where $A$ is the set of admissible controls.
and \( \mathbb{P}_1 : A \to S \) denote projection onto the first factor. Given \( F \in \mathcal{C}^1(T^*S) \), let \( X_F \) denote the Hamiltonian vector field determined by \( F \) and the canonical symplectic structure on \( T^*S \). Define \( H : \mathbb{P} \to \mathbb{R} \) and \( \chi : T^*S \to \mathbb{R} \) by

\[
H(\psi, u) := \psi \cdot X(z, u) - C(z, u) \quad \text{and} \quad \chi(\psi) := \max_{u \in A_z} H(\psi, u). \tag{16}
\]

**Definition 3** If for every \((\psi, u_*) \in \mathbb{P}\) satisfying \( H(\psi, u_*) = \chi(\psi) \) there is a neighborhood \( \mathcal{V} \) of \( \psi \) such that the restrictions of \( X(\cdot, u_*) \) and \( C(\cdot, u_*) \) to \( \mathcal{V} \) are restrictions to \( \mathcal{V} \cap \mathbb{P}_1(A) \) of \( \mathcal{C}^1 \) maps on \( \mathcal{V} \), then \( X \) and \( C \) are synthesizable.

A curve \((\Psi, v) : [0, t_f] \to \mathbb{P}\) satisfying

\[
\dot{\Psi}(t) = X_{H_t}(\Psi(t)) \quad \text{and} \quad H_t(\Psi(t)) = \chi(\Psi(t))
\]

for the local time-dependent Hamiltonians \( H_t(\psi) := H(\psi, v(t)) \) determined by synthesizable \( X \) and \( C \) is a solution of the synthesis problem determined by \( X \), \( C \) and the boundary data \( z_0 = \pi(\Psi(0)) \) and \( z_f = \pi(\Psi(t_f)) \) if \( H_0(\Psi(0)) = 0 \).

If \((\Psi, v)\) satisfies all of the above conditions except the condition that \( H_0(\Psi(0)) = 0 \), then \((\Psi, v)\) is a solution of the fixed time synthesis problem of duration \( t_f \).

We will focus on finding solutions of the synthesis problem, and will not formulate general conditions under which such solutions are in fact global minimizers.

One of the advantages of the ‘kinetic energy’ control cost term widely used in geometric optimal control for systems with controlled velocities and unbounded admissible control regions is that the optimal control value is straightforward to compute—it is simply the inverse Legendre transform of \( \psi \)—and hence solutions of the synthesis problem can be computed directly as a non-parametrized Hamiltonian system on \( T^*S \). However, if the admissible control regions are bounded, a ‘kinetic energy’ cost term can lead to non-differentiable controls. (See, e.g. \([9]\).) We now identify conditions under which a control-dependent cost function determines solutions of the synthesis problem corresponding to solutions of a traditional Hamiltonian system

Nondegeneracy of the symplectic structure guarantees that two Hamiltonian vector fields agree at \( \psi \) iff \( \psi \) is a critical point of the difference of the Hamiltonians. For fixed \( \psi \in T^*S \), we can define \( h_{\psi} \in \mathcal{C}^1(A_z, \mathbb{R}) \) by \( h_{\psi}(u) := H(\psi, u) \). If \( H \) is \( \mathcal{C}^1 \) and \( h_{\psi} \) achieves its maximum at \( u_* \in A^c_z \), then \( u_* \) is a critical point of \( h_{\psi} \). It follows that if \( H \) is \( \mathcal{C}^1 \) and there is a \( \mathcal{C}^1 \) map \( v \) such that \( H(\cdot, v(\cdot)) = \chi \) and \( v(\psi) \in A^c_z \) for every \( \psi \in T^*S \), then

\[
d(\chi - H(\cdot, u_*))(\psi_z)(w_{\psi_z}) = dH(\psi_z, u_*)(0, d_{\psi_z}v(w_{\psi_z})) = (dh_{\psi_z}(u_*)(d_{\psi_z}v(w_{\psi_z}))) = 0
\]

for \( u_* = v(\psi) \) and all \( w_{\psi_z} \in T_{\psi_z}T^*S \).

If there is a \( \mathcal{C}^0 \) map \( v \) such that \( H(\cdot, v(\cdot)) = \chi \), but \( v \) is not everywhere differentiable, or can take values on the boundaries of the admissible control region, then the above argument is not applicable, but we may still be able to replace the parametrized Hamiltonians \( H(\cdot, u_*) \) with the function \( \chi \).
The solution of the synthesis problem can be simplified given a feedback law that allows replacement of the control-parametrized Hamiltonian with a conventional autonomous Hamiltonian on the cotangent bundle $T^*S$. We now show that the moderation incentives $\tilde{C}_{\alpha,p}$ have such control laws. The key concerns are formulation of relatively simple expressions for the Hamiltonians and verification of the differentiability of the Hamiltonian on the boundaries of the admissible control regions.

**Proposition 2** If

(i) $X$ and $C$ are synthesizable,

(ii) $\chi$ given by (16) is $C^1$,

(iii) $\Psi : [0,t_f] \to T^*S$ is a solution of the canonical Hamiltonian system with Hamiltonian $\chi$, and

(iv) there is a curve $u : [0,t_f] \to P$ such that $(\Psi(t), u(t)) \in P$ and

\[
H(\Psi(t), u(t)) = \chi(\Psi(t)) \quad \text{and} \quad d(H(\cdot, u(t)) - \chi)(\Psi(t)) = 0
\]

for $0 \leq t \leq t_f$, then $(\Psi, u)$ is a solution of the fixed time synthesis problem of duration $t_f$ determined by $X$, $C$, and the boundary data $z_0 = \pi(\Psi(0))$ and $z_f = \pi(\Psi(t_f))$.

If, in addition, $H(\Psi(0)) = 0$, then $\Psi$ determines a solution of the synthesis problem.

**Proof:** For each $t \in [0,t_f]$, synthesizability of $X$ and $C$ implies that there is a neighborhood $\mathcal{V}_t$ of $\pi(\Psi(t))$ such that $H_t := H(\cdot, \nu(t)) \in C^1(\pi^{-1}(\mathcal{V}_t))$.

\[
0 = d(\chi - H_t)(\Psi(t))
\]

implies

\[
\dot{\Psi}(t) = X_\chi(\Psi(t)) = X_{H_t}(\Psi(t))
\]

and

\[
H_t(\Psi(t)) = H(\Psi(t), \nu(t)) = \chi(\Psi(t))
\]

for $0 \leq t \leq t_f$. Hence $(\Psi, \nu)$ is a solution of the fixed time synthesis problem. If the Hamiltonian is identically zero along the trajectory, $\blacksquare$

**Definition 4** If

(i) $\tilde{C}$ is a moderation incentive,

(ii) the pair $X$ and $-\tilde{C}$ is synthesizable,

(iii) $\chi$ given by (16) for $C = -\tilde{C}$ is $C^1$
(iv) there is a unique map $\nu \in \mathcal{C}_0(T^*S)$ such that

(a) $\text{graph}(\nu) \subseteq \mathcal{P}$,
(b) $H(\cdot, \nu(\cdot)) = \chi$, and
(c) for every $\psi_z \in T^*S$, $\psi_z$ is a critical point of $\chi - H(\cdot, \nu(\psi_z))$,

then we will say that $\chi$ is a moderation potential for $\tilde{C}$ and $X$.

It follows immediately from Proposition 2 and Definition 4 that if $\chi$ is a moderation potential for synthesizable $X$ and $\tilde{C}$, $\tilde{C} \in \mathcal{C}^1(S)$, and $\Psi : [0, t_f] \rightarrow T^*S$ is a solution of Hamilton’s equations for the Hamiltonian $\chi - \tilde{C} \circ \pi$, then $(\Psi, \nu \circ \Psi)$ is a solution of the synthesis problem determined by $X, C = \tilde{C} \circ \mathbb{P}_1 - \tilde{C}$, and the boundary data $z_0 = \pi(\Psi(0))$ and $z_f = \pi(\Psi(t_f))$. If, in addition, $H(\Psi(0)) = \tilde{C}(\pi(\Psi(0)))$, then $\Psi$ determines a solution of the synthesis problem.

We now show that moderation potentials exist for the family of moderation incentives constructed in Theorem 1. For some subfamilies, the moderation potentials have particularly simple expressions.

**Theorem 2** The moderation incentives (11) have moderation potentials

$$
\chi_{\alpha,p}(\psi_z; \mu) := a_0(\psi_z) + \mu(z) \tilde{\chi}_{\alpha,p}(\ell_\mu(\psi_z)),
$$

where $a_0(\psi_z) := \psi_z \cdot f(z)$ is the contribution of the drift field,

$$
\tilde{\chi}_{\alpha,p}(r) := r s \left( 1 + \frac{1}{\alpha p} \left( r s^{1-\alpha p} \right)^{\frac{1}{\alpha-1}} \right)_{s=\rho_{\alpha,p}^{-1}(r)}
$$

if $0 < \alpha < 1$, and

$$
\tilde{\chi}_{1,p}(r) := \begin{cases} 
\frac{1}{p} + \frac{1}{q} r^q & r < 1 \\
r & r \geq 1
\end{cases}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1. \quad (18)
$$

**Proof:** Setting $r = \ell_\mu(\psi_z)$ for notational simplicity, Theorem 1 implies that

$$
\left. \frac{\chi_{\alpha,p}(\psi_z) - a_0(\psi_z)}{\mu(z)} \right|_{s=\rho_{\alpha,p}^{-1}(r)} = s r + \frac{1}{\alpha p} \left( 1 - s^p \right)^{\alpha}.
$$

For $0 < \alpha < 1$, we can simplify this expression as follows:

$$
r = \rho_{\alpha,p}(s) = s^{p-1} \left( 1 - s^p \right)^{\alpha-1}
$$

implies that

$$
\left( 1 - s^p \right)^{\alpha} = \left( r s^{p-1} \right)^{\alpha-1}.
$$

Substituting this into (19) and regrouping terms yields (17).
In the case $\alpha = 1$, 
\[ s \mapsto rs + \frac{1}{p} (1 - s^p) \tag{20} \]
is the restriction of a polynomial to $[0, 1]$. The maximum of the polynomial \((20)\) occurs at $r^{p-1}$; hence if $r < 1$, the maximum is 
\[ r^{\frac{1}{p-1}} + \frac{1}{p} \left( 1 - r^\frac{p}{p-1} \right) = \frac{1}{p} + \left( 1 - \frac{1}{p} \right) r^\frac{p}{p-1}. \]
If $1 \leq r$, the maximum occurs at one of the endpoints; since $\frac{1}{p} < 1$, in this case, \((20)\) achieves its maximum of $r$ at 1. 

Direct utilization of conservation of the Hamiltonian $H$ can simplify the analysis of the synthesis problem in many situations. However, when numerically approximating solutions of Hamiltonian systems, discretion must be used when combining conservation laws with discretization to avoid artificial accelerations and related errors. We now focus on explicit use of the conservation law not to recommend it as a general purpose strategy, but to emphasize the role of the specific value of the Hamiltonian in determining solutions satisfying given boundary conditions. The following results play a pivotal role in our analysis of the projectile problem in §5.

We can express the optimal scalings for the moderation incentives $\tilde{C}_{\alpha, p}$ as functions $\hat{\sigma}_{\alpha, p}$ of 
\[ \phi(z; h) := \frac{\tilde{C}(z) + h}{\mu(z)}, \tag{21} \]
where $h$ denotes the difference of the Hamiltonian and the drift potential at $\psi_z$.

**Proposition 3** The optimal scaling for the moderation incentive $\tilde{C}_{\alpha, p}$ and associated Hamiltonian \((27)\) satisfies 
\[ \sigma_{\alpha, p}(\psi_z; \mu) = \hat{\sigma}_{\alpha, p}(\phi(z; H(\psi_z) - a_0(\psi_z))), \]
where $\hat{\sigma}_{\alpha, p} : \mathbb{R} \to (0, 1)$ and $\tau_{\alpha, p} : (0, 1) \to \mathbb{R}$ are given by 
\[ \hat{\sigma}_{\alpha, p}(\phi) = (1 - \tau_{\alpha, p}^{-1}(\phi)))^{\frac{1}{p}} \quad \text{for} \quad \tau_{\alpha, p}(w) := w^{\alpha-1} \left( 1 + \left( \frac{1}{\alpha p} - 1 \right) w \right) \tag{22} \]
if $0 < \alpha < 1 \leq p$ and 
\[ \hat{\sigma}_{1, p}(\phi) = \begin{cases} \left( \frac{p \phi - 1}{p - 1} \right)^\frac{1}{p} & \frac{1}{p} \leq \phi < 1 \\ 1 & \phi \geq 1 \end{cases} \tag{23} \]
if $p > 1$.

**Proof:** If $0 < \alpha < 1$ and $\lambda(\psi_z) \neq 0$, \((12)\) and \((21)\) imply that 
\[ \tilde{\chi}_{\alpha, p}(\rho_{\alpha, p}(\sigma_{\alpha, p}(\psi_z; \mu))) = \phi(z, h). \tag{24} \]
The composition $\hat{\chi}_{\alpha,p} \circ \rho_{\alpha,p}$ can be simplified as follows. Substituting
\[
\rho_{\alpha,p}(s) s^{1-\alpha p} = s^{(1-\alpha)p} (1 - s^p)^{\alpha-1} = (s^p - 1)^{\alpha-1}
\]
into (17) and regrouping terms yields
\[
\hat{\chi}_{\alpha,p}(\rho_{\alpha,p}(s)) = \rho_{\alpha,p}(s) \left( 1 + \frac{1}{\alpha p} (s^p - 1) \right) = (1 - s^p)^{\alpha-1} s^p \left( 1 + \frac{1}{\alpha p} (s^p - 1) \right) = (1 - s^p)^{\alpha-1} \left( \frac{1}{\alpha p} + \left( 1 - \frac{1}{\alpha p} \right)s^p \right) = \tau_{\alpha,p}(1 - s^p).
\]
$\tau_{\alpha,p}$ is strictly decreasing for $0 < \alpha \leq 1$, and hence is invertible. Solving (24) for $\sigma_{\alpha,p}(\psi_z; \mu)$ yields (22).

If $\alpha = 1 < p$, then (18) implies
\[
\hat{\chi}^{-1}_{1,p}(\phi) = \begin{cases} 
\left( q \left( \phi - \frac{1}{p} \right) \right)^{\frac{1}{q}} & \text{if } \phi < 1, \\
\phi & \text{if } \phi \geq 1
\end{cases}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.
\]
Substituting $\ell_{\mu}(\psi_z) = \hat{\chi}^{-1}_{1,p}(\phi)$ in (13) and simplifying yields (23). $\blacksquare$

In the case $\alpha = \frac{1}{p}$, the moderation potential and optimal scaling are particularly simple. We will make use of the following expressions in §5 when analyzing generalizations of the projectile example from §1.

**Corollary 2** If $\alpha = \frac{1}{p} < 1$, then
\[
\chi_{\frac{1}{p},p}(\psi_z; \mu) - a_0(\psi_z) = \| (\ell(\psi_z), \mu(z)) \|_q = (\ell(\psi_z)^q + \mu(z)^q)^{\frac{1}{q}},
\]
where $\frac{1}{p} + \frac{1}{q} = 1$, and
\[
\tilde{\sigma}_{\frac{1}{p},p}(\phi) = \left( 1 - \phi^{-q} \right)^{\frac{1}{p}}.
\]

**Proof:** Setting $q = \frac{p}{p-1}$ and substituting $\alpha = \frac{1}{p}$ and (14) into (17) yields
\[
\hat{\chi}_{\frac{1}{p},p}(r) = r s \left( 1 + r^{-q} \right)^{-\frac{1}{p}} = r \left( 1 + r^{-q} \right)^{\frac{1}{p}} = (r^q + 1)^{\frac{1}{q}}.
\]
Hence
\[
\chi_{\frac{1}{p},p}(\psi_z) - a_0(\psi_z) = \mu(z) \hat{\chi}_{\frac{1}{p},p}(\ell_{\mu}(\psi_z)) = \mu(z) (\ell_{\mu}(\psi_z)^q + 1)^{\frac{1}{q}} = (\ell(\psi_z)^q + \mu(z)^q)^{\frac{1}{q}}.
\]
(26) follows immediately from (22). $\blacksquare$
5 Vertical take-off interception with controlled velocity

To illustrate some features of moderation potentials, we return to the two dimensional system with controlled velocities briefly considered in §1, generalizing the cost functions and admissible control regions. The control problem is designed to result in an integrable Hamiltonian system—we can express the height of the projectile and the elapsed time as definite integrals of functions of the horizontal position. Further specialization yields situations in which these definite integrals have closed form expressions in terms of elliptic integrals or logarithms, facilitating comparison of solutions with different moderation incentives, admissible control regions, and targets. In particular, we shall see that the solutions change in a highly nontrivial way as the level set of Hamiltonian containing the solution changes; this illustrates the crucial difference between the optimal control problem of unspecified duration, for which solutions must lie in the zero level set, from the fixed time problem, for which the appropriate level set is determined in part by the time constraint.

We first briefly recap the projectile problem from §1 and describe the generalizations considered here. The task is to hit a target \((x_f, y_f)\), starting from an unspecified position \((x_0, 0)\) on the horizontal axis to the right of the target, with vertical initial velocity; the state space is \(S = I \times \mathbb{R}\) for a closed interval \(I \subset \mathbb{R}^+\) of the form \([x_f, \infty)\) or \([x_f, x_{\text{max}}]\). The velocity is the control, i.e. \((\dot{x}, \dot{y}) = u)\.

The admissible control region \(A(x, y)\) associated to \((x, y)\) is the closed ball of radius \(r(x)\) centered at the origin for a given function \(r \in \mathcal{C}^0(I, \mathbb{R}^+)\). The unmoderated position-dependent cost term \(\hat{C} \in \mathcal{C}^0(I, \mathbb{R}^+)\) is a function of the horizontal component of the position. The moderation incentives have the form \(\mu(x) \tilde{C}_{\alpha, p}\) for \(\mu \in \mathcal{C}^0(I, \mathbb{R}^+)\) satisfying \(\mu(x) < \hat{C}(x)\) for all \(x \in I\), \(\alpha\) and \(p\) as in §4.

We identify \(T^*S\) with a subset of \(I \times \mathbb{R}^2\), and denote \(\tilde{\psi}_z = ((x, y), \psi)\). We abuse notation, in the interest of reminding the reader of the invariance of key constructs, and denote quantities depending on the state variables as depending on \(x\), rather than the pair \((x, y)\), and drop the base point from \(\psi_z\). The quadratic form determining the admissible control regions takes the form

\[ Q_x(u) = r(x)^{-2} \|u\|^2, \]

and hence \(L_x \psi = r(x)\psi\) and \(\lambda(\psi) = r(x)\psi\) is simply a rescaling of \(\psi\). In particular, the vertical take-off condition is equivalent to the requirement that \(\psi_z(0)\) be vertical.

Our hypotheses were chosen so as to yield an integrable system: a pair of scalar conservation laws enable us to express \(\psi\) as a function of \(x\) and thus reduce the synthesis problem to a first order ODE solvable by quadrature. In Proposition 3, we showed how conservation of the Hamiltonian can be used to express the optimal scaling as a function of the state variables and the value of the Hamiltonian. The Hamiltonian

\[ H(x, \psi) = \chi_{\alpha, p}(x, \psi) - \hat{C}(x) \]

for the projectile system is independent of \(y\); hence it follows from Noether’s Theorem that the second component of \(\psi\) is a constant of the motion for the canonical Hamiltonian system.
determined by $H$. (See, e.g. [4 1].) We now show that the additional conserved quantity of this system can be used to determine the direction of the optimal control in terms of $x$ and the value $h$ of the Hamiltonian. The resulting evolution equation can be solved explicitly only in very special situations, but implicit solutions expressing $y$ and $t$ as definite integrals depending on $x$ can be formulated as follows.

**Proposition 4** Let $x_0 \in I$ and $h \in \mathbb{R}$ satisfy $\phi(x_0; h) \in \text{range}[\tilde{\chi}_{\alpha,p}]$ for $0 < \alpha < 1 \leq p$ or $\alpha = 1 < p$. Define $n\alpha,p(\cdot; h), w\alpha,p(\cdot; h) : [x_f, x_0] \rightarrow [0, \infty)$ by

$$n\alpha,p(x; h) := \frac{\mu(x)\tilde{\chi}_{\alpha,p}^{-1}(\phi(x; h))}{r(x)} \quad \text{and} \quad w\alpha,p(x, x_0; h) := \left(\frac{n\alpha,p(x_0; h)}{n\alpha,p(x; h)}\right)^2.$$ 

If $n\alpha,p(\cdot; h)$ has a strict minimum at $x_0$, then

$$y\alpha,p(x; x_0, h) := \int_{x_0}^{x} \frac{d\xi}{\sqrt{w\alpha,p(x_0, \xi, h) - 1}}$$

and

$$t\alpha,p(x; x_0, h) := \int_{x}^{x_0} \frac{d\xi}{r(\xi)\tilde{\sigma}_{\alpha,p}(\xi; h)\sqrt{1 - w\alpha,p(\xi, x_0, h)}}$$

for $x_f \leq x < x_0$ implicitly determine the state variables of a solution in $H^{-1}(h)$ of the Hamiltonian system determined by (27).

**Proof:** Invertibility of $\tilde{\chi}_{\alpha,p}$ follows from the identity

$$\tilde{\chi}_{\alpha,p}(\rho\alpha,p(s)) = \tau\alpha,p(1 - s^p)$$

and the invertibility of $\rho\alpha,p$ and $\tau\alpha,p$. Thus $\ell\mu(\psi_2) = \tilde{\chi}_{\alpha,p}^{-1}(\phi(z; h))$ and hence

$$\|\psi\| = \frac{\ell(\psi)}{r(x)} = \frac{\mu(x)\ell\mu(\psi)}{r(x)} = n\alpha,p(x; h)$$

along a solution $((x, y), \psi)$ of the canonical Hamiltonian system determined by $H$ lying in $H^{-1}(h)$.

The initial condition $\dot{x}(0) = 0$ implies that $\psi(0) = (0, \psi_2)$, and since $\psi_2 \neq 0$ is constant and $n\alpha,p(x; h)$ has a strict minimum at $x_0$, it follows that $\psi$ is always nonzero and that $\psi_1$ equals zero only when $t = 0$, then

$$\frac{\psi}{\|\psi\|} = \frac{1}{\|\psi\|} \left(-\sqrt{\|\psi\|^2 - \psi_2^2}, \psi_2\right) = \left(-\sqrt{1 - \frac{\|\psi(0)\|^2}{\|\psi\|^2}}, \frac{\|\psi(0)\|}{\|\psi\|}\right).$$

(The signs are determined by the conditions $x_f < x_0$ and $y_f > 0$, which imply that $\dot{x}$ must be negative and $\dot{y}$ must be positive.)
Figure 5: Sample plots for $r(x) = \frac{1}{x}$. Colored dots indicate positions at $t_j := \frac{j}{4}$. Left: $c = \frac{2}{3}$; green: $\mu = 0.05$, gold: $\mu = \frac{2}{\sqrt{3}} - 0.05$; $x_f = \frac{1}{10}$. Center and right: $c = 2$; green: $\mu = 0.05$, gold: $\mu = 1$, red: $\mu = 1.95$; $x_f = \frac{1}{10}$ (center) or $\frac{1}{2}$ (right). The gray dashed line indicates the normalized unmoderated cost $\hat{C}(x)$. It follows that there are functions $y_{\alpha,p}(x; x_0, h)$ and $t_{\alpha,p}(x; x_0, h)$ such that

$$y'_{\alpha,p}(x; x_0, h) = \frac{X_{\alpha,p}(x; h)_{2}}{X_{\alpha,p}(x; h)_{1}} = -\frac{\sqrt{w_{\alpha,p}(x, x_0, h)}}{1 - w_{\alpha,p}(x, x_0, h)} = -\frac{1}{\sqrt{w_{\alpha,p}(x_0, x, h) - 1}},$$

and hence [28] holds. Analogously,

$$t'_{\alpha,p}(x; x_0, h) = \frac{1}{X_{\alpha,p}(x; h)_{1}}$$

implies [29].

Remark: If we introduce the angle $\theta_{\alpha,p}(x; x_0, h) := \sin^{-1} \sqrt{w_{\alpha,p}(x, x_0, h)}$, $\theta_{\alpha,p}(x; x_0, h) \in \left(\frac{\pi}{2}, \pi\right)$, then [31] implies that $\psi$ has polar coordinates $(n_{\alpha,p}(x; x_0, h), \theta_{\alpha,p}(x; x_0, h))$ at $(x, y)$ and

$$y'_{\alpha,p}(x; x_0, h) = \tan \theta_{\alpha,p}(x; x_0, h).$$

However, we have found it more convenient in specific calculations to work with $w_{\alpha,p}$.

Proposition [4] provides implicit equations for solutions of the Hamiltonian system with Hamiltonian [27]. Such solutions only qualify as solutions of the synthesis problem if additional conditions on the parameters $x_0$ and $h$ are satisfied.

**Synthesis problem.** The projectile must strike the target and lie in the zero level set of the Hamiltonian. The initial position $x_0$ determines a solution of the synthesis problem $\iff y_{\alpha,p}(x_f; x_0, 0) = y_f$. 
Figure 6: Sample trajectories for \( r \equiv 1 \). Colored dots indicate positions at \( t_j := \frac{j}{4} \). Left: \( c = \frac{2}{3} \); green: \( \frac{\mu}{r} = 0.05 \), red: \( \frac{\mu}{r} = \frac{2}{\sqrt{3}} - 0.05 \); \( x_f = \frac{1}{10} \). Center and right: \( c = 2 \); green: \( \frac{\mu}{r} = 0.05 \), gold: \( \frac{\mu}{r} = 1 \), red: \( \frac{\mu}{r} = 1.95 \); \( x_f = \frac{1}{10} \) (center) or \( \frac{1}{2} \) (right). The gray dashed line indicates the normalized unmoderated cost \( \hat{C}(\cdot) / \hat{C}(x_f) \).

**Fixed time synthesis problem.** The projectile must strike the target at the specified time \( t_f \). The initial position \( x_0 \) and Hamiltonian value \( h \) determine a solution of the time \( t_f \) synthesis problem \( \iff y_{\alpha,p}(x_f; x_0, h) = y_f \) and \( t_{\alpha,p}(x_f; x_0, h) = t_f \).

**Remark:** Given \( x_0 \) and \( h \) such that \( y_{\alpha,p}(\cdot; x_0, h) \) and \( t_{\alpha,p}(\cdot; x_0, h) \) are well-defined on \([x_f, x_0]\) and \( y_{\alpha,p}(x_f; x_0, h) = y_f \), one can, of course, *a posteriori* specify \( t_{\alpha,p}(x_f; x_0, h) \) as the desired duration, thereby obtaining a solution of the corresponding fixed time optimal control problem. However, if there is a family of pairs \((x_0, h)\) determining solutions of different durations that all strike the target and only one of these solutions will be implemented, some criterion for selecting that solution must be established.

5.1 \( \alpha = \frac{1}{2}, \ p = 2, \ \text{and constant} \ \frac{\mu}{r} \)

The expressions for \( y_{\alpha,p} \) and \( t_{\alpha,p} \) as functions of \( x \) take a particularly simple form if \( \alpha = \frac{1}{2}, \ p = 2, \) and \( \mu \) is a positive rescaling of \( r \).

**Corollary 3** If \( \phi(x; h) < 1 \) for \( x_f \leq x < x_0 \), define

\[
 v(x_0, h) := \sqrt{1 - \phi(x_0; h)^{-2}}  
\]

and

\[
 \tilde{y}(x; x_0, h) := \int_x^{x_0} \frac{d\xi}{\sqrt{\left( \frac{\phi(\xi; h)}{\phi(x_0; h)} \right)^2 - 1}}  
\]

for \( x_f \leq x < x_0 \). Then

\[
 y_{\frac{1}{2},2}(x; x_0, h) = v(x_0, h) \tilde{y}(x; x_0, h)  
\]
and

\[ t_{\frac{1}{2},2}(x; x_0, h) = \int_{x_0}^{x} \frac{d\xi}{r(u)\sqrt{1 - \left(\frac{\phi(x_0; h)}{\phi(\xi; h)}\right)^2}}. \]  

(35)

Proof: \( \tilde{\chi}^{-1}_{\frac{1}{2},2}(\phi) = \sqrt{1 - \phi^2} \) and \( \frac{\mu}{r} \) = constant imply that

\[ w_{\frac{1}{2},2}(x_0, x; h) = \left(\frac{\tilde{\chi}^{-1}_{\frac{1}{2},2}(\phi(x; h))}{\tilde{\chi}^{-1}_{\frac{1}{2},2}(\phi(x_0; h))}\right)^2 = \frac{1 - \phi(x; h)^2}{1 - \phi(x_0; h)^2} = \frac{\left(\frac{\phi(x; h)}{\phi(x_0; h)}\right)^2 - 1}{1 - \phi(x_0; h)^2} + 1. \]

Analogously,

\[ 1 - w_{\frac{1}{2},2}(x, x_0; h) = \frac{1 - \left(\frac{\phi(x_0; h)}{\phi(x; h)}\right)^2}{1 - \phi(x_0; h)^2} = \frac{1 - \left(\frac{\phi(x_0; h)}{\phi(x; h)}\right)^2}{\tilde{\sigma}_{\frac{1}{2},2}(x; h)^2}. \]

Substituting these expressions and \( p = q = 2 \) into (28) and (29) yields (34).

The vertical component of the velocity at position \( (x, y(x)) \) is given by

\[ \frac{y'_{\frac{1}{2},2}(x)}{v'_{\frac{1}{2},2}(x)} = \frac{1 - \left(\frac{\phi(x_0; h)}{\phi(x; h)}\right)^2}{\left(\frac{\phi(x; h)}{\phi(x_0; h)}\right)^2 - 1} = \frac{\phi(x_0; h)}{\phi(x; h)}v(x_0, h)r(x). \]

\[ \frac{v'_{\frac{1}{2},2}(x)}{v_{\frac{1}{2},2}(x)} = v(x_0, h)r(x)\sqrt{1 - \left(\frac{\phi(x_0; h)}{\phi(x; h)}\right)^2 - 1} = v(x_0, h)r(x)\frac{\phi(x_0; h)}{\phi(x; h)}. \]

\[ \frac{y'_{\frac{1}{2},2}(x)}{v'_{\frac{1}{2},2}(x)} = \frac{1 - \left(\frac{\phi(x_0; h)}{\phi(x; h)}\right)^2}{\left(\frac{\phi(x; h)}{\phi(x_0; h)}\right)^2 - 1} = \frac{\phi(x_0; h)}{\phi(x; h)}v(x_0, h)r(x). \]

Corollary 3 reveals several distinctive features of this special situation.

- The relationship between \( t \) and \( x \) does not depend on the value of the constant ratio \( \frac{\mu}{r} \).
- The function \( \tilde{y} \) determines the optimal solution of the unmoderated problem, corresponding to \( \mu \equiv 0 \), with initial velocity on the boundary of the admissible control region.
- The relationship between \( y \) and \( x \) depends on the ratio \( \frac{\mu}{r} \) only via the scaling factor \( v(x_0, h) \).
- \( x_0 \) determines a trajectory with energy \( h \) passing through the point \( (x_f, y_f) \) iff \( (1, y_f) \) lies on the positive quadrant of the ellipsoid with principal axes \( \phi(x_0; h) \) and \( \tilde{y}(x_f; x_0, h) \).

If \( \phi(x_0; 0) \) and \( \tilde{y}(x_f; x_0, 0) \) determine a family of non-intersecting ellipsoids parametrized by \( x_0 \), then the synthesis problem with target \( (x_f, y_f) \) has a unique solution for each admissible value of the ratio \( \frac{\mu}{r} \), with initial position \( (x_0, 0) \) and initial velocity \( (0, v(x_0, 0)\rho(x_0)) \), for the unique value of \( x_0 \) such that the ellipse with principal axes \( \phi(x_0; 0) \) and \( \tilde{y}(x_f; x_0, 0) \) passes through \( (1, y_f) \).
We now further specialize, considering the position-dependent cost term $\tilde{C}(x) = \frac{c^2}{2x^2} + 1$ and admissible control region radius functions $r(x) = \frac{1}{2}$ or $r(x) = 1$. In these cases we can explicitly express $\gamma$ and $t$ as functions of $x$ in terms of logarithms (for $\rho(x) = \frac{1}{2}$) or elliptic integrals (for $\rho(x) \equiv 1$). We present the solutions only for $h = 0$, corresponding to solutions of the synthesis problem; the expressions for nonzero $h$ are similar, but involve somewhat messier coefficients.

If we set $\eta(x; x_0) := \sqrt{\left(\frac{c}{2x_0}\right)^2 - x^2}$, then

$$\tilde{y}(x; x_0, 0) = \left(\frac{c}{2x_0} + x_0\right) \left(\ln \left(\eta(x, x_0) + \sqrt{x_0^2 - x^2}\right) - \ln \eta(x_0, x_0)\right)$$

and

$$2 t_{\frac{1}{2}}(x; x_0, 0) = \left(\frac{c}{2x_0} + x_0\right) \tilde{y}(x; x_0, 0) - \eta(x, x_0)\sqrt{x_0^2 - x^2}$$

if $r(x) = \frac{1}{2}$.

Trajectories for some representative values of $c$, $\frac{\mu}{r}$, $x_f$, and $x_0$, with $r(x) = \frac{1}{2}$ and $y_f = 1$, are shown in Figure 5. Note that the more moderate the strategy, the further $x_0$ is from the target and the slower the initial ascent. Trajectories with moderation factor near the maximum allowable value show a slow, nearly vertical early phase, executed in relatively ‘safe’ territory (i.e. relatively small values of $\tilde{C}(x)$) followed by a rapid, nearly horizontal late phase; those with low moderation factor launch closer to the target and rapidly pursue a more rounded path. The moderation factor does not correspond to increased or reduced sensitivity to risk, but influences the approach to reducing risk—the more moderate solution takes more time and travels a longer path overall, but in doing so, is able to devote most of its (constrained) speed to nearly horizontal motion when moving through the high-risk zone near the target. The differences as the moderation factor are smaller if the risk is lower, either due to a smaller value of the risk factor $c$ or to relatively large $x_f$, and hence relatively small variation in risk from $x_0$ to $x_f$. Finally, note that for this specific system, changes in the moderation factor $\frac{\mu}{r}$ result in relatively small changes in the optimal trajectory until $\frac{\mu}{r}$ is close to the maximum value.

If we let $E_E$ and $E_F$ denote the incomplete elliptic integrals of the first and second kind, and define

$$\gamma_\pm(u; k) := E_F(\sin^{-1}u; k) \pm E_E(\sin^{-1}u; k), \quad k(x_0) := -\left(1 + \frac{4x_0^2}{c}\right),$$

and

$$\gamma_\pm(x; x_0) := x_0 \left(\gamma_\pm\left(\frac{x}{x_0}; k(x_0)\right) - \gamma_\pm(1; k(x_0))\right),$$

then

$$\tilde{y}(x; x_0, 0) = \frac{\gamma_-(x; x_0)}{1 + \frac{1}{C(x_0)}} \quad \text{and} \quad 2 t_{\frac{1}{2}}(x; x_0, 0) = \gamma_+(x; x_0) + \frac{1}{k(x_0)} \gamma_-(x; x_0)$$

if $r \equiv 1$. 
Trajectories for some representative values of \( c, \frac{\mu}{r}, x_f, \) and \( x_0 \), with \( r(x) \equiv 1 \) and \( y_f = 1 \), are shown in Figure 6. As before, the more moderate the strategy, the further \( x_0 \) is from the target and the slower the ascent. However, since the admissible control region is the unit ball for all values of \( x \), there is much less dramatic variation in the speed along any given solution and in the paths of the different solutions. All of the trajectories trace follow paths that are nearly, but not exactly, elliptical.

### 5.2 \( \alpha = 1, p = 2, \) and constant \( \frac{\mu}{r} \)

We now consider the parameters values and functions used in §2: \( C_{mi} = \tilde{C}_{1,2}(\cdot, \mu) - \tilde{C} \) for constant \( \mu \) and \( \tilde{C} \) as above and \( C_{ke} = C_{mi} - 1 + \frac{5}{2} \). Thus solutions of the synthesis problem for \( C_{ke} \) correspond to solutions of an appropriate fixed time synthesis problem for \( C_{mi} \). We derive the solutions of the synthesis problem for a general \( \tilde{C} \) depending only on \( x \) and Hamiltonian value \( h \) before specializing to \( C_{mi} \) and \( C_{ke} \).

When \( \alpha = 1 \) and \( p = 2 \), the condition that the instantaneous cost be positive everywhere imposes the inequality \( \phi(x; 0) > \frac{1}{2} \), and (23) takes the form

\[
\tilde{\chi}_{1,2}^{-1}(\phi) = \begin{cases} 
\frac{\sqrt{2\phi - 1}}{\phi} & \text{if } \frac{1}{2} \leq \phi < 1 \\
1 & \text{if } \phi \geq 1
\end{cases}
\]  

and

\[
\tilde{\sigma}_{1,2}(x; h) = \min \left\{ \sqrt{2\phi(x; h) - 1}, 1 \right\}.
\]

For simplicity, we consider only trajectories such that either \( \frac{1}{2} \leq \phi(x; h) \leq 1 \) for \( x_f \leq x \leq x_0 \) or \( 1 \leq \phi(x; h) \) for \( x_f \leq x \leq x_0 \); determining more general solutions involves patching together solutions of these kinds. (36) implies that

\[
w_{1,2}(x; x_0, h) = \begin{cases} 
\frac{2\phi(x_0; h) - 1}{2\phi(x; h) - 1} & \frac{1}{2} \leq \phi(x_0; h) < \phi(x; h) \leq 1 \\
\left( \frac{\phi(x_0; h)}{\phi(x; h)} \right)^2 & 1 \leq \phi(x_0; h)
\end{cases}
\]

for constant \( \frac{\mu}{r} \).

If \( 1 \leq \phi(x_0; h) \), comparing (28) to (33) and (29) to (34) shows that in this situation

\[
y_{1,2}(x; x_0, h) = \tilde{y}(x; x_0, h) = \frac{y_{\frac{1}{2}, 2}(x; x_0, h)}{v(x_0; h)} \quad \text{and} \quad t_{1,2}(x; x_0, h) = t_{\frac{1}{2}, 2}(x; x_0, h).
\]

If \( \frac{1}{2} \leq \phi(x; h) \leq 1 \) for \( x_f \leq x \leq x_0 \), then

\[
\frac{y_{1,2}(x; x_0, h)}{\tilde{\sigma}_{1,2}(x_0; h)} = t_{1,2}(x; x_0, h) = \int_{x}^{x_0} \frac{d\xi}{\sqrt{2(\phi(\xi; h) - \phi(x_0; h))}}.
\]  

(38)
If we further specialize to the case $\hat{C}(x) = 1 + \frac{c}{2x^2}$, $r \equiv 1$, then
\[
t_{1,2}(x; x_0, h) = \sqrt{\frac{x_0^2 - x^2}{\zeta(x_0)}} \quad \text{and} \quad \hat{\sigma}_{1,2}(x_0; h) = \zeta(x_0) + \frac{2(1+h)}{\mu} - 1 \quad \text{for} \quad \zeta(x_0) := \frac{c}{\mu x_0^2},
\]
where $\mu$ denotes the constant value of the moderation factor. It follows that the projectile paths are segments of ellipses centered at the origin. We can easily express $z = (x, y)$ explicitly as a function of $t$ in this case:
\[
z_{1,2}(t; h) = \left(\sqrt{x_0^2 - \zeta(x_0)t^2}, \hat{\sigma}_{1,2}(x_0; h)t\right).
\]
Setting $h = 0$ yields the state information of the synthesis problem for $C_{mi}$, while setting $h = \frac{\mu}{2} - 1$ gives the corresponding information for $C_{ke}$.

Remark: The graphs of $y_{1,2}(\cdot; x_0, h)$ are very nearly elliptical if $1 \leq \phi(x_0; h)$, but do not exactly coincide with segments of ellipses.

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