Domain shape dependence of semiclassical corrections to energy

Grzegorz Kwiatkowski

Gdansk University of Technology, ul. Narutowicza 11/12, 80-952, Gdansk, Poland
Center for Functionalized Magnetic Materials (FunMagMa), Immanuel Kant Baltic Federal University, 236041, Kaliningrad, Russia

E-mail: grzkwiat@mail.ru

Received 17 January 2017, revised 16 May 2017
Accepted for publication 2 June 2017
Published 29 June 2017

Abstract
Stationary solution of a one-dimensional sine-Gordon system is embedded in a multidimensional theory with an explicitly finite domain in the added spatial dimensions. Semiclassical corrections to energy are calculated for a static kink solution with emphasis on the impact of the scale of the domain as well as the choice of boundary conditions on the results for a rectangular cross-section.

Keywords: solitons, semiclassical quantization, zeta function regularization, mathematical physics

1. Introduction
Since the early works of Feynman [1, 2], path integral formulation of quantum field theory has mostly been limited in application due to its nontrivial mathematical formulation, yet it has proven to be quite successful as a means of describing quantum electrodynamics in particular as well as quantum field theory in general. One of the most widely known methods for dealing with the computational difficulties of path integrals is the semiclassical approach (a term used for many different methods including the most well known, WKB [3–5]) developed by Maslov [6, 7] for quantum-mechanical path integrals, which can be naturally applied to quantum field theory as well. It is important to note that while Maslov’s form of semiclassical propagator described through a determinant of an operator connected to classical action is similar to that given earlier by Fock [8] and Pauli [9], they are different in nature. The semiclassical form of the propagator given by Pauli and Fock contains the Hessian of the classical action integral over initial and final coordinates, while in the Maslov case it is the Hessian of classical action over all trajectories connecting well-defined initial and final states. This means that for Maslov’s semiclassical method one needs only a single classical solution, whereas other semiclassical methods usually require more knowledge about the classical system. This
has proven to be very important in quantizing nonlinear field theories. The first major success was the quantization of a one-dimensional $\phi^4$ kink by Daschen et al.\cite{Daschen1974}, where Maslov’s approach was combined with Gutzwiller’s form of propagator in energy–momentum coordinates\cite{Gutzwiller1971,Gutzwiller1972}. Not long after that, Korepin and Faddeev quantized a one-dimensional sine-Gordon kink\cite{Korepin1977}. Later on, the introduction of generalized zeta-function regularization\cite{Konoplich1985} allowed for the inclusion of more spatial variables. Using this technique Konoplich quantized the $\phi^4$ kink embedded in a general, infinite $d$-dimensional space\cite{Konoplich1990}. Most notably, the results were heavily dependent on the number of spatial variables included. Although there were many different approaches to regularization and semiclassical quantization schemes, there was little progress in applying the method to nonlinear fields other then static kinks in infinite space. In recent years Pawellek has obtained energy corrections to a static, periodic solution of sine-Gordon system in $1 + 1$ dimensions\cite{Pawellek2003} through a method developed by Kirsten and Loya\cite{Kirsten1995}, and a few years later energy corrections for static, periodic solutions of both sine-Gordon and $\phi^4$ systems embedded in a multidimensional theory were obtained as a power series in elliptic parameter around the single-kink limit case\cite{Kirsten1997}.

Because the energy corrections depend heavily on the number of dimensions included two questions arise: when can a given spatially constrained system be approximated as a one- or two-dimensional system and is such a simplification at all valid? Calculations explicitly taking the finite domain into account are essential for answering the above questions. Additionally, this would also allow us to test the impact of boundary conditions on energy corrections, which are not apparent in the continuum approximation. The primary purpose of this publication is to explore the stated questions, and as such search for further directions of research.

With the growing interest in nanoscale structures it becomes more important to accurately model phenomena at such scales. While atomistic quantum simulations are possible, they are vastly limited in the sample size they can be used for due to the significant computational complexity of such methods. Therefore quasiclassical quantization offers a way of exploring micro- and nanoscale systems, but not without its own challenges. Since continuum approximation of the spectrum is only valid for sufficiently large systems, more precise calculations become necessary. As such, the results of this paper are a starting point for the investigation of quantum effects in nonlinear continuum systems in finite domains. However, the focus of this paper is on the more theoretical aspects of the problem, which should be examined and refined before the results can be applied to physical systems.

The aim of this work is to calculate energy corrections to single-kink energy in an explicitly finite domain in the added dimensions. Various types of boundary conditions and their combinations are examined, with emphasis on the differences between the behaviour of the classical system and its quantum counterpart. In the next section, the employed quantization scheme is explained along with chosen regularization procedure. The following section contains the results obtained for a few chosen boundary conditions and their analysis.

2. Methodology

2.1. The considered system

Let us consider the sine-Gordon system
\begin{equation}
\frac{1}{c^2} \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x_3^2} - m^2 \sin(\theta) = 0
\end{equation}
with variables $t, x_3 \in \mathbb{R}$; $c$ is the linear wave propagation speed and $m$ is a real-valued potential amplitude. This system admits a well-known static kink solution.
which can be directly embedded into an $N$-dimensional theory with appropriate boundary conditions in the added dimensions. For the purpose of this publication we consider a model with three spatial dimensions

$$\frac{1}{c^2} \frac{\partial^2 \theta}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2 \theta}{\partial x_i^2} + m^2 \sin(\theta) = 0$$

with $x_1 \in [0, l_1]$ and $x_2 \in [0, l_2]$, whereas the boundary conditions are assumed to preserve the solutions (2) regardless of their type (note that von Neumann and periodic conditions grant that automatically). It is of note that as long as the shape of the solution is unchanged, the classical energy of the solution also stays the same regardless of the type of boundary condition or the number of dimensions (in the latter case there is obviously a linear dependence on the size of the domain in those added dimensions).

2.2. Semiclassical quantization

Semiclassical corrections to energy are obtained through a quantization scheme derived by Maslov [6, 7] with the generalized zeta function regularization procedure presented by Konoplich [15]. If we define the classical system through an action integral

$$S(\varphi) = A \int_{0}^{T} \int_{D} \left[ \frac{1}{2c^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2} \sum_{n=1}^{3} \left( \frac{\partial \varphi}{\partial x_n} \right)^2 \right] - V(\varphi) \prod_{n=1}^{3} dx_n dt$$

with $D$ a given spatial domain, $T$ an arbitrary time period, $c$ a unitless propagation speed, $A$ a constant containing all physical units and $V$ some potential relevant to a given model, then energy corrections for static solutions (here denoted as $\varphi$) of this system are derived from the quantum propagator in the path integral form

$$\langle \psi_T | e^{-\frac{i}{\hbar} H} | \psi_0 \rangle = \int_{C_{\psi_0,\psi_T}} e^{\frac{i}{\hbar} S(\varphi)} D\varphi$$

through Taylor expansion of the action integral around the classical solution. Assuming $\varphi$ is the classic field for which we seek energy corrections and $\phi_j$ are elements of an orthonormal base we perform a substitution

$$\phi = \varphi + \sum_{j} a_j \phi_j$$

with $\phi_j$ necessarily fulfilling the same type of boundary condition as $\phi$ but always homogeneous, because of $\psi$. This allows us to write the approximation for the action integral as

$$S(\phi) \approx S(\varphi) + \frac{1}{2} \sum_{j,k} a_j a_k \frac{\partial^2 S}{\partial a_j \partial a_k} + \ldots$$

where $\frac{\partial^2 S}{\partial a_j \partial a_k}$ takes the form
\[
\frac{\partial^2 S}{\partial q_i \partial q_k} = A \int_0^T \int_D \phi_k \left( -\frac{1}{c^2} \frac{\partial^2 \phi_j}{\partial t^2} + \sum_{n=1}^3 \frac{\partial^2 \phi_j}{\partial x_n^2} - V''(\phi) \phi \right) \prod_{n=1}^3 d\phi_n \, dt \tag{8}
\]

which can be interpreted as a scalar product of \( \phi_k \) and \( \phi_j \) with an operator

\[
L = -\frac{iA}{2\pi \hbar r^2} \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \sum_{n=1}^3 \frac{\partial^2}{\partial x_n^2} - V''(\phi) \right) \tag{9}
\]

acting on \( \phi_j \) with the additional factors taken in for normalization and in order to obtain a simpler final expression (10). Assuming full separation of variables of the eigenvalue problem of \( L \) we can take \( \phi_j \) as its eigenfunctions and the path integral simplifies to a product of Gaussian functions. Thus the approximate quantum energy takes form

\[
E_q = -\frac{S(\phi)}{T} - \Re \left[ \frac{i\hbar}{2T} \ln \left( \det[L] \right) \right] \tag{10}
\]

where \(-\frac{S(\phi)}{T}\) is the classical energy. Expression (10) needs to be regularized in two steps. First by subtraction of the analogous expression for the vacuum solution as a mean of properly setting zero for energy:

\[
E_q = -\frac{S(\phi)}{T} - \Re \left[ \frac{i\hbar}{2T} \ln \left( \frac{\det[L]}{\det[L_0]} \right) \right] \tag{11}
\]

with

\[
L_0 = -\frac{iA}{2\pi \hbar r^2} \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \sum_{n=1}^3 \frac{\partial^2}{\partial x_n^2} - V''(\phi_0) \right) \tag{12}
\]

and \( \phi_0 \) as minimum energy solution of a given system which is usually a trivial constant function. Second by a choice of the norm of base functions (\( r \) in (9) and (12)) in the path integral, for which there is no direct method within zeta function regularization. The key reason or this is that the natural way of normalizing the path integrals in the propagator (let us assume the notation \( K(t_0, x_0, t_1, x_1) \)) is to use part of its definition

\[
\forall t_0 < t_1, K(t_0, x_0, t_1, x_1) = \int_D K(t_0, x_0, t_1, x_1) K(t_1, x_1, t_2, x_2) d\phi_1. \tag{13}
\]

This approach, however, is not valid in case of field theories, since there is no general definition of path integrals in such cases. The problem of normalization lies directly within the mathematical foundation. Therefore this parameter is fitted for the 1 + 1 dimensional case to results obtained with a different method [10, 13], which implies

\[
r^2 = \frac{Am^2}{2\pi \hbar}. \tag{14}
\]

Such a fitting was also proven to give consistent results, after the inclusion of additional dimensions [20], for the kink of the \( \phi^4 \) model, for which semiclassical energy corrections in three spatial dimensions were obtained by Ventura [21] with the method developed by Dashen et al [10]. While this indeed does not prove the validity of such a fitting in other cases, the strong ties between the sine-Gordon and \( \phi^4 \) models suggests that the correlation would carry over. It is also important to keep in mind that a potentially wrong choice of the \( r \) coefficient
would only result in a constant shift of the energy corrections, which can be seen in works where it is intentionally left unspecified [15]. Since \( r \) can be extracted from the equation for the Green function through scaling procedures, this property holds for arbitrary classic fields (see [18] for a detailed description of the scaling methods).

Expression (11) is calculated using zeta function regularization [15]

\[
\ln \left( \frac{\det[L]}{\det[L_0]} \right) = -\frac{d\zeta}{ds}(0)
\]

where

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \int_0^\tau \int_D (gL(\tau, t, t, x, x) - g_{L_0}(\tau, t, t, x, x)) \prod_{n=1}^3 dx_n dt d\tau
\]

(16)

with \( g_L \) and \( g_{L_0} \) as the Green functions of following heat flow equations:

\[
\left( \frac{\partial}{\partial \tau} + L \right) g_L(\tau, t, t_0, x, x_0) = \delta(\tau) \delta(t - t_0) \delta(x - x_0)
\]

(17)

\[
\left( \frac{\partial}{\partial \tau} + L_0 \right) g_{L_0}(\tau, t, t_0, x, x_0) = \delta(\tau) \delta(t - t_0) \delta(x - x_0)
\]

(18)

with \( x = [x_1, x_2, x_3] \) and boundary conditions on those variables of the same type as the classical system, with the distinction that they are homogeneous regardless of the classical case. It is convenient to define

\[
\gamma(\tau) = \int_{[0,\tau] \times D} g_L(\tau, t, t, x, x) dx dt
\]

(19)

Using this notation we can express corrections to energy as

\[
\Delta E = \Re \left\{ \frac{iM}{2\hbar} \frac{\partial}{\partial \tau} \left[ \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} (\gamma_L(\tau) - \gamma_{L_0}(\tau)) d\tau \right] \right\}
\]

(20)

Since the considered classical field is effectively one dimensional and Green functions for heat equations in the case of variable separation can be constructed as a product of Green functions for \( 1+1 \) dimensional problems [20], it is convenient to define

\[
L_1 = \frac{iA}{2\pi \hbar r^2} \frac{1}{c^2} \frac{\partial^2}{\partial t^2}
\]

(21)

\[
L_{x_1} = -\frac{iA}{2\pi \hbar r} \frac{\partial^2}{\partial x_1^2}
\]

(22)

\[
L_{x_2} = -\frac{iA}{2\pi \hbar r} \frac{\partial^2}{\partial x_2^2}
\]

(23)

\[
L_{x_3} = -\frac{iA}{2\pi \hbar r} \left\{ \frac{\partial^2}{\partial x_3^2} + V''(x_3) \left\{ \frac{\partial^2}{\partial x_3^2} \right\} \right\}
\]

(24)
\[ L_{x,0} = -\frac{iA}{2\pi \hbar r^2} \left\{ \frac{\partial^2}{\partial x_3^2} + m^2 \right\}. \] (25)

In the case of \( L_{x_1}, L_{x_3} \) and \( L_{x_2} \), we can readily write (assuming Dirichlet boundary conditions—other cases will be considered later):

\[
\gamma_x(\tau_A) = \sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{4} \tau_A} \approx \sqrt{\frac{\text{i}\tau_A}{4\pi}}. 
\] (26)

\[
\gamma_{x_1}(\tau_A) = \sum_{n=1}^{\infty} e^{\frac{\pi^2 n^2}{4} \tau_A} \]

\[
\gamma_{x_2}(\tau_A) = \sum_{n=1}^{\infty} e^{\frac{\pi^2 n^2}{8} \tau_A} 
\] (27)

with \( \tau_A = |A|\tau \). We use continuum approximation for the time-related \( \gamma \) function, since the time period \( T \) is arbitrary, so this won’t affect the results. Although a study of the effect of short time periods (related to quick consecutive measurements) on energy corrections might also be interesting, it is not the focus of this work. The spatial spectra are explicitly left in discrete form in order to study the effects of scale and boundary conditions on the results. Considering that for (11) we only need the diagonal part of the Green function, it is resolved for \( L_{x_3} \) as a solution to Drach equation with method described in [22]. For the sine-Gordon kink we obtain (after subtracting the Green function for the vacuum solution and integration of the diagonal over \( x_3 \))

\[ \gamma_{x_3}(\tau_A) = -\text{Erf}(m\sqrt{i\tau_A}). \] (29)

3. Results

The general case of

\[
\begin{cases}
a \phi(0) + b \frac{\partial \phi}{\partial x}(0) = 0 \\
c \phi(l) + d \frac{\partial \phi}{\partial x}(l) = 0
\end{cases}
\] (30)

boundary conditions is unfortunately too complicated for calculating energy corrections. This means we will only address a few basic examples of boundary conditions and discuss differences in the outcome between them.

3.1. Dirichlet boundary conditions

Since in the \( l_1 \to \infty \) and \( l_2 \to \infty \) limit respective \( \gamma \) functions should reach the continuum limit of

\[
\gamma_{x_1}(\tau_A) = \frac{\text{if}_{x_1}^2}{4\pi\tau_A} 
\] (31)

\[
\gamma_{x_2}(\tau_A) = \frac{\text{if}_{x_2}^2}{4\pi\tau_A} 
\] (32)
it is convenient to rewrite those functions in a form that explicitly shows this limit. Using the
definition of the Jacobi $\vartheta$ function and its identities we can write:

$$
\gamma_\lambda (\tau_A) = \frac{1}{2} \left[ \sqrt{\frac{i l_1^2}{\pi \tau_A}} \left( 1 + 2 \sum_{n=1}^{\infty} e^{i l_1^2 \tau_A} \right) - 1 \right]
$$

(33)

and analogously for $\gamma_\kappa$. Let us now consider the product of $\gamma_\lambda$ and $\gamma_\kappa$, which decomposes into

$$
\begin{aligned}
&\frac{1}{4} \left( \frac{i l_1^2}{4\pi \tau_A} \right) - \frac{1}{2} \left( \frac{i l_2^2}{4\pi \tau_A} \right) - \frac{1}{2} \left( \frac{i l_3^2}{4\pi \tau_A} \right) + \sqrt{\frac{-i l_1^2 l_2^2}{16\pi^2 \tau_A^2}} \\
&- \sqrt{\frac{i l_1^2}{4\pi \tau_A}} \sum_{n_1=1}^{\infty} e^{-i l_1^2 \tau_A} - \sqrt{\frac{i l_2^2}{4\pi \tau_A}} \sum_{n_2=1}^{\infty} e^{-i l_2^2 \tau_A} \\
&+ \frac{1}{2} \sqrt{\frac{-i l_1^2 l_2^2}{\pi^2 \tau_A^2}} \sum_{n_1,n_2=1}^{\infty} \left( e^{-i l_1^2 \tau_A} + e^{-i l_2^2 \tau_A} \right) \\
&+ \sqrt{\frac{-i l_1^2 l_2^2}{\pi^2 \tau_A^2}} \sum_{n_1,n_2=1}^{\infty} \frac{e^{-\frac{2i l_1^2 + 2i l_2^2}{\tau_A}}}{n_1,n_2}. 
\end{aligned}
$$

(34)

Since we have several types of terms, we will explicitly name them for clarity

$$
\begin{align*}
\gamma_\alpha (\tau_A) &= \frac{1}{4} \left( \frac{i l_1^2}{4\pi \tau_A} \right) - \frac{1}{2} \left( \frac{i l_2^2}{4\pi \tau_A} \right) + \sqrt{\frac{-i l_1^2 l_2^2}{16\pi^2 \tau_A^2}} \\
\gamma_\beta (\tau_A) &= - \sqrt{\frac{i l_1^2}{4\pi \tau_A}} \sum_{n_1=1}^{\infty} e^{-i l_1^2 \tau_A} \\
&- \sqrt{\frac{i l_2^2}{4\pi \tau_A}} \sum_{n_2=1}^{\infty} e^{-i l_2^2 \tau_A} \\
\gamma_\gamma (\tau_A) &= \sqrt{\frac{-i l_1^2 l_2^2}{\pi^2 \tau_A^2}} \sum_{n_1,n_2=1}^{\infty} \left( e^{-i l_1^2 \tau_A} + e^{-i l_2^2 \tau_A} \right) \\
\gamma_\delta (\tau_A) &= 2 \sqrt{\frac{-i l_1^2 l_2^2}{4\pi^2 \tau_A^2}} \sum_{n_1,n_2=1}^{\infty} \frac{e^{-\frac{2i l_1^2 + 2i l_2^2}{\tau_A}}}{n_1,n_2}. 
\end{align*}
$$

(35)

The first term ($\gamma_\alpha$) corresponds to continuum limit for one to three spatial dimensions ($\frac{1}{4}$, $-\frac{1}{2} \sqrt{\frac{i l_1^2}{4\pi \tau_A}} - \frac{1}{2} \sqrt{\frac{i l_2^2}{4\pi \tau_A}}$ and $\sqrt{\frac{-i l_1^2 l_2^2}{16\pi^2 \tau_A^2}}$, respectively) with the distinction that the one-dimensional term is diminished by a factor of four and the two-dimensional terms by a factor of two in comparison with the situation in which we would consider only one or two dimensions, respectively. The remaining terms do not appear in the continuum approximation. We will proceed to compute energy corrections for all the terms separately. We obtain (with $\text{Ei}$ as the exponential integral function)
The $\Delta E_{D,a}$ term, as expected, contains all the terms corresponding to the continuum limit approximation with the aforementioned quantitative differences. Of note is the difference in sign of the 2D term in comparison with the continuum approximation (see [18]).

The $\Delta E_{D,b}$ term has a singularity at both $l_1 \to 0$ and $l_2 \to 0$ of order $\frac{1}{l}$. A singularity is expected at infinitesimal system size due to the dependence of eigenvalues of $L_{10}$ and $L_{20}$ on the size of the system. It is also expected from a physical standpoint, since spatial confinement inevitably increases the minimal value of momentum, and as a consequence the energy of the system. It is also evident that this term is oscillating in both $l_1$ and $l_2$ with the amplitude decaying as a $\frac{\ln(l)}{l}$ function for large $l$, with some components decaying faster, and a quasiperiod of $\frac{\ln^2(l)}{m}$. This means we have a rather unusual scaling of the energy on a quantum level with the oscillation frequency linearly dependent on the size of the domain. The dependence on the potential amplitude $m$ is not trivial either, since it is also periodic for any given system size. It is of note that in the $l \to 0$ limit the $\Delta E_{D,b}$ term is independent of $m$. In fact, the value of $m$ only has a significant impact on the period of oscillations in $l$, with little effect on the amplitude or the local average. This becomes noteworthy for systems for which $m$ is particularly small, since all the other components of the energy corrections, as well as the classical energy, depend on $m$ at least linearly. Additionally, it suggests that this step of the corrections is a mathematical artifact rather than a proper physical result, since with $m \to 0$ the classical solution vanishes and so should the energy corrections while $\Delta E_{D,b}$ has a non-zero limit.

$\Delta E_{D,c}$ and $\Delta E_{D,d}$ are fairly similar in nature but they scale differently. Due to the presence of unusual hypergeometric functions, a full analysis of the series in $\Delta E_{D,c}$ and $\Delta E_{D,d}$ is rather difficult, and has not yet been done. The asymptotics at infinite and infinitesimal size, however, are easy to obtain and both of the considered terms vanish at sizes tending to infinity and have singularities for $l_1$ and $l_2$ tending to 0. It is also of note that the $\mathcal{F}_2$ function is oscillating in the square of its argument (while it is not periodic), which results in an oscillation in $l_1$, $l_2$
and $m$ of the whole series due to the discrete sum. The difference between $\Delta E_{D,c}$ and $\Delta E_{D,d}$ comes when we scale up only one of $l_1$ or $l_2$: the $\Delta E_{D,a}$ term vanishes in this case while $\Delta E_{D,c}$ scales linearly with the scaled up parameter in the $l \to \infty$ limit with the proportionality constant obviously dependent on the other scale parameter. This means that for thin layers $\Delta E_{D,c}$ becomes the dominant component of energy corrections unless $m$ is sufficiently large so that the bulk material term becomes the most significant. Considering the oscillatory behaviour of $\Delta E_{D,c}$, there are specific values of $m$ for a given domain size, for which this component vanishes; this would, however, require fine tuning of potential parameters.

3.2. von Neumann boundary conditions

The difference between Dirichlet and von Neumann conditions is that in the latter case zero is a valid eigenvalue. Therefore

$$\gamma_{s_1}(\tau_A) = \sum_{n_1=0}^{\infty} e^{\frac{\pi^2 n_1^2}{l_1^2} \tau_A} \quad (37)$$

$$\gamma_{s_1}(\tau_A) = \frac{1}{2} \left[ \sqrt{\frac{il_1^2}{4\pi\tau_A}} \left( 1 + 2 \sum_{n_1=1}^{\infty} e^{\frac{-\pi^2 n_1^2}{l_1^2} \tau_A} \right) + 1 \right] . \quad (38)$$

When we consider the decomposition of $\gamma_{s_1} \gamma_{s_2}$ we obtain the same terms as before with some of them having a changed sign:

$$\frac{1}{4} + \frac{1}{2} \sqrt{\frac{il_1^2}{4\pi\tau_A}} + \frac{1}{2} \sqrt{\frac{il_2^2}{4\pi\tau_A}} + \sqrt{\frac{-il_1^2}{16\pi^2\tau_A}} + \sqrt{\frac{-il_2^2}{16\pi^2\tau_A}}$$

$$+ \sqrt{\frac{il_1^2}{4\pi\tau_A}} \sum_{n_1=1}^{\infty} e^{-\frac{\pi^2 n_1^2}{l_1^2} \tau_A} + \sqrt{\frac{il_2^2}{4\pi\tau_A}} \sum_{n_2=1}^{\infty} e^{-\frac{\pi^2 n_2^2}{l_2^2} \tau_A}$$

$$+ \frac{1}{2} \sqrt{\frac{-il_1^2}{\pi^2\tau_A}} \sum_{n_1=1}^{\infty} \left( e^{-\frac{\pi^2 n_1^2}{l_1^2} \tau_A} + e^{-\frac{\pi^2 n_1^2}{l_2^2} \tau_A} \right)$$

$$+ \sqrt{\frac{-il_2^2}{\pi^2\tau_A}} \sum_{n_1,n_2=1}^{\infty} e^{-\frac{\pi^2 n_1^2}{l_1^2} \tau_A} . \quad (39)$$

This results in energy corrections of the form

$$\Delta E_N = -\frac{\hbar mc}{4\pi} + \frac{\hbar m^2 c l_1}{8\pi} + \frac{\hbar m^2 c l_2}{8\pi} + \frac{5\hbar m^3 c l_1 l_2}{72\pi^2} - \Delta E_{D,b} + \Delta E_{D,c} + \Delta E_{D,d} . \quad (40)$$

The terms that have their sign changed are those which depend explicitly only on one of the dimensions of the cross-section. In the case of

$$\frac{\hbar m^2 c l_1}{8\pi} + \frac{\hbar m^2 c l_2}{8\pi} \quad (41)$$

this means that the change will be visible for low values of either $l_2$ or $l_1$, respectively (otherwise they will be overshadowed by terms proportional to both $l_1$ and $l_2$), as in case of thin layers. The $E_{D,b}$ term decays for large values of the scale argument $l$, so the difference would only
be visible when both \( l_1 \) and \( l_2 \) are sufficiently small, which corresponds to thin wires. Bulk material properties (large \( l_1 \) and \( l_2 \)) are the same as before and depend on the \( \frac{5\hbar m^3}{2\pi^2} \) term.

### 3.3. Periodic conditions

For periodic conditions we have

\[
\gamma_{x_1}(\tau_A) = \sum_{n=-\infty}^{\infty} e^{i \frac{4\pi n}{l_2} \tau_A} \quad (42)
\]

\[
\gamma_{x_1}(\tau_A) = \frac{1}{2} \left[ \sqrt{\frac{i l_1}{\pi \tau_A}} \left( 1 + 2 \sum_{n=1}^{\infty} e^{i \frac{n l_1}{l_2}} \right) \right]. \quad (43)
\]

This results in energy corrections of the form

\[
\Delta E_p = \frac{5\hbar m^3 c l_1 l_2}{72\pi^2} + 4\Delta E_{D,c} + 4\Delta E_{D,d} \quad (44)
\]

with \( l_1 \) and \( l_2 \) halved in both \( \Delta E_{D,c} \) and \( \Delta E_{D,d} \).

In the case of periodic conditions terms dependent on only one scaling parameter or none of them have vanished completely. Again, the only unchanged term is the one corresponding to bulk material properties in the continuum approximation.

### 3.4. Other

As another example let us consider

\[
\phi(0) = 0 \quad (45)
\]

\[
\frac{\partial \phi}{\partial x}(l) = 0 \quad (46)
\]

as a set of boundary conditions. It will give us a \( \gamma \) function of the form

\[
\gamma = \sum_{n=0}^{\infty} e^{i \frac{n^2 (2n+1)^2}{2\hbar}} \quad (47)
\]

which can be expressed as

\[
\gamma = \frac{1}{2} \vartheta \left( 0; e^{i \pi \tau_A} \right). \quad (48)
\]

In order to use the same identity as before, we need to express \( \vartheta_2 \) as \( \vartheta_3 \) using another identity from [19]

\[
\vartheta_3(z; q) = \vartheta_3(2z; q^4) + \vartheta_2(2z; q^2). \quad (49)
\]

Thus

\[
\gamma = \frac{1}{2} \sqrt{\frac{i l_1^2}{\pi \tau_A}} \left( 1 + 4 \sum_{n=1}^{\infty} e^{i \frac{n l_1}{l_2}} - 2 \sum_{n=1}^{\infty} e^{i \frac{n l_1}{l_2}} \right). \quad (50)
\]

In the end we will obtain
\[ \Delta E_D = \frac{5\hbar m c l_1 l_2}{72\pi^2} + \Delta E_{D,c}(2l_1, 2l_2) - \Delta E_{D,c}(l_1, l_2) + \Delta E_{D,d}(2l_1, 2l_2) + \Delta E_{D,d}(l_1, l_2) - \Delta E_{D,d}(2l_1, l_2) - \Delta E_{D,d}(l_1, 2l_2). \]  

(51)

3.5. Mixed conditions

Apart from taking the same type of boundary conditions on both \(x_1\) and \(x_2\), we can also combine them in any arbitrary way. The results will also show a mix of previously obtained corrections, yet there are some details worth showing explicitly. For example, if we were to take Dirichlet boundary conditions on \(x_1\) and von Neumann conditions on \(x_2\), we obtain

\[ \Delta E = \frac{\hbar m c}{4\pi} + \frac{5\hbar m c l_1 l_2}{72\pi^2} + \Delta E_{D,c} + \Delta E_{D,d}. \]  

(52)

As can be seen, all the terms dependent explicitly on only one scale parameter have vanished and the constant part of the corrections has changed its sign.

In the case of Dirichlet boundary conditions on \(x_1\) and periodic conditions on \(x_2\) we will obtain

\[ \Delta E = -\frac{\hbar m c l_2}{8\pi} + \frac{5\hbar m c l_1 l_2}{72\pi^2} + 2\Delta E_{D,c} + 2\Delta E_{D,d} \]  

(53)

with \(l_2\) halved in \(\Delta E_{D,c}\) and \(\Delta E_{D,d}\).

If we were to compare the results with those presented earlier in the paper, we can see that the change of boundary conditions in one dimension directly affects the scaling in the other. Moreover, those differences do not occur at the classical level at all as long as the shape of the solution is the same.

4. Conclusions

Taking into account the discrete spectra of eigenvalues in finite domains shows us that the semiclassical energy corrections are both qualitatively and quantitatively dependent on the type of boundary condition. As a result, the scaling of energy corrections in the size of the domain is significantly affected. From calculated cases it seems that the only independent term is the one corresponding to bulk material, which is natural considering that all possible boundary conditions have the same continuum limit of eigenvalue spectra, which gives us exactly the bulk material term. The other terms vary greatly.

Another interesting result is the oscillatory behaviour of energy corrections, which is most pronounced for small domain sizes. The key question is whether such oscillating terms are realistic or are just artefacts arising from the simplification of reality. More precisely, any type of boundary condition is a gross simplification of the interaction between the modelled sample and the surroundings. Since the type of boundary condition affects the results so much, one can expect that the approximation that the boundary conditions are themselves, changes the results in a significant way.

It is also evident that spatially confined systems never behave as one- or two-dimensional systems within the context of semiclassical quantization. While energy terms relating to such approximations appear in the general solution, they are changed by a significant factor and their presence and sign depend on the choice of boundary conditions. This type of behaviour
cannot be predicted from one- or two-dimensional simplifications of the classical system. Moreover, the aforementioned terms are overshadowed by other components characteristic of finite domain solutions for small domain sizes, while for large domains the bulk term becomes naturally dominant. For this reason, even if some dimensions of a system can be disregarded on classical level they need to be included in a quantum model.

Current findings indicate two important directions for further research. On one hand, experimental measurements of soliton energy in thin layers and wires (especially for low-energy systems, since quantum corrections are independent of the energy scale of classical systems [20]; note also that the $A$ parameter of (4) does not enter the corrections). On the other, refinement of mathematical methods and the theoretical framework for classical field theory is needed in order to describe the conditions at the edge of a modelled object more realistically.

References

[1] Feynman R P 1948 Space–time approach to non-relativistic quantum mechanics Rev. Mod. Phys. 20 367–87
[2] Feynman R P 1949 Space–time approach to quantum electrodynamics Phys. Rev. 76 749–60
[3] Wentzel G 1926 Eine Verallgemeinerung der quantenbedingungen für die zwecke der wellenmechanik Z. Phys. 38 518–29
[4] Kramers H A 1926 Wellenmechanik und halbjzahlige quantisierung Z. Phys. 39 828–40
[5] Brillouin M L 1926 La mécanique ondulatoire de Schrödinger: une méthode générale de résolution par approximations successives C. R. Acad. Sci. 183 24–6
[6] Maslov V P 1961 Stationary-phase method for Feynman’s continual integral Zh. Vychislitelnoj Mat. Mat. Fiz. 1 638
[7] Maslov V P 1970 On the stationary phase method for Feynman’s continual integral Teor. Mat. Fiz. 2 30–5
[8] Fock V A 1931 Fundamentals of Quantum Mechanics (Leningrad: The State Optical Institute)
[9] Pauli W 1951 Ausgewählte Kapitel Aus der Feldquantisierung (Zurich: Verlag des Vereins der Mathematiker und Physiker)
[10] Dashen R F, Hasslacher B and Neveu A 1974 Nonperturbative methods and extended-hadron models in field theory II two-dimensional models and extended hadrons Phys. Rev. D 10 4130–8
[11] Gutzwiller M C 1967 Phase-integral approximation in momentum space and the bound states of an atom J. Math. Phys. 8 1979–2000
[12] Gutzwiller M C 1969 Phase-integral approximation in momentum space and the bound states of an atom II J. Math. Phys. 10 1004–20
[13] Korepin V E and Faddeev L D 1975 Quantisation of solitons Teor. Mat. Fiz. 25 1039–49
[14] Dowker J S and Critchley R 1976 Effective Lagrangian and energy-momentum tensor in de Sitter space Phys. Rev. D 13 3224–32
[15] Konoplich R V 1987 Calculation of quantum corrections to nontrivial classical solutions by means of the zeta-function Teor. Mat. Fiz. 73 379–92
[16] Pawelek M 2009 Quantisation of sine-Gordon solitons on the circle: semiclassical versus exact results Nucl. Phys. B 810 527–41
[17] Kirsten K and Loya P 2008 Computation of determinants using contour integrals Am. J. Phys. 76 60–5
[18] Kwiatkowski G and Leble S 2014 Quantum corrections to quasi-periodic solution of sine-Gordon model and periodic solution of $\phi^4$ model J. Phys.: Conf. Ser. 482 012023
[19] Whittaker E T and Watson G N 1990 A Course in Modern Analysis 4th edn (Cambridge: Cambridge University Press)
[20] Kwiatkowski G 2014 Quasiclassical corrections in functional integral method and applications Doctoral Thesis Gdansk
[21] Ventura I 1981 Macroscopic solitons in thermodynamics Phys. Rev. B 24 2812–6
[22] Kwiatkowski G and Leble S 2013 Green function diagonal for a class of heat equations Appl. Math. Comput. 219 6084–92