Open Quantum Random Walks: Reducibility, Period, Ergodic Properties

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Abstract. We study the analogues of irreducibility, period, and communicating classes for open quantum random walks, as defined in (J Stat Phys 147(4):832–852, 2012). We recover results similar to the standard ones for Markov chains, in terms of ergodic behaviour, decomposition into irreducible subsystems, and characterization of invariant states.

1. Introduction

Open quantum random walks were recently defined by Attal et al. [3]. These processes have a simple definition, implementing a Markovian dynamics influenced by internal degrees of freedom, and can be useful to model a variety of phenomena: quantum algorithms (see [23]), transfer in biological systems (see [17]) and possibly quantum exclusion processes. In addition, a continuous-time version can be defined (see [19]). Therefore, open quantum random walks seem to be good quantum analogues of Markov chains.

The usefulness of (classical) Markov chains, however, comes not only from the vast number of situations they can model, but also from the many properties implied by their simple definition. A textbook description of Markov chains, for instance, can start with the notion of irreducibility, which is easily characterized through the connectedness of the associated graph, and implies mean-ergodic convergence in law if an invariant probability exists (which is the case when the state space is finite). The next notion, the aperiodicity of an irreducible chain, is not as easy to characterize, but has simple sufficient conditions (e.g. the existence of loops) and implies convergence in law, at least when the state space is finite. Last, the notion of connected subsets of the initial graph allows one to decompose a Markov chain into irreducible ones and to characterize its invariant states as convex combinations of invariant states for restricted chains.

On the other hand, the only general properties of open quantum random walks proven so far are the central limit theorem for the position process
(see [2]) and the general Kümmerer–Maassen theorem for quantum trajectories (see [16]). In the present paper, we discuss an analogue of the above textbook description of Markov chains, for open quantum random walks. The non-commutative nature of the objects under study, and specifically the fact that the transition probabilities are replaced by operators acting on a Hilbert space, are the cause of higher mathematical complexity. Some intuitive aspects of classical Markov chains, however, remain true, and we can recover a vision of irreducibility, period, and accessibility, in terms of paths. This is of interest for the study of more general quantum Markov processes, as it gives indications on the relevant extensions of classical concepts, and on techniques of proofs for associated results. We view this as an additional justification for the study of open quantum random walks.

Our theory will be constructed starting from pre-existing tools:

- a notion of irreducibility for general positive maps on non-commutative algebras, together with an associated Perron–Frobenius theorem, that was developed by various authors in the late seventies and early eighties [1,7,8, 14];
- a notion of period, together with associated results on the peripheral spectrum, which were defined in the same setting by Groh [14] and extended by Fagnola and Pellier [9];
- some old and new inspiring ergodic results [12,16] and a decomposition of the support of invariant states proposed more recently by Baumgartner and Narnhofer [4] for quantum discrete time processes acting on finite-dimensional spaces.

We briefly describe the structure of the article and the main contents. Section 2 recaps the definitions, notations and basic results regarding open quantum random walks from [3]. We also introduce the classical processes associated with an OQRW. Sections 3 and 4 discuss, respectively, irreducibility and aperiodicity for OQRWs. Both follow the same structure: they start by recalling standard definitions and properties of irreducibility or aperiodicity for positive maps on operator algebras, then study the application to the special case of OQRWs. Some immediate consequences on the ergodic behaviour of the evolution are underlined. Section 5 applies the results of the previous two sections to obtain, for irreducible, or irreducible aperiodic, open quantum random walks, convergence properties of the processes described in Sect. 2. Section 6 expands on reducible open quantum random walks, characterizing in different ways their irreducible components. The resulting decomposition can be seen as related to a “quantum communication relation” among vectors of the underlying Hilbert space. Section 7 states the general form of invariant states for reducible open quantum random walks. Its central point is an extension of some results from [4]. Section 8 mentions a natural extension of open quantum random walks. For this extension, we discuss without proof a characterization of irreducibility, periodicity, communication classes, and their consequences: as we will see, all previous results will remain with paths on a graph replaced by paths on a multigraph. We conclude with Sect. 9, which is dedicated to examples and