Wavelets in Momentum-Space Scattering Calculations

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We demonstrate that wavelet bases have features that make them advantageous for solving momentum-space scattering integral equations. Using the example of two nucleons interacting with the Malfliet-Tjon V interaction, we show it is possible to reduce the size of the matrix representation of the Lippmann-Schwinger equation by 96% with a loss of accuracy of only a few parts in a million. We also demonstrate that wavelet methods provide an accurate means for treating the singularities that appear in the scattering integral equations.

We show that the use of Daubechies’ wavelet bases for solving momentum-space scattering integral equations leads to sparse matrices which simplify the numerical solution. The method is tested on nucleon-nucleon scattering with a Malfliet-Tjon V potential. The results of the test calculations indicate that a significant reduction in computational size can be achieved for realistic few-body scattering calculations.

The motivation for this work is that accurate relativistic few-body calculations are important for the few-body program at TJNAF. These calculations are naturally formulated in momentum-space because of the need to include momentum-dependent Wigner and/or Melosh rotations, and the nonlinear relation between mass and energy in relativistic theories. The computational disadvantage of momentum-space scattering integral equations is that the kernels are represented by large dense matrices and the computation of the matrix elements involves singular integrals. Our work indicates that if the solutions of the integral equations are represented as expansions in a wavelet basis then the resulting kernel can be expressed as the sum of a sparse matrix and a matrix with small norm. Setting the small matrix to zero leads to an accurate approximation. The density of the non-zero matrix elements of the sparse matrix representation of the s-wave Malfliet-Tjon V K-matrix kernel is illustrated in the two-dimensional plot in fig 1.

The Daubechies wavelets have many features of a spline basis; the basis functions have compact support and finite linear combinations of the basis functions can (pointwise) locally represent polynomials. Unlike the spline basis, the wavelet basis functions are orthonormal, and there is an automatic method for determining the most important basis elements for a given calculation. The wavelet basis functions also have natural quadrature rules that lead to efficient methods for calculating kernel matrix elements. There is a
robust wavelet-based method to accurately and efficiently compute the singular integrals that arise in momentum space-scattering calculations.

While wavelets are used extensively in signal processing, they have not been used as much in numerical analysis. Part of the reason for this is that the wavelet basis is related by a simple orthogonal transformation to the solution of a linear renormalization group equation. The solution of this equation, which is called the scaling function, has a fractal structure, having structure on all scales. This fractal structure also appears in the wavelet basis functions and limits the applicability of numerical methods which assume that functions are smooth on a sufficiently small scale. The structure of the Daubechies wavelet and scaling function of order three are shown in fig 2. Nevertheless, we show that the renormalization group equation provides a robust tool for developing alternative methods of numerical computation which overcome all of the difficulties that result from the fractal structure [4].

Our calculation of the $K$-matrix for a Malfliet-Tjon interaction has the property that if the smallest 96% of the matrix elements in the kernel of the numerical representation of the Lippmann-Schwinger equation are set to zero, the mean square error is only a few parts in a million. This allows for the replacement of the kernel matrix by a sparse matrix. We have also developed methods to limit the amount of storage necessary to transform the original equation to this sparse-matrix representation. These preliminary results strongly suggest that wavelet methods have many advantages for the numerical treatment of momentum space scattering equations.

REFERENCES

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Figure 1. Non-zero matrix elements of $s$-wave kernel of the $K$-matrix equation for the Malfliet Tjon V potential in the Daubechies wavelet basis of order three.

Figure 2. Plot of the Daubechies order-three wavelet and scaling function.