On tame embeddings of solenoids into 3-space

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Abstract

Solenoids are \"inverse limits\" of the circle, and the classical knot theory is the theory of tame embeddings of the circle into the 3-space. We give some general study, including certain classification results, of tame embeddings of solenoids into the 3-space as the \"inverse limits\" of the tame embeddings of the circle.

Some applications are discussed. In particular, there are \"tame\" embedded solenoids $\mathbb{R}^3$ which are strictly achiral. Since solenoids are non-planar, this contrasts sharply with the known fact that if there is a strictly achiral embedding $Y \to \mathbb{R}^3$ of a compact polyhedron $Y$, then $Y$ must be planar.

1 Introduction and motivations

The classical knot theory is the theory of tame embeddings of the circle into the 3-space, which has become a central topic in mathematics. The classical theory of knots has many generalizations and variations: from the circle to graphs, from the circle to higher dimensional spheres, from tame embeddings to wild embeddings, and so on.
In the present note, we try to setup a beginning of another generalization: the tame embeddings of solenoids into the 3-space. In such a study topology and dynamics interact well.

The solenoids may be first defined in topology by Vietoris in 1927 for 2-adic case \([V]\) and by many others later for general cases, and introduced into dynamics by Smale in 1967 \([S]\). Solenoids can be presented either in a rather geometric way (intersections of nested solid tori, see Definition 2.2) or in a rather algebraic way (inverse limits of self-coverings of the circle, see Definition 2.3), or in rather dynamics way (in mapping torus over the Cantor set, see [M c]).

The precise definition of tame embedding of solenoids into the 3-space \(R^3\) will be given in \(x_2\), but the intuition is quite naive. Recall we identify \(S^1\) with the centerline of the solid torus \(S^1 \times D^2\), and say an embedding \(S^1 \hookrightarrow R^3\) is tame if the embedding can be extended to an embedding \(S^1 \times D^2 \hookrightarrow R^3\). Similarly we consider a solenoid as the nested intersections of solid tori (the defining sequence of ), and say an embedding \(R^3\) is tame if the embedding can be extended to an embedding of those solid tori into \(R^3\).

Solenoids themselves are considered as "wild" set. What motivated us originally to study the tame embeddings of solenoids was trying to find a non-planar set which admits a strictly achiral embedding into the 3-space.

An embedding \(A \hookrightarrow R^3\) is called strictly achiral, if \(A\) stays in the fixed point set of an orientation reversing homeomorphism \(r:R^3 \rightarrow R^3\). Obviously any planar set has a strictly achiral embedding. Indeed there is a simple relation between the two notions of achirality and planarity for compact polyhedra: If there is a strictly achiral embedding \(Y \hookrightarrow R^3\) of a compact polyhedron \(Y\), then \(Y\) must be planar \([JW]\). In the sequel, it is natural to ask whether the relation from \([JW]\) still holds when compact polyhedra are replaced by continua (i.e. compact, connected metric spaces).

The solenoids are promising, and indeed are proved to be in \(x_4\), counterexamples to the question, because on the one hand they are continua realized as inverse limits of planar sets, on the other hand they are non-planar themselves (see [B in], and [JW Z] for a short proof). In order to design a strictly achiral embedding \(R^3\) for a solenoid , we need careful and deep discussions about the tame embeddings of solenoids.

Among other motivations, when a solenoid \(S^3\) is realized as a hyperbolic attractor of a dynamics (called Smale solenoid), the embedding \(S^3\) is automatically tame. Our study also gives some application on this aspect.

The contents of the paper are as follows.

In \(x_2\), we give the precise definitions of tame solenoids and related notions. We present a lemma about convergence of homeomorphism which will be repeatedly used in the paper. With this lemma we give an alternative description of the tame solenoids via the language of mapping torus. For comparison we also construct examples of non-
2 Tam e embeddings of solenoids, Preliminaries

2.1 Definitions of solenoids and their tame embeddings

Let \( N = D^2 \cdot S^1 \) be the solid torus, where \( D^2 \) is the unit disc and \( S^1 \) is the unit circle. Then \( N \) admits a standard metric. A meridian disk of \( N \) is a \( D^2 \) slice of \( N \). A framing of \( N \) is a circle on \( \partial N \) which meets each meridian disk of \( N \) at exactly one point.

Definition 2.1. (1) For a sequence of maps \( f_n : X_n \rightarrow X_{n-1} \) between continua, the inverse limit is defined to be the subspace

\[
= \lim_{n \to \infty} f_n \quad (x_0; x_1; \ldots; x_n; \ldots)
\]

of the product space \( \prod_{n=0}^{\infty} X_n \).

(2) The inverse limit of a sequence of covering maps \( f_n : S^1 \rightarrow S^1 \) is called a solenoid of type \( S = (w_1; w_2; \ldots; w_n; \ldots) \).

Definition 2.2. (1) Call an embedding \( e : N \rightarrow \text{int} N \), or simply call the image \( e(N) \), a thick braid of winding number \( w \) if \( e \) preserves the \( D^2 \)–beration and descends to a covering map \( S^1 \rightarrow S^1 \) given by \( e^{it} \rightarrow e^{w t} \). Note that the composition of infinitely many thick braids is also a thick braid.

(2) Let \( f_{e_n} : N \rightarrow N_{e_n} \) be an inverse sequence of thick braids of winding numbers \( w_n \neq 0 \). Let \( n = e_n \), \( \mathbb{N} = n \mathbb{N} \), and \( N_n = n \mathbb{N} \). Then we have an inverse sequence \( N = N_0 \cdot N_1 \cdot N_2 \cdot \ldots \) of thick braids. If the diameters of the meridian disks of \( N_n \) tend to zero uniformly as \( n \rightarrow \infty \) then we call \( = \mathbb{N} \) a solenoid of type \( S = (w_1; w_2; \ldots; w_n; \ldots) \).
There is quite a rich theory about solenoids developed in 1960-1990's. We just list some basic facts (see [McL], [R] and references therein) as

Theorem 2.3. (1) The above two definitions of solenoids are equivalent; each solenoid is determined by its type $S$.

(2) Two solenoids of types $S$ and $S'$ respectively are homomorphic if deleting finitely many terms from $S$ and $S'$ can make them identical. Moreover, is the circle if and only if all except finitely many $w_n$ are equal to 1.

(3) Each solenoid is connected, compact and has topological dimension one. Moreover, if is not the circle, then has uncountably many path components.

We assume below that all winding numbers involved in the definition of solenoid are greater than 1, unless otherwise specified. In particular, a solenoid is not the circle.

Definition 2.4. (1) In Definition 2.2, call $f_{n,g_n}$ a defining sequence of the solenoid, and call $N$ a standard embedding of in the solid torus $N$.

(2) $N$ is called a tame embedding of in the solid torus $N$, if there is a homeomorphism $f : (N;N) \to (N;N)$ for some standard embedding $N$; then call $f \equiv (N)g_n$ a defining sequence of, where $f^{0}g_{n+1}$ is a defining sequence of 0.

(3) An embedding $S^3$ of solenoids is called tame, if the embedding can be factored as $N N S^3$, in which $N$ is tame; then each defining sequence $f_{n,g_n}$ of $N$ is also considered as a defining sequence of $S^3$, and we have $S^3 = N N_1 N_2 \ldots n N$.

Remark 2.5. From Definition 2.4, for each defining sequence $f_{n,g_n}$ of a tame solenoids in $S^3$, we always assume that the $D^2$-slices of all $N_i$ are coherent.

Definition 2.6. Call two tame solenoids \ $0$ $S^3$ equivalent if there is an orientation preserving homeomorphism $f : S^3 ! S^3$ such that $f( ) = 0$.

Definition 2.7. Say two defining sequences $f_{n,g_n}$ and $f_{n',g_{n'}}$ of tame solenoids in $S^3$ are strongly equivalent, if there are orientation preserving homeomorphisms $f_0 : (S^3';N_0) ! (S^3;N_0)$ and orientation preserving homeomorphism $s \equiv (N_n,N_n)$ ! $(N_n,N_n)$ with $f_0 \equiv r_{N_n} = f_n \equiv r_{N_n}$ for $n \equiv 1$. Say $f_{n,g_n}$ and $f_{n',g_{n'}}$ are equivalent, if $f_{n_k+n,g_n}$ and $f_{n_k+g_{n,k'},g_{n,k'}}$ are strongly equivalent for some non-negative integers $k$ and $k'$.

Remark 2.8. (1) A defining sequence $f_{n,g_n}$ of a tame solenoid $S^3$ carries the information of the braiding of $N_n$ in $N_{n-1}$ and the knotting of $N_n$ in $S^3$. The winding numbers $w_n$, the simplest invariant of the braiding of $N_n$ in $N_{n-1}$, give rise to the type of the abstract solenoid.
(2) Suppose \( S^3 \) is a tame embedding given by a defining sequence \( f_n, g_n \), then any infinite subsequence of \( f_n, g_n \) is a defining sequence of the same embedding.

(3) Definitions 2.4-2.7 also apply to other 3-manifolds in the obvious way.

In this paper we view each braid is defined in \( D^2 \times [0;1] \) with ends stay in \( D^2 \) \( f0;1g \). The closure \( D^2 \times S^1 \) is obtained by identifying \( D^2 \times 0 \) and \( D^2 \times 1 \) via the identity. Conversely, for each closed braid \( D^2 \times S^1 \), cutting \( D^2 \times S^1 \) open along a \( D^2 \) slice yields a braid \( D^2 \times [0;1] \) up to conjugacy. Therefore, we have the 1-1 correspondence between the set of closed braids of winding number \( n \) and the set of conjugacy classes of braids on \( n \) strands.

Note that a tubular neighborhood of a closed braid in \( D^2 \times S^1 \) is a thick braid and, conversely, a framing of a thick braid in \( D^2 \times S^1 \) is a closed braid.

2.2 Convergence of homeomorphisms

The following lemma will be used repeatedly in this paper.

Lemma 2.9. Let \( X \rightarrow Y \) be compact metric spaces and \( \{f_n : X \rightarrow Y \} \) be a series of homeomorphisms. If there exist subsets \( \{U_n \} \) of \( X \) and positive numbers \( m \) such that

1. \( U_n \neq \emptyset \)
2. \( f_n \) is uniformly continuous on \( U_n \)
3. \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( x \in X \)

where \( f \) is a homeomorphism, then \( f_n \) converges uniformly to \( f \).

Proof. Since both \( X \) and \( Y \) are compact, it follows from Condition 2 that \( f_n \) and \( f_n^{-1} \) converge uniformly to the homeomorphism \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \), respectively. By Condition 1, we have \( f_j \) is uniformly continuous on \( U_n \) for \( j \neq n \).

2.3 Tame solenoids as mapping tori

Definition 2.10. Let \( C \subset D^2 \) be a Cantor set. Say an orientation preserving homeomorphism \( f : (D^2 \setminus C) \) factors through nested disks if there exist subsets...
fU_0g_0 of D^2 such that U_{n+1} \cap \text{int} U_n \setminus \text{int} U_n = C and each U_n is a disjoint union of 2-disks on which f cyclically permutes the components of U_n.

Theorem 2.11. A solenoid S^1 is tame if and only if it has a neighborhood N such that (N; ) is homeomorphic to the mapping torus of a homeomorphism f : (D^2; C) ! (D^2; C) which factors through nested disks.

The proof depends on the following lemma, which will be also used in several other places of the paper.

Proof of Theorem 2.11. Sufficiency is clear: Suppose f : (D^2; C) ! (D^2; C) is an orientation preserving homeomorphism which factors through nested disks fU_0g_0. Then for each n \geq 0, let N_n = U_n \setminus \{1\} = f. Then N_{n+1} is a thickening in N_n and \text{int} N_n = C \setminus \{1\} = f. N_0 is a tame embedding of solenoid with defining sequence fN_0g_0.

Necessity. Let N be a tame embedding with defining sequence fN_0g_0. We will show that (N; ) is homeomorphic to the mapping torus of a homeomorphism f : (D^2; C) ! (D^2; C) which factors through nested disks. Fix a meridian disk D of N. For n \geq 0, let U_n = D \setminus N_n. Clearly U_n are disjoint disks, U_{n+1} \text{int} U_n, and C = \bigcap_{n=0}^{\infty} U_n is a Cantor set in D^2.

Now recursively define a homeomorphism f : (D; U_n) ! (D; U_n) such that f_n : f_{n-1} \setminus \{1\} = f_{n-1} \setminus \{1\} and T_{f_n} = (N_0; N_n), where T_{f_n} is the mapping torus of f_n. Since N_n is connected, f_n permutes the components of U_n cyclically. Then recursively define D^2-beration preserving homeomorphism f_n : T_{f_n} ! (N_0; N_n) such that f_n is identical to f_{n-1} on the mapping torus of D \cup U_{n-1}. Since the diameters of the components of U_n and the meridian disks of N_n tend to zero uniformly as n \to \infty and f_n is D^2-beration preserving, by Lemma 2.3 f_n uniformely converges to a homeomorphism f : (D; D \setminus ) ! (D; D \setminus ) and f_n uniformely converges to a homeomorphism f : T_f ! (N_0; ). It is clear that f factors through fU_0g_0.

2.4 Non-tame embeddings of solenoids

As there are non-tame embeddings of the circle into the 3-space, there are non-tame embeddings of solenoids. In fact, any tame embedding of solenoid can be modified in a simple way to a non-tame embedding, which we illustrate below by a concrete example.

Suppose nested disks fU_0g_0 and an orientation preserving homeomorphism f : (D^2; C) ! (D^2; C) which factors through fU_0g_0 have been chosen as in previous subsection so that each U_n consists of 2^n disks. Label those disks by U_i = D_0 \setminus U_0, \ldots, U_{n-1} = D_0 \setminus U_{n-1} with U_{i_1} \ldots U_{i_n} where i_1 \leq \ldots \leq i_n and so on. For simplicity, denote by D_i^0, D_i^0 the components D_{i_1 \ldots i_n} \cup U_n respectively.
3 Classification and applications

3.1 Maximal defining sequences

As can be easily seen, a tame solenoid $N$ has a lot of defining sequences and there is no way to choose a minimial one from them. However, a maximal defining sequence of a tame solenoid can be defined in the sense that every other defining sequence is equivalent to a subsequence of it (see Proposition 3.2).

Before giving the precise definition we x a notation. For a properly embedded surface $S$ (resp. an embedded 3-manifold $P$) in a 3-manifold $M$, we use $M \setminus S$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting $M$ along $S$ (resp. removing the interior of $P$).

Definition 3.1. Call a defining sequence $f_{N_n, g_n, 0}$ of a tame solenoid $S^3$ maximal if $N_n \cap N_{n+1}$ contains no essential torus for each $n \geq 0$. 

Proposition 3.2. Every tame solenoid $S^3$ has a maximal dehnining sequence. Moreover, if $fN_n g_n \omega$ is a maximal dehnining sequence of $N_n$, then every defining sequence of $N_n$ is equivalent to a subsequence of $fN_n g_n \omega$. Hence all maximal dehnining sequences of $N_n$ are equivalent.

The following lemma will be repeatedly used in this paper.

Lemma 3.3. Suppose $N^0$ is a thick braid in $N$ and $N$ is the JSJ-decomposition tori of $N_n N^0$. Then

1. each component $T$ of $N^0$ bounds a solid torus $N$ such that $N$ is a thick braid in $N$ and $N^0$ is a thick braid in $N$;
2. each component of $(N_n N^0)_n$ contains no essential torus;
3. for each solid torus $N^0$ with $N^0 \cap N^0 = \emptyset$, $N^0$ is isotopic in $N_n N^0$ to a component of $[\partial N \cup \partial N^0]$.

Proof. (1) Let $w$ be the winding number of $N^0$ in $N$ and let $D$ be a meridian disk of $N$ which meets $N^0$ at $w$ meridian disks of $N^0$. Then $(N_n N^0)_n D = P_w I$, where $P_w$ is the $w$-punctured disc. Isotope $T$ so that $T \setminus D$ has minimum number of components. Then a standard argument in 3-manifold topology shows that each component of $T \cap D$ in $P_w I$ is a vertical annulus which separates a vertical $D^2$I from $N_n D$. Therefore $T$ bounds a solid torus $N$ which is a thick braid in $N$ and clearly $N^0$ is a thick braid in $N$.

(2) Let $Q$ be a component of $(N_n N^0)_n$. By JSJ theory [Ja], $Q$ is either simple hence contains no essential torus by definition, or a Seifert piece. Suppose $Q$ is a Seifert piece. Then $Q$ is also a Seifert piece of knot components with incompressible boundary. According to [Ja, IX 22, Lemma 4.1], $Q$ is either a torus knot space, or a $P_w S^1$ where $P_w$ is the $w$-punctured disc with $w \geq 2$, or a cable space. Since $\partial Q$ has at least two components, $Q$ is not the torus knot space. By the conclusion of (1) and the fact that $N^0$ is connected, one can verify that there is no embedding of $P_w S^1$ in $N_n N^0$ with incompressible boundary for $w \geq 2$. Therefore $Q$ is a cable space. It is known that a cable space contains no essential torus.

(3) By the conclusions of (2), it suffices to show that $\partial N^0$ is incompressible in $N_n N^0$. Suppose $\partial N^0$ has a compressing disk $D$ in $N_n N^0$. Then after surgery on $D$ we will get a separating 2-sphere $S^2$ in $N_n N^0$ such that each component of $(N_n N^0)_n S^2$ contains a boundary torus, which contradicts that $N_n N^0$ is irreducible. □

Proof of Proposition 3.2. Let $fN_n g_n \omega$ be a defining sequence of $S^3$. Then we have $S^3 = N_0 N_1 \cdots N_n$ and

$$N_0 n N_1 N_0 n N_2 \cdots n N_n = \{n-1\} N_n \cup \{n\} N_n$$

Moreover, since the winding number of $N_{n+1}$ in $N_n$ is greater than 1 for every $n$, $\partial N_n$ is not parallel to $\partial N_j$ for $n \neq j$. 

8
Now we rene the de ning sequence $fn_n g_{n_0}$ to a maximal one. Let $n$ be the JSJ-decom position tori of $N_n N_{n+1}$. Note that the set $\sum_{0 \leq j < n=m} \mathcal{N}_n$ is the JSJ-decom position tori of $N_{0} N_{n}$. Define (1) $n = 0 \in \mathcal{N}_n$. By Lemma 2.9 (1) we can re-index the components of (1) as $f_{i} g_{n_i}$ so that

1. each $T_n$ bounds a solid torus $N_n$,
2. each $N_{n+1}$ is a thick braid in $N_n$,
3. each $N_{n} N_{n+1}$ contains no essential torus.

Clearly $f_{n} g_{n_0}$ is a subsequence of $f_{n} g_{n_0}$ and by definition $f_{n} g_{n_0}$ is a maximal de ning sequence of $\mathcal{S}$.

Let $f_{n} g_{n_0}$ be another maximal de ning sequence of $\mathcal{S}$. We shall show that $f_{n} g_{n_0}$ and $f_{n} g_{n_0}$ are equivalent. Since $\mathcal{S}$ is compact, we have $N_{j} N_{k} N_{0}$ for some big integers $j, k$. By Lemma 2.9 (3) and the construction of $f_{n} g_{n_0}$, $\mathcal{S}$ is isotopic in $N_{0} N_{j}$ to some $\mathcal{S}$. Clearly, the isotopy automatically sends $N_{0}$ to $N_{m}$.

Similarly, one argues that $N_{k+1}$ can be further isotoped in $N_{m}$ relative to some $N_{n}$, in which $m_{0}$ must be $m + 1$ because $N_{k} N_{m} N_{k+1}$ contains no essential torus, and so on. Hence we verify that $f_{n} g_{n_0}$ is strongly equivalent to $f_{n} g_{n_0}$ and the conclusion follows.

### 3.2 Classification of tame solenoids

**Theorem 3.4.** Let $S^3$ be two tame solenoids. The following statements are equivalent.

1. $S^3$ are equivalent.
2. Some de ning sequences of $S^3$ are equivalent.
3. The maximal de ning sequences of $S^3$ are equivalent.

**Proof.** (2) $(\Rightarrow)$ (1). Without loss of generality, suppose the de ning sequences $f_{n} g_{n_0}$, $f_{n} g_{n_0}$ of $S^3$ are strongly equivalent. By definition there are orientation preserving homeomorphisms $f_0: (S^3; N_0)$ and $f_0: (S^3; N_{0})$, $f_n: (N_{n+1}; N_n)$ with $f_{n+1} N_{n+1} = f_n N_n$ for each $n = 1$. By Remark 2.5, we assume the $D^2$-slices of all $N_1$, (respectively of all $N_i$), are coherent. Then it is easy to see that we can first isotopy $f_0: (S^3; N_0)$ to $f_0: (S^3; N_{0})$ so that $f_0 j: N_0 = N_0$ is $D^2$-beration preserving, then inductively to isotopy $f_i: (N_{n+1}; N_n)$ to $f_i: (N_{n+1}; N_n)$ for each $n = 1$ so that $f_n j: N_n = N_n$ is $D^2$-beration preserving, and still $f_{n} j N_{n+1} = f_n j N_{n+1}$.

To apply Lemma 2.9, we set $U_n = N_n$ and extend $f_n$ onto $S^3$ by setting $f_n j N_{n+1} = f_{n+1} j N_{n+1}$. Clearly Conditions (1) and (3) of Lemma 2.9 are satisfied. Since the diameters of the meridian disks of $N_n$ and $N_{0}$ tend to zero uniformly as $n \rightarrow 1$ and since $f_n$ is $D^2$-beration preserving and $f_{n} j N_{n+1} = f_{n} j N_{n+1}$, Condition (2) of Lemma 2.9 is also satisfied. Therefore, by Lemma 2.9 $f_n$ uniformly converges to
a homomorphism \( f : (S^3; \emptyset) \to (S^3; \emptyset) \). That is, \( = \emptyset \) and \( 0 = \emptyset \) are equivalent.

(3) (2) is obvious.

(1) (3). Let \( f : S^3 \to S^3 \) be an orientation preserving homomorphism such that \( f(0) = 0 \). Clearly for each maximal defining sequence \( f_n g_n 0 \) of \( f (N) g_n 0 \) is a maximal defining sequence of \( 0 \) and is equivalent to \( f_n g_n 0 \). By Proposition 3.3, \( f (N) g_n 0 \), hence \( f_n g_n 0 \), is equivalent to every maximal defining sequence of \( 0 \).

3.3 Knotting, linking and invariants

Thanks to the classification theorem, we can talk about the knotting, linking and invariants of tame solenoids.

Definition 3.5. Call a tame embedding of solenoid \( S^3 \) with defining sequence \( f_n g_n 0 \) is knotted, if some defining solid torus \( N_n \), \( S^3 \) is knotted; otherwise call it unknotted.

Note that for a defining sequence \( f_n g_n 0 \) of a tame solenoid, if \( N_n \) is knotted then so is \( N_n 0 \) for \( n > 1 \). It follows from Theorem 3.4 that the notion of knotting is well defined for an equivalent class of tame solenoids.

Definition 3.6. Let \( ; 0 \) \( S^3 \) be disjoint tame solenoids with disjoint defining sequences \( f_n g_n 0 \) and \( f_n 0 g_j 0 \) respectively.

(1) Call \( ; 0 \) algebraically linked if some linking number \( \text{lk} (N_n ; N_j 0) \) (i.e. the linking number of their centerlines) is non-zero.

(2) Call \( ; 0 \) linked if some defining solid tori \( N_n ; N_j 0 \) are linked.

Since two disjoint tame solenoids \( ; 0 \) \( S^3 \) always have disjoint defining sequences and since \( \text{lk} (N_n ; N_j 0) \neq 0 \) implies \( \text{lk} (N_n ; N_j 0) \neq 0 \) for all \( n \neq j \), by Theorem 3.4 again the notion of algebraic linking is well defined for equivalent classes of tame solenoids.

Similarly the notion of linking is well defined, too. In particular, \( ; 0 \) are linked if and only if there are no disjoint 3-balls \( B \) and \( B 0 \) such that \( B ; 0 \) \( B 0 \).

To define invariants of tame solenoids, the proposition below will be of help.

Proposition 3.7. Up to strong equivalence, each knotted tame solenoid \( S^3 \) has a unique maximal defining sequence \( f_n g_n 0 \) such that \( N_0 \) is knotted and any other defining sequence \( f_n 0 g_n 0 \) with knotted \( N_0 0 \) is a subsequence of \( f_n g_n 0 \).

Proof. Follow the proof of Proposition 3.2. Assume \( N_0 \) is knotted and append further into ( ) all such JSJ-decomposition tori of \( S^3 \) that bounds a solid torus in \( S^3 \), in which \( N_0 \) is a thick braid. It is a routine matter to verify that the resulting maximal defining sequence is exactly what we want.
For any knot invariant $I$ (for example, the genus, the Gromov volume, the Alexander polynomial or the Jones polynomial) one has an invariant $I$ of tame solenoids as below. For a knotted tame solenoid $S^3$, let $fN_0, g_n$ be the unique maximal defining sequence from the above proposition. Then the infinite sequence $I(\ ) = fI(N_0); I(N_1); \ldots; I(N_n); \ldots$ depends only on the equivalence class of $\Gamma$. If a tame solenoid $S^3$ is unkotted, then for any defining sequence $fN_0, g_n$ of the sequence $I(\ ) = fI(N_0); I(N_1); \ldots; I(N_n); \ldots$ is identically trivial, say $f0; 0; \ldots; 0; \ldots$, if $I$ is either the genus or the Gromov volume, or $f1; 1; \ldots; 1; \ldots$, if $I$ is either the Alexander polynomial or the Jones polynomial.

In general for given numerical function $g$ and knot invariant $I$, one may organize the sequence $I(\ )$ into a formal series $I(\ ;g) = \sum_{n=0}^{\infty} g(n)I(N_n)t^n$. We wonder if $I(\ ;g)$ would have interesting properties for certain $g$ and $I$ as well as for suitable classes of solenoids.

### 3.4 Unknotted 2-adic tame solenoids

Given an unknot solid torus $N$ in $S^3$, there are exactly two kinds of thick braids of winding number two in $N$ that are unknotted in $S^3$ as shown in Figure 2, where the left one is the left-handed embedding, which we denote by $1$, and the right one is the right-handed embedding, which we denote by $+1$. Then any maximal defining sequence of a unknot 2-adic tame solenoids in $S^3$ can be presented as an infinite sequence of $1$.

Let $Z_2$ be the set of infinite sequences $(a_1; a_2; \ldots; a_n; \ldots)$ of $1$. Two such sequences are said to be equivalent, if they can be made identical by deleting finitely many terms. By Theorem 3.4 the equivalence classes of unknot 2-adic tame solenoids are in 1:1 correspondence to the equivalence classes of $Z_2$. In particular, there are uncountably many equivalence classes of unknot 2-adic tame solenoids.

![Figure 2](image-url)
3.5 Smale solenoids

The solenoids were introduced into dynamics by Smale as hyperbolic attractors in [3].

Definition 3.8. Let $M$ be a 3-manifold and $f : M \to M$ be a homeomorphism. If there is a solid torus $N \subset M$ such that $f^j$ (resp. $f^{-j}$) defines a thick braid as in Definition 2.2 (1), we call the hyperbolic attractor $\\bigcup_{n=1}^{\infty} f^n(N)$ (resp. the hyperbolic repeller $\\bigcap_{n=1}^{\infty} f^{-n}(N)$) a Smale solenoid.

Clearly each Smale solenoid $S^3$ is tame. It is known that a Smale solenoid $S^3$ must be unknotted [JNW]. Moreover, it is proved in [JNW] that if the non-wandering set $w(f)$ of a dynamical $f$ consists of finitely many disjoint Smale solenoids, then $w(f)$ consists of two solenoids (indeed they are algebraically linked).

Definition 3.9. Let $w_1;:::;w_k$ be integers greater than 1. Call a Smale solenoid $S^3$ is of type $(w_1;:::;w_k)$, if (1) there is a dynamical $f$ taking as an attractor, (2) there is a defining sequence $fN_0, g_0$ of such that $f$ sends $N_n$ to $N_{k+n}$ for all $n \geq 0$ and (3) $w_n$ is the winding number of $N_n$ in $N_{n+1}$ for $1 \leq n \leq k$.

Proposition 3.10. Any given type $(w_1;:::;w_k)$ is realized by a Smale solenoid $S^3$. Moreover, the number of Smale solenoids $S^3$ of type $(w_1;:::;w_k)$ is finite if all $w_n \geq 3$, and is countably infinite otherwise.

Proof. First, we extend the sequence $(w_1;:::;w_k)$ to a infinite one by setting $w_{k+n} = w_n$. Then choosing an unknotted solid torus $N_0 \subset S^3$ and letting $N_n = N_n \cup N_{n-1}$ be a tubular neighborhood of $\frac{1}{w_n} \in S^3$ give rise to a defining sequence of the desired Smale solenoid, where $i$'s are standard generators of the braid groups.

The "Moreover" part follows from the Lemma below and Theorem [3.4].

Lemma 3.11 ([MP]). Let $W_n$ be the set of $n$-strand braids whose closures are unknotted in $S^3$. Then

(1) $W_n$ has two conjugacy classes as pictured in Figure 2 for $n = 2$;
(2) $W_n$ has three conjugacy classes as pictured in Figure 3 for $n = 3$;
(3) $W_n$ has in finitely many conjugacy classes for $n > 3$.

Figure 3
4 Chirality of tame solenoids

Definition 4.1. Call a subset $A \subseteq S^3$ achiral if there is an orientation reversing homeomorphism $r : S^3 \to S^3$ such that $r(A) = A$. Call $A$ strictly achiral if there is an orientation reversing homeomorphism $r : S^3 \to S^3$ such that $r(x) = x$ for every $x \in 2A$.

In Definition 4.1, "achiral" means setwise achiral, and "strictly achiral" means pointwise achiral. They are two opposite extremes among various shades of chirality in the real world.

4.1 Criteria

By definition, a tame solenoid in $S^3$ is achiral if and only if it is equivalent to its mirror image. Therefore, by Theorem 3.4 we have the following criterion of the chirality of tame solenoids.

Theorem 4.2. A tame solenoid given by the maximal defining sequence $f N, g_0$ is achiral if and only if $f N, g_0$ is equivalent to its mirror image.

Example 4.3. Recall the example in $x3.4$. The mirror image of a maximal defining sequence of a unknotted 2-adic tame solenoid presented by $(a_1; a_2; \ldots; a_n; \ldots)$ is presented by $(a_1; a_2; \ldots; a_n; \ldots)$. Therefore, by Theorem 4.2, the unknotted 2-adic tame solenoid presented by $(+1; 1; +1; \ldots)$ is achiral but those solenoids presented by $(+1; +1; +1; \ldots)$ or $(-1; -1; -1; \ldots)$ are not achiral.

Below we focus on the strict chirality of tame solenoids. For a map $f : X \to X$, we use $\text{Fix}(f)$ to denote the fixed point set of $f$.

Definition 4.4. Suppose $A$ is a subset of the solid torus $N$ and $l$ is a given framing of $N$. Call $A$ is strictly achiral with respect to $l$ if there exists an orientation reversing homeomorphism $f : N \to N$ such that $A \cup -\text{Fix}(f)$.

Remark 4.5. It is well known that the orientation reversing homeomorphism of $S^3$ is unique up to isotopy, but this is not true for the solid torus. The ambiguity can be removed by posing the framing fixing condition used in the above definition.

To prove the main theorem of this subsection, we need the following two lemmas.

Lemma 4.6. Suppose $l^0$ is a framing of a thick braid in $N$. If $l^0$ is strictly achiral with respect to a framing $l$ of $N$, then the strict chirality can be given by a $D^2$-beration preserving homeomorphism of $N$. 

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Proof. Suppose the strict adhirality of $\mathbf{I}^0 N$ with respect to $l$ is defined by an orientation reversing homeomorphism $r : N \to N$. Then we can isotope $r$ relative to $\mathbf{I}^0 [1 \to D^2]$, beration preserving homeomorphism $\mathbf{I}^0 [1 \to D^2]$, and the isotopy is just the process of this sliding. (More clear way to see this sliding: let $p : N \to N$ be the infinite cyclic covering, and $\mathbf{I}^0 N$ be the preimage of $\mathbf{I}^0$, then $(N; \mathbf{I}^0 N)$ is homeomorphic to $(D^2; w)$ points.) \(\blacksquare\)

Lemma 4.7. Suppose $S^3$ is a tame embedding given by a defining sequence $fN_n g_n 0$. If $\text{Fix}(r)$ for some homeomorphism $r : S^3 \to S^3$, then there exists $k > 0$ such that $r(N_n)$ int $N_0$, and moreover $r(N_n)$, $N_n$ have the same winding number in $N_0$ for $n - k$.

Proof. Since $S^3$ is compact, $r$ is uniformly continuous.

(i) Let $d(N_1; \partial N_0) = 2$.

(ii) Choose $0 < \varepsilon$ such that if $d(x; \delta) < \varepsilon$, then $d(r(x); r(\delta)) < \varepsilon$.

(iii) Choose $k > 0$ such that $d(x; \varepsilon) < k$.

Now $x$ an integer $n - k$. For any $x \in N_n$, by (ii) we have $d(x; \partial N_0) = 2$ and by (iii) we can choose $x^0 \geq 2$ such that $d(x; x^0) < 2$, hence by (ii) we have

$$d(x; r(x)) = d(x; x^0) + d(x^0; r(x^0)) + d(r(x^0); r(x)) < 0 + < 2 + d(x; \partial N_0).$$

It follows that the unique geodesic $(x)$ connecting $x$ and $r(x)$ lies in $\text{int} N_0$. Therefore, if $x \subseteq N_n g_n 0$ gives rise to a homotopy from $N_n$ to $r(N_n)$ in $N_0$. In particular, $r(N_n)$ int $N_0$ and $r(N_n); N_n$ have the same winding number in $N_0$. \(\blacksquare\)

Theorem 4.8. Let $S^3$ be a tame solenoid with defining sequence $fN_n g_n 0$ and let $l_n$ denote a zero framing of $N_n$ in $S^3$, that is, $l_n$ is null-homologous in $S^2 n N_n$. Then is strictly adhiral if and only if there exists $k > 0$ such that the $l_n$ is strictly adhiral in $S^3$, and $l_n$, $l_n$ is strictly adhiral in $N_n$ with respect to $l_n$ for all $n - k$.

Proof. Sufficiency. Without loss of generality, we assume $k = 0$. Since $l_n$ is strictly adhiral, there is an orientation reversing homeomorphism $f_0 : (S^3; N_0) \to (S^3; N_0)$ such that $f_0 j_n = D^2$-beration preserving and $x_{e,1}$ pointwise.

Since $l_n$ is strictly adhiral in $N_0$ with respect to $l_n$, by Lemma 4.6 there is an orientation reversing and $D^2$-beration preserving homeomorphism $f_1 : (N_0; N_1) \to (N_0; N_1)$ such that $l_n$ stays in Fix($f_1$). Since both $f_0 j_n$ and $f_1$ are orientation reversing and $D^2$-beration preserving homeomorphism of $N_0$ and both $x_{e,1}$ pointwise, we may assume $f_0 j_{N_0} = f_0 j_{N_0}$. So $f_1$ may be extended onto $S^3$ by setting $f_1 j_{N_n} = f_0 j_{N_n}$. Then for $n > 1$, by the same reason we can recursively define homeomorphism $f_n : (S^3; N_n) \to (S^3; N_n)$ such that $f_n j_{N_n} = f_1 j_{N_n}$ and that $f_n j_{N_n}$ is $D^2$-beration preserving and $x_{e,1}$ pointwise.

To apply Lemma 2.23 set $U_n = N_n$. Clearly Conditions (1) and (3) of Lemma 2.29 are satisfied. Since $f_n j_{N_n}$ is $D^2$-beration preserving and the diameters of the meridian
disks of $N_n$ tend to zero uniformly as $n \to 1$. Conditions (2) and (4) of Lemma 2.9 are also satisfied. By Lemma 2.9, $f_n$ uniformly converges to an orientation reversing homeomorphism $f : S^3 \to S^3$ with the property that \( n \circ N_n \circ \text{Fix}(f) \).

Necessity. Suppose the strictly achirality is defined by an orientation reversing homeomorphism $r$ and let $k > 0$ be given by Lemma 4.7. Then for any $n \leq k$ we have that both $N_n$ and $r(N_n)$ are contained in the interior of $N_0$.

Fix an integer $n \leq k$ and choose a big integer $j$ so that $N_j \subseteq N_n \setminus r(N_n)$. By Lemma 3.3 (3), both $\mathcal{O}_n, r(\mathcal{O}_n)$ are isotopic in $N_0 \setminus N_j$ to some components of $[\mathcal{O}_n, \mathcal{O}_j]$ which is the JSJ-decomposition tori of $N_0 \setminus N_j$. Therefore, we can isotope $r$ with support in $N_0 \setminus N_j$ so that either $r(N_j) = N_k$, or by Lemma 3.3 (1) $r(N_j) \subseteq N_k$ or $r(N_j) \subseteq r(N_k)$ is a thick braid of winding number greater than 1. By Lemma 4.7, $r(N_j) \subseteq r(N_k)$ have the same winding number in $N_0$, so the latter case cannot happen and, moreover, we may assume the zero framing $l_1$ of $N_k$ lies in $\text{Fix}(r)$.

By the same argument, $r$ can be further isotope with support in $N_0 \setminus N_j$ so that $r(N_{n+1}) = N_{n+1}$ and $l_{n+1} = \text{Fix}(r)$. Therefore, $l_1$ is strictly achiral in $S^3$ and $l_{n+1}$ is strictly achiral in $N_n$ with respect to $l_1$. □

4.2 Examples

Thanks to Theorem 4.2, the strictly achirality of tame solenoids is the problem of strictly achirality of knots in $S^3$ and closed braids in the solid torus. Below we fix a point $D^2$ of $D^2$ and let $l$ denote the framing $S^1$ of $D^2 
 S^1$.

Definition 4.9. Call a braid achiral, if it is conjugate to its mirror image.

Note that the achirality is well defined for a conjugacy class of braids. Also note that the closure $D^2 \times S^1$ of a braid is connected if and only if it is cyclic, i.e. permutes its strands cyclically.

Proposition 4.10. For every cyclic braid, the closure $D^2 \times S^1$ is strictly achiral with respect to $l$ if and only if it is achiral.

Proof. Sufficiency. Suppose $n = 1$. Then $n = (k)^1$ for any integer $k$. Since and give rise to the same cyclic permutation on their strands, it follows from the equality $n = 1$ that the permutation given by is commutative with hence is a power of that given by . So we can choose $k$ so that $k$ is a pure braid. Replacing by $k$, we may assume $e$ is a pure braid.

By the definition of braid, there is a $D^2$-beration preserving and boundary crossing homeomorphism $g : D^2 \setminus \{0,1\}$ which sends 1 to 1. Then $g$ induces a homeomorphism $g : D^2 \times S^1 \to D^2 \times S^1$ which sends 1 to 1. Moreover, by canceling the part and the 1 part of 1, we can define a $D^2$-beration preserving and boundary crossing homeomorphism $f : D^2 \times S^1 \to D^2 \times S^1$ which sends 1 to 1.
Finally, let $r : D^2 S^1 \to D^2 S^1$ be the reflection about the page (assume $l$ lies on it). Then $gfr : D^2 S^1 \to D^2 S^1$ is an orientation reversing homeomorphism and sends $\overline{\cdot}$ to $\overline{\cdot}$. Since $\overline{\cdot}$ is a pure braid and all $g;f;r$ preserve each $D^2$ slice of $D^2 S^1$, it follows that $gfr$ fixes both the closure $\overline{\cdot}$ and the framing $l$ pointwise.

Figure 4

Figure 4 illustrates the case $= 1_2^1$ and $= 2_1^2 1_2^1$ (see Example 4.11 for the equality $= 1^1$).

Necessity. If $\overline{\cdot}$ is strictly achiral with respect to $l$ then $\overline{\cdot}$ is isotopic to its mirror image, therefore $\overline{\cdot}$ is conjugate to its mirror image $\overline{}$.

Example 4.11. Examples of cyclic, achiral braids.

(1) $= 1_2^1$ is cyclic and achiral. Setting $= 2_1^2 1_2^1$, one can verify the equality $= 1^1$ either by directly a braid move or by the substitution of braid relation $2_1^2 = 1_2^1$ as follows

$$1^1 = (2_1^1 1_2^1 1_2^1)(1_2^1 1_2^1) = 2_1^1 1_2^1 1_2^1 2_1^1 1_2^1 2_1^1 1_2^1 1_2^1 = 1_2^1 1_2^1 1_2^1 2_1^1 1_2^1 2_1^1 1_2^1 1_2^1 = 1_2^1 = 1^1$$

(2) For any braid $\overline{\cdot}$ is achiral, since

$$1^1(\overline{\cdot}) = 1^1 = 1^1$$

Moreover, for any cyclic braid of odd number of strands, $\overline{\cdot}$ is also cyclic. Hence for each cyclic braid of odd number of strands, $\overline{\cdot}$ is cyclic and achiral.
If is an achiral braid, then so is \( k \) for any integer \( k \). Moreover, if is cyclic, then so is \( k \) for every integer \( k \) relatively prime to the number of strands.

**Proposition 4.12.** (1) If a connected closed braid \( D^2 S^1 \) is strictly achiral with respect to 1, then the writhe of \( - \) is zero (for the definition of writhe, see [A, p.152]).

(2) Hence a connected closed braid \( D^2 S^1 \) is not strictly achiral with respect to 1 if \( - \) is either of even winding number, or a cable.

**Proof.** (1) By Proposition 4.10, \( - \) is strictly achiral implies \( = 1 \) for some braid. Clearly \( wr(\bar{-}) = wr(-1) = wr(\bar{1}) = wr(\bar{1}) \). It follows that \( wr(\bar{-}) = 0 \).

(2) Suppose \( - \) is connected and is of even winding number. It is an elementary exercise to show that the number of the crossings of \( - \) is odd. Hence \( wr(\bar{-}) \) must be odd, which contradicts (1).

Suppose \( - \) is a cable. Then all the crossings of \( - \) have the same sign, so \( wr(\bar{-}) \) is non-zero, which contradicts (1).

Now we state the main result of this subsection.

**Theorem 4.13.** A solenoid of type \( \tau = (w_1;w_2;\ldots;w_n;\ldots) \) has a strictly achiral tame embedding into \( S^3 \) if and only if all except finitely many \( w_n \) are odd.

**Proof.** Necessity is immediate from Theorem 4.8 and Proposition 4.12 (2).

Sufficiency. Assume all \( w_n \) are odd. Let \( N_0 \) be a tubular neighborhood of a strictly achiral knot in \( S^3 \) and let \( N_n \) be a tubular neighborhood of \( n_n \) in \( N_{n-1} \) where \( n \) is an arbitrary cyclic braid on \( w_n \) strands. By Theorem 4.8, Proposition 4.10 and Example 4.11 (2) the defining sequence \( f_n g_{n_0} \) gives rise to a strictly achiral tame embedding of the solenoid of type \( \tau = (w_1;w_2;\ldots;w_n;\ldots) \).

**Example 4.14.** (1) The 2-adic solenoid has no strictly achiral tame embedding into \( S^3 \).

(2) By Lemma 3.1 (2), Example 4.11 (1) and Proposition 4.12 (2), up to equivalence the 3-adic solenoid has a unique unknotted strictly achiral tame embedding into \( S^3 \), which is yielded by nesting the thick braid pictured in the middle of Figure 3.

(3) The embedding in Theorem 4.13 can be chosen to be either knotted or unknotted, by letting in the proof either \( N_0 \) be a tubular neighborhood of the figure-8 knot, or letting \( N_0 \) be a tubular neighborhood of the unknot and \( n = 1 \) (we leave it to the reader to verify that the closure \( n_n \) is unknotted in \( S^3 \)).

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