THE SYLOW SUBGROUPS OF THE ABSOLUTE GALOIS GROUP $\text{Gal}(\mathbb{Q})$

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Abstract. We describe the $\ell$-Sylow subgroups of $\text{Gal}(\mathbb{Q})$ for an odd prime $\ell$, by observing and studying their decomposition as $F \rtimes \mathbb{Z}_\ell$, where $F$ is a free pro-$\ell$ group, and $\mathbb{Z}_\ell$ are the $\ell$-adic integers. We determine the finite $\mathbb{Z}_\ell$-quotients of $F$ and more generally show that every split embedding problem of $\mathbb{Z}_\ell$-groups for $F$ is solvable. Moreover, we analyze the $\mathbb{Z}_\ell$-action on generators of $F$.

1. Introduction

The absolute Galois group $\text{Gal}(K) = \text{Aut}(\bar{K}/K)$ of a field $K$ with algebraic closure $\bar{K}$ is a central object in Galois theory. The case $K = \mathbb{Q}$, or more generally when $K$ is a number field, is the most interesting case in number theory, but is also the most difficult.

First, consider the simpler case of $\ell$-adic fields $K$: the maximal pro-$\ell$ quotient of $\text{Gal}(K)$ is completely understood by the consecutive works of Shafarevich, Demuskin, Serre, and Labute — it admits a presentation with countably many generators subject to at most one relation, see [25, §5.6]. This led Serre to ask about a larger part of $\text{Gal}(K)$, namely, its $\ell$-Sylow subgroups. Recall that profinite Sylow subgroups of general fields follow Labute’s [18, 19, 3].

When $K$ is a number field the natural analogue of the maximal pro-$\ell$ quotient $Q$ of $\text{Gal}(K)$ that is unramified outside a given finite set $S$ of primes. See [1] for a survey on these groups. Presentations of $Q$ are known up to the second term of its descending $\ell$-central series [12, §11.4] or up to $[Q^{\text{cyc}}, Q^{\text{cyc}}]$ where $Q^{\text{cyc}} \leq Q$ is the subgroup fixing the cyclotomic $\mathbb{Z}_\ell$-extension of $K$ [27], both results are under restrictive assumptions on $K$ and $S$. Thus, in this
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case, it seems hopeless to apply Labute’s strategy to study the Sylow subgroups of \(\text{Gal}(K)\), leaving them completely unknown.

We study the number field setting in analogy to the function field setting. Let \(K = K_0(C)\) be a function field of a smooth projective curve \(C\) over a finite field \(K_0\). Then we have the canonical short exact sequence

\[
1 \rightarrow \hat{F} \rightarrow \text{Gal}(K) \rightarrow \hat{Z} \rightarrow 1,
\]

where \(\hat{F} = \text{Gal}(K\overline{K_0})\), \(\hat{Z} = \text{Gal}(\overline{K_0}/K_0) \cong \hat{Z}\). Note that \(\hat{Z}\) is generated by the Frobenius automorphism and that by a theorem of Harbater [7] and Pop [23], see also [13, Theorem 9.8.5], \(\hat{F}\) is a free profinite group on countably many generators. Assume next that \(C\) has a \(K_0\) rational point \(\zeta\), then a Frobenius element at \(\zeta\) gives a splitting of (1) and hence a decomposition

\[
\text{Gal}(K) \cong \hat{F} \rtimes \hat{Z}.
\]

The action here, of Frobenius on \(\hat{F}\) is of great importance, and is known on various quotients of \(\hat{F}\). Notably, the Frobenius action on the Galois group of the maximal unramified pro-\(\ell\) abelian extension of \(K\overline{K_0}\) is dual to the action of Frobenius on the Jacobian of \(C\). The latter is a \(\hat{Z}\)-module of finite rank, and by the Riemann Hypothesis for varieties over finite fields, its eigenvalues are given in terms of the zeros of the corresponding zeta function, see [20, §11.6] (here we assume \(\ell\) is different from the characteristic of \(K_0\)).

Returning to the case where \(K\) is a number field, we start by the observation that decomposition of the form (2) exists for an \(\ell\)-Sylow subgroup of \(\text{Gal}(K)\): For an \(\ell\)-Sylow subgroup \(\Lambda\) of \(\text{Gal}(K)\) denote by \(K^{(\ell)}\) its fixed field, so that \(\Lambda = \text{Gal}(K^{(\ell)})\). Denote by \(\mu_{\ell}\) the group of \(\ell\)-power roots of unity.

**Observation 1.1.** Let \(K\) be a number field and \(\ell\) an odd prime. Then the \(\ell\)-Sylow subgroups of \(\text{Gal}(K)\) decompose as:

\[
\text{Gal}(K^{(\ell)}) = F \rtimes Z,
\]

where \(Z\) maps isomorphically under the natural projection to the cyclic group \(\text{Gal}(K^{(\ell)}(\mu_{\ell\infty})/K^{(\ell)}) \cong \mathbb{Z}_\ell\), and \(F := \text{Gal}(K^{(\ell)}(\mu_{\ell\infty}))\) is the free pro-\(\ell\) group on countably many generators.

We note that, as in the function field case, certain primes give rise to splittings of (3), see §3.3. We also note the connection with Riemann’s existence theorem, see §3.4.

This paper studies the action of the cyclic group \(Z\) on \(F\), which is the remaining ingredient in understanding \(\text{Gal}(K^{(\ell)})\). We first determine the finite quotients of \(F\) as a \(Z\)-group, and more generally study embedding problems for \(F\) which respect the \(Z\)-action.
As in profinite group theory, in which embedding problems are used to determine profinite groups, we study the $\mathbf{Z}$-group $F$ via $\mathbf{Z}$-embedding problems. A \textbf{finite $\mathbf{Z}$-embedding problem} for $F$ is a pair of $\mathbf{Z}$-epimorphisms $(\alpha: F \to \Gamma, \beta: G \to \Gamma)$, where $G, \Gamma$ are finite $\mathbf{Z}$-groups. A \textbf{proper solution} of $(\alpha, \beta)$ is a lifting of $\beta$ to a $\mathbf{Z}$-epimorphism $\gamma: F \to G$, cf. §2.1.

Analogously to the classical setting, solvability of $\mathbf{Z}$-embedding problems is reduced to solvability of Frattini $\mathbf{Z}$-embedding problems and of split $\mathbf{Z}$-embedding problems, see Proposition 2.3. Here $(\alpha, \beta)$ is \textbf{split} if $\beta$ has a section which is a $\mathbf{Z}$-homomorphism.

\textbf{Theorem 1.2.} Every finite split $\mathbf{Z}$-embedding problem for $F$ is properly solvable. In particular, every finite $\ell$-group $G$ equipped with a $\mathbf{Z}$-action is a quotient of $F$ as a $\mathbf{Z}$-group.

We note that in general Frattini $\mathbf{Z}$-embedding problems for $F$ are not solvable. Nevertheless one can reduce such problems to a classical setting over global fields, see Proposition 4.3.

The main tools allowing us to prove Theorem 1.2 are Pop’s theorem on solvability of split embedding problems for function fields over an ample field ([22] and [13, Theorem 5.9.2]), and the observation of Colliot-Thélène that fields with pro-$\ell$ absolute Galois group are ample [13, Theorem 5.8.3]. We then apply the resulting tools to make the first step towards determining the $\mathbf{Z}$-action on $F$ by describing the action on generators of $F$ up to elements in $F^\ell[F, F]$, the first level in the lower $\ell$-central series of $F$. That is, we describe the structure of the Frattini quotient $\overline{F} = F/F^\ell[F, F]$ as a $\mathbf{Z}$-module by determining its indecomposable direct $\mathbf{Z}$-summands.

A $\mathbf{Z}$-module $M$ is said to be a direct $\mathbf{Z}$-summand of $\overline{F}$ of multiplicity $\kappa$, if $\overline{F} \cong M^\kappa \times M'$, where $M^\kappa$ is the product of $\kappa$ copies of $M$, and $M'$ has no $\mathbf{Z}$-summands isomorphic to $M$. Note that since $Z$ acts on the group ring $\mathbb{F}_\ell[Z/\ell^nZ]$, it also acts on $\mathbb{F}_\ell[[Z]] = \limleftarrow \mathbb{F}_\ell[Z/\ell^nZ]$.

\textbf{Theorem 1.3.} The indecomposable direct $\mathbf{Z}$-summands of $\overline{F}$ are $\mathbb{F}_\ell[[Z]]$ and $\mathbb{F}_\ell[Z/\ell^nZ]$ for $n \in \mathbb{N} \cup \{0\}$. Each of these summands appears with multiplicity $\omega$.

We note that in general $\overline{F}$ is not a direct product of indecomposable modules and hence not all generators of $\overline{F}$ arise from Theorem 1.3. We also show that obtaining a full account of the action on the remaining generators is equivalent to determining a certain Iwasawa module. See §5.8.

In analogy to the work of Demushkin, Serre and Labute, where the relations are determined up to elements in a low level of a filtration and then lifted to the entire group, Theorem 1.3 gives relations in a presentation of $\text{Gal}(K^{(0)})$ up to elements in the first level $F^\ell[F, F]$ of the lower $\ell$-central series of $F$. Namely, letting $\sigma$ be a generator of $Z$, each summand $\mathbb{F}_\ell[Z/\ell^kZ]$ gives a subset of generators $x_1, \ldots, x_{\ell^k}$ of $F$ subject only to the relations $\sigma x_i \sigma^{-1} = x_i x_{i+1}$ for $i = 1, \ldots, \ell^k - 1$.
and $\sigma x_k \sigma^{-1} = x_k y$, for some $y \in F[[F, F]]$. Similar relations are obtained for each $F[[Z]]$ summand, see Corollary 5.12.

The proof of Theorem 1.3 uses the theory of Ulm invariants. The proof splits into finding the multiplicity of the infinite and finite indecomposable direct $Z$-summands. In contrast to the work of Mináč-Schultz-Swallow [16], [17], this approach also allows dealing with modules over an infinite group such as $Z$, cf. §5.1.

Finding the finite summands is equivalent to computing the finite Ulm invariants of $\overline{F}$ by Proposition 5.2. This computation is carried out by reducing to $Z$-embedding problems (Proposition 5.3), establishing a local global principle using the Poitou-Tate duality theorem (Proposition 5.5), solving the local embedding problems using local Iwasawa theory (Proposition 5.6), and is completed in Proposition 5.8. The infinite summands are essentially treated in Proposition 5.10 using global Iwasawa theory. The proof of Theorem 1.3 is then completed in Section 5.6.

We note that our methods are applicable and hence also stated in greater generality over global fields and for $\ell = 2$ as well. We are hopeful that the combination of our methods with Iwasawa theory and results of Efrat-Mináč [4] will shed light on the shape of relations up to higher levels of the lower $\ell$-central series of $F$, and advance us further towards a complete understanding of $\text{Gal}(\mathbb{Q}^{(\ell)})$.

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2. Embedding Problems

2.1. $Z$-embedding problems. Let $Z$ be a profinite group. A profinite $Z$-group is a profinite group $H$ together with a continuous $Z$-action. A $Z$-homomorphism $\phi: H_1 \to H_2$ is a continuous homomorphism that commutes with the $Z$-action. We say that a subgroup $H_1$ of a profinite $Z$-group $H_2$ is a $Z$-subgroup, if the inclusion map $H_1 \to H_2$ is a $Z$-homomorphism, that is, if $H_1$ is a closed subgroup of $H_2$ that is closed under the action of $Z$. A $Z$-embedding problem for a $Z$-group $H$, denoted by $(\alpha, \beta)$, is a diagram

(4)

\[ \begin{array}{ccc}
G & \xrightarrow{\beta} & \Gamma \\
\gamma \downarrow & & \downarrow \\
H & \xrightarrow{\alpha} & \Gamma 
\end{array} \]

in which $G, \Gamma$ are profinite $Z$-groups and $\alpha, \beta$ are $Z$-epimorphisms. If $Z = 1$, we recover the usual notion of embedding problems for profinite groups. A solution of the $Z$-embedding problem is a homomorphism $\gamma: H \to G$ that commutes the
above diagram. A solution is called proper if it is surjective. A $Z$-embedding problem is called split if $\beta$ has a section which is $Z$-morphism. We define the $Z$-Frattini subgroup $\Phi_Z(G)$ of a profinite $Z$-group $G$ to be the intersection of all maximal $Z$-subgroups. We call a $Z$-embedding problem, as above, Frattini if $\ker \beta \leq \Phi_Z(G)$. If $G$ is finite (and hence so is $\Gamma$) we say that the $Z$-embedding problem is finite.

Most of the basic theory of embedding problems carries over to $Z$-embedding problems. The proofs are similar to the classical case $Z = 1$. For the sake of completeness, in the following lemmas we prove the properties we shall need in the course of the proofs of Observation 3.1 and Proposition 4.3.

**Lemma 2.1.** If $U$ is an open subgroup of a profinite $Z$-group $H$, then $U_Z = \bigcap_{z \in Z} U^z$ is open in $H$.

*Proof.* Since the action map $p: H \times Z \to H$ is continuous, $p^{-1}(U)$ is open. Thus there exist open normal subgroups $H_0 \leq H$ and $Z_0 \leq Z$ such that $p^{-1}(U)$ is a finite union of cosets of $H_0 \times Z_0$, say $p^{-1}(U) = \bigcup_{i=1}^{n} H_0 h_i \times Z_0 z_i$. Thus

$$U_Z = \bigcap_{z \in Z} U^z = \bigcap_{z \in Z} \bigcup_{i=1}^{n} (H_0 h_i)^{Z_0 z_i} = \bigcap_{z \in Z} \bigcup_{i=1}^{n} (H_0 h_i)^{z_0 z_i^{-1} z_i} = \bigcap_{z \in Z} \bigcup_{i=1}^{n} (H_0 h_i) z_0 z_i^{-1} z_i.$$

We conclude that $U_Z$ is open as a finite intersection of open sets. \(\square\)

**Lemma 2.2.** If $(\alpha: H \to \Gamma, \beta: G \to \Gamma)$ is a Frattini $Z$-embedding problem and if $\gamma: H \to G$ is a solution, then $\gamma$ is proper.

*Proof.* Let $U = \gamma(H)$. If $U \neq G$, then there is a maximal $Z$-subgroup $V$ of $G$ that contains $U$. So

$$\Gamma = \alpha(H) = \beta(\gamma(H)) = \beta(U) \leq \beta(V).$$

By the third isomorphism theorem this implies that $G = V \ker \beta$. Since $(\alpha, \beta)$ is Frattini, $\ker \beta \leq \Phi_Z(G) \leq V$. So $G = V \ker \beta \leq V \Phi_Z(G) \leq V \neq G$. This contradiction implies that $U = G$, as needed. \(\square\)

The following proposition reduces the study of solvability of embedding problems to the study of Frattini and split embedding problems.

**Proposition 2.3.** Consider a $Z$-embedding problem $E = (\alpha: H \to \Gamma, \beta: G \to \Gamma)$ for a profinite $Z$-group $H$. Then there exists an open $Z$-subgroup $U$ of $G$ such that $\beta(U) = \Gamma$ and the following properties are satisfied:

(a) The $Z$-embedding problem $E_U = (\alpha: H \to \Gamma, \beta|_U: U \to \Gamma)$ is Frattini.

(b) A solution $\alpha': H \to U$ of $E_U$ induces a split $Z$-embedding problem $E' = (\alpha': H \to U, \beta': \ker \beta \rtimes U \to U)$, where $U$ acts on $\ker \beta$ by conjugation in $G$.

(c) A proper solution $\gamma': H \to \ker \beta \rtimes U$ of $E'$ induces a proper solution $\gamma: H \to G$ of $E$ by $\gamma'(h) = (\sigma, u)$ implies $\gamma(h) = \sigma u$. 


Proof. A limit argument reduces the proof to finite $Z$-embedding problems.

Let $U$ be minimal among the open $Z$-subgroups of $G$ that map onto $\Gamma$. In particular $\beta(U) = \Gamma$. Since no proper $Z$-subgroup of $U$ maps onto $\Gamma$, we have that $\ker(\beta|_U)$ is contained in each of the maximal $Z$-subgroups of $U$, hence $\ker(\beta|_U)$ is contained in $\Phi_Z(U)$. This proves (a).

If $\alpha'$ is a solution of $E_U$, then it is proper by Lemma 2.2. To prove (b), it suffices to observe that $\ker(\beta \rtimes U)$ is a profinite $Z$-group with respect to the action $(\sigma, u)^z = (\sigma^z, u^z)$ and that the projection map $\beta': \ker(\beta \rtimes U) \to U$ is a $Z$-map.

Let $\pi: \ker(\beta \rtimes U) \to G$ defined by $\pi(\sigma, u) = \sigma u$. It is a $Z$-epimorphism that commutes in the diagram of $Z$-maps.

Thus if $\gamma'$ is a proper solution of $E'$, then $\gamma$ is a proper solution of $E$, as needed for (c). \hfill \Box

Lemma 2.4. Let $H_1$ be a $Z$-subgroup of a profinite $Z$-group $H$ and let $\alpha_1: H_1 \to \Gamma$ be a $Z$-epimorphism on a finite $Z$-group $\Gamma$. Then there exists an open $Z$-subgroup $H_2$ of $H$ that contains $H_1$ and an extension $\alpha_2: H_2 \to \Gamma$ of $\alpha_1$.

In particular any finite $Z$-embedding problem for $H_1$ is the restriction of a corresponding $Z$-embedding problem for an open $Z$-subgroup of $H$ that contains $H_1$.

Proof. The subgroup $U_1 = \ker(\alpha_1)$ is an open normal $Z$-subgroup of $H_1$. Then there exists an open normal subgroup $U_2$ of $H$ such that $U_2 \cap H_1 \leq U_1$. By Lemma 2.1 we may replace $U_2$ by $\bigcap_{z \in Z} U_2^z$ to assume that $U_2$ is a $Z$-subgroup.

Let $H_2 = U_2 H_1$. Then $H_2$ is an open $Z$-subgroup of $H$ that contains $H_1$. Let $\alpha_2: H_2 \to \Gamma$ be defined by $\alpha_2(u\sigma) = \alpha_1(\sigma)$ for all $u \in U_2$ and $\sigma \in H_1$. Then $\alpha_2$ is well defined because it is trivial on $U_2 \cap H_1 \leq U_1$ and it is a $Z$-map because its kernel $U_2$ is an open normal $Z$-subgroup. By definition $\alpha_2|_{H_1} = \alpha_1$, hence the assertion. \hfill \Box

We shall need the following two basic lemmas concerning Sylow subgroups of profinite groups.

Lemma 2.5. Let $\ell$ be a prime number, $\Lambda$ an $\ell$-Sylow subgroup of $G$, and $\alpha: G \to H$ an epimorphism of profinite groups. Assume that $H$ is pro-$\ell$. Then $\alpha(\Lambda) = H$.

Proof. The notation $[A : B]$ denotes the index of a subgroup $B$ of a profinite group as a supernatural number, cf. [6, §22.8]. By the isomorphism theorems for profinite
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groups one has

\[ [H : \alpha(\Lambda)] = [G : \Lambda \ker \alpha]. \]

Since \( H \) is pro-\( \ell \) the left hand side is a (supernatural) power of \( \ell \). Since \( \Lambda \) is an \( \ell \)-Sylow subgroup, the right hand side, which divides \( [G : \Lambda] \), is prime to \( \ell \). Hence \( [H : \alpha(\Lambda)] = 1 \), as needed. \( \square \)

**Lemma 2.6.** Let \( \ell \) be a prime number and \( H \) a normal subgroup of a profinite group \( G \). Assume \( [G : H] \) is prime to \( \ell \). Then \( H \) contains all \( \ell \)-Sylow subgroups of \( G \).

**Proof.** Let \( \Lambda \) be an \( \ell \)-Sylow subgroup of \( H \). Then \( [G : \Lambda] = [G : H][H : \Lambda] \) is prime to \( \ell \) and so \( \Lambda \) is an \( \ell \)-Sylow subgroup of \( G \). Since \( H \) is normal, also \( \Lambda^\sigma \leq H \) for all \( \sigma \in G \). By the Sylow theorem every \( \ell \)-Sylow subgroup of \( G \) is of the form \( \Lambda^\sigma \), hence the assertion. \( \square \)

Next we relate embedding problems for Sylow subgroups to embedding problems for open subgroups.

**Lemma 2.7.** Let \( \ell \) be a prime number, \( H \) a profinite group, \( \Lambda \) an \( \ell \)-Sylow subgroup, and \( E = (\alpha: \Lambda \to \Gamma, \beta: G \to \Gamma) \) a finite embedding problem with \( G \) an \( \ell \)-group. Let \( U \) be the family of pairs \( (U, \alpha_U) \) where \( U \) is an open subgroup of \( H \) containing \( \Lambda \) and \( \alpha_U: U \to G \) extends \( \alpha \).

(a) If there exists \( (U, \alpha_U) \in U \) such that \( E_U = (\alpha_U: U \to \Gamma, \beta: G \to \Gamma) \) has a solution \( \gamma_U: U \to G \), then \( \gamma = (\gamma_U)|_\Lambda \) is a solution of \( E \). Moreover if \( \gamma_U \) is proper, then \( \gamma \) is proper.

(b) If \( \ker \alpha \) is abelian and if \( E \) is solvable, then \( E_U \) is solvable.

**Proof.** The first assertion of (a), that \( \gamma \) is a solution of \( E \), is trivial. The second assertion of (a) follows from Lemma 2.5.

Now we assume that \( A = \ker \alpha \) is abelian and that \( E \) is solvable. Denote by \( b \) the class in \( H^2(\Gamma, A) \) that corresponds to the group extension

\[
\begin{array}{ccc}
1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}
\]

and write \( \alpha^*: H^2(\Gamma, A) \to H^2(\Lambda, A) \) for the inflation map. Then by Hoechsmann’s theorem [20, Proposition 3.5.9], \( \alpha^*(b) = 0 \).

Let \( (U, \alpha_U) \in U \) and let \( i: \Lambda \to U \) be the inclusion map. Then \( 0 = \alpha^*(b) = (\alpha_U \circ i)^*(b) = i^* \circ \alpha_U^*(b) \). Since \( |A| \) is a power of \( \ell \) and since \( [U : \Lambda] | [H : \Lambda] \), hence prime to \( \ell \), it follows that \( i^* \) is injective. So \( \alpha_U^*(b) = 0 \) and consequently \( E_U \) is solvable by [20, Proposition 3.5.9]. \( \square \)

The following lemma is used (before Propositions 4.3 and 5.15) to relate decompositions of sylow subgroups of \( \text{Gal}(K) \) and \( \text{Gal}(M) \) for field extensions \( M/K \).

**Lemma 2.8.** Let \( G \) be a profinite group, let \( N \) and \( P \) be closed subgroups, and put \( F = N \cap P \). Assume that \( N \trianglelefteq G \), \( G = NP \), and \( P = F \rtimes Z \), for some \( Z \leq P \). Then \( G = N \times Z \).
Proof. Since \( N \cap Z = N \cap P \cap Z = F \cap Z = 1 \) and \( NZ = NFZ = NP = G \), we get the assertion. \( \square \)

3. The cyclotomic decomposition

3.1. Proof of Observation 1.1. The following is a more general form of Observation 1.1.

**Observation 3.1.** Let \( K \) be a global field and \( \ell \neq \text{char}(K) \) a prime. If \( \ell = 2 \) and \( K \) is a number field, assume further that \( K \cap \mathbb{Q}(\mu_{\ell\infty}) \) is (totally) imaginary. Then \( \text{Gal}(K^{(\ell)}) \cong F \rtimes Z \), where \( Z = \text{Gal}(K^{(\ell)}(\mu_{\ell\infty}))/K^{(\ell)} \cong \mathbb{Z}_\ell \) and \( F = \text{Gal}(K^{(\ell)}(\mu_{\ell\infty})) \) is a free pro-\( \ell \)-group on countably many generators.

**Proof.** Since \( \mu_{\ell} \subseteq K^{(\ell)} \) by Lemma 2.6, and since \( K^{(\ell)} \cap \mathbb{Q}(\mu_{\ell\infty}) \) is (totally) imaginary if \( K \) is a number field and \( \ell = 2 \), one has \( \text{Gal}(\mathbb{Q}(\mu_{\ell\infty})/K^{(\ell)} \cap \mathbb{Q}(\mu_{\ell\infty})) \cong \mathbb{Z}_\ell \). Thus, \( Z \cong \mathbb{Z}_\ell \) as a nontrivial subgroup. The restriction map gives rise to a short exact sequence

\[
1 \longrightarrow F \longrightarrow \text{Gal}(K^{(\ell)}) \alpha \longrightarrow Z \longrightarrow 1.
\]

Since \( \mathbb{Z}_\ell \) is projective in the category of pro-\( \ell \)-groups, (5) splits and its splitting gives an isomorphism \( \text{Gal}(K^{(\ell)}) \cong F \rtimes Z \).

Let \( L = K^{(\ell)}(\mu_{\ell\infty}) \). Since \( L \) is totally imaginary and \([L_p : \mathbb{Q}_p]\) is divisible by \( \ell^\infty \) as a supernatural number for every rational prime \( p \) and a prime \( p \) of \( L \) lying over \( p \), the local Galois groups \( \text{Gal}(L_p) \) has \( \ell \)-th cohomological dimension 1 for every prime \( p \) of \( L \). The Albert-Brauer-Hasse-Noether theorem then shows that \( F = \text{Gal}(L) \) has \( \ell \)-th cohomological dimension 1, see [25, Chp. II §3.3 Proposition 9]. Thus, \( F \) is free pro-\( \ell \) [25, Chp. I §4 Corollary 2]. \( \square \)

3.2. Existence of splitting maps. For \( \ell = 2 \), if \( K \) has a real prime, then the sequence

\[
1 \longrightarrow \text{Gal}(K^{(2)}(\mu_{2\infty})) \longrightarrow \text{Gal}(K^{(2)}) \alpha \longrightarrow \text{Gal}(K^{(2)}(\mu_{2\infty}))/K^{(2)} \longrightarrow 1
\]

does not split. Otherwise, there is an embedding of

\[
\text{Gal}(K^{(2)}(\mu_{2\infty}))/K^{(2)} \cong \text{Gal}(K(\mu_{2\infty}))/K \cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}
\]

into \( \text{Gal}(K) \). But this is impossible since the normalizer of an involution \( \tau \) in an absolute Galois group is exactly \( \langle \tau \rangle \), cf. [2, Proposition 19.4.3(b)].

**Corollary 3.2.** Let \( K \) be a number field equipped with a real prime. Then

\[
\text{Gal}(K^{(2)}) \cong (F \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}/2,
\]

where \( F \) is a free pro-2 group on countably many generators.
Proof. By Observation 3.1, we have $\text{Gal}(K^{(2)}(\sqrt{-1})) \cong F \rtimes \mathbb{Z}_2$. Since $K$ has a real place, there is an embedding of $\overline{K}$ into $\mathbb{C}$ such that the complex conjugation $\tau$ fixes $K$. Thus, the restriction of $\tau$ to $\overline{K}$ is an involution which restricts to the nontrivial automorphism of $K^{(2)}(\sqrt{-1})/K^{(2)}$. This gives a splitting of the extension

$$1 \longrightarrow F \rtimes \mathbb{Z}_2 \longrightarrow \text{Gal}(K^{(2)}) \longrightarrow \text{Gal}(K^{(2)}(\sqrt{-1})/K^{(2)}) \longrightarrow 1,$$

proving the desired result. □

If $K$ is totally imaginary but $K \cap \mathbb{Q}(\mu_{2\infty})$ is totally real, by Artin’s theorem $\text{Gal}(K^{(2)})$ has no involutions and hence even the sequence

$$1 \rightarrow F \rtimes \mathbb{Z}_2 \rightarrow \text{Gal}(K^{(2)}) \rightarrow \text{Gal}(K^{(2)}(\sqrt{-1})/K^{(2)}) \rightarrow 1,$$

does not split.

3.3. Henselian splitting maps. We also note that a splitting $s : \mathbb{Z} \rightarrow \text{Gal}(K^{(\ell)})$ of (5) can be chosen so that $s(\mathbb{Z})$ is generated by a lift of the Frobenius automorphism at any prime $p$ of $K$ such that $N(p) \not\equiv 1 \mod \ell^{s+1}$, where $\ell^s$ is the number of $\ell$-power roots of unity in $K^{(\mu_{\ell^s})}$. Indeed, letting $\mathfrak{p}$ be a prime of $\tilde{K}$ dividing $p$, the condition on $N(p)$ forces $\mathfrak{p}$ to be inert in $K^{(\mu_{\ell}\infty)}/K^{(\mu_{\ell^s})}$ and in $K^{(\mu_{\ell^s})}/K^{(\ell)}$. Let $\sigma$ be any lift of the Frobenius of $\mathfrak{p}$ in $K^{(\mu_{\ell^s})}/K^{(\ell)}$ to $\tilde{K}$. As $\mathfrak{p}$ is inert in $K^{(\mu_{\ell^s})}/K^{(\ell)}$, the restriction of $\sigma$ to $K^{(\mu_{\ell^s})}$ generates $\mathbb{Z}$ and hence induces a splitting $s$ of (5).

3.4. Generators of the tame part of $F$. Let $L = K^{(\ell)}(\mu_{\ell\infty})$, $P$ (resp. $T$) the set of primes of $L$ (resp. primes of $L$ lying either over $\infty$ or $\ell$), and let $L_T$ be the maximal extension of $L$ unramified away from $T$. The number theoretical analogue of Riemann’s existence theorem [20, Corollary 10.5.2] gives a canonical set of generators of $\text{Gal}(L_T)$. Namely, it shows that $\text{Gal}(L_T)$ decomposes as the free product of its local Galois groups:

$$\text{Gal}(L_T) \cong \prod_{p \in P \setminus T} \text{Gal}(L_p).$$

Here, $\prod$ denotes the free pro-$\ell$ product over the profinite index space associated to $P \setminus T$, see [20, §10.1]. Note that $\text{Gal}(L_p)$ is the inertia group, and hence is cyclic for every prime $p \not\in T$. We also note that the abelianization of the remaining part $\text{Gal}(L_T/L)$ can be studied using Iwasawa theory, cf. §5.8.

4. The action via $\mathbb{Z}$-embedding problems

In this section we study the action in the cyclotomic decomposition via $\mathbb{Z}$-embedding problems. We consider the following more general setup. Let $K$ be a Hilbertian field and $\ell \neq \text{char}(K)$ a prime number. If $\ell = 2$ and $\text{char}(K) = 0$, assume that $\sqrt{-1} \in K$. As before set $L = K^{(\ell)}(\mu_{\ell\infty})$, $Z = \text{Gal}(L/K^{(\ell)})$, and $F = \text{Gal}(L)$. Theorem 1.2 is then a special case of:
Theorem 4.1. Every finite split $Z$-embedding problem for $F$ is properly solvable.

To prove the theorem we first deal with split embedding problems for $\text{Gal}(K^{(\ell)})$:

**Proposition 4.2.** Let $(\phi: \text{Gal}(K^{(\ell)}) \to \Gamma, \pi: G \to \Gamma)$ be a finite split embedding problem for $\text{Gal}(K^{(\ell)})$ with $G$ an $\ell$-group. Then $(\phi, \pi)$ is properly solvable.

**Proof.** Let $N$ be the fixed field of $\ker \phi$, and so $N/K^{(\ell)}$ is Galois and the map $\phi$ decomposes as $\phi = \phi' \circ r$, where $r: \text{Gal}(K^{(\ell)}) \to \text{Gal}(N/K^{(\ell)})$ is the restriction map and $\phi': \text{Gal}(N/K^{(\ell)}) \to \Gamma$ is an isomorphism. We may replace $\Gamma$ by $\text{Gal}(N/K^{(\ell)})$ and the maps $\pi, \phi$ by $(\phi')^{-1} \circ \pi$ and $r$, respectively, to assume that $\Gamma = \text{Gal}(N/K^{(\ell)})$ and $\phi$ is the restriction map.

By [13, Theorem 5.8.3] $K^{(\ell)}$ is ample. Hence by [13, Theorem 5.9.2] there exist a Galois extension $F/K^{(\ell)}(x)$ such that $\text{Gal}(F/K^{(\ell)}(x)) \cong G$, $N$ is the algebraic closure of $K$ in $F$, and the restriction map $\text{Gal}(F/K^{(\ell)}(x)) \to \text{Gal}(N(x)/K^{(\ell)}(x))$ coincides with $\pi$ (after identifying $\text{Gal}(F/K^{(\ell)}(x)) = G$, $\text{Gal}(N(x)/K^{(\ell)}(x)) = \Gamma$).

Let $K_0$ be a finite subextension of $K^{(\ell)}(x)/K$ to which the above descends to as follows: there exist $N_0/K_0$ Galois with Galois group $\Gamma$ such that $N = N_0 K^{(\ell)}$ and $F_0/K_0$ Galois with group $G$ such that $F = F_0 K^{(\ell)}$, $N_0$ is the algebraic closure of $K_0$ in $F_0$, $G = \text{Gal}(F_0/K_0(x))$ and the restriction map $\text{Gal}(F_0/K(x)) \to \text{Gal}(N_0/K_0)$ coincides with $\pi$.

Note that $K_0$ is Hilbertian as a finite extension of $K$ [6, Proposition 16.11.1]. Hence there exists $a \in K_0$ such that the prime $(x-a)$ of $K_0(x)$ is inert in $F_0$. Let $M$ be the residue field of $F_0$ at $x = a$. Then $M/K_0$ is Galois with Galois group $G$, $N_0 \subseteq M$, and the restriction map $\text{Gal}(M/K_0) \to \text{Gal}(N_0/K_0)$ coincides with $\pi$. In other words, if $\phi_0: \text{Gal}(K_0) \to \text{Gal}(M/K_0) = G$ and $\psi: \text{Gal}(K_0) \to \text{Gal}(M/K_0)$ are the restriction maps, then $\psi$ is a proper solution of $(\phi_0, \pi)$. Then $\psi|_{\text{Gal}(K^{(\ell)})}$ is a solution of $(\phi, \pi)$ which is proper by Lemma 2.5. \qed

**Proof of Theorem 4.1.** Let $(\phi: F \to \Gamma, \pi: G \to \Gamma)$ be a finite split $Z$-embedding problem with $G$ an $\ell$-group. Since $\text{Gal}(K^{(\ell)}) = F \rtimes Z$, we may extend $(\phi, \pi)$ to a split embedding problem

$$(\phi': F \rtimes Z \to \Gamma \rtimes Z, \pi': G \rtimes Z \to \Gamma \rtimes Z)$$

for $\text{Gal}(K^{(\ell)})$, where $\phi'(x, z) = (\phi(x), z)$ and $\pi'(g, z) = (\pi(g), z)$, for every $x \in F$, $z \in Z$, and $g \in G$.

Since $Z$ acts on the finite group $G$ continuously, the kernel of the action is an open subgroup of $Z$, so it contains $\ell^r Z$, for some $r \geq 1$. Composing with the natural projection $Z \to Z/\ell^r Z$ we obtain a finite embedding problem

$$(\phi'': F \rtimes Z \to \Gamma \rtimes (Z/\ell^r Z), \pi'': G \rtimes (Z/\ell^r Z) \to \Gamma \rtimes (Z/\ell^r Z))$$
for $K^{(\ell)}$ and we have the commutative diagram of profinite groups

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F \rtimes Z \\
\downarrow \\
G \rtimes Z
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\phi' \\
\downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\phi'' \\
\downarrow
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Gamma \rtimes Z \\
\downarrow \\
\Gamma \rtimes (Z/\ell Z)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\phi'' \\
\downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi' \\
\downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi'' \\
\downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Gamma \rtimes (Z/\ell Z) \\
\downarrow
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

By Proposition 4.2, there exists a proper solution $\psi''$ of $(\phi'', \pi'')$. Note that as $\ker \phi'' = \ell^r \ker \phi'$, we have $\ker \psi'' \ker \phi' = \ell^k \ker \phi'$ for some $k \geq r$. We claim that $k = r$ and hence

$$
\ker \psi'' \ker \phi' = \ker \phi''.
$$

Indeed, if $k > r$, we have $\ker \psi'' \ker \phi' \subseteq \ell^{r+1} \ker \phi'$ and hence $\pi''$ factors through the natural projection $\Gamma \rtimes Z/\ell^{r+1}Z \to \Gamma \rtimes Z/\ell^rZ$. The latter does not split, contradicting the splitting of $\pi''$, and proving the claim.

Since $G \rtimes Z$ is the fiber product of $\Gamma \rtimes Z$ and $G \rtimes (Z/\ell^rZ)$ over $\Gamma \rtimes (Z/\ell^rZ)$, we obtain a solution $\psi' = \psi'' \times_{\phi''} \phi'$ of $(\phi', \pi')$. We next show that $\psi'$ is proper. We have $\ker \psi' = \ker \psi'' \cap \ker \phi'$. Hence (7) gives:

$$
\ker \phi' / \ker \psi' = \ker \phi' / (\ker \psi'' \cap \ker \phi') \cong (\ker \phi' \ker \psi'') / \ker \psi'' = \ker \phi' / \ker \psi''.
$$

Thus, $[\ker \phi' : \ker \psi'] = [\ker \phi'' : \ker \psi''] = [G : \Gamma]$, showing that $\psi'$ is surjective. Since $\pi'$ and $\phi'$ are the identity maps on $Z$, $\psi'(F) = \text{Im} \psi' \cap G$. As $\psi'$ is proper, we get $\psi'(F) = G$. Thus, the restriction of $\psi'$ to $F$ is a proper solution of the $Z$-embedding problem $(\phi, \pi)$.

As oppose to split embedding problems, Frattini $Z$-embedding problems need not be solvable. We now descend these problems to cyclotomic extensions of number fields.

For a number field $K(\mu_\ell) \subseteq K' \subseteq K^{(\ell)}$, Lemma 2.8 applied with $N = \text{Gal}(K'(\mu_{\ell\infty}))$ and $P = \text{Gal}(K^{(\ell)})$ shows that the splitting $\text{Gal}(K^{(\ell)}) = \text{Gal}(L) \rtimes Z$ induces a splitting $\text{Gal}(K') = \text{Gal}(K'(\mu_{\ell\infty})) \rtimes Z$ such that the restriction $\text{Gal}(L) \to \text{Gal}(K'(\mu_{\ell\infty}))$ is a $Z$-homomorphism.

**Proposition 4.3.** Let $(\phi : \text{Gal}(L) \to \Gamma, \pi)$ be a $Z$-embedding problem. Then there is a number field $K(\mu_\ell) \subseteq K' \subseteq K^{(\ell)}$ and a $Z$-embedding problem

$$(\phi' : \text{Gal}(K'(\mu_{\ell\infty})) \to \Gamma, \pi)$$

whose restriction to $L$ is $(\phi, \pi)$. If furthermore $\ker \pi$ is abelian, then for every such $K'$ and $\phi'$, $(\phi, \pi)$ is solvable if and only if $(\phi', \pi)$ is solvable. In particular, if $\pi$ is $Z$-Frattini, $(\phi, \pi)$ is properly solvable if and only if $(\phi', \pi)$ is properly solvable.
Proof. Let \( N := \text{Gal}(K(\mu_{\ell^\infty})) \) be a \( \mathbb{Z} \)-group via the induced splitting \( \text{Gal}(K(\mu_{\ell})) = N \times \mathbb{Z} \). By Lemma 2.4, \( \phi \) extends to \( \phi' : U \to \Gamma \) for some open \( \mathbb{Z} \)-subgroup \( U \leq N \).

Let \( K' \) be the fixed field of \( U \times \mathbb{Z} \). Since \( U \mathbb{Z} = U\text{Gal}(L)\mathbb{Z} \supseteq \text{Gal}(K(\ell)) \), we have \( K' \subseteq K(\ell) \). Since \( U \times \mathbb{Z} \) is open in \( \text{Gal}(K(\mu_{\ell})) \), \( K' \) is a number field. Since \( U \leq N \), \( \mu_{\ell^\infty} \) is fixed by \( U \) and \( K'(\mu_{\ell^\infty}) \) is the fixed field of \( U \). Thus, \( \phi' \) is the desired \( \mathbb{Z} \)-homomorphism. The equivalence for solvability follows by Lemma 2.7. Thus, the equivalence for proper solvability follows by Lemma 2.2. \( \square \)

Explicit examples of nonsolvable Frattini \( \mathbb{Z} \)-embedding problems appear in the following section (Proposition 5.8).

5. Action on \( F/F^\ell[F,F] \)

Let \( \text{Gal}(K(\ell)) = F \times \mathbb{Z} \) be the cyclotomic decomposition for a global field \( K \) and a prime \( \ell \neq \text{char}(K) \). If \( K \) is a number field and \( \ell = 2 \) we assume \( \sqrt{-1} \in K \).

Recall that \( Z = \text{Gal}(L/K(\ell)) \cong \mathbb{Z}_\ell \) and \( F = \text{Gal}(L) \) is a free pro-\( \ell \)-group, where \( L = K(\ell)(\mu_{\ell^\infty}) \).

To find the indecomposable direct \( \mathbb{Z} \)-summands of \( \overline{F} = F/F^\ell[F,F] \), we apply the theory of Ulm invariants for countably generated \( \ell \)-torsion profinite \( \mathbb{Z} \)-modules, basing on \([10, \S 11, 12]\) as described in the following section.

5.1. \( \mathbb{Z} \)-modules. Let \( M \) be a countably generated profinite \( \mathbb{Z} \)-module which is \( \ell \)-torsion, i.e. \( \ell \cdot M = 0 \). That is, \( M \) is a profinite \( \mathbb{F}_\ell[[Z]] \)-module. The ring \( \mathbb{F}_\ell[[Z]] \) is a discrete valuation ring whose maximal ideal is the augmentation ideal \( I = (\sigma - 1) \), where \( \sigma \) is a generator of \( Z \). Thus, \( I^n M, n \in \mathbb{N} \), is a fundamental system of open neighborhoods of \( 0 \in M \).

As \( M \) is profinite its (Pontryagin) dual \( \hat{M} := \text{Hom}(M, \mathbb{F}_\ell) \) is a discrete \( \mathbb{F}_\ell[[Z]] \)-module with the \( Z \)-action \( (\tau f)(m) = f(\tau^{-1}m) \) for all \( m \in M, \tau \in Z, \) and \( f \in \hat{M} \). Moreover, \( \hat{M} \) is \( \mathbb{F}_\ell[[Z]] \)-torsion since every homomorphism \( f \in \hat{M} \) factors through \( M/I^n M \) for some \( n \in \mathbb{N} \), so \( I^n f = 0 \).

**Definition 5.1.** For a discrete torsion \( \mathbb{F}_\ell[[Z]] \)-module \( N \), let \( N^Z \) be the submodule of all elements of \( N \) fixed by \( Z \), or equivalently annihilated by \( I \). Consider the descending transfinite sequence \( I^n N \) defined by \( I^{n+1} N := I(I^n N) \) for each ordinal \( n \) and \( I^n N = \bigcap_{k<n} I^k N \) for each limit ordinal \( n \). For every ordinal \( n \), the Ulm invariant \( U_n(N) \) is the cardinality of \( (I^n N)^Z/(I^{n+1} N)^Z \).

The following proposition shows that the finite Ulm invariants \( U_n(\hat{M}) \) already determine the finite \( \mathbb{Z} \)-summands of \( M \).

Since \( \mathbb{F}_\ell[[Z]] \) is a complete discrete valuation ring, there is a unique cyclic \( \mathbb{F}_\ell[[Z]] \)-module \( V_n := \mathbb{F}_\ell[[Z]]/I^n \) of dimension \( n \) over \( \mathbb{F}_\ell \).

**Proposition 5.2.** Let \( M \) be a profinite \( \mathbb{F}_\ell[[Z]] \)-module. Then \( U_{n-1}(\hat{M}) \) is the multiplicity of \( V_n \) as a direct \( \mathbb{Z} \)-summand of \( M \), for every \( n \in \mathbb{N} \). Furthermore,
for every $N \in \mathbb{N}$, $M = M_{\leq N} \times M_{>N}$, where
\[
M_{\leq N} \cong \prod_{n \leq N} V_{n}^{U_{n-1}(\hat{M})},
\]
$M_{>N}$ has no direct $Z$-summands of dimension $\leq N$ over $F_{\ell}$.

Proposition 5.2 follows from the theory of Ulm invariants and its proof is given
in §5.9.

For $\eta \in \hat{M}$ define $ht(\eta)$ to be the maximal $n$ such that $\eta \in I^n\hat{M}$ if such an $n$
exists and $\infty$ otherwise\(^1\). Thus, the Ulm invariants can be expressed using the
height function as:
\[
U_n(\hat{M}) = \left\{ \phi \in \hat{M}^Z \mid ht(\phi) \geq n \right\} / \left\{ \phi \in \hat{M}^Z \mid ht(\phi) > n \right\},
\]
for $n \in \mathbb{N} \cup \{0\}$, and
\[
I^n\hat{M} = \left\{ \phi \in \hat{M} \mid ht(\phi) = \infty \right\}.
\]

5.2. The height via $Z$-embedding problems. To compute the finite Ulm invariants of $\hat{F}$ we first interpret the height in terms of $Z$-embedding problems. Let
$\pi_{n,m} : V_n \to V_m$, and $\pi_m : F_{\ell}[[Z]] \to V_m$ denote the natural projections.

Proposition 5.3. Let $M$ be a profinite $F_{\ell}[[Z]]$-module, $k \in \mathbb{N}$, and $\eta \in \hat{M}$.
Fix an $F_{\ell}[[Z]]$-isomorphism $\hat{\eta} : \hat{V}_m \to \hat{M}$ whose image is $F_{\ell}[[Z]]\eta$, where $m = \dim_{F_{\ell}} F_{\ell}[[Z]]\eta$. Let $\pi^* : M \to V_m$ be its dual map. Then $\eta \in I^k\hat{M}$ if and only if
the embedding problem $(\hat{\eta}^*, \pi_{m+k,m})$ is solvable.

The proof is based on the following lemma:

Lemma 5.4. (a) For $0 \leq k \leq n$, $f \in I^k\hat{V}_n$ if and only if $f(I^{n-k}V_n) = 0$. In
particular, the image of the dual map $\pi_{n,m}^* : \hat{V}_m \to \hat{V}_n$ is $I^{n-m}\hat{V}_n$.
(b) The module $\hat{V}_n$ is cyclic, hence $\hat{V}_n \cong V_n$. Moreover, an element $f \in \hat{V}_n$
generates $\hat{V}_n$ if and only if $f(I^{n-1}V_n) \neq 0$.

Proof. Let $R_i := \{ f \mid f(I^iV_n) = 0 \}$, $0 \leq i \leq n$. Note that since $\dim_{F_{\ell}} I^iV_n = n - i$, one has $\dim_{F_{\ell}} R_i = i$ for $0 \leq i \leq n$.

Fix a generator $\sigma$ of $Z$. If $f = g^{(\sigma-1)^k}$ for $g \in \hat{V}_n$, then $f(I^{n-k}V_n) = g(I^nV_n) = 0$.
Hence $I^k\hat{V}_n \subseteq R_{n-k}$. Applying the dimension formula to the linear transformation $(\sigma - 1)^k : \hat{V}_n \to \hat{V}_n$ given by $x \to (\sigma - 1)^k x$, one has:
\[
\dim_{F_{\ell}} I^k\hat{V}_n = \dim_{F_{\ell}} \hat{V}_n - \dim_{F_{\ell}} \{ f \mid f^{(\sigma-1)^k} = 0 \} = n - \dim_{F_{\ell}} R_k = \dim_{F_{\ell}} R_{n-k}.
\]
Hence $R_{n-k} = I^k\hat{V}_n$. The second assertion in Part (a) follows since $f \in \text{Im} \pi_{n,m}^*$ if and only if $f(I^mV_n) = 0$.

\(^1\)This height identifies with the height function defined in [10].
Since the dimension of $\hat{V}_n$ is $n$, $F\ell[[Z]]f = \hat{V}_n$ if and only if the sequence
\[ F\ell[[Z]]f \supset I f \supset \ldots \supset I^{n-1} f \supset I^n f = 0 \]
is strictly descending. The latter condition holds if and only if $I^{n-1} f \neq 0$ or equivalently $f(I^{n-1}V_n) \neq 0$. \hfill \Box

**Proof of Proposition 5.3.** Let $n := m + k$. The $Z$-embedding problem $(\tilde{\eta}^*, \pi_{n,m})$ has a solution $\psi: M \to V_n$ if and only if its dual $\psi^*: \hat{V}_n \to \hat{M}$ satisfies $\pi_{n,m}^* \circ \psi^* = \tilde{\eta}^*$, i.e. makes the following diagram commutative:

\begin{equation}
\begin{array}{ccc}
\hat{M} & \xrightarrow{\psi^*} & \hat{V}_n \\
\downarrow & & \uparrow_{\pi_{n,m}^*}
\end{array}
\end{equation}

For the “if” implication assume there is a solution $\psi: M \to V_n$. By Lemma 5.4.(a), we have:
\[ \eta \in \text{Im} \tilde{\eta}^* = \text{Im} \psi^* \circ \pi_{n,m}^* = \psi^*(I^k V_m) = I^k \text{Im} \psi^* \subseteq I^k \hat{M}. \]

For the converse assume $\eta = (\sigma - 1)^k \eta_n$ for some $\eta_n \in \hat{M}$. Denote $f_m := (\tilde{\eta}^*)^{-1}(\eta)$. As $f_m$ is a generator of $\hat{V}_n$, it satisfies $f_m^{(\sigma-1)^{m-1}} \neq 0$. By Lemma 5.4.(a), there is an $f_n \in V_n$ such that $f_n^{(\sigma-1)^{m-1}} = \pi_{n,m}^*(f_m)$. Since
\[ f_n^{(\sigma-1)^{m-1}} = \pi_{n,m}^*(f_m^{(\sigma-1)^{m-1}}) \neq 0, \]

Lemma 5.4.(b) implies that $f_n$ generates $\hat{V}_n$. Since in addition $I^n \eta_n = 0$, we may define $\psi^*: \hat{V}_n \to \hat{M}$ to be the unique $Z$-homomorphism for which $\psi^*(f_n) = \eta_n$. Then
\[ \psi^* \circ \pi_{n,m}^*(f_m) = \psi^*(f_n^{(\sigma-1)^k}) = \eta_n^{(\sigma-1)^k} = \eta = \tilde{\eta}(f_m). \]

Since $\psi^* \circ \pi_{n,m}^*$ and $\tilde{\eta}$ agree on a generator of $\hat{V}_n$, they coincide. Hence $\psi = (\psi^*)^*$ is a solution of $(\tilde{\eta}^*, \pi_{n,m})$, as required. \hfill \Box

Following Proposition 5.3, we define the **height** $ht(\phi)$ of a $Z$-homomorphism $\phi: M \to V_m$ to be the maximal $k$ for which $(\phi, \pi_{m+k,m})$ is solvable if such a $k$ exists, and $\infty$ otherwise. Note that by Proposition 5.3, for $\eta \in \hat{M}$, $ht(\eta) = ht(\tilde{\eta}^*)$.

Also note that an element $\eta \in \hat{M}^Z$ is a $Z$-homomorphism. By identifying $F\ell$ with $V_1$, we may choose $\tilde{\eta}^*$ to be the dual map of $\eta$. Hence, the height of such $\eta$ as a $Z$-homomorphism and its height as an element of $\hat{M}$ coincide.
5.3. A local global principle. In view of Propositions 5.2 and 5.3, the finite direct summands of $\mathcal{F}$ can be computed using $\mathbb{Z}$-embedding problems of the form $(\phi : F \to V_n, \pi_{n,m} : V_n \to V_m)$. To determine the solvability of such embedding problems, we first establish a local global principle.

For a prime $p$ of $L$, let $Z_p$ be the local Galois group $\text{Gal}(L_p/K_p^{(\ell)})$. Since $\text{Gal}(L_p)$ is a $\mathbb{Z}_p$-group, the restriction $(\phi_p : \text{Gal}(L_p) \to V_n, \pi_{n,m})$ of $(\phi, \pi)$ to $L_p$ is a $\mathbb{Z}_p$-embedding problem. Furthermore, if $\psi : \text{Gal}(L) \to V_n$ is a solution of $(\phi, \pi_{n,m})$ then the restriction $\psi_p : \text{Gal}(L_p) \to V_n$ is a solution of $(\phi_p, \pi_{n,m})$ for every prime $p$ of $L$. We claim that the converse also holds:

**Proposition 5.5.** A $\mathbb{Z}$-embedding problem $(\phi : \text{Gal}(L) \to V_n, \pi_{n,m})$ is solvable if and only if $(\phi_p, \pi_{n,m})$ is solvable for every prime $p$ of $L$. In particular, $\text{ht}(\phi) = \min_p \text{ht}(\phi_p)$ where $p$ runs over all primes of $L$.

**Proof.** By Proposition 4.3, there is a global field $K(\mu_\ell) \leq K' \leq K^{(\ell)}$ such that $\phi$ extends to a $\mathbb{Z}$-homomorphism $\phi' : \text{Gal}(L') \to V_n$, where $L' := K'(\mu_{\ell\infty})$. We identify $Z = \text{Gal}(L/K^{(\ell)})$ and $\text{Gal}(L'/K')$ via the restriction map. For every prime $p$ of $L$, this gives an identification of $Z_p$ with the decomposition group of $p \cap L'$ in $L'/K'$.

Let $A := \ker \pi_{n,m}$. Then $A$ is a $\text{Gal}(K')$-module via the restriction $\text{Gal}(K') \to Z$. We claim that the map

$$\rho : H^2(\text{Gal}(K'), A) \to \prod_p H^2(\text{Gal}(K'_p), A)$$

is injective, where $p$ runs over all primes of $K'$. Let $\hat{A} = \text{Hom}(A, \mu_{\ell})$ be the dual $\text{Gal}(K')$-module with the action $f^\sigma(x) = f(x\sigma^{-1})^\sigma$ for $\sigma \in \text{Gal}(K')$, $x \in A$, and $f \in \hat{A}$. Let $K'(\hat{A})$ be the fixed field of the centralizer $H \leq \text{Gal}(K')$ of $\hat{A}$ under the action of $\text{Gal}(K')$. Since $\text{Gal}(K')$ acts trivially on $\mu_{\ell}$ and $\text{Gal}(L')$ acts trivially on $A$, the map $\text{Gal}(K') \to \text{Aut}(\hat{A})$ splits through $Z \cong \text{Gal}(L'/K')$. Thus, $H$ is an open subgroup of $\text{Gal}(K')$ which contains $\text{Gal}(L')$, and hence $G' := \text{Gal}(K'(\hat{A})/K')$ is a finite cyclic $\ell$-group as a quotient of $Z$. By the Poitou-Tate duality theorem [21, Satz 4.5] (or [20, Theorem 8.6.8]), $\rho$ is injective if and only if

$$\rho' : H^1(G', \hat{A}) \to \prod_p H^1(G'_p, \hat{A})$$

is injective, where $p$ runs over all primes of $K'$. Here $G'_p = \text{Gal}(K'(\hat{A})_{\mathfrak{p}}/K'_p)$ for some prime $\mathfrak{p}$ of $K'(\hat{A})$ lying over $p$. Since $G'$ is cyclic, by Chebotarev’s density theorem there are infinitely many primes $p$ for which $G'_p = G'$. Thus, $\rho'$ and hence $\rho$ are injective, as claimed.

Let $\phi : \text{Gal}(K') \to V_m \times Z$ be the map given by the composition of the isomorphism $\text{Gal}(K') \cong \text{Gal}(L') \times Z$ and the map $(\phi', \text{id}) : \text{Gal}(L') \times Z \to V_m \times Z$. and let $\hat{\pi}_{n,m} : V_n \times Z \to V_m \times Z$ be the map defined by $\hat{\pi}_{n,m}(x, z) = (\pi_{n,m}(x), z)$. Since the $Z$-embedding problem $(\phi', \pi_{n,m})$ is solvable if and only if the embedding problem
(\tilde{\phi}, \tilde{\pi}_{n,m}) is solvable, it suffices to show the latter. Similarly, \((\phi_p, \pi_{n,m})\) is solvable, the restriction \((\tilde{\phi}_p, \tilde{\pi}_{n,m})\) of \((\tilde{\phi}, \tilde{\pi}_{n,m})\) to \(\text{Gal}(K'_p) = \text{Gal}(L'_p) \times \mathbb{Z}_p\) is solvable.

The maps \(\tilde{\phi}, \tilde{\phi}_p\) form the following commutative diagram:

\[
\begin{array}{ccc}
H^2(V_m \times Z, A) & \longrightarrow & \prod_p H^2(V_m \times Z_p, A) \\
\tilde{\phi}^* & \downarrow & \downarrow \prod_p \tilde{\phi}_p^* \\
H^2(\text{Gal}(K'), A) & \longrightarrow & \prod_p H^2(\text{Gal}(K'_p), A),
\end{array}
\]

where \(V_m \times Z\) acts on \(A\) via the projection onto \(Z\), \(\tilde{\phi}\) is the restriction map, and \(p\) runs through all primes of \(L'\).

Since the action of \(V_m \times Z\) on \(A\) via the extension \(\tilde{\pi}_{n,m}\) factors through the projection onto \(Z\), it agrees with the above chosen action. Let \(\alpha_{n,m} \in H^2(V_m \times Z, A)\) be the class defined by \(\tilde{\pi}_{n,m}\), and \(\alpha_{n,m}^{(p)}\) be the \(p\)-th component of \(\tilde{\phi}(\alpha_{n,m})\). Since \((\tilde{\phi}_p, \tilde{\pi}_{n,m})\) is solvable, \(\tilde{\phi}_p^*(\alpha_{n,m}^{(p)}) = 0\) for all \(p\). By (11), \(\rho \circ \tilde{\phi}^*(\alpha_{n,m}) = 0\). Since \(\rho\) is injective, \(\tilde{\phi}^*(\alpha_{n,m}) = 0\) and hence \((\tilde{\phi}, \tilde{\pi}_{n,m})\) is solvable, as required. \(\square\)

### 5.4. The local height

The above local global principle reduces the computation of the global height \(ht(\phi)\) of a \(Z\)-homomorphism \(\phi: F \to V_m\), to the computation of the local heights \(ht(\phi_p)\) for all primes \(p\) of \(L\). We compute the latter using Iwasawa theory [9].

A homomorphism \(\phi: \text{Gal}(L) \to G\) is **unramified** (resp. **tamely ramified**) at a prime \(p\) of \(L\) if the fixed field of \(\text{ker(\phi)}\) is unramified (resp. tamely ramified) over \(L\) at \(p\).

**Proposition 5.6.** Let \(p\) be a prime of \(L\) and \(\ell^t := [Z: Z_p]\). Let \(\phi: F \to V_m\) a \(Z\)-homomorphism. Then:

(a) Either \(ht(\phi_p) = \infty\) or \(\ell^t - m \leq ht(\phi_p) < \ell^t\);

(b) If \(\phi\) is unramified, then \(ht(\phi_p) = \infty\);

(c) If \(\phi\) is ramified nontrivially and tamely, then \(\ell^t - m \leq ht(\phi_p) < \ell^t\).

**Proof.** If \(p\) is infinite, \(p\) is complex since \(L\) contains all \(\ell\)-power roots of unity. Hence for infinite \(p\), \(\phi_p\) is trivial and \(ht(\phi_p) = \infty\).

Assume \(p\) is a finite prime. By Proposition 4.3, \(\phi\) extends to a \(Z\)-homomorphism \(\phi': \text{Gal}(L') \to V_m\), where \(L' = K'_{(\mu_{\ell^m})}\) and \(K'/K(\mu_\ell)\) is a finite extension. Moreover, \(ht(\phi_p) = ht(\phi_p^{(L')})\) for any prime \(p\) of \(L\). Let \(G := \text{Gal}(L'_p)\) and \(G^{\text{ab}}\) (resp. \(G\)) the maximal abelian (resp. elementary abelian) quotient of \(G\) viewed as \(Z_p\)-groups.

Iwasawa’s theorem [9, Theorem 25] gives a \(Z_p\)-isomorphism \(s: G^{\text{ab}} \to T(\mu) \times \Lambda^d\), where \(T(\mu)\) is the Tate module \(T(\mu) := \text{lim}_{\mu_{\ell^m}} \mu_{\ell^m}, \Lambda := \mathbb{Z}_p[[\mathbb{Z}_p]],\) and \(d = [K'_p: \mathbb{Q}_\ell]\) if \(p\) lies over \(\ell\) and 0 otherwise. Moreover, \(s^{-1}\) is obtained as an inverse limit of the reciprocity maps \(r_E: E^\times \to \text{Gal}(E)^{ab}\) where \(E\) runs through finite intermediate extensions \(K' \subseteq E \subseteq L'\), see [9, End of Pg. 319]. Since \(r_E\) maps the units of
to the inertia subgroup of \( \text{Gal}(E)^{\text{ab}} \), the inverse limit \( T(\mu) \) of \( \ell \)-power roots of unity is mapped under \( s^{-1} \) to the inertia subgroup of \( G^{\text{ab}} \).

As \( \Lambda/\ell \Lambda \cong \mathbb{F}_\ell[[Z]] \) and \( T(\mu)/\ell T(\mu) \cong V_1 \) as \( \mathbb{Z}_\ell \)-modules, \( s \) gives a \( \mathbb{Z}_\ell \)-isomorphism
\[
\overline{G} = G^{\text{ab}}/\ell G^{\text{ab}} \cong V_1 \times \mathbb{F}_\ell[[Z]]^d.
\]

Let \( G_1 \) be the direct \( Z \)-summand of \( \overline{G} \) which corresponds to \( V_1 \) under this isomorphism. Hence, \( G_1 \) is contained in the inertia subgroup of \( \overline{G} \).

We separate into two cases as to whether \( G_1 \) is contained in \( \ker \phi'_p \). If \( G_1 \leq \ker \phi'_p \), then \( \phi'_p \) splits through \( \mathbb{F}_\ell[[Z]]^d \). As \( \mathbb{F}_\ell[[Z]]^d \) is free as an \( \mathbb{F}_\ell[[Z]] \)-module, the embedding problem \( (\phi'_p, \pi_{n+m,m}) \) is solvable for all \( n \in \mathbb{N} \). Thus, \( \text{ht}(\phi_p) = \text{ht}(\phi'_p) = \infty \). This is in particular the case if \( \phi'_p \) is unramified, proving (b).

On the other hand if \( G_1 \not\leq \ker \phi'_p \), we claim that \( \ell^t - m \leq \text{ht}(\phi'_p) < \ell^t \). To show that \( \ell^t - m \leq \text{ht}(\phi'_p) \), it suffices to show that \( (\phi'_p, \pi_{n,m}) \) is solvable if \( n - m = \ell^t - m \), that is, \( n = \ell^t \). Let \( \sigma \) be a generator of \( Z \). Since \( (\sigma^\ell - 1) = I^{\ell^t} \), and since \( [Z : \mathbb{Z}_\ell] = \ell^t \), the \( Z \)-module \( V_\mu \) is the trivial \( \mathbb{Z}_\ell \)-module \((\mathbb{F}_\ell)^a\). In particular, the \( \mathbb{Z}_\ell \)-embedding problem \((\phi'_p, \pi_{n,m})\) is solvable, as claimed.

To show \( \text{ht}(\phi'_p) < \ell^t \), assume \( n - m = \ell^t \), that is, \( n = m + \ell^t \). Furthermore, assume on the contrary that \((\phi'_p, \pi_{n,m})\) is solvable. Hence, its restriction
\[
(\phi''_p : G_1 \to V_m, \pi_{n,m})
\]
to \( G_1 \) has a solution, say \( \psi_p \). Since \( G_1 \) is fixed by \( \mathbb{Z}_\ell \) so is its image \( J := \text{Im} \psi_p \).

Thus, \( I^{\ell^t} \cdot J = (\sigma^{\ell^t} - 1) \cdot J = 0 \). Since the kernel of the map \( V_n \to V_n, x \to x^{\sigma^{\ell^t} - 1} \) is \( I^m V_n \), we have \( J \subseteq I^m V_n = \ker \pi_{n,m} \). Hence, \( \text{Im}(\pi_{n,m} \circ \psi_p) = \text{Im}(\phi''_p) = \{0\} \). But \( \text{Im}(\phi''_p) = 0 \) if \( G_1 \not\subseteq \ker \phi'_p \). This contradiction proves the claim and Part (a).

If \( \phi_p \) ramifies nontrivially and tamely, \( p \) does not divide \( \ell \), so \( d = 0 \) and \( \overline{G} = G_1 \). As \( \phi_p \) is nontrivial, this implies that \( G_1 \not\subseteq \ker \phi'_p \). In this case, the above claim gives Part (c), completing the proof.

For \( m = 1 \) we get:

\[\text{Corollary 5.7.} \text{ Let } p \text{ be a prime of } L \text{ and } \phi : F \to V_1 \text{ a } Z \text{-homomorphism. Then } \text{ht}(\phi) = [Z : \mathbb{Z}_p] - 1 \text{ or } \infty. \text{ If } \phi \text{ is unramified then } \text{ht}(\phi) = \infty. \text{ If } \phi \text{ is ramified nontrivially and tamely then } \text{ht}(\phi) = [Z : \mathbb{Z}_p] - 1.\]

5.5. **Finite Ulm invariants.** The following proposition gives the finite Ulm invariants of \( \hat{F} \), and hence in view of Proposition 5.2 the finite direct \( Z \)-summands of \( \hat{F} \). Its proof combines the above local-global principle and computation of local heights.

**Proposition 5.8.** The \( n \)-th Ulm invariant of \( \hat{F} \) is:
\[
U_n(\hat{F}) = \begin{cases} 
\omega & \text{if } n = \ell^k - 1 \text{ for } k \in \mathbb{N} \cup \{0\} \\
0 & \text{for any other } n \in \mathbb{N}
\end{cases}
\]
Proof. Since an element \( \eta \in \hat{F} \) is a \( \mathbb{Z} \)-homomorphism, its height is the maximal \( n \) such that \((\eta, \pi_{n+1,1})\) is solvable. Thus, Proposition 5.5 and Corollary 5.7 imply that the height of each element of \( \hat{F} \) is either infinite or \( \ell^k - 1 \), for some \( k \). Hence, by \((8)\), \( U_n(\hat{F}) = 0 \) for all other \( n \in \mathbb{N} \).

For \( n = \ell^k - 1 \), \( k \in \mathbb{N} \cup \{0\} \), we shall construct an infinite subgroup \( F_n \leq \hat{F} \), the nontrivial elements of which are of height \( \ell^k - 1 \).

Let \( \ell^s \) be the number of \( \ell \)-power roots of unity in \( K(\mu_\ell) \) and hence in \( K(\ell) \).

We first claim that there exist an infinite set \( \mathbb{P}_\ell \) of rational primes \( p \) such that \( p \equiv 1 \mod \ell^{k+s}, \ p \not\equiv 1 \mod \ell^{k+s+1} \), and such that there is a prime \( q \) of \( K \) of degree one over \( p \).

Let \( M \) be the Galois closure of \( K/\mathbb{Q} \) and let \( C \leq \text{Gal}(M(\mu_{\ell^k+1})/K(\mu_{\ell^k})) \) be a cyclic subgroup which does not fix \( \mu_{\ell^k+1} \). By Chebotarev’s density theorem there are infinitely many rational primes \( q' \) of \( M(\mu_{\ell^k+1}) \) whose Frobenius lies in \( C \). Since \( C \) fixes \( K \), the restriction \( q \) of such \( q' \) to \( K \) is of degree one over \((p) = q' \cap \mathbb{Q} \).

Since the restriction of \( C \) to \( \mathbb{Q}(\mu_{\ell^k+1}) \) lies in \( \text{Gal}(\mathbb{Q}(\mu_{\ell^k+1})/\mathbb{Q}(\mu_{\ell^k})) \), we get that \( p \equiv 1 \mod \ell^{k+s} \) and \( p \not\equiv 1 \mod \ell^{k+s+1} \), proving the claim.

For each \( p \in \mathbb{P}_\ell \), let \( \phi_p' : \text{Gal}(\mathbb{Q}) \to \mathbb{F}_\ell \) be a nontrivial homomorphism ramified only over \( p \), and \( \phi_p \in \hat{F} \) be its restriction to \( F \). Let \( F_n \) be the subgroup of \( \hat{F} \) generated by \( \phi_p, \ p \in \mathbb{P}_\ell \).

We claim that every nontrivial \( \phi \in F_n \) is of height \( \ell^k - 1 \). In view of Proposition 5.5, it suffices to consider the local heights. Since \( \phi_p' \) is ramified only over \( p \), \( \phi \) is ramified only over primes of \( L \) lying over primes in \( P \). Since \( p \equiv 1 \mod \ell^{k+s} \) for every \( p \in P \), one has \( \mu_{\ell^{k+s}} \subseteq \mathbb{Q}_p \subseteq L_p \), and hence \( \ell^k \mid [Z : \mathbb{Z}_p] \) for every prime \( p \) of \( L \) dividing \( p \). Thus by Corollary 5.7, \( \text{ht}(\phi_p) \geq \ell^k - 1 \) for all primes \( p \) of \( L \).

Since \( \phi \) is the restriction of a nontrivial linear combination of \( \phi_p' \), \( p \in P \), there is a prime \( q \in P \) such that \( \phi \) is ramified over all primes of \( L \) dividing \( q \). Let \( q_0 \) be a degree one prime of \( K \) over \( q \). Thus, \( \phi \) is ramified over a prime \( \Omega_0 \) of \( K(\ell) \) lying over \( q_0 \). Since \( \mu_{\ell^{k+s}} \not\subseteq \mathbb{Q}_q \cong K_q \), we have \( \mu_{\ell^{k+s}} \not\subseteq K_{\Omega_0} \) and hence \([Z : Z_{\Omega_0}] = \ell^k \). By Corollary 5.7, \( \text{ht}(\phi_{\Omega_0}) = \ell^k - 1 \). It therefore follows from Proposition 5.5 that

\[
\text{ht}(\phi) = \min_p \text{ht}(\phi_p) = \text{ht}(\phi_{\Omega_0}) = \ell^k - 1,
\]

for every \( \phi \in F_n \), proving the claim. By \((8)\), we get \( U_{\ell^k-1}(\hat{F}) = \omega \), for all nonnegative integers \( k \).

5.6. Proof of Theorem 1.3. We shall deduce the finite direct summands of \( \mathbb{F}_\ell[[Z]] \) directly from Propositions 5.2 and 5.8. The following lemma describes the only possible infinite indecomposable summands.

Lemma 5.9. Let \( P \) be a discrete countable indecomposable torsion \( \mathbb{F}_\ell[[Z]] \)-module. Then either \( P \cong V_n \) for some \( n \in \mathbb{N} \), or \( P \cong \hat{V} \) where \( V := \mathbb{F}_\ell[[Z]] \).
Proof. If \( U_n(P) \neq 0 \) for some natural number \( n \), then \( \hat{V}_n \) is a direct summand of \( P \) by Proposition 5.2. As \( P \) is indecomposable it follows that in such case \( P \cong \hat{V}_n \cong V_n \). Thus, we may assume that \( P \) has trivial finite Ulm invariants. Such \( P \) satisfies \( IP = P \), i.e. it is a divisible \( \mathbb{F}_\ell[[Z]] \)-module. By [10, Theorem 4]\(^2\) every divisible \( \mathbb{F}_\ell[[Z]] \)-module is isomorphic to a direct sum of \( \mathbb{F}_\ell[[Z]] \)-modules isomorphic to \( \hat{V} \). Thus, if \( P \) is divisible and indecomposable \( P \cong \hat{V} \). \( \square \)

The proof of Theorem 1.3 therefore reduces to finding the multiplicity of \( \hat{V} \) as an \( \mathbb{F}_\ell[[Z]] \)-summand of \( \hat{F} \), or equivalently the multiplicity of \( V \) as an \( \mathbb{F}_\ell[[Z]] \)-summand of \( \overline{F} \). This is done using the following proposition. Note that the dual of the maximal divisible \( \mathbb{F}_\ell[[Z]] \)-submodule of \( \hat{F} \) is the maximal free \( \mathbb{F}_\ell[[Z]] \)-quotient of \( \overline{F} \).

Proposition 5.10. Let \( K \) be a global field. Then the maximal free \( \mathbb{F}_\ell[[Z]] \)-quotient of \( \overline{F} \) is \( \mathbb{F}_\ell[[Z]] \)^\( \ell \)- if char\( (K) = 0 \), and is trivial if \( \ell \neq \text{char} (K) > 0 \).

Proof. First assume that \( K \) is a number field. Let \( K(\mu_\ell) \subset K' \subset K^{(\ell)} \) be a number field. By Iwasawa theory [26, Theorem 13.31] there is a \( \mathbb{Z} \)-homomorphism

\[
\text{Gal}(K'(\mu_\ell^\infty)) \to \Lambda^{\tau_2(K')}
\]

with finite cokernel, where \( \Lambda := \mathbb{Z}[[Z]] \). Let \( J \) be its image. Since \( J/\ell J \) is an \( \mathbb{F}_\ell[[Z]] \)-submodule of finite index in \( (\Lambda/\ell\Lambda)^{\tau_2(K')} \cong \mathbb{F}_\ell[[Z]]^{\tau_2(K')} \) and \( \mathbb{F}_\ell[[Z]] \) is a discrete valuation ring, \( J/\ell J \) is \( \mathbb{F}_\ell[[Z]] \)-isomorphic to \( \mathbb{F}_\ell[[Z]]^{\tau_2(K')} \). This shows that \( \mathbb{F}_\ell[[Z]]^{\tau_2(K')} \) is a \( Z \)-quotient of \( \text{Gal}(K'(\mu_\ell^\infty)) \) and hence, by Lemma 2.5, it is also a \( Z \)-quotient of \( \text{Gal}(L) \). Since \( \tau_2(K') \) is arbitrarily large for prime to-\( \ell \) extensions we get the desired result in case \( \text{char} (K) = 0 \).

Assume \( \ell \neq \text{char} (K) > 0 \). It suffices to show that the \( Z \)-embedding problem \( (\phi, \pi_1: V \to V_1) \) is nonsolvable for every \( Z \)-homomorphism \( \phi: F \to V_1 \). By Proposition 4.3, \( \phi \) extends to a \( Z \)-homomorphism \( \phi': \text{Gal}(L') \to V_1 \), where \( L' = K'(\mu_\ell^\infty) \) for some finite subextension \( K' \) of \( K^{(\ell)} / K(\mu_\ell) \). By [9, §12.4], the maximal abelian \( Z \)-quotient \( X := \text{Gal}(L')^{ab} \) is a \( \Lambda \)-torsion module for which \( X/\ell X \) has no free \( \Lambda/\ell\Lambda \cong \mathbb{F}_\ell[[Z]] \)-quotients. Thus, \( (\phi', \pi_1) \) is nonsolvable. Hence, by Proposition 4.3, \( (\phi, \pi_1) \) is nonsolvable, as required. \( \square \)

Proof of Theorem 1.3. By Lemma 5.9 it suffices to find the multiplicities of \( V_n \) and \( \hat{V} \) as summands of \( \overline{F} \). By Propositions 5.2 and 5.8, the multiplicity of \( V_n \) is \( \omega \) if \( n = \ell^k \) for \( k \in \mathbb{N} \cup \{0\} \), and 0 otherwise. Note that for a generator \( \sigma \) of \( Z \), \( (\sigma - 1)^\ell^k = \sigma^{\ell^k} - 1 \). Thus, \( \ell^k = (\sigma^{\ell^k} - 1) \) and hence \( V_{\ell^k} \cong \mathbb{F}_\ell[Z/\ell^k Z] \), for every \( k \in \mathbb{N} \cup \{0\} \). Thus, \( \mathbb{F}_\ell[Z/\ell^k Z] \) is a direct summand of \( \overline{F} \) with multiplicity \( \omega \).

\(^2\)As noted in [10, §12] the proof of [10, Theorem 4] for \( \mathbb{Z} \)-modules also holds for \( \mathbb{F}_\ell[[Z]] \)-modules when replacing the \( \ell \)-primary part \( Q/\ell^\infty Q = \lim_{\rightarrow} Z/\ell^n Z \) of \( Q/Z \) by \( \hat{V} = \lim_{\rightarrow} \hat{V}_n \).
Since $F_\ell[[Z]]$ is a free $F_\ell[[Z]]$-module, the maximal free $F_\ell[[Z]]$-quotient of $\overline{F}$ is its direct summand. Proposition 5.10 then implies that $F_\ell[[Z]]$ has multiplicity $\omega$ in $\overline{F}$.

**Corollary 5.11.** For any positive integer $N$ the $\mathbb{Z}$-group $\overline{F}$ decomposes as $\overline{F} = F_{\leq N} \times F_{>N}$ where:

$$F_{\leq N} \cong F_\ell[[Z]]^\kappa \times \prod_{0 \leq k \leq N} F_\ell[Z/\ell^kZ]^{\omega},$$

$k = \omega$ if $K$ is a number field and $\kappa = 0$ otherwise, and $F_{>N}$ has no $F_\ell[[Z]]$-summands of dimension $\leq \ell^N$ over $F_\ell$, nor $F_\ell[[Z]]$-summands isomorphic to $F_\ell[[Z]]$.

**Proof.** As in Theorem 1.3, Proposition 5.2 gives a decomposition $\overline{F} = V_{\leq N} \times V_{>N}$, where:

$$V_{\leq N} \cong \prod_{0 \leq k \leq N} F_\ell[Z/\ell^kZ]^{\omega},$$

and $V_{>N}$ has no direct $F_\ell[[Z]]$-summands of dimension $\leq \ell^N$. If $\ell \neq \text{char}(K) > 0$, this is the desired decomposition.

If $K$ is a number field, $F_\ell[[Z]]^{\omega}$ is a quotient of $\overline{F}$, and hence of $V_{>N}$. Furthermore, since $F_\ell[[Z]]^{\omega}$ is free, it is a direct summand of $V_{>N}$. Letting $F_{\leq N}$ be the product of $V_{\leq N}$ and the $F_\ell[[Z]]^{\omega}$ summand of $V_{>N}$, and letting $F_{>N}$ be a complement of the latter summand in $V_{>N}$, we obtain the desired decomposition. \qed

**5.7. Towards a presentation.** As a Corollary to Theorem 5.11, we get the following description of $\text{Gal}(K(\ell))$ in terms of generators and relations.

Let $\sigma$ be a generator of $\mathbb{Z}$ and let $x^\sigma = \sigma^{-1} x \sigma$ denote the action of $\sigma$ on $x \in F$. Recall that $X \subseteq F$ is a basis for $F$ if $X$ converges to 1, and $F$ is the free pro-$p$ group generated by $X$ [24, §3.3].

**Corollary 5.12.** Assume $K$ be a number field, and $N$ a positive integer. Then $\text{Gal}(K(\ell))$ is generated by $\sigma$ and a basis of $F$ which is a disjoint union of three subsets $X_{>N} \cup X_\infty \cup X_{\leq N}$:

(a) $X_{\leq N}$ is a disjoint union of infinitely many copies of each of the sets

$$\{x_0, \ldots, x_{\ell n-1}\}, n \leq N,$$

subject to the relations

$$x_i^\sigma = x_{i+1} y_i \text{ and } x_{i-1}^\sigma = x_0 y$$

for some $y, y_i \in \Phi(F)$, $0 \leq i \leq \ell^n - 2$;

(b) $X_\infty$ is a disjoint union of infinitely many copies of the set $\{x_n\}_{n=0}^\infty$ which converges to 1 as $n \to \infty$, and is subject to the relations:

$$x_i^\sigma = x_{i+1} x_i y_i \text{ and } x_i^{-1} = (\prod_{i=0}^\infty x_i^{(-1)} y),$$

for some $y, y_i \in \Phi(F)$, $i \in \mathbb{N} \cup \{0\}$;
Recall that a basis for $F \sim$ is an $\eta_i$, such that $\eta_1, \ldots, \eta_\ell$ is the normal closure of $\eta_i \hat{\eta}_i$ which is unramified be an $\{ \hat{\eta}_i \} = I$ as a global field, $\hat{\eta}_i = \ker \sigma$ of $\hat{\eta}_i$ in $\subseteq x$, $\hat{\eta}_1 = 0$ is a basis for $\hat{\eta}_i : F \to \n K \times V$ using the de-

Moreover, we can assume that any finite subset of the $\eta_i$’s appearing in parts (b) and (c) is trivial.

Proof. Recall that a basis for $F$ as a profinite $\mathbb{F}_\ell$-vector space is a minimal generating set which converges to 1. We first choose a basis $\mathcal{S}$ for $\mathcal{F}$ using the decomposition in Corollary 5.11 as follows. For each $\mathbb{F}_\ell[[Z]]$-summand isomorphic to $V_{\ell^n} \cong \mathbb{F}_\ell[Z/\ell^n Z]$, $n \leq N$, include in $\mathcal{S}$ the basis $\{ \bar{\eta}_i \}_{i=0}^{\ell^n-1}$ of the summand which corresponds to the basis $\sigma^i$, $i = 0, \ldots, \ell^n - 1$, of $\mathbb{F}_\ell[Z/\ell^n Z]$. For each $\mathbb{F}_\ell[[Z]]$-summand isomorphic to $\mathbb{F}_\ell[[Z]]$, include a basis $\{ \bar{\eta}_i \}_{i=0}^\infty$ which corresponds to $(\sigma - 1)^i$, $i = 0, 1, \ldots$. Include in $\mathcal{S}$ a basis of $V_{\infty}$. Note that since each of the above bases converges to 1, their union $\mathcal{S}$ converges to 1 in the product topology. Hence the set $\mathcal{S}$ is a basis for $\mathcal{F}$.

By Burnside’s basis theorem [24, Proposition 7.6.9], a basis $\mathcal{S}$ for $\mathcal{F}$ can be lifted to basis $S$ of $F$. Since for each $V_{\ell^n}$-summand we have $\bar{\eta}_{i+1} = \bar{\eta}_i$, $i = 0, \ldots, \ell^n - 2$, and $\bar{\eta}_{\ell^n} = \bar{\eta}_1$, the relations in (12) follow. The relations in (13) follow since for each $\mathbb{F}_\ell[[Z]]$-summand we have $\bar{\eta}_{i+1} = \bar{\eta}_i^{-1} := \bar{\eta}_i - \bar{\eta}_i$, $i = 0, \ldots$, and 
\[ \bar{\eta}_i = \sum_{i=0}^\infty \bar{\eta}_i^{(1-\sigma)^i} = \sum_{i=0}^\infty (-1)^i \bar{\eta}_i. \]

Moreover, by [24, Corollary 7.6.10] the basis $\mathcal{S}$ can be lifted to a basis $S$ of $F$ in which finitely many elements in $\mathcal{S}$ have prescribed liftings. Thus, we may assume that finitely many of the $\eta_i$’s in Parts (b) and (c) are 1. \[ \square \]

5.8. Infinite Ulm invariants. To completely determine the structure of $\mathcal{F}$ as a $\mathbb{Z}$-module, it remains to find the infinite Ulm invariants of $\hat{\mathcal{F}}$ or equivalently the Ulm invariants of $I^\infty \hat{\mathcal{F}}$. The latter relates to Iwasawa modules as follows.

Let $M$ be the maximal abelian pro-$\ell$ extension of $K(\mu_{\infty})$ unramified away from primes dividing $\ell$, and $M^\text{un}$ the maximal subfield of $M$ which is unramified over $K(\mu_{\infty})$. Iwasawa theory [26, §13] studies the Galois groups $X^\text{un}(K) := \text{Gal}(M^\text{un}/K)$ and $X(K) := \text{Gal}(M/K)$ as modules over $\text{Gal}(K(\mu_{\infty})/K)$.

Proposition 5.13. Let $K$ be a global field, $X := X(K(\ell))$ and $X^\text{un} := X^\text{un}(K(\ell))$. Then $X^\text{un} \subseteq I^\infty \hat{\mathcal{F}} \subseteq X$.

The proof is based on the following lemma. As in Proposition 5.3, for $\eta \in \hat{\mathcal{F}}$, let $\bar{\eta} : V_n \to \hat{\mathcal{F}}$ be an $\mathbb{F}_\ell[[Z]]$-monomorphism whose image is $\mathbb{F}_\ell[[Z]]\eta$, and $\bar{\eta}^* : F \to V_n$ its dual map.

Lemma 5.14. Let $\eta \in \hat{\mathcal{F}}$ and $E$ the fixed field of $\ker \eta$. Then the fixed field of $\ker \bar{\eta}^*$ is the normal closure of $E/K(\ell)$.

Proof. Let $U := \ker \eta$, so that $U = \text{Gal}(E)$. Since every element in $\text{Im} \bar{\eta}$ is an $\mathbb{F}_\ell$-linear combination of $\eta^i$, $i = 0, \ldots, n - 1$, $\text{Im} \bar{\eta}$ consists of all $\chi \in \hat{\mathcal{F}}$ such that
\[ \chi(\bigcap_{i=0}^{n-1} U^\sigma i) = 0. \]

By duality \( \ker \tilde{\eta}^* = \bigcap_{i=0}^{n-1} U^\sigma i \). Thus, the fixed field of \( \ker \tilde{\eta}^* \) is the compositum \( M \) of \( E^\sigma i, \ i = 0, \ldots, n - 1 \). Since the conjugates of \( E \) are contained in the normal closure of \( E/K(\mu^{1/2}) \), so is \( M \). Since \( L/K(\mu^{1/2}) \) is Galois, \( E/L \) is Galois, and since \( \sigma \) extends to \( M, M/K(\mu^{1/2}) \) is Galois. Thus, \( M \) equals the normal closure of \( E/K(\mu^{1/2}) \).

**Proof of Proposition 5.13.** Assume \( \eta \in \hat{F} \) is unramified. Since the fixed field of \( \eta \) is unramified over \( L \), so is its normal closure over \( L \). Hence, by Lemma 5.14 the map \( \tilde{\eta}^* \) is unramified. By Propositions 5.5 and 5.6, \( \text{ht}(\tilde{\eta}^*) = \infty \). Hence by Proposition 5.3.(b) one has \( \text{ht}(\eta) = \infty \), proving the first containment.

For the second containment, assume \( \text{ht}(\eta) = \infty \). By Proposition 5.3, the map \( \text{ht}(\tilde{\eta}^*) = \infty \). By Proposition 5.6.(c), the map \( \tilde{\eta}^* \) is unramified away from primes dividing \( \ell \). By Lemma 5.14, \( \eta \) is unramified away from primes dividing \( \ell \), and hence splits through \( \text{Gal}(M/L) \), proving the second containment. \( \square \)

We next use the structure of the Iwasawa modules \( X \) and \( X^{un} \) to study \( I^\omega \hat{F} \). Letting \( L_0 \) be the \( \mathbb{Z}_\ell \)-subextension of \( K(\mu_{\infty})/K \), by Lemma 2.8 we may identify \( \hat{Z} \) with \( \text{Gal}(L_0/K) \) so that the restriction \( \text{Gal}(L_0) \to \text{Gal}(L) \) is a \( \mathbb{Z} \)-homomorphism. By [9], the \( \mathbb{Z} \)-modules \( X(K) \) and \( X^{un}(K) \) are finitely generated and hence admit a \( \mathbb{Z} \)-homomorphism with finite kernel and cokernel into a unique \( \mathbb{Z} \)-module of the form:

\[ \Lambda \times \prod_{i \in I} (\Lambda/\ell^r_i \Lambda)^{r_i} \times \prod_{j=1}^k \Lambda/(g_j(x)), \]

where \( \Lambda := \mathbb{Z}_\ell[[Z]], I \subseteq \mathbb{N} \) is a finite subset, \( r, k, r_i \in \mathbb{N} \) for all \( i \in I \), and \( g_j(x), j = 1, \ldots, k \), are monic irreducible polynomials for which all nonleading coefficients are divisible by \( \ell \). The Iwasawa \( \mu \)-invariant of such a \( \mathbb{Z} \)-module is the corresponding sum \( \sum_{i \in I} r_i \).

**Proposition 5.15.** Let \( K = \mathbb{Q} \) and \( \ell \) an odd prime. Then \( I^\omega \hat{F} \) has nontrivial Ulm invariants.

**Proof.** We shall construct a \( \mathbb{Z} \)-homomorphism \( \phi: \text{Gal}(L) \to V_1 \) with \( \text{ht}(\phi) = \infty \) and such that \( (\phi, \pi_1): V \to V_1 \) is nonsolvable. This will show that \( I^\omega \hat{F} \) is not a direct sum of \( \mathbb{F}_\ell[[Z]] \)-modules isomorphic to \( \hat{V} \), as otherwise its dual would be a free \( \mathbb{F}_\ell[[Z]] \)-module. Thus by [10, Theorem 4], \( I^\omega \hat{F} \) is not divisible, and hence \( I^\omega \hat{F} \) has nontrivial Ulm invariants, as required.

By [29], there exists a real quadratic extension \( K_0/\mathbb{Q} \) whose class number is divisible by \( \ell \). Hence there is an unramified \( \mathbb{Z}/\ell \mathbb{Z} \)-extension \( M_0/K_0 \). We define \( \phi: \text{Gal}(L) \to V_1 \) as the restriction of a homomorphism \( \phi_0': \text{Gal}(K_0) \to \mathbb{F}_\ell \) whose kernel fixes \( M_0 \). Since \( \phi \) is unramified, Proposition 5.13 shows that \( \text{ht}(\phi) = \infty \).

Assume on the contrary that there is a solution \( \psi \) to \( (\phi, \pi_1) \). Let \( L_0/K_0 \) be the \( \mathbb{Z}/\ell \mathbb{Z} \)-extension inside \( K_0(\mu_{\infty}), \) and \( \phi_0 \) the restriction of \( \phi \) to \( L_0 \). By Proposition 4.3, \( \psi \) extends to a solution \( \psi_0 \) of the \( \mathbb{Z} \)-embedding problem \( (\phi_0, \pi_1) \). Let \( K_1 = K_0(\mu_{\ell}) \),
$L_1 = K_0(\mu_{\ell,\infty})$ and $\Delta := \text{Gal}(K_1/K_0)$. In particular, $\phi_0$ splits through a $\mathbb{Z}$-homomorphism $\phi_u : X(K_0)/\ell X(K_0) \to V_1$.

At primes $p$ of $L$ that are prime to $\ell$, $\text{Gal}(L_p)$ is cyclic and in particular has no free $\mathbb{F}_\ell[[Z]]$-quotients. Thus, $\psi$ and hence $\psi_0$ are unramified at primes that do not divide $\ell$. It follows that $\psi_0$ factors through $X(K_0)$ and hence through $X(K_0)/\ell X(K_0)$, showing that $(\phi_u, \pi_1)$ is solvable.

Let $M^{sc}$ be the maximal unramified pro-$\ell$ extension of $K_1(\mu_{\ell,\infty})$ in which all primes dividing $\ell$ split completely. Let $X^{sc}(K_1) := \text{Gal}(M^{sc}/K_1(\mu_{\ell,\infty}))$. By Iwasawa’s theorem [20, Corollary 11.3.17], the $\mu$-invariants $\mu(X(K_1))$ and $\mu(X^{sc}(K_1))$ are equal. By Ferrero-Washington [5], $\mu(X^{un}(K_1)) = 0$. Since $X^{un}(K_1)$ has no free $\Lambda$-quotients [26, Proposition 13.19], $\mu(X^{sc}(K_1)) \leq \mu(X^{un}(K_1)) = 0$ and hence $\mu(X^{sc}(K_1)) = \mu(X(K_1)) = 0$. The module $X(K_1)$ over $\text{Gal}(K_1(\mu_{\ell,\infty})/K_0) \cong \Delta \times \mathbb{Z}$, decomposes into a direct sum $\bigoplus_{\chi} X(K_1)$ where $\chi$ runs through idempotents that correspond to characters $\chi \in \hat{\Delta}$. Since $\varepsilon_1 X(K_1) = X(K_1)^{\Delta}$ is the maximal $\mathbb{Z}$-quotient of $X(K_1)$ that is fixed by $\Delta$, we have $X(K_0) \cong X(K_1)^{\Delta}$ as $\mathbb{Z}$-modules. Thus, $\mu(X(K_0)) = \mu(X(K_1)^{\Delta}) = 0$. As $K_0$ is totally real, [26, Theorem 13.31] implies that $X(K_0)$ has no free $\Lambda$-quotients. Since moreover $\mu(X(K_0)) = 0$, $X(K_1)/\ell X(K_1)$ has no free $\mathbb{F}_\ell[[Z]]$-quotients, contradicting the solvability of $(\phi_u, \pi_1)$. \hfill $\Box$

As a consequence it follows from Proposition 5.2 that in the case $K = \mathbb{Q}$, $\overline{F}$ is not $\mathbb{Z}$-isomorphic to a product of the $\mathbb{Z}$-modules $\mathbb{F}_\ell[[Z]]$ and $V_n, n \in \mathbb{N}$. Indeed, otherwise the dual $\overline{F}$ would be a direct sum of $\mathbb{Z}$-modules isomorphic to $V_n$ and $V$, but each such direct sum has trivial infinite Ulm invariants.

5.9. Ulm invariants and finite summands. Proposition 5.2 follows directly from the following lemma which asserts its dual. The key to its proof is the following criterion for an $\mathbb{F}_\ell[[Z]]$-submodule $E \leq D$ to be a direct summand. The submodule $E$ is called pure if $I^k E = I^k D \cap E$ for all $k \in \mathbb{N}$. By [10, Theorem 7]³, every pure submodule $E \leq D$ such that $I^N E = 0$ for some $N \in \mathbb{N}$ is a direct $\mathbb{F}_\ell[[Z]]$-summand of $D$.

We write $\text{ht}_E$ to specify that the height is taken within $E$. We shall write $V_n^{\oplus \kappa}$ to denote the direct sum of $\kappa$ copies of $V_n$.

Lemma 5.16. Let $N \in \mathbb{N} \cup \{0\}$ and let $D$ be a discrete torsion $\mathbb{F}_\ell[[Z]]$-module. Then $D = P_N \oplus Q_N$ where

$$P_N \cong \bigoplus_{1 \leq n \leq N} V_n^{\oplus U_{n-1}(D)} ,$$

and $Q_N$ has no direct $\mathbb{F}_\ell[[Z]]$-summands of dimension $1 \leq d \leq N$.

³Theorem 7 in [10] asserts the corresponding statement for $\mathbb{Z}$-modules. As noted in [10, §12] the same proof works for modules over a PID, and in particular over $\mathbb{F}_\ell[[Z]]$. 
Proof. We argue by induction on $N$ with $N = 0$ being trivial. By induction $D = P_N \oplus Q_N$, where $P_N$ is as in (14), and $Q_N$ has no summands of dimension $\leq N$. The induction hypothesis applied to $Q_N$ also shows that $U_{n-1}(Q_N) = 0$ for all $n \leq N$, as otherwise $Q_N$ would have direct $Z$-summands of dimension $1 \leq d \leq N$. Hence by (8), there are no element in $Q_N^Z$ of height $\leq N - 1$ in $Q_N$.

We shall construct $T \leq Q_N$ such that $Q_N = T \oplus Q_{N+1}$, $T = V_{N+1}^U(Q_N)$ and $Q_{N+1}$ has no $Z$-summands isomorphic to $V_{N+1}$. As $Z$ acts trivially on $Q_N^Z$, we regard $Q_N^Z$ as an $F_v$-vector space. Let $V$ be the $F_v$-subspace of $Q_N^Z$ consisting of elements of height $> N$, and $U$ a complement of it in $Q_N^Z$. In particular, $U$ is a maximal $F_v$-subspace of $Q_N^Z$ whose nontrivial elements are of height $N$ in $Q_N$.

Thus $\dim F_v U = U_N(Q_N)$. Let \{u_j\}$_{j \in J}$ be an $F_v$-basis of $U$; hence $|J| = U_N(Q_N)$.

Let $R := F_v[[Z]]$, $\sigma$ a generator of $Z$, and $x := \sigma - 1$ a generator of the augmentation ideal $I \triangleleft R$. Since each $u_j$ is of height $N$, we may pick an element $p_j \in Q_N$ such that $x^N p_j = u_j$, $j \in J$. Let $T$ be the $R$-submodule $\sum_{j \in J} R p_j$. Since $x^N p_j = u_j \neq 0$, and $x^{N+1} p_j = 0$, $R p_j$ is cyclic of dimension $N + 1$ and hence $R p_j \cong V_{N+1}$, for $j \in J$.

We claim that $T = \oplus_{j \in J} R p_j \cong \oplus_{j \in J} V_{N+1}$. Assume there is a nontrivial linear combination $\sum_{i \leq N, j \in J} a_{i,j} x^i p_j = 0$, with $a_{i,j} \in F_v$, $i \leq N$, $j \in J$. Multiplying by $x^{N-i_0}$ where $i_0$ is the minimal number for which $a_{i_0,j} \neq 0$ for some $j$, we obtain a nontrivial linear combination $\sum_{j \in J} b_{j} x^N p_j = \sum_{j \in J} b_{j} u_j = 0$. This contradicts the linear independence of $u_j$, $j \in J$, proving the claim.

We next show that $T$ is a direct $R$-summand of $Q_N$. Since all nontrivial elements of $U$ are of height $N$ in $Q_N$, the height of each element in $T$ is the same as its height in $Q_N$. Hence, $T$ is a pure submodule of $Q_N$. Since $I^{N+1} T = 0$, [10, Theorem 7] implies that $Q_N = T \oplus Q_{N+1}$ for some $R$-submodule $Q_{N+1} \leq Q_N$.

Finally, we show that $Q_{N+1}$ has no $R$-summands isomorphic to $V_n$, for $n \leq N + 1$. Since $Q_{N+1} \leq Q_N$, $\text{ht}_{Q_{N+1}}(q) \geq N$ for every $q \in Q_{N+1}^Z$. We claim that $\text{ht}_{Q_{N+1}}(q) > N$ for every $q \in Q_{N+1}^Z$. Indeed, if $\text{ht}_{Q_{N+1}}(q) = N$ then for any $u \in U$, one has $\text{ht}_{Q_n}(q + u) = \min(\text{ht}_{Q_n}(q), \text{ht}_{Q_n}(u)) = N$, contradicting the maximality of $U$ and proving the claim. As $Q_{N+1}^Z$ has no elements of height $\leq N$, $Q_{N+1}$ has no $R$-summands isomorphic to $V_n$, for $n \leq N + 1$. Setting $P_{N+1} := P_N \oplus T$, we obtain the desired decomposition $D = P_{N+1} \oplus Q_{N+1}$. $\square$

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