LOGARITHMIC CFTS CONNECTED WITH SIMPLE LIE ALGEBRAS
SIMPLY-LACED CASE

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ABSTRACT. For any root system corresponding to a semisimple simply-laced Lie algebra a logarithmic CFT is constructed. Characters of irreducible representations were calculated in terms of theta functions.

1. INTRODUCTION

The most deeply investigated examples of LCFTs are \((p, p')\) models introduced in [1]. The chiral algebra of these models [2] is based on the \(s\ell(2)\) symmetry discovered in [3]. In the paper, we generalize a construction of a chiral algebra to the case of a semisimple simply laced Lie algebra.

We suggest a method to construct LCFTs from the following data. Let \(B\) be a Lie group, \(X\) be a space with an action of \(B\) on it and let \(\mathcal{L}\) be a vertex operator algebra in which \(B\) acts by symmetries of OPEs. We construct a bundle \(\xi = \mathcal{L} \otimes_B X\) with fibers \(\mathcal{L}\). Then the cohomology of \(\xi\) bear a VOA structure. This VOAs are main object of investigation in this paper. We assume notation \(\mathcal{W}\) for this VOA. On this way we obtain VOAs that generalize \(W\)-algebras of \((1, p)\) models [4, 5, 6]. Generalization of \((p, p')\) models from [2] also can be obtained on this way.

In the paper we consider the case where \(B\) is the Borel subgroup in a simply laced Lie group \(G\), the space \(X\) is the group \(G\) itself and \(\mathcal{L}\) is a lattice VOA in which \(B\) acts by screening operators. In this case only zero-dimension cohomologies of the sheaf \(\xi\) are nonzero and therefore we can identify global sections of \(\xi\) with a vertex operator algebra \(\mathcal{W}\). Irreducible modules of \(\mathcal{W}\) can be constructed as cohomology of some bundles on \(G/B\). We consider the sheaf of sections of the bundle \(\xi(\mathfrak{g}) = \mathfrak{g} \otimes_B G\), where \(\mathfrak{g}\) is an irreducible \(\mathcal{L}\)-module. We obtain irreducible modules of \(\mathcal{W}\) as global sections of the bundle \(\xi(\mathfrak{g})\).

Most of statements in the paper can be proved using results of [7] on \(W\)-algebras.

In the case where \(G = SL(2)\) the algebra \(\mathcal{W}\) is described as a kernel of the screening [4] and its irreducible modules are described in the similar way. This allows us to calculate characters from the Felder complex [2]. The definition with screenings can be generalized to any root system. The algebra \(\mathcal{W}\) coincides with the intersection of screenings kernels in the vacuum module of \(\mathcal{L}\).

The construction in terms of the bundle \(\xi\) allows us to calculate characters of irreducible modules using the Lefschetz formula. The equivariant Euler characteristics of \(\xi\) is

\[
\sum_{i \geq 0} (-1)^i \text{Tr}_H(e^\hbar q^{L_0}) = \sum_{x \in S} \text{Tr}_{\xi_x \otimes \mathcal{F}_x}(e^\hbar q^{L_0}),
\]
where $S$ is the set of fixed points of the standard torus action on the flag manifold $G/B$, $h$ is an element of the Cartan subalgebra, $L_0$ is the element of the Virasoro subalgebra in $LX$, $H^i$ is $i$-th cohomology of $\xi$, $\xi_x$ is the fiber of $\xi$ at the point $x$ and $F_x$ is the ring of formal series in a neighborhood of $x$. Actually, all cohomology excepting $H^0$ are equal to zero and therefore the left hand side of (1.1) is equal to the character of the irreducible $WLX$-module. Thus, we obtain expressions for the characters in terms of theta functions from [10]

\begin{equation}
\chi_\lambda(q) = \frac{1}{\eta(q)^\ell} \sum_{\omega \in \Gamma^{\vee,+}} \zeta_\omega \prod_{i=1}^\ell \Theta^\omega_\lambda(q)
\end{equation}

where

\begin{equation}
\Theta^\omega_\lambda(q) = \partial^\omega \Theta_\lambda(q, z)|_{z=1}
\end{equation}

is partial derivative of the theta function and

\begin{equation}
\zeta_\omega = \sum_{\mu \in \Gamma^{\vee}} (-1)^{(\mu, \rho)} \prod_{j=1}^\ell \left( \frac{\omega_j, \omega}{\alpha_j, \mu} \right) \prod_{\alpha \in \Delta^+} \left( 1 + \frac{\alpha, \mu}{\alpha, \rho} \right), \quad \omega \in \Gamma^{\vee,+}.
\end{equation}

See explanations of the notation in Sec. 6. In the case $G = SL(2)$, (1.2) coincides with formulas for characters of irreducible $W$-modules in [4, 8].

The paper is organized as follows. In section 2, we introduce notations and recall known facts about vertex operator algebras. In section 3, we introduce screening operators. In section 4, we introduce the main object of our investigation the vertex operator algebra $WLX$ and describe its irreducible modules as cohomologies of some bundles on homogeneous space. In section 5, we introduce a quantum group that presumably centralizes $WLX$. In section 6, we calculate characters of $WLX$ irreducible modules.

2. Preliminaries

Let $g$ be a simply-laced semisimple Lie algebra of rank $\ell$, $h$ and $b$ be its Cartan and Borel subalgebras respectively. Let $G$, $H$ and $B$ be Lie groups corresponding to $g$, $h$ and $b$ respectively. Let $\Gamma$ be the root lattice, $\Pi$ be the set of simple positive roots $\alpha_i$, $i = 1, 2, \ldots, \ell$ and $\Delta$ be the set of roots. Let $(\cdot, \cdot)$ be standard scalar product in $h^*$, $c_{ij} = \frac{2(\alpha_i, \alpha_j)}{\langle \alpha_j, \alpha_j \rangle}$ be the Cartan matrix, which in the simply-laced case coincides with the Gramm matrix and $c^{ij}$ be the inverse matrix to $c_{ij}$. Let $\rho$, $\rho$, $\alpha_i = 1, 1, \ldots, \ell$ be the half of the sum of positive roots. Let $\omega_i$, $i = 1, 2, \ldots, \ell$ be fundamental weights $(\omega_i, \alpha_j) = \delta_{ij}$. Let $\Gamma^{\vee}$ be the weight lattice. We set $\Omega = \Gamma^{\vee}/\Gamma$. We set $\gamma = |\Omega|$ the order of the group $\Omega$.

We choose representatives of the elements from the abelian group $\Omega$ in $\Gamma^{\vee}$ in the following way. For algebras $A_\ell$ we choose $0, \omega_1$ with $i = 1, \ldots, \ell$; for $D_\ell$ we choose $0, \omega_1, \omega_{\ell-1}$ and $\omega_\ell$; for $E_6$ we choose $0, \omega_1$ and $\omega_3$; for $E_7$ we choose $0, \omega_2$ and for $E_8$ the group $\Omega$ contains only $0$.

We consider the free scalar fields $\varphi_\alpha(z)$ for $\alpha \in \Gamma$ with the OPE

\begin{equation}
\varphi_\alpha(z)\varphi_\beta(w) = (\alpha, \beta) \log(z - w).
\end{equation}
We note that between \( \varphi_\alpha(z) \) there are \( \ell \) linearly independent. We also use notation \( \varphi_i(z) = \varphi_{\alpha_i}(z), i = 1, \ldots, \ell \). We assume the mode decomposition

\[
\varphi_\alpha(z) = (\bar{\varphi}_\alpha)_0 + (\varphi_\alpha)_0 \log z - \sum_{n \neq 0} \frac{1}{n}(\varphi_\alpha)_n z^{-n}.
\]  

In order to have correct commutation relations between screening operators we introduce nontrivial bracket of the constant modes of \( \varphi_\alpha \)

\[
[(\bar{\varphi}_i)_0, (\bar{\varphi}_j)_0] = b_{ij},
\]  

where

\[
b_{ij} = -b_{ji} = \begin{cases} 1, & i < j \text{ and } i\text{-th and } j\text{-th nodes are connected in the Dynkin diagram,} \\ 0, & \text{otherwise.} \end{cases}
\]  

We define a bilinear form \( \{\alpha, \beta\} \) on \( \mathfrak{h}^* \) in the basis of simple roots

\[
\{\alpha_i, \alpha_j\} = c_{ij} + b_{ij}.
\]  

We note that this bilinear form is not symmetric nor antisymmetric. We fix an integer \( p \geq 2 \) and introduce the set of vertex operators

\[
V_\lambda(z) = e^{\frac{1}{\sqrt{p}}\varphi_\lambda(z)}, \quad \lambda \in \Gamma^\vee.
\]

After changing the commutation relations of the constant modes \( (2.7) \), we have the following braiding of vertex operators

\[
V_\lambda(z)V_\mu(w) \sim q^{(\lambda, \mu)}V_\mu(w)V_\lambda(z),
\]  

where

\[
q = e^{\pi i/p}.
\]

**Remark 2.1.** Usually vertex operators are defined by \( (2.10) \) with \( [(\bar{\varphi}_i)_0, (\bar{\varphi}_j)_0] = 0 \), which gives the braiding in the standard form \( q^{(\lambda, \mu)} \). After modification the braiding is changed. We note that the monodromy of the vertex operators doesn’t change and is equal to \( q^{2(\lambda, \mu)} \). See a similar construction in [11].

We consider the lattice vertex operator algebra \( \mathcal{L}X_l(p) \) corresponding to \( \bar{\Gamma} = \sqrt{p} \Gamma \), where \( \Gamma \) is the root lattice of the semisimple algebra of the type \( X \) equals to \( A, D \) or \( E \). In what follows to simplify notation we often write \( \mathcal{L}X \). The basis in \( \mathcal{L}X \) consists of elements \( P(\partial_\varphi_\beta)V_{\rho_\alpha}(z) \), where \( \alpha \in \Gamma \) and \( P \) is a differential polynomial. We choose the energy-momentum tensor of \( \mathcal{L}X \) in the form

\[
T(z) = \frac{1}{2}e^{ij}\partial_\varphi_i(z)\partial_\varphi_j(z) + Q_0\partial^2_\varphi(z),
\]  

where

\[
Q_0 = \sqrt{p} - \frac{1}{\sqrt{p}}.
\]
We choose the nonstandard background charge in order to have the energy-momentum tensor commuting with screening operators, which are introduced in the next section. The central charge is

\[ c = \ell + 12(\rho, \rho)(2 - p - \frac{1}{p}) = \ell + h \dim(p - \frac{1}{p}), \]

where \( h \) is the Coxeter number of \( \mathfrak{g} \).

The conformal dimension \( \Delta_\lambda \) of the vertex operator \( e^{\varphi_\lambda} \) with \( \lambda \in \bar{\Gamma}^\vee \) is given by the expression

\[ \Delta_\lambda = \frac{1}{2}(\lambda - Q_0 \rho, \lambda - Q_0 \rho) + \frac{c - \ell}{24}. \]

The \( L_\ell(p) \) irreducible modules are enumerated by elements of the abelian group \( \Lambda = \bar{\Gamma}^\vee / \bar{\Gamma} \). We choose the basis \( \lambda_j = \frac{1}{\sqrt{p}} \omega_j, j = 1, 2, \ldots, \ell \) in \( \bar{\Gamma}^\vee \). For each equivalence class \( <\lambda> \in \Lambda \), a unique representative \( \lambda \in \bar{\Gamma}^\vee \) of the form

\[ \lambda = \sqrt{p} \omega + \sum_{j=1}^{\ell} (1 - s_j) \lambda_j, \]

where \( \omega \in \Omega \) and \( s_j = 1, 2, \ldots, p \) can be chosen (See description of the representatives in the second paragraph of this section.). We call the form (2.17) of representatives canonical and use notation

\[ \hat{\lambda} = \omega, \]

\[ \bar{\lambda} = \sum_{j=1}^{\ell} (1 - s_j) \lambda_j. \]

We note that \( \Lambda \) can be described as an Abelian group with generators \( \mu^i \) and relations \( \sum_{j=1}^{\ell} p c_{ij} \mu^j = 0 \). For \( A_2 \) root system and \( p = 3 \) the group \( \Lambda \) generators are shown in the diagram from Appendix A.

Let \( \mathfrak{F}_\alpha, \alpha \in \Gamma^\vee \) be the Fock module corresponding to the vertex \( V_\alpha(z) \). We set

\[ \mathfrak{V}_{<\lambda>} = \bigoplus_{\alpha \in \Gamma} \mathfrak{F}_{\lambda + \alpha}. \]

Then, \( \mathfrak{V}_{<\lambda>} \) for \( <\lambda> \in \Lambda \) is an irreducible module of \( L_\ell(p) \) [13, 14].

3. SCREENING OPERATORS

We consider the screening operators

\[ e_i = \frac{1}{2\pi i} \oint dze^{\sqrt{\mathfrak{g}}_i(z)} \]

and

\[ F_i = \frac{1}{2\pi i} \oint dze^{-\sqrt{\mathfrak{g}}_i(z)} \]
which commutes with the Virasoro algebra \((2.13)\). We note that because \((2.11)\) operators \(e_i\) commute with \(F_j\) for \(i, j = 1, \ldots, \ell\).

We note that \(e_i\) acts in each \(\mathfrak{V}_\lambda\). To define the space in which operators \(F_i\) act we should introduce dressed vertex operators \([9]\). In this paper we need only that \(F_i\) is a well defined operator from \(\mathfrak{V}_0\) to \(\mathfrak{V}_{-\lambda}\).

Actually, the subalgebra of \(\mathcal{L}X_\ell(p)\) consisting of zero momentum fields commuting with screenings \((3.21)\) is the \(W\)-algebra \(\mathcal{W}X_\ell(p)\) obtained by Hamiltonian reduction from affine algebra \(\hat{\mathfrak{g}}_k\) with

\[
\rho = k + h.
\]

A description of the \(\mathcal{W}X_\ell(p)\) representation category can be found in \([7]\).

4. THE VERTEX OPERATOR ALGEBRA \(\mathcal{W}LX_\ell(p)\)

In the section, we define the main object of the paper the vertex operator algebra \(\mathcal{W}LX_\ell(p)\), where \(X\) means \(A, D\) or \(E\) type of simply-laced semisimple Lie algebra. Thus we have \(\mathcal{W}L\Lambda_\ell(p)\) for \(\ell = 1, 2, 3, \ldots\), \(\mathcal{W}L\mathcal{D}_\ell(p)\) for \(\ell = 4, 5, 6, \ldots\) and \(\mathcal{W}L\mathcal{E}_\ell(p)\) for \(\ell = 6, 7, 8\). The vacuum representation of the algebra \(\mathcal{W}LX_\ell(p)\) can be defined as an intersection of kernels of operators \(F_i\) in \(\mathfrak{V}_0\). We give another definition of \(\mathcal{W}LX_\ell(p)\) in terms of operators \(e_i\).

4.1. The action of \(\mathfrak{b}\) in the irreducible \(\mathcal{L}X_\ell(p)\)-modules.

\textbf{Theorem 4.1.}\quad (1) The \(\mathcal{L}X_\ell(p)\) module \(\mathfrak{V}_\lambda\) for \(\lambda \in \Lambda\) admits the action of \(\mathfrak{b}\) given by the standard generators

\[
e_i = \frac{1}{2\pi i} \int dz e^{\sqrt{p}\phi_i(z)},
\]

\[
h_i = \frac{1}{2i\pi \sqrt{p}} \int dz \partial \varphi_i - \frac{1}{\sqrt{p}} (\alpha_i, \lambda) + (\alpha_i, \mu)
\]

with \(i = 1, \ldots, \ell, \mu \in \Gamma^\vee\) and \(\tilde{\lambda}\) defined in \((2.19)\).

(2) The action of the Borel subalgebra \(\mathfrak{b}\) given in part (1) of the Theorem is integrated to the action of \(\mathfrak{B}\) in \(\mathfrak{V}_\lambda\).

\textbf{Remark 4.2.}\quad We note that \((4.24)\) and \((4.25)\) define an action of \(\mathfrak{b}\) on \(\mathcal{L}X_\ell(p)\) by infinitesimal symmetries. Therefore, we can construct a semidirect product \(U(\mathfrak{b}) \ltimes \mathcal{L}X_\ell(p)\) of the universal enveloping \(U(\mathfrak{b})\) of the Borel subalgebra and the VOA \(\mathcal{L}X_\ell(p)\). Moreover, we can define on \(\mathcal{L}X_\ell(p)\)-modules \(\mathfrak{V}_\lambda\) a \(U(\mathfrak{b}) \ltimes \mathcal{L}X_\ell(p)\)-module structure. We let \(\mathfrak{V}_\lambda(\mu)\) denote the \(U(\mathfrak{b}) \ltimes \mathcal{L}X_\ell(p)\)-module defined by \((4.24)\) and \((4.25)\). We introduce 1-dimensional \(\mathfrak{b}\) module \(1(\mu), \mu \in \mathfrak{h}^*\) on which \(e_i\) acts by zero and \(h_i\) by multiplication with \((\alpha_i, \mu)\). We note \(\mathfrak{V}_\lambda(\mu) = \mathfrak{V}_\lambda(0) \otimes 1(\mu)\). We also use notation \(\mathfrak{V}_\lambda\) for \(\mathfrak{V}_\lambda(0)\).

\textbf{Proof of the Theorem 4.1.}\quad (1) The relations \([h_i, e_j] = c_{ij}e_j\) are checked by simple calculation using \((2.5)\).

To check the Serre relations

\[
\text{ad}_{e_i}^{c_{ij}} e_j = 0, \quad i, j = 1, \ldots, \ell, \quad i \neq j
\]
we consider \( \text{ad}_{c_{ij}}^{1} e^{\sqrt{\varphi(z)}} \). We note that \( \text{ad}_{c_{ij}}^{1} e^{\sqrt{\varphi(z)}} = P(\partial \varphi_k) e^{\sqrt{\varphi((1-c_{ij})\varphi(z)+\varphi(z))}} \), where \( P \) is a differential polynomial in \( \partial \varphi_k, k = 1, \ldots, \ell \). This statement is true only when we chose the braiding (2.11). The polynomial \( P \) should have a nonnegative conformal weight \( \Delta_p \). The field \( \text{ad}_{c_{ij}}^{1} e^{\sqrt{\varphi(z)}} \) has conformal weight 1 because \( e_i \) are screening operators and \( e^{\sqrt{\varphi(z)}} \) is a screening current. A direct calculation with (2.13) gives the conformal weight of \( e^{\sqrt{\varphi((1-c_{ij})\varphi(z)+\varphi(z))}} \) being equal to \( 2 - c_{ij} \), which means that the balance of weights \( 1 = 2 - c_{ij} + \Delta_p \) can not be satisfied with a nonnegative weight \( \Delta_p \). Thus, \( \text{ad}_{c_{ij}}^{1} e^{\sqrt{\varphi(z)}} = 0 \).

(2) The statement follows from the observation that \( L\mathcal{X}_\ell(p) \) decomposes into a direct sum of integrable finite dimensional representations of \( \mathfrak{b} \).

\[ \square \]

4.2. Bundles on the homogeneous space \( \mathbb{G}/\mathbb{B} \). We consider the bundle

\[ (4.27) \quad \xi_\lambda(\mu) = \mathbb{G} \times_{\mathbb{B}} \mathfrak{U}_\lambda(\mu), \quad \lambda \in \Lambda, \quad \mu \in \Gamma^\vee \]

on the homogeneous space \( \mathbb{F} = \mathbb{G}/\mathbb{B} \), where the action of \( \mathbb{B} \) on \( \mathbb{G} \) is given by the right multiplication and on \( \mathfrak{U}_\lambda \) by Theorem 4.1 with the corresponding \( \mu \). We set \( \xi_\lambda = \xi_\lambda(0) \).

We let \( \mathcal{O}(\mu) \) denote the standard 1-dimensional bundle on \( \mathbb{F} \)

\[ (4.28) \quad \mathcal{O}(\mu) = \mathbb{G} \times_{\mathbb{B}} 1(\mu) \]

with \( 1(\mu) \) defined in Remark 4.2.

**Proposition 4.3.**

\[ (4.29) \quad \xi_\lambda(\mu) = \xi_\lambda \otimes \mathcal{O}(\mu) \]

**Proof.** An immediate consequence of (4.25). \( \square \)

**Theorem 4.4.**

(1) \( H^n(\xi_\lambda) = 0 \), for \( n > 0 \)

(2) \( H^0(\xi_\lambda) \) is embedded into the fiber \( \mathfrak{U}_\lambda \) of the bundle \( \xi_\lambda \) over any point.

The proof of the Theorem is based on a calculation of the cohomologies in the \( A_1 \) case. To do that we should recall notations from [16]. In [16], indecomposable modules of the quantum group \( \mathfrak{U}_q \mathfrak{s}\ell(2) \) were described. We need the following of them

- for \( a = \pm \) and \( s = 1, \ldots, p \), \( \mathcal{X}_s^a \) are \( s \) dimensional irreducible modules;
- for \( a = \pm \), \( s = 1, \ldots, p - 1 \) and integer \( n \geq 2 \), the module \( \mathcal{M}_s^a(n) \) is an indecomposable module with socle \( \oplus_1^n \mathcal{X}_s^a \) and the quotient \( \oplus_1^{n-1} \mathcal{X}_{p-s}^a \);
- for \( a = \pm \), \( s = 1, \ldots, p - 1 \) and integer \( n \geq 2 \), the module \( \mathcal{W}_s^a(n) \) is an indecomposable module with socle \( \oplus_1^n \mathcal{X}_s^a \) and the quotient \( \oplus_1^{n-1} \mathcal{X}_{p-s}^a \).

Taking an equivalence of \( \mathfrak{U}_{e_{1}\pm p}\mathfrak{s}\ell(2) \) and \( \mathcal{W}\Lambda_1(p) \) representation categories [12] into account we assume the same notations for the corresponding \( \mathcal{W}\Lambda_1(p) \) modules. In what follows we need also condensed notation. For \( \lambda = \frac{1-a}{2}\sqrt{p}\omega_1 + (1-s)\lambda_1 \), we set

\[ (4.30) \quad \mathcal{M}_\lambda(\mu) = \mathcal{X}_s^a, \quad \text{for } \mu = 0, \]

\[ (4.31) \quad \mathcal{M}_\lambda(\mu) = \mathcal{M}_{p-s}^-(\mu + 1), \quad \text{for } \mu > 0, \]

\[ (4.32) \quad \mathcal{M}_\lambda(\mu) = \mathcal{X}_s^a, \quad \text{for } \mu = -1, \]
(4.33) \( \mathcal{M}_\lambda(\mu) = \mathcal{W}_a^\alpha(-\mu), \quad \text{for} \ \mu < -1 \)

We also set \( \mathcal{X}_\lambda = \mathcal{X}_a^\alpha. \)

**Lemma 4.5.** In \( A_1 \) case we have

for \( \mu \geq 0 \)

(4.34) \( H^0(\xi_\lambda(\mu)) = \mathcal{M}_\lambda(\mu), \)

(4.35) \( H^1(\xi_\lambda(\mu)) = 0, \)

for \( \mu \leq -1 \)

(4.36) \( H^0(\xi_\lambda(\mu)) = 0, \)

(4.37) \( H^1(\xi_\lambda(\mu)) = \mathcal{M}_\lambda(\mu), \)

**Proof.** The irreducible \( \mathcal{L}A_1(p) \)-module \( \mathcal{V}_\lambda(0) \) is a reducible module of \( \mathcal{W}L\mathcal{A}_1(p) \) \([15]\) and its structure can be described by the following exact sequence

(4.38) \[ 0 \to \mathcal{X}_\lambda \to \mathcal{V}_\lambda(0) \to \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}} \to 0. \]

The spaces \( \mathcal{X}_\lambda \) and \( \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}} \) are irreducible \( \mathcal{W}L\mathcal{A}_1(p) \)-modules and at the same time they bear the action of \( b \) induced by the action of \( b \) on \( \mathcal{V}_\lambda(0) \). The action of \( b \) on \( \mathcal{X}_\lambda \) is extended to an action of \( s\ell(2) \) \([3]\). This means that the bundle \( \xi_\lambda(0) \) contains the trivial subbundle \( \bar{\xi}_\lambda(0) = \mathbb{CP}^1 \times \mathcal{X}_\lambda \). We note that the action of \( b \) on the quotient \( \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}} \) is not extended to an \( s\ell(2) \) action but the action of \( b \) on \( \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}} \otimes 1(1) \) is extended to the \( s\ell(2) \) action. For the quotient bundle we have

(4.39) \[ \xi_\lambda(0)/\bar{\xi}_\lambda(0) = (\mathbb{CP}^1 \times \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}}) \otimes O(-1). \]

(The RHS means tensor product of the trivial bundle \( \mathbb{CP}^1 \times \mathcal{X}_{\alpha_1/\sqrt{\varpi - \lambda}} \) with \( O(-1) \).) This bundle has zero cohomology because \( H^i(O(-1)) = 0 \). This shows the Lemma for \( \mu = 0 \) and \( \mu = -1 \). Other statements of the Lemma are obtained by multiplying with \( O(\mu) \). Obviously, we have \( H^0(\xi_\lambda(\mu)) = H^0(\xi_\lambda(0)) \otimes \mathbb{C}^\mu \) for \( \mu > 0 \) and \( H^1(\xi_\lambda(\mu)) = H^1(\xi_\lambda(-1)) \otimes \mathbb{C}^{-\mu} \) for \( \mu < 0 \). Indecomposability of modules appearing in the RHS of (4.34) and (4.37) follows from the observation that the cohomology has nontrivial mappings on the corresponding Verma modules. \( \square \)

**Proof of the Theorem [4.4]** The proof of the Theorem resembles a proof of Bott-Borel-Weyl theorem and we give only a brief description of it. Let \( \sigma_\alpha \) be the Weyl group element corresponding to the simple root \( \alpha \). Let \( \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_n} \) be the reduced decomposition of the longest element in the Weyl group into the product of simple reflections. We note that \( n \) is the dimension of the flag manifold \( \mathcal{F} \). We define the shifted action of the Weyl group \( \sigma_\alpha \cdot \omega = \sigma_\alpha(\omega + \sqrt{p}p) - \sqrt{p}p \). Let \( \mathcal{P}_\alpha \) be the parabolic subalgebra corresponding to the simple root \( \alpha \) and \( \mathcal{F}_\alpha = \mathcal{G}/\mathcal{P}_\alpha \). We note that \( \mathcal{F} \) is fibered over \( \mathcal{F}_\alpha \) with fibers \( \mathbb{CP}^1 \). We let \( \pi \) denote the projection \( \mathcal{F} \to \mathcal{F}_\alpha \) and \( \xi_\lambda^\alpha = \pi_* \xi_\lambda \) the direct image of \( \xi_\lambda \).

We consider two bundles \( \xi_\lambda \) and \( \xi_{\sigma_\alpha \cdot \lambda} \). Their direct images \( \xi_\lambda^\alpha \) and \( \xi_{\sigma_\alpha \cdot \lambda}^\alpha \) are the same bundle on \( \mathcal{F}_\alpha \). In more details, taking Lemma [4.5] into account, for \( \lambda \in \Lambda \), we obtain \( H^0(\xi_{\sigma_\alpha \cdot \lambda}(\mathbb{CP}^1)) = H^1(\xi_{\sigma_\alpha \cdot \lambda}(\mathbb{CP}^1)) \neq 0 \) and \( H^1(\xi_\lambda(\mathbb{CP}^1)) = H^0(\xi_{\sigma_\alpha \cdot \lambda}(\mathbb{CP}^1)) = 0 \). This gives \( H^i(\xi_\lambda) = H^{i+1}(\xi_{\sigma_\alpha \cdot \lambda}) \) by the standard Leray spectral sequence. We repeat this procedure for the
longest element in the Weyl group and obtain $H^i(\xi_\lambda) = H^{i+k}(\xi_{\sigma_1\cdot\sigma_2\cdot\ldots\cdot\sigma_k\cdot\lambda})$. But this requires $H^i(\xi_\lambda) = 0$ for $i > 0$. □

Taking Theorem 4.4 into account, we assume the following definition.

**Definition 4.6.** The vacuum module $\mathcal{X}_0$ of the vertex-operator algebra $\mathcal{W}\mathcal{L}X_\ell(p)$ is $H^0(\xi_0)$.

Other irreducible $\mathcal{W}\mathcal{L}X_\ell(p)$-modules can be obtained as follows

(4.40) $$\mathcal{X}_\lambda = H^0(\xi_\lambda), \quad \lambda \in \Lambda.$$ 

### 4.3. Generators of the vertex operator algebra $\mathcal{W}\mathcal{L}X_\ell(p)$

The system of generators $\mathcal{W}\mathcal{L}X_\ell(p)$ consists of two subsets. The first subset of generators are generators of the W-algebra $\mathcal{W}X_\ell(p)$ (see Sec. 3), which is a subalgebra in $\mathcal{W}\mathcal{L}X_\ell(p)$. The algebra $\mathfrak{g}$ acts trivially on $\mathcal{W}X_\ell(p)$. The second subset of generators span the adjoint representation of $\mathfrak{g}$.

We describe the second subset of $\mathcal{W}\mathcal{L}X_\ell(p)$ generators in details. For the generators, we introduce notation $W^\alpha$ with $\alpha \in \Delta$ and $W^{0,\alpha}$ with $\alpha \in \Pi$. Let $t^\alpha$ for $\alpha \in \Delta$ be the basis vector from weight subspace with the weight $\alpha$ and $t^{0,\alpha}$ with $\alpha \in \Pi$ be the basis in the zero weight subspace of the adjoint representation of $\mathfrak{g}$. Let $\theta \in \Delta$ be the lowest root of $\mathfrak{g}$, i.e. $\theta - \alpha \notin \Delta$ for any $\alpha \in \Delta^+$. Then $t^\theta$ be the lowest weight vector in the adjoint representation of $\mathfrak{g}$. Any root $\alpha \in \Delta$ can be written in the form

(4.41) $$\alpha = \theta + \sum_{i=1}^\ell n_i\alpha_i, \quad n_i \in \mathbb{N}_0.$$ 

Then,

(4.42) $$t^\alpha = \text{ad}_{e_j_1}\text{ad}_{e_j_2}\ldots\text{ad}_{e_j_m} t^\theta$$

where $e_i$ appears precisely $n_i$ times and

(4.43) $$t^{0,\alpha} = \text{ad}_{e_j_1}\text{ad}_{e_j_2}\ldots\text{ad}_{e_j_m} t^\theta$$

where $e_i$ appears precisely $a_i$ times with $a_i$ be labels in the Dynkin diagram. Then, we set

(4.44) $$W^\alpha(z) = \text{ad}_{e_j_1}\text{ad}_{e_j_2}\ldots\text{ad}_{e_j_m} e^{\sqrt{\Phi}(z)}$$

with precisely the same product of adjoint operators as in (4.42) and

(4.45) $$W^{0,\alpha}(z) = \text{ad}_{e_j_1}\text{ad}_{e_j_2}\ldots\text{ad}_{e_j_m} e^{\sqrt{\Phi}(z)}$$

with precisely the same product of adjoint operators as in (4.43).

The fields $W^\alpha(z)$ and $W^{0,\alpha}(z)$ have the same conformal dimensions equal to

(4.46) $$\frac{p}{2}(\epsilon^i(\alpha_i, \theta)(\alpha_j, \theta) + (1-p)(\rho, \theta)) = h(p-1) + 1.$$
5. Quantum Group

In this section we describe the quantum group $\bar{U}_q(X_\ell)$. We conjecture that its representation category is equivalent to the representation category of the vertex operator algebra $\mathcal{W}LX_\ell(p)$.

Suppose we have an algebra graded by $\mathfrak{h}^\ast$. We introduce $q$-bracket or $q$-adjoint action. Let $x$ and $y$ are two homogeneous elements from the algebra, then
\begin{equation}
qad_x y = x y - q^{\langle x, y \rangle} y x,
\end{equation}
where $q$ is given by (2.12), the scalar product $\langle \cdot, \cdot \rangle$ is defined in (2.9) and $\# x \in \mathfrak{h}^\ast$ is the weight of $x$. The most important example is the case, where $x$ and $y$ belong to the algebra generated by vertex operators and grading operators are commutators with zero modes
\begin{equation}
h_i = \sqrt{p}(\varphi_{\omega_i})_0, \quad i = 1, \ldots, \ell,
\end{equation}
where $\omega_i$ are fundamental weights.

The quantum group $\bar{U}_q(X_\ell)$ is an associative algebra with generators $E_i, F_i, K_i, K_i^{-1}$ for $i = 1, \ldots, \ell$ and 1. This algebra is graded by the root lattice $\Gamma$ and weights of generators are
\begin{equation}
\# E_i = \alpha_i, \quad \# F_i = -\alpha_i, \quad \# K_i = 0.
\end{equation}
To describe the relations, we introduce elements
\begin{equation}
L_i = \prod_{j=1}^\ell K_j^{c_{ij}}.
\end{equation}
The relations are
\begin{align}
E_i^p = F_i^p = 0, & \quad K_i K_j = K_j K_i, \quad L_i^p = 1, \\
K_i E_j K_i^{-1} = q^{2\delta_{ij}} E_j, & \quad K_i F_j K_i^{-1} = q^{-2\delta_{ij}} F_j, \\
qad_{E_i} F_j = \delta_{ij} \frac{L_i - 1}{q - q^{-1}}, & \\
qad_{E_i}^{1-c_{ij}} E_j = 0, & \quad qad_{F_i}^{1-c_{ij}} F_j = 0, \quad i \neq j.
\end{align}
This algebra is the centralizer of $\mathcal{W}LX_\ell(p)$ and we conjecture that the two algebras have equivalent representation categories. We also note that for $\ell = 1$ this algebra is isomorphic to the quantum group from [15].

The algebra $\bar{U}_q(X_\ell)$ is a braided Hopf algebra. To describe the braided Hopf algebra structure, we introduce the braided tensor product $\bar{\otimes}$. The algebra $\bar{U}_q(X_\ell) \bar{\otimes} \bar{U}_q(X_\ell)$ differs from $\bar{U}_q(X_\ell) \otimes \bar{U}_q(X_\ell)$ only by commutation relations between two multipliers. In standard tensor product elements of the form $x \otimes 1$ and $1 \otimes y$ commute while elements $x \bar{\otimes} 1$ and $1 \bar{\otimes} y$ $q$-commute
\begin{equation}
qad_{x \bar{\otimes} 1} 1 \bar{\otimes} y = 0,
\end{equation}
where $\#(x \bar{\otimes} y) = \# x + \# y$. The comultiplication $\bar{\Delta} : \bar{U}_q(X_\ell) \to \bar{U}_q(X_\ell) \bar{\otimes} \bar{U}_q(X_\ell)$ is given by
\begin{equation}
\bar{\Delta}(K_i) = K_i \bar{\otimes} K_i
\end{equation}
and
\[ (5.57) \quad \tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x \]
for \( x = E_i \) or \( F_i \).

At the end of the section, we note that the appearance of the braided Hopf algebra structure instead of Hopf algebra structure probably solves a contradiction observed in [17]. In [17] it was observed that the tensor product of \( \mathbb{U}_q\mathfrak{sl}(2) \) modules is not necessarily commutative although representation category of \( \mathbb{U}_q\mathfrak{sl}(2) \) is equivalent [12] to the representation category of \((1, p)\) model vertex operator algebra which is braided tensor category by construction.

6. Characters

In what follows we use notation
\[ (6.58) \quad z^\alpha = \prod_{j=1}^\ell z_j^{(\alpha_j, \alpha)}, \quad \alpha \in \mathfrak{h}^*. \]
We also use notation
\[ (6.59) \quad z[\alpha] = (z_1^{(\alpha_1, \alpha)}, z_2^{(\alpha_2, \alpha)}, \ldots, z_\ell^{(\alpha_\ell, \alpha)}), \]
where in the RHS we have a list of monomials. In particular, \( z[\rho] = (z_1, z_2, \ldots, z_\ell) \). For example, for a function \( f \), \( f(z[\alpha]) \) means \( f(z_1^{(\alpha_1, \alpha)}, z_2^{(\alpha_2, \alpha)}, \ldots, z_\ell^{(\alpha_\ell, \alpha)}) \). We also use notation
\[ (6.60) \quad \partial^\alpha = \prod_{j=1}^\ell \left( \frac{\partial}{\partial z_j} \right)^{(\alpha_j, \alpha)} \]
for the derivatives with respect to \( z_j \).

We define the shifted action of the Weyl group \( W \) on \( z \) by the formulas
\[ (6.61) \quad w(z^\alpha) = z^{w^{-1} \cdot \alpha}, \quad w(f(z[\alpha])) = f(z[w^{-1} \cdot \alpha]), \]
where \( w \cdot \alpha = w(\alpha - \rho) + \rho \) and \( f \) is a function in \( z \).

We introduce the denominator
\[ (6.62) \quad d(z[\rho]) = \prod_{\alpha \in \Delta^-} (1 - z^\alpha) \]
It satisfies
\[ (6.63) \quad w(d(z[\rho])) = \varepsilon(w)d(z[\rho]), \]
where \( \varepsilon(w) = (-1)^{\#w} \), where \( \#w \) is the parity of \( w \). The Weyl formula for the character of the irreducible \( g \)-module with the highest weight \( \lambda \) is
\[ (6.64) \quad \chi^0_\lambda(z^\rho) = \frac{1}{d(z^\rho)} \sum_{w \in W} \varepsilon(w)w[z^\lambda]. \]

Let \( V \) be a linear space equipped with an action of \( L_0 \) and \( h_1, \ldots, h_\ell \in \mathfrak{h} \). Then the character of \( V \) is
\[ (6.65) \quad \chi_V(q, z^\rho) = \text{Tr}_V q^{L_0 - \frac{c}{24}} z_1^{h_1} z_2^{h_2} \cdots z_\ell^{h_\ell}. \]
In what follows we take $V$ be different VOA modules.

To write the characters we introduce the theta functions

$$
\Theta_{\lambda}(q,z[\rho]) = \sum_{\alpha \in \Gamma} q^{\frac{1}{2}(\mathcal{P}_{\alpha+\lambda} - Q_{\rho})} \mathcal{P}_{\alpha+\lambda} \cdot z^{\alpha+\lambda}, \quad \lambda \in \Lambda.
$$

We note that these theta functions differs from [10] by the factor $z^{\hat{\lambda}}$. (See definitions of $\hat{\lambda}$ and $\hat{\lambda}$ in (2.18) and (2.19).)

The character of the irreducible $\mathcal{L}X_\ell(p)$-module $\mathcal{Y}_\lambda$ is

$$
\psi_\lambda(q,z[\rho]) = \frac{\Theta_{\lambda}(q,z[\rho])}{\eta(q)^{\ell}}, \quad \lambda \in \Lambda.
$$

To calculate the character $\chi_{\lambda}(q,z[\rho])$ of the irreducible $\mathcal{W}LX_\ell(p)$-module $\mathcal{X}_\lambda$, we use (4.40) and Lefschetz formula. We defined $\mathcal{X}_\lambda$ in (4.40) so the character of $\mathcal{X}_\lambda$ is

$$
\chi_{\lambda}(q,z[\rho]) = \text{Tr}_{H_0(\xi_\lambda)} q^{L_0} - \mathcal{P}_{\rho}^2 \mathcal{P}_{z_1}^{h_1} \mathcal{P}_{z_2}^{h_2} \ldots \mathcal{P}_{z_\ell}^{h_\ell} = \sum_{w \in W} \text{Tr}_{M_w} q^{L_0} - \mathcal{P}_{\rho}^2 \mathcal{P}_{z_1}^{h_1} \mathcal{P}_{z_2}^{h_2} \ldots \mathcal{P}_{z_\ell}^{h_\ell}
$$

This expression is a sum over fixed points of the standard action of the torus $H$ on the flag manifold $F$. The fixed points are in one to one correspondence with the elements of the Weyl group $W$. Contribution of each fixed point $x_w$ is determined in the following way. We consider the space of sections of the bundle $\xi_\lambda$ in the formal neighborhood of $x_w$. Contribution of the point is $\text{Tr}_{M_w} q^{L_0} - \mathcal{P}_{\rho}^2 \mathcal{P}_{z_1}^{h_1} \mathcal{P}_{z_2}^{h_2} \ldots \mathcal{P}_{z_\ell}^{h_\ell}$. The space $M_w$ as a module of $L_0$ and $h_1, \ldots, h_\ell$ is the tensor product $\mathcal{Y}_\lambda \otimes \mathcal{F}_{x_w}$, where $\mathcal{F}_{x_w}$ is the ring of formal series in $x_w$. Thus,

$$
\text{Tr}_{M_w} q^{L_0} - \mathcal{P}_{\rho}^2 \mathcal{P}_{z_1}^{h_1} \mathcal{P}_{z_2}^{h_2} \ldots \mathcal{P}_{z_\ell}^{h_\ell} = \frac{\psi_\lambda(q,z[w(\rho)])}{d(z[w(\rho)])}.
$$

Summarizing, we have

$$
\chi_{\lambda}(q,z[\rho]) = \sum_{w \in W} \frac{\psi_\lambda(q,z[w(\rho)])}{d(z[w(\rho)])} = \frac{1}{d(z[\rho])} \sum_{w \in W} \epsilon(w) \psi_\lambda(q,z[w(\rho)]).
$$

We rewrite this expression for characters in different form.

**Theorem 6.1.** The character $\chi_{\lambda}(q,z^\rho)$ of the irreducible $\mathcal{W}LX_\ell(p)$-module $\mathcal{X}_\lambda$ can be written in the form

$$
\chi_{\lambda}(q,z^\rho) = \sum_{\alpha \in \Gamma^+} \chi_{\lambda+\alpha}^0(z[\rho]) \chi_{\lambda,\alpha}^W(q),
$$

where $\Gamma^+$ is the intersection of the root lattice with the positive Weyl chamber, $\chi_{\lambda+\alpha}^0(z[\rho])$ is the character of the irreducible $\mathcal{g}$-module with the highest weight $\hat{\lambda} + \alpha$ ($\hat{\lambda}$ is defined in (2.18)) and $\chi_{\lambda,\alpha}^W(q)$ is the character of the irreducible $\mathcal{W}X_\ell(p)$-module given by [7]

$$
\chi_{\lambda,\alpha}^W(q) = \frac{1}{\eta(q)^{\ell}} \sum_{w \in W} q^{\frac{1}{2}(\mathcal{P}_{z[\rho]} + \lambda + \frac{1}{2} \mathcal{P}_{\rho}^2 + \mathcal{P}_{z[\rho]} + \lambda + \frac{1}{2} \mathcal{P}_{\rho}^2)}. $"
We note that (6.72) are $WX_\ell(p)$-characters of general type, i.e. irreducible modules with such characters exist for any value of the central charge.

The Theorem 6.1 allows us to prove that $WLX_\ell(p)$-module $x_\lambda$ is irreducible. Taking (6.71) into account, we have

\[(6.73)\]
\[
x_\lambda = \bigoplus_{\alpha \in \Gamma^+} R_{\lambda+\alpha} \otimes Y_{\lambda,\alpha},
\]

where $R_{\lambda+\alpha}$ is irreducible $g$ module and $Y_{\lambda,\alpha}$ is irreducible $WX_\ell(p)$ module. Then the absence of invariant subspaces of $WLX_\ell(p)$ action can be obtained from a consideration of the action of generators from Sec. 4.3.

We also give formulas for the characters of irreducible $WLX_\ell(p)$ modules adapted for calculating their modular properties and calculating values of characters at $z = 1$. For this, we introduce derivatives of theta functions

\[(6.74)\]
\[
\Theta_\xi^\omega (q) = \frac{\partial^{\alpha}}{\alpha!} \Theta_\lambda(q, z) \big|_{z=1}, \quad \omega \in \Gamma^{\vee+}.
\]

Then the characters can be written in the following nice form

\[(6.75)\]
\[
\chi_\lambda(q) = \frac{1}{\eta(q)} \sum_{\omega \in \Gamma^{\vee+}} \frac{\zeta_\omega}{\prod_{i=1}^{\ell} ((\alpha_i, \omega)!) \Theta_\xi^\omega (q)},
\]

where

\[(6.76)\]
\[
\zeta_\omega = \sum_{\mu \in \Gamma^{\vee}} (-1)^{(\rho,\mu)} \prod_{j=1}^{\ell} \left( \frac{\alpha_j, \omega}{\alpha_j, \mu} \right) \prod_{\alpha \in \Delta^+} \left( 1 + \frac{\alpha, \mu}{\alpha, \rho} \right), \quad \omega \in \Gamma^{\vee+}
\]

and $\binom{n}{m}$ are binomial coefficients.

7. CONCLUSIONS

A natural development of the results is a calculation of the $SL(2, \mathbb{Z})$ action on the characters (6.75). We believe that the $SL(2, \mathbb{Z})$ action will produce finite number of characters and pseudocharacters, whose linear span will give a finite dimensional center $\bar{Z}$ of the conformal model. It is also interesting to calculate the center $Z$ of the quantum group $\bar{U}_q(X_\ell)$ and the action of $SL(2, \mathbb{Z})$ on it. We believe that $\bar{Z}$ is isomorphic to $Z$ as a representation of $SL(2, \mathbb{Z})$.

It is also interesting to generalize the results of the paper to non simply laced case and to the case of Lie superalgebras. Another interesting direction of investigations is a generalization of our results to the case of the root lattice scaled not to $\sqrt{p}$ but to $\sqrt{\frac{p}{p^\prime}}$. In the case of $A_1$, the algebra constructed in [2] can be realized in the zero cohomology of a bundle on $\mathbb{C}P^1$. Analogous construction can be done in the case of arbitrary Lie algebra.

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Appendix A. Group $\Lambda$

In the following picture long segments of straight lines are roots of $A_2$, short segments of straight lines belong to the weight lattice of $A_2$, $\bullet$s correspond to elements of the group $\Lambda$. Circled $\bullet$s correspond to Steinberg modules (irreducible $\mathcal{W}LX_\ell(p)$ modules that coincide with irreducible $LX_\ell(p)$ modules).

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