Notes on Feynman path integral-like methods of quantization on Riemannian manifolds

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Abstract

We propose an alternative method for Feynman path integrals on compact Riemannian manifolds. Our method employs the action integral

$$S(t, x, y)$$

along the shortest path between two points. The corresponding oscillatory integral operator is defined by

$$U_\chi(t)f(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_M \chi(d(x, y)) \sqrt{V(t, x, y)} e^{iS(t, x, y)} f(y) \, dy,$$

where $\chi(d(x, y))$ is the bump cut-off function with small compact support and $V(t, x, y)$ denotes van Vleck determinant. In the case of rank 1 locally symmetric Riemannian manifolds, we prove the strong convergence of time slicing products

$$\lim_{N \to \infty} \{U(t/N)\}^N$$

for low energy functions. Moreover, the strong limit includes DeWitt curvature $R/6$, where $R$ denotes the scalar curvature of a Riemannian manifold. This is an alternative rigorous formulation for Feynman path integrals on Riemannian manifolds.

1 Introduction

Let $(M, g)$ be a compact, oriented, smooth Riemannian $n$-dimensional manifold without boundary. We know that the injective radius of a compact manifold $d$ is always finite and positive [3]. It is also known that the geodesic distance function $d(x, y)$ is $C^\infty$ in a neighborhood of $(x, y) \in M \times M$ if and only if $x$ and $y$ are not conjugate points along this minimizing geodesic [33]. Thus the smooth geodesic action $S(t, x, y)$ is represented as an integral over time, taken along the geodesic path between the initial time and the final time of the development of the system:

$$S(t, x, y) = \int_0^t \frac{1}{2} g_{x(t)}(\dot{x}(t), \dot{x}(t)) \, dt = \frac{|d(x, y)|^2}{2t} \quad \text{for } d(x, y) < d,$$

and the path-density between two points [42] is given by van Vleck determinant:

$$V(t, x, y) = g^{-1/2}(x)g^{-1/2}(y) \det_{ij} \left( \frac{\partial^2 S(t, x, y)}{\partial x_i \partial y_j} \right).$$

We define a reasonable candidate for the short-time quantum propagators associated to $S$ and $V$ by a family of oscillatory integral operators.

Definition 1.1 (Shortest path approximations on $M$). The shortest path approximation $U_\chi(t)$ on $M$ is defined by

$$U_\chi(t)f(x) \equiv \frac{1}{(2\pi)^{n/2}} \int_M \chi(d(x, y)) \sqrt{V(t, x, y)} e^{iS(t, x, y)} f(y) \, dy,$$

where $\chi(d(x, y))$ is the bump function with compact support contained in $d(x, y) < d$.

It is emphasized that an approximate “local” parametrix of $e^{it\triangle}$ is based on the above form for extremely short time interval (See e.g. [18]). For the purpose of local to global in time consistency, we also introduce Feynman path integral-like methods of quantization by the limit of time slicing products:

$$\lim_{N \to \infty} \{U_\chi(t/N)\}^N. \quad (1)$$
This paper aims to give the meaning of this limit and to evaluate the limit if it converges. However, we have not accomplished this task for general compact manifolds. So, as the first observation we restrict ourselves to the class of rank 1 locally symmetric Riemannian manifolds, including the $n$-dimensional sphere $S_n$ and hyperbolic manifolds etc. (See e.g. [19] and §4). Our main result is the following:

**Main theorem** (Time slicing products and the strong limit (See §4)). Let $(M,g)$ be a compact, oriented, rank 1 locally symmetric Riemannian manifold. For $f(x) \in L^2(M)$,

$$
\lim_{N \to \infty} \{|U_N(t/N)|^N \rho(N)f(x) - \sum_{j=1}^{N} e^{tE_j}u_j(x)u_j(y)\} = 0 \quad \text{in} \ L^2,
$$

where $R$ means the scalar curvature and $\rho(N)$ is a spectral measure defined by the spectral theorem $-\Delta = \int_0^\infty E \ dp(E)$.

If $f(x)$ is a low energy function (i.e. a finite sum of eigenfunctions of $-\Delta$), the convergence of time slicing products is given without spectral projectors:

**Corollary 1.2.** Under the hypothesis of Main Theorem, if $f(x) = \sum_{j=1}^{\infty} u_j(x)$ is a finite sum of Laplace eigenfunctions, then

$$
\lim_{N \to \infty} \{|U_N(t/N)|^N f(x) - \sum_{j=1}^{N} e^{tE_j}u_j(x)u_j(y)\} = 0 \quad \text{in} \ L^2.
$$

This is an analogous result for Feynman path integral proposed by means of finite dimensional approximations and Trotter type time slicing products (See e.g. [12], [13], [14], [15], [22], [23], [24], [25], [28], [29], [43]). In these papers, the stationary action trajectories are finite for fixed time $t > 0$ and $x, y \in \mathbb{R}^n$, and the kernel $E(t,x,y)$ of $e^{it(-\Delta + V(x))}$ are bounded smooth for small $t \neq 0$. Thus time slicing products converge without spectral projectors.

On compact manifolds, however, infinite many action paths exist, even if time $t > 0$ is fixed. To clarify the meaning of spectral projectors, we consider the quantum evolution $e^{it(-\Delta + \frac{\Box}{\alpha})}$ on $M$. By Stone’s theorem (See e.g. [30]), $e^{it(-\Delta + \frac{\Box}{\alpha})}$ are unitary operators and the kernel are given by

$$
E(t,x,y) = \sum_{E_j} e^{-itE_j}u_j(x)u_j(y)
$$

where $\{u_j(x)\}$ is eigenfunction expansion of $-\Delta + \frac{\Box}{\alpha}$ and $E_j$ are eigenvalues. The behavior of $E(t,x,y)$ is quite singular (See e.g. [29], [31], [40], [41], [44]). Nevertheless, when we sum a finite number of terms in $E$, $E_{\text{finite}}(t,x,y)$ are smooth and we may intuitively choose classical shortest paths for low energy $E$. Accordingly we may define the heuristic approximation for Feynman path integration by $\{U_N(t/N)\}^N \rho(N^{1/\alpha-\epsilon})$. Indeed, uniform convergences are proven in §3:

**Proposition 3.1.** For $\alpha = 2 + \frac{(n+2)}{2}$ and small $\epsilon > 0$,

$$
\lim_{N \to \infty} \|\{U_N(t/N)\}^N - \sum_{j=1}^{N} e^{tE_j}u_j(x)u_j(y)\|_{L^2} = 0.
$$

In §4 the strong convergence is assured by $L^2$ estimates, replacing $\rho(N^{1/\alpha-\epsilon})$ with $\rho(N)$. Also notice [30] that this convergence is not uniform without spectral projectors.

Another way to understand the low energy is WKB method in which well-known $h$-small semiclassical calculus gives the low energy good parametrices of Schrödinger operators (See for instance [8] p.581), [35]). So the low energy approximation is just a rewording WKB method in less $h$-small terminology.

We end this introduction with some reasons why the amplitude is considered as van Vleck determinant. In physics literature, Feynman is saying in his book [11] that each trajectory contributes to the total amplitude to go from $a$ to $b$ and that they contribute equal to the amplitude, but contribute at different phases (i.e. the amplitude is constant in the original idea). Some researchers however propose that the amplitude should be given by the density of trajectories [37]. Since the limit (1) converges to Schrödinger operator $e^{it(-\Delta + V(x))}$ in the Euclidean case and the amplitude of its kernel reduces to the expression with van Vleck determinant (See e.g. [14]). One more reason to consider van Vleck determinants is the accuracy of convergence (See §3). Thus it is natural to employ the amplitude as the density of trajectories. For the heat semi-group case, this ambiguity in the path integral is discussed from a strictly mathematical viewpoint [6].
2 Preliminaries

We start out by giving the properties of geodesic flows, van Vleck determinants and stationary phase methods on $M$. Throughout §2, we only assume $(M, g)$ is a compact, oriented, smooth Riemannian $n$-dimensional manifold without boundary.

Recall that the tangent space $T^*M$ admits a symplectic structure, which can be expressed locally as $\sum dx^i \wedge dp_i$. Consider the smooth function defined on $T^*M$ by

$$H(x, p) = \frac{1}{2}g^{ij}p_ip_j \equiv \sum_{i,j} \frac{1}{2}g^{ij}p_ip_j.$$ 

Here we often use Einstein summation convention as above, that is, in any expression containing subscripted variables appearing twice (and only twice) in any term, the subscripted variables are assumed to be summed over.

The Hamiltonian vector field of $H$ is $X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p}$ and its exponential map $\exp tX_H : T^*M \to T^*M$ is called the geodesic flow. It is well-known \[1\] that the Legendre transform gives $p_i(t) = g_{ij}(t)v^j(t) \equiv g_{ij}(t)\dot{x}^j(t)$ for any geodesic $x(t)$ and the action integral is denoted by:

$$S(t, x, y) = \int_0^t \frac{1}{2}g_{ij}(t)\dot{x}^i(t)\dot{x}^j(t) \, dt = \frac{|d(x, y)|^2}{2t}, \quad \text{for } d(x, y) < \frac{\sqrt{2}}{\sqrt{g}}.$$ 

We also recall the definition of Riemann normal coordinates (See e.g. \[33\] for details). To define a system of Riemann normal coordinates, one needs to pick a point $P$ on the manifold which will serve as origin and a basis for the tangent space at $P$. To any $n$-tuple of real numbers $(x_1, \cdots, x_n)$, we shall assign a point $Q$ of the manifold by the following procedure:

Let $v$ be the vector whose components with respect to the basis chosen for the tangent space at $P$ are $x_1, \cdots, x_n$. There exists a unique affinely-parameterized geodesic $x(t)$ such that $x(0) = P$ and $\left[\frac{dx(t)}{dt}\right]_{t=0} = v$. Set $Q = x(1)$. Then $Q$ is defined to be the point whose Riemann normal coordinates are $(x_1, \cdots, x_n)$ for $d(P, Q) < \frac{\sqrt{2}}{\sqrt{g}}$. Riemann normal coordinates enjoy several important properties:

1. The connection coefficients $\Gamma^\alpha_{\beta\gamma}$ vanish at the origin of Riemannian normal coordinates.
2. Covariant derivatives reduce to partial derivatives at the origin of Riemann normal coordinates.
3. The partial derivatives of the components of the connection evaluated at the origin of Riemann normal coordinates equals the components of the curvature tensor.

Under these circumstances, we consider the space initial data of the geodesic initial data problem on the tangent bundle $TM$. Equivalently the cotangent bundle $T^*M$ due to the existence of metric and Legendre transforms. Trading the dependence on initial momenta by the dependence on the final positions defines a map $T^*M \to M \times M$ whose Jacobian as will be shown below is the van Vleck determinant. From the derivation it will be explicit that is the inverse of Jacobian of the geodesic exponential map.

**Lemma 2.1.** Let $d(x, y) < \frac{\sqrt{2}}{\sqrt{g}}$. For Riemann normal coordinates at $y$, we have

$$V(t, x, y) = t^{-n} \{\det(g_{ij}(x))\}^{-1/2}.$$ 

**Proof.** Remark that the time $t$ means the scaling of $S(t, x, y)$, we may only prove the theorem for the case of $t = 1$. The Liouville measure on $T^*M$ is given by

$$d\mu_L = dx^1(0) \wedge dp_i(0).$$ 

The Hamilton-Jacobi function is equal to the action integral along the geodesic between the initial point and final points:

$$S(1, x, y) = S(1, x(1), x(0)) = \frac{1}{2} \int_{x(0), t=0}^{x(1), t=1} dt \sum_{i,j} g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t).$$ 

$S(t, x, y)$ satisfies the Hamilton-Jacobi equation:

$$\frac{1}{2} \sum_{i,j} g^{ij}(x(0)) \frac{\partial S}{\partial x^i(0)} \frac{\partial S}{\partial x^j(0)} = E,$$
where $E = \frac{1}{2} \sum_{i,j} g_{ij}(x(t)) \dot{p}^i(t) \dot{p}^j(t)$ is the invariant energy. The initial momentum along the geodesic can be derived from Hamilton-Jacobi function by:

$$p_i(0) = \frac{\partial S}{\partial x^i(0)}.$$ 

Thus the transformed Liouville measure on $M \times M$ is given by:

$$\wedge_i dx^i(0) \wedge_i dp_i(0) = \det \left( \frac{\partial^2 S}{\partial x^2(0) \partial x^2(1)} \right) \wedge_i dx^i(0) \wedge_i dx^i(1)$$

$$= \frac{\det(\frac{\partial^2 S}{\partial x^2(0) \partial x^2(1)})}{\sqrt{\det g(x(0)) \det g(x(1))}} \sqrt{\det g(x(0))} \wedge_i dx^i(0) \sqrt{\det g(x(1))} \wedge_i dx^i(1)$$

$$= V(1, x(1), x(0)) \sqrt{\det g(x(0))} \wedge_i dx^i(0) \sqrt{\det g(x(1))} \wedge_i dx^i(1)$$

This leads to the alternative expression of van Vleck determinant:

$$V(1, x, y) = \sqrt{\det g(x(0))}^{-1} \sqrt{\det g(x(1))}^{-1} \det(\frac{\partial p(0)}{\partial x(1)}).$$

Now by definition:

$$x(1) = \exp^1_{x(0)}(v(0))$$

where the velocity: $v(0) \in T_{x(0)}$ is given by the Legendre transform:

$$v^i(0) = \sum_j g^{ij}(x(0)) p_j(0).$$

So we have

$$\sqrt{\det g(x(1))} = \det(d \exp^1_{x(0)})$$

$$= \sqrt{\det g(x(0))}^{-1} \sqrt{\det g(x(1))} \det(\frac{\partial x(1)}{\partial v(0)})$$

$$= \sqrt{\det g(x(0))}^{-1} \sqrt{\det g(x(1))} \det g(x(0)) \det(\frac{\partial x(1)}{\partial p(0)})$$

$$= V(1, x(1), x(0))^{-1}.$$

Next, we provide calculations of the kernel of $U_\chi(t)$. In the polar normal coordinates around $y$, the Laplacian looks like

$$\triangle f = \frac{\partial^2 f}{\partial r^2}(x) + H(x, y) \frac{\partial f}{\partial r}(x) + \frac{1}{r^2} \Delta_S f(x)$$

where $r = d(y, x)$, $S$ is the geodesic sphere of radius $r$ centered at $y$, $\Delta_S$ is the Laplacian to $S$ and $H(x, y)$ is the total mean curvature at $x$ of $S$ (See e.g. [33]).

Letting $\tilde{K}(t, x, y) = \sqrt{V(t, x, y)} e^{S(t, x, y)} a(t, r) \exp \left( \frac{ir^2}{2t} \right)$,

$$\triangle \tilde{K}(t, x, y) = \frac{\partial^2}{\partial r^2} \left( a(t, r) e^{i \frac{r^2}{2t}} \right) + H(x, y) \frac{\partial}{\partial r} \left( a(t, r) e^{i \frac{r^2}{2t}} \right)$$

$$= \left\{ \frac{\partial^2 a}{\partial r^2}(t, r) + \frac{2ir}{t} \frac{\partial a}{\partial r}(t, r) - \frac{r^2 a(t, r)}{t^2} + \frac{ia(t, r)}{t} + H(x, y) \frac{\partial a}{\partial r}(t, r) + \frac{ira(t, r) H(x, y)}{t} \right\} e^{i \frac{r^2}{2t}},$$

$$i \frac{\partial}{\partial t} \tilde{K}(t, x, y) = i \frac{\partial}{\partial t} \left( a(t, r) e^{i \frac{r^2}{2t}} \right) = \left\{ \frac{\partial a}{\partial t}(t, r) + \frac{r a(t, r)}{2t^2} \right\} e^{i \frac{r^2}{2t}}.$$

Summarizing the calculations, we have

$$\left\{ \frac{\partial}{\partial t} + \frac{1}{2} \Delta_x \right\} \tilde{K}(t, x, y) = \left\{ \frac{1}{2} \frac{\partial^2 a}{\partial r^2}(t, r) + \frac{H(x, y) \partial a}{\partial r}(t, r) \right\} e^{i \frac{r^2}{2t}}$$

$$+ i \left\{ \frac{a}{2t} (1 + r H(x, y)) + \frac{r}{t} \frac{\partial a}{\partial r}(t, r) + \frac{\partial a}{\partial t}(t, r) \right\} e^{i \frac{r^2}{2t}}.$$
Recall that \( a = a(t, r) \) satisfies the transport equation (See e.g. [16, §7.3]):

\[
\frac{\partial a}{\partial t} + \nabla a \cdot \nabla S + \frac{a}{2} \Delta S = \frac{\partial a}{\partial t} + \frac{r}{t} \frac{\partial a}{\partial r} + \frac{a}{2t}(1 + rH(x, y)) = 0.
\]

It follows that

\[
\left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta_x \right) \hat{K}(t, x, y) = \left\{ \frac{1}{2} \frac{\partial^2 a}{\partial r^2}(t, r) + \frac{H(x, y)}{2} \frac{\partial a}{\partial r}(t, r) \right\} e^{\frac{1}{2} \Delta_x} = \left\{ \frac{1}{2} \Delta_x a(t, r) \right\} e^{\frac{1}{2} \Delta_x}.
\]

**Lemma 2.2.** \( \left[ \frac{1}{2} \frac{\partial^2 a}{\partial r^2}(t, r) + \frac{H(x, y)}{2} \frac{\partial a}{\partial r}(t, r) \right]_{r=0} = \frac{1}{2} \Delta_x a(t, x, y) \big|_{x=y} = \frac{1}{2} \chi \frac{R(y)}{6} \)

**Proof.** Taking Riemann normal coordinates at \( y \), it is known [33] that

\[
\left\{ \begin{array}{l}
g_{ij}(x) = \delta_{ij} - \frac{1}{2} R_{ijkl}(y) x^k x^l + O(|x|^3), \\
\sqrt{\det g(x)} = 1 - \frac{1}{2} R_{ij}(y) x^i x^j + O(|x|^3). 
\end{array} \right.
\]

Here \( R_{ijkl} \) and \( R_{ij} \) denote the purely covariant version of Riemann curvature tensor and the Ricci curvature tensor. From Lemma 2.1,

\[
a(t, x, y) = \sqrt{V(t, x, y)} = t^{-n/2} \{\det(g(x))\}^{-1/4}.
\]

Remarking that \( \Delta = \sum_i \frac{\partial^2}{\partial x_i^2} \) at \( x = y \),

\[
\Delta_x a(t, x, y) \big|_{x=y} = t^{-n/2} \Delta_x \{\det(g(x))\}^{-1/4} \big|_{x=y} = t^{-n/2} \Delta_x \left( 1 + \frac{1}{12} R_{ij}(y) x^i x^j + o(|x|^2) \right) \big|_{x=y} = t^{-n/2} \frac{R(y)}{6}
\]
as desired. \( \square \)

For \( \hat{K}(t, x, y) \equiv \chi(d(x, y)) K(t, x, y) = \chi(d(x, y)) a(t, x, y) e^{\frac{\Delta_x}{2} \chi} \), we obtain

\[
\left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta_x \right) (\chi K(t, x, y)) = \chi \left\{ \frac{1}{2} \Delta_x a(t, x, y) \right\} e^{\frac{1}{2} \Delta_x} + \frac{1}{2} (\Delta_x \chi) K(t, x, y) + \frac{1}{2} \left( 2 \nabla_x \chi \cdot \nabla_x K(t, x, y) \right)
\]

where \( \nabla_x \) is gradient with respect to \( x \). Seeing this, we define the error integral \( E_{\chi_1}(t) \) and \( E_{\chi_2}(t) \) by

\[
\begin{aligned}
E_{\chi_1}(t) f(x) &\equiv \int_M \chi \left\{ \frac{1}{2} \Delta_x a(t, x, y) \right\} e^{\frac{1}{2} \Delta_x} f(y) dy, \\
E_{\chi_2}(t) f(x) &\equiv \int_M \nabla_x \chi \cdot \nabla_x K(t, x, y) f(y) dy.
\end{aligned}
\]

From Lemma 2.2, \( \chi \left\{ \frac{1}{2} \Delta_x a(t, x, y) \right\} + \frac{1}{2} (\Delta_x \chi) a(t, x, y) \big|_{d=0} = \frac{R(t)}{12 \pi^2} \). In order to estimate \( U(t, \chi_1(t)) \) and \( E_{\chi_2}(t) \), we state the method of stationary phase where \( S(t, x, y) \) is a quadratic form, which is convenient here (See [20, Lemma 7.7.3]).

**Lemma 2.3.** Let \( A \) be a symmetric non-degenerate matrix with \( \text{Im} A \geq 0 \). Then we have for every integer \( k > 0 \) and integer \( s > n/2 \)

\[
\left| \int_{\mathbb{R}^n} u(x) e^{\frac{i}{2} A x^2} dx - (\det(A/2\pi i t))^{-\frac{1}{2}} \sum_{j=0}^{k-1} (-it)^j (A^{-1} D^j u(0))/j! \right| \leq C_k(\|A^{-1}\|^{n/2+k} \sum_{|\alpha| \leq 2k+s} \|D^\alpha u\|_{L^2(\mathbb{R}^n)}) \quad \text{for} \ \ u(x) \in S(\mathbb{R}^n).
\]
The right hand side in the above lemma is just the Sobolev norm:

$$\| \cdot \|_{H^{2k+r}(\Omega)} = \sum_{|\alpha| \leq 2k+s} \| D^\alpha \cdot \|_{L^2(\Omega)}.$$ 

Letting $A = I$ (unit matrix), we obtain stationary phase lemma in the polar coordinate system of $\mathbb{R}^n$.

**Corollary 2.4.** Let $\chi(r) \in C_0^\infty(\mathbb{R})$ be the bump function with compact support contained in $|r| < \frac{d}{s}$. Then we have for every integer $k > 0$ and integer $s > n/2$

$$\left| \int_{S^{n-1}} \int_{\mathbb{R}} \chi(r) \, u(r, \theta) e^{i \frac{2 \pi}{s} g_{st}(r, \theta) \, drd\theta} - (2\pi i)^{n/2} \sum_{j=0}^{k-1} (it \Delta_{flat}/2)^j u(0)/j! \right| \leq \tilde{C} k^{n/2+k} \| \chi u \|_{H^{2k+r}(\Omega_\alpha)}$$

for $u(r, \theta) \in C^\infty(\mathbb{R}^n, \mathbb{C})$,

where $g_{st}(r, \theta) drd\theta$ denotes the spherical volume form on $S^{n-1}(r)$ and $\Omega_\alpha = \{ x \in \mathbb{R}^n \mid r = |x| < \frac{d}{s} \}$.

To obtain the criteria for compact Riemannian manifolds, we recall Gauss’s lemma \[33\] which asserts that the line element for geodesic polar coordinates on $M$ is given by

$$ds^2 = dr^2 + g_{ij}(r, \theta) d\theta_i d\theta_j.$$ 

In particular, letting $r \to 0_+$, we know:

$$g(r, \theta) = \frac{\sqrt{\det g_{ij}(r, \theta)}}{r^{n-1}} \to 1.$$ 

So

$$g_{st}(r, \theta)/g(r, \theta) \to 1. \quad (2)$$

By Corollary 2.4 and putting $k = 1$ and $k = 2$, we have

**Proposition 2.5.** Let $\alpha = 2 + \frac{1}{2} \left[ \frac{n+2}{2} \right]$. For $d < \frac{4}{s}$ and $x \in M,$

$$|U_\chi(t) f(x) - f(x)| \leq C t \| (-\Delta + 1)^{\alpha+1} f \|_{L^2(M)},$$

$$|E_\chi_1(t) f(x) - \frac{R(x)}{12} f(x)| \leq C t \| (-\Delta + 1)^{\alpha+1} f \|_{L^2(M)}$$

and

$$|E_\chi_2(t) f(x)| \leq C t \| (-\Delta + 1)^{\alpha} f \|_{L^2(M)} \quad \text{for} \quad f(x) \in C^\infty(M).$$

**Proof.** Take $x$-centered geodesic polar coordinate and $\Omega_{R_+} = \{ y \in S^2 \mid d(x, y) < R \}$.

$$|U_\chi(t) f(x) - f(x)| = \left| \frac{1}{(2\pi i)^{n/2}} \int_M \chi(d(x, y)) \sqrt{V(t, x, y)} e^{iS(t, x, y)} f(y) \frac{g_{st}(r, \theta)}{g_{st}(r, \theta)} g_{st}(r, \theta) drd\theta - f(x) \right|$$

Note that $\frac{g_{st}(r, \theta)}{g_{st}(r, \theta)} |_{r=0} = 1$ and $\chi(d(x, y)) \sqrt{V(t, x, y)} |_{d=0} = t^{-n/2}$ from Lemma 2.1 and $g_{ij}(x)|_{d=0} = \delta_{ij}$. By Corollary 2.4 and putting $k = 1$,

$$|U_\chi(t) f(x) - f(x)| \leq C t \| \chi f \|_{H^{2k+r}_{flat}(\Omega_{R_+})}.$$ 

Similarly $| \{ x \left\{ t \Delta_a a(t, x, y) \right\} + \frac{1}{2} \left( \Delta_a \chi \right) a(t, x, y) \}|_{r=0} = R(x)/12 t^{n/2}$, it follows that

$$|E_\chi_1(t) f(x) - R(x)/12 | \leq C t \| \chi f \|_{H^{2k+r}_{flat}(\Omega_{R_+})}.$$
To estimate $E_{\chi_2}(t)$ we need to put $k = 2$, since $\nabla_x K(t, x, y)$ gives higher-order singular $1/t$ terms. $\nabla_x \chi = 0$ on the neighborhood of $d = 0$ and so Corollary 2.4 can be applied to $E_{\chi_2}(t)$:

$$\left| E_{\chi_2}(t) f(x) \right| \leq c_3 t \| \chi f \|_{H^{k+2}(\Omega_{R^2})}.$$ 

Thus we only have to use

$$\| \chi f \|_{H^{k+2}(\Omega_{R^2})} \leq c_4 \| (-\Delta + 1)^{k/2} f \|_{L^2(M)}$$

on local charts (See e.g. [7]). We mention shortly this inequality for the reader’s convenience.

Take one “atlas” $\mathcal{A}$. Making the change of variables $y = T(x)$ and using $g_{\text{flat}} \sim g$ on small local charts,

$$\| \chi f \|_{H^{k+2}(\Omega_{R^2})} \leq c_5 \| \chi f \|_{H^k(M)}.$$ 

are equivalent under changing coordinates. Furthermore, comparing with flat and manifold metric and using $g_{\text{flat}} \sim g$ on small local charts,

$$\| \chi f \|_{H^{k+2}(\Omega_{R^2})} \leq c_6 \| \chi f \|_{H^k(M)}.$$ 

Let $\phi_i$ be a partition of unity associated to $\mathcal{A}$. Recall that $\chi$ is said to be $C^\infty$ if $\chi \circ x_i^{-1} \in C^\infty$, we find

$$\| (\phi_i \chi f) \circ x_i^{-1} \|_{H^k(M)} = \| \chi \circ x_i^{-1} [\phi_i (f) \circ x_i^{-1}] \|_{H^k(M)} \leq C_i \| [\phi_i (f) \circ x_i^{-1}] \|_{H^k(M)}$$

and summing this equation on $i$ shows $\| \chi f \|_{H^k(M)} \leq c_6 \| f \|_{H^k(M)}$ holds with $c_6 \equiv \max_i C_i$. Summarizing the calculations,

$$\| \chi f \|_{H^{k+2}(\Omega_{R^2})} \leq c_6 \| f \|_{H^k(M)}. \quad (3)$$

We apply Gårding inequality of elliptic operators to (3),

$$\| \chi f \|_{H^{k+2}(\Omega_{R^2})} \leq c_7 \| f \|_{H^k(M)} \leq C\| (-\Delta + 1)^{k/2} f \|_{L^2(M)}.$$ 

\[\Box\]

**Proposition 2.6.** Let $(M, g)$ be a compact Riemannian manifold. For $\alpha = 2 + \frac{1}{2} \frac{n+2}{n-1}$

$$\| \{ U_{\chi}(t) f(x) - e^{\frac{t}{2}(-\Delta + \frac{\alpha}{2})} f(x) \} \|_{L^2} \leq \frac{C \| f \|_{L^2}}{\| (-\Delta + 1)^\alpha f \|_{L^2}}. \quad (4)$$

\[\text{Proof.}\] For $f(x) \in C^\infty(M)$ and $E(t) = E_{\chi_1}(t) + E_{\chi_2}(t)$

$$\left( i \frac{\partial}{\partial t} + \frac{1}{2} \Delta_x \right) U_{\chi}(t) f(x) = E(t) f(x).$$

and so

$$\left( i \frac{\partial}{\partial t} + \frac{1}{2} \Delta_x - \frac{R(x)}{12} \right) U_{\chi}(t) f(x) = \left( E(t) - \frac{R(x)}{12} U_{\chi}(t) \right) f(x).$$

Letting $\hat{E}(t) = E(t) - \frac{R(x)}{12} U_{\chi}(t)$,

$$U_{\chi}(t) f(x) = e^{\frac{t}{2}(-\Delta + \frac{\alpha}{2})} \left( 1 + \int_0^t e^{-\frac{t}{2}(-\Delta + \frac{\alpha}{2}) E(s) ds} \right) f(x).$$

From Proposition 2.5

$$\| \hat{E}(t) f(x) \|_{L^2(M)} \leq \left| \left( E(t) - \frac{R(x)}{12} \right) f(x) \right| + \left| \left( \frac{R(x)}{12} U_{\chi}(t) - \frac{R(x)}{12} \right) f(x) \right| \leq C t \| (-\Delta + 1)^\alpha f \|_{L^2(M)}.$$
It follows
\[
\| \int_0^t e^{i\frac{\Delta}{\alpha}(t - \frac{s}{N})} \tilde{E}(s)f(x)ds \|_{L^2} \leq \int_0^t \| e^{i\frac{\Delta}{\alpha}(t - \frac{s}{N})} \tilde{E}(s)f(x) \|_{L^2}ds = \int_0^t \| \tilde{E}(s)f(x) \|_{L^2}ds \leq \int_0^t \tilde{C} s \| (-\Delta + 1)^\alpha f(x) \|_{L^2}ds \leq \frac{\tilde{C} t^2}{2} \| (-\Delta + 1)^\alpha f(x) \|_{L^2}
\]
as desired. 

\[\square\]

3 Feynman path integral for low energy functions on rank 1 locally symmetric Riemannian manifolds

The purpose of this section is to show the products of $U_N$’s converge uniformly for low energy functions in $L^2$. From now on, till the end of §4, we only consider rank 1 locally symmetric Riemannian manifolds. These manifolds possess nice geometric properties; in particular, they are two-point homogeneous spaces. That is the isometry group on $(M, g)$ is transitive on the set of all equidisitant point pairs. Such a situation allows us that the commutator $[\Delta, d(x, y)] = 0$ locally and the scalar curvature $R$ is constant (See e.g. [19]). Moreover $\sqrt{V}$ depends only on the distance for $d < d$ and is considered as the density of paths connecting $x$ and $y$ (See e.g. [22]). We abbreviate $U_N(t) - e^{it(\Delta - \frac{\Delta}{N})}$ to $\tilde{E}(t)$ in the following sentences.

**Proposition 3.1** (Time slicing products and energy limits). Let $(M, g)$ be a compact, oriented, rank 1 locally symmetric Riemannian manifold. For small $\varepsilon > 0$, we have
\[
\lim_{N \to \infty} \| \{U_N(t/N)\}^N - e^{it(\Delta - \frac{\Delta}{N})} \rho(N^{1/\alpha - \varepsilon}) \|_{L^2} = 0.
\]

**Proof.** From (4)
\[
\| \tilde{E}(t) \|_{L^2} \leq \frac{\tilde{C} t^2}{2} \| (-\Delta + 1)^\alpha f(x) \|_{L^2}.
\]

Let $\tilde{H} = \Delta - \frac{\Delta}{N}$. If $M$ is a rank 1 locally symmetric Riemannian manifold, $R(x)$ is a constant function and $E(t)\tilde{H} = \tilde{H}E(t)$. Consequently we have
\[
\| e^{\frac{\tilde{H}t}{N}} e^{\frac{\tilde{H}t}{N}} \cdots e^{\frac{\tilde{H}t}{N}} \tilde{E}(t/N) \tilde{E}(t/N) \cdots \tilde{E}(t/N) f(x) \|_{L^2} \leq \left( \frac{C}{2} \right)^k \left( \frac{t}{N} \right)^{2k} \| (-\Delta + 1)^\alpha f(x) \|_{L^2}.
\]
The binomical coefficients bounds $\left( \begin{array}{c} N \\ k \end{array} \right) \frac{1}{N^k} < \frac{1}{N^k}$ yields the following estimates
\[
\| \{e^{it\tilde{H}/2} - U_N(t/N)^n \} f(x) \|_{L^2} = \| \left( e^{it\tilde{H}/2} - \{e^{it\tilde{H}/2N} (1 + \tilde{E}(t/N))^k \} \right) f(x) \|_{L^2} \leq \sum_{k=1}^N \left( \begin{array}{c} N \\ k \end{array} \right) \| \{e^{i(t\tilde{H}/2N)k} (1 + \tilde{E}(t/N))^k \} f(x) \|_{L^2} \leq \sum_{k=1}^N \left( \begin{array}{c} N \\ k \end{array} \right) \left( \frac{C}{2} \right)^k \left( \frac{t}{N} \right)^{2k} \| (-\Delta + 1)^\alpha f(x) \|_{L^2} \leq \sum_{k=1}^N \frac{1}{k!} \left( \frac{Ct^2}{2N} \right)^k \| (-\Delta + 1)^\alpha f(x) \|_{L^2}.
\]
By using \( \|(-\Delta + 1)^{k\alpha}\rho(E)f(x)\|_{L^2} \leq (E + 1)^{k\alpha}\|f(x)\|_{L^2} \),

\[
\|\{e^{itH/2} - U_X(t/N)^N\}\rho(E)f(x)\|_{L^2} \leq \sum_{k=1}^{N} \frac{1}{k!} \left( \frac{C(E + 1)^{\alpha}t^2}{2N} \right)^k \|f(x)\|_{L^2} 
\leq \left[ \exp \left( \frac{C(E + 1)^{\alpha}t^2}{2N} \right) - 1 \right] \|f(x)\|_{L^2} 
\leq \frac{C_2(E + 1)^{\alpha}t^2}{2N} \|f(x)\|_{L^2}.
\]

Thus for small \( \epsilon > 0 \),

\[
\lim_{N \to \infty} \left\| \{(U_X(t/N))^N - e^{\frac{i\epsilon}{N}}\rho(N^{1/\alpha - \epsilon})\}f(x) \right\|_{L^2} \leq \lim_{N \to \infty} \frac{C_2(N^{1/\alpha - \epsilon} + 1)^{\alpha}t^2}{2N} = 0.
\]

\[\square\]

**Remark 3.2.** We note that \( s\lim_{E \to \infty} e^{\frac{i\epsilon}{N}}(\Delta - \frac{\Phi}{N}) \rho(E)f(x) = e^{\frac{i\epsilon}{N}}(\Delta - \frac{\Phi}{N}) f(x) \), so

\[
s\lim_{N \to \infty} \{(U_X(t/N))^N \rho(N^{1/\alpha - \epsilon})f(x) = e^{\frac{i\epsilon}{N}}(\Delta - \frac{\Phi}{N}) f(x) \quad \text{for} \quad \forall f(x) \in L^2(M).
\]

In §4, we show the stronger result by substituting \( \rho(N) \) for \( \rho(N^{1/\alpha - \epsilon}) \).

**Remark 3.3.** Some Trotter-Kato formulas for Feynman’s operational calculus contain infinite many spectral projectors, however we used a spectral projector once only (See [21]).

### 4 Strong limits for high energy functions

In this section, we have the strong but not uniform convergence of time slicing products. To do this, we introduce the \( L^2 \) estimates known as Hörmander and Maslov’s theorem (See e.g. [35] Theorem 2.1.1 for more details).

**Lemma 4.1.** Let \( a \in C_0(\mathbb{R}^n) \) and assume that \( \Phi \in C^\infty \) satisfies \( |\nabla \Phi| \geq c > 0 \) on supp \( a \). Then for all \( \lambda > 1 \),

\[
\left| \int_{\mathbb{R}^n} a(x)e^{i\lambda \Phi(x)} \, dx \right| \leq C_N\lambda^{-N}, \quad N = 1, 2, \ldots
\]

where \( C_N \) depends only on \( c \) if \( \Phi \) and \( a \) belong to a bounded subset of \( C^\infty \) and \( a \) is supported in a fixed compact set.

**Proof.** Given \( x_0 \in \text{supp} \ a \) there is a direction \( \nu \in S^{n-1} \) such that \( |\nu \cdot \nabla \Phi| \geq \frac{c}{2} \) on some ball centered at \( x_0 \). Thus, by compactness, we can choose a partition of unity \( \alpha_j \in C_0^\infty \) consisting of a finite number of terms and corresponding unit vectors \( \nu_j \) such that \( \sum \alpha_j(x) = 1 \) on supp \( a \) and \( |\nu_j \cdot \nabla \Phi| \geq \frac{c}{2} \) on supp \( \alpha_j \). If we set \( a_j(x) = \alpha_j(x)a(x) \), it suffices to prove that for each \( j \)

\[
\left| \int_{\mathbb{R}^n} a_j(x)e^{i\lambda \Phi(x)} \, dx \right| \leq C_N\lambda^{-N}, \quad N = 1, 2, \ldots
\]

After possibly changing coordinates we may assume that \( \nu_j = (1, 0, \ldots, 0) \) which means that \( |\partial \Phi/\partial x_1| \geq c/2 \) on supp \( a_j \). If we let

\[
L(x, D) = \frac{1}{i\lambda \Phi/\partial x_1} \frac{\partial}{\partial x_1}
\]

then \( L(x, D)e^{i\lambda \Phi(x)} = e^{i\lambda \Phi(x)} \). Consequently, if \( L^* = L^*(x, D) = \frac{\partial}{\partial x_1} \frac{1}{i\lambda \Phi/\partial x_1} \) is the adjoint, then

\[
\int_{\mathbb{R}^n} a_j(x)e^{i\lambda \Phi(x)} \, dx = \int_{\mathbb{R}^n} (L^*)^Na_j(x)e^{i\lambda \Phi(x)} \, dx.
\]

Since our assumptions imply that \( (L^*)^Na_j = O(\lambda^{-N}) \), the results follows. \[\square\]
Lemma 4.2. Suppose that \( \phi(x, y) \) is a real \( C^\infty \) function satisfying the non-degeneracy condition
\[
\det \left( \frac{\partial^2 \phi}{\partial x_j \partial y_k} \right) \neq 0
\]
on the support \( a(x, y) \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n) \). Then for \( t > 0 \),
\[
\| \int_\mathbb{R}^n e^{\frac{-x \cdot a(x, y)}{2t}} a(x, y) f(y) dy \|_{L^2(\mathbb{R}^n)} \leq Ct^{n/2} \| f \|_{L^2(\mathbb{R}^n)}.
\]
where \( C \) is indep. of \( t \) and \( f(x) \).

Proof. We note that
\[
\nabla_x [\phi(x, y) - \phi(x, z)] = \left( \frac{\partial^2 \phi(x, y)}{\partial x_j \partial y_k} \right) (y - z) + O(|y - z|^2).
\]
By using a smooth partition of unity we can decompose \( a(x, y) \) into a finite number of pieces each of which has the property that
\[
|\nabla[\phi(x, y) - \phi(x, z)]| \geq c|y - z| \quad \text{on supp} \ a,
\]
for some \( c > 0 \).
To use this we notice that
\[
\| T_t f \|^2_{L^2(\mathbb{R}^n)} = \int \int K_t(y, z) f(y) f(z) \ dy \ dz,
\]
where
\[
K_t(y, z) = \int_\mathbb{R}^n e^{\frac{\phi(x, y) - \phi(x, z)}{t}} a(x, y) a(x, z) \ dx.
\]
However, (5) and Lemma 4.1 imply that
\[
|K_t(y, z)| \leq C_N(n + 1) \| y - z \|^{-N} \quad \text{for all} \ N.
\]
By applying Schur test, the operator with kernel \( K_t \) sends \( L^2 \) into itself with norm \( O(t^n) \). This along with (6) yields
\[
\| T_t f \|^2_{L^2(\mathbb{R}^n)} \leq Ct^n \| f \|^2_{L^2(\mathbb{R}^n)},
\]
as desired.

Lemma 4.3.
\[
\| \int_0^t e^{\frac{-s}{2t}} E_x(s) f(x) ds \|_{L^2} \leq C t \| f(x) \|_{L^2} + C t^2 \| (-\Delta + 1) f(x) \|_{L^2}
\]

Proof. We shall use the partition of unity \( \{ \phi_i \} \) on \( M \) with very small support \( \text{diam} \phi_i < d \).
If \( d(\text{supp}(\phi_i), \text{supp}(\phi_j)) > 2 \epsilon \),
\[
\phi_j(x) (E_x(t)(\phi_i(y)f(y))) \ |(x) = 0.
\]
So we may assume \( \phi_i \) and \( \phi_j \) are contained in one local chart. The same calculation for \( E_x \) on local charts as Lemma 4.2 implies
\[
\| T_{t,i,j,k,l} f \|^2_{L^2} = \int \int K_{t,i,j,k,l}(y, z) \{ g^{1/2}(y)f(y) \} \{ g^{1/2}(z)f(z) \} \ dy \ dz,
\]
where
\[
K_{t,i,j,k,l}(y, z) = \int_\mathbb{R}^2 e^{\frac{\phi(x, y) - \phi(x, z)}{t}} \phi_i(x) \phi_j(x) \phi_k(y) \phi_l(z) a(x, y) a(x, z) \ dx.
\]
We give a simple explanation of the boundedness of \( T_{t,i,j,k,l} \). \( M \) is compact and so \( c_1 < g(y) < c_2 \). From Lemma 2.1,
\[
\det \left( \frac{\partial^2 d^2}{\partial x_i \partial y_j} \right) \quad \text{for} \ 0 \leq d < d.
\]
is also bounded. Applying Lemma 4.2, we have \( \| T_{i,j,k,l} f \|_2 < C \). \( i, j \)'s are finite and we conclude
\[
\| E_{\chi_{1}}(t)f \|_{L^2} = \sum_{i,j} \phi_i(x)\{ E_{\chi_{1}}(t)\phi_j(y)f\}(x) \|_{L^2} < C_3\| f \|_{L^2}.
\]
(7)

For \( E_{\chi_{2}}(t) \), we have
\[
E_{\chi_{2}}(t)f(x) = \frac{1}{(2\pi i)^n/2} \int_{M} \nabla_x \chi \cdot \nabla_x K(t, x, y)f(y) \, dy
= \frac{1}{(2\pi i)^n/2} \int_{M} \nabla_x \chi \cdot \nabla_x \{a(t, x, y)e^{i\frac{d(x,y)^2}{2n}}\} f(y) \, dy
= \frac{1}{(2\pi i)^n/2} \int_{M} \{\nabla_x \chi \cdot \nabla_x a(t, x, y)\}e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy + \frac{1}{(2\pi i)^n/2} \int_{M} a(t, x, y)\nabla_x \chi \cdot \nabla_x e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy
= \frac{1}{(2\pi i)^n/2} \int_{M} \{\nabla_x \chi \cdot \nabla_x a(t, x, y)\}e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy + \frac{1}{(2\pi i)^n/2} \int_{M} a(t, x, y)\frac{\partial}{\partial d} \left( e^{i\frac{d(x,y)^2}{2n}} \right) f(y) \, dy.
\]
Since the inner product of \( \nabla_x g(d(x,y)) \cdot \nabla_x h(d(x,y)) = \frac{\partial g(d(x,y))}{\partial d} \frac{\partial h(d(x,y))}{\partial d} \) from Gauss’s lemma about normal charts. The first term is bounded by using the same method of \( E_{\chi_{1}} \). We estimate the second term. Letting \( b(x, y) = t^{n/2}a(t, x, y)\frac{\partial}{\partial d} \)
\[
= \frac{1}{(2\pi i)^n/2} \int_{M} b(x, y) \frac{\partial}{\partial d} \left( e^{i\frac{d(x,y)^2}{2n}} \right) f(y) \, dy
= \frac{1}{(2\pi i)^n/2} \int_{M} b(x, y) \frac{i d(x,y)}{t} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy
= \frac{1}{(2\pi i)^n/2} \int_{M} b(x, y) \frac{d(x,y)}{t} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy
= \frac{n-2}{(2\pi i)^n/2} \int_{M} b(x, y) \frac{d(x,y)}{d(x,y)} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy + \frac{\partial}{\partial t} \left( \frac{1}{(2\pi i)^n/2-1} \int_{M} b(x, y) \frac{d(x,y)}{d(x,y)} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy \right)
= E_{\chi_{2}}(t)f(x) + \hat{E}_{\chi_{2}}(t)f(x).
\]
\[ b(x, y) = t^{n/2}a(t, x, y)\frac{\partial}{\partial d}(d(x,y)) \] is bounded. So we have
\[
\| E_{\chi_{2}}(t)f \|_{L^2} \leq C_4 \| f \|_{L^2}
\]
(8)
\[
\| \int_{0}^{t} e^{-\frac{is\hat{H}}{2}} E_{\chi_{2}}(t)f(x) \|_{L^2} = \| e^{-\frac{is\hat{H}}{2}} \left( \frac{1}{(2\pi i)^n/2-1} \int_{M} b(x, y) \frac{d(x,y)}{d(x,y)} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy \right) \|_{L^2}
+ \| \left( \int_{0}^{t} e^{-\frac{is\hat{H}}{2}} \left( \frac{1}{(2\pi i)^n/2-1} \int_{M} b(x, y) \frac{d(x,y)}{d(x,y)} e^{i\frac{d(x,y)^2}{2n}} f(y) \, dy \right) ds \|_{L^2}
\| L^2 \]
\[ \leq C_{5} \| f \|_{L^2} + C_{6} \| (-\Delta + 1)f \|_{L^2}. \]
(9)

Summarizing (7), (8) and (9),
\[
\| \int_{0}^{t} e^{-\frac{is\hat{H}}{2}} \{ E_{\chi_{1}}(s) + E_{\chi_{2}}(s) \} f(x) ds \|_{L^2} \leq C_1 \| f \|_{L^2} + C_2 \| (-\Delta + 1)f \|_{L^2}
\]
as desired.

It follows that \( \{ U_{\chi}(t/N) \}^{N} \rho(N) \) are uniformly bounded, and so the strong limit is obtained:

**Main theorem** (Time slicing products and the strong limits).

\[
s_{N \to \infty} \{ U_{\chi}(t/N) \}^{N} \rho(N) f(x) = e^{\hat{H}(\Delta - \hat{F})} f(x) \quad \text{for } \forall f(x) \in L^2(M).
\]
Proof. By Lemma 4.3, \( \|U_\chi(t)\rho(E)f(x)\| \leq \{1 + C_1|t| + C_2t^2(E + 1)\}\|f(x)\|_{L^2} \). Consequently
\[
\|\{U_\chi(t/N)\}^N\rho(N)f(x)\| \leq (1 + C_1|t|/N + C_2(N + 1)t^2/N^2)^N\|f(x)\|_{L^2} < e^{C|t|}\|f(x)\|_{L^2}.
\]
Letting \( \hat{H} = \triangle - \frac{\rho}{\rho} \), the estimates of Proposition 3.1 yields
\[
\lim_{N \to \infty} \|e^{\frac{\alpha t}{N}} - \{U_\chi(t/N)\}^N\rho(N)f(x)\|_{L^2} \leq \lim_{N \to \infty} \|e^{\frac{\alpha t}{N}}(1 - \rho(N^{1/\alpha - \varepsilon}))f(x)\|_{L^2} \\
+ \|\{e^{\frac{\alpha t}{N}} - \{U_\chi(t/N)\}^N\}\rho(N^{1/\alpha - \varepsilon})f(x)\|_{L^2} \\
+ \|\{U_\chi(t/N)\}^N(\rho(N) - \rho(N^{1/\alpha - \varepsilon}))f(x)\|_{L^2} = 0.
\]

5 Some remarks

Remark 5.1. Main theorem holds true even for two-point homogeneous spaces. For a torus, \( R = 0 \) and
\[
s \lim_{N \to \infty} \{U_\chi(t/N)\}^N\rho(N)f(x) = e^{\frac{\alpha t}{\rho}f(x)} \quad \text{for} \quad f(x) \in L^2(M).
\]

Remark 5.2. Since \( M \) is compact, we need not to use Cotlar-Stein lemma (See e.g. [4], p.238).

Remark 5.3. Our estimates hold in Sobolev spaces (See §2), that is
\[
s \lim_{N \to \infty} \{U_\chi(t/N)\}^N\rho(N)f(x) = e^{\frac{\alpha t}{\rho}(\triangle - \frac{\rho}{\rho})}f(x) \quad \text{in} \quad H^k(M).
\]

The Sobolev imbedding theorem yields the uniformly convergence:
\[
\lim_{N \to \infty} \sup_{x \in M} \|\{U_\chi(t/N)\}^N - e^{\frac{\alpha t}{\rho}(\triangle - \frac{\rho}{\rho})}\|_{L^2(M)} = 0 \quad \text{for} \quad f(x) \in H^k(M)
\]
where \( k > \frac{n}{2} = \dim_{\mathbb{R}}M \).

Remark 5.4. We employed the shortest paths on \( M \). \( U_\chi(t) \) is defined by the action integrals and van Vleck determinants. van Vleck determinants diverge at conjugate points, thus we ignore the long paths.

On \( S^1 \), however, we can take infinite many long paths for Fresnel integrable functions. On \( M \), can one construct the analogy?

6 Conclusion

Simple WKB like formulas of Feynman integrations are discussed. Low energy approximations assure the unique classical paths. The quantum evolution is given by means of action integrals and van Vleck determinants. That is \( \{U_\chi(t/N)\}^N\rho(N) \) converges to the modified Schrödinger operator in strong topology. We would like to mention about the case for general Riemannian manifolds in the future.

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