Neutrino masses and mixings with an $S_3$ family permutation symmetry

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Abstract

Large neutrino mixing angles suggest that the Yukawa sector is invariant under permutations of the fermion families. This $S_3$ permutation symmetry is broken at a large energy scale but much below the unification scale. Assuming that the lepton mass matrix is approximately diagonal, all neutrino mixing angles naturally come from the breaking of $S_3 \rightarrow S_2$. In the neutrino sector, $S_2$ remains (approximately) unbroken. As a consequence, we have a large atmospheric neutrino angle and $U_{e3} = 0$. The $S_3$ symmetry at the unification scale can also explain the large solar mixing angle. We give an explicit expression of the solar mixing angle in terms of the left-handed neutrino masses.

We observe that this family permutation symmetry comes very naturally from a quantized theory of functionals[1], that is an extension of quantum field theory.

1 Introduction

In the Standard Model (SM) Lagrangian, the breaking of the $SU(2) \times U(1)$ gauge symmetry is due to a Higgs scalar doublet. This is the most realistic possibility since triplets or larger representations give unrealistic masses and mixings angles in the electroweak gauge boson sector. In order to give mass to the SM fermions, we introduce Yukawa interactions between fermions and the Higgs doublet.

Neutrinos are very light with respect to all other fermions. The reason is probably that the right-handed neutrino is $SU(2) \times U(1)$ singlet. It takes a Majorana mass that does not break the electroweak gauge symmetry, and so its mass is expected to be of the order of the Grand Unification scale. The left-handed neutrino has a Majorana mass that is proportional to the inverse of the right-handed neutrino mass. Through this seesaw mechanism we understand why neutrinos are so light.

Also neutrino mixings appear very different from quark mixings. The $V_{\text{ckm}}$ matrix is close to the identity matrix, mixing angles are not large. The mixing between the heaviest quarks with the lightest is very small. On the contrary, recent neutrino experiments[2-6] have shown that mixings angles between neutrinos with different lepton flavour are very large, approximately maximal. A good candidate for the MNS matrix, (the analogue of the CKM matrix) is

$$O_{\text{MNS}} = \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$ 

(1)
Since we measure $\sin^2 \theta_{\text{sol}}$ and $\sin^2 \theta_{\text{atm}}$ the choice of signs in the matrix elements (1) is not unique. The choice in (1) corresponds to neutrino mass eigenstates that are also eigenstates of the permutation symmetry $\nu_\mu \leftrightarrow \nu_\tau$. This $S_2$ symmetry directly implies $U_{e3} = 0$ and a maximal atmospheric neutrino mixing angle (see the third column in (1)). The importance of discrete symmetries in the description of neutrino masses and mixings has been observed by several authors[7-20]. In this paper we will show that if we embed $S_2$ into a $S_3$ symmetry, that is the permutation symmetry $\nu_\mu \leftrightarrow \nu_\tau$. This $S_2$ symmetry directly implies $U_{e3} = 0$ and a maximal atmospheric neutrino mixing angle (see the third column in (1)).

### 2 $S_2$ symmetry and neutrino mass and mixings

First, we discuss the most general $S_2$ symmetric real mass matrix. We require that the mass matrix is invariant when we exchange the 2nd with the 3rd row (and column),

$$P M^R_\nu P = M^R_\nu$$

where $P$ is the $S_2$ permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

The matrix equation (2) is satisfied if and only if

$$M^R_\nu = \begin{pmatrix} a & d & d \\ d & b & c \\ d & c & b \end{pmatrix}$$

where $a, b, c, d$ are four real parameters. Since all parameters are real, we need just an orthogonal matrix $O$ to diagonalize the symmetric matrix $M^R_\nu$, (3). We already know that the vector $v_3 = (0, -1/\sqrt{2}, 1/\sqrt{2})$ is an eigenvector, since it is the only state with odd $S_2$ parity: $Pv_3 = -v_3$. So, two of the three angles of the orthogonal matrix $O$ are fixed by the $S_2$ symmetry, while the third angle (the solar neutrino mixing angle) that mixes the two states orthogonal to $v_3$ is free. Any matrix in (3) can be diagonalized by the orthogonal matrix

$$O = \begin{pmatrix} -\cos \theta & \sin \theta & 0 \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta & \frac{1}{\sqrt{2}} \end{pmatrix}$$

\(^1\)However this additional fermion $\chi_L$ is not necessary. It helps in the explanation of the large splitting between $S_3$ singlet and doublet right-handed neutrinos.
and the angle $\sin \theta$ is obtained by imposing that the matrix $m_{\nu} = O^t M^{R^O}_{\nu} O$ must be diagonal.

$$m_{\nu} = O^t \begin{pmatrix} a & d & d \\ d & b & c \\ d & c & b \end{pmatrix} O$$

(5)

or more explicitly

$$m_{\nu} = \begin{pmatrix} z + x \cos 2\theta - y \sin 2\theta & -x \sin 2\theta - y \cos 2\theta & 0 \\ -x \sin 2\theta - y \cos 2\theta & z - x \cos 2\theta + y \sin 2\theta & 0 \\ 0 & 0 & b - c \end{pmatrix}$$

(6)

where

$$x = \frac{1}{2}(a - b - c)$$
$$y = \sqrt{2}d$$
$$z = \frac{1}{2}(a + b + c).$$

(7)

We have to set to zero the off diagonal entries, and from this condition we derive the solar mixing angle

$$\tan 2\theta = \frac{y}{x}$$

(8)

or

$$\sin 2\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

(9)

If we put this value of $\theta$ in the matrix (4), the matrix (6) is diagonal and the elements in the diagonal give the three masses

$$m_1^R = z - \sqrt{x^2 + y^2}$$
$$m_2^R = z + \sqrt{x^2 + y^2}$$
$$m_3^R = b - c.$$ 

(10)

3 S₃ symmetry breaking and neutrino masses and mixings

We consider a $S_3$ symmetric model with three left-handed neutrinos $\nu^i_L$ ($i$ is the family index) and three right-handed neutrinos $\nu^i_R$. We also add a left-handed Weyl fermion $\chi_L$, that is a Standard Model singlet $(SU(3) \times SU(2) \times U(1)$ singlet). It does not carry the family index, i.e. it is a $S_3$ singlet. The scalar sector includes two standard model singlets for each family, $\phi^i$ and $\varphi^i$, plus the common $SU(2)$ scalar doublet $H$. We assume that the lepton mass matrix is approximately diagonal, while for the neutrinos we consider a Yukawa interaction of the form²

$$L_{\text{yuk}}^a = g \sum_i \bar{\nu}^i_L \nu^i_R H + \lambda_1 \sum_i \bar{\nu}^c_i \nu^i_R \varphi^i + \lambda_2 \sum_i \bar{\chi}_L \nu^i_R \varphi^i + M \bar{\chi}_L \bar{\chi}_L + h.c.$$ (11)

we have added the $S_3 \times SU(3) \times SU(2) \times U(1)$ singlet, $\chi^c_L$, because we want an explicitly renormalizable Lagrangian and we want to explain the splitting among the three right-handed neutrinos, but such a singlet is not necessary since one could add such a splitting directly to the Lagrangian (without breaking $S_3$)

$$L_{\text{yuk}}^b = g \sum_i \bar{\nu}^i_L \nu^i_R H + \left( \lambda_1 \sum_i \bar{\nu}^c_i \nu^i_R \phi^i + \lambda_2 \sum_i \bar{\chi}_L \nu^i_R \varphi^i + M \bar{\chi}_L \bar{\chi}_L + h.c. \right)$$ (12)

²The scalar field $\phi^i$ does not break $S_3$: $(\langle \phi^1 \rangle = \langle \phi^2 \rangle = \langle \phi^3 \rangle)$
$L^b_{\text{yuk}}$ is simpler, while $L^a_{\text{yuk}}$ and the $S_3$ singlet $\chi_L$ can probably explain the large splitting among the right-handed neutrinos in terms of additional symmetries of the Lagrangian. In any case, there are two important ingredients that appear to be necessary in realistic examples: there must be an $S_3$ symmetry, at the energy scale where the heaviest right-handed neutrino acquires a mass\(^3\), then $S_3$ must break at a much lower scale. The breaking $S_3 \to S_2$ occurs at a much smaller scale than the grand unification scale.

To start we take the Lagrangian (12). $\psi$ and $\phi^i$ are two scalar fields, with $\langle \psi \rangle > \langle \phi^i \rangle$. $H$ is the Higgs SU(2) doublet, it does not carry any family index\(^4\). To simplify the notation we set $\lambda_3 = 1$, since it is an irrelevant scale factor. The mass matrix for the right-handed neutrinos becomes

$$M^R_\nu = \langle \psi \rangle \begin{pmatrix} \langle \psi \rangle + 1 & 1 & 1 \\ 1 & \langle \psi \rangle + 1 & 1 \\ 1 & 1 & \langle \psi \rangle + 1 \end{pmatrix}. \quad (13)$$

Now we assume $\langle \phi^2 \rangle = \langle \phi^3 \rangle$ such that the $S_2$ exchange symmetry remains unbroken, and $\langle \psi \rangle > \langle \phi^i \rangle$. The Dirac mass matrix $g\bar{\nu}^L_i \nu^R_i H$ that mixes left-handed neutrinos with their right-handed component is proportional to the identity matrix. Then, due to the seesaw, the left-handed neutrinos get a Majorana mass matrix that is proportional just to the inverse of the matrix $M^R_\nu$. Namely,

$$M^L_\nu = g^2 \langle H \rangle^2 (M^R_\nu)^{-1}. \quad (14)$$

The same orthogonal matrix will diagonalize both $M^R_\nu$ and $M^L_\nu$. This means that is sufficient to diagonalize $M^R_\nu$. If $\langle \phi^2 \rangle = \langle \phi^3 \rangle$, the matrix (13) is $S_2$ symmetric, and it is equivalent to the (3) with $c = d$. From eqs.(7,8,10) we find a relation that constrains the solar mixing angle to be a function of the right-handed neutrino masses ($m^R_1$, $m^R_2$ and $m^R_3$)

$$\sqrt{2} \sin 2 \theta_{\text{sol}} - \cos 2 \theta_{\text{sol}} = \frac{m^R_1 + m^R_2 - 2 m^R_3}{m^R_2 - m^R_1}. \quad (15)$$

We can also express the solar mixing angle in terms of the left-handed neutrino masses\(^5\)

$$\sqrt{2} \sin 2 \theta_{\text{sol}} - \cos 2 \theta_{\text{sol}} = \frac{m^L_2 m^L_3 + m^L_1 m^L_3 - 2 m^L_1 m^L_2}{m^L_1 m^L_3 - m^L_2 m^L_3} = R \quad (15)$$

or

$$\theta_{\text{sol}} = \frac{1}{2} \arcsin \left( \frac{R}{\sqrt{3}} \right) + \frac{1}{2} \arcsin \left( \frac{1}{\sqrt{3}} \right).$$

We remind that $m^L_3$ is the mass of left-handed neutrino corresponding to the $S_2$ parity odd state. Note that in the limit $m^R_3 \to \infty$, we have $m^L_3 \sim 1/m^R_3 \to 0$, and $\sin \theta_{\text{sol}} = 1/\sqrt{3}$. This limit is achieved when the v.e.v. $\langle \psi \rangle$ is much larger than $\langle \phi^i \rangle$. The first v.e.v $\langle \psi \rangle$ breaks SO(10) and leave $S_3$ unbroken, while the second one breaks $S_3$. This is the $S_3$ approximately symmetric limit, when the $S_3$ breaking scale is very small compared to the U(1) breaking scale and unification scale.

\(^{3}\)That is probably the scale at which the extra U(1) included in SO(10) is broken.

\(^{4}\)This is the most realistic situation, taking into account constraints coming from electroweak precision tests of the Standard Model.

\(^{5}\)The left-handed neutrino mass eigenvalues are proportional to the inverse of the right-handed neutrino masses.
4 Conclusions

We have studied the breaking of an $S_3$ symmetry, that is the permutation of the three fermion families. This breaking naturally explains the large mixing angles among neutrinos of different flavours. We have concluded that while the unbroken $S_2$ permutation of the $\mu$ neutrino with the $\tau$ neutrino predicts both the atmospheric mixing angle and $U_{e3} = 0$, the $S_3$ symmetry below the scale of the $U(1) \subset SO(10)$ breaking gives a simple mechanism to explain the solar neutrino mixing angle. A very simple Yukawa sector, $S_3$ symmetric like

$$I_{\text{yuk}}^b = g \sum_i \bar{\nu}_L^i \nu_R^i \langle H \rangle + \lambda_1 \sum_i \bar{\nu}_R^i \nu_R^i \langle \phi^i \rangle + \lambda_3 \sum_{i,j} \bar{\nu}_R^i \nu_R^j \langle \psi \rangle$$

after the breaking of $S_3$ into $S_2$ automatically gives

$$U_{e3} = 0$$
$$\sin \theta_{\text{atm}} = 1/\sqrt{2}$$
$$\sqrt{2} \sin 2 \theta_{\text{sol}} - \cos 2 \theta_{\text{sol}} = (m_L^2 m_L^3 + m_L^1 m_L^2 - 2 m_L^1 m_L^3)/(m_L^1 m_L^3 - m_L^2 m_L^3).$$

The last condition also gives $\sin \theta_{\text{sol}} = 1/\sqrt{3}$ in the limit $m_L^2 = 0$. This limit suggests that the breaking of $S_3$ occurs at an energy scale much below the unification breaking scale. Finally, we observe that the family permutation symmetry derives from a quantized theory of functionals[1], that is an extension of quantum field theory.

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