GLOBAL WELLPOSEDNESS OF VACUUM FREE BOUNDARY
PROBLEM OF ISENTROPIC COMPRESSIBLE
MAGNETOHYDRODYNAMIC EQUATIONS
WITH AXISYMMETRY

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Abstract. In this paper, we prove the global existence of the strong solutions
to the vacuum free boundary problem of isentropic compressible magnetohy-
drodynamic equations with small initial data and axial symmetry, where the
solutions are independent of the axial variable and the angular variable. The
solutions capture the precise physical behavior that the sound speed is $C^{1/2}$-
Hölder continuous across the vacuum boundary provided that the adiabatic
exponent $\gamma \in (1, 2)$. The main difficulties of this problem lie in the singularity
at the symmetry axis, the degeneracy of the system near the free boundary and
the strong coupling of the magnetic field and the velocity. We overcome the
obstacles by constructing some new weighted nonlinear functionals (involving
both lower-order and higher-order derivatives) and establishing the uniform-
in-time weighted energy estimates of solutions by delicate analysis, in which
the balance of pressure and self-gravitation, and the dissipation of velocity are
crucial.

1. Introduction. Recently, much attention has been given to the study of the
magnetohydrodynamic (MHD) equations due to its physical importance, complex-
ity and mathematical challenges. The dynamics is very often controlled and modeled
by intense magnetic fields coupled with or without self-gravitating and it has a wide
range of applications, such as in astrophysics, geophysics, high-speed aerodynamics,
and plasma physics, etc. The three-dimensional viscous compressible MHD equa-
tions with a self-gravitational force can be described as follows: ([6, 47, 27, 28]):

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= (\nabla \times H) \times H + \text{div}\psi - \rho \nabla \varphi, \\
H_t - \nabla \times (u \times H) &= -\nabla \times (\nu \nabla \times H), \quad \text{div}H = 0,
\end{aligned}
$$

where $\rho, u = (u^1, u^2, u^3) \in \mathbb{R}^3$ and $H = (H^1, H^2, H^3) \in \mathbb{R}^3$ denote the density,
the velocity and the magnetic field respectively; the viscous stress tensor $\psi$ has the form:

$$
\psi = \mu(\nabla u + \nabla u^T) + \lambda \text{div}u I_3,
$$

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where $I_3$ is the $3 \times 3$ identity matrix; $\mu$ and $\lambda$ are coefficients of viscosity, satisfying the following physical restrictions:

$$\mu > 0, \quad \mu + \frac{3}{2} \lambda \geq 0.$$  

The gravitational potential $G$ is given by

$$G(x, t) = -G \int_{\Omega(t)} \rho(y) \frac{x - y}{|x - y|} dy$$

satisfying $\Delta G = 4\pi G\rho,$

with the gravitational constant $G$ taken to be unity for convenience ($\Omega(t)$ represents the changing volume occupied by a fluid at time $t$). In this paper, we consider the barotropic flow that the pressure $p$ is given by:

$$p(\rho) = \rho^\gamma.$$  

(2)

The magnetic diffusivity coefficient $\nu$ is assumed to be constants for simplicity. Note that, the displacement current can be neglected ([1, 17, 27, 28]) in magnetohydrodynamics. When the solution is sufficiently regular, by direct calculations, we find that (1)-(2) can be rewritten as in [17, 30]

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
\rho(u_t + u \cdot \nabla u) + \nabla p + \frac{1}{2} \nabla(|H|^2) - H \cdot \nabla H = \mu \triangle u + (\mu + \lambda) \nabla \text{div} u - \rho \nabla x G, \\
H_t + u \cdot \nabla H - H \cdot \nabla u + H \text{div} u = \nu \triangle H, \quad \text{div} H = 0.
\end{cases}$$

(3)

Due to its physical importance, complexity and mathematical challenges, compressible MHD equations have been widely studied by physicists and mathematicians (see, for example, [1, 7, 8, 14, 17, 16, 25, 31, 40, 41] and the references therein). In the research of three dimensional fluid system, a very important case is to consider the symmetry of the system, including the spherical symmetry and the axial symmetry. Next, let’s briefly review some previous known results on the global existence theory of MHD equations with symmetry.

For spherically symmetric case of MHD system, as shown in [9, 35], the magnetic stars with finite magnetic energy do not exist due to the fact there are no magnetic point charges. For axially symmetric case of 3D incompressible MHD system in the whole space, Lei ([29]) proved the global well-posedness of axially symmetric solutions to the system in the cylindrical coordinates $(r, z, t)$ when some of the components of solutions are zero. Recently, Su, Guo and Yang ([43]) established a family of exact solutions with large initial data of finite energy, and without the restrictions that some components of the solutions were supposed to be zero. For 3D compressible MHD system in the cylindrical coordinates $(r, t)$, Li, Li and Ou ([30]) proved the global existence and uniqueness of axially symmetric classical solutions to the vacuum free boundary problem of the MHD system away from the symmetry axis with large initial data, moreover, they also showed the expanding rate of the free boundary. However, the globally-in-time regularity up to the symmetry axis is hard to handle, due to the strong degeneracy of the system near the free boundary, the singularity in symmetry axis, and the oscillation of the magnetic field. And this will be the main task of this article and we will restrict the discussion in the isentropic regime.

Before going further, we introduce some recent works on free boundary problems involving vacuum for the Euler-Poisson and Navier-Stokes-Poisson equations. Due to the high degeneracy of system near the free boundary, the local-in-time existence
Thus in the Eulerian coordinates,\( H \) called the Lane-Emden solutions, which satisfy\( \text{[5, 19]} \). In the spherically symmetric case, a typical time-independent solutions are\( \text{viscous compressible flows, was only established recently in [2, 3, 21, 23, 38] or [5, 19]. In the spherical symmetry, a typical time-independent solutions are called the Lane-Emden solutions, which satisfy [5, 19].} \)

The solutions to (4) can be characterized by the values of\( \gamma \) (cf. [32]) for given finite total mass \( M > 0 \). For inviscid flows, the nonlinear instability of Lane-Emden solutions for \( \gamma = 6/5 \) and \( \gamma = 4/3 \) was proved in [18] and [4], respectively. The nonlinear dynamical instability for \( \gamma \in (6/5, 4/3) \), was identified in [20] and [22], respectively to the Euler-Poisson and Navier-Stokes-Poisson equations with the free boundary. Recently, the theory of weak solutions was obtained in Fang and Zhang ([46]) for compressible Navier-Stokes-Poisson equations with some restrictions on adiabatic exponent \( \gamma \) and the ratio of the two viscosity coefficients. Later, the results was extended to much broader cases by Luo, Xin and Zeng ([36], [37]) in the sense that \( 4/3 < \gamma < 2 \) (the most physically relevant regime for gaseous stars). Indeed, for the free boundary problem of the compressible Navier-Stokes-Poisson equations with spherically symmetry, they proved the large time asymptotic uniform convergence of the unique global strong solution to the Lane-Emden solution with detailed convergence rates, especially, the uniform convergence up to the free boundary, for case that constant viscosities and dependent-density viscosities, respectively. We note that a global existence of weak solutions was established in [12] for the spherically symmetric motions, where the density is positive on the free boundary. For more related results of the compressible Navier-Stokes equations with vacuum boundary, one can refer to [11, 15, 33, 34, 42, 45] and references therein.

Next, we state the original free boundary problem in three space dimensions in cylindrically coordinates as follows. Let \( x = (x_1, x_2, x_3) \) be the spatial variable and \( t \) be the time variable, and set \( r = \sqrt{x_1^2 + x_2^2} \). The domain of the system (3) is inside of an infinite cylinder being parallel to and centered along the \( x_3 \) axis in \( \mathbb{R}^3 \), with the free boundary \( r = R(t) \) connecting to a surrounding vacuum continuously. We assume the following initial conditions:

\[
(\rho, u, H)|_{t=0} = (\rho_0, u_0, H_0)(r), \quad \text{div} H_0 = 0,
\]

and the boundary conditions on the fixed boundary \( r = 0 \) and the free boundary \( r = R(t) \):

\[
(u, H)|_{r=0} = 0, \quad (\rho, (\Psi - pI_3) n_r, H)|_{r=R(t)} = 0,
\]

where \( \vec{n} \) is the unit outer normal to the free boundary.

Thus, we expect to find the global axially symmetric solution. As in [13, 24, 44], let \( e_r = \frac{(x_1, x_2, 0)}{r} \), \( e_\phi = \frac{(-x_2, x_1, 0)}{r} \), \( e_z = (0, 0, 1) \) be the three orthogonal unit vectors along the radial, the angular, and the axial directions, respectively. Then we set

\[
| u(r, t) = u^r(r, t) e_r + u^\phi(r, t) e_\phi + u^z(r, t) e_z, |
\]

\[
| H(r, t) = H^r(r, t) e_r + H^\phi(r, t) e_\phi + H^z(r, t) e_z. |
\]

Thus in the Eulerian coordinates, \( H \) can be expressed equivalently by

\[
H = (H^1, H^2, H^3) = \left( \frac{x_1 H^r - x_2 H^\phi}{r}, \frac{x_2 H^r + x_1 H^\phi}{r}, H^z \right). 
\]
Moreover, we assume that the initial density $\rho_0$ satisfies:

$$\rho_0 > 0 \quad \text{for} \quad [0, R_0), \quad \rho_0(R_0) = 0 \quad \text{and} \quad -\infty < (\rho_0^{-1})_r(R_0) < 0,$$

so,

$$\rho_0^{-1} \sim \rho_0 - r \quad \text{as} \quad r \text{ close to } R_0,$$

which means that the initial sound speed $\sqrt{p'(\rho)}|_{t=0} \sim \rho_0^{-\gamma-1}$ is $C^1$-H"older continuous across the vacuum boundary. The unknowns here are $\rho, u^r, u^\phi, u^z, H^\rho, H^\phi, H^z$ and $R(t)$ (noting that $H^r \equiv 0$ in (17)).

Indeed, the requirement (8) for the initial density near the vacuum boundary is motivated by that of the steady solutions $(\rho, u^r, u^\phi, u^z, H^\rho, H^\phi, H^z) = (\bar{\rho}(r), 0, 0, 0, 0, 0)\) of (5), which satisfy

$$(\bar{\rho}^\gamma)_r = -4\pi \bar{\rho} r^{-1} \int_0^r \bar{p}(s, t)sds. \quad (9)$$

Similarly to the results of Lane-Emden solutions (cf. [32, 36, 37]), for $\gamma \in (1, 2)$, we can prove that for any given finite positive total mass, there exists a unique solution...
to Eq. (9) whose support is compact. That means, for any $M \in (0, \infty)$, there exists a unique function $\bar{\rho}(x)$ such that
\[ \bar{\rho}_0 := \bar{\rho}(0) > 0, \bar{\rho}(r) > 0 \text{ for } r \in (0, \bar{R}), \ \bar{\rho}(\bar{R}) = 0, \ M = \int_0^{\bar{R}} 2\pi \bar{\rho}(s)ds; \quad (10) \]
\[-\infty < \bar{\rho}_r < 0 \text{ for } r \in (0, \bar{R}) \text{ and } \bar{\rho}(r) \leq \bar{\rho}_0 \text{ for } r \in (0, \bar{R}); \quad (11) \]
\[ (\bar{\rho}^\gamma)_r = -2\bar{\rho}\varphi, \text{ where } \varphi := r^{-2} \int_0^r 2\pi \bar{\rho}(s)ds \in [M/\bar{R}^2, \bar{\rho}_0] \quad (12) \]
for a certain finite positive constant $\bar{R}$ (indeed, $\bar{R}$ is determined by $M$ and $\gamma$). Note that
\[ (\bar{\rho}^{\gamma-1})_r = (\gamma - 1) \times \left[ \frac{1}{\bar{\rho}^\gamma}(\bar{\rho}^\gamma)_r \right] = -\frac{2(\gamma - 1)}{\gamma} r \varphi. \]
It then follows from (10) and (12) that $\bar{\rho}$ satisfies the physical vacuum condition, i.e.,
\[ \bar{\rho}^{\gamma-1} \sim \bar{R} - r \text{ as } r \text{ close to } \bar{R}. \]
More precisely, there exists a constant $C$ depending on $M$ and $\gamma$ such that
\[ C^{-1}(\bar{R} - r) \leq \bar{\rho}^{\gamma-1}(r) \leq C(\bar{R} - r), \ r \in (0, \bar{R}). \]

The results in this work are among few results of global strong solutions to vacuum free boundary problems of the axially symmetric compressible MHD equations without any regularity assumptions on velocity and magnetic field. It’s worthy noting that $H^r = 0$ (see (17)) due to a special structure in cylindrical coordinates in our consideration. We summarize the results in this paper: we show the global existence and uniqueness of axially symmetric strong solutions to the vacuum free boundary problem of full compressible MHD equations, when the initial density satisfies (8), and the solution captures the precise physical behavior that the sound speed is $C^{1/2}$-Hölder continuous across the vacuum boundary. Unfortunately, the the large time asymptotic uniform convergence of the solutions is not obtained for our setting and nether nor done the extension to the density-dependent viscosity coefficients case as in [36, 37]. We remark that the model studied here is more complicated due to the additional strong interactions between the magnetic field and the velocity compared with [36].

Next, we comment on the analysis of this paper. Motivated by [2, 3, 23, 36, 37], we use a method of Lagrangian trajectory to transform the original free boundary problem into an initial-boundary problem in $[0, +\infty) \times [0, \bar{R}]$ (without loss of generality, we write $\bar{R} = 1$ for simplicity). It is essential to show that $0 < c_1 \leq r_x(x, t) \leq c_2$, which guarantees the equivalence of the two problems. To this aim, we use a bootstrap argument by making the following a priori assumptions:
\[ |r_x - 1| + \left| \frac{r}{x} - 1 \right| \leq \epsilon_0, \ |v^r_x| + |v^r| + |v^s_x| + |v^s| + |B^s_x| + |B^s| + \frac{B^0}{x} \leq 1. \]
We briefly introduce the main ideas of closing above assumptions as follows. First, the lower-order estimates can be obtained through the standard weighted energy method together with some pointwise estimates. In which we get the bound of $\|r_x - 1\|_{L^\infty([0,1/2])}$ by integrating equation (23) over $(x, 1/2 \times (0, t))$, $x \in (0, 1)$, and the bound of $\|v_x^r, B^s_x\|_{L^\infty((0,1/2))}$ by solving the equations (23) directly. Then, for $\|r_x - 1\|_{L^\infty([0,1/2])}$, we bound it by using the Sobolev embedding, a similar way in [36, 37], precisely, to use the the uniform bound for $\|r_{xx}, r_x - 1\|_{L^2((0,1/2))}$. To bound $\|v_{xx}\|_{L^2((0,1/2))}$ in higher-order estimates, we first define $G = \ln(\frac{|v_x|}{x})$ whose
leading term is \(r_x - 1\) by the Taylor expansion, and decompose the gradient of the pressure as:
\[
\left[ \frac{\partial \varphi}{\partial x} \right]_x = -2x \varphi \frac{\partial \varphi}{\partial x} - \gamma \varphi^2 \frac{\partial \varphi}{\partial x} \varphi, \quad \text{where } \varphi := r^{-2} \int_0^r 2 \pi \varphi(s) ds \in \left[ \frac{M}{R}, \pi \varphi_0 \right],
\]
then the equation (23) can be rewrote in terms of the viscosity term \(G_{xx}\) as follows:
\[
\varpi G_{xx} + \gamma \left( \frac{\varpi}{\partial x} \right)^2 \varphi_x = \frac{\varpi x \varphi'}{r} - \frac{\varpi x (\varphi')^2}{r^2} - 2 \left( \frac{x^2}{r^2} \right) x \varphi \varphi' + \frac{(B_x^2 - B_x^2 + B_x^2)}{r}.
\]
This form has the advantage in identifying the interplay among the viscosity, pressure and gravitational force, which will lead to desirable estimates on \(G\) and its derivatives. In particular, we get the bound of \(\|\varpi^{-\frac{1}{2}} (r_x, \varphi_x)\|_{L^2([0,1])} \sim \|\varpi^{-\frac{1}{2}} G_x\|_{L^2([0,1])}\), which gives the bound of \(\|\varpi (r_x, \varphi_x)\|_{L^2([0,1])}\) by noting that the positivity of \(\varpi\) near the center. Then, we can use Hardy’s inequality to get \(\|\varpi (r_x - 1)\|_{L^2([0,1])}\) and then \(\|\varpi (r_x - 1)\|_{L^\infty([0,1])}\), due to the Sobolev embedding. With these estimates, the bounds of \(\|\varpi^{1/2} v_x^2\|_{L^2([0,1])}\), \(\|\varpi (\varphi')_x\|_{L^2([0,1])}\), \(\|\varpi (v_x, v_x^2)\|_{L^2([0,1])}\) and \(\|\varpi (v_x, v_x^2, B_x, B_x^2, B_x^2)\|_{L^\infty([0,1])}\) can be established sequentially by the technique of truncating functions and the Sobolev embedding. Similar idea is applied to deduce \(\|\varpi (v_x, B_x^2)\|_{L^\infty([0,1])}\). The bound of \(\|\varpi (v_x, v_x^2, B_x^2)\|_{L^\infty([1/2,1])}\) can be obtained by some pointwise estimates (see (125)). Finally, \(\|\varpi (r_x - 1, \varphi_x, \varphi_x^2, v_x^2, v_x^2, B_x, B_x^2, B_x^2)\|_{L^\infty([0,1])}\) holds, due to Cauchy’s mean value theorem. Thus the bootstrap argument is closed by choosing initial data small enough.

We introduce some techniques used in this paper. First and foremost, we derive from Hardy’s inequality and its new version that
\[
\int f^k dx \leq C \int f^{k+2} (\varpi f^2 + |f_x|^2) dx, \quad k > -1, \quad \forall f \in H^1(I).
\]
(see Corollary 3.5), which improves the integrability of the function near the symmetry axis in some sense and plays an important role in the Lower-order estimates. For example, in order to bound \(\|x^{1/2} \varpi^{1/2} v_x^2\|_{L^2}\) in Lemma 4.5, it is shown that the quantity
\[
\int x^k \varpi |v_x^2|^2 dx + \int_0^t \int x |v_x^2 - \frac{v_x^2}{x}|^2 dx \leq \delta \int_0^t \int x |v_x^2|^2 dx + C(\delta) (\text{initial data}).
\]
(15)
We first choose that \(k = 1\) and \(f(x, t) = \frac{x^2}{x}(x, t)\) in (14) to get
\[
\int x |v_x^2|^2 dx \leq C_1 \left( \int \varpi x |v_x^2|^2 dx + \int x |v_x^2 - \frac{v_x^2}{x}|^2 dx \right),
\]
(16)
then substitute (16) into (15) with a suitably small \(\delta\), and apply Gronwall’s inequality to get the desired result. Besides, to close the \textit{a priori} assumption, we use \(\|v_x^2\|_{L^2([0,1])}\) to control the term \(\|v_x^2\|_{L^2([0,1])}\) by using a modified truncation function as in [36]. By contrast, due to the absence of the bound of \(\|\varpi (v_x^2, B_x^2)\|_{L^2([0,1])}\), instead we integrate (23) with \(z\) to get the bound of \(\|v_x^2, B_x^2\|_{L^\infty(I)}\) directly, as mentioned above.
The rest of this paper is organized as follows. In Section 2, we will reformulate the free boundary problem (5)-(8) in Lagrangian coordinates and state the main results of this paper. Section 3 gives some preliminaries which will be used later. In Section 4, the uniform estimates for the solution are established.

2. Lagrangian reformulation and main results.

2.1. Lagrangian reformulation. In this section, we reformulate the free-boundary value problem (5)-(8) and state the main results. We first solve the equations (5)\textsubscript{5} and (5)\textsubscript{9} for \( H' \). Substituting equation (5)\textsubscript{9} into equation (5)\textsubscript{5}, we have

\[
H'_r(x, t) = 0, \quad \text{in } I \times (0, T),
\]

which gives

\[
H'(x, t) = H'_r(x, 0), \quad \text{in } I \times (0, T).
\]

Putting the above equation into (5), then solving the resulting equality and using the boundary conditions \( H'|_{r=1} = H'|_{r=R(t)} = 0 \), we get

\[
H'(x, t) = 0. \tag{17}
\]

To fix the boundary, we adopt a particle trajectory Lagrangian formulation (cf. [36, 37]) as follows. We use \( x \) as the reference variable and define the Lagrangian variable \( r(x, t) \) by

\[
r_t(x, t) = u'(r(x, t), t) \quad \text{for } t > 0 \quad \text{and } \quad r(x, 0) = r_0(x), \quad x \in I := (0, R). \tag{18}
\]

Where \( r_0(x) \) is the initial position which maps \( I \rightarrow (0, R_0) \) satisfying

\[
\int_0^{r_0(x)} \rho_0(s)ds = \int_0^x \overline{\rho}(s)ds, \quad x \in I, \tag{19}
\]

so that

\[
\rho_0(r_0(x)) = \frac{\overline{\rho}(x)\rho_0(x)}{\overline{\rho}(x)}, \quad x \in I,
\]

The choice of \( r_0 \) can be described by

\[
r_0(x) = \psi^{-1}(\xi(x)), \quad 0 \leq x \leq \overline{R}; \tag{20}
\]

where \( \xi \) and \( \psi \) are one-to-one mappings, defined by

\[
\xi : (0, \overline{R}) \rightarrow (0, M) : x \mapsto \int_0^x \rho_0(s)ds \quad \text{and } \quad \psi : (0, R_0) \rightarrow (0, M) : z \mapsto \int_0^z \rho_0(s)ds.
\]

Moreover \( r_0(x) \) is an increasing function and the initial total mass has to be the same as that for \( \overline{\rho} \), that is,

\[
\int_0^{R_0} 4\pi \rho_0(s)ds = \int_0^{r_0(\overline{R})} 4\pi \rho_0(s)ds = \int_0^{\overline{R}} 4\pi \overline{\rho}(s)ds = M, \tag{21}
\]

to ensure that \( r_0 \) is a diffeomorphism from \( I \) to \( (0, R_0) \), therefore \( r_0 \) is well defined. It follows from (5)\textsubscript{1} that

\[
\int_0^{r(x, t)} \rho(s, t)ds = \int_0^{r_0(x)} \rho_0(s)ds, \quad x \in I.
\]

Then we introduce the Lagrangian density, velocity and magnetic field, respectively, by

\[
f(x, t) = \rho(r(x, t), t), \quad (v^r, v^\phi, v^z)(x, t) = (u^r, u^\phi, u^z)(r(x, t), t),
\]

and

\[
(B^\phi, B^z)(x, t) = (H^\phi, H^z)(r(x, t), t).
\]

Then, we write the Lagrangian version of (5)\textsubscript{1} equivalently as

\[
f_t + \frac{fu^r}{r_x} + \frac{fv^r}{r} = 0, \quad (x, t) \in I \times (0, T).
\]
Next, using (18) and solving above equation, we get

$$f(x, t) = \frac{xp(x)}{r(x, t)r_x(x, t)}, \quad (22)$$

Combining (17) and (22), the Lagrangian version of (5) on a fixed interval reads:

$$\frac{pxv_t}{r} - \frac{pxv(x)}{r^2} + \left[ \frac{px}{r r_x} \right]_x - \frac{x^2}{r^2} \left( \bar{p} \right)_x$$

$$= \bar{w} \left( \frac{r v^r}{r r_x} \right)_x - \frac{[(B^\phi)^2 + (B^\varphi)^2]_x - r_x (B^\phi)^2}{r}, \quad \text{in } Q_T, \quad (23)$$

$$\bar{p}xv_t^\phi + \bar{p}x v^\phi v^\phi = \mu \left( \frac{(r v^\phi)}{r r_x} \right)_x, \quad \text{in } Q_T, \quad \bar{p} x v_t^\phi = \mu \left( \frac{(x v^\phi)}{r r_x} \right)_x + v_x^\phi, \quad \text{in } Q_T,$$

$$r_x B^\phi_t + B^\phi v^\phi = \nu \left( \frac{(r B^\phi)}{r r_x} \right)_x, \quad \text{in } Q_T,$$

$$r r_x B^\phi_t + B^\varphi (r v^\phi + r_x v^r) = \nu \left( (r B^\phi) + B^\varphi \right)_x, \quad \text{in } Q_T,$$

with the boundary and initial conditions:

$$\begin{cases}
    v^r = v^\phi = v^\varphi = B^\phi = B^\varphi = 0, & \text{on } Q_T(0), \\
    \bar{p} = \bar{w} \frac{v^r}{r_x} + \lambda \frac{v^r}{r} = \frac{v^\phi}{r_x} = \frac{v^\varphi}{r_x} = B^\phi = B^\varphi = 0, & \text{on } Q_T(R),
\end{cases} \quad (24)$$

$$(r, v^r, v^\phi, v^\varphi, B^\phi, B^\varphi)|_{t=0} = (r_0(x), u_0^\phi(r_0(x)), u_0^\varphi(r_0(x)), H_0^\phi(r_0(x)), H_0^\varphi(r_0(x))), \quad \text{in } x \in I, \quad (25)$$

where $Q_T = I \times (0, T)$, $Q_T(0) = \{0\} \times (0, T)$, $Q_T(R) = \{R\} \times (0, T)$, and $\bar{w} = \lambda + 2\mu$.

Moreover, it can be derived from (18), (19) and (24) that

$$r(0, t) = r_0(0) + \int_0^t v^r(0, s) ds = 0, \quad t \in [0, T]. \quad (26)$$

2.2. Strong solutions and functionals. Denote,

$$\mathfrak{B} := (\mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3) := \left( \frac{v^r}{r_x} + \lambda \frac{v^r}{r}, \mu \left( \frac{v^\phi}{r_x} - \frac{v^\phi}{r} \right), \mu \frac{v^\varphi}{r_x} \right). \quad (27)$$

Then, a strong solution to problem (23)-(25) is defined as follows.

Definition 2.1. $(v^r, v^\phi, B^\phi, v^\varphi, B^\varphi) \in L^\infty(0, T; H^2_{loc}([0, R])) \cap L^\infty(0, T; W^{1, \infty}(I))$, $(v^r, B^\phi) \in L^\infty(0, T; W^{1, \infty}(I))$ satisfying the initial condition (25) with

$$r(x, t) = r_0(x) + \int_0^t v^r(x, s) ds \quad \text{for } (x, t) \in I \times [0, T], \quad (28)$$

is called a strong solution of (23)-(25) in $Q_T$, if

1) $c_1 \leq r_x(x, t) \leq c_2$, $(x, t) \in I \times [0, T]$; for some positive constants $c_1$ and $c_2$;

2) $(\bar{p}^{1/2}(v^r, v^\phi), x^{1/2} \bar{p}^{1/2}v^r, B^\phi, x^{1/2}B^\phi) \in C^1([0, T], L^2(I))$;

3) $r \in L^\infty(0, T; H^2_{loc}([0, R]))$, $(\bar{p}^{-\frac{1}{2}} \left( \frac{(r v^r)}{r r_x} \right)_x, \bar{p}^{-\frac{1}{2}} \left( \frac{(r v^\phi)}{r r_x} \right)_x, \bar{p}^{-\frac{1}{2}} x^{-\frac{1}{2}} (r \left( \frac{v^\phi}{r_x} \right)_x + v^\phi)_x, (r B^\phi)_x, x^{-\frac{1}{2}} (r \left( \frac{B^\phi}{r_x} \right)_x + B^\phi)_x) \in L^\infty(0, T; L^2(I))$, and $\bar{p}^{-\frac{1}{2}} \left( r_{xx}, (\xi) \right)$. 

\( \rho \in C([0, T], L^2(I)); \)
4) \((v^r, v^\phi, v^z)(0, t) = (B^\phi, B^z)\big|_{\partial I} = 0 \) and \((\mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3)(\overline{\mathcal{R}}, t) = 0 \) hold in the sense of \( W^{1, \infty}\)-trace and \( H^1\)-trace, respectively, for a.e. \( t \in [0, T]; \)
5) (23)\(_{1-5}\) hold for a.e. \((x, t) \in I \times [0, T].\)

**Remark 1.** Let \((r, v^r, v^\phi, v^z, B^\phi, B^z)\) be a strong solution of (23)-(25) defined in Definition 2.1. Then it holds that for any \( b \in (0, \overline{\mathcal{R}}),\)

\[
(v^r, v^\phi, B^\phi) \in L^\infty(0, T; W^{1, \infty}(I)) \cap L^\infty(0, T; H^2([0, b])),
\]

\[
(v^r, B^z) \in L^\infty(0, T; W^{1, \infty}(I)),
\]

\[
r \in C([0, T], W^{1, \infty}(I)) \cap C([0, T], H^2([0, b])),
\]

\[
(r, v^r, v^\phi, v^z, B^\phi, B^z)/x \in L^\infty(0, T; L^\infty(I)), \quad r/x \in C([0, T], L^\infty(I)),
\]

\[
(r, v^r, v^\phi, B^\phi)/x \in L^\infty(0, T; H^1(I)), \quad (v^r_{xx}, v^\phi_{xx}, B^\phi_{xx}) \in L^\infty(0, T; L^2([0, b])),
\]

\[
x^2 (v^r_{xx}, B^\phi_{xx}) \in L^\infty(0, T; L^2([0, b])),
\]

\[
((v^r, v^\phi, B^\phi)/r, (v^r, v^\phi, B^\phi)/x, \mathfrak{B}^1, \mathfrak{B}^2, x^2 \mathfrak{B}^3) \in L^\infty(0, T; H^1(I)),
\]

\[
(v^r, B^z)/r \in L^\infty(0, T; L^\infty(I)), \quad \mathfrak{B}^3 \in L^\infty(0, T; L^\infty(I)),
\]

where \(\mathfrak{B}^1, \mathfrak{B}^2\) and \(\mathfrak{B}^3\) are defined by (27).

The arguments for (29)-(33) go as follows. Clearly, (29) follows from Definition 2.1, and (30) is derived by (29) and (28). For (31), due to the facts \((r_x, v^r_x, v^\phi_x, v^z_x, B^\phi_x, B^z_x) \in L^\infty(0, T; L^\infty(I)), \quad (r, v^r, v^\phi, v^z, B^\phi, B^z)/(0, t) = 0\) and Cauchy’s mean value theorem, one has \((r_x, v^r_x, v^\phi_x, v^z_x)/x \in L^\infty(0, T; L^\infty(I)).\)

It follows from \(c_1 \leq r(x, t) \leq c_2\) and \(r(0, t) = 0\) that \(c_1 \leq x^{-1} r(x, t) \leq c_2\); which, together with (28), gives \(r/x \in C([0, T], L^\infty(I)).\) For (32), it follows from (28), \(\overline{\mathcal{R}}^{-1/2}(r_{xx}, (r/x)_x) \in L^\infty(0, T; L^2(I)),\) and \(\overline{\mathcal{R}}(x) \geq \overline{\mathcal{R}}(b)\) for \(x \in [0, b]\) that \((r_{xx}, (r/x)_x) \in C([0, T], L^2([0, b]))\) and \((v^r_{xx}, (v^r/x)_x) \in L^\infty(0, T; L^2([0, b]))\).

So, (32) holds, due to item 3) in Definition 2.1 and the two estimates below:

\[
\left\| \left( v^\phi_{xx}, \left( \frac{v^\phi}{x} \right)_x, x^2 \left( v^\phi_{xx} \right)_x, B^\phi_{xx}, \left( \frac{B^\phi}{x} \right)_x, x^2 \left( \frac{B^\phi_{xx}}{x} \right)_x \right) \right\|_{L^2([0, b])}^2 \leq c(b) \left\| \left( \frac{r v^\phi_x}{rr_x} \right)_x x^{-1/2} (r v^\phi_x)_x + v^\phi_x, \left( \frac{r B^\phi_x}{rr_x} \right)_x x^{-1/2} (r B^\phi_x)_x + B^\phi_x \right\|_{L^2(I)}^2,
\]

and

\[
| (r, v^r, v^\phi, B^\phi)/x | \leq \frac{1}{b} (| (r_x, v^r_x, v^\phi_x, B^\phi_x) | + | (r, v^r, v^\phi, B^\phi)/x |) \text{ for } x \in [b, \overline{\mathcal{R}}].
\]

Indeed, (34) can be derived by (121)-(122) in Lemma 4.5. Finally, (33) follows from (29)-(32), \(c_1 \leq r/x \leq c_2\) and 3) of Definition 2.1.

To show the well-posedness of strong solutions to problem (23) defined in Definition 2.1, we use the method of weighted energy estimates. And motivated by the analysis in [36, 37], we introduce the following higher-order functional:

\[
\mathcal{E}(t) = \| (r_x - 1, v^r_x, v^\phi_x, v^z_x, B^\phi_x, B^z_x) (\cdot, t) \|_{L^\infty}^2 + \| \overline{\mathcal{R}}^{-1/2} (r_{xx}, (r/x)_x) (\cdot, t) \|_{L^2}^2
\]

\[
+ \| (p^{1/2} (v^r, v^\phi, x^2 v^z), (B^\phi, x^2 B^z)) \|_{L^2}^2
\]
We remark that all initial values from Lemma 4.1 to Corollary 4.11 can be controlled by $\mathcal{E}(0)$.

2.3. Main theorems and remarks. Now, we are ready to state the main results.

**Theorem 2.2.** (global existence) Let $\gamma \in (1,2)$, and $\bar{p}$ be the steady solution satisfying (10)-(12). Assume that the compatibility conditions $(v^r, v^\phi, v^z, B^\phi, B^z)(0,0) = 0$ hold and the initial density $\rho_0$ satisfies (8) and (21). There exists a constant $\delta > 0$ which may depend on $\mu, \gamma, \nu$ and $M$ such that if

$$\mathcal{E}(0) \leq \delta,$$

then the problem (23)-(25) admits a unique strong solution in $I \times [0, \infty)$ with

$$\mathcal{E}(t) \leq C_T \mathcal{E}(0), \quad t \in [0, T],$$

for some constant $C_T$ which only depends on $\mu, \gamma, \nu$ and $M$, but is independent of $t \in [0, T]$.

It should be noted that $\| (\bar{p}^{1/2}(v^r, v^\phi, x^{1/2}v^z), (B^\phi, x^{1/2}B^z))(-,0) \|_{L^2}$ is given in terms of the initial data $(r_0, u_0^\phi, u_0^\phi, u_0^\phi, H_0^\phi, H_0^\phi)$ by equation (23) \[1-5\].

**Proof.** The local existence and uniqueness to (23)-(25) can be established similarly as in [36, 41] by the method of finite difference scheme, thus we omit the details here. So, the global well-posedness of strong solutions with the estimate (37) can be shown by the lower-order estimates obtained in Subsection 4.1 and the higher-order estimates obtained in Subsection 4.2, through the standard continuation argument. \(\square\)

For any $t \geq 0$, since $r_x(x,t) > 0$ for $x \in I$, $r(x,t)$ defines a diffeomorphism from the reference domain $I$ to the changing domain $\{0 \leq r \leq R(t)\}$ with the boundary

$$R(t) = r(R,t).$$

(38)

It also induces a diffeomorphism from the initial domain, $\overline{B}_{R_0}(0)$, to the evolving domain, $\overline{B}_{R(t)}(0)$, for all $t \geq 0$:

$$x \neq 0 \in \overline{B}_{R_0}(0) \rightarrow r \left( r_0^{-1}(|x|,t) \frac{x}{|x|} \right) \in \overline{B}_{R(t)}(0).$$

where $r_0^{-1}$ is the inverse map of $r_0$ defined in (20). Here $\overline{B}_{R_0}(0) := \{x \in \mathbb{R}^2 : |x| \leq R_0\}$ and $\overline{B}_{R(t)}(0) := \{x \in \mathbb{R}^2 : |x| \leq R(t)\}$.

Denote the inverse of the map $r(x,t)$ by $R_{t}$ for $t \geq 0$ so that

$$\text{if } r = r(x,t) \text{ for } 0 \leq r \leq R(t), \text{ then } x = R_{t}(r).$$

For the strong solution $(r, v^r, v^\phi, v^z, B^\phi, B^z)$ obtained in Theorem 2.2, we set for $0 \leq r \leq R(t)$ and $t \geq 0$,

$$\rho(r,t) = \frac{x\bar{p}(x)}{r(x,t) r_x(x,t)} \text{ and } (u^r, u^\phi, u^z, H^\phi, H^z)(r,t) = (u^r, u^\phi, u^z, B^\phi, B^z)(x,t)$$

with $x = R_{t}(r)$. \(39\)

Then $(\rho(r,t), u^r(r,t), u^\phi(r,t), u^z(r,t), 0, H^\phi(r,t), H^z(r,t), R(t)) \ (t \geq 0)$ defines a global strong solution to the free boundary problem (5)-(8). Furthermore, we have the following Theorem with detailed estimates.
Theorem 2.3. Under the assumptions in Theorem 2.2, $(\rho, u^\rho, u^\omega, B^\rho, B^\omega, R(\tau))$ defined by (38) and (39) is the unique global strong solution to the free boundary problem (23)-(25) satisfying $R \in W^{1,\infty}([0, +\infty))$. Moreover, the solution satisfies the following estimates:

\[
\begin{align*}
|\rho(x, t) - x|^2 &\leq C_TE(0), \\
\left(\|r_x(x, t) - 1, x^{-1} r(x, t) - 1\|\right)^2 &\leq C_TE(0), \\
\left\|\left(u^\rho(r, t), u^\omega(r, t), H^\rho(r, t), H^\omega(r, t)\right)\right\|^2 &\leq C_TE(0), \\
\left\|\left(u_x^\rho(r, t), r^{-1} u^\rho(r, t), u_x^\omega(r, t), r^{-1} u^\omega(r, t), H_x^\rho(r, t), r^{-1} H_x^\rho(r, t), H_x^\omega(r, t), r^{-1} H_x^\omega(r, t)\right)\right\|^2 &\leq C_TE(0), \\
|\rho(r(x, t), t) - \overline{\rho}(x)|^2 &\leq C_TE(0).
\end{align*}
\]

Proof. Noting that

\[
\left\{\begin{array}{l}
(u_x^\rho(r, t), u_x^\omega(r, t), H_x^\rho(r, t), H_x^\omega(r, t)) \\
r^{-1} \cdot \left((u^\rho(r, t), u^\omega(r, t), H^\rho(r, t), H^\omega(r, t)) \right)
\end{array}\right.
\]

and

\[
\left\{\begin{array}{l}
(u_x^\rho(r, t), u_x^\omega(r, t), H_x^\rho(r, t), H_x^\omega(r, t)) \\
\frac{x}{r(x, t)} \cdot \left((v^\rho(x, t), v^\omega(x, t), v_x^\rho(x, t), v_x^\omega(x, t), B^\rho(x, t), B^\omega(x, t)) \right)
\end{array}\right.
\]

Then, (40) follows from (149); (41) follows from Corollary 3 and (151); (42) follows from (96) and (150); (43) follows from (96), (150) and (151); For (44), due to the Taylor expansion and (41), we have

\[
\left|\frac{x\overline{\rho}(x)}{r(x, t)r_x(x, t)} - \overline{\rho}(x)\right| = \left|\frac{x\overline{\rho}(x)}{r(x, t)r_x(x, t)} - \overline{\rho}(x)\right| = \left|\frac{r(x)}{r_x(x, t)} - 1\right| \\
\leq C\overline{\rho}(x)\left(r_x(x, t) - r_x(0)\right) \leq C_TE(0).
\]

\]

3. Preliminaries.

Lemma 3.1. (See [44]) Assume that $\overline{\rho}(x) \geq 0$ for $x \in I = [a, b]$, $\int_I \overline{\rho} dx = M > 0$. Then for any $q > 0$, there exists a constant $C = C(M, q) > 0$, such that

\[
\|v^q\|_{L^\infty(I)} \leq C \left(\|v^q\|_{L^1(I)} + C \|\overline{\rho}v\|_{L^1(I)}\right),
\]

for any $v^q \in H^1(I)$, Here $M$ and $q$ are given positive constants, and $p = \max\{1, q\}$.

Remark 2. Similar $L^2$ estimates in multi-dimensional domains can be found in [10]. Moreover, when $q = 1$ and $\overline{\rho}(x) = constant$, this lemma reduces to Poincaré’s inequality.

As a corollary of Lemma 3.1, we have

Lemma 3.2. ([39]) Assume that $\overline{\rho}(x)$ is a continuous function in $I = [0, 1]$, satisfying $\overline{\rho}(x) > 0$ for $x \in [0, 1]$ and $\overline{\rho}(1) = 0$. Then for any $\epsilon > 0$, there exists a positive constant $C = C(\epsilon)$ depending on $\epsilon$, but not on $u$, such that

\[
\|u\|_{L^2(I)}^2 \leq \epsilon \|u_x\|_{L^2(I)}^2 + C(\epsilon) \int_I \overline{\rho} u^2 dx, \quad \forall u \in H^1(I).
\]
Lemma 3.3. (Hardy’s inequality, see [20, 26, 42]). Let $k > 1$ be a given real number and $u$ be a function satisfying
\[ \int_0^{1/2} x^k (u^2 + u_x^2) dx < \infty \]
then it holds that
\[ \int_0^{1/2} x^{k-2} u^2 dx \leq C \int_0^{1/2} x^k (u^2 + u_x^2) dx, \tag{45} \]
where $C$ is a generic constant independent of $u$.

The following corollary follows from Lemmas 3.2 and 3.3.

Corollary 1. Assume that $\bar{p}(x)$ is a continuous function in $I = [0, 1]$, satisfying $\bar{p}(x) > 0$ for $x \in [0, 1)$ and $\bar{p}(1) = 0$. Then for any $u \in H^1(I)$, there exists a constant $C = C(\bar{p})$, such that
\[ \int_I x^k u^2 dx \leq C \int_I x^{k+2} (\bar{p}u^2 + |u_x|^2) dx, \quad k > -1. \tag{46} \]

Proof. Indeed, (46) can be proved on $(0, 1/2)$ and $(1/2, 1)$ respectively. For $(0, 1/2)$, due to Hardy’s inequality (45) and the positivity of $\bar{p}$ away from vacuum boundary, one has
\[ \int_0^{1/2} x^k u^2 dx \leq C \int_0^{1/2} x^{k+2} (u^2 + |u_x|^2) dx \leq C \left( \frac{1}{\bar{p}(1/2)} \right)^{-1} \int_0^{1/2} x^{k+2} (\bar{p}u^2 + |u_x|^2) dx. \tag{47} \]

For $(1/2, 1)$, due to Lemmas 3.2, we have
\[ \int_{1/2}^1 x^k u^2 dx \leq C(\bar{p}) \int_{1/2}^1 (\bar{p}u^2 + |u_x|^2) dx \leq C(\bar{p}) \int_{1/2}^1 x^{k+2} (\bar{p}u^2 + |u_x|^2) dx. \tag{48} \]
Thus, (46) follows from (47) and (48) immediately. \hfill \Box

Notation. (i) $I = [0, T]; Q_T = I \times [0, T]$ for $T > 0$.
(ii) For $p \in [1, \infty]$, $L^p(I)$ denotes the $L^p$ space with the norm $\| \cdot \|_{L^p}$. For $k \geq 1$ and $p \in [1, \infty]$, $W^{k,p} = W^{k,p}(I)$ denotes the Sobolev space, whose norm is denoted as $\| \cdot \|_{W^{k,p}}$. $H^k = W^{k,2}(I)$, $k \geq 1$.
(iii) Throughout the rest of paper, $c$ or $C$ will denote a positive constant which does not depend on the initial data. They are referred as universal and can change from one inequality to another one. Also we use $C(\delta)$ or $C_T$ to denote a certain positive constant depending on the quantity $\delta$ or $T$.
(iv) We will employ the notation “$x \sim y$” to denote “$C_1 x \leq y \leq C_2 x$”, where $C_1$ and $C_2$ are two generic positive constants.

4. Global-in-time estimates. The local existence and uniqueness to (23)-(25) can be established similarly as in [36, 41] by the method of finite difference scheme, thus we omit the details here. To show the global existence of strong solutions, we only need to obtain uniform-in-time estimates with regularity, which is the main task of this section.

We set the a priori assumptions (ref. [36, 37]):
\[ |r_x - 1| + |r/x - 1| \leq \epsilon_0 \quad \text{for} \quad (x,t) \in [0, \bar{r}] \times [0, T], \tag{49} \]
for some \( \epsilon_0 \in (0, 1/2) \), is a sufficiently small but fixed constant; and

\[
|v_x^r| + \frac{\nu r}{x} + |v_x^\phi| + \frac{\nu^\phi}{x} + |v_z^r| + \frac{\nu^r}{x} + |B_x^\phi| + |B_x^\phi| + |B_z^\phi| \leq 1, \tag{50}
\]

for \( (x, t) \in [0, \mathcal{R}] \times [0, T] \). In particular, it holds that

\[
1/2 \leq r_x, r/x \leq 3/2 \quad \text{for} \quad (x, t) \in [0, \mathcal{R}] \times [0, T].
\]

(Indeed, the \textit{a priori} assumptions (4.1) and (4.2) will be verified by Corollaries 3 and 6.)

For the convenience of presentation, we set \( \mathcal{R} = 1 \) and \( I = (0, \mathcal{R}) = (0, 1) \). Since the total mass \( M \) and the radius \( \mathcal{R} \) of the steady solution are determined by each other uniquely for \( 1 < \gamma < 2 \), so \( M \) is fixed when we take \( \mathcal{R} = 1 \). Therefore, we omit the dependence of \( M \) in the analysis.

### 4.1. Lower-order estimates

We recall the system (23)-(24) that:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial x v_t^r}{r} - \frac{\partial x (v_r^r)^2}{r^2} + \left[ \frac{\partial x}{r^r} \right] \gamma - \frac{x^2}{r^2} \left( \frac{\partial}{r^r} \right) = \left( \mathfrak{B}^1 \right) + 2\mu \left( \frac{v_r^r}{x} \right) - \frac{\left( B_x^\phi \right)^2 + \left( B_x^\phi \right)^2}{2} - \frac{r_x (B_x^\phi)^2}{r}, \\
\left( \mathfrak{B}^3 \right) x + \mu v_x^2, \\
\frac{\partial x v_t^\phi}{r} + \frac{\partial x v^r v^\phi}{r^2} = \left( \mathfrak{B}^2 \right) x + 2\mu \left( \frac{v^\phi}{x} \right), \\
r_x B_x^\phi + B_x^\phi v_x^r = \nu \left( \frac{B_x^\phi}{r_x} + \frac{B_x^\phi}{r} \right), \\
rr_x B_x^\phi + B_x^\phi r v_x^r + r_x v_r^r = \nu \left( r (B_x^\phi) r_x + B_x^\phi \right),
\end{array} \right. \\
&\text{in} \quad QT, \\
\end{aligned}
\]

with the boundary conditions:

\[
\begin{cases}
(v^r, v^\phi, v_x^r, B_x^\phi, B_x^\phi)|_{x=0} = 0, \\
(\mathfrak{B}, \mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3, B_x^\phi, B_x^\phi)|_{x=\mathcal{R}} = 0,
\end{cases} \tag{52}
\]

where \( (\mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3) \) is defined by (27). Now we define some functions we shall use in this work. Denote

\[
\mathfrak{D}(t) := \left\| \left( x^\gamma r^2 (v^r, v^\phi, v_x^r, B_x^\phi, B_x^\phi) \right)(\cdot, t) \right\|_{L^2}^2 + \left\| x^\gamma \nabla^2 (r_x - 1, \frac{r}{x} - 1)(\cdot, t) \right\|_{L^2}^2, \tag{53}
\]

\[
\mathfrak{D}_2(t) := \left\| \left( x^\gamma r^2 (v^r_x, v^\phi_x, v_x^r, B_x^\phi, B_x^\phi) \right)(\cdot, t) \right\|_{L^2}^2 + \left\| x^\gamma \nabla^2 (v^r_x, v^\phi_x) \right\|_{L^2}^2, \tag{54}
\]

and

\[
\mathfrak{D}_3(t) := \left\| (\mathfrak{B}^2(v^r_x, v^\phi_x), B_x^\phi)(\cdot, t) \right\|_{L^2}^2 + \left\| \rho^{\gamma-\frac{3}{2}} (r_{xx}, \frac{r_x}{x})(\cdot, t) \right\|_{L^2}^2, \tag{55}
\]

**Lemma 4.1.** (Basic Energy Estimate) Suppose that (49) holds for suitably small constant \( \epsilon_0 \). Then there exist a positive constant \( C \) independent of \( T \) such that for \( t \in [0, T], \)

\[
\mathfrak{D}(t) + \sigma \int_0^t \left\| x^\gamma (v^r_x, v^r_x - v^\phi_x, v_x^r - v^\phi_x, B_x^\phi, B_x^\phi) \right\|_{L^2}^2 ds \leq C \mathfrak{D}(0), \tag{56}
\]

where \( \sigma = \min\{2(\mu + \lambda), \mu, \nu\}. \)
Proof. Integrating the sum of $rv^r \cdot (51)_1$, $rv^\phi \cdot (51)_2$, $v^z \cdot (51)_3$, $rB^\phi \cdot (51)_4$ and $B^z \cdot (51)_5$ over $I$, and using the boundary conditions (52), we have

$$\frac{d}{dt} \int_I \left[ \frac{x \rho}{2} ((v^r)^2 + (v^\phi)^2 + (v^z)^2) + \frac{rr_x}{2} ((B^\phi)^2 + (B^z)^2) \right] dx(t)$$

$$+ D_1 + D_2 + \int_I rr_x \left[ 2 \mu \left( \frac{v^r_{xx}}{r_x} \right)^2 + \left| \frac{v^r}{r_x} \right|^2 \right] + \lambda \left( \frac{v^\phi_{xx}}{r_x} + \frac{v^\phi}{r_x} \right)^2$$

$$+ \mu \left( \frac{v^z_{xx}}{r_x} - \frac{v^z}{r_x} \right)^2 + \mu \left| \frac{v^z}{r_x} \right|^2 + \nu \left( \frac{B^\phi_{xx}}{r_x} + \frac{B^\phi}{r_x} \right)^2 + \nu \left| \frac{B^z_{xx}}{r_x} \right|^2 \right] dx = 0,$$

where

$$D_1 = \int_I rv^r p_x dx = - \int_I (rv^r)_x p dx = - \int_I (rr_x)_x p dx,$$

$$= - \frac{d}{dt} \int_I rr_x p dx + \int_I rr_x p_t dx = - \frac{d}{dt} \int_I rr_x p dx - \gamma \int_I (rr_x)_x p dx,$$

thus, $D_1 = \frac{1}{\gamma - 1} \frac{d}{dt} \int_I rr_x p dx = \frac{d}{dt} \int_I x p \gamma^{-1} (\gamma - 1) r_x^{2 - \gamma} dx$, and

$$D_2 = - \int_I rv^r \cdot \frac{x^2}{r^3} (p^\gamma)_x dx = - \int_I x^2 \cdot \frac{v^r}{r} (p^\gamma)_x dx$$

$$= \int_I x^2 (v^r)_x p^\gamma dx + \int_I 2x \cdot \frac{v^r}{r} p^\gamma dx = \frac{d}{dt} \int_I x^2 p^\gamma (\frac{r^2}{x}) + x p^\gamma \log(\frac{r}{x}) dx,$$

by using the fact that

$$\left\{ \begin{array}{l}
(rv^r)_x = rv^r_x + r_x v^r = (rr_x)_t, \\
(v^r)_x = \frac{r^2}{x} (rr_x)_t \end{array} \right\}$$

$$p_t = [(\frac{x^2}{r}) \gamma]_t = \gamma (\frac{x^2}{r}) \gamma x (\frac{x}{r})_x (rr_x)_t = - \gamma \rho (\frac{rr_x}{r})_t. \quad (58)$$

Set

$$\eta(x, t) := \frac{1}{2} \left[ x \rho \left( (v^r)^2 + (v^\phi)^2 + (v^z)^2 \right) + rr_x \left( |B^\phi|^2 + |B^z|^2 \right) \right] + x \rho \gamma \eta(x, t),$$

$$\bar{\eta}(x, t) := \left[ \frac{1}{\gamma - 1} \left( \frac{x}{r} \right) \gamma - 1 \right] r_x^{2 - \gamma} + \frac{x^2}{r} r_x + 2 \log(\frac{r}{x}) - \frac{\gamma}{\gamma - 1} \right] \quad (59)$$

Then (57) yields that

$$\frac{d}{dt} \int_I \eta(x, t) dx(t) + \int_I rr_x \left[ 2 \mu \left( \frac{v^r_{xx}}{r_x} \right)^2 + \left| \frac{v^r}{r_x} \right|^2 \right] + \lambda \left( \frac{v^\phi_{xx}}{r_x} + \frac{v^\phi}{r_x} \right)^2$$

$$+ \mu \left( \frac{v^z_{xx}}{r_x} - \frac{v^z}{r_x} \right)^2 + \mu \left| \frac{v^z}{r_x} \right|^2 + \nu \left( \frac{B^\phi_{xx}}{r_x} + \frac{B^\phi}{r_x} \right)^2 + \nu \left| \frac{B^z_{xx}}{r_x} \right|^2 \right] dx = 0,$$

By the Taylor expansion, the quantity $\eta(x, t)$ can be rewritten as

$$\bar{\eta}(x, t) = \left( \gamma - 2 \right) \left( \frac{r}{x} - 1 \right) (r_x - 1) + \frac{\gamma}{2} \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 + o(\cdot),$$

$$\geq \frac{\gamma - 1}{2} \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 + o(\cdot), \quad (\text{when} \quad \gamma \in (1, 2)) \quad (61)$$

$$\geq \frac{\gamma - 1}{2} \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 + o(\cdot),$$

where $o(\cdot)$ represents the cubic terms which can be bounded by the following estimate

$$|o(\cdot)| \leq c \left( (\frac{r}{x} - 1)^3 + (r_x - 1)^3 \right) \leq c_0 \left( |\frac{r}{x} - 1|^2 + |r_x - 1|^2 \right),$$
due to (49). Then the above calculations imply that
\[
\frac{\gamma-1}{4}([\frac{r}{x} - 1]² + |r_x - 1|²) ≤ \bar{\eta}(x,t) ≤ c([\frac{r}{x} - 1]² + |r_x - 1|²), \quad \text{(when } \gamma \in (1, 2)).
\]
(62)

Integrating (60) over (0, t), we get
\[
\int \eta(x,t)dx + \int_0^t \int r x \left[2\mu (\frac{v_x^r}{r_x} + \frac{v_r^x}{r})² + \lambda (\frac{v_x^r}{r_x} + \frac{v_r^x}{r})² + \mu \left(\frac{v_x^r}{r_x} - \frac{v_r^x}{r}\right)² \right]dxds ≤ \int_0^t \eta(x,0)dx,
\]
which and (62), (59), (49) imply (56).

Now, we try to get more information about \(v^ϕ\) than Lemma 4.1. In the following sections, we denote \(C_T\) by a constant depending on the fixed time \(T\).

**Corollary 2.** Suppose that (49) holds. Then there exist a positive constant \(C_T\), such that for \(t \in [0, T]\),
\[
\int_0^t \left\| x \hat{z}\left(\frac{v^ϕ}{x}, \frac{v^ϕ}{x}\right) \right\|_{L^2}^2 ds \leq C_T \mathcal{D}(0),
\]
(64)

**Proof.** Selecting “\(k = 1, u(x,t) = \frac{v^ϕ}{x}(x,t)\)” in (46), one has
\[
\int_0^t |x|\frac{v^ϕ}{x}|^2dx \leq C \int \int x|\hat{\mathcal{D}}(x)|^2dx + C \int x|\hat{\mathcal{D}}(x)|^2dx.
\]
Next, integrating above inequality over (0, t) and using Lemma 4.1, we get (64).

In the following lemma, we use the multiplier \(r^2 - x^2\) which is motivated by [36] to refine the weighted estimate of \(\|x^{1/2}\mathcal{D}/r^{1/2}(r/x - 1, r_x - 1)\|_{L^2}\) obtained in Lemma 4.1 by improving the estimates near the vacuum.

**Lemma 4.2.** Suppose that (49) holds. Then there exist a positive constant \(C_T\) such that for \(t \in [0, T]\),
\[
\sigma_1 \left\| x^{1/2}(\frac{r}{x} - 1, r_x - 1) \right\|_{L^2}² + (\gamma - 1) \int_0^t \left\| x^{1/2}(\frac{r}{x} - 1) \right\|_{L^2}² ds \leq C_T (\mathcal{D}(0) + \left\| x^{1/2}(\frac{r_0}{x} - 1, r_0x - 1) \right\|_{L^2}²),
\]
where \(\sigma_1 = \min\{\lambda + \mu, \mu\}\) and
\[
\|r - x\|_{L^2}² ≤ C_T (\mathcal{D}(0) + \left\| x^{1/2}(\frac{r_0}{x} - 1, r_0x - 1) \right\|_{L^2}²).
\]
(66)

**Proof.** Multiplying (51) by \(r^2 - x^2\), integrating the resulting equation with respect to the spatial variable and using the boundary condition (52), we get
\[
\int \mathcal{D} \left\{ \left[ \frac{x^2}{rr} (r^2 - x^2) \right] - \left( \frac{x}{rr} \right)^γ (r^2 - x^2) \right\} dx + F_1 + F_2
\]
\[
= \int t |\hat{\mathcal{D}}\frac{|\hat{v}|^2}{r^2}(r^2 - x^2)dx + \int \left( \frac{|\hat{B}|^2}{2} + \frac{|\hat{B}|^2}{2} \right)(r^2 - x^2)dx - \int \frac{r}{r} |\hat{B}|^2(r^2 - x^2)dx,
\]
(67)
where
\[ F_1 = \int I \left[ \Phi^1(r^2 - x^2) - 2\mu \left( \frac{v^r}{r} \right)_x (r^2 - x^2) \right] dx \]

\[ = \int I \left[ 2(\lambda + \mu) \left( \frac{v^r}{r} + \frac{v^r}{r} (rr_x - x) \right) + 2\mu \left( \frac{v^r}{r} - \frac{v^r}{r} (rr_x - x) \right) ight] dx \]

\[ = 2(\lambda + \mu) \int x \left[ \frac{rr_x}{x} + \frac{r_x v^r}{r_x} - \frac{r^2}{x} v^r + v^r \right] dx + 2\mu \int x \left[ \frac{v^r}{r} - \frac{v^r}{r} + \frac{x v^r}{r} - \frac{r v^r}{x^2} \right] dx \]

\[ = 2(\lambda + \mu) \int x \left[ \frac{rr_x}{x} - \log \left( \frac{r x}{x} \right) - 1 \right] dx + 2\mu \int x \left[ \log \left( \frac{r x}{x} \right) + \frac{r x v^r}{x^2} \right] dx, \]

\[ F_2 = \int I x^2 \Phi^1 \left( \frac{r}{x} - \frac{x}{r} \right) dx = \frac{d}{dt} \int I x^2 \Phi^1 \left( \frac{r}{x} - \frac{x}{r} \right) dx - \int I x^2 \Phi^2 \left( 1 + \frac{x^2}{r^2} \right) dx \]

Then, we set
\[ \Phi_1(z) := z - \log z - 1, \quad \Phi_2(z) := \log z + \frac{1}{z} - 1, \]

\[ \eta_0(x, t) := x^2 \Phi^1 \left( \frac{r}{x} - \frac{x}{r} \right) + \tilde{\eta}_0(x, t), \quad \tilde{\eta}_0(x, t) := 2(\lambda + \mu) \Phi_1 \left( \frac{rr_x}{x} \right) + 2\mu \Phi_2 \left( \frac{r}{x} \right), \]

and it follows from (67) that
\[ \frac{d}{dt} \int I \eta_0(x, t) dx + \int I \phi \left\{ \left[ \frac{x^2}{r^2} (r^2 - x^2) \right]_x - \left( \frac{x^2}{r^2} \right)_x \right\} dx \]

\[ = \int I x^2 \Phi^1 \left( 1 + \frac{x^2}{r^2} \right) dx + \int I x^2 \Phi^2 \left( 1 - \frac{x^2}{r^2} \right) dx \]

\[ + \int I \left( \frac{B^2}{r} + |B^2| \right) (rr_x - x) dx - \int I \frac{r x}{r} |B^2| (r^2 - x^2) dx. \]

Then, we estimate quantity \( \tilde{\eta}_0(x, t) \) of \( \eta_0(x, t) \) and the term \{\} on the left-hand side of above equality by Taylor expansion as follows. Firstly,

\[ \{\} = 2x \left( \frac{x}{r} \right)_x \left( r^2 - x^2 \right) + \left( \frac{x^2}{r^2} - \frac{x}{r} \right)^2 \right] \left( r^2 - x^2 \right)_x \]

\[ = 2x \left( 1 - \left( \frac{x}{r} \right)^2 \right) \left( 1 - \left( \frac{x}{r} \right)^2 \right) + 2x \left[ \left( \frac{x}{r} \right)^{-2} - \left( \frac{x}{r} \right)^{-2} \gamma^2 \right] \left( \frac{x}{r} \right)_x (r_x - 1) \]

\[ = 2x \left[ \gamma (r_x - 1)^2 + 2(2 - \gamma) (r_x - 1) (r_x - 1) + \gamma (r_x - 1)^2 \right] + o(\cdot) \]

\[ \geq 2x (2 - |\gamma|) (r_x - 1) (r_x - 1) + o(\cdot), \quad (when \ \gamma \in (1, 2)) \]

\[ \geq 2(\gamma - 1)x \left[ (r_x - 1)^2 + (r_x - 1)^2 \right] + o(\cdot), \]

where \( o(\cdot) \leq c(|x| - 1)^2 \leq \alpha \leq c_0 [(x - 1)^2 + |x - 1|^2], \) by using the assumption (49).

For \( \tilde{\eta}_0(x, t) \), by using the Taylor expansion, we have

\[ \tilde{\eta}_0(x, t) = (\lambda + \mu) \left( \frac{rr_x}{x} - 1 \right)^2 + \mu \left( \frac{r}{x} \right)^2 + o(\cdot), \]

(71)
noting that
\[
\left(\frac{r}{x} - 1\right)^2 \leq \left(\frac{r}{x} - 1 \cdot \frac{r}{x} + 1\right)^2 = \left(\frac{r^2}{x^2} - 1\right)^2
\]
\[
= \left[\left(1 + \left(\frac{rr_x}{x} - 1\right)\right) + \left(\frac{r}{x} - 1\right) - 1\right]^2
\leq C\left(\frac{rr_x}{x} - 1\right)^2 + \left(\frac{r}{x} - 1\right)^2 \leq C\sigma_1^{-1}\bar{\eta}_0(x, t),
\]
and similarly
\[
(r_x - 1)^2 = \left[\left(1 + \left(\frac{rr_x}{x} - 1\right)\right) + \left(\frac{x}{r} - 1\right) - 1\right]^2
\leq C\left(\frac{rr_x}{x} - 1\right)^2 + \left(\frac{r}{x} - 1\right)^2 \leq C\sigma_1^{-1}\bar{\eta}_0(x, t),
\]
where \(\sigma_1 = \min\{\lambda + \mu, \mu\}\).

Thus,
\[
C^{-1}\sigma_1\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2 \leq \bar{\eta}_0(x, t) \leq C\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2. \quad (72)
\]

Integrating (69) over \((0, t)\), using (70) and (72), we have
\[
\sigma_1 \int_0^t \int x\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2 |dx| + (\gamma - 1) \int_0^t \int x\bar{\eta}^2\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2 |dx|ds
\leq C \int_0^t \int x^2|v_0| \cdot \left|\frac{r}{x} - 1\right| |dx| + C \int_0^t \int \bar{\eta}_0(x, 0)|dx| + C \int_0^t \int x\bar{\eta}(v|v|^2 + |v|^2)|dx|ds
\]
\[
+ \frac{C}{r} \int_0^t \int x|B^\phi|^2 + |B^2|^2 \left(\frac{r}{x} - 1\right)|dx|ds + \frac{C}{r} \int_0^t \int x|B^\phi|^2 + |B^2|^2 \left(\frac{r}{x} - 1\right)|dx|ds
\leq C\left|\frac{1}{r}\int x^2|v_0| \cdot |x| + C\int_0^t \int x^2|v| \cdot |B^\phi|^2 + |B^2|^2 |dx|ds,
\]
where in the last inequality we have used Cauchy’s inequality. Thus, (73) and Lemma 4.1 deduce to (65) immediately.

Obviously, it follows from (65) that
\[
\|r - x\|_{L^2} \leq C\|x^{1/2}(r/x - 1)\|_{L^2}^2 \leq C_T(0) + \left|\frac{r_0}{x} - 1, r_{0x} - 1\right|_{L^2}^2.
\]

Motivated by [36], we are able to derive a pointwise bound for \(|r/x - 1|\) and \(|r_x - 1|\) away from the symmetry axis, by integrating equation (51)_1 with respect to both x and t. In which the steady density \(\bar{\eta}\) which decreases in the radial direction outward plays an important role.

**Corollary 3.** Let \(I_2 = [1/2, 1]\). For a suitably small constant \(\epsilon_0\) in (49), it holds that for \((x, t) \in I_2 \times [0, T]\),
\[
\|r(x, t)/x - 1, r_x(x, t) - 1\|_{L^\infty(I_2)}^2 \leq C_T \left(\bar{\eta}_0 + \|x^2 \left(\frac{r_0}{x} - 1, r_{0x} - 1\right)\|^2_{L^2} + \|r_{0x} - 1\|_{L^\infty(I_2)}^2\right). \quad (74)
\]
Proof. The proof consists of two steps.

**Step 1 (bound for \( r/x - 1 \)).** A direct calculation by using Hölder’s inequality give

\[
x(r-x)^2 = \int_0^x [y(r,y,t) - y]^2 dy \leq \|r-x\|_{L^2}^2 + 2\|r-x\|_{L^2}\|x(r_x - 1)\|_{L^2},
\]

which and Lemma 4.2, deduce that for \( x \in I_2 \),

\[
\frac{r(x,t)}{x} - 1 \leq C_T \left( \mathcal{O}(0) + \|x^\frac{1}{2} (\frac{r_0}{x} - 1, r_{0x} - 1)\|_{L^2}^2 \right). \tag{75}
\]

**Step 2 (bound for \( r_x - 1 \)).** Integrating (51) from \( x \) to 1, \( x \in (0,1) \) and using of the boundary condition (51) \( r = \mathcal{O}(1) \), we get

\[
\begin{align*}
\left[ \int_x^1 \frac{\rho_0}{r} v^r dy \right]_t &+ \int_x^1 \frac{\rho_0}{r^2} \left[ (v')^2 - (v^\phi)^2 \right] dy - \int_0^t \int_x^1 \frac{\rho_0}{r^2} (\mathcal{P} v)^y dy ds \\
&= - [\varpi (\log r_x)_t + \lambda (\log r)_t - p] + 2\mu \left[ \int_x^1 \frac{\rho_0}{r} dy \right]_t \\
&\quad - \int_x^1 \left[ B^\phi B_y^\phi + B^2 B_y^2 + \frac{r_0 (B^\phi)^2}{r} \right] dy.
\end{align*}
\]

It follows from integrating with respect to time variable that

\[
\varpi \log \left( \frac{r_x}{r_{0x}} \right) = \int_0^t K ds + \mathcal{L},
\]

where \( p = (\frac{\rho v}{rr_x})^\gamma \) and

\[
\mathcal{L} = - \int_x^1 \frac{\rho_0}{r} v^r dy \bigg|_0^t - \int_0^t \int_x^1 \frac{\rho_0}{r^2} \left[ (v')^2 - (v^\phi)^2 \right] dy ds + \int_0^t \int_x^1 \frac{\rho_0}{r^2} (\mathcal{P} v)^y dy ds \\
\quad - \lambda \log (r_0(x)) + 2\mu \log \left( \frac{r(1,t)}{r_0(1)} \right) \\
\quad - \int_0^t \int_x^1 \left[ B^\phi B_y^\phi + B^2 B_y^2 + \frac{r_0 (B^\phi)^2}{r} \right] dy ds,
\]

which implies that

\[
r_x = r_{0x} \exp \left( \frac{1}{\varpi} \mathcal{U} \right) \exp \left( \frac{1}{\varpi} \mathcal{L} \right), \quad \text{where} \quad \mathcal{U} = \int_0^t (\frac{\rho v}{rr_x})^\gamma ds \tag{76}
\]

On the other hand, we derive from (76) that

\[
\mathcal{U}_t = (\frac{\rho v}{rr_x})^\gamma = (\frac{\rho v}{rr_{0x}})^\gamma \exp \left\{ - \frac{\gamma}{\varpi} \mathcal{U} \right\} \exp \left\{ - \frac{\gamma}{\varpi} \mathcal{L} \right\},
\]

then,

\[
\exp \left\{ \frac{\gamma}{\varpi} \mathcal{U} \right\} = 1 + \int_0^t \frac{\gamma}{\varpi} (\frac{\rho v}{rr_x})^\gamma \exp \left\{ - \frac{\gamma}{\varpi} \mathcal{L} \right\} ds. \tag{77}
\]

Combining (76) and (77), we have

\[
r_x = r_{0x} \left[ 1 + \int_0^t \frac{\gamma}{\varpi} (\frac{\rho v}{rr_{0x}})^\gamma \exp \left\{ - \frac{\gamma}{\varpi} \mathcal{L}_t \right\} \exp \left\{ - \frac{\gamma}{\varpi} \mathcal{L} \right\} ds \right]^{\frac{1}{\gamma}} \\
\times \exp \left\{ \frac{1}{\varpi} \mathcal{L}_t \right\} \exp \left\{ \frac{1}{\varpi} \int_0^t \int_x^1 \frac{\rho_0}{r^2} (\mathcal{P} v)^y dy ds \right\}. \tag{78}
\]
where
\[ \mathcal{L}_1 = \mathcal{L} - \int_0^t \int_x^1 \frac{y^2}{r^2} (\bar{\rho}')_y dy ds. \]

In view of Lemma 4.1 and (75), one can get that for \( x \geq 1/2 \),
\[
\begin{align*}
\mathcal{L}_1 & \leq C \left( \int_x^1 \bar{\rho}' |v'|^2 dx \int_I \bar{\rho} dx \right)^{\frac{1}{2}} + C \left( \int_x^1 \bar{\rho}' |u_0'(x)| (r_0(x)) dx \int_I \bar{\rho} dx \right)^{\frac{1}{2}} \\
& + C \left\| \frac{r}{x} - 1 \right\|_{L^\infty (I_2 \times [0, T])} + C \int_0^t \int_I \bar{\rho}' |v' + |\rho'|^2| dy ds \\
& + C \int_0^t \int_I \|B_0|^2 + x |B_0^x|^2 + x |B_0^y|^2 + x |B_0^z|^2| dy ds \\
& \leq C_T \left( \mathcal{D}(0) + \|x^\frac{1}{2} (\frac{r_0}{x} - 1, r_{0x} - 1)\|_{L^2} \right).
\end{align*}
\]

It therefore follows from (75) and (78) that
\[
\begin{align*}
\bar{r}_x & \leq \bar{r}_{0x} \left[ 1 + (1 + C \bar{\mathcal{D}}_1) \int_0^t \frac{\gamma}{\bar{\nu}} \bar{\rho}' \exp \left\{ - \frac{\gamma}{\bar{\nu}} \int_0^t \int_x^1 \frac{y^2}{r^2} (\bar{\rho}')_y dy ds \right\} d\tau \right]^{\frac{1}{\gamma}} \\
& \times (1 + C \bar{\mathcal{D}}_1) \left( \exp \left\{ \frac{\gamma}{\bar{\nu}} \int_0^t \int_x^1 \frac{y^2}{r^2} (\bar{\rho}')_y dy ds \right\} \right) \\
& \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right) \left[ \exp \left\{ \frac{\gamma}{\bar{\nu}} \int_0^t \int_x^1 \frac{y^2}{r^2} (\bar{\rho}')_y dy ds \right\} d\tau \right]^{\frac{1}{\gamma}},
\end{align*}
\]

where \( \bar{\mathcal{D}}_1 = \mathcal{D}(0) + \|x^\frac{1}{2} (\frac{r_0}{x} - 1, r_{0x} - 1)\|_{L^2} + \|r_{0x} - 1\|_{L^\infty (I_2)} \).

Notice that \((\bar{\rho}')_x < 0\). So, one can derive from (75) that
\[
\begin{align*}
\bar{r}_x & \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right) \left[ \exp \left\{ \frac{\gamma}{\bar{\nu}} \int_0^t \int_x^1 (1 - C \bar{\mathcal{D}}_1) (\bar{\rho}')_y dy ds \right\} + (1 + C \bar{\mathcal{D}}_1) \\
& \times \int_0^t \frac{\gamma}{\bar{\nu}} \bar{\rho}' \exp \left\{ \frac{\gamma}{\bar{\nu}} \int_\tau^t \int_x^1 (1 - C \bar{\mathcal{D}}_1) (\bar{\rho}')_y dy ds \right\} d\tau \right]^{\frac{1}{\gamma}} \\
& \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right) \left[ \exp \left\{ - \frac{\gamma}{\bar{\nu}} (1 - C \bar{\mathcal{D}}_1) \bar{\rho}' t \right\} + (1 + C \bar{\mathcal{D}}_1) \\
& \times \int_0^t \frac{\gamma}{\bar{\nu}} \bar{\rho}' \exp \left\{ - \frac{\gamma}{\bar{\nu}} (1 - C \bar{\mathcal{D}}_1) \bar{\rho}' (t - \tau) \right\} d\tau \right]^{\frac{1}{\gamma}} \\
& \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right) \left[ \exp \left\{ - \frac{\gamma}{\bar{\nu}} (1 - C \bar{\mathcal{D}}_1) \bar{\rho}' t \right\} \\
& + \frac{1 + C \bar{\mathcal{D}}_1}{1 - C \bar{\mathcal{D}}_1} \left. \exp \left\{ - \frac{\gamma}{\bar{\nu}} (1 - C \bar{\mathcal{D}}_1) \bar{\rho}' (t - \tau) \right\} \right|_{\tau = 0}^{t} \right]^{\frac{1}{\gamma}} \\
& \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right) \left[ \exp \left\{ - \frac{\gamma}{\bar{\nu}} (1 - C \bar{\mathcal{D}}_1) \bar{\rho}' t \right\} \left( 1 - \frac{1 + C \bar{\mathcal{D}}_1}{1 - C \bar{\mathcal{D}}_1} \right) + \frac{1 + C \bar{\mathcal{D}}_1}{1 - C \bar{\mathcal{D}}_1} \right]^{\frac{1}{\gamma}} \\
& \leq \bar{r}_{0x} \left( 1 + C \bar{\mathcal{D}}_1 \right).
\end{align*}
\]

Similarly, \( r_x \geq r_{0x} (1 - C \bar{\mathcal{D}}_1) \). These two estimates, together with (75), imply (74). \qed
Lemma 4.3. Suppose that (49) and (50) hold. Then it holds that, for $0 \leq t \leq T$,
\[
\mathcal{D}_2(t) + \sigma \int_0^t \| \partial_t^\frac{1}{2} \left( v_{tx}^r \frac{v_x^r}{x}, |v_{tx}^\phi - \frac{v_x^\phi}{x}|, v_{tx}^\phi, B_{tx}^\phi, B_{tx}^z \right) \|_{L^2}^2 \, ds \leq C_T \left( \mathcal{D}(0) + \| x^\frac{1}{2} \mathcal{P}^2 v^r_t \|_{L^2}^2 \right)
\]
where $\sigma = \min \{ 2(\mu + \lambda), \mu, \nu \}$.

Proof. The proof consists of two steps.

Step 1. In this step, we prove that
\[
\| x^\frac{1}{2} \mathcal{P}^2 v^r_t \|_{L^2}^2 + \int_0^t \| x^\frac{1}{2} v_{tx}^r \|_{L^2}^2 \, ds \leq C_T(\mathcal{D}(0) + \| x^\frac{1}{2} \mathcal{P}^2 v^r_t(\cdot, 0) \|_{L^2}^2)
\]
Applying $\partial_t$ to (23), we have
\[
\mathcal{P} x v_t^r = \mu \left( \frac{v_z^2}{r_x} + v^r \left( \frac{v_x^2}{r_x} \right) \right) + \mu v_{tx}^r, \quad (v^r, v_t^r)|_{x=0} = \left( \frac{v_z^2}{r_x}, \frac{v_x^2}{r_x} \right), \quad |x|=1 = 0.
\]
Multiplying (82) by $v_t^r$, integrating by parts over $I$, and using (49)-(50), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I \mathcal{P} x(v_t^r)^2 \, dx + \mu \int_I r \frac{(v_{tx}^r)^2}{r_x} \, dx = \mu \int_I v_x^r v_{tx}^r \left( \frac{v_x^r}{r_x} \right) dx - \mu \int_I v_x^r \left( v_r v_{tx}^r + v_{tx}^r v_t^r \right) dx
\]
\[
\leq \delta \int_I x |v_{tx}^r|^2 \, dx + C(\delta) \int_I x |v_{tx}^r|^2 \, dx.
\]
Thus, (81) follows by integrating (83) in $(0, t)$ with a suitably small $\delta$ and Lemma 4.1.

Step 2. In this step, we prove that
\[
\| x^\frac{1}{2} (\mathcal{P}^2 v^r_t, \mathcal{P}^2 v^\phi_t, B_{tx}^\phi, B_{tx}^z) \|_{L^2}^2 + \| x^\frac{1}{2} \mathcal{P}^2 (v^r_x, \frac{v_x^r}{x}) \|_{L^2}^2
\]
\[
+ \int_0^t \| x^\frac{1}{2} (v_{tx}^r \frac{v_x^r}{x}, |v_{tx}^\phi - \frac{v_x^\phi}{x}|, B_{tx}^\phi, B_{tx}^z) \|_{L^2}^2 \, ds \leq C_T \left( \mathcal{D}(0) + \| x^\frac{1}{2} (\mathcal{P}^2 v^r_t, \mathcal{P}^2 v^\phi_t, B_{tx}^\phi, B_{tx}^z) \|_{L^2}^2 + \| x^\frac{1}{2} \mathcal{P}^2 (v^r_x, \frac{v_x^r}{x}) \|_{L^2}^2 \right).
\]
Applying $\partial_t$ to (23), we obtain
\[
\mathcal{P} x v_{tt}^r - \frac{\mathcal{P} x v^\phi_t}{r_x} \left( v^r \frac{v_x^r}{x} - 2 v_{tx}^\phi \right) + r p_{tx} + v^r p_x + \frac{x^2}{r_x^2} v^r (\mathcal{P}^r)_x
\]
\[
=r \left[ \frac{v_x^r}{r_x} + \frac{\lambda r_x}{r_x} \right]_{xt} + 2 \mu r \left[ \frac{v_x^r}{r_x} + v^r \left( \frac{v_x^r}{x} + \frac{-\lambda}{r_x} \right) \right]_{xt} + 2 \mu v^r \left( \frac{v_x^r}{r_x} \right)_{xt} \]
\[
- r \left( \frac{B_x^\phi}{2} + \frac{B_x^z}{2} \right)_{tx} - v^r \left( \left( \frac{B_x^\phi}{2} + \frac{B_x^z}{2} \right)_{tx} - \left( 2 r_x B_x^\phi B_{tx}^\phi, v_{tx}^\phi B_{tx}^z \right) \right).
\]
Multiplying (85) by \(v_r^i\), and integrating by parts over I, we have

\[
\frac{d}{dt} \int_I \rho x |v_r^i|^2 dx + 2\mu \int_I r r_x \left[ \frac{|v_r^i|^2}{r} + \frac{|v_r^i|}{r} |v_r^i| \right] dx + \lambda \int_I r r_x \left( \frac{v_r^i}{r} + \frac{v_r^i}{r} \right) dx = -2\mu \int_I r v_r^i (\frac{|v_r^i|}{r})^2 dx - \int_I \left[ \frac{3|v_r^i|}{r} + \lambda \frac{|v_r^i|}{r} \right] v_r^i v_r^i \ dx \\
- \int_I v_r^i \left[ p_{rx} + v^r p_x + \frac{x^2}{r} v^r (p^\gamma)_x \right] dx + \frac{1}{2} \int_I (rv_r^i)_x \frac{(|B^\phi|^2 + |B|^2)^2}{2} dx \\
+ \int_I (v_r^i v_r^i)_x \frac{|B^\phi|^2 + |B|^2}{2} dx - \int_I (2r_x B^\phi B^\phi + v_r^i |B^\phi|^2) v_r^i dx \\
- \int_I \frac{\rho x v^\phi}{r^2} (v^r v^\phi - 2r v^\phi) v_r^i dx + 2\mu \int_I v_r^i (\frac{|v_r^i|}{r})_x dx,
\]

where

\[
\int_I \left[ \frac{p_{rx}}{r^2} + (v^r v^r)_x p + \frac{x^2}{r} v^r (p^\gamma)_x \right] dx = -\int_I \left[ (rv_r^i)_x p + (v^r v^r)_x p + \frac{x^2}{r^2} v^r (p^\gamma)_x \right] dx
\]

\[
= -\int_I \left[ \gamma (rv_r^i)_x + (v^r v^r)_x p + \frac{x^2}{r^2} v^r (p^\gamma)_x \right] dx
\]

\[
= -\int_I \left[ \frac{\gamma}{r} v_r^i v_r^i r_x + (\gamma - 1) v^r v^r t_x + \gamma \frac{v^r}{r} v_r^i \right] dx - \int_I \left[ \frac{x^2}{r^2} (v^r v^r)_x + \left( \frac{x^2}{r^2} \right)_x v_r^i \right] p^\gamma dx
\]

\[
= \frac{d}{dt} \int_I p \left( \frac{\gamma}{2} \frac{v_r^i}{r} \right)^2 + (\gamma - 1) v^r v^r + \gamma \frac{v^r}{r} v_r^i \right] dx
\]

\[
- \int_I \left[ \frac{\gamma}{2} (v^r v^r)_x (r_x P) t_x + (\gamma - 1) v^r v^r p_t + \frac{\gamma}{2} v^r (r_x P) t_x \right] dx
\]

\[
- \int_I \left[ \frac{x^2}{r^2} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v_r^i \right] p^\gamma dx + \int_I \left[ \frac{(x^2}{r^2} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v_r^i \right] p^\gamma dx,
\]

Set

\[
\tilde{\eta}_i(x,t) := \frac{1}{2} \frac{x^2}{r^2} v^r v^r + \frac{x^2}{r^2} \frac{x}{r} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v^r v^r
\]

\[
= \frac{1}{2} \frac{x^2}{r^2} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v^r v^r \right] (by \ the \ Taylor \ expansion)
\]

\[
\sim \frac{1}{2} \frac{x^2}{r^2} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v^r v^r \left( \gamma - 1 \right) v^r v^r + \frac{\gamma}{2} \frac{v^r}{r} v^r \right] - x p^\gamma \left[ \frac{v^r}{x} v_r^i \right]
\]

then it follows (49) that,

\[
\frac{d}{dt} \int_I \rho x |v_r^i|^2 dx + \frac{\gamma - 1}{4} \frac{x^2}{r^2} \frac{x}{r} v^r v^r \right] dx \leq \tilde{\eta}_i \leq \frac{1}{2} \frac{x^2}{r^2} v^r v^r + \left( \frac{x^2}{r^2} \right)_x v^r v^r \left( \gamma - 1 \right) v^r v^r + \frac{\gamma}{2} \frac{v^r}{r} v^r \right] - x p^\gamma \left[ \frac{v^r}{x} v_r^i \right],
\]

(87)
and (86) can be rewritten as

\[
\frac{d}{dt} \int_I \rho(x,t) dx + 2 \mu \int_I r r_x \left[ \frac{v_{tx}^2}{r_x^2} + \frac{v_t^2}{r^2} \right] dx + \lambda \int_I r r_x \left( \frac{v_{tx}^2}{r_x^2} + \frac{v_t^2}{r^2} \right)^2 dx
\]

\[
= \int_I \left( \frac{3}{2} \frac{v_t^2}{r} + (\gamma - 1) v^r v_t r_t + \frac{2}{r} |v^r|^2 (\frac{r x P}{r})_t \right)
\]

\[- \left[ \left( \frac{x^2}{r^2} \right) v^r v_t^r + \left( \frac{x^2}{r^7} \right) \frac{|v^r|^2}{r^2} \right] dx - \int_I \left[ \omega \frac{v_t^2}{r_x} + \lambda v^r \right] (v^r v_t^r)_x dx
\]

\[+ \int_I (v_t^r)_x \left( \frac{|B^\phi|^2 + |B^\Phi|^2}{r} \right)_t dx + \int_I \left( v^r v_t^r \right)_x \left( \frac{|B^\phi|^2 + |B^\Phi|^2}{r} \right) dx
\]

\[- \int_I (2 r_x B^\phi B_t^\phi + r_v^r |B^\phi|^2) v_t^2 dx - \frac{\rho x v^\phi}{r^2} (v^r v^\phi - 2 r v^\phi) v_t^r dx
\]

\[= J_1 + J_2 + \ldots + J_6.
\]

We estimate terms \(J_1, \ldots, J_6\) as follows under the assumptions (49) and (50).

Applying Cauchy’s inequality, we have

\[|J_1| + |J_2| \leq \delta \int_I x \left[ |v_{tx}^2| + |\frac{v_t^2}{x}| \right] dx + C(\delta) \int_I x \left[ |v_t^2|^2 + |\frac{v_t^2}{x}| \right] dx,
\]

\[|J_3| + |J_5| \leq C \int_I r r_x \left[ \left| \frac{v_{tx}^2}{r_x} \right| + \left| \frac{v_t^2}{r} \right| \right] \left( \frac{v_t^2}{r} B^\phi \cdot \frac{B^\phi}{x} + x |B^\phi|^2 + |B^\phi| + \frac{|v^r B^\phi \cdot B^\phi|}{x r_x} \right) dx
\]

\[\leq \delta \int_I x \left[ |v_{tx}^2|^2 + |\frac{v_t^2}{x}| \right] dx + C(\delta) \int_I x \left[ |B^\phi|^2 + |B^\phi|^2 + |B^\phi|^2 \right] dx,
\]

\[|J_4| \leq \int_I \frac{2}{r^2} r r_x \left( \frac{v_t^2}{r} |v_{tx}^2| + |v_t^2| \cdot \left| \frac{v_t^2}{r_x} \right| \right) \left( \frac{B^\phi}{x}^2 + \frac{B^\phi}{x} \right) dx
\]

\[\leq \delta \int_I x \left[ |v_{tx}^2|^2 + |\frac{v_t^2}{x}| \right] dx + C(\delta) \int_I x \left[ |v_{tx}^2|^2 + |\frac{v_t^2}{x}| \right] dx,
\]

and

\[|J_6| \leq C \int_I \rho x \left( |v_t^2|^2 + |v_t^2|^2 + |v_t^2|^2 \right) dx.
\]

Similarly to (85), applying \(\partial_t\) to \(r \cdot (23)_2\), we have

\[
\partial_t \left( \frac{\rho x v_t^\phi + \frac{\partial}{\partial x} (r v^\phi v_t^\phi + r v^r v_t^\phi - 2 |v^r|^2 v^\phi) \right)
\]

\[= \mu r \left( \frac{v_t^\phi}{r_x} - \frac{v^\phi}{r} \right) v_{tx}^\phi + 2 \mu r \left( \frac{v_t^\phi}{r_x} \right) v_{tx} + \mu r^2 \left( \frac{v_t^\phi}{r_x} - \frac{v^\phi}{r} \right) v_{tx}^\phi + 2 \mu r^2 \left( \frac{v_t^\phi}{r} \right) v_{tx}^\phi.
\]

Multiplying (89) by \(v_t^\phi\), and integrating by parts over \(I\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_I \rho x (v_t^\phi)^2 dx + \mu \int_I r r_x \left( \frac{v_t^\phi}{r_x} - \frac{v^\phi}{r} \right)^2 dx
\]

\[= \mu \int_I \left( \frac{v_t^\phi v_{tx}^\phi}{|v_{tx}^\phi|^2} - \frac{v^r v^\phi}{r^2} \right) (r v_t^\phi + r x v_t^\phi) dx - 2 \mu \int_I r v_t^\phi \frac{v^r v^\phi}{r^2} dx
\]

\[- \mu \int_I \left( \frac{v^r v_t^\phi}{r_x} \right) \left( \frac{v_t^\phi}{r_x} - \frac{v^\phi}{r} \right) dx + 2 \mu \int_I v^r v_t^\phi \left( \frac{v_t^\phi}{r} \right) dx
\]

\[= K_1 + K_2 + K_3 + K_4 + K_5.
\]
Before going further, we use (46) and choose that $k = 1$, $\eta(x, t) = \frac{v_x}{x}(x, t)$ to get

$$\int_I |v_x^0|^2 \, dx \leq C_1 \left( \int_I \varpi x |v_t^0|^2 \, dx + \int_I x |v_x^0 - \frac{v_t^0}{x}|^2 \, dx \right). \quad (91)$$

Now, due to (91), the a priori assumptions (49), (50) and Cauchy’s inequality, we get

$$|K_1| = \mu \int_I r r x \left( \frac{v_x^0 v_t^0}{r^2} - \frac{v_r^0 v_t^0}{r^2} \right) \left( \frac{v_x^0}{r} - \frac{v_t^0}{r} + 2 \frac{v_r^0}{r} \right) \, dx$$

$$\leq C \int_I x \left( |v_x^0| + \frac{|v_r^0|}{x} \right) \left( |v_t^0| - \frac{v_t^0}{x} \right) \, dx$$

$$\leq \frac{\delta^2}{2} \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx$$

$$+ C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx, \quad \left( \delta \leq \min\{1, C_1^{-1}\} \right)$$

$$\leq \delta \int_I x |v_x^0 - \frac{v_t^0}{x}|^2 \, dx + C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx$$

and similarly,

$$|K_2| + |K_4| \leq C \left| \int_I r x \left( \frac{v_x^0 v_t^0}{r^2} - \frac{2 r x v_r^0 v_t^0}{r^3} \right) \, dx \right| + C \left| \int_I \frac{r x v_r^0 v_t^0}{r^3} \left( \frac{v_x^0}{r} - \frac{v_t^0}{r} \right) \, dx \right|$$

$$\leq C \int_I x \left( |v_t^0| + \frac{|v_r^0|}{x} \right) \, dx$$

$$\leq \frac{\delta^2}{2} \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx$$

$$+ C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx, \quad \left( \delta \leq \min\{1, C_1^{-1}\} \right)$$

$$\leq \delta \int_I x |v_x^0 - \frac{v_t^0}{x}|^2 \, dx + C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx,$$

and

$$|K_3| = \mu \int_I \left( v_r^0 \left( v_x^0 - \frac{v_t^0}{x} \right) + v_r^0 + \frac{v_t^0}{x} \right) \left( \frac{v_x^0}{r} - \frac{v_t^0}{r} \right) \, dx$$

$$\leq \frac{\delta^2}{2} \int_I \left( |v_x^0 - \frac{v_t^0}{x}|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx$$

$$+ C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx, \quad \left( \delta \leq \min\{1, C_1^{-1}\} \right)$$

$$\leq \delta \int_I x |v_x^0 - \frac{v_t^0}{x}|^2 \, dx + C(\delta) \int_I x \left( |v_t^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx,$$

and

$$|K_5| = \left| \int_I \frac{\varpi x}{r^2} \left( r v_t^0 v_x^0 + r v_r^0 v_t^0 - 2 |v_r^0|^2 v_x^0 \right) v_t^0 \, dx \right| \leq C \int_I \varpi x \left( |v_t^0|^2 + |v_x^0|^2 + \frac{|v_r^0|^2}{x^2} \right) \, dx.$$
Next, multiplying (89) and (93) by $B_t^\phi$ and $B_t^z$ respectively, and integrating by parts over $I$, we have

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_I \|r r_x (B_t^\phi)^2 \|_{L^2} dx + \nu \int_I \|r r_x (B_t^\phi)^2 \|_{L^2} dx \\
= \nu \int_I \left( \frac{v^r r_x B_t^\phi}{|x|^2} + \frac{v^r B_t^\phi}{r} \right) (r B_t^\phi)_x dx - \nu \int_I \left( \frac{B_t^\phi}{r x} + \frac{B_t^\phi}{r} \right) (v^r B_t^\phi)_x dx \\
- \int_I \left[ \left( r v^r x + \frac{(r v^r x)_x}{2} \right) |B_t^\phi|^2 + (r v^r x)_x B_t^\phi \right] dx \\
\leq \delta \int_I x \left[ |B_{tx}^\phi|^2 + |B_t^\phi|^2 + |v^r x|^2 \right] dx + C(\delta) \int_I x (|v^r x|^2 + |B_t^\phi|^2 + |B_t^\phi|^2) dx,
\end{align*}

(94)

and

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_I \|r r_x |B_t^z|^2 \|_{L^2} dx + \nu \int_I \|r r_x |B_t^z|^2 \|_{L^2} dx \\
= \nu \int_I \left( \frac{v^r B_t^z}{|x|^2} + \frac{v^r x B_t^z}{r} \right) (r B_t^z)_x dx - \nu \int_I \left( \frac{B_t^z}{r x} + \frac{B_t^z}{r} \right) (v^r B_t^z)_x dx \\
- \int_I \frac{B_t^z}{2} \left( 3(r v^r)_x B_t^z + B^z (r v^r)_{tx} \right) dx \\
= \nu \int_I \left( \frac{v^r B_t^z}{r x} \frac{v^r}{r} - \frac{v^r}{r} \right) dx - \int_I \frac{B_t^z}{2} \left[ 3(r v^r)_x B_t^z + B^z (r v^r)_{tx} \right] dx \\
\leq \delta \int_I x (|B_{tx}^z|^2 + |v_{tx}^z|^2 + |v^r x|^2) dx + C(\delta) \int_I x (|B_t^z|^2 + |B_t^z|^2 + |v^r|^2) dx \\
+ C \int_I x |B_t^z|^2 dx.
\end{align*}

(95)

Thus, integrating the sum of (88), (90), (94) and (95) in (0, t) with suitably small $\delta$, using the estimates of $J_1-J_6$ and $K_1-K_5$, and utilizing Lemma 4.1, one has

\begin{align*}
\|x^{\frac{1}{2}} (p^2 v^z_{tx}, p^2 v^\phi_{tx}, B_t^\phi, B_t^z) \|_{L^2}^2 + \|x^{\frac{1}{2}} p^2 (v^z_\phi, \frac{v^r}{x}) \|_{L^2}^2 \\
+ \sigma \int_0^t \|x^{\frac{1}{2}} (v^z_{tx}, v^z_\phi, v^r x, B_t^\phi, B_t^z) \|_{L^2}^2 ds \\
\leq C \left( \|x^{\frac{1}{2}} (p^2 v^z_{tx}, p^2 v^\phi_{tx}, B_t^\phi, B_t^z) \|_{L^2}^2 + \|x^{\frac{1}{2}} p^2 (v^z_\phi, \frac{v^r}{x}) \|_{L^2}^2 \right) \\
+ C_T (\mathcal{D}(0) \| + C \int_0^t \|x^{\frac{1}{2}} (p^2 v^z_{tx}, p^2 v^\phi_{tx}, B_t^\phi, B_t^z) \|_{L^2}^2 ds.
\end{align*}

which, together with Gronwall’s inequality, implies (84).

Finally, (80) follows from (81) and (84) directly. \hfill \square

**Corollary 4.** Suppose that (49) and (50) hold. Then, for $0 \leq t \leq T$ and $a \in (0, 1)$,

\begin{align*}
\|(v^z, v^\phi, B_t^z, B_t^x) \|_{L^\infty}^2 \leq C_T (\mathcal{D}(0) + \mathcal{D}_2(0)),
\end{align*}

(96)

\begin{align*}
\int_0^t \|x^\frac{1}{2} \left( v^z_{tx}, v^\phi_{tx} \right) \|_{L^2}^2 ds \leq C_T (\mathcal{D}(0) + \mathcal{D}_2(0)),
\end{align*}

(97)
\[
\int_0^t \left\| x^\frac{1}{2} \left( \frac{v^\phi_x}{x^a}, \frac{B^\phi_z}{x^a} \right) \right\|^2_{L^2} ds \leq C_T(a)(\mathfrak{D}(0) + \mathfrak{D}_2(0)),
\]
(98)

\[
\left\| x^\frac{1}{2}(v^\phi_x, \frac{v^r_x}{x}, \frac{v^\phi_x}{x}, B^\phi_x, \frac{B^\phi_x}{x}) \right\|^2_{L^2} \leq C_T \left[ \left\| x^\frac{1}{2}(v^\phi_x, \frac{v^r_x}{x}, \frac{v^\phi_x}{x}, B^\phi_x, \frac{B^\phi_x}{x})(\cdot, 0) \right\|^2_{L^2} + \mathfrak{D}(0) + \mathfrak{D}_2(0) \right],
\]
(99)

where \(C_T(a)\) in (98) is a positive constant depending on \(a \in (0, 1)\).

**Proof.** For (96), note that (108) rewritten as,
\[
\mu^{-1} x \nu v^z_x = \left( \frac{r}{r_x} v^z_x \right)_x, \quad v^z_x |_{x=0} = \left( \frac{v^z_x}{r_x} \right) |_{x=1} = 0.
\]

Integrating above inequality over \((0, x), \forall x \in (0, 1),\) to get
\[
v^z_x = \mu^{-1} r x^{-1} \int_0^x y \nu v^z_x(y) dy.
\]
(100)

Hence, due to Holder’s inequality and Lemma 4.3, for \(x \in (0, 1),\) we have
\[
|v^z_x(\cdot, t)| \leq C x^{-1} \|y^{1/2}\|_{L^2(0,x)} \|y^{1/2} \nu v^z_x\|_{L^2(0,x)} \leq C \|x^{1/2} \nu v^z_x\|_{L^2(0,1)}
\]
\[
\leq \sqrt{C_T(\mathfrak{D}(0) + \mathfrak{D}_2(0))},
\]
which implies, by noting that \(v^z_x |_{x=0} = 0,\)
\[
\|v^z_x\|^2_{L^2} \leq \|v^z_x(\cdot, t)\|^2_{L^\infty} \leq C_T(\mathfrak{D}(0) + \mathfrak{D}_2(0)).
\]
(101)

Similarly to (100) and (101), we derive from (108) and Lemma 4.3 that
\[
B^z_x = \nu^{-1} r x^{-1} \int_0^x r r_x \left( B^z_i + B^z \left( \frac{v^r_x}{r} + \frac{v^r}{r} \right) \right)(y) dy
\]
(102)

and
\[
\|B^z_x\|^2_{L^\infty} \leq \|B^z_x(\cdot, t)\|^2_{L^\infty} \leq C(x^{-1} \|y^{1/2}\|_{L^2(0,x)} \|y^{1/2} B^z_i\|_{L^2(0,x)})^2
\]
\[
\leq C \|x^{1/2} (B^z_i, B^z)\|^2_{L^2(0,1)} \leq C_T(\mathfrak{D}(0) + \mathfrak{D}_2(0)).
\]
(103)

Thus, (101) and (103) give (96). Then, similarly to the proof of the Corollary 2, we prove (97) and (98) by using (46). Precisely, selecting “\(k = 1, u(x,t) = \nu \phi^o(x,t)\), “\(k = 1 - 2a, u(x,t) = v^z(x,t)\)” and “\(k = 1 - 2a, u(x,t) = B^z(x,t)\)” in (46), respectively, we have
\[
\int_I x \nu^o \frac{v^\phi_x}{x^2} dx \leq C \int_I x \nu^o |v^\phi_x|^2 dx + C \int_I x |v^\phi_x| - \nu^o \frac{v^\phi_x}{x} |^2 dx,
\]
\[
\int_I x^2 \nu^o |^2 dx = \int_I x^{1-2a} |v^\phi_x|^2 dx \leq C(a) \int_I x^{3-2a} (\nu^o |v^\phi_x|^2 + |v^\phi_x|^2) dx
\]
\[
\leq C(a) \int_I x (\nu^o |v^\phi_x|^2 + |v^\phi_x|^2) dx,
\]
and
\[
\int_I x |B^z_x|^2 dx \leq C(a) \int_I x (\nu^o |B^z_x|^2 + |B^z_x|^2) dx.
\]
Next, integrating the sum of above three inequalities over \((0, t)\), and using Lemma 4.3, we get (97) and (98). (99) can be estimated as follows. Due to Cauchy’s inequality, we have

\[
x \left| \left( v_x^r, v_x^r \right)(x, t) \right|^2 - x \left| \left( v_x^r, v_x^r \right)(x, 0) \right|^2 = \int_0^t \left[ 2x(v_x^r, v_x^r) \cdot (v_x^r, v_x^r) \right](x, s)ds \\
\leq \int_0^t x \left( v_x^r \right)^2 + (v_x^r)^2 \right](x, s)ds.
\]

Integrating above inequality over \(I\), together with (56) and (80) to obtain

\[
\| x^2 \left( v_x^r, v_x^r \right)(x, t) \|_L^2 \leq C \| x^2 \left( v_x^r, v_x^r \right)(x, 0) \|_L^2 + C \int_0^t \left\| x^2 \left( v_x^r, v_x^r, v_x^r \right) \right\|_L^2 ds \\
\leq C \left\| x^2 \left( v_x^r, v_x^r \right)(x, 0) \right\|_L^2 + C_T(\mathcal{D}(0) + \mathcal{D}_2(0)).
\]

Similarly, by using (56), (64), (80) and (97), one has

\[
\| x^2 \left( v_x^2, \phi^o, B_x^o \right) \|_L^2 \leq C \left[ \| x^2 \left( \phi^o, \phi^o, B_x^o \right) \right]_L^2 + \mathcal{D}(0) + \mathcal{D}_2(0).
\]

Combining (104) and (105), we get (99).

4.2. Higher-order estimates. In this subsection, we derive the higher-order part of the \textit{a priori} estimates for the strong solution \((r, v^r, \phi^o, v^2, B^o, B^z)\) on the time interval \([0, T]\) defined in Definition 2.1, under the assumptions (49) and (50). Inspired by [36], we define

\[
\mathcal{G} := \ln r_x + \ln \left( \frac{L}{x} \right).
\]

This transformation between \(\mathcal{G}\) and \(r\) is one-to-one, and we can solve for \(r\) in terms of \(\mathcal{G}\) by

\[
r(x, t) = \left( 2 \int_0^x y \exp \{ \mathcal{G}(y, t) \} dy \right)^{1/2} \text{ for } x \in \mathcal{T} \text{ and } t \geq 0.
\]

Indeed, we will show in Section 4.2.1 that \(\mathcal{G} \sim r_x - 1, \mathcal{G}_t \sim v_x^r, \mathcal{G}_x \sim r_{xx}\) and \(\mathcal{G}_{tx} \sim v_{xx}^r\). Then equation (23) can be written in the form of

\[
\begin{align*}
\mathbf{wG}_{tx} + \gamma \left( \frac{x \phi}{r_{rx}} \right) \gamma \mathcal{G}_x &= \frac{\mathbf{w} \mathbf{v}_x^r}{r} - \frac{\mathbf{w} \mathbf{v}^2}{r^2} - \left( \frac{x}{r_{rx}} \right)^2 \mathbf{v}_x^r \\
&+ (B^o B_x^o + B^2 B_x^o + r_x |B^o|^2)^2, & \text{in } Q_T, \\
\mu \left( \frac{v_x^o + v_x^o}{r} \right) x &= \frac{\mathbf{w} \mathbf{v}_x^o}{r} + \frac{\mathbf{w} \mathbf{v}^2}{r^2}, & \text{in } Q_T, \\
\mu \left( \frac{v_x^o}{r} + \frac{v_x^o}{r} \right) x &= \frac{\mathbf{w} \mathbf{v}_x^o}{r}, & \text{in } Q_T, \\
\nu \left( \frac{B_x^o + B_x^o}{r} \right) x &= r_x B_t^o + B^o v_x^r, & \text{in } Q_T, \\
\nu \left( \frac{B_x^o}{r} + \frac{B_x^o}{r} \right) x &= r_x B_t^o + B^2 v_x^r + r_x v^r B_x^o, & \text{in } Q_T,
\end{align*}
\]
4.2.1. Preliminaries for higher-order estimates. The main goal of this subsubsection is to derive some preliminary estimates for the strong solution \((r, v', \phi, v, B^\phi, B^z)\) on the time interval \([0, T]\) defined in Definition 2.1 under the \textit{a priori} assumptions (49) and (50).

We can identify the principal parts of \(G, G_t, G_x\) and \(G_{xt}\) as follows. Note that

\[
G = (r_x - 1) + \left(\frac{r}{x} - 1\right) + o\left(r_x - 1, r/x - 1\right),
\]

(109)

\[
G_x = \frac{r_{xx}}{r_x} + \frac{x}{r} \left(\frac{r}{x}\right)_x = \left[r_{xx} + \left(\frac{r}{x}\right)_x\right] + \left(\frac{1}{r_x} - 1\right) r_{xx} + \left(\frac{x}{r} - 1\right) \left(\frac{r}{x}\right)_x,
\]

\[
G_t = \frac{v'_{rx} + v'_{r}}{r_x} = \left(v'_x + v'\right)_x + \left(\frac{1}{r_x} - 1\right) v'_x + \left(\frac{x}{r} - 1\right) \left(v'\right)_x,
\]

\[
G_{xt} = \left(\frac{v'_{rx} + v'_{r}}{r_x}\right)_x = \left(v'_x + v'\right)_x + \left(\frac{1}{r_x} - 1\right) v'_{xx} + \left(\frac{x}{r} - 1\right) \left(v'\right)_x\right[
\]

\[
- \left[\frac{r_{xx}}{r_x^2} v' + \left(\frac{x}{r}\right)^2 \left(\frac{r}{x}\right)_x \left(v'\right)_x\right].
\]

(110)

Similarly, we have

\[
\left(\frac{v''_x + v''_r}{r_x}\right)_x = \left(v''_x + v''\right)_x + \left[\left(\frac{1}{r_x} - 1\right) v''_{xx} + \left(\frac{x}{r} - 1\right) \left(v''\right)_x\right]
\]

\[
- \left[\frac{r_{xx}}{r_x^2} v'' + \left(\frac{x}{r}\right)^2 \left(\frac{r}{x}\right)_x \left(v''\right)_x\right],
\]

(111)

\[
\left(\frac{v''_x + v''_r}{r_x}\right)_x = v''_{xx} + v''_x + \left[\left(\frac{1}{r_x} - 1\right) v''_{xx} + \left(\frac{x}{r} - 1\right) \left(v''\right)_x\right] - \frac{r_{xx}}{r_x^2} v'^2,
\]

(112)

\[
\left(\frac{B^\phi_x + B^\phi_r}{r_x}\right)_x = \left(B^\phi_x + B^\phi\right)_x + \left[\left(\frac{1}{r_x} - 1\right) B^\phi_{xx} + \left(\frac{x}{r} - 1\right) \left(B^\phi\right)_x\right]
\]

\[
- \left[\frac{r_{xx}}{r_x^2} B^\phi + \left(\frac{x}{r}\right)^2 \left(\frac{r}{x}\right)_x \left(B^\phi\right)_x\right],
\]

(113)

\[
\left(\frac{B^z_x + B^z_r}{r_x}\right)_x = B^z_{xx} + B^z_x + \left[\left(\frac{1}{r_x} - 1\right) B^z_{xx} + \left(\frac{x}{r} - 1\right) B^z_x\right] - \frac{r_{xx}}{r_x^2} B^z_x.
\]

(114)

Thus, it follows from (29)-(32) that for \(t \in [0, T]\),

\[
(r_x, r, v', v, v_x, v_r, v^\phi, v_x, B^\phi, B_x, G, G_t) \in L^\infty(I),
\]

(115)

\[
(r, v', v^\phi, B^\phi, v_x, v_r, B^\phi, B_x) \in H^1(I),
\]

(116)

\[
(G_{tx}, (\frac{v''_x + v''_r}{r_x}), x^{\frac{1}{2}} (\frac{v''_x + v''_r}{r_x}, (\frac{B^\phi_x + B^\phi_r}{r_x})_x, x^{\frac{1}{2}} ((\frac{B^z_x + B^z_r}{r_x})_x + B^z_x) \in L^2(I),
\]

(117)

\[
(r, v', v^\phi, B^\phi) \in H^2([0, b]) \text{ for } b \in (0, 1),
\]

(118)

\[
(r, v', v^\phi, B^\phi) \in H^2(I), \text{ if } G_x \in L^2.
\]

(119)

where \(B\) is defined by (27). With the regularities (115)-(119), we have the following lemmas.
Lemma 4.4. Suppose that (49) holds for a suitably small $\epsilon_0$. Then for $t \in [0, T]$, 
\[
\left\| \left( v^r_x, v^v_x, v^v_x, v^v_x, B_x^\phi, B_x^\beta, B_x^z, B_x^z, B_x^z \right) \right\|_{L^2}^2 \leq 6 \left\| \left( G_t, \frac{v^\phi_x}{r_x}, \frac{v^\phi_x}{r_x} + \frac{v^1_x}{r_x} + \frac{v^1_x}{r_x} + \frac{B_x^\phi}{r_x} + \frac{B_x^\beta}{r_x} + \frac{B_x^z}{r_x} \right) \right\|_{L^2}^2,
\]
\[
\left\| (r_x - 1, (r/x - 1)^2) \right\|_{L^2}^2 \leq 6 \|G\|_{L^2}^2,
\]
\[
\left\| (r_{xx}, (r/x)_x^2) \right\|_{L^2}^2 \leq 6 \|G_x\|_{L^2}^2,
\]
\[
\left\| \left( v^\phi_x, \frac{v^\phi_x}{r_x}, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, B_x^\phi, B_x^\beta, B_x^z \right) \right\|_{L^2}^2 \leq 6 \left\| \left( G_{tx}, \frac{v^\phi_x}{r_x} + \frac{v^\phi_x}{r_x}, \frac{v^\phi_x}{r_x} + \frac{v^\phi_x}{r_x} + \frac{B_x^\phi}{r_x} + \frac{B_x^\beta}{r_x} + \frac{B_x^z}{r_x} \right) \right\|_{L^2}^2 + c \left\| \left( v^r_x, v^r_x, v^r_x, v^r_x, B_x^\phi, B_x^\beta, B_x^z \right) \right\|_{L^\infty}^2 \|G_x\|_{L^2}^2.
\]
Here the estimates in the last two inequalities hold if $\|G_x, (\frac{v^\phi_x}{r_x})_x + \frac{v^\phi_x}{r_x}, (\frac{B_x^\phi}{r_x})_x + \frac{B_x^\phi}{r_x} \|_{L^2} < \infty$.

Proof. The idea of the proof of this lemma is similar to that of Lemma 3.10 in [36]. We just need to select the following cut-off function $\chi_\epsilon \in [0, 1]$: 
\[
\chi_\epsilon = 0 \text{ on } [0, \epsilon] \cup [1 - \epsilon], \quad \chi_\epsilon = 1 \text{ on } [\epsilon, 1 - \epsilon]
\]
\[
\chi_\epsilon = \frac{x}{3\epsilon} - \frac{1}{3} \text{ on } [\epsilon, 4\epsilon], \quad \chi_\epsilon = \frac{1 - x}{3\epsilon} - \frac{1}{3} \text{ on } [1 - 4\epsilon, 1 - \epsilon], \quad \forall \epsilon \in (0, 1/4).
\]

Then we go through the same procedure as in [36] to get the conclusion, thus we omit it here.

By a similar idea of the proof of Lemma 4.4, we give the following lemmas without proof.

Lemma 4.5. Let $\delta > 0$ be a fixed constant. Suppose that (49) holds for a suitably small $\epsilon_0$. Then, 
\[
\| \mathcal{P}^\delta (v^r_x, v^v_x, v^v_x, v^v_x, B_x^\phi, B_x^\beta, B_x^z, B_x^z, B_x^z) \|_{L^2}^2 \leq c \| \mathcal{P}^\delta (G_t, \frac{v^\phi_x}{r_x}, \frac{v^\phi_x}{r_x} + \frac{v^\phi_x}{r_x} + \frac{B_x^\phi}{r_x} + \frac{B_x^\beta}{r_x} + \frac{B_x^z}{r_x}) \|_{L^2}^2 \]
\[
\| \mathcal{P}^\delta (r_x - 1, (r/x - 1)^2) \|_{L^2}^2 \leq c \| \mathcal{P}^\delta G \|_{L^2}^2,
\]
\[
\| \mathcal{P}^\delta (r_{xx}, (r/x)_x^2) \|_{L^2}^2 \leq c \| \mathcal{P}^\delta G_x \|_{L^2}^2,
\]
\[
\| \mathcal{P}^\delta (v^\phi_x, \frac{v^\phi_x}{r_x}, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, B_x^\phi, B_x^\beta, B_x^z) \|_{L^2}^2 \leq c \| \mathcal{P}^\delta (G_{tx}, \frac{v^\phi_x}{r_x} + \frac{v^\phi_x}{r_x}, \frac{v^\phi_x}{r_x} + \frac{v^\phi_x}{r_x} + \frac{B_x^\phi}{r_x} + \frac{B_x^\beta}{r_x} + \frac{B_x^z}{r_x}) \|_{L^2}^2 + c \| (v^r_x, v^r_x, v^r_x, v^r_x, B_x^\phi, B_x^\beta, B_x^z) \|_{L^\infty}^2 \| \mathcal{P}^\delta G_x \|_{L^2}^2.
\]
\[
\| \mathcal{P}^\delta (v^\phi_x, \frac{v^\phi_x}{r_x}, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, v^\phi_x, B_x^\phi, B_x^\beta, B_x^z) \|_{L^2}^2 \leq c \| \mathcal{P}^\delta (G_x, (\frac{v^\phi_x}{r_x})_x + \frac{v^\phi_x}{r_x}, (\frac{B_x^\phi}{r_x})_x + \frac{B_x^\phi}{r_x}) \|_{L^2}^2 + c \| (v^r_x, B_x^z) \|_{L^\infty}^2 \| \mathcal{P}^\delta G_x \|_{L^2}^2.
\]
Here the estimates (120)-(122) hold if $\| \mathcal{P}^\delta (G_x, (\frac{v^\phi_x}{r_x})_x + \frac{v^\phi_x}{r_x}, (\frac{B_x^\phi}{r_x})_x + \frac{B_x^\phi}{r_x}) \|_{L^2} < \infty$. 

Lemma 4.6. Suppose that (49) holds for a suitably small \(\epsilon_0\). Then,

\[
\|r - x\|_{L^2} \leq 2\|r - x\|_{L^2} \|r_x - 1\|_{L^2},
\]

(123)

\[
\|(\varphi', \varphi, B^0)\|_{L^2} \leq 2\|(\varphi', \varphi, B^0)\|_{L^2} \|(\varphi'_x, \varphi'_x, B^0_x)\|_{L^2}.
\]

(124)

\[
\|x(\varphi'_x, \varphi'_x, B^0_x)\|_{L^2} \leq \left[ \left\| \left( \frac{\varphi'_0}{x} + \frac{\varphi'_0}{x}, \left( \frac{B^0_x}{x}, \frac{B^0_x}{x} \right) \right) \right\|_{L^2} + \|(\varphi'_x, \varphi'_x, B^0_x)\|_{L^2}
\left[ + \|(\varphi'_x, \varphi'_x, B^0_x)\|_{L^2} \right] \times \|x(\varphi'_x, \varphi'_x, B^0_x)\|_{L^2}.
\]

(125)

4.2.2. Global existence of strong solutions. In this subsection, we prove the global existence of the strong solution for suitably small \(E(0)\).

Lemma 4.7. Suppose that (49) and (50) hold. Then the following estimates hold for \(b \in (0, 1)\),

\[
\|\rho^{\gamma - \frac{1}{2}}(r_{xx}, (r/x)_x)(\cdot, t)\|_{L^2}^2
\]

\[
\leq C_T \left( \|\rho^{\gamma - \frac{1}{2}}(r_{xx}, (r/x)_x)(\cdot, 0)\|_{L^2}^2 + D(0) + D_2(0) \right),
\]

(126)

\[
\|(r_{xx}, (r/x)_x)(\cdot, t)\|_{L^2}^2
\]

\[
\leq C_T(b) \left( \|\rho^{\gamma - \frac{1}{2}}(r_{xx}, (r/x)_x)(\cdot, 0)\|_{L^2}^2 + D(0) + D_2(0) \right),
\]

(127)

\[
\int_0^t \left\| \left( v'_x, v'_x, v'_x, v'_x, \varphi'_x, \varphi'_x, \varphi'_x, B^0_x, B^0_x \right) \right\|_{L^2}^2 \, ds
\]

\[
\leq C_T \left( \|\rho^{\gamma - \frac{1}{2}}(r_{xx}, (r/x)_x)(\cdot, 0)\|_{L^2}^2 + D(0) + D_2(0) \right).
\]

(128)

\[
\left\| (\varphi'_x, \varphi'_x, B^0_x) \right\|_{L^2}^2 + \int_0^t \left\| \left( v'_x, v'_x, v'_x, v'_x, \varphi'_x, \varphi'_x, \varphi'_x, B^0_x, B^0_x \right) \right\|_{L^2}^2 \, ds
\]

\[
\leq C_T(D(0) + D_2(0) + D_3(0)).
\]

(129)

Proof. The proof consists of three steps.

Step 1. In this step, we prove (126) and (127). Integrating the product of (108) and \(\varphi^\gamma - 1\) over \(I\), using (50) and Cauchy’s inequality, one gets that

\[
\frac{\partial}{\partial t} \int_I \varphi^\gamma \left( \frac{\partial G}{\partial x} \right)^2 \, dx + \frac{\gamma}{2} \int_I \left( \frac{\partial G}{\partial x} \right)^2 \, dx \leq C \int_I \left( \left( \frac{\partial G}{\partial x} \right)^2 \right) \, dx
\]

\[
\leq C \int_I \left( \left( \frac{\partial G}{\partial x} \right)^2 \right) \, dx
\]

\[
\leq \frac{C}{\gamma} \int_I \left( \left( \frac{\partial G}{\partial x} \right)^2 \right) \, dx
\]


(130)

where we have used that \(|(\frac{\partial G}{\partial x})^2 - \frac{\partial G}{\partial x}| \leq C(|r_x - 1| + |r/x - 1|)|, due to the Taylor expansion and (49). It follows from (130), Lemmas 4.1, 4.3 and (64) that

\[
\int_I \varphi^\gamma \left( \frac{\partial G}{\partial x} \right)^2 \, dx + \int_0^t \int_I \varphi^\gamma \left( \frac{\partial G}{\partial x} \right)^2 \, dx \, ds
\]

\[
\leq C ||\rho^{\gamma - \frac{1}{2}}(r_{xx}, (r/x)_x)(\cdot, 0)\|_{L^2}^2 + C_T(D(0) + D_2(0)).
\]

(131)
due to (120). So, (126) and (127) follows directly from (131), (120) and the positivity of $\rho$ around $x = 0$.

**Step 2.** In this step, we prove (128). Integrating the square of (108) with respect to the spatial and temporal variables, we have, by using Lemmas 4.1 and 4.3, that

$$\int_0^t \int |G_{xx}|^2 dx ds \leq C \int_0^t \int [\rho^2(|v_x|^2 + |v|^2) + \rho^2 x^2(|r_x - 1|^2 + |r/x - 1|^2)
+ |\mathcal{B}^\phi|^2 + |B^\phi|^2] dx ds + \int_0^t \int \rho^{2\gamma}|G_x|^2 dx ds$$

$$\leq C\|\rho^{-\frac{1}{2}}(r_{xx}, (\frac{r}{x})_x)(\cdot, 0)\|^2_{L^2} + C_T(\mathfrak{D}(0) + \mathfrak{D}_2(0)).$$

This, together with (121) and (131), implies

$$\int_0^t \int \frac{1}{2} |v_x|^2 + |(\frac{v}{x})_x|^2 dx ds \leq C_T(\|\rho^{-\frac{1}{2}}(r_{xx}, (\frac{r}{x})_x)(\cdot, 0)\|^2_{L^2} + \mathfrak{D}(0) + \mathfrak{D}_2(0)).$$

Clearly,

$$\int_0^t \int 0^{1/2} |v_x^r|^2 + |(\frac{v}{x})_x^r|^2 dx ds \leq C_T(\|\rho^{-\frac{1}{2}}(r_{xx}, (\frac{r}{x})_x)(\cdot, 0)\|^2_{L^2} + \mathfrak{D}(0) + \mathfrak{D}_2(0)).$$

Then, it follows from Hardy’s inequality (45) and Lemma 4.1 that

$$\int_0^t \int 0^{1/2} |v_x^r|^2 + |(\frac{v}{x})_x^r|^2 dx ds \leq C_T(\|\rho^{-\frac{1}{2}}(r_{xx}, (\frac{r}{x})_x)(\cdot, 0)\|^2_{L^2} + \mathfrak{D}(0) + \mathfrak{D}_2(0)).$$

Similar to the derivation of (132)-(134), taking the square of (108)_{2-5} respectively and integrating the sum over $I \times (0, t)$, we have

$$\left\| \left[ \left( \left( \frac{\phi^\phi}{r_x} + \frac{\phi^\phi}{r} \right)_x, \left( \frac{\phi^\phi}{r_x} + \frac{\phi^\phi}{r} \right)_x + \frac{\phi^\phi}{x}, \frac{\phi^\phi}{x} \right)_x, \frac{\phi^\phi}{x} \right)_x, \right\|_{L^2(0,t;L^2(I))}^2$$

$$\leq \left\| \left( \left( \frac{\phi^\phi}{r_x} + \frac{\phi^\phi}{r} \right)_x, \left( \frac{\phi^\phi}{r_x} + \frac{\phi^\phi}{r} \right)_x + \frac{\phi^\phi}{x}, \frac{\phi^\phi}{x} \right)_x, \frac{\phi^\phi}{x} \right\|_{L^2(0,t;L^2(I))}^2$$

$$\leq C_T(\|\rho^{-\frac{1}{2}}(r_{xx}, (\frac{r}{x})_x)(\cdot, 0)\|^2_{L^2} + \mathfrak{D}(0) + \mathfrak{D}_2(0)).$$

Thus, (128) follows from (134) and (135).
Step 3. In this step, we show (129).

For \( v^r \), applying \( \partial_t \) to (108) and noting the fact that (58), we obtain

\[
\frac{\partial}{\partial r} v_t^r - \frac{\partial}{\partial r} \left[ ru^r v_t^r + 2ru^\phi v_t^\phi - 2v^r (v^\phi)^2 \right] - \gamma \left[ \left( \frac{x^\nu}{r^x} \right)^2 + \frac{v_t^r}{r} \right] r - 2 \frac{x^\nu}{r} v_t^r (\varpi t)_x \]
\[
= r \left[ \frac{v_t^r}{r} + v_t^r \right] - r \left[ \frac{v_t^r}{r} + \frac{v_t^r}{r} \right] - \left( B^\phi B_t^\phi + B^2 B_t^\phi \right)_x
\]
\[
- \left[ r v_t^r - r x^\nu \right] (B^\phi)^2 + \frac{2 r_x}{r} B^\phi B_t^\phi \right].
\]

(136)

Let \( \psi \) be a non-increasing function defined on \([0, 1]\) satisfying

\[
\psi = 1 \text{ on } [0, 1/4], \quad \psi = 0 \text{ on } [1/2, 1] \text{ and } |\psi'| \leq 32.
\]

Multiplying equation (136) by \( \psi v_t^r \) and integrating the product with respect to the spatial variable, one has, using the integration by parts and the boundary condition

\[ v^\nu(0, t) = 0 \text{ (so } v_t^\nu(0, t) = 0 \text{), that} \]

\[
\frac{d}{dt} \int \frac{1}{2} \rho v_t^r v_t^r dx + \varpi \int \left[ \frac{v_t^r}{r} + \frac{v_t^r}{r} \right] (\psi v_t^r)_x dx = L_1 + L_2 + L_3 + L_4,
\]

(137)

where

\[
L_1 = \int \left[ \frac{\partial}{\partial r} v_t^r v_t^r \right] dx + 2 \int \left( \frac{x^\nu}{r} \right)^2 \varphi \psi v_t^r dx
\]
\[
\leq C \int_0^1 (|v_t^r|^2 + |v_t^\phi|^2 + |v_t^\phi|^2) dx
\]
\[
L_2 = -\gamma \int \left( \frac{x^\nu}{r} \right)^2 (\psi v_t^r)_x dx,
\]
\[
L_3 = \varpi \int \left[ \frac{v_t^r}{r} + \frac{v_t^r}{r} \right] (\psi v_t^r)_x dx,
\]
\[
L_4 = \int \left( B^\phi B_t^\phi + B^2 B_t^\phi \right)(\psi v_t^r)_x dx - \int \left[ r v_t^r - r \right] (B^\phi)_x + \frac{2 r_x}{r} B^\phi B_t^\phi \right] \psi v_t^r dx.
\]

The second term on the left-hand side of (137) can be estimated as follows:

\[
\int \left[ \frac{v_t^r}{r} + \frac{v_t^r}{r} \right] (\psi v_t^r)_x dx
\]
\[
= \int \psi \left( |v_t^r|^2 dx + \int \psi \left( \frac{v_t^r}{r} \right)^2 dx \right) + \int \psi \left( \frac{v_t^r}{r} \right)^2 dx \psi v_t^r dx
\]
\[
\geq \int \psi \left( |v_t^r|^2 dx \right) - \frac{1}{2} \int \left( \frac{\psi}{r} \right)_x |v_t^r|^2 dx - C \int \left( x^2 |v_t^r|^2 + |v_t^r|^2 \right) dx
\]
\[
\geq \int \psi \left( |v_t^r|^2 dx \right) - \frac{1}{2} \int \left( \frac{\psi}{r} \right)_x |v_t^r|^2 dx - C \int \left( x^2 |v_t^r|^2 + |v_t^r|^2 \right) dx.
\]

Then, (137) deduce to,

\[
\frac{d}{dt} \int \frac{1}{2} \rho v_t^r v_t^r dx + \varpi \int \left[ \frac{v_t^r}{r} + \frac{r_x}{r} \right] v_t^r dx \leq C \int \left( x^2 |v_t^r|^2 + |v_t^r|^2 \right) + L_2 + L_3 + L_4.
\]
It follows from the Cauchy inequality that
\[
\frac{d}{dt} \int_1 \frac{1}{2} \rho \frac{\partial v_i^2}{\partial t} v_i^2 \, dx + \frac{\omega}{2} \int_1 \psi \left[ \frac{\partial |v_i|^2}{\partial t} r_x^2 + r_x^2 |v_i|^2 \right] \, dx \\
\leq C \int_0^{1/2} (x^2 |v_{tx}^2| + |v_r^2| + |v_r^2|)^2 + |B_i^2|^2 + |B^2| \, dx.
\]
This, together with Lemma 4.3 and (128), implies that
\[
\int_0^{1/4} \Big( |v_r^2| + \frac{\rho}{2} \Big( \frac{\rho}{x} \Big) \Big) \, dx \\
\leq C_T \left( \| (\overrightarrow{a'} \nu^r, \rho^{-1/2} (\frac{r}{x}) x) \|_{L^2}^2 + \mathcal{D}(0) + \mathcal{D}(0) \right).
\]
Using Lemma 4.3 again, we obtain
\[
\left\| \overrightarrow{a'} \nu^r \right\|_{L^2} + \int_0^t \left\| (v_r^2, \nu^r x) \right\|_{L^2} \, ds \\
\leq C_T \left[ \left( \| \overrightarrow{a'} \nu^r \|_{L^2} + \rho^{-1/2} (\frac{r}{x}) x \right) \|_{L^2}^2 + \mathcal{D}(0) + \mathcal{D}(0) \right].
\] (138)

For \( \nu^r \) and for \( B^2 \), by a similar way to deduce (136)-(138), we have
\[
\frac{\partial x^2}{\partial t} v_{tx}^2 + \frac{\partial x^2}{\partial t^2} v_{tx}^2 - \frac{2 \partial x^2}{\partial t^2} v_{tx}^2 = \mu \left[ \frac{\partial |v_r|^2}{\partial t} r_x^2 + r_x^2 |v_r|^2 \right]
\]
\[
r_x^2 B_{tx}^2 + 2 v_r^2 B_{tx}^2 + B^2 v_r^2 = \mu \left[ \frac{d^2 |v_r|^2}{d t^2} r_x^2 + r_x^2 |v_r|^2 \right].
\]
\[
\left\| (\overrightarrow{a'} \nu^r, B_t^2) \right\|_{L^2} + \int_0^t \left\| (v_r^2, \nu^r x) \right\|_{L^2} \, ds \\
\leq C_T \left[ \left( \| \overrightarrow{a'} \nu^r \|_{L^2} + \rho^{-1/2} (\frac{r}{x}) x \right) \|_{L^2}^2 + \mathcal{D}(0) + \mathcal{D}(0) \right].
\] (139)

Then, (138) and (139) give (129). Thus, we complete the proof of the Lemma. \( \square \)

**Corollary 5.** Suppose that (49) and (50) hold. Then the following estimates hold for \( b \in (0, 1) \),
\[
\left\| \left( \mathcal{G}_t \left( \frac{\nu^r}{r_x} + \frac{\nu^r}{r} \right) + \frac{B^2}{r_x} + \frac{B^2}{r} \right) \right\|_{L^2} \\
\leq C_T \left( \mathcal{D}(0) + \mathcal{D}(0) \right) + \| x^1 \left( \frac{\nu^r}{r_x} + \frac{\nu^r}{r} \right) \|_{L^2} \right),
\] (140)
\[
\left\| x^1 \left( \frac{\nu^r}{r_x} + \frac{\nu^r}{r} \right) \right\|_{L^2} \leq C_T \left( \mathcal{D}(0) + \mathcal{D}(0) \right),
\] (141)
\[
\left\| (v^r, x) + \frac{v^r}{r_x} + \frac{v^r}{r} \right\|_{L^2} \leq C_T \left( \mathcal{D}(0) + \mathcal{D}(0) \right),
\] (142)
\[
\left\| x^1 (v^r, v^r, B^2, B^2) \right\|_{L^2} \leq C_T \left( \mathcal{D}(0) + \mathcal{D}(0) \right),
\] (143)
Proof. It follows from (108)\textsubscript{1}, (131), (138), (96), (99), Lemmas 4.1 and 4.2 that
\[
\|G_{tx}\|^2_{L^2} \leq C\|\mathbf{P}^{\dagger}G_{tx}\|^2_{L^2} + C\left\|\left(\mathbf{P}(v^t_r, v^\phi_x), x\mathbf{P}(\frac{r}{x} - 1, r_x - 1), B^\phi, B_x^\phi\right)\right\|^2_{L^2}
\]
\[
\leq C_T\left(\|\mathbf{P}^{\dagger}v^t_r, \rho^\gamma - \frac{1}{2}(r_{xx}, (\frac{r}{x}))\|_{L^2}^2 + \mathbf{D}(0) + \mathbf{D}_2(0)
\right.
\]
\[
+ \left\|x^\frac{1}{2}(\frac{r}{x} - 1, r_x - 1, \frac{v^\phi}{x}, \frac{B^\phi}{x})(\cdot, 0)\right\|^2_{L^2})
\]
Similarly, we derive from equations (108)\textsubscript{2,4} that
\[
\left\|\left(\frac{v^\phi}{r_x} + \frac{v^\phi}{r}, \left(B^\phi + \frac{B^\phi}{r}\right)\right)\right\|^2_{L^2}
\]
\[
\leq c\left\|\left(\mathbf{P}(v^t_r, v^\phi_x), B^\phi, B_x^\phi\right)\right\|^2_{L^2}
\]
\[
\leq C_T\left(\|\mathbf{P}^{\dagger}v^t_r, B^\phi, \rho^\gamma - \frac{1}{2}(r_{xx}, (\frac{r}{x}))\|_{L^2}^2 + \mathbf{D}(0) + \mathbf{D}_2(0)
\right)
\]
\[
+ \left\|x^\frac{1}{2}(\frac{v^\phi}{r_x} + \frac{B^\phi}{r})(\cdot, 0)\right\|^2_{L^2})
\]
and it follows from equations (108)\textsubscript{3,5} that
\[
\left\|x^\frac{1}{2}\left(\frac{v^\phi}{r_x} + \frac{v^\phi}{r}, \left(B^\phi + \frac{B^\phi}{r}\right)\right)\right\|^2_{L^2} \leq C_T(\mathbf{D}(0) + \mathbf{D}_2(0)).
\]
Thus, (144)-(146) give to (140) and (141). Next, (142) and (143) follows from (121), (122), (131), (140) and (141) directly. Thus, we show the proof of the Corollary.

Lemma 4.8. Suppose that (49) and (50) hold. Then the following estimates hold for \(b \in (0, 1)\),
\[
\|(r_x - 1, \frac{r}{x} - 1, v^t, v^r, v^\phi, B^\phi, B_x^\phi)\|^2_{L^2} \leq C_T\mathbf{E}(0),
\]
\[
\|(r_x - 1, \frac{r}{x} - 1, v^t_x, v^r, v^\phi, B^\phi, B_x^\phi)\|^2_{H^1((0, b))} \leq C_T(b)\mathbf{E}(0).
\]
Proof. It follows from Hardy’s inequality (45), (99), (127) and (142) that
\[
\left\|\left(r_x - 1, \frac{r}{x} - 1, v^r, v^t, v^\phi, B^\phi, B_x^\phi\right)\right\|^2_{L^2((0, 1/2))}
\]
\[
\leq C\left\|x\left(r_x - 1, \frac{r}{x} - 1, v^t, v^r, v^\phi, B^\phi, B_x^\phi\right)\right\|^2_{L^2((0, 1/2))}
\]
\[
+ C\left\|x\left(r_{xx}, (\frac{r}{x})_x, v^r_{xx}, v^t_{xx}, v^\phi, B^\phi, B_x^\phi\right)\right\|^2_{L^2((0, 1/2))}
\]
\[
\leq C_T(\mathbf{D}(0) + \mathbf{D}_3(0) + \left\|x^\frac{1}{2}(\frac{r}{x} - 1, r_x - 1, v^r, v^t, v^\phi, B^\phi, B_x^\phi)(\cdot, 0)\right\|^2_{L^2})
\]
\[
\leq C_T\mathbf{E}(0),
\]
which, together with Corollary 4, gives (147). Clearly, (148) follows from (147), (127) and (142).

As a consequence of Lemma 4.8 and Sobolev inequality, we have the following Corollary.
Corollary 6. Suppose that (49) and (50) hold. Then the following estimates hold for $b \in (0, 1)$,
\[
\| (r - x) \|_{L^\infty}^2 \leq C T \mathcal{E}(0),
\]
(149)
\[
\| (v^r_x, \nu^r_x, \frac{\nu^\phi}{r}, B^\phi_x, \frac{B^\phi}{x}) \|_{L^\infty([0, b])}^2 \leq 4 \| (v^r, \nu^\phi, B^\phi, x v^r_x, x \nu^\phi_x, x B^\phi_x) \|_{L^\infty}^2 \leq C T \mathcal{E}(0),
\]
(150)
\[
\| (r_x - 1, \frac{r}{x} - 1, v^r_x, \frac{v^\phi}{x}, \frac{\nu^\phi}{x}, B^\phi_x, \frac{B^\phi}{x}) \|_{L^\infty([0, b])}^2 \leq C T(b) \mathcal{E}(0).
\]
(151)

Proof. (149) follows from (123), (147) and Lemma 4.2. For (150), it is enough to prove the second inequality. Firstly, the estimate $\|(v^r, \nu^\phi, B^\phi)\|_{L^\infty}^2$ follows from (124) and (147). Then, we derive from (125), (140) and (147) that
\[
\| x(v^r_x, \nu^\phi_x, B^\phi_x) \|_{L^\infty}^2 \leq \left[ \| x (G_{tx}, (\frac{\nu^\phi}{r_x})_x, (\frac{B^\phi}{r_x} + \frac{B^\phi}{x})_x) \|_{L^2} + \| (v^r_x, \nu^\phi_x, B^\phi_x) \|_{L^2} \right] \times \| x(v^r_x, \nu^\phi_x, B^\phi_x) \|_{L^2} \\
\leq C T \mathcal{E}(0).
\]
This verifies (150). (151) follows from (148) immediately by using the Sobolev inequality. This finishes the proof of the Corollary. \qed

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