In this work we present an inflationary mechanism based on fluid dynamics. Starting with the action for a single barotropic perfect fluid, we outline the procedure to calculate the power spectrum and the bispectrum of the curvature perturbation. It is shown that a perfect barotropic fluid naturally gives rise to a non-attractor inflationary universe in which the curvature perturbation is not frozen on super-horizon scales. We show that a scale-invariant power spectrum can be obtained with the local non-Gaussianity parameter \( f_{NL} = 5/2 \).

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I. INTRODUCTION

Cosmic inflation has emerged as a very successful paradigm for the early universe and structure formations. The basic predictions of simple models of inflation for the curvature perturbation power spectrum and bispectrum are in very good agreement with recent cosmological observations such as WMAP \cite{1} and PLANCK \cite{2,3}.

Without fully addressing the UV completion aspects of inflation, at the low energy effective field theory level, one can explore a variety of possibilities in the inflationary model building. In fact, many models of inflation based on scalar fields are constructed purely phenomenologically. Furthermore, one may add various features to such models, for example, by introducing extra phenomena such as particle creation and field annihilation, or local departures from inflation such as steps in the potential, turning trajectories and waterfall mechanisms. These additions have been used to explain the local features or glitches seen in CMB observations \cite{4–40}.

In this work we consider a different type of low energy effective field theory model for inflation. Namely, we present a formalism to obtain inflation from a fluid. Our starting point is the Lagrangian formalism for a perfect fluid in Einstein gravity, which enables us to calculate the power spectrum and bispectrum of the curvature perturbation.

Depending on the equation of state and whether it is an isentropic (barotropic) or non-isentropic fluid, different inflationary scenarios are possible. As a first step, we concentrate on an isentropic fluid in which the pressure is a given function of the energy density. In principle, one should be able to extend this formalism to a non-isentropic fluid.

This paper is organized as follows. In Section II we present a Lagrangian formalism for fluid inflation and the background equations. In Section III we present the cosmological perturbation theory in our setup and calculate the power spectrum and bispectrum of the curvature perturbation. In Section IV we present a simple scalar field model that mimics our fluid model. We then conclude the paper with a short discussion.

II. THE FORMALISM

To calculate the power spectrum and the bispectrum we need to have a Lagrangian formalism of fluid dynamics coupled with Einstein gravity. Here we use the Lagrangian for the perfect fluid in the presence of gravity proposed by Ray \cite{41,42}

\[
\mathcal{L} = \frac{1}{2} M_{Pl}^2 \sqrt{-g} R - \sqrt{-g} \rho (1 + \epsilon(\rho)) - \sqrt{-g} \lambda_1 (g_{\mu\nu} U^\mu U^\nu + 1) + \sqrt{-g} \lambda_2 (\rho U^\mu) U_\mu,
\]

where \( \rho \) is the rest mass density, \( \epsilon(\rho) \) is the specific internal energy, \( U^\mu \) is the 4-velocity and \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers for the two constraints; the first is the normalization of the 4-velocity and the second is the conservation of the rest mass density. Note that the total energy density, \( E \), is given by

\[
E = \rho (1 + \epsilon).
\]

\[ (2) \]
Below we show that the above Lagrangian gives the correct equations of motion for an isentropic perfect fluid minimally coupled to gravity. In this work we concentrate on an isentropic fluid for which $e = e(\rho)$. In principle one can consider more general situations in which $e$ is also a function of other thermodynamic variables such as entropy.

Varying the action with respect to the Lagrange multipliers $\lambda_1$ and $\lambda_2$ yields the normalization condition for $U^\mu$ and the energy conservation equations, respectively,

$$U^\mu U_\mu = -1,$$  \hspace{1cm} (3)

and

$$(\rho U^\mu)_{,\mu} = 0.$$  \hspace{1cm} (4)

Varying the action with respect to $\rho$ and $U^\mu$ yields, respectively,

$$\lambda_{2,\mu} U^\mu = -\frac{dE}{d\rho},$$

$$\lambda_1 = \frac{1}{2} \frac{\rho}{d\rho}.$$  \hspace{1cm} (5)

where the constraint Eq. (3) have been used to obtain the latter equation.

Finally, varying the action with respect to $g_{\mu\nu}$ yields the Einstein equation,

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu},$$  \hspace{1cm} (6)

where $G_{\mu\nu}$ is the Einstein tensor and the energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \rho U^\mu U^\nu + g^{\mu\nu} \left( \frac{dE}{d\rho} - E \right).$$  \hspace{1cm} (7)

In addition to the above Euler-Lagrange equations, using the second law of thermodynamics, one has

$$Tds = de + Pd\left(\frac{1}{\rho}\right),$$  \hspace{1cm} (8)

where $s$ is the entropy density and $P$ is the pressure. For an isentropic fluid we have $ds = 0$, hence

$$\frac{de(\rho)}{d\rho} = \frac{P}{\rho^2}.$$  \hspace{1cm} (9)

Knowing that $e = e(\rho)$, the above equation also implies that $P$ is a function of $\rho$. Alternatively, in terms of the energy density $E$, using Eq. (2) we obtain

$$\frac{dE}{d\rho} = \frac{E + P}{\rho}.$$  \hspace{1cm} (10)

Equations (10) and (11) imply that $P$ is a function of $E$, $P = P(E)$, which is expected for an isentropic or barotropic fluid. Plugging Eq. (11) into the definition of $T_{\mu\nu}$ yields

$$T_{\mu\nu} = (E + P)U^\mu U^\nu + Pg^{\mu\nu}.$$  \hspace{1cm} (12)

Thus we recover the standard form for the energy momentum tensor of a perfect fluid.

A. The background equations

Here we provide the background equations. As for the background, we assume a flat FLRW universe,

$$ds^2 = -dt^2 + a(t)^2 dx^2.$$  \hspace{1cm} (13)
Noting that at the background level $U^\mu = (1, 0, 0, 0)$, from Eqs. (5) and (6) one obtains the equations for the Lagrange multipliers as

$$\lambda_1 = \frac{1}{2}(E + P), \quad \dot{\lambda}_2 = -\frac{1}{\rho}(E + P).$$

Furthermore, the rest mass conservation equation (4) yields

$$\dot{\rho} + 3H\rho = 0,$$

where $H \equiv \dot{a}/a$ is the Hubble expansion rate. The background Einstein equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{E}{3M_p^2},$$

$$\frac{\ddot{a}}{a} = -\frac{E + 3P}{6M_p^2}.$$

Combining the above Einstein equations, one can easily recover the energy conservation equation in an expanding background,

$$\dot{E} + 3H(E + P) = 0.$$

Now we consider the inflationary background. First, let us look at the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$. Using the background Friedmann equation (16) and the energy conservation equation (18), one has

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{E + P}{2M_p^2H^2}.$$

The second slow-roll parameter $\eta$ is given by

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = 2\epsilon - 3(1 + c_s^2).$$

Here the speed of sound $c_s$ for our isentropic fluid is given by

$$c_s^2 = \frac{\dot{P}}{E}.$$

For an infinitesimal perturbation, this implies

$$\delta P = c_s^2\delta E = c_s^2(E + P)\frac{\delta \rho}{\rho}.$$

Note that the definition (21) is relevant since we consider an isentropic fluid.

It is important to note that for stable perturbations with $c_s^2 > 0$, the magnitude of the $\eta$ parameter is never smaller than unity as clear from Eq. (20). Indeed, taking $\epsilon \ll 1$ to sustain a long enough period of inflation, one obtains $\eta \approx -3(1 + c_s^2)$. As we shall see below, to have an almost scale-invariant power spectrum, we must require $c_s \approx 1$. So we conclude $\eta \approx -6$. This signals that our fluid inflationary system is within the domain of “ultra slow-roll inflation” scenarios [43–47]. For a nearly constant $\eta$, one obtains

$$\epsilon(t) = \epsilon_i \left(\frac{a(t)}{a_i}\right)^{\eta},$$

where $\epsilon_i$ is the value of $\epsilon$ at an initial/reference time $t = t_i$. The fact that $\eta \approx -6$ as explained above implies that $\epsilon$ decays during the ultra slow-roll inflation like $a^{-6}$.

It is also instructive to look at the equation of state parameter $w \equiv P/E$. Using the relation $\dot{P} = c_s^2\dot{E}$ and the background Friedmann and the energy conservation equations, one can easily check that

$$\dot{w} = -3H(1 + w)(c_s^2 - w).$$
We are interested in a model in which the fluid has a (nearly) constant sound speed. With a constant \(c_s\), the above equation can be integrated, yielding
\[
w = -\frac{1 - Fc_s^2}{1 + F}, \quad F \equiv \frac{1 + w_i}{c_s^2 - w_i} e^{-3N(1+c_s^2)},
\]
(25)
where \(w_i\) is the initial value of \(w\). As inflation proceeds, \(F\) rapidly decays and one has
\[
1 + w \simeq (1 + c_s^2) F \propto e^{-3N(1+c_s^2)}.
\]
(26)
This means that \(w\) approaches \(-1\) exponentially rapidly. As mentioned before, this means we are within the domain of ultra slow-roll inflation.

Finally, with the assumption that \(w \simeq -1\) and \(\epsilon\) is rapidly decaying, the background can be approximated by a pure de Sitter solution to a high accuracy,
\[
H(\tau) = \frac{\mathcal{H}_e}{1 + \mathcal{H}_e(\tau_e - \tau)}, \quad a(\tau) = \frac{a_e}{\mathcal{H}_e(\tau_e - \tau) + 1}.
\]
(27)
Here \(\tau\) is the conformal time, \(d\tau = dt/a(t)\), \(H = a'/a\) is the conformal Hubble parameter, and the subscript \(e\) denotes the value of a quantity at the end of ultra slow-roll inflation.

### III. THE PERTURBATION

Now we consider the perturbation in our fluid coupled to gravity. For relevant studies in different contexts see [48, 49]. For this purpose, we employ the ADM formalism in which the metric components are expressed as
\[
ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt).
\]
(28)
Plugging the above into the action yields
\[
S = \int dt d^3x \sqrt{h}N \left(L_G + L_m\right);
\]
\[
L_G = \frac{M_{Pl}^2}{2} \left[R^{(3)} + N^{-2}(K_{ij}K^{ij} - K^2)\right],
\]
(29)
where \(L_G\) is the gravitational part of the Langrangian, \(K_{ij}\) is the extrinsic curvature of the \(t = \text{constant}\) hypersurface,
\[
K_{ij} = \frac{1}{2} h_{ij} - (^{(3)}\nabla_j N_i - (^{(3)}\nabla_i N_j),
\]
(30)
in which \(^{(3)}\nabla\) represents the covariant derivative with respect to the three-dimensional metric \(h_{ij}\) and \(K\) is the trace of \(K_{ij}\). The matter Lagrangian \(L_m\) is given by
\[
L_m = -\rho (1 + e(\rho)) + \lambda_1 \left(g_{\mu\nu} U^\mu U^\nu + 1\right) + \lambda_2 \left(\rho U^\nu\right)_{,\mu}.
\]
(31)
Note that, by integration by parts, the above Lagrangian density is equivalent to Lagrangian density \(\tilde{L}_m\)
\[
\tilde{L}_m = -\rho (1 + e(\rho)) + \lambda_1 \left(g_{\mu\nu} U^\mu U^\nu + 1\right) - \lambda_2, (\rho U^\mu)\right).
\]
(32)
The lapse function \(N\) and the shift vector \(N_i\) are Lagrange multipliers. Varying the action with respect to them gives the Hamiltonian and momentum constraint equations,
\[
M_{Pl}^2 R^{(3)} + 2L_m + 2N \frac{\partial L_m}{\partial N} - \frac{M_{Pl}^2}{N^2} (K_{ij}K^{ij} - K^2) = 0,
\]
(33)
\[
M_{Pl}^2 \left[\frac{1}{N}(K_{ij} - Kh_{ij})\right]_{,ij} + N \frac{\partial L_m}{\partial N_i} = 0.
\]
(34)
A. Linear perturbation

Now we consider linear perturbations. To proceed further, we have to choose a gauge. Since our system is based on the fluid dynamics, it is convenient to choose the comoving gauge in which

$$ U_\mu = (-1 + u, 0, 0, 0), \quad h_{ij} = a^2(t)e^{2\mathcal{R}} \delta_{ij}. \quad (35) $$

Here $u$ represents the velocity scalar potential to all order in perturbations and $\mathcal{R}$ denotes the curvature perturbations in the comoving gauge.

As usual we decompose the lapse and the shift functions into its scalar degrees of freedom,

$$ N_i = \partial_i \psi, \quad N = 1 + \alpha. \quad (36) $$

Similarly, we perturb the Lagrange multipliers $\lambda_i$ and the density field $\rho$ as

$$ \lambda_i = \lambda_i^0 + \delta \lambda_i, \quad \rho = \rho^0 + \delta \rho. \quad (37) $$

In the above decompositions, we have focused on the scalar perturbations and neglected the tensor and vector perturbations. From now on we omit the superscript 0 from the background quantities.

Now we obtain the perturbed field equations. Perturbing the normalization condition (3) and the rest mass conservation equation (4) yields

$$ \alpha + u = 0, \quad (38) $$
$$ \delta \dot{\rho} + 3H\delta \rho + 3\rho \dot{\mathcal{R}} - \rho \frac{\nabla^2}{a^2} \psi = 0. \quad (39) $$

Perturbing the expressions for the Lagrange multipliers $\lambda_i$ in Eqs. (3) and (4) yields

$$ \delta \lambda_1 = \frac{1}{2} \frac{\delta \rho}{2\rho} (E + P), \quad (40) $$
$$ \delta \lambda_2 = -\frac{E + P}{\rho} \alpha - \frac{\delta P}{\rho}. \quad (41) $$

Furthermore, perturbing the constraint equations (33) and (34) results in

$$ \frac{\nabla^2}{a^2} (\mathcal{R} + H\psi) + 3H(H\alpha - \dot{\mathcal{R}}) + \frac{\delta \rho}{2\rho M_{\text{Pl}}^2} (E + P) = 0, \quad (42) $$
$$ H\alpha - \dot{\mathcal{R}} + \frac{\rho \delta \lambda_2}{2M_{\text{Pl}}^2} = 0. \quad (43) $$

Alternatively, one can perturb the Einstein equations. In particular, the (0i)-component of the Einstein equations gives

$$ \dot{\mathcal{R}} = \alpha H. \quad (44) $$

Comparing this equation with (42) yields $\delta \lambda_2 = 0$.

The other components of the Einstein equations are not necessary thanks to the contracted Bianchi identities, or the energy momentum conservation law $T^{\mu}_{\nu,\mu} = 0$. From the momentum conservation equation, $T^{\mu}_{\nu,\mu} = 0$, one has

$$ \delta P = -(E + P)\alpha. \quad (45) $$

Again this is consistent with the constraint (44) if $\delta \lambda_2 = 0$. Perturbing the energy conservation equation, $T^{\mu}_{0,\mu} = 0$, gives

$$ \delta \dot{E} + 3H\delta E + 3(E + P)\dot{\mathcal{R}} + 3H\delta P - (E + P) \frac{\nabla^2}{a^2} \psi = 0. \quad (46) $$

---

1 The terminology “comoving gauge” used here is somewhat different from the standard definition of the comoving gauge. As seen from its definition (35), here it is defined by a time-slicing in which the fluid 4-velocity is orthogonal to $t = \text{const.}$ hypersurfaces and the 3-metric is conformally flat.
This equation can be obtained using the constraint equations as well as the relation between \( \rho, E \) and \( P \), mentioned before.

By setting \( \delta \lambda_2 = 0 \) in the constraint equations and solving for all the variables but \( \mathcal{R} \), one obtains an equation of motion for \( \mathcal{R} \) which represents the unique propagating degree of freedom,

\[
\frac{\nabla^2}{a^2} \mathcal{R} + 3 H \dot{\mathcal{R}} - H^2 \left( \frac{\dot{\mathcal{R}}}{c_s^2 H^2} \right) = 0 ,
\]

(47)

where we recall that the sound speed \( c_s \) is defined in Eq. (21), and it appears in the perturbed relations (22), namely,

\[
\delta P = c_s^2 \delta E = c_s^2 (E + P) \frac{\delta \rho}{\rho} .
\]

(48)

B. Power spectrum

To calculate the power spectrum we need to expand the action to second order. Let us first recapitulate the action given by Eq. (29),

\[
S = \int d^4 x \left[ M_{Pl}^2 \mathcal{L}_G + N \sqrt{h} L_m \right] ,
\]

(49)

where \( \mathcal{L}_G = N \sqrt{h} L_G \). Accordingly the second order action is given in the form,

\[
S_2 = \int d^4 x \left[ M_{Pl}^3 \mathcal{L}_G^{(2)} + a^3 \left( L_m^{(2)} + (\alpha + 3 \mathcal{R}) L_m^{(1)} + (3 \alpha \mathcal{R} + \frac{9 R^2}{2}) L_m^{(0)} \right) \right] ,
\]

(50)

where we have

\[
L_m^{(0)} = - \rho(1 + e) - \dot{\lambda}_2 \rho = P ;
\]

\[
L_m^{(1)} = - \delta \rho(1 + e) - \frac{P}{\rho} \delta \rho + \lambda_1 (2 \alpha + 2 u) - \dot{\lambda}_2 \delta \rho - \delta \lambda_2 \rho + \dot{\lambda}_2 \rho(2 \alpha + u)
\]

\[
= 2 \alpha (\lambda_1 + \rho \dot{\lambda}_2) - \rho \delta \lambda_2 ,
\]

\[
L_m^{(2)} = - \frac{1}{2 \rho} \frac{dP}{d\rho} \delta \rho^2 - \lambda_1 (3 \alpha^2 + u^2) + \delta \lambda_1 (2 \alpha + 2 u) - 3 \dot{\lambda}_2 \rho a^2 + \dot{\lambda}_2 \rho(u + 2 \alpha)
\]

\[
- \delta \lambda_2 [\delta \rho - \rho(u + 2 \alpha)] + \delta \lambda_2, i R \psi \frac{a^2}{\alpha^2} ,
\]

and

\[
\frac{\mathcal{L}_G^{(2)}}{a^3} = - \mathcal{R} \frac{\nabla^2}{a^2} \mathcal{R} - 3 \mathcal{R}^2 - 18 H \mathcal{R} \dot{\mathcal{R}} + 6 H \alpha \mathcal{R} + 9 H^2 \alpha \mathcal{R} - 2 H \alpha \frac{\nabla^2}{a^2} \psi - 3 H^2 \alpha^2 - \frac{27}{2} H^2 R^2
\]

\[
+ 2 \mathcal{R} \frac{\nabla^2}{a^2} \psi - 2 \alpha \frac{\nabla^2}{a^2} \mathcal{R} .
\]

(51)

(52)

(53)

Eliminating the lagrange multipliers and the other fields in favor of \( \mathcal{R} \), we obtain the Lagrangian for \( \mathcal{R} \) as

\[
\frac{\mathcal{L}_G^{(2)}}{a^3} = M_{Pl}^2 \left[ \frac{\epsilon}{c_s^2} \mathcal{R}^2 - \frac{\epsilon}{a^2} (\partial \mathcal{R})^2 \right] ,
\]

(54)

where \( \epsilon \equiv - \dot{H}/H^2 \) is the slow-roll parameter as defined before. Note, however, that the action (51) is obtained without any slow-roll assumptions. Also one can check that the above quadratic action results in the same linear equation for \( \mathcal{R} \) as given in Eq. (47).

Now let us quantize the system. Changing the time variable \( t \) to the conformal time \( \tau \), the quadratic action (54) becomes

\[
S = \frac{1}{2} \int d^3 x d\tau z^2 [\mathcal{R}^2 - c_s^2 (\nabla \mathcal{R})^2] ,
\]

(55)
where the prime denotes a derivative with respect to the conformal time and
\[ z^2 \equiv \frac{2c \alpha^2}{c^2} M_{Pl}^2. \] (56)

The momentum conjugate to the field \( R \) is
\[ \Pi_R \equiv \frac{\delta S}{\delta R'} = z^2 \mathcal{R}'. \] (57)

They satisfy the canonical commutation relation,
\[ [\mathcal{R}(x, \tau), \Pi_R(y, \tau)] = i \delta^3(x - y). \] (58)

The quantized field can be expressed in the Fock representation,
\[ \mathcal{R}(x, \tau) = \int \frac{d^3k}{(2\pi)^3} \left[ k R_k(\tau) a_k e^{ik \cdot x} + R_k^*(\tau) a_k^\dagger e^{-ik \cdot x} \right], \] (59)
where \( R_k \) is a positive frequency mode function that satisfies the equation of motion,
\[ (z^2 \mathcal{R}')' + c_s^2 k^2 z^2 \mathcal{R} = 0, \] (60)
and the normalization condition,
\[ R_k R_k^* - R_k^* R_k = \frac{i}{z^2}. \] (61)

The annihilation and creation operators, \( a_k \) and \( a_k^\dagger \), satisfy
\[ [a_k, a_k^\dagger] = (2\pi)^3 \delta^3(k - k'). \] (62)

Assuming that \( R_k \) should approach a conventional positive frequency function at high frequencies, \( R_k \propto e^{-i c_s k \tau} \) for \( \tau \to -\infty \), the solution is uniquely determined as
\[ R_k = C_k x^\nu H^{(1)}_{\nu}(x), \] (63)
where \( H^{(1)}_{\nu} \) is the Hankel function of the first kind,
\[ x = -c_s k (\tau - \tau_i - \mathcal{H}_e^{-1}), \quad \nu = \frac{3 + \eta}{2}, \] (64)
and
\[ |C_k|^2 = \frac{\pi c_s}{8k \epsilon a_i^2 M_{Pl}^2} 2^{1-2\nu}. \] (65)

Here again the subscript \( i \) denotes an initial/reference time \( \tau = \tau_i \). One might suspect that the absolute value of \( C_k \) would depend on the choice of the initial time \( \tau_i \). However, for a nearly constant \( \eta \), one can show that it is independent of \( \tau_i \) because one has \( \epsilon a^2 \propto a^{\eta+2} \) and \( x^{1-2\nu} \propto a^{2\nu-1} = a^{\eta+2} \).

One of the important properties of our model is that the curvature perturbation is not conserved after horizon crossing. Expanding the Hankel function at \( x \ll 1 \) gives
\[ \mathcal{R}_k(\tau) \approx -C_k \frac{i2^{-\nu} e^{-i\pi \nu}}{\pi} \Gamma(|\nu|) x(\tau)^{2\nu}. \] (66)

As a result, the final curvature perturbation at the end of ultra slow-roll inflation \( \tau = \tau_e \) is given by
\[ \mathcal{R}_k(\tau_e) \approx -C_k \frac{i2^{-\nu} e^{-i\pi \nu}}{\pi} \Gamma(|\nu|) \left( \frac{c_s k}{\mathcal{H}_e} \right)^{2\nu}. \] (67)

The power spectrum of curvature perturbation at the end of ultra slow-roll inflation is given by
\[ P_\mathcal{R} = \frac{k^3}{2\pi^2 |\mathcal{R}_k(\tau_e)|^2}. \] (68)
By using Eq. (65) the above reduces to
\[ P_R \simeq \frac{\Gamma(|\nu|^2)}{\pi^{3/2}2^{\nu+4}} \left( \frac{H_e}{M_{Pl}} \right)^2 \frac{1}{c_s \epsilon_c} \left( \frac{c_s k}{H_e a_e} \right)^{3+2\nu}, \] (69)
which, using the approximation \( \eta \simeq -3(1 + c_s^2) \), further reduces to
\[ P_R \simeq \frac{\Gamma(3c_s^2/2)^2}{\pi^{3/2}2^{\nu-3c_s^2}} \left( \frac{H_e}{M_{Pl}} \right)^2 \frac{1}{c_s \epsilon_c} \left( \frac{c_s k}{H_e a_e} \right)^{(3-3c_s^2)}. \] (70)

The spectral index is easily read off as
\[ n_s - 1 \simeq 3 + 2\nu \simeq 3(1 - c_s^2). \] (71)

Interestingly, the sound speed explicitly appears in the spectral index in this model, in contrast to the standard inflationary scenarios in which only \( \dot{c}_s \) plays a role in the spectral index. In order to have a scale-invariant perturbations we require \( c_s = 1 \). The amplitude of the spectrum in this case is given by
\[ P_R = \frac{H^2}{8^{\pi^2}M_{Pl}^2 \epsilon_e}. \] (72)

A red tilted power spectrum can be achieved by a slightly superluminal sound speed. With \( c_s = 1 \), from Eq. (20) we obtain \( \eta \simeq -6 \) and from Eq. (64) \( \nu \simeq -3/2 \). This yields \( \epsilon \propto a^{-6} \) as mentioned before. Of course, recent cosmological observations by WMAP and PLANCK strongly favor a red-tilted power spectrum [3]. We see that in our model a subluminal sound speed implies \( n_s > 1 \). This is a direct consequence of the starting assumption of our considering an isentropic fluid. To obtain a red spectral index for a subluminal sound speed, perhaps one should consider a more general, non-isentropic fluid.

As for the tensor to scalar ratio, since the tensor spectrum is exactly the same as the standard case,
\[ P_T = \frac{2H^2}{\pi^2M_{Pl}^2}, \] (73)
one finds
\[ r = \frac{P_T}{P_R} = 16 \epsilon_e. \] (74)

Since \( \epsilon \) decreases exponentially during ultra slow-roll inflation, we conclude that the amplitude of the tensor perturbation is exponentially suppressed in this model.

The above simple model is not complete by itself, since there is no mechanism to terminate inflation. In principle, one can match the non-attractor phase of inflation to an attractor phase of conventional slow-roll inflation or of a hot Friedmann stage at which \( \epsilon \) is not decaying exponentially. At such a second stage, \( \mathcal{R} \) becomes frozen on super-horizon scales as usual. This implies that one can read off the final value of \( \mathcal{R} \) by computing its value at \( \tau = \tau_e \) when the transition from the non-attractor phase to an attractor phase starts. This picture was employed in the context of a single scalar field theory in [44]. The second phase of inflation is necessary also because the non-attractor inflationary phase we considered here cannot last long enough to solve the horizon problem. Because the slow-roll parameter is decreasing exponentially with time, to get \( P_R \sim 6 \times 10^{-9} \), we need a low-scale \( H \) [46]. For example, if we assume the lower-bound reheating energy to be \( \sim 1 \) GeV, we have \( \epsilon_{\min} \sim 10^{-66} \). This means that the upper bound of the inflationary efold for this non-attractor phase is 25.

C. Cubic Action and non-Gaussianity

Here we expand the action to third order which will be suitable to calculate the bispectrum. Starting with the action given in Eq. (49), one has
\[ S_3 = \int d^4x \left[ M_{Pl}^2 L_G^{(3)} + a^3 \left( \frac{L_m^{(1)}}{2} + (3aR + \frac{9R^2}{2})L_m^{(2)} + \frac{9}{2}(R^2\alpha + R^3)L_m^{(0)} \right) \right] \] (75)
where \( L_G^{(3)} \) represents the cubic order gravitational Lagrangian density, while \( L_m^{(i)} \) stands for the \( i \)-th order matter Lagrangian.
With the expansion,
\[ E(x,t) = \rho(x,t)(1 + \epsilon(x,t)) \simeq E + (E + P)\frac{\delta \rho}{\rho} + c_s^2\epsilon E \frac{\delta^2 \rho}{3\rho^2} + \frac{c_s^2\epsilon E}{27\rho}(-2s + 2\epsilon - \eta - 6)\delta \rho^3, \]  
(76)
\[ \rho(x,t)U^0(x,t) \simeq \rho - \alpha \rho + \delta \rho - \alpha \delta \rho + \alpha^2 \rho + \alpha^2 \delta \rho - \alpha^3 \rho, \]  
(77)
\[ g_{\mu\nu}U^\mu U^\nu \simeq -1, \]  
(78)
one can check that
\[ L_{(3)}^{(m)} = \frac{\bar{R}^3}{H^3}E\epsilon \left[ \frac{2\epsilon - 2s - \eta - 6}{27c_s^4} - \frac{2}{3}(1 + \frac{1}{c_s^2}) \right], \]  
(79)
where we have introduced
\[ s = \frac{\dot{c}_s}{Hc_s}. \]  
(80)
(Not to be confused with the entropy density.)

Using the constraint equations to remove non-dynamical variables, the full cubic action from the matter sector is
\[ (N\sqrt{h}L_m)_{(3)} = -\left(2\bar{\lambda} + \bar{\Sigma}\right)\frac{\bar{R}^3}{H^3} + 3\Sigma \frac{\bar{R}^2\bar{R}}{H^2} - \frac{9\bar{R}^2\dot{R}}{2H}E + \frac{9}{2}PR^3, \]  
(81)
where
\[ \bar{\Sigma} = \frac{M_P^2H^2\epsilon}{c_s^4}, \]  
(82)
\[ \bar{\lambda} = \frac{\Sigma}{18c_s^2}(\eta + 6 + 2(s - \epsilon)) = \frac{\bar{\Sigma}}{6c_s^2}\left(\frac{2s}{3} - c_s^2 + 1\right). \]  
(83)

As demonstrated in Appendix A2 similar to [52] and [51], one can check that the above cubic matter Lagrangian is equivalent to that for the theory of a scalar field with the action \( L_m = P(X) \) where \( X = -g^{\mu\nu}X_\mu X_\nu / 2, \bar{\Sigma} = XP_X + 2X^2P_{XX} \) and \( \bar{\lambda} = X^2P_{XX} + \frac{2}{3}X^3P_{XXX} \). Since the gravitational part of the action is the same by construction, this conclusion enables us to cast the cubic action for our model to the well-studied cubic action for a general \( P(X,\phi) \) theory for k-inflation [51, 52] or DBI inflation [53] with [51, 53] with [51, 55]

\[ S_3 = \int dtd^3x\{-a^3(\bar{\Sigma}(1 - \frac{1}{c_s^2}) + 2\bar{\lambda})\frac{\bar{R}^3}{H^3} + \frac{a^3\epsilon}{c^4}(\epsilon - 3 + 3c_s^2)R\bar{R}^2 \]
\[ + \frac{a^6}{c_s^4}(\epsilon - 2s + 1 - c_s^2)R(\partial R)^2 - 2a^2\epsilon c_s^2 R(\partial R)(\partial \chi) \]
\[ + \frac{a^3\epsilon}{2c_s^2} \frac{d}{dt}\left(\frac{\eta}{c_s^2}\right)R\bar{R} + \frac{\epsilon}{2a}(\partial R)(\partial \chi)\partial^2 \chi + \frac{\epsilon}{4a}(\partial^2 R)(\partial \chi)^2 + 2f(R)\frac{\delta L}{\delta R} \right\}, \]  
(84)
where the field \( \chi \) is defined by
\[ \partial^2 \chi = a^2\frac{\epsilon}{c_s^2}R, \]  
(85)
and \( f(R) \) and \( \delta L/\delta R |_1 \), respectively, by
\[ f(R) = \frac{\eta}{4c_s^2}R^2 + \frac{1}{c_s^2H}R\dot{R} + \frac{1}{4a^2H^2}[-(\partial R)(\partial R) + \partial^{-2}(\partial_\mu \partial_\nu (\partial R)\partial_\mu \partial_\nu R)] \]
\[ + \frac{1}{2a^2H}[(\partial R)(\partial \chi) - \partial^{-2}(\partial_\mu \partial_\nu (\partial R)\partial_\mu \partial_\nu \chi)], \]  
(86)
and
\[ \delta L/\delta R |_1 = a\left(\frac{d\partial^2 \chi}{dt} + H\partial^2 \chi - \epsilon \partial^2 R\right). \]  
(87)
So far our analysis of the cubic action was general and no assumption on the value of \(c_s\) has been made. However, from our power spectrum analysis, Eq. (71), we see that to obtain a scale-invariant power spectrum we need \(c_s = 1\). Therefore, from now on we concentrate on the case \(c_s = 1\). In this limit, all the interaction terms in the cubic action becomes small except for the last term involving \(f(R)\). It is known that this last term can be eliminated by the field redefinition \(R \to R_n + f(R_n)\). This means that the leading contribution to non-Gaussianity comes only from the field redefinition. As emphasized in [44] both of the first two terms in \(f(R)\) in Eq. (86) contribute to non-Gaussianity.

Following the same steps as in [44], the amplitude of local type non-Gaussianity, \(f_{NL}\), defined in the squeezed limit, \(k_1 \ll k_2 = k_3\), as

\[
\langle R_{k_1} R_{k_2} R_{k_3} \rangle \simeq (2\pi)^3 \delta^3(\sum \mathbf{k}_i) \frac{12}{5} f_{NL} P_{k_1} P_{k_3},
\]

is obtained to be

\[
f_{NL} = \frac{-5}{4} (\eta + 4) = \frac{5}{2}.
\]

This value of \(f_{NL}\) is consistent with the recent Planck constraints on primordial non-Gaussianity [56].

IV. A MODEL

Here we present a single field model which shows the behavior similar to what we pointed out in the previous sections. Consider a canonically normalized field, so \(c_s = 1\), with the potential,

\[
V(\phi) = \begin{cases} 
V_0 & \text{for } \phi < \phi_c, \\
V_1(\phi) & \text{for } \phi > \phi_c.
\end{cases}
\]

During the first stage, the system approaches rapidly towards a de Sitter universe since \(\epsilon \propto a^{-6}\). This model was originally studied in [43] as “ultra slow-roll” (USR) and was further studied in [44] as a toy single field model which can produce non-negligible local non-Gaussianity. During this phase, the curvature perturbation is not frozen on super-horizon scales, exhibitin the non-attractor nature of the system. As studied in [44], the background dynamics during the non-attractor phase is

\[
\ddot{\phi} + 3H \dot{\phi} = 0, \quad 3M_p^2 H^2 = \frac{\dot{\phi}^2}{2} + V_0 \simeq V_0.
\]

Thus \(\dot{\phi} \propto a^{-3}\) and hence

\[
\epsilon \propto a^{-6}, \quad \eta \simeq -6.
\]

The power spectrum and bispectrum were computed in [44], and the local-type non-Gaussianity with \(f_{NL} = 5/2\) was obtained.

It is instructive to look at the bispectrum in the squeezed limit using the \(\delta N\) method. One has

\[
N(\phi, \dot{\phi}) = \frac{1}{3} \ln \left[ \frac{\dot{\phi}}{\phi + 3H(\phi - \phi_c)} \right],
\]

where \(N\) is the number of \(e\)-folds counted backward from the end of ultra slow-roll inflation at which \(\phi = \phi_c\) (not to be confused with the lapse function). It is important to note that \(N\) is a function not only of \(\phi\) but also of \(\dot{\phi}\), in contrast to the conventional slow-roll inflation for which \(\dot{\phi}\) is not independent but a function of \(\phi\). Taking the variations of \(\phi\) and \(\dot{\phi}\) yields

\[
\delta N = N(\phi + \delta \phi, \dot{\phi} + \delta \dot{\phi}) - N(\phi, \dot{\phi}).
\]
On super-horizon scales, $\delta \phi$ follows the evolution of background $\phi$, and one can check that $\delta \dot{\phi} \simeq 0$ on super-horizon scales. As a result

$$
\delta N \simeq \frac{\partial N}{\partial \phi} \delta \phi + \frac{1}{2} \frac{\partial^2 N}{\partial \phi^2} \delta \phi^2
$$

$$
= - \frac{H}{\dot{\phi} + 3H(\phi - \phi_c)} \delta \phi + \frac{3H^2}{2(\dot{\phi} + 3H(\phi - \phi_c))} \delta \phi^2.
$$

(95)

This automatically yields $f_{NL} = 5/2$ in agreement with the result obtained from the in-in formalism.

As mentioned before, inflation never ends unless there is a mechanism to terminate the non-attractor phase. In the current example, we have introduced a non-trivial potential for $\phi > \phi_c$. At the second phase, inflation proceeds as in the conventional slow-roll inflation and $R$ freezes out on super-horizon scales. Therefore, the physical parameters such as $f_{NL}$ and $n_s$ can be read off by calculating these quantities at $\tau = \tau_c$ when the non-attractor phase is matched to the attractor phase.

In summary, in this work we have presented a fluid description of inflation. To be specific, we have considered the action of a single barotropic perfect fluid with appropriate Lagrange multipliers. After eliminating the Lagrange multipliers and the other non-dynamical variables we have obtained the quadratic and cubic actions for $R$. We have shown that this barotropic fluid naturally gives rise to a non-attractor inflationary phase in which $R$ is not frozen on super-horizon scales. An interesting prediction of this model is that the curvature perturbation power spectrum is scale-invariant with the value of local type non-Gaussianity given by $f_{NL} = 5/2$. We have also shown that at the level of cosmological perturbation theory this fluid model is equivalent to a scalar field theory with the Lagrangian $P(X)$. The natural question which arises is how one can extend this formalism to a non-barotropic fluid for which the pressure is not uniquely determined by the energy density. This may help to keep $n_s$ as a free parameter to obtain a slightly red-tilted power spectrum as suggested by the PLANCK data \[3\]. We have also shown that at the level of cosmological perturbation theory this fluid model is equivalent to a scalar field theory with the Lagrangian $P(X)$. However, this may also result in generating entropy perturbations which are under strong observational constraints by the PLANCK data \[3\]. We would like to come back to this issue elsewhere.

Also in this work we have considered a model with constant $c_s$. In principle one may relax this assumption and consider the case in which $c_s$ is time-dependent. As a result, this will add the new contribution $\dot{c}_s/c_s$ (in the limit where $c_s$ is changing slowly with time) into $n_s$. It is an interesting question to see if this can help to obtain a red-tilted power spectrum.

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Appendix A: The action for $P(X, \phi)$ theory

In this appendix we prove the equivalence between the perturbation theory in our isentropic fluid and a scalar field theory with the matter action,

$$
L_M = P(X, \phi), \quad X \equiv -\frac{1}{2} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi,
$$

(A1)

similar to k-inflation models \[51, 52\]. This equivalence will be used to map the bispectrum in our model to that of a well-studied $P(X, \phi)$ theory, such as in \[53\].

Our aim here is to expand the matter Lagrangian up to third order of perturbations. It is convenient to adopt the comoving gauge in which $\delta \phi = 0$ and

$$
\delta X = \frac{\delta g^{00}}{g^{00}} X \simeq (-2\alpha + 3\alpha^2 - 4\alpha^3) X.
$$

(A2)
Noting that $\alpha = \dot{R}/H$, up to third order in comoving gauge we have

$$\sqrt{h} \simeq \alpha^3 (1 + 3\mathcal{R} + \frac{9}{2} \mathcal{R}^2 + \frac{9}{2} \mathcal{R}^3),$$

$$N \simeq 1 + \frac{\dot{\mathcal{R}}}{H},$$

$$P(X, \phi) \simeq P - XP_X \left( \frac{2}{H}(\dot{\mathcal{R}}^2 - 3 \dot{\mathcal{R}}^2 + 4 \dot{\mathcal{R}}^3) \right) + 2X^2 P_{XX} \left( \frac{\dot{\mathcal{R}}^2}{H^2} - 3 \frac{\dot{\mathcal{R}}^3}{H^3} \right) - \frac{4}{3} \dot{X}^3 P_{XXX} \frac{\dot{\mathcal{R}}^3}{H^3}. \quad (A5)$$

Gathering all cubic order terms we obtain

$$(N\sqrt{h}L_M)_{(3)} = - (2\lambda + \Sigma) \frac{\dot{\mathcal{R}}^3}{H^3} + 3\Sigma \frac{\dot{\mathcal{R}}^2}{H^2} - \frac{9\dot{\mathcal{R}}^2}{2H} \dot{R} + \frac{9}{2} \mathcal{R}^3 P, \quad (A6)$$

where $E = 2XP_X - P$ is the total energy density that appears in the Friedmann equation, $3M^2_P H^2 = E$.

Comparison between Eq. (A6) and Eq. (81) demonstrates the equivalence between the above theory and the matter sector of our fluid theory with the identifications $\Sigma \leftrightarrow \Sigma$ and $\lambda \leftrightarrow \lambda$.

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