Alternative dimensional reduction via the density matrix: A test

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Abstract

We derive and analyze the perturbation series for the classical effective potential in quantum statistical mechanics, treated as a toy model for the dimensionally reduced effective action in quantum field theory at finite temperature. The first few terms of the series are computed for the harmonic oscillator and the quartic potential.

1 Introduction

Recently, a procedure was devised to rewrite the density matrix of a quantum field in thermal equilibrium at finite temperature $T$ as the exponential of an effective action in one less dimension \cite{1}. This procedure has been termed alternative dimensional reduction, to distinguish it from the conventional dimensional reduction that occurs at infinite $T$ \cite{2}, the latter being a special case of the former. The construction of such a dimensionally reduced effective action (DREA) is in fact equivalent to the construction of a Landau-Ginzburg “coarse-grained free energy” from a microscopic Hamiltonian, which can be further analyzed using the powerful methods of renormalization group theory.

The construction of the DREA cannot, in general, be performed exactly. It requires the use of perturbation theory or a dressed-loop expansion. The purpose of this work is to perform such a construction in the simpler context of quantum mechanics, in order to test the accuracy of the truncated perturbation series for the classical effective potential $V_{\text{eff}}$, the quantum mechanical version of the DREA.\textsuperscript{3} The examples considered here are the harmonic oscillator and the quartic potential.

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\textsuperscript{3}Because we have such an objective in mind, we shall not discuss the semiclassical \textsuperscript{4} or variational \textsuperscript{4} expansion of $V_{\text{eff}}$, which, although generally yielding better results than perturbation theory, are much harder to implement in quantum field theory.
2 Density matrix and effective classical potential

The diagonal elements of the thermal density matrix for a particle of mass $m$ in the presence of the potential $V(x)$ can be written as a path integral \[5, 6\] ($\hbar = m = 1$):

$$\rho(x, x; \beta) = \int_{y(0)=x}^{y(\beta)=x} Dy \exp \left\{ - \int_0^\beta d\tau \left[ \frac{1}{2} y^2 + V(y) \right] \right\}. \quad (1)$$

In the classical limit, the temperature is high and $\beta (= 1/kT)$ is small. In this case, it is reasonable to assume that the most important paths are those for which $y(\tau) \approx x$. Then Eq. (1) can be approximated by

$$\rho(x, x; \beta) \approx e^{-\beta V(x)} \rho_0(x, x; \beta), \quad (2)$$

where \[5, 6\]

$$\rho_0(x, x; \beta) = \int_{y(0)=x}^{y(\beta)=x} D\eta \exp \left\{ - \int_0^\beta \frac{1}{2} \eta^2 \, d\tau \right\} = \frac{1}{\sqrt{2\pi\beta}} \quad (3)$$

is the diagonal element of the density matrix for the free particle. Inserting Eq. (3) into Eq. (2) and integrating over $x$ one obtains the classical partition function,

$$Z_{cl} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\beta}} e^{-\beta V(x)}. \quad (4)$$

In order to improve the approximation (2) in a systematic way, we shall put $y(\tau) = x + \eta(\tau)$, with $\eta(0) = \eta(\beta) = 0$, into Eq. (1) and compute the path integral perturbatively. Thus,

$$\rho(x, x; \beta) = e^{-\beta V(x)} \int_{y(0)=x}^{y(\beta)=0} D\eta \exp \left\{ - \int_0^\beta \frac{1}{2} \eta^2 \, d\tau + \tilde{V}_x(\eta) \right\}, \quad (5)$$

where $\tilde{V}_x(\eta) \equiv V(x + \eta) - V(x)$. Expanding the exponential in powers of $\tilde{V}_x(\eta)$, we obtain

$$\rho(x, x; \beta) = e^{-\beta V(x)} \rho_0(0, 0; \beta) \left\{ 1 - \int_0^\beta \frac{1}{2} \eta^2 \, d\tau \langle \tilde{V}_x[\eta(\tau)] \rangle_0 + \frac{1}{2!} \int_0^\beta \int_0^\beta \frac{1}{2} \eta^2 \, d\tau_1 \langle \tilde{V}_x[\eta(\tau_1)] \rangle_0 \langle \tilde{V}_x[\eta(\tau_2)] \rangle_0 + \cdots \right\}, \quad (6)$$

where

$$\langle \mathcal{O} \rangle_0 \equiv \rho_0(0, 0; \beta)^{-1} \int_{y(0)=0}^{y(\beta)=0} D\eta \, \mathcal{O} \exp \left\{ - \int_0^\beta \frac{1}{2} \eta^2 \, d\tau \right\}. \quad (7)$$

If $\tilde{V}_x(\eta)$ is a polynomial in $\eta$, one can compute the expectation values in Eq. (6) with the help of Wick’s theorem:

$$\langle \eta(\tau_1) \cdots \eta(\tau_n) \rangle_0 = \sum_{\sigma} G_0(\tau_{P(1)}, \tau_{P(2)}) \cdots G_0(\tau_{P(n-1)}, \tau_{P(n)}) \quad (8)$$
if $n$ is even, and zero otherwise. The sum $\sum_P$ runs over all permutations that lead to different expressions,\(^4\) and $G_0(\tau, \tau') = \langle \eta(\tau) \eta(\tau') \rangle_0$. It satisfies
\[
- \frac{\partial^2}{\partial \tau^2} G_0(\tau, \tau') = \delta(\tau - \tau'), \quad G_0(0, \tau') = G_0(\beta, \tau') = 0. \tag{9}
\]
Eq. (9) can be easily solved, yielding
\[
G_0(\tau, \tau') = \frac{\beta - \tau'}{\beta} \theta(\tau - \tau') + \frac{\beta - \tau}{\beta} \theta(\tau - \tau'), \tag{10}
\]
where $\theta(\tau)$ is the Heaviside step function.

As a final step, we shall resum the terms in brackets in Eq. (6) into an exponential. The result can be cast as
\[
\rho(x, x; \beta) = \frac{1}{\sqrt{2\pi\beta}} \exp\{-\beta V_{\text{eff}}(x)\}, \tag{11}
\]
where
\[
\beta V_{\text{eff}}(x) = \beta V(x) + \int_0^\beta d\tau \langle \tilde{V}_x[\eta(\tau)] \rangle_{0,c} - \frac{1}{2!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle \tilde{V}_x[\eta(\tau_1)] \tilde{V}_x[\eta(\tau_2)] \rangle_{0,c} + \cdots \tag{12}
\]
The index $c$ in the expectation values means that only connected diagrams, or cumulants, should be taken into account when applying Wick’s theorem.\(^5\)

Equation (12) formally defines the effective classical potential $V_{\text{eff}}(x)$ as a power series in $x$, with coefficients that are functions of the temperature. It is the quantum mechanical analogue of the Landau-Ginzburg functional in statistical mechanics. In the next section we shall compute the first few terms of the cumulant expansion of $V_{\text{eff}}(x)$ for the harmonic oscillator and the quartic potential.

## 3 Examples

### 3.1 Harmonic oscillator

As our first example, let us consider the harmonic oscillator:
\[
V(x) = \frac{1}{2} \omega^2 x^2, \quad \tilde{V}_x(\eta) = \frac{1}{2} \omega^2 (\eta^2 + 2x\eta). \tag{13}
\]

\(^4\)Two permutations $P$ and $P'$ lead to the same expression if the sets $\{\{P(1), P(2)\}, \ldots, \{P(n-1), P(n)\}\}$ and $\{\{P'(1), P'(2)\}, \ldots, \{P'(n-1), P'(n)\}\}$ are equal.

\(^5\)For instance, $\langle \eta^2(\tau_1) \eta^2(\tau_2) \rangle_0 = 2G_0^2(\tau_1, \tau_2) + G_0(\tau_1, \tau_1) G_0(\tau_2, \tau_2)$, whereas $\langle \eta^2(\tau_1) \eta^2(\tau_2) \rangle_{0,c} = 2G_0^2(\tau_1, \tau_2)$. 

The first two cumulants give us
\[
\langle \tilde{V}_x[\eta(\tau)] \rangle_{0,c} = \frac{1}{2} \omega^2 G_0(\tau, \tau),
\]
\[
\langle \tilde{V}_x[\eta(\tau_1)] \tilde{V}_x[\eta(\tau_2)] \rangle_{0,c} = \frac{1}{4} \omega^2 \left[ \langle \eta^2(\tau_1)\eta^2(\tau_2) \rangle_{0,c} + 4x^2 \langle \eta(\tau_1)\eta(\tau_2) \rangle_{0,c} \right]
\]
\[
= \frac{1}{4} \omega^2 \left[ 2G_0^2(\tau_1, \tau_2) + 4x^2 G_0(\tau_1, \tau_2) \right].
\] (15)

Inserting these results into Eq. (12) and computing the integrals we obtain
\[
\beta V_{\text{eff}}(x) = \left( \frac{1}{12} \beta^2 \omega^2 - \frac{1}{360} \beta^4 \omega^4 + \cdots \right) + \omega x^2 \left( \frac{1}{2} \beta \omega - \frac{1}{24} \beta^3 \omega^3 + \cdots \right). \] (16)

This result should be compared with the exact result for the harmonic oscillator,
\[
\beta V_{\text{eff,ex}}(x) = \frac{1}{2} \ln \left( \frac{\sinh \beta \omega}{\beta \omega} \right) + \omega x^2 \tanh \frac{\beta \omega}{2}. \] (17)

One can easily check that Eq. (16) correctly reproduces the first few terms of the expansion of \( \beta V_{\text{eff,ex}}(x) \) in powers of \( \beta \).

### 3.2 Quartic potential

Let us now consider the quartic potential:
\[
V(x) = \frac{\lambda}{4} x^4, \quad \tilde{V}_x(\eta) = \frac{\lambda}{4} (\eta^4 + 4x\eta^3 + 6x^2\eta^2 + 4x^3\eta). \] (18)

In this case, the expansion of the effective classical potential in cumulants, Eq. (12), is also an expansion in powers of \( \lambda \). Thus, to second order in \( \lambda \), we have
\[
V_{\text{eff}}(x) = V(x) + \frac{\lambda}{4\beta} \int_0^\beta d\tau \left[ \langle \eta^4(\tau) \rangle_{0,c} + 6x^2 \langle \eta^2(\tau) \rangle_{0,c} \right] + O(\lambda^2)
\]
\[
= \frac{\lambda}{4} x^4 + \frac{\lambda}{4\beta} \int_0^\beta d\tau \left[ 3G_0^2(\tau, \tau) + 6x^2 G_0(\tau, \tau) \right] + O(\lambda^2)
\]
\[
= \frac{\lambda}{4} x^4 + \frac{1}{4} \beta x^2 + \frac{\lambda}{40} \beta^2 + O(\lambda^2). \] (19)

The second term on the r.h.s. of Eq. (19) is analogous to the radiatively induced thermal mass in the massless \( \lambda\varphi^4 \) model in field theory at finite temperature \( T \).

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\(^{6}\)Eq. (17) is obtained by expressing \( \rho(x, x; \beta) \) for the harmonic oscillator (see Refs. [5, 6]) in the form of Eq. (11).
There occurs a problem when one computes the $O(\lambda^2)$ correction to $V_{\text{eff}}(x)$. It is a sixth degree polynomial in $x$, with the coefficient of $x^6$ given by

$$\frac{\lambda^2}{24} \beta^2 < 0.$$  

(20)

It follows that if one discards cubic and higher order terms in $\lambda$ in the expansion (12), one ends up with a classical effective potential which is unbounded from below, with the disastrous consequence that $\rho(x, x; \beta)$ diverges as $|x| \to \infty$. One can remedy this problem by going to the next order in the expansion (12). At this level of approximation, $V_{\text{eff}}(x)$ is an eighth degree polynomial in $x$, the coefficient of the highest power of $x$ now being a positive number, given by

$$\frac{\lambda^3}{80} \beta^4.$$  

(21)

Instead of expanding $V_{\text{eff}}(x)$ in powers of $\lambda$, one may expand it in powers of $\beta$ — a high temperature expansion. Using Wick’s theorem and the explicit form of $G_0(\tau, \tau')$, and making the change of variables $\tau_j = \beta \tau_j'$ in the integral below, one can easily show that

$$\frac{1}{\beta} \int_{0}^{\beta} d\tau_1 \cdots \int_{0}^{\beta} d\tau_n \langle \eta^{k_1}(\tau_1) \cdots \eta^{k_n}(\tau_n) \rangle_{0,c} \propto \beta^{n-1+(k_1+\cdots+k_n)/2}.$$  

(22)

In terms of Feynman diagrams, this means that a (connected) graph with $V$ vertices and $L$ lines gives a contribution to $V_{\text{eff}}(x)$ proportional to $\beta^{V+L-1}$. Thus, we have

$$V_{\text{eff}}(x) = \frac{\lambda}{4} x^4 + \beta \frac{\lambda}{4} x^2 + \beta^2 \left( \frac{\lambda}{40} - \frac{\lambda^2}{24} x^6 \right) - \beta^3 \frac{3\lambda^2}{40} x^4 + \beta^4 \left( \frac{71\lambda^2}{3360} x^2 + \frac{\lambda^3}{80} x^8 \right) + O(\beta^5).$$  

(23)

Notice that the second and third order approximations to $V_{\text{eff}}(x)$ are unbounded from below, hence physically nonsensical. Therefore, like the expansion in powers of $\lambda$, the expansion in powers of $\beta$ cannot be truncated at an arbitrary order, but only at those orders for which the resulting approximation to $V_{\text{eff}}(x)$ is bounded from below.

### 4 Conclusions

The perturbative construction of the DREA in quantum mechanics is, therefore, equivalent to a perturbative expansion of the classical effective potential in powers of $x$. The construction yields a high-temperature series in $\beta$, or a series in the coupling constant $\lambda$, as shown in the quartic case.
The generalization to field theories \[1\] will also yield perturbative expansions in powers of the “boundary” fields, i.e., the field configurations on the euclidean time boundaries at \( \tau = 0 \) and \( \tau = \beta \), a role played by \( x \) in quantum mechanics. The coefficients will, in general, depend on temperature, couplings, and on an ultraviolet cutoff. The boundary fields are independent of the euclidean time \( \tau \) (they only depend on spatial coordinates). Thus, their effective action accomplishes the dimensional reduction advertised. For practical applications, a truncation of the series will be required, just as in quantum mechanics.

There is, however, an additional complication in field theory: the terms in the effective action are, in general, nonlocal in the boundary fields. The coefficient of the term involving \( n \) boundary fields is given by an \( n \)-point connected Green function of the fluctuations around a “background” field. The latter is defined as the solution to the free field equation of motion that interpolates between the boundary field at \( \tau = 0 \) and \( \tau = \beta \) (in our quantum mechanical examples, the background field coincides with the boundary field, both being given by \( x \); in general, the background is \( \tau \)-dependent).

Hopefully, both difficulties — the need to truncate and the nonlocality — can be circumvented in certain cases. Renormalization theory will dictate the dependence of the coefficients on the ultraviolet cutoff, so that nonrenormalizable terms will probably not contribute in the continuum limit, leading to a natural truncation. If, in addition, we investigate the infrared limit, as in the study of critical behavior near phase transitions, we might restrict our attention to the zero momentum limit of the coefficients, and recover a local structure. In summary, we hope that the combination of ultraviolet and infrared limits will restrict the form of the effective action, and give us a Landau-Ginzburg free energy of the expected form, whose parameters we will be able to compute from the underlying microscopic theory.

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References

[1] C. A. A. de Carvalho, J. M. Cornwall, and A. J. da Silva, Phys. Rev. D 64, 025021 (2001).

[2] S. Weinberg, Phys. Lett. 91B, 51 (1980); P. Ginzparg, Nucl. Phys. B170, 388 (1980); T. Applequist and R. D. Pisarski, Phys. Rev. D 23, 2305 (1981).

[3] C. DeWitt-Morette, Commun. Math. Phys. 28, 47 (1972); 37, 63 (1974); Ann. Phys. (N.Y.) 97, 367 (1976); M. M. Mizrahi, J. Math. Phys. 17, 566 (1976); 19, 298 (1978);
[4] R. Giachetti and V. Tognetti, Phys. Rev. Lett. 55, 912 (1985); R. P. Feynman and H. Kleinert, Phys. Rev. A 34, 5080 (1986); A. Cuccoli et al., J. Phys.: Condens. Matter 7, 7891 (1995); H. Kleinert, W. Küzinger, and A. Pelster, J. Phys. A: Math. Gen. 31, 8307 (1998).

[5] R. P. Feynman, *Statistical Mechanics: A Set of Lectures* (W. A. Benjamin, Reading, Massachusetts, 1972).

[6] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (World Scientific, Singapore, 1995).

[7] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989); M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).