APPROXIMATION OF ONE-DIMENSIONAL RELATIVISTIC POINT INTERACTIONS BY REGULAR POTENTIALS REVISED

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Abstract. We show that the one-dimensional Dirac operator with quite general point interaction may be approximated in the norm resolvent sense by the Dirac operator with a scaled regular potential of the form $1/\varepsilon \ h(x/\varepsilon) \otimes B$, where $B$ is a suitable $2 \times 2$ matrix. Moreover, we prove that the limit does not depend on the particular choice of $h$ as long as it integrates to a constant value.

1. Introduction

The one-dimensional Dirac operator perturbed at one point is an important exactly solvable model of relativistic quantum mechanics. Mathematically, the perturbation is described by a boundary condition at the interaction point. Following [1] we will write it as

$$\psi(0^+)=\Lambda \psi(0^-),$$

where $\psi$ is a two-component spinor and $\Lambda$ is from a four-parametric family of admissible matrices that lead to self-adjoint realizations of the Dirac operator and will be explicitly characterized later. Note that for convenience and without loss of generality the interaction point coincides with the origin.

The question how to approximate the point interactions by regular potentials is important for two major reasons. First, an approximation sequence may tell us much more about the nature of the point interaction rather then an abstract boundary condition. Second, various short range interactions may be well approximated by the point interactions and the latter are described by analytically solvable models. In the relativistic case, this question was addressed rigorously for the first time by Šeba [2]. He focused exclusively on the so-called electrostatic and Lorentz scalar point interactions. Taking the former for example, he started with the Dirac operator with the potential $1/\varepsilon \ h(x/\varepsilon) \otimes I$ for some $h \in L^1(\mathbb{R};\mathbb{R})$. Then he proved that, as $\varepsilon \to 0+$, this operator converges in the norm resolvent sense to the Dirac operator with the point interaction described by the boundary condition

$$\frac{\eta}{2}(\psi_1(0^+) + \psi_1(0^-)) = i(\psi_2(0^+) - \psi_2(0^-))$$
$$\frac{\eta}{2}(\psi_2(0^+) + \psi_2(0^-)) = i(\psi_1(0^+) - \psi_1(0^-))$$

with

$$\eta = (\mathrm{sgn}(h)|h|^{1/2}, (1 - K^2)^{-1}|h|^{1/2})_{L^2(\mathbb{R})}.$$ 

Here $K$ is the integral operator on $L^2(\mathbb{R})$ with the kernel

$$K(x, y) = \frac{i}{2} |h(x)|^{1/2} \, \text{sgn}(x - y) \, \text{sgn}(h(y)) |h(y)|^{1/2}.$$ 

Next results are due to Hughes who found smooth local approximations to all types of the point interactions but only in the strong resolvent topology [3, 4].

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She showed that if \( h \) integrates to one then the Dirac operator with the potential 
\[ \frac{i}{\varepsilon} h(x/\varepsilon) \otimes \sigma_1 A \] converges to the Dirac operator with the point interaction 
described by the boundary condition \( (1) \) with \( \Lambda = \exp A \). Of course, \( A \) must be such 
that \( \Lambda \) belongs to the admissible class.

The main purpose of this paper is to prove that the approximations found by 
Hughes converge also in the norm resolvent sense. Furthermore, explicit formulae 
for transition between \( \Lambda \)'s and \( \Lambda \)'s, which play important role in the proof, will 
be provided. Moreover, the quantity \( (3) \) will be calculated explicitly. In particular, 
it is constant once we fix the integral of \( h \). Surprisingly, this remained unnoticed 
until now although the same quantity appears also in some very recent papers 
that deal with relativistic \( \delta \)-shell interaction in \( \mathbb{R}^3 \) and its approximation by short 
range potentials \( [5,6] \), where it plays a role of the interaction strength. Essentially 
the same quantity emerges also during the limiting procedure for other types of 
the point interactions, see Corollary \( 1.6 \). Consequently, the limit operator does 
not depend on particular choice of \( h \) but only on the integral of \( h \) and, of course, 
the matrix \( A \). This was already observed by Hughes, although only in the strong 
resolvent topology. Therefore, one could have arrived to our findings about \( (3) \) 
indirectly just by combining the results from \( [2] \) and \( [4] \). It is also interesting that 
the spectrum of \( K \) is \( h \)-independent (again with \( \int_\mathbb{R} h \) fixed) and that one can find the 
eigenfunctions of \( K \) explicitly. Finally, a new observation on the renormalization 
of the coupling constant is presented.

The paper is organized as follows. After reviewing some notation in Section \( 2 \) 
we will introduce the point interaction rigorously following \( [7] \) and \( [1] \) in Section 
\( 3 \). Section \( 4 \) starts with a concise presentation of the results of Hughes. We also 
indicate some minor issues with her proof that we fix in a separate Appendix at 
the very end of the paper. Beside this, Appendix also contains very useful explicit 
formulae for passing from \( \Lambda \) from the family of admissible matrices to all \( \Lambda \)'s with 
the property \( \exp A = \Lambda \) (and vice versa). Section \( 4 \) continues with the proof of 
the main result (Theorem \( 4.1 \)) on approximation of a huge family of the point 
interactions by regular potentials. This family comprises all special types of the 
point interactions, see Corollary \( 4.6 \). Consequently, the limit operator does 
not depend on particular choice of \( h \) but only on the integral of \( h \) and, of course, 
the point interactions, see Corollary \( 4.6 \). Consequently, the limit operator does 
not depend on particular choice of \( h \) but only on the integral of \( h \) and, of course, 
the point interactions, see Corollary \( 4.6 \).

\section{Notation}

We will denote by \( L^2(M; \mathcal{H}) \) the Hilbert space of square-integrable functions on 
\( M \) with values in a Hilbert space \( \mathcal{H} \). If \( \mathcal{H} = \mathbb{C} \) we will abbreviate \( L^p(M; \mathcal{H}) \) to 
\( L^p(M) \). If, in addition, \( M = \mathbb{R} \) we will sometimes write it just \( L^p \) for brevity. We will 
use the symbol \( \langle \cdot, \cdot \rangle \) exclusively for the dot product on \( L^2(\mathbb{R}) \). When convenient 
we will identify \( L^2(M) \otimes \mathbb{C}^2 \) with \( L^2(M; \mathbb{C}^2) \) and similarly for subspaces. We set 
\( i\mathbb{R} := \{ ix \mid x \in \mathbb{R} \} \), \( \pi \mathbb{Z} := \{ \pi k \mid k \in \mathbb{Z} \} \), \( \mathbb{N} := \{ 1, 2, \ldots \} \), and \( \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \} \). The 
Pauli matrices will be denoted by \( (\sigma_j)_{j=1}^3 \) and the Hilbert-Schmidt norm by \( \| \cdot \|_2 \).

If \( K \) is an integral operator then we will write \( K(x,y) \) for its kernel.

\section{One dimensional Dirac operator with point interaction}

Let \( M \geq 0 \) be a non-negative constant. Then it is well known that 
\[ H = -i \frac{d}{dx} \otimes \sigma_1 + M \otimes \sigma_3 \]
is self-adjoint on $W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2$ and
\[
\sigma(H) = \sigma_{ac}(H) = (-\infty, -M] \cup [M, +\infty).
\]
Additional point interaction at $x = 0$ may be introduced by firstly taking a symmetric restriction of $H$,
\[
\tilde{H} := H|_{C^\infty(\mathbb{R}\setminus\{0\}) \otimes \mathbb{C}^2}
\]
and then looking for its self-adjoint extensions. The adjoint operator $\tilde{H}^*$ acts as $H$ but on the domain
\[
\text{Dom}(\tilde{H}^*) = W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2
\]
and has deficiency indices $(2, 2)$. Therefore, every self-adjoint extension of $\tilde{H}$ is described by some $2 \times 2$ unitary matrix that is employed in the second von Neumann formula. More convenient description of the extensions in terms of a boundary condition at $x = 0$ was used in [1].

Let us put
\[
\Lambda = e^{i\varphi} \begin{pmatrix} \alpha & i\beta \\ -i\gamma & \delta \end{pmatrix}
\]
with $\varphi, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfying
\[
\alpha\delta - \beta\gamma = 1.
\]
Note that one can only consider $\varphi \in [0, \pi)$ to obtain the whole class of the matrices. Then
\[
(H^\Lambda \psi)(x) := \left( -i \frac{d}{dx} \otimes \sigma_1 + M \otimes \sigma_3 \right) \psi(x) \quad (\forall x \in \mathbb{R} \setminus \{0\})
\]
with
\[
\text{Dom}(H^\Lambda) := \{ \psi \in W^{1,2}(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 | \psi(0^+) = \Lambda \psi(0^-) \}
\]
defines a self-adjoint extension of $\tilde{H}$. In this way we obtain almost all self-adjoint extensions of $\tilde{H}$, remaining being limiting cases, see [1]. There are also other ways how to parameterize the extensions that may be useful in different situations, cf. 8. 9. 10. 11. 12.

4. Approximation by regular potentials

4.1. Approximation in the strong resolvent sense. We start this section by looking closer at the results of Hughes [3. 4]. For every $H^\Lambda$, she found a family of self-adjoint Dirac operators with smooth potentials indexed by $\varepsilon > 0$ that converges to $H^\Lambda$ in the strong resolvent sense as $\varepsilon \to 0^+$. More concretely, she considered a family $\{h_\varepsilon\}_{\varepsilon > 0}$ of smooth functions such that $h_\varepsilon \geq 0$, $\int h_\varepsilon = 1$, and supp$(h_\varepsilon) \subset [0, \varepsilon]$, and a matrix $A$ such that $\Lambda = \exp(A)$. Since $A$ is regular, there is always an $A$ with this property. The approximating operators are then given by
\[
H^\varepsilon := H + ih_\varepsilon(x) \otimes \sigma_1 A,
\]
\[
\text{Dom}(H^\varepsilon) := W^{1,2}(\mathbb{R}) \otimes \mathbb{C}^2.
\]
However, as far as I can see, there is a flaw in her proof (of [4. Theorem 1]). The first part of the proof says, for a moment with arbitrary matrices $A$ and $\Lambda$, that
\[
\sigma_1 A \sigma_1 = -A^*
\]
if and only if $\Lambda = \exp(A)$ satisfies
\[
\Lambda^* \sigma_1 \Lambda = \sigma_1,
\]
in which case $\Lambda$ has the form [4]. As a counterexample, take
\[
\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix},
\]
with $\alpha > 0$. It obeys (4) (as any matrix of the form (4) does), but
\[
A := \begin{pmatrix}
\ln \alpha + i2\pi m & 0 \\
0 & -\ln \alpha + i2\pi n
\end{pmatrix},
\]
with $m, n \in \mathbb{Z}$, does not fulfill (7) unless $m = n$.

Since condition (7) is crucial in the rest of her proof, among all possible $A$’s one should only consider those that obey it. Therefore, the question is: Does for any $\Lambda$ of the form (4) exist $A$ such that $\Lambda = \exp(A)$ and (7) holds true? In Appendix, it is demonstrated that the answer is positive. In fact, there is always a whole one or two-parametric family of $A$’s with the desired properties, that can be written explicitly in terms of the entries of $\Lambda$.

4.2. Approximation in the norm resolvent sense. In this section we will show that, for a large subfamily of self-adjoint extensions of $\dot{H}$, the approximating operators $\{H^A_\epsilon\}$ converge to $H^\Lambda$ (with $\Lambda = \exp(A)$) in the norm resolvent sense as $\epsilon \to 0^+$. More concretely, let $h \in L^1(\mathbb{R}; \mathbb{R}) \cap L^\infty(\mathbb{R}; \mathbb{R})$ be such that
\[
\int_{\mathbb{R}} h(x)dx = 1.
\]
(9)

For any $\epsilon > 0$, define
\[
h_\epsilon(x) := \frac{1}{\epsilon} h\left(\frac{x}{\epsilon}\right) \quad \text{and} \quad \Theta_\epsilon(x) := \int_{-\infty}^{x} h_\epsilon(y)dy.
\]
Then $H^A_\epsilon$ given by (4) is self-adjoint for any matrix $A$ because it is a bounded perturbation of $H$. Note that $\Theta_\epsilon$ is absolutely continuous, $\lim_{x \to -\infty} \Theta_\epsilon(x) = 0$, $\lim_{x \to +\infty} \Theta_\epsilon(x) = 1$, and $\lim_{\epsilon \to 0} \Theta_\epsilon = \Theta$, where $\Theta$ stands for the Heaviside function, pointwise on $\mathbb{R} \setminus \{0\}$. In fact, the latter is sufficient for the results of Hughes to hold true. Furthermore, let either
\[
A = i\varphi I, \quad \text{where } \varphi \in \mathbb{R} \setminus \{0, \lambda^{-1}_k | k \in \mathbb{Z}\}
\]
with
\[
\lambda_k := \frac{1}{(2k+1)\pi},
\]
(12)
or
\[
A := \begin{pmatrix}
a & ib \\
ib & -a
\end{pmatrix}, \quad \text{where } a, b, c \in \mathbb{R} \quad \text{and} \quad \det A \notin \{\lambda_k^{-2} | k \in \mathbb{N}_0\}.
\]
(13)

Note that (7) is satisfied in both the cases.

Then we have

**Theorem 4.1.** Let $h \in L^1(\mathbb{R}; \mathbb{R}) \cap L^\infty(\mathbb{R}; \mathbb{R})$ obey (9). Furthermore, let $H^A_\epsilon$ be as in (4) with $h_\epsilon$ given by (10) and $A$ either of the form (11) or (13). Then $\lim_{\epsilon \to 0^+} \|(H^A_\epsilon + i)^{-1} - (H^\Lambda + i)^{-1}\| = 0$

where $\Lambda = \exp(A)$.

Let us stress that for two special cases, namely $A = -i\sigma_1$ and $A = -\sigma_2$, the theorem has been already proved by Šeba [2]. Also note that, since the norm resolvent convergence is stable with respect to adding a constant bounded perturbation [13, Theorems IV.2.23 c) and IV.2.25], we will always assume that $M = 0$ in the rest of this section, which is devoted to the proof Theorem 4.1. We will focus on the more complex case with $A$ given by (13). The other choice of $A$ is briefly discussed in Remark 4.7.
Our starting point is the resolvent formula by Kato [14], see also [15] or [16], that was used by Šeba too. Let \( G_z \) denote the integral kernel of the resolvent of \( H \) at the point \( z \in \mathbb{C} \setminus \sigma(H) = \mathbb{C} \setminus \mathbb{R} \),
\[
G_z(x, y) \equiv (H - z)^{-1}(x, y) = \frac{i}{2} \zeta(z)I + \text{sgn}(x - y)\sigma_1 e^{iz(z)|x - y|},
\]
where
\[
\zeta(z) := \text{sgn}(3z).
\]
Then the resolvent formula yields
\[
(H^A - z)^{-1} = (H - z)^{-1} - C_\varepsilon(1 + Q_\varepsilon)^{-1}D_\varepsilon
\]
with \( C_\varepsilon, Q_\varepsilon, \) and \( D_\varepsilon \) being the integral operators on \( L^2(\mathbb{R}; \mathbb{C}^2) \) with the kernels
\[
C_\varepsilon(x, y) = G_\varepsilon(x, cy)v(y),
Q_\varepsilon(x, y) = iu(x)\sigma_1 A G_\varepsilon(x, y)v(y),
D_\varepsilon(x, y) = iu(x)\sigma_1 A G_\varepsilon(x, y),
\]
where
\[
\varepsilon := \sqrt{|h|} \quad \text{and} \quad v := \text{sgn}(h)\sqrt{|h|}.
\]
One can check that \( C_\varepsilon, D_\varepsilon, \) and \( Q_\varepsilon \) are Hilbert-Schmidt operators. Pointwise limits of their kernels, i.e.,
\[
C(x, y) := \lim_{\varepsilon \to 0^+} C_\varepsilon(x, y) = G_\varepsilon(x, 0)v(y),
Q(x, y) := \lim_{\varepsilon \to 0^+} Q_\varepsilon(x, y) = iu(x)\sigma_1 A G_\varepsilon(x, y)v(y),
D(x, y) := \lim_{\varepsilon \to 0^+} D_\varepsilon(x, y) = iu(x)\sigma_1 A G_\varepsilon(x, y),
\]
are also associated with some Hilbert-Schmidt operators, say \( C, Q, \) and \( D, \) respectively.

Using the dominated convergence theorem we infer that
\[
\lim_{\varepsilon \to 0^+} \|C_\varepsilon - C\|_2 = 0, \quad \lim_{\varepsilon \to 0^+} \|Q_\varepsilon - Q\|_2 = 0, \quad \lim_{\varepsilon \to 0^+} \|D_\varepsilon - D\|_2 = 0.
\]
This implies the convergence in the uniform operator topology too. Now, by the stability of bounded invertibility [13] Theorem IV.1.16], if \(-1 \notin \sigma(Q)\) then for all \( \varepsilon \) sufficiently small \(-1 \notin \sigma(Q_\varepsilon)\) and
\[
\text{n-lim}_{\varepsilon \to 0^+}(I + Q_\varepsilon)^{-1} = (I + Q)^{-1}.
\]
Here, n-lim denotes the limit in the uniform operator topology. We conclude that
\[
\text{n-lim}_{\varepsilon \to 0^+}(H^A - z)^{-1} = (H - z)^{-1} - C(I + Q)^{-1}D.
\]
The right-hand side of (15) splits into a sum of the free resolvent and an operator of the rank two or smaller. Note that the latter is \( z \)-dependent, too. This is exactly the decomposition that appears in the Krein formula for the resolvent of \( H^A \) [1]. To compare these two expressions one has to invert the operator \((I + Q)\).

Similarly as in [2], we start by writing \( Q = Q_1 + Q_2, \) where
\[
Q_1(x, y) := -\frac{\zeta(z)}{2} u(x)v(y)\sigma_1 A \quad \text{and} \quad Q_2 := K \otimes \tilde{A}
\]
with
\[
K(x, y) := \frac{i}{2} u(x)\text{sgn}(x - y)v(y) \quad \text{and} \quad \tilde{A} := i\sigma_1 A\sigma_1.\]
Clearly, the rank of \( Q_1 \) is at most two and \( K \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}) \). We have
\[
(I + Q)^{-1} = (I + (I + Q_2)^{-1}Q_1)^{-1}(I + Q_2)^{-1},
\]
whenever the right-hand side makes sense. One can verify directly that
\[(I + Q_2)^{-1} = (I - \nu^2 K^2)^{-1} \otimes I - K(I - \nu^2 K^2)^{-1} \otimes \tilde{A},\]
where
\[\nu^2 := \det A = bc - a^2\]
is the same as in Appendix, cf. (36). The candidate for \((I + Q_2)^{-1}\) was found by writing it as a formal geometric series and taking the fact that, with our choice of \(A\), \(\tilde{A}^2 = \nu^2 I\) into the account. Note that the inverse of \((I - \nu^2 K^2)\) exists as a bounded operator if and only if \(\nu^{-2} \notin \sigma(K^2)\). Let us look at the spectral properties of \(K\) in detail.

**Proposition 4.2.** \(K\) is Hilbert-Schmidt with \(\|K\|_2 = \frac{1}{2}\|h\|_{L^1}\). For any \(k \in \mathbb{Z}\),
\[\psi_k(x) := \frac{1}{\sqrt{\|h\|_{L^1}}} u(x)e^{i(2k+1)\pi f \int_x^\infty h(y)dy}\]
is a normalized eigenfunction of \(K\) with the eigenvalue \(\lambda_k\) given by (12). Moreover, \(\sigma(K) = \{0\} \cup \{\lambda_k | k \in \mathbb{Z}\}\) and \(0 \in \sigma_p(K)\) if and only if \(h\) is not non-zero almost everywhere on \(\mathbb{R}\). In the positive case, 0 is an eigenvalue of infinite multiplicity.

If \(h \geq 0\) then \(K\) is hermitian, therefore, its eigenfunctions form an orthonormal basis.

**Proof.** The only non-trivial part is to find all eigenpairs. Let \(\lambda \in \mathbb{C}\). If \(K \psi = \lambda \psi\) then \(\psi\) is necessarily factorized as \(\psi(x) = u(x)f(x)\). Substituting this back into the eigenvalue equation we see that
\[\frac{i}{2} \left( \int_{-\infty}^x h(y)f(y)dy - \int_x^{+\infty} h(y)f(y)dy \right) = \lambda f(x).\]

Note that \(hf \in L^1(\mathbb{R})\) because we only deal with \(\psi \in L^2(\mathbb{R})\). Hence, \(f\) is absolutely continuous and so we can take a derivative of (13). We arrive at
\[ih(x)f(x) = \lambda f'(x)\]
together with the boundary condition
\[\lim_{x \to +\infty} f(x) = - \lim_{x \to -\infty} f(x) \quad \text{for} \ \lambda \neq 0.\]
If \(\lambda \neq 0\), \(f(x) = C \exp((\lambda \int_{x}^{+\infty} h(y)dy))\). Imposing (19) on the solution we obtain
\[1 = -e^{\pi}.\]
Therefore, \(\lambda\) must be as in (12).

If \(\lambda = 0\) then the eigenvalue equation reduces to \(hf = v\psi = 0\). A non-trivial solution to this equation exists if and only if \(h\) is not non-zero almost everywhere, i.e., \(\text{ess supp}(h) \neq \mathbb{R}\). In the positive case, any non-zero \(\psi \in L^2(\mathbb{R} \setminus \text{ess supp}(h))\) is an eigenfunction. \(\square\)

**Corollary 4.3.** \(\sigma(K^2) = \{0\} \cup \{\lambda_k^2 | k \in \mathbb{N}_0\}\). In more detail, \(\lambda_k^2\) are twice degenerate eigenvalues of \(K^2\) and 0 is an eigenvalue of \(K^2\) if and only if \(h\) is not non-zero almost everywhere on \(\mathbb{R}\).

This is exactly why we assumed that \(\det A \equiv \nu^2 \neq \lambda_k^2\), \(k \in \mathbb{N}_0\), in (13). Note that if \(\nu^2 \leq 0\) then this is not a restriction at all. Now, in the the next step of inverting \((I + Q)\) we need to find \((I + (I + Q_2)^{-1}Q_1)^{-1}\). Take \(g \otimes b \in L^2(\mathbb{R}) \otimes \mathbb{C}^2\) and look for a solution \(f \otimes a \in L^2(\mathbb{R}) \otimes \mathbb{C}^2\) of the equation
\[(I + (I + Q_2)^{-1}Q_1)f \otimes a = g \otimes b.\]
Substituting for $Q_1$ and $Q_2$ we arrive at

$$
(21) \quad f \otimes a - \frac{\zeta(z)}{2} \langle v, f \rangle \left( (I - \nu^2K^2)^{-1}u \otimes \sigma_1 Aa - iK(I - \nu^2K^2)^{-1}u \otimes \sigma_1 A^2a \right) = g \otimes b.
$$

If we multiply both sides of (21) by $v$ with respect to the dot product on $L^2(\mathbb{R})$ we get

$$
(22) \quad \langle v, f \rangle a = \langle v, g \rangle \left( I - \frac{\zeta(z)\eta}{2}\sigma_1 A \right)^{-1} b,
$$

where we put

$$
(23) \quad \eta := \langle v, (I - \nu^2K^2)^{-1}u \rangle
$$

and used the following observation (done already by Šeba in [2] for the special case $\nu^2 = 1$).

**Lemma 4.4.** For all $\nu^2 \in \mathbb{C} \setminus \{\lambda_k^{-2} | k \in \mathbb{N}_0\}$, we have

$$
\langle v, K(I - \nu^2K^2)^{-1}u \rangle = 0.
$$

**Proof.** Firstly, let $\nu^2 \in \mathbb{C} : |\nu^2| < 4/\|h\|_2^2$. Then $\|\nu^2K^2\| < 1$ and we have

$$
\langle v, K(I - \nu^2K^2)^{-1}u \rangle = \sum_{n=0}^{+\infty} \nu^{2n}\langle v, K^{2n+1}u \rangle.
$$

Let $S$ be the integral operator on $L^2(\mathbb{R})$ with the kernel $S(x, y) = \frac{2}{\pi} \text{sgn}(x - y)$. Then $K = uSv$ and, for any $f \in L^2(\mathbb{R}; \mathbb{R})$, $\langle f, Sf \rangle = 0$. Therefore, every term on the right-hand side of (24) is zero because

$$
\langle v, K^{2n+1}u \rangle = \langle v, (uSv)^{2n+1}u \rangle = \langle (hS)^n h, S(hS)^n h \rangle = 0.
$$

Secondly, the function $F : \nu^2 \mapsto \langle v, K(I - \nu^2K^2)^{-1}u \rangle$ is analytic on $\mathbb{C} \setminus \{\lambda_k^{-2} | k \in \mathbb{N}_0\}$. Therefore, by the identity theorem, $F = 0$ on $\mathbb{C} \setminus \{\lambda_k^{-2} | k \in \mathbb{N}_0\}$. \hfill $\Box$

Note that the inverse in (22) always exists because $\sigma(\sigma_1 A) \subset \mathbb{R}$, $\zeta(z) \in \mathbb{R}$, and $\eta \in \mathbb{R}$. The latter will follow from Proposition 1.5. Substituting (22) into (21) we get

$$
(26) \quad \langle I + (I + Q_2)^{-1}Q_1 \rangle^{-1} = I + \frac{\zeta(z)}{2}\langle v, \cdot \rangle \left( (I - \nu^2K^2)^{-1}u \otimes \sigma_1 A \left( I - \frac{\zeta(z)\eta}{2}\sigma_1 A \right) \right)^{-1} - iK(I - \nu^2K^2)^{-1}u \otimes \sigma_1 A^2 \left( I - \frac{\zeta(z)\eta}{2}\sigma_1 A \right)^{-1}.
$$

Since we solved (20) only on the decomposable vectors, one has to verify by a direct calculation that this is indeed the two-sided inverse. Now, inserting (20) together with (17) into (16) we finish the computation of $(I + Q)^{-1}$. Actually, we only need to know $C(I + Q)^{-1}D$ in (15). For its integral kernel we have

$$
\langle C(I + Q)^{-1}D \rangle(x, y) = iG_z(x, 0)(\langle v, (I + Q)^{-1}u \rangle \sigma_1 G_z(0, y).
$$

Using Lemma 4.4 we infer that

$$
\langle v, (I + Q)^{-1}u \rangle = \eta I + \frac{\zeta(z)\eta}{2}\sigma_1 A \left( I - \frac{\zeta(z)\eta}{2}\sigma_1 A \right)^{-1} \eta \left( I - \frac{\zeta(z)\eta}{2}\sigma_1 A \right)^{-1}.
$$

After some more tedious computation we deduce that

$$
\langle C(I + Q)^{-1}D \rangle(x, y) = G_z(x, 0)M(z)G_z(0, y),
$$

where

$$
(27) \quad M(z) := \frac{-1}{\zeta(z)\eta(b + c) + i(1 - \frac{1}{2}\eta^2\nu^2)} \left( i\eta c + \frac{1}{2}\zeta(z)\eta^2\nu^2 a \eta + \frac{-a\eta}{\zeta(z)\eta^2\nu^2} \right).
$$
It remains to evaluate $\eta$ as a function of $\nu$.

**Proposition 4.5.** For all $\nu \in \mathbb{C} \setminus \{0, \lambda_k^{-1} | k \in \mathbb{Z}\}$,

$$\eta = \langle \nu, (I - \nu^2 K^2)^{-1} u \rangle = \frac{2}{\nu} \tan \frac{\nu}{2}.$$ 

**Proof.** As in the proof of Lemma 4.4 we start with the case $|\nu^2| < 4/\|h\|^2_{L^1}$. Then

$$\langle \nu, (I - \nu^2 K^2)^{-1} u \rangle = \sum_{n=0}^{+\infty} \nu^{2n} \langle \nu, K^{2n} u \rangle,$$

where

$$\langle \nu, K^{2n} u \rangle = \frac{(-1)^n}{2^{2n}} \int_{\mathbb{R}^{2n+1}} h(x_1) \text{sgn}(x_1 - x_2) h(x_2) \text{sgn}(x_2 - x_3) h(x_3) \ldots \text{sgn}(x_{2n} - x_{2n+1}) h(x_{2n+1}) dx_1 \ldots dx_{2n+1} =$$

$$\frac{(-1)^n}{(2n + 1)! 2^{2n}} \int_{\mathbb{R}^{2n+1}} \prod_{j=1}^{2n+1} h(x_j) \sum_{\tau \in S_{2n+1}} \prod_{l=1}^{2n} \text{sgn}(x_{\tau(l)} - x_{\tau(l+1)}) dx_1 \ldots dx_{2n+1}.$$ 

If we define

$$C_n(x_1, x_2, \ldots, x_{2n+1}) := \sum_{\tau \in S_{2n+1}} \prod_{l=1}^{2n} \text{sgn}(x_{\tau(l)} - x_{\tau(l+1)})$$

then we observe that that $C_n$ is a constant (which will be denoted by the same letter) almost everywhere on $\mathbb{R}^{2n+1}$. Therefore, using (28) we get

$$\langle \nu, K^{2n} u \rangle = \frac{(-1)^n C_n}{(2n + 1)! 2^{2n}}.$$ 

To find the explicit value of $C_n$ is an exercise in combinatorics. Alternatively, since we have just showed that the quantity $\langle \nu, K^{2n} u \rangle$ does not depend on a particular choice of $h$, one can use the result of Šeba [2], where it is calculated for $h = \chi_{(0,1)}$. This way we obtain

$$\langle \nu, K^{2n} u \rangle = \frac{(-1)^n 4(2^{2n+2} - 1)}{(2n + 2)!} B_{2n+2},$$

where $B_{2n+2}$ stands for the $(2n + 2)$th Bernoulli number. Substituting this into (29) we arrive at (28).

By analyticity of the both sides, one can extend (28) on $\mathbb{C} \setminus \{0, \lambda_k^{-1} | k \in \mathbb{Z}\}$ using the identity theorem.

**Corollary 4.6.** If we started with $h$ that does not integrate to one then

$$\eta = \langle \nu, (I - \nu^2 K^2)^{-1} u \rangle = \frac{2}{\nu} \tan \frac{\nu}{2} \int_0 h(x) dx.$$ 

Let us now recall the formula for the resolvent of $H^A$ as was derived in [1],

$$(H^A - z)^{-1}(x, y) = G_z(x, y) - G_z(x, 0) M^A(z) G_z(0, y),$$

where

$$M^A(z) := \frac{1}{\zeta(z)(\beta - \gamma) + i(\alpha + \delta)}$$

$$\begin{pmatrix}
2i\gamma + \zeta(z)(\alpha + \delta - 2 \cos \varphi) & \alpha - \delta - 2i \sin \varphi \\
\delta - \alpha - 2i \sin \varphi & -2i\beta + \zeta(z)(\alpha + \delta - 2 \cos \varphi)
\end{pmatrix}.$$
To conclude the proof of Theorem 4.1 we have to express $M^\Lambda(z)$ in terms of the entries of $A$ and then show that $M^\Lambda(z) = M(z)$. Following Appendix we should distinguish three cases, $\nu \in \mathbb{R} \setminus \pi \mathbb{Z}$, $\nu \in \pi \mathbb{Z}$, and $\nu \in i\mathbb{R} \setminus \{0\}$. We will focus on the first case, the latter may be treated similarly. In the second case, if $\nu = 2\pi k$ with $k \in \mathbb{Z} \setminus \{0\}$ then $\Lambda = I$, $\eta = 0$, and $M = 0$, i.e., there is no point interaction in the limit as $\varepsilon \to 0^+$. If $\nu = (2k + 1)\pi = \lambda^{-1}_k$ with $k \in \mathbb{Z}$ then we cannot use our approach. If $\nu = 0$ then $\eta = 1$ and one can proceed similarly as in the first case. In particular, the corresponding $M^\Lambda(z)$ is the same as the limit of (31) as $\nu \to 0$.

Let $\nu \in \mathbb{R} \setminus \pi \mathbb{Z}$. Starting with $A$ of the form (13) we can use (38), (39), and (40) to calculate that

$$
\alpha = \cos \nu + \frac{\sin \nu}{\nu} - a, \quad \beta = \frac{\sin \nu}{\nu} b, \quad \gamma = -\frac{\sin \nu}{\nu} c, \quad \delta = \cos \nu - \frac{\sin \nu}{\nu} - a.
$$

Note that without loss of generality, we put $\varphi = 0$ and $n = 0$ in (38). After some algebra we obtain

$$
M^\Lambda(z) = \frac{-1}{\zeta(z) \frac{\tan \frac{\varphi}{2}}{\nu}(b + c) + i(1 - \tan^2 \frac{\varphi}{2})} \left( i \frac{2 \tan \frac{\varphi}{2}}{\nu} c + 2 \zeta(z) \tan^2 \frac{\varphi}{2} \frac{a}{\nu} \tan \frac{\varphi}{2} - \frac{2 \tan \frac{\varphi}{2}}{\nu} \frac{2}{\nu} \frac{2 \tan \frac{\varphi}{2}}{\nu} b + 2 \zeta(z) \tan^2 \frac{\varphi}{2} \right).
$$

If we substitute (28) into (27) we get exactly the same matrix.

**Remark 4.7.** If we start with $A$ of the form (11) then

$$(I + Q_2)^{-1} = (I - \varphi K)^{-1} \otimes I.
$$

Since $\varphi \notin \{\lambda_k^{-1} | k \in \mathbb{Z}\}$, the inverse on the right-hand side always exists due to Proposition 4.2. Next, we have

$$(32) \quad (I + (I + Q_2)^{-1} Q_1)^{-1} = I + \frac{i \zeta(z) \varphi}{2} \langle v, \cdot \rangle (I - \varphi K)^{-1} u \otimes \sigma_1 \left( I - \frac{i \zeta(z) \varphi \eta}{2} \sigma_1 \right)^{-1}
$$

with

$$\tilde{\eta} := \langle v, (I - \varphi K)^{-1} u \rangle.
$$

The inverse matrix on the very right of (32) exists because $\tilde{\eta} \in \mathbb{R}$. In fact, employing (28) and Proposition 4.2 one can show that

$$\tilde{\eta} = \langle v, (I - \varphi^2 K^2)^{-1} u \rangle = \frac{2}{\varphi} \tan \frac{\varphi}{2}.
$$

Finally, we obtain

$$\langle v, (I + Q)^{-1} u \rangle = \tilde{\eta} \left( I - \frac{i \zeta(z) \varphi \tilde{\eta}}{2} \sigma_1 \right)^{-1}
$$

and, consequently,

$$M(z) = -\sin \varphi \left( \frac{i \zeta(z) \tan \frac{\varphi}{2}}{2} \frac{1}{\zeta(z) \tan \frac{\varphi}{2}} \right).
$$

This is exactly $M^\Lambda(z)$ with $\Lambda = \exp(A) = \exp(i\varphi)I$.

5. **Renormalization of $\delta$ potentials**

Speaking about renormalization we have to specify how we measure the strength of point interactions in the first place. Let us look at the electrostatic point interaction for example. The boundary condition (2) may be rewritten as (1) with

$$\Lambda = \begin{pmatrix}
1 - \eta^2 & -i \nu \\
-i \nu & 1 - 2 \eta 
\end{pmatrix}.$$
Given appropriate approximations of the form (6) with \( h \) that does not necessarily fulfills (8), \( \eta \) is given by (30) with \( \nu = 1 \), i.e., \( \eta = 2 \tan(\int_R h/2) \). This scales non-homogeneously with respect to \( h \). However, if we decide to measure the interaction strength by a new parameter \( \theta \) that obeys \( \eta = 2 \tan(\theta/2) \) then

\[
\Lambda = \begin{pmatrix}
\cos \theta & -i \sin \theta \\
-i \sin \theta & \cos \theta
\end{pmatrix}
\]

and \( \theta = \int_R h \) is obviously linear in \( h \). The reason why \( \eta \) (and not \( \theta \)) is referred to as the coupling constant for electrostatic potential is that by formal integration of the eigenvalue equation with potential \( \eta \delta \), one arrives at (2). During this procedure one extends \( \delta \)-distribution to the functions with discontinuity at \( x = 0 \) by putting 

\[
\psi(0) := \frac{\psi(0^+) + \psi(0^-)}{2}
\]

Mathematically, this was justified in [17]. For a general point interaction, it was deduced by Hughes [4] that

\[
H^\Lambda \psi = -i \frac{d}{dx} \otimes \sigma_1 \psi + M \otimes \sigma_3 \psi + 2i \otimes \sigma_1 (\Lambda - I)(\Lambda + I)^{-1} \psi(0) \delta
\]

in the sense of distributions. Since, in (6), \( \lim_{\epsilon \to 0+} h_\epsilon = \delta \) in the sense of distributions, it is reasonable to state that renormalization of the coupling constant does not occur if and only if

\[
A = 2(\exp(A) - I)(\exp(A) + I)^{-1},
\]

provided that the inverse exists. See [4] for some necessary and sufficient conditions on \( A \) under which (33) is fulfilled.

Now, using Appendix we can deduce that the right-hand side of (33) equals

\[
W^A := \frac{2}{\nu(\cos \nu + \cos \varphi)} \begin{pmatrix}
\Re a \sin \nu + i \nu \sin \varphi & i b \sin \nu \\
-i c \sin \nu & -\Re a \sin \nu + i \nu \sin \varphi
\end{pmatrix},
\]

where we put \( \varphi := 3a \) and \( \nu \) is given by (36). For \( \nu = 0 \), one can just send \( \nu \) to zero in the formula. Firstly, consider \( A \) given by (11). A straightforward calculation yields

\[
W^A = \frac{2}{\varphi} \tan \frac{\varphi}{2} A.
\]

Next, let \( A \) be as in (13), i.e., \( a \in \mathbb{C} \). Then

\[
W^A = \frac{2}{\nu} \tan \frac{\nu}{2} A.
\]

Therefore, for both types of the studied matrices, \( W^A \) is a multiple of \( A \). However, the renormalization occurs unless \( \varphi/2 = \tan(\varphi/2) \) or \( \nu/2 = \tan(\nu/2) \), respectively.

Finally, we conclude that if the potential in (6) is multiplied by \( t \in \mathbb{R} \) then the strength of the point interaction scales as \( 2/\varphi \tan(t \varphi/2) \) and \( 2/\nu \tan(t \nu/2) \), respectively.

**APPENDIX**

Here we address the following: Does for any \( \Lambda \) of the form (4) exist \( A \) such that \( \Lambda = \exp(A) \) and (7) holds true?

Firstly, note that \( A \) obeys (7) if and only if

\[
A = \begin{pmatrix} a & ib \\ ic & -\bar{a} \end{pmatrix}, \quad \text{where } a \in \mathbb{C} \text{ and } b, c \in \mathbb{R}.
\]

Secondly, for an arbitrary \( 2 \times 2 \) matrix \( A \) such that

\[
\nu := \sqrt{\det A - \left( \frac{\text{Tr} A}{2} \right)^2} \neq 0
\]
we have
\begin{equation}
\exp(A) = \exp \left( \frac{\text{Tr} A}{2} \right) \left( \cos \nu I + \frac{\sin \nu}{\nu} \left( A - \frac{\text{Tr} A}{2} I \right) \right).
\end{equation}

Note that both branches of the square-root lead to the same formula. If \( \nu = 0 \) then we write 1 instead of \( \sin \nu / \nu \) in (35). The formula can be derived by splitting \( A \) into a sum of a traceless matrix and a multiple of identity. The decomposition is unique and its two parts commute. The exponential of a multiple of identity is trivial to calculate, the exponential of a traceless matrix can be also calculated directly from the Taylor series because its square is diagonal and, therefore, we can sum up the odd and even terms separately. Note that the right-hand side of (35) again decomposes into a unique sum of a multiple of identity and a traceless matrix.

Essentially the same formula appears in \( 18 \) but with a different proof.

Now, for \( A \) of the form \( (\mathbb{H}) \), \( \text{Tr} A / 2 = i \Re a \),
\begin{equation}
A - \frac{\text{Tr} A}{2} I = \left( \Re a \quad i b \\ ic \quad -\Re a \right),
\end{equation}
and
\begin{equation}
\nu^2 = bc - (\Re a)^2 \in \mathbb{R}.
\end{equation}
Hence, either \( \nu^2 \geq 0 \) (equivalently, \( \nu \in \mathbb{R} \)) or \( \nu^2 < 0 \) (equivalently, \( \nu = ig \) with \( g \in \mathbb{R} \setminus \{0\} \), and consequently, \( \cos \nu = \cosh g \) and \( \sin \nu = i \sinh g \)). Now, we write
\begin{equation}
\Lambda = e^{i \varphi} \left( \frac{\alpha + \delta}{2} I + \left( \frac{\alpha - \delta}{2} \right)^2 \right) = e^{i \gamma} \left( \frac{i \beta}{2} \right).
\end{equation}

and compare it with (35).

**The case \((\alpha + \delta)/2 \in (-1, 1)\).** In this case \( \nu \in \mathbb{R} \setminus \pi \mathbb{Z} \) and we have
\begin{equation}
\exists a = \varphi + n \pi, \quad n \in \mathbb{Z},
\end{equation}
\begin{equation}
(-1)^n \cos \nu = \frac{\alpha + \delta}{2},
\end{equation}
\begin{equation}
(-1)^n \frac{\sin \nu}{\nu} \Re a = \frac{\alpha - \delta}{2}, \quad (-1)^n \frac{\sin \nu}{\nu} b = \beta, \quad (-1)^n \frac{\sin \nu}{\nu} c = -\gamma.
\end{equation}

Note that (38) follows also directly from the formula \( \det(\Lambda) = \exp(\text{Tr} A \Lambda) \). Clearly, (39) has exactly two solutions in any half-closed interval of length \( 2\pi \). With any of these solutions we calculate \( \Re a, b, \) and \( c \) from (40). Since
\begin{equation}
b c - (\Re a)^2 = \frac{\nu^2}{\sin^2 \nu} \left( -\beta \gamma - \left( \frac{\alpha - \delta}{2} \right)^2 \right) = \frac{\nu^2}{\sin^2 \nu} \left( 1 - \left( \frac{\alpha + \delta}{2} \right)^2 \right) = \nu^2,
\end{equation}
the equations (39) and (40) are compatible with (40). Here, we used (3) and (39) in the second and the third equality, respectively.

**The case \((\alpha + \delta)/2 = \pm 1\).** In this case, \( \nu = m \pi \) with \( m \in \mathbb{Z} \). Since we can absorb the sign of \( (\alpha + \delta)/2 \) into the phase factor of \( \Lambda \), we may only focus on the case \( (\alpha + \delta)/2 = 1 \). Firstly, for \( m \neq 0 \), \( \exp(A) = \exp(i \Re a)(-1)^m I \). Therefore, if \( \Lambda = \exp(i \varphi I) \) then \( \exp(A) = \Lambda \) for arbitrary \( A \) that satisfies
\begin{equation}
b c - (\Re a)^2 = (m \pi)^2 \quad \text{and} \quad \exists a = \varphi + m \pi + 2n \pi
\end{equation}
with any \( n \in \mathbb{Z} \). If \( \Lambda \) is not a multiple of identity then there is no solution \( A \) with \( m \neq 0 \).

Secondly, let \( m = \nu = 0 \). Then \( n \) in (48) must be even so that (39) is fulfilled. Moreover, we have
\begin{equation}
\Re a = \frac{\alpha - \delta}{2}, \quad b = \beta, \quad c = -\gamma.
\end{equation}
instead of (40). This yields a valid solution because it is compatible with (36).

The case \((\alpha + \delta)/2 \in \mathbb{R} \setminus [-1, 1]\). In this case \(\nu = ig\), with \(g \in \mathbb{R} \setminus \{0\}\), and we have (38) together with

\[
(-1)^n \cosh g = \frac{\alpha + \delta}{2},
\]

\[
(-1)^n \sinh g a = \frac{\alpha - \delta}{2}, \quad (-1)^n \sinh g b = \beta, \quad (-1)^n \sinh g c = -\gamma.
\]

Clearly, \(n\) must be even for \((\alpha + \delta) > 0\) and odd in the other case. Now, the two solutions of (41) differ only by sign, so if we substitute either of them into (42) we arrive at the same formulae for \(Ra, b,\) and \(c\). Therefore, the only ambiguity remains in \(\Im a\) that is given modulo \(2\pi\). The compatibility of (41) and (42) with (36) follows from (5) again.

**Remark 5.1.** Using (35) one can infer that starting with \(A\) given by (34) one always ends up with \(\Lambda = \exp(A)\) of the form (4).

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