Reduction of Toda Lattice Hierarchy to Generalized KdV Hierarchies and Two-Matrix Model

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Abstract

Toda lattice hierarchy and the associated matrix formulation of the 2M-boson KP hierarchies provide a framework for the Drinfeld-Sokolov reduction scheme realized through Hamiltonian action within the second KP Poisson bracket. By working with free currents, which abelianize the second KP Hamiltonian structure, we are able to obtain an unified formalism for the reduced SL(M + 1, M − k)-KdV hierarchies interpolating between the ordinary KP and KdV hierarchies. The corresponding Lax operators are given as superdeterminants of graded SL(M + 1, M − k) matrices in the diagonal gauge and we describe their bracket structure and field content. In particular, we provide explicit free-field representations of the associated W(M, M − k) Poisson bracket algebras generalizing the familiar nonlinear W_{M+1}-algebra. Discrete Bäcklund transformations for SL(M + 1, M − k)-KdV are generated naturally from lattice translations in the underlying Toda-like hierarchy. As an application we demonstrate the equivalence of the two-matrix string model to the SL(M + 1, 1)-KdV hierarchy.

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1 Introduction

Integrable Hamiltonian systems occupy an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory of elementary particle’s interactions at ultra-high energies [1, 2, 3], including high-energy QCD in $D = 3+1$ space-time dimensions [4]. Among the most notable physically relevant integrable models is the Kadomtsev-Petviashvili (KP) hierarchy of integrable soliton nonlinear evolution equations [5, 6]. The main interest towards the KP hierarchy in the last few years stems from their deep connection with string (multi-)matrix models [7].

The general KP system is an $1+1$-dimensional integrable model containing an infinite number of fields. Its reductions with finite number of fields (“multi-boson KP hierarchies” for short) are, similarly, integrable systems which naturally appear in two different but related settings. As continuous (field-theoretic) integrable models they define a consistent Poisson reduction of the complete continuous KP system [8]. However, they can also be traced back to the discrete Toda lattice hierarchy [9, 10, 11, 12]. These two distinct formulations are in agreement due to the existence of a discrete symmetry for the continuous multi-boson KP hierarchy, which is a canonical mapping realized by a similarity transformation for the underlying Lax operators [13, 14]. The presence of this discrete transformation allows us to view the Toda lattice [15] as a union of sites, each being a gauge copy of one continuous multi-boson KP system. Throughout this paper we shall benefit essentially from this relation between the Toda lattice and the continuous multi-boson KP system.

It has long been known that the most simple example of the multi-boson KP systems, the two-boson KP system, contains the usual KdV hierarchy [16]. In [17] the Dirac reduction of the two-boson KP system was put up and the KdV hierarchy was obtained in this process. Subsequently, the Dirac reduction was applied to other multi-boson KP systems which resulted in a large family of generalized KP-KdV hierarchies [12, 18].

In this paper we investigate the reduction of the multi-boson KP hierarchies employing the Drinfeld-Sokolov (DS) reduction scheme realized in a non-conventional way – as Hamiltonian reduction within the second KP Poisson bracket structure. By working with free currents abelianizing the latter highly reducible Poisson structure, we are able to obtain an unified description of the various reduced hierarchies, their bracket structure and their field content. Let us recall that the KP hierarchy is endowed with bi-Hamiltonian Poisson bracket structures resulting from the two compatible Hamiltonian structures on the algebra of pseudo-differential operators [19]. The latter are given by:

\[
\begin{align*}
\{ \langle L|X \rangle , \langle L|Y \rangle \}_1 &= - \langle L | [X, Y] \rangle \\
\{ \langle L|X \rangle , \langle L|Y \rangle \}_2 &= \text{Tr}_A \left( (LX)_+ LY - (XL)_+ YL \right)
\end{align*}
\]  

(1.1) (1.2)

Here and below the following notations are used. $\langle \cdot | \cdot \rangle$ denotes the standard bilinear pairing via the Adler trace $\langle L|X \rangle = \text{Tr}_A (LX)$ with $\text{Tr}_A X = \int \text{Res} X$. Here $L, X, Y$ are arbitrary elements of the algebra of pseudo-differential operators of the form $L = \sum_{k \geq -\infty} u_k D^k$, $X = \sum_{k \geq -\infty} D^k X_k$, where $D = \partial / \partial x$ denotes the differential operator w.r.t. $x$. Furthermore, the subscripts $\pm$ in $X_\pm$ denote taking the purely differential or the purely pseudo-differential...
part of $X$, respectively. The Lax operator of the KP hierarchy has the following specific form $L = D + \sum_{k=1}^{\infty} u_k D^{-k}$, and therefore, the second Poisson bracket (1.2) is modified to:

$$\{ \langle L|X \rangle, \langle L|Y \rangle \}_2 = \text{Tr}_A \left( (LX)_+ LY - (XL)_+ YL \right)$$
$$+ \int dx \text{Res}(\left[ [L, X] \right]) \partial^{-1} \text{Res}(\left[ [L, Y] \right])$$

The last term is a Dirac-bracket term originating from the second-class constraint $u_0 = 0$.

The Toda lattice hierarchy in the matrix form is a central object in our study of Hamiltonian reduction. Up to a phase-gauge transformation, the Toda matrix of the associated linear problem has the form of a matrix in the DS gauge with an extra traceless diagonal part. Hence, there exists a residual gauge symmetry preserving the DS-like form of the Toda matrix. Correspondingly, the space of Toda matrices splits into orbits of this residual gauge group. The reduction is then accomplished in the final step by restricting to the symplectic quotient space (by quotienting out the residual gauge symmetry). In the case under consideration the final step involves removal of the diagonal terms (currents) of the Toda hierarchy matrix. One obtains in this process various generalized KP-KdV hierarchies with the usual KdV model corresponding to the case when all the diagonal currents are gauged away.

Such a reduction, when based on matrix calculations, becomes quickly cumbersome with the increasing rank of the matrices. The question is whether there exists a convenient way of handling the residual gauge transformations. The natural symplectic Kirillov-Kostant-Symes (KKS) form associated with the space of Toda matrices degenerates on the vector fields tangent to the orbits defined by the residual gauge transformations. Hence, one does not expect the action of the residual gauge transformation to be Hamiltonian as there is no natural KKS-type Poisson bracket associated with the linear Toda matrix problem before the last step of Hamiltonian reduction is taken. Surprisingly, it turns out that the relevant gauge group action is nevertheless Hamiltonian but with respect to the second Poisson bracket (1.3) of the multi-boson KP system. This enables us to describe the DS reduction within the framework of the Poisson manifolds and to find closed expressions for the gauge transformed quantities on the reduced manifold.

The abelianized representation, i.e., the representation in terms of free currents, of the second Poisson bracket \[20\] plays a key rôle in the above formalism. Here the underlying lattice structure is the Volterra lattice and the problem is transformed from the DS gauge of the Toda hierarchy to the diagonal gauge. In this representation the residual gauge transformations take a simple form and the constraint manifold is described directly in terms of the original abelian canonical variables. It is crucial that the second bracket structure is reducible and the residual gauge symmetry of DS problem triggers a total factorization of the bracket structure.

The corresponding Lax operators appearing on various levels of reduction are constructed in terms of currents spanning the Cartan subalgebra of the graded $SL(M+1, M-k)$ Kac-Moody algebra \[21\]. The variable $k$ labels the level of reduction with $k = 0$ corresponding to the original $2M$-boson KP system and $k = M$ describing the maximal reduction to the ordinary $SL(M+1)$-KdV hierarchy. In the latter case the Lax operator reduces to the simple determinant of the Fateev-Lukyanov type \[22\]. Hence, we obtain an unified formalism for the reduced $SL(M+1, M-k)$-KdV hierarchies (KP-KdV hierarchies), which interpolate between
the original KP systems and the ordinary KdV hierarchies. The generic Lax operators are
given as superdeterminants of the graded $SL(M + 1, M - k)$ matrices in the diagonal gauge.
We describe their bracket structure. Thanks to our abelianization technique we are also able
to give a free-field construction for all dynamical variables of the $SL(M + 1, M - k)$-KdV
hierarchies.

The generalized $SL(M + 1,1)$-KdV hierarchies have recently been encountered in the
study of the two-matrix model $[23, 24]$. In $[24]$ the simplest nontrivial Toda-like lattice
integrable system, derived from the partition function of the two-matrix model with matrix
potentials of orders $p_1 = \text{arbitrary}$ and $p_2 = 3$, was shown to be equivalent to the $1 + 1$
dimensional generalized $SL(3,1)$-KdV hierarchy. In this paper we extend the above analysis
to the case of arbitrary finite $p_2$.

The organization of the material is as follows. In Section 2 we recapitulate the basic facts
about the Toda lattice hierarchy and the matrix approach to the spectral problem versus the
continuum multi-boson KP hierarchy. In Section 3 we compare Dirac and DS reductions of
the two-boson KP hierarchy and present DS reduction for the four-boson hierarchy. Next,
in Section 4 we show that the residual gauge transformation has a Hamiltonian action with
respect to the second KP Poisson bracket and discuss how the DS reduction, described in the
previous section, is induced in this Hamiltonian manner. These results are generalized to an
arbitrary multi-boson KP hierarchy in Section 5, where use is made of a set of free currents
abelianizing the second Poisson bracket structure. These currents enter into the Lax operator
in a form which naturally leads to the graded $SL(M+1,M)$ Kac-Moody algebra. In Section
6 the reduction process is shown to be equivalent to reducing the graded $SL(M+1,M)$
algebra to $SL(M+1,M-k)$ algebra. This framework allows for a simple expression for
the second bracket structure of the $SL(M+1,M-k)$-KdV hierarchy in terms of Lax
operators being superdeterminants of the graded $SL(M+1,M-k)$ matrices in the diagonal
gauge. Also, we show that the reduced generalized KP-KdV hierarchies are integrable (bi-
Hamiltonian) and possess canonical discrete symmetries. Our construction provides explicit
free-field representations of the associated $W(M,M-k)$ Poisson bracket algebras generalizing
the well-known nonlinear $W_{M+1}$ algebra $[25]$. Discrete Bäcklund transformations for $SL(M+
1,M-k)$-KdV are generated naturally from lattice translations in the underlying Toda-like
hierarchy. Finally, as an application we demonstrate in Section 7 the equivalence of the
two-matrix string model to the $SL(p_1,1)$-KdV hierarchy, where $p_{1,2}$ ($p_1 \leq p_2$) are the orders
of the matrix-model potentials.

2 Toda Hierarchy versus Multi-Boson KP Hierarchy.

2.1 Toda Hierarchy and Matrix Approach to the Spectral Problem.

We start with the spectral equation:

$$
\begin{align*}
\partial \Psi_n &= \Psi_{n+1} + a_0(n) \Psi_n \\
\lambda \Psi_n &= \Psi_{n+1} + a_0(n) \Psi_n + \sum_{k=1}^{M} a_k(n) \Psi_{n-k}
\end{align*}
$$

(2.1)
associated with the Toda lattice hierarchy (for the most general case, see ref.[26]). Here
\[ \partial \equiv \partial_x \equiv \partial/\partial t_{1,1}, \]
where \( t_{1,1} \) denotes the first lattice evolution parameter which is now
considered as a space coordinate of an \( 1 + 1 \)-dimensional integrable system. The spectral
equation (2.1) can be rewritten as a matrix equation
\[ \left( \begin{array}{cccc}
\partial - a_0(n - M) & -1 & 0 & \cdots & 0 \\
0 & \partial - a_0(n - M + 1) & -1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \partial - a_0(n - 1) & -1 \\
a_M(n) & a_{M-1}(n) & \cdots & a_1(n) & \partial - \lambda
\end{array} \right) \begin{pmatrix}
\Psi_{n-M} \\
\Psi_{n-M+1} \\
\vdots \\
\Psi_{n-1} \\
\Psi_n
\end{pmatrix} = 0 
\]
(2.2)
Note first that by eliminating all \( \Psi_{n-i}, i \neq 0 \) from the set of equations represented by (2.2)
we obtain:
\[ \lambda \Psi_n = L_n^{(M)} \Psi_n 
\]
(2.3)
where
\[ L_n^{(M)} = \partial + \sum_{k=1}^{M} a_k(n) \frac{1}{\partial - a_0(n - k)} \cdots \frac{1}{\partial - a_0(n - 1)} 
\]
(2.4)
The matrix form (2.2) of the Toda spectral problem will be a starting point of our discussion
of DS reduction. We first perform a phase gauge transformation:
\[ \Psi_{n-k} \rightarrow \exp \left( -\frac{1}{M + 1} \sum_{i=1}^{M} \int a_0(n - i) \right) \Psi_{n-k} \quad , \quad 0 \leq k \leq M 
\]
(2.5)
which transforms the diagonal terms in (2.2) to:
\[ \partial - a_0(n - i) \rightarrow \partial + \frac{1}{M + 1} \sum_{j=1,j\neq i}^{M} a_0(n - j) - \frac{M}{M + 1} a_0(n - i) \]
(2.6)
\[ \partial - \lambda \rightarrow \partial - \lambda + \frac{1}{M + 1} \sum_{j=1}^{M} a_0(n - j) 
\]
This transformation renders the matrix \( Q \) traceless (for \( \lambda = 0 \)). Let us denote by
\[ M\Psi \equiv \left( \partial - \bar{Q} \right) \Psi = 0 \quad , \quad \bar{Q} = \mathcal{E} - \omega \quad , \quad \mathcal{E}_{ij} = \delta_{i+1,j} \quad , \quad \text{tr} \omega = 0 
\]
(2.7)
the equation obtained from (2.2) by the above phase transformation accompanied by setting
\( \lambda = 0 \). Now, one can follow the ideas of the DS Hamiltonian reduction scheme [27, 28].
Indeed, the space of Toda matrices \( \mathcal{O}_{Toda} = \left\{ \bar{Q} ; \bar{Q} \ as \ as \ in \ (2.7) \right\} \) can be viewed as a sub-
manifold of the phase space of \( SL(M+1) \) WZNW model, which is a coadjoint orbit of
\( SL(M+1) : \)
\[ \mathcal{O}_G = \left\{ S(g) = \partial g g^{-1} : g \in G = SL(M+1) \right\} 
\]
(2.8)
Thus, this submanifold \( \mathcal{O}_{Toda} \) corresponds to a partial gauge fixing of the first-class
constraints \( \Phi \equiv \partial g g^{-1}\big|_{g^+} = \mathcal{E} = 0 \quad , \quad g \in SL(M+1), \) in a gauged \( SL(M+1) \) WZNW model
whose “big” phase space is (2.8). Here as usual $G_\pm \subset G = sl(M + 1)$ denote the nilpotent upper/lower-triangular subalgebras. Therefore, there exists a residual $M$-dimensional gauge symmetry group $\Gamma \subset SL(M + 1), \ h \in \Gamma$:

$$\mathcal{M} \equiv \partial - \mathcal{E} + \omega \rightarrow h^{-1} \mathcal{M} h = \partial - \mathcal{E} + \bar{\omega}$$

(2.9)

which preserves the form of the $\omega$ matrix and defines a gauge orbit for the Toda lattice hierarchy in the matrix form. The natural symplectic form (KKS form) on the “big” phase space (2.8) degenerates on the vector fields tangent to the gauge orbits (2.9). Hence, there is no natural KKS-type Poisson bracket structure for the coefficients $a_0(n - l), a_i(n), \ l = 1, \ldots, M$ of $\bar{Q}$ associated with the linear matrix problem (2.7). Consequently, a priori one does not expect the action of the residual gauge transformations (2.9) to be Hamiltonian.

We shall find later on that this gauge action can nevertheless be realized as a Hamiltonian action, namely, it is generated by the second KP Poisson bracket (1.3) for the $2M$-boson KP Lax operator (2.4) inherent to the Toda matrix spectral problem (2.2).

The determinant of $\mathcal{M}$ in (2.7) can be written as:

$$\det \mathcal{M} = \partial^{M+1} + u_{M-1}\partial^{M-1} + \ldots + u_1\partial + u_0$$

(2.10)

and clearly is invariant under (2.9). The differential operator $\det \mathcal{M}$ can also be obtained from $\mathcal{M}\Psi = 0$ (2.1) by eliminating all $\Psi_{n-M+1}, \ldots, \Psi_n$ apart from $\Psi_{n-M}$. Generally we obtain a family of Lax operators in the process of eliminating all $\Psi$’s apart from one element of the column in (2.2), which we denote by $\Psi_{n-i_0}$. The case $i_0 = M$ is the “pure” KdV Lax operator from (2.10) while $i_0 = 0$ gives the KP Lax operator $L_n^{(M)}$ from (2.4) (up to a phase-gauge transformation). For $0 < i_0 < M$ we get a family of Lax operators invariant under various subgroups of the residual gauge symmetry (2.9). We shall implement in this paper the DS reduction scheme in the above mentioned non-conventional setting – as a Hamiltonian reduction with respect to the second KP Poisson bracket, to describe this family of Lax operators contained in the linear system of (2.2).

### 2.2 KP Hierarchy: the First Bracket

From (2.1) we obtain the consistency conditions:

$$\partial a_0(n) = a_1(n + 1) - a_1(n)$$
$$a_k(n) = a_k(n - 1) + \left(\partial + a_0(n - k) - a_0(n - 1)\right)a_{k-1}(n - 1) \quad k = 1, \ldots, M$$
$$\partial a_M(n) = a_M(n) (a_0(n) - a_0(n - M))$$

(2.11) (2.12) (2.13)

which are the Toda equations of motion. It is easy to see that all $a_k(n), \ k = 0, 1, \ldots, M$, at each lattice site $n$, are expressed as functionals of only $2M$ independent functions, which can be chosen to be, e.g.: $a_0(M - k)$ and $a_k(M), \ k = 1, \ldots, M$.

We shall now relate the Toda lattice equation (2.12) to the recurrence relations for the $2M$-boson KP Lax operators derived in (8) and used there to abelianize the first Poisson bracket structure (1.1). Let us introduce the following correspondence (with $n = M$ in
where $X, Y$ and $P_{\text{subscript}}$.

From (2.12) we find the recurrence relation

$$B_i(M) = a_0(l - 1) - a_0(M - 1)$$

(2.15)

Define now $a_M \equiv A_i^{(M)}(M)$ and $b_M \equiv B_i^{(M)} \sim a_0(M - 1)$. As a consequence of the last definition and (2.13) we find the recurrence relation $B_i^{(M)} = b_M + B_i^{(M-1)}$. Furthermore, we notice that identifications made in (2.14)-(2.15) allow to recast the lattice Toda equation of motion (2.12) in the form of recurrence relations:

$$A_i^{(M)} = A_{i-1}^{(M-1)} + (\partial + B_i^{(M-1)}) A_i^{(M-1)}$$

(2.16)

(l = 2, ..., M - 1)

Assuming furthermore that $A_0^{(M)} = a_{M+1}(M)$ we get in addition:

$$A_1^{(M)} = (\partial + B_1^{(M-1)}) A_1^{(M-1)}$$

(2.17)

The above recurrence relations have been shown [8] to be equivalent to the recursive formula for the $2M$-boson KP Lax operator valid for arbitrary $M = 1, 2, 3, \ldots$ (with $L_0 \equiv D, a_0 \equiv 0$):

$$L_M \equiv L_M(a, b) \equiv L_M(a_1, b_1; \ldots; a_M, b_M)$$

(2.18)

$$L_M = e^{b_M} [b_M + (a_M - a_{M-1}) D^{-1} + D L_{M-1} D^{-1}] e^{-b_M}$$

In fact, the solution to (2.18) is the 2M-field Lax operator of the form of (2.4) [11]:

$$L_M = D + \sum_{i=1}^{M} A_i^{(M)} (D - B_i^{(M)})^{-1} (D - B_{i+1}^{(M)})^{-1} \cdots (D - B_{M}^{(M)})^{-1}$$

(2.19)

with coefficients satisfying (2.16) and (2.17). As a result, the latter are expressed in terms of the free boson fields $(a_r, b_r)_{r=1}^{M}$ spanning Heisenberg Poisson bracket algebra:

$$\{a_r(x), b_s(y)\}_{\mathbb{L}'} = -\delta_{rs} \partial_x \delta(x - y)$$

(2.20)

as

$$B_i^{(M)} = \sum_{s=1}^{M} b_s$$

(2.21)

$$A^{(M)}_{M-r} = \sum_{n_r=r}^{M-1} \sum_{n_{r+1}=n_r}^{M-1} \sum_{n_{r+2}=n_{r+1}}^{M-1} \cdots (\partial + b_{n_r} + \cdots + b_{n_{r-r+1}}) \cdots (\partial + b_{n_2} + b_{n_1} - 1) a_{n_1}$$

This recursive construction of the Lax in (2.19) leads to the following [8]:

**Proposition.** The $2M$-field Lax operators (2.19) are consistent Poisson reductions of the full KP Lax operator for any $M = 1, 2, 3, \ldots$.

Thus, the first Poisson bracket structure for $L_M$ from (2.19) is given by:

$$\{\langle L_M \mid X \rangle, \langle L_M \mid Y \rangle\}_{\mathbb{L}'} = -\langle L_M \mid [X, Y] \rangle$$

(2.22)

where $X, Y$ are arbitrary fixed elements of the algebra of pseudo-differential operators. The subscript $P'$ in (2.22) indicates that the constituents of $L_M(a, b)$ satisfy (2.20).
3 Reductions of the Two-Boson KP Hierarchy to KdV

3.1 Two-boson KP Hierarchy and the Dirac Reduction

We shall consider here truncated elements of the KP hierarchy providing the simplest example of (2.19) and given by Lax operator of the form

\[ L_1 = D + a(D - b)^{-1} \]  

(3.1)

with two free Bose currents \((a, b)\) \(\text{[16, 29]}\). The Lax operator can be cast into the standard form

\[ L_1 = D + \sum_{n=0}^{\infty} w_n D^{-1-n} \]

with coefficients \(w_n = (-1)^n a(D - b)^n \cdot 1\). A calculation of the Poisson bracket structures using definition (1.1) and (2.22) yields the first bracket structure of two-boson \((a, b)\) system:

\[ \{a(x), b(y)\}_1 = -\delta'(x - y), \text{ and zero otherwise.} \]

This implies that the coefficients \(w_n(a,b)\) of \(L_1\), as functionals of \(a,b\), satisfy the linear \(W_1^\infty\) Poisson-bracket algebra. The second bracket structure (1.3) takes in this case the form:

\[ \{a(x), b(y)\}_2 = -2 a(x) \delta'(x - y) - a'(x) \delta(x - y) \]  

(3.2)

\[ \{b(x), b(y)\}_2 = -\delta'(x - y) \]

Now, based on this bracket, \(w_n(a,b)\) satisfy the nonlinear \(\hat{W}_\infty\) Poisson-bracket algebra.

In \([17, 30]\) we applied the Dirac reduction scheme to obtain one-boson KdV hierarchy from the two-boson KP hierarchy. The general feature is a transformation of the two-boson Hamiltonian equations of motion expressed in terms of the 2-nd bracket structure \(\delta Z/\delta t_r = \{Z, H_r\}_2\) (where \(Z\) denotes the original degrees of freedom) to one-boson Hamiltonian system according to the Dirac scheme:

\[ \frac{\partial X}{\partial t_r} = \{X, H_r^D\}_\text{Dirac} \]  

(3.3)

with \(X\) denoting a surviving one-boson degree of freedom.

Consider the Dirac constraint: \(\Theta \equiv b = 0\) for the system described by \(L_1\). First, let us discuss the resulting Dirac bracket structure. We find for the surviving variable \(a\):

\[ \{a(x), a(y)\}_2^D = \{a(x), a(y)\}_2 - \int dz dz' \{a(x), \Theta(z)\}_2 \{\Theta(z), \Theta(z')\}_2^{-1} \{\Theta(z'), a(y)\}_2 \]

\[ = -(2a(x) \partial + a'(x) + \frac{1}{2} \partial^3) \delta(x - y) \]  

(3.4)

The reduced Lax operator looks now as:

\[ l = D + aD^{-1} \]  

(3.5)

and the corresponding (non-zero) lowest Hamiltonian functions \(H_r^{KdV} \equiv \text{Tr} l^r/r\) are

\[ H_1^{KdV} = \int a \quad ; \quad H_3^{KdV} = \int a^2 \quad ; \quad H_5^{KdV} = \int \left(2a^3 + aa''\right) \]  

(3.6)
Moreover one checks that the flow equation:

$$\delta l/\delta t_r = \left[ (l^r)_+, l \right]$$

(3.7)
gives on the lowest level

$$\delta a/\delta t_1 = a'$$

and

$$\delta a/\delta t_3 = a''' + 6aa',$$

where the last equation is the well-known KdV equation.

We shall now demonstrate that the DS reduction is an alternative to the Dirac reduction of the two-boson KP hierarchy to the usual KdV hierarchy \[30\].

### 3.2 Matrix form of Two-Boson KP Hierarchy and the DS Reduction

One can associate \(sl(2)\) matrices to pseudo-differential Lax operators in the following way \[30\] :

$$L = D + A + B D^{-1} C \sim A = \begin{pmatrix} -\frac{1}{2}A & -C \\ B & \frac{1}{2}A \end{pmatrix}$$

(3.8)

so that the gauge transformation of the Lax operator \(L' = e^{-\chi}Le^\chi\) corresponds to \(SL(2)\) gauge transformation \(A' = gAg^{-1} + g\partial g^{-1}\) with a diagonal \(2 \times 2\)-real unimodular matrix \(g = \text{diag}(\exp \chi/2, \exp -\chi/2)\).

To see the connection with the matrix Toda hierarchy (with \(\lambda = 0\)):

$$\left( \begin{array}{cc} \partial - a_0(n - 1) & -1 \\ a_1(n) & \partial \end{array} \right) \left( \begin{array}{c} \Psi_{n-1} \\ \Psi_n \end{array} \right) = 0$$

(3.9)

let us introduce new variables as in (2.5):

$$\begin{pmatrix} e^{\frac{\chi}{2}} \int a_0(n-1) \Psi_{n-1} \\ e^{\frac{-\chi}{2}} \int a_0(n-1) \Psi_n \end{pmatrix}$$

(3.10)

and denote \(a_0(n - 1) = b, a_1(n) = a\). According to (3.8), we find the association:

$$L_{KP} = D + b + a D^{-1} \sim A_{KP} = \begin{pmatrix} -\frac{1}{2}b & -1 \\ a & \frac{1}{2}b \end{pmatrix}$$

(3.11)

Here, the important point is that there is a residual gauge transformation generated by:

$$g_0 \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

(3.12)

which preserves the form of \(A_{KP}\) under:

$$A' = g_0^{-1}Ag_0 + g_0^{-1}\partial g_0 = \begin{pmatrix} -\frac{1}{2}b - \gamma \\ a + \gamma b + \gamma' + \frac{1}{2}b + \gamma \end{pmatrix}$$

(3.13)

Let us analyze what is happening by using the DS formalism. Consider the space of first-order differential operators with coefficients being \(2 \times 2\) matrices:

$$M_\mathcal{E} = \left\{ \mathcal{D}^{(1)} = D - \mathcal{E} + \omega \mid \mathcal{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_2 & \omega_2 \end{pmatrix} \right\}$$

(3.14)
and the group
\[ \Gamma \equiv \left\{ \Gamma \mid \Gamma \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \right\} \] (3.15)
acting on \( M_E \) according to
\[ \Gamma^{-1} (D - \mathcal{E} + \omega) \Gamma = D - \mathcal{E} + \bar{\omega} \] (3.16)
with
\[ \bar{\omega} = \begin{pmatrix} \omega_{11} - \gamma & 0 \\ \omega_{21} + \gamma (\omega_{22} - \omega_{11}) + \gamma^2 + \gamma' & \omega_{22} + \gamma \end{pmatrix} \] (3.17)
In the spirit of Hamiltonian reduction consider the quotient space \( M_{\text{red}} = M_E / \Gamma \). There exists a convenient realization of \( M_{\text{red}} \) in terms of second-order differential operators with scalar coefficients. The procedure to obtain it goes as follows. Consider the equation:
\[ \mathcal{D}^{(1)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \] (3.18)
Eliminating \( \psi_2 \) from this equation we arrive at \( L^{(2)} \psi_1 = 0 \) with
\[ L^{(2)} \equiv \det \left( \mathcal{D}^{(1)} \right) = D^2 + (\omega_{11} + \omega_{22})D + \omega_{21} + \omega_{11}\omega_{22} + \omega'_{11} \] (3.19)
Clearly \( \det \left( \Gamma^{-1} \mathcal{D}^{(1)} \Gamma \right) = \det \left( \mathcal{D}^{(1)} \right) \) and, therefore, the space of the second order differential operators of the form (3.19) parametrizes the quotient space \( M_{\text{red}} \).

Let us study now the special case of two-boson \( sl(2) \) matrix:
\[ \omega = \begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}b & 0 \\ a & \frac{1}{2}b \end{pmatrix} \] (3.20)
For \( \Gamma \) with \( \gamma = -\frac{1}{2}b \) the transformed \( \bar{\omega} \) matrix
\[ \bar{\omega} = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a - \frac{1}{4}b^2 - \frac{1}{2}b' & 0 \end{pmatrix} \] (3.21)
has diagonal elements equal to zero. It means that, according to (3.11), the associated Lax operator is:
\[ L_{KP} = D + uD^{-1} \quad \text{with} \quad u = a - \frac{1}{4}b^2 - \frac{1}{2}b' \] (3.22)
One can check that, with \( (a, b) \) satisfying the second Poisson bracket (3.2), \( u \) commutes with \( b \) and satisfies the Virasoro algebra:
\[ \{ u(x), u(y) \} = -2u(x) \delta'(x - y) - u'(x) \delta(x - y) - \frac{1}{2} \delta'''(x - y) \] (3.23)
We also note that with \( \omega \) like in (3.20) the second-order differential operator (3.19) becomes a typical KdV operator \( L^{(2)} = D^2 + u \). Hence, the first-order DS operator \( \mathcal{D}^{(1)} \) (3.14) with \( \bar{\omega} \) as in (3.21), or its associated Lax operator \( L_{KP} \) from (3.22), represent just a special gauge choice on \( M_E \) equivalent to the KdV Lax operator \( L^{(2)} \).
3.3 Drinfeld-Sokolov Reductions of Four-Boson KP Hierarchy

Start again with the Toda matrix problem with \( \lambda = 0 \)

\[
\begin{pmatrix}
\partial - a_0(n-2) & -1 & 0 \\
0 & \partial - a_0(n-1) & -1 \\
a_2(n) & a_1(n) & \partial
\end{pmatrix}
\begin{pmatrix}
\Psi_{n-2} \\
\Psi_{n-1} \\
\Psi_n
\end{pmatrix} = 0 \tag{3.24}
\]

Consider a general space of first order differential operators with coefficients being \( 3 \times 3 \) matrices:

\[ M_{\mathcal{E}} = \{ \mathcal{D}^{(2)} \equiv D - \mathcal{E} + \omega \} \tag{3.25} \]

with

\[ \mathcal{E} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.26} \]

and

\[ \omega = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ A_1 & A_2 & J_3 \end{pmatrix} \tag{3.27} \]

Eliminating \( \psi_1, \psi_2 \) from the equation:

\[ \mathcal{D}^{(2)} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0 \tag{3.28} \]

we get the Lax operator of the four-boson KP hierarchy:

\[
\left( \partial + A_2 \frac{1}{\partial + J_2 - J_3} + A_1 \frac{1}{\partial + J_1 - J_3} \frac{1}{\partial - J_2 - J_3} \right) e^{-\int J_3 \psi_3} = 0 \tag{3.29}
\]

while eliminating \( \psi_2, \psi_3 \) we get a KdV-type Lax operator:

\[
\left( \partial^3 + u_0 \partial^2 + u_1 \partial + u_2 \right) \psi_1 = 0 \tag{3.30}
\]

\[
u_0 \equiv J_1 + J_2 + J_3
\]

\[
u_1 \equiv A_2 + 2J_1' + J_2' + J_1J_2 + J_1J_3 + J_2J_3
\]

\[
u_2 \equiv A_1 + A_2J_1 + J_1'' + (J_1J_2)' + J_1'J_3 + J_1J_2J_3
\]

Applying similar arguments as in subsection 3.2, we are led to consider a group consisting of lower triangular \( 3 \times 3 \) matrices

\[
\Gamma \equiv \left\{ \Gamma \mid \Gamma \equiv \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \alpha_2 & \alpha_3 & 1 \end{pmatrix} \right\} \tag{3.31}
\]

with an inverse

\[
\Gamma^{-1} \equiv \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_1 & 1 & 0 \\ -\alpha_2 + \alpha_3 \alpha_1 & -\alpha_3 & 1 \end{pmatrix} \tag{3.32}
\]
The action of \( \Gamma \) on \( M \) according to (3.16) produces the following transformation of \( \omega \) (3.27):

\[
\omega \longrightarrow \bar{\omega} = \begin{pmatrix} J_1 - \alpha_1 & 0 & 0 \\ \bar{\omega}_{21} & J_2 + \alpha_1 - \alpha_3 & 0 \\ \bar{\omega}_{31} & \bar{\omega}_{32} & J_3 + \alpha_3 \end{pmatrix}
\] (3.33)

where

\[
\begin{align*}
\bar{\omega}_{21} &= \alpha' + \alpha_2^2 - \alpha_2 + \alpha_1(J_2 - J_1) \\
\bar{\omega}_{31} &= A_1 + A_2 + \alpha_1 + \alpha_2 + \alpha_3 + (J_3 - J_2) + \omega_2(\alpha_1 + \alpha_3 + J_3 - J_1) - \alpha_3(\alpha_2^2 + \alpha_1') \\
\bar{\omega}_{32} &= A_2 + \alpha_3 + \alpha_3(J_3 - J_2 - \alpha_1) + \alpha_2 + \alpha_3^2
\end{align*}
\] (3.34)

\( \Gamma \) will define the little group preserving the form of \( \mathcal{D}^{(2)} \) if we impose \( \bar{\omega}_{21} = 0 \), i.e.,

\[
\alpha_2 = \alpha_1' + \alpha_1^2 + \alpha_1(J_2 - J_1)
\] (3.35)

Hence the little group has only two independent components \( \alpha_1, \alpha_3 \) and the transformed matrix \( \bar{\omega} \) takes the form:

\[
\bar{\omega} = \begin{pmatrix} J_1 - \alpha_1 & 0 & 0 \\ 0 & J_2 + \alpha_1 - \alpha_3 & 0 \\ \bar{A}_1 & \bar{A}_2 & J_3 + \alpha_3 \end{pmatrix}
\] (3.36)

with

\[
\begin{align*}
\bar{A}_2 &= A_2 + \alpha_1' + \alpha_3^2 + \alpha_1^2 + \alpha_1(J_2 - J_1 - \alpha_3) + \alpha_3(J_3 - J_2) \\
\bar{A}_1 &= A_1 + \alpha_2'' + (\alpha_1' + \alpha_1^2)(J_2 + J_3 - 2J_1) \\
&\quad + \alpha_1 \left[ A_2 + \alpha_1^2 + 3\alpha_3 + J_1' + (J_2 - J_1)(J_3 - J_1) \right]
\end{align*}
\] (3.37)

Let us return to the Toda problem (3.24) whose matrix differential operator belongs to the space \( M\mathcal{E} \) (3.25). In order to agree with future convention let us introduce the notations:

\[
a_0(n - 1) \equiv B_2 \quad a_0(n - 2) \equiv B_1 \quad a_1(n) \equiv A_2 \quad a_2(n) \equiv A_1
\] (3.38)

We can again gauge away the trace of the matrix appearing in the spectral problem by letting \( \Psi \rightarrow \exp(-f(B_2 + B_1)/3) \Psi \). This results in a differential operator \( \mathcal{D}^{(2)} \) (3.24) with traceless matrix \( \omega \) (3.27) where:

\[
J_1 = -(2B_1 - B_2)/3 \quad J_2 = -(2B_2 - B_1)/3 \quad J_3 = (B_2 + B_1)/3
\] (3.39)

We can choose the two independent parameters \( \alpha_1, \alpha_3 \) in \( \Gamma \) (3.31) to eliminate diagonal elements in (3.33) by taking:

\[
\alpha_1 = -\frac{1}{3}(2B_1 - B_2) \quad \alpha_3 = -\frac{1}{3}(B_2 + B_1)
\] (3.40)

In this case we arrive at the standard DS-gauge for \( \mathcal{D}^{(2)} \):

\[
\mathcal{D}^{(2)} = \Gamma^{-1} (D - \mathcal{E} + \omega) \Gamma = \begin{pmatrix} D & -1 & 0 \\ 0 & D & -1 \\ \bar{A}_1 & \bar{A}_2 & D \end{pmatrix} \quad \text{det} \mathcal{D}^{(2)} = \partial^3 + \bar{A}_2 \partial + \bar{A}_1
\] (3.41)
with
\[
\tilde{A}_2 = A_2 - B_1' - \frac{1}{3} \left( B_2^2 + B_1^2 - B_2 B_1 \right) \\
\tilde{A}_1 = A_1 - \frac{1}{3} (2B_1' - B_2') - \frac{1}{3} (2B_1 - B_2) (A_2 + B_1' - B_2' + (B_1 - B_2)B_1) + \frac{2}{27} (2B_1 - B_2)^3
\] (3.42)

From the second Poisson-bracket algebra of the four-boson KP hierarchy, satisfied by \( A_{1,2}, B_{1,2} \), we find that \( \tilde{A}_2, \tilde{A}_1 \) satisfy the Poisson brackets of Bussinesq hierarchy (the \( W_3 \) algebra) (see e.g. [12])

\[
\{ \tilde{A}_2(x), \tilde{A}_2(y) \}_2 = - \left( 2\tilde{A}_2 \partial + \tilde{A}_2' + 2\partial^3 \right) \delta(x - y) \\
\{ \tilde{A}_2(x), \tilde{A}_1(y) \}_2 = - \left( 3\tilde{A}_1 \partial + 2\tilde{A}_1' - \partial^2 \tilde{A}_2 - \partial^4 \right) \delta(x - y) \\
\{ \tilde{A}_1(x), \tilde{A}_1(y) \}_2 = - \left( 2\tilde{A}_1' \partial + \tilde{A}_1'' - \frac{2}{3} (\tilde{A}_2 + \partial^2)(\partial \tilde{A}_2 + \partial^3) \right) \delta(x - y)
\] (3.43)

The above reduction was complete in the sense that all the diagonal elements were removed. In general, there are plenty of possibilities for a partial reduction with, e.g., one diagonal element surviving the reduction. We now describe one such a choice. Note first that in the Toda hierarchy problem with a general \( \omega \) the diagonal terms \( J_i \) can be represented as:

\[
J_1 = -B_1 + \bar{B} \ ; \quad J_2 = -B_2 + \bar{B} \ ; \quad J_3 = \bar{B}
\] (3.44)

where the term \( \bar{B} \) indicates an addition which can be generated by applying a gauge transformation (which effectively changes the trace of the underlying matrix).

The particular choice of \( \alpha_1, \alpha_3, \bar{B} \) we now make is to obtain in (3.36):

\[
J_1 - \alpha_1 = \frac{1}{2} (B_2 + B_1) \ ; \quad J_2 + \alpha_1 - \alpha_3 = 0 \ ; \quad J_3 + \alpha_3 = 0
\] (3.45)

The solution is \( \alpha_3 = -\bar{B} = -(B_2 + B_1)/2 \) and \( \alpha_1 = -B_1 \). Inserting these parameters into (3.37) we get by partial DS reduction a hierarchy with three fields \( \tilde{A}_2, \tilde{A}_1, \bar{B} = -(B_2 + B_1)/2 \) where

\[
\tilde{A}_2 = A_2 - \frac{1}{2} (B_2' + 3B_1') - \frac{1}{4} (B_2 - B_1)^2 \\
\tilde{A}_1 = A_1 - B_1' + (B_1 B_2)' - B_1 A_2
\] (3.46)

We shall provide below (see (4.7)) the closed Poisson-bracket algebra satisfied by these fields reproducing result obtained in [12] by a Dirac method.

### 4 Drinfeld-Sokolov Reduction Induced by the KP Poisson Structure

It turns out that the action of the residual gauge symmetry discussed above in the case of two- and four-boson KP hierarchies is Hamiltonian (generated by a Poisson bracket structure). The relevant bracket structure turns out to be the second KP Poisson structure (1.3). Before presenting the general framework we start with examples of two- and four-boson KP hierarchies to reproduce results of the DS reduction.
4.1 The Case of Two-Bosons

Recall the second bracket structure for two-boson system described by (3.2) and consider the abelian group generated by

\[ G = \exp \left( - \int b \beta \right) \]  

(4.1)

which acts on \((a, b)\) through the second bracket, e.g.:

\[ G^{-1} b(x)G = b(x) - \{b(x), \int b \beta \} + \frac{1}{2} \{\{b(x), \int b \beta \}, \int b \beta \} + \ldots \]  

(4.2)

A simple calculation shows that the result of such an action on \((a, b)\) is:

\[ b \to G^{-1} b(x)G = b + 2 \beta' \quad ; \quad a \to G^{-1} a(x)G = a + b \beta'' + (\beta')^2 \]  

(4.3)

Choosing \(\beta' = \gamma\) we see that (4.3) reproduces the result in (3.13) as we set up to show. Since this transformation is generated by the current \(b(x)\) satisfying Heisenberg Poisson-bracket algebra (last eq. (3.2)), we see that transforming \(b \to 0\) by the above transformation amounts to imposition of the Dirac constraint \(b = 0\). This explains why the above DS gauge choice in (3.21) leading to KdV was equivalent to taking the Dirac bracket. This explanation is based on the curious equivalence between transformations generated by lower triangular matrices acting on matrices representing KP hierarchy and transformations generated by currents acting on the second bracket Poisson manifold of the multi-boson KP hierarchy.

Note, that the quantity 

\[ u = a - \frac{1}{4} b^2 - \frac{1}{2} b' \]  

parametrizing the reduced manifold is invariant under the transformation (4.3) since \(\{u, b\} = 0\), the point which is obvious in matrix formulation but which can only be verified a posteriori for the current generated transformations. This fact is crucial in explaining why from algebraic point of view the Dirac conditions \(b = 0, a = u\) agree with the gauge fixing as done above. We shall see that this relationship is of general nature and applies to arbitrary multi-boson KP systems.

4.2 The Case of Four-Bosons

In the case of four-boson KP hierarchy consider a group action on the KP Poisson manifold generated by:

\[ G = \exp - \int (B_2(\gamma_3 - \gamma_1) + B_1\gamma_1) \]  

(4.4)

Our choice of the \(\gamma\)-parameters is dictated by the conventions of the corresponding DS reduction as shown above. The action of \(G\) gives (e.g., \(G^{-1} B_2 G\) etc.; cf. (4.2)):

\[
\begin{align*}
B_2 &\to B_2 + 2\gamma_3' - \gamma_1' \\
B_1 &\to B_1 + \gamma_3' + \gamma_1' \\
A_2 &\to A_2 + \gamma_3'' + \gamma_1'' + B_2(\gamma_3' - \gamma_1') + B_1\gamma_1 + (\gamma_3')^2 + (\gamma_1')^2 - \gamma_1'\gamma_3' \\
A_1 &\to A_1 + A_2\gamma_1' + \gamma_1'' + (2B_1 - B_2)\gamma_1'' + (B_1' - B_2')\gamma_1' + B_1(B_1 - B_2)\gamma_1' + 3\gamma_1\gamma_1'' + (2B_1 - B_2)\gamma_1'\gamma_1' + (\gamma_1')^3
\end{align*}
\]  

(4.5)
Taking $\gamma_i' = \alpha_i$ we recover (3.37) with the choice of $B_i$’s as in (3.39). We can now eliminate $B_2$ and $B_1$ by appropriate choice of $\alpha$’s recovering the Boussinesq hierarchy of (3.42).

Now, let us discuss the problem of invariance under $\alpha_1$, $\alpha_3$ transformations on the reduced manifold. First, we notice that $\tilde{A}_1, \tilde{A}_2$ of the Bussinesq hierarchy (3.42) are invariant under transformations given in (4.5). The situation with partial gauge fixing is more subtle. We shall examine the invariance under separate $\alpha_1$ and $\alpha_3$ transformations by using the first two relations from (4.5). There are two possibilities:

1. $B_2 + B_1$ - invariant under $\alpha_1$-transformation

2. $2B_1 - B_2$ - invariant under $\alpha_3$-transformation

In case 1) we get as before three fields $\tilde{A}_1, \tilde{A}_2, B = -(B_2 + B_1)/2$, where $A_1, A_2$ are given by (3.40) both being invariant under $\alpha_1$-transformation. This can be achieved by choosing (for instance) $\alpha_3 = -(B_2 + B_1)/2$ and $\alpha_1 = -B_1$ so $B_2 \to 0$ and $B_1 \to -(B_2 + B_1)/2$.

In the second case we take $\alpha_1 = 0, \alpha_3 = -\frac{1}{2} B_2$ in order to have transformations $B_2 \to 0$ and $B_1 \to B \equiv \frac{1}{2}(2B_1 - B_2)$. With this choice we get:

$$\tilde{A}_2 = A_2 - \frac{1}{2} B_2 - \frac{1}{4} B_2^2$$

$$\tilde{A}_1 = A_1$$

which could be obtained by putting $B_1 = 0$ in (3.40). It is easy to see that (4.6) is invariant under $\alpha_3$-transformation. One easily verifies, that for $A_{1,2}, B_{1,2}$ satisfying the Poisson brackets dictated by (1.3), the fields (1.6) together with $\mathcal{B}$ satisfy a closed algebra:

$$\{\tilde{A}_2(x), \tilde{A}_2(y)\} = -(2\tilde{A}_2 \partial + \tilde{A}_2' + \frac{1}{2} \partial^3)\delta(x - y)$$

$$\{\tilde{A}_2(x), \tilde{A}_1(y)\} = -(3\tilde{A}_1 \partial + 2\tilde{A}_1')\delta(x - y)$$

$$\{\tilde{A}_2(x), \mathcal{B}(y)\} = -\left(\frac{3}{2} \partial^2 + \mathcal{B} \partial\right)\delta(x - y)$$

$$\{\tilde{A}_1(x), \tilde{A}_1(y)\} = -(2\tilde{A}_1' + 4\tilde{A}_1 \mathcal{B})\partial + \tilde{A}_1'' + 2(\tilde{A}_1 \mathcal{B})'\delta(x - y)$$

$$\{\tilde{A}_1(x), \mathcal{B}(y)\} = -\left(\tilde{A}_2 \partial + (\partial + \mathcal{B})^2 \partial\right)\delta(x - y)$$

$$\{\mathcal{B}(x), \mathcal{B}(y)\} = -\frac{3}{2} \delta'(x - y)$$

obtained first by the Dirac bracket method in [12].

5 Generalized Miura Transformation for Multi-Boson KP Hierarchies.

The two-boson hierarchy given in Sect. 3 and described by $L_1$ (3.1) is equivalent to the model based on the pseudo-differential operator (3.1):

$$L_1 = (D - e)(D - c)(D - e - c)^{-1} = D + (e' + ec)(D - e - c)^{-1}$$

(5.1)
The Miura-like connection between these hierarchies generalizes the usual Miura transformation between one-bose KdV and mKdV structures and takes the form \([17]\):

\[
a = e' + ec \quad ; \quad b = e + c
\]

(5.2)

This Miura transformation \((e, c) \rightarrow (a, b)\) can easily be seen to abelianize the second bracket \((3.2)\), meaning that whenever

\[
\{ e(x), c(y) \} = -\delta'(x - y)
\]

(5.3)

then \(a, b\), given by \((5.2)\), satisfy \((3.2)\).

The above structures naturally appear in connection with the Toda and Volterra lattice hierarchies \([15]\). Consider namely the spectral equation:

\[
\partial \Psi_n = \Psi_{n+1} + a_0(n) \Psi_n \quad ; \quad \lambda \Psi_n = \Psi_{n+1} + a_0(n) \Psi_n + a_1(n) \Psi_{n-1}
\]

(5.4)

The Miura transformed hierarchy defined by \((5.1)\) can be associated with a “square-root” lattice with respect to the original Toda lattice system \((5.4)\):

\[
\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + A_{n+1} \Psi_n \quad ; \quad \lambda^{1/2} \Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + B_n \tilde{\Psi}_{n-\frac{1}{2}}
\]

(5.5)

\[
\tilde{\Psi}_{n+\frac{1}{2}} = (\partial - B_n - A_n) \tilde{\Psi}_{n-\frac{1}{2}} \quad ; \quad \Psi_{n+1} = (\partial - B_n - A_{n+1}) \Psi_n
\]

which yields the Volterra chain equations \([15]\). Excluding the half-integer modes in \((5.5)\) we recover \((5.4)\) with:

\[
a_0(n) = A_{n+1} + B_n \quad , \quad a_1(n) = A_n B_n = B_{n-1} + \partial A_n
\]

(5.6)

where in the last eq.\((5.6)\) we have used one of the Volterra equations of motion (consistency condition for the Volterra spectral problem \((5.5)\)):

\[
\partial A_n = A_n (B_n - B_{n-1})
\]

(5.7)

Eqs.\((5.4)\) can be cast into the form:

\[
\lambda \Psi_n = L^{(1)}_n \Psi_n \quad , \quad L^{(1)}_n = (\partial - A_n) (\partial - B_{n-1}) (\partial - B_{n-1} - A_n)^{-1}
\]

(5.8)

which, upon the identification \(A_n = e, B_{n-1} = c\), agrees with \((5.1)\). Moreover, recalling the identification \(a = a_1(n) , b = a_0(n-1)\) for the coefficients of \(L_1\) \((3.1)\) (cf. \((2.4)\)), we see that the relation \((5.6)\) between the coefficients of the Toda and Volterra discrete spectral problems precisely matches the Miura relation \((5.2)\) for the coefficients of the two-boson KP Lax operators \((3.1)\) and \((5.1)\) abelianizing the first and the second KP Poisson bracket structures, respectively.

Furthermore, using again the Volterra equation \((5.7)\) we can rewrite \((5.8)\) as:

\[
L^{(1)}_n = (\partial - A_n) (\partial - B_n - A_n)^{-1} (\partial - B_n) = \partial + B_n (\partial - B_n - A_n)^{-1} A_n
\]

(5.9)

which, upon the identification \(B_n = \bar{j}, A_n = j\), takes the form \(L = D + \bar{j} (D - j - \bar{j})^{-1} j\). This is the form of the two-boson KP hierarchy which appeared in connection with the \(SL(2, \mathbb{R})/U(1)\) coset model \([31]\).
The above simple 2-boson model will now be generalized to the arbitrary multi-boson KP hierarchies.

We start by finding a “square-root” lattice formulation corresponding to the general Toda lattice hierarchy \(2.1\). In this spirit we are led to a spectral equation:

\[
\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + A^{(0)}_{n+1} \Psi_n + \sum_{p=1}^{M} A^{(p)}_{n-p+1} \Psi_{n-p}
\]

and

\[
\lambda^{1/2} \Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + B^{(0)}_n \tilde{\Psi}_{n-\frac{1}{2}}
\]

with time evolution equations:

\[
\tilde{\Psi}_{n+\frac{1}{2}} = \left( \partial - B^{(0)}_n - A^{(0)}_{n+1}\right) \tilde{\Psi}_{n-\frac{1}{2}} \quad ; \quad \Psi_{n+1} = \left( \partial - B^{(0)}_n - A^{(0)}_{n+1}\right) \Psi_n
\]

As in the two-boson case above, it is straightforward to show that, upon excluding the half-integer modes, the generalized Volterra system \((5.10)-(5.12)\) reduces to the Toda lattice spectral eqs.\((2.1)\) for \(M+1\), where:

\[
a_0(n) = A^{(0)}_{n+1} + B^{(0)}_n \quad ; \quad a_{M+1}(n) = B^{(0)}_n A^{(M)}_{n-M}
\]

\[
a_p(n) = A^{(p)}_{n-p+1} + B^{(0)}_n A^{(p-1)}_{n-p+1} \quad , \quad p = 1, \ldots, M
\]

From \((5.10)-(5.12)\) we find:

\[
\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \left( \partial - B^{(0)}_n + \sum_{p=1}^{M} A^{(p)}_{n-p+1} (\partial - B^{(0)}_{p-n} - A^{(0)}_{p-n+1})^{-1} \cdots (\partial - B^{(0)}_{n-2} - A^{(0)}_{n-1})^{-1} \right) \Psi_n
\]

and

\[
\lambda^{1/2} \Psi_n = \left( \partial - A^{(0)}_n \right) \tilde{\Psi}_{n-\frac{1}{2}}
\]

From the last two relations it follows that:

\[
\lambda \Psi_n = \left( \partial - A^{(0)}_n \right) \left( \partial - B^{(0)}_{n-1} + \sum_{p=1}^{M} A^{(p)}_{n-p} (\partial - B^{(0)}_{n-p-1} - A^{(0)}_{n-p})^{-1} \cdots (\partial - B^{(0)}_{n-2} - A^{(0)}_{n-1})^{-1} \right) \Psi_n
\]

This defines a Lax operator through \(\lambda \Psi_n = L^{(M+1)}_n \Psi_n\) where

\[
L^{(M+1)}_n = e^{\int B^{(0)}_{n-1} (\partial - A^{(0)}_p + B^{(0)}_n)} L^{(M)}_n (\partial - A^{(0)}_n)^{-1} e^{-\int B^{(0)}_{n-1}}
\]

and

\[
L^{(M)}_n = \partial + \sum_{p=1}^{M} A^{(p)}_{n-p} (\partial + B^{(0)}_{n-p-1} - B^{(0)}_{n-p} - A^{(0)}_{n-p})^{-1} \cdots (\partial + B^{(0)}_{n-2} - B^{(0)}_{n-2} - A^{(0)}_{n-1})^{-1}
\]

Eqs. \((5.17)-(5.18)\) can be identified with the recurrence relation for the 2\(M\)-boson KP Lax operators \((2.13)\) established in \([21]\):

\[
L_{M+1} = e^{\int_{M+1}^c (D + c_{M+1} - e_{M+1})} L_M (D - e_{M+1})^{-1} e^{-\int_{M+1}^c}
\]

\[M = 0, 1, 2 \ldots \quad , \quad L_0 \equiv D\]

\[\{c_k(x), c_l(y)\} = -\delta_{kl} \partial_x \delta(x - y) \quad , \quad k, l = 1, 2, \ldots, M + 1\]
The free-field pairs \((c_r, e_r)_{r=1}^{M+1}\) are the “Darboux-Poisson” canonical pairs for the second KP bracket \((1.3)\) satisfied by \(L_{M+1}\) for arbitrary \(M\) \([20]\). This defines a sequence of the multi-boson KP Lax operators in terms of Darboux-Poisson free-field pairs with respect to the second KP bracket, very much like \((2.18)\) defined a similar sequence of Lax operators in terms of Darboux-Poisson free-field pairs with respect to the first KP bracket \([8]\). This construction can be viewed as a generalized Miura transformation for multi-boson KP hierarchies, and hence “abelianization” of the second KP Hamiltonian structure \((1.3)\), i.e., expressing the coefficient fields of the pertinent KP Lax operator in terms of canonical pairs of free fields.

Eq.\((5.19)\) implies the following recurrence relations for the coefficient fields of \(L_M \((2.19)\) (see also \([20]\)):

\[
B^{(M+1)}_k = B^{(M)}_k + c_{M+1}, \quad 1 \leq k \leq M, \quad B^{(M+1)}_{M+1} = c_{M+1} + e_{M+1} \quad (5.21)
\]

\[
A^{(M+1)}_1 = \left( \partial + B^{(M)}_1 + c_{M+1} - e_{M+1} \right) A^{(M)}_1 \quad (5.22)
\]

\[
A^{(M+1)}_k = A^{(M)}_{k-1} + \left( \partial + B^{(M)}_k + c_{M+1} - e_{M+1} \right) A^{(M)}_k, \quad 2 \leq k \leq M \quad (5.23)
\]

\[
A^{(M+1)}_{M+1} = A^{(M)}_M + (\partial + c_{M+1}) e_{M+1} \quad (5.24)
\]

As found in ref.\([20]\), the recurrence relations \((5.21)-(5.24)\) have an explicit solution:

\[
B^{(M)}_k = e_k + \sum_{l=k}^{M} c_l, \quad 1 \leq k \leq M \quad A^{(M)}_M = \sum_{k=1}^{M} (\partial + c_k) e_k \quad (5.25)
\]

\[
A^{(M)}_k = \sum_{n_{M-k+1}=1}^{k} \left( \partial + e_{n_{M-k+1}} - e_{n_{M-k+1}+M-k} + \sum_{l_k=n_{M-k+1}}^{n_{M-k+1}+M-k} c_{l_k} \right) \times \\
\times \sum_{n_{M-k-1}=1}^{n_{M-k}} \left( \partial + e_{n_{M-k}} - e_{n_{M-k}+M-1-k} + \sum_{l_{k-1}=n_{M-k}}^{n_{M-k}+M-1-k} c_{l_{k-1}} \right) \times \cdots \\
\times \sum_{n_1=1}^{n_2} (\partial + e_{n_1} - e_{n_2+1} + c_{n_2} + c_{n_2+1}) \sum_{n_1=1}^{n_2} (\partial + c_{n_1}) e_{n_1}, \quad k = 1, \ldots, M-1 \quad (5.26)
\]

in terms of the free fields \((c_r, e_r)_{r=1}^{M}\).

A simple calculation, based on the explicit expressions \((5.25)-(5.26)\), gives the following general relations (valid for any \(M\)):

**Proposition.**

\[
\{ B^{(M)}_i(x), B^{(M)}_j(y) \} = -X_{ij} \partial_x \delta(x-y) \quad ; \quad X_{ij} \equiv \delta_{ij} + 1 \quad (5.27)
\]

\[
\{ A^{(M)}_i(x), B^{(M)}_k(y) \} = - \left( (M+1-k) \partial_x + B^{(M)}_k \right) \partial_x \delta(x-y), \quad 1 \leq k \leq M \quad (5.28)
\]

\[
\{ A^{(M)}_i(x), A^{(M)}_j(y) \} = - \left( A^{(M)}_j(x) \partial_x + \partial_x A^{(M)}_i \right) \delta(x-y) \quad (5.29)
\]

\[
\{ A^{(M)}_i(x), B^{(M)}_j(y) \} = 0 \quad \text{for } i < j
\]

where for brevity we only recorded the most simple relations following from \((5.25)-(5.26)\).

**Example – 2-boson KP:**

\[
L_1 = e^{\int c_1 (D+c_1-e_1) D (D-e_1)^{-1} e^{\int c_1} D + A^{(1)}_1 (D-B^{(1)}_1)^{-1} \quad (5.28)
\]

\[
A^{(1)}_1 = (\partial + c_1) e_1, \quad B^{(1)}_1 = c_1 + e_1 \quad (5.29)
\]
Here we recognize the structure of the two-boson hierarchy from (5.1) as well the generalized Miura map (5.2).

(2) Example – 4-boson KP:

\[ L_2 = e^{\int c_2 (D + c_2 - e_2)} [D + A_1^{(1)} (D - B_1^{(1)})^{-1}] (D - e_2)^{-1} e^{-\int c_2} \]

\[ = D + A_2^{(2)} (D - B_2^{(2)})^{-1} + A_1^{(2)} (D - B_1^{(2)})^{-1} (D - B_2^{(2)})^{-1} \]  

(5.30)

\[ A_2^{(2)} = A_1^{(1)} + (\partial + c_2) e_2 = (\partial + c_1) e_1 + (\partial + c_2) e_2 \]

(5.31)

\[ A_1^{(2)} = (\partial + B_1^{(1)} + c_2 - e_2) A_1^{(1)} = (\partial + e_1 + c_1 + c_2 - e_2) (\partial + c_1) e_1 \]

(5.32)

\[ B_2^{(2)} = c_2 + e_2, \quad B_1^{(2)} = B_1^{(1)} + c_2 = e_1 + c_1 + c_2 \]  

(5.33)

where \( A_1^{(1)} \) and \( B_1^{(1)} \) are substituted with their expressions (5.29). It is easy to derive the second bracket structure for the above fields directly from (5.20).

From the recursive relation (5.13) we can obtain closed expressions for the general Lax operator \( L_M, M = 1, 2, \ldots \), directly in terms of the building blocks \((c_k, e_k)_{k=1}^{M}\):

\[ L_M = (D - e_M) \prod_{k=M-1}^{1} \left( D - e_k - \sum_{l=k+1}^{M} c_l \right) \left( D - \sum_{l=1}^{M} c_l \right) \prod_{k=1}^{M} \left( D - e_k - \sum_{l=k}^{M} c_l \right)^{-1} \]

\[ = \prod_{k=M}^{1} (D + c_k - B_k) \left( D - \sum_{l=1}^{M} c_l \right) \prod_{k=1}^{M} (D - B_k)^{-1} \]  

(5.34)

where for brevity we drop from now on the superscript \( M \), so that \( B_k \equiv B_k^{(M)} = e_k + \sum_{l=k}^{M} c_l \).

Let us introduce now the new linear combinations:

\[ \bar{B}_i \equiv c_{M-i+1} - B_{M-i+1} \quad ; \quad \bar{B}_{M+1} \equiv - \sum_{l=1}^{M} c_l \quad i = 1, \ldots, M \]  

(5.35)

The advantage of this notation is that the Lax operator \( L_M \) (5.34) takes a more compact form:

\[ L_M = \prod_{j=1}^{M+1} (D + B_j) \prod_{k=1}^{M} (D - B_k)^{-1} \]  

(5.36)

This form of the Lax operator has already appeared in \[21\]. The fields (currents) \( \bar{B}_j, B_i \) satisfy by definition the condition

\[ \psi_{M+1,M} \equiv \sum_{j=1}^{M+1} \bar{B}_j + \sum_{k=1}^{M} B_k = 0 \]  

(5.37)

recognized in \[21\] as a tracelessness condition of the graded \( SL(M + 1, M) \) Kac-Moody algebra. It follows from (5.27) and from:

\[ \{ c_i(x), B_j(y) \} = -\delta_{ij} \delta(x - y) \]  

(5.38)
that the Poisson bracket algebra satisfied by the fields $\bar{B}_j, B_i$ is the Cartan subalgebra of the graded $SL(M + 1, M)$ Kac-Moody algebra:

\[
\begin{align*}
\{\bar{B}_i(x), \bar{B}_j(y)\} &= \left(\delta_{ij} - 1\right) \delta'(x - y) \quad i, j = 1, \ldots, M + 1 \\
\{B_k(x), B_l(y)\} &= -\left(\delta_{kl} + 1\right) \delta'(x - y) \quad k, l = 1, \ldots, M \\
\{B_i(x), B_l(y)\} &= \delta'(x - y)
\end{align*}
\]

We shall refer to the algebra (5.39) as $SL_c(M + 1, M)$ Kac-Moody algebra.

### 6 Reduction of $SL(M + 1, M)$ to $SL(M + 1, M - k)$

Here we present a general scheme of gauging away $k$ (out of $M$) currents $B_{M-k+1}, \ldots, B_M$ by introducing the gauge generator

\[
G = \exp \left(-\int \sum_{i=1}^M B_i \gamma_i \right)
\]

which induces the following transformations via Hamiltonian action (as in (4.2)):

\[
\begin{align*}
B_i &\rightarrow \tilde{B}_i = G^{-1} B_i G = B_i + X_{ij} \gamma'_j \\
\tilde{c}_i &\rightarrow \bar{c}_i = G^{-1} c_i G = c_i + \gamma'_i \\
\bar{e}_i &\rightarrow \bar{e}_i = G^{-1} e_i G = e_i + \sum_{l=1}^i \gamma'_l
\end{align*}
\]

Note that $\gamma'_i$ are fixed by the gauge-fixing condition $\tilde{B}_i = 0$, $i = M - k + 1, \ldots, M$ i.e.

\[
\gamma'_i = -X_{ij}^{(k)-1} B_j \quad ; \quad X_{ij}^{(k)} \equiv 1 + \delta_{ij}
\]

with $X_{ij}^{(k)}$ being restriction of $X_{ij}$ to $M - k + 1 \leq i, j \leq M$ and $\gamma'_r = 0$ for $1 \leq r \leq M - k$.

In what follows we shall need to find explicitly the inverse of the matrix $X_{ij} \equiv 1 + \delta_{ij}$. Let $U$ be a $M \times M$ matrix with elements $U_{ij} = 1$. Hence $X = \mathbb{1} + U$ and it is easy to see that:

\[
X^{-1} = \mathbb{1} - U + U^2 - \ldots = \mathbb{1} - U \left(1 - M + M^2 - M^3 + \ldots\right) = \mathbb{1} - \frac{1}{1 + M} U
\]

Correspondingly we also find

\[
X^{(k)-1} = \mathbb{1} - \frac{1}{1 + k} U^{(k)}
\]

where again the superscript $(k)$ indicates restriction of the matrix indices to $M - k + 1 \leq i, j \leq M$.

With this information we can rewrite (6.3) as:

\[
\gamma'_i = -B_i + \frac{1}{k + 1} \sum_{n=M-k+1}^M B_n \quad , \quad M - k + 1 \leq i \leq M
\]
From (6.6) and (6.2) we find the values of the gauge-rotated non-zero $B_i$ to be given by:

$$\tilde{B}_r = B_r - \frac{1}{k + 1} \sum_{n=M-k+1}^{M} B_n \quad , \quad 1 \leq r \leq M - k$$

(6.7)

For the gauge transformed $c_i$ we find:

$$\tilde{c}_i = \begin{cases} 
  c_i & 1 \leq i \leq M - k \\
  c_i - B_i + \frac{1}{k + 1} \sum_{n=M-k+1}^{M} B_n & M - k + 1 \leq i \leq M 
\end{cases}$$

(6.8)

Correspondingly, we obtain for the new gauge transformed $\bar{B}_i = (\tilde{c}_M - B_{M+1}, - \sum_{l=1}^{M} \tilde{c}_l)$ (omitting the tilde on top of $\bar{B}_i$ for brevity):

$$\bar{B}_i = \begin{cases} 
  c_{M-i+1} - B_{M+i+1} + \frac{1}{k + 1} \sum_{n=M-k+1}^{M} B_n & 1 \leq i \leq M \\
  - \sum_{l=1}^{M} c_l + \frac{1}{k + 1} \sum_{n=M-k+1}^{M} B_n & i = M + 1 
\end{cases}$$

(6.9)

Using (5.27) and (5.38) we find that $\bar{B}_i, \tilde{B}_r$ satisfy the Poisson-bracket Cartan subalgebra of the graded $SL(M+1, M-k)$ Kac-Moody algebra (cf. eqs.(5.39)):

$$\{\bar{B}_i(x), \bar{B}_j(y)\} = (\delta_{ij} - \frac{1}{k + 1}) \delta'(x - y) \quad i, j = 1, \ldots, M + 1$$

(6.10)

for which we shall use the symbol $SL_c(M+1, M-k)$. Note that the gauge transformation (6.2) maps the trace-constraint $\psi_{M+1,M}$ (5.37) to the new $SL(M+1, M-k)$ trace condition:

$$\psi_{M+1,M-k} \equiv G^{-1} \psi_{M+1,M} G = \sum_{j=1}^{M+1} \bar{B}_j + \sum_{r=1}^{M-k} \tilde{B}_r = 0$$

(6.11)

In addition to $\bar{B}_i, \tilde{B}_r$ there are $k$ gauge parameters $\gamma'_n$ (6.6) associated with the $SL(k+1)$ algebra and satisfying:

$$\{\gamma'_n(x), \gamma'_m(y)\} = -\left(\delta_{nm} - \frac{1}{k + 1}\right) \delta'(x - y) \quad n, m = M - k + 1, \ldots, M$$

(6.12)

which we shall call $SL_c(k+1)$ algebra. Note that $\gamma'_n$ are decoupled from $\bar{B}_i, \tilde{B}_r$:

$$\{\gamma'_n, \bar{B}_i\} = \{\gamma'_r, \tilde{B}_r\} = 0 \quad, \quad n = M - k + 1, \ldots, M \quad , \quad r = 1, \ldots, M - k \quad , \quad i = 1, \ldots M + 1$$

(6.13)

We therefore easily arrive at the following:
Proposition. The second bracket of the multi-boson KP is reducible under gauge fixing procedure described above and \( \{ \vec{B}_i, B_n \} = \{ \vec{B}_r, B_n \} = 0 \) where \( B_n, M - k + 1 \leq n \leq M \) are the modes gauged away.

This result generalizes the observation, we made earlier in section 3.2, concerning the decoupling of the KdV mode \( u = a - \frac{1}{2} b^2 - \frac{1}{2} b' \) from the current \( b \).

The above process of reduction can be extended to remove all currents, i.e., \( \vec{B}_i = 0 \) for \( i = 1, \ldots M \). In this case:

\[
\gamma'_i = -X_{ij}^{-1} B_j \quad ; \quad \bar{c}_i = c_i - X_{ij}^{-1} B_j
\]

(6.14)

Accordingly \( \vec{B}_i = \left( \bar{c}_{M-i+1}, -\sum_{i=1}^{M} c_i \right) \) and the underlying Poisson bracket algebra splits into two disjoint \( SL_c(M + 1) \) algebras of opposite signatures:

\[
\{ \vec{B}_i(x), \vec{B}_j(y) \} = \left( \delta_{ij} - \frac{1}{1 + M} \right) \delta'(x - y) \quad i, j = 1, \ldots M
\]

\[
\{ \gamma'_i(x), \gamma'_j(y) \} = - \left( \delta_{ij} - \frac{1}{1 + M} \right) \delta'(x - y) \quad i, j = 1, \ldots M
\]

(6.15)

\[
\{ \vec{B}_i(x), \gamma'_j(y) \} = 0
\]

Example - 2-boson KP: Recall expressions (5.2), \( A = e' + ce, B = c + e \). In this case: \( \vec{B} = B + 2\gamma', \bar{c} = c + \gamma' \) and condition \( \vec{B} = 0 \) leads to \( \bar{c} = (c - e)/2 \) and \( \gamma' = -(c + e)/2 \). Correspondingly we find: \( A \rightarrow \tilde{A} = \bar{c} + \bar{c} = (e - c)/2, (e - c)/2 - (e - c)^2/4 \), which satisfies the Virasoro algebra due to \( \{(e - c)/2, (e - c)/2 \} = -\delta'(x - y)/2 \). Note also that we can rewrite \( \tilde{A} \) as \( A - \frac{1}{4} B^2 - \frac{1}{2} B' \) obtaining agreement with the result of the DS reduction (3.22).

The above considerations show that the graded \( SL_c(M + 1, M) \) algebra is reducible and the splitting is of the form:

**Proposition.** \( SL_c(M + 1, M) = SL_c(M + 1, M - k) \oplus SL_c(k + 1) \) with \( k = 1, \ldots, M \)

Indeed, given \( \{ \bar{\beta}_i, \beta_r \} \) with \( 1 \leq i \leq M + 1, 1 \leq r \leq M - k \) spanning \( SL_c(M + 1, M - k) \) and an independent basis \( \{ \alpha_n \} \) for \( SL_c(k + 1) \) with \( M - k + 1 \leq n \leq M \), we can form an \( SL_c(M + 1, M) \) algebra by taking the following linear combination:

\[
\vec{B}_i = \begin{cases} 
\bar{\beta}_n - \alpha_n + X^{(k)}_{nm} \alpha_m \\
\beta_r + X_{rm} \alpha_m \\
\bar{\beta}_{M+1} + \sum_{n=1}^{k} \alpha_n 
\end{cases} \quad \vec{B}_i = \begin{cases} 
-X^{(k)}_{nm} \alpha_m \\
\beta_r - X_{rm} \alpha_m 
\end{cases}
\]

(6.16)

Note, that this construction, as we have seen above, is reversible. Note also, that for \( k = M \) we have a decomposition into two independent \( SL_c(M + 1) \) algebras of opposite signatures as in (6.15).

6.1 The Second Bracket Structure of the SL(M + 1, M - k)-KdV Hierarchy

Let us recall the expression (6.10) for the graded Poisson bracket algebra \( SL_c(M + 1, M - k) \). In [21] this algebra was realized as a Dirac bracket algebra obtained from the Poisson brackets
of two independent set of bose fields of opposite signatures:

$$\{ \bar{B}_i(x), \bar{B}_j(y) \}_{PB} = \delta_{ij} \delta'(x - y) \quad i, j = 1, \ldots, M + 1$$

$$\{ B_m(x), B_n(y) \}_{PB} = -\delta_{mn} \delta'(x - y) \quad 1 \leq m, n \leq M - k$$

(6.17)

by imposing the constraint:

$$\psi_{M+1,M-k} = \sum_{j=1}^{M+1} \bar{B}_j + \sum_{n=1}^{M-k} B_n = 0$$

(6.18)

which is second-class due to:

$$\{ \psi_{M+1,M-k}(x), \psi_{M+1,M-k}(y) \} = (k + 1) \delta'(x - y)$$

(6.19)

We are interested in describing the corresponding bracket structures associated with the generalization of the Lax operator (5.36) to:

$$\mathcal{L}_{M+1,M-k} = \prod_{j=1}^{M+1} (D + \bar{B}_j) \prod_{n=1}^{M-k} (D - B_n)^{-1}$$

(6.20)

In [21] it was pointed out that the Lax operator $\mathcal{L}_{M+1,M-k}$ (with no trace condition imposed) satisfies the second Gelfand-Dickey bracket provided the fields $\bar{B}_i, B_n$ satisfy (6.17). The proposition is therefore:

**Proposition.**

$$\left\{ \langle \mathcal{L}_{M+1,M-k}|X \rangle, \langle \mathcal{L}_{M+1,M-k}|Y \rangle \right\}_{PB} =$$

$$\text{Tr}_A \left( (\mathcal{L}_{M+1,M-k}X) \cdot \mathcal{L}_{M+1,M-k}Y - (X\mathcal{L}_{M+1,M-k}) \cdot Y\mathcal{L}_{M+1,M-k} \right)$$

(6.21)

where the subscript $PB$ stands for the Poisson bracket as defined by (6.17). The statement can easily be proved by induction in $M$ and $k$. It is straightforward to verify (6.21) for $M = 0, k = 0$ just by inserting $\mathcal{L}_{1,0} = (D + \bar{B})$ into the formula (6.21). The essential part of the induction proof with respect to $M$ consists in showing that (6.21) is valid for $\mathcal{L}_{M+1} = (D + \bar{B})\mathcal{L}_M$ provided it is valid for $\mathcal{L}_M$ with $k = 0$. We have:

$$\left\{ \langle \mathcal{L}_{M+1}|X \rangle, \langle \mathcal{L}_{M+1}|Y \rangle \right\}_{PB} \bigg|_{B=\text{fixed}} = \left\{ \langle \mathcal{L}_M|X(D + \bar{B}) \rangle, \langle \mathcal{L}_M|Y(D + \bar{B}) \rangle \right\}_{PB} \bigg|_{\mathcal{L}_M=\text{fixed}}$$

$$+ \left\{ \langle (D + \bar{B})|\mathcal{L}_M X \rangle, \langle (D + \bar{B})|\mathcal{L}_M Y \rangle \right\}_{PB}$$

$$= \text{Tr}_A \left( (\mathcal{L}_M X(D + \bar{B})) \cdot \mathcal{L}_M Y(D + \bar{B}) - (X(D + \bar{B})\mathcal{L}_M) \cdot Y(D + \bar{B})\mathcal{L}_M \right)$$

$$+ \int dx \text{Res}(\mathcal{L}_M X) \partial_x \text{Res}(\mathcal{L}_M Y)$$

(6.22)

Now, using the simple identity for pseudo-differential operators (valid for any $\bar{B}$):

$$(D + \bar{B}) \left( \mathcal{L}_M X(D + \bar{B}) \right) = \left( (D + \bar{B})\mathcal{L}_M X \right) + \partial \text{Res}(\mathcal{L}_M X)$$

(6.23)
we arrive at the desired result:

\[ \{ \langle \mathcal{L}_{M+1} X \rangle , \langle \mathcal{L}_{M+1} Y \rangle \}_P = \text{Tr}_A \left( (\mathcal{L}_{M+1} X) \mathcal{L}_{M+1} Y - (X \mathcal{L}_{M+1}) Y \mathcal{L}_{M+1} \right) \] (6.24)

The remaining step of the induction proof with respect to \( k \), involving the transition \( \mathcal{L}_{M+1,k} \rightarrow \mathcal{L}_{M+1,k+1} \), can be performed using the same techniques as above.

We now turn our attention to the Dirac bracket which results from (6.21) by imposing the constraint (6.18). The relevant statement is:

**Proposition.**

\[ \{ \langle \mathcal{L}_{M+1,M-k} X \rangle , \langle \mathcal{L}_{M+1,M-k} Y \rangle \}_P = \{ \langle \mathcal{L}_{M+1,M-k} X \rangle , \langle \mathcal{L}_{M+1,M-k} Y \rangle \}_P \]

\[ + \frac{1}{k+1} \int dx \ \text{Res} \left( [\mathcal{L}_{M+1,M-k}, X] \right) \partial^{-1} \text{Res} \left( [\mathcal{L}_{M+1,M-k}, Y] \right) \] (6.25)

The proof follows from the computation of the extra term of the relevant Dirac bracket:

\[ - \int \{ \langle \mathcal{L}_{M+1,M-k} X \rangle , \psi_{M+1,M-k} \}_{DB} \{ \psi_{M+1,M-k} , \psi_{M+1,M-k} \}_{DB}^{-1} \]

\[ \times \{ \psi_{M+1,M-k} , \langle \mathcal{L}_{M+1,M-k} Y \rangle \}_{DB} \] (6.26)

One easily verifies the equality of (6.26) with the second term on the r.h.s. of (6.25) using (6.19):

\[ \{ \langle \mathcal{L}_{M+1,M-k} X \rangle , \psi_{M+1,M-k}(z) \}_{P} = -\langle [\delta(x - z) , \mathcal{L}_{M+1,M-k}] | X \rangle \]

\[ = -\text{Res} ( [X , \mathcal{L}_{M+1,M-k} ] ) (z) \] (6.27)

Formula (6.25) contains as special cases the KP hierarchy, corresponding to \( k = 0 \), and the KdV hierarchy, corresponding to \( k = M \) (see e.g. [32]). For the intermediary cases \( 0 < k < M \) eq.(6.25) represents a compact expression for the Poisson bracket structure of all \( SL(M + 1, M - k) \)-KdV hierarchies defined by the Lax operators (5.20).

### 6.2 The Lax Formulation of \( SL(M + 1, M - k) \)-KdV

In this subsection we give expressions for the coefficients of the Lax operators of the generalized \( SL(M + 1, M - k) \)-KdV hierarchy in terms of free fields. Let us start with the case where all the currents \( B_i \) are gauged away according to (6.14). In this limit the expression (5.20) becomes:

\[ \tilde{A}_M \equiv \tilde{A}_M^{(M)} = \sum_{k=1}^{M} (\partial + \bar{c}_k) \left( - \sum_{l=k}^{M} \bar{c}_l \right) \] (6.28)

\[ \tilde{A}_k \equiv \tilde{A}_k^{(M)} = \sum_{n_M-k+1=1}^{k} (\partial + \bar{c}_{n_M-k+1+M-k}) \times \sum_{n_M-k=1}^{n_M-k+1} (\partial + \bar{c}_{n_M-k+M-1-k}) \times \cdots \]

\[ \times \sum_{n_1=1}^{n_3} (\partial + \bar{c}_{n_2+1}) \sum_{n_1=1}^{n_2} (\partial + \bar{c}_{n_1}) \left( - \sum_{l=n_1}^{M} \bar{c}_l \right) , \quad k = 1, \ldots, M - 1 \] (6.29)
which agrees with Fateev-Lukyanov [22] expression:

\[
\prod_{i=1}^{M+1} \left( D + B_i \right) = D^{M+1} + \tilde{A}_M D^{M-1} + \ldots + \tilde{A}_1
\]  

(6.30)

\[
\tilde{B}_{M+1} = - \sum_{i=1}^{M} \tilde{c}_i \quad ; \quad \tilde{B}_i = \tilde{c}_{M+1-i} \quad ; \quad i = 1, \ldots, M
\]

The reason for this agreement is that the Lax operator in (6.30) is equal to \( \mathcal{L}_{M+1,0} \) (eq. (6.24)) for \( k = 0 \) with the condition \( \psi_{M+1,0} = 0 \) imposed. Both approaches (Dirac bracket and gauge fixing) lead to the algebra (6.15) or, equivalently, to the formula (6.25) with \( k = M \). Hence, our construction has provided a simple proof for the Fateev-Lukyanov expression [22] (see also [1, 33]). Note that the use of the gauge fixing method has the advantage that the Lax operator in (6.30) is equal to \( \tilde{A}_M \) (6.10) (recall \( \bar{L} \) and \( \bar{B} \) in (6.23) satisfy a superdeterminant of the graded \( \text{SL}(M+1, M-k) \) matrix in a diagonal gauge, which for the ordinary KdV case \( k = M \) becomes an ordinary determinant as in (6.30).

We can alternatively rewrite the last expression (6.32) in a way, which corresponds to the DS gauge as:

\[
\mathcal{L}_{M+1, M-k} = \sum_{l=1}^{M-k} \tilde{A}_l \prod_{i=l}^{M-k} \left( D - \tilde{B}_i \right)^{-1} + \sum_{l=0}^{k-1} \tilde{A}_{l+M-k+1} D^l + D^{k+1}
\]  

(6.33)

which automatically satisfies the appropriate trace condition (6.18). We can interpret (6.32) as a superdeterminant of the graded \( \text{SL}(M+1, M-k) \) matrix in a diagonal gauge, which for the ordinary KdV case \( k = M \) becomes an ordinary determinant as in (6.30).

We can alternatively rewrite the last expression (6.32) in a way, which corresponds to the DS gauge as:

\[
\mathcal{L}_{M+1, M-k} = \sum_{l=1}^{M-k} \tilde{A}_l \prod_{i=l}^{M-k} \left( D - \tilde{B}_i \right)^{-1} + \sum_{l=0}^{k-1} \tilde{A}_{l+M-k+1} D^l + D^{k+1}
\]  

(6.33)

with the second bracket structure automatically given by formula (6.23). The coefficients \( \tilde{A}_l \) can be explicitly expressed in terms of \( \tilde{c}_i, \tilde{B}_i \) from (5.25)-(5.26) by substituting there

\[
c_l \rightarrow \tilde{c}_l \quad ; \quad e_l \rightarrow \tilde{B}_l - \sum_{i=l}^{M} \tilde{c}_i \quad , \quad l = 1, \ldots, M - k \quad , \quad e_l \rightarrow - \sum_{i=l}^{M} \tilde{c}_i \quad , \quad l = M - k + 1, \ldots, M
\]  

(6.34)

Hence, again we arrived at representation of the coefficients of the \( \text{SL}(M+1, M-k) \)-KdV Lax operators (6.33), or (6.32), in terms of the free fields (currents) whose Poisson bracket algebra is given by (6.10) (recall \( \tilde{B}_i = \tilde{c}_{M-i+1} \) for \( 1 \leq i \leq M - k \) and \( \tilde{B}_i = \tilde{c}_{M-i+1} - \tilde{B}_{M-i+1} \) for \( M - k + 1 \leq i \leq M \)). Correspondingly, the Lax coefficients \( \tilde{A}_i, \tilde{B}_j \) in (6.33) satisfy a
nonlinear Poisson bracket algebra, called \( W(M, M-k) \)-algebra in ref.\([18]\), which results from (5.23). This \( W(M, M-k) \)-algebra is a generalization of the well-known Zamolodchikov’s nonlinear \( W_{M+1} \) algebra \([27]\), in particular, \( W(M, 0) \approx W_{M+1} \). Thus, eqs.(5.23)–(5.26) with the substitutions (5.34) provide explicit free-field realization of \( W(M, M-k) \).

The Lax operator \( \mathcal{L}_{M+1,M-k} \) (6.32) (or (6.33)) possesses the following pseudo-differential series expansion:

\[
\mathcal{L}_{M+1,M-k} = D^{k+1} + \sum_{l=0}^{k-1} A_{l+M-k+1}(\tilde{c}, \tilde{B}) D^l + \sum_{n=0}^{\infty} w_n(\tilde{c}, \tilde{B}) D^{-1-n} \tag{6.35}
\]

Here, as above, \( \tilde{A}_s(\tilde{c}, \tilde{B}) \) are given by the expressions (5.25)–(5.26) with the substitutions (5.34), whereas

\[
P_n^{(r)}(\phi_1, \ldots, \phi_r) = \sum_{m_1 + \ldots + m_r = n} (\partial + \phi_1)^{m_1} \ldots (\partial + \phi_r)^{m_r} \cdot 1 \tag{6.37}
\]

denote the multiple Faá di Bruno polynomials (cf. \([8, 20]\) for analogous to (6.35)–(6.37) expressions for the multi-boson KP Lax operators). The Poisson bracket algebra of the coefficient fields \( \tilde{A}_s(\tilde{c}, \tilde{B}) \), \( s = M-k+1, \ldots, M \), and \( w_n(\tilde{c}, \tilde{B}) \), \( n = 0, 1, 2, \ldots \), which results from the free-field Poisson brackets of their constituents (6.10) generalizing the known nonlinear \( \hat{W}_\infty \) algebra \([34]\) (see also \([21]\)). In particular, \( \hat{W}^{(k=0)}_\infty \approx \hat{W}_\infty \).

**Example – SL(3, 1)-KdV hierarchy.**

It is defined by the Lax operator \( \mathcal{L}_{3,1} = \tilde{A}_1 \left( \partial - \tilde{B}_1 \right)^{-1} + \tilde{A}_2 + \partial^2 \) (cf. (6.33)) where:

\[
\tilde{A}_1 = (\partial + \tilde{B}_1 + \tilde{c}_1) \left( \tilde{B}_1 - \tilde{c}_1 - \tilde{c}_2 \right) \tag{6.38}
\]
\[
\tilde{A}_2 = (\partial + \tilde{c}_1) \left( \tilde{B}_1 - \tilde{c}_1 \right) + (\tilde{c}_1 - \tilde{c}_2) \tilde{c}_2 \tag{6.39}
\]

and with fundamental Poisson brackets:

\[
\{ \tilde{c}_1(x), \tilde{B}_1(y) \} = -\delta'(x - y) \quad , \quad \{ \tilde{c}_2(x), \tilde{c}_2(y) \} = \frac{1}{2} \delta'(x - y)
\]
\[
\{ \tilde{c}_2(x), \tilde{B}_1(y) \} = -\frac{1}{2} \delta'(x - y) \quad , \quad \{ \tilde{B}_1(x), \tilde{B}_1(y) \} = -\frac{3}{2} \delta'(x - y) \tag{6.40}
\]

The brackets (6.40) imply that \( \tilde{A}_{1,2} \) given by (6.38), (6.39), together with \( \tilde{B}_1 \) satisfy the \( W(2, 1) \) Poisson bracket algebra (6.7) (with \( B \equiv \tilde{B}_1 \)).

Now, one notices the presence of the zero-order term \( \tilde{A}_{M-k+1} \) in the Lax operator \( \mathcal{L}_{M+1,M-k} \) (5.33). This fact enables us to prove that the \( SL(M + 1, M - k) \)-KdV hierarchy is a bi-Hamiltonian hierarchy. Consider namely \( \mathcal{L}'_{M+1,M-k} = \mathcal{L}_{M+1,M-k} - \lambda \) obtained
by redefining the zero-order term in the Lax operator by addition of the constant $\lambda$. Clearly, the second bracket (6.23) for the new Lax operator becomes:

$$
\left\{ \langle L_{M+1,k}^I X \rangle, \langle L_{M+1,k}^I Y \rangle \right\}_{DB} = \text{Tr}_A \left( \left( L_{M+1,k}^I X \right)_+ L_{M+1,k}^I Y - \left( X L_{M+1,k}^I \right)_+ Y L_{M+1,k}^I \right) + \frac{1}{k+1} \int dx \text{Res} \left( \left[ L_{M+1,k}^I, X \right] \right) \partial^{-1} \text{Res} \left( \left[ L_{M+1,k}^I, Y \right] \right) - \lambda \left< L_{M+1,k}^I \mid [X, Y]_R \right> \tag{6.41}
$$

where we introduced the $R$-commutator $[X, Y]_R \equiv [X_+, Y_+] - [X_-, Y_-]$. Here again the subscripts $\pm$ denote projections on the pure differential and pseudo-differential parts of the pseudo-differential operators $X, Y$, respectively. Define next an $R$-bracket $\{\cdot, \cdot\}_R^1$ as a bracket obtained from (1.1) by substituting $R$-commutator $[X, Y]_R$ for the ordinary commutator $[X, Y]$ [19, 33]:

$$
\{ \langle L \mid X \rangle, \langle L \mid Y \rangle \}^R_1 \equiv - \langle L \mid [X, Y]_R \rangle \tag{6.42}
$$

Relation (6.41) thus shows that the linear combination of brackets $\{\cdot, \cdot\}_{DB} + \lambda \{\cdot, \cdot\}_R^1$ satisfies the Jacobi identity. We can state this result as:

**Proposition.** SL($M + 1, M - k$)-KdV hierarchy is bi-Hamiltonian with brackets $\{\cdot, \cdot\}_{DB}$ and $\{\cdot, \cdot\}_R^1$ defining a compatible pair of Hamiltonian structures.

This Proposition establishes, therefore, the fundamental criterium for integrability of the generalized SL($M + 1, M - k$)-KdV hierarchy.

### 6.3 The Discrete Symmetry of SL($M + 1, M - k$)-KdV Hierarchy.

Recently, the multi-boson KP hierarchies have been shown to possess canonical discrete symmetry realized as a similarity transformation of their Lax operator [14]. It is natural to ask whether the discrete-similarity transformation can be constructed for the reduced SL($M + 1, M - k$)-KdV hierarchy for $k \neq 0$. One suspects that the presence of $B$ currents in this reduction will allow for remnants of the discrete symmetry to survive in this system. We shall show now that this is indeed the case.

First, let us consider the simplest nontrivial example – the pseudo-differential Lax operator of the SL($3, 1$)-KdV hierarchy (for convenience here we suppress the tildes on the coefficient fields):

$$
L_{3,1} = A_1 \frac{1}{\partial - B_1} + A_2 + \partial^2 \tag{6.43}
$$

It is easy to prove its covariance under the similarity transformation:

$$
(\partial - B^0) L_{3,1} \left( \partial - B^0 \right)^{-1} = \tilde{A}_1 \frac{1}{\partial - B^0} + \tilde{A}_2 + \partial^2 \tag{6.44}
$$

provided $B^0 = B_1 + \partial \ln A_1$. Eq. (6.44) induces the following discrete transformations on the Lax coefficients which can be viewed as auto-Bäcklund transformations for the underlying SL($3, 1$)-KdV hierarchy:

$$
\begin{align*}
B_1 & \rightarrow \bar{B}_1 = B^0 = B + \partial \ln A_1 \\
A_2 & \rightarrow \bar{A}_2 = A_2 + 2\partial (B + \partial \ln A_1) \\
A_1 & \rightarrow \bar{A}_1 = A_1 + A_2' + \partial [(B + \partial \ln A_1)^2 + \partial (B + \partial \ln A_1)]
\end{align*} \tag{6.45}
$$
This can be represented by the following Toda-like lattice equations of motion:

\[ \partial a_2(n) = a_2(n)[a_0(n + 1) - a_0(n)] \quad (6.46) \]

\[ \partial a_0(n + 1) = \frac{1}{2}[a_1(n + 1) - a_1(n)] \]

\[ \partial a_1(n) = a_2(n + 1) - a_2(n) - \partial[a_0^2(n + 1) + a_0(n + 1)] \]

upon identifying:

\[ a_2(n) \simeq A_1, \quad a_1(n) \simeq A_2, \quad a_0(n) \simeq B_1 \quad (6.47) \]

\[ a_2(n + 1) \simeq \bar{A}_1, \quad a_1(n + 1) \simeq \bar{A}_2, \quad a_0(n + 1) \simeq \bar{B}_1 \]

Eqs.(6.46) can be obtained as consistency conditions of the following lattice spectral system:

\[ a_2(n)\Psi_{n-1} + a_1(n)\Psi_n + \partial^2\Psi_n = \lambda\Psi_n \]

\[ \partial\Psi_{n-1} - a_0(n)\Psi_{n-1} = \Psi_n \quad (6.48) \]

The above discrete symmetry extends to the general case given by the Lax operator (6.33). We find that the similarity transformation \((\partial - \mathcal{B}^0)\mathcal{L}_{M+1,M-k}(\partial - \mathcal{B}^0)^{-1}\) with \(\mathcal{B}^0 = B_1 + \partial \ln A_1\) again preserves the form of the Lax operator, while its coefficients undergo the following transformations:

\[ B_l \rightarrow \bar{B}_l = B_{l+1} \quad (6.49) \]

\[ A_l \rightarrow \bar{A}_l = A_l + (\partial + B_{l+1} - \mathcal{B}^0)A_{l+1} \]

for the first \(M-k-1\) coefficients labelled by \(l = 1, \ldots, M-k-1\) and behaving under similarity transformation in a way consistent with the lattice-site translations in the underlying Toda-like lattice (cf. eqs.(6.46)). For the remaining coefficients we find:

\[ B_{M-k} \rightarrow \bar{B}_{M-k} = \mathcal{B}^0 = B_1 + \partial \ln A_1 \quad (6.50) \]

\[ A_{M-k} \rightarrow \bar{A}_{M-k} = A_{M-k} + P'_{k+1}(\mathcal{B}^0) + \sum_{l=0}^{k-1} \left( A_{M-k+l+1}P_l(\mathcal{B}^0) \right) \]

\[ A_{M-k+\alpha} \rightarrow \bar{A}_{M-k+\alpha} = \sum_{m=1}^{k} \sum_{p=0}^{m-2} \left( \begin{array}{c} m-1 \\ p \end{array} \right) \left( \begin{array}{c} m-2-p \\ \alpha-1 \end{array} \right) A_{M-k+m}P_p(\mathcal{B}^0)P_{m-1-p-\alpha}(-\mathcal{B}^0) \]

\[ + \sum_{m=1}^{k} \sum_{p=0}^{m-1} \left( \begin{array}{c} m-1 \\ p \end{array} \right) \left( \begin{array}{c} m-1-p \\ \alpha-1 \end{array} \right) A_{M-k+m}P_p(\mathcal{B}^0)P_{m-p-\alpha}(-\mathcal{B}^0) \]

\[ + \sum_{p=1}^{k} \left( \begin{array}{c} k+1 \\ p \end{array} \right) P'_p(\mathcal{B}^0)P_{k+1-p-\alpha}(-\mathcal{B}^0) \]

\[ + \sum_{p=0}^{k+1} \left( \begin{array}{c} k+1 \\ p \end{array} \right) \left( \begin{array}{c} k-p+1 \\ \alpha-1 \end{array} \right) P_p(\mathcal{B}^0)P_{k+2-p-\alpha}(-\mathcal{B}^0) \]

for \(1 \leq \alpha \leq k\). The symbols \(P_n(\pm \phi) \equiv (D \pm \phi)^n \cdot 1\) denote the ordinary Faá di Bruno polynomials (cf. (6.37)). Eqs.(6.49)–(6.50) represent the auto-Bäcklund transformations for the generalized \(SL(M+1,M-k)\)-KdV hierarchies. Both, the Hamiltonians and Poisson structures of the underlying hierarchies are invariant under (6.49)–(6.50) due to the similarity character of these auto-Bäcklund transformations.
7 Two-Matrix Model as a SL(M + 1, 1)-KdV Hierarchy

Now, let us show how the formalism of the previous sections finds application in the two-matrix string model. This model is defined through the partition function:

$$Z_N[t, \tilde{t}, g] = \int dM_1 dM_2 \exp \left\{ \sum_{r=1}^{p_1} t_r \text{Tr} M_1^r + \sum_{s=1}^{p_2} \tilde{t}_s \text{Tr} M_2^s + g \text{Tr} M_1 M_2 \right\}$$  \hspace{1cm} (7.1)

Here $M_{1,2}$ are Hermitian $N \times N$ matrices, and the orders of the matrix “potentials” $p_{1,2}$ are assumed to be finite. In refs.\cite{11, 23} it was shown that, by using the method of generalized orthogonal polynomials\cite{36}, the partition function (7.1) and its derivatives w.r.t. the parameters $(t_r, \tilde{t}_s, g)$ can be explicitly expressed in terms of solutions to constrained generalized Toda lattice hierarchies associated with (7.1). The corresponding linear problem and Lax (or “zero-curvature”) representation of the latter read \cite{11, 23}:

$$Q_{nm} \psi_m = \lambda \psi_n \hspace{1cm} -g \bar{Q}_{nm} \psi_m = \frac{\partial}{\partial \lambda} \psi_n$$ \hspace{1cm} (7.2)

$$\frac{\partial}{\partial t_r} \psi_n = (Q^+_{nm}) \psi_m \hspace{1cm} \frac{\partial}{\partial \tilde{t}_s} \psi_n = - (Q^-_{nm}) \psi_m$$ \hspace{1cm} (7.3)

$$\frac{\partial}{\partial t_r} Q = [Q, \bar{Q}^-] \hspace{1cm} \frac{\partial}{\partial \tilde{t}_s} Q = [Q, \bar{Q}^+]$$ \hspace{1cm} (7.4)

$$\frac{\partial}{\partial t_r} Q = [Q^+_{nm}, Q] \hspace{1cm} \frac{\partial}{\partial \tilde{t}_s} \bar{Q} = [Q^+_{nm}, \bar{Q}]$$ \hspace{1cm} (7.5)

$$g [Q, \bar{Q}] = \mathbb{I}$$ \hspace{1cm} (7.6)

The subscripts $-/+ \,$ denote lower/upper triangular parts, whereas $(+)/(-)$ denote upper/lower triangular plus diagonal parts. The parametrization of the matrices $Q$ and $\bar{Q}$ is as follows:

$$Q_{nn} = a_0(n) \hspace{1cm} Q_{n,n+1} = 1 \hspace{1cm} Q_{n,n-k} = a_k(n) \hspace{1cm} k = 1, \ldots, p_2 - 1$$

$$Q_{nm} = 0 \hspace{1cm} \text{for} \hspace{1cm} m - n \geq 2 \hspace{1cm} n - m \geq p_2$$ \hspace{1cm} (7.7)

$$\bar{Q}_{nn} = b_0(n) \hspace{1cm} \bar{Q}_{n,n-1} = R_n \hspace{1cm} \bar{Q}_{n,n+k} = b_k(n) R_{n+1}^{-1} \cdots R_{n+k}^{-1} \hspace{1cm} k = 1, \ldots, p_1 - 1$$

$$\bar{Q}_{nm} = 0 \hspace{1cm} \text{for} \hspace{1cm} n - m \geq 2 \hspace{1cm} m - n \geq p_1$$ \hspace{1cm} (7.8)

Let us also introduce special notations for the first evolution parameters $t_1, \tilde{t}_1$ which in the sequel will be considered as space coordinates, i.e., $\tilde{t}_1 \equiv x$ and $t_1 \equiv y$. The lattice equations of motion (7.4) with $s = 1$ imply that:

(a) all matrix elements of $\bar{Q}$ can be expressed as functionals (w.r.t. $x$) of $R_{n+1}, b_0(n), \ldots, b_{p_1-2}(n)$ at a fixed lattice site $n$;

(b) all matrix elements of $Q$ are explicitly expressed through $R_{n+1}, b_0(n), \ldots, b_{p_1-2}(n)$ via the formula \cite{24}:

$$Q_{(-)} = (\bar{Q}^{p_2-1})_{(-)} + \left( \frac{1}{g} \right) x \mathbb{I}$$ \hspace{1cm} (7.9)

\footnote{Arbitrary integration constants are ignored without loss of generality.}
where the last term is due to matching with the “string” equation (7.9).

There is a complete duality under interchanging \( p_1 \leftrightarrow p_2 \) of the orders of the matrix potentials in (7.1), supplemented with interchanging \( x \equiv \tilde{t}_1 \leftrightarrow t_1 \equiv y \), \( Q_{(-)} \leftrightarrow Q_{(+)} \) [24]. Namely, we have:

(a) all matrix elements of \( Q \) can be expressed as functionals (w.r.t. \( y \)) of \( a_0(n), \ldots, a_{p_2-1}(n) \) at a fixed lattice site \( n \);

(b) all matrix elements of \( \bar{Q} \) are explicitly expressed through \( a_0(n), \ldots, a_{p_2-1}(n) \) as:

\[
\bar{Q}_{(+)} = (Q^{p_1-1})_{(+)} + \left( \frac{1}{g} \right) \mathbb{1}, \quad R_{n+1} = Q_{n+1,n}^{p_1-1} + \frac{1}{g} n
\]  

(7.10)

where the terms involving the two-matrix model coupling parameter \( g \) come from matching with the “string” equation (7.6).

Using the parametrization (7.7)–(7.8), the equations of the auxiliary linear Lax problem (7.2), (7.3) acquire the form:

\[
\lambda \psi_n = \psi_{n+1} + a_0(n) \psi_n + \sum_{k=1}^{p_2-1} a_k(n) \psi_{n-k} \tag{7.11}
\]

\[
-\frac{1}{g} \frac{\partial}{\partial \lambda} \psi_n = R_n \psi_{n-1} + b_0(n) \psi_n + \sum_{k=1}^{p_2-1} \frac{b_k(n)}{R_{n+1} \ldots R_{n+k}} \psi_{n+k} \tag{7.12}
\]

\[
\partial_x \psi_n = -R_n \psi_{n-1}, \quad \partial_y \psi_n = \psi_{n+1} + a_0(n) \psi_n \tag{7.13}
\]

Comparing eqs. (7.11)–(7.13) with (2.1), one identifies the two-matrix model as a special constrained Toda lattice hierarchy. Using (7.13), the implications of the lattice equations of motion:

\[
a_0(n+k) = a_0(n) + \partial_y \ln (R_{n+1} \ldots R_{n+k-1}), \quad a_0(n-k) = a_0(n) - \partial_y \ln (R_n \ldots R_{n-k+1})
\]

\[
b_0(n+k) = b_0(n) + \partial_x \ln (R_{n+1} \ldots R_{n+k-1}), \quad b_0(n-k) = b_0(n) - \partial_x \ln (R_n \ldots R_{n-k+1})
\]

and the string equation solutions (7.9)–(7.10), one can rewrite (7.11)–(7.13) and their compatibility conditions as continuum Lax problem at a fixed lattice site \( n \):

\[
\lambda \psi_n = L(n) \psi_n, \quad -\frac{1}{g} \frac{\partial}{\partial \lambda} \psi_n = \bar{L}(n) \psi_n, \quad \frac{\partial}{\partial t_r} \psi_n = \mathcal{L}_r(n) \psi_n, \quad \frac{\partial}{\partial t_s} \psi_n = -\bar{\mathcal{L}}_s(n) \psi_n \tag{7.14}
\]

\[
\frac{\partial}{\partial t_r} L(n) = \begin{bmatrix} \mathcal{L}_r(n) & L(n) \end{bmatrix}, \quad \frac{\partial}{\partial t_s} \bar{L}(n) = \begin{bmatrix} L(n) & \bar{\mathcal{L}}_s(n) \end{bmatrix} \tag{7.15}
\]

\[
\frac{\partial}{\partial t_r} \bar{L}(n) = \begin{bmatrix} \mathcal{L}_r(n) & \bar{L}(n) \end{bmatrix}, \quad \frac{\partial}{\partial t_s} L(n) = \begin{bmatrix} \bar{L}(n) & \mathcal{L}_s(n) \end{bmatrix} \tag{7.16}
\]

\[
\begin{bmatrix} L(n) & \bar{L}(n) \end{bmatrix} = \frac{1}{g} \mathbb{1} \tag{7.17}
\]

The explicit form of the continuum Lax operators depends on which equation in (7.13) we are using to express \( \psi_{n \pm \ell} \) at neighboring sites in terms of \( \psi_n \). In the “\( x \)-picture” (i.e., using
the first eq. (7.13)) we have:

\[
L(n) \equiv -D_x^{-1} R_{n+1} + a_0(n) + \sum_{k=1}^{p_2-1} (-1)^k \bar{Q}_{n,n-k}^{p_2-1} \left( D_x - \partial_x \ln (R_n \ldots R_{n-k+2}) \right) \times \ldots
\times \left( D_x - \partial_x \ln R_n \right) D_x \tag{7.18}
\]

\[
\bar{L}(n) \equiv -D_x + b_0(n) + \sum_{k=1}^{p_1-1} (-1)^k b_k(n) \left( D_x + \partial_x \ln (R_{n+1} \ldots R_{n+k}) \right)^{-1} \times \ldots
\times \left( D_x + \partial_x \ln R_{n+1} \right)^{-1} \tag{7.19}
\]

\[
\bar{L}_s(n) \equiv \sum_{k=1}^{s} (-1)^k \bar{Q}_{n,n-k}^s \left( D_x - \partial_x \ln (R_n \ldots R_{n-k+2}) \right) \ldots D_x \tag{7.20}
\]

\[
\bar{L}_r(n) \equiv Q_{nn}^r + \sum_{k=1}^{r} (-1)^k Q_{n,n+k}^r R_{n+1} \ldots R_{n+k} \times
\times \left( D_x + \partial_x \ln (R_{n+1} \ldots R_{n+k}) \right)^{-1} \ldots \left( D_x + \partial_x \ln R_{n+1} \right)^{-1} \tag{7.21}
\]

where all coefficients are expressed in terms of \( R_{n+1}, b_0(n), \ldots, b_{p_1-2}(n) \) at a fixed site \( n \) through the lattice equations of motion and (7.9). In the “\( y \)-picture” (i.e., using the second eq. (7.13)) we have:

\[
L(n) \equiv D_y + \sum_{k=1}^{p_2-1} a_k(n) \left( D_y - a_0(n) + \partial_y \ln (R_n \ldots R_{n-k+1}) \right)^{-1} \times \ldots
\times \left( D_y - a_0(n) + \partial_y \ln R_n \right)^{-1} \tag{7.22}
\]

\[
\bar{L}(n) \equiv (D_y - a_0(n))^{-1} R_n + b_0(n)
+ \sum_{k=1}^{p_1-1} Q_{n,n+k}^{p_1-1} \left( D_y - a_0(n) - \partial_y \ln (R_{n+1} \ldots R_{n+k-1}) \right) \ldots \left( D_y - a_0(n) \right) \tag{7.23}
\]

\[
= (D_y - a_0(n))^{-1} R_n + \sum_{k=0}^{p_1-3} \mathcal{F}_{k}^{p_1-1} (a_0, \ldots, a_{p_1-1}) D_y^k + D_y^{p_1-1} \tag{7.24}
\]

\[
\bar{L}_s(n) \equiv \sum_{k=1}^{s} \bar{Q}_{n,n-k}^s \left( D_y - a_0(n) + \partial_y \ln (R_n \ldots R_{n-k+1}) \right)^{-1} \ldots \left( D_y - a_0(n) + \partial_y \ln R_n \right)^{-1} \tag{7.25}
\]

\[
\bar{L}_r(n) \equiv Q_{nn}^r + \sum_{k=1}^{r} Q_{n,n+k}^r \left( D_y - a_0(n) - \partial_y \ln (R_{n+1} \ldots R_{n+k-1}) \right) \ldots \left( D_y - a_0(n) \right) \tag{7.26}
\]

where all coefficients are expressed in terms of \( a_0(n), \ldots, a_{p_2-1}(n) \) at a fixed site \( n \) through the lattice equations of motion and (7.10). In particular, they imply the form (7.24) of the second Lax operator \( \bar{L}(n) \). In (7.18)–(7.26) we have used notations \( D_x, D_y \) for differential operators to distinguish them from ordinary derivatives on functions.

Eqs. (7.14)–(7.26) are the continuum analogs of the constrained Toda lattice Lax equations (7.2)–(7.6) without taking any continuum (double-scaling) limit. Let us particularly stress that they explicitly incorporate the whole information from the matrix-model string equation (7.4) through (7.9) and (7.10) which were used in their derivation. Their respective flows (7.15) and (7.16) are compatible (commuting), so it is sufficient to consider only one of them.
In ref. [24] we used the “x-picture” Lax operators (7.18)–(7.19) for \( p_2 = 3 \) and arbitrary \( p_1 \) to show that the corresponding constrained Toda lattice hierarchy (7.2)–(7.6) is equivalent to the \( SL(3,1) \)-KdV hierarchy defined by:

\[
L(n) = -D_x^{-1}R_{n+1} + 2b_1(n) + b_0^2(n) - \partial_x b_0(n) - 2b_0(n)D_x + D_x^2 \quad (7.27)
\]

The Lax operator (7.27) can be cast into the form (6.43) via a simple gauge (similarity) transformation. Now both \( p_1 \) and \( p_2 \) are arbitrary and let us assume for definiteness \( p_1 \leq p_2 \).

Here, in order to conform with the formalism of the previous sections, we shall use the “y-picture” Lax operators (7.22) and (7.24). Comparing \( L(n) \) (7.22) with (2.19) we see that the former is a constrained \( 2(p_2 - 1) \)-boson KP Lax operator with coefficients:

\[
A_{p_2-k}^{(p_2-1)} = a_k(n) , \quad B_{p_2-k}^{(p_2-1)} = a_0(n) - \partial_y \ln \left( R_n(\{a\}) \cdots R_{n-k+1}(\{a\}) \right) \quad (7.28)
\]

which depend on \( p_2 \) fields only: \( \{a\} \equiv (a_0(n), \ldots, a_{p_2-1}(n)) \). Thus, there is only one independent current, e.g., \( B_{p_2-1}^{(p_2-1)} \).

On the other hand, the second Lax operator \( \bar{L}(n) \) (7.23)–(7.24) has precisely the form of a \( SL(p_1,1) \)-KdV Lax operator, cf. (3.33) for \( p_1 = M + 1 \). Moreover, when \( p_1 \leq p_2 \) its coefficients depend only on \( a_0(n), \ldots, a_{p_1-1}(n) \) which easily follows from (7.11) and the explicit parametrization (7.7)–(7.8). Thus, the constrained generalized Toda lattice hierarchy (7.2)–(7.6), describing the two-matrix string model (7.1), is equivalent to the \( SL(p_1,1) \)-KdV hierarchy (7.16) with finite number of flows.

8 Conclusions

We conclude with a list of remarks.

- We have described here the process of reduction of the multi-boson KP hierarchies analyzed in two special settings, or gauges, of the associated Toda matrix spectral problem. Our basic gauge was the Drinfeld-Sokolov gauge naturally associated with the Toda lattice hierarchy. Another gauge, which proved to be useful in our discussion, especially when dealing with the Poisson bracket structure, was the diagonal gauge related to the Volterra lattice. Modes associated with the Volterra lattice abelianized and simplified the analysis of the second Hamiltonian structure. These two gauges led to two different Lax formulations having different transformation properties under the residual gauge transformations.

- We have obtained in this paper a coherent approach to describe generalized KP-KdV (i.e., \( SL(M + 1, M - k) \)-KdV) hierarchies and their Poisson bracket structures. Variables (free currents) abelianizing the second bracket structure were instrumental tools in this analysis. They naturally lead to the appearance of the graded \( SL(M + 1, M - k) \) Kac-Moody algebras. This raises the question about the origin of the graded algebra in this setting. One possible explanation could be given in terms of the underlying lattice structure when one recalls that the transition from Toda to Volterra lattice involves separation in even and odd sites. This could be an intuitive way of seeing how the grading could have been introduced into the formalism.
The free fields (currents) abelianizing the second Hamiltonian structure of the KP-KdV hierarchies bring about another noticeable result. Namely, they yield explicit free-field representations of the nonlinear $W(M,M - k)$ algebras, isomorphic to the $SL(M + 1, M - k)$-KdV Poisson bracket algebras, which generalize Zamolodchikov’s nonlinear $W_{M+1}$ algebra.

Lattice translations in the underlying Toda and Toda-like hierarchies naturally give rise to similarity transformations of the corresponding KP-KdV Lax operators which, first, preserve the Hamiltonians and the Poisson structures, and second, systematically generate the pertinent auto-Bäcklund transformations for the generalized $SL(M + 1, M - k)$-KdV hierarchies.

The physical relevance of the structures, defined by the reduction process described in this paper, is now strongly enhanced by the natural appearance of the generalized $SL(M + 1, 1)$-KdV hierarchy within the context of the two-matrix string model. More precisely, ref. [24] identified the two-matrix model in the simplest nontrivial case with a coupled system of 2+1-dimensional KP and modified KP ((m)KP$_{2+1}$) integrable equations subject to a specific “symmetry” constraint. This constraint together with the Miura-Konopelchenko map [17] for (m)KP$_{2+1}$ are the images in the continuum of the matrix-model string equation (7.6). In particular, the two-matrix model susceptibility is a solution to the above string-constrained KP$_{2+1}$ equation. The string-constrained (m)KP$_{2+1}$ system was shown to be equivalent to the 1 + 1-dimensional generalized $SL(3, 1)$-KdV hierarchy (7.21), (6.43). Previously, the generalized KP-KdV models were obtained in [23] from two-matrix models by imposing ad hoc additional Dirac constraints on the multi-boson KP hierarchy.

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