On Lelong numbers of generalized Monge–Ampère products

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Abstract
We consider generalized (mixed) Monge–Ampère products of quasiplushisubharmonic functions (with and without analytic singularities) as they were introduced and studied in several articles written by subsets of Andersson, Wulcan, Błocki, Lärkäng, Raufi, Ruppenthal, and the author. We continue these studies and present estimates for the Lelong numbers of push-forwards of such products by proper holomorphic submersions. Furthermore, we apply these estimates to Chern and Segre currents of pseudoeffective vector bundles. Among other corollaries, we obtain the following generalization of a recent result by Wu. If the non-nef locus of a pseudoeffective vector bundle $E$ on a Kähler manifold is contained in a countable union of $k$-codimensional analytic sets, and if the $k$-power of the first Chern class of $E$ is trivial, then $E$ is nef.

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1 Introduction
For a plurisubharmonic (psh) function $u$ on a domain $D \subset \mathbb{C}^n$, the Lelong number in $x_0 \in D$ is defined by

$$\nu(u, x_0) := \liminf_{x \to x_0} \frac{u(x)}{\log \|x - x_0\|}.$$ 

This can be seen as a generalization of the vanishing order (of $e^u$) since $\nu(\log |f|, 0) = \text{ord}_{x_0} f$ for a holomorphic function $f: D \to \mathbb{C}$. Since its introduction in [23, 24], Lelong numbers of psh functions and (its generalized version) of closed positive currents have proven to be very crucial tools in many areas beside of pluripotential theory. Many of its applications involve estimates on Lelong numbers. Let us recall a particular one which gives estimates on Lelong numbers of (mixed) Monge–Ampère products by the Lelong numbers of their potentials from below, see [13, 15].

In memory of Jean-Pierre Demailly

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Generalizing a fundamental result from Bedford and Taylor in [5, 6] defining the Monge–Ampère products of bounded psh functions (cited as Theorem 2.6), Demailly proved in [13, 15] that for a closed positive current $T$ of bidegree $(p, p)$ and a plurisubharmonic function $u$ such that its unbounded locus $L(u)$ is contained in an analytic set of codimension greater than or equal to $p + k$,

$$
(dd^c u)^k \wedge T := \lim_{k \to \infty} (dd^c u_k)^k \wedge T
$$

is a closed positive current whereby $d^c := \frac{i}{2\pi} (\overline{\partial} - \partial)$ and $u_k$ is a sequence of smooth psh functions decreasing pointwise to $u$ (cited as Theorem 2.8). To shorten the notation, we will use BTD for the reference to such Monge–Ampère (MA) products.

With BTD, we can calculate the Lelong number of a closed positive $(p, p)$-current $T$ by

$$
v(T, x_0) = \int_{x_0} (dd^c \log \|x - x_0\|)^{n-p} \wedge T.
$$

We get that $v(dd^cu, x_0) = v(u, x_0)$. Demailly’s second comparison formula for Lelong numbers (see [13, 15]) implies

$$
v((dd^c u)^k \wedge T, x_0) \geq v(u, x_0)^k \cdot v(T, x_0) \quad (1.1)
$$

for $k + p$ as above. The purpose of this work is to show that such estimates hold for more general Monge–Ampère products as explained in the following.

Let $X$ and $Y$ be complex manifolds, and let $\pi : X \to Y$ be a proper holomorphic submersion with $m$-dimensional fibres. Let $q$ be a quasipsh function, let $\alpha$ be a closed real $(1, 1)$-form, and let $T$ be a closed positive $(p, p)$-current, all defined on $X$. Furthermore, we assume that for any small enough open $V \subset Y$, there is a closed positive\(^1\) $(1, 1)$-form $\gamma$ on $U := \pi^{-1}(V)$ such that $q$ is $(\alpha + \gamma)$-psh on $U$, i.e. $dd^c q + \alpha \geq -\gamma$. This is always the case if a priori $dd^c q + \alpha$ is assumed to be positive, or if $\pi$ is Kähler, see Remark 3.2. Let $q_k$ be a sequence of $(\alpha + \gamma)$-psh functions which is decreasing pointwise to $q$. Following [20], we obtain that for all $k$ such that $\pi(L(q))$ is contained in an analytic set of codimension $\geq k + p - m$, the current

$$
\pi_*(|[dd^c q + \alpha]^k \wedge T|) := \lim_{k \to \infty} \pi_*(|(dd^c q_k + \alpha)^k \wedge T|) \quad (1.2)
$$

is well-defined and locally the difference of two closed positive currents which is independent of the choice of $\alpha$ and $q_k$, see Proposition 3.1. This generalizes the definition of MA products by BTD as the $L(q)$ is allowed to be in an analytic set of codimension strictly greater than $k + p$.

The first main result of the present work gives an estimate on the Lelong numbers of currents defined by (1.2) which generalizes (1.1).

**Theorem 1.1** Let $\pi : X \to Y$ be a proper holomorphic submersion between complex manifolds $X$ and $Y$ with $m$-dimensional fibres. Fix a point $y \in Y$. Let $\theta_1, \ldots, \theta_t$ be positive $(1, 1)$-currents on $X$ such that each $\theta_i$ is in a Kähler class on a neighbourhood of $\pi^{-1}(y)$. Then, there exist positive constants $b_i$, for $i = 1, \ldots, t$, (which only depend of the Kähler class represented by $\theta_i$ on a neighbourhood of $\pi^{-1}(y)$) such that the following statements are correct.

\(^1\) Following the notation usual for currents, we call a $(1, 1)$-form $\alpha$ positive if $\alpha \geq 0$.  

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(i) If the union of all images $\pi(L(\theta_t))$ of the unbounded loci of local $dd^c$-potentials of $\theta_t$ is contained in an analytic $A$ with codim $A \geq k_1 + \cdots + k_t - m$, then

$$v\left(\pi_*([\theta_t]^{k_1}) \land \cdots \land [\theta_t]^{k_t}\right)_y \geq \prod_{i=1}^t \min\{v(\theta_t, x), \delta_i\}^{k_i} \tag{1.3}$$

for all points $x \in \pi^{-1}(y)$ as long the LHS does not vanish due to the degree of the current.

(ii) If $\bigcup_i \pi(L(\theta_t))$ is contained in an analytic $A$ with codim $A \geq k_1 + \cdots + k_t - t \cdot m$, then

$$v\left(\pi_*([\theta_t]^{k_1}) \land \cdots \land \pi_*([\theta_t]^{k_t})\right)_y \geq \prod_{i=1}^t \min\{v(\theta_t, x), \delta_i\}^{k_i} \tag{1.4}$$

for all points $x \in \pi^{-1}(y)$ as long the LHS does not vanish due to the degree of the current.

**Remark 1.2**

(a) The bound $\delta_i$ is determined by whether for each $x \in \pi^{-1}(y)$, there exists a closed positive current which is in the same cohomology class as $\theta_t$ near $\pi^{-1}(y)$, smooth everywhere except in isolated points, and whose Lelong number is greater than or equal to $\delta_i$ in $x$; see Lemma 6.1. For example, if $\pi$ is projective and $\theta_t$ is in the class induced by the Fubini-Study metric, then we can select $\delta_i = 1$.

(b) If we consider proper holomorphic submersions $\pi_i: X_i \to Y, i = 1, \ldots, t$, and if each $\theta_t$ is defined on $X_i$ with analogous properties as in Theorem 1.1, there are positive constants $\delta_i$ such that

$$v\left(\pi_{t,*}([\theta_t]^{k_1}) \land \cdots \land \pi_{1,*}([\theta_t]^{k_1})\right)_y \geq \prod_{i=1}^t \min\{v(\theta_t, x_i), \delta_i\}^{k_i} \tag{1.5}$$

for all points $x_1 \in \pi_i^{-1}(y), \ldots, x_t \in \pi_t^{-1}(y)$ and suitable $k_1, \ldots, k_t$.

Let us sketch the proof of Theorem 1.1 (i). We will use generalized Monge–Ampère products for currents with analytic singularities which were introduced and studied in [1–4, 9, 21, 22] and the present work as explained as follows.

If a quasipsh function $q$ equals locally $c \log \|F\|^2 + b$ for a positive constant $c$, holomorphic tuple $F$ and bounded function $b$, $q$ is said to have analytic singularities in $\{F = 0\}$ (given as reduced analytic set). If moreover $b$ is smooth, $q$ has so-called neat analytic singularities. We say that a closed quasipos (1, 1)-current $\theta$ has (neat) analytic singularities in $Z$ if it is locally given by $\theta = dd^c q$ for a quasipsh $q$ with (neat) analytic singularities in $Z$. For $i = 1, \ldots, t$, let $\theta_t$ be closed quasipos (1, 1)-currents with analytic singularities in $Z_i$, and let $\alpha_i$ be (closed) real (1, 1)-forms. Then, the generalized Monge–Ampère product $[\theta_t]_{\alpha_i}^{k_1} \land \cdots \land [\theta_t]_{\alpha_i}^{k_t}$ for any $k_1, \ldots, k_t$ is defined recursively by

$$[\theta_t]_{\alpha_i} \land T := \theta_t \land 1_Z; T + \alpha_i \land 1_Z T. \tag{1.6}$$

This definition works also if we replace $\alpha_i$ by quasipositive (1, 1)-currents $\eta_i$ with neat analytic singularities in isolated points since the last term in (1.6) is defined by BTD, see Definition 4.6. This will be useful to get estimates for the Lelong number of such MA products, see Proposition 5.8 and Remark 5.9. Although the generalized Monge–Ampère product lacks some properties of a product, it is a suitable extension of the BTD Monge–Ampère product to arbitrary degrees independent of the unbounded locus of $q$. By Theorem 4.7 (cf. [22, Thm 1.2]), we get that if $\eta_i$ are in the same $\partial \overline{\partial}$-class as $\theta_t$, and if $\bigcup_i \pi(L(\theta_t))$ is contained in an analytic $A$ with codim $A \geq k_1 + \cdots + k_t - m$, then

$$\pi_*([\theta_t]^{k_1} \land \cdots \land [\theta_t]^{k_t}) = \pi_*([\theta_t]_{\eta_i}^{k_1} \land \cdots \land [\theta_t]_{\eta_i}^{k_t}). \tag{1.7}$$
Therefore, the idea behind the proof of Theorem 1.1 is to use Demailly’s regularization [14, Main Thm 1.1] (see also Theorem 2.10) which gives us that the (quasi-) positive \((1,1)\)-currents \(\theta_i\) can be approximated by sequences of quasipositive \((1,1)\)-currents \(\theta_{i,k}\) with analytic singularities (and other suitable properties). By the monotone continuity result Theorem 3.5, we get
\[
\pi_\ast([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1}) := \lim_{k \to \infty} \pi_\ast([\theta_{i,k}]^{k_1} \wedge \cdots \wedge [\theta_{1,k}]^{k_1})
\]  
(1.8)

(actually for any approximation). We would like to add that Theorem 3.5 implies Corollary 3.6 which asserts that for certain \(k_1, \ldots, k_t\),
\[
\pi_\ast[\theta_i]^{k_1} \wedge \cdots \wedge \pi_\ast[\theta_1]^{k_1} := \lim_{k \to \infty} \pi_\ast[\theta_{i,k}]^{k_1} \wedge \cdots \wedge \pi_\ast[\theta_{1,k}]^{k_1}.
\]

We obtain that Theorem 1.1 is implied by (1.8), (1.7) and the following theorem, which is the second main result of the present work.

**Theorem 1.3** Let \(\pi : X \to Y\) be a proper holomorphic submersion between complex manifolds \(X\) and \(Y\) with \(m\)-dimensional fibres. Fix a point \(y \in Y\). Let \(\theta_1, \ldots, \theta_t\) be positive \((1,1)\)-currents with analytic singularities on \(X\) such that each \(\theta_i\) is in a Kähler class on a neighbourhood of \(\pi^{-1}(y)\). Then, there exist positive constants \(\delta_i, i = 1, \ldots, t\), (which only depend of the Kähler class represented by \(\theta_i\) on a neighbourhood of \(\pi^{-1}(y)\)) such that the following is correct.

(i) Existence of \(\eta_i\): For every \(x \in \pi^{-1}(y)\), there exist closed quasipositive \((1,1)\)-currents \(\eta_i = \eta_{i,x} \in \{\theta_i\}_{\alpha^\pi}\), positive near \(\pi^{-1}(y)\), with neat analytic singularities only in \(\{x\}\) (and with \(\nu(\eta_i, x) = \delta_i\)) such that
\[
\nu\left(\pi_\ast([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1}), y\right) \geq \prod_{i=1}^t \min\{\nu(\theta_i, x), \delta_i\}^{k_i}
\]  
(1.9)

for all \(m \leq k_1 + \cdots + k_t \leq \dim X\).

Thereby, \(\eta_i = \eta_{i,x}\) depends only of the Kähler class of \(\theta_i\) near \(\pi^{-1}(y)\) and \(x\).

(ii) Independency of \(\eta_i\): For \(i = 1, \ldots, t\), let \(\eta_i \in \{\theta_i\}_{\alpha^\pi}\) be closed quasipositive \((1,1)\)-currents with neat analytic singularities in isolated points on \(X\) such that there is a point \(x \in X\) with \(L(\eta_i) \cap \pi^{-1}(y) = \{x\}\) and \(\nu(\eta_i, x) \leq \delta_i\). Then,
\[
\nu\left(\pi_\ast([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1}), y\right) \geq \prod_{i=1}^t \min\{\nu(\theta_i, x), \nu(\eta_i, x)\}^{k_i}
\]  
(1.10)

for all \(m \leq k_1 + \cdots + k_t \leq \dim X\).

Remark 1.2 (a) applies to Theorem 1.3, as well.

The idea to consider \(\eta_i\) with singularities in isolated points is inspired by X. Wu’s approach to choose specific non-smooth approximations of Segre currents to obtain Lelong number estimates of them, see [27, Proof of Thm 2].

Following the arguments in the proof of [22, Thm 1.1 (3)], the Lelong numbers of \(\pi_\ast([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1})\) are independent of \(\alpha_i\) among closed forms in the same class as \(\theta_i\). Unfortunately, this is not correct if we replace \(\alpha_i\) by \(\eta_i\) with different kind of isolated singularities, see Remark 5.12. By Proposition 5.13, we get at least that \(\pi_\ast([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1})\)’s Lelong numbers are independent of \(\eta_i\) among currents with the same kind of singularities in all points.

By applying Theorem 1.3 to Segre currents defined by (7.3) as in [22, Thm 1.1], we obtain the following corollary; see Remark/Def. 7.7.
Corollary 1.4 Let \( X \) be a complex manifold, let \( E \to X \) be a (pseudoeffective) holomorphic vector bundle on \( X \) of rank \( r \), and let \( e^{-\varphi} \) be a semipositive singular metric on \( \mathcal{O}_{\mathbb{P}(E)}(1) \) with analytic singularities, i.e. the weights \( \varphi \) are psh with analytic singularities. Then, for every \( x \in X, \xi \in \pi^{-1}(x) \) and every \( k \leq \dim X \), there exists a closed real \((k, k)\)-current \( S = S_\xi \) in the Segre class \((-1)^k s_k(E)\) such that (i) locally \( S \) is the difference of two closed positive currents, (ii) \( S \) is positive close to \( x \), and (iii) its Lelong number can be estimated from below by

\[
\nu(S, x) \geq \min\{\nu(\varphi, \xi), 1\}^{k + tr - t}.
\]

Analogously, Theorem 1.1 gives us estimates for the Lelong numbers of Segre currents defined by Theorem 7.2, following [20, Prop. 4.6].

Corollary 1.5 Let \( X \) be a complex manifold of dimension \( n \), let \( E \to X \) be a (pseudoeffective) holomorphic vector bundle on \( X \) of rank \( r \), and let \( e^{-\varphi} \) be a semipositive singular metric on \( \mathcal{O}_{\mathbb{P}(E)}(1) \) such that \( \pi(L(\varphi)) \) is contained in an analytic set of codimension \( s \). Then, for all partitions \( k_1 + \cdots + k_t = k \leq s \), all \( x \in X \) and all \( \xi \in \pi^{-1}(x) \), we get

\[
(-1)^k \nu(s_{k_1}(E, \varphi) \wedge \cdots \wedge s_{k_t}(E, \varphi), x) \geq \min\{\nu(\varphi, \xi), 1\}^{k + tr - t}.
\]

Furthermore, Corollary 1.5 implies the following sufficient condition for \( E \) being nef.

Corollary 1.6 Let \( X \) be a compact complex manifold of dimension \( n \), let \( E \to X \) be a (pseudoeffective) holomorphic vector bundle on \( X \), and let \( e^{-\varphi} \) be a semipositive singular metric on \( \mathcal{O}_{\mathbb{P}(E)}(1) \) such that \( \pi(L(\varphi)) \) is contained in an analytic set of codimension \( s \). If there are \( k_1, \ldots, k_t \) with \( k_1 + \cdots + k_t \leq s \) and \( s_{k_1}(E) \cdots s_{k_t}(E) = 0 \), then \( E \) is nef.

Let us point out that \( X \) does not need to be Kähler in Corollary 1.6. Yet, the condition on \( \pi(L(\varphi)) \), which gives us the existence of the positive Segre currents, can be seen as quite strong. Assuming that \( X \) is Kähler, we can weaken this condition by using the non-nef locus which is defined by

\[
L_{nnf}(E) := \pi\left( L_{nnf}(\mathcal{O}_{\mathbb{P}(E)}(1)) \right) = \pi\left( \bigcup_{\delta > 0} \bigcap_{\vartheta \in \mathcal{O}_{\mathbb{P}(E)}} E_+(\vartheta) \right)
\]

where the intersection runs over all closed \( \delta\mathcal{O}_{\mathbb{P}(E)} \)-positive \((1, 1)\)-currents \( \vartheta \in \mathcal{O}_{\mathbb{P}(E)}(1) \) for a Kähler form \( \omega_{\mathbb{P}(E)} \) on \( \mathbb{P}(E) \), and \( E_+(\vartheta) := \{ \xi \in \mathbb{P}(E) : \nu(\vartheta, \xi) > 0 \} \). We obtain the following generalization of X. Wu’s main result in [27].

Theorem 1.7 Let \((X, \omega)\) be a compact Kähler manifold, and let \( E \) be a pseudoeffective vector bundle on \( X \) such that \( L_{nnf}(E) \) is contained in a countable union of analytic sets of codimension \( s \). If there is a \( k \leq s \) with \( c_1(E)^k = (-s_1(E))^k = 0 \), then \( E \) is nef.

If \( X \) is strongly pseudoeffective, i.e. \( L_{nnf}(E) \neq X \) (see [11, Def. 7.1]), then the assumption on \( L_{nnf}(E) \) is satisfied for \( k = 1 \). Hence, we obtain X. Wu’s original result that all strongly pseudoeffective vector bundles with trivial first Chern class are nef. To prove Theorem 1.7, we will follow X. Wu’s argumentation. Yet, instead of the analytical construction of closed positive currents tailored to prove his result (we will see that these coincide with the one defined above, see Remark 7.4), we will use Theorem 1.1 which is based on the calculus of generalized Monge–Ampère products, see Sects. 4 and 5.

The present work is organized as follows. In Sect. 2, we recall and elaborate some basic properties which are later used. We continue with the study of currents as they are defined by (1.2) in Sect. 3. In particular, we prove the mentioned monotone continuity results Theorem
3.5 and Corollary 3.6 there. In Sect. 4, we introduce so-called currents with analytic singularities and study their properties. We continue with a section about Lelong numbers of closed real currents, see Sect. 5. Section 6 is dedicated to prove the main results, Theorems 1.1 and 1.3. We conclude the article with a short discussion on Segre currents of (pseudoeffective) vector bundles and proving Corollary 1.6 and Theorem 1.7 in Sect. 7.

2 Preliminaries

In this section, we will introduce some notations and collect results from the literature. Considering the wedge product of pushforwards, the following tool is very helpful.

Lemma 2.1 (Lem. 6.3 in [22]) Let $Y$ be a complex manifold. For $i = 1, \ldots, t$, let $X_i$ be complex manifolds, let $\pi_i : X_i \to Y$ be proper holomorphic submersions, and let $\alpha_i$ be forms on $X_i$. Let $\tilde{X} := X_t \times Y \cdots \times Y X_1$ be the fibre product of $\pi_t, \ldots, \pi_1$, let $\text{pr}_i : \tilde{X} \to X_i$ denote its projections on the $i$th component, and let $T$ be a current on $X_1$. Then,

$$\pi_t, * \alpha_t \wedge \cdots \wedge \pi_2, * \alpha_2 \wedge \pi_1, * T = \pi, * \left( \text{pr}_t^* \alpha_t \wedge \cdots \wedge \text{pr}_2^* \alpha_2 \wedge \text{pr}_1^* T \right).$$

2.1 Closed positive currents

In this subsection, we recall some basic facts about closed positive currents. Throughout it, let $X$ denote a complex manifold of dimension $n$. In [20, Def. 4.2], we introduced the following notion.

Definition 2.2 A strongly positive $(k, k)$-form $\beta$ is called bump form at a point $x \in X$ if for a (or equivalently for any) Kähler form $\omega$ defined near $x$, there exists a constant $\delta > 0$ such that $\beta - \delta \omega^k$ is a strongly positive form in a neighbourhood of $x$.

Remark 2.3 (Lem. 7.2 in [22]) As presented in the proof of [20, Lem. 4.3], for every analytic $A$ with $\dim A \leq k$ and every point $x \in A$, there exists a $(k, k)$-bump form $\beta$ with arbitrarily small support s.t. the support of $dd^c \beta$ is in the complement of $A$.

The notion of bump form turns out to be very useful to obtain the uniqueness of the extensions of closed positive currents across analytic sets as follows.

Lemma 2.4 (Lem. 4.5 in [20]) Let $T$ and $S$ be two closed positive $(p, p)$-currents on $X$ such that $T = S \mid X \setminus A$ for an analytic set $A$ with codim $A \geq p$. If for every point $x \in A$, there is an $(n-p, n-p)$-bump form $\beta$ at $x$ with arbitrarily small support such that $\int T \wedge \beta$ converges to numbers $\int S \wedge \beta$ for any $(n-p, n-p)$-bump form $\beta$ with arbitrarily small support as $\kappa \to \infty$. Then, $T = S \mid X$.

Proposition 2.5 (cf. Rem. 4.7 in [20]) Let $A$ be an analytic subset of $X$ with codim $A \geq p$, and let $T_\kappa$ be a sequence of closed positive $(p, p)$-currents on $X$. We assume (i) the sequence $T_\kappa$ converges weakly to a closed positive current $T'$ on $X \setminus A$, and (ii) $\int T_\kappa \wedge \beta$ converges to numbers $T_\beta$ for any $(n-p, n-p)$-bump form $\beta$ with arbitrarily small support as $\kappa \to \infty$. Then, $T_\kappa$ converges weakly to a closed positive $(p, p)$-current $T$ on $X$ which is uniquely defined by $T'$ and $T_\beta = \int T \wedge \beta$.

Proof We follow the argumentation of the proof of Prop. 4.6 in [20].

Due to the assumptions (i) and (ii), we obtain that the trace measures of $T_\kappa$ are (locally) uniformly bounded in $\kappa$. Therefore, the Banach–Alaoglu theorem implies that there is a
subsequence $\kappa_\lambda \to \infty$ such that $T_{\kappa_\lambda}$ converges weakly to a closed positive current $T$. Let us assume that there is another subsequence $T_{\kappa_\lambda'}$ which does not converge to $T$. By passing to a subsequence, we may assume that $T_{\kappa_\lambda'}$ converges to a closed positive current $S$ different from $T$. By the assumption (i), we get that $S$ equals $T$ on $X \setminus A$. Furthermore, the assumption (ii) implies that $\int T \wedge \beta = T_\beta = \int S \wedge \beta$ for any $(n-p, n-p)$-bump form with arbitrarily small support. By Lemma 2.4, we get $T = S$ on $X$. This is a contradiction. \hfill $\square$

Next we recall the fundamental result of Bedford–Taylor defining MA products for bounded psh functions and its generalization by Demailly.

**Theorem 2.6** (cf. [5, 6]) Let $u_1, \ldots, u_t$ be bounded psh functions, and let $T$ be a closed positive $(p, p)$-current on $T$. Then, there exists a well-defined closed positive $(t + p, t + p)$-current

$$dd^c u_1 \wedge \cdots \wedge dd^c u_t \wedge T, \quad \text{locally given as the limit of } dd^c u_{t,k} \wedge \cdots \wedge dd^c u_{1,k} \wedge T$$

for sequences of smooth psh functions $u_{i,k}$ decreasing pointwise to $u_i$. Furthermore, we obtain that this is monotone continuous: Let $v_{i,k}$ be sequences of (not necessarily smooth) psh functions decreasing pointwise to $u_i$. Then,

$$u_t dd^c u_{t-1} \wedge \cdots \wedge dd^c u_1 \wedge T = \lim_{k \to \infty} v_t dd^c v_{t-1,k} \wedge \cdots \wedge dd^c v_{1,k} \wedge T, \quad \text{and}$$

$$dd^c u_t \wedge \cdots \wedge dd^c u_1 \wedge T = \lim_{k \to \infty} dd^c v_{t,k} \wedge \cdots \wedge dd^c v_{1,k} \wedge T.$$

**Definition 2.7** For a psh function $u$ on $X$, the unbounded locus $L(u)$ is defined as the closed set of all points $x \in X$ such that $u$ is unbounded near $x$. In general, $L(u)$ is strictly larger than the pole set of $u$.

**Theorem 2.8** (cf. [13, 15]; shortly BTD) Let $u$ be a psh function, and let $T$ be a closed positive $(p, p)$-current. If $L(u) \cap \text{supp } T$ is contained in an analytic $A$ with codim $A \geq k + p$, then there exists a well-defined closed positive $(k + p, k + p)$-current $(dd^c u)^k \wedge T$ (locally) given as the limit of

$$(dd^c u_{i,k})^k \wedge T$$

for any sequence of smooth psh functions $u_{i,k}$ decreasing pointwise to $u$. Furthermore, we obtain that this is monotone continuous: Let $u_1, \ldots, u_t$ be psh functions with $L(u_i) \cap \text{supp } T$ is contained in an analytic $A$ with codim $A \geq t + p$, and $v_{i,k}$ be sequences of (not necessarily smooth) psh functions decreasing pointwise to $u_i$. Then,

$$dd^c u_t \wedge \cdots \wedge dd^c u_1 \wedge T = \lim_{k \to \infty} dd^c v_{t,k} \wedge \cdots \wedge dd^c v_{1,k} \wedge T.$$

Considering the mixed Monge–Ampère product (with different potentials in each factor), the codimension condition can be weaken by considering the (Hausdorff) dimension of the intersections of the unbounded loci, see Thm 4.5 and Prop. 4.9 in [16, Chp. III].

### 2.2 Closed quasipositive currents

Let $X$ be a complex manifold of dimension $n$, and let $\gamma$ be a (positive) $(1, 1)$-form on $X$. We call a function $q$ on $X \gamma$-psh if $dd^c q + \gamma$ is positive. This is equivalent to $q$ is quasipsh with $dd^c q + \gamma \geq 0$ in the sense of currents. Let $\theta$ be a closed quasipositive $(1, 1)$-current on $X$, i.e. locally, $\theta = dd^c q$ for a quasipsh function $q$ (shortly quasipos; also called almost
positive). \(\theta\) is called \(\gamma\)-positive (\(\gamma\)-pos) if \(\theta + \gamma \geq 0\) in the sense of currents, i.e. locally, \(\theta = dd^c q\) for a \(\gamma\)-psh function \(q\). Let us pick a closed real \((1, 1)\)-form \(\alpha\) which is in the same \(\partial \bar{\partial}\)-cohomology class as the \(\gamma\)-pos \(\theta\), i.e. \(\theta - \alpha = dd^c q\) for a \((\gamma + \alpha)\)-psh function \(q\) on \(X\). We define \(L(\theta) := L(q)\) which is independent of the choice of \(q\) and \(\alpha\). Let \(q_k\) be a sequence of (smooth) \((\gamma + \alpha)\)-psh functions which is decreasing pointwise to \(q\). Set \(\theta_k := dd^c q_k + \alpha\) which are (smooth) \(\gamma\)-positive currents. Obviously, the sequence \(\theta_k\) converges weakly to \(\theta\). Let us use the following notation improperly by saying that the sequence \(\theta_k\) is decreasing pointwise to \(\theta\). This is motivated by the following. If \(u\) is a psh function on a small enough open set with \(dd^c u = \theta + \gamma \geq 0\) (assuming that \(\gamma\) is closed), then \(u_k := q_k - q + u\) are (smooth) psh functions with \(dd^c u_k = \theta_k + \gamma \geq 0\) such that the sequence \(u_k\) is decreasing pointwise to \(u\).

Locally, we may assume that \(\gamma\) is closed and positive: On any small enough open set, there is a Kähler form \(\omega\) and a constant \(C\) such that \(\gamma \leq C \omega\). In particular, \(\theta + C \omega \geq \theta + \gamma \geq 0\) (and \(\theta_k + C \omega \geq 0\)). Therefore, we may replace \(\gamma\) by \(C \omega\) on any small enough open set.

Let \(T\) be a closed positive \((p, p)\)-current, and assume that \(\theta_k\) are smooth. By BTD, for all \(k\) such that \(L(\theta)\) is contained in an analytic set of codimension \(\geq k + p\), (assuming \(\gamma\) is closed, see above)

\[
(\theta + \gamma)^k \wedge T \overset{\text{def}}{=} \lim_{k \to \infty} (\theta_k + \gamma)^k \wedge T \overset{\text{loc}}{=} \lim_{k \to \infty} (dd^c u_k)^k \wedge T
\]

is a closed positive current which is independent of the choice of \(q_k / \theta_k \wedge u_k\). We can extend this definition to (wedge) powers of the quasipositive \(\theta\) by

\[
\theta^k \wedge T = (\theta + \gamma - \gamma)^k \wedge T := \sum_{i=0}^{k} (-1)^i \binom{k}{i} \gamma^{i} \wedge (\theta + \gamma)^{k-i} \wedge T,
\]

\[
\overset{\text{def}}{=} \lim_{k \to \infty} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \gamma^{i} \wedge (\theta_k + \gamma)^{k-i} \wedge T = \lim_{k \to \infty} \theta_k^k \wedge T. \tag{2.1}
\]

Due to the last equation, we see that the definition is independent of the choice of \(\gamma\). \(\theta^k \wedge T\) is a closed real \((k+p, k+p)\)-current which is locally the difference of two closed positive currents.

At the end of this section, we recall following versions of Demailly’s regularization of quasipos (1, 1)-currents fitting to our applications.

**Theorem 2.9** ([12], Main Thm 1.1 in [14]) Let \(X\) be a compact complex manifold, and fix a Hermitian form \(\omega\) and a real \((1, 1)\)-form \(\gamma\) on \(X\). Let \(\theta\) be a \(\gamma\)-positive \((1, 1)\)-current on \(X\). Then, there are a constant \(C\) only dependent of \((X, \omega)\), continuous functions \(\lambda_\kappa\) and \((\gamma + C \lambda_\kappa \omega)\)-pos \((1, 1)\)-forms \(\theta_\kappa \in \{\theta\}_{\partial \bar{\partial}}\) such that as \(\kappa \to \infty\), (i) the sequence \(\theta_\kappa\) is decreasing pointwise to \(\theta\), and (ii) \(\lambda_\kappa(x) \searrow v(\theta, x)\) for all \(x \in X\).

**Theorem 2.10** (Main Thm 1.1 in [14]) Let \(X\) be a complex manifold, and fix a Hermitian form \(\omega\) and a real \((1, 1)\)-form \(\gamma\) on \(X\). Let \(\theta\) be a \(\gamma\)-positive \((1, 1)\)-current on \(X\). Then, for every relatively compact \(U \subseteq X\), there are positive constants \(\varepsilon_\kappa\) and \((\gamma + \varepsilon_\kappa \omega)\)-pos \((1, 1)\)-currents \(\theta_\kappa \in \{\theta\}_{\partial \bar{\partial}}\) with analytic singularities on \(U\) such that as \(\kappa \to \infty\), (i) the sequence \(\theta_\kappa\) is decreasing pointwise to \(\theta\), (ii) \(v(\theta_\kappa, x) \not\nearrow v(\theta, x)\) for all \(x \in U\) (uniformly), and (iii) \(\varepsilon_\kappa \to 0\).

In its original version, \(X\) is assumed to be compact and \(\theta_\kappa\) may not have analytic singularities (instead \(v(\theta_\kappa, x) = v(\theta, x)\)). Nevertheless, the proof also works in the relatively compact setting as explained in the paragraph after [14, Main Thm 1.1]. Furthermore, [14, Rem. 5.15]
ensures the existence of $\theta_k$ with analytic singularities (see Definition 4.1) by weakening the condition on $\theta_k$’s Lelong numbers satisfying (ii).

## 3 Pushforwards of MA products by submersions

Following [20, Sec. 4 & Prop. 5.3], we can define Monge–Ampère products for higher degrees than in BTD if one considers the pushforwards by proper holomorphic submersions.

**Proposition 3.1** Let $\pi: X \to Y$ be a proper holomorphic submersion between complex manifolds $X$ and $Y$ with $m$-dimensional fibres, let $T$ be a closed positive $(p, p)$-current on $X$, and let $\theta$ be a closed $\gamma$-pos $(1, 1)$-current for a (positive) $(1, 1)$-form $\gamma$ such that for any small enough open $V \subset Y$, there is a closed positive $(1, 1)$-form $\gamma_+$ on $\pi^{-1}(V)$ with $\gamma \leq \gamma_+$ (for example, $\pi$ is Kähler, see Remark 3.2). Let $\theta_k \in \{ \theta \}_{\partial \pi}$ be closed $\gamma$-positive $(1, 1)$-forms such that the sequence $\theta_k$ is decreasing pointwise to $\theta$, i.e. locally there are $\gamma$-psh potentials of $\theta_k$ decreasing pointwise to a $\gamma$-psh potential of $\theta$; and let $k$ be a positive integer.

If $\pi(L(\theta))$ is contained in analytic $A$ with codim $A \geq k + p - m$, then

$$S := \pi_*([\theta]^k \wedge T) := \lim_{k \to \infty} \pi_*(\theta_k^k \wedge T)$$

is a closed real $(k+p, k+p)$-current on $Y$, independent of the choice of $\theta_k$, and locally\(^2\) the difference of two closed positive currents. If $\theta$ is positive, then $S$ is positive.

We would like to stress that in general, $[\theta]^k \wedge T$ is not defined as a current on $X$ for codim $A - p < k \leq$ codim $A - p + m$ in the setting above. Later, we will see that it can be defined for currents with analytic singularities, see Sect. 4.

**Remark 3.2** Following [19, Def. 4.1], a holomorphic function $\pi: X \to Y$ is called Kähler if there is an open covering $\{U_k\}$ of $X$ and pluriharmonic functions $\varphi_k$ on $U_k$ s.t. $\varphi_k$ is strictly pluriharmonic on $\pi^{-1}(y) \cap U_k$ and $\varphi_k - \varphi_y$ is pluriharmonic on $\pi^{-1}(y) \cap U_k \cap U_y$ for all $y \in Y$. In case of that $\pi$ is a locally trivial proper holomorphic submersion (which is equivalent to all fibres of $\pi$ are biholomorphic to each other, cf. [18]), we get $\pi$ is Kähler if its fibres are Kähler. In general, deformations of Kähler manifolds are not necessarily Kähler morphisms, see [8, Exp. 3.9] due to Deligne.

By [19, Lem. 4.4], the preimages $U = \pi^{-1}(V)$ of a proper Kähler $\pi: X \to Y$ are Kähler for any small enough open $V \subset Y$, i.e. there is a Kähler form $\omega_U$ on $U$. Moreover, for any $(1, 1)$-form $\gamma$ on $X$ and any small enough open $V$, there exists a constant $C$ such that $\gamma_+ := C\omega_U \geq \gamma$ with closed positive form $\gamma_+$ on $U = \pi^{-1}(V)$. Analogously, every closed $(1, 1)$-form $\alpha$ can be decomposed as $\alpha = \alpha_+ - \alpha_-$ with closed positive forms $\alpha_\pm$ on $\pi^{-1}(V)$.

**Proof of Proposition 3.1** By transferring the argumentation in [20, Sec. 4] to our setting, we first prove the positive case as follows.

We may define $S$ local. Therefore, we can shrink $Y$ such that $\gamma_+$ is defined on $X = \pi^{-1}(Y)$. Set $\vartheta := \theta + \gamma_+ \geq 0$ and $\vartheta_k = \theta_k + \gamma_+ \geq 0$. Let $A$ denote the analytic set of codimension $k + p - m$ such that $\pi(L(\theta)) = \pi(L(\vartheta)) \subset A$. On $X \setminus \pi^{-1}(A)$, Bedford–Taylor (see Theorem 2.6) implies that we can define $\vartheta_k \wedge T := \lim_{k \to \infty} \vartheta_k \wedge T$. The direct image $S' := \pi_*(\vartheta_k \wedge T)$ is a closed positive current on $Y \setminus A$ which is the weak limit of the closed positive currents $S_k := \pi_*(\theta_k \wedge T)$.

---

\(^2\) To be more precise, on all $V$ from before.
We set $s := \dim Y - k - p + m = n - k - p$. As $A$ is of codimension $k + p - m$, there is an $(s, s)$-bump form $\beta$ at $y$ with arbitrarily small support such that $A \cap \text{supp } dd^c \beta = \emptyset$, see Remark 2.3. Let $\alpha$ be a closed real $(1, 1)$-form in the same $\ddbar$-class of $\vartheta$ (and $\vartheta_\kappa$). Since $\vartheta_\kappa = \theta_\kappa + \gamma_+$ is decreasing pointwise to $\vartheta = \theta + \gamma_+$, there is a sequence of $\alpha$-psh functions $q_\kappa$ with $dd^c q_\kappa = \vartheta_\kappa - \alpha$ which is decreasing pointwise to an $\alpha$-psh function $q$ with $dd^c q = \vartheta - \alpha$. We get

$$\int S_\kappa \wedge \beta = \int \vartheta_\kappa \wedge T \wedge \pi^* \beta$$
$$= \int \left( \alpha^k + \sum_{i=1}^{k} (-1)^i \binom{k}{i} \alpha^{k-i} \wedge (dd^c q_\kappa)^i \right) \wedge T \wedge \pi^* \beta$$
$$= \int \left( \alpha^k \wedge \pi^* \beta + q \cdot \pi^* dd^c \beta \wedge \sum_{i=1}^{k} (-1)^i \binom{k}{i} \alpha^{k-i} \wedge (dd^c q_\kappa)^{i-1} \right) \wedge T$$
$$\xrightarrow{k \to \infty} \int \left( \alpha^k \wedge \pi^* \beta + q \cdot \pi^* dd^c \beta \wedge \sum_{i=1}^{k} (-1)^i \binom{k}{i} \alpha^{k-i} \wedge (dd^c q)^{i-1} \right) \wedge T =: S_{\beta}$$

whereby the convergence follows from Bedford–Taylor as $dd^c \beta$ has no support in $A$, see Theorem 2.6.

By Proposition 2.5, we get that $S'$ uniquely extends to a closed positive current $S$ which is the weak limit of $S_\kappa$.

The quasipositive case is now a direct consequence of the following equation, cf. (2.1).

$$\pi_\ast ([\theta]^k \wedge T) = \pi_\ast ([\theta + \gamma_+ - \gamma_+]^k \wedge T) \ := \sum_{i=0}^{k} (-1)^i \binom{k}{i} \pi_\ast \left( [\theta + \gamma_+]^{k-i} \wedge \gamma_+^i \wedge T \right)$$
$$\overset{\text{def}}{=} \lim_{k \to \infty} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \pi_\ast \left( (\theta_\kappa + \gamma_+)^{k-i} \wedge \gamma_+^i \wedge T \right) = \lim_{k \to \infty} \pi_\ast (\theta_\kappa^k \wedge T).$$

(3.1)

Thereby, the terms in the second line of (3.1) are well-defined by the first part of this proof. Due to the last equation in (3.1), this definition is independent of the choice of $\gamma_+$. $\square$

**Remark 3.3** If $\theta_\kappa$ is a sequence of $\gamma$-pos currents (not necessarily smooth) which is decreasing pointwise to $\theta$, then $L(\theta_\kappa) \subset L(\theta)$. Therefore, $\pi_\ast ([\theta_\kappa]^k \wedge T)$ are also defined by Proposition 3.1 for all $\kappa$. Moreover, the proof above works also for non-smooth $\theta_\kappa$ such that $\pi_\ast ([\theta]^k \wedge T) = \lim_{k \to \infty} \pi_\ast ([\theta_\kappa]^k \wedge T)$. In other words, the conclusions of Proposition 3.1 are also correct under the exactly same assumptions within the proposition beside of that $\theta_\kappa$ are allowed to be currents instead of forms. This is worth mentioning since we do not need the extra assumption on $[\theta]_{\ddbar}$ which will be assumed to obtain the (full) monotone continuity result, Theorem 3.5 below.

**Remark 3.4** If there is a closed (positive) $(1, 1)$-form $\omega$ on $Y$ such that $\theta$ is $\pi^* \omega$-positive, then we obtain as in (3.1) that

$$\pi_\ast ([\theta]^k \wedge T) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \pi_\ast \left( [\theta + \pi^* \omega]^{k-i} \wedge \pi^* \omega^i \wedge T \right)$$
$$= \sum_{i=0}^{k-(m-p)} (-1)^i \binom{k}{i} \pi_\ast \left( [\theta + \pi^* \omega]^{k-i} \wedge T \right) \wedge \omega^i$$
where the last equation follows from the projection formula and that \( \pi \)'s fibres are of dimension \( m \). For \( k + p = m + 1 \), we get that

\[
\pi_\ast([\theta]^{m-p+1} \wedge T) = \pi_\ast([\theta + \pi^* \omega]^{m-p+1} \wedge T) - (m-p+1)\omega \wedge \pi_\ast([\theta + \pi^* \omega]^{m-p} \wedge T).
\]

If \( T = \pi^* S \) for a closed positive current \( S \) on \( Y \), then

\[
\pi_\ast([\theta]^k) \wedge S = \sum_{i=0}^{k-m} (-1)^i \binom{k}{i} \pi_\ast([\theta + \pi^* \omega]^{k-i}) \wedge \omega^i \wedge S
\]

and

\[
\pi_\ast([\theta]^{m+1}) \wedge S = \pi_\ast([\theta + \pi^* \omega]^{m+1}) \wedge S - (m+1)\omega \wedge \pi_\ast([\theta + \pi^* \omega]^m) \wedge S.
\]

We get the following monotone continuity result for the extended BTD MA products defined by Proposition 3.1.

**Theorem 3.5** Let \( \pi : X \to Y \) be a proper holomorphic submersion between complex manifolds \( X \) and \( Y \) with \( m \)-dimensional fibres, and let \( T \) be a closed positive \((p, p)\)-current on \( X \). For \( i = 1, \ldots, t \), let \( \theta_i \) be closed \( \gamma \)-pos \((1, 1)\)-currents on \( X \) for a \((p, p)\)-form \((1, 1)\)-form \( \gamma \) such that for any small enough open \( V \subset Y \), there are closed positive \((1, 1)\)-forms \( \gamma_+ \) and \( \alpha_{i, \pm} \) with \( \gamma \leq \gamma_+ \) and \( \alpha_{i, +} - \alpha_{i, -} \in \{\theta_i\}_{\partial Y}^\pi \) on \( \pi^{-1}(V) \) (for example, \( \pi \) is Kähler). Let \( \theta_{i, \kappa} \in \{\theta_i\}_{\partial Y}^\pi \) be closed \( \gamma \)-positive \((1, 1)\)-currents decreasing pointwise to \( \theta_i \), i.e. there are local \( \gamma \)-psh potentials of \( \theta_{i, \kappa} \) decreasing pointwise to a \( \gamma \)-psh potential of \( \theta_i \); and let \( k_i \) be positive integers.

If \( \bigcup_{i=1}^t \pi(L(\theta_i)) \) is contained in an analytic \( A \) with \( \text{codim} \ A \geq k_1 + \cdots + k_t - m + p \), then

\[
\pi_\ast([\theta_1]^{k_1} \wedge \cdots \wedge [\theta_1]^{k_1} \wedge T) = \lim_{k \to \infty} \pi_\ast([\theta_{1, \kappa}]^{k} \wedge \cdots \wedge [\theta_{1, \kappa}]^{k} \wedge T).
\]

**Proof** Without loss of generality, we may assume that \( k_1 = \cdots = k_t = 1 \). The assumption on the codimension of \( A \) reads then \( \text{codim} \ A \geq t + p - m \). By a recursive application of Proposition 3.1, we obtain that \( \pi_\ast([\theta_1] \wedge \cdots \wedge [\theta_1] \wedge T) \) and (in case that \( \theta_{i, \kappa} \) is not smooth) \( \pi_\ast([\theta_{1, \kappa}] \wedge \cdots \wedge [\theta_{1, \kappa}] \wedge T) \) are well-defined currents for all \( \kappa \).

As the statement is local with respect to \( Y \), we may shrink \( Y \) and \( X = \pi^{-1}(Y) \) such that \( \gamma_+ \) and \( \alpha_{i, \pm} \) are defined on \( X \). Let us set \( \theta_i := \theta_i + \gamma_+ \) and \( \theta_{i, \kappa} := \theta_{i, \kappa} + \gamma_+ \). By assumption, these are closed positive currents, and we get that the sequence \( \theta_{i, \kappa} \) is decreasing pointwise to \( \theta_i \) in the sense defined above. To infer the statement in the quasipos case from the positive one, we are going to prove

\[
\pi_\ast([\theta_1] \wedge \cdots \wedge [\theta_{i+1}] \wedge [\theta_1] \wedge \cdots \wedge [\theta_1] \wedge T)
= \lim_{k \to \infty} \pi_\ast([\theta_{1, \kappa}] \wedge \cdots \wedge [\theta_{i+1, \kappa}] \wedge [\theta_{i+1, \kappa}] \wedge \cdots \wedge [\theta_{1, \kappa}] \wedge T)
\]

by induction over the number of \( \theta_j \)-factors, \( t-i \).

First, we proof the induction step, \( t-i \mapsto t-i+1 \) for \( 1 \leq i \leq t \) (i.e. \( i \mapsto i-1 \)). By induction assumption, we have (3.3). Yet, the induction assumption also implies that

\[
\pi_\ast([\theta_1] \wedge \cdots \wedge [\theta_{i+1}] \wedge \gamma_+ \wedge [\theta_{i-1}] \wedge \cdots \wedge [\theta_1] \wedge T)
= \lim_{k \to \infty} \pi_\ast([\theta_{1, \kappa}] \wedge \cdots \wedge [\theta_{i+1, \kappa}] \wedge \gamma_+ \wedge [\theta_{i-1, \kappa}] \wedge \cdots \wedge [\theta_{1, \kappa}] \wedge T)
\]

as there are \( t-1 - (i-1) = t-i \) of \( \theta_j \)-factors. Thereby, we use that \( \gamma_+ \) is positive and closed. Summing up (3.3) and (3.4), we obtain (since \( \theta_i = \theta_i + \gamma_+ \) and \( \theta_{i, \kappa} = \theta_{i, \kappa} + \gamma_+ \))

\[
\pi_\ast([\theta_1] \wedge \cdots \wedge [\theta_1] \wedge [\theta_{i-1}] \wedge \cdots \wedge [\theta_1] \wedge T)
\]
which is the case $t - (i - 1) = (t-i) + 1$. This concludes the induction step.

Next, we prove the base case $t-i = 0$, i.e. there is not any $\theta_{j}$-factor and all factors are positive. We obtain that the currents $S = \pi_{s}(\{\vartheta_{1}\} \wedge \cdots \wedge \{\vartheta_{1}\} \wedge T)$ and $S_{\kappa} := \pi_{s}(\{\vartheta_{t,\kappa}\} \wedge \cdots \wedge \{\vartheta_{1}\} \wedge T)$ are closed positive $(t+p-m, t+p-m)$-currents. Therefore, the weak convergence of $S_{\kappa}$ to $S$ follows from Proposition 2.5 when its assumptions (i) and (ii) are satisfied which we show as follows.

(i) Since $\vartheta_{t,\kappa}$ is decreasing pointwise to $\vartheta_{i}$, we get that $L(\vartheta_{t,\kappa}) \subset L(\vartheta_{i})$ for all $i$ and $\kappa$. By Theorem 2.6, we obtain that

\[
\vartheta_{t} \wedge \cdots \wedge \vartheta_{1} \wedge T = \lim_{\kappa \to \infty} \vartheta_{t,\kappa} \wedge \cdots \wedge \vartheta_{1,\kappa} \wedge T
\]

on $X \setminus \pi^{-1}(A) \subset X \setminus \bigcup_{i} \pi(L(\vartheta_{i}))$, and

\[
S = \pi_{s}(\{\vartheta_{t}\} \wedge \cdots \wedge \{\vartheta_{1}\} \wedge T) = \lim_{\kappa \to \infty} \pi_{s}(\{\vartheta_{t,\kappa}\} \wedge \cdots \wedge \{\vartheta_{1}\} \wedge T) = \lim_{\kappa \to \infty} S_{\kappa}
\]

on $Y \setminus A$.

(ii) Since $\text{codim } A \geq s := t + p - m$, for each point $x \in A$, there is an $(n-s, n-s)$-bump form $\beta$ with arbitrarily small support and $ddc^{\beta} \cap A = \emptyset$, see Remark 2.3. By assumption, $\alpha := \alpha_{t, +} + \gamma_{+} - \alpha_{t, -}$ is the difference of two closed positive forms and in the same $\vartheta_{t}$-class as $\vartheta_{t} = \vartheta_{t} + \gamma_{+}$, i.e. there is a quasipsh function $q$ on $X$ such that $\vartheta_{i} - \alpha = ddc^{q}$. By the assumption on $\vartheta_{t,\kappa}$, we get quasipsh functions $q_{\kappa}$ on $X$ decreasing pointwise to $q$ s.t. $\vartheta_{t,\kappa} - \alpha = ddc^{q_{\kappa}}$. We get

\[
\int \pi_{s}([\vartheta_{t}] \wedge \cdots \wedge [\vartheta_{1}] \wedge T) \wedge \beta = \int \pi_{s}([ddc^{q}] \wedge [\vartheta_{t-1}] \wedge \cdots \wedge [\vartheta_{1}] \wedge T) \wedge \beta
\]

\[
+ \int \pi_{s}([\vartheta_{t-1}] \wedge \cdots \wedge [\vartheta_{1}] \wedge \alpha \wedge T) \wedge \beta
\]

where the last term is defined since $\alpha$ is the difference of two closed positive forms. Let $q^{(\lambda)}$ be smooth $\alpha$-psh functions decreasing pointwise to $q$, and let $\vartheta_{i}^{(\lambda)}$ be positive forms decreasing pointwise to $\vartheta_{i}$ for $i = 1, \ldots, t-1$. Since recursively defined by Proposition 3.1, we obtain

\[
\int \pi_{s}([ddc^{q}] \wedge [\vartheta_{t-1}] \wedge \cdots \wedge [\vartheta_{1}] \wedge T) \wedge \beta
\]

\[
= \lim_{\lambda_{t} \to \infty} \cdots \lim_{\lambda_{1} \to \infty} \int ddc^{q^{(\lambda_{t})}} \wedge \vartheta_{t-1}^{(\lambda_{t-1})} \wedge \cdots \wedge \vartheta_{1}^{(\lambda_{1})} \wedge T \wedge \pi^{*}\beta.
\]

Since $q^{(\lambda)}$ are (globally defined) functions on $X$, we can apply integration by parts (as the definition of $d$ and $ddc^{\cdot}$ on currents) in the limit-term of the RHS and get

\[
\int ddc^{q^{(\lambda_{t})}} \wedge \vartheta_{t-1}^{(\lambda_{t-1})} \wedge \cdots \wedge \vartheta_{1}^{(\lambda_{1})} \wedge T \wedge \pi^{*}\beta
\]

\[
= \int q^{(\lambda_{t})} \cdot \vartheta_{t-1}^{(\lambda_{t-1})} \wedge \cdots \wedge \vartheta_{1}^{(\lambda_{1})} \wedge T \wedge ddc^{\pi^{*}\beta}
\]

\[
= \int_{X \setminus U} q^{(\lambda_{t})} \cdot \vartheta_{t-1}^{(\lambda_{t-1})} \wedge \cdots \wedge \vartheta_{1}^{(\lambda_{1})} \wedge T \wedge ddc^{\pi^{*}\beta}
\]

\[
= \int_{X \setminus U} ddc^{q^{(\lambda_{t})}} \wedge \vartheta_{t-1}^{(\lambda_{t-1})} \wedge \cdots \wedge \vartheta_{1}^{(\lambda_{1})} \wedge T \wedge \pi^{*}\tilde{\beta}
\]
\[
\lim_{\lambda_1 \to \infty} \cdots \lim_{\lambda_t \to \infty} \int_{X \setminus U} dd^c q \wedge \vartheta_{t-1} \wedge \cdots \wedge \vartheta_1 \wedge T \wedge \pi^* \bar{\beta}
\]

where \( U = U_\delta = \pi^{-1}(V_\delta) \) for a neighbourhood \( V = V_\delta \) of \( A \) in \( X \) (so small that \( V_\delta \cap \text{supp} \, dd^c \beta = \emptyset \)) and \( \bar{\beta} = \chi \beta \) for a smooth cutoff function \( \chi \) with support on \( Y \setminus U \) and \( \chi|_{\text{supp} \, dd^c \beta} = 1 \). As \( L(\vartheta_t) = L(\vartheta_1) \subset U \), the convergence is implied by Bedford–Taylor, see Theorem 2.6. Since \( dd^c q \) equals \( \vartheta_t - \alpha \), we get

\[
\int \pi_*\left( [\vartheta_t] \wedge \cdots \wedge [\vartheta_1] \wedge T \right) \wedge \beta = \int_{X \setminus U} \vartheta_t \wedge \cdots \wedge \vartheta_1 \wedge T \wedge \pi^* \bar{\beta} + \int_U \pi_*\left( [\vartheta_{t-1}] \wedge \cdots \wedge [\vartheta_1] \wedge \alpha \wedge T \right) \wedge \beta.
\]

(3.5)

Repeating exactly the same arguments in the lines above replacing \( \vartheta_t \) by \( \vartheta_{t,\kappa} \) and \( q \) by \( q_\kappa \), we obtain the same formula for all \( \kappa \):

\[
\int \pi_*\left( [\vartheta_{t,\kappa}] \wedge \cdots \wedge [\vartheta_{1,\kappa}] \wedge T \right) \wedge \beta = \int_{X \setminus U} \vartheta_{t,\kappa} \wedge \cdots \wedge \vartheta_{1,\kappa} \wedge T \wedge \pi^* \bar{\beta} + \int_U \pi_*\left( [\vartheta_{t-1,\kappa}] \wedge \cdots \wedge [\vartheta_{1,\kappa}] \wedge \alpha \wedge T \right) \wedge \beta.
\]

(3.6)

By Theorem 2.6, we get that the first term on the RHS in (3.6) converges to the first one on the RHS in (3.5).

We obtain the same for the second term as we may assume that the proposition is already proven for \((t-1)\)-factors (i.e. by induction over \( t \)) since \( \alpha \) is the difference of two closed positive forms.

Hence, we get

\[
\int \pi_*\left( [\vartheta_t] \wedge \cdots \wedge [\vartheta_1] \wedge T \right) \wedge \beta = \lim_{\kappa \to \infty} \int \pi_*\left( [\vartheta_{t,\kappa}] \wedge \cdots \wedge [\vartheta_{1,\kappa}] \wedge T \right) \wedge \beta.
\]

\( \square \)

Using the idea of [22] to define the wedge product of pushforwards currents with the fibre product, Theorem 3.5 implies a monotone continuity result for the recursively defined mixed products:

**Corollary 3.6** For \( i = 1, \ldots, t \), let \( \pi_i : X_i \to Y \) be proper holomorphic submersions between complex manifolds \( X_i \) and \( Y \) with \( m_i \)-dimensional fibres, let \( \vartheta_t \) be closed \( \gamma_t \)-positive \((1, 1)\)-currents on \( X_i \) for \((positive) \ (1, 1)\)-forms \( \gamma_i \) on \( X_i \) such that for any small enough open \( V \subset Y \), there are closed positive \((1, 1)\)-forms \( \gamma_{i,\pm} \) and \( \alpha_{i,\pm} \) on \( \pi_{\gamma_i}^{-1}(V) \) with \( \gamma_i \leq \gamma_{i,\pm} \) and \( \alpha_{i,\pm} - \alpha_{i,\mp} \in \{ \vartheta_t \}_{\partial Y} \) (for example, \( \pi_i \) are Kähler). Let \( \vartheta_{t,\kappa} \in \{ \vartheta_t \}_{\partial Y} \) be closed \( \gamma_t \)-positive \((1, 1)\)-currents decreasing pointwise to \( \vartheta_t \), let \( T \) be a closed pos \((p, p)\)-current on \( X_1 \), and let \( k_i \) be positive integers.

If \( \bigcup_{i=1}^t \pi_i(L(\vartheta_t)) \) is contained in an analytic \( A \) with codim \( A \geq k_1 + \cdots + k_t - m_1 - \cdots - m_t + p \), then

\[
\pi_{t,\kappa}([\vartheta_t]^k_1) \wedge \cdots \wedge \pi_{1,\kappa}([\vartheta_1]^k_1 \wedge T) = \lim_{\kappa \to \infty} \pi_{t,\kappa}([\vartheta_{t,\kappa}]^k_1) \wedge \cdots \wedge \pi_{1,\kappa}([\vartheta_{1,\kappa}]^k_1 \wedge T).
\]

(3.7)
Proof Let $\sigma : \tilde{X} \to Y$ be the fibre product of $\tilde{X} = X_1 \times_Y \cdots \times_Y X_1$, let $\text{pr}_i : \tilde{X} \to X_i$ be its projection on the $i$th component, set $\tilde{\theta}_i := \text{pr}_i^* \theta_i$, set $\tilde{\theta}_{i,k} := \text{pr}_i^* \theta_{i,k}$, and set $\gamma := \sum_i \text{pr}_i^* \gamma_i$ (such that $\gamma \leq \sum_i \text{pr}_i^* \gamma_i, +$ on $\sigma^{-1}(V)$). Let $\theta^{(\lambda)}_i$ be $\gamma_i$-pos $(1, 1)$-forms decreasing pointwise to $\theta_i$, and set $\tilde{\theta}^{(\lambda)}_i := \text{pr}_i^* \theta^{(\lambda)}_i$.

As recursively defined by Proposition 3.1 (using the projection formula, as well), we get

$$\pi_{i,*}(\theta_i^{(k)})^1 \cap \cdots \cap \pi_{1,*}(\theta_1^{(1)}_1 \cap T) \equiv \lim_{\lambda_i \to 0} \cdots \lim_{\lambda_1 \to 0} \pi_{i,*}\left((\theta_i^{(\lambda_i)})^{k_i} \cap \cdots \cap \pi_{1,*}\left((\theta_1^{(\lambda_1)})^{k_1} \cap T\right)\right)$$

$$= \lim_{\lambda_i \to 0} \cdots \lim_{\lambda_1 \to 0} \sigma_*\left((\tilde{\theta}_i^{(\lambda_i)})^{k_i} \cap \cdots \cap \tilde{\theta}_1^{(\lambda_1)} \cap T\right)$$

$$= \sigma_*\left((\tilde{\theta}_i)^{k_i} \cap \cdots \cap \tilde{\theta}_1^{k_1} \cap \text{pr}_1^* T\right)$$

(3.8)

where the second equation follows from Lemma 2.1. Repeating these arguments for $\theta_{i,k}$ instead of $\theta_i$, we obtain also

$$\pi_{i,*}(\theta_{i,k})^{k_i} \cap \cdots \cap \pi_{1,*}(\theta_1^{(1)}_1 \cap T) = \sigma_*\left((\tilde{\theta}_{i,k})^{k_i} \cap \cdots \cap \tilde{\theta}_{1,k}^{k_1} \cap \text{pr}_1^* T\right)$$

(3.9)

for all $\kappa$. By Theorem 3.5, the RHS of (3.9) converges weakly to the right side of (3.8) as $\kappa \to \infty$. Hence, the same applies to the LHS of the equations what proves the claimed. \( \square \)

4 Currents with analytic singularities

Throughout this section, let $X$ be a complex manifold of dimension $n$. The main objective of this section is the study of wedge products of currents with analytic singularities.

Definition 4.1 We call $q$ a quasipsh function with analytic singularities in $Z$ if locally $q = u + \text{smth}$ whereby $u = c \log \|F\|^2 + b$ is a psh function with analytic singularities in the (reduced) analytic set $\{F = 0\}$ for a positive constant $c$, a tuple $F$ of holomorphic functions and bounded $b$. Moreover, we call $q$ a quasipsh function with neat analytic singularities if $b$ can be chosen to be smooth, i.e. locally $q = c \log \|F\|^2 + \text{smth}$.

We say that a closed positive $\theta$ has (neat) analytic singularities if its local $dd^c$-potential is psh with (neat) analytic singularities. $\theta$ is called quasipositive (shortly quasipos) with (neat) analytic singularities if it is locally the sum of a closed positive current with (neat) analytic singularities and a closed real $(1, 1)$-form. This is equivalent to that locally $\theta = dd^c q$ for a quasipsh function $q$ with (neat) analytic singularities.

Let $u_1, \ldots, u_k$ be psh functions with analytic singularities in $Z_1, \ldots, Z_k$, resp. For constructible sets $W_2, \ldots, W_k$ such that $W_j \subset Z_j^\circ := X \setminus Z_j$, we can define the MA product

$$dd^c u_k \land 1_{W_k} dd^c u_{k-1} \land 1_{W_{k-1}} \cdots dd^c u_2 \land 1_{W_2} dd^c u_1$$

(4.1)

recursively (from right to left) as a closed positive current, see [22, Prop. 3.2]. In case of $W_j = X \setminus Z_j$, we call (4.1) the (mixed) Andersson–Walcan (AW) MA product of $dd^c u_k, \ldots, dd^c u_1$. For a psh function $u$ with analytic singularities in $Z$, $(dd^c u)^k_{\text{AW}}$ denotes the $k$-power $dd^c u \land 1_{Z_1} \cdots 1_{Z_k} dd^c u$ which was introduced in [2].

Please note there is no restriction on $k$. Moreover, this definition of (mixed) Monge–Ampère products is an extension of BTD MA, see [22, Prop. 3.4], and (mixed) products of Bochner–Martinelli currents $dd^c \log \|F_k\|^2$ for holomorphic tuples $F_k$ as it was introduced in

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[3, Sec. 5]. Also, we would like to point out that (4.1) can be defined for quasipsh potentials, as well, see [9, 21] and also [22, Lem. 3.5] (i.p. for \( \gamma \)-psh potentials for a (pos) form \( \gamma \).

As in [22, Sec. 3] and [21, Sec. 2], we consider the following class of currents.

**Definition 4.2** We call a (closed) real \((p, p)\)-current \( T \) a (closed) current with analytic singularities if it is locally of the form:

\[
\sum_i \alpha_i \wedge \mathbb{I} W_{i, j_1 + 1}dd^c u_{i, k_i} \wedge \mathbb{I} W_{i, j_k} dd^c u_{i, k_i - 1} \cdots \wedge \mathbb{I} W_{i, j_1} dd^c u_{i, 1} \mathbb{I} W_{i, 1} \tag{4.2}
\]

with smooth closed real forms \( \alpha_i \) of degree \((p - k_i, p - k_i)\), psh \( u_{i, j} \) with analytic singularities, and constructible sets \( W_{i, j} \) such that \( L(u_{i, j}) \subset W_{i, j}^c \) for \( j \leq 2 \). As \( \mathbb{I} \{p\} \) \( T = 0 \) for \((p, p)\)-currents with \( p < n \), we can omit the last condition on \( W_{i, j} \) if \( L(u_{i, j}) \) is a set of isolated points.

If furthermore \( L(u_{i, 1}) \subset W_{i, 1}^c \), \( T \wedge \cdot \) denotes an operator on closed currents with analytic singularities (which is given recursively and well-defined due to the condition on \( W_{i, 1} \)).

In particular, the AW MA product of closed quasipositive \((1, 1)\)-currents with analytic singularities (see (4.1)) is a closed current with analytic singularities. Let us use the following more specific notation for such currents.

**Definition 4.3** A closed \((p, p)\)-current \( T \) with analytic singularities is called quasipositive (shortly quasipos) if \( T \) is locally of the following form:

\[
\sum_i c_i \mathbb{I} W_{i, p + 1} dd^c q_{i, p} \wedge \mathbb{I} W_{i, p} dd^c q_{i, p - 1} \mathbb{I} W_{i, p - 1} \cdots \mathbb{I} W_{i, 1} dd^c q_{i, 1} \mathbb{I} W_{i, 1} \tag{4.3}
\]

with positive constants \( c_i \), quasipsh \( q_{i, j} \) with analytic singularities, and constructible sets \( W_{i, j} \) such that \( L(q_{i, j}) \subset W_{i, j}^c \) for \( j \geq 2 \).

Obviously, a closed \((p, p)\)-current \( T \) with analytic singularities is quasipos if and only if it is locally of the form as in (4.2) with \( \alpha_i \) is either a positive constant or the wedge product of closed real \((1, 1)\)-forms. In particular, locally \( T \) is the difference of two closed positive currents.

**Remark 4.4** Let \( \pi : X \to Y \) be a proper holomorphic submersion, and let \( \gamma \) be a (pos) \((1, 1)\)-form such that for all small enough open \( V \subset Y \), there is a closed positive \((1, 1)\)-form \( \gamma_+ \) with \( \gamma_+ \geq \gamma \) on \( U = \pi^{-1}(V) \), e.g. \( \pi \) is Kähler, see Remark 3.2. Let \( \theta_1, \ldots, \theta_p \) be closed \( \gamma \)-pos \((1, 1)\) currents with analytic singularities, and let \( W_2, \ldots, W_p \) be constructible sets with \( L(\theta_j) \subset W_j^c \) for \( j \geq 2 \). Then, \( T = \theta_p \wedge \mathbb{I} W_p \cdots \mathbb{I} W_2 \mathbb{I} W_1 \) can be decomposed into the difference \( T_+ - T_- \) of two closed positive currents \( T_\pm \) with analytic singularities on \( U = \pi^{-1}(V) \) for all small enough \( V \) (to be more precise, on all \( U \) on which the \( \gamma_+ \) from above exists).

Let us prove this by induction over \( p \). The base case \( p = 1 \) is trivial. For the induction step, we may assume that there are two closed positive currents \( S_\pm \) with analytic singularities such that \( \theta_1 \wedge \mathbb{I} W_1 \cdots \mathbb{I} W_2 \mathbb{I} W_1 = S_+ - S_- \). Then,

\[
T = \theta_p \wedge \mathbb{I} W_p (S_+ - S_-) = (\theta_p + \gamma_+) \wedge \mathbb{I} W_p S_+ + \gamma_+ \wedge \mathbb{I} W_p S_-
\]

\[= (\theta_p + \gamma_+) \wedge \mathbb{I} W_p S_+ + \gamma_+ \wedge \mathbb{I} W_p S_+ \]

where minuend and subtrahend are closed and positive.

We obtain two classes of currents which are closed under the following operations:

1. wedge products with closed real forms (of bidegree \((1, 1)\));
(2) $1_W$ for any constructible set $W$; and

(3) $dd^c q \land \mathbb{I}_{Z'}$ for any quasipsh function $q$ with analytic singularities in $Z$.

If $T$ and $S$ are (quasipos) currents with analytic singularities, then $S \land T$ is well-defined and a (quasipos) current with analytic singularities where $S \land$ denotes the operator as defined in Definition 4.2.

In this sense, the AW MA product has proven to be very useful, e.g. we can define the Monge–Ampère operator for arbitrary degrees. Yet, it lacks certain properties. For example, the product is neither commutative nor distributive, see [22, Exp. 3.1]. Furthermore, although it coincides with the BTD MA product whenever this defined (see [22, Prop. 3.4]), the pushforward of an AW MA product is in general different form the (natural) extension of BTD defined by Proposition 3.1 as the following example shows; cf. [22, Exp. 8.3].

Example 4.5 Let $X$ be $\mathbb{C}_z \times \mathbb{P}^1$, let $\pi$ be the projection on $Y = \mathbb{C}_z$, let $u$ denote the psh function $\log |z|^2$ on $X$, and let $\omega_{FS}$ be the pullback of the Fubini-Study form on $\mathbb{P}^1$. Then, $\pi_*((\omega_{FS} + dd^c u)_2^{AW}) = \pi_*((\omega_{FS} \land \mathbb{I}_{\{z=0\}}) = \mathbb{I}_{\{z=0\}}$. On the other side, if $u_k$ is a sequence of smooth psh functions decreasing pointwise to $u$, then $(\omega_{FS} + dd^c u_k)^2 = (dd^c u_k)^2 + 2\omega_{FS} \land dd^c u_k + 0$ such that $\pi_*((\omega_{FS} + dd^c u_k)^2) = 2\pi_*((\omega_{FS} + dd^c u_k)$ which converges to $2\mathbb{I}_{\{z=0\}}$, i.e.

$$\pi_*((\omega_{FS} + dd^c u)^2) = 2\mathbb{I}_{\{z=0\}} \neq \pi_*((\omega_{FS} + dd^c u)_2^{AW}).$$

Roughly speaking, the AW MA product removes too much. Inspired by [4, Thm 1.2], Lärkäng, Raufi, Wulcan and the author give the following definition in [21, 22] which keeps track of the removed part by adding an extra term to the AW MA product.

Definition 4.6 For a closed quasipositive $(1, 1)$-current $\theta$ with analytic singularities in $Z$, closed real $(1, 1)$-form $\alpha$, and closed current $T$ with analytic singularities, we get that the so-called generalized Monge–Ampère product of $\theta$ and $T$ defined by

$$[\theta]_\alpha \land T := \theta \land 1_{Z'} T + \alpha \land 1_{Z'} T$$

(4.4)

is a well-defined closed real current with analytic singularities which is quasipos when $T$ is. This is also defined and a (quasipos) current with analytic singularities for $\alpha$ replaced by a quasipos $(1, 1)$-current $\eta$ with analytic singularities in isolated points due to BTD.

Generalizing [22, Thm 1.2], we obtain the following correlation between the extended BTD MA product defined by Proposition 3.1 and pushforwards of the generalized MA products defined by (4.4).

Theorem 4.7 Let $\pi : X \to Y$ be a proper holomorphic submersion between complex manifolds with $m$-dimensional fibres, let $\gamma$ be a (pos) $(1, 1)$-form, and let $T$ be a closed real $(p, p)$-current such that for any small enough $V \subset Y$, there are a closed positive form $\gamma_+$ with $\gamma \leq \gamma_+$ and closed positive currents $T_\pm$ with analytic singularities and $T = T_+ - T_-$ on $\pi^{-1}(V)$ (cf. Remark 4.4). Let $\theta$ be a $\gamma$-pos $(1, 1)$-current with analytic singularities in $Z$, and let $\eta \in [\theta]_\gamma$ be a closed $\gamma$-pos $(1, 1)$-current with neat analytic singularities in isolated points. Then, we get

$$\pi_*([\theta]^k \land T) = \pi_*([\theta]^k_\eta \land T)$$

for all $k + p \leq \text{codim} \pi(Z) + m$. In particular, it is independent of $\eta$. 

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Let us point out in contrast to the general case, \( S := \pi_*[\theta]^k \wedge T \) is actually given as a direct image of a current on \( X \) for \( \theta \) with analytic singularities. We will utilize this fact to estimate the Lelong numbers of \( S \), see Corollary 5.10.

**Remark 4.8** By Remark 4.4, we can apply Theorem 4.7 recursively. In particular, we get

\[
\pi_*([\theta_i]^{k_1} \wedge \cdots \wedge [\theta_i]^{k_1}) = \pi_*([\theta_i]^{k_1}_{\eta_i} \wedge \cdots \wedge [\theta_i]^{k_1}_{\eta_i})
\]

(4.5)

for closed \( \gamma \)-pos \((1,1)\)-currents \( \theta_1, \ldots, \theta_t \) with analytic singularities in \( Z_i \), closed \( \gamma \)-pos \((1,1)\)-currents \( \eta_1, \ldots, \eta_l \) with analytic singularities in isolated points such that \( \eta_i \in \{ \theta_i \}_{\alpha} \) and \( k_1 + \cdots + k_t \leq \text{codim} \bigcup_{i=1}^t \pi(Z_i) + m(\gamma \text{ as above}).

**Remark 4.9** Using the idea from [22] (motivated by Lemma 2.1), we can use the fibre product to define the wedge product of pushforwards of currents with analytic singularities in the following setting. For \( i = 1, \ldots, t \), let \( \pi_i : X_i \to Y \) be proper holomorphic submersions with \( m_i \)-dimensional fibres, let \( \theta_i \) be closed \( \gamma_i \)-pos \((1,1)\)-currents with analytic singularities in \( Z_i \), and let \( \alpha_i \in \{ \theta_i \}_{\alpha} \) be closed \( \gamma_i \)-pos \((1,1)\)-forms where \( \gamma_i \) are \((1,1)\)-forms on each \( X_i \).

Following [22, Thm 1.1], we define

\[
\pi_{i,*}([\theta_i]^{k_1}_{\alpha_i}) \wedge \cdots \wedge \pi_{i,*}([\theta_i]^{k_1}_{\alpha_i}) := \sigma^*([\tilde{\theta}_i]^{k_1}_{[\tilde{\alpha}_i]} \wedge \cdots \wedge [\tilde{\theta}_i]^{k_1}_{[\tilde{\alpha}_i]})
\]

(4.6)

where \( \sigma : \tilde{X} \to Y \) is the fibre product, \( \tilde{X} = X_1 \times_Y \cdots \times_Y X_t \), \( \pi_i : \tilde{X} \to X_i \) denotes the projection on the \( i \)-th component, \( \tilde{\theta}_i := \pi^* \theta_i \) and \( \tilde{\alpha}_i := \pi^* \alpha_i \). In (4.6), we cannot replace \( \alpha \) by currents \( \eta \) with neat analytic singularities in isolated points since \( \pi^* \eta \) would have singularities in sets of higher codimension than points. If for any small enough open \( V \subset Y \), there are closed positive \((1,1)\)-forms \( \gamma_i \) with \( \gamma_i \leq \gamma_i, + \) on \( \pi^{-1}_i(V) \), then (4.5) implies

\[
\pi_{i,*}([\theta_i]^{k_1}) \wedge \cdots \wedge \pi_{i,*}([\theta_i]^{k_1}) = \pi_{i,*}([\theta_i]^{k_1}_{\alpha_i}) \wedge \cdots \wedge \pi_{i,*}([\theta_i]^{k_1}_{\alpha_i})
\]

(4.7)

for all \( k_1 + \cdots + k_t \leq \text{codim} \bigcup_i \pi(Z_i) + m_1 + \cdots + m_t \) (cf. proof of Corollary 3.6).

**Proof of Theorem 4.7.** Let \( \alpha \) be in the same \( \partial \beta \)-class as \( \theta \) (and \( \eta \)), i.e., there exist a quasipsh \( q \) with analytic singularities and a quasipsh \( v \) with neat analytic singularities in isolated points such that \( \theta = dd^c q + \alpha \) and \( \eta = dd^c v + \alpha \). By the definition of the generalized MA product (4.4), we get \( [\theta] \cap \wedge T = [dd^c q + \alpha]_{dd^c v + \alpha} \cap \wedge T = ([dd^c q]_{dd^c v + \alpha} + \alpha) \cap \wedge T \). We set \( q_k := \max\{q, v - \kappa\} \). As locally, there is a constant \( C \) s.t. \( q + C\|z\|^2 \) and \( v + C\|z\|^2 \) are psh (where \( z \) denotes coordinates of \( X \)), we get \( \kappa \leq \max\{q + C\|z\|^2, v - \kappa + C\|z\|^2\} - C\|z\|^2 \) are quasipsh. Moreover, \( q_k \) have neat analytic singularities in isolated points and the sequence is decreasing pointwise to \( q \), i.e., \( \kappa \to q \) as \( \kappa \to \infty \). On any \( \lim_{\kappa \to \infty} \pi^{-1}(V) \) for small enough open \( V \subset Y \) where \( \kappa \to \infty \) by the assumption, Lemma 4.11 below implies

\[
[\theta] \cap \wedge T = ([dd^c q]_{dd^c v + \alpha} \cap \wedge T = \lim_{\kappa \to \infty} (dd^c q_k + \alpha \cap \wedge T = \lim_{\kappa \to \infty} \theta_k \cap \wedge T.
\]

By taking the pushforwards of both sides of the equation, we get the claimed as in the proof of Proposition 3.1, see Remark 3.3.

**Lemma 4.10** Let \( T \) be a closed current with analytic singularities, and let \( q \) be a quasipsh function with analytic singularities in \( Z \). If \( \rho_k : \mathbb{R} \to \mathbb{R} \) is a sequence of non-decreasing convex and bounded from below functions such that \( \rho_k \) is decreasing pointwise to the identity as \( \kappa \to \infty \), then

\[
dd^c \rho_k \circ q \cap T \xrightarrow{\kappa \to \infty} dd^c q \cap 1_{Z^c} T.
\]

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In particular, if \( v \) is a smooth function, then
\[
[dd^c q]_{dd^c v} \wedge T = \lim_{\kappa \to \infty} dd^c (\rho_\kappa \circ (q - v) + v) \wedge T.
\] (4.9)

**Proof** (4.8) is a direct consequence of the definition of the AW MA product, see [22, Prop. 3.2]. Now, (4.8) implies
\[
dd^c \rho_\kappa \circ (q - v) \wedge T \xrightarrow{\kappa \to \infty} dd^c (q - v) \wedge 1_{Z_c} T
\]
which is equivalent to (4.9) since we can add \( dd^c v \wedge T \) on both sides of the equation. \( \square \)

We get the following variant for \( v \) with neat analytic singularities in isolated points.

**Lemma 4.11** Let \( T \) be a closed quasipos \((p, p)\)-current with analytic singularities, let \( q \) be a quasipsh function with analytic singularities in \( Z \), and let \( v \) be a quasipsh function with neat analytic singularities in isolated points. We set \( q_\kappa := \max\{q, v - \kappa\} = \max\{q - v, -\kappa\} + v \)
such that \( q_\kappa \) are quasipsh functions with neat analytic singularities in isolated points and \( q_\kappa \)
is decreasing pointwise to \( q \). Then,
\[
[dd^c q]_{dd^c v} \wedge T = \lim_{\kappa \to \infty} dd^c q_\kappa \wedge T.
\] (4.10)

**Proof** As explained in the comment after Definition 4.3, \( T \) is locally the difference of two closed positive currents (with analytic singularities). By proving the convergence (4.10)
for the minuend and subtrahend separately, we may assume that \( T \) is positive. As \( \rho_\kappa = \max\{\cdot, -\kappa\} \)
are non-decreasing convex functions decreasing pointwise to the identity as \( \kappa \to \infty \), Lemma 4.10 implies the weak convergence (4.10) on the complement of \( L(v) \).

Fix an \( x \in L(v) \). Let \( \beta \) be an \((n-\beta-1, n-\beta-1)\)-bump form at \( x \) such that \( supp \, dd^c \beta \subset L(v)^c \), see Remark 2.3. Fix a constant \( C \)
such that \( \tilde{v} := \max_x \{v, -C\} \) equals \( v \) on \( supp \, dd^c \beta \)
(\( max_\varepsilon \) denotes the regularized max for a small enough \( \varepsilon \)). We set \( \tilde{q}_\kappa := \max\{q, \tilde{v} - \kappa\} = \rho_\kappa (q - \tilde{v}) + \tilde{v} \).
By Lemma 4.10, we get that
\[
\begin{align*}
dd^c \tilde{q}_\kappa \wedge T \xrightarrow{\kappa \to \infty} (dd^c q \wedge 1_{Z_c} + dd^c \tilde{v} \wedge 1_Z) T.
\end{align*}
\]
Furthermore, \( \tilde{q}_\kappa \) equals \( q_\kappa \) on \( supp \, dd^c \beta \). By applying Stokes and the facts above, we get
\[
\begin{align*}
\int \beta \wedge dd^c \tilde{q}_\kappa \wedge T &= \int \beta \wedge dd^c \tilde{q}_\kappa T = \int \beta \wedge dd^c \tilde{q}_\kappa T \\
\overset{\kappa \to \infty}{\longrightarrow} \int \beta \wedge (dd^c q \wedge 1_{Z_c} + dd^c \tilde{v} \wedge 1_Z) T &= \int \beta \wedge (q 1_{Z_c} + \tilde{v} 1_Z) T \\
&= \int \beta \wedge (dd^c q \wedge 1_{Z_c} + dd^c v \wedge 1_Z) T.
\end{align*}
\] (4.11)

Since locally there is a closed positive \((1, 1)\)-form \( \alpha \) such that \( q, v \) and \( q_\kappa \) are \( \alpha \)-psh, \( S_\kappa := (dd^c q_\kappa + \alpha) \wedge T \)
is positive, and we can apply Proposition 2.5, i.e. \( S_\kappa \) is converging weakly to a closed positive current \( S \) with \( S \) equals \([dd^c q]_{dd^c v} \wedge T + \alpha \wedge T \) on the complement of \( L(v) \). In particular, we proved the claimed by using Lemma 2.4. \( \square \)

Let us conclude this section with the following variant of Lemma 2.4.

**Lemma 4.12** Let \( T \) be a closed quasipos \((p, p)\)-current with analytic singularities such that \( T \) has support in an analytic set \( A \) with \( \text{codim} \, A \geq p \). If for all points \( x \in A \), there is an \((n-p, n-p)\)-bump form \( \beta \) such that \( \int \beta \wedge T = 0 \), then \( T \) equals zero.
Proof  By Definition 4.2, we get locally \( T = \sum_i T_i \) where
\[
T_i = \alpha_i \wedge 1_{W_{i,k_i+1}} dd^c u_{i,k_i} \wedge 1_{W_{i,k_i}} \cdots \wedge 1_{W_{i,2}} dd^c u_{i,1} 1_{W_{i,1}}
\]
with smooth closed real forms \( \alpha_i \) of degree \( (p-k_i, p-k_i) \), psh \( u_{i,j} \) with analytic singularities, and constructible sets \( W_{i,j} \) such that \( L(u_{i,j}) \subset W_{i,j}^c \) for \( j \geq 2 \). If \( \text{codim} \ A > k_i \), then we may assume that \( T_i \) vanishes due to \( T \)’s support (independently of the bump form assumption).
So, all terms in \( T \) vanish except of \( T_i \) with \( k_i = \text{codim} \ A = p \), \( i.e. \):
\[
T = \sum_i c_i 1_{W_{i,p+1}} dd^c u_{i,p} \wedge 1_{W_{i,p}} \cdots \wedge 1_{W_{i,2}} dd^c u_{i,1} 1_{W_{i,1}}.
\]
As \( T \) is quasipos, \( c_i \geq 0 \), see comment in Definition 4.3. We conclude \( T \) is positive. Therefore, Lemma 2.4 implies \( T = 0 \). \( \square \)

5 Lelong numbers of closed real currents

In this section, we are going to study Lelong numbers of closed real currents which are not necessarily positive. To be more precise, we consider currents which are locally the difference of two closed positive ones or currents with analytic singularities, see Definition 4.2. We will recall and generalize basic properties of Lelong numbers following the notation as in the Sections 5ff of [16, Chp. III]; see also [13, 15].

Let \( X \) be a complex manifold of dimension \( n \), and fix a point \( x_0 \in X \) and coordinates \( z \) in a neighbourhood of \( x_0 \) such that \( z(x_0) = 0 \). If \( u \) is a (negative) psh function on \( X \), the Lelong number of \( u \) in \( x_0 \) is defined by
\[
v(u, x_0) := \liminf_{x \to x_0} \frac{u(x)}{\log \|z\|}.
\]
This definition trivially extends to quasipsh functions \( q \). The following lemmata are direct consequences of this definition.

Lemma 5.1  If \( S \subset X \) is a submanifold s.t. \( q |_S \neq -\infty \), then \( v(q |_S, x_0) \geq v(q, x_0) \).

Lemma 5.2  If \( \pi : Y \to X \) is a holom. submersion, then \( v(q \circ \pi, y_0) = v(q, \pi(y_0)) \).

Proof  As the statement is local with respect to \( y_0 \), we may assume that \( X \subset \mathbb{C}^n \), \( Y = X \times D \) for \( D \subset \mathbb{C}^m \), \( \pi : Y \to X \) is the projection and \( y_0 = (0, 0) \). Then, for any sequence \( x_k \) in \( X \) converging to \( y_0 \),
\[
\frac{q(x_k)}{\log \|x_k\|} = \frac{q(\pi(x_k, 0))}{\log \|x_k, 0\|}.
\]
Therefore, \( v(q, 0) \geq v(q \circ \pi, (0, 0)) \). If \( y_k \) is a sequence in \( Y \) converging to \( (0, 0) \), then
\[
\frac{q(\pi(y_k, a_k))}{\log \|y_k, a_k\|} \geq \frac{q(x_k)}{\log \|x_k\|}
\]
for almost all \( k \). This implies the other inequality \( v(q, 0) \leq v(q \circ \pi, (0, 0)) \). \( \square \)

If \( T \) is a closed positive \((p, p)\)-current, then the Lelong number of \( T \) in \( x_0 \) is given by
\[
v(T, x_0) = \int 1_{\{x_0\}} (dd^c \log \|z\|)^{n-p} \wedge T.
\]
(5.1)

Thereby, we use that \( L(dd^c \log \|z\|) = \{x_0\} \) is of codimension \( n \) such that the current \( (dd^c \log \|z\|)^{n-p} \wedge T \) is defined in the sense of BTD; see e.g. [16, Def. 5.4]. As introduced in [22, Sec. 6.4], we can extend the definition of Lelong numbers to currents which locally are given as differences of two closed positive currents as follows.
Lemma 5.3 Let $T$ be a closed real current on $X$ such that in a neighbourhood of a point $x_0$, $T = T_+ - T_-$ for two closed positive currents $T_\pm$. Then, the Lelong number of $T$ in $x_0$

$$v(T, x_0) := v(T_+, x_0) - v(T_-, x_0)$$

is well-defined since it is independent of the decomposition.

We call $v(T, x_0)$ the Lelong number of $T$ in $x_0$. By this extension, $v(\cdot, x_0)$ is a linear operator on the currents which are given as the difference of two closed positive currents in a neighbourhood of $x_0$.

Proof Let $T$ equal $T_+ - T_-$ and $S_+ - S_-$ for closed positive currents $T_\pm$ and $S_\pm$ in a neighbourhood of $x_0$. Then, we get $T_+ + S_- = S_+ + T_-$ is a closed positive current. As the Lelong number of the sum of closed positive currents is the sum of the Lelong numbers of the summands, we get

$$v(T_+, x_0) - v(T_-, x_0) = v(S_+ + T_+, x_0) - v(S_+ + T_-, x_0)$$

$$= v(S_+ + T_+, x_0) - v(S_+ + T_+, x_0) = v(S_+, x_0) - v(S_-, x_0).$$

\[ \square \]

By Prop. 5.12 in [16, Chp. III], we have the following semicontinuity property.

Proposition 5.4 Let $T$ be a closed current such that $T = T_+ - T_-$ for closed positive $T_\pm$ in a neighbourhood of $x_0$. $T_k$ be a sequence of closed currents which converges weakly to $T$ such that $T_{k, +} := T_k + T_-$ are closed positive currents for (almost) all $k$. Then,

$$\lim_{k \to \infty} \sup u(T_k, x_0) \leq v(T, x_0).$$

For proper holomorphic submersions, we get the following estimates between the Lelong numbers of closed positive currents and their pushforwards or pullbacks.

Proposition 5.5 Let $\pi : X \to Y$ be a proper holomorphic submersion between complex manifolds $X$ and $Y$, with $m$-dimensional fibres, and let $T$ be a closed positive $(p, p)$-current on $X$ with $p \geq m$. Then, for all $x_0 \in X$, we get

$$v(T, x_0) \leq v(\pi_* T, \pi(x_0)).$$

Proof Let $n$ be the dimension of $X$. We may assume that $Y$ is a small enough neighbourhood of $y_0 := \pi(x_0)$ such that $Y \subset \mathbb{C}^{n-m}$ with $z' = (z_1, \ldots, z_{n-m})$ as coordinates of $Y$ and $z'(y_0) = 0$.

By the assumption on $\pi$, $F := \pi^{-1}(y_0)$ is a smooth complex manifold of dimension $m$. We pick some coordinates $z'' = (z_{n-m+1}, \ldots, z_m)$ of $F$ in a neighbourhood $D$ of $x_0$ such that $z''(x_0) = 0$. We may assume that $U = Y \times D$ is a subset of $X$ and $\pi$ is the projection on $Y$ and obtain that $z = (z', z'')$ are coordinates for $U$ with $\pi \circ z = z'$.

We set $\varphi(z) := \gamma \log \|z\|$ for constant $\gamma > 0$ and $\psi(z') := \log \|z'\|$ which are both semiexhaustive psh functions defining the Lelong number in $x_0$ and $y_0$, respectively, by using Demailly’s generalized Lelong numbers, see [16, §5 in Chp. III], i.e.

$$\gamma^{n-p}v(T, x_0) = v(T, \varphi) := \lim_{r \to -\infty} v(T, \varphi, r), \quad v(T, \psi, r) := \int_{\{\varphi < r\}} T \wedge (dd^c \varphi)^{n-p}$$

and

$$v(\pi_* T, y_0) = v(\pi_* T, \psi) := \lim_{r \to -\infty} v(\pi_* T, \psi, r), \quad v(\pi_* T, \varphi, r) := \int_{\{\psi < r\}} \pi_* T \wedge (dd^c \psi)^{n-p}.$$
We define \( \psi_\varepsilon := \frac{1}{2} \log (\|z\|^2 + \varepsilon \|z\|^2) \) whose sequence is decreasing pointwise to \( \psi \) as \( \varepsilon \to 0 \) and \( \psi_{\varepsilon,s} := \max\{\psi_\varepsilon, s\} \) whose sequence is decreasing pointwise to \( \psi_s := \max\{\psi, s\} \) as \( \varepsilon \to 0 \) and to \( \psi_s \) as \( s \to -\infty \).

Since \( \|z\|^2 + \varepsilon \|z\|^2 \leq \|z\|^2 \), we obtain \( \gamma \psi_\varepsilon \leq \varphi \), i.e. \( \frac{\varphi(z)}{\psi_\varepsilon(z)} \leq \gamma \) close to \( x \). Moreover, we have

\[
\gamma \geq \limsup_{\psi_\varepsilon(z) \to -\infty} \frac{\varphi(z)}{\psi_\varepsilon(z)} \geq \limsup_{z' \to 0} \frac{\varphi(z', 0)}{\psi_\varepsilon(z', 0)} = \gamma.
\]

By Demailly’s second comparison theorem for Lelong numbers (cf. Thm 7.8 in [16, Chp. III]), we get that

\[
\nu(T \wedge (dd^c \varphi)^{n-p}, x_0) \leq \gamma^{n-p} \nu(T \wedge (dd^c \psi_s)^{n-p}, x_0) < \nu(T \wedge (dd^c \psi_s)^{n-p}, \varphi),
\]

i.e. for all constants \( R \) there is an \( r < R \) such that

\[
\int_{B_r} T \wedge (dd^c \varphi)^{n-p} \leq \int_{B_R} T \wedge (dd^c \psi_s)^{n-p}
\]

(5.2)

where \( B_r = \{ \varphi < r \} \) and \( B_R = \{ \varphi < R \} \) are balls with radii \( e^r \) and \( e^R \), resp. As \( \psi_{\varepsilon,s} \searrow \psi_\varepsilon \), we get

\[
\int_{B_R} T \wedge (dd^c \psi_s)^{n-p} \leq \int_{B_R} T \wedge (dd^c \psi_\varepsilon,s)^{n-p}
\]

(5.3)

for all \( s \) small enough, see for example proof of Thm 3.7 in [16, Chp. III].

Furthermore,

\[
\lim_{\varepsilon \to 0} \int_{B_R} T \wedge (dd^c \psi_{\varepsilon,s})^{n-p} = \int_{B_R} T \wedge (dd^c \psi_s \circ \pi)^{n-p}
\leq \int_{\{\psi \circ \pi < R\}} T \wedge (dd^c \psi_s \circ \pi)^{n-p} = \int_{\{\psi < R\}} \pi_\ast T \wedge (dd^c \psi_s)^{n-p}
\]

(5.4)

\[
= \int_{\{\psi < R\}} \pi_\ast T \wedge (dd^c \psi)^{n-p} = \nu(\pi_\ast T, \pi, R)
\]

where the second last equation follows from Stokes, cf. Prop. 9.3 in [16, Chp. III]. (5.2), (5.3) and (5.4) combined imply for every \( R \) there is an \( r < R \) such that \( \nu(T, \varphi, r) \leq \nu(\pi_\ast T, \psi, R) \).

Since the RHS converges to \( \nu(\pi_\ast T, y_0) \) as \( R \to -\infty \) and the LHS converges to \( \gamma^{n-p} \nu(T, x_0) \) as \( r \to -\infty \) for every \( \gamma < 1 \), we proved the claimed.

**Proposition 5.6** (Prop. 5 in [25]) \( \pi : X \to Y \) be a proper holomorphic submersion between complex manifolds \( X \) and \( Y \), and let \( T \) be a closed positive \((p, p)\)-current on \( Y \). Then, for all \( x_0 \in X \), we get

\[
\nu(\pi^\ast T, x_0) \geq \nu(T, \pi(x_0))
\]

To be more precise, Meo proved that for every holomorphic \( f : D_1 \to D_2 \) with \( D_1 \subset \mathbb{C}^n \) and \( D_2 \subset \mathbb{C}^m \), and every closed positive \((p, p)\)-current \( T = \lim_{k \to -\infty} \theta_k \) on \( D_2 \) with closed positive \((p, p)\)-forms \( \theta_k \) such that \( f^\ast \theta_k := \lim_{k \to -\infty} f^\ast \theta_k \) is a closed positive current on \( D_1 \), we get \( \nu(f^\ast T, x_0) \geq \nu(T, f(x_0)) \).
Lelong numbers of currents with analytic singularities

**Remark/Def. 5.7** If $T$ is a closed positive $(p, p)$-current and $\alpha$ a closed strongly positive $(k, k)$-form ($k > 0$), then $\alpha \wedge T$ is a closed positive current by definition. For every point $x \in X$, we get that

$$v(\alpha \wedge T, x) = \int \mathbb{1}_{\{x\}}(dd^c \log \|z\|)^{n-p-k} \wedge \alpha \wedge T$$

where $z$ denotes coordinates of $X$ with $z(x) = 0$. The last equation follows from that a closed positive $(s, s)$-current with support in an analytic set of codimension strictly greater than $s$ vanishes.

Therefore, if $\alpha$ is real (not necessary strongly positive), we may trivially extend the definition of Lelong numbers to currents as $\alpha \wedge T$, and their linear combinations. In particular, this defines the Lelong number for every closed current $T$ with analytic singularities.

We get

$$v(T, x) = v\left(\sum_i c_i \cdot \mathbb{1}_{W_{i,p+1}} dd^c u_{i,p} \wedge \mathbb{1}_{W_{i,p}} \cdots \wedge \mathbb{1}_{W_{i,2}} dd^c u_{i,1}, x\right)$$

for constants $c_i$, psh functions $u_{i,1}, \ldots, u_{i,p}$ with analytic singularities and (suitable) constructible sets $W_{i,2}, \ldots, W_{i,p+1}$ (defined in a neighbourhood of $x$). This is clear since the Lelong numbers of other terms in the local form of $T$ given by (4.2) vanish.

The following is a special variant of Cor. 7.9 in [16, Chp. III] inferred from Demailly’s second comparison theorem.

**Proposition 5.8** Let $X$ be a complex manifold of dimension $n$, let $T$ be a closed quasipos $(p, p)$-current with analytic singularities for $p < n$, and let $q$ be a quasipsh function with analytic singularities in $Z$. For every point $x \in X$, we get

$$v(dd^c q \wedge \mathbb{1}_{Z^c} T, x) \geq v(q, x) \cdot v(\mathbb{1}_{Z^c} T, x).$$

(5.5)

If $\eta$ is a closed quasipos $(1, 1)$-current with analytic singularities in isolated points, then

$$v([dd^c q]_\eta \wedge T, x) \geq \min\{v(q, x), v(\eta, x)\} \cdot v(T, x).$$

(5.6)

**Remark 5.9** The cutoff $\mathbb{1}_{Z^c}$ in the RHS of (5.5) cannot be omitted as the following example shows. Let $f$ be holomorphic, let $\sigma$ be smooth, and set $q := \log |f|^2 + \sigma$. Then

$$v((dd^c q)^2_{AW}, 0) = v(dd^c q \wedge \mathbb{1}_{Z^c} dd^c q, 0) = v([f = 0] + dd^c \sigma) \wedge dd^c \sigma, 0) = 0$$

while $v(dd^c q, 0) = v([f = 0], 0) \neq 0$.

**Proof of Proposition 5.8.** The proof follows Demailly’s strategy adapted to our setting.

Without loss of generality, we may assume that $T$ is positive since the Lelong number of each non-positive term in the local form of $T$ given by (4.2) vanishes, see Definition 5.7.

Let us assume that we work on a coordinate patch of $X$ with $x = 0$ and $z$ denoting the coordinates, and that $q = u + \sigma$ for a psh $u$ with analytic singularities and smooth function $\sigma$. 

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Due to the homogeneity of the Lelong number, we may assume that $v(u, 0) = v(q, 0) = 1 - \varepsilon$ by rescaling $q$, and to prove (5.5), it is enough to show

$$v(\mathbb{1}_{\mathbb{Z}^c} T, 0) \leq v(dd^c u \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0) = v(dd^c q \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0).$$  \hspace{1cm} (5.7)

We set $v := \log \|z\|$ and $u_k := \max\{u, v - \kappa\} = \max\{u - v, -\kappa\} + v$. Since $v(u, 0) < v(v, 0)$, there is an $r_0 > 0$ such that $v - \kappa \geq u$, i.e., $u_k = v - \kappa$ on $\|z\| < r_0$. In particular, $v(u_k, 0) = v(v, 0) = 1$ and

$$v(dd^c u_k \wedge T, 0) = v(dd^c v \wedge T, 0),$$

whereby the wedge product is well-defined due to BTD.

On the other side, $u_k \searrow u$ pointwise as $\kappa \to \infty$ such that by Lemma 4.11, we get the weak convergence

$$dd^c u_k \wedge T \overset{\kappa \to \infty}{\longrightarrow} dd^c u \wedge \mathbb{1}_{\mathbb{Z}^c} T + dd^c v \wedge \mathbb{1}_{\mathbb{Z}^c} T.$$  

As the Lelong number is upper semicontinuous in its first argument (see Proposition 5.4), we get

$$v(dd^c v \wedge T, 0) = v(dd^c u_k \wedge T, 0) \leq v(dd^c u \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0) + v(dd^c v \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0).$$

By subtracting $v(dd^c v \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0)$ on both sides, we get (5.7) since $v(\mathbb{1}_{\mathbb{Z}^c} T, 0) = v(dd^c v \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0)$, see (5.1).

As $[dd^c q]_\eta \wedge T = dd^c q \wedge \mathbb{1}_{\mathbb{Z}^c} T + \eta \wedge \mathbb{1}_{\mathbb{Z}^c} T$ and the Lelong number is linear (in its first argument), (5.6) follows from (5.5) and

$$v(\eta \wedge \mathbb{1}_{\mathbb{Z}^c} T, 0) \geq v(\eta, 0) \cdot v(\mathbb{1}_{\mathbb{Z}^c} T, 0).$$

which also follows from (5.5) since $v(\mathbb{1}_{X \setminus \{0\}} \mathbb{1}_{\mathbb{Z}^c} T, 0) = v(\mathbb{1}_{\mathbb{Z}^c} T, 0)$ as $p < n$.

Combining Proposition 5.5 with Proposition 5.8, we get the following corollary.

**Corollary 5.10** Let $X$ be a complex manifold of dimension $n$, and let $\pi : X \to Y$ be a proper holomorphic submersion with $m$-dimensional fibres. Let $T$, $\theta$ and $\eta$ be closed position currents with analytic singularities on $X$, such that $T$ is of bidegree $(p, p)$, $\theta$ and $\eta$ are of bidegree $(1, 1)$ and in the same $\partial \bar{\partial}$-class, $L(\eta)$ is a set of isolated points. Then, for all $k$ with $m \leq k + p \leq n$, and for all $x \in X$,

$$\nu(\pi_* ([\theta]_\eta^k \wedge T), \pi(x)) \geq \min\{\nu(\theta, x), \nu(\eta, x)\}^k \cdot \nu(T, x).$$  \hspace{1cm} (5.8)

**Remark 5.11** In general, (5.8) cannot hold if we consider quasipos $\theta$ which are not positive near $\pi^{-1}(\pi(x))$. This is obvious since the RHS in (5.8) is always positive for such $\theta$ while the LHS can be negative as in the following simple example.

Let $Y$ be $\mathbb{C}$ with coordinate $z$, let $F$ be $\mathbb{P}^1$ with the coordinates $[\xi_1 : \xi_2]$, let $X$ be $\mathbb{C} \times \mathbb{P}^1$, let $\pi : X \to Y$ be the projection on the first component, and let $\omega_{FS}$ be the pullback to $X$ of the Fubini-Study form on $\mathbb{P}^1$. Then, $\theta = [z = 0] - \omega_{FS}$ is quasipos and $\pi_* ([\theta]_{\omega_{FS}}^2) = \pi_* ([\theta]^2) = \pi_* (-\omega_{FS}) \wedge [z = 0] = -[z = 0]$ (for any $\eta$) such that $\nu(\pi_* ([\theta]^2), 0) = -1$.

**Remark 5.12** If $\alpha_1, \alpha_2 \in \{\theta\}_{\omega_{FS}}^2$ are smooth, then $\nu(\pi_* ([\theta]_{\alpha_1}^k \wedge T), \pi_* ([\theta]_{\alpha_2}^k \wedge T))$ following the argumentation in the proof of [22, Thm 1.1 (3)]. As the following counter example shows, this is not correct if one of them has neat analytic singularities in isolated points.
Let $Y$ be $\mathbb{C}$ with the coordinate $z$, let $X$ be $\mathbb{C} \times \mathbb{P}^1$ with the coordinate $[\xi_1 : \xi_2]$, and let $\pi : X \to Y$ be the projection on the first component. Consider the current $\theta = [\xi_2 = 0]$ which has analytic singularities in $Z = (\xi_2 = 0)$, $\eta := \frac{1}{2} \partial \bar{\partial} \log (|z|^2 |\xi_1|^2 + |\xi_2|^2)$ which has only one neat analytic singularity in $(0, [1 : 0])$, and $\alpha = \omega_{FS} = \frac{1}{2} \partial \bar{\partial} \log (|\xi_1|^2 + |\xi_2|^2)$ the Fubini-Study form on $\mathbb{P}^1$. Then, $\eta - \alpha = \partial \bar{\partial} \log \frac{|z|^2 |\xi_1|^2 + |\xi_2|^2}{|\xi_1|^2 + |\xi_2|^2}$. For any ball $B \subset \mathbb{C}$, we get

$$\int_B \pi_*([\theta]^2_B) - \int_B \pi_*([\eta]^2_B) = \int_B \pi_*((\eta - \alpha) \wedge \mathbb{H} [\xi_2 = 0]) = \int_B (\eta - \alpha) = \int_B \partial \bar{\partial} \log |z|.$$ 

Therefore, $\nu(\pi_*([\theta]^2_B), 0) - \nu(\pi_*([\eta]^2_B), 0) = \nu(\log |z|, 0) = 1$.

At least, we obtain that the Lelong numbers of currents $\pi_*([\theta]_B \cap T)$ are independent of $\eta$ among all quasipos $(1, 1)$-currents with the same type of singularities as follows.

**Proposition 5.13** Let $\pi : X \to Y$ be a proper holomorphic submersion, let $T$ and $\theta$ be closed quasipos current with analytic singularities of bidegree $(p, p)$ and $(1, 1)$, resp. Let $\eta_1$, $\eta_2$ be two quasipos $(1, 1)$-currents with neat analytic singularities in isolated points both in the same $\partial \bar{\partial}$-class as $\theta$ such that $\nu(\eta_1, x) = \nu(\eta_2, x)$ for all $x \in X$. Then, for all $y \in Y$, we get

$$\nu(\pi_*([\theta]_{\eta_1}^k \cap T), y) = \nu(\pi_*([\theta]_{\eta_2}^k \cap T), y).$$

**Proof** As $\eta_1$ and $\eta_2$ are in the same $\partial \bar{\partial}$-class, we get that $\eta_1 - \eta_2 = \partial \bar{\partial} q$ for a function $q$ which is locally the difference of two quasipsh $q_1$ and $q_2$ function with neat analytic singularities in isolated points (given by $\eta_1 = \partial \bar{\partial} q_1$ and $\eta_2 = \partial \bar{\partial} q_2$). As $\nu(q_1, x) = \nu(q_2, x)$, we obtain that $q_1 - q_2 = q$ is smooth.

Now, we can repeat the argumentation in the proof of [22, Thm 1.1 (3)] to prove the claimed.

\[\square\]

6 Proofs of the main results

Before proving the second main result of the present work, let us observe the following.

**Lemma 6.1** Let $(X, \omega)$ be a Kähler manifold. For each relatively compact $U \subset X$, there exists a constant $\delta = \delta(\omega)$ such that for every $x \in U$ and positive $\tilde{\delta} \leq \delta$, there is a closed positive $(1, 1)$-current $\eta_x \in \{\omega\}$ with neat analytic singularities only in $x$ and $\nu(\eta_x, x) = \tilde{\delta}$.

If $X \subset B \times \mathbb{P}^m$ for $B \subset \mathbb{C}^{n-m}$ and if $\omega$ is given as restriction of the product of the Euclidean and the Fubini-Study metric, then we can take $\delta = 1$.

**Proof** Set $n := \dim X$. For every $x \in U$, let $B \subset U$ be a neighbourhood of $x$ with coordinates $z = (z_1, \ldots, z_n)$ such that $B \subset \mathbb{C}^n$ with $z(x) = 0$, and let $\chi$ be a cutoff function on $B$ which is identically 1 near $x$. The function $q_\chi(z) := \chi(z) \log \|z\|$ is a quasipsh function on $B$ which extends trivially to $U$. There is a constant $\delta_x > 0$ such that $\delta_x \partial \bar{\partial} q_\chi + \omega \geq 0$. We can choose $\delta_x$ continuously depending on $x$. As $U$ is relatively compact in $X$, we get $\delta := \min_{x \in U} \delta_x > 0$. Then, $\eta_x := \tilde{\delta} \partial \bar{\partial} q_\chi + \omega$ has the claimed properties for all $\delta \leq \delta_x$.

For $X = B \times \mathbb{P}^m$ with the coordinates $z'$ for $B$ and $[\xi_1 : \ldots : \xi_{m+1}]$ for $\mathbb{P}^m$, we may assume that $x = (0, [1 : 0 : \ldots : 0])$. Then, $u_x(z', [\xi]) = \frac{1}{2} \log \|z'\|^2 |\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_{m+1}|^2$ is psh. So, $\partial \bar{\partial} u_x$ is a well-defined positive current on $X$, in the same cohomology class as $\omega$, and $\nu(\partial \bar{\partial} u_x, x) = 1$. In particular, $\eta_x = \tilde{\delta} \partial \bar{\partial} u_x + (1 - \tilde{\delta}) \omega$ has the claimed properties.
For $X \subset B \times \mathbb{P}^n$, we set $\tilde{u}_x = (pr^+_2 u_x)|_X$ where $pr_2 : B \times \mathbb{P}^n \to \mathbb{P}^n$ denotes the projection on $\mathbb{P}^n$. Then, $\eta_x = \tilde{\delta}_\mu \cdot dd^c \tilde{u}_x + (1 - \tilde{\delta}_\mu)\omega$ has the claimed properties for $\mu = v(u_x, x) / v(\tilde{u}_x, x) \leq 1$ (for the last inequality, see Lemmata 5.1 and 5.2).

**Theorem 1.3** Let $\pi : X \to Y$ be a proper holomorphic submersion between complex manifolds $X$ and $Y$ with $m$-dimensional fibres. Fix a point $y \in Y$. Let $\theta_1, \ldots, \theta_t$ be positive $(1, 1)$-currents with analytic singularities on $X$ such that each $\theta_i$ is in a Kähler class on a neighbourhood of $\pi^{-1}(y)$. Then, there exist positive constants $\delta_i, i = 1, \ldots, t$ (which only depend on the Kähler class represented by $\theta_i$ on a neighbourhood of $\pi^{-1}(y)$) such that the following is correct.

(i) Existence of $\eta_i$: For every $x \in \pi^{-1}(y)$, there exist closed quasipositive $(1, 1)$-currents $\eta_i = \eta_i, x \in \{\theta_i\}_{\bar{\partial} \bar{\partial}}$, positive near $\pi^{-1}(y)$, with neat analytic singularities only in $\{x\}$ (and with $v(\eta_i, x) = \delta_i$) such that

$$v\left(\pi_*([\theta_i]^{k_i}_{\eta_i} \wedge \cdots \wedge [\theta_i]^{k_i}_{\eta_i}), y\right) \geq \prod_{i=1}^t \min\{v(\theta_i, x), \delta_i\}^{k_i} \tag{6.1}$$

for all $m \leq k_1 + \cdots + k_t \leq \dim X$.

Thereby, $\eta_i = \eta_i, x$ depends only of the Kähler class of $\theta_i$ near $\pi^{-1}(y)$ and $x$.

(ii) Independency of $\eta_i$: For $i = 1, \ldots, t$, let $\eta_i \in \{\theta_i\}_{\bar{\partial} \bar{\partial}}$ be closed quasipos (1, 1)-currents with neat analytic singularities in isolated points on $X$ such that there is a point $x \in X$ with $L(\eta_i) \cap \pi^{-1}(y) = \{x\}$ and $v(\eta_i, x) \leq \delta_i$. Then,

$$v\left(\pi_*([\theta_i]^{k_i}_{\eta_i} \wedge \cdots \wedge [\theta_i]^{k_i}_{\eta_i}), y\right) \geq \prod_{i=1}^t \min\{v(\theta_i, x), v(\eta_i, x)\}^{k_i} \tag{6.2}$$

for all $m \leq k_1 + \cdots + k_t \leq \dim X$.

**Proof** By the assumptions, there are Kähler forms $\omega_i$ on $U = \pi^{-1}(V)$ such that $\theta_i$ and $\omega_i$ are in the same cohomology class on $U$. Therefore, Lemma 6.1 gives us positive constants $\delta_i$ and, for every $x \in \pi^{-1}(y)$, closed positive $(1, 1)$-currents $\tilde{\eta}_i \in \{\omega_i\}_{\bar{\partial} \bar{\partial}} = \{\theta_i\}_{\bar{\partial} \bar{\partial}}$ with neat analytic singularities only in $x$ on $U$ such that $v(\tilde{\eta}_i, x) = \delta_i$. In particular, $\delta_i$ and $\tilde{\eta}_i$ only depend on the cohomology class represented by $\omega_i$.

Let us extend $\tilde{\eta}_i$ to the whole $X$. Pick a closed real $(1, 1)$-form $\alpha_i$ on $X$ in the same $\bar{\partial} \bar{\partial}$-class as $\tilde{\eta}_i$. In particular, there is a quasipsh function $q_i$ with neat analytic singularities in isolated points on $U$ such that $dd^c q_i = \tilde{\eta}_i - \alpha_i$. Let $\chi$ be a cutoff function with support on $V$ and identically 1 in a neighbourhood of $y$. We define $\eta_i := dd^c (\chi \circ \pi \cdot q_i) + \alpha_i$ which is in the same $\bar{\partial} \bar{\partial}$-class as $\tilde{\eta}_i$ and $\eta_i = \tilde{\eta}_i$ on a neighbourhood of $\pi^{-1}(y)$. Furthermore, $\eta_i$ is positive there and $v(\eta_i, x) = \delta_i$.

Following Definition 4.3, we get that $T := [\theta_i]^{k_i}_{\eta_i} \wedge \cdots \wedge [\theta_i]^{k_i}_{\eta_i}$ is a closed quasipos current with analytic singularities which is positive in a neighbourhood of $\pi^{-1}(y)$. In particular, Proposition 5.5 implies $v(\pi_* T, y) \geq v(T, x)$ as long $m \leq k_1 + \cdots + k_t \leq n$. By Proposition 5.8, we get that $v(T, x) \geq \prod_i \min\{v(\theta_i, x), \delta_i\}^{k_i}$. This proves (i).

(ii) is now a direct consequence of Proposition 5.13 (recursively applied): We can exchange the quasipositive $\eta_i$ in the current on the LHS of (6.2) by a positive $\tilde{\eta}_i$ with the same Lelong number in $x$ which is given by Lemma 6.1.

Let us continue with the proof of the first main result of the present work.

**Theorem 1.1** Let $\pi : X \to Y$ be a proper holomorphic submersion between complex manifolds $X$ and $Y$ with $m$-dimensional fibres. Fix a point $y \in Y$. Let $\theta_1, \ldots, \theta_t$ be positive $(1, 1)$-currents on $X$ such that each $\theta_i$ is in a Kähler class on a neighbourhood of $\pi^{-1}(y)$.
Then, there exist positive constants \( \delta_i \), for \( i = 1, \ldots, t \), (which only depend of the Kähler class represented by \( \theta_i \) on a neighbourhood of \( \pi^{-1}(y) \)) such that the following statements are correct.

(i) If the union of all images \( \pi(L(\theta_i)) \) of the unbounded loci of local \( dd^c \)-potentials of \( \theta_i \) is contained in an analytic \( A \) with codim \( A \geq k_1 + \cdots + k_t - m \), then

\[
\nu\left( \pi_*([\theta_i]^{k_i} \wedge \cdots \wedge [\theta_1]^{k_1}), y \right) \geq \prod_{i=1}^t \min\{\nu(\theta_i, x), \delta_i\}^{k_i}
\]

(6.3)

for all points \( x \in \pi^{-1}(y) \) as long the LHS does not vanish due to the degree of the current.

(ii) If \( \bigcup_i \pi(L(\theta_i)) \) is contained in an analytic \( A \) with codim \( A \geq k_1 + \cdots + k_t - t \cdot m \), then

\[
\nu\left( \pi_*([\theta_i]^{k_i}) \wedge \cdots \wedge [\theta_1]^{k_1}, y \right) \geq \prod_{i=1}^t \min\{\nu(\theta_i, x), \delta_i\}^{k_i}
\]

(6.4)

for all points \( x \in \pi^{-1}(y) \) as long the LHS does not vanish due to the degree of the current.

**Proof** Since the statement is local with respect to neighbourhoods of \( y \in Y \), and since \( \theta_i \) is in a Kähler class near \( \pi^{-1}(y) \) by assumption, we may assume that \( X \) is Kähler by shrinking \( Y \) and \( X = \pi^{-1}(Y) \). Let \( \omega \) denote a Kähler form on \( X \). By Lemma 6.1, there are constants \( \delta_i > 0 \) and for each \( x \in \pi^{-1}(y) \), closed positive \((1, 1)\)-currents \( \eta_i = \eta_i, x \in \{\tilde{\theta}_i\}_{\overline{\partial\overline{\partial}}} \) with neat analytic singularities only in \( x \).

(i) By shrinking \( Y \) and \( X = \pi^{-1}(Y) \) again, Demailly’s regularization result (see Theorem 2.10) implies that there are positive constants \( \epsilon_k \) and closed \( \epsilon_k \omega \)-positive \((1, 1)\)-currents \( \tilde{\theta}_{i,k} \in \{\theta_i\}_{\overline{\partial\overline{\partial}}} \) with analytic singularities such that \( \tilde{\theta}_{i,k} \) is decreasing pointwise to \( \theta_i \), \( \nu(\tilde{\theta}_{i,k}, x) \to \nu(\theta_i, x) \) and \( \epsilon_k \to 0 \) as \( k \to \infty \). We set \( \tilde{\eta}_{i,k} := \eta_i + \epsilon_k \omega \) and \( \tilde{\eta}_{i,k} := \eta_i, x \in \{\tilde{\theta}_{i,k}\}_{\overline{\partial\overline{\partial}}} \).

As \( \tilde{\theta}_{i,k} \) are positive, we can apply Theorem 1.3 such that

\[
\nu\left( \pi_*([\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}}^{k_i} \wedge \cdots \wedge [\tilde{\theta}_{1,k}]_{\tilde{\eta}_{1,k}}^{k_1}), y \right) \geq \prod_{i=1}^t \min\{\nu(\tilde{\theta}_{i,k}, x), \nu(\tilde{\eta}_{i,k}, x)\}^{k_i}
\]

(6.5)

By the Theorems 3.5 and 4.7, we obtain that

\[
\pi_*([\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}}^{k_i} \wedge \cdots \wedge [\tilde{\theta}_{1,k}]_{\tilde{\eta}_{1,k}}^{k_1}) = \lim_{k \to \infty} \pi_*([\theta_{i,k}]_{\eta_i}^{k_i} \wedge \cdots \wedge [\theta_{1,k}]_{\eta_1}^{k_1})
\]

Since \([\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}} \wedge \cdots \wedge [\tilde{\theta}_{1,k}]_{\tilde{\eta}_{1,k}}\) = \([\theta_{i,k}]_{\eta_i} + \epsilon_k \omega \) \wedge \cdots \wedge \([\theta_{1,k}]_{\eta_1} + \epsilon_k \omega \), we get

\[
[\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}}^{k_i} \wedge T = \sum_{l=0}^{k_i} (-1)^l \binom{k_i}{l} [\theta_{i,k}]_{\eta_i}^{k_i-l} \wedge \epsilon_k \omega^l \wedge T
\]

\[
= [\theta_{i,k}]_{\eta_i}^{k_i} \wedge T + \sum_{l=1}^{k_i} (-1)^l \binom{k_i}{l} [\theta_{i,k}]_{\eta_i}^{k_i-l} \wedge \epsilon_k \omega^l \wedge T
\]

for any closed current \( T \) with analytic singularities. By applying this recursively to each factor in \([\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}}^{k_i-l} \wedge \cdots \wedge [\tilde{\theta}_{1,k}]_{\tilde{\eta}_{1,k}}^{k_1}\), we get

\[
\pi_*([\tilde{\theta}_{i,k}]_{\tilde{\eta}_{i,k}}^{k_i} \wedge \cdots \wedge [\tilde{\theta}_{1,k}]_{\tilde{\eta}_{1,k}}^{k_1}) = \pi_*([\theta_{i,k}]_{\eta_i}^{k_i} \wedge \cdots \wedge [\theta_{1,k}]_{\eta_1}^{k_1}) + \sum_j \epsilon_k T_{j,k}
\]
for positive integers $l_j > 0$ and currents $T_{j,\kappa}$ which are pushforwards of quasipos currents with analytic singularities. Moreover, we can apply Theorem 3.5 for each term and get

$$\pi_*([\theta_l]^{k_l} \wedge \cdots \wedge [\theta_1]^{k_1}) = \lim_{\kappa \to \infty} \pi_*([\tilde{\theta}_{l,\kappa}]^{k_l}_{\eta_{l,\kappa}} \wedge \cdots \wedge [\tilde{\theta}_{1,\kappa}]^{k_1}_{\eta_{1,\kappa}}).$$

As all currents are positive, we can estimate the Lelong number of the LHS from below with the limit superior of the Lelong number of the RHS, see Proposition 5.4. This and (6.5) imply

$$v(\pi_*([\theta_l]^{k_l} \wedge \cdots \wedge [\theta_1]^{k_1}), y) \geq \lim_{\kappa \to \infty} \prod_{i=1}^{t} \min\{v(\theta_{i,\kappa}, x), \delta_i\}^{k_i} = \prod_{i=1}^{t} \min\{v(\theta_i, x), \delta_i\}^{k_i}$$

where the last equation follows from the choice of $\theta_{i,\kappa}$, cf. Theorem 2.10 (ii).

(ii) We prove (6.4) by induction over $t$.

The case $t = 1$ follows from (i). For the induction step, we will prove

$$v(\pi_*([\theta_l]^{k_l}) \wedge T, y) \geq \min\{v(\theta_1, x), \delta_1\}^{k_1} \cdot v(T, y) \tag{6.6}$$

for the closed positive current $T := \pi_*([\theta_l]^{k_l} \wedge \cdots \wedge [\theta_1]^{k_1})$.

Pick a closed real $(1, 1)$-form $\alpha \in \{\theta_l\}_{\delta_{\alpha}} = \{\eta_l\}_{\delta_{\beta}}$. There are $\alpha$-psh functions $q$ and $v$ (the latter with neat analytic singularities only in $x$) such that $\theta_l = dd^c q + \alpha$ and $\eta_l = dd^c v + \alpha$. We set $q^{(\lambda)} := \max(q, \frac{\delta}{\beta} v - \lambda)$ with $\delta < v(q, x) = v(\theta_l, x)$ and $\delta \leq \delta_l$. We get $q^{(\lambda)}$ is $\alpha$-psh and decreasing pointwise to $q$ as $\lambda \to \infty$. As in the proof of Proposition 5.8, $q^{(\lambda)}$ has neat analytic singularities only in $x$ and $v(q^{(\lambda)}, x) = \delta$. We set $\theta_l^{(\lambda)} = dd^c q^{(\lambda)}$ and get

$$\pi_*([\theta_l]^{k_l}) \wedge T = \lim_{\lambda \to \infty} \pi_*([\theta_l^{(\lambda)}]^{k_l}) \wedge T = \lim_{\lambda \to \infty} \pi_*([\theta_l^{(\lambda)}]^{k_l} \wedge \pi^* T)$$

where the first equation follows from Theorem 3.5 and the second from Proposition 3.1 and BTD (by selecting a smooth approximation of $\theta_l^{(\lambda)}$ for each $\lambda$). Semicontinuity of the Lelong number, see Proposition 5.4, implies

$$v(\pi_*([\theta_l]^{k_l}) \wedge T, y) \geq \lim_{\lambda \to \infty} \sup v(\pi_*([\theta_l^{(\lambda)}]^{k_l} \wedge \pi^* T), y).$$

By the Propositions 5.5 and 5.6, we get

$$v(\pi_*([\theta_l^{(\lambda)}]^{k_l} \wedge \pi^* T), y) \geq v([\theta_l^{(\lambda)}], x)^{k_l} \cdot v(\pi^* T, x) \geq \delta^{k_l} \cdot v(T, y).$$

Since this holds for all $\delta \leq \delta_l$ with $\delta < v(\theta_l, x)$, (6.6) is proven. \qed

7 Segre currents

In [20], Lärkäng, Raufi, Ruppenthal and the author define so-called Chern and Segre currents for singular Hermitian metrics on vector bundles which are positively curved in the sense of Griffiths\(^3\). These currents represent the Chern and Segre classes of the vector bundles. In this section, we are going to present the construction of these currents in a (slightly) more general setting.

\(^3\) As defined by [7, Def. 3.2]; also called singularly (semi-) positive in the sense of Griffiths.
Setting 7.1 Let $X$ be a complex manifold of dimension $n$, let $E$ be a holomorphic vector bundle on $X$ of rank $r$, let $\pi : \mathbb{P}(E) \to X$ denote the projectivization of hyperplanes of $E$ (i.e. $\pi^{-1}(x) = \mathbb{P}(E^*_x)$ for all $x \in X$), let $L := \mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E)$ denote the hyperplane bundle of $E$, and let $e^{-\psi}$ be a singular metric on $L$ such that its curvature current $\theta := dd^c\psi$ is quasipositive, i.e. the weights $\varphi$ are quasipsh functions. Let $L(\varphi)$ denote the unbounded locus of $\varphi$, i.e. $L(\varphi) = L(\theta)$. There is a (pos) $(1, 1)$-form $\gamma$ on $\mathbb{P}(E)$ such that $\theta + \gamma \geq 0$, i.e. $\theta$ is $\gamma$-pos. For any small enough open set $U \subset X$, we may pick $\gamma$ such that $\gamma$ is closed and positive on $\pi^{-1}(U)$, see Remark 3.2. Let $e^{-\psi_k}$ be a sequence of smooth/singular metrics on $L$ such that $\varphi_k$ is decreasing pointwise to $\varphi$ and $dd^c\varphi_k$ is $\gamma$-positive.

Let $T$ be a closed real $(p, p)$-current on $X$ which is locally given as difference of two closed positive currents.

If $H$ is a singular Hermitian metric on $E$, then it induces a singular metric $e^{-\psi}$ on $L$ whose dual metric is given by $e^\psi = \pi^*H^*|_{L^*}$ since $L^* = \mathcal{O}_{\mathbb{P}(E)}(-1)$ is a subbundle of $\pi^*E^*$. $e^{-\psi}$ has a quite special geometric behaviour on each fibre $\pi^{-1}(x)$ coming from the Hermitian structure of $H$, which turns out to be unnecessary for the definition of Chern and Segre currents. To keep the setting more general, we consider any singular metric $e^{-\psi}$ on $L$. There is a one-to-one correspondence between singular metrics on $L$ and singular metrics on $E^*$ considering so-called singular Finsler metrics\(^4\).

If the weights of $\varphi$ are psh, i.e. $dd^c\varphi$ is positive, then $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudoeffective, and so is $E$ by definition.\(^5\) Moreover, we get that $E$ is strongly pseudoeffective if $\pi(L(\varphi))$ does not equal $X$ by definition, see [11, Def. 7.1]. In particular, the following results give Segre and Chern currents for strongly pseudoeffective vector bundles and not only vector bundles positively curved in the sense of Griffiths. Let us stress that we work in the even more general Setting 7.1 where the weights $\varphi$ are just supposed to be quasipsh.

As in [20, Prop. 4.6], Proposition 3.1 implies the following result.

Theorem 7.2 In the Setting 7.1 with $\varphi_k$ smooth, if $\pi(L(\varphi))$ is contained in an analytic set of codimension $\geq k + p$, then

$$s_k(E, \varphi) \wedge T := (-1)^k \pi_* \left( (dd^c \varphi)^{k+r-1} \wedge \pi^*T \right) \overset{\text{def}}{=} (-1)^k \lim_{k \to \infty} \pi_* (dd^c \varphi_k)^{k+r-1} \wedge \pi^*T$$

(7.1)

is a well-defined closed real current which is locally the difference of two closed positive currents, and independent of the choice of the smooth approximation $\varphi_k$.

For a smooth Hermitian metric $H$ on $E$, the total Segre form $s(E, H)$ is (classically) defined as the inverse of the total Chern form $c(E, H) = 1 + c_1(E, H) + \cdots + c_n(E, H)$. Then, its $(k, k)$-component $s_k(E, H)$ can be calculated using the projectivization of the vector bundle, $s_k(E, H) = (-1)^k \pi_* (dd^c \psi)^{k+r-1}$ where $e^\psi = \pi^*H^*|_{L^*}$. Therefore, the Segre currents $s_k(E, \varphi)$ defined above are naturally extending the concept of Segre forms to singular metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$. Moreover, the so-called Chern currents

$$c_k(E, \varphi) := (-1)^k \sum_{k_1 + \cdots + k_r = k} s_{k_1}(E, \varphi) \wedge \cdots \wedge s_{k_r}(E, \varphi)$$

(7.2)

given by a recursive application of Theorem 7.2 are naturally extending the concept of Chern forms.

\(^4\) A singular Finsler metric $h$ on $E^*$ is given by $E^* \to [0, \infty]$, $(p, \xi) \mapsto \|\xi\|_h(p)$ with $\|\lambda \xi\|_h(p) = |\lambda| \|\xi\|_h(p)$.

\(^5\) In the literature, sometimes called weakly to distinguish it from strongly pseudoeffective.
Corollary 3.6 implies the following monotone continuity result generalizing [20, Thm 1.5].

**Theorem 7.3** In the Setting 7.1, if $\pi(L(\varphi))$ is contained in an analytic set of codimension $\geq k_1 + \cdots + k_t$, then

$$s_{k_i}(E, \varphi) \wedge \cdots \wedge s_{k_1}(E, \varphi) = \lim_{\kappa \to \infty} \pi_\kappa((dd^c \varphi_\kappa)^{k_i+r-1}) \wedge \cdots \wedge \pi_\kappa((dd^c \varphi_\kappa)^{k_1+r-1})$$

for any (not necessarily smooth) approximation $\varphi_\kappa$ decreasing pointwise to $\varphi$.

**Remark 7.4** In particular, the current given by [27, Thm 2] coincide with the current defined in [20].

**Remark 7.5** Theorem 7.3 generalizes [20, Thm 1.5] in various ways. Among these generalizations, let us highlight that we do not need to assume that on $X \setminus \pi(L(\varphi))$, $\varphi$ is continuous and $\varphi_\kappa$ converges locally uniformly to $\varphi$. This is particularly interesting since the well-known calculus for Chern and Segre forms can be also applied to Chern and Segre currents defined by (7.1) and (7.2) in this general setting, cf. [20, Cor. 1.9].

Furthermore, we obtain the following cohomology result.

**Theorem 7.6** In Setting 7.1 with compact $X$, if $L(\varphi)$ is contained in an analytic set of codimension $\geq k_1 + \cdots + k_t$, then we get

$$s_{k_i}(E, \varphi) \wedge \cdots \wedge s_{k_1}(E, \varphi) \in s_{k_i}(E) \cdots s_{k_1}(E) \quad \text{and} \quad c_{k_i}(E, \varphi) \wedge \cdots \wedge c_{k_1}(E, \varphi) \in c_{k_i}(E) \cdots c_{k_1}(E).$$

**Proof** The argumentation is exactly the same as in the proof of [20, Thm 1.13] by applying Demailly’s regularization, see Theorem 2.9. \(\square\)

**Remark/Def. 7.7** Let us assume that the metric $\varphi$ on $O_{\mathbb{P}(E)}(1)$ has analytic singularities. Using the generalized Monge–Ampère products, the Segre currents

$$s_k(E, \varphi, \alpha) := (-1)^k \pi_\kappa([dd^c \varphi]^k \alpha^{k+r-1})$$

and their wedge products $s_{k_i}(E, \varphi, \alpha) \wedge \cdots \wedge s_{k_1}(E, \varphi, \alpha)$ can be defined for arbitrary degrees following [22, Thm 1.1]. If $\alpha$ is a closed $(1, 1)$-form representing $O_{\mathbb{P}(E)}(1)$, we get that these currents represent their corresponding Segre classes. $s_k(E, \varphi, \eta)$ can be defined for $\eta$ with analytic singularities in isolated points, as well. For every $\xi \in \mathbb{P}(E)$, Corollary 5.10 implies

$$v(s_k(E, \varphi, \eta), \pi(\xi)) \geq \min\{v(\varphi, \xi), v(\eta, \xi)\}^{k+r-1}$$

if $k \leq n$ and if $dd^c \varphi$ and $\eta$ are positive near $\pi^{-1}(\pi(\xi))$. By Lemma 6.1, we can always find such an $\eta = \eta_\xi$ with $v(\eta, \xi) = 1$ and neat analytic singularities, see proof of Theorem 1.3.

For wedge products of Segre currents, the situation is more complicated. Let $\sigma : P := \mathbb{P}(E) \times X \cdots \times X \mathbb{P}(E) \to X$ be the t-fibre power of $\pi : \mathbb{P}(E) \to X$, and let $\varphi_i$ denote the pullback metric $pr_i^* \varphi$ on $pr_i^*O_{\mathbb{P}(E)}(1)$ where $pr_i : P \to \mathbb{P}(E)$ denotes the projection on the $i$th component. For all $k = k_1 + \cdots + k_t$, [22] defines

$$s_{k_t}(E, \varphi, \alpha) \wedge \cdots \wedge s_{k_1}(E, \varphi, \alpha) := (-1)^k \sigma_\kappa([dd^c \varphi]^k \alpha^{k+r-1} \wedge \cdots \wedge [dd^c \varphi]^k \alpha^{k+r-1})$$

where $\alpha_i = pr_i^* \alpha$ for a closed real $(1, 1)$-form $\alpha$ on $\mathbb{P}(E)$ representing $O_{\mathbb{P}(E)}(1)$; cf. Remark 4.9. This cannot be extended straightforwardly to $\eta$ with (neat) analytic singularities in isolated points since the pullbacks of such $\eta$ have singularities in sets of lower codimension.
than points and BTD cannot be applied in general. We guess that it might be possible to still define these products as the singularities of pr$_i^*$ $\eta$ are transversal to each others. An alternative approach could be the following. For every $\xi \in \mathbb{P}(E)$, there is (exactly) one $p \in \bigcap_{i=1}^t \text{pr}_i^{-1}(\xi)$. Let $\eta_i$ be a closed quasipositive $(1, 1)$-currents with analytic singularities in isolated points including $p$ such that $\eta_i$ is representing pr$_i^*$ $\mathcal{O}_{\mathbb{P}(E)}(1)$. Then,

$$S := \sigma_s([dd^c \varphi_1]^k + r - 1 \wedge \cdots \wedge [dd^c \varphi_1^k]_{\eta_i}^1 + r - 1)$$

is locally the difference of two closed positive currents and the pushforward of a closed quasipositive $(k, k)$-current with analytic singularities, and $S \in (-1)^k s_k(E) \cdots s_k(E)$. By Corollary 5.10, we obtain

$$\nu(S, \pi(\xi)) \geq \min\{\nu(\varphi, \xi), \nu(\eta_i, p)\}^{k + tr - r}$$

if $k \leq n$ and if $dd^c \varphi$ positive near $\pi^{-1}(\pi(\xi))$ and $\eta_i$ near $\sigma^{-1}(\pi(\xi))$ since $\nu(\varphi, \xi) = \nu(\varphi, p)$ by Lemma 5.2. Unfortunately, it is not clear whether such $\eta_i$ exist as pr$_i^*$ $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not Kähler near $\sigma^{-1}(\pi(\xi))$ in general.

Let us conclude this section with the proofs of Corollary 1.6 and Theorem 1.7.

**Corollary 1.6** Let $X$ be a compact complex manifold of dimension $n$, let $E \rightarrow X$ be a (pseudoeffective) holomorphic vector bundle on $X$, and let $e^{-\varphi}$ be a semipositive singular metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that $\pi(L(\varphi))$ is contained in an analytic set of codimension $s$. If there are $k_1, \ldots, k_t$ with $k_1 + \cdots + k_t \leq s$ and $s_{k_1}(E) \cdots s_{k_t}(E) = 0$, then $E$ is nef.

**Proof** We set $k = k_1 + \cdots + k_t$. Since $S := (-1)^k s_{k_1}(E) \varphi \wedge \cdots \wedge s_{k_t}(E, \varphi)$ is a closed positive current, and since

$$S \in (-1)^k s_{k_1}(E) \cdots s_{k_t}(E) = 0,$$

$S$ must vanish. Therefore, $\nu(S, \pi(\xi)) = 0$, and by Corollary 1.5, $\nu(\varphi, \xi) = 0$ for all $\xi \in \mathbb{P}(E)$.

Let $\omega_{\mathbb{P}(E)}$ denote the Hermitian form on $\mathbb{P}(E)$ (which is not necessarily closed). For every $\varepsilon > 0$, there is a singular Hermitian metric $\varphi_\varepsilon$ of $\mathcal{O}_{\mathbb{P}(E)}(1)$ with analytic singularities such that $dd^c \varphi_\varepsilon \geq -2\varepsilon \omega_{\mathbb{P}(E)}$ and $\nu(\varphi_\varepsilon, y) \leq \nu(\varphi, y) = 0$, see [14, Prop. 3.7] or Theorem 2.10. As the Lelong number in an analytic singularity would be positive, we obtain that $L(\varphi_\varepsilon) = \emptyset$. By Richberg’s approximation [26], using the version [14, Lem. 2.15], we can get a smooth approximation $\tilde{\varphi}_\varepsilon$ of $\varphi_\varepsilon$ with $dd^c \tilde{\varphi}_\varepsilon \geq -\varepsilon \omega_{\mathbb{P}(E)}$. As we get these smooth metrics for all $\varepsilon > 0$, we conclude $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef, i.e. $E$ is nef. See also [10, Prop. 3.2 (i)].

**Theorem 1.7** Let $(X, \omega)$ be a compact Kähler manifold, and let $E$ be a pseudoeffective vector bundle on $X$ such that $L_{\text{rat}}(E)$ is contained in a countable union of analytic sets of codimension $s$. If there is a $k \leq s$ with $c_1(E)^k = (-s_1(E))^k = 0$, then $E$ is nef.

**Proof** As $X$ is Kähler, so is $\mathbb{P}(E)$. Let $\omega_{\mathbb{P}(E)}$ denote its Kähler form, which is in $\mathcal{O}_{\mathbb{P}(E)}(1)$, and let $r$ denote the rank of $E$.

Fix $\varepsilon > 0$. Let $H$ be a smooth Hermitian metric on $E$, and let $e^{-\sigma}$ denote the induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, i.e. $e^{-\sigma} = \pi^* H |_{\mathcal{O}_{\mathbb{P}(E)}(-1)}$. As $dd^c \sigma$ is positive along fibres of $\pi$, there is a small enough $\delta$ such that $dd^c \sigma \geq \delta \omega_{\mathbb{P}(E)} - \delta^{-1} \pi^* \omega$. By Demailly’s regularization, see Theorem 2.10, there exist a singular Hermitian metric $\varphi_\varepsilon$ with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that $dd^c \varphi_\varepsilon \geq -\delta^2 \varepsilon \omega_{\mathbb{P}(E)}$ and $\delta_\varepsilon > 0$ (with $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$) such that

$$\nu(\varphi, \xi) - \delta_\varepsilon \leq \nu(\varphi_\varepsilon, \xi) \leq \nu(\varphi, \xi) \quad \forall \xi \in \mathbb{P}(E).$$
As in the proof of [17, Thm 11.12], we can modify \( \varphi_\varepsilon \) such that \( dd^c \varphi_\varepsilon \geq -\varepsilon \pi^* \omega \): Replace \( \varphi_\varepsilon \) by the barycentre of \( \varphi_\varepsilon \) and \( \sigma \) given by \( (1 - \delta \varepsilon) \varphi_\varepsilon + \delta \varepsilon \cdot \sigma \) with
\[
\begin{align*}
\leq \varepsilon \pi^* \omega \\
\geq -\varepsilon \pi^* \omega.
\end{align*}
\]
Since \( \varphi_\varepsilon \) has analytic singularities, we get that \( L(\varphi_\varepsilon) \) is an analytic set. Moreover, \( \xi \in L(\varphi_\varepsilon) \) if and only if \( 0 < \nu(\varphi_\varepsilon, \xi) \leq \nu(\varphi, \xi) \) such that \( \text{codim} \pi(L(\varphi_\varepsilon)) \geq k \) by assumption. This implies
\[
(c_1(E, \varphi_\varepsilon))^k = (-s_1(E, \varphi_\varepsilon))^k = \left( \pi_*([dd^c \varphi_\varepsilon]^r) \right)^k
\]
is a closed real \((k, k)\)-current in \( c_1(E)^k \). As calculated in Remark 3.4, we get
\[
\pi_*([dd^c \varphi_\varepsilon]^r) \wedge T = \left( \pi_*[dd^c \varphi_\varepsilon + \varepsilon \pi^* \omega]^r - r \varepsilon \omega \right) \wedge T.
\]
Furthermore,
\[
S_\varepsilon := \left( \pi_*[dd^c \varphi_\varepsilon + \varepsilon \pi^* \omega]^r \right)^k
\]
is a closed positive \((k, k)\)-current in \( (c_1(E) + r \varepsilon \omega)^k \). So, Theorem 1.1 implies
\[
\nu(S_\varepsilon, \pi(\xi)) \geq \left( \nu([dd^c \varphi_\varepsilon + \varepsilon \pi^* \omega], \xi) \right)^k = \left( \nu(\varphi_\varepsilon, \xi) \right)^k (7.6)
\]
for all \( \xi \in \mathbb{P}(E) \).

By weak compactness, there is a subsequence \( \varepsilon_\lambda \) such that \( S_{\varepsilon_\lambda} \) converges weakly to a closed positive current \( S \) in \( (c_1(E))^k \). As assumed, this cohomology vanishes, and so does the closed positive \( S \). By Proposition 5.4, we get that
\[
\lim_{\lambda \to \infty} \nu(S_{\varepsilon_\lambda}, x) \leq \nu(S, 0) = 0.
\]
This in combination with (7.6) and the choice of \( \varphi_\varepsilon \) implies
\[
\nu(\varphi, \xi) = \lim_{\lambda \to \infty} \nu(\varphi_{\varepsilon_\lambda}, \xi) = 0.
\]

Repeating the arguments in the proof of Corollary 1.6 (or by [10, Prop. 3.2 (i)]), we obtain that \( \mathcal{O}_{\mathbb{P}(E)}(1) \) is nef. □

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