THE IMPACT OF QUANTUM COSMOLOGY ON QUANTUM FIELD THEORY

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Abstract. The basic problem of quantum cosmology is the definition of the quantum state of the universe, with appropriate boundary conditions on Riemannian three-geometries. This paper describes recent progress in the corresponding analysis of quantum amplitudes for Euclidean Maxwell theory and linearized gravity. Within the framework of Faddeev-Popov formalism and zeta-function regularization, various choices of mixed boundary conditions lead to a deeper understanding of quantized gauge fields and quantum gravity in the presence of boundaries.
1. Introduction

While the equations of classical field theory are hyperbolic equations, for which a Cauchy problem can be studied in a globally hyperbolic space-time, the analysis of Green’s functions of a quantum field theory makes it necessary to study the so-called Euclidean formulation of field theory. In the case of Minkowski space-time, the key observation is that complex Euclidean points belong to the analyticity domain of the Wightman functions. It is then possible to define the corresponding Schwinger functions, and a relativistic quantum field theory can be reconstructed from a set of Schwinger functions providing the Laplace-transform condition, Euclidean invariance, reflection positivity, Euclidean symmetry and cluster property hold [1].

In a curved space-time, however, there is no general result [2] which ensures that its complexification admits a real section where the metric is Riemannian, i.e. positive-definite. Moreover, given a real Riemannian four-manifold, it may be impossible to recover the Lorentzian Green’s functions by means of analytic continuation. In other words, there is no smooth way to achieve a transition between a Lorentzian and a Riemannian formulation of quantum gravity for a generic space-time. It is instead more appropriate to focus on one of the two, and then derive all mathematical properties which are relevant for the given framework. In this paper, we focus on the elliptic boundary-value problems which are appropriate for the Riemannian regime. With our terminology, this reduces to the Euclidean regime in the case of flat background geometries. The analysis of elliptic operators with suitable boundary conditions on compact Riemannian three-geometries
The Impact of Quantum Cosmology on Quantum Field Theory makes it possible to give a well-defined formulation of boundary conditions for quantum gravity and quantum cosmology, which is one of the main problems of any theory of quantum gravity, and might shed new light on the problems of classical cosmology.

Our analysis is concerned with the semiclassical evaluation of the amplitudes of quantum gravity in the presence of boundaries [2]. While the path integral for quantum gravity is a formal object which remains ill-defined, its semiclassical approximation (although of limited applicability) leads to well-posed problems with a wide range of applications, i.e. the one-loop effective action and heat-kernel methods in field theory, the quantization of constrained systems and the appropriate boundary conditions for quantum fields. For a given elliptic operator $\mathcal{A}$ (e.g. the Laplace operator or the squared Dirac operator), its zeta-function

$$\zeta_{\mathcal{A}}(s) \equiv \text{Tr}\left[\mathcal{A}^{-s}\right]$$

(1.1)

has an analytic continuation to the complex-$s$ plane as a meromorphic function which is regular at the origin. While the one-loop effective action depends linearly on $\zeta_{\mathcal{A}}(0)$ and $\zeta'_{\mathcal{A}}(0)$ and is non-local, the scaling properties of the semiclassical amplitudes are indeed determined by the $\zeta_{\mathcal{A}}(0)$ value [2], which also describes the one-loop divergences of physical theories.

Section 2 describes recent progress on Euclidean Maxwell theory, and section 3 studies mixed boundary conditions for Euclidean quantum gravity. Concluding remarks and open problems are presented in section 4.
2. Euclidean Maxwell Theory

We are interested in the one-loop amplitudes of vacuum Maxwell theory in the presence of boundaries. Since in the classical theory the potential $A_\mu$ is subject to the gauge transformations

$$\hat{A}_\mu \equiv A_\mu + \partial_\mu \varphi \ ,$$

this gauge freedom is reflected in the quantum theory by a ghost zero-form, i.e. an anticommuting, complex scalar field, hereafter denoted again by $\varphi$. The two sets of mixed boundary conditions consistent with gauge invariance and BRST symmetry are magnetic, i.e. [3-4]

$$\left[ A_k \right]_{\partial M} = 0 \ , \quad (2.2a)$$

$$\left[ \Phi(A) \right]_{\partial M} = 0 \ , \quad (2.2b)$$

$$\left[ \varphi \right]_{\partial M} = 0 \ , \quad (2.2c)$$

or electric, i.e. [3-4]

$$\left[ A_0 \right]_{\partial M} = 0 \ , \quad (2.3a)$$

$$\left[ \partial A_k / \partial \tau \right]_{\partial M} = 0 \ , \quad (2.3b)$$

$$\left[ \partial \varphi / \partial \tau \right]_{\partial M} = 0 \ , \quad (2.3c)$$
where $\Phi$ is an arbitrary gauge-averaging function defined on the space of connection one-forms $A_\mu dx^\mu$. Note that the boundary condition (2.2c) ensures the gauge invariance of the boundary conditions (2.2a)-(2.2b) on making the gauge transformation (2.1). Similarly, the boundary condition (2.3c) ensures the gauge invariance of (2.3a)-(2.3b) on transforming the potential as in (2.1).

For a given choice of one of these two sets of mixed boundary conditions, different choices of background four-geometry, boundary three-geometry and gauge-averaging function lead to a number of interesting results. We here summarize them in the case of a background given by flat Euclidean four-space bounded by one three-sphere (i.e. the disk) or by two concentric three-spheres (i.e. the ring).

(i) The operator matrix acting on the normal and longitudinal modes of the potential can be diagonalized for all relativistic gauge conditions which can be cast in the form [3-5]

$$
\Phi_b(A) \equiv \nabla^\mu A_\mu - b A_0 \Tr K ,
$$

where $\nabla^\mu$ denotes covariant differentiation on the background, $b$ is a dimensionless parameter, and $K$ is the extrinsic-curvature tensor of the boundary.

(ii) In the case of the disk, the Lorentz gauge (set $b = 0$ in (2.4)) leads to a $\zeta(0)$ value

$$
\zeta_L(0) = -\frac{31}{90} ,
$$

for both magnetic and electric boundary conditions, which agrees with the geometric theory of the asymptotic heat kernel. However, the $\zeta(0)$ value depends on the gauge condition, and unless $b$ vanishes it also depends on the boundary conditions [3-5].
(iii) In the case of the ring, one finds

\[ \zeta(0) = 0, \tag{2.6} \]

for all gauge conditions [4-6], independently of boundary conditions [3-4]. This result agrees with the geometric formulae for the heat kernel, since volume contributions to \( \zeta(0) \) vanish in a flat background, while surface contributions cancel each other.

(iv) In the case of boundary three-geometries given by one or two three-spheres, the most general gauge-averaging function takes the form

\[ \Phi(A) = \gamma_1 \langle 4 \rangle \nabla^0 A_0 + \frac{\gamma_2}{3} A_0 \text{ Tr } K - \gamma_3 \langle 3 \rangle \nabla^i A_i, \tag{2.7} \]

where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are arbitrary dimensionless parameters. Thus, unless \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) take some special values (cf. (2.4)), it is not possible to diagonalize the operator matrix acting on normal and longitudinal modes of the potential.

(v) The contributions to \( \zeta(0) \) resulting from normal and longitudinal modes do not cancel the contribution of ghost modes, unless one sets \( b = \frac{2}{3} \) in (2.4) in the case of the disk, with magnetic boundary conditions. Thus, transverse modes do not provide the only contribution to one-loop amplitudes. In other words, there seem to be no unphysical modes in a manifestly gauge-invariant quantum field theory, in that all perturbative modes are necessary to recover the gauge-invariant quantum amplitudes (see also section 3).
3. Linearized Gravity

We here focus on the amplitudes of Euclidean quantum gravity within the framework of Faddeev-Popov formalism. This means that the amplitudes depend on the boundary data for the metric and for ghost fields, and are written (formally) as Feynman path integrals over all compact Riemannian four-geometries matching the data at the boundary, i.e. [7]

\[
Z[\text{boundary data}] = \int_{C} \mu_1[g] \mu_2[\varphi] e^{-\tilde{I}_E},
\]

where \( \mu_1 \) is a suitable measure on the space of Riemannian four-metrics, \( \mu_2 \) is a suitable measure on the space of ghost fields, and the full Euclidean action takes the form

\[
\tilde{I}_E = \frac{1}{16\pi G} \int_{M} R \sqrt{\det g} \, d^4x + \frac{1}{8\pi G} \int_{\partial M} \text{Tr} \, K \sqrt{\det q} \, d^3x + \frac{1}{32\pi G\alpha} \int_{M} \Phi_\nu \Phi^\nu \sqrt{\det g} \, d^4x + I_{gh}.
\]

With our notation, \( q \) is the induced three-metric, \( \Phi_\nu \) is a relativistic gauge-averaging function, and \( I_{gh} \) is the corresponding ghost action. In the one-loop approximation, the measures in (3.1) become measures on metric perturbations and ghost perturbations, respectively. Thus, denoting by \( g \) the background four-metric and by \( h \) its perturbation, the form of \( \Phi_\nu \) necessary to find the familiar form of the propagators, as well as to recover the Vilkovisky-DeWitt effective action [8-10], is the de Donder gauge-averaging function [7]

\[
\Phi_{\nu}^{dD}(h) \equiv \nabla^\mu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \right),
\]

\[
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\]
where $\nabla$ is the Levi-Civita connection on the background four-geometry. The corresponding elliptic operator in the ghost action is then found to be $-g_{\mu\nu}\Box - R_{\mu\nu}$. Note also that the boundary term in (3.2) is the one appropriate for fixing the spatial perturbations $h_{ij}$ at the boundary.

We can now understand how to generalize the magnetic boundary conditions of section 2 to pure gravity. The basic idea is to set to zero at the boundary the spatial perturbations $h_{ij}$ of the metric, and the gauge-averaging function $\Phi^{dd}(h)$. To ensure that these boundary conditions are invariant under gauge transformations of $h_{\mu\nu}$ of the form

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \nabla_{(\mu} \varphi_{\nu)} ,$$

one has then to set to zero at the boundary the whole ghost one-form [11]:

$$\left[ \varphi_{\mu} \right]_{\partial M} = 0 .$$

(3.4)

While $h_{ij}$ obeys homogeneous Dirichlet conditions at $\partial M$ as we just said, the boundary condition $[\Phi^{dd}(h)]_{\partial M} = 0$ leads to [11]

$$\left[ \frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} \left( g^{ij} h_{ij} \right) + \frac{2}{\tau^2} h_{0i}^{\mid i} \right]_{\partial M} = 0 ,$$

(3.6)

$$\left[ \frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \right]_{\partial M} = 0 .$$

(3.7)

In the case of flat Euclidean four-space bounded by a three-sphere, which is relevant for quantum cosmology in the case of four-sphere backgrounds bounded by a three-sphere of
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small radius [2], the boundary conditions described so far, which were first proposed by Barvinsky [12], lead to the full $\zeta(0)$ value [11]

$$\zeta(0) = -\frac{241}{90} .$$

(3.8)

This differs from the contribution of transverse-traceless metric perturbations, which was found to be [13]

$$\zeta_{TT}(0) = -\frac{278}{45} .$$

(3.9)

Interestingly, the detailed calculations performed in [11] and outlined in this section add evidence in favour of no cancellation being possible between ghost- and gauge-modes contributions to one-loop amplitudes in the presence of boundaries. From the point of view of constrained Hamiltonian systems and their quantization, this seems to suggest that there are no unphysical modes in a gauge-invariant quantum field theory. Reduction of a field theory with first-class constraints to its physical degrees of freedom before quantization leads to an inequivalent quantum field theory, where gauge-invariance properties are lost (cf. [14]).

The boundary conditions studied in [11-12] are not the only possible set of mixed boundary conditions for Euclidean quantum gravity [15]. By contrast, on studying BRST transformations at the boundary, one is led to consider the following boundary conditions [11,16-17]:

$$\left[ 2\text{Tr} \, K + n^\alpha \nabla_\alpha \right] n^\mu n^\nu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \right) \partial_M = 0 ,$$

(3.10)

$$\begin{align*}
\left[ h_{ij} \right]_{\partial M} = & \left[ h_{0i} \right]_{\partial M} = \left[ \phi_0 \right]_{\partial M} = 0 ,
\end{align*}$$

(3.11)
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\[
\left[ -K_{\mu}^\nu + \delta_{\mu}^\nu \ n^\alpha \nabla_\alpha \right] P^\nu_\sigma \varphi_\sigma \Gamma_{\partial M} = 0 ,
\]

(3.12)

where there is summation over repeated indices, and we have used the tangential projection operator

\[
P^\nu_\mu \equiv \delta^\nu_\mu - n^\mu n^\nu ,
\]

(3.13)

where \(n^\mu\) is the normal to the boundary.

As shown in [11], the boundary conditions (3.10)-(3.12) lead to the following \(\zeta(0)\) value in the case of flat Euclidean space bounded by \(S^3\):

\[
\zeta(0) = -\frac{758}{45} ,
\]

(3.14)

which agrees with the results deriving from the geometric theory of the asymptotic heat kernel [11,18-20]. By contrast, the boundary conditions (3.5)-(3.7) make it more difficult to use projection operators and then apply the powerful geometric techniques available in the literature. Nevertheless, the St. Petersburg group, led by Dr. D. Vassilevich, is making progress on this crucial issue.

4. Open Problems

The analysis of Euclidean Maxwell theory in the presence of boundaries raises at least three crucial issues. First, since in the one-boundary problems the Faddeev-Popov amplitudes turn out to be gauge-dependent, should we accept that not all gauges are admissible, or should we instead argue that the Hartle-Hawking program [21] is incorrect, because one
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cannot shrink to a point one of the two boundary three-surfaces? Second, how to prove explicitly the gauge invariance of quantum amplitudes in the two-boundary problems. What happens is that changing the gauge condition leads to a smooth variation of the matrix of elliptic operators acting on gauge modes [7]. One has then to prove that the resulting contributions to \( \zeta(0) \) remain unaffected by such a smooth variation, even though it is no longer possible to express the gauge modes as linear combinations of Bessel functions. Third, the lack of cancellation of gauge- and ghost-modes contributions to the full \( \zeta(0) \) points out a deeper role played by such modes in the quantum theory. They are essential to recover the full content of the semiclassical approximation, and hence cannot be regarded as non-physical, although in the classical Lorentzian theory one is naturally led to identify the transverse part of the electromagnetic potential with the physical degrees of freedom [14].

In the case of linearized Euclidean quantum gravity, the explicit proof of gauge invariance of the one-loop amplitudes is even more complicated, since there are now ten sets of perturbative modes. Moreover, it appears necessary to obtain geometric formulae for the asymptotic heat kernel in the case of Barvinsky boundary conditions [12] studied in section 3. Other relevant problems are the analysis of non-relativistic gauges for pure gravity, and the non-local nature of the one-loop effective action expressed through the \( \zeta'(0) \) value for elliptic problems with boundaries [22].

Last, but not least, the quantum state of the Lorentzian theory corresponding to the boundary conditions of section 3 remains unknown. If this problem is not thoroughly studied, we remain unable to make contact with the world we live in, unless one is ready to
accept Hawking’s view, according to which the Euclidean regime is the more fundamental [23].

In the light of the analysis presented in this paper, it seems appropriate to conclude that quantum cosmology has indeed a deep influence on the understanding of physical fields and their quantization, and hence it lies at the very heart of fundamental theoretical physics.

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References

[1] Strocchi F. (1993) Selected Topics on the General Properties of Quantum Field Theory (Singapore: World Scientific).

[2] Esposito G. (1994) Quantum Gravity, Quantum Cosmology and Lorentzian Geometries, Lecture Notes in Physics, New Series m: Monographs, Vol. m12, second corrected and enlarged edition (Berlin: Springer-Verlag).
The Impact of Quantum Cosmology on Quantum Field Theory

[3] Esposito G. (1994) Class. Quantum Grav. 11, 905.

[4] Esposito G., Kamenshchik A. Yu., Mishakov I. V. and Pollifrone G. (1994) Class. Quantum Grav. 11, 2939.

[5] Esposito G., Kamenshchik A. Yu., Mishakov I. V. and Pollifrone G. Relativistic Gauge Conditions in Quantum Cosmology (DSF preprint 95/8, to appear in Phys. Rev. D).

[6] Esposito G. and Kamenshchik A. Yu. (1994) Phys. Lett. B 336, 324.

[7] Esposito G., Kamenshchik A. Yu., Mishakov I. V. and Pollifrone G. (1994) Phys. Rev. D 50, 6329.

[8] Taylor T. R. and Veneziano G. (1990) Nucl. Phys. B 345, 210.

[9] Vilkovisky G. A. (1984) Nucl. Phys. B 234, 125.

[10] De Witt B. S. (1987) The Effective Action, in Architecture of Fundamental Interactions at Short Distances, Les Houches Session XLIV, eds. P. Ramond and R. Stora (Amsterdam: North-Holland) p. 1023.

[11] Esposito G., Kamenshchik A. Yu., Mishakov I. V. and Pollifrone G. One-Loop Amplitudes in Euclidean Quantum Gravity (DSF preprint 95/16).

[12] Barvinsky A. O. (1987) Phys. Lett. B 195, 344.

[13] Schleich K. (1985) Phys. Rev. D 32, 1889.

[14] McMullan D. and Tsutsui I. (1995) Ann. Phys. 237, 269.

[15] Esposito G. and Kamenshchik A. Yu. Mixed Boundary Conditions in Euclidean Quantum Gravity (DSF preprint 95/23).

[16] Luckock H. C. (1991) J. Math. Phys. 32, 1755.

[17] Moss I. G. and Poletti S. J. (1990) Nucl. Phys. B 341, 155.
The Impact of Quantum Cosmology on Quantum Field Theory

[18] Branson T. P. and Gilkey P. B. (1990) *Commun. Part. Diff. Eq.* **15**, 245.

[19] Vassilevich D. (1995) *J. Math. Phys.* **36**, 3174.

[20] Moss I. G. and Poletti S. J. (1994) *Phys. Lett.* B **333**, 326.

[21] Hartle J. B. and Hawking S. W. (1983) *Phys. Rev.* D **28**, 2960.

[22] Bordag M., Geyer B., Kirsten K. and Elizalde E. *Zeta-Function Determinant of the Laplace Operator on the D-Dimensional Ball* (UB-ECM-PF preprint 95/10).

[23] Gibbons G. W. and Hawking S. W. (1993) *Euclidean Quantum Gravity* (Singapore: World Scientific).