Global Adaptive Dynamic Programming for Continuous-Time Nonlinear Systems

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Abstract

This paper presents for the first time a novel method of global adaptive dynamic programming (ADP) for the adaptive optimal control of nonlinear systems. The essential strategy consists of relaxing the problem of solving the Hamilton-Jacobi-Bellman (HJB) equation to an optimization problem, which is solved via a new policy iteration method. The proposed method distinguishes from previously known nonlinear ADP methods in that the neural network approximation is avoided, giving rise to significant computational improvement. In addition, the resultant control policy is globally stabilizing, instead of semiglobally or locally stabilizing. Furthermore, in the absence of the a priori knowledge of the system dynamics, an online learning method is devised to implement the proposed policy iteration technique by generalizing the current ADP theory. Finally, three numerical examples are provided to validate the efficiency and effectiveness of the proposed method.

Index Terms

Adaptive dynamic programming, nonlinear systems, optimal control, global stabilization.

I. INTRODUCTION

Dynamic programming [4] offers a theoretical way to solve optimal control problems. However, it suffers from the inherent computational complexity, also known as the curse of dimensionality. Therefore, the need for approximative methods has been recognized as early as in the late 1950s [5]. Within all these methods, adaptive/approximate dynamic programming (ADP) [6], [7], [36], [42], [55], [57] is a class of heuristic techniques that solve the cost function...
by searching for a suitable approximation. In particular, adaptive dynamic programming [55], [56] employs the idea from reinforcement learning [45] to achieve online approximation of the cost function, without using the knowledge of the system dynamics. ADP has been extensively studied for Markov decision processes (see, for example, [7] and [36]), as well as dynamic systems (see the review papers [28] and [52]). Stability issues in ADP-based control systems design are addressed in [2], [51], [30]. A robustification of ADP, known as Robust-ADP or RADP, is recently developed by taking into account dynamic uncertainties [22].

To achieve online approximation of the cost function and the control policy, neural networks are widely used in the previous ADP architecture. Although neural networks can be used as universal approximators [18], [34], there are at least two major limitations for ADP-based online implementations. First, in order to approximate unknown functions with high accuracy, a large number of basis functions comprising the neural network are usually required. Hence, it may incur a huge computational burden for the learning system. Besides, it is not trivial to specify the type of basis functions, when the target function to be approximated is unknown. Second, neural network approximations generally are effective only on some compact sets, but not in the entire state space. Therefore, the resultant control policy may not provide global asymptotic stability for the closed-loop system. In addition, the compact set, on which the uncertain functions of interest are to be approximated, has to be carefully quantified before one applies the online learning method, such that stability can be assured during the learning process [20].

The main purpose of this paper is to develop a novel ADP methodology to achieve global and adaptive suboptimal stabilization of uncertain continuous-time nonlinear system via online learning. As the first contribution of this paper, an optimization problem, of which the solutions can be easily parameterized, is proposed to relax the problem of solving the Hamilton-Jacobi-Bellman (HJB) equation. This approach is similar to the relaxation method used in approximate dynamic programming for Markov decision processes (MDPs) with finite state space [11], and more generalized discrete-time systems [31], [53], [54], [39], [40], [44]. However, methods developed in these papers cannot be trivially extended to the continuous-time setting, or achieve global asymptotic stability of general nonlinear systems. The idea of relaxation was also used in nonlinear $\mathcal{H}_\infty$ control, where Hamilton-Jacobi inequalities are used for nonadaptive systems [16], [48].

The second contribution of the paper is to propose a relaxed policy iteration method. For
polynomial systems, we formulate each iteration step of the proposed policy iteration as a sum of squares (SOS) program [35], [8], and give its equivalent semidefinite programming (SDP) problem [49]. For nonlinear functions that cannot be parameterized using a basis of polynomials, a less conservative sufficient condition is derived to decide their non-negativity by examining the coefficients. Thus, the proposed policy iteration is formulated as a more general SDP problem. It is worth pointing out that, different from the inverse optimal control design [27], the proposed method finds directly a suboptimal solution to the original optimal control problem.

The third contribution is an online learning method that implements the proposed iterative schemes using only the real-time online measurements, when the perfect system knowledge is not available. This method can be regarded as a nonlinear variant of our recent work for continuous-time linear systems with completely unknown system dynamics [21]. This method distinguishes from previously known nonlinear ADP methods in that the neural network approximation is avoided for computational benefits and that the resultant control policy is globally stabilizing, instead of semiglobally or locally stabilizing.

The remainder of this paper is organized as follows. Section 2 formulates the problem and introduces some basic results in nonlinear optimal control and nonlinear optimization. Section 3 relaxes the problem of solving an HJB equation to an optimization problem. Section 4 develops a relaxed policy iteration technique for polynomial systems based on sum of squares (SOS) programming [8]. Section 5 develops an online learning method for applying the proposed policy iteration, when the system dynamics are not known exactly. Section 6 extends the proposed method to deal with more generalized nonlinear systems. Section 7 examines three numerical examples to validate the efficiency and effectiveness of the proposed method. Section 8 gives concluding remarks.

Notations: Throughout this paper, we use $C^1$ to denote the set of all continuously differentiable functions. $P$ denotes the set of all functions in $C^1$ that are also positive definite and proper. $\mathbb{R}_+$ indicates the set of all non-negative real numbers. For any vector $u \in \mathbb{R}^m$ and any positive definite matrix $R \in \mathbb{R}^{m \times m}$, we define $|u|^2_R$ as $u^T R u$. A feedback control policy $u$ is called globally stabilizing, if under this control policy, the closed-loop system is globally asymptotically stable at the origin [24]. For any non-negative integers $d_1, d_2$ satisfying $d_2 \geq d_1$, $[x]_{d_1, d_2}$ is the vector of all $\binom{n+d_2}{d_2} - \binom{n+d_1}{d_1}$ distinct monic monomials in $x \in \mathbb{R}^n$ with degree no less than $d_1$ and no greater than $d_2$, and arranged in lexicographic order [10]. Also, $\mathbb{R}[x]_{d_1, d_2}$ denotes the set of all
polynomials in $x \in \mathbb{R}^n$ with degree no less than $d_1$ and no greater than $d_2$. In addition, $\mathbb{R}[x]_{d_1,d_2}^m$ denotes the set of $m$-dimensional vectors, of which each entry is a polynomial in $\mathbb{R}[x]_{d_1,d_2}$. $\nabla V$ refers to the gradient of a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we first formulate the control problem to be studied in the paper. Then, we introduce some basic tools in nonlinear optimal control and optimization theories, based on which our main results in this paper will be developed.

A. Problem formulation

Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $f(x)$ and $g(x)$ are locally Lipschitz functions with $f(0) = 0$.

In conventional optimal control theory [29], the common objective is to find a control policy $u$ that minimizes certain performance index. In this paper, it is specified as follows.

$$J(x_0, u) = \int_0^\infty r(x(t), u(t))dt, \quad x(0) = x_0$$

where $r(x, u) = Q(x) + u^T R u$, with $Q(x)$ a positive definite function, and $R$ is a symmetric positive definite matrix. Notice that, the purpose of specifying $r(x, u)$ in this form is to guarantee that an optimal control policy can be explicitly determined, if it exists.

Assumption 2.1: Consider system (1). There exist a function $V_0 \in \mathcal{P}$ and a feedback control policy $u_1$, such that

$$\mathcal{L}(V_0(x), u_1(x)) \geq 0, \quad \forall x \in \mathbb{R}^n$$

where, for any $V \in \mathcal{C}^1$ and $u \in \mathbb{R}^m$,

$$\mathcal{L}(V, u) = -\nabla V^T (f(x) + g(x)u) - r(x, u).$$

Under Assumption 2.1 the closed-loop system composed of (1) and $u = u_1(x)$ is globally asymptotically stable at the origin, with a well-defined Lyapunov function $V_0$. With this property, $u_1$ is also known as an admissible control policy [3], implying that the cost $J(x_0, u_1)$ is finite,
∀x_0 ∈ \mathbb{R}^n. Indeed, integrating both sides of (3) along the trajectories of the closed-loop system composed of (1) and u = u_1(x) on the interval [0, +∞), it is easy to show that

\[ J(x_0, u_1) \leq V_0(x_0), \quad \forall x_0 ∈ \mathbb{R}^n. \]  

(5)

**B. Optimality and stability**

Here, we recall a basic result connecting optimality and global asymptotic stability in nonlinear systems [41]. To begin with, let us give the following assumption.

**Assumption 2.2:** There exists \( V^o ∈ \mathcal{P} \), such that the Hamilton-Jacobi-Bellman (HJB) equation holds

\[ \mathcal{H}(V^o) = 0 \]  

(6)

where

\[ \mathcal{H}(V) = \nabla V^T(x)f(x) + Q(x) - \frac{1}{4} \nabla V^T(x)g(x)R^{-1}g^T(x)\nabla V(x). \]

Under Assumption 2.2, it is easy to see that \( V^o \) is a well-defined Lyapunov function for the closed-loop system comprised of (1) and

\[ u^o(x) = -\frac{1}{2}R^{-1}g^T(x)\nabla V^o(x). \]  

(7)

Hence, this closed-loop system is globally asymptotically stable at \( x = 0 \) [24]. Then, according to [41, Theorem 3.19], \( u^o \) is the optimal control policy, and the value function \( V^o(x_0) \) gives the optimal cost at the initial condition \( x(0) = x_0 \), i.e.,

\[ V^o(x_0) = \min_u J(x_0, u) = J(x_0, u^o), \quad \forall x_0 ∈ \mathbb{R}^n. \]  

(8)

It can also be shown that \( V^o \) is the unique solution to the HJB equation (6) with \( V^o ∈ \mathcal{P} \). Indeed, let \( \hat{V} ∈ \mathcal{P} \) be another solution to (6). Then, by Theorem 3.19 in [41], along the solutions of the closed-loop system composed of (1) and \( u = \hat{u} = -\frac{1}{2}R^{-1}g^T\nabla \hat{V} \), it follows that

\[ \hat{V}(x_0) = V^o(x_0) - \int_0^∞ |u^o - \hat{u}|_R^2 dt, \quad \forall x_0 ∈ \mathbb{R}^n. \]  

(9)

Finally, comparing (8) and (9), we conclude that \( V^o = \hat{V} \).
C. Conventional policy iteration

The above-mentioned result implies that, if there exists a class-$\mathcal{P}$ function which solves the HJB equation (6), an optimal control policy can be obtained. However, the nonlinear HJB equation (6) is almost impossible to be solved analytically in general. As a result, numerical methods are developed to approximate the solution. In particular, the following policy iteration method is widely used [38].

1) **Policy evaluation:** For $i = 1, 2, \cdots$, solve for the cost function $V_i(x) \in C^1$, with $V_i(0) = 0$, from the following partial differential equation.

$$L(V_i(x), u_i(x)) = 0.$$ (10)

2) **Policy improvement:** Update the control policy by

$$u_{i+1}(x) = -\frac{1}{2}R^{-1}g^T(x)\nabla V_i(x).$$ (11)

The following result is a trivial extension of [38, Theorem 4], in which $g(x)$ is a constant matrix and only stabilization over compact set is considered.

**Theorem 2.1:** Suppose Assumptions 2.1 and 2.2 hold, and the solution $V_i(x) \in C^1$ satisfying (10) exists, for $i = 1, 2, \cdots$. Let $V_i(x)$ and $u_{i+1}(x)$ be the functions generated from (10) and (11). Then, the following properties hold, $\forall i = 0, 1, \cdots$.

1) $V^o(x) \leq V_{i+1}(x) \leq V_i(x), \forall x \in \mathbb{R}^n$;
2) $u_{i+1}$ is globally stabilizing;
3) $J(x_0, u_i)$ is finite, $\forall x_0 \in \mathbb{R}^n$;
4) $\{V_i(x)\}_{i=0}^\infty$ and $\{u_i(x)\}_{i=1}^\infty$ converge pointwise to $V^o(x)$ and $u^o(x)$, respectively.

Notice that finding the analytical solution to (10) is still non-trivial. Hence, in practice, the solution is approximated using, for example, neural networks or Galerkin’s method [3]. When the precise knowledge of $f$ or $g$ is not available, ADP-based online approximation method can be applied to compute numerically the cost functions via online data [50], [20].

In general, approximation methods can only give acceptable results on some compact set in the state space, but cannot be used to achieve global stabilization. In addition, in order to reduce the approximation error, huge computational complexity is almost inevitable. These facts may affect the effectiveness of the previously developed ADP-based online learning methods.
D. Semidefinite programming and sum-of-squares programming

A standard semidefinite programming (SDP) problem can be formulated as the following problem of minimizing a linear function of a variable $y \in \mathbb{R}^{n_0}$ subject to a linear matrix inequality.

**Problem 2.1 (Semidefinite programming [49]):**

$$\min_{y} \quad c^T y$$

$$F_0 + \sum_{i=1}^{n_0} y_i F_i \geq 0$$

where $c \in \mathbb{R}^{n_0}$ is a constant column vector, and $F_0, F_1, \cdots, F_{n_0} \in \mathbb{R}^{m_0 \times m_0}$ are $n_0 + 1$ symmetric constant matrices.

SDPs can be solved using several commercial or non-commercial software packages, such as the Matlab-based solver CVX [14].

**Definition 2.1 (Sum of squares [8]):** A polynomial $p(x) \in \mathbb{R}[x]_{0,2d}$ is a sum of squares (SOS) if there exist $q_1, q_2, \cdots, q_{m_0} \in \mathbb{R}[x]_{0,d}$ such that

$$p(x) = \sum_{i=1}^{m_0} q_i^2(x).$$

An SOS programming problem is a convex optimization problem of the following form

**Problem 2.2 (SOS programming [8]):**

$$\min_{y} \quad b^T y$$

s.t. $p_i(x; y)$ are SOS, $i = 1, 2, \cdots, k_0$

where $p_i(x; y) = a_{i0}(x) + \sum_{j=1}^{n_0} a_{ij}(x)y_j$, and $a_{ij}(x)$ are given polynomials in $\mathbb{R}[x]_{0,2d}$.

In [8 p.74], it has been pointed out that SOS programs are in fact equivalent to SDPs. Indeed, the constraints (16) are equivalent to the existence of symmetric matrices $Q_i \succeq 0$ satisfying

$$p_i(x; y) = [x]_{0,d}^T Q_i [x]_{0,d}, \quad i = 1, 2, \cdots, k_0.$$  (17)

Then, by extending and matching the coefficients of (17), the equations (17) reduce to linear equations between $y$ and the entries of $Q_i$. As a result, Problem 2.1 is equivalent to an SDP problem in the variables of $y$ and all the distinct entries of $Q_i$. This equivalence implies that SOS programs can be reformulated and solved as SDPs. The conversion from an SOS to an SDP can be performed either manually, or automatically using, for example, the Matlab toolbox SOSTOOLS [37], YALMIP [32], and Gloptipoly [17].
III. SUBOPTIMAL CONTROL WITH RELAXED HJB EQUATION

In general, it is extremely difficult to obtain the analytical solution to the HJB equation (6). Therefore, in this section we consider an auxiliary optimization problem, which allows us to obtain a suboptimal solution to the minimization problem (2) subject to (1). For simplicity, we will omit the arguments of functions whenever there is no confusion in the context.

**Problem 3.1 (Relaxed optimal control problem):**

\[
\min_{V} \int_{\mathbb{R}^n} w(x)V(x)dx \\
\text{s.t.} \quad \mathcal{H}(V) \leq 0 \\
V \in \mathcal{P}
\]

where \(w(x)\) is a positive semidefinite function taking positive values only on some predefined compact set \(\Omega \subset \mathbb{R}^n\).

**Remark 3.1:** Notice that Problem 3.1 is called a relaxed problem of (6). Indeed, if we restrict this problem by replacing the inequality constraint (19) with the equality constraint (6), there will be only one feasible solution left and the objective function can thus be neglected. As a result, Problem 3.1 reduces to the problem of solving (6).

**Remark 3.2:** The function \(w(x)\) can also be recognized as the state-relevance weighting function [11]. It is easy to see that better approximation to the optimal cost function \(V^o\) in a particular region of state space can be achieved by assigning relatively higher weights to the region.

Some useful facts about Problem 3.1 are given as follows.

**Theorem 3.1:** Under Assumptions 2.1 and 2.2, the following hold.

1) Problem 3.1 has a nonempty feasible set.

2) Let \(V\) be a feasible solution to Problem 3.1 Then, the control policy

\[
\bar{u} = -\frac{1}{2} R^{-1} g^T \nabla V
\]

is globally stabilizing.

3) For any \(x_0 \in \mathbb{R}^n\), an upper bound of the cost of the closed-loop system comprised of (1) and (21) is given by \(V(x_0)\), i.e.,

\[
J(x_0, \bar{u}) \leq V(x_0). \tag{22}
\]
4) Along the trajectories of the closed-loop system (1) and (21), the following inequalities hold for any \( x_0 \in \mathbb{R}^n \):

\[
V(x_0) + \int_0^\infty H(V(x(t)))dt \leq V^o(x_0) \leq V(x_0).
\]  

(23)

5) \( V^o \) as defined in (8) is a global optimal solution to Problem 3.1.

\[ \text{Proof:} \]

1) Define \( u_0 = -\frac{1}{2}R^{-1}g^T \nabla V_0 \). Then,

\[
\mathcal{H}(V_0) = \nabla V_0^T (f + gu_0) + r(x, u_0)
\]

\[
= \nabla V_0^T (f + gu_1) + r(x, u_1)
\]

\[
+ \nabla V_0^T g(u_0 - u_1) + |u_0|^2_R - |u_1|^2_R
\]

\[
= \nabla V_0^T (f + gu_1) + r(x, u_1) - |u_0 - u_1|^2_R
\]

\[
\leq 0
\]

Hence, \( V_0 \) is a feasible solution to Problem 3.1

2) To show global asymptotic stability, we only need to prove that \( V \) is a well-defined Lyapunov function for the closed-loop system composed of (1) and (21). Indeed, along the solutions of the closed-loop system, it follows that

\[
\dot{V} = \nabla V^T (f + g\bar{u}) = \mathcal{H}(V) - r(x, \bar{u}) \leq -Q(x)
\]

Therefore, the system is globally asymptotically stable at the origin [24].

3) Along the solutions of the closed-loop system comprised of (1) and (21), we have

\[
V(x_0) = -\int_0^T \nabla V^T (f + g\bar{u})dt + V(x(T))
\]

\[
= \int_0^T [r(x, \bar{u}) - \mathcal{H}(V)] dt + V(x(T))
\]

\[
\geq \int_0^T r(x, \bar{u})dt + V(x(T))
\]

(24)

By 2), \( \lim_{T \to +\infty} V(x(T)) = 0 \). Therefore, letting \( T \to +\infty \), by (24) and (2), we have

\[
V(x_0) \geq J(x_0, \bar{u}).
\]

(25)

4) By 3), we have

\[
V(x_0) \geq J(x_0, \bar{u}) \geq \min_u J(x_0, \bar{u}) = V^o(x_0).
\]

(26)
Hence, the second inequality in (23) is proved.

On the other hand,

\begin{align}
\mathcal{H}(V) &= \mathcal{H}(V) - \mathcal{H}(V^o) \\
&= (\nabla V - \nabla V^o)^T(f + g\bar{u}) + r(x, \bar{u}) - (\nabla V^o)^T g(u^o - \bar{u}) - r(x, u^o) \\
&= (\nabla V - \nabla V^o)^T(f + g\bar{u}) + |\bar{u} - u^o|^2_R \\
&\geq (\nabla V - \nabla V^o)^T(f + g\bar{u})
\end{align}

Integrating the above equation along the solutions of the closed-loop system (1) and (21) on the interval \([0, +\infty)\), we have

\begin{equation}
V(x_0) - V^o(x_0) \leq -\int_0^{\infty} \mathcal{H}(V(x))dt. \tag{27}
\end{equation}

5) By 3), for any feasible solution \(V\) to Problem 3.1 we have \(V^o(x) \leq V(x)\). So,

\begin{equation}
\int_{\mathbb{R}^n} w(x) V^o(x) dx \leq \int_{\mathbb{R}^n} w(x) V(x) dx \tag{28}
\end{equation}

which implies that \(V^o\) is a global optimal solution.

The proof is therefore complete.

**Remark 3.3:** A feasible solution \(V\) to Problem 3.1 may not necessarily be the true cost function associated with the control policy \(\bar{u}\) defined in (21). However, by Theorem 3.1, we see \(V\) can be viewed as an upper bound or an overestimate of the actual cost, inspired by the concept of underestimator in [53]. Further, \(V\) serves as a Lyapunov function for the closed-loop system and can be more easily parameterized than the actual cost function. For simplicity, \(V\) is still called the cost function, in the remainder of the paper.

**IV. SOS-based Policy Iteration for Polynomial Systems**

The inequality constraint (19) contained in Problem 3.1 provides us the freedom of specifying desired analytical forms of the cost function. However, solving (19) is non-trivial in general, even for polynomial systems (see, for example, [9], [12], [33], [43], [58]). Indeed, for any polynomial with degree no less than four, deciding its non-negativity is an NP-hard problem [35]. Fortunately, due to the developments in sum of squares (SOS) programming [8], [35], the computational burden can be significantly reduced, if inequality constraints can be restricted to SOS constraints. The purpose of this section is to develop a novel policy iteration method for polynomial systems using SOS-based methods [8], [35].
A. Polynomial parametrization

To study polynomial systems, let us first give the following Assumption.

**Assumption 4.1:** There exist integers $d > 0$, $d_1 \geq 0$, and $r > 0$, such that

1) all entries of $f(x)$ belong to $\mathbb{R}[x]_{1,d}$ and all entries of $g(x)$ belong to $\mathbb{R}[x]_{0,d_1}$;
2) in addition to being positive definite, the weighting function $Q(x)$ satisfies $Q(x) \in \mathbb{R}[x]_{2,2d}$;
3) there exist a nonlinear mappings $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ and a feedback control policy $u_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$, such that $V_0 \in \mathbb{R}[x]_{2,2r} \cap \mathcal{P}$, $u_1 \in \mathbb{R}[x]_{1,d}$, and $\mathcal{L}(V_0, u_1)$ is SOS; and
4) the inequality holds:

$$d \geq (2r - 1) + d_1.$$  \hfill (29)

**Remark 4.1:** It is easy to see that, Assumption 4.1 holds only if Assumption 2.1 holds. In addition, under Assumption 4.1, we know that $\mathcal{L}(V_0, u_1) \in \mathbb{R}[x]_{2,2d}$. Indeed, by (29), it follows that $\mathcal{L}(V_0, u_1) \in \mathbb{R}[x]_{2,\max\{2r - 1 + d_1 + d_1 + d_1\}} = \mathbb{R}[x]_{2,2d}$.

**Remark 4.2:** Notice that the inequality (29) can be assumed without loss of generality. Indeed, if it does not hold, we can always find $\tilde{d} > \max\{d, 2r - 1 + d_1\}$. As a result, Assumption 4.1 holds with $d$ replaced by $\tilde{d}$.

For notational simplicity, we denote the dimensions of $[x]_{1,r}$, $[x]_{1,d}$, $[x]_{2,2r}$, and $[x]_{2,2d}$ by $n_r$, $n_d$, $n_{2r}$, and $n_{2d}$, respectively. By [8], we know $n_r = (n + r) - 1$, $n_d = (n + d) - 1$, $n_{2r} = (n + 2r) - n - 1$, and $n_{2d} = (n + 2d) - d - 1$.

B. SOS-programming-based policy iteration

Now, we are ready to propose a relaxed policy iteration scheme. Similar as in other policy-iteration-based iterative schemes, an initial globally stabilizing (and admissible) control policy has been assumed in Assumption 4.1.

1) **Policy evaluation:** For $i = 1, 2, \cdots$, solve for an optimal solution $p_i \in \mathbb{R}^{n_{2r}}$ to the following optimization program, and denote $V_i = p_i^T [x]_{2,2r}$.

$$\min_{p \in \mathbb{R}^{n_{2r}}} \int_{\mathbb{R}^n} w(x)V(x)dx$$ \hfill (30)

s.t. \quad $V := p^T [x]_{2,2r}$ \hfill (31)

$\mathcal{L}(V, u_i) \in \Sigma_{2,2d}$ \hfill (32)

$V_{i-1} - V \in \Sigma_{2,2r}$ \hfill (33)
where $\Sigma_{2,2d}$ and $\Sigma_{2,2r}$ denote the sets of all SOS polynomials in $\mathbb{R}[x]_{2,2d}$ and $\mathbb{R}[x]_{2,2r}$, respectively.

2) **Policy improvement:** Update the control policy by

$$u_{i+1} = -\frac{1}{2}R^{-1}g^T \nabla V_i.$$  \hspace{1cm} (34)

Then, go to Step 1) with $i$ replaced by $i + 1$.

**Remark 4.3:** The optimization problem (30)-(33) is a well defined SOS program \cite{8}. Indeed, the objective function (30) is linear in $p$, since for any $V = p^T[x]_{2,2r}$, we have $\int_{\mathbb{R}^n} w(x)V(x)dx = c^T p$, with $c = \int_{\mathbb{R}^n} w(x)[x]_{2,2r}dx$. In addition, notice that since the objective function is nonnegative, its optimal value must be finite.

**Theorem 4.1:** Under Assumptions 2.2 and 4.1, the following are true, for $i = 1, 2, \cdots$.

1) The SOS program (30)-(33) has a nonempty feasible set.

2) The closed-loop system comprised of (I) and $u = u_i$ is globally asymptotically stable at the origin.

3) $V_i \in \mathcal{P}$. In particular, the following inequalities hold:

$$V^o(x_0) \leq V_i(x_0) \leq V_{i-1}(x_0), \quad \forall x_0 \in \mathbb{R}^n.$$ \hspace{1cm} (35)

4) There exists $V^*(x)$ satisfying $V^*(x) \in \mathbb{R}[x]_{2,2r} \cap \mathcal{P}$, such that, for any $x_0 \in \mathbb{R}^n$, $\lim_{i \to \infty} V_i(x_0) = V^*(x_0)$.

5) Along the solutions of the system (I) with

$$u^* = -\frac{1}{2}R^{-1}g^T \nabla V^*,$$ \hspace{1cm} (36)

the following inequalities hold:

$$0 \leq V^*(x_0) - V^o(x_0) \leq -\int_0^\infty \mathcal{H}(V^*(x(t)))dt.$$ \hspace{1cm} (37)

**Proof:** 1) We prove by mathematical induction.

i) Suppose $i = 1$, under Assumption 4.1, we know $L(V_0, u_1) \in \Sigma_{2,2d}$. Hence, $V = V_0$ is a feasible solution to the problem (30)-(33).

ii) Let $u_{j-1} \in \mathbb{R}[x]_{1,d}^m$, and $V = V_{j-1}$ be an optimal solution to the problem (30)-(33) with $i = j - 1 > 1$. We show that $V = V_{j-1}$ is a feasible solution to the same problem with $i = j$.

Indeed, by definition,

$$u_j = -\frac{1}{2}R^{-1}g^T \nabla V_{j-1} \in \mathbb{R}[x]_{1,d}^m.$$
and

\[ L(V_{j-1}, u_j) = -\nabla V^T_{j-1}(f + gu_j) - r(x, u_j) \]
\[ = L(V_{j-1}, u_{j-1}) - \nabla V^T_{j-1}g(u_j - u_{j-1}) + u^T_{j-1}Ru_{j-1} - u^T_jRu_j \]
\[ = L(V_{j-1}, u_{j-1}) + |u_j - u_{j-1}|^2. \]

Under the induction assumption, we know \( V_{j-1} \in \mathbb{R}[x]_{2,2r}, u_{j-1} \in \mathbb{R}[x]_{m_1}^{m_2}, \) and \( L(V_{j-1}, u_j) \in \Sigma_{2,2d}. \) Hence, \( L(V_{j-1}, u_j) \in \Sigma_{2,2d}. \) As a result, \( V_{j-1} \) is a feasible solution to the SOS program (30)-(33) with \( i = j. \)

2) Again, we prove by induction.

i) Suppose \( i = 1, \) under Assumption 4.1, \( u_1 \) is globally stabilizing. Also, we can show that \( V_1 \in \mathcal{P}. \) Indeed, for each \( x_0 \in \mathbb{R}^n \) with \( x_0 \neq 0, \) we have

\[ V_1(x_0) = \int_0^\infty r(x, u_1)dt > 0. \] (38)

By (38) and the constraint (33), under Assumption 2.2 it follows that

\[ V_0 \leq V_1 \leq V_0. \] (39)

Since both \( V_0 \) and \( V_0 \) are assumed to belong to \( \mathcal{P}, \) we conclude that \( V_1 \in \mathcal{P}. \)

ii) Suppose \( u_{i-1} \) is globally stabilizing, and \( V_{i-1} \in \mathcal{P} \) for \( i > 1. \) Let us show that \( u_i \) is globally stabilizing, and \( V_i \in \mathcal{P}. \)

Indeed, along the solutions of the closed-loop system composed of (1) and \( u = u_i, \) it follows that

\[ \dot{V}_{i-1} = \nabla V^T_{i-1}(f + gu_i) = -L(V_{i-1}, u_i) - r(x, u_i) \leq -Q(x). \]

Therefore, \( u_i \) is globally stabilizing, since \( V_{i-1} \) is a well-defined Lyapunov function for the system. In addition, we have

\[ V_i(x_0) = \int_0^\infty r(x, u_i)dt > 0, \quad \forall x_0 \neq 0. \] (40)

Similarly as in (39), we can show

\[ V_0(x_0) \leq V_i(x_0) \leq V_{i-1}(x_0), \quad \forall x_0 \in \mathbb{R}^n, \] (41)

and conclude that \( V_i \in \mathcal{P}. \)

3) The two inequalities have been proved in (41).
4) By 3), for each \( x \in \mathbb{R}^n \), the sequence \( \{ V_i(x) \}_{i=0}^{\infty} \) is monotonically decreasing with 0 as its lower bound. Therefore, the limit exists, i.e., there exists \( V^*(x) \), such that \( \lim_{i \to \infty} V_i(x) = V^*(x) \). Let \( \{ p_i \}_{i=1}^{\infty} \) be the sequence such that \( V_i = p_i^T[x]_{2,2r} \). Then, we know \( \lim_{i \to \infty} p_i = p^* \in \mathbb{R}^{n_2r} \), and therefore \( V^* = p^T[x]_{2,2r} \). Also, it is easy to show \( V^o \leq V^* \leq V_0 \). Hence, \( V^* \in \mathbb{R}[x]_{2,2r} \cap \mathcal{P} \).

5) By 4), we know

\[
\mathcal{H}(V^*) = -\mathcal{L}(V^*, u^*) \leq 0, \tag{42}
\]

which implies that \( V^* \) is a feasible solution to Problem \( 3.1 \). Then, the inequalities in (5) can be obtained by the fourth property in Theorem \( 3.1 \).

The proof is thus complete.

C. An equivalent SDP implementation

According to the equivalence between SOS programs and SDPs, the SOS-based policy iteration can be reformulated as SDPs. Notice that we can always find two linear mappings \( \iota : \mathbb{R}^{n_2r} \times \mathbb{R}^{m \times n_r} \to \mathbb{R}^{n_2d} \) and \( \kappa : \mathbb{R}^{n_2r} \to \mathbb{R}^{m \times n_r} \), such that given \( p \in \mathbb{R}^{n_2r} \) and \( K \in \mathbb{R}^{m \times n_r} \),

\[
\iota(p, K)^T[x]_{2,2d} = \mathcal{L}(p^T[x]_{2,2r}, K[x]_{1,2r-1}) \tag{43}
\]

\[
\kappa(p)^T[x]_{1,2r-1} = -\frac{1}{2}R^{-1}g^T \nabla(p^T[x]_{2,2r}) \tag{44}
\]

Then, by properties of SOS constraints \( [8] \), the polynomial \( \iota(p, K)^T[x]_{2,2d} \) is SOS if and only if there exists a symmetric and positive semidefinite matrix \( L \in \mathbb{R}^{n_d \times n_d} \), such that

\[
\iota(p, K)^T[x]_{2,2d} = [x]_{1,d}^T L[x]_{1,d}. \tag{45}
\]

Furthermore, there exist linear mappings \( M_P : \mathbb{R}^{n_r \times n_r} \to \mathbb{R}^{n_2r} \) and \( M_L : \mathbb{R}^{n_d \times n_d} \to \mathbb{R}^{n_2d} \), such that, for any vectors \( p \in \mathbb{R}^{n_2r} \), \( l \in \mathbb{R}^{n_2d} \), and symmetric matrices \( P \in \mathbb{R}^{n_r \times n_r} \) and \( L \in \mathbb{R}^{n_d \times n_d} \), the following implications are true.

\[
p^T[x]_{2,2r} = [x]_{1,r}^T P[x]_{1,r} \iff p = M_P(P) \tag{46}
\]

\[
l^T[x]_{2,2d} = [x]_{1,d}^T L[x]_{1,d} \iff l = M_L(L) \tag{47}
\]

Under Assumptions \( 2.2 \) and \( 4.1 \) the proposed policy iteration can be reformulated as follows.

1) Let \( i = 1 \). Let \( p_0 \in \mathbb{R}^{n_2r} \) and \( K_1 \in \mathbb{R}^{m \times n_r} \) satisfy \( V_0 = p_0^T[x]_{2,2r} \) and \( u_1 = K_1[x]_{1,d} \).
2) Solve for an optimal solution \((p_i, P_i, L_i) \in \mathbb{R}^{n_{2r}} \times \mathbb{R}^{n_r \times n_r} \times \mathbb{R}^{n_d \times n_d}\) to the following problem.

\[
\min_{p, P, L} \quad c^T p \\
\text{s.t.} \quad \iota(p, K_i) = M_L(L) \\
p_{i-1} - p = M_P(P) \\
P = P^T \geq 0 \\
L = L^T \geq 0
\] (48)-(52)

where \(c = \int_{\mathbb{R}^n} w(x)[x]_2,2r \, dx\).

3) Go to Step 2) with \(K_{i+1} = \kappa(p_i)\) and \(i\) replaced by \(i + 1\).

**Remark 4.4:** The optimization problem (48)-(52) is a well-defined semidefinite programming (SDP) problem, since it has a linear objective function subject to linear equality and inequality constraints. It can be directly solved using, for example, Matlab-based solver CVX [14]. Also, it can be rewritten in the standard form (12)-(13) by equivalently replacing each equality constraint with two inequalities constraints, and by treating \(p\) and entries in \(P\) and \(L\) as the decision variables.

**Corollary 4.1:** Under Assumptions 2.2 and 4.1, the following are true.

1) The optimization problem (48)-(52) has at least one feasible solution, for \(i = 1, 2, \cdots\).

2) Denote \(V_i = p_i^T[x]_{2,2r}, u_{i+1} = K_i[x]_{1,d}\), for \(i = 0, 1, \cdots\). Then, the sequences \(\{V_i\}_{i=0}^\infty\) and \(\{u_i\}_{i=1}^\infty\) satisfy the properties 2)-5) in Theorem 4.1.

**Proof:** Given \(p_i \in \mathbb{R}^{n_{2r}}\), there exist \(P_i\) and \(L_i\) such that \((p_i, P_i, L_i)\) is a feasible solution to the optimization problem (48)-(52) if and only if \(p_i\) is a feasible solution to the SOS program (30)-(33). Therefore, by Theorem 4.1 1) holds. In addition, since the two optimization problems share the identical objective function, we know that if \((p_i, P_i, L_i)\) is a feasible solution to the optimization problem (48)-(52), \(p_i\) is also an optimal solution to the SOS program (30)-(33). Hence, the corollary can be obtained from Theorem 4.1.

**V. Global Adaptive Dynamic Programming for Uncertain Polynomial Systems**

The proposed policy iteration method requires the perfect knowledge of the mappings \(\iota\) and \(\kappa\), which can be determined if \(f\) and \(g\) are known exactly. In practice, precise system knowledge
may be difficult to obtain. Hence, in this section, we develop an online learning method based on the idea of ADP to implement the iterative scheme with real-time data, instead of identifying the system dynamics.

To begin with, consider the system

\[ \dot{x} = f + g(u_i + e) \]  

(53)

where \( u_i \) is a feedback control policy and \( e \) is a bounded time-varying function, known as the exploration noise, added for the learning purpose.

**Lemma 5.1:** Consider system (53). Suppose \( u_i \) is a globally stabilizing control policy and there exists \( V_{i-1} \in \mathcal{P} \), such that \( \nabla V_{i-1}(f + g u_i) + u_i^T R u_i \leq 0 \). Then, the system (53) is forward complete.

**Proof:** Under Assumptions 2.2 and 4.1, by Theorem 4.1 we know \( V_{i-1} \in \mathcal{P} \). Then, by completing the squares, it follows that

\[
\nabla V_{i-1}^T (f + g u_i + g e) \leq -u_i^T R u_i - 2 u_i^T R e \\
= -|u_i + e|^2_R + |e|^2_R \\
\leq |e|^2_R \\
\leq |e|^2_R + V_{i-1}.
\]

According to [1, Corollary 2.11], the system (53) is forward complete.

By Lemma 5.1 and Theorem 4.1 we immediately have the following Proposition.

**Proposition 5.1:** Under Assumptions 2.2 and 4.1 let \( u_i \) be a feedback control policy obtained at the \( i \)-th iteration step in the proposed policy iteration algorithm (30)-(34) and \( e \) be a bounded time-varying function. Then, the closed-loop system (1) with \( u = u_i + e \) is forward complete.

Suppose there exist \( p \in \mathbb{R}^{n_{2r}} \) and \( K_i \in \mathbb{R}^{m \times n_k} \) such that \( V = p^T [x]_{2,2r} \) and \( u_i = K_i [x]_{1,d} \). Then, along the solutions of the system (53), it follows that

\[
\dot{V} = \nabla V^T (f + g u_i) + \nabla V^T B e \\
= -r(x, u_i) - \mathcal{L}(V, u_i) + \nabla V^T g e \\
= -r(x, u_i) - \mathcal{L}(V, u_i) + 2(\frac{1}{2} R^{-1} g^T \nabla V)^T R e \\
= -r(x, u_i) - \kappa(p, K_i)^T [x]_{2,2d} - 2[x]_{1,d}^T \kappa(p)^T R e
\]

(54)
where the last row is obtained by (43) and (44).

Now, integrating the terms in (54) over the interval $[t, t + \delta t]$, we have

$$p^T ([x(t)]_{2,2r} - [x(t + \delta t)]_{2,2r}) = \int_t^{t+\delta t} \left[ r(x, u_i) + \iota(p, K_i)^T [x]_{2,2r} + 2[x]_{1,d}^T \kappa(p)^T Re \right] dt \tag{55}$$

Eq. (55) implies that, given $p \in \mathbb{R}^{n_{2r}}$, $\iota(p, K_i)$ and $\kappa(p)$ can be directly calculated by using real-time online data, without knowing the precise knowledge of $f$ and $g$.

Indeed, define

$$\sigma_e = - \left[ x^T_{2,2d} 2[x]_{1,d}^T \otimes e^T R \right]^T \in \mathbb{R}^{n_{2d} + mn_d},$$

$$\Phi_i = \left[ \int_{t_{0,i}}^{t_{1,i}} \sigma_e dt \int_{t_{1,i}}^{t_{2,i}} \sigma_e dt \cdots \int_{t_{q_i,i-1,i}}^{t_{q_i,i}} \sigma_e dt \right]^T \in \mathbb{R}^{q_i \times (n_{2d} + mn_d)},$$

$$\Xi_i = \left[ \int_{t_{0,i}}^{t_{1,i}} r(x, u_i) dt \int_{t_{1,i}}^{t_{2,i}} r(x, u_i) dt \cdots \int_{t_{q_i,i-1,i}}^{t_{q_i,i}} r(x, u_i) dt \right]^T \in \mathbb{R}^{q_i},$$

$$\Theta_i = \left[ x^T_{2,2r} \left[ x^T_{2,2r} \right]_{t_{0,i}}^{t_{1,i}} \left[ x^T_{2,2r} \right]_{t_{1,i}}^{t_{2,i}} \cdots \left[ x^T_{2,2r} \right]_{t_{q_i,i-1,i}}^{t_{q_i,i}} \right]^T \in \mathbb{R}^{q_i \times n_{2r}}.$$

Then, (55) implies

$$\Phi_i \begin{bmatrix} \iota(p, K_i) \\ \vec(\kappa(p)) \end{bmatrix} = \Xi_i + \Theta_i p. \tag{56}$$

**Assumption 5.1:** For each $i = 1, 2, \cdots$, there exists an integer $q_{i0}$, such that when $q_i \geq q_{i0}$ the following rank condition holds.

$$\text{rank}(\Phi_i) = n_{2d} + mn_d. \tag{57}$$

**Remark 5.1:** Such a rank condition (57) is in the spirit of persistency of excitation (PE) in adaptive control (e.g. [19], [46]) and is a necessary condition for parameter convergence.

Given $p \in \mathbb{R}^{n_{2r}}$ and $K_i \in \mathbb{R}^{m \times n_d}$, suppose Assumption 5.1 is satisfied and $q_i \geq q_{i0}$ for all $i = 1, 2, \cdots$. Then, it is easy to see that the values of $\iota(p, K_i)$ and $\kappa(p)$ can be uniquely determined from

$$\begin{bmatrix} \iota(p, K_i) \\ \vec(\kappa(p)) \end{bmatrix} = (\Phi_i^T \Phi_i)^{-1} \Phi_i^T (\Xi_i + \Theta_i p) \tag{58}$$

Now, we are ready to develop the ADP-based online implementation algorithm for the proposed policy iteration method.
1) Initialization:
Let $p_0$ be the constant vector such that $V_0 = p_0^T[x]_{2,2r}$, and let $i = 1$.

2) Collect online data:
Apply $u = u_i + e$ to the system and compute the data matrices $\Phi_i$, $\Xi_i$, and $\Theta_i$, until the rank condition (57) in Assumption 5.1 is satisfied.

3) Policy evaluation and improvement:
Find an optimal solution $(p_i, K_{i+1}, P_i, L_i)$ to the following optimization problem

$$
\min_{p,K,P,L} c^T p \quad \text{s.t.} \quad \begin{bmatrix} M_L(L) \\ \text{vec}(K) \end{bmatrix} = (\Phi_i^T \Phi_i)^{-1} \Phi_i^T (\Xi_i + \Theta_i p)
$$

$$
p_{i-1} - p = M_P(P) \quad \text{(61)}
$$

$$
P = P^T \geq 0 \quad \text{(62)}
$$

$$
L = L^T \geq 0 \quad \text{(63)}
$$

Then, denote $V_i = p_i^T[x]_{2,2r}$, $u_{i+1} = K_{i+1}[x]_{1,d}$, and go to Step 2) with $i \leftarrow i + 1$.

**Lemma 5.2:** Under Assumption 5.1, $(p_i, K_{i+1}, P_i, L_i)$ is an optimal solution to the optimization problem (59)-(63) if and only if $(p_i, P_i, L_i)$ is an optimal solution to the optimization problem (48)-(52) and $K_{i+1} = \kappa(p_i)$.

**Proof:** If $(p_i, K_{i+1}, P_i, L_i)$ is an optimal solution to the optimization problem (59)-(63), under Assumption 5.1 we must have $K_{i+1} = \kappa(p_i)$. Then, it is easy to check $(p_i, P_i, L_i)$ is a feasible solution to the problem (48)-(52). On the other hand, if $(p_i, P_i, L_i)$ is a feasible solution to the problem (48)-(52), $(p_i, \kappa(p_i), P_i, L_i)$ must be a feasible solution to the problem (59)-(63). Finally, since the two optimization problems share the same objective function, their optimal values are the same.

By Lemma 5.2 and Theorem 4.1, we immediately have the following Theorem.

**Theorem 5.1:** Under Assumptions 2.1, 4.1 and 5.1 the following properties hold.

1) The optimization problem (59)-(63) has a nonempty feasible set.

2) The sequences $\{V_i\}_{i=1}^\infty$ and $\{u_i\}_{i=1}^\infty$ satisfy the properties 2)-5) in Theorem 4.1.

**Remark 5.2:** Notice that the above-mentioned algorithm assumes that both $V_0$ and $u_1$ satisfying Assumption 4.1 are determined without knowing exactly $f$ and $g$. In practice, upper
and lower bounds of the coefficients in $f$ and $g$ are often available, i.e., there exist polynomial mappings $\bar{f}, \underline{f}, \bar{g}, \underline{g}$, such that $\underline{f} \leq f \leq \bar{f}$ and $\underline{g} \leq g \leq \bar{g}$. Thus, it is possible to find a globally stabilizing control policy for interval systems using robust nonlinear control methods [26], [47]. Then, we can use this control policy as a candidate of $u_1$ to solve for $V_0$ from the following robust feasibility problem in SOS programming

$$-\nabla V_0^T(\tilde{f} + \tilde{g}u_1) - Q - u_1^TRu_1 \in \Sigma_{2,2d},$$

for all $\tilde{f}$ and $\tilde{g}$ such that $\underline{f} \leq \tilde{f} \leq \bar{f}$ and $\underline{g} \leq \tilde{g} \leq \bar{g}$. This problem, if solvable, can be converted into a robust linear matrix inequality and efficiently solved using MATLAB-based solvers, such as the LMI control toolbox [13] or CVX [14].

**Remark 5.3:** In practice, a stopping criterion can be set. For example, the exploration noise can be terminated and $u_i$ can be applied as the actual control policy, if $|p_i - p_{i+1}| \leq \epsilon$ or $i = i_{\max}$, with $\epsilon > 0$ is a pre-defined threshold and $i_{\max}$ a pre-defined maximum number of iterations.

**VI. Extension**

In this section, we extend the proposed global ADP method to deal with an enlarged class of nonlinear systems. First, we will give an illustrative example to show that the SOS condition is conservative for general nonlinear functions. Second, a generalized parametrization method is proposed. Third, a less conservative sufficient condition will be derived to assure the non-negativity of a given nonlinear function. Fourth, an SDP-based implementation for the proposed policy iteration technique will be presented. Finally, an online learning method will be developed.

**A. An illustrative example**

The implementation method via SOS programs developed in the previous section can efficiently handle nonlinear polynomial systems. The results can also be trivially extended to real trigonometric polynomials [8]. However, the SOS-like constraint may be conservative to be used as a sufficient condition for non-negativity of general nonlinear functions. To see this, consider the following illustrative example:

$$f(x) = ax^2 + bx \sin x + c \sin^2 x = \begin{bmatrix} x \\ \sin x \end{bmatrix}^T P \begin{bmatrix} x \\ \sin x \end{bmatrix}. \quad (65)$$
Apparently, a symmetric matrix $P$ can be uniquely determined from the constants $a$, $b$, and $c$. Similar to the polynomial case, we know $f(x)$ is positive semidefinite, if $P$ is positive semidefinite. Unfortunately, this condition is very conservative in general. For example, for the cases of $a = 1$, $b = 0$, $c = -0.5$, or $a = 0$, $b = 1$, $c = 0$, it is easy to check $f(x)$ is positive semidefinite. But in both cases, we have either $P = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$ or $P = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$, which are not positive semidefinite matrices.

This illustrative example shows that, instead of searching for a positive semidefinite matrix $P$, a less conservative sufficient condition for the non-negativity of more general nonlinear functions is desired. Deriving this condition and developing a global ADP method for more general nonlinear systems are the main objectives of this section.

**B. Generalized parametrization**

**Assumption 6.1:** The function $f$ considered in system (1) can be decomposed as

$$f = A\sigma$$  \hspace{1cm} (66)

where $A \in \mathbb{R}^{n \times l}$ is an uncertain constant matrix, and $\sigma = [\sigma_1(x), \sigma_2(x), \cdots, \sigma_l(x)]^T$ is a vector of locally Lipschitz, piecewise-continuous, and linearly independent functions, satisfying $\sigma_i(0) = 0$, $\forall i = 1, 2, \cdots, l$.

Now, we restrict each feasible solution to Problem 3.1 to take the form of $V(x) = \phi^T(x) P \phi(x)$, where $P \in \mathbb{R}^{l \times l}$ is a constant matrix and $\phi = [\phi_1(x), \phi_2(x), \cdots, \phi_N(x)]^T$ is a vector of continuously differentiable, linearly independent, functions vanishing at the origin.

**Assumption 6.2:** The following are true.

1) For each $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, N$, and $k = 1, 2, \cdots, n$, we have

$$\frac{\partial(\phi_i \phi_j)}{\partial x_k} \in \text{span}\{\sigma_1, \sigma_2, \cdots, \sigma_l\},$$

2) Let $g_i$ be the $i$-th column of $g(x)$, with $i = 1, 2, \cdots, m$. Then,

$$g_i^T \nabla(\phi_i \phi_j) \in \text{span}\{\sigma_1, \sigma_2, \cdots, \sigma_l\}.$$

3) The weighting function $Q(x)$ defined in (2) is positive definite and satisfies $Q(x) \in \text{span}\{\sigma_1^2, \sigma_1\sigma_2, \cdots, \sigma_1\sigma_j, \cdots, \sigma_l^2\}$. 


Notice that Assumption 6.2 is not restrictive, and can be satisfied by expanding the basis functions. Indeed, if 1) and 2) in Assumption 6.2 are not satisfied, we can always find locally Lipschitz functions \( \sigma_{l+1}(x), \sigma_{l+2}(x), \ldots, \sigma_{l+s}(x) \), such that \( \sigma_1, \sigma_2, \ldots, \sigma_{l+s} \) are linearly independent and vanish at the origin, satisfying \( \frac{\partial (\phi_i \phi_j)}{\partial x_k} \in \text{span}\{\sigma_1, \sigma_2, \ldots, \sigma_{l+s}\} \) and \( g^T \nabla (\phi_i \phi_j) \in \text{span}\{\sigma_1, \sigma_2, \ldots, \sigma_{l+s}\} \). Then, the decomposition (66) can be rewritten as

\[
f(x) = \tilde{A} \tilde{\sigma}
\]  

where \( \tilde{A} = \begin{bmatrix} A & 0_{n \times s} \end{bmatrix} \) and \( \tilde{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_{l+s}]^T \).

Also, if the intersection between \( \text{span}\{\sigma_1^2, \sigma_1 \sigma_2, \ldots, \sigma_i \sigma_j, \ldots, \sigma_s^2\} \) and the set of all positive definite functions is empty, we can select \( Q(x) \) such that \( \sqrt{Q(x)} \) is locally Lipschitz and positive definite. Define \( \hat{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_l, \sqrt{Q(x)}] \). Then, clearly, all the elements in \( \hat{\sigma} \) are linearly independent, and the decomposition (66) can be rewritten as \( f = \hat{A} \hat{\sigma} \), where \( \hat{A} = \begin{bmatrix} A & 0_{n \times 1} \end{bmatrix} \).

C. A sufficient condition for non-negativity

Define \( \{\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_l\} \) as the largest linearly independent subset of \( \{\sigma_1^2, \sigma_1 \sigma_2, \ldots, \sigma_i \sigma_j, \ldots, \sigma_s^2\} \), and \( \{\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_N\} \) as the largest linearly independent subset of \( \{\phi_1^2, \phi_1 \phi_2, \ldots, \phi_i \phi_j, \ldots, \phi_N^2\} \).

Then, if \( W \in \text{span}\{\phi_1^2, \phi_1 \phi_2, \ldots, \phi_N^2\} \) and \( \delta \in \text{span}\{\phi_1^2, \sigma_1 \sigma_2, \ldots, \sigma_s^2\} \), there exist uniquely constant vectors \( p \in \mathbb{R}^N \) and \( h \in \mathbb{R}^l \), such that \( W = p^T \Phi \) and \( \delta = h^T \tilde{\sigma} \), where \( \Phi = [\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_N]^T \) and \( \tilde{\sigma} = [\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_l]^T \).

Using the above-mentioned parametrization method, we now show that it is possible to decide if \( W \) and \( \delta \) are positive semidefinite functions, by studying the coefficient vectors \( p \) and \( h \).

Without loss of generality, we assume the following properties of \( \bar{\phi}_i \):

1) For \( i = 1, 2, \ldots, N_2 \), we have \( \bar{\phi}_i \geq 0 \), with \( N_2 \) an integer satisfying \( 1 \leq N_2 \leq N_1 \).

2) There exist integers \( i_r \) and \( j_r \) with \( r = 1, 2, \ldots, N_3 \), such that \( 1 \leq i_r, j_r \leq N_2 \), \( i_r \neq j_r \) and \( \bar{\phi}_{i_r} \geq \bar{\phi}_{j_r} \).

**Definition 6.1:** For any \( p \in \mathbb{R}^N \), we say \( p \in S_\Phi^+ \) if and only if there exist constants \( \gamma_1, \gamma_2, \ldots, \gamma_{N_2} \geq 0 \), \( \alpha_1, \alpha_2, \ldots, \alpha_{N_3} \geq 0 \), \( \beta_1, \beta_2, \ldots, \beta_{N_3} \), and a symmetric positive semidefinite
matrix \( P \in \mathbb{R}^{N \times N} \), such that \( \alpha_i + \beta_i \geq 0 \), for \( i = 1, 2, \cdots, N_3 \), and

\[
p = M_\phi^T \text{vec}(P) + \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_{N_3} \\
0_{N_1-N_2}
\end{bmatrix} + \sum_{r=1}^{N_3} \begin{bmatrix}
0_{i_r-1} \\
\alpha_r \\
0_{N_1-i_r} \\
\beta_r \\
0_{N_1-j_r}
\end{bmatrix}
\] (68)

where \( M_\phi \in \mathbb{R}^{N_2 \times N_1} \) is a constant matrix satisfying \( M_\phi \tilde{\phi} = \phi \otimes \phi \).

In addition, \( W \) is said to belong to the set \( S_\phi^+ \) if and only if there exists \( p \in S_\phi^+ \), such that \( W = p^T \tilde{\phi} \).

**Lemma 6.1:** If \( p \in S_\phi^+ \), then \( p^T \tilde{\phi} \) is positive semidefinite.

*Proof:* By definition, if \( p \in S_\phi^+ \), it follows that

\[
p^T \tilde{\phi} = \phi^T P \phi + \sum_{i=1}^{N_2} \gamma_i \tilde{\phi}_i + \sum_{r=1}^{N_3} (\alpha_r \tilde{\phi}_i + \beta_r \tilde{\phi}_j)
\geq \sum_{r=1}^{N_3} (\alpha_r \tilde{\phi}_i - |\beta_r| \tilde{\phi}_j) = \sum_{r=1}^{N_3} (\alpha_r - |\beta_r|) \tilde{\phi}_i
\geq 0.
\]

The proof is complete. \[\blacksquare\]

In the same way, we can find two sets \( S_\phi^+ \) and \( S_\phi^+[x] \), such that the following implications hold

\[
h \in S_\phi^+ \iff h^T \tilde{\phi} \in S_\phi^+[x] \Rightarrow h^T \tilde{\phi} \geq 0.
\] (69)

**D. Generalized policy iteration**

**Assumption 6.3:** There exist \( p_0 \in \mathbb{R}^{N_1} \) and \( K_1 \in \mathbb{R}^{m \times l} \), such that \( V_0 = p_0^T \phi, \ u_1 = K_1 \sigma \), and \( L(V_0, u_1) \in S_{\phi^+} \).

**Remark 6.1:** Under Assumptions [6.1, 6.2 and 6.3] Assumption 2.1 is satisfied.

Now, let us show how the proposed policy iteration can be practically implemented. First of all, given \( p \in \mathbb{R}^{N_1} \), since \( u_i = K_i \sigma \), we can always find two linear mappings \( \bar{\iota} : \mathbb{R}^{N_1} \times \mathbb{R}^{m_l} \rightarrow \mathbb{R}^{l_1} \) and \( \bar{\kappa} : \mathbb{R}^{N_1} \rightarrow \mathbb{R}^{l_1 \times m_l} \), such that

\[
\bar{\iota}(p, K)^T \sigma = L(p^T \phi, K_1 \sigma)
\]
\[
\bar{\kappa}(p)^T \sigma = -\frac{1}{2} R^{-1} g^T \nabla(p^T \phi)
\] (70) (71)
Then, under Assumptions 2.2, 6.1, 6.2, and 6.3 the proposed policy iteration can be implemented as follows.

1) **Initialization:**

Find \( p_0 \in \mathbb{R}^{N_1} \) and \( K_1 \in \mathbb{R}^{m \times l_1} \) satisfying Assumption 6.3, and let \( i = 1 \).

2) **Policy evaluation and improvement:**

Solve for an optimal solution \((p_i, K_{i+1})\) of the following problem.

\[
\min_{p, K} c^T p \tag{72}
\]

s.t. \( \bar{\imath}(p, K_i) \in \mathbb{S}_\sigma^+ \) \tag{73}

\[
p_{i-1} - p \in \mathbb{S}_\sigma^+ \tag{74}
\]

\[
K = \bar{\kappa}(p) \tag{75}
\]

where \( c = \int_{\mathbb{R}^n} w(x)\bar{\phi}(x)dx \). Then, denote \( V_i = p_i^T \phi \) and \( u_{i+1} = K_{i+1}\sigma \).

3) Go to Step 2) with \( i \) replaced by \( i + 1 \).

Some useful facts about the above-mentioned policy iteration algorithm are summarized in the following theorem, of which the proof is omitted, because it is nearly identical to the proof of Theorem 4.1.

**Theorem 6.1:** Under Assumptions 2.2, 6.1, 6.2, and 6.3 the following are true, for \( i = 1, 2, \ldots \).

1) The optimization problem (72)-(75) has a nonempty feasible set.

2) The closed-loop system comprised of (1) and \( u = u_i(x) \) is globally asymptotically stable at the origin.

3) \( V_i \in \mathcal{P} \). In addition, \( V^\alpha(x_0) \leq V_i(x_0) \leq V_{i-1}(x_0) \), \( \forall x_0 \in \mathbb{R}^n \).

4) There exists \( p^* \in \mathbb{R}^{N_1} \), such that \( \lim_{i \to \infty} V_i(x_0) = p^T \phi(x_0) \), \( \forall x_0 \in \mathbb{R}^n \).

5) Along the solutions of the system (1) with \( u^* = -\frac{1}{2}R^{-1}g^T \nabla (p^T \phi) \), it follows that

\[
0 \leq p^T \phi(x_0) - V^\alpha(x_0) \leq -\int_0^\infty \mathcal{H}(p^T \phi(x(t)))dt. \tag{76}
\]

**E. Online implementation via global adaptive dynamic programming**

Let \( V = p^T \phi \). Similar as in Section V over the interval \([t, t + \delta t]\), we have

\[
p^T \left[ \phi(x(t)) - \phi(x(t + \delta t)) \right] = \int_t^{t+\delta t} \left[ r(x, u_i) + \bar{\imath}(p, K_i) \sigma + 2\sigma^T \bar{\kappa}(p)Re \right] dt \tag{77}
\]
Therefore, (77) shows that, given \( p \in \mathbb{R}^{N_1} \), \( \bar{\iota}(p, K_i) \) and \( \bar{\kappa}(p) \) can be directly obtained by using real-time online data, without knowing the precise knowledge of \( f \) and \( g \).

Indeed, define
\[
\bar{\sigma}_e = - \left[ \sigma^T \quad 2\sigma^T \otimes e^T R \right]^T \in \mathbb{R}^{l_1 + ml},
\]
\[
\Phi_i = \begin{bmatrix} \int_{t_{0,i}}^{t_{1,i}} \bar{\sigma}_e dt & \int_{t_{1,i}}^{t_{2,i}} \bar{\sigma}_e dt & \cdots & \int_{t_{q_i-1,i}}^{t_{q_i,i}} \bar{\sigma}_e dt \end{bmatrix}^T \in \mathbb{R}^{q_i \times (l_1 + ml)},
\]
\[
\Xi_i = \begin{bmatrix} \int_{t_{0,i}}^{t_{1,i}} r(x, u_i) dt & \int_{t_{1,i}}^{t_{2,i}} r(x, u_i) dt & \cdots & \int_{t_{q_i-1,i}}^{t_{q_i,i}} r(x, u_i) dt \end{bmatrix}^T \in \mathbb{R}^{q_i},
\]
\[
\Theta_i = \begin{bmatrix} \bar{\phi}(x|_{t_{0,i}}^{t_{1,i}}) & \bar{\phi}(x|_{t_{1,i}}^{t_{2,i}}) & \cdots & \bar{\phi}(x|_{t_{q_i-1,i}}^{t_{q_i,i}}) \end{bmatrix}^T \in \mathbb{R}^{q_i \times N_1}.
\]

Then, (77) implies
\[
\Phi_i \begin{bmatrix} \bar{\iota}(p, K_i) \\ \text{vec}(\bar{\kappa}(p)) \end{bmatrix} = \Xi_i + \Theta_i p. \tag{78}
\]

**Assumption 6.4:** For each \( i = 1, 2, \ldots \), there exists an integer \( q_{i0} \), such that, when \( q_i \geq q_{i0} \), the following rank condition holds.
\[
\text{rank}(\Phi_i) = l_1 + ml. \tag{79}
\]

Let \( p \in \mathbb{R}^{N_1} \) and \( K_i \in \mathbb{R}^{m \times l} \). Suppose Assumption 6.4 holds and assume \( q_i \geq q_{i0} \), for \( i = 1, 2, \ldots \). Then, \( \bar{\iota}(p, K_i) \) and \( \bar{\kappa}(p) \) can be uniquely determined by
\[
\begin{bmatrix} h \\ \text{vec}(K) \end{bmatrix} = (\Phi_i^T \Phi_i)^{-1} \Phi_i^T (\Xi_i + \Theta_i p). \tag{80}
\]

Now, we are ready to develop the ADP-based online implementation algorithm for the proposed policy iteration method.

1) **Initialization:**

Let \( p_0 \) and \( K_1 \) satisfying Assumption 6.3 and let \( i = 1 \).

2) **Collect online data:**

Apply \( u = u_i + e \) to the system and compute the data matrices \( \Phi_i, \Xi_i, \) and \( \Theta_i \), until the rank condition (79) is satisfied.
3) Policy evaluation and improvement:

Find an optimal solution \((p_i, h_i, K_{i+1})\) to the following optimization problem

\[
\min_{p, h, K} \quad c^T p
\]

\[
\text{s.t.} \quad \begin{bmatrix} h \\ \text{vec}(K) \end{bmatrix} = (\Phi_i^T \Phi_i)^{-1} \Phi_i^T (\Xi_i + \Theta_i p)
\]

\[
h \quad \in \quad \mathbb{S}_d^+
\]

\[
p_{i-1} - p \quad \in \quad \mathbb{S}_o^+
\]

Then, denote \(V_i = p_i \bar{\phi}\) and \(u_{i+1} = K_{i+1} \bar{\sigma}\).

4) Go to Step 2) with \(i \leftarrow i + 1\).

Properties of the above algorithm are summarized in the following corollary.

**Corollary 6.1:** Under Assumptions 2.2, 6.1, 6.2, 6.3, and 6.4, the algorithm enjoys the following properties.

1) The optimization problem (81)-(84) has a feasible solution.

2) The sequences \(\{V_i\}_{i=1}^\infty\) and \(\{u_i\}_{i=1}^\infty\) satisfy the properties 2)-5) in Theorem 6.1

**VII. Applications**

In this section, we illustrate the proposed methodology by means of one academic example and two practical examples.

A. A scalar nonlinear polynomial system

Consider the following polynomial system

\[
\dot{x} = ax^2 + bu
\]

where \(x \in \mathbb{R}\) is the system state, \(u \in \mathbb{R}\) is the control input, \(a\) and \(b\), satisfying \(a \in [0, 0.05]\)

and \(b \in [0.5, 1]\), are uncertain constants. The cost to be minimized is defined as \(J(x_0, u) = \int_0^\infty (0.01x^2 + 0.01x^4 + u^2)dt\). An initial stabilizing control policy can be selected as \(u_1 = -0.1x - 0.1x^3\), which globally asymptotically stabilizes system (85), for any \(a\) and \(b\) satisfying the given range. Further, it is easy to see that \(V_0 = 10(x^2 + x^4)\) and \(u_1\) satisfy Assumption 4.1 with \(r = 2\). In addition, in the present case, we set \(d = 3\) and \(d_1 = 0\) in Assumption 4.1.
Only for the purpose of simulation, set $a = 0.01$, $b = 1$, and $x(0) = 2$. The proposed global ADP method is applied with the control policy updated after every five seconds, and convergence is attained after five iterations, when $|p_i - p_{i-1}| \leq 10^{-3}$. The coefficient in the objective function (48) is defined as $c = [x(1)]_{2,4} + [x(-1)]_{2,4}$, i.e., the weighting function is set to be $w(x) = \delta(x-1) + \delta(x+1)$ with $\delta(\cdot)$ denoting the impulse function. The exploration noise is set to be $e = 0.01(\sin(10t) + \sin(3t) + \sin(100t))$, which is turned off after the fifth iteration.

The suboptimal control policy and the cost function obtained after five iterations are

$$V^* = 0.1020x^2 + 0.007x^3 + 0.0210x^4,$$

$$u^* = -0.2039x - 0.02x^2 - 0.0829x^3.$$  \hfill (86)

$$V^o = \frac{x^3}{150} + \frac{(\sqrt{101}x^2 + 100)^3}{15150} - \frac{20}{303}$$

$$u^o = -\frac{x^2\sqrt{101}x^2 + 100 + 101x^4 + 100x^2}{100\sqrt{101}x^2 + 100}$$  \hfill (88)

Figure 1 shows the comparison of the suboptimal control policy with respect to the exact optimal control policy and the initial control policy.

**B. Jet engine surge model**

Consider the following model of jet engine surge dynamics [15], [25].

$$\dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3$$

$$\dot{x}_2 = \frac{1}{\beta^2}u$$  \hfill (90)

$$\dot{x}_2 = \frac{1}{\beta^2}u$$  \hfill (91)

where $x_1$ and $x_2$ represent the scaled annulus-averaged flow and plenum pressure rise in error coordinates, respectively. $u$ is the control input, and the constant $\beta$ is assumed to be unknown belonging to $[0.7, 0.9]$. The cost is specified as $J(x_0, u) = \int_0^\infty (0.1x_1^2 + x_2^2 + 0.1u^2)dt$.

In [25], it has been shown that a linear feedback control policy can globally asymptotically stabilize the system (90)-(91) at the origin, and the resultant closed-loop system has an Lyapunov function, which is a polynomial in $x_1$ and $x_2$ with degree less than or equal to four. Using the technique in [25], we are able to find an initial stabilizing control policy $u_1 = 50x_1 - 2x_2$, and a related cost function $V_0$ satisfying Assumption 4.1 is obtained by solving the feasibility problem (64), with $r = 2$, $d = 3$, and $d_1 = 0$. For simulation, select $x_1(0) = 3$ and $x_2(0) = -4$. 
The proposed online learning scheme is applied to improve the control policy every one second for four times. In this simulation, we set $\beta = 0.8$, which is assumed to be unknown to the learning system. The exploration noise is the sum of 25 sinusoidal waves with different frequencies, and it is turned off after four iterations. Simulation results are shown in Figure 2. It can be seen that the post-learning cost function is remarkably improved compared with the one obtained in the first policy evaluation step.
C. Inverted pendulum

Consider the following differential equations which are used to model an inverted pendulum:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{kl}{m} x_2 + g \sin(x_1) + \frac{1}{m} u
\end{align*}
\]

where \( x_1 \) is the angular position of the pendulum, \( x_2 \) is the angular velocity, \( u \) is the control input, \( g \) is the gravity constant, \( l \) is the length of the pendulum, \( k \) is the coefficient of friction, and \( m \) is the mass. The design objective is to find a suboptimal and globally stabilizing control policy that can drive the state to the origin. Assume the parameters are not precisely known, but they satisfy \( 0.5 \leq k \leq 1.5, 0.5 \leq m \leq 1.5, 0.8 \leq l \leq 1.2, \) and \( 9 \leq g \leq 10 \).

Notice that we can select \( \phi = [x_1, x_2]^T \) and \( \sigma = [x_1, x_2, \sin x_1]^T \). The cost is selected as:

\[
J(x_0, u) = \int_0^\infty (10x_1^2 + 10x_2^2 + u^2) dt
\]

Further, set \( \bar{\phi} = [x_1^2, x_1 x_2, x_2^2]^T \) and \( \bar{\sigma} = [x_1^2, x_2^2, x_1 \sin x_1, \sin^2 x_1, x_2 \sin x_1, x_1 x_2]^T \). Then, based on the range of the system parameters, a pair \((V_0, u_1)\) satisfying Assumption 6.3 can be obtained as \( u_1 = -10x_1^2 - x_2 - 15 \sin x_1 \), and \( V_0 = 320.1297 x_1^2 + 46.3648 x_1 x_2 + 22.6132 x_2^2 \). The coefficient vector \( c \) is defined as \( c = \bar{\phi}(1, -1) + \bar{\phi}(1, 1) \).

The initial condition for the system is set to be \( x_1(0) = -1.5 \) and \( x_2(0) = 1 \). The control policy is updated after 0.5 seconds, until convergence is attained after 4 iterations. The exploration noise
we use is the sum of sinusoidal waves with different frequencies, and it is terminated once the convergence is attained.

The resultant control policy and the cost function are $u^* = -20.9844x_1 - 7.5807x_2$ and $V^* = 86.0463x_1^2 + 41.9688x_1x_2 + 7.5807x_2^2$. Simulation results are provided in Figure 3. It can be seen that the system performance is significantly improved under the proposed ADP scheme.

VIII. CONCLUSIONS

This paper has, for the first time, proposed a global ADP method. In particular, a new policy iteration scheme has been developed. Different from conventional policy iteration, the new iterative technique does not attempt to solve a partial differential equation but a convex optimization problem at each iteration step. It has been shown that, this method can find a suboptimal solution to continuous-time nonlinear optimal control problems [29]. In addition, the resultant control policy is globally stabilizing. Also, the method can be viewed as a computational strategy to solve directly Hamilton-Jacobi inequalities, which are used in $H_\infty$ control problems [16], [48].

When the system parameters are unknown, conventional ADP methods utilize neural networks to approximate online the optimal solution, and a large number of basis functions are required to assure high approximation accuracy on some compact sets. Thus, neural-network-based ADP
schemes may result in slow convergence and loss of global asymptotic stability for the closed-loop system. Here, the proposed global ADP method has overcome the two above-mentioned shortcomings, and it yields computational benefits.

It is under current investigation to extend the proposed methodology for more general (deterministic or stochastic) nonlinear systems, as well as systems with parametric and dynamic uncertainties [23], [20], [22].

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