Perceiving light rays above and below the horizon of a black hole in Schwarzschild spacetime

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Abstract. We consider a novel kind of plot for presenting the behaviour of null geodesics in Schwarzschild spacetime. Our diagram depicts various features of the light rays as recorded by different classes of observers. The idea stems from the use of phase portraits for illustrating the null geodesics in Schwarzschild spacetime, above and below the horizon of a black hole, but they reveal unphysical characteristics. The plots we propose are free from such anomalies. Moreover they allow us to discover new under-horizon features.

1. Introduction
In general relativity, the notion of an observer becomes crucial when one intends to describe an outcome of an experiment conducted in a strong gravitational field. In our work, we focus on the perception of light rays by different kinds of observers in Schwarzschild spacetime. We present a novel kind of plot for illustrating the behaviour of null geodesics in this geometry. The spatial trajectories of light rays can be presented in a form of a phase portrait [1] but it may reveal unphysical features: the trajectories corresponding to light rays moving in the interior of a black hole away from the singularity, passing through the event horizon and leaving this region of spacetime. We present the plots where only physical null geodesics are illustrated as each point in the diagram represents a light ray that is registered by a local observer where the spatial (the direction in space) as well as temporal aspects (frequency shift) are taken into account.

2. Light rays as measured by different observers in Schwarzschild spacetime
In the static, spherically symmetric spacetime the line element can be expressed using Schwarzschild coordinates \((t, r, \theta, \phi)\) as:

\[
ds^2 = \left(1 - \frac{r_S}{r}\right)(c dt)^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \equiv g_{tt}(c dt)^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2,
\]

where \(r_S = 2GM/c^2\) and we shall use the system of units where \(G = c = 1\). Due to the spherical symmetry of the spacetime, we restrict our considerations to the \(\theta = \pi/2\) plane. We determine...
the geodesics of massive and massless geodesics, described by the four-velocity and the wave
four-vector, respectively, using the conservation laws: $g_{tt} \dot{t} = \text{const}$ and $-g_{\phi\phi} \dot{\phi} = \text{const}$, where
the dot signifies the derivative with respect to the auxiliary parameter $\lambda$. For null geodesics
$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$ and a four-vector corresponding to a light ray, denoted as $k = k^t \partial_t + k^\tau \partial_\tau + k^\phi \partial_\phi$, reads

$$k^t = \frac{\Omega}{g_{tt}}, \quad k^\tau = \pm \left( \Omega^2 - g_{tt} \frac{L^2}{r^2} \right)^{1/2}, \quad k^\phi = \frac{L}{r^2},$$

where $\Omega$ and $L$ are constants, related to the energy and angular momentum, respectively, and
$\pm$ signs correspond to outgoing/ingoing light rays.

2.1. Light rays as measured by static observers
Let us introduce a static observer SO in Schwarzschild spacetime whose velocity four-vector
reads $V = V^t \partial_t = g_{tt}^{-1/2} \partial_t$. The components of a wave vector which are measured by this
observer determined within his/her coordinate frame (see e.g. [2]) are:

$$k^0 = g_{tt}^{1/2} \Omega, \quad k^1 = \pm \left( \Omega^2 - g_{tt} \frac{L^2}{r^2} \right)^{1/2}, \quad k^3 = \frac{L}{r}.$$

The angle of incidence $\psi$ of a light ray for SO can be expressed as

$$\cot \psi = \frac{k^1}{k^3} = \pm r \left( \frac{1}{\Omega^2} - \frac{1}{r^2} \right)^{1/2},$$

where the parameter describing a specific null geodesics $b = L/\Omega$ (impact parameter) and $+$/$-$
correspond to in-/outgoing geodesics, respectively. We can illustrate the evolution of light rays,
specified by $b$, as measured by SO at each point along the trajectory, in the diagram depicting the
$\psi(r)$ dependence, the $\psi$-diagram, in Figure 1. The range of the angle $\psi$ is restricted to

$[0, \pi]$ due to the axial symmetry. We notice the symmetry with respect to the line $\psi = \pi/2$: the
ingoing trajectories fill the lower part $\psi < \pi/2$ and outgoing trajectories the upper part $\psi > \pi/2$ of the diagram. Approaching the event horizon, all trajectories tend to be oriented radially: $\psi \rightarrow 0$ or $\psi \rightarrow \pi$.

2.2. Light rays as measured above the horizon by RF – observers in a radial free fall
For an observer falling freely from $r = r_0$, $A = [g_{tt}(r_0)]^{1/2}$ and the four-velocity

$$U = U^t \partial_t + U^r \partial_r = (A/g_{tt}) \partial_t - (A^2 - g_{tt})^{1/2} \partial_r.$$

Figure 1. $\psi$-diagram for static observers. The unstable circular orbit of the photon sphere is
represented by an isolated point: $r_{\text{phs}} = 3r_S/2$, $\psi_{\text{phs}} = \pi/2$. The
blue lines, tending asymptotically to this point, are four separate curves corresponding to critical
value of $b$: $b_{\text{cr}} = 3\sqrt{3} r_S/2$. 
In the frame of reference of such an observer the angle of incidence of a light ray is determined as (see also [2, 3])

\[
\cot \psi = \frac{k_1}{k_3} = \frac{rA}{g_{tt}b} \left[ -\frac{(A^2 - g_{tt})^{1/2}}{A} \pm \left( 1 - g_{tt}\frac{b^2}{r^2} \right)^{1/2} \right] = \frac{rA}{g_{tt}b} \left[ -v \pm \left( 1 - g_{tt}\frac{b^2}{r^2} \right)^{1/2} \right],
\]

(6)

where \( v = [1 - g_{tt}/A^2]^{1/2} \) is velocity of the RF as measured by an SO (see e.g. [4]).

Let us choose a class of observers falling freely from infinity \((A = 1)\), registering the light rays of different \( b \), where the wavelength measured at \( r = 7r_S \) is \( \lambda_0 = 590 \text{ nm} \) for all the ingoing light rays. The frequency shift as measured by each RF is described by the ratio of frequencies registered at radial coordinates \( r_2 \) and \( r_1 \) (see e.g. [5, 6])

\[
\frac{\Omega(r_2)}{\Omega(r_1)} = \frac{g_{tt}(r_1)}{g_{tt}(r_2)} \frac{1 \mp v(r_2)}{1 \mp v(r_1)} \left[ 1 - g_{tt}(r_2)\frac{b^2}{r_2^2} \right]^{1/2}.
\]

(7)

In Figure 2 a \( \psi \)-diagram for RF, falling from infinity, where the differences in wavelength are indicated by the colours of the points in the plot is presented.

\[ \text{Figure 2. The } \psi \text{-diagram for observers falling freely from infinity where for each observer and each ingoing light ray } \lambda(r_1 = 7r_S) = 590 \text{ nm}. \text{ Black/grey points indicate ultraviolet/infrared range, respectively.} \]

2.3. Light rays as measured above and below the horizon by RF and NRF (non-radially falling observers)

In the interior of the Schwarzschild black hole, the angle \( \psi \) as measured by a radially falling observer satisfies:

\[
\cot \psi = \frac{rA}{g_{tt}b} \left[ \frac{(A^2 - g_{tt})^{1/2}}{A} + \left( 1 - g_{tt}\frac{b^2}{r^2} \right)^{1/2} \right],
\]

(8)

where the interchange of roles of time \( t \) and radial \( r \) coordinates below the horizon (see [7, 8]) has been taken into account. The curves corresponding to different light rays measured by RF, above and below the event horizon are presented in Figure 3.

\[ \text{Figure 3. } \psi \text{-diagram for observers in the radial free fall from infinity.} \]

In this diagram only outgoing geodesics tend to be focused, \( \psi \to \pi \), at the horizon. The ingoing rays
cover the whole range of $\psi \in [0, \pi)$. The small area in the upper left part of the diagram between the blue lines $b = b_{cr}$ comprises the trajectories corresponding to geodesics characterized by $b > b_{cr}$, i.e. in this part of the diagram big changes in $b$ induce very small changes in $\psi$.

Another type of observer, whose $\psi$-diagram can lead to interesting insights, is the observer in a free fall but with a non-zero angular momentum $J$. The velocity four-vector of such an observer is given by:

$$U = U^t \partial_t + U^r \partial_r + U^\phi \partial_\phi = \left( A/g_{tt} \right) \partial_t - \left[ A^2 - g_{tt}(1 + J^2/r^2) \right]^{1/2} \partial_r + J/r^2 \partial_\phi .$$  \hspace{1cm} (9)

In this case the range of $\psi$ is extended to $\psi \in [0, 2\pi]$. The diagram is not symmetric with respect to the horizontal line $\psi = \pi$, apart from the curves corresponding to the light rays with $|b| = \infty$ below the horizon. Radial light rays no longer correspond to $\psi = 0/\pi$. On the other hand, all the outgoing trajectories tend to the value $\psi = \pi$ at the horizon, just like in the case of RF.

![Figure 4. $\psi$-diagram for a class of observers with $A = 1$ and $J = r_s$.](image)

3. Discussion and conclusions

In this paper we present plots illustrating the trajectories of light rays as registered by different types of observers in Schwarzschild spacetime, both above and below the horizon of the black hole. Each plot, corresponding to a specific class of observers, pictures all the light rays which can be detected by this type of observer, presenting the angle at which a given light ray is registered. The obtained $\psi$-diagrams reveal the main differences in the perception of light rays by static, radially and non-radially falling observers. The most notable issues are the symmetry of the diagrams and the behaviour of null geodesics in the vicinity and below the horizon of the black hole – for RF and NRF. Originating in the vicinity of the horizon, there exist almost empty, “tear-shaped” areas between the curves $|b| = b_{cr}$. It is characteristic for these parts of the diagrams that in order to fill them up with trajectories one needs curves corresponding to light rays of very large (up to infinite) angular momentum. This “repulsion” of trajectories around the curves $|b| = \infty$ is reminiscent of the neighbourhood of the photon sphere above the horizon. The possible analogy between these parts of $\psi$-diagrams will be the subject of further investigations. The important aspect concerning the area beneath the horizon is that the coordinate $r$ describes the evolution in time, $r = 0$ being the ultimate instant. The consequences of this fact will be considered in our future studies. Moreover, the next significant step will be the construction of $\psi$-diagrams in other geometries – so far the preliminary results have been obtained in the case of a Kerr black hole.

References

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