“Warm” Tachyon Matter from Back-reaction on the Brane

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Abstract

We study a 3-brane moving in the five-dimensional bulk of the Randall-Sundrum II model. By including back-reaction of the brane on the bulk geometry we obtain a tachyon model with a linear barotropic equation of state.

1 Introduction

The nature of dark matter is an old question [1]. The large-scale successes of the WIMP-CDM paradigm come at the small scale price of overproducing satellite galaxies and giving haloes with a central cusp [2]. These problems
are alleviated for sterile neutrino warm dark matter \cite{2,3}. Avelino et al \cite{5} have recently shown that cosmological data favour a dark matter equation of state \( w_{\text{DM}} \approx 0.01 \).

Particle dark matter is not the only possibility \cite{6}. Sen \cite{7} has noted that unstable modes in string theory can be described by an effective Born-Infeld type lagrangian

\[
L = -V(\theta) \sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}
\]

for the “tachyon” field \( \theta \), where the typical potential has minima at \( \theta = \pm \infty \). Of particular interest is the inverse power law potential \( V \propto \theta^{-n} \). For \( n > 2 \), as the tachyon rolls near minimum it behaves like pressure-less matter \cite{8}, and caustics form \cite{9}.

In general any tachyon model can be mapped to the motion of a 3-brane moving in a warped extra dimension \cite{10}. Of special note is that the potential \( V(\theta) \propto 1/\theta^4 \) maps to the AdS5 geometry of the second Randall-Sundrum model \cite{11}. One thus apparently gets cold \textit{geometric} tachyon matter. There is, however, a caveat: the Randall-Sundrum solution corresponds to an empty brane at the orbifold fixed point. Placing matter on this observer brane changes the bulk geometry; this is encoded in the radion field which, in turn,
is related to the variation of the physical interbrane distance \( d_5(x) \) between \( y = 0 \) and coordinate infinity \( y = \infty \), as illustrated in Fig. 1.

Since the geometric tachyon is seen on our brane as a form of matter, it will likewise affect the bulk geometry in which it moves. What we will now show is that the back-reaction qualitatively changes the geometric tachyon: the brane and radion form a composite object endowed on average with a linear barotropic equation of state.

## 2 Back-reaction model

Our starting point is a model based upon the second Randall- Sundrum model (RS II) \[11\]. We employ the metric convention with negative spatial signature and units \( 8\pi G = c = 1 \). In the RS II the bulk is \( \text{AdS}_5/Z_2 \):

\[
S_{\text{bulk}} = \frac{1}{K^{(5)}} \int d^5x \sqrt{g^{(5)}} \left[ -\frac{R^{(5)}}{2} - \Lambda^{(5)} \right].
\]  

(2)

Observers reside on the positive tension brane at the orbifold fixed point \( y = 0 \)

\[
S^{(+)} = -\sigma^{(+)} \int d^4x \sqrt{-h}.
\]  

(3)

where they see an induced metric \( h_{\mu\nu} \). The negative tension brane is pushed off to the \( \text{AdS}_5 \) horizon at \( y = \infty \). The solution of the field equations for this empty brane is \( \text{AdS}_5 \)

\[
d s^2_{(5)} = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2.
\]  

(4)

Although the 5-th dimension has infinite coordinate size it can be integrated out to obtain a purely four-dimensional action with a well defined value for the Planck mass of the order \( m^2_{\text{Pl}} \sim (kK^{(5)})^{-1} \).

The full description must include the fact that matter on the observer brane distorts the bulk geometry \[13\]. The naive \( \text{AdS}_5 \) geometry \( 4 \) is distorted by a scalar mode, the radion field \( \Phi \), related to the interbrane distance. It is advantageous to choose coordinates \( g_{(5)\mu\nu} = 0 \). From \( R_{(5)\mu\nu} \) upon imposing the “Einstein gauge” condition that the coefficient of the four-dimensional Ricci scalar in \( S_{\text{bulk}} \) be unchanged, one arrives at the five-dimensional line element \[13\]

\[
d s^2_{(5)} = (\Phi(x) + e^{-2ky}) g_{\mu\nu}(x) dx^\mu dx^\nu - \left( \frac{e^{-2ky}}{\Phi(x) + e^{-2ky}} \right)^2 dy^2.
\]  

(5)
where the field $\Phi(x)$ may be expressed in terms of the canonically normalized radion $\phi(x)$:

$$\Phi = \sinh^2 \sqrt{\frac{1}{6}\phi}. \quad (6)$$

The physical distance to the AdS$_5$ horizon at coordinate infinity is given by

$$d_5 = \frac{1}{2k} \ln \left( \frac{1 + \Phi}{\Phi} \right). \quad (7)$$

The constant $k$ is related to the bulk cosmological constant via

$$k^2 = -\frac{\Lambda_{(5)}}{6}, \quad (8)$$

where $k^2 > 0$ for AdS$_5$. It should be noted that in the Einstein gauge matter on the positive tension brane at $y = 0$ sees a metric

$$g_{\mu\nu}^{(+)} = (1 + \Phi)g_{\mu\nu}. \quad (9)$$

With the RS II fine tuning

$$\sigma_0^{(+)} = -\sigma^{(-)} = \frac{6k}{K_{(5)}}, \quad (10)$$

the effective bulk action takes a simple form

$$S_{\text{bulk}} = \int d^4x \sqrt{-g} \left( -\frac{R}{2K} + \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}, \right) \quad (11)$$

where

$$K = kK_{(5)} \quad (12)$$

Following our convention we set $K = 8\pi G = 1$.

Consider next a 3-brane moving in the five-dimensional bulk with the geometry (5). The brane may be parameterized by $X^M = (x^\mu, Y(x))$. Then the induced metric is given by

$$g_{\mu\nu}^{\text{(ind)}} = (\Phi + e^{-2ky}) g_{\mu\nu} - \left( \frac{e^{-2ky}}{\Phi + e^{-2ky}} \right)^2 Y_{,\mu}Y_{,\nu}. \quad (13)$$

The brane action is

$$S_{\text{brane}} = -\sigma \int d^4x \sqrt{-g^{\text{(ind)}},} \quad (14)$$
where $\sigma$ is the brane tension and $g^{(\text{ind})}$ is the determinant of $g^{(\text{ind})}_{\mu\nu}$. It is convenient to introduce

$$\Theta = 3e^{-2kY}, \quad \psi = 2\Theta + 6\Phi, \quad \lambda = \sigma/(6k^2), \quad \ell = \sqrt{6}/k. \quad (15)$$

Then the combined radion and brane Lagrangian is

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{\lambda}{\ell^2}\psi^2 \sqrt{1 - \frac{\ell^2\Theta_{,\mu}\Theta_{,\nu}}{\psi^3}}. \quad (16)$$

In the absence of $\phi$, a field redefinition $\Theta = 1/(2\theta^2)$ yields the tachyon Lagrangian (1) with the inverse quartic potential.

3 FRW Universe

We will examine the model (16) assuming a homogeneous isotropic evolution to exhibit the main features. Besides, we assume $\phi \ll 1$ and approximate $\Phi \simeq \phi^2/6$. As usual we identify the pressure with the Lagrangian

$$p = \mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{\lambda}{\ell^2}\psi^2 \sqrt{1 - \frac{\ell^2\dot{\Theta}^2}{\psi^3}}, \quad (17)$$

from which we derive the energy density as the Hamiltonian:

$$\rho = \mathcal{H} = \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \dot{\Theta}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{\ell^2}\frac{\psi^2}{\sqrt{1 - \ell^2\dot{\Theta}^2/\psi^3}}. \quad (18)$$

Now, it is advantageous to introduce a non-canonical conjugate field

$$\Pi = \frac{-\ell\dot{\Theta}}{\psi \sqrt{1 - \ell^2\dot{\Theta}^2/\psi^3}}. \quad (19)$$

In terms of $\Pi$ the pressure and energy density are respectively given by

$$p = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{\ell^2}\frac{\psi^2}{\sqrt{1 + \Pi^2/\psi}}, \quad (20)$$

and

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{\ell^2}\psi^2 \sqrt{1 + \Pi^2/\psi}. \quad (21)$$
Then, the field equations read

\[ 3H^2 = 3 \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{\ell^2} \psi^2 \sqrt{1 + \Pi^2/\psi}, \quad (22) \]

\[ \dot{\Theta} = \frac{-\Pi \psi}{\ell \sqrt{1 + \Pi^2/\psi}}, \quad (23) \]

\[ \dot{\Pi} + 3H\Pi = \frac{1}{\ell} \frac{4\psi + 3\Pi^2}{\sqrt{1 + \Pi^2/\psi}}, \quad (24) \]

\[ \ddot{\phi} + 3H\dot{\phi} = -\frac{\lambda \phi}{\ell^2} \frac{4\psi + 3\Pi^2}{\sqrt{1 + \Pi^2/\psi}}. \quad (25) \]

In Fig. 2 we exhibit the results of numerical integration of the above equations, taking time in units of \( \ell \). We take \( \lambda = 1/3 \) and integrate starting from \( t = 0 \) with the initial conditions \( \Theta = 1.01, \phi = 0.1, \Pi = \dot{\phi} = 0.00001 \).

As one would anticipate from (25), the field \( \phi \) undergoes damped oscillations with the amplitude decreasing as \( 1/t \). We note in particular that, once the initial transient dies away, the quantity \( \Pi/\sqrt{\psi} \) increases quadratically whereas the quantity \( t^2\psi \) comes to oscillate about a constant value. Thus, one can effectively use the simplified equations

\[ 3 \left( \frac{\dot{a}}{a} \right)^2 \simeq \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{\ell^2} \psi^{3/2}\Pi, \quad (26) \]

\[ \dot{\psi} \simeq 2 \left[ \phi\dot{\phi} - \frac{1}{\ell} \psi^{3/2} \right], \quad (27) \]

\[ \dot{\Pi} + 3H\Pi \simeq \frac{1}{\ell} 3\Pi \sqrt{\psi}, \quad (28) \]

\[ \ddot{\phi} + 3H\dot{\phi} \simeq -\frac{\lambda \phi}{\ell^2} 3\Pi \sqrt{\psi}. \quad (29) \]

In the absence of the radion, Eqs. (26)–(29) admit a tachyon solution

\[ \psi_T = \ell^2/t^2, \quad (30) \]

\[ \Pi_T = (4t) / (3\lambda \ell). \quad (31) \]
Figure 2: Results of numerical integration of Eqs. (22)–(25) for $\lambda = 1/3$. 
In the asymptotic regime the pressure is only due to the radion, \( p \simeq \frac{1}{2} \dot{\phi}^2 \). Moreover, the equation of state

\[
\frac{p}{\rho} \simeq \frac{\dot{\phi}^2}{\phi^2 + 2(\lambda/\ell^2)\psi^{3/2}\Pi}
\]

is positive albeit oscillatory. As the oscillations are absent in \( tH \), this quantity can be used to define an effective equation of state:

\[
w_{\text{eff}} = \frac{2}{3tH} - 1.
\]

However, since the oscillations in \( w \) are rapid on cosmological timescales, it is most useful to time average co-moving quantities. The effective equation of state is then

\[
\langle p \rangle = \langle w \rangle \langle \rho \rangle,
\]

where \( \langle x \rangle \) denotes the time average of the quantity \( x \). Now, to second order in the amplitude \( A \) of \( t\phi \), a solution to (26)–(29) is approximated by

\[
\phi \simeq \frac{A}{t} \cos (2t),
\]

\[
\psi \simeq \frac{1}{t^2} \left( 1 - \frac{3A^2}{2} + A^2 \cos^2 (2t) \right),
\]

\[
\lambda \Pi \sqrt{\psi} \simeq \frac{4}{3} - 2A^2 + \frac{A^2}{2} \cos^2 (2t),
\]

and then

\[
\langle w \rangle \simeq \frac{3A^2}{4}.
\]

The value of \( A \) may be estimated by comparing (35) with the exact solution for \( t\phi \) depicted in the top right plot in Fig. 2. We find \( A=0.1518 \) which yields \( \langle w \rangle = 0.017 \).

In Fig. 3 we compare the approximation (35)–(38) with our exact numerical solution. The advantage of the approximate solution is that it can be used to set appropriate initial conditions at late times without the tedium of intermediate simulations.

We have restricted ourselves here to an FRW model for simplicity. In particular one cannot interpret \( \sqrt{\langle w \rangle} \) as the adiabatic speed of perturbations. Note also that the quantity \( \dot{p}/\dot{\rho} \) cannot be identified with the speed of sound.
squared $c_s^2$ because $\dot{\rho}/\rho$ is, in our case, not positive semi-definite owing to interactions. Now, the non-interacting radion is stiff matter, with unit speed of sound, whereas the non-interacting tachyon asymptotically has vanishing speed of sound. Following [14] we can define the sound speed squared for the composite as the sum of the components weighted by their fraction of $\rho + p$:

$$c_s^2 = \frac{\dot{\phi}^2 + (\lambda/\ell^2)\psi \Pi^2 (1 + \Pi^2/\psi)^{-3/2}}{\dot{\phi}^2 + (\lambda/\ell^2)\psi \Pi^2 (1 + \Pi^2/\psi)^{-1/2}}.$$  (39)

After transients

$$c_s^2 \simeq \frac{2w}{1 + w}.  (40)$$

Due to the rapid oscillations, it may be useful to define the effective speed of sound as the ratio of the co-moving acoustic to the co-moving particle
horizon radii:

\[ c_{\text{eff}} = \frac{\int dt c_s / a}{\int dt / a}. \]  

(41)

In Fig. 4 we plot the effective speed of sound defined in (41) together with the approximate asymptotic value

\[ c_{\text{eff}} |_{\text{app}} \simeq \sqrt{3} A. \]  

(42)

Note that \( c_{\text{eff}} |_{\text{app}} \) is twice as large as the “average” speed of sound that one would naively expect from the equation of state \( (38) \).

![Figure 4: Effective speed of sound. The horizontal red line represents the approximate asymptote given by (42).](image)

4 Conclusions

In this paper we have studied back-reaction on a geometric tachyon in the AdS\(_5\) bulk of the RS II model. We have established that the coupled tachyon – radion is greater than the sum of its parts: the composite behaves as a form of warm dark matter with an effective barotropic equation of state. Besides, we have extracted the asymptotic field equations and found a one parameter set of approximate solutions.
In closing, we note that at the linear level one expects frustrated small-scale structure formation: initially growing modes instead undergo damped oscillations once they enter the co-moving acoustic horizon. The desired cut-off scale can be directly linked to the speed of sound and hence to the amplitude. Perturbation theory is not the whole story however [10]; we will address the issue of non-linear structure formation elsewhere.

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