Towards the optimal window for the 2MASS dipole

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ABSTRACT
A comparison of the 2MASS flux dipole to the CMB dipole can serve as a method to constrain a combination of the cosmological parameter $\Omega_m$ and the luminosity bias of the 2MASS survey. For this constraint to be as tight as possible, it is necessary to maximize the correlation between the two dipoles. This can be achieved by optimizing the survey window through which the flux dipole is measured. Here we explicitly construct such a window for the 2MASS survey. The optimization in essence reduces to excluding from the calculation of the flux dipole galaxies brighter than some limiting magnitude $K_{\text{min}}$ of the near-infrared $K_s$ band. This exclusion mitigates nonlinear effects and shot noise from small scales, which decorrelate the 2MASS dipole from the CMB dipole. Under the assumption of negligible shot noise we find that the optimal value of $K_{\text{min}}$ is about five. Inclusion of shot noise shifts the optimal $K_{\text{min}}$ to larger values. We present an analytical formula for shot noise for the 2MASS flux dipole, to be used in follow-up work with 2MASS data.

The misalignment angle between the two dipoles is a sensitive measure of their correlation: the higher the correlation, the smaller the expectation value of the angle. A minimum of the misalignment is thus a sign of the optimal gravity window. We model analytically the distribution function for the misalignment angle and show that the misalignment estimated by Maller et al. is consistent with the assumed underlying model (though it is greater than the expectation value). We predict with about 90% confidence that the misalignment will decrease if 2MASS galaxies brighter than $K_{\text{min}} = 5$ mag are excluded from the calculation of the flux dipole. This prediction has been indirectly confirmed by the results of Erdogdu et al. The measured misalignment constitutes thus an alternative way of finding the optimal value of $K_{\text{min}}$: the latter corresponds to a minimum of the former.

Key words: methods: analytical – cosmology: large-scale structure of Universe – cosmology: cosmic microwave background – galaxies: general – galaxies: infrared – galaxies: Local Group

1 INTRODUCTION
The dipole anisotropy of the cosmic microwave background (CMB) is interpreted as a direct measure, via the Doppler shift, of the motion of the Local Group (LG) relative to the CMB rest frame. The components of this motion of non-cosmological origin (the motion of the Sun in the Milky Way and the motion of the Milky Way in the LG) are known and can be subtracted (e.g., Courteau & van den Bergh 1999). When transformed to the barycenter of the LG, the motion is towards $(l, b) = (273^\circ \pm 3^\circ, 29^\circ \pm 3^\circ)$, and of amplitude $v_{\text{LG}} = 627 \pm 22$ km s$^{-1}$, as inferred from the first-year WMAP data (Bennett et al. 2003).

The kinematic interpretation of the CMB dipole is strongly supported by its remarkable alignment with the dipole component of the large-scale galaxy distribution (often called the ‘clustering dipole’), inferred from various all-sky surveys. In the gravitational instability scenario, this alignment is expected: peculiar velocities of galaxies are induced gravitationally and are thus strongly coupled to the large-scale matter distribution. Linear theory predicts the peculiar velocity of the LG, $v$, to be proportional to the LG peculiar acceleration, caused by the gravitational pull of surrounding matter inhomogeneities. Let us denote by $\delta$ the mass density contrast, $\delta = \rho/\rho_b - 1$, where $\rho$ is the mass density of matter and $\rho_b$ is its average value. The clustering dipole,

$$ g \equiv \int \frac{d^3r}{4\pi} \delta(r) \frac{r}{r^3}, \quad (1) $$

is a quantity proportional to the peculiar gravitational acceleration...
(so we will call it interchangeably ‘scaled gravity’), and can be estimated from a three-dimensional all-sky galaxy survey. In the linear regime, the relation between the velocity and the scaled gravity is

\[ v = H_0 f(\Omega_m)\hat{g}_b, \]  

(2)

Here, \( H_0 \) is the Hubble constant, \( \Omega_m \) is the cosmic matter density parameter and \( f(\Omega_m) \approx \Omega_m^{0.6} \) (e.g., Peebles 1980). For a spherical survey \( \int d^3r \frac{r}{r^3} = 0 \), hence we can write

\[ \hat{g} = \frac{\int d^3r [1 + \delta(r)]}{4\pi} \frac{r}{r^3} = g_b^{-1} \frac{\int d^3r \rho(r)}{4\pi} \frac{\hat{r}}{r^2}. \]  

(3)

In the following we will assume that dark matter (DM) in the Universe is entirely locked in DM halos of luminous galaxies. Modelling galaxies as point particles, the observed density field is \( \hat{g}(r) = \sum m_i \delta(r - r_i) \), where \( \delta_D \) is Dirac’s delta; \( m_i \) and \( r_i \) are respectively the mass and the position of the \( i \)-th galaxy. Substituting this equation into Equation (3) yields for the scaled gravity

\[ \hat{g} = g_b^{-1} \sum \frac{m_i}{4\pi} \frac{\hat{r_i}}{r_i^2}; \]  

(4)

thus we see that the true gravitational acceleration equals to \( 4\pi G \rho_b \hat{g} \). We will assume further that ‘light traces mass’, or that the mass-to-light ratio for galaxies is a universal constant, \( \Upsilon \). Then we can write

\[ \hat{g} = \rho_b^{-1} \sum \frac{\Upsilon L_i}{4\pi} \frac{\hat{r_i}}{r_i^2} = \rho_b^{-1} \sum S_i \hat{F}_i. \]  

(5)

Here, \( L_i \) is the luminosity of \( i \)-th galaxy and \( S_i \) is its observed flux, \( S_i = L_i / 4\pi r_i^2 \). In other words, since both the gravity and the flux fall off as distance squared, the gravitational acceleration of the LG is proportional to the dipole of the light distribution (i.e., the flux dipole) for a constant mass-to-light ratio. The sum in Equation (5) is in principle over all galaxies in the Universe, while in practice we have at our disposal only finite, usually flux-limited, catalogs of galaxies. In such catalogs, lower-mass dark matter halos will be underrepresented by the survey galaxies. To account for this, we write

\[ \hat{g} = \frac{\Upsilon}{\rho_b \rho_L} \sum \frac{N}{i=1} S_i \hat{F}_i, \]  

(6)

where \( b_L \) is the resulting luminosity bias and \( N \) is the total number of galaxies in a given survey. Combining Equation (6) with Equation (2) we obtain finally (Erdogdu et al. 2006; hereafter E06)

\[ v = H_0 \Omega_{m}^{0.6} \sum \frac{N}{i=1} S_i \hat{F}_i. \]  

(7)

In the above we have used the fact that the mass-to-light ratio \( \Upsilon = \rho_b/\rho_L \), where \( \rho_L \) is the luminosity density of the Universe. Equation (7) shows that in the linear theory one can predict the LG peculiar velocity using solely an angular (two-dimensional) all-sky survey, bypassing the lack of radial information, i.e. distances. Specifically, a comparison between the CMB dipole and the flux dipole of a given survey can yield an estimate of the parameter \( \beta = \Omega_m^{0.6}/b_L \).

Such a comparison was first performed by Yahil, Sandage & Tamman (1980) using the revised Shapley-Ames catalogue and by Davis & Huchra (1982) using the CfA catalogue, leading to the estimates of the flux dipoles that were within 30° away from the CMB dipole. The inclusion of redshift information, usage of progressively larger redshift surveys and theoretical improvements of the analyses led to smaller measured values of the misalignment. In particular, using the IRAS 1.2 Jy survey, Strauss et al. (1992; hereafter S92) found that the clustering dipole points around 25° away from the CMB dipole. Using the further completed IRAS PSCz survey, Schmoldt et al. 1999 (hereafter S99) obtained the clustering dipole within 15° of the CMB dipole. A similar analysis, based also on the IRAS PSCz survey, performed by Rowan-Robinson et al. (2000), determined the misalignment angle to be around 13°.

Two most recent analyses of the clustering dipole employed the Two Micron All Sky Survey (2MASS; Skrutskie et al. 1997). In particular, to compute the flux dipole, Maller et al. (2003; hereafter M03) used the angular 2MASS extended source catalogue, with a limiting magnitude of \( K_s = 13.57 \). (Approximately 740,000 galaxies covering 90% of the sky.) E06 used the Two Micron All Sky Redshift Survey (2MRS): approximately 23,200 2MASS galaxies with measured redshifts, selected from a total sample of about 24,800 galaxies with (extinction-corrected) magnitudes smaller than \( K_s = 11.25 \).

2MASS is the first near-infrared (JHKs passbands) all-sky survey. While most passbands tend to be sensitive to the instantaneous star formation rate, \( K_s \) passband is most sensitive to total stellar mass (Bell & de Jong 2001; Bell et al. 2003), making this band a better tracer of total mass. 2MASS has an effective image resolution of 1\( ^\circ \) and a hundred times greater sensitivity than the far-infrared IRAS survey. The photometric uniformity of the 2MASS survey is better than 4 per cent over the entire sky including the celestial poles (e.g., Jarrett et al. 2003). The median depth of the survey is 220 \( h^{-1} \) Mpc (Bell et al. 2003), a distance past where the clustering dipole has been shown to converge\(^1\).

Given all these advantages of 2MASS over other all-sky galaxy surveys, it is perhaps surprising that the misalignment between the CMB dipole and the 2MASS flux dipole is not smaller than the corresponding one for IRAS galaxies. The value obtained by M03 is 16\(^\circ\). For the 2MRS flux dipole, E06 obtained approximately 21\(^\circ\)\(^\text{2}\) In this paper we aim at answering the following questions. First: do we understand fully the origin of this misalignment? Second: can one do better with 2MASS, and if so, how?

The answer to these questions is essential for optimal estimation of the parameter \( \beta \) by comparing the CMB dipole to the 2MASS dipole. The stronger is the correlation between the two dipoles, the smaller are statistical errors of such an estimate. Therefore, the observational window, through which the 2MASS dipole is measured, should be adapted to obtain the best correlation possible. The misalignment angle is a sensitive measure of this correlation: the higher the correlation, the smaller the angle. In other words, a minimum of the misalignment angle is a sign of the optimal 2MASS window. In this paper we will formally prove these statements. First, we will derive the 2MASS window. Next, we will optimize it under the assumption of negligible shot noise. Finally, we will demonstrate that a minimum of the expectation value of the angle corresponds to minimal variance of the resulting estimate of \( \beta \).

\(^1\) The inclusion of galaxy redshifts in the dipole analyses allowed the estimation of the convergence depth, i.e. the distance at which most of the clustering dipole is generated. There is a controversy whether this convergence depth is about 50 \( h^{-1} \) Mpc, or rather 200 \( h^{-1} \) Mpc (for details see E06). In either case, 2MASS is deep enough to provide a reliable estimate of the clustering dipole. (But see Basilakos & Plionis 2006.)

\(^2\) E06 computed two kinds of the clustering dipole. The second one, the number dipole, was even more misaligned with the CMB dipole.
Let us enumerate possible sources of the misalignment between the CMB dipole and an all-sky galaxy survey flux dipole.

- $M/L \neq \text{const}$. The constant mass-to-light ratio is probably a good assumption for (almost) all galaxies when averaged over many galaxies of the same luminosity. For individual galaxies, however, $M/L$ is expected to have some scatter. On the other hand, as stated earlier, 2MASS, unlike IRAS surveys, is mainly sensitive to total stellar mass. Consequently, the mass-to-light ratio of 2MASS galaxies is expected to have smaller scatter than that of IRAS galaxies.

- Nonlinear bias. Writing Equation 6 we have implicitly assumed that the total flux dipole and the magnitude-limited flux dipole differ in the amplitude, but not in the direction. However, if large-scale distribution of low-mass DM halos is different from the distribution of high-mass halos, then the two dipoles will not be collinear.

- Nonlinear dynamics. The peculiar velocity of the LG is equal to the temporal integral of the LG gravitational acceleration along the LG trajectory. Therefore, loosely speaking, while the response of the LG acceleration to growing nonlinearities is ‘instantaneous’, the response of the LG velocity is time-averaged and ‘retarded’. As a result, the acceleration of the LG is more non-linear and higher in amplitude (in velocity units) than the velocity of the LG (Ciecielag et al. 2003). What is more relevant here, at orders higher than linear, non-local character of gravity reveals itself and tends to misalign the velocity vector of the LG with the vector of its acceleration. However, the mean misalignment angle between the velocity and gravity of the LG-like regions simulated in numerical experiments is about $8^\circ$ (Davis et al. 1991, Ciecielag et al. 2001).

- Observational effects: shot noise, finite volume of the survey, and the mask (due to the zone of avoidance, ZoA). Shot noise and finite volume are more an issue for 2MRS, which has a median magnitude-limited flux dipole. The CMB dipole estimates the peculiar velocity of the LG, $v$. The 2MASS flux dipole, Equation 6, estimates the gravitational acceleration – more specifically, the scaled gravity – of the LG, $g$, induced by large-scale matter inhomogeneities traced by 2MASS galaxies.

Let $p(g, v)$ denote the joint PDF for the LG scaled gravity and peculiar velocity. It is a standard practice to approximate it by a multivariate Gaussian (S92; S99). Numerical simulations (Kofman et al. 1994, Ciecielag et al. 2003) show that nongaussianity of nonlinear $g$ and $v$ is indeed small. This is not surprising since, e.g. gravity is an integral of density over a large volume (Eq. 1.), so the central limit theorem can at least partly be applicable.

Using statistical isotropy of $g$ and $v$, their joint PDF can be simplified to the form (Juszkiewicz et al. 1990; Lahav, Kaiser & Hoffman 1990):

$$p(g, v) = \frac{1}{(2\pi)^{3/2}\sigma_g\sigma_v^2} \exp \left[ -\frac{x^2 + y^2 - 2\mu xy}{2(1 - \rho^2)} \right], \quad (8)$$

where $\sigma_g$ and $\sigma_v$ are the r.m.s. values of a single Cartesian component of gravity and velocity, respectively. From isotropy, $\sigma_g^2 = \langle g \cdot g \rangle / 3$ and $\sigma_v^2 = \langle v \cdot v \rangle / 3$, where $\langle \cdot \rangle$ denote the ensemble averaging. Next, $\langle x, y \rangle = (g/\sigma_g, v/\sigma_v)$, and $\mu = \cos \theta$ with $\theta$ being the misalignment angle between $g$ and $v$. Finally, $\rho$ is the cross-correlation coefficient of $g_n$ with $v_n$, where $g_n (v_n)$ denotes an arbitrary Cartesian component of $g (v)$. From isotropy,

$$\rho = \frac{\langle g \cdot v \rangle}{\langle g^2 \rangle^{1/2}\langle v^2 \rangle^{1/2}}. \quad (9)$$

Also from isotropy,

$$\langle x_n y_n \rangle = \rho \delta_{mn}, \quad (10)$$

where $\delta_{mn}$ denotes the Kronecker delta. In other words, there are no cross-correlations between different spatial components.

For a given all-sky galaxy survey, the LG gravity is measured effectively through the window of the survey, $W_g$ (cf. Eq. 1):}

$$g = \int \frac{d^3r}{4\pi} \delta(r)W_g(r) \frac{r}{r^3}. \quad (11)$$

In contrast, the LG velocity is not estimated from a velocity survey (i.e., from a catalog of peculiar velocities of galaxies), but measured directly from the dipole anisotropy of the CMB. Still, to relate it to theoretical quantities, we write:

$$v = \int \frac{d^3r}{4\pi} \vartheta(r)W_v(r) \frac{r}{r^3}. \quad (12)$$

Here $\vartheta \equiv -\nabla \cdot v$ is the (minus) velocity divergence and we assume that the velocity field is irrotational. Thus, similarly to $g$, $v$ show the resulting distribution function for the misalignment angle. A summary and conclusions will be given in Section 8.
can be expressed as a Coulomb (Newton) integral over its source, i.e. the field of the velocity divergence. Here we do not assume that we know the latter from observations, but we know from theory its statistical relation to the density field (see this Section and Section 3). This is sufficient for our purposes in this work. Since \( v \) is directly measured from the CMB dipole, the effective velocity window, \( W_v \), which we have introduced in Equation (12), is essentially unity. (Contributions from all perturbations are included.)

We modify slightly this form of the window to reflect the finite size of the LG. Following S92 and S99, we adopt

\[
W_v = \begin{cases} 
0, & r < r_{LG}, \\
1, & \text{otherwise}, 
\end{cases}
\]

which has a small-scale cutoff, \( r_{LG} = 1 \, h^{-1} \, \text{Mpc} \). This window is markedly different from those appropriate for velocity surveys: the latter are not spherical, have complicated shapes and finite depth (Sarkar, Feldman & Watkins 2007). The gravity window, \( W_g \), of the 2MASS survey is derived in Section 4.

In Fourier space, relations (11) and (12) read:

\[
g_{k} = \frac{i k}{k^2} \delta_{k} W_{g}(k),
\]

\[
v_{k} = \frac{i k}{k^2} v_{0} W_{v}(k),
\]

where the subscript \( k \) denotes the Fourier transform. The quantity \( \hat{W} \) is related to the window \( W \) by the following equation (S92):

\[
\hat{W}(k) \equiv k \int_{0}^{\infty} W(r) j_{l}(kr) dr.
\]

Here and below \( j_{l} \) represents the spherical Bessel function of first kind of order \( l \). In particular,

\[
\hat{W}_{v}(k) = j_{0}(kr_{LG}).
\]

From equations (14) and (15) we have

\[
\langle g \cdot g \rangle = \frac{1}{2\pi^2} \int_{0}^{\infty} \hat{W}_{g}^{2}(k) P(k) dk,
\]

and

\[
\langle v \cdot v \rangle = \frac{1}{2\pi^2} \int_{0}^{\infty} \hat{W}_{v}^{2}(k) P_{0}(k) dk.
\]

Here, \( P(k) \) and \( P_{0}(k) \) are respectively the power spectrum of the density and the power spectrum of the velocity divergence. Defining

\[
\mathcal{R}(k) = \frac{P_{0}(k)}{P(k)},
\]

we have

\[
\langle v \cdot v \rangle = \frac{1}{2\pi^2} \int_{0}^{\infty} \hat{W}_{v}^{2}(k) \mathcal{R}(k) P(k) dk.
\]

Furthermore,

\[
\langle g \cdot v \rangle = \frac{1}{2\pi^2} \int_{0}^{\infty} \hat{W}_{g}(k) \hat{W}_{v}(k) C(k) P_{0}^{1/2}(k) P^{1/2}(k) dk,
\]

where \( C(k) \) is the so-called coherence function (CF), or the correlation coefficient of the Fourier components of the gravity and velocity fields (S92):

\[
C(k) \equiv \frac{\langle g_{k} \cdot v_{k} \rangle}{\langle |g_{k}|^2 \rangle^{1/2} \langle |v_{k}|^2 \rangle^{1/2}} = \frac{\langle \delta_{k} \vartheta_{k} \rangle}{\langle |\delta_{k}|^2 \rangle^{1/2} \langle |\vartheta_{k}|^2 \rangle^{1/2}}.
\]

Hence, we obtain finally

\[
\rho = \frac{\int_{0}^{\infty} \hat{W}_{g}(k) W_{c}(k) C(k) \mathcal{R}^{1/2}(k) P(k) dk}{\left( \int_{0}^{\infty} \hat{W}_{g}^{2}(k) P(k) dk \right)^{1/2} \left( \int_{0}^{\infty} \hat{W}_{g}^{2}(k) \mathcal{R}^{2}(k) P(k) dk \right)^{1/2}}.
\]

Equations (18), (21) and (22) specify all the parameters (the variances and the correlation coefficient) that determine the joint PDF for \( g \) and \( v \). Equation (5), in the absence of observational errors. The deviation of the correlation coefficient from unity is then due to different windows, through which the gravity and the velocity of the LG are measured, and due to nonlinear effects. The latter are described by two functions; the CF, and the ratio of the power spectra (Ciecielg & Chodorowski 2004; hereafter C04).

The distribution for the misalignment angle can be derived from the joint distribution (5). Here we are interested in the distribution for the misalignment angle with the observed value of the LG velocity as a constraint. The conditional distribution function, \( p(g|v) \), readily results from (9):

\[
p(g|v) = (2\pi)^{-3/2} \sigma_{g}^{-3} (1 - \rho^2)^{-3/2} \exp \left\{ -\frac{(g - \mu g)^2}{2(1 - \rho^2)} \right\}
\]

(Juszkiewicz et al. 1990; Lahav et al. 1990). The distribution for the amplitude of the LG acceleration and the cosine of the misalignment angle is \( p(g, \mu|v) = 2\pi \rho^2 g \rho g|v| \). The distribution for \( \mu \) is obtained by marginalizing over \( g \),

\[
p(\mu|v) = 2\pi \int_{0}^{\infty} d g^2 p(g|v),
\]

and the distribution for the angle itself is \( p(\theta|v) = |d\mu/d\theta| p(\mu|v) = \sin(\theta) p(\mu|v) \). This yields (Juszkiewicz et al. 1990; Lahav et al. 1990)

\[
p(\theta|v) = \sin(\theta) \exp(-q^2) \left\{ \frac{q \mu}{\sqrt{\pi}} \right\}
\]

\[
+ \left\{ \frac{1}{2} + q^2 \mu^2 \right\} \exp(q^2 \mu^2) \left[ 1 + \text{erf}(q \mu) \right].
\]

where

\[
q = \frac{\rho g}{\sqrt{2(1 - \rho^2)}}.
\]

We remind that \( y \) is the amplitude of the LG peculiar velocity in units of the 1D velocity dispersion, \( y = v_{LG}/\sigma_{v} \). For \( v_{LG} = 627 \, \text{km} \cdot \text{s}^{-1} \) (Bennett et al. 2003) and the values of the cosmological parameters adopted here (as described in Section 3), \( y = 2.64 \). This might suggest that the velocity of the LG is a rare event; however, this is on the contrary. First, one should compare the amplitude of the LG velocity to the 3D velocity dispersion, \( \sigma_{v,3D} = \sqrt{3} \sigma_{v} \). Therefore, the relevant parameter here is \( y' \equiv v_{LG}/\sigma_{v,3D} = 1.52 \). Second, the probability that a randomly chosen region will have velocity greater than \( v_{LG} \) is equal to \( \int_{1.52}^{\infty} dy' h(y') \), where \( h(y) = \sqrt{2/\pi} y^2 e^{-y^2/2} \) is the Maxwellian distribution. For the lower limit of the integral equal to 2.64, the value of the integral is 0.07 = 7%. However, for 1.52, its value is 0.51 = 51%.

The misalignment of only several degrees corresponds to a strong coupling between \( g \) and \( v \). In the strong coupling limit \( 1 - \rho \ll 1 \), so \( q \gg 1 \). Also, \( \mu \) is then close to unity. Therefore, in equation (27) we can use an asymptotic formula for the error function,
erf(s) \simeq 1 - \frac{1}{\sqrt{\pi s}} e^{-s^2} \quad \text{for } s \gg 1.

We then obtain a small-angle approximation of the distribution for the misalignment angle (Lahav et al. 1990):

\[ p(\theta|\nu) \simeq \frac{\theta}{\theta_*^2} \exp \left(-\frac{\theta^2}{2\theta_*^2}\right), \tag{30} \]

with

\[ \theta_* = \sqrt{\frac{1 - \rho^2}{\rho y}}. \tag{31} \]

Thus, in the strong coupling limit the misalignment angle, given the velocity constraint, is Rayleigh-distributed. The parameter \( \theta_* \), much smaller than unity (in radians), is a characteristic measure of the misalignment.\(^5\) The expectation value of the angle is

\[ \langle \theta | \nu \rangle = \sqrt{\frac{\pi}{2}} \theta_* . \tag{32} \]

Other quantities characterizing the distribution which are of interest here are quantiles. In our, slightly modified notation, the quantile \( \theta_q \) of a distribution \( p(\theta) \) is such a number, that

\[ \int_{\theta_{\text{min}}}^{\theta_q} p(\theta) \, d\theta = \frac{q}{100}. \tag{33} \]

For the Rayleigh distribution, \( \theta_{\text{min}} = 0 \). For our purposes, interesting quantiles are

\[ \theta_{10} = \sqrt{2 \ln(10/9)} \theta_* , \quad \text{and} \quad \theta_{90} = \sqrt{2 \ln 10} \theta_* . \tag{34} \]

3 NONLINEAR EFFECTS

Using numerical simulations, C04 modelled the CF (Eq. 23) and the ratio of the power spectra (Eq. 20). The simulations were evolved from Gaussian initial conditions. As the initial power spectrum of matter fluctuations, a cold dark matter (CDM) spectrum was adopted (as in Eq. 7 of Efstathiou, Bond & White 1992), with the shape parameter \( \Gamma = 0.19 \). Both the CF and the ratio of the power spectra were modelled as functions of the wavevector, \( k \), and the amplitude of the matter fluctuations, \( \sigma_8 \). For the CF, C04 found the following fit:

\[ C(k) = \left[1 + (a_0 k - a_2 k^{1.5} + a_1 k^2)^{2.5}\right]^{-0.2}, \tag{35} \]

with the coefficients given by the following, power-law, scaling relations in \( \sigma_8 \):

\[ a_0 = 4.008 \sigma_8^{0.750} ; \]
\[ a_1 = 2.663 \sigma_8^{0.734} ; \]
\[ a_2 = 5.889 \sigma_8^{0.714}. \tag{36} \]

The fit was calculated for \( k \in [0, 1] \) h Mpc\(^{-1} \) and \( \sigma_8 \in [0.1, 1] \), with the imposed constraint \( C(k = 0) = 1 \). This constraint assures that for sufficiently large, linear scales, the relation between the gravity and the velocity is deterministic and linear (see Eq. 2). Formula (35) is a better fit to the CF than an earlier formula of Chodorowski & Ciecielag (2002), which was less accurate for low values of \( k \).

Chodorowski & Ciecielag (2002) investigated numerically also the dependence of the CF on \( \Omega_m \) and found it to be extremely weak.

Defining the scaled velocity divergence, \( \hat{\theta} \equiv \frac{\Omega_m^{0.6}}{\Omega_\Lambda} \theta = -\Omega_m^{0.6} \nabla \cdot \nu \), C04 found the following fit for the ratio of the power spectra:

\[ R(k) = \left[1 + (7.071k^4)^{1/2}\right]^{-\alpha}, \tag{37} \]

where

\[ \alpha = -0.06574 + 0.29195\sigma_8 \quad \text{for } 0.3 < \sigma_8 < 1. \tag{38} \]

C04 argued that the ratio of the power spectra practically does not depend on the background cosmological model. This ratio is unity in the linear regime \( (k \ll 1) \) but decreases in the nonlinear regime, because the velocity grows slower than it would be expected from the linear approximation.

In this paper we use Equations (35) and (37) as the formulas respectively for the CF and the ratio of the power spectra. For \( \sigma_8 \) we adopt the value obtained from a joint analysis of third-year WMAP and SDSS, \( \sigma_8 = 0.77 \) (Spergel et al. 2007). In Equations (18), (23) and (24), as the power spectrum we use a CDM spectrum. For zero baryon content, the shape parameter of the spectrum, \( \Gamma \), equals simply to \( \Omega_m h \). Non-zero baryon content of the Universe modifies the shape parameter to (Sugiyama 1995)

\[ \Gamma_{\text{eff}} = \Omega_m h \exp \left[-\Omega_\Lambda \left(1 + \sqrt{2\Gamma/\Omega_m}\right)\right]. \tag{39} \]

Here we adopt \( \Gamma_{\text{eff}} = 0.15 \), the value obtained both from first-year WMAP (Spergel et al. 2003) and a joint analysis of third-year WMAP and SDSS (Spergel et al. 2007). This value is in excellent agreement with the constraint on the shape of the power spectrum of 2MASS galaxies, \( \Gamma_{\text{eff}} = 0.14 \pm 0.02 \), obtained by Frith, Outram and Shanks (2005; assuming a flat \( \Lambda \text{CDM} \) cosmology, a primordial scale-invariant power spectrum and negligible neutrino mass). It is slightly higher than the corresponding result of Maller et al. (2005), \( \Gamma_{\text{eff}} = 0.12 \pm 0.01 \), obtained using a measure of the three-dimensional power spectrum via an inversion of the 2MASS angular correlation function.

4 GRAVITY WINDOW OF 2MASS

In this Section we derive the gravity window of the 2MASS survey. The 2MASS survey is dense, uniform and has an unprecedented sky coverage. Therefore, to a good accuracy it can be described by a spherical window. Since distances of 2MASS galaxies are unknown, the galaxies are weighted only by their fluxes and not by their distances (like, e.g., by the inverse of the selection function). This is the central assumption of the calculation below.

The background light intensity due to uniform distribution of discrete sources is

\[ I = \int S \, dN, \tag{40} \]

where \( S = L/(4\pi r^2) \) is the observed flux from the sources with intrinsic luminosity \( L \) and \( dN \) is the number of sources per steradian. In the case of uniformly distributed sources with the luminosity function \( \Phi(L) \), the contribution from a shell of thickness \( dr \) and radius \( r \) is \( dN = \Phi(L) \, dL \, r^2 \, dr \). For a flux-, or magnitude-limited survey, only galaxies with \( L > 4\pi r^2 S_{\text{min}} \) are observed, where \( S_{\text{min}} \) is the limiting (minimal) flux. Hence,

\[ I = \int_{S_{\text{min}}}^{\infty} \int_{0}^{\infty} \Theta(r, L) \frac{L}{4\pi r^2} \Phi(L) \, dL \, r^2 \, dr. \tag{41} \]

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where $\Theta(r, L)$ is the Heaviside step-function, $\Theta_H(L-4\pi r^2 S_{\text{min}})$. Writing $L_{\text{min}} \equiv 4\pi r^2 S_{\text{min}}$, this yields

$$I = \frac{\langle L \rangle N_0}{4\pi} \int_0^\infty dr W_g(r).$$

(42)

Here, $N_0 = \int_0^\infty \Phi(L) dL$, and

$$\langle L \rangle = \frac{\int_0^\infty L \Phi(L) dL}{\int_0^\infty \Phi(L) dL}$$

(43)

is the average luminosity of the population. The flux window of the survey is

$$W_g(r) = \frac{\int_{L_{\text{min}}}^{L_{\text{max}}} L \Phi(L) dL}{\int_0^\infty L \Phi(L) dL}.$$  

(44)

$W_g$ gives the percentage of the total light from distance $r$ which is included in the survey. Loosely speaking, it suppresses contributions from distances larger than $\sqrt{\langle L \rangle/(4\pi S_{\text{min}})}$. Its detailed form is determined by the LF.

The LF of 2MASS galaxies has been estimated by Bell et al. (2003), by matching a spectroscopic sample of Early Data Release SDSS galaxies with the 2MASS extended source catalog, to obtain redshifts for a subsample of 2MASS galaxies. Bell et al. fitted the 2MASS LF by the Schechter function:

$$\Phi(L) dL = \Phi^* \left(\frac{L}{L^*}\right)^\alpha \exp\left(-\frac{L}{L^*}\right) dL,$$

where $\Phi^*$ is the LF normalization, $L^*$ is the characteristic luminosity at the ‘knee’ of the LF, where the form changes from exponential to power law, and $\alpha$ is the ‘faint end slope’. For $K_s$-band, they found that $\alpha = -0.77$ and the absolute magnitude $M^*$, corresponding to the absolute luminosity $L^*$, is $M^* = -23.29 + 5 \log_{10} h$. We adopt this form of the 2MASS LF here.

The flux window of the 2MASS survey can be compared with the selection function of the survey, defined as

$$\phi(r) = \frac{\int_{L_{\text{min}}}^{L_{\text{max}}} \Phi(L) dL}{\int_0^\infty \Phi(L) dL}.$$  

(46)

The selection function gives the probability that a randomly selected galaxy at distance $r$ will be included in the survey. E06 call the selection function ‘the number-weighted selection function’, and the flux window the ‘flux-weighted (or luminosity-weighted) selection function’. They note that “the number-weighted selection function drops with the distance faster than the luminosity-weighted selection function. At large distances, we observe only the most luminous galaxies, so the amount of ‘missing’ luminosity from a volume of space is not as big as the number of ‘missing’ galaxies”. We fully agree. This implies in practice that when estimated from a galaxy survey, the flux dipole is a more robust quantity than the number dipole.

If we want to exclude also the brightest sources, then $\Theta(r, L)$ in Equation (41) becomes the product of two Heaviside functions, $\Theta(r, L) = \Theta_H(L-4\pi r^2 S_{\text{min}}) \cdot \Theta_H(4\pi r^2 S_{\text{max}} - L)$. Here, $S_{\text{max}}$ is the upper limiting (maximal) flux. It is simple to check that the survey window then becomes

$$W_g(r) = \frac{\int_{L_{\text{min}}}^{L_{\text{max}}} L \Phi(L) dL}{\int_0^\infty L \Phi(L) dL}.$$  

(47)

where $L_{\text{max}} \equiv 4\pi r^2 S_{\text{max}}$. This windows suppresses also contributions from distances smaller than about $\sqrt{\langle L \rangle/(4\pi S_{\text{max}})}$, or, for the Schechter LF, just about $\sqrt{L^*/(4\pi S_{\text{max}})}$.

What would be the gravity window for the number dipole? The answer depends on the weighting scheme. For the number dipole all galaxies are weighted equally – in a sense that they are not weighted by their fluxes or masses – but they may, or may not, be weighted proportionally to the inverse of the selection function. If they are not, it is clear from the above analysis that then the gravity window is the selection function, $\phi$. (N0, instead of missing some percentage of the total light from distance $r$, we miss some percentage of all galaxies located there.) However, if they are weighted as $1/\phi(r)$, the gravity window is simply unity. This is so because the $1/\phi(r)$ weighting corrects for, on average, missing signal from large distances. The price to pay for this correction is huge variance of such an estimator of the LG gravity. This variance is called shot noise (see Subsection 5.1) and has dominant contributions from large scales. Number-, rather than mass-, weighting of galaxies is another source of variance of this estimator, for simplicity also called shot noise. That shot noise comes predominantly from small scales.

Using the IRAS 1.2 Jy survey, S92 measured the number dipole, weighting galaxies originally as $1/\phi(r)$. To mitigate shot noise and nonlinear effects from small scales and shot noise from large scales, S92 decided to modify these weights. They did this introducing the so-called standard IRAS window,

$$W_{\text{IRAS}} = \left\{\begin{array}{ll}
(r/r_s)^3, & r < r_s , \\
1, & r_s < r < R_{\text{max}} , \\
0, & R_{\text{max}} < r.
\end{array}\right.$$  

(48)

This window is characterized by a small-scale smoothing, $r_s$, and a sharp large-scale cutoff, $R_{\text{max}}$. S99 adopted the values $r_s = 5$ h$^{-1}$ Mpc and $R_{\text{max}} = 150$ h$^{-1}$ Mpc, appropriate for the complete PSCz catalog (Saunders et al. 2000). Modified weights assigned to IRAS galaxies by S92 and S99 were

$$\text{weight}(i) = \frac{W_{\text{IRAS}}(r_i)}{\phi(r_i)}.$$  

(49)

It is clear that in this case, the gravity window of the IRAS number dipole is $W_{\Phi \text{IRAS}} = W_{\text{IRAS}}$. We will return to this point later.

To specify completely the distribution for the misalignment angle (Eq 27) or its small-angle approximation, Eq 29, we need the value of the velocity variance, Equation (21), and of the correlation coefficient, Equation (24). In order to calculate the latter, we need to derive the Fourier form (Eq. 16) of the 2MASS window, given above. In Equation (16), we can use the fact that the spherical Bessel function, $j_1(x) = -(d/dx)j_0(x)$, and integrate by parts. This yields

$$\tilde{W}_g(k) = \int_0^\infty j_0(kr) W_g(r) dr.$$  

(50)

Let’s cast Equation (27) to the form

$$W_g(r) = \frac{\int_0^{L_{\text{max}}} \Psi(L) dL}{\int_0^\infty \Psi(L) dL},$$  

(51)

where

$$\Psi(L) \equiv L \Phi(L).$$  

(52)

We can then write

$$W_g^2(r) = \frac{\partial W_g(r) dL_{\text{max}}}{\partial L_{\text{max}} dr} + \frac{\partial W_g(r) dL_{\text{min}}}{\partial L_{\text{min}} dr} = \frac{\Psi(L_{\text{max}}) 8\pi r S_{\text{max}} - \Psi(L_{\text{min}}) 8\pi r S_{\text{min}}}{\int_0^\infty \Psi(L) dL}.$$  

(53)

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Using Equations (50) and (53), and the fact that \( j_0(x) = \sin x/x \), we obtain
\[
\tilde{W}_g(k) = \frac{8\pi S_{\text{max}}}{k} \int_0^\infty \frac{\Psi(L) dL}{\Psi(L) dL} \int_0^\infty \sin(kr)\Psi(L_{\text{max}})dr \\
- \frac{8\pi S_{\text{min}}}{k} \int_0^\infty \frac{\Psi(L) dL}{\Psi(L) dL} \int_0^\infty \sin(kr)\Psi(L_{\text{min}})dr, 
\tag{54}
\]
where \( \Psi(L) \) is defined by Equation (52). Note that the flux (or gravity) window does not appear in Equation (21) for the velocity variance, and in Equation (24) for the correlation coefficient it appears in such a way that its absolute normalization cancels out. In other words, the PDF for the misalignment angle is sensitive only to the shape of the gravity window.

To relate the limiting fluxes to the limiting magnitudes, we remind that the observed minimal flux \( S_{\text{min}} \) is related to the apparent maximal magnitude \( K_{\text{max}} \) in the following way:
\[
S_{\text{min}} = S_0 10^{-0.4K_{\text{max}}},
\tag{55}
\]
where \( S_0 \) is the reference flux, which appears also in the relation between the absolute magnitude \( M^* \) and absolute luminosity \( L^* \),
\[
M^* = -2.5 \log_{10} \frac{L^*}{4\pi(10 \text{ pc})^2 S_0}.
\tag{56}
\]
In Equations (55)–(56) we can therefore eliminate \( S_0 \), obtaining
\[
S_{\text{min}} = 1.803 \times 10^{-5} \frac{L^*}{4\pi(1h^{-1} \text{ Mpc})^2}.
\tag{57}
\]
Calculating the numerical coefficient in the above equation we have adopted \( M^* = -23.29 + 5 \log_{10} h \) (Bell et al. 2003). Following M03, for \( K_{\text{max}} \) we have adopted the value 13.57. The reason for this choice of \( K_{\text{max}} \) is twofold. First, our aim here is to improve the 2MASS window used by M03 properly accounting for nonlinear effects, which affect only the choice of optimal \( K_{\text{min}} \) (the minimal magnitude). Second, M03 chose \( K_{\text{max}} = 13.57 \) because “the extended source catalog is 97.5% complete within the SDSS early data release for extinction-corrected Kron magnitudes of \( K_{\text{s}} \leq 13.57 \text{ mag} \)” (Bell et al. 2003, Jarrett 2004). The 2MASS window is the 2MASS flux-weighted selection function under the assumption that the survey is complete within the flux limits. If we wanted to go deeper, we should account for increasing incompleteness as a function of distance. However, Figure 1 of M03, showing the convergence of the 2MASS dipole as a function of the limiting magnitude, suggests that contributions from all galaxies (i.e. even from those not included in the survey) fainter than 13.57 mag are most likely negligible. Even for 2MASS galaxies brighter than 13.57 mag (where the catalog is complete), “the faintest 300,000 galaxies only change the dipole value by less than 5%” (M03).

For \( S_{\text{max}} \) we have simply
\[
S_{\text{max}} = S_{\text{min}} 10^{-0.4(13.57 - K_{\text{min}})}.
\tag{58}
\]
If the brightest galaxies are not excluded, then either directly from Equation (45), or from Equation (54), performing the limit \( S_{\text{max}} \to \infty \), we obtain
\[
\tilde{W}_g(k) = 1 - \frac{8\pi S_{\text{min}}}{k} \int_0^\infty \frac{\Psi(L) dL}{\Psi(L) dL} \int_0^\infty \sin(kr)\Psi(L_{\text{min}})dr, 
\tag{59}
\]
Figure 1 shows the 2MASS gravity windows for \( K_{\text{max}} = 13.57 \text{ mag} \) and different values of \( K_{\text{min}} \). Dotted line corresponds to Equation (59), i.e. to the case where the brightest galaxies are not excluded from the calculation of the flux dipole. Dashed and solid lines are plotted using Equation (54) and describe respectively the cases of excluding all 2MASS galaxies brighter than \( K_{\text{min}} = 8 \text{ mag} \), and \( K_{\text{min}} = 5 \text{ mag} \), respectively. Since all 2MASS windows have the same \( K_{\text{max}} \), they are similarly suppressed at large scales, i.e., at small \( k \). For reference, we also plot the standard IRAS window (Eq. 60), with \( r_s = 5 h^{-1} \text{ Mpc} \) and \( R_{\text{max}} = 150 h^{-1} \text{ Mpc} \) (dotted-dashed line).

Figure 1. Gravity windows for the 2MASS all-sky survey, for \( K_{\text{max}} = 13.57 \text{ mag} \) and different values of \( K_{\text{min}} \). Dotted line corresponds to Equation (59), i.e. to the case where the brightest galaxies are not excluded from the calculation of the flux dipole. Dashed and solid lines are plotted using Equation (54) and describe respectively the cases of excluding all 2MASS galaxies brighter than \( K_{\text{min}} = 8 \text{ mag} \) (as done by M03), and \( K_{\text{min}} = 5 \text{ mag} \) (our choice, as justified below). Let’s try to understand the influence of the limiting magnitudes on the shape of the gravity window. Since all 2MASS windows have the same \( K_{\text{max}} \), they are similarly suppressed at large scales (small \( k \)). For a given \( K_{\text{min}} \) (corresponding to maximal limiting flux), all objects brighter than \( L^* \) are excluded from distances smaller than \( r_{\text{min}} = \sqrt{L^*/(4\pi S_{\text{max}})} \), and at distances \( r < r_{\text{min}} \), all sources brighter than \( L^*(r/r_{\text{min}})^2 \) are excluded. For the limiting magnitude \( K_{\text{min}} = 5 \), \( r_{\text{min}} \approx 4.5 h^{-1} \text{ Mpc} \), while for \( K_{\text{min}} = 8 \), \( r_{\text{min}} \approx 18.1 h^{-1} \text{ Mpc} \). Consequently, the window for no exclusion of the brightest galaxies (dotted line) does not drop down at all for large \( k \) (small scales). The window for \( K_{\text{min}} = 5 \) (solid line) does drop down but is fairly wide, while the window for \( K_{\text{min}} = 8 \) (dashed line) drops very rapidly. Since, as explained earlier, \( W_k,IRAS = W_l,IRAS \), using Equation (45) we have
\[
\tilde{W}_g,IRAS(k) = \frac{3j_1(kr_s)}{kr_s} - j_0(kR_{\text{max}}) \tag{60}
\]
For reference, we plot this standard IRAS window in Figure 1 (dotted-dashed line). The small scale smoothing of the IRAS window is \( r_s = 5 h^{-1} \text{ Mpc} \), while for the 2MASS window with \( K_{\text{min}} = 5 \), the effective smoothing scale \( r_{\text{min}} \) is about 4.5 h^{-1} Mpc. It is not therefore surprising that at small scales the IRAS window, except for its oscillatory behaviour, decreases fairly similarly to this 2MASS window.

Even neglecting shot noise, suppressing contributions to the flux dipole from small scales is necessary, since nonlinear effects...
should be mitigated. For large $k$ the coherence function of velocity with gravity (Eq. [35]) drops significantly below unity, decreasing the value of the cross-correlation coefficient (Eq. [24]). Then suppressing the gravity window for large $k$ has almost no effect on the cross-term (which is the numerator of Eq. [23]), while it decreases the gravity variance, the square root of which appears in the denominator of this equation. This manipulation on the gravity window helps therefore to achieve the best possible correlation between the LG velocity and gravity. However, when one suppresses the gravity window for scales which are linear enough so that the CF is close to unity, one worsens the correlation again. This is so because even for linear fields (CF and the ratio of power spectra equal to unity) the correlation coefficient decreases for increasingly different windows of velocity and gravity. As a result, for some value of $r_{\min}$, or $K_{\min}$, the correlation coefficient will have a maximum.

We calculate the correlation coefficient, Equation (24) (using the appropriate formulas for the CF and the ratio of the power spectra, and the velocity window given by Eqs. [13] and [17]), for the 2MASS gravity window, for a range of values of the limiting magnitude $K_{\min}$. Results are shown in Figure 2. We see that $\rho$ has a maximum $(1 - \rho$ has a minimum) for $K_{\min} \simeq 4.5$. Either not suppressing small scales at all ($K_{\min} = -\infty$), or suppressing them excessively ($K_{\min}$ greater than, say, 6) clearly decreases the correlation coefficient. This implies larger statistical errors of the estimated cosmological parameters when comparing the 2MASS gravity window (Eq. [34]), or suppressing gravity (Eq. [35]) drops significantly below unity, decreasing the value of the cross-correlation coefficient (Eq. 24). Then suppressing the gravity window for large $K$ has almost no effect on the cross-term (which is the numerator of Eq. [23]), while it decreases the gravity variance, the square root of which appears in the denominator of this equation.

In the above we have accounted for the fact that the mass to light ratio for an individual galaxy, $\Upsilon_i$, may not be equal to its average value, $\Upsilon$, but may have some scatter. The quantity $\nu_i \equiv \Upsilon_i/\Upsilon$; hence $\langle \nu_i \rangle = 1$. In Equation (62) the summation is over all galaxies in the volume element, regardless whether or not they are in the 2MASS survey.

The expectation value of the estimated gravity is

$$\langle g_E \rangle = a \langle \sum_i \nu_i L_i \rangle \hat{r} = a W_g(r) \langle \sum_i \nu_i \rangle \hat{r}$$

$$= a W_g(r) \langle L \rangle N(\Delta V) \hat{r}.$$  \hspace{1cm} (63)

In the second step we have used the fact that the 2MASS gravity window we have constructed in Section 2, $W_g$, gives the percentage of the total light from distance $r$ which is included in the survey. $N(\Delta V)$ is the number of all galaxies in the volume element (regardless whether or not included in the survey). The expectation value of the modelled gravity is

$$\langle g_M \rangle = a W(r) \langle \nu_i \rangle \langle L_i \rangle \hat{r} = a W(r) \langle L \rangle N(\Delta V) \hat{r}.$$ \hspace{1cm} (64)

In Equation (64) we have assumed that the scatter in the mass-to-light ratio is independent of luminosity. However, the above result for $\langle g_M \rangle$ is also correct for fairly broad classes of luminosity-dependent scatter, e.g. for $\nu = 1 + \alpha F(L)$, where the luminosity-independent random variable $\alpha$ has zero mean. [In order for the variable $\nu$ to be always positive, we have to impose an additional constraint $|\alpha_{\min}| < F_{\max}^{-1}$, where $\alpha_{\min}$ (negative since $\langle \alpha \rangle = 0$) is the minimum value of $\alpha$ and $F_{\max}$ is the maximum value of the function $F$.]

Comparing Equation (63) to (64) we see that if we adopt $W = W_g$, then the estimated gravity is an unbiased estimator of the modelled gravity. (From a different perspective, the modelled gravity is an unbiased estimator of what we really observe.) Our 2MASS gravity window is constructed precisely in such a way to
assure this. However, the quantity $g_L$ has scatter around $g_M$. For flux (or number) dipoles estimated from flux-limited galaxy catalogs, the dilute sampling at large distances introduces significant scatter, called shot noise. The scatter in the values of $g_L$ around $g_M$ due to the scatter in the mass-to-light ratio is not, properly speaking, shot noise (S92). However, for simplicity, we will call both these effects ‘shot noise’. (S92 also follow this convention.)

A lucid derivation of shot noise for the number dipole of IRAS galaxies can be found in Appendix A of S92. Shot noise for the 2MASS dipole can be calculated in a similar way. However, except for the fact that the 2MASS dipole is a flux one, there is another important difference. While in the derivation of S92, galaxies are assigned equal weights. Still, shot noise from small scales can be introduced a window for the are similar

Similarly plagues the number dipole (see Eq. 35 of S92, where there is another important difference. While in the derivation of S92, galaxies are assigned weights essentially proportional to the inverse of the selection function, 2MASS galaxies are given equal weights (their distances are unknown). We will see below that this introduces a qualitative difference in the resulting formula for shot noise.

To compute the variance of $g_L$ and $g_M$ for a full shell of thickness $dr$, we square it and calculate its expectation value. Finally, we sum up contributions to the total variance from all shells. The result is

$$\sigma_{SN}^2 = \int L^2 \Phi(L) dL \cdot \int L^2 \Phi(L) dL \left( \int L^2 \Phi(L) dL \right)^2.$$  

(65)

Here, $Q \equiv (p^2 - 1) \geq 0$ quantifies the amount of scatter in $M/L$. Were there no scatter, the value of $Q$ would be zero. The function $F$ is

$$F(r) = \int_{r_{min}}^{l_{max}} \int_{r_{min}}^{l_{max}} \Phi(L) dL \cdot \int_{r_{min}}^{l_{max}} \Phi(L) dL.$$  

(66)

The upper limit in the above integrals is either $l_{max} = 4 \pi r^2 S_{max}$ or infinity, depending on whether we exclude the brightest objects or not (in the latter case, $S_{max} = \infty$).

Let us now investigate contributions to shot noise from small scales ($r \rightarrow 0$) and large ($r \rightarrow \infty$) scales. For $r \rightarrow 0$, consider first the case of no exclusion of the brightest objects. Then, both $\phi(r)$ and $W_g(r)$ tend to unity. From Equation (66) it is obvious that then $F(0)$ is a constant. Therefore,

$$\Delta \sigma_{SN, \text{nearby}}^2 \propto Q \sum_{\text{nearby}, i} S_i^2.$$  

(67)

If $Q$ is significantly greater than zero, then the RHS of the above proportionality blows up (since for $r_i \rightarrow 0$, $S_i \rightarrow \infty$). This is shot noise from small scales, mentioned already in Section 4. It similarly plagues the number dipole (see Eq. 35 of S92, where there are similar $r^{-2}$ divergences). As already mentioned, to mitigate shot noise from small scales S92 introduced a window for the IRAS dipole. With inclusion of the IRAS window, contributions to shot noise from small scales in Equation (35) of S92 are proportional to $W_{IRAS}^2(r_i)/r_i^2$. For the standard IRAS window (Eq. 43), they scale as $r_i^2 \rightarrow 0$ for $r_i \rightarrow 0$, so shot noise from small scales is indeed strongly suppressed. As already stated, 2MASS galaxies are assigned equal weights. Still, shot noise from small scales can be mitigated. This is achieved by excluding from the calculation of the dipole contributions from the brightest objects, as described below.

For finite $S_{max}$, the selection function is

$$\phi(r) = \int_{r_{min}}^{l_{max}} \Phi(L) dL / \int \Phi(L) dL.$$  

(68)

Therefore, for $r \rightarrow 0$ and finite $S_{max}$, both $W_g(r)$ and $\phi(r)$ tend to zero. However, although $\phi(r)$ tends to zero, it is straightforward to verify that the quantity $W_g^2(r)/\phi(r)$ also tends to zero (at least for the Schechter form of the luminosity function). Finally, $F(r)$ tends to a constant (though different from that for the case $S_{max} = \infty$). Hence,

$$\Delta \sigma_{SN, \text{nearby}}^2 \propto \sum_{\text{nearby}, i} S_i^2.$$  

(69)

The above sum is limited to objects with $S_i < S_{max}$, what prevents it to blow up. Therefore, excluding the brightest objects is a good way to mitigate shot noise from small scales having at one’s disposal angular data only.

Contributions to shot noise from large scales do not depend on the choice whether we exclude the brightest objects, or not. For $r \rightarrow \infty$, both $\phi(r)$ and $W_g(r)$ tend to zero; it is straightforward to check that then also $W_g^2(r)/\phi(r)$ tends to zero. The limit of $F(r)$ for $r \rightarrow \infty$ is unity. Hence, using Equation (65) we obtain

$$\Delta \sigma_{SN, \text{distant}}^2 \propto \int L^2 \Phi(L) dL.$$  

(70)

We see that in the case of the flux dipole calculated with equal weights assigned to all galaxies, shot noise from large scales does not blow up; on the contrary, it decreases. As a result, in an analysis of the 2MASS dipole one does not have to exclude any data from large distances. The analysis presented here assumed $K_{min} = 13.57$, but it is now clear that when calculating the 2MASS dipole one can include contributions from 2MASS galaxies fainter than this magnitude. (Although, as mentioned earlier, Fig. 1 of M03, showing the convergence of the dipole as a function of the limiting magnitude, suggests that their contribution will be negligible. See also Fig. 3 of Jarrett 2004.)

Large-scale asymptotic behaviour of shot noise for the flux dipole (Eq. 70) with equal weighting is in contrast to the corresponding behaviour of shot noise for the number dipole calculated with galaxy weights proportional to the inverse of the selection function. In the latter case, contributions to shot noise from large scales diverge as $\phi^{-2}(r_i)$ (see Eq. 35 of S92; at large distances $\phi(r_i) \ll 1$). To cure this problem, S92 introduced in their standard IRAS window a large-scale sharp-cutoff, $R_{max}$ (see Eq. [28]). With inclusion of the IRAS window, contributions from large scales are proportional to $W_{IRAS}^2/r^2$. If $W_{IRAS}$ is truncated at some $R_{max}$, then there are no contributions to shot noise from scales beyond $R_{max}$. Still, this does not imply that ‘$1/\phi(r)$’ weighting of distant galaxies is a good one. S92 were aware of this fact and constructed the optimal window for the IRAS survey, i.e. such that it minimized variance of the estimator of the LG velocity (Eq. 45 of S92). At large scales this window behaves asymptotically as $J_3(r)\phi(r)$, where $J_3(r) = \int_0^\infty d^3 r \xi(r')$ and $\xi$ is the mass two-point correlation function. Therefore, the optimal weighting at large distances is proportional to $J_3(r)$ (see Eq. [29]), so instead of increasing [as $\phi^{-1}(r)$ does] it decreases to zero, suppressing shot noise from large scales. (2MASS weighting is intermediate between these two extremes.) At small scales, in the absence of the

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6 In this paper we need a formula for shot noise only for illustrative purposes, therefore the derivation will be presented in follow-up work.

7 These divergences are due to the weighting scheme and not to the type of the dipole.
scatter in the masses of galaxies, the window approaches unity. Therefore, the $1/\phi(r)$ weighting is then indeed the optimal one.\footnote{In a related paper, Feldman, Kaiser & Peacock (1994) constructed the optimal estimator for the density power spectrum inferred from redshift surveys. They derived a formula for the optimal weighting of galaxies (Eq. 2.3.4 of Feldman et al. 1994). For small $r$ the optimal weight behaves like $\phi^{-1}(r)$, while for large $r$ it approaches asymptotically unity.}

In the presence of scatter the window filters out small scales, as desired. Surprisingly, S92 resigned from using this window in the analysis of the LG acceleration and employed instead the standard IRAS window. The reason was that in the derivation of the optimal window they also attempted to account for nonlinear effects, but the coherence function they used was wrong (Chodorowski & Ciecielga 2002). As a consequence, the resulting window filtered out small scales excessively. In the present paper, working with only angular data we have no choice: we have to assign equal weights to all galaxies. Though this is not the optimal weighting, this is still quite good: shot noise from large scales does not blow up. Moreover, excluding the brightest objects helps to mitigate shot noise from small scales.

Let us recall: we denote the estimated gravity of the LG by $g_E$ (Eq. 51) and its modelled gravity by $g_M$ (Eq. 62). Although $g_E$ is an unbiased estimator of $g_M$, it is still a biased estimator of the true gravity of the LG. Large depth of the 2MASS survey makes the estimated dipole to converge, but in order to mitigate shot noise and nonlinear effects from small scales we have to suppress contributions from small distances. (For angular data the only way to do this is to exclude the brightest galaxies, located preferentially nearby.) This reduction of the signal introduces bias in the estimate of the LG gravity. However, applying a Maximum Likelihood analysis enables one to correct for this bias and to obtain an unbiased estimator of the parameter $\beta = \Omega_{m,0}/b_0$. This will be discussed in Section 10.

How big is actual shot noise for the 2MASS survey? To answer this question, M03 performed bootstrap resampling on the 2MASS galaxy catalog (100 times). They found that the standard deviation of the dipole direction was a fraction of a degree, and of the dipole magnitude a fraction of a percent. They concluded that ‘the systematic uncertainties are much larger than the shot noise’. Shot noise is certainly less an issue for the 2MASS dipole than for the IRAS PSCz dipole (S99) and for the 2MRS dipole (E06). If it is smaller for the 2MASS dipole partly due to much bigger number of galaxies in this survey compared to IRAS PSCz and 2MRS: there are about 13,000 galaxies in the PSCz catalog and 23,000 galaxies in 2MRS, while for the limiting magnitude $K_e = 13.57$, the 2MASS catalog contains about 740,000 galaxies (M03). The main reason, however, is non-weighting of galaxies when calculating the 2MASS dipole. (E06 weighted 2MRS galaxies inversely to the ‘flux-weighted selection function’. or, in our terminology, the gravity window, $W_g$). Still, M03 analysed shot noise including all (so also the brightest) 2MASS galaxies. Therefore, it is somewhat surprising that they did not find a trace of shot noise from small scales. A forthcoming paper of some of us (Bilicki & Chodorowski, in preparation) will be devoted to the optimal measurement of the 2MASS dipole. We are planning to reexamine carefully the issue of shot noise there. Specifically, we are going to repeat the bootstrap resampling analysis and to compare its results to our analytical formula for shot noise, Equation (65).

At first sight, it may seem surprising that Equation (65) can be used in the case of only angular data, since radial functions $\phi(r), W_g(r)$ and $F(r)$ appear in it. However, these functions are uniquely determined by specifying $K_{\text{max}}$ (corresponding to $S_{\text{max}}$), $K_{\text{min}}$ (corresponding to $S_{\text{min}}$) and the luminosity function of the 2MASS galaxies. As described before, this luminosity function has been estimated e.g. by Bell et al. (2003). The only data employed in Equation (65) are fluxes, $S_i$. (One also needs an estimate of $Q$, quantifying the amount of scatter in $M/L$.) Since we do not have these data at our disposal yet, for the rest of this paper we will accept the claim of M03 that shot noise for the 2MASS flux dipole is negligible.

\section{The mask}

The source of the biggest systematic error in 2MASS remains the lack or deficit of galaxies in the Zone of Avoidance (at low Galactic latitudes). M03 masked the region of the ZOA, and repopulated it with ‘synthetic galaxies’. In one method they cloned the sky above and below the masked region. In another method, they filled ‘the masked region with randomly chosen galaxies such that it has the same surface density as the unmasked area’. The first method gave a dipole pointing towards $l = 263^\circ, b = 40^\circ$. The second method resulted in a dipole pointing towards $l = 266^\circ, b = 47^\circ$. M03 adopted the mean of these two measurements as the best-fit dipole.

However, the error bars they attributed to the mask-filling uncertainty were somewhat underestimated. We will return to this point later.

The misalignment can be fully represented as a two-dimensional vector lying on the celestial sphere. In the absence of shot noise, the total misalignment is a vectorial sum of the cosmologically-originated misalignment $\theta_e$, described in Section 2 and the misalignment due to mask, $\theta_m$:

$$\theta = \theta_e + \theta_m. \quad (71)$$

We have $\theta_m = (\Delta l, \Delta b)$, where $l$ and $b$ are respectively the Galactic longitude and latitude. Under the simplest assumption, the distribution function for $\theta_m$ is a bivariate Gaussian of two uncorrelated variables of the same variance:

$$p(\Delta l, \Delta b) = (2\pi)^{-1}\sigma_l^{-2} \exp \left( -\frac{\Delta l^2 + \Delta b^2}{2\sigma_l^2} \right). \quad (72)$$

The distribution for the modulus $\theta_m = \sqrt{\Delta l^2 + \Delta b^2}$ results immediately from Equation (72). It is a Rayleigh distribution (cf. Eq. 59).

$$p(\theta_m) = \frac{\theta_m}{\sigma_l^2} \exp \left( -\frac{\theta_m^2}{2\sigma_l^2} \right). \quad (73)$$

Let us now invert the above reasoning and apply it to the variable $\theta_e$. Since the distribution for $\theta_m$ is (approximately) Rayleigh, the distribution for $\theta_e$ is (approximately) a bivariate Gaussian. The variable $\theta$ is therefore a sum of two independent bivariate Gaussians, which itself is a bivariate Gaussian (of uncorrelated variables). Hence, the variable $\theta$ is Rayleigh-distributed, with the parameter

$$\theta_e^2 = \sigma^2 + \sigma^2. \quad (74)$$

Here, $\theta_e^2 = \langle \theta_e^2 \rangle/2$, and $\sigma^2 = \langle \theta_m^2 \rangle/2$.

The parameter $\theta_e$ is defined by Equation (51) and determined by the LG velocity variance (Eq. 24) and the correlation coefficient (Eq. 24). Let us find an estimate for the mask variance $\sigma^2$. We have
\[
\hat{\sigma} = \frac{1}{N - 1} \sum_{i=1}^{N} x_i^2,
\]

where \(x_i = \sqrt{(l_i - \bar{l})^2 + (b_i - \bar{b})^2}\), and \((\bar{l}, \bar{b})\) are the means for the sample. As stated above, M03 study the effects of two different methods of 'repopulating' the masked regions with galaxies, so \(N = 2\). Then \(x_2 = x_1\), hence \(\hat{\sigma} = \sqrt{2}x_1\); the factor \(\sqrt{2}\) mustn't be neglected. This yields (in degrees)

\[
\hat{\sigma} \simeq 5.4^\circ
\]

(6)

(82)

\[
\beta \simeq \sqrt{3} \sigma_g / b_{Lg} g_{\text{min}} = \left( \frac{b_{Lg}^2 (g^2 + \epsilon^2)^2}{g_{\text{min}}^2} \right)^{1/2} \frac{v_m}{g_{\text{min}}}.
\]

Thus, the estimate of \(\beta\) is not just the ratio of the LG gravity to its gravity: it is modified by nonlinear effects (which affect \(\langle v^2 \rangle\) through the function \(R\)), different observational windows (which affect differently \(\langle g^2 \rangle\) and \(\langle v^2 \rangle\)), and observational errors. If all these factors are properly accounted for, then the estimate of \(\beta\) is unbiased.

An optimal estimator is such that is not only unbiased but also has minimal variance. Expanding the logarithmic likelihood (Eq. 81) around its maximum up to second order in \(\beta\) enables one to find an estimator of the variance of \(\beta\). In the strong-coupling regime \((1 - \rho' \ll 1, 1 - \mu_m \ll 1)\), it is

\[
\sigma_{\beta}^2 = \frac{v_{\text{min}}^2 \left( \frac{b_{Lg}^2 \langle g^2 \rangle + \epsilon^2}{g_{\text{min}}^2} \right)^2 (1 - \rho')}{\langle v^2 \rangle g_{\text{min}}^4} \left( \frac{b_{Lg}^2 \langle g^2 \rangle + \epsilon^2}{\langle v^2 \rangle g_{\text{min}}^4} \right) \left( 1 - \rho' \right)^2 + \epsilon^2.
\]

If errors are constant, i.e. do not depend on \(K_{\text{min}}\), then a minimum of the variance corresponds to a maximum of the cross-correlation coefficient \(\rho'\). (The dependence of \(\langle g^2 \rangle\) on \(K_{\text{min}}\) is very weak.) Including higher-order corrections to the above formula does not change this fact. In the present paper, errors are indeed constant: the error due to the mask obviously does not depend on \(K_{\text{min}}\) and shot noise is assumed to be negligible. The window function of the 2MASS survey we have constructed here maximizes \(\rho'\) (see Fig. 3). This is why we call this window, under the assumption of negligible shot noise, optimal. It exactly corresponds to the minimal expectation value of the misalignment angle.

As mentioned earlier, in follow-up work we will estimate shot noise ourselves. If we find that it is in fact not negligible, then it will influence the optimal value of \(K_{\text{min}}\). Shot noise as a function of \(K_{\text{min}}\) monotonically decreases (see Eq. 59). The factor \(1 - \rho'\), starting from the value of \(K_{\text{min}}\) which maximizes \(\rho'\), monotonically increases (see Fig. 2). The interplay between these two opposing effects in Equation (84) shifts the optimal \(K_{\text{min}}\) (corresponding to a minimum of the variance of the estimator of \(\beta\)) to a larger value, compared to the case of negligible shot noise.
Table 1. Parameters of the distribution for the misalignment angle between the 2MASS and CMB dipoles, for various forms of the 2MASS gravity window. First column shows the limiting magnitude of the excluded brightest galaxies, $K_{\min}$. Second column shows the correlation coefficient of the LG velocity and gravity, $\rho$, calculated according to Equation (33). Third column shows the characteristic value of the misalignment angle (in radians), $\theta^*$, calculated using Equation (31). Fourth column shows the corresponding value of the misalignment angle (in radians), including observational errors due to the mask, $\theta^*_s$, calculated using Equation (73). Fifth column shows the expectation value of the misalignment angle (in degrees), $\langle \theta \rangle$. Sixth and seventh columns show, respectively, the quantiles $\theta_{10}$ and $\theta_{90}$ (in degrees), defining the confidence intervals of 10 and 90% (for details see text). The last column shows the observed values of the misalignment angle (in degrees), obtained with and without exclusion of the brightest galaxies.

| $K_{\min}$ | $\rho$ | $\theta^*$ | $\theta^*_s$ | $\langle \theta \rangle$ | $\theta_{10}$ | $\theta_{90}$ | $\theta_{\text{obs}}$ |
|------------|--------|------------|-------------|----------------|-------------|-------------|----------------|
|            | 0.951  | 0.123      | 0.155       | 11.1          | 4.1          | 19.0        | 16.0          |
| 8          | 0.901  | 0.183      | 0.206       | 14.8          | 5.4          | 25.3        | 5.2           |
| 5          | 0.969  | 0.097      | 0.135       | 9.7           | 3.6          | 16.6        | —             |

7 RESULTING DISTRIBUTIONS FOR THE MISALIGNMENT

Figure 3 shows the resulting PDFs for the misalignment angle, for various forms of the 2MASS gravity window. Like previously, dotted line corresponds to the case where the brightest galaxies are not excluded from the calculation of the flux dipole. The vertical stripe shows the value of the misalignment between the CMB dipole and the 2MASS flux dipole as calculated by M03, including all galaxies brighter than $K_s = 13.57$. The ‘observed’ value (16°) is greater than the expectation value for the angle (11.1°), but smaller than $\theta_{90} = 19.0$° (see Table 1).

To decrease the misalignment, in the second step M03 excluded from the analysis all galaxies brighter than $K_{\min} = 8$ mag. A dashed line is plotted for the gravity window corresponding to this case. Consistently with Figure 2 the expectation value for the angle does not decrease; on the contrary, it increases to 14.8° (Table 1). Consequently, one would then expect the misalignment rather to increase. However, M03 noticed a substantial decrease of the misalignment, to about 5.2°. This value is smaller than the corresponding $\theta_{10} = 5.4$°. Therefore, there is less than 10% chance that the decrease might have been accidental. Rather, an error in the analysis is more likely. Indeed, E06 repeated the procedure of M03 for the 2MRS data and essentially did not observe the decrease of the misalignment. 2MRS survey misses faint galaxies (fainter than $K_s = 11.25$), but does not miss bright galaxies. Therefore, if the effect was real, one should observe it also when using the 2MRS data.

Solid line in Figure 3 is plotted using the window excluding 2MASS galaxies brighter than $K_{\min} = 5$ mag. As described in Section 2 we expect this window to be close to optimal. Indeed, the resulting distribution is the narrowest among the three plotted; the expectation value of the misalignment drops to 9.7° and $\theta_{90} = 16.6$° (Table 1). Therefore, with (almost) 90% confidence we can expect the angle to decrease when performing such a preselection on 2MASS galaxies. Of course, this assumes constant mass-to-light ratio for all remaining (i.e., included) galaxies.

The window with $K_{\min} = 8$ mag is not optimal because it excessively mitigates nonlinear effects. This window excludes too many galaxies: while the window with $K_{\min} = 5$ mag excludes all $L_s$ and brighter galaxies closer to the LG than about 5 $h^{-1}$ Mpc, the window with $K_{\min} = 8$ mag does the same for the distance of about 18 $h^{-1}$ Mpc (see Sec. 2.4). The signal from scales 5–18 $h^{-1}$ Mpc is sufficiently ‘linear’ to increase (if included) the correlation between the LG velocity and its measured gravity.

M03 found 375 2MASS galaxies brighter than $K_{\min} = 8$ mag. Based on this number and the relation $N_{\text{excl}} \propto S_{\text{max}}^{-3/2}$, where $N_{\text{excl}}$ is the number of excluded galaxies, we predict about six 2MASS galaxies to be brighter than $K_{\min} = 5$ mag. To reduce the misalignment calculated using their sample, E06 excluded five the brightest galaxies in 2MRS. They noticed a significant decrease of the misalignment, from about 21° to 14°. The five most luminous galaxies in 2MRS are also the five most luminous galaxies in 2MASS. Therefore, exclusion of these galaxies should work also for denser and deeper 2MASS survey.

8 SUMMARY

- An ultimate goal of comparing the CMB dipole to the 2MASS dipole is an estimation of the cosmological parameter $\beta \equiv \Omega_m h^2 / b_L$.
- To obtain an unbiased estimate of $\beta$, a good and standard method is Maximum Likelihood.
- An important ingredient of this Likelihood analysis is the observational window through which the 2MASS flux dipole is measured, called here the gravity window of 2MASS. This window should be properly modelled.
- By definition, the optimal window minimizes variance in the...
estimate of $\beta$; optimizing the 2MASS window is therefore important.

- In this paper, we have modelled the 2MASS gravity window and optimized it under the assumption of negligible shot noise. This optimization has been achieved by excluding contributions to the dipole from the brightest galaxies (Sec. 4). Such an exclusion mitigates nonlinear effects from small scales, which decorrelate the LG velocity from the estimated LG gravity. We have found that the optimal value of the minimal limiting magnitude, $K_{\text{min}}$ (corresponding to maximal limiting flux), is about 5.

- We have also demonstrated how to optimize the window in presence of shot noise. We have shown that the optimal value of $K_{\text{min}}$ will increase.

- The misalignment angle is a sensitive measure of the correlation between the two dipoles: the higher the correlation, the smaller the expectation value of the angle (Eqs. 31–32). We have shown that the optimal value of $\beta_{\text{min}}$ will be close to that optimizing the measurement of $\beta_{\text{est}}$ under the assumption of negligible shot noise. This optimization has been indirectly confirmed by the results of E06.

- In a future work, we plan to perform the optimal measurement of the value of $\beta$ by comparing the CMB dipole to the 2MASS dipole. We will thus have to fully specify the optimal window in presence of shot noise (though it is greater than the expectation value).

- We have predicted that the misalignment is likely to decrease if 2MASS galaxies brighter than $K_{\text{min}} = 5$ mag are excluded from the calculation of the flux dipole. This prediction has been shown to be consistent with the results of E06.

- We have modelled analytically the distribution function for the misalignment angle (Sec. 2 App. A). We have shown that the misalignment estimated by M03 is consistent with the assumed underlying model (though it is greater than the expectation value). We have predicted that the misalignment is likely to decrease if 2MASS galaxies brighter than $K_{\text{min}} = 5$ mag are excluded from the calculation of the flux dipole. This prediction has been shown to be consistent with the results of E06.

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\section*{APPENDIX A: BEYOND THE SMALL-ANGLE LIMIT}

Here we check the accuracy of the small-angle approximation, Equation (30), of the distribution function for the misalignment angle, given in general by Equation (27). First, integrating by parts one can show that for $\nu \gg 1$,

$$\int_{\nu}^{\infty} e^{-\nu^2/2} d\nu' = \frac{1}{\nu^2} e^{-\nu^2/2} + O(\nu^{-5}) + O(\sqrt{\nu} e^{-\nu^2/2}).$$

(A1)

This yields, for the error function a higher-order expansion (than Eq. 29),

$$\text{erf}(s) \simeq 1 - \left(\frac{1}{\sqrt{\pi e}} - \frac{1}{2 \sqrt{\pi e^3}} s\right) e^{-s^2}, \quad s \gg 1. \quad \text{(A2)}$$

Using this expansion in Equation (27) for $\mu > 0$, we obtain

$$p(\theta|\nu) = \sin \theta \left(1 + \cos^2 \theta \frac{\theta^2}{\bar{\nu}^2} \right) \exp \left(-\sin^2 \theta \frac{\theta^2}{2 \bar{\nu}^2} \right).$$

(A3)

where $\theta_c$ is given by Equation (31) and $\theta < \pi/2$. For $\theta_c \rightarrow 0$, this distribution simplifies to the Rayleigh distribution (Eq. 30), as expected.

For $\mu < 0$, one can show in a similar way that

$$\text{erf}(q\mu) \simeq -1 + \left(\frac{1}{\sqrt{\pi e}} - \frac{1}{2 \sqrt{\pi e^3}} q \mu\right) e^{-q^2 \mu^2}, \quad q > 1. \quad \text{(A4)}$$

Using the latter expansion in Equation (27) yields

$$p(\theta|\nu) \equiv 0 \quad \text{for} \quad \pi/2 < \theta < \pi. \quad \text{(A5)}$$

Density distribution (A3)–(A5) has analytical cumulative distribution function:

$$F(\theta) = \left\{ \begin{array}{ll}
1 - \cos \theta \exp \left(-\frac{\sin^2 \theta}{2 \bar{\nu}^2} \right), & \theta \leq \pi/2, \\
1, & \pi/2 < \theta \leq \pi.
\end{array} \right. \quad \text{(A6)}$$

This allows for a straightforward estimation of the quantiles. Moreover, using the fact that $p = dF/d\theta$ and integrating by parts, the expectation value of the angle can be readily calculated:

$$\langle \theta | \nu \rangle = \sqrt{\frac{\pi^2}{2} \theta_c} \text{erf} \left(\theta_{c^{-1}} \right) \simeq \sqrt{\frac{\pi^2}{2} \theta_c} - O \left(\theta_{c^{-1}} \theta_{c^{-2}} \right).$$

(A7)

Thus, for $\theta_{c}^2 \ll 1$, distribution (A3)–(A5) has (almost) identical mean to the Rayleigh distribution (Eq. 32).

In practice, the total misalignment angle is a convolution of the cosmologically-organized misalignment, $\theta_{c}$, and the misalignment due to mask, $\theta_{m}$. This convolution is not as simple as when both $\theta_{c}$ and $\theta_{m}$ are bivariate Gaussians. However, non-Gaussianity of $\theta_{c} + \theta_{m}$ is smaller than of the variable $\theta'$ with $\theta' = \sqrt{\theta_{c}^2 + \sigma^2}$ (because $\theta_{m}$ is Gaussian). We see in Table 1 that $\theta'$ is at most 0.2. For $\theta' \lesssim 0.2$, both distribution (A3)–(A5) and the Rayleigh distribution approximate the exact one (Eq. 27) very well. Specifically, since then $\theta_{c}^2 \ll 1$, we can expand distribution (A3)–(A5), obtaining

$$p(\theta|\nu) \simeq M(\theta) \frac{\theta}{\theta_{c}^2} \exp \left( -\frac{\theta^2}{2 \theta_{c}^2} \right), \quad \text{(A8)}$$

where

$$M(\theta) = 1 + \frac{\theta_{c}^2}{6} \theta^2 + \frac{\theta_{c}^4}{60 \theta_{c}^2} + O \left(\theta_{c}^{-6} \theta \right). \quad \text{(A9)}$$

It is straightforward to check that approximate distribution (A8) is properly normalized. A simple calculation yields

$$\langle \theta^2 | \nu \rangle \simeq 2 \theta_{c}^2 \left( 1 + \frac{\theta_{c}^2}{3} \theta \right) + O \left(\theta_{c}^{0} \theta \right). \quad \text{(A10)}$$

Hence, for $\theta' = 0.2$, the Rayleigh distribution approximates the second moment of distribution (A8) (which, in turn, is then an excellent approximation of the exact distribution) to 1.3% accuracy. Similarly simple calculations can be performed for other even moments.

Summing up, the distribution function for the misalignment angle between the CMB and 2MASS dipoles can be very well approximated by its small-angle limit, Rayleigh distribution. It may be worth noting for other applications that, while for $\theta' > 0.2$ the

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exact distribution starts to deviate from the Rayleigh form, it is still well approximated by distribution \( A3 \)–\( A5 \), up to \( \theta' \sim 0.5 \) (except for the very tail). In particular, the values of the quantiles \( \theta_{10} \) and \( \theta_{90} \) and of the mean angle remain within 2% from the exact values.

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