ALGORITHMS FOR THE STRONG CHROMATIC INDEX OF HALIN GRAPHS, DISTANCE-HEREDITARY GRAPHS AND MAXIMAL OUTERPLANAR GRAPHS

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Abstract. We show that there exist linear-time algorithms that compute the strong chromatic index of Halin graphs, of maximal outerplanar graphs and of distance-hereditary graphs.

1 Introduction

Definition 1. Let $G = (V, E)$ be a graph. A strong edge coloring of $G$ is a proper edge coloring such that no edge is adjacent to two edges of the same color.

Equivalently, a strong edge coloring of $G$ is a vertex coloring of $L(G)^2$, the square of the linegraph of $G$. The strong chromatic index of $G$ is the minimal integer $k$ such that $G$ has a strong edge coloring with $k$ colors. We denote the strong chromatic index of $G$ by $s\chi'(G)$.

Recently it was shown that the strong chromatic index is bounded by

$$(2 - \epsilon)\Delta^2$$

for some $\epsilon > 0$, where $\Delta$ is the maximal degree of the graph [22].$^3$ Earlier, Andersen showed that the strong chromatic index of a cubic graph is at most ten [1].

Let $\mathcal{G}$ be the class of chordal graphs, or the class of cocomparability graphs, or the class of weakly chordal graphs. If $G \in \mathcal{G}$ then also $L(G)^2 \in \mathcal{G}$ and it follows that the strong chromatic index can be computed in polynomial time for these classes [3–5]. Also for graphs of bounded treewidth there exists a polynomial time algorithm that computes the strong chromatic index [24].$^4$

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$^3$ In their paper Molloy and Reed state that $\epsilon \geq 0.002$ when $\Delta$ is sufficiently large.

$^4$ This algorithm checks in $O(n(s + 1)^{t})$ time whether a partial $k$-tree has a strong edge coloring that uses at most $s$ colors. Here, the exponent $t = 2^{\Omega(k \log s)}$. 

Definition 2. Let $T$ be a tree without vertices of degree two. Consider a plane embedding of $T$ and connect the leaves of $T$ by a cycle that crosses no edges of $T$. A graph that is constructed in this way is called a Halin graph.

Halin graphs have treewidth at most three. Furthermore, if $G$ is a Halin graph of bounded degree, then also $L(G)^2$ has bounded treewidth and thus the strong chromatic index of $G$ can be computed in linear time. Recently, Ko-Wei Lih, et al., proved that a cubic Halin graph other than one of the two ‘necklaces’ $Ne_2$ (the complement of $C_6$) and $Ne_4$, has strong chromatic index at most 7. The two exceptions have strong chromatic index 9 and 8, respectively. If $T$ is the underlying tree of the Halin graph, and if $G \neq Ne_2$ and $G$ is not a wheel $W_n$ with $n \neq 0 \mod 3$, then Ping-Ying Tsai, et al., show that the strong chromatic index is bounded by $s\chi'(T) + 3$. (See [25, 26] for earlier results that appeared in regular papers.)

If $G$ is a Halin graph then $L(G)^2$ has bounded rankwidth. In [10] it is shown that there exists a polynomial algorithm that computes the chromatic number of graphs with bounded rankwidth, thus the strong chromatic index of Halin graphs can be computed in polynomial time. In passing, let us mention the following result. A class of graphs $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that $\chi(G) \leq f(\omega(G))$ for $G \in \mathcal{G}$. Here $\chi(G)$ is the chromatic number of $G$ and $\omega(G)$ is the clique number of $G$. Recently, Dvořák and Král showed that for every $k$, the class of graphs with rankwidth at most $k$ is $\chi$-bounded [8]. Obviously, the graphs $L(G)^2$ have a uniform $\chi$-bound for graphs $G$ in the class of Halin graphs.

In Section 2 we show that there exists a linear-time algorithm that computes the strong chromatic index of Halin graphs. In Section 3 we show that there exists a linear-time algorithm that computes the strong chromatic index of distance-hereditary graphs. In Section 4 we show that there exists a linear-time algorithm that computes the strong chromatic index of maximal outerplanar graphs.

2 The strong chromatic index of Halin graphs

The following lemma is easy to check.

Lemma 1 (Ping-Ying Tsai). Let $C_n$ be the cycle with $n$ vertices and let $W_n$ be the wheel with $n$ vertices in the cycle. Then

$$s\chi'(C_n) = \begin{cases} 3 & \text{if } n = 0 \mod 3 \\ 5 & \text{if } n = 5 \\ 4 & \text{otherwise} \end{cases} \quad s\chi'(W_n) = \begin{cases} n + 3 & \text{if } n = 0 \mod 3 \\ n + 5 & \text{if } n = 5 \\ n + 4 & \text{otherwise} \end{cases}$$

The results of Ko-Wei Lih and Ping-Ying Tsai, et al., were presented at the Sixth Cross-Strait Conference on Graph Theory and Combinatorics which was held at the National Chiao Tung University in Taiwan in 2011.
A double wheel is a Halin graph in which the tree $T$ has exactly two vertices that are not leaves.

**Lemma 2 (Ping-Ying Tsai).** Let $W$ be a double wheel where $x$ and $y$ are the vertices of $T$ that are not leaves. Then $s\chi'(T) = d(x) + d(y) - 1$ where $d(x)$ and $d(y)$ are the degrees of $x$ and $y$. Furthermore,

$$s\chi'(W) = \begin{cases} 
    s\chi'(T) + 4 = 9 & \text{if } d(x) = d(y) = 3, \text{ i.e., if } W = \bar{C}_6 \\
    s\chi'(T) + 2 = d(y) + 4 & \text{if } d(y) > d(x) = 3 \\
    s\chi'(T) + 1 = d(x) + d(y) & \text{if } d(y) \geq d(x) > 3.
\end{cases}$$

Let $G$ be a Halin graph with tree $T$ and cycle $C$. Then obviously,

$$s\chi'(G) \leq s\chi'(T) + s\chi'(C). \hspace{1cm} (1)$$

The linegraph of a tree is a claw-free blockgraph. Since a sun $S_r$ with $r > 3$ has a claw, $L(T)$ has no induced sun $S_r$ with $r > 3$. It follows that $L(T)^2$ is a chordal graph [18] (see also [3]; in this paper Cameron proves that $L(G)^2$ is chordal for any chordal graph $G$). Notice that

$$s\chi'(T) = \chi(L(T)^2) = \omega(L(T)^2) \leq 2\Delta(G) - 1 \implies s\chi'(G) \leq 2\Delta(G) + 4. \hspace{1cm} (2)$$

### 2.1 Cubic Halin graphs

In this subsection we outline a simple linear-time algorithm for the cubic Halin graphs.

**Theorem 1.** There exists a linear-time algorithm that computes the strong chromatic index of cubic Halin graphs.

**Proof.** Let $G$ be a cubic Halin graph with plane tree $T$ and cycle $C$. Let $k$ be a natural number. We describe a linear-time algorithm that checks if $G$ has a strong edge coloring with at most $k$ colors. By Equation (2) we may assume that $k$ is at most 10. Thus the correctness of this algorithm proves the theorem.

Root the tree $T$ at an arbitrary leaf $r$ of $T$. Consider a vertex $x$ in $T$. There is a unique path $P$ in $T$ from $r$ to $x$ in $T$. Define the subtree $T_x$ at $x$ as the maximal connected subtree of $T$ that does not contain an edge of $P$. If $x = r$ then $T_x = T$.

Let $H(x)$ be the subgraph of $G$ induced by the vertices of $T_x$. Notice that, if $x \neq r$ then the edges of $H(x)$ that are not in $T$ form a path $Q(x)$ of edges in $C$.

For $x \neq r$ define the boundary $B(x)$ of $H(x)$ as the following set of edges.

(a) The unique edge of $P$ that is incident with $x$.
(b) The two edges of $C$ that connect the path $Q(x)$ of $C$ with the rest of $C$.
(c) Consider the endpoints of the edges mentioned in (a) and (b) that are in $T_x$. Add the remaining two edges that are incident with each of these endpoints to $B(x)$.
Thus the boundary $B(x)$ consists of at most 9 edges. The following claim is easy
to check. It proves the correctness of the algorithm described below. Let $e$ be an
dege of $H(x)$. Let $f$ be an edge of $G$ that is not an edge of $H(x)$. If $e$ and $f$ are at
distance at most 1 in $G$ then $e$ or $f$ is in $B(x)$.$^6$

Consider all possible colorings of the edges in $B(x)$. Since $B(x)$ contains at
most 9 edges and since there are at most $k$ different colors for each edge, there
are at most

$$k^9 \leq 10^9$$
different colorings of the edges in $B(x)$.

The algorithm now fills a table which gives a boolean value for each coloring
of the boundary $B(x)$. This boolean value is $\text{TRUE}$ if and only if the coloring of
the edges in $B(x)$ extends to an edge coloring of the union of the sets of edges
in $B(x)$ and in $H(x)$ with at most $k$ colors, such that any pair of edges in this
set that are at distance at most one in $G$, have different colors. These boolean
values are computed as follows. We prove the correctness by induction on the
size of the subtree at $x$.

First consider the case where the subtree at $x$ consists of the single vertex $x$. Then $x \neq r$ and $x$ is a leaf of $T$. In this case $B(x)$ consists of three edges, namely the three edges that are incident with $x$. These are two edges of $C$ and one edge of $T$. If the colors of these three edges in $B$ are different then the boolean value is set to $\text{TRUE}$. Otherwise it is set to $\text{FALSE}$. Obviously, this is a correct assignment.

Next consider the case where $x$ is an internal vertex of $T$. Then $x$ has two
children in the subtree at $x$. Let $y$ and $z$ be the two children and consider the
two subtrees rooted at $y$ and $z$.

The algorithm that computes the tables for each vertex $x$ processes the sub-
trees in order of increasing number of vertices. (Thus the roots of the subtrees
are visited in postorder). We now assume that the tables at $y$ and $z$ are computed
correctly and show how the table for $x$ is computed correctly and in constant
time. That is, we prove that the algorithm described below computes the table
at $x$ such that it contains a coloring of $B(x)$ with a value $\text{TRUE}$ if and only if there
exists an extension of this coloring to the edges of $H(x)$ and $B(x)$ such that any
two different edges $e$ and $f$ at distance at most one in $G$, each one in $H(x)$ or in
$B(x)$, have different colors.

Consider a coloring of the edges in the boundary $B(x)$. The boolean value in
the table of $x$ for this coloring is computed as follows. Notice that

(i) $B(y) \cap B(z)$ consists of one edge and this edge is not in $B(x)$, and
(ii) $B(x) \cap B(y)$ consists of at most four edges, namely the edge $(x, y)$ and the
three edges of $B(y)$ that are incident with one vertex of $C \cap H(y)$. Likewise,
$B(x) \cap B(z)$ consists of at most four edges.

$^6$ Two edges in $G$ are at distance at most one if the subgraph induced by their endpoints
is either $P_3$, or $K_3$ or $P_4$. We assume that it can be checked in constant time if two edges
e and $f$ are at distance at most one. This can be achieved by a suitable data structure.
The algorithm varies the possible colorings of the edge in $B(y) \cap B(z)$. Colorings of $B(x)$, $B(y)$ and $B(z)$ are consistent if the intersections are the same color and the pairs of edges in

$$B(x) \cup B(y) \cup B(z)$$

that are at distance at most one in $G$ have different colors. A coloring of $B(x)$ is assigned the value TRUE if there exist colorings of $B(y)$ and $B(z)$ such that the three colorings are consistent and $B(y)$ and $B(z)$ are assigned the value TRUE in the tables at $y$ and at $z$ respectively. Notice that the table at $x$ is built in constant time.

Consider a coloring of $B(x)$ that is assigned the value TRUE. Consider colorings of the edges of $B(y)$ and $B(z)$ that are consistent with $B(x)$ and that are assigned the value TRUE in the tables at $y$ and $z$. By induction, there exist extensions of the colorings of $B(y)$ and $B(z)$ to the edges of $H(y)$ and $H(z)$. The union of these extensions provides a $k$-coloring of the edges in $H(x)$.

Consider two edges $e$ and $f$ in $B(x) \cup B(y) \cup B(z)$. If their distance is at most one then they have different colors since the coloring of $B(x) \cup B(y) \cup B(z)$ is consistent. Let $e$ and $f$ be a pair of edges in $H(x)$. If they are both in $H(y)$ or both in $H(z)$ then they have different colors. Assume that $e$ is in $H(y)$ and assume that $f$ is not in $H(y)$. If $e$ and $f$ are at distance at most one, then $e$ or $f$ is in $B(y)$. If they are both in $B(y)$, then they have different colors, due to the consistency. Otherwise, by the induction hypothesis, they have different colors. This proves the claim on the correctness.

Finally, consider the table for the vertex $x$ which is the unique neighbor of $r$ in $T$. By the induction hypothesis, and the fact that every edge in $G$ is either in $B(x)$ or in $H(x)$, $G$ has a strong edge coloring with at most $k$ colors if and only if the table at $x$ contains a coloring of $B(x)$ with three different colors for which the boolean is set to TRUE.

This proves the theorem. 

\[ \Box \]

Remark 1. The involved constants in this algorithm are improved considerably by the recent results of Ko-Wei Lih, Ping-Ying Tsai, et al.

2.2 Halin graphs of general degree

Theorem 2. There exists a linear-time algorithm that computes the strong chromatic index of Halin graphs.

Proof. The algorithm is similar to the algorithm for the cubic case.

Let $G$ be a Halin graph, let $T$ be the underlying plane tree, and let $C$ be the cycle that connects the leaves of $T$. Since $L(T)^2$ is chordal the chromatic number of $L(T)^2$ is equal to the clique number of $L(T)^2$, which is

$$s\chi'(T) = \max \{ d(u) + d(v) - 1 \mid (u,v) \in E(T) \},$$
where \(d(u)\) is the degree of \(u\) in the tree \(T\). By Formula (1) and Lemma 1 the strong chromatic index of \(G\) is one of the six possible values

\[ s\chi'(T), s\chi'(T) + 1, \ldots, s\chi'(T) + 5. \]

Root the tree at some leaf \(r\) and consider a subtree \(T_x\) at a node \(x\) of \(T\). Let \(H(x)\) be the subgraph of \(G\) induced by the vertices of \(T_x\). Let \(y\) and \(z\) be the two boundary vertices of \(H(x)\) in \(C\).

We distinguish the following six types of edges corresponding to \(H(x)\).

1. The set of edges in \(T_x\) that are adjacent to \(x\).
2. The edge that connects \(x\) to its parent in \(T\).
3. The edge that connects \(y\) to its neighbor in \(C\) that is not in \(T_x\).
4. The set of edges in \(H(x)\) that have endpoint \(y\).
5. The edge that connects \(z\) to its neighbor in \(C\) that is not in \(T_x\).
6. The set of edges in \(H(x)\) that have endpoint \(z\).

When \(x\) is adjacent to \(y\) then we make a separate type for the edge \((x, y)\) and similar in the case where \(x\) is adjacent to \(z\).

Notice that the set of edges of every type has bounded cardinality, except the first type.

Consider a 0/1-matrix \(M\) with rows indexed by the six to eight types of edges and columns indexed by the colors. A matrix entry \(M_{ij}\) is 1 if there is an edge of the row-type \(i\) that is colored with the color \(j\) and otherwise this entry is 0. Since \(M\) has at most 8 rows, the rank over \(GF[2]\) of \(M\) is at most 8.

Two colorings are equivalent if there is a permutation of the colors that maps one coloring to the other one. Let \(S \subseteq \{1, \ldots, 8\}\) and let \(W(S)\) be the set of colors that are used by edges of type \(i\) for all \(i \in S\). A class of equivalent colorings is fixed by the set of cardinalities

\[ \{ |W(S)| \mid S \subseteq \{1, \ldots, 8\} \}. \]

We claim that the number of equivalence classes is constant. The number of ones in the row of the first type is the degree of \(x\) in \(H(x)\). Every other row has at most 3 ones. This proves the claim.

Consider the union of two subtrees, say at \(x\) and \(x'\). The algorithm considers all equivalence classes of colorings of the union, and checks, by table look-up, whether it decomposes into valid colorings of \(H(x)\) and \(H(x')\). An easy way to do this is as follows. First double the number of types, by distinguishing the edges of \(H(x)\) and \(H(x')\). Then enumerate all equivalence classes of colorings. Each equivalence class is fixed by a sequence of \(2^{16}\) numbers, as above. By table look-up, check if an equivalence class restricts to a valid coloring for each of \(H(x)\) and \(H(x')\). Since this takes constant time, the algorithm runs in linear time.

This proves the theorem. \(\square\)

Actually, according to the recent results of Ping-Ying Tsai, et al., the strong chromatic index of \(G\) is at most \(s\chi'(T) + 3\) except when \(G\) is a wheel or \(\overline{C}_6\).
3 Distance-hereditary graphs

Definition 3 ([15]). A graph $G$ is distance hereditary if any two nonadjacent vertices in a component of any induced subgraph $H$ are at the same distance in $H$ as they are in the graph $G$.

In other words, any two chordless paths between two nonadjacent vertices is of the same length. Distance-hereditary graphs are exactly the graphs that have rankwidth one [6]. In this section we prove that there is a linear-time algorithm that computes the strong chromatic index of distance-hereditary graphs. Distance-hereditary graphs are perfect. They are the graphs without induced gem, house, hole or domino. Cameron proves in [5] that, for $k \geq 4$, if $G$ has no induced cycles of length more than four then also $L(G)^2$ has no such induced cycles. It follows that, if $G$ is distance hereditary then $L(G)^2$ is perfect. Therefore, to compute the chromatic number of $L(G)^2$ it suffices to compute the clique number.

A pendant vertex in a graph is a vertex of degree one. A twin is a pair of vertices $x$ and $y$ with the same open or the same closed neighborhood. When $x$ and $y$ are adjacent then the twin is called a true twin and otherwise it is called a false twin. A $P_4$ is a path with four vertices.

Theorem 3 ([2]). A graph $G$ is distance hereditary if and only if $G$ is obtained from an edge by a sequence of the following operations.

(a) Creation a pendant vertex.
(b) Creation of a twin.

Lemma 3. Let $G$ be a graph and consider the graph $G'$ obtained from $G$ by creating a false twin $x'$ of a vertex $x$ in $G$. Then $L(G')^2$ is obtained from $L(G)^2$ by a series of true twin operations.

Proof. Let $a_1, \ldots, a_s$ be the neighbors of $x$ in $G$. By definition of $L(G)^2$, each edge $(x', a_i)$ is a true twin of the edge $(x, a_i)$ in $L(G')^2$. \hfill $\square$

Definition 4. A graph $G$ is a cograph if $G$ has no induced $P_4$.

A cograph is obtained from a graph consisting of one vertex by a series of twin operations. Chordal cographs are the graphs without induced $P_4$ and $C_4$. These are also called trivially perfect.
Lemma 4. If $G$ is a cograph then $L(G)^2$ is trivially perfect.

Proof. A cograph with at least two vertices is either the join or the union of two cographs $G_1$ and $G_2$. Assume that $G$ is the join of two cographs $G_1$ and $G_2$. The set of edges with one endpoint in $G_1$ and the other in $G_2$ are a clique in $L(G)^2$. Furthermore, this set of edges is adjacent to every edge that is contained in $G_i$ for $i \in \{1, 2\}$. In other words, every component of $L(G)^2$ has a universal vertex, i.e., a vertex adjacent to all other vertices. The graphs that satisfy this property are exactly the graphs in which every component is the comparability graph of a tree and these are exactly the graphs without induced $P_4$ and $C_4$ [27].

Notice that Lemma 4 provides a linear-time algorithm for computing the strong chromatic index of cographs. A cotree decomposition can be obtained in linear time. Assume that $G$ is the join of two cographs $G_1$ and $G_2$. Then every edge with both ends in $G_1$ is adjacent in $L(G)^2$ to every edge with both ends in $G_2$. Let $X$ be the set of edges with one endpoint in $G_1$ and the other endpoint in $G_2$. By dynamic programming on the cotree, compute the clique numbers of $L(G_1)^2$ and $L(G_2)^2$. Add $|X|$ to the sum of both. If $G$ is the union of $G_1$ and $G_2$ then the strong chromatic index of $G$ is the maximum of the clique numbers of $L(G_1)^2$ and $L(G_2)^2$. This proves the following theorem.

Theorem 4. There exists a linear-time algorithm that computes the strong chromatic index of cographs.

Lemma 5. If $G$ is distance hereditary then every neighborhood in $L(G)^2$ induces a trivially perfect graph.

Proof. We prove the theorem by induction on the elimination ordering of $G$ by pendant vertices and elements of twins.

First, assume that $G'$ is obtained from $G$ by creating a false twin $x'$ of a vertex $x$ in $G$. By Lemma 3 $L(G')^2$ is obtained from $L(G)^2$ by a series of true twin operations. In that case the claim follows easily, by induction.

Secondly, consider the operation which adds a pendant vertex $x'$, made adjacent to a vertex $x$ in $G$. Let $a_1, \ldots, a_s$ be the neighbors of $x$ in $G$. Notice that the adjacencies of the edge $(x, x')$ in $L(G')^2$ are of the following types of edges in $G$.

(a) All edges $(x, a_i)$, $i \in \{1, \ldots, s\}$. Call this set of edges $X$.
(b) The edges $(a_i, a_j) \in E(G)$, for $i, j \in \{1, \ldots, s\}$.
(c) Edges $(a_i, u)$, for $i \in \{1, \ldots, s\}$ and $u \in N_G(a_i) \setminus N_G[x]$.

Call two vertices in $N_G(x)$ equivalent if they have the same neighbors in the graph $G - N_G[x]$. Since there is no house, hole, domino or gem every equivalence class is joined to or disjoint from every other equivalence class. Let $H$ be the graph with vertex set the set of equivalence classes and edge set the pairs of equivalence classes that are joined. Since $G$ has no gem, the graph $H$ has no induced $P_4$ and so it is a cograph. Furthermore, by Lemma 4 $L(H)^2$ is trivially perfect.
Consider the components of $G - N_G[x]$. For any two components $C_1$ and $C_2$ their neighborhoods $N_G(C_1)$ and $N_G(C_2)$ are either disjoint or ordered by inclusion. First consider the components that have a maximal neighborhood in $N_G(x)$ and remove all other components. Consider the equivalence classes defined by these components. The graph on these equivalence classes is a cograph and the square of the line graph is a chordal cograph. Next, consider such an equivalence class $Q$ with at least two vertices. Remove the components $C$ of $G - N_G[x]$ with $N(C) = Q$. If there are some components left of which the neighborhood is properly contained in $Q$ then partition the vertices of $Q$ into secondary equivalence classes. If there are no more components with their neighborhood contained in $Q$ then define the secondary equivalence classes as sets of single vertices. As above, for each equivalence class $Q$, the secondary equivalence classes form a cograph $H_Q$. Also, $L(H_Q)^2$ is trivially perfect. Continuation of this process defines a chordal cotree on the subgraph of $L(G)^2$ induced by the edges of types (b) and (c). Notice that the set $X$ of edges is universal in the neighborhood of $(x, x')$ in $L(G)^2$.

Finally, consider the case where $G'$ is obtained from $G$ by creating a true twin $x'$ of a vertex $x$ in $G$. Subdivide this operation into two steps. First create a false twin. Let $G^*$ be the graph obtained in this manner. We proved above that every neighborhood in $L(G^*)^2$ is trivially perfect. Secondly, adding the edge $(x, x')$ to $G^*$ is similar to the operation of adding a pendant vertex. The set $X$ of edges as described above, now consists of pairs of true twins in $L(G)^2$. The other types of adjacencies of $(x, x')$, as described in (b) and (c), are the same as above.

This proves the lemma. □

**Theorem 5.** There exists a linear-time algorithm that computes the strong chromatic index of distance-hereditary graphs.

**Proof.** Let $G$ be distance hereditary. Consider a rank decomposition of $G$ of rankwidth one. This is a pair $(T, f)$ where $T$ is a rooted binary tree and where $f$ is a bijection from the vertices in $G$ to the leaves of $T$. Consider a subtree $T_e$ of $T$ rooted at some edge $e$ of $T$. Define $G_e$ as the subgraph of $G$ induced by the vertices that are mapped to leaves in $T_e$. Let $S_e$ be the set of vertices of $G_e$ that have neighbors in $G - V(G_e)$. The set $S_e$ is called the twinset of $G_e$ [6]. All vertices of $S_e$ have the same neighbors in $G - V(G_e)$.

Consider an edge $e$ of $T$ and let $e_1$ and $e_2$ be the two children of $e$ in $T$. The graph $G_e$ is obtained from $G_{e_1}$ and $G_{e_2}$ by a join or by a union of the twinsets $S_{e_1}$ and $S_{e_2}$. The twinset $S_e$ of $G_e$ is either one of $S_{e_1}$ and $S_{e_2}$ or it is the union of the two [6].

Let $e$ be a line in $T$ with children $e_1$ and $e_2$. Let $S_1$ and $S_2$ be the twinsets of $G_{e_1}$ and of $G_{e_2}$ and assume that there is a join between $S_1$ and $S_2$. Let $X$ be the set of edges between $S_1$ and $S_2$. For $i \in \{1, 2\}$ choose a maximal clique $\Omega_i$ in each $L(S_i)^2$ such that the set of end-vertices of edges in $\Omega_i$ has a maximal number of neighbors in $G_{e_i} - S_i$. Let $\omega_i$ be the number of edges in this maximal clique. Let $N_i$ be this number of neighbors. The algorithm keeps track of the maximal value of $|X| + \omega_1 + \omega_2 + N_1 + N_2$. It is easy to see that this algorithm can be implemented to run in linear time. □
4 Maximal outerplanar graphs

A maximal outerplanar graph $G$ is a ternary tree (i.e., every vertex in $T$ has degree at most three) of triangles, where two triangles that are adjacent in the tree share an edge (see e.g., [17]).

In [14] Hocquard, et al., prove that for every outerplanar graph $G$ with maximal degree $\Delta \geq 3$,

$$s\chi'(G) \leq 3(\Delta - 1).$$

They also prove, among various other NP-completeness results, that strong edge 4-coloring is NP-complete for planar bipartite graphs with maximal degree three and any fixed girth.

**Definition 5.** An extended triangle in a maximal outerplanar graph consists of a triangle plus all the edges that are incident with some vertex of the triangle.

All edges of an extended triangle must be colored different. Let $\phi$ be the maximal number of edges of all extended triangles in $G$. In the following theorem we prove that there exists a strong edges coloring that uses $\phi$ colors.

**Theorem 6.** There exists a linear-time algorithm that computes the strong chromatic index of maximal outerplanar graphs.

**Proof.** The algorithm colors the edges in a greedy manner as follows. First we make one leaf node of $T$ as the root. Then we traverse $T$ in a breath-first manner. When we reach a node $v$, we color the uncolored edges of its corresponding extended triangle $\tau$ so that the colors used for uncolored edges are different from the colors of those colored edges in $\tau$. As the number of edges of $\tau$ is at most $\phi$, $\phi$ colors are sufficient to do the coloring of edges in $\tau$. We proceed to color the edges in other extended triangles for other nodes in $T$ via the breadth-first search traversal order. At the end of the traversal, we finish the coloring of all edges in $G$ using only $\phi$ colors.

For the correctness of the algorithm, we argue as follows. It is easy to see that for any two edges of $(e, e')$ within distance two, they must both appear in some extended triangle $\tau$ which implies that they will obtain different colors when $\tau$ is visited in the breadth-first traversal. Hence, the above coloring is thus a strong edge coloring for $G$, and we obtain that $s\chi'(G) = \phi$. \qed

5 Concluding remarks

If $G$ is a circular-arc graph then $L(G)^2$ is also a circular-arc graph [11]. Unfortunately, coloring a circular-arc graph is NP-complete [21]. When $G$ is AT-free then also $L(G)^2$ is AT-free [4]. As far as we know the complexity of coloring AT-free graphs is an open problem. There is some hope, since the maximum independent set problem is polynomial for this class of graphs.
For Halin graphs we tried to prove that there is an optimal strong edge-coloring such that the edges in the cycle can be colored with colors from a fixed set of constant size. If true, then this would probably improve the time-bound for the strong chromatic index problem on Halin graphs. Moser and Sikdar prove that the maximum induced matching problem on planar graphs is fixed-parameter tractable [23]. As far as we know the parameterized complexity of the strong chromatic index problem on planar graphs is open. Computing a maximum induced matching in planar graphs, or in bipartite graphs is NP-complete [4].

An example of a distance-hereditary graph $G$ for which $L(G)^2$ is not chordal is depicted in Figure 2.

![Fig. 2. A distance-hereditary graph G for which L(G)^2 is not chordal.](image)

Probably the following conjecture is true. If that is the case then the strong chromatic index can be computed in polynomial time for graphs of bounded rankwidth [8, 10].

Conjecture 1. There exists a function $\rho: \mathbb{N} \rightarrow \mathbb{N}$ for which the following holds. Let $G$ be a graph of rankwidth $k$. Then the rankwidth of $L(G)^2$ is at most $\rho(k)$.

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