Control of accuracy on Taylor-collocation method to solve the weakly regular Volterra integral equations of the first kind by using the CESTAC method

Samad Noeiaghdam, a, * Denis Sidorov b,c, † and Valery Sizikov d, ‡

aDepartment of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
bIrkutsk National Research Technical University, Irkutsk, Russia.
cEnergy Systems Institute of Russian Academy of Sciences.
dITMO University, St.Petersburg, Russia.

Abstract

Finding the optimal parameters and functions of iterative methods is among the main problems of the Numerical Analysis. For this aim, a technique of the stochastic arithmetic (SA) is used to control of accuracy on Taylor-collocation method for solving first kind weakly regular integral equations (IEs). Thus, the CESTAC method is applied and instead of usual mathematical softwares the CADNA library is used. Also, the convergence theorem of presented method is illustrated. In order to apply the CESTAC method we will prove a theorem that it will be our licence to use the new termination criterion instead of traditional absolute error. By using this theorem we can show that number of common significant digits (NCSDs) between two successive approximations are almost equal to NCSDs between exact and numerical solution. Finally, some examples are solved by using the Taylor-collocation method based on the CESTAC method. Several tables of numerical solutions based on the both arithmetics are presented. Comparison between number of iterations are demonstrated by using the floating point arithmetic (FPA) for different values of $\varepsilon$.

Keywords: Stochastic arithmetic, Floating point arithmetic, CESTAC method, CADNA library, Taylor-collocation method, First kind Volterra integral equation.

1 Introduction

The first kind IEs are among of the important problems in the Applied Mathematics which have many applications in mathematical sciences and engineering [24 25 28 29 39 43 44 52 62 63 58 53 50 54 55]. Many authors have investigated analytical or numerical methods to solve

*E-mail addresses: s.noeiaghdam.sci@iauctb.ac.ir; samadnoeiaghdam@gmail.com
†E-mail addresses: contact.dns@gmail.com
‡E-mail addresses: sizikov2000@mail.ru
1Controle et Estimation Stochastique des Arrondis de Calculs
2Control of Accuracy and Debugging for Numerical Applications
the first kind IEs \[16, 29, 35, 36, 43, 50, 52, 54, 55\]. Since finding the solution of the first kind IEs by using analytical methods is challenging, especially for applications, thus the numerical methods suggested to solve the approximate solutions specially for ill-posed or singular IEs of the first kind.

In order to solve the the first kind IEs we have several numerical methods such as collocation method \[21, 23, 31, 32, 33, 46\]. Adomian decomposition method \[1, 13, 15\], homotopy analysis method \[25, 28, 29, 42, 43\] and many others \[8, 10, 14, 37, 61, 58, 53, 56, 54, 55\].

The aim of this work is to investigate the following first kind classical Volterra IE

\[
\int_{0}^{t} k(r, s)v(s)ds = f(r), \quad 0 \leq r \leq T \leq 1, \tag{1}
\]

where the kernel \(k(r, s)\) is discontinuous along continuous curves \(\rho_{i}, i = 1, 2, \cdots, m - 1\) as

\[
\begin{aligned}
  k_{1}(r, s), & \quad 0 = \rho_{0}(r) \leq s \leq \rho_{1}(r), \\
  k_{2}(r, s), & \quad \rho_{1}(r) \leq s \leq \rho_{2}(r), \\
  & \quad \vdots \\
  k_{m}(r, s), & \quad \rho_{m-1}(r) \leq s \leq \rho_{m}(r) = r \leq 1.
\end{aligned} \tag{2}
\]

Finally, we can rewrite Eq. (1) with conditions (2) as follows

\[
\int_{\rho_{0}(r)}^{\rho_{1}(r)} k_{1}(r, s)v(s)ds + \int_{\rho_{1}(r)}^{\rho_{2}(r)} k_{2}(r, s)v(s)ds + \cdots + \int_{\rho_{m-1}(r)}^{\rho_{m}(r)} k_{m}(r, s)v(s)ds = f(r), \tag{3}
\]

where \(f(0) = 0\). The pressed form of Eq. (3) is obtained in the following form

\[
\sum_{p=1}^{m} \int_{\rho_{p-1}(r)}^{\rho_{p}(r)} k_{p}(r, s)v(s)ds = f(r). \tag{4}
\]

In the last decades, many mathematical schemes have been presented to solve the IEs with jump discontinuous kernels \[22, 38, 39, 47, 48, 49, 50, 51, 52\]. In this paper, by combining the collocation method with Taylor polynomials the Volterra IE (4) is solved. Recently, many linear and non-linear problems have solved by using Taylor-collocation method \[21, 23, 31, 32, 33, 46\].

We know that the mentioned researches are accomplished by using the FPA. In these papers, the numerical results were presented for a special iteration. In \[57\] (see also \[58, p. 274\]), when solving IEs of the type \(A^{*}v = f\) by the iteration method: \(v_{n} = (I - A^{*}A)v_{n-1} + A^{*}f, n = 1, 2, \cdots\), the following rules for stopping the iteration process were proposed:

1) by the discrepancy, namely, stop at such a number \(n\), for which \(\|A^{*}v_{n} - f\| \leq \delta + \xi\|v_{n}\|\), where \(\|\hat{f} - f\| \leq \delta, \|A^{*} - A\| \leq \xi\) (\(\delta\) and \(\xi\) are the errors of \(f\) and \(A\));

2) by the correction, namely, \(\|v_{n} - v_{n-1}\| \leq a_{1}\delta + a_{2}\xi\) (\(a_{1} > 0\) and \(a_{2} > 0\) are some numbers).

It is proved that \(\|v - v_{n}\| \to 0\) for \(\delta, \xi \to 0\); however, for finite \(\delta, \xi,\) finding \(\varepsilon\) is difficult. Also, generally the efficiency of numerical methods were investigated by applying the traditional absolute error as follows

\[
|v(s) - v_{n}(s)| \leq \varepsilon. \tag{5}
\]

2
which depends on the exact solution, the small tolerance value \( \varepsilon \) and the number of iterations. Now, we can ask how authors can find that which iteration is suitable to stop? If we do not know the exact solution, how can we apply the condition (5)? How can we find the proper value of \( \varepsilon \)? In this condition, for large values of \( \varepsilon \) we can not find the appropriate approximation. Because the iterations will be stopped without getting to suitable solution. Also, for small values of \( \varepsilon \) we will have useless iterations without amending the precision. Thus, according to mentioned problems we can not accept the condition (5).

In this paper, the new arithmetic is suggested instead of traditional arithmetic which is called the SA \([9, 17, 18, 19, 20, 59]\). Thus, we will apply the Taylor-collocation method to solve the first kind Volterra IE (4) based on the SA. Also, the validate and control of accuracy on numerical results are investigated by using the CESTAC method \([2, 3, 4, 5, 6, 7]\) and the CADNA library \([11, 12, 30]\). By using this method, we can find the optimal approximation, the optimal iteration, the optimal error of numerical methods \([26, 27, 41, 42]\). Also, the CESTAC method does not have the disadvantages of the FPA. The stopping condition of the CESTAC method is not similar to the FPA. It depends on two successive approximations instead of usual absolute error and in order to apply the new condition we will prove a theorem which will show the NCSDs between exact and approximate solutions are almost equal to NCSDs between two successive approximations. In new arithmetic, we are applying the CADNA library instead of mathematical softwares like Maple, Matlab, Mathematica and so on \([11, 12, 27, 30, 41, 42]\). The CADNA library is a programming environment that it has many abilities. The CADNA programs can be written by using C, C++ or FORTRAN codes \([26, 27, 41, 42]\). Also, some of numerical instabilities such as bifurcations, blow-up, mathematical operations, branching, functions and others can be found by using the CADNA library.

This paper is organized in the following form: Section 2 presents the Taylor-collocation method and employ it for solution of the first kind IE (4). Also, the convergence theorem of presented method is proved. Section 3 describes the application of the SA. The CESTAC method and the CADNA library are considered in full details. Furthermore, a sample program of the CADNA library is presented for methodological objectives. Numerical validation of Taylor-collocation method based on the CESTAC method is investigated in Section 4. In this section, the new applicable termination criteria for the SA is presented. The theorem which is proved in this section, is our license to use the new stopping condition. In this theorem, we will show the NCSDs between two iterations are almost equal to the NCSDs between exact and approximate solutions. The CADNA algorithm to solve the IE (4) is presented in Section 5. Also, several examples are solved based on the presented algorithm. The numerical results are obtained for both environments, the FPA and the SA. In this section, the optimal iteration, approximation and error of Taylor-collocation method are presented. Finally, the conclusions and some advantages of the SA than the FPA are investigated in Section 6.
2 Taylor-collocation method

Let us search for the solution in the following form

$$v_n(s) = \sum_{j=0}^{n} \frac{1}{j!} v^{(j)}(c)(s - c)^j + O(h^{n+1}), \quad (6)$$

which is the Taylor polynomial of degree $n$ at point $s = c$ where

$$e_n(s) = |v(s) - v_n(s)| = O(h^{n+1}). \quad (7)$$

Substituting Eq. (6) into Eq. (4) leads to

$$\sum_{p=1}^{m} \int_{\rho_{p-1}(r)}^{\rho_p(r)} k_p(r, s) \sum_{j=0}^{n} \frac{1}{j!} v^{(j)}(c)(s - c)^j ds = f(r). \quad (8)$$

By producing the collocation points

$$r_i = a + \left(1 - \frac{a}{n}\right)i, \quad i = 0, 1, \ldots, n, \quad (9)$$

and putting grids (9) in Eq. (8) we get

$$\sum_{j=0}^{n} \frac{1}{j!} \left[ \sum_{p=1}^{m} \int_{\rho_{p-1}(r_i)}^{\rho_p(r_i)} k_p(r_i, s) (s - c)^j ds \right] v^{(j)}(c) = f(r_i). \quad (10)$$

Now, we can write Eq. (10) in the following form

$$AV = F, \quad (11)$$

where

$$A = \begin{bmatrix} A_{00} & A_{01} & \cdots & A_{0m} \\ A_{10} & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n0} & A_{n1} & \cdots & A_{nn} \end{bmatrix}_{(n+1)(n+1)},$$

that

$$A_{ij} = \sum_{p=1}^{m} \int_{\rho_{p-1}(r_i)}^{\rho_p(r_i)} k_p(r_i, s) (s - c)^j ds,$$

and

$$V = \begin{bmatrix} \bar{v}^{(0)}(c) & \bar{v}^{(1)}(c) & \cdots & \bar{v}^{(n)}(c) \end{bmatrix}^T,$$ 

$$F = \begin{bmatrix} F(r_0) & F(r_1) & \cdots & F(r_n) \end{bmatrix}^T.$$ 

Coefficients $\bar{v}^{(j)}(c)$ are uniquely determined by system (11). So Eq. (4) has only one unique solution which can be obtained by

$$\bar{v}_n(s) = \sum_{j=0}^{n} \frac{1}{j!} \bar{v}^{(j)}(c)(s - c)^j. \quad (12)$$
2.1 Convergence analysis

The main theorem of convergence for presented method to find the approximate solution of weakly regular Volterra IE (4) is presented. At first, several lemmas are required that we should remaind. Also, we note that \(I = [0, 1]\), \(X = L^2(I)\).

**Lemma 1.** \([43]\) Let \(X = L^2(I)\) and \(W\) be a Volterra integral operator on \(X\) with square summable kernel \(k(r, s)\) where \(\int_0^1 \int_0^1 |k(r, s)|drds = M^2\), and \(M\) is a constant. Then the operator \(W\) can be defined as

\[
W(\phi(r)) = \int_0^r k(r, s)\phi(s)ds,
\]

which \(W\) is bounded and

\[
\|W(\phi(r))\| \leq M\|\phi\|.
\]

**Lemma 2.** \([40]\) Let \(v(s)\) be a sufficiently smooth function on \([0, 1]\) and \(p_n(s)\) is the interpolating polynomial to \(v(s)\) at points \(s_i\) then

\[
v(s) - p_n(s) = \frac{v^{n+1}(s)}{(n+1)!} \prod_{i=0}^{n} (s - s_i), s \in [0, 1],
\]

where \(s_i, i = 0, 1, ..., n\) are the roots of interpolating polynomial. We can write

\[
|v(s) - p_n(s)| \leq \frac{M_n}{2^{n+1}(n+1)!},
\]

where

\[
M_n = \max\{|v^{n+1}(s)}; s \in (0, 1)\}.
\]

**Lemma 3.** The linear operators \(Z_j, j = 1, 2, ..., m\) can be defined on \(X\) as

\[
Z_1(v(r)) = \int_{\rho_1(r)}^{\rho_1(r)} k_1(r, s)v(s)ds,
\]

\[
Z_2(v(r)) = \int_{\rho_2(r)}^{\rho_2(r)} k_2(r, s)v(s)ds,
\]

\[
\vdots
\]

\[
Z_m(v(r)) = \int_{\rho_m(r)}^{\rho_m(r)} k_m(r, s)v(s)ds,
\]

where \(s \in [0, 1]\) and \(v \in L^2(I)\).

According to Lemma \([\_]\), the mentioned operators \(Z_1, Z_2, \cdots, Z_m\) are bounded. For one-one and onto \(Z_1\) the inverse operator \(Z_1^{-1}\) is bounded.
Theorem 1. Let \( v_n(s) = \sum_{m=0}^{n} \frac{1}{m!} v^{(m)}(c)(s-c)^m \) be the approximate solution of Eq. (4) which is obtained from applying the Taylor-collocation method and \( \tilde{v}_n(s) = \sum_{m=0}^{n} \frac{1}{m!} \tilde{v}^{(m)}(c)(s-c)^m \) is the expansion of exact solution \( v(s) \). Thus

\[
\|e_n(s)\|_2 \leq (\mu_1 \mu_2 + \cdots + \mu_1 \mu_m) \left( \frac{M_n}{2^{2n+1}(n+1)!} + \gamma_n \|Q_n\|_2 \right),
\]

where \( M_n = \max\{v^{n+1}(s); s \in (0,1)\} \) and \( Q_n = (q_0(c), q_1(c), \cdots, q_n(c)); q_n = v^{(n)}(c) - \tilde{v}^{(n)}(c) \).

Proof: By substituting presented operators of Lemma 3 in Eq. (4) we have

\[
Z_1(v(r)) + Z_2(v(r)) + \cdots + Z_m(v(r)) = f(r),
\]

and by using inverse operator \( Z_1^{-1} \) we get

\[
v(r) + Z_1^{-1}[Z_2(v(r))] + \cdots + Z_1^{-1}[Z_m(v(r))] = Z_1^{-1}[f(r)].
\]

Also, Eq. (15) for \( n \)-th order approximation can be written in the following form

\[
v_n(r) + Z_1^{-1}[Z_2(v_n(r))] + \cdots + Z_1^{-1}[Z_m(v_n(r))] = Z_1^{-1}[f(r)].
\]

From subtracting Eqs. (15) and (16) we have

\[
e_n(r) = Z_1^{-1}[Z_2(v_n(r) - v(r))] + \cdots + Z_1^{-1}[Z_m(v_n(r) - v(r))].
\]

By applying Lemma 3 and using Eq. (17), there are constant values \( \mu_1, \cdots, \mu_m \) such that

\[
e_n(r) \leq \mu_1 \mu_2 \|v(r) - v_n(r)\|_2 + \cdots + \mu_1 \mu_m \|v(r) - v_n(r)\|_2 \leq (\mu_1 \mu_2 + \cdots + \mu_1 \mu_m) \|v(r) - v_n(r)\|_2.
\]

Now, for exact and approximate solutions \( v(s) \) and \( v_n(s) \) we can write

\[
\|v(s) - v_n(s)\|_2 = \|v(s) - \tilde{v}_n(s) + \tilde{v}_n(s) - v_n(s)\|_2 \leq \|v(s) - \tilde{v}_n(s)\|_2 + \|\tilde{v}_n(s) - v_n(s)\|_2,
\]

where

\[
v_n(s) = \sum_{m=0}^{n} \frac{1}{m!} v^{(m)}(c)(s-c)^m,
\]

and

\[
\tilde{v}_n(s) = \sum_{m=0}^{n} \frac{1}{m!} \tilde{v}^{(m)}(c)(s-c)^m.
\]

From Lemma 2 we get

\[
\|v(s) - \tilde{v}_n(s)\|_2 \leq \frac{M_n}{2^{2n+1}(n+1)!},
\]

where

\[
M_n = \max\{v^{n+1}(s); s \in (0,1)\},
\]
and
\[ \| \bar{v}_n(s) - v_n(s) \|_2 = \left[ \int_0^1 \left( \sum_{m=0}^{n} \frac{(v_n(c) - \bar{v}_n(c))}{m!} (s-c)^m \right)^2 ds \right]^{\frac{1}{2}} \leq \gamma_n \| Q_n \|_2, \tag{21} \]

where
\[ \gamma_n = \left( \sum_{m=0}^{n} \int_0^1 \left| \frac{(s-c)^m}{m!} \right|^2 ds \right)^{\frac{1}{2}}, \]

and
\[ Q_n = (q_0(c), q_1(c), \cdots, q_n(c)); q_n = v^{(n)}(c) - \bar{v}^{(n)}(c). \]

Thus from Eqs. (19), (20) and (21) we have
\[ \| v(s) - v_n(s) \|_2 \leq \frac{M_n}{2^{2n+1}(n+1)!} + \gamma_n \| Q_n \|_2. \tag{22} \]

Finally, by using Eqs. (18) and (22) we get
\[ \| e_n(s) \|_2 \leq (\mu_1 \mu_2 + \cdots + \mu_1 \mu_m) \left( \frac{M_n}{2^{2n+1}(n+1)!} + \gamma_n \| Q_n \|_2 \right). \]

Theorem is proved.

3 Stochastic arithmetic

The CESTAC method and the CADNA library are important method and tool to simulate the Taylor-collocation method for solving problem (1). In this section, the main part of CESTAC method and the CADNA library are investigated. More definitions, properties, abilities and applications of the CESTAC method and the CADNA library can be found in [9, 17, 18, 19, 20, 59].

3.1 The CESTAC methodology

Assume that \( F \) is the set of representable values which are generated by computer. So for \( g \in \mathbb{R} \) there is \( G \in \mathbb{F} \) which \( G \) shows the reproduced form of \( g \) by computer. Now, for \( P \) mantissa bits of the binary FPA we have
\[ G = g - \chi 2^{E-P} \gamma, \tag{23} \]

where the sign of \( g \) showed by \( \chi \), the lost part of the mantissa due to round-off error demonstrated by \( 2^{-P} \gamma \) and the binary exponent of the result displayed by \( E \). Also, we note that in computer systems for single and double precisions we have \( P = 24, 53 \) respectively [2, 3, 4, 5, 6, 7].

If the value \( \gamma \) considers as a stochastic variable uniformly distributed on \([-1, 1]\) then we can make perturbation on last mantissa bit of \( g \). Thus the obtained results for \( G \) will be a random variable with mean \((\mu)\) and the standard deviation \((\sigma)\). The accuracy of random variable \( G \) depends on both parameters \((\mu)\) and \((\sigma)\) [19, 20, 59].
By \( l \) times performing the process for \( G_i, i = 1, \ldots, l \) the distribution of them is in the quasi Gaussian form. Therefore, the mean of them is equal with the exact value of \( g \) and the values of \( \mu \) and \( \sigma \) can be estimated by these \( l \) samples. The following algorithm of the CESTAC method is presented where \( \tau_\delta \) is the value of \( T \) distribution with \( l - 1 \) degree of freedom and confidence interval \( 1 - \delta \).

**Algorithm 1:**

Step 1- Find \( l \) samples for \( G \) as \( \Phi = \{G_1, G_2, \ldots, G_l\} \) by means of the perturbation of the last bit of mantissa.

Step 2- Compute \( G_{ave} = \frac{\sum_{i=1}^{l} G_i}{l} \).

Step 3- Calculate \( \sigma^2 = \frac{\sum_{i=1}^{l} (G_i - G_{ave})^2}{l - 1} \).

Step 4- Find the NCSDs between \( G \) and \( G_{ave} \) by using \( C_{G_{ave},G} = \log_{10} \sqrt{l |G_{ave}| \tau_\delta \sigma} \).

Step 5- If \( C_{G_{ave},G} \leq 0 \) or \( G_{ave} = 0 \), then write \( G = @0 \).

### 3.2 The CADNA library

Since the mathematical packages such as Mathematica, Maple and others do not have this ability to produce the random variables, so we should introduce the new arithmetic and the novel software for stochastic computations. To this aim, the CADNA library is introduced. By using this software the SA can be applied instead of the FPA. The CADNA library should run on LINUX operating system and the commands of CADNA library must be written by using C, C++, FORTRAN or ADA codes [26, 27, 41, 42].

By using the CADNA library we can control the round-off error propagation. Also, detecting the numerical instabilities is one of the important applications of this package. The CADNA algorithm will be stopped when the NCSDs of numerical results equals to zero and it will be shown by informatical zero \( @0 \). Thus, we can find the optimal iteration, approximation and error by using the informatical zero [2, 3, 4, 5, 6, 7].

The following program is a sample of CADNA library that we can find some important remarks about abilities of this library.

```c
#include <cadna.h>
cadna_init(-1);
main()
{
 double_st VALUE;
do
{
 The Main Program;
 printf(" %s ",strp(VALUE));
}
```
while(v[n]-v[n-1]!=0);
cadna_end();
}

By applying #include <cadna.h> we can call the commands of CADNA library. Since all parameters are in the stochastic mode, we should define them in double, int or other forms with _st as double_st. Also, because the parameters defined based on the SA thus in output we should apply %s and strp in print line. We note that in order to stop the presented program, the new termination criterion is applied which depends on difference of two successive approximations. Finally, command cadna_end() is the end of CADNA library.

4 Numerical validation of Taylor-collocation method

In order to solve the mentioned problems of the FPA, we suggest a novel stopping condition based on the SA and the CESTAC method. This condition is in the following form

\[ |v_{n+1}(s) - v_n(s)| = @.0, \]

which depends on two successive approximations \(v_{n+1}(s)\) and \(v_n(s)\) and will be stopped when the difference of \(v_{n+1}(s)\) and \(v_n(s)\) equals to informatical zero @.0.

Now, what is our licence to apply the new termination criterion (24) instead of stopping condition (5)? In order to apply the new condition, we should prove a new theorem to show the equality of the NCSDs between exact and approximate solutions and the NCSDs between two successive approximations.

Definition 1. [2, 3, 4] For numbers \(\theta_1, \theta_2 \in \mathbb{R}\) the NCSDs can be obtained as follows

(1) for \(\theta_1 \neq \theta_2\),

\[ C_{\theta_1, \theta_2} = \log_{10} \left| \frac{\theta_1 + \theta_2}{2(\theta_1 - \theta_2)} \right| = \log_{10} \left| \frac{\theta_1}{\theta_1 - \theta_2} - \frac{1}{2} \right|, \]

(2) for all real numbers \(\theta_1, C_{\theta_1, \theta_1} = +\infty\).

Theorem 2. Let \(v(r)\) and \(v_n(r)\) be the exact and approximate solutions of weakly regular Volterra IE (4) then

\[ C_{v_n(r), v(r)} - C_{v_n(r), v_{n+1}(r)} = \mathcal{O}(h^{n+1}), \]

where \(C_{v_n(r), v(r)}\) shows the NCSDs of \(v_n(r), v(r)\) and \(C_{v_n(r), v_{n+1}(r)}\) is the the NCSDs of two iterations \(v_n(r), v_{n+1}(r)\).

Proof: According to Definition 1.
Furthermore, Therefore, thus,

\[ C_{v_n(r),v_{n+1}(r)} = \log_{10} \left| \frac{v_n(r) + v_{n+1}(r)}{2(v_n(r) - v_{n+1}(r))} \right| = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v_{n+1}(r)} - \frac{1}{2} \right| \]

\[ = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v_{n+1}(r)} \right| + \log_{10} \left| 1 - \frac{1}{2v_n(r)}(v_n(r) - v_{n+1}(r)) \right| \]

\[ = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v_{n+1}(r)} \right| + \mathcal{O}(v_n(r) - v_{n+1}(r)). \]

Since,

\[ v_n(r) - v_{n+1}(r) = v_n(r) - v(r) - (v_{n+1}(r) - v(r)) = E_n(r) - E_{n+1}(r), \]

thus,

\[ \mathcal{O}(v_n(r) - v_{n+1}(r)) = \mathcal{O}(E_n(r) - E_{n+1}(r)) = \mathcal{O}(h^{n+1}) + \mathcal{O}(h^{n+2}) = \mathcal{O}(h^{n+1}). \]

Therefore,

\[ C_{v_n(r),v_{n+1}(r)} = \log_{10} \left| \frac{v_n(r) + v_{n+1}(r)}{2(v_n(r) - v_{n+1}(r))} \right| + \mathcal{O}(h^{n+1}). \] (27)

Furthermore,

\[ C_{v_n(r),v(r)} = \log_{10} \left| \frac{v_n(r) + v(r)}{2(v_n(r) - v(r))} \right| = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v(r)} - \frac{1}{2} \right| \]

\[ = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v(r)} \right| + \mathcal{O}(v_n(r) - v(r)) \] (28)

\[ = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v(r)} \right| + \mathcal{O}(h^{n+1}). \]

By using Eqs. (27) and (28) we get,

\[ C_{v_n(r),v(r)} - C_{v_n(r),v_{n+1}(r)} = \log_{10} \left| \frac{v_n(r)}{v_n(r) - v(r)} \right| - \log_{10} \left| \frac{v_n(r)}{v_n(r) - v_{n+1}(r)} \right| + \mathcal{O}(h^{n+1}) \]

\[ = \log_{10} \left| \frac{v_n(r) - v(r)}{v_n(r) - v_{n+1}(r)} \right| + \mathcal{O}(h^{n+1}) \]

\[ = \log_{10} \left| \frac{\mathcal{O}(h^{n+1})}{\mathcal{O}(h^{n+1})} \right| + \mathcal{O}(h^{n+1}) \]

\[ = \mathcal{O}(h^{n+1}), \]

10
and finally the following formula can be obtained

\[ C_{v_n(r), v(r)} - C_{v_{n+1}(r), v_{n+1}(r)} = O(h^{n+1}). \]

When \( n \to \infty \) the right hand side of above equation tends to zero and hence we obtain

\[ C_{v_n(r), v(r)} = C_{v_{n+1}(r), v_{n+1}(r)}. \]

Theorem 2 shows that the NCSDs between two approximations \( v_n(r), v_{n+1}(r) \) are almost equal to the NCSDs of \( v_n(r), v(r) \). Thus, we can apply termination criteria (24) instead of condition (5).

5 Numerical Illustration

In this section, several examples of weakly regular Volterra IEs of the first kind are presented. These examples are solved by using the Taylor-collocation method based on the FPA and the SA. The CESTAC method is applied to find the optimal approximation, the optimal step and the optimal error of presented method. Also, both conditions (5) and (24) are investigated to compare the numerical results. It is clear that the termination criterion (5) depends on parameter \( \varepsilon \) and specially the exact solution and the number of iteration \( n \). But the algorithm of CESTAC method only depends on two successive approximations. By comparison between these results, we will show that the presented condition based on the SA is more applicable than criterion (5) which is based on the FPA. The following CESTAC algorithm is presented to solve the problems.

Algorithm 2:
Step 1- Let \( n = 1 \).
Step 2- Define the Taylor polynomials and kernels \( k_i(r, s) \).
Step 3- Enter the parameters \( a, b, t \).
Step 4- Do the following steps while \( |v_{n+1}(r) - v_n(r)| \neq 0.0 \)
  
  \{ 
  Step 4-1- Calculate the collocation points.
  Step 4-2- Approximate the coefficient matrix \( A \).
  Step 4-3- Construct the right hand side matrix \( F \).
  Step 4-4- Solve the system (11) by using the obtained results of steps (4-2) and (4-3).
  
  Step 4-5- Apply Eq. (6) to find the numerical solution of Eq. (1). 
  Step 4-6- Print \( v_n(r), |v_{n+1}(r) - v_n(r)| \) and \( |v(r) - v_n(r)| \).
  Step 4-7- \( n = n + 1 \).
  
  \}

A sample program of the CADNA library is presented in Appendix section.
Table 1: Numerical results of Example 1 based on the CESTAC method and the stopping condition (24) for $r = 0.2$.

| $n$ | $v_n(r)$ | $|v_n(r) - v_{n-1}(r)|$ | $|v(r) - v_n(r)|$ |
|-----|-----------|------------------------|-----------------|
| 2   | 0.29415165711104E+001 | 0.29415165711104E+001 | 0.28800384839052E+001 |
| 3   | -0.2374E+000  | 0.317901E+001  | 0.2989E+000  |
| 4   | 0.1509E+000  | 0.38844E+000  | 0.8946E-001  |
| 5   | 0.4230E-001  | 0.10864E+000  | 0.1917E-001  |
| 6   | 0.6480037878687E-001 | 0.2249E-001  | 0.332229158171E-002 |
| 7   | 0.6099E-001  | 0.380E-002    | 0.48E-003    |
| 8   | 0.6154E-001  | 0.54E-003    | 0.6E-004    |
| 9   | 0.6147075816137E-001 | 0.6E-004    | 0.732904378E-005 |
| 10  | 0.6147E-001  | 0.6E-004    | 0.732904378E-005 |

Example 1: Consider the following weakly regular Volterra IE [39]

$$
\int_0^x (1 + r + s)v(s)ds + \int_0^x (2 + rs)v(s)ds + \int_\frac{3x}{4}^x (r + s - 1)v(s)ds - 4 \int_\frac{3x}{4}^x v(s)ds = \frac{1}{128} \left[ -4 - \frac{1}{8} (16r + 69r^2 + 15r^3) - \exp(\frac{r}{4})(r^2 - 13r + 12) + \exp(r)(4r^2 - 16r + 28) \right]
$$

where the exact solution is $v(r) = \exp(\frac{2r}{8})$. In Table 1, the numerical results of the CESTAC method for $r = 0.2$ are shown. According to these results, the presented algorithm stopped when the NCSDs between approximations $v_n(r), v_{n-1}(r)$ are almost equal to NCSDs between $v(r), v_n(r)$ and Theorem 2 can support the obtained results theoretically. According to numerical results this equality is obtained in 10-th iteration. So the optimal iteration of Taylor-collocation method to solve the weakly regular Volterra IE of the first kind (29) is $n = 10$. As mentioned before, if we use the Taylor-collocation method based on the FPA we should apply the stopping condition (5) which it depends on parameter $\varepsilon$. The numerical results of the FPA are presented in Table 2 for $\varepsilon = 10^{-10}$ and $r = 0.2$. Also in Table 3 the number of iterations for different values of $\varepsilon$ are demonstrated. According to this table for large $\varepsilon$, the iterations are stopped before getting to a proper approximation and for small $\varepsilon$, the useless iterations are done without amending the efficiency of the results. Furthermore, in order to find the numerical approximations based on the FPA, the exact solution is important because we should apply the termination criterion (5) instead of condition (24). They are some of most important defects of the FPA than the SA. The sample program of CADNA library to solve the example 1 is presented in Appendix.

Example 2: Consider the following IE [52]

$$
2 \int_0^{\sin(\frac{x}{2})} v(s)ds - \int_0^{2\sin(\frac{x}{2})} \frac{v(s)}{\sin(\frac{x}{2})} ds + \int_0^r v(s)ds = \frac{r^3}{3} + \sin^3 \frac{r}{2} - \frac{16}{3}\sin^3 \frac{r}{3}, \quad r \in [0, \frac{3\pi}{2}],
$$
Table 2: Numerical results of Example 1 based on the condition (5) for $\varepsilon = 10^{-10}$ and $r = 0.2$.

| $n$ | $v_n(r)$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|
| 2   | 2.94151657111040654158 | 2.88003848390524774814 |
| 3   | -0.23749678750420105677 | 0.29897487470935985021 |
| 4   | 0.15094246356455431890 | 0.08946437635939552546 |
| 5   | 0.04229965285857797269 | 0.01917843434658082075 |
| 6   | 0.06480037878687468222 | 0.00332229158171588879 |
| 7   | 0.06099350668538974785 | 0.00048458051976904559 |
| 8   | 0.0614031969903288324 | 0.000622324938740880 |
| 9   | 0.06147075816137072268 | 0.0000732904378807075 |
| 10  | 0.0614789565841218517 | 0.0000081785325339173 |
| 11  | 0.0614779914393221182 | 0.0000000886122658161 |
| 12  | 0.06147809636759438146 | 0.0000000091624355880 |
| 13  | 0.06147808629013713083 | 0.0000000009150216626 |
| 14  | 0.06147808728981307008 | 0.00000000008465427664 |

Table 3: Number of iterations of Taylor-collocation method for Example 1 based on the FPA and termination criterion (5) for different values of $\varepsilon$ and $r = 0.2$.

| $\varepsilon$ | small values | $10^{-10}$ | $10^{-5}$ | $10^{-1}$ | 0.5 | large values |
|---------------|--------------|-----------|----------|----------|-----|-------------|
| $n$           | $>> 14$      | 14        | 9        | 4        | 3   | 3           |

where $v(r) = r^2$. The numerical results of Table 4 are obtained based on the SA. As we know, we should apply the stopping condition (23). So only two successive approximations are required and we do not need to have the exact solution. According to Table 4 the optimal step of presented method is $n = 4$ and the optimal approximation for $r = 0.5$ is $v_n(0.5) = 0.24999999999999E + 000$. In Table 5 the numerical results of the FPA are shown for large $\varepsilon$. Thus its algorithm will be stopped without getting to suitable approximation with high accuracy. If we apply the FPA, finding the proper $\varepsilon$ is important and it is one of faults of computations by FPA. Numerical results of Table 6 are very good paradigm to show the mentioned reality.

Example 3: Let us consider the following Volterra IE

$$\int_{0}^{r} (1+r+s)v(s)ds + \int_{0}^{r} (2+rs)v(s)ds + \int_{0}^{r} (1+r+s)v(s)ds = \frac{31r^6}{40960} + \frac{1099r^5}{20480} + \frac{271r^4}{8192} \quad (31)$$

where the exact solution is $v(r) = \frac{r^3}{8}$. The numerical results of Taylor-collocation method to solve (31) by using CESTAC method is presented in Table 7. We should note that the final

Table 4: Numerical results of Example 2 by using the CESTAC method for $r = 0.5$.

| $n$ | $v_n(r)$ | $|v_n(r) - v_{n-1}(r)|$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|-----------------|
| 2   | 0.190380238054409E+001 | 0.190380238054409E+001 | 0.165380238054409E+001 |
| 3   | 0.250000000000000E+000 | 0.165380238054409E+001 | @.0 |
| 4   | 0.249999999999999E+000 | @.0 | @.0 |
Table 5: Obtained results of Example 2 based on the condition (5) for large value of $\varepsilon$ and $r = 0.5$.

| $n$ | $v_n(r)$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|
| 2   | 1.90380238054409267612 | 1.65380238054409267612 |

Table 6: Iterations of Taylor-collocation method for Example 2 by applying the FPA and criterion (5) for different values of $\varepsilon$ and $r = 0.5$.

| $\varepsilon$ | small values | $10^{-10}$ | $10^{-5}$ | $10^{-1}$ | 0.5 | large values |
|---------------|--------------|----------|----------|----------|-----|--------------|
| $n$           | $\gg 3$      | 3        | 3        | 3        | 3   | 2            |

column of table 7 is only for comparison and we do not need to find the absolute error in the SA because in Theorem 2 we proved $C_{v_n(s),v(s)} = C_{v_n(s),v_{n+1}(s)}$. Thus we can apply the termination criterion (24) which it depends on two successive approximations. In Table 8, the results of Taylor-collocation method based on the FPA are demonstrated for $\varepsilon = 10^{-1}$ and $r = 0.8$ which we can not guarantee the obtained results because the numerical results are without proper precision. Finally, table 9 are presented to show the number of iterations of presented method for different values of $\varepsilon$.

Example 4: Abel IEs are one kind of Volterra IEs with singular kernel which have many applications in science and engineering [29, 42, 43, 55, 60]. In this example, we consider the following linear Abel IE of the first kind

$$\int_0^r \frac{v(s)}{\sqrt{r^2 - s^2}} \, ds = \frac{2}{3} \pi r^3, \quad (32)$$

where the exact solution is $v(r) = \pi r^3$. According to the numerical results of Table 10 the optimal iteration of Taylor-collocation method is $n = 15$ with optimal approximation $v(0.1) = 0.314159E - 002$. The numerical results of FPA are obtained for $\varepsilon = 10^{-5}$ and $r = 0.1$ in Table 11 that presented algorithm is stopped in 9-th iteration. Also, the number of iterations for different values of $\varepsilon$ is demonstrated in Table 12.

Example 5: Let us consider the following non-linear Abel IE of the first kind [29, 60]

$$\pi + r = \int_0^r \frac{\sin^{-1}(v(s))}{\sqrt{r^2 - s^2}} \, ds, \quad (33)$$

Table 7: Numerical results of Example 3 by CESTAC method and condition (24) for $r = 0.8$.

| $n$ | $v_n(r)$ | $|v_n(r) - v_{n-1}(r)|$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|-----------------|
| 2   | 0.187158677184107E+000 | 0.187158677184107E+000 | 0.144283677184107E+000 |
| 3   | 0.48445393318581E-001 | 0.13871328386552E+000 | 0.5570393318581E-002 |
| 4   | 0.428749999999999E-001 | 0.5570393318581E-002 | @.0 |
| 5   | 0.428749999999999E-001 | 0.1E-015 | @.0 |
| 6   | 0.428749999999999E-001 | @.0 | @.0 |
Table 8: Results of Example 3 based on the condition (5) for $\varepsilon = 10^{-1}$ and $r = 0.8$.

| $n$ | $v_n(r)$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|
| 2   | 0.18715867718410730824 | 0.14428367718410731180 |
| 3   | 0.04844539331858149778  | 0.00557039331858150827  |

Table 9: Iterations of Taylor-collocation method for Example 3 by using the FPA and criterion (5) for different values of $\varepsilon$ and $r = 0.8$.

| $\varepsilon$ | small values | $10^{-10}$ | $10^{-5}$ | $10^{-1}$ | 0.5 | 1 | large values |
|---------------|--------------|------------|----------|-----------|-----|---|--------------|
| $n$           | >> 4         | 4          | 4        | 3         | 2   | 2 | 2            |

Table 10: Numerical results of Example 4 by CESTAC method and condition (24) for $r = 0.1$.

| $n$ | $v_n(r)$ | $|v_n(r) - v_{n-1}(r)|$ | $|v(r) - v_n(r)|$ |
|-----|----------|--------------------------|-----------------|
| 2   | 0.1009201E+001 | 0.1009201E+001 | 0.100605E+001  |
| 3   | -0.368416E+000 | 0.1377617E+001 | 0.371558E+000  |
| 4   | 0.2301187E-001 | 0.3914283E+000 | 0.1987028E-001 |
| 5   | 0.585469E-002 | 0.171571E-001 | 0.271310E-002  |
| 6   | 0.285618E-002 | 0.299850E-002 | 0.28540E-003   |
| 7   | 0.3002522E-002| 0.14633E-003 | 0.13906E-003   |
| 8   | 0.3116390E-002| 0.11386E-003 | 0.2520E-004    |
| 9   | 0.313838E-002 | 0.21993E-004 | 0.320E-005     |
| 10  | 0.313945E-002 | 0.106E-005  | 0.213E-005     |
| 11  | 0.3141559E-002| 0.296E-006  | 0.31E-007      |
| 12  | 0.3141684E-002| 0.12E-006   | 0.92E-007      |
| 13  | 0.314157E-002 | 0.11E-006   | 0.1E-007       |
| 14  | 0.314159E-002 | 0.1E-007    | @.0            |
| 15  | 0.314159E-002 | @.0         | @.0            |

Table 11: Results of Example 4 based on the condition (5) for $\varepsilon = 10^{-5}$ and $r = 0.1$.

| $n$ | $v_n(r)$ | $|v(r) - v_n(r)|$ |
|-----|----------|-----------------|
| 2   | 1.00920140743255615234 | 1.0060597681573486328 |
| 3   | -0.36841639876365661621 | 0.371557907758332988 |
| 4   | 0.230118681959438324 | 0.1987027563154697418 |
| 5   | 0.00558469184443354607 | 0.0027130988921678066 |
| 6   | 0.00285618798807263374 | 0.00028540496714413166 |
| 7   | 0.00300252321176230907 | 0.00013906974345445633 |
| 8   | 0.00311639113351702690 | 0.00002520182169973850 |
| 9   | 0.00313838454894721508 | 0.00000320840626955032 |

Table 12: Iterations of Taylor-collocation method for Example 4 by using the FPA and criterion (5) for different values of $\varepsilon$ and $r = 0.1$.

| $\varepsilon$ | small values | $10^{-10}$ | $10^{-5}$ | $10^{-1}$ | 0.5 | 1 | large values |
|---------------|--------------|------------|----------|-----------|-----|---|--------------|
| $n$           | >> 9         | > 9        | 9        | 4         | 3   | 3 | 2            |
Table 13: Numerical results of Example 5 by CESTAC method and condition (24) for $r = 0.4$.

| $n$ | $v_n(r)$ | $|v_n(r) - v_{n-1}(r)|$ | $|v(r) - v_n(r)|$ |
|-----|-----------|------------------------|-----------------|
| 2   | 0.7086622E+000 | 0.7086622E+000 | 0.331989E-001 |
| 3   | 0.6878617E+000 | 0.208003E-001 | 0.12398E-001 |
| 4   | 0.6778912E+000 | 0.99705E-002 | 0.24279E-002 |
| 5   | 0.6755348E+000 | 0.23564E-002 | 0.715E-004 |
| 6   | 0.6754605E+000 | 0.743E-004 | 0.28E-005 |
| 7   | 0.6754629E+000 | 0.24E-005 | 0.3E-006 |
| 8   | 0.6754631E+000 | 0.1E-006 | 0.2E-006 |
| 9   | 0.6754631E+000 | @.0 | @.0 |

Table 14: Results of Example 5 based on the condition (5) for $\varepsilon = 10^{-5}$ and $r = 0.4$.

| $n$ | $v_n(r)$ | $|v(r) - v_n(r)|$ |
|-----|-----------|-----------------|
| 2   | 0.70866230274517871823 | 0.03319919251737557531 |
| 3   | 0.68786180640627836436 | 0.01239869617847522143 |
| 4   | 0.67789121429022713983 | 0.00242810406242399690 |
| 5   | 0.67553501264255699787 | 0.0000263712997838894 |

where the exact solution is $v(r) = \sin(r + 2)$. In order to solve the mentioned problem at first we can convert the non-linear IE (33) to the linear form by using the transformations

$v'(s) = \sin^{-1}(v(s))$, \hspace{1cm} v(s) = \sin(v'(s)). \hspace{1cm} (34)$

Thus, we should solve the following IE

$\pi + r = \int_0^r \frac{v'(s)}{\sqrt{r^2 - s^2}} ds, \hspace{1cm} (35)$

instead of IE (33). Then by substituting the obtained solution in Eq. (34) we can find the approximate solution of IE (33). In Table 13, the optimal iteration of Taylor-collocation method, the optima approximation and the optimal error are obtained for $r = 0.4$. In Table 14 because the results are based on the FPA and in this case we do not know the optimal value of $\varepsilon$ so the algorithm is stopped in step 6 without getting to suitable approximation. Finally, in Table 15, the number of iterations which are generated by applying the criterion (5) are demonstrated for $r = 0.4$ and different values of $\varepsilon$.

Table 15: Iterations of Taylor-collocation method for Example 5 by using the FPA and criterion (5) for different values of $\varepsilon$ and $r = 0.4$.

| $\varepsilon$ small values | $10^{-10}$ | $10^{-5}$ | $10^{-1}$ | 0.5 | 1 | large values |
|---------------------------|------------|-----------|-----------|-----|---|-------------|
| $n$                       | $>> 6$     | $> 6$     | 6         | 2   | 2 | 2           |
6 Conclusion

The SA and specially using the CESTAC method and the CADNA library are applicable methods and tools to solve the mathematical problems with controlling the accuracy and detecting the rate of numerical errors against applying the methods based on the FPA. In this paper, the Taylor-collocation method to solve the first kind weakly regular Volterra IEs with discontinuous kernels was illustrated based on the SA. The methods based on the FPA have some problems that in order to void them we should apply the new arithmetic. Thus we suggested to apply the SA instead of the FPA. In order to find the numerical results based on the FPA, knowing the exact solution is important but the SA depends on two successive approximations. The termination criterion of the FPA depends on small value like $\varepsilon$ but the SA independents of $\varepsilon$. In the FPA, we used the mathematical softwares but in the SA, we applied the CADNA library. By using the CADNA library, not only the optimal approximation, the optimal iteration and the optimal error can be obtained but also some of numerical instabilities can be found. Our license to use the SA instead of the FPA is to prove the main theorem which in this theorem we showed that the NCSDs between two approximations $v_n(r), v_{n-1}(r)$ are almost equal to the NCSDs between $v(r), v_n(r)$. Thus, we can apply the SA and the stopping condition (24) instead of the FPA and the termination criterion (5).

References

[1] S. Abbasbandy, Numerical solutions of the integral equations: Homotopy perturbation method and Adomian’s decomposition method, Applied Mathematics and Computation, 173 (1) 493-500 (2006).

[2] S. Abbasbandy, M.A. Fariborzi Araghi, The use of the stochastic arithmetic to estimate the value of interpolation polynomial with optimal degree, Appl. Numer. Math. 50 279-290 (2004).

[3] S. Abbasbandy, M.A. Fariborzi Araghi, The use of the stochastic arithmetic to estimate the value of interpolation polynomial with optimal degree, Appl. Numer. Math., 50, 279-290 (2004).

[4] S. Abbasbandy, M.A. Fariborzi Araghi, A reliable method to determine the ill-condition functions using stochastic arithmetic, Southwest J. Pure Appl. Math., 1, 33-38 (2002).

[5] S. Abbasbandy, M.A. Fariborzi Araghi, Numerical solution of improper integrals with valid implementation, Math. Comput. Appl., 7, 83-91 (2002).

[6] S. Abbasbandy, M.A. Fariborzi Araghi, The valid implementation of numerical integration methods, Far East J. Appl. Math., 8, 89-101 (2002).

[7] S. Abbasbandy, M.A. Fariborzi Araghi, A stochastic scheme for solving definite integrals, Appl. Numer. Math., 55, 125-136 (2005).
[8] M. Abdulkawi, Solution of Cauchy type singular integral equations of the first kind by using differential transform method, Applied Mathematical Modelling, 39 (8) 2107-2118 (2015).

[9] R. Alt, J. Vignes, Validation of results of collocation methods for ODEs with the CADNA library, Appl. Numer. Math., 21, 119-139 (1996).

[10] E. Babolian, T. Lotfi, M. Paripour, Wavelet moment method for solving Fredholm integral equations of the first kind, Applied Mathematics and Computation, 186 (2) 1467-1471 (2007).

[11] H. Barzegar Kelishami, M.A. Fariborzi Araghi, Dynamical control of accuracy in the fuzzy Runge-Kutta methods to estimate the solution of a fuzzy differential equation, Journal of Fuzzy Set Valued Analysis, 1, 71-84 (2016).

[12] H. Barzegar Kelishami, M.A. Fariborzi Araghi, T. Allahviranloo, Dynamical control of computations using the finite differences method to solve fuzzy boundary value problem, Journal of Intelligent & Fuzzy Systems, 1-12 (2018).

[13] J. Biazar, E. Babolian, R. Islam, Solution of a system of Volterra integral equations of the first kind by Adomian method, Applied Mathematics and Computation, 139 (2-3) 249-258 (2003).

[14] J. Biazar, M. Eslami, H. Aminikhah, Application of homotopy perturbation method for systems of Volterra integral equations of the first kind, Chaos, Solitons & Fractals, 42 (5) 3020-3026 (2009).

[15] L. Bougoffa, A. Mennouni, R. C. Rach, Solving Cauchy integral equations of the first kind by the Adomian decomposition method, Applied Mathematics and Computation, 219 (9) 4423-4433 (2013).

[16] H. Brunner, Volterra Integral Equations. Cambridge University Press, 2017.

[17] J.M. Chesneaux, F. Jezequel, Dynamical control of computations using the Trapezoidal and Simpson’s rules, J. Universal Comput. Sci., 4 (1), 2-10 (1998).

[18] J.M. Chesneaux, Study of the computing accuracy by using probabilistic approach, in: C. Ullrich (Ed.), Contribution to Computer Arithmetic and Self-Validating Numerical Methods, IMACS, New Brunswick, NJ, (1990).

[19] J.M. Chesneaux, The equality relations in scientific computing, Numer. Algorithms, 7, 129-143 (1994).

[20] J.M. Chesneaux, Stochastic arithmetic properties, IMACS Comput. Appl. Math., 81-91 (1992).

[21] I. Dag, A. Canivar, A. Sahin, Taylor–Galerkin and Taylor–collocation methods for the numerical solutions of Burgers’ equation using B-splines, Communications in Nonlinear Science and Numerical Simulation, 16 (7) 2696-2708 (2011).
[22] Penny J. Davies and Dugald B. Duncan, Numerical approximation of first kind Volterra convolution integral equations with discontinuous kernels, Journal of Integral Equations and Applications, 29 (1) 41-73 (2017).

[23] Y. Enesiz, Y. Keskin, A. Kurnaz, The solution of the Bagley–Torvik equation with the generalized Taylor collocation method, Journal of the Franklin Institute, 347 (2) 452-466 (2010).

[24] M.A. Fariborzi Araghi, S. Noeiaghdam, Fibonacci-regularization method for solving Cauchy integral equations of the first kind, Ain Shams Eng J., 8, 363-369 (2017).

[25] M.A. Fariborzi Araghi, S. Noeiaghdam, A novel technique based on the homotopy analysis method to solve the first kind Cauchy integral equations arising in the theory of airfoils, Journal of Interpolation and Approximation in Scientific Computing, 1, 1-13 (2016).

[26] M.A. Fariborzi Araghi, S. Noeighdam, Dynamical control of computations using the Gauss-Laguerre integration rule by applying the CADNA library, Advances and Applications in Mathematical Sciences, 16, 1-18 (2016).

[27] M.A. Fariborzi Araghi, S. Noeiaghdam, A valid scheme to evaluate fuzzy definite integrals by applying the CADNA library, International Journal of Fuzzy System Applications, 6 (4), 1-20 (2017).

[28] M.A. Fariborzi Araghi, S. Noeiaghdam, Homotopy analysis transform method for solving generalized Abel’s fuzzy integral equations of the first kind, IEEE (2016). DOI: 10.1109/CFIS.2015.7391645

[29] M.A. Fariborzi Araghi, S. Noeiaghdam, Homotopy regularization method to solve the singular Volterra integral equations of the first kind, Jordan Journal of Mathematics and Statistics (JJMS) 11(1), 2018, 1-12.

[30] M.A. Fariborzi Araghi, E. Zarei, Dynamical control of computations using the iterative methods to solve fully fuzzy linear systems, Advances in Fuzzy Logic and Technology 2017 (2017) 55-68.

[31] E. Gokmen, O. Rasit Isik, M. Sezer, Taylor collocation approach for delayed Lotka–Volterra predator-prey system, Applied Mathematics and Computation, 268 671-684 (2015).

[32] E. Gokmen, M. Sezer, Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients, Ain Shams Engineering Journal, 4 (1) 117-125 (2013).

[33] Y. Jafarzadeh, B. Keramati, Numerical method for a system of integro-differential equations and convergence analysis by Taylor collocation, Ain Shams Engineering Journal, In press (2016).
[34] K. Maleknejad, N. Aghazadeh, R. Mollapourasl, Numerical solution of Fredholm integral equation of the first kind with collocation method and estimation of error bound, Applied Mathematics and Computation, 179 (1) 352-359 (2006).

[35] K. Maleknejad, J. Rashidinia, T. Eftekhari, Numerical solution of three-dimensional Volterra–Fredholm integral equations of the first and second kinds based on Bernstein’s approximation, Applied Mathematics and Computation, 339 (15) 272-285 (2018).

[36] K. Maleknejad, E. Saeedipoor, An efficient method based on hybrid functions for Fredholm integral equation of the first kind with convergence analysis, Applied Mathematics and Computation, 304 (1) 93-102 (2017).

[37] K. Maleknejad, S. Sohrabi, Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets, Applied Mathematics and Computation, 186 (1) 836-843 (2007).

[38] I. R. Muftahov, D. N. Sidorov, Solvability and numerical solutions of systems of nonlinear Volterra integral equations of the first kind with piecewise continuous kernels, Vestnik YuUrGU. Ser. Mat. Model. Progr., 9 (1) 130–136 (2016).

[39] I. Muftahov, A. Tynda, D. Sidorov, Numeric solution of Volterra integral equations of the first kind with discontinuous kernels, Journal of Computational and Applied Mathematics, 313 119-128 (2017). http://dx.doi.org/10.1016/j.cam.2016.09.003

[40] S. Nemati, Numerical solution of Volterra–Fredholm integral equations using Legendre collocation method, J. Comput. Appl. Math., 278 29–36 (2015).

[41] S. Noeiaghdam, M.A. Fariborzi Araghi, Finding optimal step of fuzzy Newton-Cotes integration rules by using the CESTAC method, Journal of Fuzzy Set Valued Analysis, 2017 No.2, 62-85 (2017).

[42] S. Noeiaghdam, M.A. Fariborzi Araghi, S. Abbasbandy, Finding optimal convergence control parameter in the homotopy analysis method to solve integral equations based on the stochastic arithmetic, Numerical Algorithms, 1-31 (2018). https://doi.org/10.1007/s11075-018-0546-7

[43] S. Noeiaghdam, E. Zarei, H. Barzegar Kelishami, Homotopy analysis transform method for solving Abel’s integral equations of the first kind, Ain Shams Eng J., 7, 483-495 (2016).

[44] R. Novin, M.A. Fariborzi Araghi, Y. Mahmoudi, A novel fast modification of the Adomian decomposition method to solve integral equations of the first kind with hypersingular kernels, Journal of Computational and Applied Mathematics, 343 (1) 619-634 (2018).

[45] Ram P. Kanwal, Linear Integral Equations: Theory and Techniques, Academic Press, New York, 1971.
[46] M. Sezer, M. Gülsu, Polynomial solution of the most general linear Fredholm–Volterra integrodifferential-difference equations by means of Taylor collocation method, Applied Mathematics and Computation, 185 (1) 646-657 (2007).

[47] D. N. Sidorov, On Parametric Families of Solutions of Volterra Integral Equations of the First Kind with Piecewise Smooth Kernel, Differential Equations, 49 (2) 210–216 (2013).

[48] D. N. Sidorov, Solvability of systems of Volterra integral equations of the first kind with piecewise continuous kernels, Russian Mathematics (Iz. VUZ), 57 (1) 54–63 (2013).

[49] D. Sidorov, Generalized Solution to the Volterra Equations with Piecewise Continuous Kernels, Bull. Malays. Math. Sci. Soc. (2) 37(3), 757–768 (2014).

[50] N. A. Sidorov, D. N. Sidorov, On the Solvability of a Class of Volterra Operator Equations of the First Kind with Piecewise Continuous Kernels, Mathematical Notes, 96 (5) 811-826 (2014).

[51] D. N. Sidorov, A. N. Tynda, I. R. Muftahov, Numerical solution of Volterra integral equations of the first kind with piecewise continuous kernel, Vestnik YuUrGU. Ser. Mat. Model. Progr., 7 (3) 107–115 (2014).

[52] D. Sidorov, A. Tynda, I. Muftahov, Numerical Solution of Weakly Regular Volterra Integral Equations of the First Kind. arXiv:1403.3764v2

[53] V.S. Sizikov, Mathematical Methods for Processing the Results of Measurements, Polytekhnika, 2001.

[54] V.S. Sizikov, V. Evseev, A. Fateev, S. Clausen, Direct and inverse problems of infrared tomography, Appl. Opt. 55 (1) (2016) 208-220.

[55] V. Sizikov, D. Sidorov, Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography, Appl. Numer. Math. 106 (2016) 69-78.

[56] V.S. Sizikov, A.V. Smirnov, B.A. Fedorov, Numerical solution of the Abelian singular integral equation by the generalized quadrature method, Rus. Math. (Iz. VUZ) 48 (8) (2004) 59-66.

[57] G.M. Vainikko, Error estimates of the successive approximation method for ill-posed problems, Automation and Remote Control 41 (3) (1980) 356-363.

[58] A.F. Verlan, V.S. Sizikov, Integral Equations: Methods, Algorithms, Programs, Nauk. Dumka, 1986.

[59] J. Vignes, A stochastic arithmetic for reliable scientific computation, Math. Comput. Simulation, 35, 233-261 (1993).

[60] A.M. Wazwaz, Linear and nonlinear integral equations: methods and applications. Berlin: Higher Education, Beijing, and Springer; 2011.
In this section, a sample CADNA library program to solve the Example 1 is presented. This program is based on the stochastic arithmetic. The following codes are similar to C++ codes and only some parts are different. This program is presented is some parts.

- In part 1, the commands of C++ are recalled. Specially by using `#include <cadna.h>` the commands of CADNA library can be used.
- Part 2 until end of program is the main part. We should note that in order to introduce the variables or arrays the double precision is applied with _st as `double_st` because the variables will be based on the stochastic arithmetic.
- In part 3, we can calculate the nodes of Taylor-collocation method. In every part, some disable lines are exist that we can show the results of those part by enabling them.
- Part 4 is to construct the array of matrix $A$. In this part, there are some functions as `simp3`, $f$, $ff$ and others that we did not mention here.
- By using part 5, we can find the arrays of right hand matrix $F$.
- In part 6, the obtained system of equation is solved.
- Part 7 is the final part of this program that we apply to show the final results. It is important that in order to show the numerical results with significant digits we should apply `\%s` and `strp` in print line. Also command `while` depends on difference of two successive approximations. Finally command `cadna_end()` is the end of CADNA library.

**PART 1:**
```c
#include <stdio.h>
#include <math.h>
#include <cadna.h>
```

**PART 2:**
```c
main()
{
    cadna_init(-1);
    double_st z,H[50][50],s[50],a,b,M[50][50],y[50],S[50],exact,T[50],r;
```
int n, i, j, k;
a = 0; b = 1; S[0] = 0;
n = 1; z = 0.2;
exact = (1.0 / 8) * (exp(2 * z) - 1);

printf("------------------------------------------------------------------
\n")
printf(" n approximate solution difference of two term absolute error
\n")
printf("------------------------------------------------------------------
\n")
do
{ S[n] = 0;

PART 3:
for (i = 1; i <= n + 1; i++)
{
    s[i - 1] = a + ((b - a) / n) * (i - 1);
    // printf(" %s \n", strp(s[i-1]));
}

PART 4:
for (i = 1; i <= n + 1; i++)
{
    for (j = 1; j <= n + 1; j++)
    {
        H[i - 1][j - 1] = g(s[i - 1], j, n);
        //printf(" %s \n", strp(H[i-1][j-1]));
        M[i - 1][j - 1] = simp3(f, s[i - 1], a, s[i - 1] * (1.0 / 8), j, 500)
            + simp3(ff, s[i - 1], s[i - 1] * (1.0 / 8), s[i - 1] * (1.0 / 2), j, 500)
            + simp3(fff, s[i - 1], s[i - 1] * (1.0 / 2), s[i - 1] * (3.0 / 4), j, 500)
            - simp3(ffff, s[i - 1], s[i - 1] * (3.0 / 4), s[i - 1], j, 500);
        //printf(" %s \n", strp(M[i-1][j-1]));
    }
}

PART 5:
for (i = 1; i <= n + 1; i++)
{
    y[i - 1] = (1.0 / 128) * (-4 - ((1.0 / 8) * (16 * s[i - 1] + 69 * s[i - 1] * s[i - 1] + 15 * s[i - 1] * s[i - 1] * s[i - 1] * s[i - 1] - 13 * s[i - 1] * s[i - 1] + 12))
        + (exp(s[i - 1] * (1.0 / 4)) * (s[i - 1] * s[i - 1] - 13 * s[i - 1] + 12))
        + (exp(s[i - 1] * (1.0 / 4)) * (s[i - 1] * s[i - 1] - 13 * s[i - 1] + 12))
        + (exp((3.0 / 2) * s[i - 1]) * (14 * s[i - 1] + 20) - (32 * exp(2 * s[i - 1])))
    );
    // printf(" %s \n", strp(y[i-1]));
}

PART 6:
double st temp;
for (k = 0; k <= n; k++)
temp = M[k][k];
M[k][k] = 1.0;
for (j=0; j<=n; j++)
M[k][j] = M[k][j] / temp;
for (i=0; i<=n; i++)
if (i!=k)
{
  temp = M[i][k];
  M[i][k] = 0.0;
  for (j=0; j<=n; j++)
  M[i][j] = M[i][j] - temp * M[k][j];
}

//for (i=0; i<=n; i++)
//for (j=0; j<=n; j++)
//printf(" %s ",strp(M[i][j]));
//printf("\n");
double st e,c[50];
for (i=0; i<=n; i++)
{
  e = 0;
  for (j=0; j<=n; j++)
  {
    e = e + M[i][j] * y[j];
  }
  c[i] = e;
  //printf("\n %.10f \n",c[i]);
}

PART 7:
for (i=0; i<=n; i++)
{
  S[n] = S[n] + c[i] * g(z, i+1, n);
}
printf(" %d %s %s %s \n",n+1,strp(S[n]),strp(fabs(S[n]-S[n-1])),strp(fabs(exact-S[n])));
n++;
while (S[n-1] - S[n-2] != 0);
printf("----------------------------------------------------------\n\n\n");
cadna_end();