Finite enumerable but undecidable collections

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Abstract

It is well known that in Zermelo-Fraenkel (ZF) set theory any finite set is decidable. In this paper we discuss an extension of ZF where this result is no longer valid. Such an extension is quasi-set theory and it has its origin on problems motivated by quantum mechanics.

1 Introduction

In 1982 Richard P. Feynman [2] proved that a quantum system of $n$ particles cannot be simulated by an ordinary computer without an exponential slowdown in the efficiency of the simulation. On the other hand, a classical system with $n$ particles can be simulated with a polynomial slowdown. This was the starting point of a new field of scientific knowledge known today as quantum computation. For a brief review on this and further technical details see [4].

In this paper we propose another kind of computation also inspired on quantum phenomena. Although this new computation presents a lot of disadvantages from the computational point of view, it may bring some light to a better understanding of the computational aspects of the quantum world.

It is well known that in quantum mechanics (QM) elementary particles may be considered as non-individuals in a sense. Quantum particles that share the same set of state-independent (intrinsic) properties may be indistinguishable. Although classical particles can share all their intrinsic properties, we are able to follow their trajectories, at least in principle. That allows us to identify particles. In quantum physics this is not possible, i.e., it is not possible, a priori, to keep track of individual particles in order to distinguish among them. In other words, it is not possible to label quantum particles by their trajectories. And this non-individuality plays a very important role in quantum mechanics [9]. For a philosophical discussion on the problems raised by non-individuality see, for example, the references in [3].

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On the possibility that collections of such indistinguishable entities should not be considered as sets in the usual sense, Yu. Manin [7] proposed the search for axioms which should allow to deal with indiscernible objects. As he said,

I would like to point out that it [standard set theory] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even *sets* of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the *sets* of grains of sand.

It is important to settle that ‘indistinguishable’ objects are objects that share their properties, while ‘identical’ objects means ‘the very same object’. One manner to cope with the problem of non-individuality in quantum physics is by means of quasi-set theory [5, 6, 10], which is an extension of Zermelo-Fraenkel set theory that allows us to talk about certain indistinguishable objects that are not necessarily identical. Actually, in some cases there is no sense in saying that two objects are either identical or different. In quasi-set theory identity does not apply to all objects. In other words, there are some situations in quasi-set theory where the sequence of symbols $x = y$ is not a well-formed formula, i.e., it is meaningless. A weaker equivalence relation called “indistinguishability” is an extension of identity in the sense that it allows the existence of *two* objects that are indistinguishable. In standard mathematics, there is no sense in saying that two objects are identical. If $x = y$, then we are talking about one single object with two labels, namely, $x$ and $y$.

Some applications of quasi-set theory on the foundations of quantum physics have already been done (*op. cit.*). But in this paper we intend to explore some computational properties of quasi-sets, although the quantum perspective is still present. We intend to prove that there exist finite collections of objects in quasi-set theory which are enumerable but undecidable. We recall that a collection is said to be enumerable if there is an algorithm that prints all elements of $x$ and only them. Besides, a collection is said to be decidable if there exists an algorithm that determines whether an arbitrary object belongs to $x$ or not. Otherwise, the collection is undecidable. In this sense we are extending the definition given in some textbooks like [11].

Some related topic are discussed at the end of the paper.

## 2 Quasi-sets

This section is strongly based on other works [5, 6, 10]. I use standard logical notation for first-order theories without identity [8].

It is important to remark that, in contrast to the notions of set and quasi-set, the term “collection” has an intuitive meaning in this paper.

Quasi-set theory $Q$ is based on Zermelo-Fraenkel-like axioms and allows the presence of two sorts of atoms (*Urlemma*)t, termed $m$-atoms (micro-atoms) and $M$-atoms (macro-atoms). Concerning the $m$-atoms, a weaker ‘relation of
indistinguishability’ (denoted by the symbol \(\equiv\)), is used instead of identity, and it is postulated that \(\equiv\) has the properties of an equivalence relation. The predicate of equality cannot be applied to the \(m\)-atoms, since no expression of the form \(x = y\) is a formula if \(x\) or \(y\) denote \(m\)-atoms. Hence, there is a precise sense in saying that \(m\)-atoms can be indistinguishable without being identical.

The universe of \(Q\) is composed by \(m\)-atoms, \(M\)-atoms and quasi-sets. The axiomatization is adapted from that of ZFU (Zermelo-Fraenkel with "Urelemente"), and when we restrict the theory to the case which does not consider \(m\)-atoms, quasi-set theory is essentially equivalent to ZFU, and the corresponding quasi-sets can then be termed ‘sets’ (similarly, if also the \(M\)-atoms are ruled out, the theory collapses into ZFC). The \(M\)-atoms play the same role of the "Urelemente" in ZFU.

In all that follows, \(\exists_Q\) and \(\forall_Q\) are the quantifiers relativized to quasi-sets. That is, \(Q(x)\) reads as ‘\(x\) is a quasi-set’.

In order to preserve the concept of identity for the ‘well-behaved’ objects, an Extensional Equality is defined for those entities which are not \(m\)-atoms on the following grounds: for all \(x\) and \(y\), if they are not \(m\)-atoms, then

\[
x =_E y := \forall z (z \in x \Leftrightarrow z \in y) \lor (M(x) \land M(y) \land x \equiv y).
\]

It is possible to prove that \(=_E\) has all the properties of classical identity in a first order theory and so these properties hold regarding \(M\)-atoms and ‘sets’. This happens because one of the axioms of quasi-set theory says that the axiom of substitutivity of standard identity holds only for extensional equality. Concerning the more general relationship of indistinguishability nothing else is said. In symbols, the first axioms of \(Q\) are:

- \(\forall x (x \equiv x)\),
- \(\forall x \forall y (x \equiv y \Rightarrow y \equiv x)\), and
- \(\forall x \forall y \forall z (x \equiv y \land y \equiv z \Rightarrow x \equiv z)\).

And the fourth axiom says that

- \(\forall x \forall y (x =_E y \Rightarrow (A(x, x) \Rightarrow A(x, y)))\), with the usual syntactic restrictions on the occurrences of variables in the formula \(A\).

In this text, all references to ‘\(=\)’ (in quasi-set theory) stand for ‘\(=_E\)’, and similarly ‘\(\leq\)’ and ‘\(\geq\)’ stand, respectively, for ‘\(\leq_E\)’ and ‘\(\geq_E\)’. Among the specific axioms of \(Q\), few of them deserve a more detailed explanation. The other axioms are adapted from ZFU.

For instance, to form certain elementary quasi-sets, such as those containing ‘two’ objects, we cannot use something like the usual ‘pair axiom’, since its standard formulation assumes identity; we use the weak relation of indistinguishability instead:
The ‘Weak-Pair’ Axiom - For all \( x \) and \( y \), there exists a quasi-set whose elements are the indistinguishable objects from either \( x \) or \( y \).

In symbols,

\[
∀x∀y∃Qz∀t(t ∈ z ⇔ t ≡ x \lor t ≡ y).
\]

Such a quasi-set is denoted by \([x, y]\) and, when \( x ≡ y \), we have \([x]\), by definition. We remark that this quasi-set cannot be regarded as the ‘singleton’ of \( x \), since its elements are all the objects indistinguishable from \( x \), so its ‘cardinality’ (see below) may be greater than 1. A concept of strong singleton, which plays a crucial role in the applications of quasi-set theory, may be defined.

In \( Q \) we also assume a Separation Schema, which intuitively says that from a quasi-set \( x \) and a formula \( α(t) \), we obtain a sub-quasi-set of \( x \) denoted by

\[
[t ∈ x : α(t)].
\]

We use the standard notation with ‘{’ and ‘}’ instead of ‘[’ and ‘]’ only in the case where the quasi-set is a set.

It is intuitive that the concept of function cannot also be defined in the standard way, so a weaker concept of quasi-function was introduced, which maps collections of indistinguishable objects into collections of indistinguishable objects; when there are no \( m \)-atoms involved, the concept is reduced to that of function as usually understood. Relations (or quasi-relations), however, can be defined in the usual way, although no order relation can be defined on a quasi-set of indistinguishable \( m \)-atoms, since partial and total orders require antisymmetry, which cannot be stated without identity. Asymmetry also cannot be supposed, for if \( x ≡ y \), then for every relation \( R \) such that \( ⟨x, y⟩ ∈ R \), it follows that \( ⟨y, x⟩ ∈ R \), by force of the axioms of \( Q \).

It is possible to define a translation from the language of ZFU into the language of \( Q \) in such a way that we can obtain a ‘copy’ of ZFU in \( Q \). In this copy, all the usual mathematical concepts (like those of cardinal, ordinal, etc.) can be defined; the ‘sets’ (actually, the ‘\( Q \)-sets’ which are ‘copies’ of the ZFU-sets) turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) does not contain \( m \)-atoms.

Although some authors like Weyl [12] sustain that (concerning cardinals and ordinals) “the concept of ordinal is the primary one”, quantum mechanics seems to present strong arguments for questioning this thesis, and the idea of presenting collections which have a cardinal but not an ordinal is one of the most basic and important assumptions of quasi-set theory.

The concept of quasi-cardinal is taken as primitive in \( Q \), subject to certain axioms that permit us to operate with quasi-cardinals in a similar way to that of cardinals in standard set theories. Among the axioms for quasi-cardinality, we mention those below, but first we recall that in \( Q \), \( qc(x) \) stands for the ‘quasi-cardinal’ of the quasi-set \( x \), while \( Z(x) \) says that \( x \) is a set (in \( Q \)). Furthermore, \( Cd(x) \) and \( card(x) \) mean ‘\( x \) is a cardinal’ and ‘the cardinal of \( x \)’, respectively, defined as usual in the ‘copy’ of ZFU.
Quasi-cardinality - Every quasi-set has an unique quasi-cardinal which is a cardinal (as defined in the ‘ZFU-part’ of the theory) and, if the quasi-set is in particular a set, then this quasi-cardinal is its cardinal \textit{stricto sensu}:

\[ \forall Q \exists Q! y(Cd(y) \land y =_{E} qc(x) \land (Z(x) \Rightarrow y =_{E} card(x))). \]

From the fact that \( \emptyset \) is a set, it follows that its quasi-cardinality is 0 (zero). \( Q \) still encompasses an axiom which says that if the quasi-cardinal of a quasi-set \( x \) is \( \alpha \), then for every quasi-cardinal \( \beta \leq \alpha \), there is a sub-quasi-set of \( x \) whose quasi-cardinal is \( \beta \), where the concept of \textit{sub-quasi-set} is like the usual one. In symbols,

The quasi-cardinals of sub-quasi-sets -

\[ \forall Q x(qc(x) =_{E} \alpha \Rightarrow \forall \beta \leq \alpha \exists Q y(y \subseteq x \land qc(y) =_{E} \beta)). \]

Another axiom states that

The quasi-cardinal of the power quasi-set -

\[ \forall Q x(qc(P(x)) =_{E} 2^{qc(x)}). \]

where \( 2^{qc(x)} \) has its usual meaning.

These last axioms allow us to talk about the quantity of elements of a quasi-set, although we cannot count its elements in many situations.

As remarked above, in \( Q \) there may exist quasi-sets whose elements are \( m \)-atoms only, called ‘pure’ quasi-sets. Furthermore, it may be the case that the \( m \)-atoms of a pure quasi-set \( x \) are indistinguishable from one another. In this case, the axiomatization provides the grounds for saying that nothing in the theory can distinguish among the elements of \( x \). But, in this case, one could ask what it is that sustains the idea that there is more than one entity in \( x \). The answer is obtained through the above mentioned axioms (among others, of course). Since the quasi-cardinal of the power quasi-set of \( x \) has quasi-cardinal \( 2^{qc(x)} \), then if \( qc(x) = \alpha \), for every quasi-cardinal \( \beta \leq \alpha \) there exists a sub-quasi-set \( y \subseteq x \) such that \( qc(y) = \beta \), according to the axiom about the quasi-cardinality of the sub-quasi-sets. Thus, if \( qc(x) = \alpha \neq 0 \), the axiomatization does not forbid the existence of \( \alpha \) sub-quasi-sets of \( x \) which can be regarded as ‘singletons’.

Of course the theory cannot prove that these ‘unitary’ sub-quasi-sets (supposing now that \( qc(x) \geq 2 \)) are distinct, since we have no way of ‘identifying’ their elements, but quasi-set theory is compatible with this idea. In other words, it is consistent with \( Q \) to advocate that \( x \) has \( \alpha \) elements, which may be regarded as absolutely indistinguishable objects. Since the elements of \( x \) may share the relation \( \equiv \), they may be further understood as belonging to the same ‘equivalence class’ but in such a way that we cannot assert either that they are identical or that they are distinct from one another.
The collections $x$ and $y$ are defined as *similar* quasi-sets (in symbols, $\text{Sim}(x, y)$) if the elements of one of them are indistinguishable from the elements of the other one, that is, $\text{Sim}(x, y)$ if and only if $\forall z \forall t (z \in x \land t \in y \Rightarrow z \equiv t)$. Furthermore, $x$ and $y$ are $Q$-Similar ($\text{QSim}(x, y)$) if and only if they are similar and have the same quasi-cardinality. Then, since the quotient quasi-set $x/\equiv$ may be regarded as a collection of equivalence classes of indistinguishable objects, then the ‘weak’ axiom of extensionality is:

**Weak Extensionality -**

$$\forall Q x \forall y (\forall z (z \in x/\equiv \Rightarrow \exists t (t \in y/\equiv \land \text{QSim}(z, t))) \Rightarrow x \equiv y)$$

In other words, this axiom says that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable.

**Definition 1** A strong singleton of $x$ is a quasi-set $x'$ which satisfies the following property:

$$x' \subseteq [x] \land \text{qc}(x') = E 1$$

**Definition 2** A $n$-singleton of $x$ is a quasi-set $[x]_n$ which satisfies the following property:

$$[x]_n \subseteq [x] \land \text{qc}([x]_n) = E n$$

### 3 Finite enumerable but undecidable quasi-sets

This section introduces the main contributions of this paper. The next definition is crucial for our purposes. It is important to recall that if $x$ is a term, then $x'$ is a strong singleton whose only element is indistinguishable from $x$.

**Definition 3** If $x$ is a quasi-set and $y$ is indistinguishable from a given element $z$ that belongs to $x$, then

$$x \ominus y' = E x - z', \text{ where } z' \subseteq x.$$  

We call $\ominus$ the *strong difference* between quasi-sets. This operation allows us to drop one of the elements of $x$. So, if $\text{qc}(x) = n$ and $n$ is a natural number, then $\text{qc}(x \ominus y') = n - 1$.

Now we introduce an algorithm which allows us to prove that every finite quasi-set is enumerable. We recall that a set $x$ is enumerable if there is an algorithm that prints all elements of $x$ and only them. We first prove the most interesting case where all elements of the given quasi-set are micro-atoms of the same type (i.e., indistinguishable). Other cases may be proved by similar arguments.

**Theorem 1** If $[x]_n$ is a finite $n$-singleton, then it is enumerable.
Proof: Consider the following algorithm,

1. INPUT \([x]_n\)
2. DO \(y := [x]_n\)
3. DO \([x]_{n-1} := [x]_n \ominus x'\)
4. PRINT \(y - [x]_{n-1}\)
5. DO \(n := n - 1\)
6. IF \([x]_n =_E \emptyset\) THEN GO TO 8
7. GO TO 2
8. END

In the first step, we introduce a finite \(n\)-singleton \([x]_n\), i.e., a pure quasi-set with a finite quasi-cardinality (a finite number of elements) where all its elements are indistinguishable objects of the same kind. In the second step we attribute \([x]_n\) to \(y\), which means that \(y\) and \([x]_n\) are extensionally identical. Next, we perform the strong difference in order to drop one of the elements from the quasi-set \([x]_n\) and attribute this new collection to \([x]_{n-1}\). Then we print the element that was subtracted from \([x]_n\). We repeat this process until \([x]_n\) gets empty.

So, we printed all the elements of the original \([x]_n\), which means that \([x]_n\) is enumerable.

Now we will prove that even a finite enumerable quasi-set may be undecidable. We recall that a collection \(x\) is decidable if there exists an algorithm that determines whether an arbitrary object belongs or not to \(x\). Otherwise, \(x\) is said undecidable.

**Theorem 2** If \([x]_n\) is a non-empty finite \(n\)-singleton, then it is undecidable.

**Proof:** If \(y \equiv x\), then there is no way to know if \(y\) belongs to \([x]_n\) or not. Actually we cannot even know if \(x \in [x]_n\), although we always know that \(x \in [x]\). This happens because in quasi-set theory it is legitimate the existence of many indistinguishable objects. So, \([x]_n\) is undecidable.

## 4 Final Remarks

There are many results concerning undecidability in mathematics and even in physics. See, for example, [1], where the authors derive a general undecidability and incompleteness result for elementary functions within Zermelo-Fraenkel set theory (with the axiom of choice), and apply it to some important problems in Hamiltonian mechanics and dynamical systems.
But all results on undecidability, as far as we know, refer to infinite sets or collections. In this paper we believe that we are presenting for the first time an example of a finite system that is undecidable. This is due to the fact that although in standard mathematics the membership relationship seems to present some tricky features when we talk about infinite collections, in quasi-set theory there is another tricky relationship, namely, indistinguishability.

We do not know if the weak singleton \([x]\) is enumerable or not (open problem). But by using similar arguments we can easily prove that \([x]\) is undecidable, if it is not empty.

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