Including $\beta-$ and $\gamma-$ Vibrations in Evaluating the Ground State Rotational Band of Deformed Even-Even Nuclei

Mohamed E. Kelabi*
A. Y. Ahmed*
Vikram Singh†

Abstract
From the semi-empirical formalisms of Bohr-Mottelson, a new model, based on the effect of $\beta$- and $\gamma$- head energies and the variable moment of inertia, was developed to calculate the ground state rotational band of almost all deformed $e-e$ nuclei. The model can be tuned for nuclei with experimentally available $\beta$- and $\gamma$-vibrational energies as well as for those with unmeasured $\beta$- and $\gamma$-vibrations.

Introduction
The Rotational-Vibrational model RVM of Bohr-Mottelson\cite{1}

$$E(J) = AJ(J+1) - BJ^2(J+1)^2$$

has been modified to take the effect of nuclear rotations via the variable moment of inertia\cite{2}, and introducing a correction term due to $\beta$- and $\gamma$- vibrations\cite{3}

$$\hbar \omega_\beta = \left[ \frac{\frac{1}{2} A^3}{B - \frac{1}{2} A^3 / (\hbar \omega_\gamma)^2} \right]^{1/2}$$

$$\hbar \omega_\gamma = \left[ \frac{\frac{1}{2} A^3}{B - \frac{1}{2} A^3 / (\hbar \omega_\beta)^2} \right]^{1/2}$$

where, $\hbar \omega_\beta$ and $\hbar \omega_\gamma$ are the head energies of $\beta$- and $\gamma$- vibrations, respectively; and $A$ and $B$ are defined by Eq. (1). Sood\cite{2} used molecular spectra theory\cite{4} to expressed the nuclear moment of inertia dependence on the angular momentum as

$$E(J) = A(J) J(J+1)$$

where

$$A(J) = A \left[ \frac{1 + (a + b J - 1)(B / A) J(J+1)}{1 + (a + b) J(B / A) J(J+1)} \right] .$$

The parameter $(B / A)$ was calculated from the observed energy ratio $E(4) / E(2)$, and the experimental energy of the $2^+$ state was then used to determine $A$, while $a$ and $b$ were adjusted from nucleus to another to give the best fit\cite{2}.

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* Physics Department, Al-Fateh University, Tripoli, LIBYA
† B-1/1051, Vasant Kunj, New Delhi 110070.
Eq. (3) seems to work well in describing the energies of the ground state bands in rare earth region and the actinides $^{232}$Th and $^{232-238}$U. On the other hand, it does not assume the effect of $\beta$- and $\gamma$- vibrational bands, which are experimentally observed\cite{3} in some nuclei. This fact was encouraging to deduce a new more systematic model by incorporating the Softness parameter\cite{5}

$$\sigma_n = \frac{1}{n!} \left[ \frac{\partial^n J(J)}{\partial J^n} \right]_{J=0}$$  \hspace{1cm} (4)

where $J_0$ is the unperturbed nuclear moment of inertia\cite{6}. Leaving aside few “very rigid nuclei”, the inclusion of the $\beta$- and $\gamma$- vibrations along with the variable moment of inertia, has led to the form\cite{7}

$$E(J) = A(J)J(J+1) - B(J)J^2(J+1)^2$$  \hspace{1cm} (5)

here the rotational parameter $A(J) = \frac{\hbar^2}{2J(J)}$ is connected with to the nuclear variable moment of inertia $J$, and $B(J) = \frac{4A(J)^3}{(\hbar\omega_0)^2} + \frac{12A(J)^3}{(\hbar\omega_0)^2}$ expresses the correction due to $\beta$- and $\gamma$- vibrations dependence on the nuclear moment of inertia.

**Formalism**

For readability purpose we rewrite Eq. (5) of the form

$$E(J) = A_0 \frac{1}{1 + \sigma_1 J} J(J+1) - B_0 \frac{(1-2\sigma_1 J)}{1+\sigma_1 J} J^2(J+1)^2$$  \hspace{1cm} (6)

where $A_0 = \frac{\hbar^2}{2J_0}$, $\sigma_1$ is defined by Eq. (4), and $B_0$ is related to $A_0$ by Eq. (2).

This non-linear Eq. (6) contains three free parameters $A_0$, $\sigma_1$, and $B_0$ which need to be solved by fit to the experimental data. The equation can be applied to calculate the energy levels of the ground state band of deformed $e-e$ nuclei in two major cases:

a) **Three parameters expression**

When the energies of $\beta$- and $\gamma$- vibrations are not experimentally available, then they can be treated as unknown, contained in $c$, and Eq. (6) takes the form

$$E(J) = A_0 \frac{1}{1 + \sigma_1 J} J(J+1) - c A_0 \frac{(1-2\sigma_1 J)}{1+\sigma_1 J} J^2(J+1)^2.$$  \hspace{1cm} (7)

The three free parameters $A_0$, $\sigma_1$, and $c$ can be determined by fitting the first three energy levels $2^+$, $4^+$, and $6^+$ with experimental data.

b) **Two parameters expression**

In some nuclei where the head energies of $\beta$- and $\gamma$- vibrations are experimentally available, then in this case, Eq. (6) can be rewritten as

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\[ E(J) = \frac{A_0}{1 + \sigma_1 J} J(J+1) - \left[ \frac{4}{(\hbar \omega_r)^2} + \frac{12}{(\hbar \omega_i)^2} \right] A_0^2 \left( 1 - 2\sigma_1 J \right) \frac{1}{1 + \sigma_1 J} J^2 \left( J + 1 \right)^2. \] (8)

This equation contains two free parameters \( A_0 \) and \( \sigma_1 \) which can be obtained by fitting only the first two energy levels \( 2^+ \) and \( 4^+ \) with experimental data.

**Results**

The results of our three parametric calculations Eq. (7), compared with experimental data and other existing models are presented in Table 1. The corresponding model parameters \( A_0, \sigma_1, \) and \( c \) determined by fitting are given in Table 2.

Keywords: EXP = Experimental data\(^{[8]}\)
VMI = Variable Moment of Inertia model\(^{[9]}\)
VAVM = Variable Anharmonic Vibrator Model\(^{[10],[11]}\)
GVMI = Generalized VMI\(^{[10],[11]}\)
PM = Present Model (our current work).

| Table 1. Comparison of our results with Experiment and other models in [MeV]. |
|---|
| \( E(2) \) | \( E(4) \) | \( E(6) \) | \( E(8) \) | \( E(10) \) | \( E(12) \) | \( E(14) \) | \( E(16) \) | \( E(18) \) |
| Ce-150 |
| EXP | 0.0978 | 0.308 | 0.6087 | 0.9851 |
| VMI | ------ | ------ | 0.607 | 0.977 | 1.406 | 1.886 | 2.411 | 2.976 | 3.577 |
| VAVM | ------ | ------ | ------ | 0.979 | 1.406 | 1.88 | 2.394 | 2.944 | 3.527 |
| GVMI | ------ | ------ | ------ | 0.983 | 1.419 | 1.907 | 2.422 | 3.02 | 3.634 |
| PM | ------ | ------ | ------ | 0.9809 | 1.411 | 1.894 | 2.434 | 3.047 | 3.761 |
| Nd-154 |
| EXP | 0.0729 | 0.2352 | 0.4789 | 0.807 |
| VMI | ------ | ------ | 0.475 | 0.78 | 1.141 | 1.551 | 2.005 | 2.498 | 3.026 |
| VAVM | ------ | ------ | ------ | 0.792 | 1.166 | 1.591 | 2.062 | 2.574 | 3.122 |
| GVMI | ------ | ------ | ------ | 0.795 | 1.175 | 1.611 | 2.099 | 2.634 | 3.21 |
| PM | ------ | ------ | ------ | 0.796 | 1.1809 | 1.626 | 2.128 | 2.683 | 3.286 |
| Sm-154 |
| EXP | 0.08198 | 0.2667 | 0.5443 | 0.9031 |
| VMI | ------ | ------ | 0.542 | 0.897 | 1.321 | 1.805 | 2.343 | 2.929 | 3.561 |
| VAVM | ------ | ------ | ------ | 0.901 | 1.327 | 1.81 | 2.346 | 2.927 | 3.549 |
| GVMI | ------ | ------ | ------ | 0.904 | 1.336 | 1.833 | 2.387 | 2.994 | 3.649 |
| PM | ------ | ------ | ------ | 0.9032 | 1.3312 | 1.8152 | 2.342 | 2.901 | 3.481 |
| Sm-156 |
| EXP | 0.076 | (0.2501) | (0.5179) | (0.878) |
| VMI | ------ | ------ | 0.516 | 0.865 | 1.29 | 1.783 | 2.239 | 2.952 | 3.617 |
| VAVM | ------ | ------ | ------ | 0.872 | 1.304 | 1.808 | 2.375 | 3 | 3.668 |
| GVMI | ------ | ------ | ------ | 0.873 | 1.311 | 1.823 | 2.406 | 3.053 | 3.76 |
| PM | ------ | ------ | ------ | 0.874 | 1.314 | 1.832 | 2.423 | 3.08 | 3.798 |
| Sm-158 |
| EXP | 0.0728 | 0.2403 | 0.4985 | (0.8445) |
| VMI | ------ | ------ | 0.497 | 0.838 | 1.255 | 1.742 | 2.294 | 2.905 | 3.572 |
| VAVM | ------ | ------ | ------ | 0.841 | 1.259 | 1.747 | 2.298 | 2.905 | 3.566 |
| GVMI | ------ | ------ | ------ | 0.842 | 1.265 | 1.762 | 2.328 | 2.957 | 3.645 |
| PM | ------ | ------ | ------ | 0.842 | 1.265 | 1.759 | 2.317 | 2.93 | 3.588 |
|        | $E(2)$ | $E(4)$ | $E(6)$ | $E(8)$ | $E(10)$ | $E(12)$ | $E(14)$ | $E(16)$ | $E(18)$ |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| **Gd-160** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.07526 | 0.2482 | 0.514  | 0.868  |        |        |        |        |        |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Dy-158** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.09894 | 0.31726 | 0.63787 | 1.0441 | 1.5199 | 2.0492 | 2.6126 | 3.1907 | 3.7817 |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Dy-160** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.08679 | 0.28382 | 0.5812  | 0.9672 | 1.4286 | 1.9514 | 2.5152 | 3.0919 | 3.6724 |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Dy-164** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.07339 | 0.2423 | 0.50132 | 0.84367 | 1.25876 |        |        |        |        |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Er-162** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.10208 | 0.32963 | 0.66676 | 1.0968 | 1.6028 | 2.1651 | 2.7457 | 3.2923 | 3.8465 |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Yb-166** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.10238 | 0.3305 | 0.668  | 1.09829 | 1.6059 | 2.1757 | 2.7795 | 3.274  | 3.7831 |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Yb-168** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.08773 | 0.28655 | 0.5853  | 0.97006 | 1.424  | (1.936) | (2.489) | (3.073) | (3.687) |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
| **Yb-170** |        |        |        |        |        |        |        |        |        |
| EXP    | 0.08426 | 0.27745 | 0.5736  | 0.9636 | 1.4379 | 1.9837 | 2.5808 | 3.1962 | 3.8081 |
| VMI    |        |        |        |        |        |        |        |        |        |
| VAVM   |        |        |        |        |        |        |        |        |        |
| GVMI   |        |        |        |        |        |        |        |        |        |
| PM     |        |        |        |        |        |        |        |        |        |
|       | E(2) | E(4) | E(6) | E(8) | E(10) | E(12) | E(14) | E(16) | E(18) |
|-------|------|------|------|------|-------|-------|-------|-------|-------|
| Yb-176 |      |      |      |      |       |       |       |       |       |
| EXP   | 0.08213 | 0.27169 | 0.5648 | 0.9541 | 1.4312 | 1.9849 | 2.602 | 3.27 | 3.979 |
| VMI   | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| Hf-174 |      |      |      |      |       |       |       |       |       |
| EXP   | 0.09101 | 0.29745 | 0.60837 | 1.00943 | 1.487 | 2.022 | 2.599 |
| VMI   | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-174  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.1119 | 0.355 | 0.704 | 1.137 | 1.635 | 2.186 | 2.78 | 3.392 | 3.973 |
| VMI   | ----- | ----- | ----- | 0.7 | 1.14 | 1.65 | 2.22 | 2.84 | 3.51 |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-178  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.1061 | 0.3431 | 0.6947 | 1.1423 | 1.6661 | 2.2452 | 2.8593 | 3.4891 | 4.1016 |
| VMI   | ----- | ----- | ----- | 0.693 | 1.14 | 1.669 | 2.27 | 2.936 | 3.659 |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-180  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.10357 | 0.33755 | 0.68845 | 1.13847 | 1.66418 | 2.2351 | 2.825 | 3.416 | 4.021 |
| VMI   | ----- | ----- | ----- | 0.6879 | 1.1401 | 1.6812 | 2.301 | 2.9911 |       |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-182  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.10011 | 0.32942 | 0.6805 | 1.1445 | 1.712 | 2.237 | (3.113) |      |        |
| VMI   | ----- | ----- | ----- | 0.6792 | 1.139 | 1.6985 | 2.3481 | 3.0797 | 3.8863 | 4.7619 |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-184  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.11121 | 0.36406 | 0.74831 | 1.252 | 1.851 |      |        |      |        |
| VMI   | ----- | ----- | ----- | 0.7458 | 1.2424 | 1.8408 | 2.5299 | 3.3006 | 4.1455 | 5.0581 |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| W-186  |      |      |      |      |       |       |       |       |       |
| EXP   | 0.1223 | 0.39647 | 0.80847 | 1.348 | 2.002 |      |        |      |        |
| VMI   | ----- | ----- | ----- | 0.8032 | 1.3237 | 1.9424 | 2.6471 | 3.4283 | 4.2784 | 5.1913 |
| VAVM  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| GVMI  | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
| PM    | ----- | ----- | ----- | ----- | ------ | ------ | ------ | ----- | ------ |
Table 2. Fitting parameters of Eq. (7).

| Nucleus | $\sigma_1 \times 10^{-3}$ | $A_0$ KeV | $c \times 10^{-6}$ KeV$^{-2}$ |
|---------|-----------------------------|-----------|--------------------------------|
| Ce-150  | 21.6541                     | 17.14078  | 4.89                           |
| Nd-154  | 15.7048                     | 12.52187  | 0.67                           |
| Sm-154  | 7.9415                      | 13.93509  | 3.48                           |
| Sm-156  | 5.2863                      | 12.81437  | 1.12                           |
| Sm-158  | 2.7119                      | 12.22257  | 2.16                           |
| Gd-160  | 2.4274                      | 12.63628  | 2.67                           |
| Dy-158  | 13.8062                     | 17.04180  | 3.44                           |
| Dy-160  | 4.0815                      | 14.65349  | 3.79                           |
| Er-162  | 9.9757                      | 17.45010  | 3.18                           |
| Yb-166  | 9.7046                      | 17.49824  | 3.36                           |
| Yb-168  | 3.5835                      | 14.81119  | 4.41                           |
| Yb-170  | 3.0429                      | 14.16695  | 2.26                           |
| Yb-176  | 1.5835                      | 13.75813  | 1.70                           |
| Hf-174  | 3.8362                      | 15.36536  | 3.76                           |
| W-174   | 15.4814                     | 19.40646  | 4.35                           |
| W-178   | 8.9999                      | 18.10533  | 3.02                           |
| W-180   | 5.0998                      | 17.53311  | 3.01                           |
| W-182   | 3.2526                      | 16.84082  | 1.67                           |
| W-184   | 5.8207                      | 18.80577  | 1.41                           |
| W-186   | 12.0535                     | 20.91881  | 0.84                           |
| Pu-238  | -0.0313                     | 7.366827  | 8.59                           |
| Pu-240  | 0.8512                      | 7.168516  | 8.19                           |
| Pu-242  | 2.5625                      | 7.472661  | 4.55                           |
In Table 3, we also present a sample of our calculations in comparison with experimental data and Sood’s results\cite{2}. The corresponding parameters $A_0$, $\sigma_1$, and $c$ obtained by fitting are given in Table 4.

Table 3. Comparison of our calculations with the experimental data\cite{2} and Sood’s results\cite{2} in [MeV].

|         | $E(2)$ | $E(4)$ | $E(6)$ | $E(8)$ | $E(10)$ | $E(12)$ | $E(14)$ | $E(16)$ | $E(18)$ |
|---------|--------|--------|--------|--------|---------|---------|---------|---------|---------|
| Gd-158  |        |        |        |        |         |         |         |         |         |
| EXP     | 0.0796 | 0.2619 | 0.539  | 0.898  |         |         |         |         |         |
| PM      | 0.0796 | 0.2619 | 0.539  | 0.898  |         |         |         |         |         |
| Sood    | 0.0796 | 0.2617 | 0.539  | 0.901  |         |         |         |         |         |
| Er-166  |        |        |        |        |         |         |         |         |         |
| EXP     | 0.0806 | 0.2649 | 0.545  | 0.910  | 1.334   |         |         |         |         |
| PM      | 0.0806 | 0.2649 | 0.545  | 0.909  | 1.342   |         |         |         |         |
| Sood    | 0.0806 | 0.2647 | 0.544  | 0.908  | 1.344   |         |         |         |         |
| Yb-172  |        |        |        |        |         |         |         |         |         |
| EXP     | 0.0787 | 0.2603 | 0.540  | 0.910  | 1.352   |         |         |         |         |
| PM      | 0.0787 | 0.2603 | 0.540  | 0.910  | 1.361   |         |         |         |         |
| Sood    | 0.0787 | 0.2602 | 0.540  | 0.910  | 1.364   |         |         |         |         |
| Os-184  |        |        |        |        |         |         |         |         |         |
| EXP     | 0.1198 | 0.3836 | 0.774  | 1.274  | 1.871   |         |         |         |         |
| PM      | 0.1198 | 0.3836 | 0.774  | 1.275  | 1.875   |         |         |         |         |
| Sood    | 0.1180 | 0.3834 | 0.777  | 1.278  | 1.867   |         |         |         |         |

Table 4. Fitting parameters of Eq. (7).

| Nucleus | $\sigma_1 \times 10^{-3}$ | $A_0$ KeV | $c \times 10^{-6}$ KeV$^{-2}$ |
|---------|---------------------------|------------|-----------------------------|
| Gd-158  | -0.14                     | 13.33      | 5.08                         |
| Er-166  | 1.55                      | 13.54      | 4.32                         |
| Yb-172  | 0.12                      | 13.16      | 3.09                         |
| Os-184  | 19.63                     | 20.78      | 0.64                         |

Further application of our model is that, by employing Eq. (8) to calculate the energy levels of ground state bands of some deformed $e-e$ actinides where the energies of $\beta$- and $\gamma$-vibrations are experimentally available\cite{12}. The results of our calculations are listed in Table 5, and the corresponding values of $A_0$ and $\sigma_1$ determined by fitting are given in Table 6.
Table 5. Comparison of our results with the available experimental data\cite{12} in [MeV].

|          | $E(2)$ | $E(4)$ | $E(6)$ | $E(8)$ | $E(10)$ | $E(12)$ | $E(14)$ | $E(16)$ | $E(18)$ |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Pu-238   |        |        |        |        |        |        |        |        |        |
| (EXP $E_{\beta} = 0.942, E_{\gamma} = 1.029$) | 0.044  | 0.146  | 0.303  | 0.513  | 0.773  | 1.079  | 1.427  | 1.816  | 2.241  |
| PM       | -------| -------| 0.302  | 0.505  | 0.744  | 1.006  | 1.272  | 1.518  | 1.716  |
| Pu-240   |        |        |        |        |        |        |        |        |        |
| (EXP $E_{\beta} = 0.860, E_{\gamma} = 0.900$) | 0.043  | 0.142  | 0.294  | 0.498  | (0.748)|        |        |        |        |
| PM       | -------| -------| 0.292  | 0.487  | 0.713  | 0.954  | 1.189  | 1.389  | 1.521  |
| U-238    |        |        |        |        |        |        |        |        |        |
| (EXP $E_{\beta} = 0.809, E_{\gamma} = 0.927$) | 0.043  | 0.143  | 0.296  | 0.497  | 0.741  | 1.024  | 1.341  | 1.688  | 2.063  |
| PM       | -------| -------| 0.294  | 0.487  | 0.708  | 0.937  | 1.153  | 1.325  | 1.417  |

Table 6. Fitting parameters of Eq. (8)

| Nucleus | $\sigma_{\gamma} \times 10^{-3}$ | $A_0$ [KeV] |
|---------|----------------------------------|-------------|
| Pu-238  | -3.472                           | 7.337       |
| Pu-240  | -4.170                           | 7.125       |
| U-238   | -3.157                           | 7.257       |

Conclusion

The present model Eq. (7) is practically fit to almost all deformed $\varepsilon$-$\varepsilon$ nuclei. The results of 24 nuclei where the $\beta$- and $\gamma$-vibrational energies were treated as unknowns are reported in Table 1 and Table 3. A detailed comparison of our calculations with other proposed models, reveals the closest agreement to the experimental energies. At high spin states, as in the case of eg., $^{162}$Er, $^{178}$W, and $^{180}$W nuclei, a surprisingly excellent predictions which had not been satisfactorily described by any other model. Table 1 and Table 3 are direct support to the fact that the $\beta$- and $\gamma$-vibrational energies and the variation of the moment of inertia with spin $J$ can not be ignored in evaluating the energies of the ground state rotational bands.

Our model was also applied to some nuclei where the energies of $\beta$- and $\gamma$-vibrations are experimentally available. With the known values of $\hbar \omega_{\beta}$ and $\hbar \omega_{\gamma}$, Eq. (7) reduces to two parametric expression Eq. (8). Keeping in view the inherent limitations of Eq. (2), we observe a small deviation at high spin states. Because no much data is available for $\beta$- and $\gamma$-vibrations, it is hoped that experiments may be planned to populate $\beta$- and $\gamma$-vibrational bands to test the validity of our two parametric model and also to verify Eq. (2).
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