Green’s Function and Positive Solutions of a Third-Order Equation with Periodic Boundary Conditions

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We apply the fixed point index to obtain positive solutions of a nonresonant periodic boundary value problem for a third-order differential equation $u'' + \rho^3 u = \lambda f(u)$.

1. Introduction

Third-order differential equations arise in many areas of physics and engineering [1] and describe, for example, deflection of a curved beam having a constant or varying cross section, a three-layered beam, and electromagnetic waves. Boundary value problems for third-order differential equations have been studied by many authors, for example, [2–9] just to name a few. In this paper, we consider a well-known [10, 11] boundary value problem:

$$u''(t) + \rho^3 u(t) = \lambda f(u(t)), \quad t \in (0, 2\pi),$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2.$$  (2)

We improve the results of [10, 11] and obtain positive solutions using the fixed point index. The solutions to (1) and (2) will be sought in the Banach space $\mathcal{B} = C[0, 2\pi]$ endowed with the max-norm. In order to obtain positive solutions, we apply the fixed point theorem of Guo and Lakshmikantham [12] stated in Section 2.

Green’s function of $u''(t) + \mu u(t) = 0$ with $u(0) = u(2\pi)$ is

$$G_1(t, s) = \frac{1}{e^{\rho \pi} - 1} \begin{cases} e^{(2\pi + s-t)}, & 0 \leq s \leq t \leq 2\pi, \\ e^{(s-t)}, & 0 \leq t \leq s \leq 2\pi. \end{cases}$$  (3)

Green’s function of $u''(t) - \rho u'(t) + \rho^2 u(t) = 0$ with $u^{(i)}(0) = u^{(i)}(2\pi), i = 0, 1$, is

$$G_2(t, s) = \frac{2}{\sqrt{3}\rho \left(e^{\rho \pi} + e^{-\rho \pi} - 2 \cos \sqrt{3}\rho \pi\right)} \begin{cases} g_1(t-s), & 0 \leq s \leq t \leq 2\pi, \\ g_2(s-t), & 0 \leq t \leq s \leq 2\pi, \end{cases}$$  (4)

where

$$g_1(x) = e^{(1/2)x} \left(e^{-\rho \pi} - \cos \sqrt{3}\rho \pi \sin \frac{\sqrt{3}}{2} \rho x + \sin \sqrt{3}\rho \pi \cos \frac{\sqrt{3}}{2} \rho x\right),$$  (5)

$$g_2(x) = e^{- (1/2)x} \left(e^{\rho \pi} - \cos \sqrt{3}\rho \pi \sin \frac{\sqrt{3}}{2} \rho x + \sin \sqrt{3}\rho \pi \cos \frac{\sqrt{3}}{2} \rho x\right),$$  (6)

where $x \in [0, 2\pi]$. To ensure that $G_2(t, s) \geq 0$, we need $\rho \in (0, 1/\sqrt{3})$.

We maximize $g_1$ and $g_2$. Introduce, for convenience,

$$A(\rho) = e^{-\rho \pi} - \cos \sqrt{3}\rho \pi,$$

$$B(\rho) = \sin \sqrt{3}\rho \pi,$$

$$C(\rho) = e^{\rho \pi} - \cos \sqrt{3}\rho \pi.$$  (7)

We formally find that $g_1'(x_1) = 0$ if
\[ x_1(\rho) = \frac{2}{\sqrt{3}} \cos^{-1} \sqrt{3} B(\rho) - A(\rho) + \frac{B(\rho)}{\sqrt{3} A(\rho)} \] (8)

provided \( x_1 \in (0, 2\pi) \).

Note that
\[
\lim_{\rho \to 0} \frac{A(\rho)}{B(\rho)} = \frac{1}{\sqrt{3}} \\
\lim_{\rho \to (1/\sqrt{3})^-} \frac{A(\rho)}{B(\rho)} = \infty.
\]

Also
\[
\left( \frac{A'}{B} \right) (\rho) = \frac{\sqrt{3} - e^{-\pi \rho} (\sin 3\pi \rho + \sqrt{3} \cos 3\pi \rho)}{\sin^2 3\pi \rho}
\]
(10)

Denoting
\[ \alpha(\rho) = \sqrt{3} - e^{-\pi \rho} (\sin 3\pi \rho + \sqrt{3} \cos 3\pi \rho), \]
we have \( \alpha'(\rho) = 4\pi e^{-\pi \rho} \sin 3\pi \rho > 0, \rho \in (0, 1/\sqrt{3}) \), and \( \alpha(0) = 0 \) and we have \( (A/B)'(\rho) > 0 \). Hence,
\[ \phi(\rho) = \frac{\sqrt{3} B(\rho) - A(\rho)}{\sqrt{3} A(\rho) + B(\rho)} \]
(12)

is a decreasing function on \((0, 1/\sqrt{3})\), \( \lim_{\rho \to 0^-} \phi(\rho) = \infty \), and
\[ \lim_{\rho \to (1/\sqrt{3})^-} \phi(\rho) = -1/\sqrt{3}. \]
We have
\[ \lim_{\rho \to 0^+} x_1(\rho) = \lim_{\rho \to (1/\sqrt{3})^-} \frac{2}{\sqrt{3}} \cot^{-1} \phi(\rho) \]
\[ = \frac{2}{\sqrt{3}} \lim_{\rho \to 0^-} \frac{\psi(\rho)}{1 + \psi^2(\rho)} \]
\[ = \frac{2}{\sqrt{3}} \lim_{\rho \to 0^-} A'(\rho)B(\rho) - A(\rho)B'(\rho) \]
\[ = \frac{2}{\sqrt{3}} \lim_{\rho \to 0^-} \frac{\sqrt{3} - e^{-\pi \rho} (\sin 3\pi \rho + \sqrt{3} \cos 3\pi \rho)}{e^{-2\pi \rho} + 1 - 2e^{-\pi \rho} \cos 3\pi \rho} \]
\[ = \frac{2\pi}{\sqrt{3}} \lim_{\rho \to 0^-} \frac{2 \sin 3\pi \rho}{\cos 3\pi \rho + \sqrt{3} \sin 3\pi \rho - e^{-\pi \rho}} \]
\[ = \frac{2\pi}{\sqrt{3}} \lim_{\rho \to 0^-} \frac{2 \sqrt{3} \cos 3\pi \rho}{\cos 3\pi \rho + \sqrt{3} \sin 3\pi \rho + e^{-\pi \rho}} \]
\[ = \pi. \]

It is a lengthy exercise to verify that \( x_1(\rho) \) is increasing on \((0, 1/\sqrt{3})\). We infer
\[ \pi < x_1(\rho) < \frac{4\pi}{3} \]
(14)

It follows from (8) that
\[ \cos \frac{\sqrt{3}}{2} \rho x_1 = \frac{\sqrt{3} B - A}{2\sqrt{A^2 + B^2}}, \]
\[ \sin \frac{\sqrt{3}}{2} \rho x_1 = \frac{\sqrt{3} A + B}{2\sqrt{A^2 + B^2}}. \]

In fact, \( g_1''(x_1) < 0 \); thus,
\[ g_1(x) \leq g_1(x_1) = e^{(1/2)\rho x_1} \left( A \sin \frac{\sqrt{3}}{2} \rho x_1 + B \cos \frac{\sqrt{3}}{2} \rho x_1 \right) \]
\[ = \frac{\sqrt{3}}{2} e^{(1/2)\rho x_1} \sqrt{A^2 + B^2}. \]

By a similar argument, one can show that
\[ g_2(x) \leq g_2(x_2), \]
where
\[ x_2(\rho) = \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3} C(\rho) - B(\rho)}{\sqrt{3} B(\rho) + C(\rho)} \in \left( 2\pi, \frac{1}{3}, \pi \right), \]
(17)

so that
\[ g_2(x) \leq g_2(x_2) = e^{-(1/2)\rho x_2} \left( C \sin \frac{\sqrt{3}}{2} \rho x_2 + B \cos \frac{\sqrt{3}}{2} \rho x_2 \right) \]
\[ = \frac{\sqrt{3}}{2} e^{-(1/2)\rho x_2} \sqrt{C^2 + B^2}. \]

We are in position to state and prove our first lemma.

**Lemma 1.** Green's functions \( G_1(t, s) \) and \( G_2(t, s) \) satisfy
\[ M_1 = \max_{t, s \in [0, 2\pi]} G_1(t, s) = \frac{e^{2\rho s}}{e^{2\rho s} - 1}, \]
(19)
\[ m_1 = \min_{t, s \in [0, 2\pi]} G_1(t, s) = \frac{1}{e^{2\rho s} - 1}, \]
(20)
\[ M_2 = \max_{t, s \in [0, 2\pi]} G_2(t, s) = \frac{1}{\sqrt{e^{2\rho s} + e^{-2\rho s} - 2 \cos \sqrt{3}\pi \rho}} \]
\[ \cdot \max \left\{ e^{(1/2)\rho(\pi - \sigma)}, e^{-1/2)\rho(\sigma - \pi)} \right\}, \]
(21)
\[ m_2 = \max_{t, s \in [0, 2\pi]} G_2(t, s) = \frac{2 \sin \sqrt{3}\pi \rho}{\sqrt{3}\rho e^{2\rho s} + e^{-2\rho s} - 2 \cos \sqrt{3}\pi \rho} \]
(22)

**Proof.** Identities (19) and (20) are easy to check.

The following identities are useful:
\[ A^2 + B^2 = e^{-2\rho s} - 2e^{-\rho s} \cos \sqrt{3}\pi \rho + 1, \]
\[ C^2 + B^2 = e^{2\rho s} - 2e^{\rho s} \cos \sqrt{3}\pi \rho + 1, \]
(23)
\[ A^2 + B^2 = e^{-2\rho s}(C^2 + B^2), \]
\[ e^{\rho s} + e^{-\rho s} - 2 \cos \sqrt{3}\pi \rho = e^{-\rho s}(C^2 + B^2). \]
Then, using (16) and (18), we have
\[
\max_{t, s \in [0, 2\pi]} G_2(t, s) = \frac{2}{\sqrt{3} \rho (e^{\rho t} + e^{-\rho t} - 2 \cos \sqrt{3} \rho)} \\
\cdot \max \{g_1(x_1), g_2(x_2)\} = \frac{2e^{\rho t}}{\sqrt{3} (C^2 + B^2)} \max \{e^{(1/2)\rho x_1}, \sqrt{A^2 + B^2}, e^{-(1/2)\rho x_1} \sqrt{C^2 + B^2}\} = \frac{e^{\rho}}{\rho \sqrt{C^2 + B^2}} \max \{e^{(1/2)\rho x_1}, e^{-(1/2)\rho x_1}\} = \frac{1}{\rho \sqrt{e^{\rho t} + e^{-\rho t} - 2 \cos \sqrt{3} \rho}} \cdot \max \{e^{(1/2)\rho (x_1 - \pi)}, e^{(1/2)\rho (\pi - x_2)}\},
\]
where \(x_1\) and \(x_2\) are given by (8) and (17), respectively. In addition,
\[
\min_{x \in [0, 2\pi]} g_1(x) = \min \{g_1(0), g_1(2\pi)\} = \sin \sqrt{3} \rho \\
= \min \{g_2(0), g_2(2\pi)\} = \min_{x \in [0, 2\pi]} g_2(x).
\]
Hence,
\[
\min_{t, s \in [0, 2\pi]} G_2(t, s) = \frac{2 \sin \sqrt{3} \rho}{\sqrt{3} \rho (e^{\rho t} + e^{-\rho t} - 2 \cos \sqrt{3} \rho)}.
\]
Observe that (21) and (22) are exact and improve the corresponding estimates stated in Lemma 3 of [10], namely,
\[
\frac{2 \sin \sqrt{3} \rho}{\sqrt{3} \rho (e^{\rho t} + 1)^2} \leq G_2(t, s) \leq \frac{2}{\sqrt{3} \rho \sin \sqrt{3} \rho}
\]
Indeed, \((e^{\rho t} + 1)^2 > e^{\rho t} + e^{-\rho t} - 2 \cos \sqrt{3} \rho\), so (22) improves the first inequality. From (14) and (17), it is clear that
\[
\max \left\{e^{(1/2)\rho (x_1 - \pi)}, e^{(1/2)\rho (\pi - x_2)}\right\} < e^{(1/2)\rho},
\]
and
\[
\frac{1}{\sqrt{1 + e^{2\rho t} - 2e^{-\rho t} \cos \sqrt{3} \rho}} < \frac{1}{\sqrt{e^{\rho t} + e^{-\rho t} - 2 \cos \sqrt{3} \rho}} \leq \frac{2}{\sqrt{3} \rho \sin \sqrt{3} \rho},
\]
that is, (21) is preferred to the constant in the second inequality.

Now, Green’s function of \(u''(t) + \rho^2 u(t) = 0, u^{(i)}(0) = u^{(i)}(2\pi), i = 0, 1, 2\), is determined from (3)–(6) by
\[
G(t, s) = \int_0^{2\pi} G_1(t, \tau) G_2(\tau, s) d\tau = L_1 H_1(t, s) + L_2 H_2(t, s),
\]
where
\[
H_1(t, s) = \begin{cases} 
\frac{e^{(1/2)\rho (t-s)}}{\sqrt{3} \rho \sin \left(\frac{\sqrt{3} \rho (t-s) - \pi}{6}\right)} - e^{\rho t} \sin \left(\frac{\sqrt{3} \rho (t-s-2\pi) - \pi}{6}\right), & s \leq t, \\
\frac{e^{(1/2)\rho (t+s-2\pi)}}{\sqrt{3} \rho \sin \left(\frac{\sqrt{3} \rho t - \pi}{6}\right)} - e^{\rho t} \sin \left(\frac{\sqrt{3} \rho (t-s) - \pi}{6}\right), & t \leq s,
\end{cases}
\]
\[
H_2(t, s) = \begin{cases} 
e^{\rho (s-t)}, & s \leq t, \\
e^{\rho (s-t-2\pi)}, & t \leq s,
\end{cases}
\]
\[
L_1 = \frac{2}{3\rho^2 (1 + e^{2\rho t} - 2e^{\rho t} \cos \sqrt{3} \rho)}, \\
L_2 = \frac{1}{3\rho^2 (1 - e^{-2\rho t})}.
\]
Let $\text{Theorem 1}$.\quad \forall x \in [0,\infty)\longrightarrow [0,\infty)$ is continuous and $\lambda > 0$.

It is clear that the map $T: \mathcal{B} \longrightarrow \mathcal{B}$ defined by

$$
Tu(t) = \lambda \int_0^2 G(t, s)f(u(s))ds,
$$

is completely continuous. Define

$$
\mathcal{C} = \left\{ u \in \mathcal{B} : u(t) \geq \frac{m}{M}\|u\| \right\}.
$$

Obviously, $\mathcal{C}$ is a cone in $\mathcal{B}$. By (34), $T: \mathcal{C} \longrightarrow \mathcal{C}$ and $u \in \mathcal{B}$ is a solution of (1) and (2) if $u \in \mathcal{B}$ is a fixed point of $T$.

Let

$$
\mathcal{C}_R = \left\{ u \in \mathcal{C} : \|u\| < R \right\},
$$

$$
\partial \mathcal{C}_R = \left\{ u \in \mathcal{C} : \|u\| = R \right\}.
$$

We apply the following theorem [12].

Theorem 1. Let $\mathcal{B}$ be a Banach space. Assume that $T: \mathcal{C}_R \longrightarrow \mathcal{C}$ is completely continuous such that $Tu \not\equiv u$ for $u \in \partial \mathcal{C}_R$.

(A$_2$) If $\|Tu\| \geq \|u\|$ for $u \in \partial \mathcal{C}_R$, then $i(T, \mathcal{C}_R, \mathcal{C}) = 0$.

We will restrict our attention to the case.

(H$_1$) $f_0 > 0$, where $f_0 = \lim_{x \to 0} f(x)/x$ and $f_\infty = \lim_{x \to \infty} f(x)/x$.

Recall that

$$
\int_0^2 G(t, s)ds = \frac{1}{\rho^2},
$$

If there exists $\delta > 0$ such that $f(x) \geq \delta x$, then for $u \in \mathcal{C}$,

$$
Tu(t) = \lambda \int_0^2 G(t, s)f(u(s))ds \geq \lambda \delta \int_0^2 G(t, s)u(s)ds
$$

$$
\geq \lambda \delta \frac{m}{M}\|u\|\|u\|
$$

Similarly, if there exists $\delta > 0$ such that $f(x) \leq \gamma x$, then, for $u \in \mathcal{C}$,

$$
Tu(t) = \lambda \int_0^2 G(t, s)f(u(s))ds
$$

$$
\leq \lambda \gamma \int_0^2 G(t, s)u(s)ds
$$

From these inequalities, by Theorem 1, we have the following result.

Theorem 2. If (H$_1$) and (H$_2$) are satisfied, then (1) and (2) have at least one positive solution provided:

$$
\lambda \in \left( \frac{M\rho^3}{m \max\{f_\infty, f_0\}} \right) \left( \frac{\rho^3}{\min\{f_\infty, f_0\}} \right).
$$

Proof. Consider $f_\infty > f_0$. Let $f_\infty > \epsilon > 0$ be such that

$$
\lambda \in \left( \frac{M\rho^3}{m (f_\infty - \epsilon)} \right) \left( \frac{\rho^3}{f_0 + \epsilon} \right).
$$

There exists $R_1 < 0$ such that $f(x) \leq (f_0 + \epsilon)x$, $x \in [0, R_1]$, that is, $f(u(t)) \leq (f_0 + \epsilon)u(t)$, $u \in \partial \mathcal{C}_{R_1}$. Hence,

$$
\|Tu\| \leq \lambda \left( f_0 + \epsilon \right) \frac{1}{\rho^3} \|u\| < \|u\|,
$$

$u \in \partial \mathcal{C}_{R_1}$.

There exists $R > R_1$ such that $f(x) \geq (f_\infty - \epsilon)x$, $x \geq R$.

Let $R_2 = \max\{2R_1, MR/m\}$. Then, $u(t) \geq (m/M)\|u\| \geq R$, $u \in \partial \mathcal{C}_{R_2}$. As above, for $u \in \partial \mathcal{C}_{R_2}$.
\[ \|Tu\| \geq \lambda \left( f_{\infty} - \epsilon \right) \frac{m}{M\rho} \|u\| > \|u\|. \] (44)

It follows that \( i(T, \mathcal{B}_R, \mathcal{B}) = 1 \) and \( i(T, \mathcal{B}_R, \mathcal{B}) = 0 \) so that \( i(T, \mathcal{B}_R, \mathcal{B}) = -1 \), that is, \( T \) has a fixed point in \( \mathcal{B}_R \).

The case \( f_{\infty} < f_0 \) is handled similarly. \( \square \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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