Invariant measures of Markov operators associated to iterated function systems consisting of $\varphi$-max-contractions with probabilities

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Abstract. We prove that the Markov operator associated to an iterated function system consisting of $\varphi$-max-contractions with probabilities has a unique invariant measure whose support is the attractor of the system.

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1. Introduction

Iterated function systems with probabilities, which can be viewed as particular cases of random systems with complete connections (see [8], [21], and [24]), are well known for their applications in image compression or in learning theory (see [1], [2], [4], [5] and the references therein).

The problem of asymptotic stability of iterated function systems consisting of contractions with probabilities has collected a lot of attention in the last two decades (see [9], [10], [11], [26], [28] and the references therein).

The uniqueness of invariant probability measures for place-dependent random iterations is treated in [19] and [25].

The problem of the existence and uniqueness of invariant measures for Markov type operators associated to iterated function systems with probabilities (which was initiated by J. Hutchinson [7]) was also studied, in more general settings, in [3], [12], [14], [15], [17], [18], [23] and [27].

Since in one of our previous works we introduced a new kind of iterated function systems, namely those consisting of $\varphi$-max-contractions, and we prove the existence and uniqueness of their attractor (see [6]), along the lines of research previously mentioned, the next step, which is accomplished in the present paper, is to study the Markov operators associated to such systems with probabilities. We prove that each such operator has a unique invariant measure whose support is the attractor of the system. Let us point out that the invariant measure is obtained via the Riesz representation theorem from
a positive linear functional which is generated by the dual operator of the Markov operator.

2. Preliminaries

Notations and terminology

Given the sets $A$ and $B$, by $B^A$ we mean the set of functions from $A$ to $B$.

Given a set $X$, a function $f : X \rightarrow X$ and $n \in \mathbb{N}$, by $f^n$ we mean $f \circ f \circ \ldots \circ f$.

Given a metric space $(X,d)$, by:
- $\text{diam}(A)$ we mean the diameter of the subset $A$ of $X$
- $P_{cp}(X)$ we mean the set of non-empty compact subsets of $X$
- $C(X)$ we mean the set of continuous functions $f : X \rightarrow \mathbb{R}$
- $B(X)$ we mean the $\sigma$-algebra of Borel subsets of $X$
- the support of a finite positive borelian measure $\mu$ on $X$ (denoted by $\text{supp } \mu$) we mean the smallest closed subset of $X$ on which $\mu$ is concentrated; so

$$\text{supp } \mu = \bigcap_{F=\text{P}_{\subseteq X, \mu(F)=\mu(X)}} F$$

- $M(X)$ we mean the space of borelian normalized and positive measures on $X$ with compact support
- $Lip_1(X,\mathbb{R})$ we mean the set of functions $f : X \rightarrow \mathbb{R}$ having the property that $\text{lip}(f) \overset{def}{=} \sup_{x,y \in X, x \neq y} \frac{|f(x)-f(y)|}{d(x,y)} \leq 1$
- the Hausdorff-Pompeiu metric we mean $H : P_{cp}(X) \times P_{cp}(X) \rightarrow [0, +\infty)$ given by

$$H(A, B) = \max \{ \sup_{x \in A} (\inf_{y \in B} d(x, y)), \sup_{x \in B} (\inf_{y \in A} d(x, y)) \}$$

for all $A, B \in P_{cp}(X)$

- a Picard operator we mean a function $f : X \rightarrow X$ having the property that there exists a unique fixed point $\alpha$ of $f$ and the sequence $(f^n(x))_{n \in \mathbb{N}}$ is convergent to $\alpha$ for every $x \in X$. 

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The Hutchinson distance

**Definition 2.1.** Given a complete metric space \((X, d)\), the function \(d_H : \mathcal{M}(X) \times \mathcal{M}(X) \to [0, \infty)\) described by
\[
d_H(\mu, \nu) = \sup_{f \in \text{Lip}_1(X, \mathbb{R})} \left| \int_X f \, d\mu - \int_X f \, d\nu \right|
\]
for every \(\mu, \nu \in \mathcal{M}(X)\), turns out to be a distance which is called the Hutchinson distance.

**Remark 2.2** (see page 46 from [22]). Given a compact metric space \((X, d)\), \(\mu \in \mathcal{M}(X)\) and a sequence \((\mu_n)_{n \in \mathbb{N}}\) of elements from \(\mathcal{M}(X)\), the following statements are equivalent:

a) the sequence \((\mu_n)_{n \in \mathbb{N}}\) converges to \(\mu\) in the weak topology i.e. \(\int g \, d\mu = \lim_{n \to \infty} \int_X g \, d\mu_n\) for every \(g \in \mathcal{C}(X)\);

b) \(\lim_{n \to \infty} d_H(\mu_n, \mu) = 0\).

Comparison functions

**Definition 2.3.** A function \(\varphi : [0, \infty) \to [0, \infty)\) is called a comparison function provided that it satisfies the following properties:

i) \(\varphi\) is increasing;

ii) \(\lim_{n \to \infty} \varphi^{[n]}(x) = 0\) for every \(x \in [0, \infty)\).

**Remark 2.4.** For each comparison function the following two properties are valid:

a) \(\varphi(0) = 0\);

b) \(\varphi(x) < x\) for every \(x \in (0, \infty)\).

The shift space

Given a nonempty set \(I\), we denote the set \(I^\mathbb{N}\) by \(\Lambda(I)\). Thus \(\Lambda(I)\) is the set of infinite words with letters from the alphabet \(I\) and a standard element \(\omega\) of \(\Lambda(I)\) can be presented as \(\omega = \omega_1 \omega_2 \ldots \omega_n \omega_{n+1} \ldots\).

Given a nonempty set \(I\), we denote the set \(I^\{1,2,\ldots,n\}\) by \(\Lambda_n(I)\). Thus \(\Lambda_n(I)\) is the set of words of length \(n\) with letters from the alphabet \(I\) and a standard element \(\omega\) of \(\Lambda_n(I)\) can be presented as \(\omega = \omega_1 \omega_2 \ldots \omega_n\). By \(\Lambda_0(I)\)
we mean the set having only one element, namely the empty word denoted by \(\lambda\).

For \(n \in \mathbb{N}^*\), we shall use the following notation: \(V_n(I) \equiv \bigcup_{k \in \{0, 1, 2, \ldots, n-1\}} \Lambda_k(I)\).

Given a nonempty set \(I, m, n \in \mathbb{N}\) and two words \(\omega = \omega_1\omega_2\ldots\omega_n \in \Lambda_n(I)\) and \(\theta = \theta_1\theta_2\ldots\theta_m \in \Lambda_m(I)\) or \(\theta = \theta_1\theta_2\ldots\theta_m\theta_{m+1}\ldots \in \Lambda(I)\), by \(\omega\theta\) we mean the concatenation of the words \(\omega\) and \(\theta\), i.e. \(\omega\theta = \omega_1\omega_2\ldots\omega_n\theta_1\theta_2\ldots\theta_m\) and respectively \(\omega\theta = \omega_1\omega_2\ldots\omega_n\theta_1\theta_2\ldots\theta_m\theta_{m+1}\ldots\).

For a family of functions \((f_i)_{i \in I}\), where \(f_i : X \to X\), and \(\omega = \omega_1\omega_2\ldots\omega_n \in \Lambda_n(I)\), we shall use the following notation: \(f_\omega \equiv f_{\omega_1} \circ f_{\omega_2} \circ \ldots \circ f_{\omega_n}\).

For a function \(f : X \to X\), by \(f_\lambda\) we mean \(Id_X\).

**A result concerning a sequence of compact subsets of a metric space**

**Proposition 2.5** (see Proposition 2.8 from [16]). Let \((X, d)\) be a complete metric space, \((Y_n)_{n \in \mathbb{N}} \subseteq P_{cp}(X)\) and \(Y \in P_{cp}(X)\) such that \(\lim_{n \to \infty} H(Y_n, Y) = 0\). Then \(Y \cup \left( \bigcup_{n=0}^{\infty} Y_n \right) \in P_{cp}(X)\).

**3. The Markov operator associated to an iterated function system consisting of \(\varphi\)-max-contractions with probabilities**

**Definition 3.1.** An iterated function system consisting of \(\varphi\)-max-contractions (\(\varphi\)-max-IFS for short) is described by:
- a complete metric space \((X, d)\)
- a finite family of continuous functions \((f_i)_{i \in I}\), where \(f_i : X \to X\), having the property that there exist a comparison function \(\varphi : [0, \infty) \to [0, \infty)\) and \(p \in \mathbb{N}^*\) such that \(\max_{\omega \in \Lambda_p(I)} d(f_\omega(x), f_\omega(y)) \leq \varphi(\max_{\omega \in \Lambda_p(I)} d(f_\omega(x), f_\omega(y)))\) for every \(x, y \in X\).

We denote such a system by \(S = ((X, d), (f_i)_{i \in I})\).

The fractal operator \(F_S : P_{cp}(X) \to P_{cp}(X)\), associated to the \(\varphi\)-max-IFS \(S\), is given by \(F_S(K) = \bigcup_{i \in I} f_i(K)\) for every \(K \in P_{cp}(X)\).

We say that the \(\varphi\)-max-IFS \(S\) has attractor if \(F_S\) is a Picard operator (with respect to the Hausdorff-Pompeiu metric) and the fixed point of \(F_S\) is called the attractor of the system \(S\) and it is denoted by \(A_S\).

**Theorem 3.2** (see Theorem 3.2 from [6]). Each \(\varphi\)-max-IFS has attractor.
**Definition 3.3.** An iterated function system consisting of \( \varphi \)-max-contractions with probabilities (\( \varphi \)-max-IFSp for short) is described by:

- a \( \varphi \)-max-IFS \( S = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}) \)
- a system of probabilities \( (p_i)_{i \in \{1, 2, ..., m\}} \), i.e. \( p_i \in (0, 1) \) for every \( i \in \{1, 2, ..., m\} \) and \( p_1 + p_2 + ... + p_m = 1 \).

We denote such a system by \( S = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}}) \).

We associate to such a system the Markov operator \( M_S : \mathcal{M}(X) \to \mathcal{M}(X) \) given by \( M_S(\mu) = p_1 \mu \circ f_1^{-1} + ... + p_m \mu \circ f_m^{-1} \), i.e. \( M_S(\mu)(B) = p_1 \mu(f_1^{-1}(B)) + ... + p_m \mu(f_m^{-1}(B)) \) for every \( B \in \mathcal{B}(X) \) and every \( \mu \in \mathcal{M}(X) \). A fixed point of \( M_S \) is called invariant measure.

**Lemma 3.4** (see Lemma 3 from [9]). The equality \( \text{supp} \ M_S^n(\mu) = F_S^n(\text{supp} \ \mu) \) is valid for every \( \varphi \)-max-IFSp \( S \), every \( \mu \in \mathcal{M}(X) \) and every \( n \in \mathbb{N} \).

**Remark 3.5.** The operator \( M_S \) is well defined, for every \( \varphi \)-max-IFSp \( S \).

**Remark 3.6.** Given a \( \varphi \)-max-IFSp \( S = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}}) \), the Markov operator \( M_S \) is a Feller operator since \( \int_X g \cdot M_S(\mu) = p_1 \int_X g \circ f_1 d\mu + ... + p_m \int_X g \circ f_m d\mu \) for every continuous and bounded function \( g : X \to \mathbb{R} \) and every \( \mu \in \mathcal{M}(X) \).

4. The main result

Our main result states that the Markov operator associated to an IFSp is a Picard operator and the support of its fixed point is the attractor of the system. At the beginning, we shall consider for the case of a system for which the metric space is compact (see Theorem 4.9) and then we shall treat the general case (see Theorem 4.18).

A. The case of a \( \varphi \)-max-IFSp for which the metric space is compact

Let us start with some:
Notations. For a $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}})$, $g : X \to \mathbb{R}$, $x, y \in X$, $n \in \mathbb{N}$, $\omega = \omega_1 \omega_2 ... \omega_n \in \Lambda_n(\{1, 2, ..., m\})$ and $\varepsilon > 0$, we shall use the following notations:

- $X_{x,y,n} = \{d(f_\omega(x), f_\omega(y)) \mid \omega \in \Lambda_n(\{1, 2, ..., m\})\}$
- $X_{\varepsilon,n} = \{\max X_{x,y,n} \mid x, y \in X, d(x, y) \leq \varepsilon\}$
- $a_n(\varepsilon) = \sup X_{\varepsilon,n}$
- $\mathcal{O}_\varepsilon(g) = \sup_{x,y \in X, d(x,y) \leq \varepsilon} |g(x) - g(y)|$
- $B_\varepsilon(g) = p_1 g \circ f_1 + p_2 g \circ f_2 + \ldots + p_m g \circ f_m$
- $p_\omega = p_{\omega_1} p_{\omega_2} \ldots p_{\omega_n}$.  

An easy mathematical induction argument proves the following:

Remark 4.1. For every $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}})$, $g : X \to \mathbb{R}$ and $n \in \mathbb{N}$, we have $B_\varepsilon^n(g) = \sum_{\omega \in \Lambda_n(\{1, 2, ..., m\})} p_\omega g \circ f_\omega$.  

Lemma 4.2. For every $\varphi$-max-IFS $\mathcal{S}$, $g : X \to \mathbb{R}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $\mathcal{O}_\varepsilon(B_\varepsilon^n(g)) \leq \mathcal{O}_{a_n(\varepsilon)}(g)$.

Proof. Let us suppose that $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}})$. Then we have

$$\mathcal{O}_\varepsilon(B_\varepsilon^n(g)) = \sup_{x,y \in X, d(x,y) \leq \varepsilon} \left| B_\varepsilon^n(g)(x) - B_\varepsilon^n(g)(y) \right| \quad \text{Remark 4.1}$$

$$= \sup_{x,y \in X, d(x,y) \leq \varepsilon} \left| \sum_{\omega \in \Lambda_n(\{1, 2, ..., m\})} p_\omega ((g \circ f_\omega)(x) - (g \circ f_\omega)(y)) \right| \leq$$

$$\leq \sup_{x,y \in X, d(x,y) \leq \varepsilon} \sum_{\omega \in \Lambda_n(\{1, 2, ..., m\})} p_\omega |g(f_\omega(x)) - g(f_\omega(y))| \leq$$

$$\leq \sum_{\omega \in \Lambda_n(\{1, 2, ..., m\})} p_\omega \mathcal{O}_{a_n(\varepsilon)}(g) = \mathcal{O}_{a_n(\varepsilon)}(g) \sum_{\omega \in \Lambda_n(\{1, 2, ..., m\})} p_\omega = \mathcal{O}_{a_n(\varepsilon)}(g).$$

Lemma 4.3. For every $\varphi$-max-IFS $\mathcal{S}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $a_{n+p}(\varepsilon) \leq \varphi(\max\{a_{n+p-1}(\varepsilon), a_{n+p-2}(\varepsilon), \ldots, a_n(\varepsilon)\})$, where the meaning of the natural number $p$ is the one from Definition 3.1.

Proof. Let us suppose that $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, ..., m\}}, (p_i)_{i \in \{1, 2, ..., m\}})$ and suppose that $\varphi$ is the comparison function described in Definition 3.1. For
the sake of simplicity, in the framework of this proof, we denote $a_n(\varepsilon)$ by $a_n$ and \max\{a_{n+1}, a_{n+2}, \ldots, a_n\} by $M_n$. For all $x, y \in X$, $d(x, y) \leq \varepsilon$, $\theta \in V_p(\{1, 2, \ldots, m\})$ and $\omega \in \Lambda_n(\{1, 2, \ldots, m\})$, we have $d(f_\theta(f_\omega(x)), f_\theta(f_\omega(y))) \leq a_{n+|\theta|} \leq M_n$. Thus, $\max_{\theta \in V_p(\{1, 2, \ldots, m\})} \max_{x, y, n} d(f_\theta(f_\omega(x)), f_\theta(f_\omega(y))) \leq M_n$, so

$$\varphi\left(\max_{\theta \in V_p(\{1, 2, \ldots, m\})} d(f_\theta(f_\omega(x)), f_\theta(f_\omega(y)))\right) \leq \varphi(M_n), \quad (1)$$

for every $x, y \in X$, $d(x, y) \leq \varepsilon$ and $\omega \in \Lambda_n(\{1, 2, \ldots, m\})$.

For each $\omega_j = \omega_1\omega_2\ldots\omega_{p-1}\omega_p \in \Lambda_n(\{1, 2, \ldots, m\})$, $x, y \in X$, $d(x, y) \leq \varepsilon$, with the notations $v^{not} = \omega_1\omega_2\ldots\omega_{p-1}\omega_p$ and $w^{not} = \omega_{p+1}\ldots\omega_{n+p}$, we have $d(f_\omega(x), f_\omega(y)) \leq \varphi\left(\max_{\theta \in V_p(\{1, 2, \ldots, m\})} d(f_\omega(f_\omega(x)), f_\omega(f_\omega(y)))\right)$ (1) $\varphi(M_n)$. Hence $\max_{x, y, n+p} X_{x,y,n+p} \leq \varphi(M_n)$ for every $x, y \in X$, $d(x, y) \leq \varepsilon$. Therefore, we come to the conclusion that $\sup_{x, y \in X, d(x, y) \leq \varepsilon} a_{n+p} \leq \varphi(\max\{a_{n+1}, a_{n+2}, \ldots, a_n\})$. □

**Lemma 4.4.** For every $\varphi$-max-IFS $\mathcal{S}$ and $\varepsilon > 0$, we have $\lim_{n \to \infty} a_n(\varepsilon) = 0$.

**Proof.** Let us suppose that $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, \ldots, m\}}, (p_i)_{i \in \{1, 2, \ldots, m\}})$ and the meaning of $p$ and $\varphi$ is the one described in Definition 3.1. For the sake of simplicity, in the framework of this proof, we denote $a_n(\varepsilon)$ by $a_n$ and \max\{a_1, a_2, \ldots, a_p\} by $M$.

**Claim.** $0 \leq a_{pk+j} \leq \varphi^{[k]}(M)$ for every $k \in \mathbb{N}$ and every $j \in \{1, 2, \ldots, p\}$. Indeed, we have

$$a_{p+1} \leq \varphi(M) \leq M. \quad (1)$$

We also have

$$a_{p+2} \leq \varphi(\max\{a_{p+1}, a_p, \ldots, a_2\}). \quad (2)$$

The inequalities $a_2 \leq M, \ldots, a_p \leq M$ and (1) lead to the conclusion that $\max\{a_{p+1}, a_p, \ldots, a_2\} \leq M$ and in view of (2) we get $a_{p+2} \leq \varphi(M)$ and continuing the same line of reasoning we obtain that

$$a_{p+3} \leq \varphi(M), \ldots, a_{2p} \leq \varphi(M). \quad (3)$$

Moreover, we have

$$a_{2p+1} \leq \varphi(\max\{a_{2p}, a_{2p-1}, \ldots, a_{p+1}\}) \leq \varphi(\varphi(M)) = \varphi^{[2]}(M)$$

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and, as above, we come to the conclusion that \( a_{2p+2} \leq \varphi^{[2]}(M) \), ..., \( a_{3p} \leq \varphi^{[2]}(M) \). Now, inductively one can prove the claim.

Since \( \lim_{k \to \infty} \varphi^{[k]}(M) = 0 \) (see Definition 2.3), based on the Claim and the squeeze theorem, the proof is done. □

**Lemma 4.5.** For every \( \varphi \)-max-IFS \( S = ((X, d), (f_i)_{i \in \{1, \ldots, m\}}, (p_i)_{i \in \{1, \ldots, m\}}) \), with \( (X, d) \) compact, \( g : X \to \mathbb{R} \) continuous and \( \varepsilon > 0 \), we have \( \lim_{n \to \infty} O_{a_n(\varepsilon)}(g) = 0 \).

**Proof.** As \( (X, d) \) is compact and \( g \) is continuous, for every \( \varepsilon' > 0 \) there exists \( \delta_{\varepsilon'} > 0 \) such that \( |g(x) - g(y)| < \varepsilon' \) for every \( x, y \in X \) such that \( d(x, y) < \delta_{\varepsilon'} \). According to Lemma 4.4 there exists \( n_{\varepsilon'} \in \mathbb{N} \) such that \( a_n(\varepsilon) < \delta_{\varepsilon'} \) for every \( n \in \mathbb{N}, n \geq n_{\varepsilon'} \). Consequently we get \( |g(x) - g(y)| < \varepsilon' \) for every \( x, y \in X \) such that \( d(x, y) < a_n(\varepsilon) \) and \( n \in \mathbb{N}, n \geq n_{\varepsilon'} \), so \( \sup_{x, y \in X, d(x, y) \leq a_n(\varepsilon)} |g(x) - g(y)| \leq \varepsilon' \), i.e. \( O_{a_n(\varepsilon)}(g) \leq \varepsilon' \) for every \( n \in \mathbb{N}, n \geq n_{\varepsilon'} \). This means that \( \lim_{n \to \infty} O_{a_n(\varepsilon)}(g) = 0 \). □

**Lemma 4.6.** For every \( \varphi \)-max-IFS \( S = ((X, d), (f_i)_{i \in \{1, \ldots, m\}}, (p_i)_{i \in \{1, \ldots, m\}}) \), with \( (X, d) \) compact, \( g : X \to \mathbb{R} \) continuous and \( \varepsilon > 0 \), we have \( \lim_{n \to \infty} O_{\varepsilon}(B^{|n|}_S(g)) = 0 \).

**Proof.** The squeeze theorem, Lemma 4.2 and Lemma 4.5 assure us the validity of this Lemma. □

**Proposition 4.7.** For every \( \varphi \)-max-IFS \( S = ((X, d), (f_i)_{i \in \{1, \ldots, m\}}, (p_i)_{i \in \{1, \ldots, m\}}) \), with \( (X, d) \) compact, and \( g : X \to \mathbb{R} \) continuous, there exists a constant function \( c_g : X \to \mathbb{R} \) such that \( B^{|n|}_S(g) \xrightarrow{n \to \infty} c_g \).

**Proof.** We divide the proof into three steps.

**Step 1.** The sequence \( \sup_{x \in X} B^{|n|}_S(g)(x) \) is decreasing and the sequence \( \inf_{x \in X} B^{|n|}_S(g)(x) \) is increasing.

We have

\[
B^{|n+1|}_S(g)(x) = \sum_{\omega \in \Lambda_{n+1}(\{1,2,\ldots,m\})} p_{\omega}(g \circ f_{\omega})(x) =
\]

\[
= \sum_{i \in \{1,2,\ldots,m\}, v \in \Lambda_n(\{1,2,\ldots,m\})} p_i p_v(g \circ f_v \circ f_i)(x) =
\]
for every $x \in X$ and every $n \in \mathbb{N}$, so $\sup_{x \in X} B^{[n+1]}_S(g)(x) \leq \sup_{x \in X} B^{[n]}_S(g)(x)$ for every $n \in \mathbb{N}$, i.e. the sequence $(\sup_{x \in X} B^{[n]}_S(g)(x))_{n \in \mathbb{N}}$ is decreasing. In a similar manner one can prove that the sequence $(\inf_{x \in X} B^{[n]}_S(g)(x))_{n \in \mathbb{N}}$ is increasing.

Step 2. The sequences $(\inf_{x \in X} B^{[n]}_S(g)(x))_{n \in \mathbb{N}}$ and $(\sup_{x \in X} B^{[n]}_S(g)(x))_{n \in \mathbb{N}}$ are convergent and they have the same limit (which will be denoted by $c_\varphi$).

Step 1 assures us that there exist $l_1, l_2 \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \sup_{x \in X} B^{[n]}_S(g)(x) = l_2 \leq l_1 = \lim_{n \to \infty} \inf_{x \in X} B^{[n]}_S(g)(x).$$

We have $0 \leq l_1 - l_2 \leq \sup_{x \in X} B^{[n]}_S(g)(x) - \inf_{x \in X} B^{[n]}_S(g)(x) = \sup_{x \in X} \sup_{y \in X} (B^{[n]}_S(g)(x) - B^{[n]}_S(g)(y)) \leq \mathcal{O}_{diam(X)} B^{[n]}_S(g)$ for every $n \in \mathbb{N}$ and by passing to limit as $n \to \infty$, based on Lemma 4.6, we get $l_1 = l_2 \equiv c_\varphi$.

Step 3. There exists a constant function $c_\varphi : X \to \mathbb{R}$ such that $B^{[n]}_S(g) \xrightarrow{n \to \infty} c_\varphi$.

Considering the constant function $c_\varphi : X \to \mathbb{R}$ given by $c_\varphi(x) = c_\varphi$ for every $x \in X$, we have

$$-\mathcal{O}_{diam(X)} B^{[n]}_S(g) = \inf_{x \in X} B^{[n]}_S(g)(x) - \sup_{x \in X} B^{[n]}_S(g)(x) \leq \inf_{x \in X} B^{[n]}_S(g)(x) - c_\varphi \leq B^{[n]}_S(g)(x) - c_\varphi \leq \sup_{x \in X} B^{[n]}_S(g)(x) - \inf_{x \in X} B^{[n]}_S(g)(x) = \mathcal{O}_{diam(X)} B^{[n]}_S(g),$$

i.e. $|B^{[n]}_S(g)(x) - c_\varphi(x)| \leq \mathcal{O}_{diam(X)} B^{[n]}_S(g)$, for every $x \in X$ and every $n \in \mathbb{N}$. The last inequality and Lemma 4.6 assure us that $B^{[n]}_S(g) \xrightarrow{n \to \infty} c_\varphi$.

**Proposition 4.8.** For every $\varphi$-max-IFS $\mathcal{P} = (\mathcal{P}_i)_{i \in \{1,\ldots,m\}}, (\mathcal{P}_i)_{i \in \{1,\ldots,m\}}$, with $(X,d)$ compact, there exists a unique borelian positive measure $\mu_\mathcal{P}$ on $X$ such that $c_\varphi = \int f \, d\mu_\mathcal{P}$ for every continuous function $f : X \to \mathbb{R}$.

**Proof.** Let us consider the function $I : \mathcal{C}(X) \to \mathbb{R}$ given by $I(g) = c_\varphi$ for every $g \in \mathcal{C}(X)$. As, according to Remark 4.1, $B^{[n]}_S(g+h) = B^{[n]}_S(g) + B^{[n]}_S(h)$
and $B_S^{[n]}(ag) = \alpha B_S^{[n]}(g)$ for every $g, h \in \mathcal{C}(X), \alpha \in \mathbb{R}$ and every $n \in \mathbb{N}$, by passing to limit as $n \to \infty$, we get $I(g + h) = I(g) + I(h)$ and $I(\alpha g) = \alpha I(g)$ for every $g, h \in \mathcal{C}(X)$ and every $\alpha \in \mathbb{R}$. Moreover, as $B_S^{[n]}(g) \geq 0$ for every $n \in \mathbb{N}$ and every $g \in \mathcal{C}(X)$ such that $g \geq 0$, by passing to limit as $n \to \infty$, we get $I(g) \geq 0$ for every $g \in \mathcal{C}(X)$ such that $g \geq 0$. We infer that $I$ is a positive linear functional on $\mathcal{C}(X)$, so, in view of Riesz representation theorem, we conclude that there exists a unique borelian positive measure $\mu_S$ on $X$ such that $c_g = \int_X gd\mu_S$ for every $g \in \mathcal{C}(X)$. $\square$

**Theorem 4.9.** $M_S : (\mathcal{M}(X), d_H) \to (\mathcal{M}(X), d_H)$ is a Picard operator for every $\varphi$-max-$IFS_p$ $S = ((X, d), (f_i)_{i \in \{1, 2, \ldots, m\}}, (p_i)_{i \in \{1, 2, \ldots, m\}})$, with $(X, d)$ compact, and the support of the fixed point of $M_S$ is $A_S$.

**Proof.** First of all let us note that Remark 3.6 can be restated as $\int_X gd\mu_S(\nu) = \int_X B_S^{[n]}(g)d\nu$ for every $g \in \mathcal{C}(X)$ and every $\nu \in \mathcal{M}(X)$. Therefore, we get

$$\int_X gd\mu_S^{[n]}(\nu) = \int_X B_S^{[n]}(g)d\nu, \quad (1)$$

for every $g \in \mathcal{C}(X), \nu \in \mathcal{M}(X)$ and $n \in \mathbb{N}$.

Now we divide the proof into three steps.

**Step 1.** The measure $\mu_S$, provided by Proposition 4.8, belongs to $\mathcal{M}(X)$.

For the function $g_0 : X \to \mathbb{R}$ given by $g_0(x) = 1$ for every $x \in X$, we have $B_S^{[n]}g_0(x) = 1$ for every $x \in X$ and every $n \in \mathbb{N}$. Consequently we have $c_{g_0} = 1$, i.e. $\int_X d\mu_S = 1$, so $\mu_S(X) = 1$. Moreover, as $\text{supp } \mu_S$ is a closed subset of the compact set $X$, it is compact.

**Step 2.** The measure $\mu_S \in \mathcal{M}(X)$, provided by Proposition 4.8, is the unique fixed point of $M_S$ and $\text{supp } \mu_S = A_S$.

On the one hand, by passing to limit as $n \to \infty$ in the relation $B_S^{[n+1]}(g) = B_S^{[n]}(B_S(g))$ which is valid for every $n \in \mathbb{N}$ and every $g \in \mathcal{C}(X)$, taking into account Proposition 4.7, we get $\int_X gd\mu_S = \int_X B_S(g)d\mu_S$ for every $g \in \mathcal{C}(X)$.

In other words, we have $\int_X gd(\mu_S - M_S(\mu_S)) = 0$ for every $g \in \mathcal{C}(X)$ which implies that $M_S(\mu_S) = \mu_S$, i.e. $\mu_S$ is a fixed point of $M_S$. Since $\text{supp } \mu_S = \text{supp } M_S(\mu_S)$ and $M_S(\mu_S) = F_S(\text{supp } \mu_S)$, we infer that $\text{supp } \mu_S$ is the fixed point of $F_S$, so $\text{supp } \mu_S = A_S$. 

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On the other hand, if $\nu \in \mathcal{M}(X)$ has the property that $M_S(\nu) = \nu$, then, from (1), we get $\int_X gd\nu = \int_X B_S^n(g)d\nu$ for every $g \in \mathcal{C}(X)$ and every $n \in \mathbb{N}$. By passing to limit as $n \to \infty$, based on Proposition 4.7, we get $\int_X gd\nu = \int_X c_g d\nu$.

As $\nu(X) = 1$, we obtain $\int_X gd\nu = \int_X gd\mu_S$, so $\int_X gd(\nu - \mu_S) = 0$ for every $g \in \mathcal{C}(X)$. We conclude that $\nu = \mu_S$, i.e. $\mu_S$ is the unique fixed point of $M_S$.

**Step 3.** $\lim_{n \to \infty} M_S^n(\nu) = \mu_S$ for every $\nu \in \mathcal{M}(X)$.

Since, based on Proposition 4.7, we have $\lim_{n \to \infty} \int_X B_S^n(g)d\nu = \int_X c_g d\nu$, as $\nu(X) = 1$, we infer that $\lim_{n \to \infty} \int_X B_S^n(g)d\nu = \int_X gd\mu_S$. Hence, using (1), we get $\lim_{n \to \infty} \int_X gdM_S^n(\nu) = \int_X gd\mu_S$ for every $g \in \mathcal{C}(X)$. Taking into account Remark 2.2, we conclude that $\lim_{n \to \infty} d_H(M_S^n(\nu), \mu_S) = 0$.

The last two steps assure us that $M_S$ is a Picard operator. □

**B. The case of a general $\varphi$-max-IFSp**

Let us start with some:

**Notations.**

For a complete metric space $(X, d)$ and a compact subset $Y$ of $X$, we shall consider the following:

- the function $R_Y : \text{Lip}_1(X, \mathbb{R}) \to \text{Lip}_1(Y, \mathbb{R})$ given by $R_Y(f)(x) = f(x)$ for every $f \in \text{Lip}_1(X, \mathbb{R})$ and every $x \in Y$
- the function $E_Y : \text{Lip}_1(Y, \mathbb{R}) \to \text{Lip}_1(X, \mathbb{R})$ given by $E_Y(f)(x) = \sup_{y \in Y} (f(y) - \text{lip}(f)d(x, y))$ for every $f \in \text{Lip}_1(Y, \mathbb{R})$ and every $x \in X$
- the function $i_Y : \mathcal{M}(Y) \to \mathcal{M}(X)$ given by $i_Y(\mu)(B) = \mu(Y \cap B)$ for every $\mu \in \mathcal{M}(Y)$ and every $B \in \mathcal{B}(X)$
- the Hutchinson distance $d_H^S : \mathcal{M}(Y) \times \mathcal{M}(Y) \to [0, \infty)$ described by $d_H^S(\mu, \nu) = \sup_{f \in \text{Lip}_1(Y, \mathbb{R})} \left| \int_Y f d\mu - \int_Y f d\nu \right|$ for every $\mu, \nu \in \mathcal{M}(Y)$.

For a $\varphi$-max-IFSp $S = ((X, d), (f_i)_{i \in \{1, 2, \ldots, m\}}, (p_i)_{i \in \{1, 2, \ldots, m\}})$ and $\nu \in \mathcal{M}(X)$ we shall consider the set $K_\nu \overset{\text{def}}{=} A_S \cup \bigcup_{n \in \mathbb{N}} F^n_S(\text{supp } \nu)$. 

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Note that the functions $R_Y$ and $E_Y$ are well-defined (i.e. $R_Y(f) \in Lip_1(Y, \mathbb{R})$ for every $f \in Lip_1(X, \mathbb{R})$ and $E_Y(f) \in Lip_1(X, \mathbb{R})$ for every $f \in Lip_1(Y, \mathbb{R})$; moreover, $R_Y(E_Y(f)) = f$ and $lip(E_Y(f)) = lip(f)$, according to a famous result due to E.J. McShane -see Theorem 1 from [13]-).

**Remark 4.10.** Given a complete metric space $(X, d)$ and a compact subset $Y$ of $X$, for every $\mu \in \mathcal{M}(Y)$, we have:

i) $\int_Y R_Y(f) d\mu = \int_X f d(i_Y(\mu))$ for every $f \in Lip_1(X, \mathbb{R})$;

ii) $\int_X f d\mu = \int_Y E_Y(f) d(i_Y(\mu))$ for every $f \in Lip_1(Y, \mathbb{R})$.

Indeed, since $\text{supp } i_Y(\mu) = F = \{ F \subseteq X, \cap \cap \cap i_Y(\mu)(F) = i_Y(\mu)(X) \}$, $\mu(\cap \cap \cap Y = \mu(Y))$, we have $\int_X f d(i_Y(\mu)) = \int_Y R_Y(f) d\mu$ for every $f \in Lip_1(X, \mathbb{R})$ and $\int_Y E_Y(f) d(i_Y(\mu)) = \int_X f d\mu$ for every $f \in Lip_1(Y, \mathbb{R})$.

**Lemma 4.11.** Given a complete metric space $(X, d)$ and a compact subset $Y$ of $X$, we have $d_H^Y(\mu_1, \mu_2) = d_H(i_Y(\mu_1), i_Y(\mu_2))$ for every $\mu_1, \mu_2 \in \mathcal{M}(Y)$.

**Proof.** On the one hand, we have $d_H^Y(\mu_1, \mu_2) = \sup_{f \in Lip_1(Y, \mathbb{R})} \left| \int_Y f d\mu_1 - \int_Y f d\mu_2 \right|$

Remark 4.10, ii)

$$\sup_{f \in Lip_1(Y, \mathbb{R})} \left| \int_X E_Y(f) d(i_Y(\mu_1)) - \int_X E_Y(f) d(i_Y(\mu_2)) \right| \leq \sup_{f \in Lip_1(Y, \mathbb{R})} \left| \int_X f d(i_Y(\mu_1)) - \int_X f d(i_Y(\mu_2)) \right| = d_H(i_Y(\mu_1), i_Y(\mu_2)),$$

so

$$d_H^Y(\mu_1, \mu_2) \leq d_H(i_Y(\mu_1), i_Y(\mu_2)), \quad (1)$$

for every $\mu_1, \mu_2 \in \mathcal{M}(Y)$.

On the other hand $d_H(i_Y(\mu_1), i_Y(\mu_2)) = \sup_{f \in Lip_1(X, \mathbb{R})} \left| \int_X f d(i_Y(\mu_1)) - \int_X f d(i_Y(\mu_2)) \right|$

Remark 4.10, i)

$$\sup_{f \in Lip_1(X, \mathbb{R})} \left| \int_X R_Y(f) d\mu_1 - \int_X R_Y(f) d\mu_2 \right| \leq \sup_{f \in Lip_1(Y, \mathbb{R})} \left| \int_Y f d\mu_1 - \int_Y f d\mu_2 \right|$$

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The fixed point of $\mathcal{M}$ according to Theorem 4.9, 
\[
\{F((\phi_{i})_{i\in\{1,\ldots,m\}}, (p_{i})_{i\in\{1,\ldots,m\}})\} \subseteq \mathcal{F}(\mathcal{M}(Y)),
\]
for every $\mu_{1}, \mu_{2} \in \mathcal{M}(Y)$. 
From (1) and (2), we get the conclusion. $\Box$

Remark 4.12. For every $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_{i})_{i\in\{1,\ldots,m\}}, (p_{i})_{i\in\{1,\ldots,m\}})$ and $\nu \in \mathcal{M}(X)$, we have:
\begin{enumerate}
  \item $F_{\mathcal{S}}(K_{\nu}) \subseteq K_{\nu}$;
  \item $K_{\nu}$ is compact.
\end{enumerate}
Indeed, we have $F_{\mathcal{S}}(K_{\nu}) = F_{\mathcal{S}}(A_{\mathcal{S}} \cup (\cup_{n\in\mathbb{N}} F_{\mathcal{S}}^{[n]}(\text{supp } \nu))) \subseteq F_{\mathcal{S}}(A_{\mathcal{S}}) \cup (\cup_{n\in\mathbb{N}} F_{\mathcal{S}}^{[n]}(\text{supp } \nu))) = F_{\mathcal{S}}(A_{\mathcal{S}}) \cup (\cup_{n\in\mathbb{N}} F_{\mathcal{S}}^{[n+1]}(\text{supp } \nu))) \subseteq K_{\nu}$. The compactness of $K_{\nu}$ is assured, via Proposition 2.5, by the fact that $\lim_{n \to \infty} H(F_{\mathcal{S}}^{[n]}(\text{supp } \nu)), A_{\mathcal{S}}) = 0$.

Taking into account that $f_{i}(A_{\mathcal{S}}) \subseteq A_{\mathcal{S}}$ and $f_{i}(K_{\nu}) \subseteq K_{\nu}$ for every $i \in \{1, 2, \ldots, m\}$ and every $\nu \in \mathcal{M}(X)$, we can consider the $\varphi$-max-IFSp $\mathcal{S}_{A_{\mathcal{S}}} = ((A_{\mathcal{S}}, d), (\phi_{i})_{i\in\{1,\ldots,m\}}, (p_{i})_{i\in\{1,\ldots,m\}})$ and $\mathcal{S}_{K_{\nu}} = ((K_{\nu}, d), (\psi_{i})_{i\in\{1,\ldots,m\}}, (p_{i})_{i\in\{1,\ldots,m\}})$, where $\phi_{i}(x) = f_{i}(x)$ for every $x \in A_{\mathcal{S}}$ and every $i \in \{1, 2, \ldots, m\}$ and $\psi_{i}(x) = f_{i}(x)$ for every $x \in K_{\nu}$ and every $i \in \{1, 2, \ldots, m\}$. We can also consider the Markov operator $M_{\mathcal{S}}^{A_{\mathcal{S}}}: \mathcal{M}(A_{\mathcal{S}}) \to \mathcal{M}(A_{\mathcal{S}})$ associated to $\mathcal{S}_{A_{\mathcal{S}}}$ and the Markov operator $M_{\mathcal{S}}^{K_{\nu}}: \mathcal{M}(K_{\nu}) \to \mathcal{M}(K_{\nu})$ associated to $\mathcal{S}_{K_{\nu}}$. According to Theorem 4.9, $M_{\mathcal{S}}^{A_{\mathcal{S}}}$ and $M_{\mathcal{S}}^{K_{\nu}}$ are Picard operators and we denote the fixed point of $M_{\mathcal{S}}^{A_{\mathcal{S}}}$ by $\mu_{A_{\mathcal{S}}}$ and the fixed point of $M_{\mathcal{S}}^{K_{\nu}}$ by $\mu_{K_{\nu}}$.

Proposition 4.13. The Markov operator $M_{\mathcal{S}}$ associated to a $\varphi$-max-IFS $\mathcal{S}$ has a unique fixed point denoted by $\mu_{\mathcal{S}}$ whose support is $A_{\mathcal{S}}$.

Proof. The function $G_{\mathcal{S}}: \{\mu \in \mathcal{M}(A_{\mathcal{S}}) | M_{\mathcal{S}}^{A_{\mathcal{S}}} (\mu) = \mu\} \to \{\mu \in \mathcal{M}(X) | M_{\mathcal{S}} (\mu) = \mu\}$ given by $G_{\mathcal{S}}(\mu) = i_{A_{\mathcal{S}}} (\mu)$ for every $\mu \in \mathcal{M}(A_{\mathcal{S}})$ such that $M_{\mathcal{S}}^{A_{\mathcal{S}}} (\mu) = \mu$ is a bijection (whose inverse is the function $H_{\mathcal{S}}: \{\mu \in \mathcal{M}(X) | M_{\mathcal{S}} (\mu) = \mu\} \to \{\mu \in \mathcal{M}(A_{\mathcal{S}}) | M_{\mathcal{S}}^{A_{\mathcal{S}}} (\mu) = \mu\}$ given by $H_{\mathcal{S}}(\mu) = \mu_{B(\mathcal{S})}$ for every $\mu \in \mathcal{M}(X)$ such that $M_{\mathcal{S}} (\mu) = \mu$. Therefore $i_{A_{\mathcal{S}}} (\mu_{A_{\mathcal{S}}}^{\text{not}}) = \mu_{\mathcal{S}}$ is the unique fixed point of $M_{\mathcal{S}}$. In addition, $\sup \mu_{\mathcal{S}} = \sup M_{\mathcal{S}} (\mu_{\mathcal{S}})$ Lemma 3.4 $F_{\mathcal{S}} (\text{supp } \mu_{\mathcal{S}})$, so, taking into account the uniqueness of the fixed point of $F_{\mathcal{S}}$, we infer that $\sup \mu_{\mathcal{S}} = A_{\mathcal{S}}$. $\Box$

Lemma 4.14. For every $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_{i})_{i\in\{1,\ldots,m\}}, (p_{i})_{i\in\{1,\ldots,m\}})$
and \( \nu \in \mathcal{M}(X) \), the measure \( \nu_0 \), given by \( \nu_0(B) = \nu(B \cap K_\nu) \) for every \( B \in \mathcal{B}(K_\nu) = \{ B' \cap K_\nu \mid B' \in \mathcal{B}(X) \} \), has the following properties:

i) \( \nu_0 \in \mathcal{M}(K_\nu) \);

ii) \( i_{K_\nu}(\nu_0) = \nu \).

**Proof.** We start by noting that, as \( B' \setminus K_\nu \subseteq B' \setminus \text{supp } \nu \subseteq X \setminus \text{supp } \nu \), we have \( 0 \leq \nu(B' \setminus K_\nu) \leq \nu(B' \setminus \text{supp } \nu) \leq \nu(X \setminus \text{supp } \nu) \) by definition of \( \text{supp } 0 \), so

\[
\nu(B' \setminus K_\nu) = 0, \tag{1}
\]

for every \( B' \in \mathcal{B}(X) \).

Then \( \nu_{\nu} \in \mathcal{M}(X) \)

\[
\nu(X) = \nu(X \setminus K_\nu) + \nu(K_\nu) \overset{(1)}{=} \nu(K_\nu), \text{ so i) is proved.}
\]

Moreover, \( i_{K_\nu}(\nu_0)(B') = \nu_0(B' \cap K_\nu) = \nu(B' \cap K_\nu) \overset{(1)}{=} \nu(B' \cap K_\nu) + \nu(B' \setminus K_\nu) = \nu(B') \) for every \( B' \in \mathcal{B}(X) \), so ii) is also proved. □

**Lemma 4.15.** Given a \( \varphi \)-max-IFS \( \mathcal{S} = ((X, d), (f_i)_{i \in \{1, \ldots, m\}}, (\nu_i)_{i \in \{1, \ldots, m\}}) \), we have \( i_{K_\nu}((M_{K_\nu}^{[n]}(\nu))) = M_{\mathcal{S}}^{[n]}(i_{K_\nu}(\nu)) \) for every \( n \in \mathbb{N} \) and every \( \nu \in \mathcal{M}(K_\nu) \).

**Proof.** First we prove that

\[
i_{K_\nu}(M_{\mathcal{S}}^{K_\nu}(\nu)) = M_{\mathcal{S}}(i_{K_\nu}(\nu)), \tag{1}
\]

for every \( \nu \in \mathcal{M}(K_\nu) \).

In order to justify (1), it suffices to check that

\[
\int_X g d(i_{K_\nu}(M_{\mathcal{S}}^{K_\nu}(\nu))) =
\int_X g \circ f_1 d(i_{K_\nu}(\nu)) + \ldots + p_m \int_X g \circ f_m d(i_{K_\nu}(\nu)) \text{ for every continuous and bounded function } g : X \to \mathbb{R}. \text{ This is true since }
\]

\[
\int_{K_\nu} R_{K_\nu}(g) d(M_{\mathcal{S}}^{K_\nu}(\nu)) = p_1 \int_{K_\nu} R_{K_\nu}(g) \circ \psi_1 d\nu + \ldots + p_m \int_{K_\nu} R_{K_\nu}(g) \circ \psi_m d\nu =
\int_X \left( \int_{K_\nu} R_{K_\nu}(g \circ f_1) d\nu + \ldots + \int_{K_\nu} R_{K_\nu}(g \circ f_m) d\nu \right) \overset{\text{Remark 4.10, i)}{=}
p_1 \int_X g \circ f_1 d(i_{K_\nu}(\nu)) + \ldots + \int_X g \circ f_m d(i_{K_\nu}(\nu)).
\]
Now, for every $n \in \mathbb{N}$, we have $i_{K_{\nu}}((M^{K_{\nu}})^{[n]}(\nu)) = i_{K_{\nu}}((M^{K_{\nu}})^{[n-1]}(\nu)) \overset{(1)}{=} M_{S}(i_{K_{\nu}}((M^{K_{\nu}})^{[n-1]}(\nu))) \neq M_{S}(i_{K_{\nu}}((M^{K_{\nu}})^{[n]}(\nu)))$ for every $\nu \in \mathcal{M}(K_{\nu})$. □

Lemma 4.16. Given a $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_{i})_{i \in \{1, \ldots, m\}}, (p_{i})_{i \in \{1, \ldots, m\}})$, we have $\mu_{S} = i_{K_{\nu}}(\mu^{K_{\nu}}_{S})$ for every $\nu \in \mathcal{M}(X)$.

Proof. As in the proof of Proposition 4.13, one can check that the function $\mu \mapsto i_{K_{\nu}}(\mu)$ is a bijection from the fixed points of $M^{K_{\nu}}_{S}$ to the fixed point of $M_{S}$, so $\mu_{S} = i_{K_{\nu}}(\mu^{K_{\nu}}_{S})$. □

Proposition 4.17. Given a $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_{i})_{i \in \{1, \ldots, m\}}, (p_{i})_{i \in \{1, \ldots, m\}})$, we have $\lim_{n \to \infty} d_{H}(M^{[n]}_{S}(\nu), \mu_{S}) = 0$ for every $\nu \in \mathcal{M}(X)$.

Proof. According to Lemma 4.14, for every $\nu \in \mathcal{M}(X)$ there exists $\nu_{0} \in \mathcal{M}(K_{\nu})$ such that $i_{K_{\nu}}(\nu_{0}) = \nu$. Then we have $d_{H}(M^{[n]}_{S}(\nu), \mu_{S}) = d_{H}(M^{[n]}_{S}(i_{K_{\nu}}(\nu_{0})), \mu_{S}) \overset{Lemma 4.15}{=} d_{H}(i_{K_{\nu}}(M^{[n]}_{S}(\nu_{0})), \mu_{S}) \overset{Lemma 4.16}{=} d_{H}(i_{K_{\nu}}((M^{K_{\nu}}_{S})^{[n]}(\nu_{0})), i_{K_{\nu}}(\mu^{K_{\nu}}_{S})) \overset{Lemma 4.11}{=} d_{H}^{K_{\nu}}((M^{K_{\nu}}_{S})^{[n]}(\nu_{0})), \mu^{K_{\nu}}_{S})$ for every $n \in \mathbb{N}$, so $\lim_{n \to \infty} d_{H}(M^{[n]}_{S}(\nu), \mu_{S}) = \lim_{n \to \infty} d_{H}^{K_{\nu}}((M^{K_{\nu}}_{S})^{[n]}(\nu_{0})), \mu^{K_{\nu}}_{S})$ ▽ 4.9 $= 0$. □

Combining Proposition 4.13 with Proposition 4.17, we get the following

Theorem 4.18. $M_{S} : (\mathcal{M}(X), d_{H}) \to (\mathcal{M}(X), d_{H})$ is a Picard operator for every $\varphi$-max-IFS $\mathcal{S} = ((X, d), (f_{i})_{i \in \{1,2,\ldots,m\}}, (p_{i})_{i \in \{1,2,\ldots,m\}})$ and the support of the fixed point of $M_{S}$ is $A_{S}$.

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