Online pseudo Marginal Sequential Monte Carlo smoother for general state spaces. Application to recursive maximum likelihood estimation of stochastic differential equations.

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Abstract

This paper focuses on the estimation of smoothing distributions in general state space models where the transition density of the hidden Markov chain or the conditional likelihood of the observations given the latent state cannot be evaluated pointwise. The consistency and asymptotic normality of a pseudo marginal online algorithm to estimate smoothed expectations of additive functionals when these quantities are replaced by unbiased estimators are established. A recursive maximum likelihood estimation procedure is also introduced by combining this online algorithm with an estimation of the gradient of the filtering distributions, also known as the tangent filters, when the model is driven by unknown parameters. The performance of this estimator is assessed in the case of a partially observed stochastic differential equation.

1 Introduction

The data considered in this paper originate from general state space models, usually defined as bivariate stochastic processes \(\{(X_k, Y_k)\}_{1 \leq k \leq n}\) where \(\{Y_k\}_{1 \leq k \leq n}\) are the observations and \(\{X_k\}_{1 \leq k \leq n}\) are the latent states commonly assumed to be a Markov chain. When both processes take values in general spaces, the estimation of the conditional distribution of a sequence of hidden states given a fixed observation record is a challenging task required for instance to perform maximum likelihood inference. Markov chain Monte Carlo (MCMC) and sequential Monte Carlo (SMC) methods (also known as particle filters or smoothers) are widespread solutions to propose consistent estimators of such distributions. This paper sets the focus on the special case where the conditional likelihood of an observation given the corresponding latent state (also known as the emission distribution) or the transition density of the hidden Markov chain cannot be evaluated pointwise, while they are pivotal tools of both MCMC and SMC approaches. The first objective of this paper is to prove that conditional expectations of additive functionals of the hidden states may still be estimated online with a consistent and asymptotically normal SMC algorithm. A recursive maximum likelihood estimation procedure based on this algorithm and using an approximation of the gradient of the filtering distributions, referred to as the tangent filters, is then introduced.
The use of latent data models is ubiquitous in time series analysis across a wide range of applied science and engineering domains such as signal processing [6], genomics [36, 35], target tracking [33], enhancement and segmentation of speech and audio signals [31], see also [32, 14, 37] and the numerous references therein. Statistical inference for such models is likely to require the computation of conditional expectations of sequences of hidden states given observations. In this Bayesian setting, one of the most challenging problems is the approximation of expectations under the joint smoothing distribution, i.e. the posterior distribution of the sequence of states $(X_1, \ldots, X_n)$ given the observations $(Y_1, \ldots, Y_n)$ for some $n \geq 1$. This computation is not tractable in the framework of this paper where it is assumed that the transition density of the hidden process or the conditional likelihood of observations given states cannot be computed. This circumstance is somehow common for instance in the case of partially observed stochastic differential equations (SDE), or in models where the emission distributions relies on a computationally prohibitive black-box routine.

Following [18, 21], this paper concentrates on SMC methods to approximate smoothing distributions with a random set of states, the particles, associated with importance weights by combining importance sampling and resampling steps. This allows to solve the filtering problem by combining an auxiliary particle filter with an unbiased estimate of the unknown densities. Then, the online smoother of [21] extends the particle-based rapid incremental smoother (PaRIS) of [28], to approximate, processing the data stream online, smoothed expectations of additive functionals when the unknown densities are replaced by unbiased estimates. This approach is an online version of the Forward Filtering Backward Simulation algorithm algorithm [11] specifically designed to approximate smoothed additive functionals. The crucial feature which makes the PaRIS algorithm appealing is the acceptance-rejection step which benefits from the unbiased estimation. The extension of the usual alternative, named the Forward Filtering Backward Smoothing algorithm [15], is more sensitive as it involves ratios of these unknown quantities. Other smoothing algorithms such as two-filter based approaches [2, 19, 25] could be extended similarly but they are intrinsically not online procedures as they require the time horizon and all observations to be available to initialize a backward information filter.

In [21], the only theoretical guarantee is that the accept reject mechanism of the PaRIS algorithm is still correct when the transition densities are replaced by unbiased estimates. In this paper, the consistency of the algorithm as long as a central limit theorem (CLT) are established (see Proposition 4.2 and Proposition 4.3 in Section 4.2). This makes this pseudo marginal smoother the first algorithm to approximate such expectations in the general setting of this paper with theoretical guarantees and an explicit expression of the asymptotic variance. As a byproduct, the proofs of these results require to establish exponential deviation inequalities and a CLT for the PaRIS algorithm based on the auxiliary particle filter, see Section 4.1. This extends the result of [28], written only in the case of the bootstrap filter of [22]. This also extends the theoretical guarantees obtained for online sequential Monte Carlo smoothers given in [11, 9, 17, 20].

The second part of the paper is devoted to recursive maximum likelihood estimation when the emission distributions or the transition densities depend on an unknown parameter, see Section 5. Following the filter sensitivity approach of [5, Section 10.2.4], the pseudo marginal smoother is used to estimate online the gradient of the one-step predictive likelihood of an observation given past observations. This procedure allows to perform online estimation in complex frameworks and is applied in Section 6 to partially observed SDE.
2 Online Sequential Monte Carlo smoother

Let \( n \) be a positive integer and \( X \) and \( Y \) two general state spaces. Consider a distribution \( \chi \) on \( B(X) \) and the Markov transition kernels \( (Q_k)_{0 \leq k \leq n-1} \) on \( X \times B(X) \) and \( (G_k)_{0 \leq k \leq n-1} \) on \( X \times X \times B(Y) \). Throughout this paper, for all \( 0 \leq k \leq n-1 \), \( G_k \) has a density \( g_k \) with respect to a reference measure \( \mu \) on \( B(Y) \). In the following, \( F(Z) \) denotes the set of real valued measurable functions defined on the set \( Z \). Let \( (Y_k)_{1 \leq k \leq n} \) be a sequence of observations in \( Y \) and define the joint smoothing distributions, for any \( 0 \leq k_1 \leq k_2 \leq n \) and any function \( h \in F(X^{k_2-k_1+1}) \), by:

\[
\phi_{k_1:k_2}[h] := \mathcal{L}_n^{-1}(Y_{1:n}) \int \chi(dx_0) \prod_{k=0}^{n-1} Q_k(x_k, dx_{k+1}) g_k(x_k, x_{k+1}, Y_{k+1}) h(x_{k_1:k_2}),
\]

(1)

where \( a_{u:v} \) is a short-hand notation for \( (a_u, \ldots, a_v) \) and

\[
\mathcal{L}_n(Y_{1:n}) = \int \chi(dx_0) \prod_{k=0}^{n-1} Q_k(x_k, dx_{k+1}) g_k(x_k, x_{k+1}, Y_{k+1})
\]

(2)

is the observed data likelihood. For all \( 0 \leq k \leq n-1 \), \( Q_k \) has a density \( q_k \) with respect to a reference measure \( \mu \) on \( B(X) \). The initial measure \( \chi \) is also assumed to have a density with respect to \( \mu \) which is also referred to as \( \chi \). For all \( 0 \leq k \leq n \), \( \phi_k = \phi_{k:k} \) are the filtering distributions, \( \pi_{k+1} = \phi_{k+1:k+1} \) are the one-step predictive distributions, while \( \phi_{k:n} = \phi_{k:k:n} \) are the marginal smoothing distributions.

Consider a latent Markov chain \( (X_k)_{0 \leq k \leq n} \) with initial distribution \( \chi \) and Markov transition kernels \( (Q_k)_{0 \leq k \leq n-1} \). The states \( (X_k)_{0 \leq k \leq n} \) are not available so that any statistical inference procedure is performed using the sequence of observations \( (Y_k)_{1 \leq k \leq n} \) only. The observations are assumed to be independent conditional on \( (X_k)_{0 \leq k \leq n} \) and such that for all \( 1 \leq \ell \leq n \) the distribution of \( Y_\ell \) given \( (X_k)_{0 \leq k \leq n} \) has distribution \( G_k(X_\ell, \cdot) \). In this case, (1) may be interpreted as:

\[
\phi_{k_1:k_2}[h] = \mathbb{E}[h(X_{k_1:k_2})|Y_{1:n}].
\]

Figure 1: Graphical model of the general state space hidden Markov model

Figure 1 displays the graphical model associated with (2). Note that, when for all \( 0 \leq k \leq n-1 \) \( g_k \) only depends on its last two arguments, (2) is the likelihood of a standard hidden Markov model. In such models, computing (1) allows to solve classical problems such as:
i) path reconstruction, i.e. the reconstruction of the hidden states given the observations;

ii) parameter inference, i.e., when \( q_k \) and \( g_k \) depend on some unknown parameter \( \theta \), the design of a consistent estimator of \( \theta \) from the observations.

As (1) is, in general, not available explicitly, this paper focuses on a sequential Monte Carlo based approximation specifically designed for cases where \( q_k \) and/or \( g_k \) cannot be evaluated pointwise.

Partially observed diffusion processes (POD) \([27]\), where the latent process is the solution to a stochastic differential equation are widespread examples where \( q_k \) is not tractable.

Recursive formulation of (1) for additive functionals. For all \( 0 \leq k \leq n-1 \), define

\[
\begin{align*}
    r_k(x_k, x_{k+1}) &= q_k(x_k, x_{k+1})g_k(x_k, x_{k+1}, Y_{k+1}) .
\end{align*}
\]

(3)

For all \( 0 \leq k \leq n-1 \), define also the kernel \( L_k \) on \( X \times \mathcal{B}(X) \), for all \( x \in X \) and all \( f \in F(X) \) by

\[
    L_k f(x) = \int r_k(x, y) f(y) dy .
\]

In the following, \( \mathbb{1} \) denotes the constant function which equals 1 for all \( x \in X \) so that

\[
    L_k \mathbb{1}(x) = \int r_k(x, y) dy .
\]

Following for instance \([4]\), the joint smoothing distributions \( \phi_{0:n|n} \) may be decomposed using the backward Markov kernels defined, for all \( 0 \leq k \leq n-1 \), all \( x_{k+1} \in X \) and all \( f \in F(X) \), by:

\[
\begin{align*}
    \overline{Q}_{\phi_k} f(x_{k+1}) := \frac{\int f(x_k) r_k(x_k, x_{k+1}) \phi_k(dx_k)}{\int r_k(x_k, x_{k+1}) \phi_k(dx_k)} .
\end{align*}
\]

(4)

Consequently, the joint-smoothing distribution \( \phi_{0:n|n} \) may be expressed, for all \( h \in F(X^{n+1}) \), as

\[
    \phi_{0:n|n}[h] = \phi_n[T_n h] ,
\]

(5)

where

\[
    T_n := \begin{cases} 
        \overline{Q}_{\phi_{n-1}} \otimes \overline{Q}_{\phi_{n-2}} \otimes \cdots \otimes \overline{Q}_{\phi_0} & \text{for } n > 0 , \\
        \text{id} & \text{for } n = 0 , 
    \end{cases}
\]

(6)

where, for all Markov kernels \( K_1, K_2 \) on \( X \times \mathcal{B}(X) \), all \( f \in F(X^2) \) and all \( x \in X \),

\[
    (K_1 \otimes K_2) f(x) = \int f(y, z) K_1(x, dy) K_2(y, dz) .
\]

In this paper, the focus is set on additive functionals of the form

\[
    h_{0:n}(x_{0:n}) = \sum_{k=0}^{n-1} \tilde{h}_k(x_k, x_{k+1}) ,
\]

(7)
with, for all \(0 \leq k \leq n - 1\), \(\hat{h}_k : X \times X \to \mathbb{R}^p\) for some \(p \geq 1\). The additive form of the function \(h_n\) defined in (7) allows to update the backward statistics \((T_k h_k)_{k \geq 0}\) recursively, see [3] [9]. For all \(k \geq 0\),
\[
T_{k+1} h_{k+1}(x_{k+1}) = \int \{T_k h_k(x_k) + \tilde{h}_k(x_{k:k+1})\} \tilde{Q}_\phi_k(x_{k+1}, dx_k).
\] (8)

By [5] and (8), the smoothed additive functional (5) can be updated recursively each time a new observation is available. However, its exact computation is not possible in general state spaces. In this paper, we propose to approximate \(\phi_{0:n}[h_n]\) using SMC methods: \(\phi_n\) in (5) and \(\tilde{Q}_\phi_k\) in (8) are replaced by a set of random samples associated with nonnegative importance weights. These particle filters and smoothers approximations combine sequential importance sampling steps to update recursively \(\phi_n\) and importance resampling steps to duplicate or discard particles according to their importance weights.

**Sequential Monte Carlo for additive functionals.** Let \((\xi^\ell_0)^N_{\ell=1}\) be independent and identically distributed according to the instrumental proposal density \(\rho_0\) on \(X\) and define the importance weights:
\[
\omega^\ell_0 := \chi(\xi^\ell_0) / \rho_0(\xi^\ell_0).
\]

For any \(f \in F(X)\),
\[
\phi^N_0[f] := \Omega_0^{-1} \sum_{\ell=1}^N \omega^\ell_0 f(\xi^\ell_0), \quad \text{where} \quad \Omega_0 := \sum_{\ell=1}^N \omega^\ell_0
\]
is a consistent estimator of \(\phi_0[f]\), see for instance [8]. Then, for all \(k \geq 1\), once the observation \(Y_k\) is available, the weighted particle sample \(\{(\omega^\ell_{k-1}, \xi^\ell_{k-1})\}_{\ell=1}^N\) is transformed into a new weighted particle sample approximating \(\phi_k\). This update step is carried through in two steps, selection and mutation, using the auxiliary sampler introduced in [29]. New indices and particles \((I^\ell_k, \xi^\ell_k)\) are simulated independently from the instrumental distribution with density on \(\{1, \ldots, N\} \times X\):
\[
u_k(\ell, x) \propto \omega^\ell_{k-1} \vartheta_{k-1}(I^\ell_{k-1}) p_{k-1}(I^\ell_{k-1}, x),
\] (9)

where \(\vartheta_{k-1}\) is an adjustment multiplier weight function and \(p_{k-1}\) a Markovian transition density. For any \(\ell \in \{1, \ldots, N\}\), \(\xi^\ell_k\) is associated with the importance weight defined by:
\[
\omega^\ell_k := \frac{\vartheta_{k-1}(I^\ell_{k-1}, \xi^\ell_k) r_{k-1}(I^\ell_{k-1}, \xi^\ell_k)}{\vartheta_{k-1}(I^\ell_{k-1}) p_{k-1}(I^\ell_{k-1}, \xi^\ell_k)}
\] (10)

to produce the following approximation of \(\phi_k[f]\):
\[
\phi^N_k[f] := \Omega_k^{-1} \sum_{\ell=1}^N \omega^\ell_k f(\xi^\ell_k), \quad \text{where} \quad \Omega_k := \sum_{\ell=1}^N \omega^\ell_k.
\]

For all \(k \geq 0\) and all \((x, f) \in X \times F(X)\), replacing \(\phi_k\) by \(\phi^N_k\) in (4), \(\tilde{Q}_{\phi_k} f(x)\) is approximated by:
\[
\tilde{Q}_{\phi^N_k} f(x) = \sum_{\ell=1}^N \frac{\omega^\ell_k t_k(I^\ell_k, x)}{\sum_{\ell=1}^N \omega^\ell_k t_k(I^\ell_k, x)} f(\xi^\ell_k).
\] (11)
The forward-filtering backward-smoothing (FFBS) algorithm proposed in [9], \( \hat{Q}_n \), proceeds by the approximation \( \hat{Q}_n \). Proceeding recursively, this produces a sequence of estimates \( (\hat{\tau}_k^N)_{i=1}^N \) of \((T_kr_k(\xi_{i}^k,\xi_{k+1}^i))^N_{i=1}\) for \( 0 \leq k \leq n \). Starting with \( \hat{\tau}_0^0 = 0 \) for all \( 1 \leq i \leq N \), this yields for all \( 0 \leq k \leq n - 1 \):

\[
\hat{\tau}_k^i = \sum_{j=1}^N \sum_{i=1}^N \omega_k^j r_k(\xi_{i}^k, \xi_{k+1}^i) \left( \hat{\tau}_k^i + \hat{h}_k(\xi_{k}^i, \xi_{k+1}^i) \right) .
\]

Then, at each iteration \( 0 \leq k \leq n - 1 \), \( \phi_{0:k+1|[k]}[h_{k+1}] \) and \( \phi_{0:k+1|[k+1]}[h_{k+1}] \) are approximated by

\[
\phi_{0:k+1|[k]}[h_{k+1}] := \frac{1}{N} \sum_{i=1}^N \hat{\tau}_k^i \quad \text{and} \quad \phi_{0:k+1|[k+1]}[h_{k+1}] := \sum_{i=1}^N \omega_k^i \hat{\tau}_k^i .
\]

The computational complexity of the update [12] grows quadratically with the number of particles \( N \). This computational cost can be reduced following [28] by first replacing [12] by the Monte Carlo estimate

\[
\tau_k^i = \frac{1}{N} \sum_{j=1}^N \left( \tau_k^{j(i,j)} + \hat{h}_k(\xi_{k}^{j(i,j)}, \xi_{k+1}^i) \right) ,
\]

where the sample size \( \bar{N} \geq 1 \) is typically small compared to \( N \) and \( (\tau_k^{j(i,j)} \bar{N})_{j=1}^N \) are i.i.d. samples in \( \{1, \ldots, N\} \) with probabilities proportional to \( (\omega_k^j r_k(\xi_{k}^i, \xi_{k+1}^i))^N_{i=1} \). In the resulting Particle Rapid Incremental smoother (PaRIS) algorithm, the updated \( (\tau_k^i)_{i=1}^N \), estimates of \( \phi_{0:k+1|[k]}[h_{k+1}] = \pi_{k+1}[T_{k+1}h_{k+1}] \) and \( \phi_{0:k+1|[k+1]}[h_{k+1}] = \pi_{k+1}[T_{k+1}h_{k+1}] \) are obtained as:

\[
\phi_{0:k+1|[k]}[h_{k}] := \frac{1}{N} \sum_{i=1}^N \tau_k^i \quad \text{and} \quad \phi_{0:k+1|[k]}[h_{k}] := \sum_{i=1}^N \omega_k^i \tau_k^i .
\]

**Acceptance-rejection procedure.** The computational complexity of the described approach is still of order \( N^2 \) since it requires the normalising constant \( \sum_{i=1}^N \omega_k^i r_k(\xi_{k}^i, \xi_{k+1}^i) \) to sample \( (\tau_k^{j(i,j)} \bar{N})_{j=1}^N \) for all particle \( \xi_{k+1}^i, 1 \leq i \leq N \). A faster algorithm is obtained by applying the accept-reject sampling approach proposed in [11] and illustrated in [16] which presupposes that there exists a constant \( \bar{M} > 0 \) such that \( r_k(x, x') \leq \bar{M} \) for all \( x, x' \in X \times X \). Then, in order to sample from \( (\omega_k^i r_k(\xi_{k}^i, \xi_{k+1}^i))^N_{i=1} \) a candidate \( J^* \sim (\omega_k^i)^N_{i=1} \) is accepted with probability:

\[
T_k^M(J^*, i) := r_k(\xi_{k}^{J^*}, \xi_{k+1}^i)/\bar{M} .
\]

This procedure is repeated until acceptance. Under strong mixing assumptions it can be shown, see for instance [11] Proposition 2 and [28] Theorem 10, that the expected number of trials needed for this approach to update \( (\tau_k^i)_{i=1}^N \) is \( O(\bar{N}N) \).

### 3 Pseudo marginal Sequential Monte Carlo smoother

In many applications, Sequential Monte Carlo methods cannot be used as the transition densities \( q_k \) or \( g_k \), \( 0 \leq k \leq n - 1 \), are unknown. The following crucial steps which rely on \( r_k \) are not tractable:
(a) computation of the importance weights $\omega^\ell_k$ in (10); 
(b) computation of the acceptance ratio (14).

To overcome these issues, following [21], consider the following algorithm.

**Initialization.** At time $k = 0$, set for all $1 \leq \ell \leq N$

$$
\hat{\omega}^\ell_0 = \omega^\ell_0, \quad \hat{\tau}^\ell_0 = 0 \quad \text{and} \quad \hat{\tau}^0_\ell = \tau^0_\ell = 0.
$$

**Propagation.** Starting with weighted samples $\{(\xi^\ell_k, \hat{\omega}^\ell_k)\}_{\ell=1}^N$, define

$$
\tilde{F}_k^N = \sigma \left\{ (\xi^\ell_u, \omega^\ell_u, \hat{\tau}^\ell_u) ; 1 \leq \ell \leq N, 0 \leq u \leq k \right\} \quad \text{and} \quad \tilde{G}_k^N = \sigma \left\{ (\hat{\tau}^\ell_k, \xi^\ell_k) ; 1 \leq \ell \leq N \right\}.
$$

New indices and particles $\{(\hat{\tau}_{k+1}^\ell, \xi_{k+1}^\ell)\}_{\ell=1}^N$ are simulated independently from the instrumental distribution with density on $\{1, \ldots, N\} \times X$:

$$
v_{k+1}(\ell, x) \propto \hat{\omega}_k \theta_k(\xi^\ell_k)p_k(\xi^\ell_k, x),
$$

Following [18, 27], weights update can be approximated by replacing $r_k(\xi^\ell_k, \xi^i_{k+1})$ by an unbiased estimator.

**H1** There exist a Markov kernel $R_k$ on $(X \times X, \mathcal{B}(Z))$ where $(Z, \mathcal{B}(Z))$ is a general state space and a positive mapping $\hat{r}_k$ on $X \times X \times Z$ such that, for all $(x, x') \in X^2$,

$$
\int R_k(x, x';dz)\hat{r}_k(x, x';z) = r_k(x, x').
$$

Then, under H1, if conditionally on $\tilde{F}_k^N \vee \tilde{G}_k^N$, $\zeta_k^\ell$ has distribution $\hat{R}_k(\hat{\tau}_{k+1}^\ell, \xi_{k+1}^\ell; \cdot)$, then

$$
E \left[ \hat{r}_k(\hat{\tau}_{k+1}^\ell, \xi_{k+1}^\ell; \cdot) \left| \tilde{F}_k^N \vee \tilde{G}_k^N \right. \right] = r_k(\xi_{k+1}^\ell, \xi_{k+1}^\ell).
$$

The filtering weights then become:

$$
\hat{\omega}_{k+1}^\ell := \frac{\hat{r}_k(\hat{\tau}_{k+1}^\ell, \xi_{k+1}^\ell; \cdot)}{\theta_k(\zeta_{k+1}^\ell)p_k(\xi_{k+1}^\ell, \xi_{k+1}^\ell)}.
$$

For all $f \in \mathcal{F}(X)$ and all $0 \leq k \leq n$, $\phi_k[f]$ is approximated by

$$
\hat{\phi}_k^N[f] := \sum_{i=1}^N \frac{\hat{\omega}_i^\ell f(\xi_i^\ell)}{\hat{\Omega}_k}, \quad \hat{\Omega}_k = \sum_{i=1}^N \hat{\omega}_i^\ell.
$$

To solve issue [13, 21] ensured that, under several assumptions, the acceptance-rejection mechanism introduced to implement PaRIS algorithm is still valid for stochastic differential equations. Consider the following assumption,
For all $0 \leq k \leq n$, there exists a random variable $M_k$ measurable with respect to $\tilde{G}_{k+1}^N$ such that
\[
\sup_{x,y,\zeta} \tilde{r}_k(x,y;\zeta) \leq M_k.
\]
If this assumption holds, the accept-reject mechanism of PaRIS algorithm is replaced by the following steps. For all $1 \leq i \leq N$ and all $1 \leq j \leq \tilde{N}$, a candidate $J^*$ is sampled in $\{1,\ldots,N\}$ with probabilities proportional to $\hat{\omega}_{i+1}^k$ and is accepted with probability $\hat{r}_k(\xi_{i+1},\xi_{i+1};\zeta)/M_k$, where $\zeta$ has distribution $R_k(\xi_{i+1},\xi_{i+1};\cdot)$. Then, set
\[
\hat{J}_{k+1}^{(i,j)} = J^*
\]
and
\[
\hat{\tau}_{k+1}^i = \frac{1}{N} \sum_{j=1}^{\tilde{N}} \left( \hat{\tau}_{k+1}^{(i,j)} + \hat{h}_k \left( \xi_{k+1}, \xi_{i+1} \right) \right). \tag{17}
\]

\section*{Lemma 3.1.} Assume that $H1$ and $H2$ hold. Then, for all $0 \leq k \leq n-1$ and all $1 \leq i \leq N$, $(\hat{J}_{k+1}^{(i,j)})_{1 \leq j \leq \tilde{N}}$ are i.i.d. and independent of $\hat{\omega}_{k+1}^i$ given $\tilde{F}_k^N \vee \tilde{G}_{k+1}^N$ and such that for all $1 \leq \ell \leq N$,
\[
\mathbb{P} \left( \hat{J}_{k+1}^{(i,j)} = \ell \bigg| \tilde{F}_k^N \vee \tilde{G}_{k+1}^N \right) = \frac{\hat{\omega}_{k+1}^i \hat{r}_k(\xi_{k+1},\xi_{i+1})}{\sum_{m=1}^{N} \hat{\omega}_{k+1}^m \hat{r}_k(\xi_{k+1},\xi_{i+1})},
\]
where $\hat{\omega}_{k+1}^i$ is defined by (16).

\begin{proof}
The proof follows the same lines as [21, Lemma 1].
\end{proof}

The proposed algorithm therefore leads to an estimator of the expectation (1) in the general setting of this paper. The following section provides consistency and asymptotic normality results for this estimator.

\section{Asymptotic results}

\subsection*{4.1 Auxiliary Particle filter based PaRIS algorithms}

In [26], the authors established the consistency and asymptotic normality of PaRIS algorithm for the bootstrap filter, i.e. in the simple case where for all $0 \leq k \leq n-1$, $\varrho_k$ is the constant function which equals 1 and $p_k = q_k$. This section extends these convergence results to the general auxiliary particle filter based PaRIS algorithm as such filters are required for the pseudo marginal smoother. Consider the following assumptions.

\begin{itemize}
\item \textbf{H3} For all $0 \leq k \leq n-1$, $g_k$ is a positive function such that $\|g_k\|_{\infty} < \infty$. For all $0 \leq k \leq n-1$, $\|q_k\|_{\infty} < \infty$, $\|\varrho_k\|_{\infty} < \infty$ and $\|\omega_{k+1}\|_{\infty} < \infty$ where for all $(x,y) \in X \times X$,
\[
\omega_{k+1}(x,y) := \frac{r_k(x,y)}{\varrho_k(x)p_k(x,y)}.
\]
\end{itemize}
Proof. The proof follows the same lines as the proof of [26, Theorem 1]. □

Lemma 4.2. Assume that $H_b$ holds. Then, for all $0 \leq k \leq n$, $f_k \in F(X)$ and $\bar{N} \geq 0$,
\[
\sum_{i=1}^{N} \frac{\omega^i_k}{\Omega_k} (r_k^f f_k(\xi^*_k) + \bar{f}_k(\xi^*_k)) - \phi_k[T_k h_k f_k + \bar{f}_k] = 0 \quad \text{and for all } 0 \leq k \leq n - 1,
\]
\[
\eta_{k+1}[f_{k+1}] = \frac{\eta_k[L_{k} f_{k+1}] + \phi_k[L_k \{Q_{\phi_k} (T_k h_k + \bar{h}_k - T_{k+1} h_{k+1}) f_{k+1}]}{\bar{N}_k \phi_k[L_k 1]} .
\]
Proof. The proof is postponed to Section B.1 □

Following [26, Lemma 13], for all $0 \leq k \leq n$ and $f_k \in F(X)$, the recursion given in Lemma 4.2 may also be expressed as
\[
\eta_k[f_k] = \frac{\sum_{\ell=0}^{k-1} \phi_{\ell}[L_{\ell} \{Q_{\phi_{\ell}} (T_{\ell} h_{\ell} + \bar{h}_{\ell} - T_{\ell+1} h_{\ell+1})^2 L_{\ell+1} \ldots L_{k-1} f_k \}]}{N^{k-\ell} \phi_{\ell}[L_{\ell} \ldots L_{k-1} 1]} .
\]

Establishing a central limit theorem for ParIS algorithms requires to introduce the retro-prospective kernels, defined, for all $0 \leq k \leq m \leq n$, $x_k \in X$ and $h \in F(X^{m+1})$, by
\[
D_{k,m} h(x_k) := \int h(x_{0:m}) T_k(x_k, dx_{0:k-1})L_k \ldots L_{m-1}(x_k, dx_{k+1:m}) ,
\]
\[
\bar{D}_{k,m} h(x_k) := D_{k,m} \{h - \phi_{0:m}[h]\}(x_k) .
\]

Proposition 4.1. Assume that $H_b$ holds. Then, for all $0 \leq k \leq n$, $(f_k, \bar{f}_k) \in F(X)^2$,
\[
\sqrt{N} \left( \sum_{i=1}^{N} \frac{\omega^i_k}{\Omega_k} (r_k^f f_k(\xi^*_k) + \bar{f}_k(\xi^*_k)) - \phi_k[T_k h_k f_k + \bar{f}_k] \right) \xrightarrow{D} \sigma_k(f_k; \bar{f}_k) Z ,
\]
where $Z$ is a standard Gaussian random variable and for all $0 \leq k \leq n - 1$,
\[
\sigma_k^2(f_k; \bar{f}_k) = \frac{\sum_{s=0}^{k-1} \phi_{s}[\phi_{s} \{Q_{\phi_{s}} (L_{s+1} \ldots L_{k-1} 1) \}]}{\phi_{s}[L_s \ldots L_{k-1} 1]^2}
\]
\[
+ \sum_{s=0}^{k-1} \sum_{\ell=0}^{k} \phi_{s}[\phi_{s} \{Q_{\phi_{s}} (T_{\ell} h_{\ell} + \bar{h}_{\ell} - T_{\ell+1} h_{\ell+1})^2 L_{\ell+1} \ldots L_{s+1} (Q_{\phi_{s}} \bar{\omega}_s (L_{s+1} \ldots L_{k-1} f_k) \}) \}]
\]
\[
N^{s+1-\ell} \phi_{s}[L_{\ell} \ldots L_{s+1} 1] \phi_{s}[L_s \ldots L_{k-1} 1]^2
\]
Proof. The proof is postponed to Section B.2 □
Corollary 4.1. Assume that $H_3$ holds. Then, for all $0 \leq k \leq n$, $(f_k, \tilde{f}_k) \in F(X)^2$,

$$\sqrt{N} \left( \sum_{i=1}^{N} \frac{\omega^i_k}{\Omega_k} \tau^i_k - \phi_k[T_k h_k] \right) \xrightarrow{p} N \rightarrow \infty \sigma_k(h_k) Z ,$$

where $Z$ is a standard Gaussian random variable and

$$\sigma^2_k(h_k) = \sum_{s=0}^{k-1} \phi_s[\vartheta_s] \phi_s[L_s \{ \tilde{\omega}_s D^2_{s+1,k} h_k \}]$$

$$+ \sum_{s=0}^{k-1} \sum_{\ell=0}^{k-s-1} \phi_s[\vartheta_s] \phi_s[L_\ell \{ \tilde{Q}_s \phi_s(T_\ell h_\ell + \tilde{h}_\ell - T_\ell h_{\ell+1})^2 L_{\ell+1} \cdots L_s \{ \tilde{Q}_s \phi_s(L_{s+1} \cdots L_{k-1}^2 \})]$$

$$N^{s+1-\ell} \phi_s[L_\ell \cdots L_{s-1}^2] \phi_s[L_s \cdots L_{k-1}^2] .$$

4.2 Pseudo marginal PaRIS algorithms

Consider the following assumption.

$H_4$ For all $0 \leq k \leq n-1$, $\| \tilde{\omega}_{k+1} \|_{\infty} < \infty$ where for all $(x,y,z) \in X \times X \times Z$,

$$\tilde{\omega}_{k+1}(x,y; z) := \frac{\tilde{r}_k(x,y; z)}{\partial_k(x)p_k(x,y)} . \tag{20}$$

Proposition 4.2. Assume that $H_1$, $H_2$ and $H_4$ hold. Then, for all $0 \leq k \leq n$, $(f_k, \tilde{f}_k) \in F(X)^2$ and $\tilde{N} > 0$, there exist $(c_k, \tilde{c}_k) \in (R^*_+)^2$ such that for all $N \in R^*_+$ and all $\varepsilon \in R^*_+$,

$$P \left( \left| \sum_{i=1}^{N} \frac{\tilde{\omega}^i_k}{\Omega_k} \{ \tilde{r}^i_k f_k(\xi^i_k) + \tilde{f}_k(\xi^i_k) \} - \phi_k[T_k h_k f_k + \tilde{f}_k] \right| > \varepsilon \right) \leq c_k e^{-\tilde{c}_k N \varepsilon^2} .$$

Proof. The proof follows the same lines as the proof of [20 Theorem 1]. \hfill \Box

Lemma 4.3. Assume that $H_1$, $H_2$ and $H_4$ hold. Then, for all $0 \leq k \leq n$, $f_k \in F(X)$ and $\tilde{N} > 0$,

$$\sum_{i=1}^{N} \frac{\tilde{\omega}^i_k}{\Omega_k} (\tilde{r}^i_k f_k(\xi^i_k)) \xrightarrow{p} N \rightarrow \infty \eta_k[f_k] + \phi_k[T_k^2 h_k f_k] ,$$

where for all $0 \leq k \leq n$, $\eta_k[f_k]$ is defined in (19).

Proof. The proof is postponed to Section C.1. \hfill \Box

Proposition 4.3. Assume that $H_1$, $H_2$ and $H_4$ hold. Then, for all $0 \leq k \leq n$, $(f_k, \tilde{f}_k) \in F(X)^2$,

$$\sqrt{N} \left( \sum_{i=1}^{N} \frac{\tilde{\omega}^i_k}{\Omega_k} \{ \tilde{r}^i_k f_k(\xi^i_k) + \tilde{f}_k(\xi^i_k) \} - \phi_k[T_k h_k f_k + \tilde{f}_k] \right) \xrightarrow{p} N \rightarrow \infty \bar{\sigma}_k(f_k; \tilde{f}_k) Z ,$$

where $Z$ is a standard Gaussian random variable and for all $0 \leq k \leq n-1$, $\bar{\sigma}^2_{k+1}(f_{k+1}; \tilde{f}_{k+1})$ can be computed using an explicit recursive formula given in Appendix C.2.
Proof. The proof is postponed to Section C.2.

Corollary 4.2. Assume that $H_1$, $H_2$ and $H_3$ hold. Then, for all $0 \leq k \leq n$,

$$\sqrt{N} \left( \sum_{i=1}^{N} \frac{\tilde{\omega}_k^i \zeta_k^i}{\Omega_k} - \phi_k[T_k h_k] \right) \xrightarrow{D_{N \to \infty}} \tilde{\sigma}_k(h_k) Z,$$

where $Z$ is a standard Gaussian random variable and $\tilde{\sigma}^2_k(h_k)$ can be computed using an explicit recursive formula given in Appendix C.2.

5 Tangent filters and online recursive maximum likelihood

Let $\Theta$ be a parameter space. This section considers a family of transition kernels $(Q_{k;\theta})_{\theta \in \Theta; 0 \leq k \leq n-1}$ on $X \times B(X)$ and $(G_{k;\theta})_{\theta \in \Theta; 1 \leq k \leq n}$ on $X \times B(Y)$ associated with densities $g_{k;\theta}$ and $g_{k;\theta}$ with respect to $\mu$ and $\nu$. The joint smoothing distributions are then defined, for any $\theta \in \Theta$, $0 \leq k_1 \leq k_2 \leq n$ and any function $h \in F(X^{k_2-k_1+1})$, by:

$$\phi_{k_1;k_2;\theta|n}[h] := \mathcal{L}^{-1}_{n;\theta}(Y_{1:n}) \int \chi(dx_0) \prod_{k=0}^{n-1} Q_{k;\theta}(x_k, dx_{k+1}) g_{k+1;\theta}(x_{k+1}, Y_{k+1}) h(x_{k_1:k_2}),$$

where

$$\mathcal{L}_{n;\theta}(Y_{1:n}) = \int \chi(dx_0) \prod_{k=0}^{n-1} Q_{k;\theta}(x_k, dx_{k+1}) g_{k+1;\theta}(x_{k+1}, Y_{k+1}).$$

As noted for instance in [10, Section 2] and [26], for all $\theta \in \Theta$ and all $f_{0:n} \in F(X^{n+1})$,

$$\nabla_\theta \phi_{0:n;\theta|n-1}[f_{0:n}] = \phi_{0:n;\theta|n-1}[h_{n}f_{0:n}] - \phi_{0:n;\theta|n-1}[f_{0:n}] \times \phi_{0:n;\theta|n-1}[h_{n}],$$

where

$$h_{n}(x_{0:n}) = \sum_{k=0}^{n-1} \tilde{h}_{k;\theta}(x_k, x_{k+1}),$$

with, for all $0 \leq k \leq n-1$,

$$\tilde{h}_{k;\theta}(x_k, x_{k+1}) = \nabla_\theta \log g_{k+1;\theta}(x_{k+1}, Y_{k+1}) + \nabla_\theta \log g_{k;\theta}(x_k, x_{k+1}).$$

Considering an objective function $f_n \in F(X)$ which depends on the last state $x_n$ only, the tangent filter $\eta_n$ is defined as the following signed measure:

$$\eta_{n;\theta}[f_{n}] := \nabla_\theta \pi_{n;\theta}[f_{n}] = \phi_{0:n;\theta|n-1}[h_{n}f_{n}] - \pi_{n;\theta}[f_{n}] \times \phi_{0:n;\theta|n-1}[h_{n}],$$

where $\pi_{n} = \phi_{n:n|n-1}$ is the predictive measure. The particle based estimator of $\pi_{n}[f]$ is given by:

$$\pi_{n}^N[f] = \frac{1}{N} \sum_{\ell=1}^{N} f(\xi_{n}^\ell).$$

Using the tower property, [4] and the backward decomposition (6):

$$\eta_{n;\theta}[f_{n}] = \pi_{n;\theta}[(T_{n} h_{n} - \pi_{n;\theta}[T_{n} h_{n}]) f_{n}].$$

(21)
Therefore, the tangent filter \( (21) \) can be approximated on-the-fly using the statistics \( \langle \hat{\tau}_n \rangle^N_{i=1} \) and the weighted particles \( \{(\xi^i_n, \omega^i_n)\}_{i=1}^N \):

\[
\eta_{n;\theta}^{N,\text{FFBS}}[f_n] = \frac{1}{N} \sum_{i=1}^N \tilde{\tau}_n f_n(\xi^i_n) - \left( \frac{1}{N} \sum_{i=1}^N \tilde{\tau}_n \right) \left( \frac{1}{N} \sum_{i=1}^N f_n(\xi^i_n) \right). \tag{22}
\]

In cases where \( r_k, 0 \leq k \leq n - 1 \), is unknown and replaced by an unbiased estimate, the associated pseudo marginal particle-based approximation of the tangent filter is given by:

\[
\tilde{\eta}_{n;\theta}^{N}[f_n] = \frac{1}{N} \sum_{i=1}^N \tilde{\tau}_n f_n(\xi^i_n) - \left( \frac{1}{N} \sum_{i=1}^N \tilde{\tau}_n \right) \left( \frac{1}{N} \sum_{i=1}^N f_n(\xi^i_n) \right). \tag{23}
\]

Given a set of observations \( Y_{1:n} \), maximum likelihood estimation amounts at obtaining a parameter \( \hat{\theta}_n \in \Theta \) such that \( \hat{\theta}_n = \arg \max_{\theta \in \Theta} \ell_{\theta,n}(Y_{1:n}) \), where \( \ell_{\theta,n}(Y_{1:n}) = \log L_{\theta,n}(Y_{1:n}) \) is the logarithm of the likelihood given in \( 2 \). There are many different approaches to compute an estimator of \( \hat{\theta}_n \), see for instance [4, Chapter 10]. Following [12], under strong mixing assumptions, for all \( \theta \in \Theta \), the extended process \( \{ (X_n, Y_n, \pi_n, \eta_n) \}_{n \geq 0} \) is an ergodic Markov chain and for all \( \theta \in \Theta \), the normalized score \( \nabla_\theta \ell_{\theta}(Y_{1:n})/n \) of the observations may be shown to converge where:

\[
\frac{1}{n} \nabla_\theta \ell_{\theta}(Y_{1:n}) = \frac{1}{n} \sum_{k=1}^n \nabla_\theta \ell_{\theta}(Y_k \mid Y_{1:k-1}) = \frac{1}{n} \sum_{k=0}^n \frac{\pi_{k;\theta}[\nabla_\theta g_{k;\theta}] + \eta_{k;\theta}[g_{k;\theta}]}{\pi_{k;\theta}[g_{k;\theta}]}.
\]

Assuming that the observations \( Y_{1:n} \) are generated by a model driven by a true parameter \( \theta_* \) for all \( \theta \in \Theta \) this normalized score converges almost surely to a limiting quantity \( \lambda(\theta, \theta_*) \) such that, under identifiability constraints, \( \lambda(\theta_*, \theta_*) = 0 \). A gradient ascent algorithm cannot be designed as the limiting function \( \theta \mapsto \lambda(\theta, \theta_*) \) is not available explicitly. Solving the equation \( \lambda(\theta_*, \theta_*) = 0 \) may be cast into the framework of stochastic approximation to produce parameter estimates using the Robbins-Monro algorithm

\[
\theta_{n+1} = \theta_n + \gamma_{n+1} \zeta_{n+1}, \quad n \geq 0,
\]

where \( \zeta_{n+1} \) is a noisy observation of \( \lambda(\theta_n, \theta_*) \). Obtaining such an observation is not possible in practice and following [26] this noisy observation is approximated by

\[
\zeta_{n+1} := \frac{c_{n+1}^1 + c_{n+1}^2}{c_{n+1}^3}, \tag{25}
\]

where

\[
c_{n+1}^1 := \pi_{n+1;\theta_n} \left[ (\nabla_\theta g_{n+1;\theta})_{\theta=\theta_n} \right] ,
\]

\[
c_{n+1}^2 := \eta_{n+1;\theta_n} \left[ g_{n+1;\theta_n} \right] \quad \text{and} \quad c_{n+1}^3 := \pi_{n+1;\theta_n} \left[ g_{n+1;\theta_n} \right].
\]

In [26], the measures \( \pi_{n+1;\theta_n} \) and \( \eta_{n+1;\theta_n} \) depend on all the past parameter values. In the case of a finite state space \( \mathcal{X} \) the algorithm was studied in [24], which also provides assumptions under which the sequence \( \{ \theta_n \}_{n \geq 0} \) converges towards the parameter \( \theta_* \) (see also [33] for refinements). In more general cases, these measures may be estimated online using the pseudo marginal smoother presented in this paper.
6 Application to partially observed SDE

Let \((X_t)_{t \geq 0}\) be defined as a weak solution to the following Stochastic Differential Equation (SDE) in \(\mathbb{R}^d\):

\[
X_0 = x_0 \quad \text{and} \quad dX_t = \alpha_\theta(X_t)dt + dW_t,
\]

where \((W_t)_{t \geq 0}\) is a standard Brownian motion, \(\alpha_\theta : \mathbb{R} \to \mathbb{R}\) is the drift function. The inference procedure presented in this paper is applied in the case where the solution to (27) is supposed to be partially observed at times \(t_0 = 0, \ldots, t_n\), for a given \(n \geq 1\), through an observation process \((Y_k)_{0 \leq k \leq n}\) taking values in \(\mathbb{R}^m\). For all \(0 \leq k \leq n\), the distribution of \(Y_k\) given \((X_t)_{t \geq 0}\) depends on \(X_k = X_{t_k}\) only and has density \(q_{k,\theta}\) with respect to \(\nu\). The distribution of \(X_0\) has density \(\chi\) with respect to \(\mu\) and for all \(0 \leq k \leq n-1\), the conditional distribution of \(X_{k+1}\) given \((X_t)_{0 \leq t \leq k}\) has density \(q_{k+1,\theta}(X_{k},\cdot)\) with respect to \(\mu\). This unknown density can be expressed as an expectation of a Brownian Bridge functional [7].

Let \(\omega = (\omega_s)_{0 \leq s \leq t}\) be the realization of a Brownian Bridge starting at \(x\) at time \(0\) and ending in \(y\) at time \(\Delta\). The distribution of \(\omega\) is denoted by \(\mathbb{W}_\Delta^x,y\). Moreover, suppose that for all \(\theta \in \Theta, \alpha_\theta\) is of a gradient form \(\alpha_\theta = \nabla_\theta A_\theta\) where \(A_\theta : \mathbb{R} \to \mathbb{R}\) is a twice continuously differentiable function. Denoting, \(\psi_\theta : \ x \mapsto \psi_\theta(x) = (\|\alpha_\theta(x)\|^2 + \Delta A_\theta(x))/2\), by Girsanov theorem, for all \(x, y \in \mathbb{R}^d \times \mathbb{R}^d\)

\[
q_{k+1,\theta}(x,y) = \phi_\Delta(x-y)\exp(A_\theta(y) - A_\theta(x)) \mathbb{E}_{\mathbb{W}_\Delta^x,y} \left[ \exp \left( -\int_0^{\Delta_k} \psi_\theta(\omega_s)ds \right) \right],
\]

where \(\Delta_k = t_{k+1} - t_k\), for all \(a > 0\), \(\phi_\alpha\) is the probability density function of a centered Gaussian random variable with variance \(a\).

The transition density then cannot be computed as it involves an integration over the whole path between \(x\) and \(y\). To perform the algorithm proposed in this paper, we therefore have to design a positive unbiased estimator of \(q_{k+1,\theta}(x,y)\). Moreover, maximum likelihood estimation of \(\theta\) requires an unbiased estimator of \(\nabla_\theta \log q_{k+1,\theta}(x,y)\). Such two estimators can be obtained using the General Poisson Estimator (GPE, [18]).

**Unbiased GPE estimator for** \(q_{k+1,\theta}(x,y;\zeta)\). Assume that there exist random variables \(m_\theta\) and \(\overline{m}_\theta\) such that for all \(0 \leq s \leq \Delta_k\), \(m_\theta \leq \psi_\theta(\omega_s) \leq \overline{m}_\theta\). Let \(\kappa\) be a random variable taking values in \(\mathbb{N}\) with distribution \(\mu\), \(\omega = (\omega_s)_{0 \leq s \leq \Delta_k}\) be the realization of a Brownian Bridge, and \((U_j)_{1 \leq j \leq \kappa}\) be independent uniform random variables on \((0,\Delta_k)\) and \(\zeta = (\kappa, \omega, U_1, \ldots, U_\kappa)\). As shown in [18], equation (28) leads to a positive unbiased estimator given by

\[
\hat{q}_{k+1,\theta}(x,y;\zeta) = \phi_\Delta(x-y)\exp(A_\theta(y) - A_\theta(x) - m_\theta \Delta_k) \prod_{j=1}^\kappa \frac{m_\theta - \psi_\theta(U_j)}{\overline{m}_\theta - m_\theta}.
\]

**Unbiased GPE estimator of** \(\nabla_\theta \log q_{k+1,\theta}(x,y)\). Let’s denote \(\phi_\theta : \ x \mapsto \psi_\theta(x) - m_\theta\). By (28),

\[
\nabla_\theta \log q_{k+1,\theta}(x,y) = \nabla_\theta A_\theta(y) - \nabla_\theta A_\theta(x) - \nabla_\theta m_\theta \Delta_k
\]

\[
- \frac{\mathbb{E}_{\mathbb{W}_\Delta^x,y} \left[ \int_0^{\Delta_k} \nabla_\theta \phi_\theta(\omega_s)ds \right] \exp \left( -\int_0^{\Delta_k} \phi_\theta(\omega_s)ds \right)}{\mathbb{E}_{\mathbb{W}_\Delta^x,y} \left[ \exp \left( -\int_0^{\Delta_k} \phi_\theta(\omega_s)ds \right) \right]}.
\]
On the other hand, the diffusion bridge $S_{θ,x}^{Δ_k,y}$ associated with the SDE (27) is absolutely continuous with respect to $W_x^{Δ_k,y}$ with Radon-Nikodym derivative given by
\[
\frac{dS_{θ,x}^{Δ_k,y}}{dW_x^{Δ_k,y}}(ω) = [q_{k+1;θ}(x,y)]^{-1} φ_{Δ_k}(x-y)\exp \left( A_θ(y) - A_θ(x) - m_θ Δ_k - \int_0^{Δ_k} φ_θ(ω_s)ds \right),
\]
\[
= E_{W_x^{Δ_k,y}} \left[ \exp \left( - \int_0^{Δ_k} φ_θ(ω_s)ds \right) \right]^{-1} \exp \left( - \int_0^{Δ_k} φ_θ(ω_s)ds \right).
\]
This yields
\[
∇_θ \log q_{k+1;θ}(x,y) = (∇_θ A_θ(y) - ∇_θ A_θ(x) - ∇_θ m_θ Δ_k) - E_{S_{θ,x}^{Δ_k,y}} \left[ ∫_0^{Δ_k} ∇_θ φ_θ(ω_s)ds \right]
\]
and an unbiased estimator of $∇_θ \log q_{k+1;θ}(x,y)$ is given by
\[
l_{k+1;θ}(x,y,s_U^{θ,x,y,Δ_k}) = (∇_θ A_θ(y) - ∇_θ A_θ(x) - ∇_θ m_θ Δ_k) - Δ_k ∇_θ φ_θ(s_U^{θ,x,y,Δ_k}),
\]
where $U$ is uniform on $(0,1)$ and independent of $s^{θ,x,y,Δ_k} \sim S_{θ,x}^{Δ_k,y}$. In the context of GPE, $s^{θ,x,y,Δ_k}$ can be simulated exactly using exact algorithms for diffusion processes proposed in [1].

**Experiments.** Online recursive maximum likelihood using pseudo marginal SMC is illustrated when (27) has the specific form:
\[
X_0 = x_0 \quad \text{and} \quad dX_t = \sin(X_t - θ)dt + dW_t,
\]
where $θ$ is an unknown parameter ranging between 0 and $2π$. For this numerical experiments, we suppose that a realization of (29) is only observed at times $t_k = k$ for $0 ≤ k ≤ n$ with $n = 5000$ through a noisy observation process $(Y_k)_{0 \leq k \leq n}$ such for all $0 ≤ k ≤ n$,
\[
Y_k = X_{t_k} + ε_k,
\]
where $(ε_k)_{0 ≤ k ≤ n}$ are i.i.d. standard Gaussian random variables, independent of $(W_t)_{t ≥ 0}$. In this case $α_θ : x → \sin(x - θ)$ and
\[
\inf_{x ∈ R} (α_θ^2(x) + ΔA_θ(x))/2 ≥ -1/2
\]
and for all $x ∈ R$,
\[
0 ≤ φ_θ(x) = (α_θ^2(x) + ΔA_θ(x))/2 + 1/2 ≤ 9/8
\]
and a GPE estimator of both the transition density and the gradient of its logarithm associated with the SINE model is straightforward to compute.

A simulated data set is displayed in Figure 2, where $θ_∗ = π/4$. The solution to (29) is sampled at times $(t_k)_{0 ≤ k ≤ n}$ using the Exact algorithm of [1]. For all $0 ≤ k ≤ n - 1$, $q_{k,θ}$ and the GPE unbiased estimator of $∇_θ q_{k,θ}(x,y)$ are estimated using $M = 30$ independent Monte Carlo replications of the general Poisson estimator. The estimations of $θ_*$ are given for 50 independent runs started at random locations $θ_0$ with $N = 100$ particles and $N̄ = 2$ backward samples. Following [21], the
Pseudo marginal SMC

Proposal distribution of the particle filter is obtained using an approximation of the fully adapted particle filter where \( q_{k, \theta} \) is replaced by its Euler scheme approximation.

**Sensitivity to the starting point** \( \hat{\theta}_0 \). The inference procedure was performed on the same data set from 50 different starting points uniformly chosen in \((0, 2\pi)\). The gradient step size \( \gamma_k \) of equation (24) was chosen constant (and equal to 0.5) for the first 300 time steps, and then decreasing with a rate proportional to \( k^{-0.6} \). Results are given Figure 3. There is no sensitivity to the starting point of the algorithm, and after a couple of hundred observations, the estimates all concentrate around the true value. As the gradient step size decreases, the estimates stay around the true value following autocorrelated patterns that are common to all trajectories.

**Asymptotic normality.** The inference procedure was performed on 50 different data sets simulated with the same \( \theta^\ast \). The 50 estimates were obtained starting from the same starting point (fixed to \( \theta^\ast \), as Figure 3 shows no sensitivity to the starting point). Figure 4 shows the results for the raw and the averaged estimates. The averaged estimates \( \bar{\theta}_k \) consist in averaging the values produced by the estimation procedure after a burning phase of \( n_0 \) time steps (here \( n_0 = 300 \) time steps). This procedure allows to obtain an estimator whose convergence rate does not depend on the step sizes chosen by the user, see [30, 23]. For all \( 0 \leq k \leq n_0 \), \( \bar{\theta}_k = \hat{\theta}_k \) and for all \( k > n_0 \),

\[
\bar{\theta}_k = \frac{1}{k - n_0} \sum_{j=n_0+1}^{k} \hat{\theta}_j.
\]

As expected, the estimated distribution of the final estimates tends to be Gaussian, centered around the true value.

**Step size influence.** To illustrate the influence of the gradient step sizes, different settings are considered. In each scenario, the sequence \((\gamma_k)_{k \geq 0}\) is given by

\[
\gamma_k = \gamma_0 1\{k \leq n_0\} + \frac{\gamma_0}{(k - n_0)\kappa} 1\{k > n_0\},
\]

where \( \gamma_0 = 0.5 \). In this experiment \( \kappa \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1\} \). The results are shown in Figure 5. As expected, the raw estimator shows different rates of convergence depending on \( \kappa \), whereas the averaged estimator has the same behavior in all cases.

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Figure 2: Data set simulated according to the SINE process, observed with noise at discrete time steps.

Figure 3: (Left) online estimation of $\theta$ for the data set presented in Figure 2. The algorithm is performed from 50 starting points. (Right) The gradient step sizes (defined in equation 24).
Figure 4: (Left) online estimation of $\theta$ for 50 different simulated data sets as presented in Figure 2. The algorithm is performed from 1 starting point with the gradient step size shown in Figure 3. (Center) Averaged estimator, where $\hat{\theta}$ is averaged after a burning phase of 300 time steps. (Right) Empirical distribution of $\hat{\theta}$. The red line is the value of $\theta^*$. 

Figure 5: (Left) online estimation of $\theta$ for the data set presented in Figure 2, with different decreasing rates values $\kappa$. (Right) Averaged estimator, where $\hat{\theta}$ is averaged after a burning phase of 300 time steps.
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A  Additional technical results

The proof of Lemma A.1 is given in [11].

Lemma A.1. Assume that $a_N$, $b_N$, and $b$ are random variables defined on the same probability space such that there exist positive constants $\beta$, $B$, $C$, and $M$ satisfying

(i) $|a_N/b_N| \leq M$, $\mathbb{P}$-a.s. and $b \geq \beta$, $\mathbb{P}$-a.s.,

(ii) For all $\varepsilon > 0$ and all $N \geq 1$, $\mathbb{P}( |b_N - b| > \varepsilon ) \leq Be^{-CN\varepsilon^2}$,

(iii) For all $\varepsilon > 0$ and all $N \geq 1$, $\mathbb{P}( |a_N| > \varepsilon ) \leq Be^{-CN(\varepsilon/M)^2}$.

Then,

$$\mathbb{P} \left( \left\| \frac{a_N}{b_N} \right\| > \varepsilon \right) \leq B \exp \left( -CN \left( \frac{\varepsilon \beta}{2M} \right)^2 \right).$$

The proof of Theorem A.1 is given in [13, Theorem A.3].

Theorem A.1. Let $N$ be a positive integer, $(U_{N,i})_{1 \leq i \leq N}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_{N,i})_{0 \leq i \leq N}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that for all $1 \leq i \leq N$ the random variable $U_{N,i}$ is such that $E[U_{N,i}^2 | \mathcal{F}_{N,i-1}] < \infty$. Assume also that the two following conditions hold.

(i) There exists $\sigma^2 > 0$ such that

$$\sum_{i=1}^{N} \left( E[U_{N,i}^2 | \mathcal{F}_{N,i-1}] - E[U_{N,i} | \mathcal{F}_{N,i-1}]^2 \right) \xrightarrow{N \to \infty} \sigma^2 .$$

(ii) For all $\varepsilon > 0$,

$$\sum_{i=1}^{N} E[U_{N,i}^2 1_{|U_{N,i}| \geq \varepsilon} | \mathcal{F}_{N,i-1}] \xrightarrow{N \to \infty} 0 .$$

Then, for all $u > 0$,

$$E \left[ \exp \left( iu \sum_{i=1}^{N} \left\{ U_{N,i} - E[U_{N,i} | \mathcal{F}_{N,i-1}] \right\} \right) \right] \xrightarrow{N \to \infty} e^{-u^2 \sigma^2/2} .$$

The proof of Lemma A.2 follows the same lines as [26, Lemma 14].

Lemma A.2. Assume that $H3$ holds. Let $K$ be a transition kernel on $(X, \mathcal{B}(X))$ with transition density $k \in \mathcal{F}(X \times X)$ with respect to the reference measure $\mu$. Assume that $(\varphi_N)_{N \geq 1}$ is a sequence of functions in $\mathcal{F}(X)$ such that

i) there exists $\varphi \in \mathcal{F}(X)$ such that for all $x \in X$, $\varphi_N(x) \xrightarrow{N \to \infty} \varphi(x)$;

ii) there exists $0 < c_\infty < \infty$ such that for all $N \geq 1$, $\|\varphi_N\| \leq c_\infty$.

Then, for all $0 \leq k \leq n$,

$$\phi_k^N[K\varphi_N] \xrightarrow{N \to \infty} \phi_k[\varphi] \quad \text{and} \quad \hat{\phi}_k^N[K\varphi_N] \xrightarrow{N \to \infty} \phi_k[\varphi] .$$
B Convergence results for PaRIS algorithms

For all 0 ≤ k ≤ n, define the following σ-fields:

\[ F_k^N := \sigma \{ (\xi_u^\ell, \omega_u^\ell, \tau_u^\ell) ; 1 \leq \ell \leq N, 0 \leq u \leq k \} \quad \text{and} \quad G_k^N := \sigma \{ (\xi_u^\ell, \omega_u^\ell) ; 1 \leq \ell \leq N \} . \]

**Lemma B.1.** For all 0 ≤ k ≤ n − 1, (f_k+1, h_k+1) ∈ F(X)^2 and \( \bar{N} \geq 0 \), the random variables \( \omega_{k+1}^i (\tau_{k+1}^i f_k+1 (\xi_{k+1}^i) + \bar{h}_{k+1} (\xi_{k+1}^i)) \) are i.i.d. conditionally on \( F_k^N \) with

\[
E \left[ \omega_{k+1}^i (\tau_{k+1}^i f_k+1 (\xi_{k+1}^i) + \bar{h}_{k+1} (\xi_{k+1}^i)) \mid F_k^N \right] = \left( \phi_k^N[\theta_k] \right)^{-1} \sum_{i=1}^{N} \omega_{k+1}^i \left\{ \tau_k^i L_k f_k+1 (\xi_k^i) + L_k (\bar{h}_k f_k+1 + \bar{f}_k+1) (\xi_k^i) \right\} .
\]

**Proof.** The proof follows the same lines as [26, Lemma 12]. \( \square \)

B.1 Proof of Lemma 4.2

**Proof.** The proof proceeds by induction. The case \( k = 0 \) is a direct consequence of the fact that \( \mathbf{T}_0 h_0 = 0 \) and \( \tau_{0}^i = 0 \) for all \( 1 \leq i \leq N \). Assume that the result holds for some 0 ≤ k ≤ n − 1 and write

\[ \sum_{i=1}^{N} \omega_{k+1}^i (\tau_{k+1}^i f_k+1 (\xi_{k+1}^i)) = \frac{a_N}{b_N}, \]

where

\[ a_N = \frac{1}{N} \sum_{i=1}^{N} \omega_{k+1}^i (\tau_{k+1}^i f_k+1 (\xi_{k+1}^i)) \quad \text{and} \quad b_N = \frac{1}{N} \sum_{i=1}^{N} \omega_{k+1}^i . \]

Then, using that \( (\omega_{k+1}^i)_{1 \leq i \leq N} \) are i.i.d. conditionally on \( F_k^N \) and

\[ E \left[ \omega_{k+1}^i \mid F_k^N \right] = \frac{\phi_k^N[\theta_k] L_k 1}{\phi_k^N[\theta_k]}, \]

by Hoeffding inequality, since for all 1 ≤ i ≤ N, 0 ≤ \( \omega_{k+1}^i \leq \Vert \omega_{k+1} \Vert_\infty , \)

\[ P \left( \left\| b_N - \frac{\phi_k^N[\theta_k] L_k 1}{\phi_k^N[\theta_k]} \right\| > \varepsilon \right) = E \left[ P \left( \left\| b_N - \frac{\phi_k^N[\theta_k] L_k 1}{\phi_k^N[\theta_k]} \right\| > \varepsilon \mid F_k^N \right) \right] \leq 2 e^{-2N \varepsilon^2 / \Vert \omega_{k+1} \Vert_\infty^2} . \]

Therefore, by Lemma 4.1

\[ b_N \stackrel{p-a.s.}{\to} \frac{\phi_k[\theta_k]}{\phi_k[\theta_k]} \]

Since \( \phi_k[\theta_k] > 0 \) it remains to establish the convergence in probability of \( (a_N)_{N \geq 1} \). On the other hand, by Hoeffding inequality, using that for all 1 ≤ i ≤ N, \( \omega_{k+1}^i (\tau_{k+1}^i f_k+1 (\xi_{k+1}^i)) \leq \Vert \omega_{k+1} \Vert_\infty \Vert h_{k+1} \Vert_\infty^2 \Vert f_{k+1} \Vert_\infty \),

\[ P \left( \left\| a_N - E[a_N \mid F_k^N] \right\| > \varepsilon \right) \leq 2 e^{-N \varepsilon^2 / (2 \Vert \omega_{k+1} \Vert_\infty \Vert h_{k+1} \Vert_\infty^2 \Vert f_{k+1} \Vert_\infty)} , \]

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and it is enough to obtain the limit of $\mathbb{E}[a_N|F_k^N]$ as $N$ grows to infinity. Then, write

$$\mathbb{E}[a_N|F_k^N] = \mathbb{E}\left[\omega_k f_{k+1}(\xi^1_k, \xi^2_k, x)|F_k^N\right] = \tilde{a}_N + \tilde{a}_N^2,$$

where

$$\tilde{a}_N = \bar{N}^{-1} \mathbb{E}\left[\omega_k f_{k+1}(\xi^1_k, \xi^2_k, x)|F_k^N\right],$$

$$\tilde{a}_N^2 = (\bar{N} - 1)\bar{N}^{-1} \mathbb{E}\left[\omega_k f_{k+1}(\xi^1_k, \xi^2_k, x)|F_k^N\right].$$

The first term is given by

$$\tilde{a}_N^1 = \bar{N}^{-1} \sum_{j=1}^N \frac{1}{\bar{N}} \frac{\omega_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N \omega_k (\xi^m_k, x)} \frac{r_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N r_k (\xi^m_k, x)} f_{k+1}(x) \times \left( \frac{\tau^1_k + \tilde{h}_k (\xi^1_j, x)}{\sum_{m=1}^N \omega_k r_k (\xi^m_k, x)} \right)^2 \mu(dx),$$

$$= \bar{N}^{-1} (\phi_k[\phi_k]^{-1}) \int f_{k+1}(x) \frac{1}{\Omega_k} \sum_{j=1}^N \frac{\omega_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N \omega_k (\xi^m_k, x)} \left( \frac{\tau^1_k + \tilde{h}_k (\xi^1_j, x)}{\sum_{m=1}^N \omega_k r_k (\xi^m_k, x)} \right)^2 \mu(dx).$$

By the induction hypothesis and Lemma 4.1,

$$\tilde{a}_N^1 \xrightarrow{p} (\bar{N} \phi_k[\phi_k]^{-1})^{-1} \left\{ \eta_k[L_k f_{k+1}] + \phi_k[T_k h_k L_k f_{k+1}] + \phi_k[L_k (f_{k+1}\tilde{h}_k)] + 2\phi_k[T_k h_k L_k (f_{k+1}\tilde{h}_k)] \right\},$$

which yields

$$\tilde{a}_N \xrightarrow{p} (\bar{N} \phi_k[\phi_k]^{-1})^{-1} \left\{ \eta_k[L_k f_{k+1}] + \phi_k[L_k (T_k h_k + \tilde{h}_k)^2 f_{k+1}] \right\}.$$

The second term is given by

$$\tilde{a}_N^2 = (\bar{N} - 1)\bar{N}^{-1} \sum_{j=1}^N \frac{1}{\bar{N}} \frac{\omega_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N \omega_k (\xi^m_k, x)} \frac{r_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N r_k (\xi^m_k, x)} f_{k+1}(x) \times \left( \frac{\tau^1_k + \tilde{h}_k (\xi^1_j, x)}{\sum_{m=1}^N \omega_k r_k (\xi^m_k, x)} \right)^2 \mu(dx),$$

$$= (\bar{N} - 1)\bar{N}^{-1} (\phi_k[\phi_k]^{-1})^{-1} \phi_k^2[L_k \phi_N],$$

with, for all $x \in X$,

$$\phi_N(x) = f_{k+1}(x) \left( \frac{1}{\sum_{j=1}^N \frac{\omega_k (\xi^1_j, \xi^2_j, x)}{\sum_{m=1}^N \omega_k (\xi^m_k, x)} \left( \frac{\tau^1_k + \tilde{h}_k (\xi^1_j, x)}{\sum_{m=1}^N \omega_k r_k (\xi^m_k, x)} \right)^2 \mu(dx) \right)^2. $$
For all $x \in X$, by Lemma A.1
\[
\varphi_N(x) \xrightarrow{P-a.s.} f_{k+1}(x) \left( \frac{\phi_k[T_kh_k r_k(\cdot, x) + r_k(\cdot, x)\bar{h}_k(\cdot, x)]}{\phi_k[r_k(\cdot, x)]} \right)^2.
\]
In addition, for all $x \in X$, by (8),
\[
\frac{\phi_k[T_k h_k r_k(\cdot, x) + r_k(\cdot, x)\bar{h}_k(\cdot, x)]}{\phi_k[r_k(\cdot, x)]} = \tilde{Q}_{\phi_k} \left( T_k h_k + \bar{h}_k \right)(x) = T_{k+1} h_{k+1}(x)
\]
so that
\[
\varphi_N(x) \xrightarrow{P-a.s.} f_{k+1}(x) T_{k+1}^2 h_{k+1}(x).
\]
Therefore, as $\|\varphi_N\|_\infty \leq \|f_{k+1}\|_\infty \|h_{k+1}\|_\infty^2$, by the generalized Lebesgue dominated convergence theorem, see Lemma A.2
\[
\tilde{a}_N^2 \xrightarrow{P} \tilde{(N - 1)} N^{-1} (\phi_k[\varphi_N])^{-1} \phi_k[L_k \{ f_{k+1} T_{k+1}^2 h_{k+1} \}] .
\]
Using that
\[
\frac{\phi_k[L_k f_{k+1} T_{k+1}^2 h_{k+1}]}{\phi_k[L_k]} = \phi_{k+1} [ f_{k+1} T_{k+1}^2 h_{k+1} ],
\]
yields
\[
\frac{\alpha N}{\tilde{b}_N} \xrightarrow{P} \phi_{k+1} [ f_{k+1} T_{k+1}^2 h_{k+1} ] + \frac{\eta_k[L_k f_{k+1}]}{N \phi_k[L_k]} + \frac{\phi_k[L_k \{ (T_k h_k + \bar{h}_k)^2 f_{k+1} \}] - \phi_k[L_k f_{k+1} T_{k+1}^2 h_{k+1}]}{N \phi_k[L_k]} .
\]
The proof is concluded upon noting that
\[
\phi_k[L_k \{ (T_k h_k + \bar{h}_k)^2 f_{k+1} \}] - \phi_k[L_k f_{k+1} T_{k+1}^2 h_{k+1}]
\]
\[
= \phi_k[L_k \{ \tilde{Q}_{\phi_k} (T_k h_k + \bar{h}_k - T_{k+1} h_{k+1})^2 f_{k+1} \}].
\]

\[\Box\]

B.2 Proof of Proposition 4.1

Proof. The result is proved by induction on $k$. It holds for $k = 0$ as for all $1 \leq i \leq N$, $\tau_0^i = 0$.
Assume now that the result holds for some $0 \leq k \leq n-1$ and that $\phi_{k+1} [ T_{k+1} h_{k+1} f_{k+1} + \tilde{f}_{k+1} ] = 0$.
Write
\[
\sqrt{N} \sum_{i=1}^N \frac{\omega_{k+1}^i}{\Omega_{k+1}} \{ \tau_{k+1}^i f_{k+1} (\xi_{k+1}^i) + \tilde{f}_{k+1} (\xi_{k+1}^i) \} = \Omega_{k+1}^{-1} \Delta_{k+1}^N ,
\]
where $\Delta_{k+1}^N = \sqrt{N} \sum_{i=1}^N \omega_{k+1}^i \{ \tau_{k+1}^i f_{k+1} (\xi_{k+1}^i) + \tilde{f}_{k+1} (\xi_{k+1}^i) \}$ is decomposed as follows
\[
\Delta_{k+1}^N = \Delta_{k+1,1}^N + \Delta_{k+1,2}^N .
\]

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where
\[
\Delta_{k+1,1}^N = \sqrt{N} \sum_{i=1}^{N} \mathbb{E} \left[ \omega_{k+1}(\tau_{k+1} f_{k+1}(\xi_{k+1}^i) + \tilde{f}_{k+1}(\xi_{k+1}^i)) \right| F_k^N] ,
\]
\[
\Delta_{k+1,2}^N = \sqrt{N} \sum_{i=1}^{N} \left\{ \omega_{k+1} \left( \tau_{k+1} f_{k+1}(\xi_{k+1}^i) + \tilde{f}_{k+1}(\xi_{k+1}^i) \right) - \mathbb{E} \left[ \omega_{k+1}(\tau_{k+1} f_{k+1}(\xi_{k+1}^i) + \tilde{f}_{k+1}(\xi_{k+1}^i)) \right| F_k^N] \right\} .
\]
By Lemma \[B.1\]
\[
\Omega_{k+1}^{-1} \Delta_{k+1,1}^N = \frac{N}{\Omega_{k+1}} (\phi_k^N[\theta_k])^{-1} \sqrt{N} \sum_{j=1}^{N} \omega_{k} \left\{ \tau_k L_k f_{k+1}(\xi_{k+1}^j) + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi_{k+1}^j) \right\}
\]
As \[\phi_{k+1}(T_{k+1} h_{k+1} f_{k+1} + \tilde{f}_{k+1}) = 0,\]
\[
\phi_k[T_{k+1} h_{k+1} L_k f_{k+1} + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})] = 0 .
\]
Therefore, using the induction hypothesis, Slutsky’s lemma and
\[
\frac{N}{\Omega_{k+1}} (\phi_k^N[\theta_k])^{-1} \xrightarrow{p} \frac{p}{N \rightarrow \infty} (\phi_k[L_k])^{-1}
\]
yields
\[
\Omega_{k+1}^{-1} \Delta_{k+1,1}^N \xrightarrow{D} \frac{\sigma_k(L_k f_{k+1}; L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1}))}{\phi_k[L_k]} Z ,
\]
where \[Z\] is a standard Gaussian random variable. By Lemma \[B.1\]
\[
\Omega_{k+1}^{-1} \Delta_{k+1,2}^N = \frac{N}{\Omega_{k+1}} \sum_{i=1}^{N} v_N^i ,
\]
where for all \[1 \leq i,j \leq N\] and all \[x \in \mathbf{X},\]
\[
v_N^i = \frac{1}{\sqrt{N} N} \sum_{j=1}^{N} \bar{v}_N(t_{k+1}^{i,j}, r_{k}^{x,j}(x), \xi_{k+1}^i) ,
\]
\[
\bar{v}_N(i,j, x) = \frac{r_k(\xi_{k+1}^i, x)}{\theta_k(\xi_{k+1}^i)} \left\{ \left( \tau_{k}^j + \tilde{h}_k(\xi_{k+1}^i, x) \right) f_{k+1}(x) + \tilde{f}_{k+1}(x) \right\}
\]
\[
- \left( \phi_k^N[\theta_k] \right)^{-1} \sum_{j=1}^{N} \frac{\omega_{k}^{j}}{\Omega_{k}} \left\{ \tau_{k}^j L_k f_{k+1}(\xi_{k+1}^j) + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi_{k+1}^j) \right\} .
\]
First, by Lemma \[4.1\]
\[
\frac{N}{\Omega_{k+1}} \xrightarrow{p} \frac{\phi_k[\theta_k]}{N \rightarrow \infty} \phi_k[L_k] ,
\]
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The proof is then concluded by applying Slutsky’s Lemma and Theorem A.1 to the sequence \((v^i_N)_{1 \leq i \leq N}\). By construction \(\mathbb{E}[v^i_N|F^N_k] = 0\) so that the proof of \((18)\) is based on

\[
\sum_{i=1}^{N} \mathbb{E}[(v^i_N)^2|F^N_k] = \bar{N}^{-1} \mathbb{E}[\mathbb{E}[(\bar{v}_N(I_{k+1}^i, j_{k+1}^{(1,1)}), \xi_{k+1}^i)|F^N_k \vee G^N_{k+1}] | F^N_k]
\]

\[
+ (\bar{N} - 1) \mathbb{E}[\mathbb{E}[\bar{v}_N(I_{k+1}^{(1,1)}, j_{k+1}^{(1,1)}, \xi_{k+1}^i)|F^N_k \vee G^N_{k+1}]^2|F^N_k].
\]

The first term of \((30)\) is given by

\[
\mathbb{E} \left[ \mathbb{E} \left[ (\bar{v}_N(I_{k+1}^i, J_{k+1}^{(1,1)}, \xi_{k+1}^i)) | F^N_k \vee G^N_{k+1} \right] | F^N_k \right]
\]

\[
= \mathbb{E} \left[ \sum_{i=1}^{N} \frac{\omega_k^i r_k(\xi_k^i, \xi_{k+1}^i)}{\sum_{m=1}^{N} \omega_k^m r_k(\xi_k^m, \xi_{k+1}^m)} \bar{v}_N(I_{k+1}, \ell, \xi_{k+1}^i) | F^N_k \right],
\]

\[
= \sum_{j=1}^{N} \int \frac{\omega_k^j \bar{v}_N(\xi_k^j, x) r_k(\xi_k^j, x)}{\sum_{m=1}^{N} \omega_k^m \bar{v}_N(\xi_k^m, x)} \bar{v}_N(j, \ell, x)^2 \mu(dx),
\]

\[
= (\phi_N^k(\bar{v}_k))^{-1} \sum_{j=1}^{N} \int \frac{\omega_k^j \bar{v}_N(\xi_k^j, x) r_k(\xi_k^j, x)}{\sum_{m=1}^{N} \omega_k^m \bar{v}_N(\xi_k^m, x)} \bar{v}_N(j, \ell, x)^2 \mu(dx),
\]

\[
= (\phi_N^k(\bar{v}_k))^{-1} \sum_{\ell=1}^{N} \frac{\omega_k^\ell r_k(\xi_k^\ell, x)}{\sum_{m=1}^{N} \omega_k^m r_k(\xi_k^m, x)} \left( \tau_k f_{k+1}(\xi_k^\ell) + L_k(\bar{h}_k f_{k+1} + \bar{f}_{k+1})(\xi_k^\ell) \right)^2,
\]

where

\[
A_N(x) = \sum_{j=1}^{N} \frac{\omega_k^j \bar{v}_N(\xi_k^j, x) r_k(\xi_k^j, x)}{\sum_{m=1}^{N} \omega_k^m r_k(\xi_k^m, x)} \bar{v}_N(\xi_k^j, x),
\]

\[
B_N(x) = \sum_{\ell=1}^{N} \frac{\omega_k^\ell r_k(\xi_k^\ell, x)}{\sum_{m=1}^{N} \omega_k^m r_k(\xi_k^m, x)} \left( \tau_k f_{k+1}(\xi_k^\ell) + L_k(\bar{h}_k f_{k+1} + \bar{f}_{k+1})(\xi_k^\ell) \right)^2.
\]

By Lemma 4.1

\[
(\phi_N^k(\bar{v}_k))^{-2} \left( \sum_{\ell=1}^{N} \frac{\omega_k^\ell r_k(\xi_k^\ell, x)}{\sum_{m=1}^{N} \omega_k^m r_k(\xi_k^m, x)} \left( \tau_k f_{k+1}(\xi_k^\ell) + L_k(\bar{h}_k f_{k+1} + \bar{f}_{k+1})(\xi_k^\ell) \right)^2 \right)
\]

\[
\overset{p}{\to} (\phi_k[\bar{v}_k])^{-2} \left( \phi_k[T h_k L_k f_{k+1} + L_k(\bar{h}_k f_{k+1} + \bar{f}_{k+1})]^2 \right) = 0,
\]

since by assumption and [20] Lemma 11, \(\phi_k[T h_k L_k f_{k+1} + L_k(\bar{h}_k f_{k+1} + \bar{f}_{k+1})] = 0\). Then, write

\[
\int A_N(x) B_N(x) \mu(dx) = \hat{a}_N + \tilde{a}_N^2 + \tilde{a}_N^3,
\]
where

\[ \varphi_N : x \mapsto \left\{ \sum_{j=1}^{N} \frac{\omega^j}{\Omega_k} r_k(\xi^j_k, x) \omega_k(\xi^j_k, x) \right\} \left( \sum_{m=1}^{N} \frac{\omega^m}{\Omega_k} r_k(\xi^m_k, x) \right)^{-1} = \widehat{Q}_{\phi_k} \tilde{\omega}_k(x), \]

\[ \bar{\alpha}_N^1 = \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} (\tau^\ell_k)^2 \int r_k(\xi^\ell_k, x) f_{k+1}^2(x) \varphi_N(x) \mu(dx), \]

\[ \bar{\alpha}_N^2 = \sum_{j=1}^{N} \frac{\omega^j}{\Omega_k} \int r_k(\xi^j_k, x) \omega_k(\xi^j_k, x) \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} r_k(\xi^\ell_k, x) \left( h_k(\xi^\ell_k, x) f_{k+1}(x) + \bar{f}_{k+1}(x) \right)^2 \mu(dx), \]

\[ \bar{\alpha}_N^3 = 2 \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} \int r_k(\xi^\ell_k, x) \varphi_N(x) f_{k+1}(x) \left( h_k(\xi^\ell_k, x) f_{k+1}(x) + \bar{f}_{k+1}(x) \right) \mu(dx). \]

By Lemma 4.1, for all \( x \in X, \left| \varphi_N(x) - \widehat{Q}_{\phi_k} \tilde{\omega}_k(x) \right| \xrightarrow{N \to \infty} 0, \) and note that

\[ \left| \bar{\alpha}_N^1 - \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} (\tau^\ell_k)^2 \int r_k(\xi^\ell_k, x) f_{k+1}^2(x) \widehat{Q}_{\phi_k} \tilde{\omega}_k(x) \mu(dx) \right| \leq \| h_k \|^2_{\infty} \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} \int r_k(\xi^\ell_k, x) f_{k+1}^2(x) \left| \varphi_N(x) - \widehat{Q}_{\phi_k} \tilde{\omega}_k(x) \right| \mu(dx). \]

Since \( \| f_{k+1}^2(\varphi_N - \widehat{Q}_{\phi_k} \tilde{\omega}_k) \|_{\infty} \leq 2 \| \tilde{\omega}_k \|_{\infty} \| f_{k+1}^2 \|_{\infty} < \infty, \) by the generalized Lebesgue dominated convergence theorem, see also Lemma A.2

\[ \left| \bar{\alpha}_N^1 - \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} (\tau^\ell_k)^2 \int r_k(\xi^\ell_k, x) f_{k+1}^2(x) \widehat{Q}_{\phi_k} \tilde{\omega}_k(x) \mu(dx) \right| \xrightarrow{P \ N \to \infty} 0 \]

and by Lemma 4.2

\[ \bar{\alpha}_N^1 \xrightarrow{P \ N \to \infty} \eta_k L_k \{ f_{k+1}^2 \widehat{Q}_{\phi_k} \tilde{\omega}_k \} \].

On the other hand, by Lemma A.2 applied to

\[ \psi_N : x \mapsto \sum_{\ell=1}^{N} \frac{\omega^\ell}{\Omega_k} r_k(\xi^\ell_k, x) \left( h_k(\xi^\ell_k, x) f_{k+1}(x) + \bar{f}_{k+1}(x) \right)^2, \]

which is such that \( \| \psi_N \| \leq 2(\| h_k f_{k+1} \|_{\infty}^2 + \| \bar{f}_{k+1} \|_{\infty}), \)

\[ \bar{\alpha}_N^2 \xrightarrow{P \ N \to \infty} \int \phi_k [r_k(\cdot, x) \tilde{\omega}(\cdot, x)] \phi_k [r_k(\cdot, x) (h_k(\cdot, x) f_{k+1}(x) + \bar{f}_{k+1}(x))^2] \phi_k [r_k(\cdot, x)]^{-1} \mu(dx), \]

which yields

\[ \bar{\alpha}_N^2 \xrightarrow{P \ N \to \infty} \phi_k [L_k \{ (\widehat{Q}_{\phi_k} \tilde{\omega}_k)(h_k f_{k+1} + \bar{f}_{k+1})^2 \}]. \]
Finally,
\[
\begin{align*}
\left| \sum_{\ell=1}^{N} \frac{\omega^f_\ell}{\Omega_k} \int r_k(\xi^f_\ell, x)f_{k+1}(x) & \tilde{Q}_{\phi_k \omega_k}(x) \left( \tilde{h}_k(\xi^f_\ell, x)f_{k+1}(x) + \tilde{f}_{k+1}(x) \right) \mu(dx) \right| \\
& \leq \|h_k\|_\infty \|f_{k+1}\|_\infty \|\tilde{h}_k\|_\infty + \|\tilde{f}_{k+1}\|_\infty \sum_{\ell=1}^{N} \frac{\omega^f_\ell}{\Omega_k} \int r_k(\xi^f_\ell, x)f_{k+1}(x) \left| \varphi_N(x) - \tilde{Q}_{\phi_k \omega_k}(x) \right| \mu(dx),
\end{align*}
\]
so that using again Lemma A.2 and Lemma 4.1
\[
\tilde{a}^3_N \xrightarrow{p_{N \to \infty}} 2\phi_k \left( T_k h_k L_k \left( \{ (\tilde{Q}_{\phi_k \omega_k})f_{k+1} \left( \tilde{h}_k f_{k+1} + \tilde{f}_{k+1} \right) \} \right) \right).
\]
Therefore, the first term of (30) satisfies
\[
\tilde{N}^{-1} \mathbb{E}[\mathbb{E}[(\tilde{v}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi^1_{k+1}))^2 | F_k^N \lor G_{k+1}^N | F_k^N]]
\]
\[
\xrightarrow{p_{N \to \infty}} \frac{\eta_k [L_k (f_{k+1}^\infty \tilde{Q}_{\phi_k \omega_k})]}{N \phi_k[\theta_k]} + \frac{\phi_k[L_k \{ (\tilde{Q}_{\phi_k \omega_k}) f_{k+1} \left( \tilde{h}_k f_{k+1} + \tilde{f}_{k+1} \right) \}]}{N \phi_k[\theta_k]},
\]
which concludes the proof for the first term of (30). The second term of (30) is given by
\[
\mathbb{E}[\mathbb{E}[(\tilde{v}_N(I_{k+1}^1, J_{k+1}^{(1,1)}, \xi^1_{k+1})) | F_k^N \lor G_{k+1}^N | F_k^N]]
\]
\[
= \mathbb{E} \left[ \left( \sum_{\ell=1}^{N} \frac{\omega^f_\ell r_k(\xi^f_\ell, \xi^1_{k+1})}{\sum_{m=1}^{N} \omega^m_k r_k(\xi^m_k, \xi^1_{k+1})} \tilde{v}_N(I_{k+1}^1, \ell, \xi^1_{k+1}) \right)^2 \right] | F_k^N \right],
\]
\[
= \sum_{j=1}^{N} \int \frac{\omega^f_j \varphi_k(\xi^j_k, x)}{\sum_{m=1}^{N} \omega^m_k \varphi_k(\xi^m_k, x)} \left( \sum_{\ell=1}^{N} \frac{\omega^f_\ell r_k(\xi^f_\ell, x)}{\sum_{m=1}^{N} \omega^m_k r_k(\xi^m_k, x)} \tilde{v}_N(j, \ell, x) \right)^2 \mu(dx),
\]
\[
= (\phi_k^N[\theta_k])^{-1} \phi_k^N[\phi_k] \varphi_k^N(\theta_k, \xi^f_\ell, \xi^1_{k+1}),
\]
where, for all \( x \in \mathbb{X} \),
\[
\varphi_k^N(\theta_k, \xi^f_\ell, \xi^1_{k+1}) = \int p_k(x, z) \left( \frac{r_k(x, z)}{\varphi_k(\theta_k, x)} p_k(x, z) f_{k+1}(z) \sum_{\ell=1}^{N} \frac{\omega^f_\ell r_k(\xi^f_\ell, z)}{\sum_{m=1}^{N} \omega^m_k r_k(\xi^m_k, z)} \left( \tau^f_\ell + \tilde{h}_k(\xi^f_\ell, z) \right) \right) \mu(dx) + \frac{r_k(x, z)}{\varphi_k(\theta_k, x)} f_{k+1}(z) - (\phi_k^N[\theta_k])^{-1} \sum_{\ell=1}^{N} \frac{\omega^f_\ell}{\Omega_k} \left( \tau^f_\ell L_k f_{k+1}(\xi^f_\ell) + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi^f_\ell) \right) \mu(dx).
\]
By assumption and Lemma 11, \( \phi_k[T_k h_k L_k f_{k+1}(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})] = 0 \) so that by Lemma A.2
\[
\phi_k^N[\phi_k \varphi_k^N] \xrightarrow{p_{N \to \infty}} \phi_k[\phi_k] \varphi_k^N, \]
where
\[
\varphi_k(\theta_k, \xi^f_\ell, \xi^1_{k+1}) = \int p_k(x, z) \left( \frac{r_k(x, z)}{\varphi_k(\theta_k, x)} p_k(x, z) \right)^2 \left( f_{k+1}(z) \tilde{Q}_{\phi_k}(T_k h_k + \tilde{h}_k)(z) + \tilde{f}_{k+1}(z) \right) \mu(dx) \]
\[
= \int p_k(x, z) \left( \frac{r_k(x, z)}{\varphi_k(\theta_k, x)} p_k(x, z) \right)^2 \left( f_{k+1}(z) T_{k+1} h_{k+1}(z) + \tilde{f}_{k+1}(z) \right) \mu(dx) \].
Therefore,
\[
E[\mathbb{E}[\tilde{v}_N(I_{k+1}, \cdots, I_{k+1}, \xi_{k+1}) | \mathcal{F}_k^N \vee G_{k+1}^N]^2 | \mathcal{F}_k^N] = \frac{p}{N \to \infty} \left( 1 - \frac{1}{N} \right) (\phi_k[\theta_k])^{-1} \phi_k \left[ L_k \{ \tilde{\omega}_k \left( f_{k+1} T_{k+1} h_{k+1} + \bar{f}_{k+1} \right)^2 \} \right].
\]

The proof of (ii) is an immediate consequence of (i) since for all \( 1 \leq i \leq N \),
\[
v_i \leq 2 \| \tilde{\omega}_{k+1} \|_{\infty} \left( \| h_{k+1} \|_{\infty} \| \bar{f}_{k+1} \|_{\infty} + \| \bar{f}_{k+1} \|_{\infty} \right) N^{-1/2}.
\]

Then, defining \( c_k = 2 \| \tilde{\omega}_{k+1} \|_{\infty} (\| h_{k+1} \|_{\infty} \| \bar{f}_{k+1} \|_{\infty} + \| \bar{f}_{k+1} \|_{\infty}) \), for all \( \varepsilon > 0 \),
\[
\sum_{i=1}^{N} E[(v_i^k)^2 I_{v_i \geq \varepsilon} | \mathcal{F}_k^N] \leq c_k^2 \varepsilon \sqrt{K} \frac{p}{N \to \infty} \to 0.
\]

Writing
\[
\bar{f}_{k+1} = \bar{f}_{k+1} + \phi_{k+1} [T_{k+1} h_{k+1} f_{k+1} + \tilde{f}_{k+1}],
\]
yields
\[
\sigma^2_{k+1}(f_{k+1} : \bar{f}_{k+1}) = \frac{1}{\phi_k[L_k]^2} \left[ \frac{1}{N} \phi_k[\theta_k] \eta_k[L_k \{ f_{k+1}^2 \overline{Q}_{\phi_k} \tilde{\omega}_k \}] + \phi_k[\theta_k] \phi_k \left[ L_k \{ \tilde{\omega}_k \left( f_{k+1} T_{k+1} h_{k+1} + \bar{f}_{k+1} \right)^2 \} \right] \right]
\]
and then, by (19),
\[
\sigma^2_{k+1}(f_{k+1} : \bar{f}_{k+1}) = \frac{1}{\phi_k[L_k]^2} \left[ \frac{1}{N} \phi_k[\theta_k] \phi_k \left[ L_k \{ \tilde{\omega}_k \left( f_{k+1} T_{k+1} h_{k+1} + \bar{f}_{k+1} \right)^2 \} \right] \right]
\]
By definition of the kernel \( \tilde{D}_{k+1, k+1} \),
\[
\phi_k \left[ L_k \tilde{\omega}_k \left( f_{k+1} T_{k+1} h_{k+1} + \bar{f}_{k+1} \right)^2 \right] = \phi_k \left[ L_k \tilde{\omega}_k \tilde{D}_{k+1, k+1} \left\{ h_{k+1} f_{k+1} + \bar{f}_{k+1} \right\} \right].
\]
It remains to prove the explicit expression of \( \sigma^2_{k+1}(f_{k+1} : \bar{f}_{k+1}) \) from this recursion formula. First, following the proof of (26) Theorem 3, for all \( 0 \leq s < k \),
\[
\tilde{D}_{s+1, k} \left( L_k f_{k+1} + L_k \{ \tilde{h}_k f_{k+1} + \bar{f}_{k+1} \} \right) = \tilde{D}_{s+1, k} \left( h_{k+1} f_{k+1} + \bar{f}_{k+1} \right).
\]
In addition, $0 \leq s < k$,

$$\phi_k[L_k] = \frac{\phi_s[L_s \ldots L_k]}{\phi_s[L_s \ldots L_{k-1}]} \, ,$$

which concludes the proof.

\section{Convergence results for Pseudo marginal PaRIS algorithms}

\begin{lemma}
Assume that $H_{21}$ and $H_{22}$ hold. The, for all $0 \leq k \leq n - 1$, $(f_{k+1}, \tilde{f}_{k+1}) \in F(X)^2$ and $N, \bar{N} \geq 0$, the random variables $\{\tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1} f_{k+1}(\xi^1_{k+1}) + \tilde{f}_{k+1}(\xi^1_{k+1}))\}_{i=1}^N$ are i.i.d. conditionally on $\bar{F}_k$ with

$$E \left[ \tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1} f_{k+1}(\xi^1_{k+1}) + \tilde{f}_{k+1}(\xi^1_{k+1})) \mid \bar{F}_k \right] = \tilde{\omega}_{k+1}^1(\tilde{\tau}^1_{k+1}, \xi^1_{k+1}) \, ,$$

$$E \left[ \tilde{\tau}^1_{k+1} \mid \bar{F}_k \right] = \frac{\tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1}, \xi^1_{k+1})}{\sum_{m=1}^N \tilde{\omega}^m_{k+1}(\xi^m_{k+1}, \xi^1_{k+1})} \, ,$$

where $\tilde{\omega}_k$ is defined by (18) in $H_{23}$. Then, since conditionally on $\bar{F}_k \vee \tilde{G}_{k+1}^N$, $\tilde{\tau}^1_{k+1}$ is independent of $\tilde{\omega}^1_{k+1},$

$$E \left[ \tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1} f_{k+1}(\xi^1_{k+1}) + \tilde{f}_{k+1}(\xi^1_{k+1})) \mid \bar{F}_k \right] = E \left[ \tilde{\omega}^1_{k+1} \mid \bar{F}_k \right] E \left[ \tilde{\tau}^1_{k+1} \mid \bar{F}_k \right] E \left[ f_{k+1}(\xi^1_{k+1}) \mid \bar{F}_k \right] \, ,$$

$$E \left[ \tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1}, \xi^1_{k+1}) \mid \bar{F}_k \right] = \frac{\tilde{\omega}^1_{k+1}(\tilde{\tau}^1_{k+1}, \xi^1_{k+1})}{\sum_{m=1}^N \tilde{\omega}^m_{k+1}(\xi^m_{k+1}, \xi^1_{k+1})} \left( \tilde{\tau}^1_{k+1} + \tilde{\tau}^1_{k+1}(\xi^m_{k+1}, \xi^1_{k+1}) \right) \, ,$$

which concludes the proof.
C.1 Proof of Lemma 4.3

Proof. The proof proceeds by induction and follows the same lines as [26, Lemma 13]. The case $k = 0$ is a direct consequence of the fact that $T_0\tau_0 = 0$ and $\widehat{T}_0^i = 0$ for all $1 \leq i \leq N$. Assume that the result holds for some $0 \leq k \leq n - 1$ and write

$$\sum_{i=1}^{N} \frac{\omega^i_{k+1}}{\Omega_{k+1}} (\widehat{T}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) = \frac{a_N}{b_N},$$

where

$$a_N = \frac{1}{N} \sum_{i=1}^{N} \omega^i_{k+1} (\widehat{T}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) \quad \text{and} \quad b_N = \frac{1}{N} \sum_{i=1}^{N} \omega_{k+1}^i.$$

The random variables $(\omega^i_{k+1})_{i \leq N}$ are i.i.d. conditionally on $\bar{F}_k^N$ with

$$E \left[ \omega^i_{k+1} \bigg| \bar{F}_k^N \right] = E \left[ \omega^i_{k+1} \bigg| \bar{F}_k^N \vee \bar{G}_{k+1} \right],$$

where $\omega_k$ is defined by (18) in H3. Noting that by H4 for all $1 \leq i \leq N$, $|\omega^i_{k+1}| \leq \|\omega_k\|_\infty$ and

$$E \left[ \omega^i_{k+1} (\xi_{k+1}^i) \bigg| \bar{F}_k^N \right] = \sum_{i=1}^{N} \omega^i_{k+1} \left( \frac{\Phi_k^N | L_k^i |}{\Phi_k^N | \theta_k |} \right),$$

by Hoeffding’s inequality, there exist positive constants $c_k$ and $\tilde{c}_k$ such that

$$P \left( b_N - \frac{\Phi_k^N | L_k^i |}{\Phi_k^N | \theta_k |} > \varepsilon \right) \leq c_k e^{-\tilde{c}_k N \varepsilon^2}.$$

Therefore, by Proposition 4.2 and Lemma A.1

$$P \left( b_N - \frac{\Phi_k^N | L_k^i |}{\Phi_k^N | \theta_k |} > \varepsilon \right) \leq c_k e^{-\tilde{c}_k N \varepsilon^2}$$

so that,

$$b_N \xrightarrow{p-a.s.} \Phi_k^N | L_k^i | \quad N \to \infty \quad \Phi_k^N | \theta_k |.$$

Since $\Phi_k^N | L_k^i | > 0$ it remains to establish the convergence in probability of $(a_N)_{N \geq 1}$. On the other hand, by Hoeffding inequality, using that for all $1 \leq i \leq N$, $|\omega^i_{k+1} (T_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i)| \leq \|\omega_{k+1}\|_\infty \|h_{k+1}\|_\infty^2 \|f_{k+1}\|_\infty$,

$$P \left( a_N - E[a_N|\bar{F}_k^N] > \varepsilon \right) \leq c_k e^{-\tilde{c}_k N \varepsilon^2},$$

Then, write

$$E[a_N|\bar{F}_k^N] = E \left[ \omega^i_{k+1} (\widehat{T}_{k+1}^i)^2 f_{k+1}(\xi_{k+1}^i) \bigg| \bar{F}_k^N \right],$$

$$E \left[ \omega^i_{k+1} \bigg| \bar{F}_k^N \vee \bar{G}_{k+1} \right] E \left[ (\widehat{T}_{k+1}^i)^2 \bigg| \bar{F}_k^N \vee \bar{G}_{k+1} \right],$$

$$E \left[ \omega^i_{k+1} (\xi_{k+1}^i) \bigg| \bar{F}_k^N \right] E \left[ (\widehat{T}_{k+1}^i)^2 \bigg| \bar{F}_k^N \right],$$

$$= \tilde{a}_N + \tilde{a}_N^2,$$
The second term is given by
\[
\tilde{a}_N^2 = (\tilde{N} - 1) \tilde{N}^{-1} \mathbb{E}\left[ \bar{\omega}_{k+1}(\xi_{k+1}^{1} , \xi_{k+1}) f_{k+1}(\xi_{k+1}) \mathbb{E}\left[ \left( \tilde{\tau}_k^{(1)} + \tilde{h}_k(\xi_{k+1}^{1}, \xi_{k+1}) \right)^2 \left| \tilde{F}_k^N \lor \tilde{G}_{k+1}^N \right. \right] \left| \tilde{F}_k^N \right. \right] .
\]

By Lemma 3.1, the first term is given by
\[
\tilde{a}_N^1 = \tilde{N}^{-1} \sum_{j=1}^{N} \int \frac{\omega_k^j p_k(\xi_j^1, x)}{\sum_{m=1}^{N} \omega_k^m p_k(\xi_m^m, x)} r_k(\xi_j^1, x) f_{k+1}(x) \mu(dx),
\]

and
\[
\tilde{a}_N^2 = (\tilde{N} - 1) \tilde{N}^{-1} \int f_{k+1}(x) \sum_{\ell=1}^{N} \tilde{\omega}_k^\ell r_k(\xi_{\ell}^\ell, x) \left( \tilde{\tau}_k^{\ell} + \tilde{h}_k(\xi_{\ell}^\ell, x) \right)^2 \mu(dx),
\]

By the induction hypothesis and Proposition 4.2,
\[
a_N^1 \xrightarrow{p} (\tilde{N} \phi_k[\theta_k])^{-1} \left\{ \eta_k[L_k f_{k+1} + \phi_k[T_k^2 h_k L_k f_{k+1} + \phi_k[L_k(f_{k+1} h_k^2)] + 2 \phi_k[T_k h_k L_k(f_{k+1} h_k^2)]] \right\},
\]

which yields
\[
a_N^1 \xrightarrow{p} (\tilde{N} \phi_k[\theta_k])^{-1} \left\{ \eta_k[L_k f_{k+1} + \phi_k[L_k((T_k h_k + \tilde{h}_k)^2 f_{k+1})]] \right\}.
\]

The second term is given by
\[
\tilde{a}_N^2 = (\tilde{N} - 1) \tilde{N}^{-1} \int \frac{\omega_k^j p_k(\xi_j^1, x)}{\sum_{m=1}^{N} \omega_k^m p_k(\xi_m^m, x)} r_k(\xi_j^1, x) f_{k+1}(x)
\]

\[
\times \left( \sum_{\ell=1}^{N} \tilde{\omega}_k^\ell r_k(\xi_{\ell}^\ell, x) \left( \tilde{\tau}_k^{\ell} + \tilde{h}_k(\xi_{\ell}^\ell, x) \right) \right)^2 \mu(dx),
\]

and
\[
\tilde{a}_N^2 = (\tilde{N} - 1) \tilde{N}^{-1} \left\{ \eta_k[L_k f_{k+1} + \phi_k[L_k((T_k h_k + \tilde{h}_k)^2 f_{k+1})]] \right\}.
\]

with, for all \( x \in X, \)
\[
\varphi_N(x) = f_{k+1}(x) \left( \sum_{\ell=1}^{N} \tilde{\omega}_k^\ell r_k(\xi_{\ell}^\ell, x) \left( \tilde{\tau}_k^{\ell} + \tilde{h}_k(\xi_{\ell}^\ell, x) \right) \right)^2.
\]

For all \( x \in X, \) by Proposition 4.2,
\[
\varphi_N(x) \xrightarrow{p-a.s.} \int f_{k+1}(x) \left( \frac{\phi_k[T_k h_k r_k(., x) + r_k(., x) h_k(., x)]}{\phi_k[r_k(., x)]} \right)^2.
\]
Therefore, as \(\|\phi_N\|_\infty \leq \|f_{k+1}\|_\infty \|h_{k+1}\|_\infty\), by the generalized Lebesgue dominated convergence theorem, see Lemma \[A.2\]

\[
\frac{\bar{a}_N^2}{\Omega_{k+1}} \xrightarrow{P} (N-1)N^{-1}(\phi_k[\bar{\theta}_k])^{-1} \phi_k[L_k\{f_{k+1}T_{k+1}^2h_{k+1}\}].
\]

This concludes the proof following the same steps as in the proof of Lemma \[4.2\] \(\square\)

**C.2 Proof of Proposition \[4.3\]**

**Proof.** The result is proved by induction on \(k\). It holds for \(k = 0\) as for all \(1 \leq i \leq N\), \(\tilde{T}_0 = 0\).

Assume now that the result holds for some \(0 \leq k \leq n-1\) and that \(\phi_{k+1}[T_{k+1}h_{k+1}f_{k+1} + \bar{f}_{k+1}] = 0\).

Write

\[
\sqrt{N} \sum_{i=1}^N \tilde{\omega}_{k+1}^i \left\{ \tilde{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \bar{f}_{k+1}(\xi_{k+1}^i) \right\} = \tilde{\Omega}_{k+1}^{-1} \Delta_{k+1}^N,
\]

where \(\Delta_{k+1}^N = \sqrt{N} \sum_{i=1}^N \tilde{\omega}_{k+1}^i \left\{ \tilde{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \bar{f}_{k+1}(\xi_{k+1}^i) \right\}\) is decomposed as follows

\[
\Delta_{k+1}^N = \Delta_{k+1,1}^N + \Delta_{k+1,2}^N,
\]

where

\[
\Delta_{k+1,1}^N = \sqrt{N} \sum_{i=1}^N \mathbb{E} \left[ \tilde{\omega}_{k+1}^i (\tilde{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \bar{f}_{k+1}(\xi_{k+1}^i)) \bigg| \tilde{\mathcal{F}}_k^N \right],
\]

\[
\Delta_{k+1,2}^N = \sqrt{N} \sum_{i=1}^N \left\{ \tilde{\omega}_{k+1}^i \left( \tilde{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \bar{f}_{k+1}(\xi_{k+1}^i) \right) \right\}
- \mathbb{E} \left[ \tilde{\omega}_{k+1}^i (\tilde{\tau}_{k+1}^i f_{k+1}(\xi_{k+1}^i) + \bar{f}_{k+1}(\xi_{k+1}^i)) \bigg| \tilde{\mathcal{F}}_k^N \right].
\]

By Lemma \[B.1\]

\[
\tilde{\Omega}_{k+1}^{-1} \Delta_{k+1,1}^N = \frac{N}{\tilde{\Omega}_{k+1}} \left( \tilde{\omega}^N_k[\bar{\theta}_k] \right)^{-1} \sqrt{N} \sum_{i=1}^N \tilde{\omega}_{k}^i \left\{ \tilde{\tau}_{k}^i L_k f_{k+1}(\xi_{k}^i) + L_k(\tilde{h}_k f_{k+1} + \bar{f}_{k+1})(\xi_{k}^i) \right\}
\]

As \(\phi_{k+1}[T_{k+1}h_{k+1}f_{k+1} + \bar{f}_{k+1}] = 0\),

\[
\phi_k[T_kh_kL_kf_{k+1} + L_k(\tilde{h}_k f_{k+1} + \bar{f}_{k+1})] = 0.
\]

Therefore, using the induction hypothesis, Slutsky’s lemma and

\[
\frac{N}{\tilde{\Omega}_{k+1}} \left( \tilde{\omega}^N_k[\bar{\theta}_k] \right)^{-1} \xrightarrow{P} (\phi_k[L_k1])^{-1}
\]

yields

\[
\tilde{\Omega}_{k+1}^{-1} \Delta_{k+1,1}^N \xrightarrow{P} \tilde{\sigma}_k(L_k f_{k+1}; L_k(\tilde{h}_k f_{k+1} + \bar{f}_{k+1})) Z \frac{1}{\phi_k[L_k1]}.
\]
where \( Z \) is a standard Gaussian random variable. By Lemma \[ \text{Lemma B.1} \]

\[
\hat{\Omega}_{k+1}^{-1} \Delta_{k+1,2}^N = \frac{N}{\Omega_{k+1}} \sum_{i=1}^{N} u_N^i ,
\]

where for all \( 1 \leq i, j \leq N \) and all \( x \in X \),

\[
v_N^i = \frac{1}{\sqrt{NN}} \sum_{j=1}^{N} \tilde{v}_N(i, j, x, \xi_{k+1}^j) .
\]

\[
\tilde{v}_N(i, j, x) = \frac{\tilde{r}_k(\xi_{k+1}^i, \xi_{k+1}^j)}{\widetilde{\sigma}_k(\xi_{k+1}^i)} \left\{ \left( \tilde{r}_k + \tilde{h}_k(\xi_{k+1}^j, x) \right) f_{k+1}(x) + \tilde{f}_{k+1}(x) \right\} - \left( \phi_N^i[\theta_k] \right)^{-1} \sum_{\ell=1}^{N} \frac{\tilde{\sigma}_k^\ell}{\Omega_k} \left\{ \bar{\tau}_k^\ell \left( L_k f_{k+1}(\xi_{k+1}^\ell) + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi_{k+1}^\ell) \right) \right\} .
\]

First,

\[
\frac{N}{\Omega_{k+1}} \xrightarrow{P} \phi_k[\theta_k] .
\]

Then, by construction, \( E[v_N^i | \tilde{F}_k^N] = 0 \) and

\[
\sum_{i=1}^{N} E[(v_N^i)^2 | \tilde{F}_k^N] = \tilde{N}^{-1} E[E[(\tilde{v}_N(i, j, x, \xi_{k+1}^j)))^2 | \tilde{F}_k^N \vee \tilde{G}_{k+1}^N | \tilde{F}_k^N] + (\tilde{N} - 1) \tilde{N}^{-1} E[E[\tilde{v}_N(i, j, x, \xi_{k+1}^j)) | \tilde{F}_k^N \vee \tilde{G}_{k+1}^N | \tilde{F}_k^N] .
\]

The first term of (31) is given by

\[
E \left[ E \left[ \tilde{v}_N(i, j, x, \xi_{k+1}^j) | \tilde{F}_k^N \vee \tilde{G}_{k+1}^N \right] \right] = \left( \phi_N^i[\theta_k] \right)^{-1} \int A_N(x) B_N(x) \mu(dx)
\[
- \left( \phi_N^i[\theta_k] \right)^{-2} \left( \sum_{\ell=1}^{N} \frac{\tilde{\sigma}_k^\ell}{\Omega_k} \left\{ \bar{\tau}_k^\ell \left( L_k f_{k+1}(\xi_{k+1}^\ell) + L_k(\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi_{k+1}^\ell) \right) \right\} \right)^2 ,
\]

where, for all \( (x, y) \in X \times X \),

\[
\omega_k(x, y) = \int \tilde{r}_k(x, y; z) \omega_{k+1}(x, y; z) R_k(x, y, dz) ,
\]

\[
A_N(x) = \sum_{j=1}^{N} \frac{\tilde{\omega}_k^j}{\Omega_k} \int \tilde{r}_k(\xi_{k+1}^j, x; u)^2 R_k(\xi_{k+1}^j, x, du) = \sum_{j=1}^{N} \frac{\tilde{\omega}_k^j}{\Omega_k} \omega_k(\xi_{k+1}^j, x) ,
\]

\[
B_N(x) = \sum_{\ell=1}^{N} \frac{\tilde{\omega}_k^\ell}{\sum_{m=1}^{N} \tilde{\omega}_k^m} r_k(\xi_{k+1}^\ell, x) \left\{ \left( \bar{\tau}_k^\ell + \tilde{h}_k(\xi_{k+1}^\ell, x) \right) f_{k+1}(x) + \tilde{f}_{k+1}(x) \right\}^2 .
\]
By Proposition 4.2

\[
\left(\frac{\hat{N}}{N}^{\hat{N}}[\varphi_k]^{-2} \left( \sum_{\ell=1}^{N} \frac{\varphi_{k}^{\ell}}{\Omega_k} \left\{ \bar{c}_{k}^{\ell} L_k f_{k+1}(\xi_{k}^{\ell}) + L_k (\bar{h}_k f_{k+1} + \bar{f}_{k+1})(\xi_{k}^{\ell}) \right\} \right)^2
\]

\[
\xrightarrow[\to \infty \, N \to \infty]{} \left( \phi_k[\vartheta_k]^{-2} \left( \phi_k[T_{k+1} h_k L_k f_{k+1} + L_k (\bar{h}_k f_{k+1} + \bar{f}_{k+1})] \right) \right)^2 = 0,
\]

where by assumption \(\phi_{k+1}[T_{k+1} h_k f_{k+1} + \bar{f}_{k+1}] = 0\), so that \(\phi_k[T_{k+1} h_k L_k f_{k+1} + L_k (\bar{h}_k f_{k+1} + \bar{f}_{k+1})] = 0\). Then, write

\[
\int A_N(x)B_N(x)\mu(dx) = \tilde{a}_N + \tilde{a}_N^2 + \tilde{a}_N^3,
\]

where

\[
\varphi_N : x \mapsto \left\{ \sum_{j=1}^{N} \frac{\varphi_{j,k}^{\ell}}{\Omega_k} \varphi_{j,k}(\xi_{j,k}^{\ell} x) \right\} \left( \sum_{m=1}^{N} \frac{\varphi_{m,k}^{\ell}}{\Omega_k} \varphi_{m,k}(\xi_{m,k}^{\ell} x) \right)^{-1},
\]

\[
\tilde{a}_N^1 = \sum_{j=1}^{N} \frac{\varphi_{j,k}^{\ell}}{\Omega_k} \left( \varphi_{j,k}(\xi_{j,k}^{\ell} x) \right)^2 \int \varphi_{j,k}(\xi_{j,k}^{\ell} x) f_{k+1}(x) \varphi_N(x) \mu(dx),
\]

\[
\tilde{a}_N^2 = \sum_{j=1}^{N} \frac{\varphi_{j,k}^{\ell}}{\Omega_k} \varphi_{j,k}(\xi_{j,k}^{\ell} x) \sum_{j=1}^{N} \frac{\varphi_{j,k}^{\ell}}{\Omega_k} \varphi_{j,k}(\xi_{j,k}^{\ell} x) \left( \bar{h}_k(\xi_{j,k}^{\ell} x) f_{k+1}(x) + \bar{f}_{k+1}(x) \right)^2 \mu(dx),
\]

\[
\tilde{a}_N^3 = 2 \sum_{j=1}^{N} \frac{\varphi_{j,k}^{\ell}}{\Omega_k} \varphi_{j,k}(\xi_{j,k}^{\ell} x) \int \varphi_{j,k}(\xi_{j,k}^{\ell} x) \varphi_N(\xi_{j,k}^{\ell} x) f_{k+1}(x) \left( \bar{h}_k(\xi_{j,k}^{\ell} x) f_{k+1}(x) + \bar{f}_{k+1}(x) \right) \mu(dx).
\]

Following the same steps as in the proof of Proposition 4.1

\[
\tilde{a}_N^1 \xrightarrow[\to \infty]{} \eta_k[L_k \{f_{k+1}^{2} \hat{Q}_{\phi_k} \varphi_k\}] + \phi_k[T_{k+1} h_k L_k \{f_{k+1}^{2} \hat{Q}_{\phi_k} \varphi_k\}],
\]

\[
\tilde{a}_N^2 \xrightarrow[\to \infty]{} \phi_k \left[ \int \varphi_k(\cdot, x) \phi_k(\cdot, x) (\bar{h}_k(\cdot, x) f_{k+1}(x) + \bar{f}_{k+1}(x))^2 (\phi_k(\cdot, x) \mu(dx) \right],
\]

\[
\tilde{a}_N^3 \xrightarrow[\to \infty]{} 2 \phi_k \left[ \left\{ \hat{Q}_{\phi_k} \varphi_k \right\} f_{k+1} \left( \bar{h}_k f_{k+1} + \bar{f}_{k+1} \right) \right],
\]

where \(\hat{Q}_{\phi_k} \varphi_k : x \mapsto \phi_k(\varphi_k(\cdot, x)) / \phi_k(\varphi_k(\cdot, x))\). Therefore, the first term of (31) satisfies

\[
\tilde{N}^{-1} E[\mathbb{E}[\{\eta_k(f_{k+1}^{2} \hat{Q}_{\phi_k} \varphi_k\})^2] \mathcal{F}_{k}^{N} \vee \mathcal{G}_{k+1}^{N}] \xrightarrow[\to \infty]{} \int \frac{\eta_k[f_{k+1}^{2}(x) \rho_k(x)] \phi_k(\varrho_k(\cdot, x)] \mu(dx)
\]

\[
+ \int \frac{\phi_k(\varrho_k(\cdot, x)) \left\{ (\underbrace{T_{k+1} h_k + \bar{h}_k(\cdot, x) f_{k+1}(x) + \bar{f}_{k+1}(x))^2} \phi_k(\varphi_k(\cdot, x)] \mu(dx)
\]

which concludes the proof for the first term of (31). The second term of (31) is given by

\[
E[\mathbb{E}[\{\eta_k(f_{k+1}^{2} \hat{Q}_{\phi_k} \varphi_k\})^2] \mathcal{F}_{k}^{N} \vee \mathcal{G}_{k+1}^{N}] = \left( \frac{\hat{N}}{N} \right)^{-1} \hat{N}^{\hat{N}}[\varphi_k] \varphi_k^{N}[\vartheta_k] \varphi_k^{N}[\vartheta_k] \varphi_k^{N}[\vartheta_k],
\]

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where, for all \( x \in X \),

\[
\varphi^N_k(x) = \int p_k(x, z) \left( \frac{r_k(x, z)}{\theta_k(x)p_k(x, z)} f_{k+1}(z) \sum_{\ell=1}^{N} \frac{\omega^\ell_k r_k(\xi^\ell_k, z)}{\sum_{m=1}^{N} \omega^m_k r_k(\xi^m_k, z)} \left( \tilde{r}^\ell_k + \tilde{h}_k(\xi^\ell_k, z) \right) \right) \mu(dz) + \frac{r_k(x, z)}{\theta_k(x)p_k(x, z)} \tilde{f}_{k+1}(z) - (\phi^N_k[\theta_k])^{-1} \sum_{\ell=1}^{N} \frac{\omega^\ell_k}{\Omega_k} \left( \tilde{r}^\ell_k \mathbf{L}_k f_{k+1}(\xi^\ell_k) + \mathbf{L}_k (\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})(\xi^\ell_k) \right)^2 \mu(dz).
\]

By assumption, \( \phi_k[\mathbf{T}_k h_k \mathbf{L}_k f_{k+1} + L_k (\tilde{h}_k f_{k+1} + \tilde{f}_{k+1})] = 0 \) so that by Lemma 4.2

\[
\hat{\phi}^N_k[\theta_k \varphi^N_k] \xrightarrow{p} \phi_k[\theta_k \varphi_k],
\]

where

\[
\varphi_k(x) = \int p_k(x, z) \left( \frac{r_k(x, z)}{\theta_k(x)p_k(x, z)} \right)^2 \left( f_{k+1}(z) \tilde{Q}_k \phi_k (\mathbf{T}_k h_k + \tilde{h}_k)(z) + \tilde{f}_{k+1}(z) \right)^2 \mu(dz),
\]

\[
= \int p_k(x, z) \left( \frac{r_k(x, z)}{\theta_k(x)p_k(x, z)} \right)^2 \left( f_{k+1}(z) \mathbf{T}_{k+1} h_{k+1}(z) + \tilde{f}_{k+1}(z) \right)^2 \mu(dz).
\]

Therefore,

\[
E[E[\hat{u}^N_k | \mathbf{T}_{k+1}, \mathbf{J}_{k+1}, \xi_{k+1}^1, \xi_{k+1}^2] | \mathcal{F}^N_k \vee G^N_{k+1}] \xrightarrow{p} \left( 1 - \frac{1}{N} \right) (\phi_k[\theta_k])^{-1} \phi_k \left[ \int r_k(\cdot, z) \tilde{w}_k(\cdot, z) \left( f_{k+1}(z) \mathbf{T}_{k+1} h_{k+1}(z) + \tilde{f}_{k+1}(z) \right)^2 \mu(dz) \right].
\]

The proof of (11) is an immediate consequence of (10) since for all \( 1 \leq i \leq N \),

\[
v^i_N \leq 2\|\tilde{w}_{k+1}\|_\infty \left( \|h_{k+1}\|_\infty \|f_{k+1}\|_\infty + \|\tilde{f}_{k+1}\|_\infty \right) N^{-1/2}.
\]

Then, defining \( c_k = 2\|\tilde{w}_{k+1}\|_\infty (\|h_{k+1}\|_\infty \|f_{k+1}\|_\infty + \|\tilde{f}_{k+1}\|_\infty) \), for all \( \varepsilon > 0 \),

\[
\sum_{i=1}^{N} E[(v^i_N)^2 1_{|v^i_N| > \varepsilon}] \leq c_k^2 1_{\varepsilon \in \varepsilon \sqrt{N}} \xrightarrow{p} 0.
\]

Writing

\[
\tilde{f}_{k+1} = \tilde{f}_{k+1} - \phi_{k+1} [\mathbf{T}_{k+1} h_{k+1} f_{k+1} + \tilde{f}_{k+1}],
\]

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yields

\[
\sigma^2_{k+1}(f_{k+1}; \tilde{f}_{k+1}) = \frac{\bar{\sigma}^2_k(L_kf_{k+1}; L_k(h_kf_{k+1} + \tilde{f}_{k+1}))}{\phi_k[L_k1]^2} + \frac{\phi_k[\phi_k] \int \eta_k(r_k(\cdot, x))f_{k+1}^2(x)\tilde{Q}_\phi \omega_k(x)\mu(dx)}{N \phi_k[L_k1]^2}
\]
\[
+ \frac{\phi_k[\phi_k] \int r_k(\cdot, z)\tilde{\omega}_k(\cdot, z) (f_{k+1}(z)T_{k+1}h_{k+1}(z) + \tilde{f}_{k+1}(z))^2 \mu(dz)}{N \phi_k[L_k1]^2}
\]
\[
+ \frac{\phi_k[\phi_k] \int \omega_k(\cdot, z)f_{k+1}^2(z)\tilde{Q}_\phi \left(T_kh_k + \tilde{h}_k - T_{k+1}h_{k+1}\right)^2(z)\mu(dz)}{N \phi_k[L_k1]^2}
\]
\[
+ \frac{\phi_k[\phi_k] \int \text{Cov}\{\tilde{r}_k(\cdot, z; \varsigma_k)\tilde{\omega}_k(\cdot, z; \varsigma_k)\} (f_{k+1}(z)T_{k+1}h_{k+1}(z) + \tilde{f}_{k+1}(z))^2 \mu(dz)}{N \phi_k[L_k1]^2}
\]

which concludes the proof. \(\square\)