Complex polynomial vector fields having a finitely curved orbit

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Abstract

In this paper we address the following questions: (i) Let $C \subset \mathbb{C}^2$ be an orbit of a polynomial vector field which has finite total Gaussian curvature. Is $C$ contained in an algebraic curve? (ii) What can be said of a polynomial vector field which has a finitely curved transcendental orbit? We give a positive answer to (i) under some non-degeneracy conditions on the singularities of the projective foliation induced by the vector field. For vector fields with a slightly more general class of singularities we prove a classification result that captures rational pull-backs of Poincaré-Dulac normal forms.

1 Introduction and motivation

Consider a polynomial vector field $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ on $\mathbb{C}^2$. The nonsingular orbits of $X$ are holomorphic curves in $\mathbb{C}^2$ and thus are oriented minimal surfaces in $\mathbb{R}^4$ with respect to the euclidian metric. They are Riemann surfaces, and their topology can be very complicated (space of ends not denumerable, infinity genus and so on). In general, for a 2-dimensional oriented surface $L$ immersed in $\mathbb{R}^4$, and parameterized locally by isothermal coordinates $z$, that endow $L$ with a Riemann surface structure, we have a naturally associated map $\Phi$, called the holomorphic Gauss map of the immersion, from $L$ into the complex projective space $\mathbb{C}P^3$. If we consider $\mathbb{C}P^3$ with the homogeneous coordinates $(z_1, z_2, z_3, z_4)$, the image of $\Phi$ is contained in the quadric $Q_2 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ ([7]). In the case of a minimal immersion $\Phi$ is a holomorphic map. The Gauss map is algebraic if $M$ is conformally equivalent to a finitely punctured compact Riemann surface $\overline{M}$ and $\Phi$ extends as a meromorphic map to $\overline{M}$. A classical result due to Chern and Osserman states this is equivalent, for complete minimal immersions, to the finiteness of the total curvature of the immersion. In particular, an algebraic curve $C \subset \mathbb{C}^2$ has algebraic Gauss map, nevertheless there are holomorphic curves in $\mathbb{C}^2$ with finite total curvature but which are not algebraic curves, for instance orbits of suitable vector fields (see Example [1]). One of the aims of this paper is to give conditions on the vector field $X$ in order to assure that a nonsingular orbit with algebraic Gauss map (i.e., with finite total curvature) is actually algebraic. As we will see, this is related to the nature of the singularities of the corresponding singular foliation on $\mathbb{C}P^2$. Let $\mathcal{F}$ a holomorphic foliation with discrete singular set $\text{sing}(\mathcal{F})$ on a complex surface $M$. A singularity $p \in \text{sing}(\mathcal{F})$ is called irreducible if there is an open neighborhood $U$ of $p$ in $M$ where $\mathcal{F}$ is induced by a holomorphic vector field $Z$ of the form: $Z(x,y) = x\frac{\partial}{\partial x} + [\lambda y + \text{h.o.t.}]\frac{\partial}{\partial y}$, $\lambda \notin \mathbb{Q}_+$ (non-degenerated), or $Z(x,y) = x^{m+1}\frac{\partial}{\partial x} + [y(1 + \lambda x^m) + \text{h.o.t.}]\frac{\partial}{\partial y}$, $m \geq 1$ (saddle-node).

Given a polynomial vector field $X$ with isolated singularities on $\mathbb{C}^2$ we denote by $\mathcal{F}(X)$ the corresponding holomorphic foliation with singularities induced by $X$ on $\mathbb{C}P(2)$ (see Example [1]). For the case where only irreducible singularities are allowed we have the following statement:

**Theorem 1.** Let $X$ be a polynomial vector field on $\mathbb{C}^2$ and let $L$ be an orbit of $X$ with finite total curvature with respect to the metric induced by $\mathbb{C}^2$ on $L$. Suppose that the singularities of $\mathcal{F}(X)$ on $\mathbb{C}P^2$ are irreducible or that $X$ is without invariant lines on $\mathbb{C}^2$. Then $L$ is algebraic.
In the second part of this work we classify polynomial vector fields admitting a finitely curved transcendent orbit and whose singularities of $\mathcal{F}(X)$ are in the Poincaré domain (II), i.e., $\mathcal{F}(X)$ is given in a neighborhood of a singular point by a germ of vector field of the form $Z = \lambda x \frac{∂}{∂x} + \mu y \frac{∂}{∂y} + \ldots, \lambda/\mu \notin \mathbb{R}$. By Poincaré-Dulac theorem such a singularity is dicritical if and only if it is linearizable with $\lambda/\mu \in \mathbb{Q}_+$. We prove:

**Theorem 2.** Let $X$ be a polynomial vector field on $\mathbb{C}^2$ such that the singularities of $\mathcal{F}(X)$ are non-dicritical and in the Poincaré domain. If $X$ has a finitely curved non-algebraic orbit then $\mathcal{F}(X)$ is given by a closed rational 1-form on $\mathbb{C}P^2$. Indeed, either $\mathcal{F}(X)$ is a logarithmic foliation or there is a rational map $f: \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $\mathcal{F}(X)$ is the pull-back $f^* \mathcal{F}(Y)$ where $Y$ is a Poincaré-Dulac normal form $Y(x, y) = (nx + cy^n) \frac{∂}{∂x} + y \frac{∂}{∂y}$, for some $n \in \mathbb{N}$ and some $c \in \mathbb{C} \setminus \{0\}$. In particular all orbits of $X$ have finite total curvature.

## 2 Preliminaries

### 2.1 Theorems of Chow and Remmert-Stein

In this paragraph we introduce two extension theorems found in [6] that will be referred frequently in our work; the theorem of Chow and the theorem of Remmert-Stein.

**Theorem of Remmert-Stein.** Let $V$ be subvariety of the polydisc $D \subset \mathbb{C}^n$. Suppose $W$ is an irreducible subvariety of $D \setminus V$ of dimension $n$. If $n > \text{dim}(V)$, then the closure $\overline{W}$ of $W$ in $D$ is an irreducible subvariety of dimension $n$.

**Theorem of Chow.** If $X$ is closed and proper analytic subset of $\mathbb{C}P(n)$, then $X$ is algebraic, i.e., $X$ is a finite union of algebraic varieties.

### 2.2 Holomorphic foliations with singularities

Let $M$ be a complex surface. By definition a codimension one holomorphic foliation with singularities $\mathcal{F}$ on $M$ is given by an open cover $\{U_j\}_{j \in J}$ of $M$ and a collection of 1-forms $\{\omega_j\}_{j \in J}$, where $\omega_j$ is defined on $U_j$ and such that if $U_j \cap U_k \neq \emptyset$, then

$$\omega_j|_{U_j \cap U_k} = g_{jk} \cdot \omega_k|_{U_j \cap U_k},$$

for some nonvanishing holomorphic function $g_{jk}$ defined in $U_j \cap U_k$. The singular set of $\mathcal{F}$ is the analytic subset $\text{sing}(\mathcal{F})$ of $M$ defined by

$$\text{sing}(\mathcal{F}) := \{p \in U_j : \omega_j(p) = 0\}$$

It is well-known that $\text{sing}(\mathcal{F})$ may be assumed of codimension two. Thus $\text{sing}(\mathcal{F})$ is a discrete subset of $M$. The leaves of $\mathcal{F}$ are the leaves of the restriction $\mathcal{F}|_{M \setminus \text{sing}(\mathcal{F})}$. Given a singularity $p \in \text{sing}(\mathcal{F})$, a separatrix of $\mathcal{F}$ at $p$ is a germ of analytic curve $S$ at $p$ which is invariant and such that $p \in S$. Such a curve always exists (see section 1.4).

**Example 1** (Polynomial vector fields). It is well-known that, from the above definition, a polynomial vector field $X(x, y) = P(x, y) \frac{∂}{∂x} + Q(x, y) \frac{∂}{∂y}$ on $\mathbb{C}^2$ induces a holomorphic foliation with singularities $\mathcal{F}(X)$ on $\mathbb{C}P^2$. The foliation $\mathcal{F}(X)$ is characterized by the fact that if $L$ is a leaf of $\mathcal{F}(X)$ which is not contained in the line at the infinity then $L^* = L \cap \mathbb{C}^2$ is a non-singular orbit of $X$. Conversely, any foliation with singularities on $\mathbb{C}P^2$ is obtained in this way.
2.3 Resolution of singularities

Given a holomorphic foliation with singularities $\mathcal{F}$ on a complex surface $M$, a theorem of Seidenberg ([14]) gives a resolution of the singular points of $\mathcal{F}$.

**Theorem of Seidenberg.** There is finite sequence of blow-ups at the points of $\text{sing}(\mathcal{F})$ such that their composition gives a proper holomorphic map $\pi : \tilde{M} \to M$ of a complex surface $M$ and a foliation $\mathcal{F}^* = \pi^*(\mathcal{F})$ with isolated singularities such that:

(i) $\pi^{-1}(\text{sing}(\mathcal{F})) = \bigcup_{j=1}^{k} \mathbb{P}_j$ where each $\mathbb{P}_j$ is a projective line. $\pi^{-1}(\text{sing}(\mathcal{F}))$ is called the divisor of the resolution, and $\pi|_{\tilde{M}\setminus \pi^{-1}(\text{sing}(\mathcal{F}))}$ is a biholomorphism.

(ii) At any singularity $p \in \bigcup_{j=1}^{k} \mathbb{P}_j$ there is a local chart $(x, y)$ such that $x(p) = y(p) = 0$ and $\mathcal{F}^*$ is given by one of the 1-forms

\[
\begin{align*}
xy - \lambda ydx + (\text{h.o.t}), & \quad \lambda \notin \mathbb{Q}_+(\text{non-degenerated}), \\
xy^{m+1} - y(1 + \lambda x^m)dx + (\text{h.o.t}), & \quad m \geq 1(\text{saddle-node}).
\end{align*}
\]

**Remark 1.** (1) Non-degenerated and saddle-node singularities are usually called irreducible singularities.

(2) According to [2], for each $p \in \text{sing}(\mathcal{F})$, $\mathcal{F}$ admits at least one separatrix through $p$; if the number of these separatrices is finite, $p$ is called a non-dicritical singularity and dicritical otherwise. The singularity $p$ is non-dicritical if and only if all the projective lines $\mathbb{P}_j$ belonging to $\pi^{-1}(p)$ are invariant by $\mathcal{F}^*$.

2.4 The Camacho-Sad Index Theorem

Let $\mathcal{F}$ be a holomorphic foliation on a complex surface $M$ as above and $p \in \text{sing}(\mathcal{F})$ an isolated singular point. Let $S$ be a separatrix of $\mathcal{F}$ at $p$. We can assume that $S = \{ q \mid f(q) = 0 \}$, where $f$ is a holomorphic function defined in a neighborhood $U$ of $p$. We may assume that $f$ is reduced, that is, $df \neq 0$ outside $p$. Then it is well-known that given a holomorphic 1-form $\omega$ defining $\mathcal{F}$ in $U$, with $\text{sing}(\omega) = \{ q \mid \omega = 0 \}$, there are holomorphic functions $g$ and $h$ in $U$, and a holomorphic 1-form $\eta$ in $U$ such that we have $g\omega = kdf + f\eta$. The Camacho-Sad index is defined as

\[\text{CS}(\mathcal{F}, S, p) := \frac{1}{2\pi i} \int_{\partial S} \frac{\eta}{h}\]

**Example 2** (Index of an irreducible singularity). Let $\mathcal{F}$ be a nondegenerate singularity given by the 1-form $\omega(x, y) = x(\lambda + yp(x, y))dy - y(\mu + xq(x, y))dx$, with $\lambda, \mu \neq 0$, and $p, q$ holomorphic functions. $S_x = \{(x, 0) \mid x \in \mathbb{C}\}$ and $S_y = \{(0, y) \mid y \in \mathbb{C}\}$ are separatrices for $\mathcal{F}$. Then $\text{CS}(\mathcal{F}, S_x, 0) = \frac{\lambda}{\mu}$ and also $\text{CS}(\mathcal{F}, S_y, 0) = \frac{\mu}{\lambda}$.

Let now $\mathcal{F}$ be a saddle-node singularity given by the 1-form $\omega(x, y) = x^{n+1}dy - [y(1 + \lambda x^n) + q(x, y)]dx$, with $n \geq 1$ and $q$ holomorphic function. Let $S$ be the strong separatrix for $\omega

S = \{(0, y) \mid y \in \mathbb{C}\}$.

Then $\text{CS}(\mathcal{F}, S, 0) = 0$. If there is another separatrix $S'$ for $\mathcal{F}$ then we can assume that $\omega(x, y) = x^{n+1}dy - y[(1 + \lambda x^n) + q(x, y)]dx$ and $S' = \{y = 0\}$. For this we have $\text{CS}(\mathcal{F}, S', 0) = \lambda$. 

The Camacho-Sad Index theorem reads as follows ([2]):

**Index Theorem.** Let $S$ be a compact holomorphic curve in a complex surface $M$. Assume that $S$ is invariant under a holomorphic foliation $\mathcal{F}$ with isolated singularities. Then the number of singularities of $\mathcal{F}$ on $S$ is finite, and we have

$$
\sum_{p \in \text{sing}(\mathcal{F}) \cap S} CS(\mathcal{F}, S_p, p) = S \cdot S
$$

where $S_p$ is the germ of $S$ at $p$ and $S \cdot S$ is the self intersection number of the embedding $S \hookrightarrow M$.

### 2.5 The holomorphic Gauss map of a holomorphic vector field

Let $X$ be a holomorphic vector field on $\mathbb{C}^2$ and $L$ a non-singular orbit of $X$. Since $L$ is a minimal immersion we can define a holomorphic Gauss map $\Phi: L \rightarrow \mathbb{C}P(3)$ which takes values on the quadric $Q_2$:

$$
\sum_{j=1}^{4} z_j^2 = 0
$$

This map is constructed as follows: we define a tangent Gauss map $\Phi: L \rightarrow \text{Go}(2,4)$ from $L$ into the Grassmaniann of oriented two planes in $\mathbb{R}^4$ by $\Phi(p) = T_p(L)$. Then using the canonical analytical isomorphism $G^0(2, 4) \simeq Q_2$ we may consider the tangent Gauss map $\Phi: L \rightarrow Q_2 \subset \mathbb{C}P^3$. The map $\Phi$ is holomorphic and is called the holomorphic Gauss map of the minimal immersion $L \subset \mathbb{R}^4$. Now we exploit the fact that $L$ is a holomorphic curve tangent to $X$ in $\mathbb{C}^2$ to give another interpretation of the holomorphic Gauss map. Given any $p \in L$ we have $T_p(L) = \mathbb{C}.X(p) \subset \mathbb{C}^2$. Thus the map $\Phi$ takes values on the space of directions on $\mathbb{C}^2 \setminus \{0\}$ which is naturally identified with the projective line $\mathbb{C}P^1$ the line at the infinity $L_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2$. Therefore, we can consider the holomorphic Gauss map of $L$ as a map $\Phi: L \rightarrow \mathbb{C}P(1)$. The spherical image of $L$ is defined as $\Phi(L) \subset \mathbb{C}P^1$.

### 2.6 The Integral Curvature Lemma

In this section we relate the total curvature of a nonsingular orbit of a holomorphic vector field in $\mathbb{C}^2$ with the area of its image under the holomorphic Gauss map. Let $\psi: L \rightarrow \mathbb{R}^n$ an oriented minimal surface and let $\varphi: L \rightarrow \mathbb{C}P^{n-1}$ be the corresponding holomorphic Gauss map (see [7]). The sphere $S^{2n-1}$ induces on $\mathbb{C}P^{n-1}$ a metric with constant holomorphic sectional curvature called the Fubini-Study metric, which is given in homogeneous coordinates by

$$
d s^2 = \frac{|Z \wedge dZ|^2}{|Z|^4}.
$$

For $\varphi(Z) = (\varphi_1(Z), ..., \varphi_n(Z))$ we can calculate the area of $\Phi$ by evaluating the integral

$$
A(\Phi) = \int_L \Phi^* \omega
$$

where $\omega$ is the area element induced by the Fubini-Study metric restricted to the image of $\Phi$.

The following result is adapted from [10] Section 5 page 427:

**Proposition 1.** Let $\psi: L \rightarrow \mathbb{R}^n$ be a minimal immersion with holomorphic Gauss map $\Phi$ and Gaussian curvature $K$, then

$$
\int_L K dA = -A(\Phi).
$$
Denote by \( \sigma: \mathbb{C}^2 \setminus \{0\} \to L_\infty \) the canonical fibration where we identify \( L_\infty \cong \mathbb{C}P(1) \). Given a point \( q \in L_\infty \) the fiber \( \sigma^{-1}(q) \) is a complex affine puncture punctured at one point. As a corollary of Proposition 1 we have:

**Proposition 2.** Let \( L \subset \mathbb{C}^2 \) be a nonsingular orbit of a holomorphic vector field \( X \) on \( \mathbb{C}^2 \). Suppose that generically for \( q \in L_\infty \), the intersection number \( \sharp(L \cap \sigma^{-1}(q)) \) is \( \nu \in \mathbb{N} \cup \{+\infty\} \). Then:

(i) If \( \nu \in \mathbb{N} \) then the total curvature of \( L \) is finite equal to \( C(L) = -2\pi \nu \).

(ii) If \( \nu = +\infty \) then the total curvature of \( L \) is not finite.

**Remark 2.** One can deduce Propositions 1 and 2 from the co-area formula that reads as follows: given two manifolds \( M^m \) and \( N^n \) with \( m \geq n \), a smooth map \( \Phi: M \to N \) and a measurable function \( f: M \to \mathbb{R} \) we have the following:

\[
\int_M f(x) \| \text{Jac } \Phi \| dx = \int_N \int_{\Phi^{-1}(y)} f(z) d\mathcal{H}^{m-n}(z) dy
\]

where \( \mathcal{H}^{m-n} \) is the \( (m-n) \)-dimensional Hausdorff measure induced on \( \Phi^{-1}(y) \) from \( M \), and \( \| \text{Jac } \Phi \| \) is the Radon derivative of the measure on \( N \) with respect to the image under \( d\Phi \) of the \( n \)-dimensional measure on \( M \).

### 3 Examples

The total curvature of the orbits of a linear vector field is studied below.

**Example 3** (Linear vector fields). Let \( X = x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \) be a linear vector field on \( \mathbb{C}^2 \). A parametrization for the orbit passing by \( (1, y_0) \in \mathbb{C}^2 \) is given by \( \psi(z) = e^{2 \pi i} \frac{\partial}{\partial y} + y_0 e^{\lambda z} \frac{\partial}{\partial y} \). If \( \lambda \in \mathbb{Q} \) then clearly the orbits of \( X \) are contained in algebraic curves and therefore they have finite total curvature. On the other hand for \( \lambda \) not rational the map \( \psi \) is injective and computations inspired in [12] Section 5 show that the total curvature of the orbit \( L_{(1,y_0)} \) can be written as

\[
C(\psi) = \int \int_{\mathbb{R}^2} -2|\lambda|^2|y_0|^2 (1 + |\lambda|^2 - \lambda - \bar{\lambda}) e^{2(1+\alpha)u - 2\alpha v} \frac{dudv}{(e^{2u} + |\lambda|^2|y_0|^2 e^{2\alpha u - 2\alpha v})^2}
\]

Computations similar to those in [12] (compare with Proposition 6, Section 5) actually show that \( C(\psi) > -\infty \) if and only if \( \lambda \in \mathbb{Q} \).

The following is a key example in our study.

**Example 4** (Poincaré-Dulac normal form). Let \( X = (nx + y^n) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) be a Poincaré-Dulac normal form vector field on \( \mathbb{C}^2 \) ([11]). For simplicity we will assume that \( n = 1 \). The origin is a singularity exhibiting a unique separatrix given by \( \{y = 0\} \). This is necessary for the existence of finitely curved orbits not contained in algebraic curves as we will see in Lemma 2. Let us show that this is actually the case for the orbits of \( X \). A parametrization for the orbit passing by \( (0, y_0) \) is given by \( \psi(z) = y_0 e^{\lambda z} \frac{\partial}{\partial y} + y e^{\beta z} \frac{\partial}{\partial y} \). Writing \( \psi \) as map into \( \mathbb{R}^4 \), we get \( \psi(z) = (\beta y_0 e^{\lambda z} + \frac{y_0 e^{\lambda z}}{2}, \frac{y_0 e^{\lambda z}}{2}, y_0 e^{\lambda z} + \frac{y_0 e^{\lambda z}}{2}, \frac{y_0 e^{\lambda z}}{2}) \). The holomorphic Gauss map of the immersion \( \psi \) is given by \( \varphi(z) = \frac{\partial \psi}{\partial z} = (\frac{y_0 e^{\lambda z}}{2}, \frac{y_0 e^{\lambda z}}{2}, \frac{y_0 e^{\lambda z}}{2}, \frac{y_0 e^{\lambda z}}{2}) \). Therefore denoting \( z = u + iv \) and \( \lambda = \alpha + i \beta \) we have the following expression for \( F(z) = |\varphi|^2 \)

\[
F(z) = \frac{1}{4} |1 + z|^2 e^z \bar{e}^z + \frac{1}{4} |1 + z|^2 e^z e^\bar{z} + \frac{1}{4} |y|^2 e^z \bar{e}^z + \frac{1}{4} |y|^2 e^z e^\bar{z} = \frac{|y|^2 (1 + |z|^2 + 1) e^{z+\bar{z}}}{2}
\]
So \( F(z, \bar{z}) = \frac{|y|^2((1+z)(1+\bar{z})+1)e^{z+\bar{z}}}{2} \),

\[
\frac{\partial}{\partial \bar{z}} \log F = \frac{[(1+z)(1+\bar{z})+1]+(1+z)}{(1+z)(1+\bar{z})+1} + (1+z) + 1 = \frac{(1+z)(2+\bar{z})+1}{(1+z)(1+\bar{z})+1} + 1
\]

and

\[
\frac{\partial^2 \log(F)}{\partial z \partial \bar{z}} = \frac{1}{(|1+z|^2+1)^2}
\]

Hence

\[
K = -\frac{1}{F} \frac{\partial^2 \log(F)}{\partial z \partial \bar{z}} = \frac{-2}{|y|^2(|1+z|^2+1)e^{z+\bar{z}}(|1+z|^2+1)^2} = \frac{-2}{|y|^2(1+|z|^2+1)^3e^{z+\bar{z}}}
\]

In terms of the variables \( u \) and \( v \), \( K \) can be written as

\[
K = -\frac{2|\lambda|^2|y_0|^2(1+|\lambda|^2 - \lambda - \bar{\lambda})e^{2(1+\bar{\alpha})u-2\beta v}}{(e^{2u} + |\lambda|^2|y_0|^2 e^{2\alpha u-2\beta v})^3}
\]

For the immersion \( \psi(z) = (yze^z, ye^z) \) the induced metric is given by

\[
ds^2 = |y|^2e^{z+\bar{z}}(|1+z|^2+1) |dz|^2 = |y|^2e^{z+\bar{z}}(|1+z|^2+1)(|du|^2 + |dv|^2).
\]

The area element is given by

\[
dA = |y|^2e^{z+\bar{z}}(|1+z|^2+1) du \wedge dv
\]

The total curvature can be written as

\[
C(\psi) = \int \int_{\mathbb{R}^2} \frac{-2e^{z+\bar{z}}(|y|^2e^{z+\bar{z}})(|1+z|^2+1)du\,dv}{|y|^2(|1+z|^2+1)^3} = -2 \int \int_{\mathbb{R}^2} \frac{1}{(|1+z|^2+1)^2} du\,dv = -2\pi
\]

Let us give a geometric interpretation of the above computation. The vector field \( X = (x+y)\frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) admits the first integral \( f(x,y) = ye^{-\bar{z}} \). Given \( c \in \mathbb{C} \setminus \{0\} \) the orbit \( L_c \) passing through the point \((1, c) \in \mathbb{C}^2 \) is given by the level curve \( f(x, y) = ce^{-\frac{1}{z}} \). Given \( \alpha \in \mathbb{C} \setminus \{0\} \), the intersection of the line \( y = \alpha x \) with the leaf \( L_c \) is given by \( f(x, \alpha x) = e^{-\frac{1}{z}} \) corresponds to \( x e^{-\frac{1}{\alpha}} = c.e^{-\frac{1}{z}} \) and therefore it is a single point. Therefore, according to the notation of Proposition 2 we have \( \tilde{z}(L_c \cap \sigma^{-1}(q)) = 1 \) for all \( c \in \mathbb{C} \setminus \{0\} \) and generic \( q \in L_\infty \). Applying this same proposition we conclude that the area of the spherical image of \( L_c \) is \( 2\pi \) which is the negative of the total curvature of \( L_c \). This is a general fact as follows from Proposition 2.

We point out that the fact that \( X \) has transcendent (non-algebraic) orbits with finite total curvature does not contradict Theorem 1 indeed this is due to the existence of a saddle-node singularity in \( L_\infty \). This saddle-node has strong manifold contained in the projective line \( \{y = 0\} \) and central manifold contained in the line \( L_\infty \).

4 Finitely curved orbits of holomorphic vector fields

We shall now state two basic properties of finitely curved orbits of holomorphic vector fields.

**Lemma 1** ([12]). Let \( L \) be an orbit of a holomorphic vector field \( X \) defined in \( \mathbb{C}^n \). If \( L \) has finite total curvature as a real surface defined in \( \mathbb{R}^{2n} \), then it is closed in \( \mathbb{C}^n \setminus (\text{sing}(X) \cup F(X)) \), where \( F(X) \subset \mathbb{C}^n \) is the union of the flat orbits of \( X \).
By a flat orbit we mean an orbit whose Gaussian curvature vanishes identically. Such an orbit must be contained in a straight complex line. Now we study the local behavior of a finitely curved orbit in a neighborhood of an irreducible singularity, obtaining the following result:

**Lemma 2.** Let $X$ be a holomorphic vector field on $\mathbb{C}^2$ and $p \in \text{sing}(X)$ an irreducible singularity. Let $L$ be an orbit of $X$ accumulating $p$ and if the singularity $p$ is a saddle-node then its central manifold is also accumulated by $L$. If $L$ has finite total curvature then $L$ is contained in the union of separatrices of $X$ through $p$.

**Proof.** The central point is that, for an irreducible singularity, a non-separatrix leaf which accumulates on the singularity and on the central manifold in the saddle-node case, must accumulate on all the exceptional divisor after a blow-up at the singular point. By transverse uniformity, if $L$ accumulates a point of a separatrix, it must accumulates all points in the separatrix. The Gauss map for $L$ is given by $\Phi(p) = [X(p)]$, $p \in L$ where $[(x,y)]$ denotes the straight line through the origin passing through $(x,y)$ on $\mathbb{C}^2$. It is then easy to see that the map $\Phi$ is a holomorphic map from $L$ into $\mathbb{CP}^1$. As usual we denote by $\mathcal{F}(X)$ the corresponding holomorphic foliation induced by $X$ on $\mathbb{C}^2$. A blowing up at $p \in \mathbb{C}^2$ will then produce a foliation $\tilde{\mathcal{F}} = \pi^*(\mathcal{F}(X))$ on $\mathbb{C}^2$. We can endow $\mathbb{C}^2$ with a Riemannian metric via the pull-back $\pi^*(ds^2)$, where $ds^2$ is the canonical metric on $\mathbb{C}^2$. The leaf $\tilde{L} = \pi^{-1}(L)$ has the same Riemannian metric behavior of $L$. So we get

$$\int_L KdA = \int_L Kd\tilde{A}$$

In this case the projective line $\mathbb{CP}(1)$ is invariant with respect to $\tilde{\mathcal{F}}$, and since $\tilde{L}$ accumulates the separatrices, $\tilde{L}$ must accumulate all points in $\mathbb{CP}(1)$. It is also true that: outside the separatrices, the Gauss map of $\tilde{L}$ can be identified with the fiber map. Since $\tilde{L}$ accumulates $\mathbb{CP}(1)$ we have for $\tilde{S} = \text{sing}(\tilde{\mathcal{F}})$ that $\#P^{-1}(p) \cap L^* = \infty$ for $p \in \mathbb{CP}(1) \setminus \tilde{S}$ and $P : \mathbb{C}^2 \to \mathbb{CP}(1)$ being the fiber map. Thus, $P$ has infinite area $A(P) = \infty$. Using the relation

$$\int_L KdA = \int_L \tilde{K}d\tilde{A} = -A(P)$$

we get the desired conclusion. \qed

**Corollary 1.** Let $X$ be a polynomial vector field on $\mathbb{C}^2$. Assume that every singularity $q \in \text{sing}(\mathcal{F}(X))$ is irreducible. Then a finitely curved orbit $L$ of $X$ must be closed outside $\text{sing}(\mathcal{F}(X)) \cap \mathbb{C}^2$.

**Proof.** If $L$ is not closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F}(X))$ then $L$ accumulates some invariant straight line $E \subset \mathbb{C}^2$. By the Index Theorem this line $E$ contains some singularity $q$ of $X$ in $\mathbb{C}^2$ for which we have $I(\mathcal{F}(X), E, q) > 0$. Therefore, if $q$ is a saddle-node then $E$ is the central manifold of this saddle-node. Since the total curvature of $L$ is finite by Lemma 2 there is a neighborhood $U \subset \mathbb{C}^2$ of $q$ such that $L \cap U$ is contained in a local separatrix $\Gamma$ of $X$ through $q$. In particular, $L$ cannot accumulate properly on the line $E$. \qed

With the same proof of Lemma 2 we have:

**Lemma 3.** Let $L$ be a nonsingular finitely curved orbit of a polynomial vector field $X$ on $\mathbb{C}^2$. Suppose that the line at infinity is invariant by $\mathcal{F}(X)$. If $L$ accumulates on some point $p \in L_\infty$ then $p$ is a singularity.
5 Algebraicity of finitely curved orbits

In this section we prove Theorem 1 and give a version of this result for foliations with non-irreducible singularities but excluding the saddle-node case (see Theorem 3 below).

Proof of Theorem 1. Let $X$ be a polynomial vector field defined on $\mathbb{C}^2$ whose corresponding projective foliation $\mathcal{F} = \mathcal{F}(X)$ has only irreducible singularities and let $L$ be an orbit of $X$ with finite total curvature. We study the behavior of $L$ in a neighborhood of $L_\infty$. First we assume that the line $L_\infty$ is invariant. We already know that $L$ is closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F})$ (Corollary 1). If $L$ accumulates some regular point $p \in L_\infty$ then since $L_\infty$ is invariant $L$ accumulates all the line $L_\infty$ and this is not possible for $L$ has finite total curvature. Thus $L$ accumulates only at singular points in $L_\infty$ and by Remmert-Stein theorem this implies that $L$ has analytic closure of dimension one on $\mathbb{C}P^2$ and by Chow’s Theorem $L$ is algebraic. Assume now that $L_\infty$ is not invariant. Let $p \in L_\infty$ be a regular point accumulated by $L$. Then, since $L$ is closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F})$ this implies that $L \cup \{p\}$ is analytic in a neighborhood of $p$, indeed, $L \subset L_p$ the leaf through $p$. In particular, $L$ accumulates only a discrete set of regular points in $L_\infty \setminus \text{sing}(\mathcal{F})$. Let now $p \in \text{sing}(\mathcal{F}) \cap L_\infty$ be a singular point accumulated by $L$. Suppose that $p$ is non-degenerate. In this case $\mathcal{F}$ exhibits two separatrices through $p$ and either $L$ is contained in the union of separatrices or $L$ accumulates both separatrices properly. The last possibility implies that both separatrices are contained in (parallel) straight lines in $\mathbb{C}^2$ and $L$ accumulates both lines and is not closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F})$, absurd. Suppose now that $p$ is a saddle-node singularity. In this case either $L$ is contained in a local separatrix of $\mathcal{F}$ through $p$ or $L$ accumulates the strong manifold of $\mathcal{F}$ through $p$, this implies that $L$ is not closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F})$, absurd. Since the set $\text{sing}(\mathcal{F}) \cap L_\infty$ is also finite the set $L \cup \text{sing}(\mathcal{F})$ is analytic and therefore algebraic of dimension one on $\mathbb{C}P^2$. This proves the first part of the theorem. For the second part we assume that $X$ has no invariant line on $\mathbb{C}^2$. This implies, as in the first part, that $L$ is closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F}(X))$ (Lemma 1). Now, essentially the same argumentation above shows that $L$ must accumulate only on a finite set of points in $L_\infty$. This shows, via the theorems of Remmert-Stein and Chow, that $L$ is algebraic. This ends the proof of Theorem 1. 

According to [3] a singularity of a foliation in dimension two is a generalized curve if its reduction process by blow-ups exhibits no saddle-nodes. For this type of singularity finitely curved leaves are always algebraic as follows:

Theorem 3. Let $X$ be a polynomial vector field defined on $\mathbb{C}^2$ and let $L$ be a finitely curved orbit of $X$. Suppose that the singularities of $\mathcal{F}(X)$ on $\mathbb{C}P^2$ are non-dicritical generalized curves then $L$ is contained in an algebraic curve.

The proof of Theorem 3 requires the following lemma:

Lemma 4. Let $p \in \text{sing}(\mathcal{F})$ be a non-dicritical generalized curve. Then $\mathcal{F}$ has at least two separatrices through $p$. Moreover, if a leaf $L$ accumulates on $p$ and is not contained in the set of local separatrices of $\mathcal{F}$ through $p$ then $L$ accumulates properly on all separatrices of $\mathcal{F}$ through $p$.

The first part of Lemma 4 follows from [8]. The second part follows from the local study of non-degenerate irreducible singularities and the Theorem of Seidenberg (cf. Section 2.3).

Proof of Theorem 3. The proof has the same structure of the proof of Theorem 1. First we prove that $L$ is closed in $\mathbb{C}^2 \setminus \text{sing}(\mathcal{F})$. Indeed, if this is not the case, $L$ accumulates on a regular point
Claim 1. The singularities of $F$ with isolated singularities. The first step is to show the following

Let $q \in \mathbb{C}^2 \setminus \text{sing}(F)$ and therefore $L$ accumulates properly on the leaf $L_q$ that must be an invariant straight line in $\mathbb{C}^2$. If $L_q$ contains some singularity $p \in \text{sing}(F) \cap \mathbb{C}^2$ then by hypothesis $p$ is non-dicritical and it is a generalized curve. By Lemma 2 we conclude that for some neighborhood $U$ of $p$ in $\mathbb{C}^2$ the intersection $L \cap U$ is contained in the set of local separatrices of $F$ through $p$ and therefore the union $(L \cap U) \cup p$ is analytic in $U$. This implies that $L$ cannot accumulate on the line $L_q$ properly in $U$, contradiction. Therefore we must have $L_q \cap \text{sing}(F) \subset L_\infty$. In particular, either the origin of the pencil $\sigma: \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}P^1 \simeq L_\infty$ does not belong $L_q$ or it is not a singularity of $F$. In both cases, since $L_q$ is invariant and accumulated by $L$, the intersection number of $L$ with a generic fiber of $\sigma$ is infinity and therefore the total curvature of $L$ is infinite, a contradiction. This proves that $L \cup \text{sing}(F)$ is an analytic subset of $\mathbb{C}^2$. Now it remains to prove that $L$ accumulates only on singular points in $L_\infty$. If $L_\infty$ is invariant by $F(X)$ this follows from Lemma 3. Assume now that $L_\infty$ is not invariant. If there is a regular point $p \in L_\infty$ which is accumulated by the orbit $L$ then the Flow Box theorem shows that the leaf $L_p \subset L_\infty$ is properly accumulated by $L$ in $\mathbb{C}^2$, this is absurd because $L$ is closed in $\mathbb{C}^2 \setminus \text{sing}(F(X))$. This proves Theorem 3.

6 Proof of Theorem 2

In this section we prove Theorem 2. We shall use the following proposition:

Proposition 3. Let $F$ be a holomorphic foliation on $\mathbb{C}P(2)$ and assume that:

1. $F$ has an invariant (irreducible) algebraic curve $\Lambda \subset \mathbb{C}P(2)$.

2. The holonomy group of the leaf $\Lambda \setminus \text{sing}(F)$ has an orbit accumulating only at the origin.

3. The singularities of $F$ on $\mathbb{C}P(2)$ are non-dicritical and in the Poincaré domain.

Then $F$ is given by a closed rational 1-form on $\mathbb{C}P(2)$.

Proof. Let $F$ be given in the affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$ by a polynomial 1-form $\omega = Pdy - Qdx$ with isolated singularities. The first step is to show the following

Claim 1. The singularities of $F$ are either linearizable (non-resonant) of the form $\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ with $\lambda/\mu \in \mathbb{R} \setminus \mathbb{Q}$ or analytically conjugated to a Poincaré-Dulac normal form.

Proof. We take a singularity $p \in \Lambda \cap \text{sing}(F)$. Since $p$ is in the Poincaré domain we have two possibilities. Either $F$ is analytically linearizable in a neighborhood of $p$ or $F$ is analytically conjugated to a Poincaré-Dulac normal form in a neighborhood of $p$. Moreover, since a Poincaré-Dulac normal form exhibits only one separatrix, if there are two or more separatrices then the singularity is analytically linearizable. On the other hand, we are assuming that the singularities are non-dicritical, therefore a linearizable singularity in the Poincaré domain cannot be of resonant type, i.e., must be of the form $\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ with $\lambda/\mu \in \mathbb{R} \setminus \mathbb{Q}$.

Claim 2. The holonomy group $\text{Hol}(\Lambda)$ is abelian. Moreover, either $\text{sing}(F) \cap \Lambda$ consists of only of linearizable singularities or it consists only of Poincaré-Dulac type singularities.

Proof. Indeed, the first remark is that by the Index theorem $\text{sing}(F) \cap \Lambda \neq \emptyset$. Given then a singularity $p \in \Lambda \cap \text{sing}(F)$, the local holonomy map $f \in \text{Diff}(\mathbb{C},0)$ of a local separatrix contained in $\Lambda$ gives an element in the holonomy group $\text{Hol}(\Lambda)$ of the leaf $\Lambda \setminus \text{sing}(F)$ which is either an analytically linearizable non-periodic map of the form $z \mapsto e^{2\pi \sqrt{-1}} \lambda/\mu$ with $\lambda/\mu \not\in \mathbb{Q}$, or it is
tangent to the identity and analytically conjugated to a map of the form \( z \mapsto \frac{z}{(z^n + 2k\sqrt{-1})^{1/n}} \). By Nakai’s Density theorem the holonomy group \( \text{Hol}(\Lambda) \) of the leaf \( \Lambda \setminus \text{sing}(\mathcal{F}) \) is solvable, maybe abelian. An abelian subgroup of \( \text{Diff}(\mathbb{C},0) \) which contains a non-periodic linearizable map is analytically linearizable. This implies that the group \( \text{Hol}(\Lambda) \) is either abelian analytically linearizable, abelian non-linearizable (without linearizable maps) or solvable non-abelian and analytically conjugated to a subgroup of a group \( \mathbb{H}_k = \{ (z \mapsto \frac{z}{(kz^n + 1)^{1/n}}) \} \) for some \( k \in \mathbb{N} \) (\[2\]). If the group \( \text{Hol}(\Lambda) \) contains a linearizable non-periodic map and a map tangent to the identity. In the first case, the group is necessarily abelian (because a non-trivial commutator is tangent to the identity) and therefore abelian linearizable. In this first case the singularities in \( \text{sing}(\mathcal{F}) \cap \Lambda \) are all linearizable. In the second case, all singularities in \( \text{sing}(\mathcal{F}) \cap \Lambda \) are of Poincaré-Dulac type. On the other hand in a solvable subgroup of \( \text{Diff}(\mathbb{C},0) \) (maybe abelian) the subgroup of elements tangent to the identity is abelian. Thus we conclude that the group \( \text{Hol}(\Lambda) \) is always abelian.

Thus we have two possibilities for the holonomy group \( \text{Hol}(\Lambda) \) and the singularities of \( \mathcal{F} \) in \( \Lambda \). Either \( \text{Hol}(\Lambda) \) is analytically linearizable and all singularities of \( \mathcal{F} \) in \( \Lambda \) are analytically linearizable or \( \text{Hol}(\Lambda) \) is abelian non-linearizable and all singularities of \( \mathcal{F} \) in \( \Lambda \) are of Poincaré-Dulac type. In the abelian linearizable case there is a closed meromorphic 1-form \( \Omega \) with simple poles defining \( \mathcal{F} \) in a neighborhood of \( \Lambda \) and by Levi’s Extension theorem \( \Omega \) extends to a closed rational 1-form of \( \mathcal{F} \) (see \[4\], \[11\]). This extension has simple poles and can be written in a logarithmic form as \( \left. \Omega \right|_{\mathbb{C}^2} = \sum_{j=1}^{r} \alpha_j \frac{df_j}{f_j} \) for some irreducible polynomials \( f_j \) and some complex numbers \( \alpha_j \in \mathbb{C} \). In this case the foliation \( \mathcal{F} \) is a logarithmic or Darboux type foliation. Assume now that the holonomy \( \text{Hol}(\Lambda) \) is abelian non-linearizable. In this case, we can once again construct a closed meromorphic 1-form \( \omega \) in a neighborhood of \( \Lambda \) on \( \mathbb{C}P(2) \) (\[13\]). This form is obtained as follows: Fix a point \( q \in \Lambda \setminus \text{sing}(\mathcal{F}) \) and choose a transverse disc \( \Sigma \) to \( \mathcal{F} \) at \( q = \Sigma \cap \Lambda \) and a local coordinate \( z \in \Sigma \). The holonomy group \( \text{Hol}(\Lambda) \) then corresponds to a subgroup \( \text{Hol}(\mathcal{F},\Lambda,\Sigma) \subset \text{Diff}(\mathbb{C},0) \) and the fact that this group is abelian gives a germ of holomorphic vector field \( \xi(z) \) with a singularity at the origin \( 0 \in \Sigma \) which is invariant by the group \( \text{Hol}(\mathcal{F},\Lambda,\Sigma) \), i.e., \( g_*(\xi) = \xi, \forall g \in \text{Hol}(\mathcal{F},\Lambda,\Sigma) \). Let \( \omega_0(z) \) be the 1-form dual to \( \xi(z) \) in the sense that \( \omega_0(\xi) = 1 \). Then \( \omega_0 \) is also invariant by the holonomy \( \text{Hol}(\mathcal{F},\Lambda,\Sigma) \). This invariance allows us to extend \( \omega_0 \) by holonomy into a closed 1-form \( \omega_1 \) in a neighborhood of \( \Lambda \setminus \text{sing}(\mathcal{F}) \). It remains to show that \( \omega_1 \) extends to a closed meromorphic 1-form \( \omega \) which defines \( \mathcal{F} \) in a neighborhood of each singularity \( p \in \text{sing}(\mathcal{F}) \cap \Lambda \). This is done as in \[13\] as a consequence of the Poincaré-Dulac normal form of these singularities (recall the fact that the singularities of \( \mathcal{F} \) in \( \Lambda \) are all Poincaré-Dulac of same type). Again by Levi’s Extension theorem the 1-form \( \omega \) extends to a closed rational 1-form on \( \mathbb{C}P(2) \). This ends the proof of the proposition.

In the proof of Theorem \[2\] the following lemma will be useful:

**Lemma 5.** Let \( \mathcal{F} \) be a holomorphic foliation given by a closed rational 1-form \( \Omega \) on \( \mathbb{C}P(2) \). Assume that the singularities of \( \mathcal{F} \) are non-dicritical in the Poincaré domain. Then either \( \mathcal{F} \) is a logarithmic foliation or it is a rational pull-back of a Poincaré-Dulac normal form.

**Proof.** The main point is that, as we have observed above, a non-dicritical singularity in the Poincaré domain is linearizable if and only if it exhibits more than one local separatrix. By the
Integration Lemma (11), in an affine system of coordinates, \( \Omega \) can be written as
\[
\Omega|_{\mathbb{C}^2} = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d\left( \frac{g}{\prod_{j=1}^{r} f_j^{n_j-1}} \right)
\]
form some irreducible polynomials \( f_j \), some \( n_j \in \mathbb{N} \) and some complex numbers \( \lambda_j \). In the non-logarithmic case we have some \( n_j \geq 2 \). Say for instance \( n_1 \geq 1 \). Suppose that \( \lambda_1 \neq 0 \). We claim that \( \lambda_j = 0 \) for all \( j \geq 2 \). Indeed, if for instance \( \lambda_2 \neq 0 \) then an intersection point \( q \in \{ f_1 = f_2 = 0 \} \) will exhibit two local separatrices and therefore must linearizable, comparing this with the local form of \( \Omega \) at such a point we get a contradiction. The same argumentation shows that \( n_j = 1 \) for all \( j \geq 2 \). Thus we conclude that \( \Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} \) which is clearly a rational pull-back of a Poincaré-Dulac normal form. Suppose now that \( \lambda_1 = 0 \). If there is some \( \lambda_j \neq 0 \) we can assume that \( \lambda_2 \neq 0 \). In this case arguments as above show that \( \lambda_j = 0 \) for all \( j \neq 2 \) and we have \( \Omega = \lambda_2 \frac{df_2}{f_2} + d\left( \frac{g}{\prod_{j=1}^{r} f_j^{n_j-1}} \right) \). Once again we use the number of separatrices to show that we have a contradiction with the fact that \( n_1 \geq 2 \). Therefore the only possibility is that \( \mathcal{F} \) is a rational pull-back of a Poincaré-Dulac normal form.

**Proof of Theorem 2.** Let \( L \) be a nonsingular transcendent orbit of \( X \) with finite total curvature. By Lemma 1 we have two possibilities:

**Case 1.** \( L \) is closed in \( \mathbb{C}^2 \setminus \text{sing}(\mathcal{F}) \). In this case, since \( L \) is not algebraic, \( L_\infty \) is invariant by \( \mathcal{F} \). Moreover, given a small transverse disc \( \Sigma \) to \( L_\infty \) at a point \( q \in L_\infty \setminus \text{sing}(\mathcal{F}) \), \( L \) induces in \( \Sigma \) an orbit which is discrete outside the origin \( q = \Sigma \cap L_\infty \). According to Proposition 3 \( \mathcal{F}(X) \) is given by a closed rational 1-form \( \Omega \) on \( \mathbb{C}P^2 \). Using now Lemma 5 we conclude that \( \mathcal{F} \) is as stated.

**Case 2.** \( L \) is not closed in \( \mathbb{C}^2 \) and \( L \) accumulates some invariant line \( E \subset \mathbb{C}^2 \). The same argumentation of the first case can be applied to \( E \) in place of \( L_\infty \) to show that \( \mathcal{F} \) must be a rational pull-back of a Poincaré-Dulac normal form. This ends the proof of Theorem 2.

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