Quantum Data Compression of a Qubit Ensemble

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Data compression is a ubiquitous aspect of modern information technology, and the advent of quantum information raises the question of what types of compression are feasible for quantum data, where it is especially relevant given the extreme difficulty involved in creating reliable quantum memories. We present a protocol in which an ensemble of quantum bits (qubits) can in principle be perfectly compressed into exponentially fewer qubits. We then experimentally implement our algorithm, compressing three photonic qubits into two. This protocol sheds light on the subtle differences between quantum and classical information. Furthermore, since data compression stores all of the available information about the quantum state in fewer physical qubits, it could provide a vast reduction in the amount of quantum memory required to store a quantum ensemble, making even today’s limited quantum memories far more powerful than previously recognized.

The amount of information that can be extracted from a classical system is precisely the same as the amount of information required for a complete description of the system’s state. The same is not true quantum mechanically; to fully describe the state of a single quantum bit (qubit) would require an infinite amount of information, although no more than one (classical) bit of information can ever be extracted from a measurement of its quantum state. Such fundamental differences between quantum and classical mechanics open up the possibility of new kinds of data compression with no classical analogue. In quantum mechanics an ensemble of identically prepared quantum systems provides much more information than a single copy – this is not the case classically, where the information encoded in a single system’s state can be accessed repeatedly. Although quantum mechanically we cannot compress all of the information contained in an ensemble of systems down to a single quantum copy, we can achieve an exponential savings. In this paper, we show how this exponential savings can be achieved using the quantum Schur-Weyl transform [1,2], which can compress an ensemble of N identically prepared qubits into a memory of size log2(N + 1) qubits. We show how the protocol can be made practical in an optical setting, experimentally implementing a three-qubit quantum circuit to compress a three-qubit ensemble into the state of two qubits. To characterize this circuit, we show that we can perform measurements on the two compressed qubits, and still extract as much information as we would have been able to given all three original qubits. Given our ability to extract information about measurements in multiple bases, we can conclude that the compressed state faithfully encodes the “quantum information content” of the original ensemble. Our results demonstrate that quantum memories can be used to store exponentially more information about a quantum state than would normally be expected for the number of physical qubits that the memory can hold.

From the point of view of estimation theory, a quantum state is never fully knowable, just as a classical probability distribution is not (both requiring an infinite amount of resources to be completely known). Hence, for our purposes, a quantum state is best thought of as a mathematical object which allows one to make testable predictions about the statistics of potential measurements done on a large ensemble of identically prepared systems[3]. The task colloquially referred to as “quantum state estimation” is really the task of making possible predictions about the expectation values for observables which might be measured in the future. Consider for instance estimating the spin projection of a qubit along a particular direction, given a fixed number of identically prepared copies of the qubit. To do this the best strategy is quite simply to measure the the spin along the direction of interest on each copy and draw conclusions as one would do classically. Since quantum measurements are intrinsically uncertain and the state of each qubit collapses after measurement, having more copies allows one to make a better estimate. If one does not know in advance which measurement will be of interest, the standard approach – known as quantum state tomography – is to reconstruct a density matrix [4], which contains enough information to allow one to estimate the expectation value of the spin along any direction. This approach has the disadvantage that no single estimate can ever make optimal use of all of the available information [4]. For instance, in single-qubit quantum state tomography one most commonly splits an initial ensemble of identical qubits into three equally sized groups, and measures X on all the members of one group, Y on another, and Z on the remaining group. But if, for example, one later wishes to estimate ⟨Z⟩ (the expectation value of the spin along Z), the measurements of X and Y (both of which are orthogonal to Z) give no useful information, and 2/3 of the measure-
angular momentum of the multi-qubit state, and is the only information relevant to estimating expectation values of single-qubit observables. Thus it is natural to ask if the initial $N$-qubit ensemble can be mapped reversibly (unitarily) onto exponentially fewer ($\log_2(N+1)$) qubits. In fact, this mapping of the computational basis into a new basis, separating the permutation from the angular momentum information, is well understood as the quantum Schur-Weyl transform (QSWT) \[1\], and has been theoretically proposed for use in a variety of different applications \[2, 6, 10\]. In this paper we develop a practical scheme for implementing the QSWT, and experimentally demonstrate it with photonic qubits, compressing a three-qubit ensemble into two qubits. (The compression of a quantum ensemble is very different from, and should not be confused with, quantum source-coding \[11, 12\].)

A three-qubit QSWT will compress an ensemble of three qubits into two ($\log_2(3+1)$) qubits, so that one qubit can be discarded without information loss. A quantum circuit implementing the three-qubit QSWT is shown in figure 1a. In this circuit, the two single-qubit unitaries, $\hat{U}_1$ and $\hat{U}_2$, are defined so that $\hat{U}_1 \left( \frac{\sqrt{2}}{2} |0\rangle + \frac{\sqrt{2}}{2} |1\rangle \right) = |0\rangle$, $\hat{U}_2 \left( \frac{\sqrt{3}}{\sqrt{2}} |0\rangle + \frac{\sqrt{3}}{\sqrt{2}} |1\rangle \right) = |0\rangle$ and $\hat{U}_2 \hat{U}_1 = \hat{X}$. It is straightforward to show that if the three input qubits are prepared in $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ the output will be transformed into $|\phi\rangle_{1,2} |0\rangle_3$, where

$$|\phi\rangle_{1,2} = \alpha^3 |00\rangle + \sqrt{3} \alpha^2 \beta |01\rangle + \sqrt{3} \alpha^2 |10\rangle + \beta^3 |11\rangle.$$  

Since the third qubit is always in $|0\rangle$ this circuit unitarily maps all of the information onto the first two qubits. (Such circuits can be efficiently made for any value of $N$, requiring one to keep only $\log_2(N+1)$ qubits \[1, 2\].) In the case of identical pure-state qubits the final two disentangling gates (of the circuit in figure 1a) can be implemented using measurement and feed-forward, as shown in figure 1b \[13\]. Now qubit 3 is measured and an operation is performed on the first two qubits which depends on this result. This simplification produces $|\phi\rangle_{1,2}$, and thus performs as well as the full QSWT (see the Supplemental Material for a full derivation of this).

To understand why the compression of an ensemble of three identical qubits into two does not lose information, consider how one would estimate $\langle \hat{Z} \rangle$ (which we will refer to as $Z_{\text{true}}$, the “true value” of this expectation value) with and without quantum data compression. In short, without compression each qubit is measured in the same basis and an estimate is calculated from a tally of the number of spin-up measurement results, which can be 3, 2, 1, or 0, corresponding to maximum-likelihood estimates of
To demonstrate this protocol experimentally, we use three qubits, which we encode in the path and polarization degrees-of-freedom of two photons [14–15]. Such hybrid quantum systems, using multiple degrees-of-freedom of photons, have proven very useful for demonstrating quantum protocols [16–18], testing fundamental issues in quantum mechanics, [19–20] and simplifying quantum logic gates [21–22]. In the circuit of figure 1b, qubit 1 is encoded in the polarization of photon 1, qubit 2 is encoded in an additional path degree-of-freedom of the same photon, and qubit 3 is encoded in the polarization of a second photon. After the circuit is completed, the information of all three qubits is stored in the first two logical qubits, both encoded in photon 1, allowing us to discard the second photon entirely. A sketch of our optical implementation is shown in figure 2, and an in-depth explanation of how it implements the quantum circuit of figure 1b is presented in the Supplemental Material. The two compressed qubits are encoded in the path and polarization of photon 1; to perform the post-selective disentanglement, measurements of these two qubits are post-selected on a measurement of photon 2. This corresponds to a coincidence event between a measurement on photon 2 signalling $|H+IV\rangle/\sqrt{2}$ and any of the four detectors for photon 1. There are four detectors because there are two possible path outcomes and two possible polarization outcomes. These coincidence events correspond to four different estimates of $Z_{true}$: $HP_0 = |00\rangle \Rightarrow Z_{comp} = +1/2$, $HP_1 = |01\rangle \Rightarrow Z_{comp} = +1/6$, $VP_0 = |10\rangle \Rightarrow Z_{comp} = -1/6$, or $VP_1 = |11\rangle \Rightarrow Z_{comp} = -1/2$.

To test the performance of our circuit, the compressed system was measured and a number of representative single-qubit observables were estimated. For each measurement, the two qubits were found in one of four states, corresponding to expectation-value estimates of $+1/2$, $+1/6$, $-1/6$, or $-1/2$. Since a single measurement does not yield information about the statistical performance of our circuit, we ran the circuit many times for the same input state and final measurement. The number of runs was typically $M \approx 500$. For each run, $S$ (either $X$, $Y$, or $Z$) was measured on the output and the spin expectation value was estimated as $S_{comp} = (2S_1 + S_2)/3$, then the average of $S_{comp}$ over runs was calculated. This entire process formed a single trial, and was repeated about 250 times. The resulting distributions of the averages of $S_{comp}$ are plotted in figure 3a-c for $S = X$, $Y$, and $Z$ with the initial single-qubit state $cos(\theta)|0\rangle + sin(\theta)|1\rangle$ and $\theta = 13.5^\circ$. If each of the $M$ measurements encodes the information of three qubits (as we expect) the distribution should have a variance given by the single-qubit variance $(V_1 = cos^2(2\theta)sin^2(2\theta))$ divided by the total number of qubits sampled: $3M$, three times the number of runs in each trial. This prediction is shown in blue on figure 3a-c. On the other hand, a measurement made on two independent qubits would exhibit a variance of $V_1/(2M)$, 1.5 times larger; this distribution is shown in red for comparison. The narrower blue curve, describ-
ing the behaviour of three qubits, is a much better fit to our observed data than the red curve, indicating that the amount of information extractable from the two compressed qubits is close to the full information present in the three original qubits.

To further quantify the performance of our compression circuit, we measure the 'single-shot' distributions of $X_{\text{comp}}$, $Y_{\text{comp}}$, and $Z_{\text{comp}}$. To do this we again prepare each of the three input qubits in $\cos(2\theta)|0\rangle + \sin(2\theta)|1\rangle$, run our circuit, measure one of the observables $X$, $Y$, or $Z$ (each measurement results in an estimate of $+1/2$, $+1/6$, $-1/6$, or $-1/2$) and bin the results. For each observable the circuit was run approximately $10^5$ times. The resulting normalized distributions are plotted in figure 3d-f for $\theta = 13.5^\circ$. We observe very good agreement between our experimental data (dark bars) and theory (larger light bars). Next, we vary the input states, preparing a range of $\theta$ values, and measure the variance of the resulting single-shot distributions of $X_{\text{comp}}$, $Y_{\text{comp}}$, and $Z_{\text{comp}}$ for each input state. These experimentally-measured variances are the circles, plotted versus $\theta$, in figure 4a-c. The curves in figure 4a-c are theory corresponding to the variance of two independent qubits $V_1/2$ (red dashed curve) and three independent qubits $V_1/3$ (blue solid curve). For all but two data points in the $X$ measurement (discussed in the Supplemental Material), our experimental data agree very well with the three-qubit variance. In addition to these three observables, one would ideally quantify the variance averaged over all possible measurements. Such a measurement would indicate how much information could be extracted about arbitrary measurements. Conveniently, for a given state, this average measurement variance is the same as simply averaging the variances of $X$, $Y$ and $Z$. (That is to say that the uniformly distributed discrete subensemble $\mathcal{S}(X, Y, Z)$ is an averaging set for the SO(3) uniformly (Haar) distributed superensemble $\{S(\theta, \phi)\}$ for variance $\text{Var}$. We derive this in the Supplemental Material). The resulting averaged variance, for a given state, is plotted in figure 4d. This clearly demonstrates that our circuit compresses three qubits into two, and we can conclude that all of the compressed states we tested faithfully encode the information about any single-qubit measurement.

So far we have imagined that, if presented with three qubits and a two-qubit quantum memory, our strategy in the absence of a compression circuit would be to store two of the qubits and discard the third. This measurement scheme has a variance $1.5$ times larger than we obtain with compression (red curve in figure 4). A better approach would be to measure the third qubit before discarding it. The classical bit obtained would provide extra information and could be combined with the subsequent measurement of the two stored qubits in the correct basis, yielding an improved estimate of the single-qubit spin. Any compression algorithm should be compared to such a strategy, in order to quantify the performance given

![Figure 3](image3.png)

**FIG. 3. Sample raw data for an input state**

$\cos(2\theta)|0\rangle + \sin(2\theta)|1\rangle$, for $\theta = 13.5^\circ$ — a)-c) Histograms of estimates the spin along $Z$, $Y$ and $X$, after $M$ trials (defined in the text) of the data compression circuit. The bars are experimentally measured data, and the blue (red) curve is fitted in the text) of the data compression circuit. The bars $V$ of estimates the spin along $\hat{Z}$, $\hat{Y}$ and $\hat{X}$, after $M$ trials (defined in the text).

![Figure 4](image4.png)

**FIG. 4. Measurement Variances for various input states** — The solid blue lines are the theoretical variances resulting from performing a measurement on three independent qubits (and thus our two compressed qubits), the dashed red lines are for two independent qubits, the grey dashed lines are the theoretical variance when two independent qubits are measured optimally and a random measurement is performed on a third qubit, and the circles are the variances which we observe when experimentally measuring the two compressed qubits. a)-c) By sending in various different input states, parametrized by $\theta$ as $\cos(2\theta)|0\rangle + \sin(2\theta)|1\rangle$, we see that the two compressed qubits demonstrate the statistics of three independent qubits for $Z$, $Y$, and $X$ measurements. d) Averaging the variances of $Z$, $Y$, and $X$ yields the variance averaged over all possible measurements.
a limited amount of quantum memory, without placing unreasonable constraints on the classical memory. We analyze this protocol in the Supplemental Material; the result is the dotted grey curve in figure 4. Our compression scheme outperforms even this improved protocol.

Finally, it is worth mentioning that our techniques could be useful beyond compressing sets of identical input states. For instance, one could also exponentially compress any permutationally invariant pure state. Permutationally invariant states include several entangled states which have been shown to be invaluable for quantum communication and quantum computing [2–10, 21], and for some applications our feed-forward simplification performs optimally. Given the exponential reduction in the size of the required quantum memory, and the many applications of the QSWT, circuits such as the one we have demonstrated hold great promise for both future quantum computing and quantum communication architectures.

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SUPPLEMENTAL MATERIAL

I. ADDITIONAL EXPERIMENTAL DETAILS

A. State Preparation

In our implementation, qubit 1 is encoded in the polarization of photon 1, qubit 2 in the path of the same photon, and qubit 3 in the polarization of a second photon (figure 2a of the main text). We generate photon pairs using a type-I spontaneous parametric down-conversion (SPDC) source in a “sandwich-configuration” [30] (each crystal is 1mm of BBO, and they are pumped by 500nm of 404nm light, generated by frequency-doubling 808nm light from a femtosecond Ti:Sapphire laser, using a 2nm-long BBO crystal). Our source creates photons in the entangled state $\alpha|HH\rangle_{1,2} + \beta|VV\rangle_{1,2}$ with a fidelity of $\approx 94\%$, measured with standard two-photon polarization tomography. The amplitudes $\alpha$ and $\beta$ are controlled via the pump polarization.

This polarization entanglement is converted into entanglement between the polarization of photon 2 and an “auxiliary” path degree of freedom of photon 1, by passing photon 1 through a polarization beamsplitter (PBS) followed by a half-waveplate (HWP) at 45° in the reflected port. After this, the state of the system is $|H\rangle_{1}(\alpha|a_0\rangle_{1}|H\rangle_{2} + \beta|a_1\rangle_{1}|V\rangle_{2})$, where $|a_1\rangle$ ($|a_0\rangle$) refers to photon 1 being in the auxiliary path (or not). Qubit 3 is the polarization state of this second photon (whose state, defined by $\alpha$ and $\beta$, is set by the pump polarization), and it is entangled with the auxiliary mode of photon 1. This entanglement is later used to implement quantum-logic gates between qubits encoded in photon 1 and photon 2. Since photon 1 is now horizontally polarized, the state of qubit 2 (the path of photon 1) can be set by setting the polarization of photon 1 and converting it to a path qubit with a PBS and a HWP. Finally, the state of qubit 1 (the polarization of photon 1) can be set. This entire procedure results in the state

$$\begin{align*}
(\alpha|H\rangle_{1} + \beta|V\rangle_{1}) \otimes (\alpha|p_0\rangle_{1} + \beta|p_1\rangle_{1}) \\
\otimes (\alpha|a_0\rangle_{1}|H\rangle_{2} + \beta|a_1\rangle_{1}|V\rangle_{2}),
\end{align*}$$

where $|p_0\rangle$ and $|p_1\rangle$ refer to the state of the path qubit encoded in photon 1.

B. Logic gates

The quantum circuit that we implement (figure 1b of the main text) can be broken into three parts: the two-qubit QSWT (box 1), the controlled gates between qubit three and the first two qubits (box 2), and the post-selective disentangling operation (box 3). With our encoding, the gates labelled “two-qubit QSWT” can be performed deterministically using linear optics. The controlled-not (CNOT), with the polarization qubit as the control and the path qubit as the target, is implemented by using a PBS to swap the path modes only when the photon is vertically polarized (which we define as $|1\rangle$ for the polarization qubits); the controlled-Hadamard, with path as control and polarization as target, is implemented by using a half-waveplate at 22.5° to rotate the polarization only if the photon is in path 1 (defined as the $|1\rangle$ state of the path qubit). These optical elements are shown in shaded area 1 of figure 2 in the main text.

The next two gates (box 2) are experimentally more challenging, requiring photon 2’s polarization to modify the path and polarization of photon 1. As before, the “uncontrolled” implementations of these gates can be constructed between the path and polarization qubits encoded in photon 1 by using linear optics: the NOT gate on the path qubit is implemented by swapping the path modes, and the CNOT gate, with path qubit as the control and the polarization qubit as the target, is achieved by placing a HWP at 45° only in path 1. To be clear, the CNOT gate that we are referring to at this point is an “uncontrolled implementation” of the three-qubit Toffoli gate. The challenge comes in conditioning them on the polarization state of photon 2 (qubit 3). Zhou et al. showed that this can be conveniently achieved by using “controlled-path” gates [22] to “take a shortcut through a higher dimension” [21].

A controlled-path gate places qubit 2 in an auxiliary mode dependent on the state of qubit 1. It is essential that the controlled-path gate place qubit 2 in the auxiliary mode coherently, so that if qubit 1 is in a superposition of $|0\rangle$ and $|1\rangle$ entanglement will be generated between the the state of qubit 1 and the mode of qubit 2. If this is the case, then placing some gate $\hat{A}$ in the auxiliary mode and recombining the auxiliary mode with the original mode using another controlled-path gate will create entanglement between the state of qubit 1 and $\hat{A}$ either being applied to qubit 2 or not. In other words, this process has implemented a controlled-$\hat{A}$ gate with qubit 1 as the control and qubit 2 as the target (see figure 5a of the Supplemental Material).

Since controlled-path gates could be just as challenging to implement as a CNOT gate, Zhou et al. went on to show that this scheme could be simplified by using prior entanglement between the first qubit and the auxiliary mode of the second qubit. Replacing a controlled-path gate with prior entanglement is possible because the crucial effect of the controlled-path gate is the entanglement that it generates between the state of qubit 1 and the mode of qubit 2. This is an extremely useful technique for photonic logic gates since it is often easier to generate entanglement from a photon source than it is to generate entanglement between independent photons.

To understand how this works in practice, consider implementing an entanglement-driven controlled-$\hat{A}$ gate between two polarization qubits (figure 5b of the Supplemental Material). The polarization of photon 1 will be the control qubit and the polarization of photon 2
In our experiment, since the polarization of our photons is entangled, we can use PBS1 in figure 2a (of the main text) to create an auxiliary set of paths for photon 1, so optics placed only in these modes (shaded area 2 in figure 2) are effectively controlled by the polarization of photon 2.

The final step of our data compression circuit is measurement and feed-forward to disentangle qubit 3 (illustrated in shaded area 3). In Section III, we show that this is possible by measuring qubit 3 and applying a unitary on the first two qubits based on the result. Experimentally, this is accomplished by measuring the polarization of photon 2 in the circular basis, and applying birefringent phases on photon 1 with liquid-crystal waveplates (LCWPs) set to 0°. Ideally, the LCWP retardances are switched dependent on the measurement outcome of photon 2. Since our LCWPs are not fast enough we set them to correct the phases only when photon 2 is projected onto (|H⟩ + i|V⟩)/√2 and discard the other case. With faster feed-forward (using Pockels cells, for example), both cases could be corrected [31].

After the compression, we are left with a single photon encoding the two compressed qubits. The path and polarization of this photon are measured (figure 2c-d of the main text) in coincidence with the polarization measurement on photon 2. After post-selection of photon 2 in the state |H + iV⟩/√2, approximately 1000 events per second are observed.

C. Performance

The net result of our implementation is a series of four nested interferometers when measuring $\hat{Z}$, and five when measuring $\hat{X}$ or $\hat{Y}$. The phase of each interferometer was measured to be stable for at least five minutes (drifting less than 1%) so that, with our detection rates, we could collect sufficient data in one minute without significant phase drift occurring. The first four interferometers had visibilities > 98.5%, while that of the fifth was 97.4%. This can be interpreted as an error in the measurement basis setting. This is because rather than measuring $|+\rangle\langle+x|$, we measured $(1-p)|+\rangle\langle+x| + p|-\rangle\langle-x|$, where $p$ is the leakage into the interferometer’s dark port. We measured $p = 0.015$ and used this to simulate the effect on the variance (thin blue curve in figure 3c of the main text), which describes our experimental data well. This led to the deviation of variance of $\hat{X}$ as $\theta$ approached 22.5°. The $\hat{X}$-measurement was the most sensitive to this error because to measure a variance of zero (as predicted by theory), all of the photons had to have exited the final bright port, and even a small amount of leakage into the “dark-port” would increase the variance. For the states that we used, the $\hat{Y}$-measurement should never have resulted in a dark port, and was thus not sensitive to this error, while the $\hat{Z}$-measurement was made without this final interferometer in place.
II. MEASURING THE COMPRESSED SYSTEM

After the initial three-qubit state is compressed into two qubits the information needs to be read out. Since the compression is unitary, one could run an inverse QSWT with an ancillary qubit initialized in |0⟩. This would recreate the initial three qubit state. This is unnecessary, aside from being experimentally challenging. We instead show that the desired observables may be measured directly on the compressed qubits. As discussed in the text, measuring ˆZ on each of the two compressed qubits and interpreting the result as a 2-bit number yields the same information as measuring ˆZ on the input three qubits, the 2-bit number being equivalent to a tally of how many of the three qubits were found in |0⟩. As we will see below, other spin measurements are necessarily non-local, but still feasible to implement on the compressed qubits.

It is possible to map the basis states of the output qubit pair onto the four basis states of an effective spin-3/2 particle, and then measuring ˆZ on each of the two compressed qubits. Just as with any spin system, this can be accomplished by changing the measurement basis required for the basis change using only linear optics, in a manner similar to the implementation of the path/polarization logic gates discussed in the Methods section. We implement this by placing different waveplates in the two paths of the photon, and combining the two paths at a 50:50 beamsplitter. The basis change required for a ˆY measurement can be implemented by simply changing the waveplate settings.

III. MEASUREMENT AND FEED-FORWARD

The full quantum circuit of figure 1a of the main text is not necessary if the input qubits are guaranteed to be identically prepared. In this case the final two gates can be replaced by measurement and feed-forward (figure 1b of the text). To understand this, consider the state before the disentangling gates (for input qubits in state α|0⟩ + β|1⟩):

\[ α^3|00⟩ + \sqrt{3}α^2β|01⟩ + \sqrt{3}αβ^2|10⟩ + β^3|11⟩, \]

where |φ⟩ = √{(|0⟩ + i|1⟩)/√8}, the first two qubits will be left in a mixed state, and information is lost. If instead of being discarded, qubit 3 is measured in the basis (|0⟩ ± i|1⟩)/√2, the first two qubits are left in one of two possible states with equal probability, dependent on this outcome. If qubit 3 is found in (|0⟩ + |1⟩)/√2 the first two qubits will collapse into the state

\[ α^3|00⟩ + e^{-ia}√3α^2β|01⟩ - e^{ia}√3αβ^2|10⟩ + \beta^3|11⟩, \]

whereas if qubit 3 is found in (|0⟩ − i|1⟩)/√2 the first two qubits will be left in

\[ α^3|00⟩ + e^{ia}√3α^2β|01⟩ - e^{-ia}√3αβ^2|10⟩ - iβ^3|11⟩, \]

in the path and polarization degrees of freedom of a single photon, so we can deterministically implement the entangling gates required for the basis change using only linear optics, in a manner similar to the implementation of the path/polarization logic gates discussed in the Methods section. We implement this by placing different waveplates in the two paths of the photon, and combining the two paths at a 50:50 beamsplitter. The basis change required for a ˆY measurement can be implemented by simply changing the waveplate settings.
IV. MAXIMUM-LIKELIHOOD ESTIMATION WITHOUT DATA COMPRESSION

In this section we will consider the scenario wherein one initially has three qubits, no compression circuit, and a two-qubit quantum memory. In this case, rather than discarding the third qubit, one could measure it in some basis, and the classical outcome stored along with the other two qubits, which would be measured optimally later when measurement axis is known. It is easiest to explain our analysis of this scenario in terms of a game between Alice and Bob.

Imagine that Alice prepares three qubits and gives them to Bob. The state that Alice prepares is unknown to Bob and known by Alice. Sometime later, Alice is going to ask Bob to predict the value of a spin measurement along a random direction. If Bob can only store two qubits, his best option is to perform our data compression algorithm and store all of the quantum information in memory. If he is not able to perform data compression, he could still gain some information by measuring one qubit before he discards it. Given this additional classical bit of information (the outcome of his spin measurement), he must come up with an estimate of the spin about some other axis that Alice is going to tell him. The procedure we imagine Bob following is to measure the first qubit randomly (since he has no directional information with which to make his choice), then when Alice tells him what axis she is interested in, Bob will measure the remaining two qubits along her axis. From these three measurement results and the knowledge of his single qubit measurement direction, he will construct and maximize a likelihood function.

To compare to our experiment, imagine that Alice prepares three qubits in |ψ⟩ = cos θ|0⟩ + sin θ|1⟩, and then asks Bob to estimate the spin along Z. (We do not lose generality by considering only states with a relative phase of zero because we can simply consider different measurements; i.e., preparing states with zero phase and measuring X will behave the same as preparing states with a phase of π/2 and measuring Y.) We use the convention that the eigenvalue associated with |0⟩ is +1/2, and that with |1⟩ is −1/2. In the state |ψ⟩, the expectation value of Z is ⟨Z⟩ = 1/2 cos 2θ, which we will refer to as ⟨Z⟩’s ‘true value’, Ztrue. Bob’s goal is to estimate Ztrue. For clarity, we rewrite |ψ⟩ in terms of Ztrue as |ψ⟩ = √1/2 + Ztrue|0⟩ + √1/2 − Ztrue|1⟩. Bob’s protocol is now to measure the first qubit’s spin along a Haar-randomly chosen axis, parameterized as 2δξ = cos δ sin ϵX + sin δ sin ϵY + cos δ Z. ̂Sδ,ξ has spin-up and spin-down eigenstates |Sδ,ξ = 0⟩ = cos δ/2|0⟩ + e−iθ sin δ/2|1⟩ and |Sδ,ξ = 1⟩ = sin δ/2|0⟩ − e−iθ cos δ/2|1⟩, respectively (again with eigenvalues of ±1/2). One can show that, given a state with Ztrue, Bob will find his qubit to be spin-up along ̂Sδ,ξ (i.e. |ψ⟩ will be projected onto |Sδ,ξ = 0⟩) with probability:

\[
P(\hat{S}_{δ,ξ} = 0|Z_{true}) = |\langle ψ|\hat{S}_{δ,ξ} = 0⟩|^2 = \frac{1}{2} + Z_{true} \cos δ + \frac{1}{2} \sqrt{1 - 4Z_{true}^2 \sin δ \cos θ},
\]

and similarly he will find it in |Sδ,ξ = 1⟩ with probability

\[
P(\hat{S}_{δ,ξ} = 1|Z_{true}) = \frac{1}{2} - Z_{true} \cos δ - \frac{1}{2} \sqrt{1 - 4Z_{true}^2 \sin δ \cos θ}.
\]

These two equations would typically be interpreted as the probability of getting a spin-up (\hat{S}_{δ,ξ} = 0) or a spin-down (\hat{S}_{δ,ξ} = 1) outcome given a state with a specific value of Ztrue. However, since Bob’s task is to conclude things about Ztrue given an experimental outcome, here we will view equations 7 and 8 as the likelihood that the state had Ztrue, given that a specific outcome (\hat{S}_{δ,ξ} = 0 or \hat{S}_{δ,ξ} = 1) was observed. In other words, they are “likelihood functions” for Ztrue given an experimental outcome: L(Ztrue|\hat{S}_{δ,ξ} = 0) = P(\hat{S}_{δ,ξ} = 0|Ztrue) and L(Ztrue|\hat{S}_{δ,ξ} = 1) = P(\hat{S}_{δ,ξ} = 1|Ztrue).

After Bob randomly measures one qubit, Alice will tell him to estimate the spin along Z. Thus his next step is to measure Z on the remaining two qubits. Doing this will yield three possible results: he will either find one qubit spin-down (in |0⟩) and one spin-up (in |1⟩), both spin-up (in |0⟩) or both spin-down (in |1⟩). These outcomes will occur with probabilities:

\[
P(\hat{Z} = 0, 0|Z_{true}) = \cos^4 2θ = \left(\frac{1}{2} + Z_{true}\right)^2,
\]

\[
P(\hat{Z} = 0, 1|Z_{true}) = 2 \cos^2 2θ \sin^2 2θ = \left(\frac{1}{2} - 2Z_{true}\right)^2,
\]

\[
P(\hat{Z} = 1, 1|Z_{true}) = \sin^4 2θ = \left(\frac{1}{2} - Z_{true}\right)^2,
\]

respectively. He can construct likelihood functions for Ztrue, given each of these outcomes as before: L(Ztrue|\hat{Z} = 0, 0) = P(\hat{Z} = 0, 0|Ztrue), L(Ztrue|\hat{Z} = 0, 1) = P(\hat{Z} = 0, 1|Ztrue), and L(Ztrue|\hat{Z} = 1, 1) = P(\hat{Z} = 1, 1|Ztrue).

Each time Alice and Bob play this game, it will result in one of six sets of measurement outcomes for Bob: either his first random measurement will yield \hat{S}_{δ,ξ} = 0 and his last two \hat{Z} measurements can come out one of three ways, or his first measurement will yield \hat{S}_{δ,ξ} = 1, and again his last two \hat{Z} measurements can come out three ways. For each of these six outcomes, he will need to construct a different likelihood function and maximize it. He can do this readily by taking different products of the likelihood functions we just described. For example, if he finds \hat{S}_{δ,ξ} = 0 on his first qubit and \hat{Z} = 0 on the
knows he must estimate \( Z \) over \( Z \).
Now Bob must maximize his likelihood function (equation 12) with respect to \( Z \), the only fit parameter. The other curves are the variances of \( Z \) when all of the qubits are measured optimally for one (dark blue), two (light blue), and three (purple) qubits—these are shown for comparison. Bob’s \( 2+1 \) scheme outperforms the two-qubit case, but does not perform as well as if all three qubits were measured optimally or as well as if he had used our compression circuit.

other two qubits, his likelihood function for \( Z_{\text{true}} \) is:

\[
L(Z_{\text{true}} | \hat{S}_{\delta, \epsilon} = 0) \times L(Z_{\text{true}} | \hat{Z} = 0, 0) = \frac{1}{2} + Z_{\text{true}} \cos \delta + \frac{1}{2} \sqrt{1 - 4Z_{\text{true}}^2} \sin \delta \cos \epsilon \left( \frac{1}{2} + Z_{\text{true}} \right)^2.
\]

Now Bob must maximize his likelihood function (equation 12) over \( Z_{\text{true}} \), which depends on \( \delta \) and \( \epsilon \). Since he knows he must estimate \( \langle \hat{Z} \rangle \), and he knows what he measured (even though it was randomly chosen) he knows \( \delta \), which is the angle between his measurement and \( \hat{Z} \). However, since \( \epsilon \) is the azimuthal angle between his measurement and the state that Alice prepares it is unknown to him (assuming that he knows nothing about Alice’s state preparation). Therefore his best strategy will be to choose it randomly, from the Haar measure. Then he will report the value of \( Z_{\text{true}} \) at which the likelihood is maximized as his maximum-likelihood estimate of \( Z_{\text{true}}, Z_{\text{MLE}} \).

To characterize the performance of this scheme we performed a Monte-Carlo simulation. We averaged over \( \delta \) and \( \epsilon \) from the Haar measure, and observed the statistics of \( Z_{\text{MLE}} \). Ideally the figure of merit of \( Z_{\text{MLE}} \) would be its variance (the same figure of merit we used for our compression scheme). However, the average of \( Z_{\text{MLE}} \) may be different from the actual average value of \( Z_{\text{true}} \) (this is not true for the \( Z_{\text{direct}} \) and \( Z_{\text{comp}} \) estimators that we introduced in the main text, which do converge on \( Z_{\text{true}} \)). Thus we calculate the mean-squared difference of Bob’s guesses, \( V_{2+1} \) (two qubits and one classical bit), from the true value of \( \langle \hat{Z} \rangle \) (note that since \( Z_{\text{direct}} \) and \( Z_{\text{comp}} \) converge to \( Z_{\text{true}} \) their variance is equal to their mean-squared error). The result of this simulation is shown in figure 6 above (red points). Notice that when Alice prepares \( 0 \) and asks Bob to measure in \( \hat{Z} \) the variance of his \( 2+1 \) estimate is slightly larger than if he does not make use of his extra measurement. This is because, if Bob were to simply measure two qubits in \( 0 \) optimally he would always find both to be spin-up, resulting in a variance of zero. However, his random measurement sometimes pulls his guess away from spin-up, which results in a small non-zero variance for this situation. The fact that the variance is larger for a specific state is not important; what matters is that Bob’s variance is decreased on average by the additional random measurement. These simulations were also repeated for measurements of \( \hat{X} \) and \( \hat{Y} \), plotted as the grey dotted lines in all panels of figure 4 in the main text.

V. AVERAGE MEASUREMENT VARIANCE

We can calculate the variance of all possible spin measurements, for a given state, from the variances of \( \hat{X}, \hat{Y} \) and \( \hat{Z} \). This is done by simply averaging them uniformly, which we will now show explicitly.

The variance of the arbitrary spin direction operator \( \hat{S}_{\delta, \epsilon} \) from the previous section is:

\[
V(\hat{S}_{\delta, \epsilon}) = \langle \hat{S}_{\delta, \epsilon}^2 \rangle - \langle \hat{S}_{\delta, \epsilon} \rangle^2. \tag{13}
\]

Now the average variance, taken uniformly over all possible measurement directions, is:

\[
\overline{V(\hat{S}_{\delta, \epsilon})} = \frac{1}{4\pi} \int_0^{2\pi} d\delta \int_0^\pi \sin \epsilon d\epsilon V(\hat{S}_{\delta, \epsilon}). \tag{14}
\]

Substituting in the forms of \( V(\hat{S}_{\delta, \epsilon}) \) and \( \hat{S}_{\delta, \epsilon} \), the integrals can be calculated and this expression simplifies to:

\[
\overline{V(B_{\delta, \epsilon})} = \frac{1}{3} \left( \langle \hat{X}^2 \rangle + \langle \hat{Y}^2 \rangle + \langle \hat{Z}^2 \rangle - \langle \hat{X} \rangle^2 - \langle \hat{Y} \rangle^2 - \langle \hat{Z} \rangle^2 \right), \tag{15}
\]

which yields the desired result:

\[
\overline{V(\hat{S}_{\delta, \epsilon})} = \frac{1}{3} \left\{ V(\hat{X}) + V(\hat{Y}) + V(\hat{Z}) \right\}. \tag{16}
\]