HOMOTOPY TYPES OF DIFFEOMORPHISM GROUPS
OF NONCOMPACT 2-MANIFOLDS

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Abstract. Suppose $M$ is a noncompact connected smooth 2-manifold without boundary and let $\mathcal{D}(M)_0$ denote the identity component of the diffeomorphism group of $M$ with the compact-open $C^\infty$-topology. In this paper we investigate the topological type of $\mathcal{D}(M)_0$ and show that $\mathcal{D}(M)_0$ is a topological $\ell_2$-manifold and it has the homotopy type of the circle if $M$ is the plane, the open annulus or the open Möbius band, and it is contractible in all other cases. When $M$ admits a volume form $\omega$, we also discuss the topological type of the group of $\omega$-preserving diffeomorphisms of $M$. To obtain these results we study some fundamental properties of transformation groups on noncompact spaces endowed with weak topology.

1. Introduction

The purpose of this paper is the investigation of topological properties of the diffeomorphism groups of noncompact smooth 2-manifolds endowed with the compact-open $C^\infty$-topology. When $M$ is a closed smooth $n$-manifold, the diffeomorphism group $\mathcal{D}(M)$ with the compact-open $C^\infty$-topology is a smooth Fréchet manifold [13, Section I.4], and for $n = 2$, S. Smale [26] and C. J. Earle and J. Eells [10] classified the homotopy type of the identity component $\mathcal{D}(M)_0$.

In the $C^0$-category, for any compact 2-manifold $M$, the homeomorphism group $\mathcal{H}(M)$ with the compact-open topology is a topological $\ell_2$-manifold [8, 11, 20, 27], and M. E. Hamstrom [14] classified the homotopy type of the identity component $\mathcal{H}(M)_0$ (cf. [24] for PL-case). In [29] we have shown that $\mathcal{H}(M)_0$ is an $\ell_2$-manifold even if $M$ is a noncompact connected 2-manifold. We also classified its homotopy type and showed that $\mathcal{H}(M)_0$ is contractible except a few cases.

In [2] we studied topological types of transformation groups on noncompact spaces endowed with strong topology. This formulation was intended for an application to homeomorphism groups and diffeomorphism groups of noncompact manifolds endowed with the Whitney topology.

In this article we formulate the notion of weak topology for transformation groups on noncompact spaces. This notion corresponds with the compact-open topology. We see that the main arguments in [28, 29, 30] well extend to transformation groups with weak topology (cf. Theorem 3.1) and these results can be well applied to the diffeomorphism groups of noncompact 2-manifolds.

Suppose $M$ is a smooth $n$-manifold and $X$ is a closed subset of $M$. For $r = 1, 2, \cdots, \infty$ we denote by $\mathcal{D}^r_X(M)$ the group of $C^r$-diffeomorphisms $h$ of $M$ onto itself with $h|_X = \text{id}_X$, endowed with the compact-open $C^r$-topology [16, CH.2 Section 1], and by $\mathcal{D}_X^\infty(M)_0$ the identity connected component of $\mathcal{D}_X^\infty(M)$. By a compact smooth submanifold of $M$ we mean the union of a disjoint family of $\sigma$-compact manifold, Surfaces.

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closed smooth $k$-submanifold of $M$ for $k = 0, 1, \ldots, n - 1$ and a compact smooth $n$-submanifold of $M$. The following is the main result of this paper.

**Theorem 1.1.** Suppose $M$ is a noncompact connected smooth 2-manifold without boundary and $X$ is a compact smooth submanifold of $M$. Then the following hold.

1. $D_X^r(M)_0$ is a topological $\ell_2$-manifold.
2. (i) $D_X^r(M)_0 \simeq S^1$ if $(M, X) = (\text{a plane}, \emptyset), (\text{a plane}, 1\ pt)$, (an open Möbius band, $\emptyset$) or (an open annulus, $\emptyset$).
   
   (ii) $D_X^r(M)_0 \simeq *$ in all other cases.

Note that any separable infinite-dimensional Fréchet space is homeomorphic to the separable Hilbert space $\ell_2 \equiv \{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty\}$ [6, Chapter VI, Theorem 5.2]. A topological $\ell_2$-manifold is a separable metrizable space which is locally homeomorphic to $\ell_2$. Since topological types of $\ell_2$-manifolds are classified by their homotopy types, Theorem 1.1 (2) implies that $D_X^r(M)_0 \cong S^1 \times \ell_2$ in the case (i) and $D_X^r(M)_0 \cong \ell_2$ in the case (ii).

Let $\mathcal{H}_X(M)_0$ denote the identity connected component of the group of homeomorphisms $h$ of $M$ onto itself with $h|_X = id_X$, endowed with the compact-open $C^0$-topology. The comparison of the homotopy types of $D_X^r(M)_0$ and $\mathcal{H}_X(M)_0$ (29) implies the following conclusion:

**Corollary 1.1.** Suppose $X$ is a compact smooth submanifold of $M$. Then the inclusion $D_X^r(M)_0 \subset \mathcal{H}_X(M)_0$ is a homotopy equivalence.

For the subgroup of diffeomorphisms with compact supports, we have the following consequences. Let $D_X^r(M)^c$ denote the subgroup of $D_X^r(M)$ consisting of diffeomorphisms with compact supports, and let $D_X^r(M)_0^c$ denote the identity connected component of $D_X^r(M)^c$. We can also consider the subgroup of diffeomorphisms which are isotopic to $id_M$ by isotopies with compact supports. Let $D_X^r(M)_0^c*$ denote the subgroup of $D_X^r(M)^c_0$ consisting of $h \in D_X^r(M)^c$ which admits an ambient $C^r$-isotopy $h_t : M \to M$ rel $X$ such that $h_0 = h$, $h_1 = id_M$ and $h_t (0 \leq t \leq 1)$ have supports in a common compact subset of $M$.

We say that a subspace $A$ of a space $X$ is homotopy dense (or has the homotopy negligible complement) in $X$ if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ $(0 < t \leq 1)$. In this case, the inclusion $A \subset X$ is a (controlled) homotopy equivalence, and when $X$ is metrizable, $X$ is an ANR iff $A$ is an ANR (cf. §2.4).

**Theorem 1.2.** Suppose $M$ is a noncompact connected smooth 2-manifold without boundary and $X$ is a compact smooth submanifold of $M$. Then $D_X^r(M)_0^c*$ is homotopy dense in $D_X^r(M)_0$.

**Corollary 1.2.** (1) $D_X^r(M)_0^c$ and $D_X^r(M)_0^c*$ are ANR’s.

2. The inclusions $D_X^r(M)_0^c* = D_X^r(M)_0^c* \subset D_X^r(M)_0$ are homotopy equivalences.

We notice that the subgroup $D_X^r(M)_0^c$ coincides with $D_X^r(M)_0^c$ except some specific cases.

**Proposition 1.1.** Suppose $M$ is a noncompact connected smooth 2-manifold without boundary and $X$ is a compact smooth 2-submanifold of $M$. Then $D_X^r(M)_0^c* = D_X^r(M)_0^c* \iff (a) M$ has no product end or (b) $(M, X) = (\text{a plane}, \emptyset)$ or $(\text{an open Möbius band}, \emptyset)$. 


For $n$-manifolds of finite type we can deduce the following conclusion.

**Proposition 1.2.** If $M = \text{Int } N$ for some compact smooth $n$-manifold $N$ with nonempty boundary and $X$ is a compact smooth submanifold of $M$, then $\mathcal{D}_X^r(M)_0$ is an $\ell_2$-manifold.

This proposition is not so obvious, because for the subgroup

$$\mathcal{D}_{\partial N}^r(N)' = \{ h \in \mathcal{D}_{\partial N}^r(N) \mid f = \text{id on a neighborhood of } \partial N \}$$

the restriction map $\mathcal{D}_{\partial N}^r(N)' \to \mathcal{D}^r(\text{Int } N)^c$ is a continuous bijection, but not a homeomorphism.

At this point it is important to compare the above results on the compact-open $C^\infty$-topology with those on the Whitney $C^\infty$-topology in [2]. Suppose $M$ is a noncompact connected smooth $n$-manifold without boundary. Let $\mathcal{D}^\infty(M)_w$ denote the group $\mathcal{D}^\infty(M)$ endowed with the Whitney $C^\infty$-topology. In [2] we have shown that $\mathcal{D}^\infty(M)_w$ is locally homeomorphic $\square^\omega l_2$ (the countable box product of $l_2$), while $\mathcal{D}^\infty(M)_c$ is an $\mathbb{R}^\infty \times l_2$-manifold. Here, $\mathbb{R}^\infty$ is the direct limit of the tower $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$.

Since the Whitney $C^\infty$-topology is so strong, it is seen that $(\mathcal{D}^\infty(M)_w)_0 = (\mathcal{D}^\infty(M)_c)_0$ and any compact subset of $\mathcal{D}^\infty(M)_c$ has a common compact support. When $n = 2$, the difference between $\mathcal{D}^\infty(M)_0$ and $(\mathcal{D}^\infty(M)_w)_0$ is summarized as follows.

**Remark 1.1.** If $M$ is a noncompact connected smooth $2$-manifold without boundary, then

1. $\mathcal{D}^\infty(M)_0 \cong \left\{ \begin{array}{ll} S^1 \times \ell_2 & \text{if } M = \text{a plane, an open Möbius band, an open annulus,} \\
\ell_2 & \text{in all other cases,} \end{array} \right.$

   the inclusion $\mathcal{D}^\infty(M)_c^* \subset \mathcal{D}^\infty(M)_0$ is a homotopy equivalence,

2. $(\mathcal{D}^\infty(M)_w)_0 = (\mathcal{D}^\infty(M)_c)_0 \cong \mathbb{R}^\infty \times l_2$ \hspace{1cm} $((\mathcal{D}^\infty(M)_w)_0 = \mathcal{D}^\infty(M)_c^* \text{ as sets}).$

Note that any loop in $(\mathcal{D}^\infty(M)_w)_0$ has a common compact support so that it is inessential, while a loop $(h_t)_t$ in $\mathcal{D}^\infty(M)_c^*$ may not have a common compact support though each $h_t$ admits an isotopy to $\text{id}_M$ with compact support. In [30] Example 3.1 (2)) a homotopy equivalence $f : S^1 \simeq \mathcal{H}(\mathbb{R}^2)^c_0$ is constructed explicitly. This fact can be regarded as a sort of pathology of the compact-open topology.

Next we discuss the groups of volume-preserving diffeomorphisms of noncompact $2$-manifolds. Suppose $M$ is a connected oriented smooth $n$-manifold possibly with boundary and $\omega$ is a positive volume form on $M$. Let $\mathcal{D}^\infty(M;\omega)$ denote the subgroup of $\mathcal{D}^\infty(M)$ consisting of $\omega$-preserving diffeomorphisms of $M$ (endowed with the compact-open $C^\infty$-topology) and let $\mathcal{D}^\infty(M;\omega)_0$ denote the identity connected component of $\mathcal{D}^\infty(M;\omega)$. In [31] Corollary 1.1] we have shown that the group $\mathcal{D}^\infty(M)_0$ has a factorization

$$(\mathcal{D}^\infty(M)_0, \mathcal{D}^\infty(M;\omega)_0) \cong (\mathcal{V}^+(M;\omega(M), E^c_M)_{ew}, \{\omega\}) \times \mathcal{D}^\infty(M;\omega)_0,$$

where $\mathcal{V}^+(M;\omega(M), E^c_M)$ is a convex space of positive volume forms on $M$ endowed with the finite-ends weak $C^\infty$-topology $ew$ (cf. Section 6). This means that the subgroup $\mathcal{D}^\infty(M;\omega)_0$ is a strong deformation retract (SDR) of $\mathcal{D}^\infty(M)_0$. Hence Theorem [12] yields the following consequences.

**Theorem 1.3.** Suppose $M$ is a noncompact connected orientable smooth $2$-manifold without boundary and $\omega$ is a volume form on $M$. Then the following hold.

1. $\mathcal{D}^\infty(M;\omega)_0$ is a topological $\ell_2$-manifold and it is a SDR of $\mathcal{D}^\infty(M)_0$. 


Here we have shown that the homomorphism \( c^\omega \) has a continuous (non-homomorphic) section. This induces the factorizations

\[
(D^\infty (M; \omega)\), 0, \ker c_0^\omega )) \cong (S(M; \omega), 0) \times \ker c_0^\omega
\]

and implies that the kernel \( \ker c_0^\omega \) is a SDR of \( D^\infty (M; \omega)\).

Let \( D^\infty (M; \omega)^c \) denote the subgroup of \( D^\infty (M; \omega) \) consisting of \( \omega \)-preserving diffeomorphisms of \( M \) with compact support and let \( D^\infty (M; \omega)\) denote the identity connected component of \( D^\infty (M; \omega)^c \). The subgroup \( D^\infty (M; \omega)\) of \( D^\infty (M; \omega)^c \) is defined by

\[
D^\infty (M; \omega)\) = \{ h \in D^\infty (M; \omega) | h \in D^\infty_{M-K}(M; \omega)_0 \text{ for some compact subset } K \text{ of } M \}.
\]

As another application of the results in Section 3, in \( n = 2 \) we can deduce the topological relations among the subgroups

\[
D^\infty (M; \omega)\) \subset D^\infty (M; \omega)\) \subset \ker c_0^\omega
\]

**Theorem 1.4.** Suppose \( M \) is a noncompact connected orientable smooth 2-manifold without boundary and \( \omega \) is a volume form on \( M \). Then the following hold.

1. \( \ker c_0^\omega \) is an \( \ell_2 \)-manifold and it is a SDR of \( D^\infty (M; \omega)\).
2. \( D^\infty (M; \omega)^c \) is homotopy dense in \( \ker c_0^\omega \). Therefore,
   (i) \( D^\infty (M; \omega)^c \) and \( D^\infty (M; \omega)\) are ANR’s, and
   (ii) the inclusions \( D^\infty (M; \omega)^c \) \( D^\infty (M; \omega)\) \( D^\infty (M; \omega)\) are homotopy equivalences.

This paper is organized as follows. Section 2 contains some fundamental facts on ANR’s and \( \ell_2 \)-manifolds. Section 3 contains main arguments in this article. Here, we investigate some fundamental topological properties of transformation groups on noncompact spaces endowed with weak topology. In Section 4 these results are applied to the diffeomorphism groups of noncompact manifolds with the compact-open \( C^c \)-topology. Theorems 1.1 and 1.2 are proved in Section 5. In Section 6 we discuss the groups of volume-preserving diffeomorphisms of noncompact manifolds.

### 2. Basic properties of ANR’s and \( \ell_2 \)-manifolds

In Sections 3 - 6 we see that the ANR-property of diffeomorphism groups and embedding spaces is especially important to investigate the topology of diffeomorphism groups of noncompact manifolds. In this section we recall basic properties of ANR’s. We refer to [17] [23] for the theory of ANR’s. Throughout the paper we assume that spaces are separable and metrizable and maps are continuous (otherwise specified).
A metrizable space $X$ is called an ANR (absolute neighborhood retract) for metrizable spaces if any map $f : B \to X$ from a closed subset $B$ of a metrizable space $Y$ admits an extension to a neighborhood $U$ of $B$ in $Y$. If we can always take $U = Y$, then $X$ is called an AR. An AR is exactly a contractible ANR. It is well known that $X$ is an AR (an ANR) iff it is a retract of (an open subset of) a normed space. Any ANR has a homotopy type of CW-complex.

We will apply the following criterion of ANR’s [17]:

**Lemma 2.1.** (1) Any retract of of an AR (an open subset of an ANR) is an AR (an ANR).

(2) A metrizable space $X$ is an ANR iff each point of $X$ has an ANR neighborhood in $X$.

(3) If $X = \bigcup_{i=1}^{\infty} U_i$, $U_i$ is open in $X$, $U_i \subset U_{i+1}$ and each $U_i$ is an AR, then $X$ is an AR.

(4) $X \times Y$ is a nonempty ANR iff $X$ and $Y$ are nonempty ANR’s. In a fiber bundle, the total space is an ANR iff both the base space and the fiber are ANR’s.

(5) A metric space $X$ is an ANR iff for any $\varepsilon > 0$ there is an ANR $Y$ and maps $f : X \to Y$ and $g : Y \to X$ such that $gf$ is $\varepsilon$-homotopic to $id_X$ [15].

In the statement (3) the space $X$ is an ANR and has the trivial homotopy groups, thus it is contractible. Since any Fréchet space is an AR, by (2) any Fréchet manifold (an $\ell_2$-manifold) is an ANR.

In a fiber bundle, if the base space is contractible and paracompact, then this bundle is trivial. A principal bundle is trivial iff it admits a section. We use the following fact on principal bundles with AR fibers.

**Lemma 2.2.** If a principal bundle $p : E \to B$ has an AR fiber and a metrizable base space $B$, then,

(i) the bundle $p$ admits a section $s$ and so it is a trivial bundle and

(ii) the map $sp : E \to E$ is $p$-fiber-preserving homotopic to $id_E$ and so the map $p$ is a homotopy equivalence with a homotopy inverse $s$.

The notion of homotopy denseness (or homotopy negligibility) has been defined in §1. A subspace $A$ of a space $X$ is homotopy dense (HD) in $X$ if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ ($0 < t \leq 1$) ([27]). The homotopy $\varphi_t$ is called an absorbing homotopy. The map $\varphi_1 : X \to A$ is a homotopy inverse of the inclusion $A \subset X$.

**Lemma 2.3.** (1) Suppose a subspace $A$ is HD in $X$. Then

(i) the inclusion $A \subset X$ is a (controlled) homotopy equivalence, and

(ii) $X$ is an ANR iff $A$ is an ANR.

(2) A subspace $A$ is HD in $X$ iff every $x \in X$ admits an open neighborhood $U$ in $X$ and a homotopy $\varphi_t : U \to X$ such that $\varphi_0$ is the inclusion $U \subset X$ and $\varphi_t(U) \subset A$ ($0 < t \leq 1$). (cf. [28] Fact 4.1 (i)).

Lemma 2.3(1)(ii) follows from Lemma 2.1(5).

We conclude this preliminary section with the following characterization of $\ell_2$-manifold topological groups [8] [27]. An $\ell_2$-manifold is a separable metrizable space locally homeomorphic to $\ell_2$. 

Theorem 2.1. (1) A topological group is an \( \ell_2 \)-manifold iff it is a separable, non locally compact, completely metrizable ANR.
(2) Two \( \ell_2 \)-manifolds are homeomorphic iff they are homotopy equivalent.

3. Transformation groups with weak topology

This section includes some results on transformation groups with weak topology. In the next section, these results will be applied to the diffeomorphism groups of noncompact manifolds endowed with the compact-open \( C^\ell \)-topology.

3.1. Transformation groups.

A transformation group means a pair \((G, M)\) in which \( M \) is a locally compact, \( \sigma \)-compact Hausdorff space and \( G \) is a topological group acting on \( M \) continuously and effectively. Each \( g \in G \) induces a homeomorphism of \( M \), which is also denoted by the same symbol \( g \). Let \( G^c = \{ g \in H \mid \text{supp}(g) \text{ is compact} \} \). For any subsets \( K, N \) of \( M \) we obtain the following subgroups of \( G \):

\[
G_K = \{ g \in G : g|_K = \text{id}_K \}, \quad G(N) = G_{M \setminus N}, \quad G_K(N) = G_K \cap G(N), \quad G^c_K = G_K \cap G^c \quad \text{etc.}
\]

For any subgroup \( H \) of \( G \), let \( H_0 \) denote the connected component of the unit element \( e \) in \( H \). For example, the symbol \( G_0^c \) denotes the connected component of \( e \) in \( G^c \). We also consider a subgroup \( G_0^{c*} \) of \( G_0^c \) defined by

\[
G_0^{c*} = \{ h \in G^c \mid h \in G(K)_0 \text{ for some compact subset } K \text{ of } M \}.
\]

For subsets \( K \subset L \subset N \) of \( M \), we have the set of embeddings

\[
E^G_K(L, N) = \{ g|_L : L \to M \mid g \in G_K(N) \}.
\]

(If \( K = \emptyset \), the symbol \( K \) is omitted from the notation.) The group \( G_K(N) \) acts transitively on the set \( E^G_K(L, N) \) by \( g : f = gf \) \((g \in G_K(N), f \in E^G_K(L, N))\). The restriction map

\[
r : G_K(N) \to E^G_K(L, N), \quad r(g) = g|_L.
\]

coincides with the orbit map at the inclusion \( i_L : L \subset M \) under this action.

We need to pay an attention on the topology of the space \( E^G_K(L, N) \). A topology on \( E^G_K(L, N) \) is called admissible if the \( G_K(N) \)-action is continuous with respect to this topology. The strongest admissible topology on \( E^G_K(L, N) \) is the quotient topology induced by the map \( r \). Otherwise specified, the set \( E^G_K(L, N) \) is endowed with this quotient topology. For \( K \subset L_1 \subset L \subset N_1 \subset N \), one sees that \( E^G_K(L, N_1) \) is a subspace of \( E^G_K(L, N) \) and the restriction map \( E^G_K(L, N) \to E^G_K(L_1, N) \) is continuous.

When the set \( E^G_K(L, N) \) is endowed with a specific admissible topology \( \tau \), we write \( E^G_K(L, N)\tau \) to avoid the ambiguity. The map \( r : G_K(N) \to E^G_K(L, N)\tau \) is continuous. For any subset \( F \) of \( E^G_K(L, N)\tau \), let \( F^\tau \) denote the space \( F \) endowed with the subspace topology induced from \( \tau \). When \( i_L \in F \), let \( F^\tau_i \) denote the connected component of \( i_L \) in \( F^\tau \).

We say that a map \( f : X \to Y \) has a local section at \( y \in Y \) if there exists a neighborhood \( U \) of \( y \) in \( Y \) and a map \( s : U \to X \) with \( fs = i_U \).
Lemma 3.1. Suppose $H$ is a subgroup of $G_K(N)$ and the restriction map $r : G_K(N) \to \mathcal{E}_K^G(L,N)^\tau$ has a local section $s : U \to H \subset G_K(N)$ at $i_L$. Then, the following hold.

1. The restriction map $r|_H : H \to \mathcal{E}_H^H(L,N)^\tau$ is a principal bundle with the structure group $H_L$.

2. (i) The topology $\tau$ coincides with the quotient topology.

   (ii) $\mathcal{E}_H^H(L,N)^\tau$ is open in $\mathcal{E}_K^G(L,N)^\tau$.

   (iii) If $H$ is a normal subgroup of $G_K(N)$, then

      (a) each orbit of $H$ is closed and open in $\mathcal{E}_K^G(L,N)^\tau$ (in particular, $\mathcal{E}_H^H(L,N)^\tau$ is closed and open in $\mathcal{E}_K^G(L,N)^\tau$),

      (b) if $H \subset G_K(N)_0$, then $\mathcal{E}_H^H(L,N)^\tau = \mathcal{E}_K^G(L,N)^\tau_0$.

Proof. (1) Note that (a) $\mathcal{E}_H^H(L,N)^\tau = H \cdot i_L$ and the map $r|_H$ coincides with the orbit map at $i_L$ under the action of the group $H$ and (b) $U \subset \mathcal{E}_H^H(L,N)^\tau$ and $s$ is a local section of $r|_H$ at $i_L$. Hence, the statement (1) follows from the well known fact on the orbit map in the theory of group action.

(2) (i) By (1) the map $r$ itself is a principal bundle. Hence, the map $r$ is a quotient map and the topology $\tau$ coincides with the quotient topology.

(ii) We may assume that $U$ is open in $\mathcal{E}_K^G(L,N)^\tau$. Since $U \subset \mathcal{E}_H^H(L,N)^\tau$, we have $\mathcal{E}_H^H(L,N)^\tau = HU$, which is open in $\mathcal{E}_K^G(L,N)^\tau$.

(iii) Since $Hi_L$ is open, the orbit $H(g \cdot i_L) = g(Hi_L)$ is also open in $\mathcal{E}_K^G(L,N)^\tau$. \qed

In many cases, the set $\mathcal{E}_K^G(L,N)$ admits a natural admissible topology $\tau$ (for instance, the compact-open $C^r$-topology ($r = 0, 1, \cdots, \infty$)). However, this topology $\tau$ coincides with the quotient topology, once we obtain a bundle theorem under the topology $\tau$. Hence, our convention does not lose a generality of the arguments in the subsections below.

Finally we extract a behavior of the compact-open $C^\infty$ topology on diffeomorphism groups of a noncompact manifold $M$ at the ends of $M$ extend the notion of the compact-open $C^\infty$ topology on diffeomorphism groups to transformation groups.

Definition 3.1. We say that a transformation group $(G,M)$ has a weak topology if it satisfies the following condition:

$(\ast)$ For any neighborhood $U$ of $e$ in $G$ there exists a compact subset $K$ of $M$ such that $G_K \subset U$.

Remark 3.1. (1) If $G$ admits a compatible metric $\rho$, then $(G,M)$ has a weak topology iff for any $\varepsilon > 0$ there exists a compact subset $K$ of $M$ such that $\text{diam}_\rho G_K < \varepsilon$. We can always take $\rho$ to be left-invariant.

(2) If a transformation group $(G,M)$ has a weak topology, then so is $(H,M)$ for any subgroup $H$ of $G$.

The notion of weak topology is an extension of the notion of compact-open topology to transformation groups. It is readily seen that the compact-open $C^\infty$ topology on the diffeomorphism groups of smooth manifolds is also an example of weak topology.

3.2. Basic assumptions on transformation groups.
Suppose \((G, M)\) is a transformation group. Since \(M\) is locally compact and \(\sigma\)-compact, there exists a sequence \(\{M_i\}_{i \geq 1}\) of compact subsets of \(M\) such that

\[
M = \bigcup_{i=0}^{\infty} M_i \quad \text{and} \quad M_{i-1} \subset \text{Int}_M M_i \quad (i \geq 1), \quad \text{where} \quad M_0 = \emptyset.
\]

This sequence is called an *exhausting sequence* of \(M\). Let \(U_i = \text{Int}_M M_i \quad (i \geq 1)\). Consider the following basic conditions on the tuple \((G, M, \{M_i\}_{i \geq 1})\).

**Assumption (A).**

(A-0) The group \(G\) is metrizable and the transformation group \((G, M)\) has a weak topology.

(A-1) For each \(j > i > k \geq 0\), the restriction map

\[
\pi_{k,j}^{i}: G_{M_k}(U_j)_0 \to \mathcal{E}_{M_k}^G(M_i, U_j)_0
\]

is a principal bundle with the structure group \(G_{k,j}^i \equiv G_{M_k}(U_j)_0 \cap G_{M_i}^j\).

(A-2) (i) \(G(U_i)_0\) is an ANR for each \(i \geq 1\).

(ii) The spaces \(U_{k,j}^i = \mathcal{E}_{M_k}^G(M_i, U_j)_0 \quad (j > i > k \geq 0)\) satisfy the next conditions:

\[
U_{k,j}^i \text{ is an open subspace of } \mathcal{E}_{M_k}^G(M_i, M)_0, \quad \mathcal{E}_{M_k}^G(M_i, M)_0 = \bigcup_{j>i} U_{k,j}^i \text{ and } \overline{cU_{k,j}^i} \subset U_{k,j+1}^i.
\]

Below we assume that the tuple \((M, G, \{M_i\}_{i \geq 1})\) satisfies the assumption (A).

**Lemma 3.2.** (1) For each \(i > k \geq 0\), the restriction map

\[
\pi_{k}^{i}: (G_{M_k})_0 \to \mathcal{E}_{M_k}^G(M_i, M)_0
\]

is a principal bundle with the structure group \(G_{k}^i \equiv (G_{M_k})_0 \cap G_{M_i}\).

(2) The spaces \(G_{M_k}(U_i)_0\) and \(\mathcal{E}_{M_k}^G(M_i, M)_0\) are ANR’s for each \(i > k \geq 0\).

**Proof.** (1) This lemma follows from (A-1), (A-2)(ii) and Lemma 3.1.

(2) (i) By (A-1), for each \(i > k \geq 1\), the restriction map

\[
\pi_{k}^{i}: G(U_i)_0 \to \mathcal{E}^G(M_k, U_i)_0
\]

is a principal bundle with the fiber \(G_{0,i}^k \equiv G(U_i)_0 \cap G_{M_k}\). Since \(G(U_i)_0\) is an ANR by (A-2)(i), so are \(G_{0,i}^k\) and \(G_{M_i}(U_i)_0 = (G_{0,i}^k)_0\).

(ii) By (i) and (A-1) \(\mathcal{E}_{M_k}^G(M_i, U_j)_0\) is an ANR for each \(j > i\), and so is \(\mathcal{E}_{M_k}^G(M_i, M)_0\) by (A-2)(ii). \(\square\)

**Lemma 3.3.** (1)(i) If \(G_0\) is an ANR, then so is \((G_{M_i})_0\) for any \(i \geq 1\).

(ii) If \(G_{M_i}\) is an ANR for some \(i \geq 1\), then so is \(G_0\).

(2) If \((G_{M_i})_0^*\) is HD in \((G_{M_i})_0\) and \((G_{M_i})_0\) is open in \(G_{M_i}\) for some \(i \geq 1\), then \(G_{M_i}^*\) is HD in \(G_0\).

**Proof.** (1) The restriction map \(\pi_{0}^{i}: G_0 \to \mathcal{E}^G(M_i, M)_0\) is a bundle map with fiber \(G_{0}^i = G_0 \cap G_{M_i}\). Hence, there exists an open neighborhood \(U\) of the inclusion map \(i_{M_i}\) in \(\mathcal{E}^G(M_i, M)_0\) such that \((\pi_{0}^{i})^{-1}(U) \cong U \times G_{0}^i\). Since \(U\) is an ANR by (A-1)(i), it follows that \((\pi_{0}^{i})^{-1}(U)\) is an ANR if and only if \(G_{0}^i\) is an ANR. The assertions (i) and (ii) follow from these observations.

(2) By Lemma 2.3(2) it suffices to verify the following assertion:
Every $h \in G_0$ admits an open neighborhood $V$ in $G_0$ and a homotopy $\varphi : V \times [0, 1] \to G_0$ such that $\varphi_0$ is the inclusion $V \subset G_0$ and $\varphi_t(V) \subset G_0^*$ ($0 < t \leq 1$).

Under the map $\pi^i_0 : G_0 \to \mathcal{E}_i(M, M)_0$, each $h \in G_0$ induces $h|_{M_i} \in \mathcal{E}_i(M, M)_0$, and by (A-1)(ii) we have $h|_{M_i} \in \mathcal{E}_i(M, U_j)_0$ for some $j > i$. By (A-2) the restriction map

$$\pi^j_{0,j} : G(U_j)_0 \to \mathcal{E}_j(M, U_j)_0$$

is a bundle map and so there exists an open neighborhood $U$ of $h|_N$ in $\mathcal{E}_j(M, U_j)_0$ and a local section $s : U \to G(U_j)_0$ of $\pi^j_{0,j}$ such that $s(h|_{M_i}) = h$. Choose a small open neighborhood $V$ of $h$ in $G_0$ such that $\pi^i_0(V) \subset U$. For each $g \in V$ we have $s(g|_{M_i})|_{M_i} = g|_{M_i}$ and $s(g|_{M_i})^{-1}g \in G_{M_i}$. Since $s(h|_{M_i})^{-1}h = id_M \in (G_{M_i})_0$ and $(G_{M_i})_0$ is open in $G_{M_i}$, by replacing $V$ by a smaller one, we may assume that $s(g|_{M_i})^{-1}g \in (G_{M_i})_0$ ($g \in V$).

There exists an absorbing homotopy $\psi_t$ of $(G_{M_i})_0$ into $(G_{M_i})_0^*$. Then the required homotopy $\varphi_t : V \to G_0$ is defined by

$$\varphi_t(g) = s(g|_{M_i})\psi_t(s(g|_{M_i})^{-1}g).$$

Then $\varphi_0(g) = g$ and for $0 < t \leq 1$ we have $\varphi_t(g) \in G_0^*$ since $s(g|_{M_i}) \in G(U_j)_0 \subset G_0^*$ and $\psi_t(s(g|_{M_i})^{-1}g) \in (G_{M_i})_0^* \subset G_0^*$.  

3.3. Contractibility conditions.

In this subsection we deduce some conclusions under some contractibility conditions. Consider the following conditions on the tuple $(M, G, \{M_i\}_{i \geq 1})$:

**Condition (C).**

(C-1) $\mathcal{E}_i^{G} (M_i, M)_0 \simeq *$ for each $i > k \geq 0$.

(C-2) $G_{0,j}^i \simeq *$ for each $j > i \geq 1$.

Below we assume that the tuple $(G, M, \{M_i\}_{i \geq 1})$ satisfies the assumption (A). Since $G$ is metrizable by the condition (A-0), it admits a left-invariant metric $\rho$.

**Lemma 3.4.** If $(G, M, \{M_i\}_{i \geq 1})$ satisfies the condition (C-1), then the following hold.

1. For each $i > k \geq 0$, 
   (i) the bundle $\pi^i_k$ is a trivial bundle and $\mathcal{G}^i_k = (G_{M_k})_0$.
   (ii) $(G_{M_k})_0$ strongly deformation retracts onto $(G_{M_i})_0$.

2. $(G_{M_k})_0$ is an AR for each $k \geq 0$.

**Proof.** The conditions (A-1) and (C-1) imply that $\mathcal{E}_i^{G} (M_i, M)_0$ is an AR.

1. Since the base space $\mathcal{E}_i^{G} (M_i, M)_0$ is metrizable and contractible, the bundle $\pi^i_k$ is trivial. This means that there exists a fiber-preserving homeomorphism over $\mathcal{E}_i^{G} (M_i, M)_0$,

$$ (G_{M_i})_0 \simeq \mathcal{E}_i^{G} (M_i, M)_0 \times \mathcal{G}^i_k. $$

Since $(G_{M_k})_0$ is connected, so is $\mathcal{G}^i_k$ and we have $\mathcal{G}^i_k = (G_{M_k})_0$. Since the base $\mathcal{E}_i^{G} (M_i, M)_0$ is an AR, it admits a strong deformation retraction (SDR) onto the singleton $\{i_{M_k}\}$. This induces the required SDR of $(G_{M_k})_0$ onto $\mathcal{G}^i_k$.  

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(2)(i) First we show that \((G_{M_i})_0 \simeq \ast\). By (1)(ii), for each \(i \geq 0\) there exists a SDR \(h^i_1(t) (0 \leq t \leq 1)\) of \((G_{M_i})_0\) onto \((G_{M_{i+1}})_0\). A SDR \(h_t \ (k \leq t \leq \infty)\) of \((G_{M_i})_0\) onto \(\{id_M\}\) is defined by
\[
h_t(f) = h^i_{t-1} \cdots h^1_k(f) \quad (f \in (G_{M_i})_0, \ i \geq k, \ i \leq t \leq i + 1)
\]
\[
h_\infty(f) = id_M.
\]
Since \((G, M)\) has a weak topology, it follows that \(\text{diam}_\rho (G_{M_i})_0 \to 0\) and the homotopy \(h : (G_{M_i})_0 \times [k, \infty] \to (G_{M_k})_0\) is continuous.

(ii) To see that \((G_{M_i})_0\) is an ANR, we apply Lemma 24.1(5) (the Hanner’s criterion). By (1), for each \(i > k\) the restriction map
\[
\pi^i_k : (G_{M_i})_0 \to \mathcal{E}_{M_k}^G(M_i, M)_0
\]
is a trivial bundle with an ANR base space and the fiber \(G^i_k \equiv (G_{M_i})_0\). By (i) the fiber \(G^i_k\) is contractible. These imply that \(\pi^i_k\) admits a section \(s^i_k\) and that \(s^i_k \pi^i_k\) is \(\pi^i_k\)-fiber preserving homotopic to \(id\). Since the metric \(\rho\) is left-invariant, the diameter of each fiber of \(\pi^i_k\) coincides with \(\text{diam}_\rho G_{M_i}\).

Since \(\text{diam}_\rho G_{M_i} \to 0 \ (i \to 0)\), the Hanner’s criterion implies that \((G_{M_i})_0\) is an ANR.

\[\square\]

Lemma 3.5. If \((G, M, \{M_i\}_{i \geq 1})\) satisfies the condition (C-2), then \(G^c_0\) is HD in \(G_0\).

Proof. (1) For notational simplicity, for each \(j > i \geq 1\) we set \(D_j = G(U_j)_0\) and \(U_{i,j} = \mathcal{E}^G(M_i, U_j)_0\).

By (A-2) the map \(\pi^i_{0,j} : D_j \to U_{i,j}\) is a principal bundle. Since the fiber is an AR by (C-2) and the base space is metrizable, this bundle has a global section. Thus, the map \(\pi^i_{0,j}\) is a trivial bundle with an AR fiber and hence it has the following relative lifting property:

(*) If \(Y\) is a metric space, \(B\) is a closed subset of \(Y\) and \(\varphi : Y \to U_{i,j}\) and \(\varphi_0 : B \to D_j\) are maps
with \(\varphi|_B = \pi^i_{0,j} \varphi_0\), then there exists a map \(\Phi : Y \to D_j\) such that \(\pi^i_{0,j} \Phi = \varphi\) and \(\Phi|_B = \varphi_0\).

(2) Next consider the principal bundle
\[
\pi^i_k : G_0 \to \mathcal{E}^G(M_i, M)_0.
\]
For each \(j > i \geq 1\), we set \(V_{i,j} = (\pi^i_0)^{-1}(U_{i,j}) \subset G_0\). Then \(U_{i,j}, V_{i,j}\) and \(D_j\) satisfy the following conditions: for each \(i \geq 1\)

(i) \(\mathcal{E}^G(M_i, M)_0 = \cup_{j > i} U_{i,j}\), \(U_{i,j}\) is open in \(\mathcal{E}^G(M_i, M)_0\) and \(\text{cl}U_{i,j} \subset U_{i,j+1}\).

(ii) \(G_0 = \cup_{j > i} V_{i,j}\), \(V_{i,j}\) is open in \(G_0\), \(\text{cl}V_{i,j} \subset V_{i,j+1}\) and \(V_{i+1,j} \subset V_{i,j} \ (j > i + 1)\).

(iii) \(G^*_0 = \cup_{j > i} D_j\) and \(D_j \subset D_{j+1}\).

(3) We have to construct a homotopy \(F : G_0 \times [0, 1] \to G_0\) such that \(F_0 = id\) and \(F_t(G_0) \subset G^*_0\)
for \(0 < t \leq 1\). We replace the interval \([0, 1]\) by \([1, \infty]\).

(i) For each \(i \geq 1\) we can find a map \(s^i_1 : \mathcal{E}^G(M_i, M)_0 \to G^*_0\) such that
\[
s^i_1(f)|_{M_i} = f|_{M_i} \ (f \in \mathcal{E}^G(M_i, M)_0) \text{ and } s^i_1(\text{cl}U_{i,j}) \subset D_{j+1} \ (j > i).
\]

In fact, using the property (*), inductively we can construct maps \(s^i_1 : \text{cl}U_{i,j} \to D_{j+1} \ (j > i)\) such that
\[
s^i_1(f)|_{M_i} = f \ (f \in \text{cl}U_{i,j}) \text{ and } s^i_{j+1}|_{\text{cl}U_{i,j}} = s^i_j.
\]
The map \(s^i_1\) is defined by \(s^i_1|_{\text{cl}U_{i,j}} = s^i_j\). Let \(F_i = s^i_1 \pi^i_0 : G_0 \to G^*_0\). We have \(F_i(\text{cl}V_{i,j}) \subset D_{j+1}\) and \(F_i(h)|_{M_i} = h|_{M_i}\).
(ii) For each \( i \geq 1 \), we can inductively construct a sequence of homotopies

\[ H^j : \text{cl} \mathcal{V}_{i+1,j} \times [i, i+1] \to \mathcal{D}_{j+1} \ (j > i + 1) \]

such that

\[ H^j_i = F_i, \quad H^j_{i+1} = F_{i+1}, \quad H^{j+1}_i|_{\text{cl}\mathcal{V}_{i+1,j} \times [i, i+1]} = H^j \text{ and } H^j_i(h)|_{M_i} = h|_{M_i}. \]

If \( H^j \) is given, then \( H^{j+1} \) is obtained by applying the property (*) to the diagram:

\[
\begin{align*}
\mathcal{B} & \xrightarrow{\varphi} \mathcal{D}_{j+2} \\
\cap & \\
\mathcal{Y} & \xrightarrow{\varphi} \mathcal{U}_{i,j+2},
\end{align*}
\]

\( (Y, B) = (\text{cl} \mathcal{V}_{i+1,j+1} \times [i, i+1], (\text{cl} \mathcal{V}_{i+1,j} \times [i, i+1]) \cup (\text{cl} \mathcal{V}_{i+1,j+1} \times \{i, i+1\})) \).

Thus we can define a homotopy \( F : \mathcal{G}_0 \times [i, i+1] \to \mathcal{G}_0^* \) by \( F = H^j \) on \( \text{cl} \mathcal{V}_{i+1,j} \times [i, i+1] \). Since \( F_t(h)|_{M_i} = h|_{M_i} \) for \( t \geq i \), we can continuously extend \( F \) by \( F_\infty = \text{id} \).

\[ \square \]

Lemmas 3.4 and 3.5 yield the following criterions.

**Theorem 3.1.** Suppose the tuple \( (G, M, \{M_i\}_{i \geq 1}) \) satisfies the assumption (A).

1. If \( G_{M_k}(U_i)_0 \simeq * \) and \( \mathcal{G}^i_{k,j} \) is connected (i.e., \( G_{M_k}(U_j)_0 \cap G_{M_i} = G_{M_i}(U_j)_0 \)) for each \( j > i > k \geq 0 \), then \( G_0 \) is an AR and \( G^*_0 \) is HD in \( G_0 \).
2. (i) If \( G_{M_i} \simeq * \) for each \( i \geq 1 \), then \( G_0 \) is an ANR.
   (ii) If \( G_{M_i} \) is connected and \( \mathcal{G}^i_{k,j} = G_{M_i}(U_j)_0 \cap G_{M_i} \simeq * \) for each \( j > i \geq 2 \), then \( G^*_0 \) is HD in \( G_0 \).

**Proof.** (1) By Lemmas 3.4 and 3.5 it suffices to verify the following assertions: for each \( j > i > k \geq 0 \)

(i) the bundle \( \pi^i_{k,j} \) is trivial, and the spaces \( \mathcal{G}^i_{k,j}, G_{M_k}(U_j)_0 \) and \( \mathcal{E}^G_{M_k}(M_i, U_j)_0 \) are AR’s,

(ii) \( \mathcal{E}^G_{M_k}(M_i, M)_0 \) is an AR.

(i) By Lemma 3.2(2) and the assumption, both the total space \( G_{M_k}(U_j)_0 \) and the fiber \( \mathcal{G}^i_{k,j} = G_{M_k}(U_j)_0 \) are AR’s. From the homotopy exact sequence of the bundle \( \pi^i_{k,j} \), it follows that the ANR base space \( \mathcal{E}^G_{M_k}(M_i, U_j)_0 \) is contractible and the bundle \( \pi^i_{k,j} \) is trivial. (Alternatively, since the fiber is an AR, the principal bundle \( \pi^i_{k,j} \) admits a global section, which means that this bundle is trivial and the base space is an AR since it is a retract of the AR total space.)

(ii) The assertion follows from (i), (A-2)(ii) and Lemma 2.1(3).

(2) Since \( (G_{M_i})_0 = G_{M_i} \), by Lemma 3.3 it suffices to show the next assertions:

(i’) \( (G_{M_i})_0 \) is an ANR if \( G_{M_i} \simeq * \) for each \( i \geq 1 \).

(ii’) \( (G_{M_i})^*_0 \) is HD in \( (G_{M_i})_0 \) if \( \mathcal{G}^i_{k,j} = G_{M_i}(U_j)_0 \cap G_{M_i} \simeq * \) for each \( j > i \geq 2 \).

Since the tuple \( (G_{M_i}, M, \{M_i\}_{i \geq 2}) \) also satisfies the assumption (A), by Lemmas 3.4 and 3.5 it remains to verify the condition (C-1) in (i’) and (C-2) in (ii’) respectively.

(i’) \( \mathcal{E}^G_{M_k}(M_i, M)_0 \simeq * \) for each \( i > k \geq 1 \): In fact, the restriction map

\[ \pi^i_k : (G_{M_k})_0 \to \mathcal{E}^G_{M_k}(M_i, M)_0 \]
is a principal bundle with the fiber $G_k^i \equiv (G_{M_k})_0 \cap G_{M}$. Since $(G_{M_k})_0 = G_{M_k} \simeq \ast$ and $G_k^i = G_{M_i} \simeq \ast$, it follows that the ANR base space $E_{G_k}^r(M_i, M)_0$ is also contractible.

$(ii')$ $G_{1,j}^i = G_{M_l}(U_i) \cap G_{M_i} \simeq \ast$ for each $j > i \geq 2$: This is the condition $(C-2)$ itself for the tuple $(G_{M_1}, M, \{M_i\}_{i \geq 2})$. This completes the proof. □

4. DIFFEOMORPHISM GROUPS OF NON-COMPACT $n$-MANIFOLDS

4.1. General properties of diffeomorphism groups of $n$-manifolds.

Suppose $M$ is a smooth (separable metrizable) $n$-manifold without boundary and $X$ is a closed subset of $M$ with $X \neq M$. Let $r = 1, 2, \cdots, \infty$.

When $N$ is a smooth manifold, the symbol $C^r(M, N)$ denotes the space of $C^r$-maps $f : M \to N$ with the compact-open $C^r$-topology. For any map $f_0 : X \to N$, let $C^r_{f_0}(M, N)$ denote the subspace \{ $f \in C^r(M, N)$ \mid $f|_X = f_0$ \}. When $M$ is a smooth submanifold of $N$, the symbol $E^r_X(M, N)$ denotes the subspace of $C^r(M, N)$ consisting of $C^r$-embeddings $f : M \to N$ with $f|_X = id_X$ and $E^r_X(M, N)_0$ denotes the connected component of the inclusion $i_N : N \subset M$ in $E^r_X(N, M)$. For spaces $Y$ and $Z$, the symbol $C^0(Y, Z)$ denotes the space of $C^0$-maps $f : Y \to Z$ with the compact-open topology.

Consider the jet-map $j^r : C^r(M, N) \to C^0(M, J^r(M, N))$. It is a closed embedding [16] Ch2. Section 4, p.61–62]. Thus, if we choose a complete metric $d$ on the jet-bundle $J^r(M, N)$ with $d \leq 1$ and a sequence of compact $n$-submanifolds $M_i (i \geq 1)$ such that $M_i \subset \text{Int} M_{i+1}$ and $M = \cup_i M_i$, then we can define a complete metric $\rho$ on $C^r(M, N)$ by

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in M_i} d(j^r i f, j^r i g).$$

When $N = M$, it induces a metric on $D^r_X(M) \subset C^r(M, M)$. We can define a complete metric $\rho^*$ on $D^r_X(M)$ by

$$\rho^*(f, g) = \rho(f, g) + \rho(f^{-1}, g^{-1}).$$

Lemma 4.1. (1) $D^r_X(M)$ is a topological group [16] Ch2., Section 4, p. 64].

(2) $D^r_X(M)$ and $D^r_X(M)_0$ are separable, completely metrizable and not locally compact [16] Ch2., Section 4, p. 61–62, Theorems 4.3, 4.4].

(3) $D^r_X(M)$ and $D^r_X(M)_0$ are $\ell^2$-manifolds iff they are ANR’s.

If $M$ is a closed smooth $n$-manifold, then $D^r(M)$ is a smooth Fréchet manifold [13] Section I.4, Example 4.1.3, etc.]. In this paper we are only concerned with topological Fréchet manifolds (= topological $\ell_2$-manifolds). Below, the emphasis is put on relative cases.

Suppose $M$ is a compact smooth $n$-manifold (possibly with boundary) and $X$ is a closed subset of $M$. When $E \to M$ is a smooth fiber bundle over $M$ and $s_0 : X \to E$ is a section of $E$ over $X$, the symbol $\Gamma^r_{s_0}(M, E)$ denotes the space of $C^r$-sections $s$ of $E$ over $M$ with $s|_X = s_0$, endowed with the compact-open $C^r$-topology. When $E$ is a vector bundle over $M$, the space $\Gamma^r_{s_0}(M, E)$ is a Fréchet space. If $E$ is a fiber bundle over $M$ and the fiber of $E$ is a smooth manifold without boundary, then $\Gamma^r_{s_0}(M, E)$ is a Fréchet manifold. A local chart around $s \in \Gamma^r_{s_0}(M, E)$ is defined as follows: Consider the restriction $T^u(E)|_{s(M)}$ of the vertical tangent bundle $T^u(E)$ of $E$ to the image $s(M)$ in $E$. There
exists a diffeomorphism between an open neighborhood of \(s(M)\) in \(E\) and an open neighborhood of the image of zero section \(0\) in \(T^v(E)_{|s(M)}\). It induces a homeomorphism between an open neighborhood of \(s\) in \(\Gamma^r_{s_0}(M,E)\) and an open neighborhood of \(0\) in \(\Gamma^r_{s_0}(s(M),T^v(E)_{|s(M)})\). If \(N\) is a smooth manifold without boundary and \(f_0 : X \to N\) is a map, then the space \(C^r_{f_0}(M,N) = \{f \in C^r(M,N) \mid f|_X = f_0\}\) is also a Fréchet manifold since it is identified with \(\Gamma^r_{s_0}(M,E)\) for the trivial bundle \(E = M \times N \to M\) and the section \(s_0(x) = (x, f_0(x))\) over \(X\).

**Lemma 4.2.** (i) Suppose \(N\) is a smooth manifold without boundary, \(M\) is a compact smooth submanifold of \(N\) and \(X\) is a closed subset of \(M\). Then \(E^r_X(M,N)\) is a Fréchet manifold.

(ii) Suppose \(M\) is a compact smooth \(n\)-manifold and \(X\) is a closed subset of \(M\). If \(\partial M \subset X\) or \(\partial M \cap X = \emptyset\), then \(D^r_X(M)\) is a Fréchet manifold.

**Proof.** (i) \(E^r_X(M,N)\) is open in \(C^r_{\partial X}(M,N)\) [16, Ch.2 Theorem 1.4].

(ii) If \(\partial M \subset X\), then \(D^r_X(M) = E^r_X(M,M) = E^r_X(M,\hat{M})\), where \(\hat{M}\) is the open manifold obtained from \(M\) by attaching an open collar to \(\partial M\). In the case \(X \cap \partial M = \emptyset\), consider the restriction map \(\pi : D^r_X(M) \to D^r(\partial M)\), \(\pi(h) = h|_{\partial M}\). Using a collar of \(\partial M\) in \(M\), for any \(f \in D^r(\partial M)\) which is sufficiently close to \(id\), we can easily construct a canonical extension \(\bar{f} \in D^r_X(M)\) of \(f\). This implies that \(\text{Im} \pi\) is clopen subset of \(D^r(\partial M)\) and \(D^r_X(M) \to \text{Im} \pi\) is a principal bundle with fiber \(D^r_{X \cup \partial M}(M)\). Since \(D^r(\partial M)\) and \(D^r_{X \cup \partial M}(M)\) are Fréchet manifolds, so is \(D^r_X(M)\).

**Remark 4.1.** (1) A family \(h_t \in D^r_X(M)\) \((t \in [0,1])\) is called a \(C^r\)-isotopy rel \(X\) if \(H : M \times [0,1] \to M\), \(H(x,t) = h_t(x)\), is a \(C^r\)-map. In this case \(h_t\) is a path in \(D^r_X(M)\) (i.e., \([0,1] \ni t \mapsto h_t \in D^r_X(M)\) is continuous). In general, if \(h_t\) is a path in \(D^r_X(M)\), then \(h_t\) is a \(C^0\)-isotopy rel \(X\), but \(H\) is not necessarily \(C^1\) in \(t\).

(2) Since Fréchet manifolds are locally path-connected, the connected components \(E^r_X(N,M)_0\) and \(D^r_X(M)_0\) in Lemma 2.2 are path-connected. Thus, if \(h \in D^r_X(M)_0\), then there is a path \(h_t\) \((t \in [0,1])\) in \(D^r_X(M)_0\) with \(h_0 = id_M\) and \(h_1 = h\), which is a \(C^0\)-isotopy rel \(X\).

### 4.2. The bundle theorem.

The bundle theorem connecting diffeomorphism groups and embedding spaces [7, 18, 22, 25] plays an essential role in order to apply Theorem 3.1 (recall Assumption (A)). Suppose \(M\) is a smooth \(n\)-manifold without boundary. A compact smooth submanifold of \(M\) means a compact subset \(N\) of \(M\) which is the union of a disjoint family \(\{N_k\}_{k=0}^n\) such that \(N_k\) is a (possibly empty) closed smooth \(k\)-submanifold of \(M\) for \(k = 0, 1, \cdots , n-1\) and \(N_n\) is a (possibly empty) compact smooth \(n\)-submanifold of \(M\).

Suppose \(N\) is a compact smooth submanifold of \(M\) and \(X\) is a closed subset of \(N\). Let \(U\) be any open neighborhood of \(N\) in \(M\).

**Theorem 4.1.** For any \(f \in E^r_X(N,U)\) there exist a neighborhood \(U\) of \(f\) in \(E^r_X(N,U)\) and a map \(\varphi : U \to D^r_{X\cup(M\setminus U)}(M)_0\) such that \(\varphi(g)f = g\) \((g \in U)\) and \(\varphi(f) = id_M\).

Consider the restriction map \(\pi : D^r_{X\cup(M\setminus U)}(M) \to E^r_X(N,U), \pi(h) = h|_N\). The group \(D^r_{N\cup(M\setminus U)}(M)\) acts on \(D^r_{X\cup(M\setminus U)}(M)\) by right composition.
Corollary 4.1. (1) The image \( \pi(D^r_{X\cup(M\setminus U)}(M)) \) is open and closed in \( \mathcal{E}_X^r(N,U) \) and the map
\[
\pi : D^r_{X\cup(M\setminus U)}(M) \to \pi(D^r_{X\cup(M\setminus U)}(M))
\]
is a principal bundle with fiber \( D^r_{X\cup(M\setminus U)}(M) \).

(2) The restriction map
\[
\pi : D^r_{X\cup(M\setminus U)}(M)_0 \to \mathcal{E}_X^r(N,U)_0, \quad \pi(h) = h|_N,
\]
is a principal bundle with fiber \( D^r_{X\cup(M\setminus U)}(M)_0 \cap D_N(M) \).

4.3. Diffeomorphism groups of noncompact \( n \)-manifolds.

Suppose \( M \) is a noncompact connected smooth \( n \)-manifold without boundary and \( X \) is a compact smooth submanifold of \( M \). A smooth exhausting sequence of \( (M,X) \) means an exhausting sequence \( \{M_i\}_{i \geq 1} \) of \( M \) such that each \( M_i \) is a compact \( n \)-submanifold of \( M \) for each \( i \geq 1 \) and \( X \subset \text{Int} \ M_i \). Let \( M_0 = X \). Obviously \( (M,X) \) has a smooth exhausting sequence and any smooth exhausting sequence of \( (M,X) \) satisfies the assumption (A) with respect to the diffeomorphism group \( D_X(M) \). Therefore, Theorem 3.1 and Lemma 4.1(3) yield the following consequences.

Proposition 4.1. Suppose \( M \) admits a smooth exhausting sequence \( \{M_i\}_{i \geq 1} \) such that

(i) \( D_{M\cup(M-U)}(M)_0 \simeq * \) for each \( i > k \geq 0 \),

(ii) \( D_{M\cup(M-U)}(M)_0 \cap D_{M,j}(M) = D_{M\cup(M-U)}(M)_0 \) for each \( j > i > k \geq 0 \).

Then, (1) \( D_X(M)_0 \cong \ell^2 \) and (2) \( D_X(M)_0^c \) is HD in \( D_X(M)_0 \).

Proposition 4.2. If \( M = \text{Int} \ N \) for some compact \( n \)-manifold \( N \) with non-empty boundary, then

(1) \( D_X(M)_0 \) is an \( \ell^2 \)-manifold,

(2) if \( D_{\partial N \times \{0\}}(\partial N \times [0,1]) \simeq * \), then \( D_X(M)_0 \cong D_X(N)_0 \),

(3) if \( D_{\partial N \times (0,1)}(\partial N \times [0,1]) \simeq * \), then \( D_X(M)_0^c \) is HD in \( D_X(M)_0 \).

Proof. Take a smooth closed collar \( \partial N \times [0,2] \) of \( \partial N = \partial N \times \{0\} \) in \( N - X \) and let \( M_i = M - (\partial N \times [0,1/i]) \) \( (i \geq 1) \). Then \( \{M_i\}_{i \geq 1} \) is a smooth exhausting sequence of \( (M,X) \). Let \( G = D_X^r(M) \). The tuple \( (G, M, \{M_i\}_{i \geq 1}) \) satisfies the assumption (A).

(1) By Theorem 3.1(2)(i) and Lemma 4.1(3) it suffices to show that \( G_{M_i} = D^r_{M_i}(M) \simeq * \) for each \( i \geq 1 \). This is verified by the Alexander trick towards \( \infty \), since \( M - \text{Int} M_i = \partial N \times (0,1/i] \) is an open collar of \( \partial M_i = \partial N \times \{1/i\} \).

(2) By (1) and Lemma 4.1(2), both \( D_X^r(M)_0 \) and \( D_X^r(N)_0 \) are \( \ell^2 \)-manifolds. Thus, by Theorem 2.1(2) it suffices to show that \( D_X^r(M)_0 \simeq D_X^r(N)_0 \).

Note that (i) \( G_{M_i} \) is an AR and \( (G_{M_1})_0 = G_{M_1} \) by (1) and (ii) \( D^r_{M_1}(N) \) is an AR since it is an ANR and \( D^r_{M_1}(N) \simeq D_{\partial N \times \{1\}}(\partial N \times [0,1]) \simeq * \) by the assumption. The restriction maps
\[
\pi : D_X^r(M)_0 \to \mathcal{E}_X^r(M_1,M)_0 \quad \text{and} \quad \pi_1 : D_X^r(N)_0 \to \mathcal{E}_X^r(M_1,N)_0
\]
are principal bundles with the AR fibers
\[
G_0 \cap G_{M_1} = G_{M_1} \quad \text{and} \quad D_X^r(N)_0 \cap D^r_{M_1}(N) = D^r_{M_1}(N).
\]
Thus, by Lemma 2.2 the maps \( \pi \) and \( \pi_1 \) are homotopy equivalences.
Lemma 5.1. Suppose \( G_{M_i} \cap G_{M_j} \approx \mathcal{H}(\partial N \times [0,2])_0 \cap \mathcal{H}(\partial N \times [1,2]) = \mathcal{H}(\partial N \times [1,2])_0 \) and \( \mathcal{D}_{\partial N \times \{0,1\}}(\partial N \times [0,1])_0 \simeq * \). Since \( (G_{M_i})_0 = G_{M_i} \), the assertion follows from Theorem 3.1 (2)(ii). \( \square \)

Proposition 4.2 now follows from Proposition 4.2(1). In the next section, Propositions 4.1 and 4.2 are used to deduce Theorems 1.1 and 1.2.

5. Diffeomorphism groups of non-compact 2-manifolds

5.1. Fundamental facts on diffeomorphism groups of 2-manifolds.

First we recall some fundamental facts on diffeomorphism groups of compact 2-manifolds. The symbols \( S^1, S^2, T, D, A, P, K \) and \( M \) denote the 1-sphere (circle), 2-sphere, torus, disk, annulus, projective plane, Klein bottle and Möbius band, respectively.

Theorem 5.1. (\cite{9, 26} etc.) Suppose \( M \) is a compact connected smooth 2-manifold (possibly with boundary) and \( X \) is a compact smooth submanifold of \( M \).

(i) (a) \( \mathcal{D}^r(M)_0 \simeq SO(3) \) if \( M \cong S^2, P \). (b) \( \mathcal{D}^r(M)_0 \simeq T \) if \( M \cong T \).

(ii) \( \mathcal{D}^r(M)_0 \simeq \mathbb{S}^1 \) if \( (M, X) \cong (S^2, 1pt), (S^2, 2pt), (D, \emptyset), (D, 0), (A, \emptyset), (M, \emptyset), (K, \emptyset) \).

(iii) \( \mathcal{D}^r(M)_0 \simeq * \) in all other cases.

(iv) \( \mathcal{D}^r(D) \simeq \mathcal{D}^r(D) \cap \mathcal{D}^r(D) \simeq * \) and \( \mathcal{D}^r(M) \simeq * \).

By \cite{9} and a \( C^r \)-analogue of \cite{10} we have

Lemma 5.1. Suppose \( M \) is a compact smooth 2-manifold (possibly with boundary).

1. Suppose \( N \) is a closed collar of \( \partial M \). If \( h \in \mathcal{D}^r_N(M) \) is homotopic to \( id_M \) rel \( N \), then \( h \) is \( C^r \)-isotopic to \( id_M \) rel \( N \) and \( h \in \mathcal{D}^r_N(M)_0 \).

2. Suppose \( N \) is a compact smooth 2-submanifold of \( M \) with \( \partial M \subset N \).

(i) If \( h \in \mathcal{D}^r_N(M) \) is \( C^0 \)-isotopic to \( id_M \) rel \( N \), then \( h \) is \( C^r \)-isotopic to \( id_M \) rel \( N \) and \( h \in \mathcal{D}^r_N(M)_0 \).

(ii) \( h \in \mathcal{D}^r_N(M)_0 \) iff \( h \in \mathcal{D}^r_N(M) \) and \( h \) is \( C^r \)-isotopic to \( id_M \) rel \( N \) (cf. Remark 2.1(2)).

In Corollary 4.1 we have a principal bundle with fiber \( G \equiv \mathcal{D}^r_X(M)_0 \cap \mathcal{D}^r_N(M) \). The next theorem provides us with a sufficient condition which implies \( G = \mathcal{D}^r_N(M)_0 \). The symbol \( \# X \) denotes the number of elements (or cardinal) of a set \( X \).

Theorem 5.2. Suppose \( M \) is a compact connected smooth 2-manifold (possibly with boundary), \( N \) is a compact smooth 2-submanifold of \( M \) with \( \partial M \subset N \) and \( X \) is a subset of \( N \). Suppose \( (M, N, X) \) satisfies the following conditions:

(i) \( M \neq T, P, K \) or \( X \neq \emptyset \).

(ii) (a) if \( H \) is a disk component of \( N \), then \( \#(H \cap X) \geq 2 \),

(b) if \( H \) is an annulus or Möbius band component of \( N \), then \( H \cap X \neq \emptyset \),
(iii) (a) if \( L \) is a disk component of \( \text{cl}(M \setminus N) \), then \( \#(L \cap X) \geq 2 \),
(b) if \( L \) is a Möbius band component of \( \text{cl}(M \setminus N) \), then \( L \cap X \neq \emptyset \).

Then we have

1. if \( h \in \mathcal{D}_N^r(M) \) is \( C^0 \)-isotopic to \( \text{id}_M \) rel \( X \), then \( h \) is \( C^r \)-isotopic to \( \text{id}_M \) rel \( N \),
2. \( \mathcal{D}_N^r(M)_0 \cap \mathcal{D}_N^r(M) = \mathcal{D}_N^r(M)_0 \).

**Lemma 5.2.** Suppose \( M \) is a connected smooth 2-manifold without boundary and \( N \) is a smooth closed 2-submanifold of \( M \). Suppose \( N \neq \emptyset \), \( \text{cl}(M \setminus N) \) is compact and \((M,N)\) satisfies the following conditions:

(i) \( M \not\cong \mathbb{T}, \mathbb{P}, \mathbb{K} \),
(ii) each component \( C \) of \( \partial N \) does not bound a disk or a Möbius band,
(iii) each component of \( N \not\cong S^1 \times [0,1], S^1 \times [0,1] \).

If \( h \in \mathcal{D}_N^r(M) \) is \( C^0 \)-isotopic to \( \text{id}_M \), then \( h \) is \( C^r \)-isotopic to \( \text{id}_M \) rel \( N \) and \( h \in \mathcal{D}_N^r(M)_0 \).

**Theorem 5.3.** and Lemma 5.2 follow from Lemma 5.1 and the corresponding statements in the \( C^0 \)-case [29, Theorem 3.1, Lemma 3.4 (and a remark after Lemma 3.4)].

### 5.2. Diffeomorphism groups of noncompact 2-manifolds.

In this section we prove Theorems 1.1 and 1.2. Suppose \( M \) is a noncompact connected smooth 2-manifold without boundary and \( X \) is a compact smooth submanifold of \( M \) (i.e., a disjoint union of a compact smooth 2-submanifold of \( M \) and finitely many smooth circles and points). We need to separate the next two cases:

(I) \((M,X) = (\text{a plane}, \emptyset), (\text{a plane}, 1\text{pt}), (\text{an open Möbius band}, \emptyset)\) or \((\text{an open annulus}, \emptyset)\).

(II) \((M,X)\) is not Case (I).

Theorems 1.1 and 1.2 are rewritten as follows:

**Theorem 5.3.** (1) \( \mathcal{D}_X^r(M)_0 \) is an \( \ell_2 \)-manifold.

2. \( \mathcal{D}_X^r(M)_0 \cong S^1 \) in Case (I) and \( \mathcal{D}_X^r(M)_0 \cong * \) in Case (II).

3. \( \mathcal{D}_X^r(M)_0^* \) is HD in \( \mathcal{D}_X^r(M)_0 \).

**Proof.** Case (I): From the assumption it follows that \((M,X) = (\text{Int } N,X)\), where \((N,X) = (\mathbb{D},\emptyset), (\mathbb{D}, 1\text{pt}), (\mathbb{M}, \emptyset)\) or \((\mathbb{A}, \emptyset)\). By Theorem 5.1 it is seen that \( \mathcal{D}_X^r(N)_0 \cong S^1 \) and

\[ \mathcal{D}_{L \times \{0\}}^r(L \times [0,1]) \cong \mathcal{D}_{L \times \{0\}}^r(L \times [0,1])_0 \cong * \quad \text{and} \quad \mathcal{D}_{L \times \{0,1\}}^r(L \times [0,1])_0 \cong * \]

for any closed smooth 1-manifold \( L \) (i.e., \( L \) is a disjoint union of finitely many circles). Therefore, from Proposition 1.2 for \( n = 2 \) follows the assertions (1), (3) and (2) (I) \( \mathcal{D}_X^r(M)_0 \cong \mathcal{D}_X^r(N)_0 \cong S^1 \).

Case (II): We can write as \( M = \bigcup_{i \geq 1} M_i \), where \( M_0 = X \) and for each \( i \geq 1 \)

(a) \( M_i \) is a nonempty compact connected smooth 2-submanifold of \( M \) and \( M_{i-1} \subset \text{Int } M_i \),

(b) for each connected component \( L \) of \( \text{cl}(M \setminus M_i) \), \( L \) is noncompact and \( L \cap M_{i+1} \) is connected.

Note that \( M \) is a plane (an open Möbius band, an open annulus) if infinitely many \( M_i \)'s are disks (Möbius bands, annuli respectively), and that any subsequence of \( M_i \) \((i \geq 1)\) also satisfies the conditions (a) and (b). Thus, passing to a subsequence, we may assume that
(i) if $M$ is a plane, then each $M_i$ is a disk,
(ii) if $M$ is an open Möbius band, then each $M_i$ is a Möbius band,
(iii) if $M$ is an open annulus, then each $M_i$ is an annulus (and the inclusion $M_i \subset M_{i+1}$ is essential),
(iv) if $M$ is not a plane, an open Möbius band or an open annulus, then each $M_i$ is not a disk, an annulus or a Möbius band.

For each $i \geq 1$ let $U_i = \text{int} M_i$.

To apply Proposition 1.1 we have to verify the following conditions:

\[(z)_1 \quad D^r_{M_k \cup (M - U_j)}(M)_0 \simeq \ast \quad \text{for each } i > k \geq 0,\]
\[(z)_2 \quad G^i_{k,j} \equiv D^r_{M_k \cup (M - U_j)}(M)_0 \cap D^r_{M_j}(M) = D^r_{M_k \cup (M - U_j)}(M)_0 \quad \text{for each } j > i > k \geq 0.\]

Choose a small closed collar $E_i$ of $\partial M_i$ in $U_{i+1} \setminus U_i$ and set $M'_i = M_i \cup E_i \subset U_{i+1}$.

\[(z)_1: \quad D^r_{M_k \cup (M - U_j)}(M)_0 \simeq D^r_{M_k \cup E_j}(M'_i) \simeq \ast \quad \text{by Theorem 5.1}\]
\[(z)_2: \quad \text{It suffices to show that}\]
\[D^r_{M_k \cup E_j}(M'_i)_0 \cap D^r_{M_k \cup E_j}(M'_j) = D^r_{M_k \cup E_j}(M'_j)_0.\]

We apply Theorem 5.2 to $(\tilde{M}, \tilde{N}, \tilde{X}) = (M'_j, M_i \cup E_j, M_k \cup E_j)$. It remains to verify that this triple satisfies the conditions (i)–(iii) in Theorem 5.2.

(ii): The components of $\tilde{N}$ consists of $M_i$ and the annulus components of $E_j$. The latter components obviously satisfy the condition (ii) (b). By the choice of $\{M_j\}_{j \geq 1}$, if $M_i \cong \mathbb{D} (\mathbb{M}, A)$, then $M$ is a plane (an open Möbius band, an open annulus). Thus the assumption of Case (II) implies that $(M, X) \neq (\mathbb{D}, \emptyset)$, $(\mathbb{D}, 1 \text{ pt})$, $(\mathbb{M}, \emptyset)$, $(A, \emptyset)$ and that $M_i$ satisfies the condition (ii).

(iii): Each component $L$ of $\partial (\tilde{M} \setminus \tilde{N}) = M_j \setminus U_i$ meets both $\partial M_i$ and $\partial M_j$. Thus $\partial L$ is not connected and $L \not\cong \mathbb{D}, \mathbb{M}$. This completes the proof.

\[\square\]

**Proof of Corollary 1.2** Since $D^r_X(M)_0^* \subset D^r_X(M)_0 \subset D^r_X(M)_0$, the subgroup $D^r_X(M)_0^*$ is also HD in $D^r_X(M)_0$. Thus, the assertion (1) follows from Theorem 1.1(1) and Lemma 2.3(1)(ii), while the assertion (2) follows from Lemma 2.3(1)(i).

\[\square\]

**Proof of Proposition 1.1** For notational simplicity we set $G = D^r_X(M)$. We choose an exhausting sequence $\{M_i\}_{i \geq 1}$ of $(M, X)$ as in the proof of Theorem 5.3 Case (II).

[A] If $(M, X)$ satisfies the condition (a) or (b), then $G_0^c = G_0^c*$:

Given $h \in G_0^c$, there exists $i \geq 1$ such that $h \in G(U_i)$. We show that $h \in G(U_i)_0 \subset G_0^c*$. If $(M, X)$ satisfies the condition (b), then by Theorem 5.1(iv) $G(U_i) \cong D^r_{E_i}(M_i \cup E_i) \cong D^r_{M_i}(M_i) \cong \ast$, where $E_i$ is a small closed collar of $\partial M_i$ in $M \setminus U_i$. This means that $G(U_i) = G(U_i)_0$. Below we assume that $(M, X)$ does not satisfy the condition (b). Note that $h$ is $C^0$-isotopic to $id_M$ rel $X$ since $h \in G_0$ and $G_0$ is a connected ANR.

(1) The case (a) with $X = \emptyset$.

Let $N = M \setminus U_i$ and we apply Lemma 5.2 to $(M, N)$ and $h$. The conditions (ii) and (iii) in Lemma 5.2 are verified as follows:
(ii) If a circle component $C$ of $\partial N = \partial M_i$ bounds a disk or a Möbius band $D$, then $M_i = D$ and by the choice of the exhausting sequence $\{M_i\}_{i \geq 1}$ it follows that $M$ is a plane or an open Möbius band. This contradicts the assumption that $(M, X)$ does not satisfy the condition (b).

(iii) Any component of $N$ is noncompact and not diffeomorphic to $S^1 \times [0, 1)$ by the condition (a).

Since $h \in G_0$ is $C^0$-isotopic to $id_M$, from Lemma [5.2] it follows that $h$ is $C^r$-isotopic to $id_M$ rel $N$ and $h \in G(U_i)_0$.

(2) The case (a) with $X \neq \emptyset$:

Let $F_k$ ($k = 1, \cdots, m$) denote the connected components of the compact 2-submanifold $cl(M_i \setminus X)$ which are disks or Möbius bands let $H_\ell$ ($\ell = 1, \cdots, n$) denote the remaining components. Set $N = X \cup (\cup_k F_k) \cup (M \setminus U_i)$. There exists a $C^0$-isotopy $h_t$ rel $X$ from $h$ to $id_M$. Since $\partial F_k \subset X$, it follows that $h_t$ maps each $F_k$ onto itself. Thus, if we define $h' = h_U = h$ on $X \cup (\cup_t H_\ell) \cup (M \setminus U_i)$ and $h' = id$ on $\cup_k F_k$, then $h'$ is $C^0$-isotopic to $h$ rel $X \cup (M \setminus U_i)$ and $C^0$-isotopic to $id_M$ rel $X$.

Define $A \subset X$ by choosing two interior points from each component of $X$. Set $M' = M \setminus A$ and $N' = N \setminus A$. Then $h'|_{M'} \in D_{N'}(M')$ is $C^0$-isotopic to $id_{M'}$. We show that $(M', N')$ satisfies the conditions (i)–(iii) in Lemma [5.2]. First note that $M'$ is noncompact and $L \equiv cl(M' \setminus N') = cl(M \setminus N) = \cup_\ell H_\ell$ is compact.

(ii) Suppose a circle component $C$ of $\partial N = \partial M_i \cup (\partial X \setminus (\cup_k \partial F_k))$ bounds a disk or a Möbius band $D$ in $M'$. Then $M$ is the union of $D$ and a connected submanifold $W = cl(M - D)$ with $\partial W = C$, and it follows that $N \subset W$. In fact, (a) $X \subset W$ since $A \subset W$ and each component of $X - A$ meets $A$.

(b) each $F_k \subset W$ since $F_k$ meets $X$ but does not meet $C$ and (c) $M \setminus U_i \subset W$ since each component of $M \setminus U_i$ is noncompact. If $C \subset \partial X \setminus (\cup_k \partial F_k)$, Thus $D \subset \cup_\ell H_\ell$ and $D = H_\ell$ for some $\ell$, but this contradicts the choice of $H_\ell$. If $C \subset \partial M_i$, then $D \supset M_i \supset A$, which contradicts that $D \subset M'$.

(iii) Let $J$ be any component of $N'$. If $J \subset (X \setminus A) \cup (\cup_k F_k)$, then $J$ has two ends, since it is the union of $X_1 \setminus A$ for some component $X_1$ of $X$ and some $F_k$’s. Otherwise, $J$ is a component of $M \setminus U_i$, so it is noncompact and $J \not\subset S^1 \times [0, 1)$ by the condition (a).

Therefore, by Lemma [5.2] $h'|_{M'}$ is $C^0$-isotopic to $id_{M'}$ rel $N'$ and by the end compactification we obtain a $C^0$-isotopy $h \simeq h' \simeq id_M$ rel $X \cup (M \setminus U_i)$. This implies that $h \in G(U_i)_0$.

[B] If $(M, X)$ does not satisfy the conditions (a) and (b). Then $M$ contains a product end $E \cong S^1 \times [0, 1)$ such that $E \cap X = \emptyset$. We identify $E$ with $S^1 \times [0, 1)$. Note that $F = cl(M \setminus E)$ is a connected submanifold of $M$ and $\partial F = \partial E$. Let $h \in G^c$ denote a Dehn twist in $E$. We show that $h \in G^c \setminus G^c_0$. A $C^r$-isotopy $h_t : h \simeq id_M$ with $h_t \in G^c_0$ is obtained by sliding the Dehn twist towards $\infty$. This implies that $h \in G^c_0$. It remains to show that $h \not\in G^c_0$. On the contrary, suppose $h \in G^c_0$. Then $h \in G(M_i)_0$ for some $i \geq 1$ and there exists a $C^0$-isotopy $h_t : h \simeq id_M$ rel $X$ with supp $h_t \subset M_i$ ($0 \leq t \leq 1$).

If $F \cong S^1 \times [0, 1)$, then $M \cong S^1 \times (-1, 1)$ and $M_i$ is contained in an annulus $L$ in $M$. This means that the Dehn twist in $L$ is $C^0$-isotopic to $id_L$ rel $\partial L$, a contradiction. Thus we may assume that $F \not\cong S^1 \times [0, 1)$.
Choose \( t \in (0, 1) \) such that \( M_t \subset F \cup L \), where \( L = S^1 \times [0, t] \) is an annulus. Let \( Y = S^1 \times [t, 1] \) (= \( cl(E \setminus L) \)) and \( N = F \cup Y \). Then \( h|_N = id \) and \( h_t : h \simeq id_M \) is a \( C^0 \)-isotopy \( rel \; X \cup Y \). Define \( A \) by choosing an interior point from \( Y \) and two interior points from \( X \) if \( X \neq \emptyset \). Let \( M' = M \setminus A \) and \( N' = N \setminus A \). We show that \((M', N')\) satisfies the conditions (i)–(iii) in Lemma 5.2.

(ii) \( \partial N' = \partial N \) consists of two circles \( C_1 = S^1 \times \{0\} \) and \( C_2 = S^1 \times \{t\} \). If \( C_1 \) bounds a disk or a M"obius band \( D \) in \( M' \), then \( F = D \subset M' \), so we have \( X = \emptyset \) and \( M = D \cup E \) is a plane or an open M"obius band (i.e. the condition (b)). This contradicts the assumption in [B]. Similarly, the circle \( C_2 \) bound neither a disk nor a M"obius band in \( M' \).

(iii) Note that \( N' \) consists of the two components \( F \setminus A \) and \( Y \setminus A \). These are homeomorphic to neither \( S^1 \times [0, 1] \) nor \( S^1 \times (0, 1) \). In fact, \( Y \setminus A \) has two ends, and if \( X \neq \emptyset \), then \( F \setminus A \) has at least two ends. If \( X = \emptyset \), then \( F \setminus A = F \not\supseteq S^1 \times [0, 1] \) and \( \partial F \) is a circle.

Since \( h' = h|_{M'} \) is \( C^0 \)-isotopic to \( id_{M'} \) and \( h'|_{N'} = id \), by Lemma 5.2 we obtain a \( C^0 \)-isotopy \( h' \simeq id_{M'} \; rel \; N' \). This means that the Dehn twist \( h|_L \) on the annulus \( L \) is isotopic to \( id_L \; rel \; \partial L \). This is a contradiction. This completes the proof. \( \square \)

6. Groups of volume-preserving diffeomorphisms of noncompact 2-manifolds

In this final section we discuss topological types of groups of volume-preserving diffeomorphisms of noncompact 2-manifolds with the compact-open \( C^\infty \)-topology. Since we are only concerned with the groups of \( C^\infty \)-diffeomorphisms with the compact-open \( C^\infty \)-topology, we omit the symbol \( \infty \) from the notations.

6.1. General properties of groups of volume-preserving diffeomorphisms of \( n \)-manifolds.

Suppose \( M \) is a connected oriented smooth \( n \)-manifold possibly with boundary, \( X \subsetneq M \) is a closed subset of \( M \) and \( \omega \) is a positive volume form on \( M \). Let \( \mathcal{D}_X(M; \omega) \subset \mathcal{D}^+_X(M) \) denote the subgroups of \( \mathcal{D}_X(M) \) consisting of \( \omega \)-preserving diffeomorphisms and orientation-preserving diffeomorphisms of \( M \) respectively, that is,

\[ \mathcal{D}_X(M; \omega) = \{ h \in \mathcal{D}_X(M) \mid h^* \omega = \omega \}. \]

These subgroups are endowed with the subspace topology (i.e., the compact-open \( C^\infty \)-topology). As before, the subscript ‘0’ denotes the identity connected component. Note that \( \mathcal{D}^+_X(M)_0 = \mathcal{D}_X(M)_0 \).

The group \( \mathcal{D}_X(M; \omega) \) is a separable, completely metrizable topological group since it is a closed subgroup of \( \mathcal{D}_X(M) \). It is also seen to be infinite-dimensional and non locally compact. Hence, by Theorem 2.1 the group \( \mathcal{D}_X(M; \omega) \) (or \( \mathcal{D}_X(M; \omega)_0 \)) is an \( \ell_2 \)-manifold iff it is an ANR.

First we recall Moser’s theorem [21] and its extension to the noncompact case [31]. (We refer to [1] [5] [31] for end compactifications and related matters.) The space \( E_M \) of ends of \( M \) is a compact 0-dimensional metrizable space. Let \( E^\omega_M \) denote the subspace of \( E_M \) consisting of \( \omega \)-finite ends of \( M \).

Each \( h \in \mathcal{H}(M) \) admits a unique homeomorphic extension \( \overline{h} \) on the end compactification \( M \cup E_M \). We define \( \mathcal{D}^+(M, E^\omega_M) = \{ h \in \mathcal{D}^+(M) \mid \overline{h}(E^\omega_M) = E^\omega_M \} \).

Let \( \mathcal{V}^+(M; \omega(M), E^\omega_M)_{c/w} \) denote the space of positive volume forms \( \mu \) on \( M \) such that \( \mu(M) = \omega(M) \) and \( E^\mu_M = E^\omega_M \). This space is endowed with the finite-ends weak \( C^\infty \)-topology \( c/w \) (cf. [5] [31]).
The group $\mathcal{D}^+(M,E_M^\omega)$ acts on the space $\mathcal{V}^+(M;\omega(M),E_M^\omega)$ by the push-forward of forms and induces the orbit map at $\omega$,

$$\pi_\omega : \mathcal{D}^+(M,E_M^\omega) \rightarrow \mathcal{V}^+(M;\omega(M),E_M^\omega)_{ew}.$$ 

Moser’s theorem [21] and [31, Corollary 1.2] assert that this orbit map admits a section into $\mathcal{D}_0(M)_0$. This implies the following relation between the groups $\mathcal{D}_X(M;\omega) \subset \mathcal{D}_X(M)$.

**Proposition 6.1.**

1. $(\mathcal{D}^\infty+)(M,E_M^\omega), \mathcal{D}(M;\omega)) \cong (\mathcal{V}^+(M;\omega(M),E_M^\omega)_{ew},\{\omega\}) \times \mathcal{D}(M;\omega),$

$(\mathcal{D}(M)_0, \mathcal{D}(M;\omega)_0) \cong (\mathcal{V}^+(M;\omega(M),E_M^\omega)_{ew},\{\omega\}) \times \mathcal{D}(M;\omega)_0.$

2. Suppose $N$ is a compact $n$-submanifold of $\text{Int} M$.

   (i) $\mathcal{D}_{M-N}(M;\omega)$ is a SDR of $\mathcal{D}_{M-N}^+(M)$ and is an ANR (since $\mathcal{D}_{M-N}(M)$ is an ANR).

   (ii) $\mathcal{D}_{N}(M;\omega)$ is a SDR of $\mathcal{D}_{N}^+(M,E_M^\omega)$ and $\mathcal{D}_{N}(M;\omega)_0$ is a SDR of $\mathcal{D}_{N}(M)_0$.

In the statement (2) we apply Moser’s theorem [21] and [31, Corollary 1.1] to each $L \in \mathcal{C}(M-\text{Int} N)$.

Next we recall the definition of the end charge homomorphism introduced by S. R. Alpern and V. S. Prasad [1]. Suppose $M$ is a noncompact connected orientable smooth $n$-manifold possibly with boundary and $\omega$ is a volume form on $M$. An end charge of $M$ is a finitely additive signed measure on the algebra of clopen subsets of $E_M$. Let $\mathcal{S}(M)$ denote the topological linear space of all end charges of $M$ with the weak topology and let $\mathcal{S}(M;\omega)$ denote the linear subspace of $\mathcal{S}(M)$ consisting of end charges $c$ of $M$ with $c(E_M^0) = 0$ and $c|E_M^\omega = 0$.

Let $\mathcal{D}_{E_M}(M;\omega) = \{h \in \mathcal{D}(M;\omega) \mid \overline{h}|_{E_M} = \text{id}_{E_M}\}$. We have $\mathcal{D}_{E_M}(M;\omega)_0 = \mathcal{D}(M;\omega)_0$. For each $h \in \mathcal{D}_{E_M}(M;\omega)$ an end charge $c^\omega(h) \in \mathcal{S}(M;\omega)$ is defined by

$$c^\omega(h)(E_C) = \omega(C - h(C)) - \omega(h(C) - C),$$

where $C$ is any $n$-submanifold of $M$ such that $\text{Fr}_MC$ is compact and $E_C \subset E_M$ is the set of ends of $C$. The end charge homomorphism

$$c^\omega : \mathcal{D}_{E_M}(M;\omega) \rightarrow \mathcal{S}(M;\omega) : h \mapsto c^\omega(h)$$

is a continuous group homomorphism. Let $c^\omega_0 : \mathcal{D}(M;\omega)_0 \rightarrow \mathcal{S}(M;\omega)$ denote the restriction of $c^\omega$ to $\mathcal{D}(M;\omega)_0$.

The kernels $\ker c^\omega$ and $\ker c^\omega_0$ are separable, non locally compact, completely metrizable topological groups. Hence, by Theorem [21] the group $c^\omega$ (or $c^\omega_0$) is an $\ell_2$-manifold iff it is an ANR. In [31, Corollary 1.2] we have shown that the homomorphism $c^\omega$ has a continuous (non-homomorphic) section into $\mathcal{D}_0(M;\omega)_0$. This clarifies the relation between the groups $\ker c^\omega_0 \subset \mathcal{D}(M;\omega)_0$.

**Proposition 6.2.**

1. $(\mathcal{D}_{E_M}(M;\omega), \ker c^\omega) \cong (\mathcal{S}(M;\omega),0) \times \ker c^\omega$, $(\mathcal{D}(M;\omega)_0, \ker c^\omega_0) \cong (\mathcal{S}(M;\omega),0) \times \ker c^\omega_0$.

2. (i) $\ker c^\omega$ is a SDR of $\mathcal{D}_{E_M}(M;\omega)$.

   (ii) $\ker c^\omega_0 = (\ker c^\omega)_0$ and it is a SDR of $\mathcal{D}(M;\omega)_0$. 

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6.2. The bundle theorem.

Next we obtain the bundle theorems for groups of volume-preserving diffeomorphisms. Suppose $M$ is a connected oriented smooth $n$-manifold without boundary, $\omega$ is a positive volume form on $M$, $N$ is a compact smooth $n$-submanifold of $M$, $X$ is a closed subset of $N$ and $N_0$ is smooth $n$-submanifold of $M$ such that $N \subset U_0 \equiv \text{Int} N_0$.

For notational simplicity we set $(G,H,F) = (\mathcal{D}(M), \mathcal{D}(M;\omega), \ker \phi^*)$. Let $\mathcal{V}^+(M)$ denote the space of positive volume forms on $M$ endowed with the compact-open $C^\infty$-topology. The group $G_0$ acts continuously on $\mathcal{V}^+(M)$ from the right by the pullback $\mu \cdot h = h^*\mu$. For any subset $\mathcal{F}$ of $\mathcal{E}^\infty_X(N,M)$ the symbol $\mathcal{F}^{co}$ denotes the space $\mathcal{F}$ endowed with the subspace topology (= the compact-open $C^\infty$-topology). When $i_N \in \mathcal{F}^{co}$, the symbol $\mathcal{F}^{co}_0$ denotes the connected component of the inclusion $i_N : N \subset M$ in $\mathcal{F}^{co}$. Hence, the space $\mathcal{E}^H_X(N,U_0)$ carries the quotient topology, while $\mathcal{E}^H_X(N,U_0)^{co}$ carries the compact-open $C^\infty$-topology. Let $\mathcal{C}(Y)$ denote the set of connected components of a space $Y$.

The extension theorems for the transformation groups $H$ and $F$ are summarized as follows:

**Theorem 6.1.** (H) There exists a neighborhood $\mathcal{U}$ of $i_N$ in $\mathcal{E}^H_X(N,U_0)^{co}$ and a map $\varphi : \mathcal{U} \rightarrow H(U_0) \cap H_X$ such that $\varphi(f)|_N = f$ ($f \in \mathcal{U}$) and $\varphi(i_N) = \text{id}_M$.

(F) Suppose $\partial N_0$ is compact and $U$ is an open neighborhood of $N$ in $U_0$ such that $U \cap L$ is connected for each $L \in \mathcal{C}(N_0 - \text{Int} N)$. Then there exists a neighborhood $\mathcal{U}$ of $i_N$ in $\mathcal{E}^H_X(N,U_0)^{co}$ and a map $\varphi : \mathcal{U} \rightarrow F(U_0) \cap F_X$ such that $\varphi(f)|_N = f$ ($f \in \mathcal{U}$) and $\varphi(i_N) = \text{id}_M$.

**Proof.** In each case we may assume that $X = \emptyset$, since $\varphi(f)|_X = i_X$ if $\varphi(f)|_N = f$ and $f|_X = i_X$.

(H) Choose any compact smooth $n$-submanifold $N_1$ of $U_0$ with $N \subset U_1 \equiv \text{Int} N_1$. Consider the subspace of $\mathcal{V}^+(M)$ defined by

$$\mathcal{V}^\omega_X(U_1) = \{ \mu \in \mathcal{V}^+(M) \mid \mu = \omega \text{ on } N \cup (M - U_1) \text{ and } \mu(L) = \omega(L) \text{ for each } L \in \mathcal{C}(N_0 - \text{Int} N) \}.$$

Applying Moser’s theorem [21] or [31, Corollary 1.1] to each $L \in \mathcal{C}(N_0 - \text{Int} N)$ we obtain a map

$$\eta : \mathcal{V}^\omega_X(U_1) \rightarrow G_N(U_0)$$

such that $\eta(\mu)^*\mu = \omega$ ($\mu \in \mathcal{V}^\omega_X(U_1)$) and $\eta(\omega) = \text{id}_M$.

By Theorem [41] there exists a neighborhood $\mathcal{U}_1$ of $i_N$ in $\mathcal{E}^\infty(N,U_1)$ and a map

$$\psi : \mathcal{U}_1 \rightarrow G(U_1)$$

such that $\psi(f)|_N = f$ ($f \in \mathcal{U}_1$) and $\psi(i_N) = \text{id}_M$. Then $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{E}^H(N,U_0)^{co}$ is a neighborhood of $i_N$ in $\mathcal{E}^H(N,U_0)^{co}$.

The map $\psi$ induces a map

$$\chi : \mathcal{U} \rightarrow \mathcal{V}^\omega_X(U_1), \quad \chi(f) = \psi(f)^*\omega.$$

We verify that $\chi(f) \in \mathcal{V}^\omega_X(U_1)$ ($f \in \mathcal{U}$). Since $f \in \mathcal{E}^H(N,U_0)$ there exists a $h \in H(U_0)$ with $h|_N = f$. Since $\psi(f)|_N = f$ and $\psi(f) = \text{id}$ on $M - U_1$, we have $\chi(f)|_N = f^*\omega = (h^*\omega)|_N = \omega|_N$ and $\chi(f)|_{M - U_1} = \omega|_{M - U_1}$. Since $h^{-1}\psi(f) \in G_N(U_0)$, for each $L \in \mathcal{C}(N_0 - \text{Int} N)$ it follows that $h^{-1}\psi(f)(L) = L$ and $\psi(f)(L) = h(L)$, and that $\chi(f)(L) = \omega(\psi(f)(L)) = \omega(h(L)) = \omega(L)$.
Finally, the required map \( \varphi : U \to H(U_0) \) is defined by

\[
\varphi(f) = \psi(f)\eta(\chi(f)) \quad (f \in U).
\]

By Proposition 6.1(2)(ii) we have \( \varphi(f) \in H(U_0) \cap G(U_0) = H(U_0) \).

(F) By the assumption we can find a compact smooth \( n \)-submanifold \( N_1 \) of \( M \) such that \( N \subset \text{Int} N_1 \), \( N_1 \subset U \) and \( N_1 \cap L \) is connected for each \( L \in C(N_0 - \text{Int} N) \). Let \( U_1 = \text{Int} N_1 \) and \( N_1^* = N_1 - \text{Int} N \).

Consider the subspace of \( V^+(M) \) defined by

\[
V^+_N(U_1) = \{ \mu \in V^+(M) \mid \mu = \omega \text{ on } N \cup (M - U_1) \text{ and } \mu(K) = \omega(K) \text{ for each } K \in C(N_1^*) \}.
\]

Applying Moser’s theorem [21] to each \( K \in C(N_1^*) \), we obtain a map

\[
\eta : V^+_N(U_1) \to G_N(U_1)
\]

such that \( \eta(\mu)^*\mu = \omega (\mu \in V) \) and \( \eta(\omega) = \text{id}_M \).

By Theorem 6.1 there exists a neighborhood \( U_1 \) of \( i_N \) in \( E^\infty(N, U_1) \) and a map

\[
\psi : U_1 \to G(U_0)
\]

such that \( \psi(f)|_N = f \ (f \in U_1) \) and \( \psi(i_N) = \text{id}_M \). Then \( U = U_1 \cap E^F(N, U_0)^{co} \) is a neighborhood of \( i_N \in E^F(N, M)^{co} \).

The map \( \psi \) induces a map

\[
\chi : U \to V^+_N(U_1), \quad \chi(f) = \psi(f)^*\omega.
\]

We show that \( \chi(f) \in V^+_N(U_1) \ (f \in U) \). Since \( f \in E^F(N, U_0) \) there exists an \( h \in F(U_0) \) with \( h|_N = f \). Since \( \psi(f)|_N = f \) and \( \psi(f) = \text{id} \) on \( M - U_1 \), we have \( \chi(f)|_N = f^*\omega = (h^*\omega)|_N = \omega|_N \) and \( \chi(f)|_{M - U_1} = \omega|_{M - U_1} \). By the choice of \( N_1 \), for each \( K \in C(N_1^*) \) there exists a unique \( L \in C(N_0 - \text{Int} N) \) with \( K = N_1 \cap L \). Since \( h^{-1}\psi(f) \in G_N(U_0) \), it follows that \( h^{-1}\psi(f)(L) = L \) and \( \psi(f)(L) = h(L) \). Since \( L \) is an \( n \)-submanifold of \( M \) and \( \text{Fr} \ L = \partial L \subset \partial N \cup \partial N_0 \) is compact, from the definition of the end charge \( \epsilon_0^\omega(h) \) it follows that

\[
\epsilon_0^\omega(h)(E_L) = \omega(L \setminus h(L)) - \omega(h(L) \setminus L).
\]

Since \( \epsilon_0^\omega(h) = 0 \), we have \( \omega(L \setminus h(L)) = \omega(h(L) \setminus L) \). Let \( L_1 = L - N_1 \). Then, \( L_1 = \psi(f)(L_1) \subset h(L) \)

and it is seen that

\[
K = L - L_1 = [L \setminus h(L)] \cup [(h(L) \cap L) \setminus L_1] \quad \text{and}
\]

\[
\psi(f)(K) = \psi(f)(L - L_1) = \psi(f)(L) - L_1 = h(L) - L_1 = [h(L) \setminus L] \cup [(h(L) \cap L) \setminus L_1].
\]

Thus we have \( \chi(f)(K) = \omega(\psi(f)(K)) = \omega(K) \) and this means that \( \chi(f) \in V^+_N(U_1) \).

Since \( H(U_1) \) is a SDR of \( G(U_1) \), we have \( H(U_1) \cap G(U_1)_0 = H(U_1)_0 \subset F(U)_0 \). Therefore, the required map \( \varphi : U \to F(U)_0 \) is defined by

\[
\varphi(f) = \psi(f)\eta(\chi(f)) \quad (f \in U).
\]

\[\square\]

**Remark 6.1.** The statement (F) does not necessarily hold when \( \partial N_0 \) is non compact. An example is easily obtained by inspecting a case where two ends of \( N_0 \) is included in an end of \( M \).
Consider the restriction map $\pi : F_X(U) \rightarrow E_X^F(N,U)^{\circ \circ}, \pi(h) = h|_N$. The group $F_N(U)$ acts on $F_X(U)$ by the right translation. The next corollary easily follows form Theorem 6.1 (F) and Lemma 3.1.

**Corollary 6.1.** Suppose $\partial N_0$ is compact and $U$ is an open neighborhood of $N$ in $U_0$ such that $U \cap L$ is connected for each $L \in C(N_0 - \text{Int } N)$. Then, the following hold.

1. The subspace $E_X^F(N,U)^{\circ \circ}$ is open in $E_X^F(N,U_0)^{\circ \circ}$ and the restriction map
   \[ \pi : F_X(U) \rightarrow E_X^F(N,U)^{\circ \circ} \]
   is a principal bundle with fiber $F_N(U)$.
2. If $F_X(U)_0$ is open in $F_X(U)$, then the subspace $E_X^F(N,U_0)^{\circ \circ}$ is closed and open in $E_X^F(N,U)^{\circ \circ}$ and the restriction map
   \[ \pi : F_X(U)_0 \rightarrow E_X^F(N,U_0)^{\circ \circ} \]
   is a principal bundle with fiber $F_X(U)_0 \cap F_N$.

We also need the following complementary results to Theorem 6.1.

**Lemma 6.1.** Suppose $\partial N_0$ is compact and $F_X(N_1)_0$ is open in $F_X(N_1)$ for any compact $n$-submanifold $N_1$ of $U_0$ with $X \subset \text{Int } N_1$.

1. Suppose $f \in E_X^F(N,U_0)^{\circ \circ}$ and $U$ is an open neighborhood of $f(N)$ in $U_0$ such that $U \cap L$ is connected for each $L \in C(N_0 - \text{Int } f(N))$. Then there exists a neighborhood $V$ of $f$ in $E_X^F(N,U_0)^{\circ \circ}$ and a map $\psi : V \rightarrow F_X(U)_0$ such that $\psi(g)f = g$ ($g \in V$) and $\psi(f) = \text{id}_M$.
2. Each $f \in E_X^F(N,U_0)_0^{\circ \circ}$ satisfies the following condition.
   (*) There exists a compact $n$-submanifold $N_1$ of $U_0$ such that $E_X^F(N,N_1)^{\circ \circ}$ is a neighborhood of $f$ in $E_X^F(N,U_0)_0^{\circ \circ}$.

**Proof.** (1) We may assume that $U = \text{Int } N_1$ for some compact $n$-submanifold $N_1$ of $U_0$. Applying Theorem 6.1 (F) to $(M,N_0,U,f(N),X)$, we obtain a neighborhood $U$ of $f(N)$ in $E_X^F(f(N),U_0)^{\circ \circ}$ and a map $\varphi : U \rightarrow F(U)_0 \cap F_X$ such that $\varphi(k)|_{f(N)} = k$ ($k \in U$) and $\varphi(i_{f(N)}) = \text{id}_M$. By the assumption $F_X(U)_0$ is open in $F_X(U)$ and so we may assume that $\varphi(U) \subset F_X(U)_0$.

Consider the homeomorphism
\[ \chi : E_X^F(f(N),U_0)^{\circ \circ} \approx E_X^F(N,U_0)^{\circ \circ}, \quad \chi(k) = kf. \]
Then $V = \chi(U)$ and $\psi = \varphi^{-1} : V \rightarrow F_X(U)_0$ satisfy the required conditions.

(2) We have to show that the subset $F = \{ f \in E_X^F(N,U_0)_0^{\circ \circ} \mid (*) \}$ coincides with $E_X^F(N,U_0)_0^{\circ \circ}$. Corollary 6.1 implies that $i_N \in F$. Therefore, it suffices to show that $F$ is closed and open in $E_X^F(N,N_1)^{\circ \circ}$. From the definition itself we see that $F$ is open in $E_X^F(N,N_1)_0^{\circ \circ}$.

To see that $F$ is closed, take any $f \in \text{cl } F$. There exists a compact $n$-submanifold $N_f$ of $U_0$ such that $f(N) \subset U_f \equiv \text{Int } N_f$ and $U_f \cap L$ is connected for each $L \in C(N_0 - \text{Int } f(N))$. By (1) we obtain a neighborhood $V$ of $f$ in $E_X^F(N,U_0)_0^{\circ \circ}$ and a map $\psi : V \rightarrow F_X(U_f)_0$ such that $\psi(g)f = g$ ($g \in V$) and $\psi(f) = \text{id}_M$. Since $V \cap F \neq \emptyset$, we can choose a $g \in V \cap F$. Since $g \in V$ it follows that $\psi(g)f = g$ and $f = \psi(g)^{-1}g$. In turn, since $g \in F$, there exists a compact $n$-submanifold $N_g$ of $U_0$
such that \(N \subset U_g = \text{Int} N_g\) and \(g \in \mathcal{E}_X^F(N, N_g)_0^\omega\). We may assume that \(U_g \cap L\) is connected for each \(L \in \mathcal{C}(N_0 - \text{Int} N)\). Then, by Corollary 6.1 the restriction map
\[
\pi : F_X(U_g)_0 \to \mathcal{E}_X^F(N, U_g)_0^\omega
\]
is a principal bundle. Hence, there exists an \(h \in F_X(U_g)_0\) such that \(g = h|_N\).

Take a compact \(n\)-submanifold \(N_1\) of \(U_0\) such that \(N_f \cup N_g \subset N_1\). Then, we have \(V \subset \mathcal{E}_X^F(N, N_1)_0^\omega\). In fact, for any \(k \in V\), it follows that \(\psi(k)\psi(g)^{-1} h \in \mathcal{E}_X^F(U_f)_0(\mathcal{E}_X^F(U_g)_0) \subset \mathcal{E}_X^F(N_1)_0\) and that \(k = \psi(k) f = \psi(k) \psi(g)^{-1} h|_N \in \mathcal{E}_X^F(N, N_1)_0^\omega\). This means that \(f \in \mathcal{F}\). This completes the proof. \(\Box\)

6.3. Groups of volume-preserving diffeomorphisms of noncompact \(n\)-manifolds.

Suppose \(M\) is a noncompact connected orientable smooth \(n\)-manifold without boundary, \(\omega\) is a volume form on \(M\) and \(X\) is a compact smooth \(n\)-submanifold of \(M\). We set \((G, H, F) = (\mathcal{D}(M), \mathcal{D}(M; \omega), \ker \omega)\). Choose a smooth exhausting sequence \(\{M_i\}_{i \geq 0}\) of \(M\) such that \(M_0 = X\) and for each \(i \geq 1\)

(a) \(M_i\) is connected,
(b) \(L\) is noncompact and \(L \cap M_{i+1}\) is connected for each \(L \in \mathcal{C}(M - \text{Int} M_i)\).

Let \(U_i = \text{Int} M_i\) \((i \geq 1)\).

Lemma 6.2. The tuple \((F_X, M, \{M_i\}_{i \geq 1})\) satisfies the assumption \((A)\).

Proof. (A-0) Since \((G, M)\) has a weak topology, so does \((F, M)\).

(A-1) Corollary 6.1 implies the following conclusions. For each \(j > i > k \geq 0\)

(a) \(\mathcal{E}_{M_k}^F(M_i, U_j)^{\omega} = \mathcal{E}_{M_i}^F(M_i, U_j)\) and \(\mathcal{E}_{M_k}^F(M_i, M)^{\omega} = \mathcal{E}_{M_k}^F(M_i, M)\), since the restriction maps \(\pi_{i,j} : F_{M_k}(U_j) \to \mathcal{E}_{M_k}^F(M_i, U_j)^{\omega}\) and \(\pi_k : F_{M_k} \to \mathcal{E}_{M_k}^F(M_i, M)^{\omega}\) are principal bundles,
(b) the restriction map \(\pi_{i,j} : F_{M_k}(U_j)_0 \to \mathcal{E}_{M_k}^F(M_i, U_j)_0\) is a principal bundle with the structure group \(F_{M_k}(U_j)_0 \cap F_{M_i}\).

(A-2)(i) \(F_X(U_i) = H_X(U_i)\) is an ANR for each \(i \geq 1\).

(A-2)(ii) By (A-1)(a) we can work under the compact-open \(C^\infty\)-topology. Let \(U_{k,j}^i = \mathcal{E}_{M_k}^F(M_i, U_j)^{\omega}\)

\((j > i > k \geq 0)\).

(a) \(U_{k,j}^i\) is an open subspace of \(\mathcal{E}_{M_k}^F(M_i, M)^{\omega}\): This follows from Corollary 6.1.

(b) \(\mathcal{E}_{M_k}^F(M_i, M)^{\omega} = \cup_{j > i} U_{k,j}^i\): By Lemma 6.1(2) for each \(f \in \mathcal{E}_{M_k}^F(M_i, M)^{\omega}\) there exists a compact \(n\)-submanifold \(N_1\) of \(M\) such that \(\mathcal{E}_{M_k}^F(M_i, N_1)^{\omega}\) is a neighborhood of \(f\) in \(\mathcal{E}_{M_k}^F(M_i, M)^{\omega}\).

If we choose a \(j > i\) such that \(N_1 \subset M_j\), then we have \(f \in U_{k,j}^i\).

(c) \(cl U_{k,j}^i \subset U_{k,j+1}^j\): Given any \(f \in cl U_{k,j}^i\). Then \(f(M_i) \subset M_j \subset U_{j+1}\). First we show that \(L \cap U_{j+1}\) is connected for each \(L \in \mathcal{C}(M - \text{Int} f(M_i))\). Take \(g \in U_{k,j}^i\) which is sufficiently close to \(f\) so that there exists a \(k \in G(U_{j+1})\) such that \(kg = f\). Since the restriction map \(\pi_{k,j} : F_{M_k}(U_j)_0 \to U_{k,j}^i\) is surjective, there exists an \(h \in F_{M_k}(U_j)_0 \subset G(U_{j+1})\) such that \(h|_{M_j} = g\). Recall the condition (b) for the exhausting sequence \(\{M_i\}_{i \geq 0}\). Then the claim is verified by the homeomorphism of tuples \(kh : (M, U_{j+1}, M_i) \approx (M, U_{j+1}, f(M_i))\).

By Lemma 6.1(1) there exists a neighborhood \(V\) of \(f\) in \(\mathcal{E}_{X}^F(N, M)_0^\omega\) and a map \(\psi : V \to F_{M_k}(U_{j+1})_0\) such that \(\psi(g)f = g (g \in V)\) and \(\psi(f) = id_M\). Since \(V \cap U_{k,j}^i \neq \emptyset\), we can choose
Remark 6.2. The tuple \((H_X, M, \{M_i\}_{i \geq 1})\) does not satisfy the assumption (A-2) (ii).

Lemma 3.5 and Theorem 3.1(2)(ii) induce the following conclusions.

**Proposition 6.3.** (1) If \(F_X(U_j)_0 \cap F_{M_1} \simeq *\) for each \(j > i \geq 1\), then \((F_X)_0^*\) is HD in \((F_X)_0\).

(2) If \(F_{M_1}\) is connected and \(F_{M_1}(U_j)_0 \cap F_{M_i} \simeq *\) for each \(j > i \geq 2\), then \((F_X)_0^*\) is HD in \((F_X)_0\)

For \(n\)-manifolds of finite type, Proposition 4.2 and Proposition 6.3 induce the following conclusions.

**Proposition 6.4.** Suppose \(M = \text{Int} N\) for some compact connected orientable \(n\)-manifold \(N\) with non-empty boundary, \(\omega\) is a volume form on \(M\) and \(X\) is a compact smooth \(n\)-submanifold of \(M\). Then the following hold.

1. Both \(\mathcal{D}_X(M; \omega)_0\) and \((F_X)_0\) are \(\ell^2\)-manifolds.
2. If \(\mathcal{D}_{\partial N \times \{0,1\}}(\partial N \times [0,1])_0 \simeq *\), then \((F_X)_0^*\) is HD in \((F_X)_0\).

**Proof.** (1) The group \((G_X)_0\) is an ANR by Proposition 4.2. Hence, so is \((H_X)_0\) by Proposition 5.1. We can apply Proposition 6.1(2)(ii) to each \(L \in \mathcal{C}(M - \text{Int} X)\) to show that \((F_X)_0\) is a SDR of \((H_X)_0\). Hence \((F_X)_0\) is also an ANR.

(2) We construct an exhausting sequence \(\{M_i\}_{i \geq 1}\) of \((M, X)\) as in the proof of Proposition 4.2. Let \(U_i = \text{Int} M_i \ (i \geq 1)\). Since this exhausting sequence satisfies the above conditions (a), (b), the tuple \((F_X, M, \{M_i\}_{i \geq 1})\) satisfies the assumption (A). Note that each \(L \in \mathcal{C}(M - \text{Int} M_1)\) is a product end. By Proposition 6.4(2) it suffices to show the following statements.

(i) \(F_{M_i}\) is connected: By Proposition 6.1 the subgroup \(H_{M_i}\) is a SDR of \(G_{M_i} = \mathcal{D}_{M_i}^+(M, E_{M_i}^X)\). We can apply Proposition 6.2(2)(i) to each \(L \in \mathcal{C}(M - \text{Int} M_1)\) to show that \(F_{M_1}\) is a SDR of \(H_{M_i}\). Since \(G_{M_i} \simeq *\) by the Alexander trick towards \(\infty\), we have \(F_{M_1} \simeq *\).

(ii) \(F_{M_1}(U_j)_0 \cap F_{M_i} \simeq *\) for \(j > i \geq 2\): In the proof of Proposition 4.2(3) we have already shown that \(G_{M_i}(U_j)_0 \cap G_{M_i} \simeq *\) for \(j > i \geq 2\). By Proposition 6.1(1) \(F_{M_i}(U_j) = H_{M_i}(U_j)\) is a SDR of \(G_{M_i}(U_j)\). Then, it follows that \(F_{M_1}(U_j)_0 \cap F_{M_i} = F_{M_1}(U_j)_0\) and it is a SDR of \(G_{M_i}(U_j)_0 \cap G_{M_i} = G_{M_i}(U_j)_0\). This implies the assertion. \(\Box\)

6.4. Groups of volume-preserving diffeomorphisms of noncompact 2-manifolds.

Suppose \(M\) is a noncompact connected orientable smooth 2-manifold without boundary and \(\omega\) is a volume form on \(M\).

**Proof of Theorem 1.3.** Since \(\mathcal{D}(M; \omega)_0\) is a SDR of \(\mathcal{D}(M)_0\) (Corollary 1.1(2)(ii)), see Section 6.1, the assertions follow from Theorem 1.1 and the observations in Section 6.1. \(\Box\)

**Proof of Theorem 1.4.** (1) Since \(\ker c^\omega_0\) is a SDR of \(\mathcal{D}(M; \omega)_0\) (Corollary 1.2(2)(ii)), see Section 6.1, the assertion follows from Theorem 1.3 and the observations in Section 6.1.

(2) Let \((G, H, F) = (\mathcal{D}(M), \mathcal{D}(M; \omega)_0, \ker c^\omega)\). We have to show that \(F_0^*\) is HD in \(F_0\). We separate the next two cases (cf. Theorem 5.3):
(I) $M$ = a plane or an open annulus.  (II) $M$ is not Case (I).

Case (I): Since $M = \text{Int} \, N$ ($N = \mathbb{D}$ or $A$), the assertion follows from Proposition 6.4(2) (cf. )

Case (II): There exists an exhausting sequence $\{M_i\}_{i \geq 1}$ of $M$ which satisfies the conditions (a), (b) in Section 6.3 and such that each $M_i$ is neither a disk, an annulus nor a Möbius band. Let $U_i = \text{int} \, M_i$ ($i \geq 1$). In the proof of Theorem 5.3, we have shown that

$$G(U_j)_0 \cap G_{M_i} = G_{M_i}(U_j)_0 \simeq * \quad \text{for each} \quad j > i \geq 1.$$

Since $H_{M_i}(U_j)$ is a SDR of $G_{M_i}(U_j)$, this implies that

$$F(U_j)_0 \cap F_{M_i} = H(U_j)_0 \cap H_{M_i} = H_{M_i}(U_j)_0 \simeq * \quad \text{for each} \quad j > i \geq 1.$$

Hence, the conclusion follows from Proposition 6.3. □

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