Statistical physics in deformed spaces with minimal length

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Abstract

We considered the thermodynamics in spaces with deformed commutation relations leading to the existence of minimal length. We developed a classical method of the partition function evaluation. We calculated the partition function and heat capacity for ideal gas and harmonic oscillators using this method. The obtained results are in good agreement with the exact quantum ones. We also showed that the minimal length introduction reduces degrees of freedom of an arbitrary system in the high temperature limit significantly.

Keywords: modified commutation relations, minimal length, thermodynamics.

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1 Introduction

Now it is widely accepted that quantum gravity and string theory lead to minimal length, i.e. there must exist a lower bound to the possible resolution of the distance [1–8]. The minimal length appears due to a modification of the standard Heisenberg uncertainty relation:

$$\Delta X \Delta P \geq \frac{\hbar}{2}(1 + \beta \Delta P^2 + \ldots),$$  \hspace{1cm} (1)

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where $\beta$ is a positive parameter. This modified relation implies that uncertainty of coordinate $\Delta X$ is always larger than $\Delta X_{\min} = \hbar \sqrt{\beta}$. It was shown that such a modified uncertainty relation can be obtained as a natural consequence of a specific modification of the usual commutation relation [9–13]. We would like to note that modification of the commutation relation is only one of the possible ways of introducing minimal length. For a review of different approaches to theories with the minimal length see [14].

The deformed commutation relation according to Kempf et al. [11–13] may read in one-dimensional space

$$[X, P] = i\hbar (1 + \beta P^2)$$

or in $D$-dimensional space ($i, j \in 1 \ldots D$)

$$[X_i, P_j] = i\hbar ((1 + \beta P^2)\delta_{ij} + \beta' P_i P_j), \quad [P_i, P_j] = 0,$$

$$[X_i, X_j] = i\hbar \frac{2\beta - \beta' + (2\beta + \beta')\beta P^2}{1 + \beta P^2} (P_i X_j - P_j X_i).$$

It is believed widely that the minimal length can be helpful in the context of divergencies regularization in different areas of theoretical physics [5]. Contrary to the previous theories, where the minimal length appears phenomenologically, in our case the minimal length appears as a natural consequence of modified commutation relations. An explicit representation of position and momentum operators is known and the existence of possible regularization can be verified from the first principles. It was shown that deformed commutation relations really might lead to regularization in quantum field theory [15]. Regularization with the help of deformed commutation relations for specific eigenvalue problems was studied in [16,17].

Other implications of non-zero minimal length were considered in the context of the following problems: harmonic oscillator [12, 13, 18–21], hydrogen atom [22–25], gravitational quantum well [26], the Casimir effect [27, 28], particles scattering [29, 30]. In several of the mentioned papers attempts to estimate the value of minimal length upper bound were made by comparing theoretical predictions and experimental data. In [22–25] considering the Lamb shift the authors estimated $\Delta X_{\min} \leq 10^{-16} \ldots 10^{-17}$ m, analysis of electron motion in a Penning trap also gives $\Delta X_{\min} \leq 10^{-16}$ m [18], consideration of neutron motion in the gravitational field [26] gives large
Due to significant experimental errors, we will use $\Delta X_{\text{min}} = 10^{-16}$ m.

Consideration of some theoretical physics topics requires the use of statistical methods. The purpose of this paper is to develop statistical physics with some general form of deformed commutation relation. There are several papers on the subject. In [31] the statistical physics is constructed by modification of elementary cell of space volume according to modification of the commutation relation. In that paper a very strong restriction $[X_i, X_j] = 0$ was assumed. A similar analysis was performed in [32, 33]. Density of states and black-body radiation were considered in [34] with the help of Liouville theorem analog. The microcanonical approach to ideal gas was considered in [35].

In this paper we develop a method for the consideration of thermodynamical properties of the system with arbitrary commutation relation between momenta and positions operators. We approximate the value of the partition function with the help of a semiclassical approach. We may expect that such an approximation is applicable since the WKB approximation gives correct predictions for one-dimensional eigenvalues problems with deformed commutation relations [36, 37]. We apply this method to the particular case, namely to the three-dimensional Kempf’s commutation relation and consider implications of the minimal length on thermodynamics.

2 A method

We consider a system consisting of $N$ identical non-interacting particles in the external field $U$. The behavior of each particle may be described by Schrödinger Hamiltonian

$$H = \frac{p^2}{2m} + U(X).$$  \hspace{1cm} (5)

We use this Hamiltonian to analyze examples, but the developed method can be applied to a system described by Hamiltonian of any form.

In this paper we assume that the partition function of a quantum system can be evaluated as

$$Z = \sum_n e^{-E_n/T},$$  \hspace{1cm} (6)

where $E_n$ are eigenvalues of the Hamiltonian, $n$ denotes different states. This formula has the same form either in the deformed or
in the non-deformed cases. In the non-deformed case one can use a
semiclassical approximation for the partition function

\[ Z = \int e^{-H/T}(dx)(dp), \]  

(7)

where \((dx)\) means \(dx_1dx_2\ldots dx_D\). Since it is the non-deformed case,
the coordinate and the momentum variables \(x_i\) and \(p_j\) are canonically
conjugated \(\{x_i, p_j\} = \delta_{ij}, \{x_i, x_j\} = \{p_i, p_j\} = 0\). Here and below
small variables \(x, p\) denote canonically conjugated variables.

We show below that the existence of deformation becomes perceptible only for very high temperature. For high temperature the
difference between Bose-Einstein, Fermi-Dirac or Boltzmann statistics is irrelevant, therefore we use Boltzmann one as the simplest.
One of the aims of this paper is to generalize formula (7), which gives
correct predictions for high temperature, to the case of the deformed
commutation relation. Peculiarities of Bose-Einstein and Fermi-Dirac
statistics for the particular form of deformed commutation relations
were considered in [32, 33].

Let us consider the general case of deformed commutation relation

\[ [X_i, P_j] = ihf_{ij}(X, P), \]  

(8)

\[ [P_i, P_j] = ihh_{ij}(X, P), \]  

(9)

\[ [X_i, X_j] = hgh_{ij}(X, P). \]  

(10)

In the above expressions operators \(X_i, P_j\) correspond to the same particle. Operators describing different particles commute. The deformation functions \(f_{ij}, g_{ij}, h_{ij}\) are restricted according to the properties of commutators: bilinearity, the Leibniz rules and the Jacobi identity. The investigation on this subject for some special cases of the
deforation function can be found in [13, 15].

In the classical limit \(\hbar \to 0\) the deformed commutation relations
(8–10) lead to the deformed Poisson bracket [34, 38, 39]

\[ \{X_i, P_j\} = f_{ij}(X, P), \quad \{P_i, P_j\} = h_{ij}(X, P), \quad \{X_i, X_j\} = g_{ij}(X, P). \]  

(11)

These Poisson brackets possess the same properties as the quantum mechanical commutators (8–10), namely, they are anti-symmetric, bi-
linear, and satisfy the Leibniz rules and the Jacobi identity [38, 39].
According to Darboux theorem [40] it is always possible to choose
such auxiliary canonically conjugated variables \(x_i\) and \(p_i\) that \(X_i\) and
$P_i$ as functions of $x_i$ and $p_i$ satisfy equations (11). Where, just as an example,

$$\{X_i, P_j\} = \sum_{k=1}^{D} \frac{\partial X_i}{\partial x_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial P_j}{\partial x_k} \frac{\partial X_i}{\partial p_k} = f_{ij}(X, P). \quad (12)$$

Then, the initial Hamiltonian (5) can be considered as the function of $(x, p)$ and the one-particle partition function can be evaluated according to formula (7). In appendix A we show that the Jacobian

$$J = \frac{\partial (X,P)}{\partial (x,p)}$$

can always be expressed as a combination of Poisson brackets (11). It is an important result since it gives a possibility to calculate the partition function without introducing canonically conjugated auxiliary variables. Namely,

$$Z = \int e^{-H(X,P)/T} (dX)(dP). \quad (13)$$

This equation is an analog of formula (7) for the deformed case. It is an essential and expected result that the partition function does not depend on auxiliary variables since $J$ depends only on the structure of the Poisson brackets (11) which, together with the Hamiltonian, define all the properties of the system.

Expression (13) can be considered as a semiclassical approximation of partition function (6). It is well known that the semiclassical approximation is good for large quantum numbers (it corresponds to high temperature regime). We will observe this situation with the help of the harmonic oscillators ensemble example: heat capacities calculated according to formulae (6) and (13) coincide for high $T$.

### 3 Examples

In this section we analyze the effect of deformation (3–4) on the thermodynamical quantities. Since Jacobian (38) does not depend on coordinates and Hamiltonian has the form (5) evaluation of the one-particle partition function (13) can be separated in two parts: integrating over $(X)$ and over $(P)$. Namely,

$$Z = \int (dX) \exp \left[ -\frac{U(X)}{T} \right] \int (dP) \frac{\exp \left[ -\frac{P^2}{2mT} \right]}{(1 + \beta P^2)^2(1 + (\beta + \beta')P^2)}. \quad (14)$$

Note, that during the transition from quantum expression for the partition function (6) to the classical one (14), we implicitly assume that
\( \beta \) and \( \beta' \) are independent of \( \hbar \), i.e. we keep the deformation parameters fixed as \( \hbar \to 0 \). Some comments on this topic can be found in [38].

It is easy to see that deformation changes the partition function perceptibly if \( \beta mT \sim 1 \), \( \beta' mT \sim 1 \). For electron gas with the minimal length \( 10^{-16} \) m \( \beta mT \sim 1 \) if \( T \sim 10^{18} \) K. The very high value of this temperature justifies our choice of Boltzmann statistics.

For low temperatures \( \beta mT \ll 1 \), \( \beta' mT \ll 1 \) the partition function of one particle (14) simplifies to

\[
Z = Z_0 (1 - 3(3\beta + \beta')mT + o(T)),
\]

where \( Z_0 = \int (dX) \exp [-U/T] \int (dP) \exp [-P^2/2mT] \) is the partition function of one particle described by the same Hamiltonian for the non-deformed case. The existence of deformation leads to a change of internal energy and heat capacity in such a way

\[
E = E_0 - 3(3\beta + \beta')NmT^2 + o(T^2),
\]
\[
C = C_0 - 6(3\beta + \beta')NmT + o(T).
\]

For large \( T \): \( \beta mT \gg 1 \) the partition function of one particle, internal energy and heat capacity are

\[
Z = Z_0 \frac{\pi^2 (2\pi mT)^{-3/2}}{\sqrt{\beta (\sqrt{\beta} + \sqrt{\beta} + \beta')}} \left( 1 + O \left( \frac{1}{T} \right) \right),
\]
\[
E = E_0 - \frac{3}{2} NT + O(1), \quad C = C_0 - \frac{3}{2} N + O \left( \frac{1}{T} \right).
\]

This result will be verified on the quantum level for two examples (ideal gas and ensemble of 3D harmonic oscillators). We see that the presence of deformation freezes three degrees of freedom at high temperature. In section 4 we show that such a freezing is a natural consequence of the minimal length introduction.

### 3.1 Ideal gas

Hamiltonian of one particle is \( H = P^2 \). We consider that the entire system is confined in volume \( V \), i.e. for each particle \( \int (dX) = V \).

For low temperatures, the one-particle partition function reads

\[
Z = V (2\pi mT)^{3/2} (1 - 3(3\beta + \beta')mT + o(T)).
\]
It leads to the following expressions for internal energy and heat capacity

\[ E = \frac{3}{2} NT - 3(3\beta + \beta')NmT^2 + o(T^2), \quad C = \frac{3}{2} N - 6(3\beta + \beta')NmT + o(T). (21) \]

In limit \( T \to \infty \) the one-particle partition function is

\[ Z = \frac{V \pi^2}{\sqrt{\beta(\sqrt{\beta} + \sqrt{\beta + \beta'})^2}} \left( 1 - \frac{1}{2m\beta} \frac{2\sqrt{\beta + \sqrt{\beta + \beta'}}}{\sqrt{\beta + \beta'}} + o\left( \frac{1}{T} \right) \right). \]  

(22)

Here we take into account the second term of the series expansion to obtain the second correction to the internal energy expression (the leading term, which must be proportional to \( T \), is zero):

\[ E = \frac{1}{2m\beta} \frac{2\sqrt{\beta + \sqrt{\beta + \beta'}}}{\sqrt{\beta + \beta'}} + o(1). \]  

(23)

The last equation can be explained with the help of the quantum analysis. Although the problem of the particle in the box in the deformed case has not been solved exactly yet, the preliminary analysis shows that there exists only a finite amount of bound states [41,42]. For high temperature it must lead to the situation that all particles occupy the highest energy level and thus, heat capacity must tend to 0 as we see from (23).

It is interesting to note that equation of state is the same as in the non-deformed case and its form does not depend on the temperature value. Namely, it reads

\[ pV = NT. \]  

(24)

### 3.2 3D harmonic oscillators

One-particle harmonic oscillator Hamiltonian is, as usual, \( H = \frac{p^2}{2m} + \frac{m\omega^2}{2} X^2 \). For low temperature expression (14) gives the following formula for the partition function

\[ Z = \left( \frac{2\pi T}{\omega} \right)^3 \left( 1 - 3(3\beta + \beta')mT + o(T) \right). \]  

(25)

The eigenvalue problem of the harmonic oscillator has been solved exactly in [18] for the first time and the spectrum reads

\[ E_{nl} = \hbar \omega \left( n + \frac{3}{2} \right) \sqrt{1 + m^2 \omega^2 \hbar^2 \left[ \beta^2 l(l + 1) + \left( \frac{3\beta + \beta'}{4} \right)^2 \right]} \]
We calculate the one-particle partition function using formula (6) in linear approximation over $\beta, \beta'$ exactly. This expression is rather cumbersome, but in limit $\hbar \to 0$ we obtain simple expression for it

$$Z = \frac{T^3}{\hbar^3 \omega^3} (1 - 3(\beta + \beta')mT) + O(1/\hbar^2).$$

(27)

The leading term of the last expression differs from (25) only by constant factor $(2\pi \hbar)^3$ which does not have any effect on thermodynamics and which we omit in the initial definitions of approximation (7, 13).

For large $T$ expression (14) gives

$$Z = \left(\frac{2\pi T}{m \omega^2}\right)^{3/2} \frac{\pi^2}{\sqrt{\beta(\sqrt{\beta} + \sqrt{\beta + \beta'})^2}} \left(1 + O\left(\frac{1}{T}\right)\right),$$

(28)

$$E = \frac{3}{2}NT + O(1).$$

(29)

This result can be easily explained and verified as follows. The energies with large $n$ give the main contribution to the partition function at high temperatures. In the deformed case for large $n$ according to expression (26) $E_{nl} \sim \frac{m \omega \hbar}{2} (\beta + \beta')n^2$. Such an energy dependence on the main quantum number is the same as for a particle in the box for the non-deformed case. Thus, it is an expected result to reproduce particles-in-the-box internal energy dependence on the temperature (29).

Heat capacity dependence on temperature is plotted on Fig. 1. For high $T$ heat capacity tends to $3/2$ what follows from (29). One can see that approximation (13) gives predictions very closely to the exact result for high $T$. For selected values of parameters $\beta = \beta' = 0.01$, $\hbar = \omega = 2m = 1$ it was expected that effects implied by the deformed commutation relation become significant for $T = \frac{1}{\beta m} \sim 100$. As one can see this implication appears to be important for a much lower temperature, namely for $T = \frac{1}{2}$. 

8
4 Heat capacity at high temperature and the minimal length

For a wide class of physically important systems, Hamiltonian can be expressed as a sum of kinetic and potential energy. In this section we estimate the contribution of kinetic energy to heat capacity in the limit $T \to \infty$.

Let us consider the non-deformed case. In the case of high temperature only large values of momenta contribute to the partition function. For large $p$ the Hamiltonian can be approximated as $H = \alpha p^n$ (for Schrödinger Hamiltonian $\alpha = \frac{1}{2m}, n = 2$; for Dirac Hamiltonian $\alpha = c, n = 1$). So,

$$Z(T \to \infty) \sim \int_0^\infty e^{-\frac{\alpha p^n}{T}} p^{D-1} dp = T^D \alpha^{\frac{D}{n}} \int_0^\infty e^{-x} x^{D-1} dx \quad (30)$$

Such a dependence on $T$ means that $C(T \to \infty) = \frac{D}{n}$, which reproduces several well-known results.

Let us consider deformed commutation relations leading to the minimal length. In a one-dimensional case

$$[X, P] = i\hbar(1 + f(X, P))$$

and Schrödinger uncertainty relation reads

$$\Delta X \geq \frac{\hbar}{2} \left( \frac{1}{\Delta P} + \frac{\langle f(X, P) \rangle}{D} \right) \geq \frac{\hbar}{2} \left( \frac{1}{P} + \frac{\langle f(X, P) \rangle}{P} \right),$$

where $P = \sqrt{\langle P^2 \rangle} \geq \Delta P$. Therefore, a nonzero minimal length exists if $\lim_{P \to \infty} f(X, P)$ grows as $P$ or faster for large momentum values. Then the Jacobian $J = 1 + f$ for large $P$ must grow as $P$ or faster. In the $D$-dimensional case it must grow as $P^D$ or faster.

Let us first consider the case $J \sim P^D$ and denote by $P_0$ such an intermediate value of momentum that $H \approx \alpha P^n, J \approx \gamma P^D$ for $P \geq P_0$, but $H \ll T$ for $P \leq P_0$. Then the partition function reads

$$Z = \int_0^\infty e^{-H/T} \frac{P^{D-1} dP}{J} \approx \int_0^{P_0} \frac{P^{D-1} dP}{J} + \frac{1}{\gamma} \int_{P_0}^\infty e^{-\alpha P^n/T} \frac{dP}{P} = \text{const} + \frac{1}{\gamma} \ln \frac{T}{\alpha P_0^n} \quad (31)$$

Heat capacity can be easily calculated and $C(T \to \infty) = 0$. 

A similar procedure can be performed when the Jacobian grows faster than $P^D$. It gives $Z(T \to \infty) = \text{const}$ and consequently $C(T \to \infty) = 0$. So, in any case leading to the minimal length $C(T \to \infty) = 0$.

We may conclude that for a wide class of Hamiltonians being proportional to a power of momentum for large momentum values and deformed commutation relations leading to the nonzero position uncertainty in the limit $T \to \infty$ the partition function of the system is as $\ln T$ (the Jacobian grows as $P^D$) or does not depend on temperature (the Jacobian grows faster). Case of the constant partition function was discussed in the context of the string theory from the duality point of view [43]. A similar effect of degrees of freedom reduction was observed for two model Hamiltonians with particular deformed commutation relations in the context of grand canonical approach [31].

\section{Conclusions}

In this paper we developed a general method for the consideration of thermodynamical properties of an arbitrary system of non-interacting particles with deformed commutation relations. To calculate a classical expression for the partition function we express the Jacobian of making a change from arbitrary canonically conjugated variables to the initial variables as the combination of deformed Poisson brackets.

We calculate the partition function for two model systems, namely ideal gas and ensemble of harmonic oscillators and considered corresponding thermodynamical properties. We showed that such a derived partition function is in qualitative agreement with the exact quantum expression in the limit of high temperature. Quite interesting consequence of the minimal length introduction is decreasing of the heat capacities of these model systems to 0 and $3/2$ for high temperatures. We managed to show that such a reduction of the heat capacity occurs (when compared with the corresponding non-deformed system) for each system with arbitrary deformed commutation relations leading to the minimal length. Namely, the minimal length introduction completely removes translation degrees of freedom for very high temperature.
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A Expressing Jacobian as a combination of the Poisson brackets

Let us denote \( X_i = A_{2i-1}, \) \( P_i = A_{2i}, \) \( A_j \) derivative with respect to \( x_i \) we denote \( A_{j,2i-1}, \) with respect to \( p_i \) as \( A_{j,2i} \) Then

\[
\{A_i, A_j\} = \sum_{k=1}^{D} (A_{i,2k-1}A_{j,2k} - A_{i,2k}A_{j,2k-1}).
\]

Let us prove the following identity

\[
J = \frac{\partial (X_1, P_1, \ldots, X_D, P_D)}{\partial (x_1, p_1, \ldots, x_D, p_D)} = \frac{1}{2^DD!} \sum_{i_1, \ldots, i_{2D} = 1}^{2D} \varepsilon_{i_1 \ldots i_{2D}} \{A_{i_1}, A_{i_2}\} \cdots \{A_{i_{2D-1}}, A_{i_{2D}}\},
\]

where \( \varepsilon_{i_1 \ldots i_{2D}} \) is the Levi-Civita symbol. The right-hand of this identity is equal

\[
\frac{1}{2^DD!} \sum_{i_1, \ldots, i_{2D} = 1}^{2D} \varepsilon_{i_1 \ldots i_{2D}} \{A_{i_1}, A_{i_2}\} \cdots \{A_{i_{2D-1}}, A_{i_{2D}}\} = \frac{1}{2^DD!} \sum_{i_1, \ldots, i_{2D} = 1}^{2D} \varepsilon_{i_1 \ldots i_{2D}}
\]

\[
\sum_{j_1=1}^{D} (A_{i_1,2j_1-1}A_{i_2,2j_1} - A_{i_1,2j_1}A_{i_2,2j_1-1}) \cdots \sum_{j_D=1}^{D} (A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D} - A_{i_{2D-1},2j_D}A_{i_{2D},2j_D-1})
\]

\[
= \frac{1}{D!} \sum_{j_1, \ldots, j_D} \sum_{i_1, \ldots, i_{2D}} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,2j_1-1}A_{i_2,2j_1} \cdots A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D}.
\]

In the latter equality we take into account that the Levi-Civita symbol is antisymmetric with respect to any indexes permutation, thus the following property holds

\[
\sum_{i_1, i_2} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,2j_1-1}A_{i_2,2j_1} = - \sum_{i_1, i_2} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,2j_1}A_{i_2,2j_1-1}.
\]

From the fact that

\[
\sum_{i_1, \ldots, i_{2D}} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,2j_1-1}A_{i_2,2j_1} \cdots A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D} =
\]
\[ \begin{vmatrix} A_{1,2j_1-1} & A_{1,2j_1} & \cdots & A_{1,2j_D-1} & A_{1,2j_D} \\ A_{2,2j_1-1} & A_{2,2j_1} & \cdots & A_{2,2j_D-1} & A_{2,2j_D} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{2D,2j_1-1} & A_{2D,2j_1} & \cdots & A_{2D,2j_D-1} & A_{2D,2j_D} \end{vmatrix} \]

and using the determinant properties it becomes obvious that in the last formula of equation (33) only terms with different \( j \) give the contribution to the final result. All the terms with \( j_1 \neq j_2 \neq \ldots \neq j_{2D} \) are equal. The total number of these terms equals \( D! \). Thus,

\[
\frac{1}{D!} \sum_{j_1,\ldots,j_D} \sum_{i_1,\ldots,i_{2D}} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,2j_1-1}A_{i_2,2j_1} \cdots A_{i_{2D-1},2j_D-1}A_{i_{2D},2j_D} = \\
\sum_{i_1,\ldots,i_{2D}} \varepsilon_{i_1 \ldots i_{2D}} A_{i_1,1}A_{i_2,2} \cdots A_{i_{2D-1},2D-1}A_{i_{2D},2D} = \det(A_{ij}), (34)
\]

which proves identity (32).

The right-hand side of expression (32) contains \((2D)!\) terms, each of them is a product of \( D \) Poisson brackets. Due to the skew-symmetry of Poisson bracket some of the terms are equal and the total amount of terms can be reduced to \((2D-1)!!\) terms. Below we enlist Jacobians for one-, two- and three-dimensional cases.

In a one-dimensional case the following expression

\[
\frac{\partial (X, P)}{\partial (x, p)} = \{X, P\}
\]

is obvious.

In a two-dimensional case

\[
\frac{\partial (X_1, P_1, X_2, P_2)}{\partial (x_1, p_1, x_2, p_2)} = \{X_1, P_1\} \{X_2, P_2\} - \{X_1, X_2\} \{P_1, P_2\} - \{X_1, P_2\} \{X_2, P_1\} (35)
\]

can be checked by hand.

A three-dimensional case:

\[
\frac{\partial (X_1, P_1, X_2, P_2, X_3, P_3)}{\partial (x_1, p_1, x_2, p_2, x_3, p_3)} = \{X_1, P_1\} \{X_2, P_2\} \{X_3, P_3\} - \\
\{X_1, P_3\} \{P_1, P_2\} \{X_2, X_3\} - \{X_1, P_2\} \{X_2, P_1\} \{X_3, P_3\} - \\
\{X_1, P_2\} \{X_2, P_3\} \{X_3, P_1\} - \{X_1, P_1\} \{X_2, P_3\} \{X_3, P_2\} + \\
\{X_1, X_2\} \{P_1, P_3\} \{X_3, P_2\} + \{X_1, P_3\} \{X_2, P_1\} \{X_3, P_2\} - \\
\{X_1, X_2\} \{P_2, P_3\} \{X_3, P_1\} + \{X_1, P_2\} \{X_2, X_3\} \{P_1, P_3\} - 
\]

(36)
This expression can be checked with the help of the computer. It is easy to see that to obtain such a formula one needs to start from the term \( \{X_1, P_1\} \cdot \ldots \cdot \{X_D, P_D\} \) and add to it all possible permutations. Factor multiplying each term is either +1 for even permutation and −1 for odd permutation. In the above expression all variables are ordered, i.e. in each Poisson bracket \( X_i \) is before \( P_j \), \( X_i \) is before \( X_j \) if \( j > i \), and \( P_i \) is before \( P_j \) if \( j > i \).

For deformation (3–4) Jacobian (37) simplifies to

\[
\frac{\partial (X_1, P_1, X_2, P_2, X_3, P_3)}{\partial (x_1, p_1, x_2, p_2, x_3, p_3)} = (1 + \beta P^2)^2 (1 + (\beta + \beta') P^2). \tag{38}
\]

In the case of small deviation of the deformed Poisson brackets (11) from the canonical ones \( f_{ij} - \delta_{ij}, h_{ij}, g_{ij} \approx 0 \) expression (32) can be simplified significantly in the linear approximation over these deviations. It is easy to see that in this case only the first term \( \{X_1, P_1\} \cdot \ldots \cdot \{X_D, P_D\} \) contributes to the Jacobian and

\[
J = \prod_{i=1}^{D} f_{ii}(X, P) = 1 + \sum_{i=1}^{D} (f_{ii}(X, P) - 1).
\]

Such an expression is useful for the analysis of low temperature behavior.

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Figure 1: Temperature dependence of heat capacity of the harmonic oscillators ensemble per one particle. $\beta = \beta' = 0.01$, $\hbar = \omega = 2m = 1$, $T$ is given in $\hbar\omega$ units. Dashed line — heat capacity in the non-deformed case ($\beta = \beta' = 0$), solid line — heat capacity calculated according to the direct definition of the partition function (6), dotted line — according to classical approximation (13).