Cauchy-characteristic Evolution of Einstein-Klein-Gordon Systems: The Black Hole Regime

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The Cauchy-characteristic matching (CCM) problem for the scalar wave equation is investigated in the background geometry of a Schwarzschild black hole. Previously reported work developed the CCM framework for the coupled Einstein-Klein-Gordon system of equations, assuming a regular center of symmetry. Here, the time evolution after the formation of a black hole is pursued, using a CCM formulation of the governing equations perturbed around the Schwarzschild background.

An extension of the matching scheme allows for arbitrary matching boundary motion across the coordinate grid. As a proof of concept, the late time behavior of the dynamics of the scalar field is explored. The power-law tails in both the time-like and null infinity limits are verified.

I. INTRODUCTION

The evolution of black hole binaries, after the initial inspiral phase and into the late time ring-down stage, requires the numerical solution of the full set of Einstein equations. A major scientific effort, the Binary Black Hole Grand Challenge Collaboration [1], is currently underway for the study of this problem. An important aspect of this investigation is the condition to be imposed at the outer boundary of the computational domain. Assuming a physically correct prescription of initial data on a Cauchy surface, the evolution is unambiguous in the domain of dependence of the Cauchy surface; problems arise when the space region in which the equations are to be evolved is of fixed finite size, as it is necessarily the case in numerical simulations. The practical relevance of this issue increases as longer evolution times are achieved and strong, non-symmetric field configurations are being probed.

At present, a research program is unfolding [1–6], in which the problem of radiative outer boundary conditions is dealt with at its heart, by extending the integration domain to null infinity. The method combines a characteristic initial value problem, with a regular Cauchy initial value problem. The characteristic initial value problem is specified on null cones, emerging from the world-tube boundary of the Cauchy surface. In previous work by Gómez et. al. [2] (henceforth GLPW), the CCM framework was applied to a system widely used as a model for more complicated relativistic systems, namely the coupled Einstein-Klein-Gordon set of equations in spherical symmetry. At the core of the matching strategy, continuity conditions on metric, extrinsic curvature and scalar field variables ensure smoothness across the matching interface (which is assumed to be at fixed spherical surface of area $4\pi R^2$). These conditions provide the practical “handle” for transforming geometric and field information between the two different space-time foliations. Accuracy tests and comparison of CCM evolutions with reference solutions performed in GLPW, show a remarkably transparent propagation of information across the matching interface, even for strongly non-linear waves.

The choice of gauge in GLPW did not permit the integration of the equations beyond the formation of a black hole, which is the generic conclusion of strong field evolutions. Even so, in a collapse scenario where the newly formed black hole has a mass $M_{BH} > 2R$, the exterior characteristic code allows for the computation of all relevant signal at null infinity, although the computation stops at a finite coordinate time [7,8]. In realistic three dimensional computations though, especially those planned for the computation of the binary black hole inspiral, the matching radius $R$ is expected to be at least larger than $M_{BH}$. In such a case, the CCM code developed in GLPW halts during the formation of a black hole, and the signal at null infinity is obtained only up to some finite asymptotic (Bondi) time.

In this paper we focus on this specific aspect of the CCM for the coupled Einstein-Klein-Gordon system, namely the accurate long-term evolution after a black hole has formed. To this end, it is assumed that a Schwarzschild black hole of mass $M_{BH}$ is present in the space-time, and the remaining scalar field stress-energy does not further modify the background space-time. This is a situation resembling the later stages of the black hole collision process, or, indeed, an arbitrary stage, provided the matching radius is chosen as $R >> M_{BH}$.

The fixed curved background crystallizes a basic feature of the colliding black hole space-times, namely the relativistic potential encompassing the dynamic inner region. The effects of the potential on the propagation of waves are well known [9], and can be probed analytically in a number of different ways [10–12]. Numerically, the detection of the late time features can be a challenging effort for multi-dimensional algorithms, as it requires a significant dynamical range.
Nevertheless, the full dynamical calculations of black hole collisions must continue into the quasi-normal ringing and, possibly, the tail regime to achieve a complete mapping of the anticipated signal.

The presence of late time power-law tails has been extensively verified numerically in one-dimensional problems using both Cauchy [12] and null [10] initial value formulations in fixed geometric backgrounds. The existence of tails has also been investigated in dynamical self-gravitating systems [11,14] and rotating space-times [13]. In this paper the tail behavior in the Schwarzschild potential is investigated using the CCM approach. This serves both as a test of the long term accuracy of the method, and, importantly, as an illustration of the intrinsic economy and versatility of the CCM, which adds significant dynamic range to the basic Cauchy evolution algorithm.

The paper is organized as follows: In section II, the CCM for scalar waves in a Schwarzschild background is presented, as well as the coordinate choices for both the space-like and null region. Section III describes the numerical techniques involved in performing the matching, in particular, the extensions to the interpolation procedures employed in GLPW. Section IV reports tests of the CCM algorithm and results. The stability and accuracy of the curved background evolution are examined using energy conservation and Cauchy convergence tests. The case of a moving matching boundary is illustrated and tested. Lastly, the focus is turned on the propagation phenomena associated with large evolution times. The efficiency of the scheme is demonstrated in the calculation of late-time power law tails at both null and time-like infinity with moderate resolution requirements. Established analytic results are verified readily.

II. THE CCM IN A SCHWARZSCHILD SPACE-TIME

Following the notation of GLPW, the spacetime domains covered by the Cauchy and characteristic foliations are denoted $M^-$ and $M^+$, respectively. Coordinate systems laid out in the two domains are assumed to overlap. In the overlap region, a time-like world-tube serves as the outer boundary of the interior spacelike hypersurfaces and the inner boundary of the exterior null hypersurfaces.

A coordinate system $(t, \eta)$ is adopted in region $M^-$, in which the line element is written as

$$ds^2 = \alpha(\eta)^2(-dt^2 + d\eta^2) + r(\eta)^2d\Omega^2,$$

where $r$ is an area scalar, $t$ defines a slicing in which the metric is time independent, $\alpha^2 = 1 - 2M/r$ and $d\eta/dr = 1/\alpha^2$ defines a "tortoise" coordinate $\eta$.

The matching world-tube is assumed to be located at $\eta_+$. For $r \geq r(\eta_+)$ a coordinate system $(u, x)$ is adopted, where the compactified radial coordinate $x = r/(1+r)$, and the retarded time $u = t - \eta$ are employed. The line element in the $M^+$ region takes the corresponding "null" form

$$ds^2 = -\alpha(x)^2du^2 - 2(1-x)^2dudx + r(x)^2d\Omega^2.$$

The wave equation $\Box \phi \equiv \nabla_\mu \nabla^\mu \phi = 0$ in the background metric described by either (1) or (2) is separable and reduces to 1 + 1 partial differential equations. In $M^-$, the wave equation takes the form

$$g_{,tt} - g_{,\eta\eta} = (1 - \frac{2M}{r})V(r)g,$$

where

$$V(r) = \frac{2M}{r^3} + \frac{l(l+1)}{r^2},$$

and $g = r\phi$.

In $M^+$, the wave equation is converted into an integral identity. The procedure is an application of the method outlined in [8] and we do not elaborate here further. A surface integration over a bounded region $C$, defined by $u_1 < u < u_2$, $v_1 < v < v_2$, i.e., it by two outgoing null geodesics $(u = u_1, u = u_2)$ and two incoming null geodesics $(v = v_1, v = v_2)$ leads to the identity

$$g_{u_2v_2} = g_{u_2v_1} + g_{u_1v_2} - g_{u_1v_1} - \frac{1}{2} \int_C Vg dudr,$$

which relates the values of the scalar field at the vertices of $C$ with the integral of the potential term. An important fact deserves mentioning here, the radial variable $x$ introduces a "compactification" of the radial variable $r$ in the null patch $M^+$. This transformation locates the future null infinity of the space-time at a finite coordinate distance. The value of the field at infinity is hence obtained as the last step of the radial integration of (3).

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The initial value problem in the $M^-$ domain is completed with the specification of the inner and outer boundary conditions. The inner boundary condition is imposed at a point $\eta_-$ sufficiently close to the horizon. For the purposes of this investigation a value $\eta_- = -25$ is sufficient. The boundary condition imposed at $\eta_-$ is that of a purely in-going wave. The exponential fall-off of the Schwarzschild effective potential $V$, as $\eta \to -\infty$, ensures that purely in-going wave solutions converge to the exact solution exponentially. This situation is in sharp contrast with the power law fall-off in the outer region. The outer boundary condition is the subject of section III. Algorithmically, the matching condition at the outer boundary completes the evolution of the inner region $M^-$ by supplying field information at $\eta_+$, the coordinate location of the outer boundary.

The next step is to produce discrete approximations of the wave equations (3) and (5). In the coordinate system $(t, \eta)$ and with the definition $p \equiv g_{t\eta}$, the wave equation (3) is written as a first order system,

$$g_{t} = p,$$  

$$p_{t} = g_{\eta\eta} - \left(1 - \frac{2M}{r}\right)Vg.$$  

The tortoise coordinate $\eta$ is given in the case of the Schwarzschild metric by

$$\eta = r + 2M \ln\left(\frac{r}{2M} - 1\right).$$

(8)

The inverse function $r(\eta)$ is formally given in terms of the principal part of the Lambert function $W(y)$

$$r(\eta) = 2M\{1 + W(e^{\eta/2M-1})\},$$

where $W(y)$ solves the equation

$$W(y)e^{W(y)} = y.$$  

(10)

A direct numerical iteration of equation (3) using the Newton-Raphson method provides accurate values for the inverse function at any required point $r$. The temporal discretization of the wave equation (5) and (6) in $M^-$ employs a modified staggered leapfrog scheme. The field variable $g(t, \eta)$ is approximated by values $g_i^k$ at the discrete time levels $t^n$ and spatial points $\eta_i$, with $i$ ranging between 0 (at $\eta_-$) and $N$ (at $\eta_+$). The momentum variable $p(t, \eta)$ is approximated at the staggered time levels $t^{n+1/2}$. The discrete time-step and grid spacing are denoted $\Delta t$ and $\Delta \eta$ respectively. The spatial discretization of the second derivative $g_{\eta\eta}$ is of fourth order accuracy for $1 < i < N - 1$ and of second order accuracy otherwise. The condition for stable evolution of smooth initial data, i.e., the suppression of non-physical degrees of freedom, is given by the standard stability condition $\Delta t < K \Delta \eta$, where $K = 0.5$ for marginal stability.

In the $M^+$ region too, following the finite difference paradigm, a spatio-temporal grid $(u^m, x_i)$ is introduced, with discrete time-steps and length-scales of $\Delta u$ and $\Delta r$ respectively. The field variable $g$ is approximated by a discrete set of values $g^m_k$ at those points. In order to obtain a discrete version of (5), the region of integration $C$ must be suitably placed on the grid. The null parallelogram $C$ is placed with the inward characteristics centered on grid points $x_i$. The equation of the in-going characteristics of the metric is

$$\frac{dx}{du} = -\frac{1}{2} \left(1 - \frac{2M}{r}\right)(1 - x)^2, \quad (11)$$

where $r = x/(1 - x)$. The integration of the null geodesic (11) provides the coordinates of the vertices. The values of the field at the vertices are to be interpolated from the grid values $g^m_k, g^m_{k+1}$ where $k$ spans the range $[i - 2, i + 1]$, $i$ being the index of the new point under evaluation. (See 13).

The integral in the right hand side of equation (5) is approximated as

$$\int_C Vgdu dr \approx -g_* \left[l(l + 1) \int_C \frac{du dr}{r^2} + 2M \int_C \frac{du dr}{r^3}\right], \quad (12)$$

where $g_* \equiv g(r_*).$ The point $r_*$ is at the center of the null parallelogram, hence $g_*$ can be obtained at $r_*$ by averaging $g_* = (g_{u_1u_2} + g_{u_2u_1})/2$. The remaining integrals can be evaluated analytically, yielding

$$\int_C \frac{du dr}{r^2} = 2 \log \frac{(r_Q - 2M)(r_R - 2M)}{(r_P - 2M)(r_S - 2M)}, \quad (13)$$

$$\int_C \frac{du dr}{r^3} = \frac{1}{2M} \log \frac{(1 - 2M/r_Q)(1 - 2M/r_R)}{(1 - 2M/r_P)(1 - 2M/r_S)}, \quad (14)$$

where

$$g(x) \equiv \frac{\partial}{\partial x} \left[\frac{1}{2} \int_C Vg du dr\right] \quad (15)$$

The integration of equation (5) provides the derivatives $g(x)$, and hence $g(\eta)$, for the inner region, $\eta \leq -\infty$, and for the outer region, $\eta \geq -\infty$, to be matched on the null parallelogram. The inner boundary conditions are imposed at $\eta_-$, and the matching conditions at the outer boundary completes the evolution of the inner region $M^-$.
where $P, Q, R, S$ denote the points $(u_2, v_1), (u_2, v_2), (u_1, v_1), (u_1, v_2)$ respectively. Substitution of these integrals into (14) yields the algorithm for updating the field $g$ in the $M^+$ sector.

The domain of stability for this updating algorithm is dictated by the CFL condition, which requires that

$$\Delta t < 2 \frac{\Delta x}{(1 - x_m)^2} \frac{1}{1 + 2M - 2/M_x},$$

where $x_m$ is the location of the innermost point where the method is applied.

### III. THE MATCHING INTERFACE

A smooth matching interface between the two coordinate domains requires that the scalar field, and any geometric quantities constructed from it, transform properly under the coordinate transformation connecting the two spacetime patches. Again, following the notation and exposition of GLPW, scalar quantities evaluated at the interface must be invariant:

$$[\phi]^+ = [\phi]^-, \quad (16)$$

$$[k \cdot \nabla \phi]^+ = [k \cdot \nabla \phi]^-, \quad (17)$$

where the $k$ is an arbitrary vector at the world-tube boundary and $\nabla$ is the metric connection. The first condition is sufficient for the construction of a matching interface, yet practical matching schemes can be based on combinations of (14) with various forms of (17).

The strategy for the matching must take into account technical issues arising from the adoption of a finite difference framework. The numerical solution to the initial value problem replaces the continuum domain with a finite set of points where the solution is sought to be approximated. The boundary on which conditions like (16) and (17) are true, need not necessarily lie on grid points, and hence the enforcement of those conditions calls for a systematic procedure for interpolating information between the two domains. The numerical interpolation scheme developed here, in addition to enforcing the matching conditions, also provides the flexibility of a moving, in coordinates, boundary of the two domains. Such an arrangement may be of importance in future CCM investigations as it brings to the CCM method a certain degree of adaptivity. In this work though, the moving boundary case is studied only as a proof of principle.

The collection of $M^-$ grid points that cannot be evolved using the evolution equation are denoted as boundary points. In the present one-dimensional problem, and for second order spatial discretization near the boundary, there is only a single time sequence of such points, namely $(t^n, \eta^n)$. At boundary points, one may generally need information about both the fields and their first derivatives in time and space. The basic technique of the matching procedure is to generate this information by transforming field values from the $M^+$ domain: The algorithm starts with cubic interpolations along the radial null $x$ directions on each of the retarded time levels $u^{n-1}, u^{n-2}, \ldots$. The radial locations at which interpolation is performed are at the cross-sections of the $t^n$ level with each of the $u^n$ levels. This computation provides an $O(\Delta t^4)$ accurate approximation of the field values on the extension of the $t^n$ level outside the boundary. Next, a cubic spline interpolation combines values from the regular $M^-$ grid at the $t^n$ level, with the newly interpolated values outside the boundary. Thus, an $O(\Delta t^4)$ accurate approximation of the field can be obtained anywhere in the neighborhood of the boundary. In the problem at hand, the interpolant is used to provide the value for the boundary point of the $t^n$ level and the first point of the $u^n$ level. The use of spline interpolation enforces smooth first order radial derivatives of the field. Thus, the interpolant satisfies identically both (16) and (17) conditions.

The practical implementation of the interpolation procedure asks for a minor departure from pure evolution algorithms, in that field values in the neighborhood of the boundary are stored for a sufficient number of iterations. In the present context, i.e., a fixed static geometric background, more minimalistic schemes are possible. Nevertheless, the interpolation procedure has been constructed in a way that will allow the handling of some more complicated matching scenarios.

### IV. TESTS AND RESULTS

The stability and convergence characteristics of the CCM algorithm described in the previous sections rely in a significant way on the corresponding properties of the individual components, namely the Cauchy and characteristic evolutions. The standard discretization scheme used in the space-like $M^-$ region assures that, within the stability constraints of the explicit scheme used, the numerical evolution in $M^-$ is stable. The characteristic algorithm, is
also an application of a carefully calibrated, conditionally stable, scheme \cite{13}. The matching of the two evolution algorithms introduces, on the other hand, a new approximate operation that must be validated accordingly. In GLPW, extensive testing of the algorithm established its stability in the presence of local non-linear wave propagation. The tests presented here suggest further, that in the presence of a black hole, for a variety of initial data and for long evolution times, no growing modes become manifest.

Energy conservation and Cauchy convergence are two standard tools for assessing the accuracy characteristics of an evolution algorithm. In the case under consideration, the conservation law can be used to assess the accuracy with which energy is transported across the matching surface. The total energy content \( E \) of a space-like, hypersurface in \( M^- \) labeled by \( t \) is given by

\[
E(t) = \frac{1}{2} \int_{\eta_-}^{\eta_+} (g_{\eta\eta}^t + g_{\eta n}^2 + V(\eta)g^2) d\eta.
\]

The change of total energy in a time interval \( t_1 - t_0 \) is equal to the time integral of the flux across the boundaries; that is,

\[
E(t_1) - E(t_0) = F(t_1, t_0) = \int_{t_0}^{t_1} g_{\eta g} \eta g dt \bigg|_{\eta_-}^{\eta_+} - \int_{t_0}^{t_1} g_{\nu g} g_{\nu} dt \bigg|_{\eta_-}^{\eta_+}.
\]

This identity allows a robust check on the consistency and accuracy of the matching. In Fig. 3, the second order accuracy of energy conservation is displayed. The plot is produced by grid sizes of \((200 \times 2^k + 200 \times 2^k)\) points, where \( k = 1, 2, 3, 4 \). (I.e., the coarsest grid resolution for both regions has been taken the same). The initial data are an outgoing polynomial pulse, with compact support in \(-4 < \eta < -1\). The matching of the two evolution regions is given by

\[
-E(\eta_1) = E(\eta_0) + \frac{1}{2} \int_{\eta_1}^{\eta_0} (g_{\eta\eta} + V(\eta)) g^2 d\eta + \int_{\eta_0}^{\eta} g_{\nu g} g_{\nu} dt d\eta.
\]

For the given Schwarzschild potential, the power of the late-time tail is characteristic of generic compact support initial data, and depends only on the multipole index \( l \). At the time-like and null infinities, the tail dependence on time is shown \cite{10,12} to be respectively \( t^{-2l-3} \) and \( u^{-l-2} \). Verifying this behavior constitutes a good accuracy and
stability test of the CCM algorithm, as it follows the physical behavior of the system over a large amplitude range, over long periods of time.

The null infinity signal is read at \( x = 1 \), i.e., the compactified location of null infinity. The time-like infinity signal is read at the matching boundary. In Fig. 3 the focus is on the final power law behavior at null infinity. The run parameters are a grid of 1500 points in each of the regions, a matching radius at \( r = 20M \), and an initial pulse with compact support between \( \eta \in (-15, 15) \). In Fig. 4 the focus is on the final power law behavior at time-like infinity, with the same run parameters as Fig. 3.

The late time power law tails at null infinity for various multipoles \( (l = 0, 1, 2) \) are measured to be \(-1.89, -2.91\) and \(-3.9\), in agreement with the expected values of \(-2, -3, -4\). Similarly, the late time power law tails at time-like infinity for the multipoles \( l = 0, 1 \), are measured to be \(-3.04\) and \(-4.98\) respectively, which should be compared to the theoretically expected values of \(-3\) and \(-5\) respectively. To increase the range of multipole values whose decay is observable, the grid resolution must be increased.

The monitoring of tails at both timelike and null infinity, with a modest number or radial grid points, illustrates vividly the physical and algorithmic economy achieved with the CCM approach. The late time behavior of the field is directly related to the asymptotic fall-off of the potential. This fall-off is captured easily in the compactified description of the spacetime.

V. CONCLUSIONS

The results presented in this paper accompany the discussion of GLPW and complete the investigation of the CCM framework for the model problem of a spherically symmetric scalar field. These results lend further support to the idea that a Cauchy + characteristic approach for solving initial value problems is a potentially useful approach to numerical relativity. The completion of the computational domain with a null foliation extending to infinity eliminates the problem of backscattered waves in a natural way, even for arbitrary long evolution times. It hence provides a significant increase in the dynamic range of standard Cauchy codes. There is, however, considerable amount of work to be carried out before one can reach a definite answer as to whether CCM approaches are computationally efficient in three dimensional problems.

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FIG. 1. Energy conservation. The upper plot displays the energy content inside the world-tube as a function of time (diamonds) along with the integrated flux of radiation across the world-tube (circles). The lower plot demonstrates the convergence of the conserved total energy as a function of grid spacing. The convergence rate is measured as 1.995.

FIG. 2. Space-time evolution of a scalar field in the case of a moving boundary. An oscillating source is located at \( r = 20 \). The original position of the boundary is at \( r = 50 \) and it subsequently moves inwards. To the left of the boundary (small \( r \)) the field values are obtained from the Cauchy domain. To the right of the boundary (large \( r \)) the values are given by the null code. The flattening of the evolution profile to the right of the boundary is the typical manifestation of outgoing radiation in retarded time coordinates.

FIG. 3. The scalar field at null infinity for initial data of compact support. The polynomial pulse is seen to register first (at finite retarded time) and is then followed by damped oscillations. After the amplitude of the normal modes decays below the long range tail signal, the latter becomes the dominant feature.

FIG. 4. Late time power law tails at null infinity. The measured exponents for \( l = 0, 1, 2 \) are \(-1.89, -2.91\) and \(-3.9\) respectively, which compare well with the analytic values of \(-2, -3, -4\).

FIG. 5. Late time power law tails at time-like infinity. The measured exponents for \( l = 0, 1 \) are \(-3.04\) and \(-4.98\) respectively. The predicted values are \((-3, -5)\).