Dynamics of twin bubbles formed by ultrasonic cavitation in a liquid

Jinfu Liang\(^a,^*\), Xueyou Wu\(^a\), Yupei Qiao\(^a\)

\(^a\) School of Physics and Electronic Science, Guizhou Normal University, Guiyang 550001, China

**ARTICLE INFO**

**Keywords:**
- Twin bubbles
- Ultrasonic cavitation
- Dynamics of bubbles

**ABSTRACT**

Based on potential flow and perturbation theory, a theoretical model is derived to describe the pulsation, translation, and deformation of twin bubbles in an ultrasound field. The amplitudes of radial oscillation, translation, and deformation of twin bubbles are found to depend on initial translation velocities. The radii, translation, and deformation of twin bubbles also exhibit periodic behavior. As the initial translation velocities increase, the periods of two bubbles’ oscillations reduce, and the instable area in the phase space of \(R_{01} - R_{02}\) enlarges because of the interaction between bubbles.

1. Introduction

Acoustic cavitation refers to the generation of numerous micro-sized gas bubbles when an intense ultrasonic wave propagates in a liquid [1]. Each bubble repeatedly expands, contracts, and then quickly collapses during cavitation. The intense compression of gas inside the bubble may result in high temperature and pressure [2]. When the temperature is sufficiently high, light may be emitted from a bubble, which is referred to as sonoluminescence [3,4]. Acoustic cavitation has been applied in ultrasound cleaning [5], preparation of nanostructured materials [6], and catalytic chemical reaction [7].

A study on the dynamics of bubbles is necessary to explain sonoluminescence and explore novel applications of ultrasonic cavitation. Rayleigh [8] investigated the pressure developed in the interior of a fluid during the collapse of a spherical bubble. Plesset [9] investigated deformation in a bubble based on hydrodynamics formulation. Keller and Miksis [10] proposed a model for large amplitude forced radial oscillations of a bubble in a compressible liquid. In 1992, Gaitan et al. [4] observed the single bubble sonoluminescence (SBSL) and obtained the time variation in the radius of a bubble using laser scattering. This has resulted in a novel experimental technique for exploring the dynamics of bubbles and the relation to sonoluminescence. Madrazo [11] determined the shape of a cavitation bubble to be an ellipsoid with eccentricity close to 0.2 by studying the angular correlations of dipole radiation. Flannigan et al. [12] observed the line emission from a rapidly translating bubble in solutions of sulfuic acid with alkali-metal salts irradiated with ultrasound wave. Cui et al. [13] experimentally investigated the radial and translational oscillation of a levitated bubble in aqueous ethanol solutions.

In addition to the single bubble, twin bubbles have been observed in ultrasonic cavitation experiments. The dynamics of twin bubbles and multi-bubbles are more complicated because of the interaction between bubbles and the effect of a complicated sound field. Barbat et al. [14] observed stable, periodic translational motion of twin bubbles. Shirōta et al. [15] employed a high-speed video camera to record the motion of twin bubbles revolving along an elliptical orbit about their center of mass for several minutes. Recently, Regnault et al. [16] have used a dual-frequency levitation chamber to trap oscillating twin bubbles at a close fixed distance and measured the force between bubbles. Wu et al. [17] observed the dynamic behavior of pulsation, translation, and deformation of bubbles at the water-oil interface using a high-speed camera. These experimental results show that the dynamics behaviors of both single and twin bubbles mainly include radial pulsation, translation and deformation.

Feng and Leal [18] formulated a theoretical model coupling the pulsation, translation, and deformation of a single bubble. Further, Reddy and Szeri [19] derived an expression for the time evolution of the pulsation and translation of a single bubble and then analyzed its instability. Doinikov [20] proposed the nonlinear coupling model with the pulsation, translation, and deformation of a single bubble. Mettin and Doinikov [21] analyzed the instability of a single bubble driven by high-frequency ultrasound. Wu and Liang [22] developed the model describing the radius, translation and deformation of a single gas bubble in ultrasonic field.

However, to the best of our knowledge, a theoretical model coupling the pulsation, translation, and deformation of twin bubbles has not been...
reported in the literature. Doinikov [23] analyzed radial and translational motions of twin bubbles without deformation based on Lagrangian formalism. Kurihara et al. [24] and Liang et al. [25,26] formulated a model of radial motion and surface deformation of twin bubbles without translational motion. In this study, we propose a model that couples the pulsation, translation, and deformation of twin bubbles driven by ultrasound wave based on the perturbation and potential flow theories. Our study illustrates the effects of the initial translation velocities of bubbles on the radial pulsation, translation, deformation, non-spherical oscillations, and instabilities of twin bubbles.

2. Governing equations of twin bubbles motion involving pulsation, translation, and deformation

Consider two cavitation gas bubbles in a perfect incompressible liquid (Fig. 1). Suppose that the liquid flow is irrotational, spacing between the bubbles is large compared with the bubbles’ sizes, and local spherical coordinates have their origins at the moving centers (O₁ or O₂) of two bubbles, then in the spherical coordinates the equations of bubbles’ surfaces are described as

\[ F_j(r_j, \theta_j, \phi_j, t) = r_j - S_j(\theta_j, \phi_j, t) = 0. \]  
(1)

Suppose that the shapes of twin bubbles are spherical and symmetrical about the Z-axis, and the surface functions of twin bubbles can be expressed approximately as

\[ S_j(\theta, t) = R_j(t) + a_0(t) P_j(\cos \theta), \]  
(2)

where \( R_j(t) \) and \( a_0(t) \) are the spherical components and the coefficients of aspherical components of the \( j \)th bubble, respectively; \( j = 1 \) or \( 2 \); \( \varepsilon \) is a small parameter, which is set to be less than 1 to guarantee the aspherical nature of bubble.

Based on the potential flow theory, in the liquid, the velocity potential \( \phi \) satisfies the Laplace equation \( \nabla^2 \phi = 0 \) near the two bubbles and can be expressed as [23]

\[ \phi = \phi_1 + \phi_2, \]  
(3)

where \( \phi_j \) is the scattered potential of the \( j \)th bubble.

\[ \phi_j = \sum_{n=6}^{\infty} A_{n,j}(t)r_j^{-(n+1)} P_n(\cos \theta), \]  
(4)

\[ \phi_j = \sum_{n=3}^{\infty} B_{n,j}(t)r_j^{n-3} P_n(\cos \theta), \]  
(5)

where \( P_n(\cdot) \) is the Legendre polynomial. Eq. (4) gives \( \phi_j \) in the proper coordinate of the \( j \)th bubble and Eq. (5) in the coordinate of the other bubble. The relation between \( A_{n,j}(t) \) and \( B_{n,j}(t) \) is as follows [27,23]:

\[ B_n(t) = \left( \frac{(-1)^n}{D^n} r_0 \right) \sum_{m=0}^{[n/2]} \frac{(-1)^m (n+m)!}{n! m! D^n} A_n(t). \]  
(6)

where \( D \) is the distance between bubbles. Combined with Eqs. (4)-(6), the total velocity potential near the \( j \)th bubble is expressed as

\[ \phi = \sum_{n=0}^{\infty} A_n(t) r_j^{-(n+1)} + B_n(t) r_j^n P_n(\cos \theta). \]  
(7)

Consider the pulsation, translation, and small deformation of twin bubbles, and according to the Eqs. (1), (2), and (7), the total velocity potential near the \( j \)th bubbles is written as

\[ \phi \approx B_0 + \frac{A_0}{r} P_0(\mu_t) + \left( B_1 r + \frac{A_1}{r^2} \right) P_1(\mu_t) + \epsilon \left( B_2 + \frac{A_2}{r^2} \right) P_2(\mu_t). \]  
(8)

The normal velocities at the surfaces of the \( j \)th bubbles obey Eq. (9)

\[ \frac{\partial F_j}{\partial r} - \dot{r}_j \cos \theta_j + \left( \nabla \phi \right) \frac{\partial F_j}{\partial \theta_j} = 0, \quad \text{at} \quad r = S_j, \]  
(9)

where \( F_j = F_j(r_j, \theta_j, t), S_j = S_j(\theta_j, t), \dot{r}_j \) denotes the translation velocity of the center of the \( j \)th bubble, \( \epsilon \) is the unit vector of normal velocity, and \( \nabla \) denotes the gradient with respect to \( r \).

Substituting Eqs. (2) and (8) into Eq. (9), and then expanding Eq. (9) based on Taylor’s theorem with respect to \( \epsilon \), we obtain:

\[ \epsilon^0 : \quad \rho_0 = -R_j^3 \ddot{R}_j; \]  
(10)

\[ \epsilon^1 : \quad A_1 = \frac{1}{3} R_j^3 \left( 3 \dot{R}_j B_1 - \dot{B}_1 R_j \right); \]  
(11)

\[ \epsilon^1 : \quad B_2 = \frac{2}{3} B_0 R_j^2 - R_j \ddot{B}_0 - 2 \dot{R}_j \dot{B}_0. \]  
(12)

Under the condition of the irrotational flow in the incompressible liquid, the generalized Bernoulli equation is [28]

\[ \frac{p_t - p_\infty}{\rho} = \frac{1}{2} \left( |\nabla \phi|^2 - \frac{\partial \phi}{\partial r} \right), \]  
(13)

where \( \rho \) is the density of the liquid, and \( p_t \) and \( p_\infty \) denote the pressures at the position \( r \) and infinity, in liquid, respectively.

Substituting Eqs. (2) and (8) into Eq. (13) and expanding Eq. (13) based on Taylor’s theorem with respect to \( \epsilon \), then combining Eqs. (10)-(12), we obtain

\[ \epsilon^0 : \quad R_j \ddot{R}_j + \frac{3}{2} R_j \dot{R}_j \left( \frac{3}{2} B_1 - \frac{p_t - p_\infty}{\rho} \right) = 0; \]  
(14)

\[ \frac{3}{2} R_j \dot{R}_j - \frac{p_t - p_\infty}{\rho} = 0. \]  
(15)

\[ \epsilon^1 : \quad \frac{3}{2} R_j \dot{R}_j + \frac{1}{2} R_j \ddot{R}_j + \frac{3}{2} R_j \dot{B}_1 - \frac{3}{2} \ddot{R}_j B_1 - \frac{1}{2} \dddot{R}_j = 0. \]  
(16)

\[ \frac{4}{3} R_j \ddot{R}_j + \dot{R}_j \dot{B}_1 + \frac{4}{3} \ddot{R}_j B_1 - \frac{3}{2} R_j \dot{B}_1 = 0, \]  
(17)

Fig. 1. Geometry for the twin interacting bubbles with pulsation, translation, and small shape deformation. \( \rho \) and \( c \) are the density of the liquid and speed of sound in liquid, respectively.
where
\[ p_1 = p_0 - \frac{2\gamma}{R_0} + \frac{4\rho}{R_0^2} \frac{d}{dt} (R_0^2 - \gamma^2), \]
\[ p_0 = \left( p_0 + \frac{2\gamma}{R_0} \right)^2 \left( \frac{R_0^2}{R_0^2 - \gamma^2} \right), \]
\[ p_\alpha = -p_0 \sin(2\pi ft), \]

where \( p_0 \) is the static pressure in the liquid; \( \sigma \) is the surface tension; \( \eta \) is the viscosity of the liquid; \( \gamma \) is the ratio of specific heats; \( h_1 \) is the van der Waals hard-core radius, with \( h_1 = R_{\theta}/8.54 \) for air; \( R_0 \) is the equilibrium radius of the jth bubble; \( p_\alpha \) and \( f \) are the amplitude and frequency of driving ultrasound, respectively.

Based on Eq. (6), \( B_0, B_1, \) and \( B_2 \) are written with accuracy up to the 4th term with respect to \( D^3 \):
\[ B_0 \approx -\frac{R_1^3 \dot{R}_{3,j}}{D} + \frac{R_1^3 \dot{\dot{R}}_{3,j}}{2D^2}, \]
\[ B_1 \approx -\frac{R_1^3 \dot{R}_{3,j}}{D} + \frac{R_1^3 \dot{\dot{R}}_{3,j}}{D^3}, \]
\[ B_2 \approx -\frac{R_1^3 \dot{R}_{3,j}}{D^3} + \frac{R_1^3 \dot{\dot{R}}_{3,j}}{D^4}. \]

When \( D \to \infty \), Eqs. (14)-(16) become the uncoupled model of two single bubbles with pulsation, translation, and deformation similar to that in Ref. [20,17]. When \( 1/D \neq 0 \), and \( 1/D^3 \to 0 \), Eqs. (14) and (15) reduce to the coupled equations of twin bubbles with translation as in Ref. [23,29]. When containing \( x \) and \( x \) cannot be expressed, Eqs. (14) and (16) transform into coupled equations of twin bubbles with deformation as in Ref. [25], and Eq. (15) is omitted.

Notably, in the process of deriving Eqs. (14)-(16), the terms containing \( e^0 \dot{P}_0 (\cdot), e^0 \dot{P}_1 (\cdot), \) and \( e^0 \dot{P}_2 (\cdot) \) are selected to describe the radial vibrations, translation, and deformation of twin bubbles, respectively, and the terms containing \( e^1 \dot{P}_2 (\cdot), e^1 \dot{P}_1 (\cdot), e^1 \dot{P}_3 (\cdot), \) and \( e^1 \dot{P}_4 (\cdot) \) are omitted.

### 3. Effect of initial translation velocity on the dynamics of twin bubbles

Eqs. (14)-(16) describe the radial pulsation, translation and deformation of two bubbles, respectively. By numerically calculating the coupled Eqs. (14)-(16), the time evolutions of radial, pulsation and deformation of twin bubbles are obtained. In present paper, we mainly analyze the effect of the initial translation velocities of twin bubbles on the instability, radii, translations, deformations of twin bubbles, as well as the non-spherical oscillations and interactions of twin bubbles. The physical parameters used in the calculation are shown in Table 1, wherein \( x_0 \) denotes the initial position of the center of the jth bubble.

#### 3.1. Pulsation, translation, and deformation

To understand the evolution of the radii, translation, and deformation of twin bubbles versus time, we numerically calculate Eqs. (14)-(16). Fig. 2 shows the evolutions of \( R_{11}, x_1, \) and \( x_2 \) of bubble 1 and \( R_{22}, x_2, \) and \( x_2 \) of bubble 2 for five periods under different initial translation velocities of two bubbles.

Figs. 2 (a) and (b) show that the radii of two bubbles periodically oscillate with times. When \( v_01 = 5 \text{ m/s}, v_02 = 0 \), the amplitudes of two bubbles enlarge starting at the 4th acoustic period while they barely change under \( v_01 = 0, 0.5, \) and \( 1 \text{ m/s}, \) respectively. Fig. 2 (c)-(f) show that the translation and deformation of two bubbles increase with increasing \( v_01 \), especially when \( v_01 = 5 \text{ m/s}, \) the magnitudes of the translation and deformation of two bubbles increase more strongly than those under \( v_01 = 0, 0.5, \) and \( 1 \text{ m/s}, \) respectively.

#### 3.2. Oscillations of twin aspherical bubbles

The shape of one of the twin bubbles is taken to be that of an axisymmetric ellipsoid with two semi-axes of equal length. The extent of elongation of the ellipsoid is measured by the ratio \( L = a_1/\rho_1 \) for bubble 1 and \( L = a_2/\rho_2 \) for bubble 2, whereas the length of the third semi-axis corresponds to the axis of symmetry. When \( L = 0 \), the bubble is perfectly spherical; When \( 0 < L < 1 \), the bubble is an ellipsoid; when \( L > 1 \), the bubble breaks. The bubble shape can be projected onto the XY plane following the method described in Refs. [26,30,31].

Figs. 3 and 4 show the shapes of bubbles 1 and 2 at the different instants, and under different initial translation velocities, \( v_01 \) and \( v_02 \). In Figs. 3 and 4, the black line denotes the shape of the bubble at the initial moment (\( t = 0 \)), blue line denotes the shape of the bubble with the maximum radius, magenta line denotes the shape of the bubble with the minimum radius, and red line denotes the shape of the bubble at the moment of deformation. With an increasing \( v_01 \), the moment of deformation reduces for two bubbles. However, the moment of deformation for bubble 1 at \( v_01 = 5 \text{ m/s} \) and \( v_02 = 0 \) is approximately 21.80 \( \mu \text{s} \) less than 27.68 \( \mu \text{s} \) of bubble 2 at the same \( v_01 \) and \( v_02 \). This could be because the interaction between two bubbles increases as the distance \( D \) decreases when bubble 1 approaches bubble 2.

The force \( F_{ij} \) exerted by one bubble on the other is given by [32]
\[ F_{ij} = \frac{\rho}{4\pi D^2} \left( \frac{V_i}{V_j} \right) \dot{V}_j, \]

where \( D \) and \( \epsilon \) denote the distance between the twin bubble’s centers and the radial unit vector directed from the ith bubble toward the other one, respectively; \( V_i \) and \( V_j \) denote the volumes of the ith bubble and the other bubble, respectively.

Fig. 5(a)-(f) show that the magnitude of \( F_{12} \) is not equivalent to that of \( F_{21} \), and both vary periodically over time. The net force of one bubble on the other bubble is the time average of \( F_{ij} \) over one period, which is known as the Bjerknes force \( F_{B} \) and is given by [32]
\[ F_B = \langle F_{ij} \rangle_{\epsilon} = -\frac{\rho}{4\pi D^2} \left( \frac{V_i}{V_j} \right) \epsilon, \]
\[ = -f_B \frac{D}{\epsilon} \epsilon, \]

where \( \langle \cdot \rangle \) denotes the time average, and \( \langle F_{12} \rangle = -\langle F_{21} \rangle \) in the same coordinate system.

\[ f_B = \frac{\rho}{4\pi} V_1 V_2 \]

denotes the coefficient of \( F_B \).

Figs. 5(g)-(i) show that \( V_1 V_2 \) varies periodically over time. \( f_B \) is
obtained in each of the first five periods by averaging of $\dot{V}_1$ and $\dot{V}_2$, respectively, that is, $4.79 \times 10^4\, \mu N \times \text{mm}^2$, $4.80 \times 10^4\, \mu N \times \text{mm}^2$, $5.45 \times 10^4\, \mu N \times \text{mm}^2$, $6.25 \times 10^4\, \mu N \times \text{mm}^2$, and $6.59 \times 10^4\, \mu N \times \text{mm}^2$. The two bubbles attract each other in the first five periods because the values of $f_{\text{B}}$ are positive.

3.3. Shape instability of twin bubbles

The instability of twin bubbles is closely related to Rayleigh–Taylor (RT) instability, parametric instability, and after bounce instability [33]. In this study, we consider only the effect on RT instability. In RT instability, bubble surface deformation increases faster than the radii of the bubbles after several cycles of pulsation, and the bubbles burst. Thus, the criterion of the RT instability is given as

$$\max_{n(0),N(2n)/w} \left| \alpha_n / R \right| > 1,$$

where $N$ is an integer greater than zero. Based on Eq. (27), we may obtain the $R_{01} - R_{02}$ phase picture by numerically calculating Eqs. (14)-(16). Fig. 6 depicts the $R_{01} - R_{02}$ phase picture of two bubbles under different initial translation velocities. The blue area ($A_b$) denotes the instability space, and the gray area ($A_c$) denotes the constant area. We introduce a quantity ($\beta$) to describe the contrast of the blue and gray areas:

$$\beta = \frac{A_b}{A_c} \times 100\%.$$

In the phase diagram of $R_{01} - R_{02}$, it can be observed that as $v_{01}$ increases and $v_{02} = 0$, the instable area enlarges. When $v_{01} = 3\, \text{m/s}$, twin bubbles are instable in the area of $R_{01} - R_{02}$, which is one reason why twin bubbles hardly survive in an ultrasound field.
4. Conclusion

In this study, a new model is derived based on potential flow and perturbation theory to describe the pulsation, translation, and deformation of twin bubbles in an ultrasound field. The amplitudes of radial oscillation, translation, and deformation of twin bubbles are found to depend on initial translation velocities. The radii, translation, and deformation of twin bubbles also exhibit periodic behavior. In addition, twin bubbles show unstable behavior according to their initial radii and initial translation velocities. This analysis provides insight into the complex dynamics of twin bubbles in ultrasonic cavitation.
CRediT authorship contribution statement

Jinfu Liang: Methodology, Resources, Formal analysis, Investigation, Visualization, Writing - review & editing. Xueyou Wu: Software, Visualization. Yupei Qiao: Methodology, Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grants Nos. 11864007, and 11564006).

References

[1] E.A. Neppiras, Acoustic cavitation, Phys. Rep. 61 (3) (1980) 159–251.
[2] K. Yasui, Hot-spot model of single-bubble sonoluminescence, J. Acoust. Soc. Am. 103 (5) (1998) 2974.
[3] H. Frenzel, H. Schultes, Lumineszenz im ultraschall-beschickten wasser, Z. Phys. Chem. B 27 (1934) 421–424.
[4] D.F. Gaitan, L.A. Crum, C.C. Church, R.A. Roy, Sonoluminescence and bubble dynamics for a single, stable, cavitation bubble, J. Acoust. Soc. Am. 91 (1992) 3166–3183.
[5] N.V. Thombe, A.P. Gadhkar, A.V. Patwardhan, P.R. Gogate, Ultrasound induced cleaning of polymeric nanofiltration membranes, Ultras. 162 (2020), 104891.
[6] Z.A. Sahar, B. Mahin, S.N. Masoud, Enhanced visible-light-driven photocatalytic performance for degradation of organic contaminants using phw4 nanostructure fabricated by a new, simple and green sonochemical approach, Ultrasound 72 (2021), 105420.
[7] E. Khokhina, N. Kumar, K. Ertan, M. Peurla, H. Palonen, J. Salonen, J. Lehtonen, D.Y. Murzin, Ultrasound irradiation as an effective tool in synthesis of the slag-based catalysts for carboxymethylation, Ultrasonics Sonochemistry 73 (2021), 105503.
[8] L. Rayleigh, On the pressure developed in a liquid during the collapse of a spherical cavity, Philos. Mag. 34 (200) (1917) 94–98.
[9] M.S. Plesset, On the stability of fluid flows with spherical symmetry, J. Appl. Phys. 25 (1) (1954) 96–98.
[10] J.B. Keller, M. Mikos, Bubble oscillations of large amplitude, J. Acoust. Soc. Am. 68 (2) (1980) 628–633.
[11] A. Madrazo, N. García, M. Nieto-Vesperinas, Determination of the size and shape of a sonoluminescent single bubble: Theory on angular correlations of the emitted light, Phys. Rev. Lett. 80 (1998) 4590–4593.
[12] D.J. Flannigan, K.S. Suslick, Emission from electronically excited metal atoms during single-bubble sonoluminescence, Phys. Rev. Lett. 99 (2007) 134301.
[13] W. Cui, W. Chen, S. Qi, C. Zhou, J. Tu, Radial and translational oscillations of an acoustically levitated bubble in aqueous ethanol solutions, J. Acoust. Soc. Am. 132 (1) (2012) 138–143.
[14] T. Barbat, N. Adighriz, C.S. Liu, Dynamics of two interacting bubbles in an acoustic field, J. Fluid Mech. 389 (1999) 137–168.
[15] M. Shiroti, K. Yamashita, T. Inamura, Orbital motions of bubbles in an acoustic field, AlP Conf. Proc. 1474 (1) (2012) 155–158.
[16] G. Regnault, C. Mauger, P. Blanc-Benon, C. Iserra, Secondary radiation force between two closely spaced acoustic bubbles, Phys. Rev. E 102 (2020), 033110.
[17] W. Wu, D. Eskin, A. Priyadarshi, T. Subroto, I. Tzanakis, W. Zhai, New insights into the mechanisms of ultrasonic emulsification in the oil water system and the role of gas bubbles, Ultrason. 73 (2021), 105501.
[18] Z.C. Feng, L.G. Leal, Translational instability of a bubble undergoing shape oscillations, Phys. Fluids 7 (6) (1995) 1325–1336.
[19] A.J. Reddy, A.J. Szeri, Shape stability of unstably translating bubbles, Phys. Fluids 14 (7) (2002) 2216–2224.
[20] A.A. Doinkov, Translational motion of a bubble undergoing shape oscillations, J. Fluid Mech. 501 (2004) 1–24.
[21] R. Mertin, A.A. Doinkov, Translational instability of a spherical bubble in a standing ultrasound wave, Appl. Acoust. 70 (10) (2009) 1330–1339.
[22] X. Wu, J. Liang, Translation and nonspherical oscillation of single bubble in ultrasound field, Acta Physica Sinica 70 (18) (2021), 184301.
[23] A.A. Doinkov, Translational motion of two interacting bubbles in a strong acoustic field, Phys. Rev. E 64 (2001), 026301.
[24] E. Kurihara, T.A. Hay, Y.A. Ilinskii, E.A. Zabolotskaya, M.F. Hamilton, Model for the dynamics of two interacting axi symmetric spherical bubbles undergoing small shape oscillations, J. Acoust. Soc. Am. 130 (5) (2011) 3357–3369.
[25] J. Liang, W. Chen, W. Shao, S. Qi, Aspherical oscillation of two interacting bubbles in an ultrasound field, Chinese Phys. Lett. 29 (7) (2012), 074701.
[26] J. Liang, X. Wang, J. Yang, L. Gong, Dynamics of two interacting bubbles in a nonspherical ultrasound field, Ultrasonics 75 (2017) 58–62.
[27] E. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press, London, 1931.
[28] J.P. Franc, J.M. Michel, Fundamentals of Cavitation, Kluwer Academic Publisher, Nether-lands, 2004.
[29] L. Zhang, W. Chen, Y. Zhang, Y. Wu, X. Wang, G. Zhao, Bubble translation driven by pulsation in a double-bubble system, Chin. Phys. B 29 (3) (2020), 034303.
[30] K. Tsiglifis, N.A. Pelekakis, Nonlinear oscillations and collapse of elongated bubbles subject to weak viscous effects, Phys. Fluids. 17 (10) (2005), 102101.
[31] K. Tsiglifis, N.A. Pelekakis, Numerical simulations of the spherical collapse of laser and acoustically generated bubbles, Ultrason. Sonochemistry 14 (4) (2007) 456–469.
[32] R. Mertin, I. Akhatov, U. Parlitz, C.D. Ohl, W. Lauterborn, Bjerknes forces between small cavitation bubbles in a strong acoustic field, Phys. Rev. E 56 (1997) 2924–2931.
[33] S. Hilgenfeldt, D. Lohse, M.P. Brenner, Phase diagrams for sonoluminescing bubbles, Phys. Fluids 8 (11) (1996) 2808–2826.