Abstract

Consider a discrete locally finite subset $\Gamma$ of $\mathbb{R}^d$ and the complete graph $(\Gamma, E)$, with vertices $\Gamma$ and edges $E$. We consider Gibbs measures on the set of sub-graphs with vertices $\Gamma$ and edges $E' \subset E$. The Gibbs interaction acts between open edges having a vertex in common. We study percolation properties of the Gibbs distribution of the graph ensemble. The main results concern percolation properties of the open edges in two cases: (a) when the $\Gamma$ is a sample from homogeneous Poisson process and (b) for a fixed $\Gamma$ with exponential decay of connectivity.

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1 Introduction

Let a sample $\Gamma \subset \mathbb{R}^d$ of a point process be a locally finite set of $\mathbb{R}^d$. We consider an ensemble of graphs whose vertices are the points of $\Gamma$ and whose edges are the set of unordered pairs of points in $\Gamma$. Each edge can be open or close; we study probability distributions on the set of configurations of open edges. The classical example is the Erdős-Rényi’s random graph where each edge is open independently of the others with some probability. Many works have studied this model and correlates (see for example [1], [2]). In this paper we introduce interactions between edges and/or vertices and study the associated Gibbs measures.

Given a configuration of open edges, we say that two edges collide if both of them are open and they have a vertex in common. We call monomers those vertices that are extreme of no open edge. A positive energy is paid by each collision and by each monomer. Furthermore to any open edge it is assigned a positive energy proportional to its length. This energy function is described explicitly in (2.1) below. Roughly speaking, measures associated to this energy function give more weight to configurations with few monomers, no collisions and short edges.

In Theorem 1 we prove the existence of an infinite volume Gibbs measure associated to the energy function. Call dimer an open edge not colliding with any other edge. In Theorem 2 we show that the ground states for a typical configuration $\Gamma$ is composed only by monomers and dimers. Theorem 3 gives conditions for the uniqueness of the ground state. Then we consider the problem of percolation for two distributions of the point process. In the first case we consider that $\Gamma$ is a sample of Poisson process and prove that if the density is small, then there is no percolation (Theorem 4). Second we ask $\Gamma$ to satisfy a $\epsilon_0$-hard core condition (that is, the ball of radius $\epsilon_0$ around each point of $\Gamma$ has no other point of $\Gamma$), where $\epsilon_0$ is an arbitrary small positive number and prove that there is no percolation with probability 1 in this case (Theorem 5).

To prove Theorem 5 we construct a process called cluster branching pro-
cess and use a coupling of the paths of the process and the configurations of the Gibbs random graph.

The open question is the non-percolation for Poisson vertices set having arbitrary large rate $\lambda$. Of course, the large density of Poisson vertices must be compensated by a large temperature.

2 Definitions

Let $\Gamma$ be sample of a point process and consider the complete graph $G = (\Gamma, E)$, where $\Gamma$ is the set of vertices and $E$ is the set of all unordered pairs $[\gamma_1, \gamma_2] \subset \Gamma$. The set $E$ can be represented as $E = \bigcup_{\gamma \in \Gamma} E_{\gamma}$ where $E_{\gamma}$ is the set of edges $e \in E_{\gamma}$ such that $\gamma \in e$. The length of edge $e = [\gamma_1, \gamma_2]$ is defined by $L(e) = |\gamma_1 - \gamma_2|$.

Let $\Omega = \{0, 1\}^E$ be the set of configurations $\omega : E \to \{0, 1\}$. The edge $e \in E$ is called open with respect to $\omega$ if $\omega(e) = 1$, and the edge is closed if $\omega(e) = 0$. Often we shall use the term open/close with no mentions the configuration $\omega$.

We define the graph $G(\omega) = (\Gamma, E(\omega))$ as the subgraph of $G$ whose edges are $E(\omega)$, the open edges of $\omega$. The degree $d_{\omega}(\gamma)$ of a vertex $\gamma \in \Gamma$ with the respect of $\omega \in \Omega$ is $d_{\omega}(\gamma) = \# \{ e \in E_{\gamma} : \omega(e) = 1 \}$, the number of open edges containing $\gamma$. We shall use both the configuration $\omega$ and the graph $G(\omega)$ as the synonyms.

Our goal is to define a Gibbs distribution on the ensemble

$$\Omega = \{ \omega : \omega \in \Omega, d_{\omega}(\gamma) < \infty \text{ for all } \gamma \}$$

the set of graphs whose vertices have finite degree. Introduce the following formal Hamiltonian

$$H(\omega) = \sum_{e : \omega(e) = 1} L(e) + \sum_{\gamma \in \Gamma} \phi_{\omega}(\gamma), \quad (2.1)$$
where $\phi_\omega(\gamma)$ is a “penalty” energy function defined by

$$
\phi_\omega(\gamma) = \begin{cases} 
h_0, & \text{if } d_\omega(\gamma) = 0; \\
0, & \text{if } d_\omega(\gamma) = 1; \\
h_1\left(d_\omega(\gamma)^2\right) & \text{if } d_\omega(\gamma) > 1.
\end{cases}
$$

(2.2)

where $h_0$ and $h_1$ are fixed positive parameters. Notice that $\phi_\omega(\gamma)$ depends only on the degree $d_\omega(\gamma)$. It defines the energy of a pair interaction between open edges from $E_\gamma$.

An aspect which is not standard for Gibbs field constructions is that the potential function $\phi_\omega$ depends on infinite number of ‘sites’. The edges play here the role of sites in a lattice. To know that $\phi_\omega(\gamma) = 0$ we have to check that $\omega(e) = 0$ for infinite many $e \in E_\gamma$. However the generalization of the usual Gibbs construction for this case is rather easy and does not cause special considerations. Therefore further we do not concern this peculiarity.

The description of the infinite volume Gibbs measure with Hamiltonian $H$ requires the definition of finite volume Gibbs measures. Taking a finite volume $\Lambda \subset \mathbb{R}^d$ the set of points $\Gamma_\Lambda = \Gamma \cap \Lambda$ is finite. The complete finite graph $G_\Lambda = (\Gamma_\Lambda, E^\Lambda)$ has edges $E^\Lambda$ connecting all pairs of points of $\Gamma_\Lambda$. Let $\Omega_\Lambda$ be the set of configurations $\omega : E^\Lambda \to \{0, 1\}$. The Gibbs state $P_\Lambda$ on $\Omega_\Lambda$ with the ‘free’ boundary condition is defined by

$$
P_\Lambda(\omega) = \frac{\exp\{-\beta H_\Lambda(\omega)\}}{Z_\Lambda},
$$

(2.3)

for $\omega \in \Omega_\Lambda$, where the parameter $\beta$ is the inverse temperature, $Z_\Lambda$ is the normalizing constant and

$$
H_\Lambda(\omega) = \sum_{e \in E^\Lambda(\omega)} L(e) + \sum_{\gamma \in \Gamma_\Lambda} \phi_\omega(\gamma),
$$

(2.4)

where the set $E^\Lambda(\omega) \subseteq E^\Lambda$ is the set of all open edges in $\Lambda$.

Since $\Omega$ is compact, there exists a Gibbs distribution on $\Omega$ which may have infinite-degree vertices. We show later finiteness of the degrees with probability 1. This implies that the Hamiltonian $H$ generates an infinite volume Gibbs field concentrated on $\Omega$. 

4
3 Main results

3.1 Existence

A point set \( \Gamma \) is weakly homogeneous if for any \( \gamma \in \Gamma \) and any \( \beta > 0 \)
\[
T_\gamma(\beta) = \sum_{e \in E_\gamma} e^{-\beta L(e)} < \infty,
\] (3.1)
where \( E_\gamma \) is the set of all possible edges with common extreme \( \gamma \). A point set \( \Gamma \) is strongly homogeneous if
\[
\sup_{\gamma \in \Gamma} T_\gamma(\beta) < \infty.
\] (3.2)

Since the possible unbounded contribution to the sum (3.1) comes from accumulation of vertices that are close to \( \gamma \), if \( \Gamma \) consists of hard core ball centers of a fixed radius, then \( \Gamma \) is strongly homogeneous. We show later in Lemma 4 that for a Poisson process with law \( \pi_\lambda \), almost all \( \Gamma \) is weakly homogeneous but not strongly homogeneous.

**Theorem 1.** For any weakly homogeneous \( \Gamma \) and any \( 0 < h_0 < h_1 \) the Gibbs random graph distributions associated to the Hamiltonian \( H \) defined in (2.1) are concentrated on \( \Omega \).

Notice that the theorem concerns all possible Gibbs measures \( \mathbb{P} \) associated to the Hamiltonian \( H \); the problem of uniqueness is not discussed in this article.

3.2 Ground states

A configuration \( \tilde{\omega} \) is a local perturbation of \( \omega \in \Omega \) if \( \tilde{\omega} \neq \omega \) and there exists a finite volume \( \Lambda \subset \mathbb{R}^d \) such that \( \tilde{\omega} \) coincides with \( \omega \) for edges not included in \( \Lambda \): \( \tilde{\omega}(e) = \omega(e) \) for all \( e \not\in \Lambda \). A configuration \( \omega \in \Omega \) is a ground state if for any local perturbation \( \tilde{\omega} \) of \( \omega \)
\[
H(\tilde{\omega}) - H(\omega) \geq 0.
\]
(The difference is well defined because all but a finite number of terms vanish.)

**Theorem 2.** For any finite-local $\Gamma$ and any $0 < h_0 < h_1$ there exists at least one ground state. Furthermore if $\omega$ is a ground state of the Gibbs random graph distribution, then

$$d_\omega(\gamma) \leq 1$$

for every $\gamma \in \Gamma$. Moreover, the length of edges in a ground state are less than $2h_0$.

**Theorem 3.** Let $\pi_\lambda$ be the distribution of a homogeneous Poisson process with rate $\lambda > 0$. There exists $\lambda_g$ such that if $\lambda < \lambda_g$, then for $\pi_\lambda$-almost all $\Gamma$ the ground state of the Gibbs random graph with vertices $\Gamma$ is unique.

### 3.3 Non-percolation at low rate or low temperature

Let $\omega \in \Omega$. The set $E(\omega)$ is split into a set of maximal connected components which we call *clusters*. For a locally finite $\Gamma$, we say that the associated to $\Gamma$ Gibbs random graph measure $P$ on $\Omega$ percolates if there exists an infinite cluster in $E(\omega)$ with $P$ probability 1. Otherwise we say that the Gibbs random graph measure $P$ on $\Omega$ does not percolate.

In the next theorem we establish non percolation of $P$ when $\Gamma$ is a Poisson process with small intensity $\lambda$.

**Theorem 4.** Let $\pi_\lambda$ be the distribution of a Poisson process with rate $\lambda > 0$ and $\Gamma$ chosen with $\pi_\lambda$. Then in the region

$$F = \left\{ (\lambda, T) : \lambda \leq \frac{1}{2h_0 + J(T)} \right\},$$

where

$$J(T) = \int_{2h_0}^{\infty} \frac{e^{-x/T}}{e^{-x/T} + e^{-2h_0/T}} \, dx,$$

the Gibbs measure $P$ associated to $\Gamma$ does not percolate, $\pi_\lambda$-almost surely.
The next theorem is stronger but for more restricted sets $\Gamma$.

**Theorem 5.** Let $\Gamma$ be strongly homogeneous. Then there exists a critical temperature $T_c(\Gamma)$ such that for all $T < T_c(\Gamma)$ the Gibbs measure $P$ associated to $\Gamma$ does not percolate.

## 4 Proofs

### 4.1 Main Lemma

Let $\gamma$ be a point of $\Gamma$, and $\Sigma_\gamma$ be a set of all configurations defined on $E_\gamma$ and having a finite degree at $\gamma$. That is, any $\sigma \in \Sigma_\gamma$ is the restriction of a configuration $\omega \in \Omega$ to $E_\gamma$, $\sigma = \omega_{E_\gamma}$; in this case we call $\sigma$ a star centered at $\gamma$, or simply a star. Clearly $\sigma \subset \omega$. Let $d_\sigma := d_\omega(\gamma)$.

Let $\Lambda \subset \mathbb{R}^d$ be a finite volume and $\gamma \in \Lambda$; let $E^\Lambda_\gamma$ be the set of edges contained in $\Lambda$ having $\gamma$ as its end, and $\Sigma^\Lambda_\gamma$ be the set of the configurations on $E^\Lambda_\gamma$ which are restrictions of the configurations from $\Omega^\Lambda$.

**Lemma 1.** Let the point $\gamma \in \Lambda$, consider $\sigma \in \Sigma^\Lambda_\gamma$ and $\Omega^\Lambda_\sigma = \{\omega \in \Omega^\Lambda : \sigma \subset \omega\}$, and assume that the number of open edges $|\sigma|$ in $\sigma$ is greater or equal than 2 then

$$P^\Lambda(\Omega^\Lambda_\sigma) \leq e^{-\beta \sum_{e, \sigma(e) = 1} L(e) - \beta h_1(\frac{d_\sigma}{2})} e^{\beta h_0(d_\sigma + 1)} \quad (4.1)$$

**Proof.** Let $V(\sigma)$ be the set of all vertices belonging to open edges of the star-configuration $\sigma$

$$V(\sigma) = \{\gamma' : [\gamma, \gamma'] \in E^\Lambda_\gamma(\sigma)\}.$$  

To any configuration $\omega \in \Omega^\Lambda_\sigma$ we associate a configuration $\tilde{\omega}$ without the star $\sigma$; in $\tilde{\omega}$ the point $\gamma$ is isolated:

$$\tilde{\omega}(e) = \begin{cases} 
\omega(e) & \text{if } e \notin \sigma, \\
0 & \text{if } e \in \sigma.
\end{cases} \quad (4.2)$$

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The transformation $\omega \mapsto \tilde{\omega}$ (taking out the star $\sigma$ from $\omega$) changes the penalty weights only at the vertices in $\{\gamma\} \cup V(\sigma)$. Namely, for any point $\gamma'$, the penalty weight in configuration $\tilde{\omega}$ is

$$
\phi_{\tilde{\omega}}(\gamma') = \begin{cases} 
\phi_{\omega}(\gamma') & \text{if } \gamma' \notin V(\sigma) \cup \{\gamma\}, \\
h_1(d_{\omega}(\gamma')-1) + h_0d_0(d_{\omega}(\gamma') - 1) & \text{if } \gamma' \in V(\sigma), \\
h_0 & \text{if } \gamma' = \gamma,
\end{cases}
$$

(4.3)

where $\delta_0(\cdot)$ is Kronecker symbol. Consider the possible changes of the energy caused by the star $\sigma$ removal from configuration $\omega$. The difference between energies is

$$
H(\tilde{\omega}) - H(\omega) = -\sum_{e \in \sigma} L(e) + \sum_{\gamma' \in V(\sigma) \cup \{\gamma\}} (\phi_{\tilde{\omega}}(\gamma') - \phi_{\omega}(\gamma'))
$$

(4.4)

and

$$
\phi_{\tilde{\omega}}(\gamma') - \phi_{\omega}(\gamma') = \begin{cases} 
h_0 - h_1\left(\frac{d_{\omega}}{2}\right) & \text{if } \gamma' = \gamma, \\
-h_1(d_{\omega}(\gamma') - 1) & \text{if } \gamma' \in V(\sigma) \text{ and } d_{\omega}(\gamma') \geq 2, \\
h_0 & \text{if } \gamma' \in V(\sigma) \text{ and } d_{\omega}(\gamma') = 1.
\end{cases}
$$

(4.5)

We used here the fact that for any $\gamma' \in V(\sigma)$ its degree in configurations $\omega$ and $\tilde{\omega}$ satisfy equality $d_{\tilde{\omega}}(\gamma') = d_{\omega}(\gamma') - 1$, and we used that $\binom{k}{2} - \binom{k-1}{2} = k-1$ for $k \geq 2$. Let us denote

$$
\Delta \phi_{\omega}(\gamma') := \phi_{\tilde{\omega}}(\gamma') - \phi_{\omega}(\gamma').
$$
Then the probability of the star $\sigma$ is
\[
\mathcal{P}_\Lambda(\Omega_\Lambda(\sigma)) = \sum_{\omega \in \Omega_\Lambda(\sigma)} \mathcal{P}_\Lambda(\omega) = \frac{1}{Z_\Lambda} \sum_{\omega \in \Omega_\Lambda(\sigma)} e^{-\beta H(\omega)}
\]
\[
= \frac{1}{Z_\Lambda} \sum_{\omega \in \Omega_\Lambda(\sigma)} e^{-\beta (H(\omega) - H(\tilde{\omega}))} e^{-\beta H(\tilde{\omega})}
\]
\[
= \frac{1}{Z_\Lambda} \sum_{\omega \in \Omega_\Lambda(\sigma)} e^{-\beta \sum_{e \in \sigma} L(e)} + \beta \sum_{\gamma' \in V(\sigma) \cup \{\gamma\}} \Delta \phi_\omega(\gamma') e^{-\beta H(\tilde{\omega})}
\] (4.6)

Here and further instead of the sum $\sum_{e \in \sigma} L(e) = 1$ we write simply $\sum_{e \in \sigma}$.

In the last expression of (4.6) the factor $e^{-\beta H(\tilde{\omega})}$ does not depend on $\sigma$ and the factor $e^{-\beta \sum_{e \in \sigma} L(e)}$ does not depend on $\omega$. However the factor $e^{\beta \sum_{\gamma' \in V(\sigma) \cup \{\gamma\}} \Delta \phi_\omega(\gamma')}$ depends on both $\sigma$ and $\omega$. To find an upper bound depending only on $\sigma$ we represent the energy of the difference as
\[
\sum_{\gamma' \in V(\sigma) \cup \{\gamma\}} \Delta \phi_\omega(\gamma') = h_0 - h_1 \left( \frac{d_\sigma}{2} \right) + \sum_{\gamma' \in V(\sigma)} h_0 - \sum_{\gamma' \in V(\sigma)} h_1 (d_\omega(\gamma') - 1)
\]
\[
\leq h_0 + h_0 d_\sigma - h_1 \left( \frac{d_\sigma}{2} \right).
\]

The above inequality we obtain if we assume that all vertices in $V(\sigma)$ have its degrees equal to 1.

Thus we obtain the following estimate for $\mathcal{P}_\Lambda(\Omega_\Lambda(\sigma))$. Let $\Omega(\gamma)$ denote the set of configurations where $\gamma$ is isolate point, then it follows from (4.6) that
\[
\mathcal{P}_\Lambda(\Omega_\Lambda(\sigma)) \leq e^{-\beta \sum_{e \in \sigma} L(e) + \beta (h_0 - h_1 \left( \frac{d_\sigma}{2} \right)) + \beta h_0 d_\sigma} \frac{1}{Z_\Lambda} \sum_{\tilde{\omega} \in \Omega(\gamma)} e^{-\beta H(\tilde{\omega})}.
\] (4.7)

Noting that
\[
\frac{1}{Z_\Lambda} \sum_{\tilde{\omega} \in \Omega(\gamma)} e^{-\beta H(\tilde{\omega})} = \mathcal{P}_\Lambda(\Omega(\gamma)) < 1
\]
we obtain the estimation (4.1) of the lemma.

\[ \square \]

**Remark 4.1.** The estimate (4.1) does not depend on \( \Lambda \) when \( V(\sigma) \subset \Lambda \).

We can generalize the lemma for the case when there is an “environment”. Let \( B \) and \( F \) be some nonempty sets of edges of \( E_{\gamma}^{\Lambda} \) without intersection \( B \cap F = \emptyset \). Let \( B \) be the set of open edges and \( F \) be the set of closed edges. Introduce a configuration of the “environment” \( \mu \) on \( B \cup F \)

\[ \mu(e) = \begin{cases} 
1, & \text{if } e \in B \\
0, & \text{if } e \in F 
\end{cases} \]

Consider the following sets of edges \( G_{\gamma,B \cup F}^{\Lambda} = E_{\gamma}^{\Lambda} \setminus (B \cup F) \). And let \( \Sigma_{\gamma,B \cup F}^{\Lambda} \) be the set of all configurations on \( G_{\gamma,B \cup F}^{\Lambda} \). If \( \sigma \in \Sigma_{\gamma,B \cup F}^{\Lambda} \) then the degree of the point \( \gamma \) is equal to the number of open edges on \( \sigma \) (we denote it by \( d_\sigma \)) plus the number of the open edges \( |B| \) in the “environment” \( \mu \) (denote it by \( d_\mu \)). As before denote \( \Omega_\Lambda(\sigma) \) and \( \Omega_\Lambda(\mu) \) the sets of all configurations which include the star-configuration \( \sigma \) and the configuration \( \mu \) correspondingly.

**Lemma 2.** Let \( \sigma \in \Sigma_{\gamma_0,B \cup F}^{\Lambda} \) then

\[ P_{\Lambda}(\Omega_\Lambda(\sigma) \mid \Omega_\Lambda(\mu)) \leq e^{-\beta \sum_{e:\sigma(e)=1} L(e) - \beta h_1 \left( \frac{d_\sigma}{2} \right)} e^{\beta h_0 d_\sigma}. \]  

(4.8)

The proof of Lemma 2 is similar to the proof of Lemma 1. The difference in the right hand side between (4.8) and (4.1) can be explained in the following way. The set \( B \) is nonempty, thus the number of interacted pairs is at least \( \left( \frac{d_\sigma}{2} \right) + d_\sigma = \left( \frac{d_\sigma+1}{2} \right) \). That provides the energy \( h_1 \left( \frac{d_\sigma+1}{2} \right) \). Removing the star \( \sigma \) we can obtain at most \( d_\sigma \) isolated points, but not \( d_\sigma+1 \) as in the (4.1), because now the point \( \gamma \) cannot become isolated point.

**4.2 Proof of Theorem 1 on the existence**

Let \( \Gamma \) be weakly homogeneous.
Lemma 3. The following inequality

$$E_\Lambda d_\sigma(\gamma) \leq \sum_{k=0}^{\infty} e^{-\beta h_1(k/2) + \beta h_0(k+1)} \frac{(T_\gamma(\beta))^k}{(k-1)!}$$

(4.9)

holds for the mean value of the vertex degrees. Where $E_\Lambda$ is the expectation with respect to the probability $P_\Lambda$ and $T_\gamma(\beta)$ is defined in (3.1).

Proof. The assertion of the lemma follows from the inequalities

$$E_\Lambda d_\sigma(\gamma) = \sum_{\sigma} d_\sigma(\gamma) P(\Omega_\Lambda(\sigma)) = \sum_{k=0}^{\infty} k \sum_{\sigma: |E_\gamma(\sigma)|=k} P(\Omega_\Lambda(\sigma))$$

$$\leq \sum_{k=0}^{\infty} k e^{-\beta h_1(k/2) + \beta h_0(k+1)} \sum_{\sigma: |E_\gamma(\sigma)|=k} e^{-\beta \sum_{e \in E_\gamma(\sigma)} L(e)}$$

$$\leq \sum_{k=0}^{\infty} k e^{-\beta h_1(k/2) + \beta h_0(k+1)} \frac{(T_\gamma(\beta))^k}{k!}.$$

Remark 4.2. We note that, when $\Lambda$ contains the star, then the estimation does not depend on $\Lambda$. Thus it gives an uniform over $\Lambda$ upper estimation, which holds when $\Lambda \not\rightarrow \mathbb{R}^d$

The theorem follows now from the finiteness of $E_\Lambda d_\sigma(\gamma)$ (see (4.9) and Remark 4.2).

Any sample of Poisson process is weakly homogeneous. It shows the next

Lemma 4. Almost all samples $\Gamma$ from Poisson distribution $\pi_\lambda$ are weakly homogeneous.

Proof. Let $\gamma \in \Gamma$. Consider a sequence of rectangles $U_n = [-l_n, l_n]^d$, $n \geq 0$ centered in $\gamma$ with size length $l_n$. We chose $l_n = (n + 1)^{1/d}$. Then any ring
\( W_n = U_n - U_{n-1}, \ n \geq 0, \) except \( U_{-1} = \emptyset, \) has its volume equal to 1. If \( \gamma \in W_n \) then for \( e = \langle \gamma, \gamma' \rangle \) the inequality \( L(e) \geq l_{n-1} \) holds. Let \( \xi_n \) be a number of points from \( \Gamma \) located in \( W_n. \) The variables \( \xi_n \) are independent random variables having Poisson distribution with the parameter \( \lambda \) (since the volume of \( W_n \) is equal to 1). Therefore the following series converges

\[
T_\gamma = \sum_{e \in E_\gamma} e^{-L(e)} \leq \sum_{n=0}^{\infty} \xi_n e^{-l_{n-1}} = \sum_{n=0}^{\infty} \xi_n e^{-n\frac{1}{d}} < \infty \quad \text{a.s.}
\]

The convergence with probability 1 follows from the convergence of the series of the expectations and the variances of the random variables \( \xi_n e^{-n\frac{1}{d}} \) (Theorem of "two series", [5]).

4.3 Proof of Theorem 2 and 3 on the ground states

Proof of Theorem 2 First we prove the property of the ground states if there exists at least one. Assume the inverse. Let \( \omega \) be a ground state and there be a vertex \( \gamma_1 \in \Gamma \cap \Lambda \) such that \( d_\omega(\gamma_1) \geq 2. \) Let \( e = [\gamma_1, \gamma_2] \) be the incident to the vertex \( \gamma_1 \) in graph \( \omega, \) \( \omega(e) = 1. \) Let \( \tilde{\omega} \) be the new configuration such that \( \tilde{\omega} \) is the same as \( \omega \) with the exception that the edge \( e \) is now removed: \( \tilde{\omega}(e) = 0. \) Then we have

\[
H_\Lambda(\omega) - H_\Lambda(\tilde{\omega}) = \begin{cases} 
L(e) + (d_\omega(\gamma_1) + d_\omega(\gamma_2)) h_1 & \text{if } d_\omega(\gamma_2) \geq 1, \\
L(e) + d_\omega(\gamma_1) h_1 - h_0 & \text{if } d_\omega(\gamma_2) = 0.
\end{cases}
\]

Since \( 0 < h_0 < h_1, \) we have

\[
H_\Lambda(\omega) - H_\Lambda(\tilde{\omega}) > 0.
\]

There is no edges in a ground with its length \( L \) greater than \( 2h_0, \) since the energy of two monomers is \( 2h_0 < L. \)
Proving the existence at least one of the ground states consider a sequence $(V_n)$ of increasing cubes covering $\mathbb{R}^d = \bigcup_n V_n$. We build a ground state of the model by a sequence of reconstructions of an initial configuration. It is reasonable to take the initial configuration satisfying the property proved above. For example, we can take the configuration $\omega_0$ with no edges, that is the configuration of all monomers. Let $\omega_n$ be a configuration in $V_n$ having the minimal energy over all configurations in $V_n$. There exists a sequence $(\omega'_n)$ of configurations which is a subsequence of $(\omega_n)$, that is $\omega'_n = \omega_{n_1}$, such that there exists a limit $\lim_{i \to \infty} \omega'_i(e)$ for every $e \in E$. Moreover, the sequence $(\omega'_n)$ can be chosen such that $\omega'_j(e) \equiv \text{const}$ for all $j \geq i$ when $e \in E_{V_i}$. The configuration $\omega' = \bigcup_i \omega'_i$ is one of the ground states. Indeed, let $\hat{\omega}$ be a local perturbation of $\omega'$. There exists $V_{i_0}$ such that $\{ e : \hat{\omega}(e) \neq \omega'(e) \} \subseteq E_{V_{i_0}}$. For any $i > i_0$ let $\hat{\omega}_i$ be the configuration equal to the restriction of $\hat{\omega}$ on $E_{V_i}$. The configuration $\hat{\omega}_i$ is the perturbation of $\omega'_i$ therefore

$$H_{V_i}(\hat{\omega}_i) - H_{V_i}(\omega'_i) \geq 0.$$ 

Moreover, the fact, that for any $i$ there is no edges with length greater that $2h_0$, means that there exists $i_1 \geq i_0$ such that

$$H(\hat{\omega}) - H(\omega') = H_{V_{i_1}}(\hat{\omega}_{i_1}) - H_{V_{i_1}}(\omega'_{i_1}) \geq 0.$$ 

That proves that the any local perturbation of $\omega'$ increase the energy. Thus $\omega'$ is really the ground state. \hfill \Box

**Proof of Theorem** The uniqueness follows from two observations. The first one is that there is no edges in the ground state with the length greater than $2h_0$. Another observation is that there exists a critical intensity $\lambda_c$ such that there is no boolean percolation with radius $h_0$ for all $\lambda < \lambda_c$ (see [3], Theorem 3.3).

Thus, for $\lambda < \lambda_c$ any process configuration $\Gamma$ is an union of finite clusters $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$, $|\Gamma_i| < \infty$, and for any $i \neq j$ and for any $\gamma \in \Gamma_i, \gamma' \in \Gamma_j$ the distance $|\gamma - \gamma'| > 2h_0$. There are open edges only inside of the clusters $\Gamma_i$. Since $\Gamma_i$ are finite there exists a unique configuration of open edges in every $\Gamma_i$ minimizing the energy. \hfill \Box
4.4 Proof of Theorem 4 and 5 on non-percolation

Proof of Theorem 4. The method of the proof is based on the domination principle. Namely, we construct a Bernoulli measure $\nu$ on $\Omega$ which does not percolate and stochastically dominates the Gibbs measure $P$. We can apply this method for small rates $\lambda$ of Poisson measure $\pi_\lambda$ and low temperature of the distribution of Gibbs random graph.

On the set $\Omega$ of the configurations we define the following Bernoulli measure $\nu$

$$\nu(\omega(e) = 1) = \frac{e^{-\beta L(e)}}{e^{-\beta L(e)} + e^{-2\beta h_0}}$$

(4.10)

independently for any $e \in E$. This measure forms the random - connected model (see [3], ch. 6), which is driven by Poisson process with the rate $\lambda$ and connected function

$$g(x) = \frac{e^{-\beta x}}{e^{-\beta x} + e^{-2\beta h_0}},$$

(4.11)

the probability of two points to be connected on the distance $x$. The proof of Theorem 4 is a direct application of Holley’s inequality (see [4], Theorem 4.8). It is shown in the next two lemmas.

Lemma 5. The following inequality

$$P(\omega(e) = 1 \mid \omega_e) \leq \nu(\omega(e) = 1)$$

(4.12)

holds for any $\omega_e$, where $P(\omega(e) = 1 \mid \omega_e)$ is the Gibbs conditional probability of $(\omega(e) = 1)$ given a configuration $\omega_e$ out of the edge $e$.

Proof of Lemma 5. Let $e = [\gamma_1, \gamma_2]$. Then the conditional probability in (4.12) depends on a configuration on $(E_{\gamma_1} \cup E_{\gamma_2}) \setminus \{e\}$:

$$P(\omega(e) = 1 \mid \omega_e) = P(\omega(e) = 1 \mid \omega_{\gamma_1} \cup \omega_{\gamma_2}),$$

where $\omega_{\gamma_1}$ and $\omega_{\gamma_1}$ are configurations on $E_{\gamma_1} \setminus \{e\}$ and $E_{\gamma_2} \setminus \{e\}$ respectively, and $\cup$ means the conjugation of the configurations.

Consider three cases:
1. \( \omega_{\gamma_1} = \omega_{\gamma_2} \equiv 0 \),

2. \( \omega_{\gamma_1} \neq 0, \omega_{\gamma_2} \equiv 0 \). This case has the symmetrical version \( \omega_{\gamma_1} \equiv 0, \omega_{\gamma_2} \neq 0 \).

3. \( \omega_{\gamma_1} \neq 0, \omega_{\gamma_2} \neq 0 \)

Case 1. We have

\[
P(\omega(e) = 1 \mid \omega_{\gamma_1} \cup \omega_{\gamma_2} \equiv 0) = \frac{e^{-\beta L(e)}}{e^{-\beta L(e)} + e^{-\beta h_0}} = g(L(e)) = \nu(\omega(e) = 1)
\]

which means that for the case 1 Holly’s inequality holds.

Case 2. Let \( E_{\gamma_1}(\omega_{\gamma_1}) \) ( \( E_{\gamma_2}(\omega_{\gamma_2}) \) ) be the set of the open edges of configurations \( \omega_{\gamma_1} (\omega_{\gamma_2}) \). Let \( m := |E_{\gamma_1}(\omega_{\gamma_1})| \). Since the edge \( e \) is open then it interacts with \( m \) open edges from \( E_{\gamma_1}(\omega_{\gamma_1}) \). Then

\[
P(\omega(e) = 1 \mid \omega_{\gamma_1} \cup \omega_{\gamma_2}) = \frac{e^{-\beta L(e) - \beta mh_1}}{e^{-\beta L(e) - \beta mh_1} + e^{-\beta h_0}} < \frac{e^{-\beta L(e)}}{e^{-\beta L(e) + e^{-\beta h_0}}} < \frac{e^{-\beta L(e)}}{e^{-\beta L(e)} + e^{-2\beta h_0}} = \nu(\omega(e) = 1).
\]

Case 3.

\[
P(\omega(e) = 1 \mid \omega_{\gamma_1} \cup \omega_{\gamma_2}) = \frac{e^{-\beta L(e) - 2\beta mh_1}}{e^{-\beta L(e) - \beta mh_1} + 1} < \frac{e^{-\beta L(e)}}{e^{-\beta L(e) + 1}} < \frac{e^{-\beta L(e)}}{e^{-\beta L(e)} + e^{-2\beta h_0}} = \nu(\omega(e) = 1)
\]

where \( m = |E_{\gamma_1}(\omega_{\gamma_1})| + |E_{\gamma_2}(\omega_{\gamma_2})| \).

In the next lemma we find the condition for the non-percolation of the random - connected model, which dominate the Gibbs distribution.

**Lemma 6.** In the region (3.3) there is no percolation in the random - connected model with Poisson rate \( \lambda \) and the connection function (4.11).
Proof. The assertion of the lemma is a consequence of the Theorem 6.1 of \[3\], which claims that a random - connected model with the connection function (4.11) does not percolate if

$$\lambda \int_0^\infty g(x)dx < 1.$$  \hspace{1cm} (4.13)

Note that for any $\beta > 0$ and any $h_0 > 0$ the integral of $g(x)$ in (4.13) is finite. We represent the integral in (4.13) as

$$\int g(x)dx = \int_0^{2h_0} e^{-\beta x} dx + \int_{2h_0}^\infty e^{-\beta x} dx + \int_{2h_0}^\infty e^{-2\beta x} dx,$$

where $T = 1/\beta$. The first integral $J_1(T)$ on the right side of the above equality is increasing and tends to $2h_0$ as $\beta \to \infty$. The second integral tends to 0 as $\beta \to \infty$. Choosing $\lambda$ such that

$$\lambda < \frac{1}{2h_0 + J_2(T)} \leq \frac{1}{\int g(x)dx}$$

we obtain the claim of the lemma. 

Proof of Theorem 4 By Holley inequality, Lemma 5 implies that the Gibbs measure on $\Omega$ is dominated by the product measure, Lemma 6 implies that the product measure does not percolate under the conditions of the theorem. 

The Cluster Branching Process

The proof of Theorem 5 is based on the construction of a non-homogeneous cluster branching process of the edges.

An informal description of the cluster branching process is the following. Let $B$ be some connected set of open edges which forms a cluster and let $V$
be the set of vertices in the cluster. Consider the pair \((V, B)\) as a connected graph. The graph distance \(\rho\) between two vertices is the number of edges in a shortest path connecting them. Fix a vertex \(\gamma_0\) in \(V\). For any \(n \in \mathbb{N}\) a sphere with radius \(n\) and center \(\gamma_0\) is
\[
V^{(n)} = \{ \gamma \in V : \rho(\gamma, \gamma_0) = n \}, \quad \text{where} \quad V^{(0)} = \{ \gamma_0 \}.
\]
The sequence \(\{V^{(i)} : i = 1, 2, \ldots\}\) is a partition of \(V = \bigcup_{i=0}^{\infty} V^{(i)}\). Then \(B = \bigcup_{i=1}^{\infty} B^{(i)}\), where
\[
B^{(n)} = \{ e = [w, v] \in B : w \in V^{(n-1)} \text{ and } v \in V^{(n-1)} \cup V^{(n)} \}.
\]
We interpret the set \(B^{(n)}\) as \(n\)-th offspring generation of the ancestor set \(V^{(n-1)}\). The set \(B^{(n)}\) is a set of 'plant branches' growing from a set of 'buds' \(V^{(n-1)}\). We think \(B^{(n)}\) as the state of a branching cluster process at “time” \(n\).

This construction leads to an ambiguity since the edge \([w, v] \in B^{(n)}\) can be the offspring of two ancestors \(v\) and \(w\) if \(v, w \in V^{(n-1)}\). This problem can be solved by introducing an order along which the embranchment is controlled. The order of the branching induces a dependence of the offsprings. Another peculiarity of the branching cluster process is interactions of the offsprings having different ancestors. These properties differ the branching cluster process from the standard branching processes.

The formal definition of the branching cluster process can be made in the following way.

Construction of Cluster Branching Process. Recall that \(E_\gamma\) is the set of all edges incident with the point \(\gamma \in \Gamma\). As before we denote \(\Sigma_\gamma = \{0, 1\}^{E_\gamma}\) and \(\Sigma_{\gamma,D} = \{0, 1\}^{E_\gamma \setminus D}\) the set of all configurations on \(E_\gamma\) and \(E_\gamma \setminus D\) correspondingly, where \(D\) is some set of the edges.

The path of the cluster branching process is a sequence of triples \((B^{(n)}, V^{(n)}, S_n)\). The distribution of the cluster branching process is denoted by \(\mathbb{P}\). The precise definition is the following. Let \(\gamma_0 \in \Gamma\) be the starting point of a branching process path.
Initial stage. $B^{(0)} := \emptyset$, $V^{(0)} := \{\gamma_0\}$ and $S_0 := \emptyset$.

First stage. Let us choose some set of edges $B^{(1)} \subseteq E_{\gamma_0}$ which are the offsprings of $\gamma_0$. With help of $B^{(1)}$ we construct the next objects

$$V^{(1)} := \{\gamma: [\gamma, \gamma_0] \in B^{(1)}\},$$
$$S_1 := E_{\gamma_0}.$$

In order to define the offspring probability $\mathbb{P}(B^{(1)})$ of the ancestor $\gamma_0$ we introduce the star configuration

$$\sigma_{\gamma_0}(e) = \begin{cases} 1, & \text{if } e \in B^{(1)}, \\ 0, & \text{if } e \in S_1 \setminus B^{(1)} \end{cases}$$

and

$$\mu_1(e) = \sigma_{\gamma_0}(e).$$

The path of the one step embranchment is $B_1 = B^{(1)}$.

Then

$$\mathbb{P}(B^{(1)}) := \mathbb{P}(\Omega(\sigma_{\gamma_0})), $$

where $\Omega(\sigma_{\gamma_0})$ is the set of all configurations of $\Omega$ such that its projection on $E_{\gamma_0}$ coincide with the star-configuration $\sigma_{\gamma_0}$. It follows from Theorem 1 that the number of the offsprings from one point is finite.

Second stage. Having $B^{(1)}, V^{(1)}, S_1$ we construct the next generation. Namely, we shall define the objects $B^{(2)}, V^{(2)}, S_2$. We shall do it successively according to an order in $V^{(1)}$. The order is arbitrary. We need it to avoid the ambiguity in the definition of ancestors of an offspring $e = [w, v]$ when $w, v \in V^{(1)}$. Let $k_1 = |V^{(1)}|$. Suppose that the points in $V^{(1)}$ are enumerated in some way, $V^{(1)} = \{\gamma^{(1)}_1, \ldots, \gamma^{(1)}_{k_1}\}$. We construct successively $B^{(2)}_i, V^{(2)}_i, S_{2,i}, i = 1, \ldots, k_1$. Let us begin with the first point $\gamma^{(1)}_1$. Let $B^{(2)}_1$ be a subset of $E_{\gamma^{(1)}_1} \setminus S_1$ which is a offspring set of $\gamma^{(1)}_1$. Then
\(V_1^{(2)} := \{ \gamma : [\gamma, \gamma_1^{(1)}] \in B_1^{(2)} \};\)
\(S_{2,1} := S_1 \cup E_{\gamma_1^{(1)}}.\)

Since the set \(B_1^{(2)}\) is from \(E_{\gamma_1^{(1)}} \setminus S_1\) the initial point \(\gamma_0\) can not belong to \(V_1^{(2)}\). However the points from \(V^{(1)}\) may belong to \(V_1^{(2)}\).

In order to define the offspring probability we introduce two configurations:

\[
\sigma_{\gamma_1^{(1)}}(e) = \begin{cases} 
1, & \text{if } e \in B_1^{(2)}, \\
0, & \text{if } e \in E_{\gamma_1^{(1)}} \setminus B_1^{(2)}
\end{cases}
\]

and

\[
\mu_{2,1}(e) = \begin{cases} 
1, & \text{if } e \in B_{2,1}, \\
0, & \text{if } e \in S_{2,1} \setminus B_{2,1}
\end{cases}
\]

where the path \(B_{2,1} = B^{(1)} \cup B_1^{(2)}\). We have described two steps of the process: branching from \(\gamma_0\) and from \(\gamma_1^{(1)}\).

Further the upper index denotes the number of a stage and the lower index if it single denotes the number of a step in the stage. Double lower indices contain both the step and the stage.

The conditional probability of the offsprings \(B_1^{(2)}\) of the ancestor \(\gamma_1^{(1)}\) given the environment \(B_1\) is

\[
P(B_1^{(2)} \mid B_1) := P(\Omega(\sigma_{\gamma_1^{(1)}}) \mid \Omega(\mu_1)). \quad (4.14)
\]

Assume we have constructed \(B_i^{(2)}, V_i^{(2)}, S_{2,i}\) and also we have \(B_{2,i}, \mu_{2,i}, i = 1, \ldots, m\), where \(m < k_1\). Doing the next branching of the point \(\gamma_1^{(1)}\) choose some set \(B_{m+1}^{(2)}\) from the set \(E_{\gamma_{m+1}^{(1)}} \setminus S_{2,m}\) which means the offspring set of \(\gamma_{m+1}^{(1)}\). Then

\[
V_{m+1}^{(2)} = \{ \gamma : [\gamma, \gamma_{m+1}^{(1)}] \in B_{m+1}^{(2)} \};
\]

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Now we obtain the path $B_{2,m+1} = B_{2,m} \cup B_{m+1}$. 
To define the offspring probability introduce the configuration

$$
\sigma_{(2)}^{(m+1)}(e) = \begin{cases} 
1, & \text{if } e \in B_{m+1}^{(2)}; \\
0, & \text{if } e \in E_{(1)}^{(m+1)} \setminus S_{2,m+1}
\end{cases}
$$

and configuration

$$
\mu_{2,m+1} := \mu_{2,m} \lor \sigma_{(1)}^{(m+1)}.
$$

We use the sign $\lor$ to notate the concatenation of two configurations defined on non-intersected sets.

The conditional probability of offsprings $B_{m+1}^{(2)}$ of the ancestor $\gamma_{m+1}^{(1)}$ given $B_{2,m}$ is

$$
P(B_{m+1}^{(2)} \mid B_{2,m}) := P(\Omega(\sigma_{(1)}^{m+1}) \mid \Omega(\mu_{2,m})) \quad (4.15)
$$

Having done the construction for $i = 1, \ldots, k_1$ we obtain

$$
B^{(2)} := \bigcup_{i=1}^{k_1} B_i^{(2)}; \\
V^{(2)} := \bigcup_{i=1}^{k_1} V_i^{(2)} \setminus V^{(1)} = \{\gamma_1^{(2)}, \ldots, \gamma_{k_2}^{(2)}\}; \\
B_2 := B_{2,k_1} \text{ and } \mu_2 := \mu_{2,k_1}; \\
S_2 := S_{2,k_1}.
$$

Remark that the set $\bigcup_{i=1}^{k_1} V_i^{(2)}$ can include points from $V^{(1)}$. The points from the set $(\bigcup_{i=1}^{k_1} V_i^{(2)}) \cap V^{(1)}$ can not have offsprings. Therefore they are excluded from the next branching generation.

$(n + 1)th$ stage. Assume we have constructed

$$
B^{(n)}, V^{(n)} = \{\gamma_1^{(n)}, \ldots, \gamma_{k_n}^{(n)}\}, S_n \text{ and } B_n, \mu_n.
$$
Then the next generation $B^{(n+1)} = \bigcup_{i=1}^{k_n} B_i^{(n+1)}$ is constructed in the same way as in the second stage with objects $V_i^{(n+1)}, S_{n+1,i}, B_{n+1,i}, \mu_{n+1,i}$ and $\sigma_{\gamma_{n+1}}$. The offspring probabilities are defined in the same way

$$P(B_{m+1}^{(n+1)} \mid B_{n+1,m}) := P(\Omega(\sigma_{\gamma_{m+1}}) \mid \Omega(\mu_{n+1,m})). \quad (4.16)$$

It completes the construction of the cluster branching process.

We show next that the cluster branching processes posses the main feature of the usual branching processes, namely, if the expectation of the offspring number of one ancestor is less than 1 then the processes extinct.

**Lemma 7.** Assume that there exists $\varepsilon > 0$ such that for $n > 1$ either

$$\mathbb{E}(\vert B_i^{(n)} \mid \vert B_{n,i-1}) \leq 1 - \varepsilon \text{ when } i > 1$$

or

$$\mathbb{E}(\vert B_i^{(n)} \mid \vert S_{n-1}) \leq 1 - \varepsilon \text{ when } i = 1$$

then

$$\mathbb{E}(\vert B \mid) < \infty,$$

where $B = \bigcup_{n=1}^{\infty} B^{(n)}$.
Proof follows from the following equalities

\[
\mathbb{E}[|B^{(n)}|] = \mathbb{E}\left[\sum_{k=1}^{\infty} I_{\{|B^{(n-1)}| = k\}} |B^{(n)}|\right]
\]

\[
= \sum_{k=1}^{\infty} \mathbb{E}\left[I_{\{|B^{(n-1)}| = k\}} \sum_{i=1}^{k} |B^{(n)}_i|\right]
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}\left[I_{\{|B^{(n-1)}| = k\}} |B^{(n)}_i| \right]\mathbb{E}\left[I_{\{|B^{(n)}_i| = k\}} |B^{(n)}_i| | B_{n,i-1}\right]
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{E}\left[I_{\{|B^{(n-1)}| = k\}} \mathbb{E}\left[I_{\{|B^{(n)}_i| = k\}} |B^{(n)}_i| | B_{n,i-1}\right]\mathbb{E}\left[I_{\{|B^{(n)}_i| = k\}} |B^{(n)}_i|\right].
\]

We used the measurability of the event (\{|B^{(n-1)}| = k\}) with respect to the \(\sigma\)-algebra generated by \(B_{n,i-1}\). We adopt the above that \(B_{n,0} = B_{n-1}\).

Next we obtain

\[
\mathbb{E}[|B^{(n)}|] \leq (1 - \varepsilon) \sum_{k=1}^{\infty} k \mathbb{E}\left[I_{\{|B^{(n-1)}| = k\}}\right]
\]

\[
= (1 - \varepsilon) \mathbb{E}[|B^{(n-1)}|]
\]

Thus

\[
\mathbb{E}[|B^{(n)}|] \leq \mathbb{E}[|B^{(1)}|]/\varepsilon.
\]

The definition of the cluster branching process is done in a way such that any maximal component of any configuration \(\omega\) can be obtained as a cluster process path. It means the following.

Let \(\gamma_0 \in \Gamma\), and \(\omega_0\) be some configuration from \(\Omega\). Let \(C_{\gamma_0}(\omega)\) be the maximal connected component of open edges of \(\omega_0\) containing \(\gamma_0\). We construct a cluster branching process along \(\omega_0\) where \(\gamma_0\) is the initial point of the cluster path. The only freedom in the cluster process path deriving is in the
choice of the offsprings. Doing the coupling with chosen configuration \( \omega_0 \) we define \( \sigma_{\gamma_i^{(n)}} \) as a projection of \( \omega_0 \) on the set of edges \( E_{\gamma_i^{(n)}} \setminus S_{n,i-1} \). Then the probability to have a finite connected component \( C_{\gamma_0}(\omega_0) \) can be obtained as the probabilities (4.13) of the branching cluster process path made along \( \omega_0 \). As a consequence the following equality holds: for any point \( \gamma_0 \in \Gamma \)

\[
\mathbb{P}(C_{\gamma_0} \text{ is finite}) = \mathbb{P}(B^{(n)} \text{ not survives}) \quad (4.17)
\]

The following lemma finish the prove of the theorem.

**Lemma 8.** Let \( \Gamma \) be strongly homogeneous (see (3.2)). Then for any small \( \epsilon > 0 \) there exists \( \beta_0 = \beta_0(\epsilon) \) such that for all \( \beta > \beta_0 \)

\[
\mathbb{E}(|B^{(n)}_i| \mid B_{n,i-1}) < 1 - \epsilon. \quad (4.18)
\]

uniformly over \( i, n > 1 \). Here \( B_{n,0} = B_{n-1} \).

**Proof.** Let \( \gamma_i^{(n-1)} \) be the branching point of which offsprings are \( B^{(n)}_i \). Let the previous path be \( B_{n,i-1} \). It follows from (4.18) that:

\[
\mathbb{P}(|B^{(n)}_i| = m \mid B_{n,i-1}) \leq e^{-\beta \sum_{e \in B^{(n)}_i} L(e) - \beta h_0 m} \quad (4.19)
\]

Thus,

\[
\mathbb{E}(|B^{(n)}_i| \mid B_{n,i-1}) = \sum_{m=1}^{\infty} m \sum_{|B^{(n)}_i| = m} \mathbb{P}(|B^{(n)}_i| = m \mid B_{n,i-1})
\]

\[
\leq \sum_{m=1}^{\infty} m \exp\left\{-\beta h_1 \left( m + \frac{1}{2} \right) + \beta h_0 m \right\} \times \sum_{|B^{(n)}_i| = m} \exp\left\{-\beta \sum_{e \in B^{(n)}_i} L(e) \right\}
\]

\[
< \sum_{m=1}^{\infty} m \exp\left\{-\beta h_1 \left( m + \frac{1}{2} \right) + \beta h_0 m \right\} \frac{(T_{\gamma_0^{(n+1)}}(\beta))^m}{m!}, \quad (4.20)
\]

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where

\[ T_{\gamma}(\beta) = \sum_{e \in E_{\gamma}} e^{-\beta L_e}. \]

Since \( T_{\gamma}(\beta) \) are uniformly bounded over \( \gamma \in \Gamma \) the choice of large enough \( \beta \) leads to (4.18).

\[ \square \]

5 Conclusions

1. Since the ground state of Gibbs Random Graph do not percolate the theorems about the non-percolation show a kind of ”stability” of the ground states.

2. Condition of the existence of an infinite cluster is an open problem.

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References

[1] Bela Bollobas. Random graphs. Cambridge studies in advanced mathematics. Second Edition. Cambridge University Press, 2001.

[2] Rick Durrett. Random graph Dynamics. Cambridge University Press, October 2006.
[3] Ronald Meester and Rahul Roy. Continuum Percolation. *Series: Cambridge Tracts in Mathematics (No. 119).* Cambridge University Press, 1996.

[4] Hans-Otto Georgii, Olle Haggstrom, Christian Maes. The random geometry of equilibrium phases. [arXiv:math.PR/9905031](http://arxiv.org/abs/math.PR/9905031) v1 5 May 1999.

[5] A.N. Shiryaev. Probability. Springer 1996.