Efficiency of discrete-time ratchets

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Recently there has been much interest in discrete forms of Brownian ratchets, using a game-theoretic formalism. Using the approach pioneered by Parrondo, we develop a new method for obtaining the stationary probabilities and probability current for the case of discrete-time and discrete-space ratchets. We then use this result to calculate the Parrondian ratchet efficiency in two cases: firstly, for Parrondo’s original system and, secondly, for a set of probabilities derived from a discretized ratchet potential.

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Since the field of Brownian ratchets acquired its importance, there have been numerous studies on the energetics of these microscopic devices 1,2. Parrondo 3 devised a discrete-time and discrete-space version of the flashing ratchet model 4 within a game-theoretic framework, and recent work has established the connection between game probabilities and physical variables that describe the motion of a Brownian particle 5. For extensive reviews of this area that show the relationship of the Parrondo effect to classical random walks, quantum walks, fractal pattern formation and spin systems, for example, see 6. The significance of game-theoretic approaches in physical systems has recently been reviewed 7 and the Parrondo effect in transport processes has recently been reviewed 8. There is also emerging interest in the so-called continuous-discrete interface 9, and the Parrondo approach is also of relevance in that arena.

The original Parrondo’s paradox involves the alternation of two games. The first game, game A, has a winning probability given by \( p = \frac{1}{2} - \epsilon \). It is easy to check that for \( \epsilon = 0 \) game A is fair, whereas for small \( \epsilon \) game A turns into a losing game. Game B is a capital dependent game, \( \epsilon = 0 \) game B is a fair game, and a losing one for \( \epsilon > 0 \). As in the case of game A, for \( \epsilon = 0 \) game B is a fair game, and a losing one for \( \epsilon > 0 \). This set of probabilities can be summarized as

\[
p = \frac{1}{2} - \epsilon, \quad p_1 = \frac{1}{10} - \epsilon, \quad p_2 = \frac{3}{4} - \epsilon.
\]

Parrondo conjectured that this system is a discrete form of the flashing ratchet, and this has recently been rigorously demonstrated within a Fokker-Planck framework 10. However, finding the correct formalism for describing the efficiency of the discrete ratchet and relating it back to the continuous case, has been problematic 11—this has motivated a solution, which is now presented for the first time in this paper.

The flashing ratchet consists of a Brownian particle under the influence of a potential that can be switched on and off either stochastically or periodically. Its dynamics can be described through the following Langevin equation, \( \dot{x} = -V'(x(t)) \cdot \zeta(t) + F_{\text{ext}} + D \cdot \xi(t) \), where (i) \( \zeta(t) \) accounts for white noise, (ii) \( \zeta(t) \) is a form of dichotomous noise that switches on (state B) and off (state A) the potential \( V(x) \), and (iii) \( F_{\text{ext}} \) is a force acting on the particle, that can be exerted by an external agent. It is required that the potential has a certain degree of spatial asymmetry in order to obtain directed motion from these fluctuations. Usually a ratchet-like potential is used,

\[
V(x) = V_0 \left( \sin \left( \frac{2\pi x}{L} \right) + \frac{1}{4} \sin \left( \frac{4\pi x}{L} \right) \right),
\]

where \( V_0 \) denotes the amplitude of the potential and \( L \) its spatial periodicity. Although other similar potentials perform the same task, Eq. (2) is convenient for analytical purposes.

The corresponding set of Fokker-Planck equations related to the Langevin equation that describes the transitions of the particle between states A and B are 12,

\[
\frac{\partial P_A(x,t)}{\partial x} = - \frac{\partial (J_A(x,t) P_A(x,t))}{\partial x} - \omega_{A\rightarrow B} P_A(x,t) + \omega_{B\rightarrow A} P_B(x,t)
\]

\[
\frac{\partial P_B(x,t)}{\partial x} = - \frac{\partial (J_B(x,t) P_B(x,t))}{\partial x} - \omega_{B\rightarrow A} P_B(x,t) + \omega_{A\rightarrow B} P_A(x,t),
\]

where \( J_A(x,t) \) (resp. \( J_B(x,t) \)) denotes the probability of finding the particle in state A (resp. B) at a given position \( x \) and time \( t \). The term \( \omega_{\alpha\rightarrow \beta} \) accounts for the transition rate between state \( \alpha \) and \( \beta \). The probability currents \( J_A \) and \( J_B \) are given by

\[
J_A(x,t) = F_{\text{ext}} P_A(x,t) - \frac{\partial (D(x) P_A(x,t))}{\partial x}
\]

\[
J_B(x,t) = \left( F_{\text{ext}} - \frac{\partial V(x)}{\partial x} \right) P_B(x,t) - \frac{\partial (D(x) P_B(x,t))}{\partial x}.
\]

In the stationary regime we have \( P_A(x,t) = P_A(x) \) and \( P_B(x,t) = P_B(x) \). The current in this regime is
constant \( \mathcal{J} \) and given by \( \mathcal{J} = \mathcal{J}_A(x) + \mathcal{J}_B(x) \). With this in mind, and using the same notation as in \[ \mathcal{J} \], we propose a model that describes the dynamics of the capital of the player when alternating between games A and B. The set of Master Equations (ME) describing this process are

\[
P_i^A(\tau + 1) = (1 - \omega_{AB})[a_{i-1}^A P_{i-1}^A(\tau) + a_0^A P_i^A(\tau)] + \omega_{BA}[b_{i-1}^B P_{i-1}^B(\tau) + b_0^B P_i^B(\tau)] + b_i^A P_i^A(\tau)
\]

(3)

\[
P_i^B(\tau + 1) = (1 - \omega_{BA})[b_{i-1}^B P_{i-1}^B(\tau) + b_0^B P_i^B(\tau)] + \omega_{BA}[a_{i-1}^A P_{i-1}^A(\tau) + a_0^A P_i^A(\tau)] + a_i^B P_i^B(\tau)
\]

(4)

The terms \( P_i^A(\tau) \), \( P_i^B(\tau) \) account for the probability that the player plays game A or B with a capital \( i \) at time \( \tau \) respectively. Also the term \( a_{i-1}^A \) (resp. \( b_{i-1}^A \)) denotes the probability of winning if the player plays game A (resp. B) with a capital \( i-1 \); \( a_0^A \) (resp. \( b_0^A \)) is the so-called self-transition probability, that is, the probability that the player will remain with the same capital after a round played for a given capital \( i \); and finally \( a_i^B \) (resp. \( b_i^B \)) is the probability of losing when the player has a capital \( i+1 \) and is playing game A (resp. B). The transitions between game states are shown in Fig. 1.

In order to preserve normalization, Eqs. (3) & (4) must accomplish the condition \( \sum_i [P_i^A(\tau) + P_i^B(\tau)] = 1 \) for the stationary probabilities, and \( a_{i-1}^A + a_0^A + a_i^A = 1 \) together with \( b_{i-1}^B + b_0^B + b_i^B = 1 \) for the transition probabilities in games A and B respectively. Making use of the normalization condition for the transition probabilities, Eqs. (3) & (4) can be rewritten in the form of continuity equations for \( P_i^A(\tau) \) and \( P_i^B(\tau) \) as

\[
P_i^A(\tau + 1) - P_i^A(\tau) = (1 - \omega_{AB})[a_{i-1}^A P_{i-1}^A(\tau) - (a_1^A - a_i^A) P_i^A(\tau)] + \omega_{BA}[b_{i-1}^B P_{i-1}^B(\tau) - (b_1^B - b_i^B) P_i^B(\tau)]
\]

where the lhs of both expressions must equal zero, in the stationary regime. These equations, combined with the normalization condition for the stationary probabilities, form a set of equations that must be solved for the variables \( P_i^A \), \( P_i^B \) for each \( i \). For the case of Parrondo’s games it is necessary only to consider the 0, 1 and 2 states due to the presence of a modulo three periodicity. Therefore, in matrix form we must solve the \( \mathbf{C} \cdot \mathbf{P} = \mathbf{U} \), where \( \mathbf{P} = \{ P_0^A, P_1^A, P_2^A, P_0^B, P_1^B, P_2^B \} \), \( \mathbf{U} = \{ 1, 1, 1, 1, 1, 1 \} \), and

\[
\mathbf{C} = \begin{pmatrix}
1 - A_0 - \omega_{AB} & 1 + (1 - \omega_{AB})q_1^A & 1 + (1 - \omega_{AB})q_2^A & 1 + \omega_{BA} q_0^A & 1 + \omega_{BA} q_1^A & 1 + \omega_{BA} q_2^A \\
1 + (1 - \omega_{AB})p_0^A & 1 - A_1 - \omega_{AB} & 1 + (1 - \omega_{AB})q_2^A & 1 + \omega_{BA} p_0^A & 1 + \omega_{BA} p_1^A & 1 + \omega_{BA} p_2^A \\
1 + (1 - \omega_{AB})q_0^A & 1 + (1 - \omega_{AB})p_1^A & 1 - A_2 - \omega_{AB} & 1 + \omega_{BA} q_0^A & 1 + \omega_{BA} q_1^A & 1 + \omega_{BA} q_2^A \\
1 + \omega_{AB} r_0^A & 1 + \omega_{AB} r_1^A & 1 + \omega_{AB} r_2^A & 1 + \omega_{AB} p_0^A & 1 + \omega_{AB} p_1^A & 1 + \omega_{AB} p_2^A \\
1 + \omega_{AB} q_0^A & 1 + \omega_{AB} q_1^A & 1 + \omega_{AB} q_2^A & 1 + \omega_{AB} q_0^A & 1 + \omega_{AB} q_1^A & 1 + \omega_{AB} q_2^A \\
1 + \omega_{AB} q_0^A & 1 + \omega_{AB} q_1^A & 1 + \omega_{AB} q_2^A & 1 + \omega_{AB} q_0^A & 1 + \omega_{AB} q_1^A & 1 + \omega_{AB} q_2^A
\end{pmatrix},
\]

and where \( A_i = (1 - \omega_{AB})(p_i^A + q_i^A) \) and \( B_i = (1 - \omega_{BA})(p_i^B + q_i^B) \). The solution can be obtained through \( \mathbf{P} = \mathbf{C}^{-1} \cdot \mathbf{U} \). The analytical solutions for \( P_i^A \) and \( P_i^B \) are too lengthy to be presented here—however, we will show some results concerning the original Parrondo’s games A and B. As we already stated before, these games are alternated using a mixing probability \( \gamma \). This means that independently of the previously played game, we have always a probability \( \gamma \) of playing game A and a probability \( 1 - \gamma \) of playing game B, in the next round. It can be easily checked that the latter condition is equivalent to \( \omega_{AB} = 1 - \gamma \) and \( \omega_{BA} = \gamma \).

Making use of the previous expressions for \( \omega_{AB}, \omega_{BA} \) together with the set of probabilities already defined in Eq. (1), and by letting \( \eta = -169 + 11 \gamma^2 + 16 \epsilon - 240 \epsilon^2 - 2 \gamma (11 + 8 \epsilon) \), we obtain the following expressions for the stationary probabilities for game A and B

\[
P_0^A = \frac{3 \eta}{\eta}, \quad P_1^A = \frac{2 \eta}{\eta}, \quad P_2^A = \frac{2 \eta}{\eta}, \quad P_0^B = \frac{2 \eta}{\eta}, \quad P_1^B = \frac{2 \eta}{\eta}, \quad P_2^B = \frac{2 \eta}{\eta},
\]

where \( \eta_2 = (13 + 3 \gamma^2 - 8 \epsilon + 16 \epsilon^2 + 2 \gamma (1 + 4 \epsilon)) \), \( \eta_3 = (-13 + 4 \gamma^2 - 6 \epsilon + 40 \epsilon^2 + 3 \gamma (7 + 2 \epsilon)) \) and \( \eta_4 = (-39 + 4 \gamma^2 - 6 \epsilon - 40 \epsilon^2 + 5 \gamma + 6 \epsilon) \).

They are expressed in terms of the mixing probability \( \gamma \) and the biasing term \( \epsilon \). In Fig. 2, we plot both stationary probabilities \( P_i^A, P_i^B \) versus the mixing probability \( \gamma \).
FIG. 2: Plot of the evolution of the stationary probabilities for game A and game B versus the mixing variable \( \gamma \). The inset shows that the sum of both probabilities, \( P^A_i + P^B_i \), agrees with the expressions obtained for the stationary probabilities \( P_0, P_1, P_2 \) obtained for the mixed game AB.

The previous result allows us to evaluate the efficiency in two different cases. On one hand, given two probability sets defining games A and B, we will obtain their related potentials using Eq. (7) and then the efficiency can be derived. On the other hand, we can calculate the efficiency given two potentials, from which we can obtain their respective probability sets inverting Eq. (7). For the latter case, we use a flat potential for state A, that is, \( V_A = 0 \), and state B, \( V_B \), will be obtained from Eq. (2) with \( L = 3 \).

In [11] a new expression for the efficiency is presented. It is based on a new definition of the energy output \( \mathcal{E}_{\text{out}} \): ‘...We define the energy output \( \mathcal{E}_{\text{out}}(i) \) as the minimum energy input \( \mathcal{E}_{\text{in}} \) required to accomplish the same task as the engine.’ The novelty of this definition is that permits the evaluation of the efficiency for a Brownian particle even in the absence of an external load \( F \).

So we can evaluate the energy output (or \( \mathcal{P}_{\text{out}} \)) using the expression, \( \mathcal{P}_{\text{out}} = \mathcal{P}_{\text{in}}^{\text{min}} = F_L \cdot v + \gamma v^2 \). Recalling that in our system, \( v = J \cdot L \) and \( \gamma = 1 \), we obtain \( \mathcal{P}_{\text{out}} = F_L J_L + J^2 L^2 \). This is the equation to be used for determining the energy output of our system.

The energy input of the system is the energy that we must supply to the system when switching between the two potentials. For evaluating this energy input in our system we need a potential related to each of the two games. Therefore, if we are dealing with probabilities defining our games A and B, we will make use of Eq. (7) for obtaining the potential for each game. The energy input can be calculated theoretically by means of a probability flux balance. In the stationary regime, the net flux from a given game, say game A, and state \( i \), towards the other game B and the same state \( i \) can be calculated through the difference equation \( J_{\text{net}}(i) = J^{A}_{i-1} - J^{A}_{i+1} = (p_i - p_{i+1} - q_{i+1} \cdot p_{i+1} - q_{i} \cdot p_{i}) - (p_i - p_{i+1} - q_{i+1} \cdot p_{i+1} - q_{i} \cdot p_{i+1}) \). Clearly the net current \( J_{\text{net}}^{AB}(i) \) equals the opposite current from game B to game A, that is, \( |J_{\text{net}}^{AB}(i)| = |J_{\text{net}}^{BA}(i)| \), where \( J_{\text{net}}^{AB}(i) = J^B_{i} - J^B_{i+1} \). The input power can now be obtained and is given by \( \mathcal{P}_{\text{input}} = \sum_{i=1}^{L-1} J_{\text{net}}^{AB}(i) \cdot (V_B(i) - V_A(i)) \). For the simplest case when \( F_{\text{ext}} = 0 \), for the original Parrondo games, we obtain \( \eta = 1.1078 \cdot 10^{-2} \) for \( \gamma = 0.36 \). For the alternation with the potentials \( V_A \) and \( V_B \), we obtain a maximum value for the efficiency of \( \eta = 1.061 \cdot 10^{-3} \) when \( \gamma = 0.35 \). The system possesses a low efficiency mainly because it works in an irreversible manner, far from its equilibrium state. The magnitude obtained for the efficiency agrees with other studies for the on-off ratchets [11]. Now that we have obtained an efficiency metric, it now provides a basis for comparing different Parrondo or discrete-time ratchets, and provides a basis in the search for higher efficiency systems.

Now we evaluate the efficiency, when \( F_{\text{ext}} \neq 0 \), which is difficult when considering alternation between games A and B. Clearly, the applied external force \( F_{\text{ext}} \) must bias the potential with a slope equal to \(-F_{\text{ext}}\). Whereas
it can be demonstrated that the biasing parameter, $\epsilon$, introduced earlier in Eq. (1), is exactly equivalent to $F_{\text{ext}}$ only for the case of game A—due to the non-constant values of the probabilities in case of game B, $\epsilon$ is no longer equivalent to the force.

Because of this, we proceed as follows: given a set of probabilities $\{p_i, r_i, q_i\}$ defining our game B, through Eq. (7) we calculate its discretized potential $V_i$. Then, for $F_{\text{ext}} \neq 0$ we add this extra bias through $V_i^{\text{new}} = V_i - F_{\text{ext}} \cdot i$, where $i$ denotes the state. Finally the new set of probabilities will be obtained inverting Eq. (7), using the new values for the potential $\{p_i^n, r_i^n, q_i^n\}$. If we want to calculate the efficiency when alternating between the potentials $V_A$ and $V_B$, we need only to add the slope to these potentials and then invert Eq. (7) for obtaining the probability sets. In Fig. 3a, the efficiency for the original Parrondo’s games is evaluated for different values of the mixing probability $\gamma$. In conclusion, for the first time we provide a method that permits an analytical solution for the stationary probabilities $P_i^A$, $P_i^B$, and the average gain $G = 3J$ of a player that alternates between two games A and B with different transition probabilities $\omega_{AB}$, $\omega_{BA}$. Using this approach, we then evaluated the efficiency in the case of the original Parrondo games, as well as for a given set of potentials $V_A$, $V_B$, both of them sharing the essential features with the continuous model of a flashing Brownian ratchet.

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