The $A_m^{(1)}$ Q-system

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Abstract

We propose a Q-system for the $A_m^{(1)}$ quantum integrable spin chain. We also find compact determinant expressions for all the Q-functions, both for the rational and trigonometric cases.
1 Introduction

Q-systems provide an efficient way of solving Bethe equations corresponding to quantum integrable models. Such Q-systems were first introduced for rational Bethe equations (corresponding to isotropic integrable models) in [1] (see also [2]), and have since been exploited in e.g. [3–9]. Generalizations to rank-1 trigonometric Bethe equations (corresponding to anisotropic integrable models) were recently formulated in [11,12]. One of the principal aims of this work is to formulate trigonometric Q-systems for higher rank. We also solve the rank-m Q-systems (for both the rational and trigonometric cases) in terms of determinants of a set of m + 1 functions, which are generalizations of functions introduced by Pronko and Stroganov to describe integrable spin chains with $SU(2)$ [13] and $SU(3)$ [14] symmetry. (Throughout this paper, $m = 1, 2, \ldots$)

We first consider the $SU(m+1)$-invariant integrable spin chain in Section 2. After briefly reviewing its Bethe ansatz solution and Q-system [1], we show that all the Q-functions can be expressed in terms of compact determinant expressions of $m + 1$ functions $F_0, \ldots, F_m$ \((2.18)-(2.21)\). The proof relies on a Plücker identity [15]. We then consider the $A^{(1)}_m$ integrable spin chain, which is a q-deformation of the $SU(m+1)$-invariant model, in Section 3. We propose its Q-system \((3.7)-(3.8)\), and again obtain determinant expressions for all the Q-functions \((3.9)\). We close with a brief conclusion and a list of some interesting remaining open problems in Section 4.

2 The $SU(m+1)$-invariant spin chain

We begin by briefly reviewing the closed $SU(m+1)$-invariant integrable quantum spin chain of length $N$ with periodic boundary conditions and with “spins” in the vector ($(m + 1)$-dimensional) representation of $SU(m+1)$ at each site, see e.g. [16, 17]. The R-matrix (solution of the Yang-Baxter equation) is given by the $(m + 1)^2 \times (m + 1)^2$ matrix

$$R(u) = uI + iP,$$  \hspace{1cm} (2.1)

where $I$ is the identity matrix, and $P$ is the permutation matrix, which is given in terms of the elementary $(m + 1) \times (m + 1)$ matrices $e_{ab}$ by

$$P = \sum_{a,b=1}^{m+1} e_{ab} \otimes e_{ba}, \quad (e_{ab})_{ij} = \delta_{a,i} \delta_{b,j}.$$  \hspace{1cm} (2.2)

The transfer matrix $T(u)$, which is defined by

$$T(u) = \text{tr}_0 R_{01}(u) R_{02}(u) \ldots R_{0N}(u),$$  \hspace{1cm} (2.3)

satisfies the commutativity property

$$[T(u), T(v)] = 0.$$  \hspace{1cm} (2.4)

\footnote{T-systems without spectral parameter are also called Q-systems [10], and should not be confused with the subject of this paper.}
The corresponding spin-chain Hamiltonian is proportional to \( \frac{d}{du} \left( \log T(u) \right) \bigg|_{u=0} \), up to an additive constant.

The eigenvalues \( T(u) \) of the transfer matrix can be expressed in terms of Bethe roots \( \{u_{j,k}\} \) where \( k = 1, \ldots, M_j \) and \( j = 1, \ldots, m \), which satisfy the following set of Bethe equations

\[
\left( \frac{u_{1,k} + \frac{1}{2}}{u_{1,k} - \frac{1}{2}} \right)^N = \prod_{l=1; l \neq k}^{M_j} \frac{u_{1,k} - u_{1,l} + i}{u_{1,k} - u_{1,l} - i} \prod_{l=1}^{M_j} \frac{u_{1,k} - u_{2,l} - \frac{i}{2}}{u_{1,k} - u_{2,l} + \frac{i}{2}}, \quad k = 1, \ldots, M_1, \tag{2.5}
\]

\[
1 = \prod_{l=1; l \neq k}^{M_j} \frac{u_{j,k} - u_{j,l} + i}{u_{j,k} - u_{j,l} - i} \prod_{l=1}^{M_j} \frac{u_{j,k} - u_{j+l+1,k} - \frac{i}{2}}{u_{j,k} - u_{j+l+1,k} + \frac{i}{2}}, \quad k = 1, \ldots, M_j, \quad j = 2, \ldots, m - 1, \tag{2.6}
\]

\[
1 = \prod_{l=1; l \neq k}^{M_m} \frac{u_{m,k} - u_{m,l} + i}{u_{m,k} - u_{m,l} - i} \prod_{l=1}^{M_m} \frac{u_{m,k} - u_{m-1,l} - \frac{i}{2}}{u_{m,k} - u_{m-1,l} + \frac{i}{2}}, \quad k = 1, \ldots, M_m. \tag{2.7}
\]

We define polynomials \( Q_1(u), \ldots, Q_m(u) \) by

\[
Q_j(u) = \prod_{k=1}^{M_j} (u - u_{j,k}), \quad j = 1, \ldots, m, \tag{2.8}
\]

so that their zeros are given by corresponding Bethe roots.

### 2.1 The \( SU(m + 1) \) Q-system

The QQ-relations for the \( SU(m + 1) \) Q-system are given by

\[
Q_{j,n}(u) Q_{j+1,n-1}(u) \propto Q_{j+1,n}(u) Q_{j,n-1}(u) - Q_{j+1,n}(u) Q_{j,n-1}(u),
\]

\[
\quad j = 0, 1, \ldots, m, \quad n = 1, 2, \ldots, \tag{2.9}
\]

where \( f^\pm(u) = f(u \pm \frac{i}{2}) \), and

\[
Q_{0,0}(u) = u^N,
\]

\[
Q_{j,0}(u) = Q_j(u), \quad j = 1, \ldots, m,
\]

\[
Q_{m+1,0}(u) = 1, \tag{2.10}
\]

where \( Q_j(u) \) are defined in \([2.8]\). The nontrivial Q-functions are in fact defined on a Young diagram, on whose boundary the Q-functions (including \( Q_{m+1,0} \)) are set to 1 \([1]\).

Let us verify that this Q-system indeed leads to the Bethe equations. Setting \( n = 1 \) in \([2.9]\) and using \([2.10]\), we obtain

\[
Q_{j,1}(u) Q_{j+1}(u) \propto Q_{j+1,1}(u) Q_j(u) - Q_{j+1,1}(u) Q_j(u), \quad j = 1, \ldots, m - 1. \tag{2.11}
\]
Shifting $j \mapsto j - 1$ in (2.11) and then setting $u = u_{j,k}$, we obtain
\[ 0 = Q^+_{j,1}(u_{j,k}) Q^-_{j-1}(u_{j,k}) - Q^+_j(u_{j,k}) Q^-_{j-1}(u_{j,k}), \quad (2.12) \]

since $Q_j(u_{j,k}) = 0$. Shifting $u \mapsto u \pm \frac{1}{2}$ in (2.11) and then setting $u = u_{j,k}$, we obtain the pair of equations
\[
\begin{align*}
Q^+_{j,1}(u_{j,k}) Q^+_{j+1}(u_{j,k}) &\propto -Q_{j+1,1}(u_{j,k}) Q^+_{j+1}(u_{j,k}), \\
Q^-_{j,1}(u_{j,k}) Q^-_{j+1}(u_{j,k}) &\propto Q_{j+1,1}(u_{j,k}) Q^-_{j+1}(u_{j,k}).
\end{align*}
\quad (2.13)
\]

Using (2.13) to eliminate $Q^+_{j,1}(u_{j,k})$ in (2.12), we arrive at the relations
\[
\frac{Q^+_{j+1}(u_{j,k})}{Q^-_{j+1}(u_{j,k})} = -\frac{Q^-_{j}(u_{j,k})}{Q^+_{j}(u_{j,k})} Q^-_{j-1}(u_{j,k}), \quad j = 2, \ldots, m - 1,
\quad (2.14)
\]

which is equivalent to the Bethe equations (2.6).

To get the first Bethe equation (2.5), we start from (2.9) with $n = 1$ and $j = 0$
\[ Q_{0,1}(u) Q_1(u) \propto Q^+_{1,1}(u) Q^-_{0,0}(u) - Q^-_{1,1}(u) Q^+_{0,0}(u). \quad (2.15) \]

Evaluating this relation at $u = u_{1,k}$ gives
\[
0 = Q^+_{1,1}(u_{1,k}) Q^-_{0,0}(u_{1,k}) - Q^-_{1,1}(u_{1,k}) Q^+_{0,0}(u_{1,k}).
\quad (2.16)
\]

Using (2.13) with $j = 1$ to eliminate $Q^+_{1,1}(u_{1,k})$ in (2.16), we arrive at (2.14) with $j = 1$, except with $Q_0$ replaced by $Q_{0,0}$, which is equivalent to the Bethe equation (2.5).

To get the final Bethe equation (2.7), we start from (2.9) with $n = 1$ and $j = m$
\[ Q_{m,1}(u) \propto Q^+_{m+1,1}(u) Q^-_{m}(u) - Q^-_{m+1,1}(u) Q^+_{m}(u). \quad (2.17) \]

Shifting $u \mapsto u \pm \frac{1}{2}$ in (2.17) and then setting $u = u_{m,k}$, we obtain (2.13) with $j = m$, except with $Q_{m+1}$ replaced by 1. Using these relations to eliminate $Q^+_{m+1,1}(u_{m,k})$ in (2.12) with $j = m$, we arrive at (2.14) with $j = m$, except with $Q_{m+1}$ replaced by 1, which is indeed equivalent to the Bethe equation (2.7).

### 2.2 Determinant representation for all the Q-functions

We now show that the Q-system (2.9)-(2.10) can be solved in terms of a set of $m+1$ functions $F_0(u), \ldots, F_m(u)$, whose interpretation will be discussed later. Explicitly, all the Q-functions can be expressed in terms of determinants as follows:
\[ Q_{m,n} = F_0^{(n)}, \quad (2.18) \]

\footnotetext[2]{Eq. (2.17) holds if $M_m > 1$; however, $Q_{m,1} = 1$ if $M_m = 1$, which is consistent with (2.18).}

\footnotetext[3]{Somewhat similar formulas have been known in the context of supersymmetric spin chains [18, 19].}
Hence, we make use of Plücker identities [15], which we first briefly review. Let $g$ be a regular matrix with $n_f$ discrete derivative of any function $u$. In order to show that the expressions (2.18)-(2.21) indeed satisfy the QQ-relations (2.9), an important consequence of the result (2.18)-(2.21) is that the functions $r(0) = 0$, $(u_i) = f_1$ are polynomials in $u$.

Throughout Section 2 we use the notation $f^{(n)}(u)$ to denote the $n^{th}$ discrete derivative of any function $f(u)$, i.e.

$$f^{(n)}(u) = f^{(n-1)+}(u) - f^{(n-1)-}(u) = f^{(n-1)}(u + \frac{i}{2}) - f^{(n-1)}(u - \frac{i}{2}), \quad n = 1, 2, \ldots,$$

with $f^{(0)}(u) = f(u)$. Moreover, we use the notation $f^{[k]}(u)$ to denote a $k$-fold shift by $\frac{i}{2}$, i.e.

$$f^{[k]}(u) = f(u + k\frac{i}{2}).$$

Hence, $f^{[1]} = f^+$ and $f^{[-1]} = f^-$, etc. Note that (2.10) and (2.18) imply that

$$Q_0 = Q_{m,0} = F_0.$$  (2.24)

An important consequence of the result (2.18)-(2.21) is that the functions $F_0(u), \ldots, F_m(u)$ are polynomials in $u$ if and only if all the Q-functions are polynomials in $u$.

In order to show that the expressions (2.18)-(2.21) indeed satisfy the QQ-relations (2.9), we make use of Plücker identities [15], which we first briefly review. Let $X$ denote a rectangular matrix with $r$ rows and $c$ columns, with $c > r$,

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,c} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,c} \\ \vdots & \vdots & \ddots & \vdots \\ X_{r,1} & X_{r,2} & \cdots & X_{r,c} \end{pmatrix}. \quad (2.25)$$
Furthermore, let the symbol \((i_1, i_2, \ldots, i_r)\) denote the determinant of the square matrix formed by the \(r\) columns \(i_1, i_2, \ldots, i_r\) of \(X\)

\[
(i_1, i_2, \ldots, i_r) = \begin{vmatrix}
X_{1,i_1} & X_{1,i_2} & \cdots & X_{1,i_r} \\
X_{2,i_1} & X_{2,i_2} & \cdots & X_{2,i_r} \\
\vdots & \vdots & \ddots & \vdots \\
X_{r,i_1} & X_{r,i_2} & \cdots & X_{r,i_r}
\end{vmatrix},
\]

(2.26)

which is antisymmetric in all indices. The particular Plücker identity that we need is

\[
(i_1, i_2, k_3, \ldots, k_r) (j_1, j_2, k_3, \ldots, k_r) = (j_1, i_2, k_3, \ldots, k_r) (i_1, j_2, k_3, \ldots, k_r) + (j_2, i_2, k_3, \ldots, k_r) (j_1, i_1, k_3, \ldots, k_r),
\]

(2.27)

where all indices take values in \(\{1, 2, \ldots, c\}\).

For the problem at hand, we choose \(X\) to be a rectangular matrix with \(r = m + 1 - j\) and \(c = m + 3 - j\) (where \(j = 0, 1, \ldots, m\)) given by

\[
X = \begin{pmatrix}
1 & 2 & \cdots & m + 1 - j & m + 2 - j & m + 3 - j \\
F_0^{(n)[m-j]} & F_0^{(n)[m-j-2]} & \cdots & F_0^{(n)[j-m]} & F_0^{(n-1)[m-j-1]} & 0 \\
F_1^{(n)[m-j]} & F_1^{(n)[m-j-2]} & \cdots & F_1^{(n)[j-m]} & F_1^{(n-1)[m-j-1]} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
F_{m-j-1}^{(n)[m-j]} & F_{m-j}^{(n)[m-j-2]} & \cdots & F_{m-j}^{(n)[j-m]} & F_{m-j-1}^{(n-1)[m-j-1]} & 0 \\
F_m^{(n)[m-j]} & F_{m-j-2}^{(n)[m-j-2]} & \cdots & F_m^{(n)[j-m]} & F_m^{(n-1)[m-j-1]} & 1
\end{pmatrix}.
\]

(2.28)

Moreover, we choose the indices in (2.27) as follows

\[
i_1 = 1, \quad i_2 = m + 1 - j, \quad j_1 = m + 2 - j, \quad j_2 = m + 3 - j,
\]

(2.29)

and

\[
k_l = l - 1 \quad \text{for} \quad l = 3, 4, \ldots, m + 1 - j.
\]

(2.30)

With the choices (2.29)-(2.30), the Plücker identity (2.27) reads

\[
(1, 2, \ldots, m - j, m + 1 - j) (2, \ldots, m - j, m + 2 - j, m + 3 - j) \\
\times (2, \ldots, m - j, m + 1 - j, m + 2 - j) (1, 2, \ldots, m - j, m + 3 - j) \\
- (2, \ldots, m - j, m + 1 - j, m + 3 - j) (1, 2, \ldots, m - j, m + 2 - j),
\]

(2.31)

where we have made use of the antisymmetry properties of the symbols. Using the following identifications

\[
(1, 2, \ldots, m - j, m + 1 - j) = Q_{j,n},
\]

(2.32)

\[
(2, \ldots, m - j, m + 2 - j, m + 3 - j) = Q_{j+1,n-1},
\]

(2.33)
\[ (2, \ldots, m - j, m + 1 - j, m + 2 - j) = Q_{j,n-1}, \quad (2.34) \]
\[ (1, 2, \ldots, m - j, m + 3 - j) = Q_{j+1,n}^+, \quad (2.35) \]
\[ (2, \ldots, m - j, m + 1 - j, m + 3 - j) = Q_{j+1,n}^-, \quad (2.36) \]
\[ (1, 2, \ldots, m - j, m + 2 - j) = Q_{j,n-1}^+, \quad (2.37) \]

we immediately obtain from the identity (2.31) the QQ-relations (2.9). The first (2.32), fourth (2.35) and fifth (2.36) relations follow directly from (2.20) and the expression (2.28) for \( X \); while the second (2.33), third (2.34) and sixth (2.37) relations, which involve Q-functions with \( n - 1 \) instead of \( n \), require also (2.22).

The determinant expressions (2.18)-(2.21) for the Q-functions and their proof constitute some of the main results of this paper. We remark that the result (2.20) with \( n = 0 \) is similar to Eq. (9.21) in [10].

In order to understand how to interpret the functions \( F_0(u), \ldots, F_m(u) \), it is helpful to begin by analyzing the simplest cases \( m = 1, 2 \).

**m = 1** For the \( SU(2) \) case (\( m = 1 \)), the results (2.18)-(2.21) reduce to

\[ Q_{1,n} = F_0^{(n)} , \quad Q_{0,n} = \left| \begin{array}{c} F_0^{(n)+} \\ F_1^{(n)+} \end{array} \right| = F_0^{(n)+} F_1^{(n)-} - F_0^{(n)-} F_1^{(n)+} . \quad (2.38) \]

In view of (2.24), we can identify \( F_0(u) \) as the “fundamental” Q-function \( F_0(u) = Q_1(u) \equiv Q(u) \). Moreover, we can identify \( F_1(u) \) as the “dual” Q-function, which is denoted by \( P(u) \) in [13] and [11], see also [12]. Indeed, with these identifications, (2.38) coincides with Eq. (2.23) in [11]. Note that (2.38) with \( n = 0 \) implies that

\[ Q_{0,0} = F_0^+ F_1^- - F_0^- F_1^+ , \quad (2.39) \]

which we recognize as the important discrete Wronskian relation in [13].

It was proved in [20, 21] that polynomiality of \( Q \) and \( P \) (i.e., \( F_0 \) and \( F_1 \)) is equivalent to the admissibility of the Bethe roots. It follows from (2.38) that polynomiality of the Q-system is equivalent to the admissibility of the Bethe roots, as already noted in [11].

**m = 2** For the \( SU(3) \) case (\( m = 2 \)), the results (2.18)-(2.21) reduce to

\[ Q_{2,n} = F_0^{(n)} , \quad (2.40) \]
\[ Q_{1,n} = \left| \begin{array}{c} F_0^{(n)+} \\ F_1^{(n)+} \end{array} \right| = F_0^{(n)+} F_1^{(n)-} - F_0^{(n)-} F_1^{(n)+} . \quad (2.41) \]

We remind the reader that some solutions of the Bethe equations, most notably the so-called unphysical singular solutions, do not lead to eigenvalues and eigenvectors of the transfer matrix, see e.g. [11] and references therein. Here we call admissible those solutions of the Bethe equations that do give rise to genuine eigenvalues and eigenvectors of the transfer matrix.

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4We remind the reader that some solutions of the Bethe equations, most notably the so-called unphysical singular solutions, do not lead to eigenvalues and eigenvectors of the transfer matrix, see e.g. [11] and references therein. Here we call admissible those solutions of the Bethe equations that do give rise to genuine eigenvalues and eigenvectors of the transfer matrix.
The functions $F_0, F_1, F_2$ correspond to the functions $P, Q, R$ in [14]. In particular, (2.42) with $n = 0$ can be recognized as Eq. (4) in [14].

General $m$ We now see that the functions $F_0(u), \ldots, F_m(u)$ are generalizations of the functions introduced by Pronko and Stroganov to describe integrable spin chains with $SU(2)$ [13] and $SU(3)$ [14] symmetry. These functions satisfy a generalized discrete Wronskian relation given by (2.21) with $n = 0$. We conjecture, generalizing the $m = 1$ result of [20,21], that polynomiality of $F_0(u), \ldots, F_m(u)$ (and hence, by (2.18)-(2.21), polynomiality of all the Q-functions) is equivalent to the admissibility of the Bethe roots $\{u_{j,k}\}$.

3 The $A_m^{(1)}$ spin chain

We turn now to the closed $A_m^{(1)}$ spin chain with periodic boundary conditions, which is a $q$-deformation of the $SU(m+1)$-invariant model considered in Section 2. The R-matrix is now given by (see e.g. [22], with $x = e^{2u}$ and $k = e^{-\eta}$)

$$
\mathbb{R}(u) = \frac{1}{2} e^{-\eta u} \left\{ (e^{2u} - e^{-2\eta}) \sum_{a=1}^{m+1} e_{aa} \otimes e_{aa} + e^{-\eta} (e^{2u} - 1) \sum_{a \neq b} e_{aa} \otimes e_{bb} \\
- (e^{-2\eta} - 1) \left( \sum_{a<b} + e^{2u} \sum_{a>b} \right) e_{ab} \otimes e_{ba} \right\},
$$

(3.1)

where $\eta$ is the anisotropy parameter. The transfer matrix $\mathbb{T}(u)$ is again given by (2.3)

$$
\mathbb{T}(u) = \text{tr}_0 \mathbb{R}_0(u) \mathbb{R}_0(u) \ldots \mathbb{R}_N(u).
$$

(3.2)

The Bethe equations are now given by

$$
\left( \frac{\sinh(u_{1,k} + \frac{\eta}{2})}{\sinh(u_{1,k} - \frac{\eta}{2})} \right)^N = \prod_{l=1,l \neq k}^{M_1} \frac{\sinh(u_{1,k} - u_{1,l} + \eta)}{\sinh(u_{1,k} - u_{1,l} - \eta)} \prod_{l=1}^{M_2} \frac{\sinh(u_{1,k} - u_{2,l} - \frac{\eta}{2})}{\sinh(u_{1,k} - u_{2,l} + \frac{\eta}{2})}, \quad k = 1, \ldots, M_1,
$$

(3.3)

$$
1 = \prod_{l=1,l \neq k}^{M_j} \frac{\sinh(u_{j,k} - u_{j,l} + \eta)}{\sinh(u_{j,k} - u_{j,l} - \eta)} \prod_{l=1}^{M_{j+1}} \frac{\sinh(u_{j,k} - u_{j+1,l} - \frac{\eta}{2})}{\sinh(u_{j,k} - u_{j+1,l} + \frac{\eta}{2})} \\
\times \prod_{l=1}^{M_{j-1}} \frac{\sinh(u_{j,k} - u_{j-1,l} - \frac{\eta}{2})}{\sinh(u_{j,k} - u_{j-1,l} + \frac{\eta}{2})}, \quad k = 1, \ldots, M_j, \quad j = 2, \ldots, m - 1,
$$

(3.4)
\begin{align*}
1 = \prod_{l=1; l \neq k}^{M_m} \frac{\sinh(u_m,k - u_{m,l} + \eta)}{\sinh(u_m,k - u_{m,l} - \eta)} \prod_{l=1}^{M_{m-1}} \frac{\sinh(u_{m,k} - u_{m-1,l} - \frac{\eta}{2})}{\sinh(u_{m,k} - u_{m-1,l} + \frac{\eta}{2})}, \quad k = 1, \ldots, M_m, \\
\text{and we now define functions } Q_1(u), \ldots, Q_m(u) \text{ by}
\end{align*}

\begin{align*}
Q_j(u) = \prod_{k=1}^{M_j} \sinh(u - u_{j,k}), \quad j = 1, \ldots, m,
\end{align*}

which are polynomials in \( t = e^u \) and \( t^{-1} \).

### 3.1 The \( A^{(1)}_m \) Q-system

We propose that the \( A^{(1)}_m \) spin chain has the same QQ-relations as the isotropic case, namely

\begin{align*}
Q_{j,n}(u) Q_{j+1,n-1}(u) \propto Q_{j+1,n}(u) Q_{j,n-1}(u) - Q_{j+1,n}(u) Q_{j,n-1}^+(u),
\end{align*}

\begin{align*}
j = 0, 1, \ldots, m, \quad n = 1, 2, \ldots,
\end{align*}

but where now \( f^\pm(u) = f(u \pm \frac{\eta}{2}) \). Moreover,

\begin{align*}
Q_{0,0}(u) = \sinh^N(u), \\
Q_{j,0}(u) = Q_j(u), \quad j = 1, \ldots, m, \\
Q_{m+1,0}(u) = 1,
\end{align*}

where the functions \( Q_j(u) \) are defined in (3.6).

We can easily verify that this Q-system indeed leads to the \( A^{(1)}_m \) Bethe equations. Indeed, starting from (3.7)- (3.8) and repeating the steps in Section 2.1 we arrive at the Bethe equations (3.3)- (3.5).

As in the isotropic case, the \( A^{(1)}_m \) Q-system can be solved in terms of functions \( F_0(u), \ldots, F_m(u) \) by

\begin{align*}
Q_{j,n} = \begin{vmatrix}
F_0^{(n)[m-j]} & F_0^{(n)[m-j-2]} & \cdots & F_0^{(n)[j-m]} \\
F_1^{(n)[m-j]} & F_1^{(n)[m-j-2]} & \cdots & F_1^{(n)[j-m]} \\
\vdots & \vdots & \ddots & \vdots \\
F_m^{(n)[m-j]} & F_m^{(n)[m-j-2]} & \cdots & F_m^{(n)[j-m]}
\end{vmatrix}_{(m+1-j) \times (m+1-j)}, \quad j = 0, 1, \ldots, m, \quad \text{and } n = 0, 1, \ldots, \text{except now}
\end{align*}

\begin{align*}
f^{(n)}(u) = f^{(n-1)+}(u) - f^{(n-1)-}(u) \\
= f^{(n-1)}(u + \frac{\eta}{2}) - f^{(n-1)}(u - \frac{\eta}{2}), \quad n = 1, 2, \ldots,
\end{align*}
and
\[ f^{[k]}(u) = f(u + k\eta_2). \tag{3.11} \]

Indeed, the same proof from Section 2.2 carries over to the anisotropic case.

The \( \mathcal{A}_m^{(1)} \) Q-system (3.7)-(3.8) and its determinant representation (3.9) constitute the other main results of this paper.

We conjecture that polynomiality of \( F_0 \) in \( t \equiv e^u \) and \( t^{-1} \), together with quasi-polynomiality (polynomial plus \( \log t \) times a polynomial) of \( F_1, \ldots, F_m \), is equivalent to the admissibility of the Bethe roots; and is also equivalent to polynomiality of all the Q-functions. Evidence supporting this conjecture for the simplest cases \( m = 1, 2 \) is provided below.

\( m = 1 \) The \( \mathcal{A}_1^{(1)} \) case (\( m = 1 \)) corresponds to the spin-1/2 XXZ spin chain, whose Q-system was recently formulated in [11]. As in the isotropic case, we identify \( F_0(u) \) as the “fundamental” Q-function \( Q(u) \), and \( F_1(u) \) as the “dual” Q-function \( P(u) \). The relations (3.9) with \( m = 1 \) coincide with Eq. (3.13) in [11].

It was argued in [11] that polynomiality of \( Q \), together with quasi-polynomiality of \( P \), is equivalent to the admissibility of the Bethe roots. It follows from (3.9) with \( m = 1 \) that polynomiality of the Q-system is equivalent to the admissibility of the Bethe roots.

\( m = 2 \) For the \( \mathcal{A}_2^{(1)} \) case (\( m = 2 \)), we have verified numerically for small values of \( N \) that the polynomial (in \( t \equiv e^u \) and \( t^{-1} \)) solutions of the Q-system (3.7)-(3.8) give the complete spectrum of the transfer matrix \( \mathcal{T}(u) \) (3.2). Indeed, we have used this Q-system to numerically obtain the admissible Bethe roots for some generic value of \( \eta \), as in the \( \mathcal{A}_1^{(1)} \) case [11]; and we have verified that the corresponding eigenvalues \( \mathcal{T}(u) \) of the transfer matrix computed using

\[ \mathcal{T}(u) = Q_{0,0}^+ \frac{Q_{1}^{-}(u)}{Q_{1}^{+}(u)} + Q_{0,0}(u) \frac{Q_{2}^{[3]}(u)}{Q_{1}^{+}(u)} \frac{Q_{2}^{-}(u)}{Q_{2}^{+}(u)} + Q_{0,0}(u) \frac{Q_{2}^{[3]}(u)}{Q_{2}^{+}(u)} \tag{3.12} \]

match with the results obtained by direct diagonalization of \( \mathcal{T}(u) \).

We report in Table 1 for given values of chain length \( N \), the numbers of Bethe roots of each type \((M_1, M_2)\), the corresponding number of admissible solutions \((n_{M_1, M_2})\) of the Bethe equations (3.3)-(3.5) obtained by solving the Q-system (3.7)-(3.8), and the degeneracies \((d_{M_1, M_2})\) of the corresponding transfer-matrix eigenvalues \( \mathcal{T}(u) \) (3.12) obtained by direct diagonalization of \( \mathcal{T}(u) \). (We refrain from displaying the Bethe roots themselves, which would require much bigger tables.) The completeness is demonstrated by the fact (easily confirmed from the data in Table 1) that all \( 3^N \) transfer-matrix eigenvalues are accounted for, i.e.

\[ \sum_{M_1, M_2} n_{M_1, M_2} d_{M_1, M_2} = 3^N. \tag{3.13} \]

The unusual pattern of degeneracies exhibited in the last column of Table 1 will be discussed elsewhere [23].
### Table 1

| $N$ | $M_1$ | $M_2$ | $n_{M_1,M_2}$ | $d_{M_1,M_2}$ |
|-----|-------|-------|---------------|---------------|
| 2   | 0     | 0     | 1             | 3             |
|     | 1     | 0     | 2             | 3             |
| 3   | 0     | 0     | 1             | 3             |
|     | 1     | 0     | 3             | 6             |
|     | 2     | 1     | 6             | 1             |
| 4   | 0     | 0     | 1             | 3             |
|     | 1     | 0     | 4             | 6             |
|     | 2     | 0     | 6             | 3             |
|     | 2     | 1     | 12            | 3             |
| 5   | 0     | 0     | 1             | 3             |
|     | 1     | 0     | 5             | 6             |
|     | 2     | 0     | 10            | 6             |
|     | 2     | 1     | 20            | 3             |
|     | 3     | 1     | 30            | 3             |

Table 1: Chain length ($N$), numbers of Bethe roots ($M_1, M_2$), number of admissible solutions of the Bethe equations ($n_{M_1,M_2}$) and degeneracies ($d_{M_1,M_2}$) for the $A^{(1)}_2$ spin chain.

### 4 Conclusions

We have proposed a Q-system for the $A^{(1)}_m$ spin chain (3.7)-(3.8), which provides an efficient way of obtaining solutions of the trigonometric Bethe equations (3.3)-(3.5). We have also found compact determinant expressions for all the Q-functions, both for the rational (2.18)-(2.21) and trigonometric (3.9) cases. In so doing, we have established links between Q-systems and the works by Kuniba et al. [10] and by Pronko and Stroganov [13,14].

Several interesting related problems remain to be addressed. The fact that all the Q-functions can be expressed in terms of the $m+1$ F-functions $F_0, \ldots, F_m$ suggests that the F-functions are of fundamental importance for the $SU(m+1)$ and $A^{(1)}_m$ models, and merit further investigation. In particular, it would be desirable to have proofs that polynomiality in $u$ (or quasi-polynomiality in $t$ and $t^{-1}$ for the trigonometric case) of the F-functions is equivalent to the admissibility of the Bethe roots. For $SU(m|n)$ graded (supersymmetric) spin chains [24], Q-systems were also formulated in [11,12]; it should be possible to formulate similar determinant expressions for these Q-systems and for the corresponding q-deformed models, and to relate them to results of Tsuboi [18,19]. We restricted here to periodic boundary conditions; it should be possible to generalize the $A^{(1)}_m$ Q-system (and the corresponding determinant representation) to open diagonal boundary conditions [25,28], thereby generalizing the corresponding rank-1 results [11,12] to higher rank. Only A-type Q-systems are so far known; it would be very interesting to construct such Q-systems for other algebras.
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