A NOTE ABOUT INVARIANTS OF ALGEBRAIC CURVES

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ABSTRACT. For affine algebraic curves we reduce a calculation of its invariants to calculation of the intersection of kernels of some derivations.

1. Introduction

Consider an affine algebraic curve

\[ C : F(x, y) = \sum_{i+j \leq d} a_{i,j} x^i y^j = 0, \quad a_{i,j} \in k, \]

defined over field \( k \), char \( k > 0 \). Let \( k[C] \) and \( k(C) \) be the algebras of polynomial and rational function of coefficients of the curve \( C \). Those affine transformations of plane whose preserve the algebraic form of equation \( F(x, y) \) generate a group \( G \) which is subgroup of the group of plane affine transformations \( A(2) \). A function \( \phi(a_{0,0}, a_{1,0}, \ldots, a_{d,0}) \in k(C) \) is called \( G \)-invariant if \( \phi(\tilde{a}_{0,0}, \tilde{a}_{1,0}, \ldots, \tilde{a}_{d,0}) = \phi(a_{0,0}, a_{1,0}, \ldots, a_{d,0}) \), where \( \tilde{a}_{0,0}, \tilde{a}_{1,0}, \ldots, \tilde{a}_{d,0} \) are defined from the condition

\[ F(gx, gy) = \sum_{i+j \leq d} a_{i,j} (gx)^i (gy)^j = \sum_{i+j \leq d} \tilde{a}_{i,j} x^i y^j, \]

for all \( g \in G \). Curves \( C \) and \( C' \) are said to be \( G \)-isomorphic if they lies on the same \( G \)-orbit.

The algebras of all \( G \)-invariant polynomial and rational functions denotes by \( k[C]^G \) and by \( k(C)^G \), respectively. One way to find elements of the algebra \( k[C]^G \) is a specification of invariants of associated ternary form of order \( d \). In fact, consider a vector space \( T_d \) generated by the ternary forms \( \sum b_{i,j} x^{d-(i+j)} y^j z^j, \quad b_{i,j} \in k \) endowed with the natural action of the group \( GL_3 := GL_3(k) \) and \( G \)-invariant function of \( k(T_d)^{GL_3} \), the specification \( f : b_{i,j} \mapsto a_{i,j} \) or \( b_{i,j} \mapsto 0 \) in the case when \( a_{i,j} \notin k(C) \), gives us an element of \( k(C)^G \).

But \( SL_3 \)-invariants(thus and \( GL_3 \)-invariants) of ternary form are known only for the cases \( d \leq 4 \), see [1]. Furthermore, analyzing of the Poincare series of the algebra of invariants of ternary forms, [2], we see that the algebras are very complicated and no chance to find theirs minimal generating sets.

Since \( k(T_d)^{GL_3} \) coincides with \( k(T_d)^{SL_3} \) it implies that the algebra of invariants is an intersection of kernels of some derivations of the algebra \( k(T_d) \). Then in place of the specification of coefficients of the form we may use a "specification" of those derivations.

First, consider the motivating example. Let

\[ C_3 : y^2 + a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0, \]

and let \( G_0 \) be a group generated by the translations \( x \mapsto \alpha x + b \). It is easy to show that \( j \)-invariant of the curve \( C_3 \) equals ([3], p. 46):

\[ j(C_3) = 6912 \frac{(a_0 a_2 - a_1^2)^3}{a_0^2 (4 a_1^3 a_3 - 6 a_3 a_0 a_1 a_2 - 3 a_1^2 a_2^2 + a_3^2 a_0^2 + 4 a_0 a_2^3)}. \]
Up to constant factor the \( j(C_3) \) equal to \( \frac{S^3}{T} \) where \( S \) and \( T \) are the specification of two \( SL_3 \)-invariants of ternary cubic, see [4], p.173.

From another side a direct calculation yields that the following is true: \( D(j(C_3)) = 0 \) and \( H(j(C_3)) = 0 \) where \( D, H \) denote the derivations of the algebra of rational functions \( k(C_3) = k(a_0, a_1, a_2, a_3) : \)
\[
D(a_i) = ia_i - 1, H(a_i) = (3 - i)a_i, i = 0, 1, 2, 3.
\]

From computing point of view, the calculation of \( \ker D \cap \ker H \) is more effective than the calculating of the algebra of invariants of the ternary cubic. We will derive further that

\[
\ker D_3 \cap \ker H_3 = k \left( \frac{(a_0a_2 - a_1^2)^3}{a_0^3}, \frac{a_3a_0^2 + 2a_1^3 - 3a_1a_2a_0}{a_0^2} \right).
\]

In the notes we prove that for arbitrary algebraic curve \( C \) and its preserved form group \( G \) there exist derivations \( D_i, i \leq 6 \) of \( k(C) \) such that \( k(C)^G = \bigcap_i \ker D_i \) (Theorem 3.2).

In section 1 we give full description of the algebras of polynomial and rational invariants for the curve \( y^2 = f(x) \). In section 2 we give exactly form for the derivations of the action of the Lie algebra \( gl_3 \) on \( k(C) \) and give a specification of such action for a curve of the form \( y^2 + g(x)y = f(x) \).

2. Invariants of \( y^2 = f(x) \).

In some simple cases we may obtain the defining derivation directly.

Consider the curve
\[
C_d : y^2 = a_0x^d + da_1x^{d-1} + \cdots + a_d = \sum_{i=0}^d a_d \left( \begin{array}{c} d \\ i \end{array} \right) x^{d-i},
\]
and let \( G \) be the group generated by the following transformations
\[
x = \alpha \bar{x} + b, y = \bar{y}.
\]
The algebra \( k(C_d)^G \) consists of functions \( \phi(a_0, a_1, \ldots, a_d) \) that have the invariance property
\[
\phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d) = \phi(a_0, a_1, \ldots, a_d).
\]
Here \( \tilde{a}_i \) denote the coefficients of the curve \( \tilde{C}_d : \)
\[
\tilde{C}_d : \sum_{i=0}^d \tilde{a}_d \left( \begin{array}{c} d \\ i \end{array} \right) (\alpha \bar{x} + b)^{d-i} = \sum_{i=0}^d \tilde{a}_d \left( \begin{array}{c} d \\ i \end{array} \right) \bar{x}^{d-i}.
\]
The coefficients \( \tilde{a}_i \) are given by the formulas
\[
(1) \quad \tilde{a}_i = \alpha^{-i} \sum_{k=0}^k \binom{i}{k} b^k a_{n-k}.
\]
The following statement holds

**Theorem 2.1.** We have
\[
k(C)^G = \ker D_d \cap \ker E_d,
\]
where \( D_d, E_d \) denote the following derivations of the algebra \( k(C) : \)
\[
D_d(a_i) = ia_{i-1}, E_d(a_i) = (d - i)a_i.
\]
Recall that a linear map $D : \mathbf{k}(C) \to \mathbf{k}(C)$ is called a derivation of the algebra $\mathbf{k}(C)$ if $D(fg) = D(f)g + fD(g)$, for all $f, g \in \mathbf{k}(C)$. The subalgebra $\ker D := \{ f \in \mathbf{k}(C) \mid D(f) = 0 \}$ is called the kernel of the derivation $D$. The above derivation $D_d$ is called the basic Weitzenböck derivation.

**Proof.** Acting in classical manner, we differentiate with respect to $b$ both sides of the equality

$$
\phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d) = \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d),
$$

and obtain in this way

$$
\frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_0} \frac{\partial \tilde{a}_0}{\partial b} + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_1} \frac{\partial \tilde{a}_1}{\partial b} + \cdots + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_d} \frac{\partial \tilde{a}_d}{\partial b} = 0.
$$

Substitute $\alpha = 1, b = 0$ to $\phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)$ and taking into account that $\frac{\partial \tilde{a}_i}{\partial b} \bigg|_{b=0} = i a_{i-1}$, we get:

$$
\tilde{a}_0 \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_1} + 2a_1 \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_2} + \cdots + d a_{d-1} \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_d} = 0
$$

Since the function $\phi(\tilde{a}_0, \ldots, \tilde{a}_d)$ depends on the variables $\tilde{a}_i$ in the exact same way as the function $\phi(a_0, a_1, \ldots, a_d)$ depends on the $a_i$ then it implies that $\phi(a_0, a_1, \ldots, a_d)$ satisfies the differential equation

$$
a_0 \frac{\partial \phi(a_0, a_1, \ldots, a_d)}{\partial a_1} + 2a_1 \frac{\partial \phi(a_0, a_1, \ldots, a_d)}{\partial a_2} + \cdots + d a_{d-1} \frac{\partial \phi(a_0, a_1, \ldots, a_d)}{\partial a_d} = 0
$$

Thus, $D_d(\phi) = 0$. Now we differentiate with respect to $a$ both sides of the same equality

$$
\frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_0} \frac{\partial \tilde{a}_0}{\partial a} + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_1} \frac{\partial \tilde{a}_1}{\partial a} + \cdots + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_d} \frac{\partial \tilde{a}_d}{\partial a} = 0.
$$

Substitute $\alpha = 1, b = 0$, to $\phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)$ and taking into account $\frac{\partial \tilde{a}_i}{\partial a} \bigg|_{a=b=0} = (d - i) a_i$, we get:

$$
\tilde{a}_0 \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_0} + (d - 1) a_1 \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_1} + \cdots + (d - d - 1) a_{d-1} \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_d)}{\partial \tilde{a}_{d-1}} = 0
$$

It implies that $\mathcal{E}_d(\phi(a_0, a_1, \ldots, a_d)) = 0$. □

The derivation $\mathcal{E}_d$ sends the monom $a_0^{m_0} a_1^{m_1} \cdots a_d^{m_d}$ to the term

$$(m_0 d + m_1 (d-1) + \cdots + m_{d-1}) a_0^{m_0} a_1^{m_1} \cdots a_d^{m_d}.
$$

Let the number $\omega(a_0^{m_0} a_1^{m_1} \cdots a_d^{m_d}) := m_0 d + m_1 (d-1) + \cdots + m_{d-1}$ be called the weight of the monom $a_0^{m_0} a_1^{m_1} \cdots a_d^{m_d}$. In particular $\omega(a_i) = d - i$.

A homogeneous polynomial $f \in \mathbf{k}[C]$ be called isobaric if all their monomial have equal weights. A weight $\omega(f)$ of an isobaric polynomial $f$ is called a weight of its monomials. Since $\omega(f) > 0$, then $\mathbf{k}[C]^G = 0$. It implies that $\mathbf{k}[C]^G = 0$.

If $f, g$ are two isobaric polynomials then

$$
\mathcal{E}_d \left( \frac{f}{g} \right) = (\omega(f) - \omega(g)) \frac{f}{g}.
$$

Therefore the algebra $\mathbf{k}(C)^\mathcal{E}_d$ is generated by rational fractions which both denominator and numerator has equal weight.

The kernel of the derivation $D_d$ also is well-known, see [5], [6], and

$$
\ker D_d = \mathbf{k}(a_0, z_2, \ldots, z_d),
$$

where
\[ z_i := \sum_{k=0}^{i-2} (-1)^k {i \choose k} a_{i-k} a_i^{k} a_i^{k-1} + (i - 1) (-1)^{i+1} a_i, \quad i = 2, \ldots, d. \]

In particular, for \( d = 5 \), we get
\[
\begin{align*}
z_2 &= a_2 a_0 - a_1^2 \\
z_3 &= a_3 a_0^2 + 2 a_1^3 - 3 a_1 a_2 a_0 \\
z_4 &= a_4 a_0^3 - 3 a_1^4 + 6 a_1^2 a_2 a_0 - 4 a_1 a_3 a_0^2 \\
z_5 &= a_5 a_0^4 + 4 a_1^5 - 10 a_1^3 a_2 a_0 + 10 a_1^2 a_3 a_0^2 - 5 a_1 a_4 a_0^3.
\end{align*}
\]

It is easy to see that \( \omega(z_i) = i(n - 1) \). The following element \( \frac{z_i^d}{a_0^{i(d-1)}} \) has the zero weight for any \( i \). Therefore the statement holds:

**Theorem 2.2.**
\[
k(C_d)^G = k \left( \frac{z_2^d}{a_0^{2(d-1)}}, \frac{z_3^d}{a_0^{3(d-1)}}, \ldots, \frac{z_d^d}{a_0^{d(d-1)}} \right).
\]

For the curve
\[
C_d^0: y^2 = x^d + d a_1 x^{d-1} + \cdots + a_d = x^d + \sum_{i=1}^{d} a_d \binom{d}{i} x^{d-i}.
\]
and for the group \( G_0 \) generated by translations \( x = \bar{x} + b \), the algebra of invariants becomes simpler:
\[ k \left( C_d^0 \right)^{G_0} = k(z_2, z_3, \ldots, z_d). \]

**Theorem 2.3.** (i) For arbitrary set of \( d - 1 \) numbers \( j_2, j_3, \ldots, j_d \) there exists a curve \( C \) such that \( z_i(C) = j_i \).

(ii) For two curves \( C \) and \( C' \) the equalities \( z_i(C) = z_i(C') \) hold for \( 2 \leq i \leq d \), if and only if these curves are \( G_0 \)-isomorphic.

**Proof.** (i). Consider the system of equations
\[
\begin{align*}
&\begin{cases}
a_2 - a_1^2 = j_2 \\
a_3 + 2 a_1^3 - 3 a_1 a_2 = j_3 \\
a_4 - 3 a_1^4 + 6 a_1^2 a_2 - 4 a_1 a_3 = j_4 \\
\vdots \\
a_d + \sum_{k=1}^{d-2} (-1)^k {d \choose k} a_{d-k} a_1^k + (d - 1)(-1)^{d+1} a_1^d = j_d
\end{cases}
\end{align*}
\]
By solving it we obtain
\[
a_n = j_n + \sum_{i=2}^{n} \binom{n}{i} a_i^k j_{n-k}
\]
Put \( a_1 = 0 \) we get \( a_n = j_n \), i.e. the curve
\[
C: y^2 = x^d + \binom{d}{2} j_2 x^{d-2} + \cdots + j_d,
\]
is desired one.
(ii). We may assume, without loss of generality, that the curve \( C \) has the form

\[
C : y^2 = x^d + \left( \frac{d}{2} \right) j_2 x^{d-2} + \cdots + j_d.
\]

Suppose that for a curve

\[
C' : y^2 = x^d + a_1 x^{d-1} + \cdots + a_d = x^d + \sum_{i=1}^{d} a_d \left( \frac{d}{i} \right) x^{d-i}.
\]

holds \( z_i(C') = z_i(C) = j_i \). Comparing (2) with (1) we deduce that the curve \( C' \) is obtained from the curve \( C \) by the translation \( x + a_1 \).

\[ \square \]

3. General case and invariants of \( y^2 + g(x)y = f(x) \)

Consider the vector \( \mathbf{k} \)-space \( T_d \) of ternary form of degree \( d \):

\[
u(x, y, z) = \sum_{i+j \leq d} \frac{d!}{i!j!(d-(i+j))!} a_{i,j} x^{d-(i+j)} y^i z^j,
\]

where \( a_{i,j} \in k \). Let us identify in the natural way the algebra of rational function \( k(T_d) \) on the vector space \( T_d \) with the algebra of polynomials of the \( \frac{1}{2}(d+1)(d+2) \) variables. The natural action of the group \( GL_3 \) on \( T_d \) induced the action of \( GL_3 \) (and the Lie algebra \( \mathfrak{gl}_3 \)) on \( k[T_d] \). The corresponding algebra of invariants \( k(T_d)^{GL_3} = k(T_d)^{\mathfrak{gl}_3} \) is called the algebra of \( GL_3 \)-invariants (or absolute invariants) of ternary form of degree \( d \). The following statement holds:

**Theorem 3.1.**

\[
k(T_d)^{GL_3} = \ker D_1 \cap \ker D_2 \cap \ker \hat{D}_1 \cap \ker \hat{D}_2 \cap \ker E_1 \cap \ker E_2 \cap \ker E_3
\]

where

\[
D_1(a_{i,j}) = i a_{i-1,j}, \quad D_2(a_{i,j}) = j a_{i+1,j-1},
\]

\[
\hat{D}_1(a_{i,j}) = (n-(i+j)) a_{i+1,j}, \quad \hat{D}_2(a_{i,j}) = i a_{i-1,j+1},
\]

\[
\hat{D}_3(a_{i,j}) = (n-(i+j)) a_{i,j+1}, \quad D_3(a_{i,j}) = j a_{i,j-1},
\]

\[
E_1(a_{i,j}) = (n-(2i+j)) a_{ij}, \quad E_2(a_{i,j}) = i a_{i,j},
\]

\[
E_3(a_{i,j}) = j a_{i,j}.
\]

**Proof.** The Lie algebra \( \mathfrak{gl}_3 \) acts on the vector space of ternary form \( T_d \) by derivations, namely

\[
D_1 = -y \frac{\partial}{\partial x}, \quad D_2 = -z \frac{\partial}{\partial y},
\]

\[
E_1 = -x \frac{\partial}{\partial x}, \quad E_2 = -y \frac{\partial}{\partial y},
\]

\[
\hat{D}_1 = -x \frac{\partial}{\partial y}, \quad \hat{D}_2 = -y \frac{\partial}{\partial z},
\]

\[
D_3 = -z \frac{\partial}{\partial x}, \quad E_3 = -z \frac{\partial}{\partial z},
\]

\[
\hat{D}_3 = -x \frac{\partial}{\partial z}.
\]

To extend the actions \( \mathfrak{gl}_3 \) to the algebra \( k(T_d) \) we use the well-known fact of classical invariant theory that the generic form \( u(x, y, z) \) is a covariant. It means that any of above derivation
Proof. It follows from the fact that for a nonzero derivation $D$ of polynomial ring $R$ of $n$ variables $\text{tr deg ker } D \leq n - 1$, see [5], Proposition 7.1.1. □

An obvious consequence of the theorem is the following:

**Theorem 3.2.** For affine algebraic curves $d$

$$C : \sum_{i+j=d} \frac{d!}{i!j!(d - (i + j))!} a_{i,j} x^{d-(i+j)} y^i = 0, a_{d,0} = 0, a_{i,j} \in k, \text{char } k > 0,$$

the following holds:

(i) if $a_{d,0} \neq 0$ and $a_{0,0} \neq 0$ then

$$k(C)^G = \ker D_1 \cap \ker D_2 \cap \ker \hat{D}_1 \cap \ker E_1 \cap \ker E_2,$$

(ii) if $a_{d,0} = 00 a_{0,0} \neq 0$ then

$$k(C)^G = \ker E_2 \cap \ker D_3 \cap \ker \hat{D}_1 \cap \ker E_1 \cap \ker E_2,$$

where

$$D_1(a_{i,j}) = ia_{i-1,j}, D_2(a_{i,j}) = ja_{i+1,j-1}, \hat{D}_1(a_{i,j}) = (d - (i + j))a_{i+1,j},$$

$$E_1(a_{i,j}) = (d - (i + j))a_{i,j}, E_2(a_{i,j}) = ia_{i,j}, D_3(a_{i,j}) = ja_{i,j-1}.$$

Proof. (i) Consider the associate projective plane curve in $\mathbb{P}^2$:

$$\sum_{i+j=d} \frac{d!}{i!j!(d - (i + j))!} a_{i,j} x^{d-(i+j)} y^i z^j = 0, a_{d,0} \neq 0, a_{0,0} \neq 0.$$

The transformations $X \mapsto \alpha X + \beta Y + bZ, Y \mapsto \gamma X + \delta Y + aZ, Z \mapsto Z$ generate of a subgroup of $GL_3$ which preserve the algebraic form of the equation of the curve. Therefore the algebra of invariants of the curve (and corresponding affine curve) coincides with the intersection of the kernels of the five derivations $D_1, D_2, \hat{D}_1, E_1, E_2, ([D_1, D_2] = D_3)$.

(ii) For this case the transformations are as follows: $X \mapsto \alpha X + bZ, Y \mapsto \gamma X + \delta Y + aZ, Z \mapsto Z$ and we have to exclude the derivation $D_1$. □
For the curve
\[ C'_d : \frac{d(d-1)}{2} a_{2,d-2} y^2 + \sum_{i=0}^{d-1} \frac{d!}{i! (d-(1+i))!} a_{1,i} x^{d-(i+1)} y + \sum_{i=0}^{d} \frac{d!}{i! (d-i)!} a_{0,i} x^{d-i} = 0, \]
and for the group \( G \) generated by \( x \mapsto \alpha x + a, y \mapsto \beta y + b \) we have
\[ k(C'_d)^G = \ker D_2 \cap \ker D_3 \cap \ker D_4 \cap \ker E_1 \cap \ker E_2, \]
and \( \text{tr deg}_G k(C'_d)^G \leq 2d - 3. \)

**Example.** Let us calculate the invariants of curve \( C'_5 \)
\[ 10 y^2 + (5 a_{1,0} x^4 + 20 a_{1,1} x^3 + 30 a_{1,2} x^2 + 20 a_{1,3} x + 5 a_{1,4}) y = \]
\[ = x^5 + 5 a_{0,1} x^4 + 10 a_{0,2} x^3 + 10 a_{0,3} x^2 + 5 a_{0,4} x + a_{0,5}, \]
with respect to the group \( G_0 \) generated by the translations \( x = \tilde{x} + a, y = \tilde{y} + b. \) Theorem 3.2 implies that \( C'_5 = \ker D_2 \cap \ker D_3, \) where the derivations \( D_2, D_3, \) act by
\[ D_2 (a_{1,1}) = 0, \quad D_2 (a_{1,0}) = 0, \quad D_2 (a_{0,1}) = -a_{1,0}, \quad D_2 (a_{1,2}) = 0, \quad D_2 (a_{0,2}) = -2 a_{1,1}, \]
\[ D_2 (a_{0,3}) = -3 a_{1,2}, \quad D_2 (a_{0,5}) = -5 a_{1,4}, \quad D_1 (a_{0,4}) = -4 a_{1,3}, \quad D_2 (a_{1,3}) = 0, \quad D_2 (a_{1,4}) = 4. \]
and
\[ D_3 (a_{1,2}) = 2 a_{1,1}, \quad D_3 (a_{1,1}) = a_{1,0}, \quad D_2 (a_{1,0}) = 0, \quad D_2 (a_{0,2}) = 2 a_{1,1}, \quad D_3 (a_{0,1}) = 1, \]
\[ D_2 (a_{0,3}) = 3 a_{0,2}, \quad D_3 (a_{0,5}) = 5 a_{0,4}, \quad D_2 (a_{0,4}) = 4 a_{0,3}, \quad D_2 (a_{1,3}) = 3 a_{1,2}, \quad D_3 (a_{1,4}) = 4 a_{1,3}. \]
By using the Maple command `pdsoi()` we obtain that
\[ k(C'_5)^{G_0} = k(g_1, g_2, g_3, g_4, g_5, g_6, g_7), \]
\[ k(C'_3)^{G_0} = k(g_1, g_2, g_3, g_4, g_5, g_6, g_7), \]
where
\[ g_1 = a_{1,0}, \]
\[ g_2 = a_{1,0}^2 a_{0,2} + a_{1,1}^2 - 2 a_{1,1} a_{1,0} a_{0,1}, \]
\[ g_3 = a_{1,2} - 2 a_{1,1} a_{0,1} + a_{1,0} a_{0,2}, \]
\[ g_4 = 6 a_{1,1} a_{0,1} a_{1,0} - 4 a_{1,1}^3 - 3 a_{1,0}^2 a_{2,1,1} - 3 a_{1,2} a_{1,0}^2 a_{0,1} + 3 a_{1,0} a_{1,1} a_{1,2} + a_{0,3} a_{1,0}^3, \]
\[ g_5 = 2 a_{1,1}^3 - 3 a_{1,0} a_{1,1} a_{1,2} + 3 a_{1,3} a_{1,0}^2, \]
\[ g_6 = 3 a_{1,0}^4 a_{0,2}^2 + a_{1,0}^4 a_{0,4} - 12 a_{1,0}^3 a_{1,1} a_{0,1} a_{0,2} - 4 a_{1,0}^3 a_{1,1} a_{0,3} - 4 a_{1,0}^3 a_{1,3} a_{0,1}, \]
\[ -12 a_{1,0}^3 a_{0,2} a_{1,2} + 12 a_{1,1} a_{1,0}^2 a_{0,1}^2 + 24 a_{1,1} a_{1,0}^2 a_{0,2} + 4 a_{1,1} a_{1,0}^2 a_{1,3} + 36 a_{1,1} a_{1,0}^2 a_{0,1} a_{1,2} - \]
\[ -24 a_{1,0} a_{1,2} a_{1,1}^2 - 48 a_{1,0} a_{1,1}^3 a_{0,1} + 24 a_{1,1}^4, \]
\[ g_7 = a_{1,0}^4 a_{0,2}^2 + a_{1,0}^3 a_{1,1} - 4 a_{1,0}^3 a_{1,1} a_{0,1} a_{0,2} + 6 a_{1,0}^2 a_{0,2} a_{1,2} + a_{1,0}^2 a_{0,1}^2 - \]
\[ -4 a_{1,1} a_{0,2} a_{1,2} - 12 a_{1,1} a_{1,0}^2 a_{0,1} a_{1,2} - 4 a_{1,1} a_{1,0}^2 a_{1,3} + 4 a_{1,1} a_{0,1}^2 a_{0,1} - 4 a_{1,1} a_{1,0} + \]
\[ + 12 a_{1,0} a_{1,2} a_{1,1}^2 + 8 a_{1,0} a_{1,1} a_{0,1} - 8 a_{1,1}^4. \]

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