Abstract
Markov Decision Processes (MDPs) are the underlying model for optimal planning for decision theoretic agents in stochastic environments. Although much research focuses on solving MDP problems both in tabular form or using factored representations, none focused on tensor decomposition methods. Solving MDPs using tensor algebra offers the prospect of leveraging advances in tensor-based computations to further increase solver efficiency. In this paper, we develop an MDP solver for a multidimensional problem using a tensor decomposition method to compress the transition models and optimize the value iteration and policy iteration algorithms. We empirically evaluate our approach against tabular methods and show our approach can compute much larger problems using substantially less memory, opening up new possibilities for tensor-based approaches in stochastic planning.

Introduction
Markov Decision Processes (MDPs) have been widely studied as an elegant mathematical formalism to model stochastic domains (Bellman 1957a). The solution to an MDP problem, given a stochastic state transition system and a reward function, is a policy that defines the optimal (maximum expected utility) policy for every state in the domain. Most approaches to solve tabular representations of these problems require a large number of mathematical operations and substantial memory. Such tabular approaches, while mathematically sound, have limited applicability, because of the curse of dimensionality, when the required computational resources scale exponentially with the number of state variables (Bellman 1957b).

Subsequent research developed methods to factorize the transition model, such as Factored MDPs (Boutilier, Darden, and Goldszmidt 1995; Guestrin et al. 2003; Delgado, Sanner, and de Barros 2011), and produce compact representations of complex uncertain systems allowing exponential reduction in representation complexity. Such factored approaches are grounded on representing the transition model as dynamic Bayesian networks (DBNs). Methods to solve DBN-based representations, however, do not leverage advances in tensor decomposition methods to represent large MDPs or to improve solver runtime. In this paper, we develop an efficient tensor representation for n-dimensional problems characterized by a small number of parameters to represent the transition models, enabling solvers to scale up and mitigate the curse of dimensionality. We call the resulting approach CP-MDP, addressing the challenges of (1) reducing the necessary memory and the computational cost required by tabular methods to compute the solution, and (2) leveraging advances in tensor processing to further increase solver efficiency of MDP solvers by representing the transition models as tensor components. Our main contributions are: (1) an expanded formalization of (Kuinchtn, Meneguzzi, and Sales 2020) to multidimensional model-based MDP problems formalization; and (2) a novel implementation of the GRIDWORLD using the CANDECOMP-PARAFAC (Carroll and Chang 1970) decomposition method.

Background
Markov Decision Process
A Markov decision process is a sequential decision problem for a fully observable and stochastic environment with a Markovian transition model. An MDP consists as a tuple \( M = (S, A, P, R, \gamma) \) (Sutton and Barto 2018, Ch. 3), where: \( S \) is the state space; \( A \) is the action space; \( P \) is a transition probability function \( P(s' \mid s, a) \); \( R \) is a reward function; and \( \gamma \in [0, 1] \) is a discount factor. Most MDP solvers seek to find a policy that assigns an (optimal) action choice for each agent at each state (Guestrin 2003). A policy with highest expected reward or with expected reward equal to all other policies is called the optimal policy (\( \pi^* \)).

Value Iteration and Policy Iteration are common dynamic programming algorithms to solve MDPs. The basic idea of value iteration is to calculate the utility of each state and then use the state utilities to select an optimal action in each state (Bellman 1954). The utility of a state is the immediate reward for this state plus the expected discounted utility of neighboring states, assuming the agent chooses the optimal action. The Bellman Equation (Bellman 1957a) formalizes this model in Equation 1. Value iteration propagates information through the state space iteratively by means of local updates until it converges to the optimal value, from which the optimal policy is extracted.

\[
V(s) = \max_{a \in A} \left( R(s, a) + \gamma \sum_{s'} P(s' \mid s, a) V(s') \right)
\]

\[
\pi(s) = \arg\max_{a \in A} \left( R(s, a) + \gamma \sum_{s'} P(s' \mid s, a) V(s') \right)
\]
\[ V(s) = \mathcal{R}(s) + \gamma \max_{a \in \mathcal{A}(s)} \sum_{s'} \mathcal{P}(s' | s, a)V(s') \] (1)

Unlike value iteration, the execution of policy iteration alternates in two steps: policy evaluation and policy improvement. Policy evaluation is the process of computing the value function for a fixed policy \( \pi \). Whereas policy improvement is the process of generating a new policy, such that \( V_\pi(s) \geq V_\pi(s) \), by acting greedily with respect to \( \pi \) (Sutton and Barto 2018, Ch. 4). The algorithm terminates when the policy improvement step yields no change in the policy. At this point, the policy is a solution to the Bellman equation, and \( \pi \) must be an optimal policy.

**Tensor Decomposition**

A tensor is a multidimensional array, where the \( n \)th-order tensor is an element of the tensor product of \( n \) vector spaces, each of which has its own coordinate system (Kolda and Bader 2009). We apply tensor decomposition to data arrays to extract and explain their properties, and we can consider these methods to be higher-order generalizations of matrix singular value decomposition (SVD) and principal component analysis (PCA) (Kolda and Bader 2009). Increasing computing capacity has enabled tensor-based approaches to decompose higher-order problems, leading to a number of applications in signal processing, computer vision, data mining, neuroscience and machine learning (Kolda and Bader 2009; Sidiropoulos et al. 2017).

Canonical Polyadic Decomposition with Parallel Factors, also known as CANDECOMP-PARAFAC decomposition or CP decomposition (Carroll and Chang 1970; Harshman 1977; Kiers 1998), is an example of tensor decomposition method. CANDECOMP-PARAFAC is the process that factorizes a tensor into sums of individual components, providing a parallel proportional analysis and an idea of multiple axes for analysis (Kolda and Bader 2009). This method expresses a tensor as sum of the outer product of vectors, and we illustrate a CANDECOMP-PARAFAC decomposition of a third-order tensor \( \mathcal{X} \) in Figure 1.

![Figure 1: A CANDECOMP-PARAFAC decomposition.](image)

**Tensor Decomposition of MDPs**

In order to improve solution’s runtime and memory usage, our method decomposes transition models into small tensor components, by using the CANDECOMP-PARAFAC decomposition key idea as the basic semantics of this work: to express the tensor as a sum of rank-one tensors.

**MDP Tensor-based Formalization**

We now formalize MDPs in terms of \( n \)-dimensional values, which can be applied to any discrete, stochastic and finite MDP problem.

**Definition 1** Let \( \mathcal{D} \) be the set of environment’s dimensions, where \( \mathcal{D} = \{ d_1, d_2, d_3, ..., d_D \} \) and \( D = |\mathcal{D}| \) is the number of dimensions. Each component of this set can be analyzed individually and comprises a set of states as follows:

- \( d_1 \) set of states of dimension 1, where \( d_1 = \{ s_{1d_1}, s_{2d_1}, s_{3d_1}, ..., s_{S_1d_1} \} \) and \( S_1d_1 = |d_1| \) is the number of states of dimension 1;

- \( d_2 \) set of states of dimension 2, where \( d_2 = \{ s_{1d_2}, s_{2d_2}, s_{3d_2}, ..., s_{S_2d_2} \} \) and \( S_2d_2 = |d_2| \) is the number of states of dimension 2;

... 

- \( d_D \) set of states of dimension \( D \), where \( d_D = \{ s_{1d_D}, s_{2d_D}, s_{3d_D}, ..., s_{S_Dd_D} \} \) and \( S_Dd_D = |d_D| \) is the number of states of the dimension \( D \).

**Definition 2** Given the set of dimensions \( \mathcal{D} \), it contains \( |\mathcal{D}| \) dimensions \( d \), respectively, where \( d \in \{ 1, D \} \), and the statespace \( \mathcal{S} \) is set of all states of all dimensions (such that \( \mathcal{S} = \{ \mathcal{S} \} \)), where:

\[
\mathcal{S} = \{(s_{1d_1}, s_{1d_2}, ..., s_{1d_D})s_1, ..., (s_{1d_1}, s_{1d_2}, ..., s_{2d_D})s_2, ..., (s_{3d_1}, s_{3d_2}, ..., s_{3d_D})s_3, ..., (s_{S_Dd_D} s_{S_Dd_D})s \}
\]

Finally, we define the CANDECOMP-PARAFAC tensor components representation by \( \mathcal{C}(s) \):

**Definition 3** \( \mathcal{C}(s) \) is the set of tensor components of each state \( s \), where \( |\mathcal{C}(s)| = |\mathcal{C}(s)| \) is the number of tensor components of each state \( s \):

\[
\mathcal{C}(s) = \{ [(s, s', \mathcal{P}(s' | s, a))], ..., (s, s', \mathcal{P}(s' | s, a)) \} \} \}
\]

We define each component by: \( a \) a current state \( s \); \( b \) a successor state \( s' \); and \( c \) a probability of state transition \( \mathcal{P}(s' | s, a) \). The number of tensor components \( \mathcal{C}(s) \) of each state is that of transitions with \( \mathcal{P}(s' | s, a) > 0 \).

**MDP Scenario and Problem Statement**

We evaluate our approach using standard GRIDWORLD problems, where grid size is defined by the number of states \( \mathcal{S} \). The number of actions is determined \( 2s + |\mathcal{D}| \), i.e., we consider two actions for each Cartesian plane. Obstacles \( \mathcal{O} \) are states an agent has no access to. The interaction with the environment terminates when the agent reaches one of the terminal states \( \mathcal{T} \). For example, for a two-dimensional grid, the environment is defined by \( x \times y \) states and the agent’s actions in a given state are North and South in the \( x \)-axis; and in the \( y \)-axis are West and East. For a three-dimensional grid, the environment is defined by \( x \times y \times z \) states and the actions are Forward and Backward in the \( z \)-axis; and so on.
Experimental Results

We evaluate the Python implementation of our approach using an Intel(R) Xeon(R) CPU @ 2.30GHz with 13 GB RAM from a Python 3 Google Colabary notebook. We compare our results against a standard tabular implementation of value iteration and policy iteration algorithms available in Python `pymdptoolbox`. The plotted values consist of the average of 6 executions of the GridWorld problem for each grid size configuration. We use a discount factor of $\gamma = 0.9$ and $\text{maxIter} = 1,000$. We define non-terminal states with a -3 reward, and set terminal states with an additive +100 reward or a discounted -100 reward. The grid contains randomly placed obstacles and terminal states. Table 1 shows the number of actions $A$, states $S$, obstacles $O$ and terminals $T$ we consider for each test environment. For example, for a 2D test with 4 actions, we use 6 terminals and 50 obstacles to solve a 4,900-state problem, where the total number of states are divided into two dimensions (e.g. $70 \times 70 = 4,900$).

Figure 2a shows the runtime (in seconds) to solve the GridWorld problem using the value iteration (Tabular-PI) and policy iteration (Tabular-PI) algorithms both with a tabular representation and our CP-MDP method, due to memory limitations to compute the solution for these grid sizes using tabular approaches. In terms of memory, CP-MDP-VI and CP-MDP-PI are very similar for smaller grid sizes, but for more than 500,000 states, CP-MDP-VI requires 50% or less memory than CP-MDP-PI. For problems with 2 or 3 dimensions, CP-MDP-VI runtime and memory improve by approximately 65% compared to higher-dimensional problems with the same number of states. For example, CP-MDP-VI computes the solution of a 1,000,000-state 9-dimensional problem in 5,895.82 seconds using 1.56 GB, whereas the complexity for Tabular-PI is $O(|S|^3 + |S|^2 \times |A|)$, where $S$ is the number of states and $A$ is the number of actions.

Figures 3a and 3b show runtime and memory results for tests performed only by the CP-MDP method, due to memory limitations. For example, CP-MDP-VI requires a number of multiplications equal to: $\prod_{i=1}^{S} s_i \times \prod_{j=1}^{C_{(s)}} c_j \times \prod_{k=1}^{A} a_k$, whereas the complexity for Tabular-PI is $O(|S|^2 \times |A|)$, where $S$ is the number of states and $A$ is the number of actions.

Finally, in order to compare our CP-MDP-VI and CP-MDP-PI against tabular methods, we compute their computational cost. Since CP-MDP-VI requires a number of multiplications equal to: $\prod_{i=1}^{S} s_i \times \prod_{j=1}^{C_{(s)}} c_j \times \prod_{k=1}^{A} a_k$, we arrive at a complexity of $O(|S| \times |C_{(s)}| \times |A|)$, where $C_{(s)}$ is the number of tensor components for the state $s$. By contrast, CP-MDP-PI requires a number of multiplications equal to: $\prod_{i=1}^{S} s_i \times \prod_{j=1}^{C_{(s)}} c_j \times \prod_{k=1}^{A} a_k + \prod_{l=1}^{S} s_l \times \prod_{m=1}^{C_{(s)}} c_m \times \prod_{n=1}^{A} a_n$, resulting in a $O(|S| \times |C_{(s)}|^2 + |S| \times |C_{(s)}| \times |A|)$ complexity. In consequence, these methods require much less computation, and substantially less memory than traditional tabular methods.

| $D$ | 2 | 3 | 5 | 7 | 9 |
|-----|---|---|---|---|---|
| $A$ | 4 | 6 | 10 | 14 | 18 |
| $T$ | 6 | 50 | 4,900 | 4,000 | 3,125 | 2,048 | 3,888 |
|     | 8 | 100 | 10,000 | 8,000 | 7,000 | 5,184 | 5,832 |
|     | 10 | 200 | 14,400 | 12,500 | 10,000 | 9,216 | 8,748 |
|     | 12 | 300 | 19,600 | 18,750 | 12,500 | 10,368 | 9,216 |
|     | 14 | 400 | 22,500 | 24,000 | 19,200 | 18,432 | 17,496 |
|     | 16 | 500 | 90,000 | 60,000 | 100,000 | 78,125 | 82,944 |
|     | 18 | 600 | 250,000 | 125,000 | 200,000 | 232,800 | 196,008 |
|     | 20 | 700 | 640,000 | 512,000 | 600,000 | 605,052 | 491,520 |
|     | 22 | 800 | 1,000,000 | 1,000,000 | 1,200,000 | 823,543 | 1,000,000 |

Table 1: Grid configuration.

Footnotes:
1The notation adopted is $|X|$ to define the cardinality of a set.
2Available at: https://github.com/danielakuinchtner/cp-mdp
3https://colab.research.google.com/
4https://pypi.org/project/pymdptoolbox/
We developed a tensor decomposition method, called CP-MDP, which uses the idea of the CANDECOMP-PARAFAC decomposition, to solve an MDP multidimensional problem using value iteration and policy iteration algorithms. To our knowledge our work is the first one employing tensor decomposition, to solve an MDP multidimensional problem. Our empirical analysis shows that CP-MDP method solve more efficiently than CP-MDP-PI and tabular methods in both runtime and memory. First CP-MDP-VI achieves runtime improvements of up to 80% in the best cases compared to tabular approaches. Second, both methods require substantially less memory to compute these solutions, decreasing memory use by more than 90% for large multidimensional problems.

In order to leverage advances in GPU computation libraries, we tried several different ways to parallelize our approach to run on GPUs with Tensorflow\(^5\), Pytorch\(^6\) and Numba\(^7\), however the runtime did not improve in any of these libraries. Such negative result seems to stem from the communication overhead between CPU and GPU, as our methods rely on multiplications between small tensor components.

As future work, we intend to use other methods of tensor decomposition, such as Tensor-Train Decomposition (Oseledets 2011) and Tucker Decomposition (Tucker 1963) to perform the transition models factorization, aiming to improve runtime on GPUs. We also intend to generalize the CP-MDP implementation to solve several types of problems already described in Relational Dynamic Influence Diagram Language (RDDL) (Sanner 2010), aiming to compare our approach against the state-of-art of factored MDPs solvers for different problems.

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\(^5\)https://www.tensorflow.org/
\(^6\)https://pytorch.org/
\(^7\)https://numba.pydata.org/
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