On matrices whose exponential is a P-matrix

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Abstract

A matrix is called a P-matrix if all its principal minors are positive. P-matrices have found important applications in functional analysis, mathematical programming, and dynamical systems theory. We introduce a new class of real matrices denoted $\mathcal{EP}$. A matrix is in $\mathcal{EP}$ if and only if its matrix exponential is a P-matrix for all positive times. In other words, $A \in \mathcal{EP}$ if and only if the transition matrix of the linear system $\dot{x} = Ax$ is a P-matrix for any positive time $t$. We analyze the properties of this new class of matrices and describe an application of our theoretical results to opinion dynamics.

Index Terms

P-matrices, matrix exponential, compound matrices, totally non-negative matrices, linear dynamical systems, consensus algorithms.

I. INTRODUCTION

A matrix $A \in \mathbb{R}^{n \times n}$ is called a P-matrix if every principal minor of $A$ is positive. In particular, the diagonal entries of $A$ and the determinant of $A$ are positive. The class of P-matrices, denoted $\mathcal{P}$, includes important matrix classes such as positive-definite matrices, non-singular M-matrices [25, Chapter 2], B-matrices [44], totally positive matrices [45], [13], and diagonally dominant matrices with positive diagonal entries.

Fiedler and Ptak [15] presented the first systematic study of P-matrices. These matrices have found many applications in economics [42], dynamical systems [51], and mathematical programming [41]. For a survey on P-matrices, see [28, Ch. 4].

We briefly review some of these applications. P-matrices have been used to analyze the injectivity of nonlinear mappings. Consider a $C^1$ mapping $f : \Omega \to \mathbb{R}^n$ with $\Omega \subseteq \mathbb{R}^n$. Let $J(x) := \frac{\partial}{\partial x} f(x)$ denote the Jacobian of $f$. It is natural to speculate that if $\det(J(x)) \neq 0$ for all $x \in \mathbb{R}^n$ then $f$ is injective. But this is not so in general. Gale and Nikaido [17] proved that if $J(x) \in \mathcal{P}$ for all $x \in \Omega$ and $\Omega$ is a rectangle then $f$ is injective in $\Omega$. For generalizations of the Gale and Nikaido theorem, see e.g., [36], [19].

Another important application of P-matrices is in the field of mathematical programming, in particular, the linear complementarity problem (LCP), which is a generalization of both linear programming and quadratic programming. For a comprehensive treatment of the LCP and, in particular, its numerous applications, see [9]. Some recent results on the relations between LCP and totally positive matrices are given in [6]. Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the LCP($q, M$) is: find (or conclude there is no) $z \geq 0$ such that

\begin{align}
    w &:= Mz + q \geq 0, \\
    w^Tz &= 0.
\end{align}

Condition (1) is called the complementarity condition, since it implies that if $z_i > 0$ [\(w_i > 0\)] for some index $i$ then $w_i = 0$ [$z_i = 0$]. The LCP admits a unique solution for every $q \in \mathbb{R}^n$ if and only if (iff) $M \in \mathcal{P}$ [40].

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P-matrices have also found applications in dynamical systems theory. We discuss their role in a specific model that highlights an intuitive interpretation of $P$-matrices. A fundamental model in mathematical ecology is the Lotka-Volterra equations \([24\text{, Part 5]}\):

$$
\dot{x}_i = x_i(b_i + \sum_{j=1}^{n} a_{ij}x_j), \quad i = 1, \ldots, n.
$$

(2)

Here $x_i(t)$ is the biomass of species $i$ at time $t$, $b_i$ is the growth rate of species $i$, and $a_{ij}$ describes the interconnection between species $j$ and species $i$. It is clear that the non-negative orthant $\mathbb{R}_+^n$ is an invariant set of the dynamics, and we assume throughout that $x(0) \in \mathbb{R}_+^n$. Let $A := (a_{ij})_{i,j=1}^{n}$ and $b := [b_1 \ldots b_n]^T$. Any equilibrium point $e$ of (2) satisfies

$$
\text{diag}(e_1, \ldots, e_n)(b + Ae) = 0,
$$

(3)

where $\text{diag}(c_1, \ldots, c_n)$ is the $n \times n$ diagonal matrix with diagonal entries $c_1, \ldots, c_n$. The Jacobian of the vector field in (2) is

$$
J(x) = \text{diag}(g_1(x), \ldots, g_n(x)) + \text{diag}(x_1, \ldots, x_n)A,
$$

(4)

where

$$
g_i(x) := b_i + \sum_{j=1}^{n} a_{ij}x_j.
$$

(5)

If $e$ is an equilibrium of (2), and $e_i = 0$ for some index $i$ then (3), (4), and (5) imply that

$$
(\zeta^i)^T J(e) = g_i(e)(\zeta^i)^T,
$$

where $\zeta^i$ is the $i$th canonical vector in $\mathbb{R}^n$. Thus, a necessary condition for the stability of $e$ is that

$$
g_i(e) = b_i + \sum_{j=1}^{n} a_{ij}e_j \leq 0,
$$

(6)

for any $i$ such that $e_i = 0$. Of course, (6) also holds (with an equality) for any $i$ such that $e_i > 0$. Consider the $LCP(-b, -A)$, that is, find a vector $y \geq 0$ such that $-Ay - b \geq 0$ and $y^T(-Ay - b) = 0$. Then any stable equilibrium point $e$ of (2) is a solution of this $LCP$. If $(-A) \in \mathbb{P}$ then the LCP admits a unique solution for any $b$, so (2) has no more than a single stable equilibrium, for any $b$.

Here, we introduce and analyze a new class of matrices: we call a matrix $A \in \mathbb{R}^{n \times n}$ an exponential $P$-matrix if $\exp(AT)$ is a $P$-matrix for all $t \geq 0$. We denote this class of matrices by $\mathbb{EP}$. To the best of our knowledge, this type of matrices has not been studied before. Our work is motivated in part by the work of Binyamin Schwarz [48] on totally positive differential systems (TPDSs). Recall that a matrix is called totally positive (TP) if all its minors are positive [45], [12], [18]. Schwarz considered the linear time-varying system

$$
\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0,
$$

(7)

where $t \to A(t)$ is continuous. The solution of this system is $x(t) = \Phi(t,t_0)x_0$ where $\Phi(t,t_0)$, the transition matrix from $t_0$ to $t$, is the solution at time $t$ of the matrix differential equation

$$
\frac{d}{ds} \Phi(s, t_0) = A(s)\Phi(s, t_0), \quad \Phi(t_0, t_0) = I.
$$

System (7) is called a TPDS if $\Phi(t, t_0)$ is TP for any pair $t_0, t$ with $t > t_0 \geq 0$. Schwarz showed that if (7) is a TPDS then the variation diminishing property of TP matrices [18] can be used to analyze the asymptotic behaviour of $x(t)$. If $A(t) \equiv A$ then (7) is a TPDS iff $A$ is Jacobi, that is, $A$ is tri-diagonal.
with positive entries on the super- and sub-diagonals. In other words,\[
\exp(At) \text{ is TP for all } t > 0 \text{ iff } A \text{ is Jacobi.}
\]
It was recently shown [54] that TPDSs have important applications in the asymptotic analysis of the time-varying nonlinear dynamical systems in the form \( \dot{x} = f(t, x) \) whose Jacobian \( J(t, x) := \frac{\partial f(t, x)}{\partial x} \) is a Jacobi matrix for all \( t, x \). See also [30] for the analysis of discrete-time totally positive dynamical systems. The analysis of such systems builds on the use of the multiplicative- and additive-compounds of \( J \) [3].

Note that if \( \Phi(t, t_0) \) is TP then in particular it is a P-matrix. Thus, if \( A \) is Jacobi then \( A \in \text{EP} \). However, the analysis of matrices whose exponential is a P-matrix seems to be more complicated than that of matrices whose exponential is TP due to the fact that TP matrices are closed under multiplication, whereas P-matrices are not. Another important difference is that the class of P-matrices is closed under matrix inversion, whereas the class of TP matrices is not.

The contributions in this paper include the following. We introduce the new class of matrices \( \text{EP} \), provide various conditions for a matrix to be in \( \text{EP} \), describe transformations that preserve \( \text{EP} \), describe an application to consensus systems, and analyze existing consensus algorithms in this new framework.

The remainder of this paper is organized as follows. The next section reviews some results on P-matrices that are used later on. Section III describes our main results. Section IV describes an application of our theoretical results in the context of consensus algorithms that are used later on. Section III describes our main results. Section IV describes an application to consensus systems, and analyze existing consensus algorithms in this new framework.

**Notation.** Small [capital] letters denote column vectors [matrices]. For \( A, B \in \mathbb{R}^{n \times m} \), the notation \( A \geq B \ [A \gg B] \) implies that \( a_{ij} \geq b_{ij} \ [a_{ij} > b_{ij}] \) for all \( i,j \). We call \( A \in \mathbb{R}^{n \times m} \) a non-negative [positive] matrix if \( A \geq 0 \ [A \gg 0] \). The non-negative orthant in \( \mathbb{R}^n \) is \( \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n \mid x \geq 0 \} \).

Consider a matrix \( A \in \mathbb{R}^{n \times m} \) and fix an integer \( k \in \{1, \ldots, \min\{n,m\} \} \). Let \( Q^{k,\ell} \) denote the set of increasing sequences of \( k \) numbers from \( \{1, \ldots, \ell\} \) ordered lexicographically. For example,

\[
Q^{2,3} = \{ \{1,2\}, \{1,3\}, \{2,3\} \}.
\]

A \( k \)-minor of \( A \) is the determinant of some \( k \times k \) submatrix of \( A \). Each such submatrix is defined by a set of row indices \( \alpha \in Q^{k,n} \) and column indices \( \beta \in Q^{k,m} \). This submatrix is denoted by \( A[\alpha|\beta] \), and the corresponding minor is

\[
A(\alpha|\beta) := \det(A[\alpha|\beta]).
\]

In particular, \( A(\alpha|\beta) \) is called a principle minor if \( \alpha = \beta \), and a leading principle minor if \( \alpha = \beta = \{1,2,\ldots,j\} \), for some \( j \geq 1 \). Similarly, \( A[\alpha|\alpha] \) is called a principal submatrix of \( A \). For \( \alpha \in Q^{k,n} \), let \( \bar{\alpha} := \{1,\ldots,n\} \setminus \alpha \) (we use set notation here, but we always assume that the elements in \( \bar{\alpha} \) are ordered in increasing order).

The \( k \)-th multiplicative compound matrix of \( A \in \mathbb{R}^{n \times m} \), denoted \( A^{(k)} \), is the \( \binom{n}{k} \times \binom{m}{k} \) matrix that includes all the \( k \)-minors ordered lexicographically. For example, if \( n = m = 3 \) and \( k = 2 \) then

\[
A^{(2)} = \begin{bmatrix}
A(\{1\},\{2\}) & A(\{1\},\{3\}) & A(\{2\},\{3\}) \\
A(\{1\},\{2\}) & A(\{1\},\{3\}) & A(\{2\},\{3\}) \\
A(\{2\},\{3\}) & A(\{2\},\{3\}) & A(\{2\},\{3\})
\end{bmatrix}.
\]

Note that, by definition, \( A^{(1)} = A \) and if \( n = m \) then \( A^{(n)} = \det(A) \). The \( k \)-th additive compound matrix of \( A \in \mathbb{R}^{n \times n} \) is

\[
A^{(k)} := \frac{d}{d\varepsilon}(I + \varepsilon A)^{(k)} |_{\varepsilon=0}.
\]

Note that this implies that \( A^{(k)} = \frac{d}{d\varepsilon} (\exp(A\varepsilon))^{(k)} |_{\varepsilon=0} \). There is an explicit formula for the entries of \( A^{(k)} \) in terms of the entries of \( A \) (see e.g. [48]). See [39], [55] for more information on compound matrices.
and their applications to dynamical systems described by ordinary differential equations. See also [3] for a recent tutorial on applications of compound matrices in systems and control theory.

A matrix \( A \in \mathbb{R}^{n \times n} \) is called totally positive (TP) [totally non-negative (TN)] if \( A^{(k)} \geq 0 \) for all \( k \in \{1, \ldots, n\} \). A matrix \( A \in \mathbb{R}^{n \times n} \) is called sign-symmetric if for any \( k \in \{1, \ldots, n\} \) and any \( \alpha, \beta \in \mathbb{Q}^{k,n} \) we have \( A(\alpha|\beta)A(\beta|\alpha) \geq 0 \). We say that \( A \in \mathbb{R}^{n \times n} \) is sign-pattern symmetric if \( a_{ij}a_{ji} \geq 0 \) for all \( i, j \in \{1, \ldots, n\} \). Then \( A \) is sign-symmetric iff \( A^{(k)} \) is sign-pattern symmetric for all \( k \in \{1, \ldots, n-1\} \).

II. PRELIMINARIES: P-matrices

A matrix \( A \in \mathbb{R}^{n \times n} \) is called a P-matrix if all its principal minors are positive, that is, \( A^{(k)} \) has positive diagonal entries for all \( k \in \{1, \ldots, n\} \). We list some transformations that preserve the P-matrix property. Recall that \( S \in \mathbb{R}^{n \times n} \) is called a signature matrix if it is a diagonal matrix, and every diagonal entry is either one or minus one. Then \( S^{-1} = S \). A matrix \( V \in \mathbb{R}^{n \times n} \) is called a permutation matrix if it has exactly one entry of 1 in each row and each column, and zeros elsewhere. Then \( V^{-1} = V^T \).

**Theorem 1.** [52] Suppose that \( A \in \mathbb{R}^{n \times n} \) is a P-matrix. Then

1) \( A^T \in \mathbb{P} \).
2) If \( V \) is a permutation matrix then \( VAV^T \in \mathbb{P} \).
3) If \( D_1, D_2 \) are diagonal matrices and \( D_1D_2 \) has positive diagonal entries then \( D_1AD_2 \in \mathbb{P} \).
4) If \( S \) is a signature matrix then \( SAS \in \mathbb{P} \).
5) If \( D \) is a non-negative diagonal matrix then \( A + D \in \mathbb{P} \).
6) If \( D \) is a diagonal matrix with \( 0 \leq D \leq I \) then \( D + (I - D)A \in \mathbb{P} \).
7) \( A^{-1} \in \mathbb{P} \).

Since \( \exp(At) = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} \), \( \exp(At) \in \mathbb{P} \) for all \( t > 0 \) sufficiently small. The next result provides more information on the value \( t \) for which \( \exp(At) \in \mathbb{P} \) when \( A \) has non-negative entries.

**Theorem 2.** [52] If \( A \geq 0 \) then \( \exp(At) \in \mathbb{P} \) for all \( t \in [0, \frac{1}{\rho(A)}] \), where \( \rho(A) \) is the spectral radius of \( A \).

The next result summarizes several necessary and sufficient conditions for a matrix to be a P-matrix.

**Theorem 3.** [15], [38] Let \( A \in \mathbb{R}^{n \times n} \). The following six conditions are equivalent.

1) Any principal minor of \( A \) is positive, that is, \( A \in \mathbb{P} \);
2) For any \( x \in \mathbb{R}^n \setminus \{0\} \) there exists an index \( k \in \{1, \ldots, n\} \) such that \( x_k(Ax)_k > 0 \);
3) For any \( x \in \mathbb{R}^n \setminus \{0\} \) there exists a positive diagonal matrix \( D(x) \) such that \( x^TD(x)Ax > 0 \);
4) For any \( x \in \mathbb{R}^n \setminus \{0\} \) there exists a non-negative diagonal matrix \( H(x) \) such that \( x^TH(x)Ax > 0 \);
5) Every real eigenvalue of \( A \) and of any principal submatrix of \( A \) is positive;
6) For any signature matrix \( S \) there exists a vector \( x \gg 0 \) such that \( SASx \gg 0 \).

**Remark 1.** For \( A \in \mathbb{R}^{n \times n} \), let

\[
\text{rev}(A) := \{ x \in \mathbb{R}^n : x_i(Ax)_i \leq 0 \text{ for all } i \in \{1, \ldots, n\} \}.
\]

Roughly speaking, this is the set of vectors that \( A \) “sign reverses”. Condition 2 in Theorem 3 implies that \( A \) is a P-matrix iff \( \text{rev}(A) = \{0\} \). This is known as the sign non-reversal property (see, e.g., [7]).

**Remark 2.** Recall that \( A \in \mathbb{R}^{n \times n} \) is called stable (or Hurwitz) if \( \text{Re}(\lambda) < 0 \) for any eigenvalue \( \lambda \) of \( A \). It is called diagonally stable if there exists a positive diagonal matrix \( D \) such that \( DA + A^TD \) is negative-definite [29]. Condition 3 in Theorem 3 implies that if \( A \) is diagonally stable then \( -A \in \mathbb{P} \). Recall [46] that if \( A \) is Hurwitz and Metzler then it is diagonally stable, so \( -A \in \mathbb{P} \) (see [11] for an application of this property to determining box invariance of dynamical systems). The class of matrices

\[
\{ A \in \mathbb{R}^{n \times n} : -A \in \mathbb{P} \}
\]
is sometimes denoted by \( \mathbb{P}^{(-)} \) (see [2]).

Recall that \( A \in \mathbb{R}^{n \times n} \) is called a Q matrix if for any \( k \in \{1, \ldots, n\} \) the sum of all the \( k \) principal minors of \( A \) is positive. In particular, any P matrix is a Q matrix. The next result provides a necessary spectral condition for \( A \in \mathbb{R}^{n \times n} \) to be a Q-matrix. For \( z \in \mathbb{C} \), let \( \arg(z) \in (-\pi, \pi] \) denote the argument of \( z \).

**Theorem 4.** [31] Let \( A \in \mathbb{R}^{n \times n} \), with \( n \geq 2 \), be a Q matrix (and thus a P matrix). Then any eigenvalue \( \lambda \) of \( A \) satisfies

\[
|\arg(\lambda)| < \pi - \frac{\pi}{n}.
\]

In other words, there is a wedge around the negative \( x \)-axis which is free from eigenvalues of \( A \). In particular, a P-matrix cannot have a real and negative eigenvalue (the latter is also implied by Condition 2 in Theorem 3).

It is well-known that the problem of testing whether a given \( n \times n \) matrix is a P-matrix, formulated as a decision problem, is co-NP-complete in \( n \) [10].

The next section describes our main results.

### III. MAIN RESULTS

We begin by introducing a new class of matrices that we call *exponential P-matrices*.

**Definition 1.** A matrix \( A \in \mathbb{R}^{n \times n} \) is called an EP-matrix if

\[
\exp(At) \in \mathbb{P} \text{ for all } t \geq 0.
\]

Let \( \mathbb{E}\mathbb{P} \) denote the class of exponential \( P \) matrices.

One motivation for the notion of EP-matrices is a dynamical systems generalization of the sign-reversal property described in Remark 1. Indeed, if \( \dot{x} = Ax \) then the following two properties are equivalent:

1. \( A \) is an EP-matrix;
2. for any \( x(0) \in \mathbb{R}^n \setminus \{0\} \) and any time \( t \geq 0 \) there exists at least one index \( i = i(t, x(0)) \) such that

\[
x_i(0)x_i(t) > 0.
\]

In other words, at each time \( t \) there is at least one state-variable that has the same sign at time \( t \) as in time \( 0 \). We will describe an application of this property to opinion dynamics in Section IV.

Definition 1 only considers non-negative times. However, using Condition 7 in Theorem 1 yields the following result.

**Corollary 1.** A matrix \( A \in \mathbb{R}^{n \times n} \) is an EP-matrix iff

\[
\exp(At) \in \mathbb{P} \text{ for all } t \in \mathbb{R}.
\]

**Proof:** If (10) holds then clearly \( A \in \mathbb{E}\mathbb{P} \). Now suppose that \( A \in \mathbb{E}\mathbb{P} \), that is, \( \exp(At) \in \mathbb{P} \) for any \( t \geq 0 \). Fix \( s < 0 \). Then \( \exp(sA) = (\exp(-sA))^{-1} \), and since \((-s) > 0\) and the inverse of a P-matrix is a P-matrix, we conclude that \( \exp(sA) \in \mathbb{P} \).

**Example 1.** Consider \( A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix} \). Note that \((-A) \) is Jacobi. A calculation gives

\[
\exp(At) = \begin{bmatrix} \cosh^2(t) & -\sinh(t)\cosh(t) & \sinh^2(t) \\ -\sinh(2t) & \cosh(2t) & -\sinh(2t) \\ \sinh^2(t) & -\sinh(t)\cosh(t) & \cosh^2(t) \end{bmatrix},
\]

and
\[
(\exp(At))^{(2)} = \begin{bmatrix}
\cosh^2(t) & -\sinh(2t) & \sinh^2(t) \\
-\sinh(t) \cosh(t) & \cosh(2t) & -\sinh(t) \cosh(t) \\
\sinh^2(t) & -\sinh(2t) & \cosh^2(t)
\end{bmatrix}.
\]

The diagonal entries of \((\exp(At))^{(k)}\) are the principal minors of order \(k\) of \(\exp(At)\). We see that the principal minors of order one and two of \(\exp(At)\) are positive for all \(t \geq 0\), so \(A \in \mathbb{EP}\). Note that the principal minors are positive for any \(t \leq 0\) as well. However, some non-principal minors of \(\exp(At)\) may take negative values, as \(A\) is not Jacobi. \(\Box\)

More generally, since \(\exp(-At)\) is the inverse of \(\exp(At)\), it follows from Jacobi’s identity \([26, \text{Chapter } 0]\) that for any \(k \in \{1, \ldots, n-1\}\) and any \(\alpha \in Q^{k,n}\), we have
\[
(\exp(-At))(\alpha|\alpha) = \frac{(\exp(At))(\bar{\alpha}|\bar{\alpha})}{\det(\exp(At))},
\]
where \(\bar{\alpha} := \{1, \ldots, n\} \setminus \alpha\). This shows, in particular, that \(A \in \mathbb{EP}\) iff \(-A \in \mathbb{EP}\).

The next result lists some transformations that preserve the EP-property.

**Theorem 5.** Suppose that \(A \in \mathbb{R}^{n \times n}\) is an EP-matrix. Then

1) \(cA \in \mathbb{EP}\) for any \(c \in \mathbb{R}\).
2) \(A^T \in \mathbb{EP}\).
3) If \(Q\) is a permutation matrix then \(QAQ^T \in \mathbb{EP}\).
4) If \(D\) is a positive diagonal matrix then \(DAD^{-1} \in \mathbb{EP}\).
5) If \(S\) is a signature matrix then \(SAS \in \mathbb{EP}\).
6) If \(D\) is a diagonal matrix and \(DA = AD\) then \(A + D \in \mathbb{EP}\).

**Proof:** To prove 1), fix \(c \in \mathbb{R}\). Then \(\exp((cA)t) = \exp(A(ct))\), and since \(A \in \mathbb{EP}\), \(\exp(A(ct)) \in \mathbb{P}\). The proof of 2) follows from the fact that \(\exp(A^Tt) = (\exp(At))^T\). To prove 3), note that if \(Q\) is a permutation matrix then
\[
\exp(QAQ^Tt) = Q \exp(At)Q^T,
\]
and since \(\exp(At) \in \mathbb{P}\), this implies that \(QAQ^T \in \mathbb{EP}\). The proofs of 4) and 5) are similar to the proof of 3). To prove 6), let \(B := A + D\). Then \(\exp(Bt) = \exp(Dt)\exp(At)\). Since \(\exp(Dt)\) is a positive diagonal matrix, and \(\exp(At) \in \mathbb{P}\), we conclude from Condition 5) in Theorem 1 that \(\exp(Bt) \in \mathbb{P}\). \(\square\)

The following example shows that the EP property is in general not invariant under similarity transformations.

**Example 2.** The matrix \(A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\) is EP, as \(\exp(At) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\), but for \(T = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\) we have
\[
B := TAT^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}
\]
and \(B\) is not EP, as
\[
\exp(Bt) = T \exp(At)T^{-1} = \begin{bmatrix} 1 + t & t \\ -t & 1 - t \end{bmatrix}
\]
and for any \(t \geq 1\) this matrix has a non-positive diagonal entry, so it is not a \(P\) matrix. \(\square\)

**A. Some classes of EP-matrices**

In general, it seems difficult to determine if a given matrix is \(\mathbb{EP}\). However, we show that some specific classes of matrices are \(\mathbb{EP}\). We first study the case of \(2 \times 2\) matrices where it is possible to give a necessary and sufficient sign-pattern condition for \(\mathbb{EP}\).
**Proposition 1.** Suppose that $A \in \mathbb{R}^{2 \times 2}$. Then $A \in \mathbb{EP}$ iff $a_{12}a_{21} \geq 0$.

In other words, $A \in \mathbb{EP}$ iff $A$ has the sign pattern

$$
\begin{bmatrix}
* & \geq 0 \\
\geq 0 & *
\end{bmatrix} \quad \text{or} \quad 
\begin{bmatrix}
* & \leq 0 \\
\leq 0 & *
\end{bmatrix},
$$

where * denotes “don’t care”, that is, iff $A \in \mathbb{R}^{2 \times 2}$ sign-pattern symmetric. Note that the first [second] sign pattern here corresponds to the linear dynamical system $\dot{x} = Ax$ being a cooperative [competitive] system [49].

**Proof:** If $a_{12}a_{21} = 0$ then $A$ is triangular and so is $\exp(At)$. In this case, the diagonal entries of $\exp(At)$ are $\exp(a_{ii}t)$, $i = 1, 2$. Hence, $A \in \mathbb{EP}$.

If $a_{12} > 0$ and $a_{21} > 0$ then $A$ is Jacobi, so $A \in \mathbb{EP}$. If $a_{12} < 0$ and $a_{21} < 0$, then $-A \in \mathbb{EP}$ and thus $A \in \mathbb{EP}$.

To complete the proof, we need to show that if

$$a_{12}a_{21} < 0$$

then $A \notin \mathbb{EP}$.

Let $B(t) := \exp(At)$. Since $A \in \mathbb{R}^{2 \times 2}$, $B(t)$ is a P-matrix iff its diagonal entries $b_{11}(t)$ and $b_{22}(t)$ are positive. Let $s := \text{trace}(A)/2$, and write $A = sI + (A - sI)$. Note that $\text{trace}(A - sI) = 0$. Then

$$B(t) = \exp(st) \exp((A - sI)t).$$

The term $\exp(st)$ has no effect on the signs of $b_{11}(t)$, $b_{22}(t)$, so we may assume that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}.$$

The eigenvalues of $A$ are $\pm \sqrt{a_{11}^2 + a_{12}a_{21}}$. Assume that (12) holds. We consider three cases.

**Case 1.** Suppose that $a_{12}a_{21} < -a_{11}^2$. In this case, the eigenvalues of $A$ are purely imaginary and Theorem 7 implies that $A \notin \mathbb{EP}$.

**Case 2.** Suppose that $a_{12}a_{21} = -a_{11}^2$. Then $\det(A) = \text{trace}(A) = 0$, so $A^2 = 0$ and thus

$$B(t) = I + At = \begin{bmatrix} 1 + a_{11}t & a_{12}t \\ a_{21}t & 1 - a_{11}t \end{bmatrix}.$$

Since $a_{12}a_{21} < 0$, $a_{11} \neq 0$, and we conclude that $B(t) \notin \mathbb{P}$ for any $t \geq 0$ sufficiently large, so $A \notin \mathbb{EP}$.

**Case 3.** Suppose that $a_{12}a_{21} > -a_{11}^2$. Then $\det(A) = -a_{11}^2 - a_{12}a_{21} < 0$, and the diagonal entries of $B(t)$ are

\begin{align*}
    b_{11}(t) &= \cosh(st) \left( 1 + \frac{a_{11}}{s} \tanh(st) \right), \\
    b_{22}(t) &= \cosh(st) \left( 1 - \frac{a_{11}}{s} \tanh(st) \right),
\end{align*}

where $s := \sqrt{-\det(A)}$. Since $a_{12}a_{21} < 0$, $|a_{11}| > 1$, so $B(t) \notin \mathbb{P}$ for all $t > 0$ sufficiently large, so $A \notin \mathbb{EP}$.

**Example 3.** Consider the matrix $A(w) := \begin{bmatrix} -1 & w \\ -w & -1 \end{bmatrix}$, with $w > 0$. Then

$$\exp(At) = \exp(-t) \begin{bmatrix} \cos(wt) & \sin(wt) \\ -\sin(wt) & \cos(wt) \end{bmatrix},$$
so \(\exp(At) \in \mathbb{P}\) for \(t \in [0, \frac{n}{2w}]\), but

\[
\exp\left(A\frac{\pi}{2w}\right) = \exp\left(-\frac{\pi}{2w}\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \not\in \mathbb{P},
\]

so \(A \not\in \mathbb{EP}\). Note that when \(w \to 0\), the \(L_2\) norm of \(A\) converges to one, yet the first time where we “loose” the \(\mathbb{EP}\) property, namely, \(\frac{n}{2w}\) converges to infinity. \(\square\)

The next result describes several classes of matrices that are \(\mathbb{EP}\).

**Theorem 6.** Let \(A \in \mathbb{R}^{n \times n}\). Any one of the following three conditions implies that \(A \in \mathbb{EP}\).

1. \(A\) is lower triangular or upper triangular.
2. \(A\) is symmetric.
3. \(A\) is Jacobi.

**Proof:** Pick an arbitrary \(t \geq 0\) and let \(B := \exp(At)\). We will show that if any one of the conditions in the theorem holds then \(B \in \mathbb{P}\).

To prove 1), note that the (lower or upper) triangular structure is preserved under summation and multiplication. Hence, \(B = \sum_{\ell=0}^{\infty} A^\ell t^\ell\) is also a triangular matrix. Since the eigenvalues of a triangular matrix are its diagonal entries, the diagonal entries of \(B\) are \(\exp(a_{ii}t)\), \(i = 1, \ldots, n\). Pick \(k \in \{1, \ldots, n\}\) and \(\alpha \in \mathbb{Q}^{k,n}\). Then \(B(\alpha\alpha) = \prod_{i \in \alpha} \exp(a_{ii}t) > 0\), so \(B \in \mathbb{P}\).

To prove 2), suppose that \(A\) is symmetric. Then all the eigenvalues \(\lambda_i\) of \(A\) are real, \(B\) is symmetric and the eigenvalues of \(B\) are \(\exp(\lambda_i t) > 0\), \(i = 1, \ldots, n\). Thus, \(B\) is positive-definite, and by Theorem 3 \(B \in \mathbb{P}\).

To prove 3), suppose that \(A\) is Jacobi. Then \(\exp(At)\) is TP for all \(t > 0\), and in particular \(A \in \mathbb{EP}\). Here, we also give a more direct proof. Since \(A\) is Jacobi, it has the form

\[
A = \begin{bmatrix}
    a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 \\
    0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\
    0 & 0 & 0 & 0 & \cdots & 0 & c_{n-1} & a_n
\end{bmatrix},
\]

with \(b_i, c_i > 0, i = 1, \ldots, n-1\). Consider the positive diagonal matrix \(D := \text{diag}(1, \sqrt{\frac{b_1}{c_1}}, \ldots, \sqrt{\frac{b_1b_2\cdots b_{n-1}}{c_1c_2\cdots c_{n-1}}} )\). Then \(DAD^{-1}\) is symmetric. Hence, \(\exp(DAD^{-1}t) = D\exp(At)D^{-1} \in \mathbb{P}\). Theorem 1 implies that \(\exp(At) \in \mathbb{P}\).

The three classes of matrices in Theorem 6 all have real eigenvalues. One may conjecture that if all the eigenvalues of \(A \in \mathbb{R}^{n \times n}\) are real then \(A \in \mathbb{EP}\). However, the matrix \(B\) in Example 2 above shows that this conjecture is false.

The next result shows that having all real eigenvalues is a necessary condition for EP-matrices.

**Theorem 7.** If \(A \in \mathbb{R}^{n \times n}\) has a complex (non-real) eigenvalue then \(A \not\in \mathbb{EP}\).

**Proof:** Let \(\lambda = a + jb\), where \(b \neq 0\) and \(j := \sqrt{-1}\), be a complex eigenvalue of \(A\). Since the complex eigenvalues of a real matrix always occur in conjugate pairs, we can assume that \(b > 0\). Now, \(\exp(At)\) has the eigenvalue

\[
\exp(\lambda t) = \exp(at) \exp(jbt).
\]

For \(t = \frac{(n-1)\pi}{nb} > 0\), we have \(\arg(\exp(\lambda t)) = \pi - \frac{\pi}{n}\), and Theorem 4 implies that \(\exp(At) \not\in \mathbb{P}\), so \(A \not\in \mathbb{EP}\). \(\square\)
Remark 3. An immediate implication of Theorem 7 is that if $A \in \mathbb{R}^{n \times n}$ is skew-symmetric then it is not an EP-matrix. If $A \in \mathbb{EP}$, then Theorem 7 also implies that the linear dynamical system $\dot{x} = Ax$ does not have limit cycles.

The following result provides a necessary and sufficient condition for EP in terms of the diagonal entries of additive compound matrices. Recall that if $A \in \mathbb{R}^{n \times n}$ then $A[k] \in \mathbb{R}^{r \times r}$, with $r := \binom{n}{k}$.

Proposition 2. Let $A \in \mathbb{R}^{n \times n}$. Then $A \in \mathbb{EP}$ iff for any $t \geq 0$ and any $k \in \{1, \ldots, n-1\}$, we have

$$(\exp(A[k]t))_{ii} > 0, \text{ for all } i \in \{1, \ldots, \binom{n}{k}\}.$$

Proof: Recall (see e.g. [39]) that $$(\exp(A))^{(k)} = \exp(A[k]).$$

Thus, if $\alpha$ is the $i$th element in $Q^k,n$ then the corresponding principal minor of $\exp(At)$ satisfies $$(\exp(A))(\alpha|\alpha) = (\exp(A[k]))_{ii},$$

and this completes the proof. ■

The next result provides a sufficient condition for EP based on sign-symmetry of $\exp(At)$.

Proposition 3. Suppose that $A \in \mathbb{R}^{n \times n}$ has only real eigenvalues and that $\exp(At)$ is sign-symmetric for all $t \geq 0$.

Then $A \in \mathbb{EP}$.

Note that by Theorem 7 the assumption that $A$ has only real eigenvalues is a necessary condition for $\mathbb{EP}$.

Proof: Fix $t \geq 0$. Since $A$ has real eigenvalues, $\exp(At)$ has real and positive eigenvalues. In particular $\exp(At)$ is positively stable (i.e., $(-\exp(At))$ is a Hurwitz matrix). Recall that for the class of sign-symmetric matrices, positivity of principal minors and positive stability are equivalent (see, e.g., [23]), so we conclude that $\exp(At) \in \mathbb{P}$. ■

Remark 4. Fix $k \in \{1, \ldots, n\}$. It follows from (8) that

$$(\exp(At))^{(k)} = I_r + tA[k] + o(t),$$

with $r := \binom{n}{k}$, so if condition (13) holds then $I_r + tA[k]$ is sign-pattern symmetric for any $t > 0$ sufficiently small, that is, $A[k]$ is sign-pattern symmetric.

Prop. 3 suggests the following question: suppose that $A \in \mathbb{R}^{n \times n}$ has only real eigenvalues and is sign-symmetric. Is $\exp(At)$ sign-symmetric for all $t \geq 0$? The following example shows that in general the answer is no.

Example 4. Consider the matrix $A = \begin{bmatrix} -1/2 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 0 & 20 \end{bmatrix}$. It is straightforward to verify that: $A$ has three real eigenvalues, $A$ is sign-symmetric, but $\exp(0.1A)$ is not sign-symmetric. □

There are however special cases where sign-symmetry of $A$ does allow us to apply Prop. 3. Recall that a matrix is called $P_0$ if all its principal minors are non-negative (see, e.g., [23]).

Corollary 2. Suppose that $A \in \mathbb{R}^{n \times n}$ is sign-symmetric, and is a projection matrix (i.e., $A^2 = A$). Then $A \in \mathbb{EP}$. 
Note that we do not require here explicitly that $A$ has only real eigenvalues. Indeed, since $A$ is a projection matrix, all its eigenvalue either one or zero.

Proof: Fix $t \geq 0$, and let $B := \exp(At) - I$. Since $A^2 = A$,

$$B = \sum_{k=1}^{\infty} \frac{A^k}{k!} = (\exp(t) - 1)A.$$ 

Since $A$ is sign-symmetric, $A^2$ is a $P_0$-matrix [23 Lemma 2.1], and since $A^2 = A$, $A$ is a $P_0$-matrix. Hence, $B$ is also a $P_0$-matrix. Now [14, Theorem 1.5] implies that $\exp(At) = I + B$ is a P-matrix. □

Example 5. Consider the matrix $A = \frac{1}{4} \begin{bmatrix} 4 & -3 & -3 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$. This is a sign-symmetric projection matrix.

A calculation gives

$$\exp(At) = \frac{1}{4} \begin{bmatrix} 4 \exp(t) & 3 - 3 \exp(t) & 3 - 3 \exp(t) \\ 0 & 3 + \exp(t) & 3 - 3 \exp(t) \\ 0 & 1 - \exp(t) & 1 + 3 \exp(t) \end{bmatrix},$$

and

$$\left(\exp(At)\right)^{(2)} = \frac{1}{4} \begin{bmatrix} 3 + \exp(t) & * & * \\ * & 1 + 3 \exp(t) & * \\ * & * & 4 \end{bmatrix},$$

where '* denotes values that are not relevant. Since all the diagonal entries of $\exp(At)$ and $\left(\exp(At)\right)^{(2)}$ are positive for all $t$, $A \in \text{EP}$. □

B. Generalizations of Theorem 6

This section provides some generalizations of Theorem 6. Specifically, it is straightforward to verify that the three classes of matrices in Theorem 6 are special cases of Theorem 8, Theorem 9, and Theorem 10, respectively. Furthermore, all the matrix classes in Theorem 6 are included in Theorem 11.

Theorem 8. Suppose that

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where $B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$, and $0$ denotes an $m \times n$ matrix of zeros. Then $A \in \text{EP}$ iff $B, D \in \text{EP}$.

Proof: First, we show $A \in \text{EP}$ implies $B, D \in \text{EP}$. Note that

$$\exp(At) = \begin{bmatrix} \exp(Bt) & * \\ 0 & \exp(Dt) \end{bmatrix},$$

where * denotes “don’t care”. Since $\exp(At)$ is a P-matrix, every principal submatrix of $\exp(At)$ is also a P-matrix. Hence, $B, D \in \text{EP}$.

To prove the converse implication, assume that $B, D \in \text{EP}$. Fix $t \geq 0$ and $v \in \mathbb{R}^{n+m} \setminus \{0\}$. Let $w := \exp(At)v$, i.e. the solution at time $t$ of $\dot{x} = Ax$, $x(0) = v$. By Theorem 3, it is enough to show that there exists a $k \in \{1, \ldots, n + m\}$ such that $w_k v_k > 0$. We consider two cases.

Case 1. Suppose that at least one of $v_{n+1}, \ldots, v_{n+m}$ is non zero. Then

$$\begin{bmatrix} w_{n+1} \\ \vdots \\ w_{n+m} \end{bmatrix} = \exp(Dt) \begin{bmatrix} v_{n+1} \\ \vdots \\ v_{n+m} \end{bmatrix},$$

(15)
and the fact that \( D \in \mathbb{E} \mathbb{P} \) implies that there exists \( i \in \{ n + 1, \ldots, n + m \} \) such that \( w_i v_i > 0 \).

Case 2. Suppose that \( v_{n+1} = \cdots = v_{n+m} = 0 \). Then
\[
\begin{bmatrix}
w_1 \\
\vdots \\
w_n
\end{bmatrix} = \exp(Bt) \begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix},
\]
and since \( v \neq 0 \) and \( B \in \mathbb{E} \mathbb{P} \), there exists \( i \in \{ 1, \ldots, n \} \) such that \( w_i v_i > 0 \). Hence, \( A \in \mathbb{E} \mathbb{P} \).

Remark 5. Any reducible matrix is similar to (14) up to a permutation similarity. Since the EP-property is preserved under permutation similarity (see Theorem 5), Proposition 8 shows that when analyzing EP-matrices it is enough to study irreducible matrices.

Example 6. Let
\[
A = \begin{bmatrix}
-4 & 1 & -6 \\
2 & -5 & 6 \\
0 & 0 & -9
\end{bmatrix}.
\]

Note that \( A \) does not belong to any of the three matrix classes in Theorem 6 but has the form (14) with \( B = \begin{bmatrix}
-4 & 1 \\
2 & -5
\end{bmatrix} \) and \( D = -9 \). Since \( B, D \in \mathbb{E} \mathbb{P} \), so is \( A \). Indeed, a calculation gives that
\[
3 \exp(At) = \begin{bmatrix}
2 \exp(-3t) + \exp(-6t) & * & * \\
* & \exp(-3t) + 2 \exp(-6t) & * \\
* & * & 3 \exp(-9t)
\end{bmatrix},
\]
and that \( 3(\exp(At))^{(2)} \) is equal to
\[
\begin{bmatrix}
3 \exp(-9t) & * & * \\
* & 2 \exp(-12t) + \exp(-15t) & * \\
* & * & \exp(-12t) + 2 \exp(-15t)
\end{bmatrix},
\]
so all the principal minors of \( \exp(At) \) are positive for all \( t \). \( \square \)

The next result provides a sufficient condition for \( A = T D T^{-1} \), with \( D \) a diagonal matrix, to be \( \mathbb{E} \mathbb{P} \). This condition is a kind of symmetry condition for the minors of \( T \). For \( \alpha, \beta \in Q^{k,n} \), let
\[
s(\alpha, \beta) := (-1)^{\sum_{j \in \alpha} j + \sum_{i \in \beta} i},
\]
that is, the signature of \( \alpha + \beta \).

Theorem 9. Suppose that \( A \in \mathbb{R}^{n \times n} \) has real eigenvalues \( \lambda_1, \ldots, \lambda_n \) and is diagonalizable, that is, there exists a non-singular matrix \( T \in \mathbb{R}^{n \times n} \) such that \( A = T D T^{-1} \) with \( D := \text{diag}(\lambda_1, \ldots, \lambda_n) \). If for any \( k \in \{ 1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil \} \) and any \( \alpha \in Q^{k,n} \) we have
\[
\det(T)s(\alpha, \beta)T(\alpha|\beta)T(\bar{\alpha}|\bar{\beta}) \geq 0 \quad \text{for all } \beta \in Q^{k,n},
\]
and this holds with an inequality for at least one \( \beta \in Q^{k,n} \), then \( A \in \mathbb{E} \mathbb{P} \).

Example 7. Let \( T = \begin{bmatrix}
3 & 3 & -3 \\
3 & -1 & -1 \\
1 & 1 & 1
\end{bmatrix} \). Then \( \det(T) = -24 < 0 \). Calculating
\[
r(\alpha, \beta) := s(\alpha, \beta)T(\alpha|\beta)T(\bar{\alpha}|\bar{\beta})
\]
Thus, condition \(18\) holds, and Theorem \(9\) implies that \(TDT^{-1} \in \mathbb{E}_{\mathbb{P}}\) for any diagonal matrix \(D \in \mathbb{R}^{3 \times 3}\).

**Proof of Theorem 9.** Since \(A = TDT^{-1}\), \(\exp(At) = T \exp(Dt)T^{-1}\). Pick \(k \in \{1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\}\) and \(\alpha \in Q^{k,n}\). Then the corresponding principal minor of \(\exp(At)\) satisfies

\[
(\exp(At))(\alpha|\alpha) = \sum_{\beta, \gamma \in Q^{k,n}} T(\alpha|\beta)(\exp(Dt))(\beta|\gamma)T^{-1}(\gamma|\alpha)
\]

\[
= \sum_{\beta \in Q^{k,n}} T(\alpha|\beta)(\exp(Dt))(\beta|\beta)T^{-1}(\beta|\alpha)
\]

\[
= \sum_{\beta \in Q^{k,n}} \left( T(\alpha|\beta)T^{-1}(\beta|\alpha) \prod_{i \in \beta} \exp(\lambda_i t) \right),
\]

where we used the fact that \(D\) is diagonal. By Jacobi’s identity [26, Chapter 0],

\[
T^{-1}(\beta|\alpha) = s(\alpha, \beta)T(\bar{\alpha}|\bar{\beta})/\det(T).
\]

Note that \(s(\alpha, \beta) = s(\bar{\alpha}, \bar{\beta})\). Thus,

\[
(\exp(At))(\alpha|\alpha) = \frac{1}{\det(T)} \sum_{\beta \in Q^{k,n}} \left( s(\alpha, \beta)T(\alpha|\beta)T(\bar{\alpha}|\bar{\beta}) \prod_{i \in \beta} \exp(\lambda_i t) \right),
\]

and \(18\) implies that \((\exp(At))(\alpha|\alpha) > 0\).

**Theorem 10.** Suppose that \(A \in \mathbb{R}^{n \times n}\) satisfies the following property: for any \(t \geq 0\) there exists an integer \(k > 0\) such that

\[
I + \frac{At}{\ell} \text{ is TN for any } \ell \geq k.
\]

Then \(A \in \mathbb{E}_{\mathbb{P}}\).

For example, we say that \(A \in \mathbb{R}^{n \times n}\) is a weak Jacobi matrix if it is tri-diagonal and all the entries on the super- and sub-diagonals are non-negative. If \(A\) is a weak Jacobi matrix then it satisfies the property in Theorem \(10\) (see [12] p. 6), so \(A \in \mathbb{E}_{\mathbb{P}}\). In particular, weak Jacobi matrices are the generators (in the Lie-algebraic sense) of the group of non-singular TN matrices [33].

**Proof:** Fix \(t \geq 0\). Recall that

\[
\exp(At) = \lim_{\ell \to \infty} \left( I + \frac{At}{\ell} \right)^\ell
\]
(see e.g. [20]). For any \( \ell \) sufficiently large, \( I + \frac{At}{\ell} \) is TN, and using the fact that the product of two TN matrices is a TN matrix implies that \( (I + \frac{At}{\ell})^j \) is TN for any \( j \geq 1 \). Let \( B := (I + \frac{At}{\ell})^\ell \). Using the Cauchy-Binet formula gives that any principal minor of \( B \) satisfies

\[
B(\alpha|\alpha) \geq \left( (I + \frac{At}{\ell})(\alpha|\alpha) \right) ^\ell,
\]

so

\[
B(\alpha|\alpha) \geq \left( 1 + \frac{t}{\ell} \sum _{i \in \alpha} a_{ii} + o(\frac{1}{\ell}) \right) ^\ell,
\]

and taking \( \ell \to \infty \) gives

\[
(\exp(At))(\alpha|\alpha) \geq \exp(\frac{t}{\ell} \sum _{i \in \alpha} a_{ii}) > 0,
\]

so \( A \in \mathbb{EP} \).

**Theorem 11.** Let \( A \in \mathbb{R}^{n \times n} \). If there exists an \( \bar{\varepsilon} > 0 \) such that for any \( \varepsilon \in [0, \bar{\varepsilon}] \) and any integer \( \ell > 0 \),

\[
(I + \varepsilon A)^\ell \text{ is sign symmetric.}
\]

Then \( A \in \mathbb{EP} \).

**Proof:** Suppose that \( B \in \mathbb{R}^{n \times n} \) is a P-matrix and that \( B^\ell \) is sign symmetric for any integer \( \ell > 0 \). Pick \( k \in \{1, \ldots, n\} \) and \( \alpha \in \mathcal{Q}^{k,n} \). Then the Cauchy-Binet formula gives:

\[
(B^2)(\alpha|\alpha) = \sum _{\gamma \in \mathcal{Q}^{k,n}} B(\alpha|\gamma) B(\gamma|\alpha) \geq (B(\alpha|\alpha))^2 > 0,
\]

so \( B^2 \) is also a P-matrix and, by assumption, also sign symmetric. Continuing in this fashion implies that

\[
(B^{2\ell})(\alpha|\alpha) \geq (B(\alpha|\alpha))^{2\ell} \text{ for any integer } \ell > 0. \tag{22}
\]

Now suppose that (21) holds. Fix \( t \geq 0 \). Then for any \( \ell > 0 \) sufficiently large the matrix \( B := I + \frac{At}{2^\ell} \) is a P-matrix and (21) implies that \( B^\ell \) is sign-symmetric, so (22) gives

\[
((I + \frac{At}{2^\ell})^{2\ell})(\alpha|\alpha) \geq ((I + \frac{At}{2^\ell})(\alpha|\alpha))^{2\ell} \geq (1 + \frac{t}{2^\ell} \sum _{i \in \alpha} a_{ii} + o(\frac{1}{2^\ell}))^{2\ell}.
\]

Taking \( \ell \to \infty \) gives

\[
(\exp(At))(\alpha|\alpha) \geq \exp(\frac{t}{\ell} \sum _{i \in \alpha} a_{ii}) > 0,
\]

which completes the proof.

Using the relation between sign-symmetric matrices and sign-pattern symmetric matrices, we can restate Theorem 11 as follows.

**Corollary 3.** Suppose that \( A \in \mathbb{R}^{n \times n} \) satisfies the following property: there exists an \( \varepsilon > 0 \) such that \( ((I + \varepsilon A)^{(k)})^\ell \) is sign-pattern symmetric for any \( \varepsilon \in [0, \bar{\varepsilon}] \), \( k \in \{1, \ldots, n - 1\} \), and any integer \( \ell > 0 \). Then \( A \in \mathbb{EP} \).
Remark 6. It follows from (8) that
\[(I + \varepsilon A)^{(k)})^\ell = I + \varepsilon \ell A^k + o(\varepsilon),\]
so the condition in the corollary implies that
\[A^k\] is sign-pattern symmetric for any \(k \in \{1, \ldots, n - 1\}.\] (23)

The next example demonstrates that Theorem 11 is indeed more general than Theorem 6.

Example 8. Consider again the matrix \(A\) in (17). Note that \(A\) does not belong to any of the three matrix classes in Theorem 6. A computation yields
\[
(I + \varepsilon A)^\ell = \frac{1}{3} \begin{bmatrix}
2a_1^\ell - 2a_2^\ell & a_1^\ell - a_2^\ell & * \\
0 & 0 & * \\
0 & a_3^\ell - a_4^\ell & *
\end{bmatrix},
\]
\[
((I + \varepsilon A)^{(2)})^\ell = \frac{1}{3} \begin{bmatrix}
* & * & * \\
0 & * & a_3^\ell - a_4^\ell \\
0 & 2a_3^\ell - 2a_4^\ell & *
\end{bmatrix},
\]
where \(a_1 := 1 - 3\varepsilon, a_2 := 1 - 6\varepsilon, a_3 := 1 - 12\varepsilon + 27\varepsilon^2,\) and \(a_4 := 1 - 15\varepsilon + 54\varepsilon^2.\) Note that both matrices are sign-pattern symmetric. By Corollary 3, \(A \in \mathbb{EP}.\) \(\Box\)

The next example shows that (21) is not a necessary condition for \(\mathbb{EP}.\)

Example 9. Consider the matrix
\[
A = \begin{bmatrix}
15 & -9 & -18 \\
1 & 3 & -12 \\
-1 & 3 & 12
\end{bmatrix}. \tag{24}
\]
Let \(D = \text{diag}(0, 12, 18)\) and consider the matrix \(T \in \mathbb{R}^{3 \times 3}\) given in Example 7. Then \(A = TDT^{-1}.\) Hence, Theorem 9 implies that \(A \in \mathbb{EP}.\) However, \(A\) is not sign-pattern symmetric, that is, condition (21) is not satisfied. \(\Box\)

Remark 7. Recall that any principal submatrix of a \(P\)-matrix is also a \(P\)-matrix. A similar conclusion generally does not hold for the \(\mathbb{EP}\) case. For example, the matrix \(A\) in (24) is \(\mathbb{EP}\) and has the principal submatrix \(A[\{1, 2\} \{1, 2\}] = \begin{bmatrix}
15 & -9 \\
1 & 3
\end{bmatrix},\) which is not \(\mathbb{EP}\) according to Proposition 7. \(\Box\)

IV. AN APPLICATION TO OPINION DYNAMICS

Opinion dynamics studies how local social interactions between social actors lead to the formation of opinions. Applications include group decision making, the propagation of rumors, successful marketing, the emergence of extremism, and reaching consensus. For a recent survey, see e.g., [43].

Opinion dynamics models can be classified into several groups. There are models where the opinions are considered discrete (often accepting only two different results, say, voting for the Republicans or the Democrats). Examples include using the Ising model to describe the behavior of laborers in a strike, where the two options are either to work or to strike [16], and other models for binary-state dynamics on a network like the Sznajd model [50].

A second group of models considers opinions as continuous variables that can take values in an interval. The value of the variables may represent the worthiness of a choice. Examples include the well-known Deffuant model [11] and the Hegselmann-Krause [21] models.

A third group of models considers opinions that are observed as discrete actions, but are represented internally by each agent as a continuous variable. A typical example is to model the opinion of agent \(i\) as a variable \(x_i\) taking values in \([-1, 1]\), such that \(x_i > 0\) is interpreted as voting for one candidate and \(x_i < 0\) represents the voting for the other candidate [33].
Many algorithms have been presented to solve distributed problems with many cooperating agents, such as average consensus, rendezvous, and sensor coverage [47], [8], [32], [37].

Here, we consider the well-known linear consensus algorithm
\[ \dot{x} = -Lx, \tag{25} \]
where \( L \in \mathbb{R}^{n \times n} \) is a Laplacian matrix. Note that these algorithms have also appeared in many other domains, e.g., average consensus, rendezvous, and sensor coverage [47], [8], [32], [37].

We introduce the following requirement.

**Definition 2.** We say that (25) reaches consensus without sign-reversal if for any \( x(0) \in \mathbb{R}^n \setminus \{0\} \) the following properties hold:
1) The solution \( x(t) \) converges, as \( t \to \infty \), to some \( c \in \mathbb{R} \);
2) for any \( t \geq 0 \) there exists an index \( i \in \{1, \ldots, n\} \) (that may depend on \( x(0) \) and \( t \)) such that \( x_i(t)x_i(0) > 0 \).

The second requirement aims to prevent a situation where the vectors \( x(0) \) and \( x(t) \) change their sign along the course of reaching consensus. For example, if \( n = 3 \) and \( x(0) \) has the sign pattern \([+ - +]^T\) at any time \( t \geq 0 \), this has a natural interpretation. We do not allow that all the agents “change their mind” along the course of reaching consensus. This new requirement is related to a well-known notion in opinion dynamics called “stubborn agents” [57], that is, an agent that does not change its opinion over time (the terminology in this field is not uniform and such agents are also called a leader [56], social media [4], closed-minded [5], and inflexible agent [27]). However, the requirement in Definition 2 is novel, as it does not pinpoint a specific agent, but rather focuses on the global behaviour.

The conditions needed for reaching consensus are well-known, so we focus on the second requirement in Definition 2. The next result follows immediately from the fact that the solution of (25) is \( x(t) = \exp(-Lt)x(0) \).

**Proposition 4.** The second requirement in Definition 2 holds iff \( L \in \mathbb{EP} \).

If \( L \) represents an undirected connectivity graph then it is symmetric, and Theorem 6 implies that \( L \in \mathbb{EP} \). But a non-symmetric Laplacian matrix \( L \) may have complex eigenvalues and then Theorem 7 implies that \( L \not\in \mathbb{EP} \), so there exists \( t > 0 \) such that \( \exp(-Lt) \not\in \mathbb{P} \), and by Theorem 3 there exists \( x(0) \in \mathbb{R}^n \setminus \{0\} \) such that \( x_i(0)x_i(t) \leq 0 \) for all \( i \).

**A. Discrete-time consensus**

Consider the discrete-time linear consensus system
\[ x(j + 1) = Ax(j), \]
and assume that we require that for any \( x(0) \in \mathbb{R}^n \setminus \{0\} \) the following property holds: for any time \( k = 0, 1, \ldots \) there exists an index \( i \) (that may depend on \( x(0) \) and \( k \)) such that
\[ x_i(k)x_i(0) > 0. \tag{26} \]
This is the discrete-time analogue of condition 2 in Definition 2. Since \( x(k) = A^kx(0) \), (26) is equivalent to the requirement that \( A^k \in \mathbb{P} \). Recall that a matrix \( A \in \mathbb{R}^{n \times n} \) is called a PM-matrix if \( A^k \in \mathbb{P} \) for all \( k = 0, 1, \ldots \) (see, e.g., [22]). In this sense, EP-matrices can be viewed as the continuous-time analogue of PM-matrices.

**V. Conclusion**

P-matrices play an important role in many fields of applied mathematics. Here, we defined the new notion of an EP-matrix i.e., a matrix \( A \) such that \( \exp(At) \) is a P-matrix for all \( t \geq 0 \) (and then also
for all $t \leq 0$). We show that EP-matrices must have all real eigenvalues and provide several conditions guaranteeing that a matrix is EP. For $A \in \mathbb{R}^{2 \times 2}$, a simple sign pattern condition on the matrix entries is proved to be necessary and sufficient for $A \in \mathbb{EP}$. Using the sign non-reversal property of P-matrices, we also described a natural application of $\mathbb{EP}$ matrices to linear consensus algorithms.

An interesting topic for further research is to identify sign patterns that guarantee that a matrix is EP. Several authors studied dynamical systems that include complementarity constraints [54], [53], and the notion of EP-matrices may also find applications in such systems.

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REFERENCES

[1] A. Abate, A. Tiwari, and S. Sastry, “Box invariance in biologically-inspired dynamical systems,” *Automatica*, vol. 45, no. 7, pp. 1601–1610, 2009.

[2] M. Banaji, P. Donnell, and S. Baigent, “P matrix properties, injectivity, and stability in chemical reaction systems,” *SIAM J. Applied Math.*, vol. 67, no. 6, pp. 1523–1547, 2007.

[3] E. Bar-Shalom, O. Dalin, and M. Margaliot, “Compound matrices in systems and control theory: a tutorial,” 2022, submitted. [Online]. Available: https://arxiv.org/abs/2204.00676

[4] H. Z. Brooks and M. A. Porter, “A model for the influence of media on the ideology of content in online social networks,” *Phys. Rev. Research*, vol. 2, p. 023041, 2020.

[5] B. Chazelle and C. Wang, “Inertial Hegselmann–Krause systems,” *IEEE Trans. Automat. Control*, vol. 62, no. 8, pp. 3905–3913, 2017.

[6] P. N. Choudhury, “Characterizing total positivity: single vector tests via linear complementarity, sign non-reversal, and variation diminution,” 2021. [Online]. Available: https://arxiv.org/abs/2103.05624

[7] P. N. Choudhury, M. R. Kannan, and A. Khare, “Sign non-reversal property for totally non-negative and totally positive matrices, and testing total positivity of their interval hull,” *Bull. London Math. Soc.*, pp. 1–10, 2021.

[8] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, “Coverage control for mobile sensing networks,” *IEEE Trans. Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[9] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*. Society for Industrial and Applied Mathematics, 2009.

[10] G. E. Coxson, “The P-matrix problem is co-NP-complete,” *Mathematical Programming*, vol. 64, pp. 173–178, 1994.

[11] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, “Mixing beliefs among interacting agents,” *Advances in Complex Systems*, vol. 03, no. 01n04, pp. 87–98, 2000.

[12] D. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*. Princeton, NJ: Princeton University Press, 2011.

[13] S. Fallat, C. R. Johnson, and A. D. Sokal, “Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples,” *Linear Algebra Appl.*, vol. 520, pp. 242–259, 2017.

[14] M. Fiedler and V. Ptak, “Some generalizations of positive definiteness and monotonicity,” *Numerische Mathematik*, vol. 9, pp. 163–172, 1966.

[15] ——, “On matrices with non-positive off-diagonal elements and positive principal minors,” *Czechoslovak Mathematical Journal*, vol. 12, no. 3, pp. 382–400, 1962.

[16] S. Galam, Y. Gefen (Feigenblat), and Y. Shapir, “Sociophysics: A new approach of sociological collective behaviour: I. mean-behaviour description of a strike,” *J. Math. Sociology*, vol. 9, no. 1, pp. 1–13, 1982.

[17] D. Gale and H. Nikaido, “The Jacobian matrix and global univalence of mapping,” *Mathematische Annalen*, vol. 159, p. 81–93, 1965.

[18] F. R. Gantmacher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*. Providence, RI: American Mathematical Society, 2002, translation based on the 1941 Russian original.

[19] C. B. Garcia and W. I. Zangwill, “On univalence and P-matrices,” *Linear Algebra Appl.*, vol. 24, pp. 239–250, 1979.

[20] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer, 2003.

[21] R. Hegselmann and U. Krause, “Opinion dynamics and bounded confidence: models, analysis and simulation,” *J. Artificial Societies and Social Simulation*, vol. 5, no. 3, 2002.

[22] D. Herskovits and C. R. Johnson, “Spectra of matrices with P-matrix powers,” *Linear Algebra Appl.*, vol. 80, pp. 159–171, 1986.

[23] D. Herskovits and N. Keller, “Positivity of principal minors, sign symmetry and stability,” *Linear Algebra Appl.*, vol. 364, pp. 105–124, 2003.

[24] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*. Cambridge University Press, 1988.

[25] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[26] ——, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2013.

[27] F. Jacobs and S. Galam, “Two-opinions-dynamics generated by inflexibles and non-contrarian and contrarian floaters,” *Adv. Complex Syst.*, vol. 22, no. 04, p. 1950008, 2019.

[28] C. R. Johnson, R. L. Smith, and M. J. Tsatsomeros, *Matrix Positivity*, ser. Cambridge Tracts in Mathematics. Cambridge University Press, 2020, vol. 221.

[29] E. Kaszkurewicz and A. Bhaya, *Matrix Diagonal Stability in Systems and Computation*. Basel: Birkhauser, 2000.

[30] R. Katz, M. Margaliot, and E. Fridman, “Entrainment to subharmonic trajectories in oscillatory discrete-time systems,” *Automatica*, vol. 116, p. 108919, 2020.
