Emergence of scale-free networks from optimization process

F. L. Forgerini\textsuperscript{1,2}, S. N. Dorogovtsev\textsuperscript{1,3} and J. F. F. Mendes\textsuperscript{1}
\textsuperscript{1}Departamento de Física and I3N, Universidade de Aveiro, 3810-193 Aveiro, Portugal
\textsuperscript{2}ISB, Universidade Federal do Amazonas, 69460-000 Coari-AM, Brazil
\textsuperscript{3}A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia
E-mail: fforgerini@ua.pt

Abstract. Preferential attachment is a standard mechanism producing power-laws in growing networks. Thanks to its simplicity this mechanism is realized in most of the models of scale-free network, but, unfortunately, it rather mimics scale-free networks and not explains them. Optimization based mechanisms have a much greater potential to explain the evolution of scale-free networks. We consider one of the simplest optimization based models generating power-law growing networks. Our model is defined as follows. At each time step, a new node is created and connected to \( m \) previous nodes in the network, which are selected to minimize the product \( s^\alpha r \), where \( s \) is the birth time of the node and \( r \) is a random number drawn from some distribution. In the case of complete optimization, the networks generated by this model have a power-law degree distribution with the exponent \( \gamma = 1 + 1/\alpha \) for a wide range of the random number distributions. For partial optimization, including a finite fraction of nodes in a network, we observe an exponential degree distribution.

1. Introduction
The joint feature of numerous real-world networks including the Internet, scientific collaborations, WWW, protein and gene interaction networks, etc., is their scale-free organization. In all these examples, the number of nodes of a degree \( q \) follows a power-law distribution, \( P(q) \sim q^{-\gamma} \) with the exponent \( \gamma \) typically in the range \( 2 \leq \gamma \leq 3 \). One of the most studied mechanisms producing power-laws in networks is the preferential attachment mechanism [1, 2]. In this simple mechanism, more popular nodes with many connections receive new connections preferentially [3, 4]. Preferential attachment can also be considered as a consequence of various processes on networks, e.g., optimization processes aiming at an optimal network design. Moreover, optimization can be treated as an alternative mechanism explaining complex network architectures [5, 6, 7, 8, 9].

Although the idea of preferential attachment is simple and elegant, often the preferential attachment itself cannot be explained. In addition, sometimes it is simply impossible to know about the degrees of all vertices in a network. In this respect, standard preferential attachment models are often not quite realistic. It was shown recently that a refined optimization model, incorporating trade-off between popularity and similarity of nodes, can describe real-world network architectures remarkably well [10]. In the present work we demonstrate that even within a very simple optimization based evolution model one can obtain a scale-free network with exponent \( \gamma \) in a wide range of values. This result is valid only for complete optimization, in
which information about all nodes in the network is taken into account. For partial optimization, including a finite fraction of the network at each step, or even a few randomly selected nodes, the degree distribution of a growing network has an exponential form.

2. Optimization based model for growing networks

Our model of a growing network is formulated as follows. At each time step we create a new node \( t \) and attach it to \( m \) previous nodes in the network. Each of these \( m \) nodes is selected from the set of all existing nodes by minimization of the product \( s^\alpha r_s \), where the label \( s \) is the birth time of a node, \( 1 \leq s \leq t \), \( r_s \) is a random number taken from a given distribution \( p(r) \), and \( \alpha \geq 0 \) is a new exponent. Here the random variable \( r \) actually plays the role of multiplicative noise.

Our simulations demonstrate that if the optimization process incorporates all existing nodes at each step (complete optimization), then the growing network is scale-free. In Fig. 1 we show the cumulative degree distribution of a network generated by the complete optimization model, in which \( \alpha \) is set to 1 and \( m = 1, 2, 5, 10 \), after averaging over 100 samples. The random numbers are uniformly distributed and the probability density function is

\[
p(r) = \begin{cases} 
1 & \text{for } 0 < r < 1 \\
0 & \text{otherwise.}
\end{cases}
\]

One can see that the exponent \( \gamma \) of the power-law node degree distribution in our model approaches 2 for any \( m \). Introducing \( \alpha < 1 \) leads to \( \gamma > 2 \). For the sake of brevity here we mostly discuss the simplest case of \( \alpha = 1 \).

3. Generation of power-law degree distributions

Let us derive the degree distribution for our model. Here we only present a simple estimate in the case of \( \alpha = 1 \), \( m = 1 \), and for a random number uniformly distributed from 0 to 1. A straightforward derivation for an arbitrary \( \alpha \) and other distributions \( p(r) \) will be given elsewhere.

In the case of the uniform \( p(r) \) defined above, the distribution of the product \( P(sr_s) \) is also uniform, namely \( P(0 < sr_s < s) = 1/s \) and \( P(sr_s > 0) = 0 \). Consequently, for small values of \( sr_s \), we have \( P(sr_s) = 1/s \). This means that the probability that node \( s \) has the smallest product \( sr_s \) is proportional to \( 1/s \). Therefore, the mean degree \( \langle q \rangle(s,t) \) of node \( s \) increases according to the following relation:

\[
\frac{\partial \langle q \rangle(s,t)}{\partial t} \sim \frac{1}{s \ln t}.
\]
So we have $\langle q \rangle(s) \sim 1/s$. In the continuum approximation which is applicable to scale-free networks, this corresponds to the degree distribution

$$P(q) = -\frac{1}{t} \left( \frac{\partial q(s)}{\partial s} \right)^{-1} \bigg|_{s=s(q)\sim 1/q} \sim \frac{1}{q^2},$$  \hspace{1cm} (2)

where, as is usual in the continuous approximation, we set $\langle q \rangle(s,t) = q(s,t)$. This result agrees with our simulation, Fig. 1. Generally, for $\alpha \leq 1$, assuming $p(r = 0) \neq 0$, one can obtain the relation $\gamma = 1 + 1/\alpha$.

In the work [11], attachment to a node of the maximal degree selected from a random sample of a few nodes was studied. The degree distributions of the resulting networks was found to be rapidly decaying. Inspired by these ideas we modify our model and consider a partial optimization process, in which at each step, the optimal node for attachment is selected from a finite fraction of the existing networks, namely from a uniformly randomly chosen fraction $f$ of all nodes.

In Fig. 2 we show a linear-log plot of the cumulative degree distribution for the result of the partial optimization process, $f = 0.01$. One can see that for various values of $m$, the degree distribution decays exponentially. In Fig. 3 we show the cumulative degree distribution obtained for various values of $f$, where $f = 1$ corresponds to complete optimization resulting in the scale-free network having $\gamma = 2$, while $f = 0$ actually corresponds to the standard random recursive tree in which new nodes are attached to uniformly randomly chosen existing nodes. The random recursive graph is well-known to have an exponentially decaying degree distribution. On the other hand, we find that $0 < f < 1$ produces an exponential cutoff of the power-law degree distribution, Fig. 3.

![Figure 2](image1.png) \hspace{1cm} ![Figure 3](image2.png)

**Figure 2.** Linear-log plot of the cumulative degree distribution of the network of $10^5$ nodes generated by a partial optimization process, $f = 0.01$, $m = 1, 2, 5, 10$, $\alpha = 1$.

**Figure 3.** Log-log plot of the cumulative degree distribution of the network of $10^5$ nodes generated by a partial optimization process, $f = 0, 0.1, 0.5, 1$, $m = 1$, $\alpha = 1$.

4. **Conclusions**

In summary, we considered, maybe, the simplest example of the optimization driven evolution of complex networks. The resulting networks are scale-free if at each step, the optimization involves all existing nodes in a network. If the optimization is partial, i.e., it includes only a finite fraction of a network or a few nodes, the result is an exponential cut-off of a power-law degree distribution or even an exponential degree distribution. We suggest that the optimization driven evolution is a widespread mechanism generating complex networks architectures.
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