The Hamilton-Jacobi Approach to Teleparallelism

B. M. Pimentel\(^{(a)}\), P. J. Pompeia\(^{(a,c)}\), J. F. da Rocha-Neto\(^{(b)}\)

\(^{(a)}\) Instituto de Física Teórica - Universidade Estadual Paulista, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil.
\(^{(b)}\) Campus Universitário de Araras - Universidade Federal do Tocantins, Rua Universitária s/n\(^{\text{B}}\) Centro, 77.330-000, Araras, TO, Brazil.
\(^{(c)}\) Centro Técnico Aeroespacial - Instituto de Fomento e Coordenação Industrial - Divisão de Confiabilidade Metrológica Praça Mal. Eduardo Gomes, 50, 12228-901, São José dos Campos, SP, Brazil.

Abstract:

We intend to analyse the constraint structure of Teleparallelism employing the Hamilton-Jacobi formalism for singular systems. This study is conducted without using an ADM 3+1 decomposition and without fixing time gauge condition. It can be verified that the field equations constitute an integrable system.

Keywords: Hamilton-Jacobi; Teleparallelism; Constraints.

1 Introduction

The standard approach to study singular (constrained) systems is that one developed by Dirac [1], [2], [3] in which the hamiltonian structure is employed. The success of this approach cannot be denied if we consider the wide number of cases in which it has been successfully applied. Among these cases we can cite Podolsky Electrodynamics [4], General Relativity [5], [6] and Teleparallelism [7], [8], [9]. One important feature of this formalism lies in the physical arguments provided by consistency conditions, which tell that the constraints of the theory must be conserved, i.e., their time derivatives must be null. Although Dirac approach is widely accepted [10], [11], [12], [13], it did not avoid the emerging of other approaches, that always provide new points of view for the same problems.

One of these approaches is the Hamilton-Jacobi (HJ) formalism, that makes use of Carathéodory Equivalent Lagrangian method [15]. This method is an alternative way to obtain Hamilton-Jacobi equation starting from lagrangian formalism. It was originally developed by Carathéodory to treat first order regular systems. Generalizations to treat constrained systems were developed by Güler [16], [17], who dealt with lagrangians of first order, and more recently...
by Pimentel and Teixeira [13], [19], who worked with lagrangians of higher order. An approach to treat Berezinian singular systems can also be found in literature in a work of Pimentel, Teixeira and Tomazelli [20]. This formalism does not make use of the physical arguments, as it happens in the hamiltonian one, but it uses the rigorous mathematical structure of the partial differential equation theory, being this a significant feature of the formalism. Although the development of Hamilton-Jacobi approach is quite recent, it can be found in literature some important applications of this formalism [21], [22], [23], including an application to Teleparallelism [24].

The Teleparallel Equivalent of General Relativity (TEGR or Teleparallelism) [25], [26], as the name suggests, is a theory of gravitation that was built to be equivalent to General Relativity (GR). In this way is expected that TEGR, as GR, be a singular system. Moreover, some important aspects of Teleparallelism, as geodesics and “force” equations and energy-momentum tensor, have been studied [27] (and references cited therein), as well as the interactions of spin0, spin1 and spinor fields in TEGR [28], [29], [30], [31], [32] and the gauge symmetries of this theory [33]. In [24] an application of Hamilton-Jacobi formalism to TEGR was made using a gauge fixing and an ADM 3+1 decomposition of space time [34].

In this work we intend to analyse the structure of Teleparallelism without using an ADM decomposition and without fixing the time gauge condition, since none of these conditions are requirements of the HJ approach. Accordingly, first we will make a review of Hamilton-Jacobi approach to singular systems and expose some important results of TEGR. Afterwards we analyse the constraint structure of the theory, and we make some comments at last.

Here we use the following conventions: $\mu, \nu, \ldots = 0$, $i (i = 1, 2, 3)$ are space-time indices; $a, b, \ldots = 0$, $(i) (i = 1, 2, 3)$ are $SO(3, 1)$ indices. The Minkowski metric is fixed as $\left( - , + , + , + \right)$. When the notation differs from this (as it happens in the next section) we shall indicate explicitly.

2 The Hamilton-Jacobi Formalism

According to Carathéodory Equivalent Lagrangian method, given a Lagrangian $L = L \left( q^i, \dot{q}^i \right)$, $i = 1, ..., N$, another Lagrangian $L' = L' \left( q^i, \dot{q}^i \right)$,

$$L' \left( q^i, \dot{q}^i \right) = L \left( q^i, \dot{q}^i \right) - \frac{d}{dt}S \left( q^i, t \right),$$  

(1)
can be found, in such a way that both action integrals, $A = \int dt L$ and $A' = \int dt L'$, have simultaneous extremes $^{4}\delta A = \delta A'$. Since the variational problem of finding an extreme of the action integral is the same for both Lagrangians, then $L$ and $L'$ are said to be equivalent.

$^{4}$It is important to notice that if $L$ is a singular Lagrangian, i.e. $\text{det} \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) = 0$, then $L'$ will also be singular because $\frac{\partial^2 L'}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$.
To find an extreme of the action $A'$ it is sufficient to find a set of equations $eta^i(q^j, t)$ such that

$$L'(q^i, \dot{q}^i = \beta^i(q^j, t)) = L(q^i, \dot{q}^i) - \frac{\partial}{\partial t} S(q^i, t) - \frac{\partial S(q^i, t)}{\partial q^j} \dot{q}^j = 0, \quad (2)$$

and in a neighbourhood of $\dot{q}^i = \beta^i(q^j, t)$ the condition $L'(q^i, \dot{q}^i) > 0$ is satisfied. From the condition above it follows that

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}(\dot{q}^i = \beta^i) = \frac{\partial S}{\partial q^i}(\dot{q}^i = \beta^i). \quad (3)$$

For a singular Lagrangian $L$ the condition $\det H_{ij} = 0$, where $H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial p_i}{\partial \dot{q}^i}$ is the Hessian matrix, is satisfied. If the rank of this matrix is $P = N - R$, then the coordinates $q^i$ may be ordered in such a way that the $P \times P$ matrix on the right bottom corner of the Hessian matrix is nonsingular:

$$\det H_{ab} = \det \frac{\partial p_a}{\partial \dot{q}^b} \neq 0, \quad a, b = R + 1, ..., N. \quad (4)$$

This guarantees that $N - R$ velocities $\dot{q}^b$ can be written as $\dot{q}^b = f^b(q^i, p_a)$. The other momenta $p_\alpha (\alpha = 1, ..., R)$ do not depend on any velocity, but they may depend on the coordinates, the time and the other momenta $p_a$, so that $R$ relations can be written:

$$p_\alpha = -H_\alpha(t, q^\beta, t_0, q, p_a). \quad (5)$$

It can also be seen that, using the standard definition $H_0 \equiv p_i \dot{q}^i - L$, the expression $\dddot{5}$ can be rewritten as

$$p_0 + H_0(t, t^0, q, p_a) = 0, \quad (6)$$

where $p_0 \equiv \frac{\partial S}{\partial t}$. The equations $\dddot{6}$ and $\dddot{5}$ can be written in an unified way

$$H'_\alpha \equiv p_\alpha + H_\alpha(t^\beta, q^\alpha, p_a) = 0, \quad \alpha, \beta = 0, 1, ..., R, \quad (7)$$

where $t^0 \equiv t$. This is a set of first order differential partial equations, called Hamilton-Jacobi Partial Differential Equations (HJPDE).

### 2.1 Integrability Conditions

It is known from the theory of partial differential equations $\dddot{15}$ that a set of total differential equations, named as the characteristics equations, is associated with a set of partial differential equations. It can be verified, using the independence

$$\dddot{5}$$

It can be directly verified that $\frac{\partial H_0}{\partial \dot{q}^\beta} = 0.$
of the momenta and the fact that \( S = S(t^\alpha, q^\alpha) \) \((\alpha = 0, 1, ..., R; a = R + 1, ..., N)\),
that the characteristics equations of the Hamilton-Jacobi equations are

\[
\begin{align*}
d\eta^I &= E^{IJ} \frac{\partial H'_I}{\partial q^J} dt^\alpha, \quad I, J = (\xi; i), \xi = 1, 2; \\
dS &= \frac{\partial S}{\partial q^i} dq^i = p_i \frac{\partial H'_i}{\partial p_i} dt^\alpha, \quad \alpha = 0, ..., R; i = 0, ..., N.
\end{align*}
\]

(8)

where \( \{\eta^{1i}\} = \{q^i\} \) and \( \{\eta^{2i}\} = \{p_i\} \), \( E^{IJ} = \delta^i_j \left[ \delta^\xi_\sigma \delta^2_\sigma - \delta^2_\xi \delta^1_\sigma \right], \) \((I = (\xi, i), J = (\sigma, j))\). We also have \( \{F, G\} = \frac{\partial F}{\partial q^I} E^{IJ} \frac{\partial G}{\partial p^J} \), which is the Poisson Brackets of \( F \) and \( G \). The equations of the first line of (8) will be called from now on as equations of motion.

To assure that a unique solution to the Hamilton-Jacobi equations can be found, it is enough to prove that the characteristics equations constitute a set of integrable equations. In fact it is enough to prove that only the equations of motion is a set of integrable equations, because if this condition is satisfied, then the last of the characteristics equations will be integrable as a consequence.

From the theory of differential equations it is also known that, associated with a set of total equations, \( dx^J = b^I_J \left( x^J \right) dt^\alpha, \) there are linear operators \( X_\alpha \) such that

\[
X_\alpha F \left( x^J \right) = b^I_\alpha \frac{\partial F}{\partial x^I} = 0.
\]

(9)

As a consequence, for any function \( F \) at least twice differentiable, the condition

\[
[X_\alpha, X_\beta] F = (X_\alpha X_\beta - X_\beta X_\alpha) F = 0
\]

will be satisfied. If the calculation of each quantity \( [X_\alpha, X_\beta] F \) is a linear combination of (9), \( [X_\alpha, X_\beta] F = C_{\alpha\beta\gamma} X_\gamma F \), the partial differential equations \( X_\alpha F = 0 \) is said to be complete. Otherwise a new operator \( X \) can be defined, such that \( XF = 0 \), and it can be added to the initial set of operators \( X_\alpha \). This procedure can be repeated until a complete set of partial differential be obtained. The total differential equations will be integrable if the set of associated partial equations is complete.

In the case of the equations of motion, we have \( X_\alpha F = \{F, H'_\alpha\} \), and \( [X_\alpha, X_\beta] F = -\left( \{F, H'_\alpha, H'_\beta\} \right) \). Since the relation (10) must be satisfied for any function \( F \) then it follows that

\[
\{H'_\alpha, H'_\beta\} = 0,
\]

(11)
or equivalently, considering the independence of \( t^\alpha \),

\[
dH'_\alpha = \{H'_\alpha, H'_\beta\} dt^\beta = 0.
\]

(12)

While these conditions are not satisfied, new relations of the type \( H' = 0 \) can be established, and the integrability conditions must be tested until a complete set of partial differential equations is obtained.
3 The Teleparallel Equivalent of General Relativity

While General Relativity is a theory of gravitation built on Riemann’s space-time, that is a particular case of Riemann-Cartan manifold with a null torsion tensor, Teleparallel Equivalent of General Relativity (TEGR) is a theory of gravitation built on Weitzenböck space-time, which is also a particular case of Riemann-Cartan manifold, where the torsion tensor is non-vanishing and the curvature is null. In this theory, usually studied in its Lagrangian formulation, the dynamical fields are tetrads, $e_a^\mu$, that constitute a set of four vectors under General Coordinate transformation on space-time and four vectors under Lorentz transformation on tangent space.

To assure the vanishing of curvature of Weitzenböck space-time, we impose the absolute parallelism condition, which implies that the spin connection is null. With this condition, Cartan (or Weitzenböck) connection is completely determined by tetrads, $\Gamma^\lambda_{\mu\nu} = e_\lambda^a \partial_\mu e_\nu^a$, and the torsion, which is the antisymmetric part of the connection, is given by $T^a_{\mu\nu} = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a$. As it is well known, in TEGR the torsion carries the information about the dynamics of the system. This can be directly verified from the Lagrangian density,

$$L = -k e \Sigma^{abc} T_{abc},$$

where

$$\Sigma^{abc} \equiv \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c),$$

$$e \equiv \det (e_\mu^a),$$

$T^b \equiv \eta_{ab} T^{abc}$, and $k = 1/(16\pi G)$ ($G$ is the gravitational constant). The equations of motion of the theory (in the absence of matter’s field) can be obtained from $L$ with the use of a variational principle:

$$\partial_\mu (e \Sigma^{abc}_g) + \Sigma^a_{\alpha\mu} T^{a}_{g\mu} - e_{\alpha}^{\mu} \Sigma^{abc}_{\alpha} T_{abc} = 0.$$  \hspace{1cm} (14)

It is important to remind that this Lagrangian density is built to be equivalent, up to a surface term, to Einstein-Hilbert Lagrangian density of General Relativity.

In order to study the constraints of this theory using Hamilton-Jacobi formalism, we must know the structure of TEGR in phase space. Following the steps of Carathéodory Lagrangian Equivalent method, first we must choose some coordinate to perform the role of time, $t$. The choice employed here is the same that it was used by Dirac in his study of General Relativity and consists of taking the zeroth coordinate as $t$.

Secondly we must obtain the momenta canonically conjugated to $e_{a\mu}$. This can be done with the definition

$$\Pi^{a\mu} = \frac{\delta L}{\delta \dot{e}_{a\mu}},$$

where

$$L = -k e \Sigma^{abc} T_{abc}.$$
from where it follows
\[ \Pi^{\alpha \mu} = -4k e \Sigma^{a_0 \mu} \Rightarrow \]
\[ \left\{ \begin{array}{l}
\Pi^{\alpha k} = ke \left[ g^{00} \left( -g_{kj} T_{0j}^a - e^{aj} T_{0j}^k + 2 e^{ak} T_{0j}^j \right) \right] + \\
+ g^{0k} \left( g_{0j} T_{aj}^a + e^{aj} T_{0j}^a \right) + e^{a0} \left( g_{0j} T_{0j}^k + g_{kj} T_{0j}^j \right) + \\
- 2 \left( e^{a0} g^{0k} T_{0j}^j + e^{ak} g^{0j} T_{0j}^0 \right) + \\
- g^{00} g_{kj} T_{ij}^a + e^{ai} \left( g^{0j} T_{ij}^a - g^{kj} T_{ij}^a \right) + \\
- 2 \left( g^{00} e^{ak} - g^{0k} e^{a0} \right) T_{ij}^j \\
\Pi^{a0} = 0 . \end{array} \right. \] (16)

This last result arises from the fact that there is no “time” derivative of \( e_{a0} \) in \( L \), and as a consequence we have four constraints,
\[ H^a_{1} = \int d^3 y \Pi^{a0} (y) = 0 \] (17)

The next step is to obtain the Hamiltonian density and this may be achieved with the employment of the prescription \( "L = p q - H_0" \), and, obviously, with the definitions (15):
\[ H_0 = -e_{a0} \partial_k \Pi^{ak} - \frac{1}{4 g^{00}} ke \left( g_{ik} g_{jl} P^{ij} P^{kl} - \frac{1}{2} P^2 \right) + \]
\[ + ke \left( \frac{1}{4} g^{im} g^{nj} T_{mn}^a T_{aij} + \frac{1}{2} g^{nj} T_{mn}^a T_{ij}^m - g^{ij} T_{ij}^m T_{mn}^a \right) , \] (18)

where
\[ P^{ik} \equiv \frac{1}{ke} \Pi^{(ik)} - \Delta^{ik} , \]
\[ P \equiv g_{ik} P^{ik} , \]

with
\[ \Delta^{ik} \equiv - g^{0m} \left( g^{kj} T_{mj}^i + g^{ij} T_{mj}^k \right) - 2 g^{nk} T_{mj}^i - \left( g^{km} g^{0i} + g^{im} g^{0k} \right) T_{mj}^j , \]

and \( \Pi^{(ik)} \) being the symmetric part of \( \Pi^{ik} \).

As it was seen in the previous section, the canonical Hamiltonian, that is obtained from the spatial integration of the Hamiltonian density, satisfies the Hamilton-Jacobi equation that may be rewritten as
\[ H_0^a = \int d^3 y \left( P_0 (y) + H_0 (y) \right) = 0 , \] (19)

if we define a density of momenta canonically conjugated to time, \( P_0 \).

The equations (17) and (18) constitute the set of Hamilton-Jacobi Partial Differential Equations (HJPDE) of TEGR. With these results we can write the equations of motion as
\[ d \eta^I = \left\{ \eta^I, H_0^a \right\} dt + \left\{ \eta^I, H_1^a \right\} d e_{a0} , \] (20)

where \( \eta^I = \{ e_{a\mu}, t; \Pi^{a\mu}, P_0 \} \), and we shall verify if they constitute an integrable system.
3.1 Integrability Conditions

The integrability conditions of the system of total differential equations (20) are given by

\[
\begin{align*}
\frac{d}{dt} H'_0 &= \{ H'_0, H'_0 \} \quad dt + \{ H'_0, H'^a_i \} \, de_{a0} \\
\frac{d}{dt} H'^b_i &= \{ H'^b_i, H'_0 \} \quad dt + \{ H'^b_i, H'^a_i \} \, de_{a0}.
\end{align*}
\]

These Poisson Brackets can be evaluated if we notice that in this theory there is no explicit dependence on time which implies

\[
\{ F, P_0 \} = 0,
\]

and that

\[
\{ F, H'^a_i \} = \int d^3x \, \delta F \delta \epsilon_{a0} (x) \quad .
\]

Both expressions are valid for an arbitrary \( F \). With these results we see that

\[
\begin{align*}
\{ H'_0, H'_0 \} &= \int d^3y d^3z \, \{ H_0 (y), H_0 (z) \} = - \int d^3y d^3z \, \{ H_0 (z), H_0 (y) \} = 0, \\
\{ H'^b_i, H'^a_i \} &= \int d^3y d^3z \, \frac{\delta \Pi^{b0} (z)}{\delta \epsilon_{a0} (y)} = 0, \\
\{ H'_0, H'^a_i \} &= \int d^3y C^a (y),
\end{align*}
\]

where

\[
C^a = \frac{\delta H_0}{\delta \epsilon_{a0}} = e^{a0} H_0 + e^{ai} F_i,
\]

with

\[
\begin{align*}
F_i &\equiv H_i + \Gamma^m T_{0mi} + \Gamma^{km} T_{kmi} + \frac{1}{2g^{00}} \left( g^{ik} g_{ij} P^{kj} - \frac{1}{2} g_{ij} P \right) \Gamma^j, \\
H_i &\equiv - e_{bi} \partial_k \Pi^{bk} - \Pi^{bk} T_{bki}, \\
\Gamma^k &\equiv \Pi^{0k} + 2ke \left( g^{kj} g^{0i} T^{0}_{ij} - g^{0k} g^{0i} T_{ij} + g^{00} g^{kj} T_{ij} \right), \\
\Gamma^{ik} &\equiv - \Gamma^{ki} = \Pi^{[ik]} + ke \left[ -g^{im} g^{jk} T^0_{mj} + (g^{im} g^{0k} - g^{km} g^{0i}) T^j_{mj} \right].
\end{align*}
\]

This decomposition of \( C^a \) is very important at this stage [7] and showed to be essential for the further steps.

If we want the system of total differential equations to be integrable, the condition

\[
C^a = 0
\]

must be satisfied. Moreover, if we recall the orthogonality of the tetrads, we have:

\[
\begin{align*}
H_0 &= e^0_a C^a = 0, \\
F^i &= e^i_a C^a = 0.
\end{align*}
\]
The condition $C^a = 0$ lead us to define four new constraints,

$$H_2^a = \int d^3y C^a(y) = 0 . \quad (25)$$

These constraints must be incorporated to the previous set of constraints and the integrability conditions must be tested again:

$$dH_0' = \{H_0', H_0^a\} dt + \{H_0^a, H_1^a\} de_{0a} + \{H_0^a, H_2^a\} d\lambda_a ,$$

$$dH_1^b = \{H_1^b, H_0^a\} dt + \{H_1^b, H_1^a\} de_{0a} + \{H_1^b, H_2^a\} d\lambda_a ,$$

$$dH_2^b = \{H_2^b, H_0^a\} dt + \{H_2^b, H_1^a\} de_{0a} + \{H_2^b, H_2^a\} d\lambda_a ,$$

where $\lambda_a$ are four parameters that could not be associated with specific fields of the theory, because the constraint $H_2^a$ is not a constraint of the type "$H + p = 0$". We can evaluate those Poisson Brackets one by one:

$$\{H_0', H_2^a\} = \int d^3y d^3z \{H_0(y), C^a(z)\} ,$$

$$\{H_0^a, H_2^a\} = \int d^3y d^3z \{H_0(y), C^a(z)\} = \int d^3y d^3z \frac{\delta C^b(z)}{\delta e_{0a}(y)} ,$$

$$\{H_1^b, H_2^a\} = \int d^3y d^3z \{C^b(y), C^a(z)\} .$$

It can be verified that

$$\frac{\delta \Gamma^{ik}}{\delta e_{0a}} = -\frac{1}{2} (e^{ai} \Gamma^k - e^{ak} \Gamma^i) , \quad (26)$$

$$\frac{\delta \Gamma^j}{\delta e_{0a}} = -e^{a0} \Gamma^k . \quad (27)$$

These results lead us to conclude that

$$\{H_1^b, H_2^a\} = -\int d^3y e^{ai} e^{bn} \left( \frac{1}{2k} g_{in} g_{jl} \Gamma^l + (T_{inj} + T_{jmi}) \right) \Gamma^j ,$$

and if we wish that these relations were nule for any $e^{a0}$, then

$$\Gamma^j = 0 . \quad (28)$$

We also have the following result

$$\{H_0', H_2^a\} = \int d^3y d^3z \{H_0(y), C^a(z)\} = \int d^3y d^3z e^{ai} \{H_0(y), F_i(z)\} ,$$

which may be evaluated, considering that $\Gamma^j = 0$, by the following Poisson Brackets:

$$\{H_0(y), H_i(z)\} = H_0(z) \frac{\partial}{\partial y^a} \delta(y - z) - C^a \partial_i e_{0a} \delta(y - z) , \quad (29)$$
\[ \{ H_0 (y), \Gamma^i (z) \} = \left[ g^{0i} H_0 - \frac{1}{g_{00}} P^{kl} \left( g_{kj} g_{ml} - \frac{1}{2} g_{kl} g_{jm} \right) g_{0j} \Gamma^m + \frac{1}{g_{00}} \partial_n e_{a0} + \frac{1}{2} \Gamma^m n_i T_{nm} + 2 \partial_n \Gamma^m + g^{in} (H_n - \Gamma^j T_{0nj} - \Gamma^m T_{mnj}) \right] \delta (y - z) + \Gamma^m (y) \frac{\partial}{\partial y} \delta (y - z) , \tag{30} \]

\[ \{ H_0 (y), \Gamma^{ij} (z) \} = \left[ - \frac{1}{2g_{00}} P^{kl} \left( g_{km} g_{nl} - \frac{1}{2} g_{kl} g_{mn} \right) (g^{mi} \Gamma^{nj} - g^{mj} \Gamma^{ni}) + \frac{1}{2} (\Gamma^{ni} e^{aj} - \Gamma^{nj} e^{ai}) \frac{\partial}{\partial y} \right] \delta (y - z) . \tag{31} \]

Again, if we want \( \{ H'_0, H''_a \} \) to vanish for any tetrad, we must take
\[ \Gamma^{ij} = 0 \tag{32} \]

The expressions (28) and (32) make us define new constraints
\[ H'^i_3 = \int d^3 y \Gamma^i (y) = 0 , \]
\[ H'^{ij}_4 = \int d^3 y \Gamma^{ij} (y) = 0 , \]
that we must add to the previous set of constraints
\[ dH'_0 = \{ H'_0, H'_0 \} dt + \{ H'_0, H''_a \} de_a + \{ H'_0, H''_a \} d\lambda_a + \{ H'_0, H'^{ij}_3 \} d\omega_i + \{ H'_0, H'^{ij}_4 \} d\omega_{ij} , \]
\[ : \]
\[ dH'^{mn}_4 = \{ H'^{mn}_4, H'_0 \} dt + \{ H'^{mn}_4, H'^m_1 \} de_a + \{ H'^{mn}_4, H'^m_2 \} d\lambda_a + \{ H'^{mn}_4, H'^{ij}_3 \} d\omega_i + \{ H'^{mn}_4, H'^{ij}_4 \} d\omega_{ij} , \]
and once more test the integrability conditions. We must observe that \( \omega_i \) and \( \omega_{ij} \) are, as \( \lambda_a \), parameters in the theory.

By direct inspection one can verify that these Poisson Brackets constitute an algebra, what may be verified with the above results and the following ones:
\[ \{ H_j (y), H_k (z) \} = - H_k (y) \frac{\partial}{\partial y} \delta (y - z) - H_j (y) \frac{\partial}{\partial y} \delta (y - z) , \tag{33} \]
\[ \{ \Gamma^i (y), \Gamma^j (z) \} = 0 , \tag{34} \]
\[ \{ \Gamma^{ij} (y), \Gamma^{kl} (z) \} = \frac{1}{2} (g^{ik} \Gamma^{jk} + g^{jk} \Gamma^{ik} - g^{ik} \Gamma^{jl} - g^{jk} \Gamma^{il}) \delta (y - z) . \tag{35} \]
\[
\{ \Gamma^{ij}(y), \Gamma^k(z) \} = (g^{ij} \Gamma^{ki} - g^{0i} \Gamma^{kj}) \delta(y - z), \quad (36)
\]

\[
\{ H_i(y), \Gamma^j(z) \} = \delta^i_j \Gamma^n(z) \frac{\partial}{\partial z^k} \delta(y - z) + \Gamma^j(y) \frac{\partial}{\partial y^i} \delta(y - z) - \Gamma^j e^{ao} \partial_i e_{an} \delta(y - z), \quad (37)
\]

\[
\{ H_k(y), \Gamma^{ij}(z) \} = \Gamma^{ij}(y) \frac{\partial}{\partial y^k} \delta(y - z) + \frac{\partial}{\partial y^k} (e^{aj} \Gamma^i - e^{ai} \Gamma^j) \partial_k e_{ao} \delta(y - z). \quad (38)
\]

As a consequence, no new constraint arises and then we conclude that the system of total differential equations of motion, that now read

\[
d\eta^I = \{ \eta^I, H_0^I \} dt + \{ \eta^I, H_1^a \} de_{a0} + \{ \eta^I, H_2^a \} d\lambda_a + \{ \eta^I, H_3^a \} d\omega_i + \{ \eta^I, H_4^{ik} \} d\omega_{ik}, \quad (39)
\]

is an integrable system.

4 Concluding Remarks

The analysis of the constraints of TEGR via Hamilton-Jacobi shows that the total differential equations of motions constitute a set of integrable system, which implies that the HJPD compose a system of partial differential equations that has precisely a solution. Moreover, if we analyse the set of equations (39) we see that phase space of the theory is a subspace of the space \{e_{a\mu}, t; \Pi_{a\mu}, P_0\}. We can affirm this because, according to Hamilton-Jacobi approach, the variables \(t, e_{a0}, \lambda_a, \omega_i, \omega_{ik}\) have the status of parameters.

We can likewise compare the results obtained in this work with those obtained by Maluf and Rocha-Neto [7], who studied the constraints of TEGR with Dirac approach. We see that the constraints obtained here are exactly the same obtained in [7] (compare equations (18), (22), (20), (19) of [7] and consequently the constraint algebra of the last is also the same algebra of this work. This is a consequence of the equivalency between HJ integrability conditions and Dirac consistency conditions (see the appendix in [20]). However we must notice that the paths that lead to these results in one and another approach are very different. One of the main differences that we point out is that in Dirac approach we need to determine the set of primary constraints to test the consistency conditions in order to obtain the secondary constraints. This procedure was used in [7]. In HJ we do not need to distinguish between primary and secondary constraints. This
characteristic allowed us to work with an incomplete set of primary constraints (in Dirac’s language), and, with the employment of the integrability conditions, we were able to find the secondary constraints and all the missing primary ones. This is a peculiar characteristic of HJ approach.

Moreover we believe that it is possible to write the equations of motion for the dynamical variables in a form independent of the parameters \( \lambda, \omega_i, \omega_{ik} \), which would reveal the gauge independence of the system’s evolution, as it happens, for example, in the study of QED.

Acknowledgements:

B. M. Pimentel thanks CNPq and Fundação de Amparo à Pesquisa do Estado de São Paulo, FAPESP, (grant number 02/00222-9) for partial support; P. J. Pompeia thanks the staff of CTA for incentive and support; J. F. da Rocha-Neto would like to thank Professor J. Geraldo Pereira for his hospitality at the Instituto de Física Teórica IFT/UNESP and FAPESP (grant number 01/00890-9) for partial support.

References

[1] P. A. M. Dirac - *Can. J. Math.* **2**, 129 (1950).
[2] P. A. M. Dirac - *Can. J. Math.* **3**, 129 (1951).
[3] P. A. M. Dirac - *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York (1964).
[4] C. A. P. Galvão and B. M. Pimentel - *Can. J. Phys.* **66**, 460 (1988).
[5] P. A. M. Dirac - *Proc. Roy. Soc.* **A246**, 333 (1958).
[6] P. A. M. Dirac - *Phys. Rev.* **114**, 924 (1959).
[7] J. W. Maluf, J. F. da Rocha-Neto - *Phys. Rev. D* **64**, 084014 (2001).
[8] J. W. Maluf - *J. Math. Phys.* **35**, 335 (1994).
[9] M. Blagojević and I. A. Nikolić - *Phys. Rev. D* **62**, 024021 (2000).
[10] A. Hanson, T. Regge and C. Teitelboim - *Constrained Hamiltonian Systems*, Accad. Naz. Lincei, Rome (1982).
[11] K. Sundermeyer - *Lecture Notes in Physics 169* - *Constrained Dynamics*, Springer-Verlag, New York/Berlin (1982).
[12] D. M. Gitman and I. V. Tyutin - *Quantization of Fields with Constraints*, Springer-Verlag, New York/Berlin (1990).
[13] J. Govaerts - *Hamiltonian Quantization and Constrained Dynamics*, Leuven University Press (1991).
[14] M. Henneaux and C. Teitelboim - *Quantization of Gauge Systems*, Princeton University Press, New York (1992).

[15] C. Carathéodory - *Calculus of Variations and Partial Differential Equations of the First Order - Part I and II*, Holden Day, Inc. (1967).

[16] Y. Güler - *Il Nuovo Cimento B* **107**, 1398 (1992).

[17] Y. Güler - *Il Nuovo Cimento B* **107**, 1143 (1992).

[18] B. M. Pimentel and R. G. Teixeira - *Il Nuovo Cimento B* **111**, 841 (1996).

[19] B. M. Pimentel and R. G. Teixeira - *Il Nuovo Cimento B* **113**, 805 (1998).

[20] B. M. Pimentel, R. G. Teixeira and J. L. Tomazelli - *Ann. Phys.* **267**, 75 (1998).

[21] Y. Güler - *Il Nuovo Cimento B* **109**, 341 (1994).

[22] Y. Güler - *Il Nuovo Cimento B* **111**, 513 (1996).

[23] Y. Güler and D. Baleanu - *Il Nuovo Cimento B* **114**, 1023 (1999).

[24] B. M. Pimentel, P. J. Pompeia, J. F. da Rocha-Neto and R. G. Teixeira - *Gen. Rel. Grav.* **35**, 877 (2003).

[25] K. Hayashi and T. Shirafuji - Phys. Rev. D **19**, 3524 (1979).

[26] F. W, Hehl - *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergavity*, Plenum, New York (1980).

[27] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira - Preprint. gr-qc/0011087 (2000).

[28] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira - *Phys. Rev. D* **64**, 027502 (2001).

[29] V. C. de Andrade and J. G. Pereira - *Gen. Rel. Grav.* **30**, 263 (1998).

[30] V. C. de Andrade and J. G. Pereira - *Int. J. Mod. Phys. D* **8**, 141 (1999).

[31] J. T. Lunardi, B. M. Pimentel and R. G. Teixeira - *Gen. Rel. Grav.* **34**, 491 (2002).

[32] R. Casana, J. T. Lunardi, B. M. Pimentel and R. G. Teixeira - *Gen. Rel. Grav.* **34**, 1941 (2002).

[33] M. Blagojević and M. Vasilić - *Class. Quantum Grav.* **18** 5143 (2001).

[34] R. Arnowitt, S. Deser, C. W. Misner - *Gravitation: an Introduction to Current Research*, (edited by L. Witten) John Wiley & Sons (1962).

[35] R. Weitzenböck - *Invariantentheorie*, Noordhoff, Gronningen (1923).