The universe is not isotropic or spatially homogeneous on local scales. The averaging of local inhomogeneities in general relativity can lead to significant dynamical effects on the evolution of the universe and on the interpretation of cosmological data. In particular, all deductions about cosmology are based on light paths; averaging can have an important effect on photon propagation and hence cosmological observations. It would be desirable to describe the physical effects of averaging in terms of observational quantities and focussing on the behaviour of light. Data (e.g., matter terms, such as the density or galaxy number counts, which are already expressed as averaged quantities) is given on the null cone. Therefore, it is observationally meaningful to consider light-cone averages of quantities. In principle, we wish to describe the cosmological equations on the null cone, and hence we need to construct the averaged geometry on the null cone. However, we argue that it is still necessary to average the full Einstein field equations to obtain suitably averaged equations on the null cone. Since it is not the geometry per se that appears in the observational relations, we discuss whether it is possible to covariantly ‘average’ just a subset of the evolution equations on the null cone, focussing on relevant observational quantities. We present an averaged version of the scalar null Raychaudhuri equation, which may be a useful first step in this regard.
1 Introduction

Cosmological observations [1, 2], based on the assumption of a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model plus small perturbations, are usually interpreted as implying that there exists dark energy, the spatial geometry is flat, and that there is currently an accelerated expansion giving rise to the so-called ΛCDM-concordance model. Although the concordance model is quite remarkable (at least if the idea of dark energy can be tolerated), it does not convincingly fit all data. Essentially, structure is more evolved on large scales than predicted by the ΛCDM-concordance model. There is strong evidence for coherent bulk flows on scales out to at least $300h^{-1}\text{Mpc}$ [3] and there is observational evidence of anisotropy in the Hubble expansion rate, suggesting large scale peculiar motions [4]. In addition, there are correlations between galaxy surveys and the cosmic microwave background (CMB) data [5], overestimates of the amplitude of the matter power spectrum [6] and non-Gaussianities in the CMB [7]. Unfortunately, if the underlying cosmological model is not a perturbation of an exact flat FLRW solution, the conventional data analysis and their interpretation is not necessarily valid. For example, the standard analysis of type Ia supernovae (SNIa) and CMB data in FLRW models cannot be applied directly when backreaction effects are present, because of the different behaviour of the spatial curvature [8].

Supernovae data dynamically requires an accelerating universe. However, this only implies the existence of dark energy if the universe is (approximately) FLRW. Thus the isotropic and spatially homogeneous ΛCDM model is a good phenomenological fit to the real inhomogeneous universe, as far as observational determinations of the expansion history of the universe [9]. However, this does not imply that a primary source of dark energy exists, but only that it exists as far as the phenomenological fit is concerned. Supernovae data can be explained without dark energy in inhomogeneous models, where the full effects of general relativity (GR) come into play. The apparent acceleration of the universe is thus
not caused by repulsive gravity due to dark energy, but rather is a dynamical result of inhomogeneities, either in an exact solution or via averaging effects (due to the back-reaction of inhomogeneities). Indeed, it has been indeed shown that the Lemaître-Tolman-Bondi (LTB) solution can be used to fit the observed data without the need of dark energy \cite{10}, although it is necessary to place the observer at the center of a rather large-scale underdensity.

Therefore, the averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. The correct governing equations on cosmological scales are obtained by averaging the Einstein field equations (EFE) of GR (plus a theory of photon propagation; i.e., information on what trajectories actual particles follow). By assuming spatial homogeneity and isotropy on the largest scales, the inhomogeneities affect the dynamics though correction (backreaction) terms, which can lead to behaviour qualitatively and quantitatively different from the FLRW models. It is necessary to use an exact covariant approach which gives a prescription for the correlation functions that emerge in an averaging of the full tensorial EFE. For example, in \cite{11} the macroscopic gravity equations were explicitly solved in a FLRW background geometry and it was found that the correlation tensor (backreaction) is of the form of a spatial curvature.

Clearly, backreaction (averaging) effects are real, but their relative importance must be determined. In the FLRW plus perturbations approach, the (backreaction) effects are assumed small (and are assumed to stay small during the evolution of the universe). But it is also possible that averaging effects are not small (i.e., perturbation theory cannot be used to estimate the effects, and real inhomogeneous effects must be included). The Wilkinson Microwave Anisotropy Probe (WMAP) \cite{2}, together with SNIa data in ΛCDM models \cite{1}, suggests a normalized spatial curvature Ω_k ≈ 0.01–0.02 (i.e., of about a percent). Combining these observations with large scale structure observations then puts stringent limits on the curvature parameter in the context of adiabatic ΛCDM models; however, these data analyses are very model- and prior-dependent \cite{8}, and care is needed in the proper interpretation of the data. There is a heuristic argument that Ω_k ∼ 10^{-3} – 10^{-2} \cite{12,13}, which is consistent
with CMB observations [13] and agrees with estimates for intrinsic curvature fluctuations using realistically modelled clusters and voids in a Swiss-cheese model [15, 16]. It must be appreciated that such a value for $\Omega_k$, at the 1% level, is relatively large and may have a significant dynamical effect on the evolution of the universe; the correction terms change the interpretation of observations so that they need to be accounted for carefully to determine if a model may be consistent with cosmological data [14, 13].

2 Null geodesics

All deductions about cosmology are based on light paths. Only the redshift and the energy flux of light arriving from a distant source are observed, rather than the expansion rate or the matter density. It is often assumed that intervening inhomogeneities average out. However, inhomogeneities affect curved null geodesics [12, 15] and can drastically alter observed distances when they are a sizable fraction of the curvature radius. In the real universe, voids occupy a much larger region as compared to structures [17], hence light preferentially travels much more through underdense regions and the effects of inhomogeneities on luminosity distance are likely to be significant.

The effect of averaging null geodesics in inhomogeneous models was discussed in [19]. GR is treated as a microscopic (classical) theory. Real photons travel on null geodesics in the microscopic geometry. However, because all observations are of finite resolution, observations necessarily involve averages of measured quantities. Therefore, in interpreting real observations, it is necessary to model properties of (not only a single photon but of) a ‘narrow’ beam or bundle of photons (i.e., a local congruence of null geodesics). From the geometric optics approximation we can obtain the optical scalar (Dyer-Roeder) equations that govern the propagation of the local shearing and expansion (of the cross-sectional area of the beam) with respect to the affine parameter along the congruence due to Ricci focussing.
and Weyl tidal focussing [18]. Since the nonlinear optical scalar equations require integration along the beam, the optics for a lumpy distribution does not average and there may be important resulting effects. Therefore, it is important to study the effect of averaging on a beam of photons in the optical limit.

The motion of photons in an averaged geometry (and the resulting effect on cosmological observations) was discussed in [19]. Assuming GR to be a microscopic theory on small scales, with local metric field $g$ (the micro-geometry) and matter fields, a photon follows a null geodesic $k$ in the local geometry. After averaging (using a covariant averaging scheme), we obtain a smoothed out macroscopic geometry (with macroscopic metric $\langle g \rangle$ and macroscopic matter fields, valid on larger scales. But what trajectories do photons follow in the macro-geometry? In general, the “averaged” vector $\langle k \rangle$ need not be null, need not be geodesic (and even if it is, need not be affinely parametrized) in the macro-geometry. This clearly affects cosmological observations. In the case of radial geodesics in a spherically symmetric geometry [19], it was found that there is an effect due to the non-affine parameterization of the null geodesics in the averaged geometry, together with additional small corrections due to the fact that the trajectories are not exactly null geodesics, the correlation tensor is not precisely due to a spatial curvature and since a beam of photons will experience expansion and shearing during its evolution [18]. Each of these effects are typically of order of about 1%, which can add up and perhaps produce a significant observational effect.

Similar issues have been discussed recently from a different point of view. It was conjectured that in a statistically homogeneous and isotropic dust universe, light propagation can be treated in terms of the overall geometry (meaning the average expansion rate and average spatial curvature) if the structures are realistically small and the observer is not in a special location [12]. The relationship between the expansion rate, the redshift and the distance scale in such a universe containing non-linear structures was further investigated in [20], and it was shown that light propagation can be expressed in terms of averaged geometrical quantities, up to a term related to the null geodesic shear. In general, the null shear
is not negligible, and thus the Dyer-Roeder equations do not correctly describe the effect of clumping. Instead, the redshift and the distance are determined by the average expansion rate, the matter density today and the null geodesic shear. This implies that a clumpy model can be consistent with the observed position of the CMB acoustic peaks even when there is significant spatial curvature [2], provided that the expansion history is sufficiently close to the spatially flat $\Lambda$CDM model [20].

In addition, the propagation of photons in a particular (toy) Swiss-cheese model, where the cheese consists of a spatially flat, matter only FLRW solution and the holes are constructed out of a LTB solution of the EFE, and the phenomenological effects of large-scale nonlinear inhomogeneities on observables such as the luminosity-distance–redshift relation, were discussed in [9]. Following a fitting procedure based on light-cone averages, it was found that the expansion scalar is unaffected by the inhomogeneities (essentially due to the spherical symmetry of the model), but the light-cone average of the density as a function of redshift (which is not affected by the spherical symmetry) is affected by inhomogeneities. (The effect arises because, as the universe evolves, a photon spends more and more time in the (large) voids than in the (thin) high-density structures.) The phenomenological homogeneous model describing the light-cone average of the density is similar to the $\Lambda$CDM concordance model, and behaves as if it has a dark-energy component.

Clearly averaging can have an important effect on photon propagation and hence observations. Ideally, we would like to describe the physical effects of averaging in terms of observational quantities, focussing on the behaviour of light. In particular, spatial averages at constant time are not directly related to observations; it is observationally more meaningful to consider light-cone averages of quantities. Consequently, we wish to attempt to describe the evolution equations on the null cone, focussing on observational quantities. Let us first consider the formulation of the EFE on the null cone.
3 EFEs in Observational Coordinates

The cosmological data representing galaxy redshifts, observer area distances and galaxy number counts as functions of redshift are given, not on a space-like surface of constant time, but rather on our past light cone $C^{-}(p_0)$, which is centered at our observational position $p_0$ “here and now” on our world line $C$. By using observational coordinates (OC) [21], the EFE can be formulated in a way which reflects both the geodesic flow of the cosmological fluid and the null geometry of $C^{-}(p_0)$, along which all of the observational information reaches us. In this formulation the EFE split naturally into a set of equations which can be solved on $C^{-}(p_0)$, that is on our past light cone, and a second set which evolves these solutions off $C^{-}(p_0)$ to other light cones into the past or into the future. The solution to the first set is directly determined from the data, and those solutions constitute the “initial conditions” for the solution of the second set.

For example, the OC $x^i = \{w, y, \theta, \phi\}$ for a spherically symmetric metric are centered on the observer’s world line $C$ and defined in the following way: (i) $w$ is constant on each past light cone along $C$; each observational time coordinate $w = constant$ specifies a past light cone along $C$ and our past light cone is designated as $w = w_0$. (ii) $y$ is the null radial coordinate, which measures distance down the null geodesics with affine parameter $v$, generating each past light cone centered on $C$, so that $y$ increases as one moves down a past light cone away from $C$. (iii) $\theta$ and $\phi$ are usual the latitude and longitude of observation, respectively. In OC the spherically symmetric metric takes the form:

$$ds^2 = -A(w, y)^2dw^2 + 2A(w, y)B(w, y)dwdy + C(w, y)^2d\Omega^2,$$  \hspace{1cm} (3.1)

where the comoving fluid 4-velocity is $u^a = A^{-1}\delta^a_w$. There is remaining coordinate freedom which preserves the observational form of the metric. It is also important to specify the central conditions for the metric variables $A(w, y), B(w, y)$ and $C(w, y)$ in equation (3.1) (as they approach $y = 0$). The FLRW metric is obtained in OC by effectively specifying $A = B = a(\eta), C = a(\eta)k^{-2}sin^2(ky)$ in conformal time $\eta = w - y$. It was shown in [22] how to
construct flat dust-filled $\Lambda \neq 0$ FLRW cosmological models from FLRW cosmological data on our past light cone, by integrating the exact spherically symmetric EFE in OC first on the light cone (integrating, e.g., $\frac{C''}{C}$) and then off of it (integrating, e.g., $\frac{\dot{C}}{C}$).

The basic observational quantities on $C$ are then defined as follows:

(i) The redshift $z$ at time $w_0$ on $C$ for a comoving source a null radial distance $y$ down $C^-(p_0)$ is given by $1 + z = A(w_0, 0)/A(w_0, y)$.

(ii) The luminosity distance $d_L$ [21], measured at time $w_0$ on $C$ for a source at a null radial distance $y$, is given by $d_L = (1 + z)^2C(w_0, y)$.

(iv) The number of galaxies counted by a central observer out to a null radial distance $y$ is given by $N(y) = 4\pi \int_0^y \mu(w_0, \tilde{y}) m^{-1} B(w_0, \tilde{y}) C(w_0, \tilde{y})^2 d\tilde{y}$, where $\mu$ is the mass-energy density and $m$ is the average galaxy mass.

Data is given on the null cone (in a suitably averaged form). In principle, we need to construct the metric on the null cone. For example, in the spherically symmetric case averaged quantities are given as simple integrations with respect to $y$ (or redshift); matter terms, such as $N(y)$, are already expressed as averaged quantities. To obtain the averaged geometry on the null cone (i.e., in terms of averaged metric functions $\langle A \rangle$, $\langle B \rangle$ and $\langle C \rangle$), we still need to average the EFE on the null cone. Thus we still need to be able to average the tensorial EFE using some covariant averaging scheme, and then separate the equations into evolution equations on the null cone (in terms of $\langle A \rangle$, $\langle B \rangle$ and $\langle C \rangle$ and their derivatives), and constraints and evolution equations off the null cone. Therefore, we need to project the suitably covariantly averaged full EFE onto the null cone to compare with data averaged on the null cone.

Thus, in this approach we are no further ahead, unless we can covariantly ‘average’ just an appropriate subset of the evolution equations on the null cone. Again we note that it is
not really the geometry that needs to be averaged, since it is not the geometry per se that
is observed (or appears in the observational relations). That is, we observe the averaged
matter on the null cone; if we average the geometry both on the null cone, and inside (and
outside) of the null cone, we are averaging quantities that are not directly observable and
may not play any role in determining the properties of the averaged matter on the null cone
(either directly via the energy momentum tensor or the Ricci tensor through the EFE, or
indirectly through the effects on the motion of matter in response to the geometry). Thus
we would like to be able to focus on the effects of the averaged geometry that only affect
cosmological observations.

Suppose that it is possible to covariantly ‘average’ just a subset of the evolution equations
on the null cone, focusing on relevant quantities (and not necessarily the complete averaged
spacetime geometry). Perhaps we can focus just on scalar equations (and not the full set
of EFE) in the spirit of the Buchert approach. In this approach to the averaging problem
(for irrotational dust) only scalar quantities are averaged, yielding the averaged Hamilto-
nian constraint (or generalized Friedmann equation), the averaged Raychaudhuri equation
(plus an integrability condition) \[^{15}\]. The Buchert approach is ‘heuristic’ since the Buchert
equations are not closed. Indeed, since only scalar quantities are averaged not all of the
EFE have been averaged in Buchert’s approach, and consequently any solutions for these
equations may not be consistent with the full set of EFE.

4 Scalar Equations

Let $k^a$ represent the tangent vector of the bundle of affinely parametrized null geodesics
(optical rays) with scalar expansion ($\hat{\theta} = \frac{1}{2} k^a a_a$) and shear ($\hat{\sigma}$) \[^{15}\]. The null version of the
Raychaudhuri equation is then \[^{23}\]

$$D\hat{\theta} = -\hat{\theta}^2 - \hat{\sigma}^2 - \frac{1}{2} R_{ab} k^a k^b,$$  \hspace{1cm} (4.2)
where $D\hat{\theta} = \hat{\theta}_a k^a$ is a directional derivative along the null geodesic (and can be expressed in terms of the affine parameter, $v$, or the redshift) and $\hat{\sigma}^2$ is a scalar. In the Newman-Penrose formalism, this is effectively eqn. (7.21a) in [23] and constitutes the first of the optical scalar equations (there is also an equation for $D(\hat{\sigma}^2)$).

Let us assume that the matter is dust:

$$T_{ab} = \rho u_a u_b. \quad (4.3)$$

We can decompose the fluid four-velocity as $u^a = k^a + v^a$, where $V \equiv k^a v_a = k^a u_a$ ($v^a v_a = -1 - 2V$) and is related to the redshift by $1 + z = V/V_0$. We can then rewrite eqn. (4.2) as:

$$D\hat{\theta} = -\hat{\theta}^2 - \hat{\sigma}^2 - \frac{1}{2}\rho V^2. \quad (4.4)$$

The projected part of the conservation equation, $T_{ab}^{\ b} = 0$, becomes:

$$D\rho = -\rho \hat{\theta} - A, \quad (4.5)$$

where $A \equiv \rho v^b v_b + \rho v_b v^b$. We can write an “effective” first integral in the form

$$\frac{1}{2} \hat{\sigma}^2 = \rho V^2 + B, \quad (4.6)$$

where $D(B)$ is obtained from eqns. (4.4) and (4.5).

Equation (4.4) concerns the propagation of $\hat{\theta}$ down the null cone. Let us average this scalar equation along the null cone:

$$\frac{d}{dv} \langle \hat{\theta} \rangle = -\langle \hat{\theta} \rangle^2 - \frac{V_0^2}{2} \langle \rho \rangle (1 + z)^2 + \hat{Q}, \quad (4.7)$$

where $\hat{Q} \equiv \langle \hat{\theta} \rangle^2 - \langle \hat{\sigma}^2 \rangle - \langle \hat{\sigma}^2 \rangle$. We can write an averaged version of (4.6) in the form

$$\frac{1}{2} \langle \hat{\theta} \rangle^2 = V_0^2 \langle \rho \rangle (1 + z)^2 - \frac{1}{2} \hat{R} - \frac{1}{2} \hat{Q}, \quad (4.8)$$

where $\hat{R}$ is defined through this equation.
Equation (4.7) is an important result; it is a scalar equation which formally represents how the averaged expansion varies along the null cone. It (together with an equation like (4.8)) represents a null version of the Buchert equations. It suffers the same disadvantages as the Buchert equations in that not all of the appropriate EFE have been averaged and the system is not closed. However, it does have the advantage of relating physically observed qualities (such as the density averaged on the null cone), and any assumptions to close the system now relate correlations on the null cone that may be physically better motivated.

Alternatively, we can define a characteristic length scale $\hat{\ell}$ of the average area behaviour of the geodesics by:

$$\frac{1}{2} \hat{\theta} = \hat{\ell}^{-1} \frac{d\hat{\ell}}{dv}.$$  

Eqn. (4.4) can then be written in the form

$$\frac{d^2\hat{\ell}}{dv^2} = -2\hat{\theta}^2 - \rho V^2.$$  

4.0.1 FLRW models

The spatially homogeneous and isotropic FLRW metric can be written in the form:

$$ds^2 = -dt^2 + a^2(t) \left[ dr^2/(1 - kr^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right],$$  \hspace{1cm} (4.9)

where $a(t)$ is the cosmic scale factor and $k$ is the curvature of 3-dimensional space (e.g., $k = 0$ corresponds to a spatially flat universe). The wavelength $\lambda$ of photons moving through the universe scale with $a(t)$, and the redshift of light emitted from a distant source at time $t_{\text{em}}$ is defined by $1 + z = \lambda_{\text{obs}}/\lambda_{\text{em}} = 1/a(t_{\text{em}})$; thus $dt = -dz/H(z)(1 + z)$, where $H \equiv \dot{a}/a$ is the Hubble parameter and an overdot denotes a time derivative. For an object of intrinsic luminosity $L$, the measured energy flux $F$ defines the luminosity distance $d_L$ to the object, where $d_L(z) \equiv \sqrt{L/4\pi F} = (1 + z)r(z)$, and $r(z)$ is the comoving distance to an object at redshift $z$. The key equations of cosmology are the Friedmann and Raychaudhuri equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}.$$  \hspace{1cm} (4.10)
\[
\frac{\ddot{a}}{a} = -\frac{1}{6} (\rho + 3p) + \frac{\Lambda}{3} \tag{4.11}
\]

where \(\rho\) is the total energy density of the universe, \(p\) is the total pressure, and \(\Lambda\) is the cosmological constant. For each matter component \(i\), the separate conservation of energy is expressed by \(d(a^{3} \rho_{i}) = -p_{i} da^{3}\). The deceleration parameter, \(q(z)\), is defined as \(q(z) \equiv -\frac{\ddot{a}}{aH^2}\).

As an illustration, let us consider equation (4.4) in the case of a pressure-free, flat \((k = 0)\) FLRW model with metric (4.9), where \(1+z = 1/a\). For

\[
k_{a} = \left(\frac{1}{a}, 1, 0, 0\right), \tag{4.12}
\]

we have by direct calculation that

\[
\hat{\theta} \equiv \frac{1}{2} k_{ab}q^{ab} = -\frac{1}{a^2}(\dot{a} - \frac{1}{r}), \tag{4.13}
\]

and

\[
\hat{\sigma}^2 \equiv \frac{1}{2} k_{(a:b)}k^{(a:b)} - \hat{\theta}^2 = 0, \tag{4.14}
\]

whence equation (4.3) yields

\[
a\dddot{a} - \dot{a}^2 = -\frac{1}{2} \rho a^2, \tag{4.15}
\]

as expected (and corresponding to eqns. (4.10 - 4.11)).

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