Heuristic Rating Estimation approach to the pairwise comparisons method

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4 September 2013
Outline

- Pairwise comparisons – motivation
- Classical approach
- Heuristic Rating Estimation (HRE) approach – motivation
- HRE method
- Possible areas of application
- Extensions
- Bibliography
Pairwise comparison
motivation

- FED Museum – Atlanta
  
  http://www.frbatlanta.org/about/tours/virtual/
  
  To start the story from the very beginning
Pairwise comparison
motivation

- Before money - barter
  - (see FED Museum: http://www.frbatlanta.org/about/tours/virtual/money/)

- History of trade is in fact history of pairwise comparisons
Pairwise comparison motivation

- Barter – comparing incomparable
  - It’s hard to judge the actual value of things, hence the judgment is always subjective
Pairwise comparison
motivation

- Experts judgment implies relative value of goods:

\[ \approx \frac{1}{2} \]
Pairwise comparison
motivation

- More comparisons:

  ≈ a / ≈ b

  ≈ c / ≈ d

  ....

  ≈ ?

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Pairwise comparison

motivation

The need for methods of synthesis of partial results
Pairwise comparison

motivation

- Problems with expert judgments
  - Reciprocity
    \[
    \frac{a}{b} = \frac{b}{a} \quad \text{？}
    \]
  - Consistency
    \[
    \frac{a}{b} = a, \quad \frac{b}{a} = b \quad \text{？}
    \Rightarrow \frac{b}{a} = ab
    \]

(See: A new definition of consistency of pairwise comparisons, Koczkodaj 1993),
http://www.sciencedirect.com/science/article/pii/0895717793900598
Pairwise comparison

Method
Pairwise comparison (PC) method
Result synthesis

- Pairwise comparisons matrix

\[ M = \begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix} \]

where \( m_{ij} \in \mathbb{R}_+ \)
PC method
Classical approach

- Classical approach by Thomas Saaty
  - Assumption – matrix must be reciprocal i.e.
    \[ m_{ij} = \frac{1}{m_{ji}} \]
  - Value quantification:
    \[ m_{ij} \in \left\{ \frac{a}{b} : a, b \in \{1, \ldots, 10\} \right\} \]
  - Method of synthesis:
    - Finding principal eigenvector \( x \)
      \[ Mx = \lambda_{\text{max}} x \]

(See: The Analytic Network Process, Saaty, 2006, http://link.springer.com/chapter/10.1007/0-387-33987-6_1)
PC method
Classical approach

- Relative value of is given as:

  \[ x = \begin{bmatrix} x_1, x_2, \ldots, x_5 \end{bmatrix} \]

or even better, as normalized principal eigenvector:

  \[ \hat{x} = \begin{bmatrix} \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_5 \end{bmatrix} \]

where:

  \[ \hat{x}_i = \frac{x_i}{\sum_{j=1}^{5} x_j} \]
PC method
Classical approach

It is possible to determine the relative value of goods
PC method

Problem

- What happen when some goods are exchanged for money?
  - E.g.:
PC method

Problem

- What happen when some goods are exchanged for money?
  - E.g.:
    - $\text{rabbit} = $10
    - $\text{fox} = $20

- Then:
  - $\frac{\text{rabbit}}{\text{fox}} = \frac{1}{2}$
PC method

Problem

- Following Prof. Słowiński:
  Si l’ordre apparaît quelque part dans la qualité, pourquoi chercherions-nous à passer par l’intermédiaire du nombre?” (G.Bachelard 1934)

See: Dominance-based Rough Set Approach to Multiple Criteria Decision Aiding, Słowiński, 2013, Open lecture given at The Jagiellonian University, 17 May 2013, http://www.lgi.ecp.fr/~mousseau/mcda-ss/doc/SlidesSowinski2mod.pdf

- Do not introduce the order when it is given…
PC method

Heuristic Rating Estimation (HRE) approach

- Let us do not (re) introduce the order where it is given...
  - (here the order is introduced by value in Dollars)

  but

- Adopt the given values as reference and estimate the others
PC method

HRE approach

- Set of concepts:

\[ C = C_K \cup C_U \]

- where
  - means \( C_K \) concepts for which the value \( \mu \) is initially known \( \mu(c_i) \in C_K \)
  - means \( C_U \) concepts for which \( \mu \) need to be determined
PC method

HRE approach

- HRE Input
  - Known concepts:
    \[ C_K = \{ \text{rabbit, fox} \} \]
    \[ \mu(\text{rabbit}) = 10, \quad \mu(\text{fox}) = 20 \]
  - Unknown concepts:
    \[ C_U = \{ \text{salmon, trout, shrimp} \} \]
PC method
HRE approach – result synthesis

- Pairwise comparisons matrix

\[ M = \begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} \\
 m_{21} & 1 & m_{23} & m_{24} & m_{25} \\
 m_{31} & m_{32} & 1 & 2 & 1 \\
 m_{41} & m_{42} & 1/2 & 1 & \frac{1}{2} \\
 m_{51} & m_{52} & m_{53} & m_{54} & 1 \\
\end{bmatrix} \]

where \( m_{ij} \in \mathbb{R}_+ \)
PC method
HRE approach – result synthesis algorithm

- Input data:

\[ \mu(c_3) = 20 \]
\[ \mu(c_4) = 10 \]
PC method
HRE approach – result synthesis algorithm

- First step

We may expect that:

\[
\mu(c_1) = \frac{1}{2} \left( m_{13} \mu(c_3) + m_{14} \mu(c_4) \right)
\]

\[
\mu(c_2) = \frac{1}{2} \left( m_{23} \mu(c_3) + m_{24} \mu(c_4) \right)
\]

\[
\mu(c_5) = \frac{1}{2} \left( m_{53} \mu(c_3) + m_{54} \mu(c_4) \right)
\]
PC method
HRE approach – result synthesis algorithm

- Second step and further
  - The values of $c_1, \ldots, c_5$ are known (defined)

$$\mu(c_1) = \frac{1}{4} \sum_{i=2}^{5} m_{1i} \mu(c_i)$$

- and in general:

$$\mu(c_j) = \frac{1}{4} \sum_{i \in \{1, \ldots, 5\} \setminus \{j\}} m_{ji} \mu(c_i)$$
PC method
HRE approach – result synthesis algorithm

- Update equation (general form):

\[
\mu_r(c_j) = \frac{1}{|C_j^{r-1}|} \sum_{c_i \in C_j^{r-1}} m_{ji} \mu_{r-1}(c_i)
\]

where:

\[
C_j^{r-1} = \{ c \in C : \mu_{r-1}(c) \text{ is defined and } c \neq c_j \}, \text{ and } C_j^0 = C_K
\]
PC method
HRE approach – result synthesis algorithm

- Relative value of concepts is given as:

\[ \mu = [\mu(c_1), \ldots, \mu(c_k), \mu(c_{k+1}), \ldots, \mu(c_n)] \]

\begin{align*}
\text{HRE procedure} & \quad \text{a priori given} \\
& \quad \text{(assume } C_K = \{c_{k+1}, \ldots, c_n\})
\end{align*}

- Normalization:

\[ \hat{\mu} = [\hat{\mu}(c_1), \ldots, \hat{\mu}(c_n)] \]

\[ \hat{\mu}(c_i) = \frac{\mu(c_i)}{\sum_{j=1}^{n} \mu(c_j)} \]
What is $\mu$ when the number of iterations tends to $\infty$?
PC method
From algorithm to linear equation

- The algorithm as shown above follows the Jacobi iterative method for the linear equation system in form:

\[ A\mu = b \]

where
- \( A \) matrix of \( r \times r \) where \( r = |C| - |C_K| = |C_U| \)
- \( b \) vector of constant terms
- \( \mu \) vector of values

\[ \mu^T = [\mu(c_1), \ldots, \mu(c_k)] \]
PC method
From algorithm to linear equation

- Assuming (for the sake of simplicity) that
  \[ C = \{ c_1, \ldots, c_n \} \quad \text{and} \quad C_K = \{ c_{k+1}, \ldots, c_n \} \]
- \( b \) - vector of constant terms is given as

\[
b = \begin{bmatrix}
\frac{1}{n-1} m_{1,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{1,n} \mu(c_n) \\
\frac{1}{n-1} m_{2,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{2,n} \mu(c_n) \\
\vdots \\
\frac{1}{n-1} m_{k,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{k,n} \mu(c_n)
\end{bmatrix}
\]
PC method

From algorithm to linear equation

- The matrix $A$ used by the HRE algorithm is:

$$A = \begin{bmatrix}
1 & -\frac{1}{n-1}m_{1,2} & \cdots & \cdots & -\frac{1}{n-1}m_{1,k} \\
-\frac{1}{n-1}m_{2,1} & 1 & -\frac{1}{n-1}m_{2,3} & \cdots & \cdots & -\frac{1}{n-1}m_{2,k} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\frac{1}{n-1}m_{k,1} & \cdots & \cdots & -\frac{1}{n-1}m_{k,k-1} & 1
\end{bmatrix}$$
PC method
Direct solving

- The equation:

$$A\mu = b$$

can also be solved directly...
PC method
Direct solving – theorems and facts

- Theorem 1
  - Equation $A\mu = b$ has exactly one solution if $\det(A) \neq 0$

- Theorem 2
  - The Jacobi method is convergent if $A$ is strictly diagonally dominant by rows i.e.

$$1 > \sum_{j=1, j \neq i}^{k} |a_{ij}|$$

for $i = 1, \ldots, k$
PC method
Direct solving – admissibility of solution

- Observation - result must be positive
  - i.e. $\mu(c_i) > 0$ for $i = 1, \ldots, n$
- That is because of the form of update equation:
  $$
  \mu_r(c_j) = \frac{1}{|C_j^{r-1}|} \sum_{c_i \in C_j^{r-1}} m_{ji} \mu_{r-1}(c_i)
  $$
- Thus $\mu$ obtained by direct solving $A\mu = b$ is admissible if all $\mu(c_i)$ are strictly positive
- Observation - $A$ is strictly diagonally dominant, then $\mu$ obtained using direct method is admissible
PC method

Direct solving – solution admissibility

- Is $A$ diagonally dominant?
  - i.e.
    $$1 > \sum_{j=1, j\neq i}^{k} |a_{ij}| \quad \text{where} \quad a_{ij} = -\frac{1}{n-1} m_{ij}?$$

- Observation
  - $A$ has a high chance to be diagonally dominant when $a_{ij}$ are not too large.

- Conclusion
  - The more concepts in $C_K$ and the more similar to each other the better (the more likely $A$ is diagonally dominant)
The more concepts in $C_K$ ....
- Corresponds to the natural desire to have more than the lower number of reference concepts

The more similar to each other ....
- Humans (experts) are able to compare (judge) similar things but not different.

In most cases tested the matrix $A$ was diagonally dominant
PC method, HRE approach
Areas of application

Decision Support Systems
PC method, HRE approach

How to?

- Define the problem
  - Specify problem domain, the set of concepts and the partial function $\mu$
- Define the reference set $C_K$
  - assign the $\mu$ values to the concepts from $C_K$
- Gathering comparative data
- Apply the HRE method
- To remember – this is only heuristics
  - The most important is common sense
An optimal drug (treatment) selection

- Reference set: $C_K$
  - Group of drugs with proven efficacy in clinical trials
- Other concepts $C_U = C \setminus C_K$ and the rest of $M$
  - Comparative opinions of experts (physicians) based on their clinical experience
PC method, HRE approach

Possible areas of application

- Introducing new products into the market
  - Reference set: $C_K$
    - existing products with sales statistics
- Other: $C_U = C \setminus C_K$
  - Comparative opinion survey of the target group
PC method, HRE approach
Possible areas of application

- Support for assessment of the value of companies
  - Joint stock companies – reference set
    - an actual value is determined by stock exchange
  - Companies outside the exchange trading – the rest of concepts
    - Comparative company value estimation
PC method, HRE approach
Possible areas of application

and many more...
PC method, HRE approach
Possible areas of application

- Green AGH Campus
PC method, HRE approach

Additional flavors
Minimizing estimation error heuristics

- Average absolute estimation error

\[ \hat{e}_\mu = \frac{1}{|C_U|} \sum_{c \in C_U} e_\mu(c) \]

- Absolute estimation error

\[ e_\mu(c_j) = \frac{1}{|C_j^{r-1}|} \sum_{c_i \in C_j^{r-1}} \left| \mu(c_j) - \mu(c_i) \cdot m_{ji} \right| \]
Non-reciprocal PC matrix heuristics

Lack of reciprocity:

\[ m_{ij} \neq \frac{1}{m_{ji}} \]

Let us transform: \( M \rightarrow \hat{M} \)

where:

\[ \hat{m}_{ij} = \left( m_{ij} \frac{1}{m_{ji}} \right)^{1/2} \]

Observation:

- \( \hat{M} \) is reciprocal
- If \( M \) is reciprocal then \( M = \hat{M} \)
PC method, HRE approach

Supporting heuristics

- Incomplete PC matrix heuristics:
  \[
  \mu(c_1) = \frac{1}{2}(m_{13}\mu(c_3) + m_{15}\mu(c_5))
  \]

- Impacts the lack of reciprocity heuristics
K. Kułakowski, *Heuristic Rating Estimation approach to the pairwise comparisons method*, electronic preprint, arXiv:1309.0386, Aug 2013. http://arxiv.org/abs/1309.0386

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Questions