Abstract

We present extremal constructions connected with the property of simplicial collapsibility.

(1) For each \( d \geq 2 \), there are collapsible simplicial \( d \)-complexes with only one free face. Also, there are non-evasive \( d \)-complexes with only two free faces. (Both results are optimal in all dimensions.)

(2) Optimal discrete Morse vectors need not be unique. We explicitly construct a contractible, but non-collapsible 3-dimensional simplicial complex with face vector \( f = (106, 596, 1064, 573) \) that admits two distinct optimal discrete Morse vectors, \((1, 1, 1, 0)\) and \((1, 0, 1, 1)\). Indeed, we show that in every dimension \( d \geq 3 \) there are contractible, non-collapsible simplicial \( d \)-complexes that have \((1, 0, \ldots, 0, 1, 1, 0)\) and \((1, 0, \ldots, 0, 0, 1, 1)\) as distinct optimal discrete Morse vectors.

(3) We give a first explicit example of a (non-PL) 5-manifold, with face vector \( f = (5013, 72300, 290944, 495912, 383136, 110880) \), that is collapsible but not homeomorphic to a ball.

Furthermore, we discuss possible improvements and drawbacks of random approaches to collapsibility and discrete Morse theory. We will see that in many instances, the \texttt{random-revlex} strategy works better than Benedetti–Lutz’s (uniform) random strategy in various practical instances.

On the theoretical side, we prove that after repeated barycentric subdivisions, the discrete Morse vectors found by randomized algorithms have, on average, an exponential (in the number of barycentric subdivisions) number of critical cells asymptotically almost surely.

1 Introduction

Collapsibility was introduced by Whitehead in 1939 \cite{Whitehead}, as a “simpler” version of the topological notion of contractibility. Roughly speaking, collapsible simplicial complexes can be progressively retracted to a single vertex via some sequence of elementary combinatorial moves. Each of these

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moves reduces the size of the complex by deleting exactly two faces. The only requirements are
that these two faces should be of consecutive dimension, and the larger of the two should be the
unique face properly containing the smaller one (which is usually called “free face”).

In dimension one, for example, the free faces of a graph are simply its leaves. Every tree can
be reduced to a point by recursively deleting one leaf; thus all contractible 1-complexes are
collapsible. In dimension $d \geq 2$, however, some contractible $d$-complexes have no free faces; therefore,
collapsible $d$-complexes are a proper subset of the contractible ones. Contractibility is not algorithm-
ically decidable in general, cf. [28]. Collapsibility is, though the decision problem is NP-complete
if $d \geq 3$ [27]. Relatively fast random approaches with good practical behavior have been described
in [8].

Here we present a few examples that are extremal with respect to the collapsibility property.
The first one shows that one can find collapsible complexes where the beginning of any sequence
of deletions is forced.

Main Theorem 1 (Theorems 2 and 4). For every $d \geq 2$, there are

(i) collapsible (and even shellable) simplicial $d$-complexes with only 1 free face, and

(ii) non-evasive (and even vertex-decomposable) simplicial $d$-complexes with only 2 free faces.

Both results are optimal, compare Lemma 3.

Whitehead showed that all collapsible PL-triangulations of $d$-manifolds are homeomorphic to
the $d$-ball. “PL” stands for piecewise-linear and refers to the technical requirement that the closed
star of every face should be itself piecewise-linearly homeomorphic to a standard ball; every smooth
manifold admits PL triangulations (though in dimension $\geq 5$, it also admits non-PL ones). The PL
assumption in Whitehead’s theorem is really necessary: It is a consequence of Ancel–Guilbault’s
work, cf. also [1], that any contractible manifold admits some collapsible triangulation (which
cannot be PL, by Whitehead’s theorem).

A priori, finding an explicit description of a collapsible triangulation of a manifold different
than a ball seems a hard challenge, for three reasons:

(1) the size of (non-PL) triangulations of manifolds with non-trivial topology;

(2) the absence of effective upper bounds on the number of barycentric subdivisions needed to
achieve collapsibility;

(3) the algorithmic difficulty of deciding collapsibility.

Rather than bypassing these problems, we used a direct approach, and then relied on luck.
Specifically, first we tried to realize one triangulation of a collapsible manifold different than a
ball using as few simplices as possible. Once our efforts resulted in a triangulation with 1358186
faces, we fed it to the heuristic algorithm random-discrete-Morse from [8]. To our surprise, the
algorithm was able to digest this complex and show its collapsibility directly — so that in fact no
further barycentric subdivisions were necessary.

Main Theorem 2 (Theorem 14). There is a simplicial 5-dimensional manifold with face vector

\[ f = (5013, 72300, 290944, 495912, 383136, 110880) \]

that is collapsible but not homeomorphic to the 5-ball.

To the best of our knowledge, this is the largest non-trivial example of a complex proven to be
collapsible.

We then turn to an issue left open in Forman’s discrete Morse theory [18, 19]. In this extension
of Whitehead’s theory, in order to progressively deconstruct a complex to a point, we are allowed
to perform not only deletions of a free face (which leave the homotopy type unchanged), but also
deletions of the interior of some top-dimensional simplex (which change the homotopy type in a
very controlled way). The simplices deleted with steps of the second type are called “critical cells.”

\footnote{By a result of Adiprasito–Benedetti [2], if some subdivision of a complex $C$ is collapsible, then also
some iterated barycentric subdivision of $C$ is collapsible.}
Forman’s approach works with complexes of arbitrary topology, but to detect such topology, one must keep track of the critical cells progressively deleted and of the way they were attached. A coarse way to do this is to store in a vector the numbers $c_i$ of critical $i$-faces. The resulting vector $c = (c_0, \ldots, c_d)$ is called a discrete Morse vector of the complex. It depends on the homotopy type of the complex, and also on the particular sequence of deletions chosen. The discrete Morse vector does not determine the homotopy type of the complex (basically because it forgets about attaching maps). However, it yields useful information; for example, it provides upper bounds for the Betti numbers, or for the rank of the fundamental group.

The “mission” is thus to find sequences of deletions that keep the vector $c = (c_0, \ldots, c_d)$ as small as possible. An optimal discrete Morse vector is one that minimizes the number $|c| = c_0 + \ldots + c_d$ of critical faces. At this point, one issue was left unclear, namely, whether the set of discrete Morse vectors admits a componentwise minimum, or if instead some complex might have more than one optimal discrete Morse vector. We answer this dilemma as follows.

**Main Theorem 3** (Theorems 5 and 7). For every $d \geq 3$ there is a contractible, but non-collapsible $d$-dimensional simplicial complex that has two distinct optimal discrete Morse vectors $(1, 0, \ldots, 0, 1, 1, 0)$ and $(1, 0, \ldots, 0, 0, 1, 1)$ with three critical cells each.

**Random models in discrete Morse theory and homology computations**

Discrete Morse theory for simplicial (or cubical) complexes is the basis for fast (co-)reduction techniques that are used in modern homology and persistent homology packages such as CHomP [14], RedHom [13] and Perseus [26]. The main aim in these packages is to first reduce the size of some given complex before eventually setting up boundary matrices for (expensive) Smith normal form homology computations.

The problem of finding optimal discrete Morse functions is NP-hard [20, 22]. However, in practice one can find optimal (or close to optimal) discrete Morse vectors even for huge inputs [8] — and it is, on the contrary, non-trivial to construct complicated triangulations on which the search for good discrete Morse vectors produces poor results.

In an effort to measure how complicated a given triangulation is, a randomized approach to discrete Morse theory was introduced in [8]. In this approach, a given complex is deconstructed level-wise, working from the top dimension downwards. The output is a discrete Morse vector $(c_0, c_1, c_2, \ldots, c_d)$. The algorithm has two important features:

- A $k$-face is declared critical only if there are no free $(k-1)$-faces available, or in other words, only if we cannot go further with collapsing steps. This tends to keep the number of critical faces to a minimum, and to speed up the algorithm.
- The decision of which free $(k-1)$-face should be collapsed away, or (if none) of which $k$-face should be removed as critical, is performed uniformly at random. Randomness allows for a fair analysis of the triangulation, and leaves the door open for relaunching the program multiple times, thus obtaining a whole spectrum of outputs, called experienced discrete Morse spectrum.

The collection of discrete Morse vectors that could a priori be reached this way is called discrete Morse spectrum; see Definition 9.

It was observed in [8] that the boundary spheres of many simplicial polytopes, even with a large number of vertices, have an experienced (i.e., after, say, 10000 runs of the program) discrete Morse spectrum consisting of the sole discrete Morse vector $(1, 0, \ldots, 0, 1)$. In the case of “complicated” triangulations it even might happen that their barycentric subdivisions become “easier” in the sense that on average fewer critical cells are picked up in a random search. This, however, is the exception.

**Main Theorem 4** (Theorem 18). Let $K$ be any simplicial complex of dimension $d \geq 3$. Then the random discrete Morse algorithm, applied to the $n$-th barycentric subdivision $sd^n K$, yields an expected number of $\Omega(e^n)$ critical cells a.a.s.
It is known that simplicial polytopal $d$-spheres of dimension $d \geq 3$ can have a non-trivial discrete Morse spectrum \cite{1}. Theorem 12 of Section 5 shows that the average discrete Morse vector for the discrete Morse spectrum of a high barycentric subdivision of any simplicial $d$-complex, $d \geq 3$, becomes arbitrarily large; in particular, for (polytopal) barycentric subdivisions of simplicial polytopal spheres.

Finally, Section 6 is devoted to a comparison of the random-discrete-Morse strategy introduced in \cite{8} and two variations of it, the random-lex and the random-revlex strategies. In the latter two strategies, we first randomly relabel the vertices of some given complex and then perform all choices of free faces and critical faces in lexicographical and reverse lexicographical order as in \cite{8}, respectively. The experiments we include in the Appendix indicate a surprising superior efficiency of random-revlex.

2 Collapsible complexes with fewest free faces

For the definition of collapsible, nonevasive, shellable, and vertex-decomposable, we refer the reader to \cite{6}. A triangulation $B$ of a ball is called endocollapsible if $B$ minus a facet collapses onto the boundary $\partial B$ \cite{4}. A collapsible complex always has at least one free face, and it is easy to see that every nonevasive complex always has at least two free faces (Lemma 3). The aim of this section is to construct collapsible resp. nonevasive complexes with exactly one resp. exactly two free faces.

Our construction will be based on a lemma from convex combinatorial geometry. Let $P$ be a simplicial polytope. The antiprism subdivision of $P$ is obtained by placing in the interior of $P$ a constricted copy of its polar dual $P^*$. The subdivision is then completed by taking cones from each vertex $v$ of $P$ onto the corresponding facet $v^*$ of $P^*$. The antiprism subdivision is regular, but in general it is not a triangulation, unless $P^*$ is simplicial. However, we can always refine it to a regular triangulation without adding further vertices; cf. \cite{10}. From now on, by antiprism triangulation we mean any regular triangulation obtained from an antiprism subdivision union a pulling triangulation of $P^*$ w.r.t. one of its vertices. (For the construction we have in mind, it does not matter which vertex of $P^*$ is chosen.)

**Lemma 1.** Let $C_d^\Delta$ denote the $d$-dimensional regular cross-polytope centered at the origin. The antiprism triangulation of $C_d^\Delta$ is shellable and endocollapsible.

**Proof.** Every regular triangulation is shellable \cite{11}, and every shellable triangulation is endocollapsible \cite{4}. \hfill $\square$

With Lemma 1 we are now ready to construct a series of collapsible complexes with one free face only. The main idea is to start from the cross-polytope, stack all of its facets but one, triangulate its interior using an antiprism triangulation, and finally perform identifications on the boundary until “only one facet is left” — thus maintaining the contractibility of the complex, and so the collapsibility.

**Theorem 2.** For every $d \geq 2$ there is a collapsible (and even shellable) $d$-dimensional simplicial complex $\Sigma_d$ with $2^d + d + 1$ vertices that has only one free face.

**Proof.** The proof consists of three parts: (i) we construct a $d$-dimensional CW complex $S_d$ with only one free $(d - 1)$-dimensional cell; (ii) we subdivide $S_d$ appropriately to obtain a simplicial complex $\Sigma_d$ with $2^d + d + 1$ vertices and only one free face; and (iii) we show via Lemma 1 that $\Sigma_d$ is collapsible and even shellable.

(i) Let $C_d^\Delta$ denote the $d$-dimensional regular cross-polytope centered at the origin. Let us label the vertices of $C_d^\Delta$ by $\{1, 2, \ldots, d, 1', 2', \ldots, d'\}$ such that the antipodal map $x \mapsto -x$ maps each vertex $i$ to $i'$ and $i'$ to $i$, respectively. If $\{n_1, \ldots, n_k, n'_1, \ldots, n'_{d-k}\}$ is a boundary facet of $C_d^\Delta$ different from $\{1, 2, \ldots, d\}$, we identify it with the facet $\{1', 2', \ldots, d'\}$ by mapping each $n_i$ to $n'_i$. This way, the $(d-1)$-dimensional face $\{1, 2, \ldots, d\}$ is the only boundary facet of $C_d^\Delta$ that is not identified with another boundary facet of $C_d^\Delta$. We call the resulting cell complex $S_d$.\hfill $\square$
(ii) Let us go back to the cross polytope $C^\Delta_d$ before the identification and subdivide the interior of $C^\Delta_d$ according to an antiprism triangulation. Formally, we place a $d$-dimensional cube $C_d$ with $2^d$ vertices, labeled by $\{d + 2, \ldots, 2^d + d + 1\}$, in the interior of $C^\Delta_d$, such that the cubical $(d - 1)$-faces of the cube $C_d$ correspond to the vertices of $C^\Delta_d$. The interior cube is then triangulated without adding further vertices, by using a pulling triangulation. The layer between the cube and the boundary of $C^\Delta_d$ is triangulated in a standard fashion without adding additional vertices.

Next, let us stack all the boundary facets of $C^\Delta_d$ different from $\{1, 2, \ldots, d\}$. Under the identification of the (stacked) boundary facets, the boundary of $C^\Delta_d$ gets mapped to the simplex $\{1, 2, \ldots, d\}$ union a subdivision of it — thus to the boundary of a $d$-simplex with an extra vertex $d + 1$ used for the stacking; see Figure 1 for the complex $\Sigma_2$ with the free edge in red and the identified stacked boundary of $C^\Delta_2$ in blue. Figure 2 displays (a “Schlegel diagram” of) the identified stacked boundary of $C^\Delta_3$, where the back triangle $\{1, 2, 3\}$ is not stacked.

We denote the resulting complex under the boundary identifications as explained in item (i) by $\Sigma_d$. Unlike $S_d$, this $\Sigma_d$ is a simplicial complex.
(iii) By Lemma [1], the subdivided cross-polytope from above collapses onto its boundary minus the facet \( \{1, 2, \ldots, d\} \). The same sequence of collapses, after the identification, collapses \( \Sigma_d \) onto the boundary of the \( d \)-simplex \( \{1, 2, \ldots, d, d+1\} \) minus \( \{1, 2, \ldots, d\} \), which is obviously collapsible. The proof of shellability is here omitted, but it can be derived from a line shelling of the \( d \)-dimensional cross-polytope and the techniques in [2].

By construction, the complexes \( \Sigma_d \) have \((d-1)\)-dimensional faces that are contained in more than two facets. In particular, the examples \( \Sigma_d \) are not manifolds. For a shellable 3-dimensional ball with only one ear see [23]. None of the complexes \( \Sigma_d \) is non-evasive, as the next lemma shows.

**Lemma 3.** A non-evasive \( d \)-complex, \( d \geq 1 \), has at least two free faces.

**Proof.** We proceed by induction on the dimension \( d \). For \( d = 1 \) the claim is equivalent to the well-known fact that every tree has at least two leaves. If \( d \geq 2 \), fix a non-evasive \( d \)-complex \( C \) and a vertex \( v \) of \( C \) whose link and deletion are both non-evasive. By induction, the link of \( v \) has two free faces \( \sigma \) and \( \tau \), which implies that \( v * \sigma \) and \( v * \tau \) are two free faces of \( C \).

As a consequence of Theorem [2] we show now that the bound given by Lemma [3] is sharp in all dimensions:

**Theorem 4.** For every \( d \geq 1 \), there is a non-evasive (and even vertex-decomposable) \( d \)-complex \( E_d \) with exactly two free faces, which share a codimension-one face.

**Proof.** The claim is obvious for \( d = 1 \) (a path has two endpoints) and easy for \( d = 2 \) (it suffices to consider the barycentric subdivision of the shellable 2-complex \( \Sigma_2 \) with only one free face). In higher dimensions, one has to be more careful. We explain how to deal with dimension 3, leaving it to the reader to extend our construction by induction on the dimension. Let us start with a non-evasive (and vertex-decomposable) 2-complex \( E_2 \) with 2 free edges, \( e \) and \( f \), say. The suspension

\[ S = \text{susp} \ E_2 = (v * E_2) \cup (w * E_2) \]

is also non-evasive, but it has 4 free triangles, pairwise adjacent. All we have to do is to “kill the freeness” of the triangles \( w * e \) and \( w * f \). Before proceeding with this, observe that \( w * e \) and \( w * f \) together can be thought of as one larger triangle \( T = w * (e \cup f) \) stellarly subdivided into two. Since any barycentric subdivision can be obtained by a sequence of stellar subdivisions, there is a way to stellarly subdivide \( S \) into a triangulation \( S' \) that restricted to \( (w * e) \cup (w * f) \) is combinatorially equivalent to the barycentric subdivision of \( T \). Stellar subdivisions maintain non-evasiveness; so \( S' \) is a non-evasive 3-complex with \( 2 + 3!) \) free triangles.

Now, let \( \Sigma_3 \) be the collapsible 3-complex with only one free face \( \sigma \) constructed in Theorem [2]. Let \( \text{sd} \ \Sigma_3 \) be its barycentric subdivision. Let us glue \( \text{sd} \ \Sigma_3 \) to \( S' \) by identifying the subdivision of \( (w * e) \cup (w * f) \) with \( \text{sd} \ \sigma \). Let \( E_3 \) be the resulting complex. \( E_3 \) has now only two free faces, \( v * e \) and \( v * f \). Moreover, it is non-evasive: We can start by deleting \( v \), then \( w \), and so on, until we are left with \( \text{sd} \ \Sigma_3 \), which is non-evasive. The same sequence also shows vertex-decomposability.

### 3 Complexes with two different optimal Morse vectors

Here, we address the question of whether a simplicial complex must have a unique optimal discrete Morse vector. The answer is negative in general, as we shall now see. We construct an explicit 3-dimensional simplicial complex \texttt{twooptima} with 106 vertices, which has two distinct optimal discrete Morse vectors \((1, 1, 1, 0)\) and \((1, 0, 1, 1)\). The construction of \texttt{twooptima} can be generalized to every dimension \( d \geq 3 \) to yield (up to suitable subdivisions) \( d \)-dimensional complexes with exactly two optimal discrete Morse vectors \((1, 0, \ldots, 0, 1, 1, 0)\) and \((1, 0, \ldots, 0, 0, 1, 1)\). It also follows that not all optimal discrete Morse vectors need to be contained in the discrete Morse spectrum of a complex.
The construction of two optima involves a copy of $\Sigma_2$ (with relabeled vertices), a modified copy $\Sigma'_3$ of $\Sigma_3$ and nine further copies of $\Sigma_3$. We relabel the vertices of $\Sigma_2$ of Figure 1 such that the unique free edge 1 2 is contained in the triangle 1 2 3; see Figure 3. The complex two optima is non-pure with (the relabeled copy of) $\Sigma_2$ contributing 12 triangles as facets, whereas on top of the 13th triangle 1 2 3 (in grey) we glue the modified copy $\Sigma'_3$ of $\Sigma_3$.

In order to obtain $\Sigma'_3$ from $\Sigma_3$, we again start as in Figure 2 with the boundary of the octahedron (the 3-dimensional crosspolytope), but this time with a finer subdivision and an additional identification on the (identified) boundary. Figure 4 displays this extra identification, where we glue the edge 1 2 to a segment 1 2 (in red) within (each of the seven duplicates of) the triangle 1 3 4 (on the identified boundary of $\Sigma_3$ of Figure 2). Since the edge 1 2 and its image intersect in the single vertex 1, $\Sigma'_3$ (after triangulating its 3-dimensional “interior”) and $\Sigma_3$ are homotopy equivalent. To realize $\Sigma'_3$ as an explicit 3-dimensional simplicial complex, we first subdivide the triangle 1 3 4 inside the stacked triangle 1 2 3 to host the segment 1 2 as a proper edge; see Figure 5.
Figure 5: The subdivision of the triangle $1 3 4$ within the triangle $1 2 3$.

Figure 6: Projection of the lower hull of the seven two-fold cones over the identified and subdivided boundary of the octahedron (left) and one layer further below (right).

We then glue the complex $\Sigma'_3$ on top of $\Sigma_2$ along the back-side triangle $1 2 3$ of $\Sigma'_3$. We call $1 2 3$ the bottom triangle of $\Sigma'_3$, whereas on the top side of $\Sigma'_3$ we see Figure 4 with each triangle $1 3 4$ subdivided as shown in Figure 5. The 3-dimensional solid body between the top side and the back triangle can be thought of as a 3-dimensional ball with identifications on the boundary. For triangulating this body, we first shield off the extra seven copies of the vertex 2 inside (the seven copies of) the triangle $1 3 4$ by taking seven cones

$$1 2 5 x, \ 1 2 6 x, \ 1 3 5 x, \ 1 4 6 x, \ 2 5 6 x, \ 3 4 6 x, \ 3 5 6 x,$$

with respect to the apices $x = 1, \ldots, 7$. Below these seven cones, we place a next layer of cones

$$1 3 x y, \ 1 4 x y, \ 3 4 x y, \ 1 2 4 x, \ 2 3 4 y,$$

with apices $y = x + 7$ (so that the vertices $x = 1, \ldots, 7$ are shielded from other copies of the vertex 2).

The seven two-fold cones can be assigned freely to the seven copies of the subdivided triangle $1 2 3$; our assignment is displayed in Figure 6 (left). We connect the cones (and, en passant, shield off the multiple copies of the edges 12, 13 and 14 via the tetrahedra

$$2 3 1 4 1 5, \ 1 2 1 5 1 6, \ 1 3 1 6 1 7, \ 2 3 1 7 1 8, \ 1 2 1 8 1 9, \ 1 3 1 4 1 9,$$
and

\[131520, 121720, 231920.\]

Next, we shield off the central copies of the vertices 1, 2 and 3 by gluing in the tetrahedra

\[1151617, 1151720, 2171819, 2171920, 3141519, 3151920, 15171920,\]

and obtain as a lower envelope (i.e., boundary) of the above tetrahedra 13 triangles, which, together with the back triangle 123, give a triangulated 2-sphere; see Figure 6 (right). We close this void by taking the cone over these 13 + 1 triangles with respect to the additional vertex 21:

\[121421, 131821, 1141921, 1181921, 231621, 2141521, 2151621, 3161721, 3171821, 14151921, 15161721, 15171921, 17181921, 12321.\]

The resulting complex \(\Sigma_2 \cup \Sigma'_3\) has face vector \(f = (25, 128, 218, 114)\) and admits the discrete Morse vectors \((1, 1, 1, 0)\) and \((1, 0, 1, 1)\).

For the vector \((1, 1, 1, 0)\), we have to collapse away all tetrahedra, where the only way to begin with is via the unique free triangle 123. Once we have entered into the solid 3-dimensional body we layer-wise collapse away the “interior”, so that we are left with the identified (upper) boundary. Under the identifications, the seven copies of the subdivided triangle 123 fold up to just one copy, as displayed in Figure 5, which, in turn, replaces the 123 in Figure 3. However, the once free edge 12 then is still glued to an interior edge and thus is not free. As a consequence, a triangle, say, 124 has to be marked as critical before we can continue with collapses. But then we are also forced to pick up a critical edge, yielding \((1, 1, 1, 0)\) as discrete Morse vector.

For the vector \((1, 0, 1, 1)\), we initially remove a tetrahedron, say, 12321 as critical and then empty out the interior of the solid 3-dimensional body. We then are left with \(\Sigma_2\) of Figure 5 union the “upper” triangles of Figure 5 which together is a space homotopy equivalent to the 2-sphere. If we remove one of the triangles of Figure 5 as critical, we can collapse down to \(\Sigma_2\), which is collapsible, thus resulting in \((1, 0, 1, 1)\) as a discrete Morse vector.

The vectors \((1, 1, 1, 0)\) and \((1, 0, 1, 1)\) are not optimal for \(\Sigma_2 \cup \Sigma'_3\) — in fact, \(\Sigma_2 \cup \Sigma'_3\) is collapsible with optimal discrete Morse vector \((1, 0, 0, 0)\). A respective collapsing is not obvious (we found some with the computer). It has to start via the unique free triangle 123. Instead of emptying out the whole interior of the solid body and not touching its identified boundary, we have to drill tunnels so that one of the nine triangles of Figure 5 becomes free. Then we can perforate the membrane of triangles of Figure 5 and, eventually, free the edge 12.

In order to avoid that the resulting space is collapsible, but maintaining the vectors \((1, 1, 1, 0)\) and \((1, 0, 1, 1)\), we glue on top of each of the triangles

\[t_1 t_2 t_3 \in \{125, 126, 124, 135, 146, 234, 256, 346, 356\}\]

a copy of \(\Sigma_3\) along its free face, thus contributing 12 − 3 = 9 additional vertices for each copy. The encoding of the copies of \(\Sigma_3\) is as follows. For \(a = 26 + 9j, j = 0, \ldots, 8\), and \(b = a + k, k = 1 \ldots 7\), we add the tetrahedra

\[t_1 t_2 a b, \ t_1 t_3 a b, \ t_2 t_3 a b\]

to shield off the nine stacking vertices \(a = 26 + 9k, j = 0, \ldots, 8\). We then glue in the tetrahedra

\[t_2 t_3 a + 1 a + 2, \ t_1 t_2 a + 2 a + 3, \ t_1 t_3 a + 3 a + 4, \ t_2 t_3 a + 4 a + 5, \ t_1 t_2 a + 5 a + 6, \ t_1 t_3 a + 6 a + 1\]

as well as

\[t_1 t_3 a + 2 a + 7, \ t_1 t_2 a + 4 a + 7, \ t_2 t_3 a + 6 a + 7\]
to join the cones over the stacked triangles and then the tetrahedra
\[
\begin{align*}
    & t_1 a + 2 a + 3 a + 4, \quad t_1 a + 2 a + 4 a + 7, \\
    & t_2 a + 4 a + 5 a + 6, \quad t_2 a + 4 a + 6 a + 7, \\
    & t_3 a + 1 a + 2 a + 6, \quad t_3 a + 2 a + 6 a + 7
\end{align*}
\]
over the “boundary” edges, with
\[
a + 2 a + 4 a + 6 a + 7
\]
as a final shielding piece. In a last step, we glue in the cone with respect to the vertex \(a + 8\) to fill in the central void:
\[
\begin{align*}
    & t_1 t_2 t_3 a + 8, \\
    & t_1 t_2 a + 1 a + 8, \\
    & t_1 a + 1 a + 6 a + 8, \\
    & t_2 a + 1 a + 2 a + 8, \\
    & t_2 a + 2 a + 3 a + 8, \\
    & t_3 a + 3 a + 4 a + 8, \\
    & a + 1 a + 2 a + 6 a + 8, \\
    & a + 2 a + 3 a + 4 a + 8, \\
    & a + 4 a + 5 a + 6 a + 8, \\
    & a + 2 a + 4 a + 6 a + 8.
\end{align*}
\]
For simplicity, the resulting complex \(\Sigma_2 \cup \Sigma_3' \cup_{i=1,...,9} \Sigma_3\) will be called \texttt{two_optima}; a list of facets of the complex is available online at [9].

**Theorem 5.** There is a contractible, but non-collapsible 3-dimensional simplicial complex \texttt{two_optima} with face vector \(f = (106, 596, 1064, 573)\) that has two distinct optimal discrete Morse vectors, \((1,1,1,0)\) and \((1,0,1,1)\).

*Proof.* To see that the complex \texttt{two_optima} admits the discrete Morse vectors \((1,1,1,0)\) and \((1,0,1,1)\) we can proceed as above. For \((1,1,1,0)\), we enter via the unique free triangle \(123\), empty out the interior of \(\Sigma_3'\). At this point, we can enter the nine copies of \(\Sigma_3\) via the freed triangles \(125, 126, 124, 135, 146, 234, 256, 346, 356\) and empty out the interiors of the copies. By construction, the edge \(12\) then is blocked, but can be freed once we declare one of the boundary triangles of \(\cup_{i=1,...,9} \Sigma_3\) to be critical. For the vector \((1,0,1,1)\) we can proceed similarly, once we have marked as critical and removed one of the interior tetrahedra of \(\Sigma_3'\).

Finally, we need to see that \texttt{two_optima} is not collapsible. Since each copy of \(\Sigma_3\) can only be entered via one free triangle, say, \(abc\) that is glued on top of \(\Sigma_3'\), we need to first empty out all tetrahedra of \(\Sigma_3'\) that touch the triangle \(abc\) before we can enter the respective copy of \(\Sigma_3\). At that point, the copy of \(\Sigma_3\) is glued to the remainder via the three boundary edges \(ab, ac\) and \(bc\) of the triangle \(abc\). These three edges stay fixed, in particular, cannot be free. Once we have collapsed the copy of \(\Sigma_3\), we obtain a membrane, which is (homotopy equivalent to) a 2-dimensional disc glued in along the boundary of the triangle \(abc\). It follows that none of the edges \(ab, ac\) and \(bc\) as well as none of the boundary edges of the other copies \(\Sigma_3\) can become free, before we mark some triangle as critical and perforate the collection of 2-dimensional membranes this way. \(\square\)

Our construction can be generalized to dimensions \(d > 3\).

**Lemma 6.** Let \(D\) be a \(d\)-disk, and let \(\gamma\) be any \((d-2)\)-loop in \(\partial D\). Let \(D' \subset D\) be any CW complex such that \(D\) deformation retracts to \(D'\) which contains \(\gamma\). Then there is a \((d-1)\)-disk \(d' \hookrightarrow D'\) with \(\partial d' \hookrightarrow \gamma\).

*Proof.* Choose a disk \(d \in \partial D\) with \(\partial d = \gamma\). Then the deformation retract \(f_t : D \longrightarrow D'\) deforms \(d\) to \(d'\). \(\square\)

**Theorem 7.** For every \(d \geq 3\) there is a contractible, but non-collapsible simplicial \(d\)-complex that has two distinct optimal discrete Morse vectors \((1,0,\ldots,0,1,1,0)\) and \((1,0,\ldots,0,0,1,1)\).
Proof. As before in the 3-dimensional case, for $d \geq 4$, we start with a copy of $\Sigma_{d-1}$ for which its free face is labeled $1 2 \ldots d-1$ and is contained in the $(d-1)$-simplex $1 2 \ldots d-1$. On top of the $(d-1)$-simplex $1 2 \ldots d-1$ of $\Sigma_{d-1}$ we glue a modified copy $\Sigma_d'$ of $\Sigma_d$ along the unique free face $1 2 \ldots d-1$ of $\Sigma_d'$. Here, the modified version $\Sigma_d'$ is obtained from $\Sigma_d$ by identifying the face $1 2 \ldots d-1$ with a $(d-2)$-dimensional disk on the identified boundary of $\Sigma_d$ such that the face $1 2 \ldots d-1$ and its image intersect in the vertex 1. On top of the $(d-1)$-dimensional faces of the identified boundary of $\Sigma_d'$ (up to a suitable subdivision), we glue copies of $\Sigma_d$ to avoid (by Lemma 6) perforation.

The previous proof can be adapted to show something stronger. In fact, each of the algorithms random-lex, random-revlex, random-discrete-Morse does not remove any $d$-face as critical as long as nontrivial collapses of $d$-faces are possible. Therefore, when applied to two-optima, none of the proposed algorithms can see a discrete Morse function with a critical 3-cell.

**Definition 8** (Monotone discrete Morse function). A monotone discrete Morse function on a simplicial complex $C$ is a map $f : C \to \mathbb{Z}$ satisfying the following six axioms:

1. if $\sigma \subseteq \tau$, then $f(\sigma) \leq f(\tau)$;
2. the cardinality of $f^-(q)$ is at most 2 for each $q \in \mathbb{Z}$;
3. if $f(\sigma) = f(\tau)$, then either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$;
4. for any $\sigma \subseteq \tau$ and $\sigma' \subseteq \tau'$, if $f(\sigma) = f(\tau) \leq f(\sigma') = f(\tau')$ then $\dim \tau \leq \dim \tau'$;
5. $f(C) = [0, M]$, for some $M \in \mathbb{N}$;
6. for any critical face $\Delta$ (that is, a face such that $f(\sigma) \neq f(\Delta)$ for each face $\sigma \neq \Delta$), the complex $\{ \sigma \in C \text{ s.t. } f(\sigma) \leq f(\Delta) \}$ has no free $(\dim \Delta - 1)$-dimensional face.

In [8], the discrete Morse spectrum of a simplicial complex was defined as the set of all possible outcomes of the random discrete Morse algorithm along with the respective probabilities for the individual vectors. We next give an alternative definition in terms of monotone discrete Morse functions.

**Definition 9** (Discrete Morse spectrum). Let $C$ be any simplicial complex. The discrete Morse spectrum of $C$ is the set of all discrete Morse vectors of all monotone discrete Morse functions on $C$.

**Remark 10.** Given an arbitrary discrete Morse function $f$ in the sense of Forman, it is easy to produce a function $g$ that induces the same Morse matching of $f$ and in addition satisfies (i), (ii), (iii), (iv) and (v). In fact, as explained by Forman [19], the function $f$ induces a step-by-step deconstruction of the complex: in the $i$-th step we either delete a critical face $\Delta_i$, or we delete a pair of faces $(\sigma_i, \Sigma_i)$, with $\sigma_i$ free. In the first case, let us set $\tilde{g}(\Delta_i) = -i$; in the second case, we set $\tilde{g}(\sigma_i) = \tilde{g}(\Sigma_i) = -i$. We leave it to the reader to verify that the resulting map $\tilde{g}$ satisfies axioms (i) to (iv). If $M = \min\{\tilde{g}(\sigma) : \sigma \in C\}$, the function $g$ we seek is then defined by $g(\sigma) = \tilde{g}(\sigma) + M$ for each $\sigma$.

**Remark 11.** The sixth axiom is instead “really restrictive”. If $C$ is any complex of dimension $\geq 1$, the $f$-vector of $C$ is not in the discrete Morse spectrum, though it is obtainable by some discrete Morse function in the sense of Forman (corresponding to the “empty matching”, which leaves all faces critical). The sixth axiom in fact forbids us to declare a $k$-face critical if there are still free $(k-1)$-faces available.

Any vector output from any of the strategies random-discrete-Morse, random-revlex, random-lex, is by construction the discrete Morse vector of a monotone discrete Morse function. Therefore, none of them will ever output a vector corresponding to the empty matching. However, this is not at all a drawback: As we said, the essence of Forman’s theory is to come up with discrete Morse vectors that are as small as possible. So it is actually time-saving if some very non-optimal matchings (like the empty one, which is the worst possible) are systematically neglected by our algorithm.
We observed experimentally that these algorithms often find optimal matchings. This naturally triggers the question if the discrete Morse spectrum always contains an optimal vector. (If this is the case, then it might actually be more efficient, for all sorts of discrete-Morse-theoretic computations, to switch to the “monotone discrete Morse function” setup.)

What we can derive from Theorem 7 is that not all optimal discrete Morse vectors belong to the spectrum:

Corollary 12. On the complex two optima of Theorem 7, no monotone discrete Morse function reaches the optimal Morse vector \((1, 0, 1, 1)\).

Problem 13. Is at least one of the optimal discrete Morse vector reachable via a monotone discrete Morse function? If so, does the lexicographically-largest among the optimal discrete Morse vectors always belong to the spectrum?

4 A collapsible 5-manifold different from the 5-ball

According to Whitehead [29], every collapsible combinatorial \(d\)-manifold is a combinatorial \(d\)-ball. On the other hand, every contractible \(d\)-manifold, if \(d \neq 4\), admits a collapsible triangulation [1]; and in each dimension \(d \geq 5\), there exist non-PL triangulations of contractible \(d\)-manifolds different from \(d\)-balls.

In this section, our aim is to construct a first explicit “small” example of a collapsible non-PL triangulation of a contractible 5-manifold different from the 5-ball. Our construction is in seven steps and is based on ideas from [1]. In every step of the construction, we try to save on the size of the intermediate complexes.

1. Start with a (small) triangulation of a non-trivial homology 3-sphere. The smallest known triangulation of a non-trivial homology 3-sphere is the 16-vertex triangulation \(\text{poincare} \) of the Poincaré homology 3-sphere \(\Sigma^3\). In fact, the triangulation \(\text{poincare} \) with \(f = (16, 106, 180, 90)\) is conjectured to be the smallest triangulation of \(\Sigma^3\) and to be the unique triangulation of \(\Sigma^3\) with this \(f\)-vector [10, Conj. 6]. Triangulations of other non-trivial homology 3-spheres are believed to require more vertices and faces than \(\text{poincare} \); cf. [25].

2. Remove a (large) triangulated ball. The triangulation \(\text{poincare} \) has vertex-valence-vector \(\text{val} = (14, 14, 11, 14, 14, 11, 14, 12, 13, 15, 15, 14, 15, 15, 15, 6)\). In particular, there are five vertices of valence 15 whose vertex-stars have 26 tetrahedra each. We remove the star of vertex 15 from \(\text{poincare} \) and relabel vertex 16 to 15. The resulting 15-vertex triangulation \(\text{poincare} \text{minus ball} \) has 64 tetrahedra.

3. Take the cross product with an interval. The cross product of \(\text{poincare} \text{minus ball} \) with an interval \(I\) is a prodsimplicial complex with \(2 \cdot 15 = 30\) vertices. Any prodsimplicial complex can be triangulated without adding new vertices; see [24] and references therein. This way, we obtain a 4-dimensional simplicial complex \(\text{poincare} \text{minus ball} \times I\), which is a 30-vertex triangulation of a 4-manifold \(K^4\) that has the connected sum \(\Sigma^3 \# \Sigma^3\) as its boundary.

4. Add cone over boundary. If we add to \(K^4\) the cone over its boundary with respect to a new vertex 31, then the resulting 4-dimensional simplicial complex \(L^4\) is a combinatorial 4-pseudomanifold, i.e., all of its vertex-links are combinatorial 3-manifolds. In fact, the links of the vertices 1, . . . , 30 in \(L^4\) are triangulated 3-spheres, whereas the link of vertex 31 in \(L^4\) is a triangulation of the homology 3-sphere \(\Sigma^3 \# \Sigma^3\).

5. Perform a one-point suspension. By the double suspension theorem of Edwards [17] and Cannon [12], the double suspension \(\text{susp}(\text{susp}(H^d))\) of any homology \(d\)-sphere \(H^d\) is homeomorphic to the standard sphere \(S^{d+2}\). In our case, \(L^4\) is homeomorphic to the single suspension \(\text{susp}(\Sigma^3 \# \Sigma^3)\) and therefore

\[
\text{susp}(L^4) \cong \text{susp}(\text{susp}(\Sigma^3 \# \Sigma^3)) \cong S^5.
\]
The triangulation $\text{susp}(L^4)$ of $S^5$ with $31 + 2$ vertices is non-PL, since $\Sigma^4 \# \Sigma^3$ occurs as the link of the edge $31–33$ (and of the edge $31–32$) in $\text{susp}(L^4)$. Instead of taking the standard suspension $\text{susp}(L^4)$ with respect to two new vertices 32 and 33, the one-point suspension uses one vertex less \cite{10} — for this, we simply contract the edge $31–33$ in $\text{susp}(L^4)$. The resulting complex $\Sigma^5_{32}$ then has face-vector $f(\Sigma^5_{32}) = (32, 349, 1352, 2471, 2154, 718)$.

6. **Subdivide barycentrally.** Let $\text{sd}(\Sigma^5_{32})$ be the barycentric subdivision of $\Sigma^5_{32}$ with

$$f(\text{sd}(\Sigma^5_{32})) = (7076, 152540, 807888, 1696344, 1550880, 516960).$$

This step is expensive, but paves the way for the final part of our construction.

7. **Collar the PL singular set in $\text{sd}(\Sigma^5_{32})$.** In the barycentric subdivision $\text{sd}(\Sigma^5_{32})$, there are precisely three vertices, $v_{31}$, $v_{31–32}$ and $v_{32}$, that do not have combinatorial 4-spheres as their vertex-links — these three vertices are connected by the edges $e_{31,31–32}$ and $e_{31–32,32}$ for which their links are triangulations of $\Sigma^5 \# \Sigma^4$. The (contractible!) subcomplex of $\text{sd}(\Sigma^5_{32})$ formed by these two edges is the $\text{PL singular set}$ of $\text{sd}(\Sigma^5_{32})$. Let the collar of the contractible PL singular set in $\text{sd}(\Sigma^5_{32})$ be the simplicial complex $\text{contractible non 5 ball}$; see \cite{9} for lists of facets of the complex and of its boundary $\text{contractible non 5 ball boundary}$. Then $\text{contractible non 5 ball}$ is a non-PL triangulation of a contractible 5-manifold different from the 5-ball $B^5$ with face-vector

$$f(\text{contractible non 5 ball}) = (5013, 72300, 290944, 495912, 383136, 110880).$$

**Theorem 14.** There is a collapsible 5-manifold $\text{contractible non 5 ball}$ different from the 5-ball with $f = (5013, 72300, 290944, 495912, 383136, 110880)$.

**Proof.** By construction, the simplicial complex $\text{contractible non 5 ball}$ triangulates a contractible 5-manifold different from the 5-ball. The boundary of the resulting 5-manifold is PL homeomorphic to the double over the homology ball $\Sigma^3 \setminus \Delta \times I$, where $\Sigma^3$ is the starting Poincaré homology sphere and $\Delta$ is any facet. In particular, the boundary of $\text{contractible non 5 ball}$ is a combinatorial 4-manifold $\text{contractible non 5 ball boundary}$ with trivial homology, but is not simply connected — it has the binary icosahedral group as its fundamental group, in contrast to the boundary of the 5-ball. Moreover, as a regular neighborhood of a tree, it is immediate that the complex is PL collapsible, i.e., some suitable subdivision of it is collapsible. Here, we used the random-discrete-Morse implementation of \cite{8} to directly find an explicit collapsing sequence. It took 60:17:33 h:min:sec to build the Hasse diagram of the example containing

$$2 \cdot 72300 + 3 \cdot 290944 + 4 \cdot 495912 + 5 \cdot 383136 + 6 \cdot 110880 = 5582040 \text{ edges}.$$ 

In a single random run of the program in 21:41:31 h:min:sec, a discrete Morse vector $(1, 0, 0, 0, 0, 0)$ was achieved. This proves the collapsibility of the example. \hfill \square

**Corollary 15.** The boundary of $\text{contractible non 5 ball}$ is a combinatorial 4-dimensional homology sphere $\text{contractible non 5 ball boundary}$ with $f = (5010, 65520, 212000, 252480, 100992)$ that has the binary icosahedral group as its fundamental group.

### 5 Asymptotic complicatedness of barycentric subdivisions

Mesh refinements are often used in discrete geometry to force nice combinatorial properties, while at the same time maintaining the existing ones. The price to pay is of course an increase (linear, polynomial, or even exponential) in the computation.

A particularly effective refinement, from this point of view, is the “barycentric subdivision”, as the following results suggest:

1. given an arbitrary PL ball, some iterated derived subdivision of it is collapsible \cite{11};
2. for every PL sphere, some iterated derived subdivision of it is polytopal \cite{3};
(3) for every PL triangulation of any smooth manifold that has a handle decomposition of $c_i$
i handles, some iterated barycentric subdivision of it admits $(c_0, \ldots, c_d)$ as optimal discrete Morse vector [3].

In this section we focus on the average discrete Morse vector, rather than on the smallest one. In [8, p. 13] it was observed experimentally that the (observed) average number of critical cells in the random computation of discrete Morse vectors decreases for “complicated” triangulations after performing a single barycentric subdivision. Here we show that this experiment is misleading, in the sense that the observed average should rapidly increase after a finite number of barycentric subdivisions.

2 We show that after a constant number of barycentric subdivisions (a number larger than 1, but universally bounded, i.e., independent of the 3-complex chosen) every 3-complex $C$ will contain many disjoint, 2-dimensional copies of the dunce hat as subcomplexes. As a consequence, the expected number of critical cells in a discrete Morse vector for the $n$-th barycentric subdivision of any 3-ball, grows exponentially in $n$.

**Lemma 16.** There are universal constants $\alpha \in \mathbb{N}$ and $p \in (0, 1]$ such that

(i) the $\alpha$-th iterated barycentric subdivision $\text{sd}^\alpha T$ of the tetrahedron $T$ contains a copy of the dunce hat in its 2-skeleton;

(ii) and with probability $p > 0$, any of the algorithms random-discrete-Morse, random-revlex and random-lex collapses $\text{sd}^\alpha T$ to a 2-complex containing the dunce hat.

**Proof.** A well known result in PL topology (cf. [30]) is that any regular neighborhood of the dunce hat in $\mathbb{R}^3$ is a (PL) 3-ball; moreover, any regular neighborhood contains the original complex in its interior, and collapses in the PL category onto the original complex. In other words, there is a 3-ball $C$ such that

(1) $C$ is PL, that is, we can find a PL homeomorphism that takes $C$ to the tetrahedron $T$, and

(2) some simplicial subdivision of $C$ collapses onto (a subdivision of) the dunce hat $D$.

In [1], it is shown that if $A$ and $B$ are simplicial complexes, $B \subset A$, and $\varphi$ is a PL homeomorphism such that $\varphi A$ collapses to $\varphi B$, then there is a non-negative integer $n$ such that $\text{sd}^n A$ collapses to $\text{sd}^n B$, and such that $\varphi$ is facewise linear on $\text{sd}^n A$.

Let us apply this to the ball $C$ above and the dunce hat $D$ inside it, with $\varphi$ chosen as any PL homeomorphism that maps $C$ to the tetrahedron $T$. If $\ell = m + n$; then $\text{sd}^{\ell} C$ collapses to $\text{sd}^{\ell} D$, and $\varphi(\text{sd}^{\ell} C)$ is a subdivision of $T$.

Applying again the result of [1] (with $\varphi = \text{id}$), we conclude that for some integer $\alpha$, $\text{sd}^{\alpha-1} T$ collapses onto a subdivision of the dunce hat. Therefore, there is a labeling of the vertices of $\text{sd}(\text{sd}^{\alpha-1} T)$ such that it collapses to a triangulation of the dunce hat with the lex and revlex orders. In particular, the probability that any of the algorithms above collapses $\text{sd}(\text{sd}^{\alpha-1} T)$ to the dunce hat is strictly positive.

Thus, if we have a 3-complex $C$ with $N$ facets, by subdividing $C$ barycentrically $\alpha$ times, we obtain a 3-complex that contains in its 2-skeleton $N$ disjoint copies of the dunce hat (one per each tetrahedron of $C$). To study how discrete Morse theory behaves on such a complex, we start with a simple lemma. In the following, let $|\beta(K)|$ denote the sum of the (unreduced) Betti numbers of a complex $K$.

**Lemma 17.** Let $K$ be a 2-dimensional simplicial complex that contains $r$ disjoint copies of the dunce hat. Then $K$ does not admit a discrete Morse function with less than $2r + |\beta(K)|$ critical cells.

---

2Of course, observing this phenomenon experimentally may be difficult, due to (1) computational expense of treating barycentric subdivisions after the first or second iteration, (2) the possibility that the probability $p$ determined by Lemma 16 is so small that one needs an extremely large $n$ to appreciate the effect.
Lemma 17, inside sd

Proof. We prove the theorem by double induction, on $r$ and on the number of faces ("size") of $K$. The case $r = 0$ is a straightforward consequence of the Morse inequalities. Thus, let $r \geq 1$ and let us assume the claim is proven for all 2-complexes $K'$ with at most $r$ disjoint embeddings of the dunce hat and less faces than $K$. Let $f$ be any optimal discrete Morse function, and let $\sigma$ be the facet at which $f$ attains its maximum. There are three cases to consider:

- $\sigma$ is not critical. In this case there is a free face $\rho$ of $\sigma$ such that $f(\rho) = f(\sigma)$. Note that the face $\rho$ cannot belong to a copy of the dunce hat, because in the dunce hat every edge belongs to two or three triangles. The simultaneous removal of both $\rho$ and $\sigma$ is then an elementary collapse; let $K'$ be the complex obtained. $K'$ has the same homology of $K$, and it still has $r$ copies of the dunce hats embedded. The claim follows by applying the induction assumption to $K'$.

- $\sigma$ is a critical face of $K$ not intersecting any copy of the dunce hat. In this case we simply delete $\sigma$. With a simple cellular homology argument, one can show that $|\beta(K - \sigma)| \geq |\beta(K)| - 1$. The restriction of $f$ to $K - \sigma$ has fewer than $2r + |\beta(K)| - 1$ critical cells (we deleted one), and $2r + |\beta(K - \sigma)| \geq 2r + |\beta(K)| - 1$. The claim follows by applying the induction assumption to $K' = K - \sigma$.

- $\sigma$ is a critical face and belongs to an embedding $D$ of the dunce hat. We have $\beta_1(D - \sigma) = \beta_1(D) + 1$ (and $\beta_i(D - \sigma) = \beta_i(D)$ otherwise). Therefore, by cellular homology, we obtain that $\beta_1(K - \sigma) = \beta_1(K) + 1$ (and $\beta_i(K - \sigma) = \beta_i(K)$ otherwise). Hence $|\beta(K - \sigma)| \geq |\beta(K)| + 1$, and $(K - \sigma)$ is a 2-complex containing at least $r - 1$ copies of the dunce hat. The claim follows again by applying the induction assumption to $K' = K - \sigma$. \qed

For the next result, we need to fix some notation first. Given a simplicial complex $K$, we use $f_i(K)$ to denote the number of $i$-dimensional faces of $K$. We denote by $X_{c_{rdM}}(K)$, $X_{c_{lex}}(K)$ resp. $X_{c_{rev}}(K)$ the random variables for the number of critical faces of the random-discrete-Morse, random-lex and random-revlex strategies on $K$, respectively, and let $E_{c_{rdM}}(K)$, $E_{c_{lex}}(K)$ resp. $E_{c_{rev}}(K)$ denote the associated expected values. We write $X_*(K)$ resp. $E_*(K)$ when a statement applies to all three strategies.

Recall that we write $f = \Omega(g)$ resp. $f = O(g)$ for real valued functions $f : \mathbb{N} \rightarrow \mathbb{R}$ and $g : \mathbb{N} \rightarrow \mathbb{R}$ if there is a $c > 0$ such that for all $x \in \mathbb{R}$ large enough, $f(x) \geq cg(x)$ resp. $f(x) \leq cg(x)$. We write $f = \Theta(g)$ if $f = \Omega(g)$ and $f = O(g)$.

Theorem 18. Let $K$ be a simplicial 3-complex. Then, for all $n \geq 0$

$$\mathbb{P}[X_*(sd^{n+3}K) \leq \ell] \leq (1 - p)^f_3(K) \cdot 24^n - \ell.$$  

In particular, $\log E_*(sd^nK) = \Omega(n)$.

Proof. For any nonnegative integer $n$, one has $f_3(sd^nB) = 24^n \cdot f_3(B)$. On the other hand, by Lemma [17] inside $sd^nB$ each of the tetrahedra of $B$ is subdivided sufficiently fine enough to contain a copy of the dunce hat. \qed

The bound provided by Theorem [18] is asymptotically tight:

Corollary 19. Let $K$ be any simplicial complex of dimension $d \geq 3$. Then

$$\log E_*(sd^nK) = \Theta(n).$$

Proof. By Theorem [18] we have $\log E_*(sd^nK) = \Omega(n)$. Furthermore, it is clear that $\log E_*(sd^nK)$ is bounded above by $\log f(sd^nK) = \log(2^d(d!)^n f(K)) = \log n$, where $f(\cdot)$ denotes the total number of faces of a simplicial complex. \qed

A similar behavior for the randomized algorithms with respect to simplicial polytopes can be observed when we increase the dimension of the polytopes concerned.
Theorem 20. There is a constant \( c \) such that for any simplicial polytope \( P \) we have
\[
\mathbb{E}_c(P) \geq \Omega(\dim P).
\]

Proof. Let \( P \) be a simplicial \( d \)-polytope. Then \( P \) contains a subdivision of the \((d-1)\)-dimensional simplex. Therefore, if \( T \) is the 3-dimensional tetrahedron, then \( P \) contains a subdivision of \( sd^n T \), where \( n = \lfloor \log_4 \frac{d}{4} \rfloor \). The assertion now follows from Theorem 18. \( \square \)

6 Random-lex and random-revlex strategies

In [8], a library of 45 triangulations was provided as a (non-trivial) testing ground for discrete Morse (heuristical) algorithms. For 39 out of the 45 examples, optimal discrete Morse vectors were found, either with the random-discrete-Morse search or with the random-lex and random-revlex strategies. For 4 of the 6 open cases,

- hyperbolic dodecahedral space, \((1,4,4,1)\),
- non_PL, \((1,0,0,2,2,1)\),
- S2xPoincare, \((1,2,3,3,2,1)\),
- contractible vertex homogeneous, \((1,0,0,4,8,4,0,0,0,0,0)\),

we miss appropriate lower bounds to show optimality of the found discrete Morse vectors. In the case of the 3-ball triangulation knot, \((1,1,1,0)\) was the best vector obtained in [8], while Lewiner in [21] claims to have found \((1,0,0,0)\). For the example triple trefoil bsd, \((1,1,1,1)\) was reached in [8].

Theorem 21. The triangulated 3-sphere example triple trefoil bsd has the perfect discrete Morse vector \((1,0,0,1)\) in its discrete Morse spectrum.

We found \((1,0,0,1)\) for triple trefoil bsd with both the randomized versions of the lex and rev-lex strategies; see Table 1 for the respective sampled spectra (over 10000 runs).

Table 1 displays two further examples, nc_sphere and bing, for which the optimal vector \((1,0,0,1)\) was found way more often with the random-lex and the random-revlex strategies, compared to random-discrete-Morse. Also, for the examples poincare and non_PL (and for other examples as well) we see a dramatic simplification of the spectrum when using random-revlex (and random-lex). For the library examples listed in Table 2, we highlighted (in bold), which of the three random strategies yields the smallest average of critical cells. In most cases, the random-revlex strategy scores best — and even in the case triple trefoil bsd where it did not, it revealed the optimum. The last column of Table 2 lists the best known theoretical lower bound for the number of critical cells in a discrete Morse vector. The number is in bold if we can prove that it is actually achievable, i.e. there is a discrete Morse function with that many critical cells.
### Appendix: Tables

**Table 1:** Distribution of discrete Morse vectors in 10000 rounds.

| Morse Vectors | random | random-lex | random-revlex |
|---------------|--------|------------|---------------|
| poincare      |        |            |               |
| (1,2,2,1)     | 9073   | (1,2,2,1)  | 9992          |
| (1,3,3,1)     | 864    | (1,3,3,1)  | 8             |
| (1,4,4,1)     | 45     | (1,3,3,1)  | 1             |
| (2,4,3,1)     | 7      |             |               |
| (2,3,2,1)     | 6      |             |               |
| (1,5,5,1)     | 5      |             |               |
| non_PL        |        |            |               |
| (1,0,0,2,2,1) | 9383   | (1,0,0,2,2,1)| 9991          |
| (1,0,0,3,3,1) | 441    | (1,0,0,3,3,1)| 9             |
| (1,0,1,3,2,1) | 134    |             |               |
| (1,0,0,4,4,1) | 25     |             |               |
| (1,0,1,4,3,1) | 12     |             |               |
| (1,0,2,4,2,1) | 2      |             |               |
| (1,0,0,5,5,1) | 2      |             |               |
| (1,0,2,5,3,1) | 1      |             |               |
| (1,1,2,3,2,1) | 1      |             |               |
| (1,0,4,6,2,1) | 1      |             |               |
| nc_sphere     |        |            |               |
| (1,1,1,1)     | 7902   | (1,1,1,1)  | 7550          |
| (1,2,2,1)     | 1809   | (1,2,2,1)  | 1660          |
| (1,3,3,1)     | 234    | (1,0,0,1)  | 720           |
| (1,4,4,1)     | 25     | (1,3,3,1)  | 64            |
| (1,0,0,1)     | 12     | (2,3,2,1)  | 4             |
| (2,3,2,1)     | 9      | (1,4,4,1)  | 2             |
| (1,6,6,1)     | 3      |             |               |
| (2,4,3,1)     | 3      |             |               |
| (2,5,4,1)     | 2      |             |               |
| (1,5,5,1)     | 1      |             |               |
| bing          |        |            |               |
| (1,1,1,0)     | 9764   | (1,1,1,0)  | 9484          |
| (1,2,2,0)     | 217    | (1,0,0,0)  | 280           |
| (1,0,0,0)     | 7      | (1,2,2,0)  | 233           |
| (1,3,3,0)     | 6      | (2,3,2,0)  | 2             |
| (2,3,2,0)     | 6      | (1,3,3,0)  | 1             |
| triple_trefoil_berd | | | |
| (1,2,2,1)     | 4793   | (1,2,2,1)  | 5193          |
| (1,1,1,1)     | 3390   | (1,3,3,1)  | 2531          |
| (1,3,3,1)     | 1543   | (1,1,1,1)  | 1966          |
| (1,4,4,1)     | 208    | (1,4,4,1)  | 278           |
| (1,5,5,1)     | 22     | (1,5,5,1)  | 19            |
| (2,3,2,1)     | 20     | (1,0,0,1)  | 7             |
| (2,4,3,1)     | 17     | (2,3,2,1)  | 3             |
| (1,6,6,1)     | 3      | (2,4,3,1)  | 3             |
| (2,5,4,1)     | 3      | (2,5,4,1)  | 2             |
| (1,8,8,1)     | 1      | (1,6,6,1)  | 2             |
Table 2: Average numbers of critical cells in 10000 runs.

| Name of example                      | random | random-lex | random-revlex | ≥       |
|--------------------------------------|--------|------------|---------------|---------|
| d2n12g6                              | 14.0558| 14.0000    | 14.0000       | 14      |
| regular_2_21_23_1                    | 32.1354| 32.0000    | 32.0000       | 32      |
| rand2_n25_p0.328                     | 482.9612| 476.7076  | 477.7182      | 476     |
| trefoil_arc                          | 1.0952 | 1.6246     | 1.2180        | 1       |
| trefoil                              | 2.0778 | 2.2164     | 2.0330        | 2       |
| double_trefoil_arc                   | 3.6298 | 3.7690     | 3.2718        | 3       |
| poincare                             | 6.1978 | 6.0016     | 6.0002        | 6       |
| double_trefoil                       | 3.5356 | 3.7664     | 3.0778        | 2       |
| triple_trefoil                       | 5.9558 | 5.9376     | 5.2700        | 5       |
| triple_trefoil                       | 6.0882 | 5.9408     | 5.0584        | 4       |
| hyperbolic_dodecahedral_space        | 11.5128| 10.1582    | 10.0906       | 8       |
| S_3_50_1033 (random)                 | 3.2114 | 3.2108     | 2.6104        | 2       |
| non_4_2_colorable                    | 30.6   | 25.3232    | 16.8154       | 2       |
| Hom_C5K4 (RP^3)                      | 4.0508 | 4.0300     | 4.0090        | 4       |
| trefoil_bsd                           | 2.0202 | 2.2388     | 2.1052        | 2       |
| knot                                 | 3.1228 | 3.1262     | 3.1034        | 1       |
| nc_sphere                            | 4.4788 | 4.2164     | 4.2728        | 2       |
| double_trefoil_bsd                   | 3.3426 | 3.8968     | 3.4406        | 2       |
| bing                                 | 3.0468 | 2.9918     | 2.9346        | 2       |
| triple_trefoil_bsd                   | 5.7432 | 6.2346     | 5.8460        | 2       |
| Hom_n9_655_compl_K4 ((S^2\times S^1)^#13) [100 runs] | 28.84 | 28.46     | 28.28        | 28      |
| CP2                                  | 3.0012 | 3.0000     | 3.0000        | 3       |
| RP4                                  | 5.0492 | 5.0000     | 5.0000        | 5       |
| K3_16 (unknown PL type)              | 24.8228| 24.0000    | 24.0000       | 24      |
| K3_17 (standard PL type)             | 24.8984| 24.0004    | 24.0000       | 24      |
| RP4\times K3_17                      | 28.58  | 27.0046    | 27.0018       | 27      |
| RP4\times S2\times S2               | 28.48  | 27.0088    | 27.0024       | 27      |
| Hom_n6_compl_K5_small ((S^2\times S^2)^#29) | 63.94 | 60.2824  | 60.0066      | 60      |
| Hom_n6_compl_K5 ((S^2\times S^2)^#29) [10 runs] | 83.2  | 65.0      | 61.6         | 60      |
| SU2\times S3                         | 4.1354 | 4.0000     | 4.0000        | 4       |
| non_PL                               | 6.1328 | 6.0018     | 6.0000        | 2       |
| RP5_24                               | 6.1770 | 6.0004     | 6.0000        | 6       |
| S2xPoincare                          | 15.70  | 12.078     | 12.028        | 4       |
| JHP2                                 | 3.1212 | 3.0000     | 3.0000        | 3       |
| contractible_vertex_homogeneous [10 runs] | 273.6 | 17.0      | 17.0         | 1       |

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