ON THE COVARIANT RELATIVISTIC SEPARABLE KERNEL

S.G. Bondarenko†, V.V. Burov, E.P. Rogochaya, and Y. Yanev

JINR, 141980, Dubna, Moscow region, Russia
† E-mail: bondarenko@jinr.ru

Abstract

Two different methods of the covariant relativistic separable kernel construction in the Bethe-Salpeter approach are considered. One of them leads in the center-of-mass system of two particles to the quasipotential equation. The constructed 4-dimensional covariant functions are used to reproduce neutron-proton phase shifts for total angular momenta $J = 0, 1$. Obtained results are compared with other model calculations.

1 Introduction

The relativistic description of the interaction between two nucleons can be carried out using the Bethe-Salpeter (BS) equation \[1\] and different quasipotential approximations to it \[2\]. One of the ways to solve the BS equation is using the separable presentation of its interaction kernel. The BS formalism with such kernel have been successfully applied to a covariant description of the elastic lepton-deuteron scattering, deuteron electro- and photodisintegration, deep inelastic scattering (DIS) of leptons on light nuclei. It has also provided good results for electromagnetic reactions with the deuteron. The approach facilitates the analysis of the deuteron properties and its comparison with nonrelativistic results (see, review \[3\] and references therein).

The consideration of the deuteron break up reaction demands the final state interaction between the outgoing nucleons to be taken into account. If the separable kernel of interaction is used it leads to the calculation of integrals with separable functions. Usually they represent a generalization of nonrelativistic separable functions \[4\]. The presence of second- and higher-order poles in the integrand containing these functions makes such calculation impossible. In the paper \[5\] we constructed the new parametrization with functions suggested in \[6\], which do not contain poles on the real axis in the relative energy complex plane, and apply them to the description of the neutron-proton (np) scattering. Within this approach we have achieved a good agreement with experimental data in the region of high kinematic energies of the nucleons $T_{\text{Lab}} \sim 1.2$ GeV. But as it was resumed in \[5\] it is necessary to improve an agreement with experimental data in a whole energy region up to $T_{\text{Lab}} < 3$ GeV by increasing the rank of the separable kernel of the nucleon-nucleon (NN) interaction. This work is in progress.

Before that we investigate a case when the separable NN kernel which represents a 3-dimensional function in the center-of-mass system (CMS). The use of such functions leads to the approaches called quasipotential where the two-body Green’s function in the BS equation is replaced by a suitable function which allows to reduce the 4-dimensional equation to 3-dimensional one, for example, like in \[2\].

In this work the one-rank separable kernels of NN interaction for the partial states with total momenta $J = 0, 1$ are presented for functions in 4-dimensional and 3-dimensional
forms. Parameters of the kernels are fitted by the calculation of phase shifts and low-energy observables (scattering length, effective range, deuteron bound state energy) using the constructed $T$ matrix.

2 Methods of a covariant relativistic generalization

One of the common methods of a covariant relativistic generalization of the Yamaguchi- and Tabakin-type functions is to replace 3-momentum squared by 4-momentum squared:

$$p^2 \rightarrow -p^2 = -p_0^2 + p^2. \quad (1)$$

This formal procedure converts 3-dimensional covariant functions to 4-dimensional ones.

Another method is based on the introduction of the formal 4-vector $Q$ through the relation:

$$Q = p - \frac{P \cdot p}{s} P,$$

with total momentum squared $s = P^2$.

Note that in CMS where $P = (\sqrt{s}, 0)$ the 4-vector $Q$ is defined by components $Q = (0, p)$ and thus

$$p^2 = -Q^2. \quad (3)$$

can be formally converted to the Lorentz invariant.

Let us consider the methods described above in application to the nonrelativistic Yamaguchi-type function

$$g(|p|) = \frac{1}{p^2 + \beta^2}. \quad (4)$$

In the first case using the substitution (1) we obtain the covariant function in the form:

$$g_p(p, P) = \frac{1}{-p^2 + \beta^2} \left[ \frac{1}{-p_0^2 + p^2 + \beta^2 + i\varepsilon} \right]. \quad (5)$$

In the second case we use the relation (3) and obtain the function:

$$g_Q(p, P) = \frac{1}{-Q^2 + \beta^2} \left[ \frac{1}{p^2 + \beta^2} \right]. \quad (6)$$

The presented functions have rather different properties in the relative energy $p_0$ complex plane in CMS. The function $g_p$ has two poles on the real axis for $p_0$ at $\pm \sqrt{p^2 + \beta^2} \mp i\varepsilon$ while the function $g_Q$ has no poles on it.

In practical calculations of the reactions with the high momentum transfer the $p_0$ integration can lead to singular expressions in functions $g_{p,Q}$ on $|p|$ or $\cos \theta_p$. Such problem can be easily solved by calculating the $|p|$ or $\cos \theta_p$ principal value integral. However, another form of functions $g_{p,Q}$ with odd powers in a denominator leads to nonintegrable singularities. Therefore we introduce functions $g_{p,Q}$ of type without poles on the real axis.
of the relative energy $p_0$ complex plane. As an example of such function we introduce the covariant form factors in the following form (see also the section 3 of the paper [5]):

\[
g_p(p, P) = \frac{p_c - p^2}{(p^2 - \beta^2)^2 + \alpha^4} \quad \text{CMS} \rightarrow \frac{p_c - p_0^2 + p^2}{(p_0^2 - p^2 - \beta^2)^2 + \alpha^4},
\]

and in the second case we use the relation (3) and obtain the function

\[
g_Q(p, P) = \frac{p_c - Q^2}{(Q^2 - \beta^2)^2 + \alpha^4} \quad \text{CMS} \rightarrow \frac{p_c + p^2}{(p^2 + \beta^2)^2 + \alpha^4}.
\]

We note that the function $g_Q$ still has no poles on the $p_0$ real axis while the $g_p$ has poles at $p_0$: $\pm \sqrt{p^2 + \beta^2 + i\alpha^2}$, $\pm \sqrt{p^2 + \beta^2 - i\alpha^2}$.

The two methods of a covariant relativistic generalization described above can be investigated by solving the Bethe-Salpeter equation for specific partial states.

3 Solution of the Bethe-Salpeter equation

The details of the solution of the BS equation with the one-rank separable kernel of interaction can be found in the paper [5]. The value to be calculated is $h(s)$:

\[
h(s) = -\frac{i}{4\pi^3} \int dp_0 \int |p|^2 d|p| S(p_0, |p|; s) g(p_0, |p|)^2
\]

with

\[
S(p_0, |p|; s) = \frac{1}{(\sqrt{s}/2 - E_p + i\varepsilon)^2 - p_0^2}
\]

being the two-particle Green’s function; here $E_p = \sqrt{p^2 + m^2}$, $m$ is the nucleon mass.

To obtain the function $h(s)$ the 2-dimensional integral on $p_0$ and $|p|$ should be calculated. To perform the integration on $p_0$ the Cauchy theorem is used. As it can be seen from Eq.(9) there are two types of singularities in the $p_0$ complex plane: first ones are poles of the function $S(p_0, |p|; s)$:

\[
p_0^{(1,2)} = \pm \sqrt{s}/2 \mp E_p \mp i\varepsilon,
\]

and other - poles of the function $g(p_0, |p|)$.

The function $g_p$ has four poles:

\[
\begin{align*}
p_0^{(3,4)} &= \pm \sqrt{p^2 + \beta^2 + i\alpha^2}, \\
p_0^{(5,6)} &= \pm \sqrt{p^2 + \beta^2 - i\alpha^2}
\end{align*}
\]

and to perform the $p_0$ integration residues in three poles of Eqs.(11) and (12) should be calculated. These calculations are performed analytically.

The function $g_Q$ has no poles on the $p_0$ real axis and therefore the only poles of Eq.(11) should be taken into account. The result for $h(s)$ can be written as:

\[
h(s) = \frac{1}{2\pi^2} \int |p|^2 d|p| \frac{g_Q(0, |p|)^2}{\sqrt{s} - 2E_p + i\varepsilon}.
\]
This equation formally coincides with that could be obtained within the Blankenbeckeler-Sugar-Logunov-Tavkhelidze (BSLT) approximation [2] which amounts to replacing the Green’s function in Eq. (9) by the expression

\[ S_{BSLT}(p_0, |p|; s) = -2\pi i (\sqrt{s} - 2E_p + i\varepsilon)^{-1}\delta(p_0). \] (14)

Although the solutions of the equation with functions \( g_Q \) and in the BSLT approximation coincide in CMS the difference becomes evident when the reaction with the two-particle system is considered. In that case the arguments of the function \( g_Q \) are calculated with the help of the Lorentz transformations in the system different from CMS.

4 One-rank kernel

We analyze two covariant relativistic generalizations of the Yamaguchi form factors: modified Yamaguchi (MY) functions and modified extended Yamaguchi (MEY) functions.

4.1 Modified Yamaguchi functions

For the description of the chosen partial states we use the following covariant expressions for both cases:

\[ g_p^{[S]}(p_0, p) = \frac{(p_{c_1} - p_0^2 + p^2)}{(p_0^2 - p^2 - \beta^2)^2 + \alpha^4}, \] (15)

\[ g_p^{[P]}(p_0, p) = \sqrt{-p_0^2 + p^2} \frac{1}{(p_0^2 - p^2 - \beta^2)^2 + \alpha^4}. \] (16)

\[ g_Q^{[S]}(p_0, p) = g_p^{[S]}(Q_0, Q) \xrightarrow{CMS} \frac{(p_{c_1} + p^2)}{(p^2 + \beta^2)^2 + \alpha^4}, \] (17)

\[ g_Q^{[P]}(p_0, p) = g_p^{[P]}(Q_0, Q) \xrightarrow{CMS} \frac{|p|}{(p^2 + \beta^2)^2 + \alpha^4}. \] (18)

It should be noted that the 4-vector \( Q \) is defined by the equation (2). The functions are numerated by angular momenta \( L = 0([S]), 1([P]) \). The numerator in \( g_p^{[S]} \) is introduced to compensate an additional dimension in the denominator to provide the total dimension as GeV\(^{-2}\). This form was chosen because at \( p_{c_1} = \beta^2, \alpha = 0 \) we get the function \( g_p^{[S]} \) in the standard Yamaguchi form [7]. We prefer not to consider the case with the square root in the denominator as it is used in [6] for \( S \) waves to avoid the calculations with cuts on the real axis in the \( p_0 \) complex plane.
4.2 Modified extended Yamaguchi functions

To extend the form of functions with increasing number of parameters we introduce the following $g_{p,Q}$ functions:

$$g^{[S]}_p(p_0, p) = \frac{(p_{c1} - p_0^2 + p^2)}{(p_0^2 - p^2 - \beta_1^2)^2 + \alpha_1^2} + \frac{C_{12}^2 (p_0^2 - p^2)(p_{c2} - p_0^2 + p^2)^2}{(p_0^2 - p^2 - \beta_2^2)^2 + \alpha_2^2}, \quad (19)$$

$$g^{[P]}_p(p_0, p) = \frac{\sqrt{-p_0^2 + p^2}}{(p_0^2 - p^2 - \beta_1^2)^2 + \alpha_1^2} + \frac{C_{12} \sqrt{(-p_0^2 + p^2)^3(p_{c3} - p_0^2 + p^2)}}{(p_0^2 - p^2 - \beta_2^2)^2 + \alpha_2^2}. \quad (20)$$

$$g^{[S]}_Q(p_0, Q) = g^{[S]}_p(Q_0, Q)_{\text{CMS}} \frac{(p_{c1} + p^2)}{(p^2 + \beta_1^2)^2 + \alpha_1^2} - \frac{C_{12} p^2 (p_{c2} + p^2)^2}{((p^2 + \beta_2^2)^2 + \alpha_2^2)^2}. \quad (21)$$

$$g^{[P]}_Q(p_0, Q) = g^{[P]}_p(Q_0, Q)_{\text{CMS}} \frac{|p|}{(p^2 + \beta_1^2)^2 + \alpha_1^2} + \frac{C_{12} |p|^3 (p_{c3} - p_0^2 + p^2)}{((p^2 + \beta_2^2)^2 + \alpha_2^2)^2}. \quad (22)$$

The denominators with $p_{c1}, p_{c2}, p_{c3}$ are intended for a dimension compensation as $p_{c1}$ for $g^{[S]}_p$ in the previous case \cite{15}, \cite{17}.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{$T_{\text{max}}$} & \multicolumn{2}{|c|}{$1S_0^+$} & \multicolumn{2}{|c|}{$3S_1^+$} \\
\hline
\multicolumn{2}{|c|}{$0.999 \text{ GeV}$} & \multicolumn{2}{|c|}{$1.1 \text{ GeV}$} & \multicolumn{2}{|c|}{$1.1 \text{ GeV}$} \\
\hline
\hline
$\lambda$, GeV$^2$ & 351.73 & -0.048422 & $\lambda$, GeV$^4$ & 9.2481 & -72.92 \\
$\beta$, GeV & 1.8005 & 0.19973 & $\beta$, GeV & 0.56446 & 0.72278 \\
$\alpha$, GeV & 0.24919 & 1.1891 & $\alpha$, GeV & 0.2 & 0.68577 \\
$p_{c1}$, GeV$^2$ & 1.1891 & 0.24919 & $p_{c3}$, GeV$^2$ & 0.2 & 0.68577 \\
$s_0$, GeV$^2$ & 4.0279 & 4.2020 & & & \\
\hline
\end{tabular}
\caption{Parameters of the one-rank kernels with MYQ functions.}
\end{table}
Table 3: The low-energy scattering parameters and binding energy for the singlet (s) and triplet (t) S waves.

|       | \(a_s\)(Fm) | \(r_{0s}\)(Fm) | \(a_t\)(Fm) | \(r_{0t}\)(Fm) | \(E_d\)(MeV) |
|-------|-------------|-------------|-------------|-------------|-------------|
| MYQ   | -23.69      | 2.22        | 5.37        | 1.71        | 2.2246      |
| MEYQ  | -23.49      | 2.30        | 5.37        | 1.71        | 2.2246      |
| Experiment | -23.748(10) | 2.75(5)    | 5.424(4)    | 1.759(5)    | 2.224644(46)|

Figure 1: Phase shifts for the \(1P^+_1\) wave. Dash (short dash) line corresponds to our parametrization with MY (MYQ) functions, eq. (16)(18), respectively; the solid (dash dot dot) line illustrates the extended case MEY (MEYQ), eq. (20)(22); the dotted and dash-dotted lines describe the CD-Bonn and SP07 calculations, correspondingly.

5 Discussion

Using the np scattering data we analyze the parameters of the separable kernels as it was earlier presented in [5].

The calculated parameters of the one-rank kernels with MYQ and MEYQ functions and \(T_{\text{Lab}}^{\text{max}}\) are listed in Tables 1 and 2. In Table 3 the calculated low-energy scattering parameters for S waves are compared with experimental data.

The calculated phase shifts are presented in Figs. 13 where MY (MYQ) case is denoted by the dash (short dash) line, MEY (MEYQ) one - by the solid (dash dot dot) line. Parameters for the first case (functions MY and MEY) have been presented in [5]. The calculations with the nonrelativistic CD-Bonn [8] (dotted line) potential and the SP07 solution [9] (dash dot line) are included for comparison.
In Fig. 1 we can see that all the calculations give a reasonable agreement with experimental data up to the energy $T_{\text{Lab}} \sim 1.1 \text{ GeV}$ but at larger energies their behavior becomes different drastically. Note that the CD-Bonn and SP07 results for phase shifts even change sign at $1.5 < T_{\text{Lab}} < 2.5 \text{ GeV}$. So we can stress that it is very desirable to obtain more exact determination of experimental data for the $^{1}P_{1}^+$ channel in this energy range. Now keeping in mind that the discrepancies between CD-Bonn, MEY, MEYQ and SP07 calculations at $T_{\text{Lab}} < 1.2 \text{ GeV}$ are not so large we can trust our calculations (in both cases) in this range of energies.

In Fig. 2 CD-Bonn and MY (MYQ) calculations demonstrate an opposite behavior for $^{3}P_{0}^+$ phase shifts at $T_{\text{Lab}} > 1 \text{ GeV}$. The MEY (MEYQ results practically coincide with MEY) and SP07 calculations give a very good agreement with experimental data in a wide range of energies $T_{\text{Lab}} < 3 \text{ GeV}$. Thus, both variants of the kernels of the NN interaction in the $^{3}P_{0}^+$ channel are acceptable to be used in various relativistic calculations of reactions.

Calculations for the $^{3}P_{1}^+$ channel (see Fig. 3) show us that MY (MYQ), CD-Bonn, and MEY have a similar behavior in a wide energy range, but are able to explain experimental data up to $T_{\text{Lab}} = 0.55, 0.6$ and $1.2 \text{ GeV}$, respectively. It should be noted that MEYQ calculations describe phase shifts up to $1.5 \text{ GeV}$. The SP07 calculations give a suitable agreement with experimental data up to $3 \text{ GeV}$. Calculations for the cases 1 (MY, MEY) and 2 (MYQ, MEYQ) coincide up to $0.55 \text{ GeV}$, but at larger energies their behavior becomes significantly different. So it is clear that to explain the data for the $^{3}P_{1}^+$ channel in a wider energy range within our approach we need to increase the rank of the separable kernel of the NN interaction.

![Figure 2: The same as in Fig.1 for the $^{3}P_{0}^+$ state.](image-url)
Fig. 4 demonstrates a close agreement with experimental data for CD-Bonn up to the energy $T_{\text{Lab}} < 0.6$ GeV, MY and MEYQ - up to $T_{\text{Lab}} < 1.3$ GeV in the $^3S_0^+$ channel (note that these calculations give a very similar behavior of phase shifts in a wide energy range up to 3 GeV). The MEY calculations give a very good description of phase shifts up to 1 GeV. So we can conclude that in this case as well as for $^3P_1^+$ we need to increase the rank of the kernel. The SP07 calculations describe experimental data for the $^1S_0^+$ channel up to 3 GeV.

Fig. 5 for the $^3S_1^+$ wave shows us that CD-Bonn calculations agree with experimental data up to 0.6 GeV. All the other compared results demonstrate a similar behavior in a wide energy range and explain known experimental data up to $T_{\text{Lab}} = 1.1$ GeV. It should be noted that calculations of low-energy parameters with MY (MYQ) and MEY (MEYQ) form factors give us a reasonable agreement with experimental data (see [5] for the first case and Table 3 - for the second).

6 Conclusion

We present two new different parameterizations (4-dimensional and 3-dimensional in CMS) of the separable kernel of the NN interaction which are adopted for calculations at large energies. As it was expected the MEY(MEYQ) functions give a better description of the scattering data than the MY(MYQ) ones. Using the suggested MEY (MEYQ) form factors the phase shifts are described in a whole range of measured energies for the following partial states: $^1P_1^+$, $^3S_1^+$ ($T_{\text{Lab}} < 1.2$ GeV) and $^3P_0^+$ ($T_{\text{Lab}} < 3$ GeV). The phase shifts for the $^1S_0^+$
Figure 4: The same as in Fig. 1 for the singlet partial state $^1S^+_0$, but the description by MY (MYQ) functions is defined by eq.(15)(17) and extended one MEY(MEYQ) - by eq.(19)(21).

and $^3P^+_1$ partial states can be described in our approach up to $T_{\text{Lab}} < 1.1$ GeV. To improve their agreement with experimental data up to $T_{\text{Lab}} < 3$ GeV it is necessary to increase the rank of the separable kernel of the NN interaction. This work is in progress. Finally we can say that 4-dimensional (Bethe-Salpeter type) as well as 3-dimensional (quasipotential type) in CMS parameterizations give a good description of the considered shift phases. Both of them are to be used in calculations of various electromagnetic reactions.

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Figure 5: The same as in Fig.4 for the triplet partial state $^3S_1^+$. 

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