ENRIQUES INVOLUTIONS ON PENCILS OF K3 SURFACES

DINO FESTI AND DAVIDE CESARE VENIANI

Abstract. The three pencils of K3 surfaces of minimal discriminant whose general element covers at least one Enriques surface are Kondo’s pencils I and II, and the Apéry–Fermi pencil. We enumerate and investigate all Enriques surfaces covered by their general elements.

1. Introduction

Any complex Enriques surface is doubly covered by a K3 surface. On the other hand, a K3 surface $X$ can cover infinitely many Enriques surfaces. The set $\text{Enr}(X)$ of isomorphism classes of Enriques surfaces doubly covered by $X$, though, is always finite by a result of Ohashi [20]. We call its cardinality $|\text{Enr}(X)|$ the Enriques number of the K3 surface $X$. The Enriques number $|\text{Enr}(X)|$ only depends on the transcendental lattice of $X$. Shimada and the second author [23] described a procedure to determine $|\text{Enr}(X)|$ and applied it to K3 surfaces of maximal Picard rank 20.

A K3 surface $X$ of Picard rank 19 can be seen as the generic element of a pencil of K3 surfaces. Its transcendental lattice $T_X$ is an even lattice of signature $(2,1)$. By a result of Brandhorst, Sonel and the second author [3], the surface $X$ covers an Enriques surface only if $4$ divides $\det(T_X)$, but this condition is not sufficient. In this paper we analyze in detail what happens when $|\det(T_X)|$ is small, more precisely

\begin{equation}
(1) \quad |\det(T_X)| < 16.
\end{equation}

Henceforth, let $X$ be a K3 surface of Picard rank 19 with transcendental lattice $T_X$. In the case $T_X \cong U \oplus [2n], n \geq 1$, it was already noted by Hulek and Schütt [8] that $\text{Enr}(X) \neq \emptyset$ if and only if $n$ is even. Indeed, we prove in Lemma 2.4 under assumption (1) that $\text{Enr}(X) \neq \emptyset$ if and only if

$T_X \cong U \oplus [4], U \oplus [8]$ or $U \oplus [12]$.

The main reason for bound (1) is to keep computations feasible. In particular the enumeration of jacobian elliptic fibrations on K3 surfaces with $T_X \cong U \oplus [16]$ already becomes quite hard. Moreover, the pencil of K3 surfaces with $T_X \cong U(2) \oplus [4]$ is not of the form $U \oplus [2n]$, but it still holds $\text{Enr}(X) \neq \emptyset$, as its generic element is a Kummer surface [11].

Quite interestingly, the first two pencils already feature prominently in Kondo’s classification of Enriques surfaces with finite automorphism group [13], which we briefly recall. There are seven families of such Enriques surfaces, numbered I to VII. Families I and II are 1-dimensional, while families III to VII are 0-dimensional. The K3 surfaces covering the generic Enriques surface of type I and II have transcendental lattice $T_X \cong U \oplus [4]$ and $T_X \cong U \oplus [8]$, respectively.
The third pencil with $T_X \cong U \oplus [12]$ has also been extensively studied, because of its arithmetical properties and its appearance in several seemingly unrelated physical contexts (see [6, 21]). Following Bertin and Lecacheux [2], who classified the elliptic fibrations on its generic element (Table 3), we call it the Apéry–Fermi pencil.

The aim of this paper is to enumerate and investigate the Enriques surfaces covered by these three pencils. More precisely, for each $m \in \{1, 2, 3\}$ we consider a K3 surface $X$ with $T_X \cong U \oplus [4m]$ and do the following:

- we compute the Enriques number $|\text{Enr}(X)|$;
- we classify all jacobian elliptic fibrations on $X$ using the extension of the Kneser–Nishiyama method explained in [7];
- we relate the special elliptic pencils on the Enriques quotients to the elliptic fibrations on $X$.

We summarize here our findings.

Fix $m \in \mathbb{Z}$, $m \geq 1$, and let $\omega$ be the number of prime divisors of $2m$ and $X$ a K3 surface with $T_X \cong U \oplus [4m]$, $m \geq 1$. Among the Enriques quotients of $X$ there are $2^{\omega-1}$ which we call of Barth–Peters type (Lemma 2.7). Such quotients admit a cohomologically trivial involution (see [15, 16]) and their presence is explained by the fact that our pencils are subfamilies of the 2-dimensional Barth–Peters family, a fact already noted by Hulek and Schütt [8, 9].

It turns out that if $m = 1$, then $X$ covers only one Enriques surface $Y$ (Theorem 3.1). Therefore, the Enriques surface $Y$ is of Barth–Peters type and, moreover, coincides with Kondo’s quotient, so it has finite automorphism group. The list of the 9 elliptic fibrations on $X$ appears in other papers by Scattone [22], Dolgachev [4] and Elkies and Schütt [5], and we confirm it here (Table 1).

If $m = 2$, then $X$ covers two Enriques surfaces $Y’, Y”$, of which only one, say $Y’$, is of Barth–Peters type. We show that the other surface $Y”$ is Kondo’s quotient with finite automorphism group. We include the classification of elliptic fibrations on $X$ up to automorphisms (Table 2). One subtlety arises in this case: two of the 17 elliptic fibrations on $X$ (No. 12 and 13 in Table 2) have the same Mordell-Weil group and two singular fibers of type $I_1$. Nonetheless, the two fibrations are not equivalent under the action of $\text{Aut}(X)$, as they have different frames. We determine which one is the pullback of a special elliptic pencil on $Y’$ and which one is the pullback of a special elliptic pencil on $Y”$ (Remark 3.6).

Finally, if $m = 3$, then $X$ covers three Enriques surfaces $Y’, Y”, Y’’’$, of which two, say $Y’$ and $Y”$ are of Barth–Peters type (Theorem 3.8). Applying a construction by Hulek and Schütt [8, §3] and using a particular configuration of curves on $X$ found by Peters and Stienstra, we determine a simple description of an explicit Enriques involution for $Y’’’$. In this way we find a description of smooth rational curves on $Y”$ whose dual graph is the union of a tetrahedron and a complete graph of degree 6 (Remark 3.9).

Acknowledgments. We warmly thank Simon Brandhorst, Klaus Hulek, Matthias Schütt and Ichiro Shimada for their valuable comments. We are also grateful to the anonymous referee for carefully reading the manuscript and for their useful remarks.

2. Preliminary results

In this section, after explaining our conventions on lattices in §2.1, we collect results regarding K3 surfaces with transcendental lattice $T_X \cong U \oplus [2m]$, $m \in \mathbb{Z}$, especially regarding their jacobian elliptic fibrations in §2.2. In §2.3 we recall the enumeration formula for Enriques quotients contained in [23] and we prove the lemma that motivates the whole paper. Finally, in §2.4 we introduce the notion of Enriques quotient of Barth–Peters type.
2.1. Lattices. In this paper, a lattice of rank $r$ is a finitely generated free $\mathbb{Z}$-module $L \cong \mathbb{Z}^r$ endowed with an integral symmetric bilinear form $L \times L \to \mathbb{Z}$ denoted $(v, w) \mapsto v \cdot w$. The signature of $L$ is the signature of the induced real symmetric form on $L \otimes \mathbb{R}$. We say that $L$ is even if $v^2 := v \cdot v \in 2\mathbb{Z}$ for every $v \in L$. The dual $L^{\vee} := \text{hom}(L, \mathbb{Z})$ of $L$ can be identified with \{ $w \in L \otimes \mathbb{Q}$ | $w \cdot v \in \mathbb{Z}$ for all $v \in L$ \}. The discriminant group of $L$ is defined as

$$L^{\sharp} := L^{\vee} / L,$$

which is a finite abelian group. We denote by $\ell(L)$ its length, i.e. the minimal number of generators. For a prime number $p$ we denote by $\ell_p(L^{\sharp})$ its $p$-length, i.e. the minimal number of generators of its $p$-part.

If $L$ is an even lattice, then $L^{\sharp}$ acquires naturally the structure of a finite quadratic form $L^{\sharp} \to \mathbb{Q}/2\mathbb{Z}$. There is a natural homomorphism $O(L) \to O(L^{\sharp})$ denoted $\gamma \mapsto \gamma^{\sharp}$.

We write $U$ for the indefinite unimodular even lattice of rank 2, and $A_n, D_n, E_n$ for the negative definite ADE lattices. The notation $[m]$, with $m \in \mathbb{Z}$, denotes the lattice of rank 1 generated by a vector of square $m$. We adopt Miranda–Morrison’s notation [14] for the elementary finite quadratic forms $u_k, v_k, w_{\varepsilon,k}$. We recall that $u_k$ (resp. $v_k$) is generated by two elements of order $2^k$, both of square 0 in $\mathbb{Q}/2\mathbb{Z}$ (resp. 1 in $\mathbb{Q}/2\mathbb{Z}$), such that their product is equal to $1/9^k \in \mathbb{Q}/\mathbb{Z}$. The forms $w_{\varepsilon,k}$, with $\varepsilon \in \{1, 3, 5, 7\}$, are generated by one element of order $2^k$ and square $\varepsilon/2^k \in \mathbb{Q}/2\mathbb{Z}$. For an odd prime $p$ the forms $w_{p,k}^{\varepsilon}$, with $\varepsilon \in \{\pm 1\}$, are generated by one element of order $p^k$ and square $a/p^k \in \mathbb{Q}/2\mathbb{Z}$, where $a$ is a square modulo $p$ if and only if $\varepsilon = 1$.

The genus of a lattice $L$ is defined as the set of isomorphism classes of lattices $M$ with $\text{sign}(L) = \text{sign}(M)$ and $L^{\sharp} \cong M^{\sharp}$. A genus is always a finite set (see [12, Satz 21.3]).

An embedding of lattices $\iota : M \hookrightarrow L$ is called primitive if $L/\iota(M)$ is a free group. We denote by $\iota(M)^{\perp} \subset L$ the orthogonal complement of $M$ inside $L$. We quickly summarize Nikulin’s theory of primitive embeddings [17].

By [17, Prop. 1.5.1] a primitive embedding of even lattices $M \hookrightarrow L$ is given by a subgroup $H \subset L$ and an isometry

$$\gamma : H \to H' := \gamma(H) \subset (\iota(M)^{\perp}(-1)^{\sharp}).$$

If $\Gamma$ denotes the graph of $\gamma$ in $M^{\sharp} \oplus (\iota(M)^{\perp}(-1)^{\sharp})$, the following identification between finite quadratic forms holds (the finite quadratic form on the right side being induced by the one on $M^{\sharp} \oplus (\iota(M)^{\perp}(-1)^{\sharp})$):

$$L^{\sharp} \cong (\iota(M)^{\perp}(-1)^{\sharp})^- \Gamma.\quad (2)$$

In this paper we call $H$, $\gamma$ resp. $\Gamma$ the gluing subgroup, gluing isometry resp. gluing graph of $M \hookrightarrow L$.

Equivalently by [17, Prop. 1.15.1], assuming that $L$ is unique in its genus, a primitive embedding $M \hookrightarrow L$ is given by a subgroup $K \subset L^{\sharp}$ and an isometry

$$\xi : K \to K' := \xi(K) \subset M(-1)^{\sharp}.$$ 

If $\Xi$ denotes the graph of $\xi$ in $L^{\sharp} \oplus M(-1)^{\sharp}$, the following identification between finite quadratic forms holds (the finite quadratic form on the right side being induced by the one on $L^{\sharp} \oplus M(-1)^{\sharp}$):

$$(\iota(M)^{\perp})^{\sharp} \cong \Xi^\perp / \Xi.\quad (3)$$

In this paper we call $K$, $\xi$ resp. $\Xi$ the embedding subgroup, embedding isometry resp. embedding graph of $M \hookrightarrow L$. 

2.2. Elliptic fibrations. Given a K3 surface $X$, we denote $T_X$ its transcendental lattice, $S_X$ its Néron–Severi lattice, and $\mathcal{J}_X$ the set of jacobian elliptic fibrations on $X$. The frame genus of $X$ is defined as the genus $W_X$ of negative definite lattices $W$ with $\text{rk}(W) = \text{rk}(S_X) - 2$ and $W^\perp \cong S_X^\perp$. The lattices in $W_X$ are called frames. The classes of a fiber and a section of a jacobian elliptic fibration induce a primitive embedding $\iota: U \hookrightarrow S_X$. As explained in [7], there is a well-defined function

$$\text{fr}_X : \mathcal{J}_X / \text{Aut}(X) \to W_X$$

which sends each jacobian fibration to the isomorphism class of $\iota(U)^\perp \subset S_X$.

Lemma 2.1. If $X$ is a K3 surface with transcendental lattice $T_X \cong U \oplus [2n]$, $n \geq 1$, then on $X$ there are exactly $2^{\omega - 1}$ jacobian elliptic fibrations with frame $W := E_8 \oplus [-2n]$ up to automorphisms, where $\omega$ is the number of prime divisors of $2n$.

Proof. Essentially by [7, Thm. 2.8] we want to prove that

$$|O^\perp_{T_X}(T_X) \backslash O(T_X^4)/O(W)| = 2^{\omega - 1},$$

If $n = 1$, then $O(T_X^4) = \{ \text{id} \}$ and we conclude immediately.

Suppose that $n \geq 2$. Since $\iota(T_X^4) = 1$, the discriminant form $T_X^4$ is the direct sum of forms $w^\varepsilon_{p,k}$. If $O(q + q') \cong O(q) \times O(q')$ if $q$ and $q'$ are finite quadratic forms with $|q|$ and $|q'|$ coprime, and $|O(w^\varepsilon_{p,k})| = 2$ if $p$ is odd or $p = 2$ and $k \geq 2$. Hence, $O(T_X^4)$ is a 2-elementary group of length $\omega$. In particular,

$$|O(T_X^4)| = 2^\omega,$$

As $\text{rk}(T_X)$ is odd, it holds $O^\perp_{T_X}(T_X) = \{ \pm \text{id} \}$ (see for instance [10, Cor. 3.3.5]). Note, moreover, that $\text{id} \neq -\text{id}$ in $T_X^4$. The orthogonal group of $W$ is the direct sum of $O(E_8)$, which has trivial action on the discriminant group because $E_8$ is unimodular, and $O([-2n]) = \{ \pm \text{id} \}$. Therefore, it also holds $O^\perp(W) = \{ \pm \text{id} \}$, so we have

$$|O^\perp_{T_X}(T_X) \backslash O(T_X^4)/O^\perp(W)| = |O(T_X^4)/\{ \pm \text{id} \}| = |O(T_X^4)/\{ \pm \text{id} \}| = 2^{\omega - 1}. \tag*{\square}$$

The Mordell–Weil group, i.e., the group of sections of a jacobian elliptic fibration, is naturally endowed with a rational symmetric bilinear form denoted by $\langle P, Q \rangle \in \mathbb{Q}$, called the Mordell–Weil lattice. The height of a section is defined as $\text{ht}(P) := \langle P, P \rangle$. For a clear exposition of this topic we refer to Shioda’s original paper [24].

Remark 2.2. Consider one of the elliptic fibrations $\pi: X \to \mathbb{P}^1$ as in Lemma 2.1 and for simplicity assume that $n \geq 2$. Since $W_{\text{root}} \cong E_8^\perp$, the fibration $\pi$ has two singular fibers of Kodaira type $I^\text{\#}$. As already remarked by Hulek and Schütt [8, §4.2.2], starting from the fibration $\pi$ we can construct an involution on $X$ which turns out to be an Enriques involution if $n$ is even. We repeat here their construction directly on the lattice $S_X \cong U \oplus E_8^\perp \oplus [-2n]$. In the following computations we let $O(S_X)$ act on $S_X$ from the right, so the composition of two isometries in $O(S_X)$ corresponds to the product of their associated matrices in reversed order.

Let $s_1, \ldots, s_{19}$ be a system of generators of $S_X$ such that the corresponding Gram matrix is the standard one. Then, $S_X^\perp$ is generated by $s_{19}/(2n)$. In these coordinates, consider the vectors

- $F := (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$,
- $O := (-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$,
- $P := (n - 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$. 


Note that $F^2 = 0, O^2 = P^2 = -2, F \cdot O = F \cdot P = 1, P \cdot O = n - 2$.

We can suppose that $s_3, \ldots, s_{18}$, generating the two copies of $E_8$, correspond to the components of the singular fibers which do not intersect $O$, $F$ to the class of a fiber, $O$ to a section which we take as origin and $P$ to a section of height (cf. [24, eq. (8.19)])
\[
\text{ht}(P) = \langle P, P \rangle = 2 \chi(O_X) + 2P \cdot O = 2 \cdot 2 + 2 \cdot (n - 2) = 2n.
\]

Let $Q := \exists P$ be the inverse section of $P$. Then $\text{ht}(Q) = \text{ht}(P) = 2n$, hence $Q \cdot O = n - 2$. It follows from $\langle P, Q \rangle = -\langle P, P \rangle$ that (cf. [24, eq. (8.18)])
\[
P \cdot Q = \chi(O_X) + P \cdot O + Q \cdot O - \langle P, Q \rangle = 2 + (n - 2) + (n - 2) + 2n = 4n - 2.
\]

Recalling moreover that $F \cdot Q = 1$ and $Q \cdot Q = -2$, we see that in our basis we can write
\[
Q = (n - 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1).
\]

Let $t_P \in O(S_X)$ be the pullback of the automorphism induced by the translation by $P$. We have $t_P(Q) = O, t_P(O) = P, t_P(F) = F$ and $t_P$ acts trivially on the other components of the fibers of type $\Pi^*$. Therefore, $\tau_P$ is given by the following matrix:
\[
t_P = \begin{pmatrix}
1 & & \\
& 1 & 1 \\
2n & & 1
\end{pmatrix}
\]

Let now $i \in O(S_X)$ be the isometry given by
\[
i = \begin{pmatrix}
I_2 & \\
I_8 & -1
\end{pmatrix}.
\]

Note that $i$ swaps the two fibers of type $\Pi^*$ and that $i(F) = F, i(P) = Q$ and $i(Q) = P$. Constructing an ample divisor as in [7, Prop. 2.7], it is easy to see that $i$ preserves the ample cone. Moreover, $i^2 = -\text{id} \in S_X^2$. Therefore, by the Torelli theorem $i$ is the pullback of a non-symplectic involution (whose quotient is a rational surface).

Consider $\varepsilon := i \circ t_P$, whose matrix is then given by
\[
\varepsilon = \begin{pmatrix}
1 & & \\
& 1 & -1 \\
2n & I_8 & -1
\end{pmatrix}.
\]

A computation shows that for even $n$, the invariant lattice of $\varepsilon$ is isometric to $E_8(2) \oplus U(2)$, hence it corresponds to an Enriques involution by Nikulin’s classification [18, Thm. 4.2.2]. The coinvariant lattice is isometric to $E_8(2) \oplus [-2n]$.

2.3. **Enriques numbers.** Let $X$ be a K3 surface with Néron–Severi lattice $S_X$ and transcendental lattice $T_X$. We recall briefly the formula for $|\text{Enr}(X)|$ proved in [23]. We define
\[
M := U(2) \oplus E_8(2).
\]

By Nikulin’s classification [18], if $\varepsilon \in O(S_X)$ is the pullback of an Enriques involution, then the invariant sublattice $S_X^\varepsilon := \{ x \in S_X \mid \varepsilon(x) = x \}$ is isomorphic to $M$. We denote by $(S_X)_{\varepsilon} := (S_X^\varepsilon)^\perp$ the coinvariant lattice, whose isometry class depends on the involution $\varepsilon$. 
We classify such finite quadratic forms.

Given a primitive embedding $\iota : M \to S_X$, we put

$$O(S_X, \iota) := \{ \varphi \in O(S_X) \mid \varphi(\iota(M)) = \iota(M) \}.$$ 

The Hodge structure on $H^2(X, \mathbb{Z})$ induces a Hodge structure on $T_X$. We write $O_h(T_X)$ for the group of Hodge isometries of $T_X$. We fix an anti-isometry $T^\perp_X \cong S_X^\perp$ (cf. [17, Prop. 1.6.1]), so that we can identify $O(T^\perp_X) \cong O(S_X^\perp)$. We denote the images of $O_h(T_X)$ and $O(S_X, \iota)$ under the natural morphisms $O(T_X) \to O(T^\perp_X)$ and $O(S_X) \to O(S_X^\perp)$ by $O_h^\perp(T_X)$ and $O^\perp(S_X, \iota)$, respectively.

**Theorem 2.3** ([23, Thm. 3.1.9]). For any K3 surface $X$ it holds

$$|\text{Enr}(X)| = \sum |O_h^\perp(T_X)\setminus O(T^\perp_X)|/|O(S_X, \iota)|,$$

where the sum runs over all primitive embeddings $\iota : M \to S_X$ up to the action of $O(S_X)$ such that there exists no $v \in \iota(M)^\perp$ with $v^2 = -2$.

The main topic of the present paper are K3 surfaces of Picard rank 19 covering an Enriques surface. For computational reasons, we restrict ourselves to K3 surfaces whose transcendental lattice has discriminant $|\det(T_X)| < 16$. By the next lemma, we have three cases to consider.

**Lemma 2.4.** Let $X$ be a K3 surface of Picard rank 19 and transcendental lattice $T_X$ and suppose that $|\det(T_X)| < 16$. Then, $\text{Enr}(X) \neq \emptyset$ if and only if $T_X \cong U \oplus [4m]$ with $m \in \{1, 2, 3\}$.

**Proof.** If $T_X \cong U \oplus [4m], m \geq 1$, then $X$ covers an Enriques surface by [8, Proposition 4.2].

Conversely, suppose that $\text{Enr}(X) \neq \emptyset$. By [3, Thm. 1.1] the lattice $T_X$ has a Gram matrix of the form

$$\begin{pmatrix}
2a_{11} & a_{12} & a_{13} \\
2a_{12} & 4a_{22} & 2a_{23} \\
a_{13} & 2a_{23} & 4a_{33}
\end{pmatrix}, \quad a_{ij} \in \mathbb{Z}.$$

Therefore, $\det(T_X)$ is divisible by 4. Now, the discriminant group $T^\perp_X$ is a finite quadratic form on an abelian group of order $|\det(T_X)|$ and of signature $2 - 1 = 1$, because $T_X$ has signature $(2, 1)$. We classify such finite quadratic forms $q$ using Miranda and Morrison’s normal form [14]. For each $q$ in the list we find a lattice $T$ such that $T^\perp \cong q$, obtaining the following table.

In all cases except the second one, $T$ is unique by [17, Thm. 1.14.2]. In the case $T = U(2) \oplus [2]$, $T$ is unique because $T = T'(2)$, with $T'$ a unimodular indefinite lattice.

If $T_X \cong T$ is such that $T^\perp \cong w_{2,2}$ or $T^\perp \cong w_{2,2}^2 \oplus w_{3,1}^{-1}$, then $\text{Enr}(X) = \emptyset$ because of [3, Prop. 3.9] (in the notation of [3], the two forms do not satisfy condition $C(1)$). The case $T_X \cong U(2) \oplus [2]$ is excluded because of [3, Thm. 1.1] (the lattice is an ‘exceptional lattice’).

Therefore, the only cases left are $T_X \cong T \in \{ U \oplus [4], U \oplus [8], U \oplus [12] \}$. □
2.4. Enriques quotients of Barth–Peters type. Let now \( X \) be a K3 surface with \( T_X \cong U \oplus [4m], \) \( m \geq 1. \) A primitive embedding \( \iota: M \to S_X \) depends in general on several data (cf. §2.1 and [17, Prop. 1.15.1]), but in this case one only has to consider the orthogonal complement of the image, thanks to the next lemma.

**Lemma 2.5.** Let \( X \) be a K3 surface with transcendental lattice \( T_X \cong U \oplus [4m], \) \( m \geq 1. \) If \( \iota: M \to S_X \) is a primitive embedding, then \( \iota(M)^\perp \cong N(2), \) where \( N \) is a lattice in the genus of \( E_8 \oplus [-2m]. \) Conversely, for each such lattice \( N \) there exists exactly one primitive embedding \( \iota: M \to S_X \) with \( \iota(M)^\perp \cong N(2) \) up to the action of \( O(S_X). \)

**Proof.** The Neron–Severi lattice \( S_X \) is isomorphic to \( U \oplus E_8^2 \oplus [-4m]. \) Consider a primitive embedding \( \iota: M \to S_X \) with embedding subgroup \( K \subset S_X^4 \) and embedding graph \( \Xi \) (see §2.1).

Since \( M^4 \cong 5u_1 \) is 2-elementary and \( S_X^4 \) has length 1, it holds either \(|K| = 1 \) or \(|K| = 2. \) The first case, though, is impossible, as otherwise \(|\ell_2(\iota(M)^\perp)| = 9 = \text{rk}(\iota(M)^\perp). \) Therefore, it must be \(|K| = 2, \) so there is only one choice for the subgroup \( K \subset S_X^4, \) which is generated by an element of order 2 and square 0 in \( \mathbb{Q}/2\mathbb{Z}. \)

Moreover, when taking \( \Xi^\perp/\Xi \) in the identification (3) one copy of \( u_1 \) gets killed. Hence, it holds

\[
\iota(M)^\perp \cong 4u_1 \oplus [-4m]^d,
\]

and in particular \( \ell_2(\iota(M)^\perp) = 9 = \text{rk}(\iota(M)^\perp). \) Therefore (see for instance [3, Lemma 3.10]), it holds \( \iota(M)^\perp \cong N(2), \) with \( N \) an even lattice. The genus of \( N(2) \) determines the genus of \( N, \) so we see that \( N \) is in the genus of \( E_8 \oplus [-2m]. \)

The converse holds by [17, Prop. 1.15.1], because \( S_X \) is unique in its genus, \( K \) is uniquely determined and \( O(M) \to O(M^2) \) is surjective (see for instance [1, p. 388]). \( \square \)

Note that a lattice \( N' \cong N(2), \) with \( N \) an even lattice, does not contain vectors of square \(-2. \) Therefore, Lemma 2.5 essentially says that the terms in the sum of Theorem 2.3 are in one-to-one correspondence with the lattices in the genus of \( E_8 \oplus [-2m]. \) In particular, one of them corresponds to \( E_8 \oplus [-2m] \) itself, which we presently consider more in detail.

Barth–Peters introduced a 2-dimensional family of K3 surfaces, whose general element \( X \) has transcendental lattice \( T_X \cong U \oplus U(2) \) and Neron–Severi lattice \( S_X \cong U(2) \oplus E_8^2. \) Ohashi [20, Remark 4.9(2)] proved that \(|\text{Enr}(X)| = 1. \) The coinvariant lattice of an Enriques involution on \( X \) is isomorphic to \( E_8(2). \)

In the situation of Lemma 2.5, if \( E_8(2) \) embeds into \( (S_X)_\varepsilon \cong \iota(M)^\perp, \) then \( \iota(M)^\perp \cong E_8(2) \oplus [-4m]. \) This motivates the following definition.

**Definition 2.6.** We say that an Enriques involution \( \varepsilon \in \text{Aut}(X) \) on \( X \) is of Barth–Peters type if \( (S_X)_\varepsilon \cong E_8(2) \oplus [-4m]. \) The corresponding Enriques quotient is also called of Barth–Peters type.

The following lemma provides the number of Enriques quotients of Barth–Peters type up to isomorphisms.

**Lemma 2.7.** If \( X \) is a K3 surface with transcendental lattice \( T_X \cong U \oplus [4m], \) \( m \geq 1, \) and \( \iota: M \to S_X \) is a primitive embedding with \( \iota(M)^\perp \cong E_8(2) \oplus [-4m], \) then it holds

\[
|O^\omega_k(T_X)/O^\omega(S_X, \iota)| = 2^{\omega-1},
\]

where \( \omega \) is the number of prime divisors of \( 2m. \)

**Proof.** As in the proof of Lemma 2.1, it holds \(|O(T_X)^\omega| = 2^\omega. \) As \( \text{rk}(T_X) \) is odd, it holds \( O^\omega_k(T_X) = \{ \pm \text{id} \} \) (see for instance [10, Cor. 3.3.5]). Note, moreover, that \( \text{id} \neq -\text{id} \) in \( T_X^\omega. \)
We now want to determine $O^e_h(S_X,\iota)$ using the identification (2). Let $s \in \mathbb{E}_8(2) \oplus [-4m]$ be the generator of the copy of $[-4m]$, $H$ be the gluing subgroup, $\gamma: H \to H'$ be the gluing isometry, and $\Gamma$ the gluing graph of $\iota$ (see §2.1). By the identification (2), it holds $|H| = |H'| = 2^9$. Therefore, $S^1_X \cong \Gamma^\perp/\Gamma$ is generated by an element of the form $(\alpha, s/4m)$, with $\alpha \in \mathbb{M}_4$.

Recall now that $O(M) \to O(M^2)$ is surjective (see [1, p. 388]) and that an isometry of a definite lattice preserves its decomposition in irreducible lattices up to order (see for instance [12, Satz 27.2]), so $O(\mathbb{E}_8(2) \oplus [-4m]) \cong O(\mathbb{E}_8(2) \times O([-4m]))$. These facts imply that the group $O_h(S_X,\iota)$ can only act as $\pm \id$ on $(\alpha, s/4m)$, i.e. $O^e_h(S_X,\iota) = \{ \pm \id \}$. Therefore, we have

$$|O^e_h(T_X)\setminus O(T^4_X)/O^e(S_X,\iota)| = |O(T^4_X)/\{ \pm \id \}| = |O(T^4_X)/\{ \pm \id \}| = 2^{2m-1}. \quad \square$$

**Remark 2.8.** Recall that any elliptic pencil on an Enriques surface has exactly two multiple fibers $2F, 2F'$. The divisors $F$ and $F'$ are called half-pencils (necessarily of type $I_m$ for some $m \geq 0$). An elliptic pencil on an Enriques surface is said to be special if it has a 2-section which is a smooth rational curve.

As noted by Kondō [13, Lem. 2.6], the pullback of a special elliptic pencil induces a jacobian elliptic fibration $\pi$ on the K3 surface $X$. Such pullbacks satisfy the following condition: if the fibration $\pi$ has exactly $n_i$ fibres of type $J_i$ (for $i = 1, \ldots, r$), where $J_i, J_j$ are pairwise distinct Kodaira types if $i \neq j$, then at most two coefficients $n_i$ can be odd; moreover, if $n_i$ is odd, then $J_i = I_{2m}$ for some $m \geq 0$.

The last sentence comes from the fact that one of the fibers of type $J_i$ is necessarily the pullback of a half-pencil.

**Remark 2.9.** Let $n = 2m$ be an even integer and consider one of the $2^{2m-1}$ elliptic fibrations with two fibers of type $\Pi^*$ given in Lemma 2.1. By the construction of Remark 2.2, we obtain one of the $2^{2m-1}$ Enriques quotients $Y$ of Barth–Peters type of Lemma 2.7. In the notation of Remark 2.2, the vector

$$R := (m + 1, 2, -4, -5, -7, -10, -8, -6, -4, -2, -3, -4, -6, -5, -4, -3, -2, 1)$$

has square $-2$, satisfies $R \cdot F = 2$ and has intersection number 1 with $e_3$ and $e_{18}$. Therefore, it represents a smooth rational curve. Moreover, $R \cdot \varepsilon(R) = 0$.

Thus, the surface $Y$ contains ten smooth rational curves which are the images of $R$ and of the components of the fibers of type $\Pi^*$. They form the following dual graph, where the white vertex represents the image of $R$, which is a 4-section of the highlighted elliptic pencil.

![Graph](image)

This graph appears in [13, Thm. 1.7(i)] and is related to the fact that $Y$ has a cohomologically trivial automorphism (such automorphisms were studied by Mukai and Namikawa [15, 16]).

On the above graph we can recognize three more special elliptic pencils up to symmetries (dashed lines indicates half-pencils):
In our case we retrieve the jacobian elliptic fibrations on $X$ with respectively two fibres of type $I_4^*$, two fibres of type $III^*$, and one fibre of type $I_{16}$ (hence the corresponding root lattices $W_{\text{root}}$ of the frame contain the sublattices $D_8^2$, $E_7^2$ and $A_{15}$, respectively).

3. The three pencils

This section is divided into three subsections, in which we study K3 surface $X$ with transcendental lattice $T_X \cong U \oplus [4]$ (§3.1), $T_X \cong U \oplus [8]$ (§3.2), and $T_X \cong U \oplus [12]$ (§3.3). In each case we determine $|\text{Enr}(X)|$ and $|J_X/\text{Aut}(X)|$, then we focus on their Enriques quotients, especially those not of Barth–Peters type (because those of Barth–Peters type were already considered in §2.4).

Moreover, we show that all jacobian elliptic fibrations satisfying the condition in Remark 2.8 are indeed pullbacks of elliptic pencils on some Enriques quotient.

3.1. Kondō’s pencil I. Let $X$ be a K3 surface with transcendental lattice

$$T_X \cong U \oplus [4].$$

**Theorem 3.1.** It holds $|\text{Enr}(X)| = 1$.

**Proof.** The lattice $E_8 \oplus [-2]$ is unique in its genus by the mass formula. By Lemma 2.5, the sum in Theorem 2.3 has only one term, which is equal to 1 by Lemma 2.7. □

Therefore, the surface $X$ admits only one Enriques quotient $X \to Y$. Necessarily, the Barth–Peters quotient of Lemma 2.7 coincides with Kondō’s quotient [13] (in particular, $Y$ has a finite automorphism group). Indeed, the graph of nodal curves contained in $Y$, which is pictured in [13, Fig. 1.4], contains the Barth–Peters graph as a subgraph. This Enriques quotient was also studied by Hulek and Schütt [8, §4.6].

For the sake of completeness, we enumerate all jacobian elliptic fibrations on $X$ up to automorphisms (the same list is contained in an unpublished paper by Elkies and Schütt [5]).

**Proposition 3.2.** The frame genus $\mathcal{W}_X$ contains exactly 9 isomorphism classes, listed in Table 1, whose Gram matrices are contained in the arXiv ancillary file `genus_Kondo_I.sage`. Moreover, it holds

$$|J_X/\text{Aut}(X)| = 9.$$

**Proof.** It holds $|J_X/\text{Aut}(X)| = |\mathcal{W}_X|$ by [7, Cor. 2.10]. In order to determine $\mathcal{W}_X$, we apply the Kneser–Nishiyama method with $T_0 = D_7$. The list is complete because the mass formula holds:

$$\sum_{i=1}^9 \frac{1}{|O(W_i)|} = \frac{642332179}{18881368343036559360000} = \text{mass}(\mathcal{W}_X).$$

□
### Table 1. Lattices in the frame genus $W_X$ of a K3 surface $X$ with transcendental lattice $T_X \cong U \oplus [4]$.

| $W$  | $N_{\text{root}}$ | $W_{\text{root}}$ | $W/W_{\text{root}}$ | $|\Delta(W)|$ | $|O(W)|$ | $|fr_X^1(W)|$ | Rmk. |
|------|-------------------|--------------------|----------------------|---------------|-------------|----------------|------|
| $W_1$ | $D_{16}E_8$ | $D_9E_8$ | 0 | 384 | 129448569470976000 | 1 | – |
| $W_2$ | $D_{24}$ | $D_{17}$ | 0 | 544 | 46620662575398912000 | 1 | – |
| $W_3$ | $D_{10}E_7^2$ | $A_3E_7^2$ | $\mathbb{Z}/2\mathbb{Z}$ | 264 | 809053559193600 | 1 | 2.9 |
| $W_4$ | $D_{12}^2$ | $D_5D_{12}$ | $\mathbb{Z}/2\mathbb{Z}$ | 304 | 3767021862912000 | 1 | – |
| $W_5$ | $A_{11}D_7E_6$ | $A_{11}E_6$ | $\mathbb{Z}/3\mathbb{Z}$ | 204 | 49662885888000 | 1 | – |
| $W_6$ | $A_{15}D_9$ | $A_7^2A_{15}$ | $\mathbb{Z}/4\mathbb{Z}$ | 244 | 334764638208000 | 1 | 2.9 |
| $W_7$ | $E_8^3$ | $E_8^2$ | $\mathbb{Z}$ | 480 | 19417285420640000 | 1 | 2.9 |
| $W_8$ | $D_8^3$ | $D_8^2$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 224 | 106542032486400 | 1 | 2.9 |
| $W_9$ | $D_{16}E_8$ | $D_{16}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 480 | 13711959589968000 | 1 | – |

### 3.2. Kondō’s pencil II.

Let $X$ be a K3 surface with transcendental lattice $T_X \cong U \oplus [8]$.

**Theorem 3.3.** It holds $|\text{Enr}(X)| = 2$.

**Proof.** There is only one more lattice in the genus of $E_8 \oplus [-4]$, namely $D_9$, as the mass formula shows. By Lemma 2.5 the sum in Theorem 2.3 has two terms, both equal to 1 as it holds $O_{\text{ch}}^I(T_X) = O(T_X^I)$.

By Lemma 2.7, one of the two Enriques quotient, say $X \to Y'$, is of Barth–Peters type. The corresponding coinvariant lattice is isometric to $E_8(2) \oplus [-8]$ and contains 240 vectors of square $-4$.

By Kondō’s classification, the surface $X$ admits an Enriques quotient $X \to Y''$ with finite automorphism group. Kondō’s quotient $Y''$ was also studied by Hulek and Schütt [8, §4.7 and §4.8]. We argue that $Y'$ is not isomorphic to $Y''$.

Geometrically, this follows from the fact that $Y''$ contains exactly 12 smooth rational curves whose dual graph is pictured on [13, p. 207, Fig. 2.4]. This dual graph does not contain the graph pictured in Remark 2.9 as a subgraph.

Algebraically, we can distinguish the two quotients in the following way.

1. The surface $X$ contains 24 smooth rational curves $F_i^+, F_i^-, i = 1, \ldots, 12$, which intersect as in [13, p. 207, Fig. 2.3] and generate the Néron–Severi lattice $S_X$.

2. Kondō’s Enriques involution exchanges $F_i^+$ with $F_i^-$, $i = 1, \ldots, 12$.

3. Computing explicitly the coinvariant lattice of Kondō’s Enriques involution in $S_X$, we see that it contains 144 vectors of square $-4$, so it must be isomorphic to $D_9(2)$. In particular, Kondō’s quotient is not of Barth–Peters type.

We now enumerate all jacobian elliptic fibrations on $X$ up to automorphisms.

**Proposition 3.4.** The frame genus $W_X$ contains exactly 17 isomorphism classes, listed in Table 1, whose Gram matrices are contained in the arXiv ancillary file genus_Kondo_II.sage. Moreover, it holds $|\mathcal{J}_X/\text{Aut}(X)| = 17$. 
Remark 3.6. The list is complete because the mass formula holds:

\[ \text{mass}(\mathcal{W}_X) = \sum_{i=1}^{17} \frac{1}{|O(W_i)|} = \frac{642332179}{73755345089986560000} = \text{mass}(\mathcal{W}_X). \]

**Proof.** It holds \(|\mathcal{J}_X/\text{Aut}(X)| = |\mathcal{W}_X|\) by [7, Cor. 2.10]. In order to determine \(\mathcal{W}_X\), we apply the Kneser–Nishiyama method with \(T_0 = \mathbf{A}_7\). Note that there are two different primitive embeddings \(\mathbf{A}_7 \hookrightarrow \mathbf{D}_8\) (cf. [19, Lem. 4.2]), leading to two distinct frames \(W\) with \(W_{\text{root}} \hookrightarrow \mathbf{D}_8^3\), namely \(W_{11}\) and \(W_{12}\) (cf. Remark 3.6). The list is complete because the mass formula holds:

\[ \text{mass}(\mathcal{W}_X) = \sum_{i=1}^{17} \frac{1}{|O(W_i)|} = \frac{642332179}{73755345089986560000} = \text{mass}(\mathcal{W}_X). \]  

\[ \square \]

**Remark 3.5.** The surface \(Y''\) contains 12 curves on whose dual graph one can recognize the following elliptic pencils (dashed lines indicate half-pencils):

![Diagram of elliptic pencils](image)

The first three pencils are special pencils and correspond to the elliptic fibrations on \(X\) with frames \(W_1, W_{11}\) (see Remark 3.6) and \(W_{16}\), respectively.

The fourth pencil is not special: the highlighted curves on \(Y''\) form a half-pencil and the white vertices represent 4-sections. Indeed, the pullback on \(X\) correspond to an elliptic fibration with a fiber of type \(I_{18}\), namely

\[ F_1^+ + F_2^- + F_3^+ + F_4^- + F_5^- + F_6^- + F_7^- + F_8^- + F_9^- + F_{10}^- + F_{11}^- + F_{12}^- + F_{13}^+ + F_{14}^- + F_{15}^- + F_{16}^+ + F_{17}^+ + F_{18}^+ + F_{19}^+ + F_{20}^+ + F_{21}^+ + F_{22}^+ + F_{23}^+. \]
This fibration is not jacobian, as it does not appear in Table 2.

**Remark 3.6.** The two frames $W_{11}$ and $W_{12}$ are not isometric, but they can be distinguished neither by the pair $(W_{\text{root}}, W/W_{\text{root}})$ nor by their number of automorphisms $|O(W)|$.

Using the command `is_globally_equivalent_to` of the Sage class `QuadraticForm`, we can check that the frame $W_{11}$ corresponds to the fibration with fiber

$$F = F^+_6 + F^+_8 + 2F^+_3 + 2F^+_2 + 2F^+_1 + 2F^+_1 + F^+_1 + F^+_1 + F^+_1 + F^+_1$$

which is the pullback of the second special pencil on $Y$ listed in Remark 3.5.

On the other hand, with the same command we can check that the frame $W_{12}$ corresponds to the fibration with fiber

$$F^+_2 + F^+_5 + 2F^+_3 + 2F^+_2 + 2F^+_2 + 2F^+_2 + 2F^+_2 + F^+_4 + F^+_5,$$

which is then the pullback of a special pencil on $Y'$ (cf. Remark 2.9).

### 3.3. Apéry–Fermi pencil

Let $X$ be a K3 surface with transcendental lattice

$$T_X \cong U \oplus [12].$$

The classification of the jacobian elliptic fibrations on $X$ was carried out by Bertin and Lecacheux [2] and then refined in [7]. For the reader’s convenience we reproduce in Table 3 the same table as [7, Table 7].

**Theorem 3.7.** It holds $|\text{Enr}(X)| = 3$.

**Proof.** The genus of $E_8 \oplus [-6]$ contains two lattices, namely $A_2 \oplus E_7$ and $E_8 \oplus [-6]$ itself, as the mass formula shows. Thus, by Lemma 2.5, the sum in Theorem 2.3 consists of two terms, one of which is equal to 2 by Lemma 2.7.

Fix a primitive embedding $\iota: M \to S_X$ with $\iota(M)^+ \cong A_2(2) \oplus E_7(2)$. Note that it holds

$$S_X^2 \cong w_{2,2}^5 \oplus w_{3,1}^{-1}, \quad A_2(2)^{\sharp} \cong v_1 \oplus w_{3,1}^{-1}, \quad E_7(2)^{\sharp} = 3u_1 \oplus w_{2,2}^1.$$

Let $H \subset M^2$ be the gluing subgroup (see §2.1). By the identification (2) we have $|H| = 2^5$. Thus, the image $H' := H(\iota(M)^{-1}(-1))^{\sharp}$ of the gluing isometry is the sum of the copy of $v_1$ in $A_2(2)$ and the whole group $E_7(2)^{\sharp}$ (with inverted sign).

Consider the isometry $\alpha \in O(\iota(M)^{-1})$ defined as $-\text{id}$ on the copy of $A_2(2)$ and as $\text{id}$ on the copy of $E_7(2)$. Since the natural homomorphism $O(M) \to O(M^2)$ is surjective, $\alpha$ extends to an isometry $\tilde{\alpha} \in O(S_X, \iota)$ by [17, Cor. 1.5.2].

By construction of $\alpha$ and by the above description of the gluing isometry $\gamma$, the element $\tilde{\alpha}^v$ acts as $-\text{id}$ on the 3-part of $S_X^3$ and as $\text{id}$ on the 2-part of $S_X^2$. In particular, $O^2(S_X, \iota)$ contains at least three different elements, namely $\text{id}$, $-\text{id}$ and $\tilde{\alpha}^v$. On the other hand, $O(T_X^2)$ contains exactly four elements, as it is generated by multiplication by $-1$ and by $5$. Therefore, we have $O^2(S_X, \iota) = O(T_X^2)$, which implies

$$|O^2_h(T_X)\setminus O(T_X^2)/O^2(S_X, \iota)| = 1.$$

In total we get $|\text{Enr}(X)| = 3$. \hfill \Box

Let $Y', Y'', Y'''$ be the three Enriques quotients of $X$ up to automorphisms. We can suppose that $Y', Y''$ are of Barth–Peters type (see §2.4). Here we are interested in studying $Y := Y'''$. 
Table 3. Lattices in the frame genus $\mathcal{W}_X$ of a K3 surface $X$ with transcendental lattice $T_X \cong U\oplus [12]$, numbered according to Bertin and Lecacheux (cf. [2, Tables 2 and 3]).

| $W$  | $N_{\text{root}}$ | $W_{\text{root}}$ | $W/W_{\text{root}}$ | $|\Delta(W)|$ | $|O(W)|$ | $|\text{fr}_{\mathcal{X}}(W)|$ | Rmk. |
|------|------------------|------------------|------------------|-------------|--------|------------------|------|
| $W_3$ | $D_{10}E_8$ | $D_{11}E_6$ | 0 | 292 | 8475799191552000 | 1 | – |
| $W_1$ | $E_6^2$ | $A_3^2E_8$ | 0 | 324 | 3467372396544000 | 1 | – |
| $W_7$ | $D_{10}E_7^2$ | $A_5D_5E_7$ | $\mathbb{Z}/2\mathbb{Z}$ | 196 | 16052649984000 | 1 | – |
| $W_{20}$ | $A_1D_7E_6$ | $A_4^2A_6^2A_{11}$ | $\mathbb{Z}/6\mathbb{Z}$ | 148 | 551809843200 | 1 | 3.9 |
| $W_{27}$ | $A_4^2D_5$ | $A_4A_7D_5$ | $\mathbb{Z}$ | 116 | 18579456000 | 2 | – |
| $W_{21}$ | $A_4A_7E_6$ | $A_4^2A_8E_6$ | $\mathbb{Z}$ | 148 | 300987187200 | 1 | – |
| $W_{18}$ | $A_{12}D_9$ | $A_{12}D_{14}$ | $\mathbb{Z}$ | 180 | 478235197440 | 1 | – |
| $W_{13}$ | $D_{12}^2$ | $D_9D_{17}$ | $\mathbb{Z}$ | 228 | 119859786547200 | 1 | – |
| $W_5$ | $D_{10}E_8$ | $A_5D_{13}$ | $\mathbb{Z}$ | 324 | 2448564210892800 | 1 | – |
| $W_6$ | $D_{16}E_8$ | $D_6E_8$ | $\mathbb{Z}$ | 352 | 1438317438564000 | 1 | – |
| $W_2$ | $E_8^2$ | $E_8^2$ | $\mathbb{Z}$ | 480 | 194172854206464000 | 2 | 2.9 |
| $W_{12}$ | $D_{24}$ | $D_{16}$ | $\mathbb{Z}$ | 480 | 2742391916199936000 | 1 | – |
| $W_{15}$ | $D_4^2$ | $A_5D_{12}D_8$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 164 | 951268147200 | 1 | – |
| $W_8$ | $D_{10}E_7^2$ | $A_5A_7D_{10}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 212 | 1070176656000 | 1 | – |
| $W_{16}$ | $D_6^2$ | $D_9^2$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 224 | 10654203486400 | 2 | 2.9 |
| $W_9$ | $D_{10}E_7^2$ | $A_5^2E_7^2$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 256 | 269684519731200 | 1 | 2.9 |
| $W_{14}$ | $D_4^2$ | $D_1D_{12}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 288 | 753404372582400 | 1 | – |
| $W_4$ | $D_{16}E_8$ | $D_{16}$ | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ | 480 | 1371195958099968000 | 2 | – |
| $W_{19}$ | $E_6^3$ | $A_5^2E_6^2$ | $\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})$ | 156 | 773967052800 | 1 | 3.9 |
| $W_{26}$ | $A_7^2D_5^2$ | $A_7^2A_5^2$ | $\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})$ | 116 | 52022476800 | 1 | 3.9 |
| $W_{25}$ | $A_5^2D_8$ | $A_6A_9$ | $\mathbb{Z}^2$ | 132 | 73156608000 | 1 | – |
| $W_{22}$ | $A_1D_7E_6$ | $A_4D_7$ | $\mathbb{Z}^2$ | 156 | 234101146000 | 2 | – |
| $W_{10}$ | $D_{10}E_7^2$ | $A_1D_7E_7$ | $\mathbb{Z}^2$ | 212 | 7491236659200 | 1 | – |
| $W_{11}$ | $A_7E_7$ | $A_1A_{14}$ | $\mathbb{Z}^2$ | 212 | 10461394944000 | 1 | – |
| $W_{24}$ | $D_6^2$ | $A_5D_6^2$ | $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})$ | 132 | 1019215582000 | 1 | 3.9 |
| $W_{23}$ | $A_1A_7E_6$ | $A_1A_{11}D_4$ | $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})$ | 156 | 36787328800 | 1 | – |
| $W_{17}$ | $A_5D_9$ | $A_{15}$ | $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})$ | 240 | 167382319104000 | 1 | 2.9 |

Peters and Stienstra [21] showed that $X$ contains $20 + 12$ smooth rational curves, called $L$-lines and $M$-lines, forming a particular configuration, which we call the Peters–Stienstra cube. The dual graph of the $L$-lines is pictured in [21, Fig. 1]. We do not reproduce it here, but we follow the same notation. The intersection numbers of the $M$-lines are described in [21, Lem. 1].

In order to make a connection with the construction of Remark 2.2, we first look for a fibration with two fibers of type $\Pi$ or, equivalently, with frame $W_2$ in Table 3. We can suppose

$$F = 2L_{+-0} + 3M_{2-} + 4L_{+++} + 6L_{+0+} + 5L_{++-} + 4L_{0--} + 3M_{1--} + 2L_{0-} + L_{--} = 2L_{--0} + 3M_{1++} + 4L_{---} + 6L_{0--} + 5L_{++-} + 4L_{0-} + 3M_{2--} + 2L_{0-} + L_{--}$$
as pictured below in the Peters–Stienstra cube (note that $M_{1++}$ and $M_{2+-}$ and the other $M$-lines are not displayed):

In the coordinate system of Remark 2.2, up to substituting $P$ with $Q$, we can suppose that

\begin{align*}
L_{-+0} &= O, \\
L_{++0} &= (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
L_{-+0} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \\
L_{++-} &= (4, 4, -6, -8, -11, -16, -13, -10, -7, -4, -6, -9, -12, -18, -15, -12, -8, -4, 1).
\end{align*}

The coordinates of all other $L$-lines and $M$-lines are determined by these choices.

In order to construct the Enriques involution corresponding to $Y$, we now consider a fibration with frame $W_{19}$ in Table 3. As $(W_{19})_{\text{root}} \cong \mathbb{A}_2^2 \mathbb{E}_6^3$, the fibration has two fibers of type $I_3$ (or $IV$) and two fibers of type $IV^*$. As pictured below in the Peters–Stienstra cube (omitting the $M$-lines) we choose

\begin{align*}
F_{19} := L_{+-+} + L_{++-} + 2L_{0++} + 2L_{++0} + 3L_{+++} + 2L_{0++} + L_{++-} &= M_{3+-} + M_{1+-} + M_{2+-} \\
&= L_{+-+} + L_{++-} + 2L_{0++} + 2L_{0+-} + 3L_{--} + 2L_{--0} + L_{++-} &= M_{3-} + M_{1-} + M_{2-}.
\end{align*}

Moreover, we choose $O_{19} := L_{+-0}$ as origin. Then, $P_{19} := L_{0+-}$ and $Q_{19} := L_{-0+}$ become the two 3-torsion sections, because it holds $\langle P_{19}, P_{19} \rangle = \langle Q_{19}, Q_{19} \rangle = 0$, whereas $R_{19} := L_{-+0}$ becomes a section of infinite order. From [24, eq. (8.12) and Table (8.16)] it follows

\[ \langle R_{19}, R_{19} \rangle = 2\chi + 2L_{+0-} \cdot L_{+-0} - \sum \text{contr}(R_{19}) = 2 \cdot 2 + 2 \cdot 0 - 2 \cdot \frac{4}{3} = \frac{4}{3}. \]

**Theorem 3.8.** There is an Enriques involution $\varepsilon \in \text{Aut}(X)$ which acts on the $L$-lines by exchanging all subscripts $'+'$ with $'-'$ and on the $M$-lines by exchanging $M_{k+\beta}$ with $M_{k-\beta}$, for all $k \in \{1, 2, 3\}$, $\beta \in \{+,-\}$. Moreover, $\varepsilon$ is not of Barth–Peters type.
Proof. Let $S_{19} := \square R_{19}$ be the section given by the inverse of $R_{19}$ in the Mordell–Weil group. Then we clearly have $\langle S_{19}, S_{19} \rangle = \langle R_{19}, R_{19} \rangle$ and $\langle S_{19}, R_{19} \rangle = -\langle R_{19}, R_{19} \rangle$. From these equalities, using [24, Theorem 8.6] we obtain $O_{19} \cdot S_{19} = 0$ and $R_{19} \cdot S_{19} = 2$. These intersection numbers explicitly determine $S_{19}$ in the coordinate system of Remark 2.2:

$$S_{19} = (19, 17, -27, -42, -54, -81, -66, -51, -34, -17, -27, -42, -54, -81, -66, -51, -34, -17, 4).$$

We are then able to compute the translation by $R_{19}$, denoted by $t$, and involution $\iota$ as in Hulek and Schütt’s construction [8, §3]. Explicit computations show that the invariant lattice of $\varepsilon := t \circ \iota$ is isomorphic to $\mathbf{M}$, so that $\varepsilon$ is the pullback of an Enriques involution. We can verify directly that $\varepsilon$ acts on the $L$-lines and $M$-lines as described in the statement of the theorem. By Lemma 2.5 and the proof of Theorem 3.7, we know that the coinvariant lattice $(S_X)_{\varepsilon}$ is isomorphic to either $\mathbf{A}_2(2) \oplus \mathbf{E}_7(2)$ or $\mathbf{E}_8(2) \oplus [-12]$. An explicit computation shows that $(S_X)_{\varepsilon}$ contains 132 vectors of square $-4$, so it is necessarily isomorphic to $\mathbf{A}_2(2) \oplus \mathbf{E}_7(2)$, i.e. $\varepsilon$ is not of Barth–Peters type. We refer to the ancillary file calc.Apery_Fermi.sage for the actual computations in Sage. □

Remark 3.9. Thanks to the description of the Enriques involution in Theorem 3.8, it is immediate to see that the images of the $L$-lines in $Y$ form a tetrahedron, while the images of the $M$-lines form a complete graph with 6 vertices in which three pairs of curves intersect doubly. The tetrahedron and the complete graph are connected in the following way, where double intersections are marked with a double edge.

The following pencils (we omit here the images of the $M$-lines) are special pencils on $Y$ whose pullbacks correspond to the elliptic fibrations on $X$ with frames $W_{19}$, $W_{20}$, $W_{24}$ and $W_{26}$, respectively.
References

1. Wolf Barth and Chris Peters, Automorphisms of Enriques surfaces, Invent. Math. 73 (1983), no. 3, 383–411. MR 718937
2. Marie José Bertin and Odile Lecacheux, Apéry-Fermi pencil of K3-surfaces and 2-isogenies, J. Math. Soc. Japan 72 (2020), no. 2, 599–637. MR 4090348
3. Simon Brandhorst, Serkan Sonel, and Davide Cesare Veniani, Idoneal genera and K3 surfaces covering an Enriques surface, preprint, arXiv:2003.08914v5, 2021.
4. Igor V. Dolgachev, Mirror symmetry for lattice polarized K3 surfaces, J. Math. Sci. 81 (1996), no. 3, 2599–2630, Algebraic geometry, 4. MR 1420220
5. Noam D. Elkies and Matthias Schütt, K3 families of high Picard rank, unpublished draft, http://www2.iag.uni-hannover.de/~schuett/K3-fam.pdf, 2008.
6. Dino Festi and Duco van Straten, Bhabha scattering and a special pencil of K3 surfaces, Commun. Number Theory Phys. 13 (2019), no. 2, 463–485. MR 3951114
7. Dino Festi and Davide Cesare Veniani, Counting elliptic fibrations on K3 surfaces, preprint, arXiv:2102.09411v2, 2021.
8. Klaus Hulek and Matthias Schütt, Enriques surfaces and Jacobian elliptic K3 surfaces, Math. Z. 268 (2011), no. 3-4, 1025–1056. MR 2817842
9. Shigeyuki Kondō, Enriques surfaces with finite automorphism groups, Japan. J. Math. (N.S.) 12 (1986), no. 2, 191–282. MR 914299
10. Rick Miranda and David R. Morrison, Embeddings of integral quadratic forms, preliminary draft, web.math.ucsb.edu/~drm/manuscripts/eiqf.pdf, 2009.
11. Ken-ichi Nishiyama, The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups, Japan. J. Math. (N.S.) 22 (1996), no. 2, 181–200. MR 1432379
12. Hisanori Ohashi, On the number of Enriques quotients of a K3 surface, Publ. Res. Inst. Math. Sci. 43 (2007), no. 1, 1–31. MR 2319542
13. Chris Peters and Jan Stienstra, A pencil of K3-surfaces related to Apéry’s recurrence for \( \zeta(3) \) and Fermi surfaces for potential zero, Arithmetic of complex manifolds (Erlangen, 1988), Lecture Notes in Math., vol. 1399, Springer, Berlin, 1989, pp. 110–127. MR 1034260
14. Ichiro Shimada and Davide Cesare Veniani, Enriques involutions on singular K3 surfaces of small discriminants, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XXI (2020), 1667–1701.
15. Tetsuji Shioda, On the Mordell-Weil lattices, Comment. Math. Univ. St. Paul. 39 (1990), no. 2, 211–240. MR 1081382
(Dino Festi) Dipartimento di matematica Federigo Enriques, Università degli Studi di Milano, via Saldini 50, 20133 Milan, Italy
Email address: dino.festi@unimi.it

(Davide Cesare Veniani) Institut für Topologie und Geometrie, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
Email address: davide.veniani@mathematik.uni-stuttgart.de