APPLICATION OF A MODIFIED CES PRODUCTION FUNCTION MODEL BASED ON IMPROVED FIREFLY ALGORITHM

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Abstract. The conventional CES production function model fails to consider the influences of policy factors on economic growth in different stages. This paper proposes a modified model of the CES production function. Regarding model parameter estimation, the paper proposes a modern intelligent algorithm, the firefly algorithm (FA). The paper improves conventional FA to enhance the convergence rate and precision. To overcome the shortcomings of the conventional method in model application, the paper presents a new method of calculating the contribution rates of factors influencing economic growth and provides examples.

1. Introduction. The general form of the production function is $Y = f(X_1, X_2, \cdots, X_n)$ in which $X_1, X_2, \cdots, X_n$ are input factors and $Y$ is the output. There are many forms of the production function in which the constant elasticity of substitution (CES) production function is most commonly used [1, 11, 12, 21]. Researchers often apply the CES production function model to calculate and measure the contribution rates of influencing factors on economic growth. The form of the production function with $m$ input factors is $Y = A(\delta_1 X_1^{-\rho} + \delta_2 X_2^{-\rho} + \cdots + \delta_m X_m^{-\rho})^{-\frac{\mu}{\rho}}$ in which $A$ is the coefficient of efficiency and $A > 0$, $\delta_i$ ($i = 1, 2, \cdots, m$) represents the technical intensity of the factor, $\mu$ represents the homogeneous or return to scale of the function and $\mu > 0$, and $\rho$ is a substitution parameter and $\rho \geq -1$. $A, \delta_i (i = 1, 2, \cdots, m), \rho, \mu$ are the parameters to be estimated. The generalized form of CES production function currently is $Y = A[\delta_1 (\lambda_1 X_1)^{-\rho} + \delta_2 (\lambda_2 X_2)^{-\rho} + \cdots + \delta_m (\lambda_m X_m)^{-\rho}]^{-\frac{\mu}{\rho}}$ in which the newly added $\lambda_i$ is the gain factor of the input factor. In this formula, $\lambda_i$ is constant but exhibits different characteristics in various stages under the influence of the policy factor.

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This paper offers corresponding modifications [27, 30, 33] to remain consistent with reality.

The modified production function model is essentially a nonlinear model that cannot be linearized. If we use the general nonlinear least squares method for parameter estimation, then the convergence rate will be slow, the precision will be poor, and it may be difficult to discern a globally optimal solution. Since the early 1990s, with the rise of research on intelligent optimization algorithms, many scholars have studied the parameter estimation of mathematical models using intelligent optimization methods such as the simulated annealing algorithm (SAA), genetic algorithm (GA), artificial fish swarm algorithm (AFSA), and particle swarm optimization (PSO).

Peng et al. [22] estimated the parameters of fuzzy linear and nonlinear regression models using improved SAA. The results of the calculated example and contrastive analysis revealed that the method has a good convergence rate and a good regression result, making it suitable for complicated fuzzy regression models. Based on SAA, Zheng and Zhang [34] provided a parameter estimation method for a semi-parametric linear regression model and applied the new algorithm to multivariate linear regression models using typical examples. Tang and Wu [25] translated the Chen system multi-parameter estimation problem into a multi-parameter optimal searching problem and solved it using global GA optimization. The simulation results show that compared with single-population GA applied to estimate segmented Chen system parameters, the multi-population GA has advantages of precision and robustness.

Long et al. [18] translated the parameter estimation problem of a residual oil hydrofining reaction kinetic model into a multi-dimensional optimization problem and proposed a combinatorial optimization GA to solve the optimization problem; simulation results demonstrated the effectiveness of combinatorial GA. Li et al. [16] put forth a new general method for parameter estimation in production functions based on AFSA’s characteristics of overcoming the local extreme value and strong robustness. The algorithm featured a fast optimal searching speed in parameter estimation of the production function and obtained the least regression residual sum of squares. According to cubic Hermite interpolation polynomials, Liu et al. [17] proposed a high-precision parameter estimation method based on PSO to estimate parameters of the predator-prey model. The numerical simulation experiment indicated that the parameter estimation method could calculate related parameters more precisely.

To solve the practical problem of difficult parameter estimation in Richards’ model, Yan et al. [31] suggested translating the parameter estimation of this model into a multi-dimensional unconstrained function optimization problem. They estimated the four parameters of Richards’ model using PSO and constructed the growth fitting curve and variation curve of the optimal value. Their research results showed that PSO had a good fitting result for Richards’ model and good applicability to parameter estimation.

After 2005, several researchers constructed a stochastic optimization algorithm by simulating the swarm behaviors of fireflies. Indian scholars Krishnanand and Ghose [13] proposed glowworm swarm optimization (GSO), and Cambridge scholar Yang [32] proposed the firefly algorithm (FA). The two algorithms with the same bionic principle were both called FA, which has since been applied to many fields
such as function optimization, parameter identification, and combinatorial optimization. Considering the problem of insufficient parameter precision and its effects on related soil-water movement modeling and simulation results, Fu et al. [7] introduced FA and improved the fixed random step size into the variable step size based on the distance between fireflies, aiming to solve the nonlinear fitting problem of parameter optimization in the Van Genuchten model of the soil-water characteristic curve. FA demonstrated high precision in the simulation result and good curve fit with a maximum relative error of only 2% or so compared with PSO and GA; thus, it was found to be a high-precision optimization algorithm and exhibited higher convergence efficiency after improvements. To realize better parameter setting of the PID controller, Li et al. [15] introduced the swarm intelligent firefly algorithm of self-adaptation with a variable step size strategy. They conducted a simulation experiment using MATLAB and compared the algorithm with PSO and the classical Z-N parameter setting method. Experimental results revealed that the proposed algorithm possessed high precision and simple principles and could set PID controller parameters with high efficiency. Wang et al. [28] investigated the control parameters of FA and proposed a modified FA called FA with adaptive control parameters. Erdal [5] addressed the minimum weight design of new-generation steel beams with sinusoidal openings using a metaheuristic search technique, namely the firefly method. The algorithm has also been used to compare the optimum design results of sinusoidal web-expanded beams with steel castellated and cellular beams; numerical examples were presented, where the suggested algorithm was implemented to achieve a minimum weight design of beams subjected to various loading combinations.

However, FA is relatively new, and little relevant research has focused on parameter estimation of the production function model. The FA can overcome the local extreme value and obtain the global extreme value; additionally, it can be implemented without information such as the gradient of the objective function. Even so, the conventional FA has shortcomings in the rate and precision of convergence. To improve the convergence rate and precision of algorithm estimation, this paper uses the improved FA to estimate parameters of the production function model. Results show that the proposed algorithm improves the rate and precision of convergence substantially compared with the findings of Lv et al. [19].

In terms of calculating the contribution rates of factors within the CES production function model, many scholars have used Solow’s formula. Actual data consisted of discrete time series and were generally calculated by substituting the difference expression for the differential expression, leading to deviation. The paper theoretically offered a precise general formula for calculating contribution rates of economic growth factors with the production function and proposed an accurate and specific measurement and calculation method for practical applications [4, 8, 14, 29]. Using the modified CES production function model and the proposed calculation method, this paper calculates the contribution rates of factors influencing Chinese economic growth.

2. Form and parameter estimation of modified CES production function model.

2.1. Form of modified CES production function model. The generalized CES production function can be expressed as

\[
Y = A\left[\delta_1(\lambda_1X_1)^{-\rho} + \delta_2(\lambda_2X_2)^{-\rho} + \cdots + \delta_m(\lambda_mX_m)^{-\rho}\right]^{-\frac{1}{\rho}},
\]
where \( \lambda_i \) is a constant. Actually, \( \lambda_i \) varies at different stages under the effects of policy; thus, this paper makes modifications accordingly and proposes the following modified production function model:

\[
\begin{align*}
Y &= A_0 e^{\sigma t} \left\{ \sum_{i=1}^{k} (\alpha_i D_i + \beta_i D_i + \cdots + \gamma_i D_i) X_i \right\}^{-\rho} \\
&\quad + \sum_{i=1}^{m} (\gamma_i D_i + \cdots + \gamma_i D_i) X_i \}^{-\rho},
\end{align*}
\]

where \( A(t) = A_0 e^{\sigma t} \) denotes the technological progress level, and \( D_i \) is a dummy variable at different stages:

\[
D_i = \begin{cases} 
1, & \text{the } i^{\text{th}} \text{ stage}, \\
0, & \text{others}.
\end{cases}
\]

Each stage can be considered a period of China’s five-year plan, namely the period in which China implements certain policy; it can also be considered another special period, such as that of specific policy.

Clearly, if \( \alpha_1 = \alpha_2 = \cdots = \alpha_k = \lambda_1, \beta_1 = \beta_2 = \cdots = \beta_k = \lambda_2 \) and \( \gamma_1 = \gamma_2 = \cdots = \gamma_k = \lambda_m \), then the formula is the generalized CES production function model.

2.2. Parameter estimation of modified CES production function model.

The modified CES production function model can be written as

\[
Y = f(X, \eta) + \varepsilon,
\]

where \( X = (X_1, X_2, \cdots, X_m), \eta = (A, \sigma, \delta_1, \delta_2, \cdots, \delta_m, \alpha_1, \alpha_2, \cdots, \alpha_k, \beta_1, \beta_2, \cdots, \beta_k, \gamma_1, \gamma_2, \cdots, \gamma_k, \rho, \mu) \) and \( \varepsilon \) is the fitting error of the model.

\[
RSS(\eta) = \sum_{t=1}^{N} \varepsilon_t^2 = \sum_{t=1}^{N} [Y_t - f(X_{1t}, X_{2t}, \cdots, X_{mt}, \eta)]^2 \]

can have the minimum value.

The \( RSS(\eta) \) function is complicated with many parameters, and the general nonlinear regression method involves a vast number of calculations with the results easily affected by the initial value; therefore, the method is not suitable here. We could also use the nonlinear optimization method, but general optimization methods have a low rate of convergence and low precision and cannot be used to find globally optimal solution easily. This paper employs a modern intelligent optimization algorithm, FA, to estimate parameters. The algorithm has strong robustness and a fast convergence rate. It can also be realized easily and applied flexibly to obtain a globally optimal solution. To improve the rate of convergence and precision, this paper improves conventional FA.

2.2.1. Standard FA. The basic idea of standard FA [2, 20, 23, 26] is that each firefly’s position represents a solution for the problem to be solved; firefly luminance depends on the objective function value of the problem to be solved, and the better the value, the stronger the luminance. A firefly with stronger luminance compels fireflies with weaker luminance to move towards it. In the iterative process, fireflies with weaker luminance in the population continuously approach the firefly with
stronger luminance until most fireflies gather around the one with the strongest luminance, whose position is the optimal solution.

First, we establish the relationship between firefly \( i \)’s absolute luminance \( I_i \) and the objective function value. The luminance, which is \( r \) away from an individual, is generally expressed as

\[ I = I_0 e^{-\gamma r^2}, \]

\( I_0 \) is the luminance of the firefly:

\[ I_0 = g(\eta) = \frac{1}{1 + \text{RSS}(\eta)}. \]

Suppose firefly \( i \) has a larger absolute luminance than firefly \( j \), and firefly \( j \) is attracted by firefly \( i \) and moves towards it. Firefly \( i \)’s attractiveness \( \beta_{ij} \) is

\[ \beta_{ij} = \beta_0 e^{-\gamma r_{ij}^2}, \]

where \( \beta_0 \) is the maximum attractiveness, \( \gamma \) is the absorption coefficient of light (generally, \( \beta_0 = 1 \) and \( \gamma \in [0.01, 100] \)), and \( r_{ij} \) is the Cartesian distance between \( i \) and \( j \) such that

\[ r_{ij} = \|\eta_i - \eta_j\| = \sqrt{\sum_{k=1}^{d} (\eta_{ik} - \eta_{jk})^2}, \]

where \( d \) is the variable dimension.

Because firefly \( j \) moves towards firefly \( i \) under the attraction of firefly \( i \), the update formula for firefly \( j \)’s position is

\[ \eta_j(t + 1) = \eta_j(t) + \beta_{ij} (\eta_i(t) - \eta_j(t)) + \alpha \varepsilon_j, \]

where \( t \) is the number of iterations; \( \eta_i \) and \( \eta_j \) are the spatial positions of firefly \( i \) and firefly \( j \); \( \beta_{ij} \) is firefly \( i \)’s relative attractiveness to firefly \( j \); \( \alpha \) is a constant generally within \([0, 1]\); and \( \varepsilon_j \) is a vector of a random number obtained from the uniform distribution.

2.2.2. Improved FA. This paper makes improvements in the following three aspects [3, 6, 9, 10, 24]:

(1) Adaptive variable weight

For the standard FA, as the distance between fireflies decreases in later iterations, the relative attractiveness between fireflies increases; thus, FA’s local search ability weakens and even demonstrates repeated oscillations near the extreme point, potentially requiring multiple intonations to meet the necessary accuracy. In this case, the algorithm may fail to meet the accuracy requirement within the limited number of iterations. To improve FA’s local and global search abilities, the inertia weight can be introduced into the position update formula as follows:

\[ \eta_j(t + 1) = w(t) \eta_j(t) + \beta_0 e^{-\gamma r_{ij}^2} (\eta_i(t) - \eta_j(t)) + \alpha [\text{rand} - 0.5]. \]

A weight \( w \) that is either too large or small may affect the speed and accuracy of parameter estimation. The adaptive variable weight adopts a appropriately large \( w \) in the earlier stage to enhance the local search ability and a appropriately small \( w \) in the later stage to focus on the local search and enhance the algorithm’s local search ability, thus avoiding repeated oscillations near the extreme point and increasing the solution accuracy. Accordingly, this paper uses an inertia weight nonlinear
decreasing strategy that allows $w$ to decline nonlinearly as the number of iterations $t$ increases:

$$w = \left\{ 1 - h \sin\left(\frac{\pi}{2} \cdot \frac{t}{T_{\text{max}}} \right) \right\}^{\alpha_1} \cdot \alpha_2 \cdot w_{\text{init}},$$

where $w_{\text{init}}$ is the initial value of the inertia weight, $T_{\text{max}}$ is the maximum number of iterations, and $0 \leq \alpha_1 \leq 1$ and $0 \leq h \leq 1$, $\alpha_2 \geq 1$.

(2) Adaptive attractiveness

The algorithm’s position update formula shows that attractiveness affects the moving step size of nonrandom fireflies. In the original algorithm, $\beta_0$ and $\gamma$, the coefficients affecting attractiveness, are constants and cannot realize adaptive changes through iterations, which may lead to a weak search ability in early iterations and solution oscillation near the optimal solution in the later stage. To overcome these problems and enable the algorithm to quickly approach the optimal solution, this paper changes the attractiveness coefficient with algorithm iterations using the following formula:

$$\beta_0 = \left\{ 1 - b_1 \sin\left(\frac{\pi}{2} \cdot \frac{t}{T_{\text{max}}} \right) \right\}^{\sigma_1} \cdot \beta_{\text{init}},$$

$$\gamma = \left\{ 1 - b_3 \sin\left(\frac{\pi}{2} \cdot \frac{t}{T_{\text{max}}} \right) \right\}^{\sigma_2} \cdot \gamma_{\text{init}},$$

where $t$ is the current number of iterations; $T_{\text{max}}$ is the maximum number of iterations; and $b_1, b_2, b_3, b_4, \sigma_1, \sigma_2$ are the constants set originally, and $0 \leq b_1, b_2, b_3, b_4 \leq 1$, $\sigma_1, \sigma_2 \geq 1$.

(3) Adaptive constant $\alpha$

To accelerate population convergence, this paper proposes an update mode of $\alpha$, which allows $\alpha$ to decline with the number of iterations. The following is the updated formula:

$$\alpha(t) = e^{-c_1 \left( \frac{t}{T_{\text{max}}} \right)^2},$$

where $t$ is the number of iterations, $T_{\text{max}}$ is the maximum number of iterations, and constants $c_1, c_2 \geq 0$.

2.2.3. Steps of parameter estimation based on improved AF. The six steps are as follows:

**Step 1.** Initialize the basic algorithm parameters. Set the number of fireflies, initial inertia weight $w_{\text{init}}$, initial attractiveness $\beta_{\text{init}}$, initial light absorption coefficient $\gamma_{\text{init}}$, maximum number of iterations $T_{\text{max}}$, constants $\alpha_1, \alpha_2, \beta_0, \gamma, b_1, b_2, b_3, b_4, \sigma_1, \sigma_2, c_1, c_2$, etc.

**Step 2.** Initialize firefly positions. In order to improve the convergence speed and accuracy of the algorithm, the upper and lower limits of the initial firefly positions in a small range and the appropriate initial values can be given.

**Step 3.** Calculate the objective function value and firefly’s luminance $I$.

**Step 4.** Calculate adaptive variable weight $w$ and coefficients $\beta_0$ and $\gamma$, which influence attractiveness.

**Step 5.** Compare each firefly with all others in terms of initial luminance and update the position when meeting a firefly with stronger luminance.

**Step 6.** Upon attaining either accuracy or the maximum number of iterations, end; otherwise, return to Step 3.
3. **Improvements to the method to calculate contribution rate to economic growth.** Suppose the modified CES production function model is

\[ Y = A_0 e^{\sigma t} \{ \delta_1 [(\alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_k D_k) X_1]^{-\rho} + \delta_2 [(\beta_1 D_1 + \beta_2 D_2 + \cdots + \beta_k D_k) X_2]^{-\rho} + \cdots + \delta_m [(\gamma_1 D_1 + \gamma_2 D_2 + \cdots + \gamma_k D_k) X_m]^{-\rho} \}^{-\frac{\sigma}{\rho}}, \]

where \( X_i \) represents factor input, \( A(t) = A_0 e^{\sigma t} \) is the technological progress level, \( Y \) is the output, and \( \rho \) is the dummy variable.

Let \( \lambda_1 = \alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_k D_k \), \( \lambda_2 = \beta_1 D_1 + \beta_2 D_2 + \cdots + \beta_k D_k \), \( \lambda_m = \gamma_1 D_1 + \gamma_2 D_2 + \cdots + \gamma_k D_k \), then the equation above is written as

\[ Y = A[\delta_1 (\lambda_1 X_1)^{-\rho} + \delta_2 (\lambda_2 X_2)^{-\rho} + \cdots + \delta_m (\lambda_m X_m)^{-\rho}]. \]

Assume that economic vector \((A, X_1, X_2, \cdots, X_m, Y)\) changes in the form of curve \( L_j(t) \) from period \( j \) to period \( j + 1 \). Thus, the absolute influence value of technological progress on economic growth from period 1 to period \( n \) is

\[ \Delta Y_A = \sum_{j=1}^{n-1} \int_{L_j(t)} \frac{\partial Y}{\partial A} dA, \]

and factor \( X_i \)'s absolute influence value on economic growth from period 1 to period \( n \) is

\[ \Delta Y_{X_i} = \sum_{j=1}^{n-1} \int_{L_j(t)} \frac{\partial Y}{\partial X_i} dX_i. \]

Then, from period 1 to period \( n \), the contribution rate of technological progress to economic growth is

\[ \frac{\Delta Y_A}{\Delta Y} = \frac{\Delta Y_A}{\Delta Y_A + \Delta Y_{X_1} + \Delta Y_{X_2} + \cdots + \Delta Y_{X_m}}; \]

the contribution rate of the \( i \)'th factor is

\[ \frac{\Delta Y_{X_i}}{\Delta Y} = \frac{\Delta Y_{X_i}}{\Delta Y_A + \Delta Y_{X_1} + \Delta Y_{X_2} + \cdots + \Delta Y_{X_m}}. \]

The following presents a case of the CES production function with three factors.

Suppose the CES production function with three factors is

\[ Y = A_0 e^{\sigma t} \{ \delta_1 [(\alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3) + \alpha_4 D_4 + \alpha_5 D_5) K]^{-\rho} + \delta_2 [(\beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + \beta_5 D_5) L]^{-\rho} + \delta_3 [(\gamma_1 D_1 + \gamma_2 D_2 + \gamma_3 D_3 + \gamma_4 D_4 + \gamma_5 D_5) E]^{-\rho} \}^{-\frac{\sigma}{\rho}}, \]

where \( Y \) is the output, \( K \) is the capital input, \( L \) is the labor input, \( E \) is the energy input, and \( A=A_0 e^{\sigma t} \) is the technological progress level.

\[ D_1 = \begin{cases} 1, & \text{Period of the 9th five-year plan}, \\ 0, & \text{others}. \end{cases} \]

\[ D_2 = \begin{cases} 1, & \text{Period of the 10th five-year plan}, \\ 0, & \text{others}. \end{cases} \]

\[ D_3 = \begin{cases} 1, & \text{Period of the 11th five-year plan}, \\ 0, & \text{others}. \end{cases} \]
\[ D_4 = \begin{cases} 1, & \text{Period of the } 12^{th} \text{ five-year plan}, \\ 0, & \text{others}. \end{cases} \]

\[ D_5 = \begin{cases} 1, & \text{Period of the } 13^{th} \text{ five-year plan}, \\ 0, & \text{others}. \end{cases} \]

The CES production function with three factors is written simply as

\[ Y = A \left( \delta_1 (\lambda_1 K)^{-\rho} + \delta_2 (\lambda_2 L)^{-\rho} + \delta_3 (\lambda_3 E)^{-\rho} \right)^{-\frac{1}{\rho}}, \]

where \( \lambda_1 = \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 + \alpha_5 D_5 \), \( \lambda_2 = \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + \beta_5 D_5 \), and \( \lambda_3 = \gamma_1 D_1 + \gamma_2 D_2 + \gamma_3 D_3 + \gamma_4 D_4 + \gamma_5 D_5 \).

Then, for the differential, we get

\[ \frac{\partial Y}{\partial A} = [\delta_1 (\lambda_1 K)^{-\rho} + \delta_2 (\lambda_2 L)^{-\rho} + \delta_3 (\lambda_3 E)^{-\rho}]^{-\frac{1}{\rho}} \frac{\partial}{\partial A} \left( \frac{Y}{A_0 e^{\sigma t}} \right), \]

\[ \frac{\partial Y}{\partial K} = \frac{\partial}{\partial K} \left( [\delta_1 (\lambda_1 K)^{-\rho} + \delta_2 (\lambda_2 L)^{-\rho} + \delta_3 (\lambda_3 E)^{-\rho}]^{-\frac{1}{\rho}} \frac{Y}{A_0 e^{\sigma t}} \right), \]

\[ \frac{\partial Y}{\partial L} = \frac{\partial}{\partial L} \left( [\delta_1 (\lambda_1 K)^{-\rho} + \delta_2 (\lambda_2 L)^{-\rho} + \delta_3 (\lambda_3 E)^{-\rho}]^{-\frac{1}{\rho}} \frac{Y}{A_0 e^{\sigma t}} \right), \]

\[ \frac{\partial Y}{\partial E} = \frac{\partial}{\partial E} \left( [\delta_1 (\lambda_1 K)^{-\rho} + \delta_2 (\lambda_2 L)^{-\rho} + \delta_3 (\lambda_3 E)^{-\rho}]^{-\frac{1}{\rho}} \frac{Y}{A_0 e^{\sigma t}} \right), \]

Let \( L_j \) be the curve between point \((K_j, L_j, E_j, Y_j)\) and \((K_{j+1}, L_{j+1}, E_{j+1}, Y_{j+1})\); its parameter equation is

\[ \begin{align*}
K_t &= K_j \left( \frac{K_{j+1}}{K_j} \right)^t = K_j e^{t \ln \left( \frac{K_{j+1}}{K_j} \right)}, \\
L_t &= L_j \left( \frac{L_{j+1}}{L_j} \right)^t = L_j e^{t \ln \left( \frac{L_{j+1}}{L_j} \right)}, \\
E_t &= E_j \left( \frac{E_{j+1}}{E_j} \right)^t = E_j e^{t \ln \left( \frac{E_{j+1}}{E_j} \right)}, \\
Y_t &= Y_j \left( \frac{Y_{j+1}}{Y_j} \right)^t = Y_j e^{t \ln \left( \frac{Y_{j+1}}{Y_j} \right)}. \end{align*} \]

Next, from period 1 to period \( n \), the influence value of technological progress on economic growth is

\[ \Delta Y_A = \sum_{j=1}^{21} \int_{L_j(t)}^{L_{j+1}(t)} \frac{\partial Y}{\partial A} dA = \sum_{j=1}^{21} \int_{L_j(t)}^{L_{j+1}(t)} \frac{Y}{A_0 e^{\sigma t}} dA. \]
\[\Delta Y_K = \sum_{j=1}^{21} \int_{L_j(t)} \frac{Y_j e^{\ln\left(\frac{Y_j+1}{Y_j}\right)}}{A_0 e^{\sigma t}} d(A_0 e^{\sigma t})
= \sum_{j=1}^{21} \int_{0}^{1} \sigma Y_j e^{\ln\left(\frac{Y_j+1}{Y_j}\right)} dt
= \sum_{j=1}^{21} \frac{\sigma(Y_j+1-Y_j)}{\ln Y_j+1-Y_j},\]

and the influence value of factor \(K\) is

\[\Delta Y_K = \sum_{j=1}^{21} \int_{L_j(t)} \frac{\partial Y}{\partial K} dK\]

\[= \sum_{j=1}^{21} \int_{L_j(t)} A_0 \frac{\partial Y}{\partial K} e^{\ln\left(\frac{Y_j+1}{Y_j}\right)} e^{-\frac{\sigma}{\rho} t \mu_2 \lambda_2^{-\rho} K^{-\rho} - \rho - 1} dK\]

\[= \sum_{j=1}^{21} \int_{0}^{1} A_0 Y_j \frac{\partial Y}{\partial K} e^{\ln\left(\frac{Y_j+1}{Y_j}\right)} e^{\frac{\sigma}{\rho} t \mu_2 \lambda_2^{-\rho} \ln\left(\frac{K_j+1}{K_j}\right) + \frac{\sigma}{\rho} - \rho \ln\left(\frac{K_j+1}{K_j}\right)} dK e^{\ln\left(\frac{Y_j+1}{Y_j}\right) - \frac{\sigma}{\rho} - \rho \ln\left(\frac{K_j+1}{K_j}\right)} dt\]

\[\lambda_1 = \begin{cases} 
\alpha_1, & 1 \leq j \leq 4, \\
\alpha_1(1-t) + \alpha_2 t, & j = 5, \\
\alpha_2, & 6 \leq j \leq 9, \\
\alpha_2(1-t) + \alpha_3 t, & j = 10, \\
\alpha_3, & 11 \leq j \leq 14, \\
\alpha_3(1-t) + \alpha_4 t, & j = 15, \\
\alpha_4, & 16 \leq j \leq 19, \\
\alpha_4(1-t) + \alpha_5 t, & j = 20, \\
\alpha_5, & j = 21.
\end{cases}\]

We can calculate \(\Delta Y_K\) via numerical integration.

The influence value of factor \(L\) is

\[\Delta Y_L = \sum_{j=1}^{21} \int_{L_j(t)} \frac{\partial Y}{\partial L} dL\]

\[= \sum_{j=1}^{21} \int_{L_j(t)} A_0 \frac{\partial Y}{\partial L} e^{\ln\left(\frac{Y_j+1}{Y_j}\right)} e^{-\frac{\sigma}{\rho} t \mu_2 \lambda_2^{-\rho} L^{-\rho} - \rho - 1} dL\]

\[= \sum_{j=1}^{21} \int_{0}^{1} A_0 Y_j \frac{\partial Y}{\partial L} e^{\ln\left(\frac{Y_j+1}{Y_j}\right)} e^{\frac{\sigma}{\rho} t \mu_2 \lambda_2^{-\rho} \ln\left(\frac{L_j+1}{L_j}\right) + \frac{\sigma}{\rho} - \rho \ln\left(\frac{L_j+1}{L_j}\right)} dL e^{\ln\left(\frac{Y_j+1}{Y_j}\right) - \frac{\sigma}{\rho} - \rho \ln\left(\frac{L_j+1}{L_j}\right)} dt\]

\[\lambda_2 = \begin{cases} 
\beta_1, & 1 \leq j \leq 4, \\
\beta_1(1-t) + \beta_2 t, & j = 5, \\
\beta_2, & 6 \leq j \leq 9, \\
\beta_2(1-t) + \beta_3 t, & j = 10, \\
\beta_3, & 11 \leq j \leq 14, \\
\beta_3(1-t) + \beta_4 t, & j = 15, \\
\beta_4, & 16 \leq j \leq 19, \\
\beta_4(1-t) + \beta_5 t, & j = 20, \\
\beta_5, & j = 21.
\end{cases}\]

We can calculate \(\Delta Y_L\) via numerical integration.
The influence value of factor $E$ is
\[
\Delta Y_E = \sum_{j=1}^{21} \int_{L_j(t)}^{} \frac{\partial Y}{\partial E} dE \\
= \sum_{j=1}^{21} \int_{L_j(t)}^{Y_j} A_0 \frac{dY}{Y} e^{-\frac{\rho}{\sigma} \mu \delta \lambda^\rho E} E^{-\rho} - dE \\
= \sum_{j=1}^{21} \int_{L_j(t)}^{Y_j} A_0 \frac{dY}{Y} e^{-\frac{\rho}{\sigma} \mu \delta \lambda^\rho E} (E_j e^{\int \frac{L_j(t)}{E_j} \ln \left( \frac{E_j}{E_j} \right)} - 1)^{-\rho - 1} dE_j e^{\int \frac{L_j(t)}{E_j} \ln \left( \frac{E_j}{E_j} \right)} \int \frac{L_j(t)}{E_j} \ln \left( \frac{E_j}{E_j} \right)} \lambda^\rho dt,
\]
where \(\lambda_3 = \begin{cases} 
\gamma_1, 1 \leq j \leq 4, \\
\gamma_1(1 - t) + \beta_2 t, j = 5, \\
\gamma_2, 6 \leq j \leq 9, \\
\gamma_2(1 - t) + \gamma_3, j = 10, \\
\gamma_3, 11 \leq j \leq 14, \\
\gamma_3(1 - t) + \gamma_4 t, j = 15, \\
\gamma_4, 16 \leq j \leq 19, \\
\gamma_4(1 - t) + \gamma_5 t, j = 20, \\
\gamma_5, j = 21.
\end{cases}\)

We can calculate \(\Delta Y_E\) via numerical integration.

Then, from period 1 to period \(n\), the contribution rate of technological progress to economic growth is
\[
\frac{\Delta Y_A}{\Delta Y} = \frac{\Delta Y_A}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E};
\]
the contribution rate of capital is
\[
\frac{\Delta Y_K}{\Delta Y} = \frac{\Delta Y_K}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E};
\]
the contribution rate of labor is
\[
\frac{\Delta Y_L}{\Delta Y} = \frac{\Delta Y_L}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E};
\]
And the contribution rate of energy is
\[
\frac{\Delta Y_E}{\Delta Y} = \frac{\Delta Y_E}{\Delta Y_A + \Delta Y_K + \Delta Y_L + \Delta Y_E}.
\]

4. Empirical analysis of contribution rates of influencing factors on China’s economic growth. To study Chinese economic growth, we must explore the manner of growth and calculate the respective contribution rates of individual input factors to economic growth. This paper selects GDP \(Y\) (0.1 billion) as output. Fixed-asset investment \(K\) (0.1 billion), number of employees \(L\) (10,000 people), and the total energy consumed \(E\) (10,000 tons of standard coal) are considered economic influence factors in the corresponding analysis; data are listed in Table 1.

Suppose the CES production function with three factors is
\[
Y = A_0 e^{\rho t} \{ \delta_1 [(\alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 + \alpha_5 D_5) K]^{-\rho} + \delta_2 [(\beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + \beta_5 D_5) L]^{-\rho} + \delta_3 [(\gamma_1 D_1 + \gamma_2 D_2 + \gamma_3 D_3 + \gamma_4 D_4 + \gamma_5 D_5) E]^{-\rho} \}^{-\frac{1}{\rho}},
\]
Table 1. Data on Chinese economic growth.

| Period                  | Year | Y   | L   | K   | E   |
|-------------------------|------|-----|-----|-----|-----|
| Period of the 9th plan  | 1996 | 71176.6 | 68950 | 22913.5 | 135192 |
|                         | 1997 | 78973.0 | 69820 | 24941.1 | 135909 |
|                         | 1998 | 84402.3 | 70637 | 28406.2 | 136184 |
|                         | 1999 | 89677.1 | 71394 | 29854.7 | 140569 |
|                         | 2000 | 99214.6 | 72085 | 32917.7 | 145531 |
| Period of the 10th plan | 2001 | 109655.2 | 72797 | 37213.5 | 150406 |
|                         | 2002 | 120332.7 | 73280 | 43499.9 | 159431 |
|                         | 2003 | 135822.8 | 73736 | 55566.6 | 183792 |
|                         | 2004 | 159878.3 | 74264 | 70477.4 | 213456 |
|                         | 2005 | 183217.5 | 74647 | 88773.6 | 235997 |
| Period of the 11th plan | 2006 | 211923.5 | 74978 | 109998.2 | 258676 |
|                         | 2007 | 249529.9 | 75321 | 137239.0 | 280508 |
|                         | 2008 | 316228.8 | 75564 | 172828.4 | 291448 |
|                         | 2009 | 343464.7 | 75828 | 224598.8 | 306647 |
|                         | 2010 | 401512.8 | 76105 | 251683.8 | 324939 |
| Period of the 12th plan | 2011 | 473104.0 | 76420 | 311485.1 | 348002 |
|                         | 2012 | 519470.1 | 76704 | 374694.7 | 361732 |
|                         | 2013 | 568845.0 | 76977 | 447074.0 | 375252 |
|                         | 2014 | 636462.7 | 77253 | 512760.7 | 426000 |
|                         | 2015 | 676780.0 | 77451 | 562000.0 | 430000 |
| Period of the 13th plan | 2016 | 744127.0 | 77603 | 606466.0 | 436000 |
|                         | 2017 | 827122.0 | 77640 | 641238.0 | 449000 |

where

\[
D_1 = \begin{cases} 
1 , & \text{Period of the 9th five-year plan}, \\
0 , & \text{others}.
\end{cases}
\]

\[
D_2 = \begin{cases} 
1 , & \text{Period of the 10th five-year plan}, \\
0 , & \text{others}.
\end{cases}
\]

\[
D_3 = \begin{cases} 
1 , & \text{Period of the 11th five-year plan}, \\
0 , & \text{others}.
\end{cases}
\]

\[
D_4 = \begin{cases} 
1 , & \text{Period of the 12th five-year plan}, \\
0 , & \text{others}.
\end{cases}
\]

\[
D_5 = \begin{cases} 
1 , & \text{Period of the 13th five-year plan}, \\
0 , & \text{others}.
\end{cases}
\]

Choose a population size of 10, and \(w_{\text{init}} = 1, \beta_{\text{init}} = 1, \gamma_{\text{init}} = 1, T_{\text{max}} = 800, \alpha_1 = 0.5, \alpha_2 = 10, h = 0.01, b_1 = 0.01, b_2 = 0.5, b_3 = 0.01, b_4 = 0.5, \sigma_1 = \sigma_2 = 5, \) and \(c_1 = 50, c_2 = 20.\)

When completing calculations using the method proposed in this paper, we get

\[
A = 0.7500, \sigma = 0.0572, \delta_1 = 0.6502, \delta_2 = 0.3297, \delta_3 = 0.1834, \alpha_1 = 0.5184, \\
\alpha_2 = 0.7164, \alpha_3 = 0.1986, \alpha_4 = 0.3223, \alpha_5 = 0.1538, \beta_1 = 0.1565, \beta_2 = 0.1170, \\
\beta_3 = 0.2027, \beta_4 = 0.2498, \beta_5 = 0.2239, \gamma_1 = 0.1137, \gamma_2 = 0.0901, \gamma_3 = 0.1002, \\
\gamma_4 = 0.0482, \gamma_5 = 0.1306, \rho = 1.5310, \mu = 1.2203.
\]

The correlation coefficient of the model is

\[
R^2 = 1 - \frac{\sum (Y_i - \hat{Y}_i)^2}{\sum (Y_i - \bar{Y})^2} = 0.9990.
\]
This model has high fitting precision, and the coefficient of determination is close to 1.

Next, we completed calculations to compare the conventional FA, PSO [19], and improved FA in terms of convergence rate and precision. Table 2 displays the comparison results of three algorithms. The improved FA was superior to the other two algorithms in terms of convergence rate and precision.

**Table 2. Comparison of results of three algorithms.**

| Algorithm  | Improved PSO | Conventional firefly algorithm | Improved firefly algorithm |
|------------|--------------|---------------------------------|---------------------------|
| \( \Delta Y_A \) | 0.7638 | 0.8268 | 0.7500 |
| \( \sigma \) | 0.0518 | 0.0700 | 0.0572 |
| \( \delta_1 \) | 0.6612 | 0.6460 | 0.6502 |
| \( \delta_2 \) | 0.3229 | 0.3827 | 0.3297 |
| \( \delta_3 \) | 0.1852 | 0.1712 | 0.1834 |
| \( \alpha_1 \) | 0.5352 | 0.5080 | 0.5184 |
| \( \alpha_2 \) | 0.7295 | 0.6825 | 0.7164 |
| \( \alpha_3 \) | 0.1683 | 0.2149 | 0.1986 |
| \( \alpha_4 \) | 0.3224 | 0.3464 | 0.3223 |
| \( \alpha_5 \) | 0.1511 | 0.1970 | 0.1538 |
| \( \beta_1 \) | 0.1431 | 0.1878 | 0.1565 |
| \( \beta_2 \) | 0.1072 | 0.1228 | 0.1170 |
| \( \beta_3 \) | 0.1971 | 0.1916 | 0.2027 |
| \( \beta_4 \) | 0.2472 | 0.2645 | 0.2498 |
| \( \beta_5 \) | 0.2183 | 0.1981 | 0.2239 |
| \( \gamma_1 \) | 0.1102 | 0.0988 | 0.1137 |
| \( \gamma_2 \) | 0.0710 | 0.1352 | 0.0901 |
| \( \gamma_3 \) | 0.1006 | 0.1478 | 0.1002 |
| \( \gamma_4 \) | 0.0478 | 0.0375 | 0.0482 |
| \( \gamma_5 \) | 0.1205 | 0.1945 | 0.1306 |
| \( \rho \) | 1.5457 | 1.4934 | 1.5310 |
| \( \mu \) | 1.2312 | 1.1995 | 1.2203 |

From 1996 to 2017, the contribution rates of factors to economic growth were as follows:

**Technological progress \( A \),**

\[
\frac{\Delta Y_A}{\Delta Y} = 32.12\%;
\]

**Factor \( K \),**

\[
\frac{\Delta Y_K}{\Delta Y} = 40.95\%;
\]

**Factor \( L \),**

\[
\frac{\Delta Y_L}{\Delta Y} = 4.00\%;
\]

**Factor \( E \),**

\[
\frac{\Delta Y_E}{\Delta Y} = 22.93\%.
\]
5. **Conclusion.** This paper proposes a modified CES production function model by considering the influence of policy factors to render the model more consistent with reality. With regard to parameter estimation, the paper uses an improved FA with high precision and fast convergence. The paper details an accurate measurement and calculation method to determine the contribution rates of influencing factors to economic growth. Calculation results indicate that economic growth in China depends heavily on capital input followed by technological progress and energy input; labor force is less influential. These results coincide with the economic reality in China. The provided calculation and measurement example demonstrates that the method presented in this paper is scientific and reliable.

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