High-precision determination of the electric and magnetic radius of the proton

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Abstract

Using dispersion theory with an improved description of the two-pion continuum based on the precise Roy-Steiner analysis of pion-nucleon scattering, we analyze recent data from electron-proton scattering. This allows for a high-precision determination of the electric and magnetic radius of the proton, \( r_E = (0.838^{+0.005}_{-0.004} \pm 0.004) \) fm and \( r_M = (0.847 \pm 0.004 \pm 0.004) \) fm, where the first error refers to the fitting procedure using bootstrap and the data while the second one refers to the systematic uncertainty related to the underlying spectral functions.

1. Introduction

The electric radius, \( r_E \), and the magnetic radius, \( r_M \), of the proton are fundamental quantities of low-energy QCD, as they are a measure of the probe-dependent size of the proton. While the electric radius of the proton has attracted much attention in the last decade (see, e.g., Refs. \cite{1, 2, 3} for recent reviews), this is not true for its magnetic counterpart, which is not probed in the Lamb shift in electronic or muonic hydrogen. A major source of information on the proton form factors (and the corresponding radii) is elastic electron-proton (ep) scattering. These data can be best analyzed in the time-honored framework of dispersion theory \cite{4, 5, 6, 7}, which includes all constraints from unitarity, analyticity and crossing symmetry and is consistent with the strictures from perturbative QCD at very large momentum transfer \cite{8}. Of particular importance for the proper extraction of the radii is the isovector two-pion continuum on the left shoulder of the \( \rho \)-resonance \cite{9, 10}, which can be worked out model-independently using dispersively constructed pion-nucleon scattering amplitudes combined with data of the pion vector form factor. Based on the recent Roy-Steiner analysis of pion-nucleon scattering \cite{11}, an improved determination of the two-pion continuum was given in Ref. \cite{12}, which also includes thorough error estimates. Using a sum rule for the isovector charge radius, in that paper a squared isovector charge radius, \( (r_{E}^{v})^{2} = 0.405(36) \) fm\(^{2} \), was obtained which is in perfect agreement with a recent state-of-the-art lattice QCD calculation at physical pion masses, \( (r_{E}^{v})^{2} = 0.400(13)_{\text{stat}}(11)_{\text{sys}} \) fm\(^{2} \) \cite{13}. This underscores the importance of the isovector two-pion continuum for the form factors and demonstrates the consistency of the new and improved representation from Ref. \cite{12} with QCD. This new representation of the two-pion continuum has so far not been employed in any dispersion-theoretical analysis of form factor data.

With the advent of new and precise electron-scattering data at low momentum transfer from Jefferson Laboratory (PRad collaboration) \cite{14}, it is timely to analyze these together with the precise data from the AI collaboration at the Mainz Microtron (MAMI) \cite{15} using the improved two-pion continuum contribution. Given the precision of these data and of the underlying formalism, this will allow for a high-precision determination of both the electric and the magnetic form factors and the corresponding
2. Formalism

Here, we briefly summarize the underlying formalism, which is detailed in Refs. [19, 20]. The differential cross section for $ep$ scattering can be expressed through the electric ($G_E$) and magnetic ($G_M$) Sachs form factors (FFs) as

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \frac{\tau}{\epsilon(1 + \tau)} \left[ G_M^2(t) + \frac{\epsilon}{\tau} G_E^2(t) \right],$$

where $\epsilon = [1 + 2(1 + \tau) \tan^2(\theta/2)]^{-1}$ is the virtual photon polarization, $\theta$ is the electron scattering angle in the laboratory frame, $\tau = -t/4m_N^2$, with $t$ the four-momentum transfer squared and $m_N$ the nucleon mass. Moreover, $(d\sigma/d\Omega)_{\text{Mott}}$ is the Mott cross section, which corresponds to scattering off a point-like spin-1/2 particle. Since $-t \equiv Q^2 > 0$ is spacelike in $ep$ scattering, the form factors are often displayed as a function of $Q^2$. Equation (1) will be our basic tool to analyze the data together with the two-photon corrections from Ref. [20].

The electric and magnetic radii of the proton, which are at the center of this investigation, are given by

$$r_{E/M} = \sqrt{\frac{6}{G_E/M(0) \left. \frac{dG_{E/M}(t)}{dt} \right|_{t=0}}}.$$  

(2)

For the theoretical analysis, it is advantageous to use the Dirac ($F_1$) and Pauli ($F_2$) FFs, which are linear combinations of the Sachs FFs:

$$G_E(t) = F_1(t) - \tau F_2(t), \quad G_M(t) = F_1(t) + F_2(t).$$

(3)

The FFs for spacelike momentum transfer, $t < 0$, are given in terms of an unsubtracted dispersion relation,

$$F_i(t) = \frac{1}{\pi} \int_{t_0}^\infty \frac{\text{Im} F_i(t') dt'}{t' - t}, \quad i = 1, 2,$$

(4)

with $t_0 = 4M^2_N (9M^2_N)$ the isovector (isoscalar) threshold, and $M_N$ is the charged pion mass. The spectral functions are expressed in terms of (effective) vector meson poles and continua, which leads to the following representation of the FFs:

$$F_i^s(t) = \sum_{V = \omega, \phi, s_1, s_2, \ldots} \frac{q_i^V}{m_V^2 - t} + +F_i^{\pi\rho}(t) + F_i^{\bar{K}K}(t),$$

$$F_i^v(t) = \sum_{V = \pi_1, \pi_2, \ldots} \frac{q_i^V}{m_V^2 - t} + F_i^{2\pi}(t),$$

(5)

with $i = 1, 2$, in terms of the isoscalar ($s$) and isovector ($v$) components, $F_i^{(s/v)} = (F_i^p \pm F_i^n)/2$. This representation is advantageous for dispersion analyses since the intermediate states contributing to the spectral function have good isospin. In the isoscalar spectral function, the first two poles correspond to the $\omega(782)$ and the $\phi(1020)$ mesons, so these masses are fixed and the residua are bounded as in Ref. [20]. Furthermore, we take into account the $\pi\rho$ and $KK$ continua as explained in detail in Ref. [19]. The last term in the isovector form factor corresponds to the parameterization of the two-pion continuum taken from Ref. [11]. This is the essential new theory input compared to earlier dispersive analyses. The higher mass poles are effective poles that parameterize the spectral function at large $t$. We explicitly check in our analysis that the radii are insensitive to the details of this parameterization. The fit parameters are therefore the various vector meson residua $a_i^V$ and the masses of the additional vector mesons $s_i, v_i$. 

radii, $r_E$ and $r_M$, respectively. Clearly, this is an important step in pinning down these fundamental quantities with high precision. Other recent analyses of the PRad and Mainz data can be found in Refs. [16, 17, 18] and will be discussed below.
In addition, we fulfill the normalization conditions $F_1(0) = 1$ (in units of the elementary charge $e$) and $F_2(0) = \mu_p$, with $\mu_p$ the anomalous magnetic moment of the proton. To ensure the stability of the fit \cite{21}, we demand that the residua of the vector meson poles are bounded, $|q_i^V| < 5$ GeV$^2$, and that no effective poles with masses below 1 GeV appear. Finally, the FFs must satisfy the superconvergence relations

$$\int_{t_0}^{\infty} \text{Im} F_i(t)t^n dt = 0, \quad i = 1, 2,$$

with $n = 0$ for $F_1$ and $n = 0, 1$ for $F_2$, corresponding to the fall-off with inverse powers of $Q^2$ at large momentum transfer as demanded by perturbative QCD \cite{8}.

These parameterizations (5) are used in Eq. (1) and the number of isoscalar and isovector poles is determined by the condition to obtain the best fit to the data. The quality of the fits is measured in terms of the traditional $\chi^2$,

$$\chi^2 = \sum_k \frac{(n_k C_i - C(Q_i^2, \theta_i, \vec{p}))^2}{(\mu_i + \nu_i)^2},$$

where $C_i$ are the cross section data at the points $Q_i^2, \theta_i$ and $C(Q_i^2, \theta_i, \vec{p})$ are the cross sections for a given FF parameterization for the parameter values contained in $\vec{p}$. Moreover, $n_k$ are normalization coefficients for the various data sets (labelled by the integer $k$), while $\sigma_i$ and $\nu_i$ are their statistical and systematical errors, respectively. A more refined definition of the $\chi^2$ is given by

$$\chi^2 = \sum_k (n_k C_i - C(Q_i^2, \theta_i, \vec{p}))|V^{-1}|_{ij}(n_k C_j - C(Q_j^2, \theta_j, \vec{p})),$$

in terms of the covariance matrix $V_{ij} = \sigma_i \sigma_j \delta_{ij} + \nu_i \nu_j$. Theoretical errors will be calculated on the one hand using a bootstrap method. We simulate a large number of data sets by randomly varying the points in the original set within the given errors assuming their normal distribution. We then fit to each of them separately, derive the radius from each fit, and analyze the distribution of these radius values (see App. D of Ref. \cite{20} for details). On the other hand theoretical errors are estimated by varying the number of effective vector meson poles. The first error thus gives the uncertainty due to the fitting procedure and the data while the second one reflects the accuracy of the spectral functions underlying the dispersion-theoretical analysis.

3. Results

As a first validation of our method, we only consider the PRad data \cite{14}. These can be best fitted with the lowest two isoscalar mesons (the $\omega$ and the $\phi$) and two additional isovector ones ($2s + 2v$ poles). Fitting with statistical errors only (as in Ref. \cite{14}), we have a $\chi^2/\text{dof} = 1.33$, completely consistent with the results reported there. Including also the systematic errors, the reduced $\chi^2$ is slightly improved and we find as central values

$$r_E = (0.829 \pm 0.012 \pm 0.001) \text{ fm}, \quad r_M = (0.843 \pm 0.007^{+0.018}_{-0.012}) \text{ fm},$$

consistent with the PRad result, $r_E = (0.831 \pm 0.007_{\text{stat}} \pm 0.012_{\text{syst}}) \text{ fm}$ for the electric radius. In our case, the first error is obtained by bootstrap using 1000 samples and the second error is obtained by varying the number of poles from two isoscalar and two isovector ones (which gives the best solution) up to 5 isoscalar plus 5 isovector poles. While the absolute $\chi^2$ of these 8 different solutions is almost the same, the $\chi^2/\text{dof}$ increase from 1.33 to 1.61. Note also that the uncertainty in the magnetic radius is sizeably larger than the one of the electric radius, which is due to the fact that at the very low $Q^2$ probed by PRad, the electric form factor dominates the cross section.

Next, we turn to the combined analysis of the Mainz and the PRad data. We note that we increase the weight of the PRad data by a factor ten in the combined $\chi^2$. This is legitimate as the PRad data probe much smaller momentum transfer than the Mainz data and thus should be enhanced. Changing this
Figure 1: Combined fit to the PRad (upper panel) and the Mainz data (lower panel) as described in the text. The dipole cross section $\sigma_{\text{dip}}$ is obtained by using the dipole approximation to the FFs.

weight by a factor of two leads to changes in the proton radii that are well covered by the uncertainties discussed below. The best fit to these data is shown in Fig. 1 for the PRad data and the Mainz data (normalized to the dipole cross section $\sigma_{\text{dip}}$). To best describe these combined data sets requires $5s + 5v$ poles (as in Ref. [20] for the MAMI data alone) with a $\chi^2/\text{dof} = 1.25$. The corresponding vector meson parameters (masses, residua) and the normalization constants of the various data sets are collected in Appendix A. In such a combined fit, the PRad data are described slightly worse than before, as the
Ref. [16] 0 0 0 — Ref. [17] 0 0 Ref. [18] 0 0

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In the right panel of Fig. 2, the FF ratio $\mu_p G_E(Q^2)/G_M(Q^2)$ is displayed together with the data of Refs. [24, 25]. Our ratio is consistent with these data, which were not included in the fits.

1For a detailed discussion of previous lattice QCD calculations we refer to Refs. [13, 28].
4. Summary

In this paper, we have continued the dispersion-theoretical analysis of the proton form factors triggered by two main developments. On the theoretical side, a much improved representation of the two-pion-continuum contribution to the isovector spectral function based on the precision results from the Roy-Steiner analysis of pion-nucleon scattering has been presented [11]. On the experimental side, new $ep$ scattering data at very low $Q^2$ from the PRad collaboration [14] have become available. Using our improved spectral functions and employing the two-photon corrections worked out in Ref. [20], we have analyzed these new data as well as the combination of the PRad and the Mainz data [15] which allowed us to extract the proton’s electric and magnetic radius with unprecedented precision, as given in Eq. (10). Theoretical uncertainties from the fit procedure and from variations in the spectral functions have been worked out. In the future, Bayesian methods will be used to further improve these uncertainty estimates. Furthermore, fits including also the time-like proton form factor data should be performed. Finally, data for the scattering off the neutron should be included, as these can be used to precisely pin down the corresponding neutron radii using chiral effective field theory for few-nucleon systems [27].

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Appendix A. Fit parameters

We collect the various vector meson masses and couplings that appear in the spectral functions Eqs. (5) and the normalization constants of the various data sets (see Ref. [20] for precise definitions) in Table A.1.

| \(\omega\) | 0.7830 | 0.8572 | 0.0177 | \(v_1\) | 1.0426 | 0.6876 | −1.5086 |
| \(\phi\) | 1.0190 | −1.3155 | 0.9955 | \(v_2\) | 2.3839 | −4.3848 | 4.8535 |
| \(s_1\) | 1.4790 | 2.6928 | −4.8054 | \(v_3\) | 3.5665 | 2.4907 | −3.0026 |
| \(s_2\) | 2.2381 | 1.6155 | 4.8620 | \(v_4\) | 3.3482 | −3.6869 | −4.6070 |
| \(s_3\) | 3.4614 | −2.9454 | −1.2582 | \(v_5\) | 4.7887 | 4.7612 | 2.7502 |

| \(n_1\) | 0.9965 | \(n_6\) | 0.9909 | \(n_{11}\) | 1.0000 | \(n_{16}\) | 1.0019 | \(n_{21}\) | 0.9999 | \(n_{26}\) | 1.0041 | \(n_{31}\) | 0.9980 |
| \(n_2\) | 1.0066 | \(n_7\) | 0.9983 | \(n_{12}\) | 1.0036 | \(n_{17}\) | 1.0013 | \(n_{22}\) | 0.9900 | \(n_{27}\) | 1.0100 | \(\tilde{n}_1\) | 0.9989 |
| \(n_3\) | 1.0028 | \(n_8\) | 0.9937 | \(n_{13}\) | 1.0039 | \(n_{18}\) | 1.0026 | \(n_{23}\) | 1.0033 | \(n_{28}\) | 1.0100 | \(\tilde{n}_2\) | 1.0056 |
| \(n_4\) | 1.0011 | \(n_9\) | 1.0080 | \(n_{14}\) | 1.0057 | \(n_{19}\) | 1.0014 | \(n_{24}\) | 1.0075 | \(n_{29}\) | 0.9992 |
| \(n_5\) | 1.0037 | \(n_{10}\) | 1.0000 | \(n_{15}\) | 1.0065 | \(n_{20}\) | 1.0053 | \(n_{25}\) | 1.0088 | \(n_{30}\) | 1.0069 |

Table A.1: The parameters obtained from the fit to the combined PRad and MAMI-data based on dispersion relations: Vector meson (upper panel) and normalization (lower panel) parameters. The normalization constants \(n_1, \ldots, n_{31}\) refer to the MAMI data sets, whereas \(\tilde{n}_1, \tilde{n}_2\) normalize the PRad data. Masses \(m_V\) are given in GeV and couplings \(a_i^V\) in GeV².

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