The phase transition of the four dimensional Ising spin glass in presence of a magnetic field is well described by a replica-symmetric field theory

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(Dated: July 15, 2021)

Which is the field-theory for the spin-glass phase transition in a magnetic field? This is an open question in less than six dimensions. So far, perturbative computations have not found a stable fixed-point for the renormalization group flow. We tackle this problem through a numerical analysis of the Ising spin glass in four spatial dimensions (data obtained from the Janus collaboration) and in the Bethe lattice. We find strong numerical evidence supporting that the phase transition of the four dimensional Ising spin glass is described by a replica-symmetric Hamiltonian.

In spite of their deceptive simplicity, the disordered magnetic alloys named spin glasses (SG) [1-3] are influential in a large variety of fields. Just to mention a few applications among many, let us recall computational complexity (both classical [1] or quantum [5]), resource dynamics in ecosystems [6], portfolio optimization in finance [7], or the study of aging non-linear responses in condensed matter physics [8,9]. The specific aspect that we shall be considering here is the SG response to an externally applied magnetic field. Interest in this classic problem [10] has been renewed by its connection to the Gardner transition [11] which is strongly suspected to be present in structural glasses. This connection between spin and structural glasses, somewhat anticipated twenty years ago [12], is now attracting a great deal of theoretical [13] and experimental [14-16,17] attention. In this context, the space dimension $D$ is a crucial variable (just as important as temperature or pressure). Of course, considering the $D$ dependence lies at the heart of the Wilsonian Renormalization Group (RG) [17], but the role of $D$ for the Gardner transition is even more important than usual [13]. Unfortunately, even in the comparatively simpler SG setting, the situation is still far from clear as we now explain.

Mean-field (MF) theory predicts the existence of a second-order SG phase transition in a field $H$ with a transition line, the de Almeida-Thouless (dAT) line, which separates the paramagnetic phase (which is stable at high temperature) from the SG phase (that dominates at low temperatures).

One expects that the critical exponents deviate from their MF value below the upper critical dimension $D_{U}$ = 6. In a classic paper, Bray and Roberts [18] tried to compute the critical exponents below $D_{U}$ by means of a perturbative Wilsonian-RG treatment of the Replica-Symmetric (RS) Hamiltonian of the problem. They found in a one-loop computation that, when the Gaussian fixed-point (FP) corresponding to MF behavior becomes unstable at $D = D_{U}$, no other stable FP appears (at variance with the usual scenario of second-order phase transitions). Due to this failure, alternative scenarios must be explored. Some of them are consistent with the presence of a SG phase-transition in a field. For instance, the phase transition could be of the first order (see below), or the perturbative RG could need extension to higher orders [13]. The fixed point could be of non-perturbative nature (implying the need to resort to non-perturbative methods) or the FP could be a zero-temperature one, that appears to display a higher $D_{U} = 8$ [19]. A completely different scenario is expected by the droplet theory, that predicts that the SG phase becomes unstable in the presence of even the smallest magnetic field [20,21].

The above mentioned conflicting scenarios prompted doubts about the existence of the dAT line below some critical dimension $D_{c}$. As a consequence, a long-standing debate has sparked, in which analytical and numerical arguments have been invoked (on the numerical side, see for instance Refs. [27-43]). Nevertheless, we now have solid evidences for the existence of the dAT line from numerical simulations in $D = 4$ [44] and in proxy models of a finite-dimensional SG (such as the one dimensional SG with long-range interactions [45,46]).

Recently, Höller and Read have suggested that the dAT transition actually becomes first-order [48]. The argument resolves around a parameter $\lambda$, see Eq. (7) below. In MF theory, $\lambda$ must be smaller than one. Otherwise, the dAT line is replaced by a first-order SG transition-line. Indeed, Höller and Read conjecture is that, for
\( D < D_{\nu} = 6 \), \( \lambda \) becomes larger than one.

In this work, we measure \( \lambda \) in the 4D Edwards-Anderson model (EA) and show that it is definitively smaller than one on the putative dAT line, thus disproving the suggested scenario of a first-order transition \[48\]. In fact, the emerging scenario is consistent with previous findings on the existence of the dAT line in 4D \[44\]. Our analysis focus on correlation functions suggested by the analysis of the RS field theory (as far as we know, these correlation functions were not considered in previous numerical work). We compute these correlation functions from the equilibrated 4D configurations obtained with the dedicated Janus 1 supercomputer \[44\].

The model. We study the Edwards-Anderson model in a field \( h \) where \( N \) Ising spins interact via
\[
\mathcal{H} = - \sum_{\langle xy \rangle} J_{xy} S_x S_y + h \sum_x S_x ,
\]
where the first sum is over nearest-neighbour pairs and \( J_{xy} = \pm 1 \) with 50\% probability. We indicate thermal averages as \( \langle \ldots \rangle \). The average over the coupling, indicated by an overline, is performed after the thermal average \( \langle \ldots \rangle \). In our 4D computation, the spins are located in the nodes of a hypercubic lattice with periodic boundary conditions and linear size \( L \) (\( N = L^4 \)). In our Bethe-lattice computations, spins occupy the vertices of a random-regular graph with connectivity 4.

The replica-symmetric field theory. Standard arguments \[18, 49\] tell us that the model in Eq. (1) is described at criticality by the following RS Hamiltonian
\[
\mathcal{H} = \frac{1}{2} \int d^D x \left[ m_1 \sum_{ab} \phi_{ab}(x) \phi_{ab}(x) + \frac{1}{2} \sum_{ab} (\nabla \phi_{ab})^2 + m_2 \sum_{abc} \phi_{ab} \phi_{ac} + m_3 \sum_{abcd} \phi_{ab} \phi_{cd} + \frac{1}{6} w_1 \sum_{abc} \phi_{ab} \phi_{bc} \phi_{ca} - \frac{1}{6} w_2 \sum_{ab} \phi^3_{ab} \right] .
\]
At the MF level, \( m_1 \) vanishes linearly on the dAT line and, in the SG phase, the solution displays Replica-Symmetry-Breaking (RSB) with a breaking point at a value equal to \( w_2/w_1 \) \[50, 51\]: it follows that \( \lambda \equiv w_2/w_1 \) must be smaller than one for consistency. It should be also noted that the parameter \( \lambda \) controls the MF values of equilibrium and off-equilibrium dynamical exponents in a variety of contexts \[52, 53\].

The idea of Höller and Read (who started from \[55\]), is to apply the RG to the above replicated overlap Hamiltonian until the mass term \( m_1 \) (which is initially small because we start close to the dAT line) becomes equal to one, then the RG flow is stopped and the new Hamiltonian is analyzed at the MF level. They suggest that below the upper critical dimension, \( \lambda \) becomes larger than one under the RG flow on the whole dAT line and therefore the transition becomes first-order. One should note that treating a Wilson Hamiltonian at the MF level is always an approximation, although it may be accurate close to the upper critical dimension. Essentially, one is approximating the true Gibbs free energy with the Wilson’s Hamiltonian, i.e., fluctuations are neglected. While the coefficients of the Wilson’s Hamiltonian are bare parameters that cannot be measured, the coefficients of the Gibbs free energy (the renormalized coupling constants) can be expressed in terms of physical observables and thus are directly accessible to measurements \[56–58\]. The corresponding expressions for the renormalized coupling \( w_1 \) and \( w_2 \) are (see sec. IV of Ref. \[53\]):
\[
w_{1,r} = \frac{\omega_1}{\chi_{SG} \xi_2^{D/2}} , \quad w_{2,r} = \frac{\omega_2}{\chi_{SG} \xi_2^{D/2}} .
\]
where (the subindex \( c \) stands for connected correlation function, see e.g. Ref. \[54\])
\[
\chi_{SG} = \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle_{c}^{2} ,
\]
\[
\omega_1 = \frac{1}{N} \sum_{ijk} \langle S_i S_j S_k \rangle_c \langle S_j S_k S_i \rangle_c ,
\]
\[
\omega_2 = \frac{1}{2N} \sum_{ijkl} \langle S_i S_j S_k S_l \rangle_c^2 ,
\]
and \( \xi_2 \) is the second-moment correlation length \[59\]. Finally we have
\[
\lambda \equiv \frac{w_2}{w_1} = \frac{w_{2,r}}{w_{1,r}}.
\]
The renormalized coupling constant \( w_{1,r} \) and \( w_{2,r} \) have finite and model-dependent values except at the critical temperature where, if scaling holds, they have finite universal values \( w_{1,r}^{*} \) and \( w_{2,r}^{*} \). The spin-glass susceptibility and correlation length diverge as \( \chi_{SG} \propto | T - T_c | ^{-\gamma} \) and \( \xi_2 \propto | T - T_c | ^{-\nu} \), and consistently \( \omega_1 \) and \( \omega_2 \) diverge as:
\[
\omega_{1,2} \propto | T - T_c | ^{-\gamma_3} , \quad \gamma_3 = 3n - \frac{3}{2} \nu \eta + \frac{\nu D}{2} .
\]
We note that renormalized coupling constants \( w_{1,r}^{*} \), \( w_{2,r}^{*} \) are universal quantities at criticality and play a key role in computations of critical exponents \[55, 56\], being the zeroes of the \( \beta \)-functions.

The observables \( \omega_1 \) and \( \omega_2 \) are suitable for numerical
evaluation once expressed as

$$\omega_1 = \mathcal{W}_1 - 3\mathcal{W}_5 + 3\mathcal{W}_7 - \mathcal{W}_8,$$
$$\omega_2 = \frac{1}{2} \mathcal{W}_2 - 3\mathcal{W}_3 + \frac{3}{2} \mathcal{W}_4 + 3\mathcal{W}_5 + 2\mathcal{W}_6 - 6\mathcal{W}_7 + 2\mathcal{W}_8,$$
$$\mathcal{W}_i \equiv N^2\langle \delta\mathcal{Q}_{12}\delta\mathcal{Q}_{23}\delta\mathcal{Q}_{31} \rangle,$$
$$\omega_i \equiv N^2\langle \delta\mathcal{Q}_{12}^2 \rangle,$$
$$\omega_i \equiv N^2\langle \delta\mathcal{Q}_{12}\delta\mathcal{Q}_{13} \rangle,$$
$$\omega_i \equiv N^2\langle \delta\mathcal{Q}_{12}\delta\mathcal{Q}_{13}\delta\mathcal{Q}_{14} \rangle,$$
$$\omega_i \equiv N^2\langle \delta\mathcal{Q}_{12}\delta\mathcal{Q}_{13}\delta\mathcal{Q}_{14}\delta\mathcal{Q}_{15} \rangle,$$
$$\omega_i \equiv N^2\langle \delta\mathcal{Q}_{12}\delta\mathcal{Q}_{13}\delta\mathcal{Q}_{14}\delta\mathcal{Q}_{15}\delta\mathcal{Q}_{16} \rangle,$$

where overlap fluctuations can be written in terms of independent real replicas with the same quenched disorder

$$\delta\mathcal{Q}_{ab} \equiv \frac{1}{N} \sum_i s_i^a s_i^b - \frac{1}{N} \sum_i \langle s_i \rangle^2.$$  \hspace{1cm} (9)

Each correlator \(\mathcal{W}_i\) requires a number of different real replicas equal to the largest index in its expression.

Scaling implies that each of the \(\mathcal{W}_i\) diverges at criticality with the same critical exponent \(\gamma_1\) of \(\omega_1\) and \(\omega_2\), but remarkably the above RS theory predicts that the amplitudes of the \(\{\mathcal{W}_1, \ldots, \mathcal{W}_8\}\) set are not independent. More precisely one can form six linear combination of the \(\mathcal{W}_i\)'s that diverge less than the \(\mathcal{W}_i\) separately. Using these linear relationships one can express the eight coefficients in terms of only the three-replicas \((R = 3)\) estimators \[63]\n
$$\omega_1^{(3)} = \frac{11}{30} \mathcal{W}_1 - \frac{2}{15} \mathcal{W}_2,$$
$$\omega_2^{(3)} = \frac{4}{15} \mathcal{W}_1 - \frac{1}{15} \mathcal{W}_2.$$  \hspace{1cm} (10)

Alternatively one can express \(\mathcal{W}_7\) and \(\mathcal{W}_8\) as a function of the remaining cumulants obtaining the four-replicas \((R = 4)\) estimators \[63]\n
$$\omega_1^{(4)} = \frac{23}{30} \mathcal{W}_1 + \frac{2}{20} \mathcal{W}_2 - \frac{3}{5} \mathcal{W}_3 + \frac{9}{20} \mathcal{W}_4 - \frac{6}{5} \mathcal{W}_5 + \frac{2}{2} \mathcal{W}_6,$$
$$\omega_2^{(4)} = \frac{7}{15} \mathcal{W}_1 - \frac{2}{5} \mathcal{W}_2 - \frac{9}{5} \mathcal{W}_3 + \frac{3}{5} \mathcal{W}_4 - \frac{3}{5} \mathcal{W}_5 + \mathcal{W}_6.$$  \hspace{1cm} (11)

Within the RS theory, the three- and four-replicas estimators are different from the true \(\omega_1\) and \(\omega_2\) at any given temperature but coincide with them at the critical temperature. More precisely one can show that close to the critical point

$$\omega_i^{(3)} - \omega_i = O(|T - T_c|^\gamma_\Delta), \quad \omega_i - \omega_i^{(4)} = O(|T - T_c|^\gamma_3),$$

where the exponent \(\gamma_\Delta\) is expected to be smaller than \(\gamma_3\) (e.g. in MF one finds \(\gamma_\Delta = 1\) and \(\gamma_3 = 3\)).

\[Figure 1. Temperature dependence of the ratio of renormalized coupling constants \(\lambda\), see Eq. (7), computed at magnetic field \(h = 0.7\) on a Bethe lattice (random regular graph with fixed degree 4). The critical temperature is marked with a vertical line. We plot the data obtained with the three-, four- and six-replicas estimators. The black dot reports the value of \(\lambda(T_c)\) \(\simeq 0.47\), see Eq. (12), that has been computed analytically in \[61]. All three estimators take the same value \(\lambda^* \simeq 0.55\) at the critical temperature. The continuous lines, marked with \(N = \infty\), are the extrapolations of the data considering scaling corrections \[60\] and are compatible with the analytical computation \(\lambda(T_c)\).

\[Numerical results in the Bethe-Lattice\]. To study the behavior of the three- and four-replicas estimators in a controlled setting, we studied numerically the Bethe lattice SG. In this case the fact that the dAT transition is present and is described by the above RS Hamiltonian (treated at the MF level i.e. neglecting fluctuations) can be established more solidly than in finite dimensions.

In Fig. 1 we plot the parameter \(\lambda\) as obtained from the exact expression together with the three and four replica estimators \(\lambda^{(3)} \equiv \omega^{(3)}_1/\omega^{(3)}_i\) and \(\lambda^{(4)} \equiv \omega^{(4)}_1/\omega^{(4)}_i\) for the Bethe lattice. In this case \(T_c\) and \(\lambda(T_c)\) are known analytically \[61\] and we see that the estimators extrapolate to the correct value at the critical temperature, although close to the critical point there are finite size corrections. Note as well that the finite-size corrections of the true \(\lambda\) (i.e. the six-replica estimator) and of the four-replica estimator coincide in the critical region. The same effect is expected for the three-replica estimator but it is masked by pre-asymptotic effects at the sizes considered.

At any rate, we find that the deviations are consistent with the predicted MF values \(\omega^{(3)}_i - \omega_i = O(|T - T_c|)\) and \(\omega_i - \omega^{(4)}_i = O(|T - T_c|^3)\) \[60\].

\[Numerical results in four dimensions\]. The discussion of the three- and four-replica estimators is of great practical and theoretical importance in this case. The theoretical importance relies on the fact that, at variance with the Bethe lattice case, one cannot take for granted that the transition is described by the RS theory.
outlined above. For instance, we could have a continuous transition described by a different theory and therefore the three- and four-replica estimators would yield conflicting results, thus indicating a wrong choice for the starting field-theory.

Furthermore, due to the lack of a perturbative RG fixed-point below six dimensions, one could even question the very existence of such a theory for $D < 6$. Thus the fact that the three- and four-replica expressions yield consistent estimates provides a non trivial indication that the transition is actually described by the RS field theory.

On the practical side, the importance of the three- and four-replica estimators lies in that fact that in the present study we have analyzed equilibrium configurations obtained earlier [41] using the Janus-I supercomputer [62]. Those equilibrium configurations were obtained only for four real replicas. As a consequence, $W_7$ and $W_8$ cannot be measured and $\lambda$ cannot be computed away from the critical temperature.

We start by checking in Fig. 2 the scaling behavior of the four-replica estimator for $\omega_2$ computed for the largest simulated magnetic field $h = 0.3$ [44]. It diverges at the critical temperature and the curve display a good collapse when plotted in terms of the rescaled RG variables, $L^{-5+3\eta/2} \omega_2$ against $L^{1/\nu}(T-T_c)/T_c$, with the values $T_c = 0.906(43)$, $\nu = 1.46(13)$, and $\eta = -0.30(5)$ computed in [44]. We found that the three- and four-replica estimators are hardly distinguishable in this region.

Once we know that $\omega_{1,2}$ behave as expected, we can consider their ratio $\lambda$, which is the main quantity of interest. Fig. 3 shows the three- and four-replica estimators for magnetic fields $h = 0.075, 0.15$ and $0.30$. We see that both estimators for $\lambda$ for the three simulated magnetic fields are consistent with a universal value $\lambda^* \approx 0.55$ at the critical temperature. Furthermore, data suggest that $\lambda^{(3)}$ and $\lambda^{(4)}$ converge to their thermodynamic limit from opposite sides, in agreement with what is observed on the Bethe lattice, see Fig. 1. This fortunate fact makes it particularly simple to bracket the (common) thermodynamic limit for both quantities.

Discussion. One should note that there are nonequivalent ways of taking the relevant limit for $\lambda(L,T)$:

$$
\lambda^* = \lim_{L \to \infty} \lim_{T \to T_c} \lambda(L,T), \quad \lambda(T_c^+) = \lim_{T \to T_c^+} \lim_{L \to \infty} \lambda(L,T).
$$

The fact that $\lambda^* \neq \lambda(T_c^+)$ is hardly surprising (see, for instance [43]), but it is often overlooked. What we are computing here is $\lambda^* \approx 0.55$. Similarly, the corresponding limits for renormalized coupling constants $w_{1,r}$ and $w_{2,r}$ do not commute. $\lambda(T_c^+)$ is in general more difficult to estimate than $\lambda^*$, but the former could be more desirable given that the $\beta$ functions mentioned above are typically expressed in terms of the thermodynamic quantities in analytical computations.

The difference between $\lambda(T_c^+)$ and $\lambda^*$ can be clearly seen on the Bethe lattice, with $\lambda^*$ being larger than the thermodynamic limit value $\lambda(T_c^+)$ in Fig. 1. On the contrary, our data for the 4D case shown in Fig. 3 do not manifest large finite size effects approaching the critical point: data barely depend on temperature for $T < T_c(h=0)$, thus suggesting $\lambda^*$ and $\lambda(T_c^+)$ should be very close. The only visible finite size effect in 4D data is a monotonic in $L$ decrease for $R = 3$ and increase for $R = 4$, that actually helps in bracketing $\lambda^*$ between the values measured on the largest lattice $L = 16$, both verifying $\lambda(L,T) < 1$. Hence, we conclude $\lambda(T_c^+) < 1$ in 4D spin glasses in a field which is the main result of this paper.

The authors wish to thank the Janus Collaboration for allowing us to analyze their data. We would like also to thank E. Marinari for interesting discussions. The analysis of the Janus configurations was performed at ICCAEEx supercomputer center in Badajoz, we thank its staff for their assistance.

This work was supported by the European Research Council under the European Unions Horizon 2020 research and innovation programme (grant No. 694925, G. Parisi), by Ministerio de Economía y Competitividad, Agencia Estatal de Investigación, and Fondo Europeo de Desarrollo Regional (FEDER) (Spain and European Union) through grants No. PID2020-112936GB-I00 and No. PGC2018-094684-B-C21, and by Junta de Extremadura (Spain) through grants No. GRU18079 and No. IB20079 (partially funded by FEDER). IGAP was supported by MCIU (Spain) through FPU grant No. FPU18/02665.
Figure 3. Three- and four-replicas estimators for $\lambda$ as a function of the temperature in the four dimensional Ising spin glass (the value of the magnetic field is indicated above each panel). Vertical lines report the three critical temperatures taken from [44]. The band around $\lambda^{*} \approx 0.55$ is our best $L \to \infty$ extrapolation, assuming three- and four-replicas estimators converge to a common value for all the three simulated values of the magnetic field (the width of the band represents the uncertainty in our extrapolation for $h = 0.075$).

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