SHADOWING, INTERNAL CHAIN TRANSITIVITY AND
α-LIMIT SETS

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Abstract. Let \( f : X \to X \) be a continuous map on a compact metric space \( X \) and let \( \alpha_f, \omega_f \) and \( ICT_f \) denote the set of \( \alpha \)-limit sets, \( \omega \)-limit sets and nonempty closed internally chain transitive sets respectively. We show that if the map \( f \) has shadowing then every element of \( ICT_f \) can be approximated (to any prescribed accuracy) by both the \( \alpha \)-limit set and the \( \omega \)-limit set of a full-trajectory. Furthermore, if \( f \) is additionally c-expansive then every element of \( ICT_f \) is equal to both the \( \alpha \)-limit set and the \( \omega \)-limit set of a full-trajectory.

In particular this means that shadowing guarantees that \( \alpha_f = \omega_f = ICT(f) \) (where the closures are taken with respect to the Hausdorff topology on the space of compact sets), whilst the addition of c-expansivity entails \( \alpha_f = \omega_f = ICT(f) \).

We progress by introducing novel variants of shadowing which we use to characterise both maps for which \( \alpha_f = ICT(f) \) and maps for which \( \alpha_f = ICT(f) \).

1. Introduction

Let \( f : X \to X \) be a dynamical system, so that \( f \) is a continuous map on the compact metric space \( X \). Given a point \( x \in X \), its \( \omega \)-limit set is the set of accumulation points of the sequence \( x, f(x), f^2(x), \ldots \). Calculating the \( \omega \)-limit set of a given point is often relatively easy. Conversely one may ask if a given set is a \( \omega \)-limit set: this can be quite difficult to answer. As such, various authors have either studied, or attempted to characterise, the set of all \( \omega \)-limit sets, denoted here by \( \omega_f \), in a variety of settings. For example it has been shown that \( \omega_f \) is closed (with respect to the Hausdorff topology) for maps of the circle \([38]\), the interval \([7]\) and other finite graphs \([29]\). It is known \([26]\) that every \( \omega \)-limit set is internally chain transitive: briefly a set \( A \subseteq X \) is internally chain transitive if for any \( a, b \in A \) and any \( \varepsilon > 0 \) there exists a finite sequence \( \langle x_0, x_1, \ldots, x_n \rangle \) in \( A \) such that \( x_0 = a, x_n = b \) and \( d(f(x_i), x_{i+1}) < \varepsilon \) for each \( i \). We denote the set of nonempty closed internally chain transitive sets by \( ICT_f \). The map \( f \) is said to have shadowing if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for any sequence \( \langle x_i \rangle_{i=0}^{\infty} \) with \( d(f(x_i), x_{i+1}) < \delta \) for each \( i \), there is a point \( z \in X \) such that \( d(f^i(z), x_i) < \varepsilon \) for each \( i \). In this case we say \( z \) shadows or \( \varepsilon \)-shadows the sequence \( \langle x_i \rangle_{i=0}^{\infty} \). Shadowing has both numerical and theoretical importance and has been studied extensively in a variety of settings; in the context of Axiom A diffeomorphisms \([8]\), in numerical analysis \([12, 13, 53]\), as an important factor in stability theory \([36, 39, 43]\), in understanding the structure of \( \omega \)-limit sets and Julia sets \([4, 5, 6, 9, 30]\), and as a property in and of itself \([13, 21, 22, 28, 31, 34, 36, 40]\). A variety of variants of shadowing have also been studied including, for example, ergodic, thick and Ramsey shadowing.
limit, or asymptotic, shadowing [3] [24] [37], s-limit shadowing [3] [24] [28], orbital shadowing [23] [37] [35], and inverse shadowing [13] [27].

Of particular importance to us is a result of Meddaugh and Raines [30] who establish that, for maps with shadowing, \( \overline{\alpha} = ICT_f \). More recently, using novel variants of shadowing, Good and Meddaugh [28] precisely characterised maps for which \( \overline{\alpha} = ICT_f \) and \( \omega_f = ICT_f \).

Whilst the \( \omega \)-limit set of a point can be thought of as its target - it is where the point \textit{ends up} - an \( \alpha \)-limit set concerns where a point came from - its source, so to speak. However, whilst the definition of an \( \omega \)-limit set is fairly natural, giving an appropriate definition of an \( \alpha \)-limit set is less straightforward. This is because a point may have multiple points in its preimage (or indeed, if the map is not surjective, it may have empty preimage). Various approaches to this difficulty have been taken; these will be discussed in more detail in Section 3. We follow the approach taken in [11] and [25] by refraining from defining such sets for individual points, but rather defining them for \textit{backward trajectories}. Given a point \( x \in X \) an infinite sequence \( \langle x_i \rangle_{i \leq 0} \) is called a \textit{backward trajectory} of \( x \) if \( f(x_i) = x_{i+1} \) for all \( i \leq -1 \) and \( x_0 = x \). The \( \alpha \)-limit set of \( \langle x_i \rangle_{i \leq 0} \) is the set of accumulation points of this sequence. We denote the set of all \( \alpha \)-limit sets by \( \alpha_f \). Although \( \alpha \)-limit sets have not been studied quite as extensively as there \( \omega \) counterparts, interest in them has been growing (see, for example, [11] [15] [16] [25] [26]).

As with \( \omega \)-limit sets, it is known that \( \alpha \)-limit sets are internally chain transitive [26]. In this paper we seek to provide a characterisation of maps for which \( \alpha_f \) and \( ICT_f \) coincide. We start with the preliminaries in Section 2. Section 3 is a standalone section in which we briefly explain the various types of \( \alpha \)-limit sets that have been studied in the literature. In Section 4 we show that, for maps with shadowing, for any \( \varepsilon > 0 \) and any \( A \in ICT_f \) there is a full trajectory whose \( \alpha \)-limit set and \( \omega \)-limit set both lie within \( \varepsilon \) of \( A \) (with respect to the Hausdorff distance). Furthermore, we show that the addition of c-expansivity entails that there is a full trajectory whose limit sets equal \( A \). In particular this means that for maps with shadowing \( \overline{\alpha} = \overline{\omega} = ICT(f) \), whilst the addition of c-expansivity means that \( \alpha_f = \omega_f = ICT(f) \). We progress in Section 5 by introducing novel types of shadowing which we use to characterise both maps for which \( \overline{\alpha} = ICT(f) \) and maps for which \( \alpha_f = ICT(f) \), complementing the work of the first and second author in [28].

2. Preliminaries

A dynamical system is a pair \((X, f)\) consisting of a compact metric space \( X \) and a continuous function \( f : X \to X \). We say the \textit{positive orbit} of \( x \) under \( f \) is the set of points \( \{x, f(x), f^2(x), \ldots\} \); we denote this set by \( \text{Orb}_f^+(x) \). A \textit{backward trajectory} of the point \( x \) is a sequence \( \langle x_i \rangle_{i \leq 0} \) for which \( f(x_i) = x_{i+1} \) for all \( i \leq -1 \) and \( x_0 = x \). We say a bi-infinite sequence \( \langle x_i \rangle_{i \in \mathbb{Z}} \) is a full orbit (of each \( x_i \) if \( f(x_i) = x_{i+1} \) for each \( i \in \mathbb{Z} \). We emphasise that a full orbit of a point need not be unique. Note further that we do not assume that the map \( f \) is a surjection. (NB. Because we will be particularly concerned with backward accumulation points of individual trajectories, for clarity we will say that a point which does not have an infinite backward trajectory does not have a full orbit. Whenever we say \textit{full orbit}, we mean a bi-infinite trajectory.)
For a sequence \( \langle x_i \rangle_{i>N} \) in \( X \), where \( N \geq -\infty \), we define its \( \omega \)-limit set, denoted \( \omega(\langle x_i \rangle_{i>N}) \), or simply \( \omega(\langle x_i \rangle) \), to be the set of accumulation points of the positive tail of the sequence. Formally:

\[
\omega(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \{x_n \mid n > M\}.
\]

For \( x \in X \), we define the \( \omega \)-limit set of \( x \): \( \omega(x) := \omega(\langle f^n(x) \rangle_{n=0}^\infty) \). In similar fashion, for a sequence \( \langle x_i \rangle_{i<N} \) in \( X \), where \( N \leq \infty \), we define its \( \alpha \)-limit set, denoted \( \alpha(\langle x_i \rangle_{i<N}) \), or simply \( \alpha(\langle x_i \rangle) \), to be the set of accumulation points of the negative tail of the sequence. Formally:

\[
\alpha(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \{x_n \mid n < -M\}.
\]

We denote by \( \omega_f \) the set of all \( \omega \)-limit sets of points in \( X \). We denote by \( \alpha_f \) the set of all \( \alpha \)-limit sets of full trajectories in \( (X, f) \). Note that since \( X \) is compact it follows that elements of \( \alpha_f \) and \( \omega_f \) are closed, compact and nonempty.

We denote by \( 2^X \) the hyperspace of nonempty compact subsets of \( X \). This is a (compact) metric space in its own right with the Hausdorff metric induced by the metric \( d \). For \( A, B \in 2^X \) the Hausdorff distance between \( A \) and \( B \) is given by

\[
d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon(A') \text{ and } A' \subseteq B_\varepsilon(A)\}.
\]

Note that, as collections of nonempty compact sets, \( \alpha_f \) and \( \omega_f \) are subsets of \( 2^X \).

A set \( A \subseteq X \) is said to be invariant if \( f(A) \subseteq A \). It is strongly invariant if \( f(A) = A \). A nonempty closed set \( A \) is minimal if \( \omega(x) = A \) for all \( x \in A \).

A finite or infinite sequence \( \langle x_i \rangle_{i=0}^N \) is said to be an \( \varepsilon \)-chain if \( d(f(x_i), x_{i+1}) < \varepsilon \) for all indices \( i < N \). If \( N = \infty \) then we say the sequence is an \( \varepsilon \)-pseudo-orbit.

A set \( A \) is internally chain transitive if for any pair of points \( a, b \in A \) and any \( \varepsilon > 0 \) there exists a finite \( \varepsilon \)-chain \( \langle x_i \rangle_{i=0}^N \) in \( A \) with \( x_0 = a \), \( x_N = b \) and \( N \geq 1 \). We denote by \( ICT_f \) the set of all nonempty closed internally chain transitive sets.

Notice that \( ICT_f \subseteq 2^X \). Meddaugh and Raines [30] establish the following result.

**Lemma 2.1**. [30] Let \( (X, f) \) be a dynamical system. Then \( ICT_f \) is closed in \( 2^X \).

Hirsch et al [20] show that the \( \alpha \)-limit set (resp. \( \omega \)-limit set) of any pre-compact backward (resp. forward) trajectory is internally chain transitive. Since our setting is a compact metric space all \( \alpha \)- and \( \omega \)-limit sets are internally chain transitive. We formulate this as Lemma 2.2 below.

**Lemma 2.2**. [20] Let \( (X, f) \) be a dynamical system. Then \( \alpha_f \subseteq ICT_f \) and \( \omega_f \subseteq ICT_f \).

**Remark 2.3.** When one first encounters positive and negative limit sets of trajectories, it is natural to ask (for a surjective map) if every \( \omega \)-limit set is also an \( \alpha \)-limit set, along with the converse. The following is an example of a homeomorphism for which neither is true. Take two copies of the interval and embed them side by side in the plane (i.e. one on the left and one on the right). Snake one infinite line between them which has each interval as an accumulation set - akin to how the topologist’s sine curve approaches the \( y \)-axis. Define a continuous map as follows: Let every point on each of the two intervals be fixed whilst points on the line move continuously along it, away from the left interval and towards the right. It follows that the left interval is the \( \alpha \)-limit set of the unique backward trajectory of any point on the line, whilst the right left interval is the \( \omega \)-limit set of any point on the
line. However it is clear that the left interval is not an $\omega$-limit set, whilst the right interval is not an $\alpha$-limit set.

**Remark 2.4.** As stated in [1], a minimal set is both an $\omega$-limit set and an $\alpha$-limit set.

Whilst it may be the case that $\alpha_f \neq \omega_f$, it is true that every $\alpha$-limit set contains the $\omega$-limit set of every one of its points and, similarly, every $\omega$-limit set contains an $\alpha$-limit set of a backward trajectory of each of its points. To show this we recall the well-known fact that the $\omega$-limit sets in compact systems are strongly invariant (e.g. [13, Theorem 3.1.9]). The same is true of the $\alpha$-limit sets of backward trajectories (e.g. [1, Lemma 1]).

**Proposition 2.5.** Let $x,y \in X$ and suppose that $(z_i)_{i \leq 0}$ is a backward trajectory of a point $z = z_0 \in X$. Then:

1. If $x \in \alpha((z_i))$ then $\text{Orb}^+_i(x) \subseteq \alpha((z_i))$.
2. If $y \in \omega(x)$ then there is a backward trajectory $(y_i)_{i \leq 0}$, with $y_0 = y$, which lies in $\omega(x)$ and such that $\alpha((y_i)) \subseteq \omega(x)$.

**Proof.** Condition (1) is immediate from the fact that $\alpha$-limit sets are closed and invariant under $f$.

Now suppose $y \in \omega(x)$ and let $y_0 = y$. Since $\omega$-limit sets are strongly invariant $y$ has a preimage in $\omega(x)$, call it $y_{-1}$. This itself has a preimage in $\omega(x)$; call it $y_{-2}$. Continuing in this manner gives a backward trajectory $(y_i)_{i \leq 0}$ of $y$ which lies in $\omega(x)$. The result now follows by observing that $\omega(x)$ is closed. 

**Remark 2.6.** In [25] the author proves condition (1) in Proposition 2.5 holds for interval maps.

A point $x$ is said to $\varepsilon$-shadow a sequence $(x_i)_{i=0}^{\infty}$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$. We say the system $(X,f)$ has the shadowing property, or simply shadowing, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo-orbit is $\varepsilon$-shadowed.

**Definition 2.7.** Suppose that $(X,f)$ is a dynamical system.

1. The sequence $(x_i)_{i \leq 0}$ is a backward $\delta$-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$ for each $i \leq -1$.
2. The sequence $(x_i)_{i \in \mathbb{Z}}$ is a two-sided $\delta$-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \mathbb{Z}$.
3. The system $(X,f)$ has backward shadowing if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward $\delta$-pseudo-orbit $(x_i)_{i \leq 0}$ there exists a backward trajectory $(z_i)_{i \leq 0}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \leq 0$.
4. The system $(X,f)$ has two-sided shadowing if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two-sided $\delta$-pseudo-orbit $(x_i)_{i \in \mathbb{Z}}$ there exists a full trajectory $(z_i)_{i \in \mathbb{Z}}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \in \mathbb{Z}$.

A sequence $(x_i)_{i=0}^{\infty}$ is called an asymptotic pseudo-orbit if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$. Similarly a sequence $(x_i)_{i \leq 0}$ is a backward asymptotic pseudo-orbit if $d(f(x_i), x_{i+1}) \to 0$ as $i \to -\infty$. Finally a sequence $(x_i)_{i \in \mathbb{Z}}^{\infty}$ is called a two-sided asymptotic pseudo-orbit if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm \infty$.

The system $(X,f)$ has $s$-limit shadowing if, in addition to having shadowing, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any asymptotic $\delta$-pseudo orbit $(x_i)_{i=0}^{\infty}$ there exists $z \in X$ which asymptotically $\varepsilon$-shadows $(x_i)_{i=0}^{\infty}$ (i.e. $d(f^i(z), x_i) \to 0$ as $i \to \infty$).
as \( i \to \infty \) and \( d(f^i(z), x_i) < \varepsilon \) for all \( i \in \mathbb{N}_0 \). The system has two-sided \( s \)-limit shadowing if, in addition to two-sided shadowing, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any two-sided asymptotic \( \delta \)-pseudo orbit \( \langle x_i \rangle_{i \in \mathbb{Z}} \) there exists a full trajectory \( \langle z_i \rangle_{i \in \mathbb{Z}} \) which asymptotically \( \varepsilon \)-shadows \( \langle x_i \rangle_{i \in \mathbb{Z}} \) (i.e. \( d(f^i(z), x_i) \to 0 \) as \( i \to \pm \infty \) and \( d(f^i(z), x_i) < \varepsilon \) for all \( i \in \mathbb{Z} \)).

2.1. Shift spaces. Given a finite set \( \Sigma \) considered with the discrete topology, the one-sided full shift with alphabet \( \Sigma \) consists of the set of infinite sequences in \( \Sigma \), that is \( \Sigma^{\mathbb{N}_0} \), which we consider with the product topology. This forms a dynamical system with the shift map \( \sigma \), given by

\[
\sigma(\langle a_i \rangle_{i \geq 0}) = \langle a_i \rangle_{i \geq 1}.
\]

A one-sided shift space is some compact strongly invariant (under \( \sigma \)) subset of some one-sided full shift.

In similar fashion, the two-sided full shift with alphabet \( \Sigma \) consists of the set of bi-infinite sequences in \( \Sigma \), that is \( \Sigma^{\mathbb{Z}} \), which we consider with the product topology. As before, this forms a dynamical system with the shift map \( \sigma \), which we define by saying that, for each \( i \in \mathbb{Z} \),

\[
\pi_i(\sigma(\langle a_i \rangle_{i \in \mathbb{Z}})) = a_{i+1},
\]

where \( \pi_i \) is the projection map for each \( i \). A two-sided shift space is some compact strongly invariant (under \( \sigma \)) subset of some two-sided full shift. If \( (X, \sigma) \) is a two-sided shift space and \( x = \langle a_i \rangle_{i \geq 0} \in X \) then we refer to the sequences \( \langle a_i \rangle_{i \geq 0} \) and \( \langle a_i \rangle_{i \leq 0} \) as the right-tail and left-tail of \( x \) respectively.

Given an alphabet \( \Sigma \), a word in \( \Sigma \) is a finite sequence \( a_0a_1 \ldots a_m \), made up of elements of \( \Sigma \). Let \( \mathcal{F} \) be a finite set of words in \( \Sigma \). The one-sided shift of finite type associated with \( \mathcal{F} \) is the dynamical system \((X_{\mathcal{F}}, \sigma)\) where \( X_{\mathcal{F}} \) is the set of all infinite sequences which do not contain any occurrence of any word from \( \mathcal{F} \). The two-sided shift of finite type associated with \( \mathcal{F} \) is the dynamical system \((Z_{\mathcal{F}}, \sigma)\) where \( Z_{\mathcal{F}} \) is the set of all bi-infinite sequences which do not contain any occurrence of any word from \( \mathcal{F} \). A shift space \((X, \sigma)\) is said to be a one-sided (resp. two-sided) shift of finite type if there exists a finite set of words \( \mathcal{F} \) such that \( X = X_{\mathcal{F}} \) (resp. \( X = Z_{\mathcal{F}} \)).

If \((X, \sigma)\) is a one-sided shift space and \( x = \langle a_i \rangle_{i \geq 0} \in X \) and \( n \in \mathbb{N}_0 \), we refer to the word \( a_0a_1 \ldots a_n \) as an initial segment of \( x \). In similar fashion, if \((X, \sigma)\) is a two-sided shift space and \( x = \langle a_i \rangle_{i \in \mathbb{Z}} \in X \) and \( n \in \mathbb{N}_0 \), we refer to the word \( a_{-n} \ldots a_{-1}a_0a_1 \ldots a_n \) as a central segment of \( x \). In the two-sided case, when writing out an element of \( X \) in full we use a “·” to indicate the position of the middle of the central segment:

\[
x = \ldots a_{-3}a_{-2}a_{-1}a_0 \cdot a_1a_2a_3 \ldots.
\]

The following two theorems concerning limit sets in shift spaces are folklore.

**Theorem 2.8.** Let \((X, \sigma)\) be a one-sided shift space. Let \( x, y \in X \). Then \( y \in \omega(x) \) if and only if every initial segment of \( y \) occurs infinitely often in \( x \). Given a backward trajectory \( \langle x_i \rangle_{i \leq 0} \) consider the backward infinite sequence \( \langle a_i \rangle_{i \leq 0} \) where \( a_i = \pi_0(x_i) \). Then \( y \in \alpha(\langle x_i \rangle) \) if and only if every initial segment of \( y \) occurs infinitely often in \( \langle a_i \rangle_{i \leq 0} \).

**Theorem 2.9.** Let \((X, \sigma)\) be a two-sided shift space. Let \( x, y \in X \). Then \( y \in \omega(x) \) if and only if every central segment of \( y \) occurs infinitely often in the right-tail of
Given a backward trajectory \((x_i)_{i<0}\) then \(y \in \alpha((x_i))\) if and only if every central segment of \(y\) occurs infinitely often in the left-tail of \(x_0\).

For those wanting more information about shift systems, [18, Chapter 5] provides a thorough introduction to the topic.

As stated in Lemma 2.2, \(\alpha_f\) and \(\omega_f\) are both subsets of \(ICT_f\). Example 2.10 gives a surjective shift space \((X, \sigma)\) where \(\alpha_\sigma\), \(\omega_\sigma\) and \(ICT_\sigma\) are all distinct, complementing the discussion in Remark 2.3.

**Example 2.10.** Let \(x = 1010^210^3\ldots\), and \(y = 2020^220^3\ldots\). Let
\[
P(x) = \{30^n30^{n−1} \ldots 30x \mid n \in \mathbb{N}\}.
\]
Take
\[
X = \bigcup_{z \in P(x)} \text{Orb}^+(z) \cup \text{Orb}^−(y) \cup \{0^ny \mid n \in \mathbb{N}\},
\]
where the closure is taken with regard to the one-sided full shift on the alphabet \(\{0, 1, 2, 3\}\). Then \(\omega(x) = \{0^\infty, 0^n10^n \mid n \geq 0\}\) and \(\omega(y) = \{0^\infty, 0^n20^n \mid n \geq 0\}\). It is easy to see that the only other \(\omega\)-limit set is \(\{0^\infty\}\). Thus
\[
\omega_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n10^n \mid n \geq 0\}, \{0^\infty, 0^n20^n \mid n \geq 0\}\}.
\]
Meanwhile
\[
\alpha_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n30^n \mid n \geq 0\}\}.
\]
Finally whilst \(ICT_\sigma \supseteq \alpha_\sigma \cup \omega_\sigma\) it additionally contains \(\{0^\infty, 0^n10^n, 0^n20^n \mid n \geq 0\}\), \(\{0^\infty, 0^n10^n, 0^n30^n \mid n \geq 0\}\), \(\{0^\infty, 0^n20^n, 0^n30^n \mid n \geq 0\}\), and \(\{0^\infty, 0^n10^n, 0^n20^n, 0^n30^n \mid n \geq 0\}\).

Hence \(\alpha_\sigma \not\supseteq \omega_\sigma \neq ICT_\sigma\). Furthermore \(\alpha_\sigma \not\subseteq \omega_\sigma\) and \(\omega_\sigma \not\subseteq \alpha_\sigma\).

### 3. Various notions of negative limit sets

In the previous section we defined what we mean by the term \(\alpha\)-limit set: it was defined for backward sequences. Meanwhile the definition of an \(\omega\)-limit set was extended to individual points. This was done in the only natural way: any given point only has one forward orbit. If one wishes to define the \(\alpha\)-limit set of a point, say \(x\), the best way forward is less obvious; there are multiple approaches one might reasonably take when defining negative limit sets of points. In this standalone section we give a brief outline of several different approaches taken in the literature and give two examples which serve to illustrate their differences.

For homeomorphisms one can define \(\alpha\)-limit sets (or negative limit sets) in precisely the same way as \(\omega\)-limit sets. With non-invertible maps, however, a seemingly natural definition is less obvious. One approach is to take the set of accumulation points of the sequence of sets \(f^{-k}(\{x\})\); this is done in [15] and [16]. Call this Approach 1 (A1). Two further approaches are motivated by considering the accumulation points of backward trajectories of the point in question. One might say that \(y\) is in the negative limit set of a point \(x\) if there exists a sequence \((y_i)_{i=0}^{\infty}\) such that \(y_i \in \text{Orb}^+(y_{i+1})\) for each \(i\), \(x = y_0\) and \(\lim_{i \to \infty} y_i = y\): that is, the negative limit set of \(x\) is the union of all accumulation points of backward trajectories from \(x\). In [25] the author defines this set as the special \(\alpha\)-limit set of \(x\) and examines them for interval maps. These sets are investigated in [42] and [41] for graph maps and dendrites. Call this Approach 2 (A2). The final approach, A3, used in [25], is to say \(y\) is in the \(\alpha\)-limit set of a point \(x\) if there exists a sequence
\[ \langle y_i \rangle_{i=1}^\infty \] and a strictly increasing sequence \( \langle n_i \rangle_{i=1}^\infty \) such that \( f^{n_i}(y_i) = x \) for each \( i \) and \( \lim_{i \to \infty} y_i = y \). Clearly this set contains the one given by A3. The converse is not true (see Example 3.2).

By means of demonstrating some of the differences A1-3 yield we provide the following two examples.

**Example 3.1.** Define a map \( f: [-1, 1] \to [-1, 1] \) by

\[
 f(x) = \begin{cases} 
 2x + 1 & \text{if } x \in [-1, -1/2), \\
 0 & \text{if } x \in [-1/2, 1/2), \\
 2x - 1 & \text{if } x \in [1/2, 1].
\end{cases}
\]

Under A1, the negative limit set of 0 can be seen to be the whole interval \([-1, 1]\). Under A2 and A3 the negative limit set of 0 is simply \( \{-1, 0, 1\} \). Notice that the negative limit set of any backward trajectory from 0 will be either \( \{-1\} \) or \( \{0\} \) or \( \{1\} \).

**Example 3.2.** Define a map \( f: [-1, 2] \to [-1, 2] \) by

\[
 f(x) = \begin{cases} 
 2x + 2 & \text{if } x \in [-1, 0), \\
 2 - 2x & \text{if } x \in [0, 1), \\
 2x - 2 & \text{if } x \in [1, 2].
\end{cases}
\]

Under A2 the negative limit set of 0 is \( \{2/3, 2\} \). Consider the backward trajectory of 0 given by the increasing sequence \( \langle x_i \rangle_{i \geq 0} \), where \( x_0 = 0 \) and \( x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{2}, ... \) This sequence approaches 2. However each point \( x_i \) in this sequence has a preimage \( y_i \) in the interval \([-1, 0)\). Each of these \( y_i \) thereby eventually map onto 0 but they do not themselves have preimages. Furthermore, if \( f^n(y_i) = 0 \) and \( f^m(y_{i+1}) = 0 \) then by construction \( m > n \). This, together with the fact that
lim_{i \to \infty} y_i = 0 \text{ implies that 0 is in the negative limit set of itself under A3. Under A3 the negative limit set of 0 is } \{0, 2/3, 2\}. (\text{NB. Hero } [25] \text{ provides an example illustrating this same difference. For Hero, 0 would be an } \alpha \text{-limit point of itself but not a special } \alpha \text{-limit point of itself: these would only be } 2/3 \text{ and } 2.\)

As stated previously, in this paper we will not define \(\alpha\)-limit sets of individual points, instead we focus on the accumulation points of individual backward trajectories. Note that this is the approach taken in [1] and [26].

4. Shadowing, ICT and \(\alpha_f\)

The following lemma is a recent observation of ours et al (see [20]).

**Lemma 4.1.** [20] Let \((X, f)\) be a dynamical system with \(X\) compact. Then the following are equivalent:

1. \(f\) has shadowing;
2. \(f\) has backward shadowing;
3. \(f\) has two-sided shadowing.

**Theorem 4.2.** Let \((X, f)\) be a dynamical system with shadowing. Then for any \(\varepsilon > 0\) and any \(A \in ICT_f\) there is a full trajectory \(\langle x_i \rangle_{i \in \mathbb{Z}}\) such that

1. \(d_H(\omega(x_0), A) < \varepsilon\)
2. \(d_H(\alpha((x_i)), A) < \varepsilon.\)

In particular every element of \(ICT_f\) is a limit point of both \(\alpha_f\) and \(\omega_f.\)

**Proof.** Let \(A \in ICT_f\) and let \(\varepsilon > 0\) be given. By Lemma 4.1 there exists \(\delta > 0\) such that every two-sided \(\delta\)-pseudo-orbit is \(\varepsilon/2\)-shadowed by a full orbit. Let \(n_0 \in \mathbb{N}\) be such that \(1/2^{n_0} < \delta.\) For each \(k \in \mathbb{N}\) let \(n_k = n_{k-1} + 1.\) Pick \(a_0 \in A\) and let \(N_0 = 0.\) Since \(A\) is internally chain transitive, for each \(k \in N_0,\) there exists a finite \(1/2^{n_k}\)-pseudo-orbit \(\langle a_i \rangle_{i = N_k}^{N_{k+1}-1}\) in \(A\) such that

i). For each \(k \in \mathbb{N}, a_{N_k} = a_0.\)

ii). For any \(b \in A\) and each \(k \in \mathbb{N}_0\) there exists \(N_k \leq i \leq N_{k+1} - 1\) such that \(d(b, a_i) \leq 1/2^{n_k}.\)
iii). For each \( k \in \mathbb{N} \), \( d(f(a_{N_{k+1}-1}), a_0) < \varepsilon_n \).

Notice that \( \langle a_i \rangle_{i=0}^{\infty} \) is an asymptotic \( \delta \)-pseudo-orbit in \( A \) and

\[
A = \bigcap_{n \geq 0} \{ a_i \mid i \geq n \}.
\]

We can now extend this sequence into a two-sided \( \delta \)-pseudo-orbit in \( A \) as follows:

i). For \( 0 > i \geq 1 - N_1 \) let \( a_i = a_{N_1+i} \).

ii). For \( k \in \mathbb{N} \) and for \( -N_k \geq i \geq 1 - N_{k+1} \) let \( a_i = a_{N_{k+1}+N_{k+i}} \).

Notice that we have

\[
A = \bigcap_{n \leq 0} \{ a_i \mid i \leq n \}.
\]

Let \( \langle x_i \rangle_{i \in \mathbb{Z}} \) be a full trajectory such that \( d(x_i, a_i) < \varepsilon / 2 \) for all \( i \in \mathbb{Z} \). We claim that \( d_H(\alpha(\langle x_i \rangle), A) < \varepsilon \). Indeed, pick \( a \in A \). Then there is a decreasing sequence \( \langle i_n \rangle_{n \in \mathbb{N}} \) of negative integers such that \( a = \lim_{n \to \infty} a_{i_n} \). Thus there is \( N \in \mathbb{N} \) such that \( d(a, a_{i_n}) < \varepsilon / 3 \) for all \( n > N \). Since \( d(x_{i_n}, a_{i_n}) < \varepsilon / 2 \) for all \( n \), it follows that \( x_{i_n} \in B_{\varepsilon / 3}(a) \) for \( n > N \). By compactness the sequence \( \langle x_{i_n} \rangle_{n>N} \) has a limit point \( z \in B_{\varepsilon / 3}(a) \); in particular \( d(z, a) < \varepsilon \). Hence \( z \in \alpha(\langle x_i \rangle) \) and

\[
A \subseteq \bigcup_{y \in \alpha(\langle x_i \rangle)} B_\varepsilon(y).
\]

Now take \( z \in \alpha(\langle x_i \rangle) \). Then there is a decreasing sequence \( \langle i_n \rangle_{n \in \mathbb{N}} \) of negative integers such that \( z = \lim_{n \to \infty} x_{i_n} \). Let \( k \in \mathbb{N} \) be such that \( d(z, x_{i_k}) < \varepsilon / 2 \). By shadowing \( d(a_{i_k}, x_{i_k}) < \varepsilon / 2 \). By the triangle inequality \( d(z, a_{i_k}) < \varepsilon \). Since \( a_{i_k} \in A \) it follows that

\[
\alpha(\langle x_i \rangle) \subseteq \bigcup_{a \in A} B_\varepsilon(a).
\]

By Equations 1 and 2 it follows that \( d_H(\alpha(\langle x_i \rangle_{i \in \mathbb{Z}}), A) < \varepsilon \).

The fact that \( d_H(\omega(x_0), A) < \varepsilon \) follows by similar argument.

The following example shows that the converse to Theorem 4.2 is false.

**Example 4.3.** Define a map \( f: [-1, 2] \to [-1, 2] \) by

\[
f(x) = \begin{cases} 
(x+1)^2 - 1 & \text{if } x \in [-1, 0), \\
x^2 & \text{if } x \in [0, 1].
\end{cases}
\]

Then \( f \) does not have shadowing but \( ICT_f = \alpha_f = \omega_f \).

It is easy to see that \( ICT_f = \alpha_f = \omega_f = \{ \{-1\}, \{0\}, \{1\} \} \). However \( f \) does not have shadowing. Let \( \varepsilon = 1/3 \). For any \( \delta > 0 \) we can construct a \( \delta \)-pseudo-orbit which is not \( \varepsilon \)-shadowed. Indeed, fix \( \delta > 0 \) and let \( n \in \mathbb{N} \) be such that \( 1/n < \delta \). Now pick \( z \in (2/3, 1) \) such that \( 1/n \in Orb^+(z) \). Let \( m \in \mathbb{N} \) be such that \( f^m(z) = 1/n \). Now let \( k \in \mathbb{N} \) be such that \( f^k(-1/n) \in (-1, -3/4) \). Then

\[
\langle z, f(z), \ldots, f^m(z), 0, -1/n, f(-1/n), \ldots, f^k(-1/n) \rangle
\]

is a finite \( \delta \)-pseudo-orbit. Suppose \( x \varepsilon \)-shadows this pseudo-orbit. Then \( x \in B_{\varepsilon}(z) \subseteq (1/3, 1) \). But \([0, 1] \) is strongly invariant under \( f \), hence \( Orb^+(x) \subseteq [0, 1] \). Since \((-1, -3/4) \cap B_{\varepsilon}(0, 1) = \emptyset \) this is a contradiction: \( f \) does not exhibit shadowing.

Hirsch et al. [26] showed that the \( \alpha \)-limit set (resp. \( \omega \)-limit set) of any backward (resp. forward) pre-compact trajectory is internally chain transitive. As our phase
Figure 3. Example 4.3

space is compact, it follows that in our case all such limit sets are internally chain transitive. Since such limit sets are closed and nonempty it follows that $\omega_f \subseteq ICT_f$ and $\alpha_f \subseteq ICT_f$.

**Corollary 4.4.** Let $(X, f)$ be a dynamical system with shadowing. Then $\overline{\alpha f} = ICT_f = \overline{\omega f}$.

**Proof.** As previously stated, we have $ICT_f \supseteq \alpha_f$ and $ICT_f \supseteq \omega_f$. By Theorem 4.2 we have that every element of $ICT_f$ is a limit point of each set. □

**Remark 4.5.** The fact that $\overline{\omega f} = ICT_f$ for systems with shadowing has been proved previously by Meddaugh and Raines in [30].

Since $ICT_f$ is always closed in the hyperspace $2^X$ (see Lemma 2.1), we also get the following corollary.

**Corollary 4.6.** Let $(X, f)$ be a dynamical system for which $\alpha_f = ICT_f$. Then $\alpha_f$ is closed.

Theorem 4.2 suggests the following question: when is it the case that every element of $ICT_f$ is both the $\alpha$-limit set and the $\omega$-limit set of the same full trajectory? The next result gives a sufficient condition for this to be the case.

**Theorem 4.7.** Let $(X, f)$ be a dynamical system with two-sided s-limit shadowing. Then for any $A \in ICT_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In particular $\alpha_f = \omega_f = ICT_f$.

**Proof.** Let $A \in ICT_f$ and let $\varepsilon > 0$ be given. By two-sided s-limit shadowing there exists $\delta > 0$ such that every two-sided asymptotic $\delta$-pseudo-orbit is asymptotically $\varepsilon/2$-shadowed by a full trajectory (without loss of generality we assume $\delta < \varepsilon/2$).

Now follow the construction of the two-sided asymptotic $\delta$-pseudo orbit $\langle a_i \rangle_{i \in \mathbb{Z}}$ in the proof of Theorem 4.2. Recall that

$$A = \bigcap_{n \geq 0} \{ a_i \mid i \geq n \}.$$
and

\[ A = \bigcap_{n \leq 0} \{ a_i \mid i \leq n \}. \]

Let \( (x_i)_{i \in \mathbb{Z}} \) be a full trajectory that

1. \( d(x_i, a_i) < \varepsilon/2 \) for all \( i \in \mathbb{Z} \),
2. \( \lim_{i \to \pm\infty} d(x_i, a_i) = 0. \)

It follows that \( \alpha((x_i)) = \omega((x_i)) = A \). The fact that \( \alpha_f = \omega_f = ICT_f \) now follows from Lemma 2.2. \( \square \)

**Remark 4.8.** We did not use the fact that \( (x_i)_{i \in \mathbb{Z}} \) \( \varepsilon/2 \)-shadows \( (a_i)_{i \in \mathbb{Z}} \) in the proof of Theorem 4.7. Therefore, we could replace the hypothesis of “two-sided \( s \)-limit shadowing” with the weaker condition: “there exists \( \delta > 0 \) such that for any two-sided \( \delta \)-pseudo-orbit \( (y_i)_{i \in \mathbb{Z}} \) there exists a full trajectory \( (z_i)_{i \in \mathbb{Z}} \) such that \( \lim_{i \to \pm\infty} d(y_i, z_i) = 0. \)”

A system \((X, f)\) is \( c \)-expansive if there exists \( \eta > 0 \) (referred to as an expansivity constant) such that given any two distinct full trajectories \((x_i)_{i \in \mathbb{Z}}\) and \((y_i)_{i \in \mathbb{Z}}\) there exists \( i \in \mathbb{Z} \) such that \( d(x_i, y_i) > \eta \). In [4] the first author et al showed that a \( c \)-expansive map has shadowing if and only if it has \( s \)-limit shadowing. We extended that result in [20] to show that a \( c \)-expansive map has shadowing if and only if it has two-sided \( s \)-limit shadowing. Combining this result with Theorem 4.9 we immediately obtain the following.

**Theorem 4.9.** Let \((X, f)\) be a dynamical system with shadowing. If \( f \) is \( c \)-expansive then for any \( A \in ICT_f \) there is a full trajectory \((x_i)_{i \in \mathbb{Z}}\) such that \( \alpha((x_i)) = \omega((x_i)) = A. \) In particular \( \alpha_f = \omega_f = ICT_f. \)

**Corollary 4.10.** Let \((X, \sigma)\) be a shift of finite type (whether one- or two-sided). Then for any \( A \in ICT_\sigma \) there is a full trajectory \((x_i)_{i \in \mathbb{Z}}\) such that \( \alpha((x_i)) = \omega((x_i)) = A. \) In particular \( \alpha_\sigma = \omega_\sigma = ICT_\sigma. \)

**Proof.** Shifts of finite type are precisely the shift systems that exhibit shadowing [43]. By Theorem 4.9 it now suffices to note that all shift spaces are \( c \)-expansive. \( \square \)

**Remark 4.11.** Corollary 4.10 enhances a result of Barwell et al [2] who show that \( ICT_\sigma = \omega_\sigma \) for shifts of finite type.

### 4.1. A remark on \( \gamma \)-limit sets

At this point we digress from our main topic to make a brief foray into \( \gamma \)-limit sets. First introduced by Hero [25] who studied them for interval maps, \( \gamma \)-limit sets have since been further examined by Sun et al in [42] and [41] for graph maps and dendrites respectively.

The \( \gamma \)-limit set of a point \( x \), denoted \( \gamma(x) \), is defined by saying that, for any \( y \in X \), \( y \in \gamma(x) \) if and only if \( y \in \omega(x) \) and there exists a sequence \( (y_i)_{i=1}^\infty \) and a strictly increasing sequence \( (n_i)_{i=1}^\infty \) such that \( f^{n_i}(y_i) = x \) for each \( i \) and \( \lim_{i \to \infty} y_i = y \). Note that it is possible that \( \gamma(x) = \emptyset \). We denote by \( \gamma_f \) the set of all \( \gamma \)-limit sets of \( X \).

**Remark 4.12.** Whilst we have refrained from defining the \( \alpha \)-limit set of a point, if one were to use Hero’s definition of such (see Section 3), then it would follow that \( \gamma(x) = \alpha(x) \cap \omega(x) \).
Remark 4.13. For a dynamical system \((X, f)\), if \(f\) is a homeomorphism it is easy to see that, for any \(x \in X\), \(\gamma(x) = \alpha(\langle x_i \rangle) \cap \omega(x)\), where \(\langle x_i \rangle_{i \in \mathbb{Z}}\) is the unique backward trajectory of \(x\).

Unlike \(\alpha\)- and \(\omega\)-limit sets, \(\gamma\)-limit sets are not necessarily internally chain transitive. The example below demonstrates this.

Example 4.14. Let \((X, \sigma)\) be the full two-sided shift with alphabet \(\{0, 1, 2\}\). Consider the point \(x\):
\[
x = \ldots 0^n1^n0^{n-1}1^{n-1} \ldots 0^21^20^12^11^32^31^3 \ldots 0^n21^n \ldots .
\]
Let \(\langle x_i \rangle_{i \leq 0}\) be the unique backward trajectory of \(x\). By Theorem 2.9 we can observe that:
\[
\alpha(\langle x_i \rangle) = \{0^n, 1^n, \sigma^n(0^n1^n) \mid n \in \mathbb{Z}\},
\]
\[
\omega(x) = \{0^n, 1^n, \sigma^n(0^n2^n1^n) \mid n \in \mathbb{Z}\}.
\]
Since \(\sigma\) is a homeomorphism, by Remark 4.13,
\[
\gamma(x) = \{0^n, 1^n\}.
\]
It is obvious that \(\gamma(x)\) is not internally chain transitive.

Nevertheless, each \(\gamma\)-limit set is closed and is contained in a single chain component of the dynamical system, i.e. for each \(\varepsilon > 0\) and for all \(a, b \in \gamma(x)\) there is an \(\varepsilon\)-chain from \(a\) to \(b\) in \(X\) (as opposed to in \(\gamma(x)\)).

Proposition 4.15. Let \((X, f)\) be a dynamical system. For any \(x \in X\), \(\gamma(x)\) is closed and contained in a single chain component of \((X, f)\).

Proof. If \(\gamma(x) = \emptyset\) then the closedness holds and chain transitivity is vacuous.

Let \(a, b \in \gamma(x)\). Let \(\delta > 0\) be given. Let \(y \in X\) be such that \(d(f(y), f(a)) < \delta\) and there exists \(n > 1\) such that \(f^n(y) = x\): such a point exists by the continuity of \(f\) combined with the fact that \(a \in \gamma(x)\). Now let \(m \in \mathbb{N}\) be such that \(d(f^m(x), b) < \delta\).

It follows that \(\langle a, f(y), f^2(y), \ldots, f^n(y) = x, f(x), f^2(x), \ldots, f^m(x), b \rangle\) is a \(\delta\)-chain from \(a\) to \(b\).

Now suppose \(z \in \overline{\gamma(x)}\). Then there is a sequence \(\langle y_i \rangle_{i=1}^{\infty}\) in \(\gamma(x)\) such that \(\lim_{i \to \infty} y_i = z\). Note that, since \(\omega(x)\) is closed and \(y_i \in \omega(x)\) for each \(i\) it follows that \(z \in \omega(x)\). Now, for each \(i \in \mathbb{N}\), let \(z_i \in B_{\delta/3}(y_i)\) and \(n_i \in \mathbb{N}\) be such that \(f^{n_i}(z_i) = x\) and \(\langle n_i \rangle_{i=1}^{\infty}\) is an increasing sequence. Then, as \(\lim_{i \to \infty} z_i = z\), it follows that \(z \in \gamma(x)\).

Using theorems 4.17 and 4.9 we obtain the following corollaries concerning the nonempty closed internally chain transitive sets in systems with two-sided s-limit shadowing.

Corollary 4.16. If \((X, f)\) is a system with two-sided s-limit shadowing then ICT\(_f\) \(\subseteq\) \(\gamma_f\).

Proof. Let \(A \in ICT\(_f\)\). By Theorem 4.17 there is a full trajectory \(\langle x_i \rangle_{i \in \mathbb{Z}}\) through \(x_0 = x\) such that \(\alpha(\langle x_i \rangle) = \omega(x) = A\). Notice that \(\gamma(x) \subseteq \omega(x)\) by definition. Since \(\alpha(\langle x_i \rangle) = \omega(x)\), and \(\langle x_i \rangle_{i \leq 0}\) is a backward trajectory of \(x\), it follows that \(\gamma(x) = A\). Hence ICT\(_f\) \(\subseteq\) \(\gamma_f\).

Corollary 4.17. If \((X, f)\) is a c-expansive system with shadowing then ICT\(_f\) \(\subseteq\) \(\gamma_f\).
5. Characterising $\overline{\alpha_f} = ICT_f$ and $\alpha_f = ICT_f$

In $[23]$ the authors characterise systems for which $\overline{\alpha_f} = ICT_f$ and $\omega_f = ICT_f$ in terms of novel shadowing properties. In this section we show that the natural backward analogues of these shadowing properties characterise when $\overline{\alpha_f} = ICT_f$ and $\alpha_f = ICT_f$. We also demonstrate by way of examples that, in contrast to the shadowing property, there is no general entailment between the backward and forward versions of these types of shadowing.

In $[23]$ it is shown that the property of $\overline{\alpha_f} = ICT_f$ is characterised by a variation on shadowing the authors term cofinal orbital shadowing. A system $f: X \to X$ has the cofinal orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\langle x_i \rangle_{i=0}^\infty$ there exists a point $z \in X$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\{f^{N+i}(z)\}_{i=0}^\infty, \{x_{N+i} \}_{i=0}^\infty) < \varepsilon.$$ 

The authors additionally demonstrate that this form of shadowing is equivalent to one which seems prima facie stronger: the eventual strong orbital shadowing property. A system $f: X \to X$ has the eventual strong orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\langle x_i \rangle_{i=0}^\infty$ there exists a point $z \in X$ and $K \in \mathbb{N}$ such that

$$d_H(\{f^{N+i}(z)\}_{i=0}^\infty, \{x_{N+i} \}_{i=0}^\infty) < \varepsilon$$

for all $N \geq K$.

**Definition 5.1.** A system $f: X \to X$ has the backward cofinal orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward $\delta$-pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a point backward orbit $\langle z_i \rangle_{i \leq 0}$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\{z_{i-N} \}_{i \leq 0}, \{x_{i-N} \}_{i \leq 0}) < \varepsilon.$$ 

**Definition 5.2.** A system $f: X \to X$ has the backward eventural strong orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward $\delta$-pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a point backward orbit $\langle z_i \rangle_{i \leq 0}$ and there exists $K \in \mathbb{N}$ such that

$$d_H(\{z_{i-N} \}_{i \leq 0}, \{x_{i-N} \}_{i \leq 0}) < \varepsilon$$

for all $N \geq K$.

**Theorem 5.3.** Let $(X, f)$ be a dynamical system. The following are equivalent:

(1) $f$ has the backward cofinal orbital shadowing property;

(2) $f$ has the backward eventural strong orbital shadowing property;

(3) $\overline{\alpha_f} = ICT_f$.

**Proof.** From the definitions it is easy to see that (2) $\implies$ (1). We will show (1) $\implies$ (3) and that (3) $\implies$ (2).

Suppose that $f$ has the backward cofinal orbital shadowing property. Recall that $\overline{\alpha_f} \subseteq ICT_f$, hence it will suffice to show $ICT_f \subseteq \overline{\alpha_f}$. Let $A \in ICT_f$. Let $\varepsilon > 0$ be given. It will suffice to show there exists $B \in \alpha_f$ with $d_H(A, B) < \varepsilon$. Let $\delta > 0$ correspond to $\varepsilon/2$ for cofinal orbital shadowing. Now, follow the construction of the sequence $\langle a_i \rangle_{i \in \mathbb{Z}}$ in Theorem $[12]$ (but for $\varepsilon/2$ and $\delta$ as here) and let $x_i = a_i$ for all $i \leq 0$. Recall that this means

$$A = \alpha(\langle x_i \rangle_{i \leq 0}).$$
Let \( \langle z_i \rangle_{i \leq 0} \) be given by backward cofinal orbital shadowing so that for any \( K \in \mathbb{N} \) there exists \( N \geq K \) such that

\[
d_H(\langle z_i^{-N} \rangle_{i \leq 0}, \langle x_i^{-N} \rangle_{i \leq 0}) < \varepsilon/2.
\]

Notice that in particular this means that

\[
d_H(\alpha(\langle x_i \rangle), \alpha(\langle z_i \rangle)) < \varepsilon.
\]

Since \( \alpha(\langle x_i \rangle_{i \leq 0}) = A \) it follows that \( A \in \overline{\alpha f} \).

Now suppose that \( (X, f) \) does not have backward eventual strong orbital shadowing and let \( \varepsilon > 0 \) witness this. (We will show \( ICT_f \neq \alpha_f \).) This means that for each \( n \in \mathbb{N} \) there is a backward \( 1/2^n \)-pseudo-orbit \( \langle x^n_i \rangle_{i \leq 0} \) such that for any backward orbit \( \langle z_i \rangle_{i \leq 0} \) and any \( K \in \mathbb{N} \) there exists \( N \geq K \) with

\[
d_H(\langle z_i^{-N} \rangle_{i \leq 0}, \langle x_i^{-N} \rangle_{i \leq 0}) \geq \varepsilon.
\]

It follows that, in particular, for each backward orbit \( \langle z_i \rangle_{i \leq 0} \) and any \( n \in \mathbb{N} \)

\[
d_H(\alpha(\langle x_i \rangle), \alpha(\langle x^n_i \rangle)) \geq \varepsilon/2.
\]  \hspace{1cm} (3)

For each \( n \in \mathbb{N} \) let \( A_n = \alpha(\langle x^n_i \rangle_{i \leq 0}) \). The sequence of compact sets \( \langle A_n \rangle_{n \in \mathbb{N}} \) has a convergent subsequence which converges in the hyperspace \( 2^X \). Without loss of generality we may assume the sequence itself is convergent; let \( A \) be its limit. We claim \( A \) is internally chain transitive but that \( A \notin \overline{\alpha f} \).

Let \( a, b \in A \) and let \( \xi > 0 \) be arbitrary. By the uniform continuity of \( f \), there exists \( \eta > 0 \) such that for any \( x, y \in X \) if \( d(x, y) < \eta \) then \( d(f(x), f(y)) < \xi/2 \). Without loss of generality take \( \eta < \xi/2 \). Let \( M \in \mathbb{N} \) be such that \( 1/2^M < \eta/3 \) and \( d_H(A_M, A) < \eta/3 \). Now take \( K \in \mathbb{N} \) such that

\[
d_H(\langle x^M_i \rangle_{i \leq 0}, A_M) < \eta/3.
\]

Thus

\[
d_H(\langle x^M_i \rangle_{i \leq 0}, A) < 2\eta/3.
\]

Let \( m \in \mathbb{N} \) be such that \( d(x^M_{m-K}, b) < 2\eta/3 \) and let \( l > m \) be such that \( d(x^M_{l-K}, a) < 2\eta/3 \). Let \( y_0 = a \) and \( y_{l-m} = b \). For each \( j \in 1, \ldots, l-m-1 \) pick \( y_j \in A \) with \( d(y_j, x_{l-K+j}) < 2\eta/3 \). We claim \( \langle y_0, y_1, \ldots, y_{l-m} \rangle \) is a \( \xi \)-chain from \( a \) to \( b \). Indeed, for \( j \in \{0, \ldots, l-m-1\} \)

\[
d(f(y_j), y_{j+1}) \leq d(f(y_j), f(x_{l-K+j})) + d(f(x_{l-K+j}), x_{l-K+j+1}) + d(x_{l-K+j+1}, y_{j+1})
\]
\[
\leq \xi/2 + 1/2^M + 2\eta/3
\]
\[
\leq \xi/2 + \eta/3 + 2\eta/3
\]
\[
\leq \xi.
\]

Since \( a \) and \( b \) were chosen arbitrarily in \( A \) we have that \( A \) is internally chain transitive. Thus, since \( A \) is nonempty and closed, \( A \in ICT_f \).

Suppose for a contradiction that \( A \in \overline{\alpha f} \). Then there exists a backward trajectory \( \langle z_i \rangle_{i \leq 0} \in X \) such that \( d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A) < \varepsilon/4 \). Let \( M \in \mathbb{N} \) be such that \( d_H(A_M, A) < \varepsilon/4 \). Then \( d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A_M) < \varepsilon/2 \), which contradicts Equation (3). Therefore \( A \notin ICT_f \setminus \overline{\alpha f} \). Thus \( \overline{\alpha f} \neq ICT_f \). \( \square \)
Remark 5.4. Unlike with shadowing (see Lemma 4.1), none of the shadowing properties in Theorem 5.3 imply their forward analogues (nor vice-versa). To see this, by Theorem 5.3 and [23, Theorem 13], it suffices to give an example where $\alpha_f = ICT_f$ but $\omega_f \neq ICT$ and an example where $\omega_f = ICT_f$ but $\alpha_f \neq ICT$. Examples 5.5 and 5.6 provide this.

**Example 5.5.** Let $x = 101^210^3 \ldots$. Take

$$X = \text{Orb}^+(x) \cup \{0^n x \mid n \in \mathbb{N}\},$$

where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1\}$. Then $\omega(x) = \{0^\infty, 0^n10^n \mid n \geq 0\}$. It is easy to see that the only other $\omega$-limit set is $\{0^\infty\}$. Thus

$$\omega_f = \{\{0^\infty\}, \{0^\infty, 0^n10^n \mid n \geq 0\}\}.$$ 

Meanwhile

$$\alpha_f = \{\{0^\infty\}, \{0^\infty, 0^n30^n \mid n \geq 0\}\}.$$ 

Finally $ICT_f = \omega_f \neq \alpha_f$. Hence the system has cofinal orbital shadowing and eventual strong orbital shadowing by Theorem 5.3 but the system does not have their backward analogues by Theorem 5.3.

**Example 5.6.** Take

$$X = \{10^n10^{n-1} \ldots 10^210^n \mid n \in \mathbb{N}\},$$

where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1\}$. Then

$$\omega_f = \{\{0^\infty\}\}.$$ 

Meanwhile

$$\alpha_f = \{\{0^\infty\}, \{0^\infty, 0^n10^n \mid n \geq 0\}\}.$$ 

Finally $ICT_f = \alpha_f \neq \omega_f$. Hence the system $(X, \sigma)$ has backward cofinal orbital shadowing and backward eventual strong orbital shadowing by Theorem 5.3 but it does not have their forward analogues by Theorem 5.3.

In [23, Theorem 22] the authors show that the property of $\omega_f = ICT_f$ is characterised by several equivalent asymptotic variants of shadowing: These are asymptotic orbital shadowing, asymptotic strong orbital shadowing and orbital limit shadowing. The system $(X, f)$ has then has the asymptotic orbital shadowing property if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\{x_{N+i} \}_{i \geq 0}, \{f^N+x(z) \}_{i \geq 0}) < \varepsilon.$$ 

The system has the asymptotic strong orbital shadowing property if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$d_H(\{x_{N+i} \}_{i \geq 0}, \{f^{N+i}(z) \}_{i \geq 0}) < \varepsilon$$

for all $N \geq K$. Finally, the system has the orbital limit shadowing property, as introduced by Pilyugin [37], if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that $\omega(z) = \omega(\langle x_i \rangle)$.

Before characterising $\omega_f = ICT_f$ by these notions of shadowing, the authors [23] note that asymptotic shadowing, also known as limit shadowing, is sufficient
but not necessary for $\omega_f = ICT_f$: a system has asymptotic shadowing if for each asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that
\[
\lim_{i \to \infty} d(f^i(z), x_i) = 0.
\]
We observe here that the backward analogue of this is sufficient but not necessary for $\alpha_f = ICT_f$.

**Definition 5.7.** A system $f : X \to X$ has the backward asymptotic shadowing property if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that
\[
\lim_{i \to -\infty} d(z_i, x_i) = 0.
\]

We shall see (Corollary 5.13) that backward asymptotic shadowing is sufficient for $\alpha_f = ICT_f$, however it is not necessary. The irrational rotation of the circle satisfies $\alpha_f = ICT_f$ (as a minimal map, both are equal to $\{X\}$) however it fails to have backward asymptotic shadowing. To see this one can observe that for any irrational rotation $f$ of the circle, the inverse function $f^{-1}$ is also an irrational rotation of the circle. It thereby suffices to note that no irrational rotation of the circle has asymptotic shadowing [37].

**Definition 5.8.** A system $f : X \to X$ has the backward asymptotic strong orbital shadowing property if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
\[
H\left(\{z_{i-N}\}_{i \leq 0}, \{x_{i-N}\}_{i \leq 0}\right) < \varepsilon.
\]

**Definition 5.9.** A system $f : X \to X$ has the backward orbital limit shadowing property if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that
\[
H\left(\{z_{i-N}\}_{i \leq 0}, \{x_{i-N}\}_{i \leq 0}\right) < \varepsilon
\]
for all $N \geq K$.

The following is a backward version of the orbital limit shadowing property, studied by Pilyugin et al [37].

**Definition 5.10.** A system $f : X \to X$ has the backward orbital limit shadowing property if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that
\[
\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle).
\]

As mentioned previously, Hirsch et al [26] showed that the $\alpha$-limit set (resp. $\omega$-limit set) of any backward (resp. forward) pre-compact trajectory is internally chain transitive. In the same paper, the authors show that the $\omega$-limit set of any pre-compact asymptotic pseudo-orbit is internally chain transitive [26, Lemma 2.3].Whilst we omit the proof, the same is true of pre-compact backward asymptotic pseudo-orbits. We formulate this as Lemma 5.11 below.

**Lemma 5.11.** [26] Let $(X, f)$ be a dynamical system where $X$ is a (not necessarily compact) metric space. The $\alpha$-limit set (resp. $\omega$-limit set) of any backward (resp. forward) pre-compact asymptotic pseudo-orbit is internally chain transitive. In particular, when $X$ is compact, all such limit sets are in $ICT_f$.

**Theorem 5.12.** Let $(X, f)$ be a dynamical system. The following are equivalent:
Proof. Clearly (4) \(\implies\) (3). It is also easy to see that (2) \(\implies\) (4). We will show (3) \(\implies\) (1) \(\implies\) (2).

To this end, suppose that \(f\) has backward asymptotic orbital shadowing. Let \(A \in \text{ICT}_f\). Form a backward asymptotic-pseudo orbit \(\langle x_i \rangle_{i \leq 0}\) by following the construction as in the proof of Theorem 4.2 and taking \(x_i = a_i\) for all \(i \leq 0\). (We may ignore the \(\varepsilon\) and \(\delta\) in the construction, we can simply take \(n_0 = 0\).) Recall that this means

\[
A = \bigcap_{n \leq 0} \{ x_i | i \leq n \},
\]

or equivalently,

\[
A = \alpha(\langle x_i \rangle).
\]

Let \(\langle z_i \rangle_{i \leq 0}\) be given by backward asymptotic orbital shadowing. Now let \(\varepsilon > 0\) be given and let \(N \in \mathbb{N}\) be such that \(d_H(\alpha(\langle z_i \rangle), \{ z_{i-N} \}_{i \leq 0}) < \varepsilon/3\), \(d_H(\{ z_{i-N} \}_{i \leq 0}, \{ x_{i-N} \}_{i \leq 0}) < \varepsilon/3\), and

\[
d_H(\{ x_{i-N} \}_{i \leq 0}, \alpha(\langle x_i \rangle)) < \varepsilon/3.
\]

By the triangle inequality it follows that \(d_H(\alpha(\langle z_i \rangle), A) < \varepsilon\). Since \(\varepsilon > 0\) was picked arbitrarily this implies that \(A = \alpha(\langle z_i \rangle)\). Hence \(\text{ICT}_f \subseteq \alpha_f\). As remarked previously, \(\alpha_f \subseteq \text{ICT}_f\), thus (1) holds.

Now suppose that \(\alpha_f = \text{ICT}_f\). Let \(\langle x_i \rangle_{i \leq 0}\) be an backward asymptotic-pseudo orbit. By Lemma 5.11 \(\alpha(\langle x_i \rangle) \in \text{ICT}_f\). Since \(\alpha_f = \text{ICT}_f\) there exists a backward trajectory \(\langle z_i \rangle_{i \leq 0}\) with \(\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)\). Hence \(f\) has the backward orbital limit shadowing property, i.e. (2) holds.

\[ \blacksquare \]

**Corollary 5.13.** If \((X, f)\) has backward asymptotic shadowing then \(\alpha_f = \text{ICT}_f\).

**Proof.** By Theorem 5.3 it suffices to note that backward asymptotic shadowing implies backward orbital limit shadowing.

\[ \blacksquare \]

**Remark 5.14.** Combining theorems 5.12 and 5.13 we have that if \(\alpha_f\) is closed then the following are equivalent:

1. \(f\) has the backward orbital limit shadowing property;
2. \(f\) has the backward eventual strong orbital shadowing property;
3. \(f\) has the backward asymptotic (strong) orbital shadowing property;
4. \(f\) has the backward cofinal orbital shadowing property.

**Remark 5.15.** Examples 5.5 and 5.6, together with Theorem 5.12 and [23, Theorem 22], show that, unlike shadowing (see Lemma 5.11), neither the backward orbital limit shadowing property nor the backward asymptotic orbital shadowing nor the backward asymptotic strong orbital shadowing is equivalent to its forward analogue.
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