ON THE MIXED \((\ell_1, \ell_2)\)-LITTLEWOOD INEQUALITY FOR REAL SCALARS AND APPLICATIONS

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ABSTRACT. In this paper we obtain the sharp estimates for the mixed \((\ell_1, \ell_2)\)-Littlewood inequality for real scalars with exponents \(\{2, 1, 2, 2, \ldots, 2\}\). These results are applied to find sharp estimates for the constants of a family of 3-linear Bohnenblust–Hille inequalities with multiple exponents.

1. Introduction

The mixed \((\ell_1, \ell_2)\)-Littlewood inequality for real scalars asserts that for all continuous real \(m\)-linear forms \(U : c_0 \times \cdots \times c_0 \to \mathbb{R}\) we have

\[
\left( \sum_{j_1=1}^N \left( \sum_{j_2, \ldots, j_m=1}^N |U(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{1/2} \right)^2 \leq \left( \frac{\sqrt{2}}{m} \right)^{m-1} \|U\| ,
\]

for all positive integers \(N\). From this inequality, using the intrinsic symmetry of the context, it is not difficult, using a Minkowski-type inequality, to prove that in fact for each \(k \in \{2, \ldots, m\}\) we have

\[
\left( \sum_{j_1, \ldots, j_{k-1}=1}^N \left( \sum_{j_k=1}^N \left( \sum_{j_{k+1}, \ldots, j_m=1}^N |U(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{1/2} \right)^2 \right)^{1/2k} \leq \left( \frac{\sqrt{2}}{m} \right)^{m-1} \|U\| ,
\]

which is also called mixed \((\ell_1, \ell_2)\)-Littlewood inequality. When we replace \(\mathbb{R}\) by \(\mathbb{C}\) it is well known that \(\left( \frac{\sqrt{2}}{m} \right)^{m-1}\) can be replaced by \(\left( \frac{2}{\sqrt{\pi}} \right)^{m-1}\). In a more simplified notation we have inequalities with “multiple” exponents \(\{1, 2, 2, \ldots, 2\}, \ldots, (2, \ldots, 2, 1)\). These inequalities are a crucial tool to prove the Bohnenblust–Hille inequality for multilinear forms. Recall that the multilinear Bohnenblust–Hille inequality \((1)\) for \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\) asserts that there exists a sequence of positive scalars \((B^\mathbb{K}_m)_{m=1}^\infty\) in \([1, \infty)\) such that

\[
\left( \sum_{i_1, \ldots, i_m=1}^N |U(e_{i_1}, \ldots, e_{i_m})|^{2m/m+1} \right)^{m+1/2m} \leq B^\mathbb{K}_m \sup_{\{\|z_i\|=1; j=1, \ldots, m\}} |U(z_1, \ldots, z_m)|
\]

for all continuous \(m\)-linear forms \(U : c_0 \times \cdots \times c_0 \to \mathbb{K}\) and every positive integer \(N\), where \((e_i)_{i=1}^\infty\) denotes the sequence of canonical vectors of \(c_0\). The Bohnenblust–Hille inequality can be seen as a predecessor of the multilinear theory of absolutely summing operators (see, for instance, \([7, 15, 18, 20]\) and the references therein).

The connections between the mixed \((\ell_1, \ell_2)\)-Littlewood inequality and the Bohnenblust–Hille inequality are well-known and can be easily explained with the interpolative approach from \([1]\) Section 2. To obtain the Bohnenblust–Hille inequality from the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities it suffices to observe that the exponent \(\frac{2m}{m+1}\) can be seen as a multiple exponent \(\left( \frac{2m}{m+1}, \frac{2m}{m+1}, \ldots, \frac{2m}{m+1} \right)\) and this exponent is precisely the interpolation of the exponents

\[
(1, 2, 2, \ldots, 2), \ldots, (2, \ldots, 2, 1)
\]

2010 Mathematics Subject Classification. 11Y60, 47H60.
Key words and phrases. Mixed \((\ell_1, \ell_2)\)-Littlewood inequality, multiple summing operators.
D. Pellegrino is supported by CNPq.
with $\theta_1 = \cdots = \theta_m = 1/m$.

It was recently proved in [17] that for real scalars the values $(\sqrt{2})^m$ are the sharp constants for (1). However, the proof of [17] cannot be straightforwardly extended to the general family of inequalities (2). Of course, a natural question, whose answer can be useful in other inequalities, is whether the upper estimates $(\sqrt{2})^{m-1}$ can be improved as long as the exponent 1 moves from the left to the right in (4). In this paper, among other results we show that for the multiple exponent $(2, 1, 2, 2, \ldots, 2)$ the sharp constants are still $(\sqrt{2})^{m-1}$. It is worth mentioning that our approach seems to be not effective to cover all the remaining cases $(2, 2, 1, 2, \ldots, 2), \ldots, (2, 2, \ldots, 2, 1)$ and this some open problems remain waiting for an answer.

The exact values for the optimal constants $B^K_m$ satisfying (3) are still unknown, although many progresses have been made in the last few years. Having nice estimates for these constants is, in general, crucial for applications (for instance in Quantum Information Theory (see [16]), and Complex Analysis [5]). The first estimates for $(B^K_m)_{m=1}^{\infty}$ [6, 8, 12, 19] suggested an exponential growth but only few years ago very different results have appeared. In fact, for $m \geq 2$ the most recent estimates for the optimal values for the constants satisfying (3) show a sublinear growth:

$$B^C_m < m^{0.21392}, \text{ and}$$
$$B^R_m < 1.3 \cdot m^{0.36481}. \tag{5}$$

More specifically, for complex scalars (see [5]),

$$1 \leq B^C_m \leq \prod_{j=2}^{m} \Gamma \left(2 - \frac{1}{j} \right)^{\frac{1}{j-2}},$$

where $\Gamma$ denotes the gamma function. For real scalars (see [5] [10]),

$$2^{1-\frac{1}{p}} \leq B^R_m \leq \prod_{j=2}^{m} A^{-1}_{j-2},$$

where the constants $A_p$ denote the best constants satisfying Khinchine’s inequality (see [11]), which are given by

$$A_p := \sqrt{2} \left( \frac{\Gamma(p+1/2)}{\sqrt{\pi}} \right)^{1/p},$$

for $p > p_0 \approx 1.85$ and

$$A_p := 2^{\frac{1}{2} - \frac{1}{p}}$$

for $p \leq p_0 \approx 1.85$. More precisely, the number $p_0 \in (1, 2)$ is the solution of the following equality

$$\Gamma \left( \frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2}.$$

It is still an open problem, for real scalars, if the optimal constants $B^R_m$ are $2^{1-\frac{1}{p}}$ or $\prod_{j=2}^{m} A^{-1}_{j-2}$ or whether they lie strictly between these bounds. The only known exact value appears in the real bilinear case, since $2^{1-\frac{1}{2}} = B^R_2$ (see [10]). For the complex case, similar questions remain open.

In [11] it is proved that the Bohnenblust–Hille inequality is a very particular case of a large family of sharp inequalities. More precisely, the following general result was proved in [11, Theorem 1.1]:

**Theorem 1.1** (Generalized Bohnenblust–Hille inequality, [11]). Let $m \geq 2$ be a positive integer and let $q := (q_1, \ldots, q_m) \in [1, 2]^m$. The following assertions are equivalent:

1. The sequence $(q_1, \ldots, q_m)$ satisfies

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}.$$
(2) There exists a constant $C^R_{m,(q_1,\ldots,q_m)} \geq 1$ such that

$$\left( \sum_{j_1=1}^N \left( \sum_{j_2=1}^N \left( \cdots \left( \sum_{j_m=1}^N \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{q_2}{q_2}} \right)^{\frac{q_1}{q_1}} \right)^{\frac{1}{m}} \leq C^R_{m,q} \|T\|$$

for all continuous $m$-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$, and every positive integer $N$.

Observe that the Bohnenblust–Hille inequality is curiously the particular case $q_1 = q_2 = \cdots = q_m = 2$. In the recent years, some works have provided optimal estimates for some particular constants $C^R_{m,q_i}$. For instance, $C^R_{2,q} = 2^{\frac{1}{2}}$, (see \cite{1}) and $C^R_{m,q} = 2^{\frac{m-1}{m}}$, for $q = (1,2,\ldots,2)$ and all $m \geq 2$, (see \cite{17}).

This paper is organized as follows. In Section 2 and Section 3 we state and prove our first main results related to the mixed $(\ell_1,\ell_2)$-Littlewood inequalities. In Section 4 we prove some estimates for the upper bounds of the generalized Bohnenblust–Hille inequality and the results of the sections 2,3,4 are used in the final section to obtain sharp estimates for the generalized Bohnenblust–Hille inequality for certain $3$-linear forms.

2. First main result

For $m \geq 2$ and $1 \leq i \leq m$, we define

$$\hat{q}_{i,m} := \frac{q_1 q_2 \cdots q_m}{q_i}.$$

Our first main result is the following theorem that extends the main result of \cite{17}, as we shall see in \cite{9} and \cite{10}:

**Theorem 2.1.** If $m \geq 2$ is a positive integer, and $q := (q_1,\ldots,q_m) \in [1,2]^m$ are such that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2},$$

then there exists a constant

$$C^R_{m,q} \geq 2^{\frac{(m-1)\hat{q}_{1,m} + (\sum_{i=2}^m \hat{q}_{i,m}) - (m-1)q_1 q_2 \cdots q_m}{q_1 q_2 \cdots q_m}}$$

such that

$$\left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \left( \cdots \left( \sum_{j_m=1}^n \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{q_2}{q_2}} \right)^{\frac{q_1}{q_1}} \right)^{\frac{1}{m}} \leq C^R_{m,q} \|T\|$$

for all continuous $m$–linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{R}$ and all positive integers $n$.

**Proof.** In \cite{10}, for all positive integers $m \geq 2$, the $m$-linear forms $T_m$ are inductively defined as

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

$$(x^{(1)},x^{(2)}) \mapsto x^{(1)}_1 x^{(2)}_1 + x^{(1)}_1 x^{(2)}_2 + x^{(1)}_2 x^{(2)}_1 - x^{(1)}_2 x^{(2)}_2$$
and
\[ T_m : \ell^2_{\infty} \times \cdots \times \ell^2_{\infty} \to \mathbb{R} \]
\[ (x^{(1)}, \ldots, x^{(m)}) \mapsto \left( x_1^{(m)} + x_2^{(m)} \right) T_{m-1} \left( x^{(1)}, \ldots, x^{(m-1)} \right) \]
\[ + \left( x_1^{(m)} - x_2^{(m)} \right) T_{m-1} \left( x^{(1)}, \hat{x}^{(2)}, \ldots, \hat{x}^{(m-1)} \right) , \]
where, for \( 1 \leq k \leq m \),
\[ x^{(k)} = \left( x \left( k \right) \right)^{2^{m-1}}_{j=1} \in \ell^2_{\infty} , \]
and
\[ \hat{x}^{(1)} = B^{2^{m-2}} \left( x^{(1)} \right) , \]
\[ \hat{x}^{(2)} = B^{2^{m-2}} \left( x^{(2)} \right) , \]
\[ \hat{x}^{(3)} = B^{2^{m-3}} \left( x^{(3)} \right) , \]
\[ \hat{x}^{(4)} = B^{2^{m-4}} \left( x^{(4)} \right) , \]
\[ \vdots \]
\[ \hat{x}^{(m-2)} = B^{2^1} \left( x^{(m-2)} \right) , \]
\[ \hat{x}^{(m-1)} = B^{2^0} \left( x^{(m-1)} \right) , \]
where \( B \) is the backward shift operator in \( \ell^2_{\infty} \). As a matter of fact, we can observe that the domain of \( T_m \) can be chosen as \( \ell^2_{\infty} \times \ell^2_{\infty} \times \ell^2_{\infty} \times \cdots \times \ell^2_{\infty} \times \ell^2_{\infty} \).

Let us see that for \( m \geq 2 \) and \( 1 \leq i \leq m \) we have
\[
\left( \sum_{j_1=1}^{2^m-1} \left( \sum_{j_2=1}^{2^m-1} \left( \sum_{j_m=1}^{2^m-1} \left| T_m (e_{j_1}, \ldots, e_{j_m}) \right|^{q_{m-1}} \right)^{\frac{q_3}{q_2}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = 2^{\frac{(m-1)q_1 + \sum_{j_2=1}^{2^m-1} q_{j_{m-1}}}{q_1 q_2 - q_{m-1}}}.
\]

In fact, for \( m = 2 \) it is immediate since
\[
\left( \sum_{j_1=1}^{2} \left( \sum_{j_2=1}^{2} \left| T_2 (e_{j_1}, e_{j_2}) \right|^{q_2} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = 2^{\frac{1}{q_1} + \frac{1}{q_2}} = 2^{\frac{q_1 + q_2}{q_1 q_2}} .
\]
Let us prove by induction. Suppose that it is valid for \( m - 1 \) and let us prove for \( m \). In other words we shall prove that if
\[
\left( \sum_{j_1=1}^{2^m-2} \left( \sum_{j_2=1}^{2^m-2} \left( \sum_{j_m=1}^{2^m-2} \left| T_m-1 (e_{j_1}, \ldots, e_{j_m-1}) \right|^{q_{m-1}} \right)^{\frac{q_3}{q_2}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = 2^{\frac{(m-2)q_1 + \sum_{j_2=1}^{2^m-2} q_{j_{m-1}}}{q_1 q_2 - q_{m-1}}}.
\]
is valid, then

\[
\left( \sum_{j_1=1}^{m-1} \left( \sum_{j_2=1}^{m-1} \left( \sum_{j_m=1}^{m-1} \left| T_m(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} \right) \right)^{\frac{1}{q_m}} = 2^{\frac{m-1}{q_m}} \sum_{j_{m-1}=1}^{m-1} \left( \sum_{j_{m-1}=1}^{m-1} \left| T_m(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}}.
\]

Note that

\[
\sum_{j_{m-1}=1}^{m-1} \left( \sum_{j_{m-1}=1}^{m-1} \left| T_m(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} = \sum_{j_{m-1}=1}^{m-1} \left( T_m(e_{j_1}, \ldots, e_{j_{m-1}}, e_{1}) \right)^{q_m} + T_m(e_{j_1}, \ldots, e_{j_{m-1}}, e_{2}) \right)^{q_m} = \left( \left| T_{m-1}(e_{j_1}, \ldots, e_{j_{m-1}}, e_{1}) + T_{m-1}(B^{m-2}(e_{j_1}, \ldots, B^2(e_{j_{m-1}})) \right|^{q_m} + \left| T_{m-1}(e_{j_1}, \ldots, e_{j_{m-1}}, e_{2}) - T_{m-1}(B^{m-2}(e_{j_1}, \ldots, B^2(e_{j_{m-1}})) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}}
\]

\[
= \left( \left| T_{m-1}(e_{j_1}, \ldots, e_{j_{m-1}}, e_{1}) + T_{m-1}(B^{m-2}(e_{j_1}, \ldots, B^2(e_{j_{m-1}})) \right|^{q_m} + \left| T_{m-1}(e_{j_1}, \ldots, e_{j_{m-1}}, e_{2}) - T_{m-1}(B^{m-2}(e_{j_1}, \ldots, B^2(e_{j_{m-1}})) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}}
\]

By making

\[
\sum_{j_{m-1}=1}^{m-1} \left| T_m(e_{j_1}, \ldots, e_{j_{m-1}}, e_{j_{m-1}}) \right|^{q_m} := A_1
\]
and

\[ \sum_{j_m=3}^{4} |T_{m-1}(B^{2^{m-2}}(e_{j_1}), \ldots, B^2(e_{j_{m-1}}))|^{q_{m-1}} := A_2, \]

and using the induction hypothesis it follows that

\[
\left( \sum_{j_1=1}^{2^{m-1}} \left( \sum_{j_2=1}^{2^{m-1}} \left( \cdots \left( \sum_{j_{m-2}=1}^{2^{m-1}} \left( \sum_{j_{m-1}=1}^{2^{m-1}} \left( T_m(e_{j_1}, \ldots, e_{j_{m-1}}) \right)^{q_{m-1}} \right)_{q_m} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)^{\frac{1}{q_1}}
\]

\[
= 2^{\frac{1}{q_m}} \left( \sum_{j_1=1}^{2^{m-1}} \left( \sum_{j_2=1}^{2^{m-1}} \left( \cdots \left( \sum_{j_{m-2}=1}^{2^{m-1}} \left( \sum_{j_{m-1}=1}^{2^{m-1}} \left( A_1 + A_2 \right)_{q_m} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)^{\frac{1}{q_1}}
\]

\[
= 2^{\frac{1}{q_m}} \left( \sum_{j_1=1}^{2^{m-1}} \left( \sum_{j_2=1}^{2^{m-1}} \left( \cdots \left( \sum_{j_{m-2}=1}^{2^{m-1}} \left( \sum_{j_{m-1}=1}^{2^{m-1}} \left( A_1 \right)_{q_m} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)^{\frac{1}{q_1}}
\]

\[
= 2^{\frac{1}{q_m}} \left( \sum_{j_1=1}^{2^{m-2}} \left( \sum_{j_2=1}^{2^{m-1}} \left( \cdots \left( \sum_{j_{m-2}=1}^{2^{m-1}} \left( \sum_{j_{m-1}=1}^{2^{m-1}} \left( A_1 \right)_{q_m} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)_{q_2} \right)^{\frac{1}{q_1}}
\]

\[
= 2^{\frac{1}{q_m}} + \frac{2^{(m-2)q_1} + \sum_{j=1}^{2^{m-1}} q_{j,m-1}}{q_{1,2} \cdots q_{m-1} q_{1,2} \cdots q_{m-1}}
\]

From [10] we know that \( \|T_m\| = 2^{m-1} \), and therefore

\[ C_{m,q} \geq \frac{2^{(m-1)q_1 + \sum_{j=1}^{m} q_{j,m-1}}}{2^{m-1}} \]

and the proof is done.

Note that when \( q_1 = \ldots = q_m = \frac{2m}{m+1} \), it follows that for all \( i \in \{1, \ldots, m\} \),

\[ \hat{q}_{i,m} = \left( \frac{2m}{m+1} \right)^{m-1} \]
and thus, when \(q_1 = \ldots = q_m = \frac{2m}{m+1}\), the inequality (6) recovers

\[
C_{m, q}^R \geq 2^{(m-1)\alpha_{m}^{m-1} + (\sum_{j=2}^{m} 2^{m-1} - (m-1)(\frac{2m}{m+1})^{m-1})\beta_{m}^{m-1} - (m-1)(\frac{2m}{m+1})^{m-1}}
\]

(8)

\[
C_{m, q}^R \geq 2^{(m-1)\beta_{m}^{m-1} + (\sum_{j=2}^{m} 2^{m-1} - (m-1)\alpha_{m}^{m-1} - (m-1)(\frac{2m}{m+1})^{m-1}}
\]

(9)

that is precisely the lower estimate from [10]. Besides, for \(q = (\alpha, \beta, \ldots, \beta)\), we have

\[
C_{m, q}^R \geq 2^{(m-1)\beta_{m}^{m-1} + (\sum_{j=2}^{m} 2^{m-1} - (m-1)\alpha_{m}^{m-1} - (m-1)(\frac{2m}{m+1})^{m-1}}
\]

(10)

which is the estimate from [17], if we use \(\beta_m = \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}\). In particular for \(q = (1, 2, \ldots, 2)\), we have

recovering the main result of [17].

3. Second main result

Our second main result shows that the same estimate obtained in [17] also holds for exponents of the type \((2, 1, 2, 2, \ldots, 2)\).

For \(m = 2\) let us define the bilinear operator \(L_2\) as \(T_2\) from the previous section. For \(m \geq 3\), consider

\[
L_m : \ell_\infty^{2m-1} \times \cdots \times \ell_\infty^{2m-1} \rightarrow \mathbb{R}
\]

\[
(x^{(1)}, \ldots, x^{(m)}) \mapsto \left( x_1^{(1)} + x_2^{(1)} \right) T_{m-1} (x^{(2)}, \ldots, x^{(m)}) \]

\[
+ \left( x_1^{(1)} - x_2^{(1)} \right) T_{m-1} (\hat{x}^{(2)}, \hat{x}^{(3)}, \ldots, \hat{x}^{(m)}),
\]

where

\[
\hat{x}^{(2)} = B^{2m-2} \left( x^{(2)} \right),
\]

\[
\hat{x}^{(3)} = B^{2m-2} \left( x^{(3)} \right),
\]

\[
\hat{x}^{(4)} = B^{2m-3} \left( x^{(4)} \right),
\]

\[
\hat{x}^{(5)} = B^{2m-4} \left( x^{(5)} \right),
\]

\vdots

\[
\hat{x}^{(m-1)} = B^2 \left( x^{(m-1)} \right),
\]

\[
\hat{x}^{(m)} = B^2 \left( x^{(m)} \right),
\]

and \(B\) is the backward shift operator in \(\ell_\infty^{2m-1}\). Using the previous theorem we get the following:
Theorem 3.1. If \( m \geq 2 \) is a positive integer, and \( \mathbf{q} := (q_1, \ldots, q_m) \in [1, 2]^m \) is such that
\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m + 1}{2},
\]
then there exists a constant
\[
C_{m,q}^R \geq 2^{-\frac{(m-1)\hat{q}_2 \cdot m + \sum_{i=1}^{m-1} \hat{q}_i \cdot m}{\hat{q}_1 \hat{q}_2 \cdots q_m}}
\]
such that
\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \left( \cdots \sum_{j_m=1}^{n} \left| T(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \leq C_{m,q}^R \| T \|
\]
for all continuous \( m \)-linear forms \( T : c_0 \times \cdots \times c_0 \to \mathbb{R} \) and all positive integers \( n \).

Proof. For \( m = 2 \) the result is encompassed by Theorem 2.1. Recall that for \( m \geq 3 \), we have
\[
L_m : \ell_\infty^{2m-1} \times \cdots \times \ell_\infty^{2m-1} \to \mathbb{R}
\]
\[
(x^{(1)}, \ldots, x^{(m)}) \mapsto \left( x^{(1)}_1 + x^{(1)}_2 \right) T_{m-1} \left( x^{(2)}, \ldots, x^{(m)} \right) + \left( x^{(1)}_1 - x^{(1)}_2 \right) \left( B^{2m-2} x^{(2)}, B^{2m-2} x^{(3)}, \ldots, B^2 x^{(m)} \right),
\]
where
\[
x^{(k)} = \left( x^{(k)}_j \right)_{j=1}^{2m-1} \in \ell_\infty^{2m-1},
\]
\(1 \leq k \leq m\), and \( B \) is the backward shift operator in \( \ell_\infty^{2m-1} \). We can again realize that we could consider the domain of \( L_m \) as \( \ell_\infty^2 \times \ell_\infty^{2m-1} \times \cdots \times \ell_\infty^{2m-1} \times \ell_\infty^{2m-3} \times \cdots \times \ell_\infty^2 \). Note also that
\[
\| L_m \| = \| T_m \| = 2^{m-1}.
\]
Let us see that for \( m \geq 2 \) and \( 1 \leq i \leq m \) we have
\[
\left( \sum_{j_1=1}^{2m-1} \left( \sum_{j_2=1}^{2m-1} \left( \cdots \sum_{j_m=1}^{2m-1} \left| L_m(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \leq 2^{-\frac{(m-1)\hat{q}_2 \cdot m + \sum_{i=1}^{m-1} \hat{q}_i \cdot m}{\hat{q}_1 \hat{q}_2 \cdots q_m}}.
\]
In fact, note that
\[
\left( \sum_{j_1=1}^{2m-1} \left( \sum_{j_2=1}^{2m-1} \left( \cdots \sum_{j_m=1}^{2m-1} \left| L_m(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \]
\[
= \left( \sum_{j_2=1}^{2m-1} \left( \sum_{j_1=1}^{2m-1} \left( \cdots \sum_{j_m=1}^{2m-1} \left| L_m(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \]
\[
+ \left( \sum_{j_2=1}^{2m-1} \left( \sum_{j_1=1}^{2m-1} \left( \cdots \sum_{j_m=1}^{2m-1} \left| L_m(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \]
\[
\leq \left( \sum_{j_1=1}^{2m-1} \left( \sum_{j_2=1}^{2m-1} \left( \cdots \sum_{j_m=1}^{2m-1} \left| L_m(e_{j_1, \ldots, e_{j_m}}) \right|^{\frac{q_m-1}{q_m}} \right) \right) \right) \leq 2^{-\frac{(m-1)\hat{q}_2 \cdot m + \sum_{i=1}^{m-1} \hat{q}_i \cdot m}{\hat{q}_1 \hat{q}_2 \cdots q_m}}.
\]
and also that

\[
\left( \sum_{j_2=1}^{2^{m-1}} \cdots \sum_{j_m=1}^{2^2} \right) \left| T_{m-1} \left( e_{j_2}, \ldots, e_{j_m} \right) \right|^{q_m} \frac{q_m-1}{q_m} \left( \frac{q_m}{q_m-1} \right)^{q_2} \left( \frac{q_2}{q_1} \right)^{\frac{1}{q_1}} \right)
\]

Moreover, observe that

\[
T_{m-1} \left( B^{2m-2} (e_{j_2}), B^{2m-2} (e_{j_3}), B^{2m-3} (e_{j_4}), \ldots, B^4 (e_{j_{m-1}}), B^2 (e_3) \right) = T_{m-1} \left( e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_1 \right)
\]

and

\[
T_{m-1} \left( B^{2m-2} (e_{j_2}), B^{2m-2} (e_{j_3}), B^{2m-3} (e_{j_4}), \ldots, B^4 (e_{j_{m-1}}), B^2 (e_4) \right) = T_{m-1} \left( e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_2 \right).
\]

For the sake of simplicity, let us define

\[
A_1 := \left| T_{m-1} \left( e_{j_2}, \ldots, e_1 \right) \right|^{q_m} + \left| T_{m-1} \left( e_{j_2}, \ldots, e_2 \right) \right|^{q_m}
\]

and

\[
A_2 := \left| T_{m-1} \left( e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_1 \right) \right|^{q_m} + \left| T_{m-1} \left( e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_2 \right) \right|^{q_m}.
\]

According to the definition of \( T_{m-1} \) we know that \( A_1 \) is non null only when \( j_{m-1} \in \{1, 2, 3, 4\} \), and \( j_{m-2} \in \{1, 2, \ldots, 2^3\} \), \( \ldots, j_3, j_2 \in \{1, 2, \ldots, 2^{m-2}\} \). Analogously, \( A_2 \) is non null only when \( j_{m-1} \in \{5, 6, 7, 8\} \), and \( j_{m-2} \in \{9, 2, \ldots, 2^4\} \), \( \ldots, j_3, j_2 \in \{2^{m-2} + 1, \ldots, 2^{m-1}\} \).
Therefore

\[
\sum_{j_2=1}^{2^{m-1}} \cdots \sum_{j_{m-1}=1}^{2^3} \begin{pmatrix}
|T_{m-1}(e_{j_2}, \ldots, e_1)|^q_m \\
+ |T_{m-1}(e_{j_2}, \ldots, e_2)|^q_m \\
+ |T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_1)|^q_m \\
|T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_2)|^q_m
\end{pmatrix} \frac{q_m-1}{q_m} \frac{q_{m-1}}{q_{m-1}} \frac{q_{m-2}}{q_{m-2}} \cdots \frac{q_3}{q_3}
\]

\[
\sum_{j_2=2^{m-2}+1}^{2^{m-1}} \cdots \sum_{j_{m-1}=2^2+1}^{2^3} \begin{pmatrix}
|T_{m-1}(e_{j_2}, \ldots, e_1)|^q_m \\
+ |T_{m-1}(e_{j_2}, \ldots, e_2)|^q_m \\
+ |T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_1)|^q_m \\
|T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_2)|^q_m
\end{pmatrix} \frac{q_m-1}{q_m} \frac{q_{m-1}}{q_{m-1}} \frac{q_{m-2}}{q_{m-2}} \cdots \frac{q_3}{q_3}
\]

and re-writing the indices of the last sum we have

\[
\sum_{j_2=1}^{2^{m-2}} \cdots \sum_{j_{m-1}=1}^{2^2} \begin{pmatrix}
|T_{m-1}(e_{j_2}, \ldots, e_1)|^q_m \\
+ |T_{m-1}(e_{j_2}, \ldots, e_2)|^q_m \\
+ |T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_1)|^q_m \\
|T_{m-1}(e_{j_2-2^{m-2}}, e_{j_3-2^{m-2}}, e_{j_4-2^{m-3}}, \ldots, e_{j_{m-1}-4}, e_2)|^q_m
\end{pmatrix} \frac{q_m-1}{q_m} \frac{q_{m-1}}{q_{m-1}} \frac{q_{m-2}}{q_{m-2}} \cdots \frac{q_3}{q_3}
\]
Corollary 3.2.

\[ C_{m,q} \geq 2 \left( \sum_{i=1}^{q} \left( \sum_{j=1}^{m-1} \frac{q_{i,m}}{q_{i+1,j} - q_{i,j}} \right) \right)^{-1} \times \left( q_{1,q_2...q_m} \right)^{-1}. \]

Hence

\[
\begin{align*}
2^\frac{1}{m-1} \left( \sum_{j=1}^{m-1} \sum_{j=m-1}^{m-2} \left( |T_{m-1}(e_{j_2}, \ldots, e_{1})|^{q_m} \right) \frac{q_{m-1}}{q_m} \right)^{\frac{1}{q_2}}
\end{align*}
\]

Finally, since \( \|L_m\| = 2^{m-1} \), we have

\[ C_{m,q} \geq 2 \left( \sum_{i=1}^{m-2} \frac{q_{i,m}}{q_{i+1,j} - q_{i,j}} \right)^{-1} \times \left( q_{1,q_2...q_m} \right)^{-1}. \]

Corollary 3.2. The optimal constants of the mixed \((\ell_1, \ell_2)-Littlewood-type inequality for q = (2, 1, 2, \ldots, 2) \) is \( C_{m,(2,1,2,...,2)} = 2^{\frac{m-1}{2}}. \)

Proof. We have

\[ C_{m,q} \geq 2 \left( \frac{(m-1)q_{2,m} + \sum_{i=1}^{m-1} \frac{q_{i,m}}{q_{i+1,j} - q_{i,j}}}{2^{m-1}} \right) \times \left( q_{1,q_2...q_m} \right)^{-1}. \]

On the other hand, since \( C_{m,q} \leq 2^{\frac{m-1}{2}} \) (see [11]), we conclude that

\[ C_{m,(2,1,2,...,2)} = 2^{\frac{m-1}{2}}. \]
4. Some remarks on the upper estimates of the general Bohnenblust–Hille inequality

In this section we extend some recent results providing upper estimates for the generalized Bohnenblust–Hille inequality. These results will be used in the final section. We begin by recalling two results:

**Lemma 4.1 [4 Lemma 2.1].** Let \((q_1, \ldots, q_m) \in [1, 2]^m\) such that \(\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}\). If \(q_i \geq 2m - 2\) for some index \(i\) and \(q_k = q_l\) for all \(k \neq i\) and \(l \neq i\), then

\[
B_{m,(q_1,\ldots,q_m)}^K \leq \prod_{j=2}^{m} A_{\frac{2^j-2}{j}}^{-1}
\]

where \(A_{\frac{2^j-2}{j}}\) are the respective constants of the Khinchine inequality.

**Theorem 4.2 [4 Theorem 2.3].** If \(m \geq 2\) is a positive integer, and \(q := (q_1, \ldots, q_m) \in [1, 2]^m\) are such that

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2},
\]

and

\[
\max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1}
\]

then

\[
C_{m,q}^K \leq \prod_{j=2}^{m} A_{\frac{2^j-2}{j}}^{-1}.
\]

Combining these two results and using an interpolative approach (see [1, Proposition 2.1] and [3, Lemma 2.1]) we can prove the following slightly more general result.

**Proposition 4.3.** Let \(m \geq 2\) and \(N\) be positive integers and \(q, q(1), \ldots, q(N) \in [1, 2]^m\) be such that \((\frac{1}{q_1}, \ldots, \frac{1}{q_m})\) belong to the convex hull of \((\frac{1}{q_1(k)}, \ldots, \frac{1}{q_m(k)})\), \(k = 1, \ldots, N\), where

\[
\frac{1}{q_1(k)} + \cdots + \frac{1}{q_m(k)} = \frac{m+1}{2}, \text{ for all } k = 1, \ldots, N.
\]

If, for each \(k = 1, \ldots, N\),

\[
\max_i q_i(k) < \frac{2m^2 - 4m + 2}{m^2 - m - 1}
\]

or

\[
q_i(k) \geq \frac{2m - 2}{m}
\]

for some index \(i\) and \(q_j(k) = q_l(k)\) for all \(j \neq i\) and \(l \neq i\), then

\[
C_{m,q}^K \leq \prod_{j=2}^{m} A_{\frac{2^j-2}{j}}^{-1}.
\]

**Proof.** Let us suppose that for each \(q_i(k)\), with \(k = 1, \ldots, N\),

\[
\max_i q_i(k) < \frac{2m^2 - 4m + 2}{m^2 - m - 1}
\]

or

\[
q_i(k) \geq \frac{2m - 2}{m}
\]

for some index \(i\) and \(q_j(k) = q_l(k)\) for all \(j \neq i\) and \(l \neq i\). No matter what situation happens, for each \(k \in \{1, \ldots, N\}\), we have from Lemma 4.1 or from Theorem 4.2 that

\[
C_{m,q(k)}^K \leq \prod_{j=2}^{m} A_{\frac{2^j-2}{j}}^{-1}.
\]
Now, suppose that $\mathbf{q} = \left(\frac{1}{q_1}, ..., \frac{1}{q_m}\right)$ belongs to the convex hull of $\left(\frac{1}{q_1(k)}, ..., \frac{1}{q_m(k)}\right)$, $k = 1, ..., N$, i.e.

$$\frac{1}{q_1} = \theta_1 \left(1\right) + \cdots + \theta_N \left(\frac{1}{q_i}\right),$$

with $\sum_{k=1}^{N} \theta_k = 1$ and $\theta_k \in [0, 1]$ for all $k$. So, by the interpolation procedure from [1, Proposition 2.1], we have

$$\|T(e_{j_1}, ..., e_{j_m})\| \leq \prod_{k=1}^{N} \|T(e_{j_1}, ..., e_{j_m})\|_{q_k}^{\theta_k}.$$ 

From Lemma 4.1 and/or from Theorem 4.2 we have

$$\prod_{k=1}^{N} \|T(e_{j_1}, ..., e_{j_m})\|_{q_k}^{\theta_k} \leq \prod_{k=1}^{N} \left( \prod_{t=2}^{m} A_{2t-2}^{-1} \|T\| \right)$$

and thus

$$\left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \left( \cdots \left( \sum_{j_m=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = \|T(e_{j_1}, ..., e_{j_m})\|_{\mathbf{q}} \leq \prod_{t=2}^{m} A_{2t-2}^{-1} \|T\|.$$

\[\square\]

**Example 4.4.** Note that the above result encompasses cases not covered by the previous results, and we still have

$$C_{m, \mathbf{q}}^{K} \leq \prod_{t=2}^{m} A_{2t-2}^{-1}.$$ 

For instance, suppose that $m = N = 3$ and $\mathbf{q}(1) = (2, \frac{4}{3}, \frac{4}{3})$, $\mathbf{q}(2) = (\frac{4}{3}, 2, \frac{4}{3})$, $\mathbf{q}(3) = (\frac{4}{3}, \frac{4}{3}, 2)$. So, from the Proposition 4.3 for

$$\mathbf{q} = \left(\frac{4}{3 - \theta_1}, \frac{4}{3 - \theta_2}, \frac{4}{3 - \theta_3}\right); \theta_1, \theta_2, \theta_3 \in [0, 1] \text{ and } \theta_1 + \theta_2 + \theta_3 = 1$$

we have

$$C_{3, \mathbf{q}}^{K} \leq \prod_{t=2}^{3} A_{2t-2}^{-1}.$$ 

Considering $\left(\theta_1, \theta_2, \theta_3\right) = \left(\frac{7}{17}, \frac{1}{20}, \frac{2}{29}\right)$ we have

$$\mathbf{q} = \left(\frac{40}{23}, \frac{40}{29}, \frac{40}{28}\right)$$

and, of course, $\mathbf{q}$ does not satisfy the hypotheses of Lemma 4.1 and since

$$\frac{40}{23} > \frac{2 (3)^2 - 4 (3) + 2}{(3)^2 - (3) - 1} = \frac{8}{5},$$

$\mathbf{q}$ also does not satisfy the hypotheses of Theorem 4.2.
5. Application: Sharp estimates for the general Bohnenblust–Hille inequality for 3-linear forms

In this final section we use the results of the previous sections to obtain sharp estimates for the general Bohnenblust–Hille inequality for 3-linear forms.

**Proposition 5.1.** Let \( \tau, \theta \in [0, 1]^2 \). If

\[
q = \left( \frac{4}{\theta + 3}, \frac{4}{2 + \tau - \theta \tau}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)
\]

then, the optimal constant of the generalized Bohnenblust–Hille inequality for real scalars is

\[
C_{3,q}^{\mathbb{R}} = 2^{\frac{\theta + 3}{\tau}}.
\]

**Proof.** When \( m = 3 \) and \( q = (\alpha, \beta, \gamma) \), from Theorem 2.1, we have

\[
C_{3,q}^{\mathbb{R}} \geq 2^{\frac{2\alpha + \alpha \beta + \alpha \gamma - 2\alpha \beta}{\alpha \beta}}.
\]

Let us first consider the case \( \theta = 0 \). We can verify that the values of \( (\alpha, \beta, \gamma) \in [1, 2]^3 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2 \) for which

\[
2^{\frac{2\alpha + \alpha \beta + \alpha \gamma - 2\alpha \beta}{\alpha \beta}} = 2^\frac{3}{2}
\]

is precisely \( \left( \frac{4}{5}, \frac{4}{5 \gamma - 4}, \gamma \right) \) with \( \gamma \in \left[ \frac{4}{5}, 2 \right] \). Equivalently, \( \left( \frac{4}{5}, \frac{4}{2 + \tau}, \frac{4}{3 - \tau} \right) \) with \( \tau \in [0, 1] \). Thus, using (11), if

\[
q = \left( \frac{4}{3}, \frac{4}{2 + \tau}, \frac{4}{3 - \tau} \right)
\]

with \( \tau \in [0, 1] \), then

\[
C_{3,q}^{\mathbb{R}} \geq 2^\frac{3}{2}.
\]

On the other hand, from [4, Lemma 2.1] we know that for \( q(1) = \left( \frac{4}{5}, \frac{4}{5}, \frac{4}{5} \right) \), \( q(2) = \left( \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) \), we have \( C_{3,q(1)}^{\mathbb{R}} \leq 2^\frac{3}{2} \) and \( C_{3,q(2)}^{\mathbb{R}} \leq 2^\frac{3}{2} \), and since \( q = \left( \frac{4}{3}, \frac{4}{2 + \tau}, \frac{4}{3 - \tau} \right) \) belongs to the convex hull of \( q(2) \) and \( q(1) \) for \( \tau \in [0, 1] \), from Proposition 4.3 with \( k = 2 \) we conclude that

\[
C_{3,q}^{\mathbb{R}} \leq 2^\frac{3}{2}.
\]

Thus, if \( q = \left( \frac{4}{3}, \frac{4}{2 + \tau}, \frac{4}{3 - \tau} \right) \) with \( \tau \in [0, 1] \), then

\[
C_{3,q}^{\mathbb{R}} = 2^\frac{3}{2}.
\]

This proves the result for \( \theta = 0 \).

Let us prove the case \( \theta \in (0, 1] \). We can verify that the values \( (\alpha, \beta, \gamma) \in [1, 2]^3 \) with \( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2 \) for which

\[
2^{\frac{2\gamma + 2 + 2 \gamma - 2 \gamma}{\gamma}} = 2^{\frac{\phi}{\gamma}}
\]

are \( \left( \frac{4}{\gamma + 3}, \frac{4}{\gamma - \theta \gamma - 3}, \gamma \right) \) and \( \left( \frac{4}{\gamma + 3}, \beta, \frac{4 \beta}{\beta - \theta - \gamma} \right) \) for \( \gamma, \beta \in \left[ \frac{4}{5}, 2 \right] \) and \( \theta \in [0, 1] \). Equivalently,

\[
\left( \frac{4}{\theta + 3}, \frac{4}{2 + \tau - \theta \tau}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)
\]

for \( \tau, \theta \in [0, 1] \).
Thus, we invoke (11) to conclude that for $\mathbf{q} = \left( \frac{4}{\theta + 3}, \frac{4}{2 + \tau - \theta \tau}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)$ with $\tau, \theta \in [0, 1]$, we have

(12) \quad C_{3,\mathbf{q}}^R \geq 2^{\frac{\theta + 3}{\tau}}.

On the other hand, from [17, Theorem 2.1] we know that for $\mathbf{q}(1) = (1, 2, 2)$, we have $C_{3,\mathbf{q}(1)}^R = 2$. Moreover, from what we just did we have, for $\tau \in [0, 1]$ and $\mathbf{q}(2) = \left( \frac{4}{3}, \frac{4}{2 + \tau}, \frac{4}{3 + \tau} \right)$, that $C_{3,\mathbf{q}(2)}^R = 2^2$. Interpolating $\mathbf{q}(1) = (1, 2, 2)$ and $\mathbf{q}(2) = \left( \frac{4}{3}, \frac{4}{2 + \tau}, \frac{4}{3 + \tau} \right)$, we obtain

$$
\begin{align*}
\frac{1}{q_1} &= \frac{\theta}{1} + \frac{1 - \theta}{4} \Rightarrow q_1 = \frac{4}{\theta + 3} \\
\frac{1}{q_2} &= \frac{\theta}{2} + \frac{1 - \theta}{4} \Rightarrow q_2 = \frac{4}{2 + \tau - \theta \tau} \\
\frac{1}{q_3} &= \frac{\theta}{2} + \frac{1 - \theta}{4} \Rightarrow q_3 = \frac{4}{3 + \theta \tau - \tau - \theta}
\end{align*}
$$

i.e., $\mathbf{q} = \left( \frac{4}{\theta + 3}, \frac{4}{2 + \tau - \theta \tau}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)$, and thus

$$
C_{3,\mathbf{q}}^R \leq 2^{\theta^2 \frac{3}{2}} \frac{(1 - \theta)}{\theta + 3} = 2^{\frac{\theta + 3}{\tau}},
$$

and from (12) we conclude that

$$
C_{3,\mathbf{q}}^R = 2^{\frac{\theta + 3}{\tau}},
$$

for $\mathbf{q} = \left( \frac{4}{\theta + 3}, \frac{4}{2 + \tau - \theta \tau}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)$ with $\tau \in [0, 1]$ and $\theta \in [0, 1]$. \hfill \square

We note that in the above theorem, in order to get the lower estimates, we have just used [11], which is a consequence of the Theorem 2.1. If we use Theorem 3.1 instead of Theorem 2.1, we can prove that for $m = 3$ and $\mathbf{q} = (\alpha, \beta, \gamma)$ we have

$$
C_{3,\mathbf{q}}^R \geq 2^{\frac{2 \alpha + \beta + \gamma - 2 \alpha \beta \gamma}{\alpha \beta \gamma}}.
$$

Using an analogous argument we can prove the following:

**Proposition 5.2.** Let $\tau \in [0, 1]$ and $\theta \in [0, 1]$. If $\mathbf{q} = \left( \frac{4}{2 + \tau - \theta \tau}, \frac{4}{\theta + 3}, \frac{4}{3 + \theta \tau - \theta - \tau} \right)$

then, the optimal constant of the generalized Bohnenblust–Hille inequality for real scalars is

$$
C_{3,\mathbf{q}}^R = 2^{\frac{\theta + 3}{\tau}}.
$$
Figure 1. The 3-linear case: The graph, whose origin is (1, 1, 1), represents points such that \((\alpha, \beta, \gamma) \in [1, 2]^3\) with \(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2\). In the less dark region we have the points where the optimality was proved (Proposition 5.1 and Proposition 5.2).

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