Absolutely maximally entangled states in tripartite heterogeneous systems

Yi Shen\textsuperscript{1,2} · Lin Chen\textsuperscript{1,3}

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Abstract
Absolutely maximally entangled (AME) states are closely related to quantum error correction codes. They are typically defined in homogeneous systems. However, the heterogeneous system is very common in a practical setup. In this work, we focus on the AME states in tripartite heterogeneous systems. We first introduce irreducible AME states as the basic elements to construct AME states in the systems with large local dimensions. Then, we introduce an array called magic solution array and show it is related to the AME states in the systems whose local dimensions are $l$, $m$, $n$ with $3 \leq l < m < n \leq m + l - 1$. Furthermore, we identify in which kinds of heterogeneous systems the AME states are irreducible. We also indicate some applications of our results. First, we propose a protocol to prepare a kind of tripartite heterogeneous AME states. Second, we present a method to construct $k$-uniform states of more parties from two AME states. In addition, we establish the connection between heterogeneous AME states and multi-isometry matrices, and apply heterogeneous AME states to realize quantum steering.

Keywords Absolutely maximally entangled (AME) states · Tripartite heterogeneous systems · Irreducible AME states

\textsuperscript{1} School of Mathematical Sciences, Beihang University, Beijing 100191, China
\textsuperscript{2} Department of Mathematics and Statistics, Institute for Quantum Science and Technology, University of Calgary, Calgary, AB T2N 1N4, Canada
\textsuperscript{3} International Research Institute for Multidisciplinary Science, Beihang University, Beijing 100191, China
1 Introduction

Quantum correlations play a central role in the foundation of quantum mechanics, kinds of quantum information processing tasks and the physics of strongly correlated systems [1]. The nonclassicality of quantum correlations, in particular entanglement, challenges our understanding of the relation between local and global properties of quantum states. For instance, the Bell state has the property that each reduced density matrix is maximally mixed. It implies that even if we have complete knowledge of the global state, we may know nothing about its local properties. The absolutely maximally entangled (AME) states share the property similar to the Bell state. That is if an \( n \)-partite system is in an AME state, then all information about this system is lost after removal of any \( \lceil \frac{n}{2} \rceil \) subsystems. It requires that AME states are maximally entangled in any bipartition. By relaxing this restriction, AME states can be generalized to \( k \)-uniform states which only require that every density matrix reduced to \( k \leq \lceil \frac{n}{2} \rceil \) parties is maximally mixed. Thus, \( k \)-uniform states can be regarded as an approximate characterization of AME states as \( k \) approaches to \( \lceil \frac{n}{2} \rceil \). It is not difficult to verify that AME states are pure genuinely multipartite entangled (GME) states. Since genuine multipartite entanglement is valuable resource for various experimental tasks [2–4], it is essential from a practical point of view to study AME states. Not only that, AME states also have been extensively studied for their close connection with quantum error correction codes (QECCs) [5–11].

Typically, an AME state is defined in a Hilbert space whose local dimensions are all equal. We call such a Hilbert space a homogeneous space, and call the AME states in this space homogeneous AME states. In addition to the application to QECCs [5], homogeneous AME states have been shown to be a resource for open-destination and parallel teleportation [7], for threshold quantum secret sharing schemes [8] and can be used to design holographic quantum codes [12]. In recent work, a series of quantum circuits that generate homogeneous AME states have been designed to benchmark a quantum computer [13]. Therefore, whether a homogeneous space contains AME states has aroused great interest. It has been shown that \( n \)-qubit AME states exist only when \( n = 2, 3, 5, 6 \) [10,14,15]. Thus, not all homogeneous spaces contain AME states. For the homogeneous spaces with larger local dimensions, Goyeneche et al. proposed a method to construct AME states by establishing the connection with irredundant orthogonal arrays (IrOAs) [16,17]. However, this method is not suitable for all cases. One can refer to the table in [18] for unknown cases. A famous unknown case is whether the four-partite space \( \mathbb{C}^6 \otimes \mathbb{C}^6 \otimes \mathbb{C}^6 \otimes \mathbb{C}^6 \) contains AME states [19,20]. Moreover, homogeneous \( k \)-uniform states can be constructed from graph states [21], orthogonal arrays [16], mutually orthogonal Latin squares and Latin cubes [17] and symmetric matrices [22].

The non-homogeneous systems are also very common in practical scenarios, especially when the systems are high dimensional [23]. For instance, the physical systems for encoding may have different numbers of energy levels [24]. Therefore, it is necessary to extend the study of AME states to non-homogeneous systems. We shall refer to non-homogeneous systems as heterogeneous ones, and similarly call the AME states in heterogeneous systems the heterogeneous AME states. There are several motivations to study heterogeneous AME states. First, heterogeneous systems have high research
value for multipartite entanglement from the theoretical point of view as the research explains how many degrees of freedom (i.e., quantum levels) need to be effectively entangled to prepare a state [25]. Second, heterogeneous systems are more compact than homogeneous ones. For example, if we need to prepare a qubit–qutrit state and only consider homogeneous systems, then the system has to be at least a two-qutrit system. It is not an optimal choice from a practical point of view as the number of quantum levels is greater than what we actually need. Third, we may explore a wide range of applications of heterogeneous AME states from those of homogeneous AME states. Take the promising application to QECCs as an example. Heterogeneous AME states can analogously be used to construct QECCs over mixed alphabets [24,26].

As we discussed above, if the errors act on a heterogeneous space, it is optimal to exploit a heterogeneous AME state rather than a homogeneous one. Finally, recent experiments are focusing on creating multipartite entanglement in heterogeneous systems [23,27,28]. Particularly, several tripartite heterogeneous AME states have been realized in experiments [29–33]. These exciting achievements lead us to pay more attention to tripartite heterogeneous AME states.

It is natural to ask whether there exist AME states in heterogeneous systems. There are two definitions for heterogeneous AME states. One directly follows from the definition of homogeneous AME states [26], which requires that every marginal of \( \lfloor \frac{n}{2} \rfloor \) parties is maximally mixed, where \( n \) is the total number of parties. The other one requires that every subsystem whose dimension is not larger than that of its complement must be maximally mixed [10, Sect. 10]. Unless stated otherwise, we select the former as the standard definition of heterogeneous AME states. It is more challenging to determine the existence of AME states in heterogeneous systems than in homogeneous systems because heterogeneous systems are lack of effective mathematical tools to deal with. There are only a few results on the existence of heterogeneous AME states. The concept of irredundant mixed orthogonal arrays (IrMOAs) is extended from that of IrOAs. It was introduced by Goyeneche et al. to study heterogeneous AME states [26]. In recent work [34], the authors exploited the method of IrMOAs to construct \( k \)-uniform states in many unknown heterogeneous systems. Felix Huber et al. constructed an AME state in the space \( \mathbb{C}^2 \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \) according to the second definition above-mentioned [10]. Then, in the same definition several four-partite heterogeneous AME states were numerically constructed in [35, Sect. 5b].

In this paper, we investigate the existence of heterogeneous AME states and \( k \)-uniform states with a focus on tripartite AME states. First of all, we introduce the irreducible AME states which are regarded as basic elements to generate AME states in the systems with large local dimensions. Using this concept, we propose an approach in Fig. 1 to produce every tripartite AME state possessing one qubit. In Theorem 7, we show the existence of AME states in several kinds of heterogeneous systems through constructing their expressions. Then, we explore the AME states in the remaining heterogeneous systems whose local dimensions are \( l, m, n \) with \( 3 \leq l < m < n \leq m + l - 1 \). We introduce an array called magic solution array (MSA) as a tool to study it. In Theorem 8, we show that an MSA is corresponding to an AME state. Next, we numerically construct several tripartite heterogeneous AME states using the concept of reducible AME states and Theorem 8. The results are presented in Table 1. Moreover, in Theorem 10 we discuss in which kinds of multipartite heterogeneous systems the
AME states are irreducible. We also show some applications of heterogeneous AME states. First, we propose a protocol to prepare a kind of tripartite heterogeneous AME states. Second, we present a method to construct \( k \)-uniform states of more parties from two AME states. Third, we establish the connection between heterogeneous AME states and multi-isometry matrices. Fourth, we apply tripartite heterogeneous AME states to realize quantum steering.

The remainder of this paper is organized as follows. In Sect. 2, we clarify the notations in the whole paper, present the mathematical definition of heterogeneous AME states and introduce the concept of irreducible AME states. Lastly, we recall two useful lemmas. In Sect. 3, we determine the existence of tripartite AME states in various heterogeneous systems. In Sect. 4, we identify in which kinds of multipartite heterogeneous systems the AME states are irreducible. In Sect. 5, we indicate some applications of our results. The concluding remarks are given in Sect. 6. In addition, an alternative proof of Lemma 3 is provided in Appendix A, and the proofs of main results are presented in Appendix B.

2 Notations and preliminaries

We first clarify the notations that will be used throughout the paper. We use \( \lfloor \cdot \rfloor \) to represent the floor function, and \( U(d) \) to represent the set of \( d \times d \) unitary matrices. Denote by \( \mathcal{H}_d \) a \( d \)-dimensional Hilbert space. Then, an \( n \)-partite quantum system can be represented as \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \cdots \otimes \mathcal{H}_{d_n} \). For brevity, we shall refer to such a heterogeneous system as \( d_1 \times d_2 \times \cdots \times d_n \) system.

We first present the mathematical definitions of AME states and \( k \)-uniform states.

**Definition 1** Suppose \( |\psi\rangle \) is a pure \( n \)-partite state in the system \( d_1 \times d_2 \times \cdots \times d_n \). Denote by \( \mathcal{A}(d_1, d_2, \cdots, d_n) \) the set of AME states in this system. If \( d_1 = \cdots = d_n = d \), this set is simply denoted by \( \mathcal{A}_d(n) \).

(i) \( |\psi\rangle \) is called \( k \)-uniform, if each \( k \)-partite marginal of \( |\psi\rangle \langle \psi| \) is maximally mixed, i.e., for any \( \{j_1, \cdots, j_k\} \subset \{1, \cdots, n\} \),

\[
\text{Tr}_{\{j_1, \cdots, j_k\}^c} |\psi\rangle \langle \psi| = \frac{1}{d_{j_1} \times \cdots \times d_{j_k}} I_{d_{j_1} \times \cdots \times d_{j_k}},
\]

where \( \{j_1, \cdots, j_k\}^c \) means \( \{1, \cdots, n\} \setminus \{j_1, \cdots, j_k\} \).

(ii) \( |\psi\rangle \in \mathcal{A}(d_1, d_2, \cdots, d_n) \), if it is \( \lfloor \frac{n}{2} \rfloor \)-uniform.

If \( n \) is an even number, every \( n \)-partite AME state can be regarded as a bipartite maximally entangled state in all bipartitions. However, this is impossible by Definition 1 if systems are heterogeneous. Thus, by Definition 1 there is no AME state in the heterogeneous systems which have even number of parties. In the following when it comes to heterogeneous AME states, we shall suppose that the systems possess odd number of parties. In general, the following lemma presents a necessary condition for the heterogeneous systems which contain \( k \)-uniform states.
Lemma 2 [26] A $k$-uniform state does not exist if the product of the size of $k$ local Hilbert spaces is larger than the dimension of the complementary system.

We next introduce the concept of irreducible AME states. Suppose

$$|\psi\rangle_{A_1}\cdots A_m \in \mathcal{A}(k_1, k_2, \cdots, k_m), \quad \text{and} \quad |\phi\rangle_{B_1}\cdots B_m \in \mathcal{A}(l_1, l_2, \cdots, l_m).$$

One can verify that $|\psi\rangle \otimes |\phi\rangle_{(A_1 B_1)\cdots (A_mB_m)}$ is an AME state in $\mathcal{H}_{k_1 l_1} \otimes \cdots \otimes \mathcal{H}_{k_m l_m}$, i.e.,

$$|\psi\rangle \otimes |\phi\rangle_{(A_1 B_1)\cdots (A_mB_m)} \in \mathcal{A}((k_1 l_1), (k_2 l_2), \cdots, (k_m l_m)).$$

This property enables us to construct an AME state in the system with larger local dimensions using two AME states in the systems with smaller local dimensions. For example, three-qubit GHZ state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \in \mathcal{A}_2(3)$ is an irreducible AME state, while $\frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |333\rangle) \in \mathcal{A}_4(3)$ is a reducible AME state as it is the tensor product of two GHZ states. The concept of irreducible AME states is very useful in experiments, since the tensor product which combines the corresponding subsystems of two states can be well realized by experiments.

In [36], the authors consider locally maximally entangled (LME) states, i.e., 1-uniform states in kinds of heterogeneous systems. For tripartite heterogeneous systems, LME states are indeed AME states. The existence of AME states in systems $2 \times m \times (m+n)$ and $3 \times m \times (m+n)$ has been characterized by [36, Fig. 2]. In particular, the existence of AME states in systems $2 \times m \times (m+n)$ can be explicitly characterized as follows.

Lemma 3 [36] There exist tripartite AME states in systems $2 \times m \times (m+n)$ if and only if $n = 0$, or $m = kn$, $\forall k \geq 1$.

Lemma 3 was derived using the geometric invariant theory in [36]. We present an alternative proof in Appendix A to show it directly. Note that every tripartite AME states in the system $2 \times kl \times (kl + l)$ is reducible. In Fig. 1, we illustrate how to generate an AME state in the system $2 \times kl \times (kl + l)$ with an irreducible AME state and a bipartite maximally entangled state. Different from the system $2 \times kl \times (kl + l)$ we will show there exist both reducible and irreducible AME states in the systems $2 \times m \times m$ in Sect. 4.

Finally, we shall recall two useful lemmas which will be used in our results.

Lemma 4 [37] Let

$$|\alpha\rangle = \sum_{i,j} a_{i,j} |i, j\rangle, \quad A = \sum_{i,j} a_{i,j} |i\rangle\langle j|.$$ (2)

Then, the Schmidt rank of $|\alpha\rangle$ is equal to the rank of matrix $A$.

Lemma 5 [38,39] Suppose $S$ is a subspace of $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$. There is a product vector in $S$ if $\text{Dim}(S) > (d_1 - 1)(d_2 - 1)$.

Specifically, in a two-qubit system, there is a product vector in any two-dimensional subspace.
Fig. 1 \(\mathcal{A}(2, k, (k + 1))\) and \(\mathcal{A}_l(2)\) are the sets of AME states in systems \(2 \times k \times (k + 1)\) and \(l \times l\), respectively. The existence of \(|\psi\rangle_{AB_1C_1}\) follows from Lemma 3, and \(|\phi\rangle_{B_2C_2}\) is indeed a bipartite maximally entangled state. Then, \(|\Psi\rangle_{A(B_1B_2)(C_1C_2)}\) is a tripartite AME state shared with \(A, (B_1B_2), (C_1C_2)\). The tensor product combining the corresponding subsystems \(B_1, B_2, \) and \(C_1, C_2\) has been widely used in experiments.

3 Tripartite AME states in heterogeneous systems

In this section, we focus on tripartite heterogeneous AME states. Tripartite heterogeneous systems are the first non-trivial systems when considering AME states. The authors proposed a criterion in [36] to determine whether a tripartite heterogeneous system contains AME states. However, they didn’t formulate the explicit expressions for AME states. Here, we would like to study the existence of tripartite heterogeneous AME states by constructing concrete examples. This is more useful in experiments. As far as we know, there is no systematic way to construct tripartite heterogeneous AME states. In [26], the authors constructed heterogeneous AME states and \(k\)-uniform states by constructing the corresponding IrMOAs. This method has a disadvantage that the systems which don’t have the corresponding IrMOAs may still contain AME states. For example, the corresponding IrMOA for the system \(2 \times 2 \times 3\) doesn’t exist, while this system contains AME states from Lemma 3. We will propose several effective methods to construct tripartite AME states, which are described by Theorems 7 and 8. Furthermore, one can exploit the concept of reducible AME states and these methods to construct more tripartite AME states in unknown systems.

We first formulate special orthonormal bases for bipartite systems in Lemma 6. Such orthonormal bases are helpful to construct tripartite heterogeneous AME states.

**Lemma 6** (i) The following is an orthonormal basis of system \(m \times m\), and each element is a maximally entangled state.

\[
\left\{ \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \omega^{kj} |j, j+l\rangle \right\}_{k,l=0}^{m-1},
\]

where \(\omega = e^{\frac{2\pi i}{m}}\), and \(j+l = (j + l) \mod m\).
(ii) Let $\omega = e^{\frac{2\pi i}{l}}$, and $|\Omega_l\rangle = \frac{1}{\sqrt{l}} \sum_{x=0}^{l-1} |x, x\rangle$. Define

$$D_t = \text{diag}(1, \omega^t, \ldots, \omega^{(l-1)t}), \quad 0 \leq t \leq l - 1,$$

$$P_s = \sum_{j=0}^{l-1} |j + s \mod m\rangle\langle j|, \quad 0 \leq s \leq m - 1. \tag{4}$$

Then, the following vectors form an orthonormal basis of system $l \times m$ with $l < m$.

$$|\Psi_{t,s}\rangle = (D_t \otimes P_s)|\Omega_l\rangle, \quad 0 \leq t \leq l - 1, \quad 0 \leq s \leq m - 1. \tag{5}$$

Based on the orthonormal bases given by Lemma 6, we can show the existence of tripartite AME states in several kinds of heterogeneous systems by formulating their expressions.

**Theorem 7** (i) For any $m \geq 2$ and $n \leq m^2$, the system $m \times m \times n$ contains an AME state in the form as

$$|\Psi\rangle = \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} |\Psi_x, x\rangle, \tag{6}$$

where each $|\Psi_x\rangle$ is an element in (3).

(ii) For any $l < m$ and $n = km \leq lm$, the system $l \times m \times n$ contains an AME state in the form as

$$|\Psi\rangle = \frac{1}{\sqrt{km}} \sum_{t=0}^{k-1} \sum_{s=0}^{m-1} |\Psi_{t,s}, mt + s\rangle, \tag{7}$$

where each $|\Psi_{t,s}\rangle$ is given by Eq. (5).

(iii) Suppose both $|\psi\rangle_{ABC}$ and $|\phi\rangle_{ABC}$ belong to $A(d_A, d_B, d_C)$ where $d_A, d_B, d_C$ are dimensions of the subsystems $A, B, C$, respectively. Then, the system $(2d_A) \times (2d_B) \times (d_C)$ contains an AME state in the form as

$$|\psi\rangle \oplus_{AB} |\phi\rangle := \frac{1}{\sqrt{2}} (|00\rangle_{A'B'}|\psi\rangle + |11\rangle_{A'B'}|\phi\rangle). \tag{8}$$

One can verify Theorem 7 directly by checking every density matrix reduced to one party is maximally mixed.

Next, we investigate the existence of tripartite AME states in other kinds of systems denoted by $l \times m \times n$. For this purpose, we propose a kind of arrays whose restrictions are similar to magic squares. We call an $l \times m$ array as a magic solution array (MSA) if its elements $y_{k,j}$, $0 \leq k \leq l - 1$, $0 \leq j \leq m - 1$, constitute a nonnegative solution of the following system of linear equations:
Then, we reveal the relation between such MSAs and the construction of $l \times m \times n$ AME states as follows.

**Theorem 8** Suppose $3 \leq l < m < n \leq m + l - 1$. If the magic solution array given by Eqs. (9)–(11) exists, then the system $l \times m \times n$ contains AME states in the form as

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{l}} \sum_{k=0}^{l-1} \sum_{j=0}^{m-1} \sqrt{y_{k,j}} |k, j, j + k \mod n\rangle. \quad (12)$$

It follows from calculation that Eqs. (9)–(11) guarantee the three single-party marginals are, respectively, maximally entangled. We present the detailed proof of Theorem 8 in Appendix B. When $3 \leq l < m < n \leq m + l - 1$, a given MSA is corresponding to an AME state in the system $l \times m \times n$ by Theorem 8. For generic $l, m, n$ there is no rule to express nonnegative solutions for Eqs. (9)–(11). Nevertheless, for specific $l, m, n$ one can numerically formulate nonnegative solutions if they exist.

As an example, we would like to construct an AME state in the system $3 \times 4 \times 5$ to further explain Theorem 8. One important reason to propose this example is that AME states in the system $3 \times 4 \times 5$ are all irreducible because the system $3 \times 4 \times 5$ cannot be written as a tensor product of two systems. Thus, the system $3 \times 4 \times 5$ becomes the first interesting case when each local dimension is greater than 2. We will discuss in which systems the AME states should be irreducible in Sect. 4.

**Example 9** Let

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \sum_{j=0}^{3} x_{k,j} |k, j, j + k \mod 5\rangle, \quad \text{and} \quad \rho_{ABC} = |\psi\rangle\langle\psi|_{ABC}. \quad (13)$$

A straightforward calculation yields that

$$\rho_A = \frac{1}{3} \left( \sum_{k=0}^{2} |x_{k,0}|^2 \right) |0\rangle\langle 0| + \left( \sum_{k=0}^{2} |x_{k,1}|^2 \right) |1\rangle\langle 1| + \left( \sum_{k=0}^{2} |x_{k,2}|^2 \right) |2\rangle\langle 2| + \left( \sum_{k=0}^{2} |x_{k,3}|^2 \right) |3\rangle\langle 3| \quad (14)$$

$$\rho_B = \frac{1}{3} \left( \sum_{k=0}^{2} |x_{k,0}|^2 \right) |0\rangle\langle 0| + \left( \sum_{k=0}^{2} |x_{k,1}|^2 \right) |1\rangle\langle 1| + \left( \sum_{k=0}^{2} |x_{k,2}|^2 \right) |2\rangle\langle 2| + \left( \sum_{k=0}^{2} |x_{k,3}|^2 \right) |3\rangle\langle 3| \quad (15)$$
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\[ \rho_C = \frac{1}{3} \left( \left( \sum_{k+j=0}^5 |x_{kj}|^2 \right) |0\rangle\langle 0| + \left( \sum_{k+j=1}^5 |x_{kj}|^2 \right) |1\rangle\langle 1| + \left( \sum_{k+j=2}^5 |x_{kj}|^2 \right) |2\rangle\langle 2| \right. \\
\left. + \left( \sum_{k+j=3}^5 |x_{kj}|^2 \right) |3\rangle\langle 3| + \left( \sum_{k+j=4}^5 |x_{kj}|^2 \right) |4\rangle\langle 4| \right), \quad (16) \]

where \( k + j = (k+j) \mod 5 \). Let \( y_{k,j} = |x_{kj}|^2 \), and define an array \( Y := [y_{k,j}] \). Then, the three requirements \( \rho_A = \frac{1}{3} I_3 \), \( \rho_B = \frac{1}{4} I_4 \), \( \rho_C = \frac{1}{5} I_5 \) are, respectively, equivalent to the following three equations.

\[ \sum_{j=0}^3 y_{k,j} = 1, \quad \forall 0 \leq k \leq 2, \tag{17} \]
\[ \sum_{k=0}^2 y_{k,j} = \frac{3}{4}, \quad \forall 0 \leq j \leq 3, \tag{18} \]
\[ \sum_{k+j=n} y_{k,j} = \frac{3}{5}, \quad \forall 0 \leq n \leq 4. \tag{19} \]

One can verify the following array

\[ Y = \begin{bmatrix}
12 & 24 & 4 & 0 \\
18 & 40 & 40 & 40 \\
0 & 2 & 20 & 18 \\
18 & 4 & 6 & 12 \\
40 & 40 & 40 & 40
\end{bmatrix} \tag{20} \]

is an MSA corresponding to the system \( 3 \times 4 \times 5 \). Therefore, \( |\psi\rangle_{ABC} \) expressed in Eq. (13) with coefficients given by the MSA \( Y \) in Eq. (20) is an AME state in the system \( 3 \times 4 \times 5 \). \hfill \Box

In the final part of this section, we present more tripartite heterogeneous AME states by exploiting the concept of reducible AME states and Theorem 8. We illustrate our results in Table 1. First, we explicitly formulate the corresponding MSAs in Table 1 as follows.

\[ Y_1 = \frac{1}{20} \begin{bmatrix}
5 & 9 & 1 & 3 & 2 \\
1 & 3 & 7 & 4 & 5 \\
6 & 0 & 4 & 5 & 5
\end{bmatrix}, \quad Y_2 = \frac{1}{105} \begin{bmatrix}
45 & 36 & 16 & 7 & 1 \\
9 & 20 & 31 & 28 & 17 \\
9 & 16 & 28 & 45
\end{bmatrix}, \quad Y_3 = \frac{1}{30} \begin{bmatrix}
8 & 6 & 6 & 5 & 5 \\
6 & 9 & 5 & 5 & 5 \\
5 & 5 & 6 & 8 & 6 \\
5 & 4 & 7 & 6 & 8
\end{bmatrix}, \quad (21) \]

\[ Y_4 = \frac{1}{70} \begin{bmatrix}
20 & 20 & 15 & 7 & 8 \\
20 & 17 & 15 & 10 & 8 \\
8 & 10 & 13 & 19 & 20 \\
8 & 9 & 13 & 20 & 20
\end{bmatrix}, \quad Y_5 = \frac{1}{21} \begin{bmatrix}
4 & 4 & 5 & 3 & 3 & 2 \\
4 & 4 & 3 & 3 & 3 & 4 \\
3 & 3 & 3 & 4 & 4 & 4 \\
3 & 3 & 3 & 4 & 4 & 4
\end{bmatrix}. \]

Second, one can divide the reducible systems in Table 1 as follows.
Table 1  Several AME states in systems \(l \times m \times n\)

| \((l, m)\) | \(n\) | 5 | 6 | 7 | 8 | 9 |
|----------|------|---|---|---|---|---|
| (3, 4)   | Example 9 | Reducible | ? | Reducible | ? |
| (3, 5)   | Lemma 7   | \(Y_1\) | \(Y_2\) | ? | ? |
| (4, 5)   | Lemma 7   | \(Y_3\) | \(Y_4\) | Reducible | ? |
| (4, 6)   | \(Y_3\)   | Lemma 7 | \(Y_5\) | Reducible | Reducible |

1. Here, “reducible” means the corresponding AME state can be written as a tensor product of two existing AME states.
2. Here, “?” means the corresponding AME state is irreducible, and its existence cannot be determined by Theorem 8.
3. Here, “\(Y_i\)” is the corresponding MSA from Theorem 8.

\[(3, 4, 6) = (1, 2, 2) \otimes (3, 2, 3), \quad (3, 4, 8) = (1, 2, 2) \otimes (3, 2, 4), \quad (4, 5, 8) = (2, 5, 4) \otimes (2, 1, 2),\]
\[(4, 6, 8) = (2, 2, 4) \otimes (2, 3, 2), \quad (4, 6, 9) = (2, 3, 3) \otimes (2, 2, 3).\]  

(22)

It follows from Lemma 3 and Theorems 7 and 8 that every tripartite system in (22) contains AME states. Hence, we obtain the results in Table 1.

According to the examples in Table 1, it is reasonable to check whether it is reducible first. If it is reducible, the problem can be reduced to the existence of AME states with smaller local dimensions. Otherwise, if it is irreducible, we may numerically find corresponding MSAs by searching the generic solution in terms of Eqs. (9)–(11). Then, we can apply Theorem 8 to construct corresponding AME states.

4 Irreducible AME states in heterogeneous systems

The irreducible AME states are essential blocks to construct AME states. In this section, we investigate in which kinds of heterogeneous systems the AME states are irreducible. In Theorem 10 (i), we focus on tripartite AME states, and in Theorem 10 (ii) and (iii), we study multipartite heterogeneous systems.

**Theorem 10**  (i) Suppose \(p\) is prime and \(m, n\) are coprime. If \(|\psi\rangle \in \mathcal{A}(p, m, n)\), then \(|\psi\rangle\) is irreducible.

(ii) Suppose \(|\psi\rangle\) is an AME state in the system \(p \times q \times d_1 \times \cdots \times d_{2n-1}\), where \(p\) and \(q\) are prime. Then, \(|\psi\rangle\) is irreducible if there exists \(d_i \neq pq\). Furthermore, for \(n = 1\), \(|\psi\rangle\) is irreducible if and only if \(d_1 < pq\).

(iii) Suppose \(|\psi\rangle\) is an AME state in the system \(d_1 \times \cdots \times d_{2n+1}\). If \(|\psi\rangle\) is reducible, then there are at most two primes in \(\{d_1, \cdots, d_{2n+1}\}\). Further, suppose \(|\psi\rangle\) is locally unitarily equivalent to \(|\phi_1\rangle \otimes |\phi_2\rangle\), where \(|\phi_1\rangle\) is an AME state in the system \(p_1 \times \cdots \times p_{2n+1}\), \(|\phi_2\rangle\) is an AME state in the system \(q_1 \times \cdots \times q_{2n+1}\), and \(d_i = p_i q_i\). We have
In this section, we indicate some applications of our results. In Sect. 5.1, we propose
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(iii. a) if there is no prime in \{d_1, \ldots, d_{2n+1}\}, then the number of 1 in
\{p_1, \ldots, p_{2n+1}\} is at most one, so is \{q_1, \ldots, q_{2n+1}\};
(iii. b) if only d_1 is prime up to a permutation of subsystems, then p_1 = 1, q_1 = d_1,
and p_i, q_j > 1, \forall i, j > 1;
(iii. c) if only d_1 and d_2 are prime up to a permutation of subsystems, then p_1 = 1,
p_2 = d_2, q_1 = 2, and p_i, q_j > 1, \forall i, j > 2.

We present the proof of Theorem 10 in Appendix B. It follows from Theorem 10
(iii) that if there are at least three primes among \{d_1, \ldots, d_{2n+1}\}, then every AME state
in the system \(d_1 \times \cdots \times d_{2n+1}\) is irreducible. Here, we want to emphasize that there
could be both reducible and irreducible AME states in some multipartite systems. The
following example supports our claim. First, it is known that there exist reducible AME
states in the system \(2 \times 4 \times 4\), e.g., \((|000⟩ + |111⟩)_{A1} \otimes (|00⟩ + |11⟩)_{B2C2}\. Second,
We show that \(|ψ⟩_{ABC} = |0, x⟩ + |1, y⟩, where |x⟩ = \frac{1}{2}(|00⟩ + |11⟩ + |22⟩ + |33⟩\)
and \(|y⟩ = \frac{1}{4}(|01⟩ + |12⟩ + |23⟩ + |30⟩\)\) is an irreducible AME state in the system
\(2 \times 4 \times 4\). It follows from Lemma 4 that the Schmidt rank of \(λ_1|x⟩ + λ_2|y⟩\) is
equal to the rank of \(λ_1 \sum_{j=0}^{3} |j⟩⟨j| - λ_2 \sum_{j=0}^{3} |j⟩⟨j+1 \mod 4|\). If one of \(λ_1, λ_2\)
is zero, the Schmidt rank of \(λ_1|x⟩ + λ_2|y⟩\) is four. One can verify the rank of
the above matrix is at least three if \(λ_1 \cdot λ_2 \neq 0\). It follows that the range of \(|x⟩\)
and \(|y⟩\) has no bipartite state of Schmidt rank two in \(H_{BC}\). Next, we assume that
\(|ψ⟩\) is reducible. It follows that \(|ψ⟩ = (|0, a⟩ + |1, b⟩)_{AB} \otimes |c⟩_{B2C2} such that
system \(B = B_1 B_2\) and system \(C = C_1 C_2\). It implies that \((|0, a⟩ + |1, b⟩\)\) is
three-qubit state and \(|c⟩\) is a two-qubit state. Since \(|a⟩, |b⟩\) are both two-qubit state
and they are linearly independent, it follows from Lemma 5 that the span of \(|a⟩\)
and \(|b⟩\) has a product vector. Hence, the span of \(|a, c⟩\) and \(|b, c⟩\) has a bipartite
state of Schmidt rank two in \(H_{BC}\). Since \(|a, c⟩ = |x⟩, |b, c⟩ = |y⟩\), we obtain
the contradiction. Thus, \(|ψ⟩\) is an irreducible AME state in the system \(2 \times 4 \times 4\).
Therefore, the system \(2 \times 4 \times 4\) contains both reducible and irreducible AME
states.

5 Applications

In this section, we indicate some applications of our results. In Sect. 5.1, we propose
5.1 Protocol to prepare tripartite heterogeneous AME states

Tripartite heterogeneous AME states are the most possible heterogeneous ones to
realize. As an example, we propose a protocol to prepare the states in Eq. (6) using
the current experimental techniques. Suppose Alice, Bob and Charlie want to prepare
the target state
\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\Psi_1\rangle|0\rangle + |\Psi_2\rangle|1\rangle \right). \] (23)

For this purpose, Alice, Bob and Charlie may first prepare the following state
\[ |\Phi\rangle = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)|1\rangle \right). \] (24)

Then, we propose a protocol to prepare the state \(|\Phi\rangle\) as follows. Alice and Charlie may prepare the Bell state \(|\alpha\rangle_{AC} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AC}\), and similarly Bob and Charlie may also prepare the Bell state \(|\alpha\rangle_{BC} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{BC}\). By setting \(jk := 2j + k\) on system C, we obtain the tripartite state \(|\psi\rangle := |\alpha\rangle \otimes |\beta\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |102\rangle + |113\rangle)\). Next Charlie measures system C using the POVM \(|0\rangle\langle 0| + |3\rangle\langle 3|, |1\rangle\langle 1| + |2\rangle\langle 2|\). The result is that Charlie obtains the state \(\frac{1}{\sqrt{2}}(|000\rangle + |113\rangle)\) or \(\frac{1}{\sqrt{2}}(|011\rangle + |102\rangle)\). In either case, Charlie may inform Alice and Bob of his measurement result, so that the latter may perform local unitary operations, so that the final tripartite state is the standard three-qubit GHZ state. This state is equivalent to the target state \(|\Phi\rangle\) by local unitary operations. After preparing the state \(|\Phi\rangle\) one may obtain the target state by \(|\Phi\rangle = |\Phi\rangle \otimes |\psi\rangle_{A'B'}\), where \(|\psi\rangle_{A'B'}\) is a bipartite maximally entangled state in the composite system of Alice and Bob. Similarly, one may experimentally prepare more complex states as Eq. (6).

Furthermore, the quantum setups to produce tripartite AME states in both homogeneous and heterogeneous systems have recently been designed [33]. These quantum setups were designed by exploiting a recent connection between quantum experiments and graph theory [30–32]. This method can be applied to experimentally realize quantum states which have the same amplitude. It has been shown that this method is effective to create the following two AME states [33]

\[ |\psi\rangle_{ABC} = \frac{1}{2}(|000\rangle + |011\rangle + |102\rangle + |113\rangle) \in \mathcal{A}(2, 2, 4), \]
\[ |\phi\rangle_{ABC} = \frac{1}{\sqrt{6}}(|000\rangle + |011\rangle + |102\rangle + |203\rangle + |214\rangle + |225\rangle) \in \mathcal{A}(3, 3, 6). \] (25)

The authors also derived the following condition. If AME states with the same amplitude in the system \(d_A \times d_B \times d_C (d_A \geq d_B \geq d_C)\) can be created with probabilistic photon pair sources in such a way, then the following inequality holds [33].

\[ 1 + \min((1 + (d_A - d_B), d_C) + \min((1 + (d_A - d_C), d_B - 1)) \geq d_A. \] (26)

Note that the two AME states in Eq. (25) are in the form of Eq. (6). Thus, it is reasonable to believe that more tripartite heterogeneous AME states can be created in this way.
5.2 Construction of $k$-uniform states in multipartite heterogeneous systems

The $k$-uniform state is an approximate characterization of AME states. Several constructions of heterogeneous $k$-uniform states based on mixed orthogonal arrays were proposed in [34]. In this subsection, we study multipartite AME states and $k$-uniform states. In particular, we provide a method to construct $k$-uniform states of more parties from two AME states in Lemma 11 (iii). This method is inspired by the construction of GME states proposed in [40].

**Lemma 11** (i) If $|\psi\rangle$ is a $(2n)$-partite AME state in the homogeneous system composed of $(AB), C_1, C_2, \cdots C_{2n-1}$, it is also a $(2n + 1)$-partite AME state in the system composed of $A, B, C_1, C_2, \cdots C_{2n-1}$.

(ii) Suppose $|\psi\rangle$ is a $(2n)$-partite AME state in the homogeneous system composed of $A, C_{1,1}, \cdots , C_{1,2n-1}$, and $|\phi\rangle$ is a $(2n)$-partite AME state in the homogeneous system composed of $B, C_{2,1}, \cdots , C_{2,2n-1}$. Then, $|\psi\rangle \otimes |\phi\rangle$ is a $(2n + 1)$-partite AME state in the system composed of $A, B, C_1, \cdots , C_{2n-1}$, where $C_j = (C_{1,j}C_{2,j})$.

(iii) Suppose $|\psi\rangle$ is a $(2n + 1)$-partite AME state in the system composed of $A, C_{1,1}, \cdots , C_{1,2n}$, and $|\phi\rangle$ is a $(2n + 1)$-partite AME state in the system composed of $B, C_{2,1}, \cdots , C_{2,2n}$. Then, $|\psi\rangle \otimes |\phi\rangle$ is an $n$-uniform state in the system composed of $A, B, C_1, \cdots , C_{2n}$, where $C_j = (C_{1,j}C_{2,j})$.

We present the proof of Lemma 11 in Appendix B. The method described in Lemma 11 (iii) is similar to the construction of reducible AME states. Thus, it can be realized in experiments.

5.3 Heterogeneous AME states and multi-isometry matrices

The authors introduced the concept of *multiunitary matrices* in [17] and establish the connection with homogeneous AME states. In this subsection, we first recall the multiunitarity property and then generalize it to the concept of multi-isometry matrices for heterogeneous AME states.

The square matrix $A$ of order $d^k$ acting on a composed Hilbert space $\mathcal{H}_d^\otimes k$, and represented by

$$
(A)_{\mu_1, \cdots, \mu_k}^{v_1, \cdots, v_k} = \langle \mu_1, \cdots, \mu_k | A | v_1, \cdots, v_k \rangle = a_{\mu_1, \cdots, \mu_k}^{v_1, \cdots, v_k}
$$

is called $k$-unitary if it is unitary for all possible $\binom{2k}{k}$ reordering of its indices, corresponding to all possible choices of $k$ indices out of $2k$ [17]. Here $\mu_i, v_j = 0, \cdots , d-1$, and each forms an orthonormal basis of $\mathcal{H}_d$. Then, one can construct the following unnormalized pure state in the Hilbert space $\mathcal{H}_d^\otimes 2k$

$$
|\phi\rangle = \sum_{\mu_1, \cdots, \mu_k, v_1, \cdots, v_k = 0}^{d-1} a_{\mu_1, \cdots, \mu_k}^{v_1, \cdots, v_k} |\mu_1, \cdots, \mu_k, v_1, \cdots, v_k\rangle.
$$

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It follows from the multiunitarity property that $|\phi\rangle$ in Eq. (28) is an unnormalized AME state in $\mathcal{H}_d^{\otimes 2k}$. In special, if $A$ is a 1-unitary matrix, it is a standard unitary matrix. So the first interesting case is 2-unitary matrices. Goyeneche et al. constructed a 2-unitary matrix using an AME state in $A_3(4)$, and derived several properties of 2-unitary matrices [17].

As the above discussion, the multiunitary matrices are closely related to AME states in homogeneous systems possessing even number of parties. In a direct generation, we introduce the multi-isometry matrices which are connected to AME states shared with odd number of parties. A matrix $M$ is called isometry if $M^\dagger M = I$. Then, we can analogously define the multi-isometry matrices.

The matrix $A$ represented by

\[ (A)_{\mu_1,\cdots,\mu_{k+1}} = \langle \mu_1, \cdots, \mu_{k+1} | A | v_1, \cdots, v_k \rangle = a_{\mu_1,\cdots,\mu_{k+1}}^{v_1,\cdots,v_k} \]  

is called $k$-isometry if $\text{Tr}(A^\dagger A)$ is constant and $A^\dagger A \propto I$ for all possible $\binom{2k+1}{k}$ reordering of its indices, corresponding to all possible choices of $k$ indices out of $(2k + 1)$. Here, $\mu_i = 0, \cdots, d_i - 1$, which is an orthonormal basis of $\mathcal{H}_d$, and $\nu_j = 0, \cdots, l_j - 1$, which is an orthonormal basis of $\mathcal{H}_l$. As an extension, we present the following lemma.

**Lemma 12** Suppose $A$ is a $k$-isometry matrix whose elements $a_{\mu_1,\cdots,\mu_{k+1}}^{v_1,\cdots,v_k}$ are given by Eq. (29). Then, the following $(2k + 1)$-partite state

\[ |\phi\rangle = \sum_{\mu_1,\cdots,\mu_{k+1}, v_1,\cdots,v_k = 0}^{d-1} a_{\mu_1,\cdots,\mu_{k+1}}^{v_1,\cdots,v_k} |\mu_1, \cdots, \mu_{k+1}, v_1, \cdots, v_k\rangle \]  

is an unnormalized AME state in the system $d_1 \times \cdots \times d_{k+1} \times l_1 \times \cdots \times l_k$.

This lemma follows from the fact that each $k$-partite marginal of $|\phi\rangle \langle \phi|$ is equal to $A^\dagger A$, where $A$ is represented in a product basis corresponding to a choice of $k$ indices out of $(2k + 1)$. Since $A$ is $k$-isometry, each $k$-partite marginal of $|\phi\rangle \langle \phi|$ is proportional to the identity. Thus, $|\phi\rangle$ in Eq. (30) is an unnormalized AME state.

It is different from the multiunitarity property that 1-isometry matrices are not equivalent to the standard isometry matrices. In other words, there exist standard isometry matrices which are not 1-isometry. The 1-isometry property requires that the matrices constructed by 3 different bipartitions are all proportional to identities. Hence, it’s interesting to study the relation between 1-isometry matrices and tripartite AME states. In order to better understand the multi-isometry property, we construct a concrete 1-isometry matrix using the tripartite AME state in the system $3 \times 4 \times 5$ which is formulated by Example 9. By multiplying a coefficient, we write it as

\[ |\psi\rangle_{ABC} = \sum_{\mu_1=0}^{2} \sum_{\mu_2=0}^{3} \sum_{v_1=0}^{4} a_{\mu_1,\mu_2}^{v_1} |\mu_1, \mu_2, v_1\rangle \]
Absolutely maximally entangled states...

\[ \begin{align*}
A^0 &= \sum_{\mu_1=0}^{2} \sum_{\mu_2=0}^{3} \sum_{v_1=0}^{4} a_{\mu_1,\mu_2} |\mu_1, \mu_2\rangle \langle v_1| \mu_1, \mu_2|, \quad (A^0)^\dagger (A^0) = I_5, \\
A^1 &= \sum_{\mu_1=0}^{2} \sum_{\mu_2=0}^{3} \sum_{v_1=0}^{4} a_{\mu_2, v_1} |\mu_2, v_1\rangle \langle v_1| \mu_2, v_1|, \quad (A^1)^\dagger (A^1) = \frac{5}{3} I_3, \\
A^2 &= \sum_{\mu_1=0}^{2} \sum_{\mu_2=0}^{3} \sum_{v_1=0}^{4} a_{v_1, \mu_1} |\mu_1, \mu_2\rangle \langle v_1| \mu_1, \mu_2|, \quad (A^2)^\dagger (A^2) = \frac{5}{4} I_4.
\end{align*} \] (31)

By definition, each $A^0$, $A^1$, $A^2$ in (32) is 1-isometry. To sum up, the one-one corresponding relation between heterogeneous AME states and multi-isometry matrices implies a possible direction to study multipartite heterogeneous AME states further.

5.4 Quantum steering in heterogeneous systems

Steering has been found useful in a number of applications such as subchannel discrimination and one-sided device-independent quantum key distribution. Thus, detection and characterization of steering have recently attracted increasing attention. In [41], authors propose a general framework for constructing universal steering criteria that are applicable to arbitrary measurement settings of the steering party.

Here, we introduce the quantum steering as an application of tripartite AME states. We will explain how to convert a tripartite AME state in Eq. (6) into a bipartite maximally entangled state. This process is the so-called steering. Suppose the tripartite AME state is controlled by the system Alice, Bob and Charlie. If $n = m^2$, then Alice and Bob are in the maximally mixed state $\rho_{AB} = \frac{1}{m^2} I_{m^2}$. This is a separable state, and also a classical–classical state [42]. Using the projective POVM \{\langle j | j \rangle, j = 0, ..., m^2 - 1\}, Charlie can steer the state $\rho_{AB}$ into the maximally entangled state $|\Psi_j\rangle$ with probability $1/m^2$. The process is as follows. For any $j$,

\[ P^j = \left( \frac{1}{m^2} \left( \sum_{j=0}^{m^2-1} |\Psi_j, j\rangle \langle j| \right) \right)^\dagger \left( \sum_{j=0}^{m^2-1} |\Psi_j, j\rangle \langle j| \right) = \frac{1}{m^2} |\Psi_j\rangle \langle \Psi_j|_{AB} \otimes |j\rangle \langle j|_C. \] (33)

$P^j = I_{AB} \otimes |j\rangle \langle j|_C$. Since any two bipartite maximally entangled states are LU equivalent, Alice and Bob can convert $|\Psi_j\rangle$ into the standard maximally entangled states $\frac{1}{\sqrt{m}} \sum_{j=1}^{n} |jj\rangle$. One can show that the same argument works when $n < m^2$, though $\rho_{AB}$ may be not separable.
Similarly, Alice (or Bob) may perform the projective POVM \( \{|j\rangle\langle j|, j = 1, ..., n\} \) on system \( A \) (or \( B \)), so as to steer the state \( \rho_{BC} \) of Bob and Charlie (or \( \rho_{AC} \) of Alice and Charlie) into the standard maximally entangled state up to LU equivalence. Since \( \rho_{BC} \) and \( \rho_{AC} \) are both rank-\( n \) mixed entangled states, the steering is a kind of system-assisted and one-copy entanglement distillation with probability one [43].

6 Concluding remarks

This work systematically analyzed the existence of tripartite heterogeneous AME states through constructing their expressions and explored multipartite heterogeneous AME states and \( k \)-uniform states. First, we introduced the irreducible AME states as the essential elements to construct AME states in the systems with large local dimensions. Second, we showed the existence of AME states in kinds of heterogeneous systems. In particular, we introduced the so-called magic solution arrays which are effective to construct tripartite heterogeneous AME states. Third, we discussed irreducible AME states in multipartite heterogeneous systems. We presented sufficient conditions for multipartite heterogeneous systems such that the AME states in them are irreducible. We also indicated some applications of our results. We first proposed a protocol to prepare a kind of tripartite heterogeneous AME states constructed in this paper. Second, we presented a method to construct \( k \)-uniform states of more parties from two AME states. Then, we revealed the connection between heterogeneous AME states and multi-isometry matrices. Moreover, we applied tripartite heterogeneous AME states to realize quantum steering.

There are some possible directions for further research. The first one is how to construct tripartite heterogeneous AME states if the corresponding magic solution array does not exist. The second one is to extend the methods in this paper to determine the existence of multipartite heterogeneous AME states. For example, one can try to construct 2-isometry matrices in order to construct five-partite heterogeneous AME states. Finally, it is also interesting to determine whether a heterogeneous system contains both reducible and irreducible AME states.

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Appendix A: Alternative Proof of Lemma 3

For convenience, we first present the following lemma as it will be used in the proof of Lemma 3.

Lemma 13 Suppose \( |\psi\rangle \) is a tripartite AME state in the system \( 2 \times m \times (m+n) \). Then, up to a local unitary \( |\psi\rangle \) is equivalent to the following state with proper coefficients.
\[
\frac{1}{\sqrt{2}} \left[ |0\rangle \left( \sum_{j=0}^{m-1} x_j |j, j\rangle \right) + |1\rangle \left( \sum_{j=0}^{m-1} y_j |\alpha_j, \beta_j\rangle \right) \right],
\]  
(A1)

where \(x_j, y_j \geq 0\) with \(\sum_j x_j^2 = \sum_j y_j^2 = 1\), and \(|\alpha_j\rangle\}_{j=0}^{m-1}\) and \(|\beta_j\rangle\}_{j=0}^{m-1}\) are two sets of orthonormal vectors in Hilbert spaces \(\mathcal{H}_m\) and \(\mathcal{H}_{(m+n)}\), respectively.

**Proof** Using Schmidt decomposition and a proper local unitary \(U \otimes I_m \otimes I_{m+n}\), where \(U \in \mathcal{U}(2)\), we have

\[
|\psi\rangle \simeq_{LU} \frac{1}{\sqrt{2}} (|0, \phi\rangle + |1, \phi^\perp\rangle),
\]  
(A2)

where \(|\phi\rangle\) and \(|\phi^\perp\rangle\) are two orthonormal states in the system \(m \times (m+n)\). The coefficient \(\frac{1}{\sqrt{2}}\) ensures that the reduction to the first subsystem is \(\frac{I_2}{2}\). Using Schmidt decomposition and a proper local unitary \(V \otimes W\), where \(V \in \mathcal{U}(m)\) and \(W \in \mathcal{U}(m+n)\), one can similarly obtain

\[
V \otimes W|\phi\rangle = \sum_{j=0}^{m-1} x_j |j, j\rangle, \quad \text{and} \quad V \otimes W|\phi^\perp\rangle = \sum_{j=0}^{m-1} y_j |\alpha_j, \beta_j\rangle,
\]  
(A3)

where \(x_j, y_j \geq 0\) with \(\sum_j x_j^2 = \sum_j y_j^2 = 1\), and \(|\alpha_j\rangle\}_{j=0}^{m-1}\) and \(|\beta_j\rangle\}_{j=0}^{m-1}\) are two sets of orthonormal vectors in Hilbert spaces \(\mathcal{H}_m\) and \(\mathcal{H}_{(m+n)}\), respectively. This completes the proof. \(\square\)

**Proof of Lemma 3** First, if \(n = 0\), the existence of AME states in systems \(2 \times m \times m\) follows directly from Lemma 7 (i).

Second, we investigate the case when \(n \geq 1\). Suppose \(|\psi\rangle_{ABC}\) is an arbitrary tripartite AME state in the system \(2 \times m \times (m+n)\). From Lemma 13, we can assume \(|\psi\rangle_{ABC}\) as

\[
\frac{1}{\sqrt{2}} \left[ |0\rangle \left( \sum_{j=0}^{m-1} x_j |j, j\rangle \right) + |1\rangle \left( \sum_{j=0}^{m-1} y_j |\alpha_j, \beta_j\rangle \right) \right],
\]  
(A4)

where \(x_j, y_j \geq 0\) with \(\sum_j x_j^2 = \sum_j y_j^2 = 1\), and \(|\alpha_j\rangle\}_{j=0}^{m-1}\) and \(|\beta_j\rangle\}_{j=0}^{m-1}\) are two sets of orthonormal vectors in Hilbert spaces \(\mathcal{H}_m\) and \(\mathcal{H}_{(m+n)}\), respectively. Let \(\rho = |\psi\rangle\langle \psi|\) be the density matrix of \(|\psi\rangle_{ABC}\). A straightforward computing yields that

\[
\rho_A = \frac{1}{2} I_2, \quad \text{and} \quad \rho_B = \frac{1}{2} \left( \sum_{j=0}^{m-1} x_j^2 |j\rangle\langle j| \right) + \frac{1}{2} \left( \sum_{j=0}^{m-1} y_j^2 |\alpha_j\rangle\langle \alpha_j| \right),
\]  
(A5)  
(A6)
\[ \rho_C = \frac{1}{2} \left( \sum_{j=0}^{m-1} x_j^2 |j\rangle \langle j| \right) + \frac{1}{2} \left( \sum_{j=0}^{m-1} y_j^2 |\beta_j\rangle \langle \beta_j| \right). \] (A7)

By the definition of AME states, we have \( \rho_B = \frac{1}{m} I_m \) and \( \rho_C = \frac{1}{m+n} I_{m+n} \). From Eqs. (A6) and (A7), we have

\[ \frac{1}{2} \left( \sum_{j=0}^{m-1} y_j^2 |\alpha_j\rangle \langle \alpha_j| \right) = \rho_B - \frac{1}{2} \left( \sum_{j=0}^{m-1} x_j^2 |j\rangle \langle j| \right) = \frac{1}{2} \left( \sum_{j=0}^{m-1} \left( \frac{2}{m} - x_j^2 \right) |j\rangle \langle j| \right). \] (A8)

and

\[ \frac{1}{2} \left( \sum_{j=0}^{m-1} y_j^2 |\beta_j\rangle \langle \beta_j| \right) = \rho_C - \frac{1}{2} \left( \sum_{j=0}^{m-1} x_j^2 |j\rangle \langle j| \right) = \frac{1}{2} \left( \sum_{j=0}^{m-1} \left( \frac{2}{m} - x_j^2 \right) |j\rangle \langle j| \right) + \frac{1}{m+n} \sum_{j=m}^{m+n-1} |j\rangle \langle j|. \] (A9)

Since \( \frac{1}{2} \left( \sum_{j=0}^{m-1} y_j^2 |\beta_j\rangle \langle \beta_j| \right) \) is a positive semidefinite matrix with rank at most \( m \), it follows from the second equality in Eq. (A9) that there are at least \( n \) elements of \( \{x_j^2\}_{j=0}^{m-1} \) equal to \( \frac{2}{m+n} \). Up to a local permutation on \( \mathcal{H}_B \otimes \mathcal{H}_C \), we can assume that

\[ x_0^2 = x_1^2 = \cdots = x_{n-1}^2 = \frac{2}{m+n}. \]

It follows from Eqs. (A8) and (A9) that \( \sum_{j=0}^{m-1} y_j^2 |\alpha_j\rangle \langle \alpha_j| \) and \( \sum_{j=0}^{m-1} y_j^2 |\beta_j\rangle \langle \beta_j| \) are both diagonal matrices. Since \( \{ |\alpha_j\rangle \}_{j=0}^{m-1} \) and \( \{ |\beta_j\rangle \}_{j=0}^{m-1} \) are two sets of orthonormal vectors, we have \( y_j^2 \)'s are diagonal entries of the two matrices. Thus, from Eq. (A8) we obtain the set \( \{ y_0^2, \cdots, y_{m-1}^2 \} \) is equal to the following set

\[ \left\{ \frac{2n}{m(m+n)}, \cdots, \frac{2n}{m(m+n)} \right\} = \left\{ \frac{2}{m+n} x_n^2, \frac{2}{m+n} x_{n+1}^2, \cdots, \frac{2}{m+n} x_{m-1}^2 \right\}. \] (A10)

Similarly, from Eq. (A9) we obtain the set \( \{ y_0^2, \cdots, y_{m-1}^2 \} \) is also equal to the following set

\[ \left\{ \frac{2}{m+n} - x_n^2, \frac{2}{m+n} - x_{n+1}^2, \cdots, \frac{2}{m+n} - x_{m-1}^2 \right\}. \] (A11)
We first assume \( d = m \mod n \), and \( d > 0 \). Without loss of generality, we may assume
\[
\frac{2}{m+n} - x_n^2 = \cdots = \frac{2}{m+n} - x_{2n-1}^2 = \frac{2n}{m(m+n)}.
\]
Thus, \( x_n^2 = \cdots = x_{2n-1}^2 = \frac{2m-2n}{m(m+n)} \). If \( \frac{2}{m} - x_n^2 = \cdots = \frac{2}{m} - x_{2n-1}^2 = \frac{2}{m+n} \), it implies \( \frac{4n}{m(m+n)} = \frac{2}{m+n} \) which contradicts with \( d > 0 \). Hence, we may further assume
\[
\frac{2}{m} - x_n^2 = \cdots = \frac{2}{m} - x_{2n-1}^2 = \frac{2}{m+n} - x_{2n-1}^2 = \frac{2}{m+n} - x_{3n-1}^2.
\]
It implies the following recursive relation \( x^2_{(j-1)n} - x^2_{jn} = \frac{2n}{m(m+n)} \). Since the two sets given by (A10) and (A11) are equal, it requires that \( \frac{2}{m} - x^2_{sn} = \cdots = \frac{2}{m} - x^2_{(s+1)n-1} = \frac{2(s+2)n}{m(m+n)} = \frac{2}{m+n} \) for some integer \( s \). It follows that \( m = (s + 1)n \) which contradicts with \( d > 0 \). Hence, the two sets (A10) and (A11) cannot be equal if \( d > 0 \). Next, we suppose \( m = kn \). We may assume \( |\alpha_j, \beta_j\rangle = |j, j + n, \rangle, \forall j \). In the following, we will show there exist \( x_j \)'s and \( y_j \)'s such that \( |\psi\rangle_{ABC} \) is a tripartite AME state. From Eqs. (A6)–(A7), we formulate the system of equations as follows.
\[
\begin{align*}
x^2_j + y^2_j &= \frac{2}{m}, & 0 \leq j \leq m - 1; \\
x^2_0 &= \cdots = x^2_{n-1} = \frac{2}{m+n}; \\
x^2_{n+k} + y^2_k &= \frac{2}{m+n}, & 0 \leq k \leq m - n - 1; \\
y^2_{m-n} &= \cdots = y^2_{m-1} = \frac{2}{m+n}.
\end{align*}
\]
(A12)

One can verify the system of equations (A12) has the system of solutions as follows. For any \( 0 \leq j \leq k - 1 \),
\[
\begin{align*}
x^2_{jn} &= \cdots = x^2_{(j+1)n-1} = \frac{2m - 2jn}{m(m+n)} = \frac{2k - 2j}{k(k+1)n}, \\
y^2_{jn} &= \cdots = y^2_{(j+1)n-1} = \frac{2(j+1)n}{m(m+n)} = \frac{2j + 2}{k(k+1)n}.
\end{align*}
\]
(A13)

Therefore, when \( m = kn \), the tripartite state \( |\psi\rangle_{ABC} \) in (A4) with \( |\alpha_j, \beta_j\rangle = |j, j + n, \rangle, \forall j \) and coefficients in Eq. (A13) is a tripartite AME state in \( 2 \times m \times (m+n) \). This completes the proof. \( \square \)

Appendix B: Proofs of Main Results

Proof of Theorem 8 Suppose \( \{y_{k,j}\} \) is a nonnegative solution of Eqs. (9)–(11), where \( 0 \leq k \leq l - 1 \), and \( 0 \leq j \leq m - 1 \). With these \( y_{k,j} \)'s, we construct a tripartite
state $|\psi\rangle_{ABC}$ in the form as Eq. (12). Let $\rho_{ABC} = |\psi\rangle\langle\psi|_{ABC}$. A straightforward calculation yields that

\[\rho_A = \frac{1}{l} \left( \sum_{k=0}^{l-1} \sum_{j=0}^{m-1} y_{k,j} |k\rangle\langle k| \right),\]

(B1)

\[\rho_B = \frac{1}{l} \left( \sum_{j=0}^{m-1} \sum_{k=0}^{l-1} y_{k,j} |j\rangle\langle j| \right),\]

(B2)

\[\rho_C = \frac{1}{l} \left( \sum_{k=0}^{l-1} \sum_{j=0}^{m-1} y_{k,j} |k+j\rangle\langle k+j| \right),\]

(B3)

where $|k+j\rangle = |k + j \mod n\rangle$. Then, one can verify that Eq. (9) makes $\rho_A = \frac{1}{l} I_l$, Eq. (10) makes $\rho_B = \frac{1}{m} I_m$, and Eq. (11) makes $\rho_C = \frac{1}{n} I_n$. Therefore, $|\psi\rangle_{ABC}$ expressed in Eq. (12) is a tripartite AME state in the system $l \times m \times n$. This completes the proof.

Proof of Theorem 10

(i) We prove it by contradiction. Suppose $|\psi\rangle$ is a reducible AME state. Then, there exists a local unitary $U \otimes V \otimes W$ such that $(U \otimes V \otimes W)|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$, where $|\phi_1\rangle$ is an AME state in the system $1 \times m_1 \times n_1$ and $|\phi_2\rangle$ is an AME state in $p \times m_2 \times n_2$. It implies that $m_1 = n_1$. This contradicts with $m$ and $n$ are coprime. Therefore, $|\psi\rangle$ is irreducible.

(ii) We prove it by contradiction. Suppose $|\psi\rangle$ is a reducible AME state. By definition, $|\psi\rangle$ is locally unitarily equivalent to $|\phi_1\rangle \otimes |\phi_2\rangle$, where $|\phi_1\rangle$ is an AME state in the system $p \times 1 \times k_1 \times \cdots \times k_{2n-1}$, $|\phi_2\rangle$ is an AME state in the system $1 \times q_1 \times l_1 \times \cdots \times l_{2n-1}$, and $d_i = k_i q_i l_i$. So $|\phi_1\rangle$ and $|\phi_2\rangle$ can be taken as $2n$-partite AME states. Since heterogeneous AME states don’t exist if the number of parties is even, it follows that $k_i = p_i$, $l_j = q_i$, $\forall i$, $j$. It is equivalent to that $d_i = p q_i$. So we obtain the contradiction, and thus the assertion holds. Furthermore, for $n = 1$, we have $|\psi\rangle$ is a tripartite reducible AME state if and only if $d_1 = p q$.

(iii) We prove it by contradiction. Suppose $d_1, d_2, d_3$ are three primes, and $|\psi\rangle$ is a reducible AME state in the system $d_1 \times \cdots \times d_{2n+1}$. By definition, $|\psi\rangle$ is locally unitarily equivalent to $|\phi_1\rangle \otimes |\phi_2\rangle$, where $|\phi_1\rangle$ is an AME state in the system $p_1 \times \cdots \times p_{2n+1}$, $|\phi_2\rangle$ is an AME state in the system $q_1 \times \cdots \times q_{2n+1}$, and $d_i = p_i q_i$. If $p_1 = p_2 = 1$, then $|\phi_1\rangle$ can be taken as a $(2n - 1)$-partite state, and thus it isn’t a $(2n + 1)$-partite AME state. One can similarly exclude that $q_1 = q_2 = 1$. Therefore, there is at most one 1 in the set $\{p_1, p_2, p_3\}$, the same for the set $\{q_1, q_2, q_3\}$. It contradicts with the three $d_1, d_2, d_3$ are prime. So the assertion (iii) holds. When there are at most two primes in $\{d_1, \cdots, d_{2n+1}\}$, one can verify the three specific cases (iii.a)–(iii.c) with the same idea.

This completes the proof.

Proof of Lemma 11

(i) Suppose $\rho = |\psi\rangle\langle\psi|$. A simple calculation gives that

$$\rho_{C_{j_1} \cdots C_{j_n}} \propto I,$$
\[ \rho_{ABC_{j_1} \cdots C_{j_{n-2}}} \propto I, \]
\[ \rho_{AC_{j_1} \cdots C_{j_{n-1}}} = \text{Tr}_B \rho_{ABC_{j_1} \cdots C_{j_{n-1}}} \propto I, \]
\[ \rho_{BC_{j_1} \cdots C_{j_{n-1}}} = \text{Tr}_A \rho_{ABC_{j_1} \cdots C_{j_{n-1}}} \propto I. \]

Therefore, |ψ⟩ is also a (2n + 1)-partite AME state in the system composed of A, B, C1, C2, \cdots C_{2n-1}.

(ii) Suppose \( \sigma = |\psi⟩⟨\psi|, \gamma = |\phi⟩⟨\phi|, \) and \( \rho \) is the density matrix of \( |ψ⟩ \otimes |ϕ⟩ \). A straightforward computing yields that

\[ \rho_{C_{j_1} \cdots C_{j_n}} = \sigma_{C_{j_1} \cdots C_{j_{n-1}}} \otimes \gamma_{C_{j_{n-1}}} \propto I, \]
\[ \rho_{AC_{j_1} \cdots C_{j_{n-1}}} = \sigma_{AC_{j_1} \cdots C_{j_{n-1}}} \otimes \gamma_{C_{j_{n-1}}} \propto I, \]
\[ \rho_{BC_{j_1} \cdots C_{j_{n-1}}} = \sigma_{BC_{j_1} \cdots C_{j_{n-1}}} \otimes \gamma_{BC_{j_{n-1}}} \propto I, \]
\[ \rho_{ABC_{j_1} \cdots C_{j_{n-2}}} = \sigma_{ABC_{j_1} \cdots C_{j_{n-2}}} \otimes \gamma_{ABC_{j_{n-2}}} \propto I. \]

Therefore, \( |ψ⟩ \otimes |ϕ⟩ \) is a (2n + 1)-partite AME state in the system composed of A, B, C1, \cdots, C_{2n-1}, where \( C_j = (C_{1,j}C_{2,j}) \).

(iii) The proof is similar to (ii). \( \square \)

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