First principles derivation of NLS equation for BEC with cubic and quintic nonlinearities at non zero temperature. Dispersion of linear waves.

P. A. Andreev
Department of General Physics, Physics Faculty, Moscow State University, Moscow, Russian Federation.

In this work we presented a derivation of the quantum hydrodynamic equations for neutral bosons. We considered short range interaction between particles. This interaction consists binary interaction $U(r_i, r_j)$ and three particle interaction $U(r_i, r_j, r_k)$, the last one does not include binary interaction between particles. From the quantum hydrodynamic (QHD) equations for Bose-Einstein condensate we derive nonlinear Schrödinger equation. This equation includes the nonlinearities of third and fifth degree. It is at zero temperature. Explicit form of the constant of three-particle interaction was taken. First of all, developed method we used for studying of dispersion of linear waves. Dispersion characteristics of linear waves were compared for the cases. It were of two-particle interaction in approximation third order to interaction radius (TOIR) and three-particle interaction, at zero temperature. We consider influence of temperature on dispersion of elementary excitations. For this aim we derive a system of QHD equations at non-zero temperature. Obtained system of equation is an analog of well-known two-fluid hydrodynamics. Moreover, it is generalization of two-fluid hydrodynamics equations due to three-particle interaction. Evident expressions of the velocities of the first and second sound via the concentrations of superfluid and noncondensate components is calculated.

I. INTRODUCTION

At the theoretical investigation of Bose-Einstein condensation (BEC) with account of the three-particle interaction (TPI) the nonlinear Schrödinger equation (NLSE) is used. This equation contain cubic and quintic nonlinearities [1, 2]. For describing inelastic scattering imaginary part of interaction constants is used in [3]. Three-particle interaction leads to forming of dimers. This process leads to loss of atoms from BEC [4, 5].

Opportunity of obtaining of the Gross-Pitaevkii (GP) equation from the microscopic many particle Schrödinger equation is demonstrated in [6]. The GP equation is the example of NLSE. In [7] the direct derivation of the GP equation from many particle Schrödinger equation has done.

In article [7] authors derived NLSE for system of bosons being in the state of BEC, with the short-range interaction potential, in third order in interaction radius. If one keeps only the term of the first order in interaction radius then one gets well-known GP equation from many-particle Schrödinger equation. There taking into account only potential of binary interaction. At derivation authors do not use suggestion about density of considered system. Therefore, there occur interaction whether by means of scattering or during finite interval of time. Presented in [7] derivation does not exclude interaction of a few particles at the same moment.

In this article we use the same method for studying of both a three particle interaction (TPI) and influence of temperature on dynamic of neutral Bose particles.

 Fundamental and detailed distribution of influence of non-zero temperature on BEC dynamic is presented in paper [8], where were considered the two-particle interaction only. The influence of the noncondensate atoms on BEC dynamic is considered in [8], but dynamic of noncondensed atoms do not included. Interference of superfluid and normal components usually described by two-fluid hydrodynamic [9]. Present day examples of using of two-fluid hydrodynamics can be found in papers [10-13]. Connection of two-fluid hydrodynamics with the kinetic equations is presented in papers [13, 14]. In cited papers authors used Boltzmann like kinetic equation with the collision term.

Due to studying of three particle interaction the ground state energy density in second order in $\sqrt{\rho a^3}$ is calculated in [15-21], where $\rho$ is the particles density, $a$ is the s-wave scattering length. The first calculation of the constant under the logarithm is presented in [15]. The constant depends on $\alpha$ and from the parameter described the lower energy $3 \rightarrow 3$ scattering of the particles.

The explicit three-body contact potential for a dilute condensed Bose gas is derived from microscopic theory in [22]. The derivation is based on the quantum expectation values of products of single mode annihilation and creation operators. The three-body coupling constant exhibits the general form predicted by Wu [19]. It depends on s-wave scattering length $a$, defined via binary potential like in [19]. For describing the properties of BEC at zero temperature, with three-body interaction, in [22] there were used nonlinear Schrödinger equation contained nonlinearity of fifth degree.

Here we present some physical effects there is in BEC due to TPI. Stability properties of BEC with TPI are considered in [13, 23, 25]. The modulation instability of the BEC trapped in an external parabolic potential, is investigated in [23]. The explicit time-dependent criterion for the modulation instability of the condensate was established.

The influence of initial conditions on stability of BEC was studied in [3] using a Gaussian variational approach and numerical simulations, in three-dimensional trapped BEC. Abdullaev et.al. [3] discussed the validity of the
criterion of stability suggested by Vakhitov and Kolokolov. In the works [3, 23] for studying of properties of BEC with TPI there were used nonlinear equation with cubic and quintic nonlinearities.

Abdullaev et al. [24] discussed localized ground states of BEC in optical lattices with attractive and repulsive TPI. For this aim a quintic nonlinear Schrödinger equation has used. In [4] the existence of unstable localized excitations which are similar to Townes solitons of the cubic nonlinear Schrödinger equation in two dimensions is shown.

Marklund et al. [25] found the Vlasov-like equation for BEC. They used Wigner function and NLSE with cubic-quintic nonlinearities.

Under the condition, that two-particle scattering length tends to zero, it may be realized by manipulating external magnetic field, far off-Feshbach resonance [26]. Y tends to zero too. In this situation main role has TOIR and TPI. In this case TOIR and TPI are compared.

There are various generalizations of GP equation. Here we interesting in nonlinear NLSE. We present brief review of known nonlinear NLSE generalization of GP equation. For studying of BEC the NLSE with nonlinearities containing spatial derivatives of wave functions is used. The NLSE for BEC with nonlocal nonlinearity was obtained in [27]. In that article NLSE contains second spatial derivatives of square of wave function module. In [7] NLSE in third order on interaction radius (TOIR) approximation is derived. In this case NLSE is integro-differential equation with spatial derivatives of wave functions. Comparison between nonlocal nonlinearity [27] and TOIR [7] for two dimensional space is considered in [28]. Local Lagrangian density, for two-body and three-body effective short-range interaction, contained dependence on \((\psi^\ast \psi)^2, (\psi^\ast \psi)^3\) and \(\nabla (\psi^\ast \psi)\nabla (\psi^\ast \psi)\) is presented in [29]. There are given a determination of the strength of the three-body contact interaction for various model potentials.

Generalization of GP equation to the large-gas-parameter regime is suggested in [30]. It is NLSE with nonlinearities of fourth degree.

In this article we use method of quantum hydrodynamic (QHD) of many particles system developed in works [7, 31, 32]. In [31] equation of QHD of charged particles system was found in external electromagnetic field taking into account Coulomb interaction of particles. In [32] for system of charged and spinning particles there were derived equations of quantum hydrodynamic, i.e., continuity, balance of momentum, magnetic moment and energy. In [31] there were developed methods of calculation of many-particle functions arising in equations of QHD. In [7] there was developed method of quantum hydrodynamics for system of bosons, fermions and mixtures, with short-range interaction. In this article there were given corresponding equations of continuity and balance of momentum. On the base of system QHD equations the NLSE describing the dynamics of boson-fermion mixtures is obtained. Particular case NLSE for bosons noticed in [7] is the GP equation.

For boson systems with two- and three-particle interaction, the problem of finding the quantum stress tensor can be solved and conditions of its existence can be clarified. Here, we will limit ourselves by the development of the quantum stress tensor and show that under the standard assumptions this tensor can be transformed so that the momentum balance equations for bosons coincide with the analogous NLSE contained cubic and quintic nonlinearities. Using this momentum balance equation derivation, we can obtain the equation (for wave function in medium or the order parameter), which is NLSE with nonlinearities of fifth degrees. Quantum stress tensor for boson systems is symmetric. It in first order in interaction radius consists of two parts, namely, the term coincides with interaction via binary interaction potential and the term coincides with TPI which does not contain interaction by means binary potential. The first term leads to equation coinciding with the analogous GP equation. The second term includes explicit expression for three particle interaction constant and coincides with the term analogous to the nonlinearity of fifth degrees in NLSE.

In dilute alkali gases, when interaction may be considered like scattering process, the constant of interaction in first order on interaction radius (FOIR) has the form \(\Upsilon = -4\pi\hbar^2 a/m; a-\) is scattering length. In TOIR approximation there arise the second constant of interaction \(\Upsilon_2\). In general case, parameter \(\Upsilon_2\) is independent from \(\Upsilon\) and has to considered like supplementary. In [7] an approximate estimate of \(\Upsilon_2\) via \(\Upsilon\) is considered.

In the paper [36] were shown the account of interaction up to TOIR leads to finding of new physical effects in BEC. In [36] were found new type of solitons in BEC. Frequency dependence of eigenwave in boson-fermion mixture in TOIR is derived in [7]. Therefore, generalization of Bogoliubov spectrum [34], [35] was obtained in TOIR. Also, analytical dependence \(\omega(k)\) for degenerate fermions is obtained. Using system of QHD equations (derived in [7]) in works [28, 33] was investigated dynamics of nonlinear wave in TOIR. In [28] analytical solution for bright soliton is obtained in uniform BEC within the TOIR approximation. In [33] nonlinear frequency shift in uniform BEC is analytically investigated within the TOIR approximation.

At experimental investigation of BEC in magnetic traps work with number of particles \(N\) is order \(10^4-10^5\). At quantum-mechanical description of such system it is necessary to solve Schrödinger equation determining wave function that depends from \(3N\) coordinate and time. Wave process, process of transfer, exchange by energy and momentum at interaction take place in three-dimensional physical space. In this connection it is necessary to convert Schrödinger equation to equation is determined dynamic of functions in three-dimensional physical space. This task is solved with method of quantum hydrodynamics, further development of this method is made in this work.

Moreover, in this work problem of finding of method al-
lowlow to build NLSE on the base system of QHD equations is solved.

Contribution of noncondensate particles was shown to be arise in QHD equations for BEC and degenerate fermions\(^2\).

Our paper is organized as follows. In Sect.2 we derive QHE’s for BEC from many-particle Schrödinger equation with two- and three-particle interaction. In Sect.3 we calculate quantum stress tensor due to three-particle interaction. In Sect.4 we derive system of QHD equations for system of bosons at non-zero temperature. We consider separate dynamic of two type of bosons, it is particles in BEC state and noncondensate bosons. We obtain the continuity equations and Euler equations for each type of bosons. This equations include interaction between different type of bosons. The two- and three-particle interaction are included too. In Sect.5 from QHD equations finding in sect.3 we obtain NLSE with nonlinearity of third and fifth degree. We give special attention for interaction BEC with temperature excited particles, but part of this presented in the Appendix 3. In Sect.6 we construct equations for BEC including two-particle interaction in FOIR, TOIR and contribution of first term of three-particle interaction. On this background we obtain frequency dependence of element tensor of density of the momentum flux. This tensor is solved.

Atoms in the traps are kept by means interaction of their magnetic moments with trapping magnetic field. In that theory this interaction implicitly take into account by means\(V_{\text{ext}}\). In the magnetic field of third particle may exist effect like Feshbach resonance\(^26\). Due to this effect scattering length of two-particles interaction is changed. It may be considered like mechanism of three particle interaction, which is not provided by binary potentials, only\(^1\).

The concentration of particles in the vicinity of the point \(r\) of the physical space is determined as the operator \(\sum_{i=1}^{N} \delta(r - r_i)\) averaged over the quantum-mechanical states:

\[
n(r, t) = \int dR \sum_{i} \delta(r - r_i) \psi^+(R, t) \psi(R, t),
\]

where \(dR = \prod_{i=1}^{N} dr_i\).

Differentiating this function over time and using the Schrödinger equation with the Hamiltonian\(^3\), we derive the continuity equation, in which the current density vector appears in the form:

\[
j^\alpha(r, t) = \int dR \sum_{i} \delta(r - r_i) \frac{1}{2m_i} \times \left( (\hat{p}_i^\alpha \psi)^+(R, t) \psi(R, t) + \psi^+(R, t) (\hat{p}_i^\alpha \psi)(R, t) \right).
\]

The momentum balance equation for the system of particles under consideration is obtained similarly, i.e., via differentiation of current density\(^4\) and applying the Schrödinger equation\(^3\) and\(^5\). As a result, we obtain:

\[
m \partial_t j^\alpha(r, t) + \partial_\beta \Pi^{\alpha\beta}(r, t)
\]

\[
= - \int dr' (\nabla^\alpha U(r, r')) n_2(r, r', t)
- \int dr' \int dr'' (\partial^\alpha U(r, r', r'')) n_3(r, r', r'', t)
- n(r, t) \nabla^\alpha V_{\text{ext}}(r, t).
\]

In the momentum balance equation, \(\Pi^{\alpha\beta}(r, t)\) is the quantum tensor of the density of the momentum flux. This tensor has the form

\[
\Pi^{\alpha\beta}(r, t) = \int dR \sum_i \delta(r - r_i)
\]
\[ \times \frac{1}{4m_i} \left( \psi^+(R, t)(\hat{p}_i^\alpha \hat{p}_i^\beta \psi)(R, t) + (\hat{p}_i^\alpha \psi)^+(R, t)(\hat{p}_i^\beta \psi)(R, t) + c.c. \right). \] (6)

The interaction between the particles in (5) is expressed through the two-particle probability density \( n_2(\mathbf{r}, \mathbf{r}', t) \) normalized over \( N(N-1) \) and having the form
\[
n_2(\mathbf{r}, \mathbf{r}', t) = \int dR \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \times \delta(\mathbf{r}' - \mathbf{r}_j) \psi^+(R, t)\psi(R, t),
\]
and three-particle probability density \( n_3(\mathbf{r}, \mathbf{r}', \mathbf{r}'', t) \), normalized over \( N(N-1)(N-2) \) and having the form
\[
n_3(\mathbf{r}, \mathbf{r}', \mathbf{r}'', t) = \int dR \sum_{i,j \neq i,k \neq i,j} \delta(\mathbf{r} - \mathbf{r}_i) \times \delta(\mathbf{r}' - \mathbf{r}_j) \delta(\mathbf{r}'' - \mathbf{r}_k) \psi^+(R, t)\psi(R, t). \] (8)

Let us represent the first term in the second member of equation (5), i.e., the density of the interaction force of the particles, in the form
\[
-\frac{1}{2} \int dR \sum_{i,j \neq i} \left( \delta(\mathbf{r} - \mathbf{r}_i) - \delta(\mathbf{r} - \mathbf{r}_j) \right) \times (\nabla_i^\alpha U(\mathbf{r}_{ij}))\psi^+(R, t)\psi(R, t),
\]
which is possible by virtue of symmetry (antisymmetry) of the wave function, and let us proceed in (9) to variables of the center of gravity and variables of the relative distance of the particles:
\[
\mathbf{R}_{ij} = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j), \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \] (10)

Rewrite the three-particle interaction potential in the form:
\[
\partial_\alpha^\beta U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}) = -\partial_\beta^\alpha U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}) - \partial_\beta^\alpha U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}). \] (11)

Second term on the right-hand side eq. (5) is represented in the form
\[
-\frac{1}{9} \int dR \sum_{i,j,k \neq i,j} \left( \delta(\mathbf{r} - \mathbf{r}_i) \partial_\alpha^\beta U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}) + \delta(\mathbf{r} - \mathbf{r}_j) \partial_\alpha^\beta U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}) + \delta(\mathbf{r} - \mathbf{r}_k) \partial_\alpha^\beta U(\mathbf{r}_{ij}, \mathbf{r}_{jk}, \mathbf{r}_{ki}) \right) \psi^+(R, t)\psi(R, t).
\]

Using (11) we obtain
\[
-\frac{1}{9} \int dR \sum_{i,j,k \neq i,j} \left( [\delta(\mathbf{r} - \mathbf{r}_i) - \delta(\mathbf{r} - \mathbf{r}_k)] \partial_\alpha^\beta U_{ijk} \right) \psi^+(R, t)\psi(R, t) + [\delta(\mathbf{r} - \mathbf{r}_j) - \delta(\mathbf{r} - \mathbf{r}_k)] \partial_\beta^\alpha U_{ijk} \psi^+(R, t)\psi(R, t). \] (12)

For the case of three particles we can use variables of center of mass and relative motion too.
\[
\mathbf{R}_{ijk} = \frac{1}{3} (\mathbf{r}_i + \mathbf{r}_j + \mathbf{r}_k), \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad \mathbf{r}_{ik} = \mathbf{r}_i - \mathbf{r}_k, \quad \mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k = \mathbf{r}_{ik} - \mathbf{r}_{ij}. \] (13)

Since the interaction forces between the particles rapidly descend at distances of the order of the interaction radius, small \( |\mathbf{r}_{ij}| \) give the main contribution to the integral (9). Therefore, in expressions (9) and (12) we can replace the multipliers at the interaction potential by their expansion in series by \( r_{ij}^\alpha \). We have concluded that the density of the interaction force for bosons with a short-range interaction potential can be represented in the form of divergence of the tensor field \( \partial_\beta \sigma^{\alpha\beta}(\mathbf{r}, t) \). Here, \( \sigma^{\alpha\beta}(\mathbf{r}, t) \) is the quantum stress tensor due to inter-particle interaction.

Therefore, the momentum balance equation will take the form
\[
\partial_t j^\alpha(\mathbf{r}, t) + \frac{1}{m} \partial_\beta (\Pi^{\alpha\beta}(\mathbf{r}, t) + \sigma^{\alpha\beta}(\mathbf{r}, t)) = -\frac{1}{m} n(\mathbf{r}, t) \nabla^\alpha V_{ext}(\mathbf{r}). \] (14)

Let us now consider the tensor \( \Pi^{\alpha\beta}(\mathbf{r}, t) \) and isolate in it the contributions to the momentum flow density for the convective and thermal motions and the purely quantum part. For this purpose, let us introduce velocities by formulas
\[
\mathbf{v}_i(\mathbf{R}, t) = \frac{1}{m_i} \nabla_i S(\mathbf{R}, t), \quad \psi(\mathbf{R}, t) = a(\mathbf{R}, t) \exp \left( \frac{iS(\mathbf{R}, t)}{\hbar} \right).
\]
Velocity field \( \mathbf{v}(\mathbf{r}, t) \) is determined by the formula
\[
\mathbf{j}(\mathbf{r}, t) = n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t). \] (16)

Then \( \mathbf{u}_i(\mathbf{R}, \mathbf{R}, t) = \mathbf{v}_i(\mathbf{R}, t) - \mathbf{v}(\mathbf{r}, t) \) is the quantum analog of the velocity of thermal motion. Isolating the explicitly thermal motion of the particles with velocities \( \mathbf{u}_i \) and the motion with the velocity \( \mathbf{v}(\mathbf{r}, t) \) in continuity and momentum balance equations (14), we come to the following equations:
\[
\partial_t n(\mathbf{r}, t) + \nabla (n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)) = 0, \quad mn(\mathbf{r}, t)(\partial_t + \mathbf{v} \nabla) v^\alpha(\mathbf{r}, t) + n(\mathbf{r}, t) \nabla^\alpha V_{ext}(\mathbf{r}, t) \]
\[+ \partial_\beta (p^{\alpha\beta}(\mathbf{r}, t) + \sigma^{\alpha\beta}(\mathbf{r}, t) + T^{\alpha\beta}(\mathbf{r}, t)) = 0. \] (18)

In equation (18)
\[
p^{\alpha\beta}(\mathbf{r}, t) = \int dR \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) a^2(\mathbf{R}, t) m_i u_i^\alpha u_i^\beta. \] (19)
This tensor tends to zero along with equality to zero of velocities of thermal motion \( \mathbf{u} \) of the particles. Therefore, it has the meaning of the kinetic pressure.

The tensor \( T^{\alpha \beta}(r, t) \) is proportional to \( h^2 \) and has a purely quantum origin. For the system of numerous noninteracting particles, this tensor is

\[
T^{\alpha \beta}(r, t) = -\frac{\hbar^2}{4m} \left( \partial^\alpha \partial^\beta n(r, t) - \frac{1}{n(r, t)} (\partial^\alpha n(r, t))(\partial^\beta n(r, t)) \right). \tag{20}
\]

Therefore, in this section we have obtained general form of QHD equations for system of particles with short-range interaction.

### III. CALCULATION OF QUANTUM STRESS TENSOR

One have the aim to calculate quantum stress tensor \( \sigma^{\alpha \beta}(r, t) \) it is necessary to write explicit form of \( \sigma^{\alpha \beta}(r, t) \) through wave functions.

The first terms of series in quantum stress tensor \( \sigma^{\alpha \beta}(r, t) \) gives the main contribution. Writing this terms alone, we have

\[
\sigma^{\alpha \beta}(r, t) = \frac{1}{2} \int dR \sum_{i \neq j} \delta(r - R_{ij})
\]

\[
\times \int dr_{12} dr_{13} \left( r_{12}^2 \left( \frac{\partial}{\partial r_{12}^2} (U(r_{12}, r_{13}, |r_{12} - r_{13}|) + r_{13}^2 \left( \frac{\partial}{\partial r_{13}^2} (U(r_{12}, r_{23}, |r_{12} - r_{13}|) \right) \right) + r_{12}^3 \frac{\partial}{\partial r_{12}} (U(r_{12}, r_{23}, |r_{12} - r_{13}|) \right) \right), \tag{22}
\]

where

\[
Tr f(r, r') = f(r, r),
\]

\[
Tr f(r, r', r'') = f(r, r, r).
\]

It is evident that the first term in the second series of \( \sigma^{\alpha \beta}(r, t) \) (related with TPI) for the case of fermions equals to zero. For investigation the influence of TPI on dynamics of fermions one needs to use the next term of this series.

Using knowledge of decomposition formulas of wave function and relations of orthogonality, which are presented in appendix I, we obtain following expression for two-particle concentration

\[
n_2(r, r', t) = n(r, t)n(r', t) + |\rho(r, r', t)|^2 + \varphi(r, r', t), \tag{23}
\]

and for three-particle concentration

\[
n_3(r, r', r'', t) = n(r, t)n(r', t)n(r'', t) + n(r', t)|\rho(r, r', t)|^2 + n(r, t)|\rho(r', r'', t)|^2
\]

\[
+ n(r', t)|\rho(r', r'', t)|^2 + [\rho(r, r', t)|\rho(r', r'', t)|\rho(r'', r, t) + c.c.]
\]

\[
+ n(r', t)\varphi(r, r'', t) + n(r'', t)\varphi(r', r', t) + n(r, t)\varphi(r', r'', t)
\]

\[
+ [\rho(r'', r', t) \sum_g n_g (n_g - 1) \left( \varphi_g(r, t)\varphi_g(r', t)\varphi_g(r'', t) \right) + c.c.]
\]
+ \sum_g n_g (n_g - 1)(n_g - 2) |\varphi_g(r, t)|^2 |\varphi_g(r', t)|^2 |\varphi_g(r'', t)|^2. \tag{24}

In (23), (24) there is not distinction functions describing behavior of BEC and of particles in exited states.

Here

\begin{equation}
\begin{aligned}
n(r, t) &= \sum_g n_g \varphi_g^*(r, t) \varphi_g(r, t), \\
\rho(r, r', t) &= \sum_g n_g \varphi_g^*(r, t) \varphi_g(r', t),
\end{aligned}
\end{equation}

where \( \varphi_g(r, t) \) are the arbitrary single-particle wave functions.

The last two terms in formula (23) represent part of stress tensor arose via exchange interaction. Substituting expressions (23) in (22) for quantum stress tensor of bosons system, taking into account \( T \rho(r, r', t) = n(r, t) \), one obtains formula

\begin{equation}
\begin{aligned}
\sigma_{\alpha\beta}(r, t) &= -\frac{1}{2} \Upsilon \delta_{\alpha\beta} (2n^2(r, t) + \varphi(r, t))
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\chi^{\alpha\beta} &= \frac{1}{6} \int dr_1 dr_2 \left( \sum_{i=1}^{3} \frac{r_{i}^{\alpha} r_{i}^{\beta}}{r_{i}} \partial_{i} + \frac{r_{1}^{\alpha} r_{2}^{\beta}}{r_{1} r_{2}} \partial_{2} + 2 \frac{(r_{1}^{\alpha} - r_{2}^{\alpha})(r_{1}^{\beta} - r_{2}^{\beta})}{|r_{1} - r_{2}|} \partial_{3} \right) \\
&\times U(r_{1}, r_{2}, \sqrt{r_{1}^{2} + r_{2}^{2} + 2r_{1}r_{2} \cos \Omega})
\end{aligned}
\end{equation}

where \( \Omega \) is angle between \( r_{1} \) and \( r_{2} \) and \( \partial_{1}, \partial_{2}, \partial_{3} \) are derivatives of function \( U \) on its arguments. We can see that \( \chi^{\alpha\beta} = \chi^{\beta\alpha} \).

Assuming that the potential satisfies the condition that the quantity \( r^3 U(r) \) tends to zero at \( r \) tending to zero and infinity, for \( \Upsilon \) from (30), by integration by parts, we obtain

\begin{equation}
\begin{aligned}
\Upsilon &= -\int dr U(r)
\end{aligned}
\end{equation}

which coincides with the result for the interaction constant \( g \) found by Gross and Pitaevskii allowing for the sign in (27) \( \Upsilon = -g \).

Taking into account the notions given after formula (62), we obtain the following relation for the quantum tensor of stress of the system of bosons close to the BEC state:

\begin{equation}
\begin{aligned}
\sigma_{B,n}(r, t) &= -\frac{1}{2} \Upsilon \delta_{\alpha\beta} \\
&\times \left( 2n_B(r, t)n_n(r, t) + 2n_n^2(r, t) + \varphi(r, t) \right)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
-\frac{2}{3} \chi^{\alpha\beta} \left( 18n_B(r, t)n_n^2(r, t) + 5n_n(r, t) \varphi(r, t) \\
+ 6n_n^3(r, t) + 5n_B(r, t) \varphi_n(r, t) + \tilde{m}(r, t) \right),
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
\tilde{m}(r, t) &= \sum_g n_g (n_g - 1)(n_g - 2) |\varphi_g(r, t)|^6.
\end{aligned}
\end{equation}

Here, we used the notations \( n_B(r, t) \) for the concentration of particles situating in the BEC state and \( n_n(r, t) \) for the concentration of excited particles. Notation \( g_0 \) designates ground state of system of particles corresponding to BEC.

The stress tensor that depends on inter-particle interactions contains the single-particle functions \( \varphi_g(r, t) \). By these functions, expansion of the unknown N-particles wave function is actually performed. If we neglect the...
inter-particle interaction, the N-particle problem will be reduced to the single-particle one, and the set of functions \( \varphi_g(r, t) \) will be determined by the single-particle Schrödinger equation. In this case, the momentum balance equation will also not contain interactions and, along with the continuity equation, will determine the single-particle wave function in the polar form. Such single-particle functions, which are simultaneously solutions of equations of quantum hydrodynamics of the system of noninteracting particles and the single-particle Schrödinger equation, can be used as the first approximation at calculating of the stress tensor. Thus obtained quantum balance equations will determine the set of field functions that determine the stress tensor. Therefore in this section we have obtained the QHD equations for BEC with two- and three-particle interaction.

IV. EULER EQUATIONS FOR A BOSON SYSTEM WITH NONZERO TEMPERATURE

If we interesting in separate dynamics of BEC and temperature excited bosons we can divide the concentration \( n(r, t) \) and current \( j(r, t) \) on two part

\[
  n(r, t) = n_B(r, t) + n_n(r, t)
\]

and

\[
  j(r, t) = j_B(r, t) + j_n(r, t).
\]

Here we consider a case there are no exchange of particles between BEC and noncondensate component. In this situation we can write the continuity equations for each kinds of particles, for the BEC

\[
  \partial_t n_B(r, t) + \nabla (n_B(r, t)v_B(r, t)) = 0
\]

and, for the noncondensate bosons

\[
  \partial_t n_n(r, t) + \nabla (n_n(r, t)v_n(r, t)) = 0.
\]

In previous section we derive the momentum balance equation for whole system of bosons, i.e. for the mixture of BEC and noncondensate particles. Now we need to obtain momentum balance equations for each kinds of bosons. For separation of contributions of different sorts of bosons we must to consider of derivation of quantum stress tensor \( \sigma^\alpha_{\beta, n} \) and \( \sigma^\alpha_{\beta, ext} \), in more detailed way.

For understanding the detail of evolution of \( j_B(r, t) \) and \( j_n(r, t) \) we need to consider formulas \( \sigma^\alpha_{\beta, B} \) and \( \sigma^\alpha_{\beta, ext} \) in detail. The first multiplier in this formula, which has argument \( (\text{r}, t) \) related to the particle whose motion we consider. Another one particle wave functions are related to the particles that influence on dynamic of considered current. This is give us ability to obtain the separate equation of dynamic atoms in BEC state and noncondensate ones. We consider each term in \( \sigma^\alpha_{\beta, B} \) and \( \sigma^\alpha_{\beta, ext} \). If one-particle wave function with argument \( (\text{r}, t) \) describe BEC state (has subindex "B"), we put this term in momentum balance equation for BEC. In the case one-particle wave function with argument \( (\text{r}, t) \) describe noncondensate state we put this term in momentum balance equation for noncondesate particles. In this way we get following expressions for quantum stress tensors of atoms in both BEC and noncondensed states.

The corresponding momentum balance equation of the quantum hydrodynamics for BEC alone takes the form:

\[
  mn(r, t)(\partial_t v^\alpha(r, t) + v^\beta(r, t)\nabla^\beta v^\alpha(r, t)) + \partial^\beta p^\alpha(r, t)
  \]

\[
  - \frac{\hbar^2}{2m} n(r, t)\partial_\alpha \frac{\nabla n(r, t)}{\sqrt{n(r, t)}} = \nabla n(r, t)\partial^\alpha n(r, t)
  \]

\[
  - 2\chi^{\alpha\beta} n^2(r, t)\partial^\beta n(r, t) = -n(r, t)\nabla^\alpha V_{\text{ext}}(r, t).
\]

Therefore in this section we have obtained the QHD equations for BEC with two- and three-particle interaction.

\[
  \sigma^\alpha_{\beta, B}(r, t) = -\frac{1}{2}\chi^{\alpha\beta} n_B^2(r, t) - \frac{2}{3}\chi^{\alpha\beta} n_n^3(r, t).
\]
\[ \sigma_B^{\alpha\beta} = -\frac{1}{2} \Upsilon \delta^{\alpha\beta} (n_B n_n + \phi_B) \]
\[ -\frac{2}{3} \chi^{\alpha\beta} (6n_B n_n^2 + 4n_n \phi_B + n_B \phi_n + \hat{m}_B) \] (40)

and
\[ \sigma_n^{\alpha\beta} = -\frac{1}{2} \Upsilon \delta^{\alpha\beta} (n_B n_n + 2n_n^2 + \phi_n) \]
\[ -\frac{2}{3} \chi^{\alpha\beta} (6n_n^3 + 12n_B n_n^2 + 5n_n \phi_n + 4n_B \phi_n + n_n \phi_B + \hat{m}_n). \] (41)

In appendix 2 we calculate \( \phi_n \) and \( \hat{m}_n \) and receive
\[ \phi_n = 0, \quad \hat{m}_n = 0. \]

From formulas (33) and (34) we see \( \phi_B(r, t) = n_B^2(r, t) \) and \( \hat{m}_B(r, t) = n_B^3(r, t) \). In this case we have \( \sigma_B^{\alpha\beta} \) and \( \sigma_n^{\alpha\beta} \) has more simple form and expressed in terms of \( n_B \) and \( n_n \).

Obtained expression (28) for \( \sigma^{\alpha\beta}(r, t) \) in the approximation under consideration should be substituted into momentum balance equation [18].

Using this results and formulas (33), (34) we can rewrite equations (40) and (41) in the form
\[ mn_B(r, t) (\partial_t v_B^\alpha(r, t) + v_B^\beta(r, t) \nabla^\beta v_B^\alpha(r, t)) \]
\[ -\frac{\hbar^2}{2m} n_B(r, t) \partial_\alpha \frac{\Delta \sqrt{n_B(r, t)}}{\sqrt{n_B(r, t)}} \]
\[ = -n_B(r, t) \nabla^\alpha V_{ext}(r, t) + \frac{1}{2} \Upsilon \partial^\alpha (n_B n_n + n_{BEC}^2) \]
\[ + \frac{2}{3} \chi^{\alpha\beta} \partial^\beta \left( 6n_B n_n^2 + 4n_n n_{BEC}^2 + n_{BEC}^3 \right) \] (42)

and
\[ mn_n(r, t) (\partial_t v_n^\alpha(r, t) + v_n^\beta(r, t) \nabla^\beta v_n^\alpha(r, t)) \]
\[ -\frac{\hbar^2}{2m} n_n(r, t) \partial_\alpha \frac{\Delta \sqrt{n_n(r, t)}}{\sqrt{n_n(r, t)}} \]
\[ = -n_n(r, t) \nabla^\alpha V_{ext}(r, t) + \frac{1}{2} \Upsilon \partial^\alpha (n_B n_n + 2n_n^2) \]
\[ + \frac{2}{3} \chi^{\alpha\beta} \partial^\beta \left( 6n_n^3 + 12n_B n_n^2 + n_n n_{BEC}^2 \right). \] (43)

where evident form of interaction constants \( \Upsilon \) and \( \chi^{\alpha\beta} \) are presented by formulas (30) and (31) correspondingly.

**V. DERIVATION OF NLS EQUATION FOR ZERO TEMPERATURE**

There is used the NLSE along with QHD equations in the literature. From QHD equations one can obtain the NLSE [7, 31]. Macroscopic wave function may be defined via hydrodynamic variables: the concentration of particle number \( n(r, t) \) and potential of velocities field \( \phi(r, t) \). For determination of evolution of velocity field the integral of Cauchy-Lagrange following of the momentum balance equation is used.

Therefore, for eddy-free motion \( v(r, t) = \nabla \phi(r, t) \) and barotropicity condition in terms of tensor field
\[ \frac{\partial^\beta \mu^{\alpha\beta}(r, t)}{mn(r, t)} = \partial^\beta \mu^{\alpha\beta}(r, t), \] (44)

where \( \mu^{\alpha\beta}(r, t) \)-tensor of chemical potential, momentum balance equation has the first tensor integral:
\[ \delta^{\alpha\beta} \partial_t \phi(r, t) + \frac{1}{2} \delta^{\alpha\beta} v^2(r, t) + \mu^{\alpha\beta}(r, t) \]
\[ - \frac{1}{m} \Upsilon \delta^{\alpha\beta} n(r, t) - \frac{1}{m} \chi^{\alpha\beta} n^2(r, t) + \frac{1}{m} V_{ext}(r, t) \delta^{\alpha\beta} \]
\[ - \frac{\hbar^2}{2m^2} \delta^{\alpha\beta} \frac{\Delta \sqrt{n(r, t)}}{\sqrt{n(r, t)}} = (const)^{\alpha\beta}. \] (45)

Continuity equation (17) and momentum balance equation (37) can be associated with the equivalent one-particle Schrödinger equation for some effective wave function \( \Phi(r, t) \). Represent this function in the form:
\[ \Phi(r, t) = \sqrt{n(r, t)} \exp \left( \frac{i}{\hbar} m \phi(r, t) \right). \] (46)

Differentiating it by time and using equations (17), (45) we take equivalent one-particle Schrödinger equation:
\[ i\hbar \delta^{\alpha\beta} \partial_t \Phi(r, t) = \left( -\delta^{\alpha\beta} \frac{\hbar^2 \nabla^2}{2m} + \mu^{\alpha\beta}(r, t) + V_{ext}(r, t) \delta^{\alpha\beta} \right) \Phi(r, t), \]
\[ - \frac{\Upsilon}{m} |\Phi(r, t)|^2 - \chi^{\alpha\beta} |\Phi(r, t)|^4 \Phi(r, t), \] (47)

under conditions \( \chi^{\alpha\beta} = \chi \delta^{\alpha\beta}, \mu^{\alpha\beta}(r, t) = p(r, t) \delta^{\alpha\beta} \) isotropic kinetic pressure and barotropic condition:
\[ \frac{\nabla p(r, t)}{mn(r, t)} = \nabla \mu(r, t), \]
\[ \frac{\nabla^2 p(r, t)}{m^2 n(r, t)} + \mu(r, t) \]

where \( p(r, t) \)-chemical potential, we obtain:
\[ \frac{\nabla^2 \mu(r, t)}{m^2 n(r, t)} + \mu(r, t) \]
+ V_{ext}(r, t) - \Upsilon |\Phi(r, t)|^2 - \chi |\Phi(r, t)|^4 \right) \Phi(r, t). \tag{49}

That is well known Gross-Pitaevskii equation \cite{35, 38–40}, with the nonlinearity of fifth degree \cite{1, 2}, which arise due to TPI.

The isotropic constant of three-particle interaction is:

\[
\chi \equiv -{1 \over 3} \int d\mathbf{r}_1 d\mathbf{r}_2 \left( (r_1 \partial_{r_1} + |\mathbf{r}_1 - \mathbf{r}_2| \partial_3) \times U(r_1, r_2, \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos \Omega}) \right). \tag{50}
\]

Wave function $\Phi(r, t)$ is normalized with condition:

\[
\int d\mathbf{r} \Phi^*(\mathbf{r}, t) \Phi(\mathbf{r}, t) = N,
\]

where $N$ is number of particles in the system.

Within this approximation the equation of momentum balance has form:

\[
\begin{align*}
\rho \partial_t v^\alpha(r, t) + & \frac{1}{2} m \partial^\alpha v^2(r, t) + m \partial^\alpha \mu(r, t) \\
& - \frac{\hbar^2}{2m} \partial^\alpha \Delta \sqrt{n(r, t)} - \Upsilon \partial^\alpha n(r, t) \\
& - \chi \partial^\alpha n^2(r, t) = -\partial^\alpha V_{ext}(r, t). \tag{51}
\end{align*}
\]

If system of particles is dense (e.g. Bose liquid) then two more particles can interaction simultaneously. Basic contribution in interaction three and more particles arise from binary interactions $U_{ij}$. Part of interactions in macroscopic system of particles which is described via binary potential is the origin of first series in \cite{21, 7} and, consequently, this part is also the origin of the term which is proportional $\Upsilon$ in \cite{49}.

\section*{VI. LINEAR WAVE DISPERSION IN BEC WITH TPI

\section*{ZERO TEMPERATURE LIMIT}

In this section we have the aim both to consider dispersion properties of linear wave and to compare two corrections to GP approximation. For this aim we write system of quantum hydrodynamic equations in third order to the interaction radius, for only binary interaction which was given in the work \cite{7}. Taking in account binary and three-particle interaction and including results of the previous section we obtain equation:

\[
\begin{align*}
\rho \partial_t v^\alpha(r, t) + & \frac{1}{2} mn(r, t) \partial^\alpha v^2(r, t) + mn \partial^\alpha \mu(r, t) \\
& - \frac{\hbar^2}{2m} n(r, t) \partial^\alpha \Delta \sqrt{n(r, t)} - \frac{1}{2} \Upsilon \partial^\alpha n^2(r, t) - \chi \partial^\alpha n^3(r, t) \\
& \quad - \frac{1}{2} \Upsilon \partial^\alpha n^2(r, t) = -n(r, t) \nabla^\alpha V_{ext}(r, t). \tag{52}
\end{align*}
\]

where

\[
\Upsilon_2 = \frac{\pi}{30} \int d\mathbf{r} (\mathbf{r})^3 \partial U(\mathbf{r}) \partial \mathbf{r}.
\]

Following to the article \cite{41} we redefined $\Upsilon_2$, in compare with the paper \cite{7}.

Let us consider the eigenmodes, which can propagate in one dimensional geometry, based on the Eqs. of continuity \cite{17} and momentum balance \cite{52}.

We consider the small perturbation of equilibrium state like

\[
n = n_0 + \delta n, \quad v^\alpha = 0 + v^\alpha, \tag{53}
\]

Substituting this relations into system of equations \cite{17} and \cite{53} and neglecting nonlinear terms, we obtain a system of linear homogeneous equations in partial derivatives with constant coefficients. Passing to the following representation for small perturbations $\delta f$

\[
\delta f = f(\omega, \mathbf{k}) \exp(-i\omega - i\mathbf{k} \mathbf{r}) \tag{54}
\]

yields the homogeneous system of algebraic equations. The magnitude of concentration of BEC is assumed to have a nonzero value. Expressing all the quantities entering the system of equations in terms of the concentration of BEC, we come to the dispersion equation for elementary excitations

\[
\omega^2 = \left( \frac{\hbar^2}{4m^2} + \frac{n_0 \Upsilon_2}{m} \right) k^4 - \left( \frac{\Upsilon n_0}{m} + \frac{2\chi n_0^2}{m} \right) k^2. \tag{55}
\]

In the absence of $\Upsilon_2$ the result \cite{55} is in accordance with the real part of solution $\omega(\mathbf{k})$ obtained in \cite{25}. If we compare terms which proportional to $\chi$ and $\Upsilon_2$ we can see functional dependence $\omega(\mathbf{k})$ for the cases three-particle interaction and binary interaction in third order on interaction radius. When we calculate binary interaction in third order to interaction radius there arise linear dependence on $n_0$. Term proportional to $k^2$ in \cite{55}, las in GP approxi- mation, is linear on concentration $n_0$. There is additional dependence on $n_0^2$ in \cite{55} at the expense of three-particle interaction.

\section*{VII. LINEAR WAVE DISPERSION IN BEC WITH TPI

\section*{NONZERO TEMPERATURE LIMIT}

In this section we consider a Bose particle system contained particles being in the BEC state and noncondensate atoms. We interested the dispersion of small amplitude elementary excitation in such system. For this aim we use system of equations \cite{39, 38, 42} and \cite{43}. Equilibrium state is described by constant values of concentrations of each sorts of bosons $n_{0D}$ and $n_{0N}$. The flows in equilibrium
state are absent $v_B = v_n = 0$. For small perturbations of equilibrium state we have

$$n_i = n_0 + \delta n_i, \quad v_i^\alpha = 0 + v_i^\alpha, \quad (56)$$

where $i$-mark $B$ or $n$. Using formula (54) for small perturbations we get the homogeneous system of algebraic equations. A condition of existence on nontrivial solution lead to dispersion equation. A solution of dispersion equation has form

$$\omega^2 = \frac{\hbar^2 k^4}{4 m^2} - \frac{\Upsilon k^2}{4 m} (3 n_{0B} + 5 n_{0n}) - \frac{4}{3m} k^\alpha k^\beta \chi^{\alpha\beta} (n_{0B}^2 + 6 n_{0n}^2 + 8 n_{0B} n_{0n})$$

$$\pm \frac{1}{2m} \left( \frac{\Upsilon^2 k^4}{4} (n_{0B}^2 + 9 n_{0n}^2 - 2 n_{0B} n_{0n}) + \frac{16}{9} (k^\alpha k^\beta \chi^{\alpha\beta})^2 (n_{0B}^4 + 36 n_{0n}^4 + 124 n_{0B}^2 n_{0n}^2 + 240 n_{0B} n_{0n}^3 - 8 n_{0B} n_{0n}) \right) \right)^{1/2}. \quad (57)$$

In the case $\chi^{\alpha\beta} = \chi \delta^{\alpha\beta}$ we have $k^\alpha k^\beta \chi^{\alpha\beta} = k^2 \chi$ and we can make $k^2$ out of the square root. Then, all terms, apart from first term, are proportional to $k^2$. A coefficient at $k^2$ is a square of velocity of sound. Due to two sign at the square root we have two sound velocity, for the low temperature first and second sounds.

The first sound is a usual sound, one take place in classic gases and in Bose system far of BEC condition. For the low temperature quantum velocity of first sound coincide with Bogoliubov’s, our result account three particle interaction and influence of noncondensate particles.

In the absence of TPI $k^\alpha k^\beta \chi^{\alpha\beta} = 0$, for repulsive two-particle interaction $\Upsilon < 0$, from (57) we have

$$\omega^2 = \frac{\hbar^2 k^4}{4 m^2} + \frac{\Upsilon|k^2|}{4 m} \times \left( 3 n_{0B} + 5 n_{0n} \right)$$

$$\pm \sqrt{n_{0B}^2 + 9 n_{0n}^2 - 2 n_{0B} n_{0n}} \right) \quad (58)$$

where the quantity in bracket is positive.

For the case $\Upsilon = 0$ and $k^\alpha k^\beta \chi^{\alpha\beta} < 0$ we obtain

$$\omega^2 = \frac{\hbar^2 k^4}{4 m^2} + \frac{k^2 |k^\alpha k^\beta \chi^{\alpha\beta}|}{3m} \left( 2(n_{0B}^2 + 6 n_{0n}^2 + 8 n_{0B} n_{0n}) \right)$$

$$\pm \sqrt{n_{0B}^4 + 36 n_{0n}^4 + 124 n_{0B}^2 n_{0n}^2 + 240 n_{0B} n_{0n}^3 - 8 n_{0B} n_{0n}} \right) \quad (59)$$

where the quantity in bracket is positive.

**VIII. CONCLUSION**

In this article we gave derivation of the NLSE for BEC with cubic and quintic nonlinearities from microscopic quantum theory. For the derivation we used method of quantum hydrodynamics. In original Schrödinger equation we took into account two-particle and three-particle interactions. In the article, arising force field of whole accounted interaction in the form of divergence of tensor field, is demonstrated. We obtained the expression of quantum stress tensor by means of microscopic wave function. Using its representation we derived equation of state for boson systems in BEC state including two- and three-particle interaction. The momentum balance equation obtained coincides with the analogous NLSE containing nonlinearities of third and fifth degrees. Method of finding of NLSE for the wave function in the medium for the case three-particle interaction are developed.
tained NLSE coincides with well-known NLSE with cubic and quintic nonlinearities. Explicit form for the constant of three-particle interaction was obtained. In particular, here represent derivation of GP equation from many-particle Schrödinger equation. Tensor form of constant of three-particle interaction is shown.

From derivation of QHD equations including three-particle interaction, obtained in this article, and from derivation QHD equations including TOIR, obtained in [7], we can see terms due to TOIR and TPI additional.

In this article, frequency dependence of elementary excitation on wave vector was calculated for the case including FOIR, TOIR and three particle interaction. Comparison of contributions from TOIR and three particle interaction is discussed.

A special attention we got for influence of the temperature on dynamic of Bose particles. We made a microscopic derivation of two-fluid hydrodynamic equation. We confine oneself by continuity equations and momentum balance equations for each type of bosons, i.e. particles in BEC state and noncondensate particles, due to two- and three-particle interaction, in first order on interaction radius. In described approximation we studied a spectrum of elementary excitations. We obtain the dispersion for two waves. One of them it is generalization of Bogoliubov’s mode, another it is low temperature approximation for the usual sound.

\[ \langle \mathbf{r}, \mathbf{r}', R_{N-2}, t | n_1, n_2, \ldots \rangle = \sum_f \sqrt{\frac{n_f}{N}} \langle \mathbf{r}, t | f \rangle \langle \mathbf{r}', R_{N-2}, t | n_1, \ldots (n_f - 1), \ldots \rangle = \]

\[ = \sum_f \sum_{f', f \neq f} \sqrt{\frac{n_f}{N}} \sqrt{\frac{n_{f'}}{N-1}} \langle \mathbf{r}, t | f \rangle \langle \mathbf{r}', t | f' \rangle \times \]

\[ \times \langle R_{N-2}, t | n_1, \ldots (n_{f'} - 1), \ldots (n_f - 1), \ldots \rangle \]

\[ + \sum_f \sqrt{\frac{n_f(n_f - 1)}{N(N-1)}} \langle \mathbf{r}, t | f \rangle \langle \mathbf{r}', t | f \rangle \langle R_{N-2}, t | n_1, \ldots (n_f - 2), \ldots \rangle. \] (62)

Where \( \langle \mathbf{r}, t | f \rangle = \varphi_f(\mathbf{r}, t) \) — one-particle wave function.

The first term in formula (62) represents the particles situating in two different quantum states, while the second term is referred to particles in the same quantum state.

ACKNOWLEDGMENTS

The author wish to thank L.S. Kuz’menkov for discussion of the results obtained.

APPENDIX 1

For two-particle density of probability in correspondence with the definition (7) we have:

\[ n_2(\mathbf{r}, \mathbf{r}', t) = N(N-1) \int dR_{N-2} \langle n_1, n_2, \ldots | \mathbf{r}, \mathbf{r}', R_{N-2}, t \rangle \times \langle \mathbf{r}, \mathbf{r}', R_{N-2}, t | n_1, n_2, \ldots \rangle, \] (60)

where \( dR_{N-2} = \prod_{k=3}^{N} d\mathbf{r}_k \).

For three-particle density of probability we have at the same way:

\[ n_3(\mathbf{r}, \mathbf{r}', \mathbf{r}'', t) = N(N-1)(N-2) \times \]

\[ \int dR_{N-3} \langle n_1, n_2, \ldots | \mathbf{r}, \mathbf{r}', \mathbf{r}'', R_{N-3}, t \rangle \times \]

\[ \times \langle \mathbf{r}, \mathbf{r}', \mathbf{r}'', R_{N-3}, t | n_1, n_2, \ldots \rangle, \] (61)

where \( dR_{N-3} = \prod_{k=4}^{N} d\mathbf{r}_k \).

Using knowledge of decomposition formulas of wave function \( \langle \mathbf{r}, \mathbf{r}', R_{N-2}, t | n_1, n_2, \ldots \rangle \) in formulas (33) and (34) we use following decomposition formulas [27].

Therefore, for the particles in the BEC state, it is sufficient to take into account the second term in formula (62). In consideration of the system of bosons with the temperature differing from zero, where the certain number of the parti-
particles is out of the condensate, the first summand of formula (62) gives the contribution both in the case of interaction of excited particles with each other and in the case of their interaction with the particles appearing in the BEC state. In this case, the second term of formula (62) gives the contribution in the interaction both between the particles appearing in the BEC state and between the excited particles appearing in the same quantum state.

Moreover using decomposition formulas of wave function, analogously with previous case, for three particle we have:

\[
\langle r, r', r'', R_{N-3}, t | n_1, n_2 \ldots \rangle =
\sum_f \sum_{f' \neq f} \sum_{f'' \neq f', f''} \sqrt{\frac{n_f}{N}} \sqrt{\frac{n_{f'}}{N-1}} \sqrt{\frac{n_{f''}}{N-2}} \langle r, t | f \rangle \langle r', t | f' \rangle \langle r'', t | f'' \rangle \times
\]

\[
\langle R_{N-3}, t | n_1, \ldots (n_{f'} - 1), \ldots (n_f - 2), \ldots \rangle
\]

\[
+ \sum_f \sum_{f' \neq f} \sqrt{\frac{n_f(n_f - 1)n_{f'}}{N(N-1)(N-2)}} \langle r, t | f \rangle \langle r', t | f' \rangle \langle r'', t | f'' \rangle \times
\]

\[
\langle R_{N-3}, t | n_1, \ldots (n_{f'} - 1), \ldots (n_f - 2), \ldots \rangle
\]

\[
+ \sum_f \sqrt{\frac{n_f(n_f - 1)(n_f - 2)}{N(N-1)(N-2)}} \langle r, t | f \rangle \langle r', t | f' \rangle \langle r'', t | f'' \rangle 
\times
\]

\[
\langle R_{N-3}, t | n_1, \ldots (n_{f'} - 1), \ldots (n_f - 2), \ldots \rangle,
\]

The first term in formula (64) represents the particles situating in three different quantum states. The second and third terms in (64) represents case when two particles situating in one quantum state and one particle situating in another quantum state. The last term represents the three particles situating in the same quantum state. For the particles in the BEC state, it is sufficient to take into account the last term in (64). For the consideration of the system of bosons with the temperature different from zero, where the significant number of the particles is out of the condensate or in the case of the temperature when there are no macroscopic number of particles in BEC state, the terms in formula (64) has the following influence. The first summand of formula (64) gives the contribution both in the case of interaction of excited particles with each other and in the case of interaction of two excited particles with the particles appearing in the BEC state. The second and the third summands represent case which one from two states may be BEC state, but another is always excited state. The last term of formula (64) gives the contribution in the interaction both between the particles appearing in the BEC state and between the excited particles appearing in the same quantum state.

Using relation of orthogonality

\[
\langle n_1, \ldots (n_{f'} - 1), \ldots (n_f - 1), \ldots | n_1, \ldots, n_{f'} - 1, \ldots, n_q - 1, \ldots \rangle =
\delta(f - q)\delta(f' - q')\delta(f'' - q') + \delta(f - q)\delta(f' - q'')\delta(f'' - q') + 
\]

\[
+ \delta(f - q')\delta(f - q')\delta(f'' - q'') + \delta(f - q)\delta(f - q')\delta(f'' - q') + 
\]

and

\[
\langle n_1, \ldots (n_{f'} - 1), \ldots (n_f - 1), \ldots | n_1, \ldots, n_{f'} - 1, \ldots, n_q - 1, \ldots \rangle =
\delta(f - q)\delta(f' - q') + \delta(f - q')\delta(f' - q),
\]

we obtain the expression for two- and three-particle con-
centation presented in Sect. 3.

**APPENDIX 2**

Details of calculations of \( \psi_n(r, t) \) and \( \hat{m}_n(r, t) \) are presented here. The quantity \( \psi_n(r, t) \) and \( \hat{m}_n(r, t) \) reads

\[
\varphi_n(r, t) = \sum_{g \neq g_0} n_g(n_g - 1)|\varphi_g(r, t)|^4
\]

and

\[
\hat{m}_n(r, t) = \sum_{g \neq g_0} n_g(n_g - 1)(n_g - 2)|\varphi_g(r, t)|^6.
\]

where \( g_0 \) is a label for state with lowest energy and coincide to the BEC state.

We can calculate \( \varphi_n(r, t) \) and \( \hat{m}_n(r, t) \) approximately.

We use method of calculation of correlations described in papers [7] and [31]. At the first step we suppose the particles is free. In this case motion of particles described by plane-wave.

For the case of plane waves

\[
\varphi_p(r, t) = \frac{1}{\sqrt{V}} \exp\left(-\frac{i}{\hbar}(\varepsilon_p t - \mathbf{p} \cdot \mathbf{r})\right),
\]

where \( \varepsilon_p \) is the energy of wave with momentum \( \mathbf{p} \). In this case we have \( |\varphi_p(r, t)| = 1/\sqrt{V} \). Consequently, for \( \varphi_n(r, t) \), in the many-particle system we get

\[
\varphi_n(r, t) = \frac{1}{V^{1/2}} \sum_g n_g(n_g - 1)
\]

\[
= \frac{g m^{3/2}}{\sqrt{2\pi^2 \hbar^3 V}} \int \sqrt{\varepsilon} d\varepsilon n_\varepsilon(n_\varepsilon - 1)
\]

\[
\simeq \frac{g m^{3/2}}{\sqrt{2\pi^2 \hbar^3 V}} \int \sqrt{\varepsilon} d\varepsilon n_\varepsilon^2.
\]

The quantity \( \varphi_n(\hat{m}_n) \) expressed in terms of sum of squares (cubes) of occupation numbers of quantum states [66], [68]. Since sum of squares (cubes) is much less than square (cube) of sum, we obtain \( \varphi_n \ll n_n^2, \hat{m}_n \ll n_n^3 \).

Thus, we neglect by \( \varphi_n \) and \( \hat{m}_n \) in equation of quantum hydrodynamics [41]. In formal way, we can write

\[
\varphi_n = 0, \quad \hat{m}_n = 0.
\]

**APPENDIX 3**

Analogously to section (5) we can enter the macroscopic one-particle wave function for each subsystem of bosons, for BEC

\[
\Phi_B(r, t) = \sqrt{n_B(r, t)} \exp\left(\frac{i}{\hbar} m \phi_B(r, t)\right)
\]

and for noncondensate bosons

\[
\Phi_n(r, t) = \sqrt{n_n(r, t)} \exp\left(\frac{i}{\hbar} m \phi_n(r, t)\right).
\]

Using equations (38), (39), (42) and (43) we have gotten NLSEs for each type of bosons. For particles in BEC state the NLSE arise in the form

\[
\hbar \partial_t \Phi_B(r, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + \mu_B(r, t) + V_{\text{ext}}(r, t) - \Phi_B \left( n_B + \frac{1}{2} n_n + \frac{1}{2} \int n_B \, dn_B \right) \right).
\]
and, for noncondensate component NLSE reads

\[
\imath \hbar \partial_t \Phi_n(r, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + \mu_n(r, t) + V_{ext}(r, t) - \Gamma \left( \frac{1}{2} n_B + 2n_n + \int \frac{n_B}{2n_n} dn_n \right) \right) \Phi_n(r, t),
\]

where

\[
n_B = |\Phi_B(r, t)|^2
\]

and

\[
n_n = |\Phi_n(r, t)|^2.
\]

Wave functions \(\Phi_B(r, t)\) and \(\Phi_n(r, t)\) are normalized with conditions:

\[
\int dr \Phi_B^*(r, t) \Phi_B(r, t) = N_B,
\]

\[
\int dr \Phi_n^*(r, t) \Phi_n(r, t) = N_n,
\]

where \(N_B\) and \(N_n\) are number of particles in the BEC condition and in the noncondensate particles.