Carrollian manifolds and null infinity: a view from Cartan geometry

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Abstract
We discuss three different (conformally) Carrollian geometries and their relation to null infinity from the unifying perspective of Cartan geometry. Null infinity per se comes with numerous redundancies in its intrinsic geometry and the two other Carrollian geometries can be recovered by making successive choices of gauge. This clarifies the extent to which one can think of null infinity as being a (strongly) Carrollian geometry and we investigate the implications for the corresponding Cartan geometries. The perspective taken, which is that characteristic data for gravity at null infinity are equivalent to a Cartan geometry for the Poincaré group, gives a precise geometrical content to the fundamental fact that ‘gravitational radiation is the obstruction to having the Poincaré group as asymptotic symmetries’.

Keywords: null infinity, Cartan geometry, Carroll, conformal geometry, tractors, classical gravity, asymptotic symmetries

(Some figures may appear in colour only in the online journal)

1. Introduction/summary
Carrollian geometries naturally emerge as the ultra-relativistic1 limit $c \to 0$ of pseudo-Riemannian manifolds [1, 2]. This parallels the appearance of Galilean geometry in the non-relativistic limit $c \to \infty$ and in some sense these are dual to each others [3]. Loosely speaking, while the Galilean limit forces null cones to spread out into degenerate planes of constant absolute time, in the Carrollian limit null cones collapse into degenerate lines of constant absolute space (figure 1).

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1If $v$ is the typical velocity of the system, one can easily argue that ‘ultra-relativistic’ should be kept for situations where $c/v \to 1$ (resulting in conformal geometry where only the causal structure is preserved). From this perspective the Carrollian limit $c/v \to 0$ should perhaps be called the ultra-local limit.
Accordingly, the fundamental building block of any Carrollian geometry consists of a manifold \( \mathcal{I} \) foliated by lines. The space of lines \( \Sigma \) typically is a manifold and this results in a fibre bundle \( \mathcal{I} \to \Sigma \). One obtains a Carrollian geometry \( (\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu}) \) if the foliation \( \mathcal{I} \to \Sigma \) is generated by a preferred vector field \( n^\mu \) and is complemented by a degenerate metric \( h_{\mu\nu} \) with one-dimensional kernel spanned by \( n^\mu \).

This type of structure naturally appears in many different contexts of physical interest. They typically result of a limiting process or appear as the geometry induced on a null hypersurface. Applications include Hamiltonian analysis [4], Carrollian particles [5, 6], ultra-relativistic fluid mechanics [7–13], cosmology [14], Carrollian [15–19] and conformally Carrollian [20–23] field theories, higher-spin [24–26], super-symmetric [27–32] and non-commutative [33] extensions, null hypersurfaces and isolated horizons [34–45], geometry of space/time-like infinities [46–48] and, finally, geometry of null infinity which is the main concern of this article.

It is a well-known fact that the conformal boundary of an asymptotically flat spacetime, which we will generically refer to as 'null infinity', is a (conformal) Carrollian manifold. There is in fact a precise sense in which flat limits of an asymptotically AdS spacetime correspond to ultra-relativistic contractions of the boundary [49].

The gravitational characteristic data however induce on null infinity much more than a mere Carrollian geometry [50–52]. In four dimensions this geometry is especially rich and e.g. is closely related to asymptotic twistors [53, 54] and Newman’s H-space [55, 56].

Depending on the perspective taken, especially the extent to which one wishes to preserve covariance and conformal invariance, there are various ways to describe the induced geometry. The purpose of this article is to discuss these in a coherent form emphasising their differences and relationships by framing them in the context of Cartan geometry.
Cartan geometry is a powerful organising tool in this context because the different views on the geometry of null infinity can in fact be captured by various homogeneous spaces which are all (conformal) Carrollian geometries with topology $\mathbb{R} \times S^2$. Cartan geometry amounts to making these models curved and thus capture the corresponding local geometry (this is closely related to the ‘gauging’ procedure common in physics, see [57, 58] for pedagogical physicist-oriented introductions and [59, 60] for standard references). These homogeneous models are simple enough to be presented in this introduction; the detailed discussion of the corresponding curved geometries will form the core of this article.

Before discussing null infinity, it is instructive to consider the simpler model given by three-dimensional homogeneous Carrollian–de Sitter spacetime [2, 61]:

$$\text{Carr}_{\text{ds}}^{(3)} := \text{Carr}_{\text{ds}}(3)/\text{ISO}(2).$$  \hspace{1cm} (1.1)

This is an homogeneous space for the Carroll–de Sitter group $\text{Carr}_{\text{ds}}^{(3)} := \mathbb{R}^3 \rtimes \text{SO}(3)$ (with the stabiliser $\text{ISO}(2)$ corresponding to a combination of rotations and Carrollian boosts). It can be obtained as the ultra-relativistic limit of $dS(3)$, and therefore by construction is a Carrollian geometry. It is also equipped with a preferred compatible connection $\nabla$ and altogether this defines, in the terminology of [3], a strongly Carrollian geometry $\mathcal{F} \to \Sigma, [n^\mu, h_{\mu\nu}, \nabla]$. In fact, as we will show in the first section of this article, strongly Carrollian geometries are in one-to-one correspondence with Cartan geometry modelled on (1.1). What is more, we will obtain a result which applies uniformly to all dimensions $n \geq 2$ and all possible cosmological constant i.e. Carroll–de Sitter, Carroll and Carroll–anti de Sitter. Taking the cosmological constant to zero one recovers the results from [62], restricting to dimension $2 + 1$ those of [24, 28, 63] and imposing a flatness condition the ones from [64].

The model for null infinity is, without surprise, an homogeneous space for the (connected component of the) Poincaré group $\text{ISO}(3, 1) = \mathbb{R}^4 \rtimes \text{SO}(3, 1)_0$,

$$\mathcal{F}^{(3)} := \mathbb{R}^4 \rtimes \text{SO}(3, 1)_0/\mathbb{R}^3 \rtimes (\mathbb{R}^* \rtimes \text{ISO}(2)).$$  \hspace{1cm} (1.2)

The stabiliser now is a combination of rotations, Carrollian boosts and dilatations, corresponding to $\mathbb{R}^* \rtimes \text{ISO}(2)$, in semi-direct product with Carrollian special conformal transformations $\mathbb{R}^3$. There is an essential difference with the previous model: the appearance of $\text{SO}(3, 1)_0$, the conformal group of the two-sphere, means that this second model is only equipped with a conformal Carrollian geometry [65]. This well-known fact is related to the appearance of the BMS group [66–69] as symmetries of asymptotically flat spacetimes. On top of being conformally Carrollian, the model for null infinity (1.2) is equipped with a Poincaré operator [70]: this is a differential operator $\mathcal{P}$ acting on densities and generalizing the Möbius operator of two-dimensional conformal geometry [71, 72]. This operator essentially corresponds to the possibility of defining (generalised) good-cut equations [55, 73]. In the spirit of [3] the data $\mathcal{F} \to \Sigma, [n^\mu, h_{\mu\nu}, \mathcal{P}]$ will be called strongly conformally Carrollian. As we will recall in our second section, strongly conformally Carrollian geometries are in one-to-one correspondence [70] with Cartan geometry modelled on (1.2). We will in fact recover these results from a more general (as compare to [70]) first order formalism that will encompass previous results and allows for a closer comparison with GHP-type formalisms [74–76] and closely related Ehresmann connections [10–12, 68].

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2In this introduction we restrict to three dimensions for concreteness but in the core of the article we will treat uniformly all dimension $n \geq 2$ as well as all the possible cosmological constant for the Carrollian geometries, see table 1.

3I.e. satisfying $\nabla h_{\mu\nu} = 0$, $\nabla n^\mu = 0$. 

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To further compare similitudes and differences between (1.1) and (1.2), it is instructive to introduce a third homogeneous Carrollian manifold with topology \( \mathbb{R} \times S^2 \):

\[ \mathcal{M}^{(3)} := \mathbb{R}^4 \times \text{SO}(3) / \mathbb{R} \times \text{ISO}(2). \]  

The abelian factor in the denominator is a remaining Carrollian special conformal transformation; we will soon discuss its geometrical meaning. This model bridges between the two previous ones in the sense that we have the following maps

\[ \mathcal{M}^{(3)} \to \mathcal{M}^{(3)}_B \to \text{Carr}_{dS}(n). \]

The first map is rather natural: it is obtained by choosing an isometry subgroup \( \text{SO}(3) \) inside the conformal group \( \text{SO}(3, 1) \) i.e. by picking up a round sphere representative in the conformal class of metric on \( S^2 \). Accordingly \( \mathcal{M}^{(3)}_B \) is a model for \textit{null infinity in Bondi gauge} (by which we really mean null infinity with a fixed constant curvature metric representative for the 2D metric). The second map is obtained by noting that

\[ \mathcal{M}^{(3)}_B = \mathbb{R} \times \text{Car}_{dS}(3) / \mathbb{R} \times \text{ISO}(2). \]  

and ‘forgetting’ about the overall abelian factors. Note that, contrary to what the above expression might suggests, the \( \mathbb{R} \) factor does act non trivially on the homogeneous space and there is no canonical way to perform this last step. The significance of this is the following: working in Bondi gauge, the geometry of the characteristic data at null infinity can be understood as a Carrollian geometry supplemented by an equivalence class \([\nabla]\) of compatible connections \([51, 67, 77]\) and the extra abelian factor in the denominator of (1.4) takes one representative on another \( \nabla \mapsto \nabla + fh_{\mu\nu}n^\nu, f \in C^\infty(\mathcal{M}) \). Going from (1.4) to (1.1) amounts to choosing a preferred connection with constant curvature in the equivalence class. This sequence of gauge choices, leading from strongly conformally Carrollian to strong Carrollian geometries, is summarized in table 1. More details, especially on how these choices can be understood from the perspective of the corresponding Cartan geometries, will be given in the last section of this article.

Let us close this introduction by emphasising the conceptual implications of these results for the physics of gravitational radiation. As we already discussed, characteristic data for gravity at null infinity precisely correspond to Cartan geometries modelled on (1.2), prior to any gauge

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**Table 1. Summary.**

| Null-Infinity | Null-infinity in Bondi gauge | Carrollian space-time |
|---------------|------------------------------|----------------------|
| \((\mu^\mu, h_{\mu\nu}, \nabla)\) | \((\mu^\mu, h_{\mu\nu}, [\nabla])\) | \((\mu^\mu, h_{\mu\nu}, [\nabla])\) |
| \(\text{ISO}(n, 1)\) | \(\mathbb{R} \times \text{Car}_{dS}(n)\) | \(\text{Car}_{dS}(n)\) |

Where \(\nabla\) is a compatible connection on \(\mathcal{M}\).
fixing, or to other models (1.1) and (1.4) after suitable choice of gauge. The curvature of the Cartan connection in fact matches \cite{78} the Newman–Penrose coefficients $\Psi^0_0$, $\Psi^0_3$, $\text{Im}(\Psi^0_2)$ and therefore invariantly encodes the presence of gravitational radiation at null infinity. The significance of this fact comes from the fundamental theorem of Cartan geometry:

**Theorem 1.1.** **Fundamental theorem of Cartan geometry** (see e.g. \cite{59}). Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry\footnote{Let us recall to the reader that a Cartan geometry modelled on $G/H$ is the data of a $H$-principal bundle $\mathcal{G} \to M$ together with a $g$-valued connection such that $\omega : T\mathcal{G} \to g$ is invertible. Important examples are given by the projection $\mathcal{G} \to G/H$, together with the Maurer–Cartan connection $\omega_G$ on $G$.} modelled on $G/H$, the curvature $d\omega + \frac{1}{2}[\omega, \omega]$ vanishes if and only if the geometry is locally isomorphic to the model $(\mathcal{G} \to G/H, \omega_G)$.

In other terms, gravitational radiation is the obstruction to the existence of an isomorphism between null infinity and the model (1.2). The use of Cartan geometry thus reveals the deep connection between gravitational waves and the BMS group: radiation at null infinity is the obstruction (in the precise sense of theorem 1.1) to being able to single out a preferred Poincaré group inside the BMS group. This well-known fact, which is of fundamental importance \cite{79–81} here appears in a transparent form and has the value of a mathematical theorem.

It also follows that in the absence of gravitational radiation the geometry of null infinity is equivalent to a flat Cartan connection, which by the fundamental theorem of Cartan geometry, is equivalent to an isomorphism from null infinity to the model. These are not unique and the space of all such isomorphisms physically correspond to inequivalent gravity vacua on which the BMS group act transitively \cite{70, 77, 82}. This degeneracy of gravity vacua is tightly related to the existence of memory effect in general relativity \cite{67, 83–85} and has deep implications for the quantum gravity $S$-matrix \cite{77, 86, 87} as well as the black-hole information paradox \cite{88}.

Finally, since gravity vacua correspond to flat Cartan connections on $\mathcal{I}$ it is natural to think of the corresponding Chern–Simon functional at null infinity \cite{89} as some sort of effective boundary theory (see also \cite{90} for more tractor actions for gravity).

This article is organised as follows: in a first section we discuss the correspondence between strong Carrollian geometries and Cartan geometry modelled on (1.1). In a second section we investigate the invariant geometry of null infinity as Cartan geometry modelled on (1.2). Finally we compare these two realisations by considering null infinity in Bondi gauge, i.e. for which one has singled out a preferred constant curvature representative from the conformal class.

### 2. Carrollian spacetimes

Let $\mathcal{I}$ be a manifold foliated by lines. This will be convenient (but not necessary) to suppose that the space of lines $\Sigma$ is a manifold and that consequently $\mathcal{I} \xrightarrow{\pi} \Sigma$ is a fibre bundle\footnote{This is a global requirement and can be overlooked if one is only interested in the local geometry.}. We will also suppose that the lines are generated by a (nowhere vanishing) vector field $n^\mu$. The flow generated by this vector field equips (at least locally) Carrollian geometry with a preferred time $u : U \subset \mathcal{I} \to \mathbb{R}$ which is unique up to ‘super-translations’

$$u \mapsto u + \pi^* \xi, \quad \xi \in C^\infty(\Sigma).$$

**Definition 2.1.** A **Carrollian geometry** \cite{3} $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu})$ is the data of a one-dimensional fibre bundle $\mathcal{I} \to \Sigma$ together with a nowhere vanishing vector field $n^\mu$ generating the foliation and a degenerate metric $h_{\mu\nu}$ with one-dimensional kernel such that
\( h_{\mu
u}h^{\nu} = 0, \quad \mathcal{L}_h h_{\mu\nu} = 0. \) \tag{2.2}

The above requirements mean that \( h_{\mu\nu} \) defines a genuine metric (i.e. invertible) on \( \Sigma \). This definition has the advantage that Carrollian geometries then always have a non-trivial automorphism algebra given by the semi-direct product of super-translations \( T \cong \mathcal{C}^\infty(\Sigma) \) with isometries of \( (\Sigma, h_{\mu\nu}) \)

\[
\text{Lie}(\mathcal{F} \to \Sigma, n^\mu, h_{\mu\nu}) \cong \text{Lie}(T \rtimes \text{ISO}(\Sigma, h_{\mu\nu})). \tag{2.3}
\]

This isomorphism is obtained by making a choice of local trivialisation (i.e. time). For instance, if \( (\Sigma, h_{\mu\nu}) = (\mathbb{R}^{n-1}, dy^A dy^B \delta_{AB}) \) is flat Euclidean space and \( \mathcal{F} = \mathbb{R} \times \Sigma \) then symmetries are the semi-direct product of super-translations with Euclidean isometries \( \text{ISO}(n-1) = \mathbb{R}^{n-1} \rtimes \text{SO}(n-1) \).

Even though the second condition in (2.2) is too restrictive to encompass all geometries obtained by ultra-relativistic contractions, essential gravitational realisations are nonetheless of this type: prototypical examples are Killing horizons and null infinity in Bondi gauge. It should however be noted that the above definition does not encompass generic (non-Killing) null hypersurfaces for which there is no preferred vector field \( n^\mu \), see [35] for a detailed discussion on alternative definitions modelling various type of horizons. A close inspection shows that none of the results in this section actually relies on the second requirement in (2.2).

In this section we will discuss strongly Carrollian geometries [3] (see [91] for a comprehensive discussion) and describe the related Cartan geometries for a generic dimension and cosmological constant i.e. for Cartan geometries modelled on Carroll–de Sitter, Carroll and Carroll–anti de Sitter. Taking the cosmological constant to zero we will recover results from [62], restricting the arbitrary dimension to \( 2 + 1 \) results of [24, 28, 63] and imposing a flatness condition those of [64]. The main results are summarized in table 2.

### 2.1. Homogeneous model

The ‘Carrollian isometry groups’ are obtained by ultra-relativistic contractions of the isometry group of de Sitter, flat, and anti de Sitter spacetimes in \( n \) dimensions [1, 2]. The corresponding groups are respectively called Carroll–de Sitter, Carroll and Carroll–anti de Sitter. We will use the compact notation \( \text{Carr}_\Lambda(n) \) for representing these three groups,

\[
\text{Carr}_{\text{dS}}(n) = \text{Carr}_{\Lambda > 0}(n), \quad \text{Carr}(n) = \text{Carr}_{\Lambda = 0}(n), \quad \text{Carr}_{\text{AdS}}(n) = \text{Carr}_{\Lambda < 0}(n).
\]

We then have

\[
\text{Carr}_\Lambda(n) := \mathbb{R}^n \rtimes \text{ISO}(n-1) \tag{2.4}
\]

where it is convenient to introduce a single notation

\[
\text{ISO}_{\Lambda > 0}(n-1) = \text{SO}(n), \quad \text{ISO}_{\Lambda = 0}(n-1) = \text{ISO}(n-1), \quad \text{ISO}_{\Lambda < 0}(n-1) = \text{SO}(n-1, 1)_0 \tag{2.5}
\]

for the (connected component of) isometry groups of \( S^{n-1}, \mathbb{R}^{n-1} \) and \( H^{n-1} \).

It is useful to think of \( \text{ISO}_\Lambda(n-1) \) as the subgroup of \( \text{SO}(n, 1)_0 \) stabilising a vector of norm \(-2\Lambda\) (the factor of two is only introduced for future convenience). Making use of this fact, the Carrollian isometry groups can be represented matricially as

\[
\text{Carr}_\Lambda(n) = \left\{ \begin{pmatrix} m^I_j & 0 \\ t_i & 1 \end{pmatrix} \middle| m^I_j \in \text{SO}(n, 1)_0, \ t_i \in \mathbb{R}^{n+1} \right. \\
\text{s.t.} \quad m^I_j t^I_j = I^j, \ t^I_j t^I_j = 0. \tag{2.6}
\right\}
\]
| Model                              | Cartan geometry of Carrollian spacetimes. |
|-----------------------------------|------------------------------------------|
| Carroll—De Sitter $\Lambda > 0$  | $\text{Carr}_{\text{ds}}(n)/\text{ISO}(n-1)$ $\simeq \mathbb{R} \times S^{n-1}$ |
| Carroll $\Lambda = 0$             | $\text{Carr}(n)/\text{ISO}(n-1)$ $\simeq \mathbb{R} \times \mathbb{R}^{n-1}$ |
| Carroll—anti De Sitter $\Lambda < 0$ | $\text{Carr}_{\text{ads}}(n)/\text{ISO}(n-1)$ $\simeq \mathbb{R} \times \mathbb{H}^{n-1}$ |

(Weak) Carrollian geometry

$\mathcal{F} \to \Sigma, n^{\mu}, h_{\mu
u}$

Defines a canonical $\text{ISO}(n-1)$-principal bundle with symmetry group $C^\infty(\Sigma) \rtimes \text{ISO}(\Sigma, h_{\mu\nu})$.

Strong Carrollian geometry

$\mathcal{F} \to \Sigma, n^{\mu}, h_{\mu\nu}, \nabla$

 Defines a unique $\text{carr}_\Lambda(n)$-valued connection $D$

Flatness of $D$

Reduces the symmetry group to $\text{Carr}_\Lambda(n)$.
with \( I^i \in \mathbb{R}^{n+1} \) a fixed vector of norm \(-2\Lambda\). Introducing a null basis on \( \mathbb{R}^{n+1} \) we write \( I^i = (I^+, I^A, I^0) \) with \( I^A \in \mathbb{R}^n \) and
\[
I^iI^j_{\delta AB} = 2I^+I^- + I^AI^B \delta_{AB}.
\]
Without loss of generality we can take \( I_i = (1, 0, -\Lambda) \). The Lie algebra \( \text{carr}_\Lambda(n) \) is then generated by matrices of the form
\[
\text{carr}_\Lambda(n) = \left\{ \begin{pmatrix}
0 & -m_B & 0 & 0 \\
\Lambda m_A & m^A_B & m^A & 0 \\
0 & -\Lambda m_B & 0 & 0 \\
\Lambda t_- & -t_B & t_0 & 0
\end{pmatrix} \mid m_A^B \in \mathfrak{so}(n-1), t_B \in \mathbb{R}^{n-1}, (m^A, t_-) \in \mathbb{R}^n \right\}
\]
(2.7)

Spatial rotations and ‘Carrollian boosts’ are respectively generated by \( m_A^B \) and \( t_B \), while the \( n \) ‘Carrollian translations’ are generated by \( m^A \) and \( t_- \). In this representation, the Carroll isometry groups naturally act on \( \mathbb{R}^{n+2} \) with coordinates \( Y^I = (Y^+, Y^A, Y^-) \) and degenerate metric
\[
Y^I Y^J h_{IJ} = 2Y^+Y^- + Y^AY^B \delta_{AB}.
\]
(2.8)

The homogeneous Carrollian spacetimes are then defined to be the quotient \([2, 61, 64]\)
\[
\text{Carr}_\Lambda(n) := \text{carr}_\Lambda(n) / \text{ISO}(n-1).
\]
(2.9)
Where \( \text{ISO}(n-1) \) is identified with the subgroup of \( \text{carr}_\Lambda(n) \) stabilising \( X^I = (0, 0, 1, 0) \),
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & m_A^B & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & t_B & 0 & 1
\end{pmatrix} \mid m_A^B \in \text{SO}(n-1), t_B \in \mathbb{R}^{n-1}
\subseteq \text{carr}_\Lambda(n),
\]
(2.10)
and therefore is generated by Carrollian boosts and rotations.

These models are the total space of a (trivial) line bundle over sphere/flat/hyperbolic spaces respectively (the degenerate metric is obtained by pull-back of the metric on the base) i.e. we have the isomorphisms
\[
\text{Carr}_\Lambda(n)^{\text{dS}} \cong \mathbb{R} \times S^{n-1}, \quad \text{Carr}_\Lambda(n)^{\text{AdS}} \cong \mathbb{R} \times H^{n-1}.
\]

Alternatively, the model Carrollian spacetimes can be realised as
\[
\text{Carr}_\Lambda(n)^{J} = \left\{ Y^i = \begin{pmatrix} Y^+ \\ Y^A \\ Y^- \\ Y^0 \end{pmatrix} \in \mathbb{R}^{n+2} \mid Y^I Y^J h_{IJ} = 0, Y^I I^i_{\Lambda} h_{IJ} = 1 \right\}
\]
(2.11)
where \( I^i_{\Lambda} = (1, 0, -\Lambda, 0) \) is a fixed vector of norm \(-2\Lambda\) for the degenerate metric (2.8) (the Carrollian vector field is then proportional to \( \partial_{\phi} \)). In particular, note that \( X^I = (0, 0, 1, 0) \) always belongs to \( \text{Carr}_\Lambda(n)^{J} \). In order to identify (2.11) with the homogeneous space (2.9) it suffices to note that the Carrollian isometry groups act transitively and that the stabiliser of a point \( X^I \) is \( \text{ISO}(n-1) \).
2.2. Tetrad formalism adapted to Carrollian geometry

Let \((\mathcal{J} \to \Sigma, n^\mu, h_{\mu\nu})\) be a Carrollian geometry as in definition 2.1. Let \(U\) be an open set of \(\mathcal{J}\) and \(\{e_A^\mu\}_{A=1,...,n-1}\) be vector fields on this set. We will say that \(\{e_A^\mu, n^\mu\}_{A=1,...,n-1}\) is an orthogonal frame or ‘tetrad’ if it forms a basis of the tangent bundle at any \(x \in U\) and satisfies

\[
e_A^\mu e_B^\nu h_{\mu\nu} = \delta_{AB}
\]

where \(\delta_{AB}\) is the identity matrix. Tetrad have a ISO\((n-1)\)-ambiguity

\[
e_A^\mu \mapsto m_A^B (e_B^\mu - t_B n^\mu),
\]

\[
n^\mu \mapsto n^\mu,\quad t_B = \frac{1}{2} C_{A^B},
\]

\[
\begin{pmatrix}
m_A^B & 0 \\
t_B & 1
\end{pmatrix} \in \text{ISO}(n-1).
\]

Carrollian geometries are therefore canonically equipped with a ISO\((n-1)\)-principal bundle obtained from the reduction of the frame bundle of \(\mathcal{J}\) to such tetrads. These local symmetries are respectively interpreted as spatial rotations (for those generated by SO\((n-1)\)) and ‘Carrollian boosts’ (generated by \(R^{n-1}\)). We will write the corresponding dual tetrad \(\{\theta^A^\mu, l^\mu\}_{A=1,...,n-1}\):

\[
e_B^\mu \theta^A^\mu = \delta^A_B,\quad n^\mu \theta^A^A = 0,\quad e_B^\mu l^\mu = 0,\quad n^\mu l^\mu = 1.
\]

In what follows, it will be important that, under the action of ISO\((n-1)\), dual tetrads transform as

\[
\theta^A^\mu \mapsto m_A^B \theta^B^\mu,\quad l^\mu \mapsto l^\mu + t_C \theta^C^\mu.
\]

2.3. Strongly Carrollian geometry

Let \((\mathcal{J} \to \Sigma, n^\mu, h_{\mu\nu})\) be a Carrollian geometry. We will say that it is strongly Carrollian \([3, 91]\) if, on top of this, it is equipped with a torsion-free connection \(\nabla_\rho\) on the tangent bundle satisfying the compatibility relations

\[
\nabla_\rho n^\mu = 0,\quad \nabla_\rho h_{\mu\nu} = 0.
\]

The components of this connection in a tetrad \(\{e_A^\mu, n^\mu\}\) are

\[
\begin{pmatrix}
\theta^A_\mu \nabla_\rho e^\mu_B \\
l^\rho \nabla_\rho e^\rho_B
\end{pmatrix}
= \begin{pmatrix}
\omega^A_\mu \rho^B & 0 \\
\frac{1}{2} C_{A^\rho_B} & 0
\end{pmatrix}
\]

\[
(\text{the minus one half factor is here added to fit with notation from the literature [78, 94]) and the torsion-free condition is}

\[
d^\nabla (e_A^\mu) = d^\nabla \theta^A_\mu e^\mu_A + \left(d^\nabla \frac{1}{2} C_{A^\rho_B} \theta^B \right) n^\mu = 0
\]
where we introduced the notation $d^i \theta^A \coloneqq d \theta^A + \omega^A_{\ B} \wedge \theta^B$. It follows that $\omega^A_{\ B}$ is uniquely determined in terms of $\theta^A_{\mu}$. On the other hand, the coefficients $^8$

$$C_{AB} \theta^B_{\mu} + C_{A0} \ l_{\mu} \coloneqq C_{A\mu}$$

are only fixed in terms of the tetrad up to

$$C_{A\mu} \mapsto C_{A\mu} + \delta C_{AB} \theta^B_{\mu}$$

where $\delta C_{AB}$ is a symmetric tensor, $\delta C_{AB} = \delta C_{(AB)}$. This remaining freedom corresponds to the information contained in the strongly Carrollian geometry $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu\nu}, \nabla)$ as opposed to the bare Carrollian geometry $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu\nu})$.

### 2.4. The Carrollian Cartan connection

Carrollian spacetimes are tightly related to the geometry of $\text{carr}_\Lambda$-valued connections (see e.g. [24, 62–64]).

Let $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu\nu})$ be a Carrollian geometry. Corresponding tetrads then form an $\text{ISO}(n - 1)$-principal bundle and we will call (Carrollian) tractors (following the terminology of [60, 95]) sections $\Phi$ of the associated bundle for the representation (2.10).

Let $D_\rho$ be a generic connection valued in the Carroll Lie algebra (2.7), it acts on tractors as

$$D \Phi^I = \left( \begin{array}{ccc} d & -\theta_B & 0 & 0 \\ \Lambda \theta^A & d^i \theta^A & 0 & 0 \\ 0 & -\Lambda \theta_B & d & 0 \\ \Lambda l & -\frac{1}{2} C_B & l & d \end{array} \right) \left( \begin{array}{c} \Phi^+ \\ \Phi^- \\ \Phi^0 \end{array} \right).$$

Changes of tetrad (2.12) are equivalent to gauge transformations with values in $\text{ISO}(n - 1)$ (in the representation (2.10)) and induce the following transformation on the connection components

$$\theta^A \mapsto m^A_B \theta^B, \quad l \mapsto l + \theta^C t_C,$$

$$\omega^A_{\ B} \mapsto m^A_C \omega^C_{\ D} m^D_B - d m^A_C m^B_C, \quad -\frac{1}{2} C_A \mapsto m^A_B \left( -\frac{1}{2} C_B - d^i t_B \right).$$

(2.16)

We will say that such a connection is compatible with the Carrollian geometry if $\{\theta^A_{\mu}, l_{\mu}\}$ defines a dual-tetrad. The curvature of the connection is

$$F^I = \left( \begin{array}{ccc} 0 & -d^i \theta_B & 0 \\ 0 & F^A_{\ B} - 2\Lambda \theta^A \wedge \theta_B & d^i \theta^A \\ \Lambda \left( dl - \frac{1}{2} C_C \wedge \theta^C \right) & -\frac{1}{2} d^i C_B - 2\Lambda l \wedge \theta_B & dl - \frac{1}{2} C_C \wedge \theta^C \end{array} \right).$$

In particular $(\omega^A_{\ B}, C_B)$ are the components of a torsion-free connection $\nabla$ if and only if $F^I_{\ -} = 0$. We will say that such $\text{carr}_\Lambda(n)$-valued connections are torsion-free.

$^8$Note that, even though we order tetrads with the degenerate element coming in last position, we write the corresponding coefficient with a zero.
We thus derived the following:

**Proposition 2.1 (generalized from [62]).** Let $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu})$ be a $n$-dimensional Carrollian geometry, there is a one-to-one correspondence between, on the one hand, $\text{carr}(n)$-valued connections which are both compatible with the Carroll geometry and torsion-free and, on the other hand, choice of strongly Carrollian geometries $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu}, \nabla)$.

It follows from the fundamental theorem of Cartan geometry (see theorem 1.1) that we have the

**Proposition 2.2.** Let $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu}, \nabla)$ be a $n$-dimensional strongly Carrollian geometry satisfying

\[
F^A_B - 2\Lambda \theta^A \wedge \theta_B = 0, \quad -\frac{1}{2} d\omega_{CB} - 2\Lambda l \wedge \theta_B = 0,
\]

then $\mathcal{I} \to \Sigma$ is locally isomorphic to the model

\[
\text{Carr}^{(n)}_\Lambda := \text{Carr}_\Lambda(n)/\text{ISO}(n-1).
\]

In particular, the algebra of local symmetries is $\text{carr}_\Lambda(n)$.

2.5. ‘Carrollian good-cuts’

2.5.1. BMS coordinates. We will call ‘BMS’ coordinates $u : \mathcal{I} \to \mathbb{R}$ a local trivialisation of $\mathcal{I} \to \Sigma$ chosen such that $n^\mu(du)_\mu = 1$ i.e. such that it defines an element of dual tetrad $l_\mu := (du)_\mu$. This defines a unique tetrad $\{e^\mu_A, n^n\}$ (up to $\text{SO}(n-1)$) by requiring that $e^\mu_A(du)_\mu = 0$. Changing the trivialisation $u \mapsto \hat{u}$ gives the following transformation rules

\[
e^\mu_A \mapsto e^\mu_A - t_A n^\mu, \quad t_B = \nabla B \hat{u},
\]

where $\nabla_B \hat{u} := e^\mu_A \nabla B \hat{u}$. In BMS coordinates, $dl = 0$ and the previous computation simplifies, for example normal Carrollian connections are then parametrised by $C_A = C_{(AB)} \theta^B$ and $\omega^A_B$ is just the pull-back of the Levi-Civita connection of $(\Sigma, h_{\mu\nu})$ to $\mathcal{I}$.

2.5.2. Carrollian good-cuts. Let $G : \Sigma \to \mathcal{I}$ be a section (or ‘cut’) of $\mathcal{I}$. Let $u$ be a BMS coordinate. We can always think of the cut $G$ as the zero set of another BMS coordinate $f$ on $\mathcal{I}$,

\[
f = u - \pi^* G
\]

where $G \in \mathcal{C}^\infty(\Sigma)$ is a function on $\Sigma$ and the cut is given by $f = 0$.

We will say that $G$ is a Carrollian good-cut for a strongly Carrollian geometry $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu}, \nabla)$ if and only if

\[
\nabla_{(\mu} \nabla_{\nu) f}|_G = 0.
\]

In a tetrad given by $u$ this is equivalent to

\[
\nabla_{(A} \nabla_B) G = \left. \frac{1}{2} C_{AB} \right|_{u=G} = 0.
\]

\[11\]

\[9\] Strictly speaking the trivialisation $\mathcal{I} \simeq \mathbb{R} \times \Sigma$ is given by $x \mapsto (u(x), \pi(x))$. 

\[10\]
(this follows from (2.15)). In the flat model (2.9) the space of Carrollian good-cut is $n$ dimensional and defines preferred time foliations on which the $n$ Carrollian translations act transitively.

These ‘Carrollian good-cuts’ crucially differ from the usual good-cut equation \[ [55, 73, 96, 97] \] of null-infinity\(^ {10} \) to a homogenous realisation of null-infinity (3.2) possess $n + 1$ of these. This can be clarified by relating good-cuts to covariantly constant sections.

2.5.3. Covariantly constant sections. Let \((\mathcal{J} \to \Sigma, n^\mu, h_{\mu\nu}, \nabla)\) be a strongly Carrollian geometry and let \(D\) be the corresponding Carroll-valued connection. Its action on dual elements is

\[
D\Phi_I = \begin{pmatrix} d & -\theta^B & 0 & -I \\ \Lambda\theta_A & d^2 & \theta_A & \frac{1}{2}C_A \\ 0 & -\Lambda\theta^B & d & -I\Lambda \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \Phi_- \\ \Phi_B \\ \Phi_+ \\ \Phi_0 \end{pmatrix}
\]

where duality is defined by \(\Phi_I \Phi^I = \Phi_+ \Phi^+ + \Phi_A \Phi^A + \Phi^- \Phi^- + \Phi_0 \Phi^0\). By construction the Carrollian connection stabilises the section \(I^{(A)} = (1, 0^A, -\Lambda, 0)\).

Let \(\Phi_I = (f, \Phi_A, \Phi_+, \Phi_0)\) be a covariantly constant dual tractor \(D\Phi_I = 0\) satisfying \(\Phi_+ - \Lambda f = \Phi_+ \mu^{(A)} = 0\). A direct computation shows that this is equivalent to having \(\Phi_I = (f, \nabla_A f, \Lambda f, \dot{f})\) and

\[
\nabla_{\mu} \dot{f} = 0, \quad \nabla_A \nabla_B f + \frac{1}{2}C_{AB} \dot{f} + 2\Lambda f \, h_{AB} = 0 \tag{2.17}
\]

where dot indicate Lie derivatives along \(n^\mu\).

Let us pick a BMS coordinate \(u : \mathcal{J} \to \mathbb{R}\). The above equations can be equivalently rewritten as

\[
f = k(u - G), \quad \nabla_A \nabla_B G - \frac{1}{2}C_{AB} - 2\Lambda(u - G) \, h_{AB} = 0, \tag{2.18}
\]

where \(G\) is a function on \(\Sigma\) and \(k \in \mathbb{R}\) is some constant. Evaluating the above equation at \(u = G\) one finds that \(\Phi_I\) is covariantly constant if and only if \(G\) solves the Carrollian good cut equation

\[\]

\(^{10}\) Strictly speaking Newman’s good-cut equations are obtained for \(n = 3\) and considering only the holomorphic part of this: one can always write the 2D metric as \(h = P(z, \bar{z})^{-2}dz d\bar{z}\) and we then have \((P(0))^2(P(z)^2)P(\nabla_z \nabla_{\bar{z}} G) = \partial_z P(0^2)q_2 = \partial_\bar{z}G\). Introducing \(\sigma = (P(z))^2(P_\bar{z})^2C_{AB}\) one recovers the canonical form \(\partial_\mu G - \sigma_{\mu = 0} = 0\) for the (generalised [73]) good-cut equations.
and
\[ \frac{1}{2} C_{AB} = -2\Lambda (\alpha - G) h_{AB} + \frac{1}{2} C_{AB} |_{\omega = G}. \]

This last condition amounts to the vanishing of one of the coefficient of the Carrollian curvature. When this is satisfied Carrollian good-cuts are in one-to-one correspondence with covariantly constant sections \( \Phi_I \) which are orthogonal to \( I^\Lambda \) and considered up to an overall scale. This is a space of finite dimension at most \( n \) and this dimension is attained when the geometry is flat.

### 3. Null infinity

Since null infinity is the conformal boundary of an asymptotically flat spacetime this must be some type of conformal manifold. This is in fact a conformal Carrollian geometry in the following sense.

**Definition 3.1.** A conformal Carrollian geometry \((\mathcal{I} \to \Sigma, [n^\mu, h_{\mu\nu}] )\) is the data of a \( n \)-dimensional fibre bundle \( \mathcal{I} \to \Sigma \) over a \((n-1)\)-dimensional manifold \( \Sigma \) together with an equivalence class of vector fields and degenerate metrics with one-dimensional kernel
\[ (n^\mu, h_{\mu\nu}) \sim (\lambda^{-1} n^\mu, \lambda^2 h_{\mu\nu}), \quad C^\infty(\mathcal{I}) \ni \lambda > 0, \]

satisfying
\[ n^\nu h_{\mu\nu} = 0, \quad \mathcal{L}_\lambda h_{\mu\nu} = \Theta h_{\mu\nu}, \]

where\(^{11}\) \( \Theta := \frac{1}{n} h^{\nu\mu} \mathcal{L}_\lambda h_{\mu\nu}. \)

The above definition means that \((\Sigma, [h_{\mu\nu}] )\) defines a conformal geometry in the usual sense. We stress, however, that in principle rescalings by functions \( \lambda \) defined on the whole of \( \mathcal{I} \) are allowed. As is well known \([66, 67]\), the algebra of automorphisms of conformal Carrollian geometry is a generalised version of the BMS algebra
\[ \text{Lie}(\text{ISO}(\mathcal{I} \to \Sigma, [n^\mu, h_{\mu\nu}] )) \cong \text{Lie}(T \rtimes \text{SO}(\Sigma, [h_{\mu\nu}])). \]

We recover the standard BMS group \([79–81]\) by considering the case where \((\Sigma, [h_{\mu\nu}] )\) is the two-dimensional conformal sphere, \( \text{BMS}_4 \cong T \rtimes \text{SO}(3,1). \) Note that choosing a representative \((\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu} )\) such that \( \mathcal{L}_\lambda h_{\mu\nu} = 0 \) is always possible (these representatives are Carrollian geometries in the sense of definition 2.1), we will call these ‘Bondi gauge’ and discuss them in the next section.

Conformal geometry is typically a difficult subject because of the necessity of dealing with Weyl rescalings. Tractor calculus \([95, 98]\) has been devised specifically to provide a manifestly ‘Weyl-rescaling invariant’ formalism. In \([70]\) these methods were extended to conformal Carrollian manifolds. We here keep developing this formalism, making use of a more general first order (tetrad-like) approach which will encompass all previous results. This allows for a closer comparison with GHP-type formalisms \([74–76]\) or (closely related) Ehresmann connections \([10–12, 68]\). We emphasise that the approach presented here is in principle far more powerful since it is manifestly Weyl invariant by construction (without requiring extra structure such as Weyl connections etc).

\(^{11}\) Here \( h^{\mu\nu} \) is any inverse of \( h_{\mu\nu} \).
The main difference between the Carrollian and the classical conformal situation is that a conformally Carrollian geometry is not associated with a unique normal Cartan connection. Rather, there is a family of possible connections corresponding to choices of strongly conformal Carrollian geometries (these results are summarised in table 3).

From the point of view of null infinity, this moduli space of Cartan connections corresponds to choices of gravitational characteristic data; a non-vanishing curvature then characterises the presence of gravitational radiation [78].

It is finally worth noticing that it has in fact long been known [53] that, in dimension \( d = 4 \), the local twist connection restricted to null infinity invariantly encodes the asymptotic shear of an asymptotically flat spacetime. However, it seems that the definition of the tractor bundle for conformally Carrollian geometry, the corresponding Cartan geometries and their relations to spacetime local-twistors have only been systematically investigated for the first time in [70, 78].

3.1. Homogeneous model

The homogeneous model for null infinity is

\[
\mathcal{S}^{(n)} := \text{ISO}(n, 1)/\mathbb{R}^+ \times (\mathbb{R}^+ \times \text{ISO}(n - 1)).
\]

It is the total space of a fibre bundle over the conformal sphere

\[
\mathcal{S}^{(n)} \simeq \mathbb{R} \times S^{n-1}.
\]

In a sense, the homogeneous realisation of null infinity (3.2) is a conformal generalisation of the Carrollian spacetimes (2.9). This is obvious at the level of the isometry groups since the Carroll groups can all be understood as subgroups of the Poincaré group. It suffices to compare the matrix representation (2.6) of \(\text{Car}_{\Lambda}(n)\) with

\[
\text{ISO}(n, 1) = \left\{ \begin{pmatrix} m^j_i & 0 \\ t_i & 1 \end{pmatrix} \middle| m^j_i \in \text{SO}(n, 1)_0, \ t_i \in \mathbb{R}^{n+1} \right\}.
\]

In particular, in the basis for \(\mathbb{R}^{n+2}\) introduced in section 2.1, the Lie algebra of the \((n+1)\)-dimensional Poincaré group is generated by matrices of the form

\[
\text{iso}(n, 1) = \left\{ \begin{pmatrix} \alpha & -m_B^A & 0 \\ r^A & m_B^A & m^A \\ 0 & -r_B & -\alpha \\ t_+ & t_B & t_- \end{pmatrix} \middle| m^A_B \in \text{so}(n - 1), \ t_B \in \mathbb{R}^{n-1}, \ \alpha \in \mathbb{R}, \ (r^A, t_-) \in \mathbb{R}^{n+1}, \ (r^A, t_+) \in \mathbb{R}^{n+1} \right\}.
\]

On top of the spatial rotations, Carrollian boosts and translations respectively generated by \(m^A_B, t_B\) and \(m^A, t_-\), we have, as compare to Carrollian isometries, extra dilatation generated by \(\alpha\) and \(n\) extra Carrollian special conformal transformations generated by \(r^A\) and \(t_+\).

The subgroup of \(\text{ISO}(n, 1)\) stabilising the line generated by \(X^I = (0, 0^A, 1, 0)\) is isomorphic to \(\mathbb{R}^n \times (\mathbb{R}^+ \times \text{ISO}(n - 1))\) and is realised as
Table 3. Cartan geometry of null infinity.

| Model                                      | ISO(n, 1)/\mathbb{R}^n \times (\mathbb{R}^* \times \text{ISO}(n-1)) ≃ \mathbb{R} \times S^{n-1} | 
|-------------------------------------------|-------------------------------------------------------------------------------------------------|
| (Weak) conformally Carrollian geometry    |                                                                                                 |
| \(\mathcal{F} \rightarrow \Sigma, [n^\mu, h_{\mu\nu}]\) | Defines a canonical \(\mathbb{R}^n \times (\mathbb{R}^* \times \text{ISO}(n-1))\)-principal bundle |
| Has symmetry group                        | \(C^\infty(\Sigma) \rtimes \text{ISO}(\Sigma, [h_{\mu\nu}])\)                                 |
| Strong conformally Carrollian geometry    |                                                                                                 |
| \(\mathcal{F} \rightarrow \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P}\) | Defines a unique \(\text{iso}(n, 1)\)-valued connection \(D\)                                 |
| Flatness of \(D\)                        | Reduces the symmetry group to \(\text{ISO}(n, 1)\)                                               |
Here spatial rotations $m^A_B$ and Carrollian boosts $r_B$, which were already present in the Carrollian model (2.10), are complemented by dilatations $\lambda$ to form a semi-direct product with respect to Carrollian special conformal transformations $(t_A, t_+ \in \mathbb{R})$. The homogeneous model for null infinity is then obtained as the quotient (3.2).

Alternatively, the model for null infinity can be realised as the space of null lines in $\mathbb{R}^{n+2}$

$$\mathcal{J}^{(o)} = \left\{ Y^I = \begin{bmatrix} Y^+ \\ Y^A \\ Y^- \\ Y^0 \end{bmatrix} \in \mathbb{R}^{P(n+1)} \middle| \ Y^I h_{IJ} = 0 \right\}$$

where the square bracket indicates homogeneous coordinates on the projective space $\mathbb{R}^{P(n+1)}$ and $h_{IJ}$ is the degenerate metric (2.8). The line generated by $X^I = (0, 0^A, 1, 0)$ is null and therefore defines a point of $\mathcal{J}^{(o)}$. The identification of (3.5) with (3.2) is obtained by noting that the Poincaré group acts transitively on (3.5) and that the stabiliser of a point $X^I$ is $\mathbb{R}^n \rtimes (\mathbb{R}^* \times \text{ISO}(n-1))$.

Comparing this construction with the one from section 2.1, it is clear that one obtains the homogeneous Carrollian spacetimes from the model for null infinity by introducing an extra vector $I^\Lambda = (1, 0^A, -\Lambda, 0)$. This extra structure breaks conformal invariance and reduces the isometry group from Poincaré to the corresponding Carroll group.

### 3.2. Conformal geometry of null infinity

#### 3.2.1. Bundle of scale

If $\mathcal{J}$ is orientable we define the bundle of scale $L \rightarrow \mathcal{J}$ as

$$L = (\Lambda^C T^\ast \mathcal{J}) = (\Lambda^A T^\ast \mathcal{J})^\ast$$

(3.6)

Let $\Lambda^\mu = (m^A_B, t_B) \in \mathbb{R} \rtimes \text{ISO}(n-1)$ be a choice of tetrad and $\theta^\mu = (l_\mu, t_\mu)$ the related dual tetrad, in particular it defines preferred representatives $(n^\mu, \delta_{CD} \theta^C \theta^D) \in [n^\mu, h_{\mu\nu}]$ and a preferred volume form which can be used to trivialise the bundle of scale: any section $f$ of $L^\ast$ can be rewritten as

$$f = f(l \wedge \theta^1 \wedge \cdots \wedge \theta^{n-1})^{-\frac{1}{2}}$$

(3.7)

where $f$ is a function on $\mathcal{J}$. Upon rescaling the tetrad

$$e_A^\mu \mapsto \frac{1}{m^A_B}, \quad n^\mu \mapsto \frac{1}{n^\mu}$$

we have the transformations rules for the dual tetrad

$$\theta^\mu_A \mapsto \lambda m^A_B \theta^B, \quad l_\mu \mapsto \lambda (l_\mu + t_\mu)$$

(3.8)
and therefore the transformation rules for (coordinates representatives of) sections of $L^k$,
\[
(h_{\mu\nu}, n^\mu) \mapsto (\lambda^2 h_{\mu\nu}, \lambda^{-1} n^\mu), \quad f \mapsto \lambda^k f.
\] (3.8)

This gives a practical definition of sections of $L^k$. In particular, one could rephrase the definition of conformal Carroll geometries as a choice $(h_{\mu\nu}, n^\mu)$ where $h_{\mu\nu}$ and $n^\mu$ respectively are valued in sections of $L^2$ and $L^{-1}$ respectively. We will typically abuse notation and identify coordinate representatives $f$ (following transformation rules (3.8)) with the corresponding section $f$ of $L^1$.

**Vertical connection.** In presence of a conformal Carroll structure $(\mathcal{I} \to \Sigma, [n^\mu, h_{\mu\nu}])$ there is a natural differential operator on the bundle of scale $\nabla_0^\Theta : \Gamma[L] \to C^\infty(\mathcal{I})$ defined as follows. Let $\{l_A^\mu, l_\nu\}$ be a choice of dual tetrad and let us introduce the quantity $\Theta$ defined as
\[
\Theta := \frac{2}{n-1} \mathcal{L}_n (\theta^1 \wedge \cdots \wedge \theta^{n-1}) = \frac{1}{n-1} h^{\mu\nu} \mathcal{L}_n h_{\mu\nu}.
\]

Under the change of tetrad (3.6) we have
\[
\Theta \mapsto \frac{1}{\lambda} \left( \Theta + 2 \frac{\dot{\lambda}}{\lambda} \right)
\]
where $\dot{\lambda} := \mathcal{L}_n \lambda$. If $f$ is a section of $L$ we now define the ‘vertical derivative’ as
\[
\nabla^\Theta_0 f := \dot{f} - \frac{1}{2} \Theta f.
\]
It is invariant under the change of tetrad (3.6) and therefore defines a function on $\mathcal{I}$. We will say that $f \in \Gamma[L]$ has constant vertical derivative if $\nabla^\Theta_0 f$ is a constant on $\mathcal{I}$, i.e.
\[
\nabla_\mu \left( \nabla^\Theta_0 f \right) = 0.
\] (3.9)

**3.2.2. Strongly conformally Carrollian geometry.** Let $(\mathcal{I} \to \Sigma, [n^\mu, h_{\mu\nu}])$ be a conformally Carrollian geometry and let $S^2 T^* \mathcal{I}/h$ be the bundle obtained by taking the quotient of symmetric tensors by $h_{\mu\nu}$, $T_{\mu\nu} \sim T_{\mu\nu} + \alpha h_{\mu\nu}$, $\alpha \in C^\infty(\mathcal{I})$.

We want to define Poincaré operators $P$ as second order differential operators
\[
P : \Gamma[L] \to \Gamma[S^2 T^* \mathcal{I}/h \otimes L].
\]
To do so let us take a dual tetrad $(\theta^A, l)$ and let $\omega^A_{\mu}$ be the one form obtained by solving $\omega^A \theta^A = 0$. We define the covariant derivative$^{12}$ as $\nabla_A V_B := e_A^\mu \left( \partial_\mu V_B - \omega^C_{\mu} B_C \right)$. It will also be useful to introduce the notations $(\mathcal{L}_n l)_A := e_A^\mu (\mathcal{L}_n l)_\mu$, $(\mathcal{L}_n l^A)_B := e_B^\mu (\mathcal{L}_n l^A)_\mu$, etc.

$^{12}$This is a useful notation but potentially misleading, note in particular that if $\nabla_\mu$ is a Carrollian connection as in (2.15) then $e_A^\mu e_A^\nu (\nabla_\mu V_\nu) = \nabla_\mu V_\nu + \frac{1}{2} C_{\alpha\beta\nu} V_\alpha$. 


Let $f$ be a section of $L$ i.e. transforming as $f \mapsto \lambda f$ under (3.6) and let us consider

$$
\mathcal{P}(f)_{\mu\nu} = \mathcal{P}_{00}(f) l_\mu l_\nu + \mathcal{P}_{A0}(f) 2 l_\mu \theta_A^\lambda l_\nu + \mathcal{P}(f)_{AB} \theta_A^\mu \theta_B^\nu
$$

(3.10a)

with

$$
\mathcal{P}_{00}(f) = \nabla_0 \nabla_0 (\Theta_f) f,
$$

(3.10b)

$$
\mathcal{P}_{A0}(f) = \nabla_A \nabla_0 (\Theta_f) f,
$$

(3.10c)

$$
\mathcal{P}(f)_{AB} = \nabla_A \nabla_B \bigg|_{t_f} f + \frac{1}{2} C_{AB} \dot{f} - \left( \frac{1}{2} \mathcal{L}_n C_{AB} - \nabla_A (\mathcal{L}_n) B_i \right)
$$

\[ + (\mathcal{L}_n) A_i (\mathcal{L}_n) B_i \bigg|_{t_f} f
$$

(3.10d)

where $|_{t_f}$ indicates ‘trace-free part’, dot indicates Lie derivative along $n$ and $C = C_{AB} \theta_A^\mu \theta_B^\nu$ is symmetric and trace-free. We stress that $C$ is here arbitrary and parametrizes the Poincaré operator. Comparing with (3.9) one sees that the above generalises the ‘constant vertical derivative operator’.

Equation (3.10a) defines a section of $S^2 T^* \mathcal{I} / h \otimes L$, i.e. transforms as

$$
\mathcal{P}(f)_{\mu\nu} \mapsto \lambda \mathcal{P}(f)_{\mu\nu} + \alpha h_{\mu\nu} \quad \alpha \in \mathcal{C}^\infty(\mathcal{I})
$$

under the change of tetrad (3.6), if and only if we take the transformation rules for $C_{AB}$ to be

$$
\frac{1}{2} C_{AB} \mapsto \lambda^{-1} m_A^C m_B^D \left( \frac{1}{2} C_{CD} + \nabla_D t_C + \left( \frac{\dot{\lambda}}{\lambda} + \frac{\Theta}{2} \right) t_C - \lambda^{-1} \nabla_C \lambda - (\mathcal{L}_n \dot{t}) C \right) t_D
$$

(3.11)

(here $\dot{n}, \dot{t}$ are the image of (3.6) and (3.7)). This results from a tedious but direct computation, we will however see that it straightforwardly follows from tractor calculus adapted to conformally Carollian geometry.

With the transformation rules (3.11) the expression (3.10a) therefore defines a differential operator $\mathcal{P}$ taking a section of $L$ to a section of the quotient bundle $S^2 T^* \mathcal{I} / h \otimes L$. These operators play in conformal Carollian geometry a similar role to the Möbius operators [71, 72] of two-dimensional conformal geometry and for this reason were called Poincaré operators in [70].

Let $(\mathcal{I} \rightarrow \Sigma, \{n^\mu, h_{\mu\nu}\})$ be a conformal Carollian geometry. We will say that it is a strong conformally Carollian geometry [70] if, on top of this, it is equipped with choice of Poincaré operator (3.10a) i.e. a choice of trace-free ‘tensor’ $C_{AB}$ transforming as (3.11).

The main interest of this definition is that the conformal boundary of an asymptotically flat spacetime is always strongly conformally Carollian [78]. This is because the asymptotic shear $C_{AB}$ of asymptotically flat spacetimes follows the transformation rules (3.11) (this can also be obtained more invariantly by the use of tractor methods). As we will discuss in details, strongly conformally Carollian geometries are also equivalent [70] to Cartan geometries modelled on (3.2).
3.3. The null-tractor bundle

3.3.1. Null-tractors: invariant definition. Let $J^2\mathcal{L}$ be the second order jet of sections of $\mathcal{L}$ on $\mathcal{I}$ and let $(a, a_\mu, a_{\mu\nu})$ be corresponding coordinates. Let us consider the sub-bundle $F \subset J^2\mathcal{L}$ of formal solutions to (3.9). Practically, $F$ is the sub-bundle of $J^2\mathcal{L}$ given by the condition

$$n'^\nu a_{\mu\nu} + \frac{1}{2} \Theta a_\mu + \frac{1}{2} \nabla_\mu \Theta a = 0.$$ 

There is always a canonical inclusion $S^2\mathcal{T}^*\mathcal{I} \otimes \mathcal{L} \rightarrow J^2\mathcal{L}$

$$a_{\mu\nu} \rightarrow (0, 0, a_{\mu\nu}),$$

and, requiring $n'^\nu a_{\mu\nu} = 0$, this gives a canonical inclusion $S^2(\mathcal{T}/n) \otimes \mathcal{L} \rightarrow F$. The dual null-tractor bundle on $\mathcal{I}$ is then defined as the quotient of $F$ by the image of symmetric trace-free tensors in $F \subset J^2\mathcal{L}$

$$\mathcal{T} := F/S^2(\mathcal{T}/n) \otimes \mathcal{L}. \quad (3.12)$$

This definition is useful because it is invariant. It clearly shows that any conformal Carrollian geometry has a canonical tractor bundle. However it is not very practical which is the reason why we now move to a more concrete realisation.

3.3.2. Null-tractor: transformation rules. Let $\sigma \in \Gamma[\mathcal{L}]$ be a section with constant vertical derivative, i.e. satisfying (3.9). For such sections, and after making a choice of tetrad, we define Thomas operator $T : \Gamma[\mathcal{L}] \rightarrow \Gamma[\mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R} \oplus \mathbb{R}]$ as:

$$\sigma \rightarrow \begin{pmatrix} \Phi \\ \Phi_A \\ \Phi_0 \\ \Phi_B \end{pmatrix} := \left( \begin{array}{c} \sigma \\ \nabla_A \sigma \\ \frac{\nabla C}{\nabla C - P} \sigma \\ \nabla^{(\omega)} \sigma \end{array} \right) \quad (3.13)$$

where, for $n \geq 3$, $P = \frac{1}{2(n-2)}$ is the trace of the Schouten tensor of $\nabla_\lambda$. (Recall that if $\{\theta^A, I\}$ is a dual tetrad and $\omega^A_B$ is the one form obtained by solving $d^\ast \theta^A = 0$, we define the covariant derivative $\nabla_A V_B$ as $\nabla_A V_B := e^A_{\mu} (\partial_\mu V_B - \omega^C_{\mu B} V_C).$)

Thomas operator depends on a choice of tetrad. Upon the change of tetrad (3.6) we have the transformation rules

$$\begin{pmatrix} \Phi_- \\ \Phi_A \\ \Phi_0 \\ \Phi_B \end{pmatrix} \rightarrow \begin{pmatrix} \lambda & 0 & 0 & 0 \\ -\lambda^{-1} \frac{1}{2} r^2 & \rho_A & 0 & -m_A C t_C \\ 0 & 0 & \lambda^{-1} (\rho_B m_B C t_C - t_+) \end{pmatrix} \times \begin{pmatrix} \Phi_- \\ \Phi_A \\ \Phi_0 \\ \Phi_B \end{pmatrix}$$

where $r_A, t_+$ satisfy

$$r_A = m_A C \left( \frac{\lambda}{\lambda} + \frac{\Theta}{2} \right) t_C, \quad t_+ = \frac{1}{n-1} \left( -\nabla_C t^C + \frac{1}{2} \mathcal{L}_A (r^2) + \mathcal{L}_C (\mathcal{L}_A)_C \right), \quad (3.14)$$

where $\nabla_C = \lambda^{-1} \nabla_A \lambda$. We can now define dual tractors as fields following the above transformation rules.
From the transformation behaviour of dual tractors, we obtain transformation rules for tractors

\[
\begin{pmatrix}
\Phi^+ \\
\Phi^A \\
\Phi^-
\end{pmatrix}
\mapsto
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & m^A_B & 0 & 0 \\
-\lambda^{-1} r^2 & -\lambda^{-1} r e^{C_B} & \lambda^{-1} & 0 \\
1 & 0 & t_B & 1
\end{pmatrix}
\begin{pmatrix}
\Phi^+ \\
\Phi^A \\
\Phi^- \\
\Phi^0
\end{pmatrix}
\] (3.15)

with \( r_A, t_+ \) required to satisfy (3.14). Here the pairing between tractors and their dual is taken to be \( \Phi^I \Phi_I := \Phi^+ \Phi^+ + \Phi^A \Phi_A + \Phi^- \Phi^- + \Phi^0 \Phi^0 \). In the spirit of this subsection we can define tractors as fields following the above transformations rules.

### 3.3.3. Splitting isomorphisms

We presented two definitions of (null-)tractors: the first is invariant but unpractical, the second is very concrete but hides the underlying geometrical content. These are in fact equivalent and having the two is a good thing.

The equivalence of the two definitions can be seen as follows: if \( \{e_A^\mu\} \) is a choice of tetrad, the Thomas operator (given by equation (3.13)) defines an isomorphism

\[
\mathcal{T}^* \xrightarrow{\cong} \mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R} \oplus \mathbb{R}
\] (3.16)

between the dual-tractor bundle (3.12) and \( \mathbb{R}^{n+2} \). By duality we also have a similar isomorphism for tractors. We will refer to such isomorphisms, one for each choice of tetrad, as splitting isomorphisms. If \( \Phi^I \) is a section of the tractor bundle, we will write

\[
\Phi^I \overset{\cong}{\mapsto} \begin{pmatrix}
\hat{\Phi}^+ \\
\hat{\Phi}^A \\
\hat{\Phi}^- \\
\hat{\Phi}^0
\end{pmatrix}
\] (3.17)

Changing the tetrad

\[
e_A^\mu \mapsto \hat{e}_A^\mu := \lambda^{-1} m_A^B \left( e_B^\mu - t_B n^\mu \right) \lambda \begin{pmatrix}
m^A_B \\
t_B \\
0 \\
1
\end{pmatrix} \in \text{ISO}(n-1) \times \mathbb{R},
\]

then induces a change of splitting isomorphism

\[
\Phi^I \overset{\cong}{\mapsto} \begin{pmatrix}
\hat{\Phi}^+ \\
\hat{\Phi}^A \\
\hat{\Phi}^- \\
\hat{\Phi}^0
\end{pmatrix}
\]

corresponding to the transformation rules (3.15).

### 3.3.4. The \( \mathbb{R}^n \times (\mathbb{R}^* \times \text{ISO}(n-1)) \)-principal bundle of null infinity

Comparing the transformation rules of tractors (3.15) with (3.4) one sees that null-tractors can in fact be thought as associated to a \( \mathbb{R}^{n-1} \times (\mathbb{R}^* \times \text{ISO}(n-1)) \)-principal bundle over \( \mathcal{I} \). This principal bundle is the bundle of ‘orthonormal frames’ for null-tractors.
It indeed follows from the transformation rules (3.15) that the null-tractor bundle is equipped with an invariant degenerate pairing \( h_{IJ} \), a preferred degenerate section \( I^I \) and a preferred null weighted section \( X^I \in \Gamma[T \otimes L] \)

\[
h_{IJ} \Phi^I \Phi^J = 2 \Phi^+ \Phi^- + \Phi^A \Phi^B \delta_{AB}, \quad I^I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X^I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

(3.18)

It suffices to check that these are invariant under the transformation rules (3.15).

We now define orthonormal tractor frames \( \{ e_+^I, e_A^I, e_-^I, I^I \} \) as tractor frames satisfying

\[
e_-^I \propto X^I, \quad e_+^I e_-^J h_{IJ} = 1, \quad e_A^I e_B^J h_{IJ} = \delta_{AB},
\]

with all other contractions vanishing. These frames form a \( \mathbb{R}^n \rtimes (\mathbb{R} \times \text{ISO}(n-1)) \)-principal bundle over \( \mathscr{I} \).

One concludes that conformal Carrollian geometries \((\mathscr{I} \to \Sigma, [n^\mu, h_{\mu\nu}] )\) are canonically equipped with a \( \mathbb{R}^n \rtimes (\mathbb{R} \times \text{ISO}(n-1)) \)-principal bundle. We stress that this bundle is as intrinsic to conformal Carrollian geometry as the \( \text{ISO}(n-1) \)-principal bundle of frames (2.12) was to Carrollian geometry.

### 3.3.5. Spacetime interpretation.

It was shown in [78] that if \((\mathscr{I} \to \Sigma, n^\mu, h_{\mu\nu})\) is the conformal boundary of an \((n+1)\)-dimensional asymptotically flat spacetime then the null-tractor bundle (3.12) can be canonically identified with a sub-bundle of the restriction at \( \mathscr{I} \) of the usual tractor bundle [95, 98] of the \((n+1)\)-dimensional spacetime.

A choice of representative \( g_{\mu\nu} \in [g_{\mu\nu}] \) for the conformal spacetime metric then defines a splitting isomorphism (3.16) for the tractor bundle. The Weyl parameter \( \omega(r) \) of a smooth Weyl rescaling \( g_{\mu\nu} \mapsto \omega^2 g_{\mu\nu} \) must have an expansion of the form

\[
\omega(r) = \omega_0 + r^{-1} \omega_1 + \mathcal{O}(r^{-2})
\]

and the tractor transformation rules (3.15) then receive the following interpretation: \( \lambda = \omega_0 \) corresponds to Weyl rescalings of the boundary degenerate metric while \( t_+ = \omega_1 \) corresponds to the action of sub-leading Weyl rescalings. Finally, the restriction (3.14) correspond to restricting to representatives \( g_{\mu\nu} \) satisfying the BMS gauge \( \partial_\nu \det(h_{AB}) = 0 \).

### 3.4. The tractor connection

#### 3.4.1. The \( \text{ISO}(n,1) \)-valued connection.

Let \((\mathscr{I} \to \Sigma, [n^\mu, h_{\mu\nu}] )\) be a conformal Carrollian geometry, recall that the corresponding tractor bundle is equipped with a degenerate metric \( h_{II} \) and degenerate direction \( I^I \) given by (3.18). Let \( D_\rho \) be a connection on the null-tractor bundle satisfying \( D_\rho h_{II} = 0, D_I^I = 0 \). In a given tetrad, it must be of the form

\[
D \Phi^I = \begin{pmatrix} d + \alpha & -\theta_B & 0 & 0 \\ -\xi_A & d^\nu & \theta^A & 0 \\ 0 & \xi_B & d - \alpha & 0 \\ -\psi & -\frac{1}{2} C_{AB} & l & d \end{pmatrix} \begin{pmatrix} \Phi^+ \\ \Phi^A \\ \Phi^- \\ \Phi^B \end{pmatrix}.
\]

Note that this amounts to requiring that \( D_\rho \) takes values in the Lie algebra of the Poincaré group (3.3).
The action of the $\mathbb{R}^{n-1} \times (\mathbb{R} \times \text{ISO}(n-1))$-valued gauge transformations (3.15) on the connection is given by

$$\theta^A \mapsto \lambda m^A \theta^B, \quad l \mapsto \lambda (l + \theta^C t_C),$$

$$\alpha \mapsto \alpha + \epsilon^C m_C \theta_B - \lambda^{-1} d\lambda,$$  \hspace{1cm} (3.19a)

$$\omega^A_B \mapsto m^A_C m_B^D - dm^A_C m_B^C + (m^A_C r_B - \lambda^A r^C) \theta^C,$$  \hspace{1cm} \(3.19b\)

$$\xi^A \mapsto \lambda^{-1} \xi^B + d_{\omega} \left( \epsilon^C m^A_B \right) + \lambda^{-1} \left( \frac{1}{2} \delta^A_B - \lambda^A r_B \right) m_B^C \theta^C,$$

$$-\frac{1}{2} C_A \mapsto m_A^B \left( -\frac{1}{2} C_B - \epsilon^D t_B - \theta_{Bt} \right) + r_A \left( l + \theta^C t_C \right),$$

$$\psi \mapsto \lambda^{-1} \left( \psi + d_a t_+ + \xi^C t_C - \epsilon^C m^B \left( \frac{1}{2} C_B + d_a t_B + \theta_{Bt} \right) + \frac{1}{2} \epsilon^C \left( l + \theta^B t_B \right) \right).$$

Just as in the Carrollian case, we will say that such a connection is compatible with the Carrollian geometry if \{ $\theta^A, l$ \} defines a null tetrad for \{ $h_{\mu\nu}, n^\mu$ \}.

Importantly the tractor transformation rules are such that $r_A$ and $t_+$ must satisfy (3.14). A direct computation shows that it implies that we in fact have

$$\alpha + \frac{\Theta}{2} l \mapsto \alpha + \frac{\Theta}{2} l - \frac{1}{2} h^{CD} C_{CD} \mapsto \lambda^{-1} \frac{1}{2} h^{CD} C_{CD} + \lambda^{-1} \left( (C_a) C - \frac{1}{2} C_{CD} \right) t^C$$

where $\Theta := \frac{1}{n^2} h^{CD} i_{AB}$. $C_A = C_{AB} \theta^B + C_{AB} l$. We will mainly consider torsion-free connections, these must satisfy $\frac{1}{2} C_{C0} - (C_a) C = 0$ and therefore for such connections we can invariantly require that

$$\alpha + \frac{\Theta}{2} l = 0, \quad \frac{1}{2} h^{CD} C_{CD} = 0.$$  \hspace{1cm} (3.21)

We stress that we can only impose (3.21) because the transformation rules for tractors satisfy (3.14). Since these particular transformation rules are given by Thomas operator, we will say that the connection is compatible with Thomas operator if and only if it satisfies (3.21). Alternatively, depending on the reader’s mindset, one can take the transformation rules (3.14) to be defined as the subset of (3.19a) and (3.19b) preserving the ‘gauge fixing conditions’ (3.21) for torsion-free connections (more generally, they are the transformation rules such that (3.20) hold). Even though valuable, we wish to highlight that this point of view might hide the important fact that the null-tractor bundle and its preferred transformation rules (3.15) with (3.14) are canonically given by the conformal Carrollian geometry.

The curvature of the connection is

$$F^A_{\, B} \equiv \left( \begin{array}{ccc}
\frac{d\alpha - \xi^C \wedge \theta^C}{-d_\omega \xi^A} & -d_{\omega+a} \theta_B & 0 \\
-d_\omega \xi^A & F^A_{\omega B} + \xi^A \wedge \theta_B + \theta^A \wedge \xi_B & d_{\omega+a} \theta^A \\
0 & d_{\omega+a} \xi_B & d_\omega \xi + \xi^C \wedge \theta^C - d_\omega l - \frac{1}{2} C^C \wedge \theta^C
\end{array} \right).$$  \hspace{1cm} (3.22)
Let us write

\[ \xi_A = \xi_{AB} \theta^B + \xi_{A0} l, \quad C_A = C_{AB} \theta^B + C_{A0} l. \]

The requirement that the connection is torsion-free i.e. satisfies \( F^I J X^J = 0 \) uniquely fixes \( \xi_{A0} \). \( C_{A0} \) and the skew symmetric parts \( \xi_{[AB]} \), \( C_{[AB]} \):

\[
\begin{align*}
\omega^A_B &= \Theta^A_B - \epsilon B^a (L_a \theta^A)_b, \\
\omega^C_B &= -\epsilon C^\mu e_B^\nu \partial_\mu \theta^A, \\
\frac{1}{2} C_{A0} &= eA^\mu (L_n l)_\mu, \\
C_{[AB]} &= -\epsilon A^\mu \epsilon B^\nu (dl)_\mu, \\
\xi_{A0} &= eA^\mu \left( \frac{1}{2} \nabla_\mu \Theta + \Theta^2 \right), \\
\xi_{[AB]} &= \frac{\Theta}{4} eA^\mu \epsilon B^\nu (dl)_\mu. 
\end{align*}
\]

One needs further ‘normality conditions’ (these generalise the normality condition of conformal geometry [59, 60, 95]) to constrain the components of \( \psi = \psi_0 l + \psi_B \theta^B \) and the symmetric part \( \xi_{(AB)} \) in terms of the other fields. In this context a set of rather natural invariant normality conditions are

\[
\begin{align*}
&n_\perp \left( F^A_B + \xi^A \wedge \theta_B + \theta^A \wedge \xi_B \right) = 0, \quad (3.23a) \\
&n_\perp \left( -\frac{1}{2} d^\nu \xi^\nu + \psi \wedge \theta_B + l \wedge \xi_B \right) = 0, \quad (3.23b) \\
&e^B_\perp \left( F^A_B + \xi^A \wedge \theta_B + \theta^A \wedge \xi_B \right) = 0, \quad (3.23c) \\
&e^B_\perp \left( -\frac{1}{2} d^\nu \xi^\nu + \psi \wedge \theta_B + l \wedge \xi_B \right) = 0. \quad (3.23d)
\end{align*}
\]

We will say that connections satisfying the above constraints are ‘normal’. In fact the first equation above, necessary for the third to be invariant under gauge transformations, is identically satisfied and does not add any further constraint on the field. Supposing that compatibility conditions (3.21) hold, equation (3.23b) is equivalent to

\[
(\xi_{AB})_{\perp f} + \frac{\Theta}{4} C_{AB} = \left( \frac{1}{2} \nabla_a (C_{AB}) + \nabla_a (L_n l)_B + (L_n l)_A (L_n l)_B \right)_{\perp f}, \quad (3.24)
\]

\[
(\n-1) \psi_0 = -\xi^C e_C - \nabla^C (L_n l)_B (L_n l)_C, \quad (3.24)
\]

where it is again convenient to write \( (L_n l)_C := eC^\mu (L_n l)_\mu \) etc and \( C = C_{AB} \theta^A \theta^B \). These equations fix \( \psi_0 \) and the trace-free symmetric tensor \( \xi_{(AB)}_{\perp f} \) (one can check that (3.24) does not impose any further constraint on the skew symmetric part). Equations (3.23c) and (3.23d) are respectively equivalent to

\[
(\n-3) \xi_{AB} = F^C_{\perp ACB}_{\perp f}, \quad (n-2) \xi^C_C = -\frac{1}{2} F^C_{\perp CD}, \quad (3.25)
\]

and

\[
(\n-2) \psi_A = -\frac{1}{2} \nabla^B C_{BA}, \quad (3.26)
\]

In dimension \( n = 3 \) the first equation in (3.25) is identically satisfied, the second determines \( \xi^C_C \) and (3.26) fixes \( \psi_A \). The only component of the connection remaining unconstrained is
therefore the trace-free symmetric part of $C_{AB}$. A direct computation then shows that the transformation rules (3.19b) for $C_{(AB)}$ coincide with (3.11). In other terms, it defines a Poincaré operator (3.10a).

We therefore obtained the following proposition:

**Proposition 3.1** [70]. Let $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}])$ be a three-dimensional conformal Carrollian geometry, there is a one-to-one correspondence between, on the one hand, $\mathfrak{iso}(3,1)$-valued connections which are compatible with the conformal Carroll geometry, compatible with Thomas operator, torsion-free and normal and, on the other hand, choice of Poincaré operators $\mathcal{P}$ as defined in section 3.2.2.

In other terms strongly conformally Carrollian geometry $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P})$ in dimension 3 are in one-to-one correspondence with such $\mathfrak{iso}(3,1)$-valued connections.

In dimension $n \geq 4$ the situation is similar but (3.24) and (3.25) together constrain the ‘news tensor’ $\mathcal{L}_n C_{AB}$ and therefore constrain the admissible Poincaré operator.

In dimension $n = 2$, the situation is radically different. Since one-dimensional trace-free symmetric tensors vanish identically there is no freedom in $C_{AB}$, however equations (3.25) and (3.26) are now identically satisfied and do not constrain $\xi^C$ and $\psi_A$. Thus $\xi^C$ and $\psi_A$ are the free parameters for the connection. In fact one can show [70, 78] that $M := \frac{2}{n-1}\xi^C$ and $N_A := -\psi_A$ then respectively correspond to choices of mass and angular momentum aspects for the corresponding three-dimensional asymptotically flat spacetime.

Finally, from the fundamental theorem of Cartan geometry (1.1), we have:

**Proposition 3.2.** Let $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P})$ be a $n$-dimensional strongly conformally Carrollian geometry, it is locally isomorphic to the model (3.2) if and only if the curvature (3.22) of the corresponding $\mathfrak{iso}(n, 1)$-valued connection vanishes.

This then implies that the algebra of local symmetries is $\mathfrak{iso}(n, 1)$.

### 3.4.2. Poincaré operator revisited.

Let $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P})$ be a strongly conformally Carrollian geometry and let $D$ be the corresponding $\mathfrak{iso}(n, 1)$-valued connection. Its action on dual tractors is

\[
D\Phi_f = \begin{pmatrix}
    d + \alpha & -\theta^B & 0 & -l \\
    -\xi_A & d_\alpha & \theta_A & \frac{1}{2} C_A \\
    0 & \xi^B & d - \alpha & \psi \\
    0 & 0 & 0 & d
\end{pmatrix}
\begin{pmatrix}
    \Phi_0 \\
    \Phi_B \\
    \Phi_0 \\
    \Phi_0
\end{pmatrix}
\]

where duality is defined by $\Phi_f \Phi_f' = \Phi^+ \Phi'^+ + \Phi_A \Phi^A + \Phi^- \Phi'^- + \Phi_0 \Phi^0$.

Let $\Phi_f = (f, \Phi_A, \Phi_+, \Phi_0)$ be a covariantly constant dual tractor $D_f \Phi_f = 0$. Making use of (3.21), the equation $D\Phi_+ = 0$ is found to be equivalent to $\Phi_A = \nabla_A f$, $\Phi_0 = \nabla_0^{(0)} f$. Then

\[
D_0 \Phi_0 = \nabla_0^{(0)} f
\]

\[
D_0 \Phi_A = D_A \Phi_0 = \nabla_A \nabla^{(0)}_0 f
\]

\[
D_0 \Phi_B = \nabla_A \nabla^B_0 f + \frac{1}{2} C_{BA} f - \left( \xi_{AB} + \frac{\Theta}{4} C_{AB} \right) f + h_{AB} \Phi_+
\]  

(3.27)

Making use of the normality condition (3.24) we find that (3.27) coincides with the Poincaré operator (3.10a) (in particular one can check, using the identity $\nabla_A \nabla_B f = -\frac{1}{2} f (d)_AB$ and
torsion-freeness, that $D_A (\Phi_B) = D_A \Phi_B$). This can be taken as a direct proof that the Poincaré operator is well-defined or as an alternative definition.

We can now give a geometrical meaning to zeros of the Poincaré operator. Solving for $\Phi_+$ with $D^C \Phi_C = 0$ and making use of the normality condition (3.25) we obtain that $\Phi_I$ is the image of $f$ by Thomas operator (3.13): $\Phi_I = T(f)_I$. It follows from the previous discussion that $T(f)_I$ is covariantly constant with respect to a normal tractor connection then $f$ must be a zero of the corresponding Poincaré operator. In fact one can prove [70] that $f$ is a zero of the Poincaré operator if and only if $T(f)_I$ is covariantly constant (this amounts to proving that $D_\rho \Phi_I$ identically vanishes when the other components of $D \Phi_I$ are zero).

3.4.3. Space–time interpretation. As was previously highlighted the null-tractor bundle of the conformal boundary $\mathcal{I} = \partial M$ of an asymptotically flat spacetime can be canonically identified with the restriction of the usual tractor bundle of $M$.

The (unique) normal tractor connection of an asymptotically flat spacetimes then induces a connection on the null-tractor bundle and, assuming standard fall-off on the energy–momentum tensor of $M$, one can prove that this connection is normal [78]. Therefore the boundary of an asymptotically flat spacetime is always equipped with a normal Cartan connection modelled on (3.2) or, equivalently, with a strongly conformally Carrollian geometry $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}], P)$. Once again, this essentially boils down to the fact that, in dimension $d = 4$, the asymptotic shear of an asymptotically flat spacetime as the correct transformation rules (3.11) to define a Poincaré operator. Note however that this results extends to any dimension $d \geq 3$, in particular it unifies the role of the asymptotic shear in $d \geq 4$ with that of the mass and angular momentum aspects in $d = 3$.

Let us now discuss asymptotic symmetries. First note that while the BMS group is the group of symmetries of a (weak) conformal Carrollian geometry, the Poincaré group is the group of symmetries of a flat strongly conformal Carrollian geometry—this follows from the fundamental theorem of Cartan geometry (1.1). Flatness is essential here, a generic geometry will have no symmetries. Now if $(\mathcal{J} \to \Sigma, [n^\mu, h_{\mu\nu}], P)$ is the strongly conformal Carrollian geometry induced at the boundary of an asymptotically flat spacetime, one can prove [78] that the curvature of the induced connection correspond in spacetime dimension $d = 4$ to the Newman–Penrose coefficients $\Psi_0^0, \Psi_0^3, \text{Im}(\Psi_0^2)$. In other terms, as was highlighted in the introduction, gravitational radiation can be geometrically understood as the curvature of the induced Cartan connections i.e. gravitational waves are the obstruction to being able to identify null infinity with the model (3.2). These identifications are equivalent to choosing an action of the Poincaré group on $\mathcal{J}$, and therefore ‘gravitational radiation is the obstruction to having a preferred Poincaré group inside the BMS group’ in the precise sense of Cartan geometry.

We stress that this interplay between gravitational radiation and the induced Cartan geometry is a specific feature of the critical dimension $d = 4$. In dimension $d = 3$, Einstein equations force the curvature of the induced Cartan geometry to vanish (the corresponding equations on the mass and angular momentum aspects are the so-called ‘conservation equation’, see [78]). This is in line with the absence of gravitational radiation for $d = 3$. In dimension $d \geq 5$, Einstein equations again imply (assuming enough differentiability) that the curvature of the induced Cartan geometry vanishes. This follows from results of [99] and corresponds to the fact that radiative degrees of freedom in these higher dimensions are not found in the asymptotic shear but rather sit at lower order in the asymptotic expansion and therefore cannot be captured by the boundary tractor connection.
3.5. Symmetries, BMS coordinates and transformation properties of the shear

By construction, diffeomorphisms of \( \mathcal{F} \) and \( \mathbb{R} \times \text{ISO}(n-1) \)-gauge transformations all naturally act on the space of tractor connections given by proposition (3.1), (this would form the symmetry group for the corresponding Chern–Simons action at null-infinity, see [89]). If one considers a fixed conformal Carrollian geometry \( (\mathcal{F} \to \Sigma, [n^\mu, h_{\mu\nu}]) \) symmetries consist of the BMS group (3.1) supplemented by \( \mathbb{R} \times \text{ISO}(n-1) \)-gauge transformations. These gauge transformations correspond to change of tetrads (3.6) and, in particular the \( \mathbb{R} \)-factor corresponds to Weyl rescalings: one can therefore think of this as an extension of the Weyl-BMS group\(^{13}\) of [94, 101, 102]. The action of this group on the asymptotic shear is for example obtained by composing diffeomorphisms of \( \mathcal{F} \) with the gauge transformations (3.19b) and (3.14).

We now discuss how this group can be reduced to the more usual symmetry groups by choice of BMS coordinates. Let \( u : \mathcal{F} \to \mathbb{R} \) be a trivialisation of \( \mathcal{F} \to \mathbb{R} \) let us choose a tetrad \( \{e_A^\mu, n^\mu\} \) such that \( e_A^\mu(du)_\mu = 0 \). Changing the trivialisation \( u \mapsto \hat{u} \) gives

\[
e_A^\mu \mapsto \lambda^{-1} e_A^\mu, \\
n^\mu \mapsto \lambda^{-1} n^\mu,
\]

This procedure thus reduces \( \mathbb{R} \times \text{ISO}(n-1) \)-gauge transformations to \( \mathbb{R} \times \text{SO}(n-1) \), with the remaining freedom corresponding to Weyl rescalings and rotations of the tetrad in the direction transverse to \( n^\mu \). We can fix the Weyl rescalings as follows: since \( u \) is a trivialisation, \( n^\mu(du)_\mu \) is nowhere vanishing and there is a unique representative \( n^\mu \in [n^\mu] \) such that \( n^\mu(du)_\mu = 1 \). To preserve this gauge fixing, the above transformation rules must then be complemented by

\[
\lambda = \mathcal{L}_\mu \hat{u}.
\]

This last step reduces the gauge freedom to \( \text{SO}(n-1) \).

In these type of gauges, the dual-tetrad element \( l_\mu = (du)_\mu \) satisfies \( dl = 0 \), \( \mathcal{L}_\mu l = 0 \) which drastically simplifies the expressions for torsion-freeness and normality of the tractor connection. In particular we then have \( C_B = C_{AB} \theta^B \), where \( C_{AB} \) is trace-free symmetric and follows the transformation rules

\[
\frac{1}{2} C_{AB} \mapsto \lambda^{-1} m_A^C m_B^D \left( \frac{1}{2} C_{CD} + \lambda^{-1} \nabla_C \nabla_D \hat{u} - 2 \lambda^{-1} \mathcal{T}_{CD} \nabla_D \hat{u} + \lambda^{-2} \left( \frac{\lambda}{\lambda} + \frac{\Theta}{2} \right) \nabla_C \hat{u} \nabla_D \hat{u} \right)_{\text{lf}},
\]

where we made use of (3.19b), (3.14) and \( t_B = \lambda^{-1} \nabla_B \hat{u} \). These are the finite (i.e. non infinitesimal) transformations property of the asymptotic shear under the Weyl-BMS group [78, 101, 102].

Let us anticipate on the next section and suppose that we choose a Bondi gauge i.e. a representative \( h_{\mu\nu} \in [h_{\mu\nu}] \) such that \( \mathcal{L}_\mu h_{\mu\nu} = 0 \). Then \( \Theta = 0 \) and Weyl rescalings preserving this choice must satisfy \( \hat{\lambda} = \lambda \). What is more if \( u \) and \( \hat{u} \) are two trivialisations satisfying \( n^\mu(du)_\mu = n^\mu(\hat{du})_\mu = 0 \) we must have \( u = \lambda u + \pi^\rho f \) with \( f \) is a function on \( \Sigma \). This reduces the above transformation rules accordingly.

\(^{13}\)With our definition for conformal Carrollian geometry diffeomorphisms of the sphere are not symmetries, one can however obtain the generalised BMS group from [100] by relaxing the definition and only requiring \( (\mathcal{F} \to \Sigma, n^\rho) \) with the equivalence class replaced by a weighted vector field \( n^\rho \in \Gamma[T_{\mathcal{F}} \otimes L^{-1}] \).
4. Null infinity in Bondi gauge

Even though the geometry of null infinity is based on a *conformal* Carrollian geometry it is customary in the literature (see e.g. the reviews [67, 77, 103]) to work in a fixed gauge $h_{\mu\nu} \in [h_{\mu\nu}]$ satisfying\(^{14}\) $L_n h_{\mu\nu} = 0$. We will call these representatives ‘Bondi gauge’ even though, strictly speaking, in the literature Bondi gauge corresponds to the situation where $h_{\mu\nu}$ has constant curvature. This will however be clear from the discussion when we need this extra requirement or not.

With our definitions 2.1 and 3.1 a conformally Carrollian geometry in Bondi gauge is in fact simply Carrollian. One might therefore be under the impression that, working in Bondi gauge, null infinity is just Carrollian geometry. This is however misleading because the strong geometries do not match, even in Bondi gauge [51, 67]. It follows from propositions 2.1 and 3.1 that the corresponding Cartan geometries cannot be the same either.

The deep reason for the discrepancy can be traced back to the existence of $n+1$ translations on the homogeneous realisation of null infinity (3.2): these $n+1$ translations must still be present in Bondi gauge and do not match the $n$ Carrollian translations of the homogeneous models (2.9). We will here discuss the implications of this fact for the underlying Cartan geometries (the main results are summarised in table 4).

4.1. Null infinity in Bondi gauge: homogeneous model

The base manifold of the homogeneous model (3.2) is the conformal sphere $S^{n-1}$. It can be thought as the conformal compactification of the constant curvature Riemannian models: the round sphere $S^{n-1}$, the Euclidean plane $\mathbb{R}^{n-1}$ and hyperbolic space $H^{n-1}$. From the homogeneous perspective, this simply corresponds to the fact that their isometry groups (2.5), which we collectively write $\text{ISO}_\Lambda(n-1)$, are all subgroups of the conformal group $\text{SO}(n, 1)$.

Restricting the $\text{SO}(n, 1)$ part of $\text{ISO}(n, 1) = \mathbb{R}^n \rtimes \text{SO}(n, 1)$ to these isometry groups, one obtains the following model spaces:

$$\mathcal{S}^{(n)}_{\text{Bondi}} = \mathbb{R}^{n+1} \rtimes \text{ISO}_\Lambda(n-1) / \mathbb{R} \rtimes \text{ISO}(n-1). \quad (4.1)$$

For $\Lambda > 0$ we obtain the homogeneous model for null infinity in Bondi gauge $\mathcal{S}^{(n)}_{\text{Bondi}, \Lambda = 0} \simeq \mathbb{R} \times S^{n-1}$. The homogeneous space (4.1) can be rewritten in a form which makes the relation to the Carrollian homogeneous model (2.9) stand out

$$\mathcal{S}^{(n)}_{\text{Bondi}} = \mathbb{R} \times \text{Car}_{\Lambda}(n) / \mathbb{R} \times \text{ISO}(n-1). \quad (4.2)$$

The geometry of null infinity in Bondi gauge is therefore very close that of a Carrollian manifold, it however enjoys an extra abelian symmetry. At the level of Lie algebras,

$$\mathbb{R} \oplus \text{car}_{\Lambda}(n) = \left\{ \left( \begin{array}{cccc} 0 & -m_B & 0 & 0 \\ \Lambda m_A & m^A_B & m^A \Lambda & 0 \\ 0 & -\Lambda m_B & 0 & 0 \\ t_+ & t_B & 0 & t_- \end{array} \right) \bigg| m^A_B \in \mathfrak{so}(n-1), \ t_b \in \mathbb{R}^{n-1}, \ (m^A, t_-) \in \mathbb{R}^{n+1}, \ t_+ \in \mathbb{R} \right\}, \quad (4.3)$$

\(^{14}\)Choosing this gauge is convenient but of course not necessary, see e.g. [75, 94, 101] for detailed discussions not assuming this gauge.
### Table 4. Cartan geometry of null infinity in Bondi gauge.

| Model | Description |
|-------|-------------|
| (Weak) conformally Carrollian geometry in Bondi gauge | $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu})$ |
| defines a canonical $\mathbb{R} \times \text{Carr}_c(n)/\mathbb{R} \times \text{ISO}(n - 1)$-principal bundle |
| has symmetry group $\mathcal{C}^\infty(\Sigma) \rtimes \text{ISO}(\Sigma, h_{\mu\nu})$ |
| Strong conformally Carrollian geometry in Bondi gauge | $(\mathcal{I} \to \Sigma, n^\mu, h_{\mu\nu}, \{\nabla_\rho\})$ |
| defines a unique $\text{iso}(n, 1)$-valued connection $D$ |
| flatness of $D$ and constant curvature $h_{\mu\nu}$ reduces the symmetry group to $\mathbb{R} \times \text{Carr}_c(n)$ |
the extra symmetry corresponds to the presence of an extra $t_+$ term as compare to the Carrollian case (2.7) and (2.10). As compare to null infinity (3.3) and (3.4) this corresponds to a remaining Carrollian special conformal transformation.

We stress that the abelian factor in $\mathbb{R} \times \text{Car} \Lambda(n)$ does act non-trivially on the homogeneous space (despite the fact that the, misleading, expression (4.2) might suggest the opposite). This can be realised explicitly by considering

\begin{equation}
\mathcal{G}^{(u)}_{\text{Bondi}, \Lambda} = \left\{ Y^I = \begin{pmatrix} Y^+ \\ Y^A \\ Y^0 \end{pmatrix} \in \mathbb{R}^{n+2} \middle| \quad Y^I Y^J h_{IJ} = 0, \quad Y^I t^I_{(\Lambda)} h_{IJ} = 1 \right\}
\end{equation}

(4.5)

where $I^I_{(\Lambda)} = (1, 0^A, -\Lambda, 0)$ explicitly breaks conformal invariance as compare to (3.5). The transitive action of $\mathbb{R} \times \text{Car} \Lambda(n)$ on (4.5) is then realised by restricting the action of the Poincaré group on (3.5) to the subgroup stabilising $I^I_{(\Lambda)}$. Note that even though (2.11) and (4.5) coincide as manifolds the group action is different and they therefore differ as homogeneous spaces.

### 4.2. Null infinity in Bondi gauge: the principal bundle

Let $(\mathcal{G} \to \Sigma, [n^\mu, h_{\mu\nu}])$ be a conformal Carrollian geometry. As discussed in section 3.3 it is canonically equipped with a $\mathbb{R}^n \rtimes (\mathbb{R}^* \times \text{ISO}(n-1))$-principal bundle—the bundle of orthonormal tractor frames. What is more, any choice of tetrad $\{e^A{}^\mu, n^\mu\}$ defines a splitting isomorphism $\mathcal{T}$ with transformation rules given by (3.15) and (3.14). If we work in Bondi gauge, i.e. choose a metric representative $h_{\mu\nu}$ such that $\mathcal{L}_n h_{\mu\nu} = 0$ then corresponding tetrads are unique up to

\begin{equation}
e^A{}^\mu \mapsto m_A{}^B (e^B{}^\mu - t_B n^\mu) \quad \text{and} \quad t^I_{(\Lambda)} h_{IJ} = 1
\end{equation}

(4.5)

and the tractor transformation rules (3.15) reduce to

\begin{equation}
\begin{pmatrix} \Phi^+ \\ \Phi^A \\ \Phi^0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_A{}^B & 0 & 0 \\ t_+ & t_B & 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi^+ \\ \Phi^B \\ \Phi^- \\ \Phi^0 \end{pmatrix}
\end{equation}

(4.7)

with

\begin{equation}
t_+ = \frac{1}{n-1} \left( -\nabla C t^C + \frac{1}{2} \mathcal{L}_n (t^2) + t^C (\mathcal{L}_n t)_C \right).
\end{equation}

(4.8)
The appearance of a non-zero $t_+$ term in the above transformation rules means that the $\mathbb{R}^n \ltimes (\mathbb{R} \times \text{ISO}(n-1))$-principal bundle of null infinity reduces to an $\mathbb{R} \times \text{ISO}(n-1)$-principal bundle. In particular the resulting principal bundle does not coincide with the obvious ISO$(n-1)$-principal bundle (4.6) of Carrollian geometry.

In consequence, there are always two principal bundles which can be constructed from a Carrollian geometry, the first is obtained as a reduction from the frame bundle and has structure group ISO$(n-1)$, the second is obtained from frames for one-jets of functions (the construction is similar to the one described in section 3.3.1) and has structure group $\mathbb{R} \times \text{ISO}(n-1)$.

4.3. From Poincaré operator to equivalence class of connections

Let $(\mathcal{M} \rightarrow \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P})$ be a strongly conformally Carrollian geometry. Let $h_{\mu\nu} \in [h_{\mu\nu}]$ be a Bondi gauge (in particular $\Theta := \frac{1}{n-1} h^{\mu\nu} h_{\mu\nu} = 0$), the Poincaré operator then correspond to a choice of trace-free symmetric tensor $C^{AB}$ following the transformation rules (3.11) with the restriction $\lambda = 1$ to preserve the Bondi gauge.

The transformation rules obtained in this way are the transformation rules for the components (2.15) of a torsion-free connection $\nabla_\rho$ compatible with the Carrollian geometry $(n^\mu, h_{\mu\nu})$. However since a Poincaré operator only is equivalent to a trace-free tensor we are missing a trace-part to uniquely define this connection. Rather a choice of Poincaré operator in a fixed Bondi gauge corresponds to an equivalence class $[\nabla_\rho]$ of connections

$$\nabla_\rho \sim \tilde{\nabla}_\rho \Leftrightarrow \nabla_\rho - \tilde{\nabla}_\rho \propto h_{\rho\mu} n^\mu.$$  

In a tetrad this equivalence relation indeed amounts to shifting the trace of $C^{AB}$.

Such equivalence classes were called radiative structure in [67] and have been identified as the initial characteristic data for gravity at null infinity in [50, 51].

From this perspective, the discrepancy between the flat model for null infinity in Bondi gauge (4.2) and the homogeneous Carroll space–times (2.9) receives a clear geometrical interpretation: while flat Carrollian geometries are equipped with a preferred torsion-free connection $\nabla_\rho$, the flat model for null infinity in Bondi gauge only posses an equivalence class $[\nabla_\rho]$ of such objects and the extra abelian symmetry corresponds to a shift from one connection representative to another.

4.4. Null infinity in Bondi gauge: the connection

Let $(\mathcal{M} \rightarrow \Sigma, [n^\mu, h_{\mu\nu}], \mathcal{P})$ be a strongly conformal Carrollian geometry. By results of the previous section this is equivalent to a choice of Cartan geometry modelled on (3.2). Let $h_{\mu\nu} \in [h_{\mu\nu}]$ be a Bondi gauge and let $(e_\mu^\lambda, n^\mu)$ be a choice of adapted tetrad. The corresponding Cartan connection is of the form

$$D\Phi^\lambda = \begin{pmatrix} d + \alpha & -\theta_B & 0 & 0 \\ -\xi_A & d^\omega & \theta^A & 0 \\ 0 & \xi_B & d - \alpha & 0 \\ -\psi & \frac{1}{2} C_B & l & d \end{pmatrix} \begin{pmatrix} \Phi^+ \\ \Phi^- \\ \Phi^A \\ \Phi^\beta \end{pmatrix}$$  

(4.9)
and the action of the $\mathbb{R} \times \text{ISO}(n-1)$-valued gauge transformations (4.7) is given by
\[ \theta^A \mapsto m^A B \theta^B, \quad l \mapsto l + \theta^C t_C, \quad \omega^{AB} \mapsto m^A C_{\rho} m^B B^D - d m^A C_{\rho} C^D, \]
\[ \xi^A \mapsto m^A B \xi^B, \quad \alpha \mapsto \alpha, \quad \frac{1}{2} C_A \mapsto m^A B \left( \frac{1}{2} C_B - d^2 t_B - \theta \beta C \right), \]
(4.10)
\[ \psi \mapsto \psi + \xi^C t_C + d^{-\alpha} t_+. \]

Supposing that the connection is torsion-free, the ‘compatibility with Thomas operator’ conditions (4.11) now are
\[ \alpha = 0, \quad \frac{1}{2} h^{CD} C_{CD} = 0. \]
(4.11)

Here again the constraints (4.8) on the transformation rules can be understood as the subspace of gauge transformations preserving the gauge fixing condition (4.11).

The appearance of the extra $t_+$ gauge parameter in the gauge transformation (4.10) as compare to those of a Carrollian-valued connection (2.16) implies an essential difference with the strongly Carrollian case: the pair $(\omega^{AB}, C_B)$ does not define a unique connection $\nabla$ on the tangent bundle $T \mathcal{I}$, rather we have an equivalence class of connections $[\nabla]$ where representatives are related by gauge transformations
\[ \nabla_\rho \mapsto \nabla_\rho - (t_+ \lambda) h_{\mu \nu} n^\mu. \]

This is in line with the preceding discussion on Poincaré operators.

A choice of equivalence class $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu \nu}, [\nabla])$ always defines a strongly conformally Carrollian geometry and therefore always defines a Cartan geometry modelled on (3.2). However there does not seem to have any Cartan geometry modelled on (4.1) generically equivalent to such gauged-fixed geometry $\left( \mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu \nu}, [\nabla] \right)$. Nevertheless, if the Cartan connection (4.9) is flat and the representative $h_{\mu \nu}$ satisfies the constant curvature condition
\[ F^A_{\omega B} - 2 \lambda \theta^A \wedge \theta_B = 0, \]
then it follows from the normality conditions (3.25) that the connection in fact takes values in the Lie algebra (4.3) of $\mathbb{R} \times \text{Carr}_A(n)$:
\[
D \Phi^I = \begin{pmatrix}
    d & -\theta_B & 0 & 0 \\
    \Delta \theta^A & d^2 & \theta^A & 0 \\
    0 & \Delta \theta_B & d & 0 \\
    -\psi & \frac{1}{2} C_B & l & d
\end{pmatrix}
\begin{pmatrix}
    \Phi^+ \\
    \Phi^A \\
    \Phi^0
\end{pmatrix}.
\]

From the fundamental theorem of Cartan geometry we therefore have:

**Proposition 4.1.** Let $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu \nu}, [\nabla])$ be a strongly conformally Carrollian geometry such that the metric representative satisfies
\[ F^A_{\omega B} - 2 \lambda \theta^A \wedge \theta_B = 0 \]
and such that the corresponding Cartan connection is flat. Then $(\mathcal{I} \rightarrow \Sigma, n^\mu, h_{\mu \nu}, [\nabla])$ is locally isomorphic to the homogeneous space model (4.1).

In particular the algebra of local symmetries is $\mathbb{R} \times \text{Carr}_A(n)$. 

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4.5. Poincaré operator, BMS coordinates and good-cuts

Let \( \mathcal{F} \rightarrow \Sigma, [\mu^\mu, h_{\mu\nu}], P \) be a strong conformally Carrollian geometry. In the previous section we saw that covariantly constant dual-tractors are essentially equivalent to zeros of the Poincaré operator. We will here briefly investigate what these are in a Bondi gauge.

Let \( h_{\mu\nu} \in [h_{\mu\nu}] \) be a Bondi gauge, \( u \) a BMS coordinates i.e. satisfying \( du(u) = 1 \) and let us pick a compatible dual tetrad \( \{ l = du, \theta^A \} \). The Poincaré operator (3.10a) then takes the simpler form

\[
\begin{align*}
P_{00}(f) &= \nabla_0 \dot{f}, \\P_{0\alpha}(f) &= \nabla_{\alpha} \dot{f}, \\P(f)_{AB} &= \nabla_A \nabla_B \bigg|_{u=G} f + \frac{1}{2} C_{AB} \dot{f} - \frac{1}{2} (L_u C)_{AB} f,
\end{align*}
\]

where \( f \) is a function on \( \mathcal{F} \). Let \( f \) be a zero of \( P \). In particular the first two equations are equivalent to

\[
f = k(u - \pi^* G), \quad G \in C^\infty(\Sigma), \quad k \in \mathbb{R},
\]

and the third to

\[
\nabla_A \nabla_B \bigg|_{u=G} f + \frac{1}{2} C_{AB} + (u - G) \frac{1}{2} (L_u C)_{AB} = 0.
\]

Taking an overall Lie derivative \( L_u \) one finds that \( (L_u L_u C)_{AB} = 0 \) which is an integrability condition for this equation (in fact this equivalent to the vanishing of a tractor curvature coefficient and correspond to \( \Psi^4_0 = 0 \) from the space–time point of view). Assuming this, \( f = k(u - G) \) is a zero of the Poincaré operator if and only if the good-cut equation

\[
\nabla_A \nabla_B \bigg|_{u=G} f + \frac{1}{2} C_{AB} \bigg|_{u=G} = 0
\]

is satisfied.

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No new data were created or analysed in this study.
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