An alternative way to achieve Kepler’s laws of equal areas and ellipses for the Earth

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Abstract
Kepler’s laws of planetary motion are acknowledged as highly significant to the construction of universal gravitation. This paper demonstrates different ways to derive the law of equal areas for the Earth by general geometrical and trigonometric methods, which are much simpler than the original derivation depicted by Kepler. The established law of equal area for the Earth was applied to analyse the angular velocity or the reciprocal of the distance—for the Earth’s orbit around the Sun—and can be defined as a periodic function by analysing the available data, which help explain the law of ellipses for the Earth.

1. Introduction

It has been 400 years since Johannes Kepler (1571–1630) published his major masterpiece—New Astronomy [1]. This book proposed a concise description of the orbit of Mars. Mars orbits the Sun in an ellipse with the Sun located at one focus rather than in accordance with the long-held belief that Mars moved in a perfect circle. Also, the line connecting the Sun and Mars sweeps an equal area in an equal period of time. These discoveries are known as Kepler’s first and second laws of planetary motion or the laws of ellipses and equal areas, respectively. These findings, of course, are dependent on the legacy of precise observations by Tycho Brahe (1546–1601). More importantly, it was Kepler’s creativity that led him to apply geometric methods to convert astronomical data based on an Earth-centred model to a Sun-centred model.

Most of the relevant studies about Kepler’s laws were derived algebraically from conservation principles of energy and angular momentum, which were not yet known to...
Kepler [2–7]. The first part of this paper is intended to demonstrate a geometrical method, which is much simpler than the original depicted by Kepler, to obtain the law of equal areas using mainly the law of sines that every high school student learns. In the second part of this paper, the established law of equal areas for the Earth is applied to calculate the angular velocity of the Earth around the Sun. It can be shown to be a periodic function by analysing the data using current mathematical methods, and this characteristic may assist in understanding the law of ellipses for the Earth.

2. The law of equal areas for the Earth

To construct the law of equal areas for the Earth, we must first understand the position of the Earth relative to the Sun. However, how do we determine the Earth’s location in the Universe? From thousands of years ago until the era of Kepler, all of the understanding about the motions of the planets and the Sun was based on observations taking the Earth as the origin from which the angles or positions of the planets and the Sun were measured. Specifically, man could only record the positions of the planets and the Sun as longitudes and latitudes and was unable to measure their distances from the Earth or calculate corresponding distance ratios. For this reason, Kepler, as a believer in the heliocentric model, needed to convert observational data based on the Earth as the origin to data based on the Sun as the origin, which required that he come up with a new approach. In Kepler’s era, it was known that planets moved in the same plane. To locate an object on a plane, one must have at least two reference points as a basis. However, if the Sun is taken to be one fixed point, what can be used as a second reference point? If a fixed star is chosen, its distance cannot be determined, and if a nearby planet is used, its location in the sky is not stationary.

Nevertheless, Kepler noted that the time interval from the one opposition—where the Sun, Earth, and Mars are aligned—to the next opposition was about 780 days. Based on these data, he then determined the period of Mars’ orbit around the Sun to be approximately 687 days. Because the position of Mars in the sky repeats every 687 days, this position can be used as a reference point. As a result, there are two fixed points to use as a benchmark. Finally, by applying the data of angular positions observed from the Earth, one can calculate the location of the Earth relative to the Sun.

The position is denoted by its ecliptic longitude where the longitude of the spring equinox is 0°, that of the summer solstice is 90° and so on. Kepler then chose the date of the opposition of Mars as a basic reference point. For convenience, one may select 5 am on 25 March 1950, when Mars was in opposition, i.e. the Sun (S), Earth (E) and Mars (M) were in a straight line as shown in figure 1. M will return to its original position every Martian year, and this can be treated as a second fixed point. Ei and Ej represent the positions of the Earth 1 Martian year before and after the opposition of Mars, respectively. From the observational data, these configurations occurred on 5 May 1948 and 8 February 1952, respectively. As a result, $SE\angle ME\angle$ forms a quadrilateral. If $r_i$ and $r_j$ are the lengths of the segments $SE_i$ and $SE_j$, then $r_i$ and $r_j$ represent distances from the Sun to the Earth at two different times (figure 2).

The quadrilateral $SE\angle ME\angle$ can be treated as being made of $\angle SE_iM$ and $\angle SE_jM$, in which $\angle SE_iM$, $\angle SE_jM$, $\angle E_iMS$ and $\angle E_jMS$ can be observed as the longitudes of the Sun and Mars as seen from the Earth. The longitudes of the Sun and Mars were 44.7° and 144.9°, respectively, as seen from the Earth, $E_i$ [8]. This implies the following equation:

$$\angle SE_iM = \mu_i = 144.9^\circ - 44.7^\circ = 100.2^\circ.$$
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![Figure 1](image1.png)

**Figure 1.** The positions of the Sun (S), Earth (E) and Mars (M). E\(_i\) and E\(_j\) represent the positions of the Earth 1 Martian year before and after the opposition of Mars, respectively.

![Figure 2](image2.png)

**Figure 2.** The quadrilateral formed by SE\(_i\)ME\(_j\). The angles \(\mu_i\), \(\mu_j\), \(\alpha_i\) and \(\alpha_j\) are observed.

Similarly, the longitudes of the Sun and Mars were 218.2\(^\circ\) and 318.3\(^\circ\), respectively, as seen from the Earth, E\(_j\) [8]. This implies the following equation:

\[ \angle SE_j M = \mu_j = 318.3^\circ - 218.2^\circ = 100.1^\circ. \]

However, the longitude of Mars at the opposition of 25 March 1950 was 182.0\(^\circ\), and the longitude of Mars as seen from the Earth, E\(_i\), on 5 May 1948 was 144.9\(^\circ\). Hence,

\[ \angle E_i MS = \alpha_i = 182.0^\circ - 144.9^\circ = 37.1^\circ. \]

Similarly, the longitude of Mars as seen from the Earth, E\(_j\), on 8 February 1954 was 218.2\(^\circ\).

\[ \angle E_j MS = \alpha_j = 218.2^\circ - 182.0^\circ = 36.2^\circ. \]

The three interior angles in \(\angle SE_i M\) and \(\angle SE_j M\) are then all known, and SM is a common side. Therefore, by the law of sines,

\[
\frac{r_i}{\sin \alpha_i} = \frac{SM}{\sin \mu_i}, \quad \frac{r_j}{\sin \alpha_j} = \frac{SM}{\sin \mu_j}.
\]
The relationship between \( r_i \) and \( r_j \) is as follows:

\[
\frac{r_j}{r_i} = \frac{\sin \mu_j \sin \alpha_j}{\sin \mu_i \sin \alpha_i}.
\]

Originally, it was very hard to measure the actual distance between the Sun and the Earth. Now their ratio can be obtained from the corresponding angles \( \angle SE_iM \) and \( \angle SE_jM \) spanned by the lines connecting the Sun and Mars to the Earth 1 Martian year before and after the date of the opposition of Mars, respectively, and from the angles \( \angle E_iMS \) and \( \angle E_jMS \) spanned by the line of opposition of Mars and by the line connecting the Earth to Mars 1 Martian year before and after the date of the opposition of Mars, respectively. If \( \omega_i \) and \( \omega_j \) are the angular velocities of the Earth at \( E_i \) and \( E_j \) relative to the Sun, they can be calculated by the angles swept by the Earth in one day after the dates at \( E_i \) and \( E_j \), respectively. Since the longitudes of the Earth as seen from the Sun on 5 May 1948 and 6 May 1948 are 224.897° and 225.865°, respectively, the angular speed \( \omega_i = 225.866 - 224.897 = 0.969 \). Similarly, \( \omega_j = 139.561 - 138.549 = 1.012 \).

The law of equal areas for the Earth means the line connecting the Earth to the Sun sweeps an equal area in the same period of time, as shown in figure 3. Namely,

\[
\Delta A = \frac{1}{2} r^2 \Delta \theta
\]

and

\[
\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t}.
\]

So the areal velocity is

\[
\frac{dA}{dr} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega,
\]

where \( \omega \) is the angular speed of the Earth around the Sun.

Hence, in order to prove the law of equal areas it is only necessary to show that the product of the square of the distance from the Earth to the Sun and the corresponding angular speed of the Earth is a constant, i.e.

\[
r_i^2 \omega_i = r_j^2 \omega_j. \tag{1}
\]

The position of the Earth \( E_i \), which occurred 1 Martian year before the opposition of Mars as a reference point, is selected. Then six more positions of the Earth, \( E_j \), are chosen, with three of them occurring 1, 2 and 3 Martian years before the time of the reference point, \( E_i \). From observations of these \( E_j \)'s, one can find the corresponding values of the ratios \( r_j^2/r_i^2 \) and \( \omega_i/\omega_j \), as shown in table 1.

From the results in the last two columns in the table, one may see that the difference between \( r_j^2/r_i^2 \) and \( \omega_i/\omega_j \) is very small, less than 1%, and can be treated as equal. In other words, \( r_j^2 \omega_i = r_i^2 \omega_j \).
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**Table 1.** \( \frac{r_j^2}{r_i^2} \) and \( \frac{\omega_j}{\omega_i} \) obtained from the observatory data.

| Time              | \( \mu_i \) | \( \mu_j \) | \( \alpha_i \) | \( \alpha_j \) | \( \omega_i \) | \( \omega_j \) | \( \frac{r_j^2}{r_i^2} \) | \( \frac{\omega_j}{\omega_i} \) |
|-------------------|-------------|-------------|----------------|----------------|---------------|---------------|----------------|-----------------|
| 13 September 1942 | 7.5°        | 4.6°        | 0.974          | 1.005          | 0.995         |
| 31 July 1944      | 33.8°       | 20.2°       | 0.956          | 1.026          | 1.014         |
| 18 June 1946      | 62.4°       | 33.2°       | 0.954          | 1.016          | 1.016         |
| 5 May 1948        | 100.2°      | 37.1°       | 0.969          |               |               |
| 8 February 1952   | 100.1°      | 36.2°       | 1.012          | 0.959          | 0.958         |
| 26 December 1953  | 60.6°       | 31.5°       | 1.019          | 0.958          | 0.951         |
| 13 November 1955  | 30.3°       | 17.7°       | 1.007          | 0.967          | 0.962         |

**Table 2.** The values of \( \omega_k \), \( r_k^2/r_i^2 \) and \( \omega_j/\omega_i \) obtained on 30 different dates.

| Dates             | \( \omega_i \) | \( \omega_k \) | \( r_k^2/r_i^2 \) | \( \omega_j/\omega_i \) | Dates             | \( \omega_i \) | \( \omega_k \) | \( r_k^2/r_i^2 \) | \( \omega_j/\omega_i \) |
|-------------------|---------------|---------------|----------------|----------------------|-------------------|---------------|---------------|----------------|----------------------|
| 5 May 1948        | 0.969         |               |               |                      | 5 May 1948        | 0.969         |               |               |                      |
| 26 October 1940   | 0.998         | 0.959         | 0.971          | 30 September 1957    | 0.983             | 0.982         | 0.986         |               |                      |
| 9 December 1938   | 1.016         | 0.951         | 0.954          | 18 August 1959       | 0.961             | 1.008         | 1.008         |               |                      |
| 21 January 1937   | 1.017         | 0.953         | 0.952          | 5 July 1961          | 0.953             | 1.016         | 1.016         |               |                      |
| 6 March 1935      | 1.001         | 0.968         | 0.968          | 23 May 1963          | 0.962             | 1.005         | 1.008         |               |                      |
| 18 April 1933     | 0.977         | 0.991         | 0.992          | 9 April 1965         | 0.982             | 0.988         | 0.987         |               |                      |
| 1 June 1931       | 0.958         | 1.008         | 1.012          | 25 February 1967     | 1.005             | 0.958         | 0.964         |               |                      |
| 14 July 1929      | 0.954         | 1.019         | 1.016          | 12 January 1969      | 1.019             | 0.950         | 0.951         |               |                      |
| 27 August 1927    | 0.966         | 1.011         | 1.003          | 29 November 1970     | 1.014             | 0.953         | 0.956         |               |                      |
| 9 October 1925    | 0.988         | 0.993         | 0.980          | 16 October 1972      | 0.992             | 0.974         | 0.976         |               |                      |
| 22 November 1923  | 1.010         | 0.953         | 0.959          | 3 September 1974     | 0.969             | 0.999         | 1.000         |               |                      |
| 4 January 1922    | 1.019         | 0.951         | 0.951          | 21 July 1976         | 0.955             | 1.012         | 1.015         |               |                      |
| 17 February 1920  | 1.009         | 0.957         | 0.960          | 8 June 1978          | 0.957             | 1.010         | 1.013         |               |                      |
| 1 April 1918      | 0.986         | 0.980         | 0.983          | 25 April 1980        | 0.973             | 0.991         | 0.996         |               |                      |
| 14 May 1916       | 0.964         | 1.005         | 1.005          | 13 March 1982        | 0.997             | 0.974         | 0.972         |               |                      |
| 27 June 1914      | 0.954         | 1.017         | 1.016          | 29 January 1984      | 1.016             | 0.951         | 0.954         |               |                      |

All of the data shown in table 1 are based upon the positions of the Earth and its corresponding angular speed at the opposition of Mars on 25 March 1950, and 1 to 4 Martian years before and after, respectively. Similarly, these procedures can be continued for up to 30 more positions of the Earth, \( E_k \), and its corresponding angular speed, \( \omega_k \), by including the data from 5 to 20 Martian years before and after the aforesaid opposition (table 2). The relationship \( r_j^2/\omega_j = r_i^2/\omega_i \) still holds.

Because the periods of the Martian year, 687 days, and that of the Earth year, 365 days, are mutually prime, the positions of the Earth recorded every Martian year before and after 23 March 1950 will cover almost every position along the orbit of the Earth around the Sun. The positions selected by the method shown above are so dense as to approach generality. Due to this ergodicity, the law of equal areas may be established: \( r^2 \omega = \text{constant} \).

Certainly, other oppositions of Mars can be examined, for example, 10 February 1916 and 12 February 1995. The difference between these two oppositions is only 1.8°. The procedures may be repeated as in table 2, and the relation of equal areas may be verified: \( r^2 \omega = \text{constant} \). In retrospect, this method for establishing the law of equal areas is, on one hand, to apply the information on the oppositions of Mars as well as the period of Mars and, on the other hand,
to use the mutual prime property between the 687 and 365 day periods of Mars and Earth, respectively, to guarantee ergodic distributions of the selected positions of the Earth.

It is very hard to directly measure the distances from the Earth or other planets to the Sun, or the ratio of the distances at two different positions. Nevertheless, it is relatively easy to observe the angles swept by the Earth. The establishment of the law of equal areas helps us, through measuring the angular speed of the planets, to derive the ratio of distances from the planets to the Sun at different times. This is the implicit meaning of the law of equal areas, which can be used effectively to determine distances from the planets to the Sun.

3. The law of ellipses for the Earth

After constructing the law of equal areas for the Earth, the next task is to establish the law of elliptical orbits for the Earth. Since the motion of the Earth around the Sun is regularly periodic, the distance function $r(\theta)$ from the Earth to the Sun, or the reciprocal of $r(\theta)$, can also be expressed as a periodic function of $\theta$. Namely, it can be expressed as an infinite series of sines and cosines with different multiple angles as follows [9]:

$$\frac{1}{r} = a_0 + \sum_{n} a_n \cos n\theta + \sum_{n} b_n \sin n\theta \quad (n = 1, 2, 3, \ldots).$$

In the ideal case, this function can be approximated by a single period of the trigonometric functions, i.e.

$$\frac{1}{r} = a_0 + a_1 \cos \theta + b_1 \sin \theta.$$  \hspace{1cm} (2)

By applying the law of equal areas as shown in (1), or

$$\frac{1}{r} = c \sqrt{\omega},$$

where $c$ is a proportional constant, the periodic function of the reciprocal of the distance may be expressed as follows:

$$\sqrt{\omega} = c_0 + c_1 \cos \theta + c_2 \sin \theta.$$

By doing this, the observable angular speeds $\omega$ may be used to replace the unobservable distances $r$. In order to find the three unknown coefficients $c_0, c_1$ and $c_2$, as shown in the above equation, one has to choose three sets of data to set up simultaneous linear equations with three unknowns. One may randomly select three sets of data on 23 April 1998, 31 July 1998 and 2 October 1998 to form

$$\sqrt{\omega_1} = c_0 + c_1 \cos \theta_1 + c_2 \sin \theta_1$$

$$\sqrt{\omega_2} = c_0 + c_1 \cos \theta_2 + c_2 \sin \theta_2$$

$$\sqrt{\omega_3} = c_0 + c_1 \cos \theta_3 + c_2 \sin \theta_3,$$

where $\theta_1, \theta_2$ and $\theta_3$ are the inclined angles between the line connecting the Earth to the Sun at three different dates and the $x$-axis, which is set along the line connecting the Earth to the Sun on 27 January 1998. Hence $\theta_1 = 213.2^\circ - 127.4^\circ = 85.8^\circ$, as shown in table 3.

From the values of $\theta_1, \theta_2, \theta_3, \omega_1, \omega_2$ and $\omega_3$ in table 3, the solutions to the above simultaneous linear equations in three unknowns can be found. Their solutions are as follows:

$$c_0 = 0.993 \quad c_1 = 0.015 \quad c_2 = -0.007.$$
Hence, the periodic function for the square root of angular speed at three distinct positions of the Earth, or the reciprocal of the distances from three different positions of the Earth to the Sun, is as follows:

\[
\sqrt{\omega} = 0.993 + 0.015 \cos \theta - 0.007 \sin \theta.
\]  

(3)

Furthermore, seven different positions of the Earth are randomly selected for seven different dates, and set \( d = 0.993 + 0.015 \cos \theta - 0.007 \sin \theta \). By comparing the difference of \( \sqrt{\omega} \) and \( d \) from the corresponding angular speed \( \omega \), whether the selected position of the Earth satisfies the periodic function of the distance can be verified, as shown in table 4. This table shows that \( \sqrt{\omega} \) is identical to \( d \), and this supports the validity of the periodic function in (3).

In fact, the periodic function shown in (2) and (3) is exactly the same as that for the path of an elliptical orbit. If one takes the line connecting the two foci to be the \( x \)-axis, then the path of elliptical orbit can be expressed as follows [10]:

\[
r = \frac{a(1-e^2)}{1+e \cos \theta} \quad \text{or} \quad \frac{1}{r} = B + A \cos \theta,
\]

where \( e \) is eccentricity, \( a \) is the semi-major axis, \( B = 1/a(1-e^2) \) and \( A = e/a(1-e^2) \). If the \( x \)-axis is not the line connecting the two foci, then the equation for the ellipse is as follows:

\[
\frac{1}{r} = B + A \cos (\theta - \theta_0) = B + d_1 \cos \theta + d_2 \sin \theta,
\]

(4)

where \( \theta_0 \) is the angle between the line connecting the perihelion to the origin and the \( x \)-axis, \( d_1^2 + d_2^2 = A^2 \), and \( \sqrt{d_1^2 + d_2^2 / B} = A/B = e \). Hence, the periodic function of the reciprocal of the distance shown in (2) or (3) has the form for the equation of ellipse, and the ratio of \( \sqrt{c_1^2 + c_2^2} / c_0 \) and \( c_0 \) is as follows:

\[
\frac{\sqrt{c_1^2 + c_2^2}}{c_0} = \sqrt{(-0.007)^2 + 0.015^2} = 0.017.
\]
which is exactly the same as the accepted value 0.017 for the eccentricity of the Earth’s orbit. These can obviously show the equivalence of the periodicity of the reciprocal of the distance of the Earth to the equation of an ellipse, and verify firmly that the orbit of the Earth is exactly elliptical.

4. Conclusions

Applying the specific properties of the oppositions of Mars and the Martian year, the position of Mars relative to the Sun can be fixed in the celestial sphere. By means of the geometric relationship formed by these two fixed points and the motions of the Earth, one may overcome the obstacles due to the unobservable changes in the distances between the Earth and the Sun, and express them in terms of relatively easily measurable angles. Through these, the laws of equal areas and elliptical orbits for the Earth can be concisely established, which should confer a very deep and sincere admiration upon Johannes Kepler and the insights he contributed some 400 years ago.

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