AUTOMORPHISMS OF NONSPLIT COVERINGS OF $PSL_2(q)$
IN ODD CHARACTERISTIC DIVIDING $q - 1$

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Abstract. We classify the nonsplit extensions of elementary abelian $p$-groups by $PSL_2(q)$, with odd $p$ dividing $q - 1$, for an irreducible induced action, calculate the relevant low-dimensional cohomology groups, and describe the automorphism groups of such extensions.

Keywords: Automorphism group, nonsplit extension, cohomology.

1. Introduction

Given a short exact sequence of groups

$$0 \rightarrow V \rightarrow G \rightarrow L \rightarrow 1,$$

where $V$ is abelian (written additively), we say that $G$ is an extension of $V$ by $L$, or a covering of $L$ with kernel $V$. Such extensions arise naturally in inductive arguments or when constructing minimal examples and counterexamples. We will be interested in the case where $G$ is finite and nonsplit and $V$ acquires the structure of an irreducible $FL$-module (for a suitable finite field $F$ of characteristic $p$) from the conjugation in $G$. Such extensions can only exist if $p$ divides $|L|$. We also restrict ourselves to the case $L \cong PSL_2(q)$. Extensions of this form for $p = 2$ and $q$ odd were explicitly constructed in [1], and their automorphism groups were described in [11]. Some results in the case of $q$ being a power of $p$ were obtained in [2].

The aim of this paper is to classify such extensions in the case $2 \neq p | (q - 1)$ and describe their automorphism groups. In this case, we can use the fact that the natural permutation $FL$-module arising from the action of $L$ on the projective line over $F_q$ is completely reducible. This is not so if $2 \neq p | (q + 1)$ which case will be a subject of future research. We now state the main results.

Theorem 1. Up to isomorphism there is a unique nonsplit extension of an elementary abelian $p$-group $V$ by $L = PSL_2(q)$ with irreducible induced action of $L$ on $V$, where $2 \neq p | (q - 1)$. In this extension, $|V| = p^q$.

The group $V$ from Theorem 1 as an $F_p L$-module can be identified with the unique nonprincipal irreducible module in the principal $p$-block of $L$. The low-dimensional cohomology of $V$ is as follows.

Theorem 2. In the above notation, we have $H^1(L, V) \cong H^2(L, V) \cong F_p$.

Recall that $PGL_2(q)$ denotes the extension of $PGL_2(q)$ by its field automorphisms. The automorphism group of the nonsplit extension from Theorem 1 is described by

Theorem 3. Let $G$ fit in the nonsplit exact sequence (1), where $V$ is an irreducible $F_p L$-module for $L = PSL_2(q)$ and $2 \neq p | (q - 1)$. Then there is a short exact sequence

$$0 \rightarrow V \rightarrow G \rightarrow L \rightarrow 1,$$
sequence

$$0 \to W \to \text{Aut}(G) \to \text{PGL}_2(q) \to 1,$$

where $W$ is elementary abelian of order $p^{q+1}$.

2. Auxiliary facts

Basic notation and facts of homological algebra can be found in [7, 13]. For abelian groups $A$ and $B$, we denote $\text{Hom}(A,B) = \text{Hom}_\mathbb{Z}(A,B)$ and $\text{Ext}(A,B) = \text{Ext}_1^\mathbb{Z}(A,B)$.

Lemma 4 (The Universal Coefficient Theorem for Cohomology). [7, Ch. 3, Theorem 3] For all $i \geq 1$, every group $G$, and every trivial $G$-module $A$,

$$H^i(G, A) \cong \text{Hom}(H_i(G, \mathbb{Z}), A) \oplus \text{Ext}(H_{i-1}(G, \mathbb{Z}), A).$$

Lemma 5. [7, §3.5] For a trivial $G$-module $A$, we have

(i) $H^1(G, A) \cong \text{Hom}(G/G', A)$.
(ii) $H_1(G, A) \cong G/G' \otimes \mathbb{Z} A$.

Lemma 6 (Shapiro’s lemma). [13, §6.3] Let $H \leq G$ with $|G : H|$ finite. If $V$ is an $H$-module and $i \geq 0$ then $H^i(G, V^G) \cong H^i(H, V)$, where $V^G$ is the induced $G$-module.

Lemma 7. [5, p. 322] $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where $d = (m, n)$.

Lemma 8. [13, Proposition 3.3.4]. $\text{Ext}_R^i(A, B_1 \oplus B_2) \cong \text{Ext}_R^i(A, B_1) \oplus \text{Ext}_R^i(A, B_2)$ for all rings $R$, $R$-modules $A, B_1, B_2$, and all $i \geq 0$.

The Schur multiplier of a group $G$ is denoted by $\text{Sch}(G)$. If $A$ is a finite abelian group and $p$ a prime then $A(p)$ denotes the $p$-primary component of $A$. Henceforth, we assume that $G$ is finite.

Lemma 9. [9, Theorem 25.1] Let $p$ be a prime and let $S \in \text{Syl}_p(G)$. Then $\text{Sch}(G)(p)$ is isomorphic to a subgroup of $\text{Sch}(S)$.

Lemma 10. [6] Let $F$ be a field of characteristic $p > 0$ and let $V$ be an irreducible $FG$-module that does not belong to the principal $p$-block of $G$. Then $H^n(G, V) = 0$ for all $n \geq 0$.

Let $\theta$ be an irreducible character of $G$. If $Z(G) = 1$ then $G \cong \text{Aut}(G)$ and we may speak of the inertia group $I_{\text{Aut}(G)}(\theta) = \{g \in \text{Aut}(G) \mid \theta^g = \theta\}$.

Proposition 11. [11] Proposition 4] Let $F$ be a field and $\mathcal{X}$ a faithful irreducible $F$-representation of a group $G$ with Brauer character $\theta \in \text{iBr}_F(G)$ of degree $n$. Suppose that $Z(G) = 1$ and denote

$$N = N_{\text{GL}_n(F)}(\mathcal{X}(G)) \quad \text{and} \quad Z = C_{\text{GL}_n(F)}(\mathcal{X}(G)).$$

Then $N/Z \cong I_{\text{Aut}(G)}(\theta)$.
3. Isomorphic extensions

Let $Q$ be a group, $K$ a commutative ring with 1, and $M$ a right $KQ$-module. The pair $(\nu, \mu) \in \text{Aut}(Q) \times \text{Aut}_K(M)$ is \textit{compatible} if
\[(mg)\mu = (m\mu)(g\nu)\]
for all $m \in M$, $g \in Q$. The set of all compatible pairs forms a group $\text{Comp}(Q, M)$ under composition. Given $\tau \in Z^2(Q, M)$, one can define
\[\tau^{(\nu, \mu)}(g, h) = \tau(g\nu^{-1}, h\nu^{-1})\mu\]
for all $g, h \in Q$. Then the map $\tau \mapsto \tau^{(\nu, \mu)}$ is an action of $\text{Comp}(Q, M)$ on $Z^2(Q, M)$ which preserves $B^2(Q, M)$ and so yields an action on $H^2(Q, M)$.

A $KQ$-module extension on $M$ by $Q$ is a group $E$ that fits in the short exact sequence
\[0 \to M \xrightarrow{i} E \xrightarrow{\pi} Q \to 1\]
so that the conjugation of $M$ (identified with $M_1$) by elements of $E$ agrees with the $KQ$-module structure of $M$, i.e. $m^e = m(e\pi)$ for all $m \in M$, $e \in E$.

\textbf{Proposition 12.} \cite{3} [2.7.4] \textit{The classes of those isomorphisms of $KQ$-module extensions of $M$ by $Q$ that leave $M$ invariant as a $K$-module are in a one-to-one correspondence with the orbits of $\text{Comp}(Q, M)$ on $H^2(Q, M)$.}

In Proposition 12, an isomorphism leaving $M$ invariant as a $K$-module means one that induces on $M$ an element of $\text{Aut}_K(M)$.

The $KQ$-module structure on $M$ gives rise to the representation homomorphism $C : Q \to \text{Aut}_K(M)$ by the rule $C(g) : m \mapsto mg$ for all $m \in M$, $g \in Q$. Let $C$ be the centraliser of $C(Q)$ in $\text{Aut}_K(M)$. Then $(1, \gamma) \in \text{Comp}(Q, M)$ for every $\gamma \in C$, because
\[(mg)\gamma = mC(g)\gamma = m\gamma C(g) = (m\gamma)g\]
for all $m \in M$, $g \in Q$. Hence, we also have an action of $C$ on both $Z^2(Q, M)$ and $H^2(Q, M)$ by setting $\tau^{\gamma} = \tau^{(1, \gamma)}$ for $\tau \in Z^2(Q, M)$, $\gamma \in C$, i.e. $\tau^{\gamma}(g, h) = \tau(g, h)^\gamma$. By Proposition 12 this yields the following:

\textbf{Lemma 13.} \textit{The elements of $H^2(Q, M)$ that are in the same $C$-orbit correspond to isomorphic $KQ$-module extensions.}

In particular, we have the following fact, where two elements of $H^2(Q, M)$ are called scalar multiples if they differ by a factor in $K^\times$.

\textbf{Corollary 14.} \textit{$KQ$-module extensions of $M$ by $Q$ corresponding to scalar multiples in $H^2(Q, M)$ are isomorphic.}

4. Automorphisms of extensions

Fix an extension
\[e : 0 \to M \xrightarrow{i} E \to Q \to 1\]
with abelian kernel $M$. Let $C : Q \to \text{Aut}(M)$ be the induced representation and let $\tau \in H^2(Q, M)$ be the element that corresponds to $e$. We assume that $C$ is faithful. In particular, $Q \cong C(Q)$ and the conjugation of $C(Q)$ by any $\mu \in N_{\text{Aut}(M)}(C(Q))$
induces an element \( \mu' \in \text{Aut}(Q) \), i.e. \( C(g)\mu = C(g\mu') \) for all \( g \in Q \). One defines an action of \( N_{\text{Aut}(M)}(C(Q)) \) on \( H^2(Q, M) \) given by

\[
\overline{\psi} \mapsto (\mu')^{-1}\overline{\psi}\mu
\]

for every \( \mu \in N_{\text{Aut}(M)}(C(Q)) \) and \( \overline{\psi} \in H^2(Q, M) \), which should be understood modulo \( B^2(Q, M) \) for representative cocycles, see [12] for details. We denote by \( N_{\text{Aut}(M)}(C(Q)) \) the stabilizer of \( \overline{\psi} \) with respect to this action. Let \( \text{Aut}(e) \) denote the group of those automorphisms of \( E \) that leave \( M \) invariant as a set.

**Proposition 15.** [12] Statements (4.4),(4.5) Let the extension (5) have an abelian kernel \( M \) and let it determine an element \( \overline{\psi} \in H^2(Q, M) \) and an injective induced representation \( C : Q \to \text{Aut}(M) \). Then there exists a short exact sequence of groups

\[
0 \to Z^1(Q, M) \to \text{Aut}(e) \to N_{\text{Aut}(M)}(C(Q)) \to 1.
\]

**Remark.** It is easy to see that, in the notation above, there is an embedding \( N_{\text{Aut}(M)}(C(Q)) \to \text{Comp}(Q, M) \), \( \mu \mapsto (\mu', \mu) \), where we view \( M \) as a \( \mathbb{Z}Q \)-module, under which action (6) becomes a particular case of (3), and that this embedding is in fact an isomorphism in case \( C \) is faithful (which we assume).

5. **Cohomology of \( PSL_2(q) \) in characteristic dividing \( q - 1 \)**

The aim of this section is to classify up to group isomorphism nonsplit extensions (1), where \( L = PSL_2(q) \), \( V \) is an elementary abelian \( p \)-group with irreducible induced action of \( L \), and \( p \neq 2 \) is a divisor of \( q - 1 \).

By Lemma [10], \( V \) must belong to the principal \( p \)-block of \( L \). This block contains only one nonprincipal module with Brauer character \( \chi \), see [3]. The values of characters in the principal block are shown in Table 1.

| \( q \) odd | 1a | 2a | la | lb | \((x^r)^L\) | \((y^r)^L\) | \( q \) even | 1a | 2a | \((x^r)^L\) | \((y^r)^L\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 \( \chi \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| q | -1 | 0 | 0 | 1 | -1 | 0 | q | 1 | 1 | 1 | -1 |

We first note that \( V \) is not the principal module. Indeed, extension (1) would otherwise be central, but \( \text{Sch}(L) \) has no \( p \)-torsion, because

\[
\text{Sch}(L) = \left\{ \begin{array}{ll}
\mathbb{Z}_2, & q \neq 9 \text{ odd or } q = 4; \\
\mathbb{Z}_6, & q = 9; \\
1, & q \neq 4 \text{ even}
\end{array} \right.
\]

as follows from [4]. Therefore, \( V \) must be the \( \mathbb{F}_p \)-module with character \( \chi \).

We can now prove Theorem 2 stated in the introduction.

**Proof.** Let \( P \) be the permutation \( \mathbb{F}_p \)-module of dimension \( q + 1 \) that corresponds to the natural permutation action of \( L \) on the projective line over \( \mathbb{F}_q \). We have \( P = I_L \oplus V \), where \( I_L \) is the principal \( \mathbb{F}_p \)-module. This can be deduced either by considering the Brauer character \( \chi \) of \( V \) or from [10] Table 1. In particular, by Lemma 8 we have

\[
H^i(L, P) \cong H^i(L, I_L) \oplus H^i(L, V)
\]
Consider the extension $\phi$ follows. By Corollary 14, they correspond to isomorphic nonsplit extensions. The claim has Brauer character $\chi$ from Table 1. By Proposition 15, we have the short exact sequence $0 \to Z^1(L, V) \to \text{Aut}(e) \to N_{\text{Aut}(V)}(\chi(L)) \to 1$, \hfill (16)

where the representation $\chi : L \to \text{Aut}(V)$ and the element $\overline{\varphi} \in H^2(L, V)$ are determined by \hfill (15). First, note that $\text{Aut}(e) = \text{Aut}(G)$ as $V$ is characteristic in $G$. Denote $W = Z^1(L, V)$. Since $B^3(L, V) \cong V/C_V(L)$ and $L$ acts on $V$ irreducibly

for $i = 1, 2$, since $H^i(L, B) \cong \text{Ext}^i_{\mathbb{F}_p}(F_p, B)$ for every $\mathbb{F}_pL$-module $B$, see \hfill [13]

Exercise 6.1.2]. Since $P$ is a permutation module, we have $P \cong (I_H)^G$, where $I_H$ is the principal $\mathbb{F}_pH$-module for a point stabiliser $H \leq L$. Hence, Lemma 3 implies

$$H^i(L, P) \cong H^i(H, I_H)$$ \hfill (10)

for $i = 1, 2$. By Lemma 5(i), we have

$$H^1(L, I_L) \cong \text{Hom}(L/L', I_L) = 0,$$ \hfill (11)

since $L = L'$. Also,

$$H^1(H, I_H) \cong \text{Hom}(H/H', I_H) \cong \mathbb{F}_p,$$ \hfill (12)

since $I_H \cong \mathbb{F}_p$, $H \cong \mathbb{F}_q \times \mathbb{Z}_{q-1}/(2,q-1)$, and $p \mid (q-1)$. By Lemma 6 we have

$$H^2(L, I_L) \cong \text{Hom}(H_2(L, \mathbb{Z}), I_L) \cong \text{Ext}(H_1(L, \mathbb{Z}), I_L),$$

where the first summand vanishes, since $H_2(L, \mathbb{Z}) \cong \text{Sch}(L)$ has no $p$-torsion by \hfill [5], and the second summand vanishes by Lemma 5(ii), since $L/L' = 1$. Thus

$$H^2(L, I_L) = 0.$$ \hfill (13)

Finally, Lemma 4 also yields

$$H^2(H, I_H) \cong \text{Hom}(H_2(H, \mathbb{Z}), I_H) \cong \text{Ext}(H_1(H, \mathbb{Z}), I_H).$$ \hfill (14)

By Lemma 9 the $p$-part of $H_2(H, \mathbb{Z}) \cong \text{Sch}(H)$ is isomorphic to a subgroup of $\text{Sch}(S)$ for a $p$-Sylow subgroup $S$ of $H$. However, $S$ is cyclic and cyclic groups have trivial Schur multiplier. Thus, the first summand in (14) vanishes, because $I_H \cong \mathbb{F}_p$. Since $H_1(H, \mathbb{Z}) \cong H/H' \cong \mathbb{Z}_{(q-1)/d}$ and $\text{Ext}(\mathbb{Z}_{(q-1)/d}, \mathbb{F}_p) \cong \mathbb{F}_p$ by Lemma 7, we have

$$H^2(H, I_H) \cong \mathbb{F}_p.$$ \hfill (15)

The claim follows by combining \hfill (9) through \hfill (15).

We can now prove Theorem 1 stated in the introduction.

Proof. As we explained in the beginning of this section, $V$ viewed as an $\mathbb{F}_pL$-module must be the unique nonprincipal module in the principal $p$-block of $L$. This module has dimension $q$ and can be written over $\mathbb{F}_p$, since it is a direct summand of a permutation module. Therefore, $|V| = p^q$. By Theorem 2, we have $H^2(V, L) \cong \mathbb{F}_p$ and so all nonzero elements of $H^2(V, L)$ are scalar multiples of one another. By Corollary 14 they correspond to isomorphic nonsplit extensions. The claim follows. \hfill $\square$

6. The Automorphism Group

In this section, we prove that the structure of the automorphism group of the unique nonsplit extension from Theorem 1 is as stated in Theorem 3.

Proof. Consider the extension $e$ given by \hfill (11). Theorem 1 implies that $G$ is unique up to isomorphism and $V$ has order $p^q$. Moreover, viewed as an $\mathbb{F}_pL$-module, $V$ has Brauer character $\chi$ from Table 1. By Proposition 15, we have the short exact sequence

$$0 \to Z^1(L, V) \to \text{Aut}(e) \to N_{\text{Aut}(V)}(\chi(L)) \to 1,$$ \hfill (16)

where the representation $\chi : L \to \text{Aut}(V)$ and the element $\overline{\varphi} \in H^2(L, V)$ are determined by \hfill (15). First, note that $\text{Aut}(e) = \text{Aut}(G)$ as $V$ is characteristic in $G$. Denote $W = Z^1(L, V)$. Since $B^3(L, V) \cong V/C_V(L)$ and $L$ acts on $V$ irreducibly
and nontrivially, we have $C_V(L) = 0$ and $B^1(L, V) \cong V$. Now, since $H^1(L, V) = Z^1(L, V) / B^1(L, V)$, we have $|Z^1(L, V)| = p^{r+1}$ in view of Theorem 2.

Denote $N = \bar{N}_{GL(V)}(\mathcal{X}(L))$ and $Z = C_{GL(V)}(\mathcal{X}(L))$. By Proposition 11, we have $N/Z \cong I_{\text{Aut}(L)}(\chi)$. Since $\chi$ is the only irreducible character of $L$ of dimension $q$, it must be invariant under any automorphism; in particular, $I_{\text{Aut}(L)}(\chi) = \text{Aut}(L)$. By [11], $\text{Aut}(L) \cong \text{PTL}_2(q)$. Since $V$ is absolutely irreducible as an $\mathbb{F}_p$-$L$-module, by Schur’s lemma, we see that $Z \cong \mathbb{F}_p^r \cong \mathbb{Z}_{p-1}$ consists of scalars.

In order to determine the structure of the stabiliser $N_0 = N_{\text{Aut}(V)}(\mathcal{X}(L))$, we consider the action of $N$ on $H^2(L, V)$ as explained in Section 4. Let $H^\times$ denote the set of $p - 1$ nonzero elements of $H^2(L, V)$. The elements of $H^\times$ correspond to nonsplit extensions and so we have an action homomorphism $\alpha : N \to \text{Sym}(H^\times)$ to the symmetric group on $H^\times$. Since all nonsplit extension of $V$ by $L$ are isomorphic by Theorem 11 we may assume that $\mathcal{V}$ is an arbitrary element of $H^\times$. The subgroup $Z \leq N$ acts on $H^\times$ by scalar multiplication, cf. Corollary 11 and so the image $\alpha(Z)$ is a cyclic subgroup of $\text{Sym}(H^\times)$ generated by a full cycle of length $p - 1$. Since $Z$ is central in $N$, $\alpha(N)$ must centralise $\alpha(Z)$. However, a full cyclic subgroup is self-centralising in $\text{Sym}(H^\times)$ and so $\alpha(Z)$ must be the entire image $\alpha(N)$. Thus, $\text{Ker}(\alpha)$ is a normal subgroup of $N$ of index $p - 1$ which intersects trivially with $Z$ and is thus isomorphic to $N/Z \cong \text{PTL}_2(q)$. Furthermore, $\ker(\alpha)$ coincides with the stabiliser of every element of $H^\times$ which yields $N = N_0 \times Z$ and $N_0 \cong \text{PTL}_2(q)$ as claimed.

It also follows from this proof that the representation $\mathcal{X} : L \to \text{Aut}(V)$ with character $\chi$ extends to a representation of $I_{\text{Aut}(L)}(\chi) \cong \text{PTL}_2(q)$. This fact does not hold in general for a simple group $L$ and its irreducible character $\chi$, see [11] Example 1.

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