New Conditions for the Oscillation of Second-Order Differential Equations with Sublinear Neutral Terms

Shyam Sundar Santra $^{1,†,}$, Omar Bazighifan $^{2,*†,}$ and Mihai Postolache $^{3,4,5,6,*†,}$

Abstract: In continuous applications in electrodynamics, neural networks, quantum mechanics, electromagnetism, and the field of time symmetric, fluid dynamics, neutral differential equations appear when modeling many problems and phenomena. Therefore, it is interesting to study the qualitative behavior of solutions of such equations. In this study, we obtained some new sufficient conditions for oscillations to the solutions of a second-order delay differential equations with sublinear neutral terms. The results obtained improve and complement the relevant results in the literature. Finally, we show an example to validate the main results, and an open problem is included.

Keywords: neutral; oscillation; non-oscillation; non-linear; Lebesgue’s dominated convergence theorem

1. Introduction

It is well known that the differential equations have many applications to the study of population growth, decay, Newton’s law of cooling, glucose absorption by the body, the spread of epidemics, Newton’s second law of motion, and interacting species (competition), to name a few. They appear in the study of many real-world problems (see, for instance, [1–3]). We also stress that the modeling of these phenomena is suitably formulated by evolutive partial differential equations, and moreover, moment problem approaches also appear as a natural instrument in the control theory of neutral-type systems; see [4–7], respectively.

Next, we highlight some current developments in oscillation theory for second-order differential equations of the neutral type.

Santra et al. [8] studied the qualitative behavior of the following highly nonlinear neutral differential equations:

\[ \left( b(\vartheta) \left( u(\vartheta) + p(\vartheta)u(\vartheta - \alpha) \right) \right)' + \sum_{i=1}^{m} q_i(\vartheta) G_i(u(\vartheta - \beta_i)) = 0 \]  

(1)

and established some new conditions for the oscillation of the solution of (1) under a non-canonical operator with various ranges of the neutral coefficient $p$. In another paper [9], Santra et al. established some new oscillation theorems for the differential equations of the neutral type with mixed delays under the canonical operator with $0 \leq p < 1$. By using different methods, the following papers, which have the same research topic as that of...
this paper, were concerned with the oscillation of various classes of half-linear/Emden–Fowler differential equations with different neutral coefficients, e.g.: the paper [10] was concerned with neutral differential equations assuming that \(0 \leq p(\vartheta) < 1\) and \(p(\vartheta) > 1\) where \(p\) is the neutral coefficient; the paper [11] was concerned with neutral differential equations assuming that \(0 \leq p(\vartheta) < 1\); the paper [12] was concerned with neutral differential equations assuming that \(p(\vartheta)\) is nonpositive; the papers [13,14] were concerned with neutral differential equations in the case where \(p(\vartheta) > 1\); the paper [15] was concerned with neutral differential equations assuming that \(0 \leq p(\vartheta) \leq p_0 < \infty\) and \(p(\vartheta) > 1\); the paper [16] was concerned with neutral differential equations in the case where \(0 \leq p(\vartheta) \leq p_0 < \infty\); the paper [17] was concerned with neutral differential equations in the case when \(0 \leq p(\vartheta) = p_0 \neq 1\); whereas the paper [18] was concerned with differential equations with a nonlinear neutral term assuming that \(0 \leq p(\vartheta) \leq p < 1\).

For more details on the oscillation theory for second-order neutral differential equations/impulsive differential equations, we refer the reader to the papers [4,8,9,19–42].

Motivated by the above studies, in this paper, we established some new sufficient conditions for the oscillation of solutions to second-order non-linear differential equations in the form:

\[
\left( b(\vartheta)(h'(\vartheta))\Delta_{1}\right)' + q(\vartheta)u^{\Delta}(\beta(\vartheta)) = 0, \quad \vartheta \geq \vartheta_0, \tag{2}
\]

where \(h(\vartheta) = u(\vartheta) + \sum_{k=1}^{n} p_k(\vartheta)u^{v_k}(a_k(\vartheta)), v_k, \) for all \(k = 1, 2, ..., n, \) and \(\Delta_1\) and \(\Delta\) are quotients of odd positive integers such that:

(a) \(\beta, v_k \in C([\vartheta_0, \infty), \mathbb{R}_+), a_k \in C^2([\vartheta_0, \infty), \mathbb{R}_+), \beta(\vartheta) < \vartheta, a_k(\vartheta) < \vartheta, \lim_{\vartheta \to \infty} \beta(\vartheta) = \infty, \lim_{\vartheta \to \infty} a_k(\vartheta) = \infty, \) for all \(k = 1, 2, \ldots, n;\)

(b) \(\beta(\vartheta) \geq \vartheta_0 \geq 0, \) for all \(\vartheta \geq 0;\)

(c) \(\lim_{\vartheta \to \infty} B(\vartheta) = \infty, \) where \(B(\vartheta) = \int_{\vartheta}^{\infty} b^{-1/\Delta_1}(v) dv;\)

(d) \(p_k \in C([\vartheta_0, \infty), \mathbb{R}_+) \) for \(k = 1, 2, \ldots, n;\)

(e) There is a function \(\beta_0 \in C^1([\vartheta_0, \infty), \mathbb{R}) \) such that \(0 < \beta_0(\vartheta) \leq \beta(\vartheta) \) and \(\beta_0(\vartheta) \geq \beta_0 > 0 \) for \(\vartheta \geq \vartheta^* > \vartheta_0.\)

2. Preliminaries

In this section, we provide some Lemmas that are needed later.

**Lemma 1.** If \(a_1 \) and \(b_1 \) are nonnegative numbers, then:

\[
a_1^{v_1}b_1^{1-v_1} \leq v_1a_1 + (1 - v_1)b_1 \quad \text{for} \quad 0 < v_1 \leq 1,
\]

and:

\[
a_1^{v_1}b_1^{1-v_1} = v_1a_1 + (1 - v_1)b_1 \quad \text{if and only if} \quad a_1 = b_1.
\]

**Lemma 2.** Let (a)–(d) be satisfied for \(\vartheta \geq \vartheta_0.\) If \(u\) is the eventually positive solution of (2), then we have:

\[
h(\vartheta) > 0, \quad h'(\vartheta) > 0, \quad \text{and} \quad (b(h'(\vartheta))\Delta_{1})'(\vartheta) \leq 0 \quad \text{for} \quad \vartheta \geq \vartheta_1. \tag{3}
\]

**Proof.** Let \(u\) be an eventually positive solution of (2). Hence, \(h(\vartheta) > 0,\) and for \(\vartheta_0 \geq 0,\) we have \(u(\vartheta) > 0, u(a_k(\vartheta)) > 0\) and \(u(\beta(\vartheta)) > 0,\) for all \(\vartheta \geq \vartheta_0\) and for all \(k = 1, 2, \ldots, n.\)

From (2), we obtained:

\[
\left( b(\vartheta)(h'(\vartheta))\Delta_{1}\right)' = -q(\vartheta)u^{\Delta}(\beta(\vartheta)) \leq 0 \quad \text{for} \quad \vartheta \geq \vartheta_0.
\]
Therefore, \( b(\theta)(h'(\theta))^{\Delta_1} \) is non-increasing for \( \theta \geq \theta_0 \). Assume that \( b(\theta)(h'(\theta))^{\Delta_1} < 0 \) for \( \theta \geq \theta_1 > \theta_0 \). Hence,

\[
\quad b(\theta)(h'(\theta))^{\Delta_1} \leq b(\theta_1)(h'(\theta_1))^{\Delta_1} < 0 \quad \text{for all } \theta \geq \theta_1,
\]

that is,

\[
h'(\theta) \leq \left( \frac{b(\theta_1)}{b(\theta)} \right)^{1/\Delta_1} h'(\theta_1) \quad \text{for } \theta \geq \theta_1.
\]

Using integration from \( \theta_1 \) to \( \theta \), we have:

\[
h(\theta) \leq h(\theta_1) + \left( \frac{b(\theta_1)}{b(\theta)} \right)^{1/\Delta_1} h'(\theta_1) B(\theta) \to -\infty
\]

as \( \theta \to \infty \) due to (c), which is a contradiction to \( h(\theta) > 0 \).

Therefore, \( b(\theta)(h'(\theta))^{\Delta_1} > 0 \), for all \( \theta \geq \theta_1 \). From \( b(\theta)(h'(\theta))^{\Delta_1} > 0 \) and \( b(\theta) > 0 \), we have \( h'(\theta) > 0 \). Thus, the lemma is proven. \( \square \)

**Lemma 3.** Let (a)–(d) be satisfied for \( \theta \geq \theta_0 \). If \( u \) is the eventually positive solution of (2), then we have:

\[
h(\theta) \geq \left( b(\theta) \right)^{1/\Delta_1} h'(\theta) B(\theta) \quad \text{for } \theta \geq \theta_1.
\]

and:

\[
\frac{h(\theta)}{B(\theta)} \quad \text{is decreasing for } \theta \geq \theta_1.
\]

**Proof.** Proceeding as in the proof of Lemma 2, we have (3) for \( \theta \geq \theta_1 \). Since \( b(\theta)(h'(\theta))^{\Delta_1} \) is decreasing, we have:

\[
h(\theta) \geq \int_{\theta_1}^{\theta} \left( b(s) \right)^{1/\Delta_1} h'(s) \frac{1}{\left( b(s) \right)^{1/\Delta_1}} ds
\]

\[
\geq (b(\theta))^{1/\Delta_1} h'(\theta) \int_{\theta_1}^{\theta} \frac{1}{\left( b(s) \right)^{1/\Delta_1}} ds
\]

\[
\geq (b(\theta))^{1/\Delta_1} h'(\theta) B(\theta).
\]

Again, using the previous inequality, we have:

\[
\left( \frac{h(\theta)}{B(\theta)} \right)' = \frac{b(\theta)^{1/\Delta_1} h'(\theta) B(\theta) - h(\theta)}{(b(\theta))^{1/\Delta_1} B^2(\theta)} \leq 0.
\]

We concluded that \( \frac{h(\theta)}{B(\theta)} \) is decreasing for \( \theta \geq \theta_1 \). Hence, the lemma is proven. \( \square \)

**Lemma 4.** Let (a)–(d) be satisfied for \( \theta \geq \theta_0 \). If \( u \) is the eventually positive solution of (2), then we have:

\[
u(\theta) \geq Q(\theta) h(\theta) \quad \text{for } \theta \geq \theta_1
\]

where:

\[
Q(\theta) = \left( 1 - \sum_{k=1}^{n} v_k p_k(\theta) - \frac{1}{\omega(\theta)} \sum_{k=1}^{n} (1 - v_k) p_k(\theta) \right) \geq 0
\]

for any function \( \omega \in C([\theta_0, \infty), \mathbb{R}^+) \), which is decreasing to zero.
Therefore, there exists $\vartheta$ such that:

$$\lim_{\vartheta \to 0} h(\vartheta) = \infty,$$

using Lemma 1. Since $h(\vartheta) > 0$, $h'(\vartheta) > 0$, $\omega(\vartheta) > 0$ and $\omega'(\vartheta) < 0$, there is a $\vartheta_0 \geq \vartheta_1$ such that:

$$h(\vartheta) \geq \omega(\vartheta) \quad \text{for} \quad \vartheta \geq \vartheta_1. \quad (6)$$

Using (6) in (5), we obtained:

$$u(\vartheta) \geq Q(\vartheta)h(\vartheta).$$

Hence, the lemma is proven. $\Box$

**Lemma 5.** Let (a)–(d) be satisfied for $\vartheta \geq \vartheta_0$. If $u$ is the eventually positive solution of (2), then for $\vartheta_1 > \vartheta_0$ and $\delta > 0$, we have that:

$$0 < h(\vartheta) \leq \delta B(\vartheta) \quad \text{and} \quad B(\vartheta)U^{1/\Lambda_1}(\vartheta) \leq h(\vartheta) \quad (7)$$

hold for all $\vartheta \geq \vartheta_1$ where:

$$U(\vartheta) = \int_{\vartheta}^{\infty} q(\zeta)u^\Lambda(\beta(\zeta))\,d\zeta \geq 0.$$

**Proof.** Let $u$ be an eventually positive solution of (2). Then, for $\vartheta_0 > 0$, we obtained that $u(\vartheta) > 0$, $u(\alpha_k(\vartheta)) > 0$ and $u(\beta(\vartheta)) > 0$ for all $\vartheta \geq \vartheta_0$ and for all $k = 1, 2, \cdots, n$. Therefore, there exists $\vartheta_1 > \vartheta_0$ such that Lemma 2 holds true and $h$ satisfies (3) for $\vartheta \geq \vartheta_1$. From $b(\vartheta)(h'(\vartheta))^{\Delta_1} > 0$ and being non-increasing, we have:

$$h'(\vartheta) \leq \left(\frac{b(\vartheta_1)}{b(\vartheta)}\right)^{1/\Lambda_1} h'(\vartheta_1) \quad \text{for} \quad \vartheta \geq \vartheta_1.$$

Integrating this inequality from $\vartheta_1$ to $\vartheta$,

$$h(\vartheta) \leq h(\vartheta_1) + (b(\vartheta_1))^{1/\Lambda_1} h'(\vartheta_1) B(\vartheta).$$

Since $\lim_{\vartheta \to \infty} B(\vartheta) = \infty$, we have that (7) holds, for some positive constant $\delta$. Next, $\lim_{\vartheta \to \infty} b(\vartheta)(h'(\vartheta))^{\Delta_1}$ exists, and integrating (2) from $\vartheta$ to $l$, we obtained:

$$b(1) (h'(1))^{\Delta_1} - b(\vartheta) (h'(\vartheta))^{\Delta_1} = - \int_{\vartheta}^{l} q(\zeta)u^\Lambda(\beta(\zeta))\,d\zeta.$$
Theorem 1. Suppose that there is a quotient of odd positive integer \( \vartheta \) and 5 for \( \vartheta \) can find a \( \vartheta \) and 5 for \( \vartheta \). Let \( u(\vartheta) \geq \vartheta > 0 \) for all \( \vartheta \geq \vartheta_0 \), where \( \vartheta_0 \) is an odd positive integer. Then, \( u(\vartheta) \geq \vartheta > 0 \) and \( u(\vartheta) \geq \vartheta > 0 \) for all \( \vartheta \geq \vartheta_0 \), where \( \vartheta_0 \) is an odd positive integer.

Proof. Let \( u \) be a solution of (2), and positive eventually. Therefore, for \( \vartheta_0 > 0 \), we have \( u(\vartheta) > 0 \), \( u(\vartheta) \geq \vartheta > 0 \), and \( u(\vartheta) \geq \vartheta > 0 \) for all \( \vartheta \geq \vartheta_0 \), where \( \vartheta_0 \) is an odd positive integer. We find:

\[
\int_0^\infty q(\vartheta)Q^A(\beta(\vartheta))B^A(\beta(\vartheta)) \, dg = \infty.
\]

Thus, the proof of the lemma is complete. \( \square \)

3. Oscillation Theorems

Theorem 1. Suppose that there is a quotient of odd positive integer \( \nabla_1 \) with \( 0 < \Delta < \nabla_1 < \nabla_1 \) and (a)–(d) hold for \( \vartheta \geq \vartheta_0 \). If:

(f) \( \int_0^\infty q(\vartheta)Q^A(\beta(\vartheta))B^A(\beta(\vartheta)) \, dg = \infty \)

holds, then all solutions of (2) are oscillatory.

Proof. Let \( u \) be a solution of (2), and positive eventually. Therefore, for \( \vartheta_0 > 0 \), we have \( u(\vartheta) > 0 \), \( u(\alpha_k(\vartheta)) > 0 \), and \( u(\alpha_k(\vartheta)) > 0 \) for all \( \vartheta \geq \vartheta_0 \), where \( \vartheta_0 \) is an odd positive integer. From Lemmas 2 and 5 for \( \vartheta \geq \vartheta_1 > \vartheta_0 \), we concluded that \( h \) satisfies (3), (4), (7), and (8) for all \( \vartheta \geq \vartheta_1 \). We can find a \( \vartheta_1 > 0 \) such that:

\[
h(\vartheta) \geq B(\vartheta)U^{1/\Delta_1}(\vartheta) \geq 0 \quad \text{for } \vartheta \geq \vartheta_1.
\]

Using (4), (7), \( \Delta - \nabla_1 < 0 \), and (10), we have:

\[
u^A(\vartheta) \geq Q^A(\vartheta)h^{A - \nabla_1}(\vartheta)h^{\nabla_1}(\vartheta) \geq Q^A(\vartheta)(\delta B(\vartheta))^{A - \nabla_1}h^{\nabla_1}(\vartheta)
\] \[
\geq Q^A(\vartheta)(\delta B(\vartheta))^{A - \nabla_1}(B(\vartheta)U^{1/\Delta_1}(\vartheta))^{\nabla_1} = Q^A(\vartheta)\delta^{A - \nabla_1}B^A(\vartheta)U^{1/\Delta_1}(\vartheta).
\]

for \( \vartheta \geq \vartheta_2 \). Since, \( U'(\vartheta) = -q(\vartheta)u^A(\beta(\vartheta)) \leq 0 \) for \( \vartheta \geq \vartheta_2 \), the last inequality yields:

\[
u^A(\beta(s)) \geq Q^A(\beta(s))\delta^{A - \nabla_1}B^A(\beta(s))U^{1/\Delta_1}(\beta(s)) \geq Q^A(\beta(s))\delta^{A - \nabla_1}B^A(\beta(s))U^{1/\Delta_1}(\beta(s)).
\]

Therefore,

\[
(U^{1 - \nabla_1/\Delta_1}(\vartheta))' = \left(1 - \frac{\nabla_1}{\Delta_1}\right)U^{-\nabla_1/\Delta_1}(\vartheta)U'(\vartheta).
\]

Integrating (3) from \( \vartheta_2 \) to \( \vartheta \) and using \( U > 0 \), we find:

\[
\infty > U^{1 - \nabla_1/\Delta_1}(\vartheta) \geq \left(1 - \frac{\nabla_1}{\Delta_1}\right)\left[\int_{\vartheta_2}^{\vartheta} U^{1 - \nabla_1/\Delta_1}(s)U'(s) \, ds\right]
\]

\[
= \left(1 - \frac{\nabla_1}{\Delta_1}\right)\left[\int_{\vartheta_2}^{\vartheta} q(s)U^{1/\Delta_1}(s)B^A(\beta(s)) \, ds\right]
\]

\[
\geq \left(1 - \frac{\nabla_1}{\Delta_1}\right)\left[\int_{\vartheta_2}^{\vartheta} q(s)Q^A(\beta(s))B^A(\beta(s)) \, ds\right]
\]
which is a contradiction (f) as \( \theta \to \infty \). Thus, the theorem is proven. \( \square \)

**Theorem 2.** Suppose that there is a quotient of odd positive integer \( \nabla_2 \) with \( \Delta_1 < \nabla_2 < \Delta \). Furthermore, assume that (a)–(e) hold for \( \theta \geq \theta_0 \) and \( b(\theta) \) is non-decreasing. If:

\[(g) \quad \int_0^\infty \left[ \frac{1}{\theta \psi} \int_s^\infty q(s)Q^\theta(\beta(s)) \, ds \right]^{1/\Delta_1} \, ds = \infty \]

holds, then all solutions of (2) are oscillatory.

**Proof.** Let \( u \) be an eventually positive solution of (2). Then, for \( \theta_0 > 0 \), we have \( u(\theta) > 0 \), \( u(\alpha_k(\theta)) > 0 \) and \( u(\beta(\theta)) > 0 \) for all \( \theta \geq \theta_0 \), \( k = 1, 2, \ldots, n \). Applying Lemmas 2 and 4 for \( \theta \geq \theta_1 > \theta_0 \), we concluded that \( h \) satisfies (3), that \( h \) is increasing, and that \( u(\theta) \geq Q(\theta)h(\theta) \), for all \( \theta \geq \theta_1 \). Therefore,

\[
u^\Delta(\theta) \geq Q^\Delta(\theta)h^\Delta(\theta) \geq Q^\Delta(\theta)h^{\Delta - \nabla_2}(\theta)h^{\nabla_2}(\theta) \geq Q^\Delta(\theta)h^{\Delta - \nabla_2}(\theta_1)h^{\nabla_2}(\theta) \]

implies that:

\[
u^\Delta(\beta(\theta)) \geq Q^\Delta(\beta(\theta))h^{\Delta - \nabla_2}(\theta_1)h^{\nabla_2}(\beta(\theta)) \quad \text{for} \quad \theta \geq \theta_2 > \theta_1. \tag{12} \]

Using (9) and (12), we have:

\[
\begin{align*}
b(\theta)\left( h'(\theta) \right)^{\Delta_1} & \geq h^{\Delta - \nabla_2}(\theta_1) \left[ \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right] h^{\nabla_2}(\beta(\theta)) \\
& \geq h^{\Delta - \nabla_2}(\theta_1) \left[ \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right] h^{\nabla_2}(\beta_0(\theta)) \tag{13}
\end{align*}
\]

for \( \theta \geq \theta_2 \). As \( b(\theta)\left( h'(\theta) \right)^{\Delta_1} \) is non-increasing and \( \beta_0(\theta) \leq \theta \), we have:

\[
b(\beta_0(\theta))\left( h'(\beta_0(\theta)) \right)^{\Delta_1} \geq b(\theta)\left( h'(\theta) \right)^{\Delta_1}. \tag{14}
\]

From (13) and (14), dividing by \( b(\beta_0(\theta))h^{\nabla_2}(\beta_0(\theta)) > 0 \), then taking the power \( 1/\Delta_1 \) on both sides, we obtained:

\[
b'(\beta_0(\theta)) \left[ h^{\Delta - \nabla_2}(\theta_1) \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \geq \frac{\theta}{\beta_0(\theta)} \left[ h^{\Delta - \nabla_2}(\theta_1) \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \]

for \( \theta \geq \theta_2 \). Multiplying the left-hand side by \( \beta'(\theta) / \beta_0(\theta) \geq 1 \) and integrating from \( \theta_2 \) to \( \theta \), we obtained:

\[
\int_{\theta_2}^{\theta} \frac{1}{\beta_0(\theta)} \left[ h^{\Delta - \nabla_2}(\theta_1) \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \, ds \geq \int_{\theta_2}^{\theta} \frac{1}{\beta_0(\theta)} \left[ h^{\Delta - \nabla_2}(\theta_1) \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \, ds \tag{15}
\]

since \( \Delta_1 < \nabla_2 \), \( b(\beta_0(s)) \leq b(s) \), and:

\[
\frac{1}{\beta_0(\theta)} \left[ h^{\Delta - \nabla_2}(\theta_1) \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \leq \frac{1}{\beta_0(\nabla_2/\Delta_1 - 1)} h^{\Delta - \nabla_2}(\theta_2) \]

then (15) becomes:

\[
\int_{\theta_2}^{\theta} \frac{1}{\beta_0(\theta)} \left[ \int_{\theta}^{\infty} q(s)Q^\Delta(\beta(s)) \, ds \right]^{1/\Delta_1} \, ds < \infty
\]

which contradicts (g). Thus, the theorem is complete. \( \square \)

Next, we give one example to verify the main results.
Example 1. Consider the neutral delay differential equation:
\[
\left( (\alpha + 1) \left( u(\theta) + \frac{1}{\beta^2} u^{\frac{1}{3}} \left( \frac{\vartheta}{2} \right) + \frac{1}{\beta^2} u^{\frac{1}{5}} \left( \frac{\vartheta}{3} \right) \right)^{\prime} + \vartheta^{12} u^{\frac{1}{3}} \left( \frac{\vartheta}{2} \right) \right) = 0 \quad \text{for} \quad \vartheta \geq 2, \tag{16}
\]
where \(b(\theta) := \theta + 1 \in \mathbb{R}_+\) and \(B(\theta) = \log \left( \frac{\theta + 1}{\theta^2 + 1} \right)\); \(q(\theta) := \vartheta^{12} \geq 0\); \(b(\theta) := \frac{\vartheta}{2} < \theta\) with \(\lim_{\theta \to \infty} b(\theta) = \infty\); \(b_0(\theta) > \frac{1}{2} = b_0\); \(\Delta = 3 > \Delta_1 = 1\); \(p_k(\theta) := \frac{1}{\theta^2} \in \mathbb{R}_+\); \(v_k := \frac{k-2}{2k+1}\) and \(a_k(\theta) := \frac{\vartheta}{k+1} < \theta\) with \(\lim_{\theta \to \infty} a_k(\theta) = \infty\) for \(k = 1, 2, \ldots, n\) and \(\vartheta \geq 2\). For the index \(k = 1, 2\) and \(\omega(\theta) = \frac{1}{\beta^2}\), we have:
\[
B^\Delta(b(\theta)) = 3 \log \left( \frac{\theta + 2}{2(\theta + 1)} \right)
\]
and:
\[
Q(\theta) = \left( 1 - \sum_{k=1}^{\infty} \frac{2k - 1}{2k + 1} \right) - \frac{1}{\beta^2} \sum_{k=1}^{\infty} \frac{2}{2k + 1}.
\]

Clearly, the condition (f) holds true, and by Theorem 1, all solutions of (16) are oscillatory.

4. Conclusions and Open Problem

In this work, we established some new sufficient conditions for the oscillation of second-order neutral differential equations with sub-linear neutral terms. In [10–16,18,35], the authors established sufficient conditions for the various types of neutral delay differential equations. We can claim that our methods could be applicable for any neutral delay differential equations when the neutral coefficient is positive. The problem undertaken is incomplete for negative neutral coefficients. It would be of interest to investigate the oscillation of (2) with a negative neutral coefficient using Lemmas 1–5. It would be of interest to examine the oscillation of (2) with different neutral coefficients; see, e.g., the papers [10,12–17] for more details. Furthermore, it would also be interesting to analyze the oscillation of (2) with a nonlinear neutral term; see, e.g., the paper [18] for more details.

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