Vortex solutions of the Liouville equation

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Abstract. The most general vortex solution of the Liouville equation (which arises in non-relativistic Chern-Simons theory) is associated with rational functions, \( f(z) = \frac{P(z)}{Q(z)} \) where \( P(z) \) and \( Q(z) \) are both polynomials, \( \deg P < \deg Q \equiv N \). This allows us to prove that the solution depends on \( 4N \) parameters without the use of an index theorem, as well as the flux quantization: \( \Phi = -4N\pi \text{sign } \kappa \).

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1. Vortex solutions

Non-relativistic Chern-Simons theory supports vortices [1]. (See also [2] for reviews). For suitable values of the parameters, these vortices arise as solutions of the first-order “self-duality” (SD) equations,

\[(D_1 \pm i D_2) \psi = 0, \quad \kappa B = -\varrho,\]

where $\psi$ is a complex scalar field minimally coupled to a static gauge potential $\vec{A} = (A_x, A_x)$ in the plane; $D_j \psi = (\partial_j - i A_j) \psi$ is the covariant derivative, $B = \partial_x A_y - \partial_y A_x$ is the magnetic field, and $\varrho = \psi^* \psi$ is the particle density. The solutions we are interested in are “non-topological” in that $\varrho \to 0$ at infinity.

Using the complex notations $\partial = \frac{1}{2} (\partial_x - i \partial_y)$, $A = A_x - i A_y$, $z = x + iy$, the first SD equation are solved by $A = -2i \partial \ln \psi$. Setting $\psi = \varrho^{1/2} e^{i\omega}$, we get

\[A = -i \partial \ln \varrho + 2 \partial \omega.\]

Reinserting $A$ into the second SD equation, we end up with the Liouville equation,

\[\triangle \ln \varrho = -\frac{2}{|\kappa|} \varrho,\]

whose all real solutions are known. They are in fact given in terms of an arbitrary analytic function on the complex plane,

\[\varrho = 4|\kappa| |f'(z)|^2 \left(1 + |f(z)|^2\right)^2.\]

For $f(z) = z^{-N}$, for example, we get the radially symmetric solution

\[\varrho = 4|\kappa| N^2 \frac{r^{-2(N+1)}}{(1 + r^{-2N})^2}.\]

To be regular, $N$ has to be here an integer at least 1 [1], [2].

A striking feature of these non-topological vortex solutions is that their magnetic flux is an even multiple of the elementary flux quantum,

\[\Phi \equiv \int B \, d^2x = 2N \times \Phi_0, \quad \Phi_0 = -2\pi, \quad N = 0, \pm 1, \ldots\]

This can be seen easily in the radial case, taking into account the asymptotic behaviour, $\varrho \propto r^{-2(N+1)}$ as $r \to \infty$, of the particle density. We are not aware of a general proof, though.
Another peculiarity of these vortices is that, for fixed $N$, the solution depends on $4N$ parameters. The proof given by Kim et al. [3] uses the Atiyah-Singer index theorem. If such an approach is perfectly justified for “Nielsen-Olesen” vortices or “BPS” monopoles [4], it seems too-powerful when, as in our case, all solutions are known explicitly.

A $4N$-parameter family of solutions can be written down at once: consider

\begin{equation}
(1.7) \quad f(z) = \sum_{i=1}^{N} \frac{c_i}{z - z_i}, \quad c_i, z_i \in \mathbb{C}, \quad i = 1, \ldots, N.
\end{equation}

Plotting the particle density allows us to interpret the associated solution as representing $N$ separated vortices located at the points $z_i$, with individual scales and phases $c_i$ [5]. With some work, the associated flux is found to be $\Phi = -4N\pi \text{sign} \kappa$ [1].

The questions we address ourselves sounds: what is the most general $f$ yielding physically admissible solutions? What is the associated magnetic flux? Can we count the parameters without an index theorem? Below, we show that imposing suitable regularity conditions only allows rational functions. In detail, we prove

**Theorem 1.** Let us consider a vortex solution of the Liouville equation with finite magnetic flux $\Phi < \infty$. If the magnetic field $B$ is regular on the complex plane and is such that $r^{2+\delta}B$ $(\delta > 0)$ is bounded when $r \to \infty$, then $f$ is a rational function,

\begin{equation}
(1.8) \quad f(z) = \frac{P(z)}{Q(z)},
\end{equation}

where $P(z)$ and $Q(z)$ are polynomials with $\deg P < \deg Q$. The coefficient of the highest-order term in $Q(z)$ can be normalized to 1.

It follows that the general solution does indeed depend on $4N$ parameters, namely on the $2 \times N$ complex coefficients of the polynomials with $P(z)$ and $Q(z)$. The particular form (1.7) is recovered by expanding $f$ into partial fractions, provided $Q(z)$ only has simple zeroes.

Next we prove

**Theorem 2.** The magnetic flux of the vortex associated with the rational function $f(z)$ in Eq. (1.8) is $\Phi = -4N\pi \text{sign} \kappa$, where $N$ is the degree of the denominator,

\begin{equation}
(1.9) \quad N = \deg Q(z)
\end{equation}

and is hence an integer.

As a corollary, we also get the general theorem on flux quantization, as in Eq. (1.6).
We establish our theorems by elementary complex analysis [6], [7], as the result of a series of Lemmas. We start with proving the

**Lemma 1:** Let \( f(z) \) be a complex function which only has isolated singularities. Let \( \gamma \) be a curve in the complex plane which avoids the singularities of \( f \); set \( z_0 = f(0) \) and \( z_1 = f(1) \). Then

\[
\frac{|f(z_0) - f(z_1)|}{\sqrt{1 + |f(z_0)|^2} \sqrt{1 + |f(z_1)|^2}} \leq \int_{\gamma} \frac{2|f'(z)|}{1 + |f(z)|^2} |dz|,
\]

where \( |dz| = |d\gamma/dt|dt \).

This proposition has a nice geometric meaning: the left-hand side is the length of \( \gamma \) with respect to a metric inherited by stereographic projection, while the right-hand-side is the integral of \( \rho^{1/2} \), the square-root of the particle density.

Next, using Lemma 1, we demonstrate

**Lemma 2.** The function \( f \) can not have an essential singularity in the complex plane.

This lemma eliminates functions like \( f(z) = e^{1/z} \). This function would yield in fact a density which is unbounded at the origin. Writing \( z = r \exp(i\theta) \), the density becomes

\[
\rho = \frac{4 \exp\left(\frac{2\cos \theta}{r}\right)}{r^4 \left(1 + \exp\left(\frac{2\cos \theta}{r}\right)\right)^2},
\]

so that on the imaginary axis, \( \theta = \pi/2 \), \( \lim_{r \to 0} \rho = +\infty \).

Next, we prove

**Lemma 3.** The function \( f \) can not have an essential singularity at infinity.

Lemma 3. rules out the functions \( f(z) \) as \( e^z \). For this choice, the particle density reads

\[
\rho = \frac{4}{\left(\exp(-r \cos \theta) + \exp(r \cos \theta)\right)^2}.
\]

On the imaginary axis \( \theta = \pi/2 \), we have \( \rho = 4 \): the particle density is not localized.

Now a theorem found in Whittaker and Watson [6], 5.64., p.105. says that the only one-valued functions which have no singularities, except poles, at any point (including \( \infty \)), are rational functions. This allows us to conclude that our \( f(z) \) is indeed rational. Then,

**Lemma 4.** The polynomials in Eqn. (1.8) can be chosen so that \( \deg P < \deg Q \), the highest term in \( Q \) having coefficient equal to 1.

This proves our Theorem 1.
Now, to evaluate the magnetic flux of the vortex associated with the rational function \( f \) in Eq. (1.8), we show that

**Lemma 5.** Let \( z_1, \ldots z_{N_Q} \) denote the distinct roots of the denominator \( Q(z) \), each having multiplicity \( n_i \). The particle density (1.4) can also be written as

\[
\varrho = 4|\kappa|\bar{\partial} \left[ \left( \frac{\partial f}{\overline{f}} \right) \frac{|f|^2}{1+|f|^2} + \sum_{i=1}^{N_Q} \frac{n_i}{z-z_i} \right],
\]

where the bracketed quantity is a regular function on the plane.

Then the flux (1.6) is converted into a contour integral at infinity, by Stokes’ theorem,

\[
\Phi = 2i(\text{sign } \kappa) \oint_S \left[ \left( \frac{\partial f}{\overline{f}} \right) \frac{|f|^2}{1+|f|^2} + \sum_{i=1}^{N_Q} \frac{n_i}{z-z_i} \right] \, dz,
\]

where \( S \equiv S_\infty \) is the cercle at infinity.

The integrand of (1.12) is related to the vector potential. Using (1.2), this latter reads in fact

\[
A = 2i \left( \frac{\partial f}{\overline{f}} \right) \frac{|f|^2}{1+|f|^2} - i \left( \frac{\partial^2 f}{\partial f} \right) + 2\omega.
\]

To get a regular \( A \), the phase \( \omega \) has to be chosen so that

\[
2\partial \omega = \sum_{i=1}^{N_Q} \frac{n_i-1}{z-z_i} + \sum_{i=1}^{N_P} \frac{m_i-1}{z-Z_i},
\]

so that the integrand in (1.12) is

\[
A + \left\{ i \left( \frac{\partial^2 f}{\partial f} \right) + i \left( \sum_{i=1}^{N_Q} \frac{n_i+1}{z-z_i} - \sum_{i=1}^{N_P} \frac{m_i-1}{z-Z_i} \right) \right\};
\]

the integral of the terms in the curly bracket on the circle at infinity vanishes.

Now using

**Lemma 6.**

\[
\oint_S \left( \frac{\partial f}{\overline{f}} \right) \frac{|f|^2}{1+|f|^2} = 0,
\]
the second term in (1.12) is evaluated at once,

$$\Phi = 2i \oint_S \sum_{i=1}^{N_Q} n_i \frac{dz}{z - z_i} = -4\pi \left( \sum_{i=1}^{N_Q} n_i \right) = -4N\pi \text{(sign } \kappa\text{)},$$

where $N$ is the degree of $Q(z)$. This yields Theorem 2.

The flux has been previously related to the inversions [8]. Their argument goes as follows: the particle density behaves as $\rho \sim r^{2(N-1)}$ when $r \to 0$. The regularity of the vector potential requires the phase to be chosen as $\omega = (N-1)\theta$. Then the inversion symmetry implies the behaviour $\rho \sim r^{-2(N+1)}$ at infinity, so that the flux is indeed $2\pi(N-1) + 2\pi(N+1) = 4\pi N$.

This argument is only valid in the radial case, though. To see this, let observe that the choice $f(z) = (1 + z)^{-2} - 2(z - 1)^{-1}$, yields, for example, flux $\Phi = -12\pi$ (i.e. $N = 3$) and is interpreted as a 2-vortex sitting at $z = -1$ and a 1-vortex sitting at $z = 1$. The particle density does not behave as claimed by Kim et al [8], rather as $\rho \sim r^6$ (instead of $r^4$) when $r \to 0$, and as $\rho \sim r^{-4}$ (instead of $r^{-8}$) when $r \to \infty$.

Where does the error come from? On the one hand, the behaviour at the origin assumed by Kim et al. is consistent with our formulæ (1.4), (1.14) in the radial case only. On the other hand, albeit the Liouville equation is indeed inversion-invariant (indeed invariant with respect to any conformal transformation) this is not true for individual, non-radial solutions. Therefore, the large-$r$ behaviour of a solution can not be inferred from that for small $r$.

In this paper, we only considered the case of single-valued functions $f$. It seems however, that multiple-valued function do not qualify. For example, the charge density $\rho$ associated with $f(z) = \ln z$, is also multiple-valued, and hence physically inadmissible; remember also that in the radial case $f(z) = z^{-N}$, the regularity requires $N$ to be an integer.
2. Proofs.

Proof of Lemma 1. Stereographic projection carries over the natural metric from the Riemann sphere to the complex plane. The scalar product of two tangent vectors, $u$ and $v$, at a point $p$ of the plane is

\( g_p(u, v) = \frac{4}{(1 + |p|^2)^2} u \cdot v, \)

where “$\cdot$” is the ordinary scalar product in $\mathbb{R}^2$, $|p|^2 = p \cdot p$. The length of a curve \([0, 1] \ni t \mapsto \Gamma(t) \in \mathbb{C}\) w. r. to this metric is

\( L(\Gamma) = \int_0^1 \| \frac{d\Gamma}{dt} \| dt, \)

\( \| \frac{d\Gamma}{dt} \| = \left[ g_{\Gamma(t)} \left( \frac{d\Gamma}{dt}, \frac{d\Gamma}{dt} \right) \right]^{1/2} = \frac{2|\Gamma'(t)|}{1 + |\Gamma(t)|^2}. \)

Then the distance of two points, $w_0$ et $w_1$, in the complex plane is the l’infinum of length of the curves between the points,

\( d(w_0, w_1) = \inf_{\Gamma} \left\{ L(\Gamma) \mid \Gamma(0) = w_0, \Gamma(1) = w_1 \right\}. \)

Let us now consider an analytic function $w = f(z)$. $f$ can also be viewed as a mapping of the $z$-plane into the $w$-plane; the latter is endowed with the distance defined here above. If $\gamma(t)$ is an arbitrary curve in the $z$-plane with end-points $z_0$ and $z_1$, its image by $f$ is a curve $\Gamma = f \circ \gamma$ in the $w$-plane with end-points $w_0 = f(z_0)$ and $w_1 = f(z_1)$. By (2.2), the length of $\Gamma$ is the r. h. s. of (1.10),

\( L(\Gamma) = \int_\gamma \frac{2|f'(z)|}{1 + |f(z)|^2} |dz|. \)

Thus

\( d(w_0, w_1) = d(f(z_0), f(z_1)) \leq \int_\gamma \frac{2|f'(z)|}{1 + |f(z)|^2} |dz|. \)

But the distance on the $w$-plane is just the distance on the Riemann sphere. But this latter sits in $\mathbb{R}^3$, so that the (geodesic) distance on the sphere is greater or equal to the natural distance in $\mathbb{R}^3$:

\( d(w_0, w_1) \geq \frac{|w_0 - w_1|}{\sqrt{1 + |w_0|^2} \sqrt{1 + |w_1|^2}}, \)

equality being only achieved for $w_1 = w_0$. Setting $w_i = f(z_i)$, $i = 0, 1$, the inequality (1.10) is obtained.

Q. E. D.
Proof of Lemma 2. Let us assume that \( f \) has an isolated essential singularity at a point \( z_0 \). Then it is analytic in some disk \( D \equiv D(z_0; \epsilon) \setminus \{ z_0 \} \). Now, according to Picard’s Theorem ([7], p. 90): If \( z_0 \) is an isolated singularity of a holomorphic function \( f(z) \), then for each \( r > 0 \), l’image of the annular region \( \{ z \in \mathbb{C} | 0 < |z - z_0| < r \} \) is either the whole of \( \mathbb{C} \) or \( \mathbb{C} \) without a single point.

Let us first assume that \( z_1 \) is a point in \( D \) such that \( f(z_1) = 0 \). Then, since the particle density, \( \rho \), is a regular function on the plane which goes to zero at infinity, there is a real number \( M \) such that
\[
\rho(z) = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2} \leq M^2, \quad \forall z \in \mathbb{C}.
\]
Eqn. (1.10) in Lemma 1. with \( z_0 = z \), yields, using \( f(z_1) = 0 \),
\[
\frac{|f(z)|}{\sqrt{1 + |f(z)|^2}} \leq M \int_\gamma |dz|
\]
for all curve \( \gamma \) s. t. \( \gamma(0) = z \) et \( \gamma(1) = z_1 \). For the straight line \( \gamma(t) = z + t(z_1 - z) \) in particular, the r. h. s. becomes \( |z_1 - z|M < 2\epsilon M \), since \( z_1 \) and \( z \) both belong to \( D \). Thus, choosing \( \epsilon \) to have \( 4M\epsilon \leq 1 \), \( |f(z)|/\sqrt{1 + |f(z)|^2} < 1/2 \), which implies that \( |f(z)| \leq 1 \). The function \( f(z) \) is hence bounded in \( D \), which contradicts hypothesis that \( z_0 \) is an essential singularity.

Now if \( f \) does not vanish in \( D \equiv D(z_0; \epsilon) \), one can chose \( z'_0 \) with \( f(z'_0) \) sufficiently small so that \( f(z) \) is still bounded in \( D \).

Q. E. D.

Proof of Lemma 3. Let us assume, on the contrary, that infinity is essential singularity of \( f(z) \). From the large-\( r \) behaviour \( B = o(r^2) \) we shall deduce that \( f \) is again bounded at infinity, a contradiction.

In fact, if \( \infty \) is an isolated singularity, then \( f(z) \) is holomorphic in some neighbourhood \( D \equiv \{ z \in \mathbb{C} | |z| > N \} \), of infinity. In this neighbourhood, one can find a point \( z_0 \) where \( f \) vanishes \( f(z_0) = 0 \) by Picard’s theorem (1).

Now, due to the imposed groth condition on \( B = -\varrho/\kappa \), \( N \) can be taken so that
\[
(2.6) \quad \varrho \leq C^2|z|^{-2-\delta}, \quad \text{for all} \quad |z| > N,
\]
where \( C > 0 \) is a constant. Let us chose \( N \) such that \( 4C(\pi + \frac{1}{\delta}) < N^\delta/2 \). Then, for all complex number \( z, |z| > |z_0| \), \( f \) is bounded, \( |f(z)| \leq 1 \). To see this, consider \( z_1 \), the

\( (1) \) If \( f(z) \) never vanishes in the region \( D \), it is enough to chose \( f(z_0) \) small enough.
intersection of \( C \), the cercle around 0 with radius \(|z|\), with the straight half-line \([0, z_0]\).

Then

\begin{equation}
(2.7) \quad d(0, f(z)) \leq d(0, f(z_1)) + d(f(z_1), f(z)).
\end{equation}

Now, applying the inequality (2.4) to the circular arc \((z_1, z)\) we get

\begin{equation}
(2.8) \quad d(f(z_1), f(z)) \leq \int_{arc} \varrho(f(z))^{1/2}|dz|.
\end{equation}

From this we deduce, using (2.6), that

\begin{equation}
(2.9) \quad d(f(z_1), f(z)) \leq \frac{2\pi C}{N^{\delta/2}}.
\end{equation}

On the other hand, applying (2.4) to the segment \( \gamma(t) = z_0 + t(z_1 - z_0), \ t \in [0, 1] \), we get

\begin{equation}
(2.10) \quad d(0, f(z_1)) \leq \int_{\gamma} \varrho(f(z))^{1/2}|dz| \leq \int_{\gamma} \frac{C}{|z|^{1+\delta/2}}|dz| \leq \frac{2C}{\delta N^{\delta/2}},
\end{equation}

when the condition (2.6) is used again. The inequalities (2.7)–(2.9)–(2.10) imply that

\begin{equation}
(2.11) \quad d(0, f(z)) \leq \frac{2C}{N^{\delta/2}}(\pi + \frac{1}{\delta}) < \frac{1}{2};
\end{equation}

Now, by (2.5), we have

\[ d(0, f(z)) \geq \frac{|f(z)|}{\sqrt{1 + |f(z)|^2}}; \]

using (2.11), we get finally \( |f(z)| \leq 1 \). The function \( f(z) \) is hence bounded in some neighbourhood of infinity, so that this point cannot be an essential singularity.

Q. E. D.

Now (as explained in Chapter 1), Theorem 5.64 of Whittaker and Watson [6] allows us to deduce that \( f \) is a rational function, \( f(z) = P(z)/Q(z) \), where \( P(z) \) and \( Q(z) \) are both polynomials.

**Proof of Lemma 4.** Now, since \( f \) and \( f^{-1} \) are readily seen to yield the same solutions, we can assume that \( \deg P \leq \deg Q \). The case \( \deg P = \deg Q \) is eliminated by a simple redefinition, as in Ref. [1]. In fact,

\[ f(z) = f_0 + \frac{A(z)}{B(z)}, \]

where \( A(z) \) are \( B(z) \) polynomials s. t. \( \deg A(z) < \deg B(z) \) and \( f_0 \neq 0 \), is readily seen to yield the same density (1.4) as

\[ \tilde{f}(z) = \frac{\tilde{A}(z)}{\tilde{B}(z)}, \]
with the polynomials $\tilde{A}(z)$ and $\tilde{B}(z)$ defined as

$$\tilde{A}(z) = \frac{A(z)}{1 + |f_0|^2}, \quad \tilde{B}(z) = B(z) + \left(\frac{\bar{f}_0}{1 + |f_0|^2}\right) A(z).$$

Then the coefficient of the highest-order term in $Q(z)$ can be normalized to unity.

Q. E. D.

**Proof of Lemma 5.** In complex notations, the general solution (1.4) is expressed as

$$\varrho = 4|\kappa|\bar{\partial}\left[\left(\frac{\partial f}{f}\right) \frac{|f|^2}{1 + |f|^2}\right],$$

where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Let us now denote by $z_i$, $i = 1, \ldots, N_Q$ the distinct roots of the denominator, $Q(z)$, each having a multiplicity $n_i$. [Since $P(z)$ and $Q(z)$ have no common roots, these are the same as the poles of $f(z)$]. Then the function

$$\left(\frac{\partial f}{f}\right) \frac{|f|^2}{1 + |f|^2} + \sum_{i=1}^{N_Q} \frac{n_i}{z - z_i}$$

is regular on the complex plane. Indeed, in the neighbourhood of a root, $z_i$, of order $n_i$ of $Q(z)$,

$$f \sim \frac{c_i}{(z - z_i)^{n_i}} \implies \frac{\partial f}{f} \sim -\frac{n_i}{(z - z_i)}, \quad \frac{|f|^2}{1 + |f|^2} \sim 1.$$  

In contradistinction, in the neighbourhood of a zero (denoted by $Z_0$) of order $k \geq 1$, of $P(z)$ [which is the same as a zero of $f(z)$], we have:

$$f \sim (z - Z_0)^k \implies \frac{\partial f}{f} \sim \frac{k}{z - Z_0}, \quad \frac{|f|^2}{1 + |f|^2} \sim |z - Z_0|^{2k},$$

so that (2.13) is regular. Now, since

$$\bar{\partial}\left(\sum_{i=1}^{N_p} \frac{n_i}{z - z_i}\right) = 0,$$

the second term in (2.13) can be added to the expression (2.12) of the density, yielding (1.11).

Q. E. D.
The integrand in (2.13) being a regular function, the flux $\Phi = -(1/\kappa) \int \varrho \, d^2x$ can be converted, by Stokes theorem, into an integral at infinity, to yield (1.12).

Proof of Lemma 6. Let us now denote by $z_i, \, i = 1, \ldots, N_P$, the distinct roots of the numerator, $P(z)$ [and hence those of $f(z)$]. Using the theorem on the residues,

$$\oint_{S} \left( \frac{\partial f}{\partial z} \right) \frac{|f|^2}{1 + |f|^2} \, dz = \left( \lim_{|z| \to \infty} \frac{|f(z)|^2}{1 + |f(z)|^2} \right) 2\pi i \left( \sum_{i=1}^{N_P} m_i - \sum_{i=1}^{N_Q} n_i \right) = 0,$$

since $\lim_{|z| \to +\infty} f(z) = 0$.

Q. E. D.

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