Existence of Positive Solution for a Nonlinear Weighted Bi-Harmonic System of Elliptic Partial Differential Equations via Fixed-Point Argument

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ABSTRACT

In this paper, we establish an existence criterion of positive solution for a nonlinear weighted bi-harmonic system of elliptic partial differential equations in the unit ball in \( \mathbb{R}^n \) (dimensional euclidean space). The analysis of this paper is based on a topological method (a fixed-point argument). Initially, we establish a priori solution estimates, and then use a fixed-point theorem for deducing the existence of positive solutions. Finally, we prove a non-existence criterion as the complement of existence criterion.

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1. Introduction

The nonlinear partial differential equations (PDEs for short) have proved to be valuable tools for the modeling of many physical, chemical and biological phenomena, see for instance [27, 32, 34] and references therein. In the last few decades, there has been a noticeable interest on the study of existence of solution to nonlinear elliptic
systems, especially when the nonlinear term appears as a source in the equation with the Dirichlet’s or Neumann’s boundary conditions, see for instance [13, 28, 32] and references therein. Nonlinear systems are divided into two broad classes, first one is with a variational structure, namely Hamiltonian or gradients systems, see for instance [1, 16, 18] and second one is the class of non-variational problems, which can be maintained through the topological methods (fixed-point arguments), see for instance [2, 4, 9, 10]. In this paper, we deal with the existence of non-variational boundary value problem (BVP for short) for a given system using a topological method (a fixed-point argument).

The single fourth order nonlinear PDEs arise in various physical phenomenon such as study of travelling waves in suspension bridges [20], micro electro mechanical systems [24], radar imaging [3], bending behaviour of a thin elastic rectangular plate [31], geometric and functional design [5, 6, 29] etc. In this paper, we consider the following nonlinear weighted bi-harmonic system of elliptic PDEs

$$\begin{align*}
\Delta^2 u &= \lambda a(x) g(v), \quad v > 0 \text{ in } B, \\
\Delta^2 v &= \lambda b(x) f(u), \quad u > 0 \text{ in } B, \\
u &= 0 = v, \quad \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 \text{ on } \partial B,
\end{align*}$$

where $B$ denotes the unit ball in $\mathbb{R}^N$, $N > 4$ with boundary $\partial B$, $\Omega$ is open, smooth and bounded subset of $\mathbb{R}^N$, $\lambda$ is a positive parameter, $a, b : \Omega \to \mathbb{R}$ (set of real numbers) are sign changing weights, $f, g : 0, \infty \to \mathbb{R}$ are continuous nonlinearities with $f(0) = 0, g(0) = 0$.

The system designated by (1.1) is omnipresent in physics and chemistry where steady-states are answers to problematic questions in a great diversity of systems of reaction-diffusion equations. This system interacts everywhere in nature and this interaction takes place in such unequal phenomena as the proliferation of virile mutants over a substantially wide habitat, the dispersion of fire flames in roomy forests, in combustion chambers, or in nuclear reactors where neutron populations evolve and develop. Lions [21] studied the existence of a positive solution to the following Dirichlet problem

$$\begin{align*}
-\Delta u &= \lambda a(x) f(u) \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}$$

with weight function and nonlinearity satisfy $a \geq 0, f \geq 0$, respectively. Problem with indefinite weights was studied by Brown et al. [7], Brown and Tertikas [8], Cac et al. [11], Hai [19] and their references. Dalmasso [14] studied the following BVP

$$\begin{align*}
-\Delta u &= \lambda a(x) f(v) \text{ in } \Omega, \\
-\Delta v &= \lambda b(x) g(u) \text{ in } \Omega, \\
u &= 0 = v \text{ on } \partial \Omega,
\end{align*}$$

for $a(x) = 1$, $b(x) = 1$, $\lambda = 1$, and established the existence of a positive solution to the BVP (1.3) using Schauder fixed-point theorem [30]. After Dalmasso [14], several authors studied the BVP (1.3) using different techniques and theorems for different values of $a(x), b(x), \lambda = 1$, see for instance Chen [12] and their references. In 2014 Dwivedi [13] studied the existence of positive solutions to BVP (1.1) by using Leray-Schauder fixed-point theorem. Recently, Soltani and Yazidi [4] established the existence and non-existence of positive solutions to the following system of bi-Laplacian equations

$$\begin{align*}
\Delta^2 u &= g(v), \quad v > 0 \text{ in } B, \\
\Delta^2 v &= f(u), \quad u > 0 \text{ in } B, \\
u &= 0 = v, \quad \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 \text{ on } \partial B,
\end{align*}$$
applying a topological method (fixed-point argument) \([15, 25]\).

Inspired by above mentioned works, in the present paper we study the existence and non-existence of positive solutions to \((1.1)\) by using a topological method (fixed-point argument) \([15, 25]\). The priori estimates of positive solutions for elliptic partial differential equations gives a sufficient information about the existence of positive solutions, see for instance \([9, 22, 23]\). From this context, first we establish a priori estimates of solutions to our problem given by \((1.1)\) and then we prove the existence and non-existence criteria of positive solutions depending on that priori estimate. The rest of this work is furnished as follows:

In Section 2, we establish some preliminary facts which will be needed to prove our main results. Section 3 is devoted to state and prove the existence and non-existence criteria of solutions to the nonlinear biharmonic system of elliptic PDEs given by \((1.1)\). In the end of Section 3 we provide two illustrative examples to support our analytic proof.

2. Preliminary Notes

In this section, we recall some preliminary results related to the bi-Laplacian problem and state a fixed-point theorem which will help us to prove our main results.

Let us consider the problem \((1.1)\) for radially symmetric solutions, and let \(r = |x|, u = u(r), v = v(r)\),

\[
\begin{align*}
\begin{cases}
(4) & \quad u^{(4)}(r) + \frac{2(N-1)}{r}u^{''} + \frac{(N-1)(N-3)}{r^2}u^{'} - \frac{(N-1)(N-3)}{r^3}u = \lambda a(x)g(v), \quad v > 0 \text{ for } r \in (0,1), \\
(4) & \quad v^{(4)}(r) + \frac{2(N-1)}{r}v^{''} + \frac{(N-1)(N-3)}{r^2}v^{'} - \frac{(N-1)(N-3)}{r^3}v = \lambda b(x)f(u), \quad u > 0 \text{ for } r \in (0,1), \\
(2.1) & \quad u'(0) = v'(0), \quad u''(0) = v''(0), \quad u'(1) = v'(1), \quad u(1) = 0 = v(1).
\end{cases}
\end{align*}
\]

Then any solution \((u(r), v(r)) \in C^4(0,1) \times C^4(0,1)\) of \((2.1)\) is a radial symmetric solution of \((1.1)\).

Now, we recall the following lemma from \([23, \text{Lemma 2}]\), which gives more information regarding the eigenvalue problem for the operator \(\Delta^2\).

**Lemma 2.1** \([23]\): There is a \(\mu > 0\) such that the problem

\[
\Delta^2 v = \mu v \text{ in } B, \quad v = \frac{\partial v}{\partial v} \text{ on } \partial B
\]

possesses a positive, radial symmetric solution \(\varphi(x)\) which satisfies for some positive constants \(c_1\) and \(c_2\),

\[
c_1 (1-|x|)^2 \leq \varphi(x) \leq c_2 (1-|x|)^2, \quad x \in \overline{B}.
\]

\[
(2.2)
\]

From \([23]\) as well as \([17]\), we recall the Green’s function \(G(r,s)\) for the corresponding linear problem of \((2.1)\) is given by

\[
G(r,s) = \begin{cases}
A_N(s) + r^2 B_N(s), & \text{for } 0 \leq r \leq s \leq 1, \\
(s/r)^{-1} \left( A_N(r) + s^2 B_N(r) \right), & \text{for } 0 \leq s \leq r \leq 1,
\end{cases}
\]

\[
(2.3)
\]

where

\[
A_N(t) = \frac{t^3}{4(N-2)(N-4)} \left[ 2 + (N-4)t + (N-2)t^{N-4} \right]
\]

\[
(2.4)
\]
\[ B_N(t) = \frac{t}{4N(N-2)} \left[ Nt^{N-2} - (N-2)t^{N-2} \right]. \]  
(2.5)

From [23] we also observed that the Green’s function \( G(r,s) \) has the following properties:

There exists a positive constant \( d \) such that
\[ 0 \leq G(r,s) \leq ds^{N-1}(1-s)^{2} \left( \max (r,s) \right)^{4-N}, \]  
(2.6)
\[ \frac{\partial}{\partial r} G(r,s) \leq 0, \]  
(2.7)
and
\[ \frac{\partial^2}{\partial r^2} G(r,s) \bigg|_{r=1} = \frac{1}{2}s^{N-1}(1-s^2). \]  
(2.8)

Hence, the BVP (2.1) is transformed into the following integral equations:
\[
\begin{align*}
\begin{cases}
u(r) = \int_0^1 G(r,s) \lambda a(s) g(v(s)) ds, \\
v(r) = \int_0^1 G(r,s) \lambda b(s) f(u(s)) ds.
\end{cases}
\end{align*}
\]  
(2.9)

It is well established that the BVP (2.1) and the problem (2.9) are equivalent. For the study of BVP (1.1), we need the following eigenvalue problem:
\[
\begin{align*}
\begin{cases}
\Delta^2 \phi = \lambda_1 \phi \quad \text{in} \ B, \\
\Delta^2 \psi = \lambda_2 \psi \quad \text{in} \ B, \\
\phi = 0 = \psi, \quad \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \ \partial B,
\end{cases}
\end{align*}
\]  
(2.10)

where \( \lambda_1, \lambda_2 > 0 \).

Now we establish following lemma to comment on the solution of the eigenvalue problem (2.10).

**Lemma 2.2:** Let \( \varphi_1 \) be the corresponding eigenfunction of \( \mu \) which is the first eigenvalue of \( \Delta^2 \) on the unit ball \( B \) and \( \lambda_1 \lambda_2 = \mu^2 \). Then the eigenvalue problem (2.10) has a positive solution \( (\phi, \psi) \) satisfying (modulo a constant) \( \phi = \frac{1}{\sqrt{\lambda_1}} \varphi_1 \) and \( \psi = \frac{1}{\sqrt{\lambda_2}} \varphi_1 \).

**Proof.** Using the idea established in [33] for a Laplacian eigenvalue problem, we define
\[
\begin{align*}
w_1 &= \sqrt{\lambda_1} \phi, \\
w_2 &= \sqrt{\lambda_2} \psi.
\end{align*}
\]  
(2.11), (2.12)

Combining (2.11), (2.12) and the eigenvalue problem (2.10), we obtain
\[
\begin{align*}
\begin{cases}
\Delta^2 w_1 = \sqrt{\lambda_1 \lambda_2} a(x) w_2 \quad \text{in} \ B, \\
\Delta^2 w_2 = \sqrt{\lambda_1 \lambda_2} b(x) w_1 \quad \text{in} \ B, \\
w_1 = 0 = w_2, \quad \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \quad \text{on} \ \partial B,
\end{cases}
\end{align*}
\]  
(2.13)
Adding and subtracting the equations and the equation of boundary conditions of the problem (2.13), we have
\[
\begin{align*}
\Delta^2 (w_1 + w_2) &= \sqrt{\lambda_1 \lambda_2} \left( a(x) w_2 + b(x) w_1 \right) \text{ in } B, \\
w_1 + w_2 &= 0, \quad \frac{\partial (w_1 + w_2)}{\partial \nu} = 0 \quad \text{on } \partial B.
\end{align*}
\] (2.14)

and
\[
\begin{align*}
\Delta^2 (w_1 - w_2) &= \sqrt{\lambda_1 \lambda_2} \left( a(x) w_2 - b(x) w_1 \right) \text{ in } B, \\
w_1 - w_2 &= 0, \quad \frac{\partial (w_1 - w_2)}{\partial \nu} = 0 \quad \text{on } \partial B.
\end{align*}
\] (2.15)

Multiply both sides of (2.15) by \( w_1 - w_2 \) and take integration by parts, we obtain
\[
\int_B \left| \Delta (w_1 - w_2) \right|^2 \, dx = -\sqrt{\lambda_1 \lambda_2} \int_B (a(x) w_2 - b(x) w_1)(w_1 - w_2) \, dx.
\] (2.16)

Since \( a(x) \) and \( b(x) \) are sign changing weights, then (2.16) can be written as
\[
\int_B \left| \Delta (w_1 - w_2) \right|^2 \, dx = -\sqrt{\lambda_1 \lambda_2} \int_B |w_1 - w_2|^2 \, dx.
\] (2.17)

which proves that \( w_1 = w_2 \) in \( \overline{B} \). Since \( \sqrt{\lambda_1 \lambda_2} = \mu \) and by the properties of eigenvalue problem for the bi-Laplacian, we obtain that the first eigenfunction \( \varphi_1 \) as the only solution. Then for any positive constant \( c \), we have \( w_1 = w_2 = c \varphi_1 \). Thus \( \phi = c \frac{1}{\sqrt{\lambda_1}} \varphi_1 \) and \( \psi = c \frac{1}{\sqrt{\lambda_2}} \varphi_1 \).

This completes the proof.

Lemma 2.3: Let \( F \) and \( G \) be primitives of \( f \) and \( g \) respectively with \( F(0) = 0 \) and \( G(0) = 0 \). Suppose that \((u, v)\) be a solution of the system given by (1.1) and \( \alpha, \beta \) are some positive constants. Then we have the following identity:
\[
\int_{\partial B} (\Delta u, \Delta v)(x, v) \, d\sigma_x = \int_B \left( NF(u) + NG(v) - \alpha \lambda au + (\alpha + \beta) b v g(v) \right) \, dx
\] + \( (N - 4 - (\alpha + \beta)) \int_B (\Delta u, \Delta v)(x, v) \, dx.
\] (2.18)

Proof. Using [26, proposition 4], [33, theorem 2.1] and after some easy computations, we obtain the following identity:
\[
\begin{align*}
\frac{\partial}{\partial x_i} &\left[ x_j L \left(x_k \frac{\partial u_l}{\partial x_k} + a_{ij} u_l \right) \left( L - p_i - \frac{\partial}{\partial x_j} L_{\nu_j} \right) - \frac{\partial}{\partial x_j} \left( x_k \frac{\partial u_l}{\partial x_k} + a_{ij} u_l \right) L_{\nu_j} \right] \\
&= NL + x_j L a_{ij} u_l L_{\nu_j} - (a_i + 1) \frac{\partial u_l}{\partial x_i} L_{\nu_j} - (a_i + 2) \frac{\partial^2 u_l}{\partial x_i \partial x_j} L_{\nu_j},
\end{align*}
\] (2.19)

where \( L = L(x, U, p, r) \) is a Lagrangian with \( U = (u_1, u_2), \quad p = (p_i), \quad p_i^k = \frac{\partial u_k}{\partial x_i}, \quad r = (r_{ij}), \quad i = 1, 2, \ldots, N \) and \( a_1, a_2 \) are some constants.
Applying the identity (2.19) to the Lagrangian associate with problem (1.1), we get

\[ L = L(x, U, \nabla U, \Delta U) = (\Delta u, \Delta v) + F(u) + G(v), a_1 = \alpha, a_2 = \beta. \]

Integrating (2.19) over the unit ball \( B \) and using the conditions \( u = 0 = v, \ \frac{\partial u}{\partial v} = 0 = \frac{\partial v}{\partial v} \) on \( \partial B \), we get (2.18).

This completes the proof.

**Remark 2.1:** Putting \( \alpha + \beta = N - 4 \) in (2.18), we obtain the critical conditions on \( f \) and \( g \) are

\[ NF(u) - a f(u) = 0 \quad \text{and} \quad NG(v) - (N - 4 - \alpha) v g(v) = 0, \]

then we have

\[ f(u) = \frac{N/\alpha}{u} \quad \text{and} \quad g(v) = \frac{N/(N - 4 - \alpha)}{v}. \tag{2.20} \]

Hence, for some positive constants \( c_1 \) and \( c_2 \), we obtain

\[ f(u) = c_1 u^{\alpha-1} \quad \text{and} \quad g(v) = c_2 v^{N-4-\alpha-1}. \tag{2.21} \]

**Definition 2.1** ([35]): Let \( (X, \|\|) \) be a real Banach space and \( C \) be a nonempty closed convex subset of \( X \).

This subset \( C \) is called a cone of \( X \) if it satisfies the following conditions:

(i) \( x \in C, \mu > 0 \) implies \( \mu x \in C \); (ii) \( x \in C, -x \in C \) implies \( x = 0 \).

Now, we state a fixed-point theorem due to Figueiredo et al. [15] and Peletier and van der Vorst [25], which will be used as the main tool to prove our main results.

**Theorem 2.1** ([15, 25]): Let \( C \) be a cone in a Banach space \( X \) and \( T : X \to X \) be a compact map such that \( T(0) = 0 \). Assume that there exist the numbers \( 0 < r \leq R \) such that

(a) \( x \neq \delta T(x) \) for \( \delta \in [0,1] \) and \( \|x\| = r \),

(b) there exists a compact map \( S : B_R \times [0,\infty) \to C \) such that

\[ S(x, 0) = T(x), \ \text{if} \ \|x\| = R, \]

\[ S(x, \mu) \neq x, \ \text{if} \ \|x\| = R \text{ and } 0 \leq \mu < \infty, \]

\[ S(x, \mu) \neq x, \ \text{if} \ x \in B_R \text{ and } \mu \geq \mu_0. \]

Then if \( U = \{x \in C : \|x\| < R\} \) and \( B_\rho = \{x \in C : \|x\| < \rho\} \), we have

\[ i_C(T, B_R) = 0, \quad i_C(T, B_\rho) = 1, \quad i_C(T, U) = -1, \]

where \( i_C(T, \Omega) \) denotes the index of \( T \) with respect to \( \Omega \). In particular, \( T \) has a fixed point in \( U \).

3. Results and Discussions

This section is devoted to establish the existence and non-existence criterion of positive solutions to nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1).

Assume that the nonlinearities \( f \) and \( g \) satisfies the following hypothesis:

\[ (H_1) \quad \liminf_{s \to +\infty} f(s)/s > \lambda_1, \quad \limsup_{s \to 0} f(s)/s < \lambda_1, \]

\[ \liminf_{s \to +\infty} g(s)/s > \lambda_2, \quad \limsup_{s \to 0} g(s)/s < \lambda_2, \]
\( (H_2) \quad NF(s) - \alpha sf(s) \geq \theta_1 sf(s), \ s > 0 \) for some \( \theta_1 \geq 0, \ NG(s) - \beta sg(s) \geq \theta_2 sg(s), \ s > 0 \) for some \( \theta_2 \geq 0, \) and \( \alpha \) and \( \beta \) are positive real numbers satisfying \( \alpha + \beta = N - 4. \)

To establish our main results, we need the following priori estimates:

\( (H_3) \) there exists a constant \( d > 0 \) such that for every positive solution \((u, v)\) of the problem given (1) verifies \( \|u\|_\infty \leq d \) and \( \|v\|_\infty \leq d. \)

Now we are in position to present and prove our main results.

**Theorem 3.1:** If the hypothesis \((H_1), (H_2), \) and \((H_3)\) hold, then the nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) has a positive solution.

**Proof.** We prove this theorem by applying theorem 2.1. Consider a Banach space \( X = C^* (0,1) \times C^* (0,1), \) where \( C^* (0,1) \) denote the space of continuous bounded functions defined on \((0,1),\) endowed with the norm \( \|u\| = \sup_{t \in (0,1)} |u(t)|. \) We define a cone \( C \) on \( X \) by

\[
C = \{ w \in X : w(t) \geq 0, \ \text{for all} \ t \in (0,1) \},
\]

where \( w = (y, z) \geq 0 \) means that \( y \geq 0 \) and \( z \geq 0. \) We also define a compact map \( T : X \to X \) by

\[
T(w)(r) = \int_0^r G(r, s) |h(w(s))| ds, \ \text{where} \ h(w) = (a(x)g(v), b(x)f(u)). \quad (3.1)
\]

It is clear that a fixed point of \( T \) is a solution of the integral equations given by (2.9) and hence a fixed point of \( T \) is a solution of our problem (1.1).

Now, we will prove that all the conditions of theorem 2.1 for hypothesis \((H_1), (H_2), \) and \((H_3)\).

From the hypothesis \((H_1),\) we have there exists positive constants \( q_1 < 1 \) and \( q_2 < 1 \) such that \( f(u(x)) \leq q_1 \lambda_1 u(x) \) and \( g(v(x)) \leq q_2 \lambda_2 v(x). \) Then we have

\[
\lambda_2 \int v \Delta^2 \psi dx = \int v \Delta^2 \phi dx = \int \lambda b f(u) \phi dx \leq \lambda q_1 \lambda_1 \int u \phi dx. \quad (3.2)
\]

On the other hand,

\[
\lambda_1 \int u \phi dx = \int u \Delta^2 \psi dx = \int \Delta^2 u \psi dx = \int \lambda a g(v) \psi dx \leq \lambda q_2 \lambda_2 \int v \psi dx. \quad (3.3)
\]

Combining the inequalities (3.2) and (3.3), we get

\[
\lambda_2 \int v \Delta^2 \psi dx \leq \lambda^2 q_1 q_2 \lambda_2 \int v \psi dx, \quad (3.4)
\]

\[
\lambda_1 \int u \phi dx \leq \lambda^2 q_1 q_2 \lambda_4 \int u \phi dx. \quad (3.5)
\]

Since \( \lambda \) is a positive parameter, then from (3.4) and (3.5), we obtain \( q_1 q_2 < 1. \) Hence the inequalities (3.4) and (3.5) give a contradiction as because the integrals are nonzero. Moreover, if \( u \) and \( v \) are replaced by \( \delta u \) and \( \delta v \) respectively, where \( \delta \in [0,1], \) then the inequalities (3.4) and (3.5) also give a contradiction.

Therefore, we have \( w(t) \neq \delta T(w(t)) \) with \( \delta \in [0,1], \|w\| = r, \ w \in C, \)

Hence the condition (a) of theorem 2.1 holds.

Now, define a compact map \( S : C \times [0, \infty) \to C \) by

\[
S(w, \gamma)(r) = T(w + \gamma)(r). \quad (3.6)
\]
Then it is easy to see that $S(w,0) = T(w)$. This prove the first condition of (b) of theorem 2.1.

Again, the hypothesis $(H_1)$, gives that there exists positive constants $k_1 > \lambda_1$, $k_2 > \lambda_2$ and $\mu_0 > 0$ such that $f(y + \mu) \leq k_1 y$ and $g(z + \mu) \leq k_2 z$, for all $\mu \geq \mu_0$ and $(y, z) \geq (0, 0)$. Then we have

$$\lambda_2 \int \nabla \psi \phi dx = \int \Delta \psi \phi dx = \int \lambda_k f(u) \phi dx \geq \lambda k_1 \int u \phi dx \geq \lambda \lambda_1 \int u \phi dx . \quad (3.7)$$

Or

$$\lambda_1 \int u \phi dx = \int \nabla \psi \phi dx = \int \Delta \psi \phi dx = \int \lambda k g(v) \psi dx \geq \lambda k_2 \int \nabla \psi dx . \quad (3.8)$$

Combining the inequalities (3.7) and (3.8), we get

$$\lambda_2 a \int \nabla \phi dx \geq \lambda^2 k_2 a \int \nabla \phi dx . \quad (3.9)$$

Similarly, we can prove that

$$\lambda_2 b \int u \phi dx \geq \lambda^2 k_2 b \int u \phi dx . \quad (3.10)$$

Since $\lambda$ is a positive parameter, $a$ and $b$ are sign changing weights, the integrals $\int \nabla \phi dx$ and $\int u \phi dx$ are nonzero and $k_1 > \lambda_1$, $k_2 > \lambda_2$, then the inequalities (3.4) and (3.5) give a contradiction. Therefore, there exists a constant $\mu_0 > 0$ such that

$$S(w,\mu)(t) \neq w(t), \text{ for all } w \in C \text{ and } \mu \geq \mu_0. \quad (3.11)$$

This proves the third condition of (b) of theorem 2.1. Now, we prove the second condition of (b) of theorem 2.1 and to prove it, we consider the family of nonlinearities $f(y + \mu), g(z + \mu)$ for $\mu \in [0, \mu_0]$, using a priori estimates hypothesis $(H_3)$ which does not depend on $\mu$. Hence, for $R > r$ large enough, we get

$$S(w,\mu)(r) \neq w(r), \text{ for all } w \in C \text{ and } \mu \in [0, \mu_0], \|w\| = R. \quad (3.12)$$

Therefore, the relations (3.11) and (3.12) prove the second condition of (b) of theorem 2.1. That is all the conditions of theorem 2.1 are satisfied. Thus, by theorem 2.1 and hypothesis $(H_2)$, we conclude that the problem (2.9) exists a nontrivial positive solution and hence the nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) has a positive solution.

This completes the proof.

Now, we introduce the following critical exponents associated to the nonlinear weighted bi-harmonic system (1.1) by

$$p^* = \frac{N - \alpha}{\alpha}, \quad q^* = \frac{4 + \alpha}{N - 4 - \alpha}, \quad \text{where } \alpha \in ((N - 4)/2, N/2). \quad (3.13)$$

Then it is clear that

$$\frac{1}{p^* + 1} + \frac{1}{q^* + 1} = \frac{N - 4}{4}. \quad (3.14)$$

**Remark 3.1:** From the hypothesis $(H_1)$ and $(H_2)$, we have

$$\lim_{t \to \infty} f(t)/t^{p^*} = 0 \text{ and } \lim_{t \to \infty} g(t)/t^{q^*} = 0.$$

Certainly, from $(H_1)$, we have there exists $t_0 > 0$ such that $f(t) > 0$ and $g(t) > 0$ for $t > t_0$. Hence, for $t > t_0$, from $(H_2)$ we can write

$$NF(t) \geq -\theta_1 + \eta f(t) \text{ and } NG(t) \geq -\theta_2 + \mu g(t), \quad (3.15)$$
where \( \eta = \alpha + \theta_1 \) and \( \mu = \beta + \theta_2 \). Therefore,
\[
F'(t) - \frac{N}{\eta} F(t) \leq \frac{\theta_1}{\eta} \text{ and } G'(t) - \frac{N}{\mu} G(t) \leq \frac{\theta_2}{\mu}.
\] (3.16)

Multiplying the inequalities of (3.16) by \( t^{\eta} \) and \( t^{\mu} \) respectively, we get
\[
\frac{d}{dt}\left( -\frac{N}{t^{\eta}} F(t) \right) \leq \frac{\theta_1}{\eta} - t^{-\frac{N}{\eta}} \text{ and } \frac{d}{dt}\left( -\frac{N}{t^{\mu}} G(t) \right) \leq \frac{\theta_2}{\mu} - t^{-\frac{N}{\mu}}.
\]

Hence for some positive constants \( d_1 \) and \( d_2 \), we deduce that
\[
F(t) \leq d_1 t^{\frac{N}{\eta}} \text{ and } G(t) \leq d_2 t^{\frac{N}{\mu}}.
\] (3.17)

For large enough \( t \) and for some positive constants \( D \) and \( D' \), from (3.15) and (3.17), we have
\[
f(t) \leq D t^{\frac{N-1}{\eta}} \text{ and } g(t) \leq D' t^{\frac{N-1}{\mu}}.
\] (3.18)

Or since \( \eta = \alpha + \theta_1 \), \( \mu = \beta + \theta_2 \), \( \theta_1, \theta_2 > 0 \), and \( \alpha + \beta = N - 4 \), then we have \( \eta + \mu > N - 4 \).

The following theorem verify the priori estimates hypothesis given by \( (H_3) \).

**Theorem 3.2:** Under the hypothesis \( (H_1) \) and \( (H_2) \) on nonlinearities \( f \) and \( g \), the priori estimates hypothesis \( (H_3) \) is satisfied, namely, every positive solution of nonlinear weighted bi-harmonic system given by (1.1) is bounded in \( L^\infty \).

**Proof.** We prove this theorem using following four steps.

**Step 1.** In this step we claim that there exist four positive constants \( c_1, c_2, c_3, c_4 \) such that
\[
\int_B \lambda bf(u) \phi dx \leq c_1, \quad \int_B \lambda ag(v) \psi dx \leq c_2,
\]
\[
\int_B \lambda bu \phi dx \leq c_3, \quad \int_B \lambda a v \psi dx \leq c_4.
\] (3.19) (3.20)

Using the first two equations of our problem (1.1), we have
\[
\int_B \lambda bf(u) \phi dx = \int_B \Delta^2 v \phi dx = \int_B v \Delta^2 \phi dx = \lambda_2 \int_B v a \psi dx.
\] (3.21)

According to the hypothesis \( (H_1) \), there exist \( k_2 > \lambda_2 \) and \( A > 0 \) such that \( g(v) \geq k_2 v - A \). Hence, for a positive constant \( c \), from (3.21) we have
\[
\int_B \lambda bf(u) \phi dx = \lambda_2 \int_B v a \psi dx \leq c + \frac{\lambda_2}{k_2} \int_B ag(v) \psi dx.
\] (3.22)

Similarly, using hypothesis \( (H_1) \) and for a positive constant \( d \), we obtain
\[
\int_B \lambda ag(v) \psi dx = \lambda_1 \int_B bu \phi dx \leq d + \frac{\lambda_1}{k_1} \int_B bf(u) \phi dx.
\] (3.23)

Combining (3.22) and (3.23), and for positive constants \( m_1 \) and \( m_2 \), we have
Since \( \frac{\lambda_1}{k_1} < 1 \), hence we deduce (3.19) form (3.24). In the similar way by using the hypothesis \((H_1)\), we can easily obtain (3.20).

**Step 2.** In this step we claim that there exist four positive constants \( d_1, d_2, d_3, d_4 \) such that

\[
\begin{align*}
 u(r) \leq d_1, \quad v(r) \leq d_2, \quad &\text{for } \frac{2}{3} \leq r \leq 1, \\
u^*(1) \leq d_3, \quad v^*(1) \leq d_4. 
\end{align*}
\]

(3.25) (3.26)

Using (2.6) and (2.7), we observed that \( r \to G(r, s) \) is decreasing, hence we deduce that \( u(r) \) and \( v(r) \) are decreasing in \( r \) and for arbitrary \( \frac{2}{3} \leq r \leq 1 \), we have

\[
\begin{align*}
u(r) \leq u\left(\frac{2}{3}\right) & \leq 3 \int_{1/3}^{2/3} \lambda a(s) u(s) s ds \\
& \leq d \int_0^1 s^{N-1} (1-s)^2 \lambda b(s) u(s) s ds \\
& \leq d + \int_0^1 s^{N-1} (1-s)^2 \lambda b(s) u(s) s ds,
\end{align*}
\]

(3.27)

where \( d \) is a positive constant.

From (2.2) and lemma 2.2 and (3.27), we obtain

\[
u(r) \leq u\left(\frac{2}{3}\right) \leq 1 + \int_0^1 s^{N-1} (1-s)^2 \lambda b(s) u(s) s ds \leq d \left(1 + \int_B \lambda b \phi u(s) ds\right).
\]

Thus by (3.20), we yield that \( u(r) \leq d_1 \), for \( \frac{2}{3} \leq r \leq 1 \), and in the similar way we can prove that \( v(r) \leq d_2 \), for \( \frac{2}{3} \leq r \leq 1 \).

Now, differentiating the relations of (2.9) two times with respect to \( r \), we have

\[
\begin{align*}
u^*(r) = \int_0^r \frac{\partial^2 G(r, s)}{\partial r^2} \lambda a(s) g(v(s)) ds, \\
v^*(r) = \int_0^r \frac{\partial^2 G(r, s)}{\partial r^2} \lambda b(s) f(u(s)) ds.
\end{align*}
\]

(3.28)

Since the integrals of (3.28) converge, then taking limit as \( r \to 1 \) and we obtain

\[
\begin{align*}
u^*(1) = \int_0^1 \frac{\partial^2 G(r, s)}{\partial r^2} \lambda a(s) g(v(s)) ds, \\
v^*(1) = \int_0^1 \frac{\partial^2 G(r, s)}{\partial r^2} \lambda b(s) f(u(s)) ds.
\end{align*}
\]

Using (2.8), we get

\[
\begin{align*}
u^*(1) = \frac{1}{2} \int_0^1 s^{N-1} (1-s^2) \lambda a(s) g(v(s)) ds, \\
v^*(1) = \frac{1}{2} \int_0^1 s^{N-1} (1-s^2) \lambda b(s) f(u(s)) ds.
\end{align*}
\]

(3.29)

Now, for some positive constant \( d \), from (2.2) and lemma 2.2 and (3.29), we obtain that

\[
\begin{align*}
u^*(1) & \leq d \int_B \lambda a \phi g(v) ds, \\
v^*(1) & \leq d \int_B \lambda b \phi f(u) ds.
\end{align*}
\]

(3.30)

Combining (3.15) and (3.30), we have
This proves (3.26).

**Step 3.** In this step we claim that for a small number $0<\kappa<1$, there exists positive constants $e_1,e_2,e_3,e_4$ such that

\[
\int_0^x s^{N-1} \lambda b(s) f(u(s)) ds \leq e_1, \quad \int_0^x s^{N-1} \lambda a(s) g(v(s)) ds \leq e_2,
\]

\[
\int_B \lambda buf(u) dx \leq e_3, \quad \int_B \lambda avg(v) dx \leq e_4.
\]

For proving (3.31), we follow the proof of (3.19) and (3.20), and using lemma 2.1 and lemma 2.2 and for the small number $0<\kappa<1$, we get

\[
\int_0^x s^{N-1} \lambda b(s) f(u(s)) ds \\
\leq \int_0^x s^{N-1} \left( \frac{(1-s)^2}{(1-\kappa)^2} \right) \lambda b(s) f(u(s)) ds \\
\leq d \int_0^x s^{N-1} \lambda b(s) \phi(s) f(u(s)) ds = d \int_B \lambda b \phi f(u) dx \leq dc_1 = e_1,
\]

\[
\int_0^x s^{N-1} \lambda a(s) g(v(s)) ds \\
\leq \int_0^x s^{N-1} \left( \frac{(1-s)^2}{(1-\kappa)^2} \right) \lambda a(s) g(v(s)) ds \\
\leq d \int_0^x s^{N-1} \lambda a(s) \psi(s) g(v(s)) ds = d \int_B \lambda a \psi g(v) dx \leq dc_2 = e_2,
\]

where $d, \overline{d}$ are some constants. These prove (3.31).

For proving (3.32), rewrite the identity (2.18) of lemma 2.3 and considering the fact that $\alpha + \beta = N - 4$, we have

\[
\int_B \left( NF(u) - \alpha \lambda buf(u) \right) dx + \int_B \left( NG(v) - \beta \lambda avg(v) \right) dx = \int_{\partial B} (\Delta u, \Delta v)(x,v) d\sigma_x.
\]

Using the hypothesis $(H_2)$ in the left-hand side of (3.33) and after easy computation, we get

\[
\theta_1 \int_B \lambda buf(u) dx + \theta_2 \int_B \lambda avg(v) dx \leq e u^*(1) v^*(1),
\]

where $e$ is generic constant and $\theta_1$ and $\theta_2$ are constants given in hypothesis $(H_2)$. Thus (3.34) can be written as

\[
\theta_1 \int_B \lambda buf(u) dx + \theta_2 \int_B \lambda avg(v) dx \leq e.
\]

Since the two integrals of (3.35) are positive separately, hence we obtain (3.32) directly from (3.35).

**Step 4.** In this step using the hypothesis $(H_1)$ and $(H_2)$, we claim that there exist two positive constants $c_1$ and $c_2$ such that, for any solution $(u,v)$ of our problem given by (1.1),

\[
\|u\|_{\infty} \leq c_1, \quad \|v\|_{\infty} \leq c_2.
\]

For $u$, we have
\[ \|u\|_c \leq u(0) \leq \int_0^1 G(0,s) \lambda a(s) g(v(s)) ds \leq c \int_0^1 s^3 (1-s)^2 \lambda a(s) g(v(s)) ds \]
\[ \leq c \int_0^1 s^3 \lambda a(s) g(v(s)) ds \leq c \int_0^1 s^3 \lambda a(s) g(v(s)) ds + c \int_0^1 s^3 \lambda a(s) g(v(s)) ds, \]
where \( t \in (0,1) \) is arbitrary and \( c \) is a positive constant whose value may vary.

Let \( g(m) = \max_{s \in [0,m]} g(s) \) for \( m \in (0, \infty) \), and applying Hölder’s inequality in (3.37), we get
\[ \|u\|_c \leq c \lambda a t^4 g \left( \|v\|_c \right) + c \lambda a \left( \int_t^1 s^{\gamma_1(q^*+1)} ds \right)^{\frac{1}{q^*+1}} \left( \int_t^1 s^{-N+1} g(v(s)) ds \right)^{\frac{q^*}{q^*+1}} \]
\[ \leq c \lambda a t^4 g \left( \|v\|_c \right) + c \lambda a \left( \int_t^1 s^{\gamma_1(q^*+1)} ds \right)^{\frac{1}{q^*+1}} \left( \int_t^1 s^{-N+1} g(v(s)) \left( g(v(s)) \right)^{\frac{1}{q^*}} ds \right)^{\frac{q^*}{q^*+1}} \],
where \( \gamma_1 = 3 - \left( N - 1 \right) \frac{q^*}{q^* + 1} \).

From remark 3.1, we have a positive constant \( D \) such that
\[ g(s) < D (1+s)^{q^*}, \text{ for all } s \geq 0, \]  
\[ f(s) < D (1+s)^{p^*}, \text{ for all } s \geq 0. \]  
Hence, we get
\[ \|u\|_c \leq c \lambda a t^4 g \left( \|v\|_c \right) + D^{q^*} c \lambda a \left( \int_t^1 s^{\gamma_1(q^*+1)} ds \right)^{\frac{1}{q^*+1}} \left( \int_t^1 s^{-N+1} g(v(s)) (1+v(s)) ds \right)^{\frac{q^*}{q^*+1}} \]
\[ \leq c \lambda a t^4 g \left( \|v\|_c \right) + D^{q^*} c \lambda a \left( \int_t^1 s^{\gamma_1(q^*+1)} ds \right)^{\frac{1}{q^*+1}} \left( \int_B g(v) dx + \int_B g(v) v(x) dx \right)^{\frac{q^*}{q^*+1}}. \]
Using (3.31) and (3.32) in (3.39), we get
\[ \|u\|_c \leq c \lambda a t^4 g \left( \|v\|_c \right) + c \lambda a \left( \int_t^1 s^{\gamma_1(q^*+1)} ds \right)^{\frac{1}{q^*+1}}. \]  
Similarly, for \( v \), we get
\[ \|v\|_c \leq c \lambda b t^4 f \left( \|u\|_c \right) + c \lambda b \left( \int_t^1 s^{\gamma_2(p^*+1)} ds \right)^{\frac{1}{p^*+1}}, \]
where \( \gamma_2 = 3 - \left( N - 1 \right) \frac{p^*}{p^* + 1} \).

We consider that in all the next inequalities \( c \) will always represent a positive constant.

Now, after some easy computations from (3.40) and (3.41), we get
\[ \|v\|_c \leq c \lambda a t^4 g \left( \|v\|_c \right) + c \lambda a t^{\frac{q^*}{q^*+1}}, \]  
(4.32)
\[ \|v\|_\infty \leq c\lambda bt^4 f\left(\|u\|_\infty\right) + c\lambda bt^{\frac{4+(4-N)p^*}{p^*+1}}. \] (3.43)

Note that if \( f \) and \( g \) are bounded then (3.36) comes directly from (3.42) and (3.43). But if \( g \) is not bounded then, by remark 3.1, we have there exists a positive constant \( K \), as like (3.38) such that \( g\left(r\right) \leq K\left(r\right)^{q^*} \), for all \( r \geq 1 \). Thus, we can write (3.42) as

\[ \|u\|_\infty \leq c\lambda at^{4\left(\|u\|_\infty\right)^{q^*}} + c\lambda at^{\frac{4+(4-N)q^*}{q^*+1}}. \] (3.44)

Now, inserting (3.43) in (3.44), and using the inequality \( (a+b)^n \leq c_n \left(a^n + b^n\right) \) for \( a, b, n \geq 0 \), where \( c_n \) is a positive constant depending on \( n \), we yield

\[ \|v\|_\infty \leq c\lambda at^4\left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda at^{\frac{4+(4-N)q^*}{q^*+1}}. \] (3.45)

But it is easy to show that

\[ \frac{4+(4-N)p^*}{p^*+1}q^* + 4 = \frac{4+(4-N)q^*}{q^*+1} + \frac{N}{p^*+1} = 4 - \frac{Nq^*}{q^*+1}. \] (3.46)

Hence from (3.45), we have

\[ \|v\|_\infty \leq c\lambda at^4\left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda at^{\frac{4-Nq^*}{q^*+1}}. \] (3.47)

Let \( r = 4\left(q^* + 1\right) \). Since \( t \in (0,1) \), then from (3.47) we have

\[ \|v\|_\infty \leq c\lambda at^r\left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda at^{\frac{4-Nq^*}{q^*+1}}. \]

Now, if we let \( h(t) = t^r\left(f\left(\|u\|_\infty\right)\right)^{q^*} + t \), then the function \( h \) gives its minimum value at

\[ t_0 = c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{-4-Nq^*}{q^*+1}}, \]

and we have

\[ \|v\|_\infty \leq c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{-q^*\left(q^* + 1\right)}{(r-4)\left(q^* + 1\right)+Nq^*}} + c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{-q^*\left(q^* + 1\right)}{(r-4)\left(q^* + 1\right)+Nq^*}\left(4 - \frac{Nq^*}{q^*+1}\right)}. \] (3.48)

But it is easy to show that

\[ \frac{-q^*\left(q^* + 1\right)}{(r-4)\left(q^* + 1\right)+Nq^*} + q^* = \frac{-q^*\left(q^* + 1\right)}{(r-4)\left(q^* + 1\right)+Nq^*}\left(4 - \frac{Nq^*}{q^*+1}\right) = \frac{1}{p}. \]

Hence from (3.48), we get

\[ \|u\|_\infty \leq c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{1}{p}}. \] (3.49)
On the other hand, from remark 3.1, we have $f(x) = o(x^\nu)$ for $x \to \infty$, hence the inequality (3.49) becomes
\[
\|u\|_{\infty} \leq c\lambda a\left(1+o(\|u\|_{\infty})\right),
\]
which proves that $\|u\|_{\infty}$ is bounded. Therefore, by (3.43) we have $\|v\|_{\infty}$ is bounded.

This completes the proof of this step.

**Theorem 3.3:** If the nonlinearities $f$ and $g$ satisfies the following conditions:
\[
NF(t) - \alpha f(t) \leq 0 \text{ and } NG(t) - \beta g(t) \leq 0, \text{ for } t > 0,
\]
where $\alpha$ and $\beta$ are positive real numbers satisfying $\alpha + \beta = N - 4$, then the problem given by (1.1) has no any nontrivial solution $(u, v) \in C^4(B) \times C^4(B)$.

**Proof.** If we put $\alpha + \beta = N - 4$ in the identity (2.18), then by remark 2.1 and the boundary conditions of (1.1), we obtain that
\[
\left(\Delta u, \Delta v\right) = \frac{\partial^2 u}{\partial v^2} \frac{\partial^2 v}{\partial v^2}.
\]

Now, if $(u, v) \in C^4(B) \times C^4(B)$ is a nontrivial solution of (1.1) and since $B$ is star-shaped domain about 0, then we have $u, v \geq 0$ on $\partial B$. In this case if we apply the given conditions $NF(t) - \alpha f(t) \leq 0$ and $NG(t) - \beta g(t) \leq 0$, for $t > 0$, then the identity (2.18) raised a contradiction. Hence the problem given by (1.1) may not have any nontrivial solution $(u, v) \in C^4(B) \times C^4(B)$.

This completes the proof.

Now, we discuss two illustrative examples to verify our main results.

**Example 3.1:** Let $B$ denotes the unit ball in $\mathbb{R}^n$. Consider the weight functions $a$ and $b$ are defined as
\[
a(x) = |x|^2 \text{ and } b(x) = |x|^2, \text{ for all } x \in B,
\]
and the nonlinear functions $f$ and $g$ are defined as
\[
f(x) = x^3 \text{ and } g(x) = x^2, \text{ for all } x \in [0, \infty).
\]
Then there exists a positive parameter $\lambda$ such the following nonlinear weighted bi-harmonic system of elliptic PDEs
\[
\begin{align*}
\Delta^2 u &= \lambda |x|^2 v^2, \quad v > 0 \text{ in } B, \\
\Delta^2 v &= \lambda |x|^2 u^3, \quad u > 0 \text{ in } B, \\
u &= 0 = v, \quad \frac{\partial u}{\partial v} = 0 = \frac{\partial v}{\partial v} \text{ on } \partial B,
\end{align*}
\]
has a positive solution.

**Proof.** From the definition of $f$ and $g$, it is clear that the hypothesis $(H_1)$ holds. If we fix the dimension of the space $N = 5$, then the corresponding problem for radially symmetric solution is as follows:
\[
\begin{aligned}
&u^{(4)} + \frac{2(5-1)}{r} u'' + \frac{(5-1)(5-3)}{r^2} u' - \frac{(5-1)(5-3)}{r^3} u = \lambda |x|^2 v^2, \text{ for } r \in (0,1), \\
v^{(4)} + \frac{2(5-1)}{r} v'' + \frac{(5-1)(5-3)}{r^2} v' - \frac{(5-1)(5-3)}{r^3} v = \lambda |x|^2 u^3, \text{ for } r \in (0,1), \\
&u'(0) = 0 = v'(0), u''(0) = 0 = v''(0), u'(1) = 0 = v'(1), u(1) = 0 = v(1).
\end{aligned}
\] (3.51)

Then any solution \((u(r), v(r)) \in C^4(0,1) \times C^4(0,1)\) of (3.51) is a radial symmetric solution of (3.50).

Now, it is easy to verify that for a suitable choice of values of \(\theta_1, \theta_2, \alpha\) and \(\beta\), the hypothesis \((H_2)\) and \((H_3)\) are holds. Since, all the hypothesis of theorem 3.1 are satisfied, then an application of theorem 3.1 gives a positive solution of the problem given by (3.50).

**Example 3.2**: Let \(B\) denotes the unit ball in \(\mathbb{R}^N\). Consider the weight functions \(a, b\) same as to example 3.1 and the nonlinear functions \(f\) and \(g\) are defined as

\[
f(x) = x^3 + x^2 + x \text{ and } g(x) = x^2 + x, \text{ for all } x \in [0, \infty).
\]

Then there exists a positive parameter \(\lambda\) such the following nonlinear weighted bi-harmonic system of elliptic PDEs

\[
\begin{aligned}
\Delta^2 u &= \lambda |x|^2 v^2 + v, \quad v > 0 \text{ in } B, \\
\Delta^2 v &= \lambda |x|^2 u^3 + u^2 + u, \quad u > 0 \text{ in } B, \\
u &= 0 = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial B,
\end{aligned}
\] (3.52)

has a positive solution.

**Proof**. From the definition of \(f\) and \(g\), it is clear that the hypothesis \((H_1)\) holds. If we fix the dimension of the space \(N = 5\), then the corresponding problem for radially symmetric solution is as follows:

\[
\begin{aligned}
&u^{(4)} + \frac{2(5-1)}{r} u'' + \frac{(5-1)(5-3)}{r^2} u' - \frac{(5-1)(5-3)}{r^3} u = \lambda |x|^2 v^2 + v, \text{ for } r \in (0,1), \\
v^{(4)} + \frac{2(5-1)}{r} v'' + \frac{(5-1)(5-3)}{r^2} v' - \frac{(5-1)(5-3)}{r^3} v = \lambda |x|^2 u^3 + u^2 + u, \text{ for } r \in (0,1), \\
&u'(0) = 0 = v'(0), u''(0) = 0 = v''(0), u'(1) = 0 = v'(1), u(1) = 0 = v(1).
\end{aligned}
\] (3.53)

Then any solution \((u(r), v(r)) \in C^4(0,1) \times C^4(0,1)\) of (3.53) is a radial symmetric solution of (3.52).

Now, it easy to verify that for a suitable choice of values of \(\theta_1, \theta_2, \alpha\) and \(\beta\), the hypothesis \((H_2)\) and \((H_3)\) are holds. Since, all the hypothesis of theorem 3.1 are satisfied, then an application of theorem 3.1 gives a positive solution of the problem given by (3.52).

**4. Conclusion**

In this study, we have proven the existence and non-existence criteria for the solution of nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) applying a fixed-point theorem due to Figueiredo et al. [15]
and Peletier and van der Vorst [25]. A priori solution estimates of the considered problem have been established which is proved in theorem 3.2. Using theorem 3.1, one can easily be checked the existence of positive solution of nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) and theorem 3.3 ensure the case of non-existence of solutions of that same system. The established results provide an easy and straightforward technique to check the existence and non-existence of solution to nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1). Furthermore, the results of this research extend the corresponding results of Soltani and Yazidi [28] and Dwivedi [13].

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