Casimir effect for mixed fields

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Abstract

We analyze the Casimir effect for a flavor doublet of mixed scalar fields confined inside a one-dimensional finite region. In the framework of the unitary inequivalence between mass and flavor representations in quantum field theory, we employ two alternative approaches to derive the Casimir force: in the first case, the zero-point energy is evaluated for the vacuum of fields with definite mass, then similar calculations are performed for the vacuum of fields with definite flavor. We find that signatures of mixing only appear in the latter context, showing the result to be independent of the mixing parameters in the former.

1. Introduction

The concept of vacuum in quantum field theory (QFT) is as fascinating as puzzling. In several situations from both particle physics and condensed matter, the non-trivial condensate structure of the vacuum is crucial to explain a variety of both theoretical and observable phenomena \cite{1,2,3,4}. In this connection, one of the most eloquent examples is provided by the Casimir effect \cite{5}, which occurs whenever a quantum field is enclosed in a finite region; such a confinement gives rise to a net attractive force between the boundaries, the entity of which is closely related to the nature of the vacuum itself \cite{6}.

In line with these findings, in Refs. \cite{7,8} it was shown that vacuum also plays a central role within the framework of flavor mixing and oscillations in QFT. In Refs. \cite{7}, in particular, it was found that the vacuum for fields with definite mass (mass vacuum) is unitarily inequivalent \cite{9,10} to the one for fields with definite flavor (flavor vacuum), as they are related by a non-trivial Bogoliubov transformation. In light of this, it is reasonable to expect that vacuum effects in the context of QFT mixing may, in principle, depend on which of these states represents the physical vacuum. This is indeed a matter of open debate \cite{11}: an interesting test bench in this sense has been recently provided...
by the analysis of the weak decay of accelerated protons (inverse $\beta$-decay) with mixed neutrinos [12, 13, 14].

Led by these considerations, here we analyze the Casimir effect for a system of two mixed scalar fields, showing that the force is sensitive to the choice of the vacuum state. In particular, we find that the result obtained using the flavor vacuum exhibits corrections that explicitly depend on the mixing angle and the mass difference of fields, in contrast with the case of the mass vacuum.

We remark that, although limited to scalar fields in $1 + 1$ dimensions, our analysis contains all the essential features of the problem, thus giving general validity to our results. We also stress that the local nature of the Casimir force prevents our calculations from being affected by the choice of a particular regularization scheme. Such a characteristic is not present in other contexts where effects of the flavor vacuum were studied [15].

The paper is organized as follows: Sec. 2 is devoted to briefly review the derivation of the Casimir force for a massive scalar field in $1 + 1$ dimensions. In Sec. 3 we analyze how the standard expression gets modified in the presence of mixed fields by performing calculations on both mass and flavor vacua. Sec. 4 contains conclusions and an outlook for future developments. Throughout the paper, we use natural units and the metric in the conventional timelike signature.

2. Casimir effect for a massive scalar field

Let us start by deriving the Casimir force for a massive charged scalar field $\hat{\phi}$ in $1 + 1$ dimensions (to this aim, we basically follow the treatment of Ref. [16]). In this framework, the free Lagrangian density $\hat{\mathcal{L}}$ takes the form

$$\hat{\mathcal{L}} = \partial_{\mu} \hat{\phi}^\dagger \partial^{\mu} \hat{\phi} - m^2 \hat{\phi}^\dagger \hat{\phi},$$

where $m$ is the mass of the field.

The Dirichlet boundary conditions imposed by the presence of the Casimir plates read

$$\hat{\phi}(t, 0) = \hat{\phi}(t, L) = 0,$$

with $L$ being the distance between the two confining surfaces. These constraints only allow modes with momentum $k_n = \pi n / L$ to give a non-vanishing contribution to the field expansion, yielding

$$\hat{\phi}(t, x) = \frac{1}{\sqrt{2L}} \sum_{n = 1}^{\infty} \sin \frac{k_n x}{\sqrt{\omega_n}} \left[ a_n e^{-i\omega_n t} + b_n^\dagger e^{i\omega_n t} \right],$$

where $n = 1, 2, ...$ and $\omega_n = \sqrt{k_n^2 + m^2}$. Here $a_n$ ($b_n^\dagger$) are the usual annihilation (creation) operators of a particle (antiparticle) with momentum $k_n$ and frequency $\omega_n$. They are assumed to satisfy the canonical bosonic algebra

$$\left[ a_n, a_n^\dagger \right] = \left[ b_n, b_n^\dagger \right] = \delta_{nn'}, \quad \forall n, n',$$

\footnote{To simplify the notation, we shall omit the $(t, x)$-dependence of the field when unnecessary.}
with all other commutators vanishing. The vacuum state is defined by
\[ |0\rangle = \hat{a}_n|0\rangle = \hat{b}_n|0\rangle = 0, \quad \forall n. \]  
(5)

In order to compute the Casimir force, let us now evaluate the zero-point energy density of the field as
\[ \varepsilon_0 = \langle 0|\hat{T}_{00}|0\rangle, \]  
(6)

where \( \hat{T}_{\mu\nu} \) is the stress-energy tensor derived from the Lagrangian density Eq. (1) [17]. A straightforward calculation leads to
\[ \varepsilon_0 = \frac{1}{2L} \sum_{n=1}^{\infty} \omega_n. \]  
(7)

Using the standard definition of Casimir force [6, 18]
\[ F_0 = -\frac{\partial}{\partial L}(L \varepsilon_0), \]  
(8)

and exploiting a suitable renormalization scheme [16], we finally obtain the following finite expression for the net force between the plates:
\[ F = -\frac{m^2}{\pi} \sum_{n=1}^{\infty} \left[ K_2(2mLn) - \frac{K_1(2mLn)}{2mLn} \right], \]  
(9)

where \( K_\nu(x) \) is the modified Bessel function of the second kind [19]. Notice that, in the limit \( m \to 0 \), Eq. (9) correctly reproduces the more familiar expression of the Casimir force for a massless field [6, 16, 18].

3. Casimir effect for mixed fields

Let us now generalize the above formalism to the context of field mixing. For this purpose, consider the following Lagrangian density describing two charged scalar fields with a mixed mass term [8]:
\[ \hat{L} = \sum_{\sigma=A,B} \left( \partial_\mu \hat{\phi}_\sigma^\dagger \partial^\mu \hat{\phi}_\sigma - m_\sigma^2 \hat{\phi}_\sigma^\dagger \hat{\phi}_\sigma \right) - m_{AB}^2 \left( \hat{\phi}_A^\dagger \hat{\phi}_B + \hat{\phi}_B^\dagger \hat{\phi}_A \right), \]  
(10)

where \( \hat{\phi}_\sigma \) (\( \sigma = A, B \)) are the fields with definite flavor \( \sigma \).

It is a trivial matter to check that the mixing transformations
\[ \begin{pmatrix} \hat{\phi}_A \\ \hat{\phi}_B \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix}, \]  
(11)

allow to recast the quadratic form Eq. (10) into a diagonal Lagrangian density for two free charged scalar fields \( \hat{\phi}_j \) (\( j = 1, 2 \)) with mass \( m_j \):
\[ \hat{L} = \sum_{j=1,2} \left( \partial_\mu \hat{\phi}_j^\dagger \partial^\mu \hat{\phi}_j - m_j^2 \hat{\phi}_j^\dagger \hat{\phi}_j \right), \]  
(12)
where the two sets of mass parameters $m_\sigma$ and $m_j$ are related by
\begin{align}
m_A^2 &= \cos^2\theta m_1^2 + \sin^2\theta m_2^2, \\
m_B^2 &= \sin^2\theta m_1^2 + \cos^2\theta m_2^2,
\end{align}
and $m_{AB}^2$ in Eq. (11) is given by $m_{AB}^2 = (m_2^2 - m_1^2) \sin\theta \cos\theta$. The mixing angle $\theta$ is defined as $\tan 2\theta = 2m_{AB}^2/(m_B^2 - m_A^2)$.

Note that each of the two fields $\phi_j$ ($j = 1, 2$) in Eq. (11) can be expanded as in Eq. (5). Thus, according to Eq. (5), one can define the vacuum for fields with definite mass (mass vacuum) as
\begin{equation}
\hat{a}_{n,j}|0\rangle_{1,2} = \hat{b}_{n,j}|0\rangle_{1,2} = 0, \quad \forall n, \quad j = 1, 2.
\end{equation}

To derive the corresponding relation for fields with definite flavor, it is worth rewriting Eq. (11) in terms of the mixing generator $K_{\theta,\mu}(t)$ [20] as:
\begin{equation}
\hat{\phi}_\chi(t, x) = K_{\theta,\mu}(t)^{-1} \hat{\phi}_\chi(t, x) K_{\theta,\mu}(t), \quad (\chi, \mu) = (A, 1), (B, 2),
\end{equation}
where $K_{\theta,\mu}(t) = G_\theta(t)I_\mu(t)$, with
\begin{align*}
G_\theta(t) &= \exp \left[-i\theta \int_0^L \! dx \left( \hat{\pi}_1(t, x) \hat{\phi}_2(t, x) + \hat{\phi}_1(t, x) \hat{\pi}_1(t, x) \right. \right. \\
&\quad \left. \left. - \hat{\pi}_2(t, x) \hat{\phi}_1(t, x) - \hat{\phi}_1(t, x) \hat{\pi}_2(t, x) \right) \right],
\end{align*}
and
\begin{equation}
I_\mu(t) = \exp \sum_{n=1}^\infty \sum_{\sigma, j} \xi_{\sigma, j}^n \left( a_{n,\sigma}(t) b_{n,\sigma}(t) - b_{n,\sigma}(t) a_{n,\sigma}(t) \right).
\end{equation}

Here $\hat{\pi}_j \equiv \partial_t \hat{\phi}_j^\dagger$ ($j = 1, 2$) is the canonical momentum conjugate to the field $\hat{\phi}_j$,
\begin{equation}
\xi_{\sigma, j}^n \equiv \begin{cases} \frac{1}{2} \log \left( \frac{\omega_{n,\sigma}}{\omega_{n,j}} \right) & \text{and} \\
\omega_{n,\sigma} = \sqrt{k_n^2 + \mu_\sigma^2} & (\sigma = A, B). \end{cases}
\end{equation}

For $\mu_A = m_1$ and $\mu_B = m_2$, one can easily check that $I_\mu(t) = 1$, and the field expansions for definite flavor fields read
\begin{equation}
\hat{\phi}_\chi(t, x) = \frac{1}{\sqrt{2L}} \sum_{n=1}^\infty \sin k_n x \sqrt{\omega_{n,\chi}} \left[ \hat{a}_{n,\chi}(t) e^{-i\omega_{n,\chi} t} + \hat{b}_{n,\chi}(t) e^{i\omega_{n,\chi} t} \right],
\end{equation}
with $(\chi, \mu) = (A, 1), (B, 2)$. The corresponding vacuum (flavor vacuum) is defined by
\begin{equation}
|0(t)\rangle_{A, B} = G_\theta^{-1}(t)|0\rangle_{1, 2},
\end{equation}
with $\hat{b}_{n,\sigma}(t)|0(t)\rangle_{A, B} = \hat{b}_{n,\sigma}(t)|0(t)\rangle_{A, B} = 0, \forall n, t$, as expected\footnote{In the following, we will work in the Heisenberg picture: this is particularly convenient in the present context since special care has to be taken with the time dependence of flavor vacuum (see the discussion in Ref. [21]).}.
We stress that the action of the mixing generator $K_{\theta,\mu}(t)$ on the mass vacuum is non-trivial: Eq. (16), indeed, hides a Bogoliubov transformation at the level of ladder operators, which induces a rich condensate structure into the flavor vacuum. The crucial point is that, in the infinite volume limit, mass and flavor vacua become orthogonal to each other, thus giving rise to unitarily inequivalent Fock spaces [7] (this is a well-known feature of QFT [9, 10], reflecting in the non-unitary nature of the generator of Bogoliubov transformations in the infinite volume limit).

Note that the expansions Eq. (19) rely on a particular choice of the wave function basis, namely that referring to the free field masses $m_1, m_2$. However, a natural alternative put forward in Ref. [22] would be to expand flavor fields in the basis corresponding to $\mu_A = m_A$ and $\mu_B = m_B$, with $m_A$ and $m_B$ given in Eqs. (13) and (14), respectively. For the sake of completeness, in what follows we shall deal with both these two cases; consistently with the previous notation, the vacuum associated to the last expansion will be referred to as tilde flavor vacuum and denoted by $|\tilde{0}(t)\rangle_{A,B}$.

On the basis of the above discussion, it seems obvious that different candidates for the rôle of fundamental vacuum must be taken into account in the context of field mixing: the mass vacuum $|0\rangle_{1,2}$, the flavor vacuum $|0(t)\rangle_{A,B}$ and the tilde flavor vacuum $|\tilde{0}(t)\rangle_{A,B}$. The question naturally arises as to which of these states has indeed physical meaning: to this end, in the following we evaluate the Casimir force in the aforementioned cases, showing that the result carries footprints of the particular choice of vacuum.

3.1. Mass vacuum

To begin with, we investigate the Casimir effect by assuming the mass vacuum $|0\rangle_{1,2}$ to be physical. In this case, it is a trivial matter to check that calculations closely follow the ones of Sec. 2 giving the following expression for the Casimir force:

$$F_m = - \sum_{j=1,2} \frac{m_j^2}{\pi} \sum_{n=1}^{\infty} \left[ K_2(2m_j Ln) - \frac{K_1(2m_j Ln)}{2m_j Ln} \right],$$

(21)

where the subscript $m$ is a reminder that we are dealing with the mass vacuum. By comparison with Eq. (9), we find out that the result is the same we would obtain for two non-interacting (unmixed) fields. In other words, the Casimir force evaluated with respect to the vacuum $|0\rangle_{1,2}$ is insensitive to the mixing, being independent of the mixing angle $\theta$ (this could be somehow expected, since by definition the mass vacuum can be factorized into the product of the vacuum for the field $\phi_1$ times the vacuum for the field $\phi_2$).

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3 As shown in Ref. [22], this setting is singled out by the requirement that flavor states must be simultaneous eigenstates of the 4-momentum and flavor charge operators.
Figure 1: The behavior of $F_m$ (blue line) and $\tilde{F}_f$ (red line) as functions of the distance $L$, for sample values of $\theta$, $m_1$ and $m_2$ ($\theta = \pi/3$, $m_1 = 0.8$ eV, $m_2 = 0.9$ eV).

3.2. Tilde flavor vacuum

Let us now analyze how the Casimir force gets modified when referring to the tilde flavor vacuum $|\tilde{0}(t)\rangle_{A,B}$. Following the same line of reasoning of Sec. 3.1 and exploiting Eqs. (19) and (20) with $\mu_A = m_A$, $\mu_B = m_B$, we have

$$\tilde{F}_f = -\sum_{\sigma = A,B} \frac{m_{\sigma}^2}{\pi} \sum_{n=1}^{\infty} \left[ K_2(2m_{\sigma}Ln) - \frac{K_1(2m_{\sigma}Ln)}{2m_{\sigma}Ln} \right],$$

(22)

where the subscript $f$ stands for flavor.

The forces $F_m$ and $\tilde{F}_f$ have been numerically evaluated and plotted in Fig. 1 as functions of the distance $L$ between the plates. We can readily see that their overall behaviors do not differ significantly from each other. To enhance the validity of this result, we can also compare their analytic expressions, at least in the approximation of $\delta m^2 L^2 \ll 1$, where $\delta m^2 \equiv m_2^2 - m_1^2$. In this regard, let us recast Eqs. (13) and (14) in the form

$$m_{\sigma}^2 = m_{1\sigma}^2 + \sin^2 \theta \delta m^2,$$

(23)

$$m_{\sigma}^2 = m_{2\sigma}^2 - \sin^2 \theta \delta m^2.$$

(24)

For $\delta m^2 L^2 \ll 1$, we are allowed to expand Eq. (22) as follows

$$\tilde{F}_f \approx F_m + \delta F_1,$$

(25)

where the first-order correction $\delta F_1$ can be obtained by using the asymptotic form of the modified Bessel function [19] and the zeta function regulari-
By explicit calculation, we have
\[ \delta F_1 \approx \frac{\pi}{12} \frac{\sin^2 \theta}{L^2} \left( \frac{\delta m^2}{m_1^2} \right)^2. \] (26)

By comparing with Eq. (21), we see that the magnitude of the force derived with respect to the tilde flavor vacuum is higher than the one calculated on the mass vacuum (of course, the gap between \( \tilde{F}_f \) and \( F_m \) narrows as \( L \) increases, since \( \delta F_1 \to 0 \) for \( L \to \infty \), as expected). Remarkably, \( \tilde{F}_f \) explicitly depends on the mixing parameters \( \theta \) and \( \delta m^2 \), a feature which is absent in the previous framework.

### 3.3. Flavor vacuum

Finally, let us evaluate the Casimir force on the flavor vacuum \( |0(t)\rangle_{A,B} \). In this case, using Eqs. (19) and (20) with \( \mu_A = m_1 \), \( \mu_B = m_2 \), the zero-point energy density is given by
\[ \varepsilon_0 = \frac{1}{2L} \sum_{n=1}^{\infty} \sum_{j=1,2} \omega_{n,j} \left( 1 + 2 |V_n|^2 \sin^2 \theta \right), \] (27)
where
\[ |V_n| \equiv \frac{1}{2} \left( \sqrt{\omega_{n,1}} - \sqrt{\omega_{n,2}} \right). \] (28)

Thus, inserting the previous expression into Eq. (8), the Casimir force \( F_f \) can be cast in the form
\[ F_f = F_m + \delta F_2, \] (29)
where \( F_m \) has been defined in Eq. (21) and
\[ \delta F_2 = -\frac{\partial}{\partial L} \left[ \sum_{n=1}^{\infty} (\omega_{n,1} + \omega_{n,2}) |V_n|^2 \sin^2 \theta \right]. \] (30)

The correction \( \delta F_2 \) has been numerically evaluated and plotted for small values of \( L \) (see Fig. 2).

By comparison with Eq. (29), we also provide its analytic expression in the limit of \( m_j L \ll 1 \) (\( j = 1, 2 \))
\[ \delta F_2 = -\frac{3}{2\pi^3} \frac{\sin^2 \theta L^2 \zeta(3) \left( \delta m^2 \right)^2}{L^2}, \] (31)
where \( \zeta(3) \) is the Apéry’s constant [23]. Note that, within such an approximation, the magnitude of \( \delta F_2 \) grows as \( L \) increases, as it is evident from Fig. 2.

Clearly, this is a subdominant contribution to the total Casimir force \( F_f \), which indeed decreases as \( L \) grows, as it can be easily checked. On the other hand, the behavior of \( \delta F_2 \) drastically changes for large distances, where the approximation \( m_j L \ll 1 \) fails, and indeed \( \delta F_2 \) correctly vanishes.

Therefore, by comparing Eqs. (21), (25) and (29), we realize that the predicted value of the Casimir force for mixed fields varies depending on which of the three aforementioned vacua – the mass, the flavor and the tilde flavor vacua – is indeed the physical one.
Figure 2: The behavior of $\delta F_2$ as function of $L$ (in the regime of small $L$), for $\theta = \pi/3$, $m_1 = 0.8$ eV and $m_2 = 0.9$ eV.

4. Discussion and Conclusions

In this paper, we have addressed the Casimir effect for a doublet of mixed scalar fields confined inside a one-dimensional finite region. In the framework of the long-standing discussion on the unitary inequivalence between mass and flavor representations for mixed fields, we have analyzed to what extent different choices of the vacuum lead to different predictions for the Casimir force. Specifically, we have found that, whilst the force computed with respect to the mass vacuum $|0\rangle_{1,2}$ does not exhibit mixing signatures, the use of either the flavor vacuum $|0(t)\rangle_{A,B}$ or the tilde flavor vacuum $|\tilde{0}(t)\rangle_{A,B}$ shows off an explicit dependence on the characteristic mixing parameters $\theta$ and $\delta m^2$. Underpinned by future experimental results, the arising discrepancy may provide us with the possibility to discriminate which of these states does indeed represent the physical vacuum for mixed fields.

Besides its considerable formal interest, the issue of inequivalent vacua in the context of flavor mixing in QFT is intimately related with a series of controversial problems that are currently being put forward in literature. Recently, for instance, the analysis of the inverse $\beta$-decay with mixed neutrinos has been fiercely taken back in the spotlight [13,14] after a possible disagreement between the decay rates in the laboratory and accelerated frames was highlighted [12]. As pointed out in Refs. [13], the origin of the contention is rooted in the choice of asymptotic neutrino states (and, thus, of vacuum) as mass or flavor eigenstates. Divergent views have been expressed on the subject [13,14], and precious insights toward the ultimate answer may be indeed provided by results presented here.
As partially highlighted in the above consideration, the context of accelerated frames is very promising, thus being potentially useful in several other ways. For instance, from the analysis contained in Ref. [24], it is possible to come up with another interesting application for the formalism developed in the present work. Indeed, in Ref. [24] it was found that the Unruh radiation acquires a non-thermal contribution in the case of mixed fields. Applied to the Casimir effect, this occurrence can be employed to quantitatively verify whether and how the aforementioned non-thermal corrections affect the mean vacuum energy density and – as a consequence – the pressure between the plates.

Further, we want to point out that flavor vacua can be regarded as \textit{time crystals} [25]. In fact, they exhibit a time dependence which is also reflected on physical quantities, such as the fluctuations of the flavor charge.

Finally, it should be remarked that (extended to arbitrary spin fields) our formalism may be exploited to fix more stringent constraints on the axion-photon and axion-nucleon coupling constants, as well as on the range of the axion mass [26] in a context different than the one considered so far [27]. Work in this direction is currently under active investigation [28].

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