Categorical Pairs and the Indicative Shift

Louis H. Kauffman
Department of Mathematics, Statistics
and Computer Science (m/c 249)
851 South Morgan Street
University of Illinois at Chicago
Chicago, Illinois 60607-7045
<kauffman@uic.edu>
Tel: 1-312-996-3066
Fax:1-312-996-1491.

Abstract
The paper introduces the notion of a categorical pair, a pair of categories \((C, C')\) such that every morphism in \(C\) is an object in \(C'\). Arrows in \(C'\) can express relationships between the morphisms of \(C\). In particular we show that by using a model of the linguistic process of naming, we can ensure that morphisms \(F\) in \(C\) can have an indirect self-reference of the form \(a \rightarrow Fa\) where this arrow occurs in the category \(C'\). This result is shown to complement and clarify known fixed point theorems in logic and categories, and is applied to Gödel’s Incompleteness Theorem, the Cantor Diagonal Process and the Lawvere Fixed Point Theorem.

Keywords: category; categorical pair; 2-category; indicative shift; self-reference; indirect self reference.

1. Introduction
The purpose of this paper is to introduce a categorical pattern that complements the Lawvere Fixed Point Theorem. We produce a construction for indirect self-reference that applies directly both to situations in ordinary language and to Gödel’s Theorem on the incompleteness of formal systems. Our construction can be summarized very succinctly and so we begin the paper with a self-contained account of the construction, and then devote the rest of the paper to discussion about how this indicative shift
can be seen in a number of different contexts. The indicative shift is defined in Section 2. The shift formalizes an operation on names that can also be regarded as an expansion of a name in the sense that if “A” is the name of A then the expansion E“A” refers to A“A”, the result of appending the contents of the name to the name. Thus if we regard the name as pointing to its contents as in

“A” → A

then the expansion of the name points to the concatenation of the contents with the name

E“A” → A“A”.

Self-reference results when one expands the name of the expansion operator.

E“E” → E“E”.

The arrow of reference occurs at a different level than the concatenations of names and their contents. In this paper there will be two categories $C$ and $C'$, where the morphisms in $C'$ are arrows between morphisms in $C$. We refer to $C'$ as the second category or the higher category in the pair $(C, C')$. The arrows of reference are in the higher category $C'$.

In the case of Gödel’s Theorem, the pattern is similar. One has a method to assign natural numbers (Gödel numbers) to formulas. Letting $g → F(u)$ denote the assignment of the Gödel number $g$ to the formula $F(u)$ with free variable $u$, we let $g$ denote the Gödel number that is assigned to $F(g)$, the formula obtained by evaluating $F(u)$ at its own Gödel number. Thus given

$g → F(u),$

we have

$g → F(g).$

Indirect self-reference is obtained by starting with a formula of the form $F(g)$. Then we have

$g → F(g)$

and

$g → F(g).$
where $g$ is the Gödel number of $F(\sharp u)$. The formula $F(\sharp g)$ refers to its own Gödel number. This is the key indirect self-reference behind the Gödel theorem which then proceeds to take $F(u) = NB(u)$ where the formula $NB(u)$ asserts (within a formal system $\mathcal{F}$) that there is no proof of the statement with Gödel number $u$. ($NB$ is short for “nicht beweis”.) Using this method of creating indirect self-reference we get

$$\sharp g \rightarrow NB(\sharp g),$$

a statement that asserts its own unprovability in the formal system $\mathcal{F}$. If the formal system is consistent (and capable of handling these representations of arithmetic), then the statement $NB(\sharp g)$ is true, but unprovable in $\mathcal{F}$.

The paper is organized as follows. In Section 2 we construct a categorical context for the indicative shift by considering a pair of categories $(\mathcal{C}, \mathcal{C}')$ where every morphism in the first category is an object in the second category. Arrows in the second category can be interpreted as references between arrows in the first category. In this sense the second category of the categorical pair defined Section 2 takes the place of a meta-language in a logical context. We prove two basic results about self-reference in this section that we call the First and Second Self-Reference Theorems. The First Self-Reference Theorem gives conditions under which an indirect self-reference can occur.

The formality of the indicative shift is as follows. Suppose that $a$ and $F$ are morphisms in $\mathcal{C}$ such that the composition $Fa$ is defined. Suppose that

$$a \rightarrow F$$

is an arrow in $\mathcal{C}'$. Then it is either given or constructed (First and Second Self-Reference Theorems) that there is an morphism $\sharp$ such that

$$\sharp a \rightarrow Fa.$$

This is the indicative shift. It follows that if

$$g \rightarrow F\sharp$$

then

$$\sharp g \rightarrow F\sharp g,$$

producing the desired indirect self-reference.
The Second Self-Reference Theorem assumes that the pair of categories $(C, C')$ is a 2-category, and that $a$ can be composed with itself. We show that the indicative shift with $\sharp a = aa$ follows naturally from the properties of composition in a 2-category. We end Section 2 with an application of its ideas to an example of Raymond Smullyan. Smullyan’s example is a miniature version of Gödel’s incompleteness theorem. In Section 3 we show how the Self-Reference Theorems apply to Gödel’s Theorem when that theorem is seen as the production of a statement that asserts its own unprovability in a given formal system. In Section 4 we discuss the relationship of these ideas with the Lawvere Fixed Point Theorem, and we discuss how the Lawvere Theorem relates to Gödel’s Theorem via its analog with the Cantor diagonal process. The difference between our categorical approach and that of the Lawvere Theorem is that we formalize indirect self-reference.

In Section 5 we discuss how these category ideas and the indicative shift apply to ordinary language. More work needs to be done in relating these formalisms to ordinary language, for it is in ordinary language that the line between the categories (between level and meta-level) is easily erased. Names of names are still names in ordinary language, and in the language of categories, objects and morphisms can become interchangeable. From the mathematical side one can approximate the situation of language by using higher categories or even reflexive categories (where ideally there is a 1-1 correspondence between objects and morphisms) rather than the categorical pairs of Section 2. We give an example of a reflexive category (in that every object is a morphism) by taking the generating arrows and the objects to be the arcs of an oriented knot diagram. Section 6 is an Epilogue that reviews and discusses the ideas and results of the paper. The paper ends with Section 7, a return to self-reference and a discussion of the nature of self-reference in the use of the word I.

2. The Indicative Shift

The indicative shift defined in this section formalizes an operation on names that can also be regarded as an expansion of a name in the sense that if “A” is the name of $A$ then the expansion $E“A”$ refers to $A“A”$, the result of appending the contents of the name to the name. Thus if we regard the name as pointing to its contents as in

“$A$” $\rightarrow A$
then
\[ E"A" \rightarrow A"A". \]

Self-reference results when one expands the name of the expansion operator.

\[ E"E" \rightarrow E"E". \]

In those contexts where one thinks of expanding a name to its contents it is convenient to use the symbol \( E \) for the shift operator. In this section we shall adopt the symbol \( \sharp \) for the shift. When we use the symbol \( \sharp \) we are thinking of the shift at the point where a name is given to a contents. At that point, there is an initial pointing of the name to the contents before the name is directly associated with the contents. Then a shift occurs where the name is associated with the contents and the abstract name is associated with the reference to these contents. These points of language are further discussed in Section 5.

The reader should recall that a category consists in a collection of objects and a collection of morphisms. To each morphism \( f \) there is associated an ordered pair of objects \((A, B)\). We write \( f : A \rightarrow B \) to denote the morphism and call \( A \) the domain of \( f \) and \( B \) the codomain of \( f \). Given morphisms \( f : A \rightarrow B \) and \( g : B \rightarrow C \), there is a morphism \( g \circ f : A \rightarrow C \), called the composition of \( f \) and \( g \). Composition of morphisms is associative. Every object \( A \) comes equipped with an identity morphism \( 1_A \) whose composition (with \( A \) in the role of domain or codomain) with another morphism does not affect that morphism. This is the complete definition of a category.

We take as given that in a category one can say whether two objects are equal and whether two morphisms are equal. We wish to model situations where equality is replaced by reference. We speak this way for motivation and use the word reference as it is used in ordinary language where one may say that the name of a person refers to that person, or that the title of a paper refers to the text or to the contents of the paper. We wish to model situations where one distinguishes between the morphisms of a given category and certain patterns of reference that are seen among these morphisms at a second level. An example that we shall consider later is the reference of a Gödel number to its corresponding decoded text.
Let there be given a category $C$ and suppose that the set of morphisms of $C$ are seen as the objects in another category $C'$. We shall call the morphisms in $C'$ reference arrows for the morphisms of $C$, and we shall call the pair of categories $(C, C')$ a categorical pair. We make no further restrictions on a categorical pair other than that there are two categories $C, C'$ with the morphisms in the first category forming the objects in the second category. A categorical pair is not constrained to be a 2-category (definition given below).

There is a notion of 2-category (and of higher categories) [15, 17]. A 2-category is a categorical pair with extra structure. In the notation of our reference arrows, the extra structure is as follows. One may have
\[ \alpha : a \to b \]
and
\[ \beta : d \to e, \]
arrows in $C'$ where $ad$ and $be$ are both legal compositions in the base category $C$ of the categorical pair. Then it is natural that there should be a referential arrow
\[ \alpha \circ_0 \beta : ad \to be, \]
usually called horizontal composition of these arrows in $C'$.

Along with horizontal composition we have vertical composition which is simply the given composition of arrows in $C'$. We can denote vertical composition by $\circ_1$. Thus if
\[ \alpha : a \to b, \gamma : b \to c \]
then
\[ \gamma \circ_1 \alpha : a \to c. \]
Now suppose that we have two possible vertical compositions
\[ \alpha : a \to b, \gamma : b \to c \]
\[ \beta : d \to e, \delta : e \to f \]
where $ad$, $be$ and $cf$ are each legal compositions in the base category $C$ of the categorical pair. Then it is natural to demand the compatibility
\[ (\alpha \circ_0 \beta) \circ_1 (\gamma \circ_0 \delta) = (\gamma \circ_1 \alpha) \circ_0 (\delta \circ_1 \beta). \]
A categorical pair \((C, C')\) that satisfies this compatibility (called the \textit{interchange law}) is called a \textit{2-category}.

In the discussion below we shall formulate three self-reference theorems, one just using categorical pairs and the other two assuming that the categorical pair is a 2-category. For most applications the reader will only need to assume given a categorical pair.

Consider a categorical pair \((C, C')\). Let \(a\) and \(b\) be morphisms in \(C\) and let \(a \rightarrow b\) be a morphism in \(C'\) with domain \(a\) and codomain \(b\). Remember that while \(a\) and \(b\) are morphisms in the initial category \(C\), they are objects in the referential category \(C'\). We call this arrow a \textit{reference} from \(a\) to \(b\).

We assume that for every object \(X\) in \(C\) there is a morphism \(\sharp_X : X \rightarrow X\). Thus each object in \(C\) is indexed with a special morphism \(\sharp_X\) to itself that need not be the identity morphism. Given an arbitrary morphism \(a\) in \(C\) we then have the compositions \(\sharp_X\) and \(\sharp a\), morphisms obtained by composing with \(\sharp_X\) where \(X\) is either the domain or codomain of \(a\). In this way, we will denote composition with \(\sharp_X\) without explicitly writing \(X\).

We shall say that an arrow \(a \rightarrow b\) in the category \(C'\) is a \textit{composable reference} if the composition \(ba\) is defined.

We assume that the special morphisms \(\sharp_X\) in \(C\) have the the following property:

\textbf{The Indicative Shift.} \textit{If a \rightarrow b is a composable reference arrow in C', then there is an associated reference arrow \(\sharp a \rightarrow ba\). Here \(\sharp a\) and \(ba\) denote the compositions of these morphisms in the initial category \(C\).}

A categorical pair with these properties is called a \textit{referential pair}.

\textbf{Remark.} Note that since \(C'\) is a category whose objects are the morphisms of \(C\), then given any morphism \(a : A \rightarrow B\) (\(A\) and \(B\) are objects in \(C\)), there is an identity morphism \(a \rightarrow a\) in \(C'\). Unless \(A = B\) in \(C\), this arrow will not be composable and so the shift will not apply to it. When \(A = B\), then we have a shift to \(\sharp a \rightarrow aa\). Because we demand compositability of reference in order to have the indicative shift, not all morphisms in \(C'\) can be shifted. If the base category \(C\) has only one object, or if all morphisms have the same domain and codomain, then any reference arrow can be shifted. In many
examples, this simple circumstance is satisfied. Note also that we may have
\( a \to b \) composable with
\[
a : A \to B
\]
and
\[
b : B \to Z
\]
so that the shift \( \sharp a \to ba \) exists, but no further shift is possible unless
\( Z = A \).

**First Self-Reference Theorem (SRT1).** Let \((C, C')\) be a referential pair. Let \( F \) be any morphism in the category \( C \), and assume that there is a composable reference \( g \to F\sharp \) in the category \( C' \). Then there exists a morphism \( h \) in \( C \) and a reference arrow in \( C' \) such that \( h \to Fh \).

**Proof.** We are given
\[
g \to F\sharp
\]
in \( C' \). This means that the codomain \( X \) of \( g \) is the domain of \( F\sharp \), and this is the same as the domain of \( F \), and that \( \sharp = \sharp_X \). Then apply the indicative shift and obtain:
\[
\sharp g \to F\sharp g.
\]
Thus with \( h = \sharp g \), we have \( h \to Fh \). This completes the proof. //

**Remark.** We interpret an arrow of the form \( h \to Fh \) as a model of an expression \( Fh \) that is talking about (in the internal language of compositions in \( C \)) its own “name” (which is the morphism \( h \) from the point of view of the category \( C' \)).

**Remark.** Note that, in the above proof, if \( F : X \to X \) (i.e. if \( F \) was a morphism from an object to itself) then we could take the identity morphism in \( C' \) for \( F\sharp \),
\[
F\sharp \to F\sharp
\]
since then \( F\sharp \) would be composable with itself. Then the indicative shift produces
\[
\sharp F\sharp \to F\sharp F\sharp
\]
and we have \( a = \sharp F\sharp \) with \( a \to Fa \).
Remark. Suppose that $g : X \to X$ in $C$ and that $F : X \to Y$ in $C$. Then if we have $g \to F$ in $C'$ then the indicative shift gives an infinite sequence of morphisms:

$$
\sharp g \to Fg \\
\sharp\sharp g \to Fg\sharp g \\
\sharp\sharp\sharp g \to Fg\sharp g\sharp g
$$

and continuing in this fashion in the pattern

$$
\sharp^n g \to Fg\sharp g\sharp^2 g\sharp^3 g \cdots g\sharp^{n-1} g.
$$

On the other hand, if $g : Z \to X$ where the object $Z$ is distinct from $X$, then we can be given that $g \to F$ and one shift to $\sharp g \to Fg$ is possible. But the sequence of shifts stops here since the composition of $g$ with itself or with $\sharp g$ is not given to exist. This is one of the reasons for formulating this shift in categorical terms. The properties of the base category $C$ determine limits or lack of limits on the recursion of reference that is implicit in the indicative shift.

Remark. For the next Theorem we will concentrate on references $a \to F$ where the domain and codomain of $a$ are identical, so that there is no limit on the recursion of the shift. We call a morphism $a$ in $C$ a self morphism if it has the form $a : X \to X$. We can regard the referential arrows of the category $C'$ as generalizations (categorifications) of the equality of morphisms in the base category $C$. If the referential arrows are themselves taken to be equalities then the indicative shift would state that if $a = b$ as self morphisms in $C$, then $\sharp a = ab$. In other words, in this degenerate form, we would have $\sharp a = aa$ for all self morphisms $a$ in $C$.

The First Self-Reference Theorem would then correspond to the following calculation. If

$$
g = F\sharp
$$

then

$$
\sharp g = F\sharp g.
$$

Hence

$$
gg = Fgg.
$$
The reader will recognise that this is exactly the form of the proof of the Church-Curry Fixed Point Theorem for Lambda Calculus \[1\]. See the Epilogue (Section 6 of this paper) for more discussion of this point. The Indicative Shift generalizes the Church-Curry Fixed Point Theorem to a context that encompasses indirect self-reference.

We formulate a second self-reference theorem that is close to the flavor of the lambda calculus. We assume that the pair \((C, C')\) is a referential pair that is moreover a 2-category in which \(\sharp\) is defined in a particular way. For a self morphism \(a\) in \(C\) we define

\[ \sharp a = aa, \]

as the composition of \(a\) with itself. We call the 2-category a lambda pair if this condition is met for each self morphism \(a\).

**Second Self-Reference Theorem (SRT2).** Let \((C, C')\) be a 2-category that is a lambda pair as defined above with \(\sharp a = aa\) for each self morphism in \(C\). Then, given a composable reference arrow \(a \rightarrow F\) in \(C'\), there is a corresponding morphism \(\sharp a \rightarrow Fa\). With respect to this indicative shift we obtain indirect self-reference from any composable reference \(a \rightarrow F\sharp\) by taking the corresponding shift to \(\sharp a \rightarrow F\sharp a\). Note that this morphism is the same as \(aa \rightarrow Fa\). This final conclusion is a direct generalization of the Church-Curry Fixed Point Theorem.

**Proof.** Suppose we have a morphism

\[ a \rightarrow F \]

in \(C'\). Let \(a \rightarrow a\) be the identity morphism for \(a\) in \(C'\). Then we have the horizontal composition of these two morphisms:

\[ aa \rightarrow Fa. \]

Note that the composition \(Fa\) exists since there is an arrow from \(a\) to \(F\) in \(C'\). Hence we have, as desired, the shift morphism

\[ \sharp a \rightarrow Fa. \]

The rest of the Theorem follows in the same pattern as the proof of SRT1.

//
The Smullyan Categorical Pair. An exercise related to Gödel’s Theorem due to Raymond Smullyan [18] can be naturally formulated in terms of categorical pairs. In this case we only use the structure of categorical pairs. We do not apply the indicative shift, but the Smullyan example contains its own indirect self-reference. The first category $C$ consists in (as morphisms) all words in the alphabet $\{\sim, P, R, [, ]\}$. where a word is any ordered string of these symbols. such words include the empty word which is the identity morphism in this category. The category $C$ has a single object. Composition in $C$ consists in concatenation of strings. The objects in the second category $C'$ consist in strings $X$ in the alphabet $\{\sim, P, R, [, ]\}$. Thus every morphism in $C$ is an object in $C'$. Other than the identity arrows, the following types of arrow in $C'$ are allowed, where $X$ is an arbitrary string in that alphabet.

1. $PX \rightarrow P[X]$
2. $\sim PX \rightarrow \sim P[X]$
3. $RX \rightarrow P[XX]$
4. $\sim RX \rightarrow \sim P[XX]$

Of course, once we allow these arrows in $C'$, we allow a host of possible compositions such as the composition of $PX \rightarrow P[X]$ and $P[X] \rightarrow P[[X]]$ to form

$$PX \rightarrow P[[X]].$$

The reader will note that by substituting $R$ for $X$ in item 3, we obtain the indirect self-reference

$$RR \rightarrow P[RR].$$

By substituting $\sim R$ for $X$ in item 4, we obtain the indirect self-reference.

$$\sim R \sim R \rightarrow \sim P[\sim R \sim R].$$

Smullyan has an amusing interpretation of this formalism. He tells the story of a machine that prints strings from the category $C$ (he does not use categorical terminology, but we will describe it that way). *Only the special itemized arrows (above) in $C'$ are interpreted as restrictions and descriptions of the machine’s actions.* For codomains of arrows in $C'$, $P[X]$ means printability of $X$ and $\sim P[X]$ means unprintability of $X$. The category $C'$ contains the semantics for the categorical pair, but it also contains many expressions that are not interpreted semantically with regard to the machine’s actions.
1. If the machine can print the string \( PX \) then it can print the string \( X \). In other words
\[
P X \rightarrow P[X]
\]
means that \( X \) is individually printable if the string \( PX \) is printable.

2. If the machine can print the string \( \sim PX \) then the string \( X \) is not printable (as an isolated string) by the machine. The string \( \sim P[X] \) means that \( X \) (alone) is not printable.
\[
\sim PX \rightarrow \sim P[X].
\]

3. The printing of the string \( RX \) means that \( XX \) is printable.
\[
RX \rightarrow P[XX].
\]

4. The printing of the string \( \sim RX \) means that \( XX \) is not printable.
\[
\sim RX \rightarrow \sim P[XX].
\]

Thus we can interpret the Smullyan Machine in terms of the category \( C' \) by saying that certain special morphisms in \( C' \) are interpreted as statements about printability. Each of the special string types (lets us call them interpretable strings) \( \{PX, \sim PX, RX, \sim RX\} \) might be printable by the machine, and if printed, they each tell what the machine can further print. It is given that whenever the machine prints one of these special strings then it tells the truth. We deduce that the machine cannot print the string
\[
\sim R \sim R,
\]
for this string asserts its own unprintability. Thus, while the Smullyan Machine always tells the truth when it prints an interpretable string, there are interpretable strings that are true but unprintable! This Smullyan categorical pair is an intriguing miniature version of Gödel’s Incompleteness Theorem, with printability replacing provability.

**A Simplest Example.** Let \( C \) be a category with one object \( O \), the identity morphism \( 1_O \), and one other morphism \( \sharp : O \rightarrow O \). We can take the morphisms in \( C \) to be the set of strings \( \{,\sharp, \#\sharp, \cdots \} \) including the empty string identified as \( 1_O \). In \( C' \) we take as given the morphism
\[
\rightarrow
\]
from the empty string to the empty string. Then the shift (represented by \(\#\)) produces sequentially:

\[
\begin{align*}
\# & \rightarrow \text{ }, \\
\#\# & \rightarrow \# , \\
\#\#\# & \rightarrow \#\# .
\end{align*}
\]

Self-reference appears at the third departure from the empty string. After that we have

\[\#^n \rightarrow \#^{n(n-1)/2}.
\]

**A Next Simplest Example.** Let \(C\) be any category with one object \(O\) and morphisms \(F\) other than the identity morphism \(1_O\) and the morphism \(\#: O \rightarrow O\) representing the indicative shift.

In \(C'\) we have the identity morphism \(X \rightarrow X\) for each morphism \(X\) in \(C\). Then the shift produces

\[\#X \rightarrow XX .\]

This pattern is analogous to the pattern of reference (by repetition) in the Smullyan Machine. In particular we have the self-reference

\[\#\# \rightarrow \#\# ,
\]

not necessarily the same as the identity morphism in \(C'\) for \(\#\).

**The Universal Building Machine.** We can interpret the expansion operator \(E\) described at the beginning of this section as a universal building machine. Then “\(X\)” designates a blueprint for the construction of \(X\). (Of course here we indulge in a hierarchy of names. Really \(X\) is the name of an actuality and “\(X\)” is the name of the blueprint for constructing this actuality.) Then we have

\[E“X” \rightarrow X“X” ,
\]

meaning that the universal builder \(E\) takes the blueprint “\(X\)” and produces the actuality \(X\) appended to a copy of its blueprint. The higher categorical morphism is a morphism between the composition of the building machine and the blueprint and the composition of the actuality and its blueprint. The universal building machine will build itself when supplied with its own blueprint.

\[E“E” \rightarrow E“E” .
\]
Remark. In the examples we have given so far, the category $C$ can be replaced with a monoid of strings under concatenation. In this $C$ there are many morphisms and only one object. Nevertheless, we have formulated the results in this section to include categories $C$ with more than one object and the possibility of morphisms between distinct objects in $C$. Certainly in linguistic and other referential situations there are many examples where a given entity has a multiplicity of references to it. This is modeled in a more general category $C$ by a morphism $F$ such that there is a multiplicity of morphisms $a$ such that the composition $Fa$ is defined. Under these circumstances the indicative shift still holds and we may obtain a multiplicity of indirect self-references in the form $\sharp a \rightarrow F\sharp a$. An interesting source of abstract categories to consider for examples is found by starting with any directed graph $G$ and making a category $C$ whose objects are the nodes of $G$ and whose morphisms are generated by one identity morphism for each node, one $\sharp$ morphism for each node and all the edges in the graph are interpreted as morphisms between their initial and final nodes. We allow multiple edges and loops in the graph.

3. Gödel’s Theorem

In order to discuss Gödel’s Incompleteness Theorem from the point of view of the indicative shift, we first start with the more general situation of a formal language $L$ that is susceptible to Gödel numbering. The basic notions of formal language and Gödel numbering are explained in many books on logic. The interested reader can consult [18, 19, 20, 21]. We will assume that the formal language $L$ has the capacity to make statements about natural numbers involving a free variable $x$ such as “The natural number $x$ is greater than 2.” We will denote statements involving a single free variable $x$ in the form $S(x)$. Such a statement gives rise to infinitely many specialized statements that may be either true or false by substituting specific numbers for $x$. Thus one could write

$$S(x) = "x > 2"$$

and

$$S(3) = "3 > 2."$$

It is also understood that one can substitute the name or reference to a specific number for $x$ in a statement $S(x)$. Thus, instead of the numeral 3,
we could substitute for \( x \) the statement “the first odd prime number” written in the language of \( L \) (assuming that \( L \) is rich enough to express this notion).

Given a formal system, one can set up Gödel numbering, a method that associates a unique natural number to each formula or sequence of formulas in \( L \). We write \( g \rightarrow S(x) \) to denote the Gödel number \( g \) that is associated with a formula \( S(x) \) with free variable \( x \). At this point the arrow is just a notation to indicate the association of the Gödel number with its corresponding formula. We assume that there is a well-defined notion for substituting a Gödel number \( g \) into a formula \( S(x) \) to obtain a new formula \( S(g) \). The new formula no longer has a free variable. What is substituted has to be a specific expression for the Gödel number in the language \( L \). Otherwise one would obtain a collection of formulas \( S(h) \), one for each way to express the number \( g \). Once this choice has been made, then \( S(g) \) has a specific Gödel number. In particular, we can start with \( S(x) \), obtain its Gödel number \( g \) and then further obtain the Gödel number \( h \) of the result of substituting \( g \) into \( S(x) \). We shall let \( \sharp g \) denote a formula in \( L \) that describes the process of computing the Gödel number of the result of substituting \( g \) into \( S(x) \). Thus \( \sharp g \) is a formula that stands for (the computation of) the Gödel number \( h \). We shall write

\[
g \rightarrow S(x)
\]

and

\[
\sharp g \rightarrow S(g).
\]

Note that in the second equation, we use \( \sharp g \) rather than \( h \) on the left side of the arrow. It is understood that \( \sharp g \) stands for \( h \). The reader should note that while we use the arrow notation, no categories have yet been defined.

We continue the story of these substitutions. We can assume that the formal system \( L \) is rich enough to express in its own language the operation that takes the \( g \) to \( \sharp g \) and that whenever one writes a formula of the form \( S(x) \) one can also write the formula \( S(\sharp x) \). Here, as in the previous paragraph, \( \sharp x \) stands for a formula in \( L \) that describes the process of computing the Gödel number of the result of substituting the Gödel number of \( S(x) \) into the free variable in \( S(x) \). Under these circumstances, we have

\[
g \rightarrow S(\sharp x),
\]

and

\[
\sharp g \rightarrow S(\sharp g).
\]
We need to distinguish clearly between Gödel numbers and expressions in
the language $L$ that refer to the construction of such numbers. We assume
that the Gödel numbers are written in a standard numeral form like 3 and
not expressed indirectly as in “the smallest odd prime number”. The formula
$S(\sharp g)$ refers to its own Gödel number and hence achieves an indirect form of
self-reference. In this formula, $g$ is a number written in the language $L$ and
$\sharp g$ is a formula in $L$ that is applied to $g$. Thus the expression $\sharp g$ refers to the
Gödel number of the formula $S(\sharp g)$.

This background is a short description of how indirect self-reference is
accomplished in the context of proving Gödel’s Incompleteness Theorem.
The rest of the well-known proof of the Incompleteness Theorem uses this
form of indirect self-reference applied to a statement $S(x) = \sim B(x)$ that
informally says “The statement whose Gödel number is $x$ has no proof in the
formal system $L$.” The making of such a statement within the formal system
$L$ requires that $L$ be sufficiently expressive so that it can internally encode
the notion of a proof. Once this is accomplished, one uses the construction
of indirect self reference as shown below.

$$g \rightarrow \sim B(\sharp x)$$

and

$$\sharp g \rightarrow \sim B(\sharp g).$$

The final statement $\sim B(\sharp g)$ asserts its own unprovability in $L$. If $L$ is consistent,
one concludes, by reasoning that occurs outside $L$, that $\sim B(\sharp g)$ is not
provable within $L$. Thus $\sim B(\sharp g)$ is a statement that is true but unprovable
by $L$. If the formal system $L$ is consistent, then it is incomplete. It has long
been assumed that known formal systems for elementary number theory are
consistent. Under this assumption, such systems are incomplete.

We now indicate the categories $C$ and $C'$ that will place Gödel’s Theorem
in our context. A caution to the reader: These categories do not prove the
Incompleteness Theorem. The proof still depends upon the careful construc-
tion of a formal system $L$ as described above. We obtain a description of how
the indirect self-reference in the structure of the Incompleteness Theorem can
be seen in a categorical framework.

Let the base category $C$ have a single object, call it $O$. Generating mor-
phisms in $C$, other than the identity morphism and a special morphism $\sharp$, are
formulas in $L$ that have less than or equal to one free variable and natural numbers expressed outside the system $L$. The natural numbers outside the system are candidates for Gödel numbers and will be composed according to those expressions inside $L$ for which they are code numbers. A formula without a free variable may or may not define a number (integer) in $L$. We shall call a formula *numerical* if it designates an integer. It is assumed that the language $L$ has a special category of formulas that designate numbers directly. These will be called *numerals*. For example one might use $|||\rangle$ as the numeral for 3 in the formal system $L$. When a Gödel number is substituted into a formula the number is translated to the corresponding numeral in $L$. We will use usual decimal notation for numbers outside the formal system. The coding method could depend upon the decimal system (as in the example at the end of this section) or it could just depend upon number theoretic properties (such as the unique decomposition of a natural number into prime factors). Let $G$ and $H$ be numerical formulas and let $n$ and $m$ denote numerals in $L$. Let $S(x)$ and $T(y)$ denote formulas with one free variable. Composition in $C$ will primarily correspond to substituting one formula into the free variable of another formula. We define (non-identity) compositions in $C$ as follows:

1. It is given that composition is associative.
2. $S \circ T$ is a formal composition with no specified relation if $S$ has no free variable and $S$ and $T$ are formulas in $L$.
3. $S(y) \circ T(x) = S(T(x))$ for $S$ a formula with a free variable and $T$ a formula in $L$. Note that if $T$ has a free variable, then so does $S \circ T$.
4. $S(x) \circ n = S(n)$ whenever $n$ is a numeral in $L$.
5. $S(x) \circ g = S([g])$ whenever $g$ is any natural number. Here $[g]$ denotes the numeral in $L$ that corresponds to $g$.
6. If $G$ and $H$ are formulas that represent numbers, but are not themselves numerals then $G \circ H$ is a formal composition with no specified relation.
7. If $n$ and $m$ are numerals in $L$, then $n \circ m$ is a formal composition with no specified relation.
8. If $g$ and $h$ are Gödel numbers such that $g$ is the Gödel number of a formula with one free variable $S(x)$ and $h$ is the Gödel number of a formula $T$, then $g \circ h$ is the Gödel number of $S([h])$. Here we distinguish between Gödel numbers outside the system $L$ and numerical expressions inside that system.
9. If \( g \) is the Gödel number of a formula with one free variable, then we define \( \sharp \circ g = \sharp g \) as above. That is, if \( g \) is the Gödel number of \( S(x) \), then \( \sharp g \) stands for the Gödel number of \( S([g]) \). Otherwise we take \( \sharp \circ g \) formally with no specified relation.

10. If \( n \) is a numeral in \( L \) that stands for the Gödel number \( g \) of a formula with one free variable, then \( \sharp n \) stands for \( \sharp g \) expressed in \( L \) as \( [\sharp g] \). It is understood that \( \sharp n \) is an expression in \( L \) that refers to a specific numeral in \( L \).

11. \( \sharp \circ S(x) \) is taken formally with no specified relation.

12. \( g \circ \sharp \) is taken formally with no specified relation for any natural number \( g \).

13. \( T \circ \sharp \) is taken formally with no specified relation for any formula \( T \) in \( L \).

14. \( S(x) \circ \sharp \) is taken formally with no specified relation, but note that \( (S(x) \circ \sharp) \circ g = S(x) \circ (\sharp \circ g) = S(x) \circ (\sharp g) = S(\sharp g) \) when \( \sharp g \) is numerical. Similarly, \( (S(y) \circ \sharp) \circ x = S(\sharp x) \).

This defines the category \( C \). We then define admissible arrows in \( C' \) to be the identity arrows and arrows of the form

\[
g \rightarrow F
\]

where \( F \) is a formula in \( L \) with at most one free variable, and \( g \) is the Gödel number of \( F \). If \( S \) is a formula in \( L \) that has no free variable and \( S \) represents the Gödel number \( g \) of \( F \), then we also allow the morphism

\[
S \rightarrow F.
\]

Compositions of morphisms in \( C' \) are formal with no specified relations other than associativity. This completes the definition of the categorical pair corresponding to a given formal language \( L \) with Gödel numbering. In this way, the construction of indirect self-reference in Gödel’s Incompleteness Theorem can be regarded as an application of the First Self-Reference Theorem (SRT1).

**Remark.** We can place Gödel’s Theorem in the context of the Second Self-Reference Theorem SRT2. Regard Gödel numbers \( g \) as morphisms in a category by defining \( g \circ h \), as above, to be the result of substituting \( h \) in the free variable of the decoding of \( g \) (if there is such a free variable). Then
we see that $g \circ g$ is a concise description of the $\sharp$ operator as we have defined it above. With this definition of composition of Gödel numbers we have a category $C$ and can construct that category $C'$ of arrows from Gödel numbers to texts in the formal system just as we did in the above paragraphs. Now, if $g \rightarrow F$ where $F$ is the decoding of $g$, then by definition $gg = \sharp g$ is the Gödel number of $Fg$ where $Fg$ denotes the result of substituting $g$ into the free variable in $F$. Thus we have $gg \rightarrow Fg$ as the horizontal composition of $g \rightarrow F$, and the identity arrow $g \rightarrow g$ and $gg \rightarrow Bgg$ as the horizontal composition of $g \rightarrow B\sharp g$ and the identity arrow $g \rightarrow g$. This gives exactly the 2-categorical structure of the Second Self-Reference Theorem. The reader should note that in the category $C$ we have both Gödel numbers and formulas as morphisms. In the category $C'$ we have identity morphisms that carry numbers to numbers and formulas to formulas, but otherwise arrows in $C'$ carry numbers or representatives of numbers to formulas.

In this way we see clearly that the categorification of the Church-Curry Fixed Point Theorem that is implicit in the Second Self-Reference Theorem applies to Gödel’s Theorem, showing how the indirect self-reference central to the Gödel construction comes from changing an equality to an arrow.

**Example.** In this example we give a small formal language that has Gödel numbering and use it to illustrate our categorical constructions. Let $\mathcal{L}$ denote a language with the following alphabet

$$\mathcal{A} = \{(,), \sim, P, x, |, \sharp\}.$$  

The words in $\mathcal{L}$ are all possible strings of these symbols, and we will interpret them in a way that is similar to the Smullyan Machine described in the previous section. Accordingly, we let $X$ denote any finite string of symbols in this alphabet. We interpret $x$ as a variable. We interpret $|$ as the number 1, $||$ as the number 2 and generally a string $||| \cdots ||$ with cardinality $n$ vertical slashes as the *numeral* $n$. The interpretation of $\sharp$ will be explained below. We will introduce Gödel numbering and show how to produce a string that refers to itself. If $X$ is a Gödel number of a statement in $\mathcal{L}$, we will take that statement to be the *referent* of $X$. We will interpret $P(X)$ to assert the printability of the referent of $X$ and $\sim P(X)$ to assert the unprintability of the referent of $X$. We will construct a statement that asserts its own unprintability. Then we will point out how the categories $C$ and $C'$ are constructed for the language $\mathcal{L}$.  

19
The Gödel numbers assigned to the individual members of the alphabet are as follows.

\[
\begin{align*}
1 & \rightarrow ( \\
2 & \rightarrow ) \\
3 & \rightarrow \sim \\
4 & \rightarrow P \\
5 & \rightarrow x \\
6 & \rightarrow | \\
7 & \rightarrow \# \\
\end{align*}
\]

The Gödel number assigned to a string of signs from the alphabet is the ordered list of these corresponding digits, interpreted as a natural number in the decimal system. Thus we have

\[
34152 \rightarrow \sim P(x)
\]

so that \(g = 34152\) is the Gödel number of \(\sim P(x)\). If we wish to insert \(g\) into the free variable in \(\sim P(x)\), we must translate \(g\) into the language of numbers in this formal system. This means that we replace 34152 by ||| · · · | where there are thirty four thousand one hundred fifty two vertical slashes in this numeral. We can write the abbreviation \(\sim P(g)\), but in fact the actual expression is very large indeed. Thus we have

\[
34152 \rightarrow \sim P(x)
\]

and

\[
341666 \cdots 62 \rightarrow \sim P(||| \cdots |)
\]

where there are 34152 slashes in the right hand side, and there are the same number of 6’s in the Gödel number on the left hand side. Thus 341666 · · · 62 is the Gödel number of the formula obtained from \(\sim P(x)\) by substituting its own Gödel number for the free variable \(x\). In our previous terminology we would write that, given \(g = 34152\), we have \(\sharp g = 341666 \cdots 62\) where there are 34152 repeated 6’s in the second number.
We can now explain the interpretation of ♯ in the formal system $L$. If a formula contains $♯x$ for the variable $x$ or $♯n$ for some numeral $n$, then ♯ is interpreted as that function that assigns to one Gödel number another Gödel number by decoding the first number and placing its Gödel numeral in the free variable of the decoded formula. If there is no such free variable, then no action is taken and there is no interpretation of $♯|||· · ·||$. In a full formal system for number theory, ♯ would be an abbreviation for an algorithm written in the language of that theory. Here we have only an external interpretation for the meaning of ♯. In the situation of the full formal system one has both the internal algorithm and the external interpretation, and one understands that these two ways of looking at the process describe the same function.

Returning to our formula

$$♯34152 = 341666 · · · 62$$

where there are 34152 repeated 6’s in the second number, we note that one can read the function ♯ directly in the decimal formalism. If the decimal number $n$ contains the digit 5 replace every occurrence of that digit by cardinality $n$ consecutive 6’s. In this way $♯n$ is explicitly described as a way to insert the number $n$ into itself by using two levels of coding the number (decimal and slash/numeral). When we speak of $♯n$ we can think about it as a direct function on decimal numbers, or we can understand it via its definition through coding and decoding in relation to the formal system $L$.

We are now in a position to see directly the composition of Gödel numbers defined in this section for the category $C$. This category has one object and each string in the formal system $L$ is a morphism. Along with this, all natural numbers in the decimal system are also morphisms in $C$. Note that it is given that all natural numbers written in vertical slash numerals are morphisms in $C$. We distinguish between slash numerals and decimal numerals. Here are some examples of compositions of strings and numbers viewed as morphisms in the category $C$.

1. Let $n$ and $m$ be any two decimal natural numbers. We define $n \circ m$ to be the decimal number $n$ obtained by substituting cardinality $m$ consecutive occurrences of the digit 6 in for every instance of the digit 5 in $n$. If $n$ does not contain the digit 5 then no action is required for the composition of $n$ and $m$. For example $4152 \circ 3 = 426661$. Note that

$$4152 \longrightarrow P(x)$$
and

\[ 426661 \rightarrow P(|||). \]

Thus, if \( n \) and \( m \) are Gödel numbers of formulas \( A \) and \( B \), and \( A \) has a free variable \( x \), then \( n \circ m \) is the Gödel number of the result of substituting \( B \) into the free variable in \( A \). If \( n \) is the (decimal) Gödel number of \( A \), then we have that \( \sharp n = n \circ n \).

2. We have that \( 341752 \rightarrow \sim P(\sharp x) \) so that \( \sharp 341752 = 341766 \cdots 62 \) with 341752 consecutive 6’s. Thus

\[ 341766 \cdots 62 \rightarrow \sim P(\sharp ||| \cdots |||) \]

where there are 341752 consecutive vertical slashes in the expression on the right. This expression, \( \sim P(\sharp ||| \cdots |||) \) refers to its own Gödel number. Hence, if the formal system \( L \) tells the truth, then the formula

\[ \sim P(\sharp ||| \cdots |||) \]

is not printable, since it says that it is unprintable. Since it is unprintable we find that it does tell the truth, and truth and printability are distinct for the system \( L \).

4. Lawvere’s Fixed Point Theorem

This section is a brief discussion of Lawvere’s fixed point theorem. We discuss how the Lawvere Theorem arises as a generalization of the the Cantor diagonal argument, and we illustrate the Lawvere Theorem in the category of sets. For the reader interested in seeing the full formulation of this Theorem in cartesian closed categories, we refer him to [13, 14].

Lawvere’s Theorem [13, 14] is a direct generalization of Cantor’s diagonal argument. Recall Cantor’s argument. We work in the category of sets. Let \([A, B]\) denote the collection of set theoretic mappings from \( A \) to \( B \). Let \( Z = \{0, 1\} \) and note that a subset \( A \) of a set \( X \) can be regarded as a mapping \( A : X \rightarrow Z \) where the elements of the subset are those \( x \in X \) such that \( Ax = 1 \).
Cantor. Cantor gave a proof that there is no surjective mapping from $X$ to $[X, Z]$. His proof goes as follows. Let $F : X \to [X, Z]$ be any mapping. Define a subset $C$ of $X$ by the formula

$$ Cx = \sim F(x)x $$

where it is understood that $\sim 0 = 1$ and $\sim 1 = 0$. $C$ cannot be of the form $F(a)$ for any $a \in X$. For if $C = F(a)$, then $F(a)x = \sim F(x)x$ for all $x \in X$. Hence $F(a)a = \sim F(a)a$. This is a contradiction since the negation $\sim$ has no fixed points. From this Cantor concludes that for $X$ infinite we have a higher infinity for $[X, Z]$ and so a hierarchy of infinities:

$$ X < [X, Z] < [[X, Z], Z] < \cdots. $$

Lawvere. Lawvere turns this scenario on its head by considering a more general case where $Z$ could be other than the set of two elements.

Lawvere’s Fixed Point Theorem for Sets. Let $Z$ and $X$ be sets. Suppose that there exists a function

$$ F : X \to [X, Z] $$

that is surjective. Let

$$ \alpha : Z \to Z $$

be any mapping from $Z$ to itself. Then $\alpha$ has a fixed point.

Proof. Define $C : X \to Z$ by the formula

$$ Cx = \alpha(F(x)x). $$

Then by surjectivity of $F$, we have $C = F(a)$ for some $a$ and consequently

$$ F(a)a = \alpha(F(a)a). $$

Hence any mapping $\alpha : Z \to Z$ must have a fixed point. //

Remark. Note that if we define a diagonal mapping

$$ \Delta : X \to X \times X $$

by

$$ \Delta(x) = (x, x), $$

23
then \( Cx = \alpha(F(x)x) = \alpha(\text{eval}((F \times I)(\Delta(x)))) \) where \( I \) denotes the identity map on \( X \) and \( \text{eval}(F(x), y) = F(x)y. \) Thus

\[
C = \alpha \circ \text{eval} \circ (F \times I) \circ \Delta.
\]

In this way the map constructed in Lawvere’s Theorem can be seen to work in any category with products and a terminal object. The terminal object serves to define the notion of a “point”. If \( A \) is an object in the category and \( \mathbf{1} \) is the terminal object, then a point in \( A \) is a morphism \( t : \mathbf{1} \rightarrow A \). Surjectivity of \( F : X \rightarrow [X, Z] \) then means that for every \( g : X \rightarrow Z \) there is a \( t : \mathbf{1} \rightarrow X \) such that \( g = F \circ t \). The diagonal map is crucial to the general construction. See [13] page 316 for a discussion of these points.

Lawvere’s Fixed Point Theorem can be used to place Cantor’s original argument in different contexts. For example, let \( Z = \{0, 1, J\} \) where \( \sim 0 = 1, \sim 1 = 0, \sim J = J \). In this example we can interpret \( Z \) as the set of values in Łukasiewicz three-valued logic [16]. Then generalized subsets of \( X \) are described by maps into \( Z \). In such a generalized set \( D : X \rightarrow Z \), elements of \( x \in X \) are either definite members of \( D \) (\( D(x) = 1 \)), definite non-members of \( D \) (\( D(x) = 0 \)), or indeterminate with respect to \( D \) (\( D(x) = J \)). If we have a mapping \( F : X \rightarrow [X, Z] \), then we can define a new mapping by the formula \( Cx = \sim F(x)x \) and we find that if \( C = F(z) \), then \( F(z)z = \sim F(z)z \), and we conclude that it must be the case that \( F(z)z = J \). We cannot conclude that \( C \) is not of the form \( F(z) \) for any \( z \in X \). Thus Cantor’s argument about higher cardinalities does not generalize to a set theory based on the three-valued logic.

Return to Self-Reference Now return to our First Self-Reference Theorem. In this context, for the Russell Set, \( Rx = \sim xx \), we would generalize to a reference arrow

\[
R \rightarrow \sim \#.
\]

Applying the shift, we obtain

\[
\#R \rightarrow \#\#R.
\]

Instead of a contradiction, we obtain a referential arrow from \( \#R \) to its negation. By changing equality to reference we have avoided the paradox. This is exactly how such paradox is resolved in computer languages where the referential step is often interpreted as a step in a recursive process. Of course we do not assert that this recursion solves the paradox in its original context.
We end this section with a discussion of Gödel’s Incompleteness Theorem in the Lawvere context and its relationship with our treatment of Gödel in the context of the First Self-Reference Theorem for categorical pairs.

**Gödel Revisited.** Here is how Gödel’s Theorem is related to the Lawvere Fixed Point Theorem. Let \( \{ \phi(n, x) | n = 1, 2, 3, ... \} \) denote a list of all syntactically valid formulas involving a single variable \( x \) in the formal system \( L \) (as described in Section 3). Suppose that \( L \) is strong enough to be able (by proving or invalidating) to determine the truth or falsehood of each particular formula \( \phi(n, m) \) for all natural numbers \( n \) and \( m \). We define a new formula by

\[
C x = \sim \phi(x, x).
\]

Assuming that the list of all formulas and the ability of the formal system to determine their truth or falsity is complete, we then have \( C x = \phi(N, x) \) for some natural number \( N \). Thus we have

\[
\phi(N, x) = \sim \phi(x, x)
\]

for each natural number \( x \) and hence

\[
\phi(N, N) = \sim \phi(N, N).
\]

Since negation has no fixed point in the standard logic of \( L \), we conclude that any list that we make of statements for the system will be of necessity incomplete with respect to the notion of truth within the system in terms of provability. Provability within, and truth from outside the system are distinct under the assumption that the system \( L \) is consistent.

When we describe Gödel’s Theorem this way it is clear that it can be seen as an application of the Lawvere Fixed Point Theorem. We simply take \( F(x)y = \phi(x, y) \) and the patterns match. Note that in this form of Gödel’s Theorem we did not encode directly a statement that asserts its own unprovability. This approach to Gödel via a diagonal argument sidesteps the issue of self-reference, and instead shows the contradictory nature of completeness. This is the difference between the approach to Gödel via the First and Second Self-Reference Theorems and the Lawvere Fixed Point Theorem. Using the Self-Reference Theorems we construct an abstract framework for the Gödel numbering and the indirect self-reference that is in back of the incompleteness phenomenon, and we show that this phenomenon is directly related to the higher categorical step of shifting from equality to arrow.
5. Ordinary Language

In this section we consider an interpretation for the First Self-Reference Theorem in terms of ordinary language. In this interpretation the morphisms of category $C$ are all texts in ordinary discourse and all referents for these texts. Thus we regard perceptions and objects in the world as corresponding to texts in a language that encompasses the written and spoken languages that are commonly used. In this way, if I meet another person, that other person would be regarded as a text whose name I come to learn in the course of meeting him or her. Then if I meet person P (a text) and learn his or her name N then at the beginning of that process there is indicated an arrow from N to P.

$$N \rightarrow P$$

but shortly thereafter, when the naming process is more complete, the text that is P has become modified to contain its name in a prominent place and the name has been shifted to indicate that it is a name of that person. In actual practice this process is the one that includes our ability to recognise a person P as that person with the name N. We indicate this shift of reference by the indicative shift of Section 2.

$$\#N \rightarrow PN.$$

In terms of our perception, a text P that has undergone this shift is now known to have the name N. The name N appears in our representational space along with the (text representing) the person P.

Thus we see that the notion of categorical pair and indicative shift is a model of the referential shift inherent in the naming and referring of texts in ordinary language and in language in a very general context.

The First Self-Reference Theorem then becomes a model for how self-reference occurs in language. For we see that the simplest instance of the Theorem is the act of naming the shift operation $\#$.

$$M \rightarrow \#$$

Let $M$ denote the name of the shift operation $\#$. Then $M$ is the name of the linguistic ability to combine a name with the text to which that name
refers. And we see that once that name of the shift is itself shifted, then a self-reference occurs.
\[ \sharp M \rightarrow \sharp M. \]
The completion of the naming process for the process of naming is self-referential. When we refer to ourselves in language we refer to our own ability to make and complete the act of naming.

Note how the rest of the First Self-Reference Theorem works in this context. If we have a reference
\[ G \rightarrow F^\sharp, \]
this is a reference to a text \( F^\sharp \) that talks about the naming process. Shifting this reference we obtain
\[ \sharp G \rightarrow F^\sharp G, \]
a naming of a text that discusses its own name.

We see that in the context of ordinary language a correct modeling must be flexible enough to allow even more hierarchies of reference and, at the same time to allow all these hierarchies to work at the same level since in language the name of a name is still a name. We see therefore that the splitting into two categories \( C \) and \( C' \) can lead to higher splittings (higher categories) and if these categories are all to be seen at a level, one may need to consider categories of infinite height where every object is a morphism and every morphism is an object. We call such categories reflexive and hope that they will be useful in an extension of this work to problems in mathematics, linguistics and philosophy.

To clarify these last remarks, consider a sequence of categories
\[ C \ C' \ C'' \ldots \ C^{(n)} \ C^{(n+1)} \ldots \]
where the objects in \( C^{(n+1)} \) are the morphisms in \( C^{(n)} \). We shall say that the category \( C^{(n)} \) is of type \( n \). There are a number of competing definitions for the notion of \( n- \) category (recall our specific definition of 2-category in Section 2). All \( n- \) categories are of type \( n \). All the pair constructions in this paper apply in the transition between \( C^{(n)} \) and \( C^{(n+1)} \). A reflexive category is at level \( C^\infty \) where any finite descent from morphism to object will reveal only further morphisms.
It might seem that a reflexive category would be a huge undertaking, requiring some sort of limiting construction from a hierarchy of categories. That this is not so is illustrated in Figure 1. Here we show, at the top of the Figure, a morphism between two morphisms. If one were to draw a diagram of morphisms such that every morphism of the diagram occurred between two morphisms, then the diagram could be interpreted as describing a reflexive category whose objects are the morphisms depicted in the diagram, and whose morphisms are generated under composition by the morphisms shown in the diagram (and the unwritten identity morphisms, one for each pictured morphism). In this way, certain special diagrams can represent reflexive categories.

In Figure 1 we depict such a diagram. We show a diagram $T$ of a trefoil knot and take the oriented arcs of that diagram to be morphisms in a category that we shall call the *Trefoil Category*. Knot diagrams have just the right properties, as described above, to generate reflexive categories. The generating morphisms are the arcs in the diagram and we take the objects of the category to also be this set of arcs. A morphism begins at one arc and ends at another arc. Every morphism in this category is a morphism of morphisms. Knot diagrams are of independent interest as they are formalizations of projections of curves in three space and can be used to faithfully study the topology of curve embeddings in three dimensional space. For this purpose one usually takes the knot diagrams up to an equivalence relation generated by the graphical moves shown in Figure 2 (the Reidemeister moves). It is not our purpose here to dwell on the theory of knots, but in fact this association of a category to a knot diagram can, in principle, be used to obtain topological information about the knot. We will treat this aspect in a separate paper.

We also illustrate in Figure 1 a diagram $T'$ that is not quite a knot diagram that has the same formal characteristic of generating a reflexive category. Each arc is seen to be an arrow originating on one of the arcs and terminating on another. If the reader examines the Figure, it will be apparent that we have a category with objects \{A, B, C\} and each of these objects is a morphism with

1. $A : C \rightarrow B$,
2. $B : A \rightarrow C$,
3. $C : B \rightarrow A$.  

28
Compositions of these morphisms are available, so this category has more morphisms than it has objects, but it is certainly reflexive in that all its objects are morphisms. Reflexive categories of this sort can be associated with knots and links. We shall study them in a separate paper. A second example is shown in Figure 1 with the link diagram $L$. Here the associated reflexive category has two objects $A$ and $B$ that are also generating morphisms for the category. We have

1. $A : B \rightarrow B$,
2. $B : A \rightarrow A$.

The distinct morphism/objects $A$ and $B$ are “linked” categorically in that each plays the role of a morphism for the other. It is clear that this notion of linking is close to the way we speak of linking in ordinary language where a linkage of plans, ideas or persons involves how each is a process for the other. One reason for bringing in this example of a reflexive category in a section on ordinary language is that we see that the Trefoil Category and the Link Category (and indeed the diagrams as mathematical structures) arise from the language of sketching of three dimensional forms. But also, when we translate these diagrammatic forms into the corresponding reflexive categories we see that the categories themselves contain patterns of mathematical/topological language. These are topics to be pursued elsewhere.

Figure 1: Trefoil Category and Link Category
6. Epilogue

Both the Self-Reference Theorems of this paper and the Lawvere Fixed Point Theorem come from generalizing the Cantor diagonal process, and both can also be seen as ways to generalize the Church-Curry Fixed Point Theorem. In the Church-Curry Theorem we are given an algebra with a binary operation that is not associative and an axiom of reflexivity that states that functions of a single variable expressed in that algebra can be named and regarded as elements of the algebra. Thus in such an algebra $\Lambda$ one might define $G[x] = a((bx)x)$ as a function from the algebra to itself. One is then guaranteed that there exists an element $g$ such that for all $x$ in the algebra, $gx = a((bx)x)$. This reflexive assumption of a correspondence between elements of the algebra and mappings of the algebra to itself is very strong.

The simplest instance of this strength is the Church-Curry Fixed Point Theorem which states that every element $F$ of $\Lambda$ has a fixed point in the sense that there is an $a$ in the algebra such that $Fa = a$. The proof goes as follows. Define $G[x] = F(xx)$ for all $x$ in $\Lambda$. Then, by the axiom of reflexivity there exists $g$ in $\Lambda$ such that $gx = F(xx)$ for all $x$. Letting $x = g$ we obtain $gg = F(gg)$. So $gg$ is the fixed point for $F$. 
At the formal level, the Lawvere Fixed Point Theorem can be seen as a categorical generalization of the \( \Lambda \) algebra formalism \( C[x] = \alpha(F(x)x) \) where it is known that such a \( C \) must be represented algebraically by an element of the form \( F(a) \) (the surjectivity hypothesis for \( F \)). Then we have \( F(a)x = \alpha(F(x)x) \) and consequently \( F(a)a = \alpha(F(a)a) \), giving \( \alpha \) a fixed point with a specific structure. The generality of the pattern allows it to be applied to many situations beyond the original Cantor argument. The application of the Fixed Point Theorem to Gödel’s Theorem works best when we do not think of Gödel’s Theorem as depending on indirect self-reference.

The First and Second Self-Reference Theorems are generalizations of the Church Curry Fixed Point Theorem where we replace equality signs by arrows of reference and we correspondingly generalize the operator \( \sharp x = xx \) to an arrow of reference
\[
\sharp x \rightarrow xx.
\]
We then generalize the fundamental repetition operator \( \sharp \) a notch further to the indicative shift where, if
\[
a \rightarrow b
\]
then
\[
\sharp a \rightarrow ba
\]
and the Church-Curry Fixed Point Theorem is transformed into our First Self-Reference Theorem. In fact we could take the initial category \( C \) to have one object and its morphisms the elements of the lambda algebra having either no free variable or a single free variable. Composition \( ab \) of morphisms \( a \) and \( b \) is defined whenever \( a \) has a free variable. Then \( ab \) stands for the substitution of \( b \) into the free variable in \( a \). With this we have both the indirect reference given by the First Self-Reference Theorem (and/or the Second Self-Reference Theorem) and the fixed point results of the lambda algebra available in the one categorical pair \((C, C')\).

7. Self-Reference

Finally we return to self-reference in the form of the expansion of a name. Recall the expansion operator as described in Section 1. We have an operation \( E \) on names that \textit{expands} a name in the sense that if “A” is the name of A then the expansion \( E“A” \) refers to A “A”, the result of appending the
contents of the name to the name. Thus if we regard the name as pointing to its contents as in

\[ "A" \rightarrow A \]

then

\[ E"A" \rightarrow A"A". \]

Self-reference results when one expands the name of the expansion operator.

\[ E"E" \rightarrow E"E". \]

How is this self-reference related to the self-reference we are all familiar with in our personal experience?

To begin to see an answer to this question, consider the use of the pronoun “I”. When I say I then I refer to myself. I alone does not refer to itself. It is required that there be a contents related to the one who uses the word I. I am the one who says I, and this can be said by anyone. So in a sense we can say that I is really the expansion operator and the self-reference associated with I occurs when we apply I to “I”, forming I“I” which is self-referent. In other words, we each make a personal identification

\[ I = I"I", \]

that says “I am the operation of expanding myself to my content (which is myself).” This was said more eloquently by Heinz von Foerster \[22 \]: “I am the observed relation between myself and observing myself.” We encourage the reader to expand further on these themes.

References

[1] H. P. Barendregt, “The Lambda Calculus - Its Syntax and Semantics”, North Holland, Elsevier, 1984.

[2] L.H. Kauffman, Self-reference and recursive forms, Vol. 10, Journal of Social and Biological Structures (1987), 53-72.

[3] L.H. Kauffman, Knot Logic. In Knots and Applications ed. by L. Kauffman, World Scientific Pub. (1994), pp. 1-110.
[4] L.H. Kauffman, Space and time in computation and discrete physics, Intl. J. Gen. Systems, Vol. 27, Nos 1-3 (1998), pp. 249-273.

[5] L.H. Kauffman, Virtual logic, Systems Research, Vol. 13 (Festschrift for Heinz von Foerster), No. 3, pp. 283-310 (1996).

[6] L.H. Kauffman, Time imaginary value, paradox sign and space. in Computing Anticipatory Systems, CASYS - Fifth International Conference, Liege, Belgium (2001) ed. by Daniel Dubois, AIP Conference Proceedings Volume 627 (2002).

[7] L.H. Kauffman, Biologic. AMS Contemporary Mathematics Series, Vol. 304, (2002), pp. 313-340.

[8] L.H. Kauffman, Biologic II, in “Woods Hole Mathematics” edited by Nils Tongring and R. C. Penner, World Scientific Series on Knots and Everything Vo.34 (2004), p. 94-132.

[9] L.H. Kauffman, Fibonacci form and beyond. Forma 19 (special issue edited by Jay Kappraff) (2004), no. 4, 315–334.

[10] L.H. Kauffman, Eigenform, Kybernetes - The Intl J. of Systems and Cybernetics 34, No. 1/2 (2005), Emerald Group Publishing Ltd, p. 129-150.

[11] L.H. Kauffman, Reflexivity and Eigenform – The Shape of Process. Constructivist Foundations, Vol 4, No. 3, July 2009, p. 121-137.

[12] L.H. Kauffman, Reflexivity, Eigenform and Foundations of Physics. In Reflexivity, Proceedins of ANPA 30, Arleta D. Ford, Editor , Published by ANPA, June 2010, pp. 158-222.

[13] F. W. Lawvere and S. H. Shamucl, “Conceptual Mathematics - A First Introduction to Categories”, Cambridge University Press 1997,2009.

[14] F. W. Lawvere, Diagonal arguments and cartesian closed categories, Reprints in Theory and Applications of Categories, No. 15, 2006, pp. 1-13.

[15] T. Leinster, “Higher Operads, Higher Categories”, London Mathematical Society Lecture Notes Series No. 298, Cambridge University Press, 2004.
[16] Lukasiewicz J., 1920, O logice trjwartociowej (in Polish). Ruch filozoficzny 5:170171. English translation: On three-valued logic, in L. Borkowski (ed.), Selected works by Jan Lukasiewicz, NorthHolland, Amsterdam, 1970, pp. 8788.

[17] S. Mac Lane, “Categories for the Working Mathematician”, Graduate Texts in Mathematics, Springer-Verlag (1971).

[18] R. Smullyan, “Gödel’s Incompleteness Theorems”, Oxford University Press, 1992.

[19] E. Mendelson, “Introduction to Mathematical Logic”, D. Van Nostrand Co., Inc., 1964.

[20] E. Nagel, and J. Newman, “Gödel’s Proof”, New York University Press, 1960.

[21] C. J. Ash, J. N. Crossley, C. J. Brickhill, J. C. Stillwell, N. H. Williams, “What is Mathematical Logic?”, Oxford University Press, 1972.

[22] H. von Foerster, “Understanding Understanding - Essays on Cybernetics and Cognition,” Springer-Verlag (2003), p. 257.