Exact solutions for non-Hermitian Dirac–Pauli equation in an intensive magnetic field

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Abstract
We consider modified Dirac–Pauli equations that are entered using $\gamma_5$-mass factorization, $m \rightarrow m_1 \pm \gamma_5 m_2$, of an ordinary Klein–Gordon operator. We also consider the interaction of fermions with an intensive uniform magnetic field, focusing on their $(g-2)$-gyromagnetic factor. Due to effective research procedures, we derive the exact solutions of the energy spectra of pseudo-Hermitian Hamiltonians, taking into account the spin of the fermions. The basic research methods are the elucidation of the new border areas of the unbroken $\mathcal{P}\mathcal{T}$ symmetry of non-Hermitian Hamiltonians. In particular, it is shown that the reality energy spectrum of fermions at rest can be expressed by limiting the intensity of the magnetic field, $H \leq H_{\text{max}} = m^2/(2\Delta \mu m_1)$, where $\Delta \mu$ is an anomalous magnetic moment of particles.

Keywords: non-Hermitian, relativistic, Hamiltonian, Dirac–Pauli type, maximal mass, magnetic field, $(g-2)$-factor

1. Introduction

It is a well-known fact that the reality of the spectrum in models with a non-Hermitian Hamiltonian is a consequence of $\mathcal{P}\mathcal{T}$ invariance of the theory (i.e., a combination of spatial and temporary parity of the total Hamiltonian, $[H, \mathcal{P}\mathcal{T}]\psi = 0$). When the $\mathcal{P}\mathcal{T}$ symmetry is unbroken, the spectrum of the quantum theory is real. These surprising results explain the growing interest in this problem, which was initiated by Bender and Boettcher’s observations [1].

In the last few years a lot of the new non-Hermitian $\mathcal{P}\mathcal{T}$-invariant systems [2–27] have been studied. The non-Hermitian $\mathcal{P}\mathcal{T}$-symmetric model with a $\gamma_5$ extension of the mass parameter in the Dirac equation

$$m \rightarrow m_1 \pm \gamma_5 m_2,$$

which is the consequence of the relation for the physical mass, $m$,

$$m_1^2 - m_2^2 = m^2. \quad (3)$$

However, it should be noted that the model with extension (1) is not new, because it was proposed long before [2] in [28], which was developed by Kadychevsky. As shown in [28], the Hamiltonians with extension (1) appear in the geometrical theory with a fundamental mass in the anti-de Sitter space, when describing the fermion sector of the quantum field theory (QFT).

As is established, the idea about the existence of a maximal mass in a spectrum mass of elementary particles corresponding to the Planck mass was suggested by Markov in 1965 [29]

$$m \leq m_{\text{Planck}} \cong 10^{19} \text{GeV}. \quad (4)$$

The particles with the limiting mass $m = m_{\text{Planck}}$, called ‘maximons’ by the author, should play a special role in the world of elementary particles. However, Markov’s original condition (4) was purely phenomenological, and he used standard field theoretical techniques even for the
description of the maximon. In the late 1970s, a more radical approach was suggested in [28]. This model contained a limiting mass, \( \mathcal{M} \), as a new fundamental physical constant. This condition of finiteness of the mass spectrum should be introduced by the relation

\[ m \leq \mathcal{M}. \quad (5) \]

where a new constant, \( \mathcal{M} \), was called the fundamental mass.

In [28, 30, 31], the existence of mass \( \mathcal{M} \) was understood as a new principle of nature, similar to the relativistic and quantum postulates, which was put into the ground of the new QFT. At the same time, the new constant \( \mathcal{M} \) was introduced in a purely geometric way, like the velocity of light is the maximal velocity in special relativity.

We really do not know the true value of the maximal mass. Perhaps it is equal to the Planck’s mass, but it may be more or low or high. The exact answer to this question can be found by experimental investigations. However, we can now ask the question: What values of mass \( m_{\text{max}} \) can be used when the concept of local field is still applied? Formally, the standard QFT remains a logically impeccable scheme when in the elementary act of interaction of the particles involved, the masses are comparable, say, to the masses of cars. This far extrapolation of local field theory in the area of macroscopic values of the masses looks like pathology, and elementary particle physics will not have to deal with such absurdity. But we should emphasize that modern QFT does not prohibit such physically meaningless extrapolation. Thus, one should think about modification tactics to eliminate these physically meaningless possibilities. This program was suggested by Kadyshevsky in [28] (see also [30, 31]).

The purpose of this paper is to continue studying the examples of pseudo-Hermitian relativistic Hamiltonians [5–10]. Here, we investigate non-Hermitian systems with \( \gamma_5 \)-mass extension, taking into account the external magnetic field. We also study the spectral and polarization properties of such systems, and with this aim we consider solutions of the modified Dirac equation for free fermions. After that, we take into account interaction with the intensive magnetic fields of charged and neutral particles that have anomalous magnetic moments (AMM). The novelty of our approach is associated with predictions of new phenomena caused by a number of additional terms of the non-Hermitian Hamiltonians, which radically change the picture of interactions. It not only refers to processes with large energies, but also may be observed in the region of low energies when one takes into account the interaction of the AMM of fermions with intensive magnetic fields.

This paper is organized as follows. Section 2 is devoted to the comparison of algebraic and geometric approaches to the constraint of mass parameters. In section 3, the non-Hermitian approach to the construction of plane-wave solutions is formulated for the case of free massive particles. In section 4, we study the basic characteristics of modified Dirac models with \( \gamma_5 \)-massive contributions in the external magnetic field. Then, in section 5, we consider the modified Dirac–Pauli model in the magnetic field with a non-Hermitian extension. This section also contains a discussion of the effects of the possible observability of parameters \( m_1, m_2 \), taking into account the interaction of fermions (with regard to their AMM) with the magnetic field. Section 6 contains the summary and conclusions.

2. The comparison of the algebraic and geometric approaches with the limitation of the mass parameters

If one chooses a geometrical formulation of QFT, the adequate realization of the limiting mass hypothesis is reduced to the choice of the de Sitter geometry as the geometry of the four-momentum space of the constant curvature, with a radius equal to \( \mathcal{M} \) [28]. It was also noted that in addition to the de Sitter space, there is another space of constant curvature, breaking into a Minkowski space in the case of a small four-momentum, which is called the anti-de Sitter space \( (\hbar = c = 1) \),

\[ p_0^2 - p_1^2 - p_2^2 - p_3^2 = n^2. \quad (6) \]

It is easy to see that for a free particle, \( p_0^2 - p^2 = m^2 \), condition (5) is automatically performed on the surface (6). In the approximation

\[ |p_0|, \quad |p| \ll \mathcal{M}, \quad p_3 \equiv \mathcal{M} \quad (7) \]

anti-de Sitter geometry goes into Minkowski geometry of a four-dimensional pseudo-Euclidean \( p \)-space (‘flat limit’). Objects with a mass greater than \( \mathcal{M} \) cannot be considered as elementary particles, because any local fields are absent for them.

The detailed analysis of the different aspects of the construction of the modified QFT with the maximal mass in the curved momentum anti-de Sitter space has allowed us to obtain a number of interesting results. In particular, it has been shown that non-Hermitian fermionic equations of motion with a \( \gamma_5 \)-dependent mass term must arise in the modified field theory (see, for example, [30, 31]). Indeed, it follows that in the modified QFT, the role of the Klein–Gordon operator is played by the operator \( K(p, \mathcal{M}) = 2 \mathcal{M}(p_3 - \mathcal{M} \cos \mu), \) while a new expression for the Dirac operator has the form [28],

\[ D(p, \mathcal{M}) = p_\gamma + (p_3 - \mathcal{M}) \gamma^5 - 2 \mathcal{M} \sin (\mu/2), \quad (8) \]

where \( \cos \mu = \sqrt{1 - \frac{m^2}{\mathcal{M}^2}} \). It is easy to verify that in the ‘flat limit’ (7) (formally, when \( \mathcal{M} \to \infty \), expression (8) converts into an ordinary Dirac operator.

But it is not a single form of the equation of motion. It is very important that the new Klein–Gordon operator in the geometrical approach, \( K(p, \mathcal{M}) \), can be separated into matrix multipliers using another method, completely independent from (8). As a result, we have the second operator of motion, which may be represented as [30, 31]

\[ D_{\text{exotic}}(p, \mathcal{M}) = p_\gamma + (p_5 + \mathcal{M}) \gamma^5 - 2 \mathcal{M} \cos (\mu/2). \quad (9) \]
Thus in the geometric approach that is being developed, we encounter a certain ‘exotic’ fermion field that has no analog in the ordinary theory (i.e., the conversion to an Hermitian expression with $\mathfrak{M} \rightarrow \infty$ does not occur for it$^1$).

The Hamiltonians associated with (8) and (9) may be expressed by the relations:

$$H = \alpha \vec{p} + \beta (m_1 + \gamma_5 m_2), \quad (10)$$

$$H^{\text{exotic}} = \alpha \vec{p} + \beta (m_3 + \gamma_5 m_4), \quad (11)$$

where in these modified Hamiltonians, the matrices $\beta = \gamma_0, \alpha_i = \beta \gamma_i, i = 1-3$. It is important to note that on the mass surface, $p_5 = \mathfrak{M} \cos \mu$, and from (8), (9) we have [28]

$$m_1 = 2 \mathfrak{M} \sin \mu/2, \quad m_2 = 2 \mathfrak{M} \sin^2 \mu/2, \quad (12)$$

$$m_3 = 2 \mathfrak{M} \cos \mu/2, \quad m_4 = 2 \mathfrak{M} \cos^2 \mu/2. \quad (13)$$

Taking into account that $\sin \mu = m/\mathfrak{M}$, one can obtain

$$\sin \mu/2 = \frac{1}{\sqrt{2}} \sqrt{1 - \sqrt{1 - m^2/\mathfrak{M}^2}};$$

$$\cos \mu/2 = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 - m^2/\mathfrak{M}^2}},$$

and hence we can find

$$m_{1,3} = \sqrt{2} \mathfrak{M} \sqrt{1 \mp \sqrt{1 - m^2/\mathfrak{M}^2}}; \quad (14)$$

$$m_{2,4} = \mathfrak{M} \left(1 \mp \sqrt{1 - m^2/\mathfrak{M}^2}\right). \quad (15)$$

Here, $m_1^2 - m_2^2 = m^2$ and $m_3^2 - m_4^2 = m^2$.

It is obvious that equations (10) and (11) prove to be non-Hermitian due to the appearance of the $\gamma_5$-mass summands in them ($H \neq H^\dagger; H^{\text{exotic}} \neq H^\dagger_{\text{exotic}}$). Thus, the conclusion can be drawn that mass-spectrum restriction (5), which underlies the geometric approach in the development of the new QFT with maximal mass [28, 30, 31] leads to the appearance of non-Hermitian contributions to Hamiltonians in the fermion sector of modified theory.

Thus, the opposite assumption arises: can a $\gamma_5$-extension serve as a basis for a possible search of a purely algebraic way to obtain a limitation of mass? This program was successfully implemented, and it turned out that the maximal mass in the case of the non-Hermitian $\gamma_5$-extension of Hamiltonians actually exists [5–10]. In particular, according to [9, 10] we can use the simple mathematical theorems and obtain

$$m \leq \frac{m_1^2}{2m_2} = M, \quad (16)$$

where one uses

$$m^2 + m_2^2 = m_1^2 \quad (17)$$

and the new mass parameter, $M$, is introduced.

Previous investigations of algebraic $\gamma_5$-mass extension [2, 3] have been carried out for the case of a single particle. At the same time, the ordinary Dirac equations are written in such a way that these equations are valid for the case of fermions with any mass values, $m, m'...$. The question is whether there exist adequate similar descriptions of the various fermions in the framework of the non-Hermitian theory. We can obtain, according to (3), a number of different pairs of non-Hermitian parameters, $m_1, m_2; m'_1, m'_2; ...$, corresponding to physical masses. According to (16), this allows us to define a set of maximal masses values, $m_{\text{max}}, m'_{\text{max}}...$. The physical consideration suggests that it is most preferable to consider a model with a single maximal mass that is equal to the maximal value among the considered particles. The geometric Kadyshhevsky theory [28] describes the particle mass spectrum, the maximal mass of which is equal to the mass of the maximon (i.e., the mass of particles with maximal value). Thus, since the mass of the maximon by definition is unique, it seems to be reasonable to put

$$m_{\text{max}} = m'_{\text{max}} = \cdots = M = \mathfrak{M} \quad (18)$$

(for more detailed information, see [9]). Thus the solution of the system (16) and (17), relative to parameters $m_1$ and $m_2$, may be represented in the form

$$m_1^\pm = \sqrt{2} \mathfrak{M} \sqrt{1 \mp \sqrt{1 - m^2/\mathfrak{M}^2}}; \quad (18)$$

$$m_2^\pm = \mathfrak{M} \left(1 \mp \sqrt{1 - m^2/\mathfrak{M}^2}\right) \quad (19)$$

from which it follows that expressions (14) and (15) coincide with (18) and (19), respectively, if we take $M = \mathfrak{M}$.

A full correspondence between the parameters that are used in Hamiltonians and in the equations of motion of the models under study can be obtained in [9]. Additionally, with $\mathfrak{M} \rightarrow \infty$ (‘flat limit’ (7)), we can see that the anti-de Sitter geometry does not differ from the Minkowski geometry in the pseudo-Euclidean $P$-space. This inference also entirely agrees with the interpretation of the correspondence principle in algebraic theory, when $M \rightarrow \infty$ and we have the transition to Hermitian theory.

Expressions (18) and (19) make it possible to consider a real physical approach in the given algebraic model (i.e., to find out whether the particles can be described using Hamiltonians (10) or (11)). In other words, we can determine which parameters, $m_1$ and $m_2$, must be found to describe a given particle using its known physical mass, $m$. Clearly it is impossible to answer this question unambiguously using only condition (2). However, if $M$ is introduced, then we have the answer, which follows from equations (18) and (19). Actually, we have the transition from two parametric tasks with parameters $m_1$ and $m_2$ to the problem with parameters $m$ and $M$. It should be noted that the parametric region of unbroken $PT$ symmetry is now defined by the condition

$$m \leq M. \quad (20)$$
However, if $\Omega = M$ and taking into account relations (14), (15), (18), (19), one can see that all combinations of the parameters coincide with each other: $m_1 = m_1^-, m_2 = m_2^-$ and $m_3 = m_3^+, m_4 = m_4^+.

Note that using (16), we can establish the limits of change of the different mass parameters. In particular, one can find the dependence of the mass parameters as a function of $\xi = 2m_2/m_1$. Thus, using (3) we can write

$$-2 \leq \xi \leq 2.$$ 

Note that if it is possible to express $\xi$ in the form $\xi = m_1/M$ (see (16)), we can obtain

$$m \leq m_1 \leq 2M; \quad -2M \leq m_2 \leq 2M.$$ 

Taking into account (3), we can also write

$$m/m_1 = \sqrt{1 - \xi^2/4}; \quad m/M = \xi\sqrt{1 - \xi^2/4}.$$ 

Comparing the expression $\xi = m_1/M$ with $\xi = 2m_2/m_1$, we can see that $\xi = 0$ is reached if either $m_2 = 0$ or $M \to \infty$, which correspond to each other and define the Hermitian limit.

In particular, the maximum value of the particle mass, $m = M$, is achieved at the ratio of the masses equal to $m_2 = m_1/\sqrt{2}$. Till to this value for each mass of ordinary particles, one can find the parameters $m_1$ and $m_2$, for which a limit transition to regular Dirac theory exists. Further increases of $m_2$ lead to the descending branch of the $m/M$, where the Dirac limit does not exist. At the point $m_2 = m_1 = 2M$, the value of $m$ is equal to zero. Thus, there is the region, $m_1 > m_2 > m_1/\sqrt{2}$, corresponding to the description of the ‘exotic particles,’ where there is no transition to the Hermitian limit. For example, for massless fermions we have on the one hand $m_1 = 0$, $m_2 = 0$, and on the other hand, we have $m_1 = 2M$, $m_2 = 2M$. In the first case, there are ordinary particles, and in the second case, we are dealing with exotic fermions.

It is important to note that if we exclude from consideration exotic fermions as artifacts, then we need to exclude from the realm of unbroken $\mathcal{PT}$ symmetry the components corresponding to these particles, and instead of $m_2 \leq m_1$ we must have

$$m_2 \leq m_1/\sqrt{2}.$$ 

This conclusion is not trivial, because in contrast to the geometric approach, where the emergence of new, unusual properties of particles associated with the presence in the theory of a new degree of freedom (sign of the fifth component of the momentum $e = p_5/|p|$) [30], in the case of an algebraic extension of the free Dirac equation due to the additional $\gamma_5$-mass term, a satisfactory explanation of this fact does not yet exist.

3. Non-Hermitian extensions of plane waves

Let us now consider the solutions of modified Dirac equations for free massive particles using the $\gamma_5$-factorization of the ordinary Klein–Gordon operator. In this case, similar to the Dirac procedure, one can represent the Klein–Gordon operator in the form of a product of two commuting matrix operators:

$$\left(\hat{p}_\mu^2 + m^2\right) = \left(i\hat{p}_\mu^\mu - m_1 - \gamma_5m_2\right) \times \left(-i\hat{p}_\mu^\mu - m_1 + \gamma_5m_2\right).$$ 

where designations $\hbar = c = 1$ are used and the physical mass of particles, $m$, is expressed through parameters $m_1$ and $m_2$.

$$m^2 = m_1^2 - m_2^2.$$ 

So the function would obey the Klein–Gordon equations

$$\left(\hat{p}_\mu^2 + m^2\right)\psi(x, t) = 0$$ 

one can demand that it also satisfies one of the first-order equations

$$\left(i\hat{p}_\mu^\mu - m_1 - \gamma_5m_2\right)\psi(x, t) = 0 \quad \times \left(-i\hat{p}_\mu^\mu - m_1 + \gamma_5m_2\right)\psi(x, t) = 0.$$ 

Equation (26), of course, are less common than (25), and although every solution of equation (26) satisfies (25), reverse approval has not been designated. It is also obvious that the Hamiltonians associated with (26) are not Hermitian, because the $\gamma_5$-dependent mass components appear ($H \neq H^\dagger$):

$$H = \mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot (m_1 + \gamma_5m_2)$$ 

and

$$H^\dagger = \mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot (m_1 - \gamma_5m_2).$$ 

Here, as early matrices, $a_5 = \gamma_0 \cdot \gamma_5$, $\beta = \gamma_0 \cdot \gamma_5 = -i\gamma_0\gamma_5\gamma_0\gamma_5$. It is easy to see from (24) that the mass, $m$, appearing in equation (25) is real when the inequality (2) is accomplished.

Mustafazadeh identified the necessary and sufficient requirements of the reality of the eigenvalues for pseudo-Hermitian and $\mathcal{PT}$-symmetric Hamiltonians and formalized the use of these Hamilton operators in [11, 12] (see also [20–26]). According to the recommendations of these studies, we can define the Hermitian operator $\eta$, which transforms non-Hermitian Hamiltonian by means of an invertible transformation to Hermitian-conjugated Hamiltonians. It is easy to see that with Hermitian operator

$$\eta = e^{i\vartheta/2},$$ 

where $\vartheta = \arctanh(m_2/m_1)$, we can obtain

$$\eta H \eta^{-1} = H^\dagger.$$ 

In addition, multiplying the Hamilton operator $H$ from left to $e^{i\vartheta/2}$, and taking into account that matrices $\gamma_5$ commute with matrices $a_5$ and anti-commute with $\beta$, we can obtain

$$e^{i\vartheta/2}H = H_0 e^{i\vartheta/2},$$ 

where $H_0 = \alpha p + \beta m$ is an ordinary Hermitian Hamiltonian of a free particle.

The mathematical sense of the action of operator (29) turns out, if we not that according to the properties of $\gamma_5$
matrices, all the even degrees of $\gamma$ are equal to 1, and all the odd degrees are equal to $\gamma$. Given that $\cosh (x)$ decomposes on even degrees and $\sinh (x)$ decomposes on odd degrees of $x$, expressions (30) and (31) can be obtained by representing non unitary exponential operator $\eta$ in the form

$$\eta = e^{r\theta} = \cosh \theta + \gamma \sinh \theta,$$  \hspace{1cm} (32)

where

$$\cosh \theta = m_1/m; \quad \sinh \theta = m_2/m.$$  \hspace{1cm} (33)

The region of the unbroken $PT$ symmetry of (10) can be found in the form (2). However, it is not apparent that the area without undisturbed $PT$ symmetry defined in such a way does not include the regions corresponding to some unusual particles, the description of which is radically different from the traditional one.

If we use the standard representation of $\gamma$-matrices, then the non-Hermitian Hamiltonian $H$ can be written in the following matrix form:

$$H = \begin{pmatrix}
    m_1 & 0 & p_3 - m_2 & p_1 - ip_2 \\
    0 & m_1 & p_1 + ip_2 & -m_2 - p_3 \\
    m_2 + p_3 & p_1 - ip_2 & -m_1 & 0 \\
    p_1 + ip_2 & m_2 - p_3 & 0 & -m_1
\end{pmatrix},$$

where $p_i$ are the components of momentum.

We consider

$$H\tilde{\psi} = E\tilde{\psi}.$$  \hspace{1cm} (34)

The condition $\det (H - E) = (-E^2 + m_1^2 - m_2^2 + p_1^2 + p_2^2) = 0$ results in the eigenvalues of $E$, which are represented in the form

$$E = \pm \sqrt{m_1^2 - m_2^2 + p_1^2 + p_2^2},$$  \hspace{1cm} (35)

where $p_\pm = \sqrt{p_1^2 + p_2^2}$ and $m_1^2 - m_2^2 = m^2$, which coincide with the eigenvalues of the energy of the Hermitian operator, $H_0$.

Let us now consider the state of a free particle with certain values of momentum and energy, which is described by a plane wave and can be written as

$$\tilde{\psi} = \frac{1}{\sqrt{2E}} \tilde{u} e^{-\omega t}.$$  \hspace{1cm} (36)

It is easy to see that the wave amplitude, $\tilde{u}$, is determined by a bispinor, whose normalization now needs an additional explanation.

Using (26) and taking into account the properties of matrices $\gamma$, $\tilde{\gamma}$, $\gamma$, we can also write the complex-conjugate equation

$$(-p_0 \tilde{\gamma}_0 - \tilde{p} \tilde{\gamma} - m_1 - ip_2 m_2) \tilde{\psi} = 0,$$  \hspace{1cm} (37)

where $\tilde{\gamma}_0$ are transpose matrices. Rearranging function $\tilde{\psi}$ and introducing a new bispinor, $\tilde{\tilde{\psi}} = \tilde{\psi} \gamma_0$, we can obtain

$$\tilde{\tilde{\psi}} = \tilde{\psi} \gamma_0 + \gamma_3 m_2 = 0.$$  \hspace{1cm} (38)

The operator, $p_\mu$, is assumed here to act on the function standing on the function on the left of it. Using (29) we can write equations (26) and (37) in the following form:

$$\left( p\gamma + m \right) \tilde{\psi} = 0,$$  \hspace{1cm} (39)

Multiplying (38) on the left by $\tilde{\psi} e^{-\theta_0}$ and equation (39) on the right by $e^{\theta_0} \psi$ and summing up the resulting expressions, one can obtain

$$e^{\theta_0} \tilde{\psi} e^{-\theta_0} \gamma_0 \gamma_0 e^{\theta_0} \psi = 0.$$  \hspace{1cm} (40)

Here, brackets indicate which functions are subjected to the action of the operator, $p_\mu$. The obtained equation has the form of the continuity equation

$$\partial_\mu j_\mu = 0,$$  \hspace{1cm} (41)

where

$$j_\mu = \tilde{\psi} e^{-\theta_0} \gamma_0 \gamma_0 e^{\theta_0} \psi = \left( \begin{array}{c}
    \psi e^{\theta_0} \psi \\
    \psi \gamma_0 e^{\theta_0} \psi
\end{array} \right).$$  \hspace{1cm} (42)

Thus, here the value of $j_\mu$ is a 4-vector of the current density of particles in the model with $\gamma$-mass extension. It is very important that its temporal component

$$j_0 = \tilde{\psi} e^{\theta_0} \psi$$  \hspace{1cm} (43)

is not changed in time (see (41)) and is positively defined. This is easy to see in the following procedure. Let us construct the norm of any state of the considered model for an arbitrary vector, taking into account the weight operator, $\eta$ (32):

$$\tilde{\psi} = \begin{pmatrix}
    x + iy \\
    u + iv \\
    z + iw \\
    t + ip
\end{pmatrix}.$$  \hspace{1cm} (44)

Using (33) and (43), in the result we have

$$\tilde{\psi} \eta = \begin{pmatrix}
    m_1 + m_2 \\
    m_1 + m_2 \\
    m_1 - m_2 \\
    m_1 - m_2
\end{pmatrix} \begin{pmatrix}
    x - iy \\
    u - iv \\
    z - iw \\
    t - ip
\end{pmatrix}.$$  \hspace{1cm} (45)

Then

$$\begin{pmatrix}
    \tilde{\psi} \eta \tilde{\psi} \\
    \tilde{\psi} \eta \tilde{\psi}
\end{pmatrix} = \begin{pmatrix}
    m_1 + m_2 \\
    m_1 + m_2 \\
    m_1 - m_2 \\
    m_1 - m_2
\end{pmatrix} \begin{pmatrix}
    x^2 + y^2 \\
    u^2 + v^2 \\
    z^2 + w^2 \\
    t^2 + p^2
\end{pmatrix}.$$  \hspace{1cm} (46)

is explicitly non-negative, because $m_1 \geq m_2$ in the area of unbroken $PT$ symmetry (2).

With the help of (38), (75), and the properties commutation of the $\gamma$-matrix, one sees that the components of the
new bispinor amplitudes must satisfy the following system of algebraic equations:

\[
\left( \gamma p - m e^{\gamma_5 \mathcal{A}} \right) \tilde{u} = 0; \quad (45)
\]
\[
\tilde{u} \left( \gamma p - m e^{-\gamma_5 \mathcal{A}} \right) = 0, \quad (46)
\]
where \( \tilde{u} = \bar{u} \gamma_f \).

According to (29) and (31), one can write the bispinor amplitudes in the form

\[
\tilde{u} = \sqrt{2m} \left( A_1 w^1, \right); \quad (47)
\]
\[
\tilde{u} = \sqrt{2m} \left( A_1^* w^a, -A_2^* w^a \right), \quad (48)
\]
where the following notations are used:

\[
A_1 = \cosh \frac{p}{m} \cosh \frac{\beta}{2} + \sinh \frac{p}{m} \sinh \frac{\beta}{2} (n \sigma);
\]
\[
A_2 = \sinh \frac{p}{m} \cosh \frac{\beta}{2} + \cosh \frac{p}{m} \sinh \frac{\beta}{2} (n \sigma).
\]

In addition, we have relations (32), (33), and the parameters \( \cosh \beta = E/m, \sinh \beta = p/m \). Also, \( w \) is a two-component spinor, satisfying the normalization condition

\[
w^a w = 1.
\]

In addition, note that \( \sigma \) are ordinary \( 2 \times 2 \) Pauli matrices and \( n = \mathbf{p}/p \) is a unit vector in the direction of the momentum.

The explicit form of these spinors can be found using the condition that spiral states correspond to the plane wave in which spinors, \( w \), are the eigenfunctions of the operator \( (n \sigma) \),

\[
\tilde{\sigma} n w = \zeta w^\ast.
\]

Therefore we get

\[
w^1 = \left( e^{-i \varphi/2} \cos \theta/2, e^{i \varphi/2} \sin \theta/2 \right),
\]
\[
w^{-1} = \left( -e^{-i \varphi/2} \sin \theta/2, e^{i \varphi/2} \cos \theta/2 \right),
\]
where \( \theta \) and \( \varphi \) are polar and azimuthal angles, respectively, that determine the direction \( n \) corresponding to the axes \( x_1, x_2, x_3 \).

It is easy to verify by the direct multiplication that

\[
\tilde{u} u = 2m.
\]

This result is obvious because there are connections between the bispinor amplitudes of modified equations, \( \tilde{u}, \tilde{u} \), and the corresponding solutions of the ordinary Dirac equations:

\[
\tilde{u} = e^{-i \varphi/2} w, \quad (49)
\]
\[
\tilde{u} = u, \quad e^{i \varphi/2}.
\]

Hence we have

\[
\tilde{u} u = \mu u = 2m.
\]

By using (45), (46), and (49), we can also obtain

\[
\tilde{u} e^{-i \varphi} \eta \tilde{u} = 2p_\xi.
\]

Taking into account (35), (42), and (43), one can easily find

\[
j_\mu = \frac{1}{2E} \tilde{u} \gamma_\mu \eta \tilde{u} = \{1, p/E \},
\]

from which it follows that the operator, \( \eta = e^{i \varphi} \), in full compliance with Mostafazadeh’s result (see, for example, [11, 12]) induces the inner product

\[
\tilde{\psi}^\ast \psi \tilde{\psi} = 1,
\]

for \( \tilde{\psi} \neq 0 \).

4. Dirac modified models with \( \gamma_5 \)-massive contributions in the external homogenous magnetic field

As is known, the wave Dirac equations provide a basis for the relativistic quantum mechanics and quantum electrodynamics of spinor particles in external electromagnetic fields. The exact solutions of the relativistic wave equation refer to one-particle wave functions, which allow the development of the approach known as the Furry picture. This method incorporates a study of the interactions with external field exactly, regardless of the field intensity [32]. Without knowledge of the exact solutions, there is no regular method to explicitly describe such interactions with an arbitrary intensity of field. The most physically important exact solutions of the ordinary Dirac equations are: an electron in a Coulomb field, in a uniform magnetic field, and in the field of a plane wave. In this connection, it is interesting to investigate non-Hermitian Dirac models that describe an alternative formulation of relativistic quantum mechanics where the Furry picture also may be realized.

Consider a uniform magnetic field \( \mathbf{H} = (0, 0, H) \), directed along the \( x_3 \) axis \( (H > 0) \). The electromagnetic potentials are chosen in the gauge [32]:

\[
A_0 = 0, \quad A_1 = 0, \quad A_2 = H x_1, \quad A_3 = 0. \quad (50)
\]

We can write the modified Dirac equations in the form

\[
\left( \gamma_\mu P^\mu - m e^{\gamma_5 \mathcal{A}} \right) \tilde{\psi} = 0, \quad (51)
\]

where \( P^\mu = \mathbf{i} \partial_\mu - e A_\mu \); \( e = -|e| \) and the \( \gamma \)-matrices are still chosen in the standard representation. In the field under consideration, the operators \( P_\xi, P_2, \) and \( P_3 \) are mutually commuting integrals of motion, \( [D, P_\xi] = 0, [D, P_2] = 0, [D, P_3] = 0 \), where \( D = (\gamma_\mu P^\mu - m e^{\gamma_5 \mathcal{A}}) \).
Let us present the function $\Psi$ in the form
\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} e^{-iEt}
\]
and use Hamilton’s form of Dirac equations
\[
H\Psi = E\Psi
\]
where
\[
H = \left( \alpha \mathbf{P} \right) + \beta m_1 + \gamma \beta\gamma m_2.
\]
It is useful to introduce the change of variables [32]
\[
\psi_i(x_1, x_2, x_3) = e^{i\xi_12 + i\xi_31} \Phi_i(x_1),
\]
where $i = 1–4$. We can obtain the following system of equations:
\[
(E \mp m_1) \Phi_{1,3} + i R_2 \Phi_{4,2} - (p_3 \mp m_2) \Phi_{3,1} = 0,
\]
where $R_2 = \left[ \frac{\partial}{\partial x_1} + (p_2 + eH) \right]$;
\[
(E \mp m_1) \Phi_{2,4} + i R_1 \Phi_{3,1} + (p_3 \pm m_2) \Phi_{4,2} = 0.
\]
Here, $R_1 = \left[ \frac{\partial}{\partial x_1} - (p_2 + eH) \right]$, the upper sign relates to the components of the wave function with the first index, and the lower mark relates to the components with the second index.

Next, it is convenient to go to the dimensionless variable
\[
\rho = \sqrt{\gamma} x_1 + p_2/\sqrt{\gamma},
\]
where $\gamma = |e|H$, and equations (53) and (54) take the form
\[
(E \mp m_1) \Phi_{1,3} + i \sqrt{\gamma} \left( \frac{d}{d\rho} + \rho \right) \times \Phi_{4,2} - (p_3 \mp m_2) \Phi_{3,1} = 0;
\]
\[
(E \mp m_1) \Phi_{2,4} + i \sqrt{\gamma} \left( \frac{d}{d\rho} - \rho \right) \times \Phi_{3,1} + (p_3 \pm m_2) \Phi_{4,2} = 0.
\]
The general solution of this system can be represented in the form of the Hermite functions
\[
u_n(\rho) = \left( \frac{\gamma^{1/2}}{2^n n^{1/2}} \right) e^{-\rho^2/2} \mathcal{H}_n(\rho),
\]
where $H_n(x)$ standardizes the Hermite polynomials:
\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},
\]
and $n = 0, 1, 2...$. Note that the Hermite functions are satisfied to the recurrent relations:
\[
\left( \frac{d}{d\rho} + \rho \right) \nu_n = (2n+1)\nu_{n+1} - 2n \nu_n;
\]
\[
\left( \frac{d}{d\rho} - \rho \right) \nu_{n-1} = -2n \nu_n.
\]
It is easy to see from (58) and (59) that
\[
\left( \frac{d}{d\rho} - \rho \right) \nu_n = -2n \nu_n
\]
and hence (see, for example [32])
\[
R_1 R_2 = -2m.
\]
Substituting next in (56) and (57)
\[
\Phi = \begin{pmatrix} C_1 u_{n-1}(\rho) \\ i C_2 u_n(\rho) \\ C_3 u_{n+1}(\rho) \\ i C_4 u_n(\rho) \end{pmatrix},
\]
one can find that coefficients $C_i (i = 1–4)$ are determined by algebraic equations
\[
(E \mp m_1) C_{1,3} - (2m)^{1/2} C_{2,4} - (p_3 \mp m_2) C_{3,1} = 0;
\]
\[
(E \mp m_1) C_{2,4} - (2m)^{1/2} C_{1,3} + (p_3 \pm m_2) C_{4,2} = 0.
\]
The equality to zero of the determinant of this system leads to a spectrum of energy of the non-Hermitian Hamiltonian in the form
\[
E = \pm \sqrt{m_1^2 - m_2^2 + 2m + p_3^2},
\]
where $n = 0, 1, 2...$, and taking into account $m^2 = m_1^2 - m_2^2$, we can see the result, which also (see (34)) coincides with the eigenvalues of the Hermitian Hamiltonian that is described by the relativistic Landau levels (see, for example [32]). Note that the use of exact solutions in the magnetic field still continues to give a number of interesting results. In particular, in [33], it was possible to establish an exact mapping between this relativistic model on one hand and different combinations of Jaynes-Cummings and anti-Jaynes-Cummings interactions [34] on the other. This nontrivial application allowed one to obtain as a consequence the important results, which were widely used by the quantum optics community.

The coefficients, $C_i$, may be determined if one uses the operator of polarization in the form of the third component of the polarization tensor in the direction of the magnetic field,
\[
\mu_3 = m_1 \sigma_3 + \mu_2 \left[ \sigma_3 \right]_3
\]
where matrices
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.
\]
It is easy to see that bispinor $C$ can be written as
\[
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \left( \frac{1}{2\sqrt{2}} \right) \begin{pmatrix} \cosh (\theta/2) \Phi_1 + \sinh (\theta/2) \Phi_3 \\ \cosh (\theta/2) \Phi_2 + \sinh (\theta/2) \Phi_4 \\ \sinh (\theta/2) \Phi_1 + \cosh (\theta/2) \Phi_3 \\ \sinh (\theta/2) \Phi_2 + \cosh (\theta/2) \Phi_4 \end{pmatrix},
\]
where
\[
\Phi_1 = \sqrt{1 + \zeta m_1} \sin (\pi/4 + \lambda/2),
\]
\[
\Phi_2 = \sqrt{1 - \zeta m_1} \sin (\pi/4 - \lambda/2).
\]
\[ \Phi_3 = \zeta \sqrt{1 + \zeta m/p_\perp} \sin(\pi/4 - \lambda/2), \]
\[ \Phi_4 = \sqrt{1 - \zeta m/p_\perp} \sin(\pi/4 + \lambda/2). \]

Here, \( \mu_3\psi = \zeta \kappa \psi \), \( k = \sqrt{\beta^2 + m^2} \) and \( \zeta = \pm 1 \), which corresponds to the orientation of the fermion spin: along (+1) or opposite (−1) to the magnetic field, and parameter \( \lambda \) obeys the condition \( \cos \lambda = p_\perp/E \). The functions \( \sin(\theta/2) \) and \( \cosh(\theta/2) \) are defined by relation (33).

5. Non-Hermitian modified Dirac–Pauli model in the magnetic field

In this section we will explore ways to describe the motion of Dirac particles if their own magnetic moments are different from the Bohr magneton. As shown by Schwinger [35], the Dirac equation of particles in the external electromagnetic field, \( A^\text{ext} \), taking into account the radiative corrections, may be represented in the form

\[ (P_\gamma - m)\Psi(x) - \int M(x, y)A^\text{ext}(y)\Psi(y)dy = 0, \tag{64} \]

where \( M(x, y)A^\text{ext} \) is the mass operator of the fermion in an external field. From equation (64), by means of expansion of the mass operator in series according to \( eA^\text{ext} \) with precision not over the linear field terms, one can obtain the modified equation (see, for example, [32]). This equation preserves the relativistic covariance and is consistent with pauli’s phenomenological equation, which was obtained in his early papers.

Now let us consider the model of massive fermions with \( g_5 \)-extension of mass \( m \rightarrow m_1 + \gamma_5 m_2 \), taking into account the interaction of their charges and AMM with the electromagnetic field, \( F^\mu_\nu \):

\[ \left( \gamma^\mu P_\mu - m_1 - \gamma_5 m_2 - \frac{\Delta \mu}{2} \sigma^{\mu \nu} F^\nu_{\phantom{\nu} \mu} \right) \Psi(x) = 0, \tag{65} \]

where \( \Delta \mu = (\mu - \mu_0) = \mu_0 (g - 2)/2 \). Here \( \mu \) is the magnetic moment of a fermion, \( g \) is the fermion gyromagnetic factor, \( \mu_0 = |e|/2m \) is the Bohr magneton, and \( \sigma^{\mu \nu} = i/2(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \). Thus phenomenological constant \( \Delta \mu \), which was introduced by Pauli, is part of the equation and gets the interpretation with the point of view of QFT.

The Hamiltonian form of (65) in the homogenou magnetic field is as follows:

\[ i\frac{\partial}{\partial t} \bar{\Psi}(r, t) = H_{\Delta \mu} \bar{\Psi}(r, t), \tag{66} \]

where

\[ H_{\Delta \mu} = \vec{\alpha} \vec{p} + \beta \left( m_1 + \gamma_5 m_2 \right) + \Delta \mu \beta (\vec{\sigma} \vec{H}). \tag{67} \]

Given the quantum electrodynamic contribution to the AMM of an electron with accuracy up to \( e^2 \) order, we have \( \Delta \mu = \alpha \mu_0 \), where \( \alpha = e^2/137 \) is the fine-structure constant, and we still believe that the potential of an external field satisfies the free Maxwell equations.

Note that now the projection operator of the fermion spin at the direction of its movement, \( \vec{\sigma} \vec{p} \), does not commute with the Hamiltonian (67), and hence it is not the integral of motion. The operator, which commutes with this Hamiltonian, remains \( \mu_3 \) (see (62)). Subjecting the wave function, \( \tilde{\Psi} \), to the requirement that it be an eigenfunction of the operator polarization (62) and the Hamilton operator (67), we can obtain

\[ \mu_3 \psi = \zeta \kappa \psi, \quad \mu_3 = \begin{pmatrix} m_1 & 0 & 0 & -p_1 - i p_2 \\ 0 & -m_1 & -p_1 + i p_2 & 0 \\ 0 & m_1 & 0 & 0 \\ p_1 + i p_2 & 0 & 0 & -m_1 \end{pmatrix}, \tag{68} \]

where \( \zeta = \pm 1 \) are characterized by the projection of fermion spin in the direction of the magnetic field:

\[ \Delta \mu \psi = E \tilde{\Psi}, \]

\[ H_{\Delta \mu} = \begin{pmatrix} m_1 + \Delta \mu & 0 & 0 & p_1 - i p_2 \\ 0 & m_1 - \Delta \mu & -p_1 + i p_2 & m_2 + p_3 \\ 0 & m_1 - \Delta \mu & -p_1 + i p_2 & m_2 + p_3 \\ 0 & m_1 & 0 & -m_1 \end{pmatrix}. \tag{69} \]

Performing calculations here is in many ways reminiscent of similar calculations carried out in the previous section. As a result of using a modified Dirac–Pauli equation, one can also find the exact solution for the energy spectrum:

\[ E(\zeta, \psi, 2\gamma_\mu, H) = \pm \sqrt{p_3^2 - m_3^2 + \sqrt{m_1^2 + 2\gamma_\mu + \zeta \Delta \mu H}} \tag{70} \]

and for eigenvalues of the operator polarization, \( \mu_3 \), we can write the form

\[ k = \sqrt{m_1^2 + 2\gamma_\mu}. \tag{71} \]

Note that formula (70) is valid not only for charged fermions, but also for the neutral particles possessing AMM. In this case, one must simply replace the value of the quantized transverse momentum of a charged particle in a magnetic field with the ordinary value, \( 2m \rightarrow p_1^2 + p_3^2 = p_1^2 \), and as a result we have

\[ E(\zeta, \psi, \psi, H) = \pm \sqrt{p_3^2 - m_3^2 + \sqrt{m_1^2 + p_3^2 + \zeta \Delta \mu H}}. \tag{72} \]

It is easy to see that in the case of \( \zeta = -1 \) (fermion spin is oriented against the magnetic field) in the linear approximation of the intensity of the magnetic field, we can find that the real values of the energy spectrum may be obtained only if the intensity of the magnetic field is constrained by the value

\[ H \leq \frac{p_0^2}{2 \Delta \mu k}, \tag{73} \]

where \( p_0 = \sqrt{m_1^2 - m_3^2 + p_3^2 + p_4^2} \) is the ordinary Hermitian energy of the particles.
Hence, from (72) in the case \( \zeta = -1 \) and \( p_+ = p_3 = 0 \), which correspond to the fermion at rest, we can obtain \( H \leq H_{\text{max}} \). In the linear approximation on the intensity of the magnetic field from (73), we can find

\[
H_{\text{max}} = \frac{m^2}{2m_1\Delta \mu}.
\]

(74)

From (74), we can easily verify that a direct consequence of expression (70) will be the reality of the eigenvalues of energy when values for \( H \leq H_{\text{max}} \) must be accomplished. Using (74), we can also formulate a new condition of unbroken \( PT \) symmetry for the problem of a fermion with a non-Hermitian \( \gamma_5 \)-extension of the mass and a non zero AMM in the intensive magnetic field. This condition replaces condition (3) and now may be represented in the form

\[
m_1^2 - m_2^2 \geq 2m_1\Delta \mu H.
\]

(75)

So we can see that there exists a maximum value for the magnetic field, \( H_{\text{max}} \), the excess of which in the case where \( \zeta = -1 \) leads to a full loss of the reality-spectrum of energies.

It is easy to see that in the case where \( \Delta \mu = 0 \) in (70), one can obtain expression (61). In addition, it should be emphasized that in the expression analogous to (70) in the frame of the ordinary Dirac–Pauli approach, one can obtain putting \( m_2 = 0 \) and \( m_1 = m \) (Hermitian limit):

\[
E(\zeta, p_3, 2\gamma m, H) = \pm \sqrt{p_3^2 + \left[ \sqrt{m^2 + 2\gamma m + \zeta \Delta \mu H} \right]^2}.
\]

(76)

Note that in [36] a result analogical to (76) was obtained by of using the ordinary Dirac–Pauli approach. Directly comparing the modified formula (70) in the Hermitian limit with the result in [36] shows their coincidence. It is easy to see that expression (70) is dependent on parameters \( m_1 \) and \( m_2 \) separately, which are not combined into a mass of particles, \( m = \sqrt{m_1^2 - m_2^2} \), that essentially differs from the examples that were previously considered in sections 3 and 4.

Thus, in contrast to (34) and (61), here the calculation of the interaction AMM of the fermions with the magnetic field allows one to consider the possibility of experimentally studying the effects \( \gamma_5 \)-extensions of a fermion mass. In particular, if we suggest that \( m_2 = 0 \) and hence \( m_1 = m \), we obtain, as noted earlier, the Hermitian limit. But taking into account expressions (18) and (19), we find that the energetic spectrum (70) is expressed through the fermion mass, \( m \), and the value of the maximal mass, \( M \). Thus, taking into account that the interaction of the AMM with the magnetic field removes the degeneracy of the spin variable, we can obtain the energy of the ground state

\[
E(-1, 0, 0, H, x) = m - \frac{\Delta \mu}{\mu_0 H_e} \left( 1 + x^2/8 + 7x^4/128 \right) + \left( \frac{\Delta \mu H}{m} \right)^2,
\]

(77)

where \( x = m/M \), the upper sign corresponds to the ordinary particle, and the lower sign defines their ‘exotic’ partners.

Let us now turn to a more detailed consideration of the fermion energy in the ground state in the external field. The nontrivial function (70) depends on the parameters \( x = m/M \) and \( H \). For reasons outlined above, we can obtain the effect of the magnetic field on the energy state of the ordinary fermion with small mass \( x \ll 1 \) (see figure 1) we can obtain in the form

\[
E(-1, 0, 0, H, x) = m - \frac{\Delta \mu}{\mu_0 H_e} \left( 1 + x^2/8 + 7x^4/128 \right) + \left( \frac{\Delta \mu H}{m} \right)^2,
\]

(78)

where \( H_e = m^2/e = 4.41 \cdot 10^{13} \) Gauss is the quantizing magnetic field for an electron [32].

On the other hand, in the case of ‘exotic’ particles in a similar limit, \( x \ll 1 \), the result is significantly different (see figure 2):

\[
E(-1, 0, 0, H, x) = m - \frac{\Delta \mu}{\mu_0 H_e} \left( 1 + 2x^2/8 + (\Delta \mu H/m)^2 \right),
\]

(79)
that the Dirac Hamiltonian of a particle with a

In the research presented in previous sections, it was shown

6. Summary and conclusions

From (79) one can see that the field corrections in this case are substantially increased as $1/x = M/m \gg 1$.

One can also see that the changes in parameters $\gamma_{m_1}$ and $\gamma_{m_2}$ occur in such a way that at the point $x = 1$ ($m = M$), the branches of ordinary and exotic particles are crossed. In figures 1 and 2, the dependencies of $m_1^2/m$ and $m_2^2/m$ on the parameter $x = m/M$ are represented, and one can clearly see the justification of this fact.

As equation (70)—and following from it formulas (78) and (79)—would fare for maximal intensities of magnetic field, we have

$$H_{\text{max}} = \frac{\mu_0}{\Delta \mu} \frac{m}{m_1} H_c. \quad (80)$$

Hence, in the intensive magnetic fields, after accounting for the vacuum magnetic moment of particles can lead to a substantial change of borders of $PT$ symmetry

$$2m_1 \Delta \mu H/m \leq m \leq M. \quad (81)$$

Note that a considerable increase in this correction is connected with the possible contributions from the so-called exotic particles. It is very nontrivial that the fermions of different categories may be significantly separated with the help of the intensity of the magnetic field.

The intriguing difference in the second type of particles is that they are described by different modified Dirac equations. So, if in the first case (82), the equation of motion under the transition $M \to \infty$ leads to the standard Dirac equation, in the second case such a transition is not there.

Thus, it is shown that the main progress of the construction of the fermion model with $\gamma$-mass term, which we obtained algebraically, consists of describing the new energetic scale, which is defined by the parameter $M = m_1^2/2m_2$. This value on the scale of the masses is a point of transition from ordinary particles to exotic particles. Furthermore, the description of the exotic particles in the algebraic approach turned out essentially the same as in the model with a maximal mass, which was investigated by Kadyshevsky et al on the basis of the geometrical approach (see, for example, [28, 30, 31]).

Note that although the energy spectra of the fermions in some cases make them indistinguishable from the spectrum of corresponding Hermitian Hamiltonian $H_0$, we found examples where the energy of the fermions is clearly dependent on non-Hermitian characteristics. We are talking about the consideration of the interaction of the AMM of fermions with a magnetic field. In this case, we obtained the exact solution for the energy of fermions (see (70) and (72)).

Thus, for the case of ultracold polarized ordinary electronic neutrinos (see (72)) with precision not over the linear field terms, we can write

$$E \left( -1, 0, 0, H, m_0/M \ll 1 \right) = m_0 \sqrt{1 - \frac{\mu_0}{\mu_0 H_c}}. \quad (84)$$

In the second area, we are dealing with so-called exotic partners of ordinary particles, for which (2) is still accomplished, but one can write

$$m_1/\sqrt{2} \leq m_2 \leq m_1. \quad (83)$$

In (82) the mass parameters are limited by the terms

$$0 \leq m_2 \leq m_1/\sqrt{2}. \quad (82)$$
However, in the case of exotic electronic neutrinos, we have

\[ E(-1, 0, 0, H, m_\nu/M \ll 1) \approx m_\nu \sqrt{1 - \frac{\mu_\nu}{\mu_0} \frac{2M_H}{m_\nu H_c}}, \quad (85) \]

It is well known [37, 38] that in the minimally extended SM one-loop radiative corrections, which form the magnetic moment of the neutrino that is proportional to the neutrino mass

\[ \mu_\nu = \frac{3}{8\sqrt{2}\pi^2} |e| G_F m_\nu = \left( 3 \times 10^{-10} \right) \mu_0 \left( \frac{m_\nu}{1\text{ eV}} \right), \quad (86) \]

where \( G_F \) is the Fermi coupling constant and \( \mu_0 \) is a Bohr magneton. In addition, the discussion of the problem of measuring the mass of neutrinos (either active or sterile) shows that for an active neutrino model, we have \( \sum m_\nu = 0.320 \text{ eV} \), whereas for a sterile neutrino, we have \( \sum m_\nu = 0.06 \text{ eV} \) [39].

One can also estimate the change in the border of the region of unbroken \( PT \) symmetry due to the shift of the lowest energy state in the magnetic field. Using formulas (84) and (85), we obtain corresponding regions of undisturbed \( PT \) symmetry in the form (see also equations (80) and (81)),

\[ H_{\nu(\text{ordinary})} \leq \frac{\mu_0}{\mu_\nu} H_c; \quad (87) \]

\[ H_{\nu(\text{exotic})} \leq \frac{m_\nu \mu_0}{2M \mu_\nu} H_c. \quad (88) \]

Indeed, let us consider the following parameters of the neutrino: The mass of the electronic neutrino is equal to \( m_\nu = 1 \text{ eV} \) and the magnetic moment is equal to (86). If we assume that the values of the mass and the magnetic moment of the exotic neutrino are identical to the parameters of ordinary neutrinos, we can obtain the estimates of the border area of undisturbed \( PT \) symmetry for (87) in the form

\[ H^{\text{max}}_{\nu(\text{ordinary})} = \frac{\mu_0}{\mu_\nu} H_c, \quad (89) \]

However in the case of (88), the situation may change radically:

\[ H^{\text{max}}_{\nu(\text{exotic})} = \frac{m_\nu \mu_0}{2M \mu_\nu} H_c. \quad (90) \]

In comparison with (89), where the experimentally possible field corrections are extremely small, one can see that the critical value of the magnetic field (90) is attainable in the sense of ordinary terrestrial experiments.

We do not know if the upper limit of the spectrum masses of elementary particles is equal to the Plank mass [29] or not, but experimental studies of this variant of theory at high energies today are hardly even discussed. However, the contemporary precision of alternative laboratory measurements at low energy in the intensive magnetic field may, in principle, allow one to achieve the required values for exotic particles, in case they really exist. Thus, the obtained formulas (85–90) allow us to be convinced not only of the existence of the non-Hermitian mass, but also of the so-called exotic particles, because these phenomena are inextricably related.

Note also that intensive magnetic fields exist near and within a number of space objects. So, a magnetic field with an intensity on the order of \( 10^{12} \) to \( 10^{13} \) Gauss has been observed near pulsars. Here we may also include the recent discovery of objects such as soft repeated gamma-ray bursts and anomalous x-ray pulsars. For these objects, magnetorotational models are proposed, and they are called magnetars. It has been shown that for such objects, magnetic fields with intensities up to \( 10^{15} \) Gauss are achievable [40]. It is very important that the share of magnetars in the general population of neutron stars reaches \( 10\% \). In this regard, we note that processes that involve ordinary neutrinos, and especially their possible ‘exotic partners,’ in the presence of such strong magnetic fields can have a significant influence on the processes that may determine the evolution of astrophysical objects.

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