Self–Fourier transform and self–Fourier beams due to parabolic potential

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Abstract

We theoretically and numerically investigate the propagation of light beams including Hermite–Gaussian, Bessel–Gaussian and finite energy Airy beams in free space with parabolic potential. Expectedly, the beams undergo harmonic oscillation during propagation, but quite unexpectedly they also perform self–Fourier transform, that is, periodic change from the beam to its Fourier transform and back. The oscillating period of parity–asymmetric beams is twice that of the parity–symmetric beams. In addition to oscillation, we find that finite–energy Airy beams exhibit periodic inversion during propagation. Based on the parabolic potential, we introduce a class of optically–interesting beams that are self–Fourier beams, i.e., the corresponding Fourier transforms are themselves. We outline interesting venues for future research.

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I. INTRODUCTION

It is well known that a light beam undergoes discrete diffraction during propagation in a waveguide array, and that such diffraction is prohibited when the index of the waveguide array is appropriately modulated [1, 2]. The phenomenon is linear, that is, obtained without invoking nonlinearity in the paraxial wave equation [3]. Likewise, in free space or a linear bulk medium a light beam will diffract unless it belongs to the family of nondiffracting beams [4, 5] – a class of linear beams that attracted a lot of attention in the past few years. A celebrated member of this class is the Airy beam [4, 6]. Very recently, a new class of paraxial optical beams was found which exhibit discrete-like diffraction in free space and can be analyzed based on the generalized Bessel functions [7–9]. Since a transverse linear potential modulating refractive index can prohibit discrete diffraction of a light beam in a waveguide array and thus strongly influence the propagation dynamics, a natural question arises: Is this possible in free space? Can diffraction be eliminated by using a transverse linear potential? If there is such a potential, of what form is it? Will the beam exhibit oscillations or recurrences? How is the Fourier optics of such beams affected? These questions are addressed in this paper.

A linear potential which affects the properties of an Airy plasmon beam [10] and is used to control acceleration of Airy beams [11] has been reported. However, oscillation was not observed. An external longitudinally-dependent transverse potential will modulate the propagating trajectory of the light beam according to the potential profile [12]. Still, recent research does not answer the questions posed above. In Ref. [13], the authors discussed the propagation and transformation of Airy–Gaussian beams in a linear medium with a parabolic potential, using the ABCD matrix method. They find that the Airy–Gaussian beam oscillates during propagation. This points to the potential of a linear potential to manage light beams effectively. It is a property deserving proper attention.

In this article, we demonstrate that the simple parabolic potential answers well the questions posed. Hence, we examine here the light beam management by parabolic potential in a linear medium, theoretically and numerically. Since parabolic potential causes harmonic oscillation of light beams, it is interesting to investigate the influence of the potential on the dynamics of useful light beams, in particular Hermite–Gaussian, Bessel–Gaussian, and finite energy Airy beams. Even though there are thousands of papers dealing with parabolic
potential and linear harmonic oscillation, we believe that results obtained here have not been reported before, to the best of our knowledge.

The organization of the article is as follows. In Sec. II, we describe the problem and introduce the theoretical model of beam propagation. In Sec. III, we discuss the repercussions of the model concerning the dynamics of beams. In Sec. IV, we solve the model, obtain analytical solutions, and analyze those solutions. Based on the theoretical model, we discover a class of interesting self–Fourier beams – the beams whose Fourier transforms are the beams themselves – and present them in Sec. V. Although the above results are obtained for one–dimensional beams, with appropriate changes they are applicable to two–dimensional beams as well. A brief introduction into two–dimensional case is presented in Sec. VI. With Sec. VII we conclude the paper.

II. THEORETICAL MODEL

The paraxial propagation of a beam in free space with external parabolic potential, is described by the normalized dimensionless linear Schrödinger equation

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \alpha x^2 \psi = 0, \]  

(1)

where \(x\) and \(z\) are the normalized dimensionless transverse coordinate and propagation distance, scaled by some typical transverse width \(x_0\) and the corresponding Rayleigh range \(kx_0^2\). Here, \(k = \frac{2 \pi n}{\lambda_0}\) is the wavenumber, \(n\) the index of refraction, and \(\lambda_0\) the wavelength in free space. Parameter \(\alpha < 0\) measures the depth of the parabolic potential. For our purposes, the values of parameters can be taken as \(x_0 = 100 \mu m\), \(n = 1.45\), and \(\lambda_0 = 600 \text{ nm}\) [14, 15].

Equation (1) has many well–known solutions; we will utilize ones that are of interest in the paraxial beam propagation. But before selecting any, we perform Fourier transform of Eq. (1), to obtain

\[ i \frac{\partial \hat{\psi}}{\partial z} - \alpha \frac{\partial^2 \hat{\psi}}{\partial k^2} - \frac{1}{2} k^2 \hat{\psi} = 0, \]  

(2)

where the Fourier transform is defined as

\[ \hat{\psi} = \int_{-\infty}^{+\infty} \psi e^{-ikx} dx, \quad \psi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\psi} e^{ikx} dk. \]
Obviously, Eq. (2) can be put in the same form as Eq. (1) by the transformation \( k = \sqrt{-2\alpha k'} \). Specifically, \( k = k' \) if \( \alpha = -1/2 \). Both equations carry the same information but are expressed in different spaces, direct and inverse. The eventual differences in solutions will come to the fore once the equations are presented as boundary-value problems in specific physical settings. Nonetheless, Eqs. (1) and (2) indicate that a localized light beam in direct \((x)\) and inverse \((k)\) space may share the same dynamics from a mathematical point of view. Therefore, it is natural to inquire about the following scenario in linear optical setting for a beam propagating according to Eq. (1): (i) The beam undergoes Fourier transform during propagation, and then experiences inverse Fourier transform, to reconstruct the initial beam; (ii) The process represented in (i) occurs periodically during propagation, and harmonic oscillation and recurrence take place along the propagation direction.

To present a realization of the scenario directly, we display the propagation of a Hermite–Gaussian beam, Bessel–Gaussian beam, and a finite energy Airy beam in Figs. 1(a)–1(c), respectively. Clearly, the parabolic potential imposes simple harmonic oscillation on a Hermite-Gaussian beam, as shown in Fig. 1(a), prevents discrete-like diffraction of a Bessel–Gaussian beam (Fig. 1(b)), and causes periodic inversion of a finite energy Airy beam (Fig. 1(c)). This inversion is different from the inversion of a finite energy Airy beam reported in a fiber, when the third–order dispersion is taken into account [16], and from double focusing of an Airy beam [17].

In Figs. 1(a) and 1(b), the initial beams are parity–symmetric, so an odd–integer multiple of half periods \( D_s \) of harmonic oscillation is needed for the beam to realize its Fourier transform. However, in Fig. 1(c) the initial beam is parity–asymmetric, so it inverts twice before reconstruction; as a result, the relation between oscillation periods of symmetric and asymmetric beams is \( D_{as} = 2D_s \). The corresponding Fourier transforms are located at an integer multiple of \((2m - 1)D_{as}/4\), with \( m \) being an even integer. If \( m \) is an odd integer, the rule gives the Fourier transform of the finite energy Airy beam at inversion. This demonstrates that Hermite–Gaussian, Bessel–Gaussian, and finite energy Airy beams perform self–Fourier transform, which means the Fourier transform of the beams will appear automatically during propagation based on this model. It is worth mentioning that a self–Fourier beam will be obtained from the initial beam during propagation. A self–Fourier beam means that the Fourier transform of the beam is itself. We will address the question of self–Fourier transform and self–Fourier beams in a latter section. Here, we continue the
FIG. 1. Propagation of beams in a linear medium with parabolic potential. (a) Hermite–Gaussian beam, with input \( \psi(x) = \exp(-x^2/2\sigma^2)H_0(x/\sigma) \) with \( \sigma = 5 \). (b) Bessel–Gaussian beam, input \( \psi(x) = \exp(-x^2/2\sigma^2)J_0(x) \) with \( \sigma = 10 \). (c) Finite energy Airy beam, input \( \psi(x) = \exp(ax)\text{Ai}(x) \). The parameters are \( a = 0.1 \) and \( \alpha = -0.05 \).

discussion of propagation of these beams.

III. DISCUSSION

In this section, we address the properties of the three beams during propagation. Specifically, in Fig. 2, we depict the periodic inversion and self–Fourier transform of a finite energy Airy beam during propagation. In Figs. 2(a1) and 2(b1), \( \alpha = -0.5 \) and \( k' = k \), so that the beam intensities in real domain and inverse domain are identical. In Figs. 2(a2) and 2(b2), \( \alpha = -0.05 \) and \( k' = \sqrt{10}k \); therefore, there is a transverse scaling between the intensities in
real and inverse domains. Using Parseval’s theorem
\[ \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{\psi}(k)|^2 dk \]
and the gauge transformation, we can obtain the Fourier transform of the three initial beams in real space.

FIG. 2. Periodic inversion and self-Fourier transform of a finite energy Airy beam during propagation. (a) Real space. (b) Inverse space. The parameters are: \( a = 0.1, \alpha = -0.5 \) and \( \alpha = -0.05 \) for left panels and right panels, respectively.

For an arbitrary order Hermite–Gaussian beam \( \psi(x, 0) = \exp(-x^2/2\sigma^2)H_n(x/\sigma) \), with \( n \) the order of Hermite polynomials [in Fig. 1(a), the order is 0],
\[ \psi\left(x, z = \frac{m}{2}D_s\right) = (-i)^n \sqrt{2\pi} \sigma \exp(\alpha \sigma^2 x^2) H_n(\sqrt{-2\alpha}x), \tag{3a} \]
where \( m \) is an odd integer. If we let \( \alpha = -1/2 \) and \( \sigma = 1 \), we find Eq. (3a) to coincide completely with the initial beam. The reason is that \( \psi(x, 0) = \exp(-x^2/2)H_n(x) \) is an eigenmode of Eq. (1), with \( \alpha = -1/2 \).

Concerning the Bessel–Gaussian beam, we also consider the general case \( \psi(x, 0) = \exp(-x^2/2\sigma^2)J_n(x) \), with \( n \) being the order of the Bessel function. The corresponding Fourier transform can be obtained as a convolution of Fourier transforms of the Gaussian and Bessel functions. After some algebra, one obtains
\[ \psi\left(x, z = \frac{m}{2}D_s\right) = (-i)^n \frac{\sigma \sqrt{-2\alpha}}{\pi} \int_{-\pi/2}^{\pi/2} T_n(\sin \theta) \exp \left[ -\frac{\sigma^2}{2} \left( \sqrt{-2\alpha}x - \sin \theta \right)^2 \right] d\theta, \tag{3b} \]
where $T_n$ is the Chebyshev polynomial of the first kind. For the case in Fig. 1(b), $n = 0$.

The case of finite energy Airy beam is more interesting, but also more involved. Since the Fourier transform of $\psi(x) = \text{Ai}(x) \exp(ax)$ is

$$\hat{\psi}(k) = \exp(-ak^2) \exp \left[ \frac{a^3}{3} + \frac{i}{3} (k^3 - 3a^2k) \right],$$

the beam envelope can be written as

$$\psi(x, z = \frac{2m - 1}{4} D_{as}) = \sqrt{\frac{\sqrt{2} \alpha}{2 \pi}} \exp \left( \frac{2 \alpha a x^2 + i \pi}{4} \right) \times$$

$$\exp \left[ \frac{a^3}{3} - \frac{s}{3} \left( \left( \sqrt{2 \alpha x} \right)^3 + 3\sqrt{2 \alpha a^2 x} \right) \right].$$

(3c)

In Eq. (3c), $s = -1$ if $m$ is odd and $s = 1$ if $m$ is even. It will be verified later in the article that the harmonic oscillation period is

$$D_s = \frac{\pi}{\sqrt{-\frac{2}{\alpha}}}$$

(4a)

for parity symmetric beams (e.g., Hermite–Gaussian and Bessel–Gaussian beams), and

$$D_{as} = \pi \sqrt{-\frac{2}{\alpha}}$$

(4b)

for beams lacking parity symmetry (e.g., finite energy Airy beams).

The analytical solutions given by Eqs. (3a)–(3c) completely agree with the numerical simulations displayed in Figs. 1 and 2, as shown in Fig. 3. From Eqs. (3a)–(3c), (4a) and (4b), we can see that the harmonic oscillation, self-Fourier transform and periodic inversion are only dependent on $\alpha$; apodization factors $\sigma$ and $a$ do not affect these phenomena.

Thus far, we have found analytical expressions of propagating beams at certain distances; the procedure may also be applied to other kinds of initial light beams, regardless of their parity symmetry. This feature is guaranteed by the theory of self–Fourier transform, according to which for any Fourier–transformable beam function $f(x)$ one can form a self–Fourier function (SFF) by using the formula $g(x) = f(x) + f(-x) + \hat{f}(x) + \hat{f}(-x)$.

IV. ANALYTICAL SOLUTIONS

We now go back to Eq. (1); as mentioned, it possesses many solutions. We select ones that are relevant for the study at hand, derived by the self–similar method [18–20]. Generally,
FIG. 3. Comparison between numerical and analytical results. (a)–(c) Corresponding to Figs. 1(a)–1(c), respectively; outputs are taken at $z = D_s/2$, $D_s/2$, and $3D_{as}/4$. Perfect agreement is visible.

The solution of Eq. (1) can be written as [13, 21–23]

$$
\psi(x, z) = \int_{-\infty}^{+\infty} \psi(\xi, 0) \sqrt{H(x, \xi, z)} \, d\xi,
$$

(5)

where

$$
H(x, \xi, z) = - \frac{i}{2\pi} \sqrt{-2\alpha} \csc \left(\sqrt{-2\alpha z}\right) \times \exp \left\{ i \sqrt{-2\alpha} \cot \left(\sqrt{-2\alpha z}\right) \left[ x^2 + \xi^2 - 2x\xi \sec \left(\sqrt{-2\alpha z}\right) \right] \right\}.
$$

(6)

Combining Eqs. (5) and (6), after some algebra one ends up with

$$
\psi(x, z) = f(x, z) \int_{-\infty}^{+\infty} \left[ \psi(\xi, 0) \exp \left( ib\xi^2 \right) \right] \exp(-iK\xi) \, d\xi,
$$

(7)

where

$$
b = \sqrt{-\alpha/2} \cot \left(\sqrt{-2\alpha z}\right),$$

$$K = \sqrt{-2\alpha x} \csc \left(\sqrt{-2\alpha z}\right),$$

8
and

\[ f(x, z) = \sqrt{\frac{-iK}{2\pi x}} \exp(ibx^2). \]

One can see that the integral in Eq. (7) is a Fourier transform of \( \psi(x, 0) \exp(ibx^2) \), which makes the analysis simple. Therefore, after choosing a certain input \( \psi(x, 0) \), we can get an analytical evolution solution by finding the Fourier transform of \( \psi(x, 0) \exp(ibx^2) \).

Corresponding to the case shown in Fig. 1(a) with \( \psi(x, 0) = \exp(-x^2/2\sigma^2)H_n(x/\sigma) \), the Hermite–Gaussian beam, the solution can be written as

\[
\psi(x, z) = f(x, z) \int_{-\infty}^{+\infty} \exp(-\beta \xi^2) \frac{H_n(\xi)}{\sigma} \exp(-iK\xi) d\xi,
\]

where \( \beta = 1/(2\sigma^2) - ib \). Different from the fundamental Hermite–Gaussian beam, the general solution of Eq. (8) is nontrivial. However, (i) when \( z = mD_s/2 \) with \( m \) an odd integer, the solution in Eq. (8) is reduced to Eq. (3a), the Fourier transform of the initial beam; (ii) given certain integer \( n \), one can calculate the corresponding analytical solution. Specifically, if \( n = 0 \), we have

\[
\psi(x, z) = f(x, z) \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{K^2}{4\beta}\right)
\]

\[
= f(x, z) \sqrt{2\pi \sigma} \left(\frac{\sqrt{1 - i\sigma^2\sqrt{-2\alpha \cot(\sqrt{-2\alpha z})}}}{1 - i\sigma^2\sqrt{-2\alpha \cot(\sqrt{-2\alpha z})}}\right) \exp\left(\frac{\alpha\sigma^2x^2 \csc^2(\sqrt{-2\alpha z})}{1 - i\sigma^2\sqrt{-2\alpha \cot(\sqrt{-2\alpha z})}}\right), \quad (9a)
\]

which corresponds to Fig. 1(a). From Eq. (9a), one can find that the beam envelope is periodic in \( z \), and the period is \( \pi/\sqrt{-2\alpha} \), which is in accordance with Eq. (4a). As a matter of fact, it is known that Hermite–Gaussian functions \( G_n(x) \) are SFFs [24], with the rule of transformation \( \hat{G}_n(x) = i^{-n}G_n(x) \).

As concerns Bessel–Gaussian beams, we also begin from the case with an arbitrary order \( \psi(x, 0) = \exp(-x^2/2\sigma^2)J_n(x) \). According to Eq. (7), we have

\[
\psi(x, z) = (-i)^n \frac{f(x, z)}{\beta \sqrt{\pi}} \int_{-\pi/2}^{\pi/2} d\theta T_n(\sin \theta) \exp\left(-\frac{(K - \sin \theta)^2}{4\beta}\right). \quad (9b)
\]

Comparing Eq. (9b) with Eq. (3b), we find that they share the same oscillating period \( D_s \), hence Eq. (9b) will reduce to Eq. (3b) at odd integer multiple of \( D_s/2 \).

In the end, for the case shown in Fig. 1(c), we can also obtain the corresponding analytical
solution with the input $\psi(x, 0) = \text{Ai}(x) \exp(ax)$:

$$
\psi(x, z) = -f(x, z) \sqrt{\frac{i}{b}} \exp \left( \frac{a^3}{3} \right) \frac{\text{Ai} \left( \frac{K}{2b} - \frac{1}{16b^2} + i \frac{a}{2b} \right)}{\exp \left[ \left( a + \frac{i}{4b} \right) \left( \frac{K}{2b} - \frac{1}{16b^2} + \frac{i a}{2b} \right) \right] \exp \left[ -i \frac{K^2}{4b} - \frac{1}{3} \left( a + \frac{i}{4b} \right)^3 \right]}.
$$

Taking into account the parity–asymmetry of the finite energy Airy beam, the period is $2D_s \equiv D_{as}$. Note that Eq. (9c) is not valid when $z = (2m - 1)D_{as}/4$, because at these positions $b = 0$. To obtain the analytical solution at $z = (2m - 1)D_{as}/4$ one must directly solve Eq. (7), which gives the result reported in Eq. (3c). Therefore, the solution for this case is a combination of Eqs. (3c) and (9c).

We have assigned an explicit physical meaning to the propagating beam at certain distances, that of the Fourier transform or of the beam reconstruction. What about the beam in between? The answer is again given by Eq. (7); the term ready to be Fourier–transformed in the integral is $\psi(x, 0) \exp(ibx^2)$, which can be viewed as the initial beam modulated by a parabolic chirp. Thus, the beam at an arbitrary distance is also explicitly given: it is the Fourier transform of the initial beam with a parabolic chirp, which depends on the distance. When at certain distances the chirp disappears, one then obtains the Fourier transform of the initial beam, which corresponds to the cases displayed in Eqs. (3a)–(3c).

V. SELF–FOURIER TRANSFORM OF BEAMS

By now, it is apparent that the propagation of beams according to the linear Schrödinger equation with parabolic potential is intimately connected with the self–Fourier transform. It is simple to find a function, the Fourier transform of which is the function itself – an SFF – for example, the Gaussian function. Except Gaussians, other nontrivial SFFs have been found [24–29], such as Hermite–Gaussian functions [30] and comb functions. In fact, there are infinitely many SFFs: each function that can be written as a sum $f(x) = g(x) + g(-x) + \hat{g}(x) + \hat{g}(-x)$, where $g(x)$ is any Fourier–transformable function, is a SFF. The problem is to determine SFFs that represent viable optical beams in paraxial optics. In this section, we report a class of such beams, based on the theoretical model discussed in this paper.

From Fig. 2, one can see that the intensity profiles in real and $k$ spaces are the same in between the points $z = mD_s$ and $z = (2m + 1)D_s/2$, where $m$ is a non-negative integer.
Considering that the system is linear, the places of interest should be \( z = (2m + 1)D_s/4 \). When the beam propagates to \( z = D_s/4 \) or \( z = D_{as}/8 \), we have \( b = \sqrt{-\alpha/2} \), \( K = 2\sqrt{-\alpha}x \), and

\[
f(x) = A \exp \left( i\sqrt{-\frac{\alpha}{2}} x^2 \right)
\]

with \( A = \sqrt{-\sqrt{\alpha}/\pi} \). Therefore, Eq. (7) can be rewritten as

\[
\psi(x) = A \exp \left( i\sqrt{-\frac{\alpha}{2}} x^2 \right) \int_{-\infty}^{\infty} \psi(\xi, 0) \exp \left( i\sqrt{-\frac{\alpha}{2}} \xi^2 \right) \exp \left( -i2\sqrt{-\alpha}x\xi \right) d\xi.
\]

(10)

We let \( g(x) = \exp(i\sqrt{-\alpha/2}x^2) \), and introduce the Fourier transform operator \( \mathcal{F}[\circ](\bullet) \), in which \( \circ \) and \( \bullet \) represent the original function and the spatial frequency, respectively. Equation (10) can be recast as

\[
\psi(x) = Ag(x) \{ \mathcal{F}[\psi(\xi)](K) \otimes \mathcal{F}[g(\xi)](K) \},
\]

(11)

where \( \otimes \) represents the convolution operation, and the spatial frequency is \( K \). Take the Fourier transform of Eq. (11) with spatial frequency \( k \), we find

\[
\mathcal{F}[\psi(x)](k) = A \mathcal{F}[g(x)](k) \otimes \mathcal{F}[\mathcal{F}[\psi(\xi)](K) \otimes \mathcal{F}[g(\xi)](K)](k) = \frac{A}{-16\alpha \pi^2} \mathcal{F}[g(x)](k) \otimes \left[ \psi \left( -\frac{k}{2\sqrt{-\alpha}} \right) g \left( -\frac{k}{2\sqrt{-\alpha}} \right) \right].
\]

(12)

After some algebra, Eq. (12) can be written as

\[
\mathcal{F}[\psi(x)](k) = \frac{A}{-16\alpha \pi^2} \sqrt{\frac{i\pi}{\sqrt{-\alpha}/2}} \left[ \exp \left( i\sqrt{-\frac{\alpha}{2}} \xi^2 \right) \psi(\xi) \right] \otimes \exp \left( -i\sqrt{-2\alpha}x^2 \right),
\]

(13)

in which we let \( k = -2\sqrt{-\alpha} \). The convolution in Eq. (13) can be written as

\[
\frac{1}{2\pi} \exp \left( -i\sqrt{-2\alpha}x^2 \right) \int_{-\infty}^{\infty} \psi(\xi) \exp \left( -i\sqrt{-\frac{\alpha}{2}} \xi^2 \right) \exp \left( i2\sqrt{-2\alpha}x\xi \right) d\xi.
\]

We let \( x = ix'/\sqrt{2} \) and \( \xi = i\xi' \), then the convolution is recasted as

\[
\frac{i}{2\pi} \exp \left( i\sqrt{-\frac{\alpha}{2}} x'^2 \right) \int_{-\infty}^{\infty} \psi(\xi') \exp \left( i\sqrt{-\frac{\alpha}{2}} \xi'^2 \right) \exp \left( -i2\sqrt{-2\alpha}x'\xi' \right) d\xi',
\]

(14)

which is just the integral in Eq. (10). Therefore, the beam envelopes in real and inverse spaces have the same profile (up to a transverse scaling), i.e. the function shown in Eq. (10) is an SFF, and the beams represented by Eq. (10) are the self–Fourier beams. In light of the fact that initial beam can be arbitrary, the number of self–Fourier beams is infinite and can
be arbitrarily made. Generally, for arbitrary $\psi(x)$ propagating at $D_{s}/4$ in free space with a parabolic potential, the beam will be the self–Fourier beam. Note that if the initial beam is parity–asymmetric, the corresponding Fourier transform still has the same profile, but with an intermediate inversion. In general, the corresponding Fourier transform pair is

$$\mathcal{F}[\psi(x)](k) = \sqrt{\frac{2\pi}{\sqrt{-2\alpha}}} \psi\left(-\frac{k}{\sqrt{-2\alpha}}\right).$$  \hspace{1cm} (15)

FIG. 4. Comparison of intensities of a finite energy Airy beam at $z = D_{s}/8$ in real space and inverse space. (a) and (b) Corresponding to Figs. 2(a) and 2(b). Intensities in real and frequency spaces reference left and right $y$ scales, respectively.

In Fig. 4, we display the case with a finite energy Airy beam as the initial beam, for instance. When $\alpha = -0.5$, as shown in Fig. 4(a), the beam profiles are the same except for an inversion. In Fig. 4(b) with $\alpha = -0.05$, there is a transverse scaling $\sqrt{-2\alpha}$ in the intensity profile in the $k$ space, but the beam still shares the same profile as the beam in free space.

VI. TWO–DIMENSIONAL CASE

In the two–dimensional case, the harmonic oscillation, periodic inversion, and self–Fourier transform continue to hold. This is the consequence of the linearity of the problem, so that the two–dimensional case can be separated into two one–dimensional cases. Hence, the
extension from one dimension to two dimensions is straightforward; the propagation proceeds according to the two-dimensional linear Schrödinger equation with a two-dimensional parabolic potential,

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \alpha(x^2 + y^2) \psi = 0, \]

whose solutions, e.g., Hermite–Gaussian beams, can be written as \( \psi_{nl}(x, y) = \psi_n(x)\psi_l(y) \). The two-dimensional Hermite–Gaussian beams are the eigenfunctions of the quadratic GRIN (graded index) media, but they are also the two-dimensional SFFs.

In this section, we take only the finite energy Airy beam as an example. The input beam can be written as

\[ \psi(x, y, 0) = \text{Ai}(x)\text{Ai}(y) \exp(ax) \exp(ay). \]

We depict the propagation of the finite energy Airy beam in Fig. 5 and in the Supplementary Material [31], with \( a = 0.1 \) and the potential depth coefficient \( \alpha = -0.05 \). In Fig. 5(a), the harmonic oscillation and periodic inversion during propagation are presented using the iso-surface plot. To better see the phenomenon, we also show in Fig. 5(b) the intensity distribution at the cross section \( \phi = \pi/4 \), where \( \phi = \arctan(y/x) \) is the angular coordinate, which is quite similar to those exhibited in Figs. 1(c) and 2(a2). In addition to the harmonic oscillation and periodic inversion, Fig. 5(b) also illustrates the self–Fourier transform during propagation, a feature manifested by the applied parabolic potential.

By using the variable separation method [20], such two-dimensional problem can be reduced into two one-dimensional problems. It can be demonstrated that the oscillating period for the two-dimensional case is still \( D_{as} \), and this can be also seen from Figs. 5(a) and 5(b). In Fig. 5(c), we display a few snapshots during propagation. Specifically, the panels in Figs. 5(c1)-5(c4) show the beam intensities at \( z = 0, z = D_{as}/4, z = D_{as}/2, \) and \( z = 3D_{as}/4 \), respectively. Figure 5(c1) is the initial two-dimensional finite energy Airy beam, the intensity of which is distributed in the third quadrant. In Fig. 5(c2), one can see that the intensity is exhibiting a Gaussian profile, which is the Fourier transform of the inverted beam at \( z = D_{as}/2 \), and the intensity is located in the first quadrant, as presented in Fig. 5(c3). When the beam propagates further and reaches \( z = 3D_{as}/4 \), the beam intensity is as shown in Fig. 5(c4), that is, a Gaussian beam, which is the Fourier transform of the initial beam, as shown in Fig. 5(c1), or the beam reconstructed at \( z = D_{as} \).
FIG. 5. Propagation of two–dimensional finite energy Airy beam in linear medium with a parabolic potential (see the Supplemental Material [31] for an animated version of the propagation). (a) Iso–surface plot of the intensity during propagation. (b) Intensity distribution at the cross section $\phi = \pi/4$. (c) Intensity distribution when the beam propagates from $z = 0$, to $z = D_{as}/4$, $z = D_{as}/2$, and $z = 3D_{as}/4$, respectively. The parameters are $a = 0.1$ and $\alpha = -0.05$.

VII. CONCLUSION AND OUTLOOK

In conclusion, we have theoretically and numerically investigated the beam propagation in free space with a parabolic potential. According to this model, beam will undergo harmonic oscillation during propagation, which will suppress the discrete–like diffraction of a Bessel–Gaussian beam. For a parity–asymmetric beam, such as finite energy Airy beam, there exists periodic inversion during propagation. The oscillation period for parity–symmetric beams is half that of parity–asymmetric beams.

We also find that an arbitrary beam can realize self-Fourier transform during propagation. At half the period for parity–symmetric beams and a quarter of the period for parity–asymmetric beams, the Fourier transform of the initial beams is obtained repeatedly; at the
places in between, the beam is the Fourier transform of the initial beam with a parabolic chirp.

Last but not least, based on the theoretical model, we discover a class of self–Fourier beams. All these properties not only exhibit the importance of parabolic potential in linear optics, but broaden the understanding of recurrence in paraxial beam propagation, and also exemplify scientific significance for signal processing, imaging, microparticle manipulation, information storage, and other applications.

In the end, we mention some of the useful extensions of the theory presented here. In addition to GRIN media, to which the theory applies directly, Bose–Einstein condensates with harmonic traps are viable candidates, when the nonlinearity is weak. Talbot effect in one and two dimensions can also be presented as a self–Fourier phenomenon – in fact, as a fractional self–Fourier transform [26]. A big challenge would be to extend these ideas to nonlinear domain, since we recently demonstrated [32] that the nonlinear Talbot effect can be presented in terms of primary and secondary recurrences, which might be interpreted as self–Fourier phenomena. This should not be a too big surprise, as it is known that the hyperbolic secant function – one of the workhorses in soliton theory – is also a self–Fourier function [28].

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