HYPERBOLIC POLYNOMIALS AND RIGID MODULI ORDERS

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Abstract. A hyperbolic polynomial (HP) is a real univariate polynomial with all roots real. By Descartes’ rule of signs a HP with all coefficients nonvanishing has exactly \( c \) positive and exactly \( p \) negative roots counted with multiplicity, where \( c \) and \( p \) are the numbers of sign changes and sign preservations in the sequence of its coefficients. We consider HPs with distinct moduli of the roots. We ask the question when the order of the moduli of the negative roots w.r.t. the positive roots on the real positive half-line completely determines the signs of the coefficients of the polynomial. When there is at least one positive and at least one negative root this is possible exactly when the moduli of the negative roots interlace with the positive roots (hence half or about half of the roots are positive). In this case the signs of the coefficients of the HP are either \((+, +, -, +, -), \ldots \) or \((+, -, -), +, -), \ldots \).

Key words: real polynomial in one variable; hyperbolic polynomial; sign pattern; Descartes’ rule of signs

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1. Introduction

We consider hyperbolic polynomials (HPs), i.e. real univariate polynomials with all roots real. We assume the leading coefficient to be positive and all coefficients to be nonvanishing. The Descartes’ rule of signs applied to such a degree \( d \) HP with \( c \) sign changes and \( p \) sign preservations in the sequence of its coefficients, \( c + p = d \), implies that the HP has \( c \) positive and \( p \) negative roots counted with multiplicity. In what follows we consider the generic case when the moduli of all roots are distinct.

Definition 1. (1) A sign pattern (SP) of length \( d + 1 \) is a sequence of \( d + 1 \) (+)- and/or (−)-signs. We say that the polynomial \( Q := x^d + \sum_{j=0}^{d-1} a_j x^j \) defines (or realizes) the SP \( \sigma(Q) := (+, \text{sgn}(a_{d-1}), \ldots, \text{sgn}(a_0)) \).

(2) A moduli order (MO) of length \( d \) is a formal string of \( c \) letters \( P \) and \( p \) letters \( N \) separated by signs of inequality <. These letters indicate the relative positions of the moduli of the roots of the HP on the real positive half-line. E.g. for \( d = 6 \), to say that a given HP \( Q \) defines (or realizes) the MO \( N < N < P < N < P < N \) means that for \( \sigma(Q) \), one has \( c = 2 \) and \( p = 4 \) and that for the positive roots \( \alpha_1 < \alpha_2 \) and the negative roots \( -\gamma_j \) of \( Q \), one has \( \gamma_1 < \gamma_2 < \alpha_1 < \gamma_3 < \alpha_2 < \gamma_4 \).

(3) We say that a given MO realizes a given SP if there exists a HP which defines the given MO and the given SP.

Example 1. For \( d = 1 \), if the SP defined by a HP with a nonzero root equals \((+, +)\) (resp. \((+, -)\), then this root is negative (resp. positive).

For \( d = 2 \), a HP with roots of opposite signs and different moduli defines the SP \((+, +, -)\) with MO \( P < N \) or the SP \((+, -, -)\) with MO \( N < P \).
Remark 1. Suppose that the MO $r$ is realizable by a HP $Q$. Denote by $rP$, $rN$ (resp. $Pr$ and $Nr$) the MOs obtained from $r$ by adding to the right the inequality $< P$ or $< N$ (resp. by adding to the left the inequality $P <$ or $N <$). For $\varepsilon > 0$ sufficiently small, the product $(x - \varepsilon)Q(x)$ (resp. $(x + \varepsilon)Q(x)$) defines the MO $Pr$ (resp. $Nr$). Indeed, the modulus of the root $\pm \varepsilon$ is much smaller than any of the moduli of the roots of $Q$. In the same way, the product $-(1 - \varepsilon x)Q(x)$ (resp. $(1 + \varepsilon x)Q(x)$) defines the MO $rP$ (resp. $rN$), because the modulus of the root $\pm 1/\varepsilon$ is much larger than any of the moduli of the roots of $Q$. When several products of the form $(x \pm \varepsilon)Q(x)$ and/or $\pm (1 \pm \varepsilon x)Q(x)$ are used, then they are performed with different numbers $\varepsilon_j$ for which one has $0 < \cdots \ll \varepsilon_j + 1 \ll \varepsilon_j$.

Definition 2. A MO is rigid if all HPs realizing this MO define one and the same SP, i.e. if the MO realizes only one SP.

The aim of the present paper is to characterize all rigid MOs. From now on we assume that $c \geq 1$ and $p \geq 1$. Indeed, when all roots are of the same sign, then there is a single SP corresponding to such a MO (this is either the all-pluses SP when the roots are negative or $(+, -, +, -, +, \ldots)$ when they are positive), so according to our definition this MO should be considered as rigid. However as it excludes the question how moduli of negative roots are placed w.r.t. the positive roots on the real positive half-line, this case should be considered as trivial.

Notation 1. (1) We introduce the following four MOs:

- $r_{PN} : P < N < P < \cdots < N$ ,
- $r_{PP} : P < N < P < \cdots < P$ ,
- $r_{NP} : N < P < N < \cdots < P$ and
- $r_{NN} : N < P < N < \cdots < N$ .

The orders $r_{PN}$ and $r_{NP}$ (resp. $r_{PP}$ and $r_{NN}$) correspond to even (resp. to odd) degree $d$. In the case of $r_{PN}$ and $r_{NP}$ there are $d/2$ positive and $d/2$ negative roots, in the case of $r_{PP}$ there are $(d + 1)/2$ positive and $(d - 1)/2$ negative roots and vice versa in the case of $r_{NN}$. The MOs $r_{PN}$, $r_{NP}$, $r_{PP}$ and $r_{NN}$ are the only ones in which there are no two consecutive moduli of roots of one and the same sign hence for $d \geq 3$, they are the ones and the only ones which contain no (sub)string of the form $P < P < N$, $N < N < P$, $N < P < P$ or $P < N < N$.

(2) We are particularly interested in the following two SPs:

- $\Sigma_+ := (+, +, -, +, +, -) \ldots$ and
- $\Sigma_- := (+, -, -, +, +, -, \ldots)$ .

The main result of the paper is the following theorem:

Theorem 1. (1) For $d \geq 3$, a MO different from $r_{PN}$, $r_{NP}$, $r_{PP}$ and $r_{NN}$ is not rigid.

(2) For $d \geq 1$, the MOs $r_{PP}$, $r_{PN}$, $r_{NP}$ and $r_{NN}$ are rigid. When the roots of a HP define one of these MOs, then the SP of the HP is one of the SPs $\Sigma_{\pm}$. The exact correspondence is given by the following table (its fourth and seventh columns contain the last three signs of the SP; the degree $d$ is considered modulo 4):
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The theorem is proved in Section 3. Our next step is to consider the possibility to have equalities between moduli of roots and zeros among the coefficients.

Remark 2. We remind that a HP \( Q \) with nonvanishing constant term cannot have two consecutive vanishing coefficients. Indeed, if \( Q \) is hyperbolic, then its nonconstant derivatives are also hyperbolic and the reverted polynomial \( x^d Q(1/x) \) is also hyperbolic. Suppose that \( Q \) is hyperbolic and has two or more consecutive vanishing coefficients. Then applying derivation and reversion to \( Q \) one can obtain a polynomial of the form \( Ax^s + B, s \geq 3, A, B \in \mathbb{R}^* \), which must be hyperbolic. This, however, is impossible.

Definition 3. A sign pattern admitting zeros (SPAZ) of length \( d+1 \) is a sequence of \( d+1 \) (+)- and/or (−)-signs and eventually zeros. The first element of the sequence must be a (+)-sign. To determine the number of sign changes and sign preservation of a SPAZ one has to erase the zeros. A moduli order admitting equalities (MOAE) of length \( d \) is a formal string of letters \( P \) and \( N \) separated by signs of inequality \( \leq \). E.g., for \( d = 6 \), saying that the HP \( Q \) defines the MOAE \( N \leq N \leq P \leq N \leq P \leq N \) means that the SPAZ defined by \( Q \) and the one defined by \( Q(-x) \) have at least two and four sign changes respectively, and the constant term of \( Q \) is nonvanishing; for the moduli of its roots (with the notation from Definition 1), one has \( \gamma_1 \leq \gamma_2 \leq \alpha_1 \leq \gamma_3 \leq \alpha_2 \leq \gamma_4 \).

Example 2. For \( d = 2 \), a HP with nonvanishing constant term and two opposite roots is of the form \( F := x^2 - a^2, a \in \mathbb{R}^* \). It defines the SPAZ (+,0,−) which has one sign change and no sign preservation. One has \( F(-x) = F(x) \).

Notation 2. We denote by \( r^0_{PN}, r^0_{PP}, r^0_{NN} \) and \( r^0_{NP} \) the MOsAE obtained from the respective MOs \( r_{PN}, r_{PP}, r_{NN} \) and \( r_{NP} \) (see Notation 1) by replacing the inequalities < by inequalities \( \leq \).

Theorem 2. (1) Suppose that \( d \geq 1 \) is odd. If a HP with nonvanishing constant term defines the MOAE \( r^0_{PP} \) or \( r^0_{NN} \), then this HP has no vanishing coefficient and defines the SP as claimed by part (2) of Theorem 1.

(2) If \( d \geq 2 \) is even and a HP with nonvanishing constant term defines the MOAE \( r^0_{PN} \) or \( r^0_{NP} \), then either

(i) this HP is even hence of the form \( A \prod_{j=1}^{d/2} (x^2 - a_j^2) \), where \( A > 0, a_j \in \mathbb{R}^* \) are not necessarily distinct and the HP defines the SPAZ (+,0,−,0,+,…,0), or

(ii) this HP has no vanishing coefficient, it defines the SP as claimed by part (2) of Theorem 1 and it is not possible to represent the set of its roots as a union of \( d/2 \) couples of the form \( \{a_j, -a_j\} \).
The theorem is proved in Section [3]. In the next section we compare the problem to characterize rigid MOs to other problems arising in the theory of real univariate polynomials.

2. OTHER RELATED PROBLEMS

A rigid MO is one which uniquely defines the SP. One could ask the inverse question, whether there exist SPs which uniquely define the corresponding MOs. This question is treated in [11] and [10].

Definition 4. Given a SP of length \( d + 1 \) we define the canonical MO corresponding to it as follows. The SP is read from the back and to each encountered couple of equal (resp. different) consecutive signs one puts in correspondence the letter \( N \) (resp. \( P \)) in the MO. E.g. for \( d = 7 \), to the SP \((+ , + , - , - , + , + , -)\) there corresponds the canonical MO \( P < N < P < P < P < N < P \). The canonical MO is obtained when one constructs a HP realizing the given SP using consecutive products of the form \((x \pm \epsilon)Q(x)\), see Remark [1]. Each SP is realizable by its canonical MO, see [11]. A SP is called canonical if it is realizable only by its canonical MO.

For SPs one can use the notation \( \Sigma_{p_1, p_2, ..., p_s} \), where \( p_i \) are the lengths of the maximal sequences of equal signs. E.g. the SP in Definition 4 is \( \Sigma_{2, 2, 1, 1} \). The following necessary condition for a SP to be canonical is proved in [11]:

Theorem 3. If the SP \( \Sigma_{p_1, p_2, ..., p_s} \) is canonical, then there are no two consecutive numbers \( p_i \) which are larger than 1, and for \( 2 \leq i \leq s - 1 \), one has \( p_i \neq 2 \).

Remarks 1. (1) Thus for \( d \geq 3 \), the SPs \( \Sigma_{\pm} \) (corresponding to rigid MOs, see Notation [1] and Theorem [1]) are not canonical. For \( d = 1 \) and 2, they are canonical, see Example [1].

(2) The SPs with \( c = d \), \( p = 0 \) and \( c = 0 \), \( p = d \), are canonical. They correspond to the trivial case when all roots are positive or negative, see the lines after Definition 2.

(3) The MO corresponding to a canonical SP for which one does not have \( s = 1 \) or \( p_1 = \cdots = p_s = 1 \), with at least one number \( p_i \) larger than 2 (or with \( p_1 = 2 \) or with \( p_s = 2 \)), is not rigid. Indeed, the presence of a number \( p_i \) larger than 2 for \( 2 \leq i \leq s - 1 \) (or of \( p_1 \) or of \( p_2 \)) implies the presence of \( p_i - 1 \geq 2 \) (or of \( p_1 \) or of \( p_s \)) consecutive letters \( P \) or \( N \) in the MO, see Definition [4]. Thus in and only in the trivial case does one have a rigid MO realizing a canonical SP.

(4) The SPs of the form \( \Sigma_{1, p_2}, \Sigma_{p_1, 1}, \Sigma_{1, p_2, 1}, p_2 \geq 3 \), and \( \Sigma_{p_1, 1, p_3} \) are canonical, see [10].

The problems treated in the present paper are part of problems about real (not necessarily hyperbolic) univariate polynomials. For such a polynomial without vanishing coefficients, Descartes’ rule of signs implies that the number \( \text{pos} \) of its positive roots is not greater than the number \( c \) of sign changes in the sequence of its coefficients, and the difference \( c - \text{pos} \) is even. In the same way, for the number \( \text{neg} \) of its negative roots, one has \( \text{neg} \leq p \) and \( p - \text{neg} \in 2\mathbb{Z} \), where \( p \) is the number of sign preservations.

The problem for which couples \((\text{pos}, \text{neg})\) compatible with these requirements can one find such a real polynomial with prescribed signs of its coefficients seems to have been formulated for the first time in [2]. For \( d = 4 \), D. Grabiner has obtained
the first nontrivial result, i.e. a compatible, but not realizable couple \( \text{pos}, \text{neg} \), see [6]. In the cases \( d = 5 \) and 6 the problem has been thoroughly studied in [1] while the exhaustive answer for \( d = 7 \) and 8 can be found in [4] and [7]. For \( d \leq 8 \), all compatible, but not realizable cases, are ones in which either \( \text{pos} = 0 \) or \( \text{neg} = 0 \).

For \( d \geq 9 \), there are examples of compatible and nonrealizable couples \( \text{pos}, \text{neg} \) with \( \text{pos} \geq 1 \) and \( \text{neg} \geq 1 \), see [8] and [3]. Various problems about HPs are exposed in [9]. A tropical analog of Descartes’ rule of signs is discussed in [5].

3. Proof of Theorem 1

**Proof of part (1).** Suppose that for \( d \geq 3 \), a MO \( r \) contains the string of inequalities \( P < P < N \). Consider the two polynomials

\[
P_1 := (x - 1)(x - 1.1)(x + 3) = x^3 + 0.9x^2 - 5.2x + 3.3 \quad \text{and}
\]

\[
P_2 := (x - 1)(x - 3)(x + 3.1) = x^3 - 0.9x^2 - 9.4x + 9.3 .
\]

They define two different SPs: \( \sigma(P_1) = (+, +, +, +) \) and \( \sigma(P_2) = (+, -, -, +) \). Hence one can realize the whole MO \( r \) by two different SPs starting with the polynomials \( P_1 \) and \( P_2 \) and using \( d - 3 \) multiplications with one and the same polynomials \( x \pm \varepsilon \) or \( 1 \pm \varepsilon x \), see Remark [1]. After each multiplication one obtains again two polynomials defining different SPs. Hence \( r \) is not rigid.

If the MO contains a string of inequalities \( N < N < P \), \( N < P < P \) or \( P < N < N \), then one can consider instead of the polynomials \( P_j \), \( j = 1, 2 \), the polynomials \( S_j := -P_j(-x), T_j := x^3P_j(1/x) \) and \( R_j := x^3P_j(-1/x) \) respectively and perform a similar reasoning. The SPs defined by these polynomials are:

\[
\sigma(S_1) = (+, -, -, -) , \quad \sigma(S_2) = (+, +, -, +) , \quad \sigma(T_1) = (+, -, +, +) ,
\]

\[
\sigma(T_2) = (+, -, -, +) , \quad \sigma(R_1) = (+, +, +, -) , \quad \sigma(R_2) = (+, +, -, -) ,
\]

hence \( \sigma(S_1) \neq \sigma(S_2) , \sigma(T_1) \neq \sigma(T_2) \) and \( \sigma(R_1) \neq \sigma(R_2) \).

**Proof of part (2).** We prove part (2) of the theorem by induction on \( d \). For \( d = 1 \) and 2, the theorem is to be checked straightforwardly, see Example [1]. Suppose that part (2) of the theorem holds true for \( d \leq d_0 \), \( d_0 \geq 2 \). Set \( d := d_0 + 1 \). The sign of the constant term of a HP realizing the given MO depends only on the signs of the roots, not on the MO. So this sign is also to be checked directly.

Consider a polynomial \( Q := x^{d_0+1} + \sum_{j=0}^{d_0} b_j x^j \) defining the given MO \( \rho \) with \( d = d_0 + 1 \), with \( \rho \) standing for \( r_{PP}, r_{PN}, r_{NP} \) or \( r_{NN} \). We represent it in the form

\[
Q := (x - \varphi)(x - \psi)V , \quad \text{where} \quad V := \prod_{j=1}^{d_0-1} (x - \xi_j) = x^{d_0-1} + \sum_{j=0}^{d_0-2} c_j x^j .
\]

Here \( \varphi \) and \( \psi \) are the two roots of \( Q \) of least moduli, \( |\varphi| < |\psi| \), and \( \xi_j \) are its other roots. The signs of \( \varphi \) and \( \psi \) are opposite. Denote by \( r \) the MO defined by the polynomial \( V \). Using the notation of Remark [1] one can say that the MO defined by the polynomial \( R := (x - \psi)V \) is either \( Pr \) or \( Nr \) depending on the sign of the root \( \psi \), and the MO \( \rho \) is either \( NPr \) or \( PNr \).
We denote by $\Sigma$ the SP $\Sigma_+$ or $\Sigma_-$ according to the case and by $\Sigma'$ and $\Sigma''$ the SPs obtained from $\Sigma$ by deleting its one or two last signs respectively.

We include $Q$ into a one-parameter family of polynomials of the form

$$Z_t := (x + t\psi)(x - \psi) \prod_{j=1}^{d_0-1} (x - \xi_j), \; t \in [0, 1].$$

As $\varphi \cdot \psi < 0$ and $|\varphi| < |\psi|,$ there exists $t_* \in (0, 1)$ such that $\varphi = -t_*\psi,$ i.e. $Z_{t_*} = Q.$

For $t = 0,$ one obtains $Z_0 = xR.$ The theorem being true for $d = d_0$ and $d = d_0 - 1,$ the polynomial $R$ defines the SP $\Sigma'$, because $R$ defines the MO $Pr$ or $Nr,$ and $V$ defines the MO $r$ and the SP $\Sigma''.$

For $t = 1,$ one has $Q = (x^2 - \psi^2)V.$ Hence

$$Z_1 = x^{d_0+1} + c_{d_0-2} x^{d_0} + \left(\sum_{j=0}^{d_0} (c_j - \psi^2 c_{j+2}) x^{j+2}\right) - \psi^2 (c_1 x + c_0).$$

The signs of $c_j$ and $c_{j+2}$ are opposite (see Notation $[1]$ for the definition of the SPs $\Sigma_{\pm}$), therefore $\text{sgn}(c_j - \psi^2 c_{j+2}) = \text{sgn}(c_j).$ Thus the first $d_0$ coefficients of $Z_1$ have the signs given by the SP $\Sigma.$ This is the case of the last two coefficients as well, because $\text{sgn}(\psi^2 c_1) = -\text{sgn}(c_1)$ and $\text{sgn}(\psi^2 c_0) = -\text{sgn}(c_0).$ Hence $Z_1$ defines the SP $\Sigma.$

The coefficients of $Z_t$ are linear functions in $t \in [0, 1].$ If their signs for $t = 0$ and $t = 1$ are the corresponding components of the SP $\Sigma,$ then this is the case for any $t \in [0, 1].$ (For the constant term, one has to consider its values for $t = 1$ and for $t > 0$ close to 0.) In particular, for $t = t_*,$ the signs are the ones of the SP $\Sigma.$ This proves part (2) of the theorem.

4. Proof of Theorem $[2]$

Without loss of generality we limit ourselves to the case of monic HPs. We prove the theorem by induction on $d.$ The cases $d = 1$ and 2 are considered in Examples $[1]$ and $[2].$ For $d = 1,$ no coefficient of the HP equals 0.

Suppose now that $d \geq 3.$ We assume that there is at least one equality between a modulus of a negative root and a positive root, otherwise one can apply Theorem $[1].$ So suppose that the HP has roots $\pm\alpha, \; \alpha \neq 0,$ and the HP is of the form $S := (x^2 - a^2)Q,$ where $Q$ is a degree $d - 2$ HP without root at 0. Thus the roots of $Q$ define one of the MOs $r_{PN}^0, r_{PP}^0, r_{NN}^0$ and $r_{NP}^0,$ so one can use the inductive assumption.

If $d$ is odd, then $Q$ has no vanishing coefficient and defines one of the SPs $\Sigma_{\pm}.$ Set $Q := \sum_{j=0}^{d-2} q_j x^j.$ Then

$$S = q_{d-2} x^d + q_{d-3} x^{d-1} + \left(\sum_{j=0}^{d-4} (q_j - a^2 q_{j+2}) x^{j+2}\right) - a^2 q_1 x - a^2 q_0.$$

The first two and the last two of the coefficients of $S$ are obviously nonzero. For the others one can observe that as by inductive assumption $Q$ defines one of the SPs $\Sigma_{\pm}$ hence $q_j \cdot q_{j+2} < 0,$ one has $q_j - a^2 q_{j+2} \neq 0.$ This proves part (1) of the theorem.
If $d$ is even, then $Q$ can have a vanishing coefficient in which case $Q$ is of the form $\prod_{j=1}^{d/2}(x^2 - a_j^2)$ hence $S$ is of the form $\prod_{j=1}^{d/2}(x^2 - a_j^2)$ (with $a_{d/2} = a$) and defines the SPAZ $(+, 0, -0, +, 0, -0, \ldots)$.

If $d$ is even and $Q$ has no vanishing coefficient, then the set of its roots is not representable as a union of couples $\{a_j, -a_j\}$, $a_j \in \mathbb{R}^*$, so this is the case of $S$ as well. Moreover, using equality (4.1) in the same way as for $d$ odd one concludes that $S$ has no vanishing coefficient. Part (2) of the theorem is proved.

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