A Simple Practical Algorithm for a Computationally Hard Problem

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Abstract Two $n$-by-$n$ matrices $A$ and $B$ are permutation-similar, if there exists a permutation matrix $P$ such that $A = PBP^t$. Deciding whether two matrices are permutation-similar, is a computationally hard problem. Because, all possible permutations should be checked in general. This decision problem is the base of many main problems, such as graph isomorphism problem. In the graph isomorphism problem, we should decide whether the adjacency matrices of a pair of graphs are permutation-similar. Here, a simple and fast algorithm is proposed to solve this generally hard problem efficiently in practice. The computations show that this proposed simple algorithm which is based on both matrix spectrum and randomness, successfully, solves the graph isomorphism problem for studied instances of hard cases of this problem.

Keywords permutation similar matrices · graph isomorphism problem · graph spectrum · strongly regular graphs · polynomial time algorithm

1 Introduction

In this paper, any matrix is a $n$-by-$n$ matrix. $I$ and $J$ denote the identity matrix and all one matrix, respectively. A diagonal matrix is a matrix in which the entries outside the main diagonal are all zero. A block diagonal matrix is a square diagonal matrix in which the diagonal elements are square matrices of any size, and the off-diagonal elements are 0.

The spectrum of a matrix $A$, denoted by $Spec(A)$, is the set of eigenvalues of $A$. A permutation matrix $P$ is a matrix obtained by permuting the rows of an identity matrix according to some permutation of the numbers 1 to $n$. Two matrices $A$ and $B$ are permutation similar, if there exists a permutation matrix $P$ such that $A = PBP^t$.

Suppose that two matrices are given. We want to decide whether they are permutation similar. In other words, whether there is a bilateral bijection between the rows (and columns) of the two matrices. To check this fact, one should consider
each of the $n!$ bijections between rows of two matrices and check whether two matrices are the same by that permutation. If they are not permutation-similar, one would need to check all $n!$ bijections to realize this fact. But, even for relatively small values of $n$, the number $n!$ is unmanageably large.

The above decision problem is the center of several important problems such as graph isomorphism problem. If $A_{G_1}$ and $A_{G_2}$ are the adjacency matrices of two graphs $G_1$ and $G_2$, then deciding whether two graphs are isomorphic is equal to deciding whether their adjacency matrices are permutation-similar. Here, we introduce a practical method, based on matrix spectrum and randomness, to find a very fast and efficient answer for this decision problem.

For checking the isomorphism of graphs, the idea of comparison of the spectrum of adjacency matrices of graphs has been investigated in several papers, such as spectral determination [5], spectral characterization [9], spectrum as a graph invariant [6], comparing graphs via graph spectrum [15], graph clustering [4]. However, the existence of non-isomorphic graphs with the same spectrum always has been the bottleneck of this idea. For instance, strongly regular graphs which has long been known as hard cases of the graph isomorphism problem share the same spectrum with respect to adjacency matrix, Laplacian matrix, sign less Laplacian [14] and also normalized Laplacian matrix [3].

In this paper, The idea of matrix spectrum is developed by applying randomness. We see that the combination of randomness with matrix spectrum provide a simple, fast and efficient algorithm to tackle the problem of deciding the permutation similarity of matrices.

2 The main idea

To warm up, first we deal with a preliminary problem. The solution of this problem illuminates our approach to the main problem.

**Problem 1** Suppose that two persons have two square matrices $A$ and $B$ as their secrets. We want to decide whether $A = B$ or not. They tell us only the eigenvalues of their matrices. Also, they do any function that we want on their matrices. But, they tell us just the eigenvalues of the result. Can we decide that $A = B$ or not?

**Solution 1** We choose an arbitrary matrix $X$ (which is not a scalar matrix) and ask these two persons to add $X$ to their matrices, i.e. $A$ and $B$. Clearly, matrices $A + X$ and $B + X$ are the same, if $A = B$. In opposite, if $A \neq B$, then $A + X$ and $B + X$ are two different matrices and $\mathbb{P}\{\text{Spec}(A + X) = \text{Spec}(B + X)\} = 0$ for randomly chosen matrix $X$. Consequently, if $\text{Spec}(A + X) \neq \text{Spec}(B + X)$, then we can deduce that $A \neq B$. Because, if $A = B$, surely we have $\text{Spec}(A + X) = \text{Spec}(B + X)$. If we have $\text{Spec}(A + X) = \text{Spec}(B + X)$, we deduce that $A = B$ with probability 1. Because, the probability that two randomly chosen matrices share the same eigenvalues is zero.

The diagram of this algorithm is depicted in Fig. 1.

**Problem 2** Two matrices $A$ and $B$ are given. We want to decide whether $A$ and $B$ are permutation-similar, i.e. is there a permutation matrix $P$ such that $A = PBP^t$? Similar to the Problem 1, we are restricted to compare the eigenvalues of the matrices.
Fig. 1 Deciding whether two matrices $A$ and $B$ are equal. $X$ is a random matrix.

We want to solve the problem similar to the problem. Finding a solution for the Problem 2 is important. Because, deciding whether two matrices are permutation-similar is a hard problem, that is in worst cases all possible permutations should be checked which needs exponentially time. Additionally, some known problems, such as graph isomorphism problem, are a special case of this problem. Therefore, finding a fast and easy algorithm to solve this problem in practice is valuable. In the graph isomorphism problem, the matrices $A$ and $B$ are restricted to be symmetric zero-one matrices.

As it is depicted in Fig. 2 we need to find a suitable function $f()$. Finding such a suitable function is discussed in the next section.

Fig. 2 A simple algorithm to decide whether two matrices $A$ and $B$ are permutation-similar
3 Finding a suitable function

Here, we want to find a suitable function \( f \) for the algorithm depicted in Fig 2. Two matrices \( A \) and \( B \) are given. We should decide whether two matrices \( A \) and \( B \) are permutation-similar. We require a function \( f() \) acting on matrices \( A \) and \( B \) such that \( \text{Spec}(f(A)) = \text{Spec}(f(B)) \) holds with probability 0, if two matrices \( A \) and \( B \) are not permutation-similar. In addition, function \( f() \) should preserve the equality of the matrices spectrum, if they are permutation-similar. That is, \( \text{Spec}(f(A)) = \text{Spec}(f(B)) \) if there exists a permutation matrix \( P \) such that \( A = PBP^t \).

Therefore, the desired function \( f() \) should satisfy the following two properties,

- Property 1: \( \text{Spec}(f(A)) = \text{Spec}(f(B)) \), if \( A = PBP^t \),
- Property 2: \( \mathbb{P}\{\text{Spec}(f(A)) = \text{Spec}(f(B))\} = 0 \), if \( A \neq PBP^t \).

Is it possible to find such function \( f() \)?

Clearly, a polynomial function satisfy in the first property, but it does not satisfy the second one. The function of adding a randomized matrix, i.e. \( f(A) = A + X \), similar to what we have done for the Problem 1 satisfies the second property, but it does not satisfy the first one. Now, we attempt to find a function which satisfies both of the above properties. Similar to solution of problem 1, we want to add a randomized matrix \( X \) to matrices \( A \) and \( B \) such that the equality of their spectrums is preserved if they are permutation similar. In opposite, if they are not permutation similar, we have \( \mathbb{P}\{\text{Spec}(A + X) = \text{Spec}(B + X)\} = 0 \).

In Problem 1 it can be easily checked that if \( X \) was a randomized diagonal matrix, i.e. a matrix whose all entries except the diagonal are zero, then, again, the result remains the same. Here, we suggest a randomized diagonal matrix to be added to matrices \( A \) and \( B \).

**Definition 1** Let \( A \) be a matrix. We define \( \text{diag}(A) \) as a square diagonal matrix with the main diagonal of square matrix \( A \).

Let \( c \) be a real and \( q(x) = \sum c_i x^i \) be a finite order polynomial with randomly chosen coefficients \( c_i \). Clearly, any entry of matrix \( q(A + cJ) \) is a randomized function of all entries of \( A \). If \( A \) is a block diagonal matrix, then it is possible that some entries of \( A^k \) does not depend on all entries of \( A \). Thus, \( cJ \) is added to \( A \) to be sure that \( A + cJ \) is not a block diagonal matrix.

**Proposition 1** Two square matrices \( A \) and \( B \) are permutation similar, if and only if \( A + cJ \) and \( B + cJ \) are permutation similar for any arbitrary value of \( c \).

Therefore, without loose of generality, we can assume that matrices \( A \) and \( B \) are not block diagonal matrices. It means that any entry of \( A^k \) is a function of all entries of \( A \) for any integer \( k > 2 \).

Matrix \( \text{diag}(q(A)) \) is a diagonal matrix in which any diagonal entry is a multivariate polynomial in terms of all entries of \( A \) with randomized coefficients. We show that matrix \( \text{diag}(q(A)) \) can play the same role of matrix \( X \) in the solution of Problem 1 except some special matrices studied in the next section.
Thus, there exist a polynomial

\[ \text{Spec}(q(A + cJ) - \text{diag}(q(A + cJ))) = \text{Spec}(q(B + cJ) - \text{diag}(q(B + cJ))) \]

We know that any matrix satisfies in its characteristic polynomial and the characteristic polynomial of an \( n \)-by-\( n \) matrix is of degree \( n \). Thus, any polynomial in terms of an \( n \)-by-\( n \) matrix can be reduced to a polynomial with degree at most \( n \). Therefore, it is sufficient to suppose that the degree of polynomial \( q() \) in the above definition is at most \( n \) where \( n \) is the size of matrices \( A \) and \( B \).

According to the above definition, the strongly cospectral matrices are very special matrices. Clearly, \( A^k \) and \( \text{diag}(A^k) \) are two different matrices with different eigenspace. Thus, for any integer \( k \), \( A^k - \text{diag}(A^k) \) is a matrix whose eigenspace is neither the eigenspace of \( A^k \) nor the eigenspace of \( \text{diag}(A^k) \) in general. Therefore, strongly cospectral matrices which share the same spectrum of \( A^k - \text{diag}(A^k) \) for any \( k \), have a special structure.

**Theorem 1** We define function \( f() \) acting on \( n \)-by-\( n \) matrices, by \( f(A) = q(A) - \text{diag}(q(A)) \) where \( q(x) = \sum_{i=1}^{n} c_i x^i \) is a polynomial with randomly chosen coefficients \( c_i \). The function \( f \) satisfies in the properties 1 and 2, except the case that two matrices \( A \), \( B \) are strongly-cospectral.

**Proof** We have

\[ f(A) = q(A) - \text{diag}(q(A)) \]

If \( A \) and \( B \) are permutation similar, i.e. there is a permutation matrix \( P \) such that \( A = PB P^t \), then

\[ f(A) = q(PB P^t) - \text{diag}(q(PB P^t)) \]

For any polynomial function \( q() \), we have \( q(PA P^t) = Pq(A)P^t \). In addition, For any permutation matrix \( P \) and matrix \( A \), we have \( \text{diag}(PA P^t) = P\text{diag}(A)P^t \). Thus,

\[ = Pq(B)P^t - P\text{diag}(q(B))P^t = P(q(B) - \text{diag}(q(B)))P^t \]

Therefore, \( \text{Spec}(f(A)) = \text{Spec}(P f(B)P^t) = \text{Spec}(f(B)) \). It means that the function \( f \) satisfies property 1.

Now, we show the satisfaction of property 2.

Let \( A_{q()} = q(A) - \text{diag}(q(A)) \) and \( B_{q()} = q(B) - \text{diag}(q(B)) \) where \( q(x) = \sum_{i=1}^{n} c_i x^i \).

Let \( \lambda_1 \) be an eigenvalue of \( A_{q()} \) for a randomly chosen polynomial \( q_1() \). We show that the probability that \( \lambda_1 \) is an eigenvalue of \( B_{q()} \) is zero.

According to the assumption, the matrices \( A \) and \( B \) are not strongly-cospectral. Thus, there exist a polynomial \( q_0() \) and real \( c_0 \) such that \( \text{Spec}(q_0(A + c_0J) - \text{diag}(q_0(A + c_0J))) \neq \text{Spec}(q_0(B + c_0J) - \text{diag}(q_0(B + c_0J))) \). Using proposition \( \square \) we substitute \( A \) by \( A + c_0J \) and \( B \) by \( B + c_0J \). Now, we have \( \text{Spec}(q_0(A) - \text{diag}(q_0(A))) \neq \text{Spec}(q_0(B) - \text{diag}(q_0(B))) \). Thus,

\[ F = \det(A(q) - \lambda I) - \det(B(q) - \lambda I) \]
is a non-zero finite order multivariate polynomial in terms of $c_1, \ldots, c_n$ and $\lambda$. Due to Schwartz-Zippel lemma, for a finite order non-zero multivariate polynomial $F(x_1, \ldots, x_k)$, the probability that $F(r_1, \ldots, r_k) = 0$ holds for randomly chosen $r_1, \ldots, r_k$ is zero. Therefore, the probability that $F(q_1(\lambda), \lambda_1) = 0$ holds is zero. Since $\det(B(q_1) - \lambda I) = \det(A(q_1) - \lambda I) = F(q_1(\lambda), \lambda_1)$ and $\det(A(q_1) - \lambda I) = 0$, the probability that $\lambda_1$ is also an eigenvalue of $B(q_1(\lambda))$ is zero. Consequently, for a polynomial $g()$ with randomly chosen coefficients, we have

$$\mathbb{P}\{\text{Spec}(q(A) - \text{diag}(q(A))) = \text{Spec}(q(B) - \text{diag}(q(B)))\} = 0$$

, if $A$ and $B$ are not strongly-cospectral and $A \not= PBP^t$. □

Here, $\text{diag}(A)$ plays the role of a random matrix similar to matrix $X$ in the problem $\text{P}$ Thus,

$$\mathbb{P}\{\text{Spec}(q(A) - \text{diag}(q(A))) = \text{Spec}(q(B) - \text{diag}(q(B)))\} = \mathbb{P}\{\text{Spec}(q(A) - X) = \text{Spec}(q(B) - Y)\} = 0$$

, if $A$ and $B$ are not strongly cospectral and $A \not= PBP^t$.

**Corollary 1** The computational complexity of checking that two matrices $A_1$ and $A_2$ are permutation-similar is equal to the computational complexity of checking the equality of $\text{Spec}(f(A_1)) = \text{Spec}(f(A_2))$, provided that $A_1$ and $A_2$ are not strongly-cospectral.

Since the spectrum of a symmetric $n$-by-$n$ matrix is computable in time $O(n^3)$, the permutation similarity can be checked in time $O(n^3)$ for symmetric matrices except the case that the two matrices are strongly-cospectral.

The adjacency matrix of a graph is symmetric and the spectrum of a symmetric matrix is computable in time $O(n^3)$. Thus, graph isomorphism problem is solvable in time $O(n^3)$ for any two graphs which are not strongly-cospectral. The computations shows that using $q(x) = x + .1x^2 + .01x^3$, all graphs with at most 9 vertices are distinguished by $\text{Spec}(f(A))$. That is, $\text{Spec}(f(A))$ discriminates all graphs with at most 9 vertices. The spectrum of adjacency matrix $A$ can distinguish just 81 percent of 9-vertex graphs.

### 4 Graph Isomorphism Problem

We call two graphs are strongly cospectral, if their adjacency matrices are strongly cospectral. Here, for a graph $G$, $\text{Spec}(G)$ denotes $\text{Spec}(f(A))$ where $A$ is its adjacency matrix and $f()$ is a function defined in Theorem 1. We saw that the graph isomorphism problem can be solved using $\text{Spec}(G)$ for any two graphs which are not strongly cospectral. Therefore, it is sufficient to find a solution for strongly cospectral graphs. According to the suggested method in the previous section, to decide whether two matrices $A$ and $B$ are permutation-similar, it is sufficient to compare the spectrum of $f(A)$ and $f(B)$. Therefore, the computational complexity of solving the graph isomorphism problem for any two $n$-vertex graphs, except the exceptional cases, would be the same complexity of computing the eigenvalues of the symmetric $n$-by-$n$ matrices, that is $O(n^3)$. Therefore, except the cases that
matrices are strongly-cospectral, the graph isomorphism is solvable in time $O(n^3)$. First, we ask whether there exist strongly-cospectral graphs. According to the definition of strongly cospectral graphs, they have a special structure. The following lemma shows that strongly regular graphs with the same parameters provide such graphs. A $k$-regular graph with $n$ vertices is strongly regular $\text{SRG}(n,k,r,s)$, if every two adjacent vertices have $r$ common neighbors and every two non-adjacent vertices have $s$ common neighbors \cite{2}. It has long been recognized that strongly regular graphs are hard cases of the graph isomorphism problem.

**Lemma 1** Strongly regular graphs with the same parameters are strongly-cospectral. 

Let $G_1$ and $G_2$ be two strongly regular graphs with the same parameters $(n,k,r,s)$ and with, respectively, adjacency matrices $A_1$ and $A_2$. Now, we show that $A_1$ and $A_2$ are strongly-cospectral. First, we show that for any integer $k$ and real $c$, $\text{diag}((A_1 + cJ)^k) = \text{diag}((A_2 + cJ)^k)$ is a multiple of identity matrix, i.e. $\alpha c s J$.

We know that the adjacency matrix $A$ of a strongly regular graph $G$ with parameters $(n,k,r,s)$ satisfies in $A^2 + (s-r)A + (s-k)I = sJ$. Using this relation, we have $J = 1/s(A^2 + (s-r)A + (s-k)I)$. Thus, $A + cJ = c/s(A^2 + (s-r+s/c)A + (s-k)I)$. Consequently, $(A + cJ)^k$ is a polynomial in terms of $A$. In addition, according to $A^2 + (s-r)A + (s-k)I = sJ$, any power of $A$, i.e. $A^k (k > 1)$, can be represented in terms of $A$, $I$ and $J$. Therefore, $(A + cJ)^k$ can be stated in the form of $a_k c A + b_k c I + d_k c J$. All the main diagonal entries of $A$ are zero and all the main diagonal of $I$ and $J$ are equal to $1$. Therefore, all the main diagonal entries of $A^k$ are equal to $b_k c + d_k c$ and are the same for all adjacency matrices of strongly regular graphs with the same parameters. Therefore, $\text{diag}((A + cJ)^k)$ does not depend on matrix $A$ and is the same for all strongly regular graphs with the same parameters. Therefore, if $A_1$ and $A_2$ are adjacency matrices of two strongly regular graphs with the same parameters, then $\text{diag}((A_1 + cJ)^k) = \text{diag}((A_2 + cJ)^k) = (b_k + c_k)I$ for any real $c$ and integer $k$. It results that $\text{diag}(q(A_1 + cJ)) = \text{diag}(q(A_2 + cJ)) = \alpha I$ for any $\alpha \in \mathbb{R}$. Therefore,

$$\text{Spec}(q(A + cJ)) - \text{diag}(q(A + cJ)) = \text{Spec}(q(A + cJ)) - \alpha I$$

Thus, we have $\text{Spec}(q(A_1 + cJ)) - \text{diag}(q(A_1 + cJ)) = \text{Spec}(q(A_1 + cJ)) - \alpha I = \text{Spec}(q(A_2 + cJ)) - \alpha I = \text{Spec}(q(A_2 + cJ)) - \text{diag}(q(A_2 + cJ))$ for any polynomial $q()$ and real $c$, i.e. $A_1$ and $A_2$ are strongly-cospectral.\square

According to the above lemma, strongly regular graphs with the same parameters are exceptional cases of Theorem \cite{1}. Thus, the suggested method based on eigenvalues is not solely sufficient for such graphs. However, we will show that the suggested method by some modification can be applied for such graphs, too.

What makes such graphs to be the exceptional cases of Theorem \cite{1} is their special structure. Therefore, we can overcome this problem by manipulating their structure. Fixing a vertex $v$ of graph $G$ and dividing graph $G$ into two parts: the induced subgraph on adjacent vertices to $v$ and the induced subgraph on vertices which are not adjacent to $v$, is an approach to destroy the special structure of such graphs. The computations show that this key idea has been successful to overcome the studied instances of exceptional cases of Theorem \cite{1}.
For any vertex \( v \) of graph \( G \), let \( \text{Spec}(G, v) = \{ \text{Spec}(H_v), \text{Spec}(H_{\pi}) \} \) where \( H_v \) and \( H_{\pi} \) are, respectively, the induced subgraph on adjacent vertices to \( v \) and the induced subgraph on non-adjacent vertices to \( v \). Clearly, if there is an isomorphism mapping between two graphs \( G \) and \( G' \) which maps \( v \in V(G) \) to \( v' \in V(G') \), then \( \text{Spec}(G, v) = \text{Spec}(G', v') \). Therefore, we have \( \{ \text{Spec}(G, v) \mid v \in V(G) \} = \{ \text{Spec}(G', v) \mid v \in V(G') \} \), if \( G \) and \( G' \) are isomorphic.

**Definition 3** Let \( G \) be a graph, we define the split spectrum of \( G \) as

\[
\text{SplitSpec}(G) = \{ \text{Spec}(G, v) \mid v \in V(G) \}
\]

Clearly, if \( G \) and \( G' \) are isomorphic, then \( \text{SplitSpec}(G) = \text{SplitSpec}(G') \).

The successless of Split Spectrum for separating the exceptional cases is verified by the studied instance of strongly regular graphs. Strongly regular graphs provide relatively large sets of cospectral graphs with respect to adjacency matrix \( A \), Laplacian matrix, signless Laplacian \(^5\) and normalized Laplacian \(^3\). Also, the family of strongly regular graphs has long been identified as a hard case for the graph isomorphism problem \(^11\) and the best existing graph isomorphism algorithms for them are exponential \(^4\). Therefore, they are proper and challenging choices to check the efficiency of Split Spectrum.

In Table 1 some sets of strongly regular graphs and their numbers are given. They are obtained from \(^{10,13}\). The number of each family with the same parameters are given in the table. The split spectrum is computed to distinguish the strongly regular graphs with the same parameter which are given in the Table 1.

As computations shows the defined split spectrum based on graph spectrum which was defined by the Theorem \(^1\) can easily separate all graphs of Table 1. For example, the split spectrum based on polynomial \( q(x) = 2A + A^2 + .58A^3 + .37A^4 + .248x^5 + .17A^6 + .118A^7 + .0826A^8 + .0577A^9 + .0404A^{10} + .0283A^{11} \) can easily separate all graphs of Table 1 (it is \( q(x) = q_{a=3,k=11}(x) + q_{a=7,k=11}(x) \) where \( q_{a,k}(A) = \sum_{1 \leq i \leq k} a^{k-1}A^i \) \(^{24}\)).

The split spectrum computed by the polynomial \( q(x) = x + .3x^2 + .09x^3 + .027x^4 \) can successfully discriminate all strongly regular graphs of Table 1 except families e.g and j. It distinguishes, respectively 92%, 80% and 72% of families e, g and j.

We saw that it is possible to use \( \text{Spec}(f()) \) for distinguishing non-isomorphic strongly-cospectral graphs by destroying their special structure. However, the study of successfulness of this method for all exceptional cases, i.e. strongly-cospectral graphs, is suggested for the future work.

The proposed graph isomorphism algorithm is depicted in Fig. 3. According to Theorem 1 this algorithm solves the graph isomorphism problem for all graphs which are not strongly cospectral. This algorithm is improved by applying Split

| \( \text{SRG}(n,k,r,s) \) | \# | \( \text{SRG}(n,k,r,s) \) | \# |
|----------------|----|----------------|----|
| a (26,10,3,4) | 10 | i (37,18,8,9) | 676 |
| b (29,14,6,7) | 41 | g (40,12,2,4) | 28  |
| c (35,16,6,8) | 3854| h (45,12,3,3)| 78  |
| d (35,18,9,9) | 227 | i (50,21,8,9)| 18  |
| e (36,14,4,6) | 180 | j (64,18,2,6)| 167 |

Table 1 The strongly regular families of graphs that the Split Spectrum has been checked for them. The SplitSpec is successful to distinguish all graph isomorphism classes of each family.
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Spectrum to be able to solve this problem for strongly cospectral graphs. As computations show split spectrum has been successful to solve graph isomorphism for studied instance of strongly cospectral graphs. Consequently, the study of successfulness of split spectrum for all strongly cospectral graphs is suggested for the future work.

For any n-vertex graph $G$, the complexity of computing $\text{Spec}(G)$ and $\text{SplitSpec}(G)$ are, respectively, $O(n^3)$ and $O(n^4)$.

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