Hyperbolicity of Force-Free Electrodynamics

Harald P. Pfeiffer
Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, Ontario M5S 3H8, Canada

Andrew I. MacFadyen
Physics Department, New York University, New York, NY, 10003, USA

We analyze the equations of relativistic magnetized plasma dynamics in the limiting case that electromagnetic stress-energy is dominant over pressure and rest mass energy density. The naive formulation of these equations is shown to be not hyperbolic. Modifying the equations by terms that vanish for all physical solutions, we obtain a symmetric hyperbolic evolution system, which should exhibit improved numerical behavior.

1. Introduction

Magnetically dominated regions surrounding neutron stars and black holes, can possess magnetic fields of $\sim 10^{12}$ Gauss (pulsars) and $\sim 10^{15}$ Gauss (magnetars) and extremely low mass densities \[1, 2\]. The magnetic energy density, $B^2/8\pi$, in these magnetospheres can exceed the rest mass energy density, $\rho_m c^2$, by many orders of magnitude. These are the astrophysical environments responsible for relativistic flows observed as pulsars, magnetars, quasars and gamma-ray bursts (GRBs)\[3\]. Even in the center of massive stars, regions of density of $\sim 10^6$ g cm$^{-3}$ are magnetically dominated for magnetar-like magnetic fields of $10^{15}$ Gauss.

It is therefore of interest to examine the equations governing the dynamics of relativistic plasma in the limit of extremely strong magnetic field. Here, we study the mathematical properties of the resulting evolutionary equations, with the underlying motivation of performing computer simulations of astrophysically interesting situations.

Computational codes solving this system have previously been implemented using a variety of numerical techniques including e.g. finite difference \[4, 5, 6\], pseudospectral \[7, 8\] and Godunov schemes \[9, 10, 11, 12, 13\].

We first derive the most immediate formulation of these equations. We show that this “naive” formulation is not hyperbolic, so that its initial value problem is not well-posed. We then modify the evolution equations and obtain a symmetric hyperbolic evolution system.
2. Force-Free Condition & Ohm’s Law

Force-free electrodynamics is a time-evolutionary system [14, 15] and can be formulated in terms of Ohm’s law [16].

The momentum equation for a magnetized fluid can be written as

$$\rho_m \frac{D\vec{v}}{Dt} = -\nabla P + \rho \vec{E} + \vec{j} \times \vec{B}$$

(1)

where $\rho_m$ is the mass density, $\vec{v}$ the velocity, $P$ the pressure, $\rho$ the charge density, $\vec{j}$ the current density and $\vec{E}$ and $\vec{B}$ the electric and magnetic field. We do not take the non-relativistic limit ($v \ll c$) so we retain the electric field in all equations, including the displacement current in Maxwell’s equations (Eq. 4). This differs from non-relativistic magneto-hydrodynamics where the electric field is neglected.

In the limit that the energy density of the electromagnetic field is much larger than the fluid pressure and rest mass energy density ($E^2 + B^2)/8\pi \gg P, \rho_m$ (we henceforth use units in which the speed of light $c \equiv 1$) the pressure gradient term $\nabla P$ and the inertial term $\rho_m \frac{D\vec{v}}{Dt}$ are negligibly small compared to the Lorentz force which therefore must also vanish, defining the force-free condition:

$$\rho \vec{E} + \vec{j} \times \vec{B} = 0.$$ 

(2)

It is assumed that while the mass density of the plasma is small there are still sufficient charges present at all times to carry any current necessary to enforce the force-free condition. In particular, current is assumed to flow instantaneously along the magnetic field lines to short out any component of electric field parallel to the magnetic field, as follows by taking the dot product of Eq. (2) with $\vec{B}$:

$$\vec{E} \cdot \vec{B} = 0.$$ 

(3)

We further assume that $B^2 - E^2 > 0$. If this is not true, there exists a frame of reference in which $B = 0$ and the field is entirely electric. This is a regime where effects involving particle properties will become important and the force-free equations no longer describe the dynamics.

An expression for the current density in terms of the electric and magnetic fields can be derived from the force free condition (Eq. 2) and Maxwell’s equations,

$$\partial_t \vec{E} = \nabla \times \vec{B} - \vec{j},$$

(4)

$$\partial_t \vec{B} = -\nabla \times \vec{E},$$

(5)

$$\nabla \cdot \vec{E} = \rho,$$

(6)

$$\nabla \cdot \vec{B} = 0,$$

(7)

as follows. From Eq. (3),

$$\partial_t (\vec{E} \cdot \vec{B}) = \vec{E} \cdot \partial_t \vec{B} + \partial_t \vec{E} \cdot \vec{B} = 0.$$ 

(8)

Substituting Maxwell’s equations (4) and (5) in Eq. (8) yields

$$-\vec{E} \cdot (\nabla \times \vec{E}) + \vec{B} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot \vec{j} = 0.$$ 

(9)

Taking the cross product of the force-free condition (Eq. 2) with $\vec{B}$, expanding the triple cross product and using $\rho = \nabla \cdot \vec{E}$ we get

$$\vec{B} (\vec{j} \cdot \vec{B}) = -((\nabla \cdot \vec{E}) \vec{E} \times \vec{B} + \vec{j} B^2).$$

(10)
Multiplying Eq. (9) by $\vec{B}$ and substituting Eq. (10) we finally obtain

$$\vec{j} = \frac{\vec{B}}{B^2} \left[ \vec{B} \cdot (\nabla \times \vec{B}) - \vec{E} \cdot (\nabla \times \vec{E}) \right] + \frac{\vec{E} \times \vec{B}}{B^2} \nabla \cdot \vec{E}. \quad (11)$$

Since all particle properties of the plasma (pressure, inertia) are assumed negligibly small, the current depends only on the electromagnetic fields themselves. The first term on the right-hand side of Eq. (11) represents the current along the magnetic field. We assume perfect conductivity (resistivity $\eta = 0$) so this term can be non-zero even though $\vec{E} \cdot \vec{B} = 0$. The second term is the advective current due to charge density ($\rho = \nabla \cdot \vec{E}$) moving at the plasma drift velocity.

The current is a non-linear function of the fields which together with Maxwell’s equations Eqs. (4) and (5) yields a set of time-evolution equations for the electromagnetic fields $\vec{E}$ and $\vec{B}$. We will refer to this set of equations as the “naive system,” and analyze it in the next section. Physically realistic fields, of course, must always satisfy the constraints $\vec{E} \cdot \vec{B} = 0$ and $\nabla \cdot \vec{B} = 0$ (Eqs. 3 and 7). The $\nabla \cdot \vec{E} = \rho$ equation of Maxwell’s equations (Eq. 4) is not a constraint in force-free electrodynamics, but the definition of charge density $\rho$.

3. Hyperbolicity of the Naive Formulation

Any system of evolution equations must be well-posed, this means, it must have a unique solution, and, roughly speaking, small perturbations of the initial conditions must lead to small perturbations at later times. This idea is captured by the mathematical concept of hyperbolicity (e.g., [17, 18]).

Denoting the set of evolved fields by $u = \{\vec{E}, \vec{B}\}$, the force-free equations (4), (5) with (11) have the structure of a first order system of evolution equations,

$$\partial_t u + A^i \partial_i u = 0,$$

(12)

with matrices $A^i$ depending on the fields $u$ but not their derivatives. A system of this form is called strongly hyperbolic, if, for each choice of unit-vector $\hat{n}$, the characteristic matrix

$$\sum_i \hat{n}_i A^i$$

(13)

has all real eigenvalues and a complete set of eigenvectors. Strong hyperbolicity is a necessary criterion for well-posedness [19].

Below, we also use the concept of symmetric hyperbolicity (see, e.g. [18]): The evolution system (12) is symmetric hyperbolic if there exists a positive definite matrix $S$ which simultaneously symmetrizes all derivative matrices, i.e. $SA^i$ is symmetric for each $i$. The symmetrizer $S$ can depend on the fields $u$ but not their derivatives. Symmetric hyperbolicity ensures well-posedness. Furthermore, because symmetric matrices have all real eigenvectors and complete sets of eigenvalues, every symmetric hyperbolic system is also strongly hyperbolic [1].

The eigenvector analysis of the characteristic matrix Eq. (13) is not only important for establishing well-posedness, but also for posing boundary conditions

‡ A third concept is strict hyperbolicity (e.g. [18]), which asserts that, for every $\hat{n}$, the characteristic matrix $\sum_i \hat{n}_i A^i$ has all real and distinct eigenvalues. This implies immediately that the eigenvectors form a complete set, so that every strictly hyperbolic evolution system is also strongly hyperbolic. Strict hyperbolicity is not applicable for the force-free equations, because the characteristic speeds are not distinct in all cases.
in numerical simulations. One must apply boundary conditions precisely to those characteristic modes that are *entering* the computational domain, for example by the algorithm presented in [20].

We now examine the eigenvalue problem

\[ \hat{n}_i A^i e_{(\hat{\alpha})} = v_{(\hat{\alpha})} e_{(\hat{\alpha})}, \]  

(14)

for the naive force-free equations. Here, \( e_{(\hat{\alpha})} \) denotes the right eigenvectors, labeled by \( \hat{\alpha} = 1, \ldots, 6 \), and \( v_{(\hat{\alpha})} \) are the eigenvalues, or characteristic speeds.

From Eqs. (4), (5) and (11) we find

\[ A^i = \begin{pmatrix} A_{EE}^i & A_{EB}^i \\ A_{BE}^i & 0 \end{pmatrix}, \]  

(15)

where each entry represents a \( 3 \times 3 \) submatrix:

\[ (A_{EE}^i)_{jk} = -\varepsilon_{ikl} E_l B_j + \varepsilon_{jlm} B_l B_m \delta_{ik}, \]

(16)

\[ (A_{EB}^i)_{jk} = \varepsilon_{ijk} B_j B_l + \varepsilon_{ikl} B_l B_m \delta_{jk}, \]

(17)

\[ (A_{BE}^i)_{jk} = -\varepsilon_{ijk}. \]

(18)

Solving Eq. (14) results in the characteristic speeds

\[ v_{(1)} = -1, \]

(19)

\[ v_{(2)} = 1, \]

(20)

\[ v_{(3)} = v - w, \]

(21)

\[ v_{(4)} = v + w, \]

(22)

\[ v_{(5)} = v_{(6)} = 0, \]

(23)

with

\[ v = \frac{\hat{n} \cdot (\vec{E} \times \vec{B})}{B^2}, \]

(24)

\[ w = \frac{1}{B^2} \sqrt{(\hat{n} \cdot \vec{B})^2 (B^2 - E^2)}. \]

(25)

Here, and below, we have used \( \vec{E} \cdot \vec{B} = 0 \) to simplify the expressions. \( v_{(1,2)} \) represent the fast modes, \( v_{(3,4)} \) are the Alfvén modes, and \( v_{(5,6)} \) are unphysical. The modes 5 and 6 are present because the evolution system has more variables than physical degrees of freedom, owing to the conditions Eqs. (3) and (7).

We note that \( v_{(3,4)} \) become complex when \( B^2 - E^2 < 0 \), so that in this regime hyperbolicity is lost. This reflects the breakdown of the force-free approximation when \( B^2 - E^2 < 0 \).

The eigenvectors can most easily be written using the projection operator orthogonal to \( \hat{n} \); its action on an arbitrary vector \( \vec{a} \) is defined by

\[ P\vec{a} \equiv \vec{a} - (\hat{n} \cdot \vec{a})\hat{n}. \]  

(26)

Generically, we find

\[ e_{(1)} = \begin{pmatrix} -P \vec{E} + \hat{n} \times \vec{B}, & P \vec{B} + \hat{n} \times \vec{E} \end{pmatrix}^t, \]

(27)

\[ e_{(2)} = \begin{pmatrix} -P \vec{E} - \hat{n} \times \vec{B}, & P \vec{B} - \hat{n} \times \vec{E} \end{pmatrix}^t, \]

(28)

\[ e_{(3,4)} = \begin{pmatrix} \vec{E}_{(3,4)}, & \vec{B}_{(3,4)} \end{pmatrix}^t, \]

(29)
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\[ e^{(5)} = \left\{ (\hat{n} \cdot \vec{B})\hat{n}, -P\vec{E} \right\}^t, \]
\[ e^{(6)} = \{ 0, \hat{n} \}^t. \]

with

\[ \vec{E}^{(3,4)} = -P\vec{B} + v^{(3,4)}\hat{n} \times \vec{E} + (1 - v^2)\hat{\vec{B}}, \]
\[ \vec{B}^{(3,4)} = -P\vec{E} - v^{(3,4)}\hat{n} \times \vec{B}. \]

It is interesting to note that because \( v^{(1,2)} = \pm 1 \), the fast modes can be written in a form similar in structure to the Alfvén modes. Writing the fast mode eigenvectors as

\[ e^{(1,2)} = \left\{ E^{(1,2)}, \vec{B}^{(1,2)} \right\}^t \]

they have the same form as the Alfvén modes Eqs. (32) and (33) with the replacements \( \vec{E} \rightarrow \vec{B}, \vec{B} \rightarrow -\vec{E} \), the duality transformation between \( \vec{E} \) and \( \vec{B} \).

The left eigenvectors of \( \hat{n}_i A^i \), defined by

\[ e^{(\hat{\alpha})}_i = v^{(\hat{\alpha})} e^{(\hat{\alpha})}_i, \quad \hat{\alpha} = 1, \ldots, 6, \]

are given by

\[ e^{(1)} = \left\{ -\vec{E} + \hat{n} \times \vec{B}, P\vec{B} + \hat{n} \times \vec{E} \right\}, \]
\[ e^{(2)} = \left\{ -\vec{E} - \hat{n} \times \vec{B}, P\vec{B} - \hat{n} \times \vec{E} \right\}, \]
\[ e^{(3,4)} = \left\{ \vec{E}^{(3,4)}, \vec{B}^{(3,4)} \right\}, \]
\[ e^{(5)} = \{ \vec{B}, P\vec{E} \}, \]
\[ e^{(6)} = \{ 0, \hat{n} \}, \]

with

\[ \vec{E}^{(3,4)} = \frac{(P\vec{E})^2 - \hat{n} \cdot (\vec{E} \times \vec{B}) v^{(3,4)} \hat{n} \cdot \vec{B} + P\vec{B} - v^{(3,4)}\hat{n} \times \vec{E}}, \]
\[ \vec{B}^{(3,4)} = P\vec{E} + v^{(3,4)}\hat{n} \times \vec{B}. \]

The left and right eigenvectors are orthogonal to each other:

\[ e^{(\hat{\alpha})} \cdot e^{(\hat{\beta})} = 0, \quad \hat{\alpha} \neq \hat{\beta}. \]

In order to save space, the expressions given above are not normalized.

3.1. Breakdown of Hyperbolicity

Whenever two or more characteristic speeds are equal, it is not guaranteed that a full set of eigenvectors exists, so those cases must be examined in detail. For the naive force-free equations, in many of these degenerate cases a complete set of eigenvectors does exist, as detailed in the appendix.
However, in certain cases the eigenvectors are not complete. For example, when \( \vec{E} = 0 \) and \( \vec{B} \cdot \hat{n} = 0 \), there exist four zero-speed eigenvalues. The eigenvector equation (14) reduces in this case to

\[
\hat{n} \times \vec{B}_{(\hat{\alpha})} - \hat{b} \left[ \hat{b} \left( \hat{n} \times \vec{B}_{(\hat{\alpha})} \right) \right] = 0,
\]

(45)

\[
\hat{n} \times \vec{E}_{(\hat{\alpha})} = 0,
\]

(46)

where \( \vec{E}_{(\hat{\alpha})} \) and \( \vec{B}_{(\hat{\alpha})} \) denote the electric and magnetic components of the desired eigenvector, and where \( \hat{b} = \vec{B} / B \). Equations (45) and (46) are solved by \( \vec{B}_{(\hat{\alpha})} = C_1 \hat{n} + C_2 \hat{n} \times \hat{b}, \vec{E}_{(\hat{\alpha})} = C_3 \hat{n} \) with arbitrary constants \( C_{1,2,3} \), so that the corresponding eigenspace is three-dimensional only, and the system is not strongly hyperbolic for these values of the variables.

More generally, no complete set of zero-speed eigenvectors exists whenever (at least) one Alfvén-speed vanishes, i.e. when \( v_3 = 0 \) and \( v_4 = 0 \), or equivalently,

\[
|\hat{n} \cdot \vec{B}| = |P \vec{E}|.
\]

(47)

Condition (47) is very restrictive. Consider an arbitrary point in space with values \( \vec{E} \) and \( \vec{B} \) satisfying \( B^2 - E^2 > 0 \). At this point, if \( \hat{n} \) is chosen parallel to \( \vec{B} \), then \( |\hat{n} \cdot \vec{B}| - |P \vec{E}| > 0 \), whereas for \( \hat{n} \) perpendicular to \( \vec{B} \) we have \( |\hat{n} \cdot \vec{B}| - |P \vec{E}| \leq 0 \). Therefore, if \( \hat{n} \) changes continuously between these two directions, Eq. (47) must be satisfied at least once. At each point in space, no matter what the values of \( \vec{E} \) and \( \vec{B} \), there exists at least one direction \( \hat{n} \) such that \( \hat{n} \cdot A \) has no complete set of eigenvectors. Thus, the naive force-free equations are not strongly hyperbolic.

Komissarov [9] examined hyperbolicity of a related, but not identical formulation of force-free electrodynamics. We remark that his system behaves similarly to the naive system considered here: Whenever Eq. (47) holds, no complete set of eigenvectors exists.

One might argue that even when Eq. (47) holds, there are “enough” eigenvectors to represent any physical solution satisfying \( \nabla \cdot \vec{B} = 0 \) and \( \vec{E} \cdot \vec{B} = 0 \). Such an observation, however, is irrelevant because in any numerical simulation, these constraints will not be satisfied exactly, but only to truncation error, or at best to roundoff error. If the evolution system is not well-posed, this small constraint-violation may grow on arbitrarily small timescales. We conclude that the naive formulation of force-free electrodynamics is highly unsatisfactory, at best.

4. Constraint Addition — Augmented Evolution System

The six-dimensional system of force-free dynamics must satisfy the constraints \( \vec{E} \cdot \vec{B} = 0 \) and \( \nabla \cdot \vec{B} = 0 \) (Eqs. [8] and [11]). Addition of terms to the evolution equations, which are proportional to these constraints, will not change the physical solutions of the system. However, if the new terms contain derivatives, they will modify the \( A^4 \)-matrices and influence the hyperbolicity of the system. Our strategy is to add multiples of such terms to the naive force-free equations, and choose the coefficients to achieve hyperbolicity. We augment the naive force-free equations [4], [5] and [11] as follows:

\[
\partial_t \vec{E} = \nabla \times \vec{B} - \vec{j} - \gamma_1 \frac{\vec{E}}{B^2} \times \nabla (\vec{E} \cdot \vec{B}),
\]

(48)

\[
\partial_t \vec{B} = - \nabla \times \vec{E} - \gamma_2 \frac{\vec{E} \times \vec{B}}{B^2} \nabla \cdot \vec{B} - \gamma_3 \frac{\vec{B}}{B^2} \times \nabla (\vec{E} \cdot \vec{B})
\]

(49)
with constants $\gamma_1, \gamma_2, \gamma_3$, and with $j$ given by (11). The particular form of the new terms was chosen to have appropriate dimensions and parity, as well as a form similar to the terms contained in $j$. The augmented system retains the same structure as Eq. (12). The choice $\gamma_1 = \gamma_2 = \gamma_3 = 0$ recovers the naive system.

The eigenvalues of the augmented system are
\begin{align*}
\tilde{v}(\tilde{\alpha}) &= v(\tilde{\alpha}), \quad \tilde{\alpha} = 1, 2, 3, 4, \\
\tilde{v}(3) &= \gamma_2 v, \\
\tilde{v}(6) &= (\gamma_3 - \gamma_1) v,
\end{align*}
with $v$ given by Eq. (24). Here, and below we denote quantities associated with the augmented system with tildes. The characteristic speeds $\tilde{v}(1), \ldots, \tilde{v}(4)$ are unchanged, as expected for the physical modes, but those for the unphysical modes depend on $\gamma_1, \gamma_2$ and $\gamma_3$.

### 4.1. Choice of Free Parameters

We can gain insight into the choices for $\gamma_1, \gamma_2$ and $\gamma_3$ by considering the zero-speed eigenspace for $\vec{E} = 0$, $\hat{\delta} \cdot \hat{n} = 0$, which was found above to be incomplete without constraint addition (cf. Eqs. (15) and (13)). The eigenvalue equations for the augmented system reduce in this case to
\begin{align*}
\hat{n} \times \hat{B}(\hat{\alpha}) - \hat{b} \left[ \hat{b} \left( \hat{n} \times \hat{B}(\hat{\alpha}) \right) \right] &= 0, \tag{53} \\
\hat{n} \times \left[ \hat{E}(\hat{\alpha}) - \gamma_3 \hat{b} \left( \hat{b} \cdot \hat{E}(\hat{\alpha}) \right) \right] &= 0. \tag{54}
\end{align*}

Equation (53) is unchanged by the constraint addition (cf. Eq. (15)) and is solved by $\hat{B}(\hat{\alpha}) = C_1 \hat{n} + C_2 \hat{n} \times \hat{b}$. The structure of Eq. (54) is most interesting: If and only if $\gamma_3 = 1$, the square-bracket represents the projection of $\hat{E}(\hat{\alpha})$ perpendicular to $\hat{b}$. Hence, if and only if $\gamma_3 = 1$, Eq. (53) has a two-dimensional solution space, $\hat{E}(\hat{\alpha}) = C_3 \hat{n} + C_4 \hat{b}$. The demand of a complete set of eigenvectors thus implies $\gamma_3 = 1$.

The more general case $\vec{E}$ parallel to $\hat{n}$, $\hat{b} \cdot \hat{n} = 0$ can be dealt with similarly. The demand of a complete set of eigenvectors in this case fixes uniquely $\gamma_2 = 1$.

To fix the remaining parameter $\gamma_1$, consider the derivative-matrices
\begin{equation}
\hat{A} = \begin{pmatrix} \hat{A}_{EE} & \hat{A}_{EB} \\ \hat{A}_{BE} & \hat{A}_{BB} \end{pmatrix}. \tag{55}
\end{equation}
For $\gamma_1 = 0$ (and $\gamma_2 = \gamma_3 = 1$), the off-diagonal blocks are symmetric, i.e. $
\hat{A}_{EB} = \left( \hat{A}_{BE} \right)^t$. Moreover, for $\gamma_1 = 0$ (and $\gamma_2 = \gamma_3 = 1$), the characteristic speeds $\tilde{v}(3), \ldots, \tilde{v}(6)$ are distributed symmetrically around $v$. Thus, we will choose
\begin{equation}
\gamma_1 = 0, \quad \gamma_2 = 1, \quad \gamma_3 = 1. \tag{56}
\end{equation}

### 4.2. Symmetric Hyperbolicity

With the choices $\gamma_1 = 0, \gamma_2 = \gamma_3 = 1$, the augmented system is not only strongly hyperbolic, but even symmetric hyperbolic. A symmetrizer for this system is given by the $6 \times 6$ matrix
\begin{equation}
S = \frac{1}{B^2} \begin{pmatrix} (B^2 - E^2) \delta_{ij} + (z^2 - 2\Delta) B_i B_j & -\Delta E_i B_j + z_7 B_i E_j \\ -\Delta B_i E_j + z_7 E_i B_j & (B^2 - E^2) \delta_{ij} + z_7 E_i E_j \end{pmatrix}, \tag{57}
\end{equation}
with $\Delta = 1 - E^2/B^2$ and $z_\gamma > 1$ arbitrary. The choice $z_\gamma = 2$ is natural, because then $S$ reduces to the identity matrix for $E = 0$. We have not found a symmetrizer for parameters different from $[29]$, and therefore believe that this choice is the only one that makes the augmented system symmetric hyperbolic.

Symmetric hyperbolicity is a very convenient property; in particular, real eigenvalues and a complete set of eigenvectors are guaranteed. The remaining part of this section lists these eigenvectors, beginning with the generic case. The right eigenvectors of the physical modes $e_{(1)}, \ldots, e_{(4)}$ are unchanged:

$$e_{(\alpha)} = e_{(\alpha)}, \quad \alpha = 1, 2, 3, 4.$$  \hfill (58)

The eigenvectors associated with $\tilde{v}_{(5)} = \tilde{v}_{(6)} = \nu$ are

$$ \tilde{e}_{(5)} = \left\{ - \left( B^2 - \frac{E^2 B_+^2}{B^2} \right) \vec{B} - (\hat{n} \cdot \vec{E}) (\hat{n} \cdot \vec{B}) \vec{E}, \right. $$

$$ \left. \left( B^2 - \frac{E^2 B_+^2}{B^2} \right) P \vec{E} - \frac{E^2 (\hat{n} \cdot \vec{E}) (\hat{n} \cdot \vec{B})}{B^2} P \vec{B} \right\}^t, \hfill (59)$$

$$ \tilde{e}_{(6)} = \left\{ Q_1 \vec{E} + Q_2 \vec{B}, \right. $$

$$ \left. Q_3 \hat{n} + Q_4 \vec{E} + Q_5 \vec{B} \right\}^t, \hfill (60)$$

where

$$Q_1 = B_+^2 \left( \frac{E^2 (\hat{n} \cdot \vec{E})^2 (\hat{n} \cdot \vec{B})^2}{B^2} + B^2 (B^2 - E^2) (1 - \nu^2) \right), \hfill (61)$$

$$Q_2 = \frac{E^2 B_+^2 (\hat{n} \cdot \vec{E}) (\hat{n} \cdot \vec{B})}{B^2} \left( B^2 - \frac{E^2 B_+^2}{B^2} \right), \hfill (62)$$

$$Q_3 = \frac{(B^2 - E^2) B_+^2}{\hat{n} \cdot \vec{B}} \left( B^2 - E^2 + (\hat{n} \cdot \vec{E})^2 \right) \left( B^2 - \frac{E^2 B_+^2}{B^2} \right), \hfill (63)$$

$$Q_4 = -\frac{E^2 B_+^2 (\hat{n} \cdot \vec{E}) (\hat{n} \cdot \vec{B})}{B^2} \left( B^2 - \frac{E^2 B_+^2}{B^2} \right), \hfill (64)$$

$$Q_5 = \frac{E^2 B_+^2}{B^2} \left( B^2 - E^2 + (\hat{n} \cdot \vec{E})^2 \right) \left( B^2 - \frac{E^2 B_+^2}{B^2} \right), \hfill (65)$$

with $B_+^2 \equiv (P \vec{B})^2 = (\hat{n} \times \vec{B})^2$. The left eigenvectors can be written as

$$\tilde{e}^{(1)} = \left\{ - (\hat{n} \cdot \vec{E}) \hat{n} + (1 + 2 \nu) \hat{n} \times \vec{B} - \left( v + \frac{(\hat{n} \cdot \vec{B})^2}{B^2} \right) \vec{E}, \right.$$\n
$$\left. \left( 1 + v + \frac{(\hat{n} \times \vec{E})^2}{B^2} \right) \vec{B} + \hat{n} \times \vec{E} - (\hat{n} \cdot \vec{B}) \hat{n} \right\}, \hfill (66)$$

$$\tilde{e}^{(2)} = \left\{ - (\hat{n} \cdot \vec{E}) \hat{n} + (-1 + 2 \nu) \hat{n} \times \vec{B} + \left( -\frac{(\hat{n} \cdot \vec{B})^2}{B^2} + v \right) \vec{E}, \right.$$\n
$$\left. \left( 1 - v + \frac{(\hat{n} \times \vec{E})^2}{B^2} \right) \vec{B} - \hat{n} \times \vec{E} - (\hat{n} \cdot \vec{B}) \hat{n} \right\}, \hfill (67)$$

$$\tilde{e}^{(3)} = \left\{ (\hat{n} \cdot \vec{B}) (B_+^2 - E^2) \hat{n} - (\hat{n} \cdot \vec{B})^2 P \vec{B} - B^2 w \hat{n} \times \vec{E}, \right.$$\n
$$\left. - (\hat{n} \cdot \vec{B}) \vec{E} + B^2 w \hat{n} \times \vec{B} \right\}, \hfill (68)$$
\[ \hat{e}^{(4)} = \left\{ - (\hat{n} \cdot \vec{B})(B_{\perp}^2 - E^2)\hat{n} + (\hat{n} \cdot \vec{B})^2 PB - B^2 \hat{n} \times \vec{E}, \right. \]
\[ \left. (\hat{n} \cdot \vec{B})^2 \vec{E} + B^2 \hat{n} \times \vec{B} \right\}, \]
\[ \hat{e}^{(5)} = \left\{ \vec{B}, \ P\vec{E} \right\}, \]
\[ \hat{e}^{(6)} = \{0, \ \hat{n}\} \].

We now turn our attention to the degenerate cases:

(i) For \( \vec{E} = \hat{n} \times \vec{B} \), \( \hat{v}(1) = \hat{v}(3) = -1 \). Given any \( \hat{q} \) perpendicular to \( \hat{n} \), this two-dimensional eigenspace is spanned by
\[ \hat{e}^{(1)} = Q \{\hat{q}, \ -\hat{n} \times \hat{q}\}^t, \]
\[ \hat{e}^{(3)} = \{\hat{n} \times \hat{q}, \ \hat{q}\}^t, \]
\[ \hat{e}^{(1)} = \left\{ - 2(\hat{n} \cdot \vec{B})(\hat{q} \cdot \vec{B})\hat{n} + (\hat{n} \cdot \vec{B})^2 \hat{q} - (\hat{q} \cdot \vec{B})PB, \right. \]
\[ \left. \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] \vec{B} - B^2 \hat{n} \times \hat{q} \right\}, \]
\[ \hat{e}^{(3)} = \left\{ - 2(\hat{n} \cdot \vec{B}) \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] \hat{n} + (\hat{n} \cdot \vec{B})^2 \hat{n} \times \hat{q} - \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] PB, \right. \]
\[ \left. - (\hat{q} \cdot \vec{B}) \vec{B} + B^2 \hat{q} \right\}. \]

(ii) For \( \vec{E} = -\hat{n} \times \vec{B} \), \( \hat{v}(2) = \hat{v}(4) = 1 \). Given any \( \hat{q} \) perpendicular to \( \hat{n} \), this two-dimensional eigenspace is spanned by
\[ \hat{e}^{(2)} = \{\hat{q}, \ \hat{n} \times \hat{q}\}^t, \]
\[ \hat{e}^{(4)} = \{\hat{n} \times \hat{q}, \ -\hat{q}\}^t, \]
\[ \hat{e}^{(2)} = \left\{ - 2(\hat{n} \cdot \vec{B})(\hat{q} \cdot \vec{B})\hat{n} + (\hat{n} \cdot \vec{B})^2 \hat{q} - (\hat{q} \cdot \vec{B})PB, \right. \]
\[ \left. - \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] \vec{B} + B^2 \hat{n} \times \hat{q} \right\}, \]
\[ \hat{e}^{(4)} = \left\{ - 2(\hat{n} \cdot \vec{B}) \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] \hat{n} + (\hat{n} \cdot \vec{B})^2 \hat{n} \times \hat{q} - \left[ (\hat{n} \times \hat{q}) \cdot \vec{B} \right] PB, \right. \]
\[ \left. + (\hat{q} \cdot \vec{B}) \vec{B} - B^2 \hat{q} \right\}. \]

Cases 1. and 2. occur simultaneously if \( \vec{E} = 0 \) and \( \hat{n} \times \vec{B} = 0 \).

(iii) If \( \hat{n} \cdot \vec{B} = 0 \), then \( \hat{e}^{(3)} = \hat{v}(4) = \hat{v}(5) = \hat{v}(6) = v \). One can use
\[ \hat{e}^{(3)} = \left\{ (1 - v^2)B^2 \hat{n} - v(\hat{n} \cdot \vec{E})\hat{n} \times \vec{B}, \ (\hat{n} \cdot \vec{E})\vec{B} \right\}^t, \]
\[ \hat{e}^{(4)} = \left\{ 0, \ \hat{n} \times \vec{B} \right\}^t, \]
\[ \hat{e}^{(5)} = \{\vec{B}, \ 0\}^t, \]
\[ \hat{e}^{(6)} = \{0, \ \hat{n}\}^t. \]
and
\[ \tilde{\varepsilon}^{(3)} = \{ \hat{n}, 0 \}, \] (84)
\[ \tilde{\varepsilon}^{(4)} = \{ 0, \hat{n} \times \vec{B} \}, \] (85)
\[ \tilde{\varepsilon}^{(5)} = \{ \vec{B}, 0 \}, \] (86)
\[ \tilde{\varepsilon}^{(6)} = \{ 0, \hat{n} \}. \] (87)

This case cannot occur simultaneously with cases 1. or 2. above, because \( \hat{n} \cdot \vec{B} = 0 \) and \( \vec{E} = \pm \hat{n} \times \vec{B} \) imply that \( E^2 = B^2 \), contradicting the assumption \( B^2 - E^2 > 0 \).

(iv) Finally, \( \tilde{\varepsilon}^{(5)} \) and \( \tilde{\varepsilon}^{(6)} \) are always equal. This case has already been incorporated into the general expressions, Eqs. (59) and (60).

5. Conclusion

We have performed a hyperbolicity analysis of force-free electrodynamics in the E-B formulation. The naive evolution system for this formulation was found to be not hyperbolic, and therefore it is not well-posed. We then modified the naive system by addition of constraints. The augmented system Eqs. (48)–(49) was shown to be symmetric hyperbolic for a certain choice of parameters, \( \gamma_1 = 0, \gamma_2 = \gamma_3 = 1 \).

We expect the augmented system to exhibit better behavior in numerical studies of force-free electrodynamics.

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Appendix: Degenerate cases for the naive formulation

This appendix summarizes those degenerate cases of the naive E-B system, for which complete sets of eigenvectors exist.

(i) For \( \vec{E} = \hat{n} \times \vec{B}, v_1 = v_3 = -1 \). Given any \( \hat{q} \) perpendicular to \( \hat{n} \), this two-dimensional eigenspace is spanned by
\[ e_1 = \{ \hat{q}, -\hat{n} \times \hat{q} \}, \] (88)
\[ e_3 = \{ \hat{n} \times \hat{q}, \hat{q} \}, \] (89)
\[ e_4 = \left\{ \hat{q} - \frac{\hat{q} \cdot \vec{B}}{\hat{n} \cdot \vec{B}} \hat{n}, -\hat{n} \times \hat{q} \right\}, \] (90)
\[ e_5 = \left\{ \hat{n} \times \hat{q} - \frac{(\hat{n} \times \hat{q}) \cdot \vec{B}}{\hat{n} \cdot \vec{B}} \hat{n}, \hat{q} \right\}. \] (91)

Note that \( \vec{E} = \hat{n} \times \vec{B} \) implies \( (\hat{n} \cdot \vec{B})^2 = B^2 - E^2 > 0 \).

(ii) For \( \vec{E} = -\hat{n} \times \vec{B}, v_2 = v_4 = +1 \). This case is analogous to \( \vec{E} = \hat{n} \times \vec{B} \). Explicit orthogonal eigenvectors are given by Eqs. (88)–(91) with opposite signs of the cross-product terms.
(iii) In the case $\hat{n} \cdot \vec{B} = 0$, $v_{(3)} = v_{(4)} = v$.

If $\vec{E}$ is not parallel to $\hat{n}$, then $v \neq 0$. We can choose
\begin{align}
e_{(3)} &= \left\{ v \vec{B}, \hat{n} \times \vec{B} \right\}, \\
e_{(4)} &= \left\{ \vec{E}, \vec{B} \right\}, \\
e_{(3)} &= \left\{ P \vec{B}, 0 \right\}, \\
e_{(4)} &= \left\{ \hat{n}, 0 \right\}
\end{align}

where
\begin{align}
\vec{E}^{(4)} &= (1 - v^2) B \hat{n} - v \frac{\hat{n} \cdot \vec{E}}{B} \hat{n} \times \vec{B}, \\
\vec{B}^{(4)} &= \frac{\hat{n} \cdot \vec{E}}{B} \vec{B}.
\end{align}

If, however, $\vec{E}$ is parallel to $\hat{n}$ then $v = 0$, leading to a four-dimensional eigenspace for the eigenvalue 0. No complete set of eigenvectors exists, as discussed in the main text.

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