MATHEMATICAL BASIS FOR LARGE SCALE GIS 
AND TERRESTRIAL DIGITAL PRODUCTS

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ABSTRACT The problem of spatial mathematical basis has been encountered by both large scale GIS and spatial digital products theoretically and practically. It is also a basic problem in the development of the whole geo-information science. After analyzing the status quo and the limitations of the space mathematical base of GIS, this paper points out definitely that the geodetic coordinate system is uniform, which can show the location of any point of the global exactly and uniquely in form of \((B, L, H)\) and is the most proper reference system of large-scale GIS and Digital Earth. Moreover, this paper also puts forward a set of practical model of the standard "map projection". Finally, this paper introduces a DRG system based on this model.

1  Map projection and the inverse transformation of map projection

Nowadays, nearly all GISs take certain map projection as their own spatial mathematical basis. That is, the plane measure space is expressed by the following formula:

\[
\begin{align*}
X &= f_1(\Phi, \lambda) \\
Y &= f_2(\Phi, \lambda)
\end{align*}
\]

(1)

Where \(\Phi, \lambda\) are the latitude and longitude on the sphere, \(X, Y\) are the rectangular coordinates on the plane and \(f_1, f_2\) are singly valued, limited and continued functions.

However, Eq. (1) alone is not rigorous. Theoretically, Jacobi equation of Eq. (1) is:

\[
J = \frac{\partial(X, Y)}{\partial(\Phi, \lambda)} = \begin{vmatrix}
\frac{\partial X}{\partial \Phi} & \frac{\partial X}{\partial \lambda} \\
\frac{\partial Y}{\partial \Phi} & \frac{\partial Y}{\partial \lambda}
\end{vmatrix} = H
\]

\[
\frac{\partial X}{\partial \Phi} \cdot \frac{\partial Y}{\partial \lambda} - \frac{\partial X}{\partial \lambda} \cdot \frac{\partial Y}{\partial \Phi} = H 
\]

\(H \neq 0\) should hold true in the mapping area.

The condition (2) can keep the one to one transformation from the curved surface \(\Phi\lambda\) to the plane \(XOY\), and vise versa. Therefore, map projection could be only established by Eqs. (1) and (2)

The inverse transformation of map projection is defined as the transformation of plane measure space \(XOY\) to the earth ellipsoidal surface \(\Phi\lambda\).

When there exists a second map projection in the same area, i.e.

\[
\begin{align*}
X &= f_3(\Phi, \lambda) \\
Y &= f_4(\Phi, \lambda)
\end{align*}
\]

(3)

where the relevant \(H \neq 0\) holds good in the whole area. Then, when the transformation is from the second map projection \(xy\) to the former map projection \(XOY\), the exchanged Jacobi Eq. (4) will exist everywhere. Therefore, it is also a one to one and continuous transformation.

\[
J = \frac{\partial(X, Y)}{\partial(x, y)} = \frac{\partial(X, Y)}{\partial(\Phi, \lambda)} = H
\]

\(H \neq 0\) (4)
2 Types of map projection transformation

Map projection transformation is defined as a topological transformation between two 2D fields. If the earth surface is taken as an opened 2D field with curved coordinates \((\Phi, \lambda)\), then, map projection and its inverse transformation is a special case of map projection transformation.

Generally, map projection transformation takes four types:

1) transformation between explicit projections,
2) transformation from explicit projection to practical projection,
3) transformation from practical projection to explicit projection,
4) transformation between practical projections.

From the view of practical application, the transformation of Type 1) is just a small part of map projection transformation, and in this type there is only a small part can be expressed by analytical formulas. Therefore, numerical transformation of map projection in broad sense by using numerical approximation theory is a rigorous method suitable for general use.

If we say that map projection is the spatial mathematical basis of maps, then map projection and its transformation is the spatial mathematical basis of GIS. Various GISs select referential system and projection suitable for the area and scale. Gauss projection is applied by GISs with related resolution in accordance with state basic scale topomaps.

However, it is not necessary for large scale GISs to use more Gauss projection as their spatial mathematical basis. This is because that the object of large scale GIS is a huge area with certain broadness or even the whole earth. The general three elements of map projection, i.e. map use, scale and the position and shape of the area can not be defined properly.

From the view of the basic demands for the digital earth, the continuing multi-resolution of the whole earth and their various applications show clearly that the traditional selection of map projection for large scale GIS is no more suitable. From the following analysis we can recognize the deficiency for large scale GIS caused by Gauss projection.

For digital earth and large scale GIS, the basic deficiency in applying Gauss projection is that it can not realize entire continuous visualization, both for sliced projection or broad zone projection, especially for the latter the deformation is too big and complex. This will make the following work more difficult and cause bad effect.

Essentially, Gauss projection is not suitable for multi-resolution orientation construction and expression. In Gauss projection system, it is rather complex to define the accurate length and its measurement, and it is also difficult in application. The GIS of an area faces many dynamic variations, such as the expansion and coincidence of boundary, and interchange with outside areas, etc.

Furthermore, the fact that different GISs use self-defined different spatial referential system is no good for the development of GIS and the object of digital earth.

Therefore, it is very clear that a new spatial mathematical basis for this purpose is urgent and necessary. Thus the spatial data and information will have a highly common data base of sharing.

3 Spatial coordinate system suitable for large scale GIS

The most suitable referential system for GIS is the geodesy coordinate system, which can be easily and solely used to depict the position of any point in the spatial zone.

At present the most widely used coordinate systems are the geocentric coordinate system and the geodetic system. These two systems are widely used in geodetic science. However, there exists a small difference. For large scale GIS and the digital earth, this difference can be neglected, and the correction can be calculated precisely, if necessary.

4 The map projection most suitable for large scale GIS and digital earth

The aim of large scale GIS and digital earth leads
to the research field of this subject concerning the whole China territory or even larger area. In addition to the above mentioned ellipsoidal coordinate system, in order to realize interchangeable information application and operation, the standardization of map projection is very important.

4.1 The basic demands for the standardization of map projection

1) Simple structure, rigorous basis, precise and convenience in measurement

Map projection is the spatial mathematical basis of maps, while map is the 2D symbolic model of geographic information on its main projection direction. It defines a 2D plane in certain area and certain precise measurement, and people can see, measure, realize, analyse and apply the geographical information more precisely in a broad area.

2) Large area with multi-resolution and continual visualization for even the whole world

This work is without any problem for small scale maps. However, under the past technical condition for high resolution it is nearly impossible. Today, one meter even one foot resolution is not difficult for the whole world data.

3) Smooth and convenient input and output, suitable for dynamic variation

It can be easily used for adding 3D or multi-dimensional data.

4.2 Standardized map projection model

Here the selection of 2D measure space and the definition of measurement will be discussed.

The original conception of map projection is just the transformation of curvilinear coordinate \((B, L)\) on the 3D ellipsoid to the plane \((XOY)\). The main reason lies on that the map carrier is a plane paper and plane measurement technique. Now another choice can be taken as long as the basic spatial characteristics can be maintained.

People can directly select the projection from the practical 3D space \((B, L, H)\) to \((B, L)\). And the selection of the measure space decided by 2D field \((B, L)\). And its principle measure is solely decided by its positional coordinates according to the following definition.

The known values are Point 1 \((B_1, L_1)\), Point 2 \((B_2, L_2)\), flattening ratio \(a\), the major axis \(a\), the first excentricity \(e\) and the second excentricity \(e'\), the distance from Point 1 to the Point 2 is

\[
S_{12} = K_2 b (\Delta \varphi - d \Delta \varphi) \tag{5}
\]

The direction angle from Point 1 to Point 2 is

\[
\tan u_1 = (1 - a) \tan B_1, \quad \tan u_2 = (1 - a) \tan B_2, \quad d \Delta \varphi = 0
\]

The following formulas are used for calculating conditions in circulation.

\[
\Delta \varphi = L_2 - L_1 + d \Delta \varphi
\]

\[
\tan \Delta \varphi = \frac{[\cos u_2 \sin \Delta \varphi]^2 + (\cos u_1 \sin u_2 - \sin u_1 \sin u_2)}{\sin u_1 \cos u_2 \Delta \varphi}^{1/2} \tag{6}
\]

\[
\cos u_n = \frac{\cos u_1 \cos u_2 \sin \Delta \varphi / \sin \Delta a}{\cos u_1 \cos u_2 \Delta \varphi}
\]

\[
\cos 2 \sigma_m = \cos \Delta \sigma - 2 \sin u_1 \sin u_2 / \sin^2 u_n
\]

\[
V = 1/4a \sin^2 u_n
\]

\[
K_3 = V[1 + a + a^2 - V(3 + 7a - 13V)]
\]

\[
d \Delta \varphi = (1 - K_3) a \cos u_n [\Delta \varphi + K_3 \sin \Delta \varphi \cdot (\cos 2 \sigma_m - K_3 \cos \Delta \sigma \cos 4 \sigma_m)]
\]

When the circulating calculation made the variation of \(a \Delta \varphi\) smaller than the demanded deviation, then the following formulas can be solved.

\[
t = 1/4e^2 \sin^2 u_n, \quad K_1 = 1 + t \{1 - t/4[3 - t(5 - 11t)]\}
\]

\[
K_2 = t \{1 - t[2 - t/(37 - 94t)]\}
\]

Investigating the above distances, clearly in the set of \(|B, L| K\), the distance \(S\) is a transformation which transforms \(|B, L| \times |B, L|\) into the real number field \(R\). For any points 1, 2, 3, there exist

\[
S_{12} > 0, \text{only if } l = 2, S_{12} = 0
\]

\[
S_{12} = S_{21}, \text{then } S_{12} + S_{23} \geq S_{13}
\]

Therefore, \((B, L)\) is a measure space and distance \(S\) is a measurement scale.

The above 2D measure space is better than the plane rectangular coordinate space of any map projection and its distance measuring can reach the required accuracy, while the directional conception is also more accurate than any plane rectangular coordinate space. Moreover, its area without any limitation can be used for the whole world. For example, it is better than the Gauss projection, and there is no deviation of meridional convergence in its direc-
tion and there is not any error of 0.28% in the 6 degree zone.

The projection in the 3D space is very simple, and the measurement is just needed. It can be obtained from computer in real time.

Accordingly, we define the area of polygon on this projection and thus a precise visualization measurement system on the whole world can be organized.

4.3 Model for practical application

As a 2D field in the above \( \{ B, L \} \) space, \( B, L \) can be taken as two axes and they are perpendicular to each other. Radian measure or degree can be used as units. Because the map sizes are different for different units, we may introduce a constant \( K \) into the formulas. So the practical model is as follows:

\[
\begin{align*}
X &= K \cdot L \\
Y &= K \cdot B
\end{align*}
\]

(8)

where \( B, L \) take radian measure as unit. Obviously, if \( X, Y \) take meter as their length unit, then the practical physical meaning of \( K \) is the number of meters relative to one radian.

It is clear that constant \( K \) is related to the demanded resolution of image symbol, i.e. the scale, while the precisin of \( B \) and \( L \) is related to the surveying or the primary precision. The resolution and precision of these two sides should match each other. Let \( K \) be the radius of mean curvature of the earth \( (R) \), Eq. (8) can be used to discuss the problem of planar image expression with the real earth's spatial size, i.e.

\[
\begin{align*}
X &= R \cdot L \\
Y &= R \cdot B
\end{align*}
\]

(9)

We can take it as the detailed expression of curvilinear coordinates on the plan.

4.4 Spatial property analysis of the practical mode

1) Discussion on the geometrical characteristics

Eq. (9) is equal to the spherical equidistant tangential cylindrical projection with \( R \) as the radius of mean curvature. However, the practical concepts are different. The former is the planar image expression of the geodetic coordinate system \( (B, L) \), and it is a 2D measure space with enough ellipsoidal geometrical measurement precision for the whole world. Here, the 2D image expression is apart from the measurement, but they are closely united each other, while the latter is a 2D planar expression of a map projection and a planar measurement system. It takes the measure of 2D Euclidean spatial distance of \( \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \). This is just equal to the 2D planar paper and ruler, compass, protractor and other measure work cases. And the former does not have nor can use the measure work as the latter. There exist deformations in length, angle and area controlled in a definite range. But in the former situation, there is not any notable deformation in the whole world area, and it keeps good measurement properties.

It may be said that the coordinates of this model is of those as the equidistant tangential cylindrical projection’, and its measurement is of the geometrical system of the ellipsoid. Obviously the expression on the screen is a kind of relative partial distribution. It is impossible to take planar measurement. Therefore, from the view of conception, it is a “map projection”, but not an ordinary projection.

Traditionally, a map projection represents a 2D planar expression and planar measurement. These two are connected with each other. When the expression can not meet the need of measurement, extra notes are added. This is a historical difficulty of mapping. However, these two are divided on the computer. The expression represents the geometrical and topological characteristics of an object, and it also keeps the spatial characteristics, while the measurement can precisely state the situation of spatial measuring values.

2) Discussion on the topological properties

The geographical layers on the earth surface form a solid space. Theoretically, the topological properties of 3D and 2D are different. But to certain extent we can postulate that the geographical layers and zones are topological homeomorphism with an open sphere, i.e. homeomorphism with an ellipsoidal surface which is deleted by a point, or it can be looked as a homeomorphism with a plane.

Thus, theoretically, the adopted model and the applied model can keep the topological property of the earth surface or the information world of the earth. And the map projection also can keep the ordinary topological property.

3) Discussion on the definite resolution and geo-
metrical errors

The situations discussed in 1) and 2) are theoretically precise and strict. However, the practical geographical information data are discrete under definite resolution and precision. Therefore, there exist two conceptions of geometrical and topological resolutions. In other words, the errors will affect the identity between geometry and topology. And the bigger the error is, the bigger the effect will be. It is evident that in Gauss projection the seam produced between neighbouring projection zones may be seen as the topological non-identity caused by the error produced by the projection, at the same time, the wrong directional impression caused by some bigger meridional convergence may be seen as the variation of geometrical characteristics.

However, the modes (8) -- (9) mentioned above are theoretically and practically perfect in comparison with ordinary projections. Thus, the space and the projection selected by us can strictly and reliably guarantee the spatial geometrical and topological characteristics of the tremendous geographical space and even of the whole world geographical entities, and it is also suitable for the unified models of multi-resolution or even unclassified resolution.

4.5 The relationship between the geographical and geodetic coordinates

The above mentioned coordinates we used are geodetic ones, but in practice the coordinates drawn on the maps are geographical ones. We know the value of $\lambda$ is equal to $L$, but the value of $\Phi$ differs from $B$. In the situation of large scale maps or high precision is needed, the following formula can be used, where the geographic coordinates are known.

$$B = \Phi + B_2 \sin 2\Phi + B_4 \sin 4\Phi + B_6 \sin 6\Phi + B_8 \sin 8\Phi$$

(10)

where $B_2, B_4, B_6, B_8$ are constants.

Thus, the relevant $B$ in Eqs. (8) and (9) should be changed into $\Phi$. And the values in Eq. (7) should take the coordinates on the map and calculated according to Eq. (10).

5 The problem of data accommodation and output

The data source of the products of large GIS and digital earth is very wide and complicated. It can mainly be generalized into the following cases.

1) Several ellipsoids in different periods. The reduction methods of several coordinate systems into a united or special ellipsoid have been clearly solved in many literatures, and there are not many problems left.

2) The vector data in vector mode or partial vector data in combined mode are generally organized in the form of discrete geometrical data and topological data. Such as DLG and DEM can be included in this species. In this situation, only geometrical data should be transformed, no matter it is in the form of geodetic coordinates or map projection. The transformation of them can be fulfilled by use of ellipsoidal projection or map projection, point by point. The various analytical and numerical methods can be found in References [4~6].

3) For the digital products, the transformation of figures, DRG, DOR and other source materials is different from that of the data of discrete point. The demands for their transformation is that the geometrical coordinates of all of their pixels are correct and keep the continuity of figures, images, DRG and DOR. Among the sources DRG is probably the representative, especially the nationwide database of 1:250 000 has been constructed, and the DRG of 1:50 000 and 1:10 000 and even larger scale (some part of the territory) is being developed. In view of this situation, the difference of points in the whole map should not be too large. Otherwise, the quality of the images would be evidently lower.

Therefore model (9) should be partly reconstructed to satisfy the above demands, i.e.

$$X = R_1 \cdot \lambda$$
$$Y = R_1 \cdot \Phi$$
$$R_1 = \cos(P_m) \cdot R$$

(11)

where $R_1$ is the defined mean radius of curvature of the area, $P_m$ is the radius of the relevant latitude. For the area of whole China its territory ranges from 0 to 55 in north semi-sphere. Let $R_1$ be the radius of curvature at the latitude 38, i.e. 0.788R. Thus in the area of whole China the images of DRG product at the scale 1:50 000 the maximum ratio of transverse condensation is 1/5 only, and the vertical
variation is very small, and it would not affect the quality of the primary DRG in Gauss projection.

The Wanxiang Map Transformation Software can be used for DRG transformation according to Eq. (11), which is a general transformation tool based on the broad sense numerical map projection transformation method. The application of this method in past several years proves that it is stable and reliable.

For the transformation of general used maps at scale 1:10,000 to 1:1,000,000, the suggested control points for each map are listed in Table 1.

For DEM interpolation in the model of Eq. (13), it is natural to use the longitudinal and latitudinal differences. Generally speaking, for 1:50,000 scale map sheet we take 0.5" or 1" interval and the transformation is seamless no matter how large the area is. For DOM, DLG, and DRG which are similar to the topographical maps, the transformation can be processed in the same way.

| Scale     | Map sheet | Density of cont. point | No. of c. p. |
|-----------|-----------|------------------------|--------------|
|           | Long. diff | Lat. diff.             | Long. diff.  | Lat. diff. |               |
| 1:10 000  | 3'45"     | 2'30"                  | 2'30"        | 4          |
| 1:25 000  | 7'30"     | 5'                     | 7'30"        | 9          |
| 1:50 000  | 15'       | 10'                    | 15'          | 49         |
| 1:100 000 | 30'       | 20'                    | 15'          | 9          |
| 1:250 000 | 1'30"     | 1'                     | 1'           | 49         |
| 1:500 000 | 3'        | 2'                     | 30'          | 49         |
| 1:1 000 000 | 6'     | 4'                     | 1'           | 49         |

6 The advantages of the model

1) The coordinate system of GIS in this model uses the united geodetic coordinate system. Its structure is simple, rigorous and keeps precise measurement and preserves the geometric and topological characteristics.

2) It keeps continuing visualization in a very large area and even in the whole world, and is suitable for different resolution and scales.

3) It is suitable for various data sources, outside systems, geographical data in different mode and dynamic situation. Its spatial data can be either geodetic coordinates that may directly be used, or rectangular coordinates in a known map projection which can be inversely transformed into geodetic coordinates. Furthermore, in case that some control points are known, the planar coordinates can be transformed by using the transformation method in broad sense and related software to solve the input problem.

Therefore, this mode can connect smoothly the different static and dynamic data body and data sources and various applications.

7 Experiments of large scale GIS on the spatial mathematical basis

On the basis of the above mentioned spatial mathematical model, since 1995, we have been undertaking the making of DRG, DEM and the collection of vector and raster data in multi-resolution. And Wanxiang GIS software has been developed. In the mean time, industrial experiments of practical GIS have been carried out.

1) We have accomplished the water supply information system of Wuhan city. In which more than 3,300 sheets of topomaps at scales of 1:250,000, 1:50,000, 1:10,000, 1:1,000 (partial) and 1:500 and the information data of water supply installations were collected.

2) The border guard information system for Yunnan armed police.

3) The flood prevention and drought-resistance information system for Xiaogan City.

4) The state ocean information system.

8 Conclusion

The practice shows that a specialized GIS system
for the vector data needed by DRG and DEM for a large area up to the whole country and even the whole world can be constructed quickly. The continuing visualization of 2-D and 3-D in multi-resolution in the whole area can be realized just as in the real place, as well as the query and analysis of various informations. This combination of vector and raster data for GIS is a technical trend for international advanced GIS. And the spatial mathematical basis put forward in this study is the important basis and the promise of success.

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