Oscillatory Behavior of Advanced Difference Equations with General Arguments

Şeyda Öcalan¹, Özkan Öcalan², Mustafa Kemal Yildiz³

¹T.C. Ministry of Education, Şehit Cengiz Topel Mithatpaşa Middle School, 24000, Erzincan, Turkey
or
Afyon Kocatepe University, Institute of Science, 03200 Afyonkarahisar, Turkey
²Akdeniz University, Faculty of Science
Department of Mathematics, 07058 Antalya, Turkey
³Afyon Kocatepe University, Faculty of Science and Arts
Department of Mathematics, ANS Campus, 03200 Afyonkarahisar, Turkey

Abstract. In this paper, we introduce some oscillation criteria for the first-order advanced difference equations with general arguments

\[ \nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \geq 1, \quad n \in \mathbb{N}, \]

where \( \{p_i(n)\} \) \( (i = 1, 2, \ldots, m) \) are sequences of positive real numbers, \( \{\tau_i(n)\} \) \( (i = 1, 2, \ldots, m) \) are sequences of integers and are not necessarily monotone such that \( \tau_i(n) \geq n \) \( (i = 1, 2, \ldots, m) \). An example illustrating the results is also given.

1. Introduction

In this paper, we study the oscillatory behavior of all solutions of the first-order advanced difference equations

\[ \nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1, \]  

where \( \{p_i(n)\} \) \( (i = 1, 2, \ldots, m) \) are sequences of positive real numbers, \( \{\tau_i(n)\} \) \( (i = 1, 2, \ldots, m) \) are sequences of integers and are not necessarily monotone such that

\[ \tau_i(n) \geq n \quad \text{for} \quad n \geq 1. \]
Here, $\nabla$ denotes the backward difference operator $\nabla x(n) = x(n) - x(n - 1)$. By a solution of (1), we mean a sequence of real numbers $\{x(n)\}$ which is defined for $n \geq 0$ and satisfies (1) for all $n \geq 1$.

Recently, there are too many studies in literature on the oscillation theory of advanced (or delay) type differential or difference equations. See, for example, [1-18] and the references cited therein. As usual, a solution $\{x(n)\}$ of (1) is said to be oscillatory, for every positive integer $n_0$, there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $\{x(n)\}$ is oscillatory if it is neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

Throughout this paper, we are going to use the notation: $\sum_{i=0}^{k-1} A(i) = 0$.

Now, let’s recall some well-known oscillation results on this subject. For $m = 1$, equation (1) reduces to the following equation.

$$\nabla x(n) - p(n)x(\tau(n)) = 0, \ n \in \mathbb{N}, \ n \geq 1. \quad (3)$$

In 2002, Li and Zhu [15] proved that, when $\tau(n) = n + k$, if there exists an integer $n_1 \geq 0$ and a positive integer $l$ such that

$$\sum_{n=n_1+k}^{\infty} p(n) \left( \frac{k+1}{k} \right)^l q_{-l+1}^{1/k+1}(n) - 1 = \infty,$$

where

$$q_1(n) = \sum_{i=n-k}^{n-1} p(i), \ n \geq k,$$

$$q_{j+1}(n) = \sum_{i=n-k}^{n-1} p(i)q_j(n), \ j \geq 1, \ n \geq (j+1)k,$$

then all solutions of (3) oscillate.

In 1991, Győri and Ladas [12] studied the following first order linear difference equation with advanced argument $\tau(n) = n + \sigma$.

$$\Delta x(n) - p(n)x(n + \sigma) = 0, \ n \geq 0, \quad (4)$$

where $\Delta$ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, $\sigma \geq 2$ is a positive integer and the authors proved that if

$$\limsup_{n \to \infty} \sum_{i=n}^{n+\sigma-1} p(i) > 1, \quad (5)$$

or

$$\liminf_{n \to \infty} \sum_{i=n+1}^{n+\sigma} p(i) > \left( \frac{\sigma - 1}{\sigma} \right)^{\sigma}, \quad (6)$$

then all solutions of (4) oscillate.

In 2007, Ocalan and Akın [16] analyzed the following first order linear difference equations

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(n-k_i) = 0, \ n \geq 0, \quad (7)$$

where $p_i(n) \leq 0$ and $k_i \leq -1$ for $i = 1, 2, \ldots, m$, and obtained some results for the oscillation of all solutions of (7) (See also [17]). Furthermore, when $p_i(n) = p_i \ (i = 1, 2, \cdots, m)$ in (7), see [12, Theorems 7.2.1 and 7.3.1].
In 2012, Chatzarakis and Stavroulakis [1] proved that if \( \{\tau(n)\} \) is nondecreasing and
\[
\limsup_{n \to \infty} \sum_{j=n}^{\tau(n)} p(j) > 1,
\]
then all solutions of (3) oscillate. We note that, in [1], the authors assumed that \( \tau(n) \geq n + 1, \ n \geq 1 \). We would like to state that, in fact, if \( \tau(n) \geq n, \ n \geq 1 \) is taken, then all results are valid in [1].

Also, in 2012, Chatzarakis and Stavroulakis [1] proved that if \( \{\tau(n)\} \) is not necessarily monotone and
\[
\limsup_{n \to \infty} \sigma(n) \sum_{j=n}^{\tau(n)} p(j) > 1,
\]
where
\[
\sigma(n) = \max_{1 \leq s \leq n} \{\tau(s)\}, \ s \in \mathbb{N},
\]
then all solutions of (3) oscillate. Unfortunately, we consider this result is not applicable. Indeed, if we examine this result, it can not be proved like Theorem 2.1 in [1]. To see this, by using the proof of Theorem 2.1 in [1], since \( \sigma(n) \geq \tau(n) \) and \( \{\tau(n)\}, \{\sigma(n)\} \) are eventually nondecreasing, from equation (3), we have
\[
\nabla x(n) - p(n)x(\sigma(n)) \leq 0, \ n \geq 1.
\]

Now, summing up (11) from \( n \) to \( \sigma(n) \), we obtain
\[
x(\sigma(n)) - x(n-1) - \sum_{j=n}^{\sigma(n)} p(j)x(\sigma(j)) \leq 0,
\]
and the proof is stopped here (see the proof of Theorem 2.1 in [1]). Hence, Theorem 2.1” and Theorem 2.4” are not applicable in [1].

In 2016, Ocalan and Ozkan [18] proved that if \( \{\tau(n)\} \) is not necessarily monotone and
\[
\limsup_{n \to \infty} \sum_{j=n}^{h(n)} p(j) > 1,
\]
where \( h(n) = \min_{n \leq s} \{\tau(s)\}, \ s \in \mathbb{N} \), then all solutions of (3) oscillate. Also, the authors [18], regarding the lim inf condition, tried to obtain a condition for the oscillatory solution of the equation (3) when \( \{\tau(n)\} \) is not necessarily monotone. Unfortunately, the authors have made a mistake in the proof of Theorem 2.4 in [18], caused by induction. That is, the proof of Theorem 2.4 in [18] is invalid. Therefore, one of the aim of this paper is to obtain the lim inf condition for the equation (3) to be oscillatory.

2. Main Results

In this section, we introduce a new sufficient condition, regarding the condition lim inf, for the oscillation of all solutions of (3) when \( \{\tau(n)\} \) is not necessarily monotone. Set
\[
h(n) := \min_{n \leq s} \{\tau(s)\}, \ s \in \mathbb{N}.
\]

Obviously, \( \{h(n)\} \) is nondecreasing and \( \tau(n) \geq h(n) \) for all \( n \geq 1 \). The following lemmas will be needed in the proof of the Theorem 2.3.

The following one was given in [18].
Lemma 2.1. [18] Assume that (13) holds and \( m > 0 \). Then, we have
\[
m = \lim_{n \to \infty} \inf \sum_{j=n+1}^{h(n)} p(j) = \lim_{n \to \infty} \inf \sum_{j=n+1}^{\tau(n)} p(j),
\]
where \( \{h(n)\} \) is defined by (13).

Lemma 2.2. Suppose \( p(n) > 0 \) and \( \{x(n)\} \) is positive solution of the following inequalities
\[
\nabla x(n) - p(n)x(n) \geq 0, \quad n \geq s. \tag{14}
\]
Then
\[
x(n) \geq \exp \left( \sum_{j=s+1}^{n} p(j) \right)x(s), \quad n \geq s. \tag{15}
\]

Proof. Dividing (14) by \( x(n) \), we have
\[
\nabla x(n) x(n) - p(n) \geq 0, \quad n \geq s. \tag{16}
\]
Summing up (16) from \( s + 1 \) to \( n \), we obtain
\[
\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} - \sum_{j=s+1}^{n} p(j) \geq 0. \tag{17}
\]
Now, we get
\[
\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} = \sum_{j=s+1}^{n} \frac{x(j) - x(j - 1)}{x(j)} = (n - s) - \sum_{j=s+1}^{n} \frac{x(j - 1)}{x(j)} \leq (n - s) - \sum_{j=s+1}^{n} \left( 1 + \ln \frac{x(j - 1)}{x(j)} \right) = \sum_{j=s+1}^{n} \ln \frac{x(j)}{x(j - 1)},
\]
where we have used the \( e^x \geq 1 + x \) for \( x \geq 0 \). So, we obtain
\[
\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} \leq \sum_{j=s+1}^{n} \ln \frac{x(j)}{x(j - 1)} = \ln x(n) - \ln x(s) = \ln \frac{x(n)}{x(s)}.
\]
Finally, from (17), we have
\[
\ln \frac{x(n)}{x(s)} = \sum_{j=s+1}^{n} p(j) \geq 0,
\]
or
\[
x(n) \geq \exp \left( \sum_{j=s+1}^{n} p(j) \right)x(s),
\]
which is desirable. \( \Box \)
Theorem 2.3. Assume that (2) holds. If $\{\tau(n)\}$ is not necessarily monotone and

$$\liminf_{n \to \infty} \sum_{j=n+1}^{\tau(n)} p(j) > \frac{1}{e},$$

then all solutions of (3) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a positive nonoscillatory solution $x(n)$ of (3). Since $-x(n)$ is also a solution of (3), we can confine our discussion only to the case where the solution $x(n)$ is eventually positive. Then, there exists $n_1 > n_0 \geq 1$ such that $x(n), x(\tau(n)) > 0$, for all $n \geq n_1$. Thus, from (3) we have

$$\nabla x(n) = p(n)x(\tau(n)) \geq 0,$$

which means that $\{x(n)\}$ is an eventually nondecreasing. In view of this and taking into account that $\tau(n) \geq h(n) \geq n$, (3) gives

$$\nabla x(n) - p(n)x(h(n)) \geq 0, \quad n \geq n_1$$

and

$$\nabla x(n) - p(n)x(n) \geq 0, \quad n \geq n_1.$$  (20)

On the other hand, by using Lemma 2.1 and from (18), it follows that there exists a constant $c > 0$ such that

$$\sum_{j=n+1}^{h(n)} p(j) \geq c > \frac{1}{e}, \quad n \geq n_2 > n_1.$$  (21)

So, by Lemma 2.2 and (20), we obtain

$$x(h(n)) \geq \exp\left(\sum_{j=n+1}^{h(n)} p(j)\right) x(n) \quad \text{for all } h(n) \geq n.$$  (22)

Since $e^x \geq cx$ for $x \in \mathbb{R}$, from (21) and (22), we get

$$x(h(n)) \geq c^n x(n) \geq (ec)n x(n),$$  (23)

where $ec > 1$. Thus, from (19) and (23), we have

$$\nabla x(n) - p(n)(ec)x(n) \geq 0, \quad n \geq n_2.$$  

Let $p_1(n) := (ec)p(n)$. So, we obtain

$$\nabla x(n) - p_1(n)x(n) \geq 0, \quad n \geq n_2.$$  (24)

By using Lemma 2.2, we get

$$x(h(n)) \geq \exp\left(\sum_{j=n+1}^{h(n)} p_1(j)\right) x(n) \quad \text{for all } h(n) \geq n.$$  (25)
Thus, from (21) and (25), we have

\[
x(h(n)) \geq \exp \left\{ \sum_{j=n+1}^{h(n)} (ec) p(j) \right\} x(n)
\]

\[= \exp \left\{ (ec) \sum_{j=n+1}^{h(n)} p(j) \right\} x(n) \geq \exp \left\{ ec^2 \right\} x(n)
\]

\[\geq (ec)^2 x(n) .
\]

Repeating the above procedures, it follows that by induction for any positive integer \(k\), we obtain

\[
\frac{x(h(n))}{x(n)} \geq (ec)^k \text{ for sufficiently large } n. \tag{26}
\]

On the other hand, from (21), there exists \(n^* \in (n, h(n)]\), \(n^* \in \mathbb{N}\) such that

\[
\sum_{j=n}^{n'} p(j) \geq \frac{c}{2} \text{ and } \sum_{j=n^*}^{h(n)} p(j) \geq \frac{c}{2}. \tag{27}
\]

Summing up (19) from \(n + 1\) to \(n^*\), we obtain

\[
x(n^*) - x(n) - \sum_{j=n+1}^{n'} p(j)x(h(j)) \geq 0.
\]

Now, using (27) and the fact that the functions \(\{x(n)\}\) and \(\{h(n)\}\) are nondecreasing, we have

\[
x(n^*) \geq x(h(n + 1)) \sum_{j=n+1}^{n'} p(j) \geq x(h(n)) \sum_{j=n+1}^{n'} p(j),
\]

or

\[
x(n^*) \geq x(h(n)) \frac{c}{2}. \tag{28}
\]

Summing up (19) from \(n^*\) to \(h(n)\), and using the same arguments we get

\[
x(h(n)) - x(n^* - 1) - \sum_{j=n^*}^{h(n)} p(j)x(h(j)) \geq 0,
\]

or

\[
x(h(n)) - x(h(n^*)) \sum_{j=n^*}^{h(n)} p(j) \geq 0,
\]

or

\[
x(h(n)) \geq x(h(n^*)) \frac{c}{2}. \tag{29}
\]

Combining the inequalities (28) and (29), we obtain

\[
x(n^*) \geq x(h(n)) \frac{c}{2} \geq x(h(n^*)) \left( \frac{c}{2} \right)^2,
\]
or
\[
\frac{x(h(n'))}{x(n')} \leq \left(\frac{2}{c}\right)^2 < +\infty,
\]

i.e., \(\liminf_{n \to \infty} \frac{x(h(n))}{x(n)}\) exists. This contradicts with (26). So, the proof of the theorem is completed. \(\square\)

A slight modification in the proofs of Theorem 2.3 and [18, Theorem 2.3] leads to the following result.

**Theorem 2.4.** Assume that all the conditions of Theorem 2.3 or (12) hold. Then

(i) the difference inequality
\[
\nabla x(n) - p(n)x(\tau(n)) \geq 0, \quad n \in \mathbb{N}, \quad n \geq 1
\]

has no eventually positive solutions,

(ii) the difference inequality
\[
\nabla x(n) - p(n)x(\tau(n)) \leq 0, \quad n \in \mathbb{N}, \quad n \geq 1
\]

has no eventually negative solutions.

**Example 2.5.** Consider
\[
\nabla x(n) - p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1.
\]  

(30)

We take \(p(n) = 0.19\) and \(\tau(n) = n + 2\). We observe that
\[
\limsup_{n \to \infty} \sum_{j=n}^{n+2} p(j) = 0.57 \neq 1.
\]

shows that condition (12) fails. However, since
\[
\liminf_{n \to \infty} \sum_{j=n+1}^{n+2} p(j) = 0.38 > \frac{1}{e},
\]

every solution of (30) is oscillatory.

3. Equations with several arguments

Now, we consider the first-order advanced difference equations with several arguments and coefficients
\[
\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1
\]  

(31)

where \(\{p_i(n)\} (i = 1, 2, \cdots, m)\) are positive sequences, \(\{\tau_i(n)\} (i = 1, 2, \ldots, m)\) are sequences of integers and are not necessarily monotone such that
\[
\tau_i(n) \geq n \quad \text{for all} \quad n \in \mathbb{N}, \quad n \geq 1.
\]  

(32)

In this section, we present some new sufficient conditions for the oscillation of all solutions of (31).

In 2014, Chatzarakis et al. [2] proved that if \(\{\tau_i(n)\} (i = 1, 2, \cdots, m)\) are nondecreasing and
\[
\limsup_{n \to \infty} \sum_{j=n}^{n+n+1} \sum_{i=1}^{m} p_i(j) > 1,
\]

(33)
where \( \tau(n) = \min_{1 \leq i \leq m} \{ \tau_i(n) \} \), then all solutions of (31) oscillate.

Set
\[
h_i(n) := \inf_{s \geq n} \tau_i(s) \quad \text{and} \quad h(n) = \min_{1 \leq i \leq m} h_i(n), \quad n \geq n_0.
\]
Clearly, \( \{ h_i(n) \} \) \((i = 1, 2, \cdots, m)\) are nondecreasing and \( \tau_i(n) \geq h_i(n) \geq h(n) \) for all \( n \geq n_0 \). Now, we have the following result.

**Theorem 3.1.** Assume that (32) holds. If \( \{ \tau_i(n) \} \) \((i = 1, 2, \cdots, m)\) are not necessarily monotone and
\[
\limsup_{n \to \infty} \sum_{j=n}^{m} \sum_{i=1}^{m} p_i(j) > 1,
\]

or
\[
\liminf_{n \to \infty} \sum_{j=n}^{m+1} \sum_{i=1}^{m} p_i(j) > \frac{1}{e},
\]

where \( \tau(n) = \min_{1 \leq i \leq m} \{ \tau_i(n) \} \) and \( h(n) \) is defined by (34), then all solutions of (31) oscillate.

**Proof.** Assume, for the sake of contradiction, that there exists a positive nonoscillatory solution \( x(n) \) of (31). Then there exists \( n_1 > n_0 \) such that \( x(n), x(\tau_i(n)) > 0 \) for all \( n \geq n_1 \). Thus, from (31) we have
\[
\nabla x(n) - \left( \sum_{i=1}^{m} p_i(n) \right) x(\tau_i(n)) \geq 0.
\]
Comparing (35) and (36), we obtain a contradiction to Theorem 2.4. Here, we have used the following equality
\[
\liminf_{n \to \infty} \sum_{j=n}^{m+1} \sum_{i=1}^{m} p_i(j) = \liminf_{n \to \infty} \sum_{j=n}^{m} \sum_{i=1}^{m} p_i(j),
\]
which is easily obtained as similar to the proof of Lemma 2.1. \( \square \)

A slight modification in the proof of Theorem 3.1 leads to the following result.

**Theorem 3.2.** Assume that all the conditions of Theorem 3.1 hold. Then
(i) the difference inequality
\[
\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \geq 0, \quad n \in \mathbb{N}, \quad n \geq 1
\]
has no eventually positive solutions,
(ii) the difference inequality
\[
\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad n \in \mathbb{N}, \quad n \geq 1
\]
has no eventually negative solutions.
References

[1] G. E. Chatzarakis, I. P. Stavroulakis, Oscillations of difference equations with general advanced argument, Cent. Eur. J. Math. 10(2) (2012) 807–823.
[2] G. E. Chatzarakis, S. Pinelas, I. P. Stavroulakis, Oscillations of difference equations with several deviated arguments, Aequat. Math. 88 (2014) 105–123.
[3] G. E. Chatzarakis, I. Jadlovská, Oscillations in deviating difference equations using an iterative technique, J. Inequal. Appl. (2017) Paper No. 173, 24 pages.
[4] G. E. Chatzarakis, I. Jadlovská, Improved iterative oscillation tests for first-order deviating difference equations, Int. J. Difference Equ. 12(2) (2017) 185–210.
[5] G. E. Chatzarakis, L. Shaikhet, Oscillation criteria for difference equations with non-monotone arguments, Adv. Difference Equ. (2017) Paper No. 62, 16 pages.
[6] G. E. Chatzarakis, L. Horvat-Dmitrovic, M. Pasic, Oscillation tests for difference equations with several non-monotone deviating arguments, Math. Slovaca 68(5) (2018) 1083–1096.
[7] G. E. Chatzarakis, I. Jadlovská, Oscillations in difference equations with several arguments using an iterative method, Filomat 32(1) (2018) 255–273.
[8] G. E. Chatzarakis, I. Jadlovská, Difference equations with several non-monotone deviating arguments: Iterative oscillation tests, Dynamic Systems and Applications 27(2) (2018) 271-298.
[9] G. E. Chatzarakis, I. Jadlovská, Oscillations of deviating difference equations using an iterative method, Mediterr. J. Math. 16(1) (2019) Paper No. 16, 20 pages.
[10] L. H. Erbe, B. G. Zhang, Oscillation of discrete analogues of delay equations, Differential Integral Equations 2(3) (1989) 300–309.
[11] L. H. Erbe, Q. Kong, B. G. Zhang, Oscillation theory for functional differential equations, Marcel Dekker, New York, 1995.
[12] I. Győri, G. Ladas, Oscillation theory of delay differential equations with applications, Clarendon Press, Oxford, 1991.
[13] G. S. Ladde, Oscillations caused by retarded perturbations of first order linear ordinary differential equations, Atti Acad. Naz. Lincei Rendiconti 63 (1978) 351–359.
[14] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, Oscillation theory of differential equations with deviating arguments, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, 1987.
[15] X. Li, D. Zhu, Oscillation of advanced difference equations with variable coefficients, Ann. Differential Equations 18(3) (2002) 254–263.
[16] O. Ocalan, O. Akın, Oscillation properties for advanced difference equations, Novi Sad J. Math. 37(1) (2007) 39–47.
[17] O. Ocalan, Linearized oscillation of nonlinear difference equations with advanced arguments, Arch. Math. (Brno) 45(3) (2009) 203–212.
[18] O. Ocalan, U. M. Özkan, Oscillations of dynamic equations on time scales with advanced arguments, Int. J. Dyn. Syst. Differ. Equ. 6(4) (2016) 275–284.