Comments on “Quantum Control by Decompositions of SU(2)"

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ABSTRACT: The purpose of this note is to i) point out some typographical errors in [1], and ii) to observe that the main result of [1] is valid without the restriction that the matrices $A$ and $B$ in [1] be orthogonal.

1 Some Typos and Improvements

In [1] a constructive procedure for factorizing any $SU(2)$ matrix $S$ in the form:

$$S = \prod_{k=1}^{Q} e^{a_k A + b_k B}$$

with: i) $O1$: $a_k > 0$; and ii) $O2$: $|b_k| \leq C$ for an a priori prescribed bound $C$, was given. The only restrictions on the matrices $A$ and $B$ in the published version of [1] was that they be i) linearly independent; and ii) expressible as $A = aX + bY$ and $B = cX + dY$ for any two orthogonal matrices $X$ and $Y$ in $su(2)$ (the set of all $2 \times 2$ skew-Hermitian matrices with zero trace). Here orthogonality is with respect to the inner product $\langle A, B \rangle = \text{Trace} (AB^*)$ (equivalently, the vectors in $R^3$ corresponding to $A$ and $B$ are orthogonal with respect to the usual inner product on $R^3$). There was no requirement that the vectors $(a, b)$ and $(c, d)$ in $R^2$ should be orthogonal.

The first purpose of this note is to observe that the only requirement that $A$ and $B$ need satisfy is linear independence. Indeed, given two linearly independent such $A$ and $B$ one can always find (constructively) a unitary transformation $V \in U(2)$ such that $VAV^*$ and $VBV^*$ are expressible as possibly non-orthogonal linear combinations of $i\sigma_x$ and $i\sigma_y$. 
We leave the technical details to interested readers - there are several choices for such $V$’s. It is useful, in this regard, to note that for any $W \in SU(2)$, the transformation $R_V : su(2) \rightarrow su(2), R_V(A) = VAV^*$ is in $SO(3)$ when $su(2)$ is identified with $R^3$. It then remains to find an $SO(3)$ matrix which simultaneously nulls out the $i\sigma_z$ component of $A$ and $B$. This can be easily found, for instance, by modifying the construction behind Givens rotations. Now any one of the two $SU(2)$ matrices corresponding to this $SO(3)$ matrix will achieve the desired purpose. Further as $V$ can be found completely constructively, this coordinate change is known. So indeed, one can proceed by assuming that $A = aX + bY$ and $B = cX + dY$ for any two orthogonal matrices $X$ and $Y$ in $su(2)$ and still obtain a fully explicit technique for preparing any desired unitary transformation in $SU(2)$ via controls whose pulse area (or power, depending on the interpretation of the $b_k$) is bounded $a\ priori$. The existence of such matrices, $V$, was missed due to an oversight in [1] and thus the paper concluded (incorrectly) that the general case of arbitrary linearly independent $A, B$ perhaps needed more work. Indeed, all of the above ingredients were already present in [1] and the related paper, [2]. The paper, [3], also provides an explicit $V$ which can be used to null out the $i\sigma_x$ component, but without relating it to $R_V$ or Givens rotations.

One additional “improvement” that is immediate in [1] is that the values of the number of factors, $Q$, given in Table I on Page 7 can obviously be lowered, in many instances, by concatenating exponentials of matrices which are constant multiples of each other.

The second purpose of this note is to rectify certain typographical errors which seem to have crept in during the typesetting process. The main error is that $\sigma_z$ seems to have been replaced by $\sigma_x$ at several points in the published version. These junctures are as follows:

1. Everywhere on Page 4, except the headings for subsections 1 and 2 and the headings for Algorithms I and II on Page 4, $\sigma_z$ should read as $\sigma_x$.

2. On Page 5, everywhere in Algorithm II (continued from Page 4) $\sigma_x$ should be replaced by $\sigma_z$.

3. On Page 5, in the heading for Algorithm III, $\sigma_x$ should be replaced by $\sigma_z$. 

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4. In addition, everywhere on Page 3, $\sigma_a$ should be replaced by $\sigma_z$.

References

[1] V. Ramakrishna, K. Flores, H. Rabitz and R. Ober, Phys. Rev. A, 62, 054309, 2000.

[2] V. Ramakrishna, R. J. Ober, X. Sun, O. Steuernagel, J. Botina and H. Rabitz, Phys. Rev. A, 61, 032106, 2000.

[3] D. D’Alessandro, Systems-Control Letters, 41, 213, 2000.