Dynamic complexity of a microbial pesticide model with fear effect

Xiaoxiao Cui, Yonghui Xia

College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua, 321004, China

Abstract

Microbial pesticides can avoid many of negative effects of traditional chemical pesticides. This paper considers a model of entomopathogenic nematodes killing the target insects, and the target insects can perceive fear, which will affect the birth rate of the target insects. In the model, we achieve the purpose of restricting or eliminating pests by continuously releasing nematodes. By analyzing the stability of the equilibrium of the model and the stability of the Hopf bifurcating periodic solution, the best solution to control pests is obtained. Our theoretical results are verified by examples and numerical simulations. Finally, a comparison with the previous work on the model without fear effect (Wang and Chen [10]) is presented. A conclusion ends the paper.

Keywords. microbial pesticide model; fear effect; Hopf bifurcation.

2020 Mathematics Subject Classification. 92D25; 34D20; 37G15.
1. Introduction

Pesticides can control agricultural pests and increase food production, but they also bring many disadvantages [1–3]. After pesticides are applied, part of them are adhere to plants, or penetrate into the plant body and remain, contaminating grains, vegetables, fruits, etc., and the other part of them are scatter on the soil or evaporate, escape into the air, or flow into the rivers with rainwater, polluting water bodies and aquatic organisms, and eventually entering the human body, causing various chronic or acute diseases. The unreasonable use of pesticides, especially organic pesticides, not only poses a serious threat to human health, but also causes crop phytotoxicity, human and livestock poisoning, excessive residues of agricultural products, pest resistance and environmental pollution, etc.

Microbial pesticides are made from living microorganisms. In nature, there are many microorganisms that have pathogenic effects on pests, and using this pathogenicity to control pests is an effective biological control method. From these pathogenic microorganisms, select bacteria that are convenient to use, stable in efficacy, safe to humans, animals, and the environment to make microbial insecticides. Compared with chemical synthetic pesticides, microbial pesticides have many advantages [4–6], including: (1) they are harmless to organisms other than the target; (2) pests are not easy to develop resistance; (3) they can protect natural enemies of pests; (4) they do not pollute the environment. These characteristics make microbial pesticides a class of pesticides suitable for integrated pest control.

Entomopathogenic nematode is a kind of new-type and promising microbial pesticide [7–9]. It releases a kind of symbiotic bacteria in its intestine into the blood cavity of the host insect, and then the symbiotic bacteria multiply in the blood cavity and produce antibacterial substances and toxins, causing the host insect to suffer from sepsis and die. Guangjun Ren, deputy dean and researcher of Sichuan Academy of Agricultural Sciences, said, “Entomopathogenic nematode, as specialized parasitic natural enemies of insects, is a kind of microbial pesticide
with the dual characteristics of natural enemies and pathogenic microorganisms, and is an important biological control factor for pests. It can effectively control pests, and it is safe for non-target organisms and the environment. Therefore, it has great application potential in the sustainable management of pests”.

In 2009, Wang and Chen [10] formulated the following mathematical model in order to investigate the dynamics of nematodes attacking pests:

$$\begin{align*}
\frac{dx}{dt} &= rx - cxy, \\
\frac{dy}{dt} &= cxy^2 - my,
\end{align*}$$

(1)

where $x(t), y(t)$ denotes the density of pests and entomopathogenic nematodes, respectively. $r$ denotes the birth rate of pests and $m$ denotes the death rate of entomopathogenic nematodes. Moreover, the effects of nematodes’ predation behavior on pests and nematodes are expressed as $-cxy$ and $+cxy^2$. Subsequently, Wang and Chen [11] used the Poincaré map to analyze dynamic behaviors of the impulsive state of model (1). In 2017, Wang et al. [12] considered system (1) with the Monod growth rate. And in 2021, Wang [13] studied model (1) with density dependent for pests.

There is a lot of data showing that a species’ fear of its natural predator can affect its birth rate, sometimes it’s a bigger impact than just killing them [14–20]. In 2016, Wang [17] investigated the model with the fear effect in predator-prey interactions, and in quick succession, Wang and Zou [18] considered the above model with adaptive avoidance of predators. Motivated by Wang and Chen [10], Wang [17], Wang and Zou [18], in this paper, we propose a model (1) with fear factor having an impact on the pests birth rate. The new model is formulated as follows:

$$\begin{align*}
\frac{dx}{dt} &= \frac{rx}{1 + ky} - cxy, \\
\frac{dy}{dt} &= cxy^2 - my,
\end{align*}$$

(2)

where $\frac{1}{1+ky}$ is the fear function, $k$ is the level of fear, it means that if the density of nematodes or the level of fear was zero, there was no effect on pests; with the
increase of the density of nematodes or the level of fear, the birth rate of pests would decrease.

Through the analysis of system (2), we derive that in any case, there will be not a steady state, that is, the density of pests will keep increasing, reach destructive numbers and take a toll on the economy. Therefore, we will through continuous release of nematodes to control the density of pests. And then we have the following model:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{rx}{1 + ky} - cxy, \\
\frac{dy}{dt} &= cxy^2 - my + u,
\end{align*}
\]

where \(u\) is the release rate of entomopathogenic nematodes, other parameters are the same as in systems (1) and (2). And all parameters \(r, k, c, m, u\) are positive.

We pay particular attention to the influence of fear effect on the model. By analyzing the stability of the equilibrium of the model and the stability of the Hopf bifurcation periodic solution, the best solution to control pests is obtained. A comparison with the previous work on the model without fear effect (Wang and Chen [10]) is given in our conclusion.

The structure of this paper is organized as follows. Next section is to study the stability of the equilibria and the nonexistence of limit cycle of system (2). In Section 3, we prove the existence of Hopf bifurcation of system (3). In Section 4, some numerical simulations are presented to show the feasibility of the main results. Finally, a conclusion ends the paper.

2. Dynamic behavior of system (2)

**Theorem 1.** System (2) always has two equilibria, a boundary equilibrium \(E_0(0, 0)\) and a positive equilibrium \(E_1(x_1, y_1)\). Furthermore, \(E_0\) is a saddle, \(E_1\) is an unstable node or focus.
Proof. The equilibria of \((2)\) satisfy the equations
\[
\begin{align*}
rx \frac{1}{1 + ky} - cxy &= 0, \\
cxy^2 - my &= 0.
\end{align*}
\]
Obviously, equation \((4)\) has nonnegative solutions
\[x_0 = 0, \quad y_0 = 0,\]
and
\[x_1 = \frac{2km}{-c + \sqrt{c^2 + 4ck}}, \quad y_1 = \frac{-c + \sqrt{c^2 + 4crk}}{2ck}.
\]
Consider the Jacobian matrix of system \((2)\)
\[
J = \begin{pmatrix}
r & cy \\ cy^2 - r & 2cxy - m
\end{pmatrix}.
\]
The Jacobian matrix at \(E_0(0, 0)\) is
\[
J(E_0) = \begin{pmatrix}
r & 0 \\ 0 & -m
\end{pmatrix}.
\]
It is easy to see that
\[
\lambda_1(E_0) = r > 0, \quad \lambda_2(E_0) = -m < 0,
\]
thus, \(E_0\) is a saddle. The Jacobian matrix at \(E_1(x_1, y_1)\) is
\[
J(E_1) = \begin{pmatrix}
0 & -r & cx \\ cy^2 & 0 & m
\end{pmatrix}.
\]
According to the relationship between the matrix and its corresponding eigenvalues, it can be known
\[
\lambda_1(E_1) + \lambda_2(E_1) = tr(J(E_1)) = m > 0,
\]
\[
\lambda_1(E_1) \cdot \lambda_2(E_1) = detJ(E_1) = cy^2 \left( \frac{r & cx}{1 + ky} \right) > 0,
\]
\[
\begin{align*}
\lambda_1(E_0) &= r > 0, \\
\lambda_2(E_0) &= -m < 0,
\end{align*}
\]
\[
\begin{align*}
x_1 &= \frac{2km}{-c + \sqrt{c^2 + 4ck}}, \\
y_1 &= \frac{-c + \sqrt{c^2 + 4crk}}{2ck}.
\end{align*}
\]
\[
J = \begin{pmatrix}
r & cy \\ cy^2 - r & 2cxy - m
\end{pmatrix}.
\]
\[
J(E_0) = \begin{pmatrix}
r & 0 \\ 0 & -m
\end{pmatrix}.
\]
\[
\begin{align*}
\lambda_1(E_0) + \lambda_2(E_0) &= tr(J(E_1)) = m > 0, \\
\lambda_1(E_0) \cdot \lambda_2(E_0) &= detJ(E_1) = cy^2 \left( \frac{r & cx}{1 + ky} \right) > 0,
\end{align*}
\]
and then, we have
\[ \text{Re}(\lambda_1(E_1)) > 0, \quad \text{Re}(\lambda_2(E_1)) > 0. \]
Thus, \( E_1 \) is an unstable node or focus.

**Theorem 2.** System (2) has no limit cycle in the first quadrant.

**Proof.** Set \[ P(x, y) = \frac{rx}{1 + ky} - cxy, \quad Q(x, y) = cxy^2 - my. \]
Choosing Dulac function \[ B(x, y) = \frac{1}{xy}. \]
Functions \( P, Q, B \) are continuously differentiable in the first quadrant, and
\[ \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = \frac{\partial}{\partial x} \left[ \frac{1}{y(1 + ky)} - c \right] + \frac{\partial}{\partial y} \left( cy - \frac{m}{x} \right) = c > 0. \]
It follows from Bendixon-Dulac Theorem that there is no limit cycle in the first quadrant.

From the above analysis, we observe that system (2) will not approach a stable state at any time and under any conditions. This is really bad from a biological standpoint. The increasing number of pests will cause damage to the environment and the economy. Therefore, it is necessary to take necessary measures to improve this situation. Below we discuss dynamic behavior of model (3) under the condition of continuous release of nematodes.

### 3. Dynamic behavior of system (3)

At first, we nondimensionalise system (3) by writing
\[ \bar{y} = \frac{c}{r}y, \quad \tau = rt, \]
then system (5) can be turned into
\begin{align*}
    \frac{dx}{d\tau} &= \frac{x}{1 + kr\bar{y}} - x\bar{y}, \\
    \frac{d\bar{y}}{d\tau} &= x\bar{y}^2 - m\bar{y} + cu\bar{y}^2.
\end{align*}
(5)

Taking \( \bar{k} = \frac{kr}{c} \), \( \bar{m} = \frac{m}{r} \), \( \bar{u} = \frac{cu}{r^2} \), and still replace \( \bar{y}, \tau, \bar{k}, \bar{m}, \bar{u} \) with the original variable \( y, t, k, m, u \), and then, system (5) becomes
\begin{align*}
    \frac{dx}{dt} &= \frac{x}{1 + ky} - xy, \\
    \frac{dy}{dt} &= xy^2 - my + u.
\end{align*}
(6)

**Theorem 3.** System (6) always has a boundary equilibrium \( E_2(0, y_2) \). In addition,

(i) if \( u < u_0 \), then system (6) has a positive equilibrium \( E_3(x_3, y_3) \);

(ii) if \( u \geq u_0 \), then system (6) has no other equilibrium except \( E_2 \).

**Proof.** The equilibria of system (6) satisfy
\begin{align*}
    \frac{x}{1 + ky} - xy &= 0, \\
    xy^2 - my + u &= 0.
\end{align*}
(7)

By calculations, the above system has a fixed solution
\( x_2 = 0, \ y_2 = \frac{u}{m} \).

In addition, from \( \frac{1}{1 + ky} - y = 0 \), we have
\( y_3 = \frac{\sqrt{1 + 4k} - 1}{2k} \),
and then, it follows from the second equation of system (7) that
\( x_3 = \frac{my_3 - u}{y_3^2} \).

It can be seen from non-negativity of the equilibria that \( x_3 > 0 \) i.e., \( u < my_3 := u_0 \).  \[\square\]
Theorem 4. For equilibrium $E_2(0, y_2)$,

(i) if $u < u_0$, then $E_2$ is a saddle;
(ii) if $u > u_0$, then $E_2$ is a stable node;
(iii) if $u = u_0$, then $E_2$ is an attracting saddle node.

We remark that we need use the theorem of [21][Theorem 7.1] to prove (iii). In fact, it is a powerful tool to study the bifurcation of planar systems and applications to the biological systems (see e.g. [25–31]).

Proof. The Jacobian matrix of system (6) is

$$J = \begin{pmatrix} \frac{1}{1+ky} - y & -\frac{ky}{(1+ky)^2} - x \\ y^2 & 2xy - m \end{pmatrix}.$$ 

Thus, at $E_2(0, y_2)$,

$$J(E_2) = \begin{pmatrix} \frac{1}{1+ky_2} - y_2 & 0 \\ y_2^2 & -m \end{pmatrix}.$$ 

The eigenvalues of $J(E_2)$ are

$$\lambda_1(J(E_2)) = \frac{1}{1+ky_2} - y_2, \quad \lambda_2(J(E_2)) = -m < 0.$$

(i) If $y_2 < y_3$, i.e., $u < u_0$, $\lambda_1(J(E_2)) = \frac{1}{1+ky_2} - y_2 > 0$, $E_2$ is a saddle;
(ii) if $y_2 > y_3$, i.e., $u > u_0$, $\lambda_1(J(E_2)) = \frac{1}{1+ky_2} - y_2 < 0$, $E_2$ is a stable node;
(iii) if $y_2 = y_3$, i.e., $u = u_0$, in where $E_2$ and $E_3$ coincide as a point, at this time,

$$\lambda_1(J(E_2)) = \frac{1}{1+ky_2} - y_2 = 0.$$

In order to recognize the type and stability of $E_2$, at first, translating $E_2$ to the origin by transformation $(X, Y) = (x, y - y_2)$, and performing Taylor expansion of system (6) at the origin to the third order, and noticing that

$$\frac{1}{1+ky_2} - y_2 = 0, \quad my_2 - u = 0.$$
Thus, we have
\[
\begin{align*}
\frac{dX}{dt} &= -(ky_2^2 + 1)XY + k^2y_2^3XY^2 + o(|X, Y|^4), \\
\frac{dY}{dt} &= y_2^2X - mY + 2y_2XY + XY^2.
\end{align*}
\] (8)

Taking transformation \((\tilde{X}, \tilde{Y}) = (X, X - \frac{m}{y_2^2}Y)\), system (8) gives
\[
\begin{align*}
\frac{d\tilde{X}}{dt} &= -\frac{y_2^2(ky_2^2 + 1)}{m} \tilde{X}^2 + \frac{y_2^2(ky_2^2 + 1)}{m} \tilde{X} \tilde{Y} + \frac{k^2y_2^3}{m^2} \tilde{X}^3 - \frac{2k^2y_2^7}{m^2} \tilde{X}^2 \tilde{Y} + \frac{k^2y_2^7}{m^2} \tilde{X} \tilde{Y}^2 + o(|X, Y|^4), \\
\frac{d\tilde{Y}}{dt} &= -m\tilde{Y} - \left(\frac{y_2^2(ky_2^2 + 1)}{m} + 2y_2\right) \tilde{X}^2 + \left(\frac{y_2^2(ky_2^2 + 1)}{m} + 2y_2\right) \tilde{X} \tilde{Y} + \left(\frac{k^2y_2^7}{m^2} - \frac{y_2^2}{m}\right) \tilde{X}^3 \\
&\quad - \left(\frac{2k^2y_2^7}{m^2} - \frac{2y_2^2}{m}\right) \tilde{X}^2 \tilde{Y} + \left(\frac{k^2y_2^7}{m^2} - \frac{y_2^2}{m}\right) \tilde{X} \tilde{Y}^2 + o(|X, Y|^4).
\end{align*}
\] (9)

Now we apply time rescaling \(\tau = -mt\), and system (10) transformed into the standard form
\[
\begin{align*}
\frac{d\tilde{X}}{d\tau} &= \frac{y_2^2(ky_2^2 + 1)}{m^2} \tilde{X}^2 - \frac{y_2^2(ky_2^2 + 1)}{m^2} \tilde{X} \tilde{Y} - \frac{k^2y_2^7}{m^3} \tilde{X}^3 + \frac{2k^2y_2^7}{m^3} \tilde{X}^2 \tilde{Y} - \frac{k^2y_2^7}{m^3} \tilde{X} \tilde{Y}^2 + o(|X, Y|^4), \\
\frac{d\tilde{Y}}{d\tau} &= \tilde{Y} + \left(\frac{y_2^2(ky_2^2 + 1)}{m^2} + 2y_2\right) \tilde{X}^2 - \left(\frac{y_2^2(ky_2^2 + 1)}{m^2} + 2y_2\right) \tilde{X} \tilde{Y} - \left(\frac{k^2y_2^7}{m^3} - \frac{y_2^2}{m^2}\right) \tilde{X}^3 \\
&\quad + \left(\frac{2k^2y_2^7}{m^3} - \frac{2y_2^2}{m^2}\right) \tilde{X}^2 \tilde{Y} - \left(\frac{k^2y_2^7}{m^3} - \frac{y_2^2}{m^2}\right) \tilde{X} \tilde{Y}^2 + o(|X, Y|^4).
\end{align*}
\] (10)

From \(\frac{d\tilde{Y}}{d\tau} = 0\), we have implicit function \(\tilde{Y} = \phi(\tilde{X}) = 0\), then
\[
\frac{d\tilde{X}}{d\tau} = \frac{y_2^2(ky_2^2 + 1)}{m^2} \tilde{X}^2 - \frac{k^2y_2^7}{m^3} \tilde{X}^3 + o(|X|^4).
\]

In view of \(\frac{y_2^2(ky_2^2 + 1)}{m^2} > 0\), \(E_3\) is an attracting saddle node, which can be obtained from [21][Theorem 7.1].

**Theorem 5.** For equilibrium \(E_3(x_3, y_3)\),

(i) if \(u < \frac{u_0}{2}\), then \(E_3\) is an unstable focus;

(ii) if \(\frac{u_0}{2} < u < u_0\), then \(E_3\) is a stable focus;

(iii) if \(u = \frac{u_0}{2}\), then \(E_3\) is a center type stable focus, and at the moment, system \(E_3\) undergoes a Hopf bifurcation.
Proof. The Jacobian matrix of system (6) in $E_3$ is

$$J(E_3) = \begin{pmatrix} 0 & -kx_3y_3^2 - x_3 \\ y_3^2 & 2x_3y_3 - m \end{pmatrix}.$$  \hfill (11)

According to the relationship between the matrix and its corresponding eigenvalues, we have

$$\lambda_1(E_3) + \lambda_2(E_3) = \text{tr}(J(E_3)) = 2x_3y_3 - m,$$

$$\lambda_1(E_3) \cdot \lambda_2(E_3) = \text{det}(J(E_3)) = y_3^2(kx_3y_3^2 + x_3) > 0.$$

We first prove (iii), $u = \frac{9}{2}$, i.e., $y_3 = \frac{2u}{m}$, in this state,

$$\lambda_1(E_3) + \lambda_2(E_3) = 0.$$

Translating $E_3$ to the origin by transformation $(X, Y) = (x - x_3, y - y_3)$, and performing Taylor expansion of system (6) at the origin to the third order, and noticing that $2x_3y_3 = m$. Thus, we have

$$\begin{cases}
\frac{dX}{dt} = -x_3(ky_3^2 + 1)Y - (ky_3^2 + 1)XY + k^2x_3y_3^3Y^2 + k^2y_3^3XY^2 - k^3x_3y_3^4Y^3 + o(|X, Y|^4), \\
\frac{dY}{dt} = y_3^2X + 2y_4XY + x_3Y^2 + XY^2.
\end{cases}$$

Noting $\omega := \sqrt{x_3(ky_3^2 + 1)}$, and taking transformation $(\tilde{X}, \tilde{Y}) = \left(\frac{y_4}{\omega} X, Y\right)$, system (12) gives

$$\begin{cases}
\frac{d\tilde{X}}{dt} = -y_3\omega\tilde{Y} - (ky_3^2 + 1)\tilde{X}\tilde{Y} + \frac{k^2}{\omega}x_3y_3^4\tilde{Y}^2 + k^2y_3^3\tilde{X}\tilde{Y}^2 - \frac{k^3}{\omega}x_3y_3^5\tilde{Y}^3 + o(|\tilde{X}, \tilde{Y}|^4), \\
\frac{d\tilde{Y}}{dt} = y_3\omega\tilde{X} + 2\omega\tilde{X}\tilde{Y} + x_3\tilde{Y}^2 + \frac{\omega}{y_3}\tilde{X}\tilde{Y}^2.
\end{cases}$$

(13)

Replacing the coefficients of $\tilde{X}^i\tilde{Y}^j$ $(i, j = 0, 1, 2, 3)$ in $\frac{d\tilde{X}}{dt}$ and $\frac{d\tilde{Y}}{dt}$ with $A_{ij}$ and $B_{ij}$ respectively, then system (13) becomes

$$\begin{cases}
\frac{d\tilde{X}}{dt} = -y_3\omega\tilde{Y} + A_{11}\tilde{X}\tilde{Y} + A_{02}\tilde{Y}^2 + A_{12}\tilde{X}\tilde{Y}^2 + A_{03}\tilde{Y}^3 + o(|\tilde{X}, \tilde{Y}|^4), \\
\frac{d\tilde{Y}}{dt} = y_3\omega\tilde{X} + B_{11}\tilde{X}\tilde{Y} + B_{02}\tilde{Y}^2 + B_{12}\tilde{X}\tilde{Y}^2.
\end{cases}$$

(14)
According to the calculation method of the third focus value, we obtain the third focus value of system (14) at the origin

\[
\frac{\pi}{4y_3\omega} A_{12} - \frac{\pi}{4(y_3\omega)^2} (2A_{02}B_{02} - A_{11}A_{02} + B_{11}B_{02})
= \frac{\pi}{4y_3\omega} \times k^2 y_3^3 - \frac{\pi}{4(y_3\omega)^2} \left[ \frac{2k^2}{\omega} x_3^2 y_3^4 + \frac{k^2}{\omega} x_3 y_3^2 (ky_3 + 1) + 2\omega x_3 \right]
= - \frac{\pi}{2\omega} \left( \frac{k^2}{\omega^2} x_3^2 y_3^3 + \frac{x_3}{y_3} \right) < 0,
\]

which implies \(E_3\) is a center type stable focus \([22]\)[Chapters 2.3 and 7.1]. In this state, the eigenvalues of its Jacobian matrix \(J(E_3)\) are a pair of conjugate pure virtual eigenvalues \(\lambda_{1,2} = \pm iy_3\omega\). When \(u\) changes near \(\frac{m_0}{2}\), \(J(E_3)\) has a pair of conjugate complex eigenvalues \(\lambda_{1,2} = \alpha(u) \pm i\beta(u)\), where

\[
\alpha(u) = \frac{1}{2} \text{tr}(J(E_3)) = \frac{1}{2} (2x_3y_3 - m) = \frac{m}{2} - \frac{u}{y_3}, \quad \beta(u) = \sqrt{\det J(E_3) - \alpha^2(u)}.
\]

Since \(\alpha'(u) \big|_{u=\frac{m_0}{2}} = -\frac{1}{y_3} < 0\), the transversality condition holds, it follows from Poincaré-Andronov-Hopf bifurcation theory \([23]\)[Theorem 3.1.3] that system (6) undergoes a Hopf bifurcation in \(E_3\) when \(u = \frac{m_0}{2}\):

(i) if \(y_3 > \frac{2u}{m}\), i.e., \(u < \frac{m_0}{2}\), \(\lambda_1(E_3) + \lambda_2(E_3) > 0\), \(J(E_3)\) has a pair of conjugate complex eigenvalues, and the real part is greater than 0, then \(E_3\) is an unstable focus;

(ii) if \(\frac{u}{m} < y_3 < \frac{2u}{m}\), i.e., \(\frac{m_0}{2} < u < u_0\), \(\lambda_1(E_3) + \lambda_2(E_3) < 0\), \(J(E_3)\) has a pair of conjugate complex eigenvalues, and the real part is less than 0, then \(E_3\) is a stable focus.

\[\square\]

Table 1: Equilibria and their stability in system (6)
| Equilibrium | Existence | Type                        |
|-------------|-----------|-----------------------------|
| $E_2(0, y_2)$ | Always exists | $u < u_0$, saddle         |
|             |           | $u > u_0$, stable node      |
|             |           | $u = u_0$, attracting saddle node |
| $E_3(x_3, y_3)$ | $u < u_0$ | $u < \frac{u_0}{2}$, unstable focus |
|             |           | $\frac{u_0}{2} < u < u_0$, stable focus |
|             |           | $u = \frac{u_0}{2}$, center type stable focus |

**Theorem 6.** System (6) undergoes a Hopf bifurcation at $E_3$ when $u = \frac{u_0}{2}$, furthermore, the Hopf bifurcation is subcritical, and bifurcation periodic solution is orbitally asymptotic stable.

**Proof.** Translating $E_3$ to the origin by transformation $(X, Y) = (x - x_3, y - y_3)$, and performing Taylor expansion of system (6) at the origin to the third order,

\[
\frac{dX}{dt} = -x_3(ky_3^2 + 1)Y - (ky_3^2 + 1)XY + k^2x_3y_3^3Y^2 + k^2y_3^3XY^2 - k^3x_3y_3^4Y^3 + o(|X, Y|^4),
\]

\[
\frac{dY}{dt} = y_3^2X + (2x_3y_3 - m)Y + 2y_3XY + x_3Y^2 + XY^2.
\]

Rewriting the above system as

\[
\begin{pmatrix}
\frac{dX}{dt} \\
\frac{dY}{dt}
\end{pmatrix} = J(E_3) \begin{pmatrix}
X \\
Y
\end{pmatrix} + \begin{pmatrix}
 f(x, y, u) \\
 g(x, y, u)
\end{pmatrix},
\]

(15)

where $J(E_3)$ is as in (11), and

\[
f(x, y, u) = -(ky_3^2 + 1)XY + k^2x_3y_3^3Y^2 + k^2y_3^3XY^2 - k^3x_3y_3^4Y^3 + o(|X, Y|^4),
\]

\[
g(x, y, u) = 2y_3XY + x_3Y^2 + XY^2.
\]

Define a matrix $P = \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix}$, where $N = -\frac{a(u)}{x_3(ky_3^2 + 1)}$, $M = \frac{b(u)}{x_3(ky_3^2 + 1)}$. When $u = \frac{u_0}{2}$, $M = \frac{y_3}{\sqrt{x_3(ky_3^2 + 1)}} > 0$. Then when $u$ changes near $\frac{u_0}{2}$, $P$ is invertible, and
\[ P^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{N}{M} & \frac{M}{N} \end{pmatrix}, \text{ in addition,} \]

\[ P^{-1} J(E_3) P = \begin{pmatrix} \alpha(u) & -\beta(u) \\ \beta(u) & \alpha(u) \end{pmatrix}. \]

By transformation \((X, Y)^T = P(\xi, \eta)^T\), system (15) becomes

\[
\begin{pmatrix}
\frac{d\xi}{dt} \\
\frac{d\eta}{dt}
\end{pmatrix} = \begin{pmatrix} \alpha(u) & -\beta(u) \\ \beta(u) & \alpha(u) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} F(\xi, \eta, u) \\ G(\xi, \eta, u) \end{pmatrix}, \tag{16}
\]

where

\[ F(\xi, \eta, u) = f(\xi, \eta, u) = - (k_y^2 + 1)\xi(N\xi + M\eta) + k^2 x_3 y_3^3(N\xi + M\eta)^2 + k^2 y_3^3\xi(N\xi + M\eta)^2 \\
- k^3 x_3 y_3^3(N\xi + M\eta)^3 + o(|\xi, \eta|^4), \]

\[ G(\xi, \eta, u) = - \frac{N}{M} f(\xi, \eta, u) + \frac{1}{M} g(\xi, \eta, u) \\
= - \frac{N}{M} [- (k_y^2 + 1)\xi(N\xi + M\eta) + k^2 x_3 y_3^3(N\xi + M\eta)^2 + k^2 y_3^3\xi(N\xi + M\eta)^2 \\
- k^3 x_3 y_3^3(N\xi + M\eta)^3 + o(|\xi, \eta|^4)] + \frac{1}{M} [2 y_3\xi(N\xi + M\eta) + x_3(N\xi + M\eta)^2 \\
+ \xi(N\xi + M\eta)^2]. \]

Rewriting system (16) in polar form

\[
\begin{cases}
\dot{r} = \alpha(u)r + \alpha_1(u)r^3 + o(|r|^5), \\
\dot{\theta} = \beta(u) + \beta_1(u)r^2 + o(|r|^4),
\end{cases}
\]

performing Taylor expansion of the above system at \(u = \frac{u_0}{2}\), we have

\[
\begin{cases}
\dot{r} = \alpha' \left( \frac{u_0}{2} \right) \left( u - \frac{u_0}{2} \right) r + \alpha_1 \left( \frac{u_0}{2} \right) r^3 + o \left( \left( u - \frac{u_0}{2} \right)^2, \left( u - \frac{u_0}{2} \right)^3, r^4 \right), \\
\dot{\theta} = \beta' \left( \frac{u_0}{2} \right) \left( u - \frac{u_0}{2} \right) + \beta_1 \left( \frac{u_0}{2} \right) r^2 + o \left( \left( u - \frac{u_0}{2} \right)^2, \left( u - \frac{u_0}{2} \right)^3, r^4 \right).
\end{cases}
\]
In order to examine the direction of the Hopf bifurcation and the stability of Hopf bifurcation periodic solution, we have to determine the sign of \( \alpha_1 \left( \frac{u_0}{2} \right) \), where

\[
\alpha_1 \left( \frac{u_0}{2} \right) = \frac{1}{16} \left( F_{\xi\xi} + F_{\xi\eta} + G_{\xi\xi} + G_{\eta\eta} \right) + \frac{1}{16\beta \left( \frac{u_0}{2} \right)^2} \left[ F_{\xi\eta} (F_{\xi\xi} + F_{\eta\eta}) - G_{\xi\eta} (G_{\xi\xi} + G_{\eta\eta}) \right. \\
\left. - F_{\xi\xi} G_{\eta\eta} + F_{\eta\eta} G_{\xi\xi} \right].
\]

All partial derivatives in the above formula are calculated at \((\xi, \eta, u) = (0, 0, \frac{u_0}{2})\).

By calculating,

\[
F_{\xi\xi} = F_{\xi\xi} = G_{\xi\xi} = G_{\eta\eta} = 0,
\]

\[
F_{\xi\eta} = 2k^2y_3^2M^2 \left( \frac{u_0}{2} \right), \quad F_{\xi\eta} = -(ky_3^2 + 1)M \left( \frac{u_0}{2} \right),
\]

\[
G_{\eta\eta} = 2k^2x_3y_3^2M^2 \left( \frac{u_0}{2} \right), \quad G_{\xi\eta} = 2y_3, \quad G_{\eta\eta} = 2x_3M \left( \frac{u_0}{2} \right).
\]

Noticing that

\[
M \left( \frac{u_0}{2} \right) = \frac{\beta \left( \frac{u_0}{2} \right)}{x_3( ky_3^2 + 1) }, \quad \beta \left( \frac{u_0}{2} \right) = y_3\sqrt{x_3( ky_3^2 + 1) }, \quad y_3 = \frac{u_0}{m} = \sqrt{1 + 4k - 1}.
\]

Therefore, we have

\[
\alpha_1 \left( \frac{u_0}{2} \right) = \frac{1}{16} F_{\xi\eta\eta} + \frac{1}{16\beta \left( \frac{u_0}{2} \right)^2} \left[ F_{\xi\eta} F_{\eta\eta} - G_{\xi\eta} G_{\eta\eta} + F_{\eta\eta} G_{\xi\xi} \right]
\]

\[
= \frac{1}{16} \left[ 2k^2y_3^2M^2 \left( \frac{u_0}{2} \right) + \frac{1}{\beta \left( \frac{u_0}{2} \right)^2} \left( -2k^2x_3y_3^2(ky_3^2 + 1)M^3 \left( \frac{u_0}{2} \right) - 4x_3y_3M \left( \frac{u_0}{2} \right) \right) \right.
\]

\[
+ 4k^2x_3y_3^2M^3 \left( \frac{u_0}{2} \right) \right]
\]

\[
= \frac{x_3y_3M \left( \frac{u_0}{2} \right)}{4\beta \left( \frac{u_0}{2} \right)^2} \left( \frac{k^2y_3^4}{ky_3^2 + 1} - 1 \right)
\]

\[
= \frac{x_3y_3M \left( \frac{u_0}{2} \right)}{4\beta \left( \frac{u_0}{2} \right)^2} \left( \frac{(\sqrt{1 + 4k - 1})^4}{4k((\sqrt{1 + 4k - 1})^2 + 4k - 1)} - 1 \right),
\]

taking variable substitution \( \sqrt{1 + 4k - 1} := \kappa \), then

\[
\alpha_1 \left( \frac{u_0}{2} \right) = \frac{x_3y_3M \left( \frac{u_0}{2} \right)}{4\beta \left( \frac{u_0}{2} \right)^2} \left( \frac{\kappa^4}{((\kappa + 1)^2 - 1)(\kappa^2 + (\kappa + 1)^2 - 1)} - 1 \right)
\]

\[
= \frac{x_3y_3M \left( \frac{u_0}{2} \right)}{4\beta \left( \frac{u_0}{2} \right)^2} \left( \frac{\kappa^2}{(\kappa + 2)(2\kappa + 2)} - 1 \right) < 0.
\]
The first Lyapunov coefficient

\[ l_1 \left( \frac{u_0}{2} \right) = -\frac{\alpha_1 \left( \frac{u_0}{2} \right)}{\alpha' \left( \frac{u_0}{2} \right)} < 0. \]

Therefore, the direction of the Hopf bifurcation is subcritical [24][Chapter 3.4], and bifurcation periodic solution is orbitally asymptotic stable.

4. Examples and their numerical simulations

In this section, we give an example and figures to illustrate our results.

Example 1. Consider the following system

\[
\begin{align*}
\frac{dx}{dt} &= \frac{2x}{1 + 0.5y} - 2xy, \\
\frac{dy}{dt} &= 2xy^2 - 0.4y + u.
\end{align*}
\] (17)

Comparing system (17) with system (3), we see that \( r = 2, k = 0.5, c = 2, m = 0.4 \). Furthermore, \( \bar{y} = \frac{c}{r}y = y, \tau = 2t, \bar{k} = \frac{k}{c} = 0.5, \bar{m} = \frac{m}{r} = 0.2, \bar{u} = \frac{u}{r^2} = \frac{u}{2}, \) and system (17) becomes

\[
\begin{align*}
\frac{dx}{d\tau} &= \frac{x}{1 + 0.5\bar{y}} - x\bar{y}, \\
\frac{d\bar{y}}{d\tau} &= x\bar{y}^2 - 0.2\bar{y} + \bar{u}.
\end{align*}
\] (18)

For the convenience of marking on the graphs, we still replace \( \bar{y}, \tau, \bar{k}, \bar{m}, \bar{u} \) with the original variable \( y, t, k, m, u \) until I make a prompt below. And then, model (18) can be rewritten as

\[
\begin{align*}
\frac{dx}{dt} &= \frac{x}{1 + 0.5y} - xy, \\
\frac{dy}{dt} &= xy^2 - 0.2y + u.
\end{align*}
\] (19)

By calculating, we obtain \( y_2 = \frac{u}{m} = 5u, y_3 = \frac{\sqrt{1+4k}-1}{2k} = \sqrt{3} - 1 \), and \( u_0 = my_3 = 0.2(\sqrt{3} - 1) \). In the following, we calculate and numerically simulate the dynamic behavior of system (19) by taking different values for \( u \).
(1) Taking $u = 0.1(\sqrt{2} - 1)$, $u < \frac{u_0}{2}$, system (19) has two equilibria, $E_2(0, y_2) = (0, 5u) \approx (0, 0.207)$ is a saddle, and $E_3 = \left(\frac{0.2y_3 - u}{y_3^2}, y_3\right) \approx (0.196, 0.732)$ is an unstable focus. See Fig.1 (a);

(2) taking $u = 0.1, \frac{u_0}{2} < u < u_0$, system (19) has two equilibria, $E_2(0, y_2) = (0, 5u) = (0, 0.5)$ is a saddle, and $E_3 = \left(\frac{0.2y_3 - u}{y_3^2}, y_3\right) \approx (0.087, 0.732)$ is a stable focus. See Fig.1 (b);

(3) taking $u = 0.2(\sqrt{3} - 1)$, $u = u_0$, system (19) has a equilibrium, $E_2(0, y_2) = (0, 5u) \approx (0, 0.732)$ is an attracting saddle node. See Fig.1 (c);

(4) taking $u = 0.2, u > u_0$, system (19) has a equilibrium, $E_2(0, y_2) = (0, 5u) = (0, 1)$ is a stable node. See Fig.1 (d);

Fig.1 Dynamic behavior of system (19) in the cases that (1) – (4)
(5) taking \( u = 0.1(\sqrt{3} - 1) \), \( u = \frac{u_0}{2} \), system (19) has two equilibria, \( E_2(0, y_2) = (0, 5u) \approx (0, 0.366) \) is a saddle, and \( E_3 = \left( \frac{0.2y_3 - u}{y_3^2}, y_3 \right) \approx (0.137, 0.732) \) is a center type stable focus. In this situation, system (19) undergoes a Hopf bifurcation, and the Hopf bifurcation periodic solution is asymptotic stable. See Fig.2. Furthermore, from Fig.1(a),(b) and Fig.2, we can see the Hopf bifurcation is subcritical.

Fig.2  The phase diagram of system (19) in the case that \( u = \frac{u_0}{2} \) and the time series diagram of the Hopf bifurcation periodic solution

5. Comparison of dynamic complexity and conclusion

5.1. Comparison: Impact of fear factor on the model with or without fear effect

In order to observe the impact of fear factor on model (6), we compare our results with those of model without fear effect in the previous literature (Wang and Chen [10]). We see that the models both with fear effect and without fear effect have two equilibria — a pest-free equilibrium and a positive equilibrium. In order to intuitively compare the differences between the equilibria and stability of the two models, we give two tables in follows.

Table 2: Comparison at the pest-free equilibria
### Equilibrium Existence Stability

| Equilibrium | Existence | Stability |
|-------------|-----------|-----------|
| Without fear effect $(0, \frac{u}{m})$ | Always exists | $u < m$, unstable $u > m$, stable |
| With fear effect $(0, \frac{u}{m})$ | Always exists | $u < my_3$, unstable $u > my_3$, stable |

**Table 3: Comparison at the positive equilibria**

| Equilibrium | Existence | Stability |
|-------------|-----------|-----------|
| Without fear effect $(m - u, 1)$ | $u < m$ | $u < \frac{m^2}{2}$, unstable $\frac{m^2}{2} < u < m$, stable |
| With fear effect $(\frac{my_3 - u}{y_3}, y_3)$ | $u < my_3$ | $u < \frac{my_3}{2}$, unstable $\frac{my_3}{2} < u < my_3$, stable |

From the data in the above two tables, the fear factor makes the boundary of equilibria change from $m$ to $my_3$ (from $m^2$ to $\frac{my_3}{2}$). It can be obtained by calculation that $y_3 = \frac{\sqrt{1+4k-1}}{2k}$ decreases monotonically as $k$ increases and

$$\lim_{k \to 0} y_3 = 1, \quad 0 < y_3 < 1.$$  

When $k = 0$, that is, model (6) reduces to the model *without* fear effect, it just the model considered in Wang and Chen [10]. Our results are consistent with those of [10] when $k = 0$. As the level of fear $k$ increases, $y_3$ decreases, and model (6) can change from an unstable state to a stable state when $u$ is smaller. This suggests that the pests’ fear of natural enemies allows pests populations to be controlled with fewer nematodes released. And this is also consistent with reality.

### 5.2. Conclusion

In this paper, we discussed the microbial pesticide model with fear effect in the case that continuous release of nematodes. Through the analysis of the qualitative
and stability of the model, we found the best solution to control pests. Next, we analyze system (3), and from now on, we will no longer replace $\bar{y}, \tau, \bar{t}, \bar{m}, \bar{u}$ with $y, t, k, m, u$. From the previous analysis and example verification, we can get the following conclusions:

1. Both the pest-free equilibrium and the positive equilibrium are unstable if

$$u = \frac{c}{r^2} \bar{u} < \frac{c}{2r^2} u_0 = \frac{c^2 m}{4kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right);$$

2. the pest-free equilibrium is unstable and the positive equilibrium is stable if

$$\frac{c^2 m}{4kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right) = \frac{c}{2r^2} u_0 \leq u < \frac{c}{2r^2} u_0 = \frac{c^2 m}{2kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right);$$

3. the unique equilibrium — pest-free equilibrium is stable if

$$u \geq \frac{c}{r^2} u_0 = \frac{c^2 m}{2kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right).$$

In summary, if we want to eliminate pests completely, we need to continuously release nematodes, and the speed is not less than $\frac{c^2 m}{2kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right)$. While if we only want to control the pest density within a certain range, then we only need to continuously release nematodes, and the speed is not less than $\frac{c^2 m}{4kr^4} \left(\sqrt{1 + \frac{4kr}{c}} - 1\right).

6. Conflict of Interest

The authors declare that they have no conflict of interest.

7. Data Availability Statement

My manuscript has no associated data.
References

[1] I. Ionel, G. Mara, R. Stefania, G. Margarita, P. Corina, A hazard to human health - pesticide residues in some vegetal and animal foodstuff. Journal of Biotechnology, 2019, 305: S22-S23.

[2] Y. Mikhail, M. Mariya, V. Olga, I. Mariya, N. Vasily, The Effect of Pesticides on the Microbiome of Animals. Agriculture, 2020, 10(3): 14 pp.

[3] R. James, Pesticide Residues in Foods as Health Hazards. Food, Drug, Cosmetic Law Quarterly, 1948, 3(4): 561-565.

[4] L. Robert, L. Chi, G. Pamela, History, Use, and Future of Microbial Insecticides. American entomologist, 1993, 39(2): 83-91.

[5] M. Sarwar, Microbial Insecticides - An Ecofriendly Effective Line of Attack for Insect Pests Management. International Journal of Engineering and Advanced Research Technology, 2015, 1(2): 4-9.

[6] R. Starnes, C. Liu, P. Marrone, History and Future of Microbial Insecticides. American Entomologist, 1993, 38-40: 83-91.

[7] O. Taha, R. Gordon, Efficacy of entomopathogenic nematodes against Tuta absoluta. Biological Control, 2021, 160: 104699.

[8] F. Corné, N. Fatouros, J. Kammenga, The potential of entomopathogenic nematodes to control moth pests of ornamental plantings. Biological Control, 2022, 165: 104815.

[9] Y. El Khoury, E. Noujeim, J. Ravlić, M. Oreste, R. Addante, N. Nemer, E. Tarasco, The effect of entomopathogenic nematodes and fungi against four xylophagous pests. Biocontrol Science and Technology, 2020, 30(9): 983-995.

[10] T. Wang, L. Chen, Dynamic complexity of microbial pesticide model. Nonlinear Dynamics, 2009, 58(3): 539-552.
[11] T. Wang, L. Chen, Nonlinear analysis of a microbial pesticide model with impulsive state feedback control. Nonlinear Dynamics, 2011, 65(1-2): 1-10.

[12] T. Wang, Y. Wang, F. Liu, Dynamical analysis of a new microbial pesticide model with the Monod growth rate. Journal of Applied Mathematics and Computing, 2017, 54(1-2): 325-355.

[13] T. Wang, Microbial insecticide model and homoclinic bifurcation of impulsive control system. International Journal of Biomathematics, 2021, 14(6): 2150043.

[14] D. Barman, J. Roy, S. Alam, Dynamical behaviour of an infected predator-prey model with fear effect. Iranian Journal of Science and Technology. Transactions A: Science, 2021, 45(1): 309-325.

[15] W. Gao, B. Dai, Dynamics of a predator-prey model with delay and fear Effect. Journal of Nonlinear Modeling and Analysis. 2019, 1(1): 57-72.

[16] S. Sasmal, Population dynamics with multiple Allee effects induced by fear factors - A mathematical study on prey-predator interactions. Applied mathematical modelling, 2018, 64(1): 1-14.

[17] X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator-prey interactions. Journal of Mathematical Biology, 2016, 73(5): 1179-1204.

[18] X. Wang, X. Zou, Modeling the fear effect in predator-prey interactions with adaptive avoidance of predators. Bulletin of Mathematical Biology, 2017, 79(6): 1325-1359.

[19] X. Wang, Y. Tan, Y. Cai, W. Wang, Impact of the fear effect on the stability and bifurcation of a Leslie-Gower predator-prey model. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 2020, 30(14): 2050210.
[20] H. Zhang, Y. Cai, S. Fu, W. Wang, Impact of the fear effect in a prey-predator model incorporating a prey refuge. Applied Mathematics and Computation, 2019, 356(C): 328-337.

[21] Z. Zhang, T. Ding, W. Huang, Z. Dong, Qualitative Theory of Differential Equation. Science Press, Beijing, 1992 (in Chinese).

[22] J. Zhang, Geometrical Theory and Bifurcation Problem in Ordinary Differential Equations. Peking University Press, Beijing, 1987 (in Chinese).

[23] S. Wiggins, Introduction to applied nonlinear dynamical systems and chaos. Springer-Verlag, New York, 1990.

[24] A. Yuri, Elements of Applied Bifurcation Theory. Science Press, Beijing, 2010 (in Chinese).

[25] Y. Lv, L. Chen, F. Chen, Z. Li, Stability and bifurcation in an SI epidemic model with additive Allee effect and time delay. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 2021, 31(4): 2150060.

[26] X. Guan, F. Chen, Dynamical analysis of a two species amensalism model with Beddington-DeAngelis functional response and Allee effect on the second species. Nonlinear Analysis: Real World Applications, 2019, 48: 71-93.

[27] J. Huang, Y. Gong, J. Chen, Multiple bifurcations in a predator-prey system of Holling and Leslie type with constant-yield prey harvesting. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 2013, 23(10): 1350164.

[28] J. Huang, X. Xia, X. Zhang, S. Ruan, Bifurcation of codimension 3 in a predator-prey system of Leslie type with simplified Holling type IV functional response. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 2016, 26(2): 1650034.
[29] Y. Song, X. Tang, Stability, Steady-State bifurcations and Turing patterns in a predator-prey model with herd behavior and prey-taxis. Studies in Applied Mathematics, 2017, 139(3): 371-404.

[30] Z. Wei, Y. Xia, T. Zhang, Stability and bifurcation analysis of an amensalism model with weak Allee effect. Qualitative Theory of Dynamical Systems, 2020, 19(1): 23, 15 pp.

[31] Z. Wei, Y. Xia, T. Zhang, Stability and bifurcation analysis of a commensal model with additive Allee effect and nonlinear growth rate. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 2021, 31(13): 2150204.
\[ x' = x/(1 + ky) - xy \]
\[ y' = xy^2 - my + u \]

\[ k = 0.5 \quad m = 0.2 \quad u = 0.1 (\sqrt{3} - 1) \]
This figure "u=1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2201.09230v2
This figure "u=3.jpg" is available in "jpg" format from:

http://arxiv.org/ps/2201.09230v2