Quasiparticle effective mass divergence in two dimensional electron systems

Ying Zhang and S. Das Sarma
Condensed Matter Theory Center, Department of Physics,
University of Maryland, College Park, MD 20742-4111
(Dated: March 22, 2022)

Within the infinite series of ring (or bubble) diagram approximation for the electronic self-energy as appropriate for the long-range Coulomb interaction, we calculate the density-dependent \( T = 0 \) Fermi liquid quasiparticle effective mass renormalization in two dimensional (2D) electron systems in the random-phase-approximation (RPA). We find that the quasiparticle effective mass increases very strongly with increasing \( r_s \), exhibiting an unexpected divergence at a critical \( r_s \) value (~16 in the ideal 2D system). We discuss the possibility for and the consequences of such an interaction induced effective mass divergence as the electron density decreases, and critically comment on the likely theoretical scenarios which could lead to such a low density effective mass divergence in electron liquids.

PACS numbers: 71.10.-w; 71.10.Ca; 73.20.Mf; 73.40.-c

I. INTRODUCTION

An interacting homogeneous quantum electron system in a positive jellium background is characterized (at \( T = 0 \)) by the interaction parameter \( r_s \), the dimensionless average separation between electrons measured in the unit of effective Bohr radius, with large (small) \( r_s \) being the strongly (weakly) interacting regime corresponding respectively to low (high) densities. The interaction parameter \( r_s \) depends only on the density \( n \) of the electron system with \( r_s \propto n^{-1/3} \) \((n^{-1/2})\) in 3D (2D) systems. The \( r_s \)-parameter is essentially proportional to the dimensionless ratio of the (interacting) potential energy to the (noninteracting) kinetic energy, with the potential (kinetic) energy varying as \( r_s^{-1} (r_s^{-2}) \). (This simple one parameter picture of interacting electron systems is valid only at \( T = 0 \) and for clean free-electron jellium systems with little disorder.)

Alkali metals (e.g. Li, Na, K, Rb, Cs) are examples of interacting three dimensional (3D) electron systems with reasonably simple band structures so that the homogeneous free-electron jellium model may be applicable. The typical for 3D metals is in the range of \( 3 - 6 \) (e.g. \( r_s = 3.2 \) [Li], 4.0 [Na], 5.0 [K], 5.2 [Rb], 5.6 [Cs]), making metals strongly interacting electron systems since \( r_s > 1 \). Electrons (or holes, as in p-GaAs systems) confined by external electric fields near interfaces in semiconductor structures (e.g. Si MOSFETs, GaAs quantum wells and heterojunctions) are examples of 2D electron systems, which are actually quite ideal free electron jellium systems if the impurity disorder is not too high since the noninteracting energy has free electron parabolic form without any band structure complications. One particular advantage of semiconductor-based 2D electron systems is that often (as in Si MOSFETs and GaAs HIGFETs) the carrier density can be tuned over one or two orders of magnitude (by changing the external gate voltage), thus enabling the tuning of the \( r_s \)-parameter in the same sample over one order of magnitude – for example, in high quality Si MOSFETs, \( r_s \) could be tuned from 1 to 12 and in gated n-GaAs 2D HIGFET structures, where extremely low carrier densities can be achieved and \( r_s \) can be made as high as 15. In hole-doped 2D p-GaAs structures it is possible to achieve extremely high values of \( r_s \) (\( \sim 20 - 30 \)), but there are considerable band structure complications in 2D p-GaAs hole systems due to the presence of strong spin-orbit coupling and the complex multi-band structure of the GaAs valence band so that 2D p-GaAs systems may not be thought of as simple isotropic free-electron-like systems. This capability of tuning the \( r_s \) parameter of a single system over a large range has made semiconductor-based 2D electron systems attractive systems for studying Coulomb quantum many-body effects, and indeed many-body renormalization has been actively studied in 2D electron systems over the last thirty years both experimentally \( 1, 2, 3, 4, 5, 6, 7, 8, 9 \) and theoretically \( 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22 \).

In this paper we study theoretically the quasiparticle effective mass \( (m^*) \) renormalization in an interacting 2D electron system at \( T = 0 \) finding an interesting (and intriguing) divergence in the calculated renormalized effective mass at low (high) carrier densities \( (r_s) \). Our result takes on particular significance since such an effective mass divergence has recently been claimed to have been experimentally observed in 2D Si MOSFETs with \( r_s \) values close to \( (\sim 10 - 17) \) \( 23, 24, 25 \) where we find our theoretical divergence. Because of the necessarily approximate (and somewhat uncontrolled) nature of our theory we cannot be absolutely certain that the observed effective mass divergence in 2D semiconductor systems is indeed the same phenomenon as what we obtain in our theory, but it seems likely that the observed divergence is related to our theoretical findings because our calculated functional form of \( m^*(r_s) \) is very similar to the experimental observation. We emphasize that the self-energy approximation (the so-called ‘\( GW \)’ or the single-loop calculation) used in our theory has been extensively used in the literature up to \( r_s \approx 6 - 7 \), and our main contribution is to extend the calculation to even larger values of \( r_s \).
mainly motivated by the recent experimental claims we mentioned above.

The existing paradigm (and one of the key cornerstones of condensed matter physics) for interacting electron systems is Landau’s Fermi liquid theory developed fifty years ago. The Fermi liquid theory asserts a one-to-one low energy correspondence between the noninteracting Fermi gas and the interacting Fermi liquid with weakly interacting quasiparticles (with renormalized single particle properties such as effective mass, specific heat, magnetic susceptibility, compressibility, etc.) in the interacting system behaving similar to free electrons in the noninteracting system. The key idea is that the Fermi surface, i.e. the $T=0$ discontinuity in the single-particle momentum distribution function at Fermi wave-vector $k = k_F$, survives the effect of interaction (albeit with a reduced or renormalized discontinuity at $k_F$) leading to qualitatively similar leading-order low temperature behavior in the single particle thermodynamic properties such as specific heat, spin susceptibility and compressibility, for interacting and noninteracting systems. The Fermi liquid theory provides a natural explanation for the remarkable success of the noninteracting Fermi gas theory in describing thermodynamic, magnetic and even transport properties of metals, which are in fact strongly interacting Fermi liquids ($r_s \approx 3 - 6$). The question, however, naturally arises whether the Fermi liquid theory remains valid as the interaction strength is increased by going to higher values of $r_s$. This question has taken on significance in view of recent experiments in 2D electron systems at large $r_s$ values, where the measured quasiparticle effective mass is seen to increase very rapidly with increasing $r_s$. This is the motivation for our theoretical investigation of the quasiparticle effective mass in 2D electron systems as a function of $r_s$ going to low carrier densities (i.e. high $r_s$).

There are several possible theoretical scenarios for the breakdown of the Fermi liquid theory in a clean interacting electron liquid at low densities (or high $r_s$). At very large $r_s$ the system could obviously reduce its potential energy (at the cost of higher kinetic energy) by becoming an electron crystal (the so-called quantum Wigner crystal [23]). The $T=0$ transition from an electron liquid to a Wigner solid should be a first order transition in a clean system, occurring at $r_s \sim 35 - 40$ for 2D systems as deduced from quantum Monte Carlo calculations [24]. The other scenario is the possibility of an interaction driven phase transition to some other ordered phase (e.g. charge density wave, spin density wave, superconductor, ferromagnet). We do not consider the possibility of such interaction driven phase transitions in the current work, assuming that the system remains a homogeneous electron liquid for all values of $r_s$ we investigate. Our goal is to calculate the 2D zero temperature quasiparticle effective mass $m^*(r_s)$ as a function of the density parameter $r_s$ in the interacting electron liquid.

One problem, which has no rigorous solution, that immediately arises in this context is that the electron system for $r_s > 1$ is by definition a strongly interacting system with a formal failure of the usual Feynman-Dyson diagrammatic perturbation theory, and in fact there is no known controlled technique for systematically calculating exact quasiparticle properties in interacting electron systems for $r_s > 1$. In the weak coupling limit, $r_s \ll 1$, it has been known for a long time that an asymptotically exact many-body perturbation theory can be developed for quasiparticle properties by expanding in the $r_s$ parameter. In fact, in the $r_s \ll 1$ limit, the expansion of the electron self-energy (as well as the polarizability) in the infinite series of ring or bubble diagrams (the so-called ‘RPA’, the random-phase-approximation) is exact because of the long-range nature of Coulomb interaction. Thus in the $r_s \to 0$ limit it is possible to obtain exact results for the effective mass renormalization in the interacting electron system (interacting via the Coulomb interaction). Such an exact result is of course useless for real interacting electron systems (e.g. metals in 3D, semiconductor layers in 2D) which are never in the weakly interacting $r_s \to 0$ limit, but are actually in the strongly interacting $r_s > 1$ regime. Therefore, analytical results obtained by expanding in $r_s$ (assumed to be a small parameter) are completely irrelevant for any practical purpose as was already emphasized by Hedin [25] and by Rice [26] a long time ago, since real 2D and 3D electron systems are always in the $r_s > 1$ regime. There is actually a deeper (and somewhat subtle) problem with the $r_s$ expansion of quasiparticle properties in the $r_s \to 0$ limit. It has been known for a while [27] that the $r_s$-expansion for the interacting electron liquid problem formally fails at $T=0$ even in the high-density $r_s \to 0$ limit since the radius of convergence for such a power series expansion in $r_s$ is, strictly speaking, zero. In fact, a weakly interacting ($r_s \ll 1$) system of fermions does not remain a normal Fermi liquid down to zero temperature, but makes a transition to a superconducting state (arising purely out of the repulsive electron-electron interaction) in some rather high orbital angular momentum channel with an exponentially small value of $T_c$. In our discussion we assume that this so-called Kohn-Luttinger superconductivity can be ignored since $T_c$ is exponentially small for this transition. We uncritically assume the existence of a normal Fermi liquid for the purpose of this work.

We therefore need a robust (but necessarily inexact except in the $r_s \to 0$ limit) many-body approximation for calculating the quasiparticle effective mass at arbitrary values of $r_s$. Following Hedin and Rice [25,26] we use the RPA as our central approximation although we improve upon RPA in some situations by using various approximation schemes as described in details later in this paper. We do want to make an important point here about the validity (or lack thereof) of RPA in the strongly interacting $r_s > 1$ regime. It is often stated or assumed that RPA is invalid for $r_s > 1$. This assertion is false as has already been argued [25,26,28] a long time ago in the context of the applicability of RPA to 3D metals ($r_s \approx 3 - 6$). For arbitrary values of $r_s$, it is more appropriate to think
of RPA as a dynamical Hatree-Fock self-energy calculation (using the dynamically screened interaction) rather than a perturbation expansion in \( r_s \) (which makes sense only in the \( r_s \to 0 \) high density asymptotic limit). Such a dynamical mean-field calculation (called the ‘GW’ approximation [25], where \( G \) is the electron Green function and \( W \) the dynamically screened Coulomb interaction) has a long history in electronic systems, and is extensively employed in quasiparticle band structure calculations (going beyond LDA) in 3D systems (metals, insulators, and semiconductors) with very impressive empirical success. The RPA is a systematic expansion in the dynamically screened interaction (i.e. an infinite resummation formula [25], where \( G \) is the electron Green function) that has the long-range divergence making an expansion in terms of the bare unscreened Coulomb interaction impossible, and in general, the validity of RPA for arbitrary \( r_s \) is not rigorously known except that it is rigorously known to be exact in the \( r_s \to 0 \) limit. The reverse is, however, not true, i.e., RPA is not necessarily a bad approximation for \( r_s > 1 \) (where one needs to do the calculations numerically, of course, as we do in this work instead of carrying out an \( r_s \) expansion). The applicability of RPA for \( r_s > 1 \) (through exact numerical calculations without any \( r_s \) expansion) must be ascertained empirically, and in view of its widespread empirical success in both 3D (metals) and 2D (semiconductor space charge layers) electron systems in the \( r_s > 1 \) regime, we have decided to carry out an RPA-based quasiparticle effective mass calculation in 2D systems for large \( r_s \) values in the regime of current interest in 2D systems [2, 3, 7, 8]. We do provide some qualitative and quantitative justification for the possible validity of RPA self-energy calculations at large \( r_s \) in section [IV] of this paper. We argue that our RPA calculation of \( m^*(r_s) \) gives a divergent effective mass at a large \( r_s \) value (~16 in the ideal 2D system) is qualitatively valid although the precise critical \( r_s \) value where the divergence occurs may not be quantitatively accurate within RPA. Our conclusion about the qualitative validity of the RPA prediction of a divergent 2D effective mass at large \( r_s \) is further strengthened by our finding that the inclusion of higher order vertex corrections in the theory (both in the polarizability and in the self-energy in a consistent manner) does not change our qualitative conclusion at all.

We believe that the terminology ‘RPA’ is a misnomer for the leading-order dynamical screening self-energy calculation we are carrying out in this paper although this has now become the standard terminology in the literature. A more appropriate name for this particular approximation scheme (often also called the ‘GW approximation’ in the context of electronic structure calculation) is the ‘Ring Diagram Approximation’ or ‘RDA’, which emphasizes the basic physics of keeping only the infinite series of ring or bubble diagrams in calculating the self-energy. The two most crucial non-perturbative features of the interacting electron physics are preserved (in fact, emphasized) in this ‘RDA’: Screening of the long-range interaction to eliminate the Coulomb divergence in each order and the collective plasmon excitation of the interacting system. Inclusion of these two key non-perturbative aspects of the physics of the problem makes RPA (i.e. ‘RDA’) a reasonable approximation for arbitrary values of \( r_s \) as long as there is no quantum phase transition to a new ground state.

It is well-known [29] that the in \( r_s \to 0 \) extreme high density limit, the RPA calculation for the quasiparticle effective mass becomes exact giving the following asymptotic formula (in both 2D and 3D)

\[
\frac{m^*}{m} \bigg|_{r_s \to 0} = 1 + ar_s(b + \ln r_s) + \mathcal{O}(r_s^2),
\]

where \( a, b \) are constants of order unity in both 2D and 3D. This formula shows that \( m^*(r_s) \) has a nontrivial \( r_s \) expansion even in the \( r_s \to 0 \) limit where the ring diagrams of RPA asymptotically dominate the self-energy. We emphasize that \( m^*(r_s) < m \), i.e. the mass renormalization is negative for small \( r_s \) according to the high-density expansion. Our goal in this paper is to obtain the RPA effective mass numerically without resorting to any expansion – the only approximation we utilize is the expansion in effective interaction, which we will argue later in section [IV] is not an expansion in \( r_s \) but in some other effective parameter \( \delta \) (a function of \( r_s \)) that may actually be small even for \( r_s \) larger than unity. We will argue later in this paper that the actual expansion parameter in our theory is not the physical density parameter \( r_s \), but a more subtle parameter \( \delta = r_s(1 + cr_s)^{-1} \) where \( c \gg 1 \) making the theory exact in the high-density \( \delta \to 0 \) (i.e. \( r_s \to 0 \)), but reasonably accurate for \( r_s > 1 \) also.

As mentioned above, the low-density (\( r_s > 5 \)) Fermi liquid behavior of 2D electron systems has taken on special significance during the last few years due to the extensive experimental activity [3, 4, 5, 6, 7, 8, 9] in the subject generically referred to as the two-dimensional metal-insulator-transition (2D MIT), where lowering density in clean 2D electron systems causes an apparently sharp localization transition in the system. The critical density for the 2D MIT seems to be rather close (perhaps even the same) as the density for the apparent effective mass divergence [3, 4]. It may appear trivially obvious that the conductivity \( \sigma = ne^2/\pi m^* \), where \( n/\tau/m^* \) are the carrier density / transport relaxation time / effective mass respectively, will vanish if the effective mass \( m^* \) diverges (therefore causing an MIT at the critical \( r_s \) value where \( m^* \) diverges), but this is not necessarily true. In particular, in an ideal system (i.e. a translationally invariant disorder-free jellium-background electron liquid) the effective mass entering the conductivity formula is the bare (band) effective mass (and not the renormalized quasiparticle effective mass), and therefore a strong increase in the renormalized quasiparticle mass \( m^* \), as we obtain in this work, should not affect transport properties in a translationally invariant electron liquid. The reason for this is very simple: In a translationally invariant system the total Hamiltonian can be divided into...
two parts, a center of mass Hamiltonian and a relative coordinate Hamiltonian, with electron-electron interactions affecting only the relative coordinate part which does not affect long wavelength transport by virtue of Galilean invariance. Real 2D systems, even the very high-quality (i.e. low disorder) systems where the 2D MIT phenomena are typically observed, are not ideal and have considerable impurity disorder in them. It is, therefore, possible that the renormalized quasiparticle effective mass appears in the formula for the conductivity of real 2D electron systems, with the 2D metal to insulator transition at low carrier densities somehow being intrinsically connected with the interaction induced low density effective mass divergence phenomenon since translational invariance does not apply to real systems. But the issue of the relevant effective mass entering the transport theory (e.g. the low temperature conductivity) in a disordered system is extremely subtle, and it is by no means clear that the many-body quasiparticle effective mass \( m^* \) could be used in obtaining the conductivity. Much more work is needed to clarify this issue.

Our finding of a divergent quasiparticle effective mass in the strongly interacting 2D electron system is very similar to the corresponding situation \([30, 31]\) in normal He-3 which is known to be an “almost localized” Fermi liquid \([32]\) with a divergent effective mass arising from the strong inter-particle interaction (which is short-ranged in He-3). Similar to the situation in 2D electron systems \([7, 8, 9]\), He-3 exhibits a strongly enhanced, almost ferromagnetic, susceptibility. We believe that this similarity in the quasiparticle properties of normal He-3 and low-density 2D electron gas is not fortuitous, and “almost localization” (i.e. very heavy or divergent effective mass) and “almost ferromagnetism” (i.e. strongly enhanced susceptibility) may actually be the generic behavior of strongly interacting Fermi liquids, provided the interaction is sufficiently strong, independent of whether the interaction is long-ranged (as in our case) or short-ranged (as in He-3). The presence of disorder in 2D electron systems may, however, add considerable complications to the 2D semiconductor systems not present in He-3. It is interesting to note that the strong-coupling regime (with large \( m^*/m \)) is attained in the electron system at low densities whereas the strong-coupling regime with large \( m^*/m \) is achieved at high densities in He-3 with the difference arising from the long-range (short-range) nature of the interaction in electrons (He-3).

There have of course been many other calculations of \( m^*(r_s) \) in both 3D and 2D electron systems using the RPA self-energy approximation. None of these existing calculations has however been extended to large \( r_s \) values (i.e. low density), and therefore the actual divergence of the effective mass \( m^*(r_s) \) at a critical \( r_s (\sim 16.6) \) we find in this work has not been discussed before in the literature. Motivated by the 2D MIT physics, however, there have been several recent theoretical discussions \([19, 20, 22]\) of a divergent 2D effective mass at low electron-density, but the underlying physics of these works \([19, 20, 22]\) are different from the strong interaction physics discussed in our work. Although we believe (and argue in this paper) that RPA remains (at least qualitatively) a good approximation even at low carrier densities, the question remains whether the divergence of \( m^*(r_s) \) we discover within the RPA self-energy calculation is real (i.e. qualitatively valid) or an artifact. The related question is whether the Fermi liquid theory applies near the effective mass divergence, or equivalently, what the nature of this mass divergence behavior is on a fundamental level (i.e. is it some kind of a quantum phase transition, and if it is, what is the nature of the quantum phase on the other side of the transition beyond the mass divergence?). Unfortunately we are not in a position to definitively answer these questions. But we could certainly speculate on the various theoretical possibilities. First, we believe our RPA calculation to be qualitatively valid, i.e. we believe that \( m^*(r_s) \) does indeed increase very strongly at low densities, possibly diverging at a critical density. Second, the divergent effective mass obviously signifies some kind of a strong-coupling transition whose nature cannot be surmised on the basis of our Fermi liquid calculations since the basic premise of the one-to-one correspondence with the noninteracting electron gas breaks down at the transition point. These are all important and interesting questions which should be further investigated in future work. In this paper, we only report the finding of a low-density quasiparticle effective mass divergence in 2D electron liquids within the RPA self-energy calculation.

In section II we present the formalism of our theoretical approach. In section III we provide our RPA results as a function of density. In section IV we discuss going beyond RPA in the process providing theoretical justification for our calculation. We conclude in section V with a discussion.

II. FORMALISM

We describe the basic RPA self-energy formalism in II A followed by the discussion of the finite quasi-two dimensional layer width correction in II B. We discuss appropriate inclusion of higher-order vertex corrections (beyond RPA) in II C (more on this in section V), and the on-shell or off-shell calculation of the quasiparticle effective mass in II D.

A. RPA self energy

RPA effective mass calculation simply means that the quasiparticle self-energy is obtained using RPA dynamical screening approximation. Fig. II shows the Feynman diagram for RPA self-energy, from which we obtain (see,
FIG. 1: Feynman diagram for self-energy in the RPA calculation. Solid lines denote the free electron Green’s function $G_0$ and the wiggly lines the bare Coulomb potential. Note that formally the RPA self-energy is $G_{0\mu}$, where $\mu$ is the dynamically screened interaction in the ring diagram approximation.

e.g., [29] for the 2D self-energy ($\hbar = 1$)

$$\Sigma(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} \frac{d\nu}{2\pi i} v_q G_0(q+k, \nu+\omega),$$ (2)

where $v_q = 2\pi e^2/q$ is the bare Coulomb interaction between electrons (and $q, k$ appropriate 2D wavevectors),

$$G_0(k, \omega) = \frac{1 - n_F(\xi_k)}{\omega - k + i\eta} + \frac{n_F(\xi_k)}{\omega - k - i\eta}$$

is the Green’s function for free electrons with the noninteracting energy dispersion $\xi_k = k^2/m - \mu$, $\mu$ the chemical potential, and $\epsilon(k, \omega)$ is the dynamic dielectric function. We use $\eta$ to denote an infinitesimal positive number, and $n_F(x)$ the Fermi function. At zero temperature, $n_F(x) = 1$ when $x \leq 0$ and 0 otherwise. Within RPA, we have

$$\epsilon(k, \omega) = 1 - v_q \Pi(k, \omega)$$

with $\Pi(k, \omega)$ the noninteracting electron 2D polarizability (i.e. the bare bubble in Fig. 1):

$$\Pi(k, \omega) = 2 \int \frac{d^2q}{(2\pi)^2} \frac{n_F(\xi_q) - n_F(\xi_{q+k})}{\xi_q - \xi_{q+k} + \omega},$$ (5)

Previous calculation shows [31] that in 2D we have

$$\Pi(k, \omega) = -\frac{m^2}{\pi} + \frac{m^2}{\pi k^2} \left[ \sqrt{\left( \omega + \frac{k^2}{2m} \right)^2 - \frac{2\mu k^2}{2m}} - \sqrt{\left( \omega - \frac{k^2}{2m} \right)^2 - \frac{2\mu k^2}{2m}} \right].$$ (6)

Note that in Eq. (5) and Eq. (6), the frequency $\omega$ can be any complex number, and the branch cuts of the square roots are chosen such that their imaginary part is positive.

Due to the difficulty with the principal value integration and singularities of $1/\epsilon(k, \omega)$ along the real axis in Eq. (2), it is advantageous to follow the standard procedure and detour the frequency integration from the real axis to imaginary axis [33]. After choosing the contour of frequency integration as in Fig. 2, we consider the integration

$$\int \frac{d\nu}{2\pi i} w(q, \nu) G_0(q + k, \nu + \omega),$$ (7)

where $w(q, \nu)$ is any complex function that is analytic in the upper and lower half of the complex plane. On the one hand, (7) equals the integration of the integrand along real and imaginary axes since the integration along the curve part of the contour $\Gamma$ (Fig. 2) cancels out. On the other hand, (7) equals the residue due to the pole of Green’s function

$$(1 - n_F(\xi_{q+k}))\theta(\omega - \xi_{q+k}) w(q, \xi_{q+k} - \omega - i\eta) + n_F(\xi_{q+k})\theta(\xi_{q+k} - \omega) w(q, \xi_{q+k} - \omega + i\eta),$$ (8)

where $\theta(x)$ is 1 when $x > 0$ and 0 otherwise. Now by setting

$$w(k, \omega) = \frac{1}{\epsilon(k, \omega)} - 1,$$ (9)

we have

$$\Sigma(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} v_q n_F(\xi_{q+k})$$

$$- \int \frac{d^2q}{(2\pi)^2} \int \frac{d\nu}{2\pi i} v_q w(q, \nu) G_0(q + k, \nu + \omega)$$

$$= \Sigma^{ex} + \Sigma^{res} + \Sigma^{line},$$ (10)

where

$$\Sigma^{ex}(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} v_q n_F(\xi_{q+k}),$$ (11)

$$\Sigma^{res}(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} \left[ (1 - n_F(\xi_{q+k}))\theta(\omega - \xi_{q+k}) w(q, \xi_{q+k} - \omega - i\eta) + n_F(\xi_{q+k})\theta(\xi_{q+k} - \omega) w(q, \xi_{q+k} - \omega + i\eta) \right],$$ (12)

$$\Sigma^{line}(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} \int \frac{d\nu}{2\pi i} \frac{w(q, i\nu)}{v^2 + (\omega - \xi_{q+k})^2}. $$ (13)

Note that the static exchange part $\Sigma^{ex}$ is separated from the dynamical self-energy to improve the numerical convergence of the line part $\Sigma^{line}$. The exchange (or Fock) self-energy is obviously frequency independent (but depends strongly on the wave-vector). There is no Hatree-contribution to the self-energy in a uniform electron liquid.

It is easy to see that the real part of the self energy $\Sigma'$, which we need in order to calculate the renormalized effective mass, can be written as

$$\Sigma'(k, \omega) = -\int \frac{d^2q}{(2\pi)^2} v_q n_F(\xi_{q+k})$$

$$- \int \frac{d^2q}{(2\pi)^2} v_q w(q, \xi_{q+k} - \omega) \left[ n_B(\xi_{q+k} - \omega) + n_F(\xi_{q+k}) \right]$$

$$- \int \frac{d^2q}{(2\pi)^2} \int \frac{d\nu}{2\pi} \frac{w(q, i\nu)}{v^2 + (\omega - \xi_{q+k})^2},$$ (14)

where $n_B(x)$ is the Bose function at zero temperature, (i.e., $n_B(x) = -1$ when $x < 0$ and 0 otherwise)
FIG. 2: (Color online) Contour in the frequency $\nu$ plane. The cross denotes the possible position of the pole of Green’s function.

The real 2D systems have a finite layer width in the transverse direction which reduces the strength of the 2D Coulomb interaction. In order to test whether our result is model dependent, we include finite width correction in our calculation to see how the result changes compared to our ideal 2D results. The origin of this correction comes from the fact that any 2DES confined within a quantum well or heterostructure is physically a three dimensional system, with the width in the confined dimension small but finite. Due to the finite width of the quasi-2D electron gas system (such as Si MOSFETs and GaAs/AlGaAs heterojunctions), the Coulomb interaction should not be written as the ideal 2D form $v_q$, but should have a finite-width quantum form factor $F$ such that

$$v_q = \frac{2\pi e^2}{\kappa q} F(qb),$$

where

$$F(x) = (1 + \frac{\kappa_{\text{ins}}}{\kappa_{\text{sc}}}) \frac{8 + 9x + 3x^2}{8(1 + x)^3} + (1 - \frac{\kappa_{\text{ins}}}{\kappa_{\text{sc}}}) \frac{1}{2(1 + x)^6},$$

with

$$b = \left( \frac{\kappa_{\text{sc}} \hbar^2}{4\pi m \epsilon^2 n^*} \right)^{1/3}$$

denoting the width of the quasi 2D electron gas, $\kappa_{\text{sc}}$ and $\kappa_{\text{ins}}$ are the dielectric constants for the space charge layer and the insulator layer, $\kappa = (\kappa_{\text{sc}} + \kappa_{\text{ins}})/2$, $m_z$ is the band mass in the direction perpendicular to the quasi 2D layer, and $n^* = n_{\text{depl}} + \frac{1}{2} n$ with $n_{\text{depl}}$ the depletion layer charge density and $n$ the 2D electron density. We choose $n_{\text{depl}}$ to be zero in our calculations. When including finite width correction in our calculation, we replace $v_q$ by Eq. (15) in both Eq. (2) and Eq. (4). Note that we have chosen the finite width correction for a triangular potential confinement as in GaAs heterostructures or Si MOSFETs. In Si MOSFETs system,

$$\kappa_{\text{sc}} = 11.5 \text{ (Si)}, \quad \kappa_{\text{ins}} = 3.9 \text{ (SiO}_2),$$

$$m = 0.19 m_e, \quad m_z = 0.916 m_e, \quad (18)$$

while in GaAs heterostructures,

$$\kappa_{\text{sc}} = 12.9 \text{ (GaAs)}, \quad \kappa_{\text{ins}} = 10.9 \text{ (AlGaAs)},$$

$$m = 0.07 m_e, \quad m_z = 0.07 m_e. \quad (19)$$

C. Vertex correction

Even though there has been little systematic work going beyond RPA, it is still possible to include in some approximate way the effects of vertex corrections to $\Sigma(k, \omega)$. The purpose here is again to test whether our numerical result is highly model dependent. In some of our calculations we therefore approximate vertex corrections to the self-energy by including local-field corrections of the Hubbard type in the self-energy and in the dielectric function [14, 21, 26, 29]. In this approximation the self-energy $\Sigma(k, \omega)$ in Eq. (2) is modified by a vertex term $\gamma(k, \omega)$:

$$\Sigma(k, \omega) = - \int \frac{d^2q}{(2\pi)^2} \int \frac{d\nu}{2\pi i} \frac{v_\gamma(q, \omega)}{\epsilon(q, \nu)} \cdot G_0(q + k, \nu + \omega),$$

with the dielectric function now also including the same vertex (local field) correction:

$$\epsilon(k, \omega) = 1 - v_\gamma \Pi(k, \omega) \gamma(k, \omega).$$

It is important to emphasize that the vertex correction must appear both in the self-energy and in the dielectric function for the sake of consistency. Within RPA, $\gamma \equiv 1$ by definition, and in general $\gamma(k, \omega)$ is unknown. We choose

$$\gamma(k, \omega) = \frac{1}{1 + G(k)v_\gamma \Pi(k, \omega)},$$

where $G(k)$ is a static local-field correction given in the 2D Hubbard approximation by [21]

$$G(k) = \frac{k}{2\sqrt{k^2 + k_F^2}}. \quad (23)$$

If the inclusion of this local-field vertex correction substantially modifies our RPA results, then our qualitative
conclusions about the effective mass divergence become highly suspect. We will see this not to be the case. Note that in section IV.C we introduce another form of the local field factor $G(k)$ (and refer to this approximation as “HA2”), in contrast to the $G(k)$ factor given in Eq. (29) which we call “HA1” approximation) in order to compare with our RPA results and discuss the qualitative validity of our effective mass results.

D. Effective mass in on-shell and off-shell approximations

Even after an accurate calculation of $\Sigma(k, \omega)$, there are subtle issues associated with the calculation of the quasiparticle effective mass. The single particle energies are given by the positions of the poles of the interacting Green’s functions and are determined by the Dyson’s equation

$$E(k) = \frac{k^2}{2m} + \Sigma'(k, E(k) - \mu), \quad (24)$$

where $\Sigma'$ is the real part of the self-energy. The effective mass of the quasiparticles on Fermi surface can be written as

$$\frac{m^*}{m} = \frac{1 - \frac{\partial}{\partial \omega} \Sigma'(k, \omega)}{1 + \frac{\partial}{\partial \omega} \Sigma'(k, \omega)} \bigg|_{k=k_F}, \quad (25)$$

where $k_F$ denotes the Fermi momentum. Note that $\omega$ is measured from the chemical potential $\mu$, which is renormalized by the electron-electron interaction. $\mu$ can be obtained through solving the following self-consistent equation

$$\mu = \frac{k_F^2}{2m} + \Sigma'(k_F, \omega = 0; \mu), \quad (26)$$

where $\Sigma'(k, \omega; \mu)$ is given by Eq. (14), with chemical potential $\mu$ an explicit variable with the noninteracting chemical potential being $\mu \equiv E_F \equiv k_F^2/2m$, the 2D Fermi energy (i.e. $\Sigma' = 0$). Plugging the result of renormalized chemical potential $\mu$ into Eq. (25), we can then calculate the effective mass. We name this way of effective mass calculation the “off-shell approximation”, in contrast to the “on-shell approximation” which we describe in the following.

If the self-energy is calculated only in the lowest order, it is not sensible to solve Eq. (24) but rather take its first iteration so that different orders are not mixed in the perturbation theory:

$$E(k) = \frac{k^2}{2m} + \Sigma'(k, \xi_k). \quad (27)$$

This approximation has been used and justified in the literature [12, 23, 26, 28], and here we call it the “on-shell approximation”. The effective mass can then be written as

$$\frac{m^*}{m} = \frac{1}{1 + \frac{\partial}{\partial \omega} \Sigma'(k, \xi_k)} \bigg|_{k=k_F}, \quad (28)$$

where we use the un-renormalized (i.e. the noninteracting) chemical potential $E_F$ in the right hand side of the above equation.

On-shell and off-shell approximations obviously converge as $r_x \to 0$, where the interaction is weak and the higher-order iterations in Eqs. (21) and (27) vanish making Eqs. (24) and (27) as well as Eqs. (25) and (28) completely equivalent. But for $r_x > 1$, where the system is strongly interacting, there is a very large difference between the two approximations which increases with increasing $r_x$. The fact that on-shell and off-shell approximations give increasingly different (in fact, qualitatively different in our case as we will see later in section IV) results with increasing $r_x$ calls for a careful examination of the theoretical issue underlying these two alternate methods for calculating the quasiparticle effective mass (see section IV for more details on this topic) to decide which one is the correct approximation to use in a particular situation (and why). One could take the easy way out and insist that, as often is the case in solid state physics, empiricism should decide which of these approximations is “better” by comparing with experimental data. (By this empirical criterion of agreement with experiment, the on-shell approximation certainly wins out as the decisively better approximation for the 2D effective mass calculation since the off-shell RPA self-energy approximation for the 2D effective mass produces essentially a nearly density-independent effective mass with a rather small $\sim 10 - 20\%$ mass renormalization for $r_x \gtrsim 10$, which is in sharp contrast with the recent experimental claims [5, 7, 8, 8].) We feel, however, that the issue of on-shell versus off-shell approximation is sufficiently important to merit serious theoretical considerations on its own without resorting to empiricism. Arguments supporting the on-shell RPA scheme (over the off-shell one) were actually provided in rather compelling terms by Dubois [25] and by Rice [26] a long time ago (see also more recent references [11] and [12] for the corresponding 2D arguments). The basic argument, which we will expand upon and discuss in more detail in section IV is that of mixing orders in a systematic perturbation expansion. There is no question that if one calculates the exact self-energy (an obviously impossible task in our problem) then one must solve the Dyson’s equation to obtain the interacting quasiparticle Green’s function leading automatically to the off-shell effective mass formula (i.e. Eqs. (21) and (25)). The problem arises from the leading-order perturbative (in the dynamically screened interaction) nature of the RPA self-energy (formally $\Sigma \sim uG_0$ in the RPA theory, i.e. $\Sigma$ is in the leading order of $u$ and $G_0$), which implies that all quantities should be expanded only to the leading order in the interaction; otherwise one is mixing orders in an inconsistent manner in the theory rendering...
the theory highly suspect. This means that Eq. (24) for the quasiparticle energy dispersion \( E(k) \) should only be solved to the leading order in \( \Sigma \) (i.e. keeping only its first iteration, as in the on-shell approximation of Eq. (27)), and not to all orders as in the off-shell approximation. This is explicitly seen by iterating Eq. (24) in an order by order manner leading to

\[
E(k) = \frac{k^2}{2m} + \Sigma'(k, \xi_k) + \left[ \Sigma'(k, \xi_k + \Sigma'(k, \xi_k)) + \Sigma'(k, \xi_k + \Sigma'(k, \xi_k)) \right] + \cdots \tag{29}
\]

Clearly all the terms inside the square bracket are higher orders in the interaction \( u \), and should not be kept (as long as \( \Sigma' \) is obtained in the leading order \( G_0 u \) RPA approximation) for the sake of consistency, leaving us with the on-shell approximation of Eq. (27). It is important to emphasize that the distinction between off-shell and on-shell approximations disappears in the weakly interacting \( r_s \rightarrow 0 \) high-density limit since all the terms inside the square bracket are higher-order in \( r_s \) and disappear as \( r_s \rightarrow 0 \). We discuss the issue of on-shell and off-shell approximations further in section IV where the justification for (or the validity of) our RPA self-energy approximation (at large \( r_s \)) is taken up.

### III. RESULTS

#### A. Effective mass within on-shell approximation

We first present the results of our effective mass calculation using Eq. (28). We use the dimensionless quantity \( r_s \) to denote the density of the system, \( r_s = me^2/(\hbar^2 \sqrt{\pi n}) \), showing our calculated results as a function of \( r_s \) where \( n \) is the electron density. We concentrate on the high \( r_s \) (i.e. low electron density) behavior of the effective mass. We do mention, however, that we have calculated \( m^*(r_s) \) for all values of \( r_s \) reproducing the earlier results \([11, 12, 13, 14, 15, 16]\) in the literature in the \( r_s < 5 \) regime. In particular, we quantitatively re-produce the analytical 2D behavior \( \frac{m^*}{m} = 1 - \frac{r_s}{2\sqrt{\pi}} \log \frac{\sqrt{\pi}}{r_s} \) in the \( r_s \rightarrow 0 \) limit within our numerical calculations.

From Fig. 3 we can see that the 2D effective mass has a divergence at low carrier densities (\( r_s \sim 16.6 \)). Examining Eq. (28) carefully, it is not hard to identify the origin of this divergence. The dimensionless term \( \frac{1}{\hbar^2} \Sigma'(k, \xi_k) |_{k=k_F} \) in the denominator of Eq. (28) is a negative quantity for large \( r_s \), and our numerical calculation shows that its magnitude increases as \( r_s \) increases. At \( r_s \sim 16.6 \), the magnitude of this term passes through unity, which means that the calculated effective mass diverges.

We point out that the effective mass divergence we find does not in any way automatically imply a concomitant failure of the Fermi liquid theory (of course, we do not know what happens at \( r_s \) values above the critical density for the mass divergence). In particular, in Fig. 4 we present our calculated inverse Fermi liquid renormalization factor \( Z^{-1} \equiv [1 - \partial_{\omega} \Sigma(k, \omega)] |_{k=k_F, \omega=0} \) as a function of \( r_s \). We see that \( Z \) remains well-behaved around where \( m^* / m \) diverges (i.e. \( Z^{-1} \) does not manifest any divergence). Since \( Z \) denotes the Fermi surface discontinuity at \( k_F \), \( Z \neq 0 \) means that the Fermi liquid behavior is not destroyed.

To test whether our result is highly model dependent, we introduce two kinds of corrections as we explained in section III B and section III C. Fig. 5 shows the calculated effective mass including finite width corrections with RPA, and the Hubbard vertex correction in the ideal 2D case. We find that finite width correction suppresses the effective mass renormalization, while vertex correction enhances it at large \( r_s \) values. The important point is that both of these corrections show clear effective mass
divergence – for the finite width correction the critical $r_s$ for the divergence goes up since the bare interaction is softened by finite width effect whereas vertex correction actually decreases the critical $r_s$ since local field correction tends to enhance interaction effects. In fact the inverse mass passes zero as $r_s$ goes beyond critical values in each of these two cases, and the qualitative behavior of $m^*(r_s)$ is very similar to that shown in Fig. 3.

Note that in our effective mass calculation with finite width correction (results shown in Fig. 5), we have set $\bar{\kappa} = \kappa_{sc} = \kappa_{ins}$ and $m_z = m$, which is approximately applicable for GaAs heterostructure systems. We also calculate within the quasi-2D electron gas picture the effective mass of Si MOSFETs as a function of carrier density. (These results are shown in Fig. 12 and discussed in section V.)

It is reasonable to ask whether the low density effective mass divergence shown in Fig. 3 is a specific characteristic of strongly interacting 2D electron systems or is a generic property of interacting electron systems. To answer this important question we have carried out a similar calculation for a 3D electron system interacting via 3D Coulomb interaction $v_{q} = 4\pi e^2/q^2$, finding a very similar effective mass divergence at large $r_s$ (see Fig. 6). The divergence of $m^*(r_s)$ at some large critical $r_s$ is thus a generic feature (similar to He-3) of strongly interacting electron systems with system dimensionality (2D or 3D) playing no role.

B. Self consistently calculated chemical potential

The off-shell approximation effective mass calculation requires the chemical potential $\mu$ to be calculated self-consistently using Eq. (26).

In Fig. 7 we present our numerical result for the renormalized chemical potential $\mu$ using Eq. (26). $E_F$ denotes the Fermi energy for the corresponding non-interacting Fermi system.

In Fig. 4 we present our numerical result for the renormalized chemical potential. The result shows that the chemical potential decreases and approaches 0 as $r_s$ gets larger. We can also see that $\frac{d\mu}{dr_s}$, or the inverse compressibility $\kappa^{-1} = \frac{\mu^2}{E_F}$, approaches zero as density decreases or $r_s$ increases.
C. Effective mass within the off-shell approximation

Using the renormalized chemical potential, we obtain the off-shell effective mass results, which are presented in Fig. 8. Our off-shell results differ qualitatively from our on-shell results, and no mass divergence occurs within the off-shell RPA calculation at any \( r_s \) values. In fact the effective mass decreases at high \( r_s \), which is in qualitative disagreement with the experimental results. It is also strange that the calculated \( m^*(r_s) \) in Fig. 8 (i.e., the off-shell RPA) shows extremely weak many-body renormalization, typically less than 30% even for \( r_s \) as large as 10 – 20.

![Fig. 8: Calculated RPA effective mass within off-shell approximation using Eq. (25).](image)

We now provide some rather simple (essentially hand-waving) arguments to better understand the qualitative difference between the results of on- and off-shell approximations. We examine the \( r_s \) dependence of the partial derivatives of the self-energy \( \partial_\omega \Sigma'(k, \omega)|_{(k=k_F, \omega=0)} \) and \( (m/k) \partial_k \Sigma'(k, \omega)|_{(k=k_F, \omega=0)} \) appearing in Eq. (25). Both of these quantities increase in their magnitude with \( r_s \). As an oversimplified model, we make the approximation that \( \partial_\omega \Sigma'(k, \omega)|_{(k=k_F, \omega=0)} = -\alpha r_s \) and \( (m/k) \partial_k \Sigma'(k, \omega)|_{(k=k_F, \omega=0)} = \beta r_s \), where \( \alpha > 0 \) since our numerical results show that the two partial derivatives are opposite in sign and close in magnitude. Off-shell calculation for effective mass using Eq. (25) yields the result \((1+\alpha r_s)/(1+\beta r_s)\), which increases with \( r_s \) at first saturating at large \( r_s \) without any divergence. On the other hand, the on-shell approximation has both of these two derivatives appearing in the denominator of the effective mass expression (see Eq. (28), which yields \( 1/[1-(\alpha-\beta)r_s] \). (Note that in this oversimplified argument we ignore the difference between the renormalized chemical potential and non-interacting chemical potential.) Now we can manifestly see that on- and off-shell RPA approximation give the same \( m^*(r_s) \) in the \( r_s \rightarrow 0 \) limit. However, as \( r_s \) increases the on-shell effective mass keeps on increasing, and diverges at \( r_s \sim 1/(\alpha-\beta) \), whereas the off-shell effective mass essentially saturates for \( r_s > \alpha(>\beta) \), with a rather small \( (=\alpha/\beta < 30\%) \) renormalization. We have already argued that within the leading order “\( G_0u \)” RPA self-energy approximation, the on-shell effective mass calculation is more consistent (and hence, better) than the off-shell approximation. We will discuss this issue later in section IV again.

For completeness, we also mention that often in the literature the effective mass is calculated using Eq. (25) and the noninteracting chemical potential rather than interacting \( \mu \) (i.e. \( \mu = E_F \) is used in Eq. (25)). This is somewhat a mixture of our on-shell and off-shell approximations as it uses noninteracting \( \mu \) but self-consistent quasiparticle energy, and is logically inconsistent. We have also calculated effective mass using this method, and the results shown in Fig. 9 are similar to the off-shell results of Fig. 8.

![Fig. 9: Calculated 2D RPA effective mass using Eq. (25), while keeping \( \mu = E_F \).](image)

IV. VALIDITY OF RPA

We have discussed somewhat the validity of the RPA self-energy approximation in the context of our quasiparticle effective mass calculation. In this section we concentrate entirely on this important theoretical issue, and discuss it at some depth providing several complementary arguments supporting (at least) the qualitative validity of RPA even at large \( r_s \) value of primary interest to us in this work.

We start with some general observations about RPA. In the context of the quasiparticle self-energy (and effective mass) calculations it implies three distinct steps of approximations: (i) Calculate the self-energy in a single loop “\( G_0u \)” approximation (often called “\( GW \)” approximation in the literature following Hedin’s influential publication [25] dating back forty years although the first such self-energy calculation was originally carried out by Quinn and Ferrell [23], i.e. in the leading order dynamically screened interaction; (ii) Obtain the dynamically
screened interaction by using the infinite series of bare bubble diagrams, i.e., approximate the irreducible polarization by the noninteracting polarization “$G_0 G_0$” neglecting all vertex and self-energy corrections to the polarization; (iii) Calculate the effective mass using the on-shell approximation based on the rationale that in a leading order calculation of the self-energy the quasiparticle effective mass calculation even in the well-studied systems. It is important to emphasize that the interaction goes as $r_s^{-2}$ and $r_s^{-1}$ respectively in both 2D and 3D systems. It is important to emphasize that the interaction energy is manifestly non-analytic around $r_s \to 0$ (due to the $\ln r_s$ term) and therefore RPA is not a simple power series expansion in inverse-density even in the high density $r_s \to 0$ limit. We emphasize here a surprising (and apparently little-known) fact: RPA (i.e., the ring diagram approximation) again gives an asymptotically excellent result for the ground state correlation energy in the extreme low-density $r_s \to \infty$ limit as well. Although this was theoretically demonstrated by Iwata a long time ago, this fact seems to be not very well-known in the literature.

We first assert that it is incorrect to think of RPA as an expansion in the $r_s$ parameter as is commonly done. RPA is an expansion in the dynamically screened Coulomb interaction (i.e., the infinite series of ring or bubble diagrams as in Fig. 1), not a density expansion in any sense. For example, the RPA calculation for the plasmon dispersion is exact in the $q \to 0$ limit for all $r_s$, and in reality, the RPA-calculated 2D plasmon dispersion provides an accurate quantitative description for the experimental 2D plasmon dispersion up to very low densities ($r_s > 10$). (Note that the RPA plasmon dispersion is defined entirely by the polarization through the equation $\epsilon(q, \omega) = 0$, i.e., $1 - v_F \Pi(q, \omega) = 0$. ) What is true is that the RPA self-energy approximation (i.e., Fig. 1) becomes exact in the $r_s \to 0$ limit, where the quasiparticle effective mass can be written as an expansion in $r_s$, but the precise expansion parameter even in this $r_s \to 0$ limit is not $r_s$, but $\alpha = e^2/(\hbar v_F)$ where $v_F$ is the Fermi velocity. Writing the expansion parameter $\alpha$ out in terms of $r_s$, we see that it is incorrect to think of $r_s$ as the relevant expansion parameter for the quasiparticle effective mass calculation even in the well-studied $r_s \to 0$ limit since $\alpha \approx r_s/12$ (3D), $r_s/4.5$ (2D); and therefore the true expansion parameter is much less than $r_s$, making the radius of convergence of an $r_s$-expansion much larger than unity (perhaps qualitatively explaining why RPA works so well for 3D metals where $r_s \approx 3 - 6$).

What is undoubtedly true is that for arbitrary $r_s (>1)$, RPA is not exact by any means as it is in the unphysical $r_s \to 0$ limit. This statement is, however, not equivalent to claiming that RPA is necessarily a very bad approximation for large $r_s$. The argument for the failure of RPA at large $r_s$ is based entirely on the wrong premise that RPA is a systematic $r_s$-expansion even at large $r_s$.

It has long been known, since the seminal work of Gell-mann and Brueckner fifty years ago, that RPA (i.e., the ring or bubble diagrams) provides the asymptotically exact ground state energy for an interacting electron gas in the high-density $r_s \to 0$ limit. This is true both in 2D and 3D interacting electron systems (but not in 1D system where interaction effects are nonperturbatively singular and one must use entirely different approaches) and leads to a ground state energy expression of the form

$$\epsilon_{\text{corr}} = A^{(1)} + B^{(1)} \ln r_s \ (A^{(2)} + B^{(2)} r_s \ln r_s) \text{ in } 3D \ (2D)$$

electron liquids in the $r_s \to 0$ limit (where A and B are known numerical constants), where $\epsilon_{\text{corr}}$ is only the non-trivial interaction or the correlation part of the ground state energy per particle (i.e., leaving out the noninteracting kinetic energy and the Fock exchange energy which goes as $r_s^{-2}$ and $r_s^{-1}$ respectively in both 2D and 3D systems). It is important to emphasize that the interaction energy is manifestly non-analytic around $r_s \to 0$ (due to the $\ln r_s$ term) and therefore RPA is not a simple power series expansion in density even in the high density $r_s \to 0$ limit. We emphasize here a surprising (and apparently little-known) fact: RPA (i.e., the ring diagram approximation) again gives an asymptotically excellent result for the ground state correlation energy in the extreme low-density $r_s \to \infty$ limit as well. Although this was theoretically demonstrated by Iwata a long time ago, this fact seems to be not very well-known in the literature.

We show using a very crude and rather simplistic power counting arguments, that RPA is most likely not an expansion in the $r_s$-parameter for arbitrary $r_s$ values, but it becomes equivalent to an $r_s$ expansion as $r_s \to 0$. In section we revisit the on-shell versus the off-shell approximation question for the quasiparticle effective mass calculation on a more formal level, justifying on a diagrammatic basis the superiority of the on-shell approximation over the off-shell one showing that the on-shell approximation in fact systematically (albeit implicitly) includes some higher-order diagrams in the effective interaction making it therefore better than just the simple single-loop “$G_0 G_0$” approximation. Finally, in section we explicitly calculate some appropriate vertex corrections to the 2D self-energy by including different simple local field corrections to the bare polarizability showing that at least this level of an approximate inclusion of vertex corrections to the theory does not qualitatively change the 2D effective mass results compared with the RPA effective mass calculations, providing some additional justification for using the RPA self-energy at large $r_s$ value. None of our arguments establishes the RPA effective mass calculation as a theoretically rigorous approximation for large $r_s$ (and surely it is not), but we believe that our arguments are persuasive in demonstrating that this is not necessarily a qualitatively wrong approximation at large $r_s$. 

...
A. The effective expansion parameter $\delta$ in RPA

To approximately figure out the effective expansion parameter in RPA we reconsider the Feynman-Dyson diagrammatic expansion for the electron self-energy and the irreducible polarizability in Fig. 11 by concentrating on the higher-order diagrams left out of RPA. From Fig. 11 we see that in each higher order (remembering that the expansion is in the screened interaction, not the bare Coulomb interaction which diverges in the $q \to 0$ limit) we get an additional factor of $\delta \equiv \left[ G_0 G_0 u \frac{d^3 q d \omega}{(2 \pi)^3} \right]$ with the formal series for the polarizability and the self-energy being:

$$ \Pi \Rightarrow G_0 G_0 + G_0 (G_0 G_0 u) G_0 + G_0 (G_0 G_0 u G_0 G_0 u) G_0 + \cdots, $$

$$\Sigma \Rightarrow G_0 u + G_0 u (G_0 G_0 u) + G_0 u (G_0 G_0 u G_0 G_0 u) + \cdots, \quad (30)$$

where the first terms $G_0 G_0$ (for $\Pi$) and $G_0 u$ (for $\Sigma$) are what we keep in RPA (see Fig. 11).

We now carry out a simple heuristic power counting of the factor $G_0 G_0 u \frac{d^3 q d \omega}{(2 \pi)^3}$ to get:

$$ \delta \equiv \left[ G_0 G_0 u \frac{d^3 q d \omega}{(2 \pi)^3} \right] \Rightarrow \frac{1}{E_F} \left( \frac{2 \pi e^2}{k_F + q_{TF}} \right) \frac{(\pi k_F^2) E_F}{(2 \pi)^3}. \quad (31) $$

where we have used simple dimensional forms for Green’s function $G_0 \sim E_F^{-1}$, 2D screened interaction $u \sim 2 \pi e^2 / (k_F + q_{TF})^{-1}$; and phase space $d^2 q \sim \pi k_F^2$; $d \omega \sim E_F$. This leads to

$$ \delta \sim \frac{\sqrt{2 r_s}}{4 \pi (1 + \sqrt{2} r_s)}. \quad (32)$$

Thus the 2D RPA expansion parameter $\delta$ is essentially $r_s$ as $r_s \to 0$, but is actually of the form $r_s / [4 \pi (r_s + 1 / \sqrt{2})] r_s$ for arbitrary $r_s$, which is certainly not equivalent to an $r_s$ expansion except in the high-density $r_s \ll 1$ limit. Thus, this admittedly crude power counting dimensional argument leads to the conclusion that RPA remains approximately valid even at large $r_s$, where $\delta \sim 1 / (4 \pi)$ is indeed a small parameter. This power counting argument holds equally well in 3D systems where

$$ \delta \equiv G_0 G_0 u \sim \frac{r_s (1/6 \pi) (4 / \pi) (4 / 9 \pi)^{1/3}}{1 + r_s (4 / \pi)(4 / 9 \pi)^{1/3}} \quad (33)$$

using the 3D screened Coulomb interaction and 3D phase space integral. Thus, in 3D, $\delta \sim r_s$ in the $r_s \to 0$ limit, but for $r_s \gg 1$, $\delta \sim 1 / (6 \pi)$ which is small. (We note as an aside that our crude arguments fail completely if there is a quantum phase transition to a new ground state at some finite values of $r_s$ since then the form for $G_0$, $u$ etc. will change qualitatively.)

The crude dimensional arguments of this section, like all power counting arguments, are too simplistic to be taken rigorously. But, these dimensional arguments compellingly demonstrate that, contrary to the very widespread popular belief, RPA is an expansion in the $r_s$-parameter only in the $r_s \to 0$ limit but not for arbitrary values of $r_s$. In fact, for $r_s > 1$, RPA may very well be reasonably well-valid (as in commonly found on empirical grounds) because the expansion parameter $\delta$ is not necessarily large just because $r_s \gg 1$, but is in fact quite small ($\sim 0.1$) numerically. We emphasize that the dimensional argument given here is for too simplistic to be taken rigorously — our purpose here is to demonstrate that at large $r_s$ the RPA is not equivalent to an $r_s$ expansion. In particular, one serious flaw of our dimensional argument is that it completely fails to produce the $\ln r_s$ terms which are known to be present in the $r_s \to 0$ limit, but the very presence of these logarithmic terms in the high-density limit underscores the fact that RPA is not a simple power series expansion in $r_s$, which is the main point we are making here.

B. Approximate higher-order corrections through the RPA on-shell approximation

In this sub-section we show that the on-shell approximation to the quasiparticle energy, discussed in section 11 actually implicitly incorporates some higher-order corrections (i.e. beyond the single-loop $G_0 u$ self-energy of RPA), making it therefore an improvement over the off-shell approximation involving the full solution of the Dyson’s (integral) equation. The basic ideas underlying this argument go back a very long time to Dubois [28] and to Rice [29]. We outline the arguments here in the context of the 2D RPA self-energy calculation since the arguments seem not to be very well-known (or well-appreciated) in the literature. The iterative (off-shell) solution of the Dyson’s equation to the quasiparticle energy can be formally written as (Fig. 11 (d)):

$$ E_k = \xi_k + \Sigma^\prime(k, E_k) $$

$$ = \xi_k + \Sigma^\prime(k, \xi_k) + \Sigma^\prime(k, \xi_k) + \Sigma^\prime(k, \xi_k) + \cdots $$

$$ \approx \xi_k + \Sigma^\prime(k, \xi_k) + \Sigma^\prime(k, \xi_k) \frac{\partial \Sigma}{\partial \xi_k} + \cdots, \quad (34)$$

where we have only explicitly considered up to the second-order in iteration. One can now formally combine the second order iteration in the self-energy in the Dyson’s equation with the second-order terms in the diagrammatic perturbative expansion (Fig. 11 (a)) of $\Sigma$ itself (i.e. the two second order diagrams in Fig. 11 (a) beyond the $G_0 u$ single-loop leading-order RPA self-energy term) to obtain the second order correction to the basic ‘on-shell’ RPA self-energy term:

$$ \Sigma^{(2)}(k, \omega) = \Sigma^{(2)}(1) + \Sigma^{(2)}(2) + \frac{\partial \Sigma^{(1)}}{\partial \xi_k} \Sigma^{(1)} \quad (35)$$
where $\Sigma^{(2)}_1$ and $\Sigma^{(2)}_2$ are respectively the contributions from the second order exchange self-energy and the second order vertex correction diagram, and $\Sigma^{(1)}$ is the first-order RPA self-energy obtained from the second-order Taylor expansion of the Dyson’s equation expansion.

\( \Sigma^{(2)}_1 + \frac{\partial \Sigma^{(1)}}{\partial \xi_k} \Sigma^{(1)} = \int \frac{d^2q d^2q'}{(2\pi)^4} \int \frac{dv dv'}{(2\pi)^2} \times G_0(k + q, \xi_k + \nu) \gamma G_0(k + q', \xi_k + \nu') \times u(q, v) [u(q - q', \nu' - \nu) - u(q', \nu')] \) (36)

By generalizing these arguments order by order, it can be shown in a straightforward fashion that such approximate cancellations between the higher-order exchange corrections and the iterations of the Dyson’s equation for the leading-order (i.e. RPA) self-energy occur to all orders. This leads to the conclusion, already made in section 11 and 14, that the on-shell RPA self-energy is a better approximation than the off-shell one since the on-shell approximation to the leading-order RPA self-energy implicitly contains some aspects of the higher-order corrections through the order-by-order cancellation demonstrated above. In particular, the on-shell ‘$G_0 u$’ approximation simulates the self-consistent ‘$G_4 u$’ approximation, where $G$ is the full interacting Green’s function rather than just the noninteracting $G_0$.

This reinforces our earlier assertion that the RPA on-shell effective mass calculation is not a simple high-density expansion, and could have substantial qualitative (and perhaps even quantitative) validity at larger $r_s$ values although the extent of its quantitative validity is difficult to assess because it is not a controlled perturbative theory at large $r_s$.

Similar to the other arguments made in this section, we emphasize that the cancellation between the higher order diagrams (Fig. 11) and the Taylor expansion discussed here is certainly partial and approximate, and again this is by no means a rigorous argument.

C. Including vertex corrections through Hubbard
local field effects

To obtain some ideas about the model dependence of our calculated quasiparticle effective mass we have already shown results (see Fig. 5) comparing the 2D RPA effective mass with the 2D effective mass calculated in the Hubbard approximation. The basic idea is to introduce an approximate vertex correction (to both the polarizability function and the self-energy) in the theory to calculate the renormalized effective mass in order to assess the robustness of the RPA approximation. A rigorous calculation of the self-energy vertex correction in the dynamical screening expansion is essentially an impossibly formidable task; one must therefore resort to drastic approximations. One reasonably successful technique for introducing vertex corrections to the theory has been to preserve the formal structure of the RPA self-energy and the polarizability with a new function $\gamma$, the so-called local field correction, modifying the electron-electron interaction mimicking the higher-order vertex corrections. Formally, the self-energy is given by Eqs. (20) and (21), and the $\gamma$ function is given in terms of a local field factor $G(k)$ by Eqs. (22) and (23). $G(k)$ can be chosen satisfying various sum rules. The two most popular forms for the local field correction are due to Hubbard [30], and in 2D system they take the form

\[ HA1 : G(k) = \frac{k}{2\sqrt{k^2 + k_F^2}} \]
\[ HA2 : G(k) = \frac{k}{2\sqrt{k^2 + k_F^2 + q_F^2}} \]

We have already used the first model (HA1) in obtaining results in Fig. 5. We find a divergent $m^*(r_s)$ for $r_s \sim 10 - 17$ in all three approximations with HA1 giving a stronger divergence than the other two. In Fig. 11 we compare these two vertex correction approximations with RPA for the calculated 2D $m^*(r_s)$ as a function of $r_s$ in the $r_s = 0 - 17$ regime. The results show that (i) qualitatively all three approximations give similar results, and (ii) the HA2, which is considered [27] to be a superior vertex correction than HA1, actually produces $m^*(r_s)$ quantitatively very close to RPA.

The results of Fig. 11 which include vertex corrections, again confirm the essential qualitative validity of our calculated $m^*(r_s)$. The point we are making here is that our main qualitative conclusion that $m^*(r_s)$ diverges at some value of $r_s$ remains valid even when Hubbard approximation the vertex correction is made by going beyond RPA. On the other hand the precise value of $r_s$ where $m^*$ diverges may depend on the approximation.
It is well-known that the strongly interacting Coulomb system at large $r_s$, where the electron-electron interaction cannot be considered a weak perturbation to the noninteracting kinetic energy, does not allow for a controlled systematic many-body calculation since no order-by-order perturbative diagrammatic expansion is possible because there is no small parameter (in contrast to the high-density situation, $r_s \ll 1$, where the interaction is a weak perturbation and the RPA self-energy expansion is asymptotically exact). The great merit of RPA lies in the fact that it is a systematic self-consistent mean-field theory (apart from also being an exact low-$r_s$ perturbative theory) because it includes the basic physics of the dynamical screening of the long-range Coulomb interaction. As such RPA should remain qualitatively valid at large $r_s$ until there is a quantum phase transition to a new state. There have been many systematic attempts [40] to go beyond RPA in the literature in calculating the electron self-energy, and many of these attempts include the approximate vertex corrections by exploiting the self-consistent field (rather than the perturbative) nature of RPA by including the vertex corrections through various local field factors. The two Hubbard approximations we include in our work are only among the simplest such approximations incorporating vertex corrections– we refer to the literature [42] for more sophisticated treatments of vertex corrections. The important point in the context of our theory is that the incorporation of vertex corrections is unlikely to cause a qualitative change in the RPA results although quantitative changes are expected. Therefore, we can conclude that our finding of an effective mass divergence within the RPA self-energy calculation (and the Hubbard approximation) should remain valid in general although the critical $r_s$ where the mass divergence occurs is likely to be affected by the specific many-body approximation.

V. DISCUSSION AND CONCLUSION

We have presented in this paper a detailed quantitative calculation for the zero-temperature quasiparticle effective mass in a strongly interacting homogeneous electron liquid using the realistic long-range Coulomb interaction and the noninteracting parabolic kinetic energy dispersion. The theory covers a large range of the dimensionless interaction parameter space, and provides $m^*(r_s)$ down to the low-density regime $r_s \sim 15 - 20$. We have focused primarily on the (semiconductor-based) 2D electron systems since rather high (low) values of $r_s$ (density) are now routinely achieved experimentally in 2D semiconductor systems, and therefore there is considerable current interest in the many-body properties of 2DES. Our main results for $m^*(r_s)$ are qualitatively similar in 2D and 3D systems. We also provide in this paper a serious and substantive theoretical analysis of the feasible many-body approximations for strongly interacting electron liquids within the Feynman-Dyson diagrammatic perturbation theory. We argue (based on a number of approximate non-rigorous arguments) that the RPA self-energy expansion in the dynamically screened Coulomb interaction, the dynamically screened single-loop Hatree-Fock (or equivalently ‘$G_0u$’ or ‘GW’) approximation, may very well be a reasonably good (albeit somewhat uncontrolled in the sense that a systematic improvement on the single-loop ‘$G_0u$’ approximation is difficult) approximation even at large $r_s$ (while being exact in the $r_s \to 0$ limit, as has been known for a long time). In the process we establish that the RPA self-energy or effective mass calculation is not an $r_s$ expansion at arbitrary $r_s$ values, it is more likely to be an expansion in an effective parameter $\delta = (r_s/c)/(a + r_s)$, where $a \sim 1$ and $c > 1$ in both 2D and 3D. Thus, RPA is exact in the $r_s \to 0$ limit, and is reasonably well-valid for $r_s \gg 1$ provided $c \gg 1$: we find $c \approx 4\pi$ (2D), $6\pi$ (3D) showing that the RPA self-energy and effective mass calculation may have reasonable qualitative (perhaps even quantitative) validity in the large $r_s$ regime of our interest (provided that the strong interaction does not drive the system to a quantum phase transition). The precise level of the quantitative validity of RPA cannot, however, be ascertained at large $r_s$ values since we have no reliable way of estimating the terms being left out of the calculation. (In this respect RPA shares the shortcomings of other well-known electronic theory approximations such as LDA or DMFT which are very successful quantitative theories without being controlled systematic approximations.) Analyzing in some depth the structure of the infinite resummation of perturbation terms involved in the RPA expansion in the screened interaction for the self-energy function, we also establish rather compellingly that the on-shell RPA quasiparticle
energy (and effective mass) calculation is the appropriate correct approximation at arbitrary \( r_s \) values, and the off-shell approximation in fact leads to qualitatively incorrect results (by virtue of the inconsistent mixing of different orders) at large \( r_s \); for small \( r_s (\ll 1) \), of course, the on-shell and the off-shell approximations become equivalent.

We emphasize that, in spite of several crude (and at best persuasive, by no means rigorous and compelling) arguments we have provided in section [LV] in support of our approximation scheme, our results can only be considered qualitative since the ring-diagram approximation inherent in RPA is exactly only in the high density \( (r_s \to 0) \) limit whereas our finding of a low-density effective mass divergence occurs in the strongly interacting \( r_s \approx 10-15 \) regime. While agreeing that RPA is, in principle, an uncontrolled many-body approximation at large \( r_s \), we claim that there is no particular value of \( r_s \) at (or above) which the RPA self-energy fails qualitatively. In fact, the evidence is quite contrary, typically RPA predictions for qualitative behavior always turn out to be valid. For example, RPA energetics for interacting electron systems predict a ferromagnetic instability at \( r_s \approx 19(7) \) in 3D (2D) electron systems where the spin \( g \)-factor (or equivalently, the static susceptibility) diverges, and it is now universally accepted that an interacting electron system indeed undergoes an interaction-driven low-density ferromagnetic instability (the so-called Bloch ferromagnetism) albeit at \( r_s \) values larger than the RPA predictions. We see no reason why our finding of a low-density effective mass divergence within RPA should be any different from the corresponding divergence in the magnetic susceptibility. It is likely that RPA underestimates the critical \( r_s \) where the effective mass diverges, and the real effective mass divergence would occur at higher \( r_s \) values, but we see no reason for RPA to give a qualitatively incorrect answer for the effective mass divergence. An important related point in this context is that the RPA or the “GW approximation” is the only available many-body approximation for quantitative electronic self-energy calculations, and as such it is routinely used for 3D metallic systems with \( r_s \approx 5 \). In 2D electron systems, RPA effective mass calculation has earlier been carried out in the literature up to \( r_s \approx 5 \) also in the context of providing a quantitative explanation for effective mass measurements in 2D semiconductor structure. We have now extended these calculations up to \( r_s \approx 10-15 \) because recent low density 2D systems have achieved such large values by providing high-quality low-density 2D structures.

Our theory involves two distinct approximations: RPA or ring-diagram approximation for the electronic self-energy calculation and the on-shell approximation for calculating the quasiparticle energy leading to the quasiparticle effective mass. While the RPA self-energy calculation is an essential (and in some sense, uncontrolled) approximation in our theory (except for incorporating vertex corrections through crude local field approximations, e.g. the Hubbard approximation, as we do in some of our calculations), one could question the need (or even the validity) of the on-shell quasiparticle effective calculation. We have argued in this paper that for our ‘\( G_0u \)’ self-energy approximation (Fig. 1), the on-shell effective mass calculation is, in fact, the appropriate approximation (rather than the off-shell approximation which solves the full Dyson’s equation iteratively). We assert that, as shown by Rice [24] and used by Quinn and collaborators [12] a long time ago, the effective mass approximation consistent with the Landau Fermi liquid theory and our ‘\( G_0u \)’ self-energy approximation is precisely the on-shell approximation. This is simply because we use \( G_0 \), the non-interacting Green’s function in all our internal propagator, and therefore the quasiparticle energy consistent with this leading-order self-energy calculation is precisely our Eq. (27), the on-shell approximation, rather than our Eq. (24), the off-shell approximation. This has been discussed in great detail by Rice [24] almost forty years ago (and was already implemented for 2D systems in Ref. [12] almost thirty years ago). Our on-shell quasiparticle effective mass is therefore precisely the Fermi liquid quasiparticle effective mass [11].

We find that the quasiparticle effective mass \( m^*(r_s) \) at \( T = 0 \) increases rapidly (both in 2D and 3D) at low densities with increasing \( r_s \), consistent with the strongly interacting nature of the Coulomb system at large values of \( r_s \). One most important and interesting result is a true divergence of \( m^*(r_s) \) at a critical \( r_s \approx 16.6 \) in 2D (the mass divergence occurs around \( r_s \approx 48 \) in 3D) within the on-shell single-loop \( G_0u \) RPA self-energy calculation. This theoretical effective mass divergence (or at least the rapid increase of effective mass with decreasing density) seems to superficially simulate recent measurements [3, 4, 8] of low density 2D effective mass (from the temperature dependence of the SdH oscillations of the low field magnetoresistance) in n-Si MOSFETs.

There are three general questions that immediately arise in the context of our theoretical finding of a divergent \( m^*(r_s) \) for sufficiently large (but experimentally accessible in 2D, but not in 3D systems) values of the density parameter \( r_s \). These questions are: (i) Are our theoretical results of any relevance to experiments (particularly to the large body of 2D M-I-T literature where large-\( r_s \) behavior of 2DES is being explored extensively)? (ii) Is our finding of a divergent effective mass at a large critical value of \( r_s \) real? (i.e. is it a truly robust finding as we have argued in this work or just an artifact of our single-loop \( G_0u \) approximation scheme?) (iii) If the interaction-driven divergence of \( m^*(r_s) \) is real, then what does this phenomenon imply for an electron liquid? Below we discuss some partial answers to (and educated speculations on) these questions. Since an exact solution to this strongly interacting (and 50-year old) many-body problem is unlikely to come in the near future, we can only discuss various aspects of this problem qualitatively with no definitive resolution of these important questions.

For whatever it is worth we can, of course, carry out a direct comparison with the experimentally mea-
sured density-dependent effective mass with our calculated $m^*(r_s) = m^*(n)$ for the appropriate 2DES. We show our theoretical results for the $T = 0$ RPA effective mass as a function of the carrier density in Si-SiO$_2$ inversion layer in Fig. 12, including the quasi-2D form-factor effects in the Coulomb interaction by modifying the bare Coulomb interaction $v_F = 2\pi\epsilon^2/\bar{\kappa}q$ to $v_q F(qb)$ where $\bar{\kappa}$ is the effective background dielectric constant for the system, $F$ is the quasi-2D form-factor, and $b \equiv b(n)$ is the finite quasi-2D layer width parameter discussed in section 11B. For Fig. 12 results we have used the appropriate semiconductor electron band mass and lattice dielectric constant (given in Eqs. 18) in the calculation. In Fig. 12 we have shown as an inset the experimental Si MOS effective mass values for a comparison with our theory. Clearly, the theory and experiment are in excellent qualitative agreement, most notably in the rapidly rising low density $m^*(n)$ behavior. While it is certainly gratifying to see that our theory agrees well with the existing low-density effective mass measurements, one must take such agreements with more than a grain of salt. The reliability and/or the level of accuracy of the experimentally measured $m^*(n)$, particularly at the low carrier densities of interest to us, is unknown. This is particularly true in view of the experimental effective mass measurements being based on the temperature-dependent amplitude of the weak-field SdH magnetoresistance oscillations at each density. We have shown elsewhere 17 that the quasiparticle effective mass $m^*(r_s)$ has a strong temperature dependence at low densities - in fact $m^*(T/T_F) \sim m^*(0)[1 + c(T/T_F)]$ with $c > 0$. Such an intrinsic temperature dependence of the effective mass itself may introduce unknown errors in the experimental mass measurements. In addition, there is always the key problem of not quite knowing what the electron temperature really is (a particularly severe problem at low carrier densities) which could lead to larger errors in the experimental determination of $m^*(n)$. In spite of these caveats the excellent qualitative agreement between theory and experiment demonstrated in Fig. 12 is quite impressive.

In addressing the issue of the “robustness” of our finding of a divergence of $m^*(r_s)$ at a large critical $r_s$ we first assert that within our approximation scheme this interesting result is certainly real and not an artifact as demonstrated by the manifest model independence of our result – we find that $m^*(r_s)$ diverges at a large $r_s$ both in 2D and 3D systems, both within the simple RPA and the Hubbard approximation calculations, and both in the strictly 2D and quasi-2D systems. The only model dependence of our theory is in our finding that the off-shell approximation (i.e. the full solution of the Dyson’s integral equation using, however, only the leading order dynamically screened single-loop self-energy function) does not lead to a mass divergence (and in fact lead instead to an effective mass saturation at large $r_s$ in clear disagreement with experimental results), but we have given convincing arguments that the off-shell approximation is theoretically unjustified (and inconsistent) in view of the single-loop leading-order nature of our ‘$G_{0u}u$’ self-energy calculation.

Although the divergence of $m^*(r_s)$ at large $r_s$ is certainly a real result within our ‘$G_{0u}u$’ single-loop self-energy approximation scheme, the important question of whether it is a correct result (i.e. valid beyond our approximation scheme) for the interacting electron liquid remains unanswered at this stage. While we have included some approximate vertex corrections (see section 11V in the theory, a self-energy calculation going beyond the leading-order ‘$G_{0u}u$’ single-loop diagram is essentially impossible. The second-order self-energy diagrams in the dynamically screened Coulomb interaction expansion (Fig. 10) involve six-dimensional highly-singular (with overlapping singularities and branch cuts) integrals in the ‘$G_{0u}G_{0}G_{0}u$’ second-order terms. We see no hope of an accurate (numerical) evaluation of these 6-dimensional highly singular principal value integrals in order to reliably calculate $\Sigma(k, \omega)$ to the 2-loop level so that $m^*(r_s)$ can be calculated correctly (formally) to second order in the dynamically screened interaction. Such a (hopeless) calculation is necessary to see whether the divergence of $m^*(r_s)$ is truly a feature of the interacting electron systems at low carrier densities or is merely an artifact of the single-loop self-energy formalism. Even such a 2-loop calculation will not obviously definitively establish the existence of a density-driven effective mass divergence transition (because one could still wonder what would happen at the 3-loop or the 4-loop level), but we believe that finding a divergence of $m^*(r_s)$ in both 1-loop (as we do here) and 2-loop level will go a long way in establishing the very strong possibility that such a divergence does indeed occur in the ground state of an interacting quantum Coulomb Fermi system.

Short of such a 2-loop self-energy calculation (which
we believe to be impossibly difficult) we have to make do with a number of heuristic arguments supporting the reality of the effective mass divergence reported in this work. First, the single-loop $G_{0\mu}$ RPA self-energy calculation (along with its simple Hubbard approximation vertex correction) certainly leads to an effective mass divergence. Second, we give several arguments (see section IV) why we believe the $G_{0\mu}$ approximation to be at least qualitatively valid at large $r_s$ where the effective mass divergence occurs. Third, the structure of the effective mass formula (see section III) indicates that the effective mass divergence may be a generic feature of interacting electron systems in the strong interaction regime. Fifth, as mentioned in the Introduction, the strongly interacting normal He-3 (with short-range-range interactions) is also thought to be an almost-localized Fermi liquid (i.e., very large quasiparticle effective mass) akin to what we find for strongly interacting electron liquids interacting via the Coulomb interaction. Similar to our finding in this work, normal He-3 is also reported [30, 31, 32] to go through an effective mass divergence transition at sufficiently large interaction strength. It is therefore quite possible, perhaps even very likely, that both short-range-range interacting He-3 and long-range-range interacting electron liquids have quasiparticle effective mass divergence at sufficiently large interaction strength; such an effective mass divergence leading to “an almost-localized Fermi liquid” may very well be a generic feature of strongly interacting quantum Fermi systems.

Accepting that there is indeed a critical (large) value of $r_s \equiv r_c (\approx 16.6$ in the ideal 2DES according to the single-loop calculation, but in general we expect $r_c$ to be some large $r_s$ value indicating the strong interaction regime) where $m^*(r_s) \mid_{r_s \rightarrow r_c}$ diverges, the important question of the nature and implications of this mass divergence takes on great significance (both theoretical and practical since 2DES with $r_s > 10$ is routinely available with the state of the art semiconductor technology). We first note that according to the Landau’s Fermi liquid theory the low-temperature thermodynamic properties such as specific heat, spin susceptibility, and compressibility are all proportional to the quasiparticle effective mass:

$$\begin{align*}
\frac{C_v^*}{C_v} &= \frac{m^*}{m} = 1 + A_1, \\
\chi^* &= \frac{m^* g^*}{mg} = \frac{m^*}{m} \frac{1}{1 + B_0}, \\
\kappa^* &= \frac{m^*}{m} \frac{1}{1 + A_0}.
\end{align*}$$

(39)

where $A_0$, $B_0$, $A_1$ are the appropriate Landau Fermi liquid parameters. Assuming that $A_0$ and $B_0$ are well-behaved at low densities, we conclude that both the spin susceptibility and the compressibility will also diverge around the critical $r_s$ where the effective mass diverges. Such anomalous behavior in the magnetic susceptibility, with rapidly increasing susceptibility at low densities, has indeed been reported in several recent experimental studies of 2DES. Recent 2D experiments also report [42] the inverse compressibility passing through zero at low carrier densities indicating an infinite compressibility consistent with the mass divergence (assuming regular density dependent behavior of the Landau parameter $1 + A_0$). But the compressibility measurements [42] are most likely connected with disorder effects invariably present in the system, and are unlikely to be intrinsic Fermi liquid renormalization phenomena. More work is needed to settle these issues.

A key question in this context (accepting the mass divergence to be real) is whether this effective mass divergence signifies a new strong coupling fixed point of the interacting electron system or is a subtle manifestation of one of the already known quantum phase transitions which may occur in an interacting electron system. A related question of equal importance is whether this mass divergence necessarily implies a breakdown of the Fermi liquid theory. Unfortunately, none of these questions can be answered at our current stage of understanding of the subject since our calculation only leads to the effective mass divergence without telling us the nature of the new ground state, if indeed there is a quantum phase transition associated with the mass divergence phenomenon. Using understanding [32] developed in the context of the strongly interacting neutral He-3, which also undergoes [30, 31] a very similar effective mass divergence transition, we speculate that the transition associated with the effective mass divergence may very well be a continuum version of a Mott transition, where increasing Coulomb interaction in a system leads to “localization” and metal-insulator transition as the system can lower its interaction energy at the “cost” of kinetic energy by developing infinite mass (i.e., “no hopping”) – a divergent mass in a continuum system is the closest analog to the Mott transition, but it is, in principle, possible for the system to undergo a different type of localization transition through the effective mass divergence long before the Wigner crystallization low-density point is reached. The alternate possibility is that our effective mass divergence is essentially a signature or a precursor to the eventual Wigner solidification transition in the system. We also can not rule out the possibility that the RPA effective mass divergence is a precursor of a charge density wave transition in the system (related to, but not identical to, the Wigner transition). Any of these transitions of course signifies a breakdown of the normal Fermi liquid picture because the one-to-one correspondence with the non-interacting electron gas ground state is destroyed beyond the transition point. At this stage, however, we do not
rigorously know the nature of the interaction-driven effective mass divergence phenomenon (except that it happens at some large $r_s$ values). Much more work will be needed to develop an understanding of the mass divergence phenomenon reported in this work, but the very real exciting possibility exists that all strongly interacting homogeneous Fermi liquids undergo a continuum version of a Mott transition, where the renormalized quasiparticle effective mass diverges, at sufficiently high interaction strengths. At this stage we do not know whether our theoretical discovery of a low-density divergence in the quasiparticle effective mass liquefies any kind of phase transition or not. In may be worthwhile to point out in this context that the corresponding problem of a single electron strongly coupled to the lattice ("the strong coupling polaron" problem) also shows a sharp effective polaronic mass divergence at large electron-phonon coupling strength 14, which is superficially very similar to the quasiparticle effective mass divergence we find in the current interacting electron problem. A key difference between the polaronic problem and current interacting electron problem is the fact that the polaronic case deals with a single electron strongly coupled to a bosonic bath (i.e. phonons) whereas the current problem involves infinite fermionic degrees of freedom (i.e. this is a many-electron problem). Thus, the polaronic effective mass divergence (the so-called self-trapping of the polaron), being a one electron problem, cannot by definition involve any phase transition whereas the current problem being a problem with infinite degrees of freedom does (at least) allow the possibility of a phase transition. Whether the quasiparticle effective mass divergence reported in this work is indeed a phase transition or not remains to established through further work.

This work is supported by the ONR, the LPS, and the NSF.

Six months after the submission of our manuscript for publication and its posting on the archive cond-mat/0312565, a recent preprint cond-mat/0406676 has appeared which verifies our finding of the quasiparticle effective mass divergence in low density interacting electron liquids at a critical density. In this preprint cond-mat/0406676, the quasiparticle effective mass divergence is found to occur even within many-body local field approximations which are more sophisticated than the simple RPA and Hubbard approximation many-body theories utilized in our work. The theoretical finding of a quasiparticle effective mass divergence in interacting electron liquids by two different groups employing different approximation schemes give us confidence that the mass divergence is a true many-body phenomenon, and not an artifact of a specific approximation scheme.

[1] T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. 54, 437 (1982).
[2] J. L. Smith and P. J. Stiles, Phys. Rev. Lett. 29, 102 (1972).
[3] P. T. Coleridge, M. Hayne, P. Zawadzki, and A. S. Sachrajda, Surf. Sci. 361/362, 560 (1996).
[4] W. Pan, D. C. Tsui, and B. L. Draper, Phys. Rev. B 59, 10208 (1999).
[5] A. A. Shashkin, S. V. Kravchenko, V. T. Dolgopolov, and T. M. Klapwijk, Phys. Rev. B 66, 073303 (2002).
[6] A. A. Shashkin, M. Rahimi, S. Anissimova, S. V. Kravchenko, V. T. Dolgopolov, and T. M. Klapwijk, Phys. Rev. Lett. 91, 046403 (2003).
[7] A. A. Shashkin, S. V. Kravchenko, V. T. Dolgopolov, and T. M. Klapwijk, Phys. Rev. Lett. 87, 086801 (2001).
[8] V. M. Pudalov, M. E. Gershenson, H. Kojima, N. Butch, E. M. Dizhur, G. Brunthaler, A. Prinz, and G. Bauer, Phys. Rev. Lett. 88, 196404 (2002).
[9] S. A. Vitkalov, H. Zheng, K. M. Mertes, M. P. Sarachik, and T. M. Klapwijk, Phys. Rev. Lett. 87, 086401 (2001).
[10] J. Janak, Phys. Rev. 178, 1416 (1969).
[11] B. Vinter, Phys. Rev. Lett. 35, 1044 (1975).
[12] C. S. Ting, T. K. Lee, and J. J. Quinn, Phys. Rev. Lett. 34, 870 (1975); T. K. Lee, C. S. Ting and J. J. Quinn, Phys. Rev. Lett. 35, 1048 (1975).
[13] R. Jalabert and S. Das Sarma, Phys. Rev. B 40, 9723 (1989).
[14] I. K. Marmorkos and S. Das Sarma, Phys. Rev. B 44, R3451 (1991).
[15] S. Yarlagadda and G. F. Giuliani, Phys. Rev. B 38, R10966 (1988).
[16] G. E. Santoro and G. F. Giuliani, Phys. Rev. B 39, 12818 (1989).
[17] S. Das Sarma, V. M. Galitski, and Y. Zhang, Phys. Rev. B 69, 125334 (2004); Ying Zhang and S. Das Sarma, Phys. Rev. B 70, 035104 (2004); V. M. Galitski and S. Das Sarma, Phys. Rev. B 70, 035111 (2004).
[18] A. V. Chubukov and D. L. Maslov, Phys. Rev. B 68, 155113 (2003).
[19] V. T. Dolgopolov, JETP Lett. 76, 377 (2002).
[20] M. Jonson, J. Phys. C 9, 3055 (1976).
[21] M. V. Zverev, V. A. Khodel, and V. R. Shaginyan, Zh. Eksp. Teor. Fiz. 109, 1054 (1996) [Sov. Phys. JETP 82, 567 (1996)].
[22] E. Wigner, Phys. Rev. 46, 1002 (1934).
[23] B. Tanatar and D. M. Ceperley, Phys. Rev. B 39, 5005 (1989).
[24] L. Hedin, Phys. Rev. 139 (1965).
[25] T. M. Rice, Ann. Phys. (N. Y.) 31, 100 (1965).
[26] W. H. Kohn and J. M. Luttinger, Phys. Rev. Lett. 15, 524 (1965).
[27] D. F. Du Bois, Ann. Phys. (N. Y.) 7, 174 (1959); 8, 24 (1959).
[28] A. A. Abrikosov, L. P. Gor’kov, and I. E. Dzyaloshinski, Methods of quantum field theory in statistical physics (Dover Publications, New York, 1963); G. D. Mahan, Many-Particle Physics (Plenum Press, New York, 1981); A. L. Fetter and J. D. Walecka, Quantum theory of many-particle systems (McGraw-Hill, San Francisco, 1971).
[30] A. Casey, H. Patel, J. Nyeki, B. P. Cowan, and J. Saunders, Phys. Rev. Lett. 90, 115301 (2003).
[31] J. Boronat, J. Casulleras, V. Grau, E. Krotscheck, and J. Springer, Phys. Rev. Lett. 91, 085302 (2003).
[32] D. Vollhardt, Rev. Mod. Phys. 56, 99 (1984).
[33] J. J. Quinn and R. A. Ferrell, Phys. Rev. 112, 812 (1958).
[34] E. H. Hwang and S. Das Sarma, Phys. Rev. B 64, 165409 (2001).
[35] M. A. Eriksson, A. Pinczuk, B. S. Dennis, C. F. Hirjibehedin, S. H. Simon, L. N. Pfeiffer, and K. W. West, Physica E (Amsterdam) 6, 165 (2000).
[36] C. F. Hirjibehedin, A. Pinczuk, B. S. Dennis, L. N. Pfeiffer, and K. W. West, Phys. Rev. B 65, 161309(R) (2002).
[37] M. GellMann and K. A. Brueckner, Phys. Rev. 106, 364 (1957); J. Hubbard, Proc. Roy. Soc. (London) A 243, 336 (1957); W. Macke, Z. Naturforsch, A 5, 192 (1950). For 2D correlation energy calculations see, for example, Ref. [1] and references therein.
[38] G. Iwata, Prog. Theo. Phys. 24, 1118 (1960).
[39] J. Hubbard, Proc. Roy. Soc. A240, 539 (1957); A243, 336 (1957).
[40] C. A. Kukkonen and A. W. Overhauser, Phys. Rev. B 20, 550 (1979); G. Vignale and K. S. Singwi, Phys. Rev. B 32, 2156 (1985); S. Yarlagadda and G. F. Giuliani, Phys. Rev. B 49, 14188 (1994).
[41] V. Yakovenko, private communication.
[42] S. C. Dultz and H. W. Jiang, Phys. Rev. Lett. 84, 4689 (2000).
[43] D. M. Ceperley and B. J. Alder, Phys. Rev. Lett. 45, 566 (1980).
[44] S. Das Sarma, Solid State Commun. 54, 1067 (1985); B. A. Mason and S. Das Sarma, Phys. Rev. B 33, 1412 (1986); S. A. Jackson and P. M. Platzman, Phys. Rev. B 24, R499 (1981); Y. Toyozawa and Y. Shinozuka, J. Phys. Soc. Jpn. 48, 472 (1980).