Universally Composable Privacy Amplification Against Quantum Adversaries

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Abstract

Privacy amplification is the art of shrinking a partially secret string $Z$ to a highly
secret key $S$. We show that, even if an adversary holds quantum information about the
initial string $Z$, the key $S$ obtained by two-universal hashing is secure, according to a
universally composable security definition. Additionally, we give an asymptotically optimal
lower bound on the length of the extractable key $S$ in terms of the adversary’s (quantum)
knowledge about $Z$. Our result has applications in quantum cryptography. In particular,
it implies that many of the known quantum key distribution protocols are universally
composable.

1 Introduction

1.1 Privacy amplification

Consider two parties having access to a common string $Z$ about which an adversary might
have some partial information. Privacy amplification, introduced by Bennett, Brassard, and
Robert [1], is the art of transforming this partially secure string $Z$ into a highly secret key $S$
by public discussion. A good technique is to compute $S$ as the output of a publicly chosen
two-universal hash function\(^1\) $F$ applied to $Z$. Indeed, it has been shown [1, 2, 3] that, if the
adversary holds purely classical information $W$ about $Z$, this method yields a secure key $S$
and, additionally, is asymptotically optimal with respect to the length of $S$. For instance, if
both the initial string $Z$ and the adversary’s knowledge $W$ consist of many independent and
identically distributed parts, the number of extractable key bits roughly equals the conditional
Shannon entropy $H(Z|W)$.

The analysis of privacy amplification can be extended to a situation where the adversary
might hold quantum instead of only classical information about $Z$. It has been shown [4] that
two-universal hashing allows for the extraction of a secure key $S$ whose length roughly equals the

\(^1\)See Section 2.1 for a definition of two-universal functions.
difference between the entropy of $Z$ and the number of qubits stored by the adversary. This can be applied to proving the security of quantum key distribution (QKD) protocols where privacy amplification is used for the classical post-processing of the (only partially secure) raw key $S$.

1.2 Universal composability

Cryptographic protocols (e.g., for generating a secret key) are often used as components within a larger system (where, e.g., the secret key is used to encrypt messages). It is thus natural to require that the security of a protocol is not compromised when it is, e.g., invoked as a sub-protocol in any (arbitrarily complex) scheme. This requirement is captured by the notion of universal composability. Roughly speaking, a cryptographic protocol is said to be universally composable if it is secure in any arbitrary context. For instance, the universal composability of a secret key $S$ guarantees that any bit of $S$ remains secret even if some other part of $S$ is given to an adversary.\(^2\)

In the past few years, composable security has attracted a lot of interest and lead to important new definitions and proofs (see, e.g., the framework of Canetti \(^6\) or Pfitzmann and Waidner \(^7\)). Recently, Ben-Or and Mayers have generalized the notion of universal composability to the quantum case \(^8\). Universally composable security definitions are usually based on the idea of characterizing the security of a cryptographic scheme by its distance to an ideal system which (by definition) is perfectly secure. For instance, a secret key $S$ is universally composable if it is close to an independent and almost uniformly distributed string $U$. This then implies that any cryptosystem which is proven secure when using a perfect key $U$ remains secure when $U$ is replaced by the (real) key $S$.

Ben-Or, Horodecki, Leung, Mayers, and Oppenheim \(^9\) were the first to address the problem of universal composability in the context of QKD. Usually, the security of a QKD scheme is defined by the requirement that the mutual information between the final key $S$ and the outcome of an arbitrary measurement of the adversary’s quantum system be small (for a formal definition, see, e.g., \(^10\) or \(^11\)). This, however, does not necessarily imply composability. Indeed, an adversary might wait with the measurement of his quantum state until he learns some of the bits of $S$, which might allow him to obtain more information about the remaining bits.

1.3 Contributions

We analyze the security of privacy amplification in a setting where an adversary holds quantum information. We show that the key obtained by two-universal hashing is secure according to a very strong security definition which, in any context, guarantees virtually the same security as a perfect key. The security definition we use is essentially equivalent to the definition used in \(^9\) for analyzing the composability of QKD, and thus also provides universal composability with respect to the framework of \(^8\) (cf. Section \(^3\)). This extends the result of \(^11\) where a weaker (not necessarily composable) security definition has been used. Moreover, our results have

\(^2\)Note that this is not necessarily the case for many known security definitions of a secret key.
implications for quantum cryptography. In particular, it follows from the analysis in [5] (which is based on the security of privacy amplification) that many of the known QKD protocols (such as BB84 [12] or B92 [13]) are universally composable (cf. Section 4.4 for more details).

Additionally, we improve the lower bound on the length of the extractable key $S$ given in [4]. If the initial information $Z$ as well as the adversary’s (quantum) knowledge consist of $n$ independent pieces, our bound is asymptotically tight, for $n$ approaching infinity. In particular, we obtain an explicit expression (in terms of von Neumann entropy) for the rate at which secret key bits can be generated, thus generalizing a result which has only been known for the case of purely classical adversaries (cf. Section 4.3).

2 Preliminaries

2.1 Random functions and two-universal functions

A random function from $\mathcal{X}$ to $\mathcal{Y}$ is a random variable taking values from the set of functions with domain $\mathcal{X}$ and range $\mathcal{Y}$. A random function $F$ from $\mathcal{X}$ to $\mathcal{Y}$ is called two-universal if

$$\Pr[F(x) = F(x')] \leq \frac{1}{|\mathcal{Y}|},$$

for any distinct $x, x' \in \mathcal{X}$. In particular, $F$ is two-universal if, for any distinct $x, x' \in \mathcal{X}$, the random variables $F(x)$ and $F(x')$ are independent and uniformly distributed. For instance, the random function chosen uniformly from the set of all functions from $\mathcal{X}$ to $\mathcal{Y}$ is two-universal. Non-trivial examples of two-universal functions can, e.g., be found in [14] and [15].

2.2 Density operators and random states

Let $\mathcal{H}$ be a Hilbert space. We denote by $S(\mathcal{H})$ the set of density operators on $\mathcal{H}$, i.e., $S(\mathcal{H})$ is the set of positive operators $\rho$ on $\mathcal{H}$ with $\text{tr}(\rho) = 1$. A density operator $\rho \in S(\mathcal{H})$ is called pure if it has rank 1, i.e., $\rho = |\phi\rangle\langle \phi|$ for some $|\phi\rangle \in \mathcal{H}$.

Let $(\Omega, P)$ be a discrete probability space. A random state $\rho$ on $\mathcal{H}$ is a random variable with range $S(\mathcal{H})$, i.e., a function from $\Omega$ to $S(\mathcal{H})$. Let $\rho$ and $\rho'$ be two random states on $\mathcal{H}$ and $\mathcal{H}'$, respectively. The tensor product $\rho \otimes \rho'$ of $\rho$ and $\rho'$ is the random state on $\mathcal{H} \otimes \mathcal{H}'$ defined by

$$(\rho \otimes \rho')(\omega) := \rho(\omega) \otimes \rho'(\omega),$$

for any $\omega \in \Omega$.

To describe settings involving both classical and quantum information, it is often convenient to represent classical information as a state of a quantum system. Let $X$ be a random variable with range $\mathcal{X}$ and let $\mathcal{H}$ be a $|\mathcal{X}|$-dimensional Hilbert space with orthonormal basis $\{|x\rangle\}_{x \in \mathcal{X}}$. The random state representation of $X$, denoted $\{X\}$, is the random state on $\mathcal{H}$ defined by $\{X\} := |X\rangle\langle X|$, i.e., for any $\omega \in \Omega$,

$$\{X\}(\omega) = |X(\omega)\rangle\langle X(\omega)|.$$
Let $\rho$ be a random state. For an observer which is ignorant of the randomness of $\rho$, the density operator of the quantum system described by $\rho$ is given by

$$[\rho] := E_\rho[\rho] = \sum_{\omega \in \Omega} P(\omega)\rho(\omega).$$

More generally, for any event $\mathcal{E}$, the density operator of $\rho$ conditioned on $\mathcal{E}$, denoted $[\rho|\mathcal{E}]$, is defined by

$$[\rho|\mathcal{E}] := E_{\rho|\mathcal{E}}[\rho] = \frac{1}{\Pr[\mathcal{E}]} \sum_{\omega \in \mathcal{E}} P(\omega)\rho(\omega).$$

Let $\rho \otimes \{X\}$ be a random state consisting of a classical part $\{X\}$ specified by a random variable $X$. It is easy to see that the corresponding density operator $[\rho \otimes \{X\}]$ is given by

$$[\rho \otimes \{X\}] = E_X[\rho_X \otimes |X\rangle\langle X|]$$

where $\rho_x := [\rho|X = x]$. In particular, if $X$ is independent of $\rho$, then

$$[\rho \otimes \{X\}] = [\rho] \otimes [\{X\}] .$$

2.3 Distance measures and non-uniformity

The variational distance between two probability distributions $P$ and $Q$ over the same range $\mathcal{X}$ is defined as

$$\delta(P, Q) := \frac{1}{2} \sum_{x \in \mathcal{X}} |P(x) - Q(x)| .$$

The variational distance between two probability distributions $P$ and $Q$ can be interpreted as the probability that two random experiments described by $P$ and $Q$, respectively, are different. This is formalized by the following lemma.

Lemma 2.1. Let $P$ and $Q$ be two probability distributions. Then there exists a pair of random variables $X$ and $X'$ with joint probability distribution $P_{XX'}$ such that $P_X = P$, $P_{X'} = Q$, and

$$\Pr[X \neq X'] = \delta(P, Q) .$$

The trace distance between two density operators $\rho$ and $\sigma$ on the same Hilbert space $\mathcal{H}$ is defined as

$$\delta(\rho, \sigma) := \frac{1}{2} \text{tr}(|\rho - \sigma|) .$$

The trace distance is a metric on the set of density operators $S(\mathcal{H})$. We say that $\rho$ is $\varepsilon$-close to $\sigma$ if $\delta(\rho, \sigma) \leq \varepsilon$, and denote by $B^\varepsilon(\rho)$ the set of density operators which are $\varepsilon$-close to $\rho$, i.e.,

$$B^\varepsilon(\rho) = \{ \sigma \in S(\mathcal{H}) : \delta(\rho, \sigma) \leq \varepsilon \} .$$

The trace distance is subadditive with respect to the tensor product, i.e., for any $\rho, \sigma \in S(\mathcal{H})$ and $\rho', \sigma' \in S(\mathcal{H}')$,

$$\delta(\rho \otimes \rho', \sigma \otimes \sigma') \leq \delta(\rho, \sigma) + \delta(\rho', \sigma') .$$
with equality if $\rho' = \sigma'$, i.e.,
$$\delta(\rho \otimes \rho', \sigma \otimes \rho') = \delta(\rho, \sigma) .$$

Moreover, it cannot increase when the same quantum operation $\mathcal{E}$ is applied to both arguments, i.e.,
$$\delta(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \delta(\rho, \sigma) .$$

Similarly, the trace distance between $\rho$ and $\sigma$ is an upper bound for the variational distance between the probability distributions $P$ and $Q$ of the outcomes when applying the same measurement to $\rho$ and $\sigma$, respectively, i.e.,
$$\delta(P, Q) \leq \delta(\rho, \sigma) .$$

The variational distance can be seen as a (classical) special case of the trace distance. Let $X$ and $Y$ be random variables. Then the variational distance between the probability distributions of $X$ and $Y$ equals the trace distance between the corresponding density matrices $[\{X\}]$ and $[\{Y\}]$, i.e.,
$$\delta([X], [Y]) = \delta([\{X\}], [\{Y\}]) .$$

The trace distance between two density operators containing a representation of the same classical random variable $X$ can be written as the expectation of the trace distance between the density operators conditioned on $X$.

**Lemma 2.2.** Let $X$ be a random variable and let $\rho$ and $\sigma$ be random states. Then
$$\delta([\rho \otimes \{X\}], [\sigma \otimes \{X\}]) = E_X[\delta(\rho_X, \sigma_X)]$$
where $\rho_X := [\rho | X = x]$ and $\sigma_X := [\sigma | X = x])$.

**Proof.** Using (4) and the orthogonality of the vectors $|x\rangle$, we obtain
$$\delta([\rho \otimes \{X\}], [\sigma \otimes \{X\}]) = \frac{1}{2} \text{tr} \left| E_X[(\rho_X - \sigma_X) \otimes |X\rangle\langle X|] \right| = \frac{1}{2} \text{tr} \left( E_X \left[ (\rho_X - \sigma_X) \otimes |X\rangle\langle X| \right] \right).$$

The assertion then follows from the linearity of the trace and the fact that $\text{tr}[(\rho_X - \sigma_X) \otimes |x\rangle\langle x|] = \text{tr}[\rho_x - \sigma_x]$.

In Section 3 we will see that a natural measure for characterizing the secrecy of a key is its trace distance to a uniform distribution.

**Definition 2.3.** Let $X$ be a random variable with range $\mathcal{X}$ and let $\rho$ be a random state. The **non-uniformity** of $X$ given $\rho$ is defined by
$$d(X|\rho) := \delta([\{X\} \otimes \rho], [[U]] \otimes [\rho])$$
where $U$ is a random variable uniformly distributed on $\mathcal{X}$.

Note that $d(X|\rho) = 0$ if and only if $X$ is uniformly distributed and independent of $\rho$. 

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2.4 (Smooth) Rényi entropy

Let \( \rho \in \mathcal{S}(\mathcal{H}) \) be a density operator and let \( \alpha \in [0, \infty] \). The Rényi entropy of order \( \alpha \) of \( \rho \) is defined by

\[
S_{\alpha}(\rho) := \frac{1}{1 - \alpha} \log_2 \left( \operatorname{tr}(\rho^\alpha) \right)
\]

with the convention \( S_{\alpha}(\rho) := \lim_{\beta \to \alpha} S_{\beta}(\rho) \) for \( \alpha \in \{0, 1, \infty\} \). In particular, for \( \alpha = 0 \), \( S_0(\rho) = \log_2(\operatorname{rank}(\rho)) \) and, for \( \alpha = \infty \), \( S_\infty(\rho) = -\log_2(\lambda_{\max}(\rho)) \) where \( \lambda_{\max}(\rho) \) denotes the maximum eigenvalue of \( \rho \). For \( \alpha = 1 \), \( S_1(\rho) \) is equal to the von Neumann entropy \( S(\rho) \).

Moreover, for \( \alpha, \beta \in [0, \infty] \), \( \alpha \leq \beta \iff S_{\alpha}(\rho) \geq S_{\beta}(\rho) \). (7)

Note that, for a classical random variable \( X \), the Rényi entropy \( S_{\alpha}(\{X\}) \) of the quantum representation of \( X \) corresponds to the Rényi entropy \( H_{\alpha}(X) \) of \( X \) as defined in classical information theory [16].

The definition of Rényi entropy for density operators can be generalized to the notion of smooth Rényi entropy, which has been introduced in [17] for the case of classical probability distributions.

**Definition 2.4.** Let \( \rho \in \mathcal{S}(\mathcal{H}) \), let \( \alpha \in [0, \infty] \), and let \( \varepsilon \geq 0 \). The \( \varepsilon \)-smooth Rényi entropy of order \( \alpha \) of \( \rho \) is defined by

\[
S_{\alpha}^\varepsilon(\rho) := \frac{1}{1 - \alpha} \log_2 \left( \inf_{\sigma \in \mathcal{B}^\varepsilon(\rho)} \operatorname{tr}(\sigma^\alpha) \right)
\]

with the convention \( S_{\alpha}^\varepsilon(\rho) := \lim_{\beta \to \alpha} S_{\beta}^\varepsilon(\rho) \), for \( \alpha = 0 \) or \( \alpha = \infty \), and \( S_1^\varepsilon(\rho) := S(\rho) \).

The smooth Rényi entropy of order \( \alpha \) can easily be expressed in terms of conventional Rényi entropy. In particular, for \( \alpha = 0 \),

\[
S_0^\varepsilon(\rho) = \inf_{\sigma \in \mathcal{B}^\varepsilon(\rho)} S_0(\sigma)
\]

and, for \( \alpha = \infty \),

\[
S_{\infty}^\varepsilon(\rho) = \sup_{\sigma \in \mathcal{B}^\varepsilon(\rho)} S_\infty(\sigma)
\]

The following lemma is a direct generalization of the corresponding classical statement in [17], saying that, for any order \( \alpha \), the smooth Rényi entropy \( H_{\alpha}^\varepsilon(W) \) of a random variable \( W \) consisting of many independent and identically distributed pieces asymptotically equals its Shannon entropy \( H(W) \).

**Lemma 2.5.** Let \( \rho \) be a density operator. Then, for any \( \alpha \in [0, \infty] \),

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{S_{\alpha}^\varepsilon(\rho^\otimes n)}{n} = S(\rho)
\]
3 Secret keys and composability

The main idea for obtaining universally composable security definitions is to compare the behavior of a real cryptographic protocol with an ideal functionality. For a protocol which is supposed to generate a secret key $S$, this ideal functionality is simply a source which outputs an independent and uniformly distributed random variable $U$ (in particular, $U$ is fully independent of the adversary’s information). This motivates the following definition.

**Definition 3.1.** Let $S$ be a random variable, let $\rho$ be a random state, and let $\varepsilon \geq 0$. $S$ is said to be an $\varepsilon$-secure secret key with respect to $\rho$ if

$$d(S|\rho) \leq \varepsilon.$$ 

Consider a situation where $S$ is used as a secret key and where the adversary’s information is given by a random state $\rho$. The $\varepsilon$-security of $S$ with respect to $\rho$ guarantees that this situation (which is described by the density operator $[\rho \otimes \{S\}]$) is $\varepsilon$-close—with respect to the trace distance—to an ideal setting (described by $[\rho \otimes \{U\}]$) where $S$ is replaced by a perfect key $U$ which is uniformly distributed and independent of $\rho$. Since the trace distance does not increase when appending an additional quantum system (cf. (4)) or when applying any arbitrary quantum operation (cf. (5)), this also holds for any further evolution of the system. In particular, it follows from (6) and Lemma 2.1 that the real and the ideal setting can be considered to be identical with probability at least $1 - \varepsilon$.

Definition 3.1 is essentially equivalent to an intermediate definition which has been used in [9] to prove the universal composability of QKD. More precisely, if $S$ is $\varepsilon$-secure according to Definition 3.1, it satisfies the security definition of [9] for some $\varepsilon'$ depending on $\varepsilon$. It is thus an immediate consequence of the results in [9] that Definition 3.1 provides universal composability in the framework of [8].

Note that Definition 3.1 can also be seen as a natural generalization of classical security definitions based on the variational distance (which is the classical analogue of the trace distance). Indeed, if the adversary’s knowledge is purely classical, Definition 3.1 is equivalent to the security definition as it is, e.g., used in [19] or [4].

4 Main result

4.1 Theorem and proof

**Theorem 4.1.** Let $Z$ be a random variable with range $Z$, let $\rho$ be a random state, and let $F$ be a two-universal function on $Z$ with range $S = \{0, 1\}^s$ which is independent of $Z$ and $\rho$. Then

$$d(F(Z)|(F) \otimes \rho) \leq \frac{1}{2}2^{-\frac{1}{2}(S_2([Z] \otimes \rho) - S_0(\rho) - s)}.$$ 

\[4\] In [9], a key $S$ about which an adversary has information $\rho_S$ is defined to be secure (with parameter $\varepsilon'$) if the Shannon distinguishability $SD$ between $\rho_1 := \sum_s P_S(s)|s\rangle\langle s| \otimes \rho_S$ and $\rho_0 := \sum_s \frac{|s\rangle\langle s|}{2^s} \otimes \rho_S$, for $\rho' := \sum_s \frac{|s\rangle\langle s|}{2^s}$, is small, i.e., $\varepsilon' \equiv SD(\rho_1, \rho_0)$. The relation between $\varepsilon$ and $\varepsilon'$ thus follows from the relation between the trace distance and the Shannon distance (see, e.g., [15]).
The following corollary is a consequence of property (7), expressions (8) and (9), and the triangle inequality for the trace distance.

**Corollary 4.2.** Let $Z$ be a random variable with range $\mathcal{Z}$, let $\rho$ be a random state, let $F$ be a two-universal function on $\mathcal{Z}$ with range $\mathcal{S} = \{0, 1\}^s$ which is independent of $Z$ and $\rho$, and let $\varepsilon \geq 0$. Then
\[
\frac{d(F(Z)|\{F\} \otimes \rho)}{\frac{1}{2} 2^{-\frac{1}{2}(S_{\infty}(\{F\} \otimes \rho) - S_{\infty}(\rho) - s)} + 2\varepsilon}.
\]

Let us first state some technical lemmas to be used for the proof of Theorem 4.1.

**Lemma 4.3.** Let $Z$ be a random variable with range $\mathcal{Z}$, let $\rho$ be a random state, and let $F$ be a random function with domain $\mathcal{Z}$ which is independent of $Z$ and $\rho$. Then
\[
\frac{d(F(Z)|\{F\} \otimes \rho)}{E_F[\frac{d(F(Z)|\rho)}{\rho}]}.
\]

**Proof.** Let $U$ be a random variable uniformly distributed on $\mathcal{Z}$ and independent of $F$ and $\rho$. Then
\[
\frac{d(F(Z)|\rho \otimes \{F\})}{\delta((\{F\} \otimes \rho), ([\{U\} \otimes \rho] \otimes \{F\}))},
\]
Now, applying Lemma 2.2 to the random states $\{F(Z)\} \otimes \rho$ and $\{U\} \otimes \rho$ gives the desired result, since
\[
[\{F(Z)\} \otimes \rho|F = f] = [\{f(Z)\} \otimes \rho]
\]
which holds because $F$ is independent of $Z$, $\rho$, and $U$.

The following lemmas can most easily be formalized in terms of the square of the Hilbert-Schmidt distance. For two density operators $\rho$ and $\sigma$, let
\[
\Delta(\rho, \sigma) := \text{tr}((\rho - \sigma)^2).
\]
Moreover, for a random variable $X$ with range $\mathcal{X}$ and a random state $\rho$, we define
\[
D(X|\rho) := \Delta([\{X\} \otimes \rho], [\{U\} \otimes \rho])
\]
where $U$ is a random variable uniformly distributed on $\mathcal{X}$.

**Lemma 4.4.** Let $\rho$ and $\sigma$ be two density operators on $\mathcal{H}$. Then
\[
\frac{\delta(\rho, \sigma)}{\sqrt{\text{rank}(\rho - \sigma) \cdot \Delta(\rho, \sigma)}}.
\]

**Proof.** The assertion follows directly from Lemma A.2 and the definition of the distance measures $\delta(\cdot, \cdot)$ and $\Delta(\cdot, \cdot)$.

**Lemma 4.5.** Let $X$ be a random variable with range $\mathcal{X}$ and let $\rho$ be a random state. Then
\[
\frac{d(X|\rho)}{\frac{1}{2} 2^{-\frac{1}{2}(S_{\infty}(\rho) - 1)}} \cdot D(X|\rho)}.
\]

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Proof. This is an immediate consequence of the definitions and Lemma 4.4.

Lemma 4.6. Let $X$ be a random variable with range $\mathcal{X}$ and let $\rho$ be a random state. Then

$$D(X|\rho) = \text{tr} \left( \left( \sum_{x \in \mathcal{X}} P_X(x)^2 \rho_x^2 - \frac{1}{|\mathcal{X}|} [\rho]^2 \right) \right)$$

where $\rho_x := [\rho|X = x]$ for $x \in \mathcal{X}$.

Proof. From (1), we have

$$D(X|\rho) = \text{tr} \left( \left( \sum_{x \in \mathcal{X}} P_X(x)\rho_x - \frac{1}{|\mathcal{X}|} [\rho]^2 \right) \right)$$

$$= \text{tr} \left( \sum_{x \in \mathcal{X}} P_X(x)^2 \rho_x^2 - \frac{2}{|\mathcal{X}|} [\rho] \sum_{x \in \mathcal{X}} P_X(x)\rho_x + \frac{1}{|\mathcal{X}|} [\rho]^2 \right).$$

Inserting the identity

$$[\rho] = \sum_{x \in \mathcal{X}} P_X(x)\rho_x$$

concludes the proof.

Lemma 4.7. Let $Z$ be a random variable with range $\mathcal{Z}$, let $\rho$ be a random state, and let $F$ be a two-universal function on $\mathcal{Z}$ chosen independently of $Z$ and $\rho$. Then

$$E_F[D(F(Z)|\rho)] \leq 2^{-S_2(\{|Z| \otimes \rho\})}.$$

Proof. Let us define $\rho_z := [\rho|Z = z]$ for every $z \in \mathcal{Z}$ and let $\mathcal{S}$ be the range of $F$. With Lemma 4.6 we obtain

$$E_F[D(F(Z)|\rho)] = \text{tr} \left( E_F \left[ \sum_{s \in \mathcal{S}} \text{Pr}[F(Z) = s]^2 [\rho|F(Z) = s]^2 \right] \right) - \frac{1}{|\mathcal{S}|} \text{tr}([\rho]^2), \quad (10)$$

using the linearity of the expectation value and the trace. Note that

$$\text{Pr}[f(Z) = s] \cdot [\rho|f(Z) = s] = \sum_{z \in f^{-1}(\{s\})} P_Z(z)\rho_z.$$

Using this identity and rearranging the summation order, we get

$$\sum_{s \in \mathcal{S}} \text{Pr}[f(Z) = s]^2 [\rho|f(Z) = s]^2 = \sum_{z, z' \in \mathcal{Z}} P_Z(z)P_Z(z')\rho_z\rho_{z'}\delta_{f(z), f(z')}.$$
where $\delta_{x,y}$ is the Kronecker delta which equals 1 if $x = y$ and 0 otherwise. Taking the expectation value over the random choice of $F$ then gives

$$E_F \left[ \sum_{s \in S} \Pr[F(Z) = s]^2 \rho[F(Z) = s]^2 \right] = \sum_{z,z' \in Z} P_Z(z)P_Z(z')\rho_z\rho_{z'} \Pr[F(z) = F(z')] .$$

Similarly, we obtain

$$[\rho]^2 = \sum_{z,z' \in Z} P_Z(z)P_Z(z')\rho_z\rho_{z'} .$$

Inserting this into (10), we get

$$E_F [D(F(Z)|\rho)] = \sum_{z,z' \in Z} P_Z(z)P_Z(z') \left( \Pr[F(z) = F(z')] - \frac{1}{|S|} \right) \text{tr}(\rho_z\rho_{z'}) .$$

As we assumed that $F$ is two-universal, all summands with $z \neq z'$ are not larger than zero and we are left with

$$E_F [D(F(Z)|\rho)] \leq \sum_{z \in Z} P_Z(z)^2 \text{tr}(\rho_z^2) = \text{tr}(|\{Z\} \otimes \rho|^2)$$

from which the assertion follows by the definition of the Rényi entropy $S_2$. \hfill \Box

**Proof of Theorem 4.1.** Using Lemma 4.3, Lemma 4.5, we get

$$d(F(Z)|\{F\} \otimes \rho) = E_F[d(F(Z)|\rho)]$$

$$\leq \frac{1}{2} S_{\rho}(\rho) \left( \frac{1}{2} E_F[D(F(Z)|\rho)] \right)$$

$$\leq \frac{1}{2} S_{\rho}(\rho) \sqrt{E_F[D(F(Z)|\rho)]} .$$

where the last inequality follows from Jensen’s inequality and the convexity of the square root. Applying Lemma 4.7 concludes the proof. \hfill \Box

### 4.2 Privacy amplification against quantum adversaries

We now apply the results of the previous section to show that privacy amplification by two-universal hashing is secure (with respect to the universally composable security definition of Section 3) against an adversary holding quantum information. Consider two distant parties which are connected by an authentic, but otherwise fully insecure classical communication channel. Additionally, they have access to a common random string $Z$ about which an adversary has some partial information represented by the state $\rho$ of a quantum system. The two legitimate parties can apply the following privacy amplification protocol to obtain a secure key $S$: One of the parties chooses an instance of a two-universal function $F$ and announces his choice to the other party using the public communication channel. Then, both parties compute $S = F(Z)$. Since the information of the adversary after the execution of the protocol is given by $\rho \otimes \{F\}$, one wants the final key $S$ to be $\varepsilon$-secure with respect to $\rho \otimes \{F\}$ (cf. Definition 3.1),
for some small \( \varepsilon \geq 0 \). It is an immediate consequence of Corollary 4.2 that this is achieved if the key \( S \) has length at most

\[
s = S^\varepsilon_\infty ([\{Z\} \otimes \rho]) - S^\varepsilon_0 ([\rho]) - 2 \log_2 \left( \frac{1}{4\bar{\varepsilon}} \right),
\]

for \( \bar{\varepsilon} = \varepsilon/4 \).

### 4.3 Asymptotic optimality

We now show that the bound (11) is asymptotically optimal, i.e., that the right hand side of (11) is (in an asymptotic sense) also an upper bound for the number of key bits that can be extracted by any protocol. Consider a setting where both the initial information \( Z(n) \) as well as the adversary’s state \( \rho(n) \) consist of many independent pieces: For \( n \in \mathbb{N} \), let \( Z(n) = (Z_1, \ldots, Z_n) \) and \( \rho(n) = \rho_1 \otimes \cdots \otimes \rho_n \) where \( (Z_i, \rho_i) \) are independent pairs with identical probability distribution \( P(Z_i, \rho_i) = P(Z, \rho) \). Let \( s(n) \) be the length of the key \( S \) that can be extracted from \( Z(n) \) by an optimal privacy amplification protocol. Using Lemma 2.5, we conclude from (11) that

\[
s(n) \geq H(Z(n)|\rho(n)) + o(n) \quad (12)
\]

where, for any \( \bar{Z} \) and \( \bar{\rho} \), \( H(\bar{Z}|\bar{\rho}) \) is defined by

\[
H(\bar{Z}|\bar{\rho}) := S(([\bar{Z}] \otimes \bar{\rho})) - S([\bar{\rho}]).
\]

Let now \( S := F(Z(n)) \) be a key of length \( s(n) \) computed by applying any random function \( F \) to \( Z(n) \). It is a direct consequence of Definition 3.1 that the key \( S \) can only be \( \varepsilon \)-secure with respect to \( \rho(n) \otimes \{F\} \) (for \( \varepsilon \) approaching 0 as \( n \) goes to infinity) if

\[
s(n) \leq H(F(Z(n))|\rho(n) \otimes \{F\}) + o(n). \quad (13)
\]

Note that the quantity \( H(\bar{Z}|\bar{\rho}) \) can only decrease when applying any function \( f \) to its first argument, i.e., for any random function \( F \) chosen independently of \( Z(n) \) and \( \rho \),

\[
H(F(Z(n))|\rho(n) \otimes \{F\}) \leq H(Z(n)|\rho(n) \otimes \{F\}) = H(Z(n)|\rho(n)). \quad (14)
\]

Thus, combining (12), (13), and (14), we obtain an expression for the maximum number \( s(n) \) of extractable key bits,

\[
s(n) = H(Z(n)|\rho(n)) + o(n).
\]

In particular, the maximum rate \( R := \lim_{n \to \infty} \frac{s(n)}{n} \) at which secret key bits can be generated, from independent realizations of \( Z \) about which the adversary has information given by \( \rho \), is

\[
R = S([Z] \otimes \rho) - S([\rho]) = H(Z|\rho). \quad (15)
\]

This fact is already known for the special case where the adversary’s information is purely classical. Indeed, if the adversary’s knowledge about each realization of \( Z \) is given by a realization of a random variable \( W \), expression (15) reduces to the well-known classical result

\[
R = H(ZW) - H(W) = H(Z|W)
\]

(see, e.g., [20] or [21]).
4.4 Applications to QKD

Theorem 4.1 has interesting implications for quantum key distribution (QKD). Recently, a
generic protocol for QKD has been presented and proven secure against general attacks [5].
Moreover, it has been shown that many of the known protocols, such as BB84 or B92, are special
instances of this generic protocol, i.e., their security directly follows from the security of the
generic QKD protocol. Since the result in [5] is based on the security of privacy amplification,
the strong type of security implied by Theorem 4.1 immediately carries over to this generic
QKD protocol. In particular, the secret keys generated by the BB84 and the B92 protocol
satisfy Definition 3.1 and thus provide universal composability.

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A Some identities

Lemma A.1 (Schur’s inequality). Let $A$ be a linear operator on a $d$-dimensional Hilbert
space $H$ and let $\lambda_1, \ldots, \lambda_d$ be its eigenvalues. Then

$$\sum_{i=1}^{d} |\lambda_i|^2 \leq \text{tr}(AA^\dagger),$$

with equality if and only if $A$ is normal (i.e., $AA^\dagger = A^\dagger A$).

Proof. See, e.g., [22].

Lemma A.2. Let $A$ be a normal operator with rank $r$. Then

$$\text{tr}|A| \leq \sqrt{r} \sqrt{\text{tr}(AA^\dagger)}.$$

Proof. Let $\lambda_1, \ldots, \lambda_r$ be the $r$ nonzero eigenvalues of $A$. Since the square root is concave, we
can apply Jensen’s inequality leading to

$$\text{tr}|A| = \sum_{i=1}^{r} |\lambda_i| = \sum_{i=1}^{r} \sqrt{\lambda_i} \leq \sqrt{r} \sqrt{\sum_{i=1}^{r} \lambda_i^2}.$$

The assertion then follows from Schur’s inequality.
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