On nonimbeddability of Hartogs figures into complex manifolds

E. Chirka S. Ivashkovich

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§1. Introduction. In this note we whant to discuss two closely related questions about possibility of certain imbeddings of Hartogs figures into general complex manifolds. In fact, we shall construct a counterexample to both these questions.

First question was asked by Evgeny Poletsky. Let $\Delta$ denote the unit disk in $\mathbb{C}$, $\Delta(r)$ - disk of radius $r$, $\Delta^2$ - unit bidisk in $\mathbb{C}^2$ and $S^1$ - the unit circle, $A^2_{r_1}$ - an annulus $\Delta_{r_2} \setminus \Delta_{r_1}, r_1 < r_2$. Recall that:

(a) a "thin Hartogs figure" is the following set in $\mathbb{C}^2$

$$H = \{(z, w) \in \mathbb{C}^2 : z = 0, |w| \leq 1 \text{ or } |\text{Re } z| \leq 1, \text{Im } z = 0, |w| = 1\} = \{(0) \times \Delta\} \cup ([-1, 1] \times S^1);$$

(b) a "thick Hartogs figure" (or simply Hartogs figure) is a set of the form

$$H_\varepsilon := \{(z, w) \in \mathbb{C}^2 : |z| < \varepsilon, |w| < 1 + \varepsilon \text{ or } |z| < 1 + \varepsilon, 1 - \varepsilon < |w| < 1 + \varepsilon\} = \Delta_\varepsilon \times \Delta_{1+\varepsilon} \cup \Delta_{1+\varepsilon} \times A^2_{1-\varepsilon},$$

for some $\varepsilon$, $0 < \varepsilon < 1$.

Let $X$ be some complex manifold and suppose we are given a continuous map $f : H \to X$ such that $f(0, w)$ is holomorphic on the disk $\{0\} \times \Delta$.

**Question 1.** Assume in addition that $f : H \to X$ is an imbedding. Can one find a neighborhood $V \supset f(H)$ which is biholomorphic to an open set in $\mathbb{C}^n$ or, more generally, in some Stein manifold?

To formulate the second question suppose now that $X$ is of complex dimension two and is foliated by complex curves over the unit disk. More precisely, there is a holomorphic submersion $\pi : X \to \Delta$ with connected fibers $X_z := \pi^{-1}(z)$. Hartogs figures $H_\varepsilon$ are naturally foliated over the disk in the first factor $\mathbb{C}_z$ of $\mathbb{C}_{z, w}$ and we denote the corresponding projection by $\pi_1 : H_\varepsilon \to \Delta$. A holomorphic mapping $f : (H_\varepsilon, \pi_1) \to (X, \pi)$ is called
foliated if there exist a holomorphic map $\zeta : \Delta \to \Delta$ such that $\pi(f(z, w)) = \zeta(z)$.

Suppose further we are given a smooth family $\Gamma = \{\gamma_z : z \in \Delta\}$ of diffeomorphic images of a circle $S^1$ with $\gamma_z \subset X_z$ such that $\gamma_0$ doesn’t bound a disk in $X_0$ but there are $z \in \Delta$ arbitrarily close to 0 such that $\gamma_z$ bound a disk in $X_z$. Denote by $S^1_a := \{a\} \times \{|w| = 1\} \subset \mathbb{C}^2$ circles in corresponding fibers of $H_\varepsilon$.

**Question 2.** Does there exist $\varepsilon > 0$ such that a "thick" Hartogs figure $H_\varepsilon$ can be holomorphically imbedded into $X$ in the following way:

1) imbedding $f : H_\varepsilon \to X$ is foliated;
2) $f(\{0\} \times \Delta) \subset X_a$, $f(S^1_0)$ homologous to $\gamma_a$ for some $a \in \Delta$;
3) the curve $f(S^1_1)$ is contained in $X_0$ and is homologous to $\gamma_0$ in $X_0$?

In [Br-1] and [Br-2] the existence of such imbedding is used as an obvious fact, see p.124 and p.146 correspondingly.

The goal of this note is to provide an example giving the negative answer to both Questions 1 and 2. We shall construct the following

**Example.** There exists a complex surface $X$ with a holomorphic submersion $\pi$ onto the unit disk $\Delta$ such that:

1) all fibers $X_z := \pi^{-1}(z)$ are disks with possible punctures;
2) the fiber $X_0$ over the origin is a punctured disk; the subset $U \subset \Delta$ consisting of such $z$ that the fiber $X_z$ is a disk, is nonempty, open and $\partial U \ni 0$;
3) for any circle $\gamma_0$ around the puncture in $X_0$ and for any circle $\gamma_a$ in any of $X_a$, $a \in U$, there does not exist a foliated holomorphic map $f$ from any "thick" Hartogs figure $H_\varepsilon$ to $X$ such that $f(\{0\} \times \Delta) \subset X_a$ and $f(S^1_1) \subset X_0$ is homologous to $\gamma_0$ in the fiber $X_0$.

On the way of constructing this counterexample to the Question 2 we construct also a counterexample to the Question 1, which occurs to be somewhat simpler (but not essentially simpler). We whant to emphasize that while existence of a counterexample to Q.2 should be of no surprise, the existence of it to Q.1 is somewhat unexpectable, because the "thin Hartogs figure" has very fiew of a complex structure - just one complex disk.

**Acknowledgments.** We would like to thank E. Poletsky, who was the first who asked us the question about possibility of imbeddings of Hartogs figures into general complex manifolds. We would like also to asknowledge M. Brunella for sending us a preprint [Br-3] where his erroneous argument with Hartogs figures is replaced by another approach using a sort of "non-parametrized" Levi-type extension theorem.

At any rate the question about possibility of certain imbeddings of Hartogs figures into a general complex manifold seems to be of growing demand and interest.
§2. Construction of the example. Our example is based on the violation of the argument principle. Let $J_{st}$ denote the usual complex structure in $\mathbb{C}^2_{z,w}$.

Take a function $\lambda(t) \in C^\infty(\mathbb{R}), 0 \leq \lambda \leq 1$, which satisfies

$$
\lambda(t) = \begin{cases} 
0 & \text{for } t < 1/9; \\
1 & \text{for } t > 4/9.
\end{cases}
$$

For $k \in \mathbb{N}$ consider the following domain $M = M_k$ in $\mathbb{C} \times \Delta \subset \mathbb{C}^2_{z,w}$:

$$
M := (\mathbb{C} \times \Delta) \setminus \{(z, w) : \frac{1}{3} \leq |z| \leq \frac{2}{3}, w^2 = z^k \lambda(|z|^2) \text{ or } |z| \geq \frac{1}{3}, w = 0\}.
$$

Let $J = J_k$ be the (almost) complex structure on $M_k$ with the basis of $(1,0)$-forms constituted by $dz$ and $dw + akd\bar{z}$, where

$$
a_k(z, w) = \begin{cases} 
\frac{wz^{k+1}\lambda(|z|^2)}{w^2 - z^k \lambda(|z|^2)} & \text{for } \frac{1}{3} < |z| < \frac{2}{3}, \\
0 & \text{otherwise.}
\end{cases}
$$

The subspace in $\Lambda^{p+q}(M)$ consisting of $(p, q)$-forms relative to $J$ we shall denote by $\Lambda_j^{p,q}(M)$.

**Lemma 1.** $J$ is well defined on the whole of $M$, is (formally) integrable, hence $(M, J)$ is a complex manifold. Moreover

i) $J = J_{st}$ on $M \setminus (A_{1/3}^{\infty} \times \Delta)$;

ii) functions $f_k(z, w) = w + \frac{k}{w} \lambda(|z|^2)$ and $g(z, w) = z$ are $J$-holomorphic on $M$;

iii) $\text{ind}_{|w|=1-\varepsilon} f_k(z, w) = -1$ for $|z| \geq 1$ and $0 < \varepsilon < 1/6$.

**Proof.** i) Integrability condition on $J$ reads as $d\Lambda_j^{1,0} \subset \Lambda_j^{2,0} + \Lambda_j^{1,1}$ where $\Lambda_j^{2,0}$ is the linear span of $\Lambda_j^{1,0} \cap \Lambda_j^{1,0}$ and $\Lambda_j^{1,1}$ is the same for $\Lambda_j^{1,0} \cap \Lambda_j^{0,1}$. Any form $\alpha \in \Lambda_j^{1,0}$ is represented as $\alpha_1 dz + \alpha_2 (dw + akd\bar{z})$ with smooth $\alpha_1, \alpha_2$, hence, $d\alpha = \alpha_1 da_k \wedge d\bar{z} \text{mod} \Lambda_j^{2,0} + \Lambda_j^{1,1}$.

Now, $da_k \wedge d\bar{z} = \frac{\partial a_k}{\partial w} dz \wedge d\bar{z} + \frac{\partial a_k}{\partial \bar{w}} dw \wedge d\bar{z} + \frac{\partial a_k}{\partial \bar{w}} d\bar{w} \wedge d\bar{z} \equiv \frac{\partial a_k}{\partial \bar{w}} dw \wedge d\bar{z}$ mod $\Lambda_j^{1,1}$ since $\frac{\partial a_k}{\partial \bar{w}} = 0$. Finally, $dw \wedge \partial \bar{z} = (dw + ak d\bar{z}) \wedge d\bar{z} \in \Lambda_j^{1,1}$.

ii) Really, $df_k(z, w) = (k \frac{z^{k-1}}{w} \lambda + \lambda \frac{\bar{z}}{w} \bar{z}) dz + \lambda \frac{z^{k+1}}{w} d\bar{z} + (1 - \frac{k}{w} \lambda) dw = (k \frac{z^{k-1}}{w} \lambda + \lambda \frac{\bar{z}}{w} \bar{z}) dz + (1 - \frac{k}{w} \lambda) (dw + ak d\bar{z}) \in \Lambda_j^{1,0}$, i.e., $\bar{\partial} f_k = 0$. The case of $g(z, w) = z$ is obvious since $dz \in \Lambda_j^{1,0}(M)$.

iii) is obvious.

**Lemma 2.** Let $(\zeta, \eta) \mapsto (z(\zeta), w(\zeta, \eta))$ be any foliated holomorphic map $(H_\varepsilon, J_{st}) \rightarrow (M, J)$ such that

i) $|z(0)| < 1/3, w(0, 0) = 0$;

ii) $|w(\zeta, \eta)| \geq \delta$ for some $\delta > 0$ and for all $\zeta \in \Delta_1, |\eta| = 1$.

Then $z(\Delta) \subset \Delta$. 

Proof. Suppose not. Set \( U = z^{-1}(\Delta) \). Then \( U \neq \Delta \) there are \( \zeta_0 \in \Delta \cap \partial U \) and a curve \( \gamma(t) \) from \( \gamma(0) = z(0) \) to \( \gamma(1) = \zeta_0 \) such that \( \gamma(t) \in U \) for \( 0 \leq t < 1 \). The function \( F_k(\zeta, \eta) = f_k(z(\zeta), w(\zeta, \eta)) \) is holomorphic in \( H_\varepsilon \) and therefore holomorphically extends onto the bidisk \( \Delta^2 \).

Since \( F_k(\zeta, \eta) = w(\zeta, \eta) + \frac{z(\zeta)^k}{w(\zeta, \eta)} \lambda(|z(\eta)|^2) \), we see that \( \text{ind}_{|\eta| = 1} F_k(0, \eta) = \text{ind}_{|\eta| = 1} w(0, \eta) > 0 \) due to \( J_\varepsilon \)-holomorphicity of \( w(\zeta, \eta) \) on \( \{0\} \times \Delta \).

But \( |z(\zeta_0)| = 1 \), so \( \lambda(|z(\zeta_0)|^2) = 1 \) and therefore \( \frac{|z(\zeta_0)|^k}{w(\zeta_0, \eta)} \lambda(|z(\zeta_0)|^2) > 1 > |w(\zeta_0, \eta)| \) for \( |\eta| = 1 \). As \( w(\zeta, \eta) \) is holomorphic on \( \{\zeta_0\} \times \Delta \), one has \( \text{ind}_{|\eta| = 1} F_k(\zeta_0, \eta) = \text{ind}_{|\eta| = 1} \frac{1}{w(0, \eta)} < 0 \), which contradicts to the holomorphicity of \( F_k \) on \( \Delta^2 \).

\[ \square \]

Counterexample to the Question 1. Define \( f : H \to (M, J) \) as follows:
\begin{align*}
    z(0, \eta) &= 0, w(0, \eta) = (1 - \varepsilon)\eta \text{ on } \{0\} \times \Delta \text{ and } z(\zeta, e^{i\theta}) = \zeta, w(\zeta, e^{i\theta}) = (1 - \varepsilon)e^{i\theta} \text{ with } 0 < \varepsilon < 1/10 \text{ on } [-1, 1] \times S^1, \text{ i.e., } f \text{ is a scaled tautological imbedding.}
\end{align*}

Lemma 3 There is no neighborhood of \( f(H) \) biholomorphic to an open set in some Stein manifold.

Proof. Suppose that such a neighborhood \( V \supset f(H) \) exists and \( p : V \to Y \) is a biholomorphic imbedding of \( V \) into a Stein manifold. Let \( \pi \) be the projection \( (z, w) \to z \) of \( M \) to \( \mathbb{C} \). After shrinking \( V \) if necessary we can assume about the projection \( \pi|_V : V \to \Delta \) the following:

\begin{enumerate}
    \item[i)] for \( z \) in a neighborhood \( W_1 \) of the origine in \( \mathbb{C}_z \pi^{-1}(z) \) is a disk;
    \item[ii)] there is a neighborhood \( W_2 \) of \( [-1, 1] \) on \( \mathbb{C}_z \pi^{-1}(z) \) is an annulus for all \( z \in W_2 \setminus W_1 \);
    \item[iii)] for any \( z \in [-1, 1] \setminus \{0\} \) \( \pi^{-1}(z) \cap f(H) \) is a circle \( \{z\} \times \{|w| = 1 - \varepsilon\} \) denoted by \( \gamma_z \).
\end{enumerate}

Consider \((V, p)\) as a domain over \( Y \) and let \((\tilde{V}, \tilde{p})\) be its envelope of holomorphy. \( \pi|_V \) extends to a holomorphic function \( \tilde{\pi} : \tilde{V} \to \Delta \) and it follows from the continuity principle that \( \tilde{\pi}^{-1}(z) \) contains a disk adjacent to the annulus \( \pi_z^{-1} \) for all \( z \in W_2 \).

Furthermore, the \( f_k \) holomorphically extends onto \( \tilde{V} \). Therefore \( \text{ind}_{\gamma_z} f_k \geq 0 \), which contradicts Lemma 1.

\[ \square \]

Remark. Using techniques from [Lv] one can show that \( f(H) \) has no neighborhood biholomorphic to an open set in any holomorphically convex Kähler manifold.

Counterexample to the Question 2. Let now \( K_j : |z - c_j| < r_j \) be a family of mutually disjoint discs in \( \Delta \) converging to 0 and \( \Sigma_j \) be the intersection of \( K_j \times \Delta \) with \( \{r_j/3 < |z - c_j| < 2r_j/3, w^2 = \left( \frac{z - c_j}{r_j} \right)^k \lambda \left( \frac{(z - c_j)^2}{r_j^2} \right) \} \cup \{ |z - c_j| \geq \} \).
$r_j/3, w = 0 \}$. Let $X$ be the domain $\Delta^2 \setminus \left( \bigcup_j \Sigma_j \cup \{ z \not\in \bigcup_j K_j, w = 0 \} \right)$ and $J$ be the complex structure (integrable!) in $X$ with the basis of (1,0)-forms constituted by $dz$ and $dw + b \, d\bar{z}$ where $b(z, w) = a_k_j(z - a_j/r_j, w)$ for $z \in K_j$, $j = 1, 2, \ldots$ and $b = 0$ otherwise. Then $J \in C^\infty(X)$ if $k_j \uparrow \infty$ sufficiently fast. $\pi: X \to \Delta$ denotes the natural projection which $X$ inherits as a domain in $\Delta \times \Delta$.

Checking of the following Lemma is straightforward (due to Lemma 2) and is left to the reader.

**Lemma 4** Projection $\pi$ is holomorphic and therefore $(X, \pi)$ is a holomorphic fibration. Moreover:

i) $X_z$ are disks with punctures; $X_0$ is a punctured disk; for $a \in \bigcup_j \{|z - c_j| < r_j/3\}$ $X_a$ is a disk;

ii) there exists no foliated holomorphic map $(z, w): H_\epsilon \to (X, J)$ such that $|z(0) - c_j| < r_j/3$ for some $j$ and $z(1) = 0$.

**References**

[Br-1] Brunella M.: Subharmonic variation of the leafwise Poincaré metric. Invent. math. 152, 119-148 (2003).

[Br-2] Brunella M.: Plurisubharmonic variation of the leafwise Poincaré metric. Int. J. Math. 14, No. 2, 139-151 (2003).

[Br-3] Brunella M.: On entire curves tangent to a foliation. Preprint (2004).

[Iv] Ivashkovich S.: The Hartogs-type extension theorem for meromorphic mappings into compact Kähler manifolds. Invent. math. 109, 47-54 (1992).

[Po] Poletsky, E.: Private communication.

Universite de Lille-I
UFR de Mathematiques,
59655 Villeneuve d’Ascq,
France
ivachkov@math.univ-lille1.fr

Steklov Math. Institute
Russian Academi of Sci.
Gubkin st. 8,
119991 Moscow, Russia
chirka@mi.ras.ru