Prevalent Behavior of Smooth Strongly Monotone Discrete-Time Dynamical Systems

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Abstract

For $C^1$-smooth strongly monotone discrete-time dynamical systems, it is shown that “convergence to linearly stable cycles” is a prevalent asymptotic behavior in the measure-theoretic sense. The results are then applied to classes of time-periodic parabolic equations and give new results on prevalence of convergence to periodic solutions. In particular, for equations with Neumann boundary conditions on convex domains, we show the prevalence of the set of initial conditions corresponding to the solutions that converge to spatially-homogeneous periodic solutions. While, for equations on radially symmetric domains, we obtain the prevalence of the set of initial values corresponding to solutions that are asymptotic to radially symmetric periodic solutions.

1 Introduction

Prevalence is a frequently used notion, first introduced by Christensen [2] and Hunt, Sauer and Yorke [17], that describes properties of interest occurs for “almost surely” in an infinite-dimensional space from a probabilistic or measure-theoretic perspective. It is a natural generalization to separable Banach spaces of the notion of Lebesgue measure zero for Euclidean spaces (see definition in Section 2). In particular, on $\mathbb{R}^n$, it is equivalent to the notion of “Lebesgue almost everywhere”. Over decades since its development, prevalence have undergone extensive investigations. We refer to [5,17,18,20] (and references therein) for more details.

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In dynamical systems, prevalence is parallel to another more classical notion called genericity, which formulates in the topological sense the properties of interest occurs residually (i.e., on a countable intersection of open dense subsets in a Baire space). Many examples in dynamical systems are known to be both generic in the topological sense and prevalent in measure-theoretic sense; that is, these two notions of typicality could coincide with each other in many circumstances (see, e.g. [18, 20]). However, on the other hand, there are also many cases for which prevalence differs from topological genericity. As a matter of fact, in many cases properties that are generic in the topological sense are not prevalent in the measure-theoretic sense, and vice-versa (see, e.g. [18, 20]).

Monotone dynamical systems are abundant and important sources of topological genericity and prevalence. The theory of monotone dynamical systems grew out of the series of groundbreaking work of M. W. Hirsch [12–15] and H. Matano [21, 22]; largely focusing on ordinary differential equations and parabolic partial differential equations. Since then, the theory and applications have been extended to discrete-time dynamical systems, non-autonomous systems, as well as random/stochastic systems. One may refer to [3, 8, 16, 19, 27, 30–32] for the earlier and recent developments.

The core to the huge success of developing the theory and applications of a strongly monotone semiflow is the so called Hirsch’s generic convergence theorem [15], concluding that the generic precompact orbit approaches the set of equilibria (also referred as generic quasi-convergence). Later on, for $C^1$-smooth strongly monotone semiflows, the improved generic convergence was obtained by Poláčik [25, 26] and Smith and Thieme [33]. Very recently, generic Poincaré-Bendixson Theorem was established by Feng, Wang and Wu [7, 8] for smooth flows strongly monotone with respect to cone of rank-2.

For strongly monotone discrete-time systems (mappings), however, there is no result analogous to Hirsch’s generic convergence theorem is available if they are merely continuous (or even Lipschitz). Whether any characterization of typical dynamics is possible in this case or not is still an open problem, unless certain smoothness assumption is imposed. Poláčik and Tereščák [28] first proved that the generic convergence to cycles occurs provided that the mapping $F$ is of class $C^{1,\alpha}$ (i.e., $F$ is a $C^1$-map with a locally $\alpha$-Hölder derivative $DF$, $\alpha \in (0,1]$). Here, a cycle means a periodic orbit of $F$. For the lower regularity of $F$, Tereščák [36] and Wang and Yao [38] utilized different approaches to obtain the generic convergence to cycles for $C^1$-smooth strongly monotone discrete-time systems.

A drawback of topological genericity is that open dense subsets can be arbitrarily small in terms of measure. Hirsch [13] further showed that his generic convergence theorem also holds in a measure-theoretical sense of Gaussian measure. However, since it is based on the normal
distributions in statistics, Gaussian measure is not translation-invariant.

Hence, a natural interesting question arises whether the prevalent dynamics analogue hold for strongly monotone systems. To the best of our knowledge, Enciso, Hirsch and Smith [6] first tackled the problem and investigated the prevalent behavior of a strongly monotone semiflow $\Phi_t$. Among others, they [6, Theorem 1] proved that the set of points that converge to a linearly stable equilibrium is prevalent. Here, an equilibrium is linearly stable if the spectral radius of the Frechet derivative $D\Phi_t$ at the equilibrium is no more than 1 for all $t > 0$.

In the present paper, we shall focus on prevalent behavior of the strongly monotone discrete-time dynamical systems. To be more specific, we will first show that, for $C^1$-smooth strongly monotone discrete-time system, the set of points converge to a linearly stable cycle is prevalent (see Theorem A). Here, a cycle is linearly stable if the spectral radius of the derivative $DF^p$ along the cycle (of period $p$) is no more than 1 (see Section 2).

It deserves to point out that there are a few major differences of research approaches between discrete and continuous time strongly monotone systems. For instance, Limit Set Dichotomy (see, e.g. [6, Theorem 4, p.121] or [16, 31]), which is one of the fundamental building blocks for continuous-time systems, are no longer valid for discrete-time systems (see, e.g. [4, 27, 35]). Thus, the situation is quite different with discrete-time systems as one has no a priori information on the structure of limit sets of typical trajectories. Consequently, novel techniques are needed to understand the prevalent dynamics. Our approach is based on a critical insight for the inherent structure of discrete-time strongly monotone systems, called Dynamics Alternatives (see Lemma 3.6), which was first discovered by Poláčik and Tereščák [28] for $C^{1,\alpha}$-mappings and was recently improved by the two of present authors [38] for $C^1$-mappings. Together with a useful tool of upper (respectively, lower) $\omega$-unstable sets introduced by Takáč [35], we utilize the $C^1$-Dynamics Alternatives to accomplish our approach.

Moreover, motivated by our approach, we show the prevalence of asymptotic symmetry in discrete-time strongly monotone systems on which a compact connected group $G$ acts. This gives the prevalent analogues to the works by Mierczyński and Poláčik [24] and Takáč [34] on the topological genericity of asymptotic symmetry in continuous and discrete strongly monotone dynamical systems admitting group actions (see also [37]). For these results, an interesting feature of is that no smoothness assumption was imposed on the systems. Combining with Theorem A, we further obtain that, if in addition the discrete-time system is $C^1$-smooth, then the set of points that are asymptotic to $G$-symmetric cycles is prevalent (see Theorem B).

Our main results will be applied to obtain the dynamics of prevalent initial values for nonlinear time $\tau$-periodic parabolic equations with proper boundary value problems. By applying Theorem A, we first obtain the prevalence of the set of initial points corresponding to the solutions that
are asymptotic to a linearly stable subharmonic periodic solution (i.e., solution whose minimal period is $k\tau$ ($k \in \mathbb{N}$), a nontrivial multiple of the period of the equations). In particular, for the periodic parabolic equations with Neumann boundary conditions on convex domains, we proved the prevalence of the set of initial conditions corresponding to the solutions that converge to spatially homogeneous $\tau$-periodic solutions. We further apply Theorem B to obtain asymptotic symmetry of prevalent solutions for time-period parabolic equations on a radially symmetric domain. In particular, due to the close interplay between spatial geometry and the temporal evolution of solutions, we obtained the prevalence of the set of initial values corresponding to solutions that are asymptotic to radially symmetric $\tau$-periodic solutions.

This paper is organized as follow. In Section 2, we agree on some notations, give relevant definitions and state our main results. We give the proofs of our main theorems in Section 3. Section 4 is devoted to the study the prevalent dynamical behavior of the solutions for the nonlinear time-periodic parabolic equations.

2 Notations and Main Theorems

In this section, we will fix some notations, make some preliminaries and formulate our main theorems at the end of this section.

Let $(\mathbb{B}, || \cdot ||)$ be a Banach space with norm $|| \cdot ||$. A closed convex subset $K \subset \mathbb{B}$ is called a closed convex cone if (i) if $x \in K$ and $\alpha > 0$, then $\alpha x \in K$; (ii) if $x, y \in K$, then $x + y \in K$; (iii) $K \cap (-K) = \{0\}$. For $x, y \in \mathbb{B}$, we write $x \leq y$ iff $y - x \in K$; $x < y$ iff $x \leq y$ and $x \neq y$; $x \ll y$ iff $y - x \in \text{Int}K$. $\mathbb{B}$ is called an ordered Banach space if it is endowed with an order relation, induced by a cone $K$. In particular, $\mathbb{B}$ is called a strongly ordered Banach space if the interior $\text{Int}K$ of $K$ is nonempty. For $a, b \in \mathbb{B}$ with $a \leq b$, we denote the closed order interval by $[a, b] = \{x \in \mathbb{B} : a \leq x \leq b\}$ and the open order interval by $(a, b) = \{x \in \mathbb{B} : a < x < b\}$. In particular, we write $[a, \infty) = \{x \in \mathbb{B} : x \geq a\}$ and $[\infty, b] = \{x \in \mathbb{B} : x \leq b\}$. Let $X \subset \mathbb{B}$, we denote by $\hat{X}$ the set $X$ with the order-topology generated from open order interval $(a, b)$. A subset $X \subset \mathbb{B}$ is called order-convex in $\mathbb{B}$ if $[a, b] \subset X$ whenever $a, b \in X$ and $a < b$.

Let $X \subset \mathbb{B}$ and $F : X \to X$ is a continuous map. The map $F$ is called monotone if $x \leq y$ implies $Fx \leq Fy$, and is called strongly monotone if $Fx \ll Fy$ whenever $x < y$.

A semi-orbit of $x \in X$ is $\mathcal{O}^+(x) := \{F^n x\}_{n \geq 0}$, and the $\omega$-limit set of $x$ is $\omega(x) := \bigcap_{k \geq 0} \overline{\mathcal{O}^+(F^k x)}$. We say that the map $F$ is $\omega$-compact in a subset $B \subset X$ if $\mathcal{O}^+(x)$ has compact closure in $X$ for each $x \in B$ and $\cup_{x \in B} \omega(x)$ has compact closure in $X$. A point $x \in X$ is called $p$-periodic for some integer $p \geq 1$, if $F^p(x) = x$ and $F^l x \neq x$ for $l = 1, 2, \cdots, p-1$. A set $B$ is called a cycle if $B = \mathcal{O}^+(x)$ for some periodic points $x$. A cycle is linearly stable if the spectral radius of the
derivative $DF^p$ along the cycle (of period $p$) is no more than 1. A set $B \subset X$ is unordered if it does not contain points $x, y$ such that $x < y$. Given any $x \in X$, we define the upper and lower $\omega$-limit sets of $x$ in $X$ by

$$\omega_+(x) := \bigcap_{u \in X} \bigcup_{y \in X} y \geq u \geq x \omega(y) \quad \text{and} \quad \omega_-(x) := \bigcap_{u \in X} \bigcup_{y \in X} y \leq u \leq x \omega(y),$$

respectively. Clearly, $\omega_+(x)$ (resp. $\omega_-$) is non-empty if $F$ is $\omega$-compact in every closed order interval $[a, b]$ in $X$. Hereafter, we write

$$U_+ = \{ x \in X : \omega_+(x) \neq \omega(x) \} \quad \text{and} \quad U_- = \{ x \in X : \omega_-(x) \neq \omega(x) \}$$

as the upper $\omega$-unstable set and lower $\omega$-unstable set in $X$, respectively.

Throughout the paper, we will assume the following standing assumptions:

(H1) $B$ is a separable strongly ordered Banach space with a closed-convex cone $K$; and $F : B \to B$ is a strongly monotone map.

(H2) Let $X \subset B$ be an $F$-invariant order-convex open subset; and $F$ is $\omega$-compact in every closed order interval $[a, b]$ in $X$.

(H3) $F : B \to B$ is a $C^1$-map, such that for any $x \in B$ the derivative $DF(x)$ is a compact strongly positive operator, i.e., $DF(x)v \gg 0$ whenever $v > 0$.

In the following, we give the definition of prevalence (see, e.g. [6, 17, 18]). A Borel set $W \subset B$ is shy if there exists a nonzero compactly supported Borel measure $\mu$ on $B$ such that $\mu(x+W) = 0$ for every $x \in B$. More generally, a set is called shy if it is contained in a shy Borel set. A set is prevalent if its complement is shy. Given $A \subset B$, we say that a set $W$ is prevalent in $A$ if $A \setminus W$ is shy.

In general situation, shy sets have the following properties: (i) Every subset of a shy set is shy; (ii) Every translation of a shy set is shy; (iii) No nonempty open set is shy. (iv) Every countable union of shy sets is shy ( [17, 18]). In finite-dimensional spaces, a set $M$ is shy if and only if it has Lebesgue measure zero (see, e.g. [18, Proposition 2.5] or [17, Fact 6]).

Denote by $\mathcal{C}_P$ the set of states $x \in X$ such that $\omega(x)$ is a linearly stable cycle, i.e.,

$$\mathcal{C}_P := \{ x \in X : \omega(x) \text{ is a linearly stable cycle} \}.$$
**Theorem A.** Assume that (H1), (H2) and (H3) hold. Then $\mathcal{C}_P$ is prevalent in $X$.

In order to state our next main theorem, we need to introduce an additional assumption. Let $G$ be a compact metrizable topological group. A mapping $\Gamma : G \times X \to X$ is called a *group action* of $G$ on $X$ if it is jointly continuous and $g \mapsto \Gamma(g) \equiv \gamma(g, \cdot)$ is a group homeomorphism of $G$ into $\text{Hom}(X)$, the group of all homeomorphisms of $X$ onto itself. We say that $\gamma$ is *increasing* if, for each $g \in G$, the mapping $\Gamma(g) : X \to X$ is increasing, i.e., $x \leq y$ in $X$ implies $\Gamma(g)x \leq \Gamma(g)y$. We say that $\gamma$ *commutes* with a map $F$ if, for each $g \in G$, the mapping $\Gamma(g) \commutes F$. For briefly, We write $\gamma(g, x) \equiv g \cdot x$. A point $x \in X$ is called $G$-symmetric if $g \cdot x = x$ for all $g \in G$. A subset $S \subset X$ is called $G$-symmetric if all points in $S$ are $G$-symmetric.

*(H4)* $G$ is a compact connected metrizable topological group acting on $X$ in such a way that its action is increasing and commutes with $F$.

**Theorem B.** Assume that (H1), (H2) and (H4) hold. Then the subset

$$S = \{x \in X : \omega(x) \text{ is } G\text{-symmetric}\}$$

is prevalent in $X$. Moreover, if in addition (H3) holds, then the set of initial points corresponding to semi-orbits that are asymptotic to $G$-symmetric linearly stable cycles, is prevalent in $X$.

**Remark 2.1.** Under assumptions (H1)-(H3), it has been proved in Tereščák [36] and Wang and Yao [38] that $\mathcal{C}_P$ contains an open and dense subset of $X$ (hence, $\mathcal{C}_P$ is generic in $X$). Our Theorem A shows that $\mathcal{C}_P$ is also prevalent in $X$. This entails the fact that “convergence to linearly stable cycles” in $C^1$-smooth discrete-time strongly monotone systems are both prevalent in measure-theoretic sense and generic in the topological sense.

**Remark 2.2.** In [23], Mierczyński smartly constructed an example of a strongly monotone discrete-time system, in which the subset $\mathcal{M} \cup \mathcal{N}$ is open and dense (hence, generic), but not prevalent in $X$ (see [23, p.1492]). Here $\mathcal{M}$ is the set of points whose orbits are eventually monotone, and $\mathcal{N}$ is the set of those convergent $\xi$ for which there is a simply ordered arc $L \ni \xi$, $L \subset X \setminus \mathcal{M}$ (see more detail of $\mathcal{M}$ and $\mathcal{N}$ in [23, p.1477]). We here point out that $\mathcal{M} \cup \mathcal{N}$ is actually contained in $\mathcal{C}_P$, which means that $\mathcal{M} \cup \mathcal{N}$ is not large enough to guarantee its prevalence in $X$.

**Remark 2.3.** Theorem B provides the prevalent analogues to the works by Mierczyński and Poláčik [24] and Takač [34] on the topological genericity of symmetry in continuous and discrete strongly monotone dynamical systems admitting group actions.
3 Proof of the main Theorems

In this section, we will prove our main Theorems. For this purpose, we need several useful propositions, which describe the structures of the upper (resp. low) \(\omega\)-unstable sets \(\mathcal{U}_+\) (resp. \(\mathcal{U}_-\)), as well as a useful sufficient condition to guarantee a set to be shy in \(X\).

First, we focus on the structure of \(\mathcal{U}_+\) and \(\mathcal{U}_-\). Some additional notations are introduced. A pair \((A, B)\) of subsets \(A\) and \(B\) of \(X\) is called an order decomposition of \(X\) if it satisfies: (i) \(A \neq \emptyset\) and \(B \neq \emptyset\); (ii) \(A\) and \(B\) are closed; (iii) \(A\) is lower closed (i.e., \([-\infty, a] \subset A\) whenever \(a \in A\)) and \(B\) is upper closed (i.e., \([b, \infty]\) \subset B\) whenever \(b \in B\); (iv) \(A \cup B = X\); (v) \(\text{Int}(A \cap B) = \emptyset\).

An order decomposition \((A, B)\) of \(X\) is called invariant if \(F(A) \subset A\) and \(F(B) \subset B\). The set \(H = A \cap B\) (possibly empty) is called the boundary of the order decomposition \((A, B)\) of \(X\). A \(d\)-hypersurface is a nonempty subset \(H\) of \(X\) such that \(H = A \cap B\) for some order decomposition \((A, B)\) of \(X\). The boundary \(H\) of the order decomposition \((A, B)\) of \(X\) satisfies \(H = \partial A = \partial B\), where "\(\partial\)" is the boundary symbol in \(X\), and \(H\) is invariant whenever \((A, B)\) is invariant. Clearly, a \(d\)-hypersurface \(H\) never contains two strongly ordered points \((x \ll y)\). In particular, every nonempty, unordered, invariant set \(G \subset X\) is contained in some invariant \(d\)-hypersurface \(H \subset X\) (see, e.g. [35, Proposition 1.2]).

A system \(\Gamma\) of invariant order decompositions of \(X\) is called an invariant order resolution of \(X\) if \(\Gamma\) satisfies (i) If \((A_1B_1)\) and \((A_2B_2)\) \(\in \Gamma\), then either \(A_1 \subset A_2\) or \(A_2 \subset A_1\); (ii) If \(\mathcal{O}^+(x)\) is unordered, then \(\mathcal{O}^+(x) \subset H = A \cap B\) for some \((A, B) \in \Gamma\). The existence of the invariant order resolution can be guaranteed by [35, Theorem 4.2].

Now we give the following structures of the \(\mathcal{U}_+\) and \(\mathcal{U}_-\), which are critical for the proof of the prevalence.

**Lemma 3.1.** Assume that \((H_1), \ (H_2)\) hold. Let \(\Gamma\) be the invariant order resolution of \(X\). Then \(\mathcal{U}_+\) is the union of at most countably many sets \(M_i\); each \(M_i\) is relatively open in \(H_i\), where \(H_i = A \cap B\) for some \((A, B) \in \Gamma\). A corresponding result holds for \(x \in \mathcal{U}_-\).

**Proof.** See Takáč [35, Corollary 5.6]. \(\square\)

**Remark 3.2.** If, in addition, \(X\) is order-open (\(X\) is open in the order-topology of \(\hat{\mathcal{B}}\)), then each \(M_i\) is a Lipschitz manifold of codimension one in \(\hat{\mathcal{B}}\) (see the detail in Takáč [35, Corollary 5.6]).

**Proposition 3.3.** Assume that \((H_1), \ (H_2)\) hold. Then

(i). If \(x \in \mathcal{U}_+\), then there exists some \(a_x \gg x\) such that \(\omega(y) = \omega_+(x)\) for any \(y \in [[x, a_x]]\); and hence, \([[x, a_x]] \cap \mathcal{U}_+ = \emptyset\).

(ii). Let \(x, y \in \mathcal{U}_+\) and \(x \ll y\). Then the two corresponding open sets obtained in (i) are mutually disjoint, i.e., \([[x, a_x]] \cap [[y, a_y]] = \emptyset\).
A corresponding result holds for $x, y \in U_-$.

Proof. (i). See Takáč [35, Proposition 3.3]. (ii). Let $a_x, a_y$ be obtained from (i) for $x$ and $y$, respectively. Suppose that $b \in [[x, a_x]] \cap [[y, a_y]] \neq \emptyset$. Then $x \ll y \ll b \ll a_x$; hence, $y \in [[x, a_x]]$, which implies that $y \in [[x, a_x]] \cap U_+ \neq \emptyset$, contradicting (i) of Proposition 3.3. □

Proposition 3.4. Assume that (H1), (H2) hold. Then both $U_+$ and $U_-$ are Borel sets.

Proof. We only prove that $U_+$ is a Borel set, since the proof of $U_-$ is analogous. Let $\Gamma$ be the invariant order resolution of $X$. Then, by Lemma 3.1, $U_+$ is the union of at most countably many subsets $M_i$. Each $M_i$ is relatively open with respect to $H_i$, with $H_i = A_i \cap B_i$ for certain $(A_i, B_i) \in \Gamma$. In other words,

$$U_+ = \bigcup_{i \in \Lambda} M_i = \bigcup_{i \in \Lambda} (H_i \cap V_i) = \bigcup_{i \in \Lambda} ((A_i \cap B_i) \cap V_i),$$

where $\Lambda$ is a countable index set; and $V_i (i \in \Lambda)$ are open subsets in $X$. Noticing that $A_i$ and $B_i$ are closed due to the definition of order decomposition, we obtain that $U_+$ is Borel.

Next, we introduce a sufficient condition which guarantees a Borel subset of $X$ to be shy.

Lemma 3.5. Let $W \subset X$ be a Borel subset and assume that there exists a vector $v \gg 0$ such that $L \cap W$ is countable for every straight line $L$ parallel to $v$. Then $W$ must be shy in $X$.

Proof. See Enciso, Hirsch and Smith [6, Lemma 1]. □

Before we give the proof, we mention the following critical insight for the inherent structure of the system.

Lemma 3.6 (Dynamics alternatives). Assume that (H1)-(H3) hold. For any $x \in X$, one of the following situations must occurs:

(Alta). $x \in \mathcal{C}_P$;

(Altb). there exists some $\delta > 0$ such that for any $y \in X$ satisfying $y < x$ or $y > x$, one has

$$\limsup_{n \to \infty} \|F^n(y) - F^n(x)\| \geq \delta.$$

Proof. See Wang and Yao [38, Theorem 2.1] (or Poláčik and Tereščák [28, Theorem 4.1] for $C^{1,\alpha}$-version).

Now we are ready to prove the our main Theorems.

Proof of Theorem A. Let $M = X \setminus \mathcal{C}_P$ be the complement of $\mathcal{C}_P$ in $X$. We first show that $M \subset \mathcal{U} \triangle U_+ \cap U_-$. For this purpose, we utilize Dynamics Alternatives in Lemma 3.6, which
state that, for each $x \in X$, either (Alta) $x \in \mathcal{C}_P$; or (Altb) there exists some $\delta > 0$ such that for any $y \in X$ satisfying $y > x$ or $y < x$, one has $\limsup_{n \to \infty} \|F^n y - F^n x\| \geq \delta$.

By virtue of Lemma 3.6, any point $x \in M$ can only satisfy (Altb). Then, together with the definition of upper and lower $\omega$-limit sets, we obtain that $\omega(x) \neq \omega_+(x)$ and $\omega(x) \neq \omega_-(x)$, which entails that $x \in \mathcal{U}_+ \cap \mathcal{U}_- \equiv \mathcal{U}$.

Next, we will prove that both $\mathcal{U}_+$ and $\mathcal{U}_-$ are shy. Again, we just show that $\mathcal{U}_+$ is shy. The case for $\mathcal{U}_-$ is similar. By virtue of Lemma 3.5, it suffices to show that $\mathcal{U}_+ \cap L_v$ is countable, where $L_v$ is any straight line parallel to any fixed vector $v \gg 0$. To this end, fix any positive vector $v \gg 0$. Recalling that $\mathcal{U}_+$ is Borel (by Proposition 3.4), we can define a set-function $R$ from $\mathcal{U}_+ \cap L_v$ to $B(L_v)$ as

$$R : \mathcal{U}_+ \cap L_v \to B(L_v); \quad x \mapsto f(x) \triangleq L_v \cap [[x, a_x]],$$

where $B(L_v)$ are the set of all Borel subsets of $L_v$, and $[[x, a_x]]$ is from Proposition 3.3(i). By Proposition 3.3(ii), it is clear that $R$ is injective; and moreover, for each $x \in \mathcal{U}_+ \cap L_v$, $[[x, a_x]] \cap L_v$ is mutually disjoint open in $L_v$. Consequently, the image of $R$ is an at most countable subset in $B(L_v)$. Since $R$ is injective, it follows that $\mathcal{U}_+ \cap L_v$ is at most countable. Thus, by Lemma 3.5, we have proved that $\mathcal{U}_+$ is shy.

Finally, noticing that $M \subset \mathcal{U} \triangleq \mathcal{U}_+ \cap \mathcal{U}_-$, we obtain that $M$ is shy, which implies that $\mathcal{C}_P$ is prevalent in $X$. We have completed the proof. \[ \square \]

**Proof of Theorem B.** Let $M = X \setminus S$. Then one has $M \subset \mathcal{U}$ (see Takáč [34, Theorem 0.3]). By the same arguments in the proof of Theorem A, we get that $\mathcal{U}_+$ and $\mathcal{U}_-$ are shy sets; and hence, $M$ is shy in $X$, which entails that $S$ is prevalent in $X$.

If in addition (H3) holds, then $\mathcal{C}_P$ is prevalent in $X$ by Theorem A. Recall that $S$ is prevalent in $X$. Then, we have $\mathcal{C}_P \cap S$ is prevalent in $X$. \[ \square \]

4 Application to parabolic equations

In this section, we will apply our main theorems to investigate the prevalent dynamics of parabolic equations with proper boundary value problems.

**Example 1 (Prevalent dynamics of parabolic equations on bounded domains)** Consider the following initial-boundary value problem for parabolic equations

$$\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f(t, x, u, \nabla u), \quad x \in \Omega, \ t > 0, \\
Bu &= 0, \quad x \in \partial \Omega, \ t > 0, \\
u(0, x) &= u_0(x).
\end{align*}$$

(4.1)
Here $\Omega \subset \mathbb{R}^m$ is a bounded domain with boundary $\partial \Omega$ of class $C^{2+\theta}$ for some $\theta \in (0,1)$. Of course, $\Delta$ is the Laplacian. The nonlinearity $f : (t,x,u,\xi) \mapsto f(t,x,u,\xi)$ are assumed to satisfy:

(N1) $f$ is continuous and locally Hölder continuous in $(t,x)$ uniformly for $(u,\xi)$ in bounded subset of $\mathbb{R} \times \mathbb{R}^m$, and $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial \xi}(i = 1, \ldots, m)$ exist and are continuous with respect to $(u,\xi)$.

(N2) $f$ is periodic in $t$ of given period $\tau > 0$.

We consider a time-independent regular linear boundary operator $B$ on $\partial \Omega$ of Dirichlet ($Bu = u$) or Neumann ($Bu = \frac{\partial u}{\partial v}$) type. Here $v$ is the unit outward normal vector field on $\partial \Omega$.

We also assume that

(N3) There exists an $\epsilon > 0$ and a continuous function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x,u,\xi)| \leq c(p)(1 + |\xi|^{2-\epsilon}) \quad (p \geq 0, \ (t,x,u,\xi) \in [0,\tau] \times \bar{\Omega} \times [-p,p] \times \mathbb{R}^m).$$

(N4) There is some $\kappa > 0$ such that $uf(t,x,u,0) < 0$ \quad $((t,x) \in [0,\tau] \times \bar{\Omega}, |u| \geq \kappa)$.

Let $A$ be the realization of $\Delta$ under boundary condition, that is, $A$ is the operator $u \mapsto \Delta u$ with domain

$$D(A) = X^1 \triangleq \{ u \in W^{2,p}(\Omega) : Bu = 0 \}.$$  

Let $X^\alpha$, $0 \leq \alpha < 1$, be the fractional power spaces associated with $A$. We choose $\frac{1}{2} + \frac{m}{2p} < \alpha < 1$ such that $X^\alpha \subset C^{1+\gamma}(\bar{\Omega},\mathbb{R})$ with continuous inclusion for $\gamma \in [0,2\alpha - \frac{m}{p} - 1)$. Moreover, $V := (X^\alpha, \| \cdot \|_\alpha)$ is a strongly ordered Banach space with the closed-convex cone $X^\alpha_+$ consisting of all nonnegative functions in $X^\alpha$. For every $u_0 \in V$, (4.1) admits a (locally) unique regular solution $u(t,x,u_0)$ in $V$. Set $X \triangleq \{ v \in X^\alpha : -\kappa < v(x) < \kappa \text{ for all } x \in \Omega \}$ for $\kappa$ in (N4). By virtue of (N4) and the standard a priori estimates ( [9]), the solution $u$ is bounded in $V$, and hence becomes a globally defined classical solution on $X$.

Let $F$ be the $\tau$-periodic Poincaré map of (4.1) as $F : V \to V, u(t) \mapsto u(t + \tau)$. Due to Strong Maximum Principle, $F$ generates a discrete-time strongly monotone system on $V$. One further knows that $F$ is of $C^1$ (see, e.g. [9]); and moreover, $F$ is $\omega$-compact in every closed order interval in $X$ since $F$ is a compact map with $F(X) \subseteq X$ (see, e.g. [11, 27]).

Thus, the discrete-time system generated by $F$ satisfies the standing assumptions (H1)-(H3) in Section 2. Based on our Theorem A, we have the following theorem.

**Theorem 4.1.** Assume that (N1)-(N4) hold for system (4.1). Then the set of initial condition $u_0 \in X$ corresponding to solution $u(t,x)$ converges to a linearly stable $k\tau$-periodic solution (for some integer $k > 0$), is prevalent in $X$. 


Example 2 (Prevalent dynamics of parabolic equations on convex domains) Consider the periodic-parabolic Neumann problem on a smooth convex bounded domain $\Omega \subset \mathbb{R}^m$:

$$\frac{\partial u}{\partial t} - \Delta u = f(t, u, \nabla u), \quad \text{in } \Omega \times \mathbb{R}, \quad (4.2)$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega \times \mathbb{R},$$

where $f : (t, u, \xi) \mapsto f(t, u, \xi)$ is independent of $x$ and satisfies assumptions (N1)-(N4).

Again, set $V := (X^\alpha, \| \cdot \|_\alpha)$ and $X \triangleq \{ u \in X^\alpha : -\kappa < u(x) < \kappa \text{ for all } x \in \Omega \}$. Let $F$ be the $\tau$-periodic Poincaré map as $F : V \to V, u(t) \mapsto u(t + \tau)$. So $F$ also generates a $C^1$-smooth discrete-time dynamical system on $V$ satisfying (H1)-(H3). We call a solution $u(t, x)$ is spatially homogeneous if $u(t, x)$ is independent of spatial variable $x$.

**Theorem 4.2.** Assume that (N1)-(N4) hold for system (4.2). Then the set of initial condition $u_0 \in X$, whose solution $u(t, x)$ converges to a spatially homogeneous $\tau$-periodic solution, is prevalent in $X$.

**Proof.** Since $F$ satisfies (H1)-(H3) in Section 2, it follow from Theorem 4.1 that the set of initial conditions whose solutions are asymptotic to linearly stable periodic solutions, is prevalent in $X$. Recall that Hess [10] has shown that $u$ is spatially homogeneous if it is a linearly stable periodic solution of (4.2). And Poláčik and Tereščák [29] further showed that such $u$ is a fixed point of $F$, which means $u$ is $\tau$-periodic. Thus, we obtain that the set of initial condition $u_0$ of (4.2), whose solution $u(x, t)$ converges toward a spatially homogeneous $\tau$-periodic solution, is prevalent in $X$. \hfill \Box

Example 3 (Prevalent dynamics of parabolic equations on $SO(N)$-invariant domains)

Let $G$ be a compact connected subgroup of the special orthogonal group $SO(N)$. The domain $\Omega$ is said to be $G$-invariant if $a \cdot x \in \Omega$ for all $x \in \Omega, a \in G$. Now we consider the equation (4.1) with the open bounded $G$-invariant domain $\Omega \subset \mathbb{R}^m$ with a $C^{2+\mu}$-boundary $\partial \Omega$, for some $\mu \in (0, 1)$,

$$\frac{\partial u}{\partial t} - \Delta u = f(t, x, u, \nabla u), \quad x \in \Omega, \quad t > 0,$$

$$B u = 0, \quad x \in \partial \Omega, \quad t > 0,$$

$$u(0, x) = u_0(x), \quad (4.3)$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial v}$.

We assume the nonlinearity $f$ satisfies (N1)-(N4). Moreover, we assume that $f$ satisfies:

(N5) $f(t, a \cdot x, u, a \cdot \xi) = f(t, x, u, \xi)$ for all $a \in G$ and $(t, x, u, \xi) \in \mathbb{R}_+^1 \times \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m$.
Take $X = C(\bar{\Omega}, \mathbb{R})$. The action of $G$ on $X$ is defined by $a \cdot u = u \circ a$, i.e., $\gamma(a, u)(x) = u(a \cdot x)$ for all $a \in G$, $u \in X$ and $x \in \Omega$. Let $F$ be the $\tau$-periodic Poincaré map as $F : X \to X, u(t) \mapsto u(t + \tau)$, which is $C^1$. Then $\gamma$ commutes with $F$ (see Takáč [34]). We call a solution $u(t, x)$ is $G$-symmetric if $u(t, a \cdot x) = u(t, x)$ for any $a \in G$ and $t \geq 0$. According to the Theorem B, the prevalent solutions of equation (4.3) that converge to symmetric functions, i.e., we can obtain that

**Theorem 4.3.** Assume that (N1)-(N5) hold for system (4.3). Then the set of initial condition $u_0 \in X$ corresponding to solution $u(t, x)$ converges to a $G$-symmetric linearly stable $k\tau$-periodic solution $p(x, t)$ (for some integer $k > 0$), is also prevalent in $X$.

**Example 4 (Prevalent dynamics of reaction-diffusion equations on a ball)** Consider the prevalent dynamics of the following Dirichlet initial-boundary value problem on an $N$-dimensional ball $B = \{x \in \mathbb{R}^N : \|x\| < 1\}$:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f(t, u), \quad t > 0, \quad x \in B, \\
u(x, t) &= 0, \quad t > 0, \quad x \in \partial B, \\
u(0, x) &= u_0(x), \quad x \in \bar{B}.
\end{align*}
$$

(4.4)

Again, we assume that (N1)-(N4) hold. Let $V$, $X$ and $F$ be defined as in Example 4.1. Take the group $G = SO(N)$. Then, we have the following theorem.

**Theorem 4.4.** There exists a prevalent set $S \subset X$ such that for any solution $u(x, t)$ of equation (4.4) with initial value $u_0 \in S$, there exists a linearly stable $\tau$-periodic solution $p(x, t)$ such that

(i) $p(., t)$ is radially symmetric, i.e., $p(x, t) = p(\|x\|, t)$;

(ii) the solution $\|u(., t) - p(., t)\|_X \to 0$ as $t \to \infty$.

**Proof.** Let $V_{\text{rad}}$ be the space of all radially symmetric functions in $V$ and let $X_{\text{rad}} = X \cap V_{\text{rad}}$. We indentify a function $u \in X_{\text{rad}}$ with its radial version $U \in X^\circ \subset C^1[0, 1]$ with

$$
U(\|x\|, t) = u(x, t), \quad U'(0) = U(1) = 0.
$$

Define by $F_{\text{rad}}$ be the Poincaré-map associated with the following 1-D equation on $X_{\text{rad}}$:

$$
\begin{align*}
U_t &= U_{rr} + \frac{N-1}{r} U_r + F(t, U), \quad t > 0, \quad 0 < r < 1, \\
U_r(0, t) &= U(1, t) = 0, \quad t > 0, \\
U(r, 0) &= U_0(r), \quad 0 \leq r \leq 1.
\end{align*}
$$

(4.5)

where $r = \|x\|$ and $F(t, U) = f(t, u)$.
By virtue of Theorem B, one can find a prevalent set $S \subset X$ such that, for any $u(x,t)$ with initial value $u_0 \in S$, $u$ converges in $X$ to a linearly stable periodic solution $p(r,t)$ of the (4.5) in $X_{rad}$. It follows from Chen and Polácik [1, Theorem 3.5] that any chain recurrent set of $F_{rad}$ consists of fixed points of $F_{rad}$. Then $p \in X_{rad}$ and $p(\cdot, t) = p(\cdot, t + \tau)$. Thus, we obtain that $\|u(\cdot,t) - p(\cdot,t)\|_X \rightarrow 0$ as $t \rightarrow \infty$, which completed the proof. □

Remark 4.5. Chen and Polácik [1] have proved the asymptotic $\tau$-periodicity of positive solutions. Our Theorem 4.4 showed that prevalence of the asymptotic $\tau$-periodicity for solutions of (4.4).

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