Leading Infrared Logarithms from Unitarity, Analyticity and Crossing

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We derive non-linear recursion equations for the leading infrared logarithms in massless non-renormalizable effective field theories. The derivation is based solely on the requirements of the unitarity, analyticity and crossing symmetry of the amplitudes. That emphasizes the general nature of the corresponding equations. The derived equations allow one to compute leading infrared logarithms to essentially unlimited loop order without performing a loop calculation. For the implementation of the recursion equation one needs to calculate tree diagrams only. The application of the equation is demonstrated on several examples of effective field theories in four and higher space-time dimensions.

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I. INTRODUCTION

Effective Field Theories (EFT) are non-renormalizable field theories, which allow the investigation of the infrared (low-energy) behaviour of various physical systems (see recent review [1]). The standard tool for the studies of the asymptotic behaviour of renormalizable field theories is the method of renormalization group equations (RGEs). In the case of EFTs the method of RGEs must be modified as the number of counterterms increases rapidly with the loop order.

A possibility of the systematic construction of RGEs for non-renormalizable quantum field theories was demonstrated in Ref. [2]. In particular, it was shown that the series of the leading logarithms (LLs) can be obtained by calculation of one loop diagrams. However, the solution of the RGEs derived in Ref. [2] requires the calculation of non-trivial one-loop diagrams, the number of which is rapidly increases with the loop order. Therefore, the implementation of this method in practice is not an easy task. The method of Ref. [2] has been applied in Ref. [3] for the calculation of the five-loop LLs for the pion mass in the massive $O(N+1)/O(N)$ sigma model. In Ref. [3] the authors, using dispersive methods, calculated the three-loop LLs to $\pi\pi$ scattering in massless Chiral Perturbation Theory (ChPT).

Recently, a completely different method for the calculation of LLs in a wide class of non-renormalizable massless field theories was developed in Refs. [4, 6]. The non-linear recursion equations derived in Refs. [4, 6] allow one to obtain the LLs contributions without performing non-trivial loop calculations at each loop order.

In the present paper we show that the non-linear recursion equation for LLs is a consequence of analyticity, unitarity, and crossing symmetry of the $S$-matrix. The requirement of analyticity, unitarity, and crossing symmetry for massless particles allows to obtain LLs coefficients without calculation of loop integrals. [We note that these requirements in conformal field theories lead to very powerful bootstrap equations, see [7].] The form of the corresponding equations is such that one can easily generalize them in various ways – for arbitrary space-time dimension, different symmetry groups, different spins of the fields etc.

To make the presentation of our method transparent we shall restrict ourselves to a massless scalar EFT, which is described by the following generic action:

$$S = \int d^D x \left[ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi, \partial \phi) \right],$$

where the expansion of the interaction part of the action $V(\phi, \partial \phi)$ starts with four fields $\phi$, the corresponding "$\phi^{4\nu}$" part contains $2k$ derivatives. The index $a$ (it can be a multi-index as well) corresponds to a possible internal symmetry of theory $[\Pi]$. EFTs of type $[\Pi]$ have the property that the anomalous dimension of fields is zero at the leading order. The theories with "$\phi^{4\nu}$" interaction can be considered in similar framework, this a more complicated case will be considered elsewhere. We stress that the absence of mass is crucial for our discussion below.

II. GENERAL METHOD

We consider the 4-particle scattering amplitude $\phi^a + \phi^b \rightarrow \phi^c + \phi^d$ in four dimensional space (generalization to an arbitrary number of dimensions is given in Section [V]). The LLs for other physical quantities, such as form factors [6], (generalized) parton distributions [8], etc., are related to the LLs of the 4-point amplitude.

The $2 \rightarrow 2$ amplitude can be decomposed in the irreducible representations of an internal symmetry group of the EFT $[\Pi]$:

$$A^{abcd}(s,t,u) = \sum_I P_I^{abcd} A^I(s,t,u).$$

Here $P_I^{abcd}$ is a projector on an invariant subspace corresponding to the irreducible representation $I$ of an internal symmetry group. According to the Wigner-Eckart theorem the corresponding projectors can be obtained as
the convolution of two Clebsch-Gordon coefficients, see e.g.\[9\].

The renormalizability of a theory depends on the dimension of the coupling constants that enter action (1). Let us consider the case of the action (1) with one coupling constant of dimension \(4 - D/2\) \((2k)\) is the number of derivatives in the interaction part of the Lagrangian, \(D\) is the dimension of space-time). We denote the corresponding coupling constant as \(1/F^2\), the dimension of the constant \(F\) is \(k + D/2 - 2\) (the constant \(F\) corresponds to the pion decay constant in usual ChPT). The case of several couplings we consider in Section \(\ref{sec:recursion_relations}\). At \(k = 2 - D/2\) theory (1) is renormalizable, and at \(k > 2 - D/2\) it is a non-renormalizable low-energy EFT.

We shall work with the partial wave amplitudes (see all definitions in Appendix A) as they depend only on one energy variable \(s\). For \(D = 4\) (see generalization for an arbitrary space-time dimension in Section \(\ref{sec:extension_to_arbitrary_space_time_dimensions}\)) the partial wave amplitude in LL approximation can be represented in the most general form as follows:

\[
t_I^f (s) = \pi \frac{\hat{S}}{2} \sum_{n=1}^{\infty} \frac{\hat{S}^n}{2l+1} \sum_{i=0}^{n-1} \alpha_n^{ll} \ln^i \left(\frac{\mu^2}{s}\right) \ln^{n-i-1} \left(\frac{\mu^2}{-s}\right) + \mathcal{O}(NLL),
\]  

(3)

where \(\hat{S} = \frac{k}{(4\pi)^2}\) is a dimensionless expansion parameter, \(F^2\) is a coupling in the Lagrangian. The scale parameter \(\mu\) is arbitrary in the LL approximation, since its change influences the next-to-leading logs only. \(\mathcal{O}(NLL)\) stands for terms with next-to-leading logarithms (NLL). The series of LLs for the amplitude (3) takes into account the convolution of two Clebsch-Gordon coefficients, see e.g.\[9\].

\[
\omega_n^{ll} = \sum_{i=0}^{n-1} \alpha_n^{ll},
\]  

(5)

where the index \(n\) corresponds to the loop order plus one, and the angular momentum \(l\) is restricted to \(l \leq kn\). \(I\) denotes the irreducible representation of the corresponding internal symmetry group.

In order to derive the recursive equations for the coefficients \(I_n\) we shall employ the unitarity, analyticity and crossing symmetry of the scattering amplitude.

The crossing symmetry relates the amplitudes with the interchanged momenta to the amplitudes of different symmetry group representations (the summation over the repeated “representation indices” is always assumed):

\[
A^I (s, t, u) = C^{ll'I'}_{tu} A_{I'} (s, u, t),
\]  

(6)

\[
A^I (s, t, u) = C^{ll'I'}_{su} A_{I'} (u, t, s).
\]  

(6)

The crossing matrices \(C\) are defined as

\[
C^{ll'I'}_{tu} = \frac{1}{d_{l'}} \sum_{l} [P^{abcd} P^{bacd}]^{l}_{l},
\]  

(7)

\[
C^{ll'I'}_{su} = \frac{1}{d_{l'}} \sum_{l} [P^{abcd} P^{badc}]^{l}_{l}.
\]  

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where \(d_{l}\) is the dimension of the irreducible representation \(I\) (see the explicit expressions for the crossing matrices below).

The partial wave scattering amplitudes possess right and left cuts in the complex plane of the Mandelstam variable \(s\). The right cut discontinuity is fixed by the unitary relation:

\[
\text{Im} t_I^f (s) = |t_I^f (s)|^2 + \mathcal{O}(\text{Inelastic part}.
\]  

(8)

The unitarity relation is valid for the physical region \(s > 0\). Under the inelastic part we understand the part with more than two particles in an intermediate state. Obviously the LLs coefficients do not depend on this part of the expression. Indeed, by cutting more than two lines in any diagram the power of logarithms decreases and hence such cuts influence only NLLs. We can restrict ourselves to the elastic unitarity relation, since only LL coefficients are considered.

Substituting the LL expression (3) for \(t_I^f (s)\) into the unitarity relation (8) and collecting the coefficients in front of \(\hat{S}^n L^{n-2}\) \((L \equiv \ln (\mu^2/s))\), we find the relation

\[
\sum_{i=0}^{n-i-1} (n - i - 1) \alpha_n^{ll'} = \frac{1}{2(2l+1)} \sum_{i=1}^{n-1} \omega_{n-i}^{ll'} \omega_{n-i,t}. \]  

(9)

Thus, the unitarity allows us to relate \(\alpha\) and \(\omega\) coefficients. However, corresponding relations (9) do not allow us to obtain the closed form equation for the coefficients \(\omega\).

We need some additional information about the left cut. This information is provided by the dispersion relations (analyticity) and the crossing symmetry (6). One can show that if only two-particle cuts are taken into account, the following relation connects discontinuities on the left and right cuts (see derivation in the Appendix A)
This relation for the $\pi\pi$ scattering is well-known, it is a consequence of the Roy equation [10], it was used in Ref. [3] to calculate three-loop LLs in ChPT. Substituting Eq. (8) into Eq. (10) and collecting the terms in front of the LLs we obtain the following relation in addition to [4]:

$$\sum_{i=0}^{n-1} i\alpha_{n,i}^l = \sum_{l'=0}^{kn} \frac{C_{su}^{l,l'}}{2l'+1} \sum_{i=1}^{n-1} \omega_{l'}^i \omega_{n-i,l'} (-1)^{l+l'} \Omega_{kn}^{l'} \Omega_{kn}' (11)$$

where the matrices $\Omega_{kn}^{l'}$ are defined as follows

$$\left(\frac{z-1}{2}\right)^n P_l \left(\frac{z+3}{z-1}\right) = \sum_{l'=0}^{n} \Omega_{kn}^{l'} P_l(z).$$

Summing relations (9) and (11) and taking into account $t \leftrightarrow u$ crossing symmetry we arrive at the following closed non-linear recursive relation for LL coefficients $\omega_{nl}^l$:

$$\omega_{nl}^l = \frac{1}{n-1} \sum_{j=1}^{n} \sum_{i=1}^{kn} \sum_{l'=0}^{n-1} \frac{1}{2} \left( \delta_{l'}^j \delta_{IJ} + C_{st}^{l,l'} J_{kn} + C_{su}^{l,l'} (-1)^{l+l'} \Omega_{kn}^{l'} \omega_{n-i,l'}^j \right) \omega_{n-i,l'}^j \omega_{n-i,l'}^j \Omega_{kn}^{l'}.$$ (12)

[Here we introduce the matrix $C_{st}$ which is defined as a product of the crossing matrices (7) $C_{st} = C_{su} C_{tu} C_{su}$. The recursion relation (12) for the LL coefficients should be supplemented by initial conditions, i.e. by the values of $\omega_{nl}^l$ at $n=1$ and correspondingly $l=0,1$. The corresponding values can be obtained by the tree level calculation of the partial wave amplitudes $t_l^n(s)$ (see Eq. (11)) using action (1) of an EFT under consideration.

The recursion equation (12) is the main result of the present paper. This equation allows the calculation of the leading infrared logarithms to an essentially unlimited order. Furthermore, this method presents a powerful tool for the study of the general structure of the infrared logarithms. For the derivation of Eq. (12) we used only the unitarity, analyticity, and crossing symmetry of the amplitude. This fact emphasizes a general nature of the non-linear recursion equation (12). The corresponding equation can be derived for many physical problems described by a non-renormalizable effective low-energy Lagrangian, e.g. theory of critical phenomena, low-energy quantum gravity, theory of magnets, etc.

In the next Sections we apply our method to several EFTs. Also we give the generalization of the method on an arbitrary dimension and on the theories with mixed renormalizable and non-renormalizable interactions.

III. 4D MASSLESS $O(N+1)/O(N)$ $\sigma$-MODEL.

We have several reasons to consider the $O(N+1)/O(N)$ $\sigma$-model. First, this model was considered in [3] where the recursion relations for LLs were derived by a completely different method. Second, this model (at $N=3$) is equivalent to the Weinberg Lagrangian [11] in the infrared limit. The Lagrangian of the $O(N+1)/O(N)$ $\sigma$-model has the form

$$\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \sigma \partial^{\mu} \sigma + \partial_{\mu} \phi^a \partial^{\mu} \phi^a \right) = \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a - \frac{1}{8F^2} \left( \phi^a \phi^a \right) \partial^2 \phi^a \phi^b + \mathcal{O}(\phi^6),$$ (13)

where $\sigma^2 = F^2 - \sum_{a=1}^{N} \phi^a \phi^a$. The $\mathcal{O}(\phi^6)$ part does not contribute to the 4-particle amplitude due to the absence of masses. The $\sigma$-model (13) belongs to the here consid-
and hence the channels with different internal symmetry

The initial conditions for the recursion can be obtained

The straightforward calculation with help of Eq. (7) gives the crossing matrices

Substituting the crossing matrices into Eq. (12) we obtain explicit form of recursion relations for the LL coefficients. The origin of such relations is very simple – the model is a theory with one coupling constant $\omega$ and hence the channels with different internal symmetry by the trivial tree-level calculation of the scattering amplitude with the help of the Lagrangian (13). The result is $\omega_{10} = N - 1$, $\omega_{11} = -1$, and $\omega_{12} = 1$, all other coefficients $\omega_{ij}$ are zero.

The corresponding projection operators on these representations have the following form:

The equation on $\omega$ is obtained from (12) by the substitution (14), it reads

The corresponding relations have the following form:

with the initial condition $\omega_{10} = 1, \omega_{11} = 0$. The coefficients $B_{j}^{(m,i)(n-m,l)}$ are given by:

Eq. (15) coincides exactly with the equation obtained by a completely different, more complicated method in Ref. [5]. In the present paper the equation was obtained

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from a simple symmetry consideration. The detailed discussion of the properties of the recursion equation (15) (and hence of Eq. (12)) and its particular solutions can be found in Ref. [3]. For convenience of the reader we present in Appendix C [Tables I-III] the values of the LL coefficients \( \omega_{nl} \) up to the 4-loop order. In Fig. 1 we show the values for the lowest partial waves \( \omega_{nl} \) in the \( O(4)/O(3) \) sigma model up to the 137th loop order.

**IV. 4D \( O(N) \) SYMMETRIC \( \phi^4 \) THEORY**

As a next example we consider a renormalizable field theory. For simplicity we consider the same \( O(N) \) symmetry as in the previous case. A possible renormalizable Lagrangian of type (1) is:

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^{a} \partial^{\mu} \phi^{a} - \frac{1}{8 F^2} (\phi^{a} \phi^{a})^2.
\]  

Due to the same \( O(N) \) symmetry as for the \( \sigma \)-model [13] the crossing matrices are the same. The differences to the previous case are the number of derivatives in the interaction vertex, i.e., \( k = 0 \) in Eqs. (12) and the coupling constant \( F \) is dimensionless now.

The low-energy expansion of the scattering amplitude in the renormalizable theory (17) goes over increasing power of LLs (note that \( \bar{S} = 1/(4\pi F)^2 \) at \( k = 0 \) in Eq. (11):

\[
t^{l}_{s}(s) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\omega^{l}_{nl}}{2l + 1} \left( \frac{\mu^{2}}{s} \right)^{2l+1} \ln^{n-1} \left( \frac{\mu^{2}}{s} \right).
\]  

The LL coefficients \( \omega^{l}_{nl} \) satisfy the recursion relation (12) with \( k = 0 \). The initial conditions are obtained by calculation of the scattering amplitude at the tree level in theory (17)–\( \omega_{l=0} = -(N+2) \), \( \omega_{l=0} = -(N+2) \) with all other \( \omega^{l}_{nl} \) are zero.

Eq. (12) is simplified considerably at \( k = 0 \) as it contains only the matrix \( \Omega_{0} = \partial^{0} \partial^{0} \). This implies that the LL coefficients are restricted only to the lowest partial wave \( l = 0 \). Using this simplification and introducing the following notation \( \omega_{l=0}^{l} \equiv \omega_{l=0} \) we can rewrite the recursion relation (12) for the renormalizable theory (17) in a simpler form:

\[
\omega^{l}_{n} = \frac{1}{n+1} \sum_{i=1}^{n} \left[ \frac{N+2}{2N} \omega^{l}_{i} \omega_{l=0} + \frac{N^2 + N + 2}{2N} \omega^{l}_{i} \omega_{l=0} \right] + \frac{N + 1}{N} \omega^{l}_{n=1} \omega^{l}_{l=0} \omega_{l=0} + \frac{N - 1}{N} \omega^{l}_{n=1} \omega^{l}_{l=0} \omega^{l}_{l=0}.
\]  

This system of equations can be easily solved with the following result:

\[
\omega^{l}_{n=1} = 0 \quad \omega^{l}_{n=1} = -(N+2) \left( \frac{N + 8}{2} \right)^{n-1} \quad (22)
\]  

Substituting this solution into the LL expansion of the scattering amplitudes (18), we can perform the summation of the LLs with the result:

\[
t^{l=0}_{s}(s) = \frac{\pi}{2} \frac{N + 2}{(4\pi F)^2} \frac{1}{1 + b_{1} (4\pi F)^2 \ln \left( \frac{\mu^{2}}{s} \right)},
\]  

\[
t^{l=2}_{s}(s) = \frac{\pi}{2} \frac{2}{(4\pi F)^2} \frac{1}{1 + b_{1} (4\pi F)^2 \ln \left( \frac{\mu^{2}}{s} \right)}.
\]  

All other amplitudes are zero in the LL approximation. In Eq. (23) the constant \( b_{1} = \frac{N+8}{2} \) is the well known result for the one-loop beta function of the \( O(N) \) symmetric \( \phi^4 \) theory [17], see e.g. Ref. [12].

The logs enter Eq. (23) in the combination in which one can easily recognize the running coupling constant (denoted as \( 1/F^2 \) in this paper) for the \( O(N) \) symmetric \( \phi^4 \) theory. It is not surprising. Indeed, for a generic renormalizable theory (e.g. \( k = 0, D = 4 \)) the recursion equation (12) is reduced to the following form:

\[
\omega^{l}_{n} = \frac{1}{n+1} \sum_{i=1}^{n} \sum_{j} B^{lJ} \omega^{l}_{n=1} \omega^{l}_{j},
\]  

where the matrix \( B^{lJ} \) is independent of the index \( n \) and it is expressed in terms of the crossing matrices:

\[
B^{lJ} = \frac{1}{2} (1 + C_{st} + C_{su})^{lJ}.
\]  

The recursion equation (24) can be reduced to a simple system of differential equations. For that we introduce a
generating function for the LL coefficients $\omega_{nl}^J$ as follows $\omega_{nl}^J(t) \equiv \sum_{n=1}^{\infty} \omega_{nl}^J t^{n-1}$. Then the recursion relation (24) is reduced to the following differential equation:

$$\frac{d\omega_{nl}^J(t)}{dt} = \sum_J B_{lJ}^n \omega_{nl}^J(t) \omega_{nl}^J(t),$$

with the initial conditions that can be obtained by the calculation of the tree diagrams, e.g. in $O(N)$ symmetric $\phi^4$ theory $\omega_{nl}^J(0) = \{-N-2, 0, -2\}$. Eq. (26) has the form typical for the RGEs for the running coupling constant. In this way, we demonstrate that our recursion relation (24) for LL coefficients is reduced to the standard RGEs in the case of a renormalizable field theory. Eq. (12) can be considered also as the generalization of RGEs for the case of non-renormalizable EFTs.

V. THEORY WITH RENORMALIZABLE AND NON-RENORMALIZABLE INTERACTIONS

In this section we consider the theory with mixed renormalizable and non-renormalizable interactions. To be specific we consider a Lagrangian which is the sum of the $O(N+1)/O(N)$ $\sigma$-model and the $O(N)$ symmetric $\phi^4$ theory:

$$L_2 = \frac{1}{2} \left( \partial_{\mu} \sigma \partial^{\mu} \sigma + \partial_{\mu} \phi^{\sigma} \partial^{\mu} \phi^{\sigma} \right) - \frac{\lambda}{8} (\phi^{\sigma} \phi^{\sigma})^2 - \frac{1}{8 F^2} (\phi^{\sigma} \phi^{\sigma}) \partial^2 (\phi^{\sigma} \phi^{\sigma}) - \frac{1}{8} \phi^{\sigma} \phi^{\sigma}^2 + O(\phi^6),$$

where $\sigma^2 = F^2 - \sum_{N=1}^{\infty} \phi^{\sigma} \phi^{\sigma}$. In this case the expansion is performed over two parameters: a dimensionless coupling $\lambda$ and dimensional $1/F^2$. We note that at $N = 3$ the theory (27) corresponds to the famous two-flavour chiral symmetric Weinberg Lagrangian plus the chiral symmetry breaking term proportional to the constant $\lambda$. That chiral symmetry breaking term is proportional to the constant $l_3$ in the notations of Gasser and Leutwyler [13]. However, here we force this term to be of the same order as the Weinberg Lagrangian.

The series in infrared LLs for the amplitude in the theory (27) can be presented in the form of double expansion in the parameters $1/F^2$ and $\lambda$:

$$t_1^J(s) = \frac{\pi}{2} \sum_{n,m=0}^{\infty} \omega_{nlml}^J \frac{\tilde{S}^n}{2l+1} \lambda^m \ln^{n+m-1} \left( \frac{\mu^2}{s} \right).$$

$$\omega_{nlml}^J = \frac{1}{(n+m-1)} \sum_J \sum_{i,j=0}^{n,m} \sum_{l'=0}^{l} \left( \delta^J^i J^{J'} + C_{sJ}^J J^{J'} + C_{sJ}^J (-1)^{l^J} \Omega_{nJ}^J \omega_{nJ}^J \omega_{n-iJ,J}^J \right) \frac{\omega_{n-iJ,J}^J \omega_{n-iJ,J}^J}{2l'+1}.$$

As usual, the initial conditions for this recursion equation can be obtained by the tree level calculation of the scattering amplitude with the Lagrangian (27). Actually, the initial conditions are the combination of the corresponding initial conditions discussed in the previous two sections.

An important feature of the recursion equation (29) is that its dependence on the index $m$ is trivial. Eq. (29) with fixed indices $n$ and $l$ has the generic form (24). Therefore, the corresponding equation, in principle, can be solved for fixed indices $n$ and $l$. Introducing the generating function:

$$\omega^J_n(t) = \sum_{m=0}^{\infty} \omega_{nlml}^J t^m,$$

we can partially sum up the LL expansion of the amplitude (29):

$$t_1^J(s) = \frac{\pi}{2} \sum_{n=0}^{\infty} \omega_{nlml}^J \left( \frac{\hat{\lambda} \ln \left( \frac{\mu^2}{s} \right)}{2l+1} \right) \frac{\tilde{S}^n}{n-1} \ln^{n-1} \left( \frac{\mu^2}{s} \right).$$
This expansion is similar to the LL series for the theory with a single coupling constant [4], the only difference is that the LL coefficients \( \omega_{nl}^I \) are functions of \( \lambda \) and the log. Expressions for the functions \( \omega_{nl}^I(t) \) are rather complicated, therefore we give only a couple of (non-trivial) examples for \( N = 3 \), corresponding to ChPT. We consider the case of \( n = 2 \), that corresponds to the one-loop chiral log which is dressed by any number of loops with a \( \lambda \phi^4 \) interaction. The result of simple calculations gives:

\[
\begin{align*}
\omega_{20}^0(t) &= \frac{950}{663} f_1(t) - \frac{160}{117} f_2(t) - \frac{10}{153} f_3(t), \\
\omega_{20}^2(t) &= \frac{140}{663} f_1(t) - \frac{40}{117} f_2(t) + \frac{20}{153} f_3(t), \\
\omega_{21}^1(t) &= \frac{100}{221} f_1(t) + \frac{10}{39} f_2(t) + \frac{10}{51} f_3(t),
\end{align*}
\]

with the functions \( f_i(t) \) given by:

\[
f_i(t) = \frac{1}{t(i + b_1 t)^\gamma_i}, \tag{33}
\]

where \( b_1 = 11/2 \) is the coefficient of the one-loop \( \beta \)-function of the \( O(3) \) symmetric \( \phi^4 \) theory. The “anomalous dimensions” \( \gamma_i \) have the following values: \( \gamma_1 = 9/11, \gamma_2 = 40/33, \gamma_3 = 10/33 \). Using the result [32] one can see that the inclusion of the \( \phi^4 \) interaction to the massless chiral Lagrangian \( (O(4)/O(3) \sigma\text{-model}) \) leads to decreasing of the coefficients in front of the one-loop chiral log (except the \( I = 1 \) amplitude for which the one loop chiral log is zero). Possible physics implication of this will be considered elsewhere.

VI. EFFECTIVE FIELD THEORIES IN ARBITRARY DIMENSION

The derivation of the recursion equation [12] was done for \( D = 4 \). It is straightforward to generalize it for an arbitrary even dimension \( D > 4 \) \[13\]. We restrict ourselves to the even dimensions in order to avoid the appearance of power-like singularities in the amplitudes. The analytic structure of the amplitude in an odd dimensional theory should be investigated separately.

We consider the Lagrangian of type [11] with the interaction part containing \( 2k \) derivatives. For \( D > 4 \) all theories of type [11] are non-renormalizable. Simple dimensional analysis shows that an \((n-1)\)-loop diagram in a theory [11] scales with low external momenta as \( \sim (p^2)^{(n+1)(D-4)/2} \). This power behaviour is accompanied by logs with the maximal power of \((n-1)\). The low-energy expansion of the partial wave amplitude \( t^I_l(s) \) [details of partial wave decomposition in \( D \) dimensions can be found in Appendix A] in LL approximation has the following form:

\[
t^I_l(s) = \frac{2^{D-4}(4\pi)^{\frac{D}{2}}}{32\pi s^{\frac{D-4}{2}}} \sum_{n=1}^{\infty} \frac{\omega_{nl}^I}{2l + D - 3} \hat{S}^n \ln^{n-1} \left( \frac{\mu^2}{s} \right) \tag{34}
\]

We introduce here the dimensionless expansion parameter:

\[
\hat{S} = \frac{\sqrt{\pi}}{s^{k+\frac{D-4}{2}}} \frac{2^{D-4} \Gamma \left( \frac{D-2}{2} \right)}{\Gamma \left( \frac{D-4}{2} \right)} F^2. \tag{35}
\]

Obviously, the expansion [34] is reduced to Eq. (4) for \( D = 4 \).

Now, with the help of Appendix A the reader can easily repeat the derivations presented in the Section III and after simple calculations arrive at the following recursion equation for the LL coefficients \( \omega_{nl}^I \) for a \( D \)-dimensional theory:

\[
\omega_{nl}^I = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{l'=0}^{\frac{D-4}{2}} \left( 8^l l^l d^l, J_\mu O^{l'I} \right)_{k(n-l)} \frac{\omega_{nl}^I}{2l + D - 3} (D) + C_{s(I}^{IJ} (-1)^{l+l'} O_{k(n-l)}^{l'IJ} \frac{\omega_{nl}^I}{2l + D - 3} (D) \tag{36}
\]

The crossing matrices \( \Omega_{ll}^{ij} (D) \) are straightforward generalization of \( \Omega_{ll}^{ij} \) for an arbitrary dimension:

\[
\left( z - \frac{1}{2} \right)^n C_{l}^{\frac{D-4}{2}} \left( \frac{z+3}{z-1} \right) = \sum_{l'=0}^{n} \Omega_{ll}^{ij} (D) C_{l}^{\frac{D-4}{2}} (z). \tag{37}
\]

Here \( C_{l}^{\frac{D-4}{2}} (z) \) are Gegenbauer polynomials, which form a basis for the partial wave decomposition of the amplitude in \( D \) dimensions. Details on the crossing matrices \( \Omega_{ll}^{ij} (D) \) can be found in Appendix B.

Various examples of multi-dimensional theories will be considered elsewhere. Here, as an example, we give only results for the LL coefficients in the \( 6D \, O(N) \) symmetric \( \phi^4 \) theory, which corresponds to \( k = 0 \), \( D = 6 \) in Eqs. (34) [36]. The LL expansion for the partial wave amplitude in this theory has the following form:

\[
t^I_l(s) = 2^{n+2} s \sum_{n=1}^{\infty} \frac{\omega_{nl}^I}{2l + 3} \left( \frac{s}{128\pi^3 F^2} \right)^n \ln^{n-1} \left( \frac{\mu^2}{s} \right). \tag{37}
\]
The four-loop results for the coefficients $\omega_{I=0}^I$ are presented in Tables IV-VI of Appendix C.

The recursion equation (36) for the $O(N)$ symmetric $\phi^4$ theory in $D$ dimensions can be solved in the large $N$ limit. In this limit the amplitude is dominated by the S-wave and by the singlet $I = 0$ “isospin” component. Details of the large $N$ limit for the recursion equations can be found in Ref. [3]. The result for the LL coefficients is:

$$\omega_{I=0}^I = -N \left( \frac{N}{2(D-3)} \right)^{n-1}. \quad (38)$$

With help of Eq. (34) we can perform the large-$N$ summation of LLs in the $D$-dimensional $O(N)$ symmetric $\phi^4$ model:

$$t_0^0(s) = \frac{N\sqrt{\pi}}{32\Gamma\left(\frac{D-3}{2}\right)(D-3)F^2} \times \frac{1}{1 + \frac{N\sqrt{\pi}}{2^{D-3}(D-3)\Gamma\left(\frac{D-3}{2}\right)F^2} \ln \left( \frac{\mu^2}{\pi} \right)}.$$

VII. CONCLUSIONS

Using the requirements of the unitarity, analyticity and crossing symmetry of the scattering amplitude, we derive recursion equations for the coefficients in front of leading infrared logs in massless effective field theories. The corresponding equations are given by (12) for 4D theories and by (36) for an arbitrary even dimension. To implement these equations one needs to perform a calculation of tree diagrams only. One needs such calculation to find initial conditions for the recursion equations (12,36)

In Section III we demonstrate that our recursion equations for LLs (12) are equivalent to the recursion equation derived in Ref. [5] by a more complicated method. The method of Ref. [5] required non-trivial all-order analysis of the structure of possible counter-terms as well as rather complicated loop calculations. In the present paper we found a much simpler and more general way to derive the corresponding recursion equations.

In Section IV we show that our recursion equations (12,36) are reduced to usual RGEs for the case of a non-renormalizable field theory. Therefore, one can consider our recursion equation for LLs as a generalization of RGEs for the case of a non-renormalizable field theory.

Eq. (12) in the case of a renormalizable field theory is reduced to a simple first order differential equation (RGE). For a general case it can be reduced to a Hammerstein integral equation [14]. For that type of integral equations one can prove that the solution does exist and is unique. A little is known about explicit solutions of the Hammerstein integral equations, although many approximate methods have been developed. One of them is the reduction of the integral equation to the recursion equation of the type (12).

Indeed we checked on many examples of EFTs that the corresponding recursion equation is very effective and it allows one to obtain LL coefficients to essentially unlimited loop order (e.g. it take a couple of minutes on a PC to compute the 99-loop LL coefficients for massless ChPT).

The derivation of our main equations is based on general properties of a quantum field theory: the unitarity, analyticity and the crossing symmetry. The specific form of the theory enters the equation only through the crossing matrices (type of internal symmetry) and the initial conditions for the recursion equation. That shows that the recursion equation for leading infrared logs has a general nature and can be easily written for any massless EFT with fields of various spins (e.g. gravity or theory with spinor fields) and in an arbitrary dimension.

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Appendix A: Analytical properties of partial waves.

In this appendix we give a summary of definitions and relations for partial waves in arbitrary dimension $D$.

The partial waves in $D$ ($D - 1$ spatial plus 1 time) dimensions are defined as

$$A^I(s,t) = 64\pi \sum_{l=0}^{\infty} \frac{2l + D - 3}{2} \Gamma\left(\frac{D-3}{2}\right) C_{l+\frac{1}{2}}(\cos \theta) t^I_l(s),$$

$$t^I_l(s) = \frac{1}{64\pi} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\sqrt{\pi}} \frac{2^{D-4}l!}{\Gamma(l + D - 3)} \int_0^\pi d\theta \sin^{D-3} \theta A^I(s,\cos \theta) C_l^{\frac{D-3}{2}}(\cos \theta), \quad \cos \theta = 1 + \frac{2t}{s},$$

(A1)
where $C_l^\nu(z)$ are Gegenbauer polynomials. Note that $C_l^{1/2}(z) = P_l(z)$, hence the expansion (A1) for $D = 4$ is reduced to the usual partial wave expansion in Legendre polynomials. The unitarity of the $S$-matrix leads to the following relation for the partial scattering amplitudes:

$$\text{Im} t_l^I(s) = \pi \frac{2s^4}{(4\pi)^2} |t_l^I(s)|^2$$ \quad (A2)

This unitarity relation allows one to obtain the discontinuity of the amplitude on the right cut ($s > 0$) in the complex plane of the Mandelstam variable $s$. The discontinuity on the left cut can be obtained with the help of dispersion relations. If one takes into account only cuts related to a two-particles intermediate state (that is enough for the LL approximation) the analytical properties of the amplitude are quite simple. The amplitude has the $s$-channel cut from $4m^2$ to $+\infty$, and the $u$-channel cut from $0$ to $-\infty$. [We switch on the masses for a moment in order to avoid the problems with coalescing of branch points].

The usual dispersion relation at fixed $t$ with no subtractions can be written in the form, see e.g. \[^{10}\],

$$A^I(s, t) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \left( \frac{\delta l l'}{s' - s - i0} + \frac{C_{su} l l'}{s' - 4m^2 + t + s - i0} \right) \text{Im} A'^I(s', t),$$ \quad (A3)

where the matrix $C_{su}$ is the crossing matrix \[^{11}\]. In principle, we have to make subtractions in this dispersion relation, but the subtractions do not influence the imaginary part of amplitude, and therefore for our consideration we can drop them. The discontinuity of the amplitude (A3) on the left cut ($s < 0$) receives the contribution from the second term of Eq. (A3) only. Computing the corresponding discontinuity on the left cut and performing the partial wave decomposition (A1) we find the following relation between discontinuities on the right and left cuts:

$$\text{Im} t_l^I(s) = \sum_{l' = 0}^{\infty} C_{su}^{ll'} \frac{2^{l' - 3}(2l' + D - 3)}{\Gamma(l + D - 3)} \frac{\Gamma^2 \left( \frac{D-3}{2} \right)}{\pi} l!$$

$$\times \int_{4m^2}^{\infty} ds' \frac{ds'}{s - 4m^2} \frac{d^{2l' + D - 3}}{(s - 4m^2)^2} C_{l l'}^{ll'} \frac{2s + 2s' - 4m^2}{4m^2 - s} C_{l l'}^{ll'} \frac{2s + s' - 4m^2}{4m^2 - s'} \text{Im} t_l'^I(s').$$ \quad (A4)

This relation for $D = 4$ for the case of the $\pi \pi$ scattering amplitude was derived in Ref. \[^{12}\] as a consequence of the Roy equation. Taking $D = 4$ and the limit $m^2 \to 0$ we obtain to the expression (10).

In the massless limit it is more convenient to rewrite Eq. (A4) in the form

$$\text{Im} t_l^I(s) = -\sum_{l' = 0}^{\infty} C_{su}^{ll'} \frac{(2l' + D - 3)}{\Gamma(l + D - 3)} \frac{\Gamma^2 \left( \frac{D-3}{2} \right)}{\pi} 2^{l' - 4} l!$$

$$\times \int_{-1}^{1} dz (1 - z^2)^{2l + D - 4} (-1)^{l + l'} C_{l}^{l + 3} (z) C_{l'}^{l + 3} (\frac{z + 3}{z - 1}) \text{Im} t_l'^I \left( \frac{s}{2(z - 1)} \right).$$ \quad (A5)

**Appendix B: Crossing matrices $\Omega_{\nu}^{l l'}$**

In $D$ dimensions the definition of the crossing matrix in the partial wave space is the following:

$$\frac{z - 1}{2}^n C_{l}^{\frac{D-3}{2}} (z) = \sum_{l' = 0}^{n} \Omega_{\nu}^{l l'} (D) C_{l'}^{\frac{D-3}{2}} (z).$$ \quad (B.1)

Using the orthogonality relations for the Gegenbauer polynomials, we can write the crossing matrix as the fol-
lowing integral:

$$\Omega_{n}^{ll'}(D) = \frac{2l' + D - 3}{2} \frac{2^{D-4} l'!}{\Gamma(l' + D - 3)} \frac{\Gamma\left(D - \frac{3}{2}\right)}{\Gamma} \int_{-1}^{1} dz (1 - z^2)^{\frac{D-3}{2}} \left(\frac{z - 1}{2}\right)^n C_{l'}^{\frac{D-3}{2}} \left(\frac{z + 3}{z - 1}\right) C_{l'}^{\frac{D-3}{2}}(z). \quad (B2)$$

This integral can be computed in terms of hypergeometric function in the Saalschütz form:

$$\Omega_{n}^{ll'}(D) = \frac{(-1)^{l'+n}(2l' + D - 3)}{\Gamma(n + l' + D - 2)} \frac{n!}{(n - l')!} \frac{\Gamma(l + D - 3)}{A!} \left.{}_4F_3\left(-l, l + D - 3, -l' - n - D - 3, l' - n \mid 1\right)\right|_{n = -n - \frac{D-4}{2}, \frac{D-2}{2}}. \quad (B3)$$

This representation of the crossing matrices $\Omega_{n}^{ll'}(D)$ is very convenient for the numerical calculations.

The main properties of the crossing matrices $\Omega_{n}^{ll'}(D)$ are the following:

$$\sum_{j=0}^{n} \Omega_{ij}^{ll'}(D) \Omega_{ij}^{ll'}(D) = \delta_i^l, \quad (B4)$$

$$\sum_{j=0}^{n} (-1)^j \Omega_{ij}^{ll'}(D) \Omega_{ij}^{ll'}(D) = (-1)^{l+l'} \Omega_{n}^{ll'}(D),$$

$$\Omega_{00}^{00}(D) = \frac{(-1)^n 2^{D-3} \Gamma\left(\frac{D-1}{2}\right) \Gamma\left(\frac{D}{2} + n - 1\right)}{\sqrt{\pi} \Gamma\left(D + n - 2\right)}. \quad$$

**Appendix C: Tables for LL coefficients**

In this Appendix we present Tables for LL coefficients for the 4D $O(N + 1)/O(N)$ $\sigma$-model (Tables I-III) and for the 6D $O(N)$ symmetric $\phi^4$ model (Tables IV-VI). The empty entries in the Tables correspond to zeros.

**TABLE I: Table of $I = 0$ LL coefficients for the 4D $\sigma$-model, $\omega_{nl}^{l=0} \cdot (N - 1)^{-1}$**

| $n \ \backslash \ l$ | 0     | 2     | 4     |
|----------------------|-------|-------|-------|
|                      | 1     | 1     |       |
| 2                    | $\frac{N}{2} = \frac{1}{2}$ | $\frac{5N}{12} + \frac{5N}{12}$ | $\frac{5N}{12} + \frac{13N}{12}$ |
| 3                    | $\frac{N}{6} = \frac{1}{6}$ | $\frac{61N^2}{12} + \frac{63N^2}{12}$ | $\frac{61N^2}{12} + \frac{4311N^2}{12}$ |
| 4                    | $\frac{N}{16} = \frac{1}{16}$ | $\frac{131N^2}{16} + \frac{4967N^2}{16}$ | $\frac{131N^2}{16} + \frac{4311N^2}{16}$ |
| 5                    | $\frac{N}{30} = \frac{1}{30}$ | $\frac{116N^2}{16} + \frac{6198N^2}{16}$ | $\frac{116N^2}{16} + \frac{6198N^2}{16}$ |
**TABLE II**: Table of $I = 1$ LL coefficients for the 4D $\sigma$-model, $\omega_{nl}^{I=1}$

| $n \setminus l$ | 1 | 3 | 5 |
|-----------------|---|---|---|
| 1               | $- \frac{N}{2} + \frac{3}{2}$ | $- \frac{N}{2} + \frac{7}{2}$ | $- \frac{N}{2} + \frac{7}{2}$ |
| 2               | $\frac{9N^3 - 35N^2 + 49N}{10}$ | $\frac{9N^3 - 35N^2 + 49N}{10}$ | $\frac{9N^3 - 35N^2 + 49N}{10}$ |
| 3               | $- \frac{N^3 + 49N^2 - 1791N}{10} + 8543$ | $- \frac{N^3 + 49N^2 - 1791N}{10} + 8543$ | $- \frac{N^3 + 49N^2 - 1791N}{10} + 8543$ |
| 4               | $\frac{5N^3 - 52859N^2 + 1896333N^2}{10} - 2016$ | $\frac{5N^3 - 52859N^2 + 1896333N^2}{10} - 2016$ | $\frac{5N^3 - 52859N^2 + 1896333N^2}{10} - 2016$ |
| 5               | $112N^3 - 302400 + 29216000$ | $112N^3 - 302400 + 29216000$ | $112N^3 - 302400 + 29216000$ |

**TABLE III**: Table of $I = 2$ LL coefficients for the 4D $\sigma$-model, $\omega_{nl}^{I=2}$

| $n \setminus l$ | 0 | 2 | 4 |
|-----------------|---|---|---|
| 1               | $-1$ | $-1$ | $-1$ |
| 2               | $\frac{N^2 + 4N}{10}$ | $\frac{N^2 + 4N}{10}$ | $\frac{N^2 + 4N}{10}$ |
| 3               | $- \frac{N^2 + 4N - 59}{10}$ | $- \frac{N^2 + 4N - 13}{10}$ | $- \frac{N^2 + 4N - 13}{10}$ |
| 4               | $\frac{2N^3 - 59N^2 + 13214N^2}{10} - 420$ | $\frac{2N^3 - 59N^2 + 13214N^2}{10} - 420$ | $\frac{2N^3 - 59N^2 + 13214N^2}{10} - 420$ |
| 5               | $- \frac{N^3 + 1727N^2 - 123249N^2}{10} + 1619600$ | $- \frac{N^3 + 1727N^2 - 123249N^2}{10} + 1619600$ | $- \frac{N^3 + 1727N^2 - 123249N^2}{10} + 1619600$ |

**TABLE IV**: Table of $I = 0$ LL coefficients for the 6D $\phi^4$-model, $\omega_{nl}^{I=0}$

| $n \setminus l$ | 0 | 2 | 4 |
|-----------------|---|---|---|
| 1               | $-(N + 2)$ | $-(N + 2)$ | $-(N + 2)$ |
| 2               | $\frac{(N + 2)(N - 1)}{2}$ | $\frac{(N + 2)(N - 1)}{2}$ | $\frac{(N + 2)(N - 1)}{2}$ |
| 3               | $\frac{1}{30}(N + 2)^2(N - 1)$ | $\frac{1}{30}(N + 2)^2(N - 1)$ | $\frac{1}{30}(N + 2)^2(N - 1)$ |
| 4               | $\frac{(N + 2)^2(N - 1)(\frac{N}{111} + \frac{1}{3})}{2}$ | $\frac{(N + 2)^2(N - 1)(\frac{N}{111} + \frac{1}{3})}{2}$ | $\frac{(N + 2)^2(N - 1)(\frac{N}{111} + \frac{1}{3})}{2}$ |
| 5               | $\frac{(N + 2)^2(N - 1)}{2}$ | $\frac{(N + 2)^2(N - 1)}{2}$ | $\frac{(N + 2)^2(N - 1)}{2}$ |

**TABLE V**: Table of $I = 1$ LL coefficients for the 6D $\phi^4$-model, $\omega_{nl}^{I=1}$

| $n \setminus l$ | 1 | 3 | 5 |
|-----------------|---|---|---|
| 1               | 0 | 0 | 0 |
| 2               | $\frac{(N + 2)}{10}$ | $\frac{(N + 2)}{10}$ | $\frac{(N + 2)}{10}$ |
| 3               | $\frac{1}{(N + 2)^2}$ | $\frac{1}{(N + 2)^2}$ | $\frac{1}{(N + 2)^2}$ |
| 4               | $\frac{(N + 2)^2(\frac{N}{111} + \frac{1}{3})}{2}$ | $\frac{(N + 2)^2(\frac{N}{111} + \frac{1}{3})}{2}$ | $\frac{(N + 2)^2(\frac{N}{111} + \frac{1}{3})}{2}$ |
| 5               | $\frac{(N + 2)^2}{2}$ | $\frac{(N + 2)^2}{2}$ | $\frac{(N + 2)^2}{2}$ |
TABLE VI: Table of $I = 2$ LL coefficients for the 6D $\phi^4$-model, $\omega_n^{I,n^2}$

| $n \setminus I$ | 1 | 3 | 5 |
|---------------|---|---|---|
| 1             | -2 |   |   |
| 2             | $-(N+2)$ |   |   |
| 3             | $-(N+2)(\frac{1}{2} - \frac{N}{2} - \frac{1}{12})$ | $\frac{2N}{3}$ |   |
| 4             | $(N+2)(-\frac{N^4}{720} + \frac{N^3}{270} - \frac{1}{120})$ | $(N+2)(-\frac{N^4}{720} - \frac{29N}{3150} + \frac{7}{180})$ |   |
| 5             | $(N+2)(-\frac{N^3}{45} + \frac{13N^2}{360} + \frac{17N}{11400})$ | $(N+2)(-\frac{N^3}{1164} + \frac{57N}{10590} + \frac{778800}{291600})$ | $(N+2)(-\frac{N^3}{408240} - \frac{N^2}{170100})$ |   |

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