Parton distributions in radiative corrections to the cross section of electron-proton scattering

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Abstract The structure function approach and the parton picture, developed for the theoretical description of the deep inelastic electron-proton scattering, also proved to be very effective for calculation of radiative corrections in Quantum Electrodynamics. We use them to calculate radiative corrections to the cross section of electron-proton scattering due to electron-photon interaction, in the experimental setup with the recoil proton detection, proposed by A. A. Vorobyev to measure the proton radius. In the one-loop approximation, explicit expressions for these corrections are obtained for arbitrary momentum transfers. It is shown that, at momentum transfers small compared with the proton mass, various contributions to the corrections mutually cancel each other with power accuracy. In two loops, the corrections are obtained in the leading logarithmic approximation.

1 Introduction

After appearance of the papers [1–3] the “proton radius puzzle” – the striking difference in the proton radius values extracted from the 2S-2P transition in muonic hydrogen [4,5] and obtained from electron-proton scattering and hydrogen spectroscopy [6] – tends to be resolved (for a review, see Refs. [7–10]). However, there is still confusing tension between the latest electron scattering experiments [3,11,12] as well as between the hydrogen spectroscopy experiments [1,2,13].

Currently new scattering experiments are being prepared. A distinctive feature of one of them [14], which was suggested by A. A. Vorobyev and has to be performed with a low-intensity electron beam at MAMI, is that instead of detecting a scattered electron, as in previous experiments, it is supposed to detect with a high precision a recoil proton in the region of low momentum transfers \(0.04 \text{ GeV}^2 > Q^2 > 0.001 \text{ GeV}^2\). The aim is to extract the proton radius with 0.6 percent precision, which could be decisive in solving the proton radius puzzle. To this end, it is planned to achieve 0.2 percent accuracy of the cross section \(d\sigma/(dQ^2)\) measurement.

Such accuracy requires precise account of radiative corrections. Although calculation of the radiative corrections to the electron-proton scattering cross section has a long history (see, for example, Refs. [15–18]) and recent reviews [20–22]) the results obtained before cannot be completely applied to the experiment discussed above. The reason it that they were obtained for experiments in which scattered electrons were detected (honestly speaking, there was the experiment [23] where the recoil proton was detected; but calculation of the radiative corrections to this experiment was not explained). Since the radiative corrections include contributions of inelastic processes with photon emission, they depend strongly on experimental conditions, so that the corrections calculated for experiments with detection of scattered electrons are not suitable for experiments with detection of recoil proton. It occurs [24] that the radiative corrections for experiments with detection of recoil proton have a new unexpected and pleasant property – cancellation of the most important corrections, which are due to electron-photon interaction\(^2\), in the region of low momentum transfers. In [24], the cancellation of not only infrared, but also collinear singularities was shown and a simple physical explanation of this phenomenon was given. It was also argued that in the one-loop approximation the accuracy of the cancellation

\(^1\) Higher order corrections to the lepton line was considered for the standard experimental set-up with scattered electron measurement in [19].

\(^2\) Cancellation of leptonic radiative corrections to deep inelastic scattering was discussed in [25] and [26].
is higher than the logarithmic, and the terms not having the collinear singularities (constant terms) are cancelled as well.

Here we refine the results of [24] and get new ones, with a wider scope of applicability, using the structure function approach and the parton picture, developed for the theoretical description of the deep inelastic electron-proton scattering [27–31] and adopted in [32] for calculation of radiative corrections in QED.

2 Statement of the approach

Following [24], we denote four-momenta of initial and final electron (proton) as \( l (p) \) and \( l' (p') \); \( l^2 = l'^2 = m^2 \), \( p^2 = p'^2 = M^2 \), and use the designations \( Q^2 = -q^2 \), \( q = p - p' \) both for elastic and inelastic processes.

The cross section of electron-proton scattering with radiative corrections due only to electron interaction can be considered as inclusive proton-electron scattering cross section. It means that it can be written as

\[
E_p' \frac{d^3 \sigma}{d^3 p'} = \frac{(\alpha(Q^2))^2}{Q^4} \frac{J^{\mu \nu} (p, p') W_{\mu \nu} (l, q)}{\sqrt{(pl)^2 - m^2 M^2}} ,
\]

where

\[
\alpha(Q^2) = \frac{\alpha}{1 - \mathcal{P}(q^2)} ,
\]

\( \mathcal{P}(q^2) \) is the vacuum polarisation, which is real at \( q^2 = -Q^2 < 0 \); \( J^{\mu \nu} (p, p') \) is the proton current tensor

\[
J^{\mu \nu} (p, p') = \sum_{pol} J^{\mu \nu} ,
\]

\( \sum_{pol} \) means summation over final polarisations and averaging over initial ones,

\[
J^{\mu} = \bar{u}(p') \left( f_1(Q^2) \gamma^{\mu} + f_2(Q^2) \frac{[\gamma^{\mu}, \gamma^{\nu}] q^{\nu}}{4M} \right) u(p) ,
\]

\( f_1(Q^2) \) and \( f_2(Q^2) \) are the the Dirac and Pauli form factors of the proton, and \( W_{\mu \nu} (l, q) \) is the deep inelastic scattering tensor,

\[
W_{\mu \nu} (l, q) = \frac{1}{4\pi} \sum_X \langle l | j^{\nu (e)} (0) | X \rangle \langle X | j^{\mu (e)} (0) | l \rangle \times (2\pi)^4 \delta (q + l - p_X) .
\]

Here \( | l \rangle \) is the initial electron state, \( | X \rangle \) is any state which can be produced in photon-electron collisions, \( \sum_X \) means averaging over initial electron polarisations and summation over discrete and integration over continuous variables of \( | X \rangle \), \( j^{\mu (e)} \) is the electron electromagnetic current operator.

Taking into account conservation of the current, one can represent \( W_{\mu \nu} \) in the form

\[
W_{\mu \nu}^{xy} (l, q) = F_1(x, Q^2) \left( -g^{\mu \nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right) + \frac{F_2(x, Q^2)}{(lq)} \left( l^{\mu} - \frac{(lq) q^{\mu}}{q^2} \right) \left( l^{\nu} - \frac{(lq) q^{\nu}}{q^2} \right) ,
\]

where

\[
x = \frac{Q^2}{2(lq)}
\]

is the Bjorken variable and \( F_1(x, Q^2) \) are the structure functions. They are expressed in terms of the convolutions \( W_i \) of the tensor \( W_{\mu \nu}^{xy} (l, q) \)

\[
W_g = W_{\mu \nu}^{xy} (l, q) g^{\mu \nu} , \quad W_i = W_{\mu \nu}^{xy} (l, q) l^{\mu} l^{\nu}
\]

using the relations

\[
F_1(x, Q^2) = \frac{1}{2} \left( \frac{Q^2 W_i}{Q^2 m^2 + (ql)^2} - W_g \right) ,
\]

\[
F_2(x, Q^2) = \frac{1}{2} \frac{Q^2 q l}{Q^2 m^2 + (ql)^2} \times \left( \frac{3 Q^2 W_i}{Q^2 m^2 + (ql)^2} - W_g \right) .
\]

Calculating the tensor \( J^{\mu \nu} \),

\[
J^{\mu \nu} = G_M(Q^2) \left( g_{\mu \nu} q^2 - q_{\mu} q_{\nu} \right)
\]

\[
+ \frac{Q^2 G_E^2(Q^2) + 4M^2 G_M^2(Q^2)}{4M^2 + Q^2} p_{\mu} p_{\nu} ,
\]

where \( P = p + p' \), \( G_E(Q^2) \) and \( G_M(Q^2) \) are the proton electric and magnetic form factors,

\[
G_M(Q^2) = f_1(Q^2) + f_2(Q^2) ,
\]

\[
G_E(Q^2) = f_1(Q^2) - \frac{Q^2}{4M^2} f_2(Q^2) ,
\]

performing tensor convolution and using

\[
\frac{d^3 p'}{2E_p'} = \frac{\pi}{4} \frac{Q^2 dQ^2 dx}{x^2 \sqrt{(pl)^2 - m^2 M^2}} ,
\]

we obtain

\[
\frac{d\sigma}{dQ^2 dx} = \frac{\pi (\alpha(Q^2))^2}{2 x^2 Q^2 ((pl)^2 - m^2 M^2)} \times \left[ 2Q^2 G_M^2 - 4M^2 G_E^2 \right] F_1(x, Q^2)
\]

\[
+ \left( -G_M^2 (m^2 Q^2 + (ql)^2) \right) .
\]
Formula (15) gives the exact expression for the cross section of electron-photon interaction. The radiation correction due to electron-proton scattering taking into account all processes is determined by (15) with \( \alpha(Q^2) = \alpha \), and \( F_2(x, Q^2) = 2F_1(x, Q^2) = \delta(1 - x) \):

\[
\frac{d\sigma_B}{dQ^2} = \frac{\pi\alpha^2 M^2}{Q^4} \left( \epsilon G_E^2 + \tau G_M^2 \right)
\times \frac{[(4p_1l - Q^2)^2 + (Q^2 + 4M^2)(Q^2 - 4m^2)]}{(Q^2 + 4M^2)((pl)^2 - m^2 M^2)\epsilon} ,
\] (17)

where \( E_i \) is the energy of the incident electron in the rest frame of the initial proton.

In the Born approximation, the cross section \( \frac{d\sigma_B}{dQ^2} \) is determined by (15) with \( \alpha(Q^2) = \alpha \), and \( F_2(x, Q^2) = 2F_1(x, Q^2) = \delta(1 - x) \):

\[
\frac{d\sigma_B}{dQ^2} = \frac{\pi\alpha^2 M^2}{Q^4} \left( \epsilon G_E^2 + \tau G_M^2 \right)
\times \frac{[(4p_1l - Q^2)^2 + (Q^2 + 4M^2)(Q^2 - 4m^2)]}{(Q^2 + 4M^2)((pl)^2 - m^2 M^2)\epsilon} ,
\] (17)

with

\[
\epsilon = \frac{(4p_1l - Q^2)^2 - Q^2(Q^2 + 4M^2)}{(4p_1l - Q^2)^2 + (Q^2 + 4M^2)(Q^2 - 4m^2)} ,
\] (18)

\[
\tau = \frac{Q^2}{4M^2} .
\] (19)

Formula (15) gives the exact expression for the cross section of electron-proton scattering taking into account all processes of electron-photon interaction. The radiation correction due to this interaction is determined by the equation

\[
\delta_{ey} = \int_0^1 dx \left[ \frac{d\sigma_B}{dQ^2} \right]_x - 1 \quad (20)
\]

and can be written as

\[
\delta_{ey} = \left[ 1 + \frac{\delta^e}{(1 - \mathcal{P}(q^2))^2} \right] - 1 ,
\] (21)

where \( \delta^e \) is the correction associated with the electron structure, that is, with the difference \( F_2(x, Q^2) \) and \( 2F_1(x, Q^2) \) from \( \delta(1 - x) \).

Our main goal here is to calculate just this correction. As for the vacuum polarisation \( \mathcal{P}(q^2) \), it is well known and we have nothing new to say about it. For completeness, we provide the necessary information in Appendix A.

In the proposed experiments to measure the proton radius, the momentum transfers are large compared to the electron mass, \( Q^2 \gg m^2 \). Below, we will be mainly interested in this particular area. Here, it is convenient to use the following representation of the cross sections (15) and (17)

\[
\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha(Q^2)^2}{Q^4} \left[ F_2(x, Q^2) R(x, Q^2) + \frac{Q^2(2Q^2G_M^2 - 4M^2G_E^2)}{8x^2(pl)^2}\epsilon \right. \times \left( F_1(x, Q^2) - \frac{F_2(x, Q^2)}{2x} \right) \right] ,
\] (22)

where

\[
R(x, Q^2) = \left( 1 - \frac{Q^2}{2x(lp)} \right) \frac{Q^2G_M^2 + 4M^2G_E^2}{4M^2 + Q^2}
+ \frac{Q^2}{8x^2(pl)^2} \frac{Q^2(2M^2 + Q^2)G_M^2 - 8M^4G_E^2}{4M^2 + Q^2} ,
\] (23)

and

\[
\frac{d\sigma_B}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} R(1, Q^2) .
\] (24)

### 3 Elastic scattering

For the elastic scattering, when \( |X| \) in (5) are the one-electron states with the momentum \( l' \), we have

\[
\langle X | j^{(e)}_\mu (0) | l \rangle = \langle l' | j^{(e)}_\mu (0) | l \rangle
= \bar{u}(l') \left[ f_1^e(Q^2) \gamma^\mu - f_2^e(Q^2) \frac{[\gamma^\mu, \gamma^\nu]q_\nu}{4m} \right] u(l) ,
\] (25)

where \( f_1^e(Q^2) \) are the electron form factors. Using (9), (10) and (8), one obtains for the elastic contributions to the electron structure functions \( F_i \)

\[
F_1^{el}(x, Q^2) = \frac{1}{2} \delta(1 - x) \left( f_1^e(Q^2) + f_2^e(Q^2) \right) ,
\] (26)

\[
F_2^{el}(x, Q^2) = \delta(1 - x) \left[ f_1^e(Q^2) \right]^2
+ \frac{Q^2}{4m^2} \left( f_2^e(Q^2) \right)^2 \right] .
\] (27)

Eq. (15) then gives

\[
\frac{d\sigma^{el}}{dQ^2} = \frac{\pi\alpha(Q^2)^2}{4Q^2((pl)^2 - m^2 M^2)^2} \times \frac{(4p_1l - Q^2)^2}{(Q^2 + 4M^2)((pl)^2 - m^2 M^2)^2}
\times \frac{Q^2G_M^2 + 4M^2G_E^2}{(Q^2 + 4M^2)(Q^2 + 4m^2)}
\times \bar{\mathcal{F}}_M^2 - 4M^2G_E^2 \right] ,
\] (28)

where

\[
\bar{\mathcal{F}}_M = f_1^e(Q^2) + f_2^e(Q^2) .
\] (29)
and each term of the expansion requires the regularization of the Pauli form factor.

Formally, Eq. (28) gives the exact expression for the cross section of elastic electron-proton scattering with one-photon exchange. But essentially it has no physical meaning due to the infrared singularity. Taking into account all terms of the expansion in terms of the coupling constant \( \alpha \) makes it zero, and each term of the expansion requires the regularization of this singularity. If the infrared divergence is regularised by the photon mass \( \lambda \), in the one-loop approximation one has \[ f_1^e (Q^2) = 1 \frac{\alpha}{\pi \beta} \left[ \ln (\xi - \beta) \ln \left( \frac{m^2}{\lambda^2} - 1 \right) - \frac{1}{4} \ln^2 \xi + 2 \xi \right] \]

\[ f_2^e (Q^2) = \frac{\alpha}{2 \pi} \frac{1 - \beta^2}{\beta} \ln \xi , \]

where \( \beta \) is the velocity of one of the electrons in the rest frame of the other:

\[ \beta = \frac{\sqrt{Q^2 (Q^2 + 4m^2)}}{Q^2 + 2m^2} , \quad \xi = \frac{1 + \beta}{1 - \beta} . \]

and

\[ \text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln (1 - t) . \]

The part \( \delta_{\text{vertex}}^e \) of the associated with the electron structure correction \( \delta^e \) introduced by elastic scattering is determined by the difference between the vertex (25) and the Born one, in which \( f_1^e (Q^2) = 1 \) and \( f_2^e (Q^2) = 0 \). At \( Q^2 \gg m^2 \) the Pauli form factor \( f_2^e (Q^2) \) is suppressed by a power-law, \( f_1^e (Q^2) \sim m^2 / Q^2 \), so that in the one-loop approximation we have

\[ \delta_{\text{vertex}}^e = 2 \left( f_1^e (Q^2) - 1 \right) \]

\[ = \frac{\alpha}{\pi} \left[ - \left( \ln \frac{Q^2}{m^2} - 1 \right) \ln \frac{m^2}{\lambda^2} - \frac{1}{2} \ln^2 \frac{Q^2}{m^2} + \frac{3}{2} \ln \frac{Q^2}{m^2} + \frac{\pi^2}{6} - 2 \right] . \]

4 One photon emission

For one photon emission, when the states \( | X \rangle \) in (5) are states of an electron with momentum \( l' \) and photon with momentum \( k \), we have

\[ W_{\mu \nu} (l, q) = - \frac{\alpha^2}{8 \pi} \int \frac{d^3l'}{(2\pi)^3 2E_{l'}^3} \frac{d^3k}{(2\pi)^3 2\omega} \]

\[ (2\pi)^4 \delta^{(4)} (q + l - l' - k) \]

where

\[ K_{\mu \nu} = g^{\rho \sigma} \text{tr} \left[ (\hat{l} + m) L_{\mu \rho} \hat{l} (\hat{m} + m) L_{\nu \sigma} \gamma^0 \right] , \]

with

\[ L_{\mu \rho} = \gamma_\mu - \frac{\hat{l} - \hat{k} + m}{2\kappa} \gamma_\rho + \frac{\hat{l} + \hat{k} + m}{2\kappa'} \gamma_\mu , \]

and

\[ \kappa = (kl) \quad \kappa' = (kl') = Q^2 \frac{1 - x}{2x} . \]

Moving on to integration over \( \kappa \), we obtain from (36) for the convolutions (8) of \( W_{\mu \nu} (l, q) \)

\[ W_i = - \frac{\alpha^2}{8 \pi} \int_{\kappa_-}^\kappa \frac{d\kappa}{\sqrt{I_C}} A_i , \]

where

\[ A_g = K_{\mu \nu} g^{\mu \nu} , \quad A_i = K_{\mu \nu} l^\mu l^\nu , \]

\[ I_C = (Q^2 + 2\kappa')^2 + 4m^2 Q^2 , \]

and the integration limit \( \kappa_- (\kappa_+ \) corresponds to forward (backward) virtual Compton scattering:

\[ \kappa_\pm = \frac{\kappa'}{2(m^2 + 2\kappa') \left( 2m^2 + 2\kappa' + Q^2 \pm \sqrt{I_C} \right) } . \]

Direct calculation of the convolutions \( A_i \) (41) gives

\[ A_g = 4 \left( \frac{m^2}{\kappa'^2} + \frac{m^2}{\kappa'^2} - \frac{2m^2 + Q^2}{\kappa' \kappa} + \frac{2}{\kappa} \right) \]

\[ \times (2m^2 - Q^2) + \frac{\kappa}{\kappa'} \left( \frac{\kappa}{\kappa'} + \frac{\kappa'}{\kappa} \right) , \]

\[ A_i = 2m^2 \left( \frac{m^2}{\kappa'^2} + \frac{m^2}{\kappa'^2} - \frac{2m^2 + Q^2}{\kappa' \kappa} + \frac{2}{\kappa} - \frac{6}{\kappa'} \right) \]

\[ \times (4m^2 + Q^2) + \frac{4m^2 \kappa'}{\kappa'^2} + \frac{2(2m^2 + Q^2)}{\kappa'} \]

\[ + \frac{4(4m^2 - 3\kappa')}{\kappa'^2} + 4 \right] - 4(Q^2 + 2\kappa' - 2\kappa) . \]

Note that \( A_i \) should be obtained from Eqs.(7.39) of [34] with the substitutions

\[ w^2 \to m^2 + 2\kappa' , \quad q^2 \to m^2 - 2\kappa , \]

\[ \Delta^2 \to -Q^2 , \quad \Delta^1_1 \to 0 . \]

Unfortunately, in the expression for \( A_g^{(1)} \) in Eqs. (7.39) there is a misprint; it contains the extra term \( 8(p_1 p_2)m^4 / (q^2 - m^2)^2 \).
Calculation of $W_i(lq, q^2)$ (40) is performed using the integrals
\[
\begin{align*}
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\sqrt{I_C}} &= \frac{\kappa'}{m^2 + 2\kappa'}, \\
\int_{\kappa_-}^{\kappa_+} \kappa d\kappa &= \frac{\kappa'^2(2m^2 + 2\kappa' + Q^2)}{2(m^2 + 2\kappa')^2}, \\
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\kappa \sqrt{I_C}} &= \frac{1}{m^2 \kappa'}, \\
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\kappa \sqrt{I_C}} &= \mathcal{L},
\end{align*}
\]
where
\[
\mathcal{L} = \frac{1}{\sqrt{(Q^2 + 2\kappa')^2 + 4m^2Q^2}} \times \ln \left( \frac{2m^2 + 2\kappa' + Q^2 + \sqrt{I_C}}{2m^2 + 2\kappa' + Q^2 - \sqrt{I_C}} \right).
\]
It gives
\[
\begin{align*}
W_g &= -\frac{\alpha}{2\pi} \left\{ \left( Q^2 - 2m^2 - 2m^2 \right) \left( \frac{Q^2 + 2m^2}{\kappa'} + 2 \right) \mathcal{L} \\
&\quad - \left( \frac{m^2}{\kappa'^2 + 2} \right) \frac{\kappa'}{m^2 + 2\kappa'} - \frac{1}{\kappa'} \right\},
\end{align*}
\]
\[
W_l = -\frac{\alpha}{2\pi} \left\{ \frac{m^2}{2} \left( -Q^2 + 4m^2 \right) \left( \frac{Q^2 + 2m^2}{\kappa'} + 6 \right) \\
&\quad + 8m^2 - 6\kappa' \right\} \mathcal{L} + m^2 \left( 2 + \frac{Q^2 + 4m^2}{2\kappa'} \right) \\
&\quad + \left[ m^2 \left( \frac{Q^2 + 4m^2}{2\kappa'} + \frac{m^2}{\kappa'} \right) + \frac{4m^2}{\kappa'} + 6 \right] - Q^2 - 2\kappa' \right\}.
\]

The region of variation of $x$ at fixed $Q^2$ is determined by the conditions $M_X^2 = (m^2 + 2\kappa') \geq m^2$ and $(lq) \leq E_1q_0 + \sqrt{E_1^2 - m^2 \sqrt{q_0^2 + Q^2}}$, where $q_0 = M - E_\nu' = -Q^2/(2M)$, i.e.
\[
1 \geq x \geq x_- = \frac{M Q^2}{\sqrt{E_1^2 - m^2 \sqrt{Q^2(4M^2 + Q^2)} - E_1 Q^2}}.
\]
But expressions (49) and (50) can not be used arbitrarily close to $x = 1$ (i.e. for sufficiently small $\kappa'$) because of the infrared divergency. The divergency must be regularised in the same way as in the vertex correction (35), i.e. by the photon mass $\lambda$. Taking into account the photon mass changes both the measure and the limits of integration in (40):
\[
\begin{align*}
I_C \to I_C(\lambda) &= (Q^2 + 2\kappa' + \lambda^2)^2 + 4m^2 Q^2, \\
\kappa_+ \to \kappa_+(\lambda) &= \frac{(\kappa' + \lambda^2)(2m^2 + 2\kappa' + Q^2 + \lambda^2)}{2(m^2 + 2\kappa' + \lambda^2)} \\
&\quad \pm \sqrt{(\kappa' - m^2\lambda)^2 I_C(\lambda)} \\
&\quad \frac{2(\kappa' - m^2\lambda)^2 I_C(\lambda)}{2(m^2 + 2\kappa' + \lambda^2)}.
\end{align*}
\]
In the region $m^2 \gg \kappa' > m\lambda$ at $\lambda \to 0$ they can be taken as
\[
\begin{align*}
I_0 &= Q^2(Q^2 + 4m^2), \\
\kappa_0^2 &= \frac{\kappa'(2m^2 + Q^2) \pm \sqrt{(\kappa'^2 - m^2\lambda^2) I_0}}{2m^2}.
\end{align*}
\]
The singular terms in $A_i$ are
\[
\begin{align*}
A_g &= 4(2m^2 - Q^2) \left( m^2 \kappa'^2 + m^2 \kappa'^2 - 2m^2 + Q^2 \right), \\
A_l &= 2m^2(4m^2 + Q^2) \left( m^2 \kappa'^2 + m^2 \kappa'^2 - 2m^2 + Q^2 \right).
\end{align*}
\]
Corresponding integrals become
\[
\begin{align*}
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\sqrt{I_0}} &= \frac{\sqrt{(\kappa'^2 - m^2\lambda^2)}}{m^2}, \\
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\kappa^2 \sqrt{I_0}} &= \frac{4\kappa' \sqrt{(\kappa'^2 - m^2\lambda^2)}}{4m^2\kappa'^2 + \lambda^2 Q^2(4m^2 + Q^2)}, \\
\int_{\kappa_-}^{\kappa_+} \frac{d\kappa}{\kappa \sqrt{I_0}} &= L_0,
\end{align*}
\]
where
\[
L_0 = \frac{1}{\sqrt{Q^2(4m^2 + Q^2)}} \times \ln \left( \frac{\kappa'(2m^2 + Q^2) + \sqrt{(\kappa'^2 - m^2\lambda^2) I_0}}{\kappa'(2m^2 + Q^2) - \sqrt{(\kappa'^2 - m^2\lambda^2) I_0}} \right).
\]
It gives for $W_i$ (40) in the region $m^2 \gg \kappa' > m\lambda$
\[
\begin{align*}
W_g &= \frac{2m^2 - Q^2}{2} \frac{d\omega}{d\kappa'}, \\
W_l &= \frac{2m^2 + Q^2}{4} \frac{d\omega}{d\kappa'},
\end{align*}
\]
where $m d\omega/d\kappa'$ is the spectral probability density for soft photon emission with account of photon mass in the rest frame of the final electron,
\[
\frac{d\omega}{d\kappa'} = \frac{\alpha}{\pi \kappa'} \left[ \left( 2m^2 + Q^2 \right) L_0 - \frac{\sqrt{\kappa'^2 - m^2\lambda^2}}{\kappa'} \\
&\quad - \frac{4m^2\kappa' \kappa'^2 - m^2\lambda^2}{4m^2\kappa'^2 + \lambda^2 Q^2(4m^2 + Q^2)} \right].
\]
Using (9), (10) and (60), one obtains
\[
F_2(x, Q^2) = 2F_1(x, Q^2) = \frac{Q^2}{2} \frac{d\omega}{d\kappa'}.
\]
so that (15) gives for the soft photon emission cross section

\[
\frac{d\sigma_{\text{soft}}}{dQ^2dx} = \frac{d\sigma_B}{dQ^2} \delta_{\text{soft}},
\]

Integration (63) over the region \(\kappa_0 > \kappa' > m\lambda (1 - m\lambda/Q^2 > x > 1 - 2\kappa_0/Q^2 \), \(dx = -2dk'/Q^2\) at \(\kappa_0 \ll m^2, \kappa_0 \ll Q^2\) provides at \(\lambda \to 0\)

\[
\frac{d\sigma_{\text{soft}}}{dQ^2} = \frac{d\sigma_B}{dQ^2} \delta_{\text{soft}},
\]

\[
\delta_{\text{soft}} = \frac{\alpha}{\pi} \left\{ \frac{1}{\beta} \left[ \ln \left( \frac{Q^2}{m^2} \right) - 1 \right] + \frac{1}{2} \ln2 - \frac{\pi^2}{6} \right\},
\]

which together with (35) gives

\[
\delta_{\text{vertex}} + \delta_{\text{soft}} = \frac{\alpha}{\pi} \left\{ \frac{1}{\beta} \left[ \ln \left( \frac{Q^2}{m^2} \right) - 1 \right] - \ln2 \left( \frac{Q^2}{m^2} \right) + \frac{\pi^2}{6} - 1 \right\},
\]

As it should be, the dependence on \(\lambda\) disappeared in the sum of corrections from elastic scattering and soft photon emission.

To find the contribution of real photons with \(\kappa' > \kappa_0\) for arbitrary \(Q^2\) is not so easy. In the following we restrict ourselves to considering the case \(Q^2 \gg m^2\). In this case it is possible to introduce the intermediate scale \(\kappa_1\), such that \(Q^2 \gg \kappa_1 \gg m^2\), and calculate the contributions of the regions \(\kappa' < \kappa_1\) and \(\kappa' > \kappa_1\), simplifying the integrands in them as it is described in Appendix B. In the sum of these contributions dependence on the intermediate scale disappears (see (B.13)). The remaining dependence on the boundary \(\kappa_0\) between soft and hard emission vanishes in the sum of the correction (B.13) due to the hard emission with \(\delta_{\text{vertex}} + \delta_{\text{soft}}\) (67), so that for the total correction \(\delta^e\) one has in the one-loop approximation at \(Q^2 \gg m^2\)

\[
\delta^e_{\text{one-loop}} = \frac{\alpha}{2\pi} \left( \phi_0(x_-) + \frac{Q^2}{4(\rho l)^2} \left( \frac{Q^2G^2_{\gamma} + 4M^2G^2_{E}}{4M^2 + Q^2} \right) \phi_1(x_-) \right).
\]
where $\rho = (pl)/M^2$. As we can see, only the terms with power smallness in $x_-$ remain in the full correction. The terms that do not have such smallness cancel out not only if they are strengthened by powers of $\ln(Q^2/m^2)$, but also without such strengthening, as it was noted in [24].

### 5 Parton picture

In the parton picture the structure functions $F_i(x, Q^2)$ are expressed through parton distributions. In the leading logarithmic approximation (LLA)

$$ F_2(x, Q^2) = 2 x F_1(x, Q^2) = x (f_e^p(x, Q^2) + f_e^n(x, Q^2)), \quad (74) $$

where $f_e^p(x, Q^2)$ and $f_e^n(x, Q^2)$ are the electron and positron distributions in the initial electron and the first equality is the Callan-Gross relation [35], arising from the fact that the partons have spin 1/2.

In the LLA the parton distributions can be calculated using the equations [29,30]

$$ \frac{df_i^a(x, Q^2)}{d \ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \sum_b \int_x^1 \frac{dz}{z} P_{ib}^a \left( \frac{x}{z} \right) f_b^i(z, Q^2), $$

(75)

where $a, b = e, \bar{e}, \gamma$. $P_{ib}^a(z)$ are the splitting functions,

$$ P_{ie}^e(z) = P_{ie}^\gamma(z) = \frac{1 + (1-z)^2}{z}, $$

$$ P_{ie}^\gamma(z) = P_{ie}^\gamma(z) = z^2 + (1-z)^2, $$

$$ P_{i\bar{e}}^\bar{e}(z) = P_{i\bar{e}}^\bar{e}(z) = \frac{1 + z^2}{(1-z)z} + \frac{3}{2} \delta(1-z). $$

(76)

Here the generalized function \( \frac{1}{(1-z)_+} \) is defined by the relation

$$ \int_0^1 \frac{f(z)}{(1-z)_+) \, dz = \int_0^1 \frac{f(z) - f(1)}{(1-z)_+) \, dz. \quad (77) $$

The evolution equations must be complemented by the initial conditions, which can be taken as

$$ f_e^a(x, m^2) = \delta(1-x). $$

(78)

Presenting parton distributions as the sum of the distributions of valence (v) and sea (s) partons in electron

$$ f_e^v(x, Q^2) = f_e^v(x, Q^2) + f_e^s(x, Q^2), $$

$$ f_e^\bar{e}(x, Q^2) = f_e^v(x, Q^2), $$

we obtain that these distributions obey the equations

$$ \frac{df_e^v(x, Q^2)}{d \ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P_e^v \left( \frac{x}{z} \right) f_e^v(z, Q^2), $$

(81)

$$ \frac{df_e^s(x, Q^2)}{d \ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[ P_e^v \left( \frac{x}{z} \right) f_e^v(z, Q^2) + P_e^\gamma \left( \frac{x}{z} \right) f_e^\gamma(z, Q^2) \right], $$

(82)

with initial

$$ f_e^v(x, m^2) = \delta(1-x), \quad f_e^s(x, m^2) = 0 $$

(83)

and charge conservation

$$ \int_0^1 f_e^v(x, Q^2) \, dx = 1, $$

(84)

conditions. Writing with the two-loop accuracy

$$ f_e^v(x, Q^2) = \delta(1-x) + \frac{\alpha}{2\pi} L V_1(x) $$

(85)

$$ + \left( \frac{\alpha}{2\pi} \right)^2 \frac{L^2}{2} V_2(x), $$

and

$$ f_e^s(x, Q^2) = \left( \frac{\alpha}{2\pi} \right)^2 \frac{L^2}{2} S_2(x), $$

(86)

where $L = \ln(Q^2/m^2)$, and taking into account in $\alpha(Q^2)$ only the one-loop correction coming from vacuum polarisation by electrons

$$ \alpha(Q^2) = \alpha + \frac{\alpha^2}{3\pi} L, $$

(87)

we have

$$ V_1(x) = P_e^v(x) = \frac{1 + x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x), $$

(88)

$$ V_2(x) = \frac{2}{3} P_e^v(x) + \int_x^1 \frac{dz}{z} P_e^v \left( \frac{x}{z} \right) P_e^v(z) $$

$$ = \frac{2}{3} P_e^v(x) + 8 \left[ \frac{\ln(1-x)}{(1-x)_+} + \frac{1 + 4x + x^2}{(1-x)_+} \right] (1-x)_+ $$

$$ + \frac{1 + x^2}{(1-x)_+} \left[ 1 + \frac{3x^2}{4} \right] \ln x $$

$$ - 4(1 + x) \ln(1-x) - \left( \frac{2\pi^2}{3} - \frac{9}{4} \right) \delta(1-x), $$

(89)

and

$$ S_2(x) = \int_x^1 \frac{dz}{z} P_e^\gamma \left( \frac{x}{z} \right) P_e^\gamma(z) $$

$$ = 2(1 + x) \ln x + 1 + x + \frac{4(1 - x^3)}{3x}, $$

(90)
where the generalized function \( \left( \frac{\ln(1-z)}{(1-z)} \right)^+ \) are defined by the relation analogous to (77)
\[
\int_0^1 \left( \frac{\ln(1-z)}{(1-z)} \right)^+ f(z) \, dz = \int_0^1 \left( \frac{\ln(1-z)}{(1-z)} \right) (f(z) - f(1)) \, dz .
\]  
(91)

The coefficients of the delta-function terms in (88), (89) are determined by the requirements
\[
\int_0^1 dx \, V_i(x) = 0
\]  
(92)

following from the charge conservation condition (84).

Writing the cross section (15) at \( Q^2 \gg m^2 \), \( F_2(x, Q^2) = 2x F_1(x, Q^2) \) as
\[
\frac{d\sigma}{dQ^2 dx} = \frac{4\pi (\alpha(Q^2))}{Q^4} \frac{F_2(x, Q^2)}{x} R(x, Q^2) ,
\]  
(93)

where \( R(x, Q^2) \) is given by (23), we have for the radiative correction \( \delta^r \) in the leading logarithmic approximation
\[
\delta^r_{LLA} = \int_{x_0}^1 dx \frac{R(x, Q^2)}{R(1, Q^2)} (f_{e}^{1e}(x, Q^2) + 2 f_{e}^{2e}(x, Q^2)) - 1 ,
\]  
(94)

This representation permits to find the radiative correction \( \delta^r \) in any order of perturbation theory.

With the two-loop accuracy \( f_{e}^{1e}(x, Q^2) \) and \( f_{e}^{2e}(x, Q^2) \) are given by Eqs. (85) and (86) respectively. Using (23) and (84), we have
\[
\delta^r_{LLA} = - \int_0^{x_0} dx \frac{R(x, Q^2)}{R(1, Q^2)} \left( f_{e}^{1e}(x, Q^2) + \frac{2}{3} f_{e}^{2e}(x, Q^2) \right)
\]  
+ \int_{x_0}^1 dx \left[ \left( \frac{R(x, Q^2)}{R(1, Q^2)} - 1 \right) f_{e}^{1e}(x, Q^2)
\]  
+ \frac{2}{3} \frac{R(x, Q^2)}{R(1, Q^2)} f_{e}^{2e}(x, Q^2) \right] .
\]  
(95)

In the one-loop approximation only \( f_{e}^{2e}(x, Q^2) \) does contribute. Simple integration gives
\[
\delta^r_{one-loop.LLA} = \frac{\alpha}{2\pi} L \left\{ 2\ln(1-x_0) + x_0 + \frac{x_0^2}{2} \right\}
\]  
+ \frac{Q^2}{2(p_l^2)} \frac{Q^2 G_M^2 + 4m^2 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \left[ \ln x_0 - \frac{1 - x_0^2}{2} \right]
\]  
+ \frac{Q^2}{8(p_l^2)^2} \frac{Q^2 (2M^2 + Q^2) G_M^2 - 8m^4 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \left[ - \ln x_0
\]  
+ (1 - x_0) \left( \frac{1}{x_0} + \frac{3}{2} + \frac{x_0}{2} \right) \right\} ,
\]  
(96)
in accordance with (68).

The two-loop correction contains contributions of both \( f_{e}^{1e} \) and \( f_{e}^{2e} \). Using (85)–(90) and (23), one obtains from (20) the two-loop contribution in the form
\[
\delta^e_{two-loop.LLA} = \left( \frac{\alpha}{2\pi} \right)^2 \frac{L^2}{2} \left\{ - \int_0^{x_0} dx \, V_2(x)
\right\}
\]  
+ \frac{Q^2 G_M^2 + 4m^2 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \int_{x_0}^1 dx \left[ 2S_2(x)
\right]
\]  
- \frac{Q^2}{2(p_l^2)} \left( \frac{1 - x}{x} V_2(x) + \frac{2}{x} S_2(x) \right]
\]  
+ \frac{Q^2}{8(p_l^2)^2} \frac{Q^2 (2M^2 + Q^2) G_M^2 - 8m^4 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \times \int_{x_0}^1 dx \left( \frac{1 - x^2}{x^2} V_2(x) + \frac{2}{x^2} S_2(x) \right) .
\]  
(97)

so that (see Appendix C)
\[
\delta^e_{two-loop.LLA} = \left( \frac{\alpha}{2\pi} \right)^2 \frac{L^2}{2} \left\{ \chi_0(x_0) - \frac{Q^2}{2(p_l^2)} \chi_2(x_0) \right\}
\]  
+ \frac{Q^2 G_M^2 + 4m^2 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \left[ \chi_1(x_0) - \frac{Q^2}{2(p_l^2)} \chi_2(x_0) \right]
\]  
+ \frac{Q^2}{8(p_l^2)^2} \frac{Q^2 (2M^2 + Q^2) G_M^2 - 8m^4 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \chi_3(x_0) ,
\]  
(98)

where
\[
\chi_0(x_0) = -4\text{Li}_2(x_0)
\]  
+ \frac{4\ln(1 - x_0)}{x_0} \left( 3 - 2x_0 + 2x_0^2 \right) \ln(1 - x_0)
\]  
- 3x_0 \left( 1 + \frac{x_0}{2} \right) \ln x_0 + \frac{8}{3} x_0 \frac{7}{12} x_0 ,
\]  
(99)
then
\[
\chi_1(x_0) = -2 \left( \frac{4}{3} + 2x_0 + x_0^2 \right) \ln x_0
\]  
- \frac{2}{9} (1 - x_0) (22 + 13x_0 + 4x_0^2) ,
\]  
(100)

and
\[
\chi_2(x_0) = 4\text{Li}_2(x_0) - \frac{2}{3} \pi^2 - \frac{3}{2} \text{Li}_2(x_0)
\]  
+ \frac{2}{3} \text{Li}_2(x_0) (1 - x_0)^2
\]  
- \left( \frac{11}{3} + 4x_0 \right) x_0 \frac{5}{4} x_0 ,
\]  
(101)
and finally

\begin{align}
\chi_3(x_-) = 4L_2(x_-) - \frac{2}{3} \pi^2 - \frac{3}{2} \ln^2 x_- & \nonumber \\
+ 2(1 - x_-^2) \left( \frac{2}{x_-} + 1 \right) \ln(1 - x_-) & \\
+ \left( \frac{3}{x_-} + \frac{1}{3} + 3x_- + \frac{3}{2} x_-^2 \right) \ln x_- & \\
+ \left(1 - x_- \right) \left( \frac{4}{3x_-^2} + \frac{8}{x_-} + \frac{7}{12} \left(1 + x_- \right) \right). \quad (102)
\end{align}

Note that at small momentum transfer, i.e. at small \(x_\)−, the valence quark contribution is suppressed as well as in the one loop due to the charge conservation requirement (92). It is not so for the sea quark contribution [24]. The sea quark distribution is singular at \(x = 0\) and the lower limit of the integration in (C.4) can not be taken equal to 0. Therefore the two-loop correction is not suppressed at small momentum transfer for experimental conditions at which production of electron-positron pairs is not forbidden. For such conditions we have at \(x_- \ll 1\)

\begin{equation}
\delta_\text{two-loop, LLA} = \frac{\alpha}{2\pi} \left[ \frac{L_2^2}{2} - \frac{8}{3} \ln x_- - \frac{44}{9} \right]. \quad (103)
\end{equation}

At first glance it seems that more preferable are the conditions at which production of electron-positron pairs is forbidden. In this case \(\delta_\varepsilon^0\) must be omitted in Eq. (95), and the term with \(P_\varepsilon^e(x)\) must be omitted in its expression for \(V_2(x)\) in (89). However, this is not the whole truth. The term with \(P_\varepsilon^e(x)\) in \(V_2(x)\) meets contributions from not only real, but also virtual pairs, which can not be suppressed, and therefore their contribution must be restored. It means that the term

\begin{equation}
\left( \frac{\alpha}{\pi} \right)^2 \left[ - \frac{1}{36} L^3 + \frac{19}{72} L^2 \right] \delta(1 - x) \quad (104)
\end{equation}

must be added to \(f_\varepsilon^e\) (see for details [32]). Therefore, in this case

\begin{align}
\delta_\varepsilon^0_{\text{two-loop, LLA}} &= \left( \frac{\alpha}{2\pi} \right)^2 \frac{L_2^2}{2} \left[ \chi_0(x_-) \right. \\
&\left. - \frac{Q^2}{2(p_l)^2} \frac{Q^2 G_E^2 + 4M^2 G_M^2}{(4M^2 + Q^2) R(1, Q^2)} \chi_2(x_-) \right] \\
&\left. + \frac{Q^2}{8(p_l)^2} \frac{Q^2 (2M^2 + Q^2) G_M^2 - 8M^4 G_E^2}{(4M^2 + Q^2) R(1, Q^2)} \chi_3(x_-) \right]. \quad (105)
\end{align}

Fig. 1 The correction \(\delta_\varepsilon\) obtained in the one-loop approximation Eq. (68) and taking into account the two-loop correction within the leading logarithmic approximation Eq. (98)

where

\begin{align}
\chi_0'(x_-) &= -\frac{2}{9} L + \frac{19}{9} \\
&- 4L_2(x_-) + 4 \ln(1 - x_-) \frac{(1 - x_-)}{x_-} \\
&+ 2x_- (2 + x_-) \ln(1 - x_-) \\
&- 3x_- \left( 1 + \frac{x_-}{2} \right) \ln x_- + 2x_- + \frac{1}{4} x_-^2. \quad (106)
\end{align}

then

\begin{align}
\chi_2' &= 4L_2(x_-) - \frac{2}{3} \pi^2 \\
&+ \frac{1}{2} \ln^2 x_- + 2(1 - x_-^2) \ln(1 - x_-) \\
&- \left(1 - \frac{3}{2} x_-^2 \right) \ln x_- + \frac{1}{4} (1 - x_-)(9 + x_-) \quad (107)
\end{align}

and

\begin{align}
\chi_3'(x_-) &= 4L_2(x_-) - \frac{2}{3} \pi^2 + \frac{1}{2} \ln^2 x_- \\
&+ 2(1 - x_-^2) \left( \frac{2}{x_-} + 1 \right) \ln(1 - x_-) \\
&- \left( \frac{1}{x_-} + 1 - 3x_- - \frac{3}{2} x_-^2 \right) \ln x_- \\
&+ \frac{1}{4} (1 - x_-)(9 + x_-). \quad (108)
\end{align}

The relative magnitude of the corrections (98) and (105) depends on energy and momentum transfer.

On the Fig. 1 we present numerical results for the correction \(\delta_\varepsilon\) in the conditions of the experiment suggested by A.A. Vorobyev [14].
6 Conclusion

As it was shown in [24], the setting of the experiment with recoil proton detection suggested by A.A. Vorobyev [14] for measurement of proton radius, has an interesting feature – cancellation of main radiative corrections. Here we calculated radiative corrections to the cross section of electron-proton scattering for experiments of this kind in a wide range of kinematic parameters using the method of structure functions and parton distributions. We calculated the one-loop corrections due to electron interaction for momentum transfer \( Q \) limited only by the requirement \( Q \gg m, m \) being the electron mass, and proved that when at small \( Q \) the cancellation of the virtual and real radiative corrections has a power accuracy.

In the two-loop approximation we calculated these corrections with logarithmic accuracy, again for momentum transfer \( Q \) limited only by the requirement \( Q \gg m \), using the parton distribution method [27–30] developed for the theoretical description of the deep inelastic electron-proton scattering and adopted in [32] for calculation of radiative corrections in QED. We calculated the radiation corrections both for such an experiment setup when the production of additional electron-positron pairs is allowed, and for such when it is forbidden.

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Appendix A

The vacuum polarisation \( \mathcal{P}(q^2) \) contains lepton (electron, muon, \( \tau \)-lepton) and hadron contributions:

\[
\mathcal{P}(q^2) = \mathcal{P}_e(q^2) + \mathcal{P}_\mu(q^2) + \mathcal{P}_\tau(q^2) + \mathcal{P}_h(q^2). \tag{A.1}
\]

One-loop lepton contribution \( \mathcal{P}_l^{(1)}(q^2) \), \( l = e, \mu, \tau \) is well known (see, for example, [33]):

\[
\mathcal{P}_l^{(1)}(q^2) = \frac{\alpha}{\pi} \left( \frac{1}{3} \sqrt{1 - \frac{4m_l^2}{q^2}} \left( 1 + \frac{2m_l^2}{q^2} \right) \right) \times \ln \left( \frac{\sqrt{1 - \frac{4m_l^2}{q^2}} + 1}{\sqrt{1 - \frac{4m_l^2}{q^2}} - 1} \right) - \frac{4m_l^2}{3q^2} - \frac{5}{9}. \tag{A.2}
\]

At \( Q^2 = -q^2 \gg 4m_l^2 \)

\[
\mathcal{P}_l^{(1)}(q^2) = \frac{\alpha}{3\pi} \left( \ln \left( \frac{Q^2}{m_l^2} \right) - \frac{5}{3} \right), \tag{A.3}
\]

and at \( Q^2 = -q^2 \ll 4m_l^2 \)

\[
\mathcal{P}_l(q^2) = \frac{\alpha}{15\pi} \frac{Q^2}{m_l^2}. \tag{A.4}
\]

The lepton contributions are known also in higher orders of perturbation theory (see, for example, [39, 40]). For us it is enough to know that the two-loop contribution contains only the first degree of \( \ln \left( \frac{Q^2}{m_l^2} \right) \).

The hadron contribution \( \mathcal{P}_h(q^2) \) is expressed in terms of the total cross section of one-photon electron-positron pair annihilation into hadrons

\[
\mathcal{P}_h(q^2) = \frac{q^2}{4\pi^2\alpha} \int_{4m_h^2}^{\infty} ds \frac{\sigma(e^+ e^- \to \text{hadrons}(s))}{s - q^2}. \tag{A.5}
\]

This contribution is small compared with \( \alpha/\pi \) at \( Q^2 < 4m_h^2 \), becomes of order of \( \alpha/\pi \) only at \( Q^2 \sim 4m_h^2 \) and then grows logarithmically with \( Q^2 \). Recent review is given in [40].

Appendix B

To find the contribution \( \delta'_{\text{hard}} \) of the one-photon emission with \( \kappa' > \kappa_0 \) to the radiative correction \( \delta' \) at \( Q^2 >> m^2 \) it is convenient to introduce the intermediate scale \( \kappa_1 \) such that \( Q^2 >> \kappa_1 >> m^2 \). In the region \( \kappa_1 > \kappa' > \kappa_0 \) one can put

\[
W_g = -\frac{\alpha}{2\pi} \left[ \frac{Q^2}{\kappa'} \ln \left( \frac{Q^4}{m^2(m^2 + 2\kappa')} \right) - 2 \right] + \frac{Q^2\kappa'}{(m^2 + 2\kappa')^2}, \tag{B.1}
\]

\[
W_i = 0, \tag{B.2}
\]

so that

\[
F_1 = -\frac{1}{2} W_g, \quad F_2 = 2F_1. \tag{B.3}
\]

Therefore in this region we have from Eqs. (22)–(24)

\[
\frac{d\sigma'}{dQ^2 dx} = -W_g \frac{d\sigma_B}{dQ^2}. \tag{B.4}
\]

Using that in this region it is possible to put \( \kappa' = Q^2(1-x)/2 \), it is easy to obtain the part \( \delta_{\text{hard}}^{(1)} \) of the correction \( \delta_{\text{hard}} \), defined by Eqs. (21) and (95), from this region:

\[
\delta_{\text{hard}}^{(1)} = \frac{\alpha}{\pi} \int_{\kappa_0}^{\kappa_1} \frac{d\kappa'}{\kappa'} \left[ 2 \left( \ln \left( \frac{Q^2}{m^2} \right) - 1 \right) \right].
\]
Using (67), we obtain
\[
\delta_{\text{vert}}^e + \delta_{\text{soft}} + \delta_{\text{hard}}^{(1)} = \frac{\alpha}{\pi} 2 \ln \left( \frac{2 \kappa_1}{\kappa_0} \right) \left( \ln \left( \frac{Q^2}{m^2} \right) - 1 \right) - \frac{1}{2} \ln^2 \left( \frac{2 \kappa_1}{m^2} \right) + \frac{1}{4} \left( \ln \left( \frac{2 \kappa_1}{m^2} \right) - 1 \right) + \frac{\pi^2}{6}.
\]

(B.5)

In the region \( \kappa_{\text{max}} > \kappa' > \kappa_1 \), i.e. \( 1 - \frac{2 \kappa_1}{Q^2} > x > x_- \) at \( Q^2 \gg m^2 \) one can put
\[
W_i = -\frac{\alpha Q^2}{2 \pi} \left[ \frac{1 + x^2}{1 - x} \ln \left( \frac{Q^2}{m^2 x (1 - x)} \right) + \frac{1}{2} \ln \frac{x}{2 (1 - x)} \right],
\]

(B.7)

and
\[
W_i = \frac{\alpha Q^2}{2 \pi} \frac{x}{4 x},
\]

(B.8)

so that
\[
F_1 = \frac{1}{2} \left( \frac{4 x^2 W_i}{Q^2} - W_s \right),
\]

(B.9)

\[
F_2 = x \left( \frac{12 x^2}{Q^2} W_i - W_s \right) = 2 x F_1 + \frac{\alpha x^2}{\pi}.
\]

(B.10)

As it is seen, the Callan-Gross relation [35] is violated in this region. It could be expected, since this relation is valid only in the collinear approximation.

Using Eqs. (22)–(24), we have for the part \( \delta_{\text{hard}}^{(2)} \) of the correction \( \delta_{\text{hard}}^e \) from this region:
\[
\delta_{\text{hard}}^{(2)} = \frac{\alpha}{\pi} \int_{x_-}^{1} \frac{dx}{x} \left[ F_2(x, Q^2) \left( x \frac{Q^2}{Q^2} \right) - \frac{Q^2}{2 (pl)} \frac{Q^2 G^2_{M} + 4 M^2 G^2_{E}}{(4 M^2 + Q^2) R(1, Q^2)} \right. \\
\left. - \frac{x^2}{8 (pl)} \frac{Q^2}{2 (pl)} + 2 M^2 G^2_{M} - 8 M^4 G^2_{E}}{(4 M^2 + Q^2) R(1, Q^2)} \right]
\]

(B.11)

Note that in the integral (B.11) the upper limit can be set equal to 1 in all terms except the first one. It gives
\[
\delta_{\text{hard}}^{(2)} = \frac{\alpha}{\pi} \left[ 3 x \left( \ln \left( \frac{Q^2}{m^2} \right) - 4 \ln \left( \frac{2 \kappa_1}{m^2} \right) ight) - \frac{5 + x + \frac{x^2}{2} + 2 \ln(1 - x)}{2} \right] \ln \left( \frac{Q^2}{m^2} \right)
\]

(B.12)

with \( \phi_i \) defined in Eqs. (69)–(72).

The intermediate parameter \( \kappa_1 \) disappears in the sum
\[
\delta_{\text{hard}}^e = \delta_{\text{hard}}^{(2)} + \delta_{\text{hard}}^{(1)}.
\]
Appendix C

Straightforward integration gives
\[\int_{x_-}^{1} dx \frac{V_2(x)}{x} = 4L_2(x_-) - \frac{2\pi^2}{3}\]
\[-4 \ln(1-x_-) \ln \left(1-x_-\right)\]
\[-\left(\frac{4}{3} + 4x_- + 2x_-^2\right) \ln(1-x_-)\]
\[+ 3x_- \ln x_- - \frac{8}{3} x_-^3 - \frac{7}{12} x_-^2\]
\[\int_{x_-}^{1} dx \frac{V_2(x)}{x} = 4L_2(x_-) - \frac{2\pi^2}{3}\]
\[+ \frac{1}{2} \ln^2 x_- + 2(1-x_-^2) \ln(1-x_-)\]
\[-\left(\frac{5}{3} - \frac{3}{2} x_-^2\right) \ln x_-\]
\[+ \frac{1}{12} (1-x_-)(31 + 7x_-)\]
\[\int_{x_-}^{1} dx \frac{V_2(x)}{x} = 4L_2(x_-) - \frac{2\pi^2}{3}\]
\[+ \frac{1}{2} \ln^2 x_-\]
\[+ 2(1-x_-^2) \left(\frac{2}{x_-} + 1\right) \ln(1-x_-)\]
\[-\left(\frac{1}{x_-} + \frac{5}{3} - 3x_- - \frac{3}{2} x_-^2\right) \ln x_-\]
\[+ (1-x_-)(\frac{2}{3x_-} + \frac{13}{4} + \frac{7}{12} x_-)\]
\[\int_{x_-}^{1} dx S_2(x) = -\left(\frac{4}{3} + 2x_- + x_-^2\right) \ln x_-\]
\[-\frac{1}{9} (1-x_-)(22 + 13x_- + 4x_-^2)\]
\[\int_{x_-}^{1} dx \frac{S_2(x)}{x} = -\ln^2 x_- - (2x_- + 1) \ln x_-\]
\[+ \left(\frac{1}{3} x_-\right) \left(\frac{4}{x_-} - 11 - 2x_-\right)\]
\[\int_{x_-}^{1} dx \frac{S_2(x)}{x^2} = -\ln^2 x_- + \left(\frac{2}{x_-} + 1\right) \ln x_-\]
\[+ \left(\frac{1}{3} x_-\right) \left(\frac{2}{x_-^2} + 11 - \frac{4}{x_-}\right)\]

References

1. A. Beyer et al., Science 358(6359), 79 (2017)
2. N. Bezninov et al., Science 365(6457), 1007 (2019)
3. W. Xiong et al., Nature 575(7781), 147 (2019)
4. R. Pohl et al., Nature 466, 213 (2010)
5. A. Antognini et al., Science 339, 417 (2013)
6. P. J. Mohr, B. N. Taylor and D. B. Newell, Rev. Mod. Phys. 80 (2008) 633, arXiv:0801.0028 [physics.atom-ph]
7. R. Pohl, R. Gilman, G. A. Miller and K. Pachucki, Ann. Rev. Nucl. Part. Sci. 63 (2013) 175, arXiv:1301.0905 [physics.atom-ph]
8. J. C. Bernauer, in 34th International Symposium on Physics in Collision (PIC 2014), Sep 16-20 2014, Bloomington, Indiana, USA. arXiv:1411.3743 [nucl-ex]
9. C.E. Carlson, Prog. Part. Nucl. Phys. 82, 59 (2015).
10. T. Aoyama, and others, Phys. Rept. 887 (2020) 1, arXiv:2006.04822 [hep-ph]