Steady-state performance analysis of the recursive maximum
correntropy algorithm and its application in adaptive beamforming
with alpha-stable noise

Lu Lu³, Haiquan Zhao³*

a) School of Electrical Engineering, Southwest Jiaotong University, Chengdu, China.
E-mail: lulu@my.swjtu.edu.cn, hqzhao_swjtu@163.com (Corresponding author)

Abstract-As a well-established adaptation criterion, the maximum correntropy criterion (MCC) has received increased
attention due to its robustness against outliers. In this paper, a new complex recursive maximum correntropy (CRMC)
algorithm without any priori information on the noise characteristics, is proposed under the MCC. We first study the
steady-state excess mean-square-error (EMSE) behavior of the CRMC algorithm by using energy conservation relation
and some reasonable approximations. Then, the proposed algorithm is introduced to adaptive beamforming problem,
where the desired signal is contaminated by the impulsive noises. The results obtained from simulation study establish
the effectiveness of this new beamformer.

Keywords: Maximum correntropy, Recursive algorithm, Steady-state performance analysis, Adaptive beamforming,
Impulsive noise.

1 Introduction

Adaptive beamforming has been widely applied in various wireless applications, for instance, radar [1],
communication system [2], and sonar [3]. It refers to simultaneously combine the signals from the elements of an array
antenna, to suppress interference and provide target detection. Because of its low complexity, good convergence
properties and satisfactory performance, the complex least-mean-square (LMS)¹ algorithm has become one of the most
popular adaptive beamforming algorithm. Furthermore, the application of the LMS algorithm to the adaptive
beamforming and its analysis have been extensively studied [4,5]. However, it may become unstable, when the desired
signal is corrupted by impulsive noise.

¹ To avoid confusion to the constrained least-mean-square algorithm, we use ‘LMS’ to denote the complex least-mean-square
algorithm throughout this paper.
Many improved algorithms aimed at increasing the convergence speed of the LMS algorithm based on mean square error (MSE) criterion have been presented, such as continuous mixed $p$-norm (CMPN) algorithm [6], least mean square-least mean square (LMS-LMS) algorithm [7] and shrinkage LMS algorithm [8]. These algorithms achieve improved performance for Gaussian signals. However, in practical environments, there are often impulsive noises, in which may have heavy tails and even may not possess finite second-order statistics. Based on [9], these noises can be modelled by $\alpha$-stable distribution in adaptive beamforming. In such case, the abovementioned MSE-based algorithms may fail to work.

Recently, many robust beamforming techniques have been proposed for non-Gaussian signals [10-16]. Particularly, in [12], Jiang et al. proposed the minimum dispersion distortionless response (MDDR) beamforming that is robust to non-Gaussian signals. By solving the $l_p$-norm constrained optimization problem, such algorithm achieves improved performance as compared with minimum variance distortionless response (MVDR) beamformer and subspace method. After that, they further proposed the $l_1$-regularized minimum absolute distortionless response ($l_1$-MADR) beamformer [13], which is derived by minimizing the expectation of the modulus of the output and can also obtain robust performance in $\alpha$-stable noise. Note that the abovementioned methods can be regarded as a generalization form of MSE criterion. For non-Gaussian signals, it may be unreasonable to use MSE criterion for adaptive beamformer.

As a local similarity measure, the maximum correntropy criterion (MCC) has a close relationship with M-estimation, and insensitive to outliers [17]. Due to its simplicity and robustness, the MCC has been widely used in the field of signal processing [18-19]. Very recently, Peng and Wang extend MCC to the constraint algorithm and the MUSIC method, respectively [20-21]. We observe the improved performance in these algorithm and we attempt to investigate whether MCC can be adopted to develop an efficient recursive algorithm for adaptive beamforming with $\alpha$-stable noise. In this paper, a new complex recursive algorithm, called CRMC algorithm, is proposed by recursively minimizing the MCC function. Moreover, we first analyze the steady-state performance of the CRMC algorithm. Note that the proposed algorithm is useful for adaptive beamforming, especially when the signals contain large outliers or contaminated by impulsive noises. The superior performance of CRMC algorithm is confirmed by simulation results about adaptive beamforming in $\alpha$-stable noise environments.

2 Array data model

Consider the spaced linear array of $M$ sensors receiving a signal $x(n)$ with known centre frequency of the narrowband signal $\omega$ and direction-of-arrival (DOA) $\theta$. The array measurement vector $x(n)$ is expressed as [12]

$$x(n) = as(n) + \sum_{i=1}^{J} s_i(n)a_i + \varepsilon(n)$$ (1)
where \( s(n) \) is signal of interest (SOI), \( \{s_i(n)\}_{i=1}^I \) denotes the \( I \) interferences, \( a \) is the steering vectors of the SOI, \( \{a_i\}_{i=1}^I \) is the steering vectors of the interferences, and the \( M \times 1 \) vector \( \mathbf{v}(n) \) is the additive noise vector. For a uniform linear array (ULA), \( a \) is given by \( a(\theta) = [1, e^{j\tau_1}, \ldots, e^{j\tau_{M-1}}]^T \) [12], where \( \tau_i \) denotes time delay of the \( i \)th sensor relative to the first sensor, \( j = \sqrt{-1} \) is the imaginary unit.

When the noise is impulsive, it is often modeled as the symmetric \( \alpha \)-stable distribution which can be described by the characteristic function as follows [9]

\[
\varphi(t) = \exp\left\{-|t|^\alpha\right\}
\]

where \( 0 < \alpha < 2 \) is the characteristic exponent. The smaller \( \alpha \) is, the more impulsive the process is. Assume that each element \( \mathbf{v}(n) = \mathbf{v}_{\text{Re}} + j\mathbf{v}_{\text{Im}} \) of \( \mathbf{v}(n) \) follows an isotropic distribution [22] described by the following characteristic function:

\[
\mathbb{E}\{\exp(j\theta_1\mathbf{v}_{\text{Re}} + j\theta_2\mathbf{v}_{\text{Im}})\} = \exp\left\{-2^{-\alpha/2}(\theta_1^2 + \theta_2^2)^{\alpha/2}\right\}
\]

where \( \mathbb{E}\{\cdot\} \) stands for taking expectation.

### 3 Derivation of CRMC algorithm

Let \( D \) and \( Y \) be two random variables with the same dimensions, the measure of correntropy is defined as follows [17]

\[
V(D, Y) = \mathbb{E}\{\kappa(D, Y)\} = \int \kappa(d, y) d\mathbb{R}_{D,Y}(d, y)
\]

where \( \kappa(\cdot, \cdot) \) is a shift-invariant Mercer Kernel, and \( \mathbb{R}_{D,Y}(d, y) \) denotes the joint distribution function of \( (d, y) \). The most popular kernel used in correntropy is the Gaussian kernel:

\[
\kappa(d, y) = \exp\left\{-\frac{|d - y|^2}{2\sigma^2}\right\}
\]

where \( e = d - y \), and \( \sigma \) stands for the kernel size of correntropy. Here, we introduce \( d \) to denote the desired signal, and \( y \) denotes array outputs\(^2\). Then, the cost function of CRMC can be expressed as follows [17]

\[
J_{\text{CRMC}}(n) = \sum_{i=1}^{n} 2^{e-i} \kappa(d(i), y(i))
\]

\[
= \sum_{i=1}^{n} 2^{e-i} \exp\left\{-\frac{|e(i)|^2}{2\sigma^2}\right\}
\]

\(^2\) Throughout this paper, \( d(i) \) stands for desired signals \( d \) at iteration \( i \), \( y(i) \) denotes array outputs \( y \) at iteration \( i \), respectively.
where \( 0 < \lambda < 1 \) is the forgetting factor. Taking the gradient of \( J_{CRMC}(n) \) with respect to the array coefficients \( w(n) \), we obtain

\[
\frac{\partial J_{CRMC}(n)}{\partial w(n)} = \sum_{i=1}^{n} \lambda^{-i} \exp\left(\frac{|e(i)|^2}{2\sigma^2}\right) e(i)x(i). \tag{7}
\]

Letting (7) be zero, one gets

\[
\sum_{i=1}^{n} \lambda^{-i} \psi(i)x(i)x^H(i)w(n) = \sum_{i=1}^{n} \lambda^{-i} \psi(i)x(i)d(i) \tag{8}
\]

where superscript \( H \) denotes Hermitian operator (conjugate transpose). This differs from the standard solution for \( l_2 \) norm in the presence of weighting factors \( \psi(i) \) given by

\[
\psi(i) = \exp\left(\frac{|e(i)|^2}{2\sigma^2}\right). \tag{9}
\]

Then, the expression of \( w(n) \) is obtained as follows:

\[
w(n) = F(n)\pi(n) = R^{-1}(n)\pi(n) \tag{10}
\]

where \( F(n) = R^{-1}(n) \), \( R(n) = \sum_{i=1}^{n} \lambda^{-i} \psi(i)x(i)x^H(i) \) and \( \pi(n) = \sum_{i=1}^{n} \lambda^{-i} \psi(i)x(i)d(i) \). For \( \psi(i) = 1 \), the algorithm becomes the recursive least squares (RLS) algorithm. When \( \psi(n) \neq 1 \), \( R(n) \) and \( \pi(n) \) are themselves functions of the optimal weights via \( \psi(n) \). It has to recalculate (10) in each iteration. To avoid this inconvenience, a sliding window method is proposed in [23]. However, the algorithm carries the main drawback of sliding-window strategy: the algorithm requires to keep in memory all previous samples within a window, making it not a truly online algorithm.

To further derive the truly online algorithm, \( R(n) \) and \( \pi(n) \) are updated by recursive expression as follows:

\[
R(n) \approx \lambda R(n-1) + \psi(n)x(n)x^H(n), \tag{11}
\]

\[
\pi(n) \approx \lambda \pi(n-1) + \psi(n)x(n)d(n). \tag{12}
\]

An important point in (11)-(12) need to be highlighted. Note from (10-12) that the adaptation is strikingly similar to the Wiener solution [24], which required matrix inverse operation. Hence, (10) has a heavy computational burden and is seldom used in practice. By using matrix inversion lemma [25], \( F(n) \) can be updated as

\[
F(n) = \lambda^{-1}F(n-1) - \lambda^{-1} \Phi(n)x^H(n)F(n-1) \tag{13}
\]

where \( F(n) = \rho^{-1}I \), \( \rho \) is a small positive number, and the gain factor is defined by

\[
\Phi(n) = \frac{\psi(n)F(n-1)x(n)}{\lambda + \psi(n)x^H(n)F(n-1)x(n)}. \tag{14}
\]
From (10), (12), (13) and (14), \( w(n) \) can be updated as

\[
w(n) = w(n-1) + \Phi(n) \left[ d(n) - x^H(n)w(n-1) \right].
\]

(15)

where * represents conjugate operation.

**Remark 1:** Note that (9), (14), and (15) define an implicit relationship between \( w(n) \) and \( \psi(n) \) that cannot be solved in one step. Hence, the algorithm forces an iterative approximation to the solution, where \( \psi(n) \) is calculated by using \( w(n-1) \), and the new value for \( w(n) \) is obtained via the value of \( \psi(n) \).

**Remark 2:** The proposed algorithm is nearly blind since it does not require any priori information on the noise characteristics, and it can be implemented using only \( \sigma \) and \( \lambda \).

### 4 Steady-state performance

In this section, we perform the steady-state analysis of the CRMC algorithm. First, some assumptions are given as

The desired response is produced by \( d(n) = w_o^H x(n) + v(n) \), where \( w_o \) is a vector containing the optimal coefficient values, and \( v(n) \) is the measurement noise. The input signal \( x(n) \) is independent and identically distributed (i.i.d.) with zero-mean and is approximately independent of the a priori excess errors \( e_x(n) = x^H(n)\Omega(n-1) \). The noise signal \( v(n) \) is i.i.d. with zero-mean and variance \( \sigma^2_v \). Moreover, \( v(n) \) and \( x(n) \) are mutually independent.

The weight deviation vector is defined as follows:

\[
\Omega(n) = w_o - w(n).
\]

(16)

Then, the update formulation of the weight deviation vector of the proposed algorithm can be expressed by using (16)

\[
\Omega(n) = \Omega(n-1) - \frac{\psi(n)F(n-1)x(n)}{\lambda + \psi(n)x^H(n)F(n-1)x(n)} \left[ d(n) - x^H(n)w(n-1) \right].
\]

(17)

Considering (13) and (14), and applying the matrix inversion formula, we have

\[
F^{-1}(n) = \lambda^{n-1}e^{1} + \sum_{i=0}^{n-1} \lambda^{n-i-1}\psi(i)x^H(i)x(i)
\]

(18)

Then, (17) can be rewritten as

\[
\Omega(n) = \Omega(n-1) - \kappa(n)F(n)\psi^H(n)e(n)
\]

(19)

where \( \kappa(n) = \frac{1}{q(n)} \). When the exponential term is expanded with a second-order Taylor series (SOTS), we obtain

\[
\kappa(n) = \frac{1}{q(n)} \cdot \frac{1}{q'(n)} = \frac{1}{q(n)} \cdot \frac{1}{q'(n)}.
\]

In steady-state, we define the a posteriori error \( e_p(n) = x^H(n)\Omega(n) \).

5
Multiplying both sides of (19) by \( x(i) \), we obtain the relationship between the \textit{a priori} and the \textit{a posteriori} estimation errors

\[
e_p(n) = e_a(n) - \| x(n) \|^2_{\nu(e(n))} e(n)
\]

(20)

When \( x(n) \neq 0 \), the energy conservation relation (ECR) expression for CRMC algorithm can be given as

\[
\Omega(n) + \kappa(n(F(n)x(n)) e_p(n) = \Omega(n-1) + \kappa(n) F(n)x(n) e_p(n)
\]

(21)

Combining (19) and (20), and using \( F^{-1}(n)\kappa^{-1}(n) \) as a weighting matrix for the squared-weighted Euclidean norm of a vector, we obtain

\[
\| \Omega(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 + \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}} = \| \Omega(n-1) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 + \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}}
\]

(22)

Suppose the CRMC algorithm converges. Therefore in the steady state when \( n \rightarrow \infty \) the optimum (minimum) excess MSE (EMSE) can be obtained. In this condition we can assume

\[
E \left\{ \| \Omega(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} \approx E \left\{ \| \Omega(n-1) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\}
\]

(23)

Taking expectations of both sides of (22), and substituting (23) into (22) results in

\[
E \left\{ \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}} \right\} = E \left\{ \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}} \right\} - 2E \left\{ e_a(n)e(n) \right\} + E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} e^2(n)
\]

(24)

Now, substituting (20) into (24) yields

\[
E \left\{ \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}} \right\} = E \left\{ \frac{|e_a(n)|^2}{\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}} \right\} - 2E \left\{ e_a(n)e(n) \right\} + E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} e^2(n)
\]

(25)

Using expansion (25) is simplified to

\[
E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} = 2 \Re \left\{ E \left\{ e_a(\infty)e(\infty) \right\} \right\}
\]

(26)

where \( \Re \{x\} \) denotes the real part of \( x \). Consider \( e(n) = e_a(n) + v(n) \), (26) can be expressed as

\[
\sigma^2 E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} + E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} e_a(\infty)^2 = 2E \left\{ e_a(\infty)^2 \right\}
\]

(27)

Assume that \( \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \) is independent of \( e_a(\infty) \) at steady-state, we have

\[
\sigma^2 E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} + E \left\{ \| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2 \right\} E \left\{ e_a(\infty)^2 \right\} = 2E \left\{ e_a(\infty)^2 \right\}
\]

(28)

Inserting (28) into (26), we obtain

\[
\zeta = \frac{\theta \sigma^2}{2 - \theta}
\]

(29)

where \( \zeta = E \left\{ e_a(n)^2 \right\} \) and \( \theta - E[\| x(n) \|_{\kappa^{-1}(n)F^{-1}(n)}^2] - E \left\{ (2\sigma^2)^{-1} \text{Tr}(e_a(\infty)^2) \text{Tr}(x(n)F^{-1}(n)) \right\} \)
Recalling from (18), the steady-state mean value of $F^{-1}(n)$ is obtained as $F^{-1} = \lim_{n \to \infty} E[F^{-1}(n)] = \frac{E[p(n)\mathbf{R}(n)]}{1-\lambda}$. The notation $\mathbf{R}(n)$ stands for the covariance matrix $\mathbf{R}(n) = E[\mathbf{x}(n)\mathbf{x}^H(n)]$. According to the approximation in [25] and using the SOTS, the following approximation can be made:

$$E[F(x)] \approx E[F^{-1}(x)]^{-1} = \frac{(1-\lambda)\mathbf{R}^{-1}(x)}{E[\exp\left(\frac{\mathcal{e}(x)}{2\sigma^2}\right)]} \approx \frac{E[(2\sigma^2 - \mathcal{e}^2(x))^{-1}(1-\lambda)\mathbf{R}(x)]}{(2\sigma^2)}$$

(30)

Note that, $|\mathcal{e}(n)| \ll \mathcal{v}(n)$ at steady-state for $0 < \lambda < 1$. Thus, substituting (30) and into (29), we have

$$\mathcal{e} = \frac{\sigma^2(1-\lambda)ME}{2 - (1-\lambda)ME} \left[ \frac{1}{\lambda E[(2\sigma^2 - \mathcal{v}(n)^2)^{-1}(2\sigma^2 - \mathcal{v}(n)^2) + (1-\lambda)M]} \right]$$

(31)

Finally, we perform simulation verification for the EMSE formulas given by (31) in context of system identification for the CRMC algorithm. The input signal was a white Gaussian signal with zero mean and unit variance. Table 1 compares the theory EMSE from with simulation for different parameter settings. As can be seen, in all cases the theory EMSE agrees with the simulation results.

| Noise power | Kernel size $\sigma$ | Filter order $M$ | Eq. (31) | Simulation |
|-------------|---------------------|-----------------|----------|------------|
| 0.01        | 0.5                 | 16              | -30.9684dB | -30.9660dB |
| 0.01        | 1                   | 16              | -30.9715dB | -30.0888dB |
| 0.001       | 2                   | 8               | -43.9352dB | -45.2772dB |
| 0.1         | 4                   | 32              | -18.1627dB | -18.0696dB |

5 Simulation results

To evaluate the performance of the proposed algorithm, the simulations are performed in the context of adaptive beamforming. The signal-to-noise-plus-interference-ratio (SINR) $\text{SINR}(n) = \frac{\sigma_s^2 |\mathbf{w}^H(n)a|^2}{\mathbf{w}^H(n)\Sigma\mathbf{w}(n)}$ is employed to quantify the performance [16], where $\sigma_s^2$ denotes the power of the SOI and $\Sigma = \mathbf{R}(n) - \sigma_s^2\mathbf{a}\mathbf{a}^H$. Note that the $\alpha$-stable noise is excluded in computing the SINR [16]. We present the simulation results to verify the effectiveness of the proposed algorithm in comparison with the LMS, CMPN, and the RLS algorithm\(^3\). For the simulations, the following conditions are considered:

- A ULA with $M=16$ sensors is equally spaced by half-wavelength.
- The noise follows the isotropic stable distribution with $\alpha=1.4$ or $\alpha=1.6$.

\(^3\) The MVDR approach, based on the minimum variance (MV) criterion, is belongs to other types of algorithms. Hence, we decided to compare to the LMS, CMPN, and the RLS algorithm in this section.
A desired quadrature phase-shift keying (QPSK) arrives at an angle of 15°, and the QPSK interference signal arrives at 7° and 23° with the same amplitude as the desired signal (8 degree difference).

All the beamformers are shown for the 1000th iteration. In the LMS algorithm, $\mu=0.0003$ is selected, and $\mu=0.001$ is selected for CMPN algorithm. The forgetting factors of the RLS and CRMC algorithms are selected as 0.999 and 0.995, respectively. These selections guarantee the fast and stable convergence of the RLS and CRMC algorithms.

Fig. 1 illustrates the resultant beampatterns and SINRs for different kernel sizes. As can be seen, choosing $\sigma=0.1$ will only lead to a poor performance. In this case, the kernel size cannot be chosen arbitrarily small. Hence, we fix $\sigma=4$ in following simulations. Fig. 2 shows a SINR comparison for QPSK signals. It can be observed that the CRMC beamforming leads to an improved performance compared with the other beamformers. Fig. 3 illustrates the simulation results for beampatterns. It can be seen that the LMS and CMPN algorithms fail to work in this case, while
the RLS and proposed algorithms have the stable performance. An important point in Fig. 3 need to be highlighted. Note from the QPSK interference signal at $23^\circ$ that the suppression for CRMC algorithm is stronger than that of RLS algorithm. The validity of this confirms the fact that MCC estimation is much more robust against outliers than MSE estimation [17]. To test the effect of $\alpha$-stable noise on beamformer, Fig. 4 illustrates the beampatterns with $\alpha=1.6$. Again, the proposed beamformer is superior to the LMS, RLS and CMPN method.

The computational complexity of the CRMC algorithm is compared with that of the LMS, RLS, and the CMPN algorithms in terms of the total number of multiplications, additions, other operations and computational time per recursion, as shown in Table 2. The increase in the complexity of the proposed algorithm compared with that of the RLS algorithm is moderate, and still with an affordable computation time.

![Fig. 3](image1.png)

**Fig. 3** The beampatterns achieved with the algorithms when the reference signal is contaminated by $\alpha$-stable noise ($\alpha=1.4$).

![Fig. 4](image2.png)

**Fig. 4** The beampatterns achieved with the algorithms when the reference signal is contaminated by $\alpha$-stable noise ($\alpha=1.6$).

**Table 2** Computational complexity of the algorithms.

| Algorithms | Mul.  | Add. | Other operations | Computational time(sec) |
|------------|------|------|------------------|------------------------|
| LMS        | $2M+1$ | $2M$ | 0                | 0.000106               |

---

The computational complexity of the CRMC algorithm is compared with that of the LMS, RLS, and the CMPN algorithms in terms of the total number of multiplications, additions, other operations and computational time per recursion, as shown in Table 2. The increase in the complexity of the proposed algorithm compared with that of the RLS algorithm is moderate, and still with an affordable computation time.
### 6 Conclusion

Based on the MCC, a new CRMC algorithm, not requiring any *a priori* information, is proposed along with a Gaussian kernel for solving the adaptive beamforming problem. The MCC, which has been proven to be an efficient and robust optimization criterion for outliers, is used to improve the performance of beamformer. In addition, we study the steady-state behavior of the CRMC algorithm. As compared to the MSE-based criterion, the proposed algorithm achieves superior performance. Simulation results verified the efficiency of the proposed algorithm. In future study, we will apply MCC on MVDR approach. Some initial works have been done.

### Acknowledgments

This work was partially supported by National Science Foundation of P.R. China (Grant: 61571374, 61271340, 61433011).

### References

[1] L. Qiang, B. Liao, L.Huang, C. Guo, G. Liao, S. Zhu, A robust STAP method for airborne radar with array steering vector mismatch, Signal Process. 128 (2016) 198–203.

[2] A.E. Keyi, B. Champagne, Adaptive linearly constrained minimum variance beamforming for multiuser cooperative relaying using the Kalman filter, IEEE Trans. Wireless Communications 9(2) (2010) 641–651.

[3] A.E.A. Blomberg, A. Austeng, R.E. Hansen, S.A.V. Synnes, Improving sonar performance in shallow water using adaptive beamforming, IEEE Journal of Oceanic Engineering 38 (2013) 297–307.

[4] B. Widrow, P.E. Mantey, L.J. Griffiths, B.B. Goode, Adaptive antenna systems, J. Acoust. Soc. Amer. 42 (2005) 1175–1176.

[5] D.P. Mandic, S. Kanna, S.C. Douglas, Mean square analysis of the CLMS and ACLMS for non-circular signals: The approximate uncorrelating transform approach, In: IEEE Int. Conf. Acoust., Speech, Signal Process. 2015, pp. 3531–3535.

[6] H. Zayyani, Continuous mixed $p$-norm adaptive algorithm for system identification, IEEE Signal Process. Lett. 21 (2014) 1108–1110.

[7] J.A. Srar, K.S. Chung, A. Mansour, Adaptive array beamforming using a combined LMS-LMS algorithm, IEEE Trans. Antennas and Propagation 58 (2010), 3545–3557.

[8] Y.M. Shi, L. Huang, C. Qian, H.C. So, Shrinkage linear and widely linear complex-valued least mean squares algorithms for adaptive beamforming, IEEE Trans. Signal Process. 63(1) (2015) 119–131.
[9] P. Tsakalides, C.L. Nikias, The robust covariation-based MUSIC (ROC-MUSIC) algorithm for bearing estimation in impulsive noise environments, IEEE Trans Signal Process. 44 (1996) 1623–1633.
[10] P. Tsakalides, C.L. Nikias, Maximum likelihood localization of sources in noise modeled as a stable process, IEEE Trans. Signal Process. 43(11) (1995) 2700–2713.
[11] W.J. Zeng, H.C. So, L. Huang, \( l_p \)-MUSIC: Robust direction-of-arrival estimator for impulsive noise environments, IEEE Trans. Signal Process. 61, (2013) 4296–4308.
[12] X. Jiang, W.J. Zeng, A. Yasothon, H.C. So, T. Kirubarajan, Minimum dispersion beamforming for non-Gaussian signals, IEEE Trans. Signal Process. 62(7) (2014) 1879–1893.
[13] X. Jiang, A. Yasothon, T. Kirubarajan, Robust beamforming with sidelobe suppression for impulsive signals, IEEE Signal Process. 122 (2016) 123–128.
[15] J.F. de Andrade, M.L.R. de Campos, J.A. Apolinário, \( L_1 \)-constrained normalized LMS algorithms for adaptive beamforming, IEEE Trans. Signal Process. 63(24) (2015) 6524–6539.
[16] B. Liao, S.C. Chan, Robust recursive beamforming in the presence of impulsive noise and steering vector mismatch, J. Sign. Process. Syst. 73 (2013) 1–10.
[17] B. Chen, J.C. Principe, Maximum correntropy estimation is a smoothed MAP estimation, IEEE Signal Process. Lett. 19 (2012) 491–494.
[18] Z. Wu, J. Shi, X. Zhang, W. Ma, B. Chen, Kernel recursive maximum correntropy, Signal Process. 117 (2015) 11–16.
[19] L. Lu, H. Zhao, Active impulsive noise control using maximum correntropy with adaptive kernel size, Mechanical Systems and Signal Process. 87 (2017) 180–191.
[20] S. Peng, B. Chen, L. Sun, Z. Lin, W. Ser, Constrained maximum correntropy adaptive filtering, arXiv:1610.01766v1.
[21] P. Wang, T. Qiu, F. Ren, A. Song, A robust DOA estimator based on the correntropy in alpha-stable noise environments, Digit. Signal Process. 60 (2017) 242–251.
[22] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variables. New York: Chapman & Hall, 1994.
[23] M. Belge, E.L. Miller, A sliding window RLS-like adaptive algorithm for filtering alpha-stable noise, IEEE Signal Process. Lett. 7 (2000) 86–89.
[24] A. Singh, J.C. Principe, A closed form recursive solution for maximum correntropy training, ICASSP, 2010, pp. 2070–2073.
[25] A.H. Sayed, Fundamentals of Adaptive Filtering. Wiley–Interscience, New York, 2003.