AN IDENTIFICATION THEOREM FOR PSU₆(2) AND ITS AUTOMORPHISM GROUPS

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ABSTRACT. We identify the groups PSU₆(2), PSU₆(2):2, PSU₆(2):3 and Aut(PSU₆(2)) from the structure of the centralizer of an element of order 3.

1. INTRODUCTION

The aim of this article is to provide a 3-local identification of the groups PSU₆(2):3, PSU₆(2):2 and PSU₆(2):Sym(3) as such characterizations are needed in the ongoing work to classify groups with a large p-subgroup. For a prime p, a p-local subgroup of G is by definition the normalizer in G of a non-trivial p-subgroup of G. We say that a p-subgroup Q of a group G is large provided

(L1) $F^*(N_G(Q)) = Q$; and
(L2) if $U$ is a non-trivial subgroup of $Z(Q)$, then $N_G(U) \leq N_G(Q)$.

An interesting observation is that most of the groups of Lie type in characteristic $p$ contain a large $p$-subgroup. In fact the only Lie type groups in characteristic $p$ and rank at least 2 which do not contain such a subgroup are PSp$_{2n}(2^a)$, F$_4(2^a)$ and G$_2(3^a)$. It is not difficult to show that groups $G$ which contain a large $p$-subgroup are of parabolic characteristic $p$, which means that all $p$-local overgroups $N$ of a Sylow $p$-subgroup $S$ satisfy $F^*(N) = O_p(N)$ ([18, Lemma 2.1]). The work initiated in [12] starts the determination of the $p$-local overgroups of $S$ which are not contained in $N_G(Q)$. This is the first mile of a long road to showing that typically a group with a large $p$-subgroup is a group of Lie type defined in characteristic $p$ and of rank at least 2. The basic crude idea is to gather information about the $p$-local subgroups of $G$ containing a fixed Sylow $p$-subgroup so that the subgroup generated by them can be identified with a group of Lie type via its action on the chamber complex coming from these subgroups (which will in fact be the maximal parabolic subgroups). However, one is sometimes confronted with the following situation. Some (but perhaps not all) of the $p$-local subgroups of $G$ containing a given Sylow $p$-subgroup $S$ of $G$ generate a subgroup $H$ and $F^*(H)$ is known to be isomorphic to a Lie type group in characteristic $p$. The expectation (or rather hope) is that $G = H$. In the case that $H$ is a proper subgroup of $G$, one usually tries to prove that $H$ contains all the $p$-local subgroups of $G$ which contain $S$ and then in a next step to prove that $H$ is strongly $p$-embedded.

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in $G$ at which stage [17] is applicable and delivers $G = H$. The last two steps are reasonably well understood, at least for groups with mild extra assumptions. However it might be that the first step cannot be made. Typically this will occur only when $N_G(Q)$ is not contained in $H$. If $N_H(Q)$ is non-soluble and $p$ is odd, A. Seidel in his PhD thesis [22] has shown that this cannot occur. In [18] the authors use the identification theorem presented in this paper together with further identifications [14, 15, 16, 19, 20] to handle the more delicate analysis when $p = 3$ and $N_H(Q)$ is soluble. Indeed the centralizers of involutions in both $M(22)$ and $^2E_6(2)$ and their automorphism groups feature $U_6(2)$ and its automorphism groups prominently. These identifications in addition are required for our work on the sporadic simple groups $M(23)$ and $F_2$.

In earlier work [13] the first author proved the following result: let $G$ be a finite group, $S$ be a Sylow 3-subgroup of $G$ and $Z = Z(S)$. Assume that $N_G(Z)$ is similar to a 3-normalizer in $PSU_6(2)$. Then either $Z$ is weakly closed in $F^*(H)$ but not in $H$, then $G \cong PSU_6(2)$, $PSU_6(2):2$ and $PSU_6(2):Sym(3)$ from their 3-local data (here and throughout this work we use the Atlas [3] notation for group extensions). The addition of these automorphisms cause numerous difficulties.

**Definition 1.1.** We say that $X$ is similar to a 3-centralizer in a group of type $PSU_6(2)$ or $F_4(2)$ provided the following conditions hold.

(i) $Q = F^*(X)$ is extraspecial of order $3^5$ and $Z(F^*(X)) = Z(X)$; and
(ii) $X/Q$ contains a normal subgroup isomorphic to $Q_8 \times Q_8$.

A precise description of the possibilities for the group $X/Q$ will be determined in Section 3. Our theorem is as follows.

**Theorem 1.2.** Suppose that $G$ is a group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If $H$ is similar to a 3-centralizer in a group of type $PSU_6(2)$ of $F_4(2)$ and $Z$ is weakly closed in $F^*(H)$ but not in $H$, then $G \cong PSU_6(2)$, $PSU_6(2):2$, $PSU_6(2):3$ or $PSU_6(2):Sym(3)$.

In the case that $Z$ is weakly closed in $H$, then $G$ could be a nilpotent group extended by a group similar to a 3-centralizer of type $PSU_6(2)$ of $F_4(2)$. Thus the hypothesis that $Z$ is not weakly closed in $H$ is necessary to have an identification.
theorem. On the other hand, the hypothesis that \( Z \) is weakly closed in \( F^*(H) \) is there to prevent further examples related to \( F_4(2) \) arising. The methods that we use here are also be applicable to this type of configuration, however the investigation of such a possibility would take a rather different road at the very outset of our proof and so the analysis of this possibility is not included here and is the subject of [19]. Combining the work of both papers we obtain

**Theorem 1.3.** Suppose that \( G \) is a group, \( Z \leq G \) has order 3 and set \( H = C_G(Z) \). If \( H \) is similar to a 3-centralizer of a group of type \( \text{PSU}_6(2) \) or \( F_4(2) \) and \( Z \) is not weakly closed in a Sylow 3-subgroup of \( G \) with respect to \( G \), then either \( F^*(G) \cong F_4(2) \) or \( F^*(G) \cong \text{PSU}_6(2) \).

We now describe the layout of the paper and highlight a number of interesting features of the article. We begin in Section 2 with preliminary lemmas and background material. Noteworthy results in this section are Lemma 2.5 where we embellish the statement of Hayden’s Theorem [9] to give the structure of the normal subgroup of index 3 and Lemma 2.13 where we use transfer theorems to show that a group with a certain specified 2-local subgroup has a subgroup of index 2. The relevance of such results to our proof is apparent as a look at the list of groups in the conclusion of our theorem shows. Let \( G, H \) and \( Z \) be as in the statement of Theorem 1.2 and let \( S \in \text{Syl}_3(M) \) where \( M = N_G(Z) \) contains \( H \) at index at most 2. In Section 3, we tease out the structure of \( M \) and establish much of the notation that is used throughout the proof of Theorem 1.2.

In Section 4, we determine the structure the normalizer of a further 3-subgroup which we call \( J \) and turns out to be the Thompson subgroup of \( S \). The fact that \( N_G(J) \) is not contained in \( M \) is a consequence of the hypothesis that \( Z \) is not weakly closed in \( M \). We find in Lemma 4.6 that \( N_G(J)/J \cong 2 \times \text{Sym}(6) \) or \( \text{Sym}(6) \). With this information, after using a transfer theorem, we are able to apply [13] and do so in Theorem 4.8 to get that \( G \cong \text{PSU}_6(2) \) or \( \text{PSU}_6(2):3 \) if \( N_M(S)/S \cong \text{Dih}(8) \). Thus from this stage on we assume that \( N_M(S)/S \cong 2 \times \text{Dih}(8) \) and \( N_M(J)/J \cong 2 \times \text{Sym}(6) \). With this assumption, our target groups all have a subgroup of index 2. Our plan is to determine the structure of a 2-central involution \( r \), apply Lemma 2.13 and then apply Theorem 4.8 to the subgroup of index 2. The involution we focus on is contained in \( M \) and centralizes a subgroup of \( F^*(M) \) isomorphic to \( 3^{1+2} \). But before we can make this investigation we need to determine the centralizer of another subgroup (for now we will call it \( X \)) which has order either 3 or 9. It turns out we may apply the theorems of Hayden [9] and Prince [21] to get \( E(C_G(X)) \cong \text{SU}_4(2) \). At this juncture, given the 3-local information that we have gathered, we can construct an extraspecial 2-subgroup \( \Sigma \) of order \( 2^9 \) in \( K = C_G(r) \). In Theorem 5.5 we show that \( N_K(\Sigma)/\Sigma \cong \text{Aut}(\text{SU}_4(2)), (\text{SU}_4(2) \times 3):2 \) or \( \text{Sp}_6(2) \). In our target groups the possibility \( \text{Sp}_6(2) \) does not arise and we will say more about this shortly.

In Section 6 we show that \( \Sigma \) is strongly closed in \( N_K(\Sigma) \) with respect to \( K \) and then we apply Goldschmidt’s Theorem to get that \( K = N_K(\Sigma) \). At this stage
we know the centralizer of a 2-central involution and so we prove the theorem in Section 6. We mention here that when \( K/\Sigma \cong \text{Sp}_6(2) \) we apply [23] to obtain \( G \cong \text{Co}_2 \) and then eliminate this group as it does not satisfy our hypothesis on the structure of \( M \). One should wonder if the configuration involving \( \text{Sp}_6(2) \) could be eliminated at an earlier stage. However, as \( \text{Co}_2 \) contains \( \text{PSU}_6(2):2 \) as a subgroup of index 2300, these groups are intimately related. A 3-local identification of \( \text{Co}_2 \) can be found in [15].

Our notation follows that in [1, 6] and [7]. In particular we use the definition of signalizers as given in [7, Definition 23.1] as well as the notation \( \mathcal{I}_G(A, \pi) \) to denote the set of \( A \)-signalizers in \( G \) and \( \mathcal{I}_G^*(A, \pi) \) the maximal members of \( \mathcal{I}_G(A, \pi) \). As mentioned earlier we use Atlas [3] notation for group extensions. We also use [3] as a convenient source for information about subgroups of almost simple groups. Often this information can be easily gleaned from well-known properties of classical groups. For odd \( p \), the extraspecial groups of exponent \( p \) and order \( p^{2n+1} \) are denoted by \( p^{1+2n} \). The extraspecial 2-groups of order \( 2^{2n+1} \) are denoted by \( 2^{1+2n} \) if the maximal elementary abelian subgroups have order \( 2^{1+n} \) and otherwise we write \( 2^{1+2n} \). We hope our notation for specific groups is self-explanatory. In addition, for a subset \( X \) of a group \( G \), \( X^G \) denotes that set of \( G \)-conjugates of \( X \). If \( x, y \in H \leq G \), we often write \( x \sim_H y \) to indicate that \( x \) and \( y \) are conjugate in \( H \). All the groups in this paper are finite groups.

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2. Preliminaries

In this section we gather preliminary results for our proof of Theorem [12]. For a group \( G \) with Sylow \( p \)-subgroup \( P \) and \( v \in P \), \( v \) is said to be extremal in \( P \) if \( C_P(v) \) is a Sylow \( p \)-subgroup of \( C_G(v) \).

**Lemma 2.1.** Suppose that \( p \) is a prime and \( G \) is a group. Let \( P \) a Sylow \( p \)-subgroup of \( G \) and \( Q \) be a proper normal subgroup of \( P \) such that \( P/Q \) is cyclic. Assume there is \( u \in P \setminus Q \) such that

(a) no conjugate of \( u^g \) is contained in \( P \setminus Q \); and

(b) any extremal conjugate of \( u \) in \( P \) is contained in \( Q \cup Qu \).

Then either \( G \) has a normal subgroup \( N \) with \( G/N \) cyclic and \( u \not\in N \) or there is \( g \in G \) such that

(i) \( u^g \in Q \);

(ii) \( u^g \) is extremal in \( P \); and

(iii) \( C_P(u^g) \leq C_P(u^g) \).
Lemma 2.2. Suppose that $p$ is a prime, $G$ is a group and $P \in \text{Syl}_p(G)$.

(i) Assume that there is a normal subgroup $Q$ of $P$ such that $P/Q$ is cyclic and that $y \in P \setminus Q$ has order $p$. If every extremal conjugate of $y$ in $P$ is contained in $Qy$, then $G$ has a normal subgroup $N$ with $y \not\in N$ and $G/N$ cyclic.

(ii) Assume that $P \leq M \leq G$, $y \in P \setminus M'$ has order $p$ and that, if $x \in G$ with $y^x \in P$ extremal, then there is $g \in M$ such that $y^g = y^x$. Then $y \not\in G'$.

(iii) Assume that $J = J(P)$ is the Thompson subgroup of $P$. If $J$ is elementary abelian and $J \not\leq N_G(J)'$, then $J \not\leq G'$.

Proof. (i) This follows from [2, Proposition 2.1].

(ii) As $M/M'$ is abelian, there is $N \leq M$ such that $M' \leq N$, $y \not\in N$, $M = NP$ and $P/(P \cap N)$ is cyclic. Set $Q = P \cap N$. Now for $g \in M$ with $y^g \in P$ we have that $y^g \in Qy$. Hence by assumption $y^x \in Qy$ for all $x \in G$ such that $y^x$ is extremal in $P$. Now (ii) follows from (i).

(iii) Set $M = N_G(J)$ and pick $y \in J \setminus M'$. Assume that $g \in G$ and $y^g$ is extremal in $P$. Then $C_P(y^g) \in \text{Syl}_p(C_G(y^g))$. Since $C_G(y)$ contains $J$, we have $C_P(y^g)$ contains a $G$-conjugate of $J$. Since $J$ is weakly closed in $P$, we have $J \leq C_P(y^g)$. But then $y^g \in C_P(J) \leq J$. Since $M$ controls fusion in $J$, we now have that $y^g = y^m$ for some $m \in M$. Now (iii) follows from (ii).

Lemma 2.3. Suppose that $F$ is a field, $V$ is an $n$-dimensional vector space over $F$ and $G = \text{GL}(V)$. Assume that $q$ is quadratic form of Witt index at least 1 and $S$ is the set of singular 1-dimensional subspaces of $V$ with respect to $q$. Then the stabiliser in $G$ of $S$ preserves $q$ up to similarity.

Proof. See [15, Lemma 2.10].

Lemma 2.4. Suppose that $p$ is an odd prime, $X = \text{GL}_4(p)$ and $V$ is the natural $\text{GF}(p)X$-module. Let $A = \langle a, b \rangle \leq X$ be elementary abelian of order $p^2$ and assume that $[V, a] = C_V(b)$ and $[V, b] = C_V(a)$ are distinct and of dimension 2. Let $v \in V \setminus [V, A]$. Then $A$ leaves invariant a non-degenerate quadratic form with respect to which $v$ is a singular vector. In particular, $X$ contains exactly two conjugacy classes of subgroups such as $A$. One is conjugate to a $\text{Sylow}$ $p$-subgroup of $\text{GO}_4^+(p)$ and the other to a $\text{Sylow}$ $p$-subgroup of $\text{GO}_4^-(p)$.

Proof. See [15, Lemma 2.11].

Lemma 2.5. Let $H$ be a finite group and let $d \in H$ be an element of order 3 such that $X = C_H(d)$ is isomorphic to the centralizer of a non-trivial 3-central element in $\text{PSp}_4(3)$. Let $P \in \text{Syl}_3(X)$ and $E$ be the elementary abelian subgroup of $P$ of order 27. Assume that $E$ does not normalize any non-trivial $3'$-subgroup of $H$, that $d$ is not $H$-conjugate to its inverse and $H$ has a normal subgroup of index 3. Then $H = C_H(d)$. 

Proof. See [15, Proposition 15.15] or [24, Corollary 5.3.1]. \qed
Proof. Notice first of all that $P \in \text{Syl}_3(H)$. Let $H_1$ be a normal subgroup of $H$ of index 3 and set $E_1 = E \cap H$. So $C_{H_1}(d) \cong 3^{1+2}Q_8$ and $E_1$ has order 9. Suppose that $x \in E_1 \setminus \langle d \rangle$. We see that all subgroups of order three in $E_1$ different from $\langle d \rangle$ are conjugate in $O_3(C_H(d))$ and so all $x \in E_1 \setminus \langle d \rangle$ are conjugate to its inverse and $d$ is not, $d$ is the unique conjugate of $d$ in $E_1$. Furthermore, $d$ is not conjugate to any element of $E \setminus H'$ and so $d$ is the unique conjugate of $d$ in $E$. Since $x$ is not conjugate to $d$, we have that $E_1 = \langle d, x \rangle$ is a Sylow-subgroup of $C_{H_1}(x)$. As $E_1/\langle x \rangle$ is self-normalizing in $C_{H_1}(x)/\langle x \rangle$, $C_{H_1}(x)$ has a normal 3-complement $T$ by Burnside’s Theorem. However $C_{H_1}(x)$ is normalized by $E$ and so $T = 1$ by hypothesis. It follows that $C_H(x) = E$ for all $x \in E_1 \setminus \langle d \rangle$.

Let $y \in E \setminus H_1$. Then, as before, $E_1$ is a Sylow 3-subgroup of $C_{H_1}(y)$. Since $d$ is not conjugate to any non-trivial element of $E_1 \setminus \{d\}$, we have $N_H(E_1) \leq X$. So $N_{C_{H_1}(y)}(E_1) = \langle E_1, s \rangle$ where $s$ is an element of order at most two in $X$. Since $[E_1, s] < E_1$, Grün’s Theorem [6] Chapter 7, Theorem 4.4 implies that $C_{H_1}(y)$ has a subgroup $L$ of index at least $|E_1 : [E_1, s]|$ with Sylow 3-subgroup $[E_1, s]$. Since $L$ is normalized by $E$, we also have $O_3'(L) = 1$. Hence, if $s = 1$, then $C_H(y) \leq X$ which means that $C_H(y) = E$. So suppose that $[E_1, s]$ has order 3. Then, as $C_H([E_1, s]) = E$, we have $[E_1, s]$ is self-centralizing in $L$. Applying the other Feit-Thompson Theorem [3] to $L$ and using $O_3'(L) = 1$, we now have that either $L \cong \text{Sym}(3)$ with $L = N_{X \cap H_1}([E_1, s])$ or $L \cong PSL_3(2)$ or $\text{Alt}(5)$. The latter two cases are eliminated as $L$ is normalized by $E_1$ and the centralizers of all of the non-trivial elements of $E_1$ are soluble. Therefore, $C_H(y) = C_X(y) \leq X$ for all $y \in E \setminus E_1$.

Now let $R \in \text{Syl}_2(X)$ and $r \in R$ be an involution. Then $C_X(r) = R \langle d, y \rangle$ for some $y \in E \setminus E_1$. Furthermore, as $d$ is the unique conjugate of $d \in \langle d, y \rangle$, $N_{C_{H^1}(y)}(\langle d, y \rangle) = N_X(\langle d, y, r \rangle) = \langle d, y, r \rangle$ and so $C_H(r)$ has a normal 3-complement $U$ by Burnside’s Theorem. Finally $U = \langle C_U(w) \mid w \in \langle d, y \rangle \# \rangle \leq X$ as $C_H(w) \leq X$ for each $w \in \langle d, y \rangle \#$. It follows that $U = R$. But then $R \in \text{Syl}_2(H)$ and $r \in Z^3(H)$ by [2]. As $[O_3(X), r] = O_3(X)$, we conclude $O_3(X) \leq O_2'(H)$ and deduce $H = X$ from the Frattini Argument. This completes the proof of the lemma.

**Theorem 2.6** (Hayden). Let $H$ be a finite group and let $d$ be an element of order 3 in $H$ such that $X = C_H(d)$ is isomorphic to the centralizer of a non-trivial 3-central element in $\text{PSp}_4(3)$. Let $P \in \text{Syl}_3(X)$ and $E$ be the elementary abelian subgroup of $P$ of order 27. If $E$ does not normalize any non-trivial $3'$-subgroup of $H$ and $d$ is not $H$-conjugate to its inverse, then either $H = X$ or $H \cong \text{PSp}_4(3)$.

**Proof.** By [9] either $H \cong \text{PSp}_4(3)$ or $H$ has a normal subgroup of index 3. The result now follows from Lemma [2,6].
Theorem 2.7 (A. Prince). Let \( H \) be a finite group and let \( d \) be an element of order 3 in \( H \) such that \( X = C_H(d) \) is isomorphic to the centralizer of a non-trivial 3-central element in \( \text{PSp}_4(3) \). Let \( P \in \text{Syl}_3(C_H(d)) \) and \( E \) be the elementary abelian subgroup of \( P \) of order 27. If \( E \) does not normalize any non-trivial 3'-subgroup of \( H \) and \( d \) is \( H \)-conjugate to its inverse, then either

(i) \(|H : C_H(d)| = 2");
(ii) \( H \) is isomorphic to \( \text{Aut}(\text{SU}_4(2)) \); or
(iii) \( H \) is isomorphic to \( \text{Sp}_6(2) \).

Proof. See [21, Theorem 2]. \( \square \)

Lemma 2.8. Suppose that \( X \) is a group of shape \( 3^{1+2}.\text{SL}_2(3) \), \( O_2(X) = 1 \) and a Sylow 3-subgroup of \( X \) contains an elementary abelian subgroup of order 3\(^3\). Then \( X \) is isomorphic to the centralizer of a non-trivial 3-central element in \( \text{PSp}_4(3) \).

Proof. See [13, Lemma 6]. \( \square \)

Lemma 2.9. Let \( G \) be a finite group and \( S \) be a Sylow 3-subgroup of \( G \). Set \( Z = Z(S), J = J(S) \) and \( M = N_G(Z) \). Suppose that \( G^* \) is a normal subgroup of \( G \) and set \( M^* = M \cap G^* \). Assume that the following hold:

(i) \(|M^*| = 2^7.3^6\);
(ii) \( M^* \geq QR = O_{3,2}(M^*), \) where \( Q = O_3(M^*) \) is extraspecial of order \( 3^5 \)
   and \( R \in \text{Syl}_2(O_{3,2}(M^*)) \);
(iii) \( O^2(M^*) = (S \cap M^*)R \) has index \( 2 \) in \( M^* \); and
(iv) \( Q/Z \) is a \( M^* \)-chief factor.

If \( N_{G^*}(J \cap G^*) \not\leq M^* \), then \( G^* \cong \text{PSU}_6(2) \) and \( G \) is a subgroup of \( \text{Aut}(\text{PSU}_6(2)) \) such that \( G/G^* \cong M/M^* \).

Proof. Since \( N_{G^*}(J \cap G^*) \not\leq M^* \), \( Z \) is not weakly closed in \( S \cap G^* \). The conditions imposed on the structure of \( M^* \) mean that \( M^* \) is similar to a 3-normalizer in \( \text{PSU}_6(2) \) [13, Definition 1]. Hence [13, Theorem 1] gives the result. \( \square \)

Lemma 2.10. Suppose that \( E \) is an extraspecial 2-group and \( x \in \text{Aut}(E) \) is an involution. If \( C_E(x) \geq [E, x] \), then \([E, x]\) is elementary abelian.

Proof. Let \( \langle e \rangle = Z(E) \). We show that every element of \([E, x]\) has order 2. Let \( f \in [E, x] \setminus \langle e \rangle \). Then \( fe \) has the same order as \( f \). Thus we may suppose that \( f = [h, x] \) for some \( h \in E \). As \( x[h, x] = [h, x]x \) by hypothesis, we have

\[
\begin{align*}
f^2 &= [h, x][h, x] = h^{-1}xhx[h, x] = h^{-1}xh[h, x]x \\
&= h^{-1}xhh^{-1}xhx = 1
\end{align*}
\]

as required. This proves the lemma. \( \square \)

The following lemma is an easy consequence of the Three Subgroup Lemma.
Lemma 2.11. Suppose that $p$ is a prime, $P$ is a $p$-group of nilpotency class at most 2 and that $\alpha \in \text{Aut}(P)$ has order coprime to $p$. If $\alpha$ centralizes a maximal abelian subgroup of $P$, then $\alpha = 1$.

Proof. See [18, Lemma 2.3].

For use in Lemma 2.13 and Section 6, we collect some facts about the action of $\text{Sp}_6(2)$ and $\text{Aut}(\text{SU}_4(2))$ on their irreducible 8-dimensional module $V$ over GF(2). Recall that $\text{Aut}(\text{SU}_4(2)) \cong \text{O}_6^-(2)$ is a subgroup of $\text{Sp}_6(2)$ [3, page 46]. We will frequently use the fact that as $\text{SU}_4(2)$-module, $V$ is the natural 4-dimensional GF(4)$\text{SU}_4(2)$-module regarded as a module over GF(2). We will often refer to this as the natural $\text{SU}_4(2)$-module.

Proposition 2.12. Let $X \cong \text{Sp}_6(2)$ and $Y \cong \text{Aut}(\text{SU}_4(2))$. Assume that $V$ is the 8-dimensional irreducible module for $X$ (and hence $Y$) over GF(2). Then the following hold:

(i) $X$ and $Y$ both possess exactly four conjugacy classes of involutions. In Table 1 we list the four classes of involutions and give structural information about the centralizers in both groups as can be found in [3, pages 26 and 46].

(ii) $X$ and $Y$ have orbits of length 135 and 120 on the non-zero elements of $V$. We call elements of the orbits non-singular and singular vectors respectively. Suppose that $x$ is singular and $y$ is non-singular. Then

$$|C_Y(x)| = 2^7 \cdot 3, \quad |C_X(x)| = 2^9 \cdot 3 \cdot 7.$$ $C_Y(y) \cong 3_+^{1+2}.\text{SDih}(16), \quad C_X(y) \cong G_2(2).$

(iii) $X$ and $Y$ both have exactly three conjugacy classes of elements of order 3. They are distinguished by their action on $V$. They have centralizers of dimension 0, 2 and 4. The elements with centralizer of dimension 2 are 3-central and centralize only non-singular vectors in $V^*$.

(iv) For $u \in Y$ an involution, $\dim C_Y(u)$ is given in column 4 of Table 1.

(v) Let $u$ be a unitary transvection. Then $C_Y(u)$ acts on $C_V(u)/(V,u)$ with orbits of length 1, 6 and 9.

| $u_1$ | $2^{1+4}.(\text{Sym}(3) \times \text{Sym}(3))$ | $2^7.\text{Sym}(3) \times \text{Sym}(3)$ | 6 |
| $u_2$ | $2^6.3$ | $2^7.3$ | 4 |
| $u_3$ | $2 \times \text{Sym}(6)$ | $2^5.\text{Sym}(6)$ | 4 |
| $u_4$ | $2 \times (\text{Sym}(4) \times 2)$ | $2^9.3$ | 4 |

Table 1. Involutions in $\text{Sp}_6(2)$ and $\text{Aut}(\text{SU}_4(2))$. The involutions in the first row are the unitary transvections. The involutions in the last two rows are those which are in $\text{Aut}(\text{SU}_4(2)) \setminus \text{SU}_4(2)$. 
(vi) If \( u \) is a unitary transvection, \( S_2 \leq C_Y(u) \) has order 3 and \( C_{C_Y(u)/[V,u]}(S_2) \neq 0 \), then \( \dim C_Y(S_2) = 2 \).

(vii) For \( S \in \text{Syl}_2(Y) \) and \( S_1 = S \cap Y' \), every \( S \)-invariant subspace of \( W \) of dimension at least 2 contains \( C_Y(S_1) \).

(viii) \( Y \) does not contain a fours group all of whose non-trivial elements are unitary transvections.

(ix) \( C_Y(u_4) \) is generated by non-singular vectors.

(x) The 2-rank of \( Y \) is 4.

Proof. (i) From [3, page 27, page 47], we see that \( \text{Aut}(\text{SU}_4(2)) \) and \( \text{Sp}_6(2) \) both possess exactly four conjugacy classes of involutions.

(ii) By Witt’s lemma \( Y \) has exactly two orbits on the non-zero elements of \( V^\# \) and they correspond to the singular and the non-singular vectors. Since \( 2^8 - 1 \) does not divide \( |X| \), these orbits are also orbits under the action of \( X \). Since the lengths of the orbits are 135 and 120, using [3, page 26, page 46] we get the given structure of the stabilizers.

(iii) As \( Y \) contains a Sylow 3-subgroup of \( X \), we find representatives of all \( X \)-conjugacy classes of elements of order 3 in \( Y \). By [3, page 27] there are exactly three conjugacy classes of elements of order 3 in \( Y \), which we easily distinguish by their action on \( V \). We have elements, which are fixed point free, which have centralizer of dimension 2 and those which have centralizer of dimension 4. In particular, these elements are not fused in \( X \).

Let \( d \in Y \) have 2-dimensional fixed space on \( V \). Then as \( C_Y(d) \) is perpendicular to \( [V,d] \) we deduce that \( C_Y(d) \) is non-singular (a 1-dimensional non-singular GF(4)-space).

(iv) For the unitary transvection \( u \) we have that \( \dim [V,u] = 2 \). Suppose that \( u \) is not a unitary transvection but \( u \in Y' \). Then, as \( V \) supports the structure of a vector space over GF(4), we have that \( [V,u] \) is 2-dimensional and so \( \dim [V,u] = 4 \). Suppose next that \( u \) is an involution in \( Y \backslash Y' \) and let \( P \) be the stabilizer of a maximal isotropic space \( W \) of GF(4)-dimension 2 in \( V \). Then \( O_2(P) \) is elementary abelian of order 16 and \( P/O_2(P) \cong \text{Sym}(5) \cong \text{SU}_2(4) : 2 \). Since \( P \) contains a Sylow 2-subgroup of \( Y \), we may suppose that \( u \in P \). Furthermore \( W \) and \( V/W \) are natural \( \text{SL}_2(4) \)-modules. As \( u \notin O_2(P) \leq Y' \), we have that \( \dim [W,u] = 2 = \dim [V/W,u] \). Hence we get that \( \dim [V,u] \geq 4 \) and so as \( \dim V = 8 \), we have \( \dim [V,u] = 4 \).

(v) Let \( u \) be a unitary transvection. Then \( C_{Y'}(u) \) acts on \( C_Y(u)/[V,u] \) as the group \( \text{GU}_2(2) \cong \text{Sym}(3) \times 3 \) and has three orbits one of length 1, one of length 6 and one of length 9.

(vi) From (v), a Sylow 3-subgroup \( S_1 \) of \( C_{Y'}(u) \) contains two subgroups of order 3 whose centralizer in \( C_Y(u)/[V,u] \) is of order 4 and two which are fixed point free. As the elements of order three in \( C_Y(u) \) act the same way on \( [V,u] \) as on \( V/C_Y(u) \), the elements with fixed points on \( C_Y(u)/[V,u] \) have centralizer in \( V \).
of dimension 2, as by (iii) there are no elements of order three which centralize a subspace of dimension 6. Now by coprime action we get that one subgroup of order three in $S_1$ centralizes in $V$ a subspace of dimension 4 and acts fixed point freely on $C_V(u)/[V,u]$, one acts fixed point freely $V$ and the other two centralize a subspace of dimension 2 in $V$.

(vii) Let $S \in \text{Syl}_2(Y)$ and $S_1 = S \cap Y'$. Then, as $V$ is the natural 4-dimensional unitary module for $Y'$, we have that $U = C_V(S_1)$ has GF(2)-dimension 2. Assume that (vii) is false and let $W$ be an $S$-invariant subspace of dimension at least 2 with $U \not\subseteq W$. Then $W > [W,S] \neq 0$ does not contain $U$ and so $[W,S]$ must have GF(2)-dimension 1 by the minimal choice of $W$. Hence $[W,S] \leq C_V(S_1) = U$ which means that $W + U/U \leq C_{V/U}(S) \leq C_{V/U}(S_1)$ and this latter space has GF(4)-dimension 1. It follows that $W$ has GF(2)-dimension 2. Hence $S_0 = C_S(W)$ has index 2 in $S$, $S_0 \cap S_1$ has order at least $2^5$ and this subgroup centralizes $W$ and $U$ and hence centralizes the preimage of $C_{V/U}(S_1)$ which has GF(4)-dimension 2. However, this is an isotropic line in the unitary representation and its centralizer is elementary abelian of order $2^4$, a contradiction. Hence (vii) is true.

(viii) Suppose that $F = \langle x_1, x_2 \rangle$ is a fours group with all non-trivial elements unitary transvections. Then, as $x_3 = x_1 x_2$, is also a unitary transvection, we get that $C_V(x_1) = C_V(x_2)$. But then $C_V(x_1)$ is normalized by $\langle C_V(x_1), C_V(x_2) \rangle = Y$, which is impossible.

(xi) Let $y$ be a non-singular vector. By (ii), we have that $C_Y(y) \cong 3_+^{1+2}.\text{GL}_2(3)$. This group contains an involution $u$ in $Y \setminus Y'$. If $u$ is conjugate to $u_3$ (in Table [1]), then $C_Y(u) \cong \text{Sym}(6)$ acts transitively on $C_Y(u)^\#$ and so $C_Y(u)^\#$ contains only non-singular vectors. Since $\dim C_Y(u) = 4$, this is impossible. Therefore $v$ is conjugate to $u_4$ and $y \in C_Y(u) = [V,u]$. Since $C_{C_Y(u)}(u)$ has order 6, there are eight conjugates of $y$ in $C_Y(u)$. Hence $C_Y(u)$ is generated by non-singular elements.

(x) From (i) we see that the centralizers of involutions $x \in Y \setminus Y'$ have 2-rank 4. Thus we only need to see that $Y'$ has 2-rank 4. This is well-known and can be read from [3, Table 3.3.1].

In the next lemma the group denoted by $(SU_4(2) \times 3):2$ is the subgroup of index 2 in $\text{Aut}(SU_4(2)) \times \text{Sym}(3)$ which is not expressible as a direct product.

**Lemma 2.13.** Assume that $G$ is a group, $t \in G$ is an involution, $H = C_G(t)$ and $Q = F^*(H)$ is extraspecial of order $2^3$. If $H/Q \cong \text{Aut}(SU_4(2))$ or $(SU_4(2) \times 3):2$ and $Q/\langle t \rangle$ is the natural $F^*(H/Q)$-module, then $G$ has a subgroup of index 2.

**Proof.** We let $S \in \text{Syl}_2(H)$ and note that, as $Z(S) = Z(Q) = \langle t \rangle$, we have $S \in \text{Syl}_2(G)$. Let $\overline{H} = H/\langle t \rangle$. We first show that

$$t^G \cap Q = \{t\}. $$

Assume that $u \sim_G t$ with $u \in Q \setminus \langle t \rangle$. Then $\overline{u}$ is singular in $\overline{Q}$ and so we may suppose that $\langle \overline{u} \rangle = Z(\overline{S})$. Now $C_Q(u)$ contains an extraspecial group of order
As a Sylow 2-subgroup of \( H/Q \) is not extraspecial, we have that \( t \in Q_u = O_2(C_G(u)) \). Note that \( \Phi(Q_u \cap Q) \leq \langle u \rangle \cap \langle t \rangle = 1 \). Hence \( Q_u \cap Q \) is elementary abelian. As \( Q \) is extraspecial of order \( 2^9 \), we deduce that \( |Q \cap Q_u| \leq 2^5 \). Since the 2-rank of \( H/Q \) is 4 by Proposition 2.12 (x) and \( |C_{Q_u}(t)| = 2^8 \), we infer that \( |Q \cap Q_u| \) is either \( 2^4 \) or \( 2^5 \). Furthermore, because \( C_H(u)Q \geq S \), we have that \( Q \cap Q_u \) is a normal subgroup of \( S \). We know that \( Q/Q_u \) is a GF(4)-module for \( F^*(H/Q) \). Let \( \overline{U} \) be the one-dimensional GF(4)-space in \( \overline{Q} \) containing \( \overline{Q} \), \( U \) be its preimage in \( H \) and set \( R = C_H(U) \). Since \( U \), \( Q_u \cap Q \) and \( R \) are normalized by \( S \), Proposition 2.12 (vii) implies \( U \leq Q_u \cap Q \leq R \). Assume that \( |Q_u \cap Q| = 2^5 \). Then, as \( (Q_u \cap H)Q/Q \) is a normal subgroup of \( C_H(u)Q/Q \) and \( C_H(u)Q/Q \) contains \( S/Q \), we get \( Z(S/Q) \leq (Q_u \cap H)Q/Q \). Hence there exists \( w \in Q_u \cap H \) such that \( \langle w \rangle = Z(S/Q) \) is the unitary transvection group centralizing \( R \). Therefore we have

\[
|Q_u \cap Q, w| \leq |R, w| \cap [Q_u, w] \leq \langle t \rangle \cap \langle w \rangle = 1,
\]

which is impossible as \( Q_u \cap Q \) is a maximal abelian subgroup of \( Q_u \). Thus \( |Q_u \cap Q| = 2^4 \). Since \(|(Q_u \cap Q)| = 2^2 \), we now have a contradiction to the fact that \( C_{R/U}(C_H(u)) = 1 \) by Proposition 2.12 (v). Thus \([2.13.1]\) holds.

By Proposition 2.12 (i), \( H/Q \) has exactly two conjugacy classes of involutions not in \( H'/Q \). We choose representatives \( \tilde{x}, \tilde{y} \in S/Q \) for these conjugacy classes and fix notation so that \( C_{F^*(H/Q)}(\tilde{x}) \cong \text{Sp}_4(2) \) and \( C_{F^*(H/Q)}(\tilde{y}) \cong 2 \times \text{Sym}(4) \). We have that \( |\overline{Q}(\tilde{x})| = |\overline{Q}(\tilde{y})| = 2^4 \) by Proposition 2.12 (iv). Let \( z \in H \) with \( z^2 \in \langle t \rangle \) be such that \( zQ \) is either \( \tilde{x} \) or \( \tilde{y} \). Let \( T \in \text{Syl}_2(C_H(z)) \). Then \( T' \cap Z(T) \leq T \cap H' \) and \( Z(T) \cap H' \leq Q \) as \( Z(T) = \langle z, C_Q(z) \rangle \). Thus, by \([2.13.1]\) we have \( t^G \cap T' \cap Z(T) = \{t\} \). In particular, \( T \in \text{Syl}_2(C_G(z)) \). It follows that \( z \) is not conjugate to \( t \) in \( G \) and that \( t^G \cap Z(T) = \{t\} \). We record these observations as follows:

(2.13.2) Let \( z \in S \setminus (S \cap H') \) be such that \( z^2 \in \langle t \rangle \) and \( T \in \text{Syl}_2(C_H(z)) \). Then \( T \in \text{Syl}_2(C_G(z)) \), \( t^G \cap Z(T) = \{t\} \) and \( t^G \cap H \subset H' \).

Now let \( z_1 \in S \) be such that \( z_1Q = \tilde{x} \). Since \( C_{H/Q}(z_1Q) \) contains an element \( fQ \) of order 5 with \( f \) of order 5 acting fixed point freely on \( \overline{Q} \), we see that \( C_{Q(z_1)}(f) \) has order 4. Let \( z \in C_{Q(z_1)}(f) \) have minimal order so that \( zQ = z_1Q \). Then \( z^2 \in \langle t \rangle \). Suppose that \( g \in G \) and \( z^g \in S \cap H' \) is extremal in \( S \). Then \( C_S(z^g) \in \text{Syl}_2(C_G(z^g)) \). Now let \( T \in \text{Syl}_2(C_H(z)) \). Then \( T \in \text{Syl}_2(C_G(z)) \) by \([2.13.2]\) Hence \( T^g \in \text{Syl}_2(C_G(z^g)) \) and there is a \( w \in C_G(z^g) \) such that \( T^gw = C_S(z^g) \). Now, by \([2.13.2]\) \( t^G \cap Z(T^gw) = \{t^gw\} \) and of course \( t^G \cap Z(C_S(z^g)) = \{t\} \) as \( t \in Z(H) \). Thus \( gw \in H \), which is impossible as \( z \in H \setminus H' \), \( z^g \in H' \) and \( z^gw = z^g \). Hence there are no extremal conjugates of \( z \) in \( S \setminus H' \). Since also \( z^2 \in \langle t \rangle \) and \( t^G \cap H \subset H' \), Lemma 2.1 implies that \( G \) has a subgroup of index 2 as claimed. \[\square\]
3. The finer structure of $M$

Suppose that $G$ is a group, $Z \leq G$ has order 3 and set $M = N_G(Z)$. Assume that $C_M(Z)$ is similar to a 3-centralizer in a group of type $PSU_6(2)$ or $F_4(2)$. Let $S \in \text{Syl}_3(M)$ and $Q = F^*(M) = O_3(M)$. By Hypothesis $C_M(Z)$ contains a normal subgroup $R^*$ such that $R^*/Q \cong Q_8 \times Q_8$. We let $R \in \text{Syl}_2(R^*)$. Since the commutator map from $Q/Z \times Q/Z$ to $Z$ is an $C_M(Z)/Z$-invariant non-degenerate symplectic form by [10, III(13.7)] which may be negated by $M$, $M/Q$ embeds into $\text{Out}(Q) \cong GSp_4(3)$. Our first lemma locates $M/Q$ as a subgroup of $GSp_4(3)$.

**Lemma 3.1.** We have that $M/Q$ normalizes $R^*/Q$ and is isomorphic to a subgroup of the subgroup $M$ of $GSp_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space over $GF(3)$ into a perpendicular sum of two non-degenerate 2-spaces. Furthermore, $R/Q$ maps to $O_2(M)$.

**Proof.** Consider the action of $Z(R)$ on $Q/Z$. Since $\text{Out}(Q)$ is isomorphic to a subgroup of $GSp_4(3)$, $Z(R)$ acts as a fours group of $Sp_4(3)$ on $Q/Z$. Let $a \in Z(R)^\#$. Then $Q = C_Q(a)[Q,a]$ and $[C_Q(a), [Q,a]] = 1$ by the Three Subgroup Lemma. We may suppose that $C_Q(a) \neq Z$, and so we have $C_Q(a) \cong [Q,a]$ is extraspecial of order $3^3$. Since $R$ centralizes $a$, $R$ preserves this decomposition and $R_1 = C_R([Q,a])$ has order 8 and acts faithfully on $C_Q(a)$. Hence $R_1 \cong Q_8$ and similarly $R_2 = C_R(C_Q(a)) \cong Q_8$ with $R = R_1 \times R_2$. In particular, we now have $C_M(Z)/Q$ is isomorphic to a subgroup of $Sp_2(3) \wr 2$ and $R/Q$ corresponds to the largest normal 2-subgroup of this group. It follows that $|O_2(C_M(Z)) : RQ| \leq 2$. Thus $Z(R)Q/Q$ is a characteristic subgroup of $C_M(Z)$ and so $Z(R)Q/Q$ is normalized by $M/Q$. Finally, as $RS/Q$ is the centralizer of $Z(R)Q/Q$ in $C_M(Z)/Q$ we deduce that $RQ/Q$ is normalized by $M/Q$ and that $M/Q$ preserves the decomposition of $Q/Z$ as described. \hfill $\square$

For the remainder of the paper we now assume that $Z$ is weakly closed in $Q$ but not in $S$ with respect to $G$. In particular, this means that $S > Q$.

**Lemma 3.2.** The following hold.

(i) $Z = Z(S) = Z(Q)$, $N_G(S) \leq M$ and $S \in \text{Syl}_3(G)$;

(ii) $3 \leq |S/Q| \leq 3^2$; and

(iii) $Q$ has exponent 3.

**Proof.**

(i) Since $C_M(Q) \leq Q$, we have that $Z = Z(Q) = Z(S)$. Therefore $N_G(S) \leq N_G(Z) = M$ and, in particular, $S \in \text{Syl}_3(N_G(S)) \subseteq \text{Syl}_3(G)$.

(ii) This follows directly from Lemma 3.1.

(iii) Since $[Q,a]$ admits $R$ and $C_Q(a)$ admits $R$, these groups have exponent 3 and they commute. Thus (iii) holds. \hfill $\square$

We have $\widehat{M} = M/Q$ is isomorphic to a subgroup of the subgroup of $GSp_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space into a perpendicular sum of two non-degenerate 2-spaces by Lemma 3.1. We now
describe this subgroup of GSp$_4$(3). We denote it by $\overline{M}$ as in Lemma 3.1. The boldface type is supposed to indicate that this is a subgroup of GSp$_4$(3) which contains (the image of) $\overline{M}$ but may be greater than it. Similarly $\overline{S}$ is a Sylow $3$-subgroup of $\overline{M}$ which contains $\overline{S}$.

We have $\overline{M}$ contains a subgroup of index 2 which is contained in Sp$_2$(3) and is isomorphic to the wreath product of Sp$_2$(3) $\cong$ SL$_2$(3) by a group of order 2. For $i = 1, 2$, we let $\overline{M}_i \cong$ SL$_2$(3), $\overline{R}_i = O_2(\overline{M}_i) \cong Q_8$ and $\overline{S}_i = \overline{S} \cap \overline{M}_i$. We let $\overline{t}_1$ be an involution in $\overline{M}$ which negates the symplectic form and normalizes $\overline{S}_1$ and $\overline{S}_2$. Note that, for $i = 1, 2$, $\overline{M}_i(\overline{t}_1) \cong$ GSp$_2$(3) $\cong$ GL$_2$(3). Next select an involution $\overline{t}_2$ which commutes with $\overline{t}_1$, preserves the symplectic form, normalizes $\overline{S}$ and conjugates $\overline{M}_1$ to $\overline{M}_2$. With this notation we have

$$\overline{M} = \overline{M}_1\overline{M}_2(\overline{t}_1, \overline{t}_2).$$

Now $\overline{M}$ is a subgroup of $\overline{M}$ which has index at most 6. In particular, $\overline{S}$ has index at most 3 in $\overline{S}$ by Lemma 3.2 (ii). Since $\overline{R}_1\overline{R}_2 = \overline{R}$, $\overline{M}$ contains subgroups $R_1$ and $R_2$ isomorphic to $Q_8$ such that $[R_1, R_2] = 1$ and $\overline{R}_i = \overline{R}$ for $i = 1$ and 2. Moreover $R = R_1R_2$. Let $T \in$ Syl$_2(M)$ with $T \geq R$. Now we do not yet know the index of $R$ in $T$. Thus $T$ may or may not contain elements which map to $\overline{t}_2$, $\overline{t}_2$ or $\overline{t}_1\overline{t}_2$. However if such elements are contained in $T$ we denote this involution by $t_1$, $t_2$ or $t_1t_2$ as appropriate. This discussion proves the following lemma.

**Lemma 3.3.** There are exactly five possibilities for a Sylow 2-subgroup $T$ of $M$. Moreover, one of the following hold.

(i) $T = R$, $N_M(S) = SZ(R)$ and $N_M(S)/S \cong 2^2$;
(ii) $T = R(\overline{t}_1)$, $N_M(S) = SZ(R)(\overline{t}_1)$ and $N_M(S)/S \cong 2^3$;
(iii) $T = R(\overline{t}_2)$, $N_M(S) = SZ(R)(\overline{t}_2)$ and $N_M(S)/S \cong \text{Dih}(8)$;
(iv) $T = R(\overline{t}_1\overline{t}_2)$, $N_M(S) = SZ(R)(\overline{t}_1\overline{t}_2)$ and $N_M(S)/S \cong \text{Dih}(8)$; and
(v) $T = R(\overline{t}_1, \overline{t}_2)$, $N_M(S) = SZ(R)(\overline{t}_1, \overline{t}_2)$ and $N_M(S)/S \cong 2 \times \text{Dih}(8)$. \hfill \Box

For $i = 1, 2$, let $r_i \in Z(R_i)$ and set $Q_i = [Q, r_i] = [Q, R_i]$. Note that, as $\overline{r}_1\overline{r}_2 \in Z(\overline{M})$ and $Q/Z$ is irreducible as an $\overline{M}$-module, $r_1r_2$ inverts $Q/Z$. Let $A$ be the preimage of $C_{Q/Z}(S)$. So $A$ is the second centre of $S$.

**Lemma 3.4.** The following hold.

(i) $Q_1 = [Q, R_1] = C_Q(R_2)$, $Q_2 = [Q, R_2] = C_Q(R_1)$ and both are normal in $S$;
(ii) $Q_1 \cong Q_2 \cong 3^1+^2$, $[Q_1, Q_2] = 1$ and $Q = Q_1Q_2$;
(iii) $A = [Q, S] = [Q_1, S][Q_2, S]$ is elementary abelian of order $3^3$; and

**Proof.** (i) This follows directly from the action of $M$ on $Q$ as $\overline{R}_1$ and $\overline{R}_2$ are normalized by $\overline{S}$.

(ii) We have that $C_Q(r_1)$ and $[Q, r_1]$ commute by the Three Subgroup Lemma. Since, for $i = 1, 2$, $[Q, r_i] = [Q, R_i]$ has order $3^3$ it follows that $Q_i \cong 3^1+^2$. As $r_1r_2$
inverts \(Q/Z\), \(r_2\) inverts \(C_{Q/Z}(r_1)\) and so \(C_Q(r_1) = Q_2\). In particular, \(Q_1\) and \(Q_2\) commute and \(Q = Q_1Q_2\).

(iii) From the description of \(M/Q\), we have \(A = [Q_1,S][Q_2,S]\). Since, for \(i = 1, 2\), \([Q_i/Z,S] \neq 1\) and \([Q_i,S]\) is normal in \(Q_i\), we have \(Z \leq [Q_i,S]\) and so \([Q_i,S]\) has order 9. Furthermore \([Q_i,S]\) is elementary abelian. Hence \(A\) is elementary abelian of order \(3^3\) by (ii).

Because, for \(i = 1, 2\), \(r_i\) inverts \(Q_i/Z\), if \(M\) happens to contain the involution \(t_1\), we may and do adjust \(t_1\) by multiplying by elements from \(Z(R)\) so that \(t_1\) inverts \(A/Z\). Therefore

**Lemma 3.5.** If \(t_1 \in M\), then \(t_1\) inverts \(A\) and centralizes \(Q/A\).

We now define a subgroup which will play a prominent role in all the future investigations. Set

\[ J = C_S(A). \]

It will turn out that \(J\) is the Thompson subgroup of \(S\).

**Lemma 3.6.** The following hold:

(i) \(|S : J| = 3^2\), \(J \cap Q = A\) and \(S = JQ\);
(ii) \(N_M(S) = N_M(J)\); and
(iii) if \(t_1 \in M\), then \(t_1\) inverts \(J\) and \(J\) is abelian.

In particular, the structure of \(N_M(J)/S\) is as described in the five parts of Lemma 3.5.

**Proof.** By Lemma 3.4(iii), \(A\) is elementary abelian of order \(3^3\). Furthermore, by the definition of \(J\), \(J\) is a normal subgroup of \(N_M(S)\). Since \([S,A] = Z\), the 3-structure of \(GL_3(3)\) shows that \(|S/J| \leq 3^2\). As \(J \cap Q = C_Q(A) = A\), we infer that \(|S : J| = 3^2\) and \(S = JQ\). Thus (i) holds.

As \(A = Z_2(S)\), we have \(J = C_S(A)\) is normalized by \(N_M(S)\). Also because \(N_M(J)\) normalizes \(J \cap Q = A\), we know \(N_M(J) \leq N_M(A)\). Now \(N_M(A) \geq N_M(S)\) and because \(N_R(A) = Z(R)\) and \(M = RN_M(S)\), we deduce that \(N_M(A) = N_M(S)\). Therefore \(N_M(J) = N_M(S)\) as claimed in (ii).

Suppose that \(t_1 \in M\). Then \(t_1\) inverts \(S/Q\), centralizes \(Q/A\) and inverts \(A\) by Lemma 3.5. Thus \(t_1\) inverts \(J\) and so \(J\) is abelian. This concludes the proof of (ii) and completes the verification of the lemma.

Note that \(|J| = 3^4\) if \(|S/Q| = 3\) and \(|J| = 3^5\) if \(|S/J| = 3^2\).

**Lemma 3.7.** We have \(C_G(J) = J\).

**Proof.** As \(Z \leq J\), we have \(C_G(J) = C_M(J)\). Now \(C_G(J)\) centralizes \(A = J \cap Q\) and it follows from Lemma 2.11 that \(C_M(J) = C_S(J) = J\).

**Lemma 3.8.** Every element of \(Q\) is conjugate in \(M\) to an element of \(A\).
Assume that \( \text{elementary abelian} \). This completes the proof of the lemma.

Therefore \( Z \) is abelian. Set \( B \leq S \) and then \( (B \cap Q)/Z \leq C_{Q/Z}(S) = A/Z \). Thus \( B \leq C_S(A) = J \). Hence \( J \) is the Thompson subgroup of \( S \). As \( J \) is abelian and weakly closed in \( S \), it follows from \( [37.6] \) that \( N_M(J) \) controls fusion in \( J \). In particular, \( X \) and \( Z \) are conjugate in \( N_M(J) \). Since \( \Phi(J) \leq A \), \( X \not \leq \Phi(J) \) and hence \( Z \not \leq \Phi(J) \). Therefore \( Z(S) \cap \Phi(J) = 1 \). As \( \Phi(J) \) is normal in \( S \), we get \( \Phi(J) = 1 \) and \( J \) is elementary abelian. This completes the proof of the lemma.

\textbf{Lemma 4.2.} Assume that \( Z \) is not weakly closed in \( J \) and set \( J_0 = \langle Z^{N_G(J)} \rangle \). Then

(i) \( |Z^{N_G(J)}| = 10 \) and, if \( X \in Z^{N_G(J)} \) with \( X \not \equiv Z \), \( |X^Q| = 3^2 \);

(ii) \( N_G(J) \) acts 2-transitively on \( Z^{N_G(J)} \); and

(iii) \( |J_0Q/Q| = 3 \) and \( J_0Q/Q \) is normalized by \( N_M(S)/Q \).
Proof. Let \( \mathcal{Y} = Z^{N_G(J)} \) and \( X \in \mathcal{Y} \) with \( X \neq Z \). Of course \( X \not\leq Q \) as \( Z \) is weakly closed in \( Q \). If \( C_Q(X) \not\leq J \), then, as \( X \) centralizes \( A \), \( C_Q(X) \) has order at least \( 3^4 \) and consequently is non-abelian and we have \( Z = C_Q(X)^J \leq O_3(C_G(X)) \).

However \( X \) is weakly closed in \( O_3(C_G(X)) \) with respect to \( G \) and so this is impossible. Thus \( C_Q(X) = A \) has order \( 3^3 \) and, in particular (as \( J \) is abelian), \( X^J = X^Q \) has order \( 3^2 \) and so \( |\mathcal{Y}| \equiv 1 \pmod{9} \). Observe that

\[
|\mathcal{Y}| = |N_G(J)|/|N_M(J)| = |N_G(J)/J|/|N_M(J)/J|.
\]

As \( |J| = 3^4 \) or \( 3^5 \) and \( J \) is self-centralizing and elementary abelian by Lemmas 3.7 and 4.1, \( |N_G(J)/J| \) divides \( |GL_5(3)| \). If \( |J| = 3^4 \), then, as no subgroup of order \( 3 \) in \( A \) which is not \( Z \) is conjugate to \( Z \), \( J \) contains at most \( 28 \) conjugates of \( Z \). This means that \( |\mathcal{Y}| = 10, 19 \) or \( 28 \). On the other hand, \( |GL_5(3)|_3 = 2^9 \cdot 5 \cdot 13 \) and so in this case \( |\mathcal{Y}| = 10 \). So assume from now on that \( |J| = 3^5 \). Then \( J \) contains \( 121 \) subgroups of order \( 3 \) and \( 12 \) of these are contained in \( A \) and are not conjugate to \( Z \) as \( Z \) is weakly closed in \( Q \). Since \( |GL_5(3)|_3 = 2^{10} \cdot 5 \cdot 11^2 \cdot 13 \) and \( |\mathcal{Y}| \equiv 1 \pmod{9} \), the only candidates for \( |\mathcal{Y}| \) are \( 10, 55 \) and \( 64 \). We recall from Lemma 3.6 that \( |N_M(J)/J| = 2^i \cdot 3^2 \) where \( i \in \{2, 3, 4\} \) and, if \( t_1 \in M \), then \( t_1 J \in Z(N_G(J)/J) \) and \( t_1 \) inverts every element of \( J \) by Lemma 3.6 (iii). In particular, \( t_1 \) normalizes every member of \( \mathcal{Y} \).

Suppose that \( |\mathcal{Y}| = 55 \). Then, by Lemma 3.6 (ii),

\[
|N_G(J)/J| = |N_G(J) : N_M(J)||N_M(J)/J| = 2^i \cdot 3^2 \cdot 5 \cdot 11
\]

where \( i \in \{2, 3, 4\} \). Let \( E \in \text{Syl}_{11}(N_G(J)/J) \). Then, as the normalizer of a cyclic subgroup of order \( 11 \) in \( GL_5(3) \) has order \( 2 \cdot 5 \cdot 11^2 \), the normalizer in \( N_G(J)/J \) of \( E \) has order dividing \( 110 \). In particular, \( E \) is not normal in \( N_G(J)/J \). If \( |N_M(J)/J|_2 = 2^i \), then \( t_1 J \) normalizes \( E \). So in any case the number of conjugates of \( E \) in \( N_G(J)/J \) divides \( 2^3 \cdot 3^2 \cdot 5 \) and is divisible by \( 2^2 \cdot 3^2 \) and this is impossible as it must also be congruent to \( 1 \) mod \( 11 \).

Suppose that \( |\mathcal{Y}| = 64 \). Then \( |N_G(J)/J| = 2^j \cdot 3^2 \) where \( j \in \{8, 9, 10\} \). In particular, \( N_G(J) \) is soluble. Since \( |\mathcal{Y}| = 64 \), we have that \( J = \langle \mathcal{Y} \rangle \). If \( 1 \neq K \leq J \) is normal in \( N_G(J) \), then \( K \) is normal in \( S \) and consequently \( Z \leq K \). But then \( \mathcal{Y} \subseteq K \) and so \( K = J \). Thus \( N_G(J) \) acts irreducibly on \( J \). Since \( |J| = 3^5 \) and \( N_G(J)/J \) is not abelian, Schur’s Lemma implies that \( |Z(N_G(J)/J)| \) divides \( 2 \) and, additionally, \( O_3(N_G(J)/J) = 1 \). Let \( L = O_{3,2}(N_G(J)) \). By Clifford’s Theorem [6, Theorem 4.3.1], \( J \) is completely reducible as an \( L \)-module and \( N_G(J) \) acts transitively on the homogeneous summands of \( J \) restricted to \( L \). Since \( J \) has dimension \( 5 \) as a \( GF(3)N_G(J) \)-module, and \( 5 \) does not divide \( |N_G(J)| \), we have that \( J \) is homogeneous as an \( L \)-module. It follows that \( J \) is either a direct sum of five \( 1 \)-dimensional \( L \)-modules or is irreducible as an \( L \)-module. It the first case, we get that \( |L, N_G(J)| \leq J \), \( O_3(N_L(J)/J) \neq 1 \) and this contradicts \( O_3(N_G(J)) = J \).

Thus \( J \) is an irreducible \( L \)-module. However, the degrees of irreducible \( L/Q \)-modules over the algebraic closure of \( GF(3) \) are all powers of \( 2 \) [11, 15.13] and this
again implies that $L/C_L(J)$ is cyclic, and $O_3(N_G(J)) > J$, again a contradiction. Thus $|Y| \neq 64$.

Since $|Y| \neq 55$ or 64, we must have $|Y| = 10$ as claimed in the first part of (i). Because we have also shown that $C_Q(X) = A$ the remaining parts of (i) also hold.

Part (ii) follows directly from (i).

Now with $J_0 = \langle Z^{N_G(J)} \rangle$, we have $(XQ)Q = XQ$ is normalized by $N_M(S)$ and $|XQ/Q| = 3$. This is (iii). \hfill \Box

Lemma 4.3. Suppose that $X \in Z^G \setminus \{Z\}$ and $X \leq S$. Then, for $i = 1, 2$, $[X, R_i] \not\leq Q$.

Proof. We suppose that $[X, R_1] \leq Q$ and seek a contradiction. Let $Q_X = O_3(N_G(X))$ and $W$ be the full preimage of $C_{Q/X}(X)$. Since $R_1$ acts irreducibly on $Q_X$ and $[Q_1, Q_X]$ is $R_1$-invariant, we have $Q_1 \leq W$. Then $|W| = 3^4$. Hence $W = Q_1A \cong 3 \times 3^3$ and $Z(W) = A \cap Q_2$.

If $C_W(X)$ is non-abelian, then, as $C_W(X)Q_X/Q_X$ is abelian, $Z = C_W(X)' \leq Q_X$. Since $X$ is weakly closed in $Q_X$ by assumption and $Z \neq X$, we have a contradiction.

Thus $C_W(X)$ is abelian. Since $W$ is non-abelian and $XZ$ is normalized by $W$, we get that $|C_W(X)| = 3^3$. Because $C_W(X)$ is abelian and $W$ is not, it follows that $A \cap Q_2 \leq C_W(X)$. Furthermore, we have $|C_W(X) \cap Q_1| = 3^2$ and thus, as $R_1$ acts transitively on the subgroups of order 9 in $Q_1$, we may adjust $X$ by conjugating by an element of $R_1$ and arrange for $W \cap Q_1 = A \cap Q_1$. But then $W = A$ and $X \leq J$. Put $J_0 = \langle X^{N_G(J)} \rangle$. Then by Lemma 4.2 (iii) $J_0Q = XQ$ is normalized by $N_M(S)$. Since $N_M(S)$ does not normalize $R_1$, we have $[X, R_1R_2] \not\leq Q$, and this contradicts the structure of $M$. Therefore $[X, R_i] \not\leq Q$ for both $i = 1$ and 2. \hfill \Box

Lemma 4.4. Assume that $X \in Z^G$ with $X \leq S$. Then $X \leq J$. In particular, $Z$ is not weakly closed in $J$.

Proof. Suppose that $X \leq S$ and $X \not\leq J$. Then $[A, X] = Z$ and $|C_A(X)| = 3^2$. By Lemma 4.3, $XQ$ acts non-trivially on both $R_1Q/Q$ and $R_2Q/Q$ and so $C_A(X) = C_Q(X)$. On the other hand $AX$ is normalized by $Q$ and so $AX$ contains at least, and hence exactly, 28 conjugates of $Z$. In particular, $C_A(X)X$ contains 10 conjugates of $Z$ and three subgroups of order 3 which are not conjugate to $Z$. Set $Q_X = O_3(N_G(X))$. Then the only conjugate of $Z$ contained in $C_A(X)X \cap Q_X$ is $X$. Since the subgroups of order 3 in $C_A(X)$ which are not conjugate to $Z$ generate $C_A(X)$, we get $C_A(X)X \cap Q_X = X$. So $|C_A(X)Q_X/Q_X| = 3^2$. By Lemma 4.3 two of the non-trivial cyclic subgroups of $C_A(X)Q_X/Q_X$ are not images of elements from $Z^G$. Since $C_A(X)X$ contains only three subgroups of order 3 which are not conjugate to $Z$, we have a contradiction. Therefore, if $X \in Z^G$ and $X \leq S$, $X \leq J$ as claimed. \hfill \Box

Set

$$J_0 = \langle Z^{N_G(J)} \rangle.$$
By Lemmas 4.2, 4.3 and 4.4, we have $|J_0Q/Q| = 3$, $J_0 \cap Q = A$ and $J_0Q/Q$ does not centralize either $R_1Q/Q$ or $R_2Q/Q$. In particular, $|J_0| = 3^4$. We record these facts in the first part of the next lemma.

**Lemma 4.5.** The following hold.

(i) $|J_0| = 3^4$, $|J_0Q/Q| = 3$, $J_0 \cap Q = A$ and $J_0Q/Q$ acts non-trivially on both $R_1Q/Q$ and $R_2Q/Q$;

(ii) $N_G(J) = N_G(J_0)$; and

(iii) $C_G(J_0) = C_G(J) = J$.

**Proof.** From the construction of $J_0$ we have $N_G(J_0) \geq N_G(J)$. Since $N_G(J)$ is transitive on the subgroups of $J$ which are $G$-conjugate to $Z$, we get that $N_G(J_0) = N_G(J)N_M(J_0)$. Hence, as $N_M(J_0Q) = N_M(S) \leq N_G(J)$, (ii) holds. Obviously $C_G(J_0) \leq C_M(J_0) \leq C_M(A) = J$ so (iii) also holds. □

Define

$$F = O^2(N_G(J))(r_2).$$

Note that $F$ is a group as $r_2$ normalizes $A/Z$ and hence $J$. Then

**Theorem 4.6.** The following hold:

(i) The action of $N_G(J)$ on $J_0$ preserves a non-degenerate quadratic form $q$ of $(-)$-type;

(ii) $Z^{N_G(J)}$ is the set of singular one-dimensional subspaces with respect to $q$;

(iii) $N_G(J)/J \cong 2 \times \text{Sym}(6)$ or $\text{Sym}(6)$; and

(iv) $F/J \cong \text{Sym}(6)$ and $\|[J, r_2]\| = 3$. Furthermore $[r_2, J] \leq J_0$ and $[J, F] \leq J_0$.

**Proof.** Let $X \leq J$ be conjugate to $Z$ but not equal $Z$. For $i = 1, 2$, using Lemma 4.5 (i), we have that $[J_0, Q_i/Z] \neq 1$ and so, as $[J_0, Q_i]$ is normal in $Q_i$, we get $[[J_0, Q_i]] = 3^2$ and $[J_0, Q_i, Q_i] = Z$. Furthermore, $[J_0, Q_i]$ is centralized by $Q_{3-i}$. Hence we have $[J_0, Q_i] = C_{J_0}(Q_{3-i})$. By Lemma 2.4 there exists a non-degenerate quadratic form $q$ on $J_0$ which is preserved by $Q$ and such that the elements of $X$ are singular vectors. It follows that with respect to $q$, the elements of $\bigcup X^Q$ are singular. Furthermore, as $Z = C_{J_0}(Q)$, $Z$ also consists of singular vectors. Now with respect to the bilinear form $f$ associated with $q$, none of the non-trivial elements of $\bigcup X^Q$ are perpendicular to the non-trivial elements of $Z$. It follows that $XZ$ contains exactly two singular subspace, namely $X$ and $Z$. Since $N_G(J)$ acts $2$-transitively on $Z^{N_G(J)}$ by Lemma 4.2 (ii), we infer that if $X, Y \in Z^{N_G(J)}$ with $X \neq Y$, then $XY$ contains exactly two members of $Z^{N_G(J)}$. Now suppose that $a \in Q \setminus J$ is such that $aJ$ acts quadratically on $J_0$. For $X \in Z^{N_G(J)} \setminus Z$ we know $[X, a] \neq 1$ and hence $|[X, a]| = 3$ as $a$ acts quadratically on $J_0$. It follows that $X[X, a] = XX^a$ contains three members of $Z^{N_G(J)}$, namely $X, X^a$ and $X^{a^2}$. This contradiction shows that no non-trivial element of $S/J$ acts quadratically on $J_0$. If $q$ was of $(+)$-type, this would not be the case. Hence $q$ is of $(-)$-type. We now have that $Z^{N_G(J)}$ is the set of singular one spaces in $J_0$ with respect to $q$. Since $N_G(J)$ preserves this set, we have that $N_G(J)/J$ is isomorphic to
a subgroup of $CO_4(3)$, the group preserving $q$ up to negation, from Lemma 2.3. Because $N_{N_G(J)}(Z)$ has index 10 in $N_G(J)$, we deduce that $|N_G(J)| = 2^i \cdot 5 \cdot 3^2$ where $i \in \{2, 3, 4\}$ by Lemma 2.3. In particular, $O^2(N_G(J)/J) \cong \Omega_1(3) \cong \text{Alt}(6)$. Now we see that $N_{O^2(N_G(J))}(S)/S$ is a cyclic group of order 4. Consequently of the five possibilities for the structure of $N_M(S)/S$ given in Lemma 3.3 only possibilities (iii), (iv) and (v) survive and we have $N_M(S)/N_{O^2(N_G(J))}(S)$ is elementary abelian.

Now let $C = CO_4(3)$, if $D \in \text{Syl}_3(X)$ and $Y \in \text{Syl}_2(N_G(D))$, then $Y \cong 2 \times \text{SDih}(16)$ and so using the structure of $N_M(S)/S$ given in Lemma 3.3 (iii), (iv) and (v) we infer that $N_G(J)/J \cong \text{Sym}(6)$ or $GO_4(3) \cong 2 \times \text{Sym}(6)$. We have now established (i), (ii) and (iii).

We know that $S/Q = JQ/Q$ is centralized by $r_2$ and that $[Q, r_2] = Q_2$. It follows that $[J, r_2] \leq Q_2 \cap J = A \cap Q_2$ and, as $[A \cap Q_2, r_2]$ has order 3, we now have that $[J, r_2] = [A, r_2]$ is a non-central cyclic subgroup of $Q$. In particular, $[J, r_2] \leq A \leq J_0$. Since $|[J_0, r_2]| = 3$ we get that $r_2$ has determinant $-1$ on $J_0$. Hence we have $r_2 \notin O^2(N_G(J))$ and so we conclude that $F/J \cong \text{Sym}(6)$ and that all the parts of (iv) hold.

As a corollary to Theorem 4.6, we record the following observation.

**Corollary 4.7.** There are exactly three possibilities for a Sylow 2-subgroup $T$ of $M$. They are as follows:

(i) $T = R(t_2)$, $N_M(S) = SZ(R)\langle t_2 \rangle$ and $N_M(S)/S \cong \text{Dih}(8)$;
(ii) $T = R(t_1 t_2)$, $N_M(S) = SZ(R)\langle t_1, t_2 \rangle$ and $N_M(S)/S \cong \text{Dih}(8)$; and
(iii) $T = R(t_1, t_2)$, $N_M(S) = SZ(R)\langle t_1, t_2 \rangle$ and $N_M(S)/S \cong 2 \times \text{Dih}(8)$.

In particular, $Q/Z$ is a chief factor in $M$.

**Proof.** The first three statement are readily deduced from the structure of $N_G(J)/J$ and so we only need to elucidate the fact that $Q/Z$ is a chief factor. For this we simply note that $t_2 \in M$ in all cases. □

**Theorem 4.8.** If $N_M(S)/S \cong \text{Dih}(8)$, then $G \cong \text{PSU}_6(2)$ or $\text{PSU}_6(2):3$.

**Proof.** Since $N_M(S)/S \cong \text{Dih}(8)$, we have that $N_G(J)/J \cong \text{Sym}(6)$ from the proof of Theorem 4.6 (ii) and Corollary 4.7. Since $[J, r_2] \leq J_0$, we infer that $J/J_0$ is centralized by $N_M(J)$. If $J > J_0$, then, by Lemma 2.2 $G$ has a normal subgroup $G^*$ of index 3. If $J = J_0$, then set $G = G^*$. Now $M \cap G^*$ satisfies the hypothesis of Theorem 2.9. Hence $G^* \cong \text{PSU}_6(2)$ and this proves the theorem. □

In light of Theorem 4.8 and Corollary 4.7, from here on we may assume that $N_M(S) = SZ(R)\langle t_1, t_2 \rangle$. In particular from Theorem 4.6, we have

$$N_M(S)/S \cong 2 \times \text{Dih}(8);$$
$$N_G(J)/J \cong 2 \times \text{Sym}(6);$$ and
$$C_{F/J}(r_2 J) \cong 2 \times \text{Sym}(4).$$

Furthermore, as $t_1$ inverts $J$, we have $t_1 J \in Z(N_G(J)/J)$. 

Lemma 4.9. We have

\[ C_S(Q_1) = C_S(R_1) = C_S(Q_1R_1), \]
\[ C_J(Q_1) = C_J(R_1) = C_J(Q_1R_1) \]

and \(|J : C_J(Q_1)| = 3^2.\)

Proof. We have that \([Q_1, C_S(R_1)]\) is \(R_1\)-invariant and is a proper subgroup of \(Q_1.\) Therefore \([Q_1, C_S(R_1)] \leq Z.\) Hence \([Q_1, C_S(R_1)], R_1 = 1\) and \([C_S(R_1), R_1, Q_1] = 1\) and thus the Three Subgroups Lemma implies that \([Q_1, R_1, C_S(R_1)] = 1.\) Since \(Q_1 = [Q_1, R_1],\) we have \(C_S(R_1) \leq C_S(Q_1).\) Now, as \(Q_1\) is normal in \(S\) and extraspecial of order \(3^3, |S : C_S(Q_1)Q_1| = 3,\) and so \(|C_S(Q_1)| = 3^4\) if \(|S| = 3^7\) and \(|C_S(Q_1)| = 3^3\) if \(|S| = 3^6.\) Since \(R_1\) centralizes \(Q_2,\) we have \(C_S(R_1) = C_S(Q_1) = Q_2\) if \(|S| = 3^6.\) If \(|S| = 3^7,\) then, as \(R_1Q\) is normalized by \(R_1S,\) we have \(|S/C_S(R_1)Q| = 3\) and hence the equality \(C_S(Q_1) = C_S(R_1)\) holds in this case as well. Of course we now have \(C_J(Q_1) = C_J(R_1) = C_J(Q_1R_1).\)

Since \(J\) normalizes \(R_1Q\) and does not centralize \(R_1Q/Q\) by Lemma 4.3, \(Q_1\) is normalized by \(J.\) Since \(J\) is abelian and \(J \cap Q_1 = A \cap Q_1,\) we now have that \(|J : C_J(Q_1)| = 3^2.\)

Notice that \(r_1J\) and \(r_2J\) are conjugate in \(N_G(J)/J\) (by \(t_2J\) for example) and

\[ \langle r_1, r_2, Q_1 \rangle J/J \cong 2 \times \text{Sym}(3). \]

In particular, we have \(r_1 \in F.\)

Let \(U \leq F\) be chosen so that \(\langle r_1, r_2, Q_1 \rangle J \leq U\) and \(U/J \cong \text{Sym}(5).\)

Lemma 4.10. If \(J \neq J_0,\) then \(|C_J(U)| = 3\) and \(|C_J(U)^F| = |C_J(U)^{N_G(J)}| = 6.\)

Proof. Since \(O^2(U)\) is generated by two conjugates of \(Q_1J,\) and \(|J : C_J(Q_1)| = 3^2\) by Lemma 4.9, we have that \(|C_J(O^2(U))| \geq 3.\) Since the elements of order 5 in \(U\) act fixed-point-freely on \(J_0\) we have \(C_J(O^2(U)) \cap J_0 = 1.\) Thus \(|C_J(O^2(U))| = 3\) and, as \(r_2\) centralizes \(J/J_0\) and normalizes \(C_J(O^2(U)),\) we get that \(C_J(O^2(U)) = C_J(U).\) Since \(|F : U| = 6, U\) is a maximal subgroup of \(F\) and \(t_1\) inverts \(J,\) we learn that \(|C_J(U)^F| = |C_J(U)^{N_G(J)}| = 6.\)

Lemma 4.11. Suppose that \(B \leq J_0\) with \(|B| = 3^3.\) Then \(B\) contains a conjugate of \(Z.\)

Proof. Recall that \(J_0\) is a non-degenerate quadratic space by Theorem 4.6(i). Hence this result follows because every subgroup of order \(3^3\) in the \(J_0\) contains a singular vector and the singular one-spaces in \(J_0\) are \(G-\)conjugate to \(Z.\)

We now fix some further notation. First let \(W = C_F(r_2).\) By the Frattini Argument we see that \(WJ/J \cong C_{F/J}(r_2J)\) and so \(WJ/J \cong 2 \times \text{Sym}(4)\) and \(J \cap W\) has index 3 in \(J\) by Theorem 4.6(iv).

If \(J = J_0,\) set \(\tau = 1,\) whereas, if \(J > J_0,\) select \(\tau \in C_J(U)^\#).\)
Suppose that \( J > J_0 \). Then \( \tau \neq 1 \). Let
\[
\mathcal{T} = \tau^F = \{ \tau_1, \tau, \ldots, \tau_6 \}
\]
be the six \( F \)-conjugates of \( \tau \). Then, as \([ J, r_2 ]\) has order 3 by Theorem 4.6 (iv), \( r_2 \) acts as a transposition on \( \mathcal{T} \) and \( r_2 \) centralizes \( \tau \) (as \( r_2 \in U \)). Since \( WJ/J \cong 2 \times \text{Sym}(4) \) and \( W \) has orbits of length 2 and 4 on \( \mathcal{T} \). It follows that, after adjusting notation if necessary, \( \tau^W = \{ \tau_1, \tau_2, \tau_3, \tau_4 \} \) and \( \tau^{r_2}_5 = \tau_6 \). We further fix notation so that \( Q_1 \) acts as \( \langle (\tau_2, \tau_3, \tau_4) \rangle \) and, since \( r_1 \) is conjugate to \( r_2 \) in \( N_G(J) \) and inverts \( QJ/J \), we may suppose that \( r_1 \) induces the transposition \( (\tau_2, \tau_3) \) on \( \tau^W \).

For \( 1 \leq i \leq 4 \), let
\[
J_i = \langle \tau_j \mid 1 \leq j \leq 4, i \neq j \rangle.
\]
Then each \( J_i \) is centralized by \( r_2 \) and is a hyperplane of \( C_J(r_2) \). Further
\[
J_i \cap J_j = \langle \tau_k \mid 1 \leq k \leq 4, k \notin \{ i, j \} \rangle.
\]
Let \( \rho \in [ J, r_2 ]^\# \). Then \( \rho \in (A \cap Q_2) \setminus Z \) as \( ||J, r_2|| = 3 \). Since \( [ J, r_1 ] \leq A \cap Q_1 \), we know \( [ \rho, r_1 ] = 1 \). From the choice of \( \tau \) and \( \rho \), we have that \( \langle Q_1, r_1 \rangle \) and \( \langle \tau, \rho \rangle \) commute.

We now select and fix once for all
\[
\rho \in [ A \cap Q_2, r_2 ]^\#.
\]

For \( J_0 = J \) we have to define the groups \( J_1, J_2, J_3 \) and \( J_4 \) differently. Set \( J_1 = C_A(r_2) = A \cap Q_1 \). So \( J_1 \) is normalized by \( \langle r_1, r_2, Q_1, J \rangle \) which has index 4 in \( W \). Observing that \( Z \) is centralized by the Sylow 3-subgroup \( S \) of \( F \) and \( \langle W, S \rangle = F \), yields that \( W \) is not contained in \( M \). As \( Z \) is the unique element of \( Z^G \) contained in \( J_1 \), we have \( J_1^W = \{ J_1, J_2, J_3, J_4 \} \) and \( W \) acts 2-transitively on \( J_1^W \). As \( r_1 \sim_M r_2 \), all the elements in \( J_1 \setminus Z \) are conjugate to \( \rho \). Therefore, as all the subgroups \( J_i \) are centralized by \( r_2 \), we have that \( |J_i \cap J_j| = 3 \) for all \( i \neq j \) and these intersections are conjugate to \( \langle \rho \rangle \). We capture some of the salient properties of these subgroups in the next lemma.

**Lemma 4.12.** For \( 1 \leq i \leq 4 \), \( J_i \leq C_G(r_2) \) and \( N_{N_G(J)}(J_i) \) contains a Sylow 3-subgroup of \( N_G(J) \).

**Proof.** If \( J > J_0 \), this is transparent from the construction of the subgroups. In the case that \( J = J_0 \), we have already mentioned that the subgroups commute with \( r_2 \). Also we have \( J_1 = A \cap Q_1 \) is normalized by \( S \) and as \( J_i, 2 \leq i \leq 4 \) are conjugates of \( J_1 \) in \( N_G(J) \), we have \( N_{N_G(J)}(J_i) \) contains a Sylow 3-subgroup of \( N_G(J) \). \( \square \)

Note also that when \( |J| = 3^6 \), \( \rho \in \langle [\tau_5, r_2] \rangle \). It follows that \( \langle \tau_5, \tau_6 \rangle \) contains \( \rho \) in this case. When \( J = J_0 \), of course we have \( \tau_i = 1 \). Thus to handle the two possible cases simultaneously we will consider the group \( \langle \tau_5, \rho \rangle \).
Lemma 4.13. \( \langle \tau_5, \rho \rangle \) is centralized by \( JQ_1R_1 \). Further \( C_G(\langle \tau_5, \rho \rangle) \not\leq M \).

Proof. Set \( X = \langle \tau_5, \rho \rangle \). If \( |J| = 3^4 \), then \( X = \langle \rho \rangle \leq A \cap Q_2 \) and the lemma holds. So suppose that \( |J| = 3^5 \). Then \( X = \langle \tau_5, \tau_6 \rangle \) is centralized by \( J \). Further, as \( \{ \tau_5, \tau_6 \} \) is a \( W \)-orbit and \( Q_1 \leq C_F(\tau_2) \leq W \), \( Q_1 \) centralizes \( X \). Since \( C_J(Q_1) = C_J(R_1) \) by Lemma 4.9 we now have \( [X, R_1] = 1 \) and this completes the proof.

Notice that \( \langle \tau_5, \rho \rangle \) is centralized by a subgroup of index 2 in \( W \) and so \( C_G(\langle \tau_5, \rho \rangle) \) is not contained in \( M \). \( \square \)

Lemma 4.14. The following hold.

(i) \( C_M(\rho) = JQ_1R_1\langle r_2t_1 \rangle \); and

(ii) If \( J > J_0 \), \( C_M(\langle \tau_5, \rho \rangle) = JQ_1R_1 \).

Proof. We calculate that \( C_M(\rho) \) contains \( JQ_1R_1\langle r_2t_1 \rangle \). As \( JQ_1R_1\langle r_2t_1 \rangle \) covers \( C_M/Q(\rho Z) \) (i) holds.

By Lemma 4.13 \( \langle \tau_5, \rho \rangle \) is centralized by \( JQ_1R_1 \). Since, by Lemma 3.6(iii), \( r_2t_1 \) conjugates \( \langle \tau_5 \rangle \) to \( \langle \tau_6 \rangle \), part (ii) follows from (i). \( \square \)

Lemma 4.15. \( Z \) is the unique \( G \)-conjugate of \( Z \) in \( \langle \tau_5, \rho, Z \rangle \).

Proof. Since \( Z \) is weakly contained in \( J \), \( Z \) is the unique conjugate of \( Z \) in \( \langle \rho, Z \rangle \). Also, as \( \tau_5 \) is not contained in \( J_0 \) and all the \( G \)-conjugates of \( Z \) in \( J \) are contained in \( J_0 \), there are no \( G \)-conjugates of \( Z \) in \( \langle \tau_5, \rho, Z \rangle \setminus \langle \rho, Z \rangle \). This proves the claim. \( \square \)

Lemma 4.16. Assume that \( J > J_0 \). Then \( N_G(\langle r_1, r_2 \rangle)/C_G(\langle r_1, r_2 \rangle) \not\cong \text{Sym}(3) \).

Proof. Let \( U = \langle r_1, r_2 \rangle \). By a Frattini Argument \( C_M(U) \) covers \( M/Q \). Hence as \( J > J_0 \), we have \( |C_M(U)|_3 = 3^3 \) and so \( D = C_J(U) \) is a Sylow 3-subgroup of \( C_M(U) \). Since \( Z \leq D \), we have \( C_G(D) = C_M(D) = JU \) which is 3-closed. Therefore, \( N_G(D) \leq N_G(J) \). Since \( r_1 \) and \( r_2 \) act as transpositions on \( T \), \( |N_F(DU)/J| = 32 \) and so we deduce that \( D \in \text{Syl}_3(C_G(U)) \). Let \( P = N_{N_G(U)}(D) \). Then by the Frattini Argument \( PC_G(U) = N_G(U) \). Therefore, if \( N_G(U)/C_G(U) \cong \text{Sym}(3) \), then \( r_2 \) and \( r_1r_2 \) are conjugate in \( P \). But \( P \leq N_G(J), r_2 \in F \setminus F' \) and \( r_1r_2 \in F' \) which is a contradiction. Hence \( N_G(U)/C_G(U) \not\cong \text{Sym}(3) \). \( \square \)

5. A Further 3-Local Subgroup and a 2-Local Subgroup in the Centralizer of an Involution

In this section we study the normalizer of \( \langle \tau_5, \rho \rangle \) and construct a 2-local subgroup of \( C_G(r_2) \).

Lemma 5.1. We have \( \mathcal{N}_G(J_0, 3') = \{1\} \).

Proof. Suppose that \( 1 \neq Y \in \mathcal{N}_G(J_0, 3') \). Then, as every hyperplane of \( J_0 \) contains a conjugate of \( Z \) by Lemma 4.11 and by coprime action \( Y \) is generated by centralizers of hyperplanes of \( J_0 \), we may assume that \( X = C_Y(Z) \neq 1 \). So \( X \in \mathcal{U}_M(J_0, 3') \). As \( X \) is normalized by \( A = J_0 \cap Q \) and \( X \) normalizes \( Q \),

\[ [A, X] \leq Q \cap X = 1. \]
But then $X$ centralizes a maximal abelian subgroup of $Q$ and consequently $[Q, X] = 1$ by Lemma 2.11 which is a contradiction. Thus $\mathcal{N}_G(J_0, 3') = \{1\}$.

**Lemma 5.2.** Assume $J = J_0$. Then $C_G(\rho) \not\cong \langle \rho \rangle \times \text{Sp}_6(2)$.

*Proof.* Suppose that $C_G(\rho) \cong \langle \rho \rangle \times \text{Sp}_6(2)$. Set $E = E(C_G(\rho))$. Then $E \cong \text{Sp}_6(2)$. We have that $r_2$ inverts $\rho$ and centralizes $J/\langle \rho \rangle$, so as $J \cap E$ has order $3^3$ and $C_E(J \cap E) = J \cap E$, $r_2$ induces the trivial automorphism on $E$. Hence $N_G(\langle \rho \rangle) \cong \text{Sym}(3) \times E$ and $[E, r_2] = 1$. In $E \cap J$ there is an element $\tilde{\rho}$ with $N_E(\langle \tilde{\rho} \rangle) \cong \text{Sp}_2(2) \times \text{Sp}_4(2)$ (see [3] page 46). Hence $N_{N_G(J)}(\langle \rho \rangle) \cap N_{N_G(J)}(\langle \tilde{\rho} \rangle)$ contains a Sylow 2-subgroup $T$ of $N_G(J)$. Now $\langle \rho, \tilde{\rho} \rangle = C_J(i)$, where $i \in T' \leq T$. Since $O_{3^2}(3) \cong \text{Dih}(8)$ and $N_G(J)/N_G(J)' \cong 2 \times 2$, we see that the involutions in $N_G(J)'/J$ invert a $-\text{space}$ and centralize a $+\text{-space}$ with respect to the form given in Theorem 4.10(i). In particular $C_J(i)$ is a $+\text{-space}$ and so $i$ centralizes a conjugate of $Z$. Hence $\langle \rho, \tilde{\rho} \rangle$ contains a conjugate of $Z$. But $C_E(\langle \rho, \tilde{\rho} \rangle)^{(\infty)} \cong \text{Sp}_4(2)^{(\infty)} \cong \text{Alt}(6)$ contradicts the fact that $M$ is soluble. □

**Lemma 5.3.** Let $B$ be a maximal subgroup of $\langle \tau_5, \rho, Z \rangle$ and assume that $C_G(B) \not\subseteq M$. Then $B \in \langle \tau_5, \rho \rangle^{Q_2}$ and either

(i) $J > J_0$ and $C_G(B) \cong B \times \text{SU}_4(2)$; or

(ii) $J = J_0$ and $C_G(\rho) \cong \langle \rho \rangle \times \text{Aut}(\text{SU}_4(2))$.

*Proof.* Set $U = \langle Z, \tau_5, \rho \rangle$, let $B$ be a maximal subgroup of $U$, $X = C_G(B)$ and $\tilde{X} = X/B$. Assume that $X \not\subseteq M$. By Lemma 4.13 $Z$ is the unique conjugate of $Z$ in $U$ and so, as $C_G(B) \not\subseteq M$, $U = ZB$ and $N_X(Z) = N_{\tilde{X}}(\tilde{Z})$.

Assume that $J > J_0$. Then, by Lemma 4.14(ii), $N_X(Z) = X \cap M = JQ_4R_1$ and so $N_{\tilde{X}}(\tilde{Z}) = N_X(Z) = J\tilde{R}_1Q_1 \cong 3^{1+2}\cdot \text{SL}_2(3)$ which is isomorphic to the centralizer of a 3-central element in $\text{SU}_4(2)$. As $B \cap \langle \rho, Z \rangle \not\subseteq 1$, we may assume that $\rho \in B$. Then by 4.14(i) we have $C_M(B) \leq C_M(\rho) = JQ_1R_1\langle r_2t_1 \rangle$. As $|U, r_2t_1| = 9$, we get $C_M(B) \leq JQ_1R_1$ and so $z \in Z^k$ is not $X$-conjugate to its inverse by Lemma 4.14. If $C_G(J_0, 3') = \{1\}$ by Lemma 5.1 and $C_G(B) \not\subseteq M$, we may apply Hayden’s Theorem 2.7 to get that $\tilde{X} \cong \text{SU}_4(2)$. Finally, as $JQ_1$, splits over $B$, $X$ splits over $B$ by Gaschütz’s Theorem [7, 9.26]. Hence $X$ has the structure described in (i).

Assume that $J = J_0$. In this case $B$ is $Q_2$-conjugate to $\langle \rho \rangle$. By Lemma 4.14(i), $C_X(Z) = X \cap M = JQ_1R_1$, as $r_2t_1$ inverts $Z$ and so $C_X(Z)$ is isomorphic to the centralizer of a 3-central element in $\text{SU}_4(2)$. Since $r_2t_1$ inverts $z$, we may use Prince’s Theorem 2.7 to obtain $\tilde{X} \cong \text{Aut}(\text{SU}_4(2))$ or $\text{Sp}_6(2)$. Again Gaschütz’s Theorem implies that $X \cong \langle \rho \rangle \times E$ where $E \cong \tilde{X}$. Therefore, by Lemma 5.2, $X$ has the structure claimed in (ii).

Now we consider the possibilities for $B$ when $J > J_0$. We have $B \leq U$ and $C_G(B) \not\subseteq M$. Thus, by (i), $C_G(B) \cong B \times E$ where $E \cong \text{SU}_4(2)$. Consequently, $N_{C_G(B)}(J) \cong 3^2 \times (3^3:\text{Sym}(4))$. Since $N_{C_G(B)}(J) \geq Q_1$ and since there are exactly
three subgroups isomorphic to Alt(4) which contain a given 3-cycle in \( \text{Sym}(6) \), we see that \( B \) is \( Q_2 \)-conjugate to \( \langle \tau_5, \rho \rangle \) as claimed. \( \blacksquare \)

We now set \( r = r_2 \) and aim to determine
\[
K = C_G(r).
\]

We will frequently use the following observation.

**Lemma 5.4.** \( C_J(r)Q_1 \) is a Sylow 3-subgroup of \( K \).

**Proof.** Certainly \( C_J(r)Q_1 \leq K \) by Lemma 3.4 (i). Because \([Q_1, C_J(r), Q_1] = [A \cap Q_1, Q_1] = Z\), we have that \( Z \) is a characteristic subgroup of \( C_J(r)Q_1 \) and so it follows that \( N_K(C_J(r)Q_1) \leq C_M(r) \). As \( C_J(r)Q_1 \in \text{Syl}_3(C_M(r)) \), the lemma holds. \( \blacksquare \)

Define \( E = E(C_G(\langle \tau_5, \rho \rangle)) \). Then \( E \cong \text{SU}_4(2) \) by Lemma 5.3

**Lemma 5.5.** We have \( E\langle t_1, \tau_5 \tau_6 \rangle \leq K \) and \( E\langle t_1 \rangle \cong \text{Aut}(\text{SU}_4(2)) \).

**Proof.** We know that \( r \) inverts \( \rho \) and exchanges \( \tau_5 \) and \( \tau_6 \). Hence \( r \) normalizes \( B = \langle \tau_5, \rho \rangle \) and consequently \( r \) normalizes \( E \). Furthermore, \( r \) centralizes \( J \cap E \) and since no involutory automorphism of \( E \) inverts \( \tau_5 \) we have that \( J \cap E \) inverts \( \tau_6 \). Therefore \( E \leq K \).

Since \( t_1 \) inverts \( J \) and \( t_1 \) normalizes \( \langle \tau_5, \rho \rangle \) and \( t_1 \) therefore normalizes \( E \). Since \( t_1 \) inverts \( J \cap E \) and, by [3, page 26], no inner automorphism of \( \text{SU}_4(2) \) inverts an elementary abelian group of order 27, we have \( E\langle t_1 \rangle \cong \text{Aut}(\text{SU}_4(2)) \). \( \blacksquare \)

From Lemmas 4.14 and 5.3 we have \( Q_1R_1 \leq E \). Furthermore, as \( W(= C_F(r)) \) normalizes \( [J, r] = \langle \rho \rangle \), we also have that \( C_W(\rho) \leq E \). In particular, we have

**Lemma 5.6.** \( \langle \tau_5 \tau_6 \rangle E = \langle C_W(\rho), Q_1R_1C_J(r) \rangle \).

**Proof.** As \( Q_1R_1C_J(r) \) contains the maximal parabolic subgroup of shape \( 3_4^1+^2 \text{SL}_2(3) \) of \( \text{SU}_4(2) \), we have that \( Y = Q_1R_1C_J(r) \) is a maximal subgroup of \( E\langle \tau_5 \tau_6 \rangle \) and \( C_W(\rho) \nleq Y \). \( \blacksquare \)

When \( J > J_0 \), as \( N_G(J) \) acts 2-transitively on \( T \), \( \langle \tau_5, \tau_6 \rangle \) is \( G \)-conjugate to each subgroup \( J_i \cap J_j \) for \( 1 \leq i < j \leq 4 \). When \( J = J_0 \) we have the same result from the construction of \( J_1, J_2, J_3 \) and \( J_4 \) in Section 4. Hence we may apply Lemma 5.3 to obtain the following conclusion.

**Lemma 5.7.** Assume that \( 1 \leq i < j \leq 4 \).

(i) If \( J > J_0 \), then \( C_G(J_i \cap J_j) \cong (J_i \cap J_j) \times \text{SU}_4(2) \); and
(ii) If \( J = J_0 \), then \( C_G(J_i \cap J_j) \cong (J_i \cap J_j) \times \text{Aut}(\text{SU}_4(2)) \).

\( \blacksquare \)

For \( 1 \leq i < j \leq 4 \), define
\[
E_{ij} = E(C_G(J_i \cap J_j)).
\]
Lemma 5.8. For \( 1 \leq i < j \leq 4 \) and \( k \in \{i, j\} \), \( E_{ij} \cap J_k \) is conjugate to \( Z \) and is 3-central in \( E_{ij} \). In particular, \( C_G(J_i) \cong (J_i \cap J_j) \times 3^{1+2}.SL_2(3) \) if \( J > J_0 \) and \( C_G(J_i) \cong (J_i \cap J_j) \times 3^{1+2}.SL_2(3).2 \) if \( J = J_0 \).

Proof. Let \( 1 \leq i \leq 4 \). Then by Lemma 4.12, \( J_i \) is normalized by a Sylow 3-subgroup \( T_i \) of \( N_G(J) \) and \( N_{T_i}(J_i) \) has index 3 in \( T_i \). In particular, as \( |C_G(J_i \cap J_j)|_3 = 3|J| \), we see that \( C_{T_i}(J_i) \in \text{Syl}_3(C_G(J_i \cap J_j)) \). Therefore \( J_i \cap E_{ij} \) is normalized by a Sylow 3-subgroup of \( E_{ij} \). As \( |J_i \cap E_{ij}| = 3 \), we have that \( J_i \cap E_{ij} \) is 3-central in \( E_{ij} \) as \( J_i \) is normal in \( T_i \). We see that this subgroup is also normal in a Sylow 3-subgroup of \( G \).

Define
\[
\Sigma = \langle O_2(C_K(J_k)) \mid 1 \leq k \leq 4 \rangle.
\]

In the next lemma we use the fact that if \( x \in SU_4(2) = X \) is an involution which centralizes a subgroup of order 9, then \( x \) is 2-central and
\[
C_X(x) \cong 2_+^{1+4}.(3 \times \text{Sym}(3)) \cong (SL_2(3) \circ SL_2(3)).2
\]
where \( \circ \) denotes a central product (see [3] page 26).

Lemma 5.9. Assume that \( 1 \leq i < j \leq 4 \). Then

(i) \( O_2(C_K(J_i)) \cong O_2(C_K(J_j)) \cong Q_8 \), \( [O_2(C_K(J_i)), O_2(C_K(J_j))] = 1 \) and \( O_2(C_K(J_i \cap J_j)) = O_2(C_K(J_i))O_2(C_K(J_j)) \cong 2^{1+4} \); and

(ii) \( \Sigma \) is extraspecial of \(+\)-type and order \( 2^9 \).

Proof. Suppose that \( 1 \leq i < j \leq 4 \). Then \( J_i \leq C_G(r) \) by Lemma 5.4. If \( J > J_0 \), we have \( r \in E_{ij} \) by Lemma 5.5. If \( J = J_0 \), then as \( [J_1, R_2] = 1 \) and \( r \in Z(R_2) \leq C_G(J_1) \) we have \( r \in E_{ij} \) and consequently \( r \in E_{ij} \) as \( W \) acts 2-transitively on \( \{J_1, J_2, J_3, J_4\} \).

Since \( r \in E_{ij} \) and \( |C_J(r) \cap E_{ij}|_3 \geq 9 \), \( r \) is a 2-central involution in \( E_{ij} \). It follows that \( K \cap E_{ij} \) has shape \( 2_+^{1+4}.(3 \times \text{Sym}(3)) \) and, in particular, \( O_2(C_K(J_i \cap J_j)) \cong 2_+^{1+4} \). Furthermore, as \( J_i \cap E_{ij} \) is 3-central by Lemma 5.3, we get \( O_2(C_K(J_i)) \cong Q_8 \) and \( O_2(C_K(J_i \cap J_j)) = O_2(C_K(J_i))O_2(C_K(J_j)) \). Since \( O_2(C_K(J_i \cap J_j)) \) contains exactly two subgroups isomorphic to \( Q_8 \), we have that \( [O_2(C_K(J_i)), O_2(C_K(J_j))] = 1 \). This completes the proof of (i).

Part (i) shows that \( \Sigma \) is isomorphic to a central product of 4 quaternion groups. Hence \( \Sigma \) is extraspecial of \(+\)-type and order \( 2^9 \). So (ii) holds. \( \square \)

Recall from Corollary 4.7 and 4.8, \( t_2 \in N_G(S) \leq M \cap N_G(J) \) and \( R_2^{t_2} = R_2 \).

Lemma 5.10. We have \( J_1 \) is centralized by \( R_2 \), \( R_2 \leq \Sigma \) and \( R_2 = C_{\Sigma}(Z) \).

Proof. Suppose first that \( J = J_0 \). Then \( J_1 = C_A(r) \leq Q_1 = C_Q(R_2) \) by Lemma 3.3

(i). So \( [J_1, R_2] = 1 \). Hence \( R_2 = O_2(C_K(J_1)) \leq \Sigma \).

Assume that \( J > J_0 \). We have that \( \tau_1 \) commutes with \( Q_1 \) and \( [(\tau_5, \tau_6), Q_1] = 1 \) by Lemma 4.14. Hence \( C_J(Q_1) = \langle \tau_1, \tau_5, \tau_6 \rangle = \langle \tau_5, A \cap Q_2 \rangle \). Thus \( C_J(Q_2) = \)
$C_J(Q_1)^{12} = \langle \tau_2, \tau_3, \tau_4 \rangle = \langle \tau_2, A \cap Q_1 \rangle$. By Lemma 4.3, $C_J(Q_1)$ is centralized by $R_1$, thus $J_1 = \langle \tau_2, \tau_3, \tau_4 \rangle$ is centralized by $R_2 = R_1^{12}$. Hence $R_2 = O_2(C_K(J_1)) \leq \Sigma$.

Since $R_2$ commutes with $Z$, we have $R_2 \leq C_\Sigma(Z)$ and, as $C_\Sigma(Z)$ is extraspecial we have that $R_2 = C_\Sigma(Z)$ from the structure of $M$. \hfill \Box

**Lemma 5.11.** We have $W(t_1) \leq N_K(\Sigma)$.

*Proof.* Since $W(t_1)$ permutes $\{J_1, J_2, J_3, J_4\}$ and is contained in $K$, $W(t_1) \leq N_K(\Sigma)$ by the definition of $\Sigma$. \hfill \Box

**Lemma 5.12.** We have $W = N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$. In particular $N_{N_K(\Sigma)}(C_J(r))$ controls $K$-fusion in $C_J(r)$.

*Proof.* We have that $C_G(C_J(r)) = J\langle r \rangle$. Hence $J$ is normal in $N_G(C_J(r))$. Now we have that $W = N_K(C_J(r))$. By Lemma 5.11 we have $W \leq N_K(\Sigma)$ and so $N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$. Further by Lemma 4.4 we have that $N_G(J)$ controls fusion in $J$ and so $N_K(C_J(r))$ controls $K$-fusion in $C_J(r)$. As $N_K(C_J(r)) = N_{N_K(\Sigma)}(C_J(r))$ this fusion takes place in $N_K(\Sigma)$. \hfill \Box

**Lemma 5.13.** Every $J_1$-signalizer in $K$ is contained in $\Sigma$. In particular, $N_K(J_1) \leq N_K(\Sigma)$.

*Proof.* Let $\Sigma_1 \leq K$ be a $J_1$-signalizer. Let $X_1$ be a hyperplane in $J_1$ such that $C_G(X_1) \leq M$. Then $C_{\Sigma_1}(X_1) \leq M$ is normalized by $J_1$ and so $C_{\Sigma_1}(X_1) \leq R_2 \leq \Sigma$ by Lemma 5.10. In particular $[C_{\Sigma_1}(Z), J_1] = 1$.

Suppose next that $X_1$ is a hyperplane such that $C_G(X_1) \not\leq M$. Then, by Lemma 5.8, we may assume that $X_1 = J_1 \cap J_2$. Since $r$ is 2-central in $E_{12}$, $O_2(C_J(J_1 \cap J_2))$ is the unique maximal $J_1$-signalizer in $C_G(X_1)$. Hence by Lemma 5.9 (i) we have that $C_{\Sigma_1}(X_1) \leq \Sigma$ in this case as well. Because

$$\Sigma_1 = \langle C_{\Sigma_1}(X_1) \mid |J_1 : X_1| \leq 3 \rangle \leq \Sigma,$$

we have that every $J_1$-signalizer is contained in $\Sigma$. Thus $\Sigma$ is the unique maximal member of $\mathcal{U}_K(J_1, 3')$ and so $N_K(J_1) \leq N_K(\Sigma)$ as $N_K(J_1)$ acts via conjugation on the maximal elements of $\mathcal{U}_K(J_1, 3')$. \hfill \Box

**Lemma 5.14.** $C_K(\Sigma) = \langle r \rangle$.

*Proof.* If $C_K(\Sigma)$ is a $3'$-group, then $C_K(\Sigma)$ is normalized by $J_1$ and so $C_K(\Sigma) \leq Z(\Sigma) = \langle r \rangle$ by Lemma 5.13. So suppose that $C_K(\Sigma)$ has order divisible by 3. Since by Lemma 5.4 $C_J(r)Q_1 \in Syl_3(K)$ and $C_J(r)Q_1 \leq W \leq N_K(\Sigma)$ by Lemma 5.11 we have $C_J(r)Q_1 \cap C_G(\Sigma) = \text{a Sylow 3-subgroup of } C_G(\Sigma)$. As $Z$ does not centralize $\Sigma$, we have $C_J(r)Q_1 \cap C_G(\Sigma) \leq C_J(r)$. Now, for $1 \leq i < j \leq 4$

$$C_{C_J(r)}(O_2(C_K(J_i \cap J_j))) = J_i \cap J_j$$

and consequently $C_{C_J(r)}(\Sigma) \leq J_1 \cap J_2 \cap J_3 \cap J_4 = 1$ which is a contradiction. \hfill \Box

**Lemma 5.15.** $\Sigma/\langle r \rangle$ is a minimal normal subgroup of $N_K(\Sigma)/\langle r \rangle$.
Proof. Suppose that $U \leq \Sigma$ and $U/\langle r \rangle$ is a minimal normal subgroup of $N_K(\Sigma)/\langle r \rangle$ of minimal order. Aiming for a contradiction, assume that $U \neq \Sigma$. Then either $|\Sigma : U| \leq 2^4$ or $|U/\langle r \rangle| \leq 2^4$. In particular, as $Q_1$ normalizes $\Sigma$ (see Lemma 5.13) and $\text{GL}_4(2)$ has elementary abelian Sylow 3-subgroups, $Z$ centralizes one of $U$ or $\Sigma/U$. By Lemma 5.10 either $U \leq R_2$ or $|\Sigma : U| \leq 2^2$ and $U \geq [\Sigma, Z]$.

Since $C_J(r)$ acts non-trivially on $R_2$, we get $U = R_2$ or $U = [\Sigma, Z]$. In the latter case, we have $U_1 = C_\Sigma(U)$ is normalized by $N_K(\Sigma)$ and has order smaller than $U$. Hence the minimal choice of $U$ implies that $U = R_2$. However $W \leq N_G(\Sigma)$ by Lemma 5.11 and $W$ does not normalize $R_2$ and so we have a contradiction. □

Theorem 5.16. One of the following holds.

(i) $J = J_0$ and $N_G(\Sigma)/\Sigma \cong \text{Aut}(\text{SU}_4(2))$ or $\text{Sp}_6(2)$;
(ii) $J > J_0$ and $N_G(\Sigma)/\Sigma \cong (3 \times \text{SU}_4(2)) : 2$.

Furthermore, $E\langle \tau_5\tau_6, t_4 \rangle \leq N_K(\Sigma)$ and $\Sigma/\langle r \rangle$ is isomorphic to the natural $ES/\Sigma$-module.

Proof. From Lemma 5.11 we have that $W(t_4) \leq N_G(\Sigma)$. Set $L = J_1Q_1$. Then $L \leq W$ and so $L \leq N_G(\Sigma)$. By Lemma 5.13 we have that $\Sigma$ is a maximal signalizer in $K$ for $L$ and for $C_J(r)$. Hence $N_K(L)$ and $N_K(C_J(r))$ both normalize $\Sigma$.

Suppose that $J = J_0$. Then $J_1Q_1 = (A \cap Q_1)Q_1 \leq Q_1$ and so $R_1 \leq N_K(Q_1) \leq N_K(\Sigma)$. Therefore Lemma 5.6 implies that $\langle E, t_4 \rangle \leq N_K(\Sigma)$. In particular, we have $C_{N_K(\Sigma)}/\Sigma(Z\Sigma/\Sigma)$ is isomorphic to the centralizer of a 3 element in $\text{SU}_4(2)$ and is inverted by $t_4\Sigma$. Hence Theorem 2.7 shows that (i) holds.

Suppose that $J > J_0$. This time $N_K(J_1Q_1)$ does not contain $R_1$. On the other hand $N_K(\Sigma) \geq N_K(C_J(r)) = W\Sigma$ and $W\Sigma/\Sigma$ has shape $3^4:\langle\text{Sym}(4) \times 2\rangle$. By the Frattini Argument, $N_{N_K(\Sigma)}(C_J(r)\Sigma/\Sigma) = N_{N_K(\Sigma)}(C_J(r))$. Since $N_K(C_J(r)) = W$, we now have $N_{N_K(\Sigma)}/\Sigma(C_J(r)\Sigma/\Sigma) = W\Sigma/\Sigma$.

Since $C_G(\Sigma) = \langle r \rangle$ by Lemma 5.11 we have that $N_K(\Sigma)/\Sigma$ is isomorphic to a subgroup of $O^+_8(2)$. Because $N_{N_K(\Sigma)}/\Sigma(C_J(r)\Sigma/\Sigma) = W\Sigma/\Sigma$, we infer from the list of maximal subgroups of $O^+_8(2)$ given in [3, page 85] that either $N_K(\Sigma) = W\Sigma$ or $N_K(\Sigma)/\Sigma \cong (3 \times \text{SU}_4(2)) : 2$. In the latter case we have (ii) so suppose that $N_K(\Sigma) = W\Sigma$. Let $T \in \text{Syl}_2(N_K(\Sigma))$. We claim that $T \in \text{Syl}_2(K)$. Assume that $x \in N_K(T) \setminus N_K(\Sigma)$. Then, as $\Sigma^x \neq \Sigma$, $J(T/\langle r \rangle) \not\leq \Sigma/\langle r \rangle$. Hence, setting $H = \langle J(T)N_K(\Sigma) \rangle$ and noting that $|O_3(N_K(\Sigma)/\Sigma)| = 3^4$, we may apply [11, (32.5)] to get that $H/\Sigma$ is a direct product of four subgroups isomorphic to $\text{SL}_2(2)$. But then the 2-rank of $W/\Sigma$ is at least 4 contrary to $T/\Sigma \cong \text{Dih}(8) \times 2$. Hence $N_K(T) \leq N_K(\Sigma)$ and in particular, $T \in \text{Syl}_2(K)$.

From Lemma 5.5, we have $E \leq K$. Since $T \in \text{Syl}_2(K)$, $T/\Sigma \cong \text{Dih}(8) \times 2$ and $E$ contains an extraspecial subgroup of order $2^5$ with centre $\langle r_1 \rangle$, we have that $r_1$ is $K$-conjugate to an element of $\Sigma$. Thus there is some $x \in K$ such that $\langle r_1, r \rangle \leq \Sigma^x$. Since $r_1^4 = r$ and since $r_1$ and $rr_1$ are $\Sigma^x$-conjugate, we have $N_G(\langle r_1, r \rangle)/C_G(\langle r_1, r \rangle) \cong \text{Sym}(3)$. This contradicts Lemma 4.16. Hence (ii) holds.
We have already seen that $E \leq N_K(\Sigma)$ if $J = J_0$. If $J > J_0$, then we have $N_{N_K(\Sigma)}(Z)$ contains a subgroup $(3 \times 3^{1+2})^{\text{SL}_2(3)}2$. Since $N_K(Z) = C_M(r) = Q_1R_1R_2C_J(r)/t_1$, we have $C_M(r) \leq N_K(\Sigma)$. Now $E\langle \tau_5\tau_6, t_1 \rangle \leq N_K(\Sigma)$ by Lemma 5.6. Finally, as $E$ acts irreducibly on $\Sigma/\langle r \rangle$ by Lemma 5.15, we have that $\Sigma/\langle r \rangle$ is the natural $E$-module.

We need just two final details before we can move on to determine the structure of $K$.

Lemma 5.17. The following hold.

(i) $N_K(Z) \leq N_K(\Sigma)$; and
(ii) $N_K(J_i \cap J_j) \leq N_K(\Sigma)$, for $1 \leq i < j \leq 4$.

Proof. For (i) we note that $N_K(Z) = C_M(r) \leq E\langle \tau_5\tau_6, t_1 \rangle \Sigma \leq N_K(\Sigma)$ by Theorem 5.16.

By Lemma 5.9 (i) we have that $O_2(C_K(J_i \cap J_j)) \leq \Sigma$ and, as $r$ is a 2-central element in $E_{ij}$, $C_J(r) \in \text{Syl}_3(C_K(J_i \cap J_j))$. Hence

$$N_K(J_i \cap J_j) = N_{N_K(J_i \cap J_j)}(C_J(r))O_2(C_K(J_i \cap J_j)) \leq N_K(\Sigma)$$

by Lemma 5.13. \hfill \Box

6. The structure of $K$

In this section we prove Theorem 6.11 which asserts that $K = N_K(\Sigma)$. We continue the notation introduced in the previous sections. We further set $K_1 = N_K(\Sigma)$ and denote by $x$ the natural homomorphism from $K$ onto $K/\langle r \rangle$.

By Lemma 5.13, the subgroup $\tilde{\Sigma}$ can be regarded as the 8-dimensional irreducible GF(2)-module for $K_1/\tilde{\Sigma}$. Thus we may employ the results of Proposition 2.12 to obtain information about various centralizers of elements of order 2 and 3 in $\tilde{\Sigma}$. Using Proposition 2.12 (ii), we have $K_1$ has two orbits on $\tilde{\Sigma}$. We pick representatives $\tilde{x}$ and $\tilde{y}$ of these orbits with $\tilde{x}$ singular and $\tilde{y}$ non-singular. It follows that $x$ is an involution and $y$ has order 4.

Our aim is to show that $\tilde{\Sigma}$ is strongly closed in $\tilde{K}$ and then use Goldschmidt’s Theorem 6.11 to show that $K = K_1$. We now begin the proof of Theorem 6.11.

Lemma 6.1. We have $\tilde{K}_1$ contains a Sylow 2-subgroup of $C_{\tilde{K}}(\tilde{y})$. In particular $|C_{\tilde{K}}(\tilde{y})|_2 = 2^{12}$ if $E(K_1/\tilde{\Sigma}) \cong \text{SU}_4(2)$ and $|C_{\tilde{K}}(\tilde{y})|_2 = 2^{14}$ if $K_1/\tilde{\Sigma} \cong \text{Sp}_6(2)$.

Proof. Let $T$ be a Sylow 2-subgroup of $C_{\tilde{K}}(\tilde{y})$ and assume that $T_1$ is a 2-group with $|T_1 : T| = 2$. Choose $u \in T_1 \setminus T$. If $|\tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma}| \leq 2$, then $|\tilde{\Sigma}^u \cap \tilde{\Sigma}| \geq 2^7$. But by Proposition 2.12 (iv), $\tilde{K}_1$ has no 2-elements not in $\tilde{\Sigma}$ which centralize a subgroup of index two in $\tilde{\Sigma}$. Therefore $\tilde{\Sigma} = \tilde{\Sigma}^u$ and so $u \in T_1 \cap \tilde{K}_1 = T$ which is a contradiction. Hence $|\tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma}| \geq 4.$
If \( E(\tilde{K}_1/\tilde{\Sigma}) \cong \text{SU}_4(2) \), then \( T/\tilde{\Sigma} \) is a semidihedral group of order 16 by Proposition \([2.12](ii)\). Since \( \tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma} \) is a normal elementary abelian subgroup of \( T/\tilde{\Sigma} \) of order at least 2, we have a contradiction. Hence \( \tilde{K}_1/\tilde{\Sigma} \cong \text{Sp}_6(2) \) by Lemma \([5.16](ii)\).

Now Proposition \([2.12](ii)\), gives

\[
C_{\tilde{K}_1}(\tilde{y})/\tilde{\Sigma} \cong G_2(2).
\]

Since, by \([8\, Table\, 3.3.1]\), \( G_2(2) \) does not contain elementary abelian subgroups of order 16, \( 2^6 \geq |\tilde{\Sigma}^u \cap \tilde{\Sigma}| \geq 2^5 \). But then all involutions in \( \tilde{\Sigma}^u \) centralize a subgroup of order at least \( 2^5 \) in \( \tilde{\Sigma} \), and so Proposition \([2.12](i)\) and \((iv)\) shows that all the involutions in \( \tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma} \) are unitary transvections and are conjugate in \( \tilde{K}_1/\tilde{\Sigma} \). Since the two classes of involutions in \( C_{\tilde{K}_1}(\tilde{y})/\tilde{\Sigma} \cong G_2(2) \) are not fused in \( \tilde{K}_1/\tilde{\Sigma} \), we infer that

\[
\tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma} \leq (C_{\tilde{K}_1}(\tilde{y})/\tilde{\Sigma})' \cong G_2(2)' \cong \text{SU}_3(3).
\]

Since, by \([8\, Table\, 3.3.1]\), \( \text{SU}_3(3) \) has no elementary abelian groups of order 8, we have \( |\tilde{\Sigma}^u \tilde{\Sigma}/\tilde{\Sigma}| = 4 \). This means that \( |\tilde{\Sigma}^u \cap \tilde{\Sigma}| = 2^6 \) and consequently all the involutions in \( \tilde{\Sigma}^u\tilde{\Sigma}/\tilde{\Sigma} \) have the same centralizer in \( \tilde{\Sigma} \). As centralizers of involutions in \( G_2(2)' \) are maximal subgroups \([3\, page\, 14]\), we conclude that \( \tilde{\Sigma}^u \cap \tilde{\Sigma} \) is normalized by \( (C_{\tilde{K}_1}(\tilde{y})/\tilde{\Sigma})' \). Thus \( (C_{\tilde{K}_1}(\tilde{y})/\tilde{\Sigma})' \) centralizes \( \tilde{\Sigma} \) which is impossible. This contradiction proves the lemma. The order of \( T \) is calculated from Proposition \([2.12](iii)\).

\[ \square \]

**Lemma 6.2.** Let \( S_1 \) be a Sylow 3-subgroup of \( C_{\tilde{K}_1}(\tilde{x}) \) or \( C_{\tilde{K}_1}(\tilde{y}) \). Then \( N_{\tilde{K}}(S_1) \leq \tilde{K}_1 \). In particular, for \( z \in \tilde{\Sigma}^\# \), \( C_{\tilde{K}_1}(z) \) contains a Sylow 3-subgroup of \( C_{\tilde{K}}(z) \).

**Proof.** We consider \( \tilde{y} \) first. By Proposition \([2.12](iii)\), \( S_1 \) has centre of order 3 and, as faithful GF(2)-representations of extraspecial groups of type \( 3_1^{1+2} \) have dimension 6, we have \( |C_3(Z(S_1))| = 4 \). As \( |C_3(Z)| = 4 \) we may assume using Proposition \([2.12](iii)\) that \( Z = Z(S_1) \). On the other, Lemma \([5.17](i)\) gives \( C_M(r) \leq K_1 \). Hence we have that \( N_{\tilde{K}}(S_1) \leq \tilde{K}_1 \).

Now we consider \( \tilde{x} \). By Lemma \([5.9](i)\), we have \( O_2(C_K(J_1 \cap J_2)) \leq \Sigma \). Hence, comparing orders, we may assume that \( S_1 = J_1 \cap J_2 \). But then by Lemma \([5.17](ii)\), \( N_{\tilde{K}}(S_1) \leq \tilde{K}_1 \). \[ \square \]

Let \( \tilde{E} \leq \tilde{K}_1 \) such that \( \tilde{E}/\tilde{\Sigma} = E(\tilde{K}_1/\tilde{\Sigma}) \). We have that \( \tilde{E}/\tilde{\Sigma} \cong \text{SU}_4(2) \) or \( \text{Sp}_6(2) \). By Proposition \([2.12](iii)\) there are exactly three classes of elements of order three in \( \tilde{E} \). As a Sylow 3-subgroup of \( \tilde{E} \) is isomorphic to \( 3 \wr 3 \), there is a unique elementary abelian subgroup of order 27, and this subgroup contains elements from each of the conjugacy classes of elements of order 3. As \( C_{\tilde{J}(r) \cap \tilde{E}} \) is elementary abelian of order 27, there are representatives of these elements in \( C_{\tilde{J}(r) \cap \tilde{E}} \). It follows that every element of order 3 in \( \tilde{K} \) is conjugate to an element of \( C_{\tilde{J}(r)} \). So using Lemma \([6.12]\) get the following lemma.
Lemma 6.3. Two elements of order three in $\tilde{K}_1$ are conjugate in $\tilde{K}$ if and only if they are conjugate in $\tilde{K}_1$. □

Supposing that $J > J_0$, we establish some further notation. Let $\sigma \in \tilde{K}_1$ have order 3 and $\sigma\Sigma$ be centralized by $\tilde{E}/\Sigma$. Then $\sigma$ is not $\tilde{K}$-conjugate to any element in $\tilde{E}$ by Lemma 6.3.

Lemma 6.4. Suppose that $\tilde{u} \in \tilde{K}_1 \setminus \Sigma$ is an involution which is $\tilde{K}$-conjugate to some involution in $\Sigma$. Assume that $\nu \in C_{\tilde{K}_1}(\tilde{u})$ is an element of order three. Then we have

(i) $C_{\Sigma}(\nu) \neq 1$;
(ii) $\langle \nu \rangle \not\sim Z$ in $\tilde{K}$;
(iii) if $J = J_0$, then $\nu \not\sim \rho$ in $\tilde{K}$; and
(iv) $|C_{\tilde{E}}(\tilde{u})|$ is not divisible by 9.

Proof. Let $\tilde{a} \in \Sigma$ with $\tilde{a} \sim_{\tilde{K}} \tilde{u}$. By Lemma 6.2, $\tilde{K}_1$ contains a Sylow 3-subgroup of $C_{\tilde{K}}(\tilde{a})$. By Lemma 6.3, $\nu$ is conjugate to an element $\mu$ of $C_{\tilde{K}_1}(\tilde{a})$ inside of $\tilde{K}_1$. Now obviously $C_{\Sigma}(\mu) \neq 1$ and so the same holds for $\nu$ which is (i).

If $\langle \nu \rangle$ is conjugate to $Z$ in $\tilde{K}$ or to $\langle \rho \rangle$ in case of $\tau = 1$, this happens also in $\tilde{K}_1$ by Lemma 6.3. Hence we may assume that $\tilde{a}$ is conjugate to $\tilde{u}$ in $M \cap K$, or $N_{K}(\langle \rho \rangle)$, which both are contained in $K_1$ by Lemma 5.17, a contradiction. Hence also (ii) and (iii) hold.

Assume now that $S_1 \leq C_{\tilde{E}}(\tilde{u})$, $|S_1| = 9$. Then $S_1$ is conjugate into a Sylow 3-subgroup $S_2$ of $C_{\tilde{E}}(\tilde{a})$. So by Lemma 6.2 and Proposition 2.12(ii) we may assume that $\tilde{a} = \tilde{y}$ and thus $S_2$ is extraspecial of order 27. Hence $S_1$ contains some element which is conjugate into $Z(S_2)$. But $Z(S_2)$ is conjugate to $Z$, and this contradicts (ii). This finishes the proof. □

Lemma 6.5. Suppose that $\tilde{u} \in \tilde{K}_1 \setminus \Sigma$ is an involution which is $\tilde{K}$-conjugate to some involution in $\Sigma$. Then either

(i) $\tilde{u} \in \tilde{E}$, $|\Sigma, \tilde{u}| = 4$ and $C_{\tilde{E}}(\tilde{u})$ has order 2 (if $E(\tilde{K}_1/\Sigma) \cong SU_4(2)$ and order 2 if $\tilde{K}_1/\Sigma \cong Sp_6(2)$; or
(ii) $J > J_0$, $\sigma\tilde{u} = \sigma^{-1}$ and $C_{\tilde{E}/\Sigma}(\tilde{u}) \cong 2 \times Sym(4) \leq Sym(6)$, and $|\Sigma, \tilde{u}| = 16$.

Proof. If $|\tilde{u}, \Sigma| = 16$, then all involutions in $\Sigma\tilde{u}$ are conjugate by elements of $\Sigma$. Hence, by Proposition 2.12(i), $\tilde{u}$ centralizes some non-trivial 3-element $\nu \in \tilde{E}$. By Lemma 6.4(i), $C_{\Sigma}(\nu) \neq 1$. If $J = J_0$, then by Proposition 2.12(ii) $\langle \nu \rangle$ is conjugate to $Z$ or $\langle \rho \rangle$, which contradicts Lemma 6.4(ii),(iii). So assume that $J > J_0$. If $\tilde{u} \notin \tilde{E}$, we have the assertion (ii) with Proposition 2.12(i) and Lemma 6.4(iv). So assume $\tilde{u} \in \tilde{E}$. Then $C_{\tilde{E}/\Sigma}(\tilde{u})$ is contained in a parabolic subgroup of $\tilde{E}/\Sigma$ of shape $2^4:Alt(5)$ and so $\nu$ acts fixed point freely on $\Sigma$, contradicting Lemma 6.4(i).
So assume that $|[	ilde{u}, \Sigma]| = 4$. Then, by Proposition $2.12(v)$, $C_{\tilde{E}/\tilde{S}}(\tilde{u}\tilde{\Sigma})$ has orbits of length 1, 6 and 9 on $C_{\tilde{S}}(\tilde{u})/\left[\tilde{\Sigma}, \tilde{u}\right]$. Hence there are exactly three conjugacy classes of involutions in $\tilde{\Sigma} \tilde{u}$, two of which have representatives centralized by an element of order three. Assume that $\tilde{u}$ is one of these. Let $\tilde{t}$ be the involution, which is centralized by $S_1$, a preimage of a Sylow 3-subgroup of $C_{\tilde{K}_1/\tilde{S}}(\tilde{u}\tilde{\Sigma})$. Set $S_2 = C_{S_1}(\tilde{u})$. Then, using Lemmas 6.3, 6.2 and 6.4(iv), we see that $|S_2| = 3$. Therefore $\tilde{u} \not\sim \tilde{t}$. In particular we have $\tilde{u} = \tilde{u}s$ where $\tilde{s} \in C_{\tilde{S}}(\tilde{u}) \setminus \left[\tilde{\Sigma}, \tilde{u}\right]$. Hence $C_{C_{\tilde{S}}(\tilde{u})/\left[\tilde{\Sigma}, \tilde{u}\right]}(S_2) \neq 1$. By Proposition $2.12(vi)$, we get $|C_{\tilde{S}}(S_2)| = 4$. So by Proposition $2.12(iii)$ $S_2$ does not centralize involutions in $\tilde{\Sigma}$. Thus we may assume that $S_2 = Z$. But this contradicts Lemma 6.4(iii) and proves the lemma. 

Lemma 6.6. We have $\tilde{y}\tilde{K} \cap \tilde{E} \subseteq \tilde{\Sigma}$.

Proof. Assume $\tilde{y} \sim_{\tilde{K}} \tilde{u}$ for some involution $\tilde{u} \in \tilde{E} \setminus \tilde{\Sigma}$. By Lemma 6.1 we have that $|C_{\tilde{K}}(\tilde{y})| = 2^{12}$ if $\tilde{E}/\tilde{S} \cong SU_4(2)$ or $2^{14}$ if $\tilde{E}/\tilde{S} \cong Sp_6(2)$. This conflicts with that information given in Lemma 6.5. Hence no such elements exist.

Lemma 6.7. $\tilde{\Sigma}$ is weakly closed in $\tilde{K}_1$. In particular, $\tilde{K}_1$ contains a Sylow 2-subgroup of $\tilde{K}$.

Proof. Assume that $T \in \text{Syl}_2(\tilde{K}_1)$, $w \in \tilde{K}$ and $\tilde{\Sigma}^w \leq T$ with $\tilde{\Sigma} \neq \tilde{\Sigma}^w$. Then $\tilde{\Sigma}^w \cap \tilde{E}$ has order at least $2^7$ and therefore is generated by conjugates of $\tilde{y}$. Thus Lemma 6.6 implies that $\tilde{\Sigma}^w \cap \tilde{E} \leq \tilde{\Sigma}$. But then $|\tilde{\Sigma}^w \tilde{\Sigma}/\tilde{\Sigma}| = 2$ and $|\tilde{\Sigma} \cap \tilde{\Sigma}^w| = 2^7$. Since $\tilde{K}_1$ does not contain transvections, we have a contradiction.

Lemma 6.8. No element of $\tilde{\Sigma}$ is $\tilde{K}$-conjugate to an involution $\tilde{u} \in \tilde{K}_1$ with $|[\tilde{\Sigma}, \tilde{u}]| = 4$.

Proof. Assume the statement is false. Then, by Lemma 6.6 $\tilde{u} \sim_{\tilde{K}} x$. Let $T_1$ be a Sylow 2-subgroup of $C_{\tilde{K}_1}(\tilde{u})$ and $T_2$ be a Sylow 2-subgroup of $C_{\tilde{K}}(\tilde{u})$ with $T_1 \leq T_2$. By Lemma 2.12(ii), 6.5 and 6.7 $|T_2 : T_1| = 4$. Let $\tilde{\Sigma}_u$ be the group corresponding to $\tilde{\Sigma}$ in $T_2$. Then $|\tilde{\Sigma}_u \cap T_1| \geq 2^6$. As any subgroup of $\tilde{\Sigma}$ of order at least $2^6$ is generated by conjugates of $\tilde{y}$, we have that $\tilde{\Sigma}_u \cap T_1 \leq \tilde{E}$ by Lemma 6.6. In particular, by the proof of Lemma 6.5, $J > J_0$. Therefore, we may suppose that there is a $\tilde{w} \in \tilde{\Sigma}_u \cap T_1$ such that $\tilde{w}$ inverts $\sigma$. Notice that $(\tilde{\Sigma}_u \cap T_1)\tilde{\Sigma}$ is normal in $T_1\tilde{\Sigma} \in \text{Syl}_2(\tilde{K}_1)$. In particular, if $|(\tilde{\Sigma}_u \cap T_1)\tilde{\Sigma}/\tilde{\Sigma}| = 2^2$, then $\tilde{w}\tilde{\Sigma}$ is centralized by a maximal subgroup of $T_1\tilde{\Sigma}/\tilde{\Sigma}$, which is impossible. Hence $|(\tilde{\Sigma}_u \cap T_1)\tilde{\Sigma}/\tilde{\Sigma}| \geq 2^3$. In particular, we have $|(\tilde{\Sigma}_u \cap T_1 \cap \tilde{E})\tilde{\Sigma}/\tilde{\Sigma}| \geq 2^2$ and by Lemma 6.6 all the nontrivial elements are unitary transvections. This, however, contradicts Proposition 2.12(viii) and proves the lemma.

Lemma 6.9. We have $\tilde{y}\tilde{K} \cap \tilde{K}_1 \subseteq \tilde{\Sigma}$. In particular, $\tilde{\Sigma}$ is strongly closed in $\tilde{E}$.
Proof. Suppose that \( \widetilde{u} \in \widetilde{y}^{K} \cap \widetilde{K}_{1} \setminus \widetilde{\Sigma} \). Then by Lemmas 6.6 and 6.5 we get that \( J > J_{0} \) and \( \widetilde{u} \) inverts \( \sigma \). Furthermore, all involutions in \( \widetilde{\Sigma} \widetilde{u} \) are conjugate. Hence, for \( T_{1} \in \text{Syl}_{2}(C_{K}(\widetilde{u})) \), we have using Lemma 5.16 |\( T_{1} | = 2^{9} \). Let \( T_{2} \) be a Sylow 2-subgroup of \( C_{K}(\widetilde{u}) \) with \( T_{1} \leq T_{2} \) and \( \widetilde{\Sigma}_{u} \leq T_{2} \) be a \( K \)-conjugate of \( \widetilde{\Sigma} \) in \( T_{2} \). By Lemma 6.8 \( (\widetilde{\Sigma}_{u} \cap T_{1}) \setminus \widetilde{\Sigma} \) does not contain elements \( v \) with \( [[v, \widetilde{\Sigma}]] = 4 \). So by Lemma 6.3 we have that \( (\widetilde{\Sigma}_{u} \cap T_{1}) \widetilde{\Sigma} \) inverts \( \sigma \widetilde{\Sigma} \) and this yields \( \widetilde{\Sigma}_{u} \cap T_{1} \subseteq \langle \widetilde{u} \rangle \widetilde{\Sigma} \). Since \( |\widetilde{\Sigma}_{u} \cap T_{1}| \geq 2^{5} \), we now have that \( \widetilde{\Sigma}_{u} \cap T_{1} = \langle \widetilde{u} \rangle C_{K}(\widetilde{u}) \) has order \( 2^{5} \). Hence \( T_{2} = T_{1} \widetilde{\Sigma}_{u} \) and \( T_{2}/\widetilde{\Sigma}_{u} \cong T_{1}/\langle \widetilde{u} \rangle C_{K}(\widetilde{u}) \cong 2 \times \text{Dih}(8) \). But \( T_{2}/\widetilde{\Sigma}_{u} \cong \text{SDih}(16) \) by Proposition 2.12 (ii) and we thus have a contradiction. Hence \( \widetilde{y}^{K} \cap \widetilde{K}_{1} \subseteq \widetilde{\Sigma} \). □

Lemma 6.10. We have that \( \widetilde{\Sigma} \) is strongly closed in \( \widetilde{K}_{1} \).

Proof. Assume by way of contradiction that there is some involution \( \widetilde{u} \in \widetilde{K}_{1} \setminus \widetilde{\Sigma} \), which is conjugate in \( \widetilde{K} \) to some element in \( \widetilde{\Sigma} \). By Lemma 6.9 we have \( \widetilde{u} \sim_{\widetilde{K}} \widetilde{x} \). By Lemmas 6.9 and 6.10 we have that \( \tau \neq 1 \) and we may assume that \( \widetilde{u} \) inverts \( \sigma \). Furthermore we have \( C_{E/\Sigma}(\widetilde{u}) \cong 2 \times \text{Sym}(4) \).

Let \( T_{1} \) be a Sylow 2-subgroup of \( C_{K_{1}}(\widetilde{u}) \) and \( T_{2} \) be a Sylow 2-subgroup of \( C_{K}(\widetilde{u}) \), which contains \( T_{1} \). Further let \( \widetilde{\Sigma}_{u} \) be the normal subgroup of \( T_{2} \) which is \( K \)-conjugate to \( \widetilde{\Sigma} \). Since, by Proposition 2.12 (ix), \( C_{K}(\widetilde{u}) \) is generated by conjugates of \( \widetilde{y} \), we have \( C_{\widetilde{\Sigma}}(\widetilde{u}) \leq \widetilde{\Sigma}_{u} \) by Lemma 6.9. Since \( (\widetilde{\Sigma}_{u} \cap T_{1}) \widetilde{\Sigma} = \langle \widetilde{u} \rangle \widetilde{\Sigma} \), we get \( T_{3} = \widetilde{\Sigma}_{u} \cap T_{1} = C_{\widetilde{\Sigma}}(\widetilde{u}) \widetilde{T}_{1} \).

Therefore \( T_{3} \) is normalized but not centralized by \( \widetilde{\Sigma} \) and is centralized by \( \widetilde{\Sigma}_{u} \). We have that \( \widetilde{\Sigma} \) and \( \widetilde{\Sigma}_{u} \) are contained in \( N_{K}(T_{3}) \). Let \( S_{\Sigma} \) and \( S_{\Sigma_{u}} \) be Sylow 2-subgroups of \( N_{K}(T_{3}) \), which contain \( \widetilde{\Sigma} \), \( \widetilde{\Sigma}_{u} \), respectively. As by Lemma 6.7 are \( \widetilde{\Sigma} \) is weakly closed in \( S_{\Sigma} \) and \( \widetilde{\Sigma}_{u} \) is weakly closed in \( S_{\Sigma_{u}} \), we see that \( \widetilde{\Sigma} \) and \( \widetilde{\Sigma}_{u} \) are conjugate in \( N_{K}(T_{3}) \). But this is impossible as one centralizes \( T_{3} \) and the other does not. □

Theorem 6.11. We have \( K = K_{1} \).

Proof. Let \( T \in \text{Syl}_{2}(K) \). By Lemmas 6.7 and 6.10 we have that \( \widetilde{\Sigma} \) is strongly closed in \( \widetilde{T} \) with respect to \( \widetilde{K} \). Hence an application of 4 yields that \( \widetilde{L} = \langle \widetilde{\Sigma} \rangle \) is an extension of a group of odd order by a product of a 2-group and a number of Bender groups. Furthermore \( \widetilde{\Sigma} \) is the set of involutions in some Sylow 2-subgroup of \( T \cap \widetilde{L} \). By Lemma 5.4 we have that \( K_{1} \) contains a Sylow 3-subgroup of \( K \). As \( C_{K}(\widetilde{\Sigma}) = \widetilde{\Sigma} \), we get that \( O_{2}'(\widetilde{L}) = O_{3}'(\widetilde{L}) \) now as \( J_{1} \) normalizes \( O_{2}(\widetilde{L}) \) we get with Lemma 5.13 that \( O_{2}(\widetilde{L}) = 1 \). As \( K_{1} \) acts primitively on \( \widetilde{\Sigma} \), either \( \widetilde{L} = \widetilde{\Sigma} \) and we are done, or \( \widetilde{L} \) is a simple group. So suppose that \( \widetilde{L} \) is a simple group.
Then $N_\tilde{T}(\tilde{\Sigma})$ acts transitively on $\tilde{\Sigma}$, which is not possible as $\Sigma$ is extraspecial. This proves that $K = K_1$. □

7. Proof of the Theorem 1.2

We continue with all the notation established in previous sections. If $N_M(S)/S \cong \text{Dih}(8)$, Theorem 1.2 follows with Theorem 4.8. So we may assume that $N_M(S)/S \cong 2 \times \text{Dih}(8)$. Using Theorem 6.11 and Lemma 5.17 we get that $K/\Sigma \cong \text{Aut}(\text{SU}_4(2))$, $(3 \times \text{SU}_4(2)) : 2$ or $\text{Sp}_6(2)$.

Suppose that $K/\Sigma \cong \text{Sp}_6(2)$. Then [23] implies that $G \cong \text{Co}_2$ and consequently $M = N_G(Z)$ has order $2^8 \cdot 3^6 \cdot 5$ and shape $3_4^{1+4} 2^{1+4} \cdot \text{Sym}(5)$, which is not similar to a centralizer of type $\text{PSU}_6(2)$ or $\text{F}_4(2)$. This contradicts our initial hypothesis. So suppose $K/\Sigma \cong \text{Aut}(\text{SU}_4(2))$ or $(3 \times \text{SU}_4(2)) : 2$. Then Lemma 2.13 shows that $G$ possesses a subgroup $G_0$ of index two. In particular we get $C_{G_0}(r)/\Sigma \cong \text{SU}_4(2)$ or $3 \times \text{SU}_4(2)$. Now we see that $N_{G_0 \cap M}(S)/S \cong \text{Dih}(8)$. Hence Theorem 4.8 gives $G_0 \cong \text{PSU}_6(2)$ or $\text{PSU}_6(2):3$ and so Theorem 1.2 is proved.

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