Integral representation of the Ising model

Oscar Diego∗†

May 7, 2019

Abstract

The partition function of the 2D Ising model coupled to an external magnetic field is studied. We show that the sum over the spin variables can be reduced to an integration over a finite number of variables. This integration must be performed numerically. But in order to reduce the partition function we must introduce as many different coupling constants as spin variables. The total memory that we need in order to store these coupling constants imposed important restrictions on the number of spin variables.

∗EMail: odiego@apiaxxi.es
†Present Address: Apia XXI, Departamento de Topografía, Luis Martínez 21, 39005 Santander, Spain
1. Introduction

Physical models in Statistical Mechanics and Quantum Field Theory are defined by integrals or sums over an infinite number of variables like:

$$\lim_{N \to \infty} \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp \{ W(\sigma_1, \cdots, \sigma_N) \}. \tag{1}$$

The weight $W$ is the energy in Statistical Mechanics and the classical action in Quantum Field Theory. The variables $\{\sigma_i\}$ are coupled through terms like

$$\sum_i \sigma_i \sigma_{i+1}. \tag{2}$$

In general we cannot solve exactly (1). Actually these sums can be solved exactly only if they can be transformed into sums over uncoupled variables. For instance, in free field models the Fourier coefficients of the field variables are not coupled. In Statistical Mechanics the 1D Ising model\cite{1} and the 2D Ising model without magnetic field are equivalent to free fermion models \cite{2,3}.

In this paper we are going to show that we can decouple the spin variables of the partition function of the 2D Ising model with an homogeneous magnetic field. This is an interesting result because the 2D Ising model with magnetic field cannot be solved exactly.

In order to decouple the spin variables we are going to use the following trick. Let us remark that if the interacting term between the variables is given by

$$\left( \sum_i \sigma_i \right)^2, \tag{3}$$

then we can decouple the variables using the following identity

$$\int dx \exp \left\{ -x^2 + 2x \sum_i \sigma_i \right\} \propto \exp \left\{ \left( \sum_i \sigma_i \right)^2 \right\}. \tag{4}$$

Let us consider

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T d\omega \left( \sum_j e^{i\omega \theta_j} \sigma_j \right) \left( \sum_k e^{-i\omega \theta_{k-1}} \sigma_k \right), \tag{5}$$

where $\{\theta_j\}$ is a set of different constants. In other words

$$\theta_i \neq \theta_j \text{ if } i \neq j. \tag{6}$$
Let us remark that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T d\omega e^{i\omega \theta} \neq 0 \quad (7)
\]

only if \(\theta\) is zero. Hence (5) becomes

\[
\sum_j \sum_k \delta(\theta_j, \theta_{k-1}) \sigma_j \sigma_k. \quad (8)
\]

But the constants \(\{\theta_i\}\) are different. Hence the above expression becomes

\[
\sum_j \sum_k \delta_{j,k-1} \sigma_j \sigma_{k-1} = \sum_j \sigma_j \sigma_{j+1}. \quad (9)
\]

Hence we can represent the interacting term

\[
\sum_i \sigma_i \sigma_{i+1} \quad (10)
\]

as a double sum over independent indices and we can use (4) in order to decouple the spin variables.

In the next section we will apply this trick to the 2D Ising model with a magnetic field.

Once the spin variables are decoupled we can integrate over all the spin variables. Hence we will show that the partition function of the 2D Ising model is given by an integral over a finite number of variables.

From a theoretical point of view this result is very interesting because the 2D Ising model with magnetic field cannot be solved exactly but we will show that the sum over the arbitrary number of spin variables can be reduced to an integral over a finite number of variables.

But from a practical point of view the trick developed in this paper has an important drawback. We will show that the set of coupling constants \(\{\theta_i\}\) must take their values on a very wide range of numbers. Each coupling constant \(\theta_i\) has a fixed value but if the number of spin variables is large then some of the coupling constants must take very large values. Therefore the number of spin variables cannot be large.

Therefore this approach can only be used for small lattices. But with the hypothesis of finite-size scaling\(^4\) we do not need very large lattice in order to study the physics of the model at the thermodynamic limit. Moreover for some boundary conditions the model reaches the thermodynamic limit faster\(^5\)\(^6\).
In [6], it has been shown that the dimer approach [7, 8] can be used in numerical calculation for small lattices. But the dimer approach cannot be used when a magnetic field is present or in 3D. Actually the approach that we study in this paper can be generalized to the 3D Ising model.

2. The partition function of the 2D Ising model

Let us consider the set \( \{ \sigma_i \} \) of spin variables:

\[ \sigma_i = \pm 1 \]  (11)

defined over vertices of the square lattice. Index \( i \) labels columns and \( j \) labels rows. Let us define the energy:

\[ E = \sum_{i,j} \beta \sigma_{i,j} (\sigma_{i,j+1} + \sigma_{i+1,j}) + \alpha \sigma_{i,j}. \]  (12)

The 2D Ising model with coupled to an external magnetic field is defined by the partition function:

\[ Z = \sum_{\sigma_{i,j}} \exp \left[ E \right]. \]  (13)

Because the spin variables are defined by (11), the partition function is also given by

\[ Z = (\cosh \beta)^{N_L} (\cosh \alpha)^{N_V} \sum_{\sigma_{i,j}} \prod_{i,j} \]

\[ (1 + z \sigma_{i,j} \sigma_{i,j+1})(1 + z \sigma_{i,j} \sigma_{i+1,j})(1 + h \sigma_{i,j}), \]  (14)

where \( N_L \) is the number of links and \( N_V \) is the number vertices of the lattice. And

\[ z = \tanh \beta \]
\[ h = \tanh \alpha. \]  (15)

For each \( \sigma_{i,j} \) let us define four new spin variables: \( \sigma_{i,j}^U, \sigma_{i,j}^D, \sigma_{i,j}^R \) and \( \sigma_{i,j}^L \). Now let us define the partition function:

\[ \tilde{Z} = A \sum_{\sigma} \prod_{i,j} \]

\[ (1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^U)(1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^D) \]
\[ (1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^L)(1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^R) \]
\[ (1 + h \sigma_{i,j})(1 + \sigma_{i,j} \sigma_{i+1,j})(1 + \sigma_{i,j} \sigma_{i,j+1}), \]  (16)
where
\[ A = \left( \frac{1}{2} \right)^{4N_V} \left( \cosh \beta \right)^{N_L} \left( \cosh \alpha \right)^{N_V}. \] (17)

It is very easy to show that
\[
\sum_{\sigma_R i,j} \sigma_R i,j \pm 1 \sum_{\sigma_L i,j} \sigma_L i,j \pm 1 \left( 1 + \sqrt{z \sigma_R i,j \sigma_L i,j} \right) \left( 1 + \sqrt{z \sigma_L i,j \sigma_R i,j} \right) = 2^2 \left( 1 + z \sigma_R i,j \sigma_L i,j \right).
\] (18)

Hence \( Z \) and \( \tilde{Z} \) are equal.

Now let us define the following set of constants \( \{ \theta_{i,j} \} \).

\[
\theta_{1,1} = 1
\]
\[
\ldots
\]
\[
\theta_{1,n+1} = \sum_{k=1}^{n} \theta_{1,k} + 1
\]
\[
\ldots
\]
\[
\theta_{m,n+1} = \sum_{j=1}^{m-1} \sum_{k=1}^{N} \theta_{j,k} + \sum_{k=1}^{n} \theta_{m,k} + 1.
\] (19)

The solutions of these equations are
\[ \theta_{m,n} = 2^{(m-1)N+n-1}. \] (20)

Now let us consider the following linear combination
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} n_{i,j} \theta_{i,j} = 0 \quad n_{i,j} = 0, \pm 1.
\] (21)

We are going to show that (21) holds only if the integer coefficients \( n_{i,j} \) are zero. It is very easy to show that
\[ \theta_{N,N} > \left| \sum_{i=1}^{N} \sum_{j=1}^{N-1} n_{i,j} \theta_{i,j} \right| \quad \forall n_{i,j} = 0, \pm 1. \] (22)

Hence if (21) holds then \( n_{N,N} \) must be zero.

In the same way we can show that
\[ \theta_{N,N-1} > \left| \sum_{i=1}^{N} \sum_{j=1}^{N-2} n_{i,j} \theta_{i,j} \right| \quad \forall n_{i,j} = 0, \pm 1. \] (23)
Then \( n_{N,N-1} \) must be zero.

In general we can show that

\[
\theta_{m,n} > | \sum_{i=1}^{m-1} \sum_{j=1}^{N} n_{i,j} \theta_{i,j} + \sum_{j=1}^{n-1} n_{m,j} \theta_{i,j} | \quad \forall n_{i,j} = 0, \pm 1. \tag{24}
\]

Then \( n_{m,n} \) must be zero.

Hence if the linear combination defined in (21) is zero then all the coefficients \( n_{i,j} \) must be zero.

Now let us define the partition function:

\[
Z' = A \sum_{\sigma} \lim_{T_H \to \infty} \lim_{T_V \to \infty} \frac{1}{T_H T_V} \int_{0}^{T_H} d\omega_H \int_{0}^{T_V} d\omega_V \prod_{i,j} \left( 1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^{R} \right) \left( 1 + \sqrt{z} \sigma_{i,j} \sigma_{i,j}^{L} \right) \\
\left( 1 + h \sigma_{i,j} \right) \left( 1 + e^{i \omega_H \theta_{i,j} \sigma_{i,j}^{R}} \right) \left( 1 + e^{-i \omega_H \theta_{i-1,j} \sigma_{i,j}^{L}} \right) \\
\left( 1 + e^{i \omega_V \theta_{i,j} \sigma_{i,j}^{U}} \right) \left( 1 + e^{-i \omega_V \theta_{i,j-1} \sigma_{i,j}^{D}} \right), \tag{25}
\]

Let us consider periodic boundary conditions. In this case the boundary conditions of the constants \( \{ \theta_{i,j} \} \) are given by

\[
\theta_{0,j} = \theta_{N,j} \tag{26}
\]

\[
\theta_{i,0} = \theta_{i,N}. 
\]

We are going to show that this partition function is equal to \( \tilde{Z} \).

Let us remark that

\[
\prod_{i} \left( 1 + a_i \right) = 1 + \sum_{i} a_i + \sum_{i<j} a_i a_j + \sum_{i<j<k} a_i a_j a_k + \cdots. \tag{27}
\]

We are going to use this formula in order to expand the products:

\[
\prod_{i,j} \left( 1 + e^{i \omega_H \theta_{i,j} \sigma_{i,j}^{R}} \right) \left( 1 + e^{-i \omega_H \theta_{i-1,j} \sigma_{i,j}^{L}} \right) \\
\left( 1 + e^{i \omega_V \theta_{i,j} \sigma_{i,j}^{U}} \right) \left( 1 + e^{-i \omega_V \theta_{i,j-1} \sigma_{i,j}^{D}} \right). \tag{28}
\]

Hence (28) is given by a sum of terms like

\[
\cdots + e^{i \omega_H \left( \sum'_{i,j} m_{i,j} \theta_{i,j} \right)} \prod_{i,j} \prod_{j} \sigma_{i,j}^{R} e^{i \omega_V \left( \sum''_{i,j} m_{i,j} \theta_{i,j} \right)} \prod_{i,j} \prod_{K} \sigma_{i,j}^{K} + \cdots, \tag{29}
\]
where indices $J$ and $K$ can take the values
\[ J = R, L \]
\[ K = U, D. \]  
(30)

Let us remark that $n_{i,j}$ and $m_{i,j}$ can take the values
\[ n_{i,j} = 0, \pm 1 \]
\[ m_{i,j} = 0, \pm 1. \]  
(31)

The primes over the sum and product symbols in (29) means that the indices do not take all their values.

For instance, the partition function $Z'$ depends on $\theta_{i,j}$ through the factor
\[
\left(1 + e^{i\omega_H \theta_{i,j}} \sigma^R_{i,j}\right) \left(1 + e^{-i\omega_H \theta_{i,j}} \sigma^L_{i+1,j}\right)
= \left(1 + e^{i\omega_H \theta_{i,j}} \sigma^R_{i,j} + e^{-i\omega_H \theta_{i,j}} \sigma^L_{i+1,j} + \sigma^R_{i,j} \sigma^L_{i+1,j}\right). \]  
(32)

Let us remark that first and fourth term in the right hand side of the above equation correspond to $n_{i,j} = 0$. The second term is defined by $n_{i,j} = 1$ and the third term is related with $n_{i,j} = -1$.

Now we are going to perform the integration over $\omega_H$ and $\omega_V$ and to take the limits in $T_H$ and $T_V$. The only terms of the expansion (29) that survive to the limits in $T_H$ and $T_V$ are those with all the coefficients $n_{i,j}$ and $m_{i,j}$ equal to zero. Therefore factors like (32) becomes
\[
\left(1 + \sigma^R_{i,j} \sigma^L_{i+1,j}\right). \]  
(33)

Hence $Z'$ is also given by:
\[
Z' = A \sum_{\sigma} \prod_{i,j} \left(1 + \sqrt{z} \sigma_{i,j} \sigma^R_{i,j}\right) \left(1 + \sqrt{z} \sigma_{i,j} \sigma^L_{i,j}\right)
\left(1 + \sqrt{z} \sigma_{i,j} \sigma^U_{i,j}\right) \left(1 + \sqrt{z} \sigma_{i,j} \sigma^D_{i,j}\right) \left(1 + h \sigma_{i,j}\right)
\left(1 + \sigma^R_{i,j} \sigma^L_{i+1,j}\right) \left(1 + \sigma^U_{i,j} \sigma^D_{i,j+1}\right). \]  
(34)

This is the partition function $\tilde{Z}$ given in (16).

3. Integral representation of the partition function

We have shown that the partition function of the 2D Ising model with a magnetic field is equivalent to the partition function $Z'$ given in (25). Hence $Z$ can be written as:
\[
Z = A \lim_{T_H \to \infty} \lim_{T_V \to \infty} \frac{1}{T_H T_V} \int_0^{T_H} d\omega_H \int_0^{T_V} d\omega_V \sum_{\sigma} \prod_{i,j} K_{i,j}, \]  
(35)
where $K_{i,j}$ is given by:

$$
K_{i,j} = \left( 1 + \sqrt{\sigma_{i,j} R} \left( 1 + \sqrt{\sigma_{i,j} L} \right) \right) \left( 1 + \sqrt{\sigma_{i,j} U} \left( 1 + \sqrt{\sigma_{i,j} D} \right) \right) \left( 1 + e^{i\omega H \theta_{i,j}} \left( 1 + e^{-i\omega H \theta_{i-1,j}} \right) \right) \left( 1 + e^{i\omega V \theta_{i,j}} \left( 1 + e^{-i\omega V \theta_{i-1,j}} \right) \right).
$$

(36)

Let us remark that $K_{i,j}$ depends only on the spin variables defined at the vertex $(i, j)$. Hence

$$
\sum_{\sigma} \prod_{i,j} K_{i,j} = \prod_{i,j} \sum_{\sigma_{i,j}} K_{i,j}.
$$

(37)

We can perform the integration over the spin variables and the partition function $Z$ is given by

$$
Z = A \lim_{T_H \to \infty} \lim_{T_V \to \infty} \frac{1}{T_H T_V} \int_0^{T_H} d\omega_H \int_0^{T_V} d\omega_V \prod_{i,j} \tilde{K}_{i,j},
$$

(38)

where

$$
\tilde{K}_{i,j} = \sum_{\sigma_{i,j} = \pm 1} \sum_{\sigma_{i,j} = \pm 1} \sum_{\sigma_{i,j} = \pm 1} \sum_{\sigma_{i,j} = \pm 1} \left( 1 + \sqrt{z} \sigma_{i,j} R \left( 1 + \sqrt{z} \sigma_{i,j} L \right) \right) \left( 1 + \sqrt{z} \sigma_{i,j} U \left( 1 + \sqrt{z} \sigma_{i,j} D \right) \right) \left( 1 + e^{i\omega H \theta_{i,j}} \left( 1 + e^{-i\omega H \theta_{i-1,j}} \right) \right) \left( 1 + e^{i\omega V \theta_{i,j}} \left( 1 + e^{-i\omega V \theta_{i-1,j}} \right) \right).
$$

(39)

It is very easy to show that $\tilde{K}_{i,j}$ is given by

$$
\tilde{K}_{i,j} = 2^5 \hat{K}_{i,j},
$$

(40)

where

$$
\hat{K}_{i,j} = 1 + z^2 e^{i(\omega V \theta_{i,j} - \omega H \theta_{i-1,j})} + z^2 e^{-i(\omega H \theta_{i,j} - \omega V \theta_{i-1,j})} + z^2 e^{i(\omega H \theta_{i,j} + \omega V \theta_{i,j})} + z^2 e^{-i(\omega H \theta_{i,j} + \omega V \theta_{i,j})} + z^2 e^{i(\omega H \theta_{i,j} - \omega V \theta_{i,j})} + z^2 e^{-i(\omega H \theta_{i,j} - \omega V \theta_{i,j})} + z^4 e^{i(\omega H \theta_{i,j} + \omega V \theta_{i,j} - \omega H \theta_{i-1,j} - \omega V \theta_{i-1,j})} + z^4 e^{-i(\omega H \theta_{i,j} + \omega V \theta_{i,j} - \omega H \theta_{i-1,j} - \omega V \theta_{i-1,j})}.
$$
\[ + z\omega \left( e^{i\omega V\theta_{i,j}} + e^{i\omega H\theta_{i,j}} + e^{-i\omega V\theta_{i,j-1}} + e^{-i\omega H\theta_{i-1,j}} \right) \]

\[ + z^3\omega e^{i(\omega H\theta_{i,j}-\omega H\theta_{i-1,j}-\omega V\theta_{i,j-1})} \]

\[ + z^3\omega e^{i(\omega V\theta_{i,j}-\omega H\theta_{i-1,j}-\omega V\theta_{i,j-1})} \]

\[ + z^3\omega e^{i(\omega H\theta_{i,j}+\omega V\theta_{i,j}-\omega H\theta_{i-1,j})} \]

\[ + z^3\omega e^{i(\omega H\theta_{i,j}+\omega V\theta_{i,j}-\omega V\theta_{i,j-1})}. \] (41)

Hence the partition function \( Z \) has the following representation:

\[ Z = 2^{5N^2} A \lim_{T_H \to \infty} \lim_{T_V \to \infty} \frac{1}{T_H T_V} \int_{0}^{T_H} d\omega_H \int_{0}^{T_V} d\omega_V \exp \left[ \sum_{i,j} \tilde{K}_{i,j} \right], \] (42)

where

\[ \tilde{K}_{i,j} = \ln \hat{K}_{i,j} \] (43)

and \( \hat{K}_{i,j} \) is given by (41).

Hence we have transform a sum over an arbitrary number of variables into and integral over two variables, a double sum over two indices and two limits.

4. Conclusions

We have shown that the partition function of the 2D Ising model coupled to an external magnetic field can be represented by an integral over a finite number of degrees of freedom. This result is very interesting because the 2D Ising model with magnetic field cannot be solved exactly.

The remaining finite integration must be performed numerically. But it is difficult to perform numerical calculation with this representation of the partition function. The main problem is the total memory that we need in order to store the coupling constants \( \theta_{i,j} \). If \( \theta_{i,j} \) are stored as 32 bits integers then

\[ -2^{31} < \theta_{i,j} < 2^{31}. \] (44)

If they are stored as 64 bits floating point then the maximum value of \( \theta_{i,j} \) is

\[ \theta_{i,j} < 2^{1000}. \] (45)

In 2D this bound means that

\[ N < 30 \] (46)

but in 3D

\[ N < 10 \] (47)
Acknowledgments
I thank Gloria Media for her illuminating influence in this paper.
References

[1] W. Lenz, Z. Phys. 21 (1920) 613.
   E. Ising, Z. Phys. 31 (1925) 253.
[2] L. Onsager, Phys. Rev. 65 (1944) 117.
[3] P. W. Kasteleyn, J. Math. Phys. 4 (1963) 287.
[4] M. N. Barber 1983 Finite-Size Scaling Phase Transitions and
   Critical Phenomena vol 8, Ed C. Domb and J. L. Lebowitz (New
   York: Academic).
[5] A. E. Ferdinand and M. E. Fisher Phys. Rev. 185 (1969) 832.
   M. E. Fisher 1971 Critical Phenomena (Proc. Enrico Fermi Sum-
   mer School) Ed M. S. Green (New York: Academic).
   M. E. Fisher and M. N. Barber Phys. Rev. Lett. 28 (1972) 1516.
[6] J. González and M. A. Martín-Delgado 1993 Exact finite-size re-
   sults the Ising model in 2D curved space Preprint PUPT-1367
   [hep-th/9301057].
   O. Diego, J. González and J. Salas, J. Phys. A: Math. Gen. 27
   (1994) 2965.
   Ch. Hoelbling and C. B. Lang, Phys. Rev. B 54 (1996) 3434.
   W. Janke and R. Kenna, Phys. Rev. B 65 (2002) 064110.
[7] E. W. Montroll, 1968 Lectures on the Ising Model of Phase
   Transition (Brandeis University Summer Institute in Theoretical
   Physics, 1966) ed M. Cretien et al (New York: Gordon and
   Breach).
[8] B. M. McCoy and T. T. Wu, 1973 The Two-Dimensional Ising
   Model (Cambridge, MA: Harvard University Press).