ON LISBON INTEGRALS

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ABSTRACT. We introduce new complex analytic integral transforms, the Lisbon Integrals, which naturally arise in the study of the affine space $\mathbb{C}^k$ of unitary polynomials $P_s(z)$ where $s \in \mathbb{C}^k$ and $z \in \mathbb{C}$, $s_i$ identified to the $i$–th symmetric function of the roots of $P_s(z)$. We completely determine the $\mathcal{D}$-modules (or systems of partial differential equations) the Lisbon Integrals satisfy and prove that they are their unique global solutions. If we specify a holomorphic function $f$ in the $z$-variable, our construction induces an integral transform which associates a regular holonomic module quotient of the sub-holonomic module we computed. We illustrate this correspondence in the case of a 1-parameter family of exponentials $f_t(z) = exp(tz)$ with $t$ a complex parameter.

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1. Introduction

The main purpose of this paper is to understand the behaviour of functions obtained by integration of $\mathcal{C}^\infty$-forms on the fibers of a holomorphic proper fibration. This has been investigated by the first author in two extreme cases: when the basis is 1-dimensional (see [2]) and when the fibers are finite (see [3]), but also by many authors in more general settings (see for instance [4], [12], [13]).

In the present paper we look at a simple but very interesting case where the fibers are the roots of the universal monic equation of degree $k$. The general result proved in [3] says that the singularity of these functions are controlled by regular holonomic $\mathcal{D}$-modules.

Our purpose is to give a precise answer in this special context. For instance, giving two entire functions $f$ and $g$ in $\mathcal{O}(\mathbb{C})$ we want to compute the regular holonomic system whose solutions are the $k$-uples of continuous functions on $\mathbb{C}^k$ given by

$$\Psi_p(s_1, \ldots, s_k) := \sum_{P_s(z_j) = 0} z_j^p f(z_j) \bar{g}(z_j) \quad p \in [0, k-1]$$

where $P_s(z) = \sum_{h=0}^k (-1)^h s_h z^{k-h}$ is the universal monic polynomial of degree $k$.

As we are interested only in holomorphic derivative in $s$, the function $g$ is irrelevant for the $\mathcal{D}$-module we are interested in, but, more surprisingly, there exists a sub-holonomic $\mathcal{D}$-module of which all these $k$-uples of (continuous functions) are solutions (in the sense of distributions).

We determine precisely this $\mathcal{D}$-module, via formula (4), for which it is useful to consider (in the simplest form, without $g$) the complex integral representation (4) of $\Psi_p$.

In order to make this computation, a main step is to introduce the trace of differential forms $f(z)ds_1 \wedge \cdots \wedge ds_{k-1} \wedge dz$ corresponding to the natural holomorphic volume forms on $H := \{(s, z) \in \mathbb{C}^k \times \mathbb{C} / P_s(z) = 0\}$ identified to $\mathbb{C}^k$ via the map $(s, z) \mapsto (s_1, \ldots, s_{k-1}, z)$. These holomorphic traces have a very simple integral representation via the "Lisbon integrals" (see integral representation (3) below).

Here we explicit the $\mathcal{D}$-modules of which (3) are (the unique) solutions via formula (4) showing that they derive from a very simple one by the usual functorial operations on $\mathcal{D}$-modules (inverse image and direct image) as follows:

Note that the hypersurface $H$ is also defined by the equation

$$s_k = (-1)^{k-1} \sum_{h=1}^k (-1)^h s_h z^{k-h}.$$

Let $j : H \subset \mathbb{C}^{k+1}$ denote the closed embedding and let $B_{H|\mathbb{C}^{k+1}}$ denote the regular holonomic $\mathcal{D}_{\mathbb{C}^{k+1}}$-module of holomorphic distributions supported by $H$. Since the $\mathcal{D}$-modules of which (3) are (the unique) solutions via formula (4) showing that they derive from a very simple one by the usual functorial operations on $\mathcal{D}$-modules (inverse image and direct image) as follows:

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Let $j : H \subset \mathbb{C}^{k+1}$ denote the closed embedding and let $B_{H|\mathbb{C}^{k+1}}$ denote the regular holonomic $\mathcal{D}_{\mathbb{C}^{k+1}}$-module of holomorphic distributions supported by $H$. Since the
restriction of the projection
\[ \pi : \mathbb{C}^{k+1} \to \mathbb{C}^k, (s, z) \mapsto s \]
to \( H \) is proper (with finite fibers), given a coherent \( \mathcal{D}_{\mathbb{C}^{k+1}} \)-module \( \mathcal{L} \) non characteristic for \( H \), following [11] we obtain a complex in \( \mathbb{D}^b_{\text{coh}}(\mathcal{D}_{\mathbb{C}^k}) \), the integral transformed of \( \mathcal{L} \), given by the composition of the usual derived functors of direct image and inverse image for \( \mathcal{D} \)-modules,

\[ D\pi_* (B_{H_{|C^{k+1}}} \otimes_{\mathcal{O}_{C^{k+1}}} \mathcal{L}) \simeq D\pi_* (\mathcal{H}^1_{|H}(\mathcal{L})) \simeq (\pi_{|H})_* j^* \mathcal{L} \]

Note that \( B_{H_{|C^{k+1}}} = \mathcal{H}^1_{|H}(\mathcal{O}_{C^{k+1}}) \) and that for such a module \( \mathcal{L} \), we have, thanks to Kashiwara’s equivalence theorem (cf Th 4.1 and Prop 4.2, [7]), \( \mathcal{H}^1_{|H}(\mathcal{L}) \simeq \mathcal{H}^0 D j_* D j^* \mathcal{L} \simeq j_* j^* \mathcal{L} \).

We show that the \( \mathcal{D}_{\mathbb{C}^k} \)-module determined by all vector functions \( \Phi_f \) given by the integral transform (3) (\( f \) varying in the space of holomorphic functions in the \( z \)-variable) is obtained as an integral transform in the sense of Kashiwara and Schapira ([11]) of a coherent \( \mathcal{D}_{\mathbb{C}^{k+1}} \)-module \( \mathcal{L} \).

As a consequence, we show that Lisbon Integrals (3) are exactly the global solutions of \( \pi_{|H} \mathcal{L} \).

Moreover, once an entire function \( f \) is fixed, we can consider the regular holonomic \( \mathcal{D}_{\mathbb{C}^{k+1}} \)-module (denoted by \( \mathcal{L}_f \)) it defines:

\[ \mathcal{L}_f = \mathcal{D}_{\mathbb{C}^{k+1}} / \mathfrak{J} \]

where \( \mathfrak{J} \) is the coherent ideal of \( \mathcal{D}_{\mathbb{C}^{k+1}} \) of operators \( P \) such that \( Pf = 0 \); hence, according to [9, Th. 8.1], \( \pi_{|H} j^* \mathcal{L}_f \) is regular holonomic. We explicit this module in the case of the family \( f_t(z) = e^{tz} \) where \( t \) is a complex parameter.

Since integrals (3) and (4) are strongly related as explained below, for the sake of simplicity we call both Lisbon Integrals.

We also prove that Lisbon integrals (4) are global solutions of another \( \mathcal{D}_{\mathbb{C}^k} \)-module \( \tilde{\mathcal{N}} \) which shares with the first this very simple relation:

Let \( A(s) \) be the \((k, k)\)-matrix companion of the unitary polynomial \( P_s(z) \). Then if \( \Phi \) is a solution of \( \pi_{|H} j^* \mathcal{L} \) then \( \Psi = P_s'(A(s)) \Phi \), where \( P_s' \) denotes the partial derivative of \( P_s \) with respect to \( z \), is a solution of \( \tilde{\mathcal{N}} \). Furthermore, this correspondence \( \Phi \leftrightarrow \Psi \) is a bijection when restricting to the complementary of the discriminant hypersurface \( \{ \Delta(s) = 0 \} \).

Important features of the scalar components of Lisbon integrals are the following:

- They are common solutions of a particular sub-holonomic system. This aspect will be developed in another paper by the first author. Here we compute only the simplest case \( k = 2 \).
Each entire function \( f \) determines a solution of \( \widehat{N} \) which scalar component of order \( h \) is the trace with respect to \( \pi \) in the holomorphic sense of the function \( f(z)z^h \) on \( H \).

Last but not the least, these computations illustrate the fact that it is not so easy, even in a rather simple situation, to follow explicitly the computations hidden in the “yoga” of \( \mathcal{D} \)-module theory.

We warmly thank the referee for the many pertinent comments contributing to clarify this work.

2. LISBON INTEGRALS AND THE DIFFERENTIAL SYSTEM THEY SATISFY

2.a. Lisbon integrals. For \( (z_1, \ldots, z_k) \in \mathbb{C}^k \) denote \( s_1, \ldots, s_k \) the elementary symmetric functions of \( z_1, \ldots, z_k \). We shall consider in the sequel \( s_1, \ldots, s_k \) as coordinates on \( \mathbb{C}^k \simeq \mathbb{C}^k/\mathfrak{S}_k \), isomorphism given by the standard symmetric function theorem.

We shall denote \( P_s(z) := \prod_{j=1}^k (z - z_j) = \sum_{h=0}^k (-1)^h s_h z^{k-h} \) with the convention \( s_0 \equiv 1 \).

We shall often write \( P_s(z, z) \) instead of \( P_s(z) \) with no risk of ambiguity.

The discriminant \( \Delta(s) \) of \( P_s \) is the polynomial in \( s \) corresponding to the symmetric polynomial \( \prod_{1 \leq i < j \leq k} (z_i - z_j)^2 \) via the symmetric function theorem.

Lemma 2.1. For \( h \in \mathbb{N} \) and \( f \in \mathcal{O}(\mathbb{C}) \) any entire holomorphic function, let us define, for \( R \gg ||s|| \),

\[
\varphi_h(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)\zeta^h d\zeta}{P_s(\zeta)}.
\]

Then \( \varphi_h(s) \) is independent of the choice of \( R \) large enough and defines a holomorphic function on \( \mathbb{C}^k \). For \( \Delta(s) \neq 0 \) we have

\[
\varphi_h(s) = \sum_{j=1}^k \frac{z_j^h f(z_j)}{P_s'(z_j)}
\]

where \( z_1, \ldots, z_k \) are the roots of \( P_s(z) \).

Proof. The independence on \( R \) large enough when \( s \) stays in a compact set of \( \mathbb{C}^n \) is clear. For \( s \) in the interior of a compact set, \( P_s(\zeta) \) does not vanish on \( \{|\zeta|=R\} \) for \( R \) large enough, so we obtain the holomorphy of \( \varphi_h \) near any point in \( \mathbb{C}^k \). The formula (2) is given by a direct application of the Residue’s theorem.

In fact, it will be convenient to consider the \( k \) functions \( \varphi_0, \ldots, \varphi_{k-1} \) as the component of a vector valued function \( \Phi := \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{k-1} \end{pmatrix} \). Defining \( E(z) := \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{k-1} \end{pmatrix} \) we obtain

\[
\Phi(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.
\]
Definition 2.2. We call $\Phi$ (sometimes also denoted by $\Phi_f$ when precision is required) the Lisbon Integral associated to $f$. The scalar components of $\Phi$, denoted by $\varphi_h$, $h = 0, \ldots, k - 1$, are called the scalar Lisbon Integrals. One also denote by $\varphi_h$ the functions constructed by the same formula, with $h \in \mathbb{N}$, still denominated by "scalar Lisbon Integrals".

It will be also interesting to introduce another type of integrals, still called Lisbon Integrals for the sake of simplicity:

\begin{equation}
\Psi(s) := \frac{1}{2i\pi} \int_{|\zeta| = R} \frac{f(\zeta)E(\zeta)P_s(\zeta)}{P_s(\zeta)} d\zeta
\end{equation}

$\Psi$ will also be noted below by $\Psi_f$ when precision is required.

It is easy to see that this is again a vector valued holomorphic function on $\mathbb{C}^k$ and the Residue’s theorem entails that, for $\Delta(s) \neq 0$, the component $\psi_h$ of $\Psi$ is given by:

\begin{equation}
\psi_h(s) = \sum_{j=1}^{k} z_j^h f(z_j).
\end{equation}

**Proposition 2.3.** If $f$ is not identically zero then $\Phi$ and $\Psi$ are non zero vector-valued holomorphic functions on $\mathbb{C}^k$.

**Proof.** Suppose that $f$ is not identically zero. Then the statement follows as an immediate consequence of the non vanishing of the Van der Monde determinant of $z_1, \ldots, z_k$ when these complex numbers are pair-wise distinct. ■

**An example.** Take $f \equiv 1$. Then formula (5) shows that $\psi_h(s)$ is the $h-$th Newton symmetric functions of the roots of the polynomial $P_s$. So it is a quasi-homogeneous polynomial in $s$ of weight $h$ (the weight of $s_j$ is $j$ by definition).

Let us show that we have $\varphi_h(s) \equiv 0$ for $h \in [0, k - 2]$ and $\varphi_{k-1}(s) \equiv 1$ in this case. For $h \in [0, k - 2]$ the formula (1) gives the estimate (with $f \equiv 1$)

$$|\varphi_h| \leq \frac{R^{h+1}}{(R - a)^{k-1}}$$

if each root of $P_s$ is in the disc $\{|z| \leq a\}$ when $R > a > 0$. When $R \to +\infty$ this gives $\varphi_h(s) \equiv 0$ for $h \in [0, k - 2]$.

For $h = k - 1$ write

$$kz^{k-1} = P'_s(z) - \sum_{h=1}^{k-1} (-1)^h (k - h) s_h z^{k-h-1}.$$

This gives, using the previous case and formula (2), that $\varphi_{k-1}(s) \equiv 1$.

2.b. The partial differential system. Let us introduce the $(k, k)$ matrix $A$ (the companion matrix) associated to the polynomial $P_s$:
Theorem 2.b.1. The vector valued holomorphic function $\Phi$ on $\mathbb{C}^k$ satisfies the following differential system

\[(a) \quad (-1)^{k+h} \frac{\partial \Phi}{\partial s_h}(s) = \frac{\partial (A^{k-h} \Phi)}{\partial s_k}(s) \quad \forall \, s \in \mathbb{C}^k \quad \text{and} \quad \forall \, h \in [1,k-1].\]

Moreover, this system is integrable and if $\Phi$ is a solution of this system, so is $A\Phi$.

The proof of this result will use several lemmas.

Lemma 2.4. Let $A$ be a $(k,k)$ matrix with entries in $\mathbb{C}[x]$ and put $B := \lambda \frac{\partial A}{\partial x}$ where $\lambda$ is a complex number. Let $M$ be the $(2k,2k)$ matrix given by

$$M := \begin{pmatrix} A & B \\ 0 & A^p \end{pmatrix}.$$ 

Then for each $p \in \mathbb{N}$ we have

\[(a) \quad M^p = \begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \]

where $B_p := \lambda \frac{\partial (A^p)}{\partial x}$.

Proof. As the relation (a) is clear for $p = 0, 1$ let us assume that it has been proved for $p$ and let us prove it for $p + 1$. We have:

$$\begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} A^{p+1} & A^p B + B_p A \\ 0 & A^{p+1} \end{pmatrix}$$

which allows to conclude. \(\blacksquare\)

Corollary 2.b.2. For each integer $p \in [0,k-1]$ the following equality holds in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1,\ldots,s_k,z]/(P^2)$ over the $\mathbb{C}$–algebra $\mathbb{C}[s_1,\ldots,s_k,z]/(P^2)$

$$z^p E(z) = A^p E(z) + (-1)^{k-1} P_s(z) \frac{\partial (A^p)}{\partial s_k} E(z)$$

In particular, for any entire function $f$ (of the variable $z$), we have

$$\Phi_z f = A(s) \Phi_f$$

2We shall explain in the proof what we mean here.
Moreover the following identity in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \ldots, s_k]/(P^2)$ holds
\[
P_s'(z)E(z) = P_s'(A)E(z) + (-1)^{k-1} P_s(z) \frac{\partial(P_s'(A))}{\partial s_k} E(z).
\]

**Proof.** In the basis $1, z, \cdots, z^{k-1}, P_s'(z), z P_s(z), \cdots, z^{k-1} P_s(z)$ of this algebra which is a free rank $2k$ module on $\mathbb{C}[s_1, \ldots, s_k]$, the multiplication by $z$ is given by the matrix $M$ of the previous lemma with $A$ as in $(6)$ and with $B := (-1)^{k-1} \frac{\partial A}{\partial s_k}$.

As $P_s'(z) = \sum_{h=0}^{k-1} (-1)^h (k - h) z^{k-h-1}$ does not depend on $s_k$ it is enough to sum up the previous equalities with $p = k - h - 1$ with the convenient coefficients to obtain the equality $(8)$.

**Lemma 2.5.** For any $h \in [1, k]$ and any $p \in \mathbb{N}$ the matrix $A$ in $(6)$ satisfies the relation:
\[
(-1)^{k-h} \frac{\partial A^p}{\partial s_h} = \frac{\partial A^p}{\partial s_k} A^{k-h}
\]

**Proof.** The case $p = 1$ of $(9)$ is an easy direct computation on the matrix $A$. Assume that the assertion is proved for $p \geq 1$. Then Leibnitz’s rule gives:
\[
(-1)^{k-h} \frac{\partial A^{p+1}}{\partial s_h} = (-1)^{k-h} \frac{\partial A^p}{\partial s_h} A + A^p (-1)^{k-h} \frac{\partial A}{\partial s_h}
= \frac{\partial A^p}{\partial s_k} A^{k-h+1} + A^p \frac{\partial A}{\partial s_k} A^{k-h} = \frac{\partial A^{p+1}}{\partial s_k} A^{k-h}
\]
concluding the proof of $(9)$. \hfill \Box

**Proof of the theorem 2.b.1** By derivation inside the integral in $(3)$ we obtain:
\[
\frac{\partial \Phi}{\partial s_h}(s) = \frac{1}{2\pi i} \int_{|z| = R} f(\zeta) E(\zeta)(-1)^{h+1} \zeta^{k-h} \frac{d\zeta}{P_s(\zeta)^2} \quad \text{and in particular}
\frac{\partial \Phi}{\partial s_k}(s) = \frac{1}{2\pi i} \int_{|z| = R} f(\zeta) E(\zeta)(-1)^{k+1} \frac{d\zeta}{P_s(\zeta)^2}
\]
Now for $h \in [1, k - 1]$ we use the formula of corollary 2.b.2 to obtain:
\[
\frac{\partial \Phi}{\partial s_h} = (-1)^{h+1} A^{k-h} (-1)^{k-1} \frac{\partial \Phi}{\partial s_k} + (-1)^{h+1} (-1)^{k-1} \frac{\partial A^{k-h}}{\partial s_h} \Phi
\]
that is to say, we obtain $(@)$ as desired:
\[
(-1)^{k+h} \frac{\partial \Phi}{\partial s_h} = \frac{\partial (A^{k-h}\Phi)}{\partial s_k} \quad \forall h \in [1, k]
\]
By the integrability of the system $(@)$ we mean that for any $\Phi$ such that $(@)$ holds, then the computation of the partial derivatives $\frac{\partial \Phi}{\partial s_h}$ using the system $(@)$ gives a symmetric result in $(h, j)$ for any pair $(h, j)$ in $[1, k]$. Note that if $h$ or $j$ is equal to $k$ the assertion is trivial. So consider a couple $(h, j) \in [1, k - 1]^2$. Thanks to Lemma 2.5 we have:
\[
(-1)^{h+j} \frac{\partial^2 \Phi}{\partial s_j \partial s_h} = (-1)^{k-h} \frac{\partial}{\partial s_k} \left[ \partial (A^{k-h} \Phi) \right]
\]

\[
= (-1)^{k-j} \frac{\partial}{\partial s_k} \left[ \frac{\partial A^{k-h}}{\partial s_j} \Phi + A^{k-h} \frac{\partial \Phi}{\partial s_j} \right]
\]

\[
= \frac{\partial}{\partial s_k} \left[ \frac{\partial A^{k-h}}{\partial s_j} A^{k-j} \Phi + A^{k-h} \frac{\partial (A^{k-j} \Phi)}{\partial s_k} \right]
\]

\[
= \frac{\partial^2}{\partial s^2_k} [A^{2k-h-j} \Phi]
\]

which is symmetric in \((h,j)\).

To finish the proof of the theorem we have to show that \(A \Phi\) is a solution of \((@)\) when \(\Phi\) is a solution of \((@)\). This is given by the following computation

\[
(-1)^{k-h} \frac{\partial (A \Phi)}{\partial s_h} = (-1)^{k-h} \frac{\partial A}{\partial s_h} \Phi + (-1)^{k-h} A \frac{\partial \Phi}{\partial s_h}
\]

\[
= \frac{\partial A}{\partial s_k} A^{k-h} \Phi + A \frac{\partial A^{k-h} \Phi}{\partial s_k} = \frac{\partial (A^{k-h} (A \Phi))}{\partial s_h}
\]

which also uses Lemma 2.5.

\[\blacksquare\]

Remark. A consequence of our computation on the integrability of the system \((@)\) is the fact that for any solution \(\Phi\) and any pair \((h, j) \in [1, k]\) the second order partial derivative \(\frac{\partial^2 \Phi}{\partial s_j \partial s_h}\) only depends on \(h + j\). This implies that any scalar Lisbon integral \(\varphi_h\) satisfies

\[
\frac{\partial^2 \varphi_h}{\partial s_p \partial s_{q+1}} = \frac{\partial^2 \varphi_h}{\partial s_{p+1} \partial s_q} \quad \forall \ p, q \text{ such that } 1 \leq p < q \leq k - 1
\]

Let us denote by \(\Delta\) the discriminant hypersurface \(\{s \in \mathbb{C}^k; \Delta(s) = 0\}\). An easy calculation shows that away of \(\Delta\) the matrix \(P'_s(A(s))\) is invertible.

The next corollary of theorem 2.b.1 gives an analogous system to \((@)\) for the vector function \(\Psi\) defined in (4) which is singular along \(\Delta\).

Corollary 2.b.3. (1) The vector valued holomorphic function \(\Psi\) on \(\mathbb{C}^k\) satisfies the following differential system:

\[
(@@) \quad (-1)^{k+h} \frac{\partial \Psi}{\partial s_h}(s) = \frac{\partial (A^{k-h} \Psi)}{\partial s_k}(s) + (-1)^{k}(k - h)A^{k-h-1}P'_s(A)^{-1} \Psi(s)
\]

\(\forall \ h \in [1, k - 1]\), which is singular along the discriminant hyperdurface

\[\Delta := \{s \in \mathbb{C}^k; \Delta(s) = 0\}\]

(2) If \(\Psi\) is any solution of \((@@)\) then \(A \Psi\) is also a solution of \((@@)\).

(3) If \(\Phi\) is any solution of \((@)\) then \(\Psi = P'_s(A)\Phi\) is a solution of \((@@)\).

(4) If \(\Psi\) is any solution of \((@@)\) on \(\mathbb{C}^k \setminus \Delta\) then \(\Phi := P'_s(A(s))^{-1} \Psi\) is a solution of \((@)\) on \(\mathbb{C}^k \setminus \Delta\).
Statement (1) follows by (8) in Corollary 2.b.2.
Statement (3) follows by direct computation:
For such a $\Psi = P'_s(A)\Phi$, for each $h \in [1, k - 1]$
\[
(-1)^{k-h}\frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \sum_{p=0}^{k-1} (-1)^p(k-p)(-1)^{k-2h} s_p \frac{\partial (A^{k-p-1}\Phi)}{\partial s_h}(s)
\]
and using now the fact that $A^{k-p-1}\Phi$ is solution of (@) we obtain
\[
(-1)^{k-h}\frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \sum_{p=0}^{k-1} (-1)^p(k-p)s_p \frac{\partial (A^{2k-p-h-1}\Phi)}{\partial s_k}(s)
\]
\[
= (-1)^{k-2h}(k-h)A^{k-h-1}\Phi(s) + \frac{\partial (A^{k-h}P'_s(A)\Phi)}{\partial s_k}(s)
\]
\[
= (-1)^{k-2h}(k-h)A^{k-h-1}P_s'(A)^{-1}\Psi(s) + \frac{\partial (A^{k-h}\Psi)}{\partial s_k}(s)
\]
which gives the formula (@@).
Since $P'_s(A)$ commutes with $A$, the assertion (2) is easy.
\[\blacksquare\]
The proof of (4) is a simple consequence of [9] in Lemma 2.3.

2.c. Example: The case $k = 2$. In this example we will explicit the system (@) and also a partial differential operators in the Weyl algebra $\mathbb{C}[s_1, s_2]/(\partial_{s_1}, \partial_{s_2})$, which annihilates the scalar components of its solutions. The left ideals in $\mathcal{D}_{Ck}$ of differential operators annihilating respectively the scalar components of the solutions of (@) and of (@@) for arbitrary $k$ are described in [1].

Here we use the notations $s := s_1$ and $p := s_2$. In that case, with $\Phi = (\varphi_0, \varphi_1)$, the differential system (@) becomes:

\[
\frac{\partial \varphi_0}{\partial s} = \frac{\partial \varphi_1}{\partial p}
\]

\[
\frac{\partial \varphi_1}{\partial s} = -\varphi_0 - p\frac{\partial \varphi_0}{\partial p} + s\frac{\partial \varphi_1}{\partial p}
\]
corresponding to the matrix $A := \begin{pmatrix} 0 & 1 \\ -p & s \end{pmatrix}$. Differentiating (12) with respect to $p$ we obtain, after substituting via (11)
\[
\frac{\partial^2 \varphi_0}{\partial s^2} = -2\frac{\partial \varphi_0}{\partial p} - p\frac{\partial^2 \varphi_0}{\partial p^2} - s\frac{\partial^2 \varphi_0}{\partial s \partial p}
\]
and so
\[
\frac{\partial^2 \varphi_0}{\partial s^2} + s\frac{\partial^2 \varphi_0}{\partial s \partial p} + p\frac{\partial^2 \varphi_0}{\partial p^2} + 2\frac{\partial \varphi_0}{\partial p} = 0
\]
Differentiating (12) with respect to \( s \) we obtain, after substituting via (11)
\[
-\frac{\partial^2 \varphi_1}{\partial s^2} = \frac{\partial \varphi_1}{\partial p} + p \frac{\partial^2 \varphi_1}{\partial p^2} + \frac{\partial \varphi_1}{\partial s \partial p} + s \frac{\partial^2 \varphi_1}{\partial s \partial p}
\]

and so
\[
\frac{\partial^2 \varphi_1}{\partial s^2} + s \frac{\partial^2 \varphi_1}{\partial s \partial p} + p \frac{\partial^2 \varphi_1}{\partial p^2} + 2 \frac{\partial \varphi_1}{\partial p} = 0
\]

Then the second order differential operator of weight \(-2\)
\[
\Theta := \frac{\partial^2}{\partial s^2} + s \frac{\partial^2}{\partial s \partial p} + p \frac{\partial^2}{\partial p^2} + 2 \frac{\partial}{\partial p}
\]

anihilates \( \varphi_0 \) and \( \varphi_1 \) for any solution \( \Phi \) of the system ((11), (12)).

A direct proof that \( \Theta \) annihilates scalar Lisbon Integrals for all \( m \in \mathbb{N} \). We have, for any entire function \( f : \mathbb{C} \to \mathbb{C} \) and for \( R \gg \max\{|s|,|p|\} \):

(a) \( \varphi_m(s,p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{\zeta^2 - s\zeta + p} \)

This gives:

(b) \( \frac{\partial \varphi_m}{\partial s}(s,p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^2} \)

(c) \( \frac{\partial \varphi_m}{\partial p}(s,p) = -\frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^2} \)

(d) \( \frac{\partial^2 \varphi_m}{\partial s^2}(s,p) = 2 \cdot \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+2} d\zeta}{(\zeta^2 - s\zeta + p)^3} \)

(e) \( \frac{\partial^2 \varphi_m}{\partial s \partial p}(s,p) = -2 \cdot \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^3} \)

(f) \( \frac{\partial^2 \varphi_m}{\partial p^2}(s,p) = 2 \cdot \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^3} \)

Now it is easy to check that (d) + s(e) + p(f) + 2(c) = 0.

3. The left action of \( \Gamma(\mathbb{C},\mathcal{D}_\mathbb{C}) \) on Lisbon integrals

For each entire function \( f \) of the variable \( z \) we shall henceforward denote by \( \Phi_f \) the associated Lisbon integral (previously generically denoted by \( \Phi \)). Clearly the assignement

\( f \mapsto \Phi_f \)

is \( \mathbb{C} \)-linear and, according to Proposition 2.3, it is injective.

**Lemma 3.1.** Let \( f \) be an entire function on \( \mathbb{C} \) and let \( \Phi_f \) the corresponding Lisbon integral.

(1) Let \( g \) be an entire function of \( z \). Then \( \Phi_{gf} = g(A(s))\Phi_f \). In particular \( \Phi_f = f(A(s))\Phi_1 \).
(2) We have the identity

\[ \Phi_{\partial f}(s) = -\nabla \Phi_f(s) + \left( \sum_{h=0}^{k-1} (k-h)s_h \partial s_{h+1} \right)(\Phi_f)(s) \]

where \( \nabla \) is the constant \((k,k)\) matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 2 & \cdots & 0 \\
0 & \cdots & 0 & k-1 \\
0 & \cdots & 0 & 0
\end{pmatrix}.
\]

**Proof.** When \( g \) is a polynomial on \( z \), statement (1) follows easily in Corollary 2.1.2. For an arbitrary entire function \( g \), it is a consequence of [5, Lem 3.1.8]. Let us prove (2): Consider the Lisbon integral

\[ \Phi_{\partial f}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} f'(\zeta)E(\zeta) \frac{d\zeta}{P_s(\zeta)} \]

Integration by parts gives, as \( \partial_s(E)(z) = \nabla E(z) \):

\[ (*) \quad \Phi_{\partial f}(s) = -\nabla \Phi_f(s) + \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta)E(\zeta) \frac{P_s'(\zeta)}{P_s(\zeta)^2} d\zeta \]

Now, using the equalities \( P_s'(\zeta) = \sum_{h=0}^{k-1} (-1)^h(k-h)s_h \zeta^{k-h-1} \) and

\[ \frac{\partial \Phi_f}{\partial s_h}(s) = -\frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta)E(\zeta) \frac{(-1)^h \zeta^{k-h} d\zeta}{P_s(\zeta)^2} \]

we obtain the formula of the lemma. \( \blacksquare \)

**Remark.** From formula (\( *) \) and according to (8) we obtain also the formula

\[ (**) \quad \Phi_{\partial f}(s) = -\nabla \Phi_f(s) + (-1)^{k-1} \frac{\partial(P_s'(A)\Phi)}{\partial s_k}(s) \]

Note that the formula \( \Phi_{(z+f)'} = \Phi_f + \Phi_{z'} \) corresponding to the usual relation \( \partial_z z - z\partial_z = 1 \) follows from the linearity of the map \( f \mapsto \Phi_f \) and the Leibniz rule \( (z f)' = f + z f' \).

It is not obvious that when \( \Phi \) is solution of the system (\( @ \)), then

\[ (\Phi \partial_z)(s) := -\nabla \Phi(s) + (-1)^{k-1} \partial s_k(P_s'(A)\Phi)(s) \]

(given by the formula (\( ** \))) is also a solution of the same system. A direct verification of this fact is consequence of the formula given in the lemma below.

**Lemma 3.2.** We have the following identity:

\[ \nabla A^p - A^p \nabla + p A^{p-1} = (-1)^{k-1} \partial s_k(A^p) P_s'(A), \quad \forall p \geq 1, \forall s \in \mathbb{C}^k. \]
Proof. Let check the case \( p = 1 \) first. It is an easy computation to obtain that \( \nabla A - A\nabla + Id \) is the matrix which have all lines equal to 0 excepted its last one which is given by \( (x_1, \ldots, x_k) \) with \( x_h = (-1)^{k-h}x_{sk-h} \) for \( h \in [1, k] \) with \( s_0 \equiv 0 \). On the other hand, the matrix \((-1)^{k-1}\partial_{s_k}A\) has only a non zero term at the place \((k, 1)\) which equal to 1, so it is quite easy to see that \((-1)^{k-1}(\partial_{s_k}A)A^p\) has only a non zero term at the place \((k, p+1)\) with value \((-1)^{k-1}\). According to the computation of \((-1)^{k-1}\partial_{s_k}(A)P'_s(A)\) we conclude the desired formula for \( p = 1 \) Now we shall make an induction on \( p \geq 1 \) to prove the general case. Then assume \( p \geq 2 \) and the formula proved for \( p = 1 \). Then write

\[
\nabla A^p - A^p\nabla = (\nabla A^{p-1} - A^{p-1}\nabla)A + A^{p-1}(\nabla A - A\nabla).
\]

Using the induction hypothesis and the case \( p = 1 \) gives

\[
\nabla A^p - A^p\nabla = -(p - 1)A^{p-2}A - A^{p-1} + \]

\[
+ (-1)^{k-1}\partial_{s_k}(A^{p-1})P'_s(A)A + (-1)^{k-1}A^{p-1}\partial_{s_k}(A)P'_s(A)
\]

\[
= -pA^{p-1} + (-1)^{k-1}(\partial_{s_k}(A^{p-1})A + A^{p-1}\partial_{s_k}(A))P'_s(A)
\]

\[
= -pA^{p-1} + (-1)^{k-1}\partial_{s_k}(A^p)P'_s(A).
\]

Now we shall make the direct verification that \( \Phi \) solution of \((@)\) implies that

\[
-\nabla \Phi + (-1)^{k-1}\partial_{s_k}(P'_s(A)\Phi)
\]

is also solution of \((@)\)

\[
X := (-1)^{k+h}\partial_{s_k}(\nabla \Phi) - \partial_{s_k}(A^{k-h}\nabla \Phi) = \partial_{s_k}(\nabla A^{k-h} \nabla \Phi) - \partial_{s_k}(A^{k-h} \nabla \Phi)
\]

\[
= \partial_{s_k}[-(k-h)A^{k-h-1}\Phi + (-1)^{k-1}\partial_{s_k}(A^{k-h})P'_s(A)\Phi]
\]

thanks to the previous lemma. Also, using the fact that \( P'_s(A)\Phi \) is a simple linear combination of solutions of \((@)\) (with non constant coefficients, but very simple), we obtain the formula:

\[
(-1)^{k+h}\partial_{s_k}(P'_s(A)\Phi) = (-1)^{k}(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h}P'_s(A)\Phi).
\]

Then:

\[
Y := (-1)^{k+h}\partial_{s_k}(\partial_{s_k}(P'_s(A)\Phi)) - \partial_{s_k}(A^{k-h}\partial_{s_k}(P'_s(A)\Phi))
\]

\[
= \partial_{s_k}[-(-1)^{k}(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h}P'_s(A)\Phi) - A^{k-h}\partial_{s_k}(P'_s(A)\Phi)]
\]

\[
= \partial_{s_k}[-(-1)^{k}(k-h)A^{k-h-1}\Phi + \partial_{s_k}(A^{k-h})P'_s(A)\Phi]
\]

and \(-X + (-1)^{k-1}Y = 0\), as desired.

4. The \( \mathcal{D}_{Ca} \)-Module Associated to Lisbon Integrals

We shall begin by recalling some basic facts on \( \mathcal{D} \)-modules and by fixing notations.

For a morphism of manifolds \( f : Y \to X \), we use the notation of \([10]\)

\[
f_d := f^* f : T^* X \times X \times Y \to T^* Y
\]
and
\[ f_* : T^*X \times X \times Y \rightarrow T^*X \]
the associated canonical morphisms of vector bundles.

We recall that a conic involutive submanifold \( V \) of the cotangent bundle \( T^*Z \) of a manifold \( Z \) (real or complex) is \textit{regular} if the restriction \( \omega|_V \) of the canonical 1-form \( \omega \) on \( T^*Z \) never vanishes outside the 0-section. Recall also that if \( (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n) \) are canonical symplectic coordinates on \( T^*Z \), then \( \omega(x, \xi) = \sum_{i=1}^n \xi_i dx_i \).

Let us fix some \( k \in \mathbb{N}, k \geq 2 \). In \( \mathbb{C}^{k+1} = \mathbb{C}^k \times \mathbb{C} \) we consider the coordinates \((s_1, \ldots, s_k, z)\). As in the previous sections we set
\[ P(s, z) = z^k + \sum_{h=1}^{k-1} (-1)^h s_h z^{k-h} \]
Obviously
\[ P(s_1, \ldots, s_k, z) = 0 \iff s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h}, \]
where \( s_0 = 1 \). We note \( s = (s_1, \ldots, s_k) \) and \( s' := (s_1, \ldots, s_{k-1}) \).

Let \( H \) be the smooth hypersurface of \( \mathbb{C}^{k+1} \) given by the zeros of \( P(s, z) \).

Let us denote by \( \mathcal{L} \) the \( \mathcal{D}_{\mathbb{C}^{k+1}} \)-module with one generator \( u \) defined by the equations \( \partial u / \partial s_1 = \cdots = \partial u / \partial s_k = 0 \). Such module is an example of a so called \textit{partial de Rham systems}, which have the feature, among others, that their characteristic varieties are non singular regular involutive. In our case we have
\[ \text{Char } \mathcal{L} = \{(s, z); (\eta, \tau) \in \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \text{ such that } \eta = 0 \}. \]

Since \( H \subset \mathbb{C}^{k+1} \) is defined by the equation
\[ P(s, z) = (-1)^k s_k + \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h} = 0, \]
\( T^*_H \mathbb{C}^{k+1} \) is the subbundle of \( T^*\mathbb{C}^{k+1}|_H \) described by
\[ \{(s, z; \eta, \tau), (s, z) \in H, \exists \lambda \in \mathbb{C} \text{ such that } (\eta, \tau) = \lambda dP(s, z)\}. \]

This means that \( P(s, z) = 0 \) and that there exists \( \lambda \in \mathbb{C} \) with \( \eta_h = \lambda (-1)^h z^{k-h} \) for each \( h \in [1, k] \) and that \( \tau = \lambda P(s, z) \). Hence as \( \eta_k = \lambda (-1)^k \) and this implies:
\[ \text{Char } \mathcal{L} \cap T^*_H \mathbb{C}^{k+1} \subset T^*_\mathbb{C}^{k+1} \mathbb{C}^{k+1} \]
(as usual, \( T^*_\mathbb{C}^{k+1} \mathbb{C}^{k+1} \) denotes the zero section of \( T^*\mathbb{C}^{k+1} \), in other words \( H \) is non characteristic for \( \mathcal{L} \). Let us denote by \( j : H \hookrightarrow \mathbb{C}^{k+1} \) the closed embedding. By Kashiwara’s classical results (which can be found in [K]) it follows that the induced system \( D j^* \mathcal{L} \) by \( \mathcal{L} \) on \( H \) is concentrated in degree zero and \( \mathcal{N} := \mathcal{H}^0 D j^* \mathcal{L} \) is a \( \mathcal{D}_{\mathcal{H}} \)-coherent module whose characteristic variety is exactly
\[ j d j^{-1}_* \text{Char } (\mathcal{L}). \]

Recall that
\[ D j^* \mathcal{L} := \mathfrak{O}_H \otimes_{j^{-1} \mathcal{O}_{\mathbb{C}^{k+1}}} j^{-1} \mathcal{L} \]
and in this non-characteristic case
\[ \simeq j^{-1} \frac{\mathcal{O}_{C_1}^{k+1}}{P(s', s_k, z)\mathcal{O}_{C_1}^{k+1}} \otimes j^{-1} \mathcal{O}_{C_1}^{k+1} j^{-1} \mathcal{L} \]

We have
\[ N := j^{-1}\left( \frac{\mathcal{D}_{C_1}^{k+1}}{P\mathcal{D}_{C_1}^{k+1} + \mathcal{D}_{C_1}^{k+1}\partial_{s_1} + \cdots + \mathcal{D}_{C_1}^{k+1}\partial_{s_k}} \right) \simeq j^{-1}(\mathcal{O}_{C_1}^{k+1}/\mathcal{O}_{C_1}^{k+1}) \subset \partial_z \]

which is isomorphic as a \( \mathcal{D}_H \)-module to
\[ \mathcal{O}_H < \partial_z \simeq \frac{\mathcal{D}_H}{\mathcal{D}_H \partial_{s_1} + \cdots + \mathcal{D}_H \partial_{s_{k-1}}} \]

where \( \partial_{s_i} \) stands for the derivation \( \partial/\partial s_i \) on \( \mathcal{O}_H \) and \( \partial_z \) as a derivation on \( \mathcal{O}_H \) is the class of \( \partial_z \) in the quotient above.

In particular \( N \) is sub-holonomic and it is a partial de Rham system similarly to \( \mathcal{L} \). Let us now determine the image of \( N \) under the morphism \( \pi : \mathbb{C}^k \simeq H \to \mathbb{C}^k \) given by \( (s', z) \mapsto (s', s_k) \). Clearly \( \pi \) is proper surjective with finite fibers.

Recall that one denotes by \( \mathcal{D}_{C_1^{\ast -H}} \) the transfer \( (\pi^{-1}\mathcal{D}_{C_1}, \mathcal{D}_H) \)-bimodule
\[ (\pi^{-1}\mathcal{D}_{C_1} \otimes_{\pi^{-1}\mathcal{O}_{C_k}} \pi^{-1}\Omega_{C_k}^{\otimes -1}) \otimes_{\pi^{-1}\mathcal{O}_{C_k}} \Omega_H \]

Recall also that, according to the properness and the fiber finiteness of \( \pi \), we have
\[ D\pi_*N \simeq \mathcal{H}^0D\pi_*N = \pi_*(\mathcal{D}_{C_1^{\ast -H}} \otimes_{\mathcal{D}_H} N) \]

where we abusively use the notation \( \pi_* \) for the direct image functor in the category of \( \mathcal{D} \)-modules in the two left terms and for the direct image functor for sheaves in the right term. According to [S] Th. 4.25 and 4.27 (see also the comments in loc.cit. before Theorem 4.27), one knows that \( D\pi_*N \) is concentrated in degree zero and that
\[ \text{Char} \mathcal{H}^0D\pi_*N = \pi_\pi \pi^{-1} \text{Char} N. \]

So we may henceforward denote for short \( \pi_*N := D\pi_*N \) without ambiguity.

Let \( \Delta \) as above be the zero set of the discriminant of \( P \), which can also be defined as the image by \( \pi \) of the subset of \( H \) defined by \( \{ P'(s, z) = 0 \} \).

Since \( \pi_d \) is given by the \( k \times k \) matrix
\[ \left( \frac{1}{s_k/\partial s_k/\partial z} \right)^T \]

we conclude that \( \text{Char} \pi_*N \) is the image by \( \pi_* \) of the set
\[ \{(s', z); (\eta', \tau) \in \mathbb{C}^k \times \mathbb{C}^k / \eta_j = -(z)^{k-j-1}\tau, \ \forall j \in [1, k-1]\} \]

so it is given by the set
\[ \{(s, \eta) \in \mathbb{C}^k \times \mathbb{C}^k / \exists z \in \mathbb{C} \text{ such that } P_s(z) = 0 \text{ and with } \eta_j = -(z)^{k-j-1}\eta_k, \ \forall j \in [1, k-1]\} \]

Then \( \text{Char} \pi_*N \) is an involutif analytic subset of \( T^*\mathbb{C}^k \) with codimension \( k-1 \) which proves the following:

**Lemma 4.1.** \( \pi_*N \) is a subholonomic \( \mathcal{D}_{C_1^{k+1}} \)-module.

**Remark 4.2.** Let \( \tilde{N} \) denote the \( \mathcal{D}_{C_1} \)-module associated to \( \mathcal{L} \). Then \( \tilde{N} \) is clearly not subholonomic.
Proposition 4.3. The $\mathcal{D}_{C_k}$-module $\pi_*\mathcal{N}$ is the quotient of $\mathcal{D}^k_{C_k} \simeq \mathcal{D}_{C_k} \otimes_{\mathbb{C}} \mathbb{C}^k$ by the action of

$$A_h := \partial_{s_h} \otimes Id_{C_k} + (-1)^{k-1} \partial_{s_k} \otimes A(s)^{k-1} \quad \text{for } j \in [1, k-1].$$

Moreover the action of $z$ and $\partial_z$ on $\pi_*\mathcal{N}$ deduced from the action of $\mathcal{D}_H$ on $\mathcal{N}$ are given respectively by

$$A_0 := 1 \otimes A(s) \quad \text{and} \quad B := 1 \otimes \nabla + (-1)^{k-1} \partial_{s_k} \otimes P'_s(A(s))$$

where we put $P'_s(z) := (\partial_z(P_s(z)))$ and $\nabla := \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & k-1 & 0 \end{pmatrix}$

Proof. Our goal is to explicit $\pi_*\mathcal{N}$ and to check that it coincides with the $\mathcal{D}_{C_k}$-module associated to the system $(\mathbb{Q})$ in Theorem 2.4.

In a first step we explicit the transfer-module

$$\mathcal{D}_{C_\mathbb{C} \leftarrow H} := \pi^{-1}\mathcal{D}_{C_k} \otimes_{\mathbb{C}} \mathcal{O}_{C_k} (\pi^{-1}\mathcal{O}_{C_k}^\otimes \otimes \mathbb{C} \mathcal{O}_{C_k} \mathcal{H})$$

as a $(\pi^{-1}\mathcal{D}_{C_k}, \mathcal{D}_H)$-bimodule.

The next step is to determine the cokernel of

$$\alpha : (\mathcal{D}_{C_\mathbb{C} \leftarrow H})^{k-1} \rightarrow \mathcal{D}_{C_\mathbb{C} \leftarrow H}$$

$$(u_1, \ldots, u_{k-1}) \mapsto \sum_{i=1}^{k-1} u_i \partial_{s_i}$$

The last step is to apply $\pi_*$. Let us denote for short

$$\sigma := \omega_H \otimes \omega^\otimes_{C_k} := ds_1 \wedge \cdots \wedge ds_{k-1} \wedge dz \otimes (ds_1 \wedge \cdots \wedge ds_{k-1} \wedge ds_k)^\otimes$$

the generator of the line bundle $\pi^{-1}\mathcal{O}_{C_k}^\otimes \otimes \mathcal{O}_{C_k} \mathcal{H}$.

Recall that $\mathcal{O}_H = \mathcal{O}_{C_{k+1}}/\mathfrak{J}$, where $\mathfrak{J}$ is the ideal generated by $P(s, z)$. Hence $\mathcal{O}_H$ is a $\pi^{-1}\mathcal{O}_{C_k}$-free module with rank $k$ since each section $a(s', z)$ of $\mathcal{O}_H$ is equivalent, by Weierstrass Division Theorem, to a unique polynomial $\sum_{j=0}^{k-1} a_j(s', s_k)z^j$ modulo $P(s, z)$, for some sections $a_j$ of $\mathcal{O}_{C_k}$.

Hence $\mathcal{D}_{C_\mathbb{C} \leftarrow H}$ is a left $\pi^{-1}\mathcal{D}_{C_k}$-free module of rank $k$ generated by the $k$-sections $(1 \otimes z^j)\sigma_{j=0, \ldots, k-1}$. Since the right action of each operator in $\mathcal{D}_H$ is $\pi^{-1}\mathcal{D}_{C_k}$-linear, it is sufficient to calculate each $(1 \otimes z^j)\partial_{s_i}, i = 1, \ldots, k-1, j = 0, \ldots, k-1$.

Now recall that $H$ is defined in $\mathbb{C}^{k+1}$ by the equation $s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_k z^{k-h}$ with the convention $s_0 = 1$ and so $s_1, \ldots, s_{k-1}, z$ are global coordinates on $H$. Then we have in $H$

$$\frac{\partial s_k}{\partial s_h} = (-1)^{k-1}h^{-1}z^{k-h} \quad \text{and} \quad \frac{\partial s_k}{\partial z} = (-1)^{k-1}P'_s(z)$$

$^3$Note that $z$ and $\partial_z$ commute with $\partial_{s_h}$ for $h \in [1, k-1]$ in $\mathcal{D}_H$. 

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where $P_s^r(z)$ does not depend on $s_k$.

Let $F := 1 \otimes E(z)\sigma$ denote the basis $(1 \otimes z^j\sigma), j \in [0, k - 1]$ of the free rank $k$ left $\pi^{-1}(\mathcal{D}_{\mathbb{C}^k})$-module $\mathcal{D}_{\mathbb{C}^k-H}$. Recall that, according to [3] Rem. 4.18, in view of the finiteness of the fibers of $\pi$,

**Remark 4.4.**

By functoriality $\pi\mathcal{D}$ since the generator of $\mathcal{D}_H$ in $\mathcal{D}_{\mathbb{C}^k-H}$ is defined by the following formulas, where we consider $F$ as a $k$-vector and use the usual matrix product

\begin{align*}
(0^*) & \quad F\theta(s') = \theta(s')F \quad \text{where} \quad \theta \in \mathcal{O}_H \quad \text{does not depend on} \quad z \\
(1^*) & \quad Fz = A(s)F \\
(2^*) & \quad -F\partial_{s_h} = \partial_{s_h}F + (-1)^{k-h-1}\partial_{s_h}(A(s)^{k-h}F) \quad \forall h \in [1, k-1] \\
(3^*) & \quad -F\partial_z = \nabla F + (-1)^{k-1}\partial_z(F^r(A(s))^r)F
\end{align*}

where we have used the equalities $zE(z) = A(s)E(z)$ and $\partial_z(E(z)) = \nabla E(z)$.

Summing up:

- For $g \in \mathcal{O}_H$ represented by $\sum_{r=0}^{k-1}g_r(s)z^r$, the $(k \times k)$ matrix $G$ of the $\pi^{-1}\mathcal{D}_{\mathbb{C}^k}$-linear morphism defined by $g$ on $\mathcal{D}_{\mathbb{C}^k-H}$ is given by

$$G := \sum_{r=0}^{k-1}g_r(s)A(s)^r$$

- Let us consider the $\mathcal{D}_{\mathbb{C}^k}$-linear morphism $\alpha : (\mathcal{D}_{\mathbb{C}^k})^{(k-1)} \to \mathcal{D}_{\mathbb{C}^k}$ defined by the following $k-1$ $(k, k)$-matrices

$$\mathcal{A}_h := \partial_{s_h} \otimes \text{Id}_{\mathcal{C}^k} + (-1)^{k-h}\partial_{s_h} \otimes A(s)^{k-h}$$

Let $\Phi$ be in $\mathcal{O}_{\mathcal{C}^k}$. In view of the relation $(2^*)$, the map $(1 \otimes z^h\sigma) \mapsto \Phi_h$, for $h \in [0, k-1]$ will induce an element of $\mathcal{H}\text{om}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$, that is to say a solution of $\pi_*\mathcal{N}$, if and only if we have $\partial_{s_h}(\Phi) = (-1)^{k-h}\partial_{s_h}(A^{k-h}\Phi)$, that is, in and only if $\Phi$ satisfies $(4)$, since the generator of $\pi_*\mathcal{N}$ is annihilated by the action of $\partial_{s_h}$ for each $h \in [1, k-1]$. In conclusion:

$$\pi_*\mathcal{N} \simeq \text{coker} \alpha$$

by the finiteness of the fibers of $\pi$.

**Remark 4.4.** $\mathcal{N}$ is naturally endowed with a structure of right $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$-module. By functoriality $\pi_*\mathcal{N}$ is also a right $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$-module and its structure coincides with the induced by the right $\mathcal{D}_{\mathbb{C}}$ action defined by $(1^*)$ and $(3^*)$ on $\mathcal{D}_{\mathbb{C}^k-H} \otimes_{\mathcal{D}_H} \mathcal{N}$, since it commutes with each $\partial_{s_i}$, for $i = 1, \ldots, k-1$. Therefore we obtain a natural left action of $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ on $\mathcal{H}\text{om}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$.

We also conclude, according to Lemma [3.1]

**Proposition 4.5.** The left action of $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ defined by the above Remark 4.4 on $\mathcal{H}\text{om}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ coincides with the left action of $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ on Lisbon integrals.

If $\Phi$ is a solution of $\pi_*\mathcal{N}$, replacing in the formula $(3^*)$ the second term thanks to the equality obtained for $\Phi$ after applying $(2^*)$ or equivalently $(4)$, we also derive a right action of $\partial_z$ which is given by the formula

$$-\Phi\partial_z = \nabla \Phi - \sum_{h=0}^{k-1}(k-h)s_h\partial_{s_{h+1}}\Phi$$
Our next goal is to conclude in Proposition 4.7 below that there are no global holomorphic solutions of $\pi_*N$ other than those of the form $\Phi_f$, for some holomorphic function $f$ only depending on $z$. Since $j$ is non-characteristic we have an isomorphism

$$j^{-1}R\mathcal{H}\hom_{\mathcal{D}_{C^{k+1}}}(\mathcal{L}, \mathcal{O}_{C^{k+1}}) \simeq R\mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H)$$

According to Theorem 4.33 (2) of [S], making $X = H, Y = C^k, f = \pi, N = \mathcal{O}_{C^k}$ in loc.cit, we obtain

**Theorem 4.6.** For any open subset $\Omega$ of $C^k$ we have an isomorphism functorial on $N$ compatible with restrictions to open subsets

$$R\Gamma(\pi^{-1}(\Omega); R\mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H)) \simeq R\Gamma(\Omega; R\mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^{k}}))$$

Recall that this isomorphism uses as a tool the "trace morphism": $\pi_*\mathcal{O}_H \to \mathcal{O}_{C^{k}}$ constructed in [S Prop. 4.34].

Since for any open subset $\Omega$ and any $\mathcal{D}_{H}$-module $\mathcal{P}$, $\Gamma(\Omega; \cdot)$ and $\mathcal{H}\hom_{\mathcal{D}_{H}}(\mathcal{P}, \cdot)$ are left exact functors, since if $\Omega$ is a Stein open set and if $\mathcal{P}$ admits a global resolution by free $\mathcal{D}_{H}$-modules of finite rank, then $R\mathcal{H}\hom_{\mathcal{D}_{H}}(\mathcal{P}, \mathcal{O})$ is represented by a complex in degrees $\geq 0$ with $\Gamma(\Omega; -)$-acyclic entries, we conclude that

$$H^0(R\Gamma(H; R\mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H))) = \Gamma(H; \mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H))$$

and

$$H^0(R\Gamma(C^k; R\mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k}))) = \Gamma(C^k; \mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k}))$$

therefore Theorem 4.6 entails a $\mathbb{C}$-linear isomorphism

$$T : \mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H) \simeq \mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k})$$

**Proposition 4.7.** The correspondence

$$f \mapsto \Phi(f) := \Phi_f$$

defines a $\mathbb{C}$-linear isomorphism

$$\Phi : \Gamma(\mathbb{C}; \mathcal{O}_{\mathbb{C}}) \to \Gamma(C^k; \mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k})) = \mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k})$$

Moreover, this isomorphism are also $\Gamma(\mathbb{C}, \mathcal{D}_{C})$-left linear.

**Proof.** The last statement is clear from Proposition 4.5 and Lemma 3.1.

The remaining of the statement is equivalent to prove that $\Phi : f \mapsto \Phi_f$ defines an isomorphism $\mathcal{H}\hom_{\mathcal{D}_{H}}(N, \mathcal{O}_H) \to \mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N, \mathcal{O}_{C^k})$.

We already know that $\Phi$ is injective. It remains to prove that $\Phi$ is surjective. For each $f \in \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$, we introduce the regular holonomic $\mathcal{D}_{C}$-module $M_f$ (a regular flat holomorphic connection on $\mathbb{C}$) of which the constant sheaf $\mathbb{C}f$ in degree zero is the complex of holomorphic solutions.

Note that $N \simeq \mathcal{O}_{C^{k-1}} \boxtimes \mathcal{D}_{C}$ where we consider $C^{k-1}$ endowed with the coordinates $(s_1, \ldots, s_{k-1})$ and $\mathbb{C}$ with the coordinate $z$. We denote by $N_f$ the regular holonomic $\mathcal{D}_{H}$-module (a regular flat holomorphic connection on $H$)

$$N_f := \mathcal{O}_{C^{k-1}} \boxtimes M_f.$$  

It is clear that $N_f$ is a quotient of $N$, and, by the left exactness of $\pi_*$, $\pi_*N_f$ is a quotient of $\pi_*N$. Moreover, according to Proposition 4.5 and Lemma 3.1 $\Phi_f$ belongs to $\mathcal{H}\hom_{\mathcal{D}_{C^{k}}}(\pi_*N_f, \mathcal{O}_{C^{k}})$. According to Theorem 4.6 for each $f$ we have a $\mathbb{C}$-linear
isomorphism $T_f : \Hom_{\D_H}(N_f, O_H) \simeq \Hom_{\D_{C^k}}(\pi_* N_f, O_{C^k})$ which makes this last one a one dimensional $\C$-vector space. Moreover, by left exactness of $\Hom$ and the exactness of $\pi_*$, we have monomorphisms $\Hom_{\D_{C^k}}(\pi_* N_f, O_{C^k}) \subset \Hom_{\D_{C^k}}(\pi_* N, O_{C^k})$ and, by functoriality, we have $T(f) = T_f(f)$.

We shall use the following result:

**Lemma 4.8.** Suppose that $f \neq 0$. Then $\Hom_{\D_{C^k}}(\pi_* N_f, O_{C^k})$ is a one dimensional $\C$-vector space generated by $\Phi_f$.

**Proof.** The result follows by Proposition 2.3 since $\Phi_f$ is a non zero element of $\Hom_{\D_{C^k}}(\pi_* N_f, O_{C^k})$ hence it is a generator as a $\C$-vector space. ■

Let us now end the proof of Proposition 4.7.

Clearly $\Hom_{\D_H}(N, O_H) = \sum_f \Hom_{\D_H}(N_f, O_H)$ and, according to Lemma 4.8 for each $f$, $\Hom_{\D_{C^k}}(\pi_* N_f, O_{C^k})$ is the $\C$-vector space spanned by $\Phi_f$; hence $T(f) = \lambda \Phi_f$ for some $\lambda \in \C^*$. Since $\Phi_{\lambda f} = \lambda \Phi_f$ we conclude that $\Phi$ is surjective which gives the desired result. ■

As a consequence, isomorphism $\Phi$ explicits isomorphism of Theorem 4.6 since they coincide up to the multiplication by a constant $\lambda \neq 0$.

4.a. **An example.** To conclude this article, let us give an interesting example of choice of the entire function $f$ on $\C$ for which we explicit the regular holonomic system on $\C^k$ associated to the corresponding Lisbon integrals.

The case $f_t(z) := e^{tz}$. Let us fix a parameter $t \in \C^*$ and consider the entire function $f_t(z) := e^{tz}$.

First remark that according to Lemma 3.1, we have $\frac{\partial E(z)}{\partial z} = \nabla E(z)$ where $\nabla$ is the $(k, k)$ matrix given by

$$
\nabla := \begin{pmatrix}
0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
0 & 2 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & k-1 & 0
\end{pmatrix}
$$

We have $^4$

$$
\Phi_{f_t}(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{s \zeta} E(\zeta) d\zeta}{P_s(\zeta)}
$$

and, according to the linearity of $\Phi_{(\cdot)}$, we also have

$$
t \Phi_{f_t}(s) = \Phi_{\partial_s(f_t)}(s)
$$

which, applying (**), entails

$$
t \Phi_{f_t}(s) = -\nabla \Phi_{f_t}(s) + (-1)^{k-1} \frac{\partial (P_s(\zeta) \Phi_{f_t})}{\partial s_k}(s)
$$

$^4$Remember that $t$ is a fixed complex parameter
Hence
\[(14)\]
\[(tI + \nabla)\Phi_{f_t}(s) = (-1)^{k-1} \frac{\partial(P'_s(A)\Phi_{f_t})}{\partial s_k}(s).\]

This also implies the following equation for \(\Psi_{f_t}\) away of the discriminant hypersurface \(\Delta\):
\[(15)\]
\[(tI + \nabla)P'_s(A)^{-1}\Psi_{f_t}(s) = (-1)^{k-1} \frac{\partial(\Psi_{f_t})}{\partial s_k}(s)\]

for\[
\Psi_{f_t}(s) = P'_s(A)\Phi_{f_t}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{\zeta}P'_s(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.\]

Combining (14) with the system (??) it is easy to see that we obtain a meromorphic integrable connexion on the trivial vector bundle of rank \(k\) on \(\mathbb{C}^k\) with a pole along the discriminant hypersurface.

The regularity of this meromorphic connexion is then consequence of the regularity of the \(\mathcal{D}_H\)-module \(N_{xt} = \mathcal{D}_Hu\) which is given by the equations
\[\partial_{sh} u = 0, \quad \forall h \in [1, k-1] \quad \text{and} \quad (\partial_z - t)u = 0\]
which is clearly regular holonomic on \(H\). So its direct image by \(\pi\) (as a \(\mathcal{D}_{\mathbb{C}^k}\)-module) is regular holonomic on \(\mathbb{C}^k\) (see [9, Th. 8.1]).
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