GALILEAN $W_3$ ALGEBRA

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Abstract. Galilean $W_3$ vertex operator algebra $\mathcal{GW}_3(c_L, c_M)$ is constructed as a universal enveloping vertex algebra of certain non-linear Lie conformal algebra. It is proved that this algebra is simple by using determinant formula of the vacuum module. Reducibility criterion for Verma modules is given, and the existence of subsingular vectors demonstrated. Free field realisation of $\mathcal{GW}_3(c_L, c_M)$ and its highest weight modules is obtained within a rank 4 lattice VOA.

1. Introduction

Galilean $W$–algebras have been studied extensively by physicists in the past decade (see for example [1], [6], [8], [9], [16], [20], [22] and references therein). Given an infinite-dimensional $W$-algebra with generators $W_1, \ldots, W_k$ of conformal weights $w_1, \ldots, w_k$ the associated Galilean algebra is generated by $W'_1, \ldots, W'_k, \overline{W}_1, \ldots, \overline{W}_k$ of conformal weights $w'_1, \ldots, w'_k, w_1, \ldots, w_k$, such that $(\overline{W}_1, \ldots, \overline{W}_k)$ is a commutative subalgebra on which all $W'_i$ act. Moreover, the relations between $W'_i$ and $W'_j$ as well as relations between $W'_i$ and $\overline{W}_j$ resemble the original relations between $W_i$ and $W_j$. This new algebra is obtained through a process called Galilean contraction. Roughly speaking, one considers a tensor product of two copies of the original algebra (with arbitrary central charges) and takes a non-relativistic limit (cf. [20]).

The most basic example is a Galilean conformal algebra (GCA), also known as BMS$_3$-algebra (Bondi-Metzner-Sachs) which comes from contracting the Virasoro algebra. See for example [12]. In mathematical literature GCA first appeared in [24] where it was called the $W(2,2)$ (Lie and vertex operator) algebra. Here ”(2,2)” denotes conformal weights of two generators. This algebra is constructed by adjoining to the Virasoro algebra its ”commutative double”, i.e. it is a direct sum of (either Lie or vertex operator algebra) Vir and its adjoint representation. This is analogous to construction of Takiff algebras in finite-dimensional case: $\text{Vir} \otimes_{\mathbb{C}}(\mathbb{C}[x]/(x^2))$ with brackets $[a \otimes x^i, b \otimes x^j] = [a, b] \otimes x^{i+j}, a, b \in \text{Vir}$. Free field realisation of GCA was obtained by means of $\beta\gamma$ system in [9], albeit only for central charge $c_L = 26$.

Date: August 13, 2021.

2010 Mathematics Subject Classification. Primary 17B69; Secondary 17B68, 81R10.

Key words and phrases. Galilean algebras, W algebras.
Bosonic free field realisation for arbitrary non-zero central charge was later obtained in [3]-[5] and its representation theory has been developed in many papers. We recall the most important results in Subsection 4.1.

Galilean $W_3$ or BMS$_3$-$W_3$ algebra was originally introduced in [6]. In physics sense, the construction of this algebra follows the same prescription as GCA (cf. [20]) – contraction of a tensor product of two copies of Zamolodchikov’s $W_3$ algebra. Free field realisation for central charge $c_L = 100$ was obtained by double $\beta\gamma$ system in [9]. Mathematically however, things are more complicated. First of all, $W_3$ is not a Lie algebra (quadratic terms appear when commuting the operators). Furthermore, from the OPE relations immediately follows that the Galilean $W_3$ is not an extension of $W_3$. Still, there is a ”commutative double” which makes handling this algebra somewhat easier.

The aim of this paper is to give a mathematically rigorous definition of Galilean $W_3$ algebra and to initiate the study of highest weight representations. We give this definition by using Kac - De Sole language of non-linear conformal Lie algebras (NLCA) in Section 2. The universal enveloping vertex algebra of presented NLCA is precisely the Galilean $W_3$ VOA from [6] (up to normalisation). By choosing a suitably ordered basis of the Verma module we utilise the commutative part and reduce the problem of finding zeroes of determinant formula to a very simple matrix (56) of rank 2. This enables classification of irreducible Verma modules. As in the case of GCA, reducibility depends only on highest weights and central charge corresponding to the action of commuting generators (cf. Theorem 3.3). By calculating singular and subsingular vectors of conformal weights 1 we give a basis of the vacuum module. Considering its determinant formula we also prove that (universal) Galilean $W_3$ is a simple algebra (Theorem 3.6). The method used in this section should be easily applied to other Galilean algebras, and we expect that analogous results hold in general. Much like in the case of Virasoro and GCA, the structure and representation theory of Galilean $W_3$ algebra seems to be rather different than that of $W_3$ (cf. [10], [11], [18], [23]). Notably, there are no minimal models, and the structure of the Verma modules seems to be uniform for all central charges.

It is well known that Quantum Drinfeld-Sokolov reduction of an (universal) affine VOA $V^k(sl_N)$ produces the $W$-algebra $W_N$. The resulting algebra can, in turn, be realised as a subalgebra of $M(1)_{N-1}$, the Heisenberg algebra of rank $N - 1$, and $M(1)_{N-1}$ is constructed over a lattice $L_{N-1}$ with Gram matrix equal to the Cartan matrix of $sl_N$ (cf. [21]). Since Galilean $W_N$ algebra is obtained from a tensor product of two copies of $W_N$, it is natural to consider its free field realisation within a product of two copies of Heisenberg algebras used in realisation of $W_N$. In Section 4 we start with a rank 4 lattice which is a product of two lattices whose Gram matrices are equal to Cartan matrices of $sl_3$. In the associated rank 4 Heisenberg algebra we detect a family of subalgebras isomorphic to Galilean $W_3$
algebras with arbitrary non-zero central charges. Furthermore by using the associated lattice VOA, we present a realisation of highest weight modules in Section 5. The highest weights are parametrised in such a way that reducibility of Verma modules corresponds to positive integral values of the first parameter (Proposition 5.1). This resembles the GCA case which is recalled in Subsection 4.1. We expect that the positive integral values of other parameters detect subsingular vectors in general. This is verified on (sub)singular vectors at level one (Example 5.3).

Throughout the paper we work with central charge $(c_L, c_M) \in \mathbb{C}^2$ such that $c_M = 0$. In Appendix we present the definition of $G_{W_3}(c_L, 0)$ which is an extension of Virasoro VOA by an ideal generated by the remaining three fields.

The author is partially supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

The author would like to thank Dražen Adamović for useful comments and discussions and Simon Wood for bringing the OPE package for MATHEMATICA to my attention.

2. Definitions

We start by recalling the notion of non-linear Lie conformal algebra introduced in [14].

**Definition 2.1** ([14]). A Lie conformal algebra is a $\mathbb{C}[D]$-module $R$ with a $\mathbb{C}$-linear map $[\lambda] : R \otimes R \rightarrow R[\lambda]$ satisfying the following axioms

\begin{align*}
(1) & \quad [Da_\lambda b] = -\lambda[a_\lambda b] \quad [a_\lambda Db] = (\lambda + D)[a_\lambda b], \\
(2) & \quad [a_\lambda b] = -[b_{-\lambda - D}a], \\
(3) & \quad [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]] - [[a_\lambda b]_{\lambda + \mu} c] = 0.
\end{align*}
To any Lie conformal algebra $R$ one canonically associates $V(R)$, the universal enveloping vertex algebra of $R$ which is freely generated by $R$. For $a, b, c \in V(R)$ we have

\[ [a_\lambda b] = \text{Res}_z e^{z\lambda} Y(a, z)b = \sum_{n \in \mathbb{Z} \geq 0} \frac{\lambda^n}{n!} a_{(n)} b \in \mathbb{C}[\lambda], \]

\[ :ab:- :ba:= \int_{-\lambda}^{0} [a_\lambda b] d\lambda, \]

\[ :ab:)c:- :a(bc):=a \int_{0}^{D} [b_\lambda c] d\lambda: + :b \int_{0}^{D} [a_\lambda c] d\lambda: \]

\[ [a_\lambda :bc:] = [a_\lambda b] c: + :b [a_\lambda c]: + \int_{0}^{\lambda} [[a_\lambda b]_\mu c] d\mu, \]

\[ [:ab_\lambda :c] = : (e^{D\partial_\lambda} a)[b_\lambda c]: + : (e^{D\partial_\lambda} b)[a_\lambda c]: + \int_{0}^{\lambda} [b_\mu [a_\lambda -_\mu c]] d\mu. \]

where $a_{(n)} b = \text{Res}_z Y(a, z)b$ denotes the $n$-th product and $:ab:= a_{(-1)} b$ is a normally ordered product of fields $Y(a, z)$ and $Y(b, z)$. $\int$ is a formal definite integral operator on $R[\lambda]$, i.e.

\[ \int_{a}^{b} \lambda^n d\lambda = \frac{1}{n+1} (b^{n+1} - a^{n+1}), \quad n \in \mathbb{Z}_{\geq 0}. \]

Note that (4) encodes the commutator formula $[Y[a, z], Y(b, w)]$ also known as operator product expansion (OPE)

\[ a(z)b(w) \sim \sum_{n \in \mathbb{Z} \geq 0} \frac{a_{(n)} b(w)}{(z-w)^n}. \]

Infinite-dimensional Lie algebras like Virasoro, Heisenberg and affine Kac-Moody algebras give rise to Lie conformal algebras whose universal enveloping algebras are precisely the universal vertex algebras associated to starting Lie algebras.

**Example 2.2.** If $R = \mathbb{C}[D]L \oplus \mathbb{C}$ (with $D1 = 0$) such that $[L_\lambda L] = (D + 2\lambda)L + \frac{c_L}{12} \lambda^3$ then $V(R) = \text{Vir}_c$.

Let $R_G = \mathbb{C}[D]L \oplus \mathbb{C}[D]M \oplus \mathbb{C}$ such that

\[ [L_\lambda L] = (D + 2\lambda)L + \frac{c_L}{12} \lambda^3, \]

\[ [L_\lambda M] = (D + 2\lambda)M + \frac{c_M}{12} \lambda^3, \]

\[ [M_\lambda M] = 0. \]

Then $V(R_G) = L^{W(2,2)}(c_L, c_M)$ is GCA with central charge $(c_L, c_M)$.

However, many freely generated vertex algebras are not universal envelopes of finite Lie conformal algebras because the $n$-th products of some of their generators are nonlinear, i.e. they contain normally ordered products. For this reason one needs to extend the $\lambda$-bracket to $\mathcal{T}(R)$, the tensor algebra of $R$. 
For $a \in R$ and $B \in \mathcal{T}(R)$ define $aB := a \otimes B$ so that $D(1) = 0$, $D(AB) := (DA)B$, $+ : AD(B)$: for $A, B \in \mathcal{T}(R)$ and then extend the $\lambda$-bracket to $\{ \lambda \} : R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ using (68). In order to deal with the Jacobi identity (3) we assume that $R$ is $\mathbb{Z}$-graded by conformal weights $R = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} R[\Delta]$ such that

$$\Delta(Da) = \Delta(a) + 1, \quad \Delta(a_{(n)}b) = \Delta(a) + \Delta(b) - n - 1$$

for $n \in \mathbb{Z}_{\geq 0}$. Extending the grading to $\mathcal{T}(R)$ define the subspace $\mathcal{M}_{\Delta}(R) \subset \mathcal{T}(R)_{\leq \Delta}$ spanned by all elements

$$X \otimes (b \otimes c - c \otimes b) \otimes Y - X \otimes \left( \int_{-\Delta}^{0} [b_{\lambda}c] d\lambda \right) Y :$$

where $b, c \in R$, $X, Y \in \mathcal{T}(R)$ and $\Delta(X \otimes b \otimes c \otimes Y) \leq \Delta$.

Definition 2.3 ([14]). A non-linear Lie conformal algebra (NLCA) is a $\mathbb{Z}$-graded $\mathbb{C}[D]$-module $R = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} R[\Delta]$ with a $\mathbb{C}$-linear map $\{ \lambda \} : R \otimes R \to \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$ satisfying (72), (7) and

$$\Delta([a_{\lambda}b]) < \Delta(a) + \Delta(b),$$

$$[a_{\lambda}[b_{\mu}c]] - [b_{\mu}[a_{\lambda}c]] - [[a_{\lambda}b]_{\lambda+\mu}c] \in \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}_{\Delta'}(R),$$

for all $a, b, c \in R$, where $\Delta' < \Delta(a) + \Delta(b) + \Delta(c)$.

To each NLCA $R$ one associates a universal enveloping vertex algebra $V(R) = \mathcal{T}(R)/\mathcal{M}(R)$ which is freely generated by $R$. Conversely, if $V$ is a vertex algebra freely generated by a $\mathbb{C}[D]$-submodule $R \subset V$, then there is a NLCA structure on $R$ and $V \cong V(R)$. See [14] for details.

The first example of NLCA comes from the well known Zamolodchikov $W_3$ algebra. VOA $W_3(c)$ is generated by a conformal field $\omega(z) = \sum L(n)z^{-n-2}$ and a primary field $W(z) = \sum W(n)z^{-n-3}$ satisfying the following OPE:

$$\omega(z)\omega(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\omega(w)}{(z-w)^2} + \frac{\partial \omega(w)}{z-w}$$

$$\omega(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}$$

$$W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{2\omega(w)}{(z-w)^4} + \frac{\partial \omega(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \left( \frac{3}{10} \partial^2 \omega(w) + 2\beta \Lambda(w) \right) + \frac{1}{z-w} \left( \frac{1}{15} \partial^3 \omega(w) + \beta \partial \Lambda(w) \right)$$

where $\Lambda(z) := \omega(z)^2 - \frac{3}{10} \partial^2 \omega(z)$ and $\beta = \frac{16}{22+3c}$. 


Let \( R = \mathbb{C}[D]L \oplus \mathbb{C}[D]W \oplus \mathbb{C} \) be a NLCA with the following \( \lambda \)-brackets:

\[
\begin{align*}
[L_\lambda L] &= (D + 2\lambda)L + \frac{c}{12}\lambda^3, \\
[L_\lambda W] &= (D + 3\lambda)W, \\
[W_\lambda W] &= \frac{c}{360}\lambda^5 + \left(\frac{\lambda^3}{3} + \frac{\lambda^2}{2}D + \frac{3\lambda}{10}D^2 + \frac{1}{15}D^3\right)L + \\
&\quad + \frac{16}{5c + 22}(D + 2\lambda)\left(L^2 - \frac{3}{10}D^2L\right).
\end{align*}
\]

Then \( V(R) \) is precisely \( W_3(c) \) (cf. [14]).

Now we define the Galilean \( W_3 \) algebra.

**Definition 2.4.** Let \( c_L, c_M \in \mathbb{C} \), \( c_M \neq 0 \). The Galilean \( W_3 \) NLCA is defined as

\[
GW_3(c_L, c_M) = \mathbb{C}[D]L \oplus \mathbb{C}[D]W \oplus \mathbb{C}[D]M \oplus \mathbb{C}[D]V \oplus \mathbb{C},
\]

where \( \Delta(L) = \Delta(M) = 2 \), \( \Delta(W) = \Delta(V) = 3 \) and with the following non-trivial \( \lambda \)-brackets

\[
\begin{align*}
[L_\lambda L] &= (D + 2\lambda)L + \frac{c_L}{12}\lambda^3, \\
[L_\lambda M] &= (D + 2\lambda)M + \frac{c_M}{12}\lambda^3, \\
[L_\lambda W] &= (D + 3\lambda)W, \\
[L_\lambda V] &= (D + 3\lambda)V, \\
[M_\lambda W] &= (D + 3\lambda)V, \\
[W_\lambda W] &= \frac{c_L}{360}\lambda^5 + \left(\frac{\lambda^3}{3} + \frac{\lambda^2}{2}D + \frac{3\lambda}{10}D^2 + \frac{1}{15}D^3\right)L + \\
&\quad + \frac{32}{5c_M}(D + 2\lambda)\left(LM - \frac{3}{10}D^2M\right) - \frac{16}{5c_M}\left(c_L + \frac{44}{5}\right)(D + 2\lambda)M^2, \\
[W_\lambda V] &= \frac{c_M}{360}\lambda^5 + \left(\frac{\lambda^3}{3} + \frac{\lambda^2}{2}D + \frac{3\lambda}{10}D^2 + \frac{1}{15}D^3\right)M + \frac{16}{5c_M}(D + 2\lambda)M^2.
\end{align*}
\]

Proving that the axioms of NLCA (in particular the Jacobi identity) hold is a straightforward, but rather tedious task (see Appendix [A.1]). Another way of showing that this definition is consistent is by obtaining a free field realisation. This is presented in Section [4].
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For simplicity, we shall use the same notation \( G W_3(c_L, c_M) \) for the associated universal enveloping vertex algebra which is generated by fields

\[
\begin{align*}
\omega(z) &= \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \\
W(z) &= \sum_{n \in \mathbb{Z}} W(n) z^{-n-3}, \\
M(z) &= \sum_{n \in \mathbb{Z}} M(n) z^{-n-2}, \\
V(z) &= \sum_{n \in \mathbb{Z}} V(n) z^{-n-2}.
\end{align*}
\]

Also define the following fields

\[
\begin{align*}
\Lambda(z) &= :L(z)M(z): - \frac{3}{10} \partial^2 M(z) = \sum_{n \in \mathbb{Z}} \Lambda(n) z^{-n-4}, \\
\Theta(z) &= :M(z)M(z): = \sum_{n \in \mathbb{Z}} \Theta(n) z^{-n-4},
\end{align*}
\]

so

\[
\begin{align*}
\Lambda(k) &= \sum_{i \in \mathbb{Z}} :L(-i)M(k+i): - \frac{3}{10} (k+2)(k+3)M(k), \\
\Theta(k) &= \sum_{i \in \mathbb{Z}} :M(i)M(k-i): .
\end{align*}
\]

Then the components of these fields satisfy the following non-trivial commutation relations:

\[
\begin{align*}
[L(n), L(m)] &= (n-m)L(n+m) + \delta_{n+m,0} \frac{n(n^2-1)}{12} c_L \\
[L(n), W(m)] &= (2n-m)W(n+m) \\
[L(n), M(m)] &= (n-m)M(n+m) + \delta_{n+m,0} \frac{n(n^2-1)}{12} c_M \\
[L(n), V(m)] &= (2n-m)V(n+m) \\
[M(n), W(m)] &= (2n-m)V(n+m) \\
[W(n), W(m)] &= \frac{n-m}{30} \left( (2n^2+2m^2-nm-8)L(n+m) + \frac{192}{c_M} \Lambda(n+m) + \\
& \quad - \frac{96}{c_M} (c_L + \frac{44}{5}) \Theta(n+m) \right) + \delta_{n+m,0} \frac{n(n^2-1)(n^2-4)}{360} c_L \\
&W(n), V(m) \right) &= \frac{n-m}{30} \left( (2n^2+2m^2-nm-8)M(n+m) + \frac{96}{c_M} \Theta(n+m) \right) + \\
& \quad + \delta_{n+m,0} \frac{n(n^2-1)(n^2-4)}{360} c_M.
\end{align*}
\]

This agrees with algebra introduced in [6] up to a normalisation factor 1/30 (as in [9]). Notice that \( \omega(z) \) and \( M(z) \) generate a subalgebra isomorphic to the Galilean conformal algebra with
central charge \((c_L, c_M)\). However, \(\omega(z)\) and \(W(z)\) do not generate a copy of \(W_3\) due to \((10)\).

A natural question arises: can we define a Galilean algebra in such a way that both Virasoro, and \(W_3\) are its subalgebras acting on a commutative part? It turns out that this is not possible. Due to non-linearity, the Jacobi identity for such \(\lambda\)-brackets would not hold (see Appendix [A.2]).

**Corollary 2.5.** We have

\[
(42) \quad \text{char}_q \mathcal{GW}_3(c_L, c_M) = (1 - q^2)^2 \prod_{n \geq 3} (1 - q^n)^{-4}.
\]

**Proof.** We fix an ordering \(V > M > W > L\), and obtain a PBW basis of the universal enveloping vertex algebra \(\mathcal{GW}_3(c_L, c_M)\) (cf. \([14]\)) which consists of monomials

\[
(43) \quad V(-p_v) \cdots V(-p_1)M(-r_m) \cdots M(-r_1)W(-s_m) \cdots W(-s_1)L(-t_l) \cdots L(-t_1)1
\]

such that \(p_{k+1} \geq p_k \geq 3, r_{k+1} \geq r_k \geq 2, s_{k+1} \geq s_k \geq 3, t_{k+1} \geq t_k \geq 2\). Then the \(q\)-character formula is

\[
(44) \quad \text{char}_q L(c, 0) = (1 - q^2)^2 (1 - q)^4 \prod_{n \geq 1} \frac{1}{(1 - q^n)^4}
\]

which proves the assertion. \(\square\)

### 3. Highest Weight Modules

Let \(M\) be an ordinary module over the VOA \(V\). In particular \(M = \bigoplus_{h \in \mathbb{C}} M_h\), where \(M_h = \{m \in M : L(0)v = hv\}\) is a subspace of conformal weight \(h\), \(\dim M_h < \infty\) and for each \(u \in V\) and \(v \in M\) we have \(u(n)v = 0\) for \(n \gg 0\).

A homogeneous vector \(v \in M_h\) is called

- **singular** if for each \(u \in V\) and \(n \in \mathbb{Z}_{\geq 0}\) we have \(u(n)v = \delta_{n,0}h_u v\) for \(h_u \in \mathbb{C}\);
- **pseudo-singular** if for each \(u \in V\) and \(n \in \mathbb{Z}_{\geq 0}\) we have \(u(n)v = 0\);
- **subsingular** if there exists a submodule \(N \subset M\) such that \(v + N\) is singular in a quotient module \(M/N\).

If \(M\) is generated by a singular vector \(v\) we say that \(M\) is a **highest weight module**, and call \(v\) a highest weight vector. Assume that \(\mathcal{GW}_3(c_L, c_M)\)-module is generated by a pseudo-singular vector, so \(M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{h+n}\) for some \(h \in \mathbb{C}\). Since \(L(0), M(0), W(0)\) and \(V(0)\) are mutually commuting operators acting on (a finite-dimensional complex space) \(M_h\), there exists a common eigenvector i.e. a highest weight vector. We restrict our study to highest weight modules, i.e. the case when \(M_h\) is one-dimensional.

Let \(h := (h_L, h_W, h_M, h_V) \in \mathbb{C}^4\) be arbitrary scalars and \(c := (c_L, c_M)\). Verma module denoted by \(V(c, h)\) is the universal highest weight module \(\mathcal{GW}_3(c_L, c_M).v_h\) of highest weight.
Let $P^n$ as universal highest weight modules. The top levels of these modules are precisely the one-dimensional modules such that

$$\text{(47)}$$

$$L(n)v_h = \delta_{n,0}hLv_h, \quad W(n)v_h = \delta_{n,0}hWv_h,$$

and (35)-(41).

**Remark 3.1.** Verma modules in classical case are defined either as a quotient of universal enveloping algebra of a given Lie algebra $\mathfrak{g}$, or equivalently, as a module induced from Borel subalgebra, i.e. using a triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Since $GW_3(c_L, c_M)$ is not a (linear) Lie algebra and does not have a natural triangular decomposition (to subalgebras) we do not have these tools available. However, highest weight theory still works for general VOA. One way of proving existence of universal highest weight modules is by applying Zhu’s theory. We sketch the idea without going into details.

It is well known in vertex algebra theory that for each VOA $V$ there exists an associative algebra $A(V)$ called Zhu’s algebra of $V$ which controls the representation theory of $V$ in the following sense. For a $V$-module $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{h+n}$, $M_h$ is an $A(V)$-module. Conversely, every $A(V)$-module is a top level of some $V$-module. Obviously, one-dimensional $A(V)$-modules correspond to highest weight $V$-modules.

It is not difficult to show that $A(GW_3(c_L, c_M))$ is a commutative algebra with 4 generators. We will show in Section 4 that $GW_3(c_L, c_M)$ is a subalgebra of a rank 4 Heisenberg algebra $M(1)$. Highest weight $M(1)$-modules then provide a realisation of highest weight $GW_3(c_L, c_M)$-modules. The top levels of these modules are precisely the one-dimensional $A(GW_3(c_L, c_M))-modules and their existence then yields the existence of the Verma modules over $GW_3(c_L, c_M)$ as universal highest weight modules.

Since $GW_3(c_L, c_M)$ is a freely generated VOA with a natural PBW basis

$$\{ :V(z)^{n_v}M(z)^{n_m}W(z)^{n_w}L(z)^{n_l} : | n_v, m, w, n_l \in \mathbb{Z}_{\geq 0} \}$$

(cf. [11]) by universality of the Verma module we see that the set of monomials

$$\text{(47)}$$

$$V(-i_v)\cdots V(-i_1)M(-j_m)\cdots M(-j_1)W(-k_w)\cdots W(-k_1)L(-n_l)\cdots L(-n_1)v_h$$

such that $v, m, w, l \in \mathbb{Z}_{\geq 0}$, $i_v \geq \cdots \geq i_1 \geq 1$, $j_m \geq \cdots \geq j_1 \geq 1$, $k_w \geq \cdots \geq k_1 \geq 1$ and $n_l \geq \cdots \geq n_1 \geq 1$ forms a basis of $V(c, h)$. We have

$$V(c, h) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V(c, h)_n, \quad V(c, h)_n = \{ v \in V(c, h) : L(0)v = (h_L + n)v \}.$$ 

Let $P(n)$ denote the partition function on $\mathbb{Z}_{\geq 0}$. Then

$$\text{(48)}$$

$$\dim V(c, h)_n = \sum_{i,j,k \geq 0} P(i)P(j)P(k)P(n - i - j - k)$$
\begin{equation}
\text{char}_q V(c, h) = q^{h_L} \prod_{n \geq 1} \frac{1}{1 - q^n}.
\end{equation}

Let $V$ be a VOA, $M = \oplus_{n \in \mathbb{Z}_{\geq 0}} M_{h+n}$ an ordinary weight $V$-module, and $M^* = \oplus_{n \in \mathbb{Z}_{\geq 0}} M^*_{h+n}$ its restricted dual. Let $\langle \cdot, \cdot \rangle : M^* \times M \to \mathbb{C}$ denote the natural pairing and $Y_{M^*} : V \to \text{End} M^*[z, z^{-1}]$ be a linear map such that
\begin{equation}
\langle Y_{M^*}(v, z)w', w \rangle = \langle w', Y_{M^*}(e^{zL(1)}(-z^{-1})L(0)v, z^{-1})w \rangle
\end{equation}
for $v \in V$, $w \in M$, $w' \in M^*$. Then $(M^*, Y_{M^*})$ is a $V$-module, called the contragredient of $M$ (cf. [15]). In case of $V = GW_3(c_L, c_M)$ we have
\begin{align}
L(n)^* &= L(-n), \\
W(n)^* &= -W(-n), \\
M(n)^* &= M(-n), \\
V(n)^* &= -V(-n), \\
\Lambda(n)^* &= \Lambda(-n), \\
\Theta(n)^* &= \Theta(-n).
\end{align}

**Lemma 3.2.** Let $L(c, h)$ denote the irreducible quotient of $V(c, h)$. Then
\begin{equation}
L(c, h)^* = L(c, h^*)
\end{equation}
where $h^* = (h_L, -h_W, h_M, -h_V)$.

Natural pairing with the contragredient module induces a symmetric non-degenerate invariant bilinear form on $V(c, h)$ such that
\begin{equation}
\langle v_h | v_h \rangle = 1, \quad \langle x.v_h | y.v_h \rangle = \langle v_h | x^*y.v_h \rangle.
\end{equation}

In order to classify irreducible Verma modules we need to consider the determinant formula associated to this form. Since $\langle V(c, h)_n|V(c, h)_m \rangle = 0$ for $n \neq m$ we focus on $\det \langle V(c, h)_n|V(c, h)_n \rangle$. We are only interested in its zeros so we will not calculate exponents of all the different factors in this determinant. Instead we introduce an ordering on the chosen basis of $V(c, h)_n$ and decompose the matrix $\langle V(c, h)_n|V(c, h)_n \rangle$ to a tensor product of block triangular matrices, thus reducing the problem to finding determinant of much simpler matrices. In the following subsection we show that this problem ultimately reduces to
calculation of determinant

\[
D_n = \begin{vmatrix}
(L(-n)v_h|M(-n)v_h) & \langle L(-n)v_h|V(-n)v_h \rangle \\
(W(-n)v_h|M(-n)v_h) & \langle W(-n)v_h|V(-n)v_h \rangle 
\end{vmatrix}
\]

\[
= \left( \frac{64}{5c_M} \left( h_M + \frac{n^2 - 1}{24} c_M \right)^2 \left( h_M + \frac{n^2 - 4}{96} c_M \right) - 9h_M^2 \right) n^2.
\]

Furthermore, we use the same method to describe the module \(L(c,0)\) and prove simplicity of \(GW_3(c_L, c_M)\) in Subsection 3.2.

3.1. Determinant formula and classification of irreducible Verma modules. Instead of standard PBW basis \([47]\) we will work with the following basis of \(V(c,h)\):

\[
B_n = \left\{ V(-1)^{w_1} M(-1)^{m_1} \cdots V(-1)^{w_n} M(-1)^{m_n} W(-1)^{w_0} L(-1)^{l_1} \cdots W(-1)^{w_l} L(-1)^{l_i} | \sum_{i=1}^{n} i(v_i + m_i + w_i + l_i) = n \right\}
\]

Now we introduce the “commutative degree” of a basis monomial: for \(x \in B_n\) let

\[
\deg_c x = \sum_{i=1}^{n} i(v_i + m_i),
\]

\[
V(c,h)^k_n = \text{span}_C \{ x \in B_n : \deg_c x = k \},
\]

\[
B^k_n = B_n \cap V(c,h)^k_n.
\]

Then

\[
V(c,h)_n = \bigoplus_{k=0}^n V(c,h)^k_n
\]

\[
\dim V(c,h)^k_n = P_2(k)P_2(n-k) = \dim V(c,h)_{n-k}^{n-k},
\]

where \(P_2(m) = \sum_{i=0}^m P(i)P(m-i)\).

For \(x \in B_n\) we write \(x = x^c x^{nc} v_h\), where \(\deg_c x^c = \deg_c x\), and \(\deg_c x^{nc} = 0\). In other words, \(x^c\) is a product of factors \(M(-i)\) and \(V(-i)\), while \(x^{nc}\) is a product of factors \(L(-i)\) and \(W(-i)\). Then

\[
\langle x | y \rangle = \langle (y^c)^* x^{nc} v_h | (x^c)^* y^{nc} v_h \rangle
\]

so \(\langle V(c,h)^k_n | V(c,h)^l_n \rangle = 0\) if \(k + l > n\). We may order the elements of \(B_n\) so that for \(x, y \in B_n\) \(x < y\) if \(\deg_c x < \deg_c y\). Then the Gram matrix of \(\langle V(c,h)_n | V(c,h)_n \rangle\) is block triangular with (nontrivial) diagonal blocks \(\langle V(c,h)^k_n | V(c,h)^{n-k}_n \rangle\), \(k = 0, \ldots, n\).

Let \(x \in V(c,h)^k_n\) and \(y \in V(c,h)^{n-k}_n\). Then

\[
\langle x | y \rangle = \langle (y^c)^* x^{nc} v_h | v_h \rangle \cdot \langle v_h | (x^c)^* y^{nc} v_h \rangle
\]

\[
= \langle x^{nc} v_h | y^c v_h \rangle \cdot \langle x^c v_h | y^{nc} v_h \rangle.
\]
This shows that the Gram matrix of \( \langle V(c, h)^k_n | V(c, h)^{n-k}_n \rangle \) is a tensor product of matrices of the type \( \langle V(c, h)^0_{n-k} | V(c, h)^n_{n-k} \rangle \) and \( \langle V(c, h)^k_k | V(c, h)^0_k \rangle \). Therefore the problem of calculating the zeroes of \( \det \langle V(c, h)_n | V(c, h)_n \rangle \) reduces to finding determinant of matrix \( A_0 = \langle V(c, h)^0_n | V(c, h)^n_n \rangle \) which represents the action of monomials in \( W(i) \) and \( L(j) \) on monomials in \( V(-k) \) and \( M(-l) \).

Let us introduce a suitable ordering on \( B_n \) and \( B_0 \) which makes \( A_0 \) block-triangular. We exploit the following fact:

\[
(66) \quad k > i_v, j_m \Rightarrow X(k)V(-i_v) \cdots V(-j_m)M(-j_1)v_h = 0, \quad X \in \{L, W\}.
\]

Since we don’t need to distinguish \( L \) from \( W \) and \( M \) from \( V \) we consider the type of monomial based only on partition. For \( \bar{k} = (k_1, \ldots, k_n) \), \( \bar{l} = (l_1, \ldots, l_n) \in (\mathbb{Z}_{\geq 0})^n \) we define:

\[
(67) \quad \bar{k} + \bar{l} = (k_1 + l_1, \ldots, k_n + l_n)
\]

\[
(68) \quad \mathcal{P}_n = \{(\bar{k}, \bar{l}) \in (\mathbb{Z}_{\geq 0})^n \times (\mathbb{Z}_{\geq 0})^n : \sum_{i=1}^n i(k_i + l_i) = n\}
\]

and say that \((\bar{k}, \bar{l})\) is of type \( t(\bar{k}, \bar{l}) = \bar{k} + \bar{l} \).

There is a natural one to one correspondence between \( \mathcal{P}_n \) and \( B_n \)

\[
[V M](\bar{v}, \bar{m}) := V(-n)^{v_1} M(-n)^{m_1} \cdots V(-1)^{v_1} M(-1)^{m_1} v_h.
\]

\[
(V M)(\bar{v}, \bar{m}) = V(-n)^{v_1} M(-n)^{m_1} \cdots V(-1)^{v_1} M(-1)^{m_1} v_h.
\]

i.e. between \( \mathcal{P}_n \) and \( B_0 \)

\[
[W L](\bar{w}, \bar{l}) := W(-n)^{w_1} L(-n)^{l_1} \cdots W(-1)^{w_1} L(-1)^{l_1} v_h.
\]

Define order on \((\mathbb{Z}_{\geq 0})^n\) by

\[
(69) \quad \bar{k} \preceq \bar{l} \quad \text{if} \quad k_{n-i} = l_{n-i} \quad \text{for} \quad i = 0, \ldots, j - 1 \quad \text{and} \quad k_{n-j} > l_{n-j}
\]

and use it to define order on \( \mathcal{P}_n \) by type: for \((\bar{k}, \bar{l}), (\bar{k}', \bar{l}') \in \mathcal{P}_n \)

\[
(70) \quad (\bar{k}, \bar{l}) \prec (\bar{k}', \bar{l}') \quad \text{if} \quad t(\bar{k}, \bar{l}) \prec t(\bar{k}', \bar{l}')
\]

\[
(71) \quad \text{or} \quad t(\bar{k}, \bar{l}) = t(\bar{k}', \bar{l'}), \quad \text{and} \quad \bar{k} \prec \bar{k}'.
\]

This induces partial orders on \( B_n \) and \( B_0 \). It is clear from \( (66) \) that

\[
(72) \quad t(\bar{k}, \bar{l}) \prec t(\bar{k}', \bar{l'}) \Rightarrow \langle [W L](\bar{k}, \bar{l}) | [V M](\bar{k}', \bar{l'}) \rangle = 0
\]

so \( A_0 \) is block-triangular with diagonal blocks of the type

\[
(73) \quad \langle V(c, h)^0_n(\bar{k} + \bar{l}) | V(c, h)^n_n(\bar{k} + \bar{l}) \rangle
\]

where

\[
(74) \quad V(c, h)^n_n(\bar{k}, \bar{l}) = \text{span}_\mathbb{C}\{[V M](\bar{k}', \bar{l'}) \in B_n : t(\bar{k}', \bar{l'}) = t(\bar{k}, \bar{l})\},
\]

\[
(75) \quad V(c, h)^0_0(\bar{k}, \bar{l}) = \text{span}_\mathbb{C}\{[W L](\bar{k}', \bar{l'}) \in B_0 : t(\bar{k}', \bar{l'}) = t(\bar{k}, \bar{l})\}
\]
denote spans of basis elements of the same type. However, by construction we see that

\[(76) \quad \langle V(c, h)^0_n(k_1, \ldots, k_n) | V(c, h)^p_n(k_1, \ldots, k_n) \rangle\]

is a tensor product of matrices of type

\[(77) \quad \langle V(c, h)^0_n(0, \ldots, 0, k_p, 0, \ldots, 0) | V(c, h)^p_n(0, \ldots, 0, k_p, 0, \ldots, 0) \rangle, \quad p = 1, \ldots, n\]

which correspond to the action of monomials \(L(p)^r W(p)^{k_p - r}, r = 0, \ldots, k_p\) on monomials \(V(-p)^{k_p - r} M(-p)^r, r = 0, \ldots, k_i\). Denote

\[(78) \quad W(p)V(-p)v_h = av_h, \]
\[(79) \quad W(p)M(-p)v_h = L(p)V(-p)v_h = bv_h, \]
\[(80) \quad L(p)M(-p)v_h = dv_h. \]

We show by induction on \(n\) that for \(i, j \in \{1, \ldots, n + 1\}\) the element at intersection of \(i\)th row and \(j\)th column of this matrix equals

\[(81) \quad \alpha^{(n)}_{i,j} = (n + 1 - j)!(j - 1)!a^{n-i-j+2j^2+2(-1)} \sum_{k=0}^{i-1} \binom{n+1-i}{j-k-1} \binom{n+1-i}{i-k} v_h. \]

The basis for \(n = 1\) gives \((78\text{-}80)\). Denote the righthand side of \((81)\) by \(I_i\). Then

\[(82) \quad \alpha^{(n+1)}_{i,j} = (n+2-j)aI_j + (j-1)bI_{j-1} \]
\[(83) \quad = (n+2-j)!(j-1)! \times \]
\[\quad \sum_{k=0}^{i-1} a^{n-i-j+k+1} \binom{n+1-i}{j-k-1} \binom{n+1-i}{j-k-2} v_h \]

which proves the claim.

By direct calculation one can show that for each \(j\) we have

\[(84) \quad \sum_{\ell=0}^{i} \binom{i-1}{\ell} \alpha_{i-\ell,j}(-b^2)^\ell = (n + 1 - j)!(j - 1)!a^{n-i-j+2j^2+2(ad - b^2)^{i-1}} \]

By elementary transformations we obtain a triangular matrix with determinant equal to

\[(85) \quad (ad - b^2)^{\frac{u(n+1)}{2}} \frac{n!}{j!}(n-j)! \]

The brackets \((37\text{-}41)\) yield

\[(86) \quad a = \frac{15}{p} (5p^2 - 8)h_M + \frac{96}{c_M} h_M^2 + \frac{p(p^2 - 1)(p^2 - 4)}{360} c_M, \]
\[(87) \quad b = 3ph_V, \]
\[(88) \quad d = 2ph_M + \frac{p(p^2 - 1)}{12} c_M. \]
From these considerations follows:

**Theorem 3.3.** The Verma module $V(c, h)$ is reducible if and only if

$$h_V^2 = \frac{64 \left( h_M + \frac{h^2 - 1}{2M} c_M \right)^2 \left( h_M + \frac{h^2 - 4}{96} c_M \right)}{45 c_M}$$

(89)

for some $p \in \mathbb{Z}_{>0}$. In that case, there is a singular vector in $V(c, h)_p$, where $p \in \mathbb{Z}_{>0}$ is the lowest such that (89) holds.

**Remark 3.4.** This method of finding zeros of the determinant formula and thus classifying irreducible Verma modules relies on the fact that $\langle M(z), V(z) \rangle$ is a commutative subalgebra of $GW_3(c_L, c_M)$. Essentially the same method was used in classification of irreducibles over the GCA $(W(2, 2))$ in [24] and [17] where it was shown that the Verma module $V^{W(2, 2)}(h_L, h_M)$ is reducible if and only if

$$\langle L(-p)v | M(-p)v \rangle = p \left( 2h_M + \frac{p^2 - 1}{12} c_M \right) = 0$$

(90)

for some $p \in \mathbb{Z}_{>0}$. In that case there is a singular vector in $V^{W(2, 2)}(h_L, h_M)_p$. Free field realisation of highest weight modules and formula for singular vectors was obtained in [3], [4] and [19].

More importantly, this method can be applied to other Galilean W-algebras.

### 3.2. Simplicity of $GW_3(c_L, c_M)$

The submodule structure of reducible Verma module can be complicated. As an example we present formulas for (sub)singular vectors of weight $h_L + 1$. In particular, we describe the vacuum module, and prove simplicity of $GW_3(c_L, c_M)$.

**Example 3.5.** If $h_V^2 = \frac{2h_L^2(32h_M - c_M)}{45 c_M}$ (i.e. $p = 1$) the singular vector from Theorem 3.3 is given by

$$s.v_h = \left( V(-1) - \frac{3h_V}{2h_M} M(-1) \right) v_h.$$ 

(91)

We have $W(0)s.v_h = \left( h_W - \frac{3h_V}{h_M} \right) s.v_h$, $M(0)s.v_h = h_M s.v_h$, $V(0)s.v_h = h_V s.v_h$. Consider the determinant formula of the quotient module $V(c, h)/\langle s.v_h \rangle$. Factor corresponding to level one is

$$\begin{vmatrix}
\langle L(-1)v_h | L(-1)v_h \rangle & \langle L(-1)v_h | M(-1)v_h \rangle & \langle L(-1)v_h | W(-1)v_h \rangle \\
\langle M(-1)v_h | L(-1)v_h \rangle & \langle M(-1)v_h | M(-1)v_h \rangle & \langle M(-1)v_h | W(-1)v_h \rangle \\
\langle W(-1)v_h | L(-1)v_h \rangle & \langle W(-1)v_h | M(-1)v_h \rangle & \langle W(-1)v_h | W(-1)v_h \rangle
\end{vmatrix} =$$

$$= \frac{8}{5} h_M^2 \left( h_L - 16 \frac{h_M}{c_M} (3h_L + \frac{2}{5}) + 16 \frac{h_L^2}{c_M} (c_L + \frac{44}{5}) + 3h_W \sqrt{\frac{9}{2 c_M} (32h_M - c_M)} \right).$$

(92)
Remark 3.7. Proving simplicity of general VOAs is much more difficult. For example see [2] for treatment of affine VOA, and the W-algebras obtained by the generalised quantised Drinfeld-Sokolov reduction (which includes \( W_3 \)). Determinant formula for vacuum \( W_3 \)-module is considered in [13].
4. Free field realisation

We shall first recall the free field realisation of Galilean Virasoro algebra and its highest weight modules. This realisation was obtained using a rank 2 Heisenberg algebra and associated lattice VOA. Then we use the same idea to construct Galilean $W_3$ as a subalgebra of a rank 4 lattice VOA.

4.1. Realisation of Galilean conformal algebra. Free field realisation of GCA or $W(2, 2)$ was obtained in [3, 4, 5] by means of embedding it in twisted Heisenberg-Virasoro algebra at level 0. Here we recall this construction (with slightly adjusted parametrisation).

Let $L = \mathbb{Z}c + \mathbb{Z}d$ be a rank 2 lattice such that $\langle c|d \rangle = 2, \quad \langle c|c \rangle = \langle d|d \rangle = 0$.

Let $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L$ and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ its affinization. For any $h \in \mathfrak{h}$ we write $h(n)$ for $h \otimes t^n$ and we let $h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1}$.

We denote by $M(1, h)$ the induced $\hat{\mathfrak{h}}$-module

$$U(\hat{\mathfrak{h}}) \otimes U(\mathbb{C}[t] \otimes \mathbb{C}K) \mathbb{C}e^h$$

such that $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on $e^h$, $k(0)e^h = \langle k|h \rangle e^h$ for $k \in \mathfrak{h}$ and $Ke^h = e^h$. Then $M(1) := M(1, 0)$ is a rank 2 Heisenberg vertex algebra generated by the fields $h(z), \ h \in \mathfrak{h}$ and $M(1, h), \ h \in \mathfrak{h}$ are irreducible $M(1)$-modules. Furthermore, the fields

$$\omega(z) = \frac{1}{2}c(z)d(z) + \frac{c_L - 2}{24} \partial c(z) - \frac{1}{2} \partial d(z),$$

$$M(z) = -\frac{c_M}{24} (c(z)^2 - 2\partial c(z))$$

generate a vertex operator subalgebra of $M(1)$ which is isomorphic to GCA $L^{W(2, 2)}(c_L, c_M)$.

Define

$$v_{p, r} = e^{-\frac{p+1}{2}d + \left((p+1)\frac{c_L - 2}{24} - \frac{2p-r-1}{2}\right)c},$$

$$h_L[p, r] = (1 - p^2)\frac{c_L - 2}{24} + \frac{2p - r - 1}{2},$$

$$h_M[p] = \frac{1 - p^2}{24} c_M.$$ 

Then we have

$$L(0)v_{p, r} = h_L[p, r]v_{p, r}, \quad M(0)v_{p, r} = h_M[p]v_{p, r},$$

$h_L[-p, -r - 2] = h_L[p, r], \quad h_M[-p] = h_M[p],$$

$h_L[p, r] + p = h_L[p, r - 2]$.
Denote by $F_{p,r} = M(1).v_{p,r}$. We fix central charge $(c_L, c_M)$ and denote by $V[p,r]$ (resp. $L[p,r]$) the Verma (resp. irreducible) module of highest weight $(h_L[p,r], h_M[p])$. Then we have

**Theorem 4.1** ([19],[17],[3],[4]). Let $p > 0$.

1. The Verma module $V[p,r]$ is reducible if and only if $p \in \mathbb{Z}_{>0}$. In that case, there is a singular vector $u'_p \in V[p,r]$ such that $\langle u'_p \rangle \cong V[p,r-2]$.
2. $u'_p$ generates the maximal submodule in $V[p,r]$ if and only if $r \in \mathbb{Z}_{>0}$.
3. If $r \in \mathbb{Z}_{>0}$ then the maximal submodule in $V[p,r]$ is generated by a subsingular vector $u_{rp}$ of conformal weight $h_{p,r} + pr = h_{p,-r}$.
4. $F_{p,r} \cong V[p,r]$ and $L[p,r] \cong U(W(2,2)).v_{p,r} \subset F_{-p,-r-2} \cong F^*_{p,r}$.

In the following we aim to obtain analogous results for $GW_3$.

**Remark 4.2.** GCA is realised in [3], [4], [5] as a subalgebra of Heisenberg-Virasoro VOA at level zero. One may also consider the $N = 1$ super GCA. Realisation for central charge $c_L = 11$ was presented in [9]. The $N = 1$ super Heisenberg-Virasoro VOA was introduced in a recent paper [2], and the full treatment of level zero should appear soon as well. This will provide a natural framework for studying realisation of super GCA with arbitrary central charge.

### 4.2. Realisation of Galilean $W_3$ algebra.

Let $L = \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c + \mathbb{Z}d$ be a rank 4 lattice such that

\[
\langle a|b \rangle = \langle c|d \rangle = -1, \quad \langle a|c \rangle = \langle a|d \rangle = \langle b|c \rangle = \langle b|d \rangle = 0, \quad \langle x|x \rangle = 2, \quad x \in \{a, b, c, d\}.
\]

Fix $\lambda, \mu \in \mathbb{C}$ such that $\lambda + i\mu \neq 0$ and let

\[
\bar{a} = a + ic,
\]
\[
\bar{b} = b + id,
\]
\[
\bar{\lambda} = \lambda + i\mu.
\]
Now we define the fields in $M(1)$ which generate the Galilean $W_3$ algebra $\mathcal{G}W_3(c_L, c_M)$. Let

\begin{equation}
\omega(z) = \frac{1}{3} \left( a(z)^2 + a(z)b(z) + b(z)^2 + c(z)^2 + c(z)d(z) + d(z)^2 \right) + \\
+ \lambda \partial a(z) + \lambda \partial b(z) + \mu \partial c(z) + \mu \partial d(z)
\end{equation}

\begin{equation}
W(z) = \frac{i}{27 \lambda \sqrt{10}} \left( 2((a(z) - b(z))(\bar{a}(z) + 2\bar{b}(z))(2\bar{a}(z) + \bar{b}(z)) + \\
+ (\bar{a}(z) - \bar{b}(z))(a(z) + 2b(z))(2\bar{a}(z) + \bar{b}(z)) + \\
+ (\bar{a}(z) - \bar{b}(z))(\bar{a}(z) + 2\bar{b}(z))(2a(z) + b(z))) + \\
+ 9\lambda \left( \partial \bar{a}(z)(2\bar{a}(z) + \bar{b}(z)) - \partial \bar{b}(z)(\bar{a}(z) + 2\bar{b}(z)) + \\
+ \partial \bar{a}(z)(2a(z) + b(z)) - \partial \bar{b}(z)a(z) + 2\bar{b}(z)) \right) + \\
+ 18\lambda \bar{\lambda} \left( \partial^2 \bar{a}(z) - \partial^2 \bar{b}(z) \right) + 9\bar{\lambda}^2 \left( \partial^2 a(z) - \partial^2 b(z) \right) \right) + \\
\left( \frac{4}{15\bar{\lambda}} - \frac{\lambda}{\bar{\lambda}} - 1 \right) V(z)
\end{equation}

\begin{equation}
M(z) = \frac{1}{3} \left( \bar{a}(z)^2 + \bar{a}(z)\bar{b}(z) + \bar{b}(z)^2 \right) + \bar{\lambda} \left( \partial \bar{a}(z) + \partial \bar{b}(z) \right)
\end{equation}

\begin{equation}
V(z) = \frac{i}{27 \lambda \sqrt{10}} \left( 2((\bar{a}(z) - \bar{b}(z))(\bar{a}(z) + 2\bar{b}(z))(2\bar{a}(z) + \bar{b}(z)) + \\
+ 9\lambda \left( \partial \bar{a}(z)(2\bar{a}(z) + \bar{b}(z)) - \partial \bar{b}(z)(\bar{a}(z) + 2\bar{b}(z)) + \\
+ 9\lambda \bar{\lambda} \left( \partial^2 \bar{a}(z) - \partial^2 \bar{b}(z) \right) \right) \right).
\end{equation}

Direct calculation (with help of an OPE package for Mathematica) shows that fields defined above satisfy relations of Galilean $W_3$ algebra $\mathcal{G}W_3(c_L, c_M)$ with central charge

\begin{equation}
c_L = 4 - 24(\lambda^2 + \mu^2),
\end{equation}

\begin{equation}
c_M = -24\bar{\lambda}^2,
\end{equation}

which gives us realisation for all $c_L, c_M \in \mathbb{C}$, $c_M \neq 0$.

Notice that $L$ is a tensor product of two sublattices whose Gram matrices equal the Cartan matrix of $\mathfrak{sl}_3$. This kind of rank 2 lattice has been used in free field realisation of $W_3$ algebra (cf. [21]).
We also remark that the fields \( M'(z) \) and \( V'(z) \) which are obtained by substituting \( \lambda, a(z) \) and \( b(z) \) for \( \bar{\lambda}, \bar{a}(z) \) and \( \bar{b}(z) \) in \( M(z) \) and \( V(z) \) generate a copy of \( W_3(-8\lambda - 22/5) \) (cf. [21]).

5. Realisation of highest weight representations

Let \( \mathbb{C}[L] \) be a group algebra of \( L \) and \( V_L = M(1) \otimes \mathbb{C}[L] \) associated VOA. We introduce a parametrisation of highest weight vectors in \( V_L \). Let \( e[p, q, r, s] \) denote a highest weight vector \( e^k \in V_L \) where

\[
k = \left( 1 + \frac{p + q}{2} \right) \lambda + \frac{2 - r - s}{2\lambda} a + \left( 1 + q \lambda - \frac{s - 1}{\lambda} \right) b + \left( 1 + p \mu + i \frac{2 - r - s}{2\lambda} \right) c + \left( 1 + q \mu - i \frac{s - 1}{\lambda} \right) d.
\]

Weights \( h[p, q, r, s] \) of \( e[p, q, r, s] \) are given by

\[
(101) \quad h_L[p, q, r, s] = \frac{p(1 - r) + 3q(1 - s)}{2} + \frac{c_L - 4}{96} (4 - p^2 - 3q^2),
\]

\[
(102) \quad h_W[p, q, r, s] = \frac{i}{2\sqrt{10}} \left( 2pq(1 - r) + (1 - s)(p^2 - 3q^2) + q(p^2 - q^2) \right) \left( \frac{52 - 5c_L}{120} \right),
\]

\[
(103) \quad h_M[p, q, r, s] = (4 - p^2 - 3q^2) \frac{c_M}{96},
\]

\[
(104) \quad h_V[p, q, r, s] = \frac{ic_M}{48\sqrt{10}} q(q^2 - p^2).
\]

Direct calculation shows that

\[
(105) \quad h[p, q, r, s] = h[-p, q, -r + 2, s] = h[p, -q, -2, 2] = h[p + 3q, -p + q, -r + 3s, -r + s - 4],
\]

\[
= h[p + 3q, p - q, r + 3s - 2, r - s + 2] = h[p + 3q, p - q, -r + 3s - 6, r - s + 2]
\]

\[
= h[p - 3q, p + q, r - 3s + 4, -r + s - 4],
\]

i.e. parametrisation (101)(104) is \( S_3 \)-invariant under the action

\[
(106) \quad \sigma(p, q, r, s) = \left( \frac{-p + 3q}{2}, \frac{-p + q}{2}, \frac{-r + 3s}{2}, \frac{-r + s - 4}{2} \right)
\]

\[
(107) \quad \tau(p, q, r, s) = (-p, q, -r + 2, s),
\]

where \( \sigma \) and \( \tau \) are generators of \( S_3 \) of orders 3 and 2, respectively.
Proposition 5.1. Let $\mathcal{P} = \{(p, q, r, s) \in \mathbb{C}^4 : 0 < p < 3q\}$ where "<" denotes the lexicographical ordering on $\{(\text{Re}(z), \text{Im}(z)) : z \in \mathbb{C}\}$. Let $\mathbf{h} \in \mathbb{C}^4$ such that

\[ h^2_V \neq \frac{64(h_M - \frac{c_M}{24})^3}{45c_M}. \]

i) There exists a unique $(p, q, r, s) \in \mathcal{P}$ such that $\mathbf{h} = \mathbf{h}[p, q, r, s]$. 

ii) For every $r \in \mathbb{C}$ we have

\[ h_L[0, q, r, s] = \frac{3q}{2} (1 - s) + \frac{c_L - 4}{24} \left(1 - 3 \left(\frac{q}{2}\right)^2\right), \]

\[ h_W[0, q, r, s] = -i \sqrt{\frac{2}{5}} \left(3 \left(\frac{q}{2}\right)^2 (1 - s) + \left(\frac{q}{2}\right)^3 \frac{52 - 5c_L}{60}\right), \]

\[ h_M[0, q, r, s] = \frac{c_M}{24} \left(1 - 3 \left(\frac{q}{2}\right)^2\right), \]

\[ h_V[0, q, r, s] = i \sqrt{\frac{2}{5}} \left(\frac{q}{2}\right)^3 \frac{c_M}{12}. \]

iii) We have $\mathbf{h}[p, q, r, s]^* = \mathbf{h}[p, -q, r, 2 - s]$, and $\sigma^2(p, -q, r, 2 - s) \in \mathcal{P}$. 

iv) The Verma module $V[p, q, r, s]$ is reducible if and only if $p \in \mathbb{Z}_{>0}$. In that case there is a singular vector of conformal weight $h_L + p$ in $V[p, q, r, s]$.

Proof. i) From the Jacobian matrix of parametrisation (101,104) follows that $p = 0$, and $p = \pm 3q$ are critical values. Consider the action of $S_3 = \langle \sigma, \tau \rangle$ on $\mathbb{C}_4$ defined by (106,107). Then $\mathcal{P}$ is a space of coinvariants $(\mathbb{C}_4)_{S_3}$ excluding critical values. 

ii) and iii) direct calculation. 

iv) is the reducibility condition (89) stated in terms of parametrisation. \hfill $\square$

Remark 5.2. Note that the weights $\mathbf{h}$ not equal to (108,111) such that $h^2_V = \frac{64(h_M - \frac{c_M}{24})^3}{45c_M}$ are not obtained by this parametrisation. This is analogous to realisation of GCA presented in Subsection 4.2 where each $(0, r)$ produces weight $(\frac{c_L - 2}{24}, \frac{c_M}{24})$. Highest weight modules of highest weights $(h_L, \frac{c_M}{24})$ for $h_L \neq \frac{c_L - 2}{24}$ were realised by means of deformed action on certain Whittaker modules (cf. [5]) and by using the fact that these modules coincide with the highest weight modules over the Heisenberg-Virasoro algebra. We do not study realisation of remaining highest weights in this paper. However, it would be interesting to obtain these modules by some other means.

Example 5.3. Recall Example 3.5 of subsingular vectors at level 1. Then $s.e[1, q, r, s]$ is a singular vector in $\mathcal{F}_{1, q, r, s}$ while $s.e[-1, q, -r + 2, s] = 0$. Furthermore

a: $h_M[1, q, r, s] = 0$ if and only if $q \in \{\pm 1\}$.

$s_1.e[1, 1, r, s]$ subsingular in $\mathcal{F}_{1, 1, r, s}$, while $s_1.e[-1, -1, r, s] = 0$. 


Remark 5.4. As we have seen (Proposition 5.1 iv), integral values of $p$ detect positions of singular vectors in reducible Verma modules. Based on Example 5.3 and on representation theory of GCA (Theorem 4.1) we expect that integral values of each of the remaining three parameters detect positions of subsingular vectors. Different sectors (of $S_3$ action on $\mathbb{C}^4$) should produce variant subquotients of $V[p,q,r,s]$, including the Verma module itself, and the irreducible quotient $L[p,q,r,s]$.

Appendix A. $\lambda$-bracket calculation

A.1. Jacobi identity. Recall the Jacobi identity for $\lambda$-brackets (3). The most difficult calculation occurs in case $a = b = c = W$. We have (cf. [14] Lemma 3.2)

$$[W_\lambda LM] = 2(DW)M + 2L(DV) + 3\lambda(WM + LV) + \left(4\lambda^2 D + \frac{5}{2}\lambda^3\right)V$$

$$[LM_\lambda W] = (D + 3\lambda)(LV + WM) + 2((DL)V + W(DM)) + \frac{1}{2}(-3D^3 - D^2\lambda + 7D\lambda^2 + 5\lambda^3)V$$

$$[W_\lambda M^2] = 4M(DV) + 6\lambda MV$$

$$[M^2_\lambda W] = 2(D + 3\lambda)(MV) + 4(DM)V$$

so $[W_\lambda [W_\mu W]]$ equals:

$$
\left(\frac{\mu^3}{3} + \frac{\mu^2}{2}(\lambda + D) + \frac{3\mu}{10}(\lambda + D)^2 + \frac{1}{15}(\lambda + D)^3\right)(2D + 3\lambda)W + \\
+ \frac{32}{5c_5}(2\mu + \lambda + D)\left((2D + 3\lambda)(WM + LV) - 2W(DM) - 2(DL)V + \\
+ (4\lambda^2 D + \frac{5}{2}\lambda^3)V - \frac{3}{10}(\lambda + D)^2(2D + 3\lambda)V\right) + \\
- \frac{16}{5c_5^2}\left(c_5 + \frac{44\lambda}{5}\right)(2\mu + \lambda + D)(4M(DV) + 6\lambda MV);
$$
and \([[[W_\lambda W_\mu] \lambda + \mu W]]\) equals:

\[
\left(\frac{\lambda^3}{3} - \frac{\lambda^2}{2}(\lambda + \mu) + \frac{3\lambda}{10}(\lambda + \mu)^2 - \frac{1}{15}(\lambda + \mu)^3\right)(D + 3\lambda + 3\mu)W + \\
+ \frac{32}{5cM}(\lambda - \mu)\left((D + 3\lambda + 3\mu)(WM + LV) + 2(DL)V + 2W(VM) + \frac{1}{2}(3D^3 - D^2(\lambda + \mu)^7 + D(\lambda + \mu)^2 + 5(\lambda + \mu)^3)V - \frac{3}{10}(\lambda + \mu)^2(D + 3\lambda + 3\mu)V\right) + \\
- \frac{16}{5cM^2}(cL + \frac{44}{5})(\lambda - \mu)(6(\lambda + \mu + D)MV - 4M(DV)).
\]

Comparing all the coefficients one sees that (3) holds. Similarly, in case \(a = b = W, c = V\)

we have

\[ [LM_\lambda V] = 3(D + \lambda)(MV) - 2M(DV) \]

so

\[
[W_\lambda[W_\mu V]] = \left(\frac{\mu^3}{3} + \frac{\mu^2}{2}(\lambda + D) + \frac{3\mu}{10}(\lambda + D)^2 + \frac{1}{15}(\lambda + D)^3\right)(2D + 3\lambda)V + \\
+ \frac{16}{5cM}(2\mu + \lambda + D)(6\lambda MV + 4MDV)
\]

\[ [[W_\lambda W_\mu] \lambda + \mu V] = \left(\frac{\lambda^3}{3} - \frac{\lambda^2}{2}(\lambda + \mu) + \frac{3\lambda}{10}(\lambda + \mu)^2 - \frac{1}{15}(\lambda + \mu)^3\right)(D + 3\lambda + 3\mu)V + \\
+ \frac{32}{5cM}(\lambda - \mu)(3(D + \lambda + \mu)(MV) - 2M(DV)).
\]

Other cases are easier to check.

A.2. Other definitions. Suppose we want to construct a Galilean \(W_3\) algebra in such a way that \(W_3\) is its subalgebra, i.e.

\[ [W_\lambda W] = \frac{c}{360}\lambda^5 + \left(\frac{\lambda^3}{3} + \frac{\lambda^2}{2}(D + 3\lambda + 3\mu) + \frac{1}{15}(\lambda + D)^3\right)L + \frac{16}{5c + 22}(D + 2\lambda)(L^2 - \frac{3}{10}D^2L).
\]

Let us check the Jacobi identity for a triple \(W, W, M\). We see that \([W_\lambda[W_\mu M]]\) and \([W_\mu[W_\lambda M]]\) produce nonlinear terms in \(M^2\), while \([[[W_\lambda W_\mu] \lambda + \mu M]]\) produces \(LM, (DL)M\) as well, so the identity (3) can not hold.

Therefore, either \([W_\lambda V]\) must contain a nonlinear term with factor \(L\) (which means \(W, L\) don’t act on a commutative subalgebra generated by \(M\) and \(V\), or the nonlinear terms in \([W_\lambda W]\) must contain \(M\) as a factor.
APPENDIX B. THE \(c_M = 0\) CASE

Definition 2.4 of \(GW_3(c_L, c_M)\) does not allow for central charge \(c_M = 0\). If we rescale \(W'(z) = c_M W(z)\) and then let \(c_M = 0\) we obtain the following non-trivial \(\lambda\)-brackets:

\[
\begin{align*}
[L_\lambda L] &= (D + 2\lambda)L + \frac{c_L}{12} \lambda^3, \\
[L_\lambda M] &= (D + 2\lambda)M, \\
[L_\lambda W'] &= (D + 3\lambda)W', \\
[L_\lambda V] &= (D + 3\lambda)V, \\
[W'_\lambda W'] &= -\frac{16}{5} \left( c_L + \frac{44}{5} \right) (D + 2\lambda)M^2, \\
[W'_\lambda V] &= \frac{16}{5} (D + 2\lambda)M^2.
\end{align*}
\]

The resulting VOA is an extension of Virasoro VOA by the ideal generated by primary fields \(M(z), W'(z), V(z)\).

We may follow the arguments of Subsection 3.1 in obtaining the determinant formula. However in place of (79) we have \(W'(p)M(-p)v_h = 0\). Therefore

\[
\alpha^{(n)}_{i,j} = L(p)^{i-1}W'(p)^{n+1-i}V(-p)^{n+1-j}M(-p)^{j-1}v_h = 0
\]

if \(i < j\) which makes the matrix triangular with diagonal elements equal to

\[
(n + 1 - i)!/(i - 1)!a^{n+1-i}d^{-1}.
\]

Since \(a = \frac{32}{5} ph_M\) and \(d = 2ph_M\) we conclude that the Verma module \(V(c_L, c_M = 0, h)\) is reducible if and only if \(h_M = 0\). In this case, \((M(-1)v_h)\) is a submodule, and reducibility of the associated quotient module corresponds to vanishing of \(h_L, h_{W'}\) or \(h_V\).

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