One-dimensional two-component Bose gas
and the algebraic Bethe ansatz

N. A. Slavnov

Steklov Mathematical Institute, Moscow, Russia

Abstract

We apply the nested algebraic Bethe ansatz to a model of one-dimensional two-component Bose gas with \( \delta \)-function repulsive interaction. Using a lattice approximation of the \( L \)-operator we find Bethe vectors of the model in the continuous limit. We also obtain a series representation for the monodromy matrix of the model in terms of Bose fields. This representation allows us to study an asymptotic expansion of the monodromy matrix over the spectral parameter.

1 Introduction

In this paper we consider a model of one-dimensional two-component Bose gas with \( \delta \)-function repulsive interaction (TCBG model). This model is a generalization of the Lieb–Liniger model (Quantum nonlinear Schrödinger equation), in which Bose fields have two internal degrees of freedom (colors). This model was solved by C. N. Yang where the eigenvectors and the spectrum of the Hamiltonian were found. The general approach to the solution of the model with \( n \) internal degrees of freedom (multi-component Bose gas) was given in \cite{4} (see also \cite{5, 6}). The nested algebraic Bethe ansatz was applied to this model in \cite{7, 8}. The main goal of this paper is to create a base for calculating from factors of local operators in this model in the framework of the nested algebraic Bethe ansatz.

The algebraic Bethe ansatz is an efficient method for finding the spectra of quantum Hamiltonians. However, in a viewpoint of calculating form factors of local operators application of this method meets some difficulties. The main problem is to embed the local operators of the model under consideration into the algebra of the monodromy matrix entries. In some cases, this problem can be solved. However, to construct such a solution requires that the monodromy matrix of the model \( T(u) \) would be expressed in terms of the \( R \)-matrix. This is not the case of the TCBG model. On the other hand, in the framework of the traditional approach one

\footnote{nslavnov@mi.ras.ru}
can easily obtain representations for form factors of local operators and correlation functions in terms of multiple integrals of the product of the wave functions. However, the evaluation of those multiple integrals is facing serious technical difficulties, and still they have been computed only for some relatively simple special cases [11].

Recently a method of calculating form factors of local operators in models possessing $GL(3)$ symmetry was developed in [12]. This method is based on the nested algebraic Bethe ansatz and deals with partial zero modes of the monodromy matrix entries $T_{ij}(u)$ [13] in a composite model [14]. Most of the tools of this approach can be directly used in TCBG model, however some of them should be slightly modified. In particular, one should adjust a definition of the zero modes. We solve these problems in the present paper.

We consider a lattice approximation of the TCBG model. Using the $L$-operator obtained in [7, 8] we construct a monodromy matrix and Bethe vectors. We show that these vectors have a correct continuous limit. We also obtain an explicit series representation for the monodromy matrix in terms of local Bose fields. Using this representation we are able to derive an asymptotic expansion of the monodromy matrix over the spectral parameter. In this way we find the zero modes.

The paper is organized as follows. In section 2 we describe a general scheme of the algebraic Bethe ansatz. We define Bethe vectors of $GL(3)$-invariant models and give their representation in a multi-composite model. Section 3 is devoted to a brief description the TCBG model. In section 4 we give a lattice approximation of the TCBG model in the framework of the nested algebraic Bethe ansatz. In section 5 we consider continuous limit of the Bethe vectors of the lattice model. In section 6 we obtain a series representation of the TCBG monodromy matrix. Using this representation we find an antimorphism between Bose fields in section 7 and zero modes of the monodromy matrix entries in section 8. In conclusion we discuss some further applications of the results obtained.

2 Algebraic Bethe ansatz

In this section we describe an abstract scheme of the algebraic Bethe ansatz, which is valid for a wide class of quantum integrable models [15–17]. The key objects of the algebraic Bethe ansatz are a monodromy matrix and $R$-matrix. The models considered below are described by the $GL(3)$-invariant $R$-matrix [18, 19] acting in the tensor product $V_1 \otimes V_2$ of two auxiliary spaces $V_k \sim \mathbb{C}^3$, $k = 1, 2$:

$$R(x, y) = I + g(x, y)P, \quad g(x, y) = \frac{c}{x - y}. \quad (2.1)$$

In the above definition, $I$ is the identity matrix in $V_1 \otimes V_2$, $P$ is the permutation matrix that exchanges $V_1$ and $V_2$, and $c$ is a constant.

The monodromy matrix $T(w)$ satisfies the algebra

$$R_{12}(w_1, w_2)T_1(w_1)T_2(w_2) = T_2(w_2)T_1(w_1)R_{12}(w_1, w_2). \quad (2.2)$$

Equation (2.2) holds in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the Hamiltonian of the model under consideration. The matrices $T_k(u)$ act non-trivially in $V_k \otimes \mathcal{H}$. We assume that the space $\mathcal{H}$ possesses a pseudovacuum vector $|0\rangle$. Similarly the dual space
\( \mathcal{H}^* \) possesses a dual pseudovacuum vector \( \langle 0 | \). These vectors are annihilated by the operators \( T_{ij}(w) \), where \( i > j \) for \( | 0 \rangle \) and \( i < j \) for \( \langle 0 | \). At the same time both vectors are eigenvectors of the diagonal entries of the monodromy matrix

\[
T_{ii}(w)|0\rangle = \lambda_i(w)|0\rangle, \quad \langle 0| T_{ii}(w) = \lambda_i(w)\langle 0|, \quad i = 1, 2, 3,
\]

where \( \lambda_i(w) \) are some scalar functions. In the framework of the general scheme of the algebraic Bethe ansatz \( \lambda_i(w) \) remain free functional parameters. Actually, it is always possible to normalize the monodromy matrix \( T(w) \rightarrow \lambda_2^{-1}(w)T(w) \) so as to deal only with the ratios

\[
r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \quad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}.
\]

Below we assume that \( \lambda_2(w) = 1 \).

The trace in the auxiliary space \( V \sim \mathbb{C}^3 \) of the monodromy matrix \( \text{tr} T(w) \) is called the transfer matrix. It is a generating functional of the Hamiltonian and all integrals of motion of the model.

### 2.1 Notation

We use the same notation and conventions as in the papers [20, 21]. Besides the function \( g(x, y) \) we also introduce a function \( f(x, y) \)

\[
f(x, y) = 1 + g(x, y) = \frac{x - y + c}{x - y}.
\]

We denote sets of variables by bar: \( \bar{w}, \bar{u}, \bar{v} \) etc. Individual elements of the sets are denoted by subscripts: \( w_j, u_k \) etc. Notation \( \bar{u}_i \) means \( \bar{u} \setminus u_i \) etc. We also consider partitions of sets into disjoint subsets and denote them by symbol \( \Rightarrow \). Subsets are denoted by superscripts in parenthesis: \( \bar{u}^{(j)} \). For example, the notation \( \bar{u} \Rightarrow \{ \bar{u}^{(1)}, \bar{u}^{(2)} \} \) means that the set \( \bar{u} \) is divided into two disjoint subsets \( \bar{u}^{(1)} \) and \( \bar{u}^{(2)} \), such that \( \bar{u}^{(1)} \cap \bar{u}^{(2)} = \emptyset \) and \( \{ \bar{u}^{(1)}, \bar{u}^{(2)} \} = \bar{u} \).

In order to avoid too cumbersome formulas we use a shorthand notation for products of operators or functions depending on one or two variables. Namely, if the operators \( T_{ij} \) or the functions \( r_k \) depend on sets of variables, this means that one should take the product over the corresponding set. For example,

\[
T_{ij}(\bar{u}) = \prod_{u_k \in \bar{u}} T_{ij}(u_k); \quad r_3(\bar{u}^{(1)}) = \prod_{u_j \in \bar{u}^{(1)}} r_3(u_j).
\]

Similar convention is applied to the products of the functions \( f(x, y) \):

\[
f(z, \bar{u}_i) = \prod_{w_j \in \bar{u}} f(z, w_j); \quad f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k).
\]
2.2 Bethe vectors

The eigenvectors of the transfer matrix are called on-shell Bethe vectors (or simply on-shell vectors). In order to find them one should first construct generic Bethe vectors. In the framework of the algebraic Bethe ansatz generic Bethe vectors are polynomials in operators $T_{ij}$ with $i < j$ applied to the vacuum vector. We denote them by $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$, stressing that they are parameterized by two sets of complex parameters $\bar{u} = \{u_1, \ldots, u_a\}$ and $\bar{v} = \{v_1, \ldots, v_b\}$ with $a, b = 0, 1, \ldots$. Different representations for Bethe vectors were found in [22–25]. We give here one of the representations obtained in [25].

\begin{equation}
\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum \frac{K_n(\bar{u}^{(1)}|\bar{u}^{(1)})}{f(\bar{v}, \bar{u})} f(\bar{v}^{(2)}, \bar{v}^{(1)}) f(\bar{u}^{(1)}, \bar{u}^{(2)}) T_{13}(\bar{v}^{(1)})T_{23}(\bar{v}^{(2)})T_{12}(\bar{u}^{(2)})|0\rangle. \tag{2.8}
\end{equation}

Here the sums are taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}^{(1)}, \bar{u}^{(2)}\}$ and $\bar{v} \Rightarrow \{\bar{v}^{(1)}, \bar{v}^{(2)}\}$ with $0 \leq \#\bar{u}^{(1)} = \#\bar{v}^{(1)} = n \leq \min(a, b)$. We recall that the notation $T_{13}(\bar{v}^{(1)})$ (and similar ones) means the product of the operators $T_{13}(u)$ with respect to the subset $\bar{u}^{(1)}$. Finally, $K_n(\bar{u}^{(1)}|\bar{u}^{(1)})$ is the partition function of the six-vertex model with domain wall boundary conditions [26]. Its explicit representation was found in [27].

\begin{equation}
K_n(\bar{x}|\bar{y}) = \left( \prod_{1 \leq k < j \leq n} g(x_j, x_k)g(y_k, y_j) \right) \frac{f(\bar{x}, \bar{y})}{g(\bar{x}, \bar{y})} \det_n \left( \frac{g^2(x_j, y_k)}{f(x_j, y_k)} \right). \tag{2.9}
\end{equation}

In particular, $K_1(x|y) = g(x, y)$.

A generic Bethe vector becomes on-shell, if the parameters $\bar{u}$ and $\bar{v}$ satisfy a system of Bethe equations:

\begin{equation}
\begin{aligned}
r_1(u_i) &= \frac{f(u_i, \bar{u}_i)}{f(\bar{u}_i, u_i)} f(\bar{v}, u_i), \quad i = 1, \ldots, a, \\
r_3(v_j) &= \frac{f(\bar{v}_j, v_j)}{f(v_j, \bar{v}_j)} f(v_j, \bar{u}), \quad j = 1, \ldots, b. \tag{2.10}
\end{aligned}
\end{equation}

Recall that $\bar{u}_i = \bar{u} \setminus u_i$ and $\bar{v}_j = \bar{v} \setminus v_j$.

2.3 Multi-composite model

Study of the properties of local operators in the framework of the algebraic Bethe ansatz can be done by the use of a composite model [14]. Suppose that we have a lattice quantum model of $N$ sites. Then the monodromy matrix $T(u)$ is a product of local $L$-operators

\begin{equation}
T(u) = L_N(u) \ldots L_1(u). \tag{2.11}
\end{equation}

Let us fix an arbitrary site $m$, $1 \leq m \leq N$. Then (2.11) can be written as

\begin{equation}
T(u) = T^{(2)}(u)T^{(1)}(u), \tag{2.12}
\end{equation}

where

\begin{equation}
T^{(1)}(u) = L_m(u) \ldots L_1(u), \quad T^{(2)}(u) = L_N(u) \ldots L_{m+1}(u). \tag{2.13}
\end{equation}
Similarly to (2.4) we introduce ratios

$GL$ was solved in [14] for $T$ by $B$.

Due to the normalization $B$ convention (2.6) to the products of functions (2.15).

In the framework of the algebraic Bethe ansatz it is assumed that $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ possess pseudovacuum vectors $|0\rangle^{(k)}$, $k = 1, 2$, such that $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$. These vectors have the properties analogous to (2.3)

$$T_{ij}^{(k)}(u)|0\rangle^{(k)} = 0, \quad i > j, \quad T_{ii}^{(k)}(u)|0\rangle^{(k)} = \lambda_i^{(k)}(u)|0\rangle^{(k)}, \quad k = 1, 2. \quad (2.14)$$

Similarly to (2.1) we introduce ratios

$$r_1^{(k)}(w) = \frac{\lambda_1^{(k)}(w)}{\lambda_2^{(k)}(w)}, \quad r_2^{(k)}(w) = \frac{\lambda_2^{(k)}(w)}{\lambda_2^{(k)}(w)}, \quad k = 1, 2. \quad (2.15)$$

Due to the normalization $\lambda_2(u) = 1$ we can always set $\lambda_2^{(k)}(u) = 1$. Below we also extend convention (2.6) to the products of functions (2.15).

One can construct for every partial monodromy matrix $T^{(k)}(u)$ the corresponding partial Bethe vectors $B_{a,b}^{(k)}(\tilde{u}; \tilde{v})$ and $T^{(k)}(u)$ by $T_{ij}^{(k)}(u)$ and $|0\rangle$ by $|0\rangle^{(k)}$. The main problem considered in the framework of the composite model is to express total Bethe vectors $B_{a,b}(\tilde{u}; \tilde{v})$ in terms of partial $B_{a,b}^{(k)}(\tilde{u}; \tilde{v})$. This problem was solved in [14] for $GL(2)$-based models. More general case of $GL(N)$-invariant models was considered in [22, 28]. Particular case of $GL(3)$-invariant models was studied in [29], where the following representation was found:

$$B_{a,b}(\tilde{u}; \tilde{v}) = \sum r_1^{(2)}(\tilde{u}^{(1)}) r_3^{(1)}(\tilde{v}^{(2)}) f(\tilde{u}^{(2)}, \tilde{u}^{(1)}) f(\tilde{v}^{(2)}, \tilde{v}^{(1)}) \prod_{a_1,b_1} B_{a_1,b_1}^{(1)}(\tilde{u}^{(1)}; \tilde{v}^{(1)}) \prod_{a_2,b_2} B_{a_2,b_2}^{(2)}(\tilde{u}^{(2)}; \tilde{v}^{(2)}). \quad (2.16)$$

Here the sum is taken over all possible partitions $\tilde{u} \Rightarrow \{\tilde{u}^{(1)}, \tilde{u}^{(2)}\}$ and $\tilde{v} \Rightarrow \{\tilde{v}^{(1)}, \tilde{v}^{(2)}\}$. The cardinalities of the subsets are shown by the subscripts of the partial Bethe vectors.

Similarly we can define a multi-composite model, where the original interval is divided into $M > 2$ intervals

$$T(u) = T^{(M)}(u) \ldots T^{(1)}(u). \quad (2.17)$$

For each of these intervals we can define partial Bethe vectors $B_{a,b}^{(j)}$. Then the total Bethe vector can be expressed in terms of the partial ones as follows

$$B_{a,b}(\tilde{u}; \tilde{v}) = \sum_{1 \leq k < j \leq M} \left\{ r_1^{(j)}(\tilde{u}^{(k)}) r_3^{(k)}(\tilde{v}^{(j)}) f(\tilde{u}^{(j)}, \tilde{u}^{(k)}) f(\tilde{v}^{(j)}, \tilde{v}^{(k)}) \right\} \prod_{j=1}^{M} B_{a_j,b_j}^{(j)}(\tilde{u}^{(j)}; \tilde{v}^{(j)}). \quad (2.18)$$
Here the functions $r^{(j)}_1(u)$ and $r^{(j)}_3(v)$ are vacuum eigenvalues of the operators $T^{(j)}_{11}(u)$ and $T^{(j)}_{33}(v)$ respectively. The sum in (2.18) is taken over all possible partitions

\[ \bar{u} \Rightarrow \{ \bar{u}^{(1)}, \ldots, \bar{u}^{(M)} \}, \quad \# \bar{u}^{(j)} = a_j, \quad a_1 + \cdots + a_M = a, \]
\[ \bar{v} \Rightarrow \{ \bar{v}^{(1)}, \ldots, \bar{v}^{(M)} \}, \quad \# \bar{v}^{(j)} = b_j, \quad b_1 + \cdots + b_M = b. \]  

(2.19)

It is important that the number $M$ of the partial monodromy matrices is not related to the cardinalities of the Bethe parameters $a$ and $b$. In particular, we can have $M > a$ and $M > b$. In this case some of numbers $a_j$ and $b_j$ are equal to zero, that is, the corresponding subsets are empty.

Equation (2.18) can be easily proved by induction over $M$. Indeed, assuming that it is valid for $M - 1$ partial monodromy matrices we apply (2.16) to the partial Bethe vector $B^{(M-1)}_{\bar{u}, \bar{v}}(u - \xi_{M-1}, v - \xi_{M-1})$. This immediately gives (2.18) for $M$ partial monodromy matrices.

In particular cases $a = 0$ or $b = 0$ we reproduce known formulas for Bethe vectors in $GL(2)$-invariant multi-composite model [30, 31]. For instance,

\[ B_{a,0}(\bar{u}, \emptyset) \equiv B_a(\bar{u}) = \sum \prod_{1 \leq k < j \leq M} \left\{ r^{(j)}_1(\bar{u}^{(k)}) f(\bar{u}^{(j)}, \bar{u}^{(k)}) \right\} \prod_{j=1}^M B^{(j)}_a(\bar{u}^{(j)}). \]

(2.20)

The multi-composite model is a convenient way to express the Bethe vectors in terms of local operators. In the next section, we discuss the method in more detail.

### 2.4 Bethe vectors in the $SU(2)$ XXX chain

As a first application of the multi-composite model we construct Bethe vectors of the $SU(2)$ inhomogeneous XXX chain. This result will be used in section 3 for description of Bethe vectors of TCBG model.

Consider an inhomogeneous XXX chain consisting of $M$ sites. This model has a $2 \times 2$ monodromy matrix $T^{(xxx)}(u)$, therefore Bethe vectors are parameterized by only one set of the Bethe parameters, say $\bar{u}$. Respectively, considering the multi-composite model one should use (2.20).

The monodromy matrix is defined as a product of local $L$-operators

\[ T^{(xxx)}(u) = L^{(xxx)}_M(u - \xi_M) \cdots L^{(xxx)}_1(u - \xi_1), \]

where $\xi_k$ are inhomogeneities and

\[ L^{(xxx)}_n(u) = \frac{1}{u + \frac{c}{2}(1 + \sigma_n^+)} - \frac{c}{2}(u + \frac{c}{2}(1 - \sigma_n^-)). \]

(2.21)

Here $\sigma_n^+$ and $\sigma_n^-$ are spin-1/2 operators acting in the $n$-th site of the chain. They are given by the standard Pauli matrices acting in the $n$-th copy of the tensor product $(\mathbb{C}^2)^\otimes M$. The pseudovacuum vector is the state with all spins up

\[ |\bar{0}\rangle = (\frac{1}{0})_M \otimes \cdots \otimes (\frac{1}{0})_1. \]  

(2.22)
Bethe vectors with $a$ spins down and $M - a$ spins up have the form
\[
\mathbb{B}^{(xxx)}_{a}(\bar{u}) = \sum_{M \geq j_a \geq \cdots \geq j_1 \geq 1} \Omega^{(a,M)}_{j_1,\ldots,j_a}(\bar{u}; \bar{\xi}) \prod_{m=1}^{a} \sigma_{j_m} \langle \bar{0} \rangle.
\tag{2.24}
\]
where $\Omega^{(a,M)}_{j_1,\ldots,j_a}(\bar{u}; \bar{\xi})$ are coefficients depending on the Bethe parameters $\bar{u}$ and inhomogeneities $\bar{\xi}$. Let us find these coefficients explicitly.

Consider a multi-composite model with $M$ partial monodromy matrices $T^{(j)}$. It means that every $T^{(j)}$ coincides with the $L$-operator $L_j(u - \xi_j)$. Then every partial Bethe vectors $\mathbb{B}^{(j)}_{a}(\bar{u}(j))$ in (2.20) corresponds to the $j$-th site of the chain, therefore due to (2.22) we obtain
\[
\mathbb{B}^{(j)}_{a}(\bar{u}(j)) = g(\bar{u}(j), \xi_j) \left( \sigma_j^{-} \right)^{a_j} (\bar{1})_j.
\tag{2.25}
\]
Obviously $\mathbb{B}^{(j)}_{a}$ vanishes if $a_j > 1$, because $(\sigma_j^{-})^2 = 0$. Thus, we conclude that $a_j \leq 1$ and the subsets $\bar{u}(j)$ are either empty or they consist of one element. Let subsets $\bar{u}(j_k)$ ($k = 1, \ldots, a$) corresponding to the lattice sites $j_1, \ldots, j_a$ contain one element $u_k$, while other subsets are empty. Then the sum over partitions of the set $\bar{u}$ turns into the sum over permutations in $\bar{u}$ and the sum over the lattice sites $j_1, \ldots, j_a$ with the restriction $j_a > \cdots > j_1$.

It is easy to see that
\[
\frac{u - \xi_j + \frac{c}{2}(1 + \sigma_j^{-})}{u - \xi_j} (\bar{1})_j = f(u, \xi_j) (\bar{1})_j, \quad \frac{u - \xi_j + \frac{c}{2}(1 - \sigma_j^{-})}{u - \xi_j} (\bar{1})_j = (\bar{1})_j,
\tag{2.26}
\]
and thus,
\[
r^{(j)}_1(u) = f(u, \xi_j).
\tag{2.27}
\]
Then equation (2.20) takes the form
\[
\mathbb{B}^{(xxx)}_{a}(\bar{u}) = \text{Sym} \prod_{1 \leq k < j \leq a} f(u_k, u_j) \sum_{M \geq j_a \geq \cdots \geq j_1 \geq 1} \prod_{k=1}^{a} \left[ \prod_{m=j_k+1}^{M} f(u_k, \xi_m) g(u_k, \xi_{j_k}) \sigma_{j_k}^{-} \right] |\bar{0}\rangle,
\tag{2.28}
\]
where the symbol Sym means symmetrization (i.e. the sum over permutations) over the set indicated by the subscript. The symmetrization in (2.28) acts on all the expression depending on $\bar{u}$. Comparing (2.28) with (2.24) we see that
\[
\xi^{(a,M)}_{j_1,\ldots,j_a}(\bar{u}; \bar{\xi}) = \text{Sym} \prod_{1 \leq k < j \leq a} f(u_k, u_j) \prod_{k=1}^{a} \left[ \prod_{m=j_k+1}^{M} f(u_k, \xi_m) g(u_k, \xi_{j_k}) \right].
\tag{2.29}
\]
In the homogeneous limit $\xi_k = c/2$ this expression coincides with the amplitude of the Bethe vector in the coordinate Bethe ansatz representation (see [6]).
3 Two-component Bose gas

We consider the TCBG model on a finite interval \([0, L]\) with periodic boundary conditions. In the second quantized form the Hamiltonian has the form

\[
H = \int_0^L \left( \partial_x \Psi_\alpha^\dagger \partial_x \Psi_\alpha + \kappa \Psi_\alpha^\dagger \Psi_\beta \Psi_\alpha \right) dx,
\]
(3.1)

where \(\kappa > 0\) is a coupling constant, \(\alpha, \beta = 1, 2\) and the summation over repeated subscripts is assumed. Bose fields \(\Psi_\alpha(x)\) and \(\Psi_\alpha^\dagger(x)\) satisfy canonical commutation relations

\[
[\Psi_\alpha(x), \Psi_\beta^\dagger(y)] = \delta_{\alpha\beta} \delta(x - y).
\]
(3.2)

The coupling constant \(\kappa\) is related to the constant \(c\) in (2.1) by \(\kappa = ic\).

Observe that in the case of the TCBG model the pseudovacuum vector (2.3) coincides with the Fock vacuum \(|0\rangle\), therefore we use the same notation for them.

The spectral problem for the TCBG model was solved in [3] (see also [4, 6]). The Hamiltonian eigenvectors can be found in two steps. Using the terminology of the algebraic Bethe ansatz one can say that at the first stage one should construct a generic Bethe vector \(B_{a,b}(\bar{u}; \bar{v})\). In the TCBG model Bethe vectors exist for \(a \leq b\). They have the following form:

\[
B_{a,b}(\bar{u}; \bar{v}) = \sum_{b \geq k_1, \ldots, k_a \geq 1} \int_D dz_1 \ldots dz_b \chi_{k_1, \ldots, k_a}(z_1, \ldots, z_b|\bar{u}, \bar{v})
\]

\[
\times \prod_{m=1}^a \Psi_1^\dagger(z_{k_m}) \prod_{l=1}^b \Psi_2^\dagger(z_l)|0\rangle.
\]
(3.4)

Here and below we do not take care about eigenvectors normalization in all formulas for them.

Here the integration domain is \(D = L > z_b > \cdots > z_1 > 0\). In this domain the wave function \(\chi_{k_1, \ldots, k_a}(z_1, \ldots, z_b|\bar{u}, \bar{v})\) has the form

\[
\chi_{k_1, \ldots, k_a}(z_1, \ldots, z_b|\bar{u}, \bar{v}) = \text{Sym} \Omega_{k_1, \ldots, k_a}(\bar{u}; \bar{v} + c) \prod_{b \geq j > k \geq 1} f(v_j, v_k) \prod_{k=1}^b e^{iz_k v_k} |_{\scriptscriptstyle c=-i\kappa},
\]
(3.5)

where the coefficients \(\Omega_{k_1, \ldots, k_a}(\bar{u}; \bar{v} + c)\) are given by (2.29).

Generic Bethe vector (3.4) becomes an eigenvector of the Hamiltonian (3.1) if the parameters \(\bar{u}\) and \(\bar{v}\) satisfy the system of Bethe equations (2.10). In the TCBG model it has the following
Comparing this system with (2.10) we conclude that in the TCBG model $r_1(u) = 1$ and $r_3(u) = e^{i Lu}$.

4 Lattice two-component Bose gas

Quantum systems describing by the $GL(3)$-invariant $R$-matrix (2.1) were considered in [8]. There a prototype of a lattice $L$-operator of the TCBG model was found. It has the following form:

$$L^{(a)}(u) = u 1 + p,$$

where

$$p = \left( \begin{array}{ccc}
a_1^\dagger a_1 & a_2^\dagger a_2 & ia_1^\dagger \sqrt{m+\rho} \\
ap_{12}a_1 & a_{22}a_2 & ia_2^\dagger \sqrt{m+\rho} \\
i\sqrt{m+\rho} a_1 & i\sqrt{m+\rho} a_2 & -m - \rho \end{array} \right).$$

Here $m$ is an arbitrary complex number and $\rho = a_1^\dagger a_1 + a_2^\dagger a_2$. The operators $a_k$ and $a_k^\dagger$ ($k = 1, 2$) act in a Fock space with the Fock vacuum $|0\rangle$: $a_k|0\rangle = 0$. They have standard commutation relations of the Heisenberg algebra $[a_i, a_j^\dagger] = \delta_{ij}$. The $L$-operator (4.1) satisfies the algebra (2.2) with $R$-matrix (2.1) at $c = -1$. Basing on the $L$-operator (4.1) one can construct a quantum system of discrete bosons. In order to obtain a continuous quantum system one should make several transforms of (4.1). First, we introduce operators

$$\psi_j = \Delta^{-1/2} a_j, \quad \psi_j^\dagger = \Delta^{-1/2} a_j^\dagger, \quad k = 1, 2,$$

so that

$$[\psi_j, \psi_k^\dagger] = \frac{\delta_{jk}}{\Delta}.$$  (4.4)

In these formulas $\Delta$ is a lattice interval. Setting $m = \frac{4}{\Delta^2}$ we introduce a new $L$-operator as

$$L(u) = \frac{\Delta}{2} L^{(a)} \left( \frac{u + 2i/\Delta}{i\Delta} \right) \cdot J,$$

where $J = \text{diag}(1, 1, -1)$. Obviously $L(u)$ satisfies the $RTT$-relation (2.2) with $R$-matrix (2.1) at $c = -i\Delta$.

The last transformation is to make $N$ copies $L_n$ ($n = 1, \ldots, N$) of $L$-operator (4.1) by changing $\psi_j \rightarrow \psi_j(n)$ and $\psi_j^\dagger \rightarrow \psi_j^\dagger(n)$ with

$$[\psi_j(n), \psi_j^\dagger(m)] = \frac{\delta_{jk} \delta_{nm}}{\Delta}.$$  (4.6)
The operators $\psi_k(n)$ and $\psi_k^\dagger(n)$ are lattice approximations of the Bose fields $\Psi_k(x)$ and $\Psi_k^\dagger(x)$. Indeed, let us divide the interval $[0, L]$ into $N$ sites of the length $\Delta$. Setting $x_n = n\Delta$ and

$$
\psi_k(n) = \frac{1}{\Delta} \int_{x_{n-1}}^{x_n} \Psi_k(x) \, dx, \quad \psi_k^\dagger(n) = \frac{1}{\Delta} \int_{x_{n-1}}^{x_n} \Psi_k^\dagger(x) \, dx, \quad (4.7)
$$

we reproduce commutation relations (4.6). On the other hand, in the limit $\Delta \to 0$ the operators (4.7) obviously turn into the Bose fields $\Psi_k(x)$ and $\Psi_k^\dagger(x)$.

Now we can define a monodromy matrix in a standard way

$$
T(u) = L_N(u) \ldots L_1(u), \quad (4.8)
$$

where

$$
L_n(u) = \frac{1}{\mathcal{N}} \begin{pmatrix}
1 - iu\Delta & \frac{\pm\Delta^2}{2} \psi_1^\dagger(n)\psi_1(n) & -i\Delta\psi_3^\dagger(n)Q_n \\
\frac{\pm\Delta^2}{2} \psi_2^\dagger(n)\psi_1(n) & 1 - iu\Delta & \frac{\pm\Delta^2}{2} \psi_2^\dagger(n)\psi_2(n) \\
i\Delta Q_n\psi_1(n) & i\Delta Q_n\psi_2(n) & 1 + iu\Delta + \frac{\pm\Delta^2}{2}\rho_n
\end{pmatrix}, \quad (4.9)
$$

and

$$
\mathcal{N} = \left(1 - iu\Delta\right), \quad \rho_n = \psi_1^\dagger(n)\psi_1(n) + \psi_2^\dagger(n)\psi_2(n), \quad Q_n = \left(\kappa + \frac{\pm\Delta^2}{2}\rho_n\right)^{1/2}. \quad (4.10)
$$

The normalization factor $\mathcal{N}$ in (4.9) is used in order to satisfy the condition $\lambda_2(u) = 1$.

Remark. We write the number of the lattice site $n$ as the argument of the operators $\psi_i$ and $\psi_i^\dagger$. Traditionally this number is written as subscript of $\psi_i$ and $\psi_i^\dagger$, but in the case of the TCBG model it is not convenient.

$L$-operator (4.9) is a natural generalization of a $2 \times 2$ $L$-operator found in [32] for the lattice model of one-component bosons:

$$
\tilde{L}_n(u) = \frac{1}{\mathcal{N}} \begin{pmatrix}
1 - iu\Delta & \frac{\pm\Delta^2}{2} \psi^\dagger(n)\psi(n) & -i\Delta\psi_3^\dagger(n)Q_n \\
i\Delta Q_n\psi(n) & 1 + iu\Delta + \frac{\pm\Delta^2}{2}\rho_n
\end{pmatrix}. \quad (4.11)
$$

It is easy to see that $L$-operator (4.11) is the right-lower $2 \times 2$ minor of the matrix (4.9) with the identification $\psi_1(n) \equiv 0$, $\psi_2(n) \equiv \psi(n)$. It was shown by different methods in [33, 34] that in the continuous limit $\Delta \to 0$ the $L$-operator (4.11) describes the model of one-dimensional bosons with $\delta$-function interaction. We have to solve an analogous problem: to check that in the continuous limit the model with the monodromy matrix (4.8) and $L$-operator (4.9) does describe the TCBG model. For this purpose we will find Bethe vectors of the lattice model (4.8) and will show that they coincide with the states (5.5) in the continuous limit.

Let us point out sever properties of the $L$-operator (4.9). It is easy to see that

$$
(L_n(u))_{11} |0\rangle = (L_n(u))_{22} |0\rangle = |0\rangle, \quad (L_n(u))_{33} |0\rangle = r_0(u) |0\rangle; \\
(L_n(u))_{12} |0\rangle = 0, \quad (0 | L_n(u))_{21} = 0, \quad (4.12)
$$

Here and below limits of operator-valued expressions should be understood in the weak sense.  

10
where

\[ r_0(u) = \left( \frac{1 + \frac{iu\Delta}{2}}{1 - \frac{iu\Delta}{2}} \right). \tag{4.13} \]

From these properties we easily find

\[
\begin{align*}
& r_1(u) = 1, \quad r_3(u) = r_0^N(u); \\
& T_{12}(u)|0\rangle = 0, \quad \langle 0| T_{21}(u) = 0. \tag{4.14}
\end{align*}
\]

Note that in fact the condition \( r_1(u) = 1 \) implies the actions of \( T_{12}(u) \) and \( T_{21}(u) \) in the second line of (4.14). Indeed, we have from the RTT-relation (2.2)

\[
[T_{21}(v), T_{12}(u)] = g(v, u)(T_{11}(u)T_{22}(v) - T_{11}(v)T_{22}(u)). \tag{4.15}
\]

Applying this equation, for example, to the vector \(|0\rangle\) and using \( r_1(u) = 1 \) we obtain

\[
[T_{21}(v), T_{12}(u)]|0\rangle = T_{21}(v)T_{12}(u)|0\rangle = g(v, u)\bigl(T_{11}(u)T_{22}(v) - T_{11}(v)T_{22}(u)\bigr)|0\rangle = (r_1(u) - r_1(v))|0\rangle = 0. \tag{4.16}
\]

Similarly, acting with (4.15) on \(|0\rangle\) we obtain \(|0\rangle T_{21}(u) = 0\).

The property \( T_{12}(u)|0\rangle = 0 \) leads to a simplification of the explicit formula for the Bethe vector (2.8). Obviously, in this case we should consider only such partitions of the set \( \bar{e} \) that \( \bar{e}^{(2)} = \emptyset \), and \( \bar{e}^{(1)} = \bar{e} \). Then (4.15) turns into

\[
\mathbb{B}_{a,b}(\bar{e}; \bar{v}) = \sum K_a(\bar{e}; \bar{e}) f(\bar{e}; \bar{e}) T_{13}(\bar{e}^{(1)}T_{23}(\bar{e}^{(2)}))|0\rangle. \tag{4.17}
\]

Here the sum is taken over partitions of the only one set \( \bar{v} = \{\bar{v}^{(1)}, \bar{v}^{(2)}\} \) with a restriction \( \#\bar{v}^{(1)} = a \). The last restriction evidently can be satisfied if and only if \( a \leq b \). Hence, if \( a > b \), then \( \mathbb{B}_{a,b}(\bar{e}; \bar{v}) = 0 \). In particular,

\[
\mathbb{B}_{0,1}(\emptyset; \bar{v}) = T_{23}(\bar{v})|0\rangle, \quad \mathbb{B}_{1,1}(\bar{u}; \bar{v}) = \frac{g(v, u)}{f(v, u)} T_{13}(\bar{v})|0\rangle. \tag{4.18}
\]

To conclude this section we give two formulas concerning the continuous limit \( \Delta \to 0 \). The first formula gives the limit of powers of the function \( r_0(u) \)

\[
\lim_{\Delta \to 0} r_0^n(u) = \lim_{\Delta \to 0} \left( \frac{1 + \frac{iu\Delta}{2}}{1 - \frac{iu\Delta}{2}} \right)^{x_n/\Delta} = e^{iux_n}. \tag{4.19}
\]

The second formula describes a typical procedure of taking continuous limit of sums over the lattice sites. Let \( \Phi(x) \) be an integrable function on the interval \([0, L]\). Then

\[
\Delta \sum_{j=1}^{N} \Phi(x_j) \psi^j = \sum_{j=1}^{N} \Phi(x_j) \int_{x_{j-1}}^{x_j} \Psi_k(x) dx \to \int_{0}^{L} \Phi(x) \Psi_k(x) dx, \quad \Delta \to 0, \tag{4.20}
\]

and we recall that all limits of operator-valued expressions are understood in a weak sense. Thus we can formulate a general rule: a sum over the lattice sites multiplied by \( \Delta \) turns into an integral in the continuous limit. It is easy to see that if we have an \( m \)-fold sum over the lattice sites multiplied by \( \Delta^m \), then it turns into an \( m \)-fold integral in the continuous limit.
5 Bethe vectors in terms of local operators

Consider a multi-composite model with the total monodromy matrix (4.8). Let the number \( M \) of the partial monodromy matrices coincides with the number \( N \) of the lattice sites. Then every partial monodromy matrix \( T_n(u) \) is the \( L \)-operator \( L_n(u) \) (4.9). Respectively we have
\[
\rho(k_1)_{1}^{(u)} = 1, \quad \rho(k_3)_{1}^{(v)} = r_0(v).
\]

The formula for the total Bethe vector (2.18) takes the form
\[
\mathcal{B}_{a,b}(\bar{u}; \bar{v}) = \sum_{\bar{j}=1}^{N} \prod_{j=1}^{N-1} \rho(j-1)_{0}^{(\bar{u}(j))} \prod_{1 < k < j \leq N} f(\bar{u}(j), \bar{u}(k)) f(\bar{v}(j), \bar{v}(k)) \prod_{j=1}^{N} \mathcal{B}_{a_j,b_j}(\bar{u}(j); \bar{v}(j)).
\]

This is the main formula that we shall use. But before applying this formula to the TCBG model it is useful to look how it works for a more simple example of the one-component Bose gas.

5.1 One-component Bose gas

The \( L \)-operator of the one-component Bose gas is given by (4.11), however for the construction of Bethe vectors we need to know this \( L \)-operator only up to terms of order \( \Delta \):
\[
\check{L}_n(u) = \left( \frac{1}{i\Delta \sqrt{\kappa}} \psi_j(n) - i\Delta \sqrt{\kappa} \psi_j(n) \right) + O(\Delta^2).
\]

Recall that there we have set \( \psi_2(n) \equiv \psi(n) \), \( \psi_1(n) \equiv 0 \) and similarly for \( \psi_j(n) \). In the continuous limit these operators respectively turns into Bose fields \( \Psi(x) \) and \( \Psi^\dagger(x) \).

Bethe vectors of the one-component Bose gas correspond to the particular case of \( \mathcal{B}_{a,b}(\bar{u}; \bar{v}) \) at \( a = 0 \) and \( \bar{u} = \emptyset \). Then the formula (5.2) takes the form
\[
\mathcal{B}_{0,b}(\emptyset; \bar{v}) \equiv \mathcal{B}_{b}(\bar{v}) = \sum_{j=1}^{N} \prod_{j=1}^{N-1} \rho(j-1)_{0}^{(\bar{v}(j))} \prod_{1 < k < j \leq N} f(\bar{v}(j), \bar{v}(k)) \prod_{j=1}^{N} \mathcal{B}_{b_j}(\bar{v}(j)).
\]

A partial Bethe vector in the site \( j \) is
\[
\mathcal{B}_{b_j}(\bar{v}(j)) = (-i\Delta \sqrt{\kappa} \psi_j(j))^{b_j} |0\rangle,
\]
where corrections of the order \( O(\Delta^{b_j+1}) \) are neglected.

Remark. Recall that in the multi-composite model the total pseudovacuum vector \( |0\rangle \) is equal to the tensor product of the partial pseudovacuum vectors \( |0\rangle^{(j)} \) \( (j = 1, \ldots, N) \). However, in the case of the one-component Bose gas we can assume that all operators \( \psi(j) \) and \( \psi^\dagger(j) \) act in the same Fock space. Obviously, due to commutativity of \( \psi(j) \) and \( \psi^\dagger(k) \) at \( j \neq k \) such the formulation is equivalent to the original one. In the case of the TCBG model we will use the same treatment of the multi-composite model.

Consider an example \( b = 2 \). Then we have two possibilities. 

12
• There exists one $b_j$ such that $b_j = 2$, while all other $b_\ell = 0$. Then the subset $\bar{v}(j)$ coincides with the original set $\{v_1, v_2\}$, while all other subsets $\bar{v}(\ell)$ are empty.

• There exist two $b_j$ and $b_k$ such that $b_j = b_k = 1$, while all other $b_\ell = 0$. Then the subsets $\bar{v}(j)$ and $\bar{v}(k)$ consist of one element (say, $\bar{v}(j) = v_2$ and $\bar{v}(k) = v_1$ or vice versa). All other subsets $\bar{v}(\ell)$ are empty.

Consider the first case. We denote the corresponding contribution to the Bethe vector by $\mathbb{B}_{2,0}$. Then

$$\mathbb{B}_{2,0} = -\kappa \Delta^2 \sum_{j=1}^{N} (r_0(v_1)r_0(v_2))^{j-1}(\psi^\dagger(j))^2|0\rangle,$$

and due to (4.19) we obtain

$$\mathbb{B}_{2,0} = -\kappa \Delta^2 \sum_{j=1}^{N} e^{ix_j(v_1+v_2)}(\psi^\dagger(j))^2|0\rangle.$$  

This sum goes to zero, because it has the coefficient $\Delta^2$. Indeed, due to (4.20) we have

$$\Delta^2 \sum_{j=1}^{N} e^{ix_j(v_1+v_2)}(\psi^\dagger(j))^2|0\rangle \rightarrow \Delta \int_{0}^{L} e^{ix(v_1+v_2)}(\Psi^\dagger(x))^2 dx|0\rangle \rightarrow 0, \quad \Delta \rightarrow 0. \quad (5.8)$$

It remains to consider the second case. We denote the corresponding contribution to the Bethe vector by $\mathbb{B}_{1,1,0}$. Then

$$\mathbb{B}_{1,1,0} = -\kappa \Delta^2 \text{Sym}_v \sum_{1 \leq k < j \leq N} r_0^{j-1}(v_2)r_0^{k-1}(v_1)f(v_2, v_1)\psi^\dagger(j)\psi^\dagger(k)|0\rangle,$$

or due to (4.19)

$$\mathbb{B}_{1,1,0} = -\kappa \Delta^2 \text{Sym}_v \sum_{1 \leq k < j \leq N} e^{ix_k v_1 + ix_j v_2} f(v_2, v_1)\psi^\dagger(j)\psi^\dagger(k)|0\rangle. \quad (5.10)$$

This time we have again the coefficient $\Delta^2$, but the sum is double. Therefore the limit is finite

$$\lim_{\Delta \rightarrow 0} \mathbb{B}_{1,1,0} = \mathbb{B}_2(\bar{v}) = -\kappa \text{Sym}_\bar{v} f(v_2, v_1) \int_{0}^{L} dx_2 \int_{0}^{x_2} dx_1 e^{ix_1 v_1 + ix_2 v_2} \Psi^\dagger(x_2)\Psi^\dagger(x_1)|0\rangle. \quad (5.11)$$

It is clear from (5.4) and (5.5) that for general $b$ the Bethe vector $\mathbb{B}_b(\bar{v})$ is proportional to $\Delta^b$. In the continuous limit this coefficient should be compensated. The only possible way to obtain such the compensation is to have a $b$-fold sum over the lattice sites. Then $\Delta^b$ times $b$-fold sum gives a $b$-fold integral. Hence, we should consider only such partitions of the set $\bar{v} = \{v_1, \ldots, v_b\}$ that reduce to $b$-fold sums over the lattice sites. Obviously, these are such partitions in which there are exactly $b$ nonempty subsets. In this case every such subset consists of only one variable.
Thus, actually we deal with the case already considered in section 2.4. Therefore the sum over partitions reduces to the sum over the lattice sites and the sum over permutations, i.e. to the symmetrization over $\bar{v}$.

Thus, we again consider a multi-composite model with the number of the partial monodromy matrices $\mathcal{M}_n(\bar{v})$ equal to the number of the lattice sites. Then every $\mathcal{M}_n(\bar{v})$ of such the model coincides with the $L$-operator (5.15). First of all let us find how Bethe vector depends on $\Delta$. In the TCBG model Bethe vectors are given by (5.12). Using the same arguments as in the case of the one-component Bose gas we conclude that we should consider only such partitions of the set $\bar{v}$, where we have exactly $b$ nonempty subsets consisting of one

$$
\mathbb{B}_b(\bar{v}) = (-i\sqrt{\varepsilon x})^b \text{Sym}_{\bar{v}} \prod_{b \geq j > k \geq 1} f(v_j, v_k) \sum_{j_0 > \cdots > j_1}^N \prod_{k=1}^b \left( e^{i\varepsilon x_k v_k \Psi^\dagger(j_k)} \right) |0\rangle,
$$

or partly taking continuous limit

$$
\lim_{\Delta \to 0} \mathbb{B}_b(\bar{v}) = (-i\sqrt{\varepsilon x})^b \text{Sym}_{\bar{v}} \prod_{b \geq j > k \geq 1} f(v_j, v_k) \int_{\mathcal{D}} dx_1 \cdots dx_b \prod_{k=1}^b \left( e^{i\varepsilon x_k v_k \Psi^\dagger(x_k)} \right) |0\rangle,
$$

where $\mathcal{D} = L > x_b > \cdots > x_1 > 0$. Representation (5.12) coincides with well know result for the Bethe vectors in the coordinate Bethe ansatz \[1, 2, 31\]. Thus, we have constructed Bethe vectors in terms of the local Bose field $\Psi^\dagger(x)$ starting from the lattice $L$-operator (5.13).

### 5.2 Two-component Bose gas

The infinitesimal lattice $L$-operator of the TCBG model has the form \[7\]

$$
L_n(u) = \begin{pmatrix}
1 - \frac{i\Delta}{2} & 0 & -i\sqrt{\varepsilon} \psi^\dagger_1(n) \\
0 & 1 - \frac{i\Delta}{2} & -i\sqrt{\varepsilon} \psi^\dagger_2(n) \\
i\sqrt{\varepsilon} \psi_1(n) & i\sqrt{\varepsilon} \psi_2(n) & 1 + \frac{i\Delta}{2}
\end{pmatrix} + O(\Delta^2).
$$

We again consider a multi-composite model with the number of the partial monodromy matrices $T^{(n)}(u)$ equal to the number of the lattice sites. Then every $T^{(n)}(u)$ of such the model coincides with the $L$-operator (5.15). First of all let us find how Bethe vector depends on $\Delta$. In the TCBG model Bethe vectors are given by (4.17). It is easy to see that the total number of creation operators $T_{13}$ and $T_{23}$ in (4.17) is $b$. In the case of partial Bethe vectors $\mathbb{B}_{a_j,b_j}(\bar{u}(j); \bar{v}(j))$ we have

$$
T_{13}(w) = -i\Delta \sqrt{\varepsilon} \psi^\dagger_1(j), \quad T_{23}(w) = -i\Delta \sqrt{\varepsilon} \psi^\dagger_2(j).
$$

Therefore

$$
\mathbb{B}_{a_j,b_j}(\bar{u}(j); \bar{v}(j)) \sim \Delta^{b_j}, \quad \text{and thus,} \quad \mathbb{B}_{a,b}(\bar{u}; \bar{v}) \sim \Delta^b.
$$

The Bethe vectors of the multi-composite TCBG model are given by (5.12). Using the same arguments as in the case of the one-component Bose gas we conclude that we should consider only such partitions of the set $\bar{v}$, where we have exactly $b$ nonempty subsets consisting of one
element. Then the sum over such partitions of the set \( \bar{v} \) turns into the sum over permutations of \( \bar{v} \) and a \( b \)-fold sum over the lattice sites.

Consider now what happens with the partitions of the set \( \bar{u} \). In every partial Bethe vector \( b_j \geq a_j \). As we have shown above, all \( b_j \) are equal either to zero or to one. If \( b_j = 0 \), then \( a_j = 0 \). However if \( b_j = 1 \), then either \( a_j = 1 \) or \( a_j = 0 \). In the first case we obtain a partial Bethe vector of the form \( B_{1,1}^{(j)} \), in the second case a partial Bethe vector of the form \( B_{0,1}^{(j)} \). But since all nonempty subsets consist of exactly one element, the sum over partitions of the set \( \bar{u} \) also turns into the sum over permutations in \( \bar{u} \) and a sum over the lattice sites where \( a_j = 1 \).

Thus, the sum in (5.2) is organised as follows. First, we should choose a set \( J \) consisting of \( b \) numbers \( J = \{ j_1, \ldots, j_b \} \). These are the numbers of the lattice sites, where \( b_j = 1 \). In all other sites \( b_j = 0 \). We assume that the subset \( \bar{v}^{(j_k)} \) consists of one element \( v_k \). Taking symmetrization over \( \bar{v} \) and the sum over all possible \( j_k \) with the restriction \( j_k > \cdots > j_1 \) we thus reproduce the sum over partitions of the set \( \bar{v} \). More precisely, we reproduce only such partitions that eventually contribute into the continuous limit.

Up to this point everything is exactly as in the case of one-component bosons. Now we should take into account the partitions of the set \( \bar{u} \). For this we should choose among the set \( J = \{ j_1, \ldots, j_b \} \) a subset of numbers \( K \) consisting of \( a \) elements: \( K = \{ j_{k_1}, \ldots, j_{k_a} \}, K \subset J \). These are the numbers of the lattice sites, where \( a_{j_{k_m}} = 1 \). In all other sites \( a_j = 0 \). We assume that the subset \( \bar{u}^{(j_{k_m})} \) consists of one element \( u_m \). Taking symmetrization over \( \bar{u} \) and the sum over all possible \( j_{k_m} \) with the restriction \( j_{k_a} > \cdots > j_{k_1} \) we reproduce the sum over partitions of the set \( \bar{u} \).

Summarizing all above we recast (5.2) as follows:

\[
\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \text{Sym}_{\bar{u}, \bar{v}} \prod_{b_j > k \geq 1} f(v_j, v_k) \prod_{a_j > k \geq 1} f(u_j, u_k) \sum_{j_b > \cdots > j_1} \sum_{j_{k_m} \in J} \prod_{\ell \in k_{m+1}} \prod_{m=1}^{a_b} \prod_{\ell \in k_1} f^{-1}(v_{\ell}, u_m) f^{-1}(v_k) \prod_{\ell \in k_1} f_{\ell}^{(j_{k_m})}(u_m; v_{k_m}) \prod_{j \in J \setminus K} f_{j}^{(j_{k_m})}(0; v_{j}). \tag{5.18}
\]

Due to (4.18) and (5.16) we find

\[
\mathbb{B}_{0,1}(0; v) = -i \Delta \sqrt{z} \psi_2^{(j)}(j)|0\rangle, \quad \mathbb{B}_{1,1}^{(j)}(u; v) = -i \Delta \sqrt{z} \frac{g(v, u)}{f(v, u)} \psi_1^{(j)}(j)|0\rangle, \tag{5.19}
\]

and using (4.19) we obtain

\[
\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-i \Delta \sqrt{z})^b \text{Sym}_{\bar{u}, \bar{v}} \prod_{b_j > k \geq 1} f(v_j, v_k) \prod_{a_j > k \geq 1} f(u_j, u_k) \sum_{j_b > \cdots > j_1} \sum_{j_{k_m} \in J} \prod_{\ell = k_{m+1}}^{a_b} \prod_{m=1}^{a} \prod_{\ell = k_1}^{b} e^{i x_k v_k} \prod_{\ell \in k_1} f(v_{\ell}, u_m) g(u_{k_m}, u_m) \psi_1^{(j_{k_m})} \psi_2^{(j_\ell)}(j)|0\rangle. \tag{5.20}
\]
Using obvious properties of the functions \( g(x, y) \) and \( f(x, y) \)
\[
f(x, y + c) = \frac{1}{f(y, x)}, \quad g(x, y + c) = -\frac{g(y, x)}{f(y, x)}.
\]
we see that
\[
\text{Sym}_{\bar{u}} \prod_{a \geq j > k \geq 1} f(u_j, u_k) \prod_{m=1}^{a} \frac{g(v_{k_m}, u_m)}{f(v_{k_m}, u_m)} \prod_{\ell = k_{m+1}}^{b} \frac{1}{f(v_{\ell}, u_m)} = (-1)^{a} \Omega^{(a, b)}_{k_1, \ldots, k_a; \bar{u} + c}, \quad (5.21)
\]
where the coefficients \( \Omega^{(a, b)}_{k_1, \ldots, k_a} \) are given by (2.29).

Thus (5.20) takes the form
\[
\mathbb{B}_{a, b}(\bar{u}; \bar{v}) = (-1)^a (-i \Delta \sqrt{\kappa})^b \text{Sym}_{\bar{v}} \prod_{b \geq j \geq 1} f(v_j, v_k) \prod_{j_k \in J} \prod_{j \in J \setminus K} \psi^{\dagger}_1(j \ell) |0\rangle,
\]
and it becomes clear that in the continuous limit we arrive at (3.4) up to a normalization factor.

6 Representation of the monodromy matrix in terms of Bose fields

In this section we derive explicit representations of the monodromy matrix elements \( T_{ij}(u) \) in terms of the Bose fields. These representations have the form of a formal power series in the coupling constant \( \kappa \). It is worth mentioning, however, that in a weak sense, these series are cut on an arbitrary Bethe vector.

Let us present the infinitesimal \( L \)-operator \((5.15)\) as a block-matrix of the size \( 2 \times 2 \):
\[
L_n(u) = \begin{pmatrix} a & b_n \\ c_n & d \end{pmatrix} + O(\Delta^2)
\]
(6.1)
Here \( d = 1 + \frac{iu\Delta}{2} \), and \( a \) is a \( 2 \times 2 \) matrix \( a = (1 - \frac{iu\Delta}{2}) \cdot 1 \), where \( 1 \) is the identity matrix of the size \( 2 \times 2 \). A two-component vector-column \( b_n \) and two-component vector-row \( c_n \) are
\[
b_n = -i \Delta \sqrt{\kappa} \begin{pmatrix} \psi_1^{\dagger}(n) \\ \psi_2^{\dagger}(n) \end{pmatrix}, \quad c_n = i \Delta \sqrt{\kappa} \left( \psi_1(n); \psi_2(n) \right).
\]
(6.2)
It is convenient to separate the diagonal and anti-diagonal parts of the \( L \)-operator \((6.1)\) as follows:\footnote{Here and below we omit the terms \( O(\Delta^2) \) as they do not contribute to the continuous limit.}
\[
L_n(u) = \Lambda(u) + W_n
\]
(6.3)
where
\[ \Lambda(u) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad W_n = \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix}. \] (6.4)

Now representation (6.3) should be substituted in (4.8) and then developed into the series in \( W_n \). Since the anti-diagonal part \( W_n \) is proportional to \( \sqrt{\kappa} \), the monodromy matrix \( T(u) \) becomes a polynomial in \( \sqrt{\kappa} \), that turns into an infinite power series in the continuous limit:
\[ T(u) = \sum_{n=0}^{\infty} \kappa^{n/2} T_n(u), \] (6.5)

where
\[ \kappa^{n/2} T_n(u) = \left( 1 - \frac{i u \Delta}{2} \right)^{-N} \sum_{N \geq n \geq \cdots > k_1 \geq 1} \Lambda^{N-k_n} W_{k_n} \Lambda^{k_n-k_{n-1}} \cdots \Lambda^{k_2-k_1-1} W_{k_1} \Lambda^{k_1-1}. \] (6.6)

It is clear from (6.6) that the diagonal blocks of the monodromy matrix are series in integer powers of \( \kappa \), while anti-diagonal blocks are series in half-integer powers of \( \kappa \).

Let
\[ \tilde{W}_{k_i} = \Lambda^{-k_i} W_{k_i} \Lambda^{k_i-1} = \begin{pmatrix} 0 & \tilde{b}_{k_i} \\ \tilde{c}_{k_i} & 0 \end{pmatrix}, \] (6.7)

where
\[ \tilde{b}_{k_i} = \frac{b_{k_i}}{1 + \frac{i u \Delta}{2}} \left( 1 + \frac{i u \Delta}{2} \right)^{k_i}, \quad \tilde{c}_{k_i} = \frac{c_{k_i}}{1 - \frac{i u \Delta}{2}} \left( 1 - \frac{i u \Delta}{2} \right)^{k_i}. \] (6.8)

Then equation (6.6) takes the form
\[ \kappa^{n/2} T_n(u) = \left( 1 - \frac{i u \Delta}{2} \right)^{-N} \sum_{N \geq n \geq \cdots > k_1 \geq 1} \tilde{W}_{k_n} \tilde{W}_{k_{n-1}} \cdots \tilde{W}_{k_1}. \] (6.9)

Partly taking the continuous limit via (4.19) we obtain
\[ \lim_{\Delta \to 0} \left( 1 - \frac{i u \Delta}{2} \right)^{-N} \Lambda^N \sum_{N \geq n \geq \cdots > k_1 \geq 1} \tilde{W}_{k_n} \tilde{W}_{k_{n-1}} \cdots \tilde{W}_{k_1} = \begin{pmatrix} 0 & e^{i\alpha L} \\ 0 & 0 \end{pmatrix}, \] (6.10)

and
\[ \tilde{b}_{k_i} = b_k e^{i u x k_i}, \quad \tilde{c}_{k_i} = c_k e^{-i u x k_i}. \] (6.11)

It is convenient to study the operators \( T_n(u) \) separately for \( n \) even and \( n \) odd. Let \( n = 2 \ell \). The product of two matrices \( \tilde{W}_{k_1} \) and \( \tilde{W}_{k_1-1} \) gives a block-diagonal matrix
\[ \tilde{W}_{k_1} \tilde{W}_{k_1-1} = \begin{pmatrix} \tilde{b}_{k_1} \tilde{c}_{k_1-1} & 0 \\ 0 & \tilde{c}_{k_1} \tilde{b}_{k_1-1} \end{pmatrix}. \] (6.12)

Hence, we obtain
\[ \kappa^{\ell} T_{2\ell}(u) = \begin{pmatrix} A_{2\ell}(u) & 0 \\ 0 & D_{2\ell}(u) \end{pmatrix}, \] (6.13)
where
\[ A_\ell(u) = \sum_{N \geq k_{2\ell}} \tilde{b}_{k_{2\ell}} \tilde{c}_{k_{2\ell}-1} \tilde{b}_{k_{2\ell-2}} \tilde{c}_{k_{2\ell-3}} \cdots \tilde{b}_{k_2} \tilde{c}_1, \]  
(6.14)
and
\[ D_\ell(u) = e^{iuL} \sum_{N \geq k_{2\ell}} \tilde{c}_{k_{2\ell}} \tilde{b}_{k_{2\ell}-1} \tilde{c}_{k_{2\ell-2}} \tilde{b}_{k_{2\ell-3}} \cdots \tilde{c}_{k_2} \tilde{b}_1. \]  
(6.15)

Observe that all operators in these products commute, because they are from different lattice sites. Therefore
\[ \tilde{b}_{k_{2\ell}} \tilde{c}_{k_{2\ell}-1} \tilde{b}_{k_{2\ell-2}} \tilde{c}_{k_{2\ell-3}} \cdots \tilde{b}_{k_2} \tilde{c}_1 = : \tilde{b}_{k_{2\ell}} \tilde{c}_{k_{2\ell}-1} \tilde{b}_{k_{2\ell-2}} \tilde{c}_{k_{2\ell-3}} \cdots \tilde{b}_{k_2} \tilde{c}_1 : , \]  
(6.16)
where the symbol : : means normal ordering. Obviously
\[ \tilde{c}_{k_{2\ell}} \tilde{b}_{k_{2\ell}-1} = \mathcal{A}^2 e^{iu(x_{k_{2\ell}-1}-x_{k_2})} : (\psi_1^\dagger(k_{2\ell}-1) \psi_1(k_2) + \psi_2^\dagger(k_{2\ell}-1) \psi_2(k_2)) : . \]  
(6.17)

Thus, we find
\[ A_\ell(u) = \mathcal{A}^\ell \sum_{N \geq k_{2\ell}} \prod_{i=1}^{\ell} e^{iu(x_{k_{2i}}-x_{k_{2i-1}})} \]  
\[ \times : \prod_{i=1}^{\ell-1} (\psi_1^\dagger(k_{2i}) \psi_1(k_{2i+1}) + \psi_2^\dagger(k_{2i}) \psi_2(k_{2i+1})) \left( \psi_1^\dagger(k_{2\ell}) \psi_1(k_1) \quad \psi_2^\dagger(k_{2\ell}) \psi_2(k_1) \right) :, \]  
(6.18)
and
\[ D_\ell(u) = e^{iuL} \mathcal{A}^\ell \sum_{N \geq k_{2\ell}} \prod_{i=1}^{\ell} e^{-iu(x_{k_{2i}}-x_{k_{2i-1}})} \]  
\[ \times : \prod_{i=1}^{\ell-1} (\psi_1^\dagger(k_{2i}-1) \psi_1(k_{2i}) + \psi_2^\dagger(k_{2i}-1) \psi_2(k_{2i})) :, \]  
(6.19)

It remains to replace the sums over \( k_i \) by integrals via \((6.20)\). It is convenient to set \( x_{k_{2i}} = z_i \) and \( x_{k_{2i-1}} = y_i \). Then
\[ A_\ell(u) = \mathcal{A}^\ell \int_0^L \prod_{i=1}^{\ell} \left\{ e^{iu(z_i-y_i)} \right\} \Theta_\ell(z, \bar{y}) \]  
\[ \times : \prod_{i=1}^{\ell-1} \left( \Psi_1^\dagger(z_i) \Psi_1(y_{i+1}) + \Psi_2^\dagger(z_i) \Psi_2(y_{i+1}) \right) \left( \Psi_1^\dagger(z_\ell) \Psi_1(y_1) \quad \Psi_2^\dagger(z_\ell) \Psi_2(y_1) \right) :, \]  
(6.20)
and
\[ D_\ell(u) = e^{iuL} \mathcal{A}^\ell \int_0^L \prod_{i=1}^{\ell} \left\{ e^{-iu(z_i-y_i)} \right\} \Theta_\ell(z, \bar{y}) : \prod_{i=1}^{\ell} (\Psi_1^\dagger(y_i) \Psi_1(z_i) + \Psi_2^\dagger(y_i) \Psi_2(z_i)) :, \]  
(6.21)
\[ \Theta_{\ell}(\bar{z}, \bar{y}) = \theta(z_{\ell} - y_{\ell}) \prod_{i=1}^{\ell-1} \theta(y_{i+1} - z_i)\theta(z_i - y_i). \]  

(6.22)

The anti-diagonal blocks of the monodromy matrix can be found exactly in the same manner. Setting \( n = 2\ell + 1 \) in (6.9) we find

\[ \kappa_{\ell+1} \Theta_{2\ell+1}(u) = \begin{pmatrix} 0 & B_{\ell}(u) \\ C_{\ell}(u) & 0 \end{pmatrix}, \]  

(6.23)

where

\[ C_{\ell}(u) = ie^{iuL} \int_0^L \prod_{i=1}^{\ell} \left\{ e^{iz_i - y_i} dz_i dy_i \right\} e^{-iuy_{\ell+1}} dy_{\ell+1} \Theta(z, \bar{y}) \]

\[ \times \prod_{i=1}^{\ell} \left( \Psi_1^\dagger(z_i)\Psi_1(y_{i+1}) + \Psi_2^\dagger(z_i)\Psi_2(y_{i+1}) \right) \cdot \left( \Psi_1(y_1); \Psi_2(y_1) \right), \]  

(6.24)

and

\[ B_{\ell}(u) = -ie^{iuL} \int_0^L \prod_{i=1}^{\ell} \left\{ e^{-iz_i - y_i} dz_i dy_i \right\} e^{iuy_{\ell+1}} dy_{\ell+1} \Theta(z, \bar{y}) \]

\[ \times \prod_{i=1}^{\ell} \left( \Psi_1^\dagger(y_i)\Psi_1(z_i) + \Psi_2^\dagger(y_i)\Psi_2(z_i) \right) \cdot \left( \Psi_1(y_{\ell+1}); \Psi_2(y_{\ell+1}) \right) \]  

(6.25)

Thus, we have obtained the explicit series representation for the monodromy matrix entries \( T_{ij}(u) \) in terms of the local Bose fields. This series is formal, and we do not study the problem of its convergence. It is easy to see, however, that if we introduce a vector

\[ |\Phi_{a,b}\rangle = \int_0^L dx_1, \ldots, dx_a dy_1, \ldots, dy_b \Phi_{a,b}(x_1, \ldots, x_a; y_1, \ldots, y_b) \prod_{i=1}^a \Psi_1^\dagger(x_i) \prod_{j=1}^b \Psi_2^\dagger(y_j) |0\rangle, \]  

(6.26)

where \( \Phi_{a,b}(x_1, \ldots, x_a; y_1, \ldots, y_b) \) is a continuous function within the integration domain, then the action of any \( T_{ij}(u) \) on \( |\Phi_{a,b}\rangle \) turns into a finite sum.

### 7 Mapping of fields

Due to the invariance of the \( R \)-matrix under transposition with respect to both spaces, the mapping

\[ \phi(T_{jk}(u)) = T_{kj}(u) \]  

(7.1)

defines an antimorphism of the algebra (2.2) (see [21]). This mapping is a very convenient tool in studying form factors, because it allows one to relate form factors of different operators. In
the case of the TCBG model antimorphism (7.1) agrees with the following mapping of the Bose fields:

\[
\phi(\Psi_i(x)) = -\Psi_i^\dagger(L - x), \quad \phi(\Psi_i^\dagger(x)) = -\Psi_i(L - x). \quad (7.2)
\]

Indeed, consider, for example, how the mapping (7.2) acts on the matrix elements \(T_{jk}(u)\) for \(j, k = 1, 2\). Due to equations (6.13), (6.20) we have

\[
T_{jk}(u) = \sum_{\ell=0}^\infty \varphi^\ell(T_{2\ell})_{jk}(u), \quad j, k = 1, 2, \quad (7.3)
\]

where

\[
(T_{2\ell})_{jk}(u) = \int_0^L \prod_{n=1}^{\ell} \left\{ e^{iu(z_n - y_n)} \, dz_n \, dy_n \right\} \Theta_\ell(\bar{z}, \bar{y}) : \Psi_k^\dagger(z_\ell) \Psi_k(y_1) \prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_s^\dagger(z_n) \Psi_s(y_{n+1}) \right) : \quad (7.4)
\]

Recall that due to the factor \(\Theta_\ell(\bar{z}, \bar{y})\) the integral in (7.4) is taken over domain \(z_\ell > y_\ell > z_{\ell-1} > \cdots > z_1 > y_1\). Therefore all the operators in (7.4) commute with each other, and actually the normal ordering is not necessary. Acting on (7.4) with \(\phi\) as in (7.2) we obtain

\[
\phi((T_{2\ell})_{jk}(u)) = \int_0^L \prod_{n=1}^{\ell} \left\{ e^{iu(z_n - y_n)} \, dz_n \, dy_n \right\} \Theta_\ell(\bar{z}, \bar{y}) : \Psi_k^\dagger(L - y_1) \Psi_j(L - z_\ell)
\]

\[
\times \prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_s^\dagger(L - y_{n+1}) \Psi_s(L - z_n) \right) : \quad (7.5)
\]

Now it is enough to change the integration variables \(z_n \rightarrow L - y_{\ell+1-n}\) and \(y_n \rightarrow L - z_{\ell+1-n}\). Then we have

\[
\Theta_\ell(\bar{z}, \bar{y}) \big|_{z_n \rightarrow L - y_{\ell+1-n}, \ y_n \rightarrow L - z_{\ell+1-n}} = \prod_{n=1}^{\ell-1} \theta(y_{\ell-n+1} - z_{\ell-n}) \prod_{n=1}^{\ell} \theta(z_{\ell-n+1} - y_{\ell-n+1})
\]

\[
= \prod_{i=1}^{\ell-1} \theta(y_{i+1} - z_i) \prod_{i=1}^{\ell} \theta(z_i - y_i) = \Theta_\ell(\bar{z}, \bar{y}). \quad (7.6)
\]

It is also easy to see that

\[
\prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_s^\dagger(L - y_{n+1}) \Psi_s(L - z_n) \right) \big|_{z_n \rightarrow L - y_{\ell+1-n}, \ y_n \rightarrow L - z_{\ell+1-n}} = \prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_s^\dagger(z_{\ell-n}) \Psi_s(y_{\ell-n+1}) \right)
\]

\[
= \prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_s^\dagger(z_n) \Psi_s(y_{n+1}) \right). \quad (7.7)
\]
Thus, we arrive at
\[ \phi \left( \left[ T_{2k} \right]_{jk}(u) \right) = \int_0^L \prod_{n=1}^\ell \left\{ e^{iu(z_n-y_n)} \, dz_n \, dy_n \right\} \Theta_{\ell}(\bar{z}, \bar{y}) : \Psi_k^\dagger(y_1)\Psi_j(z_\ell) \times \prod_{n=1}^{\ell-1} \left( \sum_{s=1}^2 \Psi_{s}^\dagger(z_n)\Psi_s(y_{n+1}) \right) := \left( T_{2k} \right)_{kj}(u) . \] (7.8)

Similarly, using the explicit representations for other operators \( T_{jk}(u) \) one can prove that (7.2) implies (7.1).

### 8 Zero modes

A method of calculating form factors of local operators in \( GL(3) \)-invariant models was developed in [12]. This method is based on the use of partial zero modes of the monodromy matrix entries \( T_{ij}(u) \) in the composite model consisting of two partial monodromy matrices (2.12). In spite of this approach can be applied to a wide class of integrable models, it should be slightly modified in the case of the TCBG model. The matter is that it was assumed in [12] that the monodromy matrix \( T(u) \) goes to the identity operator at \( |u| \to \infty \). This restriction is not very important, however it leads to minor changes in the case of the TCBG model.

Observe that a monodromy matrix \( T(a)(u) \) constructed by the \( L \)-operator (4.1) possesses the property mentioned above. Indeed, we can define local \( L \)-operators \( L_{ij}(a)(u) \), \( (n = 1, \ldots, N) \) by equations (4.1) and (4.2), where the operators \( a_k \) and \( a_k^\dagger \) are respectively replaced with \( a_k(n) \) and \( a_k^\dagger(n) \) with the commutation relations \([a_i(n), a_k^\dagger(m)] = \delta_{nm}\delta_{ik}\). Then we can set
\[ T^{(a)}(u) = u^{-N} L_N^{(a)}(u) \ldots L_1^{(a)}(u), \] (8.1)
and this matrix obviously has an asymptotic expansion
\[ T^{(a)}(u) = I + \frac{c}{u} T^{(a)}[0] + O(u^{-2}), \quad u \to \infty. \] (8.2)

Therefore we can define zero modes of this monodromy matrix in a standard way
\[ T^{(a)}[0] = \lim_{u \to \infty} \frac{u}{c} ( T^{(a)}(u) - I ) . \] (8.3)

However, passing from \( L \)-operator (4.1) to \( L \)-operator (4.9) we have multiplied \( L^{(a)}(u) \) by the matrix \( J = \text{diag}(1, 1, -1) \) (see (4.5)). This led to the fact that the monodromy matrix \( T(u) \) in the continuous limit has essential singularity at infinity. Therefore, in the case of the TCBG model, the definition of zero modes needs to be clarified. We do it in this section and consider an asymptotic expansion of the monodromy matrix entries \( T_{ij}(u) \) at large value of the argument. For this purpose we use the integral representations for \( T_{ij}(u) \) obtained in section 6.

If \( u \to \infty \), then the expansion for the monodromy matrix contains multiple integrals of quickly oscillating exponents. Methods of calculating quickly oscillating integrals are well known.
(see e.g. [36, 37]). In our case the integration domain of every integration variable is a finite interval \([0, L]\), therefore one of the simplest ways to obtain the asymptotic expansion of \(T_{ij}(u)\) is the integration by parts. Using this method one can easily show that single and double integrals give \(1/u\)-behavior, while all the terms with \(\ell > 1\) give contributions of order \(o(u^{-1})\). Therefore, in order to find zero modes it is enough to take only the first nontrivial terms of the expansion for \(T(u)\). Then we have

\[
T_{ij}(u) = \delta_{ij} + \kappa \int_0^L e^{iu(z-y)} \theta(z-y) \Psi_{i}^\dagger(z) \Psi_{j}(y) \, dz \, dy + O(\kappa^2), \quad i, j = 1, 2, \tag{8.4}
\]

\[
T_{33}(u) = e^{iuL} + \kappa e^{iuL} \int_0^L e^{iu(y-z)} \theta(z-y) (\Psi_{1}^\dagger(y) \Psi_{1}(z) + \Psi_{2}^\dagger(y) \Psi_{2}(z)) \, dz \, dy + O(\kappa^2). \tag{8.5}
\]

\[
T_{i3}(u) = -i \sqrt{\kappa} \int_0^L e^{iuy} \Psi_{i}^\dagger(y) dy + O(\kappa^{3/2}), \quad i = 1, 2, \tag{8.6}
\]

\[
T_{3j}(u) = i \sqrt{\kappa} e^{iuL} \int_0^L e^{-iuy} \Psi_{j}(y) dy + O(\kappa^{3/2}), \quad j = 1, 2. \tag{8.7}
\]

All the terms denoted by \(O(\kappa^2)\) or \(O(\kappa^{3/2})\) give contributions \(O(u^{-2})\) as \(u \to \infty\), and therefore they are not important. Integrating by parts we obtain

\[
T_{ij}(u) = \delta_{ij} + \frac{i \kappa}{u} \int_0^L \Psi_{i}^\dagger(y) \Psi_{j}(y) dy + O(u^{-2}), \quad i, j = 1, 2, \tag{8.8}
\]

\[
T_{33}(u) = e^{iuL} - \frac{i \kappa}{u} e^{iuL} \int_0^L (\Psi_{1}^\dagger(y) \Psi_{1}(y) + \Psi_{2}^\dagger(y) \Psi_{2}(y)) \, dy + O(u^{-2}). \tag{8.9}
\]

\[
T_{i3}(u) = - \frac{\sqrt{\kappa}}{u} (e^{iuL} \Psi_{i}^\dagger(L) - \Psi_{i}^\dagger(0)) + O(u^{-2}), \quad i = 1, 2, \tag{8.10}
\]

\[
T_{3j}(u) = - \frac{\sqrt{\kappa}}{u} (\Psi_{j}(L) - e^{iuL} \Psi_{j}(0)) + O(u^{-2}), \quad j = 1, 2. \tag{8.11}
\]

Now we define zero modes as follows:

\[
T_{ij}[0] = \lim_{u \to \infty} \frac{u}{c} (T_{ij}(u) - \delta_{ij}) = - \int_0^L \Psi_{i}^\dagger(y) \Psi_{j}(y) \, dy, \quad i, j = 1, 2, \tag{8.12}
\]

(recall that \(c = -i \sqrt{\kappa}\)). This is the same definition as for the models considered in [12]. The zero mode \(T_{33}[0]\) is slightly differently:

\[
T_{33}[0] = \lim_{u \to \infty} \frac{u}{c} (e^{-iuL} T_{33}(u) - 1) = \int_0^L (\Psi_{1}^\dagger(y) \Psi_{1}(y) + \Psi_{2}^\dagger(y) \Psi_{2}(y)) \, dy, \tag{8.13}
\]

and thus, \(T_{11}[0] + T_{22}[0] = -T_{33}[0]\).
Looking at (8.10), (8.11) we see that actually we have two types of zero modes for these operators. We call them left and right zero modes and denote respectively by $T^{(L)}_{ij}[0]$ and $T^{(R)}_{ij}[0]$. Then

$$T^{(R)}_{k3}[0] = \lim_{u \to -i\infty} e^{-iuL} \frac{u}{c} T_{k3}(u) = \frac{1}{i\sqrt{\kappa}} \Psi^\dagger_k(L), \quad k = 1, 2,$$

and

$$T^{(L)}_{k3}[0] = \lim_{u \to +i\infty} e^{-iuL} \frac{u}{c} T_{k3}(u) = -\frac{1}{i\sqrt{\kappa}} \Psi_k(0),$$

The sums $T^{(L)}_{ij}[0] + T^{(R)}_{ij}[0]$ play the same role as the zero modes of the monodromy matrix of the type (8.1), (8.2). It is known, in particular [13, 38], that some of zero modes $T_{ij}[0]$ annihilate on-shell Bethe vectors:

$$T^{(a)}_{ij}[0] \mathbb{B}_{a,b}(\bar{u}, \bar{v}) = 0, \quad i > j.$$  

Similarly one can check that

$$(T^{(L)}_{3j}[0] + T^{(R)}_{3j}[0]) \mathbb{B}_{a,b}(\bar{u}, \bar{v}) = 0, \quad j \neq 3,$$  

provided $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$ is an on-shell vector. In order to prove (8.17) it is sufficient to use the formulas of the action of $T_{ij}(u)$ onto Bethe vectors [25] and to consider there the limits $u \to \pm i\infty$ like in (8.15).

Finally, the obtained formulas for the zero modes allow us to study form factors of the local operators in the framework of the composite model (2.12). Indeed, let in (2.12) the partial monodromy matrix $T^{(1)}(u)$ corresponds to an interval $[0, x]$, where $x$ is a fixed point of the interval $[0, L]$. Then the partial zero modes $T^{(1)}_{ij}[0]$ and $T^{(1, n)}_{ij}[0]$ are given by (8.12)–(8.15), where one should replace everywhere $L$ by $x$. In particular, we obtain

$$\Psi_j(x) \Psi_j(x) = -\frac{d}{dx} T^{(1)}_{ij}[0] = \frac{1}{i\sqrt{\kappa}} \lim_{u \to -i\infty} u(T^{(1)}_{ij}(u) - \delta_{ij}), \quad i, j = 1, 2,$$

$$\Psi_j(x) = i\sqrt{\kappa} T^{(1, n)}_{3j}[0] = \frac{1}{\sqrt{\kappa}} \lim_{u \to -i\infty} u T^{(1)}_{3j}(u), \quad j = 1, 2,$$

Thus, the problem of calculating the form factors of the local operators in the TCBG model is reduced to the evaluating the form factors of the partial zero modes $T^{(1)}_{ij}[0]$ and $T^{(1, n)}_{ij}[0]$.

**Conclusion**

In this paper we gave a description of the TCBG model in the framework of the algebraic Bethe ansatz. The main goal was to prove that the lattice $L$-operator (4.9) correctly describes the
TCBG model in the continuous limit and allows one to find the zero modes of the monodromy matrix $T(u)$. This goal is successfully achieved. In order to calculate the form factors of the fields $\Psi_i(x)$, $\Psi_i^\dagger(x)$, and their combinations $\Psi_i^\dagger(x)\Psi_j(x)$ we can use now the method of [12]. Actually, a part of results can be predicted already now. Indeed, the definition (8.12) of the zero modes $T_{ij}[0]$ for $i, j = 1, 2$ coincides with the definition used in [12]. Therefore the form factors of the operators $\Psi_i^\dagger(x)\Psi_j(x)$ in fact are already computed. The calculation of the form factors of the fields $\Psi_i(x)$ and $\Psi_i^\dagger(x)$ should be slightly modified. However, in this case the modification affects only the limit $u \to \infty$, but does not affect determinant representations for the partial zero modes. We consider this question in details in our further publication.

Acknowledgements

It is a great pleasure for me to thank my colleagues S. Pakuliak and E. Ragoucy for numerous and fruitful discussions. This work was supported by the RSF under a grant 14-50-00005.

References

[1] E. H. Lieb and W. Liniger, Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State, Phys. Rev. 130 (1963) 1605–1616.

[2] E. H. Lieb, Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum, Phys. Rev. 130:4 (1963) 1616–1624.

[3] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. 19:23 (1967) 1312–1315.

[4] B. Sutherland, Further results for the many-body problem in one dimension, Phys. Rev. Lett. 20:3 (1968) 98–100.

[5] B. Sutherland, A general model for multicomponent quantum systems, Phys. Rev. B 12 (1975) 3795–3805.

[6] M. Gaudin, La Fonction dOnde de Bethe, Paris: Masson, 1983.

[7] P. P. Kulish, Classical and quantum inverse problem method and generalized Bethe ansatz, Physica D 3 (1981) 246–257.

[8] P. P. Kulish, N. Yu. Reshetikhin, GL(3)-invariant solutions of the Yang-Baxter equation and associated quantum systems, Zap. Nauchn. Sem. POMI. 120 (1982) 92–121; J. Sov. Math., 34:5 (1982) 1948–1971 (Engl. transl.)

[9] N. Kitanine, J. M. Maillet and V. Terras, Form factors of the XXZ Heisenberg spin-1/2 finite chain, Nucl. Phys. B 554 (1999) 647–678, arXiv:math-ph/9807020.

[10] J. M. Maillet, V. Terras, On the quantum inverse scattering problem, Nucl. Phys. B 575 (2000) 627–644, hep-th/9911030.
[11] B. Pozsgay, W.-V. van G. Oei and M. Kormos, On Form Factors in nested Bethe Ansatz systems, J. Phys. A: Math. Gen. 45 (2012) 465007, arXiv:1204.4037

[12] S. Pakuliak, E. Ragoucy, N. A. Slavnov, GL(3)-based quantum integrable composite models: 2. Form factors of local operators, arXiv:1502.01966.

[13] S. Pakuliak, E. Ragoucy, N. A. Slavnov, Zero modes method and form factors in quantum integrable models, arXiv:1412.6037.

[14] A. G. Izergin, V. E. Korepin, The quantum inverse scattering method approach to correlation functions, Comm. Math. Phys. 94 (1981) 67–92.

[15] L. D. Faddeev, E. K. Sklyanyin and L. A. Takhtajan, Quantum Inverse Problem. I, Theor. Math. Phys. 40 (1979) 688–706.

[16] L. D. Faddeev and L. A. Takhtajan, The quantum method of the inverse problem and the Heisenberg XYZ model, Usp. Math. Nauk 34 (1979) 13–63; Russian Math. Surveys 34 (1979) 11–68 (Engl. transl.).

[17] L. D. Faddeev, in: Les Houches Lectures Quantum Symmetries, eds A. Connes et al, North Holland, (1998) 149.

[18] P. P. Kulish and N. Yu. Reshetikhin, Generalized Heisenberg ferromagnet and the Gross–Nevu model, Sov. Phys. JETP, 53:1 (1981)108–114.

[19] P. P. Kulish, N. Yu. Reshetikhin, Diagonalization of GL(N) invariant transfer matrices and quantum N-wave system (Lee model), J. Phys. A: 16 (1983) L591–L596.

[20] S. Belliard, S. Pakuliak, E. Ragoucy, N. A. Slavnov, Form factors in SU(3)-invariant integrable models, J. Stat. Mech. (2013) P04033, arXiv:1211.3968.

[21] S. Pakuliak, E. Ragoucy, N. A. Slavnov, Form factors in quantum integrable models with GL(3)-invariant R-matrix, Nucl. Phys. B, 881 (2014) 343–368, arXiv:1312.1488.

[22] V. O. Tarasov and A. N. Varchenko, Jackson integral representations for solutions of the Knizhnik-Zamolodchikov quantum equation, Algebra i Analiz 6 (1994) 90–137; St. Petersburg Math. J. 6 (1995) 275–313 (Engl. transl.), arXiv:hep-th/9311040.

[23] S. Khoroshkin, S. Pakuliak, V. Tarasov, Off-shell Bethe vectors and Drinfeld currents, J. Geom. Phys. 57 (2007) 1713, math/0610517.

[24] S. Khoroshkin, S. Pakuliak. A computation of universal weight function for quantum affine algebra $U_q(gl_N)$. J. Math. Kyoto Univ. 48 (2008) 277, math.QA/0711.2819.

[25] S. Belliard, S. Pakuliak, E. Ragoucy, N. A. Slavnov, Bethe vectors of GL(3)-invariant integrable models, J. Stat. Mech. (2013) P02020, arXiv:1210.0768.

[26] V. E. Korepin, Calculation of norms of Bethe wave functions, Comm. Math. Phys. 86 (1982) 391–418.

25
[27] A. G. Izergin, *Partition function of the six-vertex model in a finite volume*, Dokl. Akad. Nauk SSSR 297 (1987) 331–333; Sov. Phys. Dokl. 32 (1987) 878–879 (Engl. transl.).

[28] B. Enriquez, S. Khoroshkin, S. Pakuliak. *Weight Functions and Drinfeld Currents*, Commun. Math. Phys. 276 (2007) 691–725.

[29] S. Pakuliak, E. Ragoucy, N. A. Slavnov, *GL(3)-based quantum integrable composite models: 1. Bethe vectors*, arXiv:1501.07566.

[30] A. G. Izergin, V. E. Korepin, N. Yu. Reshetikhin, *Field correlation functions in a one-dimensional Bose gas*, Zap. Nauchn. Sem. LOMI, 150 (1986) 26-36; Journ. Sov. Math. 46:1 (1989) 1581–1588 (Engl. transl.)

[31] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993.

[32] A. G. Izergin, V. E. Korepin, *Lattice model related to the the nonlinear Schrödinger equation*, Sov. Phys. Dokl. 26 (1981) 653–654.

[33] A. G. Izergin, V. E. Korepin, *Lattice versions of quantum field theory models in two dimensions*, Nucl. Phys. B, 205:3 (1982) 401–413.

[34] V. O. Tarasov, L. A. Takhtadzhyan, L. D. Faddeev, *Local Hamiltonians for integrable quantum models on a lattice*, Theor. Math. Phys., 57:2 (1983) 1059–1073

[35] N. M. Bogolyubov, V. E. Korepin, *Quantum nonlinear Schrödinger equation on a lattice*, Theor. Math. Phys., 66:3 (1986) 300–305.

[36] N. G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, Amsterdam, 1961.

[37] A. Erdélyi, *Asymptotic Expansions*, Dover Publication, New-York, 1956.

[38] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. Theory Exp. 0608 (2006) P08002, arXiv:math/0605015.