The dynamics of a cylinder on a vibrating plane with friction

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Abstract. In this work the motion of a cylinder undergoing harmonic forcing is presented. The interaction between the cylinder and the base is governed by Coulomb's law for sliding and rolling friction. The research aims at finding periodic motions of the system. The obtained motions are categorised and the characteristics of these motions are presented.

1. Introduction
In this paper we consider a cylinder on a harmonically moving horizontal plane with sliding and rolling friction. The research sets out to explore periodic motions of the cylinder.

The mechanical model used in this paper originates from a study by I. I. Blekhman and G. Yu. Dzhanelidze [1] where a particle on an inclined plane with Coulomb friction was considered. The results for the particle were applied by the authors [1] to describe the motion of a cylinder on an inclined plane with Coulomb sliding friction. The equations of motions of the cylinder can be decoupled in a way that one of them is independent from the other and has the same form as the equation of motion of a particle on an inclined plane with Coulomb friction.

The problem of the dynamics of a cylinder was modified in [2] by adding rolling friction. A qualitative description of the dynamics of a cylinder on an inclined plane performing vertical harmonic excitations was given [2]. In particular, it has been shown that periodic motions on a vertically moving plane do not exist except for the trivial case of relative rest.

The model considered in the current study naturally inherits some properties of the motion of a particle examined previously [1]. In particular, in section 2 some results obtained by the ‘reversed’ method in [1] are rewritten for a cylinder with sliding and rolling friction.

The main difficulty with these problems, if we use the equations of motion provided by classical theory, is the need to switch between the formulae governing the motion (the basic framework is described in [3]). The main tool used by I. I. Blekhman and G. Yu. Dzhanelidze [1] is the ‘reversed’ analytical method suitable for construction of solutions with periodic properties. Alternatively, such problems can be approached by computational methods. One can find more than one framework worked out to overcome difficulties in construction of the motion of the system. For example, in [4] an overview of modelling of dry friction phenomenon and unilateral constraints is made. In this paper a rather simple smoothing model [4] is used to obtain a qualitative description of the dynamics of the cylinder.
2. The mechanical model

The model to be analysed is sketched in figure 1.

![Figure 1. Sketch of cylinder on a moving horizontal base.](image)

The dynamically symmetric rigid cylinder of mass $m$, radius $r$ with moment of inertia $I$ about its axis is moving on a horizontal rigid surface.

We introduce two frames of reference: a reference frame fixed to the moving base $O\xi\eta$ and an inertial frame $1O\xi\eta$. The motion of the base is translational and harmonic and is given by

$$\sin \theta = A \sin \omega t$$

where $\theta = \theta$, $\omega = \omega$, $A = A$, and $\omega = \omega$.

The equations of motion of the cylinder in $O\xi\eta$ are

$$m \ddot{x} = -m \dot{x}^2 + F$$
$$N = mg$$
$$I \ddot{\theta} = Fr + M$$

where $x$ and $y$ are the coordinates of the center of the cylinder (which owing to dynamical symmetry coincides with its center of mass), $\theta$ is the angle of rotation of the cylinder, $F = F \cdot e_x$ and $N = N \cdot e_y$ are the projections of normal and tangential components of the force of constraint, $M = M \cdot e_x$, $e_x = e_x \times e_y$ is the projection of the torque of constraint.

The tangential component of the force of constraint is governed by Coulomb’s law

$$F = \begin{cases} -fN & \text{for } \dot{x}_{gl} > 0 \text{ or for } \dot{x}_{gl} = 0, \quad F^{(0)} \leq -fN \\ F^{(0)} & \text{for } \dot{x}_{gl} = 0, \quad F^{(0)} < fN \\ fN & \text{for } \dot{x}_{gl} < 0 \text{ or for } \dot{x}_{gl} = 0, \quad F^{(0)} \geq fN \end{cases}$$

where $\dot{x}_{gl}$ is the projection of the velocity of the contact point (the Ox components of coordinates of the contact point and of the center of the cylinder coincide, while the Ox components of their velocities do not), $f$ ($0 < f \leq 1$) is the coefficient of friction, $F^{(0)}$ is the projection of a friction force for ‘stick’ phase; $F^{(0)}$ can be obtained from the equations of motion (2) if we put $\dot{x}_{gl} = 0$.

The torques of the normal reaction and sliding friction are zero as both the cylinder and the base are assumed to be rigid. The rolling friction torque is given by

$$M = \begin{cases} -\mu rN & \text{for } \dot{\theta} > 0 \text{ or for } \dot{\theta} = 0, \quad M^{(0)} \leq -\mu rN \\ M^{(0)} & \text{for } \dot{\theta} = 0, \quad M^{(0)} \leq \mu rN \\ \mu rN & \text{for } \dot{\theta} < 0 \text{ or for } \dot{\theta} = 0, \quad M^{(0)} \geq \mu rN \end{cases}$$
where $\mu > 0$ is a coefficient of rolling friction, $M^{(0)}$ is the projection of rolling friction torque for ‘stick’ phase of rolling (rolling motion in this model also has a ‘stick’ phase, as well as sliding motion). $M^{(0)}$ may likewise be obtained from the equations of motion (2) under the assumption that $\dot{\theta} \equiv 0$. Relying on experimental evidence (see for example [5]) we consider $\mu \ll f$.

It was shown in [2], for a more general case, that sliding without rolling is impossible. True, from the equations of motion (2) it follows that if sliding occurred without rolling, then $M^{(0)} = fNr \gg \mu rN$.

Further we transition to dimensionless variables considering $m = 1$, $r = 1$, $g = \lvert g \rvert = 1$. The cylinder is always in contact with the base, thus the Euler formula $\dot{\theta} = \frac{F + M}{I}$ holds and the equations of motion (2) become

$$\begin{cases}
\dot{x}_{gl} = A\omega^2 \sin \omega t + F + \frac{F + M}{I} \\
\dot{\theta} = \frac{F + M}{I}
\end{cases}$$

(5)

Equations (5) can be analytically integrated. The obtained formulae for positions and velocities will govern the motion until either of the velocities $\dot{x}$ or $\dot{\theta}$ changes its sign. In particular, for sliding motion

$$\dot{x}_{gl} = -A\omega \left( \cos \omega t - \cos \omega t_0 \right) - \frac{(I + 1) f \text{sgn} \dot{x}_{gl} + \mu \text{sgn} \dot{\theta}}{I} (t - t_0) + \dot{x}_{gl}$$

$$\dot{\theta} = -\frac{f \text{sgn} \dot{x}_{gl} + \mu \text{sgn} \dot{\theta}}{I} (t - t_0) + \dot{\theta}_0$$

(6)

(7)

where $\dot{x}_0 = \dot{x}(t_0)$ and $\dot{\theta}_0 = \dot{\theta}(t_0)$ are the initial conditions.

For rolling without sliding

$$\dot{\theta} = -\frac{\mu \text{sgn} \dot{\theta}(t - t_0) - A\omega \left( \cos \omega t - \cos \omega t_0 \right)}{I + 1} + \dot{\theta}_0$$

(8)

Rolling without sliding is possible only if $\lvert F^{(0)} \rvert < fN$ for all $t \in [0, \frac{2\pi}{\omega})$. This yields the restriction $f (I + 1) - \mu > A\omega^2$.

It should be noted that for rolling without sliding the equations of motion (5) have the same form as for a point on a horizontally moving surface with dry friction, considered in [1], where an analytical treatment for that problem was provided. It directly follows from [1] that for

$$\mu < \frac{2A\omega^2}{(\pi^2 + 4)^{1/2}}$$

(9)

there exists a periodic motion with period $\frac{2\pi}{\omega}$ consisting of rolling back and forth with two instant stops at which the sign of angular velocity changes. For

$$\frac{2A\omega^2}{(\pi^2 + 4)^{1/2}} < \mu < A\omega^2$$

(10)

there exists a periodic rolling back and forth with period $\frac{2\pi}{\omega}$ with two long stops. Sticking occurs twice during the period. In [2] it was shown that if (10) is satisfied, then for all initial conditions the motion becomes periodic in finite time. Informally speaking, this happens because the two long stops help to adjust the period during transient stage.
It follows from (6), (7) and (8) in the general case of sliding and rolling that for every motion with periodic velocity there exists at least one instant of time when sliding stops and also at least one instant of time when rolling stops. If either of the velocities \( \dot{x}_{gl} \) or \( \dot{\theta} \) is nonzero during the whole period, its absolute value will grow infinitely beginning from some finite instant of time.

A motion with periodic velocity will also be periodic in terms of positions if

\[
\int_0^{2\pi/\omega} \dot{x}_{gl}(t) \, dt = 0, \quad \int_0^{2\pi/\omega} \dot{\theta}(t) \, dt = 0
\]

This means that for any periodic motion there are at least two instants of time when sliding stops and also at least two instants of time when rolling stops.

3. The smooth model

We introduce a smooth model following [4]. We suppose

\[
\text{sgn} \, x \approx \frac{2}{\pi} \arctan(\delta x), \quad \delta \gg 1
\]

as suggested in [4], and the equations of motion become smooth and can be numerically integrated without a special algorithm needed to switch between the governing equations.

Note that the equations of motion (5) do not include positions \( x \) and \( \theta \). We will search motions with periodic velocities \( \dot{x}_{gl} \) and \( \dot{\theta} \). Looking for motions with periodic velocities is easier because the order of the system is 2 while the order of the full system equals 4. Having obtained a motion with periodic velocities we can check the periodicity for positions later, if necessary, integrating the whole system and looking whether the path forms a cycle. We denote by \( \Delta \) the distance between starting and final points

\[
\Delta = \sqrt{(\gamma_{gl}(0) - \gamma_{gl}(2\pi/\omega))^2 + (\theta(0) - \theta(2\pi/\omega))^2}
\]

We search motions with periodic velocities for a uniform cylinder \((l = 1/2)\) on a grid with a fixed step (details are described in the appendix) in a bounded region of parameter space: \( A \in [0, 4], \omega \in [0, 4], f \in [0.1, 0.4], \mu \in [0.01, 0.04] \) and \( \mu \leq f \cdot 10^{-1} \). It should be noted that greater \( f \) required much computational time (possibly because it contributes to stiffness) so it was not considered.

The parameter of smoothed model \( \delta \) was equal to 1000. Computations with other values of \( \delta \) \((\delta = 950 \text{ and } \delta = 1050)\) were also performed to ensure stability of the model.

4. Results

We do not try to discuss in this paper suitability of smooth model to represent the exact model. We rather use the exact model as a tool for mechanical interpretation of the obtained results. Also we try to see where numerical errors distort the qualitative picture and why. All the obtained motions for the region of parameter space in which we do the search were plotted in order to make visual analysis possible. Some noticeable categories of the motions are presented below.
1. Motions with small amplitude of velocities, large period and characteristics resembling viscous behavior. An example is present in figure 2.

Figure 2. The motion integrated for $x_{gl} = 3.02 \times 10^{-7}$, $\dot{x}_{gl} = 1.34 \times 10^{-7}$, $x_{gl} = 0$, $\theta = 0$.

2. Sliding back and forth and at the same time rolling with phase shift (figure 3).

Figure 3. The motion integrated for $x_{gl} = -1.05 \times 10^{-4}$, $\dot{x}_{gl} = 0.17$, $x_{gl} = 0$, $\theta = 0$.

Here $\Delta = 10^{-9}$, thus this motion with periodic velocities may be considered periodic.

We can observe that there are intervals of rather fast growth of $\dot{x}_{gl}$ when the sign of rolling velocity changes (see figure 4).
Figure 4. The velocities during a period. Two intervals around $t_1 = 2.62$ and $t_2 = 8.80$ with almost a vertical slope of $\dot{x}_{gl}(t)$ are present.

The fast growth of $\dot{x}_{gl}$ can be explained by stiffness of the smooth model. It has been observed that making $\delta$ smaller for rolling friction while $\delta$ for sliding friction remains the same helps to reduce the growth of $\dot{x}$, whereas lessening of $\delta$ for sliding friction results in even faster growth of $\dot{x}$.

3. Sliding and rolling back and forth with two instant stops. This type of motion has a rather bad numeric behaviour, as both rolling and sliding friction have zero velocities simultaneously, each of them contributing to stiffness. Example is presented in figure 5.

Figure 5. The motion integrated for $\dot{x}_{gl} = -1.13$, $\dot{\theta} = 0.28$, $x_{gl} = 0$, $\theta = 0$.

Here $\Delta = 0.026$ and is visually detectable.
4. Sliding and rolling then only rolling and returning back. An example is given in figure 6.

![Figure 6](imageurl)

**Figure 6.** The motion integrated for $\dot{x}_{gl} = -3.8 \cdot 10^{-3}$, $\dot{\theta} = 0.68$, $x_{gl} = 0$, $\theta = 0$. In this case $\Delta = 0.003$.

Analysing the plots of all the found motions leads to conclusion that all the obtained solutions with periodic velocities seem to be fully periodic. Periodicity of motions of the third type is more questionable as $\Delta$ for the motions of the third type is greater. The influence of numeric error caused by stiffness is yet to be evaluated, for example by comparison to solutions obtained by switch algorithm (the construction of switch algorithms is described, for example in [4]).

5. Discussion

The presented problem can be developed further in several ways. The first is to make smoothing method more flexible for the particular problem by varying the coefficient $\delta$ depending on stiffness in the particular locations in the time-phase space, but the consistency needs to be checked by the instruments that are provided by exact model and probably by some other computational methods.

The second way of possible development is to construct a switch algorithm. In the current problem there are not so many modes to switch between, exactly 7, but the nonlinear equations from which switch instants are obtained have sinusoidal structure and need a care in choice of initial point. The equations are of Kepler’s form. For instance, in case of rolling without sliding the switch instant $t$ is obtained from the following equation

$$t_0 - \frac{A\omega}{\mu \text{sgn } \theta} \cos \omega t_0 + \frac{\dot{\theta}_0 (I + 1)}{\mu \text{sgn } \dot{\theta}} = t - \frac{A\omega}{\mu \text{sgn } \dot{\theta}} \cos \omega t$$

or if we put it down in Kepler’s form

$$\omega t_0 - \frac{\pi}{2} - \frac{A\omega^2 \text{sgn } \dot{\theta}}{\mu} \cos \omega t_0 + \frac{\omega \dot{\theta}_0 \text{sgn } \dot{\theta}(I + 1)}{\mu} = \omega t - \frac{\pi}{2} + \frac{A\omega^2 \text{sgn } \dot{\theta}}{\mu} \sin \left( \omega t - \frac{\pi}{2} \right)$$

In case of sliding motion, sliding stops at instant $t$ given by

$$t_0 + \frac{(A\omega \cos \omega t_0 + \dot{x}_{gl})I}{(I + 1) f \text{ sgn } \dot{x}_{gl} + \mu \text{sgn } \dot{\theta}} = t + \frac{A\omega I}{(I + 1) f \text{ sgn } \dot{x}_{gl} + \mu \text{sgn } \dot{\theta}} \cos \omega t$$

while rolling may change its direction at

$$t = t_0 + \frac{I \dot{\theta}_0}{f \text{ sgn } \dot{x}_{gl} + \mu \text{sgn } \dot{\theta}}$$

From (17) we may make a general conclusion concerning this model that during sliding in one direction rolling may change its sign only if $\dot{\theta}_0$ and $\dot{x}_{gl}$ have the same sign.
The numeric procedure solving nonlinear algebraic equations (15) and (16) is standard if
$$\frac{A\omega^2}{\mu} < 1$$

and
$$\frac{A\omega I}{(I + 1) f \text{sgn} \dot{x}_g + \mu \text{sgn} \dot{\theta}} < 1$$

for equations (15) and (16) respectively. An algorithm is described, for example in [6]. For other values of parameters the choice of initial point for a numeric procedure should be specified.

There is also another way to develop the problem moving away from the assumption of rigidity using a more complicated model of the interaction between the cylinder and the plane. We may consider a spacial region of contact, as opposed to a point (or rather a line, in this paper) and friction force depending not only on sliding velocity, but also on rolling velocity and the torque of rolling friction depending not only on rolling velocity, but on sliding velocity as well. A recent review of distributed dry friction models can be found in [7]. Examples of rolling friction problems with Kelvin–Voigt underlying surface are in [8–10] and with elastic surfaces in [11].

6. Conclusions
In this paper the existing analytical treatment of the problem [1, 2] was extended by numerical results. For a reduced system (with no sliding) analytical conditions of the existence of two types of periodic motions are given. For the first type only a necessary condition is given, for the second type the suggested parametric condition turns out to be sufficient.

The grid with fixed step in parameter space was constructed and Newton–Raphson procedure was applied to all the knots of the grid. The procedure yielded a number of motions with periodic velocities. The position variables for all of these motions strongly tend to be periodic as well, the precision being represented by $\Delta$ (there is some discussion concerning $\Delta$ in section 4). In order to gain a qualitative picture, all the obtained periodic motions were plotted and categorised (there are several hundred plots for the considered grid). The 4 qualitative categories are listed in section 4. The categorisation is visual, not analytical, so no formula governing parameters and initial conditions is suggested that would describe a domain for each type of motion. The correspondence between the periodic plots and the parameters can be done and some conclusions can be made, it is a matter of further work. In section 5 ways of further development for the problem are discussed.

Acknowledgements
The author is grateful to her science advisor A. V. Karapetyan for formulation of the problem, guidance on literature, valuable and constant help and advice.

The research was supported by the Russian Foundation for Basic Research (16-01-00338).

Appendix
In this section we give an outline of the algorithm used to obtain periodic solutions of the reduced system (integration of velocities, not positions).

In the bounded region of parameter space, described in section 3, a grid with the step equal to 0.5 for $A$ and $\omega$, equal to 0.1 for $f$ and to 0.01 for $\mu$ was taken. For each point on the grid periodic motions were searched by Newton–Raphson method. The considered initial values were $\dot{x}_{g0}, \dot{\theta}_0 \in [-3, 3]$ with the step equal to 0.5. The cycles with $\sqrt{\dot{x}^2_g(t) + \dot{\theta}^2(t)} < 10^{-2}$ were skipped. The obtained matrix of initial conditions for periodic solutions was reduced by eliminating neighbouring rows corresponding to too close initial points, i.e. if $\sqrt{(\dot{x}_{g0,i} - \dot{x}_{g0,i+1})^2 + (\dot{\theta}_{0,i} - \dot{\theta}_{0,i+1})^2} < 10^{-8}$ for $i$-th and $(i+1)$-th entries, only the first remains in the list. The search was performed using ode15s
MATLAB procedure and partly using ode45 function (it seems to suit better but requires much computational time for greater values of $f$).

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