Jack polynomials and free cumulants

Michel Lassalle
Centre National de la Recherche Scientifique
Institut Gaspard-Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France
lassalle@univ-mlv.fr
http://igm.univ-mlv.fr/~lassalle

Abstract

We study the coefficients in the expansion of Jack polynomials in terms of power
sums. We express them as polynomials in the free cumulants of the transition
measure of an anisotropic Young diagram. We conjecture that such polynomials
have nonnegative integer coefficients. This extends recent results about normalized
characters of the symmetric group.

2000 Mathematics Subject Classification: 05E05, 05E35, 33C52, 46L54.

1 Introduction

The following connection between probability theory and representations of symmetric
groups is due to Kerov [8] and Biane [1].

Let $S_n$ be the symmetric group of $n$ letters. Its irreducible representations are labelled
by partitions of $n$, i.e. weakly decreasing sequences $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers
summing to $n$. We denote $l(\lambda) = l$ and $|\lambda| = n$, the respective length and weight of $\lambda$.

Given a partition $\lambda$, Kerov [8] defined the “transition measure” of its Young diagram. The “free cumulants”
of this measure are real numbers \{$R_i(\lambda)$, $i \geq 2$\} defined in the framework of free probability [28, 26]. They were introduced by Biane [1] to solve asymptotic
problems in the representation theory of $S_n$.

Later Kerov [10] and Biane [2] observed that the character of the irreducible representation labelled by $\lambda$ may be written as a polynomial in the free cumulants $R_i(\lambda)$.

More precisely let $\dim \lambda$ be the dimension of this representation and $\chi_\rho^\lambda = \chi^\lambda(\sigma)$ the value of its character at any permutation $\sigma \in S_n$ with cycle-type $\rho$. Let $r \leq n$ be a positive integer and $\rho = (r, 1^{n-r})$ the corresponding $r$-cycle in $S_n$. Kerov [10] and Biane [2] proved
that the normalized character

$$\hat{\chi}_r^\lambda = n(n-1) \cdots (n-r+1) \frac{\chi^\lambda_{r,1^{n-r}}}{\dim \lambda}$$

is a polynomial in the free cumulants $R_i(\lambda)$, with integer coefficients. They conjectured these coefficients to be nonnegative.
This property was recently proved by Féray [5], who also extended it to
\[
\hat{\chi}_\mu^\lambda = n(n-1)\cdots(n-r+1)\frac{\chi_{\mu,1^{n-r}}^\lambda}{\dim \lambda},
\]
with \(\mu\) any partition of weight \(r\) (see also [4]). Previously, a method to compute \(\hat{\chi}_\mu^\lambda\) had been given in [7, 24], together with several explicit cases (see also [3]). Stronger conjectures (yet unproved) were formulated in [18].

The purpose of this paper is to present an extension of these results in the framework of Jack polynomials.

The family of Jack polynomials \(J_\lambda(\alpha)\) is indexed by partitions. It forms a basis of the algebra of symmetric functions with rational coefficients in some positive real parameter \(\alpha\). We consider the transition matrix between this basis and the classical basis of power sums \(p_\rho\), i.e.
\[
J_\lambda(\alpha) = \sum_\rho \theta_\rho^\lambda(\alpha) p_\rho.
\]
Let \(\mu\) be a partition with \(r = |\mu| \leq |\lambda| = n\). Using multiplicities, we write \(\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)\) and \(z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!\).

Firstly, we observe that the quantity
\[
\vartheta_\mu^\lambda(\alpha) = z_\mu \theta_\mu^\lambda,1^{n-r}(\alpha)
\]
is a natural generalization of the normalized character \(\hat{\chi}_\mu^\lambda\). This is a consequence of the Frobenius formula for the Schur functions \(s_\lambda\). Actually we have
\[
s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho \quad \text{and} \quad J_\lambda(1) = h_\lambda s_\lambda,
\]
where \(h_\lambda = n!/\dim \lambda\) is the hook-length of \(\lambda\) (see [20, Examples 1.1.1 and 1.7.6] and [27, p. 78]). This yields
\[
z_\rho \theta_\rho^\lambda(1) = n! \frac{\chi_\rho^\lambda}{\dim \lambda},
\]
and for \(\rho = (\mu, 1^{n-r})\),
\[
\hat{\chi}_\mu^\lambda = \left(\frac{n-r+m_1(\mu)}{m_1(\mu)}\right) \vartheta_\mu^\lambda(1).
\]
If \(\mu\) is chosen with no part 1, which is always possible, we thus have \(\hat{\chi}_\mu^\lambda = \vartheta_\mu^\lambda(1)\). In particular \(\hat{\chi}_r^\lambda = \vartheta_r^\lambda(1)\).

Secondly, expanding Jack polynomials in terms of the “augmented” monomial symmetric functions, which are integral combinations of the power sums [20, p. 110], and using [11, Theorem 1.1], it is known that \(\vartheta_\mu^\lambda(\alpha)\) is a polynomial in \(\alpha\) with integer coefficients.

In this paper we consider the notion of “anisotropic” Young diagram of a partition \(\lambda\), introduced by Kerov [9]. This diagram is obtained from the classical Young diagram of \(\lambda\) by the dilation \((u, v) \mapsto (u, v/\alpha)\).
We define the transition measure of this anisotropic Young diagram and the free cumulants \( \{ R_i(\lambda; \alpha), i \geq 1 \} \) of this measure. We show that \( -R_k(\lambda; \alpha) \) is a polynomial in \(-1/\alpha\) with nonnegative integer coefficients. For instance, one has

\[
R_1(\lambda; \alpha) = 0, \quad R_2(\lambda; \alpha) = |\lambda|/\alpha, \quad R_3(\lambda; \alpha) = \sum_i \lambda_i^2/\alpha - \sum_i (2i-1)\lambda_i/\alpha^2.
\]

Given a partition \( \mu \) having no part 1, we present a method to express \( \vartheta^1_\mu(\alpha) \) in terms of the free cumulants \( R_i(\lambda; \alpha) \). We show that there exists a polynomial \( K_\mu \) such that for any partition \( \lambda \) with \(|\lambda| \geq |\mu|\), one has

\[
\vartheta^1_\mu(\alpha) = K_\mu(R_2(\lambda; \alpha), R_3(\lambda; \alpha), \ldots, R_{|\mu|-t(\mu)+2}(\lambda; \alpha)).
\]

The coefficients of \( K_\mu \) in the free cumulants are only known to be in \( \mathbb{Q}(\alpha) \), the field of rational functions in \( \alpha \). However applying our algorithm, extensive computer calculations support the following conjecture.

**Conjecture 1.1.** The coefficients of \( K_\mu \) are polynomials in \( \alpha \) with integer coefficients.

Following Féray [5], given a family of polynomials \( K_\mu \), we can inductively define another family \( \tilde{K}_\mu \) by

\[
K_\mu = \sum_{k=1}^{l(\mu)} (-1)^{l(\mu)-k} \sum_{(\nu_1, \ldots, \nu_k)} \prod_{i=1}^k \tilde{K}_{\nu_i},
\]

where the second sum is taken over all decompositions of the \( l(\mu) \) parts of \( \mu \) into \( k \) disjoint partitions \((\nu_1, \ldots, \nu_k)\). For instance one has \( K_r = \tilde{K}_r, K_{r,s} = K_rK_s - \tilde{K}_{r,s} \) and \( K_{r,s,t} = \tilde{K}_rK_sK_t - \tilde{K}_r\tilde{K}_{s,t} - \tilde{K}_s\tilde{K}_{r,t} - \tilde{K}_r\tilde{K}_{r,s} + \tilde{K}_{r,s,t} \).

Our computations support the following positivity conjecture.

**Conjecture 1.2.** Denote \( \beta = 1 - \alpha \). There is a “natural” expression of the coefficients of \( \tilde{K}_\mu \) as polynomials in \((\alpha, \beta)\) with nonnegative integer coefficients.

In particular it is the case for \( K_r = \tilde{K}_r \). Such a “natural” method to express \( \tilde{K}_\mu \) will be presented in the course of this paper.

Writing \( R_i \) instead of \( R_i(\lambda; \alpha) \) for clarity of display, the values obtained for \( K_r, r \leq 6 \) are as follows:

\[
\begin{align*}
K_2 & = \alpha^2 R_3 + \alpha\beta R_2, \\
K_3 & = \alpha^3 R_4 + 3\alpha^2\beta R_3 + (\alpha^2 + 2\alpha\beta^2)R_2, \\
K_4 & = \alpha^4 R_5 + \alpha^3\beta(6R_4 + R_2^2) + (5\alpha^3 + 11\alpha^2\beta^2)R_3 + (7\alpha^2\beta + 6\alpha\beta^3)R_2, \\
K_5 & = \alpha^5 R_6 + \alpha^4\beta(10R_5 + 5R_3R_2) + \alpha^4(15R_4 + 5R_2^2) + \alpha^3\beta^2(35R_4 + 10R_2^2) \\
& + (55\alpha^3\beta + 50\alpha^2\beta^3)R_3 + (8\alpha^3 + 46\alpha^2\beta^2 + 24\alpha\beta^4)R_2, \\
K_6 & = \alpha^6 R_7 + \alpha^5\beta(15R_6 + 9R_4R_2 + 6R_2^2 + R_3^2) + \alpha^5(35R_5 + 35R_3R_2) \\
& + \alpha^4\beta^2(85R_5 + 73R_3R_2) + \alpha^4\beta(238R_4 + 96R_2^2) + \alpha^3\beta^3(225R_4 + 84R_2^2) \\
& + (84\alpha^4 + 505\alpha^3\beta^2 + 274\alpha^2\beta^4)R_3 + (144\alpha^3\beta + 326\alpha^2\beta^3 + 120\alpha\beta^5)R_2.
\end{align*}
\]
The values of $\tilde{K}_\mu, |\mu| \leq 6$ are as follows:

$$\tilde{K}_{2,2} = \alpha^3(4R_4 + 2R_2^2) + 10\alpha^2\beta R_3 + (2\alpha^2 + 6\alpha\beta^2)R_2,$$

$$\tilde{K}_{3,2} = \alpha^4(6R_5 + 6R_3R_2) + \alpha^3\beta(30R_4 + 12R_2^2) + (18\alpha^3 + 48\alpha^2\beta^2)R_3$$

$$+ (24\alpha^2\beta + 24\alpha\beta^3)R_2,$$

$$\tilde{K}_{4,2} = \alpha^5(8R_6 + 8R_4R_2 + 4R_2^3) + \alpha^4\beta(68R_5 + 72R_3R_2) + \alpha^4(80R_4 + 40R_2^2)$$

$$+ \alpha^3\beta(208R_4 + 88R_2^2) + (268\alpha^3\beta + 268\alpha^2\beta^2)R_3 + (32\alpha^3 + 212\alpha^2\beta^2 + 120\alpha\beta^4)R_2,$$

$$\tilde{K}_{3,3} = \alpha^5(9R_6 + 9R_4R_2 + 9R_3^2 + 3R_2^3) + \alpha^4\beta(72R_5 + 81R_3R_2) + \alpha^4(75R_4 + 27R_2^2)$$

$$+ \alpha^3\beta(213R_4 + 90R_2^2) + (261\alpha^3\beta + 270\alpha^2\beta^2)R_3 + (36\alpha^3 + 210\alpha^2\beta^2 + 120\alpha\beta^4)R_2,$$

$$\tilde{K}_{2,2,2} = \alpha^4(40R_5 + 64R_3R_2) + \alpha^3\beta(176R_4 + 96R_2^2) + (80\alpha^3 + 256\alpha^2\beta^2)R_3$$

$$+ (104\alpha^2\beta + 120\alpha\beta^3)R_2.$$
compute $\psi^\alpha_\mu(\alpha)$ in terms of the free cumulants is presented in Sections 7 to 9. Many conjectures about the polynomials $K_\alpha$ are given in Sections 10-11. Two other bases related with free cumulants are introduced in Sections 12-13, and the corresponding expansions of $K_\alpha$ are considered. Section 14 extends the previous conjectures to $\hat{K}_\mu$. A two-parameter $(\alpha, \beta)$-generalization of our results is presented in Section 15.

2 Generalities

The standard references for symmetric functions and Jack polynomials are [20, Section 6.10] and [27].

2.1 Symmetric functions

A partition $\lambda = (\lambda_1, ..., \lambda_l)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of $\lambda$, and $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$ the weight of $\lambda$.

For any integer $i \geq 1$, $m_i(\lambda) = \text{card}\{j : \lambda_j = i\}$ is the multiplicity of the part $i$ in $\lambda$. Clearly $l(\lambda) = \sum_i m_i(\lambda)$ and $|\lambda| = \sum_i im_i(\lambda)$. We shall also write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$ and set

$$z_\lambda = \prod_{i \geq 1} t^{m_i(\lambda)} m_i(\lambda)!; \quad u_\lambda = l(\lambda)! / \prod_{i \geq 1} m_i(\lambda)!.$$ We identify $\lambda$ with its Ferrers diagram $\{(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. We denote $\lambda'$ the partition conjugate to $\lambda$. We have $m_i(\lambda') = \lambda_i - \lambda_{i+1}$, and $\lambda'_i = \sum_{j \geq i} m_j(\lambda)$.

Let $A = \{a_1, a_2, a_3, \ldots\}$ be a (possibly infinite) set of independent indeterminates (an alphabet) and $S[A]$ be the corresponding algebra of symmetric functions with coefficients in $Q$. The power sum symmetric functions are defined by $p_k(A) = \sum_{i \geq 1} a_i^k$. Elementary and complete symmetric functions $e_k(A)$ and $h_k(A)$ are defined by their generating functions

$$E_t(A) = \prod_{i \geq 1} (1 + ta_i) = \sum_{k \geq 0} t^k e_k(A), \quad H_t(A) = \prod_{i \geq 1} \frac{1}{1 - ta_i} = \sum_{k \geq 0} t^k h_k(A).$$

If $A$ is infinite, each of these three sets forms an algebraic basis of $S[A]$, which can thus be viewed as an abstract algebra $S$ generated by functions $e_k$, $h_k$ or $p_k$. For any partition $\mu$, the symmetric functions $e_\mu$, $h_\mu$ and $p_\mu$ defined by

$$f_\mu = \prod_{i=1}^{l(\mu)} f_{\mu_i} = \prod_{i \geq 1} t^{m_i(\mu)},$$

where $f_i$ stands for $e_i$, $h_i$ or $p_i$ respectively, form a linear basis of $S$. Another classical basis is formed by the monomial symmetric functions $m_\mu$, defined as the sum of all distinct monomials whose exponent is a permutation of $\mu$. 

5
2.2 Shifted symmetric functions

Although the theory of symmetric functions goes back to the early 19th century, the notion of “shifted symmetric” functions is quite recent. We refer to [21, 22, 23] and to other references given there.

Let $Q(\alpha)$ be the field of rational functions in some indeterminate $\alpha$ (which may be considered as a positive real number). A polynomial in $N$ indeterminates $x = (x_1, \ldots, x_N)$ with coefficients in $Q(\alpha)$ is said to be “shifted symmetric” if it is symmetric in the $N$ “shifted variables” $x_i - i/\alpha$.

Dealing with an infinite set of indeterminates $x = \{x_1, x_2, \ldots\}$, in analogy with symmetric functions, a “shifted symmetric function” $f$ is a family $\{f_i, i \geq 1\}$ such that $f_i$ is shifted symmetric in $(x_1, x_2, \ldots, x_i)$, with the stability property $f_{i+1}(x_1, x_2, \ldots, x_i, 0) = f_i(x_1, x_2, \ldots, x_i)$.

This defines $S^*$, the shifted symmetric algebra with coefficients in $Q(\alpha)$. An element $f \in S^*$ may be evaluated at any sequence $(x_1, x_2, \ldots)$ with finitely many non-zero terms, hence at any partition $\lambda$. Moreover by analyticity, $f$ is entirely determined by its restriction $f(\lambda)$ to partitions. Therefore $S^*$ is usually considered as a function algebra on the set of partitions.

An algebraic basis of $S^*$ is obtained as follows [15, Section 7, p. 3467]. Given a partition $\lambda$, the “$\alpha$-content” of any node $(i, j) \in \lambda$ is defined as $j - 1 - (i - 1)/\alpha$. Consider the finite alphabet of $\alpha$-contents 

$$C_\lambda = \{j - 1 - (i - 1)/\alpha, (i, j) \in \lambda\}.$$  

The symmetric algebra $S[C_\lambda]$ is generated by the power sums

$$p_k(C_\lambda) = \sum_{(i, j) \in \lambda} (j - 1 - (i - 1)/\alpha)^k \quad (k \geq 1). \quad (2.1)$$

It is known [15, Lemma 7.1, p. 3467] that the quantities $p_k(C_\lambda)$ are shifted symmetric functions of $\lambda$. As a direct consequence, the shifted symmetric algebra $S^*$ is algebraically generated by the functions $\{p_k(C_\lambda), k \geq 1\}$ together with $p_0(C_\lambda) = |\lambda|$, the cardinal of the alphabet $C_\lambda$.

3 Jack polynomials

3.1 Notations

Let $S = S \otimes Q(\alpha)$ be the algebra of symmetric functions with coefficients in $Q(\alpha)$. The parameter $\alpha$ being kept fixed, for clarity of display, we shall omit its dependence in any notation below.

The algebra $S$ may be endowed with a scalar product $\langle , \rangle$ for which we have two orthogonal bases, both indexed by partitions:
(i) the basis of power sum symmetric functions, with
\[ < p_\lambda, p_\mu >= \delta_{\lambda \mu} \alpha^{l(\lambda)} z_\lambda, \]

(ii) the basis of (suitably normalized) Jack symmetric functions, with
\[ < J_\lambda, J_\mu >= \delta_{\lambda \mu} h_\lambda h'_\mu, \]

and
\[ h_\lambda = \prod_{(i,j) \in \lambda} (\lambda'_j - i + 1 + \alpha(\lambda_i - j)), \quad h'_\lambda = \prod_{(i,j) \in \lambda} (\lambda'_j - i + \alpha(\lambda_i - j + 1)). \]

We write \( \theta^\lambda_\rho \) for the transition matrix between these two orthogonal bases, namely
\[ J_\lambda = \sum_{|\rho|=|\lambda|} \theta^\lambda_\rho p_\rho. \] (3.1)

Let \( \mu \) be a partition with weight \( r = |\mu| \leq |\lambda| = n \) and \((\mu, 1^{n-r})\) the partition obtained by adding \( n-r \) parts 1. We define
\[ \vartheta^\lambda_\mu = z_\mu \theta^\lambda_{\mu, 1^{n-r}}. \]

It is known [17, Proposition 2] that \( \vartheta^\lambda_\mu \) is a shifted symmetric function of \( \lambda \).

By restriction of Jack symmetric functions to a finite alphabet \( x = (x_1, \ldots, x_N) \) we obtain Jack polynomials, which are eigenfunctions of the differential operator
\[ D_2 = \sum_{i=1}^N x_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i,j=1 \atop i \neq j}^N \frac{x_i^2}{x_i-x_j} \frac{\partial}{\partial x_i}. \]

We have [27, p. 84]
\[ D_2 J_\lambda = 2(p_1(C_\lambda) + |\lambda|(N-1)/\alpha) J_\lambda. \] (3.2)

### 3.2 Pieri formula

For any partition \( \lambda \) and any integer \( 1 \leq i \leq l(\lambda) + 1 \), we denote by \( \lambda^{(i)} \) the partition \( \nu \) (if it exists) obtained by adding a node on the row \( i \) of the diagram of \( \lambda \), i.e. \( \nu_j = \lambda_j \) for \( j \neq i \) and \( \nu_i = \lambda_i + 1 \).

Given a partition \( \mu \), for any integer \( r \geq 1 \) we denote by \( \mu \cup r \) the partition obtained by adding a part \( r \), and \( \mu \setminus r \) the partition (if it exists) obtained by subtracting \( r \). We write \( \mu_{1(r)} = \mu \setminus r \cup (r-1) \) and \( \mu_{1^{(r)}} = \mu \setminus r \cup (r+1) \).

Jack symmetric functions satisfy the following generalization of the Pieri formula [20, 27]:
\[ p_1 J_\lambda = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) J_{\lambda^{(i)}}. \] (3.3)
The Pieri coefficients $c_i(\lambda)$ have the following analytic expression [14]:

$$c_i(\lambda) = \frac{1}{\alpha \lambda_i + l(\lambda) - i + 2} \prod_{j=1}^{l(\lambda)+1} \frac{\alpha(\lambda_i - \lambda_j) + j - i + 1}{\alpha(\lambda_i - \lambda_j) + j - i}. $$

The differential operator

$$E_2 = \sum_{i=1}^{N} x_i^2 \frac{\partial}{\partial x_i}$$

is independent of $N$. Denoting $p_1$ the multiplication operator $f \rightarrow p_1 f$, we have easily

$$E_2 = \frac{1}{2}[D_2, p_1] - \frac{1}{\alpha} (N - 1)p_1.$$ 

Due to (3.2) this implies

$$E_2 J_\lambda = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) (\lambda_i - (i-1)/\alpha) J_{\lambda(i)}. \quad (3.4)$$

We shall need the following equivalent form of equations (3.3)-(3.4). Expanding Jack polynomials in terms of power sums by (3.1), we obtain

$$\theta^\lambda_{p_1(1)} = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \theta^\lambda_{\rho(i)},$$

$$\sum_{r \geq 1} r (m_r(\rho) + 1) \theta^\lambda_{p_1(r+1)} = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) (\lambda_i - (i-1)/\alpha) \theta^\lambda_{\rho(i)}. \quad (3.5)$$

The first relation is obvious. The second is a direct consequence of

$$E_2 p_\rho = \sum_{r \geq 1} r m_r(\rho) p_{p_1(r)},$$

and the fact that if $\sigma = p_{1(r)}$, then $\rho = \sigma_{1(r+1)}$.

**Theorem 3.1.** *Up to the multiplicative constant $\theta^1_{(1)} = 1$, the system (3.5) totally determines the family of Jack polynomials $J_\lambda$.*

**Proof.** Starting from the initial case $\lambda = \rho = (1)$, we compute any $\theta^\lambda_{\rho}$ by two inductions, firstly on the length $l(\lambda)$, secondly on the value of the last part $\lambda_{l(\lambda)}$.

Our inductive assumption is the following: given a length $d \geq 1$, the quantity $\theta^\lambda_{\rho}$ is known for any $\lambda = (\lambda_1, \ldots, \lambda_{d-1}, \lambda_d)$ with $\lambda_d \leq u < \lambda_{d-1}$ and any $\rho$ with $|\rho| = |\lambda|$.

If we write equations (3.5) for $\lambda = (\lambda_1, \ldots, \lambda_{d-1}, u)$ and $|\rho| = |\lambda| + 1$, we obtain

$$c_d(\lambda) \theta^\lambda_{\rho(\lambda_1, \ldots, \lambda_{d-1}, u+1)} + c_{d+1}(\lambda) \theta^\lambda_{\rho(\lambda_1, \ldots, \lambda_{d-1}, u, 1)} = \cdots,$$

$$c_d(\lambda) \left(u - (d-1)/\alpha\right) \theta^\lambda_{\rho(\lambda_1, \ldots, \lambda_{d-1}, u+1)} + c_{d+1}(\lambda) \left(-d/\alpha\right) \theta^\lambda_{\rho(\lambda_1, \ldots, \lambda_{d-1}, u, 1)} = \cdots.$$
where both right-hand sides \([\cdots]\) are known in view of the inductive hypothesis. Actually they are linear combinations firstly of quantities \(\theta^\lambda_{\rho_{i(r)}}\) and secondly of quantities \(\theta^\lambda_{\rho} \) with \(\lambda^{(i)} = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_{d-1}, u)\).

This linear system has a discriminant equal to \(c_d(\lambda)c_{d+1}(\lambda)(u + 1/\alpha)\), which is \(\neq 0\) due to \(u < \lambda_{d-1}\). Hence the system has two unique solutions. In other words \(\theta^\lambda_{\rho}\) is known for \(\lambda = (\lambda_1, \ldots, \lambda_{d-1}, u + 1)\) and \(\lambda = (\lambda_1, \ldots, \lambda_{d-1}, u, 1)\).

This proves the second inductive property (on the value of \(\lambda_d\)), and initiates a new induction for the length \(l(\lambda) = d + 1\) with \(\lambda_{d+1} = 1\). Hence the statement. \(\square\)

Writing \(\rho = (\mu, 1^{n-r})\) the system (3.5) can be easily translated in terms of the quantities \(\vartheta^\lambda_{\mu}\). For any partition \(\mu\) having no part 1, we obtain
\[
\begin{align*}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \vartheta^\lambda_{\mu}^{(i)} &= \vartheta^\lambda_{\mu}, \\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) (\lambda_i - (i - 1)/\alpha) \vartheta^\lambda_{\mu}^{(i)} &= 2(|\lambda| - |\mu| + 2) m_2(\mu) \vartheta^\lambda_{\mu, \lambda} + \sum_{r \geq 3} r m_r(\mu) \vartheta^\lambda_{\mu_{1(r)}}.
\end{align*}
\]

In the right-hand side the case \(r = 2\) must be distinguished because the partition \(\mu_{1(2)}\) has one part 1.

The system (3.6) is our starting point for the computation of \(\vartheta^\lambda_{\mu}\).

4 Lagrange involution

4.1 Multiplication of alphabets

Given two alphabets \(A\) and \(B\), their sum \(A + B\) and their difference \(A - B\) are defined by the generating functions
\[
H_t(A + B) = H_t(A) H_t(B), \quad H_t(A - B) = H_t(A)/H_t(B).
\]

Equivalently
\[
h_n(A + B) = \sum_{k=0}^{n} h_k(A) h_{n-k}(B). \tag{4.1}
\]

Given a positive integer \(x\), the multiple \(xA\) is the alphabet formed by \(x\) copies of \(A\). Obviously we have
\[
p_n(xA) = xp_n(A), \quad p_\mu(xA) = x^{l(\mu)} p_\mu(A).
\]

And the generating functions of this alphabet
\[
E_t(xA) = E_t(A)^x, \quad H_t(xA) = H_t(A)^x \tag{4.2}
\]
are the product of \(x\) copies of \(E_t(A)\) or \(H_t(A)\).
Both properties are chosen as a definition of the “multiple” $xA$ when $x$ is any real number. The symmetric functions of $xA$ are defined accordingly.

In this framework we have two important “Cauchy formulas” [12, Section 1.6], see also [20, pp. 62-65] or [13, p. 222]. Firstly

$$h_n(xA) = \sum_{|\mu| = n} z_{\mu}^{-1} x^{|\mu|} p_{\mu}(A),$$

$$e_n(xA) = \sum_{|\mu| = n} (-1)^{n-|\mu|} z_{\mu}^{-1} x^{|\mu|} p_{\mu}(A).$$

Secondly

$$h_n(xA) = \sum_{|\mu| = n} \left( \frac{x}{l(\mu)} \right) u_{\mu} h_{\mu}(A)$$

$$= (-1)^n \sum_{|\mu| = n} \left( -\frac{x}{l(\mu)} \right) u_{\mu} e_{\mu}(A).$$

Equivalently

$$e_n(xA) = \sum_{|\mu| = n} \left( \frac{x}{l(\mu)} \right) u_{\mu} e_{\mu}(A)$$

$$= (-1)^n \sum_{|\mu| = n} \left( -\frac{x}{l(\mu)} \right) u_{\mu} h_{\mu}(A).$$

In this paper the binomial notation $\binom{x}{k}$ will always stand for $x(x-1) \cdots (x-k+1)/k!$. Multiplication of alphabets and Cauchy formulas are better understood in the language of $\lambda$-rings. See [12, Chapter 2] (or [13, Section 3] for an elementary account) and also Section 13 below.

### 4.2 Lagrange formula

Given an alphabet $A$, an involution $f \rightarrow f^*$ can be defined on $\mathbb{S}[A]$ as follows ([20, Example 1.2.24, p. 35], [12, Section 2.4]). Let

$$u = t H_t(A) = \sum_{k \geq 0} t^{k+1} h_k(A).$$

Then $t$ can be expressed as a power series in $u$, its compositional inverse, namely

$$t = u H_u^*(A) = \sum_{k \geq 0} u^{k+1} h_k^*(A).$$

The map $h_k(A) \rightarrow h_k^*(A)$ extends to an involution of $\mathbb{S}[A]$, called “Lagrange involution”.

10
The Lagrange inversion formula asserts, in one of its equivalent forms [6], that
\[
[u^n]H_k^*(A)^k = \frac{k}{n+k}[t^n]H_t(A)^{-n-k}, \tag{4.6}
\]
where \([t^n]F_t\) denotes the coefficient of \(t^n\) in the formal series \(F_t\). This fundamental result has many consequences, listed below.

### 4.3 Consequences

Firstly omitting to mention the alphabet \(A\), we have

\[
(n+1)h_n^* = \sum_{|\mu|=n} (-n-1)^{l(\mu)} z_{\mu}^{-1} p_{\mu}
= \sum_{|\mu|=n} \left(-n-1\right) \left(l(\mu)\right) u_{\mu}h_{\mu}, \tag{4.7}
\]

\[
= (-1)^n \sum_{|\mu|=n} \left(n+1\right) \left(l(\mu)\right) u_{\mu}e_{\mu}.
\]

Secondly

\[
(n-1)e_n^* = (-1)^{n-1} \sum_{|\mu|=n} (1-n)^{l(\mu)} z_{\mu}^{-1} p_{\mu}
= - \sum_{|\mu|=n} \left(n-1\right) \left(l(\mu)\right) u_{\mu}e_{\mu}, \tag{4.8}
\]

\[
= (-1)^{n-1} \sum_{|\mu|=n} \left(-n+1\right) \left(l(\mu)\right) u_{\mu}h_{\mu}.
\]

Finally we have

\[
p_n^* = \sum_{|\mu|=n} (-n)^{l(\mu)} z_{\mu}^{-1} p_{\mu}. \tag{4.9}
\]

This formula exhibits an important connection between Lagrange involution and multiplication of alphabets. By the classical Cauchy formulas

\[
h_n = \sum_{|\mu|=n} z_{\mu}^{-1} p_{\mu}, \quad e_n = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_{\mu}^{-1} p_{\mu},
\]

it reads

\[
p_n^*(A) = \sum_{|\mu|=n} z_{\mu}^{-1} p_{\mu}(-nA) = h_n(-nA) = (-1)^ne_n(nA).
\]

Similarly from (4.7) and (4.8) we obtain

\[
(n+1)h_n^*(A) = h_n((-n-1)A) = (-1)^n e_n((n+1)A),
(n-1)e_n^*(A) = (-1)^{n-1} h_n((1-n)A) = e_n((n-1)A). \tag{4.10}
\]
These properties may be extended as follows. For any real number \(x\) and any alphabet \(A\), Lagrange formula (4.6) yields

\[
(n + x)h_n(xA) = x \sum_{|\mu|=n} \left( \frac{-n - x}{l(\mu)} \right) u_\mu h^*_\mu(A) = (-1)^n x \sum_{|\mu|=n} \left( \frac{n + x}{l(\mu)} \right) u_\mu e^*_\mu(A).
\] (4.11)

Similarly

\[
(n - x)e_n(xA) = -x \sum_{|\mu|=n} \left( \frac{n - x}{l(\mu)} \right) u_\mu e^*_\mu(A) = (-1)^{n-1} x \sum_{|\mu|=n} \left( \frac{-n + x}{l(\mu)} \right) u_\mu h^*_\mu(A).
\] (4.12)

From which we obtain

\[
(n + x)h^*_n(xA) = x h_n((-n - x)A) = (-1)^n x e_n((n + x)A),
\]

\[
(n - x)e^*_n(xA) = (-1)^{n-1} x h_n((x - n)A) = -x e_n((n - x)A).
\] (4.13)

**Proof.** Relations (4.7)-(4.10) are obtained from (4.11)-(4.13) by specializing \(x = 1\). The proofs of the latter are strictly parallel consequences of the Lagrange inversion formula (4.6). As an example let us prove the second part of (4.11). We have

\[
h^*_n(xA) = [u^n]H^*_u(A)^x = \frac{x}{n + x} [t^n]H_t(A)^{-n-x} = \frac{x}{n + x} [t^n]\left(1 + \tilde{E}_{-t}(A)\right)^{n+x} = \frac{x}{n + x} [t^n]\left(\sum_{d \geq 0} \binom{n + x}{d} \tilde{E}_{-t}(A)^d\right),
\]

with \(\tilde{E}_{-t} = \sum_{k \geq 1} (-t)^k e_k\). But

\[
[t^n]\tilde{E}_{-t}(A)^d = (-1)^n \sum_{|\mu|=n \atop l(\mu)=d} u_\mu e_\mu(A).
\]

\(\square\)

### 4.4 Some identities

For any partition \(\rho\) define

\[
v_\rho = \prod_{i \geq 1} (i - 1)^{m_i(\rho)}, \quad w_\rho = v_\rho \sum_{i \geq 2} \frac{im_i(\rho)}{i - 1}.
\]

12
For any alphabet \( A \) and any real number \( z \), we have the remarkable identities

\[
\sum_{k=0}^{n} \frac{z}{z + k} h_k(-(z + k)A)h_{n-k}((z + k - 1)A) = (-1)^n e_n(A),
\]
\[
\sum_{k=0}^{n} \frac{1}{z + k} h_k(-(z + k)A)h_{n-k}((z + k)A) = 0,
\]
\[
\sum_{k=0}^{n} h_k(-(z + k)A)h_{n-k}((z + k + 1)A) = (-1)^n \sum_{|\rho|=n} v_{\rho} u_{\rho} e_{\rho}(A),
\]
\[
\sum_{k=0}^{n} h_k(-(z + k)A)h_{n-k}((z + k)A) = (-1)^n \sum_{|\rho|=n} w_{\rho} \frac{u_{\rho}}{l(\rho)} e_{\rho}(A).
\]
(4.14)

The assumption \( e_1(A) = 0 \) is needed for the last one. Observe that the right-hand sides are independent of \( z \).

Although these identities have deep connections with \( \lambda \)-ring theory, an elementary proof may be given as follows. Expand each \( h_m(xA) \) in terms of the basis \( e_{\rho}(A) \) by the Cauchy formula (4.4). Then we are left with a relation involving binomial coefficients. For instance the third identity amounts to write

\[
\sum_{\mu \cup \nu = \rho} u_{\mu} u_{\nu} \left( |\mu| t + z \right) \left( -|\mu| t - z - 1 \right) \left( \frac{1}{l(\mu)} \frac{1}{l(\nu)} \right) = u_{\rho} \prod_{i \geq 1} (ti - 1)^{m_i(\rho)},
\]

at \( t = 1 \), the sum taken over all decompositions of \( \rho \) into two partitions (possibly empty).

This result is a consequence of a more general property, obtained by replacing the parts of \( \rho \) by any alphabet \( A \), namely

\[
\sum_{X \subseteq A} (-1)^{n - \text{card}X} \prod_{\text{i}=1}^{n} (|X| t - \text{card}X + z + i) = n! \prod_{a \in A} (ta - 1),
\]

with \( n = \text{card}A \) and \( |X| = \sum_{a \in X} a \). Omitting details, the other identities are implied by

\[
\sum_{X \subseteq A} (-1)^{n - \text{card}X} \prod_{\text{i}=1}^{n-2} (|X| t - \text{card}X + z + i) = 0 \quad (n \neq 1),
\]
\[
\sum_{X \subseteq A} (-1)^{n - \text{card}X} \prod_{\text{i}=1}^{n-m} (|X| t - \text{card}X + z + i) = 0 \quad (m = 1, 2),
\]
\[
\sum_{X \subseteq A} (-1)^{n - \text{card}X} \prod_{\text{i}=1}^{n-1} (|X| t - \text{card}X + z + i) = (n - 1)! \frac{d}{dt} \prod_{a \in A} (ta - 1).
\]

Four similar identities may be obtained by applying Lagrange involution to (4.14) and
using \((4.13)\), namely

\[
\sum_{k=0}^{n} \frac{z+k-1}{z+n-1} h_k(zA) h_{n-k}(-z+n+1) A = (-1)^n e_n^*(A),
\]

\[
\sum_{k=0}^{n} \frac{(z+k)(z+k+1)}{z(z+n+1)} h_k(zA) h_{n-k}(-z+n+1) A = 0,
\]

\[
\sum_{k=0}^{n} \frac{(z+k)^2}{z(z+n)} h_k(zA) h_{n-k}(-z+n) A = (-1)^n \sum_{|\rho|=n} w_{\rho} e_\rho^*(A),
\]

(4.15)

with \(e_1(A) = 0\) assumed for the last one.

\section{Free cumulants}

The following notions are due to Kerov \([8, 9]\). Two increasing sequences \(x_1, \ldots, x_{d-1}, x_d\) and \(y_1, \ldots, y_{d-1}\) with \(x_1 < y_1 < x_2 < \cdots < x_{d-1} < y_{d-1} < x_d\) are said to form a pair of interlacing sequences. The center of this pair is \(\sum_{i=1}^{d} x_i - \sum_{i=1}^{d} y_i\).

Given a partition \(\lambda\), its Young diagram \(Y_\lambda \subset \mathbb{R}^2\) is defined as the collection of unit boxes centered on the nodes \((j-1/2, i-1/2), (i, j) \in \lambda\). An outside corner of \(Y_\lambda\) is defined as the north-east corner of a corner box. An inside corner is the south-west corner of a corner box of the complement of \(Y_\lambda\) in \(\mathbb{R}^2\).

On \(\mathbb{R}^2\) define the content function \(c(u, v) = u - v\). Then it is easily checked that a pair of interlacing sequences is formed by the contents \(x_1, \ldots, x_d\) of inside corners of \(Y_\lambda\), and the contents \(y_1, \ldots, y_{d-1}\) of its outside corners. This pair has center 0, and we have \(x_1 = -l(\lambda)\) and \(x_{d-1} = \lambda_1\).

The collection of \(1 \times 1/\alpha\) boxes obtained from \(Y_\lambda\) by the dilation \((u, v) \rightarrow (u, v/\alpha)\) is called the \textit{anisotropic} Young diagram \(Y_\lambda(\alpha)\). Its corners define similarly two interlacing sequences \(x_1(\alpha), \ldots, x_d(\alpha)\) and \(y_1(\alpha), \ldots, y_{d-1}(\alpha)\). Defining the \(\alpha\)-content function by \(c(u, v) = u - v/\alpha\), these sequences may also be understood as the \(\alpha\)-contents of inside and outside corners of \(Y_\lambda\).

For instance if \(\lambda = (4, 3, 3, 3, 1)\), the \(x\) and \(y\) sequences are respectively \((-5, -3, 2, 4)\) and \((-4, -1, 3)\). The \(x(\alpha)\) and \(y(\alpha)\) sequences are respectively \((-5/\alpha, -4/\alpha + 1, -1/\alpha + 3, 4)\) and \((-5/\alpha + 1, -4/\alpha + 3, -1/\alpha + 4)\).

We have \(x_1(\alpha) = -l(\lambda)/\alpha\) and \(x_{d-1}(\alpha) = \lambda_1\). Moreover we may assume \(d = \lambda_1 + 1\) and for \(1 \leq k \leq \lambda_1 = d - 1\) we may write

\[
x_k(\alpha) = k - 1 - \lambda'_k/\alpha, \quad y_k(\alpha) = k - \lambda'_k/\alpha,
\]

(5.1)

\textit{provided we make the convention} that \(x_i\) and \(y_{i-1}\) should be omitted whenever \(x_i = y_{i-1}\). Clearly this pair has center 0.
For instance when $\lambda = (4, 3, 3, 1)$, we have $\lambda' = (5, 4, 4, 1)$ and with this convention, the $x(\alpha)$ and $y(\alpha)$ sequences are respectively $(-5/\alpha, -4/\alpha + 1, -4/\alpha + 2, -1/\alpha + 3, 4)$ and $(-5/\alpha + 1, -4/\alpha + 2, -4/\alpha + 3, -1/\alpha + 4)$.

The “transition measure” of $Y_\lambda(\alpha)$ is a measure $\omega_\lambda$ on the real line, supported on the finite set $\{x_1(\alpha), \ldots, x_d(\alpha)\}$. It is uniquely defined by its moment generating series

$$\mathcal{M}_\lambda(z) = z^{-1} + \sum_{k \geq 1} M_k(\lambda) z^{-k-1}$$

$$= \frac{1}{z - x_d(\alpha)} \prod_{i=1}^{d-1} \frac{z - y_i(\alpha)}{z - x_i(\alpha)}.$$ 

It is known [28, 26] that the free cumulants $\{R_k(\lambda), k \geq 1\}$ of $\omega_\lambda$ are generated by the compositional inverse

$$\mathcal{R}_\lambda(u) = \mathcal{M}_\lambda^{(-1)}(u) = u^{-1} + \sum_{k \geq 1} R_k(\lambda) u^{k-1}.$$ 

The Boolean cumulants $\{B_k(\lambda), k \geq 1\}$ of $\omega_\lambda$ are generated by the inverse series

$$B_\lambda(z) = (\mathcal{M}_\lambda(z))^{-1} = z - \sum_{k \geq 1} B_k(\lambda) z^{1-k}.$$ 

Observe that $M_1(\lambda) = B_1(\lambda) = R_1(\lambda) = 0$ and $M_2(\lambda) = B_2(\lambda) = R_2(\lambda) = |\lambda|/\alpha$.

Defining the “inside” and “outside” alphabets

$$I_\lambda = \{x_1(\alpha), \ldots, x_d(\alpha)\}, \quad O_\lambda = \{y_1(\alpha), \ldots, y_{d-1}(\alpha)\},$$

and their difference $A_\lambda = I_\lambda - O_\lambda$, we then have

$$\mathcal{M}_\lambda(z) = z^{-1} H_{1/z}(A_\lambda),$$

$$B_\lambda(z) = z (H_{1/z}(A_\lambda))^{-1} = z E_{-1/z}(A_\lambda),$$

$$\mathcal{R}_\lambda(u) = u^{-1} (H_u^*(A_\lambda))^{-1} = u^{-1} E_u^*(A_\lambda).$$

Equivalently

$$M_k(\lambda) = h_k(A_\lambda), \quad B_k(\lambda) = (-1)^{k-1} e_k(A_\lambda), \quad R_k(\lambda) = (-1)^k e_k^*(A_\lambda).$$

We may specialize the results of Section 4 at $A = A_\lambda$. From (4.5) et (4.8) we obtain

$$B_n(\lambda) = - \sum_{|\mu| = n} (-1)^{|\mu|} u_\mu M_\mu(\lambda),$$

$$(n - 1)B_n(\lambda) = \sum_{|\mu| = n} \left( n - 1 \right) u_\mu R_\mu(\lambda). \quad (5.2)$$
From (4.4) et (4.7) we get

\[ M_n(\lambda) = \sum_{|\mu|=n} u_{\mu} B_{\mu}(\lambda), \]

\[ (n + 1)M_n(\lambda) = \sum_{|\mu|=n} \left( \frac{n + 1}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda). \]  \hspace{1cm} (5.3)

Conversely (4.8) becomes

\[ (n - 1)R_n(\lambda) = -\sum_{|\mu|=n} (-1)^{l(\mu)} \left( \frac{n - 1}{l(\mu)} \right) u_{\mu} B_{\mu}(\lambda), \]

\[ = -\sum_{|\mu|=n} (-1)^{l(\mu)} \left( \frac{n + l(\mu) - 2}{l(\mu)} \right) u_{\mu} M_{\mu}(\lambda). \]  \hspace{1cm} (5.4)

From (4.4) and (4.11) we have also

\[ h_n(x A_{\lambda}) = \sum_{|\mu|=n} \left( \frac{x}{l(\mu)} \right) u_{\mu} M_{\mu}(\lambda) \]

\[ = \frac{x}{n + x} \sum_{|\mu|=n} \left( \frac{n + x}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda). \]  \hspace{1cm} (5.5)

In particular for \( x = 1 - n \), we obtain

\[ (1 - n)R_n(\lambda) = h_n((1 - n) A_{\lambda}). \]  \hspace{1cm} (5.6)

Finally the specialization of (4.15) at \( A = A_{\lambda} \) and \( z = 1 \) yields

\[ \sum_{k=1}^{n} M_k(\lambda) \sum_{|\mu|=n-k} \left( \frac{-k}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda) = R_n(\lambda), \]  \hspace{1cm} (5.7)

\[ \sum_{k=1}^{n} M_{k-1}(\lambda) \sum_{|\mu|=n-k} \left( \frac{-k}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda) = 0, \]  \hspace{1cm} (5.8)

\[ \sum_{k=2}^{n} (k - 1)M_{k-2}(\lambda) \sum_{|\mu|=n-k} \left( \frac{-k}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda) = \sum_{|\rho|=n-2} v_{\rho} u_{\rho} R_{\rho}(\lambda), \]  \hspace{1cm} (5.9)

\[ \sum_{k=2}^{n} (k - 1)M_{k-1}(\lambda) \sum_{|\mu|=n-k} \left( \frac{-k}{l(\mu)} \right) u_{\mu} R_{\mu}(\lambda) = \sum_{|\rho|=n-1} w_{\rho} \frac{u_{\rho}}{l(\rho)} R_{\rho}(\lambda). \]  \hspace{1cm} (5.10)
6 The transition measure

In the previous section, the transition measure $\omega_\lambda$ was defined by its moment generating series $\mathcal{M}_\lambda(z)$. Equivalently we can write

$$\omega_\lambda = \sum_{k=1}^d \sigma_k(\alpha) \delta_{x_k(\alpha)},$$

where $\delta_u$ is the Dirac measure at $u$, and the weights $\sigma_k(\alpha)$ are the “transition probabilities”

$$\sigma_k(\alpha) = \prod_{i=1}^{k-1} \frac{x_k(\alpha) - y_i(\alpha)}{x_k(\alpha) - x_i(\alpha)} \prod_{j=k+1}^d \frac{x_k(\alpha) - y_{j-1}(\alpha)}{x_k(\alpha) - x_j(\alpha)}.$$

Both definitions are linked by the formula

$$\mathcal{M}_\lambda(z) = \sum_{k=1}^d \frac{\sigma_k(\alpha)}{z - x_k(\alpha)}.$$

Of course multiplying both sides by $z$ and taking the limit $z \to \infty$, we obtain

$$\sum_{k=1}^d \sigma_k(\alpha) = 1.$$

Now consider the Pieri coefficients $\{c_i(\lambda), 1 \leq i \leq l(\lambda) + 1\}$. If we restrict to a finite set of $N$ indeterminates $x = (x_1, \ldots, x_N)$, we have [27, Theorem 5.4]

$$J_{\lambda}(1, \ldots, 1) = \prod_{(i,j) \in \lambda} (N + \alpha(j - 1) - i + 1).$$

Writing the Pieri formula (3.3) at $(1, \ldots, 1)$, and identifying coefficients of $N$, we easily obtain

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) = 1.$$

Thus we are led to compare two discrete probability distributions $\{\sigma_k(\alpha), 1 \leq k \leq d\}$ and $\{c_i(\lambda), 1 \leq i \leq l(\lambda) + 1\}$. This has been done by Kerov in a rather different form [9, Lemma 7.2]. His following result is fundamental for our purpose since it connects Jack polynomials with the transition measure.

**Theorem 6.1.** Given a partition $\lambda$, the weights $\{\sigma_k(\alpha), 1 \leq k \leq d\}$ are the non-zero elements of $\{c_i(\lambda), 1 \leq i \leq l(\lambda) + 1\}$. More precisely we may write

$$\omega_\lambda = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \delta_{\lambda_i-(i-1)/\alpha}.$$
Equivalently we have

\[ M_\lambda(z) = \sum_{i=1}^{l(\lambda)+1} \frac{c_i(\lambda)}{z - \lambda_i + (i - 1)/\alpha}. \]

In particular \( \sigma_1(\alpha) = c_{l(\lambda)+1}(\lambda) \) and \( \sigma_d(\alpha) = c_1(\lambda) \). In this section we shall give another proof of this result.

For any partition \( \lambda \), we have the following natural generalization of the “rising factorial” in terms of \( \alpha \)-contents

\[ (x)_{\lambda} = \prod_{(i,j) \in \lambda} (x + j - 1 - (i - 1)/\alpha). \]

Given two indeterminates \( x, y \) the quantity

\[ \frac{(x + y + 1)_\lambda}{(x + y)_\lambda} \frac{(x)_{\lambda}}{(x + 1)_\lambda} \]

has been studied in [15, Section 5, p. 3463]. Its development in descending powers of \( x \) has been given there (see [16] for an important application).

It turns out that the moment generating series of the transition measure is a function of this type.

**Theorem 6.2.** We have

\[ M_\lambda(z) = z^{-1} \frac{(-z - 1/\alpha + 1)_{\lambda}}{(-z - 1/\alpha)_{\lambda}} \frac{(-z)_{\lambda}}{(-z + 1)_{\lambda}}. \]

**Proof.** Denote \( N_\lambda(z) \) the right-hand side. In view of (5.1) and the convention made there, it is enough to prove

\[ N_\lambda(z) = \frac{1}{z - x_d(\alpha)} \prod_{k=1}^{d-1} \frac{z - y_k(\alpha)}{z - x_k(\alpha)}, \]

with \( d = \lambda_1 + 1, x_d(\alpha) = \lambda_1 \) and for \( 1 \leq k \leq \lambda_1 \),

\[ x_k(\alpha) = k - 1 - \lambda_k'/\alpha, \quad y_k(\alpha) = k - \lambda_k'/\alpha. \]

We have clearly

\[ \frac{(u + 1)_{\lambda}}{(u)_{\lambda}} = \prod_{i=1}^{l(\lambda)} \frac{\lambda_i}{u + j - (i - 1)/\alpha} = \prod_{i=1}^{l(\lambda)} \frac{u + \lambda_i - (i - 1)/\alpha}{u - (i - 1)/\alpha}, \]

which yields

\[ z N_\lambda(z) = \prod_{i=1}^{l(\lambda)} \frac{z - \lambda_i + i/\alpha}{z + i/\alpha} \frac{z + (i - 1)/\alpha}{z - \lambda_i + (i - 1)/\alpha}. \]
On the right-hand side consider the product corresponding to a part \( \lambda_i = r \). Since 
\[ \lambda'_k = \sum_{i\geq k} m_i(\lambda), \]
it writes
\[
\prod_{i=\lambda'_{i+1}+1}^{\lambda'} \frac{z-r+i/\alpha}{z+i/\alpha} \frac{z+(i-1)/\alpha}{z-r+(i-1)/\alpha} = \frac{z-r+\lambda'_r/\alpha}{z-r+\lambda'_{r+1}/\alpha} \frac{z+\lambda'_r/\alpha}{z+\lambda'_r/\alpha}.
\]

Bringing all contributions together we obtain
\[
z \mathcal{N}_\lambda(z) = \frac{z}{z-\lambda_1} \prod_{r=1}^{\lambda_1} \frac{z-r+\lambda'_r/\alpha}{z-r+1+\lambda'_r/\alpha}.
\]

Equivalently if we write
\[
\frac{(-z-1/\alpha+1)_\lambda}{(-z-1/\alpha)_\lambda} \frac{(-z)_\lambda}{(-z+1)_\lambda} = \sum_{k \geq 0} s_k(\lambda) z^{-k},
\]
we have \( M_k(\lambda) = s_k(\lambda) \). Thus we may apply the results of [15], which have two important consequences. Firstly we recover Kerov’s result.

**Theorem 6.3.** For any integer \( k \geq 0 \) we have
\[
M_k(\lambda) = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \left( \lambda_i - (i-1)/\alpha \right)^k.
\]

Equivalently
\[
\mathcal{M}_\lambda(z) = \sum_{i=1}^{l(\lambda)+1} \frac{c_i(\lambda)}{z-\lambda_i+(i-1)/\alpha}.
\]

**Proof.** Using “Lagrange interpolation” [12, Section 7.8], the first statement has been proved for \( s_k(\lambda) \) in [15, Theorem 8.1, p. 3470].

Secondly \( M_k(\lambda) \), originally defined as some rational function of \( \lambda \), is actually a shifted symmetric polynomial.

**Theorem 6.4.** The moments \( \{ M_k(\lambda), k \geq 2 \} \), boolean cumulants \( \{ B_k(\lambda), k \geq 2 \} \) and free cumulants \( \{ R_k(\lambda), k \geq 2 \} \) are three algebraic bases of the shifted symmetric algebra \( S^* \). They are polynomials in \( 1/\alpha \).

**Proof.** Due to (5.2)-(5.4), it is enough to prove both statements for the moments. But they follow from [15, Corollary 5.2, p. 3464] where we have proved
\[
s_k(\lambda) = \sum_{p,q,r \geq 0 \atop p+q+2r \leq k} \frac{1}{\alpha^r} \left( 1 - \frac{1}{\alpha} \right)^p \binom{p+q+r-1}{p} \sum_{s=0}^{\min(r,k-2r-p)} \binom{|\lambda|+r-1}{r-s} F_{k-2r-p,q,s}(\lambda).
\]

Here \( F_{p,q,r}(\lambda) \) is a shifted symmetric function explicitly known in terms of the basis (2.1) (it is also a polynomial in \( 1/\alpha \)).
7 An example

Our method for the computation of \( \vartheta_\mu^\lambda \) will be best understood through its simplest example \( \mu = (2) \).

(i) Fixing the constant \( \vartheta_0^\lambda = \theta_{1,\lambda}^\lambda = 1 \) and using \( |\lambda| = \alpha R_2(\lambda) \), equations (3.6) become

\[
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \vartheta_2^{\lambda(i)} = \vartheta_2^\lambda,
\]

\[
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \left( \lambda_i - (i - 1)/\alpha \right) \vartheta_2^{\lambda(i)} = 2\alpha R_2(\lambda).
\]

The solution being unique, we search for it in the form

\[
\vartheta_2^\lambda = A R_4(\lambda) + B R_2^2(\lambda) + C R_3(\lambda) + D R_2(\lambda).
\]

(ii) We compute the quantities \( R_k(\lambda^{(i)}) \) in terms of \( R_k(\lambda) \). Here we assume (this will be proved in Theorem 8.4 below) that

\[
\alpha^3 R_4(\lambda^{(i)}) = \alpha^3 R_4(\lambda) + 3\alpha^2 x_i^2 - 3\alpha \beta x_i - 3\alpha^2 R_2(\lambda) + \beta^2 - \alpha,
\]

\[
\alpha^2 R_3(\lambda^{(i)}) = \alpha^2 R_3(\lambda) + 2\alpha x_i - \beta,
\]

\[
\alpha R_2(\lambda^{(i)}) = \alpha R_2(\lambda) + 1,
\]

with \( \beta = 1 - \alpha \) and \( x_i = \lambda_i - (i - 1)/\alpha \).

(iii) We insert these values in the system (3.6) and apply Theorem 6.3. We obtain

\[
3\alpha^2 A M_2(\lambda) - 3\alpha^2 A R_2(\lambda) + A(\beta^2 - \alpha) + 2\alpha^2 B R_2(\lambda) + \alpha B - \alpha \beta C + \alpha^2 D = 0,
\]

\[
3\alpha^2 A M_3(\lambda) + (-3\alpha \beta A + 2\alpha^2 C) M_2(\lambda) = 2\alpha^4 R_2(\lambda).
\]

(iv) We express the moments in terms of the free cumulants by (5.3). Here we have only \( M_2(\lambda) = R_2(\lambda) \) and \( M_3(\lambda) = R_3(\lambda) \).

(v) We apply Theorem 6.4 and identify the polynomials in the free cumulants on both sides. This yields

\[
B = 0, \quad A(\beta^2 - \alpha) - \alpha \beta C + \alpha^2 D = 0,
\]

\[
A = 0, \quad 2\alpha^2 C = 2\alpha^4,
\]

that is \( A = B = 0, C = \alpha^2, D = \alpha \beta \). Hence \( \vartheta_2^\lambda = \alpha^2 R_3(\lambda) + \alpha \beta R_2(\lambda) \).

Observe that after (ii) we never used the fact that \( \alpha + \beta = 1 \). For steps (iii) to (v), \( \alpha \) and \( \beta \) might be independent parameters.
8 Adding a node

As shown above, we must first obtain an expression for $R_n(\lambda(i))$, when adding a node on the row $i$ of the diagram of $\lambda$. Since we have

$$(1 - n)R_n(\lambda) = h_n((1 - n)A_\lambda),$$

this amounts to compare the alphabet $A_{\lambda(i)}$ with $A_\lambda$. From now on we denote $\beta = 1 - \alpha$ and $x_i = \lambda_i - (i - 1)/\alpha$.

Proposition 8.1. Given a partition $\lambda$ and an integer $1 \leq i \leq l(\lambda) + 1$, one has

$$A_{\lambda(i)} = A_\lambda + B(x_i),$$

with $B(v)$ the alphabet $\{v + 1, v - 1/\alpha\} - \{v, v + 1 - 1/\alpha\}$. In other words

$$H_{1/z}(A_{\lambda(i)}) = H_{1/z}(A_\lambda) \frac{z + x_i + 1 - 1/\alpha}{z + x_i - 1/\alpha} \frac{z + x_i}{z + x_i + 1}.$$

Proof. By Theorem 6.2 we have

$$H_{1/z}(A_\lambda) = \left(\frac{-1/\alpha + 1}{-1/\alpha}\right)^{\lambda}(1 - y)^{-1}.$$

But $(u)_{\lambda(i)} = (u)_\lambda(u + x_i)$.

Proposition 8.2. For any $n \geq 1$ and any real number $u$ one has

$$h_n(uB(v)) = \sum_{r,s \geq 0} v^{n-2r-s} \binom{n-1}{2r+s-1} \binom{r+s-1}{s} \binom{-u}{r} (-1/\alpha)^{r+s} \beta^s.$$

Proof. By the definition (4.2), denoting $y = (z - v)^{-1}$, we have

$$H_{1/z}(uB(v)) = (H_{1/z}(B(v)))^u = \left(\frac{1 - y(1 - 1/\alpha)}{1 + y/\alpha} \frac{1 - y}{1 - y}\right)^u.$$

Using $\beta = 1 - \alpha$ this becomes

$$H_{1/z}(uB(v)) = \left(1 - \frac{y^2/\alpha}{1 + y\beta/\alpha}\right)^{-u}.$$

Applying the binomial development three times, we have

$$H_{1/z}(uB(v)) = \sum_{r \geq 0} \binom{-u}{r} (-1/\alpha)^r y^{2r} (1 + y\beta/\alpha)^{-r}$$

$$= \sum_{r,s \geq 0} \binom{-u}{r} \binom{r+s-1}{s} (-1/\alpha)^{r+s} \beta^s y^{2r+s}$$

$$= \sum_{r,s,t \geq 0} \binom{-u}{r} \binom{r+s-1}{s} \binom{2r+s+t-1}{t} (-1/\alpha)^{r+s} \beta^s v^t z^{-2r-s-t}.$$
Important remark: This result plays a fundamental role in the sequel since it is the “natural” way making the parameter $\beta$ enter our scheme. Although the fact that $\beta = 1 - \alpha$ is crucial for its proof, this restriction will no longer be used. From now on, $\beta$ will play an independent role.

**Proposition 8.3.** For any $n \geq 2$ one has

$$M_n(\lambda^{(i)}) - M_n(\lambda) = \sum_{r \geq 1, s, t \geq 0, 2r + s + t \leq n} x_i^{n-2r-s-t} \binom{n-t-1}{2r+s-1} \binom{r+s-1}{s} \frac{1}{r} (1/\alpha)^{r+s} (-\beta)^s M_t(\lambda).$$

**Proof.** By (4.1) we have

$$M_n(\lambda^{(i)}) = h_n(A_{\lambda^{(i)}}) = \sum_{t=0}^{n} h_t(A_{\lambda}) h_{n-t}(B(x_i)).$$

We apply Proposition 8.2 with $u = 1, v = x_i$. \hfill $\square$

**Theorem 8.4.** For any $n \geq 2$ one has

$$R_n(\lambda^{(i)}) - R_n(\lambda) = \sum_{r \geq 1, s, t \geq 0, 2r + s + t \leq n} x_i^{n-2r-s-t} \binom{n-t-1}{2r+s-1} \binom{r+s-1}{s} \frac{1}{r} (1/\alpha)^{r+s} (-\beta)^s$$

$$\times \sum_{|\sigma| = t} \frac{u_\sigma}{1-n+t} \binom{1-n+t}{l(\sigma)} R_\sigma(\lambda).$$

**Proof.** Applying (5.6) and (4.1) we have

$$(1-n)R_n(\lambda^{(i)}) = h_n((1-n)A_{\lambda^{(i)}})$$

$$= \sum_{t=0}^{n} h_t((1-n)A_{\lambda}) h_{n-t}((1-n)B(x_i)).$$

We then evaluate $h_{n-t}((1-n)B(x_i))$ by Proposition 8.2 with $u = 1 - n, v = x_i$, and $h_t((1-n)A_{\lambda})$ by relation (5.5) with $x = 1 - n$. \hfill $\square$

Expanding the right-hand side in descending powers of $x_i$, the first terms are

$$R_n(\lambda^{(i)}) - R_n(\lambda) = \frac{n-1}{\alpha} x_i^{n-2} - \frac{n-1}{2} \beta/\alpha^2 x_i^{n-3}$$

$$+ (n-1)(n-3) \left( \frac{(n-2)\beta^2/6\alpha^3 - (n-2)^2/12\alpha^2 - R_2(\lambda)/\alpha} x_i^{n-4} + \ldots \right).$$

By a product over the parts of $\rho$, Theorem 8.4 yields

$$R_\rho(\lambda^{(i)}) - R_\rho(\lambda) = \sum_{|\rho|} x_i^{|\rho|-k} \sum_{|\sigma| \leq k-2} b_{k,\sigma}(\rho) R_\sigma(\lambda),$$

(8.1)

where the coefficients $b_{k,\sigma}(\rho)$ are polynomials in $(1/\alpha, \beta)$ with integer coefficients, explicitly known. We shall need the value of $b_{k,\sigma}(\rho)$ for $|\sigma| = k - 2$. 

22
Proposition 8.5. In the right-hand side of (8.1) the contribution \( |\sigma| = k - 2 \) is given by
\[
\frac{1}{\alpha} \sum_{p \text{ part of } \rho} m_p(p)(p-1) R_{\rho \setminus p}(\lambda) \sum_p 2^{p-k} \sum_{|\nu| = k-2} \left( \frac{k-p-1}{l(\nu)} \right) u_\nu R_\nu(\lambda).
\]

Proof. If we associate a triple \((r_i, s_i, t_i)\), \(2r_i + s_i + t_i \leq i\) to any part \(i\) of \(\rho\), the condition \(|\sigma| = k - 2\) reads \(\sum_i (2r_i + s_i) = 2\), which implies \(r_i = s_i = 0, t_i = i\) for all \(i\) but one, say \(p\), for which \(r_p = 1, s_p = 0\). Then \(\sigma\) is formed by the union of \(\rho \setminus p\) and a partition \(\nu\) with \(|\nu| = t_p\).

Finally let us mention a consequence of Theorem 8.4 (which is not true for moments).

Proposition 8.6. The quantity \(-R_n(\lambda)\) is a polynomial in \(-1/\alpha\) with nonnegative integer coefficients.

Proof. By induction on \(|\lambda|\), starting from \(\lambda = (0)\). If the property is true for \(R_n(\lambda)\), it is true for \(R_n(\lambda^{(i)})\).

9 An algorithm

We now present our method and its first consequences. We begin with a result following directly from Theorem 6.4 and the fact [17, Proposition 2] that \(\vartheta^\lambda_\mu\) is a shifted symmetric polynomial of \(\lambda\).

Proposition 9.1. Let \(\mu\) be a partition with no part 1. For any \(\lambda\) with \(|\lambda| \geq |\mu|\), the quantity \(\vartheta^\lambda_\mu\) may be written as a polynomial \(K_\mu\) in the free cumulants \(\{R_k(\lambda), k \geq 2\}\), with coefficients in \(Q(\alpha)\).

We shall compute this polynomial \(K_\mu\) by induction on \(|\mu| - l(\mu)\). We assume that \(K_\nu\) is known for \(|\nu| - l(\nu) = d\). We fix a partition \(\mu\) with \(|\mu| - l(\mu) = d + 1\). We write the polynomial \(\vartheta^\lambda_\mu\) as
\[
\vartheta^\lambda_\mu = \sum_\rho a^\mu_\rho R_\rho(\lambda),
\]
the coefficients \(a^\mu_\rho \in Q(\alpha)\) to be determined.

(i) In the right-hand side of the second equation (3.6), the partitions \(\nu = \mu \setminus 2\) and \(\nu = \mu_{1(\nu)}\) satisfy \(|\nu| - l(\nu) = d\). We substitute the known values of \(K_{\mu \setminus 2}\) and \(K_{\mu_{1(\nu)}}\) in this equation.

(ii) We apply Theorem 8.4 to the left-hand sides of equations (3.6). We have
\[
\vartheta^\lambda_{\mu^{(i)}} - \vartheta^\lambda_\mu = \sum_\rho a^\mu_\rho (R_\rho(\lambda^{(i)}) - R_\rho(\lambda))
\]
\[
= \sum_\rho a^\mu_\rho \sum_{k=2l(\rho)} |\rho| - k \sum_{|\sigma| \leq k-2} b_{k,\sigma}(\rho) R_\sigma(\lambda).
\]
From Theorem 6.3 we deduce

\[
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \varphi^{(i)}(\lambda) = \varphi^{(i)}(\lambda) + \sum_{\rho} a^\mu_\rho \sum_{k=2l(\rho)} M_{|\rho|-l(\rho)} \sum_{|\sigma|\leq k-2} b_{k,\sigma}(\rho) R_\sigma(\lambda),
\]

\[
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) x_i \varphi^{(i)}(\lambda) = \sum_{\rho} a^\mu_\rho \sum_{|\rho|} M_{|\rho|-l(\rho)+1} \sum_{|\sigma|\leq k-2} b_{k,\sigma}(\rho) R_\sigma(\lambda).
\]

(iii) Now we may omit \(\lambda\) for a clearer display. Equations (3.6) read

\[
\sum_{\rho} a^\mu_\rho \sum_{k=2l(\rho)} M_{|\rho|-l(\rho)} \sum_{|\sigma|\leq k-2} b_{k,\sigma}(\rho) R_\sigma = 0,
\]

\[\text{(9.1)}\]

\[
\sum_{\rho} a^\mu_\rho \sum_{k=2l(\rho)} M_{|\rho|-l(\rho)+1} \sum_{|\sigma|\leq k-2} b_{k,\sigma}(\rho) R_\sigma = 2(\alpha R_2 - |\mu| + 2) m_2(\mu) K_{\mu_2} + \sum_{r\geq3} rm_r(\mu) K_{\mu_1(r)}.
\]

\[\text{(9.2)}\]

(iv) We express the moments \(M_k\) in terms of the free cumulants by (5.3).

We are left with two equations involving polynomials in the basis of free cumulants. By identification we obtain a linear system in the unknown coefficients \(a^\mu_\rho\).

It turns out that this linear system has one and only one solution. This fact might appear mysterious, but actually it is not. It is only an alternative formulation of Theorem 3.1.

Remarks: (i) As already emphasized, this algorithm does not use the fact that \(\alpha + \beta = 1\), except at the step (ii). Otherwise \(\alpha\) and \(\beta\) may be independent parameters. This point will be further investigated in Section 15.

(ii) Our method remains valid for \(\alpha = 1\). In this case, it gives a new algorithm to compute classical Kerov polynomials. Up to now these polynomials were only computed through Frobenius formula (see for instance [24, Section 3.3.2]).

### Proposition 9.2.

(i) \(K_{\mu}\) is a polynomial in \(\{R_2(\lambda), R_3(\lambda), \ldots, R_{|\mu|-l(\mu)+2}(\lambda)\}\).

(ii) The highest weight of \(K_{\mu}\) is \(|\mu| + l(\mu)\) (with the weight of \(R_\rho\) defined as \(|\rho|\)).

**Proof.** Both properties are true for \(\mu = (2)\). They are proved by induction.

Firstly assume that (i) is true for partitions \(\nu\) with \(|\nu| - l(\nu) = d\). Let \(R_p\) occur in \(K_{\mu}\) with \(|\mu| - l(\mu) = d + 1\). By Theorem 8.4 this produces a term with \(M_{p-1}\) (hence \(R_{p-1}\)) at the left-hand side of (9.2). However by the inductive hypothesis, the right-hand side involves only free cumulants up to \(R_{d+2}\). By comparison we have \(p - 1 \leq d + 2\).
Secondly assume that $(ii)$ is true for partitions $\nu$ with $|\nu| + l(\nu) = d$. Let $p$ be the highest weight of $K_\mu$ with $|\mu| + l(\mu) = d + 1$. By the inductive hypothesis, the highest weight of the right-hand side of (9.2) is $d$. But the highest weight of the left-hand side is $p - 1$, because it is produced by some term $M_{p-k+1}R_\sigma$ with $|\sigma| = k - 2$. We conclude by comparison.

From now on, we shall omit $\lambda$ and write $(R_2, R_3, \ldots, R_{|\mu|-l(\mu)+2})$ for the indeterminates of $K_\mu$. As mentioned in the introduction, we conjecture its coefficients to be polynomials in $\alpha$ with integer coefficients.

10 The case of rows

Up to Section 14 we now restrict to the case of a row partition $\mu = (r)$. Equations (3.6) read

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \vartheta^\lambda_i = \vartheta^\lambda_r,$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) (\lambda_i - (i - 1)/\alpha) \vartheta^\lambda_i = r \vartheta^\lambda_{r-1}.$$  

With $K_r = \sum_{|\rho| \leq r+1} a_\rho^{(r)} R_\rho$ our algorithm becomes

$$\sum_{|\rho| \leq r+1} a_\rho^{(r)} \sum_{k=2l(\rho)}^{|\rho|} M_{|\rho|-k} \sum_{|\sigma| \leq k-2} b_{k,\sigma}(\rho) R_\sigma = 0, \quad (10.1)$$

$$\sum_{|\rho| \leq r+1} a_\rho^{(r)} \sum_{k=2l(\rho)}^{|\rho|} M_{|\rho|-k+1} \sum_{|\sigma| \leq k-2} b_{k,\sigma}(\rho) R_\sigma = r \sum_{|\tau| \leq r} a^{(r-1)}_{\tau} R_\tau. \quad (10.2)$$

In this section we shall identify the terms of weight $r + 1$ and $r$ in $K_r$. The following auxiliary lemma will play a crucial role.

**Proposition 10.1.** Let $\epsilon = 0, 1$. We write

$$\sum_{|\rho| = s} a_\rho \sum_{k=2l(\rho)}^{s} M_{s-k+\epsilon} \sum_{|\sigma| = k-2} b_{k,\sigma}(\rho) R_\sigma = \sum_{|\tau| = s-2+\epsilon} x_\tau R_\tau.$$

$(i)$ If $\epsilon = 1$ only partitions $\tau_1(p) = \tau \setminus p \cup (p + 1)$ contribute to $x_\tau$. More precisely

$$x_\tau = 1/\alpha \sum_{p \text{ part of } \tau} p (m_{p+1}(\tau) + 1) a_{\tau_1(p)}.$$  

$(ii)$ If $\epsilon = 0$ only $\tau \cup 2$ contributes to $x_\tau$ and

$$x_\tau = 1/\alpha (m_2(\tau) + 1) a_{\tau \cup 2}.$$  

25
Proof. In the left-hand sides, applying Proposition 8.5, we are led to evaluate
\[ \frac{1}{\alpha} \sum_{p \text{ part of } \rho} m_p(\rho)(p-1) R_{\rho p} \sum_{k=2}^{p} M_{p-k+\epsilon} \sum_{l(\nu)=k-2}^{(k-p-1)} \rho l(\nu) u_{\nu} R_{\nu}, \]
with \( \epsilon = 1, 0 \). Applying (5.7) and (5.8) the second sum is \( R_{p-1} \) for \( \epsilon = 1 \) and \( \delta_{\rho 2} \) for \( \epsilon = 0 \).

**Theorem 10.2.** The term of weight \( r + 1 \) in \( K_r \) is \( \alpha^r R_{r+1} \).

**Proof.** The statement is true for \( r = 2 \). We prove it by induction on \( r \). By Proposition 9.2 the highest weight of \( K_r \) is \( |\rho| = r + 1 \). The term of weight \( r \) in the left-hand side of (10.2) is produced by some term \( M_{|\rho|-k+1} R_\sigma \). Hence \( r \leq |\rho| - k + 1 + k - 2 = |\rho| - 1 \), which implies \( |\rho| = r + 1 \) and \( |\sigma| = k - 2 \). Taking terms of highest weight in both sides, we obtain
\[ \sum_{|\rho|=r+1} a_{\rho}^{(r)} \sum_{k=2l(\rho)}^{r+1} M_{r-k+2} \sum_{|\sigma|=k-2} b_{k,\sigma}(\rho) R_\sigma = r \alpha^{r-1} R_r. \]

Applying Proposition 10.1(i) with \( s = r + 1 \) for \( \tau = (r) \), we obtain \( r \alpha^{r-1} = r a_{(r+1)}^{(r)}/\alpha \), hence \( a_{(r+1)}^{(r)} = \alpha^r \).

It remains to prove that \( a_{\rho}^{(r)} = 0 \) for \( |\rho| = r + 1 \) and \( l(\rho) \geq 2 \). This is done by a second induction on the lowest part of \( \rho \). Firstly we treat the case of a lowest part 2. Consider the term of highest weight in (10.1). By the same argument as above, it has weight \( r - 1 \) and writes
\[ \sum_{|\rho|=r+1} a_{\rho}^{(r)} \sum_{k=2l(\rho)}^{r+1} M_{r-k+1} \sum_{|\sigma|=k-2} b_{k,\sigma}(\rho) R_\sigma = 0. \]

Applying Proposition 10.1(ii) with \( s = r + 1 \) for \( \tau = \rho \setminus 2 \), we obtain \( a_{(r+1)}^{(r+1)} = 0 \).

Then our inductive hypothesis assumes \( a_{\nu}^{(r)} = 0 \) for any \( \nu \) with lowest part \( q - 1 \). We fix a partition \( \rho \) with lowest part \( q \), so that \( \tau = \rho \setminus q \) has lowest part \( q - 1 \). Since \( \tau_{(q-1)} = \rho \), applying Proposition 10.1(i) for \( \tau \) we have
\[ (q-1)(m_q(\tau) + 1) a_{\rho}^{(r)} + \sum_{p \text{ part of } \tau \setminus p \neq q - 1} p m_{p+1}(\tau) + 1 a_{\tau_{(q)}}^{(r)} = 0. \]

In the sum, all \( a_{\tau_{(q)}}^{(r)} \) vanish because \( \tau_{(q)} \) has lowest part \( q - 1 \). Hence \( a_{\rho}^{(r)} = 0 \).

For a clearer display we now write
\[ R_{\rho} = \prod_{i \geq 2} ((i - 1) R_i)^{m_i(\rho)/m_i(\rho)!}. \]
Theorem 10.3. The term of weight \( r \) in \( K_r \) is

\[
\alpha^{r-1}\beta \frac{r}{2} \sum_{\|\rho\|=r} (l(\rho) - 1)! R_{\rho}.
\]

Proof. It is done by induction on \( r \). We assume that the property is true for \( K_{r-1} \). For \( |\rho| = r \) we have to prove

\[
a_{\rho}^{(r)} = \alpha^{r-1}\beta \frac{r}{2} \frac{u_{\rho}}{l(\rho)} v_{\rho}.
\]

Firstly, let us compare the terms of weight \( r - 1 \) on both sides of (10.2). On the left-hand side such a term is produced by \( M_{|\rho|-k+1} R_{\sigma} \). Hence \( r - 1 \leq |\rho| - 1 \) and there are two possibilities, either \( |\rho| = r + 1, |\sigma| = k - 3 \) or \( |\rho| = r, |\sigma| = k - 2 \).

(i) The contribution of the case \( |\rho| = r + 1, |\sigma| = k - 3 \) to the left-hand side of (10.2) is

\[
\alpha^r \sum_{k=3}^{r} M_{r-k+2} \sum_{|\sigma|=k-3} b_{k,\sigma}((r+1)) R_{\sigma}.
\]

But by Theorem 8.4 for \( |\sigma| = k - 3 \) we have

\[
b_{k,\sigma}((r+1)) = -\frac{\beta}{\alpha^2} \frac{r}{2} (r - k + 2) \left( \frac{k - 3 - r}{l(\sigma)} \right) u_{\sigma}.
\]

Applying (5.10), the contribution is therefore

\[
-\alpha^{r-2}\beta \frac{r}{2} \sum_{k=3}^{r} (k - 1) M_{k-1} \sum_{|\sigma|=r-k} \left( \frac{-k}{l(\sigma)} \right) u_{\sigma} R_{\sigma} = -\alpha^{r-2}\beta \frac{r}{2} \sum_{|\tau|=r-1} w_{\tau} \frac{u_{\tau}}{l(\tau)} R_{\tau}.
\]

(ii) The contribution of the case \( |\rho| = r, |\sigma| = k - 2 \) to the left-hand side of (10.2) is

\[
\sum_{|\rho|=r} a_{\rho}^{(r)} \sum_{k=2(r)}^{r} M_{r-k+1} \sum_{|\sigma|=k-2} b_{k,\sigma}(\rho) R_{\sigma}.
\]

Applying Proposition 10.1(i) with \( s = r \), this becomes

\[
1/\alpha \sum_{|\tau|=r-1} \sum_{p \text{ part of } \tau} p (m_{p+1}(\tau) + 1) a_{\tau_1(p)}^{(r)} R_{\tau}.
\]

Finally equating the coefficients of \( R_{\tau} \) with \( |\tau| = r - 1 \) on both sides of (10.2), and applying the inductive hypothesis, we obtain

\[
\sum_{p \text{ part of } \tau} p (m_{p+1}(\tau) + 1) a_{\tau_1(p)}^{(r)} = \alpha^{r-1}\beta \frac{r}{2} \frac{u_{\tau}}{l(\tau)} \left( w_{\tau} + (r - 1) v_{\tau} \right). \tag{10.3}
\]

Secondly, let us compare the terms of weight \( r - 2 \) on both sides of (10.1). On the left-hand side such a term is produced by \( M_{|\rho|-k} R_{\sigma} \). Hence \( r - 2 \leq |\rho| - 2 \) and again two possibilities, either \( |\rho| = r + 1, |\sigma| = k - 3 \) or \( |\rho| = r, |\sigma| = k - 2 \).
By the same argument as above, applying (5.9), the contribution of the case $|\rho| = r + 1, |\sigma| = k - 3$ to the left-hand side of (10.1) is

$$-\frac{r}{2} \alpha^{r-2} \beta \sum_{k=2}^{r} (k-1) M_{k-2} \sum_{|\sigma|=r-k} \left( -\frac{k}{l(\sigma)} \right) u_\sigma R_\sigma = -\frac{r}{2} \alpha^{r-2} \beta \sum_{|\tau|=r-2} v_\tau u_\tau R_\tau.$$  

The contribution of the case $|\rho| = r, |\sigma| = k - 2$ to the left-hand side of (10.1) is

$$\sum_{|\rho|=r} a_{\rho}^{(r)} \sum_{k=2l(\rho)}^{r} M_{r-k} \sum_{|\sigma|=k-2} b_{k,\sigma}(\rho) R_\sigma.$$  

Applying Proposition 10.1(ii) with $s = r$, this contribution becomes

$$1/\alpha \sum_{|\tau|=r-2} (m_2(\tau) + 1) a_{\tau\cup 2}^{(r)} R_\tau.$$  

Finally equating the coefficients of $R_\tau$ with $|\tau| = r - 2$ on both sides of (10.1), we obtain

$$(m_2(\tau) + 1) a_{\tau\cup 2}^{(r)} - \frac{r}{2} \alpha^{r-1} \beta v_\tau u_\tau = 0. \quad (10.4)$$

We are now in a position to compute $a_{\rho}^{(r)}$ for $|\rho| = r$. This is done by a second induction on the lowest part of $\rho$. Firstly we treat the case of a lowest part 2, say $\rho = \tau \cup 2$. The value of $a_{\rho}^{(r)}$ is then given by (10.4), which yields

$$a_{\rho}^{(r)} = \alpha^{r-1} \beta \frac{r}{2} \frac{u_\tau}{m_2(\tau) + 1} v_\tau = \alpha^{r-1} \beta \frac{r}{2} \frac{u_\rho}{l(\rho)} v_\rho.$$  

Secondly our inductive hypothesis assumes that $a_{\nu}^{(r)}$ is known for any $\nu$ with lowest part $q - 1$. We fix a partition $\rho$ with lowest part $q$, so that $\tau = \rho_{\uparrow(q)}$ has lowest part $q - 1$. Since $\tau_{\uparrow(q-1)} = \rho$, (10.3) may be written

$$(q-1)(m_q(\tau) + 1) a_{\rho}^{(r)} = \alpha^{r-1} \beta \frac{r}{2} \frac{u_\tau}{l(\tau)} (w_\tau + (r-1) v_\tau) - \sum_{\rho \text{ part of } \tau \atop \rho \neq \tau_{\uparrow(q-1)}} p(m_{p+1}(\tau) + 1) a_{\tau_{\uparrow(p)}}^{(r)}.$$  

In the right-hand side, all $a_{\tau_{\uparrow(p)}}^{(r)}$ are known because $\tau_{\uparrow(p)}$ has lowest part $q - 1$. Thus induction will be completed if we have the identity

$$\sum_{\rho \text{ part of } \tau} p(m_{p+1}(\tau) + 1) \frac{u_{\tau_{\uparrow(p)}}}{l(\tau_{\uparrow(p)})} v_{\tau_{\uparrow(p)}} = \frac{u_\tau}{l(\tau)} \left( w_\tau + |\tau| v_\tau \right).$$  

But $l(\tau_{\uparrow(p)}) = l(\tau)$, $u_{\tau_{\uparrow(p)}}/u_\tau = m_p(\tau)/(m_{p+1}(\tau) + 1)$, and $u_{\tau_{\uparrow(p)}}/v_\tau = p/(p-1)$. Thus it amounts to check the trivial equality

$$\sum_i i(m_{i+1} + 1) \frac{m_i}{m_{i+1} + 1} \frac{i}{i-1} = \sum_i i m_i \left( \frac{1}{i-1} + 1 \right). \quad \square$$
11 More conjectures

For \( r \leq 20 \) our computations support the following conjecture.

**Conjecture 11.1.** We have

\[
K_r = \sum_{0 \leq j \leq i \leq r-1} \sum_{2i-j \leq r-1} \alpha^{r-i} \beta^j \sum_{|\rho|=r-2i+j+1} a_{ij}(\rho) R_{\rho},
\]

where the coefficients \( a_{ij}(\rho) \) are nonnegative integers.

From now on, we denote \( K_r(i,j) \) the coefficient of \( \alpha^{r-i} \beta^j \) in \( K_r \), with weight \( r-2i+j+1 \). Theorems 10.2 and 10.3 give

\[
K_r^{(0,0)} = R_{r+1}, \quad K_r^{(1,1)} = \frac{r}{2} \sum_{|\rho|=r} (l(\rho) - 1)! R_{\rho}.
\]

We conjecture that all components \( K_r^{(i,j)} \) but one may be described in a unified way, independent of \( r \).

**Conjecture 11.2.** For any \((i,j) \neq (0,0)\) there exists an inhomogeneous symmetric function \( f_{ij} \) having maximal degree \( 4i - 2j - 2 \) such that

\[
K_r^{(i,j)} = r \sum_{|\rho|=r-2i+j+1} (l(\rho) + 2i - j - 2)! f_{ij}(\rho) R_{\rho},
\]

where \( f_{ij}(\rho) \) denotes the value of \( f_{ij} \) at the integral vector \( \rho \). This symmetric function is independent of \( r \).

This is trivial for \( i = j = 1 \) since we have \( f_{11} = 1/2 \). It is a highly remarkable fact that the conjectured nonnegativity of the coefficients of \( K_r^{(i,j)} \) reflects a much stronger property, which appears in the expansion of \( f_{ij} \) in terms of monomial symmetric functions.

**Conjecture 11.3.** The inhomogeneous symmetric function \( f_{ij} \) may be written

\[
f_{ij} = \sum_{|\rho| \leq 4i-2j-2} c^{(i,j)}_{\rho} m_{\rho},
\]

where the coefficients \( c^{(i,j)}_{\rho} \) are positive rational numbers.

The nonnegativity of the coefficients of \( K_r^{(i,j)} \) is an obvious consequence. We emphasize that the coefficients of \( f_{ij} \) in terms of any other classical basis of symmetric functions may be negative.

Conjectures 11.2 and 11.3 are supported by extensive computer calculations up to \( r = 20 \), giving many values of the positive numbers \( c^{(i,j)}_{\rho} \). We now review the more elementary cases.
Example $j = 0$

This is the only situation appearing in the classical Kerov-Biane-Féray framework. In this case one has $\alpha = 1, \beta = 0$, so that all components $K_r^{(i,j)}$ disappear for $j \neq 0$.

This classical framework has been extensively studied. Since the components $K_r^{(i,0)}$ correspond to the grading of the classical Kerov polynomial by weights $|\rho| = r - 2i + 1$, we may benefit from any result there obtained.

In particular $K_r^{(1,0)}$ corresponds to the component of weight $r - 1$ for the classical Kerov polynomial. Its value was conjectured by Biane [2] and different proofs were later given by many authors [25, 7, 24, 5].

Theorem 11.4. We have

$$K_r^{(1,0)} = \frac{1}{4} \binom{r + 1}{3} \sum_{|\rho| = r - 1} l(\rho)! R_\rho.$$

A direct proof might be given, following the line of Theorems 10.2 and 10.3, but it is rather tedious. The component of weight $r - 3$ for the classical Kerov polynomial has been also computed in [7, 24]. In [18] it is written as follows.

Theorem 11.5. We have

$$K_r^{(2,0)} = \frac{1}{5760} \binom{r + 1}{3} \sum_{|\rho| = r - 3} (l(\rho) + 2)! f(\rho) R_\rho,$$

with

$$f = 3m_4 + 8m_{31} + 10m_{22} + 16m_{212} + 24m_{14} + 20m_3 + 36m_{24} + 48m_{13} + 35m_2 + 40m_{12} + 18m_1.$$

For $\alpha = 1$ our conjectures have been already presented in [18], where the conjectural value of $f_{i,0}$ is given for $i = 3, 4$. Observe that in [18], $K_r^{(2,0)}$ is written with a factor $\binom{r + 1}{3}$ instead of $r$. Thus we have $f_{i,0} = \frac{1}{6}(r + 1)(r - 1)f_i = \frac{1}{6}(m_1 + 2i)(m_1 + 2i - 2)f_i$ with $f_i$ given in [18, Section 2].

Example $i = j$

We know $f_{11} = 1/2$. Other values of $f_{i,i}$ are conjectured to be

$$24f_{22} = 3m_2 + 4m_{12} + 2m_1,$$

$$2.6! f_{33} = 15m_4 + 40m_{31} + 60m_{22} + 90m_{212} + 144m_{14} + 60m_3 + 120m_{21} + 180m_{13} + 75m_2 + 100m_{12} + 30m_1,$$

$$6.8! f_{44} = 105m_6 + 420m_{51} + 945m_{42} + 1484m_{412} + 1176m_{32} + 2688m_{321} + 4368m_{313} + 3906m_{23} + 6216m_{222} + 10224m_{214} + 17040m_{16} + 1050m_5 + 3430m_{41} + 6132m_{32} + 9520m_{312} + 13356m_{221} + 21168m_{213} + 34080m_{15} + 4025m_4 + 10248m_{31} + 14154m_{22} + 21252m_{212} + 32592m_{14} + 7364m_3 + 13776m_{21} + 19656m_{13} + 6412m_2 + 8120m_{12} + 2128m_1.$$
These values support another conjecture giving the linear terms of $K_{r}^{(i,i)}$, namely

$$K_{r}^{(i,i)} = |s(r, r - i)|R_{r-i+1} + \text{non linear terms},$$

with $s(r, i)$ the usual Stirling number of the first kind.

**Example $i = j + 1$**

We know $24f_{10} = m_{1}(m_{1} + 2) = m_{2} + 2m_{12} + 2m_{1}$. Other values of $f_{i,i-1}$ are conjectured to be

$$2.6!f_{21} = 13m_{4} + 40m_{31} + 55m_{22} + 95m_{212} + 162m_{14}$$
$$+ 68m_{3} + 150m_{21} + 240m_{13} + 103m_{2} + 150m_{12} + 48m_{1},$$

$$30.8!f_{32} = 800m_{6} + 3430m_{51} + 7455m_{42} + 12586m_{412} + 9520m_{32} + 22540m_{321}$$
$$+ 38220m_{313} + 31640m_{32} + 53620m_{2212} + 91260m_{214} + 155850m_{16}$$
$$+ 8920m_{5} + 30870m_{41} + 54320m_{32} + 88816m_{312} + 123200m_{221} + 202440m_{213}$$
$$+ 334200m_{15} + 37480m_{14} + 100380m_{31} + 136990m_{22} + 214746m_{212} + 338520m_{14}$$
$$+ 74160m_{3} + 144970m_{21} + 213668m_{13} + 69120m_{2} + 91280m_{12} + 24320m_{1}.$$

**Example $i = j + 2$**

Theorem 11.5 gives $f_{20}$. For $f_{31}$ the values of $40.11!c_{r}^{(3,1)}$ are conjectured to be

| $r$ | 8 | 71 | 62 | 61² | 53 | 521 | 51³ | 4² |
|-----|---|----|----|-----|----|-----|-----|----|
| 13805 | 80520 | 231550 | 412500 | 418880 | 1034880 | 1830840 | 508662 |
| 431 | 42² | 421² | 41⁴ | 3²2 | 3⁴1² | 3²⁴ | 3²⁴⁴ |
| 1598520 | 2216060 | 3921280 | 6901290 | 2818200 | 4996200 | 6925600 | 12205050 |
| 31³ | 2⁴ | 2⁴¹² | 2⁴¹⁴ | 2¹⁶ | 1⁸ |
| 21411500 | 9617300 | 16938350 | 29737400 | 52047600 | 90902700 |

| $r$ | 6 | 51 | 61 | 52 | 51² | 43 | 421 | 41³ | 3²¹ |
|-----|---|----|----|----|-----|----|-----|-----|-----|
| 292160 | 1448260 | 3563340 | 6201360 | 5462072 | 13046880 | 22528440 | 16548840 |
| 3²² | 32¹⁴ | 3¹¹ | 2¹⁴ | 2²¹⁴ | 2¹⁵ | 1⁶ |
| 22705760 | 39264280 | 67457940 | 53941800 | 92776200 | 158867500 | 271263300 |

| $r$ | 5 | 41 | 42 | 41² | 3² | 3²¹ | 3¹³ | 3²¹² |
|-----|---|----|----|-----|----|-----|-----|-----|
| 2536490 | 10429540 | 21470152 | 36269860 | 27074432 | 62069700 | 103992240 |
| 3²³ | 2²¹² | 2¹⁴ | 1⁶ |
| 84217980 | 141283560 | 235465890 | 390802500 |

| $r$ | 4 | 31 | 3² | 3¹² | 2¹² | 2¹³ | 1³ |
|-----|---|----|-----|-----|-----|-----|-----|
| 11667920 | 38560412 | 64473112 | 104816800 | 140167720 | 225795680 | 361308200 |
| 4³ | 3¹² | 3¹⁴ | 2¹⁴ | 1³ |
| 30608765 | 77913572 | 102006542 | 157306160 | 239692200 |
| 3³ | 2² | 1⁴ | 2 | 1² | 1 |
| 45634160 | 84412152 | 119902200 | 35681580 | 45160632 | 11246400 |

31
12 Q-positivity

For a better understanding of the previous results, it is necessary to introduce new polynomials $C_i$ and $Q_i$ in the free cumulants.

Define $C_0 = Q_0 = 1$, $C_1 = Q_1 = 0$ and for any $n \geq 2$,

$$C_n = \sum_{|\rho|=n} l(\rho)! \mathcal{R}_\rho, \quad Q_n = \sum_{|\rho|=n} (l(\rho) - 1)! \mathcal{R}_\rho.$$

Using the notation

$$C_\rho = \prod_{i \geq 2} C_i^{m_i(\rho)}/m_i(\rho)!, \quad Q_\rho = \prod_{i \geq 2} Q_i^{m_i(\rho)}/m_i(\rho)!,$$

the correspondence between these three families of polynomials is given by

$$C_n = \sum_{|\rho|=n} Q_\rho, \quad Q_n = \sum_{|\rho|=n} (-1)^{l(\rho)} (l(\rho) - 1)! C_\rho,$$

$$(1 - n)R_n = \sum_{|\rho|=n} (-1)^{l(\rho)} Q_\rho = \sum_{|\rho|=n} (-1)^{l(\rho)} l(\rho)! C_\rho.$$

These relations are better understood by using symmetric functions. Actually let $A_\lambda$ be the alphabet defined by

$$(1 - n)R_n = h_n(A_\lambda), \quad Q_n = -p_n(A_\lambda)/n, \quad C_n = (-1)^n e_n(A_\lambda).$$

We emphasize that in spite of

$$(1 - n)R_n(\lambda) = h_n((1 - n)A_\lambda) = h_n(A_\lambda),$$

the connection between the two alphabets $A_\lambda$ and $A_\lambda = I_\lambda - O_\lambda$ is still unclear. Therefore $A_\lambda$ may only be considered as formal. Then the previous relations are merely the classical properties [20, pp. 25 and 33]

$$p_n = -n \sum_{|\rho|=n} (-1)^{l(\rho)} u_\rho h_\rho/l(\rho) = -n \sum_{|\rho|=n} (-1)^{n-l(\rho)} u_\rho e_\rho/l(\rho),$$

$$e_n = \sum_{|\rho|=n} (-1)^{n-l(\rho)} u_\rho h_\rho = \sum_{|\rho|=n} (-1)^{n-l(\rho)} z_\rho^{-1} p_\rho,$$

$$h_n = \sum_{|\rho|=n} z_\rho^{-1} p_\rho = \sum_{|\rho|=n} (-1)^{n-l(\rho)} u_\rho e_\rho.$$

Obviously Q-positivity implies R-positivity. Therefore the following conjecture is a priori stronger than the positivity statement of Conjecture 11.1. It does not however implies integrality of the $R$-coefficients.

32
Conjecture 12.1. For any \((i, j) \neq (0, 0)\) the coefficients of \(K_{r}^{(i,j)}\) in terms of the indeterminates \(Q_k\) are nonnegative rational numbers.

It is clear that \(K_{r}^{(0,0)} = R_{r+1}\) is not \(Q\)-positive. But by Theorems 10.3 and 11.4 we have

\[
K_{r}^{(1,1)} = \frac{r}{2} Q_r, \quad K_{r}^{(1,0)} = \frac{1}{4} \binom{r + 1}{3} \sum_{|\rho| = r-1} Q_\rho.
\]

Remarkably this \(Q\)-positivity appears completely analogous (and possibly equivalent) to the \(R\)-positivity studied above. We conjecture that all components \(K_{r}^{(i,j)}\) but two may be described in a unified way, independent of \(r\).

Conjecture 12.2. For any \((i, j)\) but \((0, 0)\) and \((1, 1)\), there exists an inhomogeneous symmetric function \(g_{ij}\) having maximal degree \(4i - 2j - 2\) such that

\[
K_{r}^{(i,j)} = r \sum_{|\rho| = r-2i+j+1} (2i - j - 1)^{l(\rho)} g_{ij}(\rho) Q_\rho,
\]

where \(g_{ij}(\rho)\) denotes the value of \(g_{ij}\) at the integral vector \(\rho\). This symmetric function is independent of \(r\).

As in Section 11 the conjectured positivity of the \(Q\)-coefficients of \(K_{r}^{(i,j)}\) reflects a stronger property, given by the expansion of \(g_{ij}\) in terms of monomial symmetric functions.

Conjecture 12.3. The inhomogeneous symmetric function \(g_{ij}\) may be written

\[
g_{ij} = \sum_{|\rho| \leq 4i - 2j - 2} b_{\rho}^{(i,j)} m_{\rho},
\]

where the coefficients \(b_{\rho}^{(i,j)}\) are positive rational numbers.

The assertion of Conjecture 12.1 is a direct consequence. Conjectures 12.2 and 12.3 are supported by computer calculations up to \(r = 20\).

Example \(j = 0\)

This is the only situation appearing in the classical framework \(\alpha = 1\). It has been investigated in [18], where the values of \(b_{\rho}^{(i,0)}\) are proved for \(i = 1, 2\) and conjectured for \(i = 3, 4\). There \(K_{r}^{(i,0)}\) is written with a factor \(\binom{r + 1}{3}\) instead of \(r\). Thus we have \(g_{i0} = \frac{1}{6}(r + 1)(r - 1)g_i = \frac{1}{6}(m_1 + 2i)(m_1 + 2i - 2)g_i\) with \(g_i\) as in [18, Section 5].
Example $i = j$

Values of $g_i$ are conjectured to be

$$24g_{22} = 3m_2 + 2m_{12} + 2m_1,$$
$$96g_{33} = m_4 + 2m_{31} + 3m_{22} + 3m_{212} + 3m_{14} + 4m_3 + 6m_{21} + 6m_{13} + 5m_2 + 5m_{12} + 2m_1,$$
$$274!6!g_{44} = 405m_6 + 1350m_{51} + 2835m_{42} + 3630m_{412} + 3348m_{32} + 6012m_{321} + 7512m_{313} + 8370m_{23} + 9876m_{2212} + 15952m_{214} + 12880m_{16} + 4050m_5 + 10890m_{41} + 18036m_{32} + 22536m_{312} + 29628m_{221} + 34776m_{213} + 38640m_{15} + 15525m_4 + 32004m_{31} + 41742m_{22} + 48924m_{212} + 53880m_{14} + 28404m_3 + 42444m_{21} + 45720m_{13} + 24732m_2 + 25272m_1 + 8208m_1.$$ 

Example $i = j + 1$

We know $g_{10} = m_1(m_1 + 2)/24 = (m_2 + 2m_{12} + 2m_1)/24$. Other values of $g_{i, i-1}$ are conjectured to be

$$4.6!g_{21} = 26m_4 + 68m_{31} + 87m_{22} + 123m_{212} + 147m_{14} + 136m_3 + 246m_{21} + 294m_{13} + 206m_2 + 244m_{12} + 96m_1,$$
$$486.7!g_{32} = 3240m_6 + 12042m_{51} + 24057m_{42} + 35496m_{412} + 29808m_{32} + 57636m_{321} + 79848m_{313} + 75618m_{23} + 103056m_{2212} + 132336m_{214} + 157840m_{16} + 36126m_5 + 106488m_{41} + 172908m_{32} + 239544m_{312} + 309168m_{221} + 397008m_{213} + 473520m_{15} + 151794m_4 + 339660m_{31} + 436158m_{22} + 557838m_{212} + 659376m_{14} + 300348m_3 + 482490m_{21} + 557568m_{13} + 279936m_2 + 306288m_{12} + 98496m_1.$$ 

Example $i = j + 2$

For $g_{20}$ see [18, Section 6]. For $g_{31}$ the values of $256.10!b_3^{(3,1)}$ are conjectured to be

| $n$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $8$ | 48192 | 254976 | 673056 | 1133712 | 1153152 | 2548656 | 4133592 | 1376256 |
| $9$ | 431 | 42^{2} | 421^{2} | 41^{4} | 3^{2}2 | 31^{3} | 32^{1} | 321^{4} |
| $10$ | 3727584 | 4862736 | 7769952 | 12008844 | 5941920 | 9494112 | 12127104 | 18474444 |
| $11$ | 31^{5} | 2^{4} | 2^{4}1^{2} | 2^{4}1^{4} | 21^{6} | 1^{8} |
| $12$ | 26908920 | 15434568 | 23152176 | 33262758 | 45158040 | 57215025 |
| $13$ | 7 | 61 | 52 | 51^{2} | 43 | 421 | 41^{3} | 3^{2}1 |
| $14$ | 1019904 | 4534848 | 10194624 | 16534368 | 14910336 | 31079808 | 48035376 | 37976448 |
| $15$ | 32^{4} | 321^{2} | 31^{4} | 2^{4}1 | 2^{4}1^{3} | 21^{5} | 1^{6} |
| $16$ | 48508416 | 73897776 | 106735680 | 92608704 | 133051032 | 180632160 | 228860100 |
In the classical Kerov-Biane framework, Goulden and Rattan [7, 24] considered the expansion of $K_r$ in terms of the indeterminates $C_k$. They conjectured that the components of weight $r - 2i + 1$ (which correspond to our $K_r^{i,0}$ when $\alpha = 1$) have nonnegative rational coefficients in terms of the $C_k$'s.

It might be tempting to extend this conjecture to the $C$-expansion of any $K_r^{(i,j)}$ for $(i, j) \neq (0, 0)$. However this property is not true. When $j \neq 0$ the $C$-coefficients of $K_r^{(i,j)}$ may be negative.

It is already obvious that

$$K_r^{(1,1)} = \frac{r}{2} Q_r = \frac{r}{2} \sum_{|\rho|=r} (-1)^{l(\rho)} (l(\rho) - 1)! C_\rho$$

is not $C$-positive. But the property fails also for $K_r^{(2,2)}$ which has negative coefficients for $r \geq 5$. This is also the case for $K_r^{(2,1)}$ when $r \geq 8$, and for $K_r^{(4,4)}$ or $K_r^{(3,2)}$ for $r \geq 13$. The $C$-expansion of $K_r$ is therefore very different from the $R$ or $Q$-expansions.

In the classical framework $\alpha = 1$, we already observed [18, Sections 6-8] that the $R$ and $Q$-expansions of $K_r^{(i,0)}$ are connected through the Cauchy formulas (3.3)-(3.5). This is a general fact.

Actually if we compare both expressions of $K_r^{(i,j)}/r$ the equation

$$\sum_{|\rho|=r-2i+j+1} (2i - j - 1)^{l(\rho)} g_{ij}(\rho) Q_\rho = \sum_{|\rho|=r} (l(\rho) + 2i - j - 2)! f_{ij}(\rho) R_\rho$$

is merely the abstract identity

$$\sum_{|\rho|=n} (-1)^{l(\rho)} g_{ij}(\rho) (2i - j - 1)^{l(\rho)} z_\rho^{-1} p_\rho(A) = \sum_{|\rho|=n} (-1)^{l(\rho)} f_{ij}(\rho) \frac{(l(\rho) + 2i - j - 2)!}{\prod_i m_i(\rho)!} h_\rho(A)$$

$$= \sum_{|\rho|=n} f_{ij}(\rho) \left(-2i + j + 1\right) u_\rho h_\rho(A),$$

13 \hspace{1cm} \lambda\text{-rings}

In the classical Kerov-Biane framework, Goulden and Rattan [7, 24] considered the expansion of $K_r$ in terms of the indeterminates $C_k$. They conjectured that the components of weight $r - 2i + 1$ (which correspond to our $K_r^{i,0}$ when $\alpha = 1$) have nonnegative rational coefficients in terms of the $C_k$'s.

It might be tempting to extend this conjecture to the $C$-expansion of any $K_r^{(i,j)}$ for $(i, j) \neq (0, 0)$. However this property is not true. When $j \neq 0$ the $C$-coefficients of $K_r^{(i,j)}$ may be negative.

It is already obvious that

$$K_r^{(1,1)} = \frac{r}{2} Q_r = \frac{r}{2} \sum_{|\rho|=r} (-1)^{l(\rho)} (l(\rho) - 1)! C_\rho$$

is not $C$-positive. But the property fails also for $K_r^{(2,2)}$ which has negative coefficients for $r \geq 5$. This is also the case for $K_r^{(2,1)}$ when $r \geq 8$, and for $K_r^{(4,4)}$ or $K_r^{(3,2)}$ for $r \geq 13$. The $C$-expansion of $K_r$ is therefore very different from the $R$ or $Q$-expansions.

In the classical framework $\alpha = 1$, we already observed [18, Sections 6-8] that the $R$ and $Q$-expansions of $K_r^{(i,0)}$ are connected through the Cauchy formulas (3.3)-(3.5). This is a general fact.

Actually if we compare both expressions of $K_r^{(i,j)}/r$ the equation

$$\sum_{|\rho|=r-2i+j+1} (2i - j - 1)^{l(\rho)} g_{ij}(\rho) Q_\rho = \sum_{|\rho|=r} (l(\rho) + 2i - j - 2)! f_{ij}(\rho) R_\rho$$

is merely the abstract identity

$$\sum_{|\rho|=n} (-1)^{l(\rho)} g_{ij}(\rho) (2i - j - 1)^{l(\rho)} z_\rho^{-1} p_\rho(A) = \sum_{|\rho|=n} (-1)^{l(\rho)} f_{ij}(\rho) \frac{(l(\rho) + 2i - j - 2)!}{\prod_i m_i(\rho)!} h_\rho(A)$$

$$= \sum_{|\rho|=n} f_{ij}(\rho) \left(-2i + j + 1\right) u_\rho h_\rho(A),$$

35
specialized at $A = A_\lambda$. Moreover it has obvious links with the Cauchy formulas (3.3)-(3.5)
\[
(-1)^n e_n(xA) = \sum_{|\rho|=n} (-1)^{l(\rho)} x^{l(\rho)} z_\rho^{-1} p_\rho(A) = \sum_{|\rho|=n} \left( \frac{-x}{l(\rho)} \right) u_\rho h_\rho(A).
\]

Therefore it seems plausible that the conjectured positivity of $f_{ij}$ and $g_{ij}$ are two equivalent properties, reflecting some abstract pattern of the theory of symmetric functions.

The powerful language of $\lambda$-rings seems necessary for a better understanding of the interplay between these expansions and Cauchy formulas. Here we shall not give details about $\lambda$-ring theory, and refer the reader to [12, Chapter 2] (or [13, Section 3] for a short survey).

The simplest case case $i = j = 2$ is already a very interesting example. We have conjectured
\[
24 f_{22} = 3m_2 + 4m_{12} + 2m_1 = p_2 + 2p_1^2 + 2p_1,
\]
\[
24 g_{22} = 3m_2 + 2m_{12} + 2m_1 = 2p_2 + p_1^2 + 2p_1.
\]
The equality
\[
K^{(2,2)}_\lambda / r = \sum_{|\rho|=r-1} g_{22}(\rho) Q_\rho = \sum_{|\rho|=r-1} l(\rho)! f_{22}(\rho) R_\rho
\]
is then the specialization at the alphabet $A_\lambda$ of
\[
\sum_{|\rho|=n} (-1)^{l(\rho)} g_{22}(\rho) z_\rho^{-1} p_\rho = \sum_{|\rho|=n} (-1)^{l(\rho)} f_{22}(\rho) u_\rho h_\rho,
\]
which is itself a direct consequence of the following identities.

**Proposition 13.1.** For any alphabet $A$ and any integer $n$, one has
\[
\sum_{(i,j,k)\in \mathbb{N}^3} (-1)^i h_i e_j e_k = -n e_n,
\]
\[
\sum_{(i,j,k)\in \mathbb{N}^3, i+j+k=n} (-1)^i i^2 h_i e_j e_k = \sum_{|\rho|=n} (-1)^{n-l(\rho)} (n^2 - 2p_2(\rho)) z_\rho^{-1} p_\rho
\]
\[
= - \sum_{|\rho|=n} (-1)^{n-l(\rho)} p_2(\rho) u_\rho h_\rho.
\]

**Sketch of proof.** If $f$ is a symmetric function, we denote by $f[A]$ its $\lambda$-ring action on the alphabet $A$, which should not be confused with its evaluation $f(A)$.

We consider a one-variable alphabet $z$ such that $h_k(zA) = z^k h_k(A)$. In $\lambda$-ring terminology $z$ is an “element of type 1” and we have $p_n[-z+2] = -z^n + 2$. We start from the two “Cauchy formulas”
\[
(-1)^n e_n[(-z+2)A] = \sum_{|\rho|=n} (-1)^{l(\rho)} z_\rho^{-1} p_\rho[-z+2] p_\rho[A],
\]
\[
= \sum_{|\rho|=n} m_\rho[z-2] h_\rho[A].
\]
If we \( z \)-differentiate once and fix \( z = 1 \), we obtain the first statement. If we differentiate two times and fix \( z = 1 \), we obtain at the left-hand side

\[
(-1)^n \sum_{i+j+k=n} (-1)^i(i-1)h_ie_j e_k.
\]

At the right-hand side, we compute directly

\[
\frac{\partial^2}{\partial z^2} (p_\rho[-z+2])\bigg|_{z=1} = \frac{\partial^2}{\partial z^2} \left( \prod_{i \geq 1} (-z^i + 2)^{m_i(\rho)} \right)\bigg|_{z=1} = n^2 + n - 2p_2(\rho).
\]

Since “augmented” monomial symmetric functions \( \prod m_i(\rho)! m_\rho \) are integral combinations of power sums [20, p. 110], we may obtain similarly

\[
\prod_{i \geq 1} m_i(\rho)! \frac{\partial^2}{\partial z^2} (m_\rho[z-2]) \bigg|_{z=1} = (-1)^{l(\rho)} l(\rho)! (n - p_2(\rho)).
\]

Combining both results achieves the proof. \( \square \)

By specialization of the second statement at \( A = A_\lambda \), we see that the conjectured values of \( f_{22} \) and \( g_{22} \) are equivalent with

\[
K_r^{(2,2)} = \frac{r}{24}(2r(r-1)C_{r-1} + \sum_{(i,j,k) \in \mathbb{N}^3 \atop i+j+k=r-1} i^2(i-1)R_iC_jC_k),
\]

which explains why \( K_r^{(2,2)} \) cannot be \( C \)-positive.

For a similar example with \( K_r^{(2,0)} \) see [18, Sections 6-8]. These two examples remain elementary because they only involve symmetric functions of degree 2.

In general the study of \( K_r^{(i,j)} \) may be done by differentiating \( 4i - 2j - 2 \) times the Cauchy formulas

\[
(-1)^n e_n([-z+2i-j]A) = \sum_{|\rho|=n} (-1)^{l(\rho)} z_\rho^{-1} p_\rho[-z+2i-j] p_\rho[A],
\]

\[
= \sum_{|\rho|=n} m_\rho[z-2i+j] h_\rho[A],
\]

and fixing \( z = 1 \). On the right-hand side one must compute

\[
\frac{\partial^k}{\partial z^k} (p_\rho[-z+2i-j])\bigg|_{z=1} = (2i - j - 1)^{l(\rho)} F_k(\rho),
\]

\[
\prod_{i \geq 1} m_i(\rho)! \frac{\partial^k}{\partial z^k} (m_\rho[z-2i+j]) \bigg|_{z=1} = (-1)^{l(\rho)} (l(\rho) + 2i - j - 2)! G_k(\rho),
\]

with \( k \leq 4i - 2j - 2 \) and \( F_k, G_k \) some symmetric functions of degree \( k \).

For \( i = j = 3 \) this method shows that the conjectured values of \( f_{33} \) and \( g_{33} \) are equivalent with the beautiful nonnegative expansion

\[
K_r^{(3,3)} = \frac{r}{48} \sum_{(i,j) \in \mathbb{N}^2 \atop i+j=r-2} i(i+1)^2(i+2) C_iC_j.
\]

37
14 The general case

When the partition $\mu$ is not a row, the $R$-coefficients of $K_\mu$ are still conjectured to be polynomials in $(\alpha, \beta)$ with integer coefficients. However these integers may be negative. For instance

$$K_{22} = \alpha^4 R_2^2 + 2\alpha^3 \beta R_3 R_2 - \alpha^3(4R_4 + 2R_3^2) + \alpha^2 \beta^2 R_2^2 - 10\alpha^2 \beta R_3 - (2\alpha^2 + 6\alpha \beta^2) R_2.$$ 

This fact already appears in the classical framework $\alpha = 1$. In this case, Féray [5] observed that another family of polynomials $\tilde{K}_\mu$ can be inductively defined by

$$K_\mu = \sum_{k=1}^{l(\mu)} (-1)^{l(\mu)-k} \sum_{(\nu_1, \ldots, \nu_k)} \prod_{i=1}^k \tilde{K}_{\nu_i},$$

where the second sum is taken over all decompositions of the $l(\mu)$ parts of $\mu$ into $k$ disjoint partitions $(\nu_1, \ldots, \nu_k)$. For instance one has

$$K_r = \tilde{K}_r,$$
$$K_{r,s} = K_r K_s - \tilde{K}_{r,s},$$
$$K_{r,s,t} = K_r K_s K_t - K_r \tilde{K}_{s,t} - K_s \tilde{K}_{r,t} - K_t \tilde{K}_{r,s} + \tilde{K}_{r,s,t}.$$ 

The first values of $\tilde{K}_\mu$ are listed in the introduction and those of $\tilde{K}_{rs}$ for $r+s \leq 18$ on a web page [19].

For $\alpha = 1$ Féray [5] proved the $R$-coefficients of $\tilde{K}_\mu$ to be nonnegative. Remarkably this property seems true when $\alpha \neq 1$. We conjecture the structure of $\tilde{K}_\mu$ to be very similar to the one of $K_r$.

Conjecture 14.1. The highest weight of $\tilde{K}_\mu$ is $|\mu| - l(\mu) + 2$ and we have

$$\tilde{K}_\mu = \sum_{0 \leq j \leq |\mu| - l(\mu)} \alpha^{(|\mu|-l(\mu)+1-i)\beta^j} \sum_{|\rho|=|\mu|-l(\mu)+2-2i+j} a_{ij}(\rho) R_\rho,$$

where the coefficients $a_{ij}(\rho)$ are nonnegative integers.

However $\tilde{K}_\mu^{(i,j)}$, the coefficient of $\alpha^{(|\mu|-l(\mu)+1-i)\beta^j}$, is much more complicated than $K_r^{(i,j)}$. In particular the term of highest weight $\tilde{K}_\mu^{(0,0)}$ is no longer a monomial. Also it is not clear which $\tilde{K}_\mu^{(i,j)}$ have nonnegative $Q$-coefficients. At least $\tilde{K}_\mu^{(0,0)}$ and $\tilde{K}_\mu^{(1,1)}$ may have negative ones.

15 Extension to arbitrary $\beta$

In this article $\beta$ was of course never considered as being independent of $\alpha$. However, as already emphasized, the restriction $\alpha + \beta = 1$ is totally unnecessary. All the results obtained above may be extended when $\alpha$ and $\beta$ are independent parameters.
We now present this elegant generalization. Let \( \zeta \) and \( \eta \) be two real numbers with \( \zeta < 0 \) and \( \eta > 0 \). Define
\[
\alpha = -\frac{1}{\zeta \eta}, \quad \beta = \frac{1}{\zeta} + \frac{1}{\eta}.
\]
In other words, \( \zeta \) and \( \eta \) are the roots of \( \alpha x^2 + \beta x - 1 = 0 \). When \( \beta = 1 - \alpha \) as above, we have obviously \( \eta = 1 \) and \( \zeta = -1/\alpha \).

Our advocated generalization is obtained by appropriate substitutions of \( \eta \) and \( \zeta \) instead of 1 and \(-1/\alpha\). Incidentally this makes formulas become very symmetrical.

Firstly let us introduce the \((\zeta, \eta)\)-transition measure of any Young diagram. For any partition \( \lambda \), the \((\zeta, \eta)\)-content of a node \((i, j)\) \(\in\lambda\) is defined as \((i - 1)\zeta + (j - 1)\eta\).

Accordingly the generalized rising factorial is
\[
(x)_{\lambda} = \prod_{(i,j) \in \lambda} (x + (i - 1)\zeta + (j - 1)\eta).
\]

The \((\zeta, \eta)\)-contents of the inside and outside corners of the Young diagram of \(\lambda\) define a pair of interlacing sequences
\[
x_k(\zeta, \eta) = \lambda'_k \zeta + (k - 1)\eta, \quad y_k(\zeta, \eta) = \lambda'_k \zeta + k\eta,
\]
with \(1 \leq k \leq \lambda_1 = d - 1\) and \(x_d(\zeta, \eta) = \lambda_1\eta\). Here we maintain the convention (5.1) that \(x_i\) and \(y_{i-1}\) should be omitted whenever \(x_i = y_{i-1}\). This pair has center 0.

The \((\zeta, \eta)\)-transition measure is a measure \(\omega_{\lambda}\) on the real line, supported on the set \(\{x_1(\zeta, \eta), \ldots, x_d(\zeta, \eta)\}\). It is uniquely described by its moment generating series
\[
M_{\lambda}(z) = \frac{1}{z - x_d(\zeta, \eta)} \prod_{i=1}^{d-1} \frac{z - y_i(\zeta, \eta)}{z - x_i(\zeta, \eta)} = z^{-1}H_{1/z}(A_{\lambda}),
\]
with \(A_{\lambda} = I_{\lambda} - O_{\lambda}\) the difference of the alphabets
\[
I_{\lambda} = \{x_1(\zeta, \eta), \ldots, x_d(\zeta, \eta)\}, \quad O_{\lambda} = \{y_1(\zeta, \eta), \ldots, y_{d-1}(\zeta, \eta)\}.
\]
The moments of \(\omega_{\lambda}\) are \(M_k(\lambda) = h_k(A_{\lambda})\) and its free cumulants are \(R_k(\lambda) = (-1)^k c_k^*(A_{\lambda})\).

Exactly as in Section 6, the moment series may be written
\[
M_{\lambda}(z) = z^{-1} \frac{(-z + \zeta + \eta)^{\lambda}}{(-z + \zeta)^{\lambda}} \frac{(-z)^{\lambda}}{(-z + \eta)^{\lambda}}.
\]

This expression has several consequences. Firstly the moments \(M_k(\lambda)\) and the free cumulants \(R_k(\lambda)\) are polynomials in \((\zeta, \eta)\). Secondly for any integer \(k \geq 0\) we have
\[
M_k(\lambda) = \sum_{i=1}^{i(\lambda)+1} c_i(\lambda) ((i - 1)\zeta + \lambda_i \eta)^k,
\]
where the weights \( c_i(\lambda) \) are the transition probabilities

\[
c_i(\lambda) = \frac{\zeta^{l(\lambda)+1}}{(l(\lambda) - i + 2)\zeta - \lambda_i \eta} \prod_{j \neq i} (j - i + 1)\zeta + (\lambda_j - \lambda_i)\eta.\]

Equivalently we have

\[
\mathcal{M}_\lambda(z) = \sum_{i=1}^{l(\lambda)+1} \frac{c_i(\lambda)}{z - (i - 1)\zeta - \lambda_i \eta}.
\]

Now we have the following fundamental remark. Although moments and free cumulants are polynomials in \((\zeta, \eta)\), all formulas of our algorithm keep unchanged because these formulas only involve \(\zeta + \eta\) and \(\zeta \eta\), hence \(\alpha\) and \(\beta\).

This is proved very easily. Denote \(x_i = (i - 1)\zeta + \lambda_i \eta\). As in Proposition 8.1, relation (16.1) implies

\[
H_{1/z}(A\lambda(x)) = H_{1/z}(A\lambda) \frac{-z + x_i + \zeta + \eta}{-z + x_i + \zeta - z + x_i + \eta}\]

Equivalently \(A\lambda(x) = A\lambda + B(x)\), with \(B(v)\) the alphabet \(\{v + \zeta, v + \eta\} - \{v, v + \zeta + \eta\}\). But as in Proposition 8.2, with \(y = (z - v)^{-1}\) we have

\[
H_{1/z}(uB(v)) = \left(\frac{1 - y(\zeta + \eta)}{1 - y\zeta} \frac{1 - y\eta}{1 - y\eta}\right)^u = \left(1 - \frac{y^2/\alpha}{1 + y\beta/\alpha}\right)^{-u}.
\]

In other words, the symmetric functions of \(uB(v)\) depend only on \(\alpha\) and \(\beta\). This implies that Propositions 8.1-8.5 and Theorem 8.4 keep formally the same provided the quantity \(x_i = \lambda_i - (i - 1)/\alpha\) is replaced by \(x_i = (i - 1)\zeta + \lambda_i \eta\).

Here is an elementary example. For \(\lambda = (r, s, t)\) relations (5.4) and (16.2) yield easily

\[
R_2(\lambda) = -\eta(r + s + t),
\]

\[
R_3(\lambda) = -\eta((r + 3s + 5t)\zeta + (r^2 + s^2 + t^2)\eta),
\]

\[
R_4(\lambda) = -\eta((r + 7s + 19t)\zeta^2 + (3r^2 + 6s^2 + 9t^2 + 3rs + 3st + 3rt)\zeta + (r^3 + s^3 + t^3)\eta^2).
\]

But for \(1 \leq i \leq 3\) these quantities still satisfy

\[
\alpha R_2(\lambda(i)) = \alpha R_2(\lambda) + 1,
\]

\[
\alpha^2 R_3(\lambda(i)) = \alpha^2 R_3(\lambda) + 2\alpha x_i - \beta,
\]

\[
\alpha^3 R_4(\lambda(i)) = \alpha^3 R_4(\lambda) + 3\alpha^2 x_i^2 - 3\alpha \beta x_i - 3\alpha^2 R_2(\lambda) + \beta^2 - \alpha,
\]

with \(x_1 = r\eta, x_2 = \zeta + s\eta, x_3 = 2\zeta + t\eta\).
Incidentally these formulas also show, as in Proposition 8.6, that $-R_k(\lambda)$ is a polynomial in $(\zeta, \eta)$ with \textit{nonnegative} integer coefficients. This property does not hold for moments.

Since our algorithm keeps formally the same, we may define quantities $\vartheta^\lambda_\mu(\zeta, \eta)$ inductively by the system

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \vartheta^\lambda_\mu^{(i)} = \vartheta^\lambda_\mu,$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \left((i-1)\zeta + \lambda_i \eta\right) \vartheta^\lambda_\mu^{(i)} = 2(|\lambda| - |\mu| + 2) m_2(\mu) \vartheta^\lambda_{\mu|2} + \sum_{r\geq 3} r m_r(\mu) \vartheta^\lambda_{\mu^{(r)}}.$$ 

Obviously these quantities $\vartheta^\lambda_\mu$ depend on $(\zeta, \eta)$. However their transition matrix with the free cumulants depends only on $(\alpha, \beta)$. In this context, all results and conjectures of Sections 8 to 14 keep relevant, without any change.

Finally let us mention another conjecture, which does not involve free cumulants. Consider the finite alphabet of $(\zeta, \eta)$-contents

$$C_\lambda = \{(i-1)\zeta + (j-1)\eta, (i,j) \in \lambda\}$$

and its power sums

$$p_k(C_\lambda) = \sum_{(i,j) \in \lambda} ((i-1)\zeta + (j-1)\eta)^k \quad (k \geq 1).$$

Again the quantities $\vartheta^\lambda_\mu$ and the power sums $p_\rho(C_\lambda)$ depend on $(\zeta, \eta)$, but their transition matrix depends only on $(\alpha, \beta)$.

**Conjecture 15.1.** The quantity $\vartheta^\lambda_\mu(\zeta, \eta)$ is a polynomial in power sums $p_0(C_\lambda) = |\lambda|$ and $\{p_k(C_\lambda), k \geq 1\}$. Once written in terms of $\binom{|\lambda|}{r}$ instead of $|\lambda|^r$, its coefficients are polynomials in $(\alpha, \beta)$ with integer coefficients.

For instance we have

$$\vartheta^\lambda_2 = 2\alpha p_1(C_\lambda),$$

$$\vartheta^\lambda_3 = 3\alpha^2 p_2(C_\lambda) + 3\alpha \beta p_1(C_\lambda) - 3\alpha \binom{|\lambda|}{2},$$

$$\vartheta^\lambda_{2,2} = -12\alpha^2 p_2(C_\lambda) + 4\alpha^2 p_{11}(C_\lambda) + 8\alpha \beta p_1(C_\lambda) + 8\alpha \binom{|\lambda|}{2},$$

$$\vartheta^\lambda_4 = 4\alpha^3 p_3(C_\lambda) + 12\alpha^2 \beta p_2(C_\lambda) + (8\alpha \beta^2 - 4\alpha^2(2|\lambda| - 3)) p_1(C_\lambda) - 8\alpha \beta \binom{|\lambda|}{2}. $$
16 Final remarks

We conclude by two remarks. Firstly the existence of quantities $\vartheta_{\lambda}^{\mu}$ associated with two independent parameters $(\alpha, \beta)$ may lead to a generalization of Jack polynomials. Actually these quantities $\vartheta_{\lambda}^{\mu}$ may be used in a sum extending (3.1). However the precise form of this extension is not yet obvious.

Secondly the existence of a combinatorial scheme, underlying the theory of Jack polynomials, has been suspected for a long time, though it remains mysterious. Our results give strong evidence for the existence of a pattern involving free cumulants, with $\alpha$ and $\beta$ playing similar roles. Such a combinatorial interpretation has been recently obtained by Féray [5] for $\alpha = 1$. We expect a generalization to Jack polynomials.

References

[1] P. Biane, *Representations of symmetric groups and free probability*, Adv. Math. **138** (1998), 126-181.

[2] P. Biane, *Characters of symmetric groups and free cumulants*, Lecture Notes in Math. **1815** (2003), 185-200, Springer, Berlin, 2003.

[3] P. Biane, *On the formula of Goulden and Rattan for Kerov polynomials*, Sém. Lothar. Combin. **55** (2006), article B55d.

[4] M. Dolega, V. Féray, P. Śniady, *Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations*, arXiv 0810.3209.

[5] V. Féray, *Combinatorial interpretation and positivity of Kerov’s character polynomials*, arXiv 0710.5885.

[6] I. P. Goulden, D. M. Jackson, *Combinatorial enumeration*, Wiley Interscience, New York, 1983.

[7] I. P. Goulden, A. Rattan, *An explicit form for Kerov’s character polynomials*, Trans. Amer. Math. Soc. **359** (2007), 3669–3685.

[8] S. V. Kerov, *Transition probabilities for continual Young diagrams and the Markov moment problem*, Funct. Anal. Appl. **27** (1993), 104-117.

[9] S. V. Kerov, *Anisotropic Young diagrams and Jack symmetric functions*, Funct. Anal. Appl. **34** (2000), 41-51.

[10] S. V. Kerov, talk at IHP Conference (2000).

[11] F. Knop, S. Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Invent. Math. **128** (1997), 9–22.
[12] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS Regional Conference Series in Mathematics **99**, Amer. Math. Soc., Providence, 2003.

[13] M. Lassalle, *Une q-spécialisation pour les fonctions symétriques monomiales*, Adv. Math. **162** (2001), 217–242.

[14] M. Lassalle, *Une formule de Pieri pour les polynômes de Jack*, C. R. Acad. Sci. Paris, Sér. I Math. **309** (1989), 941–944.

[15] M. Lassalle, *Jack polynomials and some identities for partitions*, Trans. Amer. Math. Soc. **356** (2004), 3455–3476.

[16] M. Lassalle, *An explicit formula for the characters of the symmetric group*, Math. Ann. **340** (2008), 383–405.

[17] M. Lassalle, *A positivity conjecture for Jack polynomials*, Math. Res. Lett. **15** (2008), 661–681.

[18] M. Lassalle, *Two positivity conjectures for Kerov polynomials*, Adv. Appl. Math. **41** (2008), 407–422.

[19] M. Lassalle, available at http://igm.univ-mlv.fr/~lassalle/free.html

[20] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, second edition, Oxford, 1995.

[21] A. Okounkov, G. Olshanski, *Shifted Jack polynomials, binomial formula and applications*, Math. Res. Lett. **4** (1997), 69–78.

[22] A. Okounkov, G. Olshanski, *Shifted Schur functions*, St. Petersburg Math. J. **9** (1998), 239–300.

[23] A. Okounkov, *Shifted Macdonald polynomials, q-integral representation and combinatorial formula*, Compos. Math. **112** (1998), 147–182.

[24] A. Rattan, *Character polynomials and Lagrange inversion*, Thesis (2005), Waterloo University.

[25] P. Śniady, *Asymptotics of characters of symmetric groups and free probability*, Discrete Math. **306** (2006), 624–665.

[26] R. Speicher, *Free probability theory and non-crossing partitions*, Sém. Lothar. Combin. **39** (1997), article B39c.

[27] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), 76–115.

[28] D. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66** (1986), 323–346.