FLAG: Fast Linearly-Coupled Adaptive Gradient Method

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Abstract

The celebrated Nesterov’s accelerated gradient method offers great speed-ups compared to the classical gradient descent method as it attains the optimal first-order oracle complexity for smooth convex optimization. On the other hand, the popular AdaGrad algorithm competes with mirror descent under the best regularizer by adaptively scaling the gradient. Recently, it has been shown that the accelerated gradient descent can be viewed as a linear combination of gradient descent and mirror descent steps. Here, we draw upon these ideas and present a fast linearly-coupled adaptive gradient method (FLAG) as an accelerated version of AdaGrad, and show that our algorithm can indeed offer the best of both worlds. Like Nesterov’s accelerated algorithm and its proximal variant, FISTA, our method has a convergence rate of $1/T^2$ after $T$ iterations. Like AdaGrad our method adaptively chooses a regularizer, in a way that performs almost as well as the best choice of regularizer in hindsight.

1 Introduction

We consider the problem of minimizing a convex function $F$ over a convex set $C \subset \mathbb{R}^d$, and we focus on first order methods, which exploit only value and gradient information about
These methods have become very important for many of the large-scale optimization problems that arise in machine learning applications. Two techniques have emerged as powerful tools for large-scale optimization: First, Nesterov’s accelerated algorithms [11] and its proximal variants, Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) [2, 3], can exploit smoothness to improve the convergence rate of a simple gradient/ISTA method from accuracy $O(1/T)$ to $O(1/T^2)$ after $T$ iterations. Second, adaptive regularization approaches such as AdaGrad [7] can optimize a gradient method’s step-size in different directions, in some sense making the optimization problem better-conditioned. These two techniques have also become popular in non-convex optimization problems, such as parameter estimation in deep neural networks (see, for example, the work of [13] on the benefits of acceleration, and AdaGrad [7, 8], RMSProp [14], ESGD [5], and Adam [9] on adaptive regularization methods in this context).

In this paper, we introduce a new algorithm, called FLAG, that combines the benefits of these two techniques. Like Nesterov’s accelerated algorithms, our method has a convergence rate of $1/T^2$. Like AdaGrad, our method adaptively chooses a regularizer, in a way that performs almost as well as the best choice of regularizer in hindsight. In addition, our improvement over FISTA is roughly a square of AdaGrad’s improvement over mirror descent (see section 2.1 for details).

There have been a number of papers in recent years dealing with ways to interpret acceleration. Our algorithm is heavily influenced by [1]. This insightful paper introduced a view of accelerated gradient descent as a linear combination of gradient and mirror descent steps. We exploit this viewpoint in introducing adaptive regularization to our accelerated method. Authors in [11] use a fixed schedule to combine gradient and mirror descent steps. However, [4] views acceleration as an ellipsoid-like algorithm, and instead, determines an appropriate ratio using a line search. This latter idea is also crucial for our algorithm, as we do not know the right ratio in advance.

The optimal stepsize for an accelerated method generally depends on both the smoothness parameter of the objective function as well as the properties of the regularizer. In FLAG, the regularizer is chosen adaptively, and we show that it is also possible to adapt the stepsize in a way that is competitive with the best choice in hindsight. In fact, our method for picking the adaptive stepsizes is closely related to the approach in [8]. There, the authors considered the problem of picking the right stepsize to adapt to an unknown strong-convexity parameter. We use a related approach, but with different proof techniques, to choose the stepsize to adapt to a changing smoothness parameter.

Finally, it should also be noted that there are some interesting papers that study the continuous-time limit of acceleration algorithms, e.g., [12, 10, 15]. Indeed, studying adaptive regularization in the continuous time setting is an interesting future direction for research.

### 1.1 Notation and Definitions

In what follows, vectors are considered as column vectors and are denoted by bold lower case letters, e.g., $\mathbf{x}$ and matrices are denoted by regular capital letters, e.g., $A$. We overload
Consider the optimization problem

$$\min_{x \in C} F(x) = f(x) + h(x),$$

where $f : \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^p \to \mathbb{R}$ are closed proper convex convex functions and $C$ is a closed convex set. We further assume that $f$ is differentiable with $L$-Lipschitz gradient, i.e.,

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L \|x - y\|_2, \quad \forall x, y \in C,$$

and we allow $h$ to be (possibly) non-smooth with sub-differential at $x$ denoted by $\partial h(x)$.

The proximal operator \[\text{prox}\] associated with $f$, $h$ and $L$ is defined as

$$\text{prox}(x) := \arg \min_{y \in C} h(y) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2$$

$$= \arg \min_{y \in C} h(y) + \frac{L}{2} \|y - (x - \frac{1}{L} \nabla f(x))\|_2^2.$$ (3a)

For a symmetric positive definite (SPD) matrix $S$, define $\psi(x) := \frac{1}{2} x^T S x$. Note that $\psi$ is 1-strongly convex with respect to the norm $\|x\|_S := \sqrt{x^T S x}$, i.e., $\forall x, y \in C$,

$$\psi(x) \geq \psi(y) + \langle \nabla \psi(y), x - y \rangle + \frac{1}{2} \|x - y\|_S^2.$$

The Bregman divergence associated with $\psi$, is define as

$$B_\psi(x, y) := \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle = \frac{1}{2} \|x - y\|_S^2.$$

We will use the fact that, for any $z \in C$,

$$\langle S(x - y), x - z \rangle = \frac{1}{2} \|x - y\|_S^2 + \frac{1}{2} \|x - z\|_S^2 - \frac{1}{2} \|y - z\|_S^2.$$ (4)

It is easy to see that the dual of $\psi(x)$ is given by

$$\psi^*(x) := \sup_{v \in \mathbb{R}^d} \langle x, v \rangle - \psi(v) = \frac{1}{2} x^T S^{-1} x.$$ (5)
2 The Algorithm and Main Result

In this section, we describe our main algorithm, FLAG, and give its convergence properties in Theorem 1. The core of FLAG consists of five essential ingredients:

(i) A proximal gradient step (Step 3)
(ii) Construction of the adaptive regularization (Steps 4-9)
(iii) Picking the Adaptive stepsize (Step 11)
(iv) A mirror descent step (Step 12)
(v) Linear combination of the proximal gradient step and the mirror descent step (Step 13)

Algorithm 1 FLAG

INPUT: $x_1 = y_1 = z_1$, $\eta_0 = 0$, $G_0 = \emptyset$, $T$

1: $\epsilon \leftarrow 1/(6dT^2)$
2: for $k=1$ to $T$ do
3: $y_{k+1} \leftarrow \text{prox}(x_k)$
4: $p_k \leftarrow -L(y_{k+1} - x_k)$
5: $g_k \leftarrow p_k/\|p_k\|_2$
6: $G_k \leftarrow [G_{k-1}, g_k]$
7: $s_k(i) \leftarrow \|G_k(i, \cdot)\|_2$
8: $S_k \leftarrow \text{diag}(s_k) + \delta I$
9: $\psi_k(x) \leftarrow \frac{1}{2}x^T S_k x$
10: $L_k \leftarrow L g_k^T S_k^{-1} g_k$
11: $\eta_k \leftarrow \frac{1}{2L_k} + \sqrt{\frac{1}{4L_k^2} + \frac{\eta_{k-1}^2 - L_{k-1}}{L_k}}$
12: $z_{k+1} \leftarrow \arg\min_{z \in C} \langle \eta_k p_k, z - z_k \rangle + \frac{1}{2}\|z - z_k\|_{S_k}^2$
13: $x_{k+1} \leftarrow \text{BinarySearch}(z_{k+1}, y_{k+1}, \epsilon)$

return $y_T$

The subroutine $\text{BinarySearch}$ is given in Algorithm 2 where $\text{Bisection}(r, 0, 1, \epsilon)$ is the usual bisection routine for finding the root of a single variable function $r(t)$ in the interval $(0, 1)$ and to the accuracy of $\epsilon$. More specifically, for a root $r^*$ such that $r(t^*) = 0$ and given $r(0)r(1) < 0$, the sub-routine $\text{Bisection}(r, 0, 1, \epsilon)$ returns an approximation $t \in (0, 1)$ to $t^*$ such that $|t - t^*| \leq \epsilon$ and this is done with only $\log(1/\epsilon)$ function evaluations.

Algorithm 2 $\text{BinarySearch}$

INPUT: $z$, $y$, and $\epsilon$

1: Define the univariate function $r(t) := \langle \text{prox}(ty + (1-t)z) - (ty + (1-t)z), y - z \rangle$
2: if $r(1) \geq 0$ then return $y$
3: if $r(0) \leq 0$ then return $z$
4: $t = \text{Bisection}(r, 0, 1, \epsilon)$
5: return $x = ty + (1-t)z$
Theorem 1 gives the convergence properties of Algorithm 1:

**Theorem 1 (Main Result: Convergence Property of FLAG)**

Let $D := \sup_{x,y \in \mathcal{C}} \|x - y\|_\infty^2$. For any $u \in \mathcal{C}$, after $T$ iterations of Algorithm 1 we get

$$F(y_{T+1}) - F(u) \leq O\left(\frac{q_T^2 LD}{T^{3/2}}\right),$$

where $q_T \in [\sqrt{T}, \sqrt{td}]$. Also each iteration takes $O(d \log(dT^3))$ time.

### 2.1 Comparison of FLAG with AdaGrad and FISTA

In this Section, we briefly describe the advantages of FLAG relative to AdaGrad and FISTA.

Let $\nabla_k$ be the subgradient of $F$ at iteration $k$, $\mathcal{V} := \{S \in \mathbb{R}^{d \times d} \mid S$ is diagonal, $S_{ii} > 0$, $\text{trace}(S) \leq d\}$, $D_2 := \sup_{x,y \in \mathcal{C}} \|x - y\|_2^2$, and $J_A := \inf_{S \in \mathcal{V}} \sum_{k=1}^T \nabla_k^T S^{-1} \nabla_k / \sum_{k=1}^T \|\nabla_k\|_2^2$. Using a similar argument to that used in Lemma 5, we see that $J_A \in [1/d, 1]$. AdaGrad achieves a rate of

$$F(\bar{y}) - F(x^*) \leq O\left(\sqrt{J_A Dd \sum_{k=1}^T \|g_k\|_2^2 / T}\right),$$

which is in contrast to mirror descent, which achieves

$$F(\bar{y}) - F(x^*) \leq O\left(\sqrt{D_2 \sum_{k=1}^T \|g_k\|_2^2 / T}\right),$$

where $\bar{y}$ is the average of all $y_k$’s. Thus, one can see that the improvement of AdaGrad compared to mirror descent is $\sqrt{J_A Dd/D_2}$.

Now, let $g_k$ be defined as in Algorithm 1. One can verify using Lemma 5 that $J_B := q_T^2 / T = \inf_{S \in \mathcal{V}} \sum_{k=1}^T g_k^T S^{-1} g_k / \sum_{k=1}^T \|g_k\|_2^2$ and $J_B \in [1/d, 1]$. Thus, from Theorem 1 we get

$$F(y_{T+1}) - F(x^*) \leq O\left(J_B LDd / T^2\right).$$

This is in contrast to FISTA, which achieves

$$F(y_{T+1}) - F(x^*) \leq O\left(\frac{LD_2 d}{T^2}\right).$$
As a result, the improvement of FLAG compared to FISTA is $J_B Dd/D_2$. FLAG’s improvement over FISTA can thus be up to a square of AdaGrad’s improvement over mirror descent. Here, we note that $J_A$ and $J_B$ though analogous, are not the same.

Finally, we can directly compare the rates of (6) and (7) to see the speed-up offered by FLAG over AdaGrad. In particular, FLAG enjoys from the optimal rate of $O(1/T^2)$, compared to the sub-optimal rate of $O(1/\sqrt{T})$ of AdaGrad. However, we do stress that AdaGrad does not make any smoothness assumptions, and thus works in more general settings than FLAG.

3 Proofs

Below we give the details for the proof of our main result. The proofs of technical lemmas in Section 3.1 are given in Appendix A.

The proof sketch of the convergence of FLAG is as follows.

(I) FLAG is essentially a combination of mirror descent and proximal gradient descent steps (see Lemmas 1 and 4).

(II) $L_k$ in Algorithm 1 plays the role of an “effective gradient lipschitz constant” in each iteration. The convergence rate of our algorithm ultimately depends on $\sum_{k=1}^{T} L_k = L \sum_{k=1}^{T} g_k^T S_k^{-1} g_k$. (see Lemmas 7 and 2)

(III) By picking $S_k$ adaptively, as in AdaGrad, we achieve a non-trivial upper bound for $\sum_{k=1}^{T} L_k$. (see Lemma 5)

(IV) Our algorithm relies on picking an $x_k$ at each iteration that satisfies an inequality involving $L_k$ (see Corollary 1). However, because $L_k$ is not known at the moment of picking $x_k$, we must choose an $x_k$ to roughly satisfy the inequality for all possible values of $L_k$. We do this by picking $x_k$ using binary search. (see Lemmas 2 and 3 and Corollary 1)

(V) Finally, we need to pick the right step sizes for each iteration. Our scheme is very similar to the one used in [1], but generalized to handle a different $L_k$ each iteration. (see Lemma 6 and 7 and Corollary 2).

Theorem 2 combines items I, II, and IV. Theorem 1 combines Theorem 2 with items III and V to get our final result.

3.1 Technical Lemmas

We have the following key result (see Lemma 2.3 [2]) regarding the vector $p = -L(\text{prox}(x) - x)$, as in Step 4 of FLAG, which is called the Gradient Mapping of $F$ on $C$. 
Lemma 1 (Gradient Mapping)
For any $x, y \in \mathcal{C}$, we have

$$F(\text{prox}(x)) \leq F(y) + \langle L(\text{prox}(x) - x), y - x \rangle - \frac{L}{2} \|x - \text{prox}(x)\|_2^2,$$

where $\text{prox}(x)$ is defined as in [3]. In particular, $F(\text{prox}(x)) \leq F(x) - \frac{L}{2} \|x - \text{prox}(x)\|_2^2$.

The following lemma establishes the Lipschitz continuity of the $\text{prox}$ operator.

Lemma 2 (Prox Operator Continuity)
$\text{prox}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a $2$-Lipschitz continuous, that is, for any $x, y \in \mathcal{C}$, we have

$$\|\text{prox}(x) - \text{prox}(y)\|_2 \leq 2\|x - y\|_2.$$

Using $\text{prox}$ operator continuity Lemma 2 we can conclude that given any $y, z \in \mathcal{C}$, if $\langle \text{prox}(y) - y, y - z \rangle < 0$ and $\langle \text{prox}(z) - z, y - z \rangle > 0$, then there must be a $t^* \in (0, 1)$ for which $w = t^*y + (1 - t^*)z$ gives $\langle \text{prox}(w) - w, y - z \rangle = 0$. Algorithm 2 finds an approximation to $w$ in $O(\log L/\epsilon)$ iterations.

Lemma 3 (Binary Search Lemma)
Let $x = \text{BinarySearch}(z, y, \epsilon)$ defined as in Algorithm 2 Then one of 3 cases happen:

(i) $x = y$ and $\langle \text{prox}(x) - x, x - z \rangle \geq 0$,
(ii) $x = z$ and $\langle \text{prox}(x) - x, y - x \rangle \leq 0$, or
(iii) $x = ty + (1 - t)z$ for some $t \in (0, 1)$ and $|\langle \text{prox}(x) - x, y - z \rangle| \leq 3\|y - z\|_2^2\epsilon$.

Using the above result, we can prove the following:

Corollary 1
Let $x_k, y_k, z_k$ and $\epsilon_k$ be defined as in Algorithm 7 and $\eta_k L_k \geq 1$. Then for all $k \geq 1$,

$$\langle p_k, x_k - z_k \rangle \leq (\eta_k L_k - 1)\langle p_k, y_k - x_k \rangle + \frac{DL\eta_k L_k}{T^3}.$$
Next, we state a result regarding the mirror descent step. Similar results can be found in most texts on online optimization, e.g. [1].

**Lemma 4 (Mirror Descent Inequality)**  
Let $z_{k+1} = \arg \min_{z \in C} \langle \eta_k p_k, z - z_k \rangle + \frac{1}{2} \|z - z_k\|_2^2$ and $D := \sup_{x,y \in C} \|x - y\|_\infty$ be the diameter of $C$ measured by infinity norm. Then for any $u \in C$, we have

$$\sum_{k=1}^{T} \langle \eta_k p_k, z_k - u \rangle \leq \sum_{k=1}^{T} \frac{\eta_k^2}{2} \|p_k\|_{S_k^*}^2 + \frac{D}{2} \|s_T\|_1$$

Finally, we state a similar result to that of [7] that captures the benefits of using $S_k$ in FLAG.

**Lemma 5 (AdaGrad Inequalities)**  
Define $q_T := \sum_{i=1}^{d} \|G_T(i,:\|_2$, where $G_k$ is as in Step 6 of Algorithm 1. We have

1. $\sum_{k=1}^{T} g_k T S_k^{-1} g_k \leq 2 q_T$,
2. $q_T^2 = \min_{S \in \mathcal{S}} \sum_{k=1}^{T} g_k T S^{-1} g_k$, where $\mathcal{S} := \{S \in \mathbb{R}^{d \times d} \mid S \text{ is diagonal, } S_{ii} > 0, \quad \text{trace}(S) \leq 1\}$, and
3. $\sqrt{T} \leq q_T \leq \sqrt{dT}$.

### 3.2 Master Theorem

We can now prove the central theorems of this section.

**Theorem 2**  
Let $D := \sup_{x,y \in C} \|x - y\|_\infty$. For any $u \in C$, after $T$ iterations of Algorithm 1 we get

$$\sum_{k=1}^{T} \left\{ \left( \eta_{k-1}^2 L_{k-1} - \eta_k^2 L_k + \eta_k \right) F(y_k) - \eta_k F(u) \right\} + \eta_T^2 L_T F(y_{T+1}) \leq \sum_{k=1}^{T} \frac{DL \eta_k^2 L_k}{T^3} + \frac{D}{2} \|s_T\|_1.$$
Proof of Theorem 2 Noting that $p_k = -L(y_{k+1} - x_k)$ is the gradient mapping of $F$ on $C$, it follows that

$$\sum_{k=1}^{T} \eta_k (F(y_{k+1}) - F(u)) = \sum_{k=1}^{T} \eta_k (F(\text{prox}(x_k)) - F(u))$$

(By the Gradient Mapping Lemma 1) \leq \sum_{k=1}^{T} \underbrace{\eta_k \langle p_k, x_k - u \rangle}_{\text{by Lemma 1}} - \frac{\eta_k}{2L} \|p_k\|_2^2

$$= \sum_{k=1}^{T} \eta_k (p_k, (z_k - u)) + \sum_{k=1}^{T} \eta_k (p_k, x_k - z_k) - \frac{\eta_k}{2L} \|p_k\|_2^2$$

(By Step 10 of Algorithm 1) = \sum_{k=1}^{T} \underbrace{\eta_k \langle p_k, (y_k - x_k) \rangle}_{\text{by Corollary 1}} - \frac{DL\eta_k L_k}{T^3} \|p_k\|_2^2

Now rearranging terms and re-indexing the summations gives the desired result. 

3.3 Choosing the Stepsize

In this section, we discuss the final piece of our algorithm: choosing the stepsize $\eta_k$ for the mirror descent step.

Lemma 6 For the choice of $\eta_k$ in Algorithm 4 and $k \geq 1$, we have
(i) \( \eta^2_k L_k = \sum_{i=1}^{k} \eta_i \).

(ii) \( \eta^2_{k-1} L_{k-1} - \eta^2_k L_k + \eta_k = 0 \), and

(iii) \( \eta_k L_k \geq 1 \).

**Proof** We prove (i) by induction. For \( k = 1 \), it is easy to verify that \( \eta_1 = 1/L_1 \), and so \( \eta^2_1 L_1 = \eta_1 \) and the base case follows trivially. Now suppose \( \eta^2_{k-1} L_{k-1} = \sum_{i=1}^{k-1} \eta_i \). Rearranging (ii) for \( k \) gives

\[
0 = \eta^2_k L_k - \eta_k - \sum_{i=1}^{k-1} \eta_i = \eta^2_k L_k - \eta_k - \eta^2_{k-1} L_{k-1}.
\]

Now, it is easy to verify that the choice of \( \eta_k \) in Algorithm 1 is a solution of the above quadratic equation. The rest of the items follow immediately from part (i).

**Corollary 2**

Let \( D := \sup_{x,y \in C} \|x - y\|_\infty^2 \). For any \( u \in C \), after \( T \) iterations of Algorithm 1 we get

\[
F(y_{T+1}) - F(u) \leq \frac{LD}{T^2} + \frac{D\|s_T\|_1}{2 \sum_{k=1}^{T} \eta_k}.
\]

**Proof of corollary 2** Combining Theorem 2 and Lemma 6, and noting that \( \eta^2_k L_k = \sum_{i=1}^{k} \eta_i \leq \sum_{i=1}^{T} \eta_i = \eta^2_T L_T \) gives the desired result.

Finally, it only remains to lower bound \( \sum_{k=1}^{T} \eta_k \), which is done in the following Lemma.

**Lemma 7**

For the choice of \( \eta_k \) in Algorithm 1 we have

\[
\sum_{k=1}^{T} \eta_k \geq \frac{T^3}{1000 \sum_{k=1}^{T} L_k}.
\]
Remark We note here that we made little effort to minimize constants, and that we used rather sloppy bounds such as $T - 1 \geq T/2$. As a result, the constant appearing above is very conservative and a mere by product of our proof technique. We have numerically verified that a much smaller constant (e.g., 10) indeed satisfies the bound above.

And now, the proof of our main result, Theorem 1, follows rather immediately:

Proof of Theorem 1 The result follows immediately from Lemma 7 and Corollary 2 and noting that $\sum_{k=1}^{T} L_k = L \sum_{k=1}^{T} \mathbf{g}_k^T S_k^{-1} \mathbf{g} \leq 2Lq_T$ by Lemma 3 and $\|\mathbf{s}_t\|_1 = q_T$ by Step 7 of Algorithm 1 and definition of $q_T$ in Lemma 5. This gives

$$F(y_{T+1}) - F(u) \leq \frac{LD}{T^2} + \frac{q_T^2}{T} \frac{1000LD}{T^2} \leq \frac{q_T^2}{T} \frac{1001LD}{T^2}$$

The run-time per iteration follows from having to do $\log_2(\frac{1}{\epsilon})$ calls to bisection, each taking $O(n)$ time.

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A Proofs of Technical Lemmas

Proof of Lemma 1 This result is the same as Lemma 2.3 in [2]. We bring its proof here for completeness.

For any \( y \in C \), any sub-gradient, \( v \), of \( h \) at \( \text{prox}(x) \), i.e., \( v \in \partial h(\text{prox}(x)) \), and by optimality of \( \text{prox}(x) \) in (3), we have

\[
0 \leq \langle \nabla f(x) + v + L(\text{prox}(x) - x), y - \text{prox}(x) \rangle \\
= \langle \nabla f(x) + v + L(\text{prox}(x) - x), y - x \rangle + \langle \nabla f(x) + v + L(\text{prox}(x) - x), x - \text{prox}(x) \rangle,
\]

and so

\[
\langle \nabla f(x), \text{prox}(x) - x \rangle \leq \langle \nabla f(x) + v + L(\text{prox}(x) - x), y - x \rangle \\
+ \langle v, x - \text{prox}(x) \rangle - L\|x - \text{prox}(x)\|^2.
\]

Now from \( L \)-Lipschitz continuity of \( \nabla f \) as well as convexity of \( f \) and \( h \), we get

\[
F(\text{prox}(x)) = f(\text{prox}(x)) + h(\text{prox}(x)) \\
\leq f(x) + \langle \nabla f(x), \text{prox}(x) - x \rangle + \frac{L}{2}\|\text{prox}(x) - x\|^2 + h(\text{prox}(x)) \\
\leq f(x) + \langle \nabla f(x) + v + L(\text{prox}(x) - x), y - x \rangle + \langle v, x - \text{prox}(x) \rangle - \frac{L}{2}\|x - \text{prox}(x)\|^2 + h(\text{prox}(x)) \\
= f(y) + \langle L(\text{prox}(x) - x), y - x \rangle + \langle v, y - \text{prox}(x) \rangle - \frac{L}{2}\|x - \text{prox}(x)\|^2 + h(\text{prox}(x)) \\
\leq F(y) + \langle L(\text{prox}(x) - x), y - x \rangle - \frac{L}{2}\|x - \text{prox}(x)\|^2.
\]

Proof of Lemma 2 By Definition (3), for any \( x, y, z, z' \in C \), \( v \in \partial h(\text{prox}(x)) \), and \( w \in \partial h(\text{prox}(y)) \), we have

\[
\langle v, z - \text{prox}(x) \rangle \geq -\langle \nabla f(x) + L(\text{prox}(x) - x), z - \text{prox}(x) \rangle, \\
\langle w, z' - \text{prox}(y) \rangle \geq -\langle \nabla f(y) + L(\text{prox}(y) - y), z' - \text{prox}(y) \rangle.
\]

In particular, for \( z = \text{prox}(y) \) and \( z' = \text{prox}(z) \), we get

\[
\langle v, \text{prox}(y) - \text{prox}(x) \rangle \geq -\langle \nabla f(x) + L(\text{prox}(x) - x), \text{prox}(y) - \text{prox}(x) \rangle, \\
\langle w, \text{prox}(y) - \text{prox}(x) \rangle \leq \langle \nabla f(y) + L(\text{prox}(y) - y), \text{prox}(x) - \text{prox}(y) \rangle.
\]
By monotonicity of sub-gradient, we get
\[ \langle v, \text{prox}(y) - \text{prox}(x) \rangle \leq \langle w, \text{prox}(y) - \text{prox}(x) \rangle. \]

So
\[ \langle \nabla f(x) + L(\text{prox}(x) - x), \text{prox}(x) - \text{prox}(y) \rangle \leq \langle \nabla f(y) + L(\text{prox}(y) - y), \text{prox}(x) - \text{prox}(y) \rangle, \]

and as a result
\[
\begin{align*}
\langle \nabla f(x) + L(\text{prox}(x) - x), \text{prox}(x) - \text{prox}(y) \rangle \\
= \langle \nabla f(x) + L(\text{prox}(x) - \text{prox}(y) + \text{prox}(y) - x), \text{prox}(x) - \text{prox}(y) \rangle \\
= L\|\text{prox}(x) - \text{prox}(y)\|_2^2 + \langle \nabla f(x) + L(\text{prox}(y) - x), \text{prox}(x) - \text{prox}(y) \rangle \\
\leq \langle \nabla f(y) + L(\text{prox}(y) - y), \text{prox}(x) - \text{prox}(y) \rangle,
\end{align*}
\]

which gives
\[
\begin{align*}
L\|\text{prox}(x) - \text{prox}(y)\|_2^2 & \leq \langle \nabla f(y) - \nabla f(x) + L(x - y), \text{prox}(x) - \text{prox}(y) \rangle \\
& \leq \left(\|\nabla f(y) - \nabla f(x)\|_2 + L\|x - y\|_2\right)\|\text{prox}(x) - \text{prox}(y)\|_2 \\
& \leq 2L\|x - y\|_2\|\text{prox}(x) - \text{prox}(y)\|_2,
\end{align*}
\]

and the result follows. \(\square\)

**Proof of Lemma 3** Items i and ii are simply Steps 2 and 3, respectively. For item iii, we have
\[
\|x - w\|_2 = \|ty + (1-t)z - t^*y - (1-t^*)z\|_2 = \|(t-t^*)y - (t-t^*)z\|_2 = |t-t^*|\|y-z\|_2 \leq \epsilon\|y-z\|_2.
\]

Now it follows that
\[
\begin{align*}
|\langle \text{prox}(x) - x, y - z \rangle| &= |\langle \text{prox}(x) - x, y - z \rangle - \langle \text{prox}(w) - w, y - z \rangle| \\
& \leq \|\langle \text{prox}(x) - \text{prox}(w), y - z \rangle\|_2 + \|\langle x - w, y - z \rangle\|_2 \\
& \leq \|\text{prox}(x) - \text{prox}(w)\|_2\|y - z\|_2 + \|x - w\|_2\|y - z\|_2 \\
& \leq 2\|x - w\|_2\|y - z\|_2 + \|x - w\|_2\|y - z\|_2 \\
& = 3\|x - w\|_2\|y - z\|_2 \\
& \leq 3\epsilon\|y - z\|_2^2.
\end{align*}
\]

\(\square\)

**Proof of Corollary 1** Note that by Step 1 of Algorithm 1, \(p_k = -L(\text{prox}(x_k) - x_k)\). For \(k = 1\), since \(x_1 = y_1 = z_1\), the inequality is trivially true. For \(k \geq 2\), we consider the three cases of Corollary 3: (i) if \(x_k = y_k\), the right hand side is \(1/T \geq 0\) and the left hand side is \(\langle p_k, x_k - z_k \rangle = \langle -L(\text{prox}(x_k) - x_k), x_k - z_k \rangle \leq 0\), (ii) if \(x_k = z_k\), the left hand side is 0 and
\[
\langle p_k, y_k - x_k \rangle = \langle -L(\text{prox}(x_k) - x_k), y_k - x_k \rangle \geq 0, \text{ so the inequality holds trivially, and (iii) in this last case, for some } t \in (0, 1), \text{ we have}
\]
\[
\langle p_k, x_k - z_k \rangle = \langle -L(\text{prox}(x_k) - x_k), ty_k + (1 - t)z_k - z_k \rangle
\]
\[
= -Lt\langle (\text{prox}(x_k) - x_k), y_k - z_k \rangle,
\]
and
\[
\langle p_k, y_k - x_k \rangle = \langle -L(\text{prox}(x_k) - x_k), y_k - ty_k - (1 - t)z_k \rangle
\]
\[
= -L(1 - t)\langle (\text{prox}(x_k) - x_k), (y_k - z_k) \rangle.
\]

Hence
\[
\langle p_k, x_k - z_k \rangle - (\eta_k L_k - 1)\langle p_k, y_k - x_k \rangle
\]
\[
\leq |\langle p_k, x_k - z_k \rangle - (\eta_k L_k - 1)\langle p_k, y_k - x_k \rangle|
\]
\[
= |(-Lt + (\eta_k L_k - 1)L(1 - t))\langle (\text{prox}(x_k) - x_k), (y_k - z_k) \rangle|
\]
\[
\leq 3|(-Lt + (\eta_k L_k - 1)L(1 - t))\|y_k - z_k\|^2_\varepsilon_k
\]
\[
= 3(\eta_k L_k - 1)L\|y_k - z_k\|^2_\varepsilon_k
\]
\[
= 3(\eta_k L_k + 1)L\|y_k - z_k\|^2_\varepsilon_k
\]
\[
= 6D\eta_k L_k L\|y_k - z_k\|^2_\varepsilon_k
\]
\[
= 6D\eta_k L_k L\|y_k - z_k\|^2_\varepsilon_k
\]
\[
= \frac{6D\eta_k L_k L\|y_k - z_k\|^2_\varepsilon_k}{D}
\]
\[
= \frac{DL\eta_k L_k}{T^3},
\]
where in the last line we used the fact that \(\|y_k - z_k\|^2_\varepsilon_k \leq Dd\)

**Proof of Lemma 4** For any \(u \in C\) and by optimality of \(z_k\), we have
\[
\langle \eta_k p_k, z_{k+1} - u \rangle \leq \langle S_k(z_{k+1} - z_k), u - z_{k+1} \rangle.
\]

Hence, using (1) and (3), it follows that
\[
\langle \eta_k p_k, z_k - u \rangle = \langle \eta_k p_k, z_k - z_{k+1} \rangle + \langle \eta_k p_k, z_{k+1} - u \rangle
\]
\[
\leq \langle \eta_k p_k, z_k - z_{k+1} \rangle - \langle S_k(z_{k+1} - z_k), z_{k+1} - u \rangle
\]
\[
= \langle \eta_k p_k, z_k - z_{k+1} \rangle - \frac{1}{2}\|z_{k+1} - z_k\|^2_{S_k} - \frac{1}{2}\|z_{k+1} - u\|^2_{S_k} + \frac{1}{2}\|u - z_k\|^2_{S_k}
\]
\[
\leq \sup_{z \in \mathbb{R}^d} \left\{ \langle \eta_k p_k, z \rangle - \frac{1}{2}\|z\|^2_{S_k} \right\} - \frac{1}{2}\|z_{k+1} - u\|^2_{S_k} + \frac{1}{2}\|u - z_k\|^2_{S_k}
\]
\[
= \frac{\eta_k^2}{2}\|p_k\|^2_{S_k} - \frac{1}{2}\|u - z_{k+1}\|^2_{S_k} + \frac{1}{2}\|u - z_k\|^2_{S_k}.
\]
Now recalling from Steps 6-8 of Algorithm 1 that $S_k = \text{diag}(s_k) + \delta I$ and $s_k \geq s_{k-1}$, we sum over $k$ to get

$$\sum_{k=1}^{T} \langle \eta_k p_k, z_k - u \rangle \leq \sum_{k=1}^{T} \langle \eta_k p_k, S_k \rangle + \frac{1}{2} \| u - z_1 \|_{S_1}^2 + \sum_{k=2}^{T} \frac{1}{2} \| u - z_k \|_{S_k}^2 \leq \frac{T}{2} \| u - z_T \|_{S_T}^2.$$ 

**Proof of Lemma 5** To prove part i, we use the following inequality introduced in the proof of Lemma 4 in [7]: for any arbitrary real-valued sequence of $\{ a_i \}_{i=1}^{T}$ and its vector representation as $a_{1:T} = [a_1, a_2, \ldots, a_T]$, we have

$$\sum_{k=1}^{T} \frac{a_k^2}{\| a_{1:k} \|_2^2} \leq \| a_{1:T} \|_2^2.$$ 

So it follows that

$$\sum_{k=1}^{T} \frac{1}{s_k} \| a_k \|_2^2 \leq \sum_{k=1}^{T} \frac{1}{s_k} \| a_k \|_2^2 \leq \sum_{k=1}^{T} \frac{1}{s_k} \| a_k \|_2^2 \leq 2q_T,$$

where the last equality follows from the definition of $s_k$ in Step 7 of Algorithm 1.

For the rest of the proof, one can easily see that

$$\sum_{k=1}^{T} \frac{1}{s_k} \| a_k \|_2^2 = \sum_{i=1}^{d} \frac{1}{s(i)} \| a(i) \|_2^2 = \sum_{i=1}^{d} \frac{1}{s(i)} \| a(i) \|_2^2,$$

where $a(i) := \sum_{k=1}^{T} g_k^2(i)$ and $s = \text{diag}(S)$. Now the Lagrangian for $\lambda \geq 0$ and $\nu \geq 0$, can be written as

$$\mathcal{L}(s, \lambda, \nu) = \sum_{i=1}^{d} \frac{a(i)}{s(i)} + \lambda \left( \sum_{i=1}^{d} s(i) - 1 \right) + \langle \nu, s \rangle.$$
Since the strong duality holds, for any primal-dual optimal solutions, $S^*, \lambda^*$ and $\nu^*$, it follows from complementary slackness that $\nu^* = 0$ (since $s^* > 0$). Now requiring that $\partial L(s^*, \lambda^*, \nu^*)/\partial s(i) = 0$ gives $\lambda^* s^*(i) = \sqrt{a_i} > 0$, which since $s^*(i) > 0$, implies that $\lambda^* > 0$. As a result, by using complementary slackness again, we must have $\sum_{i=1}^d s^*(i) = 1$. Now simple algebraic calculations gives $s^*(i) = \sqrt{a_i / \sum_{i=1}^d \sqrt{a_i}}$ and part ii follows.

For part iii, recall that $\|g_k\|_2 = 1$. Now, since $\lambda_{\min}(S^{01}) \geq 1$, one has $1 \leq g_k^T S^{-1} g_k$, and so $q_T \geq 1$. One the other hand, consider the optimization problem

$$\max \sum_{i=1}^d \|G_T(i,:)\|_2 = \sum_{i=1}^d \sqrt{\sum_{k=1}^T g_i^2(k)}$$

s.t. $\|g_k\|_2 = 1, \ k = 1, 2, \ldots, T$.

The Lagrangian can be written as

$$L(\{g_k\}_{k=1}^T, \{\lambda\}_{k=1}^T) = \sum_{i=1}^d \sqrt{\sum_{k=1}^T g_i^2(k)} + \sum_{k=1}^T \lambda_k \left(1 - \sum_{i=1}^d g_i^2(k)\right).$$

By KKT necessary condition, we require that $\partial L(\{g_k\}_{k=1}^T, \{\lambda\}_{k=1}^T)/\partial g_i(k) = 0$, which implies that

$$\lambda_k = \frac{1}{2 \sqrt{\sum_{k=1}^T g_i^2(k)}}, \ i = 1, 2, \ldots, d.$$  

Hence,

$$T = \sum_{i=1}^d \sum_{k=1}^T g_i^2(k) = \frac{d}{4\lambda_k^2},$$

and so $2\lambda_k = \sqrt{d/T}$, which gives $q_T \leq \sqrt{dT}$.

**Proof of Lemma 7** We prove by induction on $T$. For $T = 1$, we have $\eta_1 = 1/L_1$, and the base case holds trivially. Suppose the desired relation holds for $T - 1$. We have

$$\sum_{k=1}^T \eta_k = \sum_{k=1}^{T-1} \eta_k + \eta_T$$

(By induction hypothesis) \geq \frac{(T - 1)^3}{1000 \sum_{k=1}^{T-1} L_k} + \frac{1}{2LT} + \sqrt{\frac{1}{4L_T^2} + \frac{(T - 1)^3}{1000L_T \sum_{k=1}^{T-1} L_k}}$$

\geq \frac{(T - 1)^3}{1000 \sum_{k=1}^{T-1} L_k} + \sqrt{\frac{(T - 1)^3}{1000L_T \sum_{k=1}^{T-1} L_k}}$$

\geq \frac{(T - 1)^3}{1000 \sum_{k=1}^{T-1} L_k} + \sqrt{\frac{T^3}{8000L_T \sum_{k=1}^{T-1} L_k}}.$$
Now if
\[
\frac{(T - 1)^3}{1000 \sum_{k=1}^{T-1} L_k} \geq \frac{T^3}{1000 \sum_{k=1}^{T} L_k},
\]
then we are done. Otherwise denoting \(\alpha := \sum_{k=1}^{T} L_k\), we must have that
\[
L_T \leq \frac{\alpha T^3 - \alpha (T - 1)^3}{T^3} = \frac{\alpha T^3 - \alpha (T^3 - 3T^2 + 3T - 1)}{T^3} = \frac{\alpha (3T^2 - 3T + 1)}{T^3} \leq \frac{4 \sum_{k=1}^{T} L_k}{T}.
\]
Hence, we get
\[
\sum_{k=1}^{T} \eta_k \geq \frac{(T - 1)^3}{1000 \sum_{k=1}^{T-1} L_k} + \frac{T^4}{32000 L_T \left(\sum_{k=1}^{T} L_k\right)^2} \geq \frac{(T - 1)^3}{1000 \sum_{k=1}^{T} L_k} + \frac{4T^2}{1000 \sum_{k=1}^{T} L_k} \geq \frac{T^4}{1000 \sum_{k=1}^{T} L_k}.
\]