Research Article

Inner Product Fuzzy Quasilinear Spaces and Some Fuzzy Sequence Spaces

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It has been shown that the class of fuzzy sets has a quasilinear space structure. In addition, various norms are defined on this class, and it is given that the class of fuzzy sets is a normed quasilinear space with these norms. In this study, we first developed the algebraic structure of the class of fuzzy sets \( F(\mathbb{R}^n) \) and gave definitions such as quasilinear independence, dimension, and the algebraic basis in these spaces. Then, with special norms, namely, \( \|u\|_q = (\int_0^1 (\sup_{x \in [0,1]} \|x\|^q)da)^{1/q} \) where \( 1 \leq q \leq \infty \), we stated that \( (F(\mathbb{R}^n), \|u\|_q) \) is a complete normed space. Furthermore, we introduced an inner product in this space for the case \( q = 2 \). The inner product must be in the form:

\[
\langle u, v \rangle = \int_0^1 \langle [u]^a, [v]^a \rangle_{K(\mathbb{R}^n)} da = \left\{ \int_0^1 (a, b)_{\mathbb{R}^n} da : a \in [u]^a, b \in [v]^a \right\}.
\]

Finally, we showed that the parallelogram law can only be provided in the regular subspace, not in the entire of \( F(\mathbb{R}^n) \). We also proved that the parallelogram law can only be provided in the regular subspace, not in the entire of \( F(\mathbb{R}^n) \). Finally, we showed that a special class of fuzzy number sequences is a Hilbert quasilinear space.

1. Introduction

Investigation of fuzzy sets was given by Zadeh [1] and then notions of fuzzy number, fuzzy metric, fuzzy norm, and their applications have been introduced by several authors. For example, Katsaras [2] first introduced the notions of fuzzy seminorm and norm on a vector space. Independently, Felbin [3] gave the concept of fuzzy normed space (briefly, FNS) by applying the notion of fuzzy distance of Kaleva and Seikkala [4] on vector spaces. Xiao and Zhu [5] improved Felbin’s definition of the fuzzy norm of a linear operator between FNSs [6].

The notion of fuzzy quasilinear space is introduced in [6] depending on the notion of quasilinear space which was defined by Aseev in [7].

Aseev first introduced the concept of quasilinear space which allows us to investigate both linear spaces and some nonlinear spaces such as special classes of sets in some Banach spaces, such as special classes of multivalued mappings and fuzzy sets. He followed a similar way to methods in linear algebra and in functional analysis. Furthermore, he presented some results which are “quasilinear” counterparts of fundamental definitions and theorems in linear functional analysis and differential calculus in Banach spaces. This pioneering work has motivated a lot of authors to introduce new results on multivalued mappings, fuzzy quasilinear spaces and operators, and set-valued analysis [8–14]. In this way, Rojas Medar et al. [6] introduced the concept of fuzzy quasilinear spaces and defined the notion of a norm on these spaces.

In this study, we first constructed the algebraic structure of the class of fuzzy sets \( F(\mathbb{R}^n) \) and gave definitions such as quasilinear independence, dimension, and the algebraic basis in these spaces. Then, with special norms, namely, \( \|u\|_q = (\int_0^1 (\sup_{x \in [0,1]} \|x\|^q)da)^{1/q} \) where \( 1 \leq q \leq \infty \), we stated that \( (F(\mathbb{R}^n), \|u\|_q) \) is a complete normed space. Furthermore, we introduced an inner product in this space for the case \( q = 2 \). The inner product must be in the following form:
for \( u, v \in F(\mathbb{R}^n) \). Furthermore, we showed that the inner product norm is just
\[
\|\langle u, u \rangle\|_{K(\mathbb{R})} = \int_0^1 \left( \sup_{a \in [u]^a} \|a\| \right)^2 \, \alpha \, \mathrm{d} \alpha = \|u\|^2.
\] (2)

We also prove that the parallelogram law can only be provided in the regular subspace, not in the entire of \( F(\mathbb{R}^n) \). Finally, we showed that a special class of fuzzy number sequences is a Hilbert quasinilpotent space.

2. Preliminaries and Some New Results

In the universe of \( \mathbb{R}^n \) of generalization of fuzzy numbers defined by \( F(\mathbb{R}^n) = \{ u : \mathbb{R}^n \rightarrow [0, 1] : u \) satisfies (i)–(iv) below, where

(i) \( u \) is normal, that is, there exists an \( x_0 \in \mathbb{R}^n \) such that \( u(x_0) = 1 \)

(ii) \( u \) is fuzzy convex, that is, for \( x, y \in \mathbb{R}^n \) and \( 0 \leq \lambda \leq 1, u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \)

(iii) \( u \) is upper semicontinuous

(iv) The closure of \( [x \in \mathbb{R}^n : u(x) > 0] \), denoted by \([u]^0 \), is compact.

For \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set \([u]^\alpha \) is defined by \([u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\} \). Then, from (i) to (iv), it follows that the \( \alpha \)-level set \([u]^\alpha \in K_C(\mathbb{R}^n) \) and \([u]^0 = \bigcup_{\alpha \in (0, 1)} [u]^\alpha \) where \( K_C(\mathbb{R}^n) \) denote the family of all nonempty, compact, convex subsets of \( \mathbb{R}^n \).

The set of fuzzy numbers forms an important algebraic structure called quasinilpotent space. Now, let us give the definition of quasinilpotent space.

A set \( X \) is called a quasinilpotent space (briefly, QLS) [7], on the field \( \mathbb{R} \), if a partial order relation \( " \leq \) " an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for all elements \( x, y, z, v \in X \) and for all \( \alpha, \beta \in \mathbb{R} \):

(Q1) \( x \leq x \)

(Q2) \( x \leq z \) if \( x \leq y \) and \( y \leq z \)

(Q3) \( x = y \) if \( x \leq y \) and \( y \leq x \)

(Q4) \( x + y = y + x \)

(Q5) \( x + (y + z) = (x + y) + z \)

(Q6) there exists an element (zero) \( \theta \in X \) such that \( x + \theta = x \)

(Q7) \( \alpha (\beta \cdot x) = (\alpha \beta) \cdot x \)

(Q8) \( \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \)

(Q9) \( 1 \cdot x = x \)

(Q10) \( 0 \cdot x = 0 \)

(Q11) \( (\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x \)

\( (Q12) \; x + z \leq y + v \) if \( x \leq y \) and \( z \leq v \)

\( (Q13) \; \alpha \cdot x \leq \alpha \cdot y \) if \( x \leq y \)

Any linear space is a QLS with the partial order relation \(" = \) " An element \( x' \) is called (additive) inverse of \( x \in X \) if \( x + x' = \theta \). The inverse is unique whenever it exists. An element \( x \) possessing an inverse is called a regular, otherwise it is called a singular. Suppose that each element \( x \in X \) in a QLS \( X \) has an inverse element \( x' \in X \). Then, the partial order in \( X \) is determined by equality, the distributivity conditions hold, and consequently \( X \) is a linear space [7]. It will be assumed in what follows that \( -x = -(1) \cdot x \) and sometimes \( -x \) may not be the inverse of \( x \). An element \( x \) has an inverse \( x' \) if and only if \( x' = -x \). Suppose that \( X \) is a QLS and \( YE \). Then, \( Y \) is called a subspace of \( X \) whenever \( Y \) is a QLS with the same partial order and with the restriction of the operations on \( X \) to \( Y \). Furthermore, we saw that \( Y \) is a subspace of a QLS \( X \) if and only if for every \( x, y \in Y \) and \( \alpha, \beta \in \mathbb{R} \), \( \alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B \), \( (\alpha \beta) \cdot A = \alpha (\beta \cdot A) \), \( 1 \cdot A = A \) and if \( \alpha, \beta \geq 0 \), then \( (\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric.

\[
\delta_{\alpha \beta} (A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}.
\] (3)

It is well known that \( K_C(\mathbb{R}^n) \) is complete with this metric. Furthermore, \( K_C(\mathbb{R}^n) \) is a quasinilpotent space with the inclusion relation. After the introduction of normed quasinilpotent spaces, we will say that this metric comes from a norm.

Definition 1 (see [7]). Let \( X \) be a QLS. A real function \( \| \cdot \| : X \rightarrow \mathbb{R} \) is called a norm if the following conditions hold:

(NQ1) \( \|x\| > 0 \) if \( x \neq 0 \)

(NQ2) \( \|x + y\| \leq \|x\| + \|y\| \)

(NQ3) \( \|\alpha \cdot x\| = \|\alpha\| \|x\| \)
(NQ4) if \( x \leq y \), then \( \| x \|_X \leq \| y \|_X \)

(NQ5) if for any \( \epsilon > 0 \) there exists an element \( x_\epsilon \in X \) such that \( x \leq y + x_\epsilon \) and \( \| x_\epsilon \|_X \leq \epsilon \) then \( x \leq y \)

A quasilinear space \( X \) with a norm defined on it is called normed quasilinear space, briefly, normed QLS. It follows from [7] that if any \( x \in X \) has an inverse element \( x^{-1} \in X \), then the concept of normed QLS coincides with the concept of real normed linear space. The Hausdorff metric or norm metric on \( X \) is defined by the equality.

\[
h_X(x, y) = \inf\{r \geq 0: x \leq y + a_r, y \leq x + a_r \text{ and } \| a_r \| \leq r, r = 1, 2\}.
\]

(4)

Since \( x \leq y + (x - y) \) and \( y \leq x + (y - x) \), the quantity \( h_X(x, y) \) is well-defined for any elements \( x, y \in X \), and it is not hard to see that the function \( h_X \) satisfies all the metric axioms [7]. Also, we should note that \( h_X(x, y) \) may not equal to \( \| x - y \|_X \) if \( X \) is not a linear space. However, \( h_X(x, y) \leq \| x - y \|_X \) for every \( x, y \in X \).

We should note the following useful properties of the Hausdorff metric which are given by Aseev [7].

The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous and the following properties are satisfied:

(a) Suppose that \( x_n \to x_0 \) and \( y_n \to y_0 \), and that \( x_n \leq y_n \) for any positive integer \( n \). Then, \( x_0 \leq y_0 \).

(b) Suppose that \( x_n \to x_0 \) and \( z_n \to z_0 \). If \( x_n \leq z_n \leq x_0 \) for any \( n \), then \( y_n \to y_0 \).

(c) Suppose that \( x_n + y_n \to x_0 \) and \( y_n \to \theta \). Then, \( x_n \to x_0 \).

Furthermore, for each \( \alpha \in \mathbb{R} \) and for every \( x, y, u, v \in X \),

\[
h_X(\alpha \cdot x, x \cdot y) = |\alpha| \cdot h_X(x, y),
\]

\[
h_X(x + y, z + v) \leq h_X(x, z) + h_X(y, v).
\]

(5)

If \( A \in K_C(\mathbb{R}^n) \) or \( K(\mathbb{R}^n) \), where \( K(\mathbb{R}^n) \) is the family of all nonempty compact subsets of \( \mathbb{R}^n \), then \( \| A \| = \sup_{a \in A} \| a \| \) defines a norm on these quasilinear spaces. In this case the Hausdorff metric is defined by

\[
\delta_{\text{ha}}(A, B) = \max\left\{ \inf_{a \in A} \sup_{b \in B} \| a - b \|, \inf_{b \in B} \sup_{a \in A} \| a - b \| \right\}.
\]

(6)

\[
= \inf\{r \geq 0: A \subseteq B + S(\theta), B \subseteq A + S(\theta)\},
\]

where \( S(\theta) = \{ y \in \mathbb{R}^n: \| y \| \leq \theta \} \).

Let \( u, v \in F(\mathbb{R}^n) \) and \( \alpha \in (0, 1] \). The algebraic sum and scalar multiplication by a real number \( \lambda \in \mathbb{R} \) are defined by

\[
(u + v)(x) = \sup_{y \in \mathbb{R}^n} \min\{u(y), v(x - y)\}
\]

(7)

\[
(\lambda u)(x) = \begin{cases} u(\lambda^a x), & \lambda \neq 0, \\ \chi_0(x), & \lambda = 0. \end{cases}
\]

And then we have \([u + v]^a = [u]^a + [v]^a\) and \([\lambda u]^a = \lambda [u]^a\) where for any subset \( A \) of \( \mathbb{R}^n \), \( \chi_A \) denotes the characteristic function of \( A \). Furthermore, from [6, 15], the relation is obtained.

\[
u \leq \nu \circ [u]^a \circ [v]^a, \quad \forall \alpha \in (0, 1],
\]

(8)

is a partial order on \( F(\mathbb{R}^n) \) and hence it is a quasilinear space [6] with the zero element \( \chi_{\{0\}} \).

Now let us determine two important subspace of \( F(\mathbb{R}^n) \). Consider \( \chi_{\{a\}} \), the characteristic function of a singleton \( \{a\} \) for \( a \in \mathbb{R}^n \). Obviously, \( \chi_{\{a\}} \in F(\mathbb{R}^n) \) and \( -\chi_{\{a\}} = (-1) \cdot \chi_{\{a\}} \) is the additive inverse of \( \chi_{\{a\}} \) since the following is obtained:

\[
(\chi_{\{a\}} - \chi_{\{a\}})(x) = \sup_{y \in X} \min\{\chi_{\{a\}}(y), -\chi_{\{a\}}(x - y)\}
\]

\[
= \sup_{y \in X} \min\{\chi_{\{a\}}(y), \chi_{\{a\}}(y - x)\} = \chi_{\{0\}}(0).
\]

(9)

Let us warn again that some elements may have no additive inverse; for example, consider the characteristic function \( \chi_{\{a\}} \) of \( \{1, 3\} \subset \mathbb{R} \). Although, \( -\chi_{\{1,3\}} = \chi_{\{-1,3\}} \in F(\mathbb{R}) \), it is not an additive inverse of \( \chi_{\{1,3\}} \) in \( F(\mathbb{R}) \) since \( \chi_{\{1,3\}} \neq \chi_{\{1,3\}} \neq \chi_{\{0\}} \).

Theorem 1. All regular elements in \( F(\mathbb{R}^n) \) must be in the form \( \chi_{\{a\}} \), for some \( a \in \mathbb{R}^n \).

Proof. Suppose that \( u \neq \chi_{\{a\}} \) and \( u - u = \chi_{\{0\}} \). This means \( (u - u)(x) = \chi_{\{0\}}(x) \) for any \( u \in F(\mathbb{R}^n) \), and so,

\[
(u - u)(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}
\]

(10)

Hence, for \( x \neq 0 \), \( (u - u)(x) = \sup_{y \in \mathbb{R}} \min\{u(y), u(y - x)\} = 0 \) meaning that \( \min\{u(y), u(y - x)\} = 0 \). We know that \( y - x \neq y \) for \( x \neq 0 \). Therefore, for \( c \in (0, 1] \) and \( y \in \mathbb{R}^n \) we can consider the fuzzy set \( u \) defined by the following:

\[
u(y) = \begin{cases} c, & y = a, \\ 0, & y \neq a. \end{cases}
\]

(11)

Moreover, for \( x = 0 \), \( (u - u)(0) = \sup_{y \in \mathbb{R}} \min\{u(y), u(y - x)\} = c. \) By the definition of \( \chi_{\{0\}} \) for \( x = 0 \), we must be defined by \( u(y) = \begin{cases} 1, & y = a, \\ 0, & y \neq a \end{cases} \) for \( c \in (0, 1] \) and \( y \in \mathbb{R}^n \) we can consider the fuzzy set \( u \) defined by

\[
u(y) = \begin{cases} 1, & y = a, \\ 0, & y \neq a. \end{cases} \]

(12)

This contradicts with the assumption \( u \neq \chi_{\{a\}} \).

Hence, the regular subspace \( F(\mathbb{R}^n) \) is \( F(\mathbb{R}) \), \( \{\chi_{\{a\}}: a \in \mathbb{R}^n\} \). Let us give examples of singular elements in \( F(\mathbb{R}) \).

Example 1. Let \( \nu: \mathbb{R} \to [0, 1] \) be defined by

\[
u(x) = \begin{cases} 1 - x^2, & x \in [-1, \frac{1}{2}], \\ 0, & x \notin [-1, \frac{1}{2}]. \end{cases}
\]

(13)

Clearly, \( \nu \in F(\mathbb{R}) \). Because, its \( \alpha \)-level is
And \([v]^\alpha\] is convex compact for all \(\alpha \in [0, 1]\) in \(\mathbb{R}\). The function \(v\) is given in [1]. Let us try to find the inverse of \(v\). The element \(v - v\) is defined as \(v - v(x) = \sup_{y \in X} \min\{v(y), v(y - x)\}\) and if \(x = 0\) then

\[
(v - v)(0) = \sup_{y \in \mathbb{R}} \min\{v(y), v(y)\} = \sup_{y \in \mathbb{R}} v(y)
\]

\[
= \sup_{y \in \mathbb{R}} (1 - y^2) = \frac{3}{4} \neq 1.
\]

So, \(v - v \neq \chi_{[0]}\) and this means there is no additive inverse of \(v\). That is, \(v \in (F(\mathbb{R}^n))_{\sup}\).

Furthermore, for any \(V \in K_C(\mathbb{R}^n), \chi_V\) is a singular element in \(F(\mathbb{R}^n)\), that is, \(\chi_V \in F(\mathbb{R}^n)_{\mathbb{R}}\).

We have seen that two important subspaces of \(F(\mathbb{R}^n)\) are the regular subspace \(F(\mathbb{R}^n)_{\mathbb{R}}\) and the singular subspace \(F(\mathbb{R}^n)_{\mathbb{R}}\). They are only intersect at the zero \(\chi_{[0]}\). Another important subspace of \(F(\mathbb{R}^n)\) is the symmetric subspace \(F(\mathbb{R}^n)_{\mathbb{R}}\). An element \(x\) in a QLS \(X\) is called symmetric whenever \(x = -x\), and \(X_{\mathbb{R}}\) denotes the set of all symmetric elements. It is a subspace of \(X_s\) and hence of \(X\). For example, \(\chi_{[-3, 3]} \in F(\mathbb{R}^n)_{\mathbb{R}}\) since \(\chi_{[-3, 3]} = \chi_{[1, 5]}\).

### 3. On the Algebra of the Fuzzy Sets

In this section, we will establish the necessary infrastructure to define the concepts of base and dimension in fuzzy quasilinear spaces. In this context, the dimension of a fuzzy quasilinear space can be expressed as a binary number \((a, b)\) where \(a\) and \(b\) are natural numbers [10].

Let \(\{u_k\}_{k=1}^m\) be a subset of \(F(\mathbb{R}^n)\) where \(m \leq n\) and \(m\) is a positive integer. A (linear) combination of the set \(\{u_k\}_{k=1}^m\) is an element \(z\) of \(F(\mathbb{R}^n)\) in the form \(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m = z\) where the coefficients \(\alpha_1, \alpha_2, \ldots, \alpha_m\) are real scalars. On the other hand, a quasilinear combination of the set \(\{u_k\}_{k=1}^m\) is an element \(z\) in \(F(\mathbb{R}^n)\) such that \(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \leq z\) for some real scalars \(\alpha_1, \alpha_2, \ldots, \alpha_m\). Hence, the quasilinear combination, briefly ql-combination, is defined by the partial order relation on \(F(\mathbb{R}^n)\). In fact, the definition of linear combination in \(F(\mathbb{R}^n)\) is also depended on the partial order relation, and it can be defined as in the following expanded form: a linear combination of the set \(\{u_k\}_{k=1}^m\) is an element \(z\) of \(F(\mathbb{R}^n)\) such that

\[
\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \leq z \quad \text{and} \quad z \leq \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m,
\]

where the coefficients \(\alpha_1, \alpha_2, \ldots, \alpha_m\) are real scalars. Clearly, a linear combination of \(\{u_k\}_{k=1}^m\) is a quasilinear combination of \(\{u_k\}_{k=1}^m\), but not conversely. In quasilinear spaces, there are two kinds of combinations of the set \(\{u_k\}_{k=1}^m\) namely, linear combination and quasilinear combination. If our quasilinear space is a linear space, we do not encounter a combination called a quasilinear combination. According to the definition of quasilinear combination, linear combination of \(\{u_k\}_{k=1}^m\) corresponding to \(\{\alpha_k\}_{k=1}^m\) is unique but quasilinear combination is not unique. For any nonempty subset \(A\) of \(F(\mathbb{R}^n)\), span of \(A\) is given by following known definition:

\[
\text{Sp}A = \left\{ \sum_{k=1}^m \alpha_k \cdot u_k : u_1, u_2, \ldots, u_m \in A, \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}, m \in \mathbb{N} \right\}.
\]

However, \(\text{Qsp}A\), the quasispans (q-span, for short) of \(A\), is defined by the set of all possible quasilinear combinations of \(A\), that is,

\[
\text{Qsp}A = \left\{ z : \sum_{k=1}^m \alpha_k \cdot u_k \leq z, u_1, u_2, \ldots, u_m \in A, \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}, m \in \mathbb{N} \right\}.
\]

Obviously, \(\text{Sp}A \subseteq \text{Qsp}A\). Furthermore, \(\text{Sp}A = \text{Qsp}A\) for a linear QLS (linear space), hence, the notion of \(\text{Qsp}A\) is redundant in linear spaces. Moreover, we say \(A\) quasispans \(F(\mathbb{R}^n)\) whenever \(\text{Qsp}A = F(\mathbb{R}^n)\).

Let us give an example.

**Example 2.** Take \(A = \{\chi_{[1, 5]}\}\), a singleton in \(F(\mathbb{R})\). The q-span of \(A\) is
This means, for example, \( \chi_{(2,3)} \notin QspA \) since we cannot find any \( \lambda \in \mathbb{R} \) satisfying \( \lambda \cdot \chi_{(1,5)} \leq \chi_{(2,3)} \). Hence, \( QspA \neq F(\mathbb{R}) \) and we say in this case \( A \) or the element \( \chi_{(1,5)} \) cannot q-span \( F(\mathbb{R}) \). Let us consider another singleton of a regular element in \( F(\mathbb{R}) \). For \( 0 \neq a \in \mathbb{R} \), \( \{ \chi_{[a]} \} \) can q-span \( F(\mathbb{R}) \). In this respect, we can say \( \chi_{[a]} \) can q-span \( F(\mathbb{R}) \) while it cannot span \( F(\mathbb{R}) \). We should emphasize that singular element cannot q-span \( F(\mathbb{R}) \).

This example has given us important clues as to how we can define the concept of the dimension of the quasinlinear space of fuzzy sets. Let us give a result that we can easily see the proof: Let \( A = \{ x_1, x_2, \ldots, x_n \} \) be a subset of \( F(\mathbb{R}^n) \). Then, \( QspA \) is a subspace of \( F(\mathbb{R}^n) \).

**Definition 2** (see [10]). (Quasinlinear independence and dependence) A set \( A = \{ x_1, x_2, \ldots, x_n \} \) in a QLS \( F(\mathbb{R}^n) \) is called quasinlinear independent \((ql-independent, briefly)\) whenever the inequality

\[
\chi_{(0)} \leq \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \cdots + \lambda_m \cdot x_m,
\]

holds if and only if \( \lambda_1 = \lambda_2 = \ldots = \lambda_m = 0 \). Otherwise, \( A \) is called quasinlinear dependent \((ql-dependent, briefly)\).

If we recall again that every linear space is a QLS with the relation “=”, it can be seen that the notions of quasinlinear independence and dependence coincide with linear independence and dependence, respectively. The set \( A \) in the previous example is ql-independent since \( \chi_{(0)} \leq \lambda \cdot \chi_{(1,5)} \) iff \( \{ 0 \} \leq \lambda \cdot [1, 5] \) if and only if \( \lambda = 0 \). However, the singleton \( B = \{ \chi_{[-1,2]} \} \) is ql-dependent since \( \{ 0 \} \leq \beta \cdot [-2, 3] \) for \( \beta = 2 \neq 0 \). This is an unusual case since a nonzero singleton is obviously linearly independent in linear space. On the other hand, the set \( \{ \chi_{(2,3)}, \chi_{[-1,2]} \} \) is ql-dependent. In general, we can see from the definition that any subset including an element related to zero \( \chi_{(0)} \) must be ql-dependent in \( F(\mathbb{R}^n) \). This is a generalization of the well-known result: a subset including zero must be linearly dependent in linear spaces.

**Example 3.** Let us consider two dimensional vector space \( \mathbb{R}^2 \). Let \( v_1 = e_1 \cap [-1, 1] = \{(t, s): s = 0, t \in [-1, 1]\} \) and \( v_2 = e_2 \cap [-1, 1] = \{(t, s): t = 0, s \in [-1, 1]\} \), where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) are unit basis vectors in two dimensional vector space \( \mathbb{R}^2 \). Then, \( v_1 \) and \( v_2 \) are horizontal and vertical bars of equal length intersecting at zero in \( \mathbb{R}^2 \). Then, \( \{ X_1, X_2 \} \) is a ql-dependent subset of \( F(\mathbb{R}^2) \) since

\[
QspA = \{ u \in F(\mathbb{R}): \lambda \cdot \chi_{(1,5)} \leq u, \lambda \in \mathbb{R} \} = \{ u \in F(\mathbb{R}): \lambda \cdot [1, 5] \leq [u]^n, \lambda \in \mathbb{R} \}.
\]
The above mentioned definition means that \( r - \dim X \) is just the dimension of the linear subspace \( X \) of \( X^r \) and so \( r - \dim X = \dim X \). Notice that a nontrivial singular subspace of a quasilinear space cannot be a linear space. Furthermore, we can see that any quasilinear space is \((n,0)\)-dimensional if and only if it is \( n \)-dimensional linear space. In this respect, the trivial linear subspace \( \{x_0\} \) of \( F(R^n) \) is a \((0,0)\)-dimensional quasilinear space. Later, we will give an example of a \((0,0)\)-dimensional quasilinear space other than \( \{x_0\} \).

\[
\nu_1 = e_1 \ominus [1, 2] = \{(t_1, t_2, \ldots, t_n) : t_1 \in [1, 2], t_k = 0 \text{ for } k \in \{2, 3, \ldots, n\}\}
\]

\[
\nu_2 = e_2 \ominus [1, 2] = \{(t_1, t_2, \ldots, t_n) : t_2 \in [1, 2], t_k = 0 \text{ for } k \in \{1, 3, 4, \ldots, n\}\}
\]

\[\vdots\]

\[
\nu_n = e_n \ominus [1, 2] = \{(t_1, t_2, t_3) : t_n \in [1, 2], t_k = 0 \text{ for } k \in \{1, 2, 3, \ldots n - 1\}\}.
\]

Then, \( \{X_1, X_2, \ldots, X_n\} \) is a quasilinear space of all \( R^n \) which contains the zero element of the vector space \( R^n \). Otherwise, its \( n \)-dimensional would be broken. By the above-mentioned assumption, we can find \( a_1 \in [x_1]^n, a_2 \in [x_2]^n, \ldots, a_n \in [x_n]^n \) such that \( 0 = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n \). We conclude that the set \( \{a_1, a_2, \ldots, a_n\} \) is a linearly independent subset of \( R^n \) since \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \). This is a contradiction to the fact that \( r - \dim F(R^n) = n \). So, we can write \( s - \dim F(R^n) = n \).

**Theorem 2.** \( F(R^n) \) is an \((n,n)\)-dimensional quasilinear space.

**Proof.** For the set \( \{e_1, e_2, \ldots, e_n\} \), the standard basis of the linear space \( R^n \),

\[
B = \{\chi[e_1], \chi[e_2], \ldots, \chi[e_n]\},
\]

is just the basis of the linear space \( (F(R^n)) \). This means \( r - \dim F(R^n) = n \). Now let us consider bars.

**Example 5.** Consider the singular subspace \( (K_C(R^n))_h \) of the quasilinear space \( K_C(R^n) \). \( r - \dim (K_C(R^n))_h = 0 \) since \( (K_C(R^n))_h = \{0\} \). Furthermore, \( \{1, 2\} \) is a quasilinear space of \( (K_C(R^n))_h \) and so \( s - \dim (K_C(R^n))_h = 1 \). Hence, \( (K_C(R^n))_h \) is \((0,1)\)-dimensional. Obviously, \( (K_C(R^n))_h \cong R \) is a \((1,0)\)-dimensional quasilinear space. In this respect, all \( n \)-dimensional vector spaces on the field real numbers are \((n,0)\)-dimensional quasilinear space.

If \( X = (K_C(R^n))_h \cup \{(x, y) : y = 0, x \in R\} \) then \( X \) is a subspace of \( K_C(R^n) \), the quasilinear space of all compact convex subsets of \( R^n \) with the inclusion relation, and \( r - \dim X = 1 \) since \( X = \{(x, y) : y = 0, x \in R\} \). Furthermore, the set \( \{\nu_1, \nu_2\} \) in Example 3 is quasilinear and there is no 3-elements subset which is quasilinear in \( X \). This proves \( s - \dim X = 2 \). Hence, \( X \) is a \((1, 2)\)-dimensional QLS.

Consider the QLSX = \( K_C(c_0) \), the quasilinear space of all bounded closed convex subsets of \( c_0 \), the set of all sequences convergent to zero. \( X \) is equivalent to \( c_0 \) and so \( r - \dim X = \infty \). Let us define the set

\[
\{\{t, 0, 0, \ldots\} : 1 \leq t \leq 4\}, \{\{0, t, 0, \ldots\} : 1 \leq t \leq 4\}, \ldots
\]

On the other hand, the set \( \{[1, 4] \ominus e_1, [1, 4] \ominus e_2, \ldots\} \),

\[
[1, 4] \ominus e_k = \{0, 0, \ldots, 0, k \text{ } \text{term}, 0, \ldots\} : s \in [1, 4],
\]

is quasilinear in \( X \), where \( e_k \)’s are coordinate vectors of \( c_0, k = 1, 2, \ldots \). Therefore, \( s - \dim X = \infty \) and so \( X = K_C(c_0) \) is an \((\infty, \infty)\)-dimensional QLS. In general, an infinite-dimensional linear space \( E \) is an \((\infty, 0)\)-dimensional QLS.

**Definition 5.** Let \( X \) be a QLS and \( y \in X \). The set of all regular elements preceding from \( y \) is called floor of \( y \), and \( \mathcal{F}_y \) denotes the set of all such elements. Therefore, \( \mathcal{F}_y = \{x \in X : x \leq y\} \). The floor of any subset \( M \) of \( X \) is the union of floors of all elements in \( M \) and is denoted by \( \mathcal{F}_M \).

Hence, \( \mathcal{F}_M = \bigcup_{x \in M} \mathcal{F}_x \) and it is clear that \( \mathcal{F}_x = \{x\} \) for some \( x \in X \). This means that \( \mathcal{F}_X = X \), and so the notion of floor is redundant in linear spaces. Furthermore, \( \mathcal{F}_X \) is a subspace of the linear space \( X \) in the QLS \( X \).

**Theorem 3.** Suppose that \( s - \dim X \neq 0 \) in a QLS \( X \). Then, \( r - \dim X \leq s - \dim X \).

**Example 6.** Consider \( (K_C(R^n))_{sym} = \{[-a, a] : a \in R\} \), the symmetric subspace of \( K_C(R^n) \). It is interesting that
(\(K_C(\mathbb{R})\))_{sym} is a \((0,0)\)-dimensional QLS, just similar to the trivial linear space.

### 4. Norm and Inner Product on \(F(\mathbb{R}^n)\)

In [6], it was shown that fuzzy quasilinear spaces have a normed QLS structure with various norms. In this study, we will say that \(F(\mathbb{R}^n)\) is a complete normed QLS structure with these special norms. Furthermore, in the next section, we will show that a particular one of these norms comes from an inner product. For \(1 \leq q < \infty\), the function

\[
D_q(u, v) = \left( \int_0^1 \delta_\infty (\{u^n, [v]^n\})^q \, da \right)^{1/q},
\]

(28)
defines a metric on \(F(\mathbb{R}^n)\) and it is a complete metric space by this metric. Moreover, this metric comes from the norm.

**Theorem 4.** For \(1 \leq q < \infty\), \((F(\mathbb{R}^n), \|u\|_q)\) is a complete normed (Banach) quasilinear space.

**Proof.** Consider the function \(g: [0, 1] \rightarrow \mathbb{R}, g(\alpha) = \sup_{x \in [u]^n} \|x\|\). Then, from the compactness of each \(\alpha\)-level set in \(\mathbb{R}^n\), \(\sup_{x \in [u]^n} \|x\|\) exists, and from the properties of fuzzy sets the required integral also exists. Hence, the norm is well-defined.

Now, let us verify the norm axioms:

1. Clearly, if \(u \neq \theta\), then \(\|u\|_q > 0\)
2. If \(u, v \in F(\mathbb{R}^n)\), then

\[
\|u + v\|_q = \left( \int_0^1 \left( \sup_{x+y \in [u]^n+[v]^n} \|x+y\| \right)^q \, da \right)^{1/q}
\]

(30)

\[
\leq \left( \int_0^1 \left( \sup_{x+y \in [u]^n+[v]^n} (\|x\|+\|y\|) \right)^q \, da \right)^{1/q}
\]

\[
\leq \left( \int_0^1 \sup_{x \in [u]^n} \|x\| \, da \right)^{1/q} + \left( \int_0^1 \left( \sup_{y \in [v]^n} \|y\| \right)^q \, da \right)^{1/q}
\]

\[
= \|u\|_q + \|v\|_q.
\]

(3)

If \(u \in F(\mathbb{R}^n)\) and \(\lambda \in \mathbb{R}\), then

\[
\|\lambda \cdot u\|_q = \left( \int_0^1 \left( \sup_{\lambda x \in \lambda \cdot [u]^n} \|\lambda x\| \right)^q \, da \right)^{1/q} = |\lambda| \left( \int_0^1 \sup_{x \in [u]^n} \|x\| \, da \right)^{1/q}.
\]

(31)

(F(\mathbb{R}^n), \| \cdot \|_q) is a complete with \(\|u\|_q = D_q(u, \theta)\) since \(K_C(\mathbb{R}^n)\) is complete. Hausdorff metric for this norm can be computed by the following formula:

\[
D_q(u, v) = \inf \left\{ r \geq 0 : u \subseteq v + r \cdot e, v \subseteq u + r \cdot e \text{ and } \|u'\| \leq r, i = 1, 2 \right\}
\]

\[
= \left( \int_0^1 \delta_\infty (\{u^n, [v]^n\})^q \, da \right)^{1/q}.
\]

(32)

We should note here that the norm metric (Hausdorff metric) cannot be obtained by the way \(\|u - v\|_q = D_q(u, v)\). Only we can write \(D_q(u, v) \leq \|u - v\|_q\).

After this stage, we will show that the norm \(\|u\|_q\) is an inner product norm for \(q = 2\). Now, in order to describe this inner product, let us present some of the concepts about
normed quasi-linear spaces that we have obtained earlier [8, 9, 16].

Definition 6 (see [8]). $X$ is called consolidate (solid-floor ed) QLS whenever $y = \sup \{x \in X, x \leq y\}$ for each $y \in X$. Otherwise, $X$ is called a non-consolidate QLS, briefly, nc-QLS.

The supremum in this definition is taken on the order relation “$\leq$” in the definition of a QLS. Above-mentioned definition assumes $\sup \{x \in X, x \leq y\}$ exists for each $y \in X$. Implicitly, we say that $X$ is consolidate if and only if $y = \sup \mathcal{F}_{(y)}$ for each $y \in X$.

Let us first note that any linear space is a consolidate QLS. Indeed, $X_\epsilon = X$ for any linear space $X$. So,

$$y = \sup \{x \in X_\epsilon, x \leq y\} = \sup \{x \in X, x = y\} \quad (33)$$

for any element $y \in X$.

Theorem 5. $F(R^n)$ is a consolidate QLS, that is, for each $u \in F(R^n), u = \sup \{v \in F(R^n), v \leq u\}$.

Proof. Assume that $b \in F(R^n)$ is upper bound for $\{v \in F(R^n), v \leq u\}$. We should prove that $u \leq b$. Suppose that $u \neq b$. Then, we say that $[u]^n \notin [b]^n$ for all $a \in [0, 1]$. This means one can find a $t_a \in [u]^n$ such that $t_a \notin [b]^n$. Now, let us consider the characteristic function $\chi(t_a)$ of the singleton $\{t_a\}$. We should show that $\chi(t_a) \leq u$. For any $a \in [0, 1]$, $\chi(t_a)^a = [t_a] \leq [u]^a$ since $t_a \in [u]^a$ and $\chi(t_a)^a = [t_a]$. Hence, we have $\chi(t_a) \leq u$ from the definition of partial order relation. Moreover, $\chi(t_a) \neq b$. Because, if it had been $\chi(t_a) \leq b$, then we had $[\chi(t_a)]^a \subseteq [b]^a$, for any $a \in [0, 1]$, i.e., $\{t_a\} \subseteq [b]^a$.

However, this contradicts with $t_a \notin [b]^a$. Also, we know that $\chi(t_a)$ is a regular element of $F(R^n)$ Consequently, we have found the regular element $\chi(t_a)$ such that $\chi(t_a) \leq u$ and $\chi(t_a) \neq b$. But, this conflict with the admission that $b$ is an upper bound. Thus, we obtain $u \leq b$. This gives $F(R^n)$ is a consolidate QLS.

Example 7. Quasi-linear spaces $F(R)$, and $(F(R))_{sym}$ are not consolidate. Let us show that $F(R)_{sym}$ is not consolidate. Let us take $y = \chi([-1,1]) \in F(R)_{sym}$. Then, we find

$$\mathcal{F}_y = \{x \in (F(R)_{sym}), x \leq y\} = \{x \in \chi([0])); \exists x \leq y\} = \chi([0))$$

This shows $y \neq \sup \mathcal{F}_y = \chi([0))$.

It may not be possible to perform many important functions in non-consolidate quasi-linear spaces. In order to eliminate these negative situations, we have introduced a new definition under the name of consolidation of quasi-linear spaces. The consolidation concept we have given is unnecessary for linear spaces because every linear space is consolidated. If we did not have this concept, it would not be possible for us to define the concept of the inner-product for some important quasi-linear spaces. Let us give a definition.

Definition 7. For some two quasi-linear spaces $(X, \leq)$ and $(Y, \leq)$, we say $Y$ compatible contains $X$ whenever $X \subseteq Y$ and the partial order relation “$\leq$” on $X$ is the restriction of the partial order relation $\leq$ on $Y$. We briefly use the symbol $X \subseteq Y$ in this case. We write $X \equiv Y$ whenever $X \subseteq Y$ and $Y \subseteq X$ means $X$ and $Y$ are the same sets with the same partial order relations which make them a quasi-linear space. We may write $X = Y$ for $X \equiv Y$ whenever the relations are clear from the context.

Definition 8. Let $X$ be a QLS. Consolidation of $X$ is the smallest consolidate QLS $\bar{X}$ which compatibly contains $X$, that is, if there exists another consolidate QLS $Y$ which compatibly contains $X$ then $X \subseteq Y$.

Clearly, for some consolidate QLS $X, \bar{X} = X$. We do not yet know whether every quasi-linear space has a consolidation or not. Furthermore, each linear space is a consolidate QLS. Now, let us show $K_C(R^j) = K_C(R)$.

Theorem 6. Consolidation of $(K_C(R))^j$ is $K_C(R)$.

Proof. Obviously, $K_C(R)$ compatible contains $(K_C(R))^j$. Suppose that $Z$ is another consolidate QLS which contains $(K_C(R))^j$. For an arbitrary element $x$ of $K_C(R)$ we will show that $x \in Z$. If $x \in (K_C(R))^j$, then the proof is clear. If $x \notin (K_C(R))^j$, then $x \in (K_C(R))_s$. Hence, $x = (a)$ for some $a \in R$. Assume that $[a] \notin Z$. We get $[a - \epsilon, a + \epsilon] \in (\Omega_C(R))_s$ for some $\epsilon > 0$. So, we find $[a - \epsilon, a + \epsilon] \in Z$. Since $Z$ is consolidate,

$[a - \epsilon, a + \epsilon] \subseteq [a - \epsilon, a + \epsilon] = \{y \in Z, \exists y \subseteq [a - \epsilon, a + \epsilon]\}$

for any $\epsilon > 0$. This means there exists an element $u_x \in Z$, such that $u_x \subseteq [a - \epsilon, a + \epsilon]$ in $Z$. Therefore, we have $[a] \in Z$. Otherwise, the set $[a - \epsilon, a + \epsilon]$ cannot be a closed set. So, this conflicts with the fact that $[a - \epsilon, a + \epsilon]$ is an element of $(K_C(R))^j$. Thus, the assumption $[a] \notin Z$ is incorrect.

Similarly, one can see that $K_C(R)_{sym} = K_C(R)$. We can also prove that $F(R)^n = F(R^n)$ and $(F(R^n))_{sym} = F(R^n)$ with a slightly more difficult proof technique.

For any element $y$ of a QLS $X$, the set $F^X = \{z \in (\bar{X}); \exists z \leq y\}$ denotes the floor of $y$ in $\bar{X}$ and sometimes $F^X$ is said to be the floor of $y$ in the consolidation. For a consolidate QLS, this notion is unnecessary. But, this concept is important in a non-consolidate QLS.

Example 8. Let us take two elements $[-4, -1]$ and $[-2, 5]$ in non-consolidate QLS $X = (K_C(R))_s$. Then, $\mathcal{F}_{[1]}$ is empty and $\mathcal{F}_{[-2, 5]} = \emptyset$. But $\mathcal{F}_{[-1, -1]} = \{[t]; t \in [-1, -1]\}$ and $\mathcal{F}_{[4, -1]} = \{[t]; t \in [2, 5]\}$.
Definition 9. Let $X$ be a quasilinear space having a consolidation $\overline{X}$. A mapping $\cdot \cdot \cdot K(\mathbb{R})$ is called an inner product on $X$ if for any $x, y, z \in X$ and $\alpha \in \mathbb{R}$ the following conditions are satisfied:

(IPQ1) If $x, y \in X$, then $\langle x, y \rangle \in K(\mathbb{R})$, $= \mathbb{R}$

(IPQ2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IPQ3) $\langle \alpha \cdot x, y \rangle = \alpha \cdot \langle x, y \rangle$

(IPQ4) $\langle x, y \rangle = \langle y, x \rangle$

(IPQ5) $\langle x, x \rangle \geq 0$ for $x \in X$, and $\langle x, x \rangle = 0 \Rightarrow x = 0$

(IPQ6) $\|\langle x, y \rangle\|_{K(\mathbb{R})} = \sup \{\|\langle a, b \rangle\|_{K(\mathbb{R})}: a \in \mathbb{F}_x^2, b \in \mathbb{F}_y^2\}$

\[
\begin{align*}
\hat{h}(x, y) &= \inf \{r \geq 0: x \leq y + a^i_1, y \leq x + a^i_2 \text{ and } \|a^i_1\| \leq r^i_1, i = 1, 2\} \\
&= \inf \{r \geq 0: x \leq y + a^i_1, y \leq x + a^i_2 \text{ and } \|\langle a^i_1, a^i_2 \rangle\|_{K(\mathbb{R})} \leq r^i_2, i = 1, 2\}.
\end{align*}
\]

(36)

For $A, B \subseteq K_C(\mathbb{R})$ or $K(\mathbb{R}), \langle A, B \rangle = \{ab: a \in A, b \in B\}$ is an inner product and they are Hilbert quasilinear space by this inner product norm. Furthermore, if $x_n \rightarrow x$ and $y_n \rightarrow y$ in a IPQLS, then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$ [9].

Theorem 7. For some $A, B \subseteq K(\mathbb{R})^n$, $(A, B)_{K(\mathbb{R})} = \{ab: a \in A, b \in B\}$ defines an inner product on $K(\mathbb{R})^n$. Furthermore, $K(\mathbb{R})^n$ is a Hilbert quasilinear space by this inner product norm [9].

Definition 10 (see [8]). An element $x$ of an IPQLS $X$ is said to be orthogonal to an element $y \in X$ if $\langle x, y \rangle_{K(\mathbb{R})} = 0$. We also say that $x$ and $y$ are orthogonal, and we write $x \perp y$. An orthonormal set $M \subseteq X$ is an orthogonal set in $X$ whose elements have norm 1.

Definition 11 (see [8]). Let $A$ be a nonempty subset of an inner product quasilinear space $X$. An element $x \in X$ is said to be orthogonal to $A$, denoted by $x \perp A$, if $\langle x, y \rangle_{K(\mathbb{R})} = 0$ for every $y \in A$. The set of all elements of $X$ orthogonal to $A$, denoted by $A^\perp$, is called the orthogonal complement of $A$ and is indicated by $A^\perp = \{x \in X: \|\langle x, y \rangle\|_{K(\mathbb{R})} = 0, y \in A\}$.

Theorem 8. Let $X$ be a quasilinear space having a consolidation $\overline{X}$. Then, the Schwarz inequality holds, that is, $\|\langle x, y \rangle\|_{K(\mathbb{R})} \leq \|x\| \|y\|$ for any $x, y \in X$.

You can see the proof of this theorem in cites [8, 9, 16].

5. The Inner Product on $F(\mathbb{R}^n)$

Now, let us give some basic concepts which are used to define an inner-product on $F(\mathbb{R}^n)$. Let $B(\mathbb{R}^k)$ and $B(\mathbb{R}^n)$ denote the $\sigma$–algebras of Borel subsets of $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively, where $k,n$ are positive integers. Let $F: T \rightarrow K_C(\mathbb{R}^n)$ be a function and $T \subseteq \mathbb{R}^k$ where we use again the symbol big $F$ for a set-valued function. If for any $B \in B(\mathbb{R}^n)$, $\{t \in T: F(t) \subseteq B\} \in B(\mathbb{R}^k)$, then we say

(IPQ7) if $x \leq y$ and $u \leq v$ then $\langle x, u \rangle \leq \langle y, v \rangle$

(IPQ8) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leq y + x_\varepsilon$ and $\langle x, x_\varepsilon \rangle \leq S_{\varepsilon}(\theta)$ then $x \leq y$

where $\mathbb{F}_x$ denotes the floor of $x$ in the consolidation $\overline{X}$ of $X$. A quasilinear space with an inner product is called an inner product quasilinear space, briefly, IPQLS [8]. An IPQLS is called a Hilbert QLS whenever it is complete with the following inner product metric. Every IPQLS $X$ is a normed QLS with the norm defined by $\|x\| = \sqrt{\|\langle x, x \rangle\|_{K(\mathbb{R})}}$ for every $x \in X$. This norm is called inner product norm. The inner-product metric is obtained by the following formula:

$\hat{h}(x, y) = \inf \{r \geq 0: x \leq y + a^i_1, y \leq x + a^i_2 \text{ and } \|a^i_1\| \leq r^i_1, i = 1, 2\}$

(36)

that $F$ is measurable [15] and we write $F^{-1}(A) = \{t \in T: F(t) \cap A \neq \emptyset\}$ for any subset $A$ of $\mathbb{R}^n$.

Proposition 1. [15] The following results are equivalent:

(i) The function $F: T \rightarrow K_C(\mathbb{R}^n)$ is measurable

(ii) $F^{-1}(B) \in B(\mathbb{R}^n)$ for every $B \subseteq B(\mathbb{R}^k)$

(iii) $F^{-1}(O) \in B(\mathbb{R}^k)$ for every open subset $O$ of $\mathbb{R}^n$

(iv) $F^{-1}(C) \in B(\mathbb{R}^k)$ for every closed subset $C$ of $\mathbb{R}^n$

(v) The function $d(x, F(.))$: $T \rightarrow \mathbb{R}$ is measurable for every $x \in \mathbb{R}^n$

(vi) The function $\|F(\cdot)\|$ is measurable

(vii) The function $s(x, F(.))$: $T \rightarrow \mathbb{R}$ is measurable for every $x \in \mathbb{R}^n$.

The statements (v)–(vii) explain the measurability of the single-valued mapping defined from $T$ to $\mathbb{R}$ regarding to the Borel $\sigma$–algebras $B(\mathbb{R}^k)$ and $B(\mathbb{R}^n)$. If these functions are continuous, then such mappings are measurable. For some set-valued mappings we have the following informations [15]. Any function $F: T \rightarrow K_C(\mathbb{R}^n)$ is measurable if it is upper semicontinuous or lower semicontinuous and hence if it is continuous. Let $F: T \rightarrow K_C(\mathbb{R}^n)$ be a sequence of measurable functions for $i = 1, 2, \ldots$ and suppose that $\lim_{i \rightarrow \infty} F_i(t), F(t) = 0$ for every $t \in T$. Then, the limit function $F: T \rightarrow K_C(\mathbb{R}^n)$ is also measurable. A selector of a set-valued mapping $F: T \rightarrow K_C(\mathbb{R}^n)$ is a single-valued mapping $f: T \rightarrow \mathbb{R}^n$ such that $f(t) \in F(T)$ for every $t \in T$, [15]. If the function $F$ is measurable, then it’s all selector functions are also measurable. The following theorem, known as the “Castaing Representation Theorem” gives an additive characterization of the measurability of a set-valued mapping.

Theorem 9 (see [15]). The function $F: T \rightarrow K_C(\mathbb{R}^n)$ is measurable if and only if there exist a sequence $\{f_i\}$ of measurable function $F$ such that for each $t \in T$

$(T) = \bigcup_{i=1}^{\infty} f_i(t)$.
For any lower semicontinuous function $F: T \rightarrow K_C(\mathbb{R}^n)$ there exist a continuous selector $f$ of $F$ such that $f(t) = x$ for every $x \in F(t)$ and $t \in T$. Namely, if a set-valued mapping is lower semicontinuous, then it has a continuous selector.

Now, let us give the definition of “Aumann Integral” [15]. Suppose that $F: [0,1] \rightarrow K_C(\mathbb{R}^n)$ and $S(F)$ is the set of all integrable selectors of $F$ which is defined on $[0,1]$. The Aumann integral of $F$ on $[0,1]$ is defined as

$$\int_0^1 F(t)dt = \left\{ \int_0^1 f(t)dt : f \in S(F) \right\}.$$ If $S(F) \neq \emptyset$, then the Aumann integral is exist and $F$ is said to be Aumann integrable.

The function $F$ is integrably bounded on $[0,1]$ if there exists an integrable function $g: [0,1] \rightarrow \mathbb{R}$ such that $\|F(t)\| \leq g(t)$ for almost all $t \in [0,1]$.

**Theorem 10** (see [15]). If $F: [0,1] \rightarrow K_C(\mathbb{R}^n)$ is measurable and integrably bounded, then it is Aumann integrable over each $[a, s] \subset [0,1]$ with $\int_a^s F(t)dt \in K_C(\mathbb{R}^n)$ for all $s \in [a,1]$.

For a set-valued mapping $F$ as in Theorem 10, the Castaing Representation Theorem 9 applies and provides a sequence $\{f_i\}$ of integrable selectors which are pointwise dense in $F$. Moreover,

$$\int_0^1 F(t)dt = \left\{ \int_0^1 f_i(t)dt : i = 1, 2, \ldots \right\}, \quad (37)$$

for $u, v \in F(\mathbb{R}^n)$.

**Proof.** First, we must show that the equality (5.2) is well-defined. We know from the cites [8, 9, 16], the function $\langle , \rangle_{K(\mathbb{R}^n)}$ is inner product on the QLS $K(\mathbb{R}^n)$, and we proved that an inner product is a continuous function. This means it can be Aumann integrable on $[0,1]$. Hence, the function $\langle , \rangle$ is well-defined.

Now, let us verify the inner-product axioms:

and so we need only consider these selectors to evaluate $\int_0^1 F(t)dt$.

**Theorem 11** (see [15]). The Aumann integral satisfies the following properties for all Aumann integrable functions $F, G: [0,1] \rightarrow K_C(\mathbb{R}^n)$:

$$\int_0^1 (F(t) + G(t))dt = \int_0^1 F(t)dt + \int_0^1 G(t)dt,$$

$$\int_a^b F(t)dt = \int_a^b F(t)dt , 0 \leq a \leq b \leq c \leq 1,$$

$$\int_0^1 \lambda F(t)dt = \lambda \int_0^1 F(t)dt , \lambda \in \mathbb{R},$$

$$\int_0^1 F(t)dt \subseteq \int_0^1 G(t)if F(t) \subseteq G(t)for all t \in [0,1]. \quad (38)$$

In addition, the Aumann integral uniquely determines its integrand.

**Theorem 12** (see [15]). If $F: [0,1] \rightarrow K_C(\mathbb{R}^n)$ is measurable and integrably bounded, then it is Aumann integrable over each $[a, s] \subset [0,1]$ with $\int_a^s F(t)dt \in K_C(\mathbb{R}^n)$ for all $s \in [a,1]$.

Let us give a main result.

**Theorem 13.** $F(\mathbb{R}^n)$ is an inner product quasilinear space by the function $d$.

$$\langle u, v \rangle = \int_0^1 \langle [u]^n, [v]^u \rangle_{K(\mathbb{R}^n)}d\alpha = \int_0^1 \langle a, b \rangle_{\mathbb{R}^n}d\alpha : a \in [u]^n, b \in [v]^u. \quad (39)$$

(1) If $u, v \in F(\mathbb{R}^n)_p$, then we have the representations $u = X_{[a]}$ and $v = X_{[b]}$ for some $a, b \in \mathbb{R}^n$. Hence,

$$\langle u, v \rangle = \langle X_{[a]}, X_{[b]} \rangle = \int_0^1 \langle X_{[a]}^n, X_{[b]}^u \rangle_{K(\mathbb{R}^n)}d\alpha$$

$$= \int_0^1 \langle a, b \rangle_{K(\mathbb{R}^n)}d\alpha = \langle [a], [b] \rangle_{K(\mathbb{R}^n)}$$

$$= \langle a, b \rangle_{\mathbb{R}^n} \in K(\mathbb{R})_p. \quad \text{(40)}$$
(2) From Theorem 11, we write

\[
\langle u_1 + u_2, u_3 \rangle = \int_0^1 \langle [u_1 + u_2]^a, [u_3]^a \rangle_{K(R^n)} d\alpha
\]
\[
= \int_0^1 \langle [u_1]^a, [u_3]^a \rangle_{K(R^n)} + \langle [u_2]^a, [u_3]^a \rangle_{K(R^n)} d\alpha
\]
\[
= \int_0^1 \langle [u_1]^a, [u_3]^a \rangle_{K(R^n)} d\alpha + \int_0^1 \langle [u_2]^a, [u_3]^a \rangle_{K(R^n)} d\alpha
\]
\[
= \langle u_1, u_3 \rangle + \langle u_2, u_3 \rangle.
\]

(3) If we use Theorem 11, then

\[
\langle \lambda \cdot u, v \rangle = \int_0^1 \langle [\lambda u]^a, [v]^a \rangle_{K(R^n)} d\alpha = \int_0^1 \lambda \cdot \langle [u]^a, [v]^a \rangle_{K(R^n)} d\alpha
\]
\[
= \lambda \cdot \int_0^1 \langle [u]^a, [v]^a \rangle_{K(R^n)} d\alpha = \lambda \cdot \langle u, v \rangle.
\]

(4) \[\langle u, v \rangle = \int_0^1 \langle [u]^a, [v]^a \rangle_{K(R^n)} d\alpha = \int_0^1 \langle [v]^a, [u]^a \rangle_{K(R^n)} d\alpha = \langle u, v \rangle.\]

(5) Just the same as axiom (1), if \( u \in F(R^n) \), then \( u = x_{(a)} \) for \( a \in \mathbb{R}^n \) and

\[
\langle u, u \rangle = \langle x_{(a)}, x_{(a)} \rangle = \int_0^1 \langle [x_{(a)}]^a, [x_{(a)}]^a \rangle_{K(R^n)} d\alpha,
\]
\[
= \int_0^1 \langle a, a \rangle_{K(R^n)} d\alpha = \langle a, a \rangle_{\mathbb{R}^n}.
\]

Let us remember that \( \mathbb{R}^n \) is a linear inner product space. Therefore, we say that \( \langle a, a \rangle_{\mathbb{R}^n} \geq 0 \). So, \( \langle u, u \rangle \geq 0 \) since \( \langle u, u \rangle \in (\mathcal{F}(\mathbb{R}))^* = \mathbb{R}^d \) for \( u \in F(R^n) \).

(6) We shall prove that \( \|u, u\|_{K(R)} = \sup \{\|x_{(a)}\|_{K(R)}: x_{(a)} \in \mathcal{F}_u, x_{(b)} \in \mathcal{F}_v\} \). Now, for \( a \in [u]^a \), \( b \in [v]^a \), let us consider the mappings \( f_{ab}: [0,1] \to \mathbb{R} \) such that for each \( a \in [0,1] \),

\[
f_{ab}(a) = \langle a, b \rangle_{\mathbb{R}^n}.
\]

Then, \( f_{ab}(a) = \int_0^1 \langle a, b \rangle_{\mathbb{R}^n} d\alpha \). Hence, it is a selector of the function, \( \langle \cdot, \cdot \rangle: F(\mathbb{R}^m) \times F(\mathbb{R}^m) \to K(\mathbb{R}) \). This means \( f_{ab} \in S(\langle \cdot, \cdot \rangle) \) for each \( a \in [u]^a \) and \( b \in [v]^a \). Furthermore,

\[
f_{ab}(a) = \langle a, b \rangle_{\mathbb{R}^n}
\]
\[
= \int_0^1 \langle a, b \rangle_{\mathbb{R}^n} d\alpha
\]
\[
= \int_0^1 \langle [x_{(a)}]^a, [x_{(b)}]^a \rangle_{K(R^n)} d\alpha
\]
\[
= \langle x_{(a)}, x_{(b)} \rangle.
\]

From Theorem 5 and from the definition of the Aumann integral we can write

\[
\langle u, v \rangle = \sup \{\langle x_{(a)}, x_{(b)} \rangle_{K(R)}: x_{(a)} \in \mathcal{F}_u, x_{(b)} \in \mathcal{F}_v\}.
\]

where the supremum is taken over the partial order relation \( \subseteq \) on \( K(\mathbb{R}) \). Thus, we have

\[
\|u, v\| = \sup \{\|x_{(a)}, x_{(b)}\|_{K(R)}: x_{(a)} \in \mathcal{F}_u, x_{(b)} \in \mathcal{F}_v\}.
\]

(7) Suppose that \( u_1 \leq v_1 \) and \( u_2 \leq v_2 \). Then,

\[
\langle u_1, u_2 \rangle = \int_0^1 \langle [u_1]^a, [u_2]^a \rangle_{K(R^n)} d\alpha
\]
\[
\leq \int_0^1 \langle [v_1]^a, [v_2]^a \rangle_{K(R^n)} d\alpha = \langle v_1, v_2 \rangle.
\]

(8) We suppose that there exists an element \( x_{e} \in F(\mathbb{R}^n) \) such that \( u \leq v + x_{e} \) and \( \langle x_{e}, x_{e} \rangle \leq \varepsilon(\theta) \) for any \( \varepsilon > 0 \). Then, there exists an element \( [x_{e}]^a \in K(\mathbb{R}^n) \) for each \( a \in [0,1] \) such that \( [u]^a \leq [v]^a + [x_{e}]^a \) and \( \langle [x_{e}]^a, [x_{e}]^a \rangle \leq \varepsilon(\theta) \). Then, we get \( [u]^a \leq [v]^a \) for any \( a \in [0,1] \) by Theorem 7. This means that \( u \leq v \).

Now, let us determine the inner product norm for \( u \in F(\mathbb{R}^n) \).
Let us consider the set \( R \subset \mathbb{R}^n \). In order for a set to be a basis of \( F(\mathbb{R}^n) \), all its elements must first be regular elements. None of the elements of \( A \) are regular, whereas all elements of \( B \) are regular.

A striking feature of \( F(\mathbb{R}^n) \) is that some element pairs may not be able to satisfy the parallelogram law. Especially, in the singular subspace of \( F(\mathbb{R}^n) \), this law may not be valid. Let us give a striking example that proves this situation.

Example 9. Let us consider the set \( A = \{ x_1, x_2, \ldots, x_n \} \) in the proof of Theorem 2. For \( m \neq r \) and \( m, r \leq n \)

\[
\langle x_m, x_r \rangle = \int_0^1 \langle [x_m]^a, [x_r]^a \rangle \, d\alpha
\]

\[
\int_0^1 \langle [x_m]^a, [x_r]^a \rangle \, d\alpha = \int_0^1 \langle e_m, e_r \rangle \, d\alpha = 0.
\]

Furthermore, \( \langle e_m, e_r \rangle \) is an orthonormal basis for \( F(\mathbb{R}^n) \). For \( m \neq r \),

\[
\langle e_m, e_r \rangle = \int_0^1 \langle [x_m]^a, [x_r]^a \rangle \, d\alpha = \int_0^1 \langle [x_m]^a, [x_r]^a \rangle \, d\alpha = 1.
\]

There exists a copy of \( \mathbb{R}^n \) in \( F(\mathbb{R}^n) \), and this copy is just the regular subspace of \( F(\mathbb{R}^n) \). In order for a set to be a basis

\[
\|u, u\|_{K(\mathbb{R})} = \left\| \int_0^1 \langle [u]^a, [u]^a \rangle \, d\alpha \right\|_{K(\mathbb{R})} = \left\| \int_0^1 \langle a, b \rangle \, d\alpha : a, b \in [u]^a \right\|_{K(\mathbb{R})}
\]

\[
= \sup \left\{ \int_0^1 \langle a, b \rangle \, d\alpha : a, b \in [u]^a \right\}
\]

\[
= \sup \left\{ \int_0^1 \|a\|^2 \, d\alpha : a \in [u]^a \right\} = \int_0^1 \left( \sup_{\alpha \in [u]^a} \|a\| \right)^2 \, d\alpha = \|u\|^2.
\]

Example 10. Let \( u, v : \mathbb{R} \to [0, 1] \) be defined by

\[
u(x) = \begin{cases} 1 - x, & x \in [0, 1] \\ 0, & x \notin [0, 1] \end{cases}
\]

Clearly, \( u, v \in F(\mathbb{R}) \) and their \( \alpha \)-levels are \([u]^a = [v]^a = [0, 1 - \alpha] \). Furthermore,

\[
\|u\|^2 = \|v\|^2
\]

\[
= \int_0^1 \left( \sup_{\alpha \in [0,1-\alpha]} |a| \right)^2 \, d\alpha
\]

\[
= \int_0^1 (1 - \alpha)^2 \, d\alpha
\]

\[
= \frac{1}{3}.
\]

Now,

\[
[u + v]^a = [u]^a + [v]^a = [0, 2 - 2\alpha]
\]

and \([u - v]^a = [u]^a - [v]^a = [0, 1 - \alpha] - [0, 1 - \alpha] = [\alpha - 1, 1 - \alpha] \).

And so,
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\[ \|u + v\|^2 = \int_0^1 \left( \sup_{a \in [0, 2 - 2\alpha]} |a| \right)^2 da = \int_0^1 (2 - 2\alpha)^2 da = 4 \int_0^1 (1 - \alpha)^2 da = \frac{4}{3} \]

(56)

This implies

\[ \|u + v\|^2 + \|u - v\|^2 = \frac{5}{3} + \frac{4}{3} = 3. \]

Hence, the parallelogram law is not valid in \( F(\mathbb{R}^n) \).

Let us give another important result: the Schwarz inequality.

\[ \text{Theorem 15. For any two element } u, v \text{ in } F(\mathbb{R}^n), \]

\[ \|\langle u, v \rangle\| \leq \|u\| \|v\|. \]

(58)

The proof of this result can be derived from our similar result on Hilbert quasi linear spaces.

6. Classical Fuzzy Sequence Spaces

A sequence \( U = (u_k)^{\infty}_{k=1} \) of fuzzy numbers is a function \( U \) from positive integers into \( F(\mathbb{R}) \). The fuzzy number \( u_k \) is called the \( k \)-th term of the sequence. Let \( U = (u_k) \) be a sequence of fuzzy numbers. The sequence \( U \) of fuzzy numbers is said to be bounded if the set \( \{\|u_k\|\} \) is bounded in \( \mathbb{R} \) where \( \|u_k\| = \left( \int_0^1 (\sup_{a \in [0, 1]} |u(a)|)^2 da \right)^{1/2} \). Furthermore, \( U = (u_k)^{\infty}_{k=1} \) is said to be convergent to some \( u \in F(\mathbb{R}) \) whenever \( D_2(u_k, u) = \left( \int_0^1 \delta_{\infty} (|u(a)|, |u_k(a)|)^2 da \right)^{1/2} \rightarrow 0 \) as \( k \rightarrow \infty \) where \( D_2 \) is the inner-product metric which comes from the above inner-product norm \( \|\|. \) In this case, we write

\[ u_k \rightarrow u \text{ or } \lim u_k = u \text{ in } F(\mathbb{R}) \]

Note that \( \|u_k - u\| \rightarrow 0 \) implies \( u_k \rightarrow u \) but not conversely, in general. For example, the constant sequence \( u_k = \chi_{[0,1]} \), for each \( k \in \mathbb{N} \), is convergent to \( u = \chi_{[0,1]} \). Hence, the parallelogram law is not valid in \( F(\mathbb{R}^n) \).

More details about sequence spaces of fuzzy numbers are in [18–23] and references therein. Now, \( c_0^F \) and \( c^F \) denote the set of all bounded and convergent (to zero) sequences of fuzzy numbers, respectively. Hence,

\[ c_0^F = \left\{ U = (u_k)^{\infty}_{k=1} : u_k \in F(\mathbb{R}) \text{ and } \|u_k\| \rightarrow 0 \text{ as } k \rightarrow \infty \right\}, \]

\[ c^F = \left\{ U = (u_k)^{\infty}_{k=1} : u_k \in F(\mathbb{R}) \text{ and sup } \|u_k\| < \infty \right\}. \]

(60)

Furthermore, for \( 1 \leq p < \infty \),

\[ \ell^p_F = \left\{ U = (u_k)^{\infty}_{k=1} : u_k \in F(\mathbb{R}) \text{ and } \sum_{k=1}^{\infty} \|u_k\|^p < \infty \right\}, \]

is the set of all \( p \)-th order absolute summable sequences of fuzzy numbers. Furthermore, the set of all fuzzy numbers are denoted by \( w^F \) and so \( w^F = \{ U = (u_k)^{\infty}_{k=1} : u_k \in F(\mathbb{R}) \} \).

\[ \text{Theorem 16. By the coordinate wise operations and by the coordinate wise relation of fuzzy numbers, } w^F \text{ is a consolidate quasi linear spaces.} \]

\[ \text{Proof. We must verify the quasi linear axioms. For every } U, V, W, Z \in w^F \text{ and } \alpha, \beta \in \mathbb{R}, \]

\[ U + V = (u_k)^{\infty}_{k=1} + (v_k)^{\infty}_{k=1} \text{ and } u_k + v_k = [u_k]^a + [v_k]^a \text{ for every } k = 1, 2, \ldots, \]

\[ \alpha \cdot U = \alpha \cdot (u_k)^{\infty}_{k=1} \text{ and } \alpha \cdot u_k = \alpha \cdot [u_k]^a \text{ for every } k = 1, 2, \ldots, \]

(62)

\[ U \leq V \Rightarrow u_k \leq v_k \Rightarrow [u_k]^a \leq [v_k]^a \text{ for every } k = 1, 2, \ldots. \]

(1) \( U \leq U \) since \([u_k]^a \leq [u_k]^a \) for \( u_k \leq u_k \), \( k = 1, 2, \ldots, \)

(2) If \( U \leq V \) and \( V \leq W \), then we get \( u_k \leq v_k \) and \( v_k \leq w_k \) for \( k = 1, 2, \ldots \). From here, we have \([u_k]^a \leq [v_k]^a \) and \([v_k]^a \leq [w_k]^a \) for \( k = 1, 2, \ldots \). Since \([u_k]^a, [v_k]^a, [w_k]^a \in K_C(\mathbb{R}) \) and \( K_C(\mathbb{R}) \) is a quasi linear space, we obtain \([u_k]^a \leq [v_k]^a \). This gives us \( u_k \leq w_k \) for \( k = 1, 2, \ldots \). So, \( U \leq W \).

(3) If \( U \leq V \) and \( V \leq U \), then \( u_k \leq v_k \) and \( v_k \leq u_k \) for \( k = 1, 2, \ldots \). Also, we have \([u_k]^a \leq [v_k]^a \) and
Theorem 18. $\ell_p^F$ is a Banach quasilinear space with the norm $\|U\|_p = (\sum_{k=1}^{\infty} \|u_k\|_p^p)^{\frac{1}{p}}$ where $1 \leq p < \infty$ and $U = (u_k)_{k=1}^{\infty} \in \ell_p^F$.

The proof of former two theorems is not so hard. So, we rest the proof to the reader. We will try to show $\ell_2^F$ is a Hilbert quasilinear space. To accomplish this, we will define an inner product on $\ell_2^F$ and show that the norm arising from this inner product corresponds to the abovementioned norm for $p = 2$.

Theorem 19. $\ell_2^F$ is a Hilbert quasilinear space.

Proof. Let us write the equality

$$\langle U, V \rangle = \sum_{k=1}^{\infty} \langle u_k, v_k \rangle = \sum_{k=1}^{\infty} \left( \langle u_k \rangle^{\infty}, [v_k] \rangle_{K(R)} \cdot da, \right.$$  

for $U = (u_k)_{k=1}^{\infty}, V = (v_k)_{k=1}^{\infty} \in \ell_2^F$, where $u_k, v_k \in F(R)$. Obviously, is defined from $\ell_2^F$ to $K(R)$, and it is well-defined since.

$$\|\langle U, V \rangle\| \leq \sum_{k=1}^{\infty} \| u_k \| \cdot \| v_k \| \leq \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \| u_k \| \cdot \| v_k \| \leq \left( \sum_{k=1}^{\infty} \| u_k \|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \| v_k \|^2 \right)^{\frac{1}{2}} = \| U \|_2 \cdot \| V \|_2,$$

where we used Theorem 8. The verification of the inner product axioms are routine task since it is similar to Theorem 13.

7. Conclusions

In this paper, we improved the algebraic structure of the class of fuzzy sets $F(R^n)$, and we researched the some of algebraic properties such as quasilinear independence, dimension, and the algebraic basis in these spaces. Also, we proved that $F(R^n)$ is a complete normed space with special norm $\|x\|_q = (\int_0^1 (\sup_{x \in [a^n]} x)^q da)^{\frac{1}{q}}$ where $1 \leq q \leq \infty$. After that, we introduced an inner product on this space for the case $q = 2$. We proved that the parallelogram law can only be provided in the regular subspace, not in the entire of $F(R^n)$. Finally, we obtained a special class of fuzzy number sequences that is a Hilbert quasilinear space. We hope that our presented idea herein will be a source of motivation for other researchers to extend and improve these findings for their applications. In addition, we intend to develop a method to approximate the autocorrelation of nondeterministic signals containing some uncertainties with the help of the fuzzy inner product that we have defined in this article. Our plan for the next study is to define the concept of fuzzy signal.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, no. 3, pp. 338–353, 1965.

[2] A. K. Katsaras, “Fuzzy topological vector spaces II,” Fuzzy Sets and Systems, vol. 12, no. 2, pp. 143–154, 1984.
[3] C. Felbin, “Finite dimensional fuzzy normed linear space,” Fuzzy Sets and Systems, vol. 48, no. 2, pp. 239–248, 1992.

[4] O. Kaleva and S. Seikkala, “On fuzzy metric spaces,” Fuzzy Sets and Systems, vol. 12, no. 3, pp. 215–229, 1984.

[5] J.-Z. Xiao and X.-H. Zhu, “Fuzzy normed space of operators and its completeness,” Fuzzy Sets and Systems, vol. 133, no. 3, pp. 389–399, 2003.

[6] M. A. Rojas-Medar, M. D. Jiménez-Gamero, Y. Chalco-Cano, and A. J. Viera-Brandão, “Fuzzy quasilinear spaces and applications,” Fuzzy Sets and Systems, vol. 152, no. 2, pp. 173–190, 2005.

[7] S. M. Aseev, “Quasilinear operators and their application in the theory of multivalued mappings,” Proceedings of the Steklov Institute of Mathematics, vol. 167, no. 2, pp. 23–52, 1986.

[8] H. Bozkurt, Inner product quasilinear spaces and some generalizations, PhD Thesis, Inonu University, Malatya, Turkey, 2016.

[9] H. Bozkurt and Y. Yılmaz, “New inner product quasilinear spaces on interval numbers,” Journal of Function Spaces, vol. 2016, Article ID 2619271, 9 pages, 2016.

[10] S. Çakan and Y. Yılmaz, “Normed proper quasilinear spaces,” The Journal of Nonlinear Science and Applications, vol. 8, pp. 816–839, 2015.

[11] Y. Yılmaz, S. Çakan, and Ş. Aytekin, “Topological quasilinear spaces,” Abstract and Applied Analysis, vol. 2012, Article ID 951374, 10 pages, 2012.

[12] H. Levent and Y. Yılmaz, “An application: representations of some systems on non-deterministic EEG signals,” Journal of Biometrics and Its Applications, vol. 2, no. 1, pp. 101–113, 2018.

[13] H. Levent and Y. Yılmaz, “On some algebraic and topological properties of fuzzy normed quasilinear spaces,” Advances in Fuzzy Sets and Systems, vol. 22, no. 1, pp. 25–51, 2017.

[14] H. Levent and Y. Yılmaz, “Some new results on fuzzy quasilinear spaces,” Journal of Fuzzy Mathematics, vol. 27, no. 1, pp. 41–62, 2019.

[15] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets Theory and Applications, World Scientific, Singapore, 1994.

[16] Y. Yılmaz, H. Bozkurt, and S. Çakan, “On orthonormal sets in inner product quasilinear spaces,” Creative Mathematics and Informatics, vol. 25, no. 2, pp. 237–247, 2016.

[17] H. Levent and Y. Yılmaz, “Translation, modulation and dilation systems in set-valued signal processing,” Carpathian Mathematical Publications, vol. 10, no. 1, pp. 143–164, 2018.

[18] F. Nuray and E. Savas, “Statistical convergence of sequences of fuzzy real numbers,” Mathematica Slovaca, vol. 45, no. 3, pp. 269–273, 1995.

[19] E. Savas, “A note on sequence of fuzzy numbers,” Information Science, vol. 124, no. 1–4, pp. 297–300, 2000.

[20] H. Altınok, R. Colak, and M. Et, “λ-difference sequence spaces of fuzzy numbers,” Fuzzy Sets and Systems, vol. 160, pp. 3128–3139, 2009.

[21] R. Colak, Y. Altın, and M. Mursaleen, “On some sets of difference sequences of fuzzy real numbers,” Soft Computing, vol. 15, pp. 787–793, 2011.

[22] M. Matloka, “Sequences of fuzzy numbers,” BUSEFAL, vol. 28, pp. 28–37, 1986.

[23] S. Nanda, “On sequences of fuzzy numbers,” Fuzzy Sets and Systems, vol. 33, no. 1, pp. 123–126, 1989.