Phantom-like behaviour in a brane-world model with curvature effects

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Recent observational evidence seems to allow the possibility that our universe may currently be under a dark energy effect of a phantom nature. A suitable effective phantom fluid behaviour can emerge in brane cosmology; In particular, within the normal non self-accelerating DGP branch, without any exotic matter and due to curvature effects from induced gravity. The phantom-like behaviour is based in defining an effective energy density that grows as the brane expands. This effective description breaks down at some point in the past when the effective energy density becomes negative and the effective equation of state parameter blows up. In this paper we investigate if the phantom-like regime can be enlarged by the inclusion of a Gauss-Bonnet (GB) term into the bulk. The motivation is that such a GB component would model additional curvature effects on the brane setting. More precisely, our aim is to determine if the GB term, dominating and modifying the early behaviour of the brane universe, may eventually extend the regime of validity of the phantom mimicry on the brane. However, we show that the opposite occurs: the GB effect seems instead to induce a breakdown of the phantom-like behaviour at an even smaller redshift.

I. INTRODUCTION

In the context of cosmology, one of the most relevant astronomical observations of the last decade are those from distant type Ia supernova implying that the universe is in a state of accelerated expansion¹,² which has been latter on confirmed by other observational probes. The fundamental nature of what is driving the cosmic acceleration is unknown, although many theoretical propositions have been put forward.³,⁴ Invoking a cosmological constant to explain the late-time acceleration of the universe turns out to be the most economical option which moreover is in agreement with all the observational data. However, it turns out that the expected theoretical value of the cosmological constant is about 120 orders of magnitude larger than the measured one.²

Whatever it is the fuel inducing the late-time acceleration of our universe, from a phenomenological point of view and in the framework of general relativity, it can be described through a dark energy component \((\rho_d, p_d)\) with an effective equation of state \(w_{\text{eff}} = p_d/\rho_d\). The current value of \(w_{\text{eff}}\) is extremely close to \(-1\)³,⁴; i.e. a cosmological constant equation of state, but it can be larger than \(-1\)-like in quintessence models or even smaller.³

The latter case disclosed above is of no lesser importance. Quite on the contrary: If dark energy has a phantom nature, i.e., \(w_{\text{eff}} < -1\), then the scientific community is facing a most considerable twofold challenge: (i) explaining the cause of the recent speed up of our universe and (ii) also how to accommodate a phantom energy component in our theoretical framework; i.e. what can cause \(w_{\text{eff}} < -1\) without invoking a real phantom energy which is known to violate the null energy condition and induce quantum instabilities.

A possibility to mimic such a phantom-like behaviour is within the paradigm of string inspired brane-world models, where matter (standard model particles) is confined on a 4-dimensional (4D) hypersurface embedded in a higher dimensional space-time (the bulk) and where gravity is the only interaction experiencing the full bulk. More precisely, Sahni and Shtanov proposed the ADGP model with a phantom-like behaviour on the brane, without the need of any matter that violates the null energy condition on the brane.

The ADGP scenario is a 5D brane-world model with infrared modifications to general relativity caused by an induced gravity term on the brane. The model is based on the normal or non self-accelerating branch of the Dvali-Gabadadze-Porrati proposal (DGP)¹⁷,¹⁸, which, unlike the self-accelerating DGP branch is free from the ghost problem, being the brane filled with cold dark matter (CDM) and a cosmological constant which drives the late-time acceleration of the brane. The phantom-like behaviour is a consequence of the extra-dimension which screens the brane cosmological constant and it is based in mapping the brane evolution to that of an equivalent 4D general relativistic phantom energy model.¹³,¹⁶ More precisely, the basis of this mimicry is an effective energy density (in the 4D general relativistic picture) corresponding

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¹ We are referring here to a phantom energy component described through a minimally coupled scalar field with the wrong kinetic term.
to the cosmological constant corrected by the curvature effect due to the induced gravity term on the brane. This effective energy density grows as the brane expands and therefore effectively it behaves as a phantom fluid; i.e. \( w_{\text{eff}} < -1 \), where \( w_{\text{eff}} \) corresponds to the ratio between the effective energy density and the effective pressure. The ADGP model in \(^3\) is by far the simplest way to mimic a phantom-like behaviour in a brane-world setup\(^2\).

Other brane proposals aiming to produce such a mimicry are based on a bulk filled with matter and/or on an energy exchange between the brane and the bulk, therefore modifying the effective equation of state of dark energy on the brane \(^27\).

The effective description of the phantom behaviour in the ADGP model breaks down at a finite redshift \(^3\) (cf. figure 6 and section IV for a more detailed description): i.e. the effective energy density vanishes and becomes negative over a certain redshift. When the effective energy density vanishes, the effective equation of state blows up. Given that the phantom-like behaviour results from (i) induced gravity effects on the brane causing curvature corrections and (ii) describing the brane model as a 4D relativistic phantom energy setup, could the breakdown of the phantom-like behaviour be eliminated by considering further curvature effects on the brane-world scenario? This is the main question we address in this paper.

We will model such additional and new curvature effects through a Gauss-Bonnet term (GB) in the bulk \(^30\) \(^31\). The reason behind including this specific curvature terms is that it induces an ultraviolet correction on the brane \(^30\) \(^31\), as expected from high-energy stringy features, and therefore may modify the phantom-like behaviour at earlier times. Eventually, affecting its long term dynamics and even possibly preventing the mentioned break down to occur at all. The other reason for considering such a curvature term was anticipated recently \(^28\) \(^29\): even though the DGP model is characterised by an interesting infrared effect of gravity occurring with respect to general relativity, which for the self-accelerating branch can lead to a late-time acceleration on the brane even in the absence of any exotic matter invoked to produce the dark energy effect \(^18\), it would be expected that a consistent DGP brane model would have also ultraviolet modifications as well, associated to high-energy stringy effects at earlier times.

This paper is therefore outlined as follows. In section II we define a brane-world model, henceforth designated as ADGP-GB model, and constrain the set of parameters that defined in it such away that the brane is currently accelerating. We also comment on the non super-acceleration of the brane. In section III we solve the cubic Friedmann equation for the normal DGP branch with a GB term in the bulk: The reason is to obtain an accurate description of the effective energy density, that will behave like a phantom component on the brane, which depends explicitly on the Hubble rate. In section IV we subsequently show how a mimicry of a phantom behaviour takes place on the brane without considering any matter that violates the null energy condition on the brane. Then, we compare the behaviour of the ADGP-GB setting with the behaviour found on the ADGP model \(^15\) \(^16\). Finally, in section V we summarise and conclude. We also present some results related to the solutions of the cubic Friedmann equation \(^17\) in the appendix A. On the other hand, in the appendix B, we show under which conditions the ADGP model is recovered from the model we propose.

II. ACCELERATING ADGP-GB MODEL AND PARAMETER CONSTRAINTS

The generalised Friedmann equation of a brane with induced gravity embedded in a 5D Minkowski bulk with a GB term reads\(^4\) \(^28\) \(^29\)

\[
\left(1 + \frac{8}{3} \alpha H^2\right)^2 H^2 = \left(r_c H^2 - \frac{\kappa_5^2}{6} \rho\right)^2,
\]

where a mirror symmetry has been assumed across the brane. In the previous equation \(r_c\) is the crossover scale in the DGP model \(^17\) and has length unit. This parameter measures the strength of the induced gravity effect on the brane and is related to the 4D and 5D gravitational constants by

\[
r_c = \frac{\kappa_5^2}{2\kappa_4^4}.
\]

On the other hand, the parameter \(\alpha\) measures the strength of the GB curvature effect on the brane and has length square unit and is positive \(^30\).

If \(\alpha = 0\), then the induced gravity in the DGP setup modifies the late-time evolution of the brane with respect to the standard 4D relativistic case \(^18\) (for an alternative approach where the induced gravity effect corresponds to a correction to RS model at high energies see, e.g., \(^20\)). However, if it is instead \(r_c = 0\) then the GB term modifies the early-time evolution of the brane \(^30\).

Equation \(^1\) can be conveniently rewritten as

\[
H^2 = \frac{\kappa_4^2}{3} \rho \pm \frac{1}{r_c} \left(1 + \frac{8}{3} \alpha H^2\right) H,
\]

\(^4\) We restrict to the Friedmann equation that has an induced gravity limit and therefore contains the DGP model \(^23\).
which generalises the Friedmann equation of the self-accelerating DGP solution \[17, 18\] (+ sign in Eq. \[3\] with \(\alpha = 0\); i.e. we recover the self-accelerating solution when \(\alpha = 0\). On the other hand, Eq. \[3\] also has as a particular solution the DGP normal branch or non-self-accelerating solution \[17, 18\] (− sign in Eq. \[3\] when \(\alpha = 0\)).

From now on, we restrict to the normal branch; i.e. − sign in Eq. \[3\]. In addition, we consider that the energy density of the brane \(\rho\) corresponds to a CDM component with energy density \(\rho_m\) and a cosmological constant \(\Lambda\)

\[\rho = \rho_m + \Lambda, \quad (4)\]

with the latter driving the late-time acceleration of the brane. We will refer to this scenario as the ADGP-GB model. The total energy density of the brane is conserved and therefore also the sector corresponding to the CDM, which scales in the standard way with the redshift

\[\rho_m = \rho_{m0}(1 + z)^3. \quad (5)\]

From now on a subscript 0 stands for the observed current value of a given quantity. Finally, the Friedmann equation on the brane can be presented as

\[E^2(z) = \Omega_m(1 + z)^3 + \Omega_\Lambda - 2\sqrt{\Omega_{\text{DE}}} [1 + \alpha_0 E^2(z)] E(z), \quad (6)\]

where \(E(z) = H/H_0\) and

\[\Omega_m = \frac{k^2 \rho_{m0}}{3H_0^2}, \quad \Omega_\Lambda = \frac{k^2 \Lambda}{3H_0^2}, \quad \Omega_{\text{DE}} = 1 \frac{1}{4r^2 H_0^2}, \quad (7)\]

are the usual convenient dimensionless parameters while the new parameter \(\alpha_0\) is defined as

\[\Omega_{\alpha_0} = \frac{8}{3} \alpha H_0^2. \quad (8)\]

Evaluating the Friedmann equation \[6\] at \(z = 0\) gives a constraint on the cosmological parameters of the model

\[\Omega_m + \Omega_\Lambda = 1 + 2\sqrt{\Omega_{\text{DE}}(1 + \alpha_0)}. \quad (9)\]

For \(\Omega_{\alpha_0} = 0\) we recover the constraint in the ADGP model. The constraint \[9\] implies that the region \(\Omega_m + \Omega_\Lambda < 1\) is unphysical. Moreover, although the brane is spatially flat, the previous constraint can be interpreted in the sense that our model constitutes a mimic of a closed FLRW universe in the \((\Omega_m, \Omega_\Lambda)\) plane. In particular, this is likewise to what happens in ADGP, QDGP and CDGP models \[13, 16, 21, 22, 24\]. We recall that the QDGP and CDGP models correspond to variants of the ADGP scenario, where the late-time evolution of the universe is driven by a quiessence \[21\] and a Chaplygin gas \[24\], respectively, instead of a cosmological constant. Their dark energy effect is more dynamical and the phantom divide (or the \(w = -1\) line) crossing is possible in the QDGP and CDGP unlike in the ADGP model \[21, 24\]. We remind that the interest on modelling

a mimicry of a phantom divide crossing is based on the possibility (backed by recent observational data) that the equation of state may have crossed the cosmological constant barrier (\(w = -1\)).

Coming back to our model, if the dimensionless crossover energy density \(\Omega_{\text{DE}}\) is the same in a ADGP model and in our model (which is not necessarily the case), then the similarities with a spatially closed universe are made more significant from the GB effect, since \(\Omega_m > 0\). Notice that this statement also applies to the variants of the ADGP brane mentioned previously and their generalisations by the GB effect, if the acceleration of both branes is driven by the same sort of dark energy. In fact, in this case the appropriately modified constraint \[9\] would read

\[\Omega_m + \Omega_{\text{DE}} = 1 + 2\sqrt{\Omega_{\text{DE}}(1 + \alpha_0)}. \quad (10)\]

where \(\Omega_{\text{DE}}\) correspond to the current dimensionless energy density of dark energy on the brane which can be for example modelled by a quiessence or a Chaplygin gas.

Furthermore, by imposing that the universe is currently accelerating; i.e. the deceleration parameter \(q = -(\dot{H}/H^2 + 1)\) is currently negative, where

\[q_0 = - \left[ 1 - \frac{3\Omega_m}{2 + 2\sqrt{\Omega_{\text{DE}}(1 + \alpha_0)}} \right], \quad (11)\]

we obtain another constraint on the set of cosmological parameters \(\Omega_m, \Omega_{\alpha_0}\) and \(\Omega_\Lambda\), which reads

\[3\Omega_m < 2 + 2(1 + \alpha_0)\sqrt{\Omega_{\text{DE}}}. \quad (12)\]

An example of the cosmological evolution of the deceleration parameter is given in Fig. \[1\] where it can be seen that the brane accelerates at late-time.

On the other hand, the modified Raychaudhuri equation follows easily from the Friedmann equation of the brane and the conservation of the brane energy density. It can be written as

\[\frac{\dot{H}}{H_0^2} = -\frac{3}{2} \frac{\Omega_m(1 + z)^3 E(z)}{E(z) + \sqrt{\Omega_{\text{DE}}(1 + 3\Omega_{\alpha_0} E^2(z))}}. \quad (13)\]

where a dot stands for the derivative respect to the cosmic time. The key point of the previous equation is that the brane never super-accelerates; i.e. the Hubble rate decreases as the brane expands. Nevertheless, as we will show in section \[14\] a phantom-like behaviour takes place at the brane: This occurs without including any material that violates the null energy condition. The phantom-like behaviour is based in defining an effective energy density which corresponds to a balance between the cosmological constant and geometrical effects encoded on the Hubble rate evolution. Therefore, in order to get the evolution of the effective energy density with the redshift it is necessary to solve the cubic Friedmann equation of the normal branch (Eq. \[3\] with (−) sign). The solutions and an analysis of the mentioned Friedmann equation is presented on the next section.
we show the set of those parameters (14) which make the study much easier because the discriminant of the cubic equation, $\mathcal{N}$ defined in Eq. (18), is much simpler.

Then the Friedmann equation can be rewritten as

\[
\dot{H}^3 + \dot{H}^2 + b\dot{H} - \bar{\rho} = 0. \tag{17}
\]

The number of real roots is determined by the sign of the discriminant function $\mathcal{N}$ defined as

\[
\mathcal{N} = Q^3 + R^2, \tag{18}
\]

where $Q$ and $R$ read

\[
Q = \frac{1}{3} \left( b - \frac{1}{3} \right), \quad R = \frac{1}{6} b + \frac{1}{2} \bar{\rho} - \frac{1}{27}. \tag{19}
\]

It is helpful to rewrite $\mathcal{N}$ as

\[
\mathcal{N} = \frac{1}{4} (\bar{\rho} - \bar{\rho}_1) (\bar{\rho} - \bar{\rho}_2), \tag{20}
\]

where

\[
\bar{\rho}_1 = -\frac{1}{3} \left( b - \frac{2}{9} \left[ 1 + \sqrt{1 - 3b} \right]^2 \right), \tag{21}
\]

\[
\bar{\rho}_2 = -\frac{1}{3} \left( b - \frac{2}{9} \left[ 1 - \sqrt{1 - 3b} \right]^2 \right), \tag{22}
\]

for the analysis of the number of physical solutions of the modified Friedmann equation (17). If $\mathcal{N}$ is positive then there is a unique real solution. On the other hand, if $\mathcal{N}$ is negative there are 3 real solutions. Finally, if $\mathcal{N}$ vanishes, all roots are real and at least two are equal.

A. Case 1: $0 < b < \frac{1}{4}$

This is by far the most interesting physical case as we expect $b$ to be small because it is proportional to $\Omega_r$ [see Eq. (18)] and the equivalent quantity in the $\Lambda$DGP scenario is relatively small [22, 23]. We do not expect that $\Omega_r$ in our model to be very different from that in the $\Lambda$DGP model. Furthermore, from the mimicry of our model regarding a closed FLRW universe [see Eq. (19)] and the constraint on the curvature of the universe; for example from the recent WMAP 5 years data in combination with the baryon acoustic oscillations [8], $\Omega_r$ and $\Omega_\alpha$ should be small and therefore $b$ is also expected to be small.

The analysis of this case is slightly involved. The reason is essentially that if $0 < b < \frac{1}{4}$ then $\bar{\rho}_1$ and $\bar{\rho}_2$ are real [see Eqs (21) and (22)] and therefore it is not as straightforward as in the next cases to know the number of real solutions of the cubic Friedmann equation on $\dot{H}$; i.e. to know the sign of $\mathcal{N}$ [see Eq. (20)].

More precisely, in this case $\bar{\rho}_2 < 0$ and $0 < \bar{\rho}_1$ [see Eqs (21)-(22)]. Then, the number of real roots of the cubic Friedmann equation (17) depends crucially on the minimum energy density of the brane:

\[
\bar{\rho}_{\min} = 4\Omega_r \Omega_c^2 \Omega_\Lambda; \tag{23}
\]

i.e. the asymptotic value of the total energy density at $z = -1$. We enumerate next the possible different situations [see Fig. 3]:

1. $\bar{\rho}_1 < \bar{\rho}_{\min}$

The minimum energy density of the brane is such that $\bar{\rho}_1 < \bar{\rho}_{\min}$. Then the function $\mathcal{N}$ is positive and there is a unique solution. The condition $\bar{\rho}_1 < \bar{\rho}_{\min}$ implies

\[
- \frac{1}{3b} \left( b - \frac{2}{9} \left[ 1 + \sqrt{1 - 3b} \right]^2 \right) < \Omega_r \Omega_\Lambda, \tag{24}
\]

and therefore constrains the set of allowed values of $\Omega_r$, $\Omega_\alpha$ and $\Omega_\Lambda$. In Fig. 3, we show the set of those parameters that fulfill the inequality (24) as the uncoloured area while

FIG. 1: Plot of the deceleration parameter, $q = -\dot{H}/H^2 + 1$, versus the redshift. The set $(\Omega_m, \Omega_\Lambda, \Omega_r, \Omega_\alpha) = (0.26, 0.7602, 10^{-4}, 0.01)$. As can be seen the brane accelerates at late-time when $q$ gets negative.
the red coloured area corresponds to the set \((\Omega_c, \Omega_e, \Omega_n, \Omega_\Lambda)\) that does not fulfil the condition \((21)\).

Finally, the expansion of the brane is described by Eq. \((A1)\) or

\[
\bar{H}_1 = \frac{1}{3} \left[ 2\sqrt{1-3\bar{b}} \cosh \left( \frac{\eta}{3} \right) - 1 \right],
\]

where \(\eta\) is defined as

\[
\cosh(\eta) = \frac{R}{\sqrt{-Q^3}}, \quad \sinh(\eta) = \sqrt{\frac{Q^3 + R^2}{-Q^3}},
\]

and \(\eta_{\text{min}} \leq \eta\). The parameter \(\eta_{\text{min}}\) is defined as in Eq. \((20)\) with \(\bar{\rho} = \rho_{\text{min}}\) and this value of \(\eta\) is reached at \(z = -1\). It turns out that the expanding brane solution is asymptotically de Sitter in the future. On the other hand, at early time (large \(\eta\)) matter on the brane is dominated by dust though its cosmological evolution does not correspond to the standard relativistic dust case because at high redshift \(\bar{H} \sim \rho_{\text{min}}^\dagger\), where \(\rho_{\text{min}}\) is defined as in Eq. \((15)\). This is a consequence of the dominance of GB effects at high energy. This feature applies also to the high energy regime described in the next subsection.

\section{Limiting regime: \(\bar{\rho} = \rho_{\text{min}}\)}

The minimum energy density of the brane is such that \(\rho_{\text{min}} \leq \bar{\rho}_1\). Consequently, the inequality \((21)\) is not satisfied and this again restricts the set \((\Omega_c, \Omega_e, \Omega_n, \Omega_\Lambda)\) which in this case corresponds to the coloured area in Fig. 2. As this figure highlights this is the most likely situation as we expect \(\Omega_c\) and \(\Omega_n\) to be small for the reasons stated before.

As the energy density blue-shifts backward in times; i.e. the energy density grows backward in times we can distinguish three regimes:

\begin{itemize}
  \item High energy regime: \(\bar{\rho}_1 < \bar{\rho}\).
  \item Low energy regime: \(\bar{\rho}_{\text{min}} \leq \bar{\rho} < \bar{\rho}_1\).
  \item Limiting regime: \(\bar{\rho} = \rho_{\text{min}}\).
\end{itemize}

During the high energy regime, the energy density of the brane, \(\bar{\rho}\), is bounded from below by \(\bar{\rho}_1\) and therefore the function \(N\) is positive or equivalently there is a unique solution of the cubic Friedmann equation \((17)\). During this regime, the expansion of the brane is described by Eq. \((26)\) where \(0 < \eta\) and defined in Eq. \((20)\). When \(\eta \to 0\), the energy density of the brane approaches \(\bar{\rho}_1\).

During the limiting regime, \(\bar{\rho} = \rho_{\text{min}}\). Consequently \(N\) vanishes and there are two solutions:

\[
\bar{H}_1 = \frac{1}{3} \left( 2\sqrt{1 - 3\bar{b}} - 1 \right),
\]

\[
\bar{H}_2 = -\frac{1}{3} \left( \sqrt{1 - 3\bar{b}} + 1 \right).
\]

The high energy regime connects with the limiting regime through \(\bar{H}_1\). The negative solution \(\bar{H}_2\) is not relevant physically.

Finally, at the low energy regime the total energy density of the brane is bounded from above by \(\bar{\rho}_1\). Then \(N\) is negative and there are 3 different solutions [see Fig. 3]. One of this solutions corresponds to an expanding brane while the other two corresponds to contracting branes:

It can be shown that the expanding solution \((\bar{H} > 0)\) is described by Eq. \((A1)\) and more appropriately rewritten as

\[
\bar{H}_1 = \frac{1}{3} \left[ 2\sqrt{1 - 3\bar{b}} \cos \left( \frac{\theta}{3} \right) - 1 \right], \quad 0 < \theta \leq \theta_{\text{max}}
\]

where

\[
\cos(\theta) = \frac{R}{\sqrt{-Q^3}}, \quad \sin(\theta) = \sqrt{1 + \frac{R^2}{Q^3}}.
\]

For \(\theta \to 0\), the energy density \(\bar{\rho}\) approaches \(\bar{\rho}_1\); i.e. the low energy regime is connected with the high energy regime through the solution \((27)\). On the other hand, \(\theta_{\text{max}}\) is defined as in Eq. \((30)\) with \(\bar{\rho} = \rho_{\text{min}}\) where the brane reaches its asymptotic de Sitter regime at \(z = -1\). Notice that as matter redshifts on the brane, the angle \(\theta\) gets larger. On the other hand, in this model, if the cosmological constant vanishes then the maximum angle \(\theta\) is given by \(\theta_0\) where

\[
\cos \left( \frac{\theta_0}{3} \right) = \frac{1}{2\sqrt{1 - 3\bar{b}}}
\]

and therefore, the Hubble rate vanishes. This feature signals that this brane solution does not corresponds to a self-accelerating brane.

For completeness, we write down the remaining two solutions of the Friedmann equation \((17)\) when \(0 < \bar{b} < 1/4\) and \(\bar{\rho} < \bar{\rho}_1\). As it was anticipated before, these
solutions describe contracting branes and correspond to the solutions given in Eqs. (32) and (33). They read
\[
\tilde{H}_2 = -\frac{1}{3} \left[ 2\sqrt{1 - 3\cos\left(\frac{\pi - \theta}{3}\right)} + 1 \right], \quad 0 < \theta \leq \theta_{\text{max}},
\]
\[
\tilde{H}_3 = -\frac{1}{3} \left[ 2\sqrt{1 - 3\cos\left(\frac{\pi + \theta}{3}\right)} + 1 \right], \quad 0 < \theta \leq \theta_{\text{max}},
\]
respectively. Unlike the solution \( \tilde{H}_1 \), these two solutions are contracting because \( \tilde{H}_2 \) and \( \tilde{H}_3 \) are negative. It can be shown that \( \tilde{H}_2 \leq \tilde{H}_3 \). Both solutions approaches the same Hubble rate in the past at \( \theta = 0 \) corresponding to the limiting solution (28). Finally, if there is no cosmological constant on the brane, then in the far future (at \( z = -1 \)) the angle \( \theta \) is given by Eq. (31), where \( \tilde{H}_2 \) and \( \tilde{H}_3 \) approach constant negative values.

Before ending we would like to point out that it is only the expanding branch with Hubble rate \( \tilde{H}_1 \) that has a phantom-like behaviour which we will describe in the next section. An example of the three different solutions of the Friedmann equation (17) for \( 0 < b < \frac{1}{3} \) and \( \bar{\rho}_{\text{min}} \leq \bar{\rho}_1 \) can be seen in Fig. 3.

**FIG. 3:** Plot of the dimensionless Hubble rates \( \tilde{H}_1, \tilde{H}_2 \) and \( \tilde{H}_3 \) against the dimensionless energy density \( \bar{\rho} \). The red curve corresponds to \( \tilde{H}_1 \). The blue curve corresponds to \( \tilde{H}_2 \) and the other one to \( \tilde{H}_3 \). As can be seen it is only the solution corresponding to \( \tilde{H}_1 \) that can exist in the “far” past and it is also the only expanding solution. If the minimum energy density of the brane \( \bar{\rho}_{\text{min}} \) is larger than \( \bar{\rho}_1 \) only the solution \( \tilde{H}_1 \) exits. In the opposite case; i.e. \( \bar{\rho}_{\text{min}} \leq \bar{\rho}_1 \), the three solutions exist.

**B. Case 2: \( \frac{1}{3} \leq b < \frac{1}{4} \)**

If\(^6\) \( \frac{1}{3} \leq b < \frac{1}{4} \) then \( \bar{\rho}_1 \leq 0 \) and \( \bar{\rho}_2 < 0 \) which implies that \( \mathcal{N} \) is positive because the total energy density of the brane \( \bar{\rho} \) is positive. Consequently, there is a unique real solution and the cosmological evolution of the brane is unique. The dimensionless Hubble parameter is given in Eqs. (25) and (26) and \( \eta_{\text{min}} \) is defined as in Eq. (26) with \( \bar{\rho} = \bar{\rho}_{\text{min}} \). When \( \eta \) approaches its minimum value, the Hubble rate is constant and positive; i.e. the brane is asymptotically de Sitter and its expansion is dominated by the cosmological constant. If all matter on the brane redshifts and the total energy density on the brane vanishes at \( z = -1 \) then at \( \eta = \eta_0 \), where
\[
\cosh\left(\frac{\eta_0}{3}\right) = \frac{1}{2\sqrt{1 - 3b}}
\]
the Hubble rate vanishes. This feature is in agreement with the fact that this solution does not correspond to a self-accelerating branch.

At high energy (large \( \eta \)) matter on the brane is dominated by dust. However, its cosmological evolution does not correspond to the standard relativistic dust case because at high redshift \( \tilde{H} \sim \rho_{\text{min}}^0 \) due to the GB effects at high energy.

**C. Case 3: \( b = \frac{1}{3} \)**

This constitutes a marginal case where \( b = 1/3 \); i.e. \( \Omega_r = 1/(12\Omega_c) \). The modified Friedmann Eq. (17) has a unique real solution because \( \mathcal{N} > 0 \). This can be noticed easily by realising that \( \bar{\rho}_1 = \bar{\rho}_2 < 0 \) when \( b = 1/3 \) and therefore the right hand side of Eq. (20) is always positive. The dimensionless Hubble parameter given in Eq. (A1) can be rewritten in a simple way as
\[
\tilde{H} = \frac{1}{3} \left[ (1 + 27\bar{\rho})^{\frac{1}{3}} - 1 \right].
\]
(35)
The brane is therefore asymptotically de Sitter in the future (\( \bar{\rho} \to \bar{\rho}_{\text{min}} \)). At high energy/earlier time, the matter on the brane is dominated by dust although the dimensionless Hubble parameter redshift as \( \rho_{\text{m}}^0 \).

**D. Case 4: \( \frac{1}{4} < b \)**

In this case, Eq. (17) has a unique real solution because \( \mathcal{N} > 0 \) as \( \bar{\rho}_1 \) and \( \bar{\rho}_2 \) are complex conjugates when \( \frac{1}{4} < b \) and therefore the right hand side of Eq. (20) is always positive. The dimensionless Hubble parameter is given in Eq. (A1) and can be rewritten as
\[
\tilde{H} = \frac{1}{3} \left[ 2\sqrt{3b - 1} \sinh\left(\frac{\eta}{3}\right) - 1 \right], \quad \eta_{\text{min}} \leq \eta,
\]
(36)
where now \( \eta \) fulfils
\[
\cosh(\eta) = \sqrt{1 + \frac{R^2}{Q^3}}, \quad \sinh(\eta) = \frac{R}{\sqrt{Q^3}},
\]
(37)
and the parameter \( \eta_{\text{min}} \) is defined as in Eq. (37) with \( \bar{\rho} = \bar{\rho}_{\text{min}} \). The brane is therefore asymptotically de Sitter.
(\(z = -1\) corresponds to \(\eta\) approaching \(\eta_{\text{min}}\)) though there is no self-accelerating solution.

For the sake of completeness, we point out that if all matter on the brane redshifts and the total energy density on the brane vanishes at \(z = -1\) then at this redshift \(\eta = \eta_0\), where the Hubble rate vanishes, in agreement with the fact that this solution does not correspond to a self-accelerating branch, and where \(\eta_0\) satisfies

\[
\sinh \left( \frac{\eta_0}{3} \right) = \frac{1}{2\sqrt{3b-1}}.
\] (38)

Finally, we have that at high energy (large \(\eta\)) the brane is dust dominated although its cosmological evolution does not correspond to the standard relativistic dust case because \(H \sim \rho_m^{\frac{1}{2}}\) where \(\rho_m\) is defined as in Eq. (15).

The asymptotic de Sitter regime of the brane in all the cases numerated is due to the presence of a cosmological constant on the 4D hypersurface (unlike the self-accelerating solutions 28]). In addition, the fact that the brane is asymptotically de Sitter implies that there is no big rip singularity in the future despite that the brane has a phantom-like behaviour at late-time as we show in the next section.

Before proceeding into discussing how a phantom-like behaviour takes place on the brane framework we use, there is a point worthy to emphasise. The Gauss-Bonnet parameter can, in principle, have an arbitrary value. However, being also considered as a perturbative term arising from string theory, it is sensible that within the model discussed here, realistic cosmological solutions should coincide with the \(\Lambda\)DGP cosmology in the limit \(\alpha \to 0\). We therefore present in the appendix B how this is the case for one of the herein found solutions (Eq. (29)) and under which conditions the \(\Lambda\)DGP model is recovered from the model we propose. This procedure follows the analysis in Refs. 28, 31. Moreover, it will be the solution (29) that will be employed in the next section.

**IV. PHANTOM-LIKE BEHAVIOUR ON THE BRANE AND GAUSS-BONNET EFFECT**

The phantom-like behaviour on the brane is based in defining a corresponding effective energy density \(\rho_{\text{eff}}\) and an effective equation of state with parameter \(w_{\text{eff}}\). More precisely, the effective description is inspired in writing down the modified Friedmann equation of the brane as the usual relativistic Friedmann equation so that

\[
H^2 = \frac{\kappa_4^2}{3} (\rho_m + \rho_{\text{eff}});
\] (39)

i.e., to map the brane evolution in Eq. (6) to the equivalent 4D general relativistic phantom cosmology with Friedmann equation 39]. In the previous equation the effective energy density \(\rho_{\text{eff}}\) reads

\[
\rho_{\text{eff}} = \Lambda - \frac{3}{\kappa_4^2 r_c} \left( 1 + \frac{8}{3} \alpha H^2 \right) H,
\]

\[
= \frac{3H_0^2}{\kappa_4^2} \left[ \Omega_{\Lambda} - 2 \sqrt{\Omega_{r_c}(1 + \Omega_{\Lambda}E^2(z))E(z)} \right].
\] (40)

This effective energy density corresponds to a balance between the cosmological constant and geometrical effects encoded on the Hubble parameter. On the other hand, gravity leakage at late-time screens the cosmological constant like in the \(\Lambda\)DGP scenario 13, 16. This phantom-like behaviour is obtained without any matter violating the null energy condition and without any super-acceleration on the brane. We stress that the dependence of \(\rho_{\text{eff}}\) on the redshift is known analytically by means of the different solutions \(H\) of the cubic Friedmann equation presented on the previous section.

In Fig. 4 we show an example of the evolution of the dimensionless effective energy density \(\kappa_4^2 \rho_{\text{eff}}/(3H_0^2)\). In this example \(b = 4 \times 10^{-6}\). Our choice for the values of \(b\) is based on the fact that observationally the most favourable set of solutions \(H_1\) are those such that \(0 < b < \frac{1}{2}\) and \(\rho_{\text{min}} < \rho_1\) (we refer the reader to the previous section). Furthermore, the chosen value \(b\) is obtained for \(\Omega_{r_c} = 10^{-4}\) which is in agreement with the best fit of the \(\Lambda\)DGP model 28] and suitable for our model as we expect GB effects in our model to correspond to small corrections to the \(\Lambda\)DGP setup.

As in the \(\Lambda\)DGP model, it is possible to define an effective equation of state or parameter \(w_{\text{eff}}\) associated to the effective energy density as

\[
\rho_{\text{eff}} + 3H(1 + w_{\text{eff}}) \rho_{\text{eff}} = 0.
\] (41)

This effective equation of state is defined in analogy with the standard relativistic case. Then using Eq. (41), we
FIG. 5: Plot of the effective equation of state versus the redshift. The set \((\Omega_m, \Omega_\Lambda, \Omega_r, \Omega_b) = (0.26, 0.7602, 10^{-4}, 0.01)\) the same considered in Fig. 4.

obtain

\[
1 + w_{\text{eff}} = \frac{1}{\kappa^2 r_c} \frac{\dot{H}(1 + 8\alpha H^2)}{H_{\text{eff}}},
\]

\[
= \frac{2}{3} \frac{\dot{H} / H^2}{\Omega_\Lambda + 2 \sqrt{\Omega_r (1 + \Omega_a E^2(z))} E(z)}.
\]

Because the brane never super-accelerates, i.e. \(\dot{H} < 0\) [see Eq. (43)], we can then conclude that \(\rho_{\text{eff}}\) mimics the behaviour of a phantom energy component on the brane: i.e. \(1 + w_{\text{eff}} < 0\), as long as the effective energy density \(\rho_{\text{eff}}\) is positive [see Figs. 4 and 5].

We would like to point out that a phantom energy component can be defined in two ways: (i) any matter such that its equation of state fulfills \(p/\rho = w < -1\) or (ii) any matter whose energy density grows when the universe expands. Both definitions are equivalent as long as the universe expands and the energy density of the phantom component is positive. Here, we are assuming that the phantom energy is defined as in (i) and it turns out that the definition (ii) is automatically satisfied. In this model, the opposite situation does not hold, in other words if condition (ii) is satisfied (which is always the case in this model) it does not imply that condition (i) is fulfilled. The reason behind it is that \(\rho_{\text{eff}}\) is always positive,

\[
\rho_{\text{eff}} = -\frac{3}{\kappa^2 r_c} \dot{H}(1 + 8\alpha H^2),
\]

and therefore \(\rho_{\text{eff}}\) always grows as the brane expands independently of its sign. However, \(1 + w_{\text{eff}}\) changes its sign, although in an abrupt way, when \(\rho_{\text{eff}}\) vanishes and becomes negative [see Eq. (42) and Fig. 6].

When the effective energy \(\rho_{\text{eff}}\) vanishes, the mimicry of the phantom behaviour breaks down; i.e. the mapping between the ADGP-GB model and the 4D relativistic phantom cosmology model with Friedmann Eq. (39) is no longer valid, although the ADGP-GB model is well defined at any redshift. At the redshift \(z_b\), where \(\rho_{\text{eff}}(z_b) = 0\), the Hubble rate is constrained to fulfil [see Eq. (40)]

\[
2 \sqrt{\Omega_r} \left[1 + \Omega_a E^2(z_b)\right] E(z_b) - \Omega_\Lambda = 0.
\]

On the other hand, the Friedmann equation (39) implies

\[
E^2(z_b) = \Omega_m (1 + z_b)^3.
\]

By combining the last two equations, we find that the redshift at which the mimicry of a phantom behaviour breaks down reads [32]

\[
z_b = \left[\frac{S_+ - S_-}{2 \sqrt{\Omega_r \Omega_m}}\right]^{1/2} - 1,
\]

where

\[
S_{\pm} = \left[\sqrt{\frac{4 \Omega_r}{3 \Omega_\alpha}} + \sqrt{2 \Omega_r \Omega_\Lambda} \pm 2 \Omega_r \Omega_\alpha \Omega_\Lambda \right].
\]

When the dimensionless energy density, \(\Omega_\alpha\), associated to the GB geometrical effects is much smaller than \(\Omega_r\), the redshift \(z_b\) fulfills

\[
2 \sqrt{\Omega_m \Omega_r} (1 + z_b)^{3/2} = \Omega_\Lambda - \frac{1}{4} \Omega_r \Omega_\alpha + O^2 \left(\frac{\Omega_\alpha}{\Omega_r}\right).
\]

Therefore, in the limiting situation \(\Omega_\alpha = 0\), we recover that the phantom-like description breaks down also in the ADGP setup at same point in the past [7]:

\[
2 \sqrt{\Omega_m \Omega_r} (1 + z_b)^{3/2} = \Omega_\Lambda.
\]

In fact, the effective phantom-like description breaks down in the ADGP scenario when the analogous effective energy density vanishes and therefore the effective equation of state parameter blows up [32] [see Eq. (42) with \(\Omega_\alpha = 0\)]. This again points to the fact the mapping between the ADGP model and the 4D relativistic phantom cosmology model with Friedmann expansion (39) (for \(\alpha = 0\)) is no longer valid, although the ADGP brane description remains valid [8].

The above behaviour raises the following possibility: Can the phantom behaviour break down in our model at a redshift \(z_b\) [10], such that \(\tilde{z}_b > z_b\)? Our motivation is that such a GB component would model additional curvature effects on the brane setting. The subsequent aim is then to determine if the GB term, even if dominating and modifying the early behaviour of the brane universe, may eventually extend its effect towards later times. Thus, could the regime of validity of the phantom mimicry in the ADGP setting be extended by considering further curvature effects on the brane-world scenario?

The answer will depend on the cosmological parameters that characterise both models, three in the ADGP scenario namely \((\Omega_m, \Omega_\Lambda, \Omega_r)\) and four in the model we

\[\text{We use a tilde to define quantities in the ADGP model and hence distinguish them from the ones used in our model.}\]

\[\text{We thank Y. Shtanov for pointing out this to us.}\]
are analysing namely $(\Omega_m, \Omega_\Lambda, \Omega_c, \Omega_\alpha)$. In order to be able to give a definite answer, we will first assume three different cosmological situations. More general situation will be analysed by means of three dimensional plots (see Fig. 6).

### A. Fixed $\Omega_m$ and $\Omega_\Lambda$

We first assume the same amount of dark matter $\Omega_m$ and dark energy $\Omega_\Lambda$ in the ADGP model and its twin with GB effect. Therefore, the difference between the ADGP scenario and ADGP-GB in this case is encoded on the geometrical effects quantified through the crossover scale and the GB parameter. By using the constraint equation (49), it turns out that the dimensionless energy density associated to the crossover scale cannot be the same in both models. In fact, the dimensionless energy density related to the crossover scale $\Omega_{r_c}$ in the ADGP model is related to $\Omega_{r_c}$ in the ADGP-GB model by

$$\Omega_{r_c} = \Omega_{r_c}(1 + \Omega_\alpha)^2.$$  \hspace{1cm} (50)

Therefore, in this situation the crossover scale in the ADGP model will be larger than in the ADGP scenario with a GB term in the bulk. Let $\tilde{z}_b$ and $z_b$ be the redshift at which the effective energy density vanishes in the ADGP setup and in our scenario, respectively. Then, as the effective energy density vanishes at those redshifts Eq. (10) implies

$$\Omega_{\Lambda} = 2\sqrt{\Omega_{r_c}} \left[1 + \Omega_\alpha E^2(z_b)\right] E(z_b),$$

$$= 2\sqrt{\Omega_{r_c}} \tilde{E}(z_b),$$  \hspace{1cm} (51)

which by using Eq. (50) translates into

$$\tilde{E}(z_b) = \frac{1 + \Omega_\alpha E^2(z_b)}{1 + \Omega_\alpha}.$$  \hspace{1cm} (52)

Now we recall that $1 < E(z_b)$ as the brane do not super-accelerate and $0 < z_b$. Therefore, the right hand side of the previous equation is larger than one, which implies that $z_b < \tilde{z}_b$ because of Eq. (50) and the analogous relation

$$\tilde{E}(z_b) = \Omega_{\bar{m}} (1 + \tilde{z}_b)^3,$$  \hspace{1cm} (53)

in the ADGP model.

In conclusion, the phantom-like behaviour of the ADGP model with GB effect breaks down at a smaller redshift than the phantom-like behaviour on the ADGP model under the assumption that $\Omega_m$ and $\Omega_\Lambda$ are the same in the ADGP setup with and without GB effect in the bulk [see Fig. 6].

### B. Fixed $\Omega_\Lambda$ and $\Omega_{r_c}$

We next assume the same amount of dark energy $\Omega_\Lambda$ and the same dimensionless energy density $\Omega_{r_c}$ in both models. Then, the constraint equation (50) implies that the amount of dark matter is slightly larger in the ADGP-GB model than in the ADGP brane because of the GB effects

$$\Omega_m - \Omega_{\bar{m}} = 2\sqrt{\Omega_{r_c}} \Omega_\alpha.$$  \hspace{1cm} (54)

At the redshifts $\tilde{z}_b$ and $z_b$ the effective energy densities in the ADGP and the ADGP-GB models vanish, respectively, and Eq. (10) implies

$$\frac{\tilde{E}(\tilde{z}_b)}{E(z_b)} = 1 + \Omega_{r_c} E^2(z_b) > 1,$$  \hspace{1cm} (55)

because we have assumed the same $\Omega_\Lambda$ and $\Omega_{r_c}$ in both models. Now, Eqs. (49), (53) and (55) imply

$$\left(\frac{1 + \tilde{z}_b}{1 + z_b}\right)^3 > \frac{\Omega_m}{\Omega_{\bar{m}}}.$$  \hspace{1cm} (56)

Therefore, $\tilde{z}_b > z_b$ because the amount of dark matter is larger in the ADGP-GB brane than in the ADGP brane [see Eq. (54)]. In conclusion, we have that the phantom-like behaviour breaks down sooner in the ADGP-GB model than in the ADGP setup [see Fig. 6].

### C. Fixed $\Omega_m$ and $\Omega_{r_c}$

In this case we assume that both models have the same amount of dark matter $\Omega_m$ and the same dimensionless energy density $\Omega_{r_c}$. Now, we use Eq. (10) to constrain the remaining free parameters of both models

$$\Omega_\Lambda - \Omega_{\bar{\Lambda}} = 2\sqrt{\Omega_{r_c}} \Omega_\alpha.$$  \hspace{1cm} (57)

The previous constraint implies that the amount of dark energy would be slightly larger in the ADGP-GB model.
than in the ADGP model in this case. Because the effective energy density $\rho_{\text{eff}}$ vanishes at $\tilde{z}_b$ and $z_b$, it implies a constraint on how different can be the amount of dark energy in both models [see Eq. (11)]

$$\Omega_{\Lambda} - \Omega_{\Lambda} = 2\sqrt{\Omega_r} \left[ E(z_b) - \tilde{E}(\tilde{z}_b) \right] + 2\sqrt{\Omega_r} \Omega_{\alpha} E^3(z_b).$$

(58)

By combining Eqs. (57), (58) and recalling that $z_b > 0$, we can conclude that

$$\tilde{E}(\tilde{z}_b) - E(z_b) = \Omega_{\alpha} \left[ E^3(z_b) - 1 \right] > 0;$$

i.e. the dimensionless Hubble rate $\tilde{E}(\tilde{z}_b)$ is larger than the dimensionless Hubble rate $E(z_b)$. Therefore, $z_b < \tilde{z}_b$ [see Eqs. (55) and (53)]; i.e. we reach a similar conclusion to the one presented in the previous subsections [see Fig. 6].

So, far we have compared the redshifts $z_b$, $\tilde{z}_b$ where the mimicry of the phantom-like behaviour breaks down in the ADGP-GB and ADGP models, respectively, under the assumption that two of the parameters that characterise both models are equals. We relax this condition in the next subsection.

### D. More general situations

In order to compare further the redshifts $\tilde{z}_b = \tilde{z}_b(\Omega_{\Lambda}, \Omega_m, \Omega_r)$ and $z_b = z_b(\Omega_{\Lambda}, \Omega_m, \Omega_r, \Omega_{\alpha})$ at which the phantom-like behaviour breaks down in the ADGP and ADGP-GB models, respectively, we relax the conditions imposed in the previous subsection.

We consider that only one of the parameters ($\Omega_{\Lambda}$, $\Omega_m$, $\Omega_r$) is fixed in the ADGP-GB brane. We further assume that the given value of the fixed parameter is very close to that obtained by constraining the ADGP model with observational data ($H(z)$, CMB shift parameter and SNIa data)\(^9\)

$$\Omega_m = 0.26 \pm 0.05, \quad \Omega_r \lesssim 0.05.$$  

(60)

As far as we know, the DGP-GB model has been constrained by cosmological observations in Ref. [33]. However, it turns out that the mentioned paper deals with the self-accelerating branch of the model which is different from the one we explore in our paper (the one that contains the ADGP-GB model). The model analysed in Ref. [33] reduces to the self-accelerating DGP branch when $a \to 0$ while our model reduces to the ADGP model which is contained in the normal DGP branch. Therefore, as we are considering the GB term as a perturbation of the ADGP model, we take as a good approximation to consider the cosmological parameters of the ADGP-GB model close to those of the original ADGP model.

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\(^9\) As far as we know, the DGP-GB model has been constrained.
GB brane breaks down sooner (smaller redshift) than in the ADGP brane. In more precise terms, the phantom-like behaviour in the ADGP is regular in the interval $[0, \tilde{z}_b)$ but breaks down for higher redshifts. On the other hand, the phantom-like behaviour in the ADGP-GB scenario works in the interval $[0, \tilde{z}_b)$ but fails for higher redshifts. Because $\tilde{z}_b < z_b$, the GB effect actually makes smaller the interval $[0, \tilde{z}_b)$ and therefore it does not help to improve the situation. Please note (i) we are analysing the phantom-like behaviour through the mapping between the ADGP/ADGP-GB models and the 4D relativistic phantom cosmology models with Friedmann Eq. (39) from today ($z = 0$) till $\tilde{z}_b/z_b$ and (ii) the break down of the phantom-like behaviour does not imply any sort of singularity in the brane-world models.

V. SUMMARY AND CONCLUSION

In this paper we have analysed in some detail the ADGP-GB model which corresponds to a 5D brane-world model where the bulk is a 5D Minkowski space-time. The model contains a GB term in the bulk $\mathcal{L}_{GB}$ and an induced gravity term on the brane $\mathcal{L}_{Brane}$. Our analysis was performed for the normal or non self-accelerating branch which we have assumed to be filled by CDM and a cosmological constant, the latter driving the late-time acceleration of the brane. We have shown how the brane accelerates at late-times (cf. Eq. (12) and Fig. 1).

The attractive and promising feature of this model is the role of the extra-dimension: It induces a mimicry of a phantom behaviour without resorting to any matter that violates the null energy condition on the brane. This phantom-like behaviour happens without any super-acceleration of the brane (see Eq. (13)) and, therefore, the brane does not hit a big rip singularity in its future. Indeed, the brane is asymptotically de Sitter. This regime is reached when the cosmological constant dominates over the CDM component. Our model reduces to the ADGP scenario if the GB corrections in the bulk are put aside.

Our motivation for considering a GB correction to the ADGP is threefold: (i) it is known that the mimicry of a phantom behaviour on the ADGP model breaks down at some point in the past $\tilde{z}_b$. This happens when the mapping between the ADGP and the 4D relativistic phantom cosmology model breaks down. More precisely, when the effective energy density that mimics the phantom-like behaviour gets negative and therefore the corresponding effective equation of state parameter $\omega_{eff}$ blows up (see Eq. (12) for $\alpha = 0$ where $\alpha$ is the GB parameter). (ii) GB effects induce ultraviolet corrections on the brane $30$, as expected from high-energy stringy features, and therefore may modify the phantom-like behaviour at earlier times and may alleviate the shortcoming mentioned in the previous point. And (iii), even though the DGP model is characterised by an interesting infrared effect of gravity occurring with respect to general relativity, which for the self-accelerating branch can lead to a late-time acceleration on the brane even in the absence of any exotic matter invoked to produce the dark energy effect, it would be expected that a consistent DGP brane model would have ultraviolet modifications as well, associated to high-energy stringy effects at earlier times $25$.

The phantom-like behaviour on the brane is based on (i) writing down the modified Friedmann equation of the brane as the standard relativistic Friedmann equation (see Eq. (39)) and (ii) defining a corresponding effective energy density, $\rho_{eff}$, which grows as the brane expands, and an effective equation of state of the brane (cf. Eqs. (40) and (42) and Figs. 4 and 5). The effective energy density corresponds to a balance between the cosmological constant and geometrical effects encoded on the Hubble rate. This was done by generalising the way a phantom-like behaviour is obtained in the ADGP model $13, 16$ and other of its variants $21, 24$ where dark energy has a more dynamical character. As $\rho_{eff}$ depends explicitly on the Hubble rate, in order to get its evolution with the redshift it is necessary to solve the cubic Friedmann equation (6). This was done in section 11. This analysis has also allowed us to constraint the set of most likely values of the model (see Fig. 2) and therefore to pick up the suitable cubic solution of the Friedmann equation (9).

It turns out that the phantom-like behaviour also breaks down in the ADGP-GB model, namely when $\rho_{eff} = 0$: i.e. when the brane cosmological constant balances the geometrical effects described in terms of the Hubble rate (see Eq. (10)). This feature highlights that the mapping between the ADGP-GB model and the 4D relativistic phantom cosmology model ceases to be valid although the brane description remains valid. The redshift, $z_b$, at which this event happens is given in Eq. (16) and depends on the set $(\Omega_m, \Omega_b, \Omega_{\kappa}, \Omega_{\sigma})$; i.e. on the amount of CDM, the weight of the cosmological constant, as well as on the weights of the curvatures effects encoded on $\Omega_{\kappa}$ and $\Omega_{\sigma}$ (related to the crossover scale and the GB parameter $\alpha$, respectively). Concerning such feature, namely that the phantom-like behaviour in the ADGP and ADGP-GB breaks down at the redshifts $\tilde{z}_b$ and $z_b$, respectively, can it then be that $z_b > \tilde{z}_b$? i.e. can the GB effect bring the breakdown of the phantom-like behaviour in the ADGP to higher redshifts? This question has been addressed analytically (in specific situations) as well as numerically in more general situations and the answer has been always negative (cf. Figs. 6 and 7). The GB effect seems instead to induce a breakdown of the phantom-like behaviour in the ADGP scenario at an even smaller redshift. Namely, the ADGP has a regular phantom-like behaviour for $[0, \tilde{z}_b)$ whereas the phantom-like behaviour in the ADGP-GB model is regular only for $[0, z_b)$, with $z_b < \tilde{z}_b$. Thus, we conclude that the GB term does extend the regime of validity of the phantom mimicry in the ADGP model.

We were expecting that new curvature corrections would modify the regime of validity of the phantom-like
behaviour on the brane. We have shown that this is the case by considering a GB term in the bulk. However, these bulk curvature corrections rather than enlarging the regime of validity of the phantom-like behaviour on the brane, they make it smaller. It might be that we need to consider exclusively curvature corrections to the induced gravity action on the brane, modelled for example through an \( f(R) \) term on the brane action [34]. We leave this question for a future work.

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**APPENDIX A: SOLUTIONS OF THE FRIEDMANN EQUATION**

The solutions of the Friedmann equation (17) can be written as [35]

\[
\begin{align*}
H_1 &= S_1 + S_2 - \frac{1}{3}, \\
H_2 &= -\frac{1}{2}(S_1 + S_2) - \frac{1}{3} + i\frac{\sqrt{3}}{2}(S_1 - S_2), \\
H_3 &= -\frac{1}{2}(S_1 + S_2) - \frac{1}{3} - i\frac{\sqrt{3}}{2}(S_1 - S_2),
\end{align*}
\]

\begin{align}
S_1 &= \left[R + (Q^3 + R^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \\
S_2 &= \left[R - (Q^3 + R^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}.
\end{align}

(A1) (A2) (A3) (A4)

\( Q \) and \( R \) are given in Eq. (19). Only those solutions real and positive correspond to cosmologically interesting solutions. The latter depends strongly on the ratio between the GB parameter \( \alpha \) and the crossover scale \( r_c \). Once the real roots of Eq. (17) are identified, it is much more useful to rewrite them as trigonometric function (if all the solutions are real) or as hyperbolic functions if there is a unique real solution in order to analyse the cosmological evolution of the brane. This is performed in section III.

**APPENDIX B: RECOVERY OF THE ADGDP SCENARIO**

In this appendix, we show that only two of the solutions of the Friedmann equation of the normal DGP branch with GB effects presented in Sect. III have a well defined limit when \( \alpha \to 0 \). These solutions correspond to Eqs. (29) and (33). Furthermore, only the solution (29) reduces to the ADGP scenario when \( \alpha = 0 \).

As it can be noticed, it is not as straightforward to take such a limit; i.e. \( \alpha \to 0 \), of the mentioned solutions because all the dimensionless parameters involved in the problem \( \bar{H} \), \( \bar{\rho} \) and \( b \) are proportional to \( \alpha \) or \( \alpha^2 \). To make the task easier, we introduce the following definitions

\[
\begin{align*}
\bar{\rho} &= f_1 \alpha^2; \\
b &= f_2 \alpha; \\
\bar{H} &= f_3 \alpha; \\
\end{align*}
\]

(B1) (B2) (B3)

where \( f_1, f_2 \) and \( f_3 \) do not depend on \( \alpha \).

We start considering the solution (29) and we make a series expansion of \( \tan(\theta) \) at \( \alpha = 0 \), where \( \theta \) is defined in Eq. (30),

\[
\tan(\theta) = -\frac{3}{2} \left( \sqrt{12 f_1 + 3 f_2^2} \right) \alpha + O^2(\alpha).
\]

(B4)

Therefore,

\[
\theta = \pi - \frac{3}{2} \left( \sqrt{12 f_1 + 3 f_2^2} \right) \alpha + O^2(\alpha),
\]

(B5)

because \( 0 < \theta < \pi \).

On the other hand, by combining the last equation and

\[
\cos \left( \frac{\pi}{3} - c x \right) = \frac{1}{2} \left( 1 + \sqrt{3} c x \right) + O^2(x)
\]

(B6)

for \( x \to 0 \) and \( c \) a constant, we obtain

\[
\cos \left( \frac{\theta}{3} \right) = \frac{1}{2} \left[ 1 + \sqrt{3} \left( \sqrt{12 f_1 + 3 f_2^2} \right) \alpha \right] + O^3(\alpha).
\]

(B7)

Finally, we substitute the last equation in Eq. (29) and we conclude

\[
f_3 \alpha = \frac{1}{3} \left\{ \left( 1 - \frac{3}{2} f_2 \alpha \right) \left[ 1 + \frac{\sqrt{3}}{2} \left( \sqrt{12 f_1 + 3 f_2^2} \right) \alpha \right] - 1 \right\} + \ldots,
\]

\[
= \frac{1}{2} f_2 \alpha \left( -1 + \sqrt{1 + 4 \frac{f_1}{f_2}} \right) + O^2(\alpha).
\]

(B8)

We equate the lower order in \( \alpha \) of the last equation (of both handsides of the equation) and we substitute the definitions of \( f_1, f_2 \) and \( f_3 \). We finally obtain

\[
H = \frac{1}{2 r_c} \left( -1 + \sqrt{1 + \frac{2}{3} \kappa^2 r_c \rho} \right),
\]

(B9)

which is the modified Friedmann equation of the non-self-accelerating DGP branch that can be obtained from
Eqs (1) or (3) (with - sign) with \(\alpha = 0\) and contains the ADGP model as a specific solution.

Following a similar procedure, it can be shown that the limit \(\alpha = 0\) of Eq. (58)
\[
H = -\frac{1}{2r_c} \left( 1 + \sqrt{1 + \frac{2}{3} \kappa_5^2 r_c \rho} \right).
\]
(B10)
The limit is well defined although the solution it is not physical (at least it is not suitable for the late-time evolution of the universe). Please notice that Eq. (3) of the manuscript with \(\alpha = 0\) is quadratic on the Hubble rates and therefore has two roots. This second root; i.e. Eq. (B10), after a time reversal \((t \rightarrow -t)\) corresponds to the self-accelerating DGP solution.

On the other hand, the solution (32) of the paper has not a well defined \(\alpha = 0\) limit because the left hand side of the equation is proportional to \(\alpha\) while the right handside lower order in \(\alpha\) is a constant different from zero.

If one tries to take the limit \(\alpha \rightarrow 0\) of the solution presented in Eq. (25) it turns out that \(\eta\) must be complex. In fact, in that case one is getting an analytical extension of the solution which corresponds to the solution (29).

The reason behind this behaviour is that the solutions (20) and (25) are related by a sort of “Wick rotation” where \(\theta = i\eta\). Notice at this respect that the function \(N\) changes sign and the prolongation is well defined.

For the rest of solutions presented in section III, although all of them comes from the general solution (A1), the limit \(\alpha \rightarrow 0\) cannot be taken because \(1/4 < b\).

In summary, the recovery of the ADGP scenario from the normal DGP branch with GB effects requires that (i) the value of the parameter \(b\), which is proportional to \(\alpha\), to be such that \(0 < b < 1/4\) and (ii) the dimensionless amount of the brane total energy density has to fulfill \(\bar{\rho} < \bar{\rho}_1\). Therefore, in order to recover the ADGP scenario from the model we propose, the GB parameter has to be bounded from above and the brane total energy density has also to be bounded by a maximum threshold. Along section IV, we have restricted to this solution as it is the most favourable solution observationally because the parameter \(b\) is expect to be small which implies that the most likely set of cosmological parameters that characterises the model would lie on the bottom of the left handside corner of figure 2. This last feature on itself implies that the brane energy density cannot be too large, more precisely \(\bar{\rho}_{\text{min}} \leq \bar{\rho}_1\) which implies in particular that there is a range of values of \(\bar{\rho}\) such that \(\bar{\rho}_1\).

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