On infinite quon statistics and "ambiguous" statistics

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We critically examine a recent suggestion that "ambiguous" statistics is equivalent to infinite quon statistics and that it describes a dilute, nonrelativistics ideal gas of extremal black holes. We show that these two types of statistics are different and that the description of extremal black holes in terms of "ambiguous" statistics cannot be applied.

PACS numbers: 05.30.-d, 05.70.Ce, 04.70.Dy
1. Introduction.
A few years ago, it was argued \(^1\) that extremal black holes obeyed infinite quon statistics, i.e., that quantum states of extremal black holes belonged to the (quantum) Boltzmannian space \(^2,3\). This connection of extremal black holes to infinite quon statistics was later adopted by several authors \(^4,5\) and, recently, the thermodynamics of the ensemble of extremal black holes was discussed from the point of view of statistical mechanics of ”ambiguous” statistics \(^6\), which interpolates between Bose and Fermi statistics and was claimed to be equivalent to quon statistics.

The aim of this paper is to closely examine the supposed connection between ”ambiguous” statistics and infinite quon statistics and their relation to the statistics of extremal black holes. First, we briefly review quon statistics and the corresponding thermodynamics. We mention a few shortcomings of the thermodynamics of quonic systems. Then, we show that ”ambiguous” statistics is different from quon statistics and that there is no operator realization of it.

2. Quon algebra.
Recall that a free system of particles obeying infinite quon statistics is described by the following commutation relations \(^2\) (existence of the unique vacuum state \(|0>\) is assumed):

\[
a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij},
\]

\[
a_i |0> = 0 <0|0>= 1.
\]

Here \(i, j\) are discrete indices, \(i, j = 1, 2, \ldots M\), and the parameter \(q\) is a real number, \(q \in \mathbb{R}\).

The main properties of quons are as follows:
(i) Norms of the states are positive definite for \(-1 < q < 1\).

(ii) For \(q^2 \neq 1\), the commutation relations do not exist between annihilation (creation) operators \(a_i, a_j (a_i^\dagger, a_j^\dagger)\), i.e., there are \(n!\) linearly independent states \(a_{i_1}^\dagger \ldots a_{i_n}^\dagger |0\rangle\) for different permutations of fixed indices 1, 2, \ldots, \(n\).

(iii) The number operator exists in the form of an infinite series expanded in powers of creation and annihilation operators.

(iv) The particles obeying infinite quon statistics do not have a local-field theory, there is no spin-statistics restriction for such particles and they can have any spin. Nevertheless, the TCP theorem and the clustering property is valid for free infinite-statistics fields.

3. Statistical mechanics of the quon gas.

To discuss the statistical mechanics of the quon gas, we assume that quon particles are massive (with mass \(m\)), spinless, nonrelativistic and noninteracting. The Hamiltonian of an ensemble of particles is that of the free system:

\[ H = \sum_{k=1}^{M} E_k N_k, \]  

where \(E_k\) is the energy of the \(k^{th}\) level and \(N_k\) are the number operators counting particles on the \(k^{th}\) level. Recall that the statistical average of an observable, described by an operator \(O\) in a given ensemble, is defined as

\[ \langle O \rangle = \frac{\text{Tr} O e^{-\beta H}}{\text{Tr} e^{-\beta H}} = \frac{1}{Z} \text{Tr} O e^{-\beta H}. \]  

Here \(Z\) is the thermodynamical (grand-canonical) partition function for a multi-level system described by \(M\) independent creation (annihilation) operators \(a_i^\dagger (a_i)\), \(i = 1, 2, \ldots M\). The partition function \(Z\) for a free system described by the algebra, Eq.(1), is a power series in \(x_k = e^{\beta \mu - \beta E_k}\) (where \(\mu\) denotes a chemical potential)
and is given by (compare Ref. (7))

\[
Z(x_1, ...x_M) = 1 + \sum_{k=1}^{M} x_k + (\sum_{k=1}^{M} x_k)^2 + \cdots = 1 + z_1 + (z_1)^2 + \cdots \tag{4}
\]

\[
= \frac{1}{1 - z_1},
\]

for \( z_1 = \sum_{k=1}^{M} x_k < 1 \). Notice that the partition function does not depend on the interpolation parameter \( q \), which is a generic property of all generalized statistics in the second quantized approach. From \( Z(x_1, ...x_M) \) one can derive, e.g., distribution function \( n_k(E_k) \) (occupation number of the k-th level) and an average particle number \( <N> \) as

\[
n_k(E_k) = x_k \frac{\partial}{\partial x_k} \ln Z(x_1, ...x_M) = \frac{x_k}{1 - z_1}, \tag{5}
\]

\[
<N> = \frac{1}{\beta \mu} \frac{d}{d\mu} \ln Z(x_1, ...x_M) = \frac{z_1}{1 - z_1}.
\]

In the limit \( M \to \infty \) and \( E_k = E = \frac{p^2}{2m} \), transforming \( \sum_k \to \int \frac{dV}{(2\pi \hbar)^3} \), one recovers the partition functions used in Refs. (8,9), namely,

\[
(z_1)_\infty = e^{\beta \mu} \left( \frac{V}{\lambda^3} \right) \tag{6}
\]

\[
(Z)_\infty = \sum_{N=0}^{\infty} ((z_1)_\infty)^N
\]

where \( \lambda = \left( \frac{2\pi \hbar^2}{m} \right)^{\frac{1}{2}} \) is the thermal de Broglie wavelength. In this limit, Eq.(5) yields

\[
<N>_\infty = e^{\beta \mu V} \frac{V}{\lambda^3 - e^{\beta \mu V}}.
\]

It is well known \(^8\,^9\) that quon gas, described by Eq.(6), is plagued with several difficulties. The most serious ones are:

(1) The partition functions (6) exhibit Gibbs’ paradox which cannot be resolved by fixing the overall statistical weight of the \( N \)-particle phase space as in classical statistical mechanics, since the weight is unambiguously determined by quon
quantum-statistical mechanics.

(2) the entropy $S$ or an average particle number fails to be an extensive variable of the system; moreover, they both diverge for the critical volume $V = e^{-\beta \mu \lambda^3}$.

The pathological behavior of the free quon gas can also be detected for finite $M$. To show this, we apply the usual method of Lagrange’s undetermined multipliers in order to extremize the entropy $S$ under subsidiary conditions (constancy of the total number of particles, $N$, and the total energy of the gas, $E$):

$$\frac{\partial}{\partial N_k}(S - \alpha \sum_{k=1}^{M} N_k - \beta \sum_{k=1}^{M} E_k N_k) = 0, \quad (7)$$

$$\sum_{k=1}^{M} N_k = N, \quad \sum_{k=1}^{M} E_k N_k = E.$$  

Here, the entropy $S$ is related to the number of all allowed N-body states distributed over M energy levels, $W(M, N)$, as $S = ln W(M, N)$. A simple counting procedure, in agreement with (1), gives for $W(M, N)$

$$W(M, N) = \sum_{\sum_{k=1}^{M} N_k = N} \frac{N!}{N_1! N_2! \cdots N_M!} = M^N.$$  

Solving Eq.(7) gives a constraint

$$\beta E_k = lnM - \alpha = const, \quad \forall k = 1, 2, ..M. \quad (8)$$

In such a thermal equilibrium state, after identifying $\alpha = -\beta \mu$, one would have

$$x_k = \frac{1}{M}$$ and $z_1 = 1$. Hence, the partition function diverges, $Z(x_1, ...x_M) \to \infty$, and, as a consequence, quantities such as $< N > \to \infty$ also diverge. The conclusion is obvious. Regardless of finite or infinite $M$ and for any $-1 < q < 1$, the thermodynamical limit does not make sense for the free quon gas.

3. **“Ambiguous” statistics.**

The basic idea of “ambiguous” statistics is that only primary Bose and Fermi
statistics are allowed, but the statistics of the particle may fluctuate between Bose and Fermi statistics and is determined in a probabilistic way. During the interaction, the first particle identifies the second one as either a boson or a fermion, with respective probabilities $p_b$ and $p_f$. The system of $N$ particles effectively consists of $k$ bosons and $N - k$ fermions and the probability of this $N$-particle realization is $p_b^k p_f^{N-k}$. The total number of states of the system is

$$W(G_j, N_j) = \prod_j \sum_{k=0}^{N_j} \binom{N_j}{k} \binom{G_j + k - 1}{k} \binom{G_j}{N_j - k} p_b^k p_f^{N_j-k}, \quad (9)$$

where $G_j$ is the number of allowed one-particle states of type $j$. It is claimed in Ref.(6) that the operator realization of this statistics is described by Eq.(1), with the deformation parameter $q = \frac{p_b - p_f}{p_b + p_f}$. Particularly, for $p_b = p_f$, one should recover Greenberg's infinite statistics with $q = 0$ (quantum Boltzmann statistics). We have several objections to this identification.

Indeed, from Eq.(9), one can derive the distribution function $n_j$ and the thermodynamic properties by varying the entropy, as in Eq.(7). However, from the definition (9) alone, one cannot determine the operator algebra (commutation relation) for the annihilation and creation operators. First of all, it is impossible to obtain the fractional number of states, Eq.(9), in Fock space. Whether or not there exists an operator algebra, leading to the same thermodynamic properties as for "ambiguous" statistics, is an open question. Furthermore, we point out that the proposed "ambiguous" statistics is completely different from infinite statistics. For the case $p_b = p_f \equiv p$, the distribution function $n_j$, derived from Eqs.(7,9), can be rewritten as

$$N_j = \frac{2p}{e^{\beta(E_j-\mu)} + p} + \frac{2p}{e^{\beta(E_j-\mu)} - p} \equiv N_j(p) + N_j(-p)$$

and has nothing to do with infinite statistics. However, $N_j(\pm p)$ is exactly the
distribution derived from the counting rule

\[ W_j(N_j) = \frac{1}{N_j!} \left( \frac{(G_j)!}{(G_j - p N_j)!} \right)^p. \]

The thermodynamics of such a system is described in Ref.(11).

In defining "ambiguous" statistics, two unjustified assumptions are made:

(i) First, that the annihilation and creation operators satisfy Bose \((q = -1)\), Fermi \((q = +1)\), and quon \((q = 0)\) statistics:

\[
\begin{align*}
    a_i a_j^\dagger - a_j^\dagger a_i & = \delta_{ij}, \\
    a_i a_j^\dagger + a_j^\dagger a_i & = \delta_{ij}, \\
    a_i a_j & = \delta_{ij},
\end{align*}
\]

with the probabilities \(p_b^2\), \(p_f^2\), and \(2p_b p_f\), respectively. This is in contradiction with the initial assumption that only Bose and Fermi statistics are allowed.

(ii) Second, in the second quantized approach, it is not mathematically correct \(^{12}\) to add the above commutation relation with relative weights \(p_b^2\), \(p_f^2\), and \(2p_b p_f\) in order to obtain Eq.(1) with the deformation parameter \(q = \frac{p_b - p_f}{p_b + p_f}\). Instead, one can postulate Eq.(1) \textit{ab initio} and investigate the thermodynamics of particles obeying quon statistics but this thermodynamics is completely different from that studied in Ref.(6). Hence, we may conclude that "ambiguous" statistics, as defined in Ref.(6), possesses no Fock-like realization and is completely different from infinite quon statistics.

We mention that a similar idea, \textit{i.e.}, that, basically, only symmetric \((i.e.,\) Bose-like) and antisymmetric \((i.e.,\) Fermi-like) IRREP’s of the permutation group \(S_N\) are allowed in generalized statistics, was suggested and investigated in Ref.(13). In this approach, generalized statistics is described by an algebra for which there exists a
Fock-like realization. The construction is based on two vacua $|\pm\rangle$ and on the commutation rules containing a unitary operator $\hat{q}$:

$$a_i a_j^\dagger - \hat{q} a_j^\dagger a_i = \delta_{ij}, \quad \hat{q}|\pm\rangle = \pm |\pm\rangle. \quad (10)$$

The corresponding Fock-like representations are built on a linear combination of $|\pm\rangle$ vacua. In Ref.(14) the minimal generalized statistics with permutation-group invariance, interpolating between Bose and Fermi statistics, is constructed. Moreover, it is shown that $a_i a_j^\dagger$ can be brought to a normally ordered form without using the operator $\hat{q}$ and that a complete theory can be formulated using a single vacuum state $|0\rangle$ (instead of $|\pm\rangle$ vacua). We notice that this type of statistics is also different from infinite quon statistics.

4. Conclusion.

"Ambiguous" statistics, defined through the counting rule (9), although interesting as a possible generalization of Bose and Fermi statistics, is completely different from infinite statistics, and Eq.(1) does not represent its operator realization. If one seriously takes the suggestion that extremal black holes exhibit the $q = 0$ quonic behavior, then "ambiguous" statistics, not being equivalent to quon statistics, does not describe extremal black holes. Moreover, the assumption that $q = 0$ quons may be viewed as distinguishable particles with infinite internal degrees of freedom would also lead to difficulties and would not resolve Gibbs’ paradox. We believe that a correct statistics of extremal black holes is still an interesting open question.
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