The Scalar-flat Kähler Metric and Painlevé III

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Abstract
We study the anti-self-dual equation for non-diagonal $SU(2)$-invariant metrics and give an equivalent ninth-order system. This system reduces to a sixth-order system if the metric is in the conformal class of scalar-flat-Kähler metric.

1 introduction
The aim of this paper is to analyse scalar-flat Kähler metrics $g$ in real dimension four admitting an isometric action of $SU(2)$ with generically three-dimensional orbits. A scalar-flat Kähler metric is a metric with zero scalar curvature which is Kählerian with respect to a complex structure on $M$. It is automatically anti-self-dual with respect to the canonical orientation.

Hitchin[6] shows that the $SU(2)$-invariant anti-self-dual metric is generically specified by a solution of Painlevé VI type equation, and if the metric is scalar-flat Kähler it is specified by a solution of Painlevé III type equation. Hitchin used the twistor correspondence to associate the anti-self-dual equation and Painlevé equation. The lifted action of $SU(2)$ determines a pre-homogenous action of $SU(2)$ on the twistor space $Z$, and it determines a isomonodromic family of connections on $\mathbb{C}P^1$, and then we have Painlevé equations. In this way, Dancer[5] analyse the scalar-flat Kähler metric with $SU(2)$-symmetry. In [1], Hichin obtains a complete classification of anti-self-dual Einstein metrics admitting an isometric action of $SU(2)$ with three-dimensional orbits. For the completeness analysis, it is important to have the explicit form of anti-self-dual equation.

If the metric is diagonal, the explicit form of anti-self-dual equation is known, but if the metric is non-diagonal, it is known very little. For scalar-flat Kähler metric, complex structure is not known for non-diagonal metric.

In Section 2 we show how non-diagonal metric is represented, and by use of the block form of curvature tensor given by Besse[3], we have the ninth-order system equivalent to the anti-self-dual equation.

In Section 3 we establish the relationship between $SU(2)$-invariant anti-self-dual manifold and the isomonodromic deformation. It is essentially equivalent to Hitchin’s ansatz. Still in our way, we have the explicit form of the isomonodromic deformations, and we have the condition that the corresponding Painlevé equation is type III.
In Section 4, we show that the anti-self-dual equation reduce to Painlevé III if and only if the metric admits an Hermitian structure. In this case, the anti-self-dual equation is equivalent to a seventh-order system, and it also admits Kähler structure, the seventh-order system reduce to a sixth-order system.

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2 The non-diagonal anti-self-dual equations

We can write the $SU(2)$-invariant metric in the form

$$g = f(\tau) d\tau^2 + \sum_{l,m=1}^{3} h_{lm}(\tau) \sigma_l \sigma_m,$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis of left invariant one-forms on each $SU(2)$-orbit satisfying

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$

Using the Killing form, we can diagonalize the metric $g$ on each $SU(2)$-orbit. Then we can express the metric as follows:

$$g = (abc)^2 dt^2 + a^2 d\tilde{\sigma}_1^2 + b^2 \tilde{\sigma}_2^2 + c^2 \tilde{\sigma}_3^2,$$

where $t = t(\tau), a = a(t), b = b(t), c = c(t)$ and

$$d\left(\begin{array}{c} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{array}\right) = R(t) \left(\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array}\right),$$

$R(t)$ is $SO(3)$-valued function.

Since $RR^{-1}$ (where $i = \frac{d}{dt}$) is $so(3)$-valued, we can write

$$d\left(\begin{array}{c} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{array}\right) = R(t) \left(\begin{array}{c} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_1 \wedge \sigma_2 \end{array}\right) + \dot{R} dt \wedge \left(\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array}\right)$$

$$= \left(\begin{array}{c} \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \\ \tilde{\sigma}_3 \wedge \tilde{\sigma}_1 \\ \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \end{array}\right) + \left(\begin{array}{ccc} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{array}\right) dt \wedge \left(\begin{array}{c} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{array}\right),$$

for some $\xi_1 = \xi_1(t), \xi_2 = \xi_2(t), \xi_3 = \xi_3(t)$. If $\xi_1 = 0, \xi_2 = 0, \xi_3 = 0$, then the matrix $(h_{lm})$ can be chosen to be diagonal for all $\tau$, and then we say that $g$ has diagonal form.

In this paper we mainly study the non-diagonal case.

To compute the curvature tensor we choose a basis for $\bigwedge^2$

$$\{\Omega_1^+, \Omega_2^+, \Omega_3^+, \Omega_1^-, \Omega_2^-, \Omega_3^-\},$$
Theorem 2.1

The metric is anti-self-dual with vanishing scalar curvature if

\[ \alpha \text{ and only if } \]

the free parts of Ricci tensor.

We set

\[ w = 4 \text{ trace } D \]

is the scalar curvature, \( W^+ = A - \frac{1}{12} s \) and \( W^- = D - \frac{1}{12} s \) are the self-dual and anti-self-dual parts of the Weyl tensor and \( B \) is the trace free parts of Ricci tensor.

We set \( w_1 = bc, w_2 = ca, w_3 = ab \) and determine \( \alpha_1, \alpha_2, \alpha_3 \) by

\[
\begin{align*}
\dot{w}_1 &= -w_2 w_3 + w_1 (\alpha_2 + \alpha_3), \\
\dot{w}_2 &= -w_3 w_1 + w_2 (\alpha_3 + \alpha_1), \\
\dot{w}_3 &= -w_1 w_2 + w_3 (\alpha_1 + \alpha_2)
\end{align*}
\]

Calculating the condition \( A = 0 \), we have the following theorem.

**Theorem 2.1** The metric is anti-self-dual with vanishing scalar curvature if and only if \( \alpha_1, \alpha_2, \alpha_3 \) and \( \xi_1, \xi_2, \xi_3 \) satisfies the following equations:

\[
\begin{align*}
\dot{\alpha}_1 &= -\alpha_2 \alpha_3 + \alpha_1 (\alpha_2 + \alpha_3) + \frac{1}{4} (w_2^2 - w_3^2)^2 \left( \frac{\xi_1}{w_2 w_3} \right)^2 \\
&\quad + \frac{1}{4} (w_3^2 - w_1^2)(3w_1^2 + w_3^2) \left( \frac{\xi_2}{w_3 w_1} \right)^2 \\
&\quad + \frac{1}{4} (w_2^2 - w_1^2)(3w_1^2 + w_2^2) \left( \frac{\xi_3}{w_1 w_2} \right)^2, \\
\dot{\alpha}_2 &= -\alpha_3 \alpha_1 + \alpha_2 (\alpha_3 + \alpha_1) + \frac{1}{4} (w_3^2 - w_1^2)^2 \left( \frac{\xi_2}{w_3 w_1} \right)^2 \\
&\quad + \frac{1}{4} (w_1^2 - w_2^2)(3w_2^2 + w_1^2) \left( \frac{\xi_3}{w_1 w_2} \right)^2 \\
&\quad + \frac{1}{4} (w_3^2 - w_2^2)(3w_2^2 + w_3^2) \left( \frac{\xi_1}{w_2 w_3} \right)^2, \\
\dot{\alpha}_3 &= -\alpha_1 \alpha_2 + \alpha_3 (\alpha_1 + \alpha_2) + \frac{1}{4} (w_1^2 - w_2^2)^2 \left( \frac{\xi_3}{w_1 w_2} \right)^2 \\
&\quad + \frac{1}{4} (w_2^2 - w_3^2)(3w_3^2 + w_2^2) \left( \frac{\xi_1}{w_2 w_3} \right)^2 \\
&\quad + \frac{1}{4} (w_1^2 - w_3^2)(3w_3^2 + w_1^2) \left( \frac{\xi_2}{w_3 w_1} \right)^2.
\end{align*}
\]
and

\[ (w_2^2 - w_3^2) \frac{d}{dt} \left( \frac{\xi_1}{w_2 w_3} \right) = \frac{\xi_2}{w_3 w_1} \frac{\xi_3}{w_1 w_2} (-2w_2^2 w_3^2 + w_1^2 w_3^2 + w_1^2 w_2^2) \]
\[ + \frac{\xi_1}{w_2 w_3} (\alpha_2 w_2^2 - \alpha_3 w_3^2 + 3\alpha_2 w_1^2 + 3\alpha_3 w_2^2), \]
\[ (w_3^2 - w_1^2) \frac{d}{dt} \left( \frac{\xi_2}{w_3 w_1} \right) = \frac{\xi_3}{w_1 w_2} \frac{\xi_1}{w_2 w_3} (-2w_3^2 w_1^2 + w_1^2 w_2^2 + w_2^2 w_3^2) \]
\[ + \frac{\xi_2}{w_3 w_1} (\alpha_3 w_3^2 - \alpha_1 w_1^2 + 3\alpha_3 w_2^2 + 3\alpha_1 w_3^2), \]
\[ (w_1^2 - w_2^2) \frac{d}{dt} \left( \frac{\xi_3}{w_1 w_2} \right) = \frac{\xi_1}{w_1 w_3} \frac{\xi_2}{w_3 w_1} (-2w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2) \]
\[ + \frac{\xi_3}{w_1 w_2} (\alpha_1 w_1^2 - \alpha_2 w_2^2 + 3\alpha_1 w_2^2 + 3\alpha_2 w_1^2). \]

Remark 2.2 If \( \xi_1 = 0, \xi_2 = 0 \) and \( \xi_3 = 0 \) then the system (1), (2), (3) reduce to a sixth-order system given by Tod [9]. Furthermore, if \( \alpha_1 = w_1, \alpha_2 = w_2, \alpha_3 = w_3 \) then (2), (3), (4) reduce to a third-order system which determines Atiyha-Hitchin family [1], and if \( \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0 \) then the system reduce to a third-order system which determines BGPP family [4].

Remark 2.3 If \( w_2 = w_3, \) then we can set \( \xi_1 = 0, \xi_2 = 0 \) and \( \xi_3 = 0 \) by taking another flame. This is also a diagonal case. Therefore we assume \( (w_2 - w_3)(w_3 - w_1)(w_1 - w_2) \neq 0. \)

3 The Isomonodromic Deformations and Painlevé equation

Let \( (M, g) \) be an oriented Riemannian four manifold. We define \( Z \) to be the unit sphere bundle in the bundle of self-dual two-forms, and let \( \pi : Z \to M \) denote the projection. Each point \( z \) in the fiber over \( \pi(z) \) defines a complex structure on the tangent space \( T_{\pi(z)}M \), compatible with the metric and its orientation.

Using the Levi-Civita connection, we can split the tangent space \( T_zZ \) into horizontal and vertical spaces, and the projection \( \pi \) identifies the horizontal space with \( T_{\pi(z)}M \). This space has a complex structure defined by \( z \) and the vertical space is the tangent space of the fiber \( S^3 \cong \mathbb{CP}^1 \) which has its natural complex structure. The almost complex structure on \( Z \) is integrable if and only if the metric is anti-self-dual [2][8]. In this situation \( Z \) is called the twistor space of \( (M, g) \) and the fibers are called the real twistor lines.

The almost complex structure on \( Z \) can be determined by the following (1,0)-forms:

\[ \Theta_1 = z(e^2 + \sqrt{-1}e^3) - (e^0 + \sqrt{-1}e^1), \]
\[ \Theta_2 = z(e^0 - \sqrt{-1}e^1) + (e^2 - \sqrt{-1}e^3), \]
\[ \Theta_3 = dz + \frac{1}{2} z^2 (\omega_0^2 - \omega_1^2 + \sqrt{-1}(\omega_3^0 - \omega_2^1)) \]
\[ - \sqrt{-1} z (\omega_0^0 - \omega_3^0) + \frac{1}{2} (\omega_0^0 - \omega_3^0 - \sqrt{-1}(\omega_0^0 - \omega_2^1)), \]
where \( \{ e^0, e^1, e^2, e^3 \} \) is an orthonormal flame, and \( \omega^i_j \) are the connection forms determined by \( de^i + \omega^i_j \wedge e^j = 0 \) and \( \omega^i_j + \omega^i_j = 0 \). Then the anti-self-dual condition is

\[
d\Theta_1 \equiv 0, \quad d\Theta_2 \equiv 0, \quad d\Theta_3 \equiv 0 \quad (\text{mod } \Theta_1, \Theta_2, \Theta_3). \tag{5}
\]

If the metric is \( SU(2) \) invariant, we can write

\[
\begin{pmatrix}
\Theta_1 \\
\Theta_2 \\
\Theta_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} dz +
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} dt +
A
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{pmatrix},
\tag{6}
\]

where \( v_1 = v_1(z, t), \ v_2 = v_2(z, t), \ v_3 = v_3(z, t); \ A = (a_{ij}(z, t))_{i,j=1,2,3}. \)

If \( \det A \equiv 0 \), then metric is in the BGPP family \( (\square) \).

If \( \det A \neq 0 \), then we can write

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{pmatrix} \equiv -A^{-1}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} dz +
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} dt =:
\begin{pmatrix}
\varsigma_1 \\
\varsigma_2 \\
\varsigma_3
\end{pmatrix},
\tag{7}
\]

and then

\[
d
\begin{pmatrix}
\varsigma_1 \\
\varsigma_2 \\
\varsigma_3
\end{pmatrix} \equiv
\begin{pmatrix}
\varsigma_2 \wedge \varsigma_3 \\
\varsigma_3 \wedge \varsigma_1 \\
\varsigma_1 \wedge \varsigma_2
\end{pmatrix}.
\tag{8}
\]

Since \( \varsigma_1, \varsigma_2, \varsigma_3 \) are one-forms on \((z, t)\)-plane,

\[
d
\begin{pmatrix}
\varsigma_1 \\
\varsigma_2 \\
\varsigma_3
\end{pmatrix} =
\begin{pmatrix}
\varsigma_2 \wedge \varsigma_3 \\
\varsigma_3 \wedge \varsigma_1 \\
\varsigma_1 \wedge \varsigma_2
\end{pmatrix}.
\tag{9}
\]

If we set

\[
\Sigma = \frac{1}{\sqrt{2}}
\begin{pmatrix}
\sqrt{-1}\varsigma_1 \\
\varsigma_3 + \sqrt{-1}\varsigma_2 \\
\varsigma_3 + \sqrt{-1}\varsigma_1
\end{pmatrix}
\tag{10}
\]

\[
=: -B_1 \, dz - B_2 \, dt,
\tag{11}
\]

then

\[
d\Sigma + \Sigma \wedge \Sigma = 0.
\tag{12}
\]

This is the isomonodromic condition of the equation

\[
\left( \frac{d}{dz} - B_1 \right)
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = 0.
\tag{13}
\]

\( B_1 \) has poles on \( \{ z \ | \ det A = 0 \} \).

**Lemma 3.1** \( det A = 0 \) is equivalent to the following equation

\[
z^4 ((\alpha_2 + \alpha_3) - \sqrt{-1}X_1) - 2z^3 (X_2 - \sqrt{-1}X_3) + 2z^2 (-2\alpha_1 + \alpha_2 + \alpha_3)
+ 2z (X_2 + \sqrt{-1}X_3) + ((\alpha_2 + \alpha_3) + \sqrt{-1}X_1) = 0,
\tag{14}
\]

where

\[
X_1 = \frac{w_2^2 - w_3^2}{w_2 w_3} \xi_1, \quad X_2 = \frac{w_3^2 - w_1^2}{w_3 w_1} \xi_2, \quad X_3 = \frac{w_1^2 - w_2^2}{w_1 w_2} \xi_3.
\]
For this lemma, generically $B_1$ has four simple poles.

**Theorem 3.2** The anti-self-dual equation on $SU(2)$-invariant metrics generically reduce to a Painlevé VI type equation.

**Remark 3.3** If $z = \zeta$ is a solution of the equation then $z = -1/\bar{\zeta}$ is also a solution. Therefore the equation is compatible with the real structure of twistor space.

**Remark 3.4** The idea of Hitchin [6] is that the lifted action of $SU(2)$ on the twistor space $Z$ gives a homomorphism of vector bundles $\alpha: Z \times su(2)^C \rightarrow T_{\mathbf{Z}}$, and the inverse of $\alpha$ gives a flat meromorphic $SL(2, \mathbb{C})$-connection, which determine isomonodromic deformations. we can think that one-forms $\Theta_1, \Theta_2, \Theta_3$ on $Z$ are infinitesimal variations, therefore we can identify $\Sigma$ with $\alpha^{-1}$.

**Lemma 3.5** Let $g$ be a non-diagonal $SU(2)$-invariant metric. Then (14) has two solutions of order two if and only if there exists a function $f(t)$ satisfying

$$
X_1^2 = 4(f - \alpha_2)(f - \alpha_3),
$$

$$
X_2^2 = 4(f - \alpha_3)(f - \alpha_1),
$$

$$
X_3^2 = 4(f - \alpha_1)(f - \alpha_2).
$$

And then the anti-self-dual equation reduce to (1), (2) and $\dot{f} = f^2$.

Proof.

We can write (14) as

$$
\bar{a}z^4 - \bar{b}z^3 + cz^2 + bz + a = 0,
$$

(15)

where $a, b$ are complex coefficient and $c$ is a real coefficient. By an linear fractional transformation

$$
z \mapsto \frac{(b - |b|)\zeta - b + |b|}{(-b + |b|)\zeta - b + |b|}
$$

(16)

preserving the real structure, we can write (14) as

$$
\zeta^4 - \bar{b}_0\zeta^3 + c_0\zeta^2 + b_0\zeta + 1 = 0,
$$

(17)

where $b_0$ is a complex coefficient and $c_0$ is a real coefficient. Since this equation is also compatible with the real structure, if $\zeta = \zeta_0$ is a solution of order two then $\zeta = -1/\bar{\zeta}_0$ is also a solution of order two. Therefore

$$
\zeta^4 - \bar{b}_0\zeta^3 + c_0\zeta^2 + b_0\zeta + 1 = (\zeta - \zeta_0)^2(\zeta + 1/\bar{\zeta}_0)^2,
$$

(18)

then we have $\zeta_0^2(-1/\bar{\zeta}_0)^2 = 1$ and then $\zeta_0 = \pm\bar{\zeta}_0$, which implies $\zeta_0$ is real or pure-imaginary. Therefore $b_0$ must be real or pure-imaginary. Calculating this condition, we have the Lemma.
4 The Hermitian Structure

Hitchin [3] shows that if a metric is scalar-flat Kähler but not Hyper-Kähler, then the anti-self-dual equation reduce to a Painlevé III type equation. We can interpret this result as the following result.

**Corollary 4.1** If a metric is scalar-flat-Kähler but not hyper-Kähler then the equation (14) has two double zeros.

Therefore we analyze the case (14) has two double zeros.

Let \( z = z(t) \) is a solution of (14). If we restrict \((1,0)\)-forms \( \Theta_1, \Theta_2 \) on \( Z \) to \( z = z(t) \), we have \((1,0)\)-forms on \( M \), which determine an almost complex structure on \( M \). Analyzing this almost complex structure, we have the following result.

**Theorem 4.2** Let \( g \) be an \( SU(2) \)-invariant anti-self-dual scalar-flat metric. There exists a \( SU(2) \)-invariant hermitian structure \((g,I)\) if and only if (14) has solutions of order two.

**proof.**

Let \((g,i)\) be a \( SU(2) \)-invariant hermitian structure. The complex structure \( I \) is determined by \((1,0)\)-forms \( \Theta_1|_{z=z(t)} \) and \( \Theta_2|_{z=z(t)} \), where \( z = z(t) \) is a function on \( M \) depending on \( t \) only. Since the complex structure is integrable, \( \Theta_3|_{z=z(t)} \equiv 0 \) (mod \( \Theta_1|_{z=z(t)}, \Theta_2|_{z=z(t)} \)). Therefore we have

\[
dz_{|z=z(t)} = \left\{ \frac{1}{4} \left( -\alpha_2 + \alpha_3 \pm \sqrt{-1}X_1 \right) z^3 
- \frac{1}{2} \left( \frac{w_3^2}{w_1^2 - w_2^2} X_2 + \frac{w_1^2}{w_1^2 - w_2^2} X_3 \right) z^2 + \frac{1}{2} \frac{X_1 z}{w_2^2 - w_1^2} X_1 z 
- \frac{1}{2} \left( \frac{w_1^2}{w_1^2 - w_2^2} X_2 - \frac{w_2^2}{w_1^2 - w_2^2} X_3 \right) \right\} dt \tag{19}
\]

On the other hand, since \( \Theta_3|_{z=z(t)} \equiv 0 \), \( z = z(t) \) is a solution of (14). Moreover if we substitute \( z = z(t) \) and (19) into the derivative of left hand side of (14), it also becomes zero. Therefore (14) has solutions of order two.

Conversely, let \( z = z_0 \) be a solution of order two, then from lemma 3.1 we have

\[
z_0 = \frac{X_2X_3 \pm \sqrt{X_2^2X_3^2 + X_2^2X_1^2 + X_1^2X_3^2}}{X_1(X_2 - \sqrt{-1}X_3)}, \tag{20}
\]

if \( X_1X_2X_3 \neq 0 \). And then we have \( \Theta_3|_{z=z(t)} \equiv 0 \). Therefore the almost complex structure determined by the \((1,0)\)-forms \( \Theta_1|_{z=z(t)} \) and \( \Theta_2|_{z=z(t)} \) is integrable.

If \( X_1X_2X_3 = 0 \), \( f \) must be \( \alpha_1, \alpha_2 \) or \( \alpha_3 \). Let \( f = \alpha_1 \), then we have \( X_2 = 0 \) and \( X_3 = 0 \), and then

\[
z_0 = \frac{\sqrt{\alpha_3 - \alpha_1} + \sqrt{-1}\sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 + 2\alpha_1}}, \tag{21}
\]

and then \( \Theta_3|_{z=z(t)} \equiv 0 \). In this case the almost complex structure is also integrable.
Theorem 4.3 The hermitian structure \((g,I)\) determined by theorem 4.2 is Kähler if and only if

\[ X_1^2 = 4\alpha_2\alpha_3, \quad X_2^2 = 4\alpha_3\alpha_1, \quad X_3^2 = 4\alpha_1\alpha_2. \] (22)

proof.  
If \(X_1X_2X_3 \neq 0\), the Kähler form is determined by (20) as

\[
\Omega = \frac{X_2X_3}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}}\Omega_1^+ + \frac{X_3X_1}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}}\Omega_2^+ + \frac{X_1X_2}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}}\Omega_3^+.
\]

By the anti-self-dual equations (1),(2),(3), we have

\[
d\Omega = -\frac{2f w_1X_2X_3}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}} dt \wedge \sigma_2 \wedge \sigma_3 + \frac{2f w_2X_1X_3}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}} dt \wedge \sigma_3 \wedge \sigma_1 + \frac{2f w_3X_1X_2}{\sqrt{X_2^2X_3^2 + X_3^2X_2^2 + X_1^2X_2^2}} dt \wedge \sigma_1 \wedge \sigma_2.
\]

Since \(w_1w_2w_3 \neq 0\) and \(X_1X_2X_3 \neq 0\), we have \(d\Omega = 0\) if and only if \(f = 0\).

If \(X_1X_2X_3 = 0\), then \(f\) must be \(\alpha_1\), \(\alpha_2\) or \(\alpha_3\). Let \(f = \alpha_1\), then \(X_1^2 = 4(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1), \quad X_2 = 0, \quad X_3 = 0\). The Kähler form is determined by (21) as

\[
\Omega = \frac{\sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} \Omega_1^+ + \frac{\sqrt{\alpha_3 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} \Omega_2^+.
\] (23)

Then

\[
d\Omega = \frac{2w_1\alpha_1 \sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} dt \wedge \tilde{\sigma}_3 \wedge \tilde{\sigma}_1 + \frac{2w_2\alpha_1 \sqrt{\alpha_3 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 - 2\alpha_1}} dt \wedge \tilde{\sigma}_1 \wedge \alpha_2.
\] (24)

Since the metric is non-diagonal, \(X_1^2 = 4(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) \neq 0\) and then \(d\Omega = 0\) if and only if \(\alpha_1 = 0\).

Remark 4.4 If the metric is scalar-flat Kähler, the anti-self-dual equation reduces to a sixth-order equation.

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