DIFFERENTIAL EIGENFORMS

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Abstract. The aim of this paper is to show how \( \delta \)-characters of Abelian varieties (in the sense of [3]) can be used to construct \( \delta \)-modular forms of weight 0 and order 2 (in the sense of [5]) which are eigenvectors of Hecke operators. These \( \delta \)-modular forms have “essentially the same” eigenvalues as certain classical complex eigenforms of weight 2.

1. Introduction

The concept of \( \delta \)-modular form was introduced in [5]. Very roughly speaking a level one \( \delta \)-modular form of order \( r \) is a “homogeneous” function of plane elliptic curves \( y^2 = x^3 + ax + b \) (where \( a, b \in \mathbb{R} := \mathbb{Z}_p \)) that can be written as a \( p \)-adic restricted power series in \( a, b, \delta a, \delta b, \ldots, \delta^r a, \delta^r b \), \( \Delta^{-1} \), where \( \Delta := 4a^3 + 27b^2 \) and \( \delta^i a, \delta^i b \) are the iterated “Fermat quotients” of \( a, b \) with respect to \( p \). We recall that \( \delta x := (\phi(x) - x^p)/p \), where \( \phi : R \to R \) is the lift of the \( p \)-power Frobenius on \( R/pR \). Morally one may view \( \delta \) as an arithmetic analogue of a derivation (acting on “numbers” rather than “functions”) and one may view \( \delta \)-modular forms as “non-linear arithmetic differential operators of order \( r \)” acting on pairs \( (a, b) \). We shall review this concept presently, from a slightly different (but equivalent) viewpoint. There is a level \( N \) generalization of this. Also there are Hecke operators \( T(l) \) acting on \( \delta \)-modular forms (where \( l \) are primes with \( (l,Np) = 1 \)) so one can talk about \( \delta \)-eigenforms (for all these \( T(l) \)'s). Finally one can attach, to \( \delta \)-modular forms of order \( r \), \( \delta \)-Fourier expansions which are series in the variables \( q, q', \ldots, q^{(r)} \). For applications of our theory we refer to [5], [6].

There is an “easy” way to construct \( \delta \)-eigenforms by considering \( I \)-linear combinations of “\( \phi \)-powers”, \( f^{\phi^i} \), of classical (complex) eigenforms \( f \) where \( I \) is the ring generated by the isogeny covariant \( \delta \)-modular forms (in a sense generalizing that in [5]). A natural question is whether all \( \delta \)-eigenforms can be obtained in this way. As we shall see in this paper the answer is no. Indeed, we provide, in this paper, a construction of \( \delta \)-eigenforms \( f^2 \) of weight 0 and order 2 that have “essentially the same” Hecke eigenvalues as certain classical eigenforms \( f \) of weight 2 (and order 0). As we shall see, forms of weight 0 (such as \( f^2 \)) are never \( I \)-linear combinations of forms \( f^{\phi^i} \). Having constructed the forms \( f^2 \) one can ask, of course, if any \( \delta \)-eigenform is an \( I \)-linear combination of forms \( f^{\phi^i} \) and \( (f^2)^{\phi^i} \); at this point it is not clear what to expect.

The \( \delta \)-Fourier expansion of \( f^2 \) will be related in an interesting way to the Fourier expansion of \( f \). Indeed, if \( f = \sum a_n q^n \) is a (classical) newform of weight 2 on \( \Gamma_0(N) \) (which is not of “CM type”) with Fourier coefficients \( a_n \in \mathbb{Z} \), then the \( \delta \)-Fourier

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expansion of $f^\natural$ will be a series $f^\natural_{\#}(q,q',q'')$ in 3 variables $q, q', q''$ which, after the substitution $q' = q'' = 0$, becomes equal to the series

$$f^{(-1)}(q) := \sum_{(n,p)=1} a_n \frac{q^n}{n}.$$  

(A similar, but more complicated statement holds for $f$ of “CM type”.) The series $f^{(-1)}$ is, of course, not the Fourier series of any (classical) eigenform but, rather, a $p$–adic modular form in the sense of Serre; cf. [21], p. 115. Note that, viewed as a function of elliptic curves in the sense of Katz [16] the $p$–adic modular form $f^{(-1)}$ does not extend across the “supersingular disks” because, if this were the case, $f^{(-1)}$ would define a non-constant function on a projective modular curve. On the other hand, remarkably, the $\delta$–modular form $f^\natural$ does extend across the “supersingular disks” (this being the case with any $\delta$–modular form). One may ask if, in spite of this phenomenon, $f^\natural$ is, nevertheless, a linear combination, with isogeny covariant coefficients defined outside the supersingular disks, of $\phi$–powers of $f^{(-1)}$; we will show that this is not the case.

The idea in our construction of the forms $f^\natural$ is to use the Eichler-Shimura construction for the $f$’s in conjunction with our theory of $\delta$–characters introduced in [3]. Roughly speaking $\delta$–characters are homomorphisms from the group of $R$–points of an Abelian variety to the additive group of $R$ which, in coordinates, are given by expressions involving the coordinates of the points and their iterated Fermat quotients. They are arithmetic analogues of the Manin maps introduced by Manin in the context of the Mordell conjecture over function fields [14]. Then our forms $f^\natural$ will arise by composing certain $\delta$–characters of the modular Jacobians $J_1(N)$ with the Abel-Jacobi maps $X_1(N) \to J_1(N)$ that send a fixed cusp into 0.

Here is the plan of this paper. In Section 2 we review (and slightly extend) the concept of $\delta$–modular form and Hecke operators in [5]. Then we state one of our main results about the existence of the forms $f^\natural$ and their independence from $f$. In Section 3 we review results of Eichler-Shimura and Manin-Drinfeld. Section 4 reviews $\delta$–characters [3] and examines the existence of eigenvectors in the space of $\delta$–characters. In Section 5 we conclude our construction of the forms $f^\natural$; it will turn out that the forms $f^\natural$ “vanish at all the cusps”. In Section 6 we introduce $\delta$–Fourier expansions at $\infty$ and we compute them for our forms $f^\natural$. In Section 7 we use $\delta$–Fourier expansions to prove, in particular, the independence of $f^\natural$ from $f^{(-1)}$. In Section 8 we use $\delta$–Fourier expansions to compute the effect of $\delta$–Serre operators (in the sense of [3]) on $f^\natural$. In Section 9 we prove that the $\delta$–Fourier expansion of $f^\natural$ is in the domain of definition of the (partially defined) Hecke operator $T(p)_{\infty}$ and is an eigenvector of this operator. We end the paper by stating a result (whose proof will be given in a subsequent paper [7]) saying that $\delta$–modular forms which “vanish at the cusps” and are in the domain of definition of $T(p)_{\infty}$ automatically arise from composing $\delta$–characters of the modular Jacobians with Abel-Jacobi maps. This is, in some sense, a converse of our existence results for the forms $f^\natural$ in the present paper.

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2. Main concepts

2.1. Prolongation sequences. Our main reference here is [5]. We fix, throughout this paper, a prime integer \( p \geq 5 \). Let \( C_p(X,Y) \in \mathbb{Z}[X,Y] \) be the polynomial with integer coefficients

\[
C_p(X,Y) := \frac{X^p + Y^p - (X + Y)^p}{p}.
\]

A \( p \)-derivation from a ring \( A \) into an \( A \)-algebra \( \varphi : A \to B \) is a map \( \delta : A \to B \) such that \( \delta(1) = 0 \) and

\[
\begin{align*}
\delta(x + y) &= \delta x + \delta y + C_p(\varphi(x), \varphi(y)) \\
\delta(xy) &= \varphi(x)^p \cdot \delta y + \varphi(y)^p \cdot \delta x + p \cdot \delta x \cdot \delta y,
\end{align*}
\]

for all \( x, y \in A \). Given a \( p \)-derivation we always denote by \( \phi : A \to B \) the map \( \phi(x) = \varphi(x)^p + p\delta x \); then \( \phi \) is a ring homomorphism. A prolongation sequence is a sequence \( S^* \) of rings \( S^n, n \geq 0 \), together with ring homomorphisms \( \varphi_n : S^n \to S^{n+1} \) and \( p \)-derivations \( \delta_n : S^n \to S^{n+1} \) such that \( \delta_{n+1} \circ \varphi_n = \varphi_{n+1} \circ \delta_n \) for all \( n \). We usually denote all \( \varphi_n \) by \( \varphi \) and all \( \delta_n \) by \( \delta \) and we view \( S^{n+1} \) as an \( S^n \)-algbera via \( \varphi \).

A morphism of prolongation sequences, \( u^* : S^* \to S^* \) is a sequence \( u^n : S^n \to S^n \) of ring homomorphisms such that \( \delta \circ u^n = u^{n+1} \circ \delta \) and \( \varphi \circ u^n = u^{n+1} \circ \varphi \).

Let \( W \) be the ring of polynomials \( \mathbb{Z}[\phi] \) in the indeterminate \( \phi \). Then, for \( w = \sum a_i \phi^i \in W \), we set \( \text{deg}(w) := \sum a_i \). If \( a_r \neq 0 \) we set \( \text{ord}(w) = r \); we also set \( \text{ord}(0) = 0 \). For \( w \) as above (respectively for \( w \in W_+ := \{ \sum b_i \phi^i \mid b_i \geq 0 \} \)), \( S^* \) a prolongation sequence, and \( x \in (S^0)^\times \) (respectively \( x \in S^0) \) we can consider the element \( x^w := \prod_{i=0}^r \varphi^{-1}(\phi)(a_i) \in (S^r)^\times \) (respectively \( x^w \in S^r \)). We let \( W(r) := \{ w \in W \mid \text{ord}(w) \leq r \} \).

Let \( R := \mathbb{Z}_p := \mathbb{Z}[\phi]^\Gamma \) be the completion of the maximum unramified extension of \( \mathbb{Z}_p \). Then \( R \) has a unique \( p \)-derivation \( \delta : R \to R \) given by \( \delta x = (\phi(x) - x^p)/p \) where \( \phi : R \to R \) is the unique lift of the \( p \)-power Frobenius map on \( R/pR \).

One can consider the prolongation sequence \( R^* \) where \( R^n = R \) for all \( n \). By a prolongation sequence over \( R \) we understand a prolongation sequence \( S^* \) equipped with a morphism \( R^* \to S^* \). From now on all our prolongation sequences are assumed to be over \( R \).

2.2. \( \delta \)-modular forms. Our main reference here is, again, [5]. We fix, throughout this paper, an integer \( N \geq 1 \), not divisible by \( p \). For any ring \( S \) let us denote by \( \mathbf{M}(\Gamma_1(N), S) \) the set of all triples \( (E/S, \alpha, \omega) \) where \( E/S \) is an elliptic curve, \( \omega \) is an invertible 1-form on \( E \), and \( \alpha : (\mathbb{Z}/N\mathbb{Z})_S \to E \) is a closed immersion of group schemes (referred to as a \( \Gamma_1(N) \)-level structure). Fix \( w \in W \) with \( \text{ord}(w) \leq r \). A \( \delta \)-modular form of weight \( w \in W \) and order \( r \) on \( \Gamma_1(N) \) is a rule \( f \) that associates to any prolongation sequence \( S^* \) of Noetherian, \( p \)-adically complete rings and any triple \( (E/S^0, \alpha, \omega) \in \mathbf{M}(\Gamma_1(N), S^0) \) an element \( f(E/S^0, \alpha, \omega, S^*) \in S^* \) such that the following properties are satisfied:

1. \( f(E/S^0, \alpha, \omega, S^*) \) depends on the isomorphism class of \( (E/S^0, \alpha, \omega) \) only.
2. Formation of \( f(E/S^0, \alpha, \omega, S^*) \) commutes with base change \( u^* : S^* \to S^* \) i.e.

\[
f(E \otimes_{S^0} \tilde{S}^0/\tilde{S}^0, \alpha \otimes \tilde{S}^0, u^{0*} \omega, \tilde{S}^*) = u^*(f(E/S^0, \alpha, \omega, S^*)).
\]
3. \( f(E/S^0, \alpha, \lambda \omega, S^*) = \lambda^{-w} \cdot f(E/S^0, \alpha, \omega, S^*) \) for all \( \lambda \in (S^0)^\times \).
We denote by $M^r(\Gamma_1(N), R, w)$ the $R$–module of all $\delta$–modular forms over $R$ of weight $w \in W$ and order $r$ on $\Gamma_1(N)$. Then the direct sum

$$M^r(\Gamma_1(N), R, *) := \bigoplus_{w \in W(r)} M^r(\Gamma_1(N), R, w)$$

has a natural structure of graded ring. We view $M^r(\Gamma_1(N), R, *)$ as a subring of $M^{r+1}(\Gamma_1(N), R, *)$ via \( \varphi \) and we have naturally induced ring homomorphisms \( \phi : M^r(\Gamma_1(N), R, *) \to M^{r+1}(\Gamma_1(N), R, *) \) sending any $f \in M^r(\Gamma_1(N), R, w)$ into $f^\phi := \phi \circ f \in M^{r+1}(\Gamma_1(N), R, \phi w)$; for $w = \sum a_i \delta^i \in W_+$ we write $f^w := \prod (f^{\delta^i})^{a_i}$. The rings $M^r(\Gamma_1(N), R, *)$ are integral domains. Their union will be denoted by $M^\infty(\Gamma_1(N), R, *)$.

We end our discussion here by noting that, by [6], p.252, the spaces $M^r(\Gamma_1(N), R, w)$ embed into spaces of \textit{ordinary} $\delta$–modular forms, denoted by

$$M^r_{\text{ord}}(\Gamma_1(N), R, w)$$

and defined exactly as the spaces $M^r(\Gamma_1(N), R, w)$ except that instead of the set $M(\Gamma_1(N), S^0)$ one considers the set $M_{\text{ord}}(\Gamma_1(N), S^0)$ of all tuples in $M(\Gamma_1(N), S^0)$ with ordinary reduction.

2.3. $\delta$–Hecke operators. Again, our main reference here is [5]. Assume $S^*$ is a prolongation sequence of Noetherian, $p$–adically complete rings, and let $\tilde{S}$ be a finite étale over-ring of $S^0$. Then, by [5], (1.6), there is a unique structure of prolongation sequence on $S^* \otimes_{S^0} \tilde{S} := (S^n \otimes_{S^0} \tilde{S})$ compatible (in the obvious sense) with that of $S^*$. Now let $l$ be a prime integer not dividing $Np$. Let $f \in \mathcal{M}(\Gamma_1(N), R, w)$ be a $\delta$–modular form. We can define a $\delta$–modular form $T(l)f \in M^r(\Gamma_1(N), R, w)$ by the formula

$$T(l)f = \sum_{i=0}^l f(\tilde{E}_i \otimes S^* \otimes_{S^0} \tilde{S}),$$

where $\tilde{S}$ is any finite étale over-ring of $S^0$ such that the group scheme of points of order $l$ of $\tilde{E} := E \otimes_{S^0} \tilde{S}$ is isomorphic to $(\mathbf{Z}/l\mathbf{Z})^2$ (hence the elliptic curve $\tilde{E}$ has exactly $l + 1$ finite, flat subgroup schemes $H_0, ..., H_l$ of rank $l$, $\tilde{E}_i = \tilde{E}/H_i$, $u_1 : \tilde{E} \to \tilde{E}$ are the natural projections, and the $u_i \omega$’s are induced by $\omega$ via pullback to $\tilde{E}$ followed by trace to the $\tilde{E}_i$’s). In the above we can always assume $\tilde{S}$ is Galois over $S^0$. Note that $(T(l)f)(E/S^0, \alpha, \omega, S^*)$ which is, a priori, an element of $S^* \otimes_{S^0} \tilde{S}$, actually belongs to $S^*$, and does not depend on the choice of $\tilde{S}$.

We refer to the maps $T(l) : M^r(\Gamma_1(N), R, w) \to M^r(\Gamma_1(N), R, w)$ as $\delta$–Hecke operators. Clearly these maps commute with $\phi$. For $r = 0$ and $w = m \in \mathbf{Z}$ one can normalize our $T(l)$ in the classical fashion by considering the operators

$$T_m(l) := l^{m-1}T(l) : M^0(\Gamma_1(N), R, m) \to M^0(\Gamma_1(N), R, m).$$

2.4. Classical eigenforms. Our main references here are [20, 8]. Denote by $S_m(\Gamma_1(N), \mathbf{C})$ the space of (classical) cusp forms of weight $m$ on $\Gamma_1(N)$ over the complex field $\mathbf{C}$. On this space one has Hecke operators $T_m(n)$ acting, $n \geq 1$. An \textit{eigenform} $f \in S_m(\Gamma_1(N), \mathbf{C})$ is a nonzero element which is a simultaneous eigenvector for all $T_m(n), n \geq 1$. An eigenform $f = \sum_{n \geq 1} a_n q^n$, $a_n = a_n(f)$, is \textit{normalized} if $a_1 = 1$; in this case $T_m(n)f = a_n \cdot f$ for all $n \geq 1$. One associates to any eigenform $f \in S_m(\Gamma_1(N), \mathbf{C})$ its \textit{system of eigenvalues} $l \mapsto a_l$, $(l, N) = 1$. A \textit{newform} is a
The normalized eigenform whose system of eigenvalues does not come from a system of eigenvalues associated to an eigenform in \(S_n(\Gamma_1(M), \mathbb{C})\) with \(M | N, M \neq N\). For any normalized eigenform \(f \in S_n(\Gamma_1(N), \mathbb{C})\) one may consider the subring \(\mathcal{O}_f\) of \(\mathbb{C}\) generated by all \(a_n(f)\), \(n \geq 1\); then \(\mathcal{O}_f\) is a finite \(\mathbb{Z}\)-algebra and one denotes by \(K_f\) its fraction field. If \(Q \geq 1\) is any integer we denote by \(\mathcal{O}_f^{(Q)}\) the subring of \(\mathbb{C}\) generated by all \(a_l(f)\), where \(l\) is prime, not dividing \(Q\).

We will later need to consider the subspace \(S_m(\Gamma_0(N), \mathbb{C})\) of \(S_m(\Gamma_1(N), \mathbb{C})\) of all cusp forms of weight \(m\) on \(\Gamma_0(N)\). Recall that if \(f = \sum a_nq^n\) is an eigenform in \(S_m(\Gamma_0(N), \mathbb{C})\) then

\[
a_{n_1 n_2} = a_{n_1} a_{n_2} \text{ for } (n_1, n_2) = 1
\]

(2.2) \[
a_{l^i - 1} a_l = a_l + l^{m-1} a_{l^i - 2} \text{ for } l \text{ prime, } (l, N) = 1 \text{ and } i \geq 2,
\]

\[
a_{l^i - 1} a_l = a_l \text{ for } l \text{ prime, } l \nmid N \text{ and } i \geq 2.
\]

2.5. \(\delta\)-eigenforms. A non-zero \(\delta\)-modular form \(h \in M^r(\Gamma_1(N), R_p, w)\) is called a \(\delta\)-eigenform if \(T(l)h = \lambda_l \cdot h\), \(\lambda_l \in R_p\), for all primes \(l\) not dividing \(Np\). A \(\delta\)-eigenform is said to belong (outside \(Np\)) to a (classical) normalized eigenform \(f = \sum a_n q^n \in S_m(\Gamma_1(N), \mathbb{C})\) if there exist a (necessarily injective) ring homomorphism \(\chi : \mathcal{O}_f^{(Np)} \rightarrow R_p\) and an integer \(e \in \mathbb{Z}\) such that \(\lambda_l = l^e \chi(a_l)\) for all primes \(l\) not dividing \(Np\). We then say that \(f^2\) belongs to \(f\) with character \(\chi\) and exponent \(e\).

Note that \(\chi, \chi' : \mathcal{O}_f^{(Np)} \rightarrow R_p\) are ring homomorphisms and \(e, e'\) are integers such that \(l^e \chi(a_l) = l^{e'} \chi'(a_l)\) for all primes \(l\) not dividing \(Np\) and assume \(\chi \neq \chi'\). Then clearly \(e \neq e'\). Set \(L := l^{e-e'} \neq 1\) and, since \(\chi \neq \chi'\), one can choose a prime \(l\) not dividing \(Np\) such that \(a_l \neq 0\). Let \(\Phi(t) = t^d + b_1 t^{d-1} + ... + b_d \in \mathbb{Q}[t]\) be the minimal polynomial of \(a_l\) over \(\mathbb{Q}\). Then both \(\chi(a_l)\) and \(L \cdot \chi(a_l)\) are roots of \(\Phi(t)\). Hence \(\chi(a_l)\) is a root of \(\Psi(t) := \Phi(t) - L^{-d} \Phi(Lt) = \sum_{i=1}^d b_i (1 - L^{-i}) t^{d-i}\). Since \(\Psi(t)\) has degree \(\leq d - 1\) we must have \(\Psi(t) = 0\) hence \(\Phi(t) = t\) hence \(a_l = 0\), a contradiction.

2.6. \(\delta\)-eigenforms arising from classical eigenforms. There is an “easy” way to construct \(\delta\)–eigenforms belonging to classical eigenforms \(f\) by taking linear combinations of \("\phi\)–powers of \(f\) with isogeny covariant \(\delta\)–modular forms (in a sense slightly generalizing that in [5]). In what follows we explain this construction. We should point out that the forms \(f^\delta\) mentioned in the introduction will be shown not to be obtainable via this construction.

Let \(F \in M^r(\Gamma_1(N), R, w)\) be a \(\delta\)–modular form of weight \(w = \sum n_i \phi^i\) on \(\Gamma_1(N)\). Assume \(\text{deg}(w) := \sum n_i\) is even. Generalizing the level one definition in [9] we say that \(F\) is isogeny covariant if for any prolongation sequence \(S^*\), any triples \((E_1, \alpha_1, \omega_1), (E_2, \alpha_2, \omega_2) \in M(\Gamma_1(N), S^0)\), and any isogeny \(u : E_1 \rightarrow E_2\) of degree prime to \(p\), with \(\omega_1 = u^* \omega_2\) and \(u \circ \alpha_1 = \alpha_2\), we have

\[
F(E_1, \alpha_1, \omega_1, S^*) = \text{deg}(w)^{-\text{deg}(w)/2} \cdot F(E_2, \alpha_2, \omega_2, S^*).
\]

Example 2.1. By [8], p. 268 and Theorem 8.83, for each \(r \geq 1\) the \(R_p\)–module of isogeny covariant \(\delta\)–modular forms in \(M^r(\Gamma_1(N), R_p, -1 - \phi^r)\) is free of rank one. Following [8] we shall denote by \(f^r = f^r_{\text{crys}}\) a basis of this rank one module. (So the upper \(r\) is an index, not an exponent. Recall from [8] that \(f^r\) is constructed via crystalline cohomology.)
We denote by $\mathcal{I} \subset M^\infty(\Gamma_1(N), R, \ast)$ the multiplicative system of all non-zero isogeny covariant $\delta$–modular forms and by $\mathcal{J} \subset \mathcal{I}$ the multiplicative system generated by all $(f^r)^{\phi^s}$ for $r \geq 1$ and $s \geq 0$. The $R$–linear spans of $\mathcal{I}$ and $\mathcal{J}$ will be denoted by $I$ and $J$ respectively. Then $I$ is a ring, $J$ is a subring of $I$, and it is tempting to conjecture \cite{1, 2} that $J \otimes \mathbb{Q} = I \otimes \mathbb{Q}$.

**Lemma 2.2.** If $F \in M^r(\Gamma_1(N), R, w)$ is isogeny covariant and
\[ G \in M^r(\Gamma_1(N), R, v) \]
is any $\delta$–modular form then, for any prime $l$ not dividing $Np$,
\[ T(l)(F \cdot G) = l^{-\deg(w)/2} \cdot F \cdot T(l)G. \]
In particular, if $G$ is a $\delta$–eigenform belonging to the classical normalized eigenform $f \in S_m(\Gamma_1(N), \mathbb{C})$ with character $\chi$ and exponent $e$ then $F \cdot G$ is a $\delta$–eigenform belonging to $f$ with character $\chi$ and exponent $e - \frac{\deg(w)}{2}$.

**Proof.** This follows from a computation similar to the one in \cite{3}, p.125 (where the case $N = 1$ was treated). \hfill $\square$

Now let $f \in S_m(\Gamma_1(N), \mathbb{C})$, $f = \sum a_nq^n$, be a normalized eigenform of weight $m$ and let $\rho : \mathcal{O}_f[1/N, \zeta_N] \rightarrow R_p$ be any ring homomorphism, where $p$ does not divide $N$. Then, by the “$q$–expansion principle” \cite{10}, pp. 70 and 112, $f$ naturally defines (via $\rho$) a rule $f^\rho$ (compatible with base change and homogeneous of degree $-m$) that attaches to any $R_p$–algebra $S$ and any triple $(E/S, \alpha, \omega) \in \mathcal{M}(\Gamma_1(N), S)$ an element $f^\rho(E/S, \alpha, \omega) \in S$ depending only on the isomorphism class of the triple. (Here it is essential that we have a fixed primitive $N$–th root of unity $\rho(\zeta_N)$ in $R_p$.) Then $f^\rho$ induces a $\delta$–modular form (still denoted by) $f^\rho \in M^0(\Gamma_1(N), R_p, m)$, of weight $m$, defined by the formula
\[ f^\rho(E/S^0, \alpha, \omega, S^*) := f^\rho(E/S^0, \alpha, \omega). \]
The composition $f^{\rho \phi^j} := \phi^j \circ f^\rho$ is a well defined element of $M^j(\Gamma_1(N), R_p, \phi^j w)$. There is an obvious compatibility between the classical and our Hecke operators $T_m(l)$, which yields:
\[ T(l)f^{\rho \phi^j} = (T(l)f^\rho)^{\phi^j} = (l^{1-m}T_m(l)f^\rho)^{\phi^j} = l^{1-m}(T_m(l)f)^{\rho \phi^j} = l^{1-m}\phi^j(\rho(a_l)) \cdot f^{\rho \phi^j} \]
for all primes $l$ not dividing $Np$. So we see that $f^{\rho \phi^j}$ is a $\delta$–eigenform of order $j$ and weight $m\phi^j$ which belongs to $f$ with character $\phi^j \circ \rho$ and exponent $e = 1 - m$.

In particular, according to Lemma 2.2 if $\phi^b \circ \rho = \rho$ for some $b \geq 1$, and if $a \geq 0$, then any $\delta$–modular function of weight $w$ in the $I$–linear span of
\[ \{ f^{\rho \phi^a}, f^{\rho \phi^{a+b}}, f^{\rho \phi^{a+2b}}, f^{\rho \phi^{a+3b}}, \ldots \} \]
is a $\delta$–eigenform belonging to $f$ with character $\phi^a \circ \rho$ and exponent
\[ e = 1 - \frac{m + \deg(w)}{2}. \]
Lemma 2.3. Let \( f \in S_2(\Gamma_1(N), \mathbb{C}) \) be a normalized eigenform and let \( \tilde{f} \) be a non-zero \( \delta \)-modular form of weight 0. Then \( \tilde{f} \) cannot be in the \( I \)-linear span of
\[ \{ f^p, f^{\rho \phi}, f^{\rho \phi^2}, f^{\rho \phi^3}, \ldots \}. \]

Proof. Assume the conclusion is false. We may assume \( \tilde{f} = \sum F_a \cdot f^{\rho \phi^a} \), where \( F_a \) are isogeny covariant of weight \(-2\phi^a\). To get a contradiction we need to check the following:

Claim. For any \( 0 \leq a \leq r \) there are no non-zero isogeny covariant \( \delta \)-modular forms in \( M'(\Gamma_1(N), R_p, -2\phi^a) \).

We fix \( a \) and prove this claim by induction on \( r \). Assume first \( r = a \). If \( a = 0 \) then the claim follows from [6], Proposition 8.75. If \( a \geq 1 \) our claim follows from [6], Theorem 8.83, assertion 2. To perform the induction step assume \( r > a \). Then, by [6], Corollary 8.40 and Proposition 8.75, if \( h \in M'(\Gamma_1(N), R_p, -2\phi^a) \) is isogeny covariant then \( \partial_r h = 0 \) where \( \partial_r \) is the \( \delta \)-Serre operator in loc. cit. This easily implies that \( h \in M'(\Gamma_1(N), R_p, -2\phi^a) \) and we conclude by the induction hypothesis. (The various results in [6] quoted above apply to our situation in view of [6], Proposition 8.22. Note also that the special case \( N = 1 \) of our claim was proved by Barcau [1].)

2.7. Ordinary \( \delta \)-modular forms arising from \( p \)-adic modular forms. The main references here are [21] [16] [12]. Let \( g \in \mathbb{Q}_p[[q]] \) be a \( p \)-adic modular form of weight \( m \in \mathbb{Z} \) in the sense of Serre. Fix a homomorphism \( \rho : \mathbb{Z}N, 1/N \to R_p \). Then \( g \) induces a \( p \)-adic modular form \( g^\rho \) of level \( N \), weight \( m \) and growth 1 in the sense of Katz [10]; cf. [12], Theorem 6.21, p. 158. On the other hand \( g^\rho \) induces an ordinary \( \delta \)-modular form (still denoted by) \( g^\rho \in M^0_{\text{ord}}(\Gamma_1(N), R_p, m) \). So for each \( j \geq 0 \) we may consider the ordinary \( \delta \)-modular form \( g^{\rho \phi^j} \in M^j_{\text{ord}}(\Gamma_1(N), R_p, m\phi^j) \).

In particular, if \( f = \sum a_n q^n \in S_2(\Gamma_1(N), \mathbb{C}) \) is a normalized eigenform of weight 2 with \( a_n \in \mathbb{Z} \) then, by [21], p. 115, the series
\[ f^{(-1)} = \sum_{(n,p) = 1} a_n n q^n \]
is a \( p \)-adic modular form of weight 0 such that \( T_0(l) f^{(-1)} = l^{-1} a_l f^{(-1)} \) for \( l \) prime different from \( p \). It immediately follows that
\[ f^{(-1)\rho \phi^j} \in M^j_{\text{ord}}(\Gamma_1(N), R_p, 0) \]
is, in the obvious sense, an ordinary \( \delta \)-eigenform belonging to \( f \) with exponent 0. So any \( R \)-linear combination of such forms will have the same property.

Note that, if in the definition of isogeny covariant \( \delta \)-modular forms, one replaces \( M(\Gamma_1(N), S^0) \) by \( M_{\text{ord}}(\Gamma_1(N), S^0) \) one obtains the notion of ordinary isogeny covariant \( \delta \)-modular form. Let \( I_{\text{ord}} \) be the multiplicative system of all such forms and let \( I_{\text{ord}} \) be the \( R \)-linear span of \( I_{\text{ord}} \). Then \( I_{\text{ord}} \) is a ring.

Lemma 2.4. Let \( f \in S_2(\Gamma_1(N), \mathbb{C}) \) be a normalized eigenform of weight 2 with \( a_n \in \mathbb{Z} \) and let \( \tilde{f} \) be an ordinary \( \delta \)-modular form of weight 0 which is in the \( I_{\text{ord}} \)-linear span of the set
\[ \{ f^{(-1)}, f^{(-1)\rho \phi}, f^{(-1)\rho \phi^2}, f^{(-1)\rho \phi^3}, \ldots \}. \]

Then \( \tilde{f} \) is in the \( R \)-linear span of this set.
Proof. Let $\tilde{f} = \sum_{j} F_{j} \cdot f^{(-1)}(\cdot)^{\rho_{j}}$, $F_{j} \in I_{\text{ord}}$. Picking the weight 0 components we may assume $F_{j}$ have weight 0. So we are reduced to showing that any ordinary isogeny covariant $\delta$–modular form of weight 0 is a constant in $R$. This follows from [4], Propositions 7.21 (plus the Remark after it) and 7.23.

2.8. The forms $f^{3}$. The main purpose of this paper is to provide a construction of $\delta$–eigenforms $f^{3}$ of weight 0 and order 2 belonging to classical eigenforms $f$ of weight 2. As we shall see, the forms $f^{3}$ will be neither $I$–linear combinations of $\phi$–powers of $f$ nor $I_{\text{ord}}$–linear combinations of $\phi$–powers of $f^{(-1)}$. Here is one of our main results. This result will be complemented by other results later in the paper; cf. Remark 2.10 below.

**Theorem 2.5.** Let $f \in S_{2}(\Gamma_{1}(N), C)$ be a newform of weight 2 on $\Gamma_{1}(N)$, $N > 4$, and let $g := [K_{f} : Q]$. Then, for any sufficiently large prime $p$, and any embedding $\rho : \mathcal{O}_{f}[1/N, \zeta_{N}] \to R_{p}$, there exist $\delta$–eigenforms

$$f_{1}^{3}, \ldots, f_{g}^{3} \in M^{2}(\Gamma_{1}(N), R_{p}, 0),$$

of weight 0 and order 2 on $\Gamma_{1}(N)$, such that

1) $f_{j}^{3}$ belongs to $f$ with exponent 0,

2) $f_{j}^{3}$ is not in the $I$–linear span of $\{f^{p}, f^{p^{2}}, f^{p^{2}}, \ldots\}$,

3) $f_{1}^{3}, \ldots, f_{g}^{3}$ are $R_{p}$–linearly independent.

**Remark 2.6.** 1) Assertion 2 follows directly from Lemma 2.3.

2) As we shall see, $f_{j}^{3}$ themselves should be morally viewed as a kind of “$\delta$–cusp forms” in the sense that they “vanish at the cusps”; cf. Remark 2.1.

3) In case $g = 1$ (i.e. $a_{n}(f) \in Q$, equivalently $a_{n}(f) \in \mathbb{Z}$) the form $f^{3} = f_{1}^{3}$ will be essentially canonically associated to $f$ and its $\delta$–Fourier expansion will be closely related to that of $f$; cf. Theorems 5.3 and 5.4. We will show that $f^{3}$ is not an $R_{p}$–linear combination of $\phi$–powers of $f^{(-1)}$; cf. Theorem 6.1. Also we shall compute the effect of the $\delta$–Serre operators on $f^{3}$; cf. Propositions 8.3 and 8.4.

4) It would be interesting to extend the above Theorem to $f$’s of higher weight.

5) Let $g = 1$, $f^{3} := f_{1}^{3}$, and $0 \leq a < b \leq r$, $r \geq 2$; then the $\delta$–eigenforms $(f^{a})^{\delta^{-a}} \cdot f^{3}$ and $f^{a} \cdot f^{b} \cdot f^{p}$ of order $r$ have the same weight $-\phi^{a} - \phi^{b}$ and belong to $f$ with the same character and same exponent $e = 1$. One can ask if these forms are $R_{p}$–linearly dependent. The answer is negative as one will see in Theorem 7.1.

3. Review of Eichler-Shimura and Manin-Drinfeld

We need to review some basic facts about modular curves. The references for this section are [17, 10, 20, 8]. Fix an integer $N \geq 4$. Then the modular curve

$$(3.1) \quad Y_{1}(N)/\mathbb{Z}[1/N]$$

is the scheme whose $S$–points ($S$ any scheme over $\mathbb{Z}[1/N]$) identify with isomorphism classes of pairs $(E, \alpha)$ where $E/S$ is an elliptic curve and $\alpha : (\mathbb{Z}/N\mathbb{Z})_{S} \to E$ is a closed immersion of group schemes. Recall that $Y_{1}(N)$ is smooth affine of relative dimension one over $\mathbb{Z}[1/N]$, with geometrically irreducible fibers.

Let now $l$ be a prime integer not dividing $N$. There is a scheme

$$(3.2) \quad Y_{1}(N,l)/\mathbb{Z}[1/Nl]$$

whose $S$–points identify with triples $(E, \alpha, H)$ where $(E, \alpha)$ is as above and $H$ is a finite flat subgroup scheme of $E$ of rank $l$. Recall that $Y_{1}(N,l)$ is smooth affine
of relative dimension one over \( \mathbb{Z}[1/Nl] \), with geometrically irreducible fibers. Over \( \mathbb{Z}[1/Nl] \) one can consider the natural projections

\[
\sigma_1, \sigma_2 : Y_1(N, l) \rightarrow Y_1(N)
\]

\[
\sigma_1(E, \alpha, H) = (E, \alpha),
\]

\[
\sigma_2(E, \alpha, H) = (E/H, u \circ \alpha),
\]

(3.3)

where \( u : E \rightarrow E/H \) is the canonical projection. Moreover \( \sigma_1 \) and \( \sigma_2 \) are étale above \( \mathbb{Z}[1/Nl] \). For details on the discussion above see [17] pp. 87, 117, 125, 129, [10], pp. 69-72, [8], pp. 212-213. (For \( \sigma_1 \) we use the convention in [8] rather than that in [10].)

In \( N > 4 \). Recall that \( Y_1(N) \) is an open set in its Deligne-Rapoport compactification \( X_1(N)/\mathbb{Z}[1/N] \) which is a proper smooth scheme. Cf. [10], p. 79. The complex points of \( X_1(N) \) \( \setminus Y_1(N) \) are called the cusps of \( X_1(N) \) and they come from \( \mathbb{Z}[1/\mathbb{N}, \zeta_N] \) points of \( X_1(N) \). As usual we denote by \( J_1(N) \) the Jacobian of \( X_1(N) \) viewed as an Abelian scheme over \( \mathbb{Z}[1/N] \). Let \( X_1(N, l)_C \) be a smooth complete model over \( C \) of the complex curve \( Y_1(N, l)_C \); the morphisms in Equation 3.3 induce morphisms

\[
\sigma_1, \sigma_2 : X_1(N, l)_C \rightarrow X_1(N)_C.
\]

These morphisms induce endomorphisms \( T(l)_* \) of \( J_1(N)_C \) as follows: if \( D \) is a divisor of degree 0 on \( X_1(N)_C \) and \( [D] \) is the point of \( J_1(N)_C \) representing \( D \) then

\[
T(l)_*[D] := [\sigma_2, \sigma_1]D.
\]

The endomorphisms \( T(l)_* \) have models over \( \mathbb{Z}[1/N] \) (arising from Néron model theory); cf. [8]. We will need the following basic construction due to Eichler-Shimura (cf. [8], p. 215, [11], pp. 241-242):

**Theorem 3.1.** (Eichler-Shimura) Let \( f \in S_2(\Gamma_1(N)), C \) be a newform. Then there exists an Abelian variety \( A = A_{Q} \) defined over \( Q \) \( [K_f : Q] \), a ring homomorphism \( \iota : \mathcal{O}_f \rightarrow \text{End}(A_Q) \), and a dominant homomorphism \( \pi : J_1(N)_Q \rightarrow A \) defined over \( Q \) such that the following hold:

1) For all primes \( l \) we have a commutative diagram

\[
\begin{array}{ccc}
J_1(N)_Q & \xrightarrow{T(l)} & J_1(N)_Q \\
\pi \downarrow & & \downarrow \pi \\
A & \xrightarrow{\iota(a_l(f))} & A
\end{array}
\]

2) The image of the pull-back map

\[
\pi^* : H^0(A_C, \Omega) \rightarrow H^0(J_1(N)_C, \Omega) \simeq S_2(\Gamma_1(N))_C
\]

is the \( C \)-linear span of \( \{f_\sigma, | \sigma : \mathbb{Q}_f \rightarrow C \} \).

3) If \( f \in S_2(\Gamma_0(N), C), \), \( g = 1 \), and \( p \) is a sufficiently big prime then \( a_p \) equals the trace of the \( p \)-power Frobenius on the elliptic curve obtained by reducing \( A \mod p \).

**Remark 3.2.** If \( g = 1 \) then \( A \) in the above Theorem is an elliptic curve over \( Q \) (which, by condition 2 in the Theorem, is uniquely determined by \( f \) up to isogeny); we say that \( f \) is of CM type or not of CM type according as \( A \) has CM or does not have CM. Assertion 3 in the above Theorem, appropriately reformulated, holds for
Γ₁(N) and arbitrary g; for our purposes here we will not need this more general statement.

On the other hand we will need the following theorem due to Manin and Drinfeld (cf. [15] or [19], p. 62):

**Theorem 3.3.** (Manin-Drinfeld) If D is a divisor of degree 0 on X₁(N)C supported in the set of cusps then its image [D] in J₁(N)C is a torsion point.

4. δ—CHARACTERS

We start by reviewing some concepts from [3]. A δ—morphism of order r, \( f : X \to Y \), between two \( R \)—schemes is a rule that attaches to any prolongation sequence \( S^* \) of \( p \)—adically complete rings a map \( f_{S^*} : X(S^0) \to Y(S^r) \) which is functorial in \( S^* \). In the special case when \( X \) is smooth over \( R \) and \( Y = \mathbb{A}^1 \) is the affine line any δ—morphism \( f : X \to \mathbb{A}^1 \) is completely determined by the map \( f_{\mathbb{A}^1} : X(R) \to \mathbb{A}^1(R) = R \). We denote by \( \mathcal{O}^r(X) \) the ring of all δ—morphisms \( X \to \mathbb{A}^1 \) of order \( r \). Assume that \( G \) is a smooth group scheme over \( R \). A δ—morphism \( f : G \to \mathbb{A}^1 = \mathbb{G}_a \) for which \( f_{\mathbb{A}^1} : G(R) \to \mathbb{G}_a(R) = R \) is a group homomorphism into the additive group of \( R \) is called a δ—character. We denote by \( \mathbb{X}^r(G) \) the \( R \)—module of δ—characters of \( G \) of order \( r \).

**Theorem 4.1.** [3] Let \( A \) be an Abelian scheme over \( R \) of relative dimension \( g \). Then \( (r−1)g \leq rank_R \mathbb{X}^r(A) \leq rg \).

**Proof.** This is contained in [3], pp. 325-326. □

As explained in [3], the δ—characters \( \psi : A \to \mathbb{G}_a \) of an Abelian variety should be viewed as arithmetic analogues of the Manin maps in [14]; Manin maps are homomorphisms \( A(L) \to L \) defined for any Abelian variety \( A \) over a field \( L \) of characteristic zero equipped with a non-zero derivation \( D : L \to L \). In our theory the role of the derivation \( D \) is played by the \( p \)—derivation \( \delta \).

**Remark 4.2.** For \( N \geq 4 \) the description of points of \( Y₁(N) \) immediately implies that we have an identification

\[
M^r(\Gamma₁(N), R, 0) \simeq \mathcal{O}^r(Y₁(N)_R).
\]

More generally, the spaces \( M^r(\Gamma₁(N), R, w) \) and \( M^r_{\text{ord}}(\Gamma₁(N), R, w) \) identify with the spaces \( M^r_{Y₁(N)_R}(w) \) and \( M^r_{Y₁(N)_R, \text{ord}}(w) \) in [3], p. 251, respectively.

The above suggests the following:

**Definition 4.3.** A δ—modular form \( f \in M^r(\Gamma₁(N), R, 0) \simeq \mathcal{O}^r(Y₁(N)_R) \) is called δ—holomorphic if it lies in the image of \( \mathcal{O}^r(X₁(N)_R) \to \mathcal{O}^r(Y₁(N)_R) \). If \( \rho : \mathbb{Z}[1/N, \zeta_N] \to R \) is an embedding then the cusps of \( X₁(N)_R \) (with respect to \( \rho \)) are the \( R \)—points of \( X₁(N)_R \) obtained as images, via \( \rho \), of the cusps of \( X₁(N)_\mathbb{C} \) (viewed as \( \mathbb{Z}[1/N, \zeta_N] \)—points). A δ—modular form \( f \in M^r(\Gamma₁(N), R, 0) \) is a δ—cusp form (with respect to \( \rho \)) if it is δ—holomorphic and it vanished at all the cusps of \( X₁(N)_R \) (with respect to \( \rho \)).

**Remark 4.4.** For the next Proposition and its proof it is useful to introduce some terminology and record some facts about it. If \( K \) is a field, \( V \) is an \( n \)—dimensional \( K \)—linear space, and \( T \in \text{End}(V) \) then, by the eigenvalues of \( T \) on \( V \) we mean
the eigenvalues (in an algebraic closure $K^a$ of $K$) of any matrix in $Mat_n(K)$ representing $T$. If $W \subset V$ is a subspace with $TW \subset W$ then all eigenvalues of $T$ on $W$ and all eigenvalues of $T$ on $V/W$ are also eigenvalues of $T$ on $V$. If $L$ is a field extension of $K$ we say that $T \in \text{End}(V)$ is diagonalizable over $L$ if $V \otimes_K L$ has an $L$-basis consisting of eigenvectors of $T$. More generally, $T_1, \ldots, T_g \in \text{End}(V)$ are said to be simultaneously diagonalizable over $L$ if $V \otimes_K L$ has an $L$-basis consisting of common eigenvectors of $T_1, \ldots, T_g$. We will use the following trivial facts:

1) $T$ is diagonalizable over $K^a$ if and only if the minimal polynomial of $T$ on $V \otimes_K K^a$ has simple roots only.

2) If $T$ is diagonalizable over $K^a$ and all its eigenvalues are in $K$ then $T$ is diagonalizable over $K$.

3) If $T_1, \ldots, T_g$ are pairwise commuting and each of them is diagonalizable over $K$ then $T_1, \ldots, T_g$ are simultaneously diagonalizable over $K$.

**Proposition 4.5.** Let $A/R$ be an Abelian scheme of relative dimension $g$ and $\tau_1, \ldots, \tau_s \in \text{End}(A/R)$ be commuting endomorphisms each of which annihilates a polynomial with $\mathbb{Z}$-coefficients with simple complex roots only. Let $T$ be the subring of $\text{End}(A/R)$ generated by $\tau_1, \ldots, \tau_s$ and let $K$ be the fraction field of $R$. Assume there exist $R$-linearly independent $\delta$-characters $\psi_1, \ldots, \psi_g : A \to \mathbb{G}_a$ of order $2$ and ring homomorphisms $\chi^1, \ldots, \chi^g : T \to R$ such that $\psi_j \circ \tau = \chi_j(\tau) \cdot \psi_j$ for all $\tau \in T$, $j = 1, \ldots, g$.

**Proof.** Let the formal group $A^\text{for}$ of $A$ be identified with $\text{Spf} \ R[[x]]$ where $x = \{x_1, \ldots, x_g\}$ are some variables. Let $L = (L_1, \ldots, L_g) \in K[[x]]^g$ be the logarithm of the formal group law of $A$ with respect to $x$. Let $x'$ and $x''$ be additional $g$-tuples of variables and let $K[[x]] \overset{\phi}{\to} K[[x', x'']] \overset{\delta}{\to} K[[x', x''']]$ be the ring homomorphisms, extending $\phi : R \to R$, defined by $\phi(x) = x' + px'$, $\phi(x') = (x')^p + px''$. By Theorem E.1 we have $\text{rank}_K \mathbf{X}^2(A) \geq g$. Note that $\text{End}(A/R)^{op}$ acts on $\mathbf{X}^2(A)$ (and hence on $\mathbf{X}^2(A) \otimes K$) by the formula $(\tau, \psi) \mapsto \tau^* \psi := \psi \circ \tau$, $\tau \in \text{End}(A/R)$. On the other hand $\text{End}(A/R)^{op}$ naturally acts on $R[[x]]$ via $(\tau, x) \mapsto \tau^* x$ and one can extend this action uniquely to an action on $K[[x', x''']] = K[[x, \phi(x), \phi^2(x)]]$ such that the induced endomorphisms $\tau^*$ on the latter ring satisfy $\phi(\tau^* G) = \tau^* \phi(G)$ for all $G \in K[[x', x'']]$. By 3, Lemma 2.8, there exists an injective $K$-linear $\text{End}(A/R)^{op}$-equivariant map $\mathbf{X}^2(A) \otimes K \to K[[x', x''']]$ whose image lies in the $K$-linear space

\begin{equation}
V := \sum_{i=1}^g (K \cdot L_i + K \cdot \phi(L_i) + K \cdot \phi^2(L_i)).
\end{equation}

The logarithm $L : A^\text{for}_K \to (\mathbf{G}_a^\text{for})^g$ is an isomorphism of formal groups over $K$ hence, in particular, the endomorphisms $\tau_i : A^\text{for}_K \to A^\text{for}_K$ induce, via $L$, endomorphisms of $(\mathbf{G}_a^\text{for})^g$, hence matrices $M(\tau_i) = (m_{js}(\tau_i)) \in Mat_g(K)$ satisfying

\begin{equation}
\tau_i^* L_j = \sum_{s=1}^g m_{js}(\tau_i)L_s.
\end{equation}
On the other hand, taking the Lie functor we get commutative diagrams

\[
\begin{align*}
\text{Lie}(A_K/K) & \simeq \text{Lie}(A_K^{\text{for}}) \\
\downarrow \varphi & \downarrow \phi \\
\text{Lie}(A_K/K) & \simeq \text{Lie}(G_{a,K}^{\text{for}})
\end{align*}
\]

showing that the endomorphisms \( \varphi \) on \( \text{Lie}(A_K/K) \) can be represented by the matrices \( M(\tau_i) \). Since all eigenvalues of \( \varphi \) on \( \text{Lie}(A_K/K) \) lie in \( K \) the same is true about the eigenvalues of the matrices \( M(\tau_i) \). On the other hand Equation 4.9 implies that

\[
\tau_i^*(\phi^e(L_j)) = \sum_{s=1}^g \phi^e(m_{js}(\tau_i))\phi^e(L_s)
\]

for \( e = 1, 2 \). Hence \( \tau_i \) act on the space \( V \) in Equation 4.2 via a matrix of the form

\[
\begin{bmatrix}
M(\tau_i) & 0 & 0 \\
0 & \phi M(\tau_i) & 0 \\
0 & 0 & \phi^2 M(\tau_i)
\end{bmatrix}
\]

The above matrices have all their eigenvalues in \( K \). We deduce that all the eigenvalues of \( \tau_i \) on \( X^2(A) \otimes K \) are in \( K \). On the other hand, since each \( \tau_i \) annihilates a polynomial with \( \mathbb{Z} \)-coefficients having simple complex roots only, the same will be true about \( \tau_i \) acting on \( V \). Hence the minimal polynomial of \( \tau_i \) on \( V \otimes K \) has simple roots only so the minimal polynomial of \( \tau_i \) acting on \( X^2(A) \otimes K \) has simple roots only. So \( \tau_i \) acting on \( X^2(A) \otimes K \) is diagonalizable over \( K \) (by Remark 4.4 assertion 1) hence over \( K \) (by Remark 4.4 assertion 2). Then, by Remark 4.4 assertion 3, \( \tau_1, ..., \tau_s \) acting on \( X^2(A) \otimes K \) are simultaneously diagonalizable over \( K \). So there exist at least \( g \) \( K \)-linearly independent \( \delta \)-characters \( \psi_1, ..., \psi_g \in X^2(A) \otimes K \) such that \( \tau_i^* \psi_j = \lambda_{ij} \cdot \psi_j, \lambda_{ij} \in K \). Multiplying \( \psi_j \) by a power of \( p \) we may assume \( \psi_j \in X^2(A) \). Now, since \( \text{End}(A/R) \subset Mat_{2g}(\mathbb{Z}) \), \( \tau_i \) are integral over \( \mathbb{Z} \) hence so are the \( \lambda_{ij} \)'s. Since \( R \) is integrally closed, \( \lambda_{ij} \in R \). Clearly then, for \( \tau \in \mathcal{T} \), we have \( \tau^* \psi_j = \chi^1(\tau) \cdot \psi_j \) for some \( \chi^1(\tau) \in R \) and \( \chi^1 \) defines a ring homomorphism \( \mathcal{T} \to R \).

\[
5. \text{ Proof of Theorem 2.5} \]

Fix a cusp \( P^0 \) of \( X_1(N)_C \) and consider the morphisms

\[
X_1(N)_C \xrightarrow{\beta} J_1(N)_C \xrightarrow{\pi} A_C,
\]

where \( \beta \) is the Abel-Jacobi map sending \( P^0 \to 0 \) and \( A \) is as in Theorem 3.1. Let \( M \in \mathbb{Z} \) be divisible by \( N \) such that the embedding \( \iota : \mathcal{O}_F \to \text{End}(A) \), all the cusps of \( X_1(N)_C \) and all the objects and morphisms in Equation 4.1 have (compatible) models over \( \mathbb{Z}[1/M, \zeta_N] \).

Let \( p \in \mathbb{Z} \) be any prime not dividing \( M \) and unramified in \( F := \mathbb{K}_f(\zeta_N) \) where \( \mathbb{K}_f \) is the normal closure of \( K_f \) in \( \mathbb{C} \). Assume we are given an embedding \( \rho : \mathcal{O}_F[1/N, \zeta_N] \to R_p = \mathbb{Z}_p^{nr} \). The latter can be lifted to an embedding \( \tilde{\rho} : \mathcal{O}_F[1/M] \to R_p \) where \( \mathcal{O}_F \) is the ring of integers of \( F \). Then the morphisms in Equation 5.1 induce (by base change via \( \tilde{\rho} \)) morphisms of \( R_p \)-schemes

\[
X_1(N)_{R_p} \xrightarrow{\beta} J_1(N)_{R_p} \xrightarrow{\pi} A_{R_p},
\]
Select primes $l_1, \ldots, l_s$ not dividing $Np$ such that the endomorphisms $T(l_1)_*, \ldots, T(l_s)_*$ of $J_1(N)/\mathcal{Z}[1/N]$ generate the subring

$$\mathbb{Z}[T(l')_* \mid l' \text{ prime not dividing } Np]$$

of $\text{End}(J_1(N)/\mathcal{Z}[1/N])$. Let $T^{(Np)}$ be the subring of $\text{End}(A_{R_p}/R_p)$ generated by $\iota(a_{l'})$ with $l'$ prime not dividing $Np$. Clearly $T^{(Np)}$ is generated as a ring by $\iota(a_{l_1}), \ldots, \iota(a_{l_s})$.

**Claim.** Each of the maps $d(\iota(a_{l_1})), \ldots, d(\iota(a_{l_s})) \in \text{End}(\text{Lie}(A_{K_p}/K_p))$ has all its eigenvalues in $K_p$, the fraction field of $R_p$.

Indeed an eigenvalue of $d(\iota(a_{l_1}))$ on $\text{Lie}(A_{K_p}/K_p)$ is also an eigenvalue of $d(\iota(a_{l_1}))$ on $\text{Lie}(A_Q/Q)$ hence on $H^0(A_C, \Omega)$. By the Eichler-Simura Theorem\(^\text{[24]}\) the latter identifies with the $\mathbb{C}$–linear span $V_f$ of $\{f^\sigma \mid \sigma : K_f \to \mathbb{C}\}$ and the action of $d(\iota(a_{l_1}))$ corresponds to the action on $V_f$ of $T_2(l_i)$. But the eigenvalues of $T_2(l_i)$ on $V_f$ are clearly in $K_f$ and our Claim is proved.

Now each $a_{l_i}$, being an element of the field $K_f$, annihilates a polynomial with $\mathbb{Z}$–coefficients having simple complex roots only. Hence the same is true about $\iota(a_{l_i})$. By Proposition\(^\text{[25]}\) there exist $g = [K_f : Q] R_p$–linearly independent $\delta$–characters of order 2,

$$\psi_1, \ldots, \psi_g : A_{R_p} \to \mathbb{G}_{a,R_p} = \mathbb{A}_{R_p}^1$$

and ring homomorphisms $\chi^1, \ldots, \chi^g : T^{(Np)} \to R_p$ such that for any prime $l'$ not dividing $Np$ we have

$$\psi_j \circ \iota(a_{l'}) = \chi^j(\iota(a_{l'})) \cdot \psi_j, \quad j = 1, \ldots, g.$$

We may then consider the $\delta$–morphisms of order 2,

$$f^j_{\delta} := \psi_j \circ \pi \circ \beta : X_1(N)_{R_p} \to \mathbb{A}_{R_p}^1, \quad j = 1, \ldots, g.$$

Their restrictions to $Y_1(N)_{R_p}$ can be viewed as $\delta$–modular forms of weight 0.

Let now $F'$ be a number field containing $F$, let $v'$ be a valuation on $F'$ above $p$ and let $P$ be any $\mathcal{O}_{F',v'}$–point of $Y_1(N)$ where $\mathcal{O}_{F',v'}$ is the valuation ring of $F'$ at $v'$. Since $\mathcal{O}_F[1/M] \subseteq \mathcal{O}_{F',v'}$ all cusps of $X_1(N)$ are $\mathcal{O}_{F',v'}$–points and $T(l)_*$ has a model over $\mathcal{O}_{F',v'}$. So one may write

$$\sigma_2, \sigma_1^* P = \sum_{i=1}^{l+1} P_i, \quad \sigma_2, \sigma_1^* P^0 = \sum_{i=1}^{l+1} P^0_i,$$

where $P_i$ are distinct $\mathcal{O}_{F',v'}$–points of $X_1(N)$, $\mathcal{O}_{F',v'}$ an unramified finite extension of $\mathcal{O}_{F',v'}$, and $P^0_i$ are (a priori not necessarily distinct) cusps of $X_1(N)$ so they are $\mathcal{O}_{F',v'}$–points. So, for $l$ prime, not dividing $Np$, Equation\(^\text{[26]}\) yields

$$T(l)_*(\beta(P)) = \sum_{i=1}^{l+1} \beta(P_i) - \sum_{i=1}^{l+1} \beta(P^0_i).$$

Choose now an embedding $\mathcal{O}_{F',v'} \to R_p$ extending the fixed embedding $\mathcal{O}_F[1/M] \to R_p$ above; in particular the points $P_i$ and $P^0_i$ can be viewed as $R_p$–points of $X_1(N)$. 


By Theorem 5.3, $\beta(P_i^0)$ are torsion points hence their image via $\psi_j \circ \pi$ is 0. Using this and the fact that $\psi_j$ and $\pi$ are homomorphisms, we get, by Equation 5.5:

$$\psi_j \pi T(l)_*(\beta(P)) = \psi_j \pi(\sum_{i=1}^{l+1} \beta(P_i) - \sum_{i=1}^{l+1} \beta(P_i^0))$$

$$= \sum_{i=1}^{l+1} f_j^i(P_i)$$

$$= (T(l)f_j^i)(P).$$

(5.6)

On the other hand, using Theorem 3.1, we get

$$\psi_j \pi T(l)_*(\beta(P)) = \psi_j(\iota(a_i) \pi \beta(P)) = \chi^l(\iota(a_i)) \cdot f_j^i(P).$$

(5.7)

Equations 5.7 and 5.6 imply that

$$\sum_{i=1}^{l+1} f_j^i(P_i) = (T(l)f_j^i)(P).$$

(5.8)

In the above equality $P$ has coordinates in $\mathcal{O}_{F',v'}$. Since $F'$ and $v'$ are arbitrary, and since, by smoothness, the image of

$$\bigcup_{F',v'} Y_1(N)(\mathcal{O}_{F',v'}) \rightarrow Y_1(N)(R_p)$$

is $p$–adically dense in $Y_1(N)(R_p)$ it follows, by continuity, that Equation 5.8 holds for any $R_p$–point $P$ of $Y_1(N)$. This implies that $T(l)f_j^i = \chi_j(a_i) \cdot f_j^i$ where $\chi_j : \mathcal{O}_f^{(Np)} \rightarrow R_p$ is the composition

$$\mathcal{O}_f^{(Np)} \rightarrow \mathcal{T}^{(Np)} \chi_j \rightarrow R_p.$$

To conclude it is enough to check that $f_1^i, \ldots, f_g^i$ are $R_p$–linearly independent. Assume $\sum c_j f_j^i = 0$, $c_j \in R_p$. For large $n$, the image of the natural map

$$X_1(N)(R_p)^n \rightarrow J_1(N)(R_p)$$

contains all the $R$–points of an open subset $U$ of $J_1(N)$. It follows that the restriction of $(\sum c_j \psi_j) \circ \pi$ to $U(R_p)$ is 0. This immediately implies that $(\sum c_j \psi_j) \circ \pi = 0$. Hence $\sum c_j \psi_j = 0$ which implies $c_j = 0$ for all $j$. This ends our proof.

Remark 5.1. Note that the $\delta$–modular forms $f_j^i$ constructed above are $\delta$–cusp forms (with respect to $\rho$) in the sense of Definition 4.3. Indeed $f_j^i$ come from $\delta$–morphisms

$$\psi \circ \pi \circ \beta : X_1(N)_{R_p} \rightarrow \mathbb{A}_{R_p}^1,$$

so they are $\delta$–holomorphic. Now, by the Manin-Drinfeld Theorem 3.3, the images of the cusps in $J_1(N)(R)$ via the Abel-Jacobi map

$$\beta : X_1(N)(R_p) \rightarrow J_1(N)(R_p)$$

are torsion points. On the other hand

$$\psi \circ \pi : J_1(N)(R_p) \rightarrow \mathbb{G}_{a,R_p}(R_p) = R_p$$

is a homomorphism with torsion free target so it vanishes on torsion points.
6. $\delta$–Fourier expansions

We start by recalling the background of classical Fourier expansions; cf. [10], p. 112. (The discussion in loc. cit. involves the model $X_\rho(N)$ instead of the model $X_\mu(N)$ used here but the two models, and hence the two theories, are isomorphic over $\mathbf{Z}[1/N, \zeta_N]$ cf. [10], p. 113.) There is a point $s_\infty : \mathbf{Z}[1/N, \zeta_N] \to X_1(N)\mathbf{Z}[1/N, \zeta_N]$ arising from the generalized elliptic curve $\mathbf{P}^1_{\mathbf{Z}[1/N, \zeta_N]}$ with its canonical embedding of $\mu_N, \mathbf{Z}[1/N, \zeta_N] \simeq (\mathbf{Z}/N\mathbf{Z})\mathbf{Z}[1/N, \zeta_N]$; the complex point corresponding to $s_\infty$ is the cuspidal point $\Gamma_1(N) \cdot \infty$. The map $s_\infty$ is a closed immersion; denote by $X_1(N)\mathbf{Z}[1/N, \zeta_N]$ the completion of $X_1(N)\mathbf{Z}[1/N, \zeta_N]$ along the image of $s_\infty$. Now consider the Tate generalized elliptic curve

$$\text{Tate}(q)/\mathbf{Z}[1/N, \zeta_N][[q]];$$

it has a canonical immersion $\alpha_{\text{can}}$ of $\mu_N, \mathbf{Z}[1/N, \zeta_N] \simeq (\mathbf{Z}/N\mathbf{Z})\mathbf{Z}[1/N, \zeta_N]$ so there is an induced map $\text{Spec } \mathbf{Z}[1/N, \zeta_N] \to X_1(N)\mathbf{Z}[1/N, \zeta_N]$ (which further composed with $\text{Spec } \mathbf{Z}[1/N, \zeta_N] \to \text{Spec } \mathbf{Z}[1/N, \zeta_N][[q]], q \mapsto 0$, equals $s_\infty$). There is an induced isomorphism

$$(6.1) \quad S_{\text{pf}} \mathbf{Z}[1/N, \zeta_N][[q]] \to \tilde{X}_1(N)\mathbf{Z}[1/N, \zeta_N];$$

There is a canonical 1-form $\omega_{\text{can}}$ on the elliptic curve $\text{Tate}(q)/\mathbf{Z}[1/N, \zeta_N][((q))]$ over $\mathbf{Z}[1/N, \zeta_N][((q))] := \mathbf{Z}[1/N, \zeta_N][[q]][1/q]$ such that the induced map

$$S_2(\Gamma_1(N), \mathbf{C}) \to \mathbf{C}((q)), f \mapsto f_{\infty} := f_{\text{Tate}}(q) := f(\text{Tate}(q)/\mathbf{C}((q)), \alpha_{\text{can}}, \omega_{\text{can}}),$$

has image in $q\mathbf{C}[[q]]$ and is the classical Fourier expansion at the cusp $\Gamma_1(N) \cdot \infty$. (Here we interpret $f$ as a function of triples in the style of [10].)

Next we move to the $\delta$–theory. Fix a prime $p$ not dividing $N$ and fix a homomorphism $\rho : \mathbf{Z}[1/N, \zeta_N] \to R_p$. We may consider the prolongation sequence $S_{\infty}^r$ defined by

$$S_{\infty}^r := R_p((q))[[q', q'', \ldots, q^{(r)}]],$$

where $\cdot$ means $p$–adic completion and $q'$, $q''$, $\ldots$, $q^{(r)}$ are new indeterminates; $S_{\infty}^{r+1}$ is viewed as an $S_{\infty}^r$–algebra via the inclusion and the $p$–derivations $\delta : S_{\infty}^r \to S_{\infty}^{r+1}$ are the unique $p$–derivations satisfying $\delta q^{(i)} = q^{(i+1)}$. Explicitly, they are defined as follows. First extend $\phi : R_p \to R_p$ to ring homomorphisms $\phi : S_{\infty}^r \to S_{\infty}^{r+1}$ by requiring that

$$\phi(q) := q^p + pq', \phi(q') := (q')^p + pq'', \ldots$$

and then define $\delta : S_{\infty}^r \to S_{\infty}^{r+1}$ by

$$\delta F := \frac{\phi(F) - F^p}{p},$$

Finally define the $\delta$–Fourier expansion map

$$E_{\infty, \rho} : M^r(\Gamma_1(N), R_p, w) \to S_{\infty}^r, \quad f \mapsto E_{\infty, \rho}(f) = f_{\infty, \rho},$$

by the formula

$$f_{\infty, \rho} := f_{\infty, \rho}(q, q', \ldots, q^{(r)}) := f(\text{Tate}(q)/S_{\text{pf}}^0, \alpha_{\text{can}}, \omega_{\text{can}}, S_{\infty}^r).$$

Note that if $f \in S_m(\Gamma_1(N), \mathbf{C})$ is a newform and $\rho : \mathcal{O}_f[1/N, \zeta_N] \to R_p$ extends the homomorphism $\rho : \mathbf{Z}[1/N, \zeta_N] \to R_p$ above then, by functoriality, $(f^p)_{\infty, \rho}(q) = (f_{\infty}(q))^p$, where $(f^p)_{\infty, \rho}(q)$ is the $\delta$–Fourier expansion of $f^p \in M^0(\Gamma_1(N), R_p, m)$ and $(f_{\infty}(q))^p$ is the image of the classical Fourier expansion $f_{\infty}(q) \in \mathcal{O}_f[[q]]$ via the
natural map \( \rho : \mathcal{O}_f[[q]] \to R_p[[q]] \). If \( \mathcal{O}_f = \mathbb{Z} \) we simply have \((f_\infty(q))^\rho = f_\infty(q)\).

Also note that the \( \delta \)-Fourier expansion map commutes with \( \phi \) i.e. \((f_\phi)_\infty,\rho = (f_\infty,\rho)_\phi \).

**Lemma 6.1.** \( \delta \)-expansion principle

The \( \delta \)-Fourier expansion map

\[ E_{\infty,\rho} : M^r(\Gamma_1(N), R_p, w) \to S^r_{\infty} \]

is injective, with torsion free cokernel.

**Proof.** This follows from \([4], Proposition 4.43\), by “taking fractions” with denominators powers of a local equation defining the cusp \( \infty \). \( \square \)

For two elements \( F, G \in S^r_{\infty} \) we write \( F \sim G \) if \( F = \lambda \cdot G \) with \( \lambda \in R_p^\times \) and we write \( \bar{F} \sim \bar{G} \) if the images,

\[ \bar{F}, \bar{G} \in S^r_{\infty} \otimes k = k((q))[q', q'', \ldots, q^{(r)}] \]

of \( F \) and \( G \) satisfy \( \bar{F} = c \cdot \bar{G} \) for some \( c \in k^\times \), \( k := R_p/pR_p \). Let

\[ \Psi = \Psi(q, q') := \frac{1}{p} \cdot \log \left( 1 + p \frac{q'}{q''} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} p^{n-1} \left( \frac{q'}{q''} \right)^n \in R_p((q))[[q', \ldots, q^{(r)}]]. \]

Then the \( \delta \)-Fourier expansion of \( f^r = f^r_{\text{crys}} \in M^r(\Gamma_1(N), R_p, -1 - \phi^r) \) is given by:

**Lemma 6.2.** \( \square \)

\[ f^r_{\infty,\rho} \sim \Psi^{\delta r - 1} + p\Psi^{\delta r - 2} + \ldots + p^{r-1} \Psi \in R_p((q))[[q', \ldots, q^{(r)}]]. \]

In particular

\[ \overline{f^r_{\infty,\rho}} \sim \left( \frac{q'}{q''} \right)^{p^{r-1}} \in k((q))[q', \ldots, q^{(r)}]. \]

Next we would like to compute the \( \delta \)-Fourier expansion of the forms \( f^t = f^t_1 \) in Theorem 2.5 for \( g = 1 \).

**Theorem 6.3.** Let \( f = \sum a_nq^n \in S_2(\Gamma_0(N), \mathbb{C}) \) be a newform with \( K_f = \mathbb{Q} \) which is not of CM type. Then, for any sufficiently large prime \( p \) and any embedding \( \rho : \mathbb{Z}[1/N, \zeta_N] \to R_p \) there is a unique \( \delta \)-eigenform \( f^t \in M^2(\Gamma_1(N), R_p, 0) \) with \( \delta \)-Fourier expansion:

\[ f^t_{\infty,\rho}(q, q', q'') = \frac{1}{p} \sum_{n \geq 1} a_n \left( q^{n \phi^2} - a_p q^{n \phi} + p q^n \right) \in R_p((q))[[q', q'']] \].

Moreover \( f^t \) is a \( \delta \)-cusp form (with respect to \( \rho \)) belonging to \( f \) with exponent 0 and

\[ f^t_{\infty,\rho}(q, 0, 0) = \sum_{(n,p)=1} a_n q^n \in R_p[[q]]. \]

**Remark 6.4.** 1) Explicitly, in Equation 6.3 we have:

\[ q^{n \phi} = (q^p + pq')^n, \]

\[ q^{n \phi^2} = [(q^p + pq')^p + p(q')^p + p^2 q'']^n. \]

2) Uniqueness of \( f^t \) follows, of course, from the \( \delta \)-expansion principle (Lemma 6.1).
3) The series

\[
\sum_{(n,p)=1} \frac{a_n}{n} q^n
\]

is normalized and has coefficients in \(\mathbb{Z}_{(p)}\) but not all its coefficients are in \(\mathbb{Z}\). Indeed if the latter were the case then any prime \(l \neq p\) would divide \(a_l\). Since, for big enough \(l\), \(a_l\) are the traces of Frobenius of an elliptic curve \(A\) over \(\mathbb{Q}\) taken modulo \(l\) it would follow that \(A\) has supersingular reduction for sufficiently big \(l\), a contradiction. In particular the series \(6.6\) is not a (classical) eigenform.

**Proof.** We begin with a preparatory discussion; in this discussion we will not assume yet that \(f\) is not of CM type (for we will use this discussion later in case \(f\) is of CM type). We place ourselves in the context of the proof of Theorem 2.5. Since \(K_f = \mathbb{Q}\) the field \(F\) in that proof equals \(\mathbb{Q}(\zeta_N)\). Also we let \(\beta : X_1(N)_{\mathbb{C}} \to J_1(N)_{\mathbb{C}}\) be the Abel-Jacobi map that sends the cusp \(P_0\) into \(\infty\) into 0. One can choose \(A\) in Theorem 3.1, and hence in the proof of Theorem 2.5, such that

\[
\Phi := \pi \circ \beta : X_1(N)_{\mathbb{C}} \to A_{\mathbb{C}}
\]

satisfies

\[
\Phi^* \omega_A = c \cdot 2 \pi i \cdot f(z) dz = c \cdot \sum_{n \geq 1} a_n q^{n-1} dq,
\]

where \(\omega_A\) is a 1–form on \(A\) over \(\mathbb{Q}\), \(q = e^{2 \pi i z} \) and \(c \in \mathbb{Q}^\times\). Cf. [9], p. 19. Let \(T\) be an étale coordinate around the origin 0 of \(A\) such that \(T\) vanishes at 0. Let \(L(T) \in \mathbb{Q}[[T]]\) be the logarithm of the formal group of \(A\) associated to \(T\); cf. [22]. Then \(\omega_A = \omega(T) dT \in \mathbb{Q}[[T]] \cdot dT\); replacing \(\omega_A\) by a \(\mathbb{Q}^\times\)–multiple of it we may (and will) assume \(\omega(0) = 0\). Let \(m \in \mathbb{Z}\) be such that \(X_1(N)\), \(A\), \(T\), \(\omega_A\) have (compatible) models over \(\mathbb{Z}[1/m]\). Then \(P_0\) and \(\Phi\) are defined over \(\mathcal{O} := \mathbb{Z}[1/Nm, \zeta_N]\). We have an induced homomorphism \(\Phi^* : \mathcal{O}[[T]] \to \mathcal{O}[[q]]\) and we set \(\Phi^*(T) = \varphi(q) \in \mathcal{O}[[q]]\).

Since \(\omega_A = c \cdot \frac{dt}{T} \cdot dT\) we get

\[
c \cdot \sum_{n \geq 1} a_n q^{n-1} dq = \Phi^* \omega_A = c \cdot \frac{dL}{dT}(\varphi(q)) \cdot \frac{d\varphi}{dq}(q) \cdot dq = c \cdot \frac{d}{dq}(L(\varphi(q))).
\]

Setting \(q = 0\) in the coefficients of \(dq\) we get that \(\frac{d\varphi}{dq}(0) = 1\). Also we deduce that

\[
L(\varphi(q)) = \sum_{n \geq 1} \frac{a_n}{n} q^n.
\]

Now, for \(p\) sufficiently large, \(\rho : \mathbb{Z}[1/N, \zeta_N] \to R_p\) induces a homomorphism \(\rho : \mathbb{Z}[1/Nm, \zeta_N] \to R_p\).

From this point on we assume \(f\) is not of CM type. Since \(A\) is not a CM elliptic curve it follows that \(A_{R_p}\) does not have a lift of Frobenius (i.e. there is no morphism of schemes \(\phi : A \to A\) lifting the morphism \(\text{Spec} R_p \to \text{Spec} R_p\) induced by \(\phi\) such that the reduction \(p\) of \(\phi\) is the \(p\)–power Frobenius on \(A_{R_p} \otimes k\)). Since \(A_{R_p}\) does not have a lift of Frobenius, by [9], Theorem 7.22 and [4], Theorem 1.10, one can assume the \(\delta\)–character \(\psi\) in the proof of Theorem 2.5 gives rise to the series (still denoted by)

\[
\psi = \frac{1}{p} (\phi^2 - a_p \phi + p) L(T) \in R_p[[T]][[T', T'']^*],
\]
where $\phi$ is viewed here as naturally extended to series with $K-$coefficients. Then the $\delta-$Fourier expansion of the form $f^\sharp := f_l^\sharp$ provided by the proof of Theorem 6.5 equals

\[
f_{l,0}^\sharp = \delta^* \psi = \Phi^* \left( \frac{1}{p} (\phi^2 - a_p \phi + p) L(T) \right) = \frac{1}{p} \left( (\phi^2 - a_p \phi + p)(L(T)) \right)(\varphi(q), \delta(\varphi(q)), \delta^2(\varphi(q))) = \frac{1}{p} (\phi^2 - a_p \phi + p) \left( \sum_{n \geq 1} \frac{a_n}{n} q^n \right) = \frac{1}{p} \sum_{n \geq 1} \frac{\phi_n}{n} (q^{n \phi^2} - a_p q^{n \phi} + pq^n)
\]

and our Equation 6.4 follows.

Setting $q' = q'' = 0$ in this equation we get

\[
f_{l,0}^\sharp(q,0,0) = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n p^2} - a_p q^{n p} + pq^n) = \sum_{c \geq 1} c_n q^n,
\]

c_n = \frac{1}{p} \left( \frac{a_{n/p^2}}{n/p^2} - \frac{a_{n/p} a_p}{n/p} + \frac{a_n}{n} \right).
\]

Here, by definition, $a_x = 0$ for $x \in \mathbb{Q} \setminus \mathbb{Z}$. Note that for $p$ not dividing $n$ we get $c_n = a_n/n$. For $n = pm$ with $p$ not dividing $m$ we get

\[
c_n = \frac{a_n}{n} - \frac{a_m a_p}{pm} = 0.
\]

For $n = p^i m$ with $p$ not dividing $m$ and $i \geq 2$ we get, by Equation 2.2, that

\[
c_n = \frac{a_{p^i-2} a_m}{p^{i-1} m} - \frac{a_{p^i-1} a_p}{n} + \frac{a_n}{n} = \frac{a_m}{n} (p a_{p^i-2} - a_{p^i-1} a_p + a_{p^i}) = 0.
\]

Equation 6.6 follows.

In the CM case we have the following:

**Theorem 6.5.** Let $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbb{C})$ be a newform with $K_f = \mathbb{Q}$ which is of CM type (corresponding to an imaginary quadratic field $K$). Then, for any sufficiently large prime $p$ and any embedding $\rho : \mathbb{Z}[1/N, \zeta_N] \to R_p$ there is a unique $\delta-$eigenform $f^\sharp \in M^2(\Gamma_1(N), R_p, 0)$ such that the following hold:

1) If $p$ does not split in $K$ then Equation 6.3 holds (with $a_p = 0$).

2) If $p$ splits in $K$ and if $m$ is the unique root in $p \mathbb{Z}_p^*$ of the polynomial $x^2 - a_p x + p$ then the following equation holds:

\[
f_{l,0}^\sharp(q,q',q'') = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n \phi} - puq^n) \in R_p((q)) \left[ q', q'' \right].
\]

Moreover $f^\sharp$ is a $\delta-$cusp form (with respect to $\rho$), $f^\sharp$ belongs to $f$ with exponent 0, and the following hold:

1') If $p$ does not split in $K$ then Equation 6.6 holds;

2') If $p$ splits in $K$ then

\[
f_{l,0}^\sharp(q,0,0) = -u \cdot \sum_{(m,p)=1} \frac{a_m}{m} \sum_{i \geq 0} u^i q^{mp^i} \in R_p[[q]].
\]
Proof. Let us go back to the preparatory discussion in the proof of Theorem 6.3. In our case here $A$ is an elliptic curve over $\mathbb{Q}$ with CM by an order of $\mathcal{O}_K$. By [23], p. 180, there exists an elliptic curve $A'$ over $\mathbb{Q}$ and an isogeny defined over $\mathbb{Q}$, $A \rightarrow A'$, such that $A'$ has CM by the ring of integers $\mathcal{O}_K$ of $K$. Replacing $A$ by $A'$ we may assume that $A$ itself has CM by $\mathcal{O}_K$. Since $A$ is defined over $\mathbb{Q}$, $K$ has class number one; cf. [23], pp. 118, 121. By standard facts about elliptic curves with CM ([23], pp. 184, 133) we have that for $p$ large enough the following hold:

I) If $p$ does not split in $K$ then $A_{R_p}$ has supersingular reduction, $a_p = 0$, and $A_{R_p}$ doesn't have a lift of Frobenius.

II) If $p$ splits in $K$ then $A_{R_p}$ has ordinary reduction, $a_p \not\equiv 0 \pmod{p}$, and $A_{R_p}$ has a lift of Frobenius.

If we are in case I the argument in the proof of Theorem 6.3 applies. Assume we are in case II. Then, by [4], Theorem 1.10, $\psi$ gives rise to the series (still denoted by)

$$\psi = \frac{1}{p}(\phi - up)L(T) \in R_p[[T]][T']^*.$$  
(6.10)

Then the $\delta$–Fourier expansion of the form $f^2$ provided by the proof of Theorem 2.5 equals

$$f^2_{\infty, \rho} = \Phi^* \psi$$

$$= \Phi^* \left( \frac{1}{p}(\phi - up)L(T) \right)$$

$$= \frac{1}{p}\{(\phi - up)(L(T))\}(\varphi(q), \delta(\varphi(q)))$$

$$= \frac{1}{p}(\phi - up)L(\varphi(q))$$

$$= \frac{1}{p}(\phi - up) \left( \sum_{n \geq 1} \frac{a_n}{n} q^n \right)$$

$$= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^\phi - upq^n)$$

and our Equation 6.8 follows.

Setting $q' = q^m = 0$ in this equation we get

$$f^4_{\infty, \rho}(q, 0, 0) = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{np} - upq^n) = \sum_{c \geq 1} c_n q^n,$$

$$c_n = \frac{a_n/p - ua_n}{n}.$$  

Note that for $p$ not dividing $n$ we get $c_n = -ua_n/n$. For $n = p^i m$ with $p$ not dividing $m$ and $i \geq 1$ we get, by Equation 2.2 that

$$c_n = \frac{a_m}{m} \frac{a_{n/p} - ua_{n/p}}{p^i}.$$  

We claim that

$$a_{n/p} - ua_{n/p} = -p^{i+1},$$

and this will, of course, end the proof of the equality in Equation 6.9. To check the claim note that, for $i = 1$, we get $a_1 - ua_p = -pa^2$. In general we proceed by
induction, using Equations 2.2
\[
    a_{p^i} - ua_{p^{i+1}} = a_{p^i} - u(a_{p^i}a_p - pa_{p^{i-1}})
    = (1 - ua_p)a_{p^i} + upa_{p^{i-1}}
    = -pu^2a_{p^i} + upa_{p^{i-1}}
    = up(a_{p^{i-1}} - ua_p)
    = -p^{i+1}u^{i+2},
\]
and our claim is proved. \(\square\)

7. Independence of \(f^2\) from \(f\) and \(f(-1)\)

Using \(\delta\)-Fourier expansions it is possible to prove a variant of the result on the independence of \(f^2\) from \(f\) contained in assertion 2 of Theorem 2.5 and also a result on the independence of \(f^2\) from \(f(-1)\).

**Theorem 7.1.** Let \(f \in S_2(\Gamma_0(N), \mathbb{C})\) be a newform of weight 2 on \(\Gamma_0(N), N > 4\), with \(K_f = \mathbb{Q}\). For any sufficiently large prime \(p\) and any embedding \(\rho : \mathbb{Z}[1/N, \zeta_N] \to R_p\) let \(f^2 \in M^2(\Gamma_1(N), R_p, 0)\) be the unique form in Theorems 6.3 or 6.5 according as \(f\) is of non-CM or CM type respectively. Then the following hold

1) There is no \(G \in J\) such that \(G \cdot f^2\) belongs to the \(J\)-linear span of
\[
    \{f^p, f^{p^2}, f^{p^3}, ..., \}.
\]

2) \(f^2\) does not belong to the \(I_{ord}\)-linear span in \(M^r_{ord}(\Gamma_1(N), R_p, 0)\) of
\[
    \{f(-1)p, f(-1)p^2, f(-1)p^3, ..., \}.
\]

**Proof.** We prove assertion 1. Since there are infinitely many primes \(l\) of ordinary reduction for the elliptic curve \(A\) we may choose two such distinct primes \(l_1\) and \(l_2\); so \(a_{l_1}, a_{l_2} \neq 0\). Let \(p\) be a prime which is sufficiently big so that Theorem 6.5 holds in case \(A\) doesn’t have CM (respectively Theorem 6.5 holds in case \(A\) has CM) and, in addition, \(p\) does not divide \(l_1l_2(l_1 - l_2)a_{l_1}a_{l_2}\). Fix a homomorphism \(\rho : \mathbb{Z}[1/N, \zeta_N] \to R_p\) and let \(f^2\) be the unique form in Theorem 6.5 in case \(A\) doesn’t have CM (respectively the unique form in Theorem 6.5 in case \(A\) has CM).

Assume \(A\) doesn’t have CM; the case when \(A\) has CM is entirely similar and left to reader. Assume there exists \(G \in J\) such that \(G \cdot f^2\) is a \(J\)-linear combination of forms \(f^{p^d}\). Hence \(G \cdot f^2\) is a \(R_p\)-linear combination of forms \(F \cdot f^{p^d}\) with \(F \in J\). By taking \(\delta\)-Fourier expansions we get that \(G_{\infty, \rho} \cdot f^2_{\infty, \rho}\) is an \(R_p\)-linear combination of forms \(F_{\infty, \rho} \cdot f^{p^d}_{\infty, \rho}\). Reducing modulo \(p\), setting \(q'' = 0\), and using Lemma 6.2 and Theorem 6.3 we have a congruence mod \(p\) of the form
\[
    \left(\frac{q'}{q^p}\right)^s \left(\sum_{(m, p) = 1} \frac{a_m}{m} q^m + q'B(q, q')\right) \equiv \sum_{j \geq 0} \lambda_{d_j} \left(\frac{q'}{q^p}\right)^j \left(\sum_{m \geq 1} a_m q^m q^p\right)
\]
in \(R_p((q))[q']\), where \(s \geq 0\), \(\lambda_{d_j} \in R_p\), and \(B(q, q') \in R_p[[q]][q']\). Let \(l\) be either \(l_1\) or \(l_2\). Identifying the coefficients of \((q')^s q^{l-ps}\) in the above Equation we get
\[
    \frac{a_l}{T} \equiv \lambda_{0s} \cdot a_l \mod (p)
\]
in \( R_p \). Since \( a_l \not\equiv 0 \mod p \) we get
\[
1 \equiv l \cdot \lambda_0 s \mod (p).
\]
So \( l_1 \equiv l_2 \mod p \), a contradiction.

We prove assertion 2. Assume \( f^{\#} = \sum_{j \geq 0} \lambda_j j^{(-1)\rho \phi^j} \), \( \lambda_j \in I_{ord} \). By Lemma 2.4 we may assume \( \lambda_j \in R \). Looking at \( \delta \)-Fourier expansions we get one of the following equalities:
\[
\frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n \phi} - a_p q^{n \rho} + pq^n) = \sum_{j \geq 0} \sum_{(n,p)=1} \lambda_j \frac{a_n}{n} q^{n \phi^j},
\]
(7.1)
\[
\frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n \phi} - puq^n) = \sum_{j \geq 0} \sum_{(n,p)=1} \lambda_j \frac{a_n}{n} q^{n \phi^j}.
\]
Setting \( q' = q'' = ... = 0 \) and picking out the coefficient of \( q^{n \rho} \) we get \( \lambda_0 = 1 \), \( \lambda_1 = \lambda_2 = ... = 0 \) in the first case and \( \lambda_j = -u^{j+1} \) in the second case respectively. In both situations we clearly get a contradiction. \( \square \)

8. \( \delta \)-SERRE OPERATORS

Recall from [6], p. 255, that the Serre-Katz operators on modular forms [10], p. 169, induce \( R \)-derivations
\[
\partial_j : M^{\#}(\Gamma_1(N), R, \ast) \to M^{\#}(\Gamma_1(N), R, \ast), \quad j \geq 0,
\]
such that if \( w = \sum a_i \phi^i \) then
\[
\partial_j M^{\#}(\Gamma_1(N), R, w) \subset M^{\#}(\Gamma_1(N), R, w + 2\phi^j).
\]
According to [10] (or [6], p. 255) the Ramanujan form defines an element
\[
P \in M_{ord}^0(\Gamma_1(N), R, 2).
\]
(N.B: the \( P \) in [10] is 12 times the \( P \) in [6]; here we are using the \( P \) in [6].) One can consider the \( R \)-derivations
\[
\partial_j^* : M_{ord}^{\#}(\Gamma_1(N), R, \ast) := \bigoplus_{w \in W(r)} M_{ord}^{\#}(\Gamma_1(N), R, w) \to M_{ord}^{\#}(\Gamma_1(N), R, w),
\]
where, for \( w = \sum a_i \phi^i \), the restriction of \( \partial_j^* \) to \( M_{ord}^{\#}(\Gamma_1(N), R, w) \) equals
\[
\partial_j + a_j p^j P \phi^j.
\]
Recall from [6], p. 93, that one also defines \( \partial_{\ast \ast} := \sum_{j \geq 0} p^{-j} \partial_j \). On the other hand one can consider the \( R \)-derivation \( \theta := q \frac{d}{dq} \) on \( S_\infty^0 := R((q)) \). Exactly as in [6], p. 113, there exist unique \( R \)-derivations
\[
\theta_j : \bigcup_{r \geq 0} S_r^0 \to \bigcup_{r \geq 0} S_r^0,
\]
satisfying the properties
\[
(8.1) \quad \theta_j \circ \phi^s = 0, \quad \text{on } S_s^0 \quad \text{for } s \neq j,
\]
\[
\theta_j \circ \phi^j = p^j \cdot \phi^j \circ \theta \quad \text{on } S_\infty^0.
\]

Lemma 8.1. For all \( j \geq 0, r \geq 0, w \in W(r) \), we have an equality of maps
\[
E_{\infty, r} \circ \partial_j^* = \theta_j \circ E_{\infty, r} : M^{\#}(\Gamma_1(N), R, w) \to S_\infty^r.
\]
In particular we have an equality of maps
\[
E_{\infty, r} \circ \partial_j = \theta_j \circ E_{\infty, r} : M^{\#}(\Gamma_1(N), R, 0) \to S_\infty^r.
\]
Proof. Same argument as in [6], p. 259, where the case of Serre-Tate expansions (rather than Fourier expansions) was considered; to make that argument work one uses [16], p. 180.

Remark 8.2. By [6], p. 113, \( \theta_j \) sends each \( R[[q', q^{(r)}]] \) into itself hence induces a \( K \)-derivation (still denoted by) \( \theta_j \) on \( K[[q, q^{(r)}]] \) which still satisfies Equations 8.1 (with \( S_0^\infty \) replaced by \( K[[q]] \)).

In the next two Propositions we compute the effect of \( \partial_j \) on \( f^\sharp \).

Proposition 8.3. Assume the hypotheses and notation of Theorem 6.3. Then

\[
\partial_2 f^\sharp = p f^\phi^2, \quad \partial_1 f^\sharp = -a_p f^\phi, \quad \partial_0 f^\sharp = f.
\]

In particular \( \partial_\ast^\ast f^\sharp = \frac{1}{p} (f^\phi - a_p f^\phi + pf) \).

Proof. By Lemma 8.1, Theorem 6.3, and Remark 8.2 one has:

\[
(\partial_2 f^\sharp)_{\infty, \rho} = \theta_2 (f^\sharp_{\infty, \rho}) = \frac{1}{p} \sum_{n \geq 1} a_n \theta_2 (\phi^2 (q^n)) = \frac{1}{p} \sum_{n \geq 1} a_n p^2 \phi^2 (\theta (q^n)) = p \left( \sum_{n \geq 1} a_n q^n \right) \phi^2 = (pf^\phi^2)_{\infty, \rho}.
\]

By the \( \delta \)-expansion principle (Lemma 6.1) we get \( \partial_2 f^\sharp = pf^\phi^2 \). The other equalities are obtained in the same way.

In a similar way one proves:

Proposition 8.4. Assume the hypotheses and notation of Theorem 6.5. Then the following hold.

1) If \( p \) splits in \( K \) then Equations 8.2 hold (with \( a_p = 0 \)).

2) If \( p \) does not split in \( K \) then

\[
(8.3) \quad \partial_1 f^\sharp = f^\phi, \quad \partial_0 f^\sharp = -uf.
\]

In particular \( \partial_\ast^\ast f^\sharp = \frac{1}{p} (f^\phi - uf) \).

Note that Equations 8.2 and 8.3 together with the condition that \( f^\sharp \) is in \( M^r(\Gamma_1(N), R_p, 0) \) \((r = 1, 2)\) pin down \( f^\sharp \) up to an additive constant in \( R \).

9. The Hecke operator \( T(p)_{\infty} \)

A direct attempt to define the Hecke operator \( T(p) \) on \( \delta \)-modular forms along the lines of Equation 2.1 obviously fails. The “expected” definition of \( T(p) \) on arbitrary series in \( R[[q, q', ..., q^{(r)}]] \) is also easily seen to fail. We will define \( T(p)_{\infty} \) on a certain \( R \)-submodule of \( R[[q, q', ..., q^{(r)}]] \); then \( f^\ast_{\infty, \rho} \) will be in that submodule and will turn out to be an eigenvector for \( T(p)_{\infty} \) with eigenvalue \( a_p(f) \). By considering a slightly different \( R \)-module of series we will show that the \( \delta \)-Fourier expansions \( f^\ast_{\infty, \rho} \) of \( f^r = f^r_{\text{cryst}} \) are also eigenvectors of an appropriate version, \( T(p)_{\infty, 2} \), of \( T(p)_{\infty} \) with eigenvalues \( p(p + 1) \).
**Definition 9.1.** Let $q_1, \ldots, q_m$ be variables and let $S_1, \ldots, S_m$ be the fundamental symmetric polynomials in $q_1, \ldots, q_m$: so

$$S_1 = q_1 + \ldots + q_m, \ldots, S_m = q_1 \ldots q_m.$$ 

A series

$$G \in R[[q_1, \ldots, q_m, \ldots, q_1^{(r)}, \ldots, q_m^{(r)}]]$$

is $\delta-$symmetric if there exists a series

$$G(m) \in R[[q_1, \ldots, q_m, \ldots, q_1^{(r)}, \ldots, q_m^{(r)}]]$$

such that

$$(9.1) \quad G(q_1, \ldots, q_m, q_1^{(r)}, \ldots, q_m^{(r)}) = G(m)(S_1, \ldots, S_m, \ldots, \delta S_1, \ldots, \delta S_m).$$

(The series $G(m)$ is trivially seen to be unique; cf. [7].)

On the other hand, for any series $F \in R[[q_1, \ldots, q^{(r)}]]$ one can define the series

$$\Sigma_m F := \sum_{j=1}^{m} F(q_1, \ldots, q_j^{(r)}) \in R[[q_1, \ldots, q_m, \ldots, q_1^{(r)}, \ldots, q_m^{(r)}]].$$

The series $\Sigma_m F$ is not $\delta-$symmetric in general. But there are important examples when $\Sigma_m F$ is $\delta-$symmetric; for instance we have:

**Lemma 9.2.** Let $T = (T^1, \ldots, T^g)$ be a $g-$tuple of variables, let $F \in R[[T_1, T_2]]^g$ be a formal group law, and let $\psi \in R[[T, \ldots, T^{(r)}]]$ be such that

$$\psi(F(T_1, T_2), \ldots, \delta^r F(T_1, T_2)) = \psi(T_1, \ldots, T_1^{(r)}) + \psi(T_2, \ldots, T_2^{(r)}).$$

in the ring

$$R[[T_1, T_2, \ldots, T_1^{(r)}, T_2^{(r)}]].$$

Let $\varphi(q) \in R[[q]]^g$ be a $g-$tuple of series and let

$$F := \psi(\varphi(q), \ldots, \delta^r(\varphi(q))) \in R[[q, \ldots, q^{(r)}]].$$

Then $\Sigma_m F$ is $\delta-$symmetric for all $m \geq 2$.

**Proof.** An easy exercise. Cf. also [7].

**Corollary 9.3.** If $f_{\infty, \rho}^3$ is an in Theorems 6.3 and 6.5 then $\Sigma_m f_{\infty, \rho}^3$ is $\delta-$symmetric for all $m \geq 2$.

**Proof.** By Equations 6.7 and 6.10 $f_{\infty, \rho}^3$ can be written as

$$\psi(\varphi(q), \ldots, \delta^r(\varphi(q)))$$

with $\psi$ and $\varphi$ as in Lemma 9.2 and we may conclude by Lemma 9.2.

**Definition 9.4.** Let $F \in R[[q_1, \ldots, q^{(r)}]]$ be a series such that $G := \Sigma_m F$ is $\delta-$symmetric. Then define the action of the Hecke operator $T(p)_\infty$ on $F$ by

$$(9.2) \quad (T(p)_\infty F) := F(q^p, \ldots, \delta^r(q^p)) + G(p)(0, \ldots, 0, q, \ldots, 0, q^{(r)}) \in R[[q, \ldots, q^{(r)}]].$$

Morally this should correspond to the Hecke action on $\delta-$expansions of weight (of degree) 0.
Remark 9.5. If $G = \Sigma_p F$ is $\delta-$symmetric then so is $G^\phi = \Sigma_p (F^\phi)$ and we have

\[(G^\phi)_{(p)} = (G_{(p)})^\phi.\]

In particular $T(p)_\infty$ commutes with $\phi$ in the sense that

\[T(p)_\infty (F^\phi) = \phi (T(p)_\infty F)^\phi\]

for any $F$ for which $\Sigma_p F$ is $\delta-$symmetric.

Remark 9.6. If $F = \sum_{n \geq 1} c_n q^n \in R[\{q\}]$ then $\Sigma_p F$ is $\delta-$symmetric and

\[(9.3)\]

\[T(p)_\infty F = \sum_{n \geq 1} c_n q^{np} + p \sum_{n \geq 1} c_npq^n.\]

Indeed note that if

\[(9.4)\]

\[q_1^n + ... + q_p^n = P_n(S_1, ..., S_p)\]

with $P_n$ a weighted homogeneous polynomial with $Z-$coefficients of degree $n$ (with respect to the weights $1, 2, ..., p$) then $P_n(0, ..., 0, q)$ is either $mq^{np}$ (with $m \in Z$) or $0$ according as $p$ divides $n$ or not. In case $n/p \in Z$, specializing $q_i \mapsto \zeta_p$, we get $S_i \mapsto 0$ for $0 \leq i \leq p - 1$ and $S_p \mapsto 1$ so Equation 9.3 yields $p = m$. Equation 9.3 follows. Formula 9.3 is, in some sense, what one would expect the action of $T(p)_\infty$ to yield on series of order $0$ and weight $0$.

Remark 9.7. One can introduce a variant over $K$ of the above definitions. A series

\[G \in K[[q_1, ..., q_m, ..., q_1^{(r)}_m, ..., q_m^{(r)}]]\]

is $K - \delta-$symmetric if there exists a series

\[G_{(m)} \in K[[q_1, ..., q_m, ..., q_1^{(r)}_m, ..., q_m^{(r)}]]\]

such that Equation 9.1 holds. (The series $G_{(m)}$ is, again, trivially seen to be unique; cf. 7.) For any series $F \in K[[q, ..., q^{(r)}]]$ one can define the series $\Sigma_p F \in K[[q_1, ..., q_m, ..., q_1^{(r)}_m, ..., q_m^{(r)}]]$ as before. If $F$ is such that $\Sigma_p F$ is $K - \delta-$symmetric we define $T(p)_\infty F \in K[[q, ..., q^{(r)}]]$ by the formula 9.2. Then Remarks 9.5 and 9.6 hold verbatim with $R$ replaced by $K$ and the words "$\delta-$symmetric" replaced by "$K - \delta-$symmetric".

Definition 9.8. A series $F \in R[[q, ..., q^{(r)}]]$ is an eigenvector for $T(p)_\infty$ with eigenvalue $\lambda \in R$ if $\Sigma_p F$ is $\delta-$symmetric and

\[(9.5)\]

\[T(p)_\infty F = \lambda \cdot F.\]

Proposition 9.9. Let $f = \sum a_n q^n$ and $f^\phi$ be as in Theorems 6.3 and 6.4 respectively. Then $f^\phi_{\infty, \phi}$ is an eigenvector of $T(p)_\infty$ with eigenvalue $a_p = a_p(f)$.

Proof. By Corollary 3.3 $\Sigma_p f^\phi_{\infty, \phi}$ is $\delta-$symmetric. Now we think of $f^\phi_{\infty, \phi}$ as an element of $K[[q, q', q'']]$ or $K[[q, q']]$ respectively and we consider the extension of $T(p)_\infty$ “over $K$” discussed in Remark 9.7. Since $T(p)_\infty$ commutes with $\phi$ it is enough to check that

\[\sum \frac{a_n}{n} q^n\]

is an eigenvector of $T(p)_\infty$ with eigenvalue $a_p$. This can be checked directly as follows. First note that, by Equations 2.2 we have

\[pa_{np} + a_n = a_p a_n, \quad n \geq 1.\]
Then, by Remark 9.6, we have
\[ T(p)_{\infty} \left( \sum \frac{a_n}{n} q^n \right) = \sum \frac{a_n}{n} q^n + p \sum \frac{a_n}{np} q^n \]
\[ = \sum \frac{a_n}{n/p} q^n + \sum \frac{a_n}{n} q^n \]
\[ = \sum \frac{pa_n}{n+p} q^n \]
\[ = a_p \sum \frac{a_n}{n} q^n. \]

\qed

One can develop a variant of \( T(p)_{\infty} \) by allowing it to act on certain series with denominators. We need a preparation. We continue to denote by \( S \) the fundamental symmetric polynomials in \( q_1, \ldots, q_p \) and we let \( s_1, \ldots, s_p \) be variables.

**Lemma 9.10.** Consider the \( R \)-algebras

\[ A := R[[s_1, \ldots, s_p]][s_p^{-1}][s_1', \ldots, s_p', s_1^{(r)}, \ldots, s_p^{(r)}], \]
\[ B := R[[q_1, \ldots, q_p]][q_p^{-1}][q_1', \ldots, q_p', q_1^{(r)}, \ldots, q_p^{(r)}]. \]

Then the natural algebra map

\[ A \to B, \quad s_j^{(i)} \mapsto \delta^i S_j \]

is injective with torsion free cokernel.

**Proof.** Let \( \sigma_1, \ldots, \sigma_{p-1} \) be variables and let \( \Sigma_1, \ldots, \Sigma_{p-1} \in R[S_1, \ldots, S_p] \) be defined by

\[ \Sigma_i := q_1^i + \ldots + q_p^i. \]

Note that

\[ R[\Sigma_1, \ldots, \Sigma_{p-1}, S_p] = R[S_1, \ldots, S_p]. \]

So there is a natural isomorphism \( C \cong A \) where

\[ C := R[[\sigma_1, \ldots, \sigma_{p-1}, s_p]][s_p^{-1}][\sigma_1', \ldots, \sigma_{p-1}', s_p', \sigma_1^{(r)}, \ldots, \sigma_{p-1}^{(r)}], \]

such that the composition \( C \to A \to B \) is given by

\[ \sigma_j^{(i)} \mapsto \delta^i \Sigma_j, \quad s_p^{(i)} \mapsto \delta^i S_p. \]

So it is enough to prove that \( C \otimes k \to B \otimes k \) is injective. We have

\[ C \otimes k = k[[\sigma_1, \ldots, \sigma_{p-1}, s_p]][s_p^{-1}][\sigma_1', \ldots, \sigma_{p-1}', s_p', \sigma_1^{(r)}, \ldots, \sigma_{p-1}^{(r)}], \]
\[ B \otimes k = k[[q_1, \ldots, q_p]][q_p^{-1}][q_1', \ldots, q_p', q_1^{(r)}, \ldots, q_p^{(r)}]. \]

Now the morphism

\[ k[[\sigma_1, \ldots, \sigma_{p-1}, s_p]] \to k[q_1, \ldots, q_p] \]

is finite and flat, and \((q_1, \ldots, q_p)\) is the unique maximal ideal lying over \((\sigma_1, \ldots, \sigma_{p-1}, s_p)\).

Hence the ring homomorphism

\[ k[[\sigma_1, \ldots, \sigma_{p-1}, s_p]] \to k[[q_1, \ldots, q_p]] \]
is faithfully flat, hence injective, so we have an inclusion $L \subset M$ of their fraction fields. It is enough to show that the map

$$L[\sigma'_1, \ldots, \sigma'_{p-1}, \sigma'_p, \ldots, \sigma^{(r)}_1, \ldots, \sigma^{(r)}_p, s_p] \rightarrow M[q'_1, \ldots, q'_p, \ldots, q^{(r)}_1, \ldots, q^{(r)}_p]$$

in injective. We will show (and this will end our proof) that for each $i = 0, \ldots, r$

$$(9.6) \quad \delta^i \Sigma_1, \ldots, \delta^i \Sigma_{p-1}, \delta^i S_p \in R[q_1, \ldots, q_p, \ldots, q^{(i)}_1, \ldots, q^{(i)}_p]$$

in the ring

$$M[q'_1, \ldots, q'_p, \ldots, q^{(i)}_1, \ldots, q^{(i)}_p]$$

are algebraically independent over

$$M[q'_1, \ldots, q'_p, \ldots, q^{(i-1)}_1, \ldots, q^{(i-1)}_p].$$

Now one checks by induction on $i$ that for all $a = 1, \ldots, p-1$,

$$\delta^i \Sigma_a = \sum_{j=1}^p (aq_j^a(q^{p-1}) q_j^i + O(i-1) + pO(i),$$

where

$$O(i) \in R[q_1, \ldots, q_p, \ldots, q^{(i)}_1, \ldots, q^{(i)}_p],$$

and $O(i-1)$ has the corresponding meaning. Similarly one has

$$\delta^i S_p = \sum_{j=1}^p q_j^i (s_p/q_j)^p + pO(i) + pO(i-1).$$

So the images of the polynomials $9.6$ in the ring

$$k[q_1, \ldots, q_p, \ldots, q^{(i)}_1, \ldots, q^{(i)}_p]$$

are (non-homogeneous) linear polynomials in $q^{(i)}_1, \ldots, q^{(i)}_p$ with coefficients in

$$k[q_1, \ldots, q_p, \ldots, q^{(i-1)}_1, \ldots, q^{(i-1)}_p].$$

So we need to check that the matrix of the coefficients of $q^{(i)}_1, \ldots, q^{(i)}_p$ in the reductions mod $p$ of the polynomials $9.6$ is non-singular. But this matrix is the $p^i$-th power of the matrix

$$
\begin{pmatrix}
1 & \cdots & 1 \\
2q_1^p & \cdots & 2q_p^p \\
\vdots & \ddots & \vdots \\
(p-1)q_{p-2}^p & \cdots & (p-1)q_{p-2}^p \\
s_p/q_1 & \cdots & s_p/q_p
\end{pmatrix}
$$

which is clearly non-singular. \hfill \Box

**Definition 9.11.** In the notations of Lemma 9.10, an element $G \in B$ will be called *Laurent $\delta$-symmetric* if it is the image of some element $G_{(p)} \in A$ (which is then unique by Lemma 9.10). For any $F \in R((q))[q', \ldots, q^{(r)}]$ such that

$$\Sigma_p F := \sum_{j=1}^p F(q_j, \ldots, q^{(r)}_j) \in B$$
is Laurent $\delta$-symmetric we may define

$$T(p)_{\infty,m}F := F(q^p, \ldots, \delta^r(q^p)) + p^m G(p)(0, \ldots, 0, q, \ldots, 0, \ldots, 0, q^{(r)}) \in R[[q, \ldots, q^{(r)}]].$$

We write $T(p)_{\infty}F = T(p)_{\infty,0} F$. A series $F \in R((q))[q', \ldots, q^{(r)}]$ is a Laurent eigenvector of $T(p)_{\infty,m}$ with eigenvalue $\lambda \in R$ if $\sum F$ is Laurent $\delta$-symmetric and $T(p)_{\infty,m}F = \lambda \cdot F$. The various values of $m$ morally correspond to the Hecke action on $\delta$-Fourier expansions of $\delta$-modular forms of various weights $w$ (with $\deg(w) = -m$).

**Remark 9.12.** If $F \in R[[q]][q', \ldots, q^{(r)}]$ then one can consider the following conditions:

a) $\Sigma F$ is $\delta$-symmetric;

b) $\Sigma F$ is Laurent $\delta$-symmetric.

A priori none of these conditions seems to imply the other. On the other hand one can trivially see that $f^\sharp$ in Theorems 8.3 and 9.5 is not only $\delta$-symmetric but also Laurent $\delta$-symmetric and a Laurent eigenvector for $T(p)_{\infty}$ with eigenvalue $a_p(f)$.

**Proposition 9.13.** For any $r \geq 1$ the $\delta$–Fourier expansion

$$f^*_{crys} \in R((q))^\wedge[q', \ldots, q^{(r)}]$$

of the $\delta$–modular form

$$f^* = f^*_{crys} \in M^\flat(\Gamma_1(N), R, 0)$$

is a Laurent eigenvector for $T(p)_{\infty,2}$ with eigenvalue $p(p + 1)$.

**Proof.** By Lemma 6.2 it is enough to show that $\Psi^{\phi^i}$ are Laurent eigenvectors for $T(p)_{\infty,2}$ with eigenvalues $p(p + 1)$. Now $\Psi^{\phi^i}$ is Laurent $\delta$–symmetric because

$$\sum_{j=1}^{\ell} \Psi^{\phi^i}(q_j, \ldots, q_j^{(i+1)}) = \phi^i \left( \sum_{j=1}^{\ell} \frac{1}{p} \log \left( 1 + p \frac{q_j}{q_j'} \right) \right)$$

$$= \phi^i \left( \sum_{j=1}^{\ell} \frac{1}{p} \log \frac{\phi(q_j)}{q_j'} \right)$$

$$= \phi^i \left( \frac{1}{p} \log \prod_{j=1}^{\ell} \frac{\phi(q_j)}{q_j'} \right)$$

$$= \phi^i \left( \frac{1}{p} \log \frac{\phi(q)}{q'} \right)$$

$$= \Psi^{\phi^i}(s_p).$$

Moreover, since

$$\Psi^{\phi^i}(q^p, \ldots, q^{i+1}(q^p)) = \phi^i \left( \frac{1}{p} \log \frac{\phi(q^p)}{q^{(i+1)}} \right) = p \cdot \Psi^{\phi^i},$$

we have

$$T(p)_{\infty,2} \Psi^{\phi^i} = p \cdot \Psi^{\phi^i} + p^2 \cdot \Psi^{\phi^i} = p(p + 1) \Psi^{\phi^i}. \quad \square$$

We end by stating a result (to be proved in a subsequent paper) showing that there is an interesting relationship between (Laurent) $\delta$–symmetry and $\delta$–characters. This result is, in some sense, a “converse” of the existence results for $f^\sharp$ in the present paper. We assume, in what follows, that $X_1(N)$ has genus at least 2.
Theorem 9.14. Fix an embedding \( \rho : \mathbb{Z}[1/N, \zeta_N] \to R_p \) and a \( \delta \)-modular form \( G \in M'(\Gamma_1(N), R, 0) \). Assume \( G \) is \( \delta \)-holomorphic, \( G \) vanishes at \( \infty \), and \( \Sigma_p G_{\infty, \rho} \) is either \( \delta \)-symmetric or Laurent \( \delta \)-symmetric. Then \( G = \psi \circ \beta \) where \( \beta : X_1(N)_{R} \to J_1(N)_R \) is the Abel-Jacobi map (corresponding to \( \infty \)) and \( \psi : J_1(N)_R \to G_{a,R} \) is a \( \delta \)-character. (In particular \( G \) is automatically a \( \delta \)-cusp form.)

The case when \( \Sigma_p G_{\infty, \rho} \) is \( \delta \)-symmetric follows directly from the main Theorem of [5]. The case when \( \Sigma_p G_{\infty, \rho} \) is Laurent \( \delta \)-symmetric can be proved in an entirely similar way.

Remark 9.15. Given a classical newform \( f = \sum a_n q^n - \infty \in S_2(\Gamma_0(N), \mathbb{C}) \), an embedding \( \rho : \mathcal{O}_F[1/MN, \zeta_N] \to R_p \) (where the cusps are defined over \( \mathcal{O}_F[1/M] \)), and an embedding \( \chi : \mathcal{O}_f^{(Np)} \to R_p \) one is naturally lead to try to compute the \( R \)-module \( \mathcal{M} = \mathcal{M}(f, \rho, \chi, r, p) \) of all \( \delta \)-modular forms \( G \in \mathcal{M}'(\Gamma_1(N), R_p, 0) \) satisfying the following properties:

1) \( G \) is a \( \delta \)-cusp form (with respect to \( \rho \)),
2) \( G \) belongs to \( f \) (outside \( Np \)) with character \( \chi \) and exponent 0,
3) \( G_{\infty, \rho} \) is an eigenvector (respectively a Laurent eigenvector) of \( T(p)_{\infty} \) with eigenvalue \( \chi(a_p) \).

By Proposition 9.14 and Theorem 4.1 rank \( \mathcal{M} \leq rg_1(N) \) where \( g_1(N) \) is the genus of \( X_1(N)_{\mathbb{C}} \). One should expect a much better bound for rank \( B \). Of course, by Theorems 6.3 and 6.5 plus Proposition 9.9 if \( g = [K_f : Q] = 1 \) then \( f_{\infty, \rho}^{\Sigma} \in \mathcal{M} \).

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