A New Approach to QCD Sum Rules and Inclusive $\tau$ Decay

H. F. Jones$^a$, A. Ritz$^a$, and I.L. Solovtsov$^{ab}$

$^a$ Physics Department, Imperial College,
Prince Consort Rd., London, SW7 2BZ,
United Kingdom

$^b$ Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
Dubna, Moscow Region, 141980,
Russia

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Abstract

We show that renormalization group improvement, making use of analyticity and the structure of the operator product expansion, combined with a non-perturbative expansion method allows one to describe quarkonium states via QCD sum rules without the need for explicit power corrections. This technique also allows an accurate determination of parameters related to the inclusive decay of the $\tau$ lepton.

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*email: h.f.jones@ic.ac.uk
†email: a.ritz@ic.ac.uk
‡email: solovtso@thsun1.jinr.dubna.su
While the QCD sum rules technique [1] provides a powerful method for extracting hadronic characteristics from QCD, there is the drawback that a description of the large distance behaviour via power corrections associated with quark and gluon condensates introduces extra phenomenological parameters. In this Letter we shall argue that the non-perturbative expansion technique of Ref. [2], based on a new small expansion parameter, combined with a particular renormalization group improvement which respects both analyticity and the structure of the operator product expansion, enables us to describe some fundamental hadronic properties without the explicit introduction of condensates. As a consequence of the retention of the analytic properties of the running coupling, which is essentially equivalent to introducing a renormalon contribution, one may also evaluate the $R_\tau$ ratio for the inclusive semi-leptonic decay of the $\tau$-lepton both by integration over the physical region and via use of Cauchy’s theorem [3].

The non-perturbative technique we use involves the so-called “floating” or “variational” series. In quantum mechanics this series has been applied successfully to the anharmonic oscillator [4,5]. The variational series also appears in the linear $\delta$-expansion [6] and in variational perturbation theory [7,8]. In the case of zero- and one-dimensional field theories there exist rigorous proofs of the convergence of such series [9,10]. When one considers QCD, the same ideas lead to an expansion for the generating functional in terms of a new parameter $a$, related to the coupling by [2]

$$\lambda \equiv \frac{g^2}{(4\pi)^2} = \frac{1}{C} \frac{a^2}{(1-a)^3}, \quad (1)$$

where $C$ is a positive constant which can be found from meson spectroscopy. It is clear that for all values of the coupling $\lambda \geq 0$ the expansion parameter $a$ lies in the range $0 \leq a < 1$.

In order to illustrate our technique, let us consider the Adler $D$–function $D(Q^2) = -Q^2 d\Pi/dQ^2$ corresponding to the vector hadronic correlator in the massless case. The two–loop perturbative approximation is given by $D(t,\lambda) = 1 + 4\lambda(\mu^2)$, where $t = Q^2/\mu^2$. Standard renormalization group (RG) improvement leads to the substitution $\lambda(\mu^2) \to \bar{\lambda}(t,\lambda)$, which implies a summation of the leading logarithmic contributions. However, due to the Landau pole of the running coupling at $Q^2 = \Lambda^2_{QCD}$ this substitution breaks the analytic properties of the $D$–function in the complex $q^2 = -Q^2$ plane, namely that the $D$–function should only have a cut on the positive real $q^2$ axis.

We may correct this feature by noting that the above solution of the renormalization group equation is not unique. The general solution is a function of the running coupling $1 + 4\lambda$, for small $\lambda$. To maintain the analytic properties of the $D$–function we can write it as the dispersion integral of $R(s) = (1/\pi)\text{Im}\Pi(s+i\epsilon)$, namely

$$D(t,\lambda) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} R(s, \lambda), \quad (2)$$

and use RG improvement on the integrand rather than $D$ itself. This method leads to $D(t,\lambda) = 1 + 4\lambda_{\text{eff}}(t, \lambda)$, where, with $\tau = s/Q^2$,

$$\lambda_{\text{eff}} = \int_0^\infty \frac{d\tau}{(1+\tau)^2} \frac{\bar{\lambda}(t,\lambda)}{1 + \lambda(t,\lambda)\beta_0 \ln \tau}. \quad (3)$$

This has the Borel representation
\[ \lambda_{\text{eff}}(t, \lambda) = \int_0^\infty db \, e^{-b/\lambda_{\text{eff}}(t, \lambda)} B(b), \tag{4} \]

with

\[ B(b) = \Gamma(1 + b\beta_0) \Gamma(1 - b\beta_0). \tag{5} \]

Here \( \beta_0 = 11 - 2/3N_f \) is the first coefficient of the \( \beta \)-function, and \( N_f \) is the number of active flavours. Thus, in the Borel plane there are singularities at \( b\beta_0 = -1, -2, \ldots \) and \( b\beta_0 = 1, 2, \ldots \) corresponding to ultraviolet and infrared (IR) renormalons respectively.

The first IR singularity at \( b\beta_0 = 1 \) is probably absent since there is no corresponding operator in the operator product expansion. Although this issue is not currently settled, it seems reasonable to assume that the first IR renormalon occurs at \( b = 2/\beta_0 \), and we would like to use this property of the operator product expansion as an additional constraint on the choice of solution to the renormalization group equation. This can be achieved by integrating by parts in Eq. (2), using the fact that to two-loop order \( R(s) \) is a constant, and applying the same RG improvement to the new integrand, to obtain the following expression for \( \lambda_{\text{eff}} \):

\[ \lambda_{\text{eff}}(t, \lambda) = \int_0^\infty d\tau \omega(\tau) \frac{\tilde{\lambda}(kt, \lambda)}{1 + \tilde{\lambda}(kt, \lambda) \beta_0 \ln \tau}, \tag{6} \]

in which the factor \( k \) reflects the renormalization scheme ambiguity and the function

\[ \omega(\tau) \equiv \frac{2\tau}{(1 + \tau)^3} \tag{7} \]

describes the distribution of virtuality usually associated with renormalon chains. The function (7) coincides with the function used in [11] and is numerically very close to that found in [12]. The Borel transform of (6) has the form

\[ B(b) = \Gamma(1 + b\beta_0) \Gamma(2 - b\beta_0). \tag{8} \]

Thus in this representation for \( \lambda_{\text{eff}} \) the positions of all ultraviolet singularities remain unchanged, but the first IR renormalon singularity at \( b = 1/\beta_0 \) is absent.

In summary, a representation for the effective coupling, and consequently the \( D \)-function, which manifests renormalon-type characteristics can be obtained as a particular RG improvement of the lowest order radiative corrections which takes into account both the analytic properties and the structure of the operator product expansion. In order to render Eq. (6) integrable we must combine this method with the non-perturbative \( a \)-expansion of Ref. [4] in which from the beginning the running coupling has no ghost pole. Effectively, the representation for the \( D \)-function obtained in such a way coincides with a technique explicitly introducing power corrections [13], and we can, in principle, describe hadronic parameters using, say, the method of QCD sum rules.

However, retention of the correct analytic properties under RG improvement also allows for the possibility of considering inclusive \( \tau \) decay, and we shall consider this first. Using RG improvement following the procedure described above in the context of the \( a \)-expansion leads to

\[ \lambda_{\text{eff}}(q^2) = -q^2 \int_0^\infty d\sigma \frac{2\sigma}{(\sigma - q^2)^3} \tilde{\lambda}(\sigma), \tag{9} \]
where to leading order

\[ \tilde{\lambda} = \frac{a^2}{C} (1 + 3a) \]  

(10)

The renormalization group defines the running of \(a(\sigma)\) as the solution of the transcendental equation

\[ \sigma = \sigma_0 \exp \left[ \frac{C}{2 \beta_0} (f(a) - f(a_0)) \right], \]  

(11)

where

\[ f(a) = \frac{2}{a^2} - \frac{6}{a} - 48 \ln a - \frac{18}{11} \frac{1}{1 - a} + \frac{624}{121} \ln (1 - a) + \frac{5184}{121} \ln (1 + \frac{9}{2} a). \]  

(12)

The parameter \(a_0\) and the virtuality \(\sigma_0\) in Eq. (11) are defined by some renormalization point for the effective coupling.

We now apply the representation (9) to inclusive \(\tau\)-decay. The starting point is the expression

\[ R_\tau = 2 \int_0^{M^2_{\tau}} \frac{d s}{M^2_{\tau}} \left( 1 - \frac{s}{M^2_{\tau}} \right)^2 \left( 1 + \frac{2s}{M^2_{\tau}} \right) \tilde{R}(s), \]  

(13)

where \(\tilde{R}(s) = R^0_{\tau} R(s)\) and

\[ R^0_{\tau} = 3 \left( |V_{ud}|^2 + |V_{us}|^2 \right) S_{EW} \]  

(14)

in which the electroweak factor \(S_{EW} = 1.0194\) and the CKM matrix elements are \(|V_{ud}| = 0.9753, |V_{us}| = 0.221\), taken from Ref. [15]. Then, in order to isolate the QCD correction to \(R_\tau\), we write \(R_\tau = R^0_{\tau} (1 + \Delta R_\tau)\).

The effective coupling (9) is an analytic function of \(s\) in the complex \(s\)-plane with a cut along the positive real axis. \(\Delta R_\tau\) may be written as the contour integral

\[ \Delta R_\tau = \frac{d_1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} (1 - z)^3 (1 + z) \lambda_{\text{eff}} (M^2_{\tau} z), \]  

(15)

where \(d_1 = 4\) is the 2-loop coefficient of the \(D\)-function.

Substituting Eq. (9) into Eq. (15) and using Cauchy’s theorem we obtain

\[ \Delta R_\tau = 12d_1 \int_0^{M^2_{\tau}} \frac{d s}{M^2_{\tau}} \left( \frac{s}{M^2_{\tau}} \right)^2 \left( 1 - \frac{s}{M^2_{\tau}} \right) \tilde{\lambda}(ks), \]  

(16)

in which the factor \(k\) again parametrizes the renormalization scheme. In what follows we shall always use the \(\overline{MS}\) scheme, in which \(k = \exp(-5/3)\). Note that in comparison with (13), use of the representation (9) for the coupling modifies the kinematic factor so that the maximum now occurs near \(s = (2/3)M^2_{\tau}\).
Taking as input the experimental value of $R_\text{exp}^{\tau} = 3.56 \pm 0.03$ \cite{17}, three active quark flavours and the variational parameter $C = 4.1$ as in \cite{3}, we find

$$\alpha_s(M_\tau^2) = 4\pi \tilde{\lambda}(M_\tau^2) = 0.339 \pm 0.015.$$  \hspace{1cm} \text{(17)}$$

which differs significantly from that obtained ($\alpha_s(M_\tau^2) = 0.40$ in leading order \cite{3}) without the renormalon-inspired representation (9) for the coupling. The new method, applying the matching procedure in the physical $s$–channel and using standard heavy quark masses \cite{20}, leads to $R_Z = 20.90 \pm 0.03$, which agrees well with experimental data \cite{20}.

As mentioned above, the requirement of analyticity naturally generates power corrections. Consequently, it is natural to investigate whether this technique allows us to describe meson parameters using the QCD sum rules approach. Consider, for example, the hadronic correlator $\Pi(s)$ corresponding to the vector current \cite{18}. The imaginary part of $\Pi(s)$ to order $\alpha^3$ in the non-perturbative expansion of Ref. \cite{2} is given by

$$\text{Im}\Pi(s) = \frac{1}{4\pi} \left[ \Pi^{(0)}(s) + 4\tilde{\lambda}\Pi^{(1)}(s) \right],$$  \hspace{1cm} \text{(18)}$$

where $\tilde{\lambda}$ is given by Eq. (10) and for $\Pi^{(0)}$ and $\Pi^{(1)}$ we use the perturbative two-loop expressions given in \cite{19}. It should be stressed that while the $\alpha$-expansion technique makes use of the perturbative coefficients, the structure of the new expansion is fundamentally different. That it is indeed valid to use perturbative formulae in a non-perturbative calculation can be seen by noting that for the $c\bar{c}$ bound states to be considered, the dominant contribution to the moment integrals comes from an energy scale well above $\Lambda_{QCD}$.

The initial expression for the first power moment is given by

$$M_1(Q^2) = \frac{D(Q^2)}{Q^2} \equiv -\frac{d\Pi(Q^2)}{dQ^2} = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\text{Im}\Pi(u)}{(Q^2 + \sigma + 4m^2)^2},$$  \hspace{1cm} \text{(19)}$$

where $u = \sqrt{(\sigma/(\sigma + 4m^2))}$. In analogy with the massless case we shall associate the parameter $\sigma = s - 4m^2$ with virtuality. The zeroth–order contribution is given by

$$M_1^{(0)}(Q^2) = \frac{1}{4\pi^2} \int_0^\infty d\sigma \frac{\Pi^{(0)}(s)}{(Q^2 + \sigma + 4m^2)^2},$$  \hspace{1cm} \text{(20)}$$

while the order $O(\tilde{\lambda})$ correction term associated with $\Pi^{(1)}$ can be rewritten, again in analogy with the massless case, as

$$M_1^{(1)}(Q^2) = \frac{\tilde{\lambda}}{\pi^2} \int_0^\infty d\sigma \frac{2(\sigma + 4m^2)}{(Q^2 + \sigma + 4m^2)^3}\Psi(u),$$  \hspace{1cm} \text{(21)}$$

where

$$\Psi(u) = (1 - u^2) \int_u^\infty dv \frac{2v}{(1 - v^2)^2}\Pi^{(1)}(v).$$  \hspace{1cm} \text{(22)}$$
The structure of the integrand in the expression for \( M_1 \) allows us to simulate a summation of all terms by placing the term proportional to \( \tilde{\lambda} \) in the denominator (cf. the mass correction for the propagator). As a result we have the expression

\[
M_1(Q^2) = \frac{1}{4\pi^2} \int_0^\infty d\sigma \frac{\Pi^{(0)}(u)}{(Q^2 + W(\sigma))^2},
\]

(23)

where, after RG improvement, \( W(\sigma) \) has the form

\[
W(\sigma) = (\sigma + 4m^2(k\sigma)) \left[ 1 - 4\tilde{\lambda}(k\sigma) \frac{\Psi(u)}{\Pi^{(0)}(u)} \right],
\]

(24)

and now \( u = \sqrt{(\sigma/(\sigma + 4m^2(k\sigma)))} \). Note that although this procedure introduces correction terms of order \( \tilde{\lambda}^2 \), it is valid a posteriori, as the dominant contribution to the moment integral comes from the region where \( \tilde{\lambda} \) is small \[21\].

Evaluation of this expression uses the running expansion parameter \( \alpha(\sigma) \) obtained from Eq. (11), while the running mass, \( \bar{m}(\sigma) = m_0(\chi(\sigma)/\chi(\sigma_0))^{\gamma_0/\beta_0} \), has the standard perturbative form, with \( \gamma_0 \) the first coefficient of the anomalous dimension. However, in this case \( \chi(\sigma) \) is expressed via Eq. (1) in terms of the running expansion parameter \( \alpha \). The moments of the vector correlator for general \( n \) are then given by

\[
M_n(Q^2) = \frac{1}{4\pi^2} \int_0^\infty d\sigma \frac{\Pi^{(0)}(u)}{(Q^2 + W(\sigma))^{n+1}}.
\]

(25)

A crucial feature of this approach is that the function \( W(\sigma) \) achieves a minimum at some \( \sigma = \tilde{\sigma} \), and for large \( n \) the dominant contribution comes from this point. Defining the ratio of the moments, \( R_n(Q^2) \equiv M_{n-1}(Q^2)/M_n(Q^2) \), we see that for large \( n \) this ratio tends asymptotically to its saddle–point approximation, i.e.

\[
R_n(Q^2) \xrightarrow{n \to \infty} Q^2 + W(\tilde{\sigma}).
\]

(26)

On the other hand the corresponding phenomenological ratio for mesons, \( R_n^{\mathrm{had}}(Q^2) \), has the form \( Q^2 + M^2 \) at large \( n \), where \( M \) is the mass of the first resonance in the relevant channel. Consequently, we obtain

\[
M = \sqrt{W(\tilde{\sigma})}.
\]

(27)

In \[1\] the moments were considered at \( Q^2 = 0 \), while the case of \( Q^2 \neq 0 \) has also been considered in the literature (see \[19\] for a review). It is an important feature of this approach that the analysis is independent of \( Q^2 \), as the functional form of the \( Q^2 \)-dependence for large \( n \) is the same for both the QCD and phenomenological moment ratios.

Estimates for the \( \bar{c}\bar{c} \) bound state masses for the vector and axial vector channels, corresponding to \( \sqrt{W(\tilde{\sigma})} \) are shown in Fig. \[1\] for values of \( \sigma \) near \( \tilde{\sigma} \). The analysis has been presented thus far for the vector channel only. The calculation in the axial-vector case is similar, the modifications to the moment expressions being standard, and will not be discussed here (see e.g. \[14\]). Keeping \( C = 4.1 \) and taking \( \alpha_0 \) at the \( \tau \) mass scale from Eq. (17) we have only one free parameter, the quark mass \( m_0 \). With \( m_0 = 1.483 \) GeV we obtain a good fit to the experimental masses \[21\] of both \( J/\psi(1S) \) and \( \chi_{c1}(1P) \), as shown in Fig. \[4\].
FIG. 1. A plot of $\sqrt{W(\sigma)}$ versus $\sigma$ for the vector (solid curve), and axial-vector (dashed curve) currents, the minimum being the asymptotic limit of the moment ratios for large $n$. For comparison, the straight lines are the corresponding experimental $c\bar{c}$ bound state masses [20].

Note that the parameter $C = 4.1$ used here was found [2] from a fit to the $\beta$-function and corresponds to the linear growth of the quark-antiquark potential. The present calculation is consistent with this derivation, since the dominant contribution to the moment integrals arises from a region where the system is still reasonably non–relativistic ($u^2 \sim 0.3$).

In conclusion, we have presented in this Letter a technique for obtaining quarkonium and $\tau$–decay parameters which makes use of RG improvement and a non-perturbative expansion which removes the Landau pole in the running coupling. It appears that careful control of RG improvement, to ensure the correct analytic properties, naturally induces power corrections [14] required for the description of meson parameters without the need for their explicit introduction. In the case of QCD sum rules the fact that this technique involves only three parameters, two being fundamental to QCD, i.e. the coupling constant $\alpha_0$ and the quark mass $m_0$ defined at some energy scale, suggests that it is applicable to the study of other channels in the $c\bar{c}$ family and also to consideration of other heavy quarkonium states. Such an extension is currently under investigation and the results will be published elsewhere [22].

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