Dissipative Hofstadter Model at the Magic Points and Critical Boundary Sine-Gordon Model

Seungmuk Ji
Department of Physics, Kangwon National University, Chuncheon 200-701 Korea and
Korea Research Institute of Standards and Science, P.O.Box 102, Yuseong, Daejeon 305-600, Korea

Ja-Yong Koo
Korea Research Institute of Standards and Science, P.O.Box 102, Yuseong, Daejeon 305-600, Korea

Taejin Lee
Department of Physics, Kangwon National University, Chuncheon 200-701 Korea and
Pacific Institute for Theoretical Physics, University of British Columbia,
6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1, Canada

The dissipative Hofstadter model describes quantum particles moving in two dimensions subject to a uniform magnetic field, a periodic potential and a dissipative force. We discuss the dissipative Hofstadter model in the framework of the boundary state formulation in string theory and construct exact boundary states for the model at the magic points by using the fermion representation. The dissipative Hofstadter model at magic points is shown to be equivalent to the critical boundary sine-Gordon model.

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I. INTRODUCTION

The quantum mechanical description of dissipation has been one of the outstanding problems in theoretical physics. Since the well-known dissipative or frictional force term in the classical equation of motion cannot be driven from a local action, it has been a conundrum how to quantize the dissipative system for a long time. Caldeira and Leggett [1, 2] provided a proper answer to this problem by coupling a bath or environment which consists of an infinite number of harmonic oscillators to the system. Assuming that the interaction between the bath and the system is linear and imposing the Ohmic condition for the spectral function of the oscillator frequencies, they found that the effective friction term could be generated in the equation of motion. In the quantum theory the interaction with the bath produces a non-local effective interaction. This model for the dissipative system is now called the Caldeira-Leggett model.

Since the effective interaction is nonlinear, the dissipative quantum system exhibits various kinds of phase transition, unlike the quantum mechanical systems with local interactions only. One of the interesting dissipative systems, which has been extensively studied for its novel phase diagram [3, 4], is the dissipative Hofstadter model. In condensed matter physics, the dissipative Hofstadter model has a wide range of applications which include Josephson junction arrays [5, 6, 7], the Kondo problem [8, 9, 10, 11], the study of one-dimensional conductors [12], tunnelling between Hall edge states [13], and junctions of quantum wires [14]. The dissipative Hofstadter model also has an important application in string theory, since the boundary (conformal) field theories describing the dissipative Hofstadter model are solutions of classical open string field theory [15, 16, 17]. In particular, a marginal, periodic boundary interaction which is termed the “rolling tachyon” [18, 19] gives a description of the process of tachyon condensation [20, 21, 22, 23, 24, 25] in string theories with unstable D-branes. The conformal field theory for the rolling tachyon corresponds to the Schmid model [26, 27, 28] at the critical point (also called the magic point), which is the dissipative Hofstadter model in the absence of the magnetic field.

The dissipative Hofstadter model is a one dimensional quantum mechanical system which has time dimension only. In the string theory, the time of the dissipative Hofstadter model is mapped onto the boundary of the disk diagram and the bulk degrees of freedom of the string play the role of the bath to produce the dissipative non-local interaction at the boundary. As is well known, the string theory on a disk can be discussed best in the framework of the boundary state formulation. Thus, the boundary state formulation would be the most suitable framework to discuss the dissipative Hofstadter model, at least at the magic points where the boundary interaction becomes exactly marginal. Along this line the Schmid model at the critical point was first discussed by Callan and Thorlacius [29] and later the dissipative Hofstadter model at the magic points by Maldacena et al.

One of the authors has recently discussed the rolling tachyon boundary state by fermionizing the rolling tachyon boundary conformal field theory and then constructed the exact boundary states by using fermion variables [30]. Extending the previous work, we shall discuss

*Electronic address: taejin@kangwon.ac.kr
the dissipative Hofstadter model at the magic points in the framework of the boundary state formulation in the string theory using the fermion representation. The dissipative Hofstadter model at the magic points will be shown to be equivalent to the boundary sine-Gordon model and the mobility will be calculated exactly by using the fermion representation of the boundary state for the sine-Gordon model.

II. The Schmid Model

In the absence of the magnetic field, the dissipative Hofstadter model reduces to the Schmid model, of which the action is given as follows:

$$S_{SM} = \frac{\eta}{4 \pi \hbar} \int_{-T/2}^{T/2} dt dt' \frac{(X(t) - X(t'))^2}{(t - t')^2} + \frac{M}{2 \hbar} \int_{-T/2}^{T/2} dt \pi X^2 - \frac{V_0}{\hbar} \int_{-T/2}^{T/2} dt \cos \frac{2\pi X}{\alpha}.$$  \hspace{1cm} (1)

The first term is responsible for the dissipation, and the second term is the usual kinetic term for a particle with mass $M$. The third term denotes the periodic potential. The period of time is to be $T$. If we wish to describe the system in real time, we may take the limit where $T \to \infty$. Since we are only interested in the long-time behavior of the system, we may ignore the kinetic term, which only plays a role of regulator in the long-time analysis.

Mapping the Schmid model to the string theory on a disk begins with identifying the time as the boundary parameter $\sigma$ in string theory and scaling the field variable $X$:

$$t = \frac{T}{2\pi} \sigma, \quad X \to \frac{a}{2\pi} X.$$  \hspace{1cm} (2)

Then, the action for the Schmid model reads as

$$S_{SM} = \frac{\eta}{4 \pi \hbar} \left( \frac{a}{2\pi} \right)^2 \int_{-\pi}^{\pi} d\sigma d\sigma' \frac{(X(\sigma) - X(\sigma'))^2}{(\sigma - \sigma')^2} - \frac{V_0}{\hbar} \frac{T}{2\pi} \int_{-\pi}^{\pi} d\sigma \frac{1}{2} \left( e^{iX} + e^{-iX} \right).$$  \hspace{1cm} (3)

This action precisely coincides with the boundary effective action for the bosonic string subject to a boundary periodic potential on a disk with a boundary condition $X(\tau = 0, \sigma) = X(\sigma)$,

$$e^{-S_{SM}} = \int D[X] \exp \left[ -\frac{1}{4\pi \alpha'} \int d\tau d\sigma \partial_\tau X \partial_\sigma X + \frac{g}{2} \int d\sigma \left( e^{iX} + e^{-iX} \right) \right].$$  \hspace{1cm} (4)

Here, we identify the physical parameters of the two theories as

$$\frac{\eta}{4 \pi \hbar} \left( \frac{a}{2\pi} \right)^2 = \frac{1}{8\pi^2 \alpha'}, \quad \frac{V_0}{\hbar} \frac{T}{2\pi} = g.$$  \hspace{1cm} (5)

The action for the Schmid model also appears in the boundary state for the bosonic string

$$|B\rangle = \int D[X] \exp \left( g \int d\sigma \cos X \right) |X\rangle,$$  \hspace{1cm} (6)

$$|X\rangle = e^{-\frac{g}{2\pi \alpha'} \int_{-\pi}^{\pi} d\sigma d\sigma' \frac{(X(\sigma) - X(\sigma'))^2}{(\sigma - \sigma')^2}} |0\rangle.$$  \hspace{1cm} (7)

Thus,

$$\langle 0 | B \rangle = \int D[X] \exp (-S_{SM} | X \rangle \rangle.$$  \hspace{1cm} (7)

The critical point for the Schmid model corresponds to the point where $\alpha' = 1$, i.e., the boundary periodic potential becomes marginal. As you may notice, the Schmid model at the critical point is nothing but the conformal field theory for the rolling tachyon of the full-S-brane. As discussed in ref. [30], at the critical point the conformal theory can be better understood in its fermion representation [31, 32]; the fermionic representation has the advantage that the boundary interaction becomes a simple fermion current operator which is bilinear in fermion fields. As a result, one can easily construct an exact boundary state for the theory. In order to fermionize the system, we need to introduce an auxiliary free boson $Y$ which obeys a Dirichlet boundary condition on all boundaries. This allows us to form the bosonic variables

$$\phi_1 = \frac{1}{\sqrt{2}} (X + Y), \quad \phi_2 = \frac{1}{\sqrt{2}} (X - Y),$$  \hspace{1cm} (8)

with which the mapping to fermions is defined:

$$\psi_{1L}(z) = \zeta_{1L} : e^{-\sqrt{\alpha} \phi_{1L}(z)} :,$$  \hspace{1cm} (9)

$$\psi_{1R}(\bar{z}) = \zeta_{1R} : e^{\sqrt{\alpha} \phi_{1R}(\bar{z})} :,$$  \hspace{1cm} (9)

where $\zeta_{aL/R}$ are co-cycles, ensuring the anti-commutation relations between the fermion operators. In terms of the fermion fields, the string action for the Schmid model is given by

$$\int d\tau d\sigma L$$

$$= \int d\tau d\sigma \left[ \psi_L^\dagger \partial_\tau + \partial_\sigma \right] \psi_L + \psi_R^\dagger \left( \partial_\tau - i \partial_\sigma \right) \psi_R$$

$$+ ig \int d\sigma \left[ \psi_L^\dagger \left( 1 + \sigma^3 \right) \psi_R^\dagger \right] \left( 1 - \frac{\sigma^3}{2} \right) \psi_R,$$  \hspace{1cm} (10)

where $\psi_L = (\psi_{1L}, \psi_{2L})^t$, $\psi_R = (\psi_{1R}, \psi_{2R})^t$. We quote from ref. [30] the explicit expression of the boundary state. This is in the $NS$-sector

$$|BD\rangle_{NS} = \Pi_{r=\frac{1}{2}} e^{i \psi_{1L}^\dagger U^{-1} i \sigma^3 \psi_{2L}^\dagger + i \psi_{1R}^\dagger i \sigma^3 U \psi_{2R}^\dagger + i} |0\rangle.$$  \hspace{1cm} (11)
and in the R-sector

$$|BD\rangle_R = \prod_{n=1}^{\infty} \exp \left[ \psi_n^* U^{-1} i\sigma^1 \psi_n - \psi_n^* i\sigma^1 U \psi_n \right] \exp \left[ \psi_0^* U^{-1} i\sigma^1 \psi_0 \right] \mid - - -$$, \hspace{1cm} (12)

where

$$U = \left[ \begin{array}{cc} \sqrt{1 - \pi^2 g^2} & -i\pi g \\ -i\pi g & \sqrt{1 - \pi^2 g^2} \end{array} \right]. \hspace{1cm} (13)$$

Note that the matrix $U$ is not unitary if the coupling exceeds a critical strength, $|g| > 1/\pi$. (For the notation, the reader should refer to ref. [30].)

III. The Dissipative Hofstadter model at Magic Points

If we turn on the magnetic field, the Schmid model becomes the dissipative Hofstadter model, of which the action is given by

$$S_{DHM} = \frac{\hbar}{4\pi} \int_{-T/2}^{T/2} \frac{dt dt'}{(t - t')^2} \left( \frac{X(i) - X(i')}{(t - t')^2} \right)^2 + \frac{i e B}{2\hbar} e^{ij} \partial_t X^i X^j - \frac{V_0}{\hbar} \int_{-T/2}^{T/2} dt \left( \cos \frac{2\pi X^1}{a} + \cos \frac{2\pi X^2}{a} \right), \hspace{1cm} (14)$$

where $i, j = 1, 2$. As in the case of the Schmid model, mapping the model to the string theory on a disk can be accomplished by identifying the time as the boundary parameter $\sigma$ and scaling the field variables $X^i$.

$$S_{DHM} = \frac{1}{8\pi^2 \alpha'} \int_{-\pi}^{\pi} d\sigma d\sigma' \frac{(X^i(\sigma) - X^i(\sigma'))^2}{(\sigma - \sigma')^2} + \frac{i B}{4\pi} \int_{-\pi}^{\pi} d\sigma \epsilon^{ij} \partial_\sigma X^i X^j - g \int_{-\pi}^{\pi} d\sigma \left( \cos X^1 + \cos X^2 \right), \hspace{1cm} (15)$$

where $2\pi \beta = \frac{\epsilon B \sigma}{\hbar} a^2$. As in the case of the Schmid model, the action for the dissipative Hofstadter model can be interpreted as the boundary effective action for the string subject to a boundary periodic potential and the magnetic field on a disk with a boundary condition $X(\tau = 0, \sigma) = X(\sigma)$,

$$\exp (-S_{DHM}) = \int D[X] \exp \left[ -\frac{1}{4\pi \alpha'} \int d\tau d\sigma E_{ij} (\partial_\tau + \partial_\sigma) X^i (\partial_\tau - \partial_\sigma) X^j + \frac{g}{2} \int d\sigma \left( e^{iX^1} + e^{-iX^1} + e^{iX^2} + e^{-iX^2} \right) \right]. \hspace{1cm} (16)$$

where $E_{ij} = \delta_{ij} + \frac{2}{\hbar} \epsilon_{ij} \alpha$ and $\alpha = 1/\alpha'$. The relationship between the action for the dissipative Hofstadter model and the boundary state for the corresponding string theory is the same as before in Eq. (17).

$$\langle 0| B \rangle = \int D[X] \exp (-S_{DHM}[X]). \hspace{1cm} (17)$$

Here, the boundary state $|B\rangle$ may be written as

$$|B\rangle = \exp \left[ g\pi \int \frac{d\sigma}{\partial M} \left( e^{iX^1} + e^{-iX^1} + e^{iX^2} + e^{-iX^2} \right) \right]|B_E\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbf{q}_i = \pm 1} \prod_{j=1}^{n} \exp \left[ i \mathbf{q}_j \cdot \mathbf{X}(\sigma_j) \right] |B_E\rangle. \hspace{1cm} (18)$$

Integration over the zero mode imposes the constraints: $\sum_{i} q_i = 0$. In the absence of the periodic potentials, the boundary state reduces to $|B_E\rangle$, which satisfies

$$\left( \delta_{ij} \partial_\sigma X^i - \frac{\beta}{\alpha'} \epsilon_{ij} \partial_\sigma X^j \right) |B_E\rangle = 0. \hspace{1cm} (19)$$

Rewriting the boundary condition for $|B_E\rangle$ in terms of the oscillators

$$\left( E_{ij} \alpha^j_{-n} + E^T_{ij} \tilde{\alpha}^j_{n} \right) |B_E\rangle = 0, \hspace{1cm} p^j |B_E\rangle = 0, \hspace{1cm} (20)$$

we find the explicit expression for $|B_E\rangle$

$$|B_E\rangle = \sqrt{\det E} \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha^i_{-n} \left( E^{-1} E^T \right)_{ij} \tilde{\alpha}^j_{n} \right]|0\rangle, \hspace{1cm} (21)$$

where $X^i(0, \sigma) = x^i + \omega^i \sigma + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0}^{\infty} \frac{1}{n} \left( \alpha^i_{n} - \tilde{\alpha}^i_{-n} \right) e^{-i n \sigma}.$

The interaction term with the magnetic field may be completely removed by using the $O(2,2, R)$ transformation in the absence of the periodic potential; the boundary state has a simpler expression if we choose a new oscillator basis $\{\beta, \beta\}$ which is related to the basis $\{\alpha, \tilde{\alpha}\}$ by the $O(2,2, R)$ transformation

$$\alpha^i_{n} = (G(E)^{-1})^j \beta^i_{-n}, \hspace{1cm} \tilde{\alpha}^i_{n} = (G(E^T)^{-1})^j \tilde{\beta}^i_{n}. \hspace{1cm} (22)$$

In terms of the new oscillator basis the boundary condition may be transcribed into the Neumann condition

$$\left( \beta^i_{-n} + \tilde{\beta}^i_{n} \right) |B_E\rangle = 0. \hspace{1cm} (23)$$
It is noteworthy that the oscillators \( \{ \beta, \tilde{\beta} \} \) respect the worldsheet metric \( G \)

\[
G = E^T E = \begin{pmatrix}
1 + \left( \frac{\beta}{\alpha} \right)^2 & 0 \\
0 & 1 + \left( \frac{\tilde{\beta}}{\alpha} \right)^2
\end{pmatrix}.
\] (24)

Their commutation relations are

\[
\begin{align*}
[\beta_n, \beta^*_m] &= (G^{-1})^{ij} n \delta(n + m), \\
[\tilde{\beta}_n, \beta^*_m] &= (G^{-1})^{ij} \tilde{\beta}_n \delta(n + m)
\end{align*}
\] (25)

and the boundary state \( |B_E \rangle \) is rewritten as

\[
|B_E \rangle = \sqrt{\text{det} E} \prod_{n=1} \exp \left( -\frac{1}{n} \beta_n^* G_{ij} \tilde{\beta}_n^* \right) |0\rangle.
\] (26)

If we define a new string coordinate \( Z \),

\[
Z'(0, \sigma) = x' + \omega' + \sum_{n \neq 0} \sqrt{\frac{\alpha'}{2}} \sum_{n} \frac{1}{n} \left( \beta_n^* - \tilde{\beta}_n^* e^{-in\sigma}\right).
\] (27)

the relation between the two oscillator bases is summarized as

\[
X^i(\sigma) = X^i_L(\sigma) + X^i_R(\sigma) = \left( \beta^i - \frac{\beta^i}{\alpha} e^i \right) Z^i_L(\sigma) + \left( \beta^i + \frac{\beta^i}{\alpha} e^i \right) Z^i_R(\sigma).
\] (28)

Using the Baker-Hausdorff Lemma, \( e^{A e^B} = e^B e^{A e^{|B|}} \) where

\[
A = -\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} q_{1k} (2\pi \alpha' B)^k \left( \beta_n^* + \tilde{\beta}_n^* e^{-in\sigma} \right),
\]

\[
B = -\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} q_{2r} (\beta_m^* - \tilde{\beta}_m^* e^{-im\sigma}),
\] (30)

we have

\[
[A, B] = -\alpha' q_1 \cdot (2\pi \alpha' B G^{-1}) \cdot \sum_{n \neq 0} \frac{1}{n} e^{-in\sigma_1 \sigma_2} = i\pi q_1 \cdot (2\pi \alpha' B G^{-1}) \cdot \sum_{n \neq 0} e^{-in\sigma_1 \sigma_2}.
\] (31)

Note

\[
-\sum_{n \neq 0} \frac{1}{n} e^{-in\sigma_1 \sigma_2} e^{-\epsilon n} = \ln \left[ \frac{z_1 - z_2 e^\epsilon}{z_2 - z_1 e^\epsilon} \right] = i\pi \text{sign}(\sigma_1 - \sigma_2).
\] (32)

where \( z_j = e^{-i\sigma_j} \). Thus, by repetition

\[
\prod_{j=1}^n \exp [i q_j \cdot (X(\sigma_j))] |B_E \rangle
\]

\[
= \exp \left[ \sum_{i > j} i\pi q_i \cdot (2\pi \alpha'^2 B G^{-1}) \cdot q_j \text{sign}(\sigma_i - \sigma_j) \right] |B_E \rangle
\]

\[
= \prod_{j=1}^n \exp [i q_j \cdot (Z(\sigma_j))] |B_E \rangle,
\] (33)

where the boundary condition in Eq. (23) is used. Some algebra leads us to

\[
\exp \left[ \sum_{i > j} i\pi q_i \cdot (2\pi \alpha'^2 B G^{-1}) \cdot q_j \text{sign}(\sigma_i - \sigma_j) \right]
\]

\[
= \exp \left[ \sum_{i > j} \frac{\beta}{\alpha^2 + \beta^2} \left( q_i q_j^2 - q_i^2 q_j \right) \text{sign}(\sigma_i - \sigma_j) \right]
\]

\[
= \exp \left[ \sum_{i > j} 2\pi i \frac{\beta}{\alpha^2 + \beta^2} q_i \left( \sum_{\sigma_i > \sigma_j} q_j^2 \right) \right].
\] (34)

Since \( q_i^2, q_j^2 = 0, \pm 1 \) for \( i = 1, 2 \), if \( \frac{\beta}{\alpha^2 + \beta^2} \) is an integer, this phase due to the magnetic field reduces to 1. It simply implies that the periodic potential does not change under the mapping \( X \rightarrow Z \). These circles

\[
\alpha^2 + \left( \beta - \frac{1}{2n} \right)^2 = \left( \frac{1}{2n} \right)^2, \quad n \in Z
\] (35)

on the two dimensional plane of \( (\alpha, \beta) \) may be called "magic circles". Hence, on the magic circles, the dissipative Hofstadter model can be mapped into the string
theory on a disk with the potential potential only, i.e., the boundary sine-Gordon model:

\[
\exp\left(-S_{DHM}\right) = \int D[Z] \\
\exp\left[ -\frac{1}{4\pi\alpha} \int_M d\tau d\sigma \left( \partial_{\tau} + \partial_{\sigma} \right) Z^i \left( \partial_{\tau} - \partial_{\sigma} \right) Z^j \right] \\
+ \frac{g}{2} \int_{\partial M} d\sigma \left( e^{iZ^i} + e^{-iZ^i} + e^{iZ^j} + e^{-iZ^j} \right). \tag{36}
\]

It may be convenient to scale \(\beta^i, \tilde{\beta}^i\) and \(G\) such that

\[
\beta^i_n \to \sqrt{\alpha} \beta^i_n, \quad \tilde{\beta}^i \to \sqrt{\alpha} \tilde{\beta}^i, \\
G \to \frac{1}{\alpha} G = \left( \begin{array}{cc}
\frac{\alpha^2 + \beta^2}{\alpha} & 0 \\
0 & \frac{\alpha^2 + \beta^2}{\alpha}
\end{array} \right). \tag{37}
\]

The points where the effective worldsheet metric becomes a unit metric form a circle called the "critical circle":

\[
\frac{\alpha^2 + \beta^2}{\alpha} = 1. \tag{38}
\]

On the critical circle, the commutation relations between \(\beta^i_n\) and \(\tilde{\beta}^i_m\) (after scaling) respect the unit metric

\[
\left[ \beta^i_n, \beta^j_m \right] = \delta^{ij} \delta(n + m), \\
\left[ \tilde{\beta}^i_n, \tilde{\beta}^j_m \right] = \delta^{ij} n \delta(n + m). \tag{39}
\]

Accordingly, the boundary state \(|B_E\rangle\) may be written on the critical circle as

\[
|B_E\rangle = \sqrt{\det E} \prod_{n=1} \exp \left( -\frac{1}{n} \beta^i_n \tilde{\beta}^i_n \right) |0\rangle. \tag{40}
\]

The points where the magic circles meet the critical circle are termed magic points; at the magic points, the dissipative Hofstadter model is equivalent to a set of two independent critical boundary sine-Gordon models:

\[
\exp\left(-S_{DHM}\right) = \int D[Z] \exp \left[ -\frac{1}{4\pi} \int_M d\tau d\sigma \left( \partial_{\tau} + \partial_{\sigma} \right) Z^i \left( \partial_{\tau} - \partial_{\sigma} \right) Z^j \right] \\
+ \frac{g}{2} \int_{\partial M} d\sigma \left( e^{iZ^i} + e^{-iZ^i} + e^{iZ^j} + e^{-iZ^j} \right). \tag{41}
\]

In other words, the system can be described by a set of two decoupled Schmid models at the critical points, which can be mapped into the free fermion theory.

**IV. CONCLUSIONS**

We discuss the dissipative Hofstadter model in the framework of the boundary state formulation, which has been developed for the string theory. As pointed out by Callan and Thorlacius [29], if we map the quantum system of the Hofstadter model onto the boundary of the string worldsheet disk, we see that the Caldeira-Leggett coupling, responsible for the dissipation, is already built in; the bulk string degrees of freedom on the disk play the role of the bath. Therefore, the boundary state formulation, which is the most convenient tool to discuss the string theory on a disk, could be the most effective framework for the dissipative Hofstadter model. This point has been elaborated by Callan and Thorlacius [29], who dealt with the Schmid model only, and their work has been further extended to the dissipative Hofstadter model by Callan et al. [33]. However, the full advantage of the boundary state formulation has not been taken in the previous work.

One of the most useful tools for the two-dimensional theories is fermionization (or bosonization). This is particularly useful when the bosonic (fermionic) theory becomes exactly solvable if it is fermionized (bosonized). The Schmid model and the dissipative Hofstadter model come under this category. Combining the boundary state formulation and the fermionization technique, in this paper we show that the Schmid model at the critical point, being equivalent to the critical sine-Gordon model, can be exactly solvable. The exact boundary state for the Schmid model at the critical point is nothing but the fermion boundary state for the rolling tachyon constructed in ref. [30]. Thus, using the exact boundary state, one can calculate any physical quantity exactly. To illustrate, let us calculate the mobility, which determines if the ground state of the system is localized or delocalized. The mobility is a two point correlation function,

\[
M(\sigma, \sigma') = \langle \partial_{\sigma} X(\sigma) \partial_{\sigma'} X(\sigma') \rangle \\
= \langle 0 | \partial_{\sigma} X(\sigma) \partial_{\sigma'} X(\sigma') | B \rangle. \tag{42}
\]

The exact boundary state and the fermionization technique enable us to calculate the mobility exactly:

\[
M(\sigma, \sigma') = \langle 0 | \left( J^2_\sigma(\sigma) + J^2_\sigma(\sigma') \right) \left( J^2_{\sigma'}(\sigma') + J^2_{\sigma'}(\sigma') \right) \prod_{r=1}^{1/2} \exp \left[ \psi^\dagger_{-r} U^{-1} i \sigma \psi_{-r} - \psi^\dagger_{-r} i \sigma \psi_{-r} \right] |0\rangle \\
= \frac{-1}{2} \left( 1 - \pi^2 g^2 \right) \sin^{-2} \left( \sigma - \sigma' \right) \frac{\alpha}{2}, \tag{43}
\]

where \(|\sigma| \leq 1/\pi\). (Note that the Ramond sector contains only the states with half integer momenta. This implies that the vacuum state belongs to the NS-sector. Thus, there is no contribution from the Ramond sector.)

Using the boundary state formulation and the \(O(2,2,\mathbb{R})\) transformation of the string theory, we also show that the dissipative Hofstadter model on the magic circles, where flux/unit cell \(\beta\) is an integer multiple of the friction/unit cell \(\alpha\), can be mapped into the Schmid model. Hence, the equivalence between the two models is made manifest. The equivalence between the dissipative Hofstadter model on the magic circles and the Schmid model has been discussed only in the calculation of the partition function [5]. The exact calculation of the mobility of the Schmid model at the critical point can be
easily extended to the case of the dissipative Hofstadter model; at the magic points the mobility of the dissipative Hofstadter model can be calculated by using the exact fermion boundary state as follows:

\[
M^{ij}(\sigma, \sigma') = \langle \partial_\sigma X^i(\sigma) \partial_\sigma X^j(\sigma') \rangle = \langle 0 | \partial_\sigma X^i(\sigma) \partial_\sigma X^j(\sigma') | B \rangle = \langle 0 | \left( \delta^{ik} - \frac{\beta}{\alpha} \epsilon^{ik} \right) \partial_\sigma Z^k_L(\sigma) + \left( \delta^{ik} + \frac{\beta}{\alpha} \epsilon^{ik} \right) \partial_\sigma Z^k_R(\sigma) \rangle \left[ \left( \delta^{jl} - \frac{\beta}{\alpha} \epsilon^{jl} \right) \partial_\sigma Z^l_L(\sigma') + \left( \delta^{jl} + \frac{\beta}{\alpha} \epsilon^{jl} \right) \partial_\sigma Z^l_R(\sigma') \right] | B \rangle,
\]

where

\[
| B \rangle = | BD \rangle_1 \otimes | BD \rangle_2.
\]

Since \( Z^k_{L/R} \) can be represented as the fermion current operators \( J^k_{L/R} \), the exact calculation of the mobility is not a difficult task. The result of the calculation and a more detailed discussion on the dissipative Hofstadter model will be saved for future work.

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References

[1] A. O. Caldeira and A. J. Leggett, Ann. Phys. 149, 374 (1983).
[2] A. O. Caldeira and A. J. Leggett, Physica 121A, 587 (1983).
[3] C. G. Callan, Jr. and D. Freed, Nucl. Phys. B374, 543 (1992) [hep-th/9110046].
[4] C. Callan, A. Pelce and D. Freed, Nucl. Phys. B392, 551 (1993) [hep-th/9202085].
[5] D. E. Freed, Nucl. Phys. B409, 565 (1993) [hep-th/9304006], [hep-th/9311034].
[6] L. I. Glazman and A. I. Larkin, Phys. Rev. Lett. 79, 3736 (1997) [cond-mat/9809118].
[7] R. Fazio and H. van der Zant, Phys. Rep. 355, 235 (2001).
[8] D. Giuliano and P. Sodano, Nucl. Phys. B711, 480 (2005) [cond-mat/0501378].
[9] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 360, 641 (1991).
[10] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 352, 849 (1991).
[11] J. B. Hong, T. Park and Y. S. Kwon, J. Korean Phys. Soc. 47, 479 (2005).
[12] C. L. Kane and M. P. A. Fisher, Phys. Rev. B46, 15233 (1992).
[13] C. L. Kane and M. P. A. Fisher, cond-mat/9506116.
[14] M. Oshikawa, C. Chamon and I. Affleck, Phys. Rev. Lett. 91, 206403 (2003) [cond-mat/0509675].
[15] E. Witten, Phys. Rev. D 46, 5467 (1992) [hep-th/9208027].
[16] S. L. Shatashvili, Phys. Lett. B 311, 83 (1993) [hep-th/9303143].
[17] S. L. Shatashvili, Alg. Anal. 6, 215 (1994) [hep-th/9311177].
[18] A. Sen, JHEP 0204, 048 (2002) [hep-th/0203211].
[19] See for a review on the rolling tachyon: A. Sen, Int. J. Mod. Phys. A20, 5513 (2005) [hep-th/0410103].
[20] C. Kim, H. B. Kim, Y. Kim, O.-K. Kwon and C. O. Lee, J. Korean Phys. Soc. 45, 5871 (2004).
[21] J. H. Chung and W. S. L’yi, J. Korean Phys. Soc. 45, 318 (2004).
[22] T. Lee, Phys. Rev. D64, 106004 (2001) [hep-th/0105115].
[23] T. Lee, J. Korean Phys. Soc. 39, 5536 (2001).
[24] T. Lee, Phys. Rev. B520, 385 (2001) [hep-th/0105264].
[25] T. Lee, J. Korean Phys. Soc. 42, 34 (2003) [hep-th/0109092].
[26] A. Schmid, Phys. Rev. Lett. 51, 1506 (1983).
[27] M. P. A. Fisher and W. Zwerger, Phys. Rev. B32, 6190 (1985).
[28] F. Guinea, V. Hakim, and A. Muramatsu, Phys. Rev. Lett. 54, 263 (1985).
[29] C. G. Callan, Jr. and L. Thorlacius, Nucl. Phys. B329, 117 (1990).
[30] M. Hasselfield, Taejin Lee, G. W. Semenoff, P. C. E. Stamp, in press, Ann. Phys. (2006) [hep-th/0512219].
[31] J. Polchinski and L. Thorlacius, [hep-th/9404008].
[32] T. Lee and G. W. Semenoff, JHEP 0505, 072 (2005) [hep-th/0502236].
[33] C. G. Callan, I. R. Klebanov, J. M. Maldacena and A. Yegulalp, Nucl. Phys. B443, 444 (1995), [hep-th/9503014].