Homotopy groups as centres of finitely presented groups

R. V. Mikhailov and J. Wu

Abstract. For every finite Abelian group $A$ and integer $n \geq 3$ we construct a finitely presented group defined by explicit generators and relations such that its centre is isomorphic to $\pi_n(\Sigma K(A,1))$.

Keywords: homotopy theory, homotopy groups, simplicial groups, finitely presented groups.

Dedicated to I. R. Shafarevich

§ 1. Introduction

In this paper we continue the study of connections between group theory and homotopy theory started in [1]–[6]. We briefly summarize some results and ideas in these papers.

The notions of simplicial sets and simplicial groups have been widely studied since their introduction in the early 1950s when Kan laid down the foundations of simplicial homotopy theory [7], [8]. Various important results have been achieved by studying simplicial groups. For example, the lower central series of Kan’s construction yields a particular case of the Adams spectral sequence, which is an effective tool for computing homotopy groups [9]. Applying Milnor’s $F[K]$-construction [10] to the simplicial circle and using Moore’s classical theorem on simplicial groups, the second author obtained a combinatorial description of the homotopy groups of the 2-sphere [1]. We recall this description. Let $F_n$ be a free group of rank $n \geq 1$ with basis $\{x_1, \ldots, x_n\}$. Let $R_i = \langle x_i \rangle^{F_n}$ be the normal closure of $x_i$ in $F_n$ for $1 \leq i \leq n$, and let $R_{n+1} = \langle x_1 x_2 \cdots x_n \rangle^{F_n}$ be the normal closure of the product $x_1 x_2 \cdots x_n$ in $F_n$. We define the symmetric commutator subgroup

$$[R_1, R_2, \ldots, R_{n+1}]_S = \prod_{\sigma \in \Sigma_{n+1}} [\ldots [R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n+1)}].$$

(1.1)

Theorem 1.1 [1]. For $n \geq 1$ there is an isomorphism

$$\pi_{n+1}(S^2) \simeq \frac{R_1 \cap \cdots \cap R_{n+1}}{[R_1, \ldots, R_{n+1}]_S}.$$

Moreover, $\pi_{n+1}(S^2)$ is isomorphic to the centre of the group $F_n/[R_1, \ldots, R_{n+1}]_S$.

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It turns out that certain connectivity conditions on a family $R_1, \ldots, R_n$ of normal subgroups in a group $G$ (see [2] for their precise statement) imply that

$$
\pi_n(X) \simeq \frac{R_1 \cap \cdots \cap R_{n+1}}{[R_1, \ldots, R_{n+1}]_S},
$$

where $X$ is the homotopy colimit of the classifying spaces $K(G/\prod_{i \in I} R_i, 1)$ and $I$ ranges over all subsets of $\{1, \ldots, n\}$. In particular, the description (1.2) covers the description of the homotopy groups of $S^2$ given in Theorem 1.1.

Further exploration of this topic led to discovery of fruitful connections between homotopy theory, braid groups, mapping class groups, link groups, and the theory of group rings $[3, 5, 11-17]$. For example, the authors applied homotopy groups of spheres to describe generalized dimension subgroups for symmetric products of ideals (see [3]). Some purely group-theoretic statements can be proved using homotopy theory while it is unclear how else to prove them. For example, if $F_n$ and $R_i$ are as in Theorem 1.1, then the quotient group $F_n/[R_1, \ldots, R_{n+1}]_S$ is residually nilpotent, that is, the intersection of its lower central series is trivial. This follows from the Curtis convergence theorem for Milnor’s simplicial construction $F[S^1]$ and the description of Moore boundaries in $F[S^1]$ as symmetric commutators in $[1]$.

Some progress in the combinatorial description of homotopy groups was recently achieved in [6] by giving a combinatorial description of general homotopy groups of spheres and Moore spaces in group-theoretic terms. It is shown in [6] that all the homotopy groups of spheres and Moore spaces can be represented as the centres of certain finitely generated groups given by explicit generators and relations. However, the groups (constructed in [1], [6]) whose centres are isomorphic to homotopy groups need not be finitely presented. Hence the following question naturally arises.

For which spaces $X$ and integers $n \geq 2$ is there a finitely presented group $\Gamma(X)$ (given by explicit generators and relations) such that the centre of $\Gamma(X)$ is isomorphic to $\pi_n(X)$? In this paper we study this question for suspensions of classifying spaces of finite Abelian groups. For every finite Abelian group $A$ and integer $n \geq 2$ we construct a finitely presented group $J_n$ such that $2^2 Z(J_n) \approx \pi_{n+1}(\Sigma K(A, 1))$.

The main approach used in [1], [6] to the construction of finitely generated groups whose centres are isomorphic to homotopy groups is as follows. For certain simply connected spaces $X$ there are simplicial group models $G_*$ for the loops of $X$ (that is, $|G_*| \simeq \Omega X$) such that the centres of the components of $G_*$ are trivial and there is a combinatorial description of the Moore boundaries $BG_*$. Then the homotopy groups $\pi_{n+1}(X) \simeq \pi_n(G_*)$ are isomorphic to the centres of the quotient groups $G_n/\mathcal{B}G_n$. But for all such simplicial models in [1] and [6], the Moore boundaries $\mathcal{B}G_n$ are not generated as normal subgroups of $G_n$ by finitely many elements. We recall that for the two-dimensional sphere $S^2$ there is a trick based on the properties of braid groups which yields a sequence of finitely presented groups (given by explicit generators and relations) whose centres are isomorphic to $\pi_*(S^2) \times \mathbb{Z}$ [4].

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1The $n$th homotopy group is described in [2] as the intersection of the subgroups $R_i$ modulo the subgroup $(R_1, \ldots, R_n) := \prod_{i,j \in \{1,\ldots, n\}, i \neq j} \langle \cap_{i \in I} R_i, \cap_{j \in J} R_j \rangle$. However, $(R_1, \ldots, R_n)$ coincides with the symmetric commutator subgroup $[R_1, \ldots, R_n]_S$ in many important cases (see [5]).

2The centre of a group $G$ is denoted by $Z(G)$. 
This trick does not work for other spaces since it uses very specific properties of Milnor’s simplicial construction $F[S^1]$ [10]. We also note that if the commutator subgroup $[G, G]$ of a group $G$ has trivial centre, then the non-abelian tensor square $G \otimes G$ in the sense of Brown–Loday [18] has the following property:

$$\pi_3(\Sigma K(G, 1)) \simeq Z(G \otimes G).$$

However, a generalization of this construction to higher homotopy groups seems to be a hard problem.

The homotopy groups of the suspensions $\Sigma K(A, 1)$ are highly non-trivial from a computational point of view. For example, the homotopy groups $\pi_n(\Sigma K(\mathbb{Z}/2, 1))$ are currently known only for $n \leq 6$ (see [19], [20]).

Our construction is as follows. Let $A$ be an Abelian group. For $n \geq 1$ we define the free product

$$T_n := (A \times \cdots \times A) \ast (A \times \cdots \times A).$$

For $i = 1, \ldots, n$ we denote the $i$th copy of $A$ in the first (resp. second) free summand in $T_n$ by $A_i$ (resp. $A_{n+i}$). Given $a \in A$ and $j = 1, \ldots, 2n$, we write $a_j$ for the element $a$ in $A_j$. Put

$$R_1 = \langle a_1, a_{n+1} \mid a \in A \rangle^{T_n},$$
$$R_i = \langle a_i a_{i-1}^{-1}, a_{n+i} a_{n+i-1}^{-1} \mid a \in A \rangle^{T_n}, \quad 1 < i \leq n,$$
$$R_{n+1} = \langle a_n, a_{2n} \mid a \in A \rangle^{T_n}.$$

We now define the group

$$J_n(A) := T_n / (\gamma_{2n+1}([T_n, T_n])[R_1, \ldots, R_{n+1}]S),$$

where $\gamma_k(\cdot)$, $k \geq 1$, is the $k$th term of the lower central series. Here is the main result of this paper.

**Theorem 1.2.** Let $A$ be a finite Abelian group. The homotopy group $\pi_{n+1}(\Sigma K(A, 1))$ is isomorphic to the centre of the polycyclic group $J_n(A)$ for all $n \geq 2$.

All polycyclic groups are finitely presented. In particular, $J_n(A)$ is finitely presented. Moreover, it follows from the definition that $J_n(A)$ is virtually nilpotent. The group $J_n(A)$ can be obtained canonically from the simplicial group $K(A, 1) \ast K(A, 1)$. The lower central series of the free product $K(A, 1) \ast K(A, 1)$ may give some computational information on the homotopy groups of suspensions and on $J_n(A)$.

An explicit construction of finitely presented groups with centres $\pi_n(S^2) \times \mathbb{Z}$ is given in [4]. However, there are some mistakes in [4]. In § 4 we briefly review the main result of [4] and correct the mistakes. The notion of symmetric commutator subgroup (1.1) plays a central role in this construction.
\section{Simplicial models}

Let $A$ be an Abelian group. The homotopy commutative diagram of fibre sequences

$$
\begin{array}{ccc}
\Sigma K(A, 1) \land K(A, 1) & \xrightarrow{H} & \Sigma K(A, 1) \\
\downarrow & & \downarrow \\
K(A, 2) = BK(A, 1) & \xrightarrow{\Delta} & K(A, 2) \vee K(A, 2) \xleftarrow{\iota} K(A, 2) \times K(A, 2)
\end{array}
$$

(2.1)

where $H$ is the Hopf fibration, implies that for $n \geq 3$ there are isomorphisms

$$\pi_n(\Sigma K(A, 1)) \cong \pi_n(K(A, 2) \vee K(A, 2)).$$

(2.2)

We choose the simplest simplicial model for $K(A, 1)$. Applying the inverse of the normalization functor in the sense of Dold–Kan to the complex $A[1]$, we get the Abelian simplicial group $E_*$ with components

$$E_i = (A \times \cdots \times A)_i, \quad i \geq 1,$$

and the property $|E_*| \simeq K(A, 1)$. The face and degeneracy maps in $E_*$ are standard. Their structure follows from the construction of the inverse of the normalization functor.

The following fact is a corollary of a theorem of Whitehead [21] (see [22], Proposition 4.3, for a simplicial version of this theorem).

**Lemma 2.1.** For an Abelian group $A$ there is a homotopy equivalence

$$|E_* \ast E_*| \simeq \Omega(K(A, 2) \vee K(A, 2)).$$

Here $E_* \ast E_*$ is the free product of two copies of the simplicial group $E_*$.  

We recall that a simplicial group $G_*$ is said to be free if each $G_i, i \geq 0$, is free with a given basis such that the images of the basis elements under the degeneracy homomorphisms are again basis elements. The following result is due to Curtis [23].

**Theorem 2.2.** Suppose that $G_*$ is a connected simplicial group and $r \geq 2$. Then the homomorphism $G \rightarrow G/\gamma_r(G)$ of simplicial groups induces isomorphisms

$$\pi_i(G) \simeq \pi_i(G/\gamma_r(G))$$

for all $i < \log_2 r$.

We now consider the free product $T_* := E_* \ast E_*$ of simplicial groups. It has the following components:

$$T_i := (A \times \cdots \times A)_i \ast (A \times \cdots \times A)_i, \quad i \geq 1.$$

**Lemma 2.3.** The simplicial group $[T_*, T_*]$ is free.
Proof. For Abelian groups $B, C$ the commutator subgroup $[G, G]$ of the free product $G = B \ast C$ is a free group with basis given by all commutators $^3 [b, c], 1 \neq b \in B, 1 \neq c \in C$ (see, for example, [24]). Regarding these commutators as basis elements in $[T_*, T_*]$, we see from the definition of $T_*$ that the degeneracy homomorphisms map these basis elements to basis elements. □

It follows from Theorem 2.2 and Lemma 2.3 that the natural map
\[
[T_*, T_] \to [T_*, T_*/\gamma_{2n+1}([T_*, T_*)])
\]
of simplicial groups induces an isomorphism of homotopy groups
\[
\pi_i([T_*, T_*]) \simeq \pi_i([T_*, T_*/\gamma_{2n+1}([T_*, T_*)]), \quad i < n.
\]
The short exact sequence of simplicial groups
\[
1 \to [T_*, T_+] \to T_* \to T_*/[T_*, T_*] \to 1
\]
induces a long exact sequence of homotopy groups. The homotopy equivalence
\[
[T_*/[T_*, T_*]] \simeq K(A, 1) \times K(A, 1)
\]
implies that for $i \geq 3$ there is an isomorphism
\[
\pi_i([T_*, T_*]) \simeq \pi_i(T_*).
\]
Using Lemma 2.1 and the isomorphism (2.2), we see that for all $i$, $2 < i < n$, there are isomorphisms of homotopy groups
\[
\pi_i(T_*) \simeq \pi_i(T_*/\gamma_{2n+1}([T_*, T_*])) \simeq \pi_{i+1}(\Sigma K(A, 1)). \tag{2.3}
\]

§ 3. Proof of Theorem 1.2

Lemma 3.1. Let $A$ and $B$ be non-trivial Abelian groups and put $G = A \ast B$. The centre of the group $H = G/\gamma_n([G, G])$ is trivial for $n \geq 2$.

Proof. The commutator subgroup $[G, G]$ is free with basis \{\([a, b], 1 \neq a \in A, 1 \neq b \in B\)\} (see [24]). We shall prove the statement for $n = 2$. The proof for other values of $n$ is similar.

Take $h \in [G, G]$. Considering $h$ modulo $\gamma_2([G, G])$, we can write $h$ uniquely as
\[
h = \prod_{1 \neq a \in A, 1 \neq b \in B} [a, b]^{m(a, b)}, \quad m(a, b) \in \mathbb{Z}. \tag{3.1}
\]
For $c \in A$ we get
\[
h^c \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a, b)}[c, b]^{-m(a, b)} \mod \gamma_2([G, G]),
\]
\[
[h, c] \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a, b)}[a, b]^{-m(a, b)}[c, b]^{-m(a, b)} \mod \gamma_2([G, G]).
\]

\(^3\)We use the standard notation $[a, b] := a^{-1}b^{-1}ab.$
Assume that $1 \neq \alpha \in Z(H)$. We can write $\alpha$ in the form $\alpha = f dh. \gamma_2([G, G])$, $f \in A$, $d \in B$, $h \in [G, G]$. Suppose that $d \neq 1$. Writing $\alpha$ in the form (3.1) and taking $c \in A$, we have

$$[c, \alpha] \equiv [c, d] \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a, b)}[a, b]^{-m(a, b)}[c, b]^{-m(a, b)} \mod \gamma_2([G, G]).$$

Since $\alpha \in Z(H)$ and $ca \neq c$ for $a \neq 1$, we have

$$m(c, d) + \sum_{1 \neq a \in A} m(a, d) = 1. \quad (3.2)$$

Since $c \in A$ is arbitrary, the identity (3.2) holds for all $c \in A$ (but $d \in B$ is fixed). Summing (3.2) over all $c \in A$, we get

$$(1 + |A|) \sum_{1 \neq a \in A} m(a, d) = |A|.$$ 

But this is impossible since all the coefficients are integers: $m(a, d) \in \mathbb{Z}$. Hence $d = 1$.

An analogous argument shows that $f = 1$ and, therefore, $\alpha \in [G, G]. \gamma_2([G, G])$. We now represent $\alpha = h. \gamma_2([G, G])$ in the form (3.1). Since $\alpha \in Z(H)$, we have

$$[h, c] \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a, b)}[a, b]^{-m(a, b)}[c, b]^{-m(a, b)} \equiv 0 \mod \gamma_2([G, G])$$

for any $c \in A$. Therefore, for every $b \in B$, we have

$$m(c, b) + \sum_{1 \neq a \in A} m(a, b) = 0. \quad (3.3)$$

Summing (3.3) over all $c \in A$, we have

$$(1 + |A|) \sum_{1 \neq a \in A} m(a, b) = 0,$$

whence $\sum_{1 \neq a \in A} m(a, b) = 0$. It now follows from (3.3) that $m(c, b) = 0$ for all $c \in A$, $b \in B$. Therefore $h \in \gamma_2([G, G])$. □

Proof of Theorem 1.2. By Lemma 3.1, the centres of the components of the simplicial group $T_{*}/\gamma_{2^{n+1}}([T_{*}, T_{*}])$ are trivial. Using the isomorphism (2.3) and Proposition 2.14 in [1], we obtain an isomorphism

$$\pi_{n+1}(\Sigma K(A, 1)) \simeq Z(T_{*}/\gamma_{2^{n+1}}([T_{*}, T_{*}])B_{n}),$$

where $B_{n}$ is the Moore boundary. It remains to prove that the Moore boundary is given by the symmetric commutator subgroup:

$$B_{n} = [R_{1}, R_{2}, \ldots, R_{n+1}]_{S}. \quad (3.4)$$

Let $F = F(A \setminus \{1\})$ be the free group generated by all non-trivial elements $^{4}$ of $A$. Let $\varphi: F \to A$ be the canonical quotient homomorphism, that is, the unique group

$^{4}$Here we use multiplicative notation for Abelian groups.
homomorphism such that \( \varphi(a) = a \) for \( a \in A \setminus \{1\} \). Then \( \varphi \) induces a simplicial epimorphism

\[
\tilde{\varphi} : F^F[S^1] \to F^A[S^1] \to F^A[S^1]^{ab} = K(A, 1).
\]

We recall that the simplicial circle \( S^1 \) has elements that are explicitly given by

\[
S^1_n = \{ *, x_{i+1} = s_{n-1} \cdots s_{i+1}s_i \cdots s_0 \sigma_1 \mid 0 \leq i \leq n-1 \},
\]

where \( \sigma_1 \) is the non-degenerate element of \( S^1 \) and, by the definition of Carlsson’s construction [25], \( F^F[S^1]_n \) is the free product of copies of \( F \) indexed by the elements of \( S^1_n \setminus \{ * \} \). Thus,

\[
F^F[S^1]_n = (F)_{x_1} \ast \cdots \ast (F)_{x_n},
\]

where \( (F)_{x_i} \) is the copy of \( F \) labeled by \( x_i \). The epimorphism \( \tilde{\varphi} : F^F[S^1]_n \to K(A, 1)_n \) is explicitly given by the composite

\[
(F)_{x_1} \ast \cdots \ast (F)_{x_n} \mapsto (A)_{x_1} \ast \cdots \ast (A)_{x_n} \mapsto (A)_{x_1} \times \cdots \times (A)_{x_n},
\]

whence

\[
\tilde{\varphi}((a)_{x_i}) = a_i
\]

for \( a \in A \setminus \{0\} \) and \( 1 \leq i \leq n \). Consider the epimorphism

\[
\tilde{\varphi} \ast \tilde{\varphi} : F^F[S^1] \ast F^F[S^1] \to K(A, 1) \ast K(A, 1).
\]

We observe that \( F^F[S^1] \ast F^F[S^1] = F^{F \ast F}[S^1] \). Following the notation for the groups \( T_n \) used in the introduction, we let \( B \) be a copy of \( A \), so that the group \( F \ast F \) is free with basis elements \( a \in A \setminus \{1\} \) and \( b \in B \setminus \{1\} \). The epimorphism

\[
\tilde{\varphi} \ast \tilde{\varphi} : F^{F \ast F}[S^1]_n \to K(A, 1)_n \ast K(A, 1)_n = T_n
\]

is given by

\[
(\tilde{\varphi} \ast \tilde{\varphi})((a)_{x_i}) = a \in A_i, \quad (\tilde{\varphi} \ast \tilde{\varphi})((b)_{x_i}) = b \in B_i
\]

for \( 1 \leq i \leq n \). We put

\[
\tilde{R}_1 = \langle (a)_{x_1}, (b)_{x_1} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F \ast F}[S^1]_n},
\]

\[
\tilde{R}_i = \langle (a)_{x_i}(a)_{x_{i-1}^{-1}}, (b)_{x_i}(b)_{x_{i-1}^{-1}} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F \ast F}[S^1]_n}, \quad 1 < i \leq n,
\]

\[
\tilde{R}_{n+1} = \langle (a)_{x_n}, (b)_{x_n} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F \ast F}[S^1]_n}.
\]

Then

\[
\tilde{\varphi} \ast \tilde{\varphi}(\tilde{R}_i) = R_i
\]

for \( 1 \leq i \leq n \) and

\[
\tilde{\varphi} \ast \tilde{\varphi}([\tilde{R}_1, \ldots, \tilde{R}_{n+1}]_S) = [R_1, \ldots, R_{n+1}]_S. \quad (3.5)
\]

\(^5\)See [1], [25] for a description of Carlsson’s construction \( F^G[S^1] \).
Let $H_j$ be a sequence of subgroups of $G$ for $1 \leq j \leq k$. We recall that the fat commutator subgroup $[[H_1, \ldots, H_k]]$ is generated in $G$ by all commutators $eta^t(h_{i_1}^{(1)}, \ldots, h_{i_t}^{(t)})$, where

1) $1 \leq i_s \leq k$;
2) every integer in $\{1, 2, \ldots, k\}$ appears at least once among the $i_s$;
3) $h_{j}^{(s)} \in H_j$;
4) $\beta^t$ runs over all bracket arrangements of weight $t$ for every $t \geq k$.

The proof of Theorem 1.8 in [1] shows that the Moore boundary can be written as

$$B_nF^1 S^1 = [[\tilde{R}_1, \ldots, \tilde{R}_{n+1}]].$$

Following Theorem 1.1 in [5], we conclude that

$$[[\tilde{R}_1, \ldots, \tilde{R}_{n+1}]] = [\tilde{R}_1, \ldots, \tilde{R}_{n+1}]_S.$$ 

Furthermore,

$$B_nF^1 S^1 = [\tilde{R}_1, \ldots, \tilde{R}_{n+1}]_S. \tag{3.6}$$

By Lemma 5, 3.8 in [26], every simplicial epimorphism induces an epimorphism of Moore chains, whence

$$B_n(K(A, 1) * K(A, 1)) = d_0N_{n+1}(K(A, 1) * K(A, 1)) = d_0(\varphi * \varphi(N_{n+1}(F^1 S^1))) = \varphi * \varphi(d_0N_{n+1}(F^1 S^1)) = \varphi * \varphi(B_nF^1 S^1).$$

Equation (3.4) now follows from (3.5) and (3.6). This completes the proof. Note that for every pair $A, B$ of finite Abelian groups, the quotient group $A*B/G_n([A*B, A*B])$ is polycyclic. In other words, it is a soluble group all of whose subgroups are finitely generated. In particular, the groups $\mathcal{J}_n(A) = T_n/G_n([T_n, T_n])B_n$ are finitely presented for all $n \geq 2$. □

§ 4. Homotopy groups of $S^2$

 Considering the simplicial group $E_* * E_*$ in § 2, for $A = \mathbb{Z}$ we get a homotopy equivalence

$$|E_* * E_*| \simeq \Omega(K(\mathbb{Z}, 2) \vee K(\mathbb{Z}, 2)).$$

Since the Moore boundaries of $E_* * E_*$ can be described combinatorially using symmetric commutator subgroups, we obtain the following description of the homotopy groups of $S^2$, which is an alternative to the description in [1].

**Proposition 4.1.** Let $n \geq 3$, $T_n := (\mathbb{Z} \times \cdots \times \mathbb{Z}) * (\mathbb{Z} \times \cdots \times \mathbb{Z})$. There is an isomorphism

$$\pi_{n+1}(S^2) \simeq Z(T_n/[R_1, \ldots, R_{n+1}]_S),$$

where the subgroups $R_i$, $i = 1, \ldots, n+1$, are defined as in (1.3)–(1.5).

Note that the groups $T_n/[R_1, \ldots, R_{n+1}]_S$ are not finitely presented and neither are the groups in [1] whose centre is isomorphic to $\pi_{n+1}(S^2)$. A construction of a finitely presented group with centre isomorphic to $\pi_n(S^2) \times \mathbb{Z}$ was given in [4].
However, there are some mistakes in [4]. In this section we briefly review the main result of [4] and correct the mistakes.

For $1 \leq i \leq n$ let $d_i : P_n \to P_{n-1}$ be the group homomorphism obtained by removing the $i$th strand from each $n$-strand pure braid. There is a well-defined additional face operation: a group homomorphism $d_0 : P_n \to P_{n-1}$ defined by

$$d_0 A_{i,j} = A_{i-1,j-1}$$

for $1 \leq i < j \leq n$, where the braid $A_{0,j} \in P_{n-1}$ is given by

$$A_{0,j} = (A_{j,j+1} \cdots A_{j,n-1})^{-1}(A_{1,j} \cdots A_{j-1,j})^{-1} = (\sigma_j \cdots \sigma_{n-3} \sigma_{n-2} \sigma_{n-3} \cdots \sigma_j)^{-1}(\sigma_{j-1} \cdots \sigma_2 \sigma_{2} \cdots \sigma_{j-1})^{-1}.$$

Then the sequence of groups $P = \{P_n\}_{n \geq 0}$ forms a $\Delta$-group with the property (see [4], Proposition 2.5) that the Moore homotopy group of the $\Delta$-group $P$ is

$$\pi_n(P) \cong \pi_n(S^2). \quad (4.1)$$

It follows from the definition of the Moore homotopy groups of $\Delta$-groups that the Moore chains are given by

$$N_n P = \bigcap_{i=1}^{n} \ker(d_i : P_n \to P_{n-1}) = \text{Brun}_n,$$ \quad (4.2)

where $\text{Brun}_n$ is the group of $n$-strand pure Brunnian braids. Let

$$\text{Bd}_n = d_0(\text{Brun}_{n+1}) = B_n P$$

be the Moore boundaries of $P$, which are called $n$-strand boundary Brunnian braids in [4]. The main result of [4] is as follows.

**Theorem 4.2** ([4], Theorem 1.1, (2)). The group $\text{Bd}_n$ is a normal subgroup of the Artin braid group $B_n$. For $n \geq 4$ there are group isomorphisms

$$Z(P_n/\text{Bd}_n) \cong \pi_n(S^2) \times \mathbb{Z}, \quad Z(B_n/\text{Bd}_n) \cong \{\alpha \in \pi_n(S^2) \mid 2\alpha = 0\} \times \mathbb{Z},$$

where the $\mathbb{Z}$ comes from $Z(P_n) = Z(B_n) = \mathbb{Z}$.

The main results (Theorems 1, 3) of [4] are correct. It is also true that the groups $P_n/\text{Bd}_n$ and $B_n/\text{Bd}_n$ are finitely presented. The major mistake in [4] is that Lemma 3.6 is false. As a consequence of this fallacious lemma, Theorem 3.7, Lemma 3.11, Corollaries 3.12–3.14, Proposition 3.15 and the statements at the end of the second paragraph on p. 523 in [4] are also false. In the introduction to the main results of [4], the sets of normal generators for $\text{Brun}_n$ ([4], p. 522, line 3) and $\text{Bd}_n$ ([4], p. 523, line 14) are incorrect. Here we correct these mistakes by describing finite sets of normal generators for $\text{Brun}_n$ and $\text{Bd}_n$. This will confirm that the groups $P_n/\text{Bd}_n$ and $B_n/\text{Bd}_n$ are finitely presented.
Theorem 4.3. Let \( \text{Brun}_n \) be the group of \( n \)-strand Brunnian pure braids, and let \( \text{Bd}_n \) be the group of \( n \)-strand boundary Brunnian pure braids. Then the following assertions hold.

1) The group \( \text{Brun}_n \) is the normal closure in \( P_n \) of the elements

\[
\ldots [[[A_{1,n}, A_{\sigma(2),j_2}], A_{\sigma(3),j_3}], \ldots, A_{\sigma(n-1),j_{n-1}}]
\]

for \( \sigma \in \Sigma_{n-2} \) acting on \( \{2,3,\ldots,n-1\} \) and \( 1 \leq j_2, \ldots, j_{n-1} \leq n, j_s \neq \sigma(s) \) for \( 2 \leq s \leq n-1 \) and \( A_{j,i} = A_{i,j} \) if \( i > j \).

2) The group \( \text{Bd}_n \) is the normal closure in \( P_n \) of the elements

\[
\ldots [[[A_{0,n}, A_{\sigma(1),j_1}], A_{\sigma(2),j_2}], \ldots, A_{\sigma(n-1),j_{n-1}}]
\]

for \( \sigma \in \Sigma_{n-1} \) and \( 1 \leq j_1, \ldots, j_{n-1} \leq n, j_s \neq \sigma(s) \) for \( 2 \leq s \leq n-1 \), where \( A_{j,i} = A_{i,j} \) if \( i > j \).

Proof. Let \( R_{i,j} \) be the normal closure of \( A_{i,j} \) in \( P_n \) for \( 1 \leq i < j \leq n \). It follows from Theorem 1.1 in [11] that

\[
\text{Brun}_n = \prod_{\sigma \in \Sigma_{n-1}} \ldots [R_{\sigma(1),n}, R_{\sigma(2),n}], \ldots, R_{\sigma(n-1),n}].
\]

By Theorem 1.2 in [5],

\[
\prod_{\sigma \in \Sigma_{n-1}} \ldots [R_{\sigma(1),n}, R_{\sigma(2),n}], \ldots, R_{\sigma(n-1),n}
\]

\[
= \prod_{\sigma \in \Sigma_{n-2}} \ldots [R_{1,n}, R_{\sigma(2),n}], \ldots, R_{\sigma(n-1),n}].
\]

Thus,

\[
\text{Brun}_n = \prod_{\sigma \in \Sigma_{n-2}} \ldots [R_{1,n}, R_{\sigma(2),n}], \ldots, R_{\sigma(n-1),n}]. \tag{4.3}
\]

Let \( G_i, 1 \leq i \leq n \), be the subgroup of \( P_n \) generated by \( A_{i,j} \) for \( 1 \leq j \leq n, j \neq i \), where \( A_{i,j} = A_{j,i} \) if \( i > j \). We recall that \( P_n = \pi_1(F(\mathbb{C},n)) \), where

\[
F(\mathbb{C},n) = \{(z_1, \ldots, z_n) | z_i \neq z_j, i \neq j \} \tag{4.4}
\]

is the ordered configuration space. By Proposition 3.2 in [11], the operation \( d_i : P_n \to P_{n-1} \) of removing the \( i \)-th strand is induced by the coordinate projection

\[
\pi_i : F(\mathbb{R}^2, n) \to F(\mathbb{R}^2, n-1), \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).
\]

Let \((p_1, \ldots, p_n)\) be a base point in \( F(\mathbb{C},n) \). Namely, let \( p_1, \ldots, p_n \) be distinct points in the plane \( \mathbb{R}^2 \). By the classical Fadell–Neuwirth theorem [27], the coordinate projection \( \pi_i \) is a bundle with fibre \( \mathbb{R}^2 \setminus \{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n\} \). Taking the fundamental groups of the bundles (4.4), we get

\[
\ker(d_i : P_n \to P_{n-1}) = G_i = F_{n-1}.
\]
Moreover, for $1 \leq i < j \leq n$ we have

$$G_i \cap G_j = \ker(d_i \colon P_n \to P_{n-1}) \cap \ker(d_j \colon P_n \to P_{n-1}) = R_{i,j}$$

since the restriction of $d_i$ to $G_j$ is given by the projection map

$$F(A_{1,j}, \ldots, A_{j-1,j}, A_{j,j+1}, \ldots, A_{j,n}) \to F(A_{1,j-1}, \ldots, A_{j-1,n-1}) \subseteq P_{n-1},$$

where $d_i A_{i,j} = 1, d_i A_{s,j} = A_{s,j-1}$ for $s < i$. Furthermore, $d_i A_{s,j} = A_{s-1,j-1}$ for $i < s < j$ and $d_i A_{s,j} = A_{j-1,s-1}$ for $s > j$. Consider the factors of the product (4.3).

For each $\sigma \in \Sigma_{n-2}$ we have

$$[\ldots [R_{1,n}, R_{\sigma(2),n}], \ldots, R_{\sigma(n-1),n}] \subseteq [\ldots [R_{1,n}, G_{\sigma(2)}], \ldots, G_{\sigma(n-1)}].$$

Note that

$$[\ldots [R_{1,n}, G_{\sigma(2)}], \ldots, G_{\sigma(n-1)}] \subseteq \mathrm{Brun}_n.$$ 

The following assertion enables us to construct a finite set of normal generators in $P_n$ for the subgroup $[\ldots [R_{1,n}, G_{\sigma(2)}], \ldots, G_{\sigma(n-1)}]$.

**Assertion 4.4** ([11], proof of Lemma 5.2). Let $G$ be a group and let $A$, $B$ be normal subgroups of $G$. If $\{a_i \mid i \in I\}$ is a set of normal generators for $A$ in $G$ and $\{b_j \mid j \in J\}$ is a set of generators for $B$, then $\{[a_i, b_j] \mid i \in I, j \in J\}$ is a set of normal generators for the commutator subgroup $[A, B]$ in $G$.

One can construct a set of normal generators for $[\ldots [R_{1,n}, G_{\sigma(2)}], \ldots, G_{\sigma(n-1)}]$ in the following way. Observe that $R_{1,n}$ has a normal generator $A_{1,n}$ while $G_{\sigma(2)}$ is generated by $A_{\sigma(2),j_2}$ for $1 \leq j_2 \leq n, j_2 \neq \sigma(2)$. By Assertion 4.4 we get a set of normal generators of $[R_{1,n}, G_{\sigma(2)}]$ with elements

$$[A_{1,n}, A_{\sigma(2),j_2}]$$

for $1 \leq j_2 \leq n, j_2 \neq \sigma(2)$. Repeating this procedure, we get a set of normal generators for $[\ldots[R_{1,n}, G_{\sigma(2)}], \ldots, G_{\sigma(n-1)}]$ with elements

$$[\ldots[A_{1,n}, A_{\sigma(2),j_2}], A_{\sigma(3),j_3}], \ldots, A_{\sigma(n-1),j_{n-1}}]$$

for $1 \leq j_s \leq n, j_s \neq \sigma(s)$. This proves part 1).

By definition, $\mathrm{Bd}_n = d_0(\mathrm{Brun}_{n+1})$. Since

$$d_0([\ldots [A_{1,n+1}, A_{\sigma(1)+1,j_1+1}], \ldots, A_{\sigma(n-1)+1,j_{n-1}+1}]) = [\ldots [A_{0,n}, A_{\sigma(1),j_1}], \ldots, A_{\sigma(n-1),j_{n-1}}],$$

where

$$[\ldots [A_{1,n+1}, A_{\sigma(1)+1,j_1+1}], \ldots, A_{\sigma(n-1)+1,j_{n-1}+1}] \subseteq \mathrm{Brun}_{n+1},$$

the elements listed in part 2) lie in $\mathrm{Bd}_n$. Using equation (4.3), we have

$$\mathrm{Bd}_n = \prod_{\sigma \in \Sigma_{n-1}} [\ldots[d_0(R_{1,n+1}), d_0(R_{\sigma(2),n+1})], \ldots, d_0(R_{\sigma(n),n+1})]$$

$$= \prod_{\sigma \in \Sigma_{n-1}} [\ldots[R_{0,n}, R_{\sigma(2)-1,n}], \ldots, R_{\sigma(n)-1,n}],$$

(4.5)
where \( R_{0,j} \) is the normal closure of \( A_{0,j} \) in \( P_n \). For any \( \sigma \in \Sigma_{n-1} \) acting on \( \{2, \ldots, n\} \), we get
\[
\ldots [R_{0,n}, R_{\sigma(2)-1,n}], \ldots, R_{\sigma(n)-1,n}] \leq \ldots [R_{0,n}, G_{\sigma(2)-1}], \ldots, G_{\sigma(n)-1}],
\]
where \( 1 \leq \sigma(t) - 1 \leq n - 1 \) for \( 2 \leq t \leq n \). Assertion 4.4 yields a set of normal generators for \( \ldots [R_{0,n}, G_{\sigma(2)-1}, \ldots, G_{\sigma(n)-1}] \) with elements
\[
\ldots [[A_{0,n}, A_{\sigma(2)-1,j_2}], A_{\sigma(3)-1,j_3}], \ldots, A_{\sigma(n)-1,j_n}]
\]
for \( 1 \leq j_2, j_3, \ldots, j_n \leq n \) and \( j_s \neq \sigma(s) - 1 \) for \( 2 \leq s \leq n \). This proves part 2). \( \square \)

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Roman V. Mikhailov
Steklov Mathematical Institute, RAS
Chebyshev laboratory,
St.-Petersburg State University
Institute for Advanced Study,
Princeton, NJ, USA
E-mail: romanvm@mi.ras.ru

Jie Wu
Department of Mathematics,
National University of Singapore
E-mail: matwuj@nus.edu.sg

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