Large deviation generating function for energy transport in the Pauli-Fierz model

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Abstract: We consider a finite quantum system coupled to quasifree thermal reservoirs at different temperatures. Under the assumptions of small coupling and exponential decay of the reservoir correlation function, the large deviation generating function of energy transport into the reservoirs is shown to be analytic on a bounded set. Our method is different from the spectral deformation technique which was employed recently in the study of spin-boson-like models. As a corollary, we derive the Gallavotti-Cohen fluctuation relation for the entropy production and a central limit theorem for energy transport.

KEY WORDS: Gallavotti-Cohen symmetry, nonequilibrium statistical mechanics, spin-boson model

1 Introduction

1.1 Fluctuations in open quantum systems

Recently, the physics community has shown quite some interest in current fluctuations in nonequilibrium quantum systems. We mention two interesting perspectives:

1) Since the work of [14, 17], it has become clear that nonequilibrium systems, both classical and quantum, exhibit a symmetry in the fluctuations of entropy production. This symmetry, dubbed the “Gallavotti-Cohen Fluctuation Theorem” holds far for equilibrium.

2) It has been realized [29] that noise between electron contacts shows distinct signs of Fermi statistics, studies of this kind go by the name of “Full counting statistics”.

Perhaps the most important promise of fluctuation theory is in the construction of nonequilibrium statistical mechanics: Via the study of the large deviation rate function, one hopes to find a useful variational principle describing nonequilibrium stationary states. Recent papers taking part in this project are e.g. [4, 31, 12].

In this paper, we study heat current fluctuations in a nonequilibrium model of the type ‘spin-boson’. We prove that the large deviation generating function corresponding to energy transport exists in a bounded (but arbitrarily large) set around 0 and that it is analytic.

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1.2 Large deviation generating function

We briefly sketch the framework of large deviations.

Assume that we have a family of $\mathbb{R}^d$-valued random variables $A_t$, indexed by time $t \in \mathbb{R}^+$ and with distribution given by the expectation $\mathbb{E}_t$. To fix thoughts, one can think of the $A_t$ as time-integrals of some variable $a(s), 0 \leq s \leq t$, i.e.

$$A_t := \int_0^t ds \, a(s) \quad (1.1)$$

The large deviation generating function on $\mathbb{R}^d$ (if it exists) is defined as

$$f(\kappa) := \lim_{t \uparrow \infty} \frac{1}{t} \log \mathbb{E}_t(e^{-\langle \kappa | A_t \rangle}) \quad (1.2)$$

where $\langle \cdot | \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^d$.

From the function $f(\kappa)$, one can extract large time properties of the observables $A_t$, as sketched below, see [6] for precise statements and details.

1) If $f$ is analytic in a neighbourhood of 0, then $A_t$ satisfies a central limit theorem with mean $-\frac{\partial f}{\partial \kappa}(0)$ and covariance $\sigma = \frac{\partial^2 f}{\partial \kappa^2}(0)$ (the gradient and the Hessian of $f$). Let

$$b_t := \frac{1}{\sqrt{t}} \left( A_t + t \frac{\partial f}{\partial \kappa}(0) \right), \quad (1.3)$$

then

$$\lim_{t \uparrow \infty} \mathbb{E}_t(e^{-i\langle \gamma | b_t \rangle}) = e^{-\langle \gamma | \sigma \gamma \rangle}, \quad \gamma \in \mathbb{R}^d \quad (1.4)$$

2) If $f$ is differentiable on $\mathbb{R}^d$, then the family $\frac{A_t}{t}$ satisfies a large deviation principle with rate function $I(\alpha)$ given by

$$I(\alpha) := -\inf_{\kappa \in \mathbb{R}^d} \left( \langle \kappa | \alpha \rangle + f(\kappa) \right) \quad (1.5)$$

Heuristically, this means that

$$\text{Prob}_t \left( \frac{A_t}{t} \approx \alpha \right) \sim e^{-tI(\alpha)}, \quad \alpha \in \mathbb{R}^d, t \uparrow \infty \quad (1.6)$$

(in a logarithmic sense) as $t \uparrow \infty$.

In classical statistical mechanics, the existence of the large deviation generating function can usually be established through a convexity argument, see e.g. [37]. A similar general understanding is lacking in quantum statistical mechanics (see however [1, 19, 28, 33] for partial results). Another -even conceptual- problem in quantum statistical mechanics, is how to describe...
joint large deviations of several noncommuting variables. Remark that it was exactly to solve
such a conceptual problem for the central limit theorem, that the framework of the fluctuation
algebra was constructed [18].

We consider a quantum setup where $A_t$ corresponds to the total heat transport into reser-
voirs. Hence the setup is somewhat different from that in [1][19][28][33]; the expectation $\mathbb{E}(g(A_t))$
for some function $g$ can not be formulated as an expectation of some observable in a quantum
state, rather it is the probability of obtaining certain (differences of) measurement outcomes.
The problem of joint distributions for non-commuting observables does not even appear in this
context since the different reservoir Hamiltonians do mutually commute. This is discussed more
extensively in [11]. Our result will establish the existence and analyticity of $f(\kappa)$ on a compact
(but arbitrarily large) set containing $0$. Hence, we do not prove the large deviation principle (but
we do prove the central limit theorem). The Gallavotti-Cohen fluctuation theorem is a simple
corollary of our result.

1.3 Open quantum systems with finite reservoirs

Our model describes a small quantum system (an atom) interacting with a quantum system
with many degrees of freedom (a reservoir). We choose the reservoir as simple as possible: a
free field of bosons, although fermions would do just as well. The system is coupled to the
reservoirs through a term, which is linear in the field creation and annihilation operators. This
type of models are known as Pauli-Fierz models, or, in the simplest case, the spin-boson model.
These models arise as toy-models in solid state physics, were the bosons are lattice phonons,
or through the dipole approximation in QED, where the bosons are photons, see [7] for more
background.

To make the statements mathematically sharp, we consider this field in the thermodynamic
limit, or equivalently, in the limit where the modes form a continuum. However, for the sake
of distilling the right physical question addressed in this paper, we start from a finite-volume
setup.

1.3.1 Setup

Fix a finite-dimensional Hilbert space $\mathcal{E}$ with self-adjoint Hamiltonian $E$ and let $\mathcal{K}$ be a finite
set which indexes the heat reservoirs at inverse temperatures $\beta_{k \in \mathcal{K}} > 0$. The superscript $n \in \mathbb{N}$
indicates that the thermodynamic limit ($n \to \infty$) has not yet been taken. See also Section [1.4] for
specific notation and conventions. To each $k \in \mathcal{K}$, we associate

1) A finite-dimensional one-particle Hilbert space $\mathcal{h}_{k,n}$ and its bosonical second quantization
$\Gamma_s(\mathcal{h}_{k,n})$.

\[2\text{In fact, they would simplify the technical work.}\]
2) The coupling operator $V_{k,n} \in B(\mathcal{E}, \mathcal{E} \otimes \mathfrak{h}_{k,n})$.

3) A self-adjoint one-particle Hamiltonian $h_{k,n}$ acting on $\mathfrak{h}_{k,n}$ with corresponding second quantization $d\Gamma(h_{k,n})$.

4) A Gibbs state $\rho_{k,\beta_{k}}$ on $B(\Gamma_s(h_{k,n}))$ at inverse temperature $\beta_k$

\[
\rho_{k,\beta_{k}}[R] = \frac{\text{Tr} \left[ e^{-\beta_k d\Gamma(h_{k,n})} R \right]}{\text{Tr} \left[ e^{-\beta_k d\Gamma(h_{k,n})} \right]}, \quad R \in B(\Gamma_s(h_{k,n}))
\]  \quad (1.7)

We define the total interacting Hamiltonian on $\mathcal{E} \otimes_{k \in K} \Gamma_s(h_{k,n})$ as

\[
H_{\lambda,n} = E + \sum_{k \in K} d\Gamma(h_{k,n}) + \lambda \sum_{k \in K} (a^*(V_{k,n}) + a(V_{k,n})).
\]  \quad (1.8)

We take as initial state

\[
\rho_{\mathcal{E}} \otimes \rho_{\mathcal{R}_n}, \quad \rho_{\mathcal{R}_n} := \bigotimes_{k \in K} \rho_{k,\beta_{k},n}
\]  \quad (1.9)

corresponding to initially decorrelated reservoirs and an arbitrary state $\rho_{\mathcal{E}}$ on $B(\mathcal{E})$.

### 1.3.2 Transport fluctuations and their limits

We introduced the finite volume systems in order to pick the right expression for transport fluctuations, and hence, now that all tools are in place, we ask what we mean by transport fluctuations in the finite-volume models.

Note that the reservoir Hamiltonians $d\Gamma(h_{k,n})$ mutually commute and that they have discrete spectrum. Hence one can measure them simultaneously in the beginning and at the end of an experiment. To determine the transport (of energy), we look at the differences of those measurement values. Let $T := \prod_{k \in K} \text{sp}(d\Gamma(h_{k,n}))$ and let $P_{x \in T}$ be the joint spectral projections of $d\Gamma(h_{k,n})$ corresponding to the eigenvalues $x = (x_k)_{k \in K}$. The standard interpretation of quantum mechanics yields the probabilities

\[
P_{\rho_{\mathcal{E}},t,\lambda,n}(y) := \sum_{x,x' \in T, x' - x = y} \rho_{\mathcal{E}} \otimes \rho_{\mathcal{R}_n} \left[ P_x e^{-itH_{\lambda,n}} P_x^* e^{itH_{\lambda,n}} P_x \right]
\]  \quad (1.10)

for observing energy differences $y \in \mathbb{R}^{|K|}$. The Fourier-Laplace transform of this measure has a nice expression which is better suited for taking the thermodynamic limit: Using that (the density matrix corresponding to) $\rho_{\mathcal{R}_n}$ commutes with the spectral projections $P_{x'}$, one arrives at

\[
\int_{\mathbb{R}^{|K|}} dy \, P_{\rho_{\mathcal{E}},t,\lambda,n}(y) e^{-\kappa|y|} = \rho_{\mathcal{E}} \otimes \rho_{\mathcal{R}_n} \left[ \Gamma(w_{-\kappa,n}) e^{itH_{\lambda,n}} \Gamma(w_{\kappa,n}) e^{-itH_{\lambda,n}} \right]
\]  \quad (1.11)
where
\[ w_{\kappa,n} = \left( \bigoplus_{k \in K} e^{-\kappa_k h_{k,n}} \right) \] (1.12)

We will study the infinite-volume limit of this expression, given by (2.9) and introduced in Section 2.2. In Section 3, we will substantiate the claim that (2.9) is indeed the \( n \uparrow \infty \)-limit of (1.11).

This approach to fluctuations was already used in [27, 34, 24, 38] for fluctuations of heat and work, and, most widespread, in [29, 30] for fluctuations of charge transport (“Full counting statistics”), made mathematically transparent in [26, 2].

1.4 Conventions and Notation

For \( E \) a Hilbert space, we use the standard notation for \( 1 \leq p < \infty \)
\[ B_p(E) := \{ S \in B(E), \text{Tr} \left[ (S^*S)^{p/2} \right] < \infty \} \] (1.13)
and
\[ \|S\|_p := (\text{Tr} \left[ (S^*S)^{p/2} \right])^{1/p} \] (1.14)

For a Hilbert space \( \mathfrak{h} \) we write
\[ \Gamma_n^\mathfrak{h} := \text{Sym}_n \otimes \mathfrak{h}, \quad \Gamma^\mathfrak{h} := \bigoplus_{n \in \mathbb{N}} \Gamma_n^\mathfrak{h} \] (1.15)

where \( \text{Sym}_n \) projects on the fully symmetrized subspace and \( \Gamma^\mathfrak{h} \) is the bosonic Fock space built on \( \mathfrak{h} \). For operators \( C \) on \( \mathfrak{h} \), we write (whenever the RHS is well-defined as an operator on \( \Gamma^\mathfrak{h} \))
\[ \Gamma(C) = \bigoplus_{n \in \mathbb{N}} \otimes^n C \] (1.16)
\[ C \mapsto d\Gamma(C) = \bigoplus_{n \in \mathbb{N}} \sum_{i=1}^n 1 \otimes \ldots \otimes C \otimes 1 \ldots \otimes 1 \] (1.17)

For \( W \in B(E, \mathcal{E} \otimes \mathfrak{h}) \), we use the generalized creation and annihilation operators \( a(W)/a(W^*) \) on \( \mathcal{E} \otimes \Gamma^\mathfrak{h} \), (see [7] for an extensive review of this notation). If, for some \( \psi \in \mathfrak{h} \) and \( D \in B(E) \),
\[ W u = D u \otimes \psi, \quad u \in \mathcal{E} \] (1.18)
then \( a^*(W) = D \otimes a^*(\psi) \) where \( a^*(\psi) \) is the more familiar creation operator.

For a Hilbert space \( \mathfrak{h} \), we write \( \overline{\mathfrak{h}} \) for its conjugate space, which is fixed by an antiunitary map \( \mathfrak{h} \to \overline{\mathfrak{h}} : a \mapsto \bar{a} \). If \( \mathfrak{h} = L^2(\mathcal{X}, \mathbb{C}) \) for some measure space \( \mathcal{X} \), the map \( a \mapsto \bar{a} \) is identified
with the complex conjugation on functions $X \to \mathbb{C}$. If $R \in B(\mathfrak{h})$, then $\overline{R} \in B(\mathfrak{h})$ is defined by $\overline{Ra} = \overline{R}a$.

For $\kappa \in \mathbb{C}^d$, we write

$$
\mathcal{R}\kappa = (\mathcal{R}\kappa_1, \ldots, \mathcal{R}\kappa_d), \quad \mathcal{I}\kappa = (\mathcal{I}\kappa_1, \ldots, \mathcal{I}\kappa_d) \tag{1.19}
$$

For indicator functions, we use the notation $\text{Ind}(\cdot)$, i.e. for a premise $\alpha(x)$ dependent on some variable $x$

$$
\text{Ind}(\alpha(x)) = \begin{cases} 
1 & \text{if } \alpha(x) \text{ is true} \\
0 & \text{if } \alpha(x) \text{ is false}
\end{cases} \tag{1.20}
$$

1.5 Outline

We introduce the model in abstract terms in Sections 2.1 and 2.2, immediately followed by the result in Section 2.3. The physical justification of this model is given in Section 3.2, where it is explained how it emerges from the quantities discussed in Sections 1.3.1 and 1.3.2. In Section 3.3, we discuss related results in the literature. The rest of the paper is devoted to the proofs. Section 4 contains the main line of reasoning and the technical lemma’s are postponed to Section 5. The main idea is in Lemma 4.4 whose main ingredient is Lemma 4.2.

2 Model and results

2.1 Zero-temperature objects

Introduce a finite-dimensional space $\mathcal{E}$ with a self-adjoint Hamiltonian $E$ and (for each $k \in \mathcal{K}$) one-particle spaces $\mathfrak{h}_{k,\infty}$ with a self-adjoint operator $h_{k,\infty}$ on $\mathfrak{h}_{k,\infty}$. We also need coupling operators $V_{k,\infty} \in B(\mathcal{E}, \mathcal{E} \otimes \mathfrak{h}_{k,\infty})$.

One should think of these objects as defining the zero-temperature Hamiltonian of the subsystem+reservoir system, formally

$$
H_{\lambda,\infty} := E + \sum_{k \in \mathcal{K}} d\Gamma(h_{k,\infty}) + \lambda \sum_{k \in \mathcal{K}} (a(V_{k,\infty}) + a^*(V_{k,\infty})) \tag{2.1}
$$

The heavy notation with the subscript $\infty$ is because in what follows, more natural infinite-volume objects are introduced. The objects with subscript $\infty$ are relevant at $\beta = \infty$.

We anticipate the finite-temperature by introducing the Bose-density operators

$$
\zeta_{k,\infty} := (e^{\beta_k h_{k,\infty}} - 1)^{-1} \tag{2.2}
$$

We will use the above notation to build a dynamical system which represents our system at positive temperature. The connection between the finite-volume objects, introduced in Section 1.3, and the infinite-volume model, is given in Section 3.
2.2 Positive temperatures

Define
\[ h_k := h_{k,\infty} \oplus h_{k,\infty} \quad h := \bigoplus_{k \in \mathbb{K}} h_k \]
\[ h := h_{k,\infty} \oplus (-h_{k,\infty}) \quad h := \bigoplus_{k \in \mathbb{K}} h_k \]
\[ V_k := \sqrt{1 + \zeta_{k,\infty}} V_{k,\infty} \oplus \sqrt{\zeta_{k,\infty}} V_{k,\infty} \quad V := \bigoplus_{k \in \mathbb{K}} V_k \]
\[ w_{k,k} := e^{-\kappa_k h} \quad w_k := \bigoplus_{k \in \mathbb{K}} w_{k,k} \]

Let the total Hilbert space be \( \mathcal{H} := \mathcal{E} \otimes \Gamma_s(\mathfrak{h}) \) and define on \( \mathcal{E} \otimes (\mathcal{D}(d\Gamma(h)) \cap \mathcal{D}(a(V) + a^*(V))) \),
\[ H_\lambda := E + d\Gamma(h) + \lambda (a(V) + a^*(V)) \tag{2.4} \]

The following theorem comes from [10]

**Theorem 2.1.** Assume that \( \|V\| < \infty \). Let \( H_0 := E + d\Gamma(h) \) and denote
\[ I(u) := e^{iuH_0} (a(V) + a^*(V))e^{-iuH_0}. \tag{2.5} \]

The series
\[ U^\lambda_t := e^{itH_0} \sum_{n \in \mathbb{N}} \int_{0 \leq u_1 \ldots u_n \leq t} \cdots I(u_n) \ldots I(u_1), \tag{2.6} \]
originally defined on the dense subspace \( \mathcal{D}_1 \) (see Section 4.1), extends to a strongly continuous unitary group on \( \mathcal{H} \), which will also be denoted \( U^\lambda_t \). Its self-adjoint generator is an extension of \( H_\lambda \) as in (2.4) and will be simply called \( H_\lambda \) in what follows.

Let \( \rho_\mathcal{E} \) be a state on \( \mathcal{B}(\mathcal{E}) \) and let \( \rho_\Gamma \) be the state on \( \mathcal{B}(\Gamma_s(\mathfrak{h})) \) given by
\[ \rho_\Gamma[\cdot] = \langle \Omega, \cdot \Omega \rangle \tag{2.7} \]
where \( \Omega = 1 \oplus 0 \oplus 0 \ldots \in \Gamma_s(\mathfrak{h}) \) is the vacuum vector. We will take \( \rho_\mathcal{E} \otimes \rho_\Gamma \) as initial state on \( \mathcal{B}(\mathcal{H}) \) for our dynamics. Unless otherwise stated, we assume \( \rho_\mathcal{E} \) to be arbitrary.

We now introduce our main object of study

**Assumption A-1 (Bounded interaction).** For all \( \kappa \) with \( \Re \kappa \in D \),
\[ \|w_{\frac{\kappa}{2}} V\| < \infty, \quad \|w_{-\frac{\kappa}{2}} V\| < \infty \tag{2.8} \]

The following lemma follows from Section 4.1
Lemma 2.2. Assume there is an open set \( D \subset \mathbb{R}^{|K|} \) with \( 0 \in D \) such that Assumption (A-1) is satisfied and let \( U^\lambda_t \) be as in Theorem 2.1. Then the function
\[
\kappa \mapsto \rho_\mathcal{E} \otimes \rho_\mathcal{R} \left[ \Gamma(w_{-\kappa})U^\lambda_t \Gamma(w_\kappa)U^\lambda_t \right]
\]  
has an analytical continuation from \( \{ \Re \kappa = 0 \} \) into \( \{ \Re \kappa \in D \} \).

The function (2.9) should be thought of as the Fourier-Laplace transform of the probability distribution of energy transport. This is discussed and justified in Section 3.

2.3 Results

To continue, we need additional assumptions. The next assumption basically establishes that the operator \( h \) on \( \mathfrak{h} \) has absolutely continuous spectrum.

Assumption A-2. There are measure spaces \((\mathcal{X}, dx)\) and \((\mathcal{Y}, dy)\) such that \( \mathfrak{h} = L^2(\mathcal{X}, dx) \), \( \mathcal{X} = \mathcal{Y} \times \mathbb{R} \) and \( dx = dyd\xi \) where \( d\xi \) is the Lebesgue measure on \( \mathbb{R} \). For \( (y, \xi) = x \in \mathcal{X} \), we write \( \xi(x) = \xi \) for the projection on \( \mathbb{R} \). The operator \( h \) acts by multiplication with \( \xi(x) \),
\[
(h\psi)(x) = \xi(x)\psi(x), \quad \psi \in \mathfrak{h}
\]  
(2.10)

Remark that one can associate to \( V \) a measurable function \( \mathcal{X} \to \mathcal{B}(\mathcal{E}) \), which we denote \( x \mapsto V(x) \) and which satisfies
\[
\langle v \otimes \psi, Vu \rangle_{\mathcal{E} \otimes \mathfrak{h}} = \int_{\mathcal{X}} dx \overline{\psi(x)} \langle v, V(x)u \rangle_{\mathcal{E}}, \quad u, v \in \mathcal{E}, \psi \in \mathfrak{h}
\]  
(2.11)

Define the reservoir time-correlation function
\[
p_\kappa(t) := \sup_{S \in \mathcal{B}(\mathcal{E}), \|S\|=1} \| V^* S e^{-it\hbar} w_\kappa V \|
\]  
(2.12)

Assumption A-3 (Decay of bath correlations). Let \( p_\kappa \) be as defined above. There are \( C, \alpha > 0 \) such that for \( \kappa \in D \)
\[
p_\kappa(t) \leq Ce^{-\alpha|t|}, \quad t \in \mathbb{R}
\]  
(2.13)

Introduce the set of Bohr frequencies \( \mathcal{F} := \text{sp} E - \text{sp} E \) and let \( 1_e \) stand for the spectral projection of \( E \) on \( e \in \text{sp} E \).

The following assumption expresses that the coupling between system and reservoir is sufficiently effective.

Assumption A-4 (Fermi Golden Rule).

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1. For all $\omega \in \mathcal{F}$ and $dy$-almost all $y \in \mathcal{Y}$, the function $V : \mathcal{X} \mapsto \mathcal{B}(\mathcal{E})$ is continuous on the set \{ $x = (y, \xi) \mid \xi = \omega$ \}. This implies that $V(x = (y, \omega))$ is well-defined.

2. If $S \in \mathcal{B}(\mathcal{E})$ satisfies
\[
\sum_{\omega \in \mathcal{F}} \sum_{e, e' \in \text{sp}E, \omega = e - e'} \int_{\mathcal{Y}} dy \left\| [S, 1_{\mathcal{E}} V(y, \omega)] 1_{\mathcal{E}_e} \right\| = 0
\]
then $S = c_1$ for some $c \in \mathbb{C}$.

Now comes our main theorem

**Theorem 2.3.** Assume Assumptions A-1, A-2, A-3, and A-4. There is a $\lambda_0 > 0$ such that for $\lambda \in [-\lambda_0, \lambda_0]$ and $\kappa \in D$,
\[
f(\kappa, \lambda) := \lim_{t \to \infty} t^{-1} \log \rho_{\mathcal{E}} \otimes \rho_{\mathcal{R}} \left[ \Gamma(w_{-\kappa})U_{-t}^\lambda \Gamma(w_{\kappa})U_t^\lambda \right]
\]
exists, is independent of $\rho_{\mathcal{E}}$ and real-analytic in $\kappa$ and $\lambda$.

As the main corollary, we state the Gallavotti-Cohen fluctuation theorem for the entropy production. This requires an additional assumption

**Assumption A-5 (Time-reversal invariance).** There is an anti-unitary $\Theta$ on $\mathcal{H}$ such that for all $\lambda \in \mathbb{R}$,
\[
\Theta^{-1} H_{\lambda} \Theta = H_{\lambda}, \quad \Theta^{-1} \Gamma(h_{k}) \Theta = \Gamma(h_{k}), \quad \Theta(\mathcal{E} \otimes \mathcal{\Omega}) = (\mathcal{E} \otimes \mathcal{\Omega})
\]
and $\Theta$ is an involution, i.e. $\Theta^{-1} = \Theta$.

**Theorem 2.4.** Assume A-1, A-2, A-3, A-4, and A-5. Let $f(\kappa, \lambda)$ be as in Theorem 2.3 and define for $\nu \in \mathbb{R}$
\[
\kappa(\nu) = \nu(\beta_1, \beta_2, \ldots, \beta_{|\mathcal{K}|}) \in \mathbb{R}^{|\mathcal{K}|}
\]
For $\nu \in \mathbb{R}$ such that $\kappa(\nu), \kappa(1 - \nu) \in D$,
\[
f(\kappa(\nu), \lambda) = f(\kappa(1 - \nu), \lambda)
\]
By Bochner’s theorem, there is a nonnegative Borel measure $dP_{\rho_{\mathcal{E}}, t, \lambda}$ on $\mathbb{R}^{|K|}$ such that
\[
\rho_{\mathcal{E}} \otimes \rho_{\mathcal{R}} \left[ \Gamma(w_{-\kappa})U_{-t}^\lambda \Gamma(w_{\kappa})U_t^\lambda \right] = \int_{\mathbb{R}^{|\mathcal{K}|}} dP_{\rho_{\mathcal{E}}, t, \lambda}(y)e^{-\langle \kappa, y \rangle}
\]
for $\mathbb{R} \kappa \in D$. Putting $\kappa = 0$, one sees that $d\mathbb{P}_{\rho E, t, \lambda}$ is a probability measure. It is the infinite-volume analogue of the probabilities $\mathbb{P}_{\rho E, t, \lambda, n}$ introduced in Section 1.3.2. We write $\mathbb{P}_{\rho E, t, \lambda} [\cdot]$ for the expectation w.r.t. $d\mathbb{P}_{\rho E, t, \lambda}$.

The $\mathbb{R}^d$-valued random variable $y = (y_k)$ is interpreted as the energy transport into the distinct reservoirs. Remark that in thermodynamics, one interprets $S := \sum_{k \in K} \beta_k y_k$ as the entropy production.

Since
\[
f(\kappa(\nu), \lambda) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_{\rho E, t, \lambda} [e^{-\nu S}]
\]  
(2.20)
one sees that $f(\kappa(\nu), \lambda)$ is indeed related to (large) fluctuations of the entropy production.

The following corollary follows from Theorem 2.3 by [5].

**Corollary 2.5.** Assume the assumptions of Theorem 2.3. Then the $\mathbb{R}^d$-valued random variable $y$ satisfies a central limit theorem with mean $-\frac{\partial}{\partial \kappa} f(\kappa, \lambda) |_{\kappa = 0}$ and covariance $\sigma_\lambda := \frac{\partial^2}{\partial \kappa^2} f(\kappa, \lambda) |_{\kappa = 0}$. Let
\[
b_t := \frac{1}{\sqrt{t}} \left( y + t \frac{\partial}{\partial \kappa} f(\kappa, \lambda) |_{\kappa = 0} \right),
\]  
(2.21)
then
\[
\mathbb{E}_{\rho E, t, \lambda} [e^{-i(\gamma | b_t)}] \to e^{-i(\gamma | \sigma_\lambda \gamma)} , \quad \gamma \in \mathbb{R}^{|K|}
\]  
(2.22)

The expectation value $-\frac{\partial}{\partial \kappa} f(\kappa, \lambda) |_{\kappa = 0}$ and the covariance $\frac{\partial^2}{\partial \kappa^2} f(\kappa, \lambda) |_{\kappa = 0}$ can be written in a more familiar form. Introduce the operators
\[
\triangle_{k, t} := U_{-t}^\lambda d\Gamma(h_k) U_t^\lambda - d\Gamma(h_k),
\]  
(2.23)
Then
\[
-\frac{\partial}{\partial \kappa} f(\kappa, \lambda) |_{\kappa = 0} = \lim_{t \to \infty} \frac{1}{t} \rho_E \otimes \rho_R [\triangle_{k, t}] =: \langle \triangle_k \rangle
\]  
(2.24)
\[
\frac{\partial^2}{\partial \kappa_k \partial \kappa_{k'}} f(\kappa, \lambda) |_{\kappa = 0} = \lim_{t \to \infty} \frac{1}{t} \rho_E \otimes \rho_R [(\triangle_{k, t} - t \langle \triangle_k \rangle) (\triangle_{k', t} - t \langle \triangle_{k'} \rangle)]
\]  
(2.25)
where the convergence of the expressions on the RHS is a consequence of the analyticity of $f(\kappa, \lambda)$. However, it is not true in general (beyond second order in $\kappa$) that
\[
\rho_E \otimes \rho_R [\Gamma(w_{-\kappa}) U_{-t}^\lambda \Gamma(w_\kappa) U_t^\lambda] = \rho_E \otimes \rho_R [e^{-\sum_k \kappa_k \triangle_{k, t}}].
\]  
(2.26)
See [11] for a thorough discussion of different approaches to quantum fluctuations.
3 Discussion

3.1 Initial state

We formulate our result only for particular initial states, namely $\rho_E \otimes \rho_R$ with $\rho_R$ the vacuum state. One could ask whether Theorem 2.3 still holds for a different initial state. In fact, by a slight generalization of our method, one can prove (see e.g. the previous version of the present paper) that the same result holds if one replaces

$$\rho_E \otimes \rho_R \left[ \Gamma(w_{-\kappa})U_{-t}^\lambda \Gamma(w_{\kappa})U_{t}^\lambda \right]$$

by

$$\rho_E \otimes \rho_R \left[ U_{-s}^\lambda \Gamma(w_{-\kappa})U_{-t}^\lambda \Gamma(w_{\kappa})U_{t}^\lambda U_{s}^\lambda \right]$$

for arbitrary $s$. That is, $f(\kappa, \lambda)$ is independent of $s$.

However, in Section 1.3.2 one sees that the very choice of our object of study (3.1) depends on the fact that $\rho_R$ is 'diagonal' in the operators $d\Gamma(h_k)$. This (or rather, its finite-volume analogue) is used in going from (1.10) to (1.11). Expressed more dramatically, an expression like (3.2) does not appear!

3.2 Thermodynamic limit

We skipped over a thorough justification of the object (2.9), which features in our results. We remedy this by telling in which sense the dynamical system is the infinite-volume version of the finite-volume systems and how the expression (2.9) emerges. Usually, thermodynamical limits are constructed by specifying volumes which go to infinity in some sense (e.g. in the sense of Van Hove). In our case, such an explicit setup is not necessary (though of course possible). We simply demand the following relation between the finite-volume objects and the objects introduced in Section 2.1.

Assumption A-6 (Thermodynamic limit of finite-volume models). Let

$$g_1, t(x) = \frac{e^{-itx}}{e^{\beta k x} - 1}, \quad g_2, t(x) = e^{-itx} \left( 1 - \frac{1}{e^{\beta k x} - 1} \right), \quad t \in \mathbb{R}, x \in \mathbb{R}^+$$

For $S \in \mathcal{B}(\mathcal{E})$, $i = 1, 2$, we have

$$\| V_{k,\infty}^* S g_{i, t}(h_{k, \infty}) V_{k, \infty} \| < \infty$$

and

$$V_{k,n}^* S g_{i, t}(h_{k,n}) V_{k,n} \longrightarrow V_{k,\infty}^* S g_{i, t}(h_{k,\infty}) V_{k, \infty}$$

uniformly on compacts in $t \in \mathbb{R}$. 

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If Assumption A-6 is satisfied, a large class of correlation functions converges. There is quite some arbitrariness in this statement, which is usually not considered in the literature.

Define
\[ \Phi_{k,n}(t) := e^{itH_{0,n}}(a(V_{k,n}) + a^*(V_{k,n}))e^{-itH_{0,n}} \]  
(3.6)
\[ \Phi_k(t) = e^{itH_0}(a(V_k) + a^*(V_k))e^{-itH_0} \]  
(3.7)

Assume Assumption A-6, then for all \( t,t' \in \mathbb{R} \) and \( S \in B(E) \),
\[ \rho_E \otimes \omega_{R,n}\left[ \Phi_{k,n}(t)S\Phi_{k',n}(t') \right] \rightarrow_{n \uparrow \infty} \rho_E \otimes \rho_R\left[ \Phi_k(t)S\Phi_k(t') \right] \]  
(3.9)

Of course, from (3.9) one deduces also convergence of higher-order correlation functions (since the states \( \omega_{R,n} \) and \( \rho_R \) are quasifree, those are expressed in terms of the second order correlation function). In particular, one has also convergence of the same correlation functions with the time-dependence now given by the fully interacting evolution, that is, let
\[ \Phi^I_{k,n}(t) := e^{itH_{\lambda,n}}e^{-itH_{0,n}}\Phi_{k,n}(t)e^{itH_{0,n}}e^{-itH_{\lambda,n}} \]  
(3.10)
\[ \Phi^I_k(t) := e^{itH_\lambda}e^{-itH_0}\Phi_k(t)e^{itH_0}e^{-itH_\lambda}, \]  
(3.11)
then equation (3.9) holds with \( \Phi^I \) replacing \( \Phi \), as follows from a Dyson expansion, e.g. (4.3).

It is now straightforward to see that Assumptions A-1 and A-6 imply
\[ \rho_E \otimes \rho_{R,n}\left[ \Gamma(w_{-\kappa,n})e^{itH_{\lambda,n}}\Gamma(w_{\kappa,n})e^{-itH_{\lambda,n}} \right] \rightarrow_{n \uparrow \infty} \rho_E \otimes \rho_R\left[ \Gamma(w_{-\kappa})U^\lambda_s\Gamma(w_{\kappa})U^\lambda_s \right] \]  
(3.12)
where the LHS was introduced through physical considerations in Section 1.3.2.

The critical reader might wonder why there is in our presentation no mention of \( W^* \)-algebra’s, which often play a prominent role in the mathematical formulation of statistical mechanics. If one defines the Araki-Woods algebra \( \mathcal{A} \) as in Section 4.6, one finds that the dynamics
\( \mathcal{A} \ni A \mapsto U_{-s}^\lambda AU_s^\lambda, \quad s \in \mathbb{R} \)  
(3.13)
leaves \( \mathcal{A} \) invariant. Physically, one should restrict the state \( \rho_E \otimes \rho_R \), originally defined on \( B(E \otimes \Gamma_s(\mathfrak{h})) \), to \( \mathcal{A} \). However, in our approach, it is neither mathematically nor physically necessary to consider this restriction. We study the expression (3.1), which is well-defined and whose motivation is via (3.12).

For the same reasons, we do not have to ask ourselves whether the operator (2.4) is the right choice. In the literature, this operator is called the semi-standard Liouvillean, but one can also consider the standard Liouvillean. Again, the resolution of any possible ambiguity is via finite-volume limits. That being said, it might be worth remarking that (3.1) can be expressed as the expectation of powers of a relative modular operator, see [32], thus providing a more algebraic starting point for our work. Another possible approach is in [2], where the expression (3.1) is constructed (for fermions) via different, but essentially equivalent reasoning.
3.3 Comparison with other works

There has lately been a lot of work on spin-boson and spin-fermion models, or more general, Pauli-Fierz models.

We feel our work is technically closest to [22], in which one considers the spin-boson model and one proves that the generator of the dynamics has absolutely continuous spectrum for \( \lambda \neq 0 \), except for one eigenvalue which corresponds to the stationary state. The other eigenvalues of the system at \( \lambda = 0 \) turn into resonances whose location is in first nonvanishing order predicted by the Lindblad generator. The assumptions are very similar; to allow for a comparison, we assume that

\[
V u = Du \otimes \psi
\]

for some \( D \in B(\mathcal{E}) \) and \( \psi \in \mathfrak{h} \sim L^2(\mathbb{R}, L^2(\mathcal{Y}, d\gamma)) \), in which case \( a^*(V) = D \otimes a^*(\psi) \).

The basic assumption in [22] reads

**Assumption A-7** (analytic coupling). *The function \( \psi \) is analytic in a strip \( \{ \Im z \leq \delta \} \) and*

\[
\sup_{\gamma \in [-\delta, \delta]} \int_{\mathbb{R}} d\xi \| \psi(\xi + i\gamma) \|^2 < \infty
\]

(3.15)

Assumption A-7 implies that

\[
\left| \int_{\mathbb{R}} d\xi \| \psi(\xi) \|^2 e^{-it\xi} \right| \leq Ce^{-t\delta}
\]

(3.16)

which is just Assumption A-3 for \( \kappa = 0 \). However, the \( \kappa \) have no analogue in [22] and we would need to assume Assumption A-7 with \( \psi \) derived through (3.14) from \( w_{\kappa}V \) rather than from \( V \).

In contrast, we do not need any additional infrared condition on \( \xi \mapsto \psi(\xi) \), contrary to [22]. This is because we construct the dynamics via the Dyson expansion instead of via the Nelson commutator theorem. Physically speaking\(^3\), there is of course already an infrared condition present since

\[
\| V \| < \infty \quad \Rightarrow \quad \sum_{k \in K} \| (\beta_{k} h_{k,\infty})^{-1/2} V_{k,\infty} \| < \infty
\]

(3.17)

with the notation as in Section 2.1.

The technique of [22] consists of a spectral deformation of the generator \( H_{\lambda} \). We employ time-dependent perturbation theory and we rewrite the Dyson expansion as a one-dimensional polymer model. This is embodied in Lemma 4.2. Starting from that lemma, one can obtain our result through a simple cluster expansion (as in the previous version of this paper). However, since the polymer model is one-dimensional, we can apply the transfer-matrix technique. In

\(^3\)That is, in terms of the zero-temperature coupling operator, or 'form-factor' \( V_{k,\infty} \)
dealing with the transfer matrix, we use a variant of the spectral deformation technique, such that our technique is not as different from [22] as might seem.

Assumption [A-3] cannot be weakened without changing the method drastically. Note that one cannot assume Assumption [A-3] for $D = \mathbb{R}^{[K]}$ since that would imply that

$$\mathbb{R}^{[K]} \ni \kappa \mapsto p_\kappa(t)$$

(3.18)

is a bounded analytic function, hence constant.

Results that need weaker regularity properties of $\psi(\xi)$ are e.g. [3], [9, 8], [15]. In those works one employs Mourre theory or renormalization group techniques, however they do not permit to localize the resonances.

A different type of works are those using scattering theory. This approach was initiated in [36], but so far, it has not been successful for spin-boson type models, although it works well for junctions [16].

From the physical point of view, our result is closer to [23, 20] where one studies a non-equilibrium setup and one derives approach to a non-equilibrium steady state and the Green-Kubo relations, or to [21], where one studies a form of the central limit theorem. See [11] for an extensive discussion of the difference and similarities of different approaches to quantum fluctuations and central limits.

4 Proof of Theorems 2.3 and 2.4

4.1 Construction of the dynamics

Let $1_n$ be the projector on $E \otimes \Gamma_s^n(h)$ (the $n$-particle sector, see Section 1.4) and let the domain $D_1 \subset E \otimes \Gamma_s(h)$ be defined by

$$\psi \in D_1 \iff \exists C > 0 : \|1_n(\psi)\| \leq \frac{C^n}{\sqrt{n!}}$$

(4.1)

Let $H_0 := E + d\Gamma(h)$ and

$$I_\kappa(u) := e^{iuH_0}(a(w_{-\kappa}V) + a^*(w_\kappa V))e^{-iuH_0}.$$  

(4.2)

For $\Re \kappa = 0$, the series

$$e^{uH_0} \sum_{n \in \mathbb{N}} \int_{0 \leq u_1 \ldots u_n \leq t} du_1 \ldots du_n I_{\phi_+}^n(u_n) \ldots I_{\phi_+}^n(u_1),$$

(4.3)
originally defined on $D_1$, extends to the unitary group (Theorem 6.1 [10])

$$\Gamma(w_\frac{1}{2})U_t^\lambda \Gamma(w_{-\frac{1}{2}}) \tag{4.4}$$

Since the argument in [10] showing that (4.3) is a strongly continuous group on $D_1$, depends only on the assumption $\|V\| < \infty$, this remains true for $\kappa$ satisfying Assumption A-1, and (4.3) can be taken as the definition of (4.4).

In what follows and unless stated otherwise, we will assume that Assumptions A-1 and A-2 are satisfied and that $\Re \kappa \in D$.

### 4.2 Dynamics and notation on $B_1(\mathcal{H})$

It is advantageous to rewrite the object of study in a slightly more abstract way. Let $D_{1,\otimes}$ stand for the subspace of $B_1(E \otimes \Gamma_s(h))$ defined by finite linear combinations of $|\phi_1\rangle\langle\phi_2|$ for $\phi_1, \phi_2 \in D_1$. From the conclusions of Section 4.1 it follows that

$$A \mapsto (\Gamma(w_{\frac{1}{2}})U_t^\lambda \Gamma(w_{-\frac{1}{2}})) A (\Gamma(w_{-\frac{1}{2}})U_t^{\lambda^*} \Gamma(w_{\frac{1}{2}})) =: Z_{t,\kappa,\lambda}(A) \tag{4.5}$$

maps $D_{1,\otimes}$ into itself. In what follows, we write

$$M(S) := i[E, S] \tag{4.6}$$

as a bounded operator on $B(E)$. We define the embedding $\mathcal{I}_1 : B(E) \to B_1(E \otimes \Gamma_s(h))$

$$S \mapsto \mathcal{I}_1(S) = S \otimes |\Omega\rangle\langle\Omega| \tag{4.7}$$

and the compression $\mathcal{I}_1 : B_1(E \otimes \Gamma_s(h)) \to B(E)$

$$(S \otimes R) \mapsto \mathcal{I}_1(S \otimes R) = S \operatorname{Tr}_{\Gamma_s(h)}[R]. \tag{4.8}$$

with $\operatorname{Tr}_{\Gamma_s(h)}$ the trace on $B_1(\operatorname{Tr}_{\Gamma_s(h)})$ (Hence $\mathcal{I}_1$ is actually a partial trace).

We have hence rewritten

$$\rho_E \otimes \rho_R \left[ \Gamma(w_{-\kappa})U_t^{\lambda^*} \Gamma(w_{\kappa})U_t^{\lambda} \right] = \operatorname{Tr} \left[ (\mathcal{I}_1 Z_{t,\kappa,\lambda}) (\tilde{\rho}_E) \right] \tag{4.9}$$

where $\tilde{\rho}_E$ is the density matrix, corresponding to the state $\rho_E$, i.e. $\rho_E [S] = \operatorname{Tr} [\tilde{\rho}_ES]$. 

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4.3 The deformed Lindblad generator

If \( \| p_\kappa \|_1 := \int dt \, p_\kappa(t) < \infty \) for \( \kappa = 0 \), we can define

\[
\Upsilon = -i \sum_{\omega \in \mathcal{F}} \sum_{\epsilon = \epsilon'} \int_0^\infty 1_{\mathcal{E}_\epsilon} V^* \mathcal{E}_{\epsilon'} e^{-i t \omega} V 1_{\mathcal{E}_\epsilon} dt. \tag{4.10}
\]

Assuming additionally the first statement of Assumption \([A-4]\) we introduce the deformed Lindblad generator. For \( S \in \mathcal{B}(\mathcal{E}) \), let

\[
L_\kappa(S) = -i(\Upsilon S - S \Upsilon^*) + 2\pi \sum_{\omega \in \mathcal{F}} \sum_{\epsilon = \epsilon'} 1_{\mathcal{E}_\epsilon} V^* 1_{\mathcal{E}_{\epsilon'}} S \delta(h - \omega) \mathcal{W}_\kappa V 1_{\mathcal{E}_\epsilon} \tag{4.11}
\]

where the operator-valued Dirac-delta distribution \( \delta(\cdot) \) is well-defined by the continuity assumption in Assumption \([A-4]\). For example, one can take a sequence of functions converging in the sense of distributions to \( \delta(h - \omega) \), then the mentioned continuity assumption assures convergence in \( (4.11) \). One checks, see e.g. \([10]\), that for \( \kappa = 0 \), or equivalently, \( w_\kappa = 1 \), we recover the usual definition for the Lindblad generator, which satisfies

\[
\text{Tr}[L_{\kappa=0}(S)] = 0 \tag{4.12}
\]

However, since the second term in \( (4.11) \) is a completely positive map, it follows that \( e^{t L_\kappa} \) is a completely positive semigroup for \( \{\Re \kappa \in D\} \).

We need the following properties of \( L_\kappa \).

**Theorem 4.1.** Let \( L_\kappa \) be as in \( (4.11) \) and \( M \) as defined in Section 4.2.

1) Assume Assumption \([A-4]\) and fix a \( \tau < 0 \). The operator \( e^{\tau L_\kappa} \) has a maximal simple eigenvalue \( e^{\tau f_\kappa} \) with \( f(\kappa) \in \mathbb{R} \) and there is a ‘gap’ \( g_\kappa > 0 \) such that

\[
\sup\{|z|, z \in \text{sp}(e^{\tau L_\kappa}) \setminus e^{\tau f_\kappa} \} < e^{\tau f_\kappa}(1 - e^{-\tau g_\kappa}) \tag{4.13}
\]

The eigenvector corresponding to \( e^{\tau f_\kappa} \) can be chosen a positive invertible operator.

2) \( [L_\kappa, M] = 0 \) \tag{4.14}

3) Assume \( \| p_\kappa \|_1 < \infty \). For all \( \tau > 0 \),

\[
\| \mathcal{I}_1 Z^{\kappa, \lambda} \mathcal{I}_1 - e^{-i \tau (\lambda^2 M + i L_\kappa)} \| \xrightarrow{\lambda \downarrow 0} 0 \tag{4.15}
\]

where the LHS is continuous in \( \lambda, \kappa, \tau \).

Statement (1) of Theorem 4.1 is the only place where we use the second statement of Assumption \([A-4]\). It is a non-degeneracy assumption which enters the non-commutative Perron-Frobenius theorem.
4.4 Dyson expansion and transfer operator

Our basic tool is a rearranged Dyson expansion, whose properties are collected in the upcoming Lemma 4.2. Fix a parameter \( \tau > 0 \) and define on \( B(E) \) for \( n \in \mathbb{N}_0 \),
\[
W_{n}^{\kappa,\lambda,\tau} := I_{\uparrow} Z_{\lambda-2\tau}^{\kappa,\lambda} (1 - I_{\uparrow} I_{\downarrow}) \ldots (1 - I_{\uparrow} I_{\downarrow}) Z_{\lambda-2\tau}^{\kappa,\lambda} (1 - I_{\uparrow} I_{\downarrow}) I_{\downarrow} \tag{4.16}
\]
\((n - 1 \text{ factors of } (1 - I_{\uparrow} I_{\downarrow}) \text{ inserted}). The definition (4.16) makes sense since \( Z_{\kappa,\lambda}^{\tau} \) maps \( D_{1,\otimes} \) into itself (see Section 4.1) and, obviously, \( I_{\uparrow} B(E) \subset D_{1,\otimes} \). Whenever reasonable, we will abbreviate \( W_{n} = W_{n}^{\kappa,\lambda,\tau} \).

Lemma 4.2. Let \( W_{n} = W_{n}^{\kappa,\lambda,\tau} \) be as above.

1) For all \( m \in \mathbb{N}_0 \),
\[
I_{\downarrow} Z_{\lambda-2m\tau}^{\kappa,\lambda} I_{\uparrow} = \sum_{r \in \mathbb{N}} \sum_{\sum_{i=1}^{r} n_i = m} W_{n_r} \ldots W_{n_2} W_{n_1} \tag{4.17}
\]

2) Assume Assumption A-3. There is \( c := c(\kappa, \lambda, \tau) > 0 \), vanishing as \( \lambda \downarrow 0 \) and continuous in the three parameters, such that for \( n > 1 \),
\[
\|W_{n}\| \leq c^{n-1} \tag{4.18}
\]

In what follows, we use the Hilbert space \( l_2(\mathbb{N}_0) \otimes B_2(E) \). Let for \( n \in \mathbb{N}_0 \), \( e_n \) be the canonical \( n' \)th base vector in \( l_2(\mathbb{N}_0) \) and let \( S \) be the unilateral shift, defined by (setting \( e_0 := 0 \))
\[
Se_n = e_{n-1} \tag{4.19}
\]
Recall that \( E \) is finite-dimensional, which allows to define the embedding \( P_n : B(E) \to l_2(\mathbb{N}_0) \otimes B_2(E) : u \mapsto e_n \otimes u \) and compression \( P_n^* : e_n \otimes u \mapsto u \). We are led to examine the following operator on \( l_2(\mathbb{N}_0) \otimes B_2(E) \);
\[
T_{\kappa,\lambda,\tau} = \sum_{n \in \mathbb{N}_0} P_n W_n P_1^* + S \otimes 1 \tag{4.20}
\]
From Lemma 4.2(1), one has
\[
I_{\downarrow} Z_{\kappa,\lambda}^{\lambda-2m\tau} I_{\uparrow} = P_1^* (T_{\kappa,\lambda,\tau})^m P_1 \tag{4.21}
\]
If the operator \( T := T_{\kappa,\lambda,\tau} \) had a maximal eigenvalue, isolated from the rest of the spectrum, we could easily estimate the \( n \nearrow \infty \) asymptotics of (4.21). However, upon realizing that \( \text{sp} S = \{ z \in \mathbb{C}, |z| \leq 1 \} \),
\[
\text{sp} S = \{ z \in \mathbb{C}, |z| \leq 1 \} \tag{4.22}
\]
this surely fails at \( \kappa = 0 \), since the highest eigenvalue of \( e^{\tau L_{\kappa=0}} \) is 1. This difficulty is addressed in the next section.
4.5 Spectral deformation

Introduce the unbounded operator

\[ R = \sum_{n \in \mathbb{N}_0} n P_n P_n^* . \]  

(4.23)

The following statements are straightforward.

**Lemma 4.3.** For \( \delta \in \mathbb{R} \) and \( W \in \mathcal{B}(\mathcal{B}(\mathcal{E})) \),

1) \( e^{\delta R} S e^{-\delta R} = e^{-\delta S} \)
2) \( e^{\delta R} P_n W P_m^* e^{-\delta R} = e^{(n-m)\delta} P_n W P_m^* \)
3) \( P_1^* T^m P_1 = P_1^* (e^{\delta R T e^{-\delta R}})^m P_1 \)

Most importantly, the operator \( e^{\delta R T e^{-\delta R}} \) does have an isolated eigenvalue for well-chosen \( \delta \), as we show now.

**Lemma 4.4.** Let \( \hat{\delta} := -1/2 \ln c(\kappa, \lambda, \tau) \), where the latter was introduced in Lemma 4.2. There is a \( \lambda_0 > 0 \) such that for \( \lambda \in [-\lambda_0, \lambda_0], \kappa \in D \) and \( \tau \) varying in some compact set \( D_\tau \), the operator

\[ e^{\hat{\delta} R T e^{-\hat{\delta} R}} \]  

has a maximal simple eigenvalue \( e^{\tau f_{\kappa, \lambda, \tau}} \) with \( f_{\kappa, \lambda, \tau} \in \mathbb{R} \). There is \( g_{\kappa, \lambda, \tau} > 0 \) such that

\[ \sup\{ |z|, z \in \text{sp}(e^{\hat{\delta} R T e^{-\hat{\delta} R}}) \setminus e^{\tau f_{\kappa, \lambda, \tau}} \} < e^{\tau f_{\kappa, \lambda, \tau}} (1 - e^{-\tau g_{\kappa, \lambda, \tau}}) . \]  

(4.25)

The eigenvector \( G_{\kappa, \lambda, \tau} \) corresponding to this eigenvalue can be chosen such that \( P_1^* G_{\kappa, \lambda, \tau} \in \mathcal{B}(\mathcal{E}) \) is an invertible, positive operator. The function \( f_{\kappa, \lambda, \tau} \) is real-analytic in \( \kappa \in D, |\lambda| \leq \lambda_0 \) and \( \tau \in D_\tau \).

**Proof.** By Lemma 4.3

\[ e^{\delta R T e^{-\delta R}} = e^{-i\tau(\lambda-2M+iL_\kappa)} + \Delta T \]  

(4.26)

where

\[ \Delta T := e^{-\delta S} + (W_1 - e^{-i\tau(\lambda-2M+iL_\kappa)}) + \sum_{n>1} e^{(n-1)\delta} P_n W_n P_1^* \]  

(4.27)

By Lemma 4.2 (let \( c \) be as defined therein) and assuming \( |ce^\delta| < 1 \),

\[ \|\Delta T\| \leq e^{-\delta} + \|W_1 - e^{-i\tau(\lambda-2M+iL_\kappa)}\| + (\|P_1^*\| \sup_{n \in \mathbb{N}_0} \|P_n\|) \frac{ce^\delta}{1 - ce^\delta} \]  

(4.28)
The norms $\|P_n\|, \|P_n\|$ are independent of $n$ and finite since $\dim \mathcal{E}$ is finite, and hence, using Theorem 4.1(3), $\|\Delta T\|$ vanishes as $\lambda \downarrow 0$ and as $\delta = \hat{\delta}$.

Remark that $M$ is self-adjoint on $B_2(\mathcal{E})$ and that $\text{sp} M = \mathcal{F}$. By Theorem 4.1(2), we can hence decompose $L_\kappa = \bigoplus_{\omega \in \mathcal{F}} L_{\kappa, \omega}$ where $L_{\kappa, \omega}$ acts on the $\omega$-eigenspace of $M$. Hence

\[
\left( z - e^{-\im \tau(\lambda^2 + \im L_\kappa)} \right)^{-1} = \bigoplus_{\omega \in \mathcal{F}} e^{\im \tau \lambda^2 \omega} \left( e^{\im \tau \lambda^2 \omega} z - e^{\tau L_{\kappa, \omega}} \right)^{-1} \tag{4.29}
\]

Theorem 4.1(1), the expression (4.29) and compactness of the unit circle in $\mathbb{C}$ yield that there is a $\epsilon > 0, C > 0$, such that for $\epsilon > 0, C > 0$, such that for $e^{\tau f_{\kappa}} - \epsilon < |z| < e^{\tau f_{\kappa}}$, for $\kappa \in D$ and for $\tau$ varying over some compact set,

\[
\| \left( z - e^{-\im \tau(\lambda^2 + \im L_\kappa)} \right)^{-1} \| \leq C \left( |z| - e^{\tau f_{\kappa}} \right)^{-1} \tag{4.30}
\]

The existence of an isolated eigenvalue and positivity of the eigenvector now follows from (4.28) by standard perturbation theory, see e.g. [25]. Positivity of the eigenvalue follows since by (4.3), $Z_{\kappa, \lambda}$ is a completely positive map for $\Im \kappa = 0$.

Real Analyticity in $\kappa$ and $\lambda$ for $\lambda \neq 0$ follows from analyticity of $L_\kappa$ and $\Delta T$, both of which are straightforward consequences of Assumption A-1. Since $e^{\im \lambda^2 M}$ does not have a limit as $\lambda \downarrow 0$, analyticity at $\lambda = 0$ is not immediate. However, since, $f_{\kappa, \lambda, \tau}$ is analytic for $\lambda \neq 0$ and continuous at $\lambda = 0$, it is analytic.

By Lemma 4.4, we get for $m$ large enough

\[
\frac{1}{\tau m} \log I_{\lambda, \tau} = f_{\kappa, \lambda, \tau} + \frac{1}{\tau m} \log \left( P_1^* P_{G_{\kappa, \lambda, \tau}} P_1 + O(e^{-\tau m g_{\kappa, \lambda, \tau}}) \right) \tag{4.31}
\]

where $P_{G_{\kappa, \lambda, \tau}}$ is the projection on $G_{\kappa, \lambda, \tau}$.

Taking $\tau, \tau' \in D$, such that $m \tau = m' \tau'$ for some $m, m' \in \mathbb{N}$, we get from (4.31) that $f_{\kappa, \lambda, \tau} = f_{\kappa, \lambda, \tau'}$. Since $f_{\kappa, \lambda, \tau}$ is also continuous in $\tau$, it is constant and we write $f_{\kappa, \lambda} := f_{\kappa, \lambda, \tau}$. Theorem 2.3 now follows with $f(\kappa, \lambda) = \lambda^2 f_{\kappa, \lambda}$ by (4.9).

### 4.6 Proof of Theorem 2.4

Assume that

\[
\rho_\mathcal{F}[S] = \frac{1}{\dim \mathcal{E}} \text{Tr}[S] \tag{4.32}
\]

Let $\mathcal{U}$ be the $W^*$-algebra (Von Neumann-algebra) which is generated by the sets

\[
\mathcal{B}(\mathcal{E}) \otimes 1 \quad \text{and} \quad \{ e^{i(\alpha(\psi) + a^*(\psi))}, \psi \in \mathfrak{h} \} \tag{4.33}
\]
Remark that the expansion (4.3) shows that for all \( t \),
\[
e^{itH_0}e^{-itH} \in \mathfrak{U}.
\] (4.34)
(See [7] for details on \( W^* \)-algebra’s). Extend the notation \( \kappa(\nu) \) in Theorem 2.4 to \( \nu \in \mathbb{C} \). The maps of automorphisms
\[
\mathfrak{U} \ni A \mapsto \eta_s(A) := \Gamma(w_{\kappa(is)})A\Gamma(w_{\kappa(-is)}), \quad s \in \mathbb{R}
\] (4.35)
is a \( W^* \)-dynamics and \( \rho_\mathcal{E} \otimes \rho_\mathcal{R} \) is a 1-KMS state wrt. to this dynamics. This can be easily checked or read in the literature, see again [7]. Then, the KMS-condition reads that for \( A, B \in \mathfrak{U} \), the function
\[
\rho_\mathcal{E} \otimes \rho_\mathcal{R}[A\eta_s(B)]
\] (4.36)
is analytic in \( \{0 \leq \Im s \leq 1\} \) and satisfies
\[
\rho_\mathcal{E} \otimes \rho_\mathcal{R}[\eta_s(A)B] = \rho_\mathcal{E} \otimes \rho_\mathcal{R}[B\eta_{s+i}(A)]
\] (4.37)
Choosing \( A = e^{-itH_0}e^{itH} \) and \( B = e^{-itH}e^{itH_0} \), inserting \( 1 = \Theta \Theta \), using Assumption A-5, the general property \( \rho(C^*) = \overline{\rho(C)} \) (true for every state \( \rho \)), \( [e^{-itH_0}, \Gamma(w_{\kappa})] = 0 \) and the invariance of \( \rho_\mathcal{E} \otimes \rho_\mathcal{R} \) under the dynamics \( e^{-itH_0} \cdot e^{itH} \), one gets the relation
\[
\rho_\mathcal{E} \otimes \rho_\mathcal{R}[\eta_{-\nu}(U-U_i)U_i] = \rho_\mathcal{E} \otimes \rho_\mathcal{R}[\eta_{-\nu(1-\nu)}(U-U_i)U_i]
\] (4.38)
for \(-1 \leq \nu \leq 0\). This is extended by analyticity to values of \( \nu \) such that \( \kappa(\nu) \in D \). Theorem 2.4 follows since by Theorem 2.3, \( f(\kappa, \lambda) \) is independent of \( \rho_\mathcal{E} \).

5 Proof of some estimates

We prove the lemma’s that were used in Section 4. As in Section 4 we always assume Assumptions [A-1] and [A-2] and we take \( \kappa \) such that \( \{\Re \kappa \in D\} \) where \( D \) is as in Assumption [A-1].

5.1 The Wick-representation of the dynamics on \( \mathcal{B}(\mathcal{E}) \)

The aim of this section is to introduce a convenient notation to handle the Wick-ordered Dyson expansion, stated in (5.5-5.6). The result is equation (5.12).

Recall the representation of \( V \) as a function \( V : \mathcal{X} \rightarrow \mathcal{B}(\mathcal{E}) \), introduced in 2.11. Denote
\[
V_t^\#(x) := e^{itE} V^\#(x)e^{-itE} \quad t \in \mathbb{R}, x \in \mathcal{X}, \quad V^\#(x) = V(x), (V(x))^* \quad (5.1)
\]
By Assumption [A-2], both \( h \) and \( w_\kappa \) can be represented as multiplication operators with functions on \( \mathcal{X} \). We will denote these functions by respectively \( \xi(x) \) and \( w_\kappa(x) \) (consistent with the use of \( \xi \) in Assumption [A-2]).

Introduce the space \( Z = \mathcal{X} \times \{1, 2, 3, 4\} \) with elements \( z = (x, j) \) and measure \( dz = dxdj \) (\( dj \) stands for the counting measure on \( \{1, 2, 3, 4\} \)) and the maps \( Q_{u,z}(s) \in B(B(\mathcal{E})) \),

\[
Q_{u,z}(S) = \begin{cases} 
  e^{-iu\xi(x)}w_{\frac{x}{2}}(x) & V_u(x) & S & j = 1 \\
  e^{iu\xi(x)}w_{-\frac{x}{2}}(x) & V_u^*(x) & S & j = 2 \\
  e^{-iu\xi(x)}w_{\frac{x}{2}}(x) & V_u^*(x) & S & j = 3 \\
  e^{iu\xi(x)}w_{-\frac{x}{2}}(x) & V_u(x) & S & j = 4 
\end{cases}
\]  

(5.2)

We now introduce the pairing coefficient \( C(z, z') \) for \( z, z' \in Z \);

\[
C(z = x, j; z' = x', j') := \delta(x - x') \begin{cases} 
  1 & j = 1, j' = 2 \\
  0 & \text{otherwise} \\
  0 & \text{otherwise} \\
  1 & j = 3, j' = 4 \\
  0 & \text{otherwise} \\
\end{cases}
\]  

(5.3)

For \( n \in 2\mathbb{N} \), let \( \text{Pair}(n) \) denote the set of partitions of \( \{1, \ldots, n\} \) in pairs. For such a partition \( \pi \in \text{Pair}(n) \), we write

\[
(i, i') \rightarrow \pi \iff (i, i') \text{ is one of the pairs in the partition } \pi
\]  

(5.4)

The following representation for \( I_+Z_{\lambda, \kappa}^\kappa I_+ \) is our starting point.

\[
I_+Z_{\lambda, \kappa}^\kappa I_+ = e^{\lambda^{-2}tu} \sum_{n \in 2\mathbb{N}} \int_{0 \leq u_1 \leq \cdots \leq u_n \leq t} du_1 \cdots du_n \sum_{\pi \in \text{Pair}(2n)} \lambda^{-n} \int_{Z^n} dz_1 \cdots dz_n \left( \prod_{(i, i') \rightarrow \pi} C(z_i, z'_{i'}) \right) Q_{\lambda^{-2}u_n, z_n}^\kappa \cdots Q_{\lambda^{-2}u_1, z_1}^\kappa
\]  

(5.5)

(5.6)

It follows from the definition (4.5), the Dyson expansion (4.3) and the Wick theorem.

Let \( [0, t]_2 \) be the set of (unordered) couples in \( [0, t] \) and

\[
\Omega_t := \{ \sigma \subset [0, t]_2, |\sigma| < \infty \}
\]  

(5.7)

We remark that there is an identification between \( n \in 2\mathbb{N}, 0 \leq u_1 \leq \cdots \leq u_n \leq t, \pi \in \text{Pair}(n) \) and \( \sigma \in \Omega_t \) with \( |\sigma| = n/2 \), given by

\[
\sigma = \bigcup_{(i, i') \rightarrow \pi} \{ (u_i, u_{i'}) \}
\]  

(5.8)
By writing $dn$ and $d_n \pi$ for the counting measures on respectively $\mathbb{N}$ and $\text{Pair}(n)$, we define, using the above identification,

$$d\sigma := dn \times du_1 \times \ldots \times du_n \times d_n \pi,$$

This definition could be ambiguous when $|\sigma| = 0$ (hence $\sigma = \emptyset$), which we fix by defining

$$\int_{\Omega_t} d\sigma \text{Ind}(\sigma = \emptyset) = 1.$$  

(5.10)

Thus, we have made $\Omega_t$ into a measure space. Using the same identification, we define $V^{\kappa, \lambda}(\sigma) \in \mathcal{B}(\mathcal{B}(E))$ to equal the line (5.6)

$$V^{\kappa, \lambda}(\sigma) := \lambda^{-n} \int_{\mathbb{Z}^n} dz_1 \ldots dz_n \left( \prod_{(i, i') \to \pi} C(z_i, z_{i'}) \right) Q^{\kappa}_{\lambda^{-2} u_1 \ldots u_n} \ldots Q^{\kappa}_{\lambda^{-2} u_1 \ldots u_n}(\sigma)$$

(5.11)

and we again abbreviate $V(\sigma) := V^{\kappa, \lambda}(\sigma)$.

We have hence rewritten (5.5-5.6) as

$$I_1 Z_{\lambda^{-2} t} I_1^* = e^{i\lambda^{-2} t M} \int_{\Omega_t} d\sigma V(\sigma)$$

(5.12)

For convenience, we also define $\tilde{\Omega}_t \subset \Omega_t$ as the set of those $\sigma$ with $|\sigma| = 1$. Hence $\tilde{\Omega}_t$ is the set of ordered pairs in $[0, t]$. We will write the elements of this pair as $s(\tilde{\sigma}), s^{*}(\tilde{\sigma})$ with $s(\tilde{\sigma}) < s^{*}(\tilde{\sigma})$.

We stress that up to this point, nothing happened; we just cooked up a fancy notation, culminating in equation (5.12), for the Wick-ordered Dyson expansion!

5.2 Proof of Lemma 4.2

Statement (1) of Lemma 4.2 is an obvious consequence of the definition (4.16), we concentrate on Statement (2). We first establish the crude a-priori bound (5.16).

Let $(u_a)$ be a basis in $\mathcal{E}$ and define

$$q_\kappa(t) := \sum_{a, a', a'', a'''} \left| \int_X dx \langle u_a, (V(x))^* u_{a'} \rangle \langle u_{a''}, V(x) u_{a'''} \rangle e^{i\xi(x)} w_\kappa(x) \right|$$

(5.13)

Since $\mathcal{E}$ is finite-dimensional, the function $q_\kappa(t)$ is dominated by a multiple of $p_\kappa(t)$ (as defined in 2.12) and vice versa. Using the explicit expression (5.2), (5.3) and (5.11), one gets

$$\|V(\tilde{\sigma})\| \leq \lambda^{-2} (q_{R\kappa} + q_{3\kappa}) \left( \frac{\bar{s}(\tilde{\sigma}) - \bar{s}(\tilde{\sigma})}{\lambda^2} \right) + \lambda^{-2} (q_{R\kappa} + q_{3\kappa}) \left( - \frac{\bar{s}(\tilde{\sigma}) - \bar{s}(\tilde{\sigma})}{\lambda^2} \right) =: \lambda^{-2} d_\kappa \left( \frac{-s(\tilde{\sigma})}{\lambda^2} \right)$$

(5.14)

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One easily checks
\[ \| \mathcal{V}(\sigma) \| \leq \prod_{\tilde{\Omega}, \tilde{\sigma} \subset \sigma} \lambda^{-2} d_{\kappa} \left( \frac{s(\tilde{\sigma}) - s(\tilde{\sigma})}{\lambda^2} \right) =: G(\sigma) \]  
(5.15)

(For example, one can represent \( V_u^\#(x) = \sum_{a, a'} |a\rangle \langle a|, V_u^\#(x)a' \rangle \langle a'| \text{ in (5.2)} \) and then factorize (5.11)). By a change of integration variables, and summing over all values of \(|\sigma|\), we arrive at the a-priori bound
\[ \int_{\overline{\sigma}_t} d\sigma \| \mathcal{V}(\sigma) \| \leq e^{t \| d_{\kappa} \|_1} \]  
(5.16)

with \( \| d_{\kappa} \|_1 = \int_{\mathbb{R}^+} d_{\kappa}(t) dt \), which is finite since \( \| p_{\kappa} \|_1 \) is finite.

Let
\[ J_{s, \tau} (\sigma) := \text{Ind}[\exists \tilde{\sigma} \in \tilde{\Omega}_t, \tilde{\sigma} \subset \sigma, s(\tilde{\sigma}) \leq s \leq s(\tilde{\sigma}), s(\tilde{\sigma}) - s(\tilde{\sigma}) \geq \tau] \]  
(5.17)

One can easily convince oneself that (\( \lor \) stands for the maximum)
\[ W_n = e^{i n \lambda - 2 \tau M} \int_{\Omega_n} d\sigma \left( \prod_{j=1}^{n-1} J_{j \tau, \tau} \lor J_{j \tau, \tau} \right) (\sigma) \mathcal{V}(\sigma) \]  
(5.18)

In words, each \( \sigma \) contributing to \( W_n \) contains for each \( j = 1, \ldots, n-1 \) a \( \tilde{\sigma} \) which 'crosses' \( j \tau \). Or, the insertion of \( 1 - I_{\uparrow} I_{\downarrow} \) forces a pairing to occur.

**Lemma 5.1.** Assume Assumption [A-3] There are \( c_{\pm} := c_{\pm}(\kappa, \lambda, \tau) \) vanishing as \( \lambda \downarrow 0 \) and continuous in the three parameters, such that for \( J_-, J_+ \) disjoint subsets of \( \mathbb{N}_0 \),
\[ \int_{\Omega_t} d\sigma \left( \prod_{j_{\pm} \in J_{\pm}} J_{j_{\pm} \tau, \tau_{\pm}} \right) (\sigma) \| \mathcal{V}(\sigma) \| \leq (c_+)^{|J_+|} (c_-)^{|J_-|} \int_{\Omega_t} d\sigma G(\sigma) \]  
(5.19)

**Proof.** Denote by \( \{ \mathcal{G}_i \}_i \) a partition of \( J_+ \) in subsets \( \mathcal{G}_i \) satisfying \( \max \mathcal{G}_i < \min \mathcal{G}_{i+1} \) for all \( i \). Write \( \sum_{\{ \mathcal{G}_i \}_i} \) for the sum over all such partitions. Then,
\[ \text{LHS of (5.19)} \leq \left( \sum_{\{ \mathcal{G}_i \}_i} \prod_i \int_{\hat{\Omega}_t} d\tilde{\sigma} \left( \prod_{\tilde{\sigma} \in \mathcal{G}_i} J_{j_{\tau, \tau_+} (\tilde{\sigma})} G(\tilde{\sigma}) \right) \right) \times \left( \prod_{j_{\tau} \in J_-} \int_{\Omega_t} d\tilde{\sigma} J_{j_{\tau, \tau_-} (\tilde{\sigma})} G(\tilde{\sigma}) \right) \times \left( \int_{\Omega_t} d\sigma G(\sigma) \right) \]  
(5.20)

(5.21)
Since
\[ \int_{\tilde{\Omega}} d\tilde{\sigma} J_{j,\tau,\tau}(\tilde{\sigma}) G(\tilde{\sigma}) \leq \lambda^{-2} \int_{\tilde{\sigma} \leq s \leq \tilde{\sigma}} d\tilde{\sigma} ds d\kappa(\lambda^{-2}(\tilde{\sigma} - \tilde{\sigma})) \leq \lambda^{2} \int_{\mathbb{R}^+} du \, d\kappa(u) =: c_{-}, \] (5.22)

hence the first factor in (5.21) is bounded by \((c_{-})^{J_{-}}\).

By the argument following (5.13), Assumption A-3 implies that there are \(C_{\kappa}, \alpha_{\kappa} > 0\) such that \(d\kappa(t) < C_{\kappa} e^{-\alpha_{\kappa}|t|}\) (Obviously, \(C_{\kappa}; \alpha_{\kappa}\) can be chosen constant if \(\kappa\) varies in a bounded set).

One can bound
\[ \int_{\tilde{\Omega}} d\tilde{\sigma} \left( \prod_{j \in G_{i}} J_{j,\tau,\tau}(\tilde{\sigma}) \right) G(\tilde{\sigma}) \leq C_{\kappa}^{\alpha_{\kappa}} e^{-\alpha_{\kappa} \lambda^{-2}(1/2) \max G_{i}-\min G_{i}|\tau}. \] (5.24)

Using \(\sum_i |\max G_{i} - \min G_{i}| \geq |J_{+}|\), we arrive at the upper bound for the RHS of (5.20)
\[ \sum_{(G_{i})} \prod_{i} \int_{\tilde{\Omega}_{i}} d\tilde{\sigma} \left( \prod_{j \in G_{i}} J_{j,\tau,\tau}(\tilde{\sigma}) \right) G(\tilde{\sigma}) \leq C_{\kappa}^{\alpha_{\kappa}} e^{-\alpha_{\kappa} \lambda^{-2}(1/2) |J_{+}| |\tau} =: (c_{+})^{J_{+}} \] (5.25)

To conclude the proof of Lemma 4.2, we use expression (5.18), replacing \(\lor \rightarrow +\),
\[ \|W_{n}\| \leq \int_{\Omega} d\sigma \left( \prod_{j=1}^{n-1} (J_{j,\tau,\tau} + J_{j,\tau,\tau}) \right) (\sigma) \|V(\sigma)\| \leq (c_{-} + c_{+})^{n-1} (e^{\tau\|d\kappa\|_{1}})^{n} \] (5.26)

To get the last inequality, we represented the product in \(\prod_{j=1}^{n-1} (J_{j,\tau,\tau} + J_{j,\tau,\tau})\) as a sum over partitions of \(\{1, \ldots, n-1\}\) in 2 sets \(J_{-}\) and \(J_{+}\), we applied Lemma 5.1 and we resummed the sum over partitions by the binomial formula. Finally, the bound (5.16) with \(t = n\tau\) was used.

5.3 Proof of Theorem 4.1

These statements are contained in the literature. Statement (1) is a consequence of the Perron-Frobenius theorem for completely positive maps, stated in [13] and valid in our context under Assumption A-4 (This is extensively discussed in [35]). Statement (2) can be immediately checked from the explicit expressions in Section 4.11. For \(\kappa = 0\), Statement (3) is a result of the usual weak-coupling theory, see e.g. [10]. For \(\kappa \neq 0\), it is a straightforward generalization of these theorems. One can easily follow the arguments in [10] and adapt the statements.
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