Research Article

An Osgood Type Regularity Criterion for the 3D Boussinesq Equations

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We consider the three-dimensional Boussinesq equations, and obtain an Osgood type regularity criterion in terms of the velocity gradient.

1. Introduction

In this paper, we consider the following three-dimensional (3D) Boussinesq equations with the incompressibility condition:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \Delta u + \nabla \pi &= \theta e_3, \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta - \Delta \theta &= 0, \\
\nabla \cdot u &= 0, \\
\nabla \times u &= 0, \quad \theta(x,0) = \theta_0, \\
\n\end{align*}
\]

where \( u = (u_1(x,t), u_2(x,t), u_3(x,t)) \) is the fluid velocity, \( \pi = \pi(x,t) \) is a scalar pressure, and \( \theta = \theta(x,t) \) is the scalar temperature, while \( u_0 \) and \( \theta_0 \) are the prescribed initial velocity and temperature, respectively, with \( \nabla \cdot u_0 = 0 \).

In case \( \theta = 0 \), (1) reduces to the incompressible Navier-Stokes equations. The regularity of its weak solutions and the existence of global strong solutions are important open problems; see [1–3]. Starting with [4, 5], there have been a lot of literatures devoted to finding sufficient conditions (which now are called regularity criteria) to ensure the smoothness of the solutions; see [6–16] and so forth. Since the convective terms \( (u \cdot \nabla) u \) are the same in the Navier-Stokes equations and Boussinesq equations, the authors also consider the regularity conditions for (1). In particular, Qiu et al. [17] obtained Serrin type regularity condition:

\[
\|u\|_{L^p(0,T; L^q(\mathbb{R}^3))}, \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty. \tag{2}
\]

The extension to the multiplier spaces was established by the same authors in [18]. For the Besov-type regularity criterion, Fan and Zhou [19] and Ishimura and Morimoto [20] showed the following regularity conditions:

\[
\nabla \times u \in L^1(0,T; \dot{B}^{0,\infty}_{\infty,\infty}(\mathbb{R}^3)), \\
\nabla u \in L^1(0,T; L^\infty(\mathbb{R}^3)). \tag{3}
\]

Zhang [21, 22] then considers the regularity criterion in terms of the pressure or its gradient. The readers are also referred to [23] for generalized models.

Motivated by [24–26], we will improve (3) as in the following.

Theorem 1. Let \( (u_0, \theta_0) \in H^1(\mathbb{R}^3) \). Assume that \( (u, \theta) \) is the smooth solution to (1) with the initial data \( (u_0, \theta_0) \) for \( 0 \leq t < T \). If

\[
\sup_{0 \leq t < \infty} \int_0^T \left\| \nabla \times u \right\|_{L^\infty} \frac{1}{q \ln q} < \infty, \tag{4}
\]
then the solution \((u,\theta)\) can be extended after time \(t = T\). Here, \(\tilde{\Delta}_k\) denotes the Fourier localization operator and \(\Delta S_q = \sum_{l=-q}^{q} \tilde{\Delta}_l\).

Remark 2. The Osgood type condition (4) is weaker than (3). Notice that, for \(q \in [2, \infty)\), we have

\[
\left\| \sum_{l=-q}^{q} \tilde{\Delta}_l (\nabla \times u) \right\|_{L^\infty} \leq C \| \nabla \times u \|_{B^{0,\infty}_{\infty,\infty}}.
\]  

The rest of this paper is organized as follows. In Section 2, we recall the definition of Besov spaces and some interpolation inequalities. Section 3 is devoted to proving Theorem 1.

2. Preliminaries

Let \(\mathcal{S}(\mathbb{R}^3)\) be the Schwartz class of rapidly decreasing functions. For \(f \in \mathcal{S}(\mathbb{R}^3)\), its Fourier transform \(\mathcal{F} f = \tilde{f}\) is defined by

\[
\tilde{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} \, dx.
\]

Let us choose a nonnegative radial function \(\varphi \in \mathcal{S}(\mathbb{R}^3)\) such that

\[
0 \leq \varphi(\xi) \leq 1, \quad \varphi(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq 1, \\
0, & \text{if } |\xi| \geq 2,
\end{cases}
\]

and let

\[
\varphi_j(x) = 2^{3j} \varphi\left(2^j x\right), \quad \varphi_j(x) = 2^{3j} \varphi\left(2^j x\right), \quad j \in \mathbb{Z}.
\]

For \(j \in \mathbb{Z}\), the Littlewood-Paley projection operators \(S_j\) and \(\tilde{\Delta}_j\) are, respectively, defined by

\[
S_j f = \varphi_j * f, \quad \tilde{\Delta}_j f = \varphi_j * f.
\]

Observe that \(\tilde{\Delta}_j = S_j - S_{j-1}\). Also, it is easy to check that if \(f \in L^2(\mathbb{R}^3)\), then

\[
S_j f \to 0, \quad \text{as } j \to -\infty; \quad S_j f \to f, \quad \text{as } j \to +\infty,
\]

in the \(L^2\) sense. By telescoping the series, we thus have the following Littlewood-Paley decomposition:

\[
f = \sum_{j=-\infty}^{+\infty} \tilde{\Delta}_j f,
\]

for all \(f \in L^2(\mathbb{R}^3)\), where the summation is the \(L^2\) sense. Notice that

\[
\tilde{\Delta}_j f = \sum_{l=j-2}^{j+2} \Delta_j \tilde{\Delta}_l f = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f;
\]

then from Young’s inequality, it readily follows that

\[
\left\| \tilde{\Delta}_j f \right\|_{L^p} \leq C 2^{3j/p} \| f \|_{L^p},
\]

where \(1 \leq p \leq q \leq \infty\) and \(C\) is an absolute constant independent of \(f\) and \(j\).

Let \(-\infty < s < \infty\), \(1 \leq p, q \leq \infty\); the homogeneous Besov space \(\dot{B}^s_{p,q}\) is defined by the full-dyadic decomposition such that

\[
\dot{B}^s_{p,q} = \left\{ f \in \mathcal{L}'(\mathbb{R}^3); \| f \|_{\dot{B}^s_{p,q}} < \infty \right\},
\]

where

\[
\| f \|_{\dot{B}^s_{p,q}} = \left\| \left(2^{js} \| \tilde{\Delta}_j f \|_{L^p} \right)^{+\infty}_{j=-\infty} \right\| e',
\]

and \(\mathcal{L}'(\mathbb{R}^3)\) is the dual space of

\[
\mathcal{L}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3); \quad D^{\alpha} \tilde{f}(0) = 0, \quad \forall \alpha \in \mathbb{N}^3 \right\}.
\]

Also, it is well known that

\[
\dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3), \quad \forall s \in \mathbb{R}.
\]

We refer to [27] for more detailed properties.

3. Proof of Theorem 1

This section is devoted to proving Theorem 1. From standard continuity arguments, we need to only provide the uniform \(H^s\) bounds of the solution \((u,\theta)\).

Taking the inner products of (1), with \(-\Delta u, (1)_2\) with \(-\Delta \theta\), we obtain by adding together that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla (u,\theta) \|_{L^2}^2 + \| \Delta (u,\theta) \|_{L^2}^2
\]

\[
= \int_{\mathbb{R}^3} \left\langle \nabla(u \cdot \nabla) u, \Delta u dx - \int_{\mathbb{R}^3} \theta \Delta u \, dx + \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \Delta \theta \, dx \right\rangle
\]

\[
= \int_{\mathbb{R}^3} \partial_k \theta \partial_k u \, dx
\]

\[
- \int_{\mathbb{R}^3} \partial_k u_j \left( \partial_k u \partial_k u_i + \partial_j \partial_i \theta \right) \, dx
\]

\[
\equiv I + J.
\]

For \(I\), we use Hölder’s inequality to get

\[
I \leq \frac{1}{2} \| \nabla (u,\theta) \|_{L^2}^2.
\]

For \(J\), applying the Littlewood-Paley decomposition as in (11), we get

\[
\nabla u = \sum_{l=-q}^{q} \tilde{\Delta}_l \nabla u + \sum_{l=-q}^{q} \tilde{\Delta}_l \nabla u + \sum_{l=-q}^{q} \tilde{\Delta}_l \nabla u,
\]
where \( q \) is positive integral to be determined later on. Plugging (20) into \( J \), we see that

\[
J \leq \sum_{l<-q} \int_{\mathbb{R}^3} |\hat{\Delta}_l \nabla u| \cdot |\nabla (u, \theta)|^2 \, dx + \int_{\mathbb{R}^3} \sum_{l<-q} |\hat{\Delta}_l \nabla u| \cdot |\nabla (u, \theta)|^2 \, dx + \sum_{l>q} \int_{\mathbb{R}^3} |\hat{\Delta}_l \nabla u| \cdot |\nabla (u, \theta)|^2 \, dx = J_1 + J_2 + J_3.
\] (21)

For \( J_1 \), we dominate as

\[
J_1 \leq \sum_{l<-q} \|\hat{\Delta}_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \leq C \sum_{l<-q} 2^{3l/2} \|\hat{\Delta}_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \quad \text{(by (13))}
\]

\[
\leq C \left( \sum_{l<-q} 2^{3l/2} \|\hat{\Delta}_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \right)^{1/2} \left( \sum_{l<-q} \|\hat{\Delta}_l \nabla u\|_{L^2}^2 \right)^{1/2} \|\nabla (u, \theta)\|_{L^2}^2 \leq C 2^{-3q/2} \|\nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \quad \text{(by (17))}
\]

\[
= \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2}^2 \right]^3.
\] (22)

For \( J_2 \), we have

\[
J_2 = \int_{\mathbb{R}^3} \|\nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \cdot |\nabla (u, \theta)|^2 \, dx \leq \|\nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2.
\] (23)

Finally, for \( J_3 \), we estimate as

\[
J_3 \leq \sum_{l=q} \|\Delta_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \leq C \sum_{l=q} 2^{l/2} \|\Delta_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2} \|\Delta (u, \theta)\|_{L^2} \quad \text{(by (13) and Gagliardo-Nireberg inequality)}
\]

\[
\leq C \left( \sum_{l=q} 2^{l/2} \|\Delta_l \nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2} \|\Delta (u, \theta)\|_{L^2} \right)^{1/2} \left( \sum_{l=q} \|\Delta_l \nabla u\|_{L^2}^2 \right)^{1/2} \|\nabla (u, \theta)\|_{L^2} \|\Delta (u, \theta)\|_{L^2} \leq \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2} \|\Delta (u, \theta)\|_{L^2} \right]^3 \quad \text{(by (17)).}
\] (24)

Gathering (22), (23), and (24) together and plugging them into (21), we deduce

\[
J \leq \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2} \right]^3 + \|\nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \quad \text{by (13)}
\]

\[
+ \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2} \right] \|\Delta (u, \theta)\|_{L^2}^2.
\] (25)

Substituting (19) and (25) into (18), we find

\[
\frac{1}{2} \frac{d}{dt} \|\nabla (u, \theta)\|_{L^2}^2 + \|\Delta (u, \theta)\|_{L^2}^2 \leq \frac{1}{2} \|\nabla (u, \theta)\|_{L^2}^2 + \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2} \right]^3 + \|\nabla u\|_{L^2} \|\nabla (u, \theta)\|_{L^2}^2 \quad \text{by (25)}
\]

\[
+ \left[ C 2^{-q/2} \|\nabla (u, \theta)\|_{L^2} \right] \|\Delta (u, \theta)\|_{L^2}^2.
\] (26)

Taking

\[
q = \left[ \frac{2}{\ln 2 \ln (\|\nabla (u, \theta)\|_{L^2})} \right] + 1,
\] (27)

where \( [t] \) is the largest integer smaller than \( t \in \mathbb{R} \) and \( \ln^+ t = \ln(e + t) \), then (26) implies that

\[
\frac{d}{dt} \|\nabla (u, \theta)\|_{L^2}^2 \leq \|\nabla (u, \theta)\|_{L^2}^2 + C \left[ \frac{\|\nabla u\|_{L^2}}{q \ln q} \right] \ln^+ \left( \|\nabla (u, \theta)\|_{L^2} \right) \ln^+ \left( \|\nabla (u, \theta)\|_{L^2} \right) \times \|\nabla (u, \theta)\|_{L^2}^2 \quad \text{by (27)}
\] (28)

Applying Gronwall inequality three times, we deduce

\[
\|\nabla (u, \theta)\|_{L^2}^2 + \int_0^t \|\Delta (u, \theta)\|_{L^2} \, dr \leq C \exp \exp \left( \int_0^t \left[ \frac{\|\nabla u\|_{L^2}}{q \ln q} \right] \, dr \right) \quad \text{(29)}
\]

Recalling (4), we see the solution \((u, \theta)\) is uniformly bounded in \( L^\infty (0, T; H^1(\mathbb{R}^3)) \). This completes the proof of Theorem 1.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
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