MAXIMAL SYSTOLE OF HYPERBOLIC SURFACE WITH LARGEST $S^3$ EXTENDABLE ABELIAN SYMMETRY

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Abstract. We give the formula for the maximal systole of the surface admits the largest $S^3$-extendable abelian group symmetry. The result we get is $2 \arccosh K$. Here

$$K = \sqrt{\frac{1}{216} L^3 + \frac{1}{8} L^2 + \frac{5}{8} L - \frac{1}{8} + \frac{1}{108} L(L^2 + 18L + 27)}$$

$$+ \sqrt{\frac{1}{216} L^3 + \frac{1}{8} L^2 + \frac{5}{8} L - \frac{1}{8} - \frac{1}{108} L(L^2 + 18L + 27)}$$

$$+ \frac{L + 3}{6}$$

and $L = 4 \cos^2 \frac{\pi}{g + 1}$.

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1. Introduction

In the study of hyperbolic surfaces, systole is an important topic. It has a wide connection with topics on surfaces, like Teichmüller theory and differential geometry on surfaces. On Teichmüller theory, from the classical Mumford’s compactness criterion [Mum71], to some recent progress in the Weil-Peterson metric [Wol17], systole plays an important role. In differential geometry, it relates to the spectrum of Laplacian operator ([BMM16] [BMM18] [Mon14]), the optimal systolic ratio ([CL15] [CK03] [Gro83]) and so on. For a survey on the study of the systole, see Parlier [Par14].
In this paper, systole on a closed hyperbolic surface is referred to either a shortest closed geodesic or its length, and we often use it to refer to the latter. In another point of view, systole is also a function on $\mathcal{M}_g$, the moduli space of all the genus-$g$ closed hyperbolic surfaces.

An important topic on systole is to get the maximal systole on $\mathcal{M}_g$. This question is very hard. Today the only known closed surface with the maximal systole is the Bolza surface with genus 2, which was proved by Jenni [Jen84]. However, there are progresses on globally maximal systole on subspaces of $\mathcal{M}_g$ and locally maximal systoles on $\mathcal{M}_g$. On subspace-maxima, Bavard got the maximal systole of genus 2 and 5 hyperelliptic surfaces in [Bav92] in 1992. On locally maximal systole, Schmutz [Sch93] gave a necessary and sufficient condition for the surfaces with locally maximal systole and got some local-maximal-systole examples with polyhedral symmetry. Recently Bourque and Rafi constructed surfaces with locally maximal systoles and trivial symmetry in [BR18].

Another approach is estimating the lower bound of the maximal systole. P. Buser and P. C. Sarnak [BS94] got surfaces with systoles longer than $4/3 \log g$ in 1994 by arithmetic method. Later Katz, Schaps and Vishne [KSV07] obtained more surfaces with this lower bound. Part of their examples are surfaces with the Hurwitz symmetry. Recently, [PW15] and [Pet18] obtained more concrete examples with systole longer than $4/7 \log g - K$ by different method.

Inspired by [KSV07] and [Sch93], we are interested in the connections between surface symmetry and systoles and consider the maximal systole of surfaces with some big-order symmetries.

The symmetry we consider is the largest $S^3$ extendable abelian symmetry. The $S^3$-extendable symmetry on topological surface was recently defined in [WWZZ13] and [WWZZ15] and the largest $S^3$-extendable abelian symmetry is a special type of it.

Here is the definition of $S^3$-extendable symmetry on topological surface: For the finite group $G$ acting on the surface $\Sigma_g$, $G$ is $S^3$-extendable if and only if there is an embedding $i : \Sigma_g \to S^3$ such that for any $g \in G$, there is a $\tilde{g} \in SO(4)$ acting on $S^3$ and the following diagram commutes:

$$
\begin{array}{ccc}
\Sigma_g & \xrightarrow{g} & \Sigma_g \\
\downarrow i & & \downarrow i \\
S^3 & \xrightarrow{\tilde{g}} & S^3.
\end{array}
$$

Here is a definition for the largest $S^3$ extendable abelian symmetry: For the $S^3$ extendable surface $(\Sigma, G)$, when $G$ is an abelian group and the order of $G$ is maximal among all the abelian group acting on $\Sigma_g$, $S^3$-extendably, we call $(\Sigma_g, G)$ a genus $g$ surface with largest $S^3$-extendable abelian symmetry.

For convenience, we call $(\Sigma_g, G)$, the genus $g$ largest $S^3$ extendable abelian surface: $\Gamma(2,n)$-surface, as [WWZZ13] [WWZZ15]. Here $n = g + 1$.

We give the definition of hyperbolic version of this symmetry: For a hyperbolic surface $\Sigma_g$ with a group $G$ isometrically acting on it, if $(\Sigma_g, G)$ is a surface with largest $S^3$ extendable abelian symmetry when forgetting the hyperbolic structure, then we call $(\Sigma_g, G)$ a hyperbolic surface with largest $S^3$ extendable abelian symmetry. All of the genus $g$ hyperbolic surfaces with largest $S^3$ extendable abelian symmetry form a subspace of $\mathcal{M}_g$, which can be parametrized by two parameters.
Table 1. Maximal systole of surface with largest $S^3$-extendable abelian symmetry

| Genus | systole $= 2 \text{arccosh} K$ |
|-------|---------------------------------|
| 2     | 3.0571                          |
| 3     | 3.6478                          |
| 4     | 3.9078                          |
| 5     | 4.0464                          |
| 6     | 4.1291                          |

(for details, see Section 2). It is known that systole can be viewed as a function from $\mathcal{M}_g$ to $\mathbb{R}^+$. We define the maximal systole of $\Gamma(2, n)$ surface to be maximal value of the systole function on the subspace of $\mathcal{M}_g$ of all the genus $g$ hyperbolic surfaces with largest $S^3$ extendable abelian symmetry.

Our result is:

**Theorem 1.** The maximal systole of the $\Gamma(2, n)$ surface is

$$2 \text{arccosh} K.$$

Here

$$K = \sqrt[3]{\frac{1}{216}L^3 + \frac{1}{8}L^2 + \frac{5}{8}L - \frac{1}{8} + \sqrt[3]{\frac{1}{108}L(L^2 + 18L + 27)}} + \sqrt[3]{\frac{1}{216}L^3 + \frac{1}{8}L^2 + \frac{5}{8}L - \frac{1}{8} - \sqrt[3]{\frac{1}{108}L(L^2 + 18L + 27)}} + \frac{L + 3}{6}.$$

and $L = 4 \cos^2 \frac{\pi}{n}$.

The maximal systole is obtained when

$$(c, t) = \left(\text{arccosh} K, 2 \text{arccosh} \frac{K + 1}{2 \cos \frac{\pi}{n}}\right).$$

The symbol $c$ and $t$ are defined in Section 2.

Numerical results of small genera are shown in Table 1.

Compared with the work [BGW18] on the systole of surface with large cyclic symmetry, the surface with large cyclic symmetry has unique geometric structure, while surfaces studied in this paper form a subspace of $\mathcal{M}_g$ parametrized by two parameters when $g \geq 2$. Hence the method in this paper is quite different from that in [BGW18]. In this paper, we obtain our result by classifying the family of curves that are possible to be the shortest geodesics on surfaces admitting this symmetry and then getting the condition when the systole is maximal.

It is worth to note that when $g = 2$, the $\Gamma(2, n)$ surface is the Bolza surface. Bolza surface is the genus 2 surface with the maximal systole and is constructed by attaching the opposite sides of a hyperbolic regular octagon (Figure 1(b)). In Figure 1(a) the pants is one of the two pants that forms the $\Gamma(2, 3)$ surface. The points $A_1, \ldots, A_6$ are fixed points of the hyperelliptic involution. All of the segments between these points shown in Figure 1(a) have the same length (actually half of the systole). There is an isometry between $\Gamma(2, 3)$ surface (Figure 1(a)) and the
Bolza surface (Figure 1(b)). This isometry is shown in Figure 1: the $A_i$ in Figure 1(a) are mapped to Figure 1(b).

Figure 1.

In Section 2, we construct the hyperbolic $\Gamma(2,n)$ surface and describe its symmetry. In Section 3, we give some useful lemmas on the intersection of the systoles. Then in Section 4, by the hyperellipticity of the $\Gamma(2,n)$ surface, we proved that systoles must be the simple curves meeting the singular points of the orbifolds (the quotients of the surface by the symmetric group).

We divided such curves into three types in Section 4 and found the shortest one in each type in Section 5. These are the central parts of this work.

Next we prove that the surface’s systole is maximal if and only if the three curves have the same length (Proposition 4). Finally we calculate this length using the condition that the three candidates have equal length.

We give more details of the part dividing the curves into three types. We divided the curves in the orbifold by the singular points they meet. In each type, a curve corresponds to an element in $\mathbb{Z}$ (Lemma 5 and 7). Then based on this characterization, by the method of constructing cyclic covers similar to that using Seifert surface in knot theory, we construct the surface from orbifold and prove that for two curves $l$, $l'$ in the orbifold of the same type, if $l$ is longer than $l'$, then $l$’s lift is longer than $l'$’s (Theorem 2). Then in each type, we found the shortest curve in Proposition 1 together with Proposition 2 and 3 (Proposition 2 and 3 exclude two curves that cannot be lifted to a systole in the maximal surface and are not excluded by Proposition 1).

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2. The construction and symmetry of $\Gamma(2,n)$ surface

We construct the hyperbolic $\Gamma(n, 2)$ surface, which is similar to the construction of topological $\Gamma(n, 2)$ surface in [WWZZ15] and [WWZZ13].
We pick two isometric \( n \)-holed spheres with the order \( n \) rotation symmetry. Similarly to 3-holed spheres, we call a boundary component of the \( n \)-holed spheres a cuff and the common perpendicular between two neighboring cuffs a seam. (See Figure 2.) The two spheres are attached along their boundary components in the order shown in Figure 2. The twist parameter at all the boundary components are the same to give the \( \Gamma(2,n) \) surface the order \( n \) rotation symmetry frome the \( n \)-holed spheres.

![Figure 2. \( \Gamma(2,n) \) surface. The red segment is a seam. Two cuffs with the same label are attached together.](image)

### 2.1. The symmetry of \( \Gamma(2,n) \) surface

The symmetric group of the \( \Gamma(2,n) \) surface is \( \mathbb{Z}^2 \oplus \mathbb{Z}^n \). We denote the order \( n \) and order 2 generators of the group by \( \sigma \) and \( \tau \) respectively. \( \sigma \) maps each \( n \)-holed sphere to itself, and on each \( n \)-holed sphere, \( \sigma \) is the order \( n \) rotation, while \( \tau \) is the order two rotation exchanges the two \( n \)-holed spheres.

Assume \( \Sigma \) is a \( \Gamma(2,n) \) surface, then \( \Sigma/\langle \tau \rangle = S^2(2,2,\ldots,2,2) \), and \( (\Sigma/\langle \tau \rangle)/\langle \sigma \rangle = S^2(2,2,n,n) \).

We assume

\[
\pi : \Sigma \to \Sigma/\langle \tau \rangle
\]

to be the double cover induced by \( \tau \) and

\[
\pi' : \Sigma/\langle \tau \rangle \to (\Sigma/\langle \tau \rangle)/\langle \sigma \rangle
\]

is the branch cover induced by \( \sigma \) on the orbifold.

Here is the diagram of the covering:

\[
(2.1) \quad \Sigma \xrightarrow{\pi} S^2(2,2,\ldots,2) \xrightarrow{\pi'} S^2(2,2,n,n) \,.
\]

On the orbifold \( S^2(2,2,n,n) \), there is a branched double covering \( \pi'' \) from \( S^2(2,2,n,n) \) to \( S^2(2,2,2,n) \) (See Figure 3). Then we can extend the Diagram (2.1) longer:

\[
(2.2) \quad \Sigma \xrightarrow{\pi} S^2(2,2,\ldots,2) \xrightarrow{\pi'} S^2(2,2,n,n) \xrightarrow{\pi''} S^2(2,2,2,n) \,.
\]
2.2. The geometry of $\Gamma(2,n)$ surface. The $\Gamma(2,n)$ surface consists of two isometric $n$-holed spheres (Figure 2). The geometric structure of the surface is determined by two parameters similar to the Fenchel-Nielsen parameters. One of the parameters is the length of the cuffs (denoted $2c$), the other is the "twist parameter", the following is its formal definition. This parameter (denoted $t$) is the distance between two end points of seams on a cuff. The two seams are required to be in different $n$-holed spheres, and both seams’ end points are in the same two cuffs. By this definition, $t \leq 2c/2 = c$. (See Figure 4. In this Figure, the circle $c_1, c_2$ are cuffs of a $\Gamma(2,n)$ surface, $AA'$ is a seam in one $n$-holed spheres and $BB'$ is a seam in the other. Then $t$ is the distance between the points $A$ and $B$ on the cuff)

The length of the seams is denoted by $s$. We give a relation between $s$ and $c$ that will be used in latter calculation. Each $n$-holed sphere of the $\Gamma(2,n)$ surface consists of two isometric right-angled $2n$-polygons with order $n$ rotation symmetry. By the definition of seam and cuff, the edge length of the $2n$-polygon is $s$ or $c$. If an edge’s length is $s$, then its neighboring edges’ length is $c$; If an edge’s length is $c$, then its neighboring edges’ length is $s$.

By connecting the center of the polygon and the mid-points of two neighboring edges, we obtained a trirectangle $OABC$ (Figure 5) with $AB = s/2$, $BC = c/2$, $\angle O = \pi/n$ and $\angle A = \angle B = \angle C = \pi/2$. Then by formula 2.3.1(i) in [Bus10, p.471], we have
\[
\cosh \frac{s}{2} \cosh \frac{c}{2} = \cos \frac{\pi}{n}.
\]

Now we begin to describe the geometric structures of the orbifolds $S^2(2,2,\ldots,2)$, $S^2(2,2,n,n)$ and $S^2(2,2,2,n)$ induced from the $\Gamma(2,n)$ surface.

An $n$-holed sphere in the construction of the $\Gamma(2,n)$ surface is a fundamental domain of the double branched cover $\pi$. On each cuff, there is a pair of antipodal points that are the fixed points of $\pi$. Therefore the image of each cuff of the $\Gamma(2,n)$ surface is a segment on $S^2(2,2,\ldots,2)$ with length $c$. End points of the segment are the singular points. For the seams, since a seam is fully contained in
a fundamental domain, the image of a seam is the common perpendicular of two neighboring (images of) cuffs on $S^2(2,2,\ldots,2)$ with length $s$. Finally we consider the twist parameter $t$. Since $\pi$ is a locally-isometric double cover, the two seams in $\Gamma(2,n)$ surface connecting the same two cuffs are mapped to the same curve in $S^2(2,2,\ldots,2)$. (In Figure 4, $AA'$ and $BB'$ are the two seams connecting the same two cuffs, they are mapped to the same curve in $S^2(2,2,\ldots,2)$ by $\pi$. ) Therefore, the mid-point between the roots of the two seams ($A, B$ in Figure 4) is the singular point of $\pi$. We let $C$ in Figure 4 be the mid-point between $A$ and $B$. Then $AC = BC = t/2$.

Figure 5 is the orbifold $S^2(2,2,\ldots,2)$ with the geometric structure induced by the $\Gamma(2,n)$ surface and $\pi$. Here $AA'$ is the image of the seam, $|AA'| = s$. $C, C'$ (and other blue points) are singular points. The segment $CC'$ is the image of a cuff and $|CC'| = c$. Then $|AC| = t/2$.

Next we consider the orbifold $S^2(2,2,n,n)$. In Figure 6, the four regions separated by the red lines are the fundamental domains of the $n$-covering map $\pi'$. Here $O$ and $O'$ are the centers of the regular $2n$-polygon, $D$ is the mid-point of $AA'$. $ODO'$ is a red line and other red lines are $ODO'$'s image by the order-$n$ rotation.

Figure 7 is the $S^2(2,2,n,n)$ orbifold with geometric structure induced from $\Gamma(2,n)$ surface by covering map $\pi$ and $\pi'$. In Figure 7, the point $A$ is the image of the $A$ in Figure 6 and so are other points labelled by the same letter. Thus $C$ and $C'$ are the order 2 singular points, $O$ and $O'$ are the order $n$ singular points.
Then $CC'$ is the image of the cuffs and $|CC'| = c$. $AD$ is the image of the seams and $|AD| = s/2$. And therefore $|AC| = t/2$.

![Figure 7](image1.png)

![Figure 8](image2.png)

Finally we consider the orbifold $S^2(2,2,2,n)$. In Figure 7, one of the two pentagons is a fundamental domain of the double-covering map $\pi''$. The two fixed points of the covering map are $D$ and the mid-point of $CC'$.

Figure 8 is the $S^2(2,2,2,n)$ orbifold with geometric structure induced from $\Gamma(2,n)$ surface by covering map $\pi, \pi'$ and $\pi''$. In Figure 8, the point $A$ is the image of the $A$ in Figure 7 and so are other points labelled by the same letter except the point $E$. $E$ is the image of the mid-point of $CC'$ of Figure 7. $C$, $D$ and $O$ are the order 2 singular points, $O$ is the order $n$ singular point. Then $CE$ is the image of the cuffs and $|CE| = c/2$. $AD$ is the image of the seams and $|AD| = s/2$. And therefore $|AC| = t/2$.

3. The intersections of the systoles

Here is a basic and very useful claim:

**Claim 1.** Two systoles intersect at most once.

A key observation is crucial in proving the three Lemmas in this Section:

**Observation 1.** Any simple closed curve on $S^2(2,2,\ldots,2)$ (or $S^2(2,2,\ldots,2)$) is separating. Therefore if two simple closed curves on $S^2(2,2,\ldots,2)$ (or $S^2(2,2,\ldots,2)$) intersect each other, then they intersect at least twice.

Claim 1 and Observation 1 are essential tools for proving the following three Lemmas. These Lemmas are important tools to characterize the systoles on $\Gamma(2,n)$ surface.

**Lemma 1.** If $\Sigma$ is a $\Gamma(2,n)$ surface, then the image of a systole of $\Sigma$ on $S^2(2,2,\ldots,2)$ by $\pi$ does not intersect itself at any regular point of the orbifold.

The images of two systoles of $\Sigma$ on $S^2(2,2,\ldots,2)$ by $\pi$ do not intersect at any regular point of the orbifold.

**Lemma 2.** If $\Sigma$ is a $\Gamma(2,n)$ surface, then the image of a systole of $\Sigma$ on $S^2(2,2,\ldots,2)$ by $\pi' \circ \pi$ does not intersect itself at any regular point of the orbifold.

The images of two systoles of $\Sigma$ on $S^2(2,2,\ldots,2)$ by $\pi' \circ \pi$ do not intersect at any regular point of the orbifold.
Lemma 3. If $\Sigma$ is a $\Gamma(2,n)$ surface, then the image of a systole of $\Sigma$ on $S^2(2,2,2,n)$ by $(\pi'' \circ \pi' \circ \pi)$ does not intersect itself at any regular point of the orbifold.

The images of two systoles of $\Sigma$ on $S^2(2,2,2,n)$ by $\pi'' \circ \pi' \circ \pi(\Sigma)$ do not intersect at any regular point of the orbifold.

The idea to prove these Lemmas is direct: using Claim 1. We assume that a systole’s image on the orbifold has self-intersection or two systoles’ images intersect each other at regular points. Then we prove that the lift of the images always contain two simple closed curve with equal length and at least two intersections.

But the proofs are rather long, because we need to deal with all the possible shape of the images of a simple closed curve on $\Sigma$ by $\pi$ and the lifts of the curves’ images case by case.

Proof of Lemma 4

If $\alpha$ is a simple closed curve in $\Sigma$, $\pi(\alpha)$ has a self-intersection point $p$. Then $\pi^{-1}(p)$ consists of two points, both are the intersection points of $\pi^{-1} \circ \pi(\alpha)$. By the definition of double cover, $\pi^{-1}(\pi(\alpha))$ consists of either one curve or two curves with equal length. Since $\alpha$ is simple, $\pi^{-1}(\pi(\alpha))$ consists of two curves. These two curves intersect at least twice, therefore cannot be systole.

We assume $\alpha$ and $\beta$ are two simple closed curves with equal length on $\Sigma$, $p$ is the intersection point of $\pi(\alpha)$ and $\pi(\beta)$.

We recall that there are two type of order 2 isometric action on $S^1$. Therefore, the shape of $\pi(\alpha)$ and $\pi(\beta)$ has 2 possibilities: $S^1$ or a segment, whose endpoints are fixed points of $\pi$.

(a) If $\pi(\alpha)$ and $\pi(\beta)$ are simple closed curves, then $\pi(\alpha)$ intersects $\pi(\beta)$ at least twice by Observation 1. Recall that there are two types of double cover of $S^1$, namely $S^1$ and $S^1 \sqcup S^1$. Then there are three types of double cover of $\pi(\alpha) \cup \pi(\beta)$ shown in Figure 10.

In all the cases, $\alpha$ intersects $\beta$ at least twice, which contradicts to Claim 1.

(b) If $\pi(\alpha)$ is a segment while $\pi(\beta)$ is a simple closed curve (Figure 11), then there are two types of the double cover of $\pi(\alpha) \cup \pi(\beta)$ shown in Figure 12.

If the double cover of $\pi(\alpha) \cup \pi(\beta)$ is the case shown in Figure 12(a), then it is clear that the curve $\alpha$ and $\beta$ have at least two intersections. Therefore, $\alpha$ and $\beta$ cannot be systoles.

If the double cover of $\pi(\alpha) \cup \pi(\beta)$ is the case shown in Figure 12(b), we assume $\tilde{p}$ is one of the fix point of $\pi$ in Figure 12(b). Therefore $\pi_1([\beta]) = \pi_1([\beta'])$ in $\pi_1(\pi(\Sigma), \pi(\tilde{p}))$. Here $[\hat{\beta}]$ and $[\beta']$ are elements of $\pi_1(\Sigma, \tilde{p})$ represented by $\beta$ and $\beta'$. It contradicts to the injectivity of $\pi_1$ ($\pi$ is a covering map).

(c) If both $\pi(\alpha)$ and $\pi(\beta)$ are segments (Figure 13), then $|\pi^{-1}(\pi(\alpha)) \cap \pi^{-1}(\pi(\beta))| \geq 2$ since the intersection point of $\pi(\alpha)$ and $\pi(\beta)$ is a regular point. However, both

Figure 9. $\pi(\alpha) \cup \pi(\beta)$
Figure 10. The double cover of $\pi(\alpha) \cup \pi(\beta)$

Figure 11. $\pi(\alpha) \cup \pi(\beta)$ (2)

Figure 12. The double cover of $\pi(\alpha) \cup \pi(\beta)$ (2)
\[\pi^{-1}(\pi(\alpha)) \text{ and } \pi^{-1}(\pi(\beta)) \text{ are connected. Therefore } |a \cap b| \geq 2, \text{ so that } \alpha \text{ and } \beta \text{ cannot be systole.} \]

Furthermore, we have

**Proof of Lemma 2.** Recall that \(\pi' : S^2(2,2,\ldots,2) \to S^2(2,2,\ldots,2)\) is the covering map. If \(\alpha\) is a systole of \(\Sigma\) then by Lemma 1, \(\pi'(\alpha)\) has no self-intersection and won’t intersect the image of another systole at regular points. Therefore, if \(\pi'\pi(\alpha)\) has self-intersection at regular points, then it implies that either \(\pi(\alpha)\) intersects itself or it intersects another lift of \(\pi'(\alpha)\). Therefore \(\pi'\pi(\alpha)\) has no self-intersections.

By exactly the same argument, we can prove that the images of two systoles of \(\Sigma\) on \(S^2(2,2,\ldots,2)\) do not intersect at any regular point of the orbifold. \(\square\)

We omit the proof for Lemma 3 because it is exactly the same to the proof for Lemma 2.

If \(\alpha\) is a systole on \(\Sigma\) then \(\pi'' \circ \pi' \circ \pi(\alpha)\) is either a simple closed curve or a segment connecting two singular points.

Now we begin to characterize and classify the image of systoles on \(S^2(2,2,\ldots,2)\).

4. The Image of Systoles on \(S^2(2,2,\ldots,2)\)

**Lemma 4.** The image of systoles won’t pass the order \(n\) singular point.

**Proof.** The order \(n\) singular point is lifted to a regular point in \(S^2(2,2,\ldots,2)\), and a segment in the neighborhood of the order \(n\) point that passes the point is lifted to \(n\) segments intersecting at the pre-image of the order \(n\) singular point. Then by Lemma 1, the curve passing the order \(n\) singular point cannot be lifted to a systole in the surface. \(\square\)

By Lemma 3, the image of systoles in \(S^2(2,2,\ldots,2)\) is either a simple closed curve or the double of a segment connecting two singular points. We first consider the systole whose image is a segment connecting two singular points.

**Lemma 5.** For two given singular points \(p,q\) of the orbifold \(S^2(p,q,r,s)\), there is a 1-1 correspondence between the segment between \(p,q\) (up to homotopy) and the elements of the fundamental group \(\pi_1(S^1)\).

**Proof.** We construct the correspondence directly. We pick the fundamental group \(\pi_1([S^2(p,q,r,s)]\setminus\{r,s\})\), which is isomorphic to \(\pi_1(S^1)\). We notice that the segment between \(p\) and \(q\) won’t pass \(r\) or \(s\). Choosing a segment \(\alpha\) between \(p\) and \(q\), we have that for any segment \(\beta\) between \(p\) and \(q\), \(\alpha\beta^{-1}\) represents an element of \(\pi_1([S^2(p,q,r,s)]\setminus\{r,s\})\). This is the 1-1 correspondence. \(\square\)
We define a symbol $\tilde{l}_{pq}$, to be the family of segments connecting the singular points $p$ and $q$. Figure 14 are the three families on $S^2(2,2,2,n)$.

Now we begin to discuss the simple closed curve in $S^2(2,2,2,n)$.

**Lemma 6.** The simple closed curve in $S^2(2,2,2,n)$ is the curve contains exactly one order-two singular point and the other two order-two singular points are in the same side of the curve. (see Figure 15).

**Proof.** First we consider the curve with no singular point on it. Any such simple closed curve in $S^2(2,2,2,n)$ is splitting and in each side there are two singular points (Otherwise the curve corresponds to a torsion element in the fundamental group and cannot be lifted to the surface. ). Moreover, there are two order 2 singular points in one side of the curve. However, the curve cannot be realized as a geodesic in any hyperbolic structure of the orbifold $S^2(2,2,2,n)$, because the disk it bounds $D^2(2,2)$ has Euler characteristic 0. If $\partial D^2(2,2)$ is a geodesic boundary, then the interior of $D^2(2,2)$ admits a complete hyperbolic structure, which contradicts to $\chi(D^2(2,2)) = 0$.

If the curve contains one singular point, the only thing to prove is that the other two order-two singular points are in the same side of the curve. Otherwise, there is an order-$n$ singular point and order-2 singular point on one side of the curve, while only one order-2 singular point on the other side of the curve. See Figure 16 for the blue curve passing the singular point $E$ (denoted $\alpha$), the singular points $O$ (order $n$) and $C$ (order 2) are on one side of $\alpha$, while the singular point $D$ (order 2) is
on the other side. We assume $\beta$ is a segment connecting $D$ and $E$, and $\beta$ does not intersect $\alpha$ except at $E$. $\alpha$ is homotopic to the double of $\beta$ by the contractibility of a disk and $\beta$ is homotopic to a geodesic connecting $D$ and $E$. By the uniqueness of geodesics in homotopy class, $\alpha$ cannot be a geodesic and then cannot be the image of a geodesic.

The geodesic passing a order-2 singular points always returns (see Figure 17). Therefore the image of a systole that contains at least two singular points must be

![Figure 17.](image)

(a) The lift of a geodesic passing an order-2 singular point
(b) The geodesic passing an order-2 singular point

the segment connecting two singular points. The lemma is proved.

\[\square\]

**Lemma 7.** There is an injective map from the simple closed curve passing one order-two singular point to the fundamental group $\pi_1(S^1)$.

**Proof.** By Lemma 6, for a simple closed curve passing one order-two singular point, the other two order-two singular points are on the same side of the curve. There is a unique segment connecting the two singular points, not meeting the curve. Then by Lemma 5 there is a 1-1 correspondence between segments connecting the two singular points and elements in $\pi_1(S^1)$. This Lemma holds.

This segment is unique because the part of the orbifold bounded by the curve that contains the two singular points is a disk (with two singular points), and therefore any two segments connecting the two singular points in this disk are homotopic.

The map is injective because by cutting along the segment, we get a disk with two singular points. The curve passing one given singular point is unique up to homotopy.

\[\square\]

We define another symbol $\tilde{l}_p$ to be the family of simple closed geodesics in $S^2(2, 2, 2, n)$ that pass an order-two singular point $p$. Figure 15 is a curve in $\tilde{l}_p$. We define $|l(S^2(2, 2, n, n))|$ be the length of a component of $l$‘s preimage in $S^2(2, 2, n, n)$. Similarly, we can define $|l(S^2(2, 2, \ldots, 2))|$ and $|l(\Sigma)|$. This definition is well defined by the symmetry of the surface. The detail is in the proof of the following Theorem.
Theorem 2. For \( l, l' \in \tilde{l}_{pq} \) or \( \tilde{l}_p \), if \( l \) and \( l' \) can be lifted to a systole of the \( \Gamma(2, n) \) surface, then

\[
\frac{|l(S^2(2, 2, n, n))|}{|l|} = \frac{|l'(S^2(2, 2, n, n))|}{|l'|}, \\
\frac{|l(S^2(2, 2, \ldots, 2))|}{|l|} = \frac{|l'(S^2(2, 2, \ldots, 2))|}{|l'|}, \\
\frac{|l(\Sigma)|}{|l|} = \frac{|l'(\Sigma)|}{|l'|}.
\]

This Theorem has a direct Corollary:

Corollary 1. If \( l_0 \) is the shortest curve in one curve family \( \tilde{l}_{pq} \) or \( \tilde{l}_p \) in \( S^2(2, 2, 2, n) \), then \( l_0 \)'s lift in the \( \Gamma(2, n) \) surface \( \Sigma \) is not longer than the lift of other curves in the same family.

Proof. Consider the orbifold \( S^2(2, 2, 2, n) \) in Figure 18. \( O \) is the order-\( n \) singular point, \( C, D \) and \( E \) are order-two singular points. \( D \) and \( E \) will be lifted to regular points in \( S^2(2, 2, n, n) \).

First we prove that curves in \( \tilde{l}_D \) and \( \tilde{l}_E \) cannot be lifted to be a systole on \( \Gamma(2, n) \) surface. \( \forall l \in \tilde{l}_D \), \( l \) is lifted to a curve with self-intersections (See Figure 19). Therefore \( l \) cannot be lifted to a systole by Lemma 2.

Now we prepare a tool for the following proof. We construct the cyclic cover of the orbifold \( S^2(2, 2, 2, n) \), as the construction of cyclic cover of knot complement constructed by Seifert surface.
We first construct the $S^2(2, 2, n, n)$ from $S^2(2, 2, 2, n)$. Two order-two singular points of $S^2(2, 2, 2, n)$ are lifted to regular points of $S^2(2, 2, n, n)$ (points $D$ and $E$ in Figure 18). We pick a curve $l$ connecting $D$ and $E$ (in other words $l \in \tilde{l}_{DE}$). We cut the regular neighbourhood $N(\tilde{l})$ of the interior of $l$ away, obtaining an orbifold with boundary (See Figure 20). The boundary of $S^2(2, 2, 2, n) \setminus N(\tilde{l})$ are divided into two parts by $D$ and $E$, we call them $l^+$ and $l^-$ respectively (See Figure 20(b)).

![Figure 20.](image)

Then we pick two copies of $S^2(2, 2, 2, n) \setminus N(\tilde{l})$ calling them $P_1$ and $P_2$ respectively. We attach $P_1$’s $l^+$ to $P_2$’s $l^-$; $P_1$’s $l^-$ to $P_2$’s $l^+$. Therefore we get the orbifold $S^2(2, 2, n, n)$, the double cover of $S^2(2, 2, 2, n)$. (Figure 21).

![Figure 21.](image)
Similarly, we construct $S^2(2,2,\ldots,2)$, the order-$n$ cyclic cover of $S^2(2,2,n,n)$ from $S^2(2,2,n,n)$.

The two order-$n$ singular points of $S^2(2,2,n,n)$ (the points O and O' in Figure 21(b)) will be lifted to regular points in $S^2(2,2,\ldots,2)$. We pick a segment $l$ connecting O and O' (in other words $l \in l_{OO'}$) and cut away $N(l)$ (Figure 22).

The boundary of $S^2(2,2,n,n) \setminus N(l)$ is divided into two pieces by O and O'. We call the two pieces $l^+$ and $l^-$ respectively. We pick $n$ copies of $S^2(2,2,n,n) \setminus N(l)$, denoted $P_1, P_2, \ldots, P_n$ respectively. We attach $P_i$'s $l^+$ to $P_{i+1}$'s $l^-$ (assign $P_{n+1} = P_1$). Then we get $S^2(2,2,\ldots,2)$ the $n$-cyclic cover of $S^2(2,2,2n,n)$ (Figure 23).

Then we construct the surface $\Sigma$, the double cover of the orbifold $S^2(2,2,\ldots,2)$. We construct it from $S^2(2,2,n,n)$. In $S^2(2,2,n,n)$ (Figure 21(b)), we pick a curve $l$ connecting C and C' (in other words $l \in l_{CC'}$). We cut the regular neighbourhood of $l$ and get $S^2(2,2,n,n) \setminus N(l)$. The boundary of $S^2(2,2,n,n) \setminus N(l)$ are divided into two pieces by C and C'. We call the two pieces $l^-$ and $l^+$ respectively. Then we consider the pre-image $\pi^{-1}(S^2(2,2,n,n) \setminus N(l)) \subset S^2(2,2,\ldots,2)$ (Figure 24). We pick two copies $P_1$ and $P_2$ of $\pi^{-1}(S^2(2,2,n,n) \setminus N(l))$ then attach $P_1$'s $l^+$ to $P_2$'s $l^-$ and attach $P_1$'s $l^-$ to $P_2$'s $l^+$. Then we get the surface $\Sigma$. 

---

**Figure 22.**

**Figure 23.**
The tool is now prepared. Now we continue to prove this theorem.

(1) First we consider the family \( \tilde{C}_l \). In the orbifold \( S^2(2,2,2,n) \) (Figure 18), \( \forall l \in \tilde{C}_l \), by Lemma 7, there exists \( l' \in \tilde{D}_E \) such that \( l \cap l' = \emptyset \). (Figure 25(b))

Then by the construction of \( S^2(2,2,2,n) \)'s double cover using \( l' \in \tilde{D}_E \), \( l \) is lifted to two copies with the length equal to \( l \)'s length. Therefore, \( \forall l \in \tilde{C}_l \),

\[
|l(S^2(2,2,2,n))| = |l(S^2(2,2,n,n))| = |l(S^2(2,2,\ldots,2))|.
\]

For any \( l \in \tilde{C}_l \) in \( S^2(2,2,2,n) \), one side of \( l \) contains the order-\( n \) singular point \( O \), the other side contains two order-two singular points \( D \) and \( E \). Therefore, we consider \( l \)'s lift in \( S^2(2,2,2,n) \). With a little abuse of symbol, we denote the lift as \( l \) (Figure 25(b)). Then for the two order \( n \) singular points \( O \) and \( O' \), \( l \) seperates \( O \) and \( O' \).

We pick a curve \( l'' \) connecting \( O \) and \( O' \), \( l'' \) intersects \( l \) exactly once. Then \( l \) is cut into two pieces (denoted \( l_1 \) and \( l_2 \) respectively) by \( l'' \). Then in the construction of \( S^2(2,2,\ldots,2) \) by \( l'' \), for the \( n \) copies of \( S^2(2,2,n,n) \backslash N(l'') \) (denoted \( P_1, P_2, \ldots, P_n \)), \( l_1 \) in \( P_1 \) is attached to \( l_2 \) in \( P_2 \) (Figure 26). Therefore \( l \) is lifted to a curve with length equal to \( l \)'s length. Therefore \( \forall l \in \tilde{C}_l \),

\[
|l(S^2(2,2,2,n))| = |l(S^2(2,2,n,n))| = |l(S^2(2,2,\ldots,2))|.
\]
The end points of $l$ in $S^2(2,2,\ldots,2)$ are $C'$ and $C''$ respectively. Therefore by the construction of the surface $\Sigma$, the lift of $l$ in the surface $\Sigma$ consists of two pieces divided by the points $C$ and $C''$. Each piece has the length $|l(S^2(2,2,\ldots,2))|$. Therefore, $\forall l \in \tilde{l}_C,$

$$|l(\Sigma)| = 2|l(S^2(2,2,\ldots,2))|.$$

(2) Then we consider the family $\tilde{l}_{CE}$. The proof here is similar to the proof for the family $\tilde{l}_C$.}

Figure 26.

Figure 27.
For any $l \in \tilde{I}_{CE}$, $l$ is a non-separating curve in $S^2(2, 2, n)$. Then there is an $l' \in \tilde{I}_{DE}$ such that $l'$ intersects $l$ only once at $E$ (Figure 27(b)). Then we construct $S^2(2, 2, n)$'s double cover $S^2(2, 2, n)$ by $l'$. The lift of $l$ in $S^2(2, 2, n)$ is the blue curve in Figure 27(b). It consists of two pieces, $CE$ and $C'E$. Each piece has the length equal to $l$’s length. Therefore $\forall l \in \tilde{I}_{CE}$

$$|l(S^2(2, 2, n))| = 2|l(S^2(2, 2, n))|. $$

Then we lift $l$ to $S^2(2, 2, \ldots, 2)$. $\forall l \in \tilde{I}_{CC'}$ in $S^2(2, 2, n, n)$, $l$ is a non-separating curve. Thus there exists $l''$ connecting $O$ and $O'$ such that $l''$ does not intersect $l$ (Figure 27(b)). Then we use $l''$ to construct $S^2(2, 2, \ldots, 2)$, $S^2(2, 2, n, n)$. Thus we know $l$ is lifted to $n$ disjoint segments in $S^2(2, 2, \ldots, 2)$ with length equal to $l$’s. That is to say $\forall l \in \tilde{I}_{CE}$

$$|l(S^2(2, 2, \ldots, 2))| = |l(S^2(2, 2, n, n))|. $$

The last thing is to lift $l$ to the surface $\Sigma$. By exactly the same proof in (1), we have

$$|l(\Sigma)| = 2|l(S^2(2, 2, \ldots, 2))|. $$

(3) Next we consider the family $\tilde{I}_{CD}$. $\forall l \in \tilde{I}_{CD}, l$ is lifted to a curve connecting $C$ and $C'$ in $S^2(2, 2, n, n)$ (in other words, in the family $\tilde{I}_{CC'}$) (See Figure 28). Then what we need to prove has already proved in (2).

![Figure 28](image_url)

(4) Finally we consider the family $\tilde{I}_{DE}$. $\forall l \in \tilde{I}_{DE}$, we use $l$ to construct $S^2(2, 2, n, n)$, the double cover of $S^2(2, 2, n, n)$. Then the lift of $l$ in $S^2(2, 2, n, n)$ is a simple closed curve. This curve is divided into two pieces. Each piece has length equal to the length of $l$ (Figure 29(b) [29(c)]. Thus $\forall l \in \tilde{I}_{DE}$,

$$|l(S^2(2, 2, n, n))| = 2|l(S^2(2, 2, 2, n))|. $$

Then we lift $l$ to $S^2(2, 2, \ldots, 2)$ similar to the proof in (1), $l$ in $S^2(2, 2, n, n)$ separates $O$ and $O'$. We pick a segment $l'$ connecting $O$ and $O'$ such that $l'$ intersects $l$ only once. We use $l'$ to construct $S^2(2, 2, \ldots, 2)$, the $n$-cyclic cover of $S^2(2, 2, n, n)$. Since $\setminus l'$ is a segment, $l$’s lift in $S^2(2, 2, \ldots, 2)$ is a simple closed curve whose length is $n$-th the length of $l$. That is $\forall l \in \tilde{I}_{DE}$,

$$|l(S^2(2, 2, \ldots, 2))| = n|l(S^2(2, 2, n, n))|. $$

At last we lift $l$ to the surface $\Sigma$. We use $l'' \in \tilde{I}_{CC'}$ in Figure 29(b) (and its pre-image in Figure 29(a)) to construct $\Sigma$. $\pi^{-1}(S^2(2, 2, n, n) \setminus N(l''))$ is an $n$-holed
spheres. We pick two copies of the $n$-holed spheres, denoted $P_1$ and $P_2$. We attach $P_1$ to $P_2$ along their boundaries (See Figure 30). In Figure 30 by the definition of $t$, $F_i$ in $P_1$ is attached to $G_{i+1}$ in $P_2$ and $G_i$ in $P_1$ is attached to $F_{i-1}$ in $P_2$. Here $F_i$ and $G_i$ are end points of components of $l$ in the $n$-holed spheres. Therefore, the lift of $l$ in $\Sigma$ consists of simple closed curve(s). Each curve is connected segments in $P_1$ or $P_2$. If $n$ is odd the lift of $l$ is one simple closed curve consists of the segments $G_1F_1^{(1)}$, $G_2F_2^{(2)}$, \ldots, $G_nF_n^{(2)}$. If $n$ is even the lift of $l$ consists of two simple closed curves. One curve consists of the segments $G_1F_1^{(1)}$, $G_2F_2^{(2)}$, \ldots, $G_nF_n^{(2)}$. The other curve consists of the segments $G_1F_1^{(2)}$, $G_2F_2^{(1)}$, \ldots, $G_nF_n^{(1)}$.

Therefore, $\forall l \in \tilde{l}_{DE}$,

$$|l(\Sigma)| = 2|l(S^2(2,2,\ldots,2))|$$

if $n$ is odd; while

$$|l(\Sigma)| = |l(S^2(2,2,\ldots,2))|$$

if $n$ is even.

□

Now we have proved the theorem. We list the length ratio we obtained in the following Table:

Figure 29.
MAXIMAL SYSTOLE OF HYPERBOLIC SURFACE WITH LARGEST $S^3$ EXTENDABLE ABELIAN SYMMETRY

5. Shortest curve in each family

By Lemma 3 there are only four families of curves in $S^2(2, 2, 2, n)$ ($\tilde{l}_C$, $\tilde{l}_{CE}$, $\tilde{l}_{CD}$ and $\tilde{l}_{DE}$) that are possible to be lifted to a systole. By Corollary 1, in each family, only the shortest curve is possible to be lifted to a systole in the surface.

Then we characterize the shortest curve in each family in $S^2(2, 2, 2, n)$ by the Proposition 1.

Before stating Proposition 1, we give a short preparation:

We recall that the pentagon in Figure 8 is a model to describe the geometry of the orbifold $S^2(2, 2, 2, n)$. The vertices of the pentagon in Figure 8 that corresponds to the same point of $S^2(2, 2, 2, n)$ are labeled by the same letter. To avoid ambiguity, we replace one $A$ and one $D$ by $A_1$ and $D_1$ respectively (Figure 31).

A curve in the orbifold $S^2(2, 2, 2, n)$ corresponds to broken segments in the pentagon. (See for example Figure 32(a)) The pentagon is symmetric. There is a ‘reflection’ (orientation-reversing, isometric map) of the pentagon mapping $A_1$ to $A$ and $D_1$ to $D$. We reflect some components of the segments corresponding to the curve, then get a connected broken line with the same length to the segments (the dashed blue line in Figure 32(b)). By this construction, one of the endpoint of the broken line is an endpoint of the segments (point $C$ in Figure 32); while the other endpoint of the broken line is either the other endpoint of the segments (point $D$...
in Figure 32 or the reflection of the endpoint of the segments (point $D_1$ of Figure 32).

![Figure 31](image)

**Proposition 1.** The shortest curve among a family of curves ($\tilde{l}_C$, $\tilde{l}_{CE}$, $\tilde{l}_{DE}$ or $\tilde{l}_{CD}$) in $S^2(2,2,2,n)$ corresponds to the broken segments with the least number of components.

Moreover, the shortest curve in each family is shown in Figure 37.

By this Proposition, the possible systoles of the $\Gamma(2,n)$ surface are reduced to finitely many curves.

**Proof.** The proof for all these cases are similar although they are different in details.

(1) For $\tilde{l}_{CD}$, the shortest curve in this family is the segment $CD$ in the pentagon $ODAA_1D_1$ (Figure 37(a)). By the symmetry of the pentagon, $\forall l \in \tilde{l}_{CD}$, there is a connected broken line connecting $CD$ or $CD_1$, with the length of $l$ (The dashed blue line in Figure 32(b)). The broken line connects $CD$ or $CD_1$, therefore is longer than the straight line connecting $CD$ or $CD_1$. But $CD$ is always shorter than $CD_1$, because in the right-angled triangle $\triangle CAD$ and $\triangle CA_1D_1$, $\angle A = \angle A_1 = \pi/2$, $AD = A_1D_1$ while $AC = t/2 < c - t/2 = A_1C_1$ (see Section 2.2). Therefore the shortest curve in $\tilde{l}_{CD}$ is the segment connecting $C$ and $D$ in the pentagon, denoted $l_{CD}$. (See Figure 37(a)).

![Figure 32](image)

(2) For $\tilde{l}_{DE}$, the proof is exactly the same by the symmetry of the pentagon. The shortest curve in $\tilde{l}_{DE}$ is the straight segment connecting $D_1$ and $E$, denoted $l_{DE}$ (see Figure 37(b)).
(3) For $\tilde{l}_{CE}$, the proof is similar.

(3.1) For $l \in \tilde{l}_{CD}$, if $l$ (consists of segments in the pentagon) has a component connecting $AD$, $OD$ or $A_1D_1$, $OD_1$ (see Figure 33(a)), then we reflect this segment and the image of the segment connects two other segments of $l$ (Figure 33(b)). Therefore we get a curve, shorter than $l$, and has less intersections with $OD$ than $l$ (Figure 33(c)).

![Figure 33](image-url)

(3.2) If all of the components of $l$ are segments connecting $AA_1$, $OD$ or $AA_1$, $OD_1$, and there exist components not meeting $C$ or $E$, then we pick two such segments, $A_2D_2$ and $A_3D_3$ in Figure 34(a); $A_2$ and $A_3$ correspond to the same point in the orbifold. Then we replace $A_2D_2$ by $A_3D_5$ (here $A_2$ and $A_3$ correspond to the same point in the orbifold; $D_2$ and $D_5$ correspond to the same point in the orbifold) (see Figure 34(b)); or we replace $A_3D_3$ by $A_2D_4$ (here $A_2$ and $A_3$ correspond to the same point in the orbifold; $D_3$ and $D_4$ correspond to the same point in the orbifold) (see Figure 34(c)).

Segments in Figure 34(b) and Figure 34(c) correspond to a curve in the family $\tilde{l}_{CE}$. One of the two curves is shorter than $l$, the curve in Figure 34(a) (explained later). Then, without loss of generality, we assume the curve in Figure 34(b) is shorter. By replacing the broken line $ED_5A_3D_3$ in Figure 34(b) by the straight line $ED_3$, we get a curve (Figure 34(d)) homotopic to the curve in Figure 34(b). This curve is shorter than $l$ and has less intersections with $OD$ than $l$.

(3.3) $\forall l \in \tilde{l}_{CE}$ we use the operations described in (3.1) and (3.2) to change $l$ until we can not use the operations. Everytime we use the operations, we get a curve shorter and has less intersections with $OD$ than the original curve. Finally we get
the curve in Figure 37(c) (denoted $l_{CE}$) or the curve in Figure 37(d) (denoted $l'_{CE}$). We prove later in Lemma xxx that $l'_{CE}$ cannot be lifted to a systole.

3.4 One thing left to prove: in the operation described in (3.2), the curve in Figure 34(a) is longer than the curve in Figure 34(b) or the curve in Figure 34(c).

We prove it with the following formula [BGW18, (3.10)]:

\[
\cosh c = \cosh d \cosh a \cosh b - \sinh a \sinh b.
\]

The meaning of the symbols in the formula is illustrated by Figure 35. In Figure 34(a)

\[
\cosh |A_2D_2| = \cosh |AD| \cosh |AA_2| \cosh |DD_2| - \sinh |AA_2| \sinh |DD_2|.
\]
While in Figure 34(b),
\[ \cosh |A_3 D_5| = \cosh |A_1 D_1| \cosh |A_1 A_3| \cosh |D_1 D_5| - \sinh |A_1 A_3| \sinh |D_1 D_5|. \]
Here \(|AD| = |A_1 D_1|\) by the symmetry of the pentagon. \(|DD_2| = |D_1 D_5|\) because \(D_2\) and \(D_5\) correspond to the same point in the orbifold and so are \(D\) and \(D_1\).

Therefore curve in Figure 34(a) is longer than curve in Figure 34(b) if and only if
\[ |A_2 D_2| > |A_3 D_5|. \]
Similarly, in Figure 34(a)
\[ \cosh |A_3 D_3| = \cosh |A_1 D_1| \cosh |A_1 A_3| \cosh |D_1 D_3| - \sinh |A_1 A_3| \sinh |D_1 D_3|. \]

While in Figure 34(c)
\[ \cosh |A_2 D_4| = \cosh |AD| \cosh |AA_2| \cosh |DD_4| - \sinh |AA_2| \sinh |DD_4|. \]
Here \(|AD| = |A_1 D_1|\) by the symmetry of the pentagon. \(|DD_4| = |D_1 D_3|\) because \(D_4\) and \(D_3\) correspond to the same point in the orbifold and so are \(D\) and \(D_1\).

Therefore curve in Figure 34(a) is longer than curve in Figure 34(c) if and only if
\[ |A_3 D_3| > |A_2 D_4|. \]
In conclusion, the curve in Figure 34(a) is longer than either the curve in Figure 34(b) or the curve in Figure 34(c).

(4) For \(I_C\), the proof is similar to (1) and (2). \(\forall l \in \tilde{I}_C\) (Figure 36(a)), by using the reflection, we can construct two connecting broken lines. One line connects \(C\) and a point on \(OD\) \((D_6, D_7, D_1, D_1)\) in Figure 36(b), while the other connects \(C\) and a point on \(OD_1\) \((D_7, D_7, D_1, D_1)\) in Figure 36(b). \(D_6\) and \(D_7\) correspond to the same point in the orbifold. The length of the broken lines are equal to \(l\) since all the changes are reflections.

![Figure 36](image-url)

Figure 36.
Then the straight lines connecting $C, D_6$ and $C, D_7$ are shorter than the corresponding broken lines in Figure 36(b). Therefore, the shortest curve in the family $l_C$ is the closed geodesic in the orbifold consisting of two straight lines in the pentagon, one connecting $C$ and a point in $OD$, the other connecting $C$ and a point in $OD_1$ (denoted $l_C$). (See Figure 37(e)).

Figure 37.

6. Calculations

6.1. The curve $l_{DE}$ and $l'_{CE}$. In this subsection, we prove that the curve $l_{DE}$ and $l'_{CE}$ cannot be lifted to systoles.

First we give the following Proposition:
**Proposition 2.** The curve \( l'_{CE} \) in \( S^2(2, 2, 2, n) \) is not possible to lift to a systole in the \( \Gamma(2, n) \) surface \( \Sigma \).

**Proof.** We prove this Proposition by cut and paste of the pentagon. In the pentagon (Figure 38(a)), we cut along \( CD \) and \( D_1E \) then paste along \( CE \). Then we get a quadrilateral (Figure 38(b)).

In the quadrilateral in Figure 38(b) we cut along the dashed green segment \( OE \), then paste along the segments \( OD \) and \( OD_1 \). Then we get a pentagon (Figure 38(c)).

In the pentagon in Figure 38(c) we cut along the dashed yellow segment that is from \( E \) and perpendicular to \( DD_2 \), then paste along the segments \( DE_1 \) and \( D_1D \). We denote the foot of the perpendicular to be \( A_2 \). Then we get a pentagon (Figure 38(d)). The pentagon in Figure 38(d) has two right angles, \( A_2 \) and \( A_3 \).

In all the four subfigures of Figure 38 the blue segments represent the curve \( l'_{CE} \); while the segment \( CE \) always represents the curve \( l_{CE} \). Then since in Figure 38(d) \( |A_2A_3| = 2|CD| \), therefore \( |A_3C| > |A_2C| \). \( |A_2E| = |A_3E_1| \). Then by hyperbolic cosine law, \( CE < CE_1 \).

\[ \square \]
Then we give the lengths of the curves in Figure 37 by $c$, $s$ and $t$. It is a preparation for proving Proposition 3. We recall that in the pentagon $|CE| = c/2$, $|AC| = t/2$, $|EA_1| = (c - t)/2$ and $|AD| = |A_1D_1| = s/2$ (see Section 2.2).

It is direct that, for the curve in Figure 37(c),

\[(6.1) \quad |l_{CE}| = \frac{c}{2}.\]

We calculate the lengths of $l_{CD}$ (Figure 37(e)) and $l_{CD_1}$ (Figure 37(b)) by the cosine law of hyperbolic right-angled triangles:

\[(6.2) \quad \cosh |l_{CD}| = \cosh |CD| = \cosh |AC| \cosh |AD| = \cosh \frac{t}{2} \cosh \frac{s}{2}.\]

\[(6.3) \quad \cosh |l_{DE}| = \cosh |D_1E| = \cosh |EA_1| \cosh |A_1D_1| = \cosh \frac{c - t}{2} \cosh \frac{s}{2}.\]

For $l_C$ and $l_{CE}$, to calculate their lengths, we attach a copy of the pentagon to its edge $OD$ (Figure 39). The length of the segment $CE'$ is equal to the length of the curve $l_{CE}'$ in Figure 37(b) and the length of the segment $CC'$ is equal to the length of the curve $l_C$ in Figure 37(e) by symmetry. We use Formula (5.1) to calculate these lengths. Here $|AA'_1| = 2|AD| = s$, $|AC| = t$, $|A'_1E'| = |A_1E| = (c - t)/2$, and $|A'_1C'| = |A_1C| = c - t/2$. Then

\[(6.4) \quad \cosh l_{CE}' = \cosh |CE'| = \cosh |AA'_1| \cosh |AC| \cosh |A'_1E'| - \sinh |AC| \sinh |A'_1E'| = \cosh s \cosh \frac{t}{2} \cosh \frac{c - t}{2} - \sinh \frac{t}{2} \sinh \frac{c - t}{2}.\]
\[
\cosh l_C = \cosh |CC'| \\
= \cosh |AA'| \cosh |AC| \cosh |A'_1 C'| - \sinh |AC| \sinh |A'_1 C'| \\
= \cosh s \cosh \frac{t}{2} \cos - \frac{t}{2} - \sinh \frac{t}{2} \sin c - \frac{t}{2}.
\]

Now we are ready to prove the following Proposition:

**Proposition 3.** In the maximal surface, \( l_{DE} \) in \( S^2(2, 2, 2, n) \) is not possible to be lifted to a systole of the surface.

*Proof.* The curves in the orbifold \( S^2(2, 2, 2, n) \) that are possible to be lifted to the systole of the \( \Gamma(2, n) \) surface are \( l_{CD}, l_{DE}, l_{CE} \) and \( l_C \). If \( l_{DE} \) is lifted to a systole of the surface, then \( l_{CD} \) and \( l_{CE} \) cannot be lifted to a systole of this surface. This is because \( l_{CD} \) and \( l_{CE} \) intersect \( l_{DE} \) at \( D \) and \( E \) respectively. \( D \) and \( E \) are lifted to regular points in \( S^2(2, 2, n, n) \). Then by Lemma 2, since \( l_{DE} \) is lifted to a systole, \( l_{CD} \) and \( l_{CE} \) cannot be lifted to systoles.

If \( l_{DE} \) is lifted to a systole of the surface, then only \( l_{DE} \) and \( l_C \) can be lifted to systoles of the surface. The lengths of \( l_{DE} \) and \( l_C \) are given by (6.3) and (6.5) respectively. Here we give the differentials of the lengths:

First we obtain \( ds/dc \) by (2.3):

\[
\frac{ds}{dc} = \frac{\cosh c}{\cosh s} \sinh \frac{s}{2} \sinh \frac{c}{2}.
\]

Then for \( l_{DE} \)

\[
\begin{align*}
\frac{\partial |l_{DE}|}{\partial t} &= \frac{\partial}{\partial t} \left( \cosh \frac{s}{2} \cosh \frac{c-t}{2} \right) \\
&= -\frac{1}{2} \cosh \frac{s}{2} \sinh \frac{c-t}{2}.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial |l_{DE}|}{\partial c} &= \frac{\partial}{\partial c} \left( \cosh \frac{s}{2} \cosh \frac{c-t}{2} \right) \\
&= \frac{1}{2} \left( \sinh \frac{s}{2} \frac{ds}{dc} \cosh \frac{c-t}{2} + \cosh \frac{s}{2} \sinh \frac{c-t}{2} \right) \\
&= \frac{1}{2} \left( \frac{\cosh \frac{s}{2} \sinh \frac{s}{2} \frac{ds}{dc}}{\cosh \frac{s}{2} \sinh \frac{c}{2}} \cosh \frac{c-t}{2} + \cosh \sinh \frac{c-t}{2} \right).
\end{align*}
\]

\[
\begin{align*}
\frac{d|l_{DE}|}{\partial t} &= \frac{\partial |l_{DE}|}{\partial t} dt + \frac{\partial |l_{DE}|}{\partial c} dc.
\end{align*}
\]
For $l_C$

\begin{equation}
\frac{\partial |l_C|}{\partial t} = \frac{\partial}{\partial t} \left( \cosh s \cosh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) - \sinh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right) \right)
= \frac{1}{2} \left( \cosh s \left( \sinh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) - \cosh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right) \right)
- \cosh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right) + \sinh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) \right)
= \frac{1}{2} \left( \cosh s + 1 \right) \sinh \left( t - c \right).
\end{equation}

\[ \frac{\partial |l_C|}{\partial c} = \frac{\partial}{\partial c} \left( \cosh s \cosh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) - \sinh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right) \right)
= \sinh s \frac{d s}{d c} \cosh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) + \cosh s \cosh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right)
- \sinh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right)
- \cosh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right)
= \frac{1}{2} \left( \cosh s + 1 \right) \sinh \left( t - c \right).
\]

\[ \frac{d |l_C|}{dt} = \frac{\partial |l_C|}{\partial t} + \frac{\partial |l_C|}{\partial c} \frac{dc}{dt}. \]

The two tangent vectors $d |l_{DE}|$, $d |l_C|$ are non-zero vectors. $\forall c > 0, 0 \leq t \leq c$,

\[ \frac{\partial |l_{DE}|}{\partial t} < 0, \frac{\partial |l_C|}{\partial t} < 0. \]

Therefore $d |l_{DE}| \neq k d |l_C|$ $\forall k \leq 0$. Then there is a vector $(A(c, t), B(c, t))$ such that $d |l_{DE}|(A(c, t), B(c, t)) > 0, d |l_C|(A(c, t), B(c, t)) > 0$,

\[ \forall c > 0, 0 \leq t \leq c. \]

By the assumption that $l_{DE}$ is lifted to a systole of the surface, only $l_{DE}$ and $l_C$ can be lifted to a systole of the surface. Then there is another surface with systole bigger than the surface. Therefore the surface is not maximal.

\[ \square \]

6.2. $l_{CE}$, $l_{CD}$ and $l_C$ in the maximal surface. Now only $l_{CE}$, $l_{CD}$ and $l_C$ can be lifted to a systole of the maximal surface.

We have the following Proposition:

Proposition 4. If $\Sigma_0$ is the maximal $\Gamma(2, n)$ surface, then $|l_C(\Sigma_0)| = |l_{CD}(\Sigma_0)| = |l_{CE}(\Sigma_0)|$.

Proof. First we calculate the partial derivatives of the lengths. It is direct that

\begin{equation}
\frac{\partial |l_{CE}|}{\partial t} = \frac{\partial c}{\partial t} = 0.
\end{equation}
Therefore 2

We have obtained \( \frac{\partial |l_c|}{\partial t} \) in (6.6). For \( \frac{\partial |l_{CD}|}{\partial t} \), we have the following formula:

\[
(6.8) \quad \frac{\partial |l_{CD}|}{\partial t} = \frac{\partial}{\partial t} \left( \cosh \frac{t}{2} \cosh \frac{s}{2} \right) = \frac{1}{2} \sinh \frac{t}{2} \cosh \frac{s}{2}.
\]

For any fixed \( c > 0 \), \( |l_{CD}| \) is strictly increasing about \( t \), while \( |l_c| \) is strictly decreasing about \( t \) when \( 0 \leq t \leq c \) by (6.8) and (6.6) respectively. By Table 2 \( |l_{CD}(\Sigma)| = 4|l_{CD}| \), while \( |l_c(\Sigma)| = 2|l_c| \). Then we compare \( |l_{CD}(\Sigma)| \) with \( |l_c(\Sigma)| \) by comparing \( |l_{CD}| \) and \( 2|l_c| \) when \( t = 0 \) and \( t = c \).

By formulae (6.2) and (6.5), when \( t = 0 \), \( \cosh |l_{CD}| = \cosh \frac{s}{2} \cosh \frac{s}{2} = \cosh \frac{s}{2} \). Therefore \( 2|l_{CD}| = s \). \( \cosh |l_c| = \cosh s \cosh \left( c - \frac{t}{2} \right) - \sinh \left( \frac{t}{2} \right) \sinh \left( c - \frac{t}{2} \right) = \cosh s \cosh \left( c - \frac{t}{2} \right) \). Therefore \( 2|l_{CD}| < |l_c| \) when \( t = 0 \). When \( t = c \),

\[
\cosh 2|l_{CD}| = 2 \cosh^2 |l_{CD}| - 1 = 2 \cosh^2 \frac{s}{2} \cosh^2 \frac{t}{2} - 1 = 2 \cosh^2 \frac{s}{2} \cosh^2 \frac{c}{2} - 1 = \left( 2 \cosh^2 \frac{s}{2} - 1 \right) \cosh^2 \frac{c}{2} + \cosh^2 \frac{c}{2} - 1 = \cosh s \cosh^2 \frac{c}{2} + \sinh^2 \frac{c}{2}.
\]

\[
\cosh |l_c| = \cosh s \cosh \frac{t}{2} \cosh \left( c - \frac{t}{2} \right) - \sinh \frac{t}{2} \sinh \left( c - \frac{t}{2} \right) = \cosh s \cosh \left( c - \frac{t}{2} \right).
\]

Therefore, when \( t = c \), \( 2|l_{CD}| > |l_c| \).

For any fixed \( c \), \( |l_{CE}| = c \) and \( |l_{CE}(\Sigma)| = 4c \). The systole of the surface is \( \min(|l_{CE}(\Sigma)|, |l_{CD}(\Sigma)|, |l_c(\Sigma)|) = 2 \min(2|l_{CE}|, 2|l_{CD}|, |l_c|) \).

Figure 40 shows these lengths as functions about \( t \). In this Figure, \( (t_0, l_0) \) is the intersecting point of the graphs of the functions \( 2|l_{CD}| \) and \( |l_c| \) about \( t \).

(a) If \( l_0 > 2|l_{CE}| \), then \forall \in [0, c], \min(2|l_{CE}|, 2|l_{CD}|, |l_c|) \leq 2|l_{CE}| = 2c \). When \( t \in (t_1, t_2) \), \min(2|l_{CE}|, 2|l_{CD}|, |l_c|) = 2|l_{CE}| = 2c \). Here \( t_1, t_2 \) are the intersection point of \( 2|l_{CE}|, 2|l_{CD}| \) and \( 2|l_{CE}|, |l_c| \) respectively (See Figure 40(a)). That is, the systole of the surface corresponding to \( (c, t) \) is \( 4c \) when \( t \in (t_1, t_2) \). Then \forall (c, t) \) with \( t \in (t_1, t_2) \), there is an \( \varepsilon > 0 \), such that the systole of the surface corresponding to \( (c + \varepsilon, t) \) is \( 4(c + \varepsilon) \). Therefore if \( l_0 > 2|l_{CE}| \), the corresponding surface cannot be the maximal surface.

Therefore if a surface is the maximal \( \Gamma(2, n) \) surface, then \( l_0 \leq 2c \).

(b) If \( 2c > l_0 \) then we prove the corresponding surface is not the maximal surface by changing the coordinate.

By the cut-and-paste described in the proof of Proposition 2 (Figure 38), we get another pentagon representation of the \( S^2(2, 2, 2, n) \) induced by the \( \Gamma(2, n) \) surface.

We pick the pentagon representations shown in Figure 38(a) and Figure 38(d) see Figure 41. To avoid confusion, we relabel the pentagon in Figure 41(b) \( c'' \). We define a new coordinate \( (c'', t'') \) for pentagon in Figure 41(b) \( c'' = |C''E''|, t'' = |C''D''| \). By the correspondence of segments in the two pentagons, we have \( |CD| = |C''E''| \) and \( |CE| = |C''D''| \).
Then by this observation, we have the following conclusion:
For a $\Gamma(2, n)$ surface represented by the pair $(c, t)$, there exists a pair $(c'', t'')$ such that:

(1) The surfaces represented by $(c, t)$ and $(c'', t'')$ are isometric.
(2) $|l_{CD}| = |l_{C'E''}|$, $|l_{CE}| = |l_{C'D''}|$ and $|l_C| = |l_{C''}|$.

If $2c > l_0$, then for fixed $c$, the maximal systole about $t$ is realized by the coordinate $(c, t_0)$, the blue point $P$ in Figure 40(b). At this point, $|l_C| = 2|l_{CD}|$ and $2|l_{CE}| > |l_C|$ (See Figure 40(b)). Then by the conclusion, we have there is $(c'', t'')$ corresponding to the isometric surface as $(c, t)$ such that $|l''_C| = 2|l''_{CE}|$ and $2|l''_{CD}| > |l''_C|$ (the blue point $P''$ in Figure 40(a)). By the proof in (a), if $2c > l_0$, then the surface is not maximal.

By (a) and (b), if the surface is maximal then $l_0 = 2c$, It implies that $|l_C| = 2|l_{CD}| = 2|l_{CE}|$, namely $|l_C(\Sigma)| = |l_{CD}(\Sigma)| = |l_{CE}(\Sigma)|$.

$\square$

6.3. Final calculation. Finally we calculate the systole of the surface with $|l_C(\Sigma)| = |l_{CD}(\Sigma)| = |l_{CE}(\Sigma)|$. To calculate the length, we obtain a subsurface with the signature $(1, 2)$ by cutting along the red curves $c_0, c''_0$ in Figure 40. $C_1, C_2, C_3$ and $C_4$ are branch points of the branch cover $\pi$. By the proof of Theorem 2, in the $\Sigma_{(1,2)}$. 

**Figure 40.**

**Figure 41.**
the curve connecting $C_1$, $C_2$ is $l_{CE}(\Sigma)$, the curve connecting $C_1$, $C_3$ is $l_C(\Sigma)$, the curve connecting $C_2$, $C_3$ is $l_{CD}(\Sigma)$.

Next, we represent the length of $\partial \Sigma_{(1,2)}$ by $c$ and $s$. The seams and cuffs of the $\Gamma(2,n)$ surface that intersect $\Sigma_{(1,2)}$ cut $\Sigma_{(1,2)}$ into four equal hexagons. (See Figure 43). In Figure 43 $A_2A_1H_2H_1A_4A_3$ is one of the hexagons. This hexagon is a right-angled hexagon. It is clear the angles are right at the four vertices $A_1$, $A_2$, $A_3$ and $A_4$ by the definition of cuffs and seams. For $H_1$ and $H_2$, we consider the $n$-holed spheres of the $\Gamma(2,n)$ surface. If cutting all the seams of one $n$-holed sphere, we get two isometric $2n$-polygon with order $n$ rotations. In each polygon, we pick the common perpendicular between two nearest non-neighboring seams. Two such curves, each from a polygon, forms a simple closed curve. This curve is one component of $\partial \Sigma_{(1,2)}$.

**Figure 42.**

**Figure 43.** The red curves are cuffs. The blue curves are seams.
In the right-angled hexagon $A_2A_1H_2H_1A_4A_3$, $|A_1A_2| = |A_3A_4| = c$, $|A_2A_3| = s$, then by Formula 2.4.1(i) in [Bus10, p. 454],

$$\cosh |H_1H_2| = \sinh |A_1A_2| \sinh |A_3A_4| \cosh |A_2A_3| - \cosh |A_1A_2| \cosh |A_3A_4|$$

$$= \sinh^2 c \cosh s - \cosh^2 c. \hspace{1cm} (6.9)$$

We pick another set of curves to cut $\Sigma_{(1,2)}$ to obtain the length of the systole. We pick two common perpendicular segments between the two boundary components of $\Sigma_{(1,2)}$. One of the segment is homotopic to the broken segment $H_1A_3A_3'H_1'$ (see Figure 43) relative to the boundary, the other is homotopic to the broken segment $H_2A_1A_1'H_2'$ relative to the boundary. These two segments are $H_3H_3'$ and $H_4H_4'$ in Figure 44 respectively. In Figure 44 by the definition of $t$ in [2.2] and the symmetry of $\Sigma_{(1,2)}$, the mid-point of $H_3H_3'$ and $H_4H_4'$ are the branch points $C_1$ and $C_4$ respectively.

![Figure 44](image_url)

Then by cutting along $H_3H_3'$ and $H_4H_4'$, we get a surface with the topology of annulus (see Figure 45). The unique non-trivial closed geodesic is illustrated by the segment $H_5H_6$. The common perpendiculars between $H_5H_6$ and $H_iH_i'$ ($i = 1, 2$) meet the mid-points $C_1$ or $C_4$ since all such common perpendiculars have the same length by the symmetry of $\Sigma_{(1,2)}$, then by the sine law of right-angle hexagon (Formula 2.4.1 (ii) in [Bus10, p 454]), the common perpendiculars meet the mid-points of $H_iH_i'$ ($i = 1, 2$).

Then we describe the curve $l_{CD}(\Sigma)$, $l_{CE}(\Sigma)$ and $l_C(\Sigma)$ in the annulus. We recall that $l_{CD}(\Sigma)$ passes $C_2$ and $C_3$, $l_{CE}(\Sigma)$ passes $C_1C_2$ or $C_3C_4$ and $l_C(\Sigma)$ passes $C_1C_3$ or $C_2C_4$.

The curve $l_{CD}(\Sigma)$ (the curve meeting $C_2$ and $C_3$) does not touch $H_1H_1'$ or $H_2H_2'$. Therefore it is the unique non-trivial closed geodesic in the annulus, namely the curve in Figure 45 corresponding to $H_5H_6$. In Figure 45 the red segments ($C_1C_3C_1$ and $C_4C_3C_4$) are cuffs ($l_{CE}(\Sigma)$). Therefore the intersection between cuffs and $H_5H_6$
are the branch points $C_2$ and $C_3$. The curve $l_C(\Sigma)$ passes $C_1C_3$ or $C_2C_4$. Then in Figure 45, the blue curve connecting $C_1C_3C_1$ is one of the $l_C(\Sigma)$ curves.

By Proposition 4, $|l_C(\Sigma_0)| = |l_{CD}(\Sigma_0)| = |l_{CE}(\Sigma_0)|$. Then in Figure 45, the triangle $\Delta C_1C_2C_3$ is an equilateral triangle, because the edges $C_1C_2$, $C_2C_3$ and $C_3C_1$ are the half of $l_{CE}(\Sigma)$, $l_{CD}(\Sigma)$ and $l_C(\Sigma)$ respectively. When the triangle is equilateral, the shape of the annulus is shown in Figure 46.

We assume in Figure 46, $h = |C_1H_2| = |C_4H_6|$, $k = |C_1C_2| = |C_2C_3| = |C_3C_1|$. Then $|C_2H_6| = |C_3H_6| = k/2$. Then we have the following two formulae:

In the hexagon $H_1H_2C_1H_5H_6C_4$, by formula 2.4.1(i) in [Bus10, p 454] we have:

$$\cosh |H_1H_2| = \sinh |C_1H_5| \sinh |C_4H_6| \cosh |H_5H_6| - \cosh |C_1H_5| \cosh |C_4H_6|$$

$$\cosh l = \sinh^2 h \cosh k - \cosh^2 h$$

(6.10) $$\cosh^2 h = \frac{\cosh l + \cosh k}{\cosh k - 1}.$$
Then in the triangle \( \triangle C_1H_6C_3 \), by the hyperbolic cosine law (Formula 2.2.2 (i) in [Bus10, p. 454]), we have
\[
\cosh |C_1C_3| = \cosh |C_1H_6| \cosh |C_3H_6|
\]
(6.11)
\[
\cosh k = \cosh h \cosh \frac{k}{2}.
\]
Finally we can get the formula for \( \cosh k \):
\[
\frac{\cosh^2 k}{\cosh^2 \frac{k}{2}} = \cosh h \quad \text{by (6.11)}
\]
(6.12)
\[
= \frac{\cosh l + \cosh k}{\cosh k - 1} \quad \text{by (6.10)}
\]
For convenience, we assume \( K = \cosh k \), Then
\[
\frac{\cosh^2 k}{\cosh^2 \frac{k}{2}} = \frac{2 \cosh^2 k}{\cosh k + 1} = \frac{2K^2}{K + 1}.
\]
We use it on (6.12):
\[
\frac{2K^2}{K + 1} = \frac{K + \cosh l}{K - 1} = \frac{K + \sinh^2 \frac{s}{2} \cosh s - \cosh^2 \frac{s}{2}}{K - 1} \quad \text{by (6.9)}
\]
\[
= \frac{(K^2 - 1) \cosh s - K^2 + K}{K - 1} c = k \quad \text{by definition}
\]
\[
= (K + 1) \cosh s - K
\]
\[
= (K + 1)(2 \sinh^2 \frac{s}{2} + 1) - K
\]
\[
= (K + 1) \left( \frac{2 \cos^2 \frac{\pi}{n} + 1}{\sinh^2 \frac{s}{2}} \right) - K \quad \text{by (2.3)}
\]
(6.13)
\[
= (K + 1) \left( \frac{4 \cos^2 \frac{\pi}{n}}{K - 1} + 1 \right) - K.
\]
Then from (6.13), we have
\[
2K^3 - 3K^2 + 1 - 4 \cos^2 \frac{\pi}{n}(K + 1)^2 = 0
\]
The unique real solution of this equation is:
\[
K = \frac{1}{6} \sqrt[3]{\frac{1}{216} L^3 + \frac{1}{8} L^2 + \frac{5}{8} L - \frac{1}{8} + \sqrt{\frac{1}{108} L(L^2 + 18L + 27)}}
\]
\[
+ \frac{1}{2} \left( \frac{1}{2} L^3 + \frac{1}{8} L^2 + \frac{5}{8} L - \frac{1}{8} - \sqrt{\frac{1}{108} L(L^2 + 18L + 27)} \right) - \frac{L + 3}{6}.
\]
and \( L = 4 \cos^2 \frac{\pi}{n} \).
Now we calculate the coordinate \((c, t)\) when the systole is maximal.
It is clear that $c = \text{arccosh} \, K$. Then we calculate $t$ using (6.2).

\[
\cosh \frac{t}{2} = \frac{\cosh \frac{c}{2}}{\cosh \frac{s}{2}} = \frac{\cosh^2 \frac{c}{2}}{\cos \frac{\pi}{n}} \text{ by (2.3)} = \cosh c + 1 \quad 2 \cos \frac{\pi}{n} = K + 1 \quad 2 \cos \frac{\pi}{n} .
\]

Therefore we get the Theorem:

**Theorem 3.** The maximal systole of the $\Gamma(2,n)$ surface is

\[2 \text{arccosh} \, K.\]

Here

\[
K = \frac{1}{6} \left( T^3 + 27T^2 + 12\sqrt{3} \sqrt{T^3 + 18T^2 - 27T + 135T - 27} \right) ^{\frac{1}{3}} +
\frac{T^2 + 18T + 9}{6 \left( T^3 + 27T^2 + 12\sqrt{3} \sqrt{T^3 + 18T^2 - 27T + 135T - 27} \right)} + \frac{T + 3}{6} ,
\]

and $T = 4 \cos^2 \frac{\pi}{n}$.

The maximal systole is obtained when

\[(c, t) = \left( \text{arccosh} \, K, 2 \text{arccosh} \, \frac{K + 1}{2 \cos \frac{\pi}{n}} \right) .\]

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