ON UNIFORM BOUNDEDNESS PROPERTIES OF SGD AND ITS MOMENTUM VARIANTS

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ABSTRACT

A theoretical, and potentially also practical, problem with stochastic gradient descent is that trajectories may escape to infinity. In this note, we investigate uniform boundedness properties of iterates and function values along the trajectories of the stochastic gradient descent algorithm and its important momentum variant. Under smoothness and $R$-dissipativity of the loss function, we show that broad families of step-sizes, including the widely used step-decay and cosine with (or without) restart step-sizes, result in uniformly bounded iterates and function values. Several important applications that satisfy these assumptions, including phase retrieval problems, Gaussian mixture models and some neural network classifiers, are discussed in detail.

Keywords  Stochastic gradient descent · Dissipativity · Uniform boundedness

1 Introduction

We consider the stochastic optimization problem of minimizing a (possibly non-convex) function $f$

$$
\min_x f(x) = \mathbb{E}_{\xi \sim \Xi}[f(x; \xi)].
$$

(1)

Here, $\xi$ is a random variable drawn from the unknown probability distribution $\Xi$ and $f(x; \xi)$ is the instantaneous loss function over the variable $x$. We assume that we may query the gradient of $f$ by randomly sampling mini-batches of size $b$. Stochastic gradient descent (SGD [Robbins and Monro, 1951]) and its momentum (or accelerated) variants [Polyak, 1964, Sutskever et al., 2013] have been the subject of intense research, both in terms of theoretical developments and applications, particularly to training of deep neural networks and other machine learning models.

The standard SGD algorithm [Robbins and Monro, 1951] is

$$
x_{k+1} = x_k - \eta_k g_k \quad \text{for } k \in \mathbb{N}
$$

(2)

where $\eta_k > 0$ is the step-size and $g_k$ is a noisy but unbiased estimator of $\nabla f(x_k)$. In the noise-less setting (i.e., when $g_k = \nabla f(x_k)$), the function value $f(x_k)$ is decreasing with $k$ as long as $\nabla f$ is $L$-Lipschitz continuous and $\eta_k \leq 1/L$. In fact, neither $f(x_k)$ nor the distance between $x_k$ and the optimal set will expand. However, the behaviour is very different in the presence of noise. It may happen that $x_k$ tends to infinity if the correction $\eta_k g_k$ is always large even though the step-size $\eta_k$ tends to zero; or if the variance $\mathbb{E}[\|g_k - \nabla f(x_k)\|^2]$ increases rapidly with $\|x\|$. In this note, we are interested in finding simple and verifiable conditions under which $\|x_k - x^*\|$ and $f(x_k) - f^*$ remain bounded along every possible trajectory of the system.

Of course, $f(x_k)$ will remain bounded if $f$ itself is bounded on $\mathbb{R}^d$; this is the case for some truncated loss functions that have been used in the robust machine-learning literature [Park and Liu, 2011, Xu et al., 2020]. Another trivial case is when the variable $x$ is kept in the bounded region, for example by running a projected SGD. But these are not the cases that we are interested in addressing in this paper. The main focus of this work is on loss functions $f$ that may go to infinity when $\|x\|$ tends to infinity. The most common loss functions in machine learning, such as the least-squares and the logistic loss, are unbounded on $\mathbb{R}^d$, and it very common to add a weight-decay regularization
with momentum) with the help of a regularity assumption which we call “Assumption 1. For any input vector $x$, the stochastic gradient oracle $O$ returns a vector $g$ such that (a) $E[g_k] = \nabla f(x_k)$; (b) $E[||g_k - \nabla f(x_k)||^2] \leq \rho \|\nabla f(x_k)\|^2 + \sigma^2$ where $\rho \geq 0$.

**Definition 1.** ($\ell$-smooth) A function $f$ is $\ell$-smooth if $\|\nabla f(x) - \nabla f(y)\| \leq \ell \|x - y\|$ for every $x, y \in \text{dom}(f)$. The smoothness property also implies that $|f(x) - f(y) - \langle \nabla f(y), x - y \rangle| \leq \frac{\ell}{2} \|x - y\|^2$.

**Notations.** We use $x^*$ to denote the minimizer of function $f$ and $f^* = f(x^*)$. Let $\|\cdot\| := \|\cdot\|_2$, without specific mention. We use $[n]$ denote the set of $\{1, 2, \cdots, n\}$. The subgradient of the function $f$ on $x$ is denoted by $\partial f(x) := \{v : f(y) \geq f(x) + \langle v, y - x \rangle \text{ for any } y\}$. For simplicity of the notations, we use $N, T, S$ are all integers. We use $F_\sigma$ to denote $\sigma$-algebra generated by all the random information at the current iterate $x_k$ and $x_k \in F_\sigma$. 

2 Preliminaries

Throughout the paper, we make the following assumption on the stochastic gradient oracle $O$ which samples the stochastic gradient as an unbiased estimator of the true gradient given any input variable $x$.

**Assumption 1.** For any input vector $x$, the stochastic gradient oracle $O$ returns a vector $g$ such that (a) $E[g_k] = \nabla f(x_k)$; (b) $E||g_k - \nabla f(x_k)||^2 \leq \rho \|\nabla f(x_k)\|^2 + \sigma^2$ where $\rho \geq 0$. 

In this work, we study uniform boundedness properties of SGD and its momentum variant (SGD with momentum) with the help of a regularity assumption which we call “R-dissipativity”: $\langle x - x^*, \nabla f(x) \rangle \geq \theta_1 \|x - x^*\|^2 - \theta_2$ for all $x$ with $\|x - x^*\| \geq R$ where $\theta_1 > 0$, $\theta_2 \geq 0$ and $R \geq 0$. This concept is a slight variation of the “localized dissipativity” property introduced in [Yu et al., 2021] but we do not concerns the local properties of the function for $x$ with $\|x - x^*\| \leq R$. The term of “dissipativity” is originated from dynamic systems [Hale, 1988] and used in a number of papers in optimization, learning and Bayesian analysis [Mattingly et al., 2002, Raginsky et al., 2017, Erdogdu et al., 2018, Mou et al., 2019]. As we will see in Section 3.1, many important application in machine learning, including shallow neural networks and the sum of a weight-decay regularization and the function whose gradient is bounded result in R-dissipative objective functions. We make the following contributions:

- Uniform boundedness properties of SGD and its momentum variant (SGD with momentum) are established for $\ell$-smooth and R-dissipative loss functions.
- We consider a broad class of step-sizes that are only required to be smaller than a constant $O(1/L)$. Our results cover many of the most popular step-size policies, including the classical constant and polynomial decay step-sizes, as well as more recently proposed time-dependent step-sizes such as stage-wise decay [Li et al., 2021, Wang et al., 2021], cosine with or without restart [Loshchilov and Hutter, 2017], and bandwidth-based step-sizes [Wang and Johansson, 2021, Wang and Yuan, 2021].
3 Non-convex Function with its Tail Growing Quadratically

In this section, we establish uniform boundedness of SGD and its momentum variant when the objective function is potentially nonconvex and satisfies the following regularity assumption.

**Definition 2.** (R-dissipativity) A function is R-dissipative that there exist $\theta_1 > 0$, $\theta_2 \geq 0$, and $R \geq 0$ such that, for all $x$ with $\|x - x^\ast\| \geq R$, the function $f$ satisfies

$$
\langle x - x^\ast, \nabla f(x) \rangle \geq \theta_1 \|x - x^\ast\|^2 - \theta_2.
$$

(3)

When $R = 0$, this assumption reduces the standard dissipativity assumption [Raginsky et al., 2017, Yu et al., 2021] (see (A.3) in [Raginsky et al., 2017]). Dissipativity can be seen as a relaxation of strong convexity, since every $\mu$-strongly convex function is also dissipative. In particular, it satisfies Definition 2 with $\theta_1 = \mu$, $\theta_2 = 0$ and $R = 0$.

The term “dissipative” comes from the theory of dynamical systems [Hale, 1988] but the concept has found many uses in the analysis of optimization and learning algorithms [Mattingly et al., 2002, Raginsky et al., 2017, Erdogdu et al., 2018], and in Bayesian analysis [Mou et al., 2019]. R-dissipativity is a natural property of many non-convex optimization problems, such as neural network training with weight-decay regularization [Krogh and Hertz, 1992]; see § 3.1 for more details.

**Remark 1.** For $L$-smooth functions, an alternative way of verifying R-dissipativity is to find $\theta_1'$ and $\theta_2'$ such that

$$
\langle x, \nabla f(x) \rangle \geq \theta_1' \|x\|^2 - \theta_2'
$$

(4)

for $\|x - x^\ast\| \geq R$. As shown in Appendix A, this implies that $f$ is R-dissipative with parameters

$$
\theta_1 = \frac{\theta_1'}{2}, \quad \theta_2 = \left(\theta_1' + 2L + \frac{L^2}{2\theta_1'}\right) \|x^\ast\|^2 + \theta_2'.
$$

**Remark 2.** Although the definition of R-dissipativity in Definition 2 considers differentiable functions, smoothness is not an essential property of our analysis. If $f$ is non-differentiable, the gradient $\nabla f(x)$ in Definition 2 can be replaced by the sub-gradient $\partial f(x)$ and the subsequent analysis for SGD and SGD with momentum still holds true.

3.1 Applications

We now show that several important applications in machine learning, robust statistic learning, and Bayesian models are R-dissipative. Some of these are also dissipative, while others are localized dissipative, but it is only the R-dissipative concept that captures all examples.

**Least squares problems** where $f(x) = \frac{1}{2} \|Px - b\|^2$ and $P$ has full rank are R-dissipative with $\theta_1 = \lambda_{\min}(P^TP)$ (the smallest eigenvalues of $P^TP$), $\theta_2 = 0$, and $R = 0$. They are therefore also dissipative.

**ℓ₂-regularized problems**, where $f + \lambda \|x\|^2$ with $\lambda > 0$ and $f$ is a possibly non-convex whose gradient norm is bounded by $G$ also satisfy Assumption 2. For example, most of the activation functions in machine learning including the sigmoid, hyperbolic tangent, rectified linear unit (ReLU) and variants such as Leaky ReLU and ELU can be applied. In this case, we have $\theta_1 = \lambda/2, \theta_2 = G/(2\lambda)$, and $R = 0$.

**Shallow neural networks** A subset of neural network training objectives with weight-decay regularization can also be shown to satisfy Assumption 2 (or the alternative condition (4)). For example, consider the training a fully connected 2-layer neural network (with $m$ hidden nodes) for binary classification with a weight decay ($\lambda \|\cdot\|^2/2$). Let $\sigma_1$ and $\sigma_2(\cdot)$ denote the inner and output layer activation functions, respectively and use cross-entropy to measure the output with its true label. If we use $\sigma_2(y) = 1/(1 + \exp(-y))$ and $\sigma_1(x) = \max(0, x)$ (the ReLU loss function), then the resulting training objective satisfies

$$
\langle x, \nabla f(x) \rangle \geq \lambda \|x\|^2 - 2.
$$

(5)

On the other hand, if $\sigma_1(x) = 1/(1 + \exp(-x))$, we find

$$
\langle x, \nabla f(x) \rangle \geq \frac{\lambda}{2} \|x\|^2 - \left(1 + \frac{m}{2\lambda}\right).
$$

(6)

Similarly, a three layer fully connected neural network with ReLU as the activation function of the inner-layers satisfies

$$
\langle x, \nabla f(x) \rangle \geq \lambda \|x\|^2 - 3.
$$

(7)
These three examples are all dissipative and R-dissipative, but not localized dissipative; see Appendix C for details.

**Phase retrieval** problems [Tan and Vershynin, 2019], where \( f(x) = 1/(2n) \sum_{i=1}^n (\langle a_i, x \rangle - b_i)^2 \) also satisfy this condition with \( \theta_1 = \lambda_{\text{min}}(P^T P)/(2n) \) (\( P = [a_1^T, \cdots, a_n^T]^T \in \mathbb{R}^{n \times d} \)), \( \theta_2 = \sum_{i=1}^n b_i^2/(2n) \), and \( R = 0 \).

**Regularized MLE** Another important example that satisfies the R-dissipativity assumption is the regularized MLE for heavy tailed linear regression [Yu et al., 2021] arising in robust statistics. Here, \( f(x) = \frac{1}{2n} \sum_{i=1}^n \log(1 + (b_i - \langle a_i, x \rangle)^2) + \frac{1}{2} \|x\|^2 \) with \( \theta_1 = \lambda, \theta_2 = \frac{1}{n} \sum_{i=1}^n |b_i| \) and \( R = 0 \).

**Blake & Zisserman** As shown in [Yu et al., 2021], the objective function of regularized Blake-Zisserman maximum-likelihood estimation, where \( f(x) = -\frac{1}{2n} \sum_{i=1}^n \log(\nu + \exp(-b_i - \langle a_i, x \rangle)^2)) + \frac{1}{2} \|x\|^2 \) for some \( \nu > 0 \) satisfies Assumption 2 with \( \theta_1 = \lambda, \theta_2 = 1/(2n) \sum_{i=1}^n |b_i|/\sqrt{\nu(\nu+1)} \), and \( R = 0 \).

**Gaussian mixture model** As a final example, we note that over-specified Bayesian Gaussian mixture models, which have been widely used to study datasets with heterogeneity [Mou et al., 2019], can be verified to satisfy Assumption 2 with \( \theta_1 = 4c_1, \theta_2 = 4c_1 \), and \( R = \sqrt{2} \) where \( c_1 > 0 \) is a universal constant (see Section 4.3 in [Mou et al., 2019]). Gaussian mixture models are R-dissipative and localized dissipative, but not dissipative.

**Remark 3.** If the original objective function does not satisfy R-dissipativity, it can always be rendered R-dissipative by adding a smooth regularizer \( \max \left\{ \|x\|^2 - R^2, 0 \right\} \) with \( R > 0 \). By properly choosing \( R \), we do not have to change the structure of the function around global optimizer but still force the iterations in a bounded region. A similar technique has been used in matrix completion problems in [Sun and Luo, 2016].

### 3.2 Uniform Boundedness Properties of SGD

In the remaining pages of this note, we use the concept of R-dissipativity to establish bounded properties of SGD and SGD with momentum. Our first result, proven in Appendix A is the following:

**Theorem 3.1.** Let \( f \) be \( L \)-smooth and R-dissipative. If the stochastic gradient oracle that satisfies Assumption 1, then the iterates of SGD with step-size \( \eta_k \leq \frac{\theta_1}{(\rho + 1)L^2} \) satisfy

\[
\mathbb{E}[\|x_k - x^*\|^2] \leq \max \left\{ \|x_0 - x^*\|^2, \frac{2(\sigma^2 + L^2\rho^2)\theta_1^2}{(1 + \rho)^2L^4} + 2\rho^2 \right\},
\]

where \( \rho^2 = \max \left\{ R^2, \frac{2\theta_1}{\theta_2} + \frac{\sigma^2}{(1 + \rho)L^2} \right\} \).

Theorem 3.1 shows that the iterations of SGD can be guaranteed to be in a bounded region under mild conditions. The uniform bound is directly related to the initial state, the noise \( \sigma^2 \), and the properties of the function itself.

Although the result is valid for all values of \( R \) (and therefore holds for both dissipative and localized dissipative functions), the right-hand side of (8) is not tight when \( R = 0 \). We analyze dissipative functions in Appendix D and derive tighter results, recovering the well-known bounds for \( \mu \)-strongly convex functions as a special case.

**Remark 4.** Theorem 3.1 guarantees uniform boundedness of the SGD iterates for any (possibly non-monotonic) step-size upper bounded by \( \frac{\theta_1}{(\rho + 1)L^2} \). This includes common step-size policies such as constant, polynomial decay [Moulines and Bach, 2011] step-decay [Wang et al., 2021], and exponential decay [Li et al., 2021], as well as more recently proposed non-monotonic step-sizes such as the bandwidth-based [Wang and Yuan, 2021, Wang and Johansson, 2021] and cosine [Loshchilov and Hutter, 2017] step-sizes.

### 3.3 Uniform Boundedness of SGD with Momentum

We now extend the results to also cover the following momentum variant of SGD

\[
\begin{align*}
v_{k+1} &= \beta v_k + (1 - \beta)g_k \quad (9a) \\
x_{k+1} &= x_k - \eta_k v_{k+1} \quad (9b)
\end{align*}
\]

where \( \beta \in (0, 1) \) and \( x_0 = x_1 \). We analyze the three common step-size policies: constant step-size, decaying step-sizes, and bandwidth step-sizes, respectively. Our first result is the following:

**Theorem 3.2.** (Constant step-size) Let \( f \) be \( L \)-smooth and R-dissipative with a stochastic oracle satisfying Assumption 1. Consider SGD with momentum defined by (9a) and (9b) with \( \beta \in (0, 1) \), for any constant step-size

\[
\eta_k = \eta \leq \min \left\{ \frac{(1 - \beta^2)}{(1 - \beta)^2 L^2} + \beta(1 - \beta)L, \frac{\theta_1}{2((1 - \beta)^2 + 1)(\rho + 1)L^2} \right\},
\]

where \( \beta \in (0, 1) \) and \( \eta \) is a constant step-size.
The, the quantities $\mathbb{E}[\|x_k - x^*\|^2]$ and $\mathbb{E}[f(x_k) - f^*]$ are uniformly bounded for all $k \in [1, T + 1]$.

Theorem 3.2 establishes uniform boundedness properties of SGD with momentum for any constant step-size under mild conditions. The most challenging part in establishing Theorem 3.2 is how to construct a Lyapunov function that is decreasing when $\|x_k - x^*\|^2 \geq R^2$. The details of the proofs are given in Appendix B.

For the time dependent step-size, the analysis is more complicated than the constant step-size (see Theorem 3.2). But our next theorem shows that the uniform boundedness can also be guaranteed.

**Theorem 3.3. (Decaying step-sizes)** Suppose that the objective function is $L$-smooth and $R$-dissipative, and that the stochastic gradient oracle satisfies Assumption 1. Consider SGD with momentum defined by (9a) and (9b) with a time-varying step-size $\eta_k \leq \frac{1 - \beta^2 k}{4(\rho + 1)(1 + \beta^2) L^2}$. Then both $\mathbb{E}[\|x_k - x^*\|^2]$ and $\mathbb{E}[f(x_k) - f^*]$ are uniformly bounded for all $k \in [1, T + 1]$, under the following step-size policies:

1. polynomial decaying $\eta_k = \eta_1/k^p$ for $p \in (0, 1]$ and $\eta_1 \geq \frac{4}{\theta_1}$, for any $k \geq 3$
2. linearly decaying $\eta_k = A - B k$ where $\eta_1 = \eta_{\text{max}}$ and $\eta_T = c/\sqrt{T}$ for $c \geq \left(\frac{4\eta_{\text{max}}^2}{\theta_1} \right)^{1/3}$
3. cosine decaying $\eta_k = A + B \cos\left(\frac{\pi k}{T}\right)$ where $\eta_1 = \eta_{\text{max}}$ and $\eta_T = c/\sqrt{T}$ with $c \geq \max\left\{ \left(\frac{\eta_{\text{max}}^2}{\theta_1} \right)^{1/3}, \frac{\eta_{\text{max}}^2}{4T^{3/2}} \right\}$
4. exponentially decaying $\eta_k = \eta_1/\alpha k - 1$ where $\alpha = (\nu/T)^{-1/T} > 1$, $\nu \geq 1$, and $\eta_1 \geq \frac{4\eta_{\text{max}}^2}{\theta_1}$.

In the above theorem, we analyze four important families of decaying step-sizes that have excellent practical performance, see e.g. [Loshchilov and Hutter, 2017] for the cosine step-size and Li et al. [2021] for the exponential step-size. Our results demonstrate that the uniform boundedness of SGD with momentum can also be verified under these step-size policies.

Finally, we show that the uniform boundedness properties of SGD with momentum can be guaranteed also by stage-wise bandwidth step-sizes [Wang and Yuan, 2021, Wang and Johansson, 2021], a class of step-size policies that allow for non-monotonic behaviour (ot within and between stages). The bandwidth framework not only covers the most popular stage-wise step-size (sometimes referred to as the step-decay step-size) [Ge et al., 2019, Yuan et al., 2019, Wang et al., 2021], but it also includes the popular cosine with restart step-size [Loshchilov and Hutter, 2017].

**Theorem 3.4. (Stage-wise bandwidth-based step-size)** Given the total number of iterations $T$, we consider the bandwidth step-size with the form $\eta_{\text{min}}^t \leq \eta_k \leq \eta_{\text{max}}^t$ and its upper and lower bounds $\eta_{\text{min}}^t$ and $\eta_{\text{max}}^t$ are decreasing with $t$ where $t \in [N]$, $k \in [\sum_{i=1}^{t-1} S_i + 1, \sum_{i=1}^{t} S_i]$, and $\sum_{i=1}^{N} S_i = T$. We assume that the bandwidth $s = \eta_{\text{max}}^t/\eta_{\text{min}}^t$ is bounded and the step-size is decreasing at each stage $t$. Under the same setting as Theorem 3.3, we consider the constant and decaying modes discussed in Theorems 3.2 and 3.3, then the quantitites $\mathbb{E}[\|x_k - x^*\|^2]$ and $\mathbb{E}[f(x_k) - f^*]$ for all $k \in [1, T + 1]$ are uniformly bounded.

Theorem 3.4 gives us a lot of freedom to choose the lower and upper bounds $\eta_{\text{min}}^t, \eta_{\text{max}}^t$ and stage length $S_i$. For example $\eta_{\text{min}}^t$ and $\eta_{\text{max}}^t$ can be selected as $O(1/t^p)$ with $p \in (0, 1]$ or $O(1/\alpha^t)$ with $\alpha > 1$. Although we restrict $s = \eta_{\text{max}}^t/\eta_{\text{min}}^t$ is bounded, the lower bound $\eta_{\text{min}}^t$ and upper bound $\eta_{\text{max}}^t$ may be of different orders. The stage length $S_i$ can be a constant ($S_i = S \geq 1$) or time dependent with $t$.

**4 Conclusion**

We have provided uniform boundedness guarantees for SGD and SGD with momentum under the $L$-smoothness and $R$-dissipativity conditions. Our results allow for broad families of step-sizes that are only upper bounded by a constant $O(1/L)$. We have demonstrated that, $\mathbb{E}[\|x_k - x^*\|^2]$ and $\mathbb{E}[f(x_k) - f^*]$ stay bounded along every possible trajectory, even under step-sizes that do not satisfy the Robbins-Monro conditions (not square summable but summable). The uniform bounds that we have derived depend on loss function properties, noise parameters $\sigma^2$ and $\rho$ of the stochastic oracle, and the initial state of the algorithm.

$R$-dissipativity captures non-convex functions that grow quadratically when we move (possibly far) away from the global minimizer. As shown in Section 3.1, several interesting applications in machine learning satisfy this regularity condition. It would be interesting to know whether uniform boundedness properties can be guaranteed also for loss functions with slower asymptotic growth, such as Bayesian logistic regression. We leave this question for future work.
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Lemma A.1. A Supplementary Theoretical Results for SGD

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A Supplementary Theoretical Results for SGD

Lemma A.1. We consider the SGD algorithm and assume that the noise of the stochastic gradient satisfies Assumption 1. Then

$$
\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid F_k] \leq \|x_k - x^*\|^2 - 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle + \eta_k^2 \left( \left(1 + \rho \right) \|\nabla f(x_k)\|^2 + \sigma^2 \right)
$$

Proof. Applying the iteration of the SGD algorithm, we have

$$
\|x_{k+1} - x^*\|^2 = \|x_k - \eta_k g_k - x^*\|^2 = \|x_k - x^*\|^2 - 2\eta_k \langle g_k, x_k - x^* \rangle + \eta_k^2 \|g_k\|^2.
$$

By Assumption 1, we can estimate the expectation of $\|g_k\|^2$ as below

$$
\mathbb{E}[\|g_k\|^2 \mid F_k] = \mathbb{E}[\|g_k - \nabla f(x_k) + \nabla f(x_k)\|^2 \mid F_k] = \mathbb{E}[\|g_k - \nabla f(x_k)\|^2 \mid F_k] + \|\nabla f(x_k)\|^2 
\leq \sigma^2 + (\rho + 1) \|\nabla f(x_k)\|^2.
$$

Taking conditional expectation with respect to $F_k$ on the both sides of (10) and applying (11) gives

$$
\mathbb{E}\left[\|x_{k+1} - x^*\|^2 \mid F_k\right] = \|x_k - x^*\|^2 - 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle + \eta_k^2 \mathbb{E}\left[\|g_k - \nabla f(x_k) + \nabla f(x_k)\|^2\right] 
\leq \|x_k - x^*\|^2 - 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle + \eta_k^2 \left( \left(1 + \rho \right) \|\nabla f(x_k)\|^2 + \sigma^2 \right) .
$$

Proof. (of Theorem 3.1) Let $r^2 = \max\left\{R^2, \frac{\sigma^2}{n^2} + \frac{\sigma^2}{(n+1)^2} \right\}$. In this theorem, we consider two different cases based on the distance of the initial point $x_1$ with the global minimizer $x^*$. First, we consider the initial point $\|x_1 - x^*\| \leq r$. Suppose that at time $k$, $\mathbb{E}[\|x_k - x^*\|^2 \mid F_{k-1}] > r^2$ but $\mathbb{E}[\|x_k - x^*\|^2 \mid F_{k-2}] \leq r^2$. We can see that

$$
\mathbb{E}[\|x_k - x^*\|^2 \mid F_{k-1}] = \mathbb{E}[\|x_k - x_{k-1} + x_{k-1} - x^*\|^2 \mid F_{k-1}] 
\leq 2\mathbb{E}[\|x_k - x_{k-1}\|^2 \mid F_{k-1}] + 2 \|x_{k-1} - x^*\|^2 
\leq 2\eta_k^2 \mathbb{E}\left[\|g_{k-1}\|^2 \mid F_{k-1}\] + 2 \|x_{k-1} - x^*\|^2 
\leq 2\eta_k^2 \left( \sigma^2 + L^2 \|x_{k-1} - x^*\|^2 \right) + 2 \|x_{k-1} - x^*\|^2 
= 2\eta_k^2 \sigma^2 + 2(\eta_k^2 L^2 + 1) \|x_{k-1} - x^*\|^2 
< 2\eta_k^2 \sigma^2 + 2(\eta_k^2 L^2 + 1)r^2 
\leq \frac{2(\sigma^2 + L^2)^2\theta^2}{(1 + \rho)^2L^4} + 2r^2.
$$

(12)
This means that when $\mathbb{E}[\|x_{k} - x^*\|^2] > r^2$, then $R$-dissipativity holds. By Lemma A.1, we have
\[
\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] = \|x_k - x^*\|^2 - 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle + \eta_k^2 \left( (1 + \rho) \|\nabla f(x_k)\|^2 + \sigma^2 \right)
\leq \|x_k - x^*\|^2 + (-2\theta_1 \eta_k + (1 + \rho)L^2\eta_k^2) \|x_k - x^*\|^2 + 2\theta_2 \eta_k + \eta_k^2 \sigma^2. \tag{13}
\]
By $\eta_k \leq \frac{\theta_1}{(\rho + 1)L}$, we have $-2\theta_1 \eta_k + (1 + \rho)L^2\eta_k^2 \leq -\theta_1 \eta_k$. Then
\[
\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq \|x_k - x^*\|^2 - \theta_1 \eta_k \|x_k - x^*\|^2 + 2\theta_2 \eta_k + \eta_k^2 \sigma^2. \tag{14}
\]
Once $\mathbb{E}[\|x_k - x^*\|^2] > r^2$, for any kinds of step-sizes as long as $\eta_k \leq \frac{\theta_1}{(\rho + 1)L}$, if $r$ is large enough for example
\[
r^2 = \max \left\{ R^2, \frac{2\theta_2}{\theta_1} + \frac{\sigma^2}{(1 + \rho)L^2} \right\}
\]
to make sure that
\[-\theta_1 \eta_k \|x_k - x^*\|^2 + 2\theta_2 \eta_k + \eta_k^2 \sigma^2 < 0.
\]
Thus
\[
\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] < \|x_k - x^*\|^2.
\]
That is to say, once there exist an iterate $k$ such that $\mathbb{E}[\|x_k - x^*\|^2] > r^2$, for any step-size $\eta_k \leq \frac{\theta_1}{(\rho + 1)L}$, then the follow-up $\mathbb{E}[\|x_{k+1} - x^*\|^2]$ is decreasing.

If the initial point $x_1$ is far from the optimal point $x^*$, that is $\|x_1 - x^*\|^2 > r^2$ but $\|x_1 - x^*\|$ is bounded, by applying the above statements, we can see that the follow-up $\mathbb{E}[\|x_{k+1} - x^*\|^2]$ is decreasing until $\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq r^2$.

\[\square\]

**Proof.** (Proof of Remark 1) In this proof, we show that $R$-dissipativity condition can be derived from
\[
\langle x, \nabla f(x) \rangle \geq \theta_1 \|x\|^2 - \theta_2 \tag{15}
\]
for all $\|x - x^*\| \geq R$. When $x^* = 0$, the two conditions are the same. Suppose that $\|x^*\| > 0$, then
\[
\langle x, \nabla f(x) \rangle \geq \theta_1 \|x\|^2 - \theta_2 = \theta'_1 \left( \|x - x^*\|^2 + \|x^*\|^2 - 2 \|x^*\| \|x - x^*\| \right) - \theta_2 \tag{16}
\]
and
\[
\langle \nabla f(x), x^* \rangle^{(a)} \leq \|\nabla f(x)\| \|x^*\| \leq L \|x - x^*\| \|x^*\| \tag{17}
\]
where $(a)$ follows from the fact that $\|\nabla f(x)\| \leq L \|x - x^*\|$ (due to $L$-smoothness). Incorporating the above results gives that
\[
\langle x - x^*, \nabla f(x) \rangle \geq \theta'_1 \|x - x^*\|^2 - \left( 2\theta'_1 + L \right) \|x^*\| \|x - x^*\| - \theta'_2 + \theta'_1 \|x^*\|^2
\geq \frac{\theta'_1}{2} \|x - x^*\|^2 - \left( \frac{(2\theta'_1 + L)^2}{2\theta'_1} - \theta'_1 \right) \|x^*\|^2 - \theta'_2. \tag{18}
\]
Thus the $R$-dissipativity condition holds with $\theta_1 = \frac{\theta'_1}{2}$ and $\theta_2 = \left( \theta'_1 + 2L + \frac{L^2}{2\theta'_1} \right) \|x^*\|^2 + \theta'_2$.

\[\square\]

## B Supplementary Material for SGD with Momentum

In this part, we provide supplementary proofs for theorems in Section 3.3. Before giving the main proofs, we first show some extra and useful lemmas.
Lemma B.1. Suppose that the objective function $f$ is $L$-smooth and the stochastic gradient satisfies Assumption 1. Consider SGD with momentum with the momentum parameter $\beta \in (0, 1)$, for any step-size $\eta_k > 0$, we have

$$
\mathbb{E}[\|x_{k+1} - x_k\|^2 \mid \mathcal{F}_k] = \frac{\eta_k^2 \beta^2}{\eta_k^2} \|x_k - x_{k-1}\|^2 + \eta_k^2 (1 - \beta)^2 \mathbb{E}[\|g_k\|^2 \mid \mathcal{F}_k] + \frac{\eta_k^2 \beta(1 - \beta)}{\eta_k} \left( f(x_{k-1}) - f(x_k) + \frac{L}{2} \|x_k - x_{k-1}\|^2 \right).
$$

Proof. (of Lemma B.1) Applying the update recursion of the SGD with momentum algorithm in (9a) and (9b) gives

$$
\mathbb{E}[\|x_{k+1} - x_k\|^2 \mid \mathcal{F}_k] = \mathbb{E}[\|\eta_k v_{k+1}\|^2 \mid \mathcal{F}_k] = \eta_k^2 \mathbb{E}[\|\beta v_k + (1 - \beta)g_k\|^2 \mid \mathcal{F}_k]
$$

$$
= \eta_k^2 \mathbb{E} \left[ \|\beta \eta_k^{-1} (x_{k-1} - x_k) + (1 - \beta)g_k\|^2 \mid \mathcal{F}_k \right]
$$

$$
= \eta_k^2 \beta^2 \|x_k - x_{k-1}\|^2 + \eta_k^2 (1 - \beta)^2 \mathbb{E}[\|g_k\|^2 \mid \mathcal{F}_k] + \frac{2\eta_k^2 \beta(1 - \beta)}{\eta_k} \langle x_{k-1} - x_k, \nabla f(x_k) \rangle
$$

(19)

where $\mathbb{E}[g_k \mid \mathcal{F}_k] = \nabla f(x_k)$. Using the $L$-smooth assumption, for any $x, y \in \mathbb{R}^d$, we have

$$
\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|.
$$

Let $x = x_{k-1}$ and $y = x_k$, the $L$-smoothness also implies that

$$
f(x_{k-1}) \geq f(x_k) + \langle x_{k-1} - x_k, \nabla f(x_k) \rangle - \frac{L}{2} \|x_k - x_{k-1}\|^2.
$$

(20)

Applying the above inequality into (19), we have

$$
\mathbb{E}[\|x_{k+1} - x_k\|^2 \mid \mathcal{F}_k] \leq \frac{\eta_k^2 \beta^2}{\eta_k^2} \|x_k - x_{k-1}\|^2 + \eta_k^2 (1 - \beta)^2 \mathbb{E}[\|g_k\|^2 \mid \mathcal{F}_k] + \frac{2\eta_k^2 \beta(1 - \beta)}{\eta_k} \left( f(x_{k-1}) - f(x_k) + \frac{L}{2} \|x_k - x_{k-1}\|^2 \right).
$$

Then the proof is complete. □

Lemma B.2. We define $\tilde{x}_{k+1} := \frac{x_{k+1} - \beta x_k}{1 - \beta}$. Suppose that Assumption 1 and (2) hold at current iterations $x_k$ and $\tau_k = \eta_k / \eta_{k-1} \in (0, 1]$, then

(i) If $\tau_k = 1$, i.e., $\eta_k = \eta$ for all $k \geq 1$, we have

$$
\mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq \|\tilde{x}_k - x^*\|^2 - \eta \left( 2\theta_1 - \frac{\beta L^2}{(1 - \beta)\omega} \right) \|x_k - x^*\|^2 + 2\eta \theta_2 + \frac{\beta \omega \eta}{1 - \beta} \|x_{k-1} - x_k\|^2 + \eta^2 \mathbb{E}[\|g_k\|^2 \mid \mathcal{F}_k].
$$

where $\omega$ is a positive scalar.

(ii) else if $\tau_k \in (0, 1)$, we have

$$
\mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq \|\tilde{x}_k - x^*\|^2 - \left( 2\eta_1 \theta_1 - (1 - \tau_k)^2 - \frac{\beta \tau_k L^2}{1 - \beta} \omega \right) \|x_k - x^*\|^2 + 2\eta \theta_2 + \frac{\beta \tau_k}{1 - \beta} \left( \omega \eta_k + \frac{\beta \tau_k}{1 - \beta} \right) \|x_{k-1} - x_k\|^2 + \eta^2 \mathbb{E}[\|g_k\|^2 \mid \mathcal{F}_k].
$$

where $\omega > 0$. 

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Proof. (of Lemma B.2) Recalling the definition of $\tilde{x}_{k+1}$ and applying the recursion of SGD with momentum, we have

$$
\|\tilde{x}_{k+1} - x^*\|^2 = \| (1-\beta)^{-1} (x_{k+1} - \beta x_k) - x^* \|^2 = \| (1-\beta)^{-1} (x_k - \eta_k (\beta v_k + (1-\beta) g_k) - \beta x_k) - x^* \|^2
$$

$$
= \| x_k - \frac{\eta_k}{1-\beta} \left( \beta \left( \frac{x_k - x_{k-1}}{\eta_{k-1}} \right) + (1-\beta) g_k \right) - x^* \|^2
$$

$$
= \| \frac{\eta_k}{1-\beta} (x_k - x_{k-1}) + \left( 1 - \frac{\eta_k}{\eta_{k-1}} \right) x_k - \eta_k g_k - x^* \|^2
$$

$$
= \| \frac{\eta_k}{\eta_{k-1}} (\tilde{x}_k - x^*) + \left( 1 - \frac{\eta_k}{\eta_{k-1}} \right) (x_k - x^*) - \eta_k g_k \|^2
$$

$$
= \tau_k^2 \| \tilde{x}_k - x^* \|^2 + (1-\tau_k)^2 \| x_k - x^* \|^2 - 2 \tau_k \eta_k \langle \tilde{x}_k - x^*, g_k \rangle - 2(1-\tau_k) \eta_k \langle x_k - x^*, g_k \rangle
$$

$$
+ \eta_k^2 \| g_k \|^2 + 2 \tau_k (1-\tau_k) \langle \tilde{x}_k - x^*, x_k - x^* \rangle.
$$

(21)

Taking conditional expectation on both sides, we then estimate the first inner product:

$$
-\mathbb{E}[\langle \tilde{x}_k - x^*, g_k \rangle] = -\langle \tilde{x}_k - x^*, \nabla f(x_k) \rangle = -\langle (1-\beta)^{-1} (x_k - \beta x_{k-1}) - x^*, \nabla f(x_k) \rangle
$$

$$
= -\langle x_k - x^*, \nabla f(x_k) \rangle + \frac{\beta}{1-\beta} \langle x_{k-1} - x_k + x_k - x^*, \nabla f(x_k) \rangle
$$

$$
\leq -\langle \theta_1 \| x_k - x^* \|^2 - \theta_2 \rangle + \frac{\beta}{1-\beta} \left( \frac{c}{2} \| x_k - x_{k-1} \|^2 + \frac{1}{2c} \| \nabla f(x_k) \|^2 \right)
$$

$$
= -\langle \theta_1 \| x_k - x^* \|^2 - \theta_2 \rangle + \frac{\beta}{1-\beta} \left( \frac{\omega}{2} \| x_k - x_{k-1} \|^2 + \frac{1}{2\omega} \| \nabla f(x_k) \|^2 \right)
$$

(22)

where (a) follows from the $R$-dissipativity condition and the Cauchy-Schwartz inequality that $x^T y \leq \frac{\omega}{2} \| x \|^2 + \frac{1}{2\omega} \| y \|^2$ for any $\omega > 0$ and (b) applies the $L$-smooth assumption which implies that $\| \nabla f(x) \| \leq L \| x - x^* \|$.

Next, we consider two situations: (1) if $\tau_k = 1$, that is $\eta_k = \eta_{k-1}$. Incorporating the inequality (22) into (21), we have

$$
\mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid \mathcal{F}_k]
$$

$$
\leq \| \tilde{x}_k - x^* \|^2 - 2 \eta_k \left( \theta_1 \| x_k - x^* \|^2 - \theta_2 \right) + \eta_k^2 \mathbb{E}[\| g_k \|^2 \mid \mathcal{F}_k] + \frac{2\beta \eta_k}{1-\beta} \left( \frac{\omega}{2} \| x_{k-1} - x_k \|^2 + \frac{1}{2\omega} \| x_k - x^* \|^2 \right)
$$

$$
= \| \tilde{x}_k - x^* \|^2 - \eta_k \left( 2 \theta_1 - \frac{\beta}{1-\beta} \frac{L^2}{\omega} \right) \| x_k - x^* \|^2 + 2 \eta_k \theta_2 + \frac{2\beta \omega \eta_k}{1-\beta} \| x_{k-1} - x_k \|^2 + \eta_k^2 \mathbb{E}[\| g_k \|^2 \mid \mathcal{F}_k].
$$

(23)

(2) If $\tau_k \in (0,1)$, we then estimate

$$
-\mathbb{E}[\langle x_k - x^*, g_k \mid \mathcal{F}_k \rangle] = -\langle x_k - x^*, \nabla f(x_k) \rangle \leq -\theta_1 \| x_k - x^* \|^2 + \theta_2.
$$

(24)

Next we turn to estimate $\langle \tilde{x}_k - x^*, x_k - x^* \rangle$ as follows:

$$
\langle \tilde{x}_k - x^*, x_k - x^* \rangle = \langle \tilde{x}_k - x^*, \tilde{x}_k - \frac{\beta}{1-\beta} (x_k - x_{k-1}) - x^* \rangle
$$

$$
\leq \| \tilde{x}_k - x^* \|^2 + \frac{\beta}{1-\beta} \left( \frac{1}{2\omega_1} \| \tilde{x}_k - x^* \|^2 + \frac{\omega_1}{2} \| x_k - x_{k-1} \|^2 \right)
$$

(25)

for any $\omega_1 > 0$. Incorporating the above inequality into (21) and in order to make the scalar in front of $\| \tilde{x}_k - x^* \|^2$ is not larger than 1, we choose $\omega_1 = \frac{\beta \tau_k}{(1-\beta)(1-\tau_k)}$ such that

$$
\tau_k^2 + 2\tau_k (1-\tau_k) \left( 1 + \frac{\beta}{1-\beta} \frac{1}{2\omega_1} \right) \leq 1.
$$

(26)
Finally, incorporating the above results (22), (24), and (25) into (21), we can achieve that

\[
\mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid F_k]
\leq \|\tilde{x}_k - x^*\|^2 + (1 - \tau_k)^2 \|x_k - x^*\|^2 - 2\tau_k \eta_k \left( \theta_1 \|x_k - x^*\|^2 - \theta_2 \right) + \eta_k^2 \mathbb{E}[\|g_k\|^2 \mid F_k] + \frac{\beta^2 \tau_k^2}{(1 - \beta)^2} \|x_{k-1} - x_k\|^2
\]

\[
+ \frac{2\beta \tau_k \eta_k}{1 - \beta} \left( \frac{\omega}{2} \|x_{k-1} - x_k\|^2 + \frac{1}{2\omega} L_x^2 \|x_k - x^*\|^2 \right) + 2(1 - \tau_k) \eta_k \left( -\theta_1 \|x_k - x^*\|^2 + \theta_2 \right)
\]

\[
= \|\tilde{x}_k - x^*\|^2 - \left( 2\eta_k \theta_1 - (1 - \tau_k)^2 \frac{\beta \tau_k}{1 - \beta} \frac{L_x^2 \eta_k}{\omega} \right) \|x_k - x^*\|^2 + 2\eta_k \theta_2
\]

\[
+ \frac{\beta \tau_k}{1 - \beta} \left( \omega \eta_k + \frac{\beta \tau_k}{(1 - \beta)} \right) \|x_{k-1} - x_k\|^2 + \eta_k^2 \mathbb{E}[\|g_k\|^2 \mid F_k].
\]

We now complete the proof. \(\square\)

**Proof.** (of Theorem 3.2) In this case, we consider \(\eta_k = \eta\) is a constant step-size. From Lemma B.1, we have

\[
\mathbb{E}[\|x_{k+1} - x_k\|^2 \mid F_k]
\leq \beta^2 \|x_k - x_{k-1}\|^2 + \eta^2 (1 - \beta)^2 \mathbb{E}[\|g_k\|^2] + 2\eta \beta (1 - \beta) \left( f(x_{k-1}) - f(x_k) + \frac{L}{2} \|x_k - x_{k-1}\|^2 \right)
\]

\[
= (\beta^2 + \eta (1 - \beta) L) \|x_k - x_{k-1}\|^2 + \eta^2 (1 - \beta)^2 \mathbb{E}[\|g_k\|^2] + 2\eta \beta (1 - \beta) \left( f(x_{k-1}) - f(x_k) \right). \quad (27)
\]

Then we turn to estimate \(\|\tilde{x}_{k+1} - x^*\|^2\). For the constant step-size \(\eta_k = \eta\), we have \(\tau_k = \eta_k / \eta_{k+1} = 1\). By Lemma B.2(i), we can achieve that

\[
\mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid F_k] \leq \|\tilde{x}_k - x^*\|^2 - \eta \left( 2\theta_1 - \frac{\beta}{1 - \beta} \frac{L^2}{\omega} \right) \|x_k - x^*\|^2 + 2\eta \theta_2 + \frac{\beta \omega}{1 - \beta} \eta \|x_{k-1} - x_k\|^2 + \eta^2 \mathbb{E}[\|g_k\|^2 \mid F_k].
\]

Next we define a function \(W_{k+1}\):

\[
W_{k+1} = \|\tilde{x}_{k+1} - x^*\|^2 + \|x_{k+1} - x_k\|^2 + 2\eta \beta (1 - \beta) \left( f(x_k) - f^* \right).
\]

Then applying the results derived from Lemmas B.1 and B.2, we have

\[
\mathbb{E}[W_{k+1} \mid F_k] = \mathbb{E}[\|\tilde{x}_{k+1} - x^*\|^2 \mid F_k] + \mathbb{E}[\|x_{k+1} - x_k\|^2 \mid F_k] + 2\eta \beta (1 - \beta) \left( f(x_k) - f^* \right)
\]

\[
\leq \|\tilde{x}_k - x^*\|^2 + \left( \beta^2 + \eta \beta (1 - \beta) L + \eta \frac{\beta \omega}{(1 - \beta)} \right) \|x_{k-1} - x_k\|^2 - \eta \left( 2\theta_1 - \frac{\beta \omega}{(1 - \beta)} \right) \|x_k - x^*\|^2
\]

\[
+ \eta^2 \left( (1 - \beta)^2 + 1 \right) \left( \sigma^2 + (\rho + 1) L^2 \|x_k - x^*\|^2 \right) + 2\eta \beta (1 - \beta) \left( f(x_{k-1}) - f^* \right) + 2\eta \theta_2. \quad (28)
\]

where the above inequality uses the fact that

\[
\mathbb{E}[\|g_k\|^2 \mid F_k] = \mathbb{E}[\|g_k - \nabla f(x_k) + \nabla f(x_k)\|^2 \mid F_k] = \mathbb{E}[\|g_k - \nabla f(x_k)\|^2 \mid F_k] + \|\nabla f(x_k)\|^2
\]

\[
\leq (\rho + 1) \|\nabla f(x_k)\|^2 + \sigma^2 \leq (\rho + 1) \|x_k - x^*\|^2 + \sigma^2.
\]

Let \(\omega = \frac{\beta L^2}{(1 - \beta) \theta_1}\), then \(2\theta_1 - \frac{\beta \omega}{(1 - \beta) \theta_1} = \theta_1\). Suppose that

\[
\eta \leq \frac{(1 - \beta^2)}{\beta \omega (1 - \beta) \theta_1 \theta_1}, \quad (29)
\]

we have

\[
(\beta^2 + \eta \beta (1 - \beta) L + \eta \frac{\beta \omega}{(1 - \beta)}) \leq 1. \quad (30)
\]

Then (28) can be estimated as

\[
\mathbb{E}[W_{k+1} \mid F_k] \leq W_k - \eta \theta_1 \|x_k - x^*\|^2 + \eta^2 \left( (1 - \beta)^2 + 1 \right) \left( \sigma^2 + (\rho + 1) L^2 \|x_k - x^*\|^2 \right) + 2\eta \theta_2
\]

\[
= W_k - \eta (\theta_1 - \eta (((1 - \beta)^2 + 1) (\rho + 1) L^2) \|x_k - x^*\|^2 + 2\eta \theta_2 + \eta^2 ((1 - \beta)^2 + 1) \sigma^2.
\]
Furthermore, assume that \( \eta \leq \frac{\theta_1}{2(1-\beta)^2+1} \), then \( \theta_1 - \eta \theta_2 \geq \frac{\theta_1}{2} \), we have
\[
\mathbb{E}[W_{k+1} | \mathcal{F}_k] \leq W_k - \frac{\eta \theta_1}{2} ||x_k - x^*||^2 + 2\eta \theta_2 + \eta^2 ((1-\beta)^2 + 1) \sigma^2.
\]
(31)

If we further let
\[
r^2 = \max \left\{ R^2, \frac{4\theta_2}{\theta_1} + \frac{2\eta((1-\beta)^2 + 1)}{\theta_1} \right\},
\]
(32)
once \( ||x_k - x^*||^2 \geq r^2 \), we have
\[
-\frac{\eta \theta_1}{2} ||x_k - x^*||^2 + 2\eta \theta_2 + \eta^2 ((1-\beta)^2 + 1) \sigma^2 \leq 0.
\]
(33)

Then \( \mathbb{E}[W_{k+1} | \mathcal{F}_k] \leq W_k \).

If \( ||x_1 - x^*||^2 \leq r^2 \), let \( x_k^* \) be the first iteration that makes \( ||x_k^* - x^*||^2 \geq r^2 \) and \( ||x_k^* - x_k||^2 \leq r^2 \). First, we show that \( ||x_k^* - x_k||^2 \) will not be far larger than \( r^2 \).

If \( ||x_k^* - x^*||^2 \leq r^2 \) and \( ||x_k^* - x_k||^2 \leq r^2 \), then applying the recursion of SGD with momentum, we have
\[
\mathbb{E}[||x_k^* - x_k||^2] = \mathbb{E}[||x_k^* - \eta \beta x_k^*||^2] = \mathbb{E}\left[ ||x_{k-1} - \eta \right. \beta^{-1} \left( x_{k-2} - x_{k-1} \right) + (1-\beta)g_k^* - x^* ||^2\right]
\]

(34)

where \((a)\) follows from the fact that \( \frac{(x+w+y)^2}{2} \leq 3x^2 + y^2 + z^2 \) and \((b)\) applies inequality (11) that \( \mathbb{E}[||g_k||^2] \leq \sigma^2 + (\rho + 1)(\nabla f(x_k))^2 \). In the case that \( ||x_k^* - x^*||^2 \geq r^2 \) and \( ||x_{k-1} - x^*||^2 \leq r^2 \), we can estimate the Lyapunov function \( W_k \) at \( k \)-th iteration
\[
\mathbb{E}[W_k] = \mathbb{E}[\|\tilde{x}_k - x^*\|^2 + ||x_k - x_k^*||^2 + 2\eta\beta (1-\beta) (f(x_k^*) - f^*)]
\]

\[
\leq \frac{1}{(1-\beta)^2} ||x_k^* - x_k||^2 + 2\eta\beta (1-\beta) \frac{L}{2} ||x_k^* - x^*||^2 \quad \mathcal{F}_k
\]

\[
\leq \left( \frac{1}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^2} \right) \left( \mathbb{E}[||x_k^* - x^*||^2] + \mathbb{E}[||x_k^* - x_k||^2] \right) + 2 \mathbb{E}[||x_k^* - x^*||^2] + \mathbb{E}[||x_k^* - x_k||^2]
\]

\[
\leq \left( \frac{1}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^2} \right) (\Delta^2 + r^2) + \eta\beta (1-\beta) \frac{L}{2} r^2.
\]
(35)

As we discussed before, once \( ||x_k - x^*||^2 \geq r^2 \), then \( \mathbb{E}[W_k] \) is decreasing. Thus we can conclude that \( \mathbb{E}[W_k] \) is uniformly bounded by
\[
\mathbb{E}[W_k] \leq \left( \frac{1}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^2} \right) (\Delta^2 + r^2) + \eta\beta (1-\beta) \frac{L}{2} r^2.
\]
(36)

If the initial point \( ||x_1 - x^*||^2 > r^2 \) but is finite. Then we can see that \( \mathbb{E}[W_k] \) is decreasing until \( ||x_k - x^*||^2 \leq r^2 \). Thus in this case, we also can conclude that \( \mathbb{E}[W_k] \) is uniformly bounded. Due to that the three quantities \( \mathbb{E}[\|\tilde{x}_k - x^*\|], \mathbb{E}[\|x_k - x_{k-1}\|^2], \eta \mathbb{E}[f(x_{k-1}) - f^*] \geq 0 \), we can conclude that the three quantities are uniformly bounded. By the \( L \)-smoothness, we can estimate the function value \( \mathbb{E}[f(x_k) - f^*] \) as
\[
\mathbb{E}[f(x_k) - f^*] \leq \frac{L}{2} \mathbb{E}[\|x_k - x^*\|^2] = \frac{L}{2} \mathbb{E}\left[ \|\tilde{x}_k - \frac{1}{1-\beta} (x_k - x_{k-1}) - x^* \|^2 \right]
\]

\[
\leq LE[\|\tilde{x}_k - x^*\|^2] + \frac{L^2\beta^2}{(1-\beta)^2} \mathbb{E}[\|x_k - x_{k-1}\|^2] \leq L \left( 1 + \frac{\beta^2}{(1-\beta)^2} \right) \mathbb{E}[W_k]
\]
(37)

is uniformly bounded. Now we complete the proof. \( \square \)
Proof. (Proofs of Theorem 3.3) In this case, we consider the step-size \( \eta_k \) is strictly decaying which implies that \( \tau_k = \eta_k / \eta_{k-1} < 1 \). By applying Lemma B.1, we have

\[
E[\|x_{k+1} - x_k\|^2 \mid \mathcal{F}_k] \leq \tau_k^2 \beta^2 \|x_k - x_{k-1}\|^2 + \eta_k^2 (1 - \beta)^2 E[\|g_k\|^2 \mid \mathcal{F}_k] \\
+ 2\beta (1 - \beta) \tau_k \eta_k \left( f(x_{k-1}) - f(x_k) + \frac{L}{2} \|x_k - x_{k-1}\|^2 \right).
\]

(38)

By Lemma B.2(ii), we can achieve that

\[
E[\|\tilde{x}_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq \|\tilde{x}_k - x^*\|^2 - \left( 2\theta_1 \eta_k - (1 - \tau_k)^2 - \frac{\beta \omega^2}{1 - \beta} \right) \|x_k - x^*\|^2 \\
+ 2\theta_2 \eta_k + \frac{\beta \tau_k}{1 - \beta} \left( \omega \eta_k + \frac{\beta \tau_k}{1 - \beta} \right) \|x_k - x^*\|^2.
\]

(39)

We now define a Lyapunov function \( W_k \):

\[
W_{k+1} = \gamma_1 \|\tilde{x}_{k+1} - x^*\|^2 + \|x_{k+1} - x_k\|^2 + 2\eta_k \tau_k \beta (1 - \beta) (f(x_k) - f^*).
\]

(40)

for \( \gamma_1 > 0 \). Then incorporating the above inequalities (38) and (39), we have

\[
E[W_{k+1} \mid \mathcal{F}_k] \leq \gamma_1 \|\tilde{x}_k - x^*\|^2 + \left( \tau_k^2 \beta^2 + \beta (1 - \beta) L \tau_k \eta_k + \gamma_1 \frac{\beta \tau_k}{1 - \beta} \left( \omega \eta_k + \frac{\beta \tau_k}{1 - \beta} \right) \right) \|x_k - x_{k-1}\|^2 \\
+ 2\beta (1 - \beta) \eta_k \tau_k \left( f(x_{k-1}) - f^* \right) + 2\beta (1 - \beta) \left( \eta_k \tau_k - \eta_{k-1} \tau_{k-1} \right) \left( f(x_{k-1}) - f^* \right) \\
- \gamma_1 \left( 2\theta_1 \eta_k - (1 - \tau_k)^2 - \frac{\beta \omega^2}{1 - \beta} \right) \|x_k - x^*\|^2 \\
+ 2\theta_2 \eta_k + \left( (1 - \beta)^2 + \gamma_1 \right) \omega^2 \eta_k^2.
\]

(41)

If the step-size \( \eta_k \) is decreasing, then \( \tau_k = \eta_k / \eta_{k-1} \leq 1 \). Let \( \omega = \tau_k L^2 \beta / (\theta_1 (1 - \beta)) \) and \( \gamma_1 = \frac{(1 - \beta)^2 (1 - \beta^2)}{4\beta^2} \).

Letting

\[
\eta_k \leq \frac{(1 - \tau_k^2) (\beta^2 + 1)/2}{\beta (1 - \beta) L \tau_k + (\frac{1 - \beta^2) \omega^2 L^2}{2\theta_1} \tau_k^2}
\]

(42)

then

\[
\tau_k^2 \beta^2 + \beta (1 - \beta) \eta_k L + \gamma_1 \left( \frac{\beta \tau_k}{1 - \beta} \left( \omega \eta_k + \frac{\beta \tau_k}{1 - \beta} \right) \right) \leq 1.
\]

(43)

The right side of (42) is decreasing with \( \tau_k \in (0, 1] \), so we set

\[
\eta_k \leq \frac{(1 - \beta^2)}{2\beta (1 - \beta) L + \frac{(1 - \beta^2)(2L^2)}{2\theta_1}} = \frac{1}{2\beta (1 + \beta)^{-1} L + \frac{L^2}{2\theta_1}}.
\]

(44)

Furthermore, we choose

\[
\eta_k \leq \frac{\gamma_1 \theta_1}{4(\rho + 1)((1 - \beta)^2 + \gamma_1)L^2} = \frac{(1 - \beta^2) \theta_1}{4(\rho + 1)(1 + \beta^2) L^2} < \frac{1}{2\beta (1 + \beta)^{-1} L + \frac{L^2}{2\theta_1}},
\]

(45)

such that

\[
(\rho + 1) \left( (1 - \beta)^2 + \gamma_1 \right) L^2 \eta_k^2 \leq \frac{\gamma_1 \theta_1 \eta_k}{4}.
\]

(46)

That is to say: if

\[
\eta_k \leq \frac{(1 - \beta^2) \theta_1}{4(\rho + 1)(1 + \beta^2) L^2},
\]

(47)

then (41) can be re-written as

\[
E[W_{k+1} \mid \mathcal{F}_k] \leq W_k - (3\theta_1 \eta_k - (1 - \tau_k)^2) \|x_k - x^*\|^2 + 2\gamma_1 \theta_2 \eta_k + \left( (1 - \beta)^2 + \gamma_1 \right) \sigma^2 \eta_k^2 \\
+ 2\beta (1 - \beta) \left( \eta_k \tau_k - \eta_{k-1} \tau_{k-1} \right) (f(x_{k-1}) - f^*).
\]

(48)

Based on different decaying modes, we can achieve the following results.
• Polynomial decaying step-size: $\eta_k = \eta_1/k^p$ for $p \in (0, 1]$. In this case $\tau_k = \eta_k/\eta_{k-1} = (k-1)^p/k^p$, then
\[
\eta_k \tau_k - \eta_{k-1} \tau_{k-1} = \eta_1 (1 - (k-1)^p/k^p) = \eta_1 (1 - (k-1)^p/k^p)^2 < 0
\]
for all $k \geq 3$. For any $p \in (0, 1]$ and $k \geq 1$, we have
\[
k^p \leq (k-1)^p + 1
\]
then
\[
(1 - \tau_k)^2 = \left(1 - \frac{(k-1)^p}{k^p}\right)^2 \leq \left(1 - \frac{k^p - 1}{k^p}\right)^2 = \frac{1}{k^{2p}}.
\]
For $\eta_1 \geq \frac{4}{\gamma_1}$, we have $(1 - \tau_k)^2 \leq \frac{\theta_2}{2} \eta_k$, then
\[
E[W_{k+1} | F_k] \leq W_k - \frac{\gamma_1 \theta_1 \eta_k}{2} \|x_k - x^*\|^2 + 2\gamma_1 \theta_2 \eta_k + ((1 - \beta)^2 + \gamma_1) \sigma^2 \eta_k^2
\]
• Linear decay: $\eta_k = A - B k$ where $\eta_1 = \eta_{\text{max}}$ and $\eta_T = \eta_{\text{min}} = \eta_1/\sqrt{T}$, then we have
\[
A = \frac{T \eta_{\text{max}} - \eta_{\text{min}}}{T - 1}, \quad B = \frac{\eta_{\text{max}} - \eta_{\text{min}}}{T - 1}.
\]
Next we turn to estimate the sign of $\eta_k \tau_k - \eta_{k-1} \tau_{k-1}$.
\[
\eta_k \tau_k - \eta_{k-1} \tau_{k-1} = \frac{\eta_2^2 \eta_{k-2} - \eta_3^2}{\eta_{k-1} \eta_{k-2}} = \frac{(A - B k)^2 (A - B (k - 2) - (A - B (k - 1))^3}{(A - B (k - 1)) (A - B (k - 2))} 
\]
\[
\quad = \frac{(A - B (k - 1 + 1))^2 (A - B (k - 1 - 1)) - (A - B (k - 1))^3}{(A - B (k - 1)) (A - B (k - 2))} 
\]
\[
\quad = \frac{-B [(A - B (k - 1))^2 + B (A - B (k - 1))] - B^2}{(A - B (k - 1)) (A - B (k - 2))} 
\]
We know that $B > 0$ and $A - B k > 0$, then $(A - B k + B)^2 + B (A - B k) > 0$, thus we have $\eta_k \tau_k - \eta_{k-1} \tau_{k-1} < 0$. Next we estimate $(1 - \tau_k)^2$:
\[
(1 - \tau_k)^2 = \left(1 - \frac{\eta_k}{\eta_{k-1}}\right)^2 = \left(1 - \frac{A - B k}{A - B (k - 1)}\right)^2 = \frac{B^2}{(A - B (k - 1))^2}.
\]
We know that $\eta_{\text{min}} = c/\sqrt{T}$, let $c \geq \left(4 \frac{\eta_{\text{max}}^2}{\theta_1 \sqrt{T}}\right)^{1/3}$, we have
\[
(1 - \tau_k)^2 \leq \frac{B^2}{\eta_{k-1}^2} \leq \frac{(\eta_{\text{max}} - \eta_{\text{min}})^2}{(T - 1)^2 \eta_{\text{min}}^2} \leq \frac{\theta_1}{4} \eta_{\text{min}} \leq \frac{\theta_1}{4} \eta_k.
\]
Finally, we can achieve that
\[
E[W_{k+1} | F_k] \leq W_k - \frac{\gamma_1 \theta_1 \eta_k}{2} \|x_k - x^*\|^2 + 2\gamma_1 \theta_2 \eta_k + \eta_k^2 ((1 - \beta)^2 + \gamma_1) \sigma^2.
\]
• Cosine decay step-size: $\eta_k = A + B \cos(k \pi / T)$ where $A = \frac{\eta_{\text{min}} + \eta_{\text{max}}}{2}$ and $B = \frac{\eta_{\text{max}} - \eta_{\text{min}}}{2}$. We first estimate $\tau_k \eta_k - \eta_{k-1} \eta_{k-1}$:
\[
\eta_k \tau_k - \eta_{k-1} \tau_{k-1} = \eta_2^2 \eta_{k-2} - \eta_3^2 \eta_{k-1} \eta_{k-2} = \eta_{k-1} \left(\frac{\eta_k}{\eta_{k-1}} - \frac{\eta_k}{\eta_{k-1}}\right) = \eta_{k-1} \left(\eta_k - \eta_{k-1}\right) / \eta_{k-1}.
\]
In order to estimate the sign of $\eta_k \tau_k - \eta_{k-1} \tau_{k-1}$, we try to estimate $\left( \frac{\eta_k}{\eta_{k-1}} \right)^2 - \frac{\eta_{k-1}}{\eta_k}$. Then

$$
\psi_k := \left( 1 - \frac{\eta_{k-1}}{\eta_k} \right)^2 = \frac{\eta_{k-1}}{\eta_k} - \frac{\eta_{k-2}}{\eta_{k-1}} \eta_{k-1} - \eta_k
$$

For $k \in [1, T/2]$, by the graph of the step-size, we know that $\eta_k / \eta_{k-1} \leq \eta_{k-1} / \eta_{k-2} \leq 1$ and $\eta_{k-2} - \eta_{k-1} \leq \eta_{k-1} - \eta_k$, we have $\psi_k < 1$. If $k \in [T/2, T)$, we can see that $\eta_k / \eta_{k-1} \geq \eta_{k-1} / \eta_{k-2}$, then

$$
\psi_k \leq \frac{1}{2} \frac{\eta_{k-2} - \eta_{k-1}}{\eta_{k-1} - \eta_k} \leq \frac{1}{2} \left( \frac{\sin((2k-3)\pi)}{2 \sin((2k-1)\pi)} \right) = \frac{1}{2} \left( \cos\left(\frac{\pi}{T}\right) - \cos\left(\frac{2(2k-1)\pi}{2T}\right) \right) \leq \frac{1}{2} \left( \cos\left(\frac{\pi}{T}\right) + \frac{\cos\left(\frac{3\pi}{2T}\right) \sin\left(\frac{\pi}{T}\right)}{\sin\left(\frac{3\pi}{2T}\right)} \right) = \frac{1}{2} \cos\left(\frac{\pi}{T}\right) \left( 1 + \tan\left(\frac{\pi}{2T}\right) \right) < 1.
$$

That is for $k \in [1, T)$, we have $\psi_k < 1$. Thus $\left( \frac{\eta_k}{\eta_{k-1}} \right)^2 - \frac{\eta_{k-1}}{\eta_k} < 0$, then we have $\eta_k \tau_k - \eta_{k-1} \tau_{k-1} < 0$.

Next we turn to estimate $(1 - \tau_k)^2$:

$$
(1 - \tau_k)^2 = \left( 1 - \frac{\eta_k}{\eta_{k-1}} \right)^2 = \left( 1 - \frac{A + B \cos(k \pi / T)}{A + B \cos((k-1) \pi / T)} \right)^2 = \frac{A^2 + B^2 \cos((k-1) \pi / T)}{(A + B \cos((k-1) \pi / T))^2} \leq \frac{A^2 + B^2 \left( \frac{\pi}{2T} \right)^2}{(A + B \cos((k-1) \pi / T))^2} = \frac{4B^2 \left( \frac{\pi}{2T} \right)^2}{(A + B \cos((k-1) \pi / T))^2} \leq \frac{4B^2 \left( \frac{\pi}{2T} \right)^2}{(A + B \cos((k-1) \pi / T))^2} = \frac{4B^2 \left( \frac{\pi}{2T} \right)^2}{\eta_k^2}
$$

Let $\eta_{\text{min}} = c / \sqrt{T}$. To make sure that $(1 - \tau_k)^2 \leq \frac{4}{\eta_{\text{min}}^2}$, we let $c \geq \left( \frac{4 \eta_{\text{max}}^2}{\eta_{\text{max}}^2} \right)^{1/3}$. Then for any $k \in [1, T)$, we have

$$
E[W_{k+1} \mid F_k] \leq W_k - \frac{\eta_1 \theta_1 \tau_k}{2} \| x_k - x^* \|^2 + 2\gamma_1 \theta_2 \eta_k \left( (1 - \beta)^2 + \gamma_1 \right) \sigma^2 \eta_k^2.
$$

- Exponential decaying step-size $\eta_k = \eta_1 / \alpha^{k-1}$ where $\alpha = (\nu / T)^{-1/T} > 1$ and $\nu \geq 1$. In this case, we have $\tau_k = \eta_k / \eta_{k-1} = 1 / \alpha$ and

$$
(1 - \tau_k)^2 = (1 - 1 / \alpha)^2 \leq \frac{\ln^2 (T / \nu)}{T^2}
$$

where $1 - x \leq \ln(\frac{1}{x})$ for any $x > 0$.

$$
\eta_k \tau_k - \eta_{k-1} \tau_{k-1} = \eta_1 \left( \frac{1}{\alpha^{k-2} - 1} - \frac{1}{\alpha^{k-1} - 1} \right) = \frac{\eta_1}{\alpha^k} (1 - \alpha) < 0
$$

Let $\eta_1 \geq \frac{4 \ln^2 T}{\theta_1^2}$, then

$$
(1 - \tau_k)^2 \leq \frac{\ln^2 (T / \nu)}{T^2} \leq \frac{\theta_1^2}{4} \eta_k
$$

for any $1 \leq k \leq T$ and $\nu \in [1, T]$, we have

$$
E[W_{k+1} \mid F_k] \leq W_k - \frac{\gamma_1 \theta_1 \eta_k}{2} \| x_k - x^* \|^2 + 2\gamma_1 \theta_2 \eta_k \left( (1 - \beta)^2 + \gamma_1 \right) \sigma^2 \eta_k^2.
$$

For the four cases, we let

$$
r^2 = \max \left\{ R^2, \frac{4 \theta_2}{\theta_1} + \frac{\sigma^2}{2(\rho + 1) T^2} \right\},
$$

once $E[\| x_k - x^* \|^2] \geq r^2$, we have $E[W_{k+1}]$ is decreasing. Following the same discussion as the constant step-size in Theorem 3.2, for the decaying modes mentioned above, we can conclude that the quantities $E[W_k], E[\| x_k - x^* \|^2]$, and $E[\eta_k]$ decrease.
\[ E[f(x_k) - f^*] \text{ is uniformly bounded. Note that for the cosine decaying mode, the above results only hold for } k \in [1, T). \]
At the final iterate \( k = T \), we next show that \( E[W_{T+1}] \) is also bounded:
\begin{align*}
E[W_{T+1}] &\leq E[W_T] - \gamma T \left( 3\theta_1 \eta T \left( \frac{3\theta_1 \eta T}{4} - (1 - \tau_T)^2 \right) E[\|x_T - x^*\|^2] + 2\gamma_1 \theta_2 \eta T + ((1 - \beta)^2 + \gamma_1) \sigma^2 \eta^2 T \\
&\quad + 2\beta(1 - \beta) (\eta_T \tau_T - \eta_{T-1} \tau_{T-1}) E[f(x_{T-1}) - f^*] \right) \tag{68}
\end{align*}

where \( \tau_T = \eta_T / \eta_{T-1} \leq 1, \eta_T \tau_T - \eta_{T-1} \tau_{T-1} > 0 \), and \( \eta_T = \eta_{\min} \). Because \( E[W_{k+1}] \) is uniformly bounded for all \( k \in [1, T) \), by the previous discussion, we know that \( E[W_T], E[\|x_T - x^*\|^2] \), and \( E[f(x_{T-1}) - f^*] \) are uniformly bounded. Firstly, we know that
\begin{align*}
(1 - \tau_T)^2 &= \left( 1 - \frac{\eta_T}{\eta_{T-1}} \right)^2 = \left( \frac{B(\cos((T-1)\pi/T) + 1)}{A + B \cos((T-1)\pi/T)} \right)^2 = \left( \frac{B(- \cos(\pi/T) + 1)}{A - B \cos(\pi/T)} \right)^2 \\
&\leq \frac{(a)}{B (\frac{\eta_T}{A - B})^2 / 2} = \left( \frac{2 \eta_{\max} - \eta_{\min}}{2 \eta_{\min}} \right)^2 \frac{\pi^2}{2T^2} \leq \frac{(b)}{\eta_{\max} \frac{\pi^4}{16c^2 T^3}} \tag{69}
\end{align*}

where (a) follows from the fact that by the Taylor series of \( \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \), we have \( 1 - \cos(\pi/T) \leq \frac{x^2}{2T^2} \) and (b) uses the fact that \( \eta_{\min} = c / \sqrt{T} \). Recalling that \( c \geq \left( \frac{2 \eta_{\max} \pi^2}{c \sqrt{T}} \right) \), we know that this bound \( \frac{2 \eta_{\max} \pi^4}{16c^2 T^3} \ll \frac{3\theta_1 \eta T}{4} - (1 - \tau_T)^2 \) of \( E[\|x_T - x^*\|^2] \) is positive. Next we turn to estimate \( \eta_T \tau_T - \eta_{T-1} \tau_{T-1} = \tau_{T-1} \eta_{T-1} \left( \frac{\eta_T \eta_{T-1}}{\eta_{T-1} \eta_{T-1}} - 1 \right) = \tau_{T-1} \eta_{T-1} \left( \frac{\eta_{T-1} \eta_{T-1}}{\eta_{T-1} \eta_{T-1}} - 1 \right) \tag{70} \)

Recall the definition of \( \psi_k \) in (59) at \( k = T \), we have
\begin{align*}
\psi_T &= 1 - \frac{\eta_T \eta_{T-1}}{\eta_{T-1} \eta_{T-1}} \leq \frac{\eta_T \eta_{T-1}}{\eta_{T-1} \eta_{T-1}} + 1 = \frac{1}{2} \eta_T - \eta_{T-1} = \frac{1}{2} \eta_T - \eta_{T-1} \\
&\leq \frac{1}{2} \left( \cos(\frac{\pi T}{2}) + \cos(\frac{\pi T}{2}) \sin(\frac{\pi T}{2}) \right) = \frac{1}{2} \left( \cos(\frac{\pi T}{2}) + \cos(\frac{\pi T}{2}) \sin(\frac{\pi T}{2}) \right) = \frac{1}{2} \left( \cos(\frac{\pi T}{2}) + 1 \right) < \frac{3}{2} \tag{71}
\end{align*}

Then
\begin{align*}
\left( \frac{\eta_T}{\eta_{T-1}} \right)^2 \cdot \left( \eta_{T-1} \eta_{T-2} \right)^{-1} - 1 &\leq \frac{\left( \frac{\eta_T}{\eta_{T-1}} \right)^2}{1 - 3 \frac{\left( \frac{\eta_T}{\eta_{T-1}} \right)^2}{3 \left( \frac{\eta_T}{\eta_{T-1}} \right)^2 - 1} - 1 = \frac{(1 - \frac{\eta_T}{\eta_{T-1}})^2}{2(A - B)^2 T^2} \leq 1 \tag{72}
\end{align*}

where \( \eta_{T-1} / \eta_T = 1 + \frac{B}{A - B} (1 - \cos(\pi/T)) \leq 1 + \frac{B^2}{2(A - B)^2 T^2} \leq 2 \) for sufficient large \( c \geq \frac{2 \eta_{\max} \pi^2}{c \sqrt{T}} \). Then we can see that
\( \eta_T \tau_T - \eta_{T-1} \tau_{T-1} \leq \eta_{T-1} \tau_{T-1} \). Recall the definition of \( E[W_T] \), we know that \( 2\beta(1 - \beta) \eta_{T-1} \tau_{T-1} E[f(x_{T-1}) - f^*] \leq E[W_T] \). By the above analysis, we can get that
\begin{align*}
E[W_{T+1}] &\leq E[W_T] + 2\gamma_1 \theta_2 \eta T + (((1 - \beta)^2 + \gamma_1) \sigma^2 \eta^2 T + 2\beta(1 - \beta) (\eta_T \tau_T - \eta_{T-1} \tau_{T-1}) E[f(x_{T-1}) - f^*] \\
&\leq E[W_T] + 2\gamma_1 \theta_2 \eta_{\min} + ((1 - \beta)^2 + \gamma_1) \sigma^2 \eta^2_{\min} + 2\beta(1 - \beta) \eta_{T-1} \tau_{T-1} E[f(x_{T-1}) - f^*] \\
&\leq E[W_T] + 2\gamma_1 \theta_2 \eta_{\min} + ((1 - \beta)^2 + \gamma_1) \sigma^2 \eta^2_{\min} + E[W_T] \\
&= 2E[W_T] + 2\gamma_1 \theta_2 \eta_{\min} + ((1 - \beta)^2 + \gamma_1) \sigma^2 \eta^2_{\min}
\end{align*}

For the cosine decaying step-size, \( E[W_{T+1}] \) is also bounded. That is to say, for all \( k \in [1, T + 1] \), the quantities \( E[W_k] \) generated from the cosine decaying mode is uniformly bounded. \( \square \)
Proof. (Bandwidth-based Step-Size)

At each stage, we assume that the step-size is decreasing ($\eta_k \leq \eta_{k-1} \leq \eta_{k-2}$) for all $k \in ((t-1)S, tS]$. Then considering the step-size modes discussed in Theorems 3.2 and 3.3, we can see that:

- Step-decay step-size: $\eta_k = \eta_1/\alpha^{t-1}$ for $k \in (S(t-1), St)$ where $S = T/N$ and $t \in [N]$. At each stage $t$, we know that the step-size is a constant for all $k \in ((t-1)S, tS]$. By applying the results for constant step-sizes in Theorem 3.2, we can see that at each stage $t \in [N]$, the quantity $W_k$ is uniformly bounded for $k \in ((t-1)S + 2, tS]$, that is

$$E[W_{k+1}] \leq \left( \frac{1 + \beta^2}{(1 - \beta)^2} + 2 \right) (\Delta_t^2 + r^2) + \eta_k \cdot \beta (1 - \beta) \frac{L}{2} r^2$$

where $\Delta_t^2 = 3 ((1 + \beta)^2 + \beta^2) r^2 + 3\eta_k^2 (1 - \beta)^2 (\sigma^2 + (\rho + 1)L^2 r^2)$. Therefore, as we discussed in Theorem 3.2, at each stage, the quantities

$$E[\|\hat{x}_{k+1} - x^*\|^2] \leq E[W_{k+1}], \quad E[\|x_{k+1} - x_k\|^2] \leq E[W_{k+1}];$$

$$E[\|x_{k+1} - x^*\|^2] \leq 2 \left( 1 + \frac{\beta^2}{(1 - \beta)^2} \right) E[W_{k+1}], \quad E[f(x_{k+1}) - f^*] \leq L \left( 1 + \frac{\beta^2}{(1 - \beta)^2} \right) E[W_{k+1}]$$

are also uniformly bounded for all $k \in ((t-1)S + 2, tS]$. By the definition of $\tau_k$, at $k = (t-1)S + 1$, we have $\tau_{(t-1)S+1} = \eta_{(t-1)S+1}/\eta_{(t-1)S} = 1/\alpha$, then we define

$$W_{(t-1)S+2} = \gamma_1 \|\hat{x}_{(t-1)S+2} - x^*\|^2 + \|x_{(t-1)S+2} - x_{(t-1)S+1}\|^2 + 2 \frac{\eta_{(t-1)S+1}}{\alpha} \beta (1 - \beta) (f(x_{(t-1)S+1}) - f^*).$$

Similar to Theorem 3.3 (see (41)), we can estimate $E[W_{(t-1)S+2}]$ as

$$E[W_{(t-1)S+2}] = \gamma_1 E[\|\hat{x}_{(t-1)S+2} - x^*\|^2] + \|x_{(t-1)S+2} - x_{(t-1)S+1}\|^2 + 2 \frac{\eta_{(t-1)S+1}}{\alpha} \beta (1 - \beta) (f(x_{(t-1)S+1}) - f^*)$$

$$= W_{(t-1)S+1} - \gamma_1 \left[ \frac{3\gamma_1 \eta_{(t-1)S+1}}{4} - \left( 1 - \frac{1}{\alpha} \right)^2 \right] \|x_{(t-1)S+1} - x^*\|^2 + \|x_{(t-1)S+1} - x_{(t-1)S+1}\|^2 + 2 \gamma_1 \beta (1 - \beta) \left( \eta_{(t-1)S+1} \tau_{(t-1)S+1} - \eta_{(t-1)S+1} \tau_{(t-1)S} \right) (f(x_{(t-1)S+1}) - f^*)$$

$$\leq \gamma_1 \|x_{(t-1)S+1} - x^*\|^2 + \|x_{(t-1)S+1} - x_{(t-1)S+1}\|^2 + \frac{2 \gamma_1 \theta_2 \eta_{(t-1)S+1}}{\alpha} (1 - \beta)^2 + \gamma_1 \beta (1 - \beta) \left( \frac{\eta_{(t-1)S+1}}{\alpha} \right)^2$$

where $E[\|x_{(t-1)S+1} - x^*\|^2] \leq 2 \left( 1 + \frac{\beta^2}{(1 - \beta)^2} \right) E[W_{(t-1)S+1}]$ which are uniformly bounded, then we can conclude that $E[W_{(t-1)S+2}]$ can be bounded by $E[W_{(t-1)S+1}]$ plus a constant term. Thus in this case, we can derive that $E[W_{k}]$ is uniformly bounded for all $k \in [1, T+1]$.

- We then consider the three decaying modes: polynomial decay, linear decay, and cosine decay modes discussed in Theorem 3.3 at each stages. First, we consider polynomial decaying and linear decay. From the results of Theorem 3.3, we can see that $E[W_{k+1}]$ is uniformly bounded for $k \in ((t-1)S + 2, tS]$ and its bound can be derived the same as Theorem 3.2, that is

$$E[W_{k+1}] \leq \left( \frac{1 + \beta^2}{(1 - \beta)^2} + 2 \right) (\Delta_t^2 + r^2) + \eta_k \cdot \beta (1 - \beta) \frac{L}{2} r^2$$

where $\Delta_t^2 = 3 ((1 + \beta)^2 + \beta^2) r^2 + 3\eta_k^2 (1 - \beta)^2 (\sigma^2 + (\rho + 1)L^2 r^2)$. Because $\eta_{\min} \leq \eta_k \leq \eta_{\max}$ for all $k$, the upper bound is non-expanding for each stage $t$. However, at $k = (t-1)S + 1$, we have $	au_{(t-1)S+1} = \frac{1}{\alpha}$. Therefore, we conclude that $E[W_{k+1}]$ is uniformly bounded for all $k \in ((t-1)S, tS]$.
\( \eta_{(t-1)S+1}/\eta_{(t-1)S} \leq \eta_{\max}'/\eta_{\min}' \leq s_0 \). Similar to the above discussion for step-decay, we get that
\[
\mathbb{E}[W_{(t-1)S+2}] \\
:= \mathbb{E} \left[ \eta_{(t-1)S+1} \frac{3\eta_{(t-1)S+1}}{4} (1 - s_0)^2 \right] \left[ x_{(t-1)S+1} - x^* \right] + \frac{2\eta_{(t-1)S+1}}{\alpha} (1 - \beta) \left( f(x_{(t-1)S+1}) - f^* \right) \\
= W_{(t-1)S+1} - \eta_{(t-1)S+1} \left( \frac{3\eta_{(t-1)S+1}}{4} (1 - s_0)^2 \right) \left[ x_{(t-1)S+1} - x^* \right] + \frac{2\eta_{(t-1)S+1}}{\alpha} (1 - \beta) \left( f(x_{(t-1)S+1}) - f^* \right)
\]
However, for the cosine decay mode, the only difference from polynomial and linear decay modes lies on the
\( \eta_{\max}' \) is also upper bounded,
we can conclude that \( \mathbb{E}[W_{(t-1)S+2}] \) is also bounded.

As we know \( \mathbb{E}[W_{(t-1)S+1}] \) and \( \mathbb{E}[f(x_{(t-1)S}) - f^*] \) are uniformly bounded and \( \eta_{\max}' \) is also upper bounded, we can conclude that \( \mathbb{E}[W_{(t+1)S+2}] \) is uniformly bounded.
Thus, we also can make a conclusion that \( \mathbb{E}[W_{(t+1)S+2}] \) is also bounded.

\[
\mathbb{E}[W_{(t+1)S+2}] \leq 2\mathbb{E}[W_{(t-1)S}] + 2\gamma_1 \theta_2 \eta_{\min}' + \left( (1 - \beta)^2 + \gamma_1 \right) \sigma^2 (\eta_{\min}')^2.
\]

The estimation of \( \mathbb{E}[W_{(t-1)S+2}] \) for the cosine decay mode is same as polynomial and linear modes. Thus, we also can make a conclusion that \( \mathbb{E}[W_{(t+1)S+2}] \) is uniformly bounded for the bandwidth step-size with the cosine decay mode.

\[
\Box
\]

### C Applications

- A fully connected 2-layer neural network (hidden nodes \( m \)) with a weight decay: let \( \sigma_1 \) and \( \sigma_2(\cdot) \) denote the inner and output layer activation functions, respectively. We use cross-entropy to measure the output with its true label \( CE(\cdot) \). Given \( n \) pairs input data \( \{a_j, b_j\}_{j=1}^n \), we revise the input data as \( a_j = [a_j, 1] \in \mathbb{R}^d \) then
\[
F(x) = \frac{1}{n} \sum_{j=1}^n CE(\sigma_2(X_2 \sigma_1(X_1 a_j))) + \frac{\lambda}{2} \| X \|^2
\]

where \( X = [X_1, X_2] \in \mathbb{R}^{dm + sm}, X_1 \in \mathbb{R}^{m \times d} \) and \( X_2 \in \mathbb{R}^{s \times m} \). For simplicity, we consider the binary dataset \( b_j \in \{-1, +1\} \) and \( s = 1 \), and \( \sigma_2(\cdot) = 1/(1 + \exp(\cdot)) \), then
\[
F(x) = \frac{1}{n} \sum_{j=1}^n \log \left( 1 + \exp(-b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j)) \right) + \frac{\lambda}{2} \| X \|^2
\]

Then we turn to compute the gradient (sub-gradient) of each component function \( F_j \) with regarding to data \( (a_j, b_j) \)
\[
\frac{\partial F_j}{\partial X_1[i]} = \frac{b_j \exp(-b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j))}{1 + \exp(-b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j))} \cdot X_2[i] \sigma_1(X_1[i]^T a_j) a_j + \lambda X_1[i] \in \mathbb{R}^d
\]
\[
\frac{\partial F_j}{\partial X_2[i]} = \frac{b_j \exp(-b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j))}{1 + \exp(-b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j))} \cdot \sigma_1(X_1[i]^T a_j) + \lambda X_2[i] \in \mathbb{R}^d
\]

Let \( u = b_j \sum_{i=1}^m X_2[i] \sigma_1(X_1[i]^T a_j) \), we have
\[
\langle X, \nabla F_j(X) \rangle = \sum_{i=1}^m \left( \langle X_1[i], \frac{\partial F_j}{\partial X_1[i]} \rangle + \langle X_2[i], \frac{\partial F_j}{\partial X_2[i]} \rangle \right)
\]
\[
= \lambda \| X \|^2 - \frac{b_j \exp(-u)}{1 + \exp(-u)} \sum_{i=1}^m (X_2[i] \sigma_1(X_1[i]^T a_j) X_1[i]^T a_j + X_2[i] \sigma_1(X_1[i]^T a_j))
\]

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If $\sigma_1(x) = \max(0, x)$, then its gradient $\partial \sigma_1(x) = \chi_{x \geq 0}$. We get the relationship of $\sigma_1$ with its gradient $\sigma_1(x) = (x)_+ = x \cdot \chi_{x \geq 0} = x \cdot \partial \sigma_1(x)$. Then the above product can be estimated as

$$\langle X, \nabla F_j(X) \rangle = \lambda \|X\|^2 - \frac{b_j \exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} \left( X_2[i] \partial \sigma_1(X_1[i]^T a_j) X_1[i]^T a_j + X_2[i] \sigma_1(X_1[i]^T a_j) \right)$$

$$= \lambda \|X\|^2 - \frac{b_j \exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} \left( X_2[i] \sigma_1(X_1[i]^T a_j) + X_2[i] \sigma_1(X_1[i]^T a_j) \right)$$

$$= \lambda \|X\|^2 - \frac{2u \exp(-u)}{1 + \exp(-u)}.$$ 

No matter $u \geq 0$ or not, we all have 

$$\frac{\exp(-u)u}{1 + \exp(-u)} = \frac{u}{1 + \exp(u)} < 1.$$ 

In this case, we have

$$\langle X, \nabla F(X) \rangle = \frac{1}{n} \sum_{j=1}^{n} \langle X, \nabla F_j(X) \rangle \geq \lambda \|X\|^2 - 2.$$ 

If $\sigma_1(x) = 1/(1 + \exp(-x))$ is a sigmoid loss function, we have $\frac{\partial \sigma_1(x)}{\partial x} = \sigma_1(x)(1 - \sigma_1(x))$. In this case, we have

$$\langle X, \nabla F_j(X) \rangle = \lambda \|X\|^2 - \frac{b_j \exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} \left( X_2[i] \partial \sigma_1(X_1[i]^T a_j) X_1[i]^T a_j + X_2[i] \sigma_1(X_1[i]^T a_j) \right)$$

$$= \lambda \|X\|^2 - \frac{u \exp(-u)}{1 + \exp(-u)} - \frac{b_j \exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j$$

We consider different situations to show how to estimate the last term. We let $A = \{i : b_j X_2[i] \geq 0\}$.

* If $A = [m]$, then

$$\frac{\exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} b_j X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j \leq \frac{\exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} b_j X_2[i]$$

$$= \frac{u \exp(-u)}{1 + \exp(-u)},$$

where the inequality follows from the facts that $\sigma_1(x) \in [0, 1]$ and 

$$(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j = \frac{X_1[i]^T a_j}{1 + \exp(X_1[i]^T a_j)} < 1.$$ (76)

* If some of $b_j X_2[i] \geq 0$ are negative. That is $A \subseteq [m]$. Then we can split the sum into two cases

  * For the term that $i \in A$, we have $b_j X_2[i] \geq 0$. Let $B = \{i : X_1[i]^T a_j \geq 0\}$. We can also achieve that

$$\frac{\exp(-u)}{1 + \exp(-u)} \sum_{i=1}^{m} b_j X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j$$

$$= \frac{\exp(-u)}{1 + \exp(-u)} \sum_{i \in A} b_j X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j$$

$$+ \frac{\exp(-u)}{1 + \exp(-u)} \sum_{i \in [m \setminus A] \cap B} b_j X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j$$

$$+ \frac{\exp(-u)}{1 + \exp(-u)} \sum_{i \in [m \setminus A] \cap (m \setminus B)} b_j X_2[i] \sigma_1(X_1[i]^T a_j)(1 - \sigma_1(X_1[i]^T a_j)) X_1[i]^T a_j.$$ (77)
Next we choose one component function and product term. Let \( \frac{\partial X}{\partial F} \). Applying the above results gives

\[
-1 + \exp(-u) \leq \exp(-u) - \exp(-u) \leq \frac{c_1}{2} \left( \sum_{i=1}^{m} ||X[i]||^2 \right) + \frac{1}{2c_1} \left( \sum_{i \in A} \sigma_1^2(X[i]^T a_j) + \sum_{i \in (m|\setminus A) \cap (m|\setminus B)} \sigma_2^2(-X[i]^T a_j) \right)
\]

\[
\leq \left( \frac{c_1}{2} \right) ||X[i]||^2 + \frac{m}{2c_1}.
\]

Let \( c_1 = \lambda \), then

\[
\langle X, \nabla F(X) \rangle = \frac{1}{n} \sum_{j=1}^{n} \langle X, \nabla F_j(X) \rangle \geq \lambda \|X\|^2 - \frac{u \exp(-u)}{1 + \exp(-u)} - \left( \frac{\lambda}{2} \|X\|^2 + \frac{m}{2\lambda} \right)
\]

\[
\geq \frac{\lambda}{2} \|X\|^2 - \left( 1 + \frac{m}{2\lambda} \right).
\]

- If the neural networks are three layers, we consider ReLU as the activation function of the inner layers.

\[
F(x) = \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \exp \left( -b_j \sum_{i=1}^{m} X_3[i] \sigma_2 \left( X_2[i]^T a_j \right) \right) \right) + \frac{\lambda}{2} \|X\|^2.
\]

Next we choose one component function \( F_j \) which is related to the \( j \)-th dataset to analyze its gradient and product term. Let \( u = b_j \sum_{i=1}^{m_2} X_3[i] \sigma_2 \left( X_2[i]^T a_j \right) \)

\[
\frac{\partial F_j}{\partial X_1[i]} = \frac{\exp(-u)}{1 + \exp(-u)} \cdot b_j \sum_{i=1}^{m_2} X_3[i] \sigma_2 \left( X_2[i]^T a_j \right) \sigma_1 \left( X_1[i]^T a_j \right) + \lambda X_1[i]
\]

\[
\frac{\partial F_j}{\partial X_2[i]} = \frac{\exp(-u)}{1 + \exp(-u)} \cdot -b_j \sigma_2 \left( X_2[i]^T a_j \right) \sigma_1 \left( X_1[i]^T a_j \right) \sigma_1 \left( X_1[i]^T a_j \right) + \lambda X_2[i]
\]

\[
\frac{\partial F_j}{\partial X_3[i]} = \frac{\exp(-u)}{1 + \exp(-u)} \cdot -b_j \sigma_2 \left( X_2[i]^T a_j \right) \sigma_1 \left( X_1[i]^T a_j \right) + \lambda X_3[i].
\]
Then
\[
\langle X, \nabla F_j(X) \rangle = \sum_{i=1}^{m_1} \left\langle X_1[i], \frac{\partial F_j(X)}{\partial X_1[i]} \right\rangle + \sum_{i=1}^{m_2} \left\langle X_2[i], \frac{\partial F_j(X)}{\partial X_2[i]} \right\rangle + \sum_{i=1}^{m_2} \left\langle X_3[i], \frac{\partial F_j(X)}{\partial X_3[i]} \right\rangle.
\]

Let \( \sigma_1(x) = \sigma_2(x) = \max(0, x) \), then their gradients \( \partial \sigma_1(x) = \partial \sigma_2(x) = x_{x \geq 0} \). Then the activation functions with their gradient satisfy that \( \sigma_i(x) = x \partial \sigma_i(x) \) for \( i = 1 \) and \( 2 \). Then
\[
\sum_{i=1}^{m_1} \left\langle X_1[i], \frac{\partial F_j(X)}{\partial X_1[i]} \right\rangle = \sum_{i=1}^{m_1} \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot \sum_{i=1}^{m_2} X_3[i] \partial \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) X_2[i][l] \partial \sigma_1 (X_1[l]^T a_j) X_1[l]^T a_j + \lambda \|X_1[l]\|^2
\]
\[
= \sum_{i=1}^{m_1} \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot \sum_{i=1}^{m_2} X_3[i] \partial \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) X_2[i][l] \sigma_1 (X_1[l]^T a_j) + \lambda \|X_1[l]\|^2
\]
\[
= \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot \sum_{i=1}^{m_2} X_3[i] \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) + \lambda \|X_2[i]\|^2.
\]
\[
\sum_{i=1}^{m_2} \left\langle X_2[i], \frac{\partial F_j(X)}{\partial X_2[i]} \right\rangle = \sum_{i=1}^{m_2} \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot \sum_{i=1}^{m_2} X_3[i] \partial \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) X_2[i][l] \sigma_1 (X_1[l]^T a_j) + \lambda \|X_2[i]\|^2
\]
\[
= \sum_{i=1}^{m_2} \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot X_3[i] \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) + \lambda \|X_2[i]\|^2
\]
\[
= \frac{-b_j \exp(-u)}{1 + \exp(-u)} \cdot \sum_{i=1}^{m_2} X_3[i] \sigma_2 \left( X_2[i]^T \sigma_1 (X_1^T a_j) \right) + \lambda \|X_3[i]\|^2.
\]
Incorporating the above results, we can achieve that
\[
\langle X, \nabla F_j(X) \rangle = \lambda \|X\|^2 - \frac{3u \exp(-u)}{1 + \exp(-u)}.
\]

For any \( u \), we can estimate
\[
\frac{u \exp(-u)}{1 + \exp(-u)} = \frac{u}{1 + \exp(u)} \leq 1.
\]
Finally, we get that
\[
\langle X, \nabla F(X) \rangle = \frac{1}{n} \sum_{j=1}^{n} \langle X, \nabla F_j(X) \rangle \geq \lambda \|X\|^2 - 3.
\]

D Objective Function is Dissipative on its Domain

**Theorem D.1.** Under the condition of Lemma A.1, if \( f \) is \( L \)-smooth and \( R \)-dissipative for all \( x \in \mathbb{R}^d \). Consider the SGD algorithm with the total number of iterations \( T \geq 1 \), for any step-size \( \eta_k \leq \eta_1 / ((1 + \rho)L^2) \), we have
\[
\mathbb{E}[\|x_{T+1} - x^*\|^2] \leq \Pi_{k=1}^{T} (1 - \eta_k \theta_1) \|x_1 - x^*\|^2 + \sum_{k=1}^{T} \left( \theta_2 + \eta_k \sigma^2 \right) \eta_k \cdot \Pi_{s>k}^{T} (1 - \eta_s \theta_1).
\]
Proof. Applying the $L$-smooth assumption, for any $x, y \in \mathbb{R}^d$, we have
\[ \|\nabla f(y) - \nabla f(x)\| \leq L \|y - x\|, \]
\[ f(y) + \langle \nabla f(x), y - x \rangle - \frac{L}{2} \|y - x\|^2 \leq f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2. \]

Let $x = x^*$. By $\nabla f(x^*) = 0$, we get that
\[ \|\nabla f(y)\| \leq L \|x - x^*\|, \tag{79a} \]
\[ f(y) - f^* \leq \frac{L}{2} \|y - x^*\|^2. \tag{79b} \]

In this case, $R = 0$, it means that $\langle x_k - x^*, \nabla f(x_k) \rangle \geq \theta_1 \|x_k - x^*\|^2 - \theta_2$ for all $x \in \mathbb{R}^d$. Applying this condition and (79a) into Lemma A.1, we have
\[
E[\|x_{k+1} - x^*\|^2] \leq \|x_k - x^*\|^2 - 2\eta_k \theta_1 \|x_k - x^*\|^2 + \theta_2 \eta_k + \eta_k^2 \sigma^2 + \eta_k^2 (1 + \rho) L^2 \|x_k - x^*\|^2 \\
= (1 - 2\eta_k \theta_1 + \eta_k^2 (1 + \rho) L^2) \|x_k - x^*\|^2 + (\theta_2 + \eta_k \sigma^2) \eta_k
\]

Letting $\eta_k \leq \frac{\theta_1}{(1 + \rho) L^2}$ and , then $0 < 1 - 2\eta_k \theta_1 + \eta_k^2 (1 + \rho) L^2 \leq 1 - \eta_k \theta_1$. Consider the above recursion from $k = 1$ to $T$, we can achieve that
\[
E[\|x_{T+1} - x^*\|^2] \leq \prod_{k=1}^T (1 - \eta_k \theta_1) \|x_1 - x^*\|^2 + \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \prod_{s>k}^T (1 - \eta_s \theta_1).
\]

Therefore, the proof is complete. \qed

**Theorem D.2.** (Constant step-size) Under the same conditions of Theorem D.1. If the step-size is a constant $\eta_k = \eta \leq \theta_1 / ((1 + \rho) L^2)$, then for any $T \geq 1$, we have
\[
E[\|x_{T+1} - x^*\|^2] \leq (1 - \eta \theta_1)^T \|x_1 - x^*\|^2 + \frac{(\theta_2 + \eta \sigma^2)}{\theta_1}.
\]

**Proof.**
\[
E[\|x_{T+1} - x^*\|^2] \leq \prod_{k=1}^T (1 - \eta_k \theta_1) \|x_1 - x^*\|^2 + \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \prod_{s>k}^T (1 - \eta_s \theta_1) \\
= (1 - \eta \theta_1)^T \|x_1 - x^*\|^2 + (\theta_2 + \eta \sigma^2) \eta \sum_{k=1}^T (1 - \eta \theta_1)^{T-k} \\
= (1 - \eta \theta_1)^T \|x_1 - x^*\|^2 + (\theta_2 + \eta \sigma^2) \eta \cdot \frac{1 - (1 - \eta \theta_1)^T}{\eta \theta_1} \\
= (1 - \eta \theta_1)^T \|x_1 - x^*\|^2 + \frac{(\theta_2 + \eta \sigma^2)}{\theta_1} \cdot (1 - (1 - \eta \theta_1)^T) \\
\leq (1 - \eta \theta_1)^T \|x_1 - x^*\|^2 + \frac{(\theta_2 + \eta \sigma^2)}{\theta_1}.
\]

\[ \square \]

**Theorem D.3.** (Time-dependent step-size) Under the same conditions of Theorem D.1. If the step-size is time dependent on the iteration where $\eta_k \leq \theta_1 / ((1 + \rho) L^2)$, then for any $T \geq 1$, we have
\[
E[\|x_{T+1} - x^*\|^2] \leq \exp \left( -\theta_1 \sum_{k=1}^T \eta_k \right) \|x_1 - x^*\|^2 + \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \exp \left( -\theta_1 \sum_{s=k+1}^T \eta_s \right)
\]

- \* $\eta_k = \eta_1 / k^p$ for $p \in (0, 1)$

- If $p \in (0, 1)$, we have
\[
E[\|x_{T+1} - x^*\|^2] \leq \exp \left( -\frac{\theta_1 \eta_1}{1 - p} \left( (T+1)^{1-p} - 1 \right) \right) \|x_1 - x^*\|^2 + \frac{(\theta_2 + \eta \sigma^2) 2^p}{\theta_1}.
\]

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$- p = 1$, then
\[ E[\|x_{T+1} - x^*\|^2] \leq \frac{1}{(T+1)^{\theta_1 \eta_1}} \|x_1 - x^*\|^2 + 2 \left(\frac{\theta_2 + \eta_1 \sigma^2}{\theta_1}\right). \]

- **Bandwidth-based step-sizes including step-decay band and polynomial band**

$- \frac{m/k^p}{\eta_k} \leq \frac{M/k^p}{\eta_k} \leq \frac{M/k^p}{\eta_k}$

* $p \in (0, 1)$,

\[ E[\|x_{T+1} - x^*\|^2] \leq \exp \left( -\frac{\theta_1 m}{1-p} \right) \left( (T + 1)^{1-p} - 1 \right) \|x_1 - x^*\|^2 + \left( \frac{\theta_2 + M \sigma^2}{m \theta_1} \right)^{2p M}. \]

* $p = 1$

\[ E[\|x_{T+1} - x^*\|^2] \leq \frac{1}{(T+1)^{\theta_1 m}} \|x_1 - x^*\|^2 + \frac{2 (\theta_2 + M \sigma^2) M}{m \theta_1}. \]

- $m \delta(k) \leq \eta_k^k \leq M \delta(k)$ where $\delta(k) = 1/\alpha^{k-1}$ for $i \in [S]$ and $1 \leq k \leq N$

**Proof.**

\[ E[\|x_{T+1} - x^*\|^2] \leq \Pi_{k=1}^T (1 - \eta_k \theta_1) \|x_1 - x^*\|^2 + \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \Pi_{s=k+1}^T (1 - \eta_s \theta_1) \]

\[ = \exp \left( -\theta_1 \sum_{k=1}^T \eta_k \right) \|x_1 - x^*\|^2 + \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \exp \left( -\theta_1 \sum_{s=k+1}^T \eta_s \right) \quad (80) \]

If $\eta_k = \eta_1/k^p$ for $p \in (0, 1)$, we have

\[ \sum_{k=1}^T \eta_k \geq \eta_1 \int_{k=1}^{T+1} 1/k^p dk = \frac{\eta_1}{1-p} \left( (T + 1)^{1-p} - 1 \right) \quad (81a) \]

\[ \sum_{s=k+1}^T \eta_s \geq \eta_1 \int_{s=k+1}^{T+1} 1/s^p ds = \frac{\eta_1}{1-p} \left( (T + 1)^{1-p} - (k + 1)^{1-p} \right) \quad (81b) \]

Then we turn to estimate the last term of (80) as below

\[ \sum_{k=1}^T (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \exp \left( -\theta_1 \sum_{s=k+1}^T \eta_s \right) \leq \sum_{k=1}^T \left( \frac{\eta_1}{k^p} \right)^p \exp \left( \frac{-\theta_1 \eta_k}{1-p} \right) \left( T^{1-p} - (k + 1)^{1-p} \right) \]

\[ \leq \left( \frac{\theta_2 + \eta_1 \sigma^2}{\theta_1} \right)^{2p} \sum_{k=1}^T \exp \left( \frac{-\theta_1 \eta_k}{1-p} \right) \left( T^{1-p} - (k + 1)^{1-p} \right) \]

\[ \leq \left( \frac{\theta_2 + \eta_1 \sigma^2}{\theta_1} \right)^{2p} \left( \frac{1}{1-p} \right)^p \int_{k=1}^{T+1} \frac{1}{k^p} \exp \left( \frac{-\theta_1 \eta_k}{1-p} \right) \left( T^{1-p} - (k + 1)^{1-p} \right) dk \]

\[ = \left( \frac{\theta_2 + \eta_1 \sigma^2}{\theta_1} \right)^{2p} \left( \frac{1}{1-p} \right)^p \left( \exp \left( \frac{-\theta_1 \eta_k}{1-p} (T + 1)^{1-p} \right) - \exp \left( \frac{-\theta_1 \eta_k}{1-p} k^{1-p} \right) \right) \]

\[ \leq \left( \frac{\theta_2 + \eta_1 \sigma^2}{\theta_1} \right)^{2p} \frac{2p}{\theta_1 \eta_k}. \]

If $p = 1$, then

\[ \sum_{k=1}^T \eta_k \geq \eta_1 \int_{k=1}^{T+1} \frac{dk}{k} = \eta_1 \log(T + 1) \quad (82a) \]

\[ \sum_{s=k+1}^T \eta_s \geq \eta_1 \int_{s=k+1}^{T+1} \frac{ds}{s} \eta_1 \log \left( \frac{T + 1}{k + 1} \right). \quad (82b) \]
The last term of (80) can be bounded as
\[
\sum_{k=1}^{T} (\theta_2 + \eta_k \sigma^2) \eta_k \cdot \exp \left( -\theta_1 \sum_{s=k+1}^{T} \eta_s \right) \leq \sum_{k=1}^{T} (\theta_2 + \eta_1 \sigma^2) \frac{\eta_1}{k} \cdot \exp \left( -\theta_1 \eta_1 \log \left( \frac{T+1}{k+1} \right) \right)
\]
\[
\leq (\theta_2 + \eta_1 \sigma^2) \eta_1 \sum_{k=1}^{T} \frac{(k+1)^{\theta_1 \eta_1}}{k(T+1)^{\theta_1 \eta_1}}
\]
\[
\leq (\theta_2 + \eta_1 \sigma^2) \eta_1 \left( 1 + \frac{1}{k} \right) \int_{k=1}^{T} \frac{(k+1)^{\theta_1 \eta_1-1}}{(T+1)^{\theta_1 \eta_1}}
\]
\[
\leq 2 (\theta_2 + \eta_1 \sigma^2) \eta_1 \frac{(T+1)^{\theta_1 \eta_1 - 2\theta_1 \eta_1}}{\theta_1 \eta_1 (T+1)^{\theta_1 \eta_1}}
\]
\[
\leq 2 (\theta_2 + \eta_1 \sigma^2) \eta_1.
\]
If \(m \delta(k) \leq \eta_k \leq M \delta(k)\) and \(\delta(k) = 1/\alpha^{k-1}\) where \(\alpha > 1\). Then
\[
\sum_{k=1}^{N} \sum_{s=1}^{S} \eta_i^k \geq mS \sum_{k=1}^{N} \alpha^{-k+1} = \frac{mS(1 - \alpha^{-N})}{1 - \alpha^{-1}} > mS.
\]
The second term of (80) can be estimated as:
\[
\sum_{k=1}^{N} \sum_{s=1}^{S} (\theta_2 + \eta_i^k \sigma^2) \eta_i^k \cdot \exp \left( -\theta_1 \left( \sum_{s=i+1}^{S} \eta_s^k + \sum_{k=1}^{N} \sum_{l=k+1}^{N} \eta_l^k \right) \right)
\]
\[
\leq \sum_{k=1}^{N} (\theta_2 + M \sigma^2) M \alpha^{-k+1} \exp \left( -m \theta_1 \left( S \sum_{l=k+1}^{N} \alpha^{-l+1} \right) \right) \left( 1 + \sum_{i=1}^{S-1} \exp \left( -m \theta_1 (S - i) \alpha^{-k+1} \right) \right)
\]
\[
\leq (\theta_2 + M \sigma^2) M \sum_{k=1}^{N} \alpha^{-k+1} \exp \left( -m \theta_1 \left( S \sum_{l=k+1}^{N} \alpha^{-l+1} \right) \right) \left( 1 + \frac{\exp(-m \theta_1 \alpha^{-k+1})}{1 - \exp(-m \theta_1 \alpha^{-k+1})} \right)
\]
\[
\leq (\theta_2 + M \sigma^2) M \frac{1}{1 - \exp(-m \theta_1)} \left( 1 + \sum_{k=1}^{N-1} \exp \left( -m \theta_1 S \alpha^{-N+1} k \right) \right) \leq (\theta_2 + M \sigma^2) M \frac{1}{1 - \exp(-m \theta_1)} \left( 1 + \frac{\exp(-m \theta_1 S \alpha^{-N+1})}{1 - \exp(-m \theta_1 S \alpha^{-N+1})} \right)
\]
\[
\leq (\theta_2 + M \sigma^2) M \frac{1}{1 - \exp(-m \theta_1) 1 - \exp(-m \theta_1 S \alpha^{-N+1})}
\]
where (a) follows from the fact that for all \(k \in [N]\), we have
\[
\alpha^{-k+1} \frac{1}{1 - \exp(-m \theta_1 \alpha^{-k+1})} \leq \frac{1}{1 - \exp(-m \theta_1)}
\]
due to that \(h(x) = \frac{x}{1 - \exp(-m \theta_1 x)}\) is increasing with \(x\). By properly choosing \(S\) and \(N\), for example: \(N = \log_{\alpha} T/2\) and \(S = 2T/\log_{\alpha} T\), we have \(S \alpha^{-N+1} = \frac{2T^2}{\sqrt{T} \log_{\alpha} T} = \frac{2\alpha \sqrt{T}}{\log_{\alpha} T}\). For sufficient large \(T\), we can see that \(0 < \exp(-m \theta_1 S \alpha^{-N+1}) \ll 1\). Thus, \(1 - \exp(-m \theta_1 S \alpha^{-N+1}) \approx 1\), we can achieve that
\[
\mathbb{E}[\|x_{T+1} - x^*\|^2] \leq \exp(-\theta_1 mS) \|x_1 - x^*\|^2 + \frac{(\theta_2 + M \sigma^2) M}{1 - \exp(-m \theta_1)}.
\]