The Light-Front Hamiltonian formalism for two-dimensional Quantum Electrodynamics equivalent to the Lorentz-covariant approach

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Abstract

A light-front Hamiltonian reproducing the results of two-dimensional quantum electrodynamics in the Lorentz coordinates is constructed using the bosonization procedure and an analysis of the bosonic perturbation theory in all orders in the fermion mass. The resulting Hamiltonian involves a supplementary counterterm in addition to the usual terms appearing in the naive light-front quantization. This term is proportional to a linear combination of zeroth fermion modes (which are multiplied by a factor compensating the charge and fermion number). The coefficient of the counterterm has no ultraviolet divergence, depends on the value of the fermion condensate in the $\theta$-vacuum, and is linear in this value for a small fermion mass.

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1. Introduction

The Hamiltonian approach to quantum field theory in the light-front (LF) coordinates $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, $x^\perp = (x^1, x^2)$, where $x^+$ plays the role of time [1], is one of the nonperturbative approaches to solving the strong coupling problem [2, 3]. In the framework of this approach, quantization is performed in the plane $x^+ = 0$, and the generator $P_+$ of the shift along the $x^+$ axis serves as the Hamiltonian. The generator $P_-$ of the shift along the $x^-$ axis does not displace the quantization surface and is therefore kinematic (in the Dirac terminology) in contrast to the dynamic generator $P_+$. As a result, the momentum operator $P_-$ turns out to be quadratic with respect to the fields and is independent of the interaction. On the other hand, this operator is nonnegative and has a zero eigenvalue only on the physical vacuum. Therefore, the field Fourier modes corresponding to positive and negative values of $p_-$ play the role of creation and annihilation operators over the physical vacuum, and they can be used to construct the Fock space. In the LF coordinates, the physical vacuum thus coincides with the ”mathematical” vacuum. The bound-state spectrum can be found by solving the Schrödinger equation

$$P_+|\Psi\rangle = p_+|\Psi\rangle$$

(1)

in the subspace with fixed $p_-$ and $p_\perp$, which gives the expression $m^2 = 2p_+p_- - p_\perp^2$ for the mass $m$.

This search for bound states can be performed outside the framework of perturbation theory (PT), for example, using the so-called discretized LF quantization method (the discretized light-cone quantization (DLCQ) method [2, 4]). But the LF-Hamiltonian formalism involves a specific divergence at $p_- = 0$ [2, 3] and requires its regularization. The simplest regularization (we refer to it as lightlike) is the ordinary cutoff $|p_-| \geq \varepsilon > 0$ breaking the Lorentz and gauge invariances. It is also possible to apply another regularization preserving the gauge invariance, namely, the cutoff $|x^-| \leq L$ with the introduction of periodic boundary conditions (this regularization is used in the DLCQ method.) In this case, the momentum $p_-$ becomes discrete ($p_- = p_n = \pi n/L$, where $n$ is an integer), and the zeroth field mode corresponding to $n = 0$ is explicitly separated. In principle, the canonical formalism permits expressing the zeroth mode via the others by solving the constraints, which is usually a complicated problem [5].

Introducing the lightlike regularization can generate a nonequivalence between the LF-coordinate theory and the ordinary Lorentz-covariant theory.
Indeed, experience with nonperturbative calculations based on the LF-Hamiltonian formalism shows that the calculation result can differ from the corresponding result obtained in Lorentz coordinates \([6–8]\). Moreover, such differences were found even in the lowest PT orders \([9]\). This leads to the problem of ”improving” the canonical LF Hamiltonian (resulting from the naive quantization in LF coordinates), i.e., to the problem of finding counterterms for this Hamiltonian that compensate the indicated differences. If this problem can be solved in all PT orders, then the resulting improved LF Hamiltonian can be used for nonperturbative calculations.

The PT generated by the LF Hamiltonian can be represented as the Feynman PT with the same diagrams, but with the cutoff \(|p_-| \geq \varepsilon > 0\) and with a special integration rule, which we call the lightlike calculation, namely, integration is performed first with respect to \(p_+\) and then with respect to the other momentum components \([10]\). To reveal the abovementioned differences in the PT framework, it therefore suffices to compare the lightlike and the ordinary (which we call Lorentz) methods for calculating diagrams. (In what follows, the terms lightlike or Lorentz applied to diagrams and Green’s functions mean that the lightlike or Lorentz calculation methods are used to find them.)

For nongauge field theories of the type of the Yukawa model, it is possible to find the counterterms required for improving the canonical LF Hamiltonian \([9, 11]\) including all PT orders in the coupling constant \([12]\). But the direct application of the method in \([12]\) to the gauge theory using the simplest ultraviolet (UV) regularization requires adding infinitely many counterterms to the canonical LF Hamiltonian. This difficulty can be overcome by introducing a specific nonstandard regularization \([13]\) similar to the Pauli-Villars regularization, and the gauge invariance is then broken. This leads to the appearance of a large, but finite, number of counterterms with unknown coefficients. As a result, the improved LF Hamiltonian for quantum chromodynamics contains great number of undetermined coefficients, and it reproduces the results of the Lorentz-covariant theory in all PT orders only for some preliminarily unknown dependence of these coefficients on the regularization parameter in the regularization-removal limit. The presence of unknown coefficients and the complicated structure of the regularization (the regularized Hamiltonian contains many additional fields) make practical calculations with the resulting Hamiltonian extremely complicated. Moreover, because only the PT with respect to the coupling constant was analyzed, there could remain purely
nonperturbative effects that are not taken into consideration.

In this paper, a different method is suggested for constructing an LF Hamiltonian suitable for twodimensional gauge theories (the two-dimensional quantum electrodynamics (QED-2) is considered here). We first pass to the boson formulation of the theory and then analyze the boson PT (with respect to the fermion mass) to find the improved LF Hamiltonian in terms of bosons. Furthermore, we return to the fermion variables in the already constructed LF theory. We note that the boson PT is principally distinct from the PT with respect to the coupling constant in the original fermion theory (in QED-2, the latter PT does not exist at all because of infrared divergences). Therefore, the resulting LF Hamiltonian can take the nonperturbative effects (with respect to the ordinary coupling constant) into account.

The suggested method for constructing the LF Hamiltonian can be applied only to QED-2, but the information obtained in the analysis of this two-dimensional model can be used to develop new methods that take the nonperturbative vacuum effects into account in constructing the LF Hamiltonian for fourdimensional gauge theories. Such attempts to extract some information about four-dimensional LF gauge theories from an analysis of the QED-2 appeared recently [14].

2. Method for constructing the LF Hamiltonian for the QED-2

The suggested method for constructing the LF Hamiltonian for the QED-2 is based on the possibility of passing from the QED-2 to an equivalent scalar theory [15] (of the type of the sine-Gordon model). This is done using the bosonization procedure, i.e., the transformation from fermion to boson variables [8, 16]. After this transformation, the mass term of the fermion field in the QED-2 Hamiltonian becomes an interaction term for the scalar field, and the fermion mass $M$ becomes a coupling constant in the boson theory. For $M = 0$, the QED-2 is the Schwinger model, and the boson theory turns out to be a free theory. The PT for the boson theory (i.e., the PT in $M$) is a chiral PT. The nontriviality of the quantum vacuum in the QED-2 related to instantons ($\theta$-vacuum) [15, 17] is taken into account explicitly in the boson theory using the parameter $\theta$ in the interaction term.

The Lagrangian of the boson theory has the form

$$L = L_0 + L_I,$$
\[ L_0 = \frac{1}{8\pi} \left( \partial_{\mu} \varphi \partial^{\mu} \varphi - m^2 \varphi^2 \right), \quad (3) \]

\[ L_I = \frac{\gamma}{2} e^{i\theta} : e^{i\varphi} : + \frac{\gamma}{2} e^{-i\theta} : e^{-i\varphi} :, \quad \gamma = \frac{M m e C}{2\pi}, \quad m = \frac{e_{el}}{\sqrt{\pi}}, \quad (4) \]

where \( e_{el} \) is an analogue of the electron charge, \( C = 0.577216 \) is Euler’s constant, and the normal-ordering symbol means that the diagrams with closed lines are excluded from the PT with respect to \( \gamma \) (this corresponds to the usual meaning of the normal-ordering symbol in the Hamiltonian).

We consider the Feynman rules for this theory in Lorentz coordinates. There are vertices of two types with \( j \) external lines (\( j = 0, 1, 2, \ldots \) here). The factors corresponding to these vertices are \( i^{j+1} e^{i\theta} \gamma / 2 \) for the first type and \( i^{-j+1} e^{-i\theta} \gamma / 2 \) for the second type. The vertices without lines (\( j = 0 \)) are regarded as connected subdiagrams. It is convenient to relate part of the vertex factors \( i^{\pm j} \) to the lines that are external relative to the vertex (i.e., \( \pm i \) for each of the lines). The propagator \( \Delta(x) = \langle 0 | T(\varphi(x)\varphi(0)) | 0 \rangle \), where the free field \( \varphi(x) \) corresponds to the expansion of Lagrangian (2), has the form

\[ \Delta(x) = \int d^2k \ e^{ikx} \Delta(k), \quad \Delta(k) = \frac{i}{\pi} \frac{1}{(k^2 - m^2 + i0)}, \quad (5) \]

where \( d^2k = dk_0dk_1 \) and \( kx = k_0x^0 + k_1x^1 \).

On one hand, the resulting PT is simple because it is a scalar-field theory; on the other hand, it is complicated because it involves a nonpolynomial interaction. Therefore, there are infinitely many diagrams in each PT order, and their sum can have a UV divergence, although each of the diagrams is finite [17, 18]. It can be easily shown that in the second PT order with respect to \( \gamma \) (and hence also with respect to \( M \)), the sum of all diagrams contributing to the nonvacuum Green’s function is UV finite [18]. This suggests that the situation is the same in the higher PT orders as well. Indeed, analyzing the PT in the coordinate space permits proving [19] that the sum of all connected Lorentz-covariant diagrams of an arbitrary order \( n \) with respect to \( \gamma \) is UV finite, and there is no divergence for \( n > 2 \) even if the method of adjoining external lines is fixed, whereas for \( n = 2 \), this is so only after the summation using all such methods. Only the sum of second-order vacuum diagrams and also the sum of second-order nonvacuum diagrams with a fixed method of adjoining external lines remain UV divergent. We note that the indicated UV finiteness can be proved only for Lorentz-covariant Green’s functions and
not for lightlike ones, because some diagrams are zero in the lightlike calculation, which destroys the proof. The UV divergences of lightlike diagrams can be automatically regularized using the lightlike cutoff parameter $\varepsilon$. These divergences appear as $\varepsilon \to 0$, and they must be compensated by counterterms.

The exponentiality of the interaction in the model under consideration permits reformulating the PT in the language of superpropagators, i.e., of sums of contributions corresponding to the versions of connecting a pair of vertices by a different number of propagators [17, 19]. The superpropagator is equal to $e^{\Delta(x)}$ for a pair of vertices of different type and to $e^{-\Delta(x)}$ for vertices of the same type. In this approach, for a given number of vertices and a fixed method of adjoining external lines, the sum of all ordinary diagrams (including the disconnected ones) is described by a single diagram in which each pair of vertices is connected by the corresponding superpropagator (the connectedness is always understood in the usual sense).

The presence of UV divergences in the abovementioned sums of second-order nonvacuum diagrams requires introducing an intermediate UV regularization (it is intermediate because the ultimate values of nonvacuum Green’s functions are UV finite and no renormalization of the Lorentz-covariant theory is needed). For example, the Pauli-Villars regularization can be taken as the intermediate regularization. An LF boson Hamiltonian was thus constructed in [18] (a similar consideration was performed in [20] for the sine-Gordon model involving no UV divergence). This Hamiltonian involves a counterterm with a coefficient that is proportional to the chiral condensate and divergent in the limit of removing the Pauli-Villars regularization. Therefore, this regularization should be retained to the end of the calculations. Because the Hamiltonian remains Pauli-Villars regularized, it is impossible to return to the former fermion variables and obtain an LF Hamiltonian in fermion terms characteristic of the original gauge theory. Therefore, we here use a special UV regularization for the Lorentz-covariant propagator of the boson field $\varphi$,

$$
\Delta_{\text{reg}}(x) = \begin{cases} 
\Delta_{\varphi}^{lf}(x), & \{|x^-| \leq \alpha\} \cap \{|x^+| \leq \alpha\} \\
\Delta(x), & \{|x^-| > \alpha\} \cup \{|x^+| > \alpha\} 
\end{cases},
$$

(6)

where

$$
\Delta_{\varphi}^{lf}(x) = \frac{i}{\pi} \int_{|k_-| \geq \varepsilon} dk_- \int_{-\infty}^{\infty} dk_+ \frac{e^{i(k_+x^+ + k_-x^-)}}{2k_+k_- - m^2 + i0},
$$

(7)
is the lightlike propagator and $\alpha$ is the UV-regularization parameter. This regularization has a remarkable property, namely, after the transformation
to LF coordinates, it introduces no additional modifications in the theory apart from the already performed cutoff $|p_-| \geq \varepsilon > 0$ and permits the inverse transformation to the fermion variables.

Unfortunately, if regularization (6) is used, then the Lorentz-PT propagator depends essentially on the parameter $\varepsilon$ because the passage to the limit as $\varepsilon \to 0$ corresponds to removing the regularization. This is why direct comparison of the lightlike and Lorentz PTs in the momentum space using the method in [12] becomes impossible because the complicated expression for the superpropagator in the momentum space does not permit separating the full dependence on $\varepsilon$ for the difference between the calculation results for the diagrams in the Lorentz and LF coordinates. Therefore, we must perform the analysis in the coordinate space.

If the method described in [12] is used, then the contribution of the domain $p_- \approx \varepsilon, p_+ \approx \frac{1}{\varepsilon}$ is taken into account, which corresponds to the domain of large values of $x^-, x^- \approx \frac{1}{\varepsilon}$, in the coordinate space. Analyzing the convergence of exponential series in the expansions of superpropagators in this domain leads to the conclusion that these series can be truncated, i.e., it is possible to pass to ”partial” superpropagators,

$$D^m_{\pm}(x) = \sum_{m'=0}^{m} \frac{1}{m'!} (\pm \Delta(x))^{m'}, \quad e^{\pm \Delta(x)} = \lim_{m \to \infty} D^m_{\pm}(x).$$

(8)

The results obtained by the method in [12] can then be used. This procedure is performed in Sec. 4, but it must be preceded by finding the differences between the lightlike and Lorentz superpropagators for finite values of $x^-$ as $\varepsilon \to 0$. Usually no such differences between the propagators appear, but such a contribution in terms of superpropagators appears in the model under study because of the ”bad” UV behavior of the theory. We must first find precisely this contribution and compensate it using a counterterm for the LF Hamiltonian.

3. Compensating the differences between the lightlike and Lorentz superpropagators for finite values of $x^-$

The lightlike propagator completely regularized by the condition $\varepsilon \leq |k_-| \leq V$ can be written in the form [19]

$$\Delta_{\varepsilon,V}^{lf}(x) = \int_{\varepsilon}^{V} \frac{dk}{k} e^{-i(kx^-+\frac{m^2}{2k}x^+)} \text{sign}(x^+)$$

(9)
(Here, we proceed from expression (7) with the additional cutoff $|k_-| \leq V$.) On the other hand, the Lorentz propagator can be brought to a similar form with the momentum cutoff in the Lorentz coordinates, $|k_1| \leq \Lambda$,

$$\Delta_\Lambda(x) = \int_{\epsilon_\Lambda}^{V_\Lambda} \frac{dk}{k} e^{-i(kx^- + \frac{m^2}{2k}x^+)} \text{sign}(x^0), \quad (10)$$

where $\epsilon_\Lambda \xrightarrow{\Lambda \to \infty} 0$ and $V_\Lambda \xrightarrow{\Lambda \to \infty} \infty$ [19]. For $x^2 \neq 0$, the regularization in expressions (9) and (10) can be removed, after which they coincide. (It should be taken into account that the sign of the exponent in the integrand function in (10) becomes inessential after the regularization is removed.) It hence follows that the related superpropagators also coincide. For $x^2 \approx 0$ and $x^\mu \neq 0$, the behavior of the Lorentz propagator in the regularization-removal limit is described by the relation

$$\Delta(x) \sim -\ln \left(-\frac{m^2e^{2C}}{4}(x^2 - i0)\right). \quad (11)$$

This implies that the Lorentz superpropagator connecting vertices of different types in the regularization-removal limit behaves as

$$e^{-\Delta(x)} \sim -\frac{4e^{-2C}}{m^2} \frac{1}{x^2 - i0} \quad (12)$$

for the indicated values of $x$. The behavior of the Lorentz superpropagator $e^{-\Delta(x)}$ connecting vertices of the same type is described by the right-hand side of (12) to the -1st power.

The behavior of lightlike superpropagators in the domain $x^- \approx 0$, $x^+ \neq 0$ coincides with that of the Lorentz superpropagator because formulas (9) and (10) coincide in this domain. In relation to continuity, the behavior of $e^{-\Delta(x)}$ for $x^+ \approx 0$ and $x^- \neq 0$ is the same as that of $e^{-\Delta(x)}$, and $e^{\Delta(x)}$ behaves like the distribution $P_{\frac{1}{x^+}}$ in the sense of the principal value,

$$e^{\Delta(x)} \sim -\frac{4e^{-2C}}{m^2} \left(P_{\frac{1}{x^+}}\right) \frac{1}{2x^- - i0\text{sign}(x^+)} \quad (13)$$

(This can be proved by estimating the integral of $e^{\Delta(x)}$ over a small neighbourhood of the point $x^+ = 0$ [19].) Summarizing the foregoing, we conclude that

$$e^{-\Delta(x)} = e^{-\Delta(x)} \quad (14)$$
\[ e^{\Delta(x)} - e^{\Delta^G(x)} = \frac{-2\pi ie^{-2C}}{m^2|x^-|} \delta(x^+). \] (15)

Relations (14) and (15) hold in the sense of distributions on the class of test functions vanishing for \( x^\mu = 0 \) and having a support bounded with respect to \( x^- \). If the limit values of the lightlike superpropagators \( e^{\pm \Delta^G(x)} \) replaced with the regularized superpropagators \( e^{\pm \Delta^L(x)} \) then relations (14) and (15) hold as \( \varepsilon \to 0 \) if the size \( W \) of the abovementioned support with respect to \( x^- \) satisfies the condition \( W\varepsilon \xrightarrow{\varepsilon \to 0} 0 \).

The LF Hamiltonian should now be supplemented with a counterterm ensuring the improvement of the lightlike superpropagator \( e^{\Delta^L(x)} \) according to formula (15). The related counterterm for the action can be taken in the form

\[
S_c = \frac{\pi e^{-2C}}{2} \gamma^2 \int d^2x d^2y \left( e^{i\phi(x)} e^{-i\phi(y)} : -1 \right) \times \\
\times \delta(x^+ - y^+) \theta(|x^- - y^-| - \alpha) \frac{v(\varepsilon(x^- - y^-))}{|x^- - y^-|},
\] (16)

where the parameter \( \alpha \) (first introduced in formula (6)) is used to "cut out" the UV singularity and \( v(z) \) is an arbitrary additionally included rapidly decreasing continuous function satisfying the conditions \( v(0) = 1 \) and \( v^*(z) = v(-z) \).

It can be shown [19] that such an addition to the action is equivalent (to within an order-\( \varepsilon \) correction) to adding the expression

\[
- \frac{2\pi i}{m^2} e^{-2C} \delta(x^+) \frac{\theta(|x^-| - \alpha)}{|x^-|} v(\varepsilon x^-)
\] (17)
to each lightlike superpropagator \( e^{\Delta^L(x)} \). As \( \varepsilon, \alpha \to 0 \), expression (17) coincides with the right-hand side of (15) in the action on functions of the class described after formula (15). The indicated equivalence is proved by transforming the expression for an arbitrary Green’s function using superpropagators. In this case, the estimate \( e^{-\Delta^L(x)} \big|_{x^+=0} = O(\varepsilon) \) implied by (9) is essential.

We note that the presence of the delta function of \( x^+ \) in (15) and hence in (16) ensures the locality of the action with respect to the time \( x^+ \) and permits passing from the counterterm in (16) for the action to that in the LF Hamiltonian.
4. Analysis of the lightlike and Lorentz-covariant perturbation theories

We consider the lightlike PT for Green’s functions without vacuum loops that is generated by the action

\[ S = \int \frac{1}{8\pi} \left( \partial_{\mu} \varphi \partial^{\mu} \varphi - m^2 \varphi^2 \right) + B : e^{i\varphi} : + B^\ast : e^{-i\varphi} : + \frac{2\pi}{m^2} e^{-2C} |B|^2 \times \]
\[ \times \int d^2 x d^2 y \left( : e^{i\varphi(x)} e^{-i\varphi(y)} : - 1 \right) \delta(x^+ - y^+) \theta(|x^- - y^-| - \alpha) \frac{v(\varepsilon(x^- - y^-))}{|x^- - y^-|}. \] (18)

This action differs from that corresponding to Lagrangian (2) in the replacement of the coupling constant \( \frac{\gamma^2}{2} e^{i\theta} \) with the complex parameter \( B \) and in the addition of the counterterm described in the foregoing section. (The coefficient in this counterterm is related to \( B \) in the same way as the coefficient in (16) is related to \( \frac{\gamma^2}{2} e^{i\theta} \).)

We prove that the lightlike PT generated by action (18) is equivalent to the Lorentz-covariant PT for all orders in the limit as \( \varepsilon, \alpha \to 0 \). In this case, the expression \( B \) depends on \( m, \gamma, \theta, \alpha \) and \( \varepsilon \), and it is a power series in \( \gamma \),

\[ B = \frac{\gamma}{2} e^{i\theta} + \sum_{k=2}^{\infty} B_k \gamma^k. \] (19)

To analyse this PT, it is convenient to pass from its statement in terms of the superpropagators \( e^{\pm \Delta(x)} \) to the expression via the ”nonfull” superpropagators \( (e^{\pm \Delta(x)} - 1) \). In the new terms, each pair of points may or may not be connected by the corresponding nonfull superpropagator on the condition that only connected diagrams are considered. This excludes vacuum subdiagrams that would be taken into account if the full superpropagators \( e^{\pm \Delta(x)} \) were used.

In the ordinary lightlike PT, there is a class of diagrams that are always zero, namely, the diagrams all of whose external lines are adjoined to a single vertex. A diagram of this type is called a ”generalized tadpole” (or a GT diagram). From some standpoint, nonzero GT diagrams do exist in the theory under consideration, namely, in accordance with the presentation in Sec. 3, the second term in the right-hand side of formula (18) for the action can be replaced by adding expression (17) to each superpropagator \( (e^{\Delta_f(x)} - 1) \), after which the GT diagram may become nonzero. It can be shown [19] that only a second-order GT diagram with vertices of different type, which is simply a product of the two vertex factors and a superpropagator, becomes
nonzero. Only the additional term (17) contributes to this GT diagram, and this contribution is equal to \( i |B|^2 w \), where

\[
w = \frac{2\pi e^{-2C}}{m^2} \int dx^- \frac{\theta(|x^-| - \varepsilon \alpha)}{|x^-|} v(x^-).
\] (20)

This GT diagram may or may not be adjoined to each of the vertices. Therefore, its inclusion can be replaced by redefining the vertex factors, namely, the sums \((B + |B|^2 w)\) and \((B^* + |B|^2 w)\) can be regarded as the vertex factors instead of \(B\) and \(B^*\). The lightlike PT generated by action (18) thus has the form of a set of diagrams consisting of nonfull lightlike superpropagators with the addition of expression (17) (in the case of vertices of different type). Here, these diagrams contain no GT subdiagrams, and their vertex factors are \((B + |B|^2 w)\) and \((B^* + |B|^2 w)\).

With regard to intermediate UV regularization (6), the GT diagrams in the Lorentz-covariant PT (generated by Lagrangian (2)) are nonzero in all orders. Nevertheless, arguing by analogy with the above, we conclude that their inclusion can be replaced by redefining the vertex factors, namely, the vertex factors \(\frac{\gamma}{2} e^{i\theta}\) and \(\frac{\gamma}{2} e^{-i\theta}\) can be replaced with \(A\) and \(A^*\), where \(A\) is the sum of all GT diagrams with external lines adjoined to the vertex \(\frac{\gamma}{2} e^{i\theta}\).

\[
A = \frac{\gamma}{2} e^{i\theta} + \sum_{k=2}^{\infty} A_k \gamma^k.
\] (21)

Expression (21) is calculated in the Lorentz coordinates (in all orders of \(\gamma\) including the first). In view of the results in the proof of UV finiteness mentioned in Sec. 2, we can conclude that the expressions \(A_k\) are finite for \(k > 2\) and that \(A_2\), is divergent as \(\varepsilon \to 0\) or \(\alpha \to 0\) (because \(A_2\) is the sum of second-order diagrams with a fixed method for adjoining external lines). It turns out here that the divergent part can be separated as [19]

\[
A_2 = \frac{\gamma^2}{4} w + \text{const.}
\] (22)

The Lorentz-covariant PT regularized according to formula (6) is thus a set of diagrams consisting of nonfull Lorentz superpropagators, these diagrams contain no GT subdiagrams, and their vertex factors are \(A\) and \(A^*\).

The lightlike and Lorentz PTs obtained in the above form are suitable for comparison. We first require that their vertex factors be equal,

\[
B + |B|^2 w = A.
\] (23)
Furthermore, as shown in Sec. 3, the nonfull lightlike superpropagators with the addition of (17) (for vertices of different type) that constitute the lightlike PT coincide in the limit as $\varepsilon \to 0$ in the domain of finite values of $x^-$ (as $\varepsilon \to 0$) with the nonfull Lorentz superpropagators that constitute the Lorentz PT. As a consequence of regularization (6), this also holds for $|x^-| \leq \alpha$. Therefore, the difference between the lightlike and Lorentz PTs can appear only at the expense of differences in the domain of large values of $x^-$, $x^- \sim \frac{1}{\varepsilon}$.

As can be seen from formula (9), for these (and greater) values of $x^-$, the lightlike superpropagator remains finite for any $x^+$ and $\varepsilon$. Hence, series (8) for the lightlike superpropagator can be truncated, and the sum of the residual series is uniformly small with respect to $x^+$ and $\varepsilon$. It can also be shown that the contribution from the additional terms (17) is insignificant for large values of $x^-$. Analytic behavior (11) of the Lorentz superpropagator permits moving the integration contour with respect to $x^+$ away from the singularity $x^+ = 0$. Hence, the series can once again be truncated, and the sum of the residual series is uniformly small with respect to $x^+$ (and there is no dependence on $\varepsilon$). This means that the contribution to the difference between the lightlike and Lorentz diagrams coming from the domain of large values of $x^-$ does not change under the passage to "partial" superpropagators (8). This is equivalent to the passage to finite sums of diagrams in terms of the ordinary propagators. The comparison method in [12] for the lightlike and Lorentz methods for calculating diagrams can be used for these diagrams as well. This method applied to the scalar theory shows that the differences only exist for GT diagrams [19]. But the form of PTs that we consider involves no GT subdiagrams (see the comments after formulas (20) and (22)).

We can thus see that if the parameter $B$ satisfies condition (23), then the lightlike PT generated by action (18) is equivalent to the Lorentz-covariant PT for all orders in the limit as $\varepsilon, \alpha \to 0$. Hence, the theory defined by the LF Hamiltonian

$$H = \int dx^- \left( \frac{1}{8\pi m^2} \varphi^2 : -B : e^{i\varphi} : -B^* : e^{-i\varphi} : \right) - 2\pi e^{-2C} \frac{|B|^2}{m^2} \times$$

$$\times \int dx^- dy^- \left( : e^{i\varphi(x^-)} e^{-i\varphi(y^-)} : -1 \right) \theta(|x^- - y^-| - \alpha) \frac{\varv(\varepsilon(x^- - y^-))}{|x^- - y^-|} \quad (24)$$

corresponding to (18) is perturbatively equivalent in the regularization-removal limit to the Lorentz-covariant QED-2.
5. Removing the intermediate UV regularization and returning to the fermion variables

We find the parameter $B$ from Eq. (23),

$$B = -\frac{1}{2w} + \sqrt{\frac{1}{4w^2} + \frac{A'}{w} - A''^2 + iA''}, \quad (25)$$

where $A'$ and $A''$ are the real and imaginary parts of $A$ and the sign of the root is chosen in accordance with the lowest order in expansions (19) and (21). As can be seen from formula (20), the expression $w$ is a function of the product $\varepsilon\alpha$, and it diverges as $\ln(\varepsilon\alpha)$ as $\varepsilon\alpha \to 0$.

Hamiltonian (24) and Eq. (23) were derived as a result of analyzing the PT with respect to $\gamma$ for a fixed value of the regularization parameter $\alpha$. The regularization removal $\alpha \to 0$ is therefore impossible in the PT framework. But because the analysis was performed in all PT orders, we leave the PT framework in what follows and apply the resulting Hamiltonian in nonperturbative calculations. We can thus remove the intermediate UV regularization $\alpha \to 0$ and hence $w \to \infty$. Then, proceeding from the available information about the divergence of $A$ (see formula (22) and the preceding text), expression (25) can be rewritten as

$$B = \frac{\gamma^2}{4} - A''^2 + iA'' = \frac{\gamma}{2} e^{i\hat{\theta}}, \quad \sin \hat{\theta} = \frac{2A''}{\gamma}. \quad (26)$$

It is interesting that the modulus of the coupling constant turns out the same as in the original Lorentz-covariant theory. (We recall that the complex parameter $B$ plays the same role in the lightlike theory as the expression $\frac{\gamma}{2} e^{i\theta}$ in the Lorentz-covariant theory.) For large values of $\gamma$, expression (26) may involve the root of a negative number, which means that Eq. (23) has no solution; hence, the suggested scheme for constructing an LF Hamiltonian cannot be used for these values of $\gamma$.

We also note that because of the special structure of the counterterm, the zero value of the parameter $\alpha$ in Hamiltonian (24) does not result in the appearance of divergences in matrix elements.

The scalar theory defined by LF Hamiltonian (24) can be rewritten in the form of an LF fermion theory using the inverse bosonization procedure [19]. Here, the regularization $|p^-| \geq \varepsilon$ is replaced with the cutoff $|x^-| \leq L$ with periodic boundary conditions (as in the DLCQ method mentioned in
Sec. 1). In this case, the zeroth mode with respect to \( x^- \) is excluded from consideration, as a result of which the step \( \pi/L \) of the discrete momentum \( p_- \) starts to play the role of the parameter \( \varepsilon \). To perform the inverse bosonization procedure, it is convenient to change the normal ordering of the exponentials in the second term of Hamiltonian (24) according to the formula

\[
: e^{i\varphi(x^ -)} e^{-i\varphi(y^-)} : = : e^{i\varphi(x^-)} : e^{-i\varphi(y^-)} : \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} e^{-i\frac{\pi}{L} n(x^- - y^-)} \right)
\]  

and choose a function \( v(z) \) of the form

\[
v(z) = |z| \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} e^{-i\pi m z} \right) \sum_{n=-\infty}^{\infty} \frac{1}{n + \frac{1}{2}} e^{i\pi n z}
\]

(We recall the arbitrariness in the definition of \( v(z) \) mentioned in the comment after (16).) In this case, applying the inverse bosonization formula \([16, 8]\)

\[
\psi^+ (x) = \frac{1}{\sqrt{2L}} e^{-i\omega} e^{-i\frac{\pi}{L} x^-} Q e^{i\frac{\pi}{L} x^-} : e^{-i\varphi(x)} :
\]

permits using fermion variables to write LF Hamiltonian (24) defining a theory that is perturbatively equivalent to the Lorentz-covariant QED-2 in the regularization-removal limit (see \([19]\)),

\[
H = \int_{-L}^{L} dx^- \left( \frac{e_{el}^2}{2} \left( \partial_-^{-1}[\psi^+ \psi^+] \right)^2 - \frac{M}{2} \left( R e^{i\omega} d_0^+ + h.c. \right) - \frac{iM^2}{2} \psi^+ \partial_-^{-1} \psi^+ \right),
\]

where

\[
R = \frac{e_{el} e^C}{2\pi^{3/2}} e^{-i\hat{\theta}(M/e_{el}, \theta)}.
\]

The field \( \psi^+ \) satisfies antiperiodic boundary conditions with respect to \( x^- \) and can be expanded with respect to the creation and annihilation operators as

\[
\psi^+ (x) = \frac{1}{\sqrt{2L}} \left( \sum_{n \geq 1} b_n e^{-i\frac{\pi}{L} (n-\frac{1}{2}) x^-} + \sum_{n \geq 0} d_n^+ e^{i\frac{\pi}{L} (n+\frac{1}{2}) x^-} \right),
\]

\[
\{b_n, b_n^+\} = \{d_n, d_n^+\} = \delta_{nn'}, \quad b_n |0\rangle = d_n |0\rangle = 0.
\]

The expression \( Q \) in (29) is the charge operator defining the physical subspace of vectors \( |\text{phys}\rangle \),

\[
Q = \sum_{n \geq 1} b_n^+ b_n - \sum_{n \geq 0} d_n^+ d_n, \quad Q |\text{phys}\rangle = 0.
\]
The expression $\omega$ in (30) is the operator canonically conjugate to $Q$. The operator $\omega$ has the properties \[ e^{i\omega}\psi(x)e^{-i\omega} = e^{i\overrightarrow{x}^\gamma \gamma^5\psi(x)}, \quad e^{i\omega}|0\rangle = b_1^+|0\rangle, \quad e^{-i\omega}|0\rangle = d_0^+|0\rangle, \] (34) which completely define it, and the square brackets mean that the zeroth mode with respect to $x^-$ is discarded. The lightlike momentum operator $P_-$ has the form \[ P_- = \sum_{n \geq 1} b_n^+b_n \frac{\pi}{L} (n - \frac{1}{2}) + \sum_{n \geq 0} d_n^+d_n \frac{\pi}{L} (n + \frac{1}{2}). \] (35)

It is also interesting that the coefficient $R$ in the ultimate expression for LF Hamiltonian (30) and its defining parameter $\hat{\theta}$ are related to the values of the vacuum condensates in the Lorentz coordinates \[ \text{Im } R = \langle \Omega | : \overline{\Psi} \gamma^5 \Psi : |\Omega \rangle, \quad |R| = \frac{e_{el}e^C}{2\pi^{3/2}}, \] (36) \[ \sin \hat{\theta} = \langle \Omega | : \sin(\varphi + \theta) : |\Omega \rangle. \] (37) (The second relation in (36) follows from (31).) Here, $|\Omega \rangle$ is the physical vacuum, and the normal ordering is performed in the Lorentz coordinates.

6. Conclusion

Using the bosonization procedure and an analysis of the PT in all orders with respect to the fermion mass $M$, we have constructed LF Hamiltonian (30) defining a fermion theory perturbatively equivalent to the Lorentz-covariant QED-2 in the continuous limit as $L \to \infty$. This Hamiltonian involves all terms appearing in the naive quantization in the LF coordinates. In this case, one of these terms (the third term in (30)) does not participate in the quantization in the Lorentz coordinates and appears as one of the counterterms restoring the equivalence of the lightlike and Lorentz PTs. Moreover, the resulting Hamiltonian contains one more counterterm (the second term in (30)) proportional to the zeroth modes of the fermion fields times the phase operator $e^{i\omega}$, which neutralizes their charge and fermion number. The coefficient of this counterterm is determined by the value of fermion vacuum condensate (36) and is linear in this value for a small fermion mass. Under a certain relation between the condensate value and the magnitude of the charges $e_{el}$, Eqs. (36) defining
the abovementioned coefficient may be incompatible, which means that the suggested scheme for constructing the LF Hamiltonian is inapplicable in this case. But if the condensate ever assumes such values, this can only occur for sufficiently large fermion masses.

The resulting LF Hamiltonian can be used for nonperturbative calculations using the DLCQ method. We hope that the information obtained in the analysis of the given two-dimensional model will facilitate developing constructive methods for the LF Hamiltonian that take nonperturbative vacuum effects into account and are applicable to four-dimensional gauge field theories.

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