Super-horizon second-order perturbations for cosmological random fluctuations and the Hubble-constant problem

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The super-horizon second-order density perturbations corresponding to cosmological random fluctuations are considered, their non-vanishing spatial average is shown to be useful in solving the serious problem with the cosmological tension between measured Hubble constants at present and those at the early stage, and the difference from previous works on the backreaction is discussed.

1. Introduction

There are two types of gravitational instability theories in the expanding universe. One of them is general-relativistic theory. The linear theory was first derived by Lifshitz[1, 2] and the gauge-invariant version by Bardeen[3] and others[4]. The second-order nonlinear theory in the matter-dominant stage was first derived by us[5, 6] extending the Lifshitz theory, and by Russ et al. [7] and Matarrese et al. [8] in different formulations. Furthermore their gauge-invariant theories and nonlinear theories at the early stages also have been studied thereafter.[9, 10] The backreaction problems were studied by Nambu[11, 12] analyzing the second-order perturbations that include the renormalization process. Kasai et al.[13] derived a no-go theorem for an accelerating universe through the backreaction.

Another representative theory is the Newtonian instability theory, which has been studied by many workers [14, 15], in which the Newtonian gravitational potential is treated only linearly, but the hydrodynamical quantities can be treated in the second order and higher orders. This theory can be applied only to perturbations whose linear size \( L \) is smaller than \( 1/H \), where \( H \) is the Hubble parameter.

In this letter, we discuss using these theories on what scales the second-order perturbations for cosmological random fluctuations can have a non-vanishing average to solve the Hubble-constant problem, and on the difference from previous works.

For later reference we show here the condition that the inequality \( L < 1/H \) holds always at the matter-dominant stage after an epoch such as \( 1 + z_1 = 1500 \). In a flat model with the density parameter \( \Omega_M \) and the cosmological-constant parameter \( \Omega_\Lambda (= 1 - \Omega_M) \), we obtain for the present length \( L_0 \) \( (= (1 + z_1)L_1) \)

\[
L_0 < L_{0m} = 200h^{-1}/(15\Omega_M)^{1/2}
\]

from the above condition, where the present Hubble constant \( H_0 \) is \( 100h \) km s\(^{-1}\) Mpc\(^{-1}\).

For \( \Omega_M = 0.22 \) and 0.24, we have

\[
L_{0m} = 110h^{-1} \text{ and } 105h^{-1} \text{ Mpc},
\]

\[\text{(2)}\]
respectively.

In Sect. 2, we show the derivation of second-order perturbations from random fluctuations. In Sect. 3, we explain the meaning of the renormalized model parameters used in our previous papers [17, 18], and show their influence on the parameters upon the Hubble-constant problem. In Sect. 4, we have concluding remarks which include the comparison with the previous works on the backreaction.

2. Random fluctuations and second-order perturbations

Now let us consider the random fluctuations that were caused by quantum fluctuations at the very early stage, and whose amplitude and spectrum have been studied through the precise measurements of fluctuations in the cosmic microwave background (CMB) radiation by WMAP[16] and Planck Collaborations[19, 20]. Here we assume that the fluctuations as first-order perturbations $\delta_1 \rho / \rho$ have a vanishing spatial average (i.e. $\langle \delta_1 \rho / \rho \rangle = 0$), where $\rho$ is the background matter density, and derive the corresponding second-order perturbation $\delta_2 \rho / \rho$ and its spatial average $\langle \delta_2 \rho / \rho \rangle$.

For perturbations with the present size satisfying the condition $L_0 < L_{0m}$, the they can be treated in the Newtonian cosmological approximation[14, 15], and so

$$\langle \delta_2 \rho / \rho \rangle = 0$$

at the matter-dominant stage (with $1 + z < 1500$). Similarly higher-order perturbations also have vanishing average values, i.e.

$$\langle \delta_n \rho / \rho \rangle = 0$$

for $n > 2$.

For the perturbations with $L_0 > L_{0m}$, on the other hand, the linear sizes of the perturbations are always larger than the Hubble size $1/H$ or cross it once at the matter-dominant stage, and therefore the evolution should be treated using general-relativistic perturbation theories.

In one of our recent papers (I) [17], we used the second-order perturbations $\delta_2 \rho / \rho$, which were derived using our general-relativistic perturbation theory with non-zero $\Lambda$ (in the comoving and synchronous gauge) [6] and include non-Newtonian terms, and obtained the spatial average $\langle \delta_2 \rho / \rho \rangle$ in the form of an integral with respect to wave-number $k$. In this paper, the spectrum of first-order density fluctuations was given by using the BBKS transfer function[21], and the amplitude was determined using the result of the Planck measurements.

The upper limit of the wave-number $k_{max}$ in the above integral was specified as

$$L_{max} \equiv 2\pi / k_{max} = 102/h \text{ Mpc}$$

for $\Omega_M = 0.22$. This length of $L_{max}$ represents the present distance, over which smooth observations on cosmological scales may be possible. The condition that the wave-number $k$ should be smaller than $k_{max}$ (or the length $L_0$ of perturbations should be larger than $L_{max}$) is nearly equal to the super-horizon condition $L_0 > L_{0m}$ with Eq.(2), because $L_{max} \simeq L_{0m}$. Thus, in (I), we obtained $\langle \delta_2 \rho / \rho \rangle$ with a positive value from the super-horizon random fluctuations with the length $L_0 > L_{0m}$. This $\langle \delta_2 \rho \rangle$ is the average energy density of random fluctuations which we call the density of the fluctuation energy.
3. Energy density of random fluctuations and the Hubble-constant problem

In paper (I) [17], we derived the second-order perturbations corresponding to not only density perturbations but also metric perturbations, and obtained the average values of $\langle \delta_2 \rho \rangle$ and $\langle \delta_2 (H^2) \rangle$, where $H$ is the Hubble parameter.

Using these average values, we defined the cosmologically renormalized quantities

$$\rho_{\text{rem}} \equiv \rho + \langle \delta_2 \rho \rangle$$

and

$$H_{\text{rem}} \equiv \left[ H^2 + \langle \delta_2 (H^2) \rangle \right]^{1/2},$$

which were derived in the comoving and synchronous gauge. Moreover, the renormalized model parameters were defined as

$$(\Omega_M)_{\text{rem}} \equiv \Omega_M \frac{1 + \langle \delta_2 \rho / \rho \rangle}{1 + \langle \delta_2 \rho / \bar{\rho} \rangle},$$

and

$$(\Omega_\Lambda)_{\text{rem}} \equiv \Omega_\Lambda \frac{1}{1 + \langle \delta_2 \rho / \bar{\rho} \rangle},$$

where $\bar{\rho} \equiv \rho + \Lambda$. Here and in the previous papers [17, 18], the “renormalized” model parameters mean the new model parameters, which are given to homogeneous models with the average second-order perturbations. This “renormalization” used here is not connected with any dynamical renormalization processes, including the gauge-invariant property.

At present we face the cosmological tension between the direct measurements of the Hubble constant at present epoch and the Hubble constant derived from the Planck measurements of the CMB anisotropies [22–26]. We showed in the previous paper (I) the possibility of solving the problem on the above tension, using the renormalized Hubble constant (7) and the renormalized model parameters (8) and (9). For the background model with the present parameters $\Omega_M = 0.22, \Omega_\Lambda = 1 - \Omega_M$, and $H_0 = 67.3 \text{ km s}^{-1} \text{Mpc}^{-1}$, it was found that we obtain $(\Omega_M)_{\text{rem}} = 0.305$, $(\Omega_\Lambda)_{\text{rem}} = 1 - (\Omega_M)_{\text{rem}}$, and $H_{\text{rem}} = 74.0 \text{ km s}^{-1} \text{Mpc}^{-1}$ at the present epoch. On the other hand, the models at the early stage ($z \gg 1$) have values consistent with the background ones. These model parameters and Hubble constants are found to represent the observed ones.

In our next paper (II) [18], we expressed the fluctuation energy $\rho_f (\equiv \langle \delta_2 \rho \rangle)$ as a function of $\rho$ in a background model, and regarded $\rho_f$ as the density of a new constituent pressureless matter. Moreover, assuming that the total matter density $\rho_T$ is given by

$$\rho_T = \rho + \rho_f (\rho),$$

we derived a new cosmological model with

$$ds^2 = g_{\mu \nu} dx^\mu dy^\nu = a^2(\eta) [-d\eta^2 + \delta_{ij} dx^i dx^j],$$

where the Greek and Roman letters denote 0, 1, 2, 3 and 1, 2, 3, respectively. The conformal time $\eta(= x^0)$ is related to the cosmic time $t$ by $dt = a(\eta) d\eta$, and $a(\eta)$ is the new scale factor in the universe including the fluctuation energy $\rho_f$. 

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Then the velocity vector and energy-momentum tensor of pressureless matter are expressed in comoving coordinates as

\[ u^0 = 1/a, \quad u^i = 0 \]  

and

\[ T^0_0 = -\rho_T, \quad T^0_i = 0, \quad T^i_j = 0. \]

From the Einstein equations, we obtain

\[ \rho_T a^2 = 3(a'/a)^2 - \Lambda a^2, \]  

and the energy-momentum conservation \( (T^\mu_\nu; \nu = 0) \) gives the relation

\[ \rho_T a^3 = \rho_T(t_0), \]

where \( a = 1 \) at the present epoch \( (t = t_0) \) and a prime denotes \( \partial/\partial \eta \).

For comparison, the usual background model without the fluctuation energy is denoted using the suffix \( b \), so that the scale factor, the Hubble parameter and the density are denoted as \( a^b, H^b \) and \( \rho^b \), where \( a^b = 1 \) at the present epoch. Then for the present parameters \( \Lambda^b_M = 0.22, \Omega^b_M = 0.78, \) and \( H^b_0 = 67.3 \, \text{km s}^{-1}\text{Mpc}^{-1} \), it is found that the present value of \( \beta(\rho^b) \equiv \rho_f(\rho^b)/\rho^b \) is 0.55.

By comparing the solutions of \( a \) and \( a^b \), we could express \( \Omega_M, \Omega_\Lambda \) and \( H \) as the functions of \( z, \Omega^b_M, \Omega^b_\Lambda \) and \( H^b \). It is found that the correspondence between \( (\Omega_M, H) \) and \( (\Omega^b_M, H^b) \) depends on the value of the ratio of present matter densities \( (\rho/\rho^b)_0 \).

For example, for \( (\rho/\rho^b)_0 = 1.181 \), we obtain the present parameters \( \Omega_M = 0.341 \) and \( H_0 = 73.2 \, \text{km s}^{-1}\text{Mpc}^{-1} \), while for \( z \gg 1 \), we have \( (\Omega_M, H) = (\Omega^b_M, H^b) \). Their values for the other \( (\rho/\rho^b)_0 \) are shown in the paper (II) [18]. These changes in the Hubble constant and cosmological parameters may explain the observational difference between the direct measurements and the Planck CMB measurements. They reflect the situation that the universe evolves from the background one to the present one with larger Hubble constants, corresponding to the increase of the fluctuation energy \( \rho_f \).

4. Concluding remarks

Using our second-order perturbations in the comoving and synchronous gauge in the super-horizon region, it was shown that the Hubble-constant problem may be solved by introducing the renormalized parameters or building cosmological models that include the fluctuation energy as one of the constituent matters.

This conclusion is different from that of Nambu’s works[11, 12] which treated the backreaction problem. In them it is found that no large-scale motion can arise from cosmological random fluctuations. The difference between two works comes from the treatment of the perturbations: Nambu analyzed the backreaction using the theoretical renormalization process, while we used the above simple model (with second-order perturbations in the comoving and synchronous gauge) without considering any renormalization process with gauge invariant property into account. So we need to have comparative studies of these two treatments.

Our work is consistent with Kasai et al.’s no-go theorem for an accelerating universe,[13] though they used the Newtonian gauge different from our gauge, where \( \rho_T(\equiv \rho + \rho_f) \) in our paper (II) corresponds to \( \bar{\rho} \) in their paper and their main equation corresponds to our Eq.
(14). Our model is decelerating in the zero-Λ case, though it has comparatively large Hubble parameters.

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