Simple Cosmological Model with Relativistic Gas

Guilherme de Berredo-Peixoto 1, Ilya L. Shapiro 2, Flávia Sobreira 3

Departamento de Física – ICE, Universidade Federal de Juiz de Fora,
Juiz de Fora, CEP: 36036-330, MG, Brazil

ABSTRACT

We construct simple and useful approximation for the relativistic gas of massive particles. The equation of state is given by an elementary function and admits analytic solution of the Friedmann equation, including more complex cases when the relativistic gas of massive particles is considered together with radiation or with dominating cosmological constant. The model of relativistic gas may be interesting for the description of primordial Universe, especially as a candidate for the role of a Dark Matter.

Keywords: Relativistic gas, cosmological solutions, dark matter.
PACS: 98.80.Hw, 98.80.Cq, 98.80.Bq

1 Introduction

During many years most of the works about cosmological solutions assumed stationary equation of state for the matter or vacuum sources of the Friedmann equations. The conventional equation of state is a linear relation between pressure and energy density \( P = w \rho \). Different values of the parameter \( w \) correspond to different kinds of sources. For example, \( w = -1 \) holds for the cosmological constant (vacuum energy), \( w = 0 \) for the pressureless (dust-like) matter, \( w = 1/3 \) for the radiation. Recently, due to the growing amount and quality of the observational/experimental cosmological data, theoretical considerations involved more complicated equations of state, with \( w \) depending on time or/and energy density. Perhaps the first examples of this kind were related to inflation, where the variable vacuum energy density was introduced due to the electroweak phase transition [1] and the non-trivial inflaton potential [2].

Theoretical investigations of the models with variable vacuum energy were fueled recently by the new precise measurements of the expansion rate of the Universe from the type Ia supernovae experiments [3] and also from the cosmic microwave background radiation [4]. The
existing data indicate that the Universe is mainly composed by the non-luminous sources, such as Dark Matter, responsible for 20-30% of the overall energy density balance, and the Dark Energy, responsible for 65-75%. One of the main candidates to the role of the most of the Dark Matter is a gas of weakly interacting massive particles, e.g., the ones corresponding to the broken supersymmetry. Let us notice that recent theoretical and phenomenological considerations of the supersymmetric neutralino do not rule out the light DM options \([5, 6]\) (see \([7, 8]\) for the review). The main candidate to be Dark Energy is the cosmological constant with equation of state \(w = -1\). The anthropic considerations show that the total value of the cosmological constant should be positive and in fact close to the observed one \([9, 10]\).

However, since nobody can guarantee that the vacuum energy is red-shift independent, it is quite natural to meet a variety of alternative models for the Dark Energy, such as quintessence \([11]\), Chaplygin gas \([12]\) and the low-energy renormalization group running of the proper cosmological constant \([13]\). Most of these models lead to a variable \(w\) in the equation of state for the vacuum energy. This effect is achieved either by postulating this equation of state in the case of Chaplygin gas or by postulating a properly chosen quintessence potential.

There is another possibility to meet the equation of state with a variable \(w\), depending on the energy density and therefore on the red-shift. In the present article we shall consider the Universe filled by the ideal gas of relativistic massive particles\(^4\). This model may have interesting applications, e.g., in the early radiation-dominated Universe we can consider relativistic gas of massive particles as a model for the hot matter content. Furthermore, relativistic effects may be, in principle, relevant for the Dark Matter problem. Since we do not know exactly from what the Dark Matter is done, any possibility here deserves careful exploration. Indeed, when assuming that the Dark Matter is a relativistic gas, we suppose that it is composed (or has been composed in the earlier epochs) from the relatively light, massive particles weakly interacting with the baryonic matter and radiation.

The equation of state for the relativistic ideal gas is known for a long time \([15]\) (see also, e.g., \([16]\)\(^5\)). The relation between pressure and energy density involves modified Bessel functions. Obviously, this form of equation of state in not very useful for cosmological applications. At the same time one can considerably simplify the cosmological model with relativistic gas without losing much of the physical sense. In order to do so we shall assume that, instead of following Maxwell distribution, all particles have equal kinetic energy. Below we call the model of relativistic gas of massive particles with equal energies the *reduced relativistic gas*.

The “defect” in the equation of state which follows from the assumption of equal energies

\(^4\)The cosmological model taking into account the relativistic effects related to the peculiar velocities of galaxies has been developed in \([14]\).

\(^5\)The relativistic version of Maxwell distribution follows from the corresponding generalization of the Boltzmann \(H\)-theorem (see, e.g., \([17]\)).
is not very significant. The numerical comparison with the Maxwell distribution is presented in the Appendix. It turns out that the difference between the two distributions does not exceed 2.5% even in the low-energy region, being negligible for the ultrarelativistic gas. Let us remember that the Maxwell distribution is also just an approximation to the real situation. For example, when considering the Maxwell distribution for the identical massive particles in the Early Universe we are disregarding interactions between these particles and radiation, and also differences between the masses of different kinds of particles.

At the same time, the model of reduced relativistic gas provides a great advantage for cosmological applications. Starting from the reduced equation of state one can integrate the Friedmann equation analytically, leading to a nice and simple cosmological model interpolating between radiation-dominated and matter-dominated epochs of the Universe. In the present paper we shall develop this model and also consider more complicated cases of the Universe filled by reduced relativistic gas plus radiation and of the Universe dominated by the cosmological constant where matter content is the reduced relativistic gas.

The paper is organized as follows. In the next Section we shall briefly describe the conventional model of relativistic gas and our model of the reduced relativistic gas. In Section 3 we use the equation of state for the reduced model and obtain the scale dependence for the energy density and pressure. Section 4 is devoted to the solution of the cosmological model with the reduced relativistic gas for several interesting particular cases. In section 4 we draw our conclusions.

2 Reduced model for relativistic gas

Consider a single relativistic particle with the rest mass $m$ in a volume $V$. The dispersion relation for this particle has standard form

$$\epsilon^2 - c^2 p^2 = m^2 c^4,$$

where

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (1)

An elementary consideration shows that the time average of the pressure produced by the particle on the walls of the vessel is

$$P = \frac{1}{3V} \cdot \frac{mv^2}{\sqrt{1 - v^2/c^2}}.$$ \hspace{1cm} (2)

For the gas of $N$ such particles with equal kinetic energies $\epsilon$, we arrive at the following equation of state

$$P = \frac{\rho}{3} \cdot \left[ 1 - \left( \frac{mc^2}{\epsilon} \right)^2 \right], \quad \text{where} \quad \rho = \frac{N \epsilon}{V}$$ \hspace{1cm} (3)

is the energy density. Let us notice that $w = P/\rho$ tends to $1/3$ in the ultra-relativistic limit $\epsilon \to \infty$ and to zero in the non-relativistic limit $\epsilon \to mc^2$. It proves useful to introduce
the following new notations: the density of the rest energy of the particles at the initial moment \( \rho_1 = N mc^2/V_0 \), where \( V_0 \) is some fixed initial value of the volume, and also \( \rho_d = \rho_d(V) = \rho_1 V_0 / V = N mc^2/V \), which shows how the same density evolves with the change of the volume. Using these notations we can cast the equation (3) into the form

\[
P = \frac{\rho}{3} \cdot \left[ 1 - \frac{\rho_d^2}{\rho^2} \right].
\]

In order to understand better the difference between these formulas and the "correct" ones, let us consider the Maxwell distribution for the ideal gas of massive particles. The statistical integral for a single particle is given by the expression

\[
Z = \int e^{-\epsilon/kT} d^3 p d^3 q = 4\pi m^2 c V \cdot K\left(\frac{mc^2}{kT}\right),
\]

where \( K_\nu(x) \) is a modified Bessel function of index \( \nu \). The equation of state for the gas of \( N \) particles can be derived in a standard way

\[
PV = kT N \left( \frac{\partial \ln Z}{\partial \ln V} \right) = N kT,
\]

while the average energy of the particle is

\[
\bar{\epsilon} = \frac{1}{Z} \int e^{-\epsilon/kT} \epsilon d^3 p d^3 q = mc^2 \frac{K_3(mc^2/kT)}{K_2(mc^2/kT)} - kT.
\]

The energy density \( \rho = N\bar{\epsilon}/V \) and pressure \( P \) are related by an implicit functional dependence (5) and (7), which has to be compared with the formula (4) for the reduced gas case. This comparison will be performed numerically in the Appendix, where we show that (4) is an excellent approximation to the relations (6), (7). For a while we conclude this section by an obvious observation that (4) is much simpler than (5) plus (7).

### 3 Scale dependence in the reduced model

Let us use the equation of state (4) for the reduced relativistic gas and find how the energy density depends on the volume under adiabatic expansion. For this end we replace the equation (4) into the conservation law

\[
-\frac{dV}{V} = \frac{d\rho}{\rho + P}.
\]

In the cosmological setting the last equation implies that the reduced relativistic gas does not exchange energy with other entities like with radiation in the Early Universe or with baryonic matter and vacuum energy in the case when the reduced relativistic gas is considered as a model for the Dark Matter.
The differential equation (8) can be easily solved. The solution has the form

$$\rho(V) = \left[ \rho_1^2 \left( \frac{V_0}{V} \right)^2 + \rho_2^2 \left( \frac{V_0}{V} \right)^{8/3} \right]^{1/2},$$

(9)

where the initial condition has been defined as

$$\rho(V_0) = \left[ \rho_1^2 + \rho_2^2 \right]^{1/2}.$$

Remember that $\rho_1 = Nmc^2/V_0$ is the rest energy density at the initial point $V = V_0$. Hence the second component $\rho_2$ can be interpreted as the energy density of the radiation component of the reduced relativistic gas. However the total energy density is not a simple sum of the two components but the square root of the sum of their squares. The non-relativistic or ultra-relativistic limits are achieved when one takes, correspondingly, $\rho_2 = 0$ or $\rho_1 = 0$. It is easy to see that the expression (9) provides correct scaling laws in these two cases.

For the sake of cosmological applications it is better to express the energy density as a function of the conformal factor $a$, where $(a/a_0)^3 = V/V_0$. Then we arrive at the formula

$$\rho(a) = \left[ \rho_1^2 \left( \frac{a_0}{a} \right)^6 + \rho_2^2 \left( \frac{a_0}{a} \right)^{8} \right]^{1/2},$$

(10)

with the initial condition $\rho(a_0) = [\rho_1^2 + \rho_2^2]^{1/2}$. The density of the rest mass behaves like $\rho_d(a) = \rho_1 (a_0/a)^3$. The scale dependence of the pressure and parameter $w$ are given by the eq. (11) and the equation

$$w = \frac{1}{3} \left[ 1 - \frac{\rho_d^2(a)}{\rho^2(a)} \right].$$

(11)

It is easy to see that the relations (11) and (11) predict dust-like (pressureless) scaling in the limit $a \to \infty$ and the radiation-like scaling $w \approx 1/3$ in the limit $a \to 0$. Hence our model of reduced relativistic gas can be regarded as an interpolation model between the radiation-dominated and matter-dominated evolutions. If the Universe, from the very beginning, were filled by ideal hot gases of massive particles, there would not be nuclear reactions and the scale behaviour of the energy density would be close to (11).

4 Solving the Friedmann equation

Consider the cosmological model of the Universe filled by the reduced relativistic gas (11). For the sake of generality, let us start by formulating the Friedmann equation for an arbitrary $k$ and also include vacuum energy density, $\rho_\Lambda = \Lambda/(8\pi G)$, and the radiation energy density $\rho_r(a) = \rho_{r0}/a^4$. In what follows we set $a_0 = 1$. The equation of interest has the form

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \left[ \rho(a) + \rho_\Lambda + \rho_r(a) \right],$$

(12)
where $\rho(a)$ is given by (10). One can easily add other matter sources and solve the resulting equation following the examples which will be elaborated below.

### 4.1 Pure reduced gas model

As a first particular case we shall integrate eq. (12) for the pure reduced relativistic gas model, $k = \rho_\Lambda = \rho_r = 0$. After introducing a new variable $x = a^2$ the equation becomes

$$\dot{x}^2 = \frac{32\pi G \rho_1}{3} \sqrt{x + b}, \quad \text{where} \quad b = \frac{\rho_2^2}{\rho_1^2}. \quad (13)$$

The solution of this equation has the form

$$\left( a^2 + \frac{\rho_2^2}{\rho_1^2} \right)^{3/4} = \sqrt{6\pi G \rho_1} \cdot t. \quad (14)$$

The last expression shows, again, that our model interpolates between the usual matter-dominated and radiation-dominated FRW solutions. In the non-relativistic case, $\rho_2 = 0$, we obtain directly the standard behaviour $a(t) \sim t^{2/3}$. The ultrarelativistic regime can not be considered in a direct way, because the limit $\rho_1 = 0$ is singular. Let us assume $\rho_1 \ll \rho_2$ and expand until the lowest nontrivial order in the ratio $(\rho_1/\rho_2)$

$$\left( \frac{\rho_2}{\rho_1} \right)^{3/2} \left[ 1 + \frac{3}{4} \left( \frac{\rho_1}{\rho_2} \right)^2 a^2 \right] = \sqrt{6\pi G \rho_1} \cdot t. \quad (15)$$

The last expression is nothing else but the usual radiation-dominated solution with the shifted time variable

$$a(t) = \left( \frac{32\pi G \rho_2}{3} \right)^{1/4} \sqrt{t - t_0}, \quad \text{where} \quad t_0 = \frac{\rho_2^{3/2}}{\sqrt{6\pi G \rho_1^2}}. \quad (16)$$

Indeed, such time shifts are irrelevant and we will not pay attention to them in what follows.

In order to illustrate the behaviour of the conformal factor (14) in the transition period, let us choose the logarithmic parametrization for both time and conformal factor. For the sake of simplicity we fix $t_0$ according to (16) and rewrite (14) in the form

$$e^v = \ln \left[ \left( e^{2s} + \frac{\rho_2^2}{\rho_1^2} \right)^{3/4} - \left( \frac{\rho_2}{\rho_1} \right)^{3/2} \right], \quad (17)$$

where $e^v = \sqrt{6\pi G \rho_1} \cdot t$ and $a = e^s$.

The plot of $v$ versus $s$ depends on the value of a single parameter $\rho_2/\rho_1$ and in all cases it clearly demonstrates the transition between the two linear asymptotic regimes. We present an example for $\rho_2/\rho_1 = 1$ at the Figure 1. The reader can easily achieve similar plots for other values of this parameter, also in the presence of a radiation content. The solution for the last case is presented in the next subsection.
4.2 Radiation dominated epoch

Consider the reduced relativistic massive gas in the radiation-dominated epoch. According to the known estimates, cosmological constant and space curvature are not very relevant in this case [18, 19] and we can safely set $k = \rho_\Lambda = 0$.

The solution of the Friedmann equation (12) can be written as (parameter $b$ is defined in (13))

$$\frac{4}{3} \left[ \sqrt{a^2 + b + \frac{\rho_{r0}}{\rho_1}} \right]^{3/2} - \frac{4\rho_{r0}}{\rho_1} \left[ \sqrt{a^2 + b + \frac{\rho_{r0}}{\rho_1}} \right]^{1/2} = \left( \frac{32\pi G \rho_1}{3} \right)^{1/2} t. \quad (18)$$

The last relation represents exact solution, but it is too complicated for the qualitative analysis. Let us consider the special situation when the radiation energy density is strongly dominated and the effect of the reduced relativistic massive gas is a small correction to the $a \sim t^{1/2}$ law. Our purpose is to evaluate this correction. For this end one has to expand the expression (here $\gamma = 1/2$ or $3/2$)

$$\left[ \sqrt{a^2 + b + \frac{\rho_{r0}}{\rho_1}} \right]^{\gamma} = \left( \frac{\rho_{r0}}{\rho_1} \right)^{\gamma} \left[ 1 + \frac{\rho_1}{\rho_{r0}} \sqrt{a^2 + b} \right]^{\gamma}. \quad (19)$$

until the third order in the small parameter $\frac{\rho_1}{\rho_{r0}} \sqrt{a^2 + b}$.

The result has the form

$$a^2 - \frac{1}{3} \left( \frac{\rho_1}{\rho_{r0}} \right) \left[ a^2 + b \right]^{3/2} = \left( \frac{32\pi G \rho_{r0}}{3} \right)^{1/2} t, \quad (20)$$

where we disregard the initial value $t_0$. It is easy to see that the effect of the reduced relativistic gas is to accelerate the expansion of the Universe compared to the pure radiation content. The effect of relativistic gas of massive particles is weaker that the one coming from the dust-like matter with the same energy density. The illustrative plots of the reduced relativistic gas compared to radiation and dust cases are presented at the Figure 2.
4.3 Cosmological constant dominated epoch

The next relevant particular case is the reduced relativistic gas as a model for the Dark Matter, in a Universe dominated by the vacuum energy. After the usual change of variable, $x = a^2$, we arrive at the solution in the form of an integral

$$\int \frac{dx}{\left[ \rho_\Lambda x^2 + \rho_1 \sqrt{x + b} \right]^{1/2}} = \sqrt{\frac{32\pi G}{3}} t,$$

with $b$ defined in (13). This integral is rather complicated and difficult to evaluate analytically. Hence we can either apply the numerical method or use the dominant role of the vacuum energy density. The results of numerical analysis for $\Omega_\Lambda = 0.7$ and $\Omega_M = 0.3$ are shown at the Figure 3. The presence of the reduced relativistic massive gas results in the slower expansion of the Universe compared to the pure cosmological constant case.

Consider an approximate analytical solution based on the assumption of the dominant contribution of the vacuum energy density. We rewrite the integral in the l.h.s. of eq. (21) in the form

$$\frac{1}{\sqrt{\rho_\Lambda}} \int \frac{dx}{x \sqrt{1 + \gamma}}, \quad \text{where} \quad \gamma = \frac{\rho_1}{\rho_\Lambda} \frac{\sqrt{x + b}}{x^2},$$

and expand into power series in the small parameter $\gamma$. Then the integration becomes trivial and taking the lowest nontrivial order into account we arrive at the solution

$$\ln a + X[a] = \lambda t,$$

where

$$X[a] = \frac{\rho_1}{8 \rho_\Lambda} \left\{ \frac{\sqrt{a^2 + b}}{a^4} + \frac{\sqrt{a^2 + b}}{2ba^2} + \frac{1}{4b^{3/2}} \ln \left( \frac{\sqrt{a^2 + b} - \sqrt{b}}{\sqrt{a^2 + b} + \sqrt{b}} \right) \right\}$$

Figure 2: The plot for the conformal factor $a(t)$ in the following three cases: pure radiation $a(t) \sim t^{1/2}$ is represented by the dashed line, pure dust-like case by the dots line, the case of pure reduced relativistic gas is represented by the continuous line.
is a small term and $\lambda = \sqrt{8\pi G \rho \Lambda / 3}$. One can easily find an approximate explicit formula for $a(t)$ starting from the expression

$$a(t) = \left[1 + f(t)\right] \cdot e^{\lambda t}, \quad |f(t)| \ll 1, \quad (24)$$

and using the smallness of $X[a]$. Replacing (24) into (22) we arrive at the solution $f(t) = -X[e^{\lambda t}]$. This expression can be replaced into (24) to give

$$a(t) = e^{\lambda t} \left(1 - X[e^{\lambda t}]\right). \quad (25)$$

Qualitatively the behaviour of $a(t)$ fits with the numerical analysis. The presence of reduced relativistic gas slows the acceleration of the Universe caused by $\rho \Lambda$.

5 Conclusions

We constructed a simple and useful model of reduced relativistic gas of massive particles, starting from the “primitive distribution” for kinetic energies. Our model is a very good approximation to the much more complicated Maxwell distribution (see Appendix). The main advantage of the reduced model is that it admits analytical derivation of the dependence $\rho(a)$ in terms of elementary (and very simple) functions. Moreover, one can easily integrate the Friedmann equation for the Universe filled by the reduced relativistic gas, again the solution is given by elementary function. In this way we arrive at the interpolation between the cosmological solutions for the matter-dominated and radiation-dominated cases.
In the more complex situation, when the sources of the Friedmann equation include also radiation or the cosmological constant term, one can obtain the solution in the form of an integral. The integration can be performed numerically or analytically using the assumption of radiation dominance or cosmological constant dominance. In the last case the reduced relativistic gas can be viewed as a model for the Dark Matter.

The model of reduced relativistic gas may be, in principle, testable. Imagine the Dark Matter is composed by the relatively light, weakly interacted massive particles which have, at present, \( w \approx 0 \) equation of state. One can not rule out the possibility that the relativistic effects of these particles were relevant in the earlier epochs of the Universe, e.g. in the structure formation period. Then, some the traces of these relativistic effects may be eventually found in the precise CMB measurements. Therefore it would be interesting to investigate density and metric perturbations in this model. We postpone this problem for the future work.

**Appendix**

The purpose of this Appendix is to compare the results (6) and (7) of the Maxwell distribution for the relativistic gas of massive particles and the corresponding relation (11) for the reduced relativistic gas model. In the formulas (6) and (7) the temperature \( kT \) plays the role of parameter and we are in fact interested only in the dependence between \( P \) and \( \rho \). Hence we solve (6) with respect to pressure and replace it into (7)

\[
\rho_M(P) = \frac{K_3(\rho_d/P)}{K_2(\rho_d/P)} \rho_d - P,
\]

where the subscript \( M \) indicated Maxwell distribution.

It is easy to see that the non-relativistic limit \( P \to 0 \) and the ultrarelativistic limit \( P \to \rho/3 \) of the last relation coincide with the one for the reduced model (11). In order to compare the two expressions numerically at the intermediate scales, let us rewrite (11) in the form similar to (26)

\[
\rho(P) = \frac{3}{2} P + \sqrt{\frac{9P^2}{4} + \rho_d^2}.
\]

After assigning numerical value to \( \rho_1 \) (we set \( \rho_1 = 1 \), but the result does not depend on this choice.), one can plot the relative deviation

\[
\delta_\rho = \frac{|\rho - \rho_M|}{\rho_M}
\]

versus \( P \). The plot for this dependence in presented at the Figure 4. It is easy to see that the relative deviation \( \delta_\rho \) achieves its maximum of about 2.5% in the non-relativistic region and becomes completely negligible in the higher energy region.
Figure 4: Plot of $\delta_\rho$ for the $\rho_0 = 1$ case. The maximum discrepancy is about 2.5%.

Acknowledgments. I.Sh. is indebted to A. Belyaev and J. Fabris for useful discussions. The work of the authors has been supported by the scholarship from CNPq (F.S.), the fellowship from FAPEMIG (G.B.P.) and the grants from CNPq and FAPEMIG (I.Sh.).

References

[1] A.H. Guth, Phys. Rev. 23D (1981) 347.

[2] A.D. Linde, Phys. Lett. 108B (1982) 389;
   A. Albrecht, P.J. Steinhardt, Phys. Rev. Lett. 48 (1982) 1220.

[3] S. Perlmutter et al. (the Supernova Cosmology Project), Astrophys. J. 517 (1999) 565;
   A.G. Riess et al. (the High–z SN Team), Astronom. J. 116 (1998) 1009.

[4] P. de Bernardis et al., Nature 404 (2000) 955;
   C.B. Netterfield et al. (Boomerang Collab.), Astrophys. J. 571 (2002) 604,
   astro-ph/0104460;
   D.N. Spergel et al., Astrophys. J. Suppl. 148 (2003) 175, astro-ph/0302209
   WMAP collaboration: http://map.gsfc.nasa.gov/;
   C.L. Benett et al., Astrophys. J. Suppl. 148 (2003) 1, astro-ph/0302207

[5] C. Boehm, P. Fayet and J. Silk, Phys. Rev. D69 (2004) 101302 hep-ph/0311143;
   C. Boehm, Int. J. Mod. Phys. A19 (2004) 4355;
   D. Hooper, F. Ferrer, C. Boehm, J. Silk, J. Paul, N. Wyn Evans and Michel Casse,
   Phys. Rev. Lett. 93 (2004) 161302 astro-ph/0311150.
[6] H. Baer, C. Balazs, A. Belyaev, T. Krupovnickas and X. Tata, JHEP 0306 (2003) 054 [hep-ph 0304303];
H. Baer, A. Belyaev, T. Krupovnickas and X. Tata, JHEP 0402 (2004) 007 [hep-ph 0311351].

[7] P. Fayet, Proceedings of 39th Rencontres de Moriond Workshop on Exploring the Universe: Contents and Structures of the Universe, [hep-ph/0408357].

[8] A. Belyaev, 15th Topical Conference on Hadron Collider Physics (HCP2004) (AIP Conf.Proc. 753 (2005) 352 [hep-ph/0410385].

[9] S. Weinberg, Phys. Rev. Lett. 59 (1987) 2607.

[10] J. Garriga, A. Vilenkin, Phys. Rev. 61D (2000) 083502.

[11] R.R. Caldwell, R. Dave, P.J. Steinhardt, Phys. Rev. Lett. 80 (1998) 1582;
P.J.E. Peebles, B. Ratra, Rev. Mod. Phys. 75 (2003) 599.

[12] A.Yu. Kamenshchik, U. Moschella, V. Pasquier Phys. Lett. 511B (2001) 265;
M.C. Bento, O. Bertolami, A.A. Sen, Phys. Rev. 66D (2002) 043507;
J.C. Fabris, S.V.B. Gonçalves, P.E. de Souza, Gen. Rel. Grav. 34 (2002) 53.

[13] I.L. Shapiro, J. Solà, Phys. Lett. 475 B (2000) 236; JHEP 0202 (2002) 006;
A. Babic, B. Guberina, R. Horvat and H. Stefancic, Phys.Rev. D65 (2002) 085002;
I.L. Shapiro, J. Solà, C. España-Bonet, P. Ruiz-Lapuente, Phys. Lett. 574B (2003) 149;
A. Bonanno, M. Reuter, Phys. Rev. 65D (2002) 043508.

[14] H.H. Soleng, Astron. Astrophys. 237 (1990) 1.

[15] F. Jüttner, Ann. der Phys. Bd 116 (1911) S. 145.

[16] W. Pauli, *Theory of Relativity*, (Dover, 1981).

[17] C. Cercignani, G.M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications*, (Birkhäuser, Basel, 2002)

[18] S.A. Bludman and M.A. Ruderman, Phys. Rev. Lett. 38 (1977) 255.

[19] S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons. Inc., 1972).