Examples of groups whose automorphisms have exotic growth

Rémi Coulon

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Abstract

In this note we produce examples of outer automorphisms of finitely generated groups which have exotic behaviors in terms of growth of conjugacy classes.

1 Introduction

Growth of outer automorphisms. Let $G$ be a finitely generated group. The outer automorphism group of $G$, which we denote by $\text{Out}(G)$, naturally acts on the set of conjugacy classes of $G$. It is a standard strategy to study the properties of an automorphism $\Phi \in \text{Out}(G)$ by looking at the dynamics of this action. Here is an example. Endow $G$ with the word metric. Given any conjugacy class $c$ of $G$, denote by $\|c\|$ the length of the smallest element in $c$. One may wonder what is the growth of $\Phi$, i.e. the asymptotic behavior of the map $N \to \mathbb{R}$ sending $n$ to $\|\Phi^n(c)\|$. It turns out that for various intensively studied groups, one observes a strong growth dichotomy.

Theorem 1.1. Let $G$ be a finitely generated group in one of the following classes: free abelian groups, free groups, fundamental groups of closed compact orientable surfaces, torsion-free hyperbolic groups, torsion-free toric relatively hyperbolic groups. Then for every $\Phi \in \text{Out}(G)$, for every conjugacy class $c$ of $G$, the map

\[
\begin{align*}
N & \to \mathbb{R} \\
n & \to \|\Phi^n(c)\|
\end{align*}
\]

grows either polynomially or at least exponentially.

In the above statement, the growth type of a map is understood in the following sense: given two functions $f_1, f_2 : N \to \mathbb{R}$, we say that $f_1$ grows at least as fast as $f_2$.

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most like $f_2$ (or $f_2$ grows at least like $f_1$) and we write $f_1 \prec f_2$, if there exists $C > 0$ such that for every $n \in \mathbb{N}$,

$$f_1(n) \leq C f_2(n) + C.$$ 

We say that $f_1$ grows like $f_2$ (or $f_1$ and $f_2$ are equivalent) and we write $f_1 \asymp f_2$, if $f_1 \prec f_2$ and $f_2 \prec f_1$. A map $f: \mathbb{N} \to \mathbb{R}$ grows polynomially (respectively exponentially) if it grows like $n \to n^d$, for some exponent $d \in \mathbb{N}$, (respectively $n \to \lambda n$, for some $\lambda > 1$).

If $G$ is a free abelian group, then $\Phi$ can be seen as a matrix $M$ in $\text{GL}(n, \mathbb{Z})$ and Theorem 1.1 follows from the Jordan decomposition of $M$. When $G$ is a surface group, the statement is a consequence of the Nielsen-Thurston classification of mapping classes; see for instance [9, Theorem 13.2]. For non-abelian free groups, a complete classification of the possible growth types of $n \to \|\Phi^n(c)\|$ has been given by Levitt [15]. It builds on the theory of train-track representatives developed by Bestvina and Handel [3]. The case of torsion-free hyperbolic and toric relatively hyperbolic groups is handled in [8].

In all these examples the growth type of $\Phi$ directly relates to the algebraic/geometric properties of $\Phi$. Let us mention one example. Let $\Sigma$ be a closed compact surface of genus at least 2 and $G$ its fundamental group. Let $f \in \text{Mod}(\Sigma)$ be an element of the mapping class group and $\Phi$ the corresponding outer automorphism of $G$. Let $\alpha$ be a closed curve on $\Sigma$ and $c$ the associated conjugacy class of $G$. Roughly speaking, if $n \to \|\Phi^n(c)\|$ grows exponentially then $\alpha$ crosses an $f$-invariant subsurface on which $f$ behaves as a pseudo-Anosov element. In addition the exponential growth rate of $n \to \|\Phi^n(c)\|$ directly relates to the stretching factor of the underlying pseudo-Anosov homeomorphism. On the other hand, if $n \to \|\Phi^n(c)\|$ is linear then $c$ lies on a subsurface on which $f$ is equal to a product of Dehn twists along pairwise disjoint curves.

**Main results.** This article goes in the opposite direction. Our goal is to produce examples of groups where such growth dichotomy fails. More generally, we try to understand what are the possible growth types for the map $n \to \|\Phi^n(c)\|$. In order to state our main result, we first recall the definition of a length function.

**Definition 1.2.** A length function on a group $H$ is a map $L: H \to \mathbb{N}$ with the following properties

(i) $L(h) = 0$ if and only if $h = 1$;

(ii) $L(h) = L(h^{-1})$, for every $h \in H$;

(iii) $L(h_1 h_2) \leq L(h_1) + L(h_2)$, for every $h_1, h_2 \in H$;

(iv) there exists $\lambda > 0$ such that for every $r \in \mathbb{N}$, the set $\{h \in H : L(h) \leq r\}$ contains at most $\lambda^r$ elements.

For instance, the map $n \to |n|^\alpha$ is a length function on $\mathbb{Z}$, for every $\alpha \in (0, 1)$.
Theorem 1.3. Let \( \mathcal{L} \) be a finite collection of computable length functions on \( \mathbb{Z} \). There exist a finitely generated group \( G \) and an automorphism \( \Phi \in \text{Out}(G) \) with the following property. For every non-trivial conjugacy class \( c \) of \( G \), the map
\[
T_c : \mathbb{Z} \to \mathbb{R}, \quad n \to \ln \| \Phi^n(c) \|
\]
is equivalent to a linear combination of the elements of \( \mathcal{L} \). Conversely for every \( L \in \mathcal{L} \) there exists a conjugacy class \( c \) of \( G \) such that \( T_c \) is equivalent to \( L \).

Recall that there are infinitely many inequivalent computable length functions (actually countably many). Hence Theorem 1.3 provides numerous examples of exotic automorphisms. As we explain in the last section, our result is more than just a theoretical existence statement. In many cases, one can build an explicit presentation of such a group \( G \).

Unlike in Theorem 1.1, our techniques are not accurate enough to get a precise behavior of the map \( n \to \| \Phi^n(c) \| \). We only control its logarithm. For instance the last part of Theorem 1.3 can be reformulated as follows: for every \( L \in \mathcal{L} \), there exist a conjugacy class \( c \) of \( G \), \( \lambda_1, \lambda_2 > 1 \) and \( A > 0 \) such that for every \( n \in \mathbb{N} \), we have
\[
\frac{1}{A} \lambda_1^{L(n)} \leq \| \Phi^n(c) \| \leq A \lambda_2^{L(n)}.
\]

Nevertheless, as we explain below, the map \( n \to \ln \| \Phi^n(c) \| \) has a natural interpretation in terms of the Lipschitz distance on \( \text{Out}(G) \).

In the second part of our work, we stop focusing on a single outer automorphism and adopt a more global point of view. The word metric on a finitely generated group \( G \) induces an asymmetric left-invariant pseudo-metric on \( \text{Out}(G) \), called the Lipschitz metric, and defined as follows: for every \( \Phi_1, \Phi_2 \in \text{Out}(G) \),
\[
d_{\text{Lip}}(\Phi_1, \Phi_2) = \ln \left( \sup_c \frac{\| \Phi_2^{-1}(c) \|}{\| \Phi_1^{-1}(c) \|} \right)
\]
where \( c \) runs over all non-trivial conjugacy classes of \( G \). This distance is directly inspired by the Thurston metric on the Teichmüller space [21] and the Lipschitz metric on the Culler-Vogtmann outer space [10, 17].

Remark.

1. Let \( S \) and \( S' \) be two generating sets of \( G \). It is well-known that the corresponding word metrics on \( G \) are bi-Lipschitz equivalent. Thus the associated Lipschitz distances on \( \text{Out}(G) \) are quasi-isometric. As we work up to quasi-isometry, the choice of a generating set for \( G \) does not really matter.

2. Note that if \( \text{Out}(G) \) is finitely generated, the Lipschitz metric \( d_{\text{Lip}} \) is a priori not quasi-isometric to the word metric \( d \). Consider for instance the
free group $G = F(a, b)$. Let $\varphi$ be the automorphism characterized by

$$
\varphi: \quad F(a, b) \rightarrow F(a, b)
\begin{align*}
    a &\rightarrow a \\
    b &\rightarrow ba
\end{align*}
$$

and $\Phi$ its outer class. Note that $\Phi$ can be interpreted as a Dehn twist on a puncture torus. It follows that $n \rightarrow d_{\text{Lip}}(1, \Phi^n)$ grows logarithmically. However $\text{Out}(G) = \text{GL}_2(\mathbb{Z})$ is hyperbolic, thus $n \rightarrow d(1, \Phi^n)$ grows linearly. Hence $d_{\text{Lip}}$ and $d$ are not quasi-isometric.

It is known that any finitely presented group $Q$ is isomorphic to $\text{Out}(G)$ for some suitable finitely generated group $G$; see for instance Matumoto [16]. The next statement can be seen as a geometric analogue of this realization result.

**Theorem 1.4.** For every finitely presented group $Q$, there exists a finitely generated group $G$ such that $\text{Out}(G)$ endowed with the Lipschitz metric is quasi-isometric to $Q$.

**Strategy.** Our work relies on the Rips construction [20]. If $Q$ is a finitely presented group, Rips explains how to build a non-elementary hyperbolic group $G$ such that $Q$ is the quotient of $G$ by a finitely generated normal subgroup $N$. The action of $G$ on $N$ by conjugation defines a map $G \rightarrow \text{Aut}(N)$ which induces a homomorphism $\chi: Q \rightarrow \text{Out}(N)$. It turns out that this map is a quasi-isometric embedding, provided $N$ is not finite. The proof has two steps.

(i) Classical arguments show that the word metric on $Q$ “dominates” the Lipschitz metric on $\text{Out}(N)$ (Proposition 2.2).

(ii) Building on the fact that the divergence function in a hyperbolic space is exponential, one provides an estimate from below of the Lipschitz metric (Proposition 2.3).

Once the relation between the word metric on $Q$ and the Lipschitz metric on $\text{Out}(N)$ is established, we vary the group $Q$ to get the above theorems:

- According to Ol’shanskii [18], for any computable length function $L$ on $\mathbb{Z}$, there exist a finitely presented group $Q$ and an element $q$ such that $L$ is equivalent to the word metric of $Q$ restricted to $\langle q \rangle$. If $\langle q \rangle$ is distorted in $Q$, then the corresponding automorphism $\Phi = \chi(q)$ has an exotic growth type, which yields Theorem 1.3.

- Bumagin and Wise proved that the Rips construction can be refined so that the map $\chi: Q \rightarrow \text{Out}(N)$ is actually an isomorphism [6], leading to Theorem 1.4.
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2 Exploiting a short exact sequence

Notations. Let $G$ be a group and $d$ a left-invariant (pseudo-)distance on $G$. In order to lighten the notations, for every $g \in G$, we write $|g| = d(1,g)$ for the length of $g$. It induces a norm on the set of conjugacy classes of $G$: if $c$ is a conjugacy class, $\|c\|$ is the length of the shortest element in $c$. Given $g \in G$, we make an abuse of notation and write also $\|g\|$ for the norm of the conjugacy class of $g$.

In the course of the article, we will work with several metrics. If one needs to avoid ambiguity, we will use indices of the form $d_*$ to distinguish between them. The associated length function and norm will be then denoted by $|\cdot|$ and $\|\cdot\|_*$ respectively.

We now fix once for all a short exact sequence of groups

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

where $N$ and $Q$ are finitely generated, whereas $G$ is non-elementary hyperbolic in the sense of Gromov [12]. We endow $N$ and $Q$ with the word metric. The distance on $N$ induces a norm on the set of conjugacy classes in $N$, denoted by $\|\cdot\|$. The action by conjugation of $G$ on $N$ defines a map $G \rightarrow \text{Aut}(N)$, which induces a morphism $\chi: Q \rightarrow \text{Out}(N)$. The goal of this section is to prove the following statement.

Proposition 2.1. If $N$ is infinite, then the map $\chi: Q \rightarrow \text{Out}(N)$ is a quasi-isometric embedding. More precisely, there exist $\kappa \geqslant 1$ and $\ell \geqslant 0$ such that for every $q, q' \in Q$,

$$\kappa^{-1}d(q', q) - \ell \leqslant d_{\text{Lip}}(\chi(q'), \chi(q)) \leqslant \kappa d(q', q).$$

We start with a very general fact which does not require any kind of negative curvature.

Proposition 2.2. There exists $\kappa > 0$ such that for every $q, q' \in Q$,

$$d_{\text{Lip}}(\chi(q'), \chi(q)) \leqslant \kappa d(q', q).$$

(1)

Proof. Since both metrics on $Q$ and $\text{Out}(N)$ are left-invariant, it suffices to prove the inequality when $q' = 1$. Let $S$ be the generating set of $Q$ defining the word metric. We let

$$\kappa = \max_{s \in S \cup S^{-1}} |\chi(s)|_{\text{Lip}}.$$
Let \( q \in Q \). We decompose \( q \) as a geodesic word \( q = s_1s_2\ldots s_m \) over the alphabet \( S \cup S^{-1} \), i.e. \( m = |q| \). It follows from the triangle inequality that

\[
|\chi(q)|_{\text{Lip}} \leq \sum_{k=1}^{m} d_{\text{Lip}}(\chi(s_1\ldots s_{k-1}), \chi(s_1\ldots s_k)) \leq \sum_{k=1}^{m} |\chi(s_k)|_{\text{Lip}} \leq \kappa m. \hspace{1cm} \square
\]

In order to bound from below the Lipschitz metric we start by studying the behavior of a single conjugacy class under the action of \( \text{Out}(G) \).

**Proposition 2.3.** There exists \( \kappa > 0 \) with the following property. Let \( c \) be a conjugacy class of \( N \), whose elements have infinite order. Then there exists \( \ell \geq 0 \), such that for all \( q \in Q \), we have

\[
\ln \|\Phi(c)\| \geq \kappa |q| - \ell, \hspace{1cm} \text{where} \hspace{1cm} \Phi = \chi(q).
\]

**Proof.** We start by setting up all the useful objects for this proof. Let \( S_N \) and \( S_Q \) be the respective generating sets of \( N \) and \( Q \) used to define the word metric \( d_N \) and \( d_Q \) on these groups. We fix a generating set \( S_G \) of \( G \) which contains \( S_N \) and whose image in \( Q \) lies in \( S_Q \), and endow \( G \) with the corresponding word metric \( d_G \) (or simply \( d \)). It follows that the maps \( N \rightarrow G \) and \( G \rightarrow Q \) are 1-Lipschitz.

Let \( X \) (respectively \( Y \)) be the Cayley graph of \( G \) (respectively \( N \)) with respect to \( S_G \) (respectively \( S_N \)). We identify the vertex set of \( X \) with \( G \). Although it may not be unique, given two points \( x, y \in X \), we write \([x, y]\) for some geodesic joining them. Since \( S_N \) is contained in \( S_G \), there exists a (unique) 1-Lipschitz \( N \)-equivariant graph embedding \( \iota : Y \rightarrow X \) sending the identity of \( N \) to the one of \( G \). For simplicity we don’t make any distinction (as a set) between \( Y \) and its image in \( X \).

By assumption the group \( G \) is hyperbolic, i.e. there exists \( \delta \in \mathbb{R}_+ \), such that every geodesic triangle in \( X \) is \( \delta \)-thin, or equivalently for every \( x, y, z \in X \), the geodesic \([x, z]\) lies in the \( \delta \)-neighborhood of \([x, y] \cup [y, z]\). Without loss of generality we can assume that \( \delta > 1 \). We refer the reader to Gromov’s seminal paper [12] or [7, 11, 5] for the background on hyperbolic geometry.

The next statement is a straightforward reformulation of [5, Chapter III.H, Proposition 1.6] expressing that the divergence function in a hyperbolic space is exponential.

**Lemma 2.4.** There exist \( A > 0 \) and \( \lambda > 1 \) with the following property. Let \( x, y \in X \) and \( p \in [x, y] \). Let \( r \geq 0 \). Let \( \gamma \) be a rectifiable path from \( x \) to \( y \). If \( \gamma \) does not intersect the ball \( B(p, r) \), then its length is at least \( AR^\lambda \).

Let \( c \) be a conjugacy class of \( N \) whose elements have infinite order. We choose an element \( u \in N \) in \( c \) with minimal length in \( N \), i.e. \( |u|_N = ||c||_N \). It follows from the Morse Lemma, that the set

\[
A(u) = \{ x \in X : d(x, ux) \leq |u|_G + 4\delta \},
\]

is quasi-isometric to a line, on which the group \( \langle u \rangle \) roughly acts by translations. Compare for instance with [7, Chapitre 10, § 7]. In particular, there exists \( a \geq 0 \),
such that the diameter of $A(u)/u$ is less than $a$. Without loss of generality, we can assume that $a > |u|_G + 2\delta$.

Let $q \in Q$ and $\Phi = \chi(q)$. We fix $v \in N$ in the conjugacy class $\Phi(c)$ such that $|v|_N = ||\Phi(c)||_N$. It follows from the definition of $\chi: Q \to \text{Out}(N)$, that there exists a pre-image $h \in G$ of $q$ such that $v = huh^{-1}$. We fix such an element $h$ which minimizes $|h|_G$.

**Lemma 2.5.** There exists a point $x \in [uh^{-1}, h^{-1}]$ such that $d(1, x) \leq a + 2\delta$.

**Proof.** If $d(1, h^{-1}) \leq a$, one simply takes $x = h^{-1}$. Suppose that $d(1, h^{-1}) > a$. We write $z$ for the point on a geodesic $[1, h^{-1}]$ such that $d(1, z) = a$ (see Figure 1). It suffices to show that $z$ is $2\delta$-close to $[uh^{-1}, h^{-1}]$. Recall that geodesic triangles in $X$ are $\delta$-thin. Hence $z$ is $2\delta$-close to one of the following geodesics $[1, u]$, $[u, uh^{-1}]$, or $[uh^{-1}, h^{-1}]$. Let us rule out the first two cases. Assume that $z$ is $2\delta$-close to $[1, u]$. It follows from the triangle inequality that $a \leq d(1, z) \leq |u| + 2\delta$, which contradicts our choice of $a$. On the other hand, if $z$ is $2\delta$-close to a point $y$ on $[u, uh^{-1}]$, the triangle inequality yields

$$d(uz, z) \leq |d(u, uz) - d(u, y)| + 2\delta \leq |d(1, z) - d(u, y)| + 2\delta \leq d(1, u) + d(z, y) + 2\delta.$$
Consequently $d(uz, z) \leq |u| + 4\delta$, i.e. $z$ belongs to $A(u)$. According to our choice of $a$, there exists $m \in \mathbb{Z}$, such that $d(1, u^mz) < a \leq d(1, z)$. We let $h_0 = hu^{-m}$. Note that $h$ and $h_0$ have the same image in $Q$ whereas $h_0uh_0^{-1} = hhu^{-1} = v$. Moreover, it follows from the triangle inequality that

$$|h_0| \leq d(1, u^mz) + d\left(u^mz, u^m h^{-1}\right) < d(1, z) + d(z, h^{-1}) \leq |h|,$$

which contradicts our definition of $h$. 

Let $\gamma : [a, b] \to Y$ be a geodesic of $(Y, d_N)$ joining 1 to $v$. We identify $\gamma$ with its image in $X$. As a path of $X$, $\gamma$ is no more geodesic. Nevertheless its length is still $\text{Length}(\gamma) = \|\Phi(c)\|_N$.

**Lemma 2.6.** The path $h^{-1}\gamma$ does not intersect the ball of radius $r = |q|Q - a - 3\delta$ centered at $x$.

**Proof.** By construction $h\gamma^{-1}$ is contained in $h^{-1}Y$. Thus

$$d(h, Y) \leq d\left(h, h^{-1}Y\right) \leq d\left(h, h^{-1}x\right) + d\left(x, h^{-1}Y\right) \leq d\left(x, h^{-1}\gamma\right) + a + 2\delta.$$ 

Hence it suffices to prove that $d(h, Y) \geq |q|Q - \delta$. Recall that $Y$ is an embedded copy of the Cayley graph of $N$ in $G$. If $p$ is a point in $Y$, there exist $w \in N$ seen as a vertex of $Y$ such that $d(p, w) \leq \delta$. Observe that $w^{-1}h$ is a pre-image of $q$. Since the projection $G \to Q$ is 1-Lipschitz, we get $|q|Q \leq d(h, w) \leq d(h, p) + \delta$. This inequality holds for every $p \in Y$ which completes the proof.

Since $v = huh^{-1}$, the path $h^{-1}\gamma$ joins $h^{-1}$ to $uh^{-1}$ and avoid the ball centered at $x$ of radius $r$. It follows from Lemma 2.4 that

$$\|\Phi(c)\|_N = \text{Length}(\gamma) = \text{Length}(h^{-1}\gamma) \geq A\lambda.$$ 

It can be rewritten $\ln \|\Phi(c)\|_N \geq \kappa |q|Q - \ell$, where

$$\kappa = \ln \lambda \quad \text{and} \quad \ell = (a + 3\delta) \ln \lambda - \ln A$$

do not depend on $q$. Note also that $\kappa$ does not depend on $c$. Hence the proof is complete.

**Proof of Proposition 2.1.** As in Proposition 2.2, it suffices to prove the inequalities for $q' = 1$. The second inequality is provided by Proposition 2.2. Let us focus on the first one. Since $N$ is infinite, it contains a conjugacy class $c_0$ whose elements have infinite order [11, Chapitre 8, Corollaire 36]. By Proposition 2.3, there exist $\kappa > 0$ and $\ell \geq 0$ such that for every $q \in Q$,

$$\ln \|\Phi(c_0)\| \geq \kappa |q| - \ell, \quad \text{where} \quad \Phi = \chi(q).$$

In particular,

$$|\chi(q)|_{\text{Lip}} \geq \ln \left(\frac{\|\Phi(c_0)\|}{\|c_0\|}\right) \geq \kappa |q| - \left(\ell + \ln \|c_0\|\right).$$

\[\square\]
3 Applications

Our applications combine two ingredients: the Rips construction and the existence of groups with distorted elements.

Rips construction. In order to produce short exact sequences as studied in Section 2, we use the Rips construction.

**Theorem 3.1** (Rips [20]). Let $Q$ be a finitely presented group. There exists a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

where $G$ is a hyperbolic group and $N$ a finitely generated non-elementary normal subgroup of $G$.

Rips construction is very flexible. In particular, one can strengthen the conclusions by adding one of the following requirements (some of them are incompatible, hence they cannot be all simultaneously satisfied):

- The group $G$ is torsion-free; see Rips [20].
- The group $N$ is the quotient of a prescribed non-elementary hyperbolic group. In particular one can choose $N$ to have Kazhdan’s property (T); see Belegradek-Osin [2].
- The group $G$ and thus $N$ is residually finite; see Wise [22]. Actually the group $G$ is obtained by mean of small cancellation theory. It follows from the work of Wise [23, 24], Agol [1] and Haglund-Wise [14] that $G$ and thus $N$ are linear.
- The induced morphism $Q \rightarrow \text{Out}(N)$ is an isomorphism; see Bumagin-Wise [6].

**Distortion.** Let $H$ be a finitely generated group. Two length functions $L_1$ and $L_2$ on $H$ are strongly equivalent if and only if there exists $C > 0$ such that for every $h \in H$,

$$\frac{1}{C} L_1(h) \leq L_2(h) \leq CL_1(h).$$

If $H$ is a subgroup of a finitely generated group $G$, then the word metric of $G$ restricted to $H$ provides a length function on $H$. Ol’shanskii proved that every length function on $G$ essentially arises in this way [19]. Moreover he gives a complete description of the distortion of subgroups of finitely presented group.

**Theorem 3.2** (Ol’shanskii [18, Theorem 2]). Let $H$ be a finitely generated group and $L \colon H \rightarrow \mathbb{N}$ a computable length function on $H$. Then $H$ embeds in a finitely presented group $G$ such that $L$ is strongly equivalent to the word metric of $G$ restricted to $H$. 

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Exotic growth.

**Theorem 3.3.** Let $L : \mathbb{Z} \to \mathbb{N}$ be computable length function. There exist a finitely generated group $N$ and an outer automorphism $\Phi \in \text{Out}(N)$ such that for every non-trivial conjugacy class $c$ of $N$ the map

$$
N \to \mathbb{R} \\
n \to \ln \|\Phi^n(c)\|
$$

is equivalent to $L$.

**Proof.** According to Theorem 3.2, there exist a finitely presented group $Q$ and an element $q \in Q$ such that the map $L_q : \mathbb{Z} \to N$ sending $n \to |q^n|$ is equivalent to $L$. Using the Rips construction, we produce a short exact sequence

$$1 \to N \to G \to Q \to 1$$

where $G$ is a torsion-free hyperbolic group and $N$ a non-elementary finitely generated subgroup. Let $\Phi$ be the image of $q$ by the morphism $\chi : Q \to \text{Out}(N)$ induced by this short exact sequence. Combining Propositions 2.2 and 2.3 we observe that for every non-trivial conjugacy class $c$ of $N$, the map $N \to \mathbb{R}$ sending $n$ to $\ln \|\Phi^n(c)\|$ is equivalent to $L_q$ hence to $L$. $\square$

**Remark 3.4.** Using one of the aforementioned variations of the Rips construction, one can build a group $N$ as in Theorem 3.3 which satisfies one of the following additional assumptions: $N$ has Kazhdan Property (T), $N$ is linear, $N$ is residually finite, etc.

**Remark 3.5.** Ol’shanskiǐ’s Theorem 3.2 relies on a rather heavy construction, involving among others a variation on the Higmann embedding theorem. In practice though, it is easy to produce explicit examples of automorphisms with an exotic behavior. Consider for instance the discrete Heisenberg group whose presentation is

$$Q = \langle a, b \mid [a, [a, b]] = 1, [b, [a, b]] = 1 \rangle.$$  

It is known that the commutator $q = [a, b]$ is distorted. More precisely $n \to |q^n|$ behaves like $n \to \sqrt{n}$. Following Rips’ construction, one can choose for $G$ the group generated by $a, b, x, y$ subject to the following relations

$$axa^{-1} = xy^kxy^{k+1}xy^{k+2} \ldots xy^{2k-1},$$

$$a^{-1}xa = xy^{2k}xy^{2k+1}xy^{2k+2} \ldots xy^{3k-1},$$

$$bxb^{-1} = xy^{3k}xy^{3k+1}xy^{3k+2} \ldots xy^{4k-1},$$

$$b^{-1}xb = xy^{4k}xy^{4k+1}xy^{4k+2} \ldots xy^{5k-1},$$

$$[a, [a, b]] = xy^{5k}xy^{5k+1}xy^{5k+2} \ldots xy^{6k-1},$$

$$[b, [a, b]] = xy^{6k}xy^{6k+1}xy^{6k+2} \ldots xy^{7k-1},$$

where $k$ is a sufficiently large exponent. The group $N$ given by Theorem 3.3 is then the (normal) subgroup of $G$ generated by $x$ and $y$. In particular, if $\Phi$
stands for the image of \( q \) under the morphism \( \chi : Q \to \text{Out}(N) \), then for every non-trivial conjugacy class \( c \), the map \( n \to \ln \| \Phi^n(c) \| \) behaves like \( n \to \sqrt{n} \).

Other “explicit” examples of distorted elements in a finitely presented group can be found in Gromov [13, Chapter 3] or Bridson [4].

We now prove the theorems announced in the introduction.

**Proof of Theorem 1.3.** We write \( L_1, \ldots, L_m \) for the elements of \( \mathcal{L} \). For every \( i \in \{1, \ldots, m\} \) we build using Theorem 3.3 a finitely generated group \( N_i \) and an automorphism \( \Phi_i \in \text{Out}(N_i) \) such that for every non-trivial conjugacy class \( c \) of \( N_i \) the map \( N \to \mathbb{R} \) sending \( n \) to \( \ln \| \Phi^n_i(c) \| \) is equivalent to \( L_i \). We now let
\[
G = N_1 \times N_2 \times \cdots \times N_m.
\]

We endow \( G \) with the word metric relative to the generating set \( S = S_1 \cup \cdots \cup S_m \) where \( S_i \) is a finite generating set of \( N_i \). For each \( i \in \{1, \ldots, m\} \), we pick a representative \( \varphi_i \in \text{Aut}(N_i) \) of \( \Phi_i \). We consider the automorphism \( \varphi \in \text{Aut}(G) \) whose restriction to the factor \( N_i \) is \( \varphi_i \), and write \( \Phi \) for the outer class of \( \varphi \). One checks easily that if \( g = (g_1, \ldots, g_m) \) is an element of \( G \), then for every \( n \in \mathbb{Z} \),
\[
\| \varphi^n(g) \|_G = \| \varphi_1^n(g_1) \|_{N_1} + \cdots + \| \varphi_m^n(g_m) \|_{N_m}.
\]

Let \( c \) be the conjugacy class of \( g \) in \( G \). We denote by \( I \) the set which consists of all \( i \in \{1, \ldots, m\} \), such that \( g_i \) is non-trivial. It follows from the previous discussion that \( T_c \) is equivalent to
\[
\sum_{i \in I} L_i
\]

Conversely for every \( i \in \{1, \ldots, m\} \), \( L_i \) is equivalent to \( T_{c_i} \), where \( c_i \) is the conjugacy class in \( G \) of any non-trivial element in \( N_i \).

**Proof of Theorem 1.4.** According to Bumagin-Wise [6] there exists a short exact sequence
\[
1 \to N \to G \to Q \to 1
\]

where \( G \) is a hyperbolic group and \( N \) a non-elementary finitely generated subgroup such that the corresponding map \( Q \to \text{Out}(N) \) is an isomorphism. By Proposition 2.1, \( Q \to \text{Out}(N) \) is a surjective quasi-isometric embedding, hence a quasi-isometry.

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Rémi Coulon
Univ Rennes, CNRS
IRMAR - UMR 6625
F-35000 Rennes, France
remi.coulon@univ-rennes1.fr
http://rcoulon.perso.math.cnrs.fr