Standard-Model Bundles on Non-Simply Connected Calabi–Yau Threefolds

Ron Donagi
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104–6395, USA.

Burt A. Ovrut
Department of Physics, University of Pennsylvania, Philadelphia, PA 19104–6396, USA.

Tony Pantev
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104–6395, USA.

Daniel Waldram
Theory Division, CERN CH-1211, Geneva 23, Switzerland.
Department of Physics, The Rockefeller University, 1230 York Avenue, New York, NY 10021.

Abstract: We give a proof of the existence of $G = SU(5)$, stable holomorphic vector bundles on elliptically fibered Calabi–Yau threefolds with fundamental group $\mathbb{Z}_2$. The bundles we construct have Euler characteristic 3 and an anomaly that can be absorbed by M-theory five-branes. Such bundles provide the basis for constructing the standard model in heterotic M-theory. They are also applicable to vacua of the weakly coupled heterotic string. We explicitly present a class of three family models with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Keywords: Standard Model, Heterotic M-theory
1. Introduction

Hořava–Witten theory \[\text{[1]}\] can be consistently compactified on a Calabi–Yau threefold with non-vanishing four-form “G-flux” \[\text{[2]}\]. It was shown in \[\text{[3]}\] that this reduction leads to a new low-energy limit of M-theory. This limit consists of a five-dimensional “bulk” space with \(N = 1\) local supersymmetry, bounded by two, four-dimensional, \(N = 1\) supersymmetric \(\mathbb{Z}_2\)-orbifold fixed planes. Furthermore, the theory can admit BPS fivebranes located in the bulk space, each with two spacelike directions wrapped on a holomorphic curve in the Calabi–Yau threefold \[\text{[2, 4]}\]. The worldvolumes of these wrapped fivebranes possess \(N = 1\) supersymmetry. Hence, this limit of M-theory, called heterotic M-theory, provides an explicit description of a “brane” universe, derived directly from a fundamental theory.

In addition to its brane structure, heterotic M-theory can also account for much of phenomenological particle physics. In keeping with the brane context, three families of \(N = 1\) supersymmetric quarks and leptons can be shown to exist naturally on one of the orbifold fixed planes, which we call the “observable” brane. These transform under either grand unified gauge groups, such as \(SO(10)\) and \(SU(5)\), or under the standard model gauge group,
\[ SU(3)_C \times SU(2)_L \times U(1)_Y. \] The supersymmetric matter and the associated gauge groups arise from the higher dimensional Hořava–Witten theory as follows. Hořava–Witten theory consists of an eleven-dimensional \( N = 1 \) locally supersymmetric bulk space bounded by two ten-dimensional \( \mathbb{Z}_2 \)-orbifold fixed planes exhibiting \( N = 1 \) supersymmetry. It was shown in \[ ] that anomaly cancellation requires the existence of an \( N = 1 \), \( E_8 \) Yang–Mills supermultiplet on each of the orbifold fixed planes. This theory is then dimensionally reduced on a Calabi–Yau threefold, \( Z \), with a non-vanishing four-form flux as required by anomaly cancellation. On either one of the orbifold fixed planes, the result at low energy is a four-dimensional, \( N = 1 \) supersymmetric theory with matter and gauge group content arising from the “decomposition” of the \( E_8 \) supermultiplet under the dimensional reduction. This decomposition is entirely controlled by the vacuum structure of the \( E_8 \) gauge fields on the Calabi–Yau threefold. As discussed in \[ ], some subgroup \( G \subseteq E_8 \) of the gauge fields can be non-vanishing on \( Z \). These “\( G \)-instantons” will preserve \( N = 1 \) supersymmetry as long as they satisfy the Hermitian Yang–Mills equations. However, the low energy gauge group is altered by these instantons, being spontaneously broken from \( E_8 \) to \( H \), where \( H \subseteq E_8 \) is the commutant of the instanton structure group \( G \), that is, the largest subgroup in \( E_8 \) such that \([H, G] = 0\). This mechanism, originally stated in \[ ], allows GUT groups, such as \( SO(10) \) and \( SU(5) \) (the commutants of \( G = SU(4) \) and \( SU(5) \) respectively), and the standard model gauge group (the commutant of \( G = SU(5) \times \mathbb{Z}_2 \), for example) to appear on the observable brane. Furthermore, the decomposition of the \( E_8 \) Yang–Mills supermultiplet under \( G \) into \( H \) multiplets determines the structure of matter on the observable brane.

The construction of heterotic M-theory models with grand unified gauge groups \( H \) and three families of matter is relatively straightforward, and has been discussed in \[ \]]. Specifically, such theories require the construction of \( G \)-instantons on a simply connected elliptically fibered Calabi–Yau threefold with a zero section. Such instantons can be computed using the theorem of Donaldson \[ ] and Uhlenbeck–Yau \[ ], which relates them to polystable holomorphic vector bundles on \( Z \), and extensions \[ \] of the work of Friedman, Morgan and Witten \[ ], Donagi \[ ] and Bershadsky et al. \[ ], which constructs vector bundles via the method of spectral covers. However, constructing \( G \)-instantons that lead to the standard model gauge group \( H = SU(3)_C \times SU(2)_L \times U(1)_Y \) and matter content is considerably more difficult. The reason is that, in addition to vector bundles with continuous structure group \( G \), one must introduce Wilson lines into the theory \[ \]. The existence of Wilson lines, however, requires that the Calabi–Yau threefold \( Z \) have non-trivial fundamental group. Such manifolds do not admit a zero section and the vector bundle construction of \[ \], \[ \] no longer applies.

To overcome this problem, it is necessary to give a method for computing polystable, holomorphic vector bundles over a non-simply connected Calabi–Yau threefold. A major step in this direction was taken in \[ ], where a spectral cover formalism applicable to torus fibrations without zero section was presented. (Similar constructions were considered in \[ \].) The torus fibrations, \( Z \), were constructed as a quotient, \( Z = X/\tau_X \), where \( X \) is a simply connected Calabi–Yau threefold with two sections and \( \tau_X \) is a freely acting involution inter-
changing the sections. Finding phenomenologically acceptable heterotic M-theory vacua in this context amounts to finding \( \tau_X \)-invariant stable vector bundles on \( X \) satisfying certain conditions (anomaly cancellation and 3-families). These conditions involve only the charges of the corresponding vacua, which mathematically are encoded in the Chern classes. The \( \tau_X \)-invariance imposes non-trivial restrictions on the charges. In \cite{[18]} we exhibited an infinite collection of \( \tau_X \)-invariant admissible charges and showed that each of those is realized by a non-empty family of vector bundles on \( X \). The next step is to prove the existence of actual \( \tau_X \)-invariant vacua in such families. This is the subject of the present work.

Such a proof is considerably more difficult than showing the invariance of the Chern classes alone. Rather than consider the bundles presented in \cite{[18]}, for various technical reasons, it is expedient to consider a different class of vector bundles over a Calabi–Yau threefold with \( \pi_1(Z) = \mathbb{Z}_2 \). These bundles also lead to the standard model, but have a structure that lends itself more easily to discussing invariance under \( \mathbb{Z}_2 \). In this paper, we explicitly present polystable, holomorphic vector bundles of this type and prove that they are \( \mathbb{Z}_2 \)-invariant. This is a fundamental step in demonstrating that the standard model can arise in heterotic M-theory. This construction, and the proof of the \( \mathbb{Z}_2 \) invariance, is necessarily of a technical nature but is sufficiently important for phenomenological physics that we synopsize it in this paper. The complete proof, with all mathematical details, is presented in a companion paper \cite{[21]}

The types of invariant bundles discussed here can be extended to larger freely acting automorphism groups of \( X \), such as \( \mathbb{Z}_3 \), \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and so on. We will show in a future paper that standard-like models arising from \( SO(10) \) and other grand unified groups can be constructed with the possibility of suppressed nucleon decay. Finally, we would like to emphasize that, although the vector bundles discussed in \cite{[1,2,18]} and this paper are within the context of heterotic M-theory, these bundles are equally applicable to the construction of new, phenomenologically interesting vacua in the weakly coupled heterotic string.

2. Outline of the Construction

Let us start by summarizing the problem we have to solve. Recall from the introduction that to construct an \( N = 1 \) vacuum in four-dimensions, we must compactify on a Calabi–Yau threefold \( Z \) and choose \( E_8 \) gauge fields in a \( G \subseteq E_8 \) polystable holomorphic vector bundle \( \mathcal{V} \) on \( Z \).

We would like to construct models with the standard model gauge group and three families of charged matter. In this paper, we will break the gauge group through an \( SU(5) \) GUT group. We have

\[
E_8 \xrightarrow{\text{bundle}} \mathcal{V} \xrightarrow{\mathbb{Z}_2 \text{ Wilson line}} SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{2.1}
\]

By choosing an \( SU(5) \) vector bundle \( \mathcal{V} \) on the Calabi–Yau manifold \( Z \), we break the preserved gauge group to the commutant \( SU(5) \). By choosing a \( \mathbb{Z}_2 \) Wilson line, we can then further break this down to the standard model gauge group \( SU(3)_C \times SU(2)_L \times U(1)_Y \).
We recall that there are two additional conditions [6, 7]. First, the requirement of three families translates into a condition on the third Chern class

\[ N_{\text{gen}} = \frac{1}{2} c_3(\mathcal{V}) = 3. \]  

(2.2)

Second, in order to cancel anomalies the orbifold planes and the fivebranes must be sources of G-flux. The condition that the net charge vanishes becomes

\[ c_2(\mathcal{V}) + [W] = c_2(TZ), \]  

(2.3)

where \([W]\) is the total cohomology class of the holomorphic curves on which the fivebranes are wrapped. To describe physical branes, the class \([W]\) must be effective.

In general, then, we need to satisfy the following conditions

**Supersymmetry:** \(Z\) is a Calabi–Yau threefold. \(\mathcal{V}\) is a polystable, holomorphic \(SU(5)\) bundle,

**Wilson Line:** \(Z\) has, at least, \(\pi_1(Z) = \mathbb{Z}_2\),

**Anomaly Cancellation:** \(c_2(TZ) - c_2(\mathcal{V})\) must be an effective class in \(Z\),

**Three Families:** \(c_3(\mathcal{V}) = 6\),

in order to have a realistic model.

The simplest way to construct a suitable Calabi–Yau threefold \(Z\) is as a quotient [16]. Let \(X\) be a smooth Calabi–Yau threefold with trivial fundamental group. Suppose, in addition, we have an involution,

\[ \tau_X : X \to X, \]  

(2.4)

with \(\tau_X^2 = \text{id}\), which is freely acting, that is, has no fixed points, and which preserves the holomorphic 3-form. The quotient space \(Z\) formed by identifying points related by the involution,

\[ Z = X/\tau_X, \]  

(2.5)

is then a smooth Calabi–Yau threefold with \(\pi_1(Z) = \mathbb{Z}_2\).

Rather than construct the bundle \(\mathcal{V}\) directly on the quotient \(Z = X/\tau_X\), it is generally easier to construct it from a bundle \(V\) on \(X\). If \(V\) is \(\tau_X\)-invariant, that is

\[ \tau_X^* V \cong V, \]  

(2.6)

then, as long as \(V\) is stable, it will descend to a bundle \(\mathcal{V}\) on \(Z\).

We will construct the bundle \(V\) via the “spectral cover” construction [13, 14, 15] which, essentially, uses T-duality to describe \(V\) in terms of simpler T-dual data. However, this puts a constraint on the Calabi–Yau manifold \(X\). It requires that \(X\) is elliptically fibered over a
two-complex-dimensional base. This means that at each point on the base, there is a torus fiber on which one can perform the T-duality. We should note also that, in general, the spectral construction gives a $U(n)$ rather than an $SU(n)$ bundle. Thus, we will need the additional condition that $c_1(V) = 0$ in order to ensure that $V$ is an $SU(5)$ bundle.

In summary then, if $Z$ is a quotient manifold and we build the bundle via the spectral construction, we must satisfy the following conditions

(Z2) $X$ is a smooth elliptically fibered Calabi–Yau threefold admitting a freely-acting involution $\tau_X : X \to X$,

(S) $V$ is a stable rank-five vector bundle on $X$,

(I) $V$ is $\tau_X$-invariant,

(C1) $c_1(V) = 0$,

(C2) $c_2(TX) - c_2(V)$ is effective,

(C3) $c_3(V) = 12$.

Note that the final three-family condition is now $c_3(V) = 12$ because, under the quotient, we have $c_3(V) = \frac{1}{2}c_3(V)$.

Our problem, then, is to find solutions to the conditions (Z2), (C3). The procedure, which is summarized in Figure 1, will be as follows. First, in Section 3, we construct a large family of elliptically fibered Calabi–Yau manifolds $X$ satisfying the involution condition (Z2). Second, in Section 4, we construct a large family of bundles $V$ on $X$ satisfying the invariance condition (I). This is the most difficult part of the construction. Finally, in Section 5, we reduce the stability condition (S) and the conditions on the Chern classes (C1)–(C3) to numerical conditions on the parameters defining $V$. A class of solutions to these conditions is then given in Section 6.

In order to simplify the construction of $\tau_X$-invariant bundles, we specialize to Calabi–Yau manifolds $X$ built from a particular type of complex two-fold base $B$ known as a “rational elliptic surface” or a “dP9”. These are surfaces which are themselves elliptically fibered. In order to construct an involution $\tau_X$ on $X$, we need to understand the action of involutions $\tau_B$ on the surfaces $B$. The structure of $B$ and the special class of $B$ admitting $\tau_B$, are discussed in Sections 3.1 and 3.2.

In order to find solutions to the numerical conditions, we must actually specialize further and consider only those rational elliptic surfaces where some of the elliptic fibers of $B$ split. This introduces new effective classes on $B$ and, hence, new freedom in constructing $V$. This four-dimensional sub-family of surfaces is described in Section 3.3. The manifold $X$ is then constructed as the “fiber product” of two surfaces $B$ and $B'$. (Note that Calabi–Yau manifolds of this type were also considered in [19].) The involution $\tau_X$ is similarly built from involutions $\tau_B$ and $\tau_B'$ on each of the two surfaces. This is described in Section 3.4.

Turning to the construction of $V$, we may take as a first approximation to $V$ a pullback to $X$ of a bundle $W$ on $B$. The point is that on $B$, it is comparatively easy to find those
Construct a large family of elliptically fibered Calabi–Yau threefolds $X$ with a freely acting involution $\tau_X$ to satisfy the condition (Z2).

Construct a family of $\tau_X$-invariant vector bundles $V$ for each $(X, \tau_X)$ via the spectral construction, thus satisfying the condition (I).

Impose the conditions (S), (C1), (C2), and (C3), which become a set of numerical conditions on the parameters defining $V$.

1. Construct a five-dimensional class of rational elliptic surfaces $B$ with an involution $\tau_B$. (I.1), (3.2)
2. Specialize to a class of $(B, \tau_B)$ with split fibers. (3.3)
3. Construct $(X, \tau_X)$ as fiber product $B \times_{\varphi_1} B'$ of special rational elliptic surfaces. (3.4)
4. Construct a class of $\tau_X$-invariant, rank-$r$ bundles $V_r$ as pullbacks of $\tau_B$-invariant bundles on $B$, modified by Hecke transforms. (4.2)
5. Construct $V$ as an extension of $V_2$ and $V_3$ bundles. (4.3)

Figure 1: Flow diagram of the construction. The numbers refer to the relevant section.

bundles which are invariant under the corresponding involution $\tau_B$, a problem which is harder to solve on the full manifold $X$. However, bundles of this form are not general enough to satisfy all the numerical conditions. One is forced to make two generalizations. First, one modifies the bundle by means of “Hecke transforms” which, roughly speaking, modify the bundle over surfaces in the Calabi–Yau threefold. Here, it is the fact that the fibers of $B$ and $B'$ split which provides new surfaces in $X$, and the additional freedom to make Hecke transforms over these surfaces. This construction is described in Section 4.2.

Even with the additional freedom from Hecke transforms, the bundles one constructs, $V_r$ for general rank $r$, cannot satisfy both (C1) and (C3). This leads to the second generalization discussed in Section 4.3: the final bundle $V$ is actually built as the “extension” of a rank-two bundle $V_2$ and a rank-three bundle $V_3$, each built using the construction just described. This means that $V$ is a sort of “twisted” sum of $V_2$ and $V_3$.

Finally, we show in Section 6 that there is then a large class of solutions to the numerical conditions. For the example we give, the solution has four independent parameters. Other classes of examples also exist. Even given the constraints, we find that there is still a large freedom in the examples.
3. A Family of \((X, \tau_X)\)

Our first goal is to construct a large family of elliptically fibered Calabi–Yau threefolds \(\pi : X \rightarrow B'\) with a freely acting involution \(\tau_X\). Some explicit examples were described in [18]. Here, we will consider a more general class of involutions but specialize to the case where the base \(B'\) of the fibration is a rational elliptic surface. This will lead to a large class of examples for which the demonstration of the invariance of the vector bundles is particularly simple.

Let us recall some properties of the construction in [18]. First, the involution \(\tau_X\) will necessarily induce some involution \(\tau_{B'}\) of the base \(B'\). Let \(\Delta\) be the discriminant of the fibration \(\pi\). Since \(X\) is a Calabi–Yau manifold, \(\Delta\) is a section of \(K_B^{-12}\). A necessary condition for \(\tau_X\) to be freely acting is that the set of fixed points of \(\tau_{B'}\) be disjoint from the discriminant locus of \(\pi\), that is

\[
\{\text{fixed points of } \tau_{B'}\} \cap \{\Delta = 0\} = \emptyset. \tag{3.1}
\]

Consequently, the first step in the construction of \((X, \tau_X)\) is to find suitable base pairs \((B', \tau_{B'})\). Much of this section will thus concentrate on the construction of involutions on rational elliptic surfaces.

As discussed in section 3.4, the fact that the base \(B'\) is itself an elliptic fibration, \(\beta' : B' \rightarrow \mathbb{P}^1\), will allow us to construct \(X\) as the fiber product of a pair of rational elliptic surfaces \(B\) and \(B'\) over a common \(\mathbb{P}^1\). Thus, in this case, simply by understanding involutions on rational elliptic surfaces we will be able to construct involutions on \(X\). Such constructions were first considered in [22]. In fact, exactly the four dimensional subfamily of rational elliptic surfaces described in section 3.3 below happened to appear, in a different context, as an example in [22, Section 9].

3.1 Rational Elliptic Surfaces

A rational elliptic surface \(B\) is a two-dimensional complex manifold which is a fibration of elliptic curves over a sphere base \(\mathbb{P}^1\)

\[
\beta : B \rightarrow \mathbb{P}^1. \tag{3.2}
\]

It can be described as the blow-up of the projective plane \(\mathbb{P}^2\) as follows. Recall that an elliptic curve, which is topologically a torus, corresponds to a cubic curve in \(\mathbb{P}^2\). Consider a one-parameter family or “pencil” of cubics where each curve passes through nine fixed points \(A_1, \ldots, A_9\) in \(\mathbb{P}^2\). The general surface \(B\) is then the blow-up of \(\mathbb{P}^2\) at the nine points and the elliptic fibers of \(B\) just correspond to the cubic curves. Under mild general position requirements, each subset of eight of the points determines the pencil of cubics and, hence, the ninth point. This implies that the rational elliptic surfaces depend on eight complex parameters. (Fixing eight points in \(\mathbb{P}^2\) requires 16 parameters. However, since only the relative positions matter, we must subtract the dimension \(\dim \mathbb{P}GL(3, \mathbb{C}) = 8\) of the automorphism group of \(\mathbb{P}^2\), leaving eight parameters.) Although not true del Pezzo surfaces, rational elliptic
surfaces are sometimes referred to as $dP_9$. (A del Pezzo surface $dP_n$ is obtained by blowing up $n \leq 8$ points in $\mathbb{P}_2$.)

Let $e_1, \ldots, e_9$ be the classes of the exceptional divisors of $B$ corresponding to the nine blown-up points $A_i$. An additional divisor $l$ comes from the class of a line in $\mathbb{P}_2$. It is easy to see that these provide an independent basis for the cohomology of $B$, such that

$$H^2(B, \mathbb{Z}) = \mathbb{Z}l \oplus (\oplus_{i=1}^9 \mathbb{Z}e_i),$$  \hspace{1cm} (3.3)$$

and, furthermore, the intersections between classes are given by $l^2 = 1$, $l \cdot e_i = 0$ and $e_i \cdot e_j = -\delta_{ij}$. In addition, the anti-canonical class of $B$ is equal to the class of the elliptic fiber

$$-K_B = c_1(B) = 3l - \sum_{i=1}^9 e_i.$$  \hspace{1cm} (3.4)$$

which we denote by $f$. Finally, we note that since each exceptional curve $e_1, \ldots, e_9$ intersects the fiber at one point, these are all sections of the fibration. In this paper, we will always identify one section $e : \mathbb{P}_1 \to B$ as the zero section. Without loss of generality we can take $e = e_9$. Fixing the zero section determines a group law generating the translational symmetries of each smooth (toroidal) fiber of $B$.

### 3.2 Involutions of Rational Elliptic Surfaces

Let us now consider involutions of $B$. Given the group action on the fibers, one can define a natural involution $(-1)_B : B \to B$ of any rational elliptic surface as the extension to all of $B$ of

$$(-1)_B(x) = -x,$$  \hspace{1cm} (3.5)$$

where $x$ is any point on a smooth fiber. This is the usual inversion symmetry of the torus so that, on any smooth fiber, the action of $(-1)_B$ will have four fixed points. In particular, one notes that this involution leaves the zero section invariant

$$(-1)_B(e) = e.$$  \hspace{1cm} (3.6)$$

Recall the requirement (3.1) that the fixed points of $\tau_B$ be disjoint from the locus $\Delta = 0$ of the discriminant, which is a section of $K_B^{-12} = \mathcal{O}_B(12f)$. Clearly, this locus intersects the zero section 12 times, and, hence, is not disjoint from the set of fixed points of $(-1)_B$. Thus, $(-1)_B$ is unsatisfactory for construction of a freely acting $\tau_X$.

Instead, we must specialize the rational elliptic surface to a family that admits additional involutions. Since $B$ is elliptically fibered, our discussion will be much like the discussion in [18] for elliptically fibered threefolds, though with one generalization.

First note that any involution $\tau_B$ will induce an involution $\tau_{\mathbb{P}_1} : \mathbb{P}_1 \to \mathbb{P}_1$ in the base. Assuming that $\tau_{\mathbb{P}_1}$ does not act as the identity\footnote{Note that there is a whole second family of rational elliptic surfaces with $\tau_B$ built on $\tau_{\mathbb{P}_1} = \text{id}_{\mathbb{P}_1}$. We will not discuss them here, but these too could be used to find suitable $(X, \tau_X)$ for constructing particle physics vacua, very much along the lines of this paper.}, any $\tau_{\mathbb{P}_1}$ will have two fixed points, which we
will denote as $0, \infty \in \mathbb{P}^1$ and which uniquely determine the involution. Let us fix a particular $\tau_{\mathbb{P}^1}$. One can then show that any $\tau_B$ satisfying $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B$ can be built out of a pair of objects $(\alpha_B, \zeta)$. First, one needs an involution $\alpha_B$ which is a lift of $\tau_{\mathbb{P}^1}$ and so satisfies $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \alpha_B$. We also assume it leaves the zero section invariant

$$\alpha_B(e) = e. \quad (3.7)$$

Second, one needs a section $\zeta$ of $\beta$ satisfying

$$\alpha_B(\zeta) = (-1)_B(\zeta). \quad (3.8)$$

The involution $\tau_B$ is then given by

$$\tau_B = t_\zeta \circ \alpha_B, \quad (3.9)$$

where $t_\zeta$ is the translation of the elliptic fibers defined by the section $\zeta$

$$t_\zeta(x) = x + \zeta(p), \quad (3.10)$$

where $x$ is any element of a smooth fiber $\pi^{-1}(p)$ over a point $p \in \mathbb{P}^1$. The conditions (3.7) and (3.8) are required to ensure $\tau_B^2 = \text{id}_B$. Note that in the particular case where $\zeta = e$, then $\tau_B = \alpha_B$. Also note that this construction is a generalization of the involutions considered in [18]. In that paper, we required that $(-1)_B(\zeta) = \zeta$ so that (3.8) became $\alpha_B(\zeta) = \zeta$.

One finds [21] that there is a natural (and unique up to a twist by $(-1)_B$) way to construct $\alpha_B$ from the Weierstrass model of $B$. However, the construction does not exist for all rational elliptic surfaces $B$. Rather, within the eight parameter family of rational elliptic surfaces there is a five-dimensional sub-family of surfaces which admit $\alpha_B$. In addition, all surfaces in this sub-family also admit a non-trivial section $\zeta$ satisfying (3.8). Rather than discuss the details of the construction, which can be found in [21], let us simply summarize the fixed point structure of $\alpha_B$ and $\tau_B$.

Since the fixed points of $\tau_{\mathbb{P}^1}$ are at $0$ and $\infty$, any fixed points of an involution of $B$ lifting $\tau_{\mathbb{P}^1}$ must lie in the fibers $f_0 = \beta^{-1}(0)$ and $f_\infty = \beta^{-1}(\infty)$. First consider $\alpha_B$. One can show that it acts as the identity on one of the fibers, say $f_0$ and as $(-1)$ on $f_\infty$. Thus, the whole fiber $f_0$ is fixed pointwise under $\alpha_B$, as are four fixed points in $f_\infty$. This is shown schematically in Figure 2. It might appear odd that this action treats $f_0$ and $f_\infty$ asymmetrically. The point is that the Weierstrass model naturally defines two involutions preserving the zero section, $\alpha_B$ and $\alpha_B \circ (-1)_B$. Under the latter involution, the fixed point structure of $f_0$ and $f_\infty$ is reversed.

Let us now turn to $\tau_B$. First we note that the condition (3.8) implies that on $f_0$ we have $\zeta(0) = -\zeta(0)$ (that is, it is one of the four fixed points of the $(-1)$ involution on $f_0$), while there is no such condition on $f_\infty$. Provided $\zeta$ is not the zero section, translation by $\zeta$ will thus remove all the fixed points on $f_0$. The $\alpha_B$-fixed points on $f_\infty$ are simply translated by $\zeta(\infty)/2$. Thus $\tau_B$ has four fixed points as shown in Figure 2.
Clearly, in general, both $\alpha_B$ and $\tau_B$ satisfy the condition (3.1). However, as we will discuss in section 3.4 below, it is easy to show that only $\tau_B$ leads to a freely acting $\tau_X$. Nonetheless, it would appear that we have solved our problem, finding a five-dimensional family of $(B, \tau_B)$ suitable for building the threefold $X$.

3.3 Special Rational Elliptic Surfaces

It turns out that the sub-family of surfaces described in the previous section is not quite suitable for our purposes. General rational elliptic surfaces, including generic members of the sub-family, have 12 singular $I_1$ elliptic fibers where the torus pinches to a sphere. In constructing $\tau_X$-invariant bundles on $X$, we will find that it is important for $B$ (and hence $X$) to have a richer cohomology structure. In particular, we will require that $B$ has some $I_2$ fibers where the torus splits into a pair of spheres. If the split fiber is not to lie over a fixed point of $\tau_{P^1}$, we see that we actually need at least one pair of $I_2$ fibers. These will lie above a pair of points $p_1$ and $p_2$ in $\mathbb{P}^1$ exchanged by $\tau_{P^1}$.

Thus the actual rational elliptic surfaces we will use in constructing $X$ are a special four-dimensional sub-family of the family described in the previous section, with, generically, a pair of $I_2$ fibers and 8 $I_1$ fibers. These surfaces admit $\alpha_B$ and $\tau_B$ exactly as above, and, in particular, the fixed points of $\tau_B$ remain four points in $f_\infty$ (see Figure 3).

Regarded as a blow-up of $\mathbb{P}^2$, the special features of this sub-family translate into special position requirements on the nine blow-up points. This is described explicitly in [21]. Again, rather than discuss the details of the construction, let us simply note the action of the $\tau_B$ involution. We have already discussed the fixed points of $\tau_B$. What remains is to identify the cohomology classes and the induced action of $\tau_B$ on them.
Figure 3: A special rational elliptic surface $B$.

| $e_1 = \zeta$  | $\tau_B^* e_9$ |
|----------------|----------------|
| $e_j$ ($j = 2, 3$) | $f - e_j + e_1 + e_9$ |
| $e_i$ ($i = 4, 5, 6$) | $f - l + e_i + e_1 + e_7 + e_9$ |
| $e_7$ | $l - e_2 - e_3$ |
| $e_8$ | $f - l + e_1 + e_7 + e_8 + e_9$ |
| $e_9 = e$ | $e_1$ |
| $l$ | $2f + 2(e_1 + e_9) - (e_2 + e_3) + e_7$ |
| $f$ | $f$ |

Table 1: Action of $\tau_B^*$ on $H^2(B, \mathbb{Z})$

Let the $I_2$ fibers be $f_1$ and $f_2$. Then each fiber is a union of spheres

$$f_1 = n_1 \cup o_1, \quad f_2 = n_2 \cup o_2.$$  \hfill (3.11)

As in the general case (3.3), the cohomology of $B$ can be described in terms of nine exceptional divisors $e_i$ and the pre-image $l$ of the line in $\mathbb{P}^2$. As usual, we will identify $e = e_9$ as the zero section. In addition, we can identify $\zeta = e_1$ as the section defining $\tau_B$. The explicit construction then identifies the new effective classes $n_i$ and $o_i$ as follows

$$n_1 = e_8 - e_9,$$
$$o_1 = 3l - e_1 - \cdots - e_7 - 2e_8 = f - n_1,$$
$$n_2 = l - e_7 - e_8 - e_9,$$
$$o_2 = 2l - e_1 - \cdots - e_6 = f - n_2.$$  \hfill (3.12)
Under $\tau_{P1}$ the reducible $I_2$ fibers $f_1$ and $f_2$ must be exchanged. Thus, $\tau_B$ must somehow exchange $(n_1, o_1)$ and $(n_2, o_2)$. Specifically, one finds

$$\begin{align*}
\tau_B^*(n_1) &= o_2, \\
\tau_B^*(o_1) &= n_2.
\end{align*}$$

(3.13)

The full action of $\tau_B^*$ on the cohomology is given in Table 1.

This completes the description of the family of rational elliptic surfaces and the involutions $\tau_B$ which we will use to construct $(X, \tau_X)$.

### 3.4 Construction of $(X, \tau_X)$

Given our specific family of rational elliptic surfaces $B$, we can now describe the construction of a suitable family of elliptically fibered $(X, \tau_X)$. The fact that rational elliptic surfaces are themselves elliptically fibered, allows a particularly simple construction of $X$ as the fiber product,

$$X = B \times_{\mathbb{P}^1} B',$$  

(3.14)

of a pair of rational elliptic surfaces $B$ and $B'$. That is to say, $X$ fits into a commutative diagram of projections

![Diagram](image)

The space $X$ is formed by taking the union, as $p$ varies in $\mathbb{P}^1$, of the product of the fibers $\beta^{-1}(p) \times \beta'^{-1}(p)$ above $p$. This is shown schematically in Figure 4.

For generic choice of $B$ and $B'$, $X$ will be smooth. It is an elliptic fibration in two ways: either via $\pi$ or $\pi'$. Since most of our construction will center on the elliptic fibers, we will make the somewhat unconventional choice that the primed object $B'$ is the base of the fibration, simply to avoid cumbersome notation. Much of the structure of the fibration is inherited from the structure of the $\beta$ fibration of $B$. For instance, the discriminant of $\pi$ is the pull-back of the discriminant of $\beta$ and so is a section of $\mathcal{O}(12f') = K_{B'}^{-12}$. Hence, $c_1(X) = 0$ and $X$ is Calabi–Yau. Similarly, the zero section $\sigma : B' \to X$ of $\pi$ is inherited from the zero section $e : \mathbb{P}^1 \to B$, and is given by $\sigma = e \times_{\mathbb{P}^1} B'$.

Let us now assume that $B$ and $B'$ are special in the sense of section 3.3. We then have involutions $\alpha_B$, $\tau_B$ and $\alpha_B'$, $\tau_{B'}$ acting on $B$ and $B'$ from which we will construct $\tau_X$. There is still some freedom in how we choose to identify the $\mathbb{P}^1$ bases of $B$ and $B'$. However, since all involutions of $X$ must induce the same involution $\tau_{P1}$, there are only two possibilities,
depending on the identification of fixed points. Either we identify \( 0 \in \mathbb{P}^1 \) with \( 0' \in \mathbb{P}^1' \) and \( \infty \in \mathbb{P}^1 \) with \( \infty' \in \mathbb{P}^1' \), or \( 0 \) with \( \infty' \) and \( \infty \) with \( 0' \). Suppose we make the first identification. Recall that both \( \alpha_B \) and \( \tau_B \) leave four points on \( f_\infty \) fixed. Since in the first case, both \( f_\infty \) and \( f'_\infty \) lie above the same point in \( \mathbb{P}^1 \), it is clear that no combinations of involutions in \( B \) and \( B' \) can be freely acting. However, with the second identification it is easy to see that the involution

\[
\tau_X = \tau_B \times_{\mathbb{P}^1} \tau_{B'}
\]  

acts freely on \( X \). This is because the four fixed points of \( \tau_B \) and \( \tau_{B'} \) live in fibers above different points in \( \mathbb{P}^1 \), as shown in Figure 4. Note that this would never be the case if \( \tau_X \) was built from either \( \alpha_B \) or \( \alpha_{B'} \).

As for the involutions of the rational elliptic surfaces, the involution \( \tau_X \) can also be built from an involution \( \alpha_X \) preserving the zero section \( \sigma \) and a translation \( t_{\zeta_X} \) by a second section \( \zeta_X \). (This was described in [18].) Both are built out of the corresponding structures on \( B \), namely

\[
\alpha_X = \alpha_B \times_{\mathbb{P}^1} \tau_{B'},
\]  

and

\[
\zeta_X = \zeta \times_{\mathbb{P}^1} B'.
\]
In what follows, it will be useful to identify the classes of divisors $H^2(X, \mathbb{Z})$ on $X$. The fiber product structure means that

$$H^2(X, \mathbb{Z}) = \frac{H^2(B, \mathbb{Z}) \times H^2(B', \mathbb{Z})}{H^2(\mathbb{P}^1, \mathbb{Z})}$$

That is, all divisor classes are either pull-backs of classes from $B$ or of classes from $B'$, modulo the one relation on the fiber classes that $\pi^*(f') = \pi'^*(f)$.

Finally, we will also need the expression for $c_2(X)$ in order to solve the condition $(C2)$. In general, this is given by a curve in $X$. Recalling that $c_2(B) = c_2(B') = 12$ and the fibered structure of $X$, it is easy to show that $c_2(X)$ is given solely by some number of $\pi$ and $\pi'$ fibers. That is

$$c_2(X) = 12 \left( f \times \text{pt} + \text{pt} \times f' \right),$$

where pt is the class of a point in the relevant manifold ($B$ or $B'$ or later $X$).

In summary, we have described the construction of a large class of threefolds $X$ with freely acting $\tau_X$. The quotient $Z = X/\tau_X$ will thus be smooth, with non-trivial $\pi_1(Z)$. We note that it is not difficult to show that $Z$ is also a Calabi–Yau manifold, as required.

4. A Family of $\tau_X$-Invariant Bundles $V$

In this section, we will describe the construction a large family of stable bundles $V$ on $X$ which are invariant under the involution $\tau_X$. As in previous papers $[6, 7, 8, 18]$, we will use the spectral construction. One new feature is that we are forced to work with a reducible spectral cover. Rather than describing the complicated behavior of the spectral sheaf at the singularities of the spectral cover, we chose to realize the resulting vector bundle as an extension of two vector bundles $V_2$ and $V_3$ each coming from more manageable spectral data. This approach is a variation of an idea of Richard Thomas $[20]$.

The key ingredient in the spectral construction is the fact that $X$ is elliptically fibered. This allows $V$ to be constructed via the “Fourier–Mukai transform” which, in physics terms, is the action on the bundle of T-duality along the elliptic fibers. Recall that, with respect to the complex structure, the T-dual Calabi–Yau manifold is isomorphic to $X$. Formally, the Fourier–Mukai transform $FM_X$ is then an “autoequivalence” of the “derived category” $D^b(X)$ of sheaves on $X$,

$$FM_X : D^b(X) \to D^b(X).$$

Physically, we can think of a sheaf as describing a D-brane. Thus, as expected, $FM_X$ maps one configuration of D-branes to another. It is really a little subtler. The objects in $D^b(X)$ are actually not single sheaves but complexes of sheaves. Although very important for the details of the construction (see $[21]$), this subtlety will not generally concern us here.

The usefulness of the Fourier–Mukai transform is that it allows one to describe $V$ in terms of its simpler T-dual data. In particular consider a line bundle $N_2$ over a smooth
surface $i_\Sigma : \Sigma \to X$ in $X$ which is a finite $r$-fold cover of the base $B'$. The transform of the corresponding sheaf $i_\Sigma^* N_\Sigma$ on $X$,

$$V = FM_X(i_\Sigma^* N_\Sigma),$$

(4.2)
is then precisely the object we want: a stable vector bundle over $X$ of rank $r$. (We should note that the stability also depends on a choice of a suitable Kähler form on $X$.) The surface $\Sigma$ is the spectral cover, $N_\Sigma$ the spectral datum and the correspondence between $(\Sigma, N_\Sigma)$ and $V$ is commonly known as the spectral construction.

In order to find $\tau_X$-invariant bundles, we would like to translate the invariance condition into a condition on $(\Sigma, N_\Sigma)$. Since $FM_X$ is invertible, we can construct the induced action of $\tau_X$ on the spectral sheaf $i_\Sigma^* N_\Sigma$

$$T_X = FM_X^{-1} \circ \tau_X^* \circ FM_X.$$

(4.3)
The search for invariant bundles $V$ is then reduced to finding $(\Sigma, N)$ such that

$$T_X(i_\Sigma^* N_\Sigma) \cong i_\Sigma^* N_\Sigma.$$

(4.4)

In general, the action of $T_X$ is extremely hard to calculate. One particular problem is that the space $Pic(\Sigma)$ of line bundles $N_\Sigma$ on $\Sigma$ is generically not simply characterized by pullbacks of bundles from $X$. Instead, as we saw in [18], new divisor classes on $\Sigma$ appear. Calculating $T_X$ for the corresponding bundles is difficult. However, here, the fiber-product structure of $X$ will come to our aid. Just as many of the properties of $X$ were inherited from the vertical surface $B$, in the cases of interest, we will be able to build the Fourier–Mukai transform $FM_X$ from the horizontal pullback by $\pi''$ of the corresponding transform $FM_B$ on $B$.

Our first step in this section is, thus, to give some results on the action of $FM_B$ and the corresponding $T_B$ on the special rational elliptic surfaces $B$. We then use these results, together with the technique of modifying bundles by “Hecke transforms,” to construct a large class of $\tau_X$-invariant bundles.

4.1 The Fourier–Mukai Transform on $B$ and a No-Go Theorem

Since $B$ is elliptically fibered, there is also a Fourier–Mukai action $FM_B$ on sheaves on $B$. Similarly, given the involution $\tau_B$ on $B$, there is an induced action $T_B = FM_B^{-1} \circ \tau_B^* \circ FM_B$ on sheaves on $B$. Let us now simply state some results for $FM_B$ and $T_B$. Details can be found in [21].

Our main result is the following. Let $L$ be a line bundle on $B$. In general $FM_B(L)$ will be some complex of sheaves. Nonetheless, one can show, as long as $c_1(L) \cdot o_i = 0$, that $T_B(L)$ is actually still a line bundle. Explicitly, in terms of the divisor classes defined in section 3.3, one shows (see [21], Part I, Theorem 7.1):

$$T_B(L) = \tau_B^* (L) \otimes O([c_1(L) \cdot (e - f)]f + [c_1(L) \cdot f](e - \zeta - f)) \otimes O(e - \zeta - f).$$

(4.5)
Thus we see that $\mathbf{T}_B$ induces a complicated affine action on the space of line bundles $\text{Pic}(B)$. The first two terms give the underlying linear part of the transformation, while the last term gives the constant shift.

The analogue in $B$ of the spectral cover $\Sigma$ is a smooth curve $i_C : C \hookrightarrow B$ which is a finite cover of the base $\mathbb{P}^1$. The analog of $N_\Sigma$ is then a line bundle $\mathcal{N}$ on $C$. Consequently, we would also like to know the action of $\mathbf{T}_B$ on spectral sheaves $i_C^*\mathcal{N}$. Let us assume that the bundle $\mathcal{N}$ is the pullback $\mathcal{N} = i_C^*(L)$ of a global bundle $L$ in $B$ (note that, in general, this is not always the case). One can then show that

$$\mathbf{T}_B(i_C^*i_C^*(L)) = i_{D^*}i_D^*(\mathbf{T}_B(L)), \quad (4.6)$$

where

$$D = \alpha_B(C) \quad (4.7)$$

is the image of $C$ under the involution $\alpha_B$. This matches the result of [18], where it was shown that the spectral cover transforms under the involution preserving the zero section, $\alpha_B$, rather than the full involution, $\tau_B$, of the manifold.

Finally, if we restrict $C$ to be in the divisor class $re + kf$ for some integers $r$ and $f$, one specifically finds, using $(4.5)$, that

$$\mathbf{T}_B(i_C^*i_C^*(L)) = i_{D^*}i_D^*(\alpha_B^*(L) \otimes \mathcal{O}(e - \zeta - f)). \quad (4.8)$$

We can now use these results for $\mathbf{T}_B$ to show a useful no-go theorem for constructing suitable bundles on $X$. The most obvious simplification in the spectral construction is to ignore any new classes on the spectral cover $\Sigma$, and to assume that the line bundle $N_\Sigma$ is the pullback of a global line bundle $\mathcal{L}$ on $X$

$$N_\Sigma = i_\Sigma^*(\mathcal{L}). \quad (4.9)$$

Recall from equation $(3.19)$ that all the divisor classes on $X$ came as either pullbacks of classes on $B$ or pullbacks of classes on $B'$. Thus, in general, $\mathcal{L}$ can be written as

$$\mathcal{L} = \pi''^*L \otimes \pi'^*L', \quad (4.10)$$

where $L$ and $L'$ are global line bundles on $B$ and $B'$ respectively. The action of $\mathbf{FM}_X$ then splits into a Fourier-Mukai action on $B$ and the trivial action on $B'$. Specifically

$$\mathbf{FM}_X(\mathcal{L}) = \pi'''^*\mathbf{FM}_B(L) \otimes \pi'^*L'. \quad (4.11)$$

Similarly, given the form $(3.16)$ of $\tau_X$, the action of $\mathbf{T}_X$ is given by

$$\mathbf{T}_X(\mathcal{L}) = \pi'''^*\mathbf{T}_B(L) \otimes \pi'^*\tau_{B'}^*L'. \quad (4.12)$$

One notes that the action of $\mathbf{T}_X$ is simple on $L'$ but more complicated on $L$. This reflects the fact that the Fourier–Mukai transformation is the action of T-duality on the $\pi$ fibers.
From these relations it is straight-forward to deduce the action of $T_X$ on the spectral data $(C, i_C^*(L))$. The bundle is invariant under the involution provided that

$$
\alpha_X(\Sigma) = \Sigma,
\tau_B^*L' \cong L',
T_B(\Sigma) = \Sigma,
$$

(4.13)

Finding solutions of the first two conditions it relatively easy. We can then use the general result (4.5) to try and solve the $L$ condition. Rewriting the action of $T_B$ in terms of cohomology, we see that (Prop. 2.11 in [21]).

$$
c_1(T_B(L)) = \tau_B^*(c_1(L)) + [c_1(L) \cdot (e - \zeta)] f + [c_1(L) \cdot f + 1] (e - \zeta + f).
$$

(4.14)

Using the action of $\tau_B^*$ given in Table [3], it is easy to see that $c_1(T_B(L)) = c_1(L)$ if and only if $c_1(L)$ is in the affine subspace of $H^2(B, \mathbb{Q})$

$$
-\frac{1}{2}e_1 + \text{Span}(f, e_9, e_4 - e_5, e_4 - e_6, l - e_7 - 2e_8, 3l - 2(e_4 + e_5 + e_6) - 3e_7).
$$

(4.15)

However, $c_1(L)$ must be in $H^2(B, \mathbb{Z})$ and there are no integral vectors in this subspace. Thus, we see that there are no solutions to the $L$ condition.

We have derived a no-go theorem: $V$ can never be $\tau_X$-invariant if $N_{\Sigma}$ is the pullback of a global line bundle on $X$. Instead we are forced to consider cases where $N_{\Sigma}$ comes at least partly from additional classes on $\Sigma$.

4.2 Construction of $V_r$

Given the general results of the last section, we can now turn to the specific construction of a suitable family of rank $r$ bundles $V_r$. We start by defining an appropriate spectral cover $\Sigma$. Again, we can use the projection $\pi'$ to describe it as a pullback of a simpler object in $B$. Let $C$ be a smooth irreducible curve in $B$ suitable for a spectral construction in $B$. Specifically, let $C$ be irreducible and in the divisor class

$$
[C] = re + kf,
$$

(4.16)

for some integer $k$, so that it is an $r$-fold cover of the base $\mathbb{P}^1$. We then take $\Sigma$ to be the pullback

$$
\Sigma = C \times_{\mathbb{P}^1} B'.
$$

(4.17)

Given the properties of $C$, the spectral cover $\Sigma$ will be a smooth, irreducible $r$-fold cover of $B'$. By construction, $\Sigma \rightarrow C$ is an elliptic fibration over $C$. In particular, it includes some number of reducible fibers. Let $f'_1 = n'_1 \cup o'_1$ and $f'_2 = n'_2 \cup o'_2$ be the reducible fibers of $B'$ (as in equation (3.11)). Let $F_1$ and $F_2$ be the fibers of $B$ over the corresponding points in $\mathbb{P}^1$. This is shown schematically in Figure [3]. Given the way we glued the $\mathbb{P}^1$ bases of $B$ and
The fibers $F_1$ and $F_2$ are smooth. The fibers $F_1$ and $F_2$ will each intersect the curve $C$ in $r$ points in $B$. Let us label these by an index $\kappa = 1, \ldots, r$, so
\[ C \cap F_j = \{p_{j\kappa}\}_{\kappa=1}^r \quad (4.18) \]
for $j = 1, 2$. Above each of these points, the fiber of $\Sigma$ will split. Thus $\Sigma$ is an elliptic surface with $2r$ fibers of type $I_2$ given by $(n'_j \cup o'_j) \times \{p_{j\kappa}\}$ for $j = 1, 2$ and $\kappa = 1, \ldots, r$.

Next, we turn to the line bundle $\mathcal{N}_\Sigma$. Generically, there are three types of divisor on $i'_\Sigma : \Sigma \hookrightarrow X$. First, there are pullbacks under $i'_\Sigma$ of global divisors on $X$. Then, we have pullbacks of divisors (points) on $C$ and, finally, the $2r$ new divisors coming from the reducible fibers. Thus, we can take $\mathcal{N}_\Sigma$ of the form
\[ \mathcal{N}_\Sigma = \pi'_\Sigma^* \mathcal{N} \otimes \mathcal{O}_\Sigma \left( - \sum_{j, \kappa} \{p_{j\kappa}\} \times (a_{j\kappa} n'_j + b_{j\kappa} o'_j) \right) \otimes i'_\Sigma^* \pi^* L, \quad (4.19) \]
where $\mathcal{N}$ is a line bundle of degree $d$ on $C$, $a_{j\kappa}$ and $b_{j\kappa}$ are integers and $L$ is a line bundle on $B'$. The first term is precisely the pullback of a bundle on $C$. The second is a bundle corresponding to some combination of the new divisors from the reducible fibers. The last term is the pullback of a global bundle on $X$. We note that there is some redundancy in the choices of $a_{j\kappa}$ and $b_{j\kappa}$. Since $n'_j + o'_j = f'_j$ is a pullback from $\mathbb{P}^1$, it can be absorbed in $L$. Thus, we are free to take
\[ \text{either } a_{j\kappa} \geq 0, b_{j\kappa} = 0 \text{ or } a_{j\kappa} = 0, b_{j\kappa} \geq 0 \text{ for all } j \text{ and } \kappa. \quad (4.20) \]

Finally, we should stress that both the pullback of $\mathcal{N}$ and the bundles from the extra reducible fibers represent contributions to $\mathcal{N}_\Sigma$ which are not pullbacks of global line bundles.
on $X$. Thus, we can hope to avoid the no-go theorem of section 4.1. In fact, generalizing to include $\mathcal{N}$ would be sufficient to find invariant bundles. However, as we will see, in order to satisfy conditions $[S]$, one needs the additional freedom of the reducible fibers.

Having defined $(\Sigma, \mathcal{N}_\Sigma)$, we now need to understand the action of $\mathcal{F}M_X$ and $T_X$. This could be addressed directly on $X$. However, in fact, the action can be decomposed in the following, relatively simple way. We first note that, as in equation (4.11) above, we can factor off the contribution of the global line bundle $\pi^* L$ under the action of $\mathcal{F}M_X$. We have

$$V_r = \mathcal{F}M_X (i_{\Sigma^*} \mathcal{N}_\Sigma) = \tilde{W} \otimes \pi^* L,$$

where $\tilde{W}$ is the bundle constructed from $\Sigma$ and the spectral datum $\tilde{\mathcal{N}}_\Sigma$ given by the first two terms in (4.19). If, somehow, we could also remove the contribution from the reducible fibers, we would then be left with a bundle which, given the structure of $\Sigma$, is just a pullback of a bundle $W$ on $B$ (similar to the case of equation (4.11)),

$$\mathcal{F}M_X (i_{\Sigma^*} \pi'_f \mathcal{N}) = \pi'^* W,$$

where,

$$W = \mathcal{F}M_B (i_{C^*} \mathcal{N}).$$

This can then be calculated given our results on $\mathcal{F}M_B$ in section 4.1.

It turns out that there is a precise way to go from $\pi'^* W$ to $\tilde{W}$. It is the action of a series of Hecke transforms. Details can be found in [21]. Here we will simply note that, given a vector bundle $E$ on $X$, a divisor $i_D : D \hookrightarrow X$ and a short exact sequence $(\xi) : 0 \to F \to E|_D \to G \to 0$ of vector bundles on $D$, the associated Hecke transform generates a new vector bundle $\text{Hecke}_{(\xi)}(E)$ on $X$. This new bundle has two characteristic properties: the Chern character of $\text{Hecke}_{(\xi)}(E)$ is equal to $\text{ch}(E) - \text{ch}(i_D^* G)$, and $\text{Hecke}_{(\xi)}(E)$ is isomorphic to $E$ on the complement of $D$.

$$\tilde{W} = \text{Hecke}_{a_{j\kappa}, b_{j\kappa}} (\pi'^* W),$$

where $\text{Hecke}_{a_{j\kappa}, b_{j\kappa}}$ represents $a_{j\kappa}$ successive Hecke transforms on the the divisor $D = F_j \times n'_j$ together with $b_{j\kappa}$ successive Hecke transforms on $D = F_j \times o'_j$.

In summary, we have built $V_r$ as

$$V_r = \text{Hecke}_{a_{j\kappa}, b_{j\kappa}} (\pi'^* (\mathcal{F}M_B (i_{C^*} \mathcal{N}))) \otimes \pi^* L.$$  

In the remainder of this section we want to show that there are suitable $a_{j\kappa}$, $b_{j\kappa}$, $L$ and $\mathcal{N}$ such that $V_r$ is invariant under $\tau_X$.

Acting with $\tau_X$ on $V_r$, it is clear from equation (4.21) and the form (3.16) of $\tau_X$ that

$$\tau_X^* (V_r) = \tau_X^* (\tilde{W}) \otimes \pi^* \tau_B^* L.$$  

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Thus, as in equation (4.13), invariance of $V_r$ requires
\[ \tau_{B'} L \cong L, \] (4.27)
and
\[ \tau_X^* \tilde{W} \cong \tilde{W}. \] (4.28)

Recall that the Hecke transforms were on the divisors $F_j \times n'_j$ and $F_j \times o'_j$. From (3.13), we see that $\tau_{B'}$ exchanges $n'_1$ with $o'_2$ and $n'_2$ with $o'_1$. Hence, for invariance under $\tau_X$ we must perform the same number of Hecke transforms on each set of divisors paired under $\tau_{B'}$. This implies that $a_{1\kappa} = b_{2\kappa}$ and $a_{2\kappa} = b_{1\kappa}$. Given (4.20), we take
\[ a_{1\kappa} = b_{2\kappa} \cong a_\kappa \geq 0, \quad a_{2\kappa} = b_{1\kappa} = 0. \] (4.29)

After undoing all the Hecke transforms, the $\tau_X$-invariance of $\tilde{W}$ reduces, from equation (4.22), to the $\tau_X$-invariance of $\pi^*_r W$. Since this is just the pull back of a bundle from $B$, given the expression (4.23), we finally have the condition
\[ T_B(i_C^* N) \cong i_C^* N. \] (4.30)

Since $[C] = re + kf$, using the result (4.8) for global line bundles on $B$, one can show that equation (4.30) implies that
\[ C = \alpha_B(C), \] (4.31)
\[ N \cong \alpha^*_B(C) \otimes \mathcal{O}_C(e - \zeta + f). \] (4.32)

where $\alpha_B|_C$ is the restriction of the involution $\alpha_B$ to $C$.

The requirement that $\tau_X^* V_r \cong V_r$, has been reduced to the four conditions (4.27), (4.28), (4.31) and (4.32). From Table I, we see that there is a six-dimensional lattice of $\tau_{B'}$-invariant classes on $B'$, so there are many possibilities for $L$. The conditions on $a_{j\kappa}$ and $b_{j\kappa}$ simply reduce to the choice of positive integers $a_\kappa$. Since the set of divisors $C$ in the class $re + kf$ forms a projective space, there must be a solution to the condition on $C$. In general, $C$ will be smooth provided
\[ k \geq r > 1. \] (4.33)

However, it is not obvious that the general $C$ satisfying (4.31) will be smooth as well. In fact, for the values of $k$ which we need, this turns out to be true for $r = 3$ and false for $r = 2$. The relevant curves $C_2$ are actually reducible and always have a vertical component. This requires a separate analysis of the invariance properties of $W_2$ which is carried out in [21]. In [21] it is also shown that for all bundles $\mathcal{N}$ on $C$ there is some positive-dimensional torus of solutions to the condition (4.32).

We conclude, therefore, that using the construction (4.25), we can build a large family of $\tau_X$-invariant rank-$r$ bundles $V_r$ on $X$. 

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4.3 Extensions and Stability

As we shall see in section 5, it turns out that even with the additional freedom coming from the reducible classes on Σ, the bundles $V_r$ that we have just constructed are not quite general enough to satisfy all the conditions $(S)$–$(C3)$. We will actually construct our rank-five bundle $V$ as the extension

$$0 \to V_2 \to V \to V_3 \to 0,$$

(4.34)

where $V_r$, with $r = 2, 3$, are $\tau_X$-invariant rank-$r$ bundles constructed via the procedure given above, and the extension class itself is also $\tau_X$-invariant.

By construction, $V$ will be $\tau_X$-invariant. However, we also require $(S)$ that $V$ be stable given a suitable Kähler form $H$ on $X$. This puts some constraint on the extension and also on the bundle $V_2$. Again, we will simply quote the result. Stability of $V$ is equivalent, first, to the fact that $V$ is not split, that is, it is not a direct sum,

$$V \neq V_2 \oplus V_3$$

(4.35)

and, second, to a condition on the slopes

$$\mu(V_2) < \mu(V)$$

(4.36)

where, for an arbitrary bundle $E$, $\mu(E) = (\int_X c_1(E) \wedge H \wedge H) / \text{rk}(E)$. (Note that condition $(C1)$ implies that $\mu(V) = 0$.)

5. Numerical Conditions

In the previous section, we built a class of bundles $V$ satisfying the $\tau_X$-invariance condition $(I)$. We now need to find the requirements on $V$ for simultaneously satisfying all the conditions $(S)$–$(C3)$. In particular, we will reduce them to a set of numerical constraints on the parameters defining $V$.

Recall that the invariance conditions (4.31) and (4.32) were geometrical in nature, fixing a particular $C$ and $N$. Let us assume these conditions are satisfied. Recalling that $V$ is built from two bundles $V_r$, for $r = 2, 3$, the bundle is then determined by the following parameters, again indexed by $r$,

- the integers $k_r$ giving the classes of the curves $C_r$ as in (4.16),
- the integers $d_r = \text{deg}(N_r)$ giving the degrees of the bundles $N_r$,
- the integers $a_{r\kappa}$ with $\kappa = 1, \ldots, r$ determining the number of Hecke transforms (4.29),
- and the line bundles $L_r$ on $B'$.

The $\tau_X$-invariance condition only constrains $L_r$. We have

21
$(\# I)$ $\tau_B^* L_r = L_r$, for $r = 2, 3$.

Recall that there was also a condition $[4.33]$ in the construction of $V_r$ in order for a generic $C_r$ to be smooth. Here, it implies

\[ k_2 \geq 2 \text{ and } k_3 \geq 3. \tag{5.1} \]

Let us now turn to the condition of stability $[S]$. We saw that this implied that $V$ did not split $[4.35]$ and a condition on the slopes $[4.37]$. It can be shown $[21]$ that these conditions amount to

$(\# Se)$ $L_2 \cdot f' > L_3 \cdot f'$,

$(\# Ss)$ $(2L_2 + (d_2 - 2k_2 + 1)f' - S_1^1(n_1' + o_2')) \cdot h' < 0$ for some ample class $h' \in H^2(\mathbb{Z}, B')$,

respectively. Here we have introduced the notation $S_p^r$ for sums of $p$-th powers of $a_{r\kappa}$

\[ S_p^r = \sum_{\kappa=1}^r (a_{r\kappa})^p. \tag{5.2} \]

What remains are the conditions $[C1] - [C3]$ on the Chern classes. Using the explicit construction $[4.25]$, one can derive the following expression for $\text{ch}(V_r)$ for $V_2$ and $V_3$,

\begin{align*}
\text{ch}(V_r) = r + \pi^* \left( r L_r + \left( d_r - r k_r + \binom{r}{2} \right) f' - S_1^1(n_1' + o_2') \right) \\
+ \left[ \frac{r}{2} L_r^2 + \left( d_r - r k_r + \binom{r}{2} \right) L_r \cdot f' - S_1^1 (L_r \cdot n_1' + L_r \cdot o_2') - 2S_2^2 \right] (f \times pt) \\
- k_r (pt \times f') - k_r (L_r \cdot f') pt.
\end{align*} \tag{5.3}

Note that $c_1(V_r)$ is a pullback from $B'$, while the only terms in $\text{ch}_2(V_r)$ are proportional to $f \times pt$ and $pt \times f'$. More significantly, one notes that requiring $c_1(V_r) = 0$ implies that $L_r \cdot f' = 0$ in $B'$. From the last term in $(5.3)$ this, in turn, implies that $c_3(V_r)$ vanishes. Clearly, $V_r$ by itself cannot, therefore, satisfy both conditions $[C1]$ and $[C3]$. This is the reason we were forced to consider the generalization of $V$ constructed as an extension.

Given the form of $V$, we have $\text{ch}(V) = \text{ch}(V_2) + \text{ch}(V_3)$. Combining this with $(5.3)$ and $(3.2)$, the conditions $[C1] - [C3]$ imply the following numerical constraints

$(\# C1)$ $2L_2 + 3L_3 = (S_1^1 + S_3^1)(n_1' + o_1') - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f'$,

$(\# C2f)$ $k_2 + k_3 \leq 12$

$(\# C2f')$ $L_2^2 + \frac{3}{2} L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') - (S_1^1 L_2 + S_3^1 L_3)(n_1' + o_2') - 2(S_2^2 + S_3^2) \geq -12,$

$(\# C3)$ $k_2 (L_2 \cdot f') + k_3 (L_3 \cdot f') = -6.$

Note that the $c_2(V)$ condition splits into two pieces, one from the component proportional to $f \times pt$ and one from that proportional to $pt \times f'$. 
6. A Class of Solutions

What remains is to find a simultaneous solution of the equations \( (#I) \), \( (#Se) \) and \( (#Ss) \), and \( (#C1) \), \( (#C3) \) together with the inequality \( (#1.1) \). It is a straightforward, if tedious, procedure to calculate a fairly general solution. Let us summarize the result, noting only that the main constraint on finding solutions is the tension between the stability condition \( (#Se) \) and the \( c_2(V) \) conditions \( (#C2f) \) and \( (#C2f') \).

First, solving conditions \( (#C1) \), \( (#C3) \) and \( (#I) \) constrains the line bundles \( L_r \) to have the following form

\[
L_2 = \frac{9}{k} (e' + \zeta') + \frac{1}{2} (x - d_2 + 2k_2 - 1) f' + \frac{1}{2} \left( u + \frac{9}{k} + S^1 \right) (n'_1 + o'_2) + 3M \tag{6.1}
\]

\[
L_3 = -\frac{6}{k} (e' + \zeta') + \frac{1}{3} (-x - d_3 + 3k_3 - 3) f' + \frac{1}{3} \left( u - \frac{9}{k} + S^3 \right) (n'_1 + o'_2) - 2M
\]

where \( k = 2k_3 - 3k_2 \) and \( x \) and \( u \) are as yet undetermined. From Table 1 and equation \( (5.1) \), we note that \( e' + \zeta', f' \) and \( n'_1 + o'_2 \) are \( \tau_B \)-invariant classes. The parameter \( M \) represents an arbitrary \( \tau_B \)-invariant class orthogonal to \( e' + \zeta', f' \) and \( n'_1 + o'_2 \). It is then clear that the \( L_r \) satisfy \( (#I) \).

Satisfying the inequalities \( (#Se) \), \( (#C2f) \) and \( (5.1) \), and requiring that the \( L_r \) are integral classes, leaves only two possible values for \( k_2 \) and \( k_3 \),

\[
k_2 = 3, \quad k_3 = 5, \quad \text{giving} \quad k = 1
\]

or

\[
k_2 = 3, \quad k_3 = 6, \quad \text{giving} \quad k = 3 \tag{6.2}
\]

In general, \( M \) lies in a three-dimensional subspace spanned by

\[
e'_4 - e'_5, \quad e'_4 - e'_6, \quad 3l' - 2e'_4 - 2e'_5 - 2e'_6 - 3e'_7. \tag{6.3}
\]

We will restrict ourselves to the one-dimensional subspace \( M = z(e'_4 - e'_5) \) for some integer \( z \). (Other solutions exist with more general \( M \).)

Finally, we are left to satisfy the second stability condition \( (#Ss) \) and the inequality \( (#C2f') \). It is straightforward to show that there is a solution

\[
k_2 = 3, \quad k_3 = 6, \tag{6.4}
\]

together with

\[
u = 3, \quad z = 1, \quad x = 5. \tag{6.5}
\]

For the integers \( a_{r\kappa} \) we have

\[
a_{21} = a_{22} = a, \quad b_{21} = b_{22} = b_{23} = b \tag{6.6}
\]
for arbitrary non-negative integers $a$ and $b$. Finally, the degrees $d_2$ and $d_3$ have the form
\begin{equation}
  d_2 = 2p, \quad d_3 = 3q + 1, \quad (6.7)
\end{equation}
for arbitrary integers $p$ and $q$.

In conclusion, we have constructed a large new class of bundles on non-simply connected Calabi-Yau manifolds which give three-family, anomaly-free vacua with the standard model gauge group. From equations (6.6) and (6.7), we see that rather than a single solution, we have a class of solutions depending on four arbitrary parameters. Furthermore, other solutions exist, with a more general class $M$ in (6.1). This provides flexibility for discussing other physical properties of these models, such as nucleon decay and Yukawa couplings.

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