On the motion of a compact elastic body

Robert Beig
Institut für Theoretische Physik der Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria

Michael Wernig-Pichler
Institut für Theoretische Physik der Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria

October 7, 2018

Abstract

We study the problem of motion of a relativistic, ideal elastic solid with free surface boundary by casting the equations in material form (“Lagrangian coordinates”). By applying a basic theorem due to Koch, we prove short-time existence and uniqueness for solutions close to a trivial solution. This trivial, or natural, solution corresponds to a stress-free body in rigid motion.

1 Introduction

In the problem of motion for classical continua with free surface boundary, despite its obvious physical relevance, there is a surprising scarcity of rigorous results. A recent review of known results with and without gravity can be found in [14]. In the case of a nonrelativistic, non-gravitating perfect fluid local wellposedness has only been proved very recently by Lindblad [12]. In the present paper we consider the analogous problem for relativistic elastic solids without self-gravity. The nonrelativistic case of our results (which is in principle known - see [15] - although even there a detailed treatment seems to be lacking) can be proved by similar methods. The nonrelativistic case for incompressible materials has, by different methods, been treated in [7].

The plan of this paper is as follows. In Sect.2 we describe elasticity theory on an arbitrary curved background as a Lagrangian field theory. Sect.3 is devoted to the notion of “natural state”, that-is-to-say a solution to the elastic equations
corresponding to a configuration with zero stress. The existence of such a solution requires the elastic body to move along a geodesic, timelike Killing vector. We require the Killing vector to be also hypersurface-orthogonal (thus forcing spacetime to be "ultrastatic"). In the case where the metric on the space orthogonal to the Killing vector is flat, we are in Special Relativity. In Sect. 4 we perform a 3+1 decomposition of the elastic equations corresponding to the natural spacetime splitting afforded by the Killing vector. We then write the elastic equations in "material" form (often called "Lagrangian representation"). This means that the maps making up the elastic configuration space - which go from spacetime to the 3 dimensional "material space" - are replaced by time dependent maps ("deformations") from material space into physical space. In the material representation the boundary of the body is fixed, namely the boundary of material space. In Sect. 5 we state a corollary to the theorem of Koch [11], which is the version of existence theorem we are using. In Sect. 6 we state our basic constitutive assumption. This assumption, which is satisfied by elastic materials occurring in practice, implies the validity of the conditions in the Koch theorem. From this one concludes the main result, which is stated in Sect. 6. In the appendix we show that the corner conditions on the boundary, which initial data have to satisfy in order for the time evolved solution to be a classical solution, can be satisfied for a large class of initial data.

We should emphasize that our requirement of the existence of a global relaxed state, and the associated restriction on the background spacetime would be unnecessary if we were dealing with well-posedness in the absence of a boundary, or inside a region of the body causally disconnected from the boundary. However in the presence of the boundary we have to ensure corner conditions using an implicit function argument, and for this we use the existence of a state - the global relaxed one - which satisfies these corner conditions to all orders. In this sense our strong restriction on the existence of a global relaxed configuration is perhaps not completely essential, but very convenient.

We can state the central result of this paper as follows: For initial data close to initial data for the natural deformation in the appropriate function space, there exists, for sufficiently short times, a unique solution to the elastic equations. This solution depends continuously on initial data, in particular tends to the natural deformation, when the initial data tend to that of the natural deformation.

2 The theory

We treat elasticity as a Lagrangian field theory in the manner of [2] or [4], see also [10]. In the language of standard elasticity this means that the material in question is "hyperelastic". The dynamical objects of the theory are furnished by

\[1\] In Special Relativity a geodesic, timelike Killing motion has to be a time translation, so the above assumption is superfluous in that case.
sufficiently regular maps $f$ sending a closed region $\bar{\mathcal{S}} = \mathcal{S} \cup \partial \mathcal{S}$ of spacetime $\mathcal{M}$ onto $\bar{\mathcal{B}} = \mathcal{B} \cup \partial \mathcal{B}$, with $\mathcal{B}$ a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial \mathcal{B}$ called the "body" or material space. The domain $\mathcal{B}$ is to be thought of as an abstract collection of points ("labels") making up the elastic continuum, and $f$ is to be thought of as the back-to-labels map sending spacetime events to the particles by which they are occupied. We endow $\mathcal{B}$ with a volume form $\Omega$ which is smooth on $\bar{\mathcal{B}}$. Coordinates on $\mathcal{M}$ are written as $x^\mu$, with $\mu, \nu$ etc ranging from 0 to 3 and coordinates on $\mathcal{B}$ as $X^A$ with $A, B$ etc going from 1 to 3. The metric $g_{\mu \nu}$ is also taken to be smooth. There are additional requirements on the maps $f$. Namely the equation
\[
f^A_{\ \mu} u^\mu = 0, \quad g_{\mu \nu} u^\mu u^\nu = -1
\]should have a solution $u^\mu$ which is unique up to sign. Thus particles move along trajectories in $\mathcal{M}$ with unit tangent $u^\mu$. Furthermore the function $n$ on $\bar{\mathcal{S}}$ defined by the equation
\[
\Omega_{ABC}(f(x)) f^A_{\ \mu}(x) f^B_{\ \nu}(x) f^C_{\ \lambda}(x) = n(x) \epsilon_{\mu \nu \lambda \rho}(x) u^\rho(x)
\]should be everywhere positive on $\bar{\mathcal{S}}$. It is well known that (2) implies that the continuity equation
\[
\nabla_\mu (nu^\mu) = 0
\]is identically satisfied. The equations of motion are the Euler-Lagrange equations of the action
\[
S[f, \partial f] = \int_{\bar{\mathcal{S}}} \rho(f, \partial f, x) (-g)^{\frac{1}{2}} d^4x.
\]The function $\rho$ is required to be smooth in all its arguments. It plays the double role of being the Lagrangian as well as energy (i.e. rest mass plus internal elastic energy) density of the material. Diffeomorphism covariance demands that $\rho$ be of the form
\[
\rho(f, \partial f, x) = \hat{\rho}(f, H^{AB})|_{H^{AB} = h^{AB}(\partial f, x)},
\]where $h^{AB}(\partial f, x) = f^A_{\ \mu}(x) f^B_{\ \nu}(x) g^{\mu \nu}(x)$. Thus $\rho$ depends only on points $X$ on $\mathcal{B}$ and positive definite contravariant tensors $H^{AB}$ (see [2]). (In nonrelativistic elasticity materials governed by such a Lagrangian are said to satisfy the condition of "material frame indifference".) By abuse of notation we henceforth omit the hat from $\hat{\rho}$. It is useful to know that the Euler-Lagrange equations are equivalent to divergence-lessness of the symmetric energy momentum tensor. More precisely, there holds the identity (see [2])
\[
- \nabla_\nu T_{\mu}^\nu = \mathcal{E}_A f^A_{\ \mu},
\]
where
\[ T_{\mu\nu} = 2 \frac{\partial \rho}{\partial g^{\mu\nu}} - \rho g_{\mu\nu} \quad (7) \]
and
\[ -\mathcal{E}_A = (-g)^{-\frac{1}{2}} \partial_\mu \left( (-g)^{\frac{1}{2}} \frac{\partial \rho}{\partial (f^A,_{\mu})} \right) - \frac{\partial \rho}{\partial f^A}. \quad (8) \]

The relation (6) shows that of the four conservation laws \( \nabla_\nu T^{\mu\nu} = 0 \) only three are independent. This, of course, is due to the fact that within the present formalism the continuity equation is an identity.

We now turn to boundary conditions. The boundary conditions usually considered in nonrelativistic elasticity are either the so-called ”boundary conditions of place” - where, in our language, the set \( f^{-1}(\partial B) \) is prescribed, or ”traction boundary conditions” - where the normal traction, i.e. the components of the stress tensor normal to this surface are prescribed. When the normal traction is zero, one speaks of ”natural” boundary conditions: these are the boundary conditions appropriate for a freely floating elastic body considered in the present work. They are employed e.g. in geophysics for the elastic motion corresponding to seismic waves, the free boundary in question corresponding to the surface of the earth (see [1] ²). While one could in the present framework in principle consider all the above boundary conditions, it is interesting to observe that these conditions - except for the natural condition - become inconsistent once one couples to the Einstein equations. Namely the standard junction conditions, together with the Einstein equations, imply that \( T^\nu_{\mu\nu}n^\nu_{\nu}|_{f^{-1}(\partial B)} \) be zero, where \( n^\mu \) is the conormal of \( f^{-1}(\partial B) \). However these conditions are precisely the natural boundary conditions. To see this, we have to first compute the right hand side of Eq.(7). The result is
\[ T^\mu_{\nu\mu} = \rho u^\mu u^\nu + t^\mu_{\nu}, \quad (9) \]
where \( t^\mu_{\nu} \) is the (negative) Cauchy stress tensor given by
\[ t^\mu_{\nu} = n\tau_{AB}f^A_{\mu}f^B_{\nu} \quad (10) \]
and we have written \( \rho \) as
\[ \rho = n\epsilon \quad (11) \]
and \( \tau_{AB} = 2 \frac{\partial \xi}{\partial H_{AB}} \). Note this makes sense, since \( n \) - whence \( \epsilon \) - is a function purely of \( f \) and \( H^{AB} \), as is apparent from the identity
\[ 6n^2 = H^{AA'}H^{BB'}H^{CC'}\Omega_{ABC}\Omega_{A'B'C'}. \quad (12) \]

The function \( \epsilon \) is the relativistic version of the ”stored-energy-function” of standard elasticity. The quantity \( \tau_{AB} \) corresponds to the negative of the ”second

²Needless to say, these studies are confined to the linear approximation in which the problem of the free boundary disappears.
Piola-Kirchhoff stress” of nonrelativistic elasticity. Now to the boundary conditions, namely

\[ T_\mu^\nu n_\nu|_{f^{-1}(\partial B)} = 0. \]  

(13)

It follows from the tangency of \( u^\mu \) to the inverse images of points of \( \mathcal{B} \) under \( f \), that \( (u, n) = 0 \). Consequently the contraction of Eq. (13) with \( u^\mu \), using (9), is identically satisfied. The remaining components yield

\[ \tau_{AB} f^B g^{\mu\nu} n_\nu|_{f^{-1}(\partial B)} = 0 \]

(14)

Equation (14) will turn out to be a Neumann-type boundary condition, but it has a free (i.e. determined-by-\( f \)) boundary.

## 3 Natural configuration

We assume there exists a contravariant metric \( H_0^{AB}(X) \), smooth and positive definite on \( \overline{\mathcal{B}} \), such that

\[ \epsilon_{(X, H^{AB} = H_0^{AB}(X))} = \epsilon_0(X) > 0, \quad \frac{\partial \epsilon}{\partial H^{CD}} \bigg|_{(X, H^{AB} = H_0^{AB}(X))} = 0 \]

(15)

on \( \overline{\mathcal{B}} \). The quantity \( H_0^{AB} \) measures local distances in the relaxed body. The function \( \epsilon_0(X) \) is the density of rest mass in the stressfree configuration.

If there exists a configuration \( f_0 \) such that

\[ H_0^{AB}(f_0(x)) = h^{AB}(\partial f_0(x), x) = f_0^A_{\mu}(x) f_0^B_{\nu}(x) g^{\mu\nu}(x) =: h_0^{AB}(x), \]

(16)

this map \( f_0 \) is called a natural or relaxed configuration. Note that spacetime has to be special in order for a natural configuration to exist. Namely it follows from (16) that \( h_0^{AB} \) has to be constant along \( u_0^\mu \), the four velocity associated with \( f_0 \), from which one infers that \( u_0^\mu \) is a Born rigid motion (see [16]), i.e.

\[ \mathcal{L}_{u_0}(g_{\mu\nu} + u_0^\mu u_0^\nu) = 0 \]

(17)

One deduces from (6, 7, 9) and (15) that \( f_0 \) solves the equations of motion \( \mathcal{E}_A = 0 \) if and only if this rigid motion is geodesic. This in turn implies that \( u_0^\mu \) is Killing.

In this work we will assume that \( u_0^\mu \) is in addition irrotational. Using coordinates in which \( u_0^\mu \partial_\mu = \partial_t \) and \( u_0^\mu dx^\mu = -dt \), the spacetime metric has thus to be of the form \((i, j = 1, 2, 3)\)

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij}(x^k) dx^i dx^j. \]

(18)

(In particular: when the positive definite three metric \( g_{ij} \) is flat, we are in Minkowski spacetime.) With this choice of coordinates, the natural configuration \( f_0 \) is of the form \( f_0(t, x) = f_0(x) \) with \( f_0(x) \) a smooth, invertible function \( \mathcal{S} \cap \{ t = 0 \} \rightarrow \overline{\mathcal{B}} \) where, using coordinates on \( M \) and \( \overline{\mathcal{B}} \) in which \( \epsilon_{ijk} \) and \( \Omega_{ABC} \) are both positive, \( det(f_0^{A}_{\cdot \cdot}) \) is positive on \( \mathcal{S} \cap \{ t = 0 \} \), the latter condition guaranteeing that \( n_0 \) is positive and \( h_0^{AB} \) is positive definite on \( \mathcal{S} \cap \{ t = 0 \} \). The inverse of \( f_0(x) \) will be denoted by \( \Phi_0(X) \).
4 3 + 1 and the material picture

In this section we switch to the material description of the elastic medium by changing to “Lagrangian” coordinates. In a first step we replace the four velocity $u^\mu$ corresponding to the configuration $f$ by the coordinate velocity $v^\mu \partial_\mu = \partial_t + V^i \partial_i$, which is a multiple of $u^\mu$. Using (1) we find that $V^i$ is given in terms of $f^A$ by (note that $f^A_{,i}$ is non-degenerate by construction)

$$\dot{f}^A + f^A_{,i} V^i = 0,$$  \hspace{1cm} (19)

where a dot denotes partial differentiation w.r. to $t$. It follows from (18) that

$$h^{AB} = f^A_{,\mu} f^B_{,\nu} g^{\mu\nu} = f^A_{,i} f^B_{,j} (g^{ij} - V^i V^j).$$  \hspace{1cm} (20)

The timelike nature of $u^\mu$ (equivalently: the positive definiteness of $h^{AB}$) requires that

$$|V|^2 = g_{ij} V^i V^j < 1$$  \hspace{1cm} (21)

We also see that

$$n = k (1 - |V|^2)^{1/2},$$  \hspace{1cm} (22)

where $k$ is defined by

$$k \epsilon_{ijk} = f^A_{,i} f^B_{,j} f^C_{,k} \Omega_{ABC}$$  \hspace{1cm} (23)

with $\epsilon_{ijk}$ being the volume element of $g_{ij}$. Consequently the action (4) takes the form

$$S = \int_S (1 - |V|^2)^{1/2} \epsilon k (\det g_{ij})^{1/2} dt d^3x$$  \hspace{1cm} (24)

We now pass over to the material representation. By this we mean that configurations $f^A$ are replaced by "deformations" $x^i = \Phi^i(t, X)$ defined by

$$f^A(t, \Phi(t, X)) = X^A$$  \hspace{1cm} (25)

with $\det(\Phi^i_{,A})$ positive in $\bar{B}$. It follows from (23) that

$$f^A_{,i}(t, \Phi(t, X)) \Phi^i_{,B}(t, X) = \delta^A_{\ B}, \quad \Phi^i_{,A}(t, X) f^A_{,j}(t, \Phi(t, X)) = \delta^i_{\ j}$$  \hspace{1cm} (26)

and

$$\dot{f}^A(t, \Phi(t, X)) + f^A_{,i}(t, \Phi(t, X)) \dot{\Phi}^i(t, X) = 0$$  \hspace{1cm} (27)

and, from (27) and (19), that

$$V^i(t, \Phi(t, X)) = \dot{\Phi}^i(t, X)$$  \hspace{1cm} (28)

In order to study the field equations in the material representation it will be extremely convenient to change representation directly in the action (24). Using (23) there holds

$$S = \int_{\{t\} \times \bar{B}} \epsilon \gamma^{-1} dt d^3\Omega$$  \hspace{1cm} (29)
with
\[ \gamma^{-1} = (1 - |V|^2)^{\frac{1}{2}} \]  
and \( d^3\Omega = \Omega_{ABC}(X) \, dX^A \wedge dX^B \wedge dX^C = \Omega^\frac{1}{2} \, d^3X \). Now the field equations take the form
\[
\partial_t \left( \frac{\partial L}{\partial \dot{\Phi}^i} \right) + \partial_A \left( \frac{\partial L}{\partial \Phi^i_A} \right) - \frac{\partial L}{\partial \Phi^i} = 0 \quad \text{in } \mathcal{B}
\]
where \( L = \epsilon \gamma^{-1} \Omega^\frac{1}{2} \) for functions \( \Phi \) such that \( \text{det}(\Phi^i_A) \) is positive in \( \bar{\mathcal{B}} \). Note that \( \Omega \) depends only on \( X^A \), \( g_{ij} \) depends only on \( \Phi^i \) and \( \gamma^{-1} \) depends only on \( (\Phi^i, \dot{\Phi}^j) \).

Finally \( h^{AB} \), i.e.
\[
h^{AB} = F^A_i(\Phi^k_C,F^B_j(\Phi^l_D)g^{ij}(\Phi) - \dot{\Phi}^i \dot{\Phi}^j),
\]
where \( F^A_i(\Phi^k_C,F^B_j) \) is the inverse of \( \Phi^i_A \), depends on \( (\Phi^i, \dot{\Phi}^j, \Phi^k_B) \), so that \( \epsilon \) depends on \( (X^A, \Phi^i, \dot{\Phi}^j, \Phi^k_B) \). Using these facts, together with the identity
\[
\frac{\partial h^{AB}}{\partial \Phi^i_C} = -2 \epsilon C^{(A} F^{B)i},
\]
we find that
\[
\frac{\partial L}{\partial \Phi^i_A} = -2 \Omega^\frac{1}{2} N \gamma^{-1} \frac{\partial \epsilon}{\partial H^{BC}} \bigg|_{H^{DE} = h^{DE}} h^{AB} F^{C}i
\]
and
\[
\frac{\partial L}{\partial \Phi^i} = -\Omega^\frac{1}{2} \left[ \epsilon \gamma g_{ij} + 2 \frac{\partial \epsilon}{\partial H^{AB}} \bigg|_{H^{CD} = h^{CD}} F^A_i F^B_j \right] \dot{\Phi}^j.
\]
From (33) we infer that the boundary conditions (14) are simply equivalent to
\[
\left( \frac{\partial L}{\partial \Phi^i_A} \right) n_A \bigg|_{\{t\} \times \partial \mathcal{B}} = 0,
\]
where \( n_A \) is a conormal of \( \partial \mathcal{B} \).

We now turn to the natural deformation, which is the material version of the natural configuration described in Sect.3. In the chosen foliation it is time-independent, namely of the form
\[
\Phi^i(t, X) = \Phi^i_0(X),
\]
where \( \Phi_0(f_0(x)) = x \). Furthermore we have that
\[
F^A_i(\Phi^k_C,F^B_j(\Phi^l_D)g^{ij}(\Phi_0(X)) = H^{AB}_0(X),
\]
in short: \( F^A_i F^B_j g_0^{ij} = H^{AB}_0 \). We know from Sect.3 that the field equations and the boundary conditions are identically satisfied in the stress-free state. In the present context this fact takes the form
\[
\left( \frac{\partial L}{\partial \Phi^i} \right)_0 = 0, \quad \left( \frac{\partial L}{\partial \Phi^i_A} \right)_0 = 0, \quad \left( \frac{\partial L}{\partial \Phi^i} \right)_0 = 0,
\]
(39)
where the subscript 0 means evaluation in the stress-free state \((\Phi^i_i, \dot{\Phi}_i) = (\Phi_0^i, 0)\).

To derive Eq. (39) we have used \((15,34,35)\).

The causality properties of this system are essentially governed by the nature of the coefficients \(M_{ij}^{tt}\) of \(\Phi_j\) and \(M_{iAB}^{AB}\) of \(\Phi_{i,AB}\). In order to be able to apply the theorem of Koch [11], we will need a "negativity" condition on \(M_{ij}^{tt}\) and a certain positivity ("coerciveness") condition on \(M_{iAB}^{AB}\). We easily find that

\[
M_{ij}^{tt} = \left( \frac{\partial^2 L}{\partial \dot{\Phi}_i \partial \dot{\Phi}_j} \right)_0 = -\Omega^2 \epsilon_0 \eta_{0ij}
\]

(40)

For the coefficient of \(\Phi_{i,AB}\), using (34), observe that

\[
M_{iAB}^{AB} = \left( \frac{\partial^2 L}{\partial \Phi_i \partial \Phi_{j,B}} \right)_0 = 4 \Omega^2 \epsilon L_{0CDEF} H_0^{AE} H_0^{BF} F_0^C F_0^D,
\]

(41)

where

\[
L_{0ABCD} := \left( \frac{\partial^2 \epsilon}{\partial H^A \partial H^C \partial H^D} \right) \bigg|_{H^E = H_0^E}.
\]

(42)

5 Koch theorem

For convenience we state here a corollary of the theorem in [11], which is the precise statement we need.

Let \(\Phi^i(t, X)\) be maps from \(\{t\} \times B\) to \(\mathbb{R}^3\), where \(B\) is an open set in \(\mathbb{R}^3\) with smooth boundary \(\partial B\). We are given a system of the form of

\[
\partial_\alpha F_i^{\alpha} = w_i \text{ in } \{t\} \times B,
\]

(43)

where \(\alpha\) runs from 0 to 3 and with \(F_i^{\alpha}\) and \(w_j\) all being smooth functions of \((X^A, \Phi^i, \Phi^j, \alpha)\) on \(\bar{B} \times \mathbb{R}^3 \times \mathbb{R}^{12}\), together with the boundary conditions

\[
F_i^{\alpha} n_\alpha|_{\{t\} \times \partial B} = F_i^{A} n_A|_{\{t\} \times \partial B} = 0.
\]

(44)

We make the following further assumptions:

(i) Symmetry: there hold the symmetries \(M_{ij}^{\alpha \beta} = M_{ji}^{\beta \alpha}\) where \(M_{ij}^{\alpha \beta} = \frac{\partial F_i^{\alpha}}{\partial \Phi_{j,\beta}}\).

(ii) Static solution: There is given a time independent function \(\Phi^i_0(t, X) = \Phi^i_0(X) \in C^\infty(\bar{B})\) satisfying

\[
F_i^{\alpha} = 0, \quad w_i = 0
\]

(45)

and the following estimates

(iii) Time components:

\[
M_{ij}^{tt} \eta^i \eta^j \leq -\kappa |\eta|^2 \text{ in } B
\]

(46)
for a positive constant $\kappa$,

(iv) Space components:

$$
\int_B \mathcal{M}_{0}^{AB} \delta \Phi^i_{,A} \delta \Phi^j_{,B} d^3 X + ||\delta \Phi||^2_{L^2(B)} \geq \sigma ||\delta \Phi||^2_{H^1(B)}
$$

for $\sigma > 0$ and all $\delta \Phi \in C^\infty(\bar{B})$, where $H^s$ denotes the Sobolev space $H^{s;2}$. We remark that condition (47) implies the "strong ellipticity" or "Legendre-Hadamard" condition, namely that $\mathcal{M}_{0}^{AB} N_A N_B k^i k^j > 0$ in $\bar{B}$ for non-zero $N_A, k^i$. This latter condition is the relevant one for wellposedness in the pure initial value problem (see [9]).

**Theorem:** Let $(\Phi(0), \dot{\Phi}(0))$ lie in a small neighborhood of $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B})$, $s \geq 3$ and satisfy the corner conditions of order $s$ in the sense that $\partial^r F_{A} |_{t=0} \times \partial B$ is in $H^{s-r}(\mathcal{B}) \cap H^1_0(\mathcal{B})$ for $0 \leq r \leq s$. Then there exists, for sufficiently small $t_0$, a unique classical solution $\Phi$ of (43,44) in $C^2([0, t_0]) \times \bar{B}$ with $(\Phi(0), \dot{\Phi}(0))$ as initial data. Furthermore $\partial^r \Phi(t) \in L^2(\mathcal{B})$ for $0 \leq r \leq s + 1$.

In the last expression $\partial^r \Phi$ denotes all partial derivatives of order $r$. The evolved solution stays close to the static solution $\Phi_0$ in the sense that $(\Phi(t), \dot{\Phi}(t))$ remains close to $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B})$ and $\partial^r \Phi(t)$ remains close to $\partial^r \Phi_0 \in L^2(\mathcal{B})$ for $0 \leq r \leq s + 1$. (This last statement is the "stability" part of the theorem.)

### 6 Main result

We now add our final constitutive assumption, sometimes called "pointwise stability" see [13], namely that

$$
\mathcal{L}_{0}^{ABCD} N^{AB} N^{CD} \geq \sigma (H^{0CA}H^{0BD} + H^{0CB}H^{0AD}) N^{AB} N^{CD} \quad \text{in } \mathcal{B}
$$

where $\sigma$ is a positive constant which only depends on the choice of coordinates and with $H_{0,AB}$ being the inverse of $\mathcal{H}_{0}^{AB}$. Condition (48) is a convexity condition on $\rho$ as a function of $H^{AB}$. We note that this in general does not imply convexity with respect to the deformation gradient $\Phi^i_{,A}$ (see [5]). An important special case is where

$$
4\varepsilon_0 \mathcal{L}_{0}^{ABCD} = \lambda H^{0AB}H^{0CD} + \mu(H^{0CA}H^{0BD} + H^{0CB}H^{0AD})
$$

and

$$
\mu(X) > 0, \quad 3\lambda(X) + 2\mu(X) > 0 \quad \text{in } \bar{B}
$$

The quantities $\mu$ and $\lambda$, when they are independent of $X$, are the Lamé constants of homogeneous, isotropic materials. We now invoke the Korn inequality (see

---

3This condition says that $(\partial^r F_{A}) |_{t=0} \times \partial B$ is zero for $0 \leq r \leq s$ In the appendix we show that it can be met by a suitable choice of odd-order normal derivatives of $(\Phi(0), \dot{\Phi}(0))$ on $\partial B$.}
e.g. [8] or [6]) of which we need a slight generalization due to [3], in the following
form: Let $\Psi^A$ be vector field on $(\mathcal{B}, H_{0,AB})$ in some chart. Let $L(\Psi)$ be defined
by $L(\Psi)_{AB} = 2H_{0C(A}\Psi^{C=B)}$, i.e. the Lie derivative of $H_{0,AB}$ w.r. to $\Psi$ to leading
derivative-order in $\Psi^A$. Then there is a positive constant $\sigma'$ such that
\[
\|L(\Psi)\|_{L^2(\mathcal{B})}^2 + \|\Psi\|_{L^2(\mathcal{B})}^2 \geq \sigma'\|\Psi\|_{H^1(\mathcal{B})}^2.
\] (51)

We now assume coordinates $X$ in $\mathcal{B}$ to be chosen so that $\Phi_0$ is the identity map.
Combining (41) with (48) and using (51), there follows condition (iv), i.e. that
\[
\int_{\mathcal{B}} M^A_{ij} \delta \Phi^i_A \delta \Phi^j_B \, d^3X + \|\delta \Phi\|_{L^2(\mathcal{B})}^2 \geq \sigma''\|\delta \Phi\|_{H^1(\mathcal{B})}^2
\] (52)
for positive $\sigma''$. The validity of the condition (iii) in the Koch theorem is immedi-
ate from Eq.(40). Furthermore the validity of (ii) has been checked in Sect.4. The
validity of (i) is obvious from the variational character of the equations. Thus all
assumptions of the Koch theorem are met.

It remains to check the determinant condition $det(\Phi^i_A) > 0$ in $\bar{\mathcal{B}}$. But, when
$(\Phi(0), \dot{\Phi}(0))$ is close to $(\Phi_0, 0)$ in $H^{s+1}(\mathcal{B}) \times H^s(\mathcal{B})$, $s \geq 3$, this immediately follows
by Sobolev embedding and the positivity of $det(\Phi^i_A)$ in $\bar{\mathcal{B}}$. Now the precise form
of our final statement can be read off from the Koch theorem in Sect.5.

Stated somewhat informally, our final result is as follows:

**Theorem:** Let there be given a volume form $\Omega(X^A)$, a spacetime metric $g_{\mu\nu}(x^A)$
of the form (18) and an internal energy $\epsilon(X^A, H^{BC})$ satisfying (15), all smooth
functions of their respective arguments. Suppose there exists a smooth natural
(i.e. stress-less) configuration such that the corresponding vector field $u^\mu_0$ is the
static Killing field $\partial_t$. Also suppose that the elasticity tensor for this natural
configuration satisfies the pointwise stability condition (48). Let the initial data
$(\Phi(0), \dot{\Phi}(0))$ for the deformation be close to those for the natural deformation and
satisfy the corner condition of the appropriate order. Then there exists, for these
initial data, a solution $\Phi(t)$ of the dynamical equations (31) with boundary condi-
tions (36) for small times. This solution remains close to the natural deformation.

We end with a remark on the possible generalization of the results presented
here to the case of an elastic body (or bodies, if there are several) which are
self-gravitating. The presence in G.R. of constraints and gauge freedom will then
make things more complicated. Furthermore the fact that the elastic equations
are written in material form, but the Einstein equations are equations on space-
time, means that one is now not dealing with a system of partial differential
equations in the strict sense.

**Acknowledgement:** We are grateful to Bernd Schmidt for very useful remarks
on the manuscript. One of us (R.B.) thanks Tom Sideris for pointing out the
relevance of the work of Koch and Sergio Dain for helpful discussions on the Korn inequality. This work has been supported by Fonds zur Förderung der Wissenschaftlichen Forschung in Österreich, project no. P16745-NO2

7 Appendix: corner conditions

We study corner conditions for the system \((43)\) with the boundary conditions \((44)\). Suppose that \((\Phi(0)|_{\partial B}, \dot{\Phi}(0)|_{\partial B})\) is sufficiently close to \((\Phi_0|_{\partial B}, 0)\). In order for obtaining a solution of the equations of motion, one has to be able to satisfy conditions on the behaviour of certain normal derivatives of \((\Phi(0), \dot{\Phi}(0))\) on \(\partial B\), which guarantee that, not only \(\mathcal{F}^A_i n_A|_{\partial B} = 0 \) at \(t = 0\), but also a sufficient number of time derivatives at \(t = 0\) of that condition is satisfied. (It is understood that higher-than-second-order time derivatives are eliminated in terms of spatial and lower-order time derivatives, using the equations of motion: this by virtue of the negativity of \(M_{ij0}^{tt} n_A n_B|_{\partial B}\) is always possible for \((\Phi(0)|_{\partial B}, \dot{\Phi}(0)|_{\partial B})\) close to \((\Phi_0|_{\partial B}, 0)\).)

First recall that \(\Phi_0(t, X) = \Phi_0(X)\) identically satisfies the equations of motion and the boundary condition. Now we check that the corner condition of order 0, i.e. the undifferentiated-in-time Eq.\((36)\) can be satisfied by a choice of \(\partial_n \Phi(0)|_{\partial B}\) (where \(\partial_n\) denotes any derivative transversal to \(\partial B\)). To see this notice first that, by the above observation, \((\Phi_0|_{\partial B}, 0)\) solves the order-0 corner condition. Furthermore

\[
\frac{\partial^2 L}{\partial \partial_n \Phi_j \partial \Phi^i A} n_A|_{\partial B} = \frac{\partial^2 L}{\partial \Phi^i B \partial \Phi^i A} n_A n_B|_{\partial B}.
\]

Then the result follows from the (finite-dimensional) implicit function theorem using that, by virtue of \((48)\), the quadratic form

\[
M_{ij0}^{AB} n_A n_B|_{\partial B}
\]

is nonsingular, which in turn follows from the strong ellipticity condition (see Sect.5). One now performs the process of taking \((s \geq 1\) say) time derivatives of the boundary condition and eliminating \(\partial^m \Phi(0)|_{\partial B}\) for \(s + 1 \geq m > 1\). Using that \(\mathcal{F}^0_i\) is zero for \((\Phi(0)|_{\partial B}, \dot{\Phi}(0)|_{\partial B}) = (\Phi_0|_{\partial B}, 0)\), the \(s\)-th order corner condition becomes an equation of the form

\[
M_{ij0}^{AB} n_A n_B \partial_S^n \Phi^j(0)|_{\partial B} \doteq \text{lower order}
\]

for odd \(s\) and

\[
M_{ij0}^{AB} n_A n_B \partial_S^{s+1} \Phi^j(0)|_{\partial B} \doteq \text{lower order}
\]

for even \(s\), where “\(\doteq\) lower order” means expressions which depend on lower-order, even-numbered normal derivatives of \((\Phi(0)|_{\partial B}, \dot{\Phi}(0)|_{\partial B})\), modulo terms which depend on normal derivatives of the same order, but which are zero when \((\Phi(0)|_{\partial B}, \dot{\Phi}(0)|_{\partial B}) = (\Phi_0|_{\partial B}, 0)\). The equations \((55, 56)\) are identically satisfied if
(Φ(0)|∂B, ˙Φ(0)|∂B) and their normal derivatives appearing on both sides are replaced by those of (Φ₀|∂B, 0). Using the fact that $M^{AB}_{ij} n_A n_B$ depends only on first derivatives of Φ, we are now able to recursively solve the corner conditions, with Φ(0)|∂B and ˙Φ(0)|∂B and their even normal derivatives given arbitrarily, provided that (Φ(0)|∂B, ˙Φ(0)|∂B) is close to (Φ₀|∂B, 0). We have to - and by the above can - choose a large class of initial data (Φ(0), ˙Φ(0)) close to (Φ₀, 0) so that the corner conditions are fulfilled for arbitrary order and the negativity of $M^{u}_{ij}$ and the coerciveness of $M^{AB}_{ij}$ are satisfied.

References

[1] Aki K and Richards P G 2002 Quantitative Seismology (Sausalito: University Science Books)

[2] Beig R and Schmidt B G 2003 Relativistic elasticity Class. Quantum Grav. 20 889-904

[3] Chen W and Jost J 2002 A Riemannian version of Korn’s inequality Calc. Var. Partial Differential Equations 14 517-530

[4] Christodoulou D 2000 The Action Principle and Partial Differential Equations (Princeton: Princeton University Press)

[5] Ciarlet P G 1988 Mathematical Elasticity, Volume 1: Three-Dimensional Elasticity North-Holland

[6] Dain S 2005 Generalized Korn’s inequality and conformal Killing vectors arXiv:gr-qc/0505022

[7] Ebin D 1992 Deformations of Incompressible Bodies with Free Boundaries Arch. Rat. Mech. Anal. 120 61-97

[8] Horgan C O 1995 Korn’s inequalities and their applications in continuum mechanics SIAM Rev. 37 491-511

[9] Hughes T, Kato T and Marsden J 1976 Well-posed quasilinear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity Arch. Rat. Mech. Anal. 63 273-294

[10] Kijowski J and Magli G 1992 Relativistic elastomechanics as a Lagrangian field theory J. Geom. Physics 9 207-223

[11] Koch H 1993 Mixed problems for fully nonlinear hyperbolic equations Math. Z. 214 9-42
[12] Lindblad H 2005 Wellposedness for the motion of a compressible liquid with free surface boundary *Commun.Math.Phys.* 260 319-392

[13] Marsden J and Hughes T 1994 *Mathematical foundations of elasticity* (New York: Dover)

[14] Rendall A 2005 Theorems on Existence and Global Dynamics for the Einstein Equations *Living Rev. Relativity* 6 4.URL (cited on <9.December 2005>): [http://www.livingreviews/Irr-2005-6](http://www.livingreviews/Irr-2005-6)

[15] Sideris T 2004 Nonlinear hyperbolic systems and elastodynamics, preprint

[16] Trautman A 1965 Foundations and Current Problems of General Relativity in: *Lectures on General Relativity* ed S Deser and K W Ford New Jersey: Prentice-Hall