Separation-Free Spectral Super-Resolution via Convex Optimization

Zai Yang*, Yi-Lin Mo*, Gongguo Tang†, and Zongben Xu*

November 29, 2022

Abstract

Atomic norm methods have recently been proposed for spectral super-resolution with flexibility in dealing with missing data and miscellaneous noises. A notorious drawback of these convex optimization methods however is their lower resolution in the high signal-to-noise (SNR) regime as compared to conventional methods such as ESPRIT. In this paper, we devise a simple weighting scheme in existing atomic norm methods and show that the resolution of the resulting convex optimization method can be made arbitrarily high in the absence of noise, achieving the so-called separation-free super-resolution. This is proved by a novel, kernel-free construction of the dual certificate whose existence guarantees exact super-resolution using the proposed method. Numerical results corroborating our analysis are provided.

Keywords: Spectral super-resolution, separation-free, weighted atomic norm, convex optimization, dual certificate

1 Introduction

The problem of resolving several unknown frequency components from discrete samples of their mixture in the time/spatial domain is known as line spectral estimation or spectral super-resolution [1,2]. Its equivalent forms are also referred to as spectral compressed sensing or infinite-dimensional/off-the-grid/continuous compressed sensing [3–6]. Spectral super-resolution is a fundamental problem in statistical signal processing and has been studied under various topics such as direction-of-arrival estimation in array processing [7], beamforming in acoustics [8], channel estimation in wireless communications [9], and modal analysis in structural health monitoring [10].

In this paper, we consider the general multichannel spectral super-resolution problem. In particular, we have access to an $N \times L$ data matrix $Y$ composed of equispaced samples $\{y_{jl}\}$ that, in the absence of noise, is given by

$$y_{jl} = \sum_{k=1}^{K} e^{i2\pi(j-1)f_k} s_{kl}, \quad j = 1, \ldots, N, \quad l = 1, \ldots, L,$$

where $L$ is the number of channels, $N$ is the sample size per channel, $i = \sqrt{-1}$, and $s_{kl}$ denotes the unknown complex amplitude of the $k$th frequency $f_k$ in the $l$th channel. In [1], the frequencies $\{f_k\}$ are normalized

---

*School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, Shaanxi 710049, China
†School of Electrical, Computer & Energy Engineering, University of Colorado, Boulder, CO80309, US
The frequencies are exactly recovered as
\[ \Delta \]
derived from the sampling frequency (at a Nyquist rate) and belong to the unit circle \( T = [0, 1] \), where 0 and 1 are identified. Let \( \mathbf{a}(f) = [1, e^{i2\pi f}, \ldots, e^{i2\pi(N-1)f}]^T \) denote a sampled complex sinusoid of frequency \( f \) and \( s_k = [s_{k1}, \ldots, s_{kL}] \) be the coefficient vector of the \( k \)th component. The data matrix \( \mathbf{Y} \) is then expressed as
\[
\mathbf{Y} = \sum_{k=1}^{K} \mathbf{a}(f_k) s_k = \mathbf{A}(f) \mathbf{S},
\]
where \( \mathbf{A}(f) = [\mathbf{a}(f_1), \ldots, \mathbf{a}(f_K)] \) is an \( N \times K \) Vandermonde matrix and \( \mathbf{S} = [s_{11}^T, \ldots, s_{K1}^T]^T \) is the matrix of coefficients. Given \( \mathbf{Y} \) (possibly corrupted by noise), our goal is to recover/estimate \( \{f_k\} \) and \( s_{kl} \).

Spectral super-resolution is complicated by the fact that the data samples are highly nonlinear functions of the frequencies \( \{f_k\} \) of interest and also by the need of resolving close-located frequencies. To avoid solving a nonconvex optimization problem (e.g., the one induced by the maximum likelihood method), subspace-based methods were developed in the 1970s and since then have dominated the research on this topic for decades. This kind of methods enjoy good statistical properties in the presence of white Gaussian noise \([11]\). They have high resolution in the regime of high signal-to-noise ratio (SNR), and such a resolution can be made arbitrarily high in the limiting noiseless case \([12, 13]\). However, their drawbacks are also evident. For example, they need the number \( K \) of frequencies that is usually unknown in practice, and their performances deteriorate in the presence of outliers or missing data.

Sparse estimation and compressed sensing methods \([7, 14, 15]\), especially the recent atomic norm approaches \([2, 5, 16–24]\), have been proposed to overcome the aforementioned drawbacks of subspace methods. Differently from the subspace methods, sparse methods use the fact that the unknown number of frequency components is small—the so-called signal sparsity—and attempt to solve for this number jointly with the frequencies by solving a typically convex optimization problem. This optimization framework is flexible in dealing with missing or corrupted samples by taking them as optimization variables. Moreover, differently from earlier compressed sensing methods that only work for discretized frequencies, atomic norm approaches work directly on the continuum, can be cast as semidefinite programming and have provable theoretical guarantees for off-the-grid frequencies.

A key ingredient in the theoretical guarantees for atomic norm approaches is a frequency separation condition (that reflects the resolution limit), to be specific, a wrap-round distance above \( 4/N \) between any two frequencies. While such a separation condition can be improved using advanced proof techniques (see, e.g., \([21]\)), we provide a simulation result to show that it cannot be removed for atomic norm approaches even in the limiting case with noiseless data and infinitely many channels. In this case, we have access to the noiseless data covariance matrix that under mild assumption is a rank-\( K \), positive-semidefinite Hermitian Toeplitz matrix, and the atomic norm method can be implemented equivalently based on this covariance matrix \([23]\). We consider \( K = 2 \) frequency components with unit power and frequencies \( \{0.1, 0.1 + \Delta f\} \), where \( \Delta f \) varies from \( 0.05/N \) to \( 1/N \) and \( N = 20 \). Our numerical results are presented in Fig. 1. A sharp phase transition is seen due to the fact that the randomness in the sample covariance matrix diminishes as \( L \to +\infty \). The frequencies are exactly recovered as \( \Delta f > \frac{1}{2N} \) and failures occur as \( \Delta f \leq \frac{1}{2N} \).

In contrast to the atomic norm method, it is well-known that subspace methods do not suffer from a resolution limit in the absence of noise. In fact, other conventional methods such as Prony’s method or Capon’s beamforming \([25]\) are also known to be separation-free in the noiseless case, though their performances can be sensitive to noise. Therefore, it is crucial to understand and overcome the resolution limit of atomic norm approaches for their application in the high SNR regime. The study of this topic is expected to close the gap between atomic norm and subspace methods as well as to shed light on novel techniques for spectral super-resolution.
We note that separation-free, convex optimization techniques have recently been proposed in literature; see \([26-30]\). However, all these studies assume positive (as opposed to complex in our case) coefficients of the spectral components and simplify the problem at hand.

In this paper, we show that the frequency separation condition can be removed in the atomic norm method if it is modified appropriately. In particular, inspired by \([6]\), we devise a weighted atomic norm technique in which a weighting function is constructed from the data and used to assign preference of the frequency candidates in the atomic norm. The weighted atomic norm can be cast as convex programming like the standard atomic norm and it degenerates into the standard one in the case of a non-informative, constant weighting function. We provide a rigorous analysis showing that the frequency components can be exactly recovered by the weighted atomic norm method under mild conditions, without any frequency separation condition. Numerical results are provided that validate our analysis.

Our technical proof is based on a novel construction of the dual certificate which verifies optimality of the true frequencies among all feasible solutions to the proposed convex optimization problem. While such a dual certificate was constructed for the standard atomic norm based on low-pass kernels (see \([2,21]\)), we do not use any kernels in our proof. This allows us not to sacrifice the resolution due to the choice of a specific kernel.

Notations used in this paper are as follows. The sets of real and complex numbers are denoted by \(\mathbb{R}\) and \(\mathbb{C}\) respectively. Bold-face letters are reserved for vectors and matrices. For vector \(\mathbf{x}\), \(\text{diag}(\mathbf{x})\) denotes a diagonal matrix with \(\mathbf{x}\) on the diagonal. For matrix \(\mathbf{X}\), \(\text{diag}(\mathbf{X})\) forms the diagonal of \(\mathbf{X}\) as a column vector. For matrix \(\mathbf{A}\), its conjugate, transpose, conjugate transpose, inverse, pseudo-inverse, rank, trace, spectral norm and Frobenius norm are denoted by \(\overline{\mathbf{A}}, \mathbf{A}^T, \mathbf{A}^H, \mathbf{A}^{-1}, \mathbf{A}^\dagger, \text{rank}(\mathbf{A}), \text{tr}(\mathbf{A}), \|\mathbf{A}\|_2\) and \(\|\mathbf{A}\|_F\), respectively. That \(\mathbf{A}\) is Hermitian positive semidefinite is expressed as \(\mathbf{A} \succeq 0\). For matrices \(\mathbf{A}, \mathbf{B}\) of proper dimensions, their Kronecker, Khatri-Rao (or column-wise Kronecker) and Hadamard products are...
denoted by $A \otimes B$, $A \ast B$ and $A \odot B$, respectively. The $N \times N$ identity matrix is denoted by $I_N$, in which the subscript can be omitted. For function $a(f)$ of $f$, its first and second derivatives are denoted by $\frac{da(f)}{df}$ and $\frac{d^2a(f)}{df^2}$, or $a'(f)$ and $a''(f)$, or $a^{(1)}(f)$ and $a^{(2)}(f)$ for convenience. The real part of $x$ is denoted by $\Re x$.

The rest of the paper is organized as follows. Section 2 presents our main results consisting of the weighted atomic norm method and its theoretical guarantee. Section 3 presents the proof of the main theorem. Section 4 presents numerical results validating our analysis, and Section 5 concludes this paper.

2 Main Results

2.1 Atomic Norm Method

It is seen from (2) that

$$Y = \sum_{k=1}^{K} \| s_k \|_2 \cdot a(f_k) \cdot \frac{s_k}{\| s_k \|_2}, \quad (3)$$

can be written as a mixture of $K$ atoms in the set of atoms

$$A = \{ a(f) \phi : f \in T, \phi \in \mathbb{C}^{L \times 1}, \| \phi \|_2 = 1 \}. \quad (4)$$

To seek for the spectral components of $Y$ by exploiting the spectral sparsity in the sense that $K$ is small, the atomic norm of $Y$ induced by the set of atoms $A$ is defined as [16, 19, 20]:

$$\| Y \|_A = \inf \left\{ \sum c_k : Y = \sum c_k a_k, c_k > 0, a_k \in A \right\}. \quad (5)$$

It is shown in [19, 20] that the atomic norm $\| Y \|_A$ can be computed using the following semidefinite programming (SDP):

$$\| Y \|_A = \min_{X, T} \frac{1}{2} \text{tr}(X) + \frac{1}{2N} \text{tr}(T), \text{ subject to } \begin{bmatrix} X & Y^H \\ Y & T \end{bmatrix} \succeq 0 \text{ and } T \text{ is Toeplitz.} \quad (6)$$

Once (6) is solved numerically, we apply the Carathéodory-Fejér theorem [7, Theorem 11.5] to the solution $T^*$ and obtain the Vandermonde decomposition (that is unique if $T^*$ is rank deficient):

$$T^* = \sum_{k=1}^{K^*} p^*_k a(f^*_k) a^H(f^*_k) = A(f^*) \text{diag}(p^*_s) A^H(f^*), \quad K^* = \text{rank}(T^*), \quad (7)$$

where $K^*$, $f^*$ are estimates of $K$ and $f$, respectively. By using the column inclusion lemma for positive semidefinite matrices, there exist matrix $S^*$ such that

$$Y = A(f^*) S^* = \sum_{k=1}^{K^*} a(f^*_k) s^*_k, \quad (8)$$

where $s^*_k$ denotes the $k$th row of $S^*$. The atomic decomposition of $Y$ resulting from (8) achieves the atomic norm in the sense that $\sum_{k=1}^{K^*} \| s^*_k \|_2 = \| Y \|_A$ [19]. Moreover, the above process recovers exactly $f, S$ if the frequencies are mutually separated by at least $\frac{4}{N}$, an assumption that can be weakened but not removed as shown in Fig. [1].
2.2 Weighted Atomic Norm Method

Suppose that \( w(f) > 0 \) is a weighting function. The weighted atomic norm of \( Y \) associated with \( A \) and \( w(f) \) is defined as [6, 31]:

\[
\|Y\|_{A^w} = \inf \left\{ \sum_k w(f_k)c_k : \ Y = \sum_k c_k a_k, \ c_k > 0, \ a_k \in \mathcal{A} \right\}.
\] (9)

Intuitively, the smaller the weighting function \( w(f) \) is, the more likely the associated frequency \( f \) is selected by the weighted atomic norm. In the case when \( w(f) \) is constant, the weighted atomic norm degenerates into the atomic norm. Therefore, the key to a better performance by using the weighted atomic norm is to design a good weighting function.

Let

\[
w(f) = \sqrt{a^H(f)W a(f)}, \quad W \succeq 0.
\] (10)

It is shown in [6] that the weighted atomic norm \( \|Y\|_{A^w} \) can be cast as the following SDP:

\[
\|Y\|_{A^w} = \min_{X,T} \frac{1}{2} \text{tr}(X) + \frac{1}{2} \text{tr}(WT), \quad \text{subject to} \quad \begin{bmatrix} X & Y^H \\ Y & T \end{bmatrix} \succeq 0 \ \text{and} \ T \ \text{is Toeplitz}.
\] (11)

Evidently, (11) degenerates into (6) if we let \( W = \frac{1}{N} I_N \) and \( w(f) \equiv 1 \). Once the SDP in (11) is solved numerically, as in the case of the atomic norm, the atomic decomposition of \( Y \) that achieves the weighted atomic norm can be obtained from the Vandermonde decomposition of the solution to \( T \).

In this paper, we use the following weight inspired by [6, 32]:

\[
W = \begin{cases} \left( I + \epsilon^{-1} \hat{R} \right)^{-1}, & \text{if } \hat{R} \text{ is rank deficient;} \\ \left( \epsilon I + \hat{R} \right)^{-1}, & \text{otherwise}, \end{cases}
\] (12)

where \( \hat{R} = \frac{1}{L} YY^H \) is the sample covariance matrix and \( \epsilon > 0 \) is a regularization parameter. Note that the two choices of the weight differ only by a global scaling factor and thus result in the same frequency estimates. They are made different for computational convenience. By using the second expression of \( W \), we have

\[
w^{-2}(f) = \frac{1}{a^H(f) \left( \epsilon I + \hat{R} \right)^{-1} a(f)},
\] (13)

which coincides with Capon’s beamforming [25] if \( \epsilon \) is omitted. Since the spectrum of Capon’s beamforming is usually peaked around the true frequencies, the weighted atomic norm is expected to have good performance, but no rigourous analysis has been derived.

2.3 Separation-Free Theoretical Guarantee

In this paper, we mainly consider noiseless measurements and in this case \( \hat{R} \) is rank deficient. Our main result is provided in the following theorem.

Theorem 1. Assume we have the \( N \times L \) data matrix \( Y = A(f) S \), where the \( K < N \) frequencies \( \{f_k\} \) are distinct and the \( K \times L \) matrix \( S \) has full row rank. Let \( \hat{R} = \frac{1}{L} YY^H \) and \( W = \left( I + \epsilon^{-1} \hat{R} \right)^{-1} \). Then, there exists \( \epsilon_0 > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_0 \), computing the weighted atomic norm of \( Y \) by solving (11) produces the true frequencies \( \{f_k\} \).
Theorem 1 states that the frequencies \( \{ f_k \} \) can be exactly recovered by using the proposed weighted atomic norm method if \( \epsilon \) is appropriately small. In contrast to the standard atomic norm method, no frequency separation condition is required in Theorem 1. According to our proof of Theorem 1, which will be provided in the ensuing section, smaller separations among the frequencies \( \{ f_k \} \) result in more ill-conditioned matrix \( A(f) \) and a smaller \( \epsilon_0 \). This causes potential numerical instability to the SDP solving. In fact, similar numerical issues stem from the problem setup and are also encountered for other super-resolution algorithms, e.g., ESPRIT.

The assumption that \( S \) has full row rank is satisfied if \( L \geq K \) and the rows of \( S \) are at general positions, which in fact is also required for subspace methods or Capon’s beamforming. In the case when this assumption is not satisfied, e.g., due to limited number of channels (a.k.a., \( L < K \)) or presence of proportional rows of \( S \), spatial smoothing techniques have been widely studied and successfully employed to restore the performance of subspace methods by creating more, overlapping and shortened measurement vectors from the original measurements \( Y \) [13, 33]. Such spatial smoothing techniques can also be adopted in the proposed weighted atomic norm method.

3 Proof of Theorem 1

We prove Theorem 1 in this section. Our proof follows the routines for atomic norm methods (see [2, 5, 19, 21–23]): find a sufficient condition characterized by the so-called dual certificate and then construct the dual certificate. In fact, such routines are quite common in related sparse recovery problems such as compressed sensing and low-rank matrix recovery; see [14, 34]. But differently from previous kernel-based techniques for atomic norm methods, our approach is kernel-free and involves no frequency separation condition.

3.1 Dual Certificate

The following proposition is a corollary to the proof of [19, Theorem 4].

**Proposition 1.** Computing the weighted atomic norm by solving (11) produces the true frequencies \( \{ f_k \} \) if there exists a vector-valued dual certificate \( Q(f) = a^H(f)V \), \( V \in \mathbb{C}^{N \times L} \) satisfying that

\[
Q(f_k) = w(f_k)\phi_k, \quad k = 1, \ldots, K; \tag{14}
\]

\[
\|Q(f)\|_2 < w^2(f), \quad f \in T \setminus \{ f_k \}. \tag{15}
\]

**Proof.** Note that the weighted atomic norm in (9) can be identified as a specialized atomic norm induced by the atomic set

\[
A^w = \{ w^{-1}(f)a(f)\phi : \ f \in T, \ \phi \in \mathbb{C}^{L \times 1}, \ \| \phi \|_2 = 1 \}. \tag{16}
\]

It then follows from the proof of [19, Theorem 4] that computing such atomic norm produces the true frequencies \( \{ f_k \} \) if there exists a dual certificate \( \overline{Q}(f) = w^{-1}(f)a^H(f)V \), \( V \in \mathbb{C}^{N \times L} \) satisfying that

\[
\overline{Q}(f_k) = \phi_k, \quad k = 1, \ldots, K; \tag{17}
\]

\[
\|\overline{Q}(f)\|_2^2 < 1, \quad f \in T \setminus \{ f_k \}, \tag{18}
\]

or equivalently, if there exists \( Q(f) = w(f)\overline{Q}(f) = a^H(f)V \) satisfying (14) and (15), completing the proof.

The remaining task of the proof is to find such a dual certificate \( Q(f) = a^H(f)V \) in Proposition 1.
3.2 Construction of Dual Certificate

We construct the dual certificate $Q(f) = A^H(f) V$ in Proposition 1 by finding a proper $V$ in this subsection. It follows immediately from (14) that $V$ must satisfy that

$$Q(f_k) = a^H(f_k) V = w(f_k) \phi_k, \quad k = 1, \ldots, K,$$

or equivalently,

$$A^H V = \text{diag}(\alpha) \Phi,$$

where $A$ is short for $A(f)$, $\Phi$ is a $K \times L$ matrix of which the $k$th row is given by $\phi_k$, and $\alpha = [w(f_1), \ldots, w(f_K)]^T$. To make (15) hold true, a necessary condition is the following:

$$\frac{d}{df} \left( \|Q(f_k)\|_2^2 - w^2(f_k) \right) = 0, \quad k = 1, \ldots, K,$$

or equivalently,

$$\Re \left\{ Q'(f_k) Q^H(f_k) - a^H(f_k) W a'(f_k) \right\} = 0,$$

where (10) was used. Inserting (14) and (19) into (22) yields that

$$\Re \left\{ A^H(f_k) V \phi_k^H w(f_k) - a^H(f_k) W a'(f_k) \right\} = 0.$$

Consequently, one feasible choice making (23) and thus (21) hold is to impose that

$$a^H(f_k) V \phi_k^H = \frac{a^H(f_k) W a'(f_k)}{w(f_k)}, \quad k = 1, \ldots, K,$$

or equivalently,

$$\text{diag} \left( A'^H V \Phi^H \right) = \beta,$$

where $A' = [a'(f_1), \ldots, a'(f_K)]$ and $\beta$ is a $K \times 1$ vector with the $k$th entry being $\frac{a^H(f_k) W a'(f_k)}{w(f_k)}$.

Note that (20) and (25) form a system of linear equations with respect to $V$ that can be written equivalently as:

$$\left[ \begin{array}{c} I_L \otimes A^H \\ (\Phi^T \circ A')^H \end{array} \right] \text{vec}(V) = \left[ \begin{array}{c} \text{vec}(\Phi^T \ast I_K) \\ \alpha \end{array} \right].$$

We are going to show that the coefficient matrix in (26) has full row rank and thus (26) admits a solution. In order to do that, we need a few lemmas. The following result is a consequence of [35, Proposition 1.1].

**Lemma 1.** For every $k = 1, \ldots, K$, the $K + 1$ vectors $a'(f_k), a(f_1), \ldots, a(f_K)$ are linearly independent given $K < N$.

The following lemma presents results on the Kronecker, Khatri-Rao and Hadamard products of matrices that can be found in [36,37].

**Lemma 2.** Let $A$, $B$, $C$, $D$ be matrices of proper dimensions from instance to instance, and additionally $A$, $B$ nonsingular in (29). Then,

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B),$$

$$\left( A \otimes B \right)^H = A^H \otimes B^H,$$

$$\left( A \otimes B \right)^{-1} = A^{-1} \otimes B^{-1},$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD,$$

$$(C \otimes D)(A \ast B) = CA \ast DB,$$

$$(A \ast B)^H(A \ast B) = A^H A \otimes B^H B.$$
The following result is part of the Schur product theorem; see [38] and [13] Theorem 18.

**Lemma 3.** If $A$ is positive definite and $B$ is positive semidefinite with a positive diagonal, then $A \otimes B$ is positive definite.

Now we are ready to show the following result.

**Lemma 4.** If $\Phi$ has full row rank and the frequencies $\{f_k\}$ are distinct, then the $(L + 1) K \times LN$ coefficient matrix $\left[ I_L \otimes A^H \right] \left[ (\Phi^T * A')^H \right]$ has full row rank.

**Proof:** It suffices to show that the Gram matrix

$$
\begin{bmatrix}
I_L \otimes A^H \\
(\Phi^T * A')^H
\end{bmatrix}
\begin{bmatrix}
I_L \otimes A^H \\
(\Phi^T * A')^H
\end{bmatrix}^H
= \begin{bmatrix}
I_L \otimes A^H I_L \otimes A^H \\
(\Phi^T * A')^H (\Phi^T * A')^H
\end{bmatrix} \begin{bmatrix}
\Phi^T \otimes A^H A' \\
\Phi^T \otimes A^H A'
\end{bmatrix}
= \begin{bmatrix}
I_L \otimes A^H A' & \Phi^T \otimes A^H A' \\
(\Phi^T * A')^H & \Phi^T \otimes A^H A'
\end{bmatrix}
$$

(33)

is positive definite, where we have used (28), (30), (31) and (32) of Lemma 2. Since $I_L \otimes A^H A$ is positive definite, we need only to show that its Schur complement is also positive definite which is given by

$$
\Phi^T \otimes A' \begin{bmatrix}
I_L \otimes A^H A' & \Phi^T \otimes A^H A'
\end{bmatrix} = \Phi^T \otimes A' \begin{bmatrix}
I_L \otimes A^H (A^H A)^{-1} A^H A'
\end{bmatrix}
$$

(34)

where we have consecutively applied (29), (30), (31) and (32) in the first four equalities.

We next study the two matrix factors in the Hadamard product in (34). The first factor $\Phi^T \otimes A'$ is positive definite since $S$, and thus $\Phi$, has full row rank by assumption. The proof will be completed by using Lemma 3 if we can show that the second factor $A^H \begin{bmatrix}
I_N - A (A^H A)^{-1} A^H
\end{bmatrix} A'$ is positive semidefinite and has a positive diagonal. To this end, let $A = \begin{bmatrix} U & U_1 \end{bmatrix} \begin{bmatrix} A \\ A_0 \end{bmatrix}$ and $V^H = UV^H = U \Lambda V^H$ be the singular value decomposition (SVD) of $A$. It is easy to verify that the second factor

$$
A^H \begin{bmatrix}
I_N - A (A^H A)^{-1} A^H
\end{bmatrix} A' = A^H U_1 U_1^H A'
$$

(35)

is positive semidefinite. Now suppose its $(k, k)$ entry, which is given by

$$
a'(f_k) U_1 U_1^H a'(f_k) = \| U_1^H a'(f_k) \|_2^2,
$$

(36)
Lemma 5. completes the proof. We need the following proposition.

We verify that $Q$.

3.3 Validation of Dual Certificate

It is easy to see that $\parallel V \parallel_2$. It follows immediately from Lemma 4 that the $V$.

Proof. Note that $\epsilon$.

\[
\| Q^{(l)}(f) \|_2 \leq C_l \max \{ \epsilon^2, \epsilon \}, \quad l = 0, 1, 2, \tag{38}
\]

where $Q^{(l)}(f)$ denotes the $l$th derivative of $Q(f)$, with $Q^{(0)}(f) = Q(f)$, and $\{C_l\}$ are constants independent of $\epsilon$.

Proof. Note that $Q^{(l)}(f) = a^{(l)H}(f) V$ and thus, for $l = 0, 1, 2$,

\[
\| Q^{(l)}(f) \|_2 \leq \| a^{(l)H}(f) \|_2 \| V \|_2 = \| a^{(l)}(f) \|_2 \| V \|_2. \tag{39}
\]

It is easy to see that $\| a^{(l)}(f) \|_2$ is a constant depending only on $l$. Consequently, it suffices to bound $\| V \|_2$ with a constant times $\max \{ \epsilon^2, \epsilon \}$. By using (37), we have that

\[
\| V \|_2 \leq \| V \|_F \\
= \| \text{vec} (V) \|_2 \\
\leq \left\| \begin{bmatrix} I_L \otimes A^H \\ (\Phi^T + A')^H \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} (\Phi^T * I_K) \alpha \\ \beta \end{bmatrix} \right\|_2 \tag{40}
\]

(\Phi^T * I_K) \alpha_2 + \| \beta \|_2.

To complete the proof, therefore, we need only to show that both $\{\alpha_k\}$ and $\{\beta_k\}$ can be bounded from above by a constant times $\max \{ \epsilon^2, \epsilon \}$.
We first do some preparations. Recall that \( \overline{R} = \frac{1}{T} YY^H = A \left( \frac{1}{T} S S^H \right) A^H \) and let
\[
\overline{R} = \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^H = U \Lambda^2 U^H
\] (41)
be its eigen-decomposition, where \( \Lambda^2 = \text{diag}(\lambda_1, \ldots, \lambda_K) \) with \( \lambda_1 \geq \cdots \geq \lambda_K > 0 \). It is easy to see that \( A \) and \( U \) share the same range space and thus,
\[
U^H a(f_k) = a(f_k), \quad k = 1, \ldots, K, \tag{42}
\]
\[
U_\perp^H a(f_k) = 0, \quad k = 1, \ldots, K, \tag{43}
\]
\[
U_\perp^H a(f) \neq 0, \quad f \in \mathcal{F} \setminus \{ f_k \}. \tag{44}
\]

Moreover,
\[
W = (I + \epsilon^{-1} \overline{R})^{-1} = \left( I + \epsilon^{-1} \left[ U \quad U_\perp \right] \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^H \right)^{-1}
\]
\[
= \begin{bmatrix} U & U_\perp \end{bmatrix} \left( I + \epsilon^{-1} \Lambda^2 \right)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^H
\]
\[
= U \text{diag} \left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right) U^H + U_\perp U_\perp^H. \tag{45}
\]

Now we are ready to bound \( \{ \alpha_k = w(f_k) \} \). Making use of (42), (43) and (45), we have for every \( k = 1, \ldots, K \) that
\[
\alpha_k^2 = w^2(f_k)
\]
\[
= a^H(f_k) W a(f_k)
\]
\[
= a^H(f_k) U \text{diag} \left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right) U^H a(f_k) + a^H(f_k) U_\perp U_\perp^H a(f_k)
\]
\[
= a^H(f_k) U \text{diag} \left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right) U^H a(f_k)
\]
\[
= a^H(f_k) \text{diag} \left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right) a(f_k). \tag{46}
\]

It follows immediately that
\[
\frac{N \epsilon}{\lambda_1 + \epsilon} \frac{\epsilon}{\lambda_1 + \epsilon} \| a(f_k) \|_2^2 \leq \alpha_k^2 \leq \frac{\epsilon}{\lambda_K} \| a(f_k) \|_2^2 = \frac{N \epsilon}{\lambda_K} \tag{47}
\]
and thus,
\[
\sqrt{ \frac{N \epsilon}{\lambda_1 + \epsilon} } \leq \alpha_k \leq \sqrt{ \frac{N}{\lambda_K} \epsilon^2 }. \tag{48}
\]

The upper bound above is what we need.
We next bound \( \{ \beta_k = \frac{a^H(f_k)W\alpha'(f_k)}{\alpha_k} \} \). It follows from (45) and (43) that for every \( k = 1, \ldots, K \),
\[
|a^H(f_k)W\alpha'(f_k)| = \left| a^H(f_k)U\text{diag} \left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right)U^H\alpha'(f_k) \right|
\leq \|a(f_k)\|_2 \|U\|_2^2 \cdot \frac{\epsilon}{\lambda_K} \cdot \|\alpha'(f_k)\|_2
\leq \sqrt{N} \cdot \frac{\epsilon}{\lambda_K} \cdot 2\pi \sqrt{\frac{(N - 1)N(2N - 1)}{6}}
\leq \frac{2\pi N^2\epsilon}{\sqrt{3}\lambda_K}.
\]
where the second equality uses \( \|a(f_k)\|_2 = \sqrt{N} \), \( \|U\|_2 = 1 \) and
\[
\|a'(f_k)\|_2 = \|2\pi i \cdot \text{diag}(0, \ldots, N - 1) a(f_k)\|_2 = 2\pi \sqrt{\frac{(N - 1)N(2N - 1)}{6}}.
\]
Making use of (48) and (49), we have that for every \( k = 1, \ldots, K \),
\[
|\beta_k| = \frac{|a^H(f_k)W\alpha'(f_k)|}{\alpha_k} \leq \frac{2\pi N^2\epsilon}{\sqrt{3}\lambda_K} \cdot \sqrt{\frac{\lambda_1 + \epsilon}{N\epsilon}} \leq \frac{2\pi N^\frac{3}{2}}{\sqrt{3}\lambda_K} \cdot \sqrt{\lambda_1\epsilon + \epsilon^2} \leq \frac{2\pi N^\frac{3}{2}}{\lambda_K} \max \{ \sqrt{\lambda_1\epsilon^\frac{1}{2}}, \epsilon \}.
\]
Inserting (48) and (51) into (40) and then inserting (40) into (39) complete the proof.

Lemma 5 and its proof will be used to prove the following two propositions.

**Proposition 2.** There exist \( \nu_0 > 0 \) and \( \epsilon'_0 > 0 \) such that for every \( k = 1, \ldots, K \), if \( 0 < \epsilon \leq \epsilon'_0 \), then
\[
\|Q(f)\|_2^2 < w^2(f), \quad f \in [f_k - \nu_0, f_k) \cup (f_k, f_k + \nu_0].
\]

**Proof.** Consider the function
\[
g_\epsilon(f) = w^2(f) - \|Q(f)\|_2^2
\]
of \( f \) with \( \epsilon > 0 \). Since (14) and (21) are satisfied, we have immediately that
\[
g_\epsilon(f_k) = g_\epsilon'(f_k) = 0, \quad k = 1, \ldots, K.
\]
To prove the lemma, therefore, it suffices to show that \( g_\epsilon(f) \) is a strictly convex function of \( f \) on \([f_k - \nu_0, f_k + \nu_0] \) for every \( k = 1, \ldots, K \) and \( 0 < \epsilon \leq \epsilon'_0 \).

In order to do that, we first argue that it suffices to prove that for every \( k = 1, \ldots, K \),
\[
\lim_{\epsilon \to 0} g_\epsilon''(f_k) > 0.
\]
In particular, note that once (55) is shown to be true, according to the continuity of \( g_\epsilon''(f) \) as a function of \( (f, \epsilon) \in \mathbb{T} \times (0, +\infty) \), there will exist \( \nu_k, \epsilon'_k > 0 \) for every \( k = 1, \ldots, K \) such that
\[
g_\epsilon''(f) \geq \frac{1}{2} \lim_{\epsilon \to 0} g_\epsilon''(f_k) > 0, \quad (f, \epsilon) \in [f_k - \nu_k, f_k + \nu_k] \times (0, \epsilon'_k].
\]
This means that for every \( \epsilon \leq \epsilon'_0 \), \( g_{\epsilon}(f) \) is a strictly convex function of \( f \) on \([f_k - \nu, f_k + \nu]\). The proof is completed by letting \( \epsilon'_0 = \min\{\epsilon'_1, \ldots, \epsilon'_K\} \) and \( \nu_0 = \min\{\nu_1, \ldots, \nu_K\} \).

The rest of the proof is devoted to showing (55). Since

\[
\frac{d^2}{df^2} \|Q(f_k)\|_2^2 = 2 \|Q'(f_k)\|_2^2 + 2 \Re \{Q''(f_k)QH(f_k)\},
\]

it follows from Lemma 5 that

\[
\lim_{\epsilon \to 0} \frac{d^2}{df^2} \|Q(f_k)\|_2^2 = 0.
\]

Moreover, recall (10) and (45) and we have

\[
\lim_{\epsilon \to 0} \frac{d^2}{df^2} w^2(f_k) = \lim_{\epsilon \to 0} 2a''H(f_k)W a'(f_k) + 2 \Re \{a''H(f_k)Wa''(f_k)\}
\]

\[
= \lim_{\epsilon \to 0} 2a''H(f_k)W a'(f_k)
\]

\[
= \lim_{\epsilon \to 0} 2a''H(f_k)\left[ U \text{diag}\left( \frac{\epsilon}{\lambda_1 + \epsilon}, \ldots, \frac{\epsilon}{\lambda_K + \epsilon} \right) U^H + U \frac{1}{U^H} \right] a'(f_k)
\]

\[
= 2a''H(f_k)U \frac{1}{U} a'(f_k)
\]

\[
= 2 \|U \frac{1}{U} a'(f_k)\|_2^2 > 0,
\]

where the second equality holds since

\[
\lim_{\epsilon \to 0} a''H(f_k)Wa''(f_k) = 0,
\]

which can be shown similarly to (49), and the last inequality comes from the arguments around (36). Combining (58) and (59) yields that

\[
\lim_{\epsilon \to 0} g''(f_k) = \lim_{\epsilon \to 0} \frac{d^2}{df^2} w^2(f_k) - \lim_{\epsilon \to 0} \frac{d^2}{df^2} \|Q(f_k)\|_2^2 > 0,
\]

which concludes (55) and completes the proof.

\[\Box\]

**Proposition 3.** There exists \( \epsilon''_0 > 0 \) such that if \( 0 < \epsilon \leq \epsilon''_0 \), then

\[
\|Q(f)\|_2^2 < w^2(f), \quad f \in \mathbb{T} \setminus \bigcup_{k=1}^{K} [f_k - \nu, f_k + \nu],
\]

where \( \nu_0 > 0 \) was defined in Proposition 2.

**Proof.** Recall (45) and we have

\[
w^2(f) = a''H(f)Wa(f) \geq a''H(f)U \frac{1}{U} a(f) = \|U \frac{1}{U} a(f)\|_2^2.
\]

Let

\[
\zeta = \inf \left\{ \|U \frac{1}{U} a(f)\|_2^2 : f \in \mathbb{T} \setminus \bigcup_{k=1}^{K} [f_k - \nu, f_k + \nu] \right\}.
\]
We must have \( \zeta > 0 \) since, otherwise, \( U^H_1 a(f) = 0 \) for some \( f \in \mathbb{T} \setminus \bigcup_{k=1}^{K} [f_k - \nu_0, f_k + \nu_0] \), which cannot be true according to (44). It follows immediately that for every \( \epsilon > 0 \),

\[
w^2(f) \geq \zeta > 0, \quad f \in \mathbb{T} \setminus \bigcup_{k=1}^{K} [f_k - \nu_0, f_k + \nu_0].
\] (65)

On the other hand, by Lemma 5 there exists \( \epsilon_0'' > 0 \) such that if \( 0 < \epsilon \leq \epsilon_0'' \), then

\[
\| Q(f) \|_2^2 \leq \frac{1}{2} \zeta, \quad f \in \mathbb{T}.
\] (66)
Combining (65) and (66) results in (62), completing the proof.

Now we are ready to show that the constructed dual certificate \( Q(f) \) satisfies (15) under the assumption of Theorem 1 and thus complete the whole proof. In fact, (15) is a direct consequence of combining Proposition 2 and Proposition 3 under the assumption \( 0 < \epsilon < \epsilon_0 \) by defining \( \epsilon_0 = \min \{ \epsilon_0', \epsilon_0'' \} \).

4 Numerical Results

In this section we provide numerical results to validate our theoretical findings as well as to demonstrate the usefulness of the weighted atomic norm method in challenging scenarios with missing data and Gaussian noise.

We first consider the noiseless, full data case, as concerned in Theorem 1. In Experiment 1, we study the performance of the weighted atomic norm method (WANM) with respect to \( \epsilon \). We set \( N = 20, K = 2, L = 3, f_1 = 0.3 \) and \( f_2 = 0.325 \), implying that the frequency separation is \( \frac{5}{2N} = 0.025 \). The source signals in \( S \) are generated from a standard complex Gaussian distribution. We consider \( \epsilon = 10^{-4}, 10^{-3}, \ldots, 10^4 \) in WANM. We randomly generate 100 problems and solve them using WANM, in which we consider \( \epsilon = 10^{-4}, 10^{-3}, \ldots, 10^4 \). We claim that the frequencies are successfully recovered if the root mean squared error of frequency recovery is below \( 10^{-4} \). The atomic norm method (ANM) is also used to solve the problems for comparison that does not depend on \( \epsilon \). Our numerical results are presented in Fig. 2. It is seen that the performance of WANM approaches that of ANM as \( \epsilon \) grows very large, which is true since the weighting function in WANM becomes constant and WANM degenerates into ANM as \( \epsilon \to +\infty \). As \( \epsilon \) becomes small (below \( 10^{-2} \) in this case), 100\% success occurs for WANM, which validates Theorem 1.

We present in Fig. 3 the dual certificates of WANM and ANM for one problem. For the purpose of better illustration, we plot the amplitude of the scaled dual certificate \( \tilde{Q}(f) = w^{-1}(f)Q(f) \) with respect to \( f \) for different values of \( \epsilon \). It follows from (17) and (18) that the frequency estimates of WANM (and ANM) can be identified from the points of tangency between the curve \( \| \tilde{Q}(f) \|_2 \) and the straight line at unit. It is seen that WANM succeeds to recover the frequencies when \( \epsilon = 10^{-3}, 1 \), while it fails when \( \epsilon = 10^{3} \) and \( \epsilon \to +\infty \) (the latter corresponds to ANM). The dual polynomial gets sharp around the true frequencies as \( \epsilon \) becomes small. Interestingly, when \( \epsilon \) is large, we have \( \| \tilde{Q}(f) \|_2 \equiv 1 \), as shown in Fig. 3. It occurs because the solved Toeplitz matrix for WANM has full rank (see (7)) and in this case any value of frequency can be included in the Vandermonde decomposition (see [7], Theorem 11.5).

In Experiment 2, we fix \( \epsilon = 10^{-3} \) and vary the frequency separation \( \Delta_f \) from 0 to \( \frac{2}{N} \) at a step of \( \frac{0.05}{N} \) and use the other setups as in Experiment 1. Our numerical results are presented in Fig. 2. It is seen that 100\% success occurs for ANM as \( \Delta_f \geq \frac{0.85}{N} \), while this resolution limit is improved to \( \frac{0.2}{N} \) for WANM.

In Experiment 3, we study the missing data case in which only a subset of the rows of the data matrix \( Y \) is observed, as in array signal processing. We randomly select \( M \) out of \( N \) rows from \( Y \). We set \( N = 20, \)
Figure 2: Results of success rate of frequency recovery versus $\epsilon$, where $N = 20$, $K = 2$, $L = 3$, $f_1 = 0.3$ and $f_2 = f_1 + \frac{1}{2N} = 0.325$.

$L = 3$, $K = 2$ and $\Delta f = \frac{0.7}{N}$ and use $\epsilon = 10^{-3}$ in WANM. ANM, ESPRIT and Capon’s method are considered for comparison. Capon’s method is implemented as in (13), where $\epsilon$ is used for regularization in the case when $\hat{R}$ is rank deficient. Consequently, WANM is a combination of Capon and ANM. Our results on the root mean squared error of frequency recovery with varying $M$ are presented in Fig. 5. It is seen that WANM has the smallest error, followed by ANM. ESPRIT and Capon do not have satisfactory performance, except for $M = N$, since the data covariance matrix cannot be estimated accurately with missing data.

In Experiment 4, we study the noisy data case in which the measurements are corrupted by additive white Gaussian noise. We set $N = 20$, $K = 2$, $L = 10$ and $\Delta f = \frac{0.3}{N}$. We set $\epsilon = 10^{-3}$ in WANM. We vary the signal-to-noise ratio (SNR) from $-20$dB to $20$dB. We say that the frequencies are successfully resolved if the absolute estimation error for each frequency is less than $\frac{1}{2}\Delta f$. Our results on the success rate versus SNR is presented in Fig. 6. Again, it is seen that WANM has the best performance.

5 Conclusion

In this paper, we provided a theoretical analysis showing that the resolution limit of the atomic norm method for spectral super-resolution can be overcome by including an adaptive weighting strategy. Numerical results are provided that corroborate our analysis and demonstrate the usefulness of the proposed method in practical scenarios with missing data and noise.

References

[1] P. Stoica and R. L. Moses, *Spectral analysis of signals*. Upper Saddle River, NJ, US: Pearson/Prentice Hall, 2005.
Figure 3: Plots of the weighted dual polynomial $w^{-1}(f)Q(f)$ of WANM with respect to frequency $f$ for different values of $\epsilon$. WANM degenerates into ANM in the case $\epsilon \to \infty$.

[2] E. J. Candès and C. Fernandez-Granda, “Towards a mathematical theory of super-resolution,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.

[3] M. F. Duarte and R. G. Baraniuk, “Spectral compressive sensing,” *Applied and Computational Harmonic Analysis*, vol. 35, no. 1, pp. 111–129, 2013.

[4] B. Adcock and A. C. Hansen, “Generalized sampling and infinite-dimensional compressed sensing,” *Foundations of Computational Mathematics*, pp. 1–61, 2015.

[5] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, “Compressed sensing off the grid,” *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7465–7490, 2013.

[6] Z. Yang and L. Xie, “Enhancing sparsity and resolution via reweighted atomic norm minimization,” *IEEE Transactions on Signal Processing*, vol. 64, no. 4, pp. 995–1006, 2016.

[7] Z. Yang, J. Li, P. Stoica, and L. Xie, “Sparse methods for direction-of-arrival estimation,” *Academic Press Library in Signal Processing Volume 7 (R. Chellappa and S. Theodoridis, Eds.),* pp. 509–581, 2018.

[8] A. Xenaki and P. Gerstoft, “Grid-free compressive beamforming,” *The Journal of the Acoustical Society of America*, vol. 137, no. 4, pp. 1923–1935, 2015.

[9] Y. Tsai, L. Zheng, and X. Wang, “Millimeter-wave beamformed full-dimensional mimo channel estimation based on atomic norm minimization,” *IEEE Transactions on Communications*, vol. 66, no. 12, pp. 6150–6163, 2018.

[10] S. Li, D. Yang, G. Tang, and M. B. Wakin, “Atomic norm minimization for modal analysis from random and compressed samples,” *IEEE Transactions on Signal Processing*, vol. 66, no. 7, pp. 1817–1831, 2018.
Figure 4: Results of success rate of frequency recovery versus frequency separation $\Delta f$, where $N = 20$, $K = 2$, $L = 3$, $f_2 = f_1 + \Delta f$ and $\epsilon = 10^{-3}$.

[11] P. Stoica and N. Arye, “MUSIC, maximum likelihood, and Cramer-Rao bound,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 37, no. 5, pp. 720–741, 1989.

[12] W. Liao and A. Fannjiang, “MUSIC for single-snapshot spectral estimation: Stability and super-resolution,” *Applied and Computational Harmonic Analysis*, vol. 40, no. 1, pp. 33–67, 2016.

[13] Z. Yang, “Nonasymptotic performance analysis of ESPRIT and spatial-smoothing ESPRIT,” *IEEE Transactions on Information Theory, early access*, 2022.

[14] E. J. Candès, J. Romberg, and T. Tao, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,” *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[15] D. Malioutov, M. Cetin, and A. S. Willsky, “A sparse signal reconstruction perspective for source localization with sensor arrays,” *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3010–3022, 2005.

[16] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The convex geometry of linear inverse problems,” *Foundations of Computational Mathematics*, vol. 12, no. 6, pp. 805–849, 2012.

[17] E. J. Candès and C. Fernandez-Granda, “Super-resolution from noisy data,” *Journal of Fourier Analysis and Applications*, vol. 19, no. 6, pp. 1229–1254, 2013.

[18] J.-M. Azaïs, Y. De Castro, and F. Gamboa, “Spike detection from inaccurate samplings,” *Applied and Computational Harmonic Analysis*, vol. 38, no. 2, pp. 177–195, 2015.

[19] Z. Yang and L. Xie, “Exact joint sparse frequency recovery via optimization methods,” *IEEE Transactions on Signal Processing*, vol. 64, no. 19, pp. 5145–5157, 2016.
Figure 5: Results of success rate of frequency recovery versus the number $M$ of observed rows of data matrix, where $N = 20$, $K = 2$, $L = 3$, $\Delta f = \frac{0.7}{N}$ and $\epsilon = 10^{-3}$.

[20] Y. Li and Y. Chi, “Off-the-grid line spectrum denoising and estimation with multiple measurement vectors,” *IEEE Transactions on Signal Processing*, vol. 64, no. 5, pp. 1257–1269, 2016.

[21] C. Fernandez-Granda, “Super-resolution of point sources via convex programming,” *Information and Inference: A Journal of the IMA*, vol. 5, no. 3, pp. 251–303, 2016.

[22] C. Fernandez-Granda, G. Tang, X. Wang, and L. Zheng, “Demixing sines and spikes: Robust spectral super-resolution in the presence of outliers,” *Information and Inference: A Journal of the IMA*, p. iax005.

[23] Z. Yang, J. Tang, Y. Eldar, and L. Xie, “On the sample complexity of multichannel frequency estimation via convex optimization,” *IEEE Transactions on Information Theory*, vol. 65, no. 4, pp. 2302–2315, 2019.

[24] Q. Li and G. Tang, “Approximate support recovery of atomic line spectral estimation: A tale of resolution and precision,” *Applied and Computational Harmonic Analysis*, vol. 48, no. 3, pp. 891–948, 2020.

[25] J. Capon, “High-resolution frequency-wavenumber spectrum analysis,” *Proceedings of the IEEE*, vol. 57, no. 8, pp. 1408–1418, 1969.

[26] V. I. Morgenshtern and E. J. Candes, “Super-resolution of positive sources: The discrete setup,” *SIAM Journal on Imaging Sciences*, vol. 9, no. 1, pp. 412–444, 2016.

[27] T. Bendory, “Robust recovery of positive stream of pulses,” *IEEE Transactions on Signal Processing*, vol. 65, no. 8, pp. 2114–2122, 2017.
Figure 6: Results of success rate versus SNR, where \( N = 20, K = 2, L = 10, \Delta f = \frac{0.3}{N} \) and \( \epsilon = 10^{-3} \).

[28] J.-J. Fuchs, “Sparsity and uniqueness for some specific under-determined linear systems,” in Proceedings.(ICASSP’05). IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005., vol. 5. IEEE, 2005, pp. v–729.

[29] G. Schiebinger, E. Robeva, and B. Recht, “Superresolution without separation,” Information and Inference: A Journal of the IMA, vol. 7, no. 1, pp. 1–30, 2018.

[30] B. Kurmanbek and E. Robeva, “Multivariate super-resolution without separation,” arXiv preprint arXiv:2210.09979, 2022.

[31] K. V. Mishra, M. Cho, A. Kruger, and W. Xu, “Spectral super-resolution with prior knowledge,” IEEE Transactions on Signal Processing, vol. 63, no. 20, pp. 5342–5357, 2015.

[32] Z. Yang and L. Xie, “Fast convex optimization method for frequency estimation with prior knowledge in all dimensions,” Signal Processing, vol. 142, pp. 271–280, 2018.

[33] T.-J. Shan, M. Wax, and T. Kailath, “On spatial smoothing for direction-of-arrival estimation of coherent signals,” IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 33, no. 4, pp. 806–811, 1985.

[34] E. J. Candès and B. Recht, “Exact matrix completion via convex optimization,” Foundations of Computational Mathematics, vol. 9, no. 6, pp. 717–772, 2009.

[35] R. L. Ellis and D. C. Lay, “Factorization of finite rank Hankel and Toeplitz matrices,” Linear Algebra and its Applications, vol. 173, pp. 19–38, 1992.

[36] C. F. Van Loan, “The ubiquitous Kronecker product,” Journal of Computational and Applied Mathematics, vol. 123, no. 1, pp. 85–100, 2000.
[37] S. Liu and G. Trenkler, “Hadamard, Khatri-Rao, Kronecker and other matrix products,” *Int. J. Inf. Syst. Sci.*, vol. 4, no. 1, pp. 160–177, 2008.

[38] J. Schur, “Bemerkungen zur theorie der beschränkten bilinearformen mit unendlich vielen veränderlichen.” *Journal für die reine und Angewandte Mathematik*, vol. 140, pp. 1–28, 1911.