Generating two-mode squeezing and Schrödinger cat states with multimode measurement-induced nonlinearity

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Measurement-induced nonclassical effects in a two-mode interferometer are investigated theoretically using numerical simulations and analytical results. We have found that, for certain parameters, partial state measurements within the interferometer lead to the occurrence of two-mode squeezing. The results strongly depend on the phase inside the interferometer, the detection probability, and the choice of the input states. The appropriate parameters for maximized squeezing are obtained. We further demonstrate how exotic quantum states, such as Schrödinger cat states corresponding to two-mode coherent state superpositions, may be generated with high fidelity. We analyze the influence of losses and confirm that the predicted effects are within reach of current experimental techniques.

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I. INTRODUCTION

Entanglement and squeezing are two important properties of light required for the implementation of many quantum computation and quantum communication protocols [1]. One way to generate squeezed light is to utilize the nonlinear interaction in the medium via parametric down-conversion (PDC) or four-wave mixing (FWM) [2]. The amount of squeezing in such processes is proportional to the intensities of the pump fields, which limits their application at the single-photon level. Moreover, for small numbers of emitted photons, the light-matter interaction strength is very low, which leads to a small conversion efficiency from the pump to the generated nonclassical light [3]. Another way of creating squeezed light in quantum optics is the use of measurements to produce so-called measurement-induced nonlinearities (MINL), whereby nonlinear effects can be acquired by applying detection [4]. Partial detection can subtract photons from a state and may result in various nonlinear transformations [5–11]. The advantage of using detection compared to PDC or FWM is that fewer incident photons are required to generate nonclassical effects. However, the acquired effects have a probabilistic nature, i.e., the desired effect emerges only if and when a certain measurement outcome occurs.

Measurement-induced effects in optics have been considered in a number of works. Knill, Laflamme, and Milburn showed that the nonlinear photon interaction required for quantum computation can be implemented at the single-photon level by using linear optical elements and single-photon detection [12]. An experimental generation of a coherent superposition of states with using of a beam splitter and a single-photon detector has been realized by Lvovsky and Mlynek [13]. A theoretical description of multimode schemes including single-photon input states, their subsequent transformations by linear optical elements, and further detection can be found in [4]. An experimental implementation of the measurement-induced Kerr nonlinearity was considered by Costanzo et al. [14], where the setup combines the single-photon addition and the single-photon subtraction to induce a nonlinear phase shift. The generation of coherent state superpositions and amplification and manipulation of states at the single photon level has been discussed in [15–19]. In [20], the generation of nonclassical multi-photon states due to interference between a coherent state and a Fock state by using the quantum catalysis is discussed.

In this work, we present a theoretical investigation of a two-mode interferometer including detection and considering different input states, i.e., a coherent state and a single photon state. In the output channels, we analyze the acquired nonclassical effects after detection. It is shown that for certain combinations of parameters inside the interferometer the implemented detection leads to two-mode squeezing in the system. Moreover, the implemented detection allows one to generate interesting entangled states, for example, two-mode coherent state superpositions, with high fidelity.

This paper is organized as follows: In section II we present our theoretical description of the two-mode interferometer. In section III we present and discuss the results of numerical simulations which demonstrate two-mode squeezing for optimized parameters. In section IV we analyze the generation of the Schrödinger cat states in the presented scheme. We close with a brief summary in section V and furthermore provide several useful analytical results in the Appendix A.

II. THEORETICAL MODEL

The two-mode interferometer considered in this work is shown in Fig. 1. Various quantum states can be injected into the input ports of the interferometer. Here, we consider, in particular, the combination of a single photon state $|1⟩_1 = ˆa_1^† |0⟩$ in channel 1 and a coherent state $|α⟩_2 = \exp(-\frac{1}{2}|α|^2) \sum_{n=0} \frac{α^n}{n!} (ˆa_2^†)^n |0⟩$ in channel 2.
respectively, as shown in Fig. 1 where |α|^2 is the mean photon number of the coherent state. The quantum state of light injected in the two channels is thus the tensor product |ψ⟩ = |1⟩_1 ⊗ |α⟩_2.

Each (lossless) BS in Fig. 1 is characterized by its own transmission T_i = t_i^2 and reflection R_i = 1 - T_i = r_i^2 coefficients with amplitudes t_i and r_i, respectively. In general, these amplitudes define a linear transformation of creation (and annihilation) operators between input a_i = [a^\dagger_1, a^\dagger_2]^T and output b^\dagger = [b^\dagger_1, b^\dagger_2]^T modes:

\[ a_i^\dagger → Λ b_i^\dagger, \quad Λ = \begin{pmatrix} ir & t \\ t & ir \end{pmatrix}. \]

The density matrix of the quantum state can be written as a function of creation and annihilation operators \( ρ = ρ[a, a_i^\dagger] \). The transformation of the density matrix at each BS can be obtained by using the input/output relations, Eq. 1, for each operator. The relation between the density matrices before (\( ρ_b \)) and after (\( ρ_a \)) the BS transformation can be expressed using the BS operator \( B(t) \) as \( ρ_a = B(t)ρ_bB^\dagger(t) \).

The action of a phase shifter in a channel \( n \) corresponds to the following transformation: \( a_n^\dagger → e^{iϕ}a_n^\dagger \). Considering phase modulation in one channel can be described by a matrix transformation \( ρ_a = P(ϕ)ρ_b \).

In this work we consider two types of detectors: click detectors which measure the absence or presence of photons but provide no information about the photon number and photon-number-resolving (PNR) detectors [21]. PNR detection in one channel can be described by the projection of the state on the chosen Fock state with \( n \) photons in the \( i \)-th channel \( |n⟩_i; \quad |ψ⟩ = |n⟩_1⟨n|_2|ψ⟩_b \) where \( |ψ⟩_b \) is the state before detection. The probability of such an event is \( P_{det} = ⟨ψ|_a ⟨ψ⟩_a \) where \( |ψ⟩_a \) is the unnormalized state after projection. Since we inject a single-photon and a weak coherent state (\( |α| \leq 1 \)), we consider only single-photon detection, i.e., the projection onto the single photon state |1⟩ or the vacuum state |0⟩. Therefore, for the considered two detectors in channels 3 and 4, see Fig. 1 four different outcomes are possible: (i) both detectors click, (ii) only the detector in channel 4 clicks, (iii) only the detector in channel 3 clicks, and (iv) neither detector clicks:

\[ \begin{align*}
|ψ_a^{(3&4)}⟩ &= |1⟩⟨1|_3 ⊗ |1⟩⟨1|_4|ψ⟩_b \\
|ψ_a^{(4)}⟩ &= |0⟩⟨0|_3 ⊗ |1⟩⟨1|_4|ψ⟩_b \\
|ψ_a^{(3)}⟩ &= |1⟩⟨1|_3 ⊗ |0⟩⟨0|_4|ψ⟩_b \\
|ψ_a^{(none)}⟩ &= |0⟩⟨0|_3 ⊗ |0⟩⟨0|_4|ψ⟩_b,
\end{align*} \]

where \( |ψ⟩_b \) is the state in the four channels after BS2 and BS3 but before detection.

Click detectors do not resolve the number of photons and must therefore take into account all possible photon-number contributions. The action of click detectors can be described in terms of the positive operator valued measure (POVM) operators \( \hat{Π}^{-} = |0⟩⟨0| \) and \( \hat{Π}^{(+)} = \hat{I} - \hat{Π}^{-} = \sum_{n=1}^{∞} |n⟩⟨n| \) which describe the absence and presence of a click, respectively. The two detectors are again described by four possible projection operators:

\[ \begin{align*}
\hat{Π}_{3&4} &= \hat{Π}_{3}^{(+)} ⊗ \hat{Π}_{4}^{(+)} \\
\hat{Π}_{4} &= \hat{Π}_{3}^{(-)} ⊗ \hat{Π}_{4}^{(+)} \\
\hat{Π}_{3} &= \hat{Π}_{3}^{(+)} ⊗ \hat{Π}_{4}^{(-)} \\
\hat{Π}_{none} &= \hat{Π}_{3}^{(-)} ⊗ \hat{Π}_{4}^{(-)}.
\end{align*} \]

To obtain the density matrix after detection \( ρ_a \), in the click detection case we apply the POVM operators to the density matrix before detection \( ρ_b \) and take the partial trace over the detecting channels \( ρ_a = Tr_{3,4}(ρ_b\hat{Π}_{event}) \). The detection probability is given by \( P_{det}^{(event)} = Tr(ρ_b\hat{Π}_{event}) \). For both cases the detection leads to an unnormalized density matrix. Therefore, we define a new normalized detection operator \( \hat{D} = \hat{Π}_{event}/P_{det}^{(event)} \).

The whole interferometer acts on the input light as a series of transformations. First the light passes through BS1. Afterwards it is split up into the four channels at the beam splitters BS2 and BS3. Detection is possible in the channels 3 and 4. Then an inner phase modulation \( \hat{P}(ϕ_1) \) is applied in the upper channel and, finally, the last beam splitter BS4 is reached. In the end of the interferometer, the second phase modulation \( \hat{P}(ϕ_2) \) is implemented in one of the channels to modify the relative phase between the channels. Altogether, the resulting transformations define the relation between the input and output probability density matrices can be written as:

\[ ρ_{out} = \hat{P}(ϕ_2)\hat{B}(t_4)\hat{P}(ϕ_1)\hat{D}\hat{B}(t_2, t_3)\hat{B}(t_1)ρ_{in} \]

Here, \( \hat{B}(t_2, t_3) \) denotes the action of both BS2 and BS3.
The output density matrix can be represented in the general form:

$$\rho_{\text{out}} = \sum_{p_1, p_2} \rho_{p_1 p_2, p_1' p_2'} |p_1, p_2\rangle \langle p_1', p_2'|,$$

where $p_1$, $p_1'$ and $p_2$, $p_2'$ are the numbers of photons in the channels 1 and 2, respectively, and $\rho_{p_1 p_2, p_1' p_2'}$ are the corresponding matrix elements.

The output density matrix $\rho_{\text{out}}$ for arbitrary input states and system parameters can be calculated numerically by evaluating Eq. (4). For the chosen input state $|\alpha\rangle \otimes |\alpha_2\rangle$, the output density matrix can, however, be obtained analytically, see Appendix A and depends on the parameters of the system: four beam splitters BS and the channels 1 and 2, respectively, and $\rho_{p_1, p_2, p_1', p_2'}$ are the corresponding matrix elements.

III. SIMULATING TWO-MODE SQUEEZING

Single-mode squeezing is defined as the reduction of the quadrature variance below the shot noise level $\Delta^2 X_1 < \frac{1}{2}$, with quadratures defined by $X_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger)$ and $X_2 = \frac{1}{\pi} (\hat{a} - \hat{a}^\dagger)$ and the variance $\Delta^2 X_2 = (X_2^2 - X_1^2)^2$. Two-mode squeezing between modes $a$ and $b$ is connected with the mutual variance of quadratures and is described by the joint quadrature operators:

$$C_x = X_1^a - X_1^b,$$

$$C_p = X_2^a + X_2^b.$$  

Similarly to the single-mode case, two-mode light is squeezed if one of the variances Eq. (6) is lower than the shot noise level: $\Delta^2 C_i < \frac{1}{2}$. The condition $\Delta^2 C_x = \Delta^2 X_1^a + \Delta^2 X_1^b - 2\text{Cov}[X_1^a, X_1^b] < \frac{1}{2}$ can be satisfied either if the two modes are uncorrelated and, simultaneously, one or both of them are individually squeezed, or when nonclassical correlations between the modes (entanglement) exist.

To understand the influence of the detectors on the generated two-mode squeezing in the circuit depicted in Fig. 1, we first consider a simplified setup without degenerated two-mode squeezing in the circuit depicted in Fig. 1. We then consider a two-mode setup without detectors and demonstrate the absence of squeezing in the system. For all red and blue curves we fixed the critical detection probability $P_{\text{crit}} = 0.1$. The black line represents the maximized squeezing for the simplified circuit without detectors. The dashed red line corresponds to the both-detectors-click PNR detection. The solid blue line represents the maximized squeezing in the single-click detection case where only one detector clicks. The dashed blue line corresponds to the both-click detection case.

In all cases, the maximized squeezing is symmetrical around the point $\phi_1 = \pi$ such as $f(\pi - \phi_1) = f(\pi + \phi_1)$. As may be expected in the case of linear elements and non-squeezed input states, the output light is not squeezed as well. The squeezing, i.e., $S_x$, maximized over $t_1, t_2, \phi_2$ is shown by the black line as a function of $\phi_1$ in Fig. and is equal to 0 dB.

Adding detectors to the scheme may generate two-mode squeezing. Since there are several parameters in the system which influence the output state, we perform an optimization to find the minimum of the variance $\Delta^2 C_x$ in order to maximize squeezing. We use the following algorithm: choose some detection event $d = D_i$, constrain the probability to be higher than some minimum value $P_{\text{crit}}$ (to be able to undertake a reasonable experiment), fix the phase $\phi_1$, and then minimize the variance over all BS parameters and $\phi_2$:

$$\text{minimize: } \Delta^2 C_x(T, \phi_1, \phi_2, d)$$

subject to: $\phi_1 = \phi_0$, $P_{\text{det}} \geq P_{\text{crit}}$, $d = D_i$.  

Although the quantities $\Delta^2 C_x$ and $\Delta^2 C_p$ exhibit smooth continuous behavior over all parameters, it is still numerically not easy to find a global minimum of
these four-variable functions. The straightforward approach with evaluating variances over a multidimensional grid and choosing their minimal values is computationally expensive. One way to improve the situation is to use a gradient descent-based algorithms. In this work we apply the gradient-based algorithm “Adam”\textsuperscript{25} with its TensorFlow library implementation\textsuperscript{26}. To speed up the convergence of the algorithm, different starting points were chosen.

A. Photon-number-resolved detection

In the PNR detection case, the results of the variance minimization procedure Eq. \textsuperscript{7} for a single-photon detection event (red solid line) and a both-photons detection event (red dashed line) are presented in Fig.\textsuperscript{2}. It is clear that the phase difference between the channels plays a crucial role in maximizing two-mode squeezing. Due to the optimization over the second phase, the graph is symmetric around $\phi_1 = \pi$ such that $f(\pi - \phi_1) = f(\pi + \phi_1)$. A global minimum of squeezing over all parameters emerges for the phase $\phi_1 = 0.4\pi$ and is equal to $1.25$ dB. The dependence of squeezing on the phase is identical for detection in either channel 3 or 4 due to the symmetry of the circuit. Analytical equations for the output states and detection probabilities for the single- and both-channel detection are presented in Eqs. \textsuperscript{(A7) - (A10)} in the Appendix.

As can be seen, the final formulas for the output density matrices for the single-detector-click and the both-detectors-click cases can be presented in the identical form but with different sets of coefficients $\gamma_i$ and $\tilde{\gamma}_i$, see Eqs. \textsuperscript{(A7), (A9)}. However, the probabilities of detecting one and two photons in the system are different, see Eqs. \textsuperscript{(A8), (A10)}. Therefore, by fixing the certain value of the critical detection probability, for example, $P_{\text{crit}} = 0.1$, the both-detectors-click PNR detection induces less amount of squeezing compared to the single-detector-click PNR detection, see Fig. \textsuperscript{2}.

B. Click detection

When the PNR detection is replaced by click detection, the maximum squeezing is smaller, since click detection involves the sum over all photon numbers. This is shown by the solid blue and the dashed blue lines in Fig.\textsuperscript{2}. With click detectors the maximum squeezing is also obtained for the phase $\phi_1 = 0.4\pi$ and equals $-1.18$ dB for a single detector click and $-1.16$ dB when both detectors click. These values are only slightly smaller than for PNR detection, since we consider coherent state with the mean photon number $\alpha = 1$ and therefore the contribution from higher-order photon numbers is small. Analytical expressions for the output states, two-mode squeezing, and detection probabilities can be found in the Appendix.

C. Experimental considerations

When applying this procedure in an experiment, it is prudent to consider the range of experimental parameters which can be effectively optimized. In the following we consider detection via click detectors, since these are the most widely available. The required detection probability $P_{\text{crit}}$ strongly affects the amount of achievable squeezing. Typically, the achievable squeezing decreases with increasing probability, as shown in Fig.\textsuperscript{4} which implies a trade-off between measurable squeezing and success rate for the experiment. We define a minimum acceptable success probability $P_{\text{crit}}$ from which squeezing can be obtained. In the case when only one detector clicks with a probability of at least $P_{\text{crit}} = 0.1$, we vary the coherent state amplitude $\alpha$ to investigate the dependence on the potential squeezing. From Fig.\textsuperscript{4} it can be seen that $\alpha = 0.6$ leads to the largest squeezing of $S_x = -1.18$ dB. For a probability of $P_{\text{crit}} = 0.5$ the largest squeezing is obtained for the largest considered value of $\alpha = 1$ and amounts to $S_x = -0.76$ dB. For the case that both detectors click with a probability of $P_{\text{crit}} = 0.1$ the optimal value of $\alpha = 1$ gives squeezing of $S_x = -0.71$ dB, whereas for a probability of $P_{\text{crit}} = 0.3$ the same $\alpha$ re-
The two-mode squeezing maximized over all $T$ parameters as function of the detection probability for a fixed phases of $\phi_1 = 1.5\pi$, $\phi_2 = 0$. Click detection type is considered. (a) corresponds to the clicking of a single detector whereas in (b) both detectors click. The input states are a single-photon state and a coherent state with different $\alpha$ as given by the colored numbers.

Results in squeezing of $S_x = -0.17$ dB. Generally, a single detector response provides higher squeezing for the same probability because fewer photons are extracted. For the case of a single detector measurement, see Fig. 4(a), an optimal value of the mean photon number for the coherent state with $\alpha = 0.6$ exists for which the maximal amount of squeezing at small probabilities is achieved.

The structure of the output light can be revealed from the photon number distribution between two channels $P_{m,n} = \rho_{m,m,n,n}$ which is also accessible in experiment. As an example, the photon number distribution for maximally squeezed light in the single PNR detection case with parameters $\alpha = 1.0$, $S_x = -1.25$ $dB$, $T = [0.68, 0.82, 0.38, 1.0]$, $\phi_1 = 1.5\pi$, $\phi_2 = 0$, $P = 0.3$ is shown in Fig. 5.

In experimental realizations of the proposed interferometer because of losses not all input photons will propagate through the entire circuit. For the considered setup, losses related to absorption and scattering are expected to be the largest contribution. To model losses in our scheme we place additional beam splitters in both channels between BS$_1$ and BS$_4$ and consider losses before and after detection, i.e., before and after BS$_2$ and BS$_3$, as shown in Fig. 6.

Non-zero reflectivities of the additional beam splitters correspond to the removal of a certain fraction of photons from our circuit. The coefficients $R_b^\Sigma$ and $R_a^\Sigma$ are the total reflection coefficients of the additional beam splitters (losses) placed before and after detection, respectively. They are defined as the sum of the top and bottom reflection coefficients: $R_b^\Sigma = R_{b\text{top}}^\Sigma + R_{b\text{bottom}}^\Sigma$ and $R_a^\Sigma = R_{a\text{top}}^\Sigma + R_{a\text{bottom}}^\Sigma$.

We perform numerical simulations where the coefficients $R_b^\Sigma$ and $R_a^\Sigma$ are varied under the condition: $R_{b\text{top}}^\Sigma = R_{b\text{bottom}}^\Sigma$ both before and after detection. For weak losses, $R_{b\text{top}}^\Sigma \in [0, 0.1]$, the dependence of squeezing on the total reflection is shown in Fig. 7. For instance, including 5% loss before and after detection ($R_b^\Sigma = R_a^\Sigma = 0.05$), squeezing is reduced from -1.25 dB to -1.0 dB. It is worth to note that losses before detection reduce the two-mode squeezing much more.
**IV. EXOTIC STATE GENERATION AND FIDELITY**

**A. Generating two-mode coherent superposition states**

In addition to generating squeezing, measurement-induced nonlinearity can be used to prepare exotic quantum states. To estimate the effectiveness of generating a state \( \rho \), we evaluate the fidelity to particular target states with density matrix \( \sigma \), given by

\[
F(\rho, \sigma) = \left| \frac{\text{Tr} \sqrt{\rho \sigma}}{\sqrt{\text{Tr} \rho \text{Tr} \sigma}} \right|^2,
\]

which for pure states simplifies to

\[
F(\rho, \sigma) = \left| \langle \psi | \psi \rangle \right|^2.
\]

As a target state, we consider the Schrödinger’s cat state, i.e., superpositions of coherent states:

\[
|\psi_{\text{trg}}\rangle = N_t(|\alpha_t|e^{i\varphi_1}0) + e^{i\varphi_2}|0, |\alpha_t|e^{i\varphi_3}\rangle,
\]

where \(|\alpha_t,0\rangle = |\alpha_t\rangle_1 \otimes |0\rangle_2\) is the product of the coherent state \(|\alpha_t\rangle_1\) with the amplitude \(|\alpha_t|\) and the phase \(\varphi_1\) in the first channel and the vacuum state \(|0\rangle_2\) in the second channel. \(N_t = (2 + 2 \cos \varphi_2 \exp(-|\alpha_t|^2))^{-\frac{1}{2}}\) is the normalization constant, \(\varphi_2\) and \(\varphi_3\) are the phases. Such entangled states are important in quantum information and can be used for quantum teleportation protocols [27] and quantum communication processes [28–30].

In this section we consider the case of PNR detection with a single-photon-click in both detectors, which allows us to obtain analytical results. By setting the system parameters as \(T_1 = T_2 = T_3 = \frac{1}{2}\), \(T_4 = 1\), \(\varphi_1 = 1.5\pi\), \(\varphi_2 = 0\), we can produce the Schrödinger cat state, Eq. (8), inside the interferometer with high fidelity. For the given parameters the fidelity and the probability of the state generation are given by:

\[
F(|\psi_{\text{out}}, \psi_{\text{trg}}\rangle) = \frac{|\alpha_t|^2 \exp(-|\alpha_t|^2 - \frac{1}{2}|\alpha_{in}|^2)}{5 + 4 \cos \varphi_2 \exp(-|\alpha_t|^2)} \times
\]

\[
\times \left| e^{-i(\varphi_2 + \varphi_3)} \exp\left(\frac{1}{2} |\alpha_t| |\alpha_{in}e^{-i\varphi_3}|\right) - e^{-i\varphi_1} \exp\left(\frac{1}{2} |\alpha_t| |\alpha_{in}e^{-i\varphi_3}|\right)\right|^2
\]

\[
P_{\text{det}} = \frac{1}{4} \exp\left(\frac{-\alpha_{in}^2}{2}\right) \left(1 + \frac{\alpha_{in}^2}{2}\right),
\]

where \(|\alpha_{in}|^2\) is the mean photon number of the input coherent state. For coherent states with \(\alpha_{in} = \alpha_t = 1\) and phases \(\varphi_1 = \varphi_3 = 0\), \(\varphi_2 = \pm \pi\) the fidelity and the probability are given by \(F = 0.96\) and \(P_{\text{det}} = 0.23\), respectively. The fidelity for the chosen target state is maximized when \(\varphi_2 = \pm \pi\). The target and the generated states for the parameters given above are shown in Fig. 8.

When the parameter \(\alpha_{in}\) is increased significantly the effect of detection vanishes. At the same time, if \(|\alpha_{in}|\) and \(|\alpha_t|\) differ significantly, the fidelity between the output and the target states drops to zero. The fidelity, Eq. (9), as a function of real amplitudes of the input \(\alpha_{in}\) and the target \(\alpha_t\) states is presented in Fig. 9. For small \(\alpha\), the

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Fig. 6. A schematic representation of modeling losses by additional beam splitters in the top and bottom channels before and after detection.

Fig. 7. Two-mode squeezing dependence on losses in the channels. \(R_{b}^{\Sigma}\) and \(R_{a}^{\Sigma}\) are total reflection coefficients of loss BS placed before and after detection respectively defined as the sum of the top and the bottom reflection coefficients: \(R_{b}^{\Sigma} = R_{b}^{\text{before}} + R_{b}^{\text{bottom}}\) and \(R_{a}^{\Sigma} = R_{a}^{\text{after}} + R_{a}^{\text{bottom}}\), where \(R_{b}^{\Sigma} = R_{b}^{\Sigma}\).

significantly than losses after detection due to the different photon numbers before and after detection. For instance, the include 5% loss only before detection \((R_{b}^{\Sigma} = 0.05, R_{a}^{\Sigma} = 0)\) reduces squeezing from -1.25 dB to -1.01 dB, however, including 5% loss only after detection \((R_{b}^{\Sigma} = 0, R_{a}^{\Sigma} = 0.05)\), squeezing is reduced from -1.25 dB to -1.21 dB. 
Fig. 8. Comparison between the two-mode cat state (target state) and the output generated state for parameters $|\alpha_{\text{in}}| = |\alpha_t| = 1$ and $\phi_1 = \phi_3 = 0$, $\phi_2 = \pm \pi$. In the plots the real parts of coefficients $C_{n_1,n_2}$ are shown and the imaginary parts are zero $\text{Im}\{C_{n_1,n_2}\} = 0$. The coefficients $C_{n_1,n_2}$ are the state amplitudes defined in section I. To produce this state, we consider that both detectors measure one photon.

Fidelity is quite large and the maximum occurs close to $\alpha_{\text{in}} = \alpha_t$.

Generally, the input coherent state is complex, $\alpha_{\text{in}} = |\alpha_{\text{in}}|e^{i\phi_{\text{in}}}$. It is therefore interesting to understand the relationship between the phases of the input and the target states that can yield high fidelity. Here we set for simplicity $|\alpha_{\text{in}}| = |\alpha_t| = 1$ and $\phi_2 = \pm \pi$ to realize a high fidelity. It turns out that for different phases of the input state $\phi_{\text{in}}$ we need to adjust the phases of the target state $\phi_1$ and $\phi_3$ accordingly to get the optimal fidelity, see Fig. 10. The maximum fidelity is reached for $\phi_1 = \phi_3 = \phi_{\text{in}}$, as can also be deduced from Eq. (9). Thus, the phases $\phi_1$ and $\phi_3$ of the generated state can be controlled by varying $\phi_{\text{in}}$.

Due to the symmetry of the chosen target state, an asymmetric detection with only single-detector-response gives lower fidelity compared to symmetric detection. General expressions for the fidelity and the corresponding detection probability for arbitrary setup parameters are presented in the Appendix.

Fig. 9. Dependence of the fidelity given by Eq. (9) on $\alpha_{\text{in}}$ and $\alpha_t$ for parameters $\phi_1 = \phi_3 = 0$, $\phi_2 = \pm \pi$.

Fig. 10. Fidelity as a function of $\phi_1$ and $\phi_3$ with parameters $|\alpha_{\text{in}}| = |\alpha_t| = 1$ and $\phi_2 = \pm \pi$. (a), (b), (c), and (d) correspond to $\phi_{\text{in}} = 0$, $\pi$, and $3\pi$, respectively.
B. Influence of losses

We model the influence of losses on fidelity exactly as it is shown in the section about squeezing. We perform numerical simulations where the coefficients $R^\Sigma_0$ and $R^\Sigma_a$ are varied under the condition: $R^\Sigma_{\text{top}} = R^\Sigma_{\text{bottom}}$ both before and after detection. For weak losses, $R^\Sigma_{b/a} \in [0, 0.1]$, the dependence of the fidelity on the total reflection is linear and is approximately given by $F \approx F_0 - 0.91 R^\Sigma_0 - 0.48 R^\Sigma_a$, where $F_0$ is the fidelity without losses. For instance, including 5% loss before and after detection ($R^\Sigma_0 = R^\Sigma_a = 0.05$), the fidelity is reduced from 0.96 to 0.89. Losses before detection reduce the fidelity more significantly (almost twice as much) than losses after detection due to the different amount of photons before and after detection.

V. CONCLUSIONS

We present theoretical and numerical investigations of a linear two-mode interferometer with nonlinear detection operations. We analyze the influence of detection on squeezing and the generation of exotic quantum states. It is shown that by applying detection it is possible to create two-mode squeezing, even if the input states are initially uncorrelated. Moreover, it is shown that detection can be used to generate exotic states, such as the Schrödinger’s cat states, with high fidelity. To investigate the feasibility to observe the predicted effects in experiments, we analyze the influence of losses in the channels and show that fidelity and squeezing are degraded only weakly for not too high losses.

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Appendix A: Analytical results

1. No detection

The output state of light for the circuit without detection is given by

$$
|\psi\rangle = (\gamma_0 |\hat{a}_1^\dagger + \gamma_2 |\hat{a}_2^\dagger\rangle)|a_{01}, a_{02}\rangle
$$

$$
\gamma_0 = t_1 t_2 e^{i \phi_1} - r_1 r_2
$$

$$
\gamma_2 = i e^{i \phi_2} (t_1 t_2 e^{i \phi_1} + r_1 t_2) 
$$

$$
\alpha_{01} = i \alpha_{in} (t_1 r_2 + r_1 t_2 e^{i \phi_1})
$$

$$
\alpha_{02} = \alpha_{in} e^{i \phi_2} (t_1 t_2 - r_1 r_2 e^{i \phi_1})
$$

where $|a_{01}, a_{02}\rangle = |a_{01}\rangle \otimes |a_{02}\rangle$ is a product of two coherent states in different channels. For this state, the variance $\Delta^2 C_x$ can be calculated analytically and is given by

$$
\Delta^2 C_x = (C_x^2) - (C_x)^2 = \\
= \langle (X_1^a - X_1^b)^2 \rangle - \langle X_1^a - X_1^b \rangle^2 \\
= \frac{1}{2} + \frac{1}{2} \text{Re}(\langle a_1^\dagger a_1 \rangle) + \frac{1}{2} \text{Re}(\langle a_2^\dagger a_2 \rangle) + \frac{1}{2} \langle a_1^\dagger a_2 \rangle + \\
+ \frac{1}{2} \langle a_2^\dagger a_1 \rangle - \text{Re}(\langle a_1 a_2 \rangle) - \text{Re}(\langle a_1 a_2^\dagger \rangle) - \\
- (\text{Re}(a_1^\dagger) - \text{Re}(a_2^\dagger))^2 \\
= 1 + t_1 \sqrt{1 - t_1^2} \sin \phi_1 \cos \phi_2 + \\
+ \sin \phi_2 ((2t_1^2 - 1) t_2 \sqrt{1 - t_2^2} + \\
+ \cos \phi_1 (2t_1^2 - 1) t_1 \sqrt{1 - t_1^2}).
$$

2. Single-channel click detection

The output density matrix for the circuit with single-channel detection (channel 4), see the Fig. 1, is given by

$$
\rho_{out} = \frac{e^{-|\alpha_3|^2 - |\alpha_4|^2}}{P_{\text{single}}} \sum_{k=1}^{\infty} \frac{1}{k!} |\alpha_4 |^{2k-2} |\psi_k\rangle \langle \psi_k|
$$

$$
|\psi_k\rangle = (k \gamma_0 + \gamma_1 |\hat{a}_1^\dagger + \gamma_2 |\hat{a}_2^\dagger\rangle)|a_1, a_2\rangle \\
\gamma_0 = i t_1 r_3 \\
\gamma_1 = \alpha_{in} r_1 (r_1 t_2 r_4 - t_1 t_4 e^{i \phi_1}) \\
\gamma_2 = -i \alpha_{in} e^{i \phi_2} r_1 r_3 (r_1 t_2 r_4 + t_1 t_4 e^{i \phi_1}) \\
\alpha_1 = i \alpha_{in} (t_1 r_2 t_4 + r_1 t_3 t_4 e^{i \phi_1}) \\
\alpha_2 = \alpha_{in} e^{i \phi_2} (t_1 t_2 r_4 - r_1 t_3 r_4 e^{i \phi_1}) \\
\alpha_3 = i \alpha_{in} t_1 r_2 \\
\alpha_4 = -i \alpha_{in} r_1 r_3.
$$

The state after detection is not normalized due to the photon removal. To normalize the state we divide it by the detection probability, which corresponds to the probability of realizing such state. The probability of realizing the state in Eq. (A3) is:

$$
P_{\text{single}} = e^{-|\alpha_{in}|^2} \left( \sum_{n_2,n_3,n_4=0}^{\infty} |\tilde{\alpha}_{n_2} \tilde{\alpha}_{n_3} \tilde{\alpha}_{n_4} | \mid n_2 / n_3 / n_4 | \right) \\
\times (g_2 n_2 \tilde{\alpha}_1 \tilde{\alpha}_4 + g_3 n_3 \tilde{\alpha}_2 \tilde{\alpha}_4 + g_4 n_4 \tilde{\alpha}_2 \tilde{\alpha}_4)^2 \\
- \sum_{n_2,n_3,n_4=0}^{\infty} |\tilde{\alpha}_{n_2} \tilde{\alpha}_{n_3} \tilde{\alpha}_{n_4} | \mid n_2 / n_3 / n_4 | \right) \\
\times (g_2 n_2 \tilde{\alpha}_4 + g_4 n_4 \tilde{\alpha}_4)^2 \\
g_2 = i r_1 t_2 \\
g_3 = \gamma_{0} \\
g_4 = t_1 t_3 \\
\tilde{\alpha}_2 = \alpha_{in} t_1 t_2 \\
\tilde{\alpha}_3 = \alpha_4 \\
\tilde{\alpha}_4 = i \alpha_{in} r_1 t_3.
$$

3. Two-channels click detection

In the both-channels detection case the output state is given by

$$
\rho_{out} = \frac{e^{-|\alpha_3|^2 - |\alpha_4|^2}}{P_{\text{both}}} \\
\times \sum_{m=n=1}^{\infty} \sum_{m=1}^{\infty} |\alpha_3 |^{2m-2} |\alpha_4 |^{2n-2} |\psi_{m,n}\rangle \langle \psi_{m,n}|
$$

$$
|\psi_{m,n}\rangle = (m \gamma_3 + n \gamma_4 + \gamma_1 |\hat{a}_1^\dagger + \gamma_2 |\hat{a}_2^\dagger\rangle)|a_1, a_2\rangle \\
\gamma_1 = \gamma_1 \alpha_3 \\
\gamma_2 = \gamma_2 \alpha_3 \\
\gamma_3 = \alpha_{in} r_1^2 r_2 r_3 \\
\gamma_4 = i \alpha_{in} t_1 r_2 \\
\alpha_3 = \alpha_4 = 0.
$$

The probability of realizing the state in Eq. (A5) is:
\[ P_{\text{both}} = 1 - e^{-|\alpha_n|^2} \left( \sum_{n_2,n_3,n_4=0}^{\infty} \frac{\alpha_2^{n_2} \alpha_3^{n_3} \alpha_4^{n_4}}{\sqrt{n_2! n_3! n_4!}} \times \right. \\
\times \left. (g_2 n_1 \alpha_2 \alpha_3 + g_4 n_1 \alpha_2 \alpha_4 + g_3 n_1 \alpha_2 \alpha_3)^2 \right. \\
- \sum_{n_1,n_2,n_4=0}^{\infty} \frac{\alpha_2^{n_2} \alpha_4^{n_4}}{\sqrt{n_2! n_4!}} (g_2 n_1 \alpha_2 \alpha_4 + g_4 n_4 \alpha_2 \alpha_3)^2 \right) \\
g_1 = -r_1 r_2 \\
\tilde{\alpha}_1 = \alpha_3 \\
(A6) \]

4. Density matrices and probabilities

With the single photon PNR detection, expressions for the output density matrices and their probabilities of realization for single-detector-click and both-detectors click cases can be obtained from Eqs. \((A3), (A4), (A5), \) and \((A6)\) by substituting \(k = m = n = 1\). For example, for the single-detector-click can be written:

\[ \rho_{\text{out}} = |\psi_{\text{single}}\rangle \langle \psi_{\text{single}} | \\
|\psi_{\text{single}}\rangle = N_{\text{single}}(\gamma_0 + \gamma_1 \hat{a}_1^\dagger + \gamma_2 \hat{a}_2^\dagger)|\alpha_1, \alpha_2 \rangle \\
N_{\text{single}} = \tilde{P}_{\text{single}}^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (|\alpha_3|^2 + |\alpha_4|^2) \right], \]  

\[ \tilde{P}_{\text{single}} = \exp(-|\alpha_n|^2(T_1 R_2 + R_1 R_3)) \times \\
R_3 |\alpha_n|^2 R_2 T_2 + T_1 (1 - |\alpha_n|^2 R_1 (2T_2 + T_3) + (A8) \\
+ |\alpha_n|^4 R_1^2 (T_2^2 - T_3^2)), \]

for both-detectors click:

\[ \rho_{\text{out}} = |\psi_{\text{both}}\rangle \langle \psi_{\text{both}} | \\
|\psi_{\text{both}}\rangle = N_{\text{both}}(\tilde{\gamma}_0 + \tilde{\gamma}_1 \hat{a}_1^\dagger + \tilde{\gamma}_2 \hat{a}_2^\dagger)|\alpha_1, \alpha_2 \rangle \\
\tilde{\gamma}_0 = \tilde{\gamma}_3 + \tilde{\gamma}_4 \\
N_{\text{both}} = \tilde{P}_{\text{both}}^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (|\alpha_3|^2 + |\alpha_4|^2) \right], \]  

\[ \tilde{P}_{\text{both}} = \exp(-|\alpha_n|^2(T_1 R_2 + R_1 R_3)) \times \\
R_3 |\alpha_n|^2 R_2 T_2 + T_1 (1 + |\alpha_n|^2 R_1 (3T_2 - 2T_3) + \\
+ |\alpha_n|^4 T_1^2 (T_2 - T_3)^2) + \\
+ R_1 (-2T_1 + |\alpha_n|^2 T_1^2 (-2T_2 + 3T_3)), \]  

\[ \tilde{P}_{\text{both}} = \exp(-|\alpha_n|^2(T_1 R_2 + R_1 R_3)) \times \\
+ |\alpha_n|^2 R_2 R_3 |T_1^2 + R_1^2 (1 + |\alpha_n|^2 R_1 (3T_2 - 2T_3) + \\
+ |\alpha_n|^4 T_1^2 (T_2 - T_3)^2) + \\
+ R_1 (-2T_1 + |\alpha_n|^2 T_1^2 (-2T_2 + 3T_3)), \]  

where \(|\alpha_1, \alpha_2\rangle = |\alpha_1 \rangle \otimes |\alpha_2 \rangle\) is a product of two coherent states.

5. Two-mode variance

For the PNR detection case the analytical formula of two-mode squeezing can be derived:
\[ \Delta^2 C_x = \frac{1}{2} + \frac{1}{2} \text{Re}(\langle a_1^2 \rangle) + \frac{1}{2} \text{Re}(\langle a_2^2 \rangle) + \frac{1}{2} \langle a_1^2 a_1 \rangle + \frac{1}{2} \langle a_2^2 a_2 \rangle - \text{Re}(\langle a_1 a_2 \rangle) - \text{Re}(\langle a_1^2 \rangle) - \text{Re}(\langle a_2^2 \rangle) \]

\[ \langle a_1 \rangle = N^2(|\gamma_1|^2 a_1 + \gamma_0 \gamma_1 (|a_1|^2 + 1) + \gamma_0 \gamma_2 a_1 a_2 + \gamma_1^* |\gamma_0|^2 a_2^* a_2 + |\gamma_1|^2 a_1 (|a_1|^2 + 1) + \gamma_1^* \gamma_2 a_1 (a_2^* + 1) + \gamma_2^* |\gamma_0|^2 a_1 a_2 + \gamma_2^* \gamma_2 a_1 a_2) \]

\[ \langle a_2 \rangle = N^2(|\gamma_0|^2 a_2 + \gamma_0 \gamma_1 a_1 a_2 + |\gamma_1|^2 a_1 (|a_1|^2 + 1) + \gamma_1^* \gamma_2 a_1 (a_2^* + 1) + \gamma_2^* |\gamma_0|^2 a_2^* a_2 + \gamma_2^* \gamma_2 a_1 a_2) \]

where parameters \( \gamma_i, \alpha_i, \) and \( N \equiv N_{\text{single(both)}} \) are defined in Eqs. \( \mathbb{A}7 \) and \( \mathbb{A}9 \) for cases where single-detector (both-detectors) measures one photon.

6. Fidelity

An expression for the fidelity between the target state, Eq. \( \mathbb{S} \), and the output state after the single/both-click PNR detection, see Eqs. \( \mathbb{A}7 \) and \( \mathbb{A}9 \), is given by

\[ F(\psi_{\text{out}}, \psi_{\text{trg}}) = (N_{\text{single(both)}} N_i)^2 \times \times \exp(-|\alpha_1|^2 - |\alpha_2|^2) \times \times \left( |\gamma_1| |\alpha_1| e^{-i\phi_1} + \gamma_3 \right) \exp(\alpha_1 |\alpha_1| e^{-i\phi_1}) + (A12) + (\gamma_2 |\alpha_2| e^{-i\phi_2} + \gamma_3) \exp(\alpha_2 |\alpha_2| e^{-i\phi_2}) \right)^2. \]

where parameters \( \gamma_i, \) (\( \gamma_i \)), \( \alpha_1, \alpha_2 \) and \( N_{\text{single(both)}} \) can be found in Eqs. \( \mathbb{A}7 \) and \( \mathbb{A}9 \). Parameters \( N_i, \alpha_i, \phi_1, \phi_3 \) are defined in Eq. \( \mathbb{S} \).