Strong $q$-log-convexity of the Eulerian polynomials of Coxeter groups *

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Abstract

In this paper we prove the strong $q$-log-convexity of the Eulerian polynomials of Coxeter groups using their exponential generating functions. Our proof is based on the theory of exponential Riordan arrays and a criterion for determining the strong $q$-log-convexity of polynomials sequences, whose generating functions can be given by the continued fraction. As consequences, we get the strong $q$-log-convexity of the Eulerian polynomials of types $A_n$, $B_n$, their $q$-analogous and the generalized Eulerian polynomials associated to the arithmetic progression \{a, a + d, a + 2d, a + 3d, \ldots\} in a unified manner.

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1 Introduction

The Eulerian polynomials $P(W, q)$, which enumerate the number of descents of a (finite) Coxeter group $W$, is one of the classical polynomials in combinatorics. During their long history, they arised often in combinatorics and were extensively studied (see [5, 6, 7, 13] and references therein). In recent years, there has been a considerable amount of interesting extensions and modifications devoted to these polynomials (see [2, 3, 11, 18, 19, 20, 22, 25] for instance). In fact, Brenti showed that it is enough to study the Eulerian polynomials for irreducible Coxeter groups [4, 5]. For Coxeter groups of type $A_n$, it is known that these polynomials coincide with the classical Eulerian polynomials, whose properties have been well studied from a combinatorial point of view [13, 16, 18, 19, 22]. Some properties of the classical Eulerian polynomials can be generalized to the Eulerian

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polynomials of type $B_n$, such as recurrence relations, the reality of zeros, generating functions, unimodality and total positivity properties [2, 5, 12, 20]. In this paper, using their exponential generating functions, we present the strong $q$-log-convexity of many Eulerian polynomials of Coxeter groups, which on one hand also generalizes the strong $q$-log-convexity of the classical Eulerian polynomials [25], on the other hand will give the strong $q$-log-convexity of types $A_n$, $B_n$, their $q$-analogues and the generalized Eulerian polynomials associated to the arithmetic progression $\{a, a + d, a + 2d, a + 3d, \ldots\}$ in a unified manner.

Let $q$ be an indeterminate. For two real polynomials $f(q)$ and $g(q)$, denote $f(q) \succeq_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients as a polynomial of $q$. We say that a real polynomial sequence $\{f_n(q)\}_{n \geq 0}$ is called $q$-log-convex if

$$f_{n-1}(q)f_{n+1}(q) \succeq_q f_n^2(q)$$

for $n \geq 1$, and it is strongly $q$-log-convex if

$$f_{m-1}(q)f_{n+1}(q) \succeq_q f_m(q)f_n(q)$$

for all $n \geq m \geq 1$. Clearly, the strong $q$-log-convexity of polynomials sequences implies the $q$-log-convexity. However, the converse does not follow.

As we know that many famous polynomials sequences, such as the Bell polynomials [10, 19], the classical Eulerian polynomials [19, 25], the Narayana polynomials [9], the Narayana polynomials of type $B$ [8] and the Jacobi-Stirling numbers [17, 26], are $q$-log-convex. Furthermore, almost all of these polynomials sequences are strongly $q$-log-convex [10, 17, 25]. In this paper we give the strong $q$-log-convexity of many Eulerian polynomials. Our proof relies on the theory of exponential Riordan arrays and a criterion of Zhu [25] for determining the strong $q$-log-convexity of polynomials sequences, whose generating functions can be given by the continued fraction.

This paper is organized as follows. In section 2, using the theory of exponential Riordan arrays and orthogonal polynomials, we first give the continued fraction of the ordinary generating function of the polynomials sequence, whose exponential generating function generalizes the exponential generating function of many Eulerian polynomials. Then we obtain the strong $q$-log-convexity of the polynomials sequence using the continued fraction and a criterion of Zhu [25]. As applications, we obtain the strong $q$-log-convexity of the Eulerian polynomials of Coxeter groups, including the Eulerian polynomials of types $A_n$, $B_n$, their $q$-analogues defined by Foata and Schützenberger [16] and Brenti [5] respectively, and the generalized Eulerian polynomials associated to the arithmetic progression $\{a, a + d, a + 2d, a + 3d, \ldots\}$ [24] in a unified manner in section 3. In section 4, we present some conjectures and open problems. Finally, in the Appendix, we can obtain a quick introduction to the exponential Riordan arrays and the orthogonal polynomials used in this paper.

2 The strong $q$-log-convexity

In this section, we first give the continued fraction of the ordinary generating function of the polynomials sequence $\{T_n(q)\}_{n \geq 0}$, whose exponential generating function generalizes the exponential generating functions of many Eulerian polynomials. Then using the
continued fraction and a criterion of Zhu \([25]\), we prove the strong \(q\)-log-convexity of the polynomials sequence \(\{T_n(q)\}_{n \geq 0}\).

**Theorem 2.1.** Suppose that the exponential generating function of the polynomials sequence \(\{T_n(q)\}_{n \geq 0}\) has the following simple expression

\[
g(x) = \sum_{n \geq 0} \frac{T_n(q)}{n!} x^n = \left( \frac{(1-q)e^{a(1-q)x}}{1-qe^{d(1-q)x}} \right)^b,
\]

for \(a, b, d \in \mathbb{R}\). Then the ordinary generating function of \(\{T_n(q)\}_{n \geq 0}\) can be given by the continued fraction

\[
h(x) = \sum_{n \geq 0} T_n(q)x^n = \frac{1}{1 - s_0(q)x - \frac{t_1(q)x^2}{1 - s_1(q)x - \frac{t_2(q)x^2}{1 - s_2(q)x - \frac{t_3(q)x^2}{1 - s_3(q)x - \cdots}}}},
\]

where

\[
s_i(q) = (di + ab) + (di + bd - ab)q \quad \text{and} \quad t_{i+1}(q) = d^2(i + 1)(i + b)q
\]

for \(i \geq 0\).

In order to prove this theorem, we need three lemmas. Using the theory of the exponential Riordan arrays, the first lemma presents that the production matrix \(P\) of the exponential Riordan array \(L = [g(x), f(x)]\), where \(g(x)\) is the exponential generating function of \(\{T_n(q)\}_{n \geq 0}\) given by (2.1), is tri-diagonal.

**Lemma 2.1.** The production matrix \(P\) of the exponential Riordan array

\[
L = [g(x), f(x)] = \begin{bmatrix} \frac{(1-q)e^{a(1-q)x}}{1-qe^{d(1-q)x}} & b \frac{e^{d(1-q)x} - 1}{d[1-qe^{d(1-q)x}]} \end{bmatrix},
\]

for \(a, b, d \in \mathbb{R}\), is tri-diagonal.

**Proof.** In order to get the production matrix \(P\), it suffices to calculate \(r(x)\) and \(c(x)\). Recall that

\[
r(x) = f'\left(\bar{f}(x)\right), \quad c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))},
\]

where \(\bar{f}(x)\) is the compositional inverse of \(f(x)\).

By the direct calculation, we have

\[
f'(x) = \frac{(1-q)^2e^{d(1-q)x}}{[1-qe^{d(1-q)x}]^2}.
\]

Note that the compositional inverse of \(f(x)\) satisfies

\[
f(\bar{f}(x)) = \frac{e^{d(1-q)\bar{f} - 1}}{d[1-qe^{d(1-q)\bar{f}}]} = x.
\]
Then we have
\[
\bar{f}(x) = \frac{1}{d(1-q)} \ln \left( \frac{1 + dx}{1 + dqx} \right).
\]
Hence
\[
r(x) = f'(\bar{f}(x)) = (1 + dx)(1 + dqx) = 1 + (1 + q)x + d^2q^2x^2.
\]
On the other hand, we have
\[
g'(x) = b \left( \frac{(1-q)e^{a(1-q)x}}{1-ge^{d(1-q)x}} \right)^{b-1} \frac{(1-q)^2e^{a(1-q)x}[a + (d-a)qge^{d(1-q)x}]}{[1-ge^{d(1-q)x}]^2}.
\]
So
\[
c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = \frac{b(1-q)[a + (d-a)qge^{d(1-q)f}]}{1-ge^{d(1-q)f}}
\]
\[
= ab(1 + dqx) + b(d-a)q(1 + dx)
\]
\[
= b[a + (d-a)q] + bd^2qx.
\]
Thus the production matrix \(P\) of \(L\) is tri-diagonal, where
\[
P = \begin{pmatrix}
    s_0(q) & 1 & 0 & 0 & 0 & 0 & \cdots \\
    t_1(q) & s_1(q) & 1 & 0 & 0 & 0 & \cdots \\
    0 & t_2(q) & s_2(q) & 1 & 0 & 0 & \cdots \\
    0 & 0 & t_3(q) & s_3(q) & 1 & 0 & \cdots \\
    0 & 0 & 0 & t_4(q) & s_4(q) & 1 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}, \quad (2.4)
\]
with \(s_i(q)\) and \(t_{i+1}(q)\) given by \((2.3)\).

The second lemma constructs a family of orthogonal polynomials related to the production matrix \(P\) of the exponential Riordan array \(L = [g(x), f(x)]\).

**Lemma 2.2.** Suppose that the production matrix \(P\) of an exponential Riordan array \(L\) is tri-diagonal as above \((2.4)\). Then we can construct a family of orthogonal polynomials \(Q_n(x)\) defined by
\[
Q_n(x) = [x - s_{n-1}(q)]Q_{n-1}(x) - t_{n-1}(q)Q_{n-2}(x), \quad (2.5)
\]
with \(Q_0(x) = 1\) and \(Q_1(x) = x - s_0(q)\), where coefficients \(s_{n-1}(q)\) and \(t_{n-1}(q)\) are given by the expression \((2.3)\).

**Proof.** In order to construct the family of orthogonal polynomials \(Q_n(x)\), it suffices to get the coefficient matrix \(A\) such that
\[
\begin{pmatrix}
    Q_0(x) \\
    Q_1(x) \\
    Q_2(x) \\
    Q_3(x) \\
    \vdots
\end{pmatrix} = A \begin{pmatrix}
    1 \\
    x \\
    x^2 \\
    x^3 \\
    \vdots
\end{pmatrix}. \quad (2.6)
\]
And by the condition and the Favard’s Theorem 5.1 in Appendix, we will get that the orthogonal polynomials \( Q_n(x) \) satisfies the following

\[
P \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \end{pmatrix} = \begin{pmatrix} s_0(q) & 1 & 0 & 0 & 0 & 0 & \cdots \\ t_1(q) & s_1(q) & 1 & 0 & 0 & 0 & \cdots \\ 0 & t_2(q) & s_2(q) & 1 & 0 & 0 & \cdots \\ 0 & 0 & t_3(q) & s_3(q) & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \end{pmatrix} = \begin{pmatrix} xQ_0(x) \\ xQ_1(x) \\ xQ_2(x) \\ xQ_3(x) \end{pmatrix}.
\]  

(2.7)

Then we have that the coefficient matrix \( A \) satisfies

\[
PA \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = A \begin{pmatrix} x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = AI \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix},
\]

(2.8)

where \( I = (\delta_{i+1,j})_{i,j \geq 0} \). Since the polynomials sequence \( \{x^k\}_{k \geq 0} \) is linear independence. So the coefficient matrices of the first and last polynomials in (2.7) are equal, i.e., \( PA = AI \). Since \( P = L^{-1}\bar{L}, I = \bar{L}L^{-1} \). So we have that the coefficient matrix \( A \) will satisfy \( L^{-1}\bar{L}A = ALL^{-1} \). Thus we can obtain that \( L^{-1} \) is a coefficient matrix of the orthogonal polynomials \( Q_n(x) \). The proof of the lemma is complete.

\[\square\]

Remark 2.1. From the proof of Lemma 2.2, we have that the coefficient matrix of the orthogonal polynomials \( Q_n(t) \) is

\[
L^{-1} = \left[ \frac{1}{(1 + dx)^{\frac{4k}{d}}(1 + dqx)^{\frac{k-4k}{d}}}, \frac{1}{d(1 - q)} \ln \left( \frac{1 + dx}{1 + dqx} \right) \right],
\]

which has been shown by Barry [3]. However our proof is more natural and based on the algebraic method.

The last lemma, obtained by Barry [3], gave the connection between the production matrix and the moment sequence of orthogonal polynomials.

Lemma 2.3 (3). Let \( L, T_n(q) \) and \( Q_n(x) \) be as above. Then we have \( \{T_n(q)\}_{n \geq 0} \) is the moment sequence of the associated family of orthogonal polynomials \( Q_n(x) \).

Now we can obtain that the ordinary generating function of \( \{T_n(q)\}_{n \geq 0} \) is given by the continued fraction (2.2) from Theorem 5.2, which proves Theorem 2.1.

Then we can present the strong \( q \)-log-convexity of \( \{T_n(q)\}_{n \geq 0} \) using the following criterion of Zhu [25].

Theorem 2.2 (25 Proposition 3.13). Given two sequences \( \{s_i(q)\}_{i \geq 0} \) and \( \{t_{i+1}(q)\}_{i \geq 0} \) of polynomials with nonnegative coefficients, let

\[
\sum_{n \geq 0} D_n(q)x^n = \frac{1}{1 - s_0(q)x - \frac{t_1(q)x^2}{1 - s_1(q)x - \frac{t_2(q)x^2}{1 - s_2(q)x - \frac{t_3(q)x^2}{1 - s_3(q)x - \cdots}}}}.
\]
If \( s_i(q)s_{i+1}(q) \geq t_{i+1}(q) \) for all \( i \geq 1 \), then the sequence \( \{D_n(q)\}_{n \geq 0} \) is strongly \( q \)-log-convex.

The main result of this section is the following.

**Theorem 2.3.** The polynomials sequence \( \{T_n(q)\}_{n \geq 0} \) defined by (2.1) forms a strongly \( q \)-log-convex sequence for \( b \geq 0 \) and \( d \geq a \geq 0 \)

**Proof.** By Theorem 2.1 if the exponential generating function of \( \{T_n(q)\}_{n \geq 0} \) has the expression (2.1), then we have the ordinary generating function of \( \{T_n(q)\}_{n \geq 0} \) can be given by the continued fraction (2.2). Note that \( s_i(q) = (d_i + ab) + (d_i + bd - ab)q \) and \( t_{i+1}(q) = d^2(i+1)(i+b)q \) for \( i \geq 0 \). So

\[
\begin{align*}
s_i(q)s_{i+1}(q) - t_{i+1}(q) &= ((d_i + ab) + (d_i + bd - ab)q)((d_i + d + ab) + (d_i + d + bd - ab)q) - d^2(i + 1)(i + b)q \\
&= (d_i + ab)(d_i + d + ab) + (d_i + bd - ab)(d_i + d + bd - ab)q^2 \\
&\quad + ((d_i + ab)(d_i + d + bd - ab) + (d_i + bd - ab)(d_i + d + ab) - d^2(i + 1)(i + b))q \\
&= (d_i + ab)(d_i + d + ab) + (d_i + bd - ab)(d_i + d + bd - ab)q^2 \\
&\quad + ((d_i + ab)(d_i + d + bd - ab) + abd(b-1) - a^2b^2)q \\
&\geq q(d_i + ab)(d_i + d + ab) + (d_i + bd - ab)(d_i + d + bd - ab)q^2 + (ab^2d - a^2b^2)q \\
&\geq q0.
\end{align*}
\]

The first and second inequalities hold by conditions \( i, b \geq 0 \) and \( d \geq a \geq 0 \). Hence the polynomials sequence \( \{T_n(q)\}_{n \geq 0} \) forms a strongly \( q \)-log-convex sequence by Theorem 2.2.

\[\square\]

## 3. The Eulerian Polynomials of Coxeter Groups

Given a finite Coxeter group \( W \), define the Eulerian polynomials of \( W \) by

\[
P(W, q) = \sum_{\pi \in W} q^{d_W(\pi)},
\]

where \( d_W(\pi) \) is the number of \( W \)-descents of \( \pi \). We refer the reader to Björner [4] for relevant definitions.

For Coxeter groups of type \( A_n \), it is known that \( P(A_n, q) = A_n(q)/q \), the shifted Eulerian polynomials, whose strong \( q \)-log-convexity was obtained by Zhu [25]. Since the exponential generating function of \( \{A_n(q)\}_{n \geq 0} \) and \( \{P(A_n, q)\}_{n \geq 0} \) is

\[
\sum_{n \geq 0} A_n(q)\frac{x^n}{n!} = \frac{(1 - q)}{1 - qe^{x(1-q)}}
\]

and

\[
\sum_{n \geq 0} P(A_n, q)\frac{x^n}{n!} = \frac{(1 - q)e^{x(1-q)}}{1 - qe^{x(1-q)}}
\]

(3.1) and (3.2) respectively.
respectively (see [13, p. 244]). So from Theorem 2.1, we have
\[
\sum_{n \geq 0} A_n(q)x^n = \frac{1}{qx^2}\left(1 - x - \frac{4qx^2}{1 - (2 + q)x - \frac{9qx^2}{1 - (3 + 2q)x - \frac{16qx^2}{1 - (4 + 3q)x - \cdots}}}ight),
\]
with \(s_i(q) = i + (i + 1)q\) and \(t_{i+1}(q) = (i + 1)^2q\) for \(i \geq 0\) (see [2] for instance). And
\[
\sum_{n \geq 0} P(A_n, q)x^n = \frac{1}{qx^2}\left(1 - qx - \frac{4qx^2}{1 - (1 + 2q)x - \frac{9qx^2}{1 - (2 + 3q)x - \frac{16qx^2}{1 - (3 + 4q)x - \cdots}}}ight),
\]
with \(s_i(q) = (i + 1) + iq\) and \(t_{i+1}(q) = (i + 1)^2q\) for \(i \geq 0\).

Obviously, we have the following result by Theorem 2.3.

**Proposition 3.1.** The polynomials \(P(A_n, q)\) and \(A_n(q)\) form strongly \(q\)-log-convex sequences respectively.

In [16], Foata and Schützenberger introduced a \(q\)-analog of the classical Eulerian polynomials defined by
\[
A_n(q; t) := \sum_{\pi \in S_n} q^\text{exc} (\pi) + 1_c (\pi),
\]
where \(\text{exc} (\pi)\) and \(c(\pi)\) denote the numbers of excedances and cycles in \(\pi\) respectively. It is clear that \(A_n(q; 1) = A_n(q)\) is precisely the classical Eulerian polynomial. Brenti showed that the exponential generating function of \(\{A_n(q; t)\}_{n \geq 0}\) is given by
\[
\sum_{n \geq 0} A_n(q; t)x^n = \left(\frac{(1 - q)e^{x(1-q)}}{1 - qe^{x(1-q)}}\right)^t.
\]
(3.3)

So from Theorem 2.1, we have
\[
\sum_{n \geq 0} A_n(q; t)x^n = \frac{1}{tx^2}\left(1 - tx - \frac{2(t + 1)qx^2}{1 - (t + 1 + q)x - \frac{3(t + 2)qx^2}{1 - (t + 2 + 2q)x - \frac{16(q^2 + 2q)x^2}{1 - (t + 3 + 3q)x - \cdots}}}ight),
\]
Here \(s_i(q) = (t + i) + iq\) and \(t_{i+1}(q) = (i + 1)(t + i)q\) for \(i \geq 0\).

Obviously, we have the following result by Theorem 2.3.

**Proposition 3.2.** The polynomials \(A_n(q; t)\) form a strongly \(q\)-log-convex sequence for \(t \geq 0\).
For Coxeter groups of type $B_n$, suppose that the Eulerian polynomials of type $B_n$

\[ P(B_n, q) = \sum_{k=0}^{n} B_{n,k} q^k, \]

where $B_{n,k}$ is the Eulerian numbers of type $B_n$ counting the elements of $B_n$ with $k$ $B$-descents. Then the Eulerian numbers of type $B_n$ satisfy the recurrence

\[ B_{n,k} = (2k + 1)B_{n-1,k} + (2n - 2k + 1)B_{n-1,k-1}. \]  

(3.4)

Hence the Eulerian polynomials of type $B_n$ satisfy the recurrence

\[ P(B_n, q) = [(2n - 1)q + 1]P(B_{n-1}, q) + 2q(q - 1)P'(B_{n-1}, q). \]  

(3.5)

It is well known that $P(B_n, q)$ have only real zeros (see [5, 20] for instance). Note that the exponential generating function of the Eulerian polynomials of type $B_n$ has the following expression

\[ \sum_{n \geq 0} P(B_n, q) \frac{x^n}{n!} = (1 - q)e^{x(1-q)} \frac{1}{1 - qe^{x(1-q)}(1+t)} . \]  

(3.6)

(see [5, Theorem3.4] and [12, Corollary3.9]). Hence from Theorem 2.1, we have the generating function of the Eulerian polynomials of type $B_n$ is given by

\[ \sum_{n \geq 0} P(B_n, q) x^n = \frac{1}{1 - (1+q)x - \frac{4qx^2}{1 - 3(1+q)x - \frac{16qx^2}{1 - 5(1+q)x - \frac{36qx^2}{1 - 7(1+q)x - \cdots}}}}. \]

Here $s_i(q) = (2i + 1)(1 + q)$ and $t_{i+1}(q) = 4(i + 1)^2 q$ for $i \geq 0$.

Thus the strong $q$-log-convexity of $P(B_n, q)$ follows from Theorem 2.3.

**Proposition 3.3.** The polynomials $P(B_n, q)$ form a strongly $q$-log-convex sequence.

From the definitions, if a sequence of polynomials is strongly $q$-log-convex, then it is $q$-log-convex. So we have the following corollary immediately.

**Corollary 3.1.** The polynomials $P(B_n, q)$ form a $q$-log-convex sequence.

**Remark 3.1.** From Liu and Wang [19, Theorem 4.1], we can also get Corollary 3.1 using recurrences (3.4) and (3.5).

Brenti [5] defined a $q$-analogue of the polynomials $P(B_n, q)$ by

\[ B_n(q; t) := \sum_{\pi \in B_n} q^{d_B(\pi)} t^{N(\pi)}, \]

where $N(\pi) := |\{i \in [n], \pi(i) < 0\}|$. In particular, if $t = 1$, then $B_n(q; 1) = P(B_n, q)$, the Eulerian polynomials of type $B_n$. And if $t = 0$, then $B_n(q; 0) = A_n(q)$, the classical Eulerian polynomials. He showed that the exponential generating function of $\{B_n(q; t)\}_{n \geq 0}$ has the following expression

\[ \sum_{n \geq 0} B_n(q; t) \frac{x^n}{n!} = \frac{(1 - q)e^{x(1-q)}}{1 - qe^{x(1-q)(1+t)}} . \]  

(3.7)
Here \( t \geq t \)

**Proposition 3.4.** The polynomials \( B_n(q; t) \) form a strongly q-log-convex sequence for \( t \geq 0 \).

Recently, Xiong, Tsao and Hall [24] defined the general Eulerian numbers \( A_{n,k}(a, d) \) associated with an arithmetic progression \( \{a, a + d, a + 2d, a + 3d, \ldots\} \) as

\[
A_{n,k}(a, d) = (-a + (k + 2)d)A_{n-1,k}(a, d) + (a + (n - k - 1)d)A_{n-1,k-1}(a, d),
\]

where \( A_{0,1} = 1 \) and \( A_{n,k} = 0 \) for \( k \geq n \) or \( k \leq -2 \). In particular, when \( a = d = 1 \), \( A_{n,k}(1,1) = A_{n,k} \), the classical Eulerian numbers which enumerating the number of \( A_n \) with \( k - 1 \) descents. Similarly, the general Eulerian polynomials associated with an arithmetic progression \( \{a, a + d, a + 2d, a + 3d, \ldots\} \) can be defined as

\[
P_n(q, a, d) = \sum_{k=-1}^{n-1} A_{n,k}(a, d)q^{k+1}.
\]

It is shown that the exponential generating function of \( \{P_n(q, a, d)\}_{n \geq 0} \) has the following expression

\[
\sum_{n \geq 0} P_n(q, a, d) \frac{x^n}{n!} = \frac{(1 - q) e^{ax(1-q)}}{1 - q e^{dx(1-q)}}. \tag{3.8}
\]

So from Theorem 2.1 the generating function of \( \{P_n(q, a, d)\}_{n \geq 0} \) is given by

\[
\sum_{n \geq 0} P_n(q, a, d) x^n = \frac{1}{1 - s_0(q)x - \frac{t_1(q)x^2}{1 - s_1(q)x - \frac{t_2(q)x^2}{1 - s_2(q)x - \frac{t_3(q)x^2}{1 - s_3(q)x - \cdots}}}},
\]

with \( s_i(q) = (di + a) + (di + d - a)q \) and \( t_{i+1}(q) = (d(i + 1))^2q \) for \( i \geq 0 \) (see [3] for instance).

Thus the strong q-log-convexity of \( P_n(q, a, d) \) follows from Theorem 2.3.

**Proposition 3.5.** The general Eulerian polynomials \( P_n(q, a, d) \) associated with an arithmetic progression \( \{a, a + d, a + 2d, a + 3d, \ldots\} \) form a strongly q-log-convex sequence for \( d \geq a \geq 1 \).
4 Concluding remarks and open problems

Let \(a_0, a_1, a_2, \ldots\) be a sequence of nonnegative numbers. The sequence is called log-convex (respectively log-concave) if for \(k \geq 1\), \(a_k^2 \leq a_{k-1}a_{k+1}\) (respectively \(a_k^2 \geq a_{k-1}a_{k+1}\)). Let \(\{a(n,k)\}_{0 \leq k \leq n}\) be a triangular array of nonnegative numbers. Define a linear transformation of sequences by

\[ z_n = \sum_{k=0}^{n} a(n,k)x_k, \quad n = 0, 1, 2, \ldots. \]  

We say that the linear transformation (4.1) preserve log-convexity if it preserves the log-convexity of sequences, i.e., the log-convexity of \(\{x_n\}\) implies that of \(\{z_n\}\). We also say that corresponding triangle \(\{a(n,k)\}_{0 \leq k \leq n}\) preserve log-convexity. Liu and Wang [19] obtained the binomial transformation, the Stirling transformation s of the first and second kind preserve log-convexity respectively. They also proposed the following conjecture, which is still open now.

**Conjecture 4.1** ([19]). The Eulerian transformation

\[ z_n = \sum_{k=0}^{n} A_{n,k}x_k \]  

preserve log-convexity.

Similarly, we can raise the following problem related to the Eulerian polynomials of type \(B_n\).

**Conjecture 4.2.** Let

\[ z_n = \sum_{k=0}^{n} B_{n,k}x_k \]  

denote the Eulerian transformation of type \(B_n\). Then the transformation (4.2) preserves log-convexity.

5 Appendix

The exponential Riordan array [11 14 15] denoted by \(L = [g(x), f(x)]\), is an infinite lower triangular matrix whose exponential generating function of the \(k\)th column is \(g(x)(xf(x))^k/k!\) for \(k = 0, 1, 2, \ldots\), where \(g(0) \neq 0 \neq f(0)\). An exponential Riordan array \(L = (l_{i,j})_{i,j \geq 0}\) can also be characterized by two sequences \(\{c_n\}_{n \geq 0}\) and \(\{r_n\}_{n \geq 0}\) such that

\[ l_{0,0} = 1, \quad l_{i+1,0} = \sum_{j \geq 0} j!c_jl_{i,j}, \quad l_{i+1,j} = \frac{1}{j!} \sum_{k \geq j-1} k!(c_{k-j} + jr_{k-j+1})l_{i,j}, \]

for \(i, j \geq 0\) (see [14] for instance). Call \(\{c_n\}_{n \geq 0}\) and \(\{r_n\}_{n \geq 0}\) the c— and z— sequences of \(L\) respectively. Associated to each exponential Riordan array \(L = (l_{i,j})_{i,j \geq 0}\), there is a matrix \(P = (p_{i,j})_{i,j \geq 0}\), called the production matrix, whose bivariate generating function is given by

\[ e^{xy}[c(x) + r(x)y], \]

where

\[ c(x) = \frac{g'(f(x))}{g(f(x))} := \sum_{n \geq 0} c_n x^n, \quad r(x) = f'(f(x)) := \sum_{n \geq 0} r_n x^n. \]
Deutsch et al. [14] obtained the elements of production matrix $P = (p_{i,j})_{i,j \geq 0}$ satisfying 

$$p_{i,j} = \frac{i!}{j!} (c_{i-j} + j r_{i-j+1}).$$

Assume that $c_{-1} = 0$. Note that 

$$P = L^{-1} \bar{L}, \bar{I} = \bar{L}L^{-1},$$

where $\bar{L}$ is obtained from $L$ with the first row removed and $\bar{I} = (\delta_{i+1,j})_{i,j \geq 0}$, where $\delta_{i,j}$ is the usual Kronecker symbol.

The following well-known results establish the relationship among the orthogonal polynomials, three-term recurrences, recurrence coefficients and the continued fraction of the generating function of the moment sequence. The first result is the well-known "Favard’s Theorem".

**Theorem 5.1** ([21 Théorème 9 on p. I-4], or [23 Theorem 50.1]). Let $\{p_n(x)\}_{n \geq 0}$ be a sequence of monic polynomials with degree $n = 0, 1, 2, \ldots$ respectively. Then the sequence $\{p_n(x)\}_{n \geq 0}$ is (formally) orthogonal if and only if there exist sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 1}$ with $\beta_n \neq 0$ such that the three-term recurrence

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x)$$

holds, for $n \geq 1$, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

**Theorem 5.2** ([21 Propersition 1 (7) on p. V-5], or [23 Theorem 51.1]). Let $\{p_n(x)\}_{n \geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some linear functional $\mathcal{L}$. For $n \geq 1$, let

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x),$$

be the corresponding three-term recurrence which is guaran ted by Favard’s theorem. Then the generating function

$$h(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ satisfies

$$h(x) = \frac{\mu_0}{1 - \alpha_0 x - \frac{\beta_1 x^2}{1 - \alpha_1 x - \frac{\beta_2 x^2}{1 - \alpha_2 x - \frac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$  

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