On the Comparison of Perturbation-Iteration Algorithm and Residual Power Series Method to Solve Fractional Zakharov-Kuznetsov Equation

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Abstract

In this paper, we present analytic-approximate solution of a fractional Zakharov-Kuznetsov equation by means of perturbation-iteration algorithm (PIA) and residual power series method (RSPM). Basic definitions of fractional derivatives are described in the Caputo sense. Several examples are given and the results are compared to exact solutions. The results show that both methods are competitive, effective, convenient and simple to use.

Keywords: Fractional-integro differential equations, Caputo fractional derivative, Initial value problems, Perturbation-Iteration Algorithm.

1 Introduction

Fractional differential equations have received considerable interest in recent years and have been extensively investigated and applied for many real problems which are modeled in various areas and have been the focus of many studies due to their frequent appearance in various applications such as fluid mechanics, viscoelasticity, biology, physics and engineering. Therefore, great attention has been given to find numerical and Analytic-approximate solutions of FDEs. Some of the recent analytical-approximate methods for FDEs include the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational iteration method (VIM) and homotopy analysis method (HAM). The ADM was applied to fractional diffusion equations in [1] and fractional modified KdV equations in [2]. Hosseinnia et al. [3] suggested an enhanced HPM for FDEs and also Abdulaziz et al. [4] improved the application of HPM to systems of FDEs. In [5], Abdulaziz et al. solved the fractional IVPs by the HPM. The HAM was applied to fractional KDV-Burgers-Kuromoto equations [6], fractional IVPs [7], time-fractional PDEs [8], linear and nonlinear FDEs [9] and systems of nonlinear FDEs [10]. Variational iteration method was applied to solve some types of FDEs in [11,12].

In this paper, we first introduce fractional Zakharov-Kuznetsov equation, then we describe perturbation-iteration algorithm and residual power series method (RSPM) in order to implement on Zakharov-Kuznetsov equation, then we present some examples that show reliability and efficiency of two methods in order to compare their numerical results. At last, we discuss about obtained results as a section for conclusion. This paper considers the fractional version of the Zakharov-Kuznetsov equations as studied in [13]. The fractional Zakharov-Kuznetsov equations considered are of the form:
\[ D^\alpha_t u + a(u^p)_x + b(u^q)_{xx} + b(u^r)_{tt} = 0 \] (1)

where \( u = u(x, y, t) \), \( \alpha \) is a parameter describing the order of the fractional derivative \((0 < \alpha \leq 1)\), \( a \) and \( b \) are arbitrary constants and \( p, q \) and \( r \) are integers and \( p, q; r \neq 0 \) governs the behavior of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [14, 15]. The Zakharov-Kuznetsov equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [16]. The FZK equations have been studied previously by using VIM [17] and HPM [18].

2 Basic Definitions

Definition 2.1 A real function \( f(t), t > 0 \) is said to be in the space \( C_\mu, (\mu > 0) \) if there exists a real number \( p(> \mu) \) such that \( f(t) = t^p f_1(t) \) where \( f_1 \in C[0, \infty) \) and it is said to be in the space \( C^m_\mu \) if \( f^{(m)} \in C_\mu, m \in \mathbb{R} \) [19].

Definition 2.2 The Riemann-Liouville fractional integral operator \( (J^\alpha) \) of order \( \alpha \geq 0 \) of a function \( f \in C_\mu, \mu \geq -1 \) is defined as [20]:

\[ J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha, t > 0 \] (2)

and \( J^0 f(t) = f(t) \), where \( \Gamma \) is the well-known gamma function. For \( f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0 \) and \( \lambda > -1 \), the following properties hold.

- \( J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) \)
- \( J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t) \)
- \( J^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} t^{\lambda+\alpha} \).

Definition 2.3 The Caputo fractional derivative of \( f \) of order \( \alpha, f \in C^m_{\alpha+1}, m \in \mathbb{N} \cup \{0\} \), is defined as [21]:

\[ D^\alpha f(t) = J^{m-\alpha} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \quad \alpha, t > 0 \] (3)

where \( m > \alpha \) with the following properties:

- \( D^\alpha (af(t) + bg(t)) = aD^\alpha f(t) + bD^\alpha g(t), a, b \in \mathbb{R} \),
- \( D^\alpha J^\alpha f(t) = f(t) \),
- \( J^\alpha D^\alpha f(t) = f(t) - \sum_{j=0}^{k-1} f^{(j)}(0) \frac{t^j}{j!}, \quad t > 0 \).

3 Overview of the Perturbation-Iteration Algorithm PIA(1,1)

As one of the most practical subjects of physics and mathematics, differential equations create models for a number of problems in science and engineering to give an explanation for a better understanding of the events. Perturbation methods have been used for this purpose for over a century [22, 24].

But the main difficulty in the application of perturbation methods is the requirement of a small parameter or to install a small artificial parameter in the equation. For this reason, the obtained solutions are restricted by
a validity range of physical parameters. Therefore, to overcome the disadvantages come with the perturbation techniques, some methods have been suggested by several authors [34–35].

Parallel to these studies, recently a new perturbation-iteration algorithm has been proposed by Aksoy, Pakdemirli and their co-workers [34–35]. In the new technique, an iterative algorithm is established on the perturbation expansion. First, the method applied for Bratu type second order equations [34] to obtain approximate solutions. Then, the algorithms were tested on some nonlinear heat equations also [35]. The solutions of the Volterra and Fredholm type integral equations [36], ordinary differential equation and systems [37] and the mate solutions. Then the algorithms were tested on some nonlinear heat equations also [35].

The solutions of Pakdemirli and their co-workers [34, 35]. In the new technique, an iterative algorithm is established on the perturbation-iteration algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and equations for the first time. To obtain the approximate solutions of equations, the most basic perturbation-iteration algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor series expansion, i.e. $n = 1, m = 1$.

Consider the following initial value problem.

$$F \left( u_1^{(\alpha)}, u_x, u_{xxx}, u_{ttx}, \varepsilon \right) = 0$$

(4)

$$u(x, y, 0) = c$$

where $u = u(x, y, t)$ and $\varepsilon$ is a small perturbation parameter. The perturbation expansions with only one correction term is

$$u_{n+1} = u_n + \varepsilon(u_c)_n$$

(5)

where subscript $n$ represents the $n-th$ iteration.

Replacing Eq.(5) into Eq.(4) and writing in the Taylor series expansion for first order derivatives in the neighborhood of $\varepsilon = 0$ gives

$$F \left( (u_n^{(\alpha)})_t, (u_n)_x, (u_n)_{xxx}, (u_n)_{txx}, 0 \right) + F^{(\alpha)}_{u_t} \left( (u_n^{(\alpha)})_t, (u_n)_x, (u_n)_{xxx}, (u_n)_{txx}, 0 \right) \varepsilon \left( (u_c^{(\alpha)})_t \right)_n + F^{(\alpha)}_{u_x} \left( (u_n^{(\alpha)})_t, (u_n)_x, (u_n)_{xxx}, (u_n)_{txx}, 0 \right) \varepsilon \left( (u_c)_x \right)_n + F^{(\alpha)}_{u_{xxx}} \left( (u_n^{(\alpha)})_t, (u_n)_x, (u_n)_{xxx}, (u_n)_{txx}, 0 \right) \varepsilon \left( (u_c)_{xxx} \right)_n + F^{(\alpha)}_{u_{txx}} \left( (u_n^{(\alpha)})_t, (u_n)_x, (u_n)_{xxx}, (u_n)_{txx}, 0 \right) \varepsilon = 0$$

(6)

or

$$\frac{F}{\varepsilon} + \left( (u_c)_x \right)_n F^{(\alpha)}_{u_t} + (u_c)_x F^{(\alpha)}_{u_x} + (u_c)_{xxx} F^{(\alpha)}_{u_{xxx}} + (u_c)_{txx} F^{(\alpha)}_{u_{txx}} + F_x = 0$$

(7)

where $F^{(\alpha)}_{u_t} = \frac{\partial F}{\partial u_t}$, $F^{(\alpha)}_{u_x} = \frac{\partial F}{\partial u_x}$, $F^{(\alpha)}_{u_{xxx}} = \frac{\partial F}{\partial u_{xxx}}$, $F^{(\alpha)}_{u_{txx}} = \frac{\partial F}{\partial u_{txx}}$ and $F_x = \frac{\partial F}{\partial x}$.

All derivatives in the expansion are calculated at $\varepsilon = 0$. Therefore in the procedure of computations, each term is obtained when $\varepsilon$ tends to zero. The method converges in few iterations and in fact we have a saturated solution after doing computations even in the initial steps to find favorite approximate solution. Beginning with an initial function $u_0(x, y, t)$, first $(u_c)_0(x, y, t)$ has been determined by the help of Eq.(7). Then using Eq.(5), $(n + 1)$ iteration solution could be found. Iteration process is repeated using Eq.(7) and Eq.(5) until achieving an acceptable result. The ability of the method is so high that it can become convergent in just a few of computational iterations. The reliability and effectiveness of the method is shown by an example after presenting the convergence of PIA method. The purpose of this paper is to obtain approximate solutions of the fractional Zakharov-Kuznetsov equations by RSPM and PIA and to determine series solutions with high accuracy.
4 Algorithm of RPSM

In this section, we employ our technique of the RPS method to find out series solution for the Zhakarov-Kunetsov equation. The RPS method \cite{39,40} consists of expressing the solution of (1) as a fractional power-series expansion about the initial point \( t = 0 \).

\[
    u(x, y, t) = \sum_{n=0}^{\infty} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad 0 < \alpha \leq 1, \quad x \in [a, b], \quad 0 \leq t < \mathcal{R} \tag{8}
\]

Next, we let \( u_k(x, t) \) to denote the \( k \)-th truncated series of \( u(x, t) \), i.e.,

\[
    u_k(x, y, t) = \sum_{n=0}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad 0 < \alpha \leq 1, \quad x \in [a, b], \quad 0 \leq t < \mathcal{R} \tag{9}
\]

To achieve our goal, we suppose that the zeroth RPS approximate solutions of \( u(x, t) \) is as follows :

\[
    u_0(x, y, t) = f_0(x, y) = u(x, y, 0) = f(x, y) \tag{10}
\]

Also, Eq.(13) can be written as :

\[
    u_k(x, y, t) = f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \quad 0 < \alpha \leq 1, \quad x \in [a, b], \quad 0 \leq t < \mathcal{R}, \quad k = 1, 2, 3, \ldots \tag{11}
\]

Now, we define the residual functions, \( \text{Res} \), for Eq.(1) as

\[
    \text{Res}_a(x, y, t) = D_t^\alpha u + a(u^p)_x + b(u^q)_xxx + b(u^r)_{yyy} \tag{12}
\]

and, therefore, the \( k \)-th residual function, \( \text{Res}_{a,k}(x, t) \) is given as an iterative relation as

\[
    \text{Res}_{a,k}(x, y, t) = D_t^\alpha u_k + a(u_k^p)_x + b(u_k^q)_xxx + b(u_k^r)_{yyy} \tag{13}
\]

substitution of \( k \)-th truncated series \( u_k(x, y, t) \) of Eq.(13) into Eq.(16) leads to the following definition for the \( k \)-th residual function that is called \( \text{Res}^k(x, y, t) \) and in this case regarding the main Eq.(1), we have :

\[
    R^k(x, y, t) = D_t^\alpha \left( f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \right) + a \left( f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \right)_x^p + b \left( f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \right)_x^q + b \left( f(x, y) + \sum_{n=1}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \right)_x^r \tag{14}
\]

and we have :

\[
    \text{Res}^\infty(x, y, t) = \lim_{k \to \infty} \text{Res}^k(x, y, t) \tag{15}
\]

It is easy to see that \( \text{Res}^\infty(x, y, t) = 0 \) for each \( x \in [a, b] \). This show that \( \text{Res}^\infty(x, y, t) \) is infinitely many times differentiable at \( x = a \). On the other hand, \( \frac{d^k}{dx^{k-\epsilon}} R^\infty(x, y, 0) = \frac{d^k}{dx^{k-\epsilon}} R^k(x, y, 0) = 0 \). In fact, this relation is a fundamental rule in RPS method and its applications \cite{13}.

Now, in order to derive the \( k \)-th approximate solution, we consider \( u_k(x, y, t) = \sum_{n=0}^{k} f_n(x, y) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \) and we differentiate both sides of Eq.(13) with respect to \( x, y, t \) and substitute \( t = 0 \) in order to find constant parameters. After substituting these parameters in \( u_k(x, y, t) \), we can obtain \( k \)-th truncated series and by putting it in Eq.(1), we reach to our favorite approximate solution. This procedure can be repeated till the arbitrary order coefficients of RPS solutions for Eq.(1). Moreover, higher accuracy can be achieved by evaluating more components of the solution.
5 Applications

Case 5.1 PIA: Consider the following time-fractional FZK equation \[41, 42\]:

\[D_t^\alpha u(x, y, t) + (u^2(x, y, t))_x + \frac{1}{8}(u^2(x, y, t))_{xxx} + \frac{1}{8}(u^2(x, y, t))_{xyy}, \quad t > 0, \quad 0 \leq t < 1, \quad 0 < \alpha \leq 1 \quad (16)\]

with the initial condition \(u(x, y, 0) = \frac{4}{3}\rho \sinh^2(x + y)\) and the known exact solution for \(\alpha = 1\) is

\[u(x, y, t) = \frac{4}{3}\rho \sinh^2(x + y - pt) \quad (17)\]

Before iteration process rewriting Eq.(16) with adding and subtracting \(u_t(x, y, t)\) and inserting artificial parameter \(\varepsilon\) to the equation gives

\[F(u', u, \varepsilon) = \varepsilon \frac{d^n u(x, y, t)}{dt^n} + \frac{\partial}{\partial t} u(x, y, t) - \varepsilon \frac{\partial}{\partial t} u_n(x, y, t) + 2u_n(x, y, t) \frac{\partial}{\partial x} u_n(x, y, t) + \frac{3}{4} \frac{\partial}{\partial x} u_n(x, y, t) + \frac{1}{4} \frac{\partial^3}{\partial x^3} u_n(x, y, t) + \frac{1}{4} \frac{\partial^3}{\partial y^2} u_n(x, y, t) + \frac{1}{4} \frac{\partial^3}{\partial x^2 \partial y} u_n(x, y, t) + \frac{1}{4} \frac{\partial^3}{\partial x \partial y^2} u_n(x, y, t) + \frac{1}{4} \frac{\partial^3}{\partial x \partial y \partial y} u_n(x, y, t) \quad (18)\]

and for the iteration formula

\[u_t(x, y, t) + \frac{F_u}{F_u} u(x, y, t) = - \frac{F + \varepsilon}{F_u} \quad (19)\]

the terms that will be replaced in, are

\[F = (u_n)_t(x, y, t) - 1 + \frac{t}{4} \]

\[F_u = 0 \]

\[F_{u_t} = 1 \]

\[F_{\varepsilon} = -u_t'(x, y, t) + \frac{1}{\Gamma(1 - \alpha)} \varepsilon \int_0^t (u_n)_t(x, y, s) ds + 2u_n(x, y, t)(u_n)_x(x, y, t) + \frac{3}{4} (u_n)_x(x, y, t)(u_n)_{xx}(x, y, t) + \frac{1}{4} (u_n)(y)(x, y, t)(u_n)_x(x, y, t) + \frac{1}{4} (u_n)_y(x, y, t)(u_n)_x(x, y, t) + \frac{1}{2} (u_n)_y(x, y, t)(u_n)_y(x, y, t) + \frac{1}{4} u_n(x, y, t)(u_n)_{xy}(x, y, t) \quad (20)\]

After substitution the differential equation for this problem, Eq.(16) becomes

\[\frac{4}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha}(u_n)_t(x, y, s)ds + 4u_t(x, y, t) + 2(u_n)_y(x, y, t)(u_n)_xy(x, y, t) + \frac{1}{4} u_n(x, y, t)(u_n)_xy(x, y, t) + \frac{1}{4} u_n(x, y, t)(u_n)_xy(x, y, t) + \frac{1}{2} (u_n)_y(x, y, t)(u_n)_xy(x, y, t) = 0 \quad (21)\]

Appropriate to the initial condition, chosen \(u(x, y, 0) = \frac{4}{3}\rho \sinh^2(x + y)\) and solving Eq.(21) for \(n = 0\) gives

\[u_c((x, y, t))_0 = \frac{8}{9} t \left(4\rho^2 \sinh(2x + 2y) - 5\rho^2 \sinh(4x + 4y)\right) + c_1(x, y) \quad (22)\]

This expression written in iteration formula
\[ u_1 = u_0 + \varepsilon u_c((x, y, t))_0 \]  
(23)

yields

\[ u_1(x, y, t) = u_0(x, y, t) + \varepsilon \left( \frac{8}{9} t \left( 4\rho^2 \sinh(2x + 2y) - 5\rho^2 \sinh(4x + 4y) \right) + c_1(x, y) \right) \]  
(24)

or

\[ u_1(x, y, t) = \varepsilon(t - \frac{t^2}{8} + C_1) \]  
(25)

Solving this equation for initial condition

\[ u_1(x, y, 0) = \frac{4}{3} \rho \sinh^2(x + y) \]  
(26)

we obtain

\[ c_1(x, y) = 0 \]  
(27)

For this value and \( \varepsilon = 1 \) reorganizing \( u_1(x, y, t) \)

\[ u_1(x, y, t) = -\frac{4}{9} \rho (4\rho (\cosh(x + y) + 5 \cosh(3(x + y))) - 3 \sinh(x + y))) \sinh(x + y) \]  
(28)

gives the first iteration result. If the iteration procedure is continued in a similar way, we obtain the following second iteration.

\[ u_2(x, y, t) = 2^{123} \rho \left( 108 \rho^2 - \alpha \right) \left( 5 \sinh(4(x + y)) \right) \left( 3 - \alpha \right) \]  
(29)

The other iterations contain large inputs and are not given. A computational software program could help to calculate the other iterations up to any order.

**Case 5.2 RPSM:** Now let us solve FZK equation by RPSM. The \( k \)-th residual function is

\[ \text{Res}_k(x, y, t) = \frac{1}{8} (u_k^2(x, y, t)_{xxy} + \frac{1}{8} (u_k^2(x, y, t)_{xxx} + (u_k^2(x, y, t)_{x})_x \right) \]  
(30)

To determine \( f_1(x, y) \), we consider \( k = 1 \)

\[ \text{Res}_1(x, y, t) = D^\alpha_1 u_1(x, y, t) + \frac{\partial}{\partial x} u_1^2(x, y, t) + \frac{1}{8} \frac{\partial^3}{\partial x^3} u_1^2(x, y, t) + \frac{1}{8} \frac{\partial^3}{\partial x \partial y^2} u_1^2(x, y, t) \]  
(31)

since

\[ u_1(x, y, t) = f(x, y) + f_1(x, y) \frac{\rho^\alpha}{\Gamma[1 + \alpha]} \]

thus

\[ \text{Res}_1(x, y, t) = f_1(x, y) + 2u_1(x, y, t) \frac{\partial}{\partial x} u_1(x, y, t) + \frac{1}{4} \frac{\partial^2}{\partial y^2} u_1(x, y, t) \frac{\partial}{\partial x} u_1(x, y, t) + \frac{1}{2} \frac{\partial}{\partial y} u_1(x, y, t) \frac{\partial^2}{\partial x \partial y} u_1(x, y, t) + \frac{3}{4} \frac{\partial}{\partial x} u_1(x, y, t) \frac{\partial^2}{\partial x^2} u_1(x, y, t) + \frac{1}{4} \frac{\partial}{\partial x} u_1(x, y, t) \frac{\partial^3}{\partial x^3} u_1(x, y, t) \]  
(32)
from $D_t^{(k-1)\alpha} Res_k(x, y, 0) = 0$, $0 < \alpha \leq 1$, $x \in I$, $k = 1, 2, 3, \ldots$ and

$$Res_1(x, y, 0) = f_1(x, y) + 2f(x, y)f(x, y)_x + \frac{1}{8} (2f(x, y)_{yy}f(x, y)_x + 4f(x, y)_yf(x, y)_{xy} + 2f(x, y)(f(x, y)_{xyy})$$

$$+ \frac{1}{8} f(x, y)_xf(x, y)_{xx} + 2f(x, y)f(x, y)_{xxx}$$

(33)

for setting $Res_1(x, y, 0) = 0$ we obtain

$$f_1(x, y) = 2f(x, y)(f(x, y)_x - \frac{1}{4}(f(x, y)_{yy}f(x, y)_x - \frac{1}{2}f(x, y)_yf(x, y)_{xy} - \frac{1}{4}f(x, y)(f(x, y)_{xyy})$$

$$- \frac{3}{4} f(x, y)_xf(x, y)_{xx} + \frac{1}{4} f(x, y)(f(x, y)_{xxx})$$

(34)

Therefore, the first RPS approximate solution is

$$u_1(x, y, t) = f(x, y) + \frac{1}{4T[1 + \alpha]} t^\alpha (-8f(x, y)f(x, y)_x - f(x, y)_{yy}f(x, y)_x - 2f(x, y)_yf(x, y)_{xy}$$

$$- f(x, y)f(x, y)_{xyy} - 3f(x, y)_xf(x, y)_{xx} - f(x, y)f(x, y)_{xxx}$$

(35)

To obtain $f_2(x, y)$ substituting the second truncated series

$$u_2(x, y, t) = f(x, y) + f_1(x, y) \frac{t^\alpha}{\Gamma[1 + \alpha]} + f_2(x, y) \frac{t^{2\alpha}}{\Gamma[1 + 2\alpha]}$$

into the second residual function $Res_2(x, y, t)$

$$Res_2(x, y, t) = D_t^\alpha u_2(x, y, t) + \frac{\partial}{\partial x} u_2^2(x, y, t) + \frac{\partial^3}{\partial x^3} u_2^2(x, y, t) + \frac{d^3}{dx^3 dy^2} u_2^2(x, y, t)$$

(36)

Applying $D_t^\alpha$ on both sides and solving the equation $D_t^\alpha Res_2(x, y, 0) = 0$ gives

$$f_2(x, y) = \frac{1}{4} (-8f_1(x, y)f(x, y)_x - f_1(x, y)_{yy}f(x, y)_x - 8f(x, y)_yf_1(x, y)_x - f(x, y)_{yy}f_1(x, y)_x$$

$$- 2f_1(x, y)_yf(x, y)_{xy} - 2f_1(x, y)_{yy}f_1(x, y)_{xy} - f_1(x, y)f(x, y)_{xyy}$$

$$- f(x, y)f_1(x, y)_{xyy} - 3f_1(x, y)_xf(x, y)_{xx} - 3f(x, y)_xf_1(x, y)_{xx}$$

$$- f_1(x, y)f(x, y)_{xxx} - f(x, y)f_1(x, y)_{xxx}$$

(37)

so the first RPS approximate solution is

$$u_2(x, y, t) = f(x, y) + f_1(x, y) \frac{t^\alpha}{\Gamma[1 + \alpha]} + \frac{1}{4T[2\alpha + 1]} t^{2\alpha} (-8f_1(x, y)f(x, y)_x - f_1(x, y)_{yy}f(x, y)_x$$

$$- 3f_1(x, y)_{xy}f_1(x, y)_x - 8f_1(x, y)_yf_1(x, y)_{xy} - 2f_1(x, y)_{xy}f_1(x, y)_{xy}$$

$$- f_1(x, y)f_1(x, y)_{xyy} - 3f_1(x, y)_xf_1(x, y)_{xx} - f_1(x, y)f(x, y)_{xxx} - 8f(x, y)f_1(x, y)_{xxx}$$

$$- f(x, y)f_1(x, y)_{xyy} - f(x, y)f_1(x, y)_{xxx})$$

(38)

for the initial condition $f(x, y) = u(x, y, 0) = \frac{4}{3}\rho\text{sinh}^2(x + y)$. Following this manner the third iteration result, $u_3(x, y, t)$ is calculated. In Table 1, the third order approximate PIA and RPSM results are compared numerically. Figure 1, Figure 2 and Figure 3 prove that PIA and RPSM both give remarkably approximate results. We claim that the higher iterations would give closer results.

6 Conclusion

In this study, perturbation-iteration algorithm and residual power series method was introduced for time-fractional Zakharov-Kuznetsov partial differential equation. It is clear that these methods are very simple and reliable techniques and producing highly approximate results. We expect that these methods could used to calculate the approximate solutions of other types of fractional differential equations.
Figure 1: PIA solution of fractional FZK Equation

Figure 2: RPSM solution of fractional FZK Equation

Figure 3: Comparison of the PIA, RPSM and exact solutions of fractional FZK Equation
Table 1: Numerical results of Example 3.1. for different $u$ values when $\alpha = 1$

| $x$ | $y$ | $t$ | $\alpha = 0.67$ | $\alpha = 0.75$ | $\alpha = 1.00$ |
|-----|-----|-----|-----------------|-----------------|-----------------|
|     |     |     | $PIA$           | $RPSM$          | $PIA$           | $RPSM$          | $PIA$           | $RPSM$          | $PIA$           | $RPSM$          |
| 0.1 | 0.1 | 0.2 | 5.31854E-5      | 5.31244E-5      | 5.3536E-5       | 5.35536E-5      | 5.39388E-5      | 5.35536E-5      | 3.85217E-7      | 3.85217E-7      |
|     |     |     | 5.28410E-5      | 5.29757E-5      | 5.33082E-5      | 5.33082E-5      | 5.37911E-7      | 5.37911E-7      |
| 0.3 | 0.6 | 0.2 | 2.95409E-3      | 2.95185E-3      | 2.96516E-3      | 2.96516E-3      | 2.98987E-3      | 2.98987E-3      | 4.66389E-5      | 4.66389E-5      |
|     |     |     | 2.92662E-3      | 2.92709E-3      | 2.93717E-3      | 2.93780E-3      | 2.96715E-3      | 2.96715E-3      | 6.86314E-5      | 6.86314E-5      |
|     |     |     | 2.90307E-3      | 2.90522E-3      | 2.91448E-3      | 2.91561E-3      | 2.94515E-3      | 2.94515E-3      | 8.99046E-5      | 8.99046E-5      |
| 0.9 | 0.9 | 0.2 | 1.06822E-2      | 1.05506E-2      | 1.07143E-2      | 1.07143E-2      | 1.10227E-2      | 1.10227E-2      | 5.12131E-4      | 5.12131E-4      |
|     |     |     | 1.04487E-2      | 1.01199E-2      | 1.05488E-2      | 1.05488E-2      | 1.07861E-2      | 1.07861E-2      | 7.38186E-4      | 7.38186E-4      |
|     |     |     | 1.02777E-2      | 9.60606E-3      | 1.03736E-2      | 9.67434E-3      | 1.05742E-2      | 9.57942E-2      | 9.89139E-4      | 9.89139E-4      |

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