Convergence of Transition Probability Matrix in CLV-Markov Models

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Abstract. A transition probability matrix is an arrangement of transition probability from one state to another in a Markov chain model (MCM). One of interesting study on the MCM is its behavior for a long time in the future. This term is called the convergence of the n-step transition matrix for \( n \) moves to infinity. Mathematically, the convergence of the transition probability matrix is finding the limit of the transition matrix which is powered by \( n \) where \( n \) moves to infinity. The convergence form of the transition probability matrix is very interesting as it will bring the matrix to its stationary form. This form is useful for predicting the probability of transitions between states in the future. The method usually used to find the convergence of transition probability matrix is through the process of limiting the distribution. In this paper, the convergence of the transition probability matrix is searched using a simple concept of linear algebra that is by diagonalizing the matrix. This method has a higher level of complexity because it has to perform the process of diagonalization in its matrix. But this way has the advantage of obtaining a common form of power \( n \) of the transition probability matrix. This form is useful to see transition matrix before stationary. For example cases are taken from CLV model using MCM called Model of CLV-Markov. There are several models taken by its transition probability matrix to find its convergence form. The result is that the convergence of the matrix of transition probability through diagonalization has similarity with convergence with commonly used distribution of probability limiting method.

1. Introduction

In Markov Chain Model (MCM), matrix of transition probability is a main model or a main result describing characteristics of relationship inter states. The matrix contain probabilities of transition from one state to another states. One of interesting study on the MCM is its behavior for a long time in the future. This term is called convergency of n-step transition matrix for \( n \) moves to infinity. Mathematically, the convergence of the transition probability matrix is finding the limit of the transition matrix that is powered by \( n \) where \( n \) moves to infinity. In [1],[2] the long run behavior of MCM is explained by a theorem below.

Theorem 1. Let \( P \) be a regular transition probability matrix on the states 1,2,...,N. Then the limiting distribution \( \pi = (\pi_1, \pi_2, \ldots, \pi_N) \) is the unique nonnegative solution of the equations...
The theorem has implication,

\[ \lim_{n \to \infty} P^n = \begin{pmatrix} \pi_1^n & \pi_2^n & \cdots & \pi_N^n \\ \pi_1^n & \pi_2^n & \cdots & \pi_N^n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1^n & \pi_2^n & \cdots & \pi_N^n \end{pmatrix} \]

The long run behaviour of MCM is stationary form of Markov chain. The two equation in the theorem are an ordinary linear system that has simple method to get solution vector of \( \pi \). This paper will explain another method to get vector of \( \pi \). We use one of concept in linear algebra. That is a diagonalization of the transition matrix to find \( \pi \) and then limiting it for \( n \) to infinity to get the solution. This way has more advantage than the theorem. One, we can get generalization form of matrix transition to the power of \( n \) for \( n \)-steps transition probability matrix with \( n \) arbitrary. This result is useful to see transition matrix before stationary.

For cases, we take models of determination of Customer Lifetime Value (CLV) using Markov Chain, we called models of CLV-Markov. In the model, there are some characteristic of relationship interstate of customer. Every model has different size of transition matrix. It is based on the number of states. This paper will take three matrix from the models of CLV-Markov.

CLV is a value of a customer as expectation value of a firm revenue from him or her, start from present time to long time in the future\([3]\). CLV is a familiar term in marketing system. CLV is one concepts to support marketing program \([4]\) especially to manage of cost of customer, to acquire a new customer and to maintain an old customer\([5]\). Many methods can be used to determine CLV\([6]\). One of the methods using MCM \([7]\).

2. Diagonalization

Let \( P \) is matrix size \( mxm \) which has \( m \) different eigen values. Thus, \( P \) diagonalizable\([8]\), write

\[ P = U D U^{-1} = \lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_m C_m \]

Matrix \( C_j \) is come from multiplication of \( j^{th} \) column vector of \( U \) with \( j^{th} \) row vector of \( U^{-1} \). Therefore, \( P \) to the power of \( n \) can rewrite,

\[ P^n = U D^n U^{-1} = \lambda_1^n C_1 + \lambda_2^n C_2 + \cdots + \lambda_m^n C_m \]

For the simple, let \( m \) is equal two. Matrix \( C_1 \) and \( C_2 \) can be write,

\[ P^n = \lambda_1^n C_1 + \lambda_2^n C_2 \]

\[ = \left( \begin{array}{cc} k_1 & k_3 \\ k_5 & k_7 \end{array} \right) \left( \begin{array}{c} k_2 \\ k_6 \end{array} \right) \]

\[ = \begin{pmatrix} k_1 \lambda_1^n + k_2 \lambda_2^n & k_3 \lambda_1^n + k_4 \lambda_2^n \\ k_5 \lambda_1^n + k_6 \lambda_2^n & k_7 \lambda_1^n + k_8 \lambda_2^n \end{pmatrix} \]

Especially if \( P \) is a transition probability matrix, the last equation can be write,

\[ P^n = \begin{pmatrix} k_1 \lambda_1^n + k_2 \lambda_2^n & 1 - k_1 \lambda_1^n - k_2 \lambda_2^n \\ k_3 \lambda_1^n + k_4 \lambda_2^n & 1 - k_3 \lambda_1^n - k_4 \lambda_2^n \end{pmatrix} \]

Some \( n \) can be fixed to construct a linear system and solve it to get \( k_i \), \( i=1..4 \).

3. Models of CLV-Markov

Here, we take three models of CLV-Markov. The first model contain two states, customer and former customer. The transition matrix is below,

\[ P = \begin{pmatrix} r & 1 - r \\ q & 1 - q \end{pmatrix} \]

Symbol \( r \) is a retention probability. It is mean a kind of retention rate or proportion customers which succeses to be maintained. Whereas symbol \( q \) is an aquation probability. It is mean probability of a
former customer to back to customer again. The model use this matrix is called Model CF(1,1). Index
have a meaning that the number of customer state and former customer state.
The second model contain three states, prospect, Customer, and Former Customer. The matrix can be
rewriten below,
\[
P = \begin{pmatrix} 0 & a & 1-a \\ 0 & r & 1-r \\ 0 & q & 1-q \end{pmatrix}
\]
Symbol \(a\) is aquisition probability to get a new customer. The model use this matrix is called Model
PCF(1,1,1).
The last models contain three states too, one level customer state and two level former customer
state. So, the model is called Model CF(1,2), representing below
\[
P = \begin{pmatrix} r & 1-r & 0 \\ q & 0 & 1-q \\ 0 & 0 & 1 \end{pmatrix}
\]

4. Results
The first model has two eigen values, \(\lambda_1 = 1\) and \(\lambda_2 = r - q\). Thus \(P^n\) can write,
\[
P^n = \frac{1}{1 - r + q} \left[ \begin{pmatrix} q & 1-r \\ q & 1-r \end{pmatrix} + \begin{pmatrix} 1-r & r-1 \\ -q & q \end{pmatrix} (r-q)^n \right]
\]
\[
= \frac{1}{1 - r + q} \begin{pmatrix} q & 1-r \\ q & 1-r \end{pmatrix} + A\lambda_2^n
\]
Thus,
\[
\lim_{n \to \infty} P^n = \frac{1}{1 - r + q} \begin{pmatrix} q & 1-r \\ q & 1-r \end{pmatrix}
\]
The second model has three eigen values, \(\lambda_1 = 1, \lambda_2 = r - q\), and \(\lambda_3 = 0\). Thus \(P^n\) can write,
\[
P^n = \frac{1}{1 - r + q} \begin{pmatrix} 0 & q & 1-r \\ 0 & q & 1-r \\ 0 & q & 1-r \end{pmatrix} + \begin{pmatrix} 0 & a & -a \\ 0 & r & -r \\ 0 & -q & q \end{pmatrix} + \frac{q}{1 - r + q} \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} (r-q)^n
\]
\[
= \frac{1}{1 - r + q} \begin{pmatrix} 0 & q & 1-r \\ 0 & q & 1-r \\ 0 & q & 1-r \end{pmatrix} + B\lambda_2^n
\]
Thus,
\[
\lim_{n \to \infty} P^n = \frac{1}{1 - r + q} \begin{pmatrix} 0 & q & 1-r \\ 0 & q & 1-r \\ 0 & q & 1-r \end{pmatrix}
\]
Whereas the third model has three eigen values, \(\lambda_1 = \frac{r+\sqrt{r^2+4q(1-r)}}{2}, \lambda_2 = \frac{r-\sqrt{r^2+4q(1-r)}}{2}\), and \(\lambda_3 = 1\).
Thus \(P^n\) can write,
\[
P^n = \begin{pmatrix} (1-r)(\lambda_2 - r + q) & \lambda_2^n + (1-r)(r-q) & 1 - P^n_{11} - P^n_{12} \\ (1-\lambda_1)(\lambda_2 - \lambda_3) & \lambda_3^n & 1 - P^n_{21} - P^n_{22} \\ -q(1-r+\lambda_2) & \lambda_2^n + q(1-r) & 1 \end{pmatrix}
\]
\[
= G\lambda_2^n + H\lambda_2^n + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]
Thus, \[
\lim_{n \to \infty} P^n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

5. Conclusions
Convergence of transition probability matrix in Markov Chain is an interesting subject to study properties of transition object inter states in the long future time. Commonly, convergence of the matrix is found by a familiar theorem in many stochastic books. But, here we try to find convergence the matrix use approach diagonalization of the transition matrix. This method are applicated to three models of CLV-Markov. The result, we get generalization of \( P^n \) from every models.

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