AN INEQUALITY FOR ADJOINT RATIONAL SURFACES

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Abstract. We generalize an inequality for convex lattice polygons – aka toric surfaces – to general rational surfaces.

Our collaboration started when the second author proved an inequality for algebraic surfaces which, when translated via the toric dictionary into discrete geometry, yields an old inequality by Scott [5] for lattice polygons.

In a previous article [2], we were then able to refine this estimate on the discrete side. Here, we generalize the refinement to (non-toric) algebraic surfaces. We use the ideas of one of the discrete proofs.

1. Introduction

Let $S$ be a smooth complex algebraic surface, let $H$ be a big and nef divisor on $S$, and let $K$ denote the canonical class of $S$. Roughly speaking, the adjoint surface $S^{(1)}$ of $(S, H)$ is (the minimal resolution of) the image of $S$ in $|H + K|^*$, and the level of $(S, H)$ is the number of iterations of this adjunction process until $H^{(l)}$ on $S^{(l)}$ is no longer big. (See Section 2 for precise definitions.) If $S$ is rational, we prove the inequality

$$2\ell b \leq d + 9\ell^2,$$

where $d = H^2$ is the degree of $S$ in $|H|^*$ and $b = -HK$ is the anticanonical degree of $H$. Note that the inequality only makes sense for surfaces with negative Kodaira dimension. We do not know about its validity in case of irrational ruled surfaces (examples show that a much stronger inequality should hold here).

For toric surfaces, the inequality was proved in [2], using the toric dictionary in the following table.

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Research of Haase supported by DFG Heisenberg fellowship HA 4383/4.
Research of Schicho supported by the FWF: F22766-N18.
2. THE ADJOINED PAIR

We need some concepts on adjunction theory for rational surfaces. We will use [3] as the basic reference for adjunction theory for surfaces.

We consider rational surfaces $F$, possibly singular, in projective space $\mathbb{P}^N$, $N > 0$. There is a resolution of singularities $f : S \to F \subset \mathbb{P}^N$ with nonsingular $S$, and the pullback of the line bundle $\mathcal{O}(1)$ defines a nef and big divisor class $H \in \text{Pic}(S)$. Working with nonsingular surfaces and nef and big divisor classes is technically easier than working with surfaces with arbitrary singularities, so we think of $F$ as being represented by the pair $(S, H)$. Such a pair is called a polarized surface, and $H$ is called the polarization divisor class. We will always require that the polarization divisor class is nef, and the adjunction process starts with a polarization divisor which is nef and big.

As the resolution of singularities is not unique, we may have non-isomorphic polarized surfaces representing the same projective and possibly singular surface. There is, however, a minimal one $(S_0, H_0)$. For any other polarized surface $(S_1, H_1)$ representing the same singular surface, there is a morphism $g : S_1 \to S_0$ such that $g^*H_0 = H_1$. Minimality of a polarized surface $(S, D)$ is characterised by the absence of $-1$-curves $E$ such that $EH = 0$.

Adjunction is an iterative process to replace a polarized surface $(S, H)$, with $H$ nef and big, by another polarized surface $(S^{(1)}, H^{(1)})$, which is “smaller” in a certain sense. When the process ends, we have reached a particularly simple situation. More precisely, adjunction terminates if $H + K$ is not effective. Because of the formula $s(H) = h^0(H + K)$ for nef and big divisors on a rational surface, it follows that either $H$ is not big or that the genus of $H$ is 0.
In the other case, adjunction proceeds in two steps. First, we transform \((S, H)\) into a minimal pair by successively blowing down all \(-1\)-curves orthogonal to the polarization divisor. This produces a birational morphism \(f : S \to S^{(1)}\) such that \(H\) is orthogonal to the kernel of \(f_\ast\). Second, we set \(H^{(1)} := f_\ast(H) + K^{(1)}\), where \(K^{(1)}\) is the canonical divisor class of \(S^{(1)}\).

**Lemma 1.** If adjunction is defined, i.e., if \(H + K\) is effective, then \(H^{(1)}\) is again nef.

**Proof.** This is well-known (see, e.g. [3], Proof of Theorem D.3.3); we give the proof just for the sake of completeness. Assume that \(C\) is a prime divisor in \(S^{(1)}\) such that \(CH^{(1)} < 0\). Then \(CK^{(1)} < 0\). If \(C^2 \geq 0\), then Riemann-Roch implies \(h^0(C) > 1\), hence \(C\) moves in a linear system. Then it cannot have negative intersection with the effective divisor class \(H^{(1)}\). Hence \(C^2 < 0\). By the genus formula, \(C^2 + CK^{(1)} \geq -2\). This leaves just room for one case, namely \(C^2 = CK^{(1)} = -1\) and \(Cf_\ast H = 0\). But this contradicts minimality of the pair \((S^{(1)}, f_\ast H)\). □

The adjunction process is finite, because on any rational surface there is a nef divisor class \(L\) which satisfies \(LK < 0\), namely the pullback of the class of lines along the inverse of a rational parametrization. Then \(LH/(−LK)\) is an upper bound for the number of possible adjunction steps with initial polarized surface \((S, H)\).

We set

\[\ell(S, H) := \sup \left\{ \frac{p}{q} : qH + pK \text{ effective} \right\}.\]

Then the possible number of adjunction steps is \(\lfloor \ell \rfloor\), the largest integer less than or equal to \(\ell\). For the final polarized surface, we have three cases, by Theorem D.4.1 in [3]. Either

1. \(\ell \not\in \mathbb{N}\), but \(2\ell \in \mathbb{N}\) or \(3\ell \in \mathbb{N}\) and \(H^{(\lfloor \ell \rfloor)}\) is a big divisor class of genus 0, or
2. \(\ell \in \mathbb{N}\) and \((H^{(\ell)})^2 = 0\) and either
   a. \(H^{(\ell)} = 0\) or
   b. \(H^{(\ell)} = kP\) for some integer \(k > 0\) and class \(P\) of a pencil of genus 0.

For all cases, there are toric examples.

### 3. The proof

3.1. **Intersection theory.** We recall two well-known facts on rational surfaces which we will need in the proof. We write \(\rho\) for the rank of the Picard group of the rational surface under consideration.
Proposition 2. $K^2 + \rho = 10$, and $K^2 = 9 \Rightarrow S = \mathbb{P}^2$.

Proof. In a single blowing up, the number $K^2$ decreases by 1 and the number $\rho$ increases by 1. Because every birational map is a composition of blowing ups and their inverses, it follows that $K^2 + \rho$ is a birational invariant. It assumes the value 10 for $S = \mathbb{P}^2$, hence it is 10 for all rational surfaces.

If $K^2 = 9$, then the Picard rank must be 1. By the classification of minimal rational surfaces, $S$ must be $\mathbb{P}^2$. □

We will formulate the proof in terms of the genus $s$ of the intersection of $S$ with a generic hyperplane in $|H|^*$. The parameters for $(S, H)$ and $(S^{(1)}, H^{(1)})$ are related as follows.

Proposition 3. $b^{(1)} + s^{(1)} = s$.

Proof. By definition, $b^{(1)} = -H^{(1)}K^{(1)}$, and by the genus formula,

$$s^{(1)} = \frac{1}{2}H^{(1)}(H^{(1)} + K^{(1)}) + 1.$$ 

Let $f : S \to S^{(1)}$ be the minimalisation morphism. Then

$$b^{(1)} + s^{(1)} = \frac{H^{(1)}(H^{(1)} - K^{(1)})}{2} + 1 = \frac{H^{(1)}f_*(H)}{2} + 1$$

$$= \frac{f^*(H^{(1)})H}{2} + 1 = \frac{(H + K)H}{2} + 1 = b.$$ □

3.2. The induction step.

Lemma 4. Suppose $H^{(1)}$ is big, and denote $b^{(1)}$ the anti-canonical degree of $H^{(1)}$ on $S^{(1)}$. Then $b \leq b^{(1)} + 9$ with equality if and only if $S = \mathbb{P}^2$.

Proof. If we intersect

$$H = f^*f_*H = f^*(H^{(1)} - K^{(1)})$$

with $-K$, we get

$$b = -HK = -f^*(H^{(1)} - K^{(1)})K = -(H^{(1)} - K^{(1)})f_*K$$

$$= -H^{(1)}K^{(1)} + K^{(1)}K^{(1)} = b^{(1)} + 10 - \rho^{(1)}.$$ □

Theorem 5. Let $H$ be a nef and big divisor on the smooth rational surface $S$. Let $\ell$ be the level, $s$ the sectional genus of $(S, H)$, and let $b = -KH$, $d = H^2$. Then

$$2\ell b \leq d + 9\ell^2,$$

(1)
or equivalently
\[(2) \quad (2\ell - 1)b \leq 2s + 9\ell^2 - 2.\]

**Proof.** The validity of the statement is preserved if we replace \(H\) by \(qH\) for some \(q > 0\). This is apparent in (1). So we may assume that the level is integral and proceed by induction on \(\ell\).

For \(\ell = 1\), the statement is equivalent to \(-HK \leq H^2 + HK + 9\). This is equivalent to \(K^2 \leq (H + K)^2 + 9\). But \(H + K\) is nef and effective, hence \((H + K)^2 \geq 0\), so the statement is a consequence of \(K^2 \leq 9\).

If \(\ell \geq 2\), we have by Lemma 4 induction, and Proposition 3 in that order,
\[(2\ell - 1)b \leq (2\ell - 1)b^{(1)} + 9(2\ell - 1)
= 2b^{(1)} + (2(\ell - 1) - 1)b^{(1)} + 9(2\ell - 1)
\leq 2b^{(1)} + 2s^{(1)} + 9(\ell - 1)^2 - 2 + 9(2\ell - 1)
= 2s + 9\ell^2 - 2.\]

\(\square\)

4. Concluding remarks

As Wouter Castryck points out [1, (2.7)], it would probably yield much stronger bounds if one found a way to incorporate the parameter
\[v := \rho + 2 - \sum_{x \in \text{Sing } F} \text{mult}(x)\]
into the induction inequality of Lemma 4. Here, the sum runs over the singular points of the image \(F\) of \(S\) in \(|H|^*\), and the multiplicity \(\text{mult}(x)\) of such a point \(x\) is the number of exceptional divisors in its minimal resolution. In the toric case, \(v\) is just the number of vertices of the lattice polygon.

As an application of this circle of ideas, one can estimate the smallest degree of a parametrization of a rational surface. To this end, [1] bounds the level \(\ell\) in terms of the degree \(d\) of \(S\). The exponent 4 in this bound is potentially not optimal. For toric surfaces, there is even a linear bound. The area of a lattice polygon drops at least by 3 in each adjunction step, so the level is bounded by 2/3 times the degree.

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