Weakly coupled systems of semi-linear Klein-Gordon equations with memory-type dissipation

Wenhui Chen\textsuperscript{a,}∗, Abdelhamid Mohammed Djaouti\textsuperscript{a,b}

\textsuperscript{a}Institute of Applied Analysis, Faculty for Mathematics and Computer Science, Technical University Bergakademie Freiberg, Präferstraße 9, 09596 Freiberg, Germany.

\textsuperscript{b}Faculty of Exact and Computer Sciences, Hassiba Ben Bouali University, Ouled Fares, 02180 Chlef, Algeria.

Abstract

We are concerned with the existence of global in time solutions to the Cauchy problem for semi-linear Klein-Gordon equations with memory-type dissipation in $\mathbb{R}^n$. In the first place, we consider the linearized equation: applying the energy method in the Fourier space, we derive the point-wise estimate of solution in the Fourier space. In the second place, we consider the single semi-linear equation: we show that the global existence result could be proved by introducing a set of time-weighted Sobolev spaces and applying the contracting mapping theorem. Lastly, we consider the weakly coupled systems: we prove the global existence of small data Sobolev solutions, where initial data are supposed to belong to different classes of regularity.

Keywords: Klein-Gordon equation, Memory-type dissipation, Point-wise estimate, Global in time solution, Weakly coupled system, Cauchy problem

1. Introduction

The Klein-Gordon equation with memory-type dissipation for $\theta \in [0, 1]$ has been introduced by [22] in 2003

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + u - G \ast (-\Delta)^{\theta} u = 0, \quad x \in \mathbb{R}^n, \ t > 0, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\end{equation}

Concerning the case $\theta = 0$, we recall the above model can represent an ionized atmosphere. When the relaxation function $G = G(t)$ decays exponentially, we point out a result of [23]. The case where $\theta = 1$ in (1.1) with vanishing mass corresponds to an isotropic viscoelastic model.

In the first part of this paper, we consider the following Cauchy problem:

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + u - G \ast (-\Delta)^{\theta} u = |u|^p, \quad x \in \mathbb{R}^n, \ t > 0, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\end{equation}

where $\theta \in [0, 1]$ and $p > 1$. We assume that the relaxation function $G = G(t)$ satisfies the following set of hypotheses.

Hypothesis 1.1.

1. $G \in C^2([0, \infty)) \cap W^2_1([0, \infty));$

2. $G(t) > 0, \ -C_0G(t) \leq G'(t) \leq -C_1G(t), \ |G''(t)| \leq C_2G(t)$ for any $t \in (0, \infty)$;

∗Corresponding author

Email addresses: Wenhui.Chen@student.tu-freiberg.de (Wenhui Chen), djaouti_abdelhamid@yahoo.fr, a.mohammeddjaouti@univ-chief.dz (Abdelhamid Mohammed Djaouti)
where $C_0$ to $C_3$ are positive constants and $W_{p,0}^\sigma$ denotes the Sobolev space for $\sigma \in \mathbb{N}$, $p \in [1, \infty]$.

There are some papers devoted the study of a class of abstract wave equations with a strictly positive self-adjoint operator $A$ and a memory term

$$\begin{cases}
    u_{tt} + Au - G * A^\theta u = 0, & x \in \mathbb{R}^n, \quad t > 0, \\
    (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{cases}$$  \tag{1.3}

We mention, first, the works of [22], which studied the existence and the asymptotic behavior of the solutions to the model (1.3) with $\theta \in (0, 1)$ where the relaxation function satisfies Hypotheses 1.1. When the relaxation function is defined in $C^1[0, \infty) \cup L^1[0, \infty)$ such that $G \geq 0$, $G' \leq 0$, the optimal decay rate of solutions for (1.3) with $\theta \in (0, 1)$ was investigated by [24]. Assuming (1.3) with $\theta = 1$ and non-negative monotone decreasing relaxation function $G \in C^1[0, \infty)$, the authors in [17] obtained some decay estimates by using the intrinsic method. Thereafter, [14] considered the corresponding inhomogeneous model of (1.3) with $A = -\Delta$ and $\theta = 1$ and investigated, under some assumptions for the relaxation function and the right-hand side source term, the energy estimates of solutions.

For the linear second-order hyperbolic systems of viscoelastic materials with dissipation, the energy decay estimates was established in [5]. Recently, the paper [28] extended the result of [3]. They took initial data in the weighted space to derive faster decay estimates. Moreover, they used these decay estimates of linear problem combined with the weighted energy method to tackled a semi-linear problem. In [4] and [5] the authors proved sharp decay estimates and a large-time behavior of solutions to a quasi-linear second-order hyperbolic systems of viscoelasticity.

Let us turn to some semi-linear equations with power source nonlinearity. In the pioneering papers [11, 31, 32], they studied the semi-linear classical damped wave equation

$$\begin{cases}
    u_{tt} - \Delta u + u_t = |u|^p, & x \in \mathbb{R}^n, \quad t > 0, \\
    (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{cases}$$  \tag{1.4}

They proved the small data solution exists globally in time when the the exponent of power nonlinearity is above the Fujita exponent, i.e., $p > p_{Fuj}(n)$, where $p_{Fuj}(n) := 1 + \frac{2}{n}$. On the contrary, if $1 < p \leq p_{Fuj}(n)$, then every energy solution to given initial data having positive average value does not exist globally. For the semi-linear equation with memory-type dissipation, we refer to readers also [19, 18]. In 2011, the authors of [19] studied the global (in time) existence of small data solutions to the plate equation with mass and memory-type dissipation

$$\begin{cases}
    u_{tt} + \Delta^2 u + u + G * \Delta u = f(u), & x \in \mathbb{R}^n, \quad t > 0, \\
    (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{cases}$$  \tag{1.5}

where the relaxation function $G$ satisfies Hypotheses 1.1 and $f = f(u)$ such that $f \in C^\infty(\mathbb{R} \setminus \{0\})$ and $f(u) = O(|u|^p)$ as $|u| \to 0$ here $p$ is an integer satisfying $p > p_{Fuj}(n)$. Then, [18] investigated the global (in time) existence of small data solution to the model (1.5) with the rotational inertia $-\Delta u_{tt}$ and the nonlinearities $f(u, u_t, \nabla u)$, which have the property that $f(\lambda u, \lambda u_t, \lambda \nabla u) = \lambda^p f(u, u_t, \nabla u)$ for $\forall \lambda > 0$, here $p$ is an integer satisfying $p > p_{Fuj}(n)$.

The second part of this paper is devoted to apply the result proved for single semi-linear equation to the weakly coupled systems of semi-linear Klein-Gordon equations with memory-type dissipation. The model we have in mind is

$$\begin{cases}
    u_{tt} - \Delta u + u - G * (-\Delta)^\theta u = |v|^p, & x \in \mathbb{R}^n, \quad t > 0, \\
    v_{tt} - \Delta v + v - G * (-\Delta)^\theta v = |u|^q, & x \in \mathbb{R}^n, \quad t > 0, \\
    (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}$$  \tag{1.6}

where $\theta \in [0, 1]$ and $p, q > 1$. 

\[ (3) \ 1 - \int_0^t G(\tau)d\tau \geq C_3 \text{ for any } t \in (0, \infty); \]

where $C_0$ to $C_3$ are positive constants and $W_{p,0}^\sigma$ denotes the Sobolev space for $\sigma \in \mathbb{N}$, $p \in [1, \infty]$. 

We mention that for the strongly coupled system of semi-linear viscoelastic equations with distinct relaxation functions has been considered in [13].

Let us consider the weakly coupled system of semi-linear classical damped waves with \( p, q > 1 \)
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + u_t &= |v|^p, \quad x \in \mathbb{R}^n, \ t > 0, \\
\frac{\partial^2 v}{\partial t^2} - \Delta v + v_t &= |u|^q, \quad x \in \mathbb{R}^n, \ t > 0, \\
(u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]

We describe the interplay between the exponents \( p \) and \( q \) as follows:
\[
\alpha_{\text{max}} = \max\{\frac{1}{p}; \frac{1}{q}\} + 1.
\]

The authors in [30] proved the following results for \( n = 1, 3 \). If \( \alpha_{\text{max}} < \frac{n}{2} \), then the global solution uniquely exists. Moreover, the nonexistence of solution holds when \( \alpha_{\text{max}} \geq \frac{n}{2} \). In [25] the authors generalized the global existence result to \( n = 1, 2, 3 \) and improved the time decay estimates when \( n = 3 \). Recently in [26] the global existence and nonexistence results for any space dimension \( n \) was determined, where the proof of global (in time) existence of energy solutions is based on the weighted energy method.

The following weakly coupled system of semi-linear damped waves with time-dependent coefficients in the dissipation terms studied in [20, 21]:
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + b(t)u_t &= |v|^p, \quad x \in \mathbb{R}^n, \ t > 0, \\
\frac{\partial^2 v}{\partial t^2} - \Delta v + b(t)v_t &= |u|^q, \quad x \in \mathbb{R}^n, \ t > 0, \\
(u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]

where a global (in time) existence of solutions was proved for \( \alpha_{\text{max}} > \frac{2}{\min\{p, q\}} \) if only one exponent of power nonlinearity is above a modified Fujita exponent and data defined in the energy space. The system (1.8) behave like semi-linear damped wave equation with time-dependent dissipation term if both exponents of power nonlinearities are above modified Fujita exponent.

The remainder of this paper is organized as follows. In Section 2, we show the representation of solution to the corresponding linear Cauchy problem with vanishing right-hand side. In Section 3, we prepare the decay estimate of solutions to the corresponding linear Cauchy problem with vanishing right-hand side. In Section 4, we prove the global (in time) existence of Sobolev solutions to the single semi-linear equation (1.2) with small data in Sobolev spaces with different regularities. In Section 5, we show the interplay between the exponents of power nonlinearities \( p \) and \( q \) for the system (1.6). For large regular data and even not embedded in \( L^\infty \), we show the benefits of this regularity to get results for initial data with different regularities in any space dimension using now tools from harmonic analysis.

**Notations.** We introduce some notations used throughout this paper. \( \mathcal{F}(f) \) denote the Fourier transform of \( f \) defined by
\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-x \cdot \xi} f(x) dx,
\]
and its inverse transform by \( \mathcal{F}^{-1} \). Let \( \mathcal{L}(f) \) the Laplace transform of \( f \) defined by
\[
\mathcal{L}(f)(\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t) dt,
\]
and its inverse transform denoted by \( \mathcal{L}^{-1} \).

We write \( f \lesssim g \), if there exists a constant \( C > 0 \) such that \( f \leq Cg \).
We take the following notations for any real or complex-valued functions \( f = f(t) \) and \( g = g(t) \):

\[
(g * f)(t) := \int_0^t g(t - \tau)f(\tau)d\tau,
\]

\[
(g \circ f)(t) := \int_0^t g(t - \tau)(f(\tau) - f(t))d\tau,
\]

\[
(g \Box f)(t) := \int_0^t g(t - \tau)|f(\tau) - f(t)|^2d\tau.
\]

For the sake of clarity, we introduce for any \( s \geq 0 \) and \( m \in [1, 2] \) the function space

\[
\mathcal{D}_m^s := (H^{s+1} \cap L^m) \times (H^s \cap L^m),
\]

with the corresponding norm

\[
\|(u, v)\|_{\mathcal{D}_m^s} := \|u\|_{H^{s+1}} + \|u\|_{L^m} + \|v\|_{H^s} + \|v\|_{L^m}.
\]

Here we denote by \( H^s_m \) and \( \hat{H}^s_m \) the Bessel and Riesz potential spaces based on \( L^m \), respectively, and \(|D|^s \) stands for the pseudo-differential operator with the symbol \(|\xi|^s\).

2. Representation of solutions

To study global (in time) solutions to the semi-linear model (1.2) and (1.6), we begin by the corresponding Cauchy problem for the single homogeneous Klein-Gordon equation with memory-type dissipation

\[
\begin{aligned}
&u_{tt} - \Delta u + u - G*(-\Delta)^{\sigma}u = 0, \quad x \in \mathbb{R}^n, \ t > 0, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

Before investigating some estimates of solutions, we need to get the representation of solutions. Let \( K_0 = K_0(t, x) \) and \( K_1 = K_1(t, x) \) the fundamental solutions to (2.1) for \((u_0, u_1) = (\delta_0, 0)\) and \((u_0, u_1) = (0, \delta_0)\), respectively. Here \( \delta_0 \) is the Dirac distribution in \( x = 0 \) with respective to the spatial variables.

Using the partial Fourier transform and Laplace transform for (2.1) we formally get

\[
\begin{aligned}
\hat{K}_0(t, \xi) &= \tilde{C}\mathcal{L}^{-1}\left(\frac{\lambda}{(\lambda^2 + 1 + |\xi|^2 - |\xi|^{2\sigma}\mathcal{L}(G)(\lambda))}\right)(t, \xi), \\
\hat{K}_1(t, \xi) &= \tilde{C}\mathcal{L}^{-1}\left(\frac{1}{(\lambda^2 + 1 + |\xi|^2 - |\xi|^{2\sigma}\mathcal{L}(G)(\lambda))}\right)(t, \xi),
\end{aligned}
\]

where \( \tilde{C} \) is the constant.

**Lemma 2.1**. For any \( \xi \in \mathbb{R}^n \) and \( t \in [0, \infty) \), the kernels \( \hat{K}_0(t, \xi) \) and \( \hat{K}_1(t, \xi) \) exist.

**Proof.** Here, we only prove the existence of \( \hat{K}_0(t, \xi) \) and similarly we could prove that \( \hat{K}_1(t, \xi) \) exists. Let us take the notation

\[
J(\lambda) := \lambda^2 + 1 + |\xi|^2 - |\xi|^{2\sigma}\mathcal{L}(G)(\lambda).
\]

In order to prove \( \mathcal{L}\left(\frac{\lambda}{J(\lambda)}\right) \) exists, we should consider the zero point of \( J(\lambda) \). Now, we denote \( \lambda = \sigma + i\zeta \) with \( \sigma > -C_1 \) where \( C_1 \) introduced in the assumption of \( G(t) \). So \( \mathcal{L}(G)(\lambda) \) exists. We assume \( \lambda_1 = \sigma_1 + i\zeta_1 \) is a zero point of \( J(\lambda) \) and \( \sigma_1 > -C_1 \), then \( \sigma_1 \) and \( \zeta_1 \) satisfy

\[
\begin{aligned}
\Re J(\lambda_1) &= \sigma_1^2 + 1 + |\zeta_1|^2 - |\zeta_1|^{2\sigma} \int_0^\infty \cos(\zeta_1\tau)e^{-\sigma_1\tau}G(\tau)d\tau = 0, \\
\Im J(\lambda_1) &= 2\sigma_1\zeta_1 + |\zeta_1|^{2\sigma} \int_0^\infty \sin(\zeta_1\tau)e^{-\sigma_1\tau}G(\tau)d\tau = 0.
\end{aligned}
\]
To understand the zero point of $J(\lambda)$ we distinguish between two cases.

**Case 1.** $|\xi| = 0$

The condition (2.4) yields

$$\sigma_t^2 - \zeta_t^2 + 1 = 0 \quad \text{and} \quad 2\sigma_t\zeta_t = 0.$$  

So we get $\sigma = 0$, $\zeta = \pm 1$.

**Case 2.** $|\xi| > 0$

In this case we state that $\sigma > 0$. This statement can be proved by a contradiction.

Assume that $\sigma > 0$. Then, if $\zeta = 0$, we get

$$\Re J(\lambda_1) = \sigma_t^2 + 1 + |\xi|^2 - |\xi|^{2\theta} \int_0^\infty e^{-\sigma_t \tau} G(\tau) d\tau \geq \sigma_t^2 + 1 + |\xi|^2 - |\xi|^{2\theta} \geq \frac{1}{2},$$

where we apply the assumption $\int_0^\infty G(\tau) d\tau \leq 1 - C_3 < 1$. It implies a contradiction with (2.4).

If $\zeta \neq 0$, then we obtain

$$\Im J(\lambda_1) = \zeta_1 \left(2\sigma_1 + |\xi|^{2\theta} \int_0^\infty \sin(\zeta_1 \tau) e^{-\sigma_t \tau} G(\tau) d\tau\right) = 0.$$  

From [19] we know

$$\int_0^\infty \sin(\zeta_1 \tau) e^{-\sigma_t \tau} G(\tau) d\tau > 0.$$  

By a contradiction we proved our statement $\sigma_1 < 0$.

Then, following the same procedure of the proof of Lemma 2.1 in [19], we immediately complete the proof.

Finally, one can represent the solution to the linear model (2.1) by the following way:

$$u(t, x) = K_0(t, x) \ast_{(x)} u_0(x) + K_1(t, x) \ast_{(x)} u_1(x).$$

### 3. Energy estimates for solutions to homogeneous Cauchy problem

In this section, we prove the point-wise estimates using energy methods in Fourier space. Applying the partial Fourier transform to (2.1) we obtain the following ordinary differential equation of second order:

$$\begin{cases}  
\dot{u} + (1 + |\xi|^2) \ddot{u} - |\xi|^{2\theta} G \ast \dot{u} = 0, & \xi \in \mathbb{R}^n, \ t > 0, \\
(\dot{u}, \ddot{u})(0, \xi) = (\dot{u}_0, \ddot{u}_1)(\xi), & \xi \in \mathbb{R}^n. 
\end{cases} \tag{3.1}$$

By straightforward calculations one can get the following lemma, which is useful to prove the point-wise estimate of solutions.

**Lemma 3.1.** For any functions $f = f(t) \in C(\mathbb{R})$, and $g = g(t) \in W_2^1(0, T)$, we have

1. $(f \ast g)(t) = (f \ast g)(t) + \left(\int_0^t f(\tau) d\tau \right) g(t)$;
2. $\Re((f \ast g)(t) g_t(t)) = -\frac{1}{2} f(t) |g(t)|^2 + \frac{1}{2} (f' \square g)(t) - \frac{1}{2} \frac{d}{dt} \left(\int_0^t f(\tau) d\tau \right) |g(t)|^2$;
3. $|f \ast g(t)|^2 \leq \left(\int_0^t |f(\tau)|^2 d\tau \right) |f(\square g)(t)|$.

**Theorem 3.1.** (Point-wise estimates) The Fourier image $\hat{u} = \hat{u}(t, \xi)$ of the solution $u = u(t, x)$ to the Cauchy problem (2.1) satisfies the following estimates in the Fourier space:

$$|\hat{u}_t|^2 + (1 + |\xi|^2) |\hat{u}|^2 + |\xi|^{2\theta} G \square \hat{u} \leq e^{-c \omega(|\xi|)^t} |(1 + |\xi|^2) |\hat{u}_0|^2 + |\hat{u}_1|^2|,$$

where $\omega = \omega(|\xi|) = \frac{|\xi|^{2\max(1 - \sigma, \sigma)}}{1 + |\xi|^2\max(1 - \sigma, \sigma)}$ and $c$ a real positive constant.
Proof. Firstly, we multiply (3.1) by $\ddot{u}_t$ and take the real part

$$\frac{\partial}{\partial t}|\ddot{u}_t|^2 + (1 + |\xi|^2)|\dot{u}|^2 - 2|\xi|^2\Re((G * \ddot{u})\ddot{u}) = 0.$$  

Let us define the energy in the Fourier space

$$E_1(\ddot{u}) = E_1(\ddot{u})(t, \xi) = |\ddot{u}_t|^2 + (1 + |\xi|^2)|\dot{u}|^2 + |\xi|^2\Re G\dot{\ddot{u}} - |\xi|^2\Re\left(\int_0^t G(\tau)d\tau\right)|\dot{u}|^2.$$  

We introduce

$$F_1(\ddot{u}) = F_1(\ddot{u})(t, \xi) = |\xi|^2(G|\dot{u}|^2 - G'\dot{\ddot{u}}).$$  

From the second statement of Lemma 3.1, we obtain

$$\frac{\partial}{\partial t}E_1(\ddot{u})(t, \xi) + F_1(\ddot{u})(t, \xi) = 0. \tag{3.3}$$  

Multiplying (3.1) by $-(G * \ddot{u})_t$ and taking the real part, we get

$$\frac{1}{2}\frac{\partial}{\partial t}(|\xi|^2(G|\dot{u}|^2) - \Re((G * \ddot{u})_t\dot{u}_t) - \Re((G * \ddot{u})_t(1 + |\xi|^2)\ddot{u}) = 0.$$  

Thus, we use the relation $(G * \ddot{u})_t = G(0)\ddot{u} + G' * \ddot{u}$ to get

$$-\Re((G * \ddot{u})_t\dot{u}_t) = -\frac{\partial}{\partial t}\Re((G * \ddot{u})_t\dot{u}_t) + \Re((G * \ddot{u})_t\dot{u}_t)$$

$$= -\frac{\partial}{\partial t}\Re((G * \ddot{u})_t\dot{u}_t) + G(0)|\dot{u}_t|^2 + \Re((G * \ddot{u})_t\dot{u}_t).$$  

Using the following definitions:

$$E_2(\ddot{u}) = E_2(\ddot{u})(t, \xi) = \frac{1}{2}|\xi|^2(G|\dot{u}|^2 - \Re((G * \ddot{u})_t\dot{u}_t),$$

$$F_2(\ddot{u}) = F_2(\ddot{u})(t, \xi) = G(0)|\dot{u}_t|^2,$$

$$R_2(\ddot{u}) = R_2(\ddot{u})(t, \xi) = \Re(-(G * \ddot{u})_t\dot{u}_t + (G * \ddot{u})_t(1 + |\xi|^2)\ddot{u}),$$

we obtain

$$\frac{\partial}{\partial t}E_2(\ddot{u})(t, \xi) + F_2(\ddot{u})(t, \xi) = R_2(\ddot{u})(t, \xi). \tag{3.4}$$  

Similarly, multiplying (3.1) by $\dddot{u}$ and taking the real part implies

$$\frac{\partial}{\partial t}\Re(\dddot{u}\dddot{u}) - |\dddot{u}|^2 + (1 + |\xi|^2)|\dot{u}|^2 - |\xi|^2\Re((G * \dddot{u})\dddot{u}) = 0.$$  

According to the relation

$$\Re((G * \dddot{u})\dddot{u}) = \left(\int_0^t G(\tau)d\tau\right)|\dot{u}|^2 + \Re((G * \dddot{u})*\dddot{u}),$$  

we use the following notations:

$$E_3(\dddot{u}) = E_3(\dddot{u})(t, \xi) = \Re(\dddot{u}\dddot{u}),$$

$$F_3(\dddot{u}) = F_3(\dddot{u})(t, \xi) = (1 + |\xi|^2)|\dot{u}|^2 - |\xi|^2\Re\left(\int_0^t G(\tau)d\tau\right)|\dot{u}|^2,$$

$$R_3(\dddot{u}) = R_3(\dddot{u})(t, \xi) = |\dddot{u}|^2 + |\xi|^2\Re((G * \dddot{u})\dddot{u}).$$

6
Then, we obtain
\[ \frac{\partial}{\partial t} E_3(t, \xi) + F_3(t, \xi) = R_3(t, \xi). \]  
(3.5)

Using the defined function \( \omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^{2\bar{\gamma}}} \), we introduce
\[
E(\hat{u}) = E_1(\hat{u})(t, \xi) + \omega(\xi) (\gamma_1 E_2(\hat{u})(t, \xi) + \gamma_2 E_3(\hat{u})(t, \xi)),
\]
\[
F(\hat{u}) = F_1(\hat{u})(t, \xi) + \omega(\xi) (\gamma_1 F_2(\hat{u})(t, \xi) + \gamma_2 F_3(\hat{u})(t, \xi)),
\]
\[
R(\hat{u}) = R_1(\hat{u})(t, \xi) = \omega(\xi) (\gamma_1 R_2(\hat{u})(t, \xi) + \gamma_2 R_3(\hat{u})(t, \xi)),
\]
where \( \gamma_1, \gamma_2 \) are real positive constants to be defined later. Combining (3.3) to (3.5) leads to
\[ \frac{\partial}{\partial t} E(\hat{u})(t, \xi) + F(\hat{u})(t, \xi) = R(\hat{u})(t, \xi). \]  
(3.6)

Now, we should introduce the following Lyapunov functionals:
\[ E_0(\hat{u}) = E_0(\hat{u})(t, \xi) = |\hat{u}|^2 + (1 + |\xi|)|\hat{u}|^2 + |\xi|^{2\theta} G \hat{u}, \]
\[ F_0(\hat{u}) = F_0(\hat{u})(t, \xi) = G \hat{u} + G|\hat{u}|^2. \]

From the definitions of \( E_1(\hat{u})(t, \xi) \) and \( F_1(\hat{u})(t, \xi) \), we obtain
\[ l_0 E_0(\hat{u})(t, \xi) \leq E_1(\hat{u})(t, \xi) \leq E_0(\hat{u})(t, \xi), \]
\[ l_1 |\xi|^2 F_0(\hat{u})(t, \xi) \leq F_1(\hat{u})(t, \xi), \]  
(3.7)

where
\[ 0 < l_0 < \inf_{\xi \in \mathbb{R}^n} \left( 1 - \frac{(1 - C_3)|\xi|^{2\theta}}{1 + |\xi|^2} \right) \text{ and } l_1 = \min \{1; C_1\}. \]

From statements (1) and (3) of Lemma 3.1, we can estimate the energy \( E_2(\hat{u})(t, \xi) \) by using Cauchy’s inequality as follows:
\[
E_2(\hat{u})(t, \xi) \leq \frac{1}{2} |\hat{u}|^2 + \frac{1}{2} |G(0)|^2 \hat{u} + \frac{1}{2} |G^{*} \hat{u}|^2 + \frac{1}{2} |\xi|^{2\theta} |G^{*} \hat{u}|^2
\]
\[ \leq \frac{1}{2} |\hat{u}|^2 + G(0)|\hat{u}|^2 + (2 \max \{C_0; C_1\} + |\xi|^{2\theta}) \left( \int_0^t G(\tau) + d\tau \right) \left( \int_0^t G(\tau) + d\tau \right) |\hat{u}|^2 + G \hat{u}
\]
\[ \leq C(|\hat{u}|^2 + (1 + |\xi|^{2\theta})(|\hat{u}|^2 + G \hat{u})). \]

From the estimate
\[ E_3(\hat{u})(t, \xi) \leq C(|\hat{u}|^2 + |\hat{u}|^2), \]
there exists a positive constant \( l_2 \) such that
\[ |\omega(\xi)(\gamma_1 E_2(\hat{u})(t, \xi) + \gamma_2 E_3(\hat{u})(t, \xi))| \leq l_2(\gamma_1 + \gamma_2) E_0(\hat{u})(t, \xi). \]

Choosing sufficient small constants \( \gamma_1, \gamma_2 \) satisfying \( l_2(\gamma_1 + \gamma_2) \leq l_0/2 \), we obtain
\[ \frac{\partial}{\partial t} E_0(\hat{u})(t, \xi) \leq E(\hat{u})(t, \xi) \leq \frac{3}{2} E_0(\hat{u})(t, \xi). \]  
(3.8)

Similarly, we get
\[ F(\hat{u})(t, \xi) \geq l_1 |\xi|^{2\theta} F_0(\hat{u})(t, \xi) + \omega(\xi)(\gamma_1 G(0)|\hat{u}|^2 + \frac{\gamma_2}{2} (1 + |\xi|^2)|\hat{u}|^2). \]

Cauchy’s inequality implies
\[ |R_2(\hat{u})(t, \xi)| \leq \varepsilon_1 |\hat{u}|^2 + \varepsilon_2 (1 + |\xi|^2)|\hat{u}|^2 + C_{\varepsilon_1, \varepsilon_2} F_0(\hat{u})(t, \xi), \]  
(3.9)
where \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are positive constants to be chosen later.

Combining (3.9) with (3.10) we can get

\[
|R(\hat{u})(t, \xi)| \leq \omega(|\xi|) \varepsilon_1 \gamma_1 + \gamma_2 |\hat{u}|^2 + \omega(|\xi|) \varepsilon_2 (1 + |\xi|^2) \gamma_1 + \varepsilon_3 |\xi| |\hat{u}|^2
\]

\[
+ C_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \gamma_1 \varepsilon_3 \gamma_2 |\hat{u}| \omega(\hat{u})(t, \xi).
\]

Now choosing \( \gamma_2 = \frac{G(0)}{8} \gamma_1, \varepsilon_1 = \frac{G(0)}{8}, \varepsilon_2 = \frac{G(0)}{8} \) and \( \varepsilon_3 = \frac{1}{2} \) we obtain

\[
|R(\hat{u})(t, \xi)| \leq \omega(|\xi|) \frac{G(0)}{2} \gamma_1 |\hat{u}|^2 + \omega(|\xi|) \frac{G(0)}{32} \gamma_1 (1 + |\xi|^2 + |\xi|^{2\theta}) |\hat{u}|^2
\]

\[
+ C_{\varepsilon_1, \varepsilon_2, \varepsilon_3} (G(0)/4 + 1) \gamma_1 |\xi|^{2\theta} F_0(\hat{u})(t, \xi),
\]

(3.11)

where we choose \( (G(0)/4 + 1) \gamma_1 = \min \left\{ \frac{\gamma_1}{20\varepsilon_1 \varepsilon_2 \varepsilon_3 \gamma_2}, \frac{\omega(\hat{u})}{\gamma_2} \right\} \). From (3.6) and (3.11) we get

\[
\frac{\partial}{\partial t} E(\hat{u})(t, \xi) + \frac{1}{2} F(\hat{u})(t, \xi) \leq 0.
\]

Moreover, there exists the positive constant \( C \) such that

\[
F(\hat{u})(t, \xi) \geq C \omega(|\xi|) E(\hat{u})(t, \xi).
\]

Applying Grönwall’s inequality we conclude

\[
E(\hat{u})(t, \xi) \leq e^{-C\omega(|\xi|)t} E(\hat{u})(0, \xi).
\]

Together with (3.7) the proof is completed. \( \square \)

In the following theorem we show the energy estimates of solution to (2.1) with data belonging to the function space \((H^{s+1} \cap L^m) \times (H^s \cap L^m)\).

**Theorem 3.2.** Let us consider the Cauchy problem (2.1) with \( \theta \in [0, 1] \), \( n \geq 1 \) and data \((u_0, u_1) \in D^s_2 \) for \( s \geq 0 \). Then, the following estimate holds:

\[
\|u(t, \cdot)\|_{H^s}^2 + \|u(\cdot, \cdot)\|_{H^{s+1}}^2 + \int_0^t (\|u(t, \cdot)\|_{H^s}^2 + \|u(\cdot, \cdot)\|_{H^{s+1}}^2) dt \lesssim \|u_0\|_{H^{s+1}}^2 + \|u_1\|_{H^s}^2.
\]

**Proof.** From the proof of Theorem 3.1 we can obtain

\[
\frac{\partial}{\partial t} E(\hat{u})(t, \xi) + C \omega(|\xi|) E(\hat{u})(t, \xi) \leq 0,
\]

which implies

\[
E(\hat{u})(t, \xi) + \int_0^t \omega(|\xi|) E(\hat{u})(\tau, \xi) d\tau \lesssim E(\hat{u})(0, \xi).
\]

The estimates (3.8) implies

\[
E_0(\hat{u})(t, \xi) + \int_0^t \omega(|\xi|) E_0(\hat{u})(\tau, \xi) d\tau \lesssim E_0(\hat{u})(0, \xi).
\]

(3.12)

Multiplying (3.12) by \( |\xi|^{2s} \) and integrating over \( \mathbb{R}^n \) we complete the proof. \( \square \)
Theorem 3.3. Let us consider the Cauchy problem (2.1) with \( \theta \in [0,1] \), \( n \geq 1 \) and data \((u_0, u_1) \in D_m^\theta \) for \( s \geq 0, m \in [1,2] \). Then, we have the following estimates:

\[
\| D^{s+1} u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{D_m^\theta},
\]

\[
\| D^s u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{D_m^\theta}.
\]

Moreover, if we assume \((u_0, u_1) \in (L^2 \cap L^m) \times (L^2 \cap L^m) \) for \( m \in [1,2] \), then the following estimate holds:

\[
\| u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(\theta+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{(L^2 \cap L^m) \times (L^2 \cap L^m)}. \tag{3.13}
\]

Combining Theorem 3.2 with the Parseval-Plancherel theorem, the Hausdorff-Young inequality and Hölder inequality, we immediately completed the proof.

We prove the following corollary to study the semi-linear problem (1.6), where we need higher order energy estimates with data belonging to the function spaces \((\dot{H}^{s+1} \cap L^m) \times (\dot{H}^s \cap L^m) \) or \( \dot{H}^s \times \dot{H}^s \).

Corollary 3.1. Let us consider the Cauchy problem (2.1) with \( \theta \in [0,1] \) and data \((u_0, u_1) \in (\dot{H}^{s+1} \cap L^m) \times (\dot{H}^s \cap L^m) \) for \( s \geq 0, m \in [1,2] \). Then, we have the following estimates:

\[
\| D^{s+1} u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{(\dot{H}^{s+1} \cap L^m) \times (\dot{H}^s \cap L^m)},
\]

\[
\| D^s u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{(\dot{H}^{s+1} \cap L^m) \times (\dot{H}^s \cap L^m)}.
\]

Moreover, if we consider data \((u_0, u_1) \in \dot{H}^s \times \dot{H}^s \) for \( s \geq 0 \), then the following estimates hold:

\[
\| D^{s+1} u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m(\theta+1)}{4m+\max\{1-\theta, 0\}}} \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^s},
\]

\[
\| D^s u(t, \cdot) \|_{L^2} \lesssim \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^s}.
\]

Proof. The assumption for data \((u_0, u_1) \in (\dot{H}^{s+1} \cap L^m) \times (\dot{H}^s \cap L^m) \) allow us to modify the considerations for large frequencies to complete the proof. For data \((u_0, u_1) \in \dot{H}^s \times \dot{H}^s \), we may apply

\[
\| \mathcal{F}_{\xi \rightarrow x}^{-1} (\chi_{\text{int}}(\xi) \xi^{s+1} e^{-c\omega(|\xi|)^{\frac{\theta}{p}}} (\hat{u}_0(\xi) + [\hat{u}_1(\xi)]) \|_{L^2} \lesssim \sup_{|\xi| < 1} \| \mathcal{F}_{\xi \rightarrow x}^{-1} (\chi_{\text{int}}(\xi) \xi^s e^{-c\omega(|\xi|)^{\frac{\theta}{p}}} (\hat{u}_0(\xi) + [\hat{u}_1(\xi)]) \|_{L^2} \lesssim \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^s},
\]

where the function \( \chi_{\text{int}} = \chi_{\text{int}}(\xi) \in C^\infty \) is supported in \(|\xi| < 1\). These yield the desired estimates.

4. Semi-linear Klein-Gordon equations with memory-type dissipation

4.1. Strategies

In this section we study the Cauchy problem (1.2). Our main interest is to prove the global (in time) existence of small data solutions. Such results imply immediately stability results for zero solution. We mention that in papers [11, 31, 32] the authors proved the critical exponent \( p_{Fuj}(n) = 1 + \frac{\theta}{2} \) for the semi-linear wave equation with friction damping. We assume in the first case that data belong to the solution space \( D_m^\theta = (H^1 \cap L^m) \times (L^2 \cap L^m) \), where \( m \in [1,2] \) and we prove the global (in time) existence of Sobolev solution

\[
u \in C([0, \infty), H^1),
\]

provided that the exponent of power nonlinearity \( p \) is larger than the modified Fujita exponent \( p_{Fuj}(\frac{n}{m}, \frac{\theta}{m}, [\theta]) \) and smaller than some upper bound for \( n \geq 3 \).

The second case is related to data with high regularity, this means, data belong to \( D_m^\theta = (H^{s+1} \cap L^m) \times (H^s \cap L^m) \) with \( s > 0 \) and \( m \in [1,2] \). We prove the global (in time) existence of solutions by using among other things new tools from Harmonic Analysis from [27] (see Appendix A), where we require the condition \( p > 1 + [s] \) if \( s \in (0, \frac{n}{2}) \). Finally, if \( s > \frac{n}{2} \), then using fractional powers the last condition \( p > 1 + [s] \) will be relaxed to \( p > 1 + s \).
4.2. Main results and examples

Now, let us formulate the main results. In the case that data are supposed to belong to the classical energy space we have the following result.

**Theorem 4.1.** Let $n \leq \frac{4}{2-m}$ and $n < \frac{2m \max(1-\theta;\theta)}{m \max(1-\theta;\theta) - 1}$ where $\theta \in [0, 1]$. Data $(u_0, u_1)$ are supposed to belong to the function space $\mathcal{D}_m^0$ for $m \in [1, 2)$. The exponent $p$ satisfies

$$p > p_{Fuji} \left( \frac{n}{m \max \{1 - \theta; \theta\}} \right)$$

(4.1)

and

$$\frac{2}{m} \leq p < \infty \quad \text{if} \quad n \leq 2,$$

$$\frac{2}{m} \leq p \leq \frac{n}{n - 2} \quad \text{if} \quad n \geq 3.$$ (4.2)

Then, there exists a small constant $\varepsilon_0$ such that if

$$\|(u_0, u_1)\|_{\mathcal{D}_m^0} \leq \varepsilon_0,$$

then there exists a uniquely determined globally (in time) Sobolev solution to (1.2) such that

$$u \in C([0, \infty), H^1).$$

Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2-m)}{4m \max(1-\theta;\theta) + \varepsilon}} \|(u_0, u_1)\|_{\mathcal{D}_m^0},$$

$$\|\nabla_x u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2-m)}{4m \max(1-\theta;\theta) + 2m \max(1-\theta;\theta) + \varepsilon}} \|(u_0, u_1)\|_{\mathcal{D}_m^0},$$

where $\varepsilon = 0$ if $\theta \in [0, 1/2) \cup (1/2, 1]$ and $\varepsilon$ is positive and sufficiently small if $\theta = 1/2$.

**Remark 4.1.** We remark from the statement of Theorem 4.1 that for all dimensions $n \geq 1$, the solution to (1.2) with $\theta = 1/2$ globally in time exists provided that the parameter of additional regularity $m$ is close to 2 and the condition (4.1) still holds.

In the following theorem we use the generalized (fractional) Gagliardo-Nirenberg inequality used in the papers [10]. Furthermore, we shall use the fractional Leibniz rule and the fractional chain rule.

**Theorem 4.2.** Let $n \geq 3$ and $s > 0$. Data $(u_0, u_1)$ are supposed to belong to the function space $\mathcal{D}_m^s$ for $m \in [1, 2)$. Finally, the following conditions are satisfied for the exponent $p$:

$$\max \left\{ p_{Fuji} \left( \frac{n}{m \max \{1 - \theta; \theta\}} \right); 1 + |s| \right\} : 1 + |s| \right\} \leq p \leq 1 + \frac{2}{n - 2(s + 1)} \quad \text{if} \quad s \in \left(0, \frac{n}{2} - 1\right),$$

$$\max \left\{ p_{Fuji} \left( \frac{n}{m \max \{1 - \theta; \theta\}} \right); 1 + |s| \right\} \leq p < \infty \quad \text{if} \quad s \in \left[\frac{n}{2}, 1, \infty\right).$$

Then, there exists a constant $\varepsilon_0 > 0$ such that if $\|(u_0, u_1)\|_{\mathcal{D}_m^s} \leq \varepsilon_0$, then there exists a uniquely determined globally (in time) Sobolev solution to (1.2) such that

$$u \in C([0, \infty), H^{s+1}).$$

Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2-m)}{4m \max(1-\theta;\theta) + \varepsilon}} \|(u_0, u_1)\|_{\mathcal{D}_m^s},$$

$$\|D^{s+1} u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2-m+s+1)}{4m \max(1-\theta;\theta) + 2m \max(1-\theta;\theta) + \varepsilon}} \|(u_0, u_1)\|_{\mathcal{D}_m^s},$$

where $\varepsilon = 0$ if $\theta \in [0, 1/2) \cup (1/2, 1]$ and $\varepsilon$ is positive and sufficiently small if $\theta = 1/2$. 

10
Remark 4.2. We point out that for any \( \theta \in [0, 1] \) and \( m \in [1, 2) \) that the lower bound for the exponent \( p \) is determined by \( 1 + \lceil s \rceil \) if \( n \geq 5 \). For this reason we restrict ourselves to study the coupled system (1.6) with different regularities of data for \( n \geq 5 \).

We are interested now in the case of data having a large regularity such that they belong to \( L^\infty \) where we can prove a similar result as Theorem 4.2 with larger admissible range for the exponent of power nonlinearity \( p \).

**Theorem 4.3.** Let \( n \geq 3 \) and \( s > \frac{n}{2} \). Data \((u_0, u_1)\) are supposed to belong to the function space \( D_m^s \) for \( m \in [1, 2) \). Let us assume the exponent \( p \) satisfies

\[
1 + s < p.
\]

Then, there exists a constant \( \varepsilon_0 > 0 \) such that if \( \| (u_0, u_1) \|_{D_m^s} \leq \varepsilon_0 \), then there exists a uniquely determined globally (in time) Sobolev solution to (1.2) such that

\[
u \in C([0, \infty), H^{s+1}).
\]

Moreover, the solution satisfies the same estimates as those in Theorem 4.2.

**Example 4.1.** We consider the problem (1.2) with \( \theta = 1 \) in three-dimensional case.

- If data belong to the classical energy space, i.e., \( s = 0 \), then using Theorem 4.1 the admissible range for \( p \) is the following:
  \[
p \in \left[ \frac{2}{m}, 3 \right] \quad \text{if} \quad m \in [1, (\sqrt{57} - 3)/4],
  
p \in \left(1 + \frac{2m}{3}, 3 \right] \quad \text{if} \quad m \in ((\sqrt{57} - 3)/4, 2].
\]

- If \( s = 1 \), then using Theorem 4.2 the admissible range for \( p \) is the following:
  \[
p \in (2, \infty) \quad \text{if} \quad m \in [1, 3/2),
  
p \in \left(1 + \frac{2m}{3}, \infty \right) \quad \text{if} \quad m \in [3/2, 2].
\]

- If \( s = 2 \), then using Theorem 4.3 the admissible range for \( p \) is the following:
  \[
p \in (3, \infty).
\]

4.3. Philosophy of our approach

We introduce the operator \( N \) as follows

\[
N : u \in X(T) \rightarrow Nu = u^{ln}(t, x) + u^{nl}(t, x),
\]

where

\[
u^{ln}(t, x) := K_0(t, x) * (x) u_0(x) + K_1(t, x) * (x) u_1(x),
\]

is the solution to the homogeneous Cauchy problem

\[
\begin{aligned}
u_{tt} - \Delta u + u - G * (-\Delta)^s u &= 0, \quad x \in \mathbb{R}^n, \quad t \geq \tau, \\
u(\tau, x) &= (u_0, u_1)(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

and

\[
u^{nl}(t, x) := \int_{0}^{t} K_1(t - \tau, x) * (x) |u(\tau, x)|^p d\tau,
\]

\text{for} \quad t > \tau.

\text{Moreover, the solution satisfies the same estimates as those in Theorem 4.2.}
is the solution to the family of parameter-dependent Cauchy problem

\[
\begin{cases}
  u_{tt} - \Delta u + u - G * (-\Delta)^{\theta} u = 0, & x \in \mathbb{R}^n, \ t \geq \tau, \\
  (u, u_t)(\tau, x) = (0, |u(\tau, x)|^p), & x \in \mathbb{R}^n.
\end{cases}
\]

Let \( \{X(T)\}_{T > 0} \) the solution space. Our aim is to prove the following inequalities:

\[
||Nu||_{X(T)} \lesssim ||(u_0, u_1)||_{\mathcal{P}_m} + ||u||^p_{X(T)}, \quad (4.3)
\]

\[
||Nu - N\tilde{u}||_{X(T)} \lesssim ||u - \tilde{u}||_{X(T)} (||u||^p_{X(T)} + ||\tilde{u}||^{p-1}_{X(T)}), \quad (4.4)
\]

After proving these both inequalities we apply Banach’s fixed point theorem. In this way we get the local (in time) existence of large data Sobolev solutions and the global (in time) existence of small data Sobolev solutions as well. One can also remark that the last both inequalities imply for the fixed point \( u = u(t, x) \) the estimate

\[
||u||_{X(T)} \lesssim ||(u_0, u_1)||_{\mathcal{P}_m},
\]

which leads to the desired estimates for the Sobolev solution.

From the definition of the norm of the solution space \( X(T) \) which will be defined for each case, we can immediately obtain

\[
||u^{in}||_{X(T)} \lesssim ||(u_0, u_1)||_{\mathcal{P}_m}.
\]

We complete the proof of all results separately by showing the inequality

\[
||u^{in}||_{X(T)} \lesssim ||u||^p_{X(T)}, \quad (4.5)
\]

which lead leads to the desired estimate (4.3). Afterwards we shall prove (4.4). The main tools to prove (4.5) and (4.4) are the Gagliardo-Nirenberg inequalities, the fractional chain rule, the fractional Leibniz rule and the fractional powers, which have been discussed in Harmonic Analysis (cf. with Appendix A).

\[4.4. \text{Sketch of the proof of Theorem 4.1}\]

Firstly, we prove the results for the case \( \theta \in [0,1/2) \cup (1/2,1] \). For any \( T > 0 \) let us introduce the solutions space

\[X(T) = C([0,T], H^1)\]

with the corresponding norm

\[
||u||_{X(T)} = \sup_{0 \leq t \leq T} \left( (1 + t)^{\frac{n(2 - m)}{2m + (1 - \theta)p}} ||u(t, \cdot)||_{L^2} + (1 + t)^{\frac{n(2 - m)}{2m + (1 - \theta)p}} ||\nabla_x u(t, \cdot)||_{L^2} \right).
\]

Applying the classical Gagliardo-Nirenberg inequality we obtain for \( 0 \leq \tau \leq t \)

\[
||u(\tau, x)||^p_{L^m} \lesssim (1 + \tau)^{-\frac{n(2 - m)}{2m + (1 - \theta)p}} ||u||^p_{X(t)},
\]

\[
||u(\tau, x)||^p_{L^2} \lesssim (1 + \tau)^{-\frac{n(2 - m)}{2m + (1 - \theta)p}} ||u||^p_{X(t)}, \quad (4.6)
\]

provided that

\[
\frac{2}{m} \leq p < \infty \quad \text{if} \quad n \leq 2,
\]

\[
\frac{2}{m} \leq p \leq \frac{n}{n - 2} \quad \text{if} \quad n \geq 3.
\]
To prove (4.5) we have to estimate \( \|u^m(t, \cdot)\|_{L^2} \) and \( \|\nabla_x u^m(t, \cdot)\|_{L^2} \).

Using from Theorem 3.3 the derived \((L^2 \cap L^m)-L^2\) estimates for the integral over \([0, t/2]\) and \(L^2-L^2\) estimates for the integral over \([t/2, t]\) together with (4.6) we obtain

\[
\|u^m(t, \cdot)\|_{L^2} \lesssim \int_0^{t/2} (1 + t - \tau)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u(\tau, x)\|^p_{L^2 \cap L^m} d\tau + \int_{t/2}^{t} \|u(\tau, x)\|^p_{L^2} d\tau \\
\lesssim (1 + t)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)} \int_0^{t/2} (1 + \tau)^{-\frac{n(p-1)}{2m \max (1 - \theta, \eta)}} d\tau + (1 + t)^{1-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)}
\]

where we use our assumption \( p > p_{Fuj} \left( \frac{n}{m \max (1 - \theta, \eta)} \right) \). Consequently, we get

\[
\|u^m(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)} \quad (4.7)
\]

Analogously, we use from Corollary 3.1 the derived \((L^2 \cap L^m)-L^2\) estimates for the integral over \([0, t/2]\) and \(L^2-L^2\) estimates for the integral over \([t/2, t]\) to get

\[
\|\nabla_x u^m(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)} \int_0^{t/2} (1 + \tau)^{-\frac{n(p-1)}{2m \max (1 - \theta, \eta)}} d\tau \\
+ (1 + t)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)} \int_{t/2}^{t} (1 + t - \tau)^{-\frac{n(p-1)}{2m \max (1 - \theta, \eta)}} d\tau \\
\lesssim (1 + t)^{-\frac{n(2-m)+2m}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)}
\]

where \( \theta \in [0, 1/2) \cup (1/2, 1] \) and \( p > p_{Fuj} \left( \frac{n}{m \max (1 - \theta, \eta)} \right) \). Then,

\[
\|\nabla_x u^m(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m}{2m \max (1 - \theta, \eta)}} \|u\|^p_{X(t)} \quad (4.8)
\]

Therefore, combining (4.7) with (4.8) we get the desired estimate (4.5).

Next, we prove the Lipschitz condition (4.4). We assume that \( u = u(t, x) \) and \( \tilde{u} = \tilde{u}(t, x) \) are two elements from the function space \( X(T) \). Then,

\[
N u - N \tilde{u} = \int_0^t K_1(t - \tau, x) \ast (|u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p) \, d\tau.
\]

Using Hölder's inequality and the classical Gagliardo-Nirenberg inequality together with the definition of our solution space \( X(T) \) one can get

\[
\|u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p\|_{L^\infty} \lesssim (1 + \tau)^{-\frac{n(p-1)}{2m \max (1 - \theta, \eta)}} \|u - \tilde{u}\|_{X(T)} \|u\|^{p-1}_{X(T)} + \|\tilde{u}\|^{p-1}_{X(T)}, \quad (4.9)
\]

\[
\|u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p\|_{L^2} \lesssim (1 + \tau)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u - \tilde{u}\|_{X(T)} \|u\|^{p-1}_{X(T)} + \|\tilde{u}\|^{p-1}_{X(T)}. \quad (4.10)
\]

In a similar way to (4.7) and (4.8) we can prove

\[
\|N (u - \tilde{u})(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)}{2m \max (1 - \theta, \eta)}} \|u - \tilde{u}\|_{X(T)} \|u\|^{p-1}_{X(T)} + \|\tilde{u}\|^{p-1}_{X(T)},
\]

\[
\|\nabla_x (N (u - \tilde{u})(t, \cdot))\|_{L^2} \lesssim (1 + t)^{-\frac{n(2-m)+2m}{2m \max (1 - \theta, \eta)}} \|u - \tilde{u}\|_{X(T)} \|u\|^{p-1}_{X(T)} + \|\tilde{u}\|^{p-1}_{X(T)}. \]

Thus, the proof is completed.
If \( \theta = 1/2 \), then we have
\[
\int_{t/2}^{t} (1 + t - \tau)^{-1} d\tau \lesssim \log(e + t).
\]
Consequently, we modify the norm of solution space as follows:
\[
\|u\|_{X(T)} = \sup_{0 \leq t \leq T} \left( (1 + t)^{\frac{n(2-m)}{4m(1+s)q}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m(1+s)q}} \|D^{s+1} u(t, \cdot)\|_{L^2} \right).
\]
Similar as the case \( \theta \in [0, 1/2) \cup (1/2, 1] \) and using the modified norm of solution space we can prove the estimates (4.5) and (4.4) which complete the proof for the case \( \theta = 1/2 \).

4.5. Sketch of the proof of Theorem 4.2
For the same reason explained in the proof of Theorem 4.1, we restrict ourselves only to the case where \( \theta \in [0, 1/2) \cup (1/2, 1] \).
For any \( T > 0 \) we define the complete evolution solution space
\[
X(T) = C([0, T], H^{s+1})
\]
with the corresponding norm
\[
\|u\|_{X(T)} = \sup_{0 \leq t \leq T} \left( (1 + t)^{\frac{n(2-m)}{4m(1+s)q}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{-\frac{n(2-m)+2m(s+1)}{4m(1+s)q}} \|D^{s+1} u(t, \cdot)\|_{L^2} \right).
\]
We shall estimate all norms making up the norm \( \|u\|_{X(T)} \) which are \( \|u^{h}(t, \cdot)\|_{L^2} \) and \( \|D^{s+1} u^{h}(t, \cdot)\|_{L^2} \).
Let us begin to estimate the last term. From Corollary 3.1 we use the derived \( (H^s \cap L^m) - H^s \) estimates for the integral over \([t/2, t]\) and \( H^s - H^s \) estimates for the integral over \([t/2, t]\) to get
\[
\|D^{s+1} u^{h}(t, \cdot)\|_{L^2} \lesssim \int_{t/2}^{t} (1 + t - \tau)^{-\frac{n(2-m)+2m(s+1)}{4m(1+s)q}} \|u(\tau, x)\|_{H^s \cap L^m} d\tau + \int_{t/2}^{t} (1 + t - \tau)^{-\frac{2m(1+s)q}{4m(1+s)q}} \|u(\tau, x)\|_{H^s} d\tau.
\]
(4.11)
Similar as (4.6), applying the fractional Gagliardo-Nirenberg inequality we obtain for \( 0 \leq \tau \leq t \)
\[
\|u(\tau, x)\|_{L^m} \lesssim (1 + \tau)^{-\frac{n(q-1)}{m(2m+n(2+s+1))}} \|u\|_{X(t)}^p,
\]
(4.12)
where
\[
\frac{2}{m} \leq q < \infty \quad \text{if} \quad n \leq 2(s+1),
\]
\[
\frac{2}{m} \leq q \leq \frac{2n}{m(n-2(s+1))} \quad \text{if} \quad n > 2(s+1).
\]
(4.13)
Applying the fractional chain rule from Proposition A.4 we have
\[
\|u(\tau, x)\|_{H^s} \lesssim \|u(\tau, \cdot)\|_{L^2}^{p-1} \|u(\tau, \cdot)\|_{H^s_{q_2}} \|u(\tau, \cdot)\|_{H^s_{q_1}},
\]
(4.14)
where \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s} \) and \( p > [s] \). We use again the fractional Gagliardo-Nirenberg-type inequality, then we have
\[
\|u(\tau, \cdot)\|_{L^{q_1}} \lesssim \|u(\tau, \cdot)\|_{L^2}^{1-\beta_{s+1}(q_1)} \|u(\tau, \cdot)\|_{H^{s+1}_{q_1}},
\]
\[
\|u(\tau, \cdot)\|_{H^s_{q_2}} \lesssim \|u(\tau, \cdot)\|_{L^2}^{1-\beta_{s+1}(q_2)} \|u(\tau, \cdot)\|_{H^{s+1}_{q_2}},
\]
where \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s} \) and \( p > [s] \).
Then, using the estimates (4.12) and (4.15) in (4.11) we have the following estimates:

From the estimates (4.16) and (4.19), the desired estimate (4.5) is proved.

Consequently, for $0 \leq \tau \leq t$ we have

$$
||u(\tau, x)||_{L^p} \lesssim (1 + \tau)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u||_{X(t)}^p.
$$

(4.15)

Then, using the estimates (4.12) and (4.15) in (4.11) we have the following estimates:

$$
||D|^{s+1}u^{n1}(t, \cdot)||_{L^2} \lesssim (1 + t)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u||_{X(t)}^p \int_0^{t/2} (1 + \tau)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} d\tau
$$

$$
+ (1 + t)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u||_{X(t)}^p \int_{t/2}^t (1 + t - \tau)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} d\tau
$$

$$
\lesssim (1 + t)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u||_{X(t)},
$$

where we use $p > p_{Fu}$ \left(\frac{n}{m \max(1, \theta)}\right)$ for $\theta \in [0, 1/2) \cup (1/2, 1]$. Thus,

$$
||D|^{s+1}u^{n1}(t, \cdot)||_{L^2} \lesssim (1 + t)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u||_{X(t)}^p.
$$

(4.16)

For $u^{n1} = u^{n1}(t, x)$ we use the fractional Gagliardo-Nirenberg inequality and it yields for $0 \leq \tau \leq t$

$$
||u(\tau, x)||_{L^p} \lesssim (1 + \tau)^{-\frac{n(2-p-m)}{m \max(1, \theta)}} ||u||_{X(t)}^p,
$$

(4.17)

where

$$
1 \leq p < \infty \quad \text{if} \quad n \leq 2(s + 1),
$$

$$
1 \leq p \leq \frac{n}{n - 2(s + 1)} \quad \text{if} \quad n > 2(s + 1).
$$

(4.18)

Analogously, using (4.12) and (4.17) we get

$$
||u^{n1}(t, \cdot)||_{L^2} \lesssim (1 + t)^{-\frac{n(2-p-m)}{m \max(1, \theta)}} ||u||_{X(t)}^p.
$$

(4.19)

From the estimates (4.16) and (4.19), the desired estimate (4.5) is proved.

The last step is to derive the Lipschitz condition. We recall

$$
Nu - N\tilde{u} = \int_0^t K_1(t - \tau, x) * (|u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p) d\tau,
$$

where $u = u(t, x)$ and $\tilde{u} = \tilde{u}(t, x)$ are two elements from the function space $X(T)$. Now we have to control all norms making up $||Nu - N\tilde{u}||_{X(t)}$, which are $||D|^{s+1}(Nu - N\tilde{u})(t, \cdot)||_{L^2}$ and $||(Nu - N\tilde{u})(t, \cdot)||_{L^2}$. Analogously to (4.11) we have

$$
||D|^{s+1}(Nu - N\tilde{u})(t, \cdot)||_{L^2} \lesssim \int_0^{t/2} (1 + t - \tau)^{-\frac{n(2-p-m+2m+1)}{m \max(1-\theta, \theta)}} ||u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p|_{H^{s+1}} d\tau
$$

$$
+ \int_{t/2}^t (1 + t - \tau)^{-\frac{1}{\max(1-\theta, \theta)}} ||u(\tau, x)|^p - |\tilde{u}(\tau, x)|^p|_{H^{s+1}} d\tau.
$$

(4.20)
From (4.9) its remains only to estimate \( \| u(\tau, x)^p - |\tilde{u}(\tau, x)|^p \|_{H^s} \). The use of Minkowski’s inequality and the fractional Leibniz rule from Proposition A.3 shows that

\[
\| u(\tau, x)^p - |\tilde{u}(\tau, x)|^p \|_{H^s} \lesssim \int_0^1 \| (u(\tau, \cdot) - \tilde{u}(\tau, \cdot))g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{H^s} d\nu
\]

\[
\lesssim \int_0^1 \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^s_{r_1}} \| g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{L^{r_2}} d\nu
\]

\[
+ \int_0^1 \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^s_{r_3}} \| g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{H^s_{r_4}} d\nu,
\]

where \( g(f) = f|f|^{p-2} \) and \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2} \). Taking account of the first term on the right-hand side in above inequality we notice that

\[
\int_0^1 \| g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{L^{r_2}} d\nu \lesssim \| u(\tau, \cdot) \|_{L^{r_2}(p-1)}^{p-1} + \| \tilde{u}(\tau, \cdot) \|_{L^{r_2}(p-1)}^{p-1}.
\]

Actually, we use the fractional Gagliardo-Nirenberg inequality to get the following inequalities

\[
\| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^s_{r_1}} \lesssim \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{L^{r_2}}^{1-\beta_{0,s+1}(r_1)} \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^{s+1}}^{\beta_{0,s+1}(r_1)},
\]

\[
\| u(\tau, \cdot) \|_{L^{r_2}(p-1)} \lesssim \| u(\tau, \cdot) \|_{L^{r_2}}^{1-\beta_{0,s+1}(r_2)(p-1)} \| u(\tau, \cdot) \|_{H^{s+1}}^{\beta_{0,s+1}(r_2)(p-1)},
\]

\[
\| \tilde{u}(\tau, \cdot) \|_{L^{r_2}(p-1)} \lesssim \| \tilde{u}(\tau, \cdot) \|_{L^{r_2}}^{1-\beta_{0,s+1}(r_2)(p-1)} \| \tilde{u}(\tau, \cdot) \|_{H^{s+1}}^{\beta_{0,s+1}(r_2)(p-1)},
\]

\[
\| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^s_{r_3}} \lesssim \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{L^{r_2}}^{1-\beta_{0,s+1}(r_3)} \| u(\tau, \cdot) - \tilde{u}(\tau, \cdot) \|_{H^{s+1}}^{\beta_{0,s+1}(r_3)}.
\]

Using the fractional chain rule we obtain

\[
\| g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{H^s_{r_4}} \lesssim \| g(\nu u(\tau, \cdot) + (1 - \nu)\tilde{u}(\tau, \cdot)) \|_{L^{r_2}}^{p-2} \| u(\tau, \cdot) \|_{L^{r_2}} \| \tilde{u}(\tau, \cdot) \|_{L^{r_2}} \| u(\tau, \cdot) \|_{H^{s+1}} + \| \tilde{u}(\tau, \cdot) \|_{H^{s+1}} \beta_{0,s+1}(r_2)(p-2) + \beta_{0,s+1}(r_3),
\]

provided that \( p > 1 + [s] \). Moreover, the parameters appear in previous estimates must satisfy the following conditions:

\[
\frac{1}{r_4} = \frac{p-2}{r_5} + \frac{1}{r_6},
\]

and

\[
\beta_{0,s+1}(r) = \frac{n}{s+1} \left( \frac{1}{2} - \frac{1}{r} \right) \in [0,1] \quad \text{for } r = r_2(p-1), r_3, r_5,
\]

\[
\beta_{0,s+1}(r) = \frac{n}{s+1} \left( \frac{1}{2} - \frac{1}{r} + \frac{s}{n} \right) \in \left[ \frac{s}{s+1}, 1 \right] \quad \text{for } r = r_1, r_6.
\]

The parameters \( r_1 \) to \( r_6 \) can be chosen as follows:

\[
r_1 = \frac{2n}{n-2}, \quad r_2 = n, \quad r_3 = n(p-1), \quad r_4 = \frac{2n(p-1)}{n(p-1) - 2}, \quad r_5 = n(p-1), \quad r_6 = \frac{2n}{n-2},
\]

which leads to the condition

\[
1 + \frac{2}{n} \leq p \leq 1 + \frac{2}{n-2(s+1)}.
\]

Using the norm of \( X(T) \) leads to

\[
\| u(\tau, x)^p - |\tilde{u}(\tau, x)|^p \|_{H^s} \lesssim (1 + \tau) \frac{\bar{u}((2n+1)(2n)+1)}{\bar{u}L_{X(T)}} \| u - \tilde{u} \|_{X(T)} \left( \| u \|_{X(T)}^{p-1} + \| \tilde{u} \|_{X(T)}^{p-1} \right).
\]
was defined by the exponent $p$.

5.1.1. Data from the energy space

Gordon equations with memory-type dissipation.

Analogous to the condition required in papers [30] and [26] which is

5.1. Main results and examples

This completes the proof.

4.6. Sketch of the proof of Theorem 4.3

To prove this theorem we define the same solution space $X(T)$ used in the proof of Theorem 4.2 and we follow the same steps. We modify only the estimates of $\|u(\tau,x)\|^p_{H^s}$ and $\|u(\tau,x)|^p - |\tilde{u}(\tau,x)|^p\|_{H^s}$ which generate a new condition $p > 1 + s$ weaker than $p > 1 + \lceil s \rceil$. Indeed, using the fractional powers and Proposition A.7 we get for $s^* < \frac{n}{2}$ the following estimates for $0 \leq \tau \leq t$:

$$||u(\tau,x)||^p_{H^s} \leq (1 + \tau)^{-\frac{n(2 - m) + p + 2m^*(p - 1) + 2ms}{4m \max\{1 - \theta; \theta\}}} \|u\|^p_{X(t)}$$

and

$$||u(\tau,x)|^p - |\tilde{u}(\tau,x)|^p\|^p_{H^s} \leq (1 + \tau)^{-\frac{2np - m(p - 1) + 2ms(p - 1) + 2ms}{4m \max\{1 - \theta; \theta\}}} \|u - \tilde{u}\|^p_{X(t)}(\|u\|^p_{X(t)} + \|\tilde{u}\|^p_{X(t)}).$$

By choosing $s^* = \frac{n}{2} - \varepsilon$ for $\varepsilon$ sufficiently small, we obtain

$$-\frac{n(2 - m)p + 2ms(p - 1) + 2ms}{4m \max\{1 - \theta; \theta\}} < -\frac{n(p - 1)}{2m \max\{1 - \theta; \theta\}}.$$  

Using these estimates we complete the proof.

5. Weakly coupled system of semi-linear Klein-Gordon equations with memory-type dissipation

5.1. Main results and examples

In this section we apply results of the previous section to study the system of weakly coupled Klein-Gordon equations with memory-type dissipation.

5.1.1. Data from the energy space

From Theorem 4.1 we remark that the pivotal condition for the exponent $p$ in the power nonlinearity was defined by the exponent $p_{Fuj}(\frac{n}{m \max\{1 - \theta; \theta\}})$. For this reason we compare in system (1.6) the exponents $p$ and $q$ with $p_{Fuj}(\frac{n}{m \max\{1 - \theta; \theta\}})$. If the exponents $p$ and $q$ in the power nonlinearities are greater than $p_{Fuj}(\frac{n}{m \max\{1 - \theta; \theta\}})$, then we can prove the existence of solution separately for each equation of the system. Hence, we are interested in the case where only one exponent is above $p_{Fuj}(\frac{n}{m \max\{1 - \theta; \theta\}})$. We shall prove a global (in time) existence result with a loss of decay and the exponents $p$, $q$ satisfying the following condition:

$$\alpha_{\max}(m, \theta) = m \max\{1 - \theta; \theta\} \left(\max\{p; q\} + 1\right) < \frac{n}{2}.$$  

A similar condition was required in papers [30] and [26] which is

$$\alpha_{\max}(1, 1) = \max\{p; q\} + 1 < \frac{n}{2}.$$  

As an effect of different power source nonlinearities having different influences, we allow a loss of decay comparing with the decay of the solution to the corresponding linear problem for each equation. Now we present some of our results for the system (1.6).
Theorem 5.1. Let $n \leq \frac{2m^2 \max\{1, -\theta\}}{2m - 1}$ and $n < \frac{2m \max\{1, -\theta\}}{m \max\{1, -\theta\} - 1}$ if $m > \frac{1}{\max\{1, -\theta\}}$ where $\theta \in [0, 1]$. Data $(u_0, u_1)$, $(v_0, v_1)$ are supposed to belong to the function space $D^0_m \times D^0_m$ with $m \in [1, 2)$. The exponents $p$ and $q$ satisfy

$$\frac{2}{m} \leq \min\{p; q\} < p_{Fuj} \left( \frac{n}{m \max\{1, -\theta\}} \right) < \max\{p, q\} < \infty \quad \text{if} \quad n \leq 2,$$

$$\frac{2}{m} \leq \min\{p; q\} < p_{Fuj} \left( \frac{n}{m \max\{1, -\theta\}} \right) < \frac{n}{n - 2} \quad \text{if} \quad n \geq 3,$$

and

$$\alpha_{\max}(m, \theta) = m \max\{1, -\theta\} \left( \frac{\max\{p, q\} + 1}{pq - 1} \right) < \frac{n}{2}.$$

(5.1)

(5.2)

There exists a small constant $\varepsilon_0$ such that if $\|u_0\|_{D^m_0} + \|v_0\|_{D^m_0} \leq \varepsilon_0$, then there exists a uniquely determined globally (in time) Sobolev solution to (1.6) such that

$$(u, v) \in (C([0, \infty), H^1))^2.$$

Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2 - m)}{m \max\{1, -\theta\}} + \epsilon|\gamma_{n, m, \theta, \epsilon}(p)|}\left( \|u_0\|_{D^m_0} + \|v_0\|_{D^m_0} \right),$$

$$\|\nabla u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2 - m)}{m \max\{1, -\theta\}} + \epsilon|\gamma_{n, m, \theta, \epsilon}(p)|}\left( \|u_0\|_{D^m_0} + \|v_0\|_{D^m_0} \right),$$

$$\|v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2 - m)}{m \max\{1, -\theta\}} + \epsilon|\gamma_{n, m, \theta, \epsilon}(q)|}\left( \|u_0\|_{D^m_0} + \|v_0\|_{D^m_0} \right),$$

$$\|\nabla v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{- \frac{n(2 - m)}{m \max\{1, -\theta\}} + \epsilon|\gamma_{n, m, \theta, \epsilon}(q)|}\left( \|u_0\|_{D^m_0} + \|v_0\|_{D^m_0} \right),$$

where $\epsilon$ defined in Theorem 4.1 and $\gamma_{n, m, \theta, \epsilon}(p) = 1 - \frac{n(p - 1)}{2m \max\{1, -\theta\}} + \epsilon$ (resp. $\gamma_{n, m, \theta, \epsilon}(q) = 1 - \frac{n(q - 1)}{2m \max\{1, -\theta\}} + \epsilon$) with an arbitrarily small positive number $\epsilon$ is the loss of decay in comparison with the corresponding decay estimates for the solutions $u$ (resp. $v$) of the corresponding linear Cauchy problems with vanishing right-hand sides.

Remark 5.1. If we have $\min\{p, q\} = p_{Fuj} \left( \frac{n}{m \max\{1, -\theta\}} \right)$ in condition (5.1), then we get an arbitrarily small loss of decay generated by the logarithmic term.

Remark 5.2. If we take the data from the function space $D^0_{m_1} \times D^0_{m_2}$ with $m_1 \neq m_2$, we can prove a similar result as Theorem 5.1, where the exponents $p$ or $q$ can be below or equal to the modified Fujita exponent $p_{Fuj} \left( \frac{n}{m \max\{1, -\theta\}} \right)$ without any loss of decay. For more details the reader can see [21].

5.1.2. Data from Sobolev spaces with suitable regularity

Let us consider now data with high regularity but not embedded in $L^\infty$. From Section 4 we remark that the upper bound for the exponents of power nonlinearities can be dominated by the regularity of data only for $s \geq 2$. Then, we restrict data with this case with different regularities of data $s_1 \neq s_2$. For $s \in (0, 2)$ we prove a similar results as Theorem 5.1 if $p_{Fuj} \left( \frac{n}{m \max\{1, -\theta\}} \right) > 1 + [s]$.

Theorem 5.2. Let $n \geq 4$, $s_1, s_2 \in \left[2, \frac{2}{n} + 1\right], 0 < s_2 - s_1 < 1$ and $[s_1] \neq [s_2]$. The data $(u_0, u_1), (v_0, v_1)$ are supposed to belong to the function space $D_{m, s_1} \times D_{m, s_2}$, where $m \in [1, 2)$. Furthermore, we assume for the exponents $p$ and $q$ the conditions

$$[s_1] < p, \quad [s_2] < q \quad \text{if} \quad n \leq 2s_1,$$

$$[s_1] < p, \quad [s_2] < q \leq 1 + \frac{2}{n - 2s_1} \quad \text{if} \quad 2s_1 < n \leq 2s_2,$$

$$[s_1] < q \leq 1 + \frac{2}{n - 2s_2}, \quad [s_2] < q \leq 1 + \frac{2}{n - 2s_1} \quad \text{if} \quad n > 2s_2.$$

(5.3)
Then, there exists a constant $\varepsilon_0 > 0$ such that if $\|(u_0, u_1)\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m} \leq \varepsilon_0$, then there exists a uniquely determined globally (in time) Sobolev solution to (1.6) such that

$$(u, v) \in \mathcal{C}([0, \infty), H^{s_1+1}) \times \mathcal{C}([0, \infty), H^{s_2+1}).$$

Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|D^{s_1+1} u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)+2m(s_1+1)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|v(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|D^{s_2+1} v(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)+2m(s_2+1)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

where $\varepsilon = 0$ if $\theta \in [0, 1/2) \cup (1/2, 1]$ and $\varepsilon$ is positive and sufficiently small if $\theta = 1/2$.

**Example 5.1.** We consider the problem (1.6) with $\theta = 1/2$ in four-dimensional case with data belonging to $\mathcal{D}^0_{4/2} \times \mathcal{D}^0_{4/2}$. If $p = 5/4$, $q = 8/5$, then the solution to (1.6) exists globally.

5.1.3 Large regular data

In the following results we consider data with different higher regularities even embedded in $L^\infty$.

**Theorem 5.3.** Let $n \geq 4$ The data $(u_0, u_1), (v_0, v_1)$ are supposed to belong to the function space $\mathcal{D}_{m,s_1} \times \mathcal{D}_{m,s_2}$ with $m \in [1,2)$ and $s_2 \geq s_1 > \frac{n}{2} + 1$. Moreover, we assume for the exponents $p$ and $q$ the conditions

$$p > s_1 \quad \text{and} \quad q > \tilde{s}_2,$$

where $\tilde{s}_2 \in [s_1, s_1 + 1]$ and $\tilde{s}_2 < s_2$.

Then, there exists a constant $\varepsilon_0 > 0$ such that if $\|(u_0, u_1)\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m} \leq \varepsilon_0$, then there exists a uniquely determined globally (in time) Sobolev solution to (1.6) such that

$$(u, v) \in \mathcal{C}([0, \infty), H^{s_1+1}) \times \mathcal{C}([0, \infty), H^{s_2+1}).$$

Moreover, the solution satisfies the estimates

$$\|u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|D^{s_1+1} u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)+2m(s_1+1)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|v(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

$$\|D^{s_2+1} v(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n(2-m)+2m(s_2+1)}{4m \max(1-p, m)}} \varepsilon_0 \|u_0, u_1\|_{\mathcal{D}^{s_1}_m} + \|(v_0, v_1)\|_{\mathcal{D}^{s_2}_m},$$

where $\varepsilon = 0$ if $\theta \in [0, 1/2) \cup (1/2, 1]$ and $\varepsilon$ is positive and sufficiently small if $\theta = 1/2$.

5.2. Philosophy of our approach

We define the norm of the solution space $X(T)$ by

$$\|(u, v)\|_{X(T)} = \sup_{0 \leq t \leq T} (M_1(t, u) + M_2(t, v),$$

where for each theorem we choose suitable $M_1(t, u)$ and $M_2(t, v)$. We introduce the operator $N$ by

$$N : (u, v) \in X(T) \rightarrow N(u, v) = (u^{in} + u^{id}, v^{in} + v^{id}),$$
where
\[ u^0(t, x) := K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x), \]
\[ u^n(t, x) := \int_0^t K_1(t - \tau, x) *_{(x)} |v(\tau, x)|^p d\tau, \]
\[ v^0(t, x) := K_0(t, x) *_{(x)} v_0(x) + K_1(t, x) *_{(x)} v_1(x), \]
\[ v^n(t, x) := \int_0^t K_1(t - \tau, x) *_{(x)} |u(\tau, x)|^q d\tau. \]

Our aim is to prove the following inequalities which imply among other things the global (in time) existence of small data solutions:
\[ \|N(u, v)\|_{X(T)} \leq \|(u_0, u_1)\|_{D^s_{\alpha}} + \|(v_0, v_1)\|_{D^s_{\alpha}} + \|(u, v)\|_{X(T)}^p + \|(u, v)\|_{X(T)}^q, \quad (5.4) \]
\[ \|N(u, v) - N(\bar{u}, \bar{v})\|_{X(T)} \leq \|(u, v) - (\bar{u}, \bar{v})\|_{X(T)} \times \left( \|(u, v)\|_{X(T)}^{p-1} + \|(\bar{u}, \bar{v})\|_{X(T)}^{p-1} + \|(u, v)\|_{X(T)}^{q-1} + \|(\bar{u}, \bar{v})\|_{X(T)}^{q-1} \right). \quad (5.5) \]

From the definition of the norm of the solution space \( X(T) \), we can immediately obtain
\[ \|(u^n, v^n)\|_{X(T)} \leq \|(u_0, u_1)\|_{D^s_{\alpha}} + \|(v_0, v_1)\|_{D^s_{\alpha}}. \]

We complete the proof of all results separately by showing the inequality
\[ \|(u^n, v^n)\|_{X(T)} \leq \|(u, v)\|_{X(T)}^p + \|(u, v)\|_{X(T)}^q, \quad (5.6) \]
which lead to the desired estimate (5.4).

Without loss of generality, we consider in the proof only the case \( p < q \).

5.3. Sketch of the proof of Theorem 5.1

For the same reason explained in the proof of Theorem 4.1, we only prove the case \( \theta \in [0, 1/2) \cup (1/2, 1] \).

Let \( s_1 = s_2 = 0 \) and the solutions space
\[ X(T) = (C([0, T], H^1))^2 \]
with
\[ M_1(t, u) = (1 + t)^{-\frac{n(m - s)}{2m}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{n(2 - m)}{2m} + 2m} \|\nabla_x u(t, \cdot)\|_{L^2}, \]
\[ M_2(t, v) = (1 + t)^{\frac{n(m - s)}{2m} \|v(t, \cdot)\|_{L^2} + (1 + t)^{n(2 - m) + 2m} \|\nabla_x v(t, \cdot)\|_{L^2}. \]

To prove (5.6) we have to control all components of the norm with respect to \( u^n = u^0(t, x) \) and \( v^n = v^0(t, x) \). For \( u^n \) we use the estimates, which are proved in Theorem 3.3 with \( m = 2 \) for the integral over \([t/2, t] \), we obtain
\[ \|u^n(t, \cdot)\|_{L^2} \leq \int_0^t (1 + t - \tau)^{-\frac{n(2 - m)}{2m} \|v(\tau, x)\|_{L^2}} d\tau \]
\[ + \int_t^{t/2} \|v(\tau, x)\|_{L^2} d\tau. \]

Using the Gagliardo-Nirenberg inequality we get
\[ \|v(\tau, x)\|_{L^m} \leq (1 + \tau)^{-\frac{n(p-1)}{2m(1 - \theta p)}} \|(u, v)\|_{X(\tau)}^p, \]
\[ \|v(\tau, x)\|_{L^2} \leq (1 + \tau)^{-\frac{n(2p - m)}{2m(1 - \theta p)}} \|(u, v)\|_{X(\tau)}^p, \]
\[ \|u(\tau, x)\|_{L^2} \leq (1 + \tau)^{-\frac{n(2 - m)}{2m(1 - \theta p)}} \|(u, v)\|_{X(\tau)}^2. \]
where we use condition (5.1). Plugging these estimates implies
\[ ||u^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} ||(u, v)||_{X(t)}^{\frac{q}{2}} \int_0^{t/2} (1 + \tau)^{-\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) d\tau \]
\[ + (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} ||(u, v)||_{X(t)}^{\frac{q}{2}} \int_0^{t/2} (1 + \tau)^{-\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) d\tau \]
\[ \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

Then,
\[ ||u^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

(5.7)

Analogously, we obtain
\[ ||\nabla x u^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)+2m}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

(5.8)

For \( v^{nl} \) we use also the estimates proved in Theorem 3.3 with \( m = 2 \) for the integral over \([t/2, t]\) to obtain
\[ ||v^{nl}(t,\cdot)||_{L^2} \lesssim \int_0^{t/2} (1 + t - \tau)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} ||(u(\tau, x)||_{L^2}^{\frac{q}{2}} d\tau + \int_{t/2}^t ||u(\tau, x)||_{L^2}^{\frac{q}{2}} d\tau. \]

The classical Gagliardo-Nirenberg inequality leads to
\[ ||u(\tau, x)||_{L^2}^{\frac{q}{2}} \lesssim (1 + \tau)^{\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]
\[ ||u(\tau, x)||_{L^2}^{\frac{q}{2}} \lesssim (1 + \tau)^{\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

where we use our assumption (5.1) again. Consequently, we obtain
\[ ||v^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}} \int_0^{t/2} (1 + \tau)^{-\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) d\tau \]
\[ + (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}} \int_0^{t/2} (1 + \tau)^{-\frac{n(q-1)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) d\tau \]
\[ \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

where \(-\frac{n(q-1)}{2m \max(1-\theta,\theta)} + q(\gamma_{n,m,\theta}(p)) < -1 \) which is equivalent to (5.2). The, we have
\[ ||v^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

(5.9)

Analogously, we prove
\[ ||\nabla x v^{nl}(t,\cdot)||_{L^2} \lesssim (1 + t)^{\frac{n(2-n)+2m}{2m \max(1-\theta,\theta)}} + q(\gamma_{n,m,\theta}(p)) ||(u, v)||_{X(t)}^{\frac{q}{2}}. \]

(5.10)

From (5.7) to (5.10) we obtain the desired inequality (5.6). The proof of (5.5) is completely analogous to the proof of (5.6) by using the same steps from the proof of (4.4). In this way, we complete the proof.

5.4. Sketch of the proof of Theorem 5.2

Let us define the evolution space for \( T > 0 \) as follows:
\[ X(T) = C([0, T], H^{s_1+1}) \times C([0, T], H^{s_2+1}) \]
with the norm
\[ ||(u, v)||_{X(T)} = \sup_{0 \leq t \leq T} (M_1(1, t, u) + M_2(1, t, v)), \]
\[ 21 \]
Moreover, we need to estimate the nonlinear terms also in homogeneous Sobolev spaces with high regularity.

As we did in the proof of previous theorem, especially the inequality (5.6), we estimate all terms making up of the norm \( \| (u^n, v^0) \|_{X(T)} \). Using the Gagliardo-Nirenberg inequality analogously to (4.12) we can get for \( 0 \leq \tau \leq t \)

\[
\| v(\tau, x) \|^p_{L^m} \lesssim (1 + \tau)^{\frac{n(p-1)}{2m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^p,
\]

\[
\| u(\tau, x) \|^q_{L^m} \lesssim (1 + \tau)^{\frac{n(q-1)}{2m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^q,
\]

(5.11)

Moreover, we need to estimate the nonlinear terms also in homogeneous Sobolev spaces with high regularity. Then, using the fractional chain rule, analogously to (4.15) we have

\[
\| v(\tau, x) \|^p_{H^{s_1}} \lesssim \| (v(\tau, \cdot))_{L^{s_1}} \|_{L^p_{x_s}} \| (v(\tau, \cdot))_{H^{s_1}} \|_{L^p_{x_s}}\]

where \( \frac{p-1}{q} + \frac{1}{q} = \frac{1}{2} \) and \( p \equiv \lfloor s_1 \rfloor \). The application of the fractional Gagliardo-Nirenberg inequality yields

\[
\| v(\tau, \cdot) \|_{L^{s_1}} \lesssim \| (v(\tau, \cdot))_{L^p} \|^{1-\beta_{s_1, s_2+1}(q_1)}_{L^q} \| (v(\tau, \cdot))_{H^{s_1}} \|^{\beta_{s_1, s_2+1}(q_1)}_{H^{s_1}} ,
\]

\[
\| v(\tau, \cdot) \|_{H^{s_1}_{x_s}} \lesssim \| (v(\tau, \cdot))_{H^{s_1}} \|^{1-\beta_{s_1, s_2+1}(q_2)}_{L^q} \| (v(\tau, \cdot))_{H^{s_1}} \|^{\beta_{s_1, s_2+1}(q_2)}_{H^{s_1}} ,
\]

where \( s_1 - s_2 < 1 \) and

\[
\beta_{s_1, s_2+1}(q_1) = \frac{n}{s_2 + 1} \left( \frac{1}{2} - \frac{1}{q_1} \right) \in [0, 1], \quad \beta_{s_1, s_2+1}(q_2) = \frac{n}{s_2 + 1} \left( \frac{1}{2} - \frac{1}{q_2} + \frac{s_1}{n} \right) \in \left[ \frac{s_1}{s_2 + 1}, 1 \right].
\]

Consequently, under the condition (5.3) assumed in the statement we get for \( 0 \leq \tau \leq t \)

\[
\| v(\tau, x) \|^p_{H^{s_1}} \lesssim (1 + \tau)^{\frac{n(p-1)}{2m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^p,
\]

\[
\| u(\tau, x) \|^q_{H^{s_2}} \lesssim (1 + \tau)^{\frac{n(q-1)}{2m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^q,
\]

(5.12)

Plugging these estimates in the inequalities derived from Corollary 3.1, we get similarly to (4.16) and (4.17) for \( 0 \leq \tau \leq t \) the following estimates:

\[
\| u^n(t, \cdot) \|_{L^2} \lesssim (1 + t)^{\frac{n(2-m)\beta_{s_1, s_2+1}}{4m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^p,
\]

\[
\| D^{s_1+1} u^n(t, \cdot) \|_{L^2} \lesssim (1 + t)^{\frac{n(2-m)\beta_{s_1, s_2+1}}{4m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^p,
\]

\[
\| u^n(t, \cdot) \|_{L^2} \lesssim (1 + t)^{\frac{n(2-m)\beta_{s_1, s_2+1}}{4m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^q,
\]

\[
\| D^{s_2+1} u^n(t, \cdot) \|_{L^2} \lesssim (1 + t)^{\frac{n(2-m)\beta_{s_1, s_2+1}}{4m(1+\beta_0)}} \| (u, v) \|_{X(\tau)}^q.
\]

The proof is completed.

5.5. Sketch of the proof of Theorem 5.3

Due to the Sobolev embedding \( H^{s_2} \hookrightarrow \dot{H}^{\tilde{s}_2} \) for \( \tilde{s}_2 < s_2 \), we know that \( \| u(\tau, x) \|^q_{\dot{H}^{\tilde{s}_2}} \) is well-defined. Then, we assume a loss of regularity in the evolution space as follows:

\[
X(T) = C([0, T], H^{s_1+1}) \times C([0, T], H^{s_1+1})
\]

22
with the norm
\[ \|(u, v)\|_{X(T)} = \sup_{0 \leq t \leq T} \left( M_1(t, u) + M_2(t, v) \right), \]
where
\[ M_1(t, u) = (1 + t)^{\frac{n(2\alpha - m)}{m + (1 - 2\alpha)\beta}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{n(2\alpha - m) + 2\alpha + 1}{4m + (1 - 2\alpha)\beta}} \|D^{\frac{1}{2}}u(t, \cdot)\|_{L^2}, \]
\[ M_2(t, v) = (1 + t)^{\frac{n(2\alpha - m)}{m + (1 - 2\alpha)\beta}} \|v(t, \cdot)\|_{L^2} + (1 + t)^{\frac{n(2\alpha - m) + 2\alpha + 1}{4m + (1 - 2\alpha)\beta}} \|D^{\frac{1}{2}}v(t, \cdot)\|_{L^2}. \]

We repeat the same steps of the previous proof of Theorem 5.2, but now we use the following estimates by the fractional powers with \(2s < n\) to obtain for \(0 \leq \tau \leq t\)
\[ \|v(\tau, x)|^p\|_{H^\tau} \lesssim (1 + \tau)^{\frac{n(2\alpha - m) + 2\alpha s + (p - 1) + 2m_{\alpha}}{4m + (1 - 2\alpha)\beta}} \|(u, v)\|_{X(t)}, \]
and
\[ \|u(\tau, x)|^q\|_{H^\tau} \lesssim (1 + \tau)^{\frac{n(2\alpha - m) + 2\alpha s + (p - 1) + 2m_{\alpha}}{4m + (1 - 2\alpha)\beta}} \|(u, v)\|_{X(t)}, \]
where \(\tilde{s}_\tau \in [s_1, s_1 + 1]\) is used. In this way we complete the proof.

A. Tools from Harmonic Analysis

In this section we present some tools from Harmonic Analysis.

**Proposition A.1.** (Classical Gagliardo-Nirenberg inequality) Let \(j, m \in \mathbb{N}\) with \(j < m\), and let \(f \in C_0^{m}\). Let \(\beta = \beta_{j, m} \in [\frac{1}{m}, 1]\) with \(p, q, r \in [1, \infty]\) satisfy
\[ j - \frac{n}{q} = (m - \frac{n}{r})\beta - \frac{n}{p}(1 - \beta). \]
Then, we have the following inequality:
\[ \|D^j f\|_{L^q} \lesssim \|f\|^{1 - \beta}_{L^p} \|D^m f\|^{\beta}_{L^r}, \]
provided that \((m - \frac{n}{r}) - j \notin \mathbb{N}\). If \((m - \frac{n}{r}) - j \in \mathbb{N}\), then the classical Gagliardo-Nirenberg inequality holds provided that \(\beta \in [\frac{1}{m}, 1]\).

The proof of the classical Gagliardo-Nirenberg inequality can be found in [6] and [1, 7, 8, 9, 15, 16].

**Proposition A.2.** (Fractional Gagliardo-Nirenberg inequality) Let \(p, p_0, p_1 \in (1, \infty)\) and \(\kappa \in (0, s)\) with \(s > 0\). Then, for all \(f \in L^{p_0} \cap \dot{H}^\kappa\), the following inequality holds:
\[ \|f\|_{\dot{H}^\kappa} \lesssim \|f\|^{1 - \beta}_{L^{p_0}} \|f\|^{\beta}_{\dot{H}^\kappa}, \]
where \(\beta = \beta_{\kappa, s} = (\frac{1}{p_0} - \frac{1}{p_1} + \frac{s}{m}) / (\frac{1}{p_0} - \frac{1}{p_1} + \frac{s}{m})\) and \(\beta \in [\frac{s}{m}, 1]\).

The proof of this result can be found in [10].

**Proposition A.3.** (Fractional Leibniz rule) Let \(s > 0\) and \(1 \leq r \leq \infty\), \(1 < p_1, p_2, q_1, q_2 \leq \infty\) satisfy the relation
\[ \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}. \]
Then, for all \(f \in \dot{H}^\kappa \cap L^{q_2}\) and \(g \in \dot{H}^\kappa \cap L^{q_2}\) the following inequality holds:
\[ \|fg\|_{\dot{H}^\kappa} \lesssim \|f\|_{\dot{H}^\kappa} \|g\|_{L^{q_2}} + \|f\|_{L^{q_1}} \|g\|_{\dot{H}^\kappa}. \]
The proof of this inequality can be found in [8].

**Proposition A.4.** (Fractional chain rule) Let $s > 0$, $p > \lceil s \rceil$ and $1 < r, r_1, r_2 < \infty$ satisfy the relation

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Then, for all $f \in \dot{H}_r^s \cap L^{r_1}$ the following inequality holds:

$$\| \pm |f|^{p-1}f \|_{\dot{H}_r^s} + \| |f|^p \|_{\dot{H}_r^s} \lesssim \| f \|_{L^{r_1}}^{p-1} \| f \|_{\dot{H}_r^s}.$$

One can find the proof in [27].

**Proposition A.5.** (Fractional powers) Let $r \in (1, \infty)$, $p > 1$ and $s \in (0, p).$ Then, for all $f \in \dot{H}_r^s \cap L^\infty$ the following inequality holds:

$$\| \pm |f|^{p-1}f \|_{\dot{H}_r^s} + \| |f|^p \|_{\dot{H}_r^s} \lesssim \| f \|_{L^{r}}^{p-1}.$$

**Proposition A.6.** Let $r \in (1, \infty)$ and $s > 0$. Then, for all $f, g \in \dot{H}_r^s \cap L^\infty$ the following inequality holds:

$$\| fg \|_{\dot{H}_r^s} \lesssim \| f \|_{\dot{H}_r^s} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{\dot{H}_r^s}.$$

The above two propositions with their proofs can be found in [29].

**Proposition A.7.** Let $0 < 2s^* < n < 2s$. Then for any function $f \in \dot{H}_r^{s^*} \cap \dot{H}_r^s$ one has the estimate

$$\| f \|_{L^\infty} \leq \| f \|_{\dot{H}_r^{s^*}} + \| f \|_{\dot{H}_r^s}.$$

For the proof one can see [2].

**Acknowledgments**

The first author Mr. Wenhui Chen is supported by Sächsisches Landesgraduiertenstipendium. The second author Mr. Mohammed Djaotui Abdelhamid is supported by German Academic Exchange Service, DAAD (Personal identification number: 91524991). The authors express a sincere thankfulness to their supervisor Prof. Michael Reissig for numerous discussions and the Institute of Applied Analysis for their hospitality.

**References**

[1] F. Christ, M. Weinstein, Dispersion of small-amplitude solutions of the generalized korteweg-de vries Equation, J. Funct. Anal. 100 (1991) 87–109.

[2] M. D’Abbicco, M.R. Ebert, S. Lucente, Self-similar asymptotic profile of the solution to a nonlinear evolution equation with critical dissipation, Math. Methods Appl. Sci. 40 (18) (2017) 6480–6494. https://doi.org/10.1002/mma.4469

[3] P.M.N. Dharmawardane, J.E.M. Rivera, S. Kawashima, Decay property for second order hyperbolic systems of viscoelastic materials, J. Math. Anal. Appl. 366 (2) (2010) 621–635. https://doi.org/10.1016/j.jmaa.2009.12.019

[4] P.M.N. Dharmawardane, T. Nakamura, S. Kawashima, Global solutions to quasi-linear hyperbolic systems of viscoelasticity, Kyoto J. Math. 51 (2) (2011) 467–483.

[5] P.M.N. Dharmawardane, T. Nakamura, S. Kawashima, Decay estimates of solutions for quasi-linear hyperbolic systems of viscoelasticity, SIAM J. Math. Anal. 44 (3) (2012) 1976–2001. https://doi.org/10.1137/11083900X

[6] A. Friedman, Partial differential Equations, (Correct reprint of the original edition) Robert E. Krieger Publishing Co., Huntington, N. Y. (1976).

[7] L. Grafakos, Classical and modern Fourier analysis, Prentice Hall, 2004.

[8] L. Grafakos, S. Oh, The Kato-Ponce inequality, Comm. Partial Differential Equations 39 (6) (2014) 1128–1157. https://doi.org/10.1080/03605302.2013.822885

[9] A. Gulisashvili, M. Kon, Exact smoothing properties of Schrödinger semigroups, Amer. J. Math. 118 (1996) 1215–1248.

[10] H. Hajajie, L. Molinet, T. Ozawa, B. Wang, Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations, Harmonic Analysis and Nonlinear Partial Differential Equations RIMS Kökyüroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto (2011) 159–175.
[11] R. Ikehata, K. Tanizawa, Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^N$ with noncompactly supported initial data, Nonlinear Anal. 61 (7) (2005) 1189–1208. https://doi.org/10.1016/j.na.2005.01.097
[12] M. Kafini, S.A. Messaoudi, On the uniform decay in viscoelastic problems in $\mathbb{R}^n$, Appl. Math. Comput. 215 (3) (2009) 1161–1169. https://doi.org/10.1016/j.amc.2009.06.058
[13] M. Kafini, S.A. Messaoudi, A blow-up result for a viscoelastic system in $\mathbb{R}^n$, Electron. J. Differential Equations 113 (2017) 1–7.
[14] M. Kafini, M. Mustafa, On the stabilization of a non-dissipative Cauchy viscoelastic problem, Mediterr. J. Math. 13 (6) (2016) 5163–5176.
[15] T. Kato, G. Ponce, Well-posedness and scattering results for the generalized Kortewegde-Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (4) (1993) 527–629.
[16] C.E. Kenig, G. Ponce, L. Vega, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. 41 (1988) 891–907.
[17] I. Lasiecka, S.A. Messaoudi, M.I. Mustafa, Note on intrinsic decay rates for abstract wave equations with memory, J. Math. Phys. 54 (3) (2013) https://doi.org/10.1063/1.4793988
[18] Y. Liu, Decay of solutions to an inertial model for a semilinear plate equation with memory, J. Math. Anal. Appl. 394 (2) (2012) 616–632. https://doi.org/10.1016/j.jmaa.2012.04.003
[19] Y. Liu, S. Kawashima, Decay property for a plate equation with memory-type dissipation, Kinet. Relat. Models 4 (2) (2011) 531–547. doi: 10.3934/krm.2011.4.531
[20] A. Mohammed Djaouti, M. Reissig, Weakly coupled systems of semilinear effectively damped waves with time-dependent coefficient, different power nonlinearities, Nonlinear Anal. 175 (2018) 28–55. https://doi.org/10.1016/j.na.2018.05.006
[21] A. Mohammed Djaouti, On the benefit of different additional regularity for the weakly coupled systems of semilinear effectively damped waves, Mediterr. J. Math. 15 (3) (2018) 115. https://doi.org/10.1007/s00009-018-1173-1.
[22] J.E. Muñoz Rivera, M.G. Naso, F.M. Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory, J. Math. Anal. Appl. 286 (2) (2003) 692–704. https://doi.org/10.1016/S0022-247X(03)00511-0
[23] J.E. Muñoz Rivera, M.G. Naso, E. Vuk, Asymptotic behaviour of the energy for electromagnetic systems with memory, Math. Methods Appl. Sci. 27 (7) (2004) 819–841. https://doi.org/10.1002/mma.473
[24] J.E. Muñoz Rivera, M.G. Naso, Optimal energy decay rate for a class of weakly dissipative second-order systems with memory, Appl. Math. Lett. 23 (7) (2010) 745–746. https://doi.org/10.1016/j.aml.2010.02.016
[25] T. Narazaki, Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations, Discrete Contin. Dyn. Syst. (2009) 592–601.
[26] K. Nishihara, Y. Wakasugi, Critical exponent for the Cauchy problem to the weakly coupled damped wave system, Nonlinear Anal. 108 (2014) 249–259. https://doi.org/10.1016/j.na.2014.06.001
[27] A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation, II, Math. Nach. (2018) 1–34. https://doi.org/10.1002/mana.201700144
[28] R. Racke, B. Said-Houari, Decay rates for semilinear viscoelastic systems in weighted spaces, J. Hyperbolic Differ. Equ. 9 (1) (2012) 67–103. https://doi.org/10.1142/S0219891612500026
[29] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications, 3, Walter de Gruter & Co., Berlin, 1996.
[30] F. Sun, M. Wang, Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping, Nonlinear Anal. 66 (12) (2007) 2989–3010. https://doi.org/10.1016/j.na.2006.04.012
[31] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2) (2001) 464–489. https://doi.org/10.1006/jdeq.2000.3933
[32] Q.S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math. 333 (2) (2001) 109–114. https://doi.org/10.1016/S0764-4442(01)01999-1