Spinor Field Realizations of the half-integer $W_{2,s}$ Strings

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Abstract: The grading Becchi-Rouet-Stora-Tyutin (BRST) method gives a way to construct the integer $W_{2,s}$ strings, where the BRST charge is written as $Q_B = Q_0 + Q_1$. Using this method, we reconstruct the nilpotent BRST charges $Q_0$ for the integer $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings. Then we construct the exact grading BRST charge with spinor fields and give the new realizations of the half-integer $W_{2,s}$ strings for the cases of $s = 3/2$, $5/2$, and $7/2$.

Keywords: $W_{2,s}$ strings, BRST charge, Spinor realization

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1. Introduction

$W$ algebra [1, 2] was discovered in two-dimensional field theories in the middle of the 1980’s, much work has been focused on the classification of it and on the study of $W$ gravity and $W$ string theories. Furthermore, $W$ algebra plays a central role in many areas of two-dimensional physics. It appears in the quantum Hall effect [3] and black holes [4, 5], in lattice models of statistical mechanics at criticality, and in other physical models [6, 7] and so on.

In all applications of $W$ algebra, the investigation of the $W$ strings is more interesting and important. The idea of building $W$ string theories was first developed in Ref. [8]. Since $W_3$ is the simplest $W$ algebra, most of the efforts in constructing $W$ string theories have been concentrated on it [9, 10, 11, 12, 13, 14, 15, 16]. Using the method of Lagrangian Realization and Hamiltonian Reduction, other strings and superstrings also had been studied in [17, 18, 19, 20]. Later Pope et al. discovered that the Becchi-Rouet-Stora-Tyutin (BRST [21]) method provides the only viable approach to the quantization of $W$ string theories. Using a kind of canonical transformation [11], the BRST operator could be written in the form of $Q_B = Q_0 + Q_1$, which is graded. Using this method, the scalar field realizations of $W_{2,s}$ strings have been obtained for $s=3, 4, 5, 6, 7$ [22, 23]. Subsequently, the methods to construct the spinor field realization of critical $W_{2,s}$ strings and $W_N$ strings were also found in Refs. [24, 25]. Assuming the BRST charges of the $W_{2,s}$ strings and $W_N$ strings are graded, the exact spinor field realizations of $W_{2,s}(s = 3, 4, 5, 6)$ strings and $W_N(N = 4, 5, 6)$ strings...
were obtained [24, 25, 26, 27]. Recently, we constructed the nilpotent BRST charges of the spinor non-critical $W_{2,s}$ (s=3,4) strings by taking account of the property of spinor field [28]. It was shown that certain integer $W_{2,s}$ (s=3,4) algebras can be linearized as the $W_{1,2,s}$ algebras by the inclusion of a spin-1 current. This provides a way of obtaining new realizations of the integer $W_{2,s}$ strings. Based on the linear $W_{1,2,s}$ algebras, the ghost field realizations and the spinor realizations of the spinor non-critical $W_{2,s}$ strings were given in Refs. [29, 30].

The half-integer $W_{2,s}$ algebra with $s = 3/2$ is known as the $N = 1$ superconformal algebra or $SW(3/2,2)$ superconformal algebra. Recently, several constructions of the algebra were obtained in Refs. [31, 32]. Much research had been done also on $N = 2$ superconformal algebra, in which another $s = 3/2$ current is added [33, 34, 35, 36]. Correspondingly, the $N = 1$ and $N = 2$ superstrings had been investigated in [37, 38]. In addition, the case of $s = 5/2$ was investigated in Ref. [39]. It was found that this algebra can not be closed with spin-2 current $T$ and spin-5/2 current $G$ only, and so there is no realization for this algebra. But by introducing other spin currents, this algebra may be linearized [10], which implies that the realizations of the algebra could be found. The $W_{2,s}$ ($s = 7/2, 9/2, 11/2, 13/2, 15/2$) algebras were also studied in Ref. [41], in which the values of central charges for these algebras were given.

Since up to now there is no work focused on the research of spinor field realizations of the half-integer $W_{2,s}$ strings, we will construct the nilpotent BRST charges of spinor half-integer $W_{2,s}$ strings for the first time by using the grading BRST method. This method provides a more easy way to construct $W_{2,s}$ algebras and strings. And the physical states can be obtained by the grading form of BRST operators for these strings. Firstly, we reconstruct the BRST charges $Q_0$ for integer $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings. For each case we obtain four solutions. All these solutions indicate that we can construct the BRST charges $Q_B$ for both the integer $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings by using scalar field $\varphi$ together with spinor field $\psi$. But for simplicity, we only consider the realizations with spinor field $\psi$. With these solutions, we discuss the exact BRST charges for $s = 3/2, 5/2, 7/2$ in detail and we then find that many new valuable solutions will be obtained with increasing of spin $s$. All these results will be of importance for embedding the Virasoro string into the $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings, and they may provide the essential ingredients to help us better understanding the fundamental properties of the half-integer $W_{2,s}$ strings. Furthermore, after giving the explicit realizations and BRST charges, the physical states of these $W$ strings can be investigated.

This paper is organized as follows: in Section 2 we give a brief review of the grading BRST method for scalar field and spinor field realizations of the integer $W_{2,s}$ strings. In Section 3 we reconstruct the general BRST charges $Q_0$ for the integer and the half-integer $W_{2,s}$ strings. Using these results, we construct the BRST charges $Q_B$ for $W_{2,s}$ ($s = 3/2, 5/2, 7/2$) in Section 4. Finally the conclusions are drawn in the last section.

2. Review of the grading BRST method

In 1993, Pope et. al. [11] discovered a kind of canonical transformation that leads a
considerable simplification of the BRST operator and the physical states of $W_3$ string. This is a nonlinear realization involving a scalar field $\varphi$ whilst the spin-2 and spin-3 ghost fields are introduced. Then the BRST operator could be written in the form of $Q_B = Q_0 + Q_1$, where $Q_0$ has grade $(1,0)$ and $Q_1$ has grade $(0,1)$, with $(p,q)$ denoting the grading of an operator with ghost number $p$ for the redefined spin-2 $(b, c)$ ghost system and ghost number $q$ for the redefined spin-$s$ $(\beta, \gamma)$ ghost system. In particular $Q_1$ only involves $\varphi, \beta, \gamma$. This leads to an immediate generalization, the $W_{2, s}$ strings [23, 42], whose BRST operator has the similar form except that the $(\beta, \gamma)$ system is a spin-$s$ current rather than a spin-3 current.

Following Refs. [23, 42], the BRST operator for the scalar field realizations of $W_{2, s}$ strings can be written as

$$Q_B = Q_0 + Q_1, \quad (2.1)$$

$$Q_0 = \oint dz J_0 = \oint dz \ cT, \quad (2.2)$$

$$Q_1 = \oint dz J_1 = \oint dz \ \gamma F(\varphi, \beta, \gamma), \quad (2.3)$$

where $J_0, J_1$ have spin 1 and $T$ are constructed as

$$T = T^{\text{eff}} + T_\psi + \frac{1}{2} T_{bc} + T_{\beta \gamma}, \quad (2.4)$$

here the energy-momentum tensors are given by

$$T_\psi = -\frac{1}{2}(\partial \varphi)^2 - \alpha \partial^2 \varphi, \quad (2.5)$$

$$T_{bc} = -2b \partial c - \partial bc, \quad (2.6)$$

$$T_{\beta \gamma} = -s \beta \partial \gamma - (s - 1) \partial \beta \gamma, \quad (2.7)$$

$$T^{\text{eff}} = -\frac{1}{2} \eta_{\mu \nu} \partial X^\mu \partial X^\nu - ia_\mu \partial^2 X^\mu. \quad (2.8)$$

The operator $F(\varphi, \beta, \gamma)$ has spin $s$ and ghost number zero. Because of the grading character of $Q_B$, there exist the nilpotency conditions

$$Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0, \quad (2.9)$$

where the first nilpotency condition provides that the total central charge vanishes, i.e.,

$$-26 - 2(6s^2 - 6s + 1) + 1 + 12\alpha^2 + C^{\text{eff}} = 0, \quad (2.10)$$

the remaining two conditions determine the precise form of the operator $F(\varphi, \beta, \gamma)$ appearing in Eq. (2.3).

Recently, in Ref. [24], the authors generalized the scalar field grading BRST method above developed by Pope et al to the spinor field realization of $W_{2, s}$ strings. The BRST charge for the spinor field realization of $W_{2, s}$ strings takes the similar form (2.1)-(2.3) of the scalar case but with $F = F(\psi, \beta, \gamma)$, and the energy-momentum tensor $T$ was constructed as

$$T = T^{\text{eff}} + T_\psi + KT_{bc} + y T_{\beta \gamma}, \quad (2.11)$$
in which $K, y$ are pending constants and the spinor field $\psi$ has spin 1/2 and satisfies the OPE
\[ \psi(z)\psi(\omega) \sim -\frac{1}{z-\omega}. \] (2.12)

The study can be carried out by introducing the $(b, c)$ ghost system for the spin-2 current, and the $(\beta, \gamma)$ ghost system for the spin-s current, where $b$ has spin 2 and $c$ has spin $-1$ whilst $\beta$ has spin $s$ and $\gamma$ has spin $(1-s)$. The ghost fields $b, c, \beta, \gamma$ are all bosonic and commuting. The energy-momentum tensors in Eq. (2.11) take the following forms
\[ T_\psi = -\frac{1}{2}\partial \psi \psi, \] (2.13)
\[ T_{bc} = 2b\partial c + \partial bc, \] (2.14)
\[ T_{\beta\gamma} = s\beta\partial \gamma + (s-1)\partial \beta \gamma, \] (2.15)
\[ T^{\text{eff}} = -\frac{1}{2}\eta_{\mu\nu}\partial X^\mu \partial X^\nu. \] (2.16)

Using the properties of spinor field and noticing the multi-spinor $Y^\mu$ is a multi-spinor, the BRST charge here is also graded with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. Different from the scalar realizations, the first nilpotency condition $Q_0^2 = 0$ is satisfied for an arbitrary $s$, and there has no constraint on the total central charge. The remaining two conditions determine the precise form of the operator $F(\psi, \beta, \gamma)$ and the exact $y$. The particular method used to construct $F(\psi, \beta, \gamma)$ can be found in [24, 25, 26, 27], in which the solutions for $s = 3, 4, 5, 6$ have been obtained and the discussions of any $s$ were carried out.

3. Construct BRST charges $Q_0$ for the integer and the half-integer $W_{2,s}$ strings

After a brief review of the grading BRST method, we would like to construct more general BRST charges $Q_0$ for the integer $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings.

Focusing on the form of $Q_0$ in Eq. (2.2), here we write the general form of the energy-momentum tensor $T$ as
\[ T = m_1 T^{\text{eff}} + m_2 T_\psi + m_3 T_\varphi + m_4 T_{bc} + m_5 T_{\beta\gamma}, \] (3.1)
where $m_1 - m_5$ are constants which should be determined by the nilpotency of $Q_0$. The energy-momentum tensors on the right hand side of (3.1) are
\[ T_\psi = -\frac{1}{2}\partial \psi \psi, \] (3.2)
\[ T_\varphi = -\frac{1}{2}(\partial \varphi)^2 - \alpha \partial^2 \varphi, \] (3.3)
\[ T_{bc} = -2b\partial c + \partial bc, \] (3.4)
\[ T_{\beta\gamma} = -s\beta\partial \gamma - (s-1)\partial \beta \gamma, \] (3.5)
\[ T^{\text{eff}} = -\frac{1}{2}\eta_{\mu\nu}\partial X^\mu \partial X^\nu - ia_\mu \partial^2 X^\mu - \frac{1}{2}\eta_{\mu\nu}\partial Y^\mu \partial Y^\nu, \] (3.6)
here the $(b, c)$ ghost system corresponds to the spin-2 current, and the $(\beta, \gamma)$ corresponds to the spin-s one. $b$ has spin 2 and $c$ has spin $-1$ whilst $\beta$ has spin $s$ and $\gamma$ has spin $(1-s)$.
The spinor field $\psi$ has spin 1/2 and the scalar field $\varphi$ has spin 0, they satisfy the OPEs

$$\psi(z)\psi(\omega) \sim -\frac{1}{z-\omega},$$

(3.7)

$$\partial \varphi(z)\partial \varphi(\omega) \sim -\frac{1}{(z-\omega)^2}.$$  

(3.8)

The OPE of $\psi$ with itself only has the first order pole and $\partial \varphi$ only has the second order pole, they all just contain one pole in their OPEs. The OPE of the effective energy-momentum tensor $T_{\text{eff}}$ with itself is

$$T_{\text{eff}}(z)T_{\text{eff}}(\omega) \sim \frac{C_{\text{eff}}}{(z-\omega)^4} + \frac{2T_{\text{eff}}}{(z-\omega)^2} + \frac{\partial T_{\text{eff}}}{z-\omega},$$

(3.9)

where $C_{\text{eff}}$ is the central charge of $T_{\text{eff}}$. Different from the cases of $\partial \varphi$ and $\psi$, one will find that this OPE contains three poles, the first order pole, the second and the forth ones, and the third order pole is vanished.

Next, we consider the nilpotency condition $Q_0^2 = 0$. Using residue theorem, one may find this condition converts to require the vanishing of the first order pole of $J_0(z)J_0(\omega)$. Making use of the OPE relations, we could calculate the OPE of $J_0(z)J_0(\omega)$, which contains three poles, but we are only interesting in the first order pole, which has many terms. Then let the coefficients of each terms to be zero will give constraint equations to the parameters $m_1 - m_5$. The number of these equations may larger than that of the parameters, but these equations are not linearly independent. After solving these equations, we can determine the parameters $m_1 - m_5$ and get the solutions of energy-momentum tensor $T$. We find that the nilpotency of $Q_0$ also requires the total central charge of $T$ to be zero for each solution.

3.1 Solutions for the case $s = \text{integer}$

For the case $s = \text{integer}$, the ghost fields $b, c, \beta, \gamma$ are all fermionic and anticommuting. They satisfy the OPEs

$$b(z)c(\omega) \sim \frac{1}{z-\omega}, \quad \beta(z)\gamma(\omega) \sim \frac{1}{z-\omega}.$$  

(3.10)

In other cases the OPEs vanish. We can see the OPEs of $b(z)c(\omega)$ and $\beta(z)\gamma(\omega)$ only have first order pole.

Using the OPEs (3.7)-(3.10) and considering the condition $Q_0^2 = 0$, we obtain four solutions of $T$, and the constraint on the total central charge for each solution:

- Solution 1

$$T = T_{\text{eff}} + T_{\psi} + T_{\varphi} + \frac{1}{2}T_{bc} + T_{\beta\gamma},$$

$$-53 + 2C_{\text{eff}} + 24\alpha^2 + 24s - 24s^2 = 0.$$  

(3.11)

- Solution 2

$$T = T_{\text{eff}} + T_{\varphi} + \frac{1}{2}T_{bc} + T_{\beta\gamma},$$

$$-26 - 2(6s^2 - 6s + 1) + 1 + 12\alpha^2 + C_{\text{eff}} = 0.$$  

(3.12)
From above four solutions, we can see each solution contains the energy-momentum tensors $T^{eff}$, $T_{bc}$ and $T_{\varphi}$. The energy-momentum tensor $T_{\psi}$ is appeared only in solutions 1 and 3, and energy-momentum tensor $T_{\beta\gamma}$ only in solutions 1 and 2. It is worth to note that solution 2 is in accord with (2.4) and (2.10), which are the exact expression of energy-momentum tensor and the constraint condition on the central charge in [22, 23], where $W_{2,s}$ ($s=3,4,5,6$) strings were constructed with scalar field. One may also find that solution 1 is the only one expression of the energy-momentum tensor $T$ which contains five energy-momentum tensors, and this may imply that we can give new realizations of $W_{2,s}$ strings using scalar field $\varphi$ and spinor field $\psi$. Choosing the expression (3.11) of energy-momentum tensor $T$, we have constructed the $W_{2,3}$ string and found that the result of $Q_1 = \oint dz \gamma F(\varphi, \psi, \beta, \gamma)$ does not contain the spinor field $\psi$, and the result is exact the result in [22, 23]. We will discuss the more general case of $Q_1 = \oint dz \gamma F(\varphi, \psi, b, c, \beta, \gamma)$ in our later work.

### 3.2 Solutions for the case $s =$ half-integer

Next, we will consider the half-integer $W_{2,s}$ strings. It is important to note that for the half integer case, the ghost fields $\beta$, $\gamma$ corresponding to spin-$s$ current are no more fermionic and anticommuting but bosonic and commuting, they satisfy the following OPE relation

$$
\beta(z)\gamma(\omega) \sim -\frac{1}{z-\omega}.
$$

(3.15)

Following the procedure above, one can get four non-trivial solutions also for the half-integer $W_{2,s}$ strings.

- **Solution 1**

$$
T = T^{eff} + T_{\psi} + T_{\varphi} + \frac{1}{2}T_{bc} + T_{\beta\gamma},
-51 + 2C^{eff} + 4\left(1 - 6s + 6s^2\right) + 2 + 24\alpha^2 = 0.
$$

(3.16)

- **Solution 2**

$$
T = T^{eff} + T_{\psi} + T_{\varphi} + \frac{1}{2}T_{bc},
-51 + 2C^{eff} + 2 + 24\alpha^2 = 0.
$$

(3.17)
• Solution 3

\[ T = T^{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc}, \]
\[ -51 + 2 C^{\text{eff}} = 0. \]  

(3.18)

• Solution 4

\[ T = T^{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc} + T_{\beta\gamma}, \]
\[ -51 + 2 C^{\text{eff}} + 4(1 - 6s + 6s^2) = 0. \]

(3.19)

One can see the energy-momentum tensors \( T^{\text{eff}}, T_{\psi} \) and \( T_{bc} \) appear in each of these solutions. Energy-momentum tensor \( T_{\phi} \) is contained in solutions 1 and 2 only, while \( T_{\beta\gamma} \) is contained in solutions 1 and 4 only. Here, we need to point out that any realization of \( F(\psi, b, c, \beta, \gamma) \) for the half-integer \( W_{2,s} \) strings with \( T \) given by solution 3 (or 4) is also a realization for the case \( T \) is given by solution 2 (or 1).

The total central charges of the matter sector for integer and the half-integer \( W_{2,s} \) strings are given in Table 1 and Table 2 for each solution of \( T \), respectively. It’s clear that, for solutions 1 and 2 of the integer \( W_{2,s} \) strings, the total central charges of the matter sector for the integer \( W_{2,s} \) strings increase with spin \( s \). But for solutions 3 and 4, they are independent of \( s \) and equal to 26, which is resulted by the contribution of \( T_{bc} \). Similarly, for the half-integer \( W_{2,s} \) strings, the central charges of the matter sector in solutions 2 and 3 are independent of spin \( s \). But the central charge in solutions 1 and 4 are dependent of spin \( s \). When \( s \geq 5/2 \), the central charge of the matter sector will be negative. However, we can substitute solution 1 for solution 4 and set \( \alpha \to i\alpha \), which has no effect on the realizations of the half-integer \( W_{2,s} \) strings. Then by choosing the proper value of \( \alpha \), the central charge of the matter sector will be positive.

|        | \( W_{2,3} \) | \( W_{2,4} \) | \( W_{2,5} \) | \( W_{2,6} \) | \( W_{2,7} \) |
|--------|----------------|----------------|----------------|----------------|----------------|
| Solutions 1,2 | 100 | 172 | 268 | 388 | 532 |
| Solutions 3,4 | 26 | 26 | 26 | 26 | 26 |

Table 1: The total central charges of the matter sector for the integer \( W_{2,s} \) strings

|        | \( W_{2,3/2} \) | \( W_{2,5/2} \) | \( W_{2,7/2} \) | \( W_{2,9/2} \) | \( W_{2,11/2} \) |
|--------|----------------|----------------|----------------|----------------|----------------|
| Solutions 1,4 | 15 | -21 | -81 | -165 | -273 |
| Solutions 2,3 | 26 | 26 | 26 | 26 | 26 |

Table 2: The total central charge of the matter sector for the half-integer \( W_{2,s} \) strings

From above discussion, it implies that we can construct the half-integer \( W_{2,s} \) strings using scalar field \( \varphi \) and spinor field \( \psi \). But for simplicity, in the next section, with those solutions (3.18) and (3.19) for the energy-momentum tensor \( T \), we will construct the BRST charge for the half-integer \( W_{2,s} \) strings using spinor field \( \psi \) only.
4. Spinor field realizations of the half-integer $W_{2,s}$ strings

In present section, we will construct the spinor field realizations of the half-integer $W_{2,s}$ strings for the case of $s = 3/2, 5/2$ and $7/2$ by introducing the fermionic $(b_1,c_1)$ ghost system for the spin-2 current. The OPE of $b_1(z)c_1(\omega)$ is the same as (3.10). The BRST charge $Q_B$ is also graded $Q_B = Q_0 + Q_1$ with $Q_0$ taking the form of (2.2), while $T$ is given in (3.18) or (3.19), and $Q_1$ is given by

$$Q_1 = \oint dz \gamma F(\psi,b_1,c_1,\beta,\gamma).$$  \hspace{1cm} (4.1)

Using the grading BRST method, we will construct $F(\psi,b_1,c_1,\beta,\gamma)$ and discuss the spinor field realizations of the half-integer $W_{2,s}$ strings for the case of $s = 3/2, 5/2$ and $7/2$.

The general procedure is described as follows. First, write down all possible terms of $F$ with $\psi,b_1,c_1,\beta,\gamma$ by considering the spin ‘s’ of each term and ghost number zero. Then leave out all the total differential terms in $\gamma F$ since their contribution to $Q_1$ is zero. Next, get the coefficient equations by using the nilpotency conditions $Q_1^2 = \{Q_0,Q_1\} = 0$. Finally, determine the coefficients in $F$ by solving these equations.

4.1 Spinor field realizations of the $W_{2,3/2}$ string

In this case, the most extensive combinations of $F(\psi,b_1,c_1,\beta,\gamma)$ in (4.1) with correct spin and ghost number can be constructed as follows

$$F(\psi,b_1,c_1,\beta,\gamma) = f_1 \partial \psi + f_2 \psi b_1 c_1 + f_3 \psi \beta \gamma + f_4 \partial \beta c_1 + f_5 \beta \partial c_1 + f_6 \beta c_1 + f_7 \beta^2 c_1.$$  \hspace{1cm} (4.2)

Substituting (4.2) back into (4.1) and imposing the nilpotency conditions $Q_1^2 = \{Q_0,Q_1\} = 0$, we can determine $f_i$ (i=1,2,...,7).

**Case 1:** $T = T^{\text{eff}} + T_0 + \frac{1}{2} T_{bc}$

For the case, the solution is

$$f_1 = f_2 = f_3 = f_4 = f_7 = 0,$$

and the remaining $f_5, f_6$ are arbitrary constants but do not vanish at the same time.

**Case 2:** $T = T^{\text{eff}} + T_0 + \frac{1}{2} T_{bc} + T_{\beta \gamma}$

For the case we can just get

$$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = f_7 = 0,$$

this is a trivial result that all the coefficients of the terms in $F(\psi,b_1,c_1,\beta,\gamma)$ are vanished.

4.2 Spinor field realizations of the $W_{2,5/2}$ string

Similarly, for the case $s=5/2$, $F(\psi,b_1,c_1,\beta,\gamma)$ is expressed in the following form

$$F(\psi,b_1,c_1,\beta,\gamma) = g_1 \partial^2 \psi + g_2 \partial \psi b_1 c_1 + g_3 \psi \beta \partial \gamma + g_4 \psi \beta \partial c_1 + g_5 \psi b_1 c_1 + g_6 \psi \beta \partial c_1 + g_7 \psi \beta b_1 c_1 + g_8 \psi \beta \partial c_1 + g_9 \beta \partial c_1 + g_{10} \beta \partial c_1 + g_{11} \beta \partial^2 c_1 + g_{12} \partial \beta \partial c_1 + g_{13} \partial \beta \partial c_1 + g_{14} \partial \beta \partial c_1 + g_{15} \partial \beta \partial c_1 + g_{16} \partial \beta \partial c_1 + g_{17} \partial \beta c_1.$$  \hspace{1cm} (4.3)
It is clear that the number of these terms in $F(\psi, b_1, c_1, \beta, \gamma)$ for $s = 5/2$ is larger than $s = 3/2$, and this may give more solutions.

For the first case $T = T^{\psi f} + T^\psi + \frac{1}{2} T_{bc}$, there are seven solutions:

- **Solution 1**
  
  \( g_i = 0 \quad (i = 1 - 12, 15 - 17) \),

  and \( g_{13} \) and \( g_{14} \) are arbitrary constants but do not vanish at the same time.

- **Solution 2**
  
  \( g_i = 0 \quad (i = 1 - 8, 11, 14 - 17) \),
  
  \( g_9 = g_{10} = 3M_1, \quad g_{12} = M_1, \quad g_{13} = -9M_1 \),

  where \( M_1 \) is a non-zero constant.

- **Solution 3**
  
  \( g_i = 0 \quad (i = 1 - 7, 11, 15 - 17) \),
  
  \( g_8 = M_2, \quad g_9 = -36M_3, \quad g_{10} = -36M_3, \)
  
  \( g_{12} = 16M_3, \quad g_{13} = -270M_3, \quad g_{14} = -135M_3, \)

  where \( M_2, M_3 \) are arbitrary constants but do not vanish at the same time.

- **Solution 4**
  
  \( g_i = 0 \quad (i = 1 - 7, 15 - 17) \),
  
  \( g_8 = M_4, \quad g_9 = -24M_5, \quad g_{10} = -60M_5, \)
  
  \( g_{11} = 4M_5, \quad g_{12} = -8M_5, \quad g_{13} = 33M_5, \quad g_{14} = -39M_5, \)

  where \( M_4, M_5 \) are arbitrary constants but do not vanish at the same time.

- **Solution 5**
  
  \( g_i = 0 \quad (i = 1 - 7, 9, 15 - 17) \),
  
  \( g_8 = M_6, \quad g_{10} = -12M_7, \quad g_{11} = M_7, \)
  
  \( g_{12} = -2M_7, \quad g_{13} = -6M_7, \quad g_{14} = -6M_7, \)

  where \( M_6 \) and \( M_7 \) are arbitrary constants but do not vanish at the same time.

- **Solution 6**
  
  \( g_i = 0 \quad (i = 1 - 9, 11, 13, 15, 17) \),
  
  \( g_{10} = 33M_8, \quad g_{12} = 4M_8, \quad g_{14} = -18M_8, \quad g_{16} = M_9, \)

  where \( M_8 \) and \( M_9 \) are arbitrary constants but do not vanish at the same time.
• Solution 7

\[ g_i = 0 \quad (i = 1 - 8, 15 - 17), \]
\[ g_9 = -24M_{10}, \quad g_{10} = -60M_{10}, \quad g_{11} = 4M_{10}, \]
\[ g_{12} = -8M_{10}, \quad g_{13} = 33M_{10}, \quad g_{14} = -39M_{10}, \]

where \( M_{10} \) is a non-zero constant.

It is worth pointing out that every term in each of these solutions does not contain spinor field \( \psi \), but we will show that, for \( s = 7/2 \), this situation will be changed.

For the second case, i.e., \( T = T^{\text{eff}} + T_\psi + \frac{1}{2} T_{bc} + T_{\beta\gamma} \), we get

\[ f_i = 0 \quad (i = 1 - 17). \]

Like the case of \( W_{2,3/2} \), this solution is also trivial.

4.3 Spinor field realizations of the \( W_{2,7/2} \) string

For \( s = 7/2 \), \( F(\psi, b_1, c_1, \beta, \gamma) \) can be expressed in the following form:

\[
F(\psi, b_1, c_1, \beta, \gamma) = h_1 \partial^2 \psi + h_2 \partial^2 \psi \partial b_1 \partial \beta \gamma + h_3 \partial^2 \psi \partial b_1 \partial \beta \gamma + h_4 \partial \psi \beta^2 \partial \gamma^2 + h_5 \partial \psi \partial b_1 c_1 + h_6 \partial \psi \partial b_1 c_1
\]

\[ + h_7 \partial \psi \beta^2 \partial \gamma + h_8 \partial \psi \partial^2 b_1 \partial \beta \gamma + h_9 \partial \psi \partial b_1 \partial^2 \beta \gamma + h_{10} \partial \psi \partial b_1 b_1 \partial \gamma \partial \gamma
\]

\[ + h_{11} \partial \psi \beta^3 \gamma^3 + h_{12} \partial \psi \partial^2 b_1 c_1 + h_{13} \partial \psi \partial b_1 \partial c_1 + h_{14} \partial \psi \partial b_1 \partial \beta \gamma + h_{15} \partial \psi \partial^2 \beta \gamma + h_{16} \partial \psi \partial^2 \gamma
\]

\[ + h_{17} \partial \psi \partial \beta \partial \gamma + h_{18} \partial \psi \partial^3 b_1 \gamma^2 + h_{19} \partial \psi \partial^3 b_1 \partial \beta \gamma + h_{20} \partial \psi \partial^3 b_1 \partial \beta \gamma + h_{21} \partial \psi \partial^2 b_1 b_1 \partial \beta \gamma
\]

\[ + h_{22} \partial \psi b_1 b_1 \partial^3 \gamma \gamma + h_{23} \partial \psi b_1 \partial^2 \gamma \partial \gamma + h_{24} \partial \psi b_1 \partial^2 \gamma \partial \gamma + h_{25} \partial \psi b_1 c_1 \partial \beta \gamma + h_{26} \partial \psi b_1 \partial \beta \gamma
\]

\[ + h_{27} \partial \psi b_1 c_1 \partial \beta \gamma + h_{28} \partial \psi \partial \psi \partial^2 b_1 \gamma + h_{29} \partial \psi b_1 \partial^2 \gamma + h_{30} \partial \psi \partial \psi b_1 \partial \gamma + h_{31} \partial \psi \partial^2 \gamma
\]

\[ + h_{32} \partial \psi \partial \beta + h_{33} \partial \psi \partial \beta + h_{34} \partial^3 b_1 b_1 \partial \beta \gamma + h_{35} \partial b_1 b_1 \partial \beta \gamma \gamma + h_{36} \partial^3 b_1 \partial \gamma
\]

\[ + h_{37} \partial b_1 b_1 \partial^2 \gamma + h_{38} \partial b_1 \partial^2 \gamma + h_{39} \partial b_1 \partial^4 \gamma + h_{40} \partial^3 b_1 \partial^2 b_1 \gamma + h_{41} \partial^3 b_1 \partial b_1 \partial^2 \gamma
\]

\[ + h_{42} \partial^2 b_1 \partial b_1 \partial \gamma + h_{43} \partial^2 b_1 \partial b_1 \partial \beta \gamma + h_{44} \partial^2 b_1 \partial b_1 \partial b_1 \partial \gamma + h_{45} b_1 \partial^3 \beta \gamma
\]

\[ + h_{46} \partial^2 b_1 \partial b_1 \partial \gamma + h_{47} \partial b_1 \partial b_1 \partial^2 \beta \gamma + h_{48} \partial b_1 \partial b_1 \partial b_1 \partial \gamma + h_{49} \partial b_1 \partial b_1 \partial \gamma
\]

\[ + h_{50} \partial b_1 \partial b_1 \partial \gamma \partial \gamma + h_{51} \partial b_1 \partial b_1 \partial \gamma \partial \gamma + h_{52} b_1 \partial \beta \partial^2 \eta \partial \gamma + h_{53} b_1 \partial \beta \partial^2 \eta \partial \gamma + h_{54} \partial b_1 \partial^2 \beta \gamma
\]

\[ + h_{55} \partial b_1 \partial^2 \beta \partial \gamma + h_{56} \partial b_1 \partial \beta \partial \gamma + h_{57} \partial b_1 \partial b_1 \partial \gamma + h_{58} \partial b_1 \partial b_1 \partial \gamma + h_{59} b_1 \partial^3 \beta \gamma
\]

\[ = 0 \quad (i = 1 - 17). \]

For the first case \( T = T^{\text{eff}} + T_\psi + \frac{1}{2} T_{bc} \), one will get three solutions

• Solution 1

\[ h_i = 0 \quad (i = 1, 2, 4 - 7, 11 - 17, 19, 21 - 59), \]
\[ h_3 = M_{11}, \quad h_8 = M_{11}, \quad h_9 = M_{11}, \]
\[ h_{10} = 2M_{11}, \quad h_{18} = M_{12}, \quad h_{20} = 3M_{12}, \]

where \( M_{11} \) and \( M_{12} \) are arbitrary constants but do not vanish at the same time. This case is different from those above since it contains the spinor field \( \psi \) at each of the exist terms.
• Solution 2

\[ h_i = 0 \quad (i = 1 - 35, 40 - 48, 50, 51, 53, 57, 58), \]
\[ h_{49} = M_{13} - 3M_{14}, \quad h_{52} = M_{13} - 3M_{14}, \quad h_{54} = M_{14}, \]
\[ h_{55} = M_{13}, \quad h_{56} = 2M_{13} - 6M_{14}, \quad h_{59} = M_{14}, \]

where \( M_{13}, M_{14}, M_{15} \) and \( h_i (i = 36 - 39) \) are arbitrary constants but do not vanish at the same time.

• Solution 3

\[ h_i = 0 \quad (i = 1 - 35, 39 - 42, 44 - 48, 50, 51, 53, 57, 58), \]
\[ h_{38} = 2M_{16}, \quad h_{54} = 2M_{17}, \quad h_{55} = M_{18}, \quad h_{59} = 2M_{19}, \]
\[ h_{36} = 2M_{16} - 9M_{17} + 9M_{19}, \quad h_{37} = 3M_{16}, \quad h_{43} = 6(M_{19} - M_{17}), \]
\[ h_{49} = 6M_{17} + M_{18} - 12M_{19}, \quad h_{52} = M_{18} - 6M_{19}, \quad h_{56} = 2(M_{18} - 6M_{19}), \]

where \( M_{16}, M_{17}, M_{18}, M_{19} \) are arbitrary constants but do not vanish at the same time. This is a non-trivial case and it is different from that of \( W_{2,3/2} \) and \( W_{2,5/2} \). One of the main consequence of the solution may be that with increasing of spin \( s \), the number of those terms constructing \( F(\psi, b_1, c_1, \beta, \gamma) \) becomes large and may give more new solutions.

Until now, we have constructed the explicit forms of \( F(\psi, b_1, c_1, \beta, \gamma) \) for \( s = 3/2, 5/2, 7/2 \), and find that with increasing of spin \( s \), the results become more interesting and complicated.

5. Conclusion

In this paper, we first give a brief review of the grading BRST method to construct \( W_{2,s} \) strings using scalar field \( \varphi \) and spinor field \( \psi \), respectively. Then we reconstruct the BRST charge \( Q_0 \) more generally for both the integer \( W_{2,s} \) strings and the half-integer \( W_{2,s} \) strings. Each of them gives four solutions, and each solution has the condition that the total central charge must vanish. We find that for the half-integer \( W_{2,s} \) strings, when \( s \geq 5/2 \), the central charge of the matter sector will be negative, but this can be solved by setting \( \alpha \rightarrow i\alpha \), which has no effect on the realizations of the half-integer \( W_{2,s} \) strings. We also find that the energy-momentum tensor used in Refs. \cite{22, 23} is one of the special case of our results. Based our results, we construct the \( W_{2,3} \) strings using scalar field \( \varphi \) together with spinor field \( \psi \), but we find the results are the same as the ones obtained in Refs. \cite{22, 23}. But
with these results, following the procedure expressed detailedly in Section 4, we construct the nilpotent BRST charges $Q_B$ for the half-integer $W_{2,s}$ strings for $s = 3/2, 5/2, 7/2$ using spinor field $\psi$ only for the first time. When the spin $s$ takes $3/2$ and $5/2$, each solution of $F(\psi, b_1, c_1, \beta, \gamma)$ for the case $T = T_{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc}$ has not any term which contains the spinor field $\psi$, and for the case $T = T_{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc} + T_{\beta \gamma}$, there only exists the trivial solution $F(\psi, b_1, c_1, \beta, \gamma) = 0$. But for $s = 7/2$, some valuable solutions are obtained. For the case of $T = T_{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc}$, there exists one solution in which each term of it contains the spinor field $\psi$, and for the case of $T = T_{\text{eff}} + T_{\psi} + \frac{1}{2} T_{bc} + T_{\beta \gamma}$, only one non-trivial solution is found. These two solutions are different from the cases of $s = 3/2$ and $5/2$. This shows that with increasing of spin $s$, many new valuable solutions should be found. All these results obtained in this paper will be of importance for embedding of the Virasoro string and superstrings into the integer $W_{2,s}$ strings and the half-integer $W_{2,s}$ strings, and they may provide the essential ingredients to help us better understanding the fundamental properties of the half-integer $W_{2,s}$ strings. By giving the explicit realizations and BRST charges, the study on the physical states of these $W$ strings will become possible.

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