Counting the Closed Subgroups 
of Profinite Groups

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Abstract

The sets of closed and closed-normal subgroups of a profinite group carry a natural profinite topology. Through a combination of algebraic and topological methods the size of these subgroup spaces is calculated, and the spaces partially classified up to homeomorphism.

1 Introduction

In this paper we calculate the possible cardinalities of $\mathcal{S}(G)$, the set of all closed subgroups of a profinite group $G$, and find conditions on $G$ determining the cardinality of $\mathcal{S}(G)$. In summary: $\mathcal{S}(G)$ is finite if and only if $G$ is finite; $\mathcal{S}(G)$ is countably infinite if and only if $G$ is a finite central extension of $\bigoplus_{i=1}^{n} \mathbb{Z}_{p_{i}}$ where the $p_{i}$ are distinct primes and $\mathbb{Z}_{p_{i}}$ is a copy of the $p_{i}$-adic integers; and otherwise $|\mathcal{S}(G)| = 2^{w(G)}$ where $w(G)$ is the weight of $G$ (the cardinality of a minimal sized base for the topology of $G$).

We further show that for a profinite group $G$, the set $\mathcal{N}(G)$ of closed normal subgroups of $G$ is either countable or of size $2^{w(G)}$.

These results follow from a mixture of algebraic and topological considerations. In particular we use a natural topology defined on $\mathcal{S}(G)$, and inherited by $\mathcal{N}(G)$, introduced in [4] and explored further in [5]. Extending our results on the number of closed (normal) subgroups we also consider the problem of classifying the space of closed subgroups $\mathcal{S}(G)$ and closed normal

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subgroups $\mathcal{N}(G)$ up to homeomorphism. Here we give a complete solution in the case when $\mathcal{S}(G)$ is countable and when $w(G) = \aleph_1$ (the first uncountable cardinal – in this case $\mathcal{S}(G)$ is homeomorphic to $\{0,1\}^{\aleph_1}$). The situation when $w(G) > \aleph_1$ is rather mysterious: in particular $\mathcal{S}(G)$ need not be homeomorphic to $\{0,1\}^{w(G)}$. We also determine when $\mathcal{S}(G)$ is homeomorphic to the Cantor set (equivalently, homeomorphic to $\{0,1\}^{\aleph_0}$). These results all depend on determining when a subgroup space has an isolated point.

The remaining case in the topological classification of the space of closed subgroups of a profinite group is the ‘mixed’ case: the subgroup space is uncountable, has a countable base, but is not homeomorphic to the Cantor set. This case is investigated further in [6].

2 Background Material

Subgroup Spaces By definition, a profinite group $G$ is one that can be represented as a projective limit of finite groups, $\lim_{\leftarrow} G_{\lambda}$. Writing $\mathcal{S}(G)$ for the set of closed subgroups of a profinite group, $G = \lim_{\leftarrow} G_{\lambda}$, one sees that $\mathcal{S}(G) = \lim_{\leftarrow} \mathcal{S}(G_{\lambda})$. Giving the finite set $\mathcal{S}(G_{\lambda})$ the discrete topology for each $\lambda$, we see that the projective limit $\mathcal{S}(G)$ picks up a natural topology. This topology is profinite (compact, Hausdorff and zero-dimensional).

An alternative description of the topology on $\mathcal{S}(G)$ is that it is the subspace topology inherited by $\mathcal{S}(G)$ from the space of all compact subsets of $G$ with the Vietoris topology, and so the topology is independent of the particular projective representation of $G$.

We can concretely describe canonical basic open neighbourhoods of a subgroup in $\mathcal{S}(G)$ or $\mathcal{N}(G)$ for a profinite group $G$ as follows.

**Definition 2.1** Let $G$ be a profinite group. For $H \leq G$ and $N \trianglelefteq G$, we define $B(H, N) = \{K \leq G \mid KN = HN\}$.

**Lemma 2.2** Let $G$ be a profinite group, and $H \leq G$. Suppose that $(N_{\lambda})_{\lambda \in \Lambda}$ is a family of open normal subgroups of $G$, forming a base for the open neighbourhoods of 1 in $G$. Then $(B(H, N_{\lambda}))_{\lambda \in \Lambda}$ forms a base for the open neighbourhoods of $H$ in $\mathcal{S}(G)$. Also, if $H \trianglelefteq G$ then $(B(H, N_{\lambda}) \cap \mathcal{N}(G))_{\lambda \in \Lambda}$ forms a base for the open neighbourhoods of $H$ in $\mathcal{N}(G)$.

**Lemma 2.3** Let $G$ be an infinite profinite group. Then $\chi(1, \mathcal{S}(G)) = \chi(1, \mathcal{N}(G)) = w(G)$. 

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Corollary 2.4 Let $G$ be an infinite profinite group. Then $w(S(G)) = w(N(G)) = w(G)$.

Proposition 2.5 If a profinite group $G$ is topologically isomorphic to $\prod_{i=1}^{n} G_{p_i}$ where $p_1, \ldots, p_n$ are distinct primes, and $G_{p_i}$ is a pro-$p_i$ group, then $S(G) \cong \prod_{i=1}^{n} S(G_{p_i})$.

Classifying Profinite Spaces

Definition 2.6 Let $X$ be a topological space. For $Y \subseteq X$, let $Y'$ denote the set of all limit points of $Y$, that is $Y \setminus Y'$ is the set of points of $Y$ which are isolated in $Y$. We define the following transfinite sequence.

$X^{(0)} = X$,
$X^{(\alpha+1)} = (X^{(\alpha)})'$, for $\alpha$ an ordinal,
$X^{(\lambda)} = \bigcap_{\mu < \lambda} X^{(\mu)}$, for $\lambda$ a limit ordinal.

We now list some facts about this Cantor–Bendixson process.

Lemma 2.7 Let $X$ be a Hausdorff space.

(i) $X^{(\alpha)}$ is closed in $X$ for every ordinal $\alpha$.

(ii) If $\alpha \leq \beta$ then $X^{(\alpha)} \supseteq X^{(\beta)}$.

(iii) $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is the set of isolated points of $X^{(\alpha)}$ for every ordinal $\alpha$, and is countable if $X$ is countably based.

(iv) $(X^{(\alpha)})^{(\beta)} = X^{(\alpha+\beta)}$ for all ordinals $\alpha$ and $\beta$.

(v) If $Y \subseteq Z \subseteq X$ then $Y^{(\alpha)} \subseteq Z^{(\alpha)}$ for every ordinal $\alpha$.

(vi) There is a least ordinal $\lambda$ such that $X^{(\lambda)} = X^{(\lambda+1)}$ and if $\alpha \geq \lambda$ then $X^{(\alpha)} = X^{(\lambda)}$. $X^{(\lambda)}$ is perfect in itself. If $X$ is countably based then $\lambda$ is countable.

(vii) If $X$ is compact, and $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$, then there is a successor ordinal $\lambda$ such that $X^{(\lambda)} = \emptyset$ and $X^{(\lambda-1)}$ is a finite non-empty discrete space.

(ix) If $Y$ is open in $X$ then $Y^{(\alpha)} = Y \cap X^{(\alpha)}$ for every ordinal $\alpha$. 
It follows from Lemma 2.7 that every countably based Hausdorff space can be written as the disjoint union of a countable set (the **scattered part** of $X$) and a set which is perfect in itself (the **perfect hull** of $X$); this is the Cantor–Bendixon theorem.

**Proposition 2.8** Let $X$ be a non-empty profinite space. Then $X$ is homeomorphic to the Cantor set if and only if $X$ is perfect and countably based.

**Definitions 2.9** Let $X$ be a Hausdorff space. The **scattered height** of $X$, $ht(X)$ is the least ordinal $\lambda$ such that $X^{(\lambda)} = X^{(\lambda+1)}$.

Let $x \in X \setminus X^{(ht(X))}$. $ht(x, X)$ is defined to be the least ordinal $\alpha$ such that $x \notin X^{(\alpha+1)}$.

Of course to say that $x \in X \setminus X^{(ht(X))}$ has $ht(X) = \alpha$ is precisely the same as saying that $x \in X^{(\alpha)}$ and that $x$ is isolated in $X^{(\alpha)}$. The next lemma follows immediately from the definitions.

**Lemma 2.10** Let $X$ be a Hausdorff space, $Y$ an open set in $X$ and $x \in Y$. Then $ht(Y) \leq ht(X)$ and $ht(x, X) = ht(x, Y)$.

In relation to Lemma 2.7(vii), a compact Hausdorff space $X$ for which $X^{(ht(X))} = \emptyset$ is sometimes known as a **scattered** space. It is clear that a countably based profinite space is scattered if and only if it is countable.

**Lemma 2.11** Let $\alpha$ be a non-zero countable ordinal, and $n$ be a positive integer. Then $\omega^\alpha n + 1$, with the order topology, is a countably infinite profinite space. Moreover $(\omega^\alpha n + 1)^{(\alpha)} = \{\omega^\alpha, \omega^\alpha 2, \ldots, \omega^\alpha n\}$. In particular $ht(\omega^\alpha n + 1) = \alpha + 1$ and $|(\omega^\alpha n + 1)^{(\alpha)}| = n$.

Note that $\omega + 1$ is homeomorphic to a convergent sequence.

**Proposition 2.12** Let $X$ be a countable profinite space. Then $X$ is homeomorphic to $\omega^{ht(X) - 1} \cdot X^{(ht(X) - 1)} + 1$.

### 3 When $S(G')$ is Countable

In this section we characterise the profinite groups which have precisely countably infinitely many closed subgroups. A fortiori such a group has countably many open subgroups, so must be countably based. The following
theorem (originally established by Dikranjan [2] and re–discovered by Morris, Oates-Williams and Thompson [12]) and example suggests that these groups might be built up in some simple way from the \( p \)-adic integers \( \mathbb{Z}_p \). Our main result, Theorem 3.7, confirms this.

**Theorem 3.1** Let \( G \) be an infinite compact Hausdorff topological group. Then the following are equivalent.

(i) Every non-trivial closed subgroup of \( G \) is open.

(ii) \( G \) is topologically isomorphic to \( \mathbb{Z}_p \) for some prime \( p \).

**Example 3.2** (See [4]) \( S(\mathbb{Z}_p) \cong (\omega + 1) \).

More precisely, the open subgroups \( p^n\mathbb{Z}_p \) form a sequence converging to the sole non-open closed subgroup, the trivial subgroup.

For infinite profinite groups, having countably infinitely many closed subgroups turns out to be equivalent to having less than \( 2^{\aleph_0} \) closed subgroups. To show this we do not need to assume the Continuum Hypothesis.

This fact will follow immediately from the topological considerations of Section 6 (see Corollary 6.10). Here we give a direct group-theoretic argument.

**Continuum Many Closed Subgroups** First we give some conditions which ensure that a profinite group has at least \( 2^{\aleph_0} \) closed subgroups.

**Lemma 3.3**

(i) Let \( (G_i)_{i \in I} \) be an infinite family of profinite groups where infinitely many of the \( G_i \) are non-trivial. Then \( G = \prod_{i \in I} G_i \) has at least \( 2^{\aleph_0} \) closed subgroups.

(ii) Let \( G \) be an infinite abelian torsion profinite group. Then \( G \) has at least \( 2^{\aleph_0} \) closed subgroups.

(iii) Let \( G \) be a profinite group, \( N \trianglelefteq G \), and \( K \trianglelefteq G \) with \( K \leq N \). Suppose for some prime \( p \), \( K \cong \mathbb{Z}_p \) and \( G/N \cong \mathbb{Z}_p \). Then \( G \) has at least \( 2^{\aleph_0} \) closed subgroups.
Proof. For (i), let $J$ be a countably infinite subset of $I$ such that for every $i \in J$, $G_i$ is non-trivial. Then for each of the $2^\aleph_0$-many distinct subsets $J'$ of $J$, we have distinct closed subgroups $G_{J'} = \prod_{i \in J} G_i(J')$ where $G_i(J') = G_i$ if $i \in J'$ and $G_i(J') = 1$ otherwise.

(ii) follows immediately from (i) and the fact that compact abelian torsion groups are topologically isomorphic to a Cartesian sum of finite cyclic groups of bounded order.

Now, for (iii), let $(G/K)/(N/K) = \langle (N/K)Kx \rangle$, $L/K = \langle Kx \rangle$, and let $P/K$ be a Sylow $p$-subgroup of $L/K$. Since $(L/K)/(N/K \cap L/K) \cong (N/K)/(N/K \cap L/K) = (G/K)/(N/K) \cong G/N \cong \mathbb{Z}_p \times p^\infty \mid o(L/K)$. Since $L/K$ is procyclic and $P/K$ is not finite, we see that $P/K \cong \mathbb{Z}_p$. Hence without loss of generality we may assume that $K = N$ (and thus $P = G$). We now show that $G$ splits over $N$. Let $G/N = \langle a \rangle$. Then $G = N \langle a \rangle$ and so $\langle a \rangle / \langle a \rangle \cap N \cong N \langle a \rangle / N = G/N \cong \mathbb{Z}_p$. So $\langle a \rangle \cap N = 1$ and $G = N \times \langle a \rangle$.

Now define $\phi: N \to \{ H \mid H \leq C \ G \}$, by $\phi(n) = \langle na \rangle$. Suppose $n_1 \neq n_2$ but $\phi(n_1) = \phi(n_2)$. Then as $(n_1a)(n_2a)^{-1} \in \langle n_1a \rangle$ and $(n_1a)(n_2a)^{-1} = n_1n_2^{-1} \in N \setminus \{ 1 \}$, $\langle n_1a \rangle \cap N \neq 1$. As $\langle a \rangle \cap N = 1$, $n_1a \not\in N$, and so $\langle n_1a \rangle \not\subset N$. Thus $\langle n_1a \rangle N/N$ is a non-trivial subgroup of $G/N$, and as $\mathbb{Z}_p \cong \langle n_1a \rangle N/N \cong \langle n_1a \rangle / (\langle n_1a \rangle \cap N)$, $\langle n_1a \rangle \cap N = 1$, a contradiction. So $\phi$ is injective, and since $|N| = 2^\aleph_0$, $G$ has at least $2^\aleph_0$ closed subgroups. □

So if a profinite group has less than $2^\aleph_0$ closed subgroups it cannot have a closed section of any of the above three forms.

The Abelian Case and the Center  Next we shall prove a special case of part of the main result; namely, a necessary condition for an infinite abelian profinite group to have less than $2^\aleph_0$ closed subgroups. This result can be obtained using Pontryagin duality from a result of Boyer (Lemma 3 of [1]) or from a result of Berhanu, Comfort and Reid (Proposition 2.3 of [11]). Here we give a direct argument.

Lemma 3.4 Let $G$ be an infinite abelian profinite group. Suppose $G$ has less than $2^\aleph_0$ closed subgroups. Then there is a finite nonempty set of primes $\{ p_1, \ldots, p_n \}$ (with $p_i \neq p_j$ for $i \neq j$) and a finite abelian group $T$ such that $G$ is topologically isomorphic to $T \times \bigoplus_{i=1}^n \mathbb{Z}_{p_i}$.

Proof. Since $G$ is abelian it is topologically isomorphic to the Cartesian product of its Sylow subgroups. So by Lemma 3.3(i), there are only finitely many primes, $p_1, \ldots, p_n$, dividing the order of $G$, and $G = \prod_{i=1}^n G_{p_i}$, where
\(G_{p_i}\) is pro-\(p_i\). Hence \(S(G) = \prod_{i=1}^{n} S(G_{p_i})\), so it now suffices to prove the result in the case where \(G\) is a pro-\(p\) group for some prime \(p\).

By Lemma \ref{lem:3.3}(ii) \(G\) is not a torsion group. Let \(x\) be an element of \(G\) of infinite order, and let \(H = \langle x \rangle\). Then \(H \cong \mathbb{Z}_p\). Let \(y \in G\), and let \(K = \langle x, y \rangle\).

Then \(K\) is a finitely generated abelian pro-\(p\) group, so \(K = A \times B\) where \(A\) is a finitely generated torsion-free abelian pro-\(p\) group and \(B\) is a finite abelian \(p\)-group. \(A \cong \mathbb{Z}_p^n\) for some positive integer \(n\). But by Lemma \ref{lem:3.3}(iii), \(m = 1\), and so \(A = H\). Let \(T = Tor(G)\). Then \(B \leq T\), and so \(K = A \times B \leq H \times T\).

Thus for any \(y \in G\), \(y \in H \times T\). Hence \(G = H \times T\).

We now show that \(T\) is finite. For each non-negative integer \(i\), let \(T_i = \{g \in G \mid g^{p_i} = 1\}\). Then \(T_i \leq G\) as \(T_i\) is the kernel of the continuous endomorphism \(g \mapsto g^{p_i}\). As \(T = \bigcup_{i \geq 0} T_i\), \(G = \bigcup_{i \geq 0} (H \times T_i)\), and so by Baire’s Category theorem there is an \(i\) such that \(H \times T_i \leq G\). Thus there is a non-negative integer \(k\) such that \(|G : H \times T_i| = p^k\). Now for all integers \(j \geq i\), \(p^k \geq |HT_j : HT_i| = |T_j : (HT_i) \cap T_j| = |T_j : HT_i|\). It follows that \(T = T_{i+k}\). Thus \(T \leq G\). By Lemma \ref{lem:3.3}(ii), \(T\) is finite.

**Lemma 3.5** Let \(G\) be an abelian profinite group and \(N \leq G\). Suppose for some finite non-empty set of primes \(\{p_1, \ldots, p_n\}\) (with \(p_i \neq p_j\) for \(i \neq j\)) that \(N \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i}\). Then there is a finite abelian group \(T\) such that \(G\) is topologically isomorphic to \(T \times \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i}\).

**Proof.** Again as \(G\) is topologically isomorphic to the Cartesian product of its Sylow subgroups, it suffices to prove the result in the case where \(G\) is a pro-\(p\) group for some prime \(p\). Let \(N = \langle x \rangle\). As in the proof of Lemma \ref{lem:3.4} above, let \(y \in G\), and let \(K = \langle x, y \rangle\). Again \(K = A \times B\) where \(A\) is a finitely generated torsion-free abelian pro-\(p\) group and \(B\) is a finite abelian \(p\)-group. Since \(h_i(G) = 1\), \(A = N\) and as above \(K \leq N \times Tor(G)\). So \(G = N \times Tor(G)\).

\[\blacksquare\]

The next proposition is well known.

**Proposition 3.6**

(i) If \(G\) is a group with \(|G : Z(G)| < \infty\), then \(G'\) is finite.

(ii) If \(G\) is a finitely generated profinite group with \(G'\) finite, then \(Z(G) \leq G\).

**Proof.** (i) is a well-known result of Schur; see, for example 10.1.4 of \cite{9}. 

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For (ii), suppose that \( G = \langle g_1, \ldots, g_n \rangle \). Now for each \( i \), with \( 1 \leq i \leq n \), define \( \phi_i : C_G(G') \to G \) by \( \phi_i(x) = [x, g_i] \) for \( x \in C_G(G') \). Now each \( \phi_i \) is a continuous homomorphism since \( \phi_i(x_1 x_2) = \[ x_1, g \] \[ x_2, g \] = \[ x_1, g \] \[ x_2, g \] \), for \( x_1, x_2 \in C_G(G') \). Since \( G' \) is a finite normal subgroup of \( G \), \( C_G(G') \ltimes O_G \).

Hence, for each \( i \), \( \ker(\phi_i) \leq O_G \). Let \( K = \bigcap_{i=1}^n \ker(\phi_i) \). Then \( K \leq O_G \). For each \( i \), \( g_i \in C_G(K) \). Thus \( K \) is central in \( G \), and so \( Z(G) \trianglelefteq O_G \) as required.

**General Case** We are now ready to characterise the class of profinite groups with precisely countably infinitely many closed subgroups. We first mention three further facts used in the proof. Firstly, the fact that if \( G \) is a profinite group and \( H \) is a closed subgroup of \( G \) then either \( H \leq O_G \) or \( |G : H| \geq 2^{\aleph_0} \).

The second fact is about properties of the \( p \)-adic integers, \( \mathbb{Z}_p \), for some prime \( p \). As an inverse limit of rings, \( \mathbb{Z}_p \) is a profinite ring. Every automorphism of the additive group \( \mathbb{Z}_p \) is continuous. It turns out that \( \text{Aut}(\mathbb{Z}_p) \) is topologically isomorphic to \( U(\mathbb{Z}_p) \), the group of units of \( \mathbb{Z}_p \). Further, \( U(\mathbb{Z}_p) \cong \mathbb{Z}_p \times K_p \), where \( K_p = C_{p-1} \) if \( p \) is odd or \( K_p = C_2 \) if \( p = 2 \). (See section 3, Chapter II of [13].)

The third fact concerns a small amount of cohomology we use at the end of the proof of our characterisation theorem. Let \( G \) be a profinite group and \( N \trianglelefteq C_G(G) \). Suppose that \( N \) has a closed complement \( H \), i.e., \( H \leq C_G(G) \), \( G = H N \) and \( H \cap N = 1 \) and so \( G = N \times H \). By a **derivation** from \( H \) to \( N \) we mean a continuous map \( d : H \to N \) such that \( d(g_1 g_2) = (d(g_1)) g_2 (d(g_2)) \) for all \( g_1, g_2 \in H \). The set of derivations from \( H \) to \( N \) is denoted by \( \text{Der}(H, N) \). The basic result we require is that there is a bijection between the set of all complements by closed subgroups of \( N \) in \( G \) and \( \text{Der}(H, N) \). The proof which is very similar to the abstract case can be found in Lemma 6.2.3(a) of [19]. In fact since we actually encounter central split extensions, we are really using the first cohomology group, \( H^1(G, N) \).

**Theorem 3.7** Let \( G \) be an infinite profinite group. Then the following are equivalent.

(i) \( G \) has precisely countably infinitely many closed subgroups.

(ii) \( G \) has less than \( 2^{\aleph_0} \) closed subgroups.
(iii) \( Z(G) \trianglelefteq O G \) and there is a finite non-empty set of primes \( \{p_1, \ldots, p_n\} \)
(with \( p_i \neq p_j \) for \( i \neq j \)) and a finite abelian group \( F \) such that \( Z(G) \) is topologically isomorphic to \( F \times \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i} \).

(iv) \( G' \) is finite and there is a finite non-empty set of primes \( \{p_1, \ldots, p_n\} \)
(with \( p_i \neq p_j \) for \( i \neq j \)) and a finite abelian group \( F \) such that \( G/G' \) is topologically isomorphic to \( F \times \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i} \).

**Proof.** Clearly (i) \( \Rightarrow \) (ii). First, we show that (ii) \( \Rightarrow \) (iii), so suppose that \( G \) has less than \( 2^{8n} \) closed subgroups. We shall now show that \( G \) is virtually abelian. Let \( H \) be a maximal abelian subgroup of \( G \) (which exists by Zorn’s lemma). Then \( H \) is also abelian, and so \( H \leq G \) with \( H = C_G(H) \).

Let \( K = N_G(H) \). Then \( |G : K| = |G : N_G(H)| = |\{H^g \mid g \in G\}| < 2^{8n} \). So by the first fact above, \( K \trianglelefteq O G \). If \( H \) is finite, then \( K/H = N_G(H)/C_G(H) \) is finite, and thus \( G \) is finite, a contradiction. Hence \( H \) is infinite.

Now by Lemma 3.3, \( H = T \times \bigoplus_{i=1}^{n} H_i \) where \( T \) is a finite subgroup of \( H \), \( \{p_1, \ldots, p_n\} \) is a finite non-empty set of primes (with \( p_i \neq p_j \) for \( i \neq j \)), and \( H_i \cong \mathbb{Z}_{p_i} \) for each \( i \). \( T \) is a characteristic subgroup of \( H \), and so \( T \trianglelefteq C \). Also suppose \( \theta \trianglelefteq C \). Let \( C_i = C_K(TH_i/T) \), then \( C_i \trianglelefteq C \). Suppose there is an \( i \) such that \( K/C_i \) is infinite. But, by our discussion above, \( K/C_i \cong Aut(TH_i/T) \cong Aut(\mathbb{Z}_{p_i}) \cong \mathbb{Z}_{p_i} \times K_{p_i} \), where \( K_{p_i} = C_{p_i-1} \) if \( p_i \) is odd or \( K_{p_i} = C_2 \) if \( p_i = 2 \). Thus there exists \( L_i/C_i \leq C \) \( K/C_i \) such that \( L_i/C_i \cong \mathbb{Z}_{p_i} \). Also \( TH_i/T \leq C \), and \( TH_i/T \cong \mathbb{Z}_{p_i} \). Hence by Lemma 3.3 (iii), \( L_i/T \) has at least \( 2^{8n} \) closed subgroups, which is a contradiction.

Clearly \( C_K(T) \trianglelefteq O K \), so \( L = C_K(H/T) \cap C_K(T) \trianglelefteq O K \). Also \( H \leq L \), so to show that \( H \leq O G \), it suffices to show that \( H \leq O L \). For each \( x \in L \), define \( \theta_x : H \to T \), by \( \theta_x(h) = [h, x] \). Then as \( H \leq C \), \( \theta_x \) is a homomorphism and is continuous since \( H \) is a finitely generated (and otherwise very simple) profinite group. Define \( \theta : L \to Hom(H, T) \), by \( \theta(x) = \theta_x \). Then as \( L \leq C \), \( \theta \) is a homomorphism. Now \( Hom(H, T) \cong Hom(T, T) \times \bigoplus_{i=1}^{n} Hom(\mathbb{Z}_{p_i}, T) \), which is finite. So \( ker \theta \trianglelefteq O L \) since \( L \) is strongly complete. But clearly \( ker \theta = C_L(H) = H \). Hence \( H \leq O G \).

Let \( N \) be the core of \( \bigoplus_{i=1}^{n} H_i \) in \( G \). Then \( N \trianglelefteq O G \) and \( N = \bigoplus_{i=1}^{n} N_i \) where \( N_i \cong \mathbb{Z}_{p_i} \) for each \( i \). To show that \( Z(G) \trianglelefteq O G \) we show that \( N \leq Z(G) \). So suppose for a contradiction that \( N \neq Z(G) \). Then there is an \( i \) such that \( G \neq C_G(N_i) \). Let \( M = \bigoplus_{j 
eq i}^{n} N_j \). For each \( j \), \( N_j \) is characteristic in \( N \) and so \( N_j \trianglelefteq C \). Thus \( M \trianglelefteq C \). Then \( G/M \neq C_{G/M}(N/M) \) and \( N/M \cong N_i \). So we
may assume that $N_i = N \cong \mathbb{Z}_p$ for some prime $p$. Then $N \trianglelefteq G$. Let $G$ act on $N$ by conjugation. $N$ has a natural ring structure, and as per our discussion above of $\mathbb{Z}_p$, $Aut(N) \cong U(N)$, the group of units of $N$. Let $x \in G \setminus C_G(N)$, and $n \in N$. There exists $u \in U(N)$ such that $u \cdot n = n^x$ where $\cdot$ is the ring multiplication in $N$. If $n^x = n$ then $u \cdot n = n$ and $n$ is the additive identity in $N$. Thus $C_N(x)$ is trivial and $N \cap \langle x \rangle = 1$. Since $N \trianglelefteq G$ and $\langle x \rangle = \langle x \rangle/N \cap \langle x \rangle \cong N\langle x \rangle/N$, $x$ is of finite order and so $\langle x \rangle \leq C_G$. Define $\phi: N \to \{H \mid H \trianglelefteq_G G\}$ by $\phi(g) = \langle x \rangle^g$. If $\phi(g_1) = \phi(g_2)$ then $\langle x^{g_1} \rangle = \langle x^{g_2} \rangle$. So there is an integer $k$ such that $x^{g_1} = (x^{g_2})^k$. Then $N x = N x^k$ and as $N \cap \langle x \rangle = 1$, $k = 1$. Thus $g_1 = g_2$. Hence $\phi$ is injective, and as $|N| = 2^{\aleph_0}$ we have a contradiction. Thus $N \trianglelefteq Z(G)$, and $Z(G) \trianglelefteq G$. The structure of $Z(G)$ is as required by Lemma 3.4.

Now for (iii) $\Rightarrow$ (i). By Lemma 3.3 we have nothing more general than the class of groups in (iii), by supposing that $Z(G) \trianglelefteq G$, and we have an $N \trianglelefteq Z(G)$ with $N = \bigoplus_{i=1}^n N_i$ where $N_i \cong \mathbb{Z}_{p_i}$ and $\{p_1, \ldots, p_n\}$ is a finite non-empty set of primes with $p_i \neq p_j$ for $i \neq j$. Suppose for a contradiction that $G$ has uncountably many closed subgroups. Let $(H_\alpha)_{\alpha \in A}$ be a family of closed subgroups of $G$ with $A$ uncountable, and $H_\alpha \neq H_\beta$ for $\alpha \neq \beta$. Firstly we show that we may assume without loss of generality that $G$ splits over $N$. Since $N \trianglelefteq G$, there are only finitely many closed subgroups of $G$ containing $N$. So there exists an uncountable subset $B$ of $A$ such that $H_\alpha N = H_\beta N$ for every $\alpha, \beta \in B$. Let $L = H_\alpha N$ for $\alpha \in B$. Also for every $\alpha \in B$, $H_\alpha \trianglelefteq_L L$, $N \trianglelefteq_L L$ and $N \trianglelefteq Z(L)$. Thus without loss of generality we may assume that $H_\alpha N = G$ for every $\alpha \in A$.

Now $N$ has precisely countably infinitely many closed subgroups. So $\{H_\alpha \cap N \mid \alpha \in A\}$ is at most countable. Thus there exists an uncountable subset $B$ of $A$ such that $H_\alpha \cap N = H_\beta \cap N$ for every $\alpha, \beta \in B$. Let $K = H_\alpha \cap N$ for $\alpha \in B$. Then since $K \leq Z(G)$, $K \trianglelefteq_C G$. If $K \trianglelefteq_G$, then $H_\alpha \trianglelefteq_O G$ for every $\alpha \in B$, a contradiction as $G$ has only countably many open subgroups. Clearly $G/K$ has uncountably many closed subgroups, $H_\alpha/K \ (\alpha \in C)$. As $Z(G)/K \leq Z(G/K)$, $N/K \trianglelefteq_O Z(G/K) \trianglelefteq G/K$. If $K \trianglelefteq_N$, then $K \trianglelefteq_O G$, a contradiction. So there exists $M/K \leq_C N/K$ such that $M/K$ is topologically isomorphic to $\bigoplus_{p \in P} \mathbb{Z}_p$ where $P$ is a non-empty subset of $\{p_1, \ldots, p_n\}$. Clearly for each $\alpha \in C$, $H_\alpha/K \cap M/K \leq (H_\alpha \cap N)/K = K/K$. Thus we may assume without loss of generality that $H_\alpha \cap N = 1$ for every $\alpha \in A$.

So now $G$ splits over $N$ and every $H_\alpha \ (\alpha \in A)$ is a complement to $N$ in
Let $\alpha \in A$. Then

$$|\{H \leq C \, G \mid H \text{ is a complement to } N \text{ in } G\}| = |\text{Der}(H_\alpha, N)| = |\text{Hom}(H_\alpha, N)| \quad \text{(as } N \text{ is central in } G)$$

$$= |\text{Hom}(H_\alpha, \bigoplus_{i=1}^N N_i)| = |\bigoplus_{i=1}^N \text{Hom}(H_\alpha, N_i)| = 1$$

This is a contradiction as every $H_\alpha$ is a complement to $N$ in $G$. Hence $G$ has precisely countably infinitely many closed subgroups.

Now suppose that $G$ has precisely countably infinitely many closed subgroups. Then we know (iii) holds. In particular $Z(G) \trianglelefteq O_G$. So by Proposition 3.6(i), $G'$ is finite. Then by Lemma 3.4, $G/G'$ is of the required form for (iv).

Finally, suppose that (iv) holds. Then, clearly $G$ is finitely generated and so by Proposition 3.6(ii), $Z(G) \trianglelefteq O_G$. Now $Z(G)/(Z(G) \cap G') \cong Z(G)/G'/G' \trianglelefteq O_G/G'$. So for each $i$, $h_{p_i}(Z(G)/(Z(G) \cap G')) = 1$. But $Z(G) \cap G'$ is finite and so $h_{p_i}(Z(G)) = 1$ for each $i$. Consequently it is clear that $Z(G)$ is of the required form for (iii), and so (iv) $\Rightarrow$ (iii). $\blacksquare$

## 4 Topological Classification of Countable $S(G)$

Applying the classification of countable profinite spaces of Section 2.3, we are now able to describe $S(G)$ for $G$ profinite and $S(G)$ countable. Naturally we use our description (Theorem 3.7) of such groups. First we require a preliminary lemma whose proof is similar to part of the proof of that theorem.

**Lemma 4.1** Let $G$ be a profinite group with a central open subgroup $N$ such that $N = \bigoplus_{i=1}^k N_i$ where $\{p_1, \ldots, p_k\}$ is a finite non-empty set of primes (with $p_i \neq p_j$ for $i \neq j$) and for each $i$, $N_i$ is a procyclic pro-$p_i$ group. Let $H \leq C \, G$. Then $\{K \leq C \, G \mid K \cap N = H \cap N\}$ is finite.

**Proof.** Suppose for a contradiction that there is a profinite group $G$ satisfying the above conditions with a closed subgroup $H$ of $G$ and an infinite family $(K_\lambda)_{\lambda \in \Lambda}$ of closed subgroups of $G$ such that $K_\lambda \neq K_\mu$ for $\lambda \neq \mu$ and $K_\lambda \cap N = H \cap N$ for every $\lambda \in \Lambda$. We show that without loss of generality $G$ splits over $N$. Since $N \trianglelefteq O \, G$, there are only finitely many closed subgroups of $G$ containing $N$. So there exists an infinite subset $\Lambda'$ of $\Lambda$ such that
$K_\lambda N = K_\mu N$ for every $\lambda, \mu \in \Lambda'$. Now for every $\lambda \in \Lambda'$, $N \leq Z(K_\lambda N)$ and $K_\lambda \leq G K_\lambda N$. So without loss of generality we may assume that $K_\lambda N = G$ for every $\lambda \in \Lambda$. Since $N$ is central in $G$, $H \cap N \trianglelefteq G$. Clearly for every $\lambda \in \Lambda$, $(K_\lambda/(H \cap N))(N/(H \cap N)) = G/(H \cap N)$ and $(K_\lambda/(H \cap N)) \cap (N/(H \cap N))$ is trivial. Also $(N/(H \cap N)) \leq (Z(G)/(H \cap N)) \leq Z(G/(H \cap N))$. Moreover $(N/(H \cap N)) = \bigoplus_{i=1}^k (N_i/(H \cap N_i))$ and each $N_i/(H \cap N_i)$ is a pro-$p_i$ group. Hence without loss of generality we may assume that $G$ splits over $N$ and every $K_\lambda$ for $\lambda \in \Lambda$ is a complement to $N$ in $G$, and thus finite.

Then if $\lambda \in \Lambda$,

$$|\{K \leq G \mid K \text{ is a complement to } N \text{ in } G\}|$$

$$= |\text{Der}(K_\lambda, N)|$$

$$= |\text{Hom}(K_\lambda, N)| \quad \text{(as } N \text{ is central in } G)$$

$$= |\text{Hom}(K_\lambda, \bigoplus_{i=1}^k N_i)|$$

$$= |\bigoplus_{i=1}^k \text{Hom}(K_\lambda, N_i)|$$

But for each $i$, $\text{Hom}(K_\lambda, N_i)$ is finite. Thus there are only finite many complements to $N$ in $G$. But this is a contradiction since every $K_\lambda$ for $\lambda \in \Lambda$ is a complement to $N$ in $G$. ■

**Theorem 4.2** Let $G$ be a profinite group with $S(G)$ countably infinite. Let $k$ equal the number of primes $p$ such that $p^\infty \mid o(G)$. Then there exists a positive integer $n$ such that $S(G)$ is homeomorphic to $\omega^k n + 1$.

**Proof.** Firstly note that by Theorem 3.7 there are only finitely many primes $p$ such that $p^\infty \mid o(G)$. By Proposition 2.12 it suffices to show that $ht(S(G)) = k + 1$. Now by Theorem 3.7 $G$ has a central open subgroup $N$ such that $N$ is topologically isomorphic to $\bigoplus_{i=1}^k \mathbb{Z}_{p_i}$ for some finite non-empty set of primes $\{p_1, \ldots, p_k\}$ (with $p_i \neq p_j$ for $i \neq j$). Now clearly by Example 3.2 and Proposition 2.5 $S(N)$ is homeomorphic to $(\omega + 1)^k$. But $ht((\omega + 1)^k) = k + 1$ and so by Lemma 2.10 $ht(S(G)) \geq k + 1$. Thus it suffices to show that for every $H \leq C G$, $ht(H, S(G)) \leq k + 1$.

We know that $ht(H \cap N, S(N)) \leq k + 1$ and by Lemma 2.10 $ht(H \cap N, S(G)) \leq k + 1$. Hence it suffices to show that $ht(H, S(G)) \leq ht(H \cap N, S(G))$. Now by Lemma 2.2 there exists $U \trianglelefteq G$ with $U \leq N$ such that if $L \in B(H \cap N, U)$ then either $L = H \cap N$ or $ht(L, S(G)) < ht(H \cap N, S(G))$. By Lemma 4.1 $\{K \leq C G \mid K \cap N = H \cap N\}$ is finite. Hence since $S(G)$ is Hausdorff there exists $V \trianglelefteq G$ with $V \leq U$ such that $B(H, V) \cap \{K \leq C G \mid K \cap N = H \cap N\} = \{H\}$. Now let $K \in B(H, V)$. Then $K \in B(H, U)$ and
since $U \leq N$, $K \cap N \in B(H \cap N, U)$. Thus by the above choice of $U$, either $K = H \cap N$ or $ht(K \cap N, S(G)) < ht(H \cap N, S(G))$. But if $K = H \cap N$ then by the above $K = H$. Suppose that $ht(K \cap N, S(G)) < ht(H \cap N, S(G))$. Then, by induction on $ht(H \cap N, S(G))$, $ht(K, S(G)) \leq ht(K \cap N, S(G))$. So then $ht(K, S(G)) < ht(H \cap N, S(G))$. Hence $ht(H, S(G)) \leq ht(H \cap N, S(G))$ as required. $lacksquare$

**Corollary 4.3** Let $G$ be a pro-p group with $S(G)$ countably infinite. Let $n$ equal the number of closed non-open subgroups of $G$. Then $S(G)$ is homeomorphic to $\omega n + 1$.

## 5 Isolated Subgroups

In this section we characterise the profinite groups $G$ for which $S(G)$ is perfect (i.e. does not contain an isolated point). In particular we determine the profinite groups $G$ such that $S(G)$ is homeomorphic to the Cantor set. We also make some remarks on the situation for $\mathcal{N}(G)$.

### When Is $S(G)$ Perfect?

**Definition 5.1** If $G$ is a profinite group and $H \leq C G$ then $H$ is said to be an isolated subgroup of $G$ if $H$ is an isolated point of $S(G)$.

So, for example (cf. Example 3.2), for each non-negative integer $n$, $p^n\mathbb{Z}_p$ is an isolated subgroup of $\mathbb{Z}_p$ whereas 0 is not an isolated subgroup of $\mathbb{Z}_p$.

**Definition 5.2** For $G$ a profinite group, let $\Psi(G)$ be the intersection of the maximal (proper) open normal subgroups of $G$.

Recall that the Frattini subgroup, $\Phi(G)$, of $G$ is the intersection of the maximal (proper) open subgroups.

We require some well-known facts about the Frattini subgroup of a profinite group $G$ and about $\Psi(G)$. For example Lemma 5.3(i) is a profinite version of a well-known result about finite groups, cf. Lemma 11.4 of [10].

**Lemma 5.3** Let $G$ be a profinite group.

(i) If $K \leq C G$, then $K \leq \Phi(G)$ if and only if $H \leq C G$ and $HK = G$ implies that $H = G$. 


(ii) If \( K \triangleleft G \), then \( K \leq \Psi(G) \) if and only if \( H \triangleleft G \) and \( HK = G \) implies that \( H = G \).

(iii) \( \Phi(G) \leq \Psi(G) \).

(iv) If \( G \) is pronilpotent then \( \Phi(G) = \Psi(G) \).

**Proof.** For (i), first suppose that \( K \leq \Phi(G) \), and let \( H \) be a proper closed subgroup of \( G \). Then \( H \) is contained in a maximal open subgroup \( M \) of \( G \). Since \( K \leq \Phi(G) \), \( K \leq M \). But now \( HK \leq M \neq G \). Now suppose that \( K \not\leq \Phi(G) \). Now \( G \) must be non-trivial and there exists a maximal open subgroup \( M \) of \( G \) such that \( K \not\leq M \). But then \( M \) is a proper subgroup of \( MK \). Thus \( MK = G \). The argument for (ii) is basically the same.

For (iii), let \( N \) be a maximal open normal subgroup of \( G \). Then \( N \) is contained in a maximal open subgroup \( M \) of \( G \). Clearly \( N = M_G \). But each conjugate of \( M \) in \( G \) is a maximal open subgroup of \( G \). So \( N \) is an intersection of maximal open subgroups of \( G \). Thus \( \Phi(G) \leq N \), and so \( \Phi(G) \leq \Psi(G) \).

For (iv), let \( M \) be a maximal open subgroup of \( G \). Let \( N = M_G \). Then \( M/N \) is a maximal subgroup of \( G/N \). But \( G/N \) is a finite nilpotent group, and so \( M/N \triangleleft G/N \). Thus \( M \triangleleft G \), and so clearly \( \Phi(G) = \Psi(G) \). \( \blacksquare \)

Note that it follows immediately from Lemma 5.3(i) that if \( G \) is a profinite group and \( H \leq C_G \) with \( H\Phi(G) = G \) then \( H = G \) (cf. Proposition 2.5.1(a) of [19]). Of course the corresponding fact holds for \( \Psi(G) \).

We now prove some elementary facts about isolated subgroups.

**Lemma 5.4** Let \( G \) be a profinite group and \( H \leq C_G \).

(i) \( H \) is an isolated subgroup of \( G \) if and only if there exists an open normal subgroup \( N \) of \( G \) such that if \( K \leq C_G \) and \( KN = HN \) then \( K = H \).

(ii) If \( H \) is an isolated subgroup of \( G \) then \( H \leq O_G \).

(iii) If \( H \leq O_G \) then \( H \) is an isolated subgroup of \( G \) if and only if there exists an open normal subgroup \( N \) of \( G \) with \( N \leq H \) such that if \( K \leq C_G \) and \( KN = H \) then \( K = H \).

(iv) If \( H \leq O_G \) then \( H \) is an isolated subgroup of \( G \) if and only if \( \Phi(H) \leq O_H \).
Proof. By Lemma 2.2, $H$ is an isolated subgroup of $G$ if and only if $B(H, N) = \{H\}$ for some open normal subgroup $N$ of $G$. (i) now follows immediately.

For (ii), by (i) there is an open normal subgroup $N$ of $G$ such that if $K \leq C G$ and $KN = HN$ then $K = H$. Let $K = HN$. Then $KN = HN$, and so $HN = H$, that is $N \leq H$. Thus $H \leq O G$.

For (iii), let $H$ be an isolated open subgroup of $G$. Then by (i) there exists an open normal subgroup $N$ of $G$ such that if $K$ is a closed subgroup of $G$ and $KM = HN$ then $K = H$. Let $K = HN$. Then $KN = HN$, and so $HN = H$, that is $N \leq H$. Thus $H \leq O G$.

For (iv), first suppose that $H \leq O G$ and $\Phi(H) \not\sim O H$. Let $N \sim O G$ with $N \leq H$. Then $N \not\leq \Phi(H)$. So there exists a maximal open subgroup $M$ of $H$ such that $N \leq M$. Then $MN = H$ but $M \neq H$. Hence $H$ is not an isolated subgroup of $G$ by (iii).

Now suppose that $H$ is an open subgroup of $G$ and $\Phi(H) \sim O H$. Let $N = \Phi(H)_G$. Then $N \sim O G$ and $N \leq \Phi(H)$. Suppose for $K \leq C G$ that $KN = HN$. Then $KN = H$, so $H \leq K\Phi(H)$. But also $\Phi(H) \leq H$ and $K \leq H$, so $H = K\Phi(H)$. Thus $H = K$ by Lemma 5.3(i). Hence $H$ is an isolated subgroup of $G$ by (i).

The following lemma is well known.

Lemma 5.5

(i) Let $G$ be a profinite group and $p$ be a prime. Then $p \mid o(G)$ if and only if $p \mid |G : \Phi(G)|$.

(ii) Let $G$ be a profinite group. If $G$ is finitely generated, pronilpotent and only finitely many primes divide the order of $G$ then $\Phi(G) \sim O G$.

Proof. For (i), if $p \mid |G : \Phi(G)|$ then $p \mid o(G)$. So now suppose $p \mid o(G)$ but $p \nmid |G : \Phi(G)|$. Let $P$ be a Sylow-$p$ subgroup of $\Phi(G)$. Then by a profinite version of the Frattini argument (see Proposition 2.2.3(c) of [19]), $P \sim C G$. Clearly $p \nmid |G : P|$, and so $P$ is a normal Hall subgroup of $G$. Then by the profinite version of the Schur–Zassenhaus theorem (see Proposition 2.3.3 of [19]), there exists a closed subgroup $H$ of $G$ such that $G = P \rtimes H$. But then by Lemma 5.3(i), $H = G$, a contradiction.
For (ii), suppose $G$ is finitely generated, pronilpotent and only finitely many primes divide the order of $G$. Then since $G$ is pronilpotent it is topologically isomorphic to the product of its Sylow subgroups (see Proposition 2.4.3 of [19]). So there exist primes $p_1, \ldots, p_n$ such that $G$ is topologically isomorphic to $\bigoplus_{i=1}^{n} G_{p_i}$. Then $G/\Phi(G)$ is topologically isomorphic to $\bigoplus_{i=1}^{n} G_{p_i}/\Phi(G_{p_i})$ (see Lemma 20.4 of [3] for example). But for each $i$, $G_{p_i}$ is finitely generated and so $\Phi(G_{p_i}) \trianglelefteq O_{G_{p_i}}$ (see Proposition 1.14 [3]). Hence $\Phi(G) \trianglelefteq O_G$.

**Theorem 5.6** Let $G$ be a profinite group. Then the following are equivalent.

(i) $S(G)$ is not perfect.

(ii) $G$ has an isolated open subgroup.

(iii) Every open subgroup of $G$ is isolated.

(iv) $\Phi(G) \trianglelefteq O_G$.

(v) $G$ is finitely generated, virtually pronilpotent and only finitely many primes divide the order of $G$.

**Proof.** (i) $\Rightarrow$ (ii) follows immediately from Lemma 5.4(ii). Clearly (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). It is also clear by Lemma 5.4 that (iii) $\Rightarrow$ (iv) and that (iv) $\Rightarrow$ (ii).

For (ii) $\Rightarrow$ (v) let $H$ be an isolated open subgroup of $G$. Then by Lemma 5.4(iv), $\Phi(H) \trianglelefteq O_H$. So $H$ is finitely generated and hence $G$ is finitely generated. Also $\Phi(H)$ is pronilpotent and so $G$ is virtually pronilpotent. Suppose infinitely many primes divide the order of $G$. Then infinitely many primes divide the order of $H$, and so by Lemma 5.5(i), $\Phi(H) \not\trianglelefteq O_H$, a contradiction.

Finally for (v) $\Rightarrow$ (iii). Suppose that $G$ is finitely generated, virtually pronilpotent and only finitely many primes divide the order of $G$. Let $H \leq O G$. Then there exists an open normal subgroup $N$ of $H$ such that $N$ is pronilpotent. Now $N$ is finitely generated and only finitely many primes divide the order of $N$, so by Lemma 5.5(ii), $\Phi(N) \not\trianglelefteq O N$. Thus $\Phi(N) \leq O H$. But $\Phi(N) \leq \Phi(H)$. So $\Phi(H) \trianglelefteq O H$. Hence $H$ is an isolated subgroup of $G$ by Lemma 5.4(iv).

**Corollary 5.7** Let $G$ be a profinite group. Then $S(G)$ is homeomorphic to the Cantor set if and only if $G$ is countably based and not (finitely generated, virtually pronilpotent and only finitely many primes divide the order of $G$).
Proof. This follows immediately from Proposition 5.6, Corollary 2.4 and Proposition 2.8. □

Remark There is another way of associating a profinite space to a profinite group $G$. This is in terms of the Burnside algebra of $G$. Pierce has shown (see Theorem 6.6 of [8]) that the space obtained, $\Sigma(G)$ is not perfect if and only if $G$ is finitely generated, virtually pronilpotent and only finitely many primes divide the order of $G$. So by Proposition 5.6, this is precisely when $S(G)$ is not perfect. We do not know how, in general $\Sigma(G)$ is related to $S(G)$; though $\Sigma(G)$ can be constructed from the lattice of open subgroups of $G$.

When Is $\mathcal{N}(G)$ Perfect? We now consider what can be said about when $\mathcal{N}(G)$ is perfect for $G$ a profinite group. It is clear that by the same arguments we have a direct analogue of Lemma 5.4 using $\Psi(G)$. In particular we have the following.

Lemma 5.8 Let $G$ be a profinite group and $H \lhd G$.

(i) $H$ is isolated in $\mathcal{N}(G)$ if and only if there exists an open normal subgroup $N$ of $G$ such that if $K \lhd G$ and $KN = HN$ then $K = H$.

(ii) If $H$ is isolated in $\mathcal{N}(G)$ then $H \lhd \mathcal{O}G$.

(iii) If $H \lhd \mathcal{O}G$ then $H$ is isolated in $\mathcal{N}(G)$ if and only if $\Psi(H) \lhd \mathcal{O}H$.

Example 5.9 If $\kappa$ is any infinite cardinal then a profinite group with a unique maximal open normal subgroup of weight $\kappa$ can be constructed as a split extension of the Cartesian product of $\kappa$ copies of a finite simple group, by $A_5$.

For such a group $G$, clearly $\Psi(G) \lhd \mathcal{O}G$, and thus by Lemma 5.8 $G$ is isolated in $\mathcal{N}(G)$.

Hence there exist infinite profinite groups of arbitrary weight with $\mathcal{N}(G)$ not perfect. So we have no simple analogue of Theorem 5.6 for the space of normal subgroups. We can say the following.

Proposition 5.10 Let $G$ be a profinite group. If $\mathcal{N}(G)$ is perfect then $S(G)$ is perfect. If $G$ is pronilpotent and $S(G)$ is perfect then $\mathcal{N}(G)$ is perfect.
Proof. Suppose that $S(G)$ is not perfect. Then by Proposition 5.6, $\Phi(G) \triangleleft G$. Thus by Lemma 5.3(iii), $\Psi(G) \triangleleft G$. So, by Lemma 5.8(iii), $G$ is isolated in $N(G)$ and $N(G)$ is not perfect.

Now suppose that $G$ is pronilpotent. If $H \leq C_G$ then by Lemma 5.3(iv), $\Phi(H) = \Psi(H)$. It is now clear from the proof of Proposition 5.6 that $S(G)$ is perfect if and only if $N(G)$ is perfect.

6 Profinite Groups of Large Weight

In this section we examine the situation when our profinite group, $G$, is not countably based, $w(G) > \aleph_0$. By Theorem 5.6 we know $S(G)$ is perfect. Since for countably based $G$ with perfect $S(G)$, the space of subgroups is homeomorphic to $\{0, 1\}^{\aleph_0}$, the natural conjecture – solving simultaneously the problem of counting and topologically classifying $S(G)$ – is that $S(G)$ is homeomorphic to $\{0, 1\}^{w(G)}$. So we start this section by introducing some topological properties (Dugundji compactness and character homogeneity) which characterise the Cantor cube $\{0, 1\}^\kappa$. Then we note that $S(G)$ always has a property ($\kappa$-metrizability) just a little weaker than Dugundji compactness, and is always character homogeneous. We deduce that the conjecture is correct when $w(G) = \aleph_1$, but is false in general. Nevertheless from our topological considerations we have enough to deduce that $S(G)$ and $N(G)$ both have cardinality equal to $2^{w(G)}$. We conclude with a sketch of a more algebraic proof of the latter fact for $S(G)$. However we know of no algebraic proof that $|N(G)| = 2^{w(G)}$.

Topological Properties of Cantor Cubes

Definitions 6.1 Let $X$ be a topological space.

$X$ is said to be **Dugundji compact** if $X$ is compact and Hausdorff and whenever $Y$ is a profinite space, $Z$ is a closed subspace of $Y$, and $f: Z \to X$ is a continuous map, then $f$ can be extended to a continuous map $f': Y \to X$.

Now let $\mathcal{RC}(X)$ denote the family of all regularly closed subsets of $X$; that is $\mathcal{RC}(X) = \{F \subseteq X \mid F = \text{Int} \overline{F}\}$.

A $\kappa$-metric on $X$ is a non-negative function $\rho: X \times \mathcal{RC}(X) \to \mathbb{R}$ satisfying the following four conditions:

(i) $\rho(x, F) = 0$ if and only if $x \in F$;
(ii) if \( x \in X, F, F' \in \mathcal{RC}(X) \) and \( F \subseteq F' \), then \( \rho(x, F') \leq \rho(x, F) \);

(iii) given \( F \in \mathcal{RC}(X) \), the function \( \rho(\cdot, F) : X \to \mathbb{R} \) is continuous with respect to the first argument;

(iv) if \( (F_\alpha)_{\alpha<\tau} \) is an increasing transfinite sequence of regularly closed sets in \( X \), and \( x \in X \), then \( \rho(x, \bigcup_{\alpha<\tau} F_\alpha) = \inf_{\alpha<\tau} \rho(x, F_\alpha) \).

\( X \) is said to be **\( \kappa \)-metrizable** if \( X \) admits a \( \kappa \)-metric.

\( X \) is said to be **\( \kappa \)-adic** if there is a \( \kappa \)-metrizable space \( Y \) and a continuous surjection \( f : Y \to X \).

The definition of Dugundji compact spaces we have given is not the original definition due to Pe/csupr e/pelczyński. Our definition asserts that a Dugundji compact space is a compact Hausdorff space with the property of being an absolute extensor in dimension 0 (\( \text{AE}(0) \)). The equivalence of these definitions is due to Haydon, 17. Note that a profinite space is Dugundji compact if and only if it is an injective object in the category of profinite spaces. The class of \( \kappa \)-metrizable spaces and the class of \( \kappa \)-adic spaces are due to Shchepin, 16, 17. §7 of Shakhmatov, 14 is a survey of these three class of spaces.

We also require the following simple terms:

**Definitions 6.2** Let \( X \) be a topological space.

A **\( \pi \)-basis for the open neighbourhoods** of \( x \in X \) is a collection \( \mathcal{B}(x) \) of open sets in \( X \) such that every open neighbourhood of \( x \) contains an element of \( \mathcal{B}(x) \).

The **\( \pi \)-character** of \( X \) at \( x \in X \) is defined to be \( \pi \chi(x, X) = \min\{|\mathcal{B}(x)| \mid \mathcal{B}(x) \text{ is a } \pi \text{-basis for the open neighbourhoods of } x \} \).

\( X \) is said to be **character homogeneous** if \( \chi(x, X) = \chi(X) \) for every \( x \in X \).

In the next theorem we collect together all the topological results we require for this section.

**Theorem 6.3**

(Shchepin)
(i) A compact Hausdorff space $X$ is $\kappa$-metrizable if and only if $X$ is homeomorphic to the inverse limit of an inverse system $(X_i, p_{ij})$ of compact metrizable spaces indexed by a directed set $I$ such that every $p_{ij}$ is an open continuous surjective map and every countable chain of elements of $I$ has its least upper bound in $I$. 

(ii) Let $X$ be a compact Hausdorff space. If $X$ is Dugundji compact then $X$ is $\kappa$-metrizable. If $X$ is $\kappa$-metrizable and $w(X) = \aleph_1$, then $X$ is Dugundji compact. 

(iii) Let $\kappa$ be an infinite cardinal and $X$ be a profinite space with $w(X) = \kappa$. Then $X$ is homeomorphic to $\{0, 1\}^\kappa$ if and only if $X$ is Dugundji compact and character homogeneous. 

(iv) If $X$ is a compact Hausdorff $\kappa$-adic space then $\chi(X) = w(X)$. 

(v) If $X$ is a compact Hausdorff $\kappa$-metrizable space and $x \in X$ then $\pi\chi(x, X) = \chi(x, X)$.  

(Shapirovskii) 

(vi) Let $X$ be a compact Hausdorff $\kappa$-metrizable space. Then there exists a continuous surjective map $f : X \to [0, 1]^{w(X)}$ (where $[0, 1]^{w(X)}$ is the Hilbert cube of weight $w(X)$) if and only if $\pi\chi(x, X) = w(X)$ for some $x \in X$. 

Proof. For (i), see [16] and Theorem 21 of [18]. For (ii), see Corollary 1 of Theorem 4 and Theorem 5 of [17]. For (iii), see Theorem 9 of [16]. For (iv), see the Corollary to Theorem 11 of [17]. For (v), see Theorem 8 of [17]. For (vi), see [15], and also Theorem 7.21(vi) of [14]. 

By (v), it might seem unnecessary to consider $\pi$-characters in (vi). Shapirovskii proves (vi) for a wider class of spaces, where characters and $\pi$-characters may not coincide. We have introduced $\pi$-characters since in our application of (vi), we can avoid the use of (v); see the remark before Lemma 6.6. 

Topological Properties of $S(G)$ and $N(G)$ 

Theorem 6.4 (Fisher and Gartside, [5]) Let $G$ be a profinite group. Then $S(G)$ and $N(G)$ are $\kappa$-metrizable.
Lemma 6.5 Let $G$ be a profinite group with $S(G)$ perfect. Then $\chi(G, S(G)) = w(G)$. If $G$ is also pronilpotent then $\chi(G, N(G)) = w(G)$.

Proof. $\chi(G, S(G)) \leq w(S(G)) = w(G)$, Corollary 2.4. By Lemma 2.2 there exists a family $(N_\lambda)_{\lambda \in \Lambda}$ of open normal subgroups of $G$ with $|\Lambda| = \chi(G, S(G))$ and such that $(B(G, N_\lambda))_{\lambda \in \Lambda}$ is a base for the open neighbourhoods of $G$ in $S(G)$. As $S(G)$ is Hausdorff, $\bigcap_{\lambda \in \Lambda} B(G, N_\lambda) = \{G\}$. Hence if $H \leq C G$ and $HN_\lambda = G$ for every $\lambda \in \Lambda$ then $H = G$. Let $K = \bigcap_{\lambda \in \Lambda} N_\lambda$. Then if $H \leq C G$ with $HK = G$ then $HN_\lambda = G$ for every $\lambda \in \Lambda$ and so $H = G$. So by Lemma 6.6(i) $K \leq \Phi(G)$. As $S(G)$ is perfect, by Proposition 6.6 $\Phi(G) \not\leq_O G$. So $w(G/K) \geq w(G/\Phi(G)) = w(G)$ (the last equality follows from the fact that density and weight coincide in a profinite group, and properties of the Frattini subgroup, see Proposition 5.2.3(a) [19]). But $G/K$ embeds in $\prod_{\lambda \in \Lambda} G/N_\lambda$ and, $w(\prod_{\lambda \in \Lambda} G/N_\lambda) = |\Lambda|$. Hence $w(G) = |\Lambda| = \chi(G, S(G))$.

Now suppose that $G$ is also pronilpotent. Then by Lemma 5.3(iii) $\Psi(G) = \Phi(G)$. So we may use essentially the same argument above but using (ii) of Lemma 5.3 and choosing $(N_\lambda)_{\lambda \in \Lambda}$ such that $(B(H, N_\lambda) \cap N(G))_{\lambda \in \Lambda}$ is a base for the open neighbourhoods of $G$ in $N(G)$. ■

The next lemma follows immediately from Lemma 2.3, Proposition 6.4 and Theorem 6.3(v). We give a direct proof, since it is straightforward and we can then avoid using Theorem 6.3(v). The proof is very similar to the proof of Lemma 2.3.

Lemma 6.6 Let $G$ be an infinite profinite group. Then $\pi\chi(1,N(G)) = w(G)$.

Proof. $\pi\chi(1,N(G)) \leq \chi(1,N(G)) = w(G)$, by Lemma 2.3. Now there exists a family $(N_\lambda)_{\lambda \in \Lambda}$ of open normal subgroups of $G$ and a family $(H_\lambda)_{\lambda \in \Lambda}$ of closed normal subgroups of $G$ such that $(B(H_\lambda, N_\lambda) \cap N(G))_{\lambda \in \Lambda}$ is a $\pi$-base for the open neighbourhoods of $1$ in $N(G)$ and $|\Lambda| = \pi\chi(1,N(G))$. Let $N <_O G$. Then $B(1,N) \supseteq B(H_\lambda, N_\lambda)$ for some $\lambda \in \Lambda$. Clearly $H_\lambda N_\lambda \in B(H_\lambda, N_\lambda)$ and so $(N_\lambda)_{\lambda \in \Lambda}$ is a base for the open neighbourhoods of $1$ in $G$. Hence $w(G) = \chi(1,G) \leq |\Lambda| = \pi\chi(1,N(G))$, and so $\pi\chi(1,N(G)) = w(G)$ as required. ■

Proposition 6.7 Let $G$ be a profinite group with $S(G)$ perfect. Then $S(G)$ is character homogeneous.
Proof. For every $H \leq C G$, $\chi(H, S(G)) \leq w(S(G)) = w(G)$ by Corollary 2.4. Suppose for a contradiction that there exists $H \leq C G$ with $\chi(H, S(G)) < w(G)$. Then $\chi(H, H^G) < w(G)$ where $H^G$ is the space of conjugates in $G$ of $H$. Now, clearly $H^G$ is homogeneous, and is isomorphic, as a $G$-space, to the coset space $G/ N_G(H)$. Thus $\chi(G/ N_G(H)) < w(G)$. But, by Proposition 6.4, $S(G)$ is $\kappa$-metrizable, and so $G/ N_G(H)$ is $\kappa$-adic. Hence, by Theorem 6.3(iv), $w(G/ N_G(H)) < w(G)$. Thus $w(N_G(H)) = w(G)$.

Now $S(N_G(H)/H)$ embeds in $S(N_G(H))$, the trivial subgroup of $N_G(H)/H$ being mapped to $H$. So by Lemma 2.3 $w(N_G(H)/H) \leq \chi(H, S(N_G(H))) \leq \chi(H, S(G))$. Now, $w(G) = w(N_G(H)) = w(N_G(H)/H)w(H)$. If $S(H)$ is not perfect, then by Proposition 5.6 $H$ is countably based. Hence $w(G) = w(N_G(H)/H) < \chi(H, S(G))$, a contradiction. So $S(H)$ must be perfect. But, then by Lemma 6.5 $w(H) = \chi(H, S(H)) \leq \chi(H, S(G))$. Hence $w(G) = w(N_G(H)/H)w(H) \leq \chi(H, S(G))$, a contradiction. ■

Counting and Classifying Large Subgroup Spaces

Theorem 6.8 Let $G$ be a profinite group with $w(G) = \aleph_1$. Then $S(G)$ is homeomorphic to $\{0, 1\}^{\aleph_1}$.

Proof. By Proposition 6.4 and Theorem 6.3(ii), $S(G)$ is Dugundji compact. By Proposition 5.6 $S(G)$ is perfect and so by Proposition 6.7 $S(G)$ is character homogeneous. The result now follows from Theorem 6.3(iii). ■

Haydon (private communication) has shown that $S(C_2^{\aleph_2})$ is not Dugundji compact. In particular $S(C_2^{\aleph_2})$ is not homeomorphic to $\{0, 1\}^{\aleph_2}$.

Theorem 6.9 Let $G$ be a non-countably based profinite group. Then $|S(G)| = |\mathcal{N}(G)| = 2^{w(G)}$.

Proof. We know that $|\mathcal{N}(G)| \leq |S(G)| \leq 2^{w(G)}$. By Proposition 6.4, $\mathcal{N}(G)$ is $\kappa$-metrizable. Also $\pi\chi(1, \mathcal{N}(G)) = w(G)$ by Lemma 6.6 (or by Lemma 2.3 and Theorem 6.3(v)). Hence, by Theorem 6.3(vi), $\mathcal{N}(G)$ maps continuously onto $[0, 1]^{w(G)}$. The result then follows. ■

Corollary 6.10 For any profinite group $G$:
1) $S(G)$ is either countable or of size $2^{w(G)}$.
2) $\mathcal{N}(G)$ is either countable or of size $2^{w(G)}$. 

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Proof. Let $G$ be any profinite group. If $G$ is not countably based, then by the previous theorem, $|S(G)| = 2^{w(G)} \geq 2^{{\aleph}_{0}}$, as required. Hence if $G$ has $< 2^{{\aleph}_{0}}$ many closed subgroups, then it must be countably based. In which case $S(G)$ is a compact countably based space, and compact countably based spaces can only be countable or of size $2^{{\aleph}_{0}} = 2^{w(G)}$, again as required.

The proof for the space of closed normal subgroups is identical. ■

An Algebraic Proof that $|S(G)| = 2^{w(G)}$

Proposition 6.11 Let $G$ be a non-countably based profinite group. Then $G$ has precisely $2^{w(G)}$ closed subgroups. In fact $G$ has precisely $2^{w(G)}$ procyclic subgroups.

Proof. It suffices to show that $G$ has $2^{w(G)}$ procyclic subgroups. Let $(X_{\lambda})_{\lambda \in \Lambda}$ be a partition of $G$ such that for $x, y \in G$, $\langle x \rangle = \langle y \rangle$ if and only if $x, y \in X_{\lambda}$ for some $\lambda \in \Lambda$. As $G$ is uncountable, $|G| = 2^{w(G)}$. Thus $2^{w(G)} = \sum_{\lambda \in \Lambda} |X_{\lambda}| = |\Lambda| \cdot \sup_{\lambda \in \Lambda} |X_{\lambda}|$. Each procyclic subgroup is countably based and thus has cardinality at most $2^{{\aleph}_{0}}$. Hence $|X_{\lambda}| \leq 2^{{\aleph}_{0}}$ for each $\lambda \in \Lambda$. Thus $\sup_{\lambda \in \Lambda} |X_{\lambda}| \leq 2^{{\aleph}_{0}}$. Consequently, $|\Lambda| \leq 2^{w(G)} \leq |\Lambda| \cdot 2^{{\aleph}_{0}}$. But $w(G) > {\aleph}_{0}$ and so $|\Lambda| = 2^{w(G)}$ as required. ■

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