Triple Linkage of Quadratic Pfister Forms

Adam Chapman

Department of Computer Science, Tel-Hai Academic College, Upper Galilee, 12208 Israel

Andrew Dolphin

Department of Mathematics, Ghent University, Ghent, Belgium

David B. Leep

Department of Mathematics, University of Kentucky, Lexington, KY 40506

Abstract

Given a field $F$ of characteristic 2, we prove that if every three quadratic $n$-fold Pfister forms have a common quadratic $(n-1)$-fold Pfister factor then $I_{n+1}^1 F = 0$. As a result, we obtain that if every three quaternion algebras over $F$ share a common maximal subfield then $u(F)$ is either 0, 2 or 4. We also prove that if $F$ is a nonreal field with char($F$) $\neq$ 2 and $u(F) = 4$, then every three quaternion algebras share a common maximal subfield.

Keywords: Quadratic Forms, Quaternion Algebras, Linked Fields, $u$-Invariant

2010 MSC: 11E81 (primary); 11E04, 16K20, 11R52 (secondary)

1. Introduction

We say that a set of quadratic $n$-fold Pfister forms is linked if there exists a quadratic or bilinear $(n-1)$-fold Pfister form which is a common factor to all the forms in this set. By the natural identification of quaternion algebras with their norm forms which are quadratic 2-fold Pfister forms, a set of quaternion algebras is linked if they share a common maximal subfield.

A maximal subfield $K$ of a quaternion algebra over $F$ is a quadratic field extension of $F$. When char($F$) = 2, $K/F$ can be either separable or inseparable, and one can refine the definition of linkage accordingly: a set of quaternion algebras is separably (inseparably) linked if they share a common separable (inseparable) quadratic field extension of $F$. It was observed in [Dra75] that inseparable linkage for pairs of quaternion algebras implies separable linkage, and a counter example for the converse was provided in [Lam02]. This observation was extended to pairs of Hurwitz algebras in [EV05] and pairs of cyclic $p$-algebras of any prime degree in [Cha15].

Email addresses: adam1chapman@yahoo.com (Adam Chapman), Andrew.Dolphin@uantwerpen.be (Andrew Dolphin), leep@uky.edu (David B. Leep)
We extend the notion of separable and inseparable linkage of quaternion algebras to arbitrary \( n \)-fold Pfister forms in the following way: a set of quadratic \( n \)-fold Pfister forms are separably (inseparably) linked if there exists a quadratic (bilinear) \((n-1)\)-fold Pfister form which is a common factor to all the forms in the set. It can be easily concluded from the fact mentioned above (as can be seen in [Fai08]) that if two quadratic \( n \)-fold Pfister forms \( \varphi_1 \) and \( \varphi_2 \) satisfy \( \varphi_1 = B \otimes \pi_1 \) and \( \varphi_2 = B \otimes \pi_2 \) for some bilinear \((n-1)\)-fold Pfister form \( B \) and quadratic 1-fold Pfister forms \( \pi_1 \) and \( \pi_2 \), then \( \varphi_1 \) and \( \varphi_2 \) are separably linked.

It was proven in [EL73, Main Theorem] that if \( F \) is a field with \( \text{char}(F) \neq 2 \) and every two quaternion algebras over \( F \) are linked then the possible values \( u(F) \) can take are 0, 1, 2, 4 and 8, where \( u(F) \) is defined to be the supremum of the dimensions of nonsingular anisotropic quadratic forms over \( F \) of a finite order in \( W_q F \). For fields \( F \) of characteristic 2, it was shown in [Bae82, Theorem 3.1] that every two quaternion algebras over \( F \) share a quadratic inseparable field extension of \( F \) if and only if \( u(F) \leq 4 \), and in [CD, Corollary 5.2] that if every two quaternion algebras over \( F \) share a maximal subfield then \( u(F) \) is either 0, 2, 4 or 8.

In [Bec], the case of linkage of three bilinear \( n \)-fold Pfister forms in any characteristic was studied. It was shown in [Bec, Theorem 5.1] that if \( F \) is a nonreal field and every three bilinear \( n \)-fold Pfister forms are linked then \( I^{n+1}_q F = 0 \). When \( \text{char}(F) \neq 2 \), by the natural identification of quadratic forms with their underlying symmetric bilinear forms, this implies that if every three \( n \)-fold Pfister forms are linked then \( I^{n+1}_q F = 0 \). This was used to study the Hasse number of \( F \). The Hasse number, denoted \( \tilde{u}(F) \), is the supremum of the dimensions of anisotropic totally indefinite quadratic forms over \( F \). Note that if \( F \) is nonreal, then \( \tilde{u}(F) = u(F) \). It was shown in [Bec, Corollary 5.8] that if \( F \) is a field with \( \text{char}(F) \neq 2 \) and every three quaternion algebras over \( F \) are linked then \( \tilde{u}(F) \leq 4 \). The question of whether the converse held, that is whether \( \tilde{u}(F) \leq 4 \) implies that all triples of quaternion algebras over \( F \) are linked, appeared in a preliminary version of [Bec].

In this paper we prove that if \( \text{char}(F) = 2 \) and every three quadratic \( n \)-fold Pfister forms in \( F \) are linked then \( I^{n+1}_q F = 0 \). We conclude that if every three quaternion algebras over \( F \) are linked then \( u(F) \leq 4 \). In the last section we show that if \( F \) is a nonreal field with \( \text{char}(F) \neq 2 \) and \( u(F) = 4 \) then every three quaternion algebras over \( F \) are linked. Note that we shared this last result with the author of [Bec] who included a similar proof in the final version of that paper, acknowledging our contribution.

2. Bilinear and Quadratic Pfister Forms

Let \( F \) be a field with \( \text{char}(F) = 2 \). We recall what we need from the algebraic theory of quadratic forms. For general reference see [EKM08, Chapters 1 and 2].

Let \( V \) be an \( n \)-dimensional \( F \)-vector space. A symmetric bilinear form on \( V \) is a map \( B : V \times V \to F \) satisfying \( B(v, w) = B(w, v) \), \( B(cv, w) = cB(v, w) \) and \( B(v + w, t) = B(v, t) + B(w, t) \) for all \( v, w, t \in V \) and \( c \in F \). A symmetric bilinear form \( B \) is degenerate if there exists a vector \( v \in V \setminus \{0\} \) such that \( B(v, w) = 0 \) for all \( w \in V \). If such a vector does not exist, we say that \( B \) is nondegenerate. Two symmetric bilinear forms \( B : V \times V \to F \) and \( B' : W \times W \to F \) are isometric if there exists an isomorphism \( M : V \to W \) such that \( B(v, v') = B'(Mv, Mv') \) for all \( v, v' \in V \).
A quadratic form over $F$ is a map $\varphi : V \to F$ such that $\varphi(av) = a^2\varphi(v)$ for all $a \in F$ and $v \in V$ and the map defined by $B_\varphi(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$ for all $v, w \in V$ is a bilinear form on $V$. The bilinear form $B_\varphi$ is called the polar form of $\varphi$ and is clearly symmetric. Two quadratic forms $\varphi : V \to F$ and $\psi : W \to F$ are isometric if there exists an isomorphism $M : V \to W$ such that $\varphi(v) = \psi(Mv)$ for all $v \in V$. We are interested in the isometry classes of quadratic forms, so when we write $\varphi = \psi$ we actually mean that they are isometric.

We say that $\varphi$ is singular if $B_\varphi$ is degenerate, and that $\varphi$ is nonsingular if $B_\varphi$ is nondegenerate. Every nonsingular form $\varphi$ is even dimensional and can be written as

$$\varphi = [(\alpha_1, \beta_1)] \perp \cdots \perp [(\alpha_n, \beta_n)]$$

for some $\alpha_1, \ldots, \beta_n \in F$, where $[(\alpha, \beta)]$ denotes the two-dimensional quadratic form $\psi(x, y) = \alpha x^2 + xy + \beta y^2$ and $\perp$ denotes the orthogonal sum of quadratic forms.

We say that a quadratic form $\varphi : V \to F$ is isotropic if there exists a vector $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$. If no such vector exists, we say that $\varphi$ is anisotropic. The unique nonsingular two-dimensional isotropic quadratic form is $\mathbb{H} = [0, 0]$, called “the hyperbolic plane”. A hyperbolic form is an orthogonal sum of hyperbolic planes. We say that two nonsingular quadratic forms are Witt equivalent if their orthogonal sum is a hyperbolic form.

We denote by $\langle \alpha_1, \ldots, \alpha_n \rangle$ the diagonal bilinear form given by $(x, y) \mapsto \sum_{i=1}^n \alpha_i x_i y_i$. Given two symmetric bilinear forms $B_1 : V \times V \to F$ and $B_2 : W \times W \to F$, the tensor product of $B_1$ and $B_2$ denoted $B_1 \otimes B_2$ is the unique $F$-bilinear map $B_1 \otimes B_2 : (V \otimes_F W) \times (V \otimes_F W) \to F$ such that $(B_1 \otimes B_2)((v_1 \otimes w_1), (v_2 \otimes w_2)) = B_1(v_1, v_2) \cdot B_2(w_1, w_2)$ for all $v_1, w_2 \in V, v_1, v_2 \in W$.

A bilinear $n$-fold Pfister form over $F$ is a symmetric bilinear form isometric to a bilinear form

$$\langle 1, \alpha_1 \rangle \otimes \langle 1, \alpha_2 \rangle \otimes \cdots \otimes \langle 1, \alpha_n \rangle$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^\times$. We denote such a form by $\langle \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \rangle$. By convention, the bilinear 0-fold Pfister form is $\langle 1 \rangle$.

Let $B : V \times V \to F$ be a symmetric bilinear form over $F$ and $\varphi : W \to F$ be a quadratic form over $F$. We may define a quadratic form $B \circ \varphi : V \otimes_F W \to F$ determined by the rule that $(B \circ \varphi)(v \otimes w) = B(v, v) \cdot \varphi(w)$ for all $w \in W, v \in V$. We call this quadratic form the tensor product of $B$ and $\varphi$. A quadratic $n$-fold Pfister form over $F$ is a tensor product of a bilinear $(n-1)$-fold Pfister form $\langle \langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle \rangle$ and a two-dimensional quadratic form $[1, \beta]$ for some $\beta \in F$. We denote such a form by $\langle \langle \alpha_1, \ldots, \alpha_{n-1}, \beta \rangle \rangle$. Quadratic $n$-fold Pfister forms are isotropic if and only if they are hyperbolic (see [EKM08, (9.10)]).

The set of Witt equivalence classes of nonsingular quadratic forms is an abelian group with $\perp$ as the binary group operation and the class of $\mathbb{H}$ as the zero element. We denote this group by $I_qF$ or $I^q_0F$. This group is generated by scalar multiples of quadratic 1-fold Pfister forms. Let $I^q_0F$ denote the subgroup generated by scalar multiples of quadratic $n$-fold Pfister forms.

A quaternion algebra over $F$ is an $F$-algebra of the form $[\alpha, \beta]_{2,F} = F(x, y : x^2 + x = \alpha, y^2 = \beta, xyx^{-1} = x + 1)$ for some $\alpha \in F$ and $\beta \in F^\times$. Its norm form is the quadratic 2-fold Pfister form $\langle \langle \beta, \alpha \rangle \rangle$. This matching provides a 1-to-1 correspondence between
quaternion $F$-algebras and quadratic 2-fold Pfister forms over $F$. Therefore, the notions of separable and inseparable linkage translate naturally from quaternion algebras to 2-fold Pfister forms. We extend these notions of linkage to quadratic $n$-fold Pfister forms for any $n \geq 2$ in the following way: a set $S = \{\varphi_1, \ldots, \varphi_m\}$ of $m$ quadratic $n$-fold Pfister forms is separably linked if there exist a quadratic $(n-1)$-fold Pfister form $\phi$ and bilinear 1-fold Pfister forms $B_1, \ldots, B_m$ such that $\varphi_i = B_i \otimes \phi$ for all $i \in \{1, \ldots, m\}$. The set $S$ is inseparably linked if there exists a bilinear $(n-1)$-fold Pfister form $B$ and quadratic 1-fold Pfister forms $\phi_1, \ldots, \phi_m$ such that $\varphi_i = B \otimes \phi_i$ for all $i \in \{1, \ldots, m\}$.

We make use of the following well-known results on quaternion algebras, and, in the case of (1), its reinterpretation in terms of 2-fold Pfister forms.

**Lemma 2.1** ([BO13, VII.1.9]). For $\alpha \in F$ and $\beta, \gamma \in F^\times$ we have

1. $[\alpha, \beta]_2 F \cong [\alpha^2 + \beta, \beta]_2 F$. Further if $\alpha \neq 0$, then $[\alpha, \beta]_2 F \cong [\alpha, \beta\alpha]_2 F$.
2. $[\alpha, \beta]_2 F \otimes_F [\alpha, \gamma]_2 F$ is Brauer equivalent to $[\alpha, \beta\gamma]_2 F$.

**3. Triple Linkage for Quadratic Pfister Forms**

In this section we study properties of fields $F$ with $\text{char}(F) = 2$ in which every three quadratic $n$-fold Pfister forms are linked for some given integer $n \geq 3$. The case $n = 2$ is somewhat different and will be handled in the next section. We make use of the following invariant defined in [CGV, Definition 4.1]:

**Definition 3.1.** Let $n$ be an integer $\geq 2$. Consider a set $S$ of quadratic $n$-fold Pfister forms. We say $S$ is tight if every element in the subgroup $G$ of $I_n q F/I_{n+1} q F$ generated by the Witt classes of the forms in $S$ is represented by a quadratic $n$-fold Pfister form. For such a finite tight set, we define $\Sigma_S$ to be the Witt Class of the orthogonal sum of the quadratic $n$-fold Pfister representatives of the elements in $G$. By the identification of quaternion algebras with their norm forms, the notion of tightness and the associated invariant also apply for sets of quaternion algebras in the Brauer group.

A similar invariant was studied in [Siv14] when $\text{char}(F) \neq 2$ and $|S| = 3$.

**Lemma 3.2.** Let $n$ and $k$ be integers $\geq 2$ and let $S = \{\varphi_1, \ldots, \varphi_k\}$ be a tight set of $k$ quadratic $n$-fold Pfister forms over a field $F$ with $\text{char}(F) = 2$.

1. If $S$ is separably linked then $\Sigma_S$ is the Witt class of a quadratic $(n+k-1)$-fold Pfister form. More precisely, if $\varphi_i = \langle \langle a_i \rangle \rangle \otimes \phi$ for each $i \in \{1, \ldots, k\}$, then $\Sigma_S = \langle \langle a_1, \ldots, a_k \rangle \rangle \otimes \phi$.
2. If $S$ is inseparably linked then $\Sigma_S$ is the trivial Witt class.

**Proof.** The first statement is a special case of [CGV, Proposition 4.3]. The second statement is an immediate result of [CGV, Corollary 3.6 and Proposition 4.2 (2)]

**Theorem 3.3.** Let $F$ be a field with $\text{char}(F) = 2$. Then for any integer $n \geq 3$, if every set of three quadratic $n$-fold Pfister forms over $F$ is either separably or inseparably linked then $I_{q+1} F = 0$. 

4
Proof. Let $n$ be an integer $\geq 3$. Consider a quadratic $(n + 1)$-fold Pfister form $\Phi$. Then $\Phi$ can be written as $\langle \langle a, b, c \rangle \rangle \otimes \psi$ where $\psi$ is some quadratic $(n–2)$-fold Pfister form and $a, b, c$ are some elements in $F^\times$. Consider the forms $\varphi_1 = \langle \langle a, b, c \rangle \rangle \otimes \psi$, $\varphi_2 = \langle \langle a, c \rangle \rangle \otimes \psi$ and $\varphi_3 = \langle \langle b, c, ac \rangle \rangle \otimes \psi$. Then in $I_q^n/F/I_q^{n+1}F$ we have

$$\varphi_1 \perp \varphi_2 = \langle \langle b, ab, c, ac \rangle \rangle \otimes \psi = \langle \langle a, bc \rangle \rangle \otimes \psi \mod I_q^{n+1}F.$$ Similarly,

$$\varphi_1 \perp \varphi_3 = \langle \langle b, ac \rangle \rangle \otimes \psi \mod I_q^{n+1}F,$$

and

$$\varphi_2 \perp \varphi_3 = \langle \langle c, ab, c \rangle \rangle \otimes \psi \mod I_q^{n+1}F.$$

Therefore $S = \{\varphi_1, \varphi_2, \varphi_3\}$ is a tight triplet. A straightforward computation then gives that $\Sigma_S$ is Witt equivalent to $\Phi$.

Since $\varphi_1, \varphi_2, \varphi_3$ are quadratic $n$-fold Pfister forms, $S$ is separably or inseparably linked. If they are separably linked then $\Sigma_S$ is a quadratic $(n + 2)$-fold Pfister form by Lemma 3.2 (1). By the Hauptstatz theorem [EKM08, 23.7], this can happen only when $\Phi$ is hyperbolic. If $S$ is inseparably linked then $\Sigma_S$ is hyperbolic by Lemma 3.2 (2) and so $\Phi$ is hyperbolic as well. Consequently $I_q^{n+1}F = 0$. □

Corollary 3.4. For any integer $n \geq 3$, if every set of three quadratic $n$-fold Pfister forms over $F$ is separably linked then every set of two quadratic $n$-fold Pfister forms over $F$ is inseparably linked.

Proof. By the previous theorem $I_q^{n+1}(F) = 0$. By the assumption, every set of two quadratic $n$-fold Pfister forms over $F$ is separably linked. By [CGV, Corollary 5.4], $I_q^{n+1}(F) = 0$ implies that two quadratic $n$-fold Pfister forms over $F$ are separably linked if and only if they are inseparably linked. Hence every set of two quadratic $n$-fold Pfister forms over $F$ are inseparably linked. □

4. Triple Linkage for Quaternion Algebras

We again fix $F$ to be a field with char($F$) = 2. In this section we complete the picture for the case of quadratic 2-fold Pfister forms over $F$. These forms are in 1-1 correspondence with quaternion algebras and our proofs are mainly written in terms of these algebras. The proof in this particular case is somewhat different from the case of higher-fold Pfister forms.

Lemma 4.1. Let $\alpha, \beta, \lambda \in F$ with $\lambda^2 \neq \beta \neq 0$. Then $[\alpha, \beta]_{2,F} = [\alpha + \lambda^2 a \beta^{-1}, \beta + \lambda^2]_{2,F}$.

Proof. Write $[\alpha, \beta]_{2,F} = F(x, y : x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1)$. Let $w = y + \lambda$ and $z = x + \lambda y^{-1}$. Note that $w^2 = \lambda^2 + \beta \in F^\times$, and in particular $w$ is invertible. We have that $wz - zw = y + \lambda = w$, i.e. $w2w^{-1} = z + 1$. Therefore

$$[\alpha, \beta]_{2,F} = [z^2 + z, w^2]_{2,F} = [\alpha + \lambda^2 a \beta^{-1}, \beta + \lambda^2]_{2,F},$$

as required. □
Lemma 4.2. Let \( \alpha \in F, \beta, \gamma \in F^\times \) with \( \beta \neq \alpha^2 \). Then the quaternion algebras \( \psi = [\alpha^2 + \beta, \gamma]_{2,F} \) and \( \phi = [\beta + \alpha^4 \beta^{-1}, \beta + \alpha^2]_{2,F} \) are separably linked and \( \Sigma_{[\psi, \phi]} \) is Witt equivalent to \( \langle \langle \gamma, \beta, \alpha \rangle \rangle \).

Proof. Let \( \phi' = [\alpha^2 + \beta, \beta]_{2,F} \). Then by Lemma 3.1 (taking \( \lambda = \alpha \)) we have
\[
\phi' = [\alpha^2 + \beta + \alpha^2 (\alpha^2 + \beta) \beta^{-1}, \beta + \alpha^2]_{2,F} = [\beta + \alpha^4 \beta^{-1}, \beta + \alpha^2]_{2,F} = \phi.
\]
Hence \( \psi \) and \( \phi \) are separably linked. Furthermore, by Lemma 2.1 (2) we have that \([\alpha^2 + \beta, \beta \gamma]_{2,F} = \psi \otimes_F \phi \) in the Brauer group. Hence \( S = \{ \psi, \phi \} \) is a tight triplet, and by Lemma 3.2 (1) and Lemma 2.1 (1) we have \( \Sigma_S = \langle \langle \gamma, \beta, \alpha^2 + \beta \rangle \rangle = \langle \langle \gamma, \beta, \alpha^2 + \beta \rangle \rangle \).

Proposition 4.3. Let \( \alpha \in F, \beta, \gamma \in F^\times \) with \( \beta \neq \alpha^2 \). Then the quaternion algebras \( \psi = [\beta + \alpha^4 \beta^{-1}, \gamma]_{2,F}, \phi = [\beta + \alpha^4 \beta^{-1}, \gamma (\alpha^2 + \beta)]_{2,F} \) and \( \pi = [\alpha^2 + \beta, \gamma]_{2,F} \) form a tight triplet and \( \Sigma_{[\psi, \phi, \pi]} \) is Witt equivalent to \( \langle \langle \gamma, \beta, \alpha \rangle \rangle \).

Proof. The algebra \( \pi \) is evidently inseparably, and hence separably (by Lam02, Lemma 4.2), linked to \( \psi \) and to \( \phi \) using Lemma 4.1 (1). Further, \( \psi \) and \( \phi \) are separably linked. By Lemma 2.1 (2), we have that \( [\beta + \alpha^4 \beta^{-1}, \alpha^2 + \beta]_{2,F} = \xi \) is the quaternion representative of \( \psi \otimes_F \phi \) in the Brauer group. By Lemma 4.2, \( \xi \) is separably linked to \( \pi \) as well. Using Lemma 2.1 (2) we see that the tensor product of any two separably linked quaternion algebras is represented by a quaternion algebra in the Brauer group. We conclude that \( S = \{ \psi, \phi, \pi \} \) is a tight triplet of quaternion algebras.

Since the pairs \( \psi, \phi \) and \( \psi, \pi \) are separably linked, by [CGV, Lemma 11.2] we have that \( \Sigma_S \) is Witt equivalent to \( \Sigma_{\{\xi, \pi\}} \). Finally by Lemma 4.2 we have that \( \Sigma_{\{\xi, \pi\}} \) is Witt equivalent to \( \langle \langle \gamma, \beta, \alpha \rangle \rangle \).

Theorem 4.4. If every set of three quaternion algebras over \( F \) is linked, then \( I_q^3 F = 0 \).

Proof. Let \( \varphi = \langle \langle \gamma, \beta, \alpha \rangle \rangle \) be an arbitrary quadratic 3-fold Pfister form. If \( \beta = \alpha^2 \) then \( \varphi \) is hyperbolic. Otherwise, by Proposition 4.3 \( \varphi \) is Witt equivalent to \( \Sigma_S \) for some linked set \( S \) of three quaternion algebras. By the assumption, these three quaternion algebras share a common maximal subfield. If this subfield is a separable field extension of \( F \) then by Lemma 3.2 (1) we have that \( \Sigma_G \) is Witt equivalent to a quadratic 4-fold Pfister form. If this subfield is an inseparable field extension of \( F \) then by Lemma 3.2 (2) we have that \( \Sigma_G \) is hyperbolic. In both cases, \( \varphi \) must be hyperbolic, and so \( I_q^3 F = 0 \).

Corollary 4.5. If \( F \) is linked and \( I_q^3(F) = 0 \) then \( u(F) \leq 4 \). In particular, if every set of three quaternion algebras over \( F \) is linked then \( u(F) \leq 4 \).

Proof. If \( I_q^3 F = 0 \), by [Cha17, Lemma 4.3] then there exists an anisotropic nonsingular quadratic form of dimension \( u(F) \) whose Witt class is in \( I_q^3 F \). However, such a form is a direct sum of 2-fold Pfister forms (recall that when \( I_q^3 F = 0 \), 2-fold Pfister forms are isometric to their similar forms), and since every two quaternion algebras over \( F \) are linked, such a direct sum is Witt equivalent to one 2-fold Pfister form. Consequently, \( u(F) \leq 4 \). The second statement follows from Theorem 4.4.
5. Characteristic different from 2

Let $F$ be now a field of $\text{char}(F) \neq 2$. It is the case that $\tilde{u}(F) = 4$ implies that every three quaternion algebras over $F$ are linked? In this section we prove that this is indeed the case if the field is nonreal, in which case $\tilde{u}(F) = u(F)$. Note that this question is trivial if $u(F) < 4$, as then there are no division quaternion algebras over $F$.

The main tool is the complex constructed in [Pey95] and studied further in [QMT15]. For a finite subset $U$ of $Br(F)$, the Brauer group of $F$, we let $F^\times \cdot U$ denote the subgroup of $H^3(F,\mu_2)$ generated by classes $\lambda \cdot \alpha$ with $\lambda \in F^\times$ and $\alpha \in U$.

**Proposition 5.1** ([QMT15, Theorem 3.13]). Let $S = \{Q_1, Q_2, Q_3\}$ be a tight set of quaternion algebras over $F$ and $U$ be the subgroup $\langle Q_1, Q_2, Q_3 \rangle$ of the Brauer group $Br(F) \cong H^2(F, Q/[\mathbb{Z}/2])$. Further, let $M$ be the function field of the Cartesian product of the underlying Severi-Brauer varieties of $Q_1, Q_2, Q_3$. Then the homology of the complex

$$
\begin{array}{c}
F^\times \cdot U \\ H^3(F, Q/[\mathbb{Z}/2]) \\
\text{res} \\ H^3(M, Q/[\mathbb{Z}/2])
\end{array}
$$

has order 1 or 2. It is of order 1 if and only if $S$ is linked.

**Proposition 5.2.** Let $F$ be a field with $\text{char}(F) \neq 2$ and $I_q^3(F) = 0$. Then every tight set of three quaternion algebras over $F$ is linked.

*Proof.* Let $S = \{Q_1, Q_2, Q_3\}$ be a tight set of quaternion algebras over $F$, $U$ be the subgroup of $Br(F)$ generated by $S$ and $M$ be the function field of the Cartesian product of the Severi-Brauer varieties of $Q_1, Q_2, Q_3$. Since $H^3(F, Q/[\mathbb{Z}/2]) \cong I_q^3(F) = 0$, the orders of elements in $H^3(F, Q/[\mathbb{Z}/2])$ must be odd or infinite. By Proposition 5.1 the order of the homology of the complex

$$
\begin{array}{c}
F^\times \cdot U \\ H^3(F, Q/[\mathbb{Z}/2]) \\
\text{res} \\ H^3(M, Q/[\mathbb{Z}/2])
\end{array}
$$

is either 1 or 2. In particular, the kernel of the restriction map cannot contain any elements of infinite order, as elements of $F^\times \cdot U$ are of finite order. Therefore the order of the homology of the above complex must be 1. By Proposition 5.1 it follows that $S$ is linked.

**Corollary 5.3.** Let $F$ be a nonreal field with $\text{char}(F) \neq 2$ and $u(F) = 4$. Then every set of three quaternion algebras over $F$ is linked.

*Proof.* Since $u(F) = 4$ we have that $F$ is linked by [EKM08, (39.1)]. Hence for all triples $Q_1, Q_2$ and $Q_3$ of quaternion algebras over $F$, the set $S = \{Q_1, Q_2, Q_3\}$ is tight. That $F$ is nonreal and $u(F) = 4$ implies that $I_q^3(F) = 0$. The result therefore follows from Proposition 5.2.

**Question 5.4.** Let $F$ be a field with $\text{char}(F) = 2$ and $u(F) = 4$. Is every set of three quaternion algebras over $F$ linked?

Note that if a similar result to Proposition 5.1 can be shown for fields of characteristic 2, the same proof of Corollary 5.3 could be used for these fields to give a positive answer to Question 5.4.
Acknowledgements

We thank the referee for useful suggestions that improved the clarity of the paper. The second author was supported by Automorphism groups of locally finite trees (G011012) with the Research Foundation, Flanders, Belgium (F.W.O. Vlaanderen).

Bibliography

References

[Bae82] R. Baeza, Comparing u-invariants of fields of characteristic 2, Bol. Soc. Brasil. Mat. 13 (1982), no. 1, 105–114.

[Bec] K. J. Becher, Triple linkage, Ann. K-theory, to appear.

[BO13] G. Berhuy and F. Oggier, An introduction to central simple algebras and their applications to wireless communications, Mathematical surveys and monographs, vol. 191, 2013.

[CD] A. Chapman and A. Dolphin, Differential Forms, Linked Fields and the u-Invariant, Arch. Math. (Basel), to appear.

[CGV] A. Chapman, S. Gilat, and U. Vishne, Linkage of quadratic Pfister forms, Comm. Algebra, to appear.

[Cha15] A. Chapman, Common subfields of p-algebras of prime degree, Bull. Belg. Math. Soc. Simon Stevin 22 (2015), no. 4, 683–686.

[Cha17] A. Chapman, Symbol length of p-algebras of prime exponent, J. Algebra Appl. 16 (2017), no. 5, 1750136, 9.

[Dra75] P. Draxl, Über gemeinsame separabel-quadratische Zerfallskörper von Quaternionenalgebren, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1975), no. 16, 251–259.

[EKM08] R. Elman, N. Karpenko, and A. Merkurjev, The algebraic and geometric theory of quadratic forms, American Mathematical Society Colloquium Publications, vol. 56, Amer. Math. Soc., Providence, RI, 2008.

[EL73] R. Elman and T. Y. Lam, Quadratic forms and the u-invariant II, Invent. Math. 21 (1973), 125–137.

[EV05] A. Elduque and O. Villa, A note on the linkage of Hurwitz algebras, Manuscripta Math. 117 (2005), no. 1, 105–110.

[Fai06] F. Faivre, Liaison des formes de Pfister et corps de fonctions de quadriques en caractéristique 2, 2006, Thesis (Ph.D.)–Université de Franche-Comté.

[Lam02] T. Y. Lam, On the linkage of quaternion algebras, Bull. Belg. Math. Soc. Simon Stevin 9 (2002), no. 3, 415–418.
[Pey95] Emmanuel Peyre, *Products of Severi-Brauer varieties and Galois cohomology*, $K$-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 369–401.

[QMT15] Anne Quéguiner-Mathieu and Jean-Pierre Tignol, *The Arason invariant of orthogonal involutions of degree 12 and 8, and quaternionic subgroups of the Brauer group*, Doc. Math. (2015), no. Extra vol.: Alexander S. Merkurjev’s sixtieth birthday, 529–576.

[Siv14] A.S. Sivatski, *Linked triples of quaternion algebras*, Pacific J. Math. 268 (2014), no. 2, 465–476.