Minimal modeling of the intrinsic cycle of turbulence driven by steady forcing

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Abstract

Quasi-Cyclic Behavior (QCB) is a common feature of various turbulent flows. In this study, we attempt to formulate the simplest possible model mimicking this behavior using only three variables. To this end, we first conduct Direct Numerical Simulations of three-dimensional flow driven by the steady Taylor–Green forcing to find a similarity between a stable periodic flow at a small Reynolds number (Re) and turbulent QCB at higher Re. Since the temporal dynamics of these two flows are continuously connected by varying Re, a close examination of the periodic flow allows the heuristic formulation of a three-equation model, representing the evolution of Fourier modes in three distinct scales. The model reproduces the continuous bifurcation from a periodic solution to turbulence with QCB when Re is varied. We also demonstrate that, by changing model parameters, the proposed model exhibits a discontinuous transition from steady to chaotic solutions.

I. INTRODUCTION

A number of laminar and turbulent flows display (intrinsic) periodic features. An illustrative example is vortex shedding behind an obstacle. For low Reynolds number (Re), the so-called von Kármán vortex street behind a cylinder is perfectly periodic. When the flow becomes fully turbulent at higher Re, this periodicity is still present, even though the stochastic nature of turbulence motion prevents the system from being perfectly periodic. This close-to-periodic motion, embedded in turbulent fluctuations, is what we will call Quasi-Cyclic Behavior (QCB).

Another important example of QCB is the temporal behavior of turbulent channel flow, where a self-sustaining process governs the dynamics [1–4]. In particular, in small channel flow domains (the so-called minimal flow unit), close to periodic behavior is observed (Fig. 6 of Ref. [5]). The simplified descriptions of this phenomenon are specific to channel flow or the simplified case of Waleffe flow [6–12]. Non-trivial QCB was also observed [13] in a confined cylindrical flow between two counter-rotating disks (the so-called von Kármán flow). Furthermore, QCB is observed in periodic box flow with steady forcing [14–16]. These observations suggest that such dynamics might be more general than wall-bounded flow or

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flow behind obstacles. Moreover, a vast amount of recent research is dedicated to identifying unstable periodic orbits (UPOs) embedded in turbulent flows [17–24]. These studies show that UPOs reproduce statistics and dynamical properties of turbulence.

The current study attempts to formulate the simplest possible model to reproduce QCB, ideally consisting of two or three ordinary differential equations (ODEs), while retaining a close link with the governing equations. Such an approach has been tried for different systems in the literature [24–30]. Indeed, systematic methods exist to reduce the complexity of a system, to obtain a reduced model, retaining the key physics, such as proper orthogonal decomposition (POD) or Galerkin truncation [31]. As we will show, retaining the modes governing both energy and enstrophy in the flow considered in the present investigation leads to a subset of about twenty complex-valued Fourier modes, resulting in a system of forty coupled ODEs, which will not allow analytical treatment. We formulate our model using a more heuristic approach where we analyze a realistic flow, identify the key interactions between scales, and formulate the simplest model which retains these interactions and the forcing and dissipation mechanisms. This approach allows us to formulate a model containing only three degrees of freedom, reproducing certain characteristics of the investigated fluid flow.

We organize the present article as follows. In § II we show DNS results of flow driven by a steady force and describe the similarity of a temporally periodic but non-trivial flow in a low Reynolds number range and turbulent flow for higher Reynolds numbers. In § III we dissect the periodic flow and assess Fourier-mode interactions in detail to identify three groups of modes and their interactions. This allows us to propose a minimal model of three ordinary differential equations exhibiting periodic solutions and QCB. In § IV we describe a parameter survey and bifurcation analysis of the model to show similarities in the statistical behavior of the proposed model and real turbulence. § V concludes the article with a proposal of future research directions.

II. OBSERVATION OF QUASI CYCLIC BEHAVIOR

To illustrate the features we want to reproduce and guide the formulation of a minimal model reproducing these features, we conduct numerical simulations of both turbulent and temporally periodic flows with the same type of forcing. More precisely, we conduct Direct
FIG. 1. (a) A snapshot of the turbulent flow at Re = 29.7. Isosurfaces of $|\omega| = 20$ (blue) and low-pass filtered $|\omega^\leq| = 4$ (red) are visualized. (b) Time series of energy input rate $P(t)$ and energy dissipation rate $\epsilon(t)$ of the same flow. The red rectangle denotes the time interval examined in Fig. 2 (a).

Numerical Simulations (DNS) of three-dimensional incompressible flow governed by the Navier–Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f},$$

with a steady forcing of the two-dimensional Taylor–Green type \cite{14,16},

$$\mathbf{f} = (-f_0 \sin x \cos y, f_0 \cos x \sin y, 0),$$

and the continuity equation, $\nabla \cdot \mathbf{u} = 0$. Here, $\mathbf{u}$, $p$, and $\mathbf{f}$ are the velocity, pressure, and forcing fields, respectively. The forcing amplitude $f_0$ is set to unity. The only control parameter is the kinematic viscosity $\nu$. We employ a pseudo-spectral method in a $(2\pi)^3$ periodic box \cite{32}. See Appendix A for the details of the DNS. We define the Reynolds number and the characteristic timescale of large-scale flow as

$$Re \equiv \frac{\sqrt{f_0}}{|k_f|^{3/2} \nu} \quad \text{and} \quad T \equiv \frac{1}{\sqrt{|k_f| f_0}} = 0.840,$$

respectively. Here, $k_f = (1,1,0)$ is the wavenumber of the forcing \cite{2}.

We first describe the flow observed at Re = 29.7. The time-averaged Taylor scale-based Reynolds number $\langle \text{Re}_\lambda \rangle_t$ for this flow is about 90, and the flow is thus turbulent. Figure 1 (a) shows isosurfaces of $|\omega|$, where $\omega \equiv \nabla \times \mathbf{u}$, capturing small-scale structures, whereas the forcing-induced columnar vortices emerge by visualizing the isosurfaces of $|\omega^\leq|$. 
FIG. 2. (a) Parametric plots of the instantaneous values of the energy dissipation rate $\epsilon(t)$ and the energy input rate $P(t)$ for the turbulent flow at $\text{Re} = 29.7$ for $50T$, which is denoted by the red rectangle in Fig. 1 (b). (b) Phase-averaged values $\langle P \rangle_{\text{phase}}$ and $\langle \epsilon \rangle_{\text{phase}}$ at $\text{Re} = 29.7$ (same as in (a)) for $20T$. (c) Parametric plots of the phase averaged values $\langle P \rangle_{\text{phase}}$ and $\langle \epsilon \rangle_{\text{phase}}$ at four different values of $\text{Re}$ (29.7, 11.9, 8.49, and 6.61). For comparison, we plot the periodic orbit at $\text{Re} = 5.83$, which is re-plotted in (d) with a colored line. In (a), (b), and (d), the time evolves from dark to light colors, and the gap between two consecutive dots corresponds to $5T$.

Here, $\omega^\leq \equiv \nabla \times u^\leq$, which is obtained by applying a low-pass filter to the velocity field, defined as $u^\leq(x) \equiv \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int \! \! \int
the parametric plot of $\epsilon(t)$ against $P(t)$ in Fig. 2 (a) shows that within the random fluctuations one can distinguish quasi-cyclic temporal fluctuations in counter-clockwise direction, suggesting QCB and a causality between $P(t)$ and $\epsilon(t)$. Since these quantities are the large- and small-scale representatives, respectively, this causality reflects the energy cascade.

We apply a phase average to the complex time series of $P(t)$ and $\epsilon(t)$ conditioned on the local maxima of $P(t)$ in order to extract smooth, time-delayed oscillations shown in Fig. 2 (b). See Appendix B for the detailed procedure. We denote the phase-averaged quantities by $\langle \cdot \rangle_{\text{phase}}$. These results suggest that the QCB of turbulent flow driven by the steady body force (2) is robust and statistically significant. We note that such QCB is also shown in Fig. 12 of Ref. [15] for two different forcing types at even higher Re. To determine the origin of QCB, we need to understand the underlying nonlinear interactions of turbulence. Identifying the direct cause of QCB from tens of thousands of excited Fourier modes seems illusory. Thus, we decrease the Reynolds number to reduce the complexity of the flow. We observe that the QCB persists at lower Re until, at a value of Re around 5.83, a perfectly periodic flow is obtained. As will be shown in Fig. 3 below, this is not the laminar solution of the system, which would correspond to a purely two-dimensional structure resulting from a balance between viscous stress and the forcing (2).

In Fig. 2 (c) we show the phase-averaged plots of the parametric time series of $P(t)$ and $\epsilon(t)$ for four different values of the Reynolds number. The change in the shape of the parametric plots is gradual, suggesting that the quasi-periodic orbit in the turbulent flow is continuously connected to the orbit, which is also shown in Fig. 2 (d) for comparison, in the periodic non-turbulent flow. Indeed, we find that the amplitude and the period of the (quasi) periodic flow monotonically increase when we decrease Re from 29.7 to 5.83. This does not prove that the dynamics are identical, but it seems that the turbulent QCB and periodic flow share the same driving mechanism to some extent. Note that in high-Reynolds-number turbulence beyond Re = 30, the amplitude and period seem to saturate to values of the same order as in Fig. 2 (b) (See Fig. 12 of Ref. [15]).

In Fig. 3, we visualize the periodic flow. We distinguish four large-scale columnar vortices associated with the Taylor–Green force (2) and counter-rotating pairs of smaller vortices perpendicular to them. Note that we do not perform low-pass filtering as in Fig. 1 (a) since there is no significant scale separation in the periodic flow. Nevertheless, we can observe a one-step energy cascading process from the four large-scale columnar vortices to smaller-scale
FIG. 3. Visualization of vortical structures at four instances with isosurface of $|\omega| = 5$.

lateral vortices. More concretely, we observe only large-scale vortices at $t = T_0$ (Fig. 1 (a)), then the energy cascade starts to create smaller-scale vortices (Fig. 1 (b), $t = T_0 + 9.42T$), while the large-scale vortices get weaker (Fig. 1 (c), $t = T_0 + 17.0T$). Afterwards, the energy dissipation dominates to weaken smaller-scale vortices, then the entire system becomes calm (Fig. 1 (d), $t = T_0 + 19.3T$). When small-scale vortices disappear, energy input by the external force dominates dissipation to reestablish the large-scale vortices, and the system returns to the initial state (Fig. 1 (a)). We emphasize that this periodic behavior is similar to turbulent QPB observed at higher Reynolds numbers (Fig. 1 and Figs. 12-17 of Ref. [15]). This similarity is the reason why we can see the continuous change of the parametric plots in Fig. 2 between the stable periodic solution and turbulence.

In the next section, we analyze the periodic flow to unveil the essential physics behind QCB. Even though we have not shown the connection between the periodic flow and turbulence rigorously, we hope to obtain new insights into QCB in Navier–Stokes flow by dissecting the periodic flow.

III. THREE-EQUATION MODEL

A. Construction of the model

Our objective is to construct the simplest possible model capable of reproducing QCB while retaining a close connection with the structure of the Navier–Stokes equations [1]. For this purpose, we recall that in a Fourier representation of [1], the individual modes $q_i$
for the $i$th wavenumber $k_i$ are governed by [33, 34],

\[
\left( \frac{\partial}{\partial t} + \nu|k_i|^2 \right) q_i = \sum_{j,m} A_{ijm} q_j q_m + f_i,
\]

where $f_i$ is the forcing applied to the $i$th mode, and $A_{ijm}$ are the coupling constants resulting from the advection and pressure terms of (1). The nonlinear term associated with triad interactions rapidly yields an overwhelming complexity when the number of retained modes increases. Even in our periodic flow, a large number of modes is dynamically active. In order to develop an analytically tractable model, we use a coarse-graining approach where we group subsets of Fourier modes and represent each group by a single variable, leading to a sort of shell-model [35, 36].

The shells, or groups, used in our model are not regrouping modes as a function of scale using a rigorous criterion but as a function of the type of nonlinear interactions and energetic content. Indeed we investigate the Fourier decomposition of the periodic flow to find that only seven Fourier modes with wavevectors $(k_x, k_y, k_z)$,

\[(1,1,0)(= k_f), (1,0,0), (0,1,0), (0,0,2), (1,1,2), (2,0,2), \text{ and } (0,2,2), \]

are responsible for 98% of its energy. See Appendix C for the details of these energetic modes. Figure 4 (a) illustrates that the time evolution of the kinetic energy is closely reproduced, retaining only these modes.

A close inspection of the seven modes shows that all the nonlinear interactions involve the forced mode and two of the six other modes. In the following, $X \in \mathbb{R}$ denotes the characteristic velocity of the forced mode, and $Y \in \mathbb{R}$ corresponds to that of the other six modes. At this point, we suppose that there are only these two classes of modes and that we represent each class by a single, real variable. Furthermore, we assume Eq. (4) to govern the interaction of these two variables, $X$ and $Y$, yielding,

\[
\begin{cases}
\frac{dX}{dt} = -AY^2 - \nu K^2_X X + F, \\
\frac{dY}{dt} = +AXY - \nu K^2_Y Y,
\end{cases}
\]

with a coefficient $A > 0$, typical wavenumbers $K_\alpha > 0$ with $\alpha \in \{X,Y\}$, and a steady force $F > 0$. The first term on the RHS of each equation represents the nonlinear coupling between $X$ and $Y$. This interaction conserves the global energy, $(X^2 + Y^2)/2$. Note that
FIG. 4. (a) Time series of energy $E(t)$ and energy dissipation rate $\epsilon(t)$ computed from all modes (solid lines) and those of the forced plus the six primary energetic modes, denoted by $(\cdot)_{X+Y}$ (dashed lines). (b) Schematic of three different scales: “forced”, “primary”, and “secondary”. We visualize $|\omega|$ distributions of typical Fourier modes in each scale. The forced scale corresponds to $k_f = (1, 1, 0)$. In the primary scale, we visualize $k = (0, 0, 2)$ and $(0, 2, 2)$ modes. For the secondary scale, we visualize $k = (3, 1, 0)$ and $(2, 2, 2)$ modes for example. Triangles denote triad interactions between different scales.

$X$ and $Y$ are used both as the principal variables of our model and as subscripts to denote quantities associated with these variables.

We have conducted an extensive parameter scan of the two-equation model. However, the model always decays to a steady solution, and we do not observe a permanent periodic behavior or QCB. Linear stability analysis of the fixed points of Eq. (6) shows that there are only stable steady solutions (Appendix D). Thus, retaining only this simple triad interaction between the forced and most energetic modes is insufficient to reproduce permanent QCB via supercritical bifurcations. Results of the parameter scan further suggest that the subcritical route to QCB is not present either.

The additional ingredient for QCB turns out to be a small-scale representative and its associated triad interaction terms. Figure 4 (a) shows the time series of energy $E(t)$ and energy dissipation rate $\epsilon(t)$ in the periodic flow along with partial energy $E_{X+Y} \equiv \langle |u_X|^2 \rangle/2 + \langle |u_Y|^2 \rangle/2$ and partial energy dissipation rate $\epsilon_{X+Y} \equiv \nu(\langle |\omega_X|^2 \rangle + \langle |\omega_Y|^2 \rangle)$ contained by the forced and primary modes. While the energy is almost entirely contained in $E_{X+Y}$, there is a visible difference between the full and partial energy dissipation rates. This reveals that the rest of the Fourier modes, neither in $X$ nor $Y$, contribute significantly to the
dynamics of the energy dissipation, representing the small scales. We denote the ensemble of these residual modes by $Z$. The essential nonlinear interactions of $Z$ form triads with one mode of the $Y$-ensemble and another mode from either the $Z$-ensemble or the forced mode $X$. These observations lead to a refined three-equation model,

$$
\begin{align*}
\frac{dX}{dt} &= -A_1 Y^2 + A_3 Y Z - \nu K_X Y + F; \\
\frac{dY}{dt} &= +A_1 X Y - A_2 Z^2 + A_4 X Z - \nu K_Y Y; \\
\frac{dZ}{dt} &= +A_2 Y Z - (A_3 + A_4) X Y - \nu K_Z Z,
\end{align*}
$$

which is represented by a schematic in Fig. 4 (b). Here, $A_1, A_2 > 0$ and $A_3, A_4 \in \mathbb{R}$ are triad coefficients which retain the discrete Navier–Stokes structure (4). We choose the signs and the values of the triad coefficients such that the detailed balance holds in the energy transfer between the three scales. The signs of $A_1$ and $A_2$ are defined so that energy cascades towards small scales: from $X$ to $Y$ and $Y$ to $Z$. The triads with coefficients $A_3$ and $A_4$ represent the “non-local” interactions involving all three scales. Note that the presence of $F$ makes this system different from the usual polynomial models such as the Lorenz [37] or Rössler models [38, 39].

B. Determination of the parameters

The model (7) is a simplified representation of the periodic flow, where all Fourier modes are sorted into three scales; the forced mode $X$, the energetic modes $Y$ directly draining energy from $X$ through the $A_1$ interaction, and small scale modes $Z$ which couple through the local direct cascade interaction $A_2$ with $Y$. There are also scale non-local interactions represented by $A_3$ and $A_4$. Even though such a representation of the flow discards details of the actual flow obtained by the DNS, we will fit the model parameters to the DNS data to assess how the model can reproduce actual flow properties.

We can fit six out of eight model constants in Eq. (7), $A_i$ ($i = 1, 2, 3, 4$), $K_{\alpha}^2$ ($\alpha \in \{X, Y, Z\}$), and $F$, by comparing them to the DNS of the periodic flow. To do so, we use
the energy equations associated with Eq. \([7]\),
\[
\begin{align*}
\frac{dE_X}{dt} &= T_X - \epsilon_X + P, \\
\frac{dE_Y}{dt} &= T_Y - \epsilon_Y, \\
\frac{dE_Z}{dt} &= T_Z - \epsilon_Z.
\end{align*}
\] (8)

Here, \(E_\alpha \equiv \alpha^2/2\) is the energy,
\[
\begin{align*}
T_X &\equiv -A_1XY^2 + A_3XYZ, \\
T_Y &\equiv +A_1XY^2 - A_2YZ^2 + A_4XYZ, \\
T_Z &\equiv +A_2YZ^2 - (A_3 + A_4)XYZ.
\end{align*}
\] (9)

are the energy transfer terms, \(\epsilon_\alpha \equiv 2\nu K_2^2 E_\alpha\) is the energy dissipation rate, and \(P \equiv FX\) is the energy input rate. The values of \(K_2^2\) and \(F\) are determined by comparing their corresponding time-averaged DNS values. The coefficients \(A_1\) and \(A_2\) are adjusted to yield the observed energy flux. The resulting values are
\[
A_1 = 0.4, \quad A_2 = 4, \quad F = 0.7, \quad K_2^2_X = 2, \quad K_2^2_Y = 5, \quad \text{and} \quad K_2^2_Z = 15. \] (10)

We describe the details of the parameter determination procedure in Appendix \[E\]. The undetermined parameters of the model are the scale non-local interaction coefficients \(A_3\) and \(A_4\), which can be freely chosen. The only control parameter is \(Re \equiv 1/\nu\). We numerically integrate the model with a fourth-order Runge-Kutta scheme and \(\Delta t = 0.01\) starting from random initial conditions. See Ref. \[40\] for the solver information. Our numerical simulations seem to indicate that no periodic solutions exist without the complete non-local interactions: \(A_3 = 0\), \(A_4 = 0\), or \(A_3 + A_4 = 0\). Conversely, periodic behavior is observed for a wide range of values when \(A_3 \neq 0\), \(A_4 \neq 0\), and \(A_3 + A_4 \neq 0\). This observation emphasizes the importance of non-local triad interactions for periodic behavior.

### C. Comparison between the model and the DNS result

Figure \[5\] compares the periodic solution obtained by the model and the DNS. Figure \[5\](a) shows the time series of the model with the parameters \([10]\) and \((A_3, A_4) = (0.5, -0.95)\). The Reynolds number is not the same. Indeed, the model contains far fewer degrees of freedom,
FIG. 5. Time series of fluctuating energy $E_{X-X_0}(t)$ of the forced scale and residual energy $E_{Y+Z}(t)$ of periodic solutions of (a) model (7) at $Re = 14.05$ and (b) the Navier–Stokes equations (1) at $Re = 5.83$. Parameters of the model are Eq. (10) and $(A_3, A_4) = (0.5, -0.95)$. Note that time in panel (b) is normalized by $T$.

and the detailed dynamics do not change in the same manner as in the DNS. Therefore, we have chosen a Reynolds number in the model which reproduces the DNS results most closely. We compute two quantities. One is $E_{X-X_0} \equiv (X - X_0)^2/2$, which is the fluctuating energy of the forced mode around the laminar base flow $X_0 \equiv F \Re/K_X^2$. The other quantity $E_{Y+Z} \equiv Y^2/2 + Z^2/2$ is the energy of the rest of the modes. We compare them to the corresponding quantities in the DNS of the periodic flow [Fig. 5(b)], where the base flow is $u_0 \equiv f/2\nu|k_f|^2$, $E_{X-X_0} \equiv (|u_X - u_0|^2)/2$, and $E_{Y+Z}$ is defined by the energy possessed by the non-forced modes. We can observe similar periodic behavior of $E_{X-X_0}$ and $E_{Y+Z}$ in the model (7) and in the periodic flow driven by the steady forcing (2). In particular, there are predator-prey-like exponential growth and decay in both systems. Although fast oscillations are observed in the model but not in the DNS, a close analysis (Appendix C) of the DNS of the periodic flow reveals the presence of rapid oscillations in specific Fourier modes. These oscillations are compensated by modes that display the same energy oscillations with an opposite phase and do not appear in Fig. 5(b). We stress that this periodic solution is independent of the exact amplitude of the initial conditions, unlike the standard two-species Lotka–Volterra equations. In such systems, the initial condition sets the exact amplitude of the orbit in two-dimensional phase space. The phase-space trajectory of the present model is relatively stable with a large basin of attraction. This is a feature of dissipative systems,
IV. DYNAMICS OF THE MODEL

We observe a chaotic state of the model by varying Re from 14.05 to 14.1 while keeping the model parameters as in Fig. 5 (a). Figure 6 (a) shows the orbits in the phase space for both the periodic (at Re = 14.05) and chaotic (at Re = 14.1) cases. The chaotic solution remains close to the periodic solution, showing chaotic QCB. Thus, the same model reproduces periodic solutions and chaotic QCB. Incidentally, the periodic orbit resembles a Shilnikov homoclinic orbit [41]. To further assess the behavior of the system, we draw the bifurcation diagram in Fig. 6 (b) with the same parameter set as in Fig. 5 (a) and Fig. 6 (a). We observe a supercritical transition from periodic to chaotic solutions at a critical Reynolds number Re_{cr} ∈ [14.060, 14.061] [inset of Fig. 6 (b)], and, as observed in Fig. 6 (a), the chaotic orbit
FIG. 7.  (a) Time series of (X, Y, Z) of the model (7) with parameters (10) and \((A_3, A_4) = (0.4, -0.5)\) at \(Re = 32\). A random initial condition is used. (b) The bifurcation diagram for the same parameter set. The red vertical dashed line denotes \(Re = 32\), which is used for Fig. 7 (a). Inset: close-up of the diagram in the range shown by the red rectangle in the main plot.

remains close to the periodic orbit. We note that the solution becomes periodic again when we further increase \(Re\) beyond the range of Fig. 6 (b), probably because the model contains only a small number of degrees of freedom. The observed chaotic solution is permanent as in the turbulence driven by the Taylor–Green forcing (2). The inset of Fig. 6 (b) shows that there is a hysteresis in the range \(Re \in [13.82, 14.03]\), below \(Re_{cr}\), which corresponds to a subcritical bifurcation from a periodic solution to another periodic solution shown in Fig. 5 (a).

A natural question is whether the model can reproduce the features of different flows. Indeed, the system (7) has a general form associated with the discrete Navier–Stokes equations (4). From the outset, it is unclear whether it can describe features of other flow configurations or that it is restricted to the specific forcing (2). We focus on the transition properties as it is plausible that the present model exhibits different bifurcation scenarios for different parameter sets if the model can represent other types of flows. By varying the undetermined parameters of the model, we observe transient chaos at \((A_3, A_4) = (0.4, -0.5)\) as shown in Fig. 7 (a). The corresponding bifurcation diagram in Fig. 7 (b) shows a subcritical bifurcation between steady and chaotic solutions around \(Re \approx 32.3\). There are bi-stable states for \(33 \lesssim Re \lesssim 35\) of steady and chaotic solutions. The inset of Fig. 7 (b) shows that there are multiple windows of periodic solutions in the chaotic regime, probably due to the very
FIG. 8. (a) The survival probability $P_{Re}(t)$ of the transient chaos of the model (7) evaluated from 10,000 samples for each Re. Parameter set is same as Fig. 7. Dashed line denotes exponential fitting by (11) using $0.01 \leq P_{Re}(t) \leq 0.9$. (b) The escape rate $1/\tau$ as a function of Re. Dashed line denotes the exponential fitting by (12).

limited number of degrees of freedom of the model (7).

The transient behavior in Fig. 7 (a) reminds us of the sudden relaminarization observed in a linearly forced turbulence [42], turbulent Kolmogorov flow [43], pipe flow [44], and even in the Lorenz system [45, 46]. We evaluate the survival probability $P_{Re}(t)$, representing how likely the solution remains in a chaotic regime at a given time $t$, to investigate this phenomenon. To evaluate $P_{Re}(t)$, we identify the relaminarization time $t_{relaminarized}$ by the first time when the local maxima of oscillating energy $E_{Y-Y_0} \equiv (Y - Y_0)^2/2$ becomes smaller than a threshold $\delta = 1 \times 10^{-3}$. Here, $Y_0$ is the stable, steady solution. Then, the probability $P_{Re}(t)$ for given $t$ can be evaluated by the ratio of number of samples with $t_{relaminarized} < t$ against the number of the whole sample. We plot $P_{Re}(t)$ in Fig. 8 (a) to find that an exponential scaling,

$$P_{Re}(t) \propto \exp\left(-\frac{t}{\tau(Re)}\right),$$

fits the data. The characteristic time scale $\tau$ in Fig. 8 (b) also displays an exponential scaling,

$$\tau(Re) \propto \exp[a \cdot Re],$$

against Re. Although the scaling (11) of $P_{Re}(t)$ is consistent with the observations in the
previous studies [42], the exponential scaling (12) of \( \tau(\text{Re}) \) differs from a super-exponential behaviour observed in Ref. [42]. This qualitative difference may also be caused by the minimal number of degrees of freedom in the model.

Note that the Taylor–Green forcing (2) in the DNS does not permit such a transition since the laminar base flow \( u_0 \equiv f/2\nu|k_f|^2 \) is linearly unstable. However, the steady Kolmogorov forcing with a linearly stable laminar base flow exhibits sudden relaminarizations [43]. We can thus speculate that the model can reflect different forcing set-ups applied to the Navier-Stokes equations, by varying the parameters \( (A_3, A_4) \).

V. CONCLUSION

The present investigation attempts to construct a minimal model of Quasi-Periodic Behavior (QCB) in a steady-force driven flow while keeping the structure of the Navier–Stokes equations. First, we show that QCB in high Reynolds number (Re) turbulence is continuously connected to a periodic flow at small Re by extracting the intrinsic periodicity of QCB via a phase averaging technique (§ II). Next, we conduct a mode-by-mode analysis of the periodic flow to identify the flow’s forced, primary energetic, and secondary scales. We propose the three-equation model (7) describing the evolution of such three distinct scales (§ III A). By adjusting the model parameters, we observe that the model reproduces a periodic orbit similar to the periodic flow of the DNS (§ III B). We emphasize that scale non-local nonlinear interactions (interactions involving three separate scales) are mandatory for reproducing these dynamics. Then, we conduct a bifurcation analysis to show that the model also exhibits chaotic QCB via a supercritical bifurcation, which is continuously connected to the periodic orbit (§ IV). Thus, we conclude that the proposed model reproduces QCB and its relation to a periodic orbit using a minimum number of degrees of freedom.

Further analysis of the model by varying the undetermined parameters yields transient chaos with sudden relaminarization, which is also observed in turbulent flow with different forcing set-ups (§ IV). Thus, we speculate that the present model can be a minimal model of turbulence. We hope the model guides us to a better understanding of turbulence, as the Lorenz model has done for general chaotic dynamics.

An outstanding open question is how QCB survives in spatially extended flows. How will the global dynamics change when the forcing is applied to scales smaller than the domain
size? In other words, how will the modes larger than the forced scale alter QCB turbulence, and how can we model it? Investigating the relation between space and scale locality and temporal dynamics of turbulence is left for further research.

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Appendix A: Direct Numerical Simulations

This appendix describes the detail of the DNS condition. We use an in-house parallelized code \[32\] to conduct DNS. It employs a pseudo-spectral method with the 2/3 dealiasing rule for spatial discretization and the Adams-Bashforth scheme in the time domain. The initial condition is generated in the Fourier space by Rogallo’s method \[47\].

We perform DNS in a \((2\pi)^3\) triply periodic box. We focus on two distinct flows: a three-dimensional periodic and a turbulent flow. The periodic flow is obtained by the DNS with \(64^3\) Fourier modes by adjusting the viscosity to \(\nu = 0.102\). This corresponds to the value of the Reynolds number \((3)\) of \(Re = 5.83\). We use \(128^3\) Fourier modes to simulate turbulent flow at \(\nu = 0.02\) (\(Re = 29.7\)). For the phase averaging procedure (Appendix B), we use the time series in the interval \(1.28 \times 10^3 \leq t \leq 1.04 \times 10^4\) to guarantee statistical converge. Note that we discard the transient part from the analysis. This interval is approximately \(1.08 \times 10^4 T\) with \(T\) defined in Eq. \((3)\).

Appendix B: Detailed procedure of the phase averaging

This appendix explains the detailed procedure of the phase averaging shown in Fig. 2 (b). First, we pick up the local maxima of \(P(t)\) [Fig. 9 (a)] with the following two criteria: (i) It must be larger than \(\langle P \rangle_t + \sigma(P)\) where \(\langle \cdot \rangle_t\) and \(\sigma(\cdot)\) denote the time average and the standard deviation, respectively. The horizontal pink line indicates this value in Fig. 9 (a). (ii) The temporal gap between two consecutive local maxima must be larger than \(\tau_{max}/2\) where \(\tau_{max}\) is the time for the second peak of autocorrelation function of \(P\) (note that the first peak is at \(\tau = 0\)). We denote the identified local maximum of \(P(t)\) and the corresponding time by \(P_0\) and \(t_0\), respectively. Second, the segments of the time series of \(P(t)\) around the local maximum \(P_0\) are overlapped, as shown in Fig. 9 (b). Note that we normalize the segments by \(P_0\) to avoid overestimation due to huge intermittent peaks. Third, we compute the average over the overlapped and normalized time series to obtain the phase averaged time series \(\langle P \rangle_{\text{phase}}\) shown in Fig. 9 (c).

We also apply a similar procedure to \(\epsilon(t)\). However, the time is shifted for \(t_0\), and \(\epsilon(t)\) is normalized by \(P_0\) so that we can evaluate the time delay and the relative amplitude difference between the two quantities. The pink vertical dashed line shows the time delay in Fig. 9 (c),
FIG. 9. (a) Time series of $P(t)$ with its local maxima $P_0$ denoted by dots. The pink horizontal line corresponds to the threshold of the magnitude $\langle P \rangle + \sigma(P)$. (b) Overlapped segments of the time series of $P(t)$ around $t_0$ and normalized by $P_0$. (c) Phase averaged time series of $P(t)$ and $\epsilon(t)$. Shaded region represents $\langle f \rangle_t \pm \sigma(f)$ where $f$ is $P(t)$ or $\epsilon(t)$. The pink vertical dashed line indicates the maximum of $\langle \epsilon \rangle_{\text{phase}}$, denoting the average time delay between $P(t)$ and $\epsilon(t)$. which is $2.80T$. Figure 2(b) is a parametric plot of Fig. 9(c).

Appendix C: Primary energetic modes of the periodic flow

This appendix shows the detailed behavior of the seven most energetic modes. Figure 10 shows $|\omega|$ distributions of these seven modes. Triangles indicate combinations of different modes where energy transfer via triad interactions is possible. Figure 10 also compares isosurfaces of $|\omega|$ of the sum of these seven primary energetic modes and that of all modes. We find similar principal structures: the large columnar vortices and the small counter-rotating pairs of vortices. Here, we denote the velocity field consisting of the forced mode by $u_X$ and the other six primary energetic modes by $u_Y$. The corresponding vorticity fields
FIG. 10. Schematic of forced (center) plus six primary energetic (surrounding) Fourier modes in the periodic flow at Re = 5.83. Visualizations show distributions of $|\omega|$ at the same instance. Three-digit numbers on the visualizations indicate wavenumbers $k_xk_yk_z$. Triangles denote the possible triad interactions. On top, we compare the isosurfaces of $|\omega_X + \omega_Y|$ with $|\omega|$ of the full flow. Here, $\omega_X$ and $\omega_Y$ denote the vorticity of the forced and the primary energetic modes, respectively.

are denoted by $\omega_X$ and $\omega_Y$, respectively.

We plot the time series of the energy of the forced and six primary modes of the periodic flow at Re = 5.83 in Fig. [11]. Although the energy $E_{110}(t)$ of the forced mode dominates, which is approximately equal to the total energy $E(t)$, we observe a distinctive difference between $E(t)$ and $E_{110}(t)$ when the primary scale energies are excited. By summing up the contributions of these seven modes, we obtain $E_{X+Y}$ shown in Fig.[4] (a). We also note that
FIG. 11. Time series of energy of the forced and six primary energetic modes in the periodic flow. $E_{k_xk_yk_z}$ denotes the energy for the wavevector $(k_x, k_y, k_z)$. Total energy $E(t)$ is also shown for reference.

there are fast oscillations in $E_{100}(t)$ and $E_{010}(t)$. However, these two modes are compensated with each other, and such rapid dynamics are not visible in $E_{X+Y}(t)$ [Fig. 4 (a)]. This observation explains why there are no fast oscillations in the time series shown in Fig. 5 (b).

Appendix D: Linear stability analysis of the two-equation model

In this appendix, we show the results of the linear stability analysis of the fixed points of Eq. (6). There are two kinds of fixed points: namely,

$$
\mathbf{X}_1 = \left( \frac{F}{\nu K_x^2}, 0 \right) \quad \text{and} \quad \mathbf{X}_2 = \left( \frac{\nu K_Y^2}{A}, \pm \frac{1}{A} \sqrt{AF - \nu^2 K_x^2 K_Y^2} \right), \tag{D1}
$$

where $\mathbf{X} \equiv (X, Y)$. Note that the fixed points $\mathbf{X}_2$ exist only for $\nu < \sqrt{AF} / K_X K_Y$. The perturbation $(x, y)$ in the vicinity of the fixed points $(X, Y)$ obeys

$$
\begin{align*}
\frac{dx}{dt} &= -A(Y^2 + 2XY) - \nu K_X^2 (X + x) + F, \\
\frac{dx}{dt} &= +A(XY + XY + xY) - \nu K_Y^2 (Y + y), \tag{D2}
\end{align*}
$$
FIG. 12. Time series of (a) forcing coefficient \( F(t) \) and (b) scale coefficients \( K_\alpha^2(t) \) in the DNS of the periodic flow.

where we have neglected second-order terms \( x^2, y^2 \) and \( xy \). The Jacobian matrix is then expressed as

\[
J = \begin{pmatrix}
-\nu K_X^2 & -2A Y \\
A Y & A X - \nu K_Y^2
\end{pmatrix},
\]

whose eigenvalues are

\[
\lambda = -\frac{1}{2} \left[ -A X + \nu (K_X^2 + K_Y^2) \right] \pm \frac{1}{2} \sqrt{ \left[ A X + \nu (K_X^2 - K_Y^2) \right]^2 - 8A^2 Y^2 }.
\]

The eigenvalues for \( X_1 \) are

\[
\lambda_{1,2}(X_1) = -\nu K_X^2, \quad \lambda_{1,2}(X_1) = \frac{AF}{\nu K_X^2} - \nu K_Y^2,
\]

which are both negative for \( \nu > \sqrt{AF/K_X K_Y} \). Therefore, \( X_1 \) is stable for \( \nu > \sqrt{AF/K_X K_Y} \), and a pitchfork bifurcation takes place at \( \nu = \sqrt{AF/K_X K_Y} \). Then, for \( \nu < \sqrt{AF/K_X K_Y} \), \( X_2 \) exists, which is stable irrespective of \( \nu \) because the eigenvalues are

\[
\lambda_{1,2}(X_2) = -\frac{\nu K_X^2}{2} \pm \sqrt{-8AF + \nu^2 K_X^2 (1 + 8K_Y^2)}/2.
\]

Appendix E: Detailed procedure of the parameter fitting

Figure 12 (a) shows the time evolution of the forcing coefficient defined by

\[
F(t) \equiv \frac{P}{\sqrt{2E_X}}.
\]
FIG. 13. Time series of (a) energy transfer terms $T_\alpha(t)$ and (b) transfer coefficients $A_i(t)$ in the DNS of the periodic flow.

We estimate the model parameter $F = 0.7$, since the time average $\langle F(t) \rangle_t = 0.696$. The periodic drops of $F(t)$ are associated with a phase-desynchronization between the forcing and the forcing-induced velocity field, $u_X$.

We also compute the scale factors

$$K_\alpha^2(t) \equiv \frac{\epsilon_\alpha}{2\nu E_\alpha} \quad (\alpha \in \{X, Y, Z\}). \quad (E2)$$

Figure 12 (b) shows their temporal evolutions. The forced scale factor $K_X^2(t) = 2$ is constant, since it corresponds to $k_f = (1, 1, 0)$ mode. On the other hand, $K_Y^2(t)$ and $K_Z^2(t)$ fluctuate, reflecting the competition of different Fourier modes in these scales. We estimate the model parameters by $K_Y^2 = 5$ and $K_Z^2 = 15$, since $\langle K_Y^2(t) \rangle_t = 4.97$ and $\langle K_Z^2(t) \rangle_t = 15.4$, respectively.

To obtain rough estimates of the scale local coefficients $A_1$ and $A_2$, we compute the average energy transfer rate from $X$ to $Y$ and $Y$ to $Z$ while ignoring the scale non-local interactions by setting $A_3 = A_4 = 0$. In this way, the energy transfer terms (9) of the energy equation (8) of the model are approximated by

$$\begin{cases} 
T_X(t) \approx -A_1 X Y^2, \\
T_Y(t) \approx +A_1 X Y^2 - A_2 Y Z^2, \\
T_Z(t) \approx +A_2 Y Z^2.
\end{cases} \quad (E3)$$

Figure 13 (a) shows their time series by the DNS of the periodic flow. $T_X(t) < 0$ supports the energy cascade picture; the forced scale $X$ is transferring energy to smaller scales ($Y, Z$)
on average. Similarly, $T_Y(t), T_Z(t) > 0$ means that these smaller scales receive energy from the larger scales. We then evaluate the time-dependent coefficients,

$$
\begin{align*}
A_1(t) &\approx -\frac{T_X}{XY^2} = -\frac{1}{2\sqrt{2}} \frac{T_X}{\sqrt{E_X E_Y}}, \\
A_2(t) &\approx \frac{T_Z}{YZ^2} = \frac{1}{2\sqrt{2}} \frac{T_Z}{\sqrt{E_Y E_Z}}.
\end{align*}
$$

Again, we neglect the scale non-local interactions ($A_3 = A_4 = 0$) in these expressions. The result is shown in Fig. 13 (b), and we estimate $A_1 = 0.4$ and $A_2 = 4$ as the model parameters from the time-averaged values $\langle A_1(t) \rangle_t = 0.440$ and $\langle A_1(t) \rangle_t = 4.04$, respectively.

The above argument allows us to determine the model parameters as in Eq. (10). The non-local interaction coefficients $A_3$ and $A_4$ are left to be determined. In § III B we vary these two parameters to investigate the model properties.