LIE ATOMS AND THEIR DEFORMATIONS

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Abstract. A Lie atom is essentially a pair of Lie algebras and its deformation theory is that of a deformation with respect to the first algebra, endowed with a trivialization with respect to the second. Such deformations occur commonly in Algebraic Geometry, for instance as deformations of subvarieties of a fixed ambient variety. Here we study some basic notions related to Lie atoms, focussing especially on their deformation theory, in particular the universal deformation. We introduce Jacobi-Bernoulli cohomology, which yields the deformation ring, and show that, under suitable hypotheses, infinitesimal deformations are classified by certain Kodaira-Spencer data.

Many deformation-theoretic problems and results in algebraic and complex geometry can be profitably formulated in terms of Lie algebras, more specifically differential graded Lie algebras or dglas. These problems includes, notably, the Kodaira-Spencer theory of deformations of complex structures. Nevertheless, there are fundamental deformation problems in geometry for which no Lie theoretic formulation is known. These include, notably, the deformation theory of submanifolds in a fixed ambient manifold, i.e. the local theory of the Hilbert scheme in algebraic geometry or the Douady space in complex-analytic geometry. A principal purpose of this paper is to remedy this situation.

To this end, and for what we consider its own intrinsic interest, we introduce and begin to study a notion which we call Lie atom and which generalizes that of the (shifted) quotient of a Lie algebra by a subalgebra (more precisely, a pair of Lie algebras up to bracket-preserving quasi-isomorphism). Actually, it turns out to be preferable to work with a somewhat more general algebraic object, consisting of a pair of Lie algebras \( g, h^+ \), a Lie homomorphism \( g \to h^+ \), and a \( g \)-module \( h \subset h^+ \). A special case of this is a Lie pair, where \( h = h^+ \). Geometrically, a Lie atom can be used to control situations where a geometric object is deformed while some aspect of the geometry 'stays the same' (i.e. is deformed in a trivialized manner); specifically, the algebra \( g \) controls the deformation while the module \( h \) and the algebra \( h^+ \) control the trivialization. A typical example of this situation is that of a submanifold \( Y \) in an ambient manifold \( X \), where \( g \) is the Lie algebra of relative vector fields (infinitesimal motions of \( X \) leaving \( Y \) invariant), and \( h = h^+ \) is the algebra and \( g \)-module of all ambient vector fields, so that the associated Lie atom is just the shifted normal bundle \( N_{Y/X}[-1] \), and the associated deformation theory is that of the Hilbert scheme or Douady space of submanifolds of \( X \).

Our point of view is that a Lie atom possesses some of the formal properties of Lie algebras. In particular, we shall see that there is a deformation theory for Lie atoms, which generalizes the case of Lie algebras and which in addition allows us to treat some classical, and disparate, deformation problems. These include, on the one hand, the Hilbert scheme, and on the other hand heat-equation deformations, introduced in the first-order case by Welters [We]. Here we will present a systematic development of some of the rudiments of the deformation theory of Lie atoms, which are closely analogous to those of (differential graded) Lie algebras. See [Rre2] for an application of Lie atoms to the so-called Knizhnik-Zamolodchikov -Hitchin connection on the moduli space of curves.

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An essential tool in the deformation theory of Lie atoms is the Jacobi-Bernoulli complex, a comultiplicative complex whose zeroth cohomology, dualized, yields the deformation ring of the atom. This is a local ring which, in good cases, e.g. when the global automorphisms are trivial or 'nearly' so, is the base of the universal deformation. The Jacobi-Bernoulli complex is an analogue of the familiar Jacobi complex of a Lie algebra, but its differentials require twisting by Bernoulli numbers (hence the name), and this makes it slightly less than obvious that the square of the differential vanishes. The proof of this vanishing necessitates brief excursions into the realms of Lie identities and Bernoulli number identities, which occupy §0. In §1 we develop some basic algebra on Lie atoms and the Jacobi-Bernoulli complex, and introduce some fundamental examples. In §2 we give some definitions and remarks on deformation theory for Lie atoms, and introduce the Kodaira-Spencer formalism. Finally in §3 we construct universal deformations under suitable hypotheses of finiteness and automorphism-paucity. See the introductions to individual sections for additional background, motivation and more detailed descriptions.

0. Preliminaries

0.1 Lie identities. The purpose of this subsection is to write down some elementary, but possibly non-standard, identities involving iterated brackets in a Lie algebra. These identities will be technically useful in what follows. First some notation. Let \( a, b \) be elements in a Lie algebra with bracket \([,]\) and set

\[
a_{\otimes}b = [a, b] = -\text{ad}(b)(a)
\]

and inductively, for any natural number \( m \),

\[
a_{\otimes}b^m = (a_{\otimes}b^{m-1})_{\otimes}b = (-\text{ad}(b))^m(a).
\]

For a function \( f(a_1, a_2) \) with values in an abelian group, we set

\[
f(a_1, a_2)^{\text{alt}} = f(a_1, a_2) - f(a_2, a_1).
\]

Thus, for example

\[
[a_1, a_2]^{\text{alt}} = 2[a_1, a_2];
\]

from the Jacobi identity, it is easy to check that

\[
[a_1, a_2 @ b]^{\text{alt}} = [a_1, a_2] @ b.
\]

We need a generalization of the latter formula:

Lemma 0.1. We have, for all \( m \geq 0 \):

\[
[a_1, a_{2 @ b}^m]^{\text{alt}} = \sum_{i=0}^{[m/2]} (-1)^i \binom{m - i - 1}{i} \binom{m - i - 1}{i - 1} [a_1 b^i, a_{2 @ b^i} @ b^{m - 2i}]
\]

where we set \( \binom{j}{-1} = 0 \), \( j \geq 0 \).
proof. The Jacobi identity yields
\[ [a_1, a_2 @ b^m]_{\text{alt}} = [a_1, a_2 @ b^{m-1}]_{\text{alt}} b - [a_1 @ b, a_2 @ b^{m-1}]_{\text{alt}}. \]

From this, it is easy to see inductively that we may write
\[ (0.2) [a_1, a_2 @ b^m]_{\text{alt}} = \sum_{i=0}^{[m/2]} c_{i,m} [a_1 @ b^i, a_2 @ b^{m-2i}]_{\text{alt}}. \]

where the coefficients \( c_{i,m} \) satisfy
\[ c_{0,m} = c_{0,m-1}, m \geq 2 \]
so that \( c_{0,m} = 1, m \geq 1, c_{0,0} = 2 \); and the recursion
\[ (0.3) c_{i,m} = c_{i-1,m-1} - c_{i-1,m-2}, 1 \leq i \leq [m/2]. \]

To solve this recursion set formally for \( j \geq 1 \)
\[ g_j(x) = \sum c_{i,i+j} x^j. \]

Then we easily check that \( c_{1,2} = -2 \) to that \( g_1(x) = 1 - 2x \), and \( (0.3)m \) translates into
\[ g_j(x) = (1 - x)g_{j-1}(x), j > 1. \]

Thus \( g_j(x) = (1 - 2x)(1 - x)^{j-1} \)
which yields \( (0.1) \).

0.2 Bernoulli numbers. The Bernoulli numbers \( B_n \) can be defined by the generating function
\[ C(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1} = -\frac{x}{2} + \frac{x}{2} \coth(\frac{x}{2}). \]

We set
\[ c_n = \frac{B_n}{n!}, B(x) = C(x) + \frac{x}{2} = \frac{x}{2} \coth(\frac{x}{2}). \]

Thus \( c_0 = 1, c_1 = -1/2, c_{2m+1} = 0, \forall m > 0 \). Moreover,
\[ (0.4) c_n = \sum_{1 \leq i \leq m \leq n} (-1)^i \binom{m}{i} \frac{i^n}{(m+1)n!}, \forall n \geq 1. \]

[ This formula will not be needed, but may be proved as follows. Set \( y = e^x - 1 \) so
\[ C(x) = \frac{\log(1 + y)}{y} = \sum_{m=0}^{\infty} (-1)^m y^m/(m + 1). \]
The binomial expansion yields
\[ y^m = \sum_{i=0}^{\infty} \sum_{n=m}^{\infty} (-1)^{i-m} \binom{m}{i} \frac{(ix)^n}{n!}. \]
Then the fact that \( \text{ord}_x(y^m) = m \), yields for \( m \geq 1 \)
\[ y^m = \sum_{i=1}^{m} \sum_{n=m}^{\infty} (-1)^{i-m} \binom{m}{i} \frac{(ix)^n}{n!}. \]
Now the freshman calculus identity \( d \coth x/dx = 1 - \coth^2 x \) easily yields the identities
\[ C^2 = -xC' + (1 - x)C, \]
\[ B^2 = -xB' + B + x^2/4 \]
hence the quadratic recursion
\[ (2m + 1)c_{2m} = - \sum_{i=1}^{m-1} c_{2i}c_{2m-2i}, m > 1. \]
It is not hard to see from (0.6) that \( B \) has the remarkable property that the \( \mathbb{Q}[x] \)-module generated by all its derivatives is closed under multiplication. We shall not need this fact as such, but rather a precise form of a special case of it. First some notation. Set \( d = d/dx \) and
\[ D_k = \frac{1}{k!} \prod_{i=0}^{k-1} (-xd + k - i), \forall k \geq 1. \]
Explicitly, in terms of power series,
\[ D_k \sum a_i x^i = (-1)^k \sum \binom{i-1}{k} a_i x^i. \]
Multiplying \( D_1B = -xB' + B \) by \( B \) and using (0.6) to eliminate \( B^2 \) and its derivative \( 2BB' \), one can check easily that \( B \cdot D_1B = D_2B \).
We generalize this fact to higher derivatives as follows
\[ \text{Proposition 0.2. We have for } k \geq 1 \]
\[ B \cdot D_kB = \sum_{i=0}^{[k/2]} c_{2i}x^{2i}D_{k+1-2i}B, \]
\[ C \cdot D_kC = \sum_{i=0}^{k} c_{x^i}D_{k+1-i}C. \]
Equivalently, we have
\[ \sum \binom{i-1}{k} c_i c_{m-i} = - \sum_{i=0}^{[k/2]} \binom{m-2i-1}{k+1-2i} c_{2i}c_{m-2i} = - \sum_{i=0}^{k} \binom{m-i-1}{k+1-i} c_i c_{m-i} \]
\[ \text{proof. To start with, note that } D_kx = 0, k \geq 1, \text{ so } D_kB = D_kC \text{ and the two equations in (0.8) are equivalent; clearly the second equation is equivalent to (0.8bis). Next, note that } \]
\[ D_kB \equiv 1 \mod x^l, \forall l \geq 1, \]
hence
\[ x^iD_{k+1-i}B \equiv x^i \mod x^{k+1}. \]
Therefore to prove (0.8) it suffices to prove
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\( (\ast)_k. \) \( B \cdot D_k B \) is a constant linear combination of \( x^i D_{k+1-i} B, i \geq 0 \)

The coefficients of the linear combination are then determined by examining the coefficients of \( 1, x, ..., x^k \). We will prove \((\ast)_k\) by induction simultaneously with \((\ast\ast)_k\).

\( (\ast\ast)_k. \) \( B^{k+1} \) is a constant linear combination of \( x^{2i} D_{k-2i} B, i \leq \left[ \frac{k+1}{2} \right] \) where by definition \( D_0 B = B, D_{-1} B = 1 \).

**proof.** Firstly, assuming \((\ast)_k\) and \((\ast\ast)_k\), we deduce \((\ast\ast)_{k+1}\) immediately by multiplying \((\ast\ast)_k\) by \( B \).

So it remains to prove \((\ast)_{k+1}\). To this end we use \((\ast\ast)_{k+1}\) to obtain an expression

\[
B^{k+2} = \sum_{i=0}^{\left[ \frac{k+1}{2} \right]} a_i x^{2i} D_{k+1-2i} B
\]

for some constants \( a_i \). Comparing constant terms, it’s clear that \( a_0 = 1 \). Now apply \((-xd+k+2)/(k+2)\) to \((0.9)\). Using the operator identity

\[
(-xd+r)x^m = x^m(-xd+r-m),
\]

we get

\[
\frac{1}{k+2}(-xd+k+2)B^{k+2} = \sum_{i=0}^{\left[ \frac{k+1}{2} \right]} a'_i x^{2i} D_{k+2-2i} B
\]

with \( a'_0 = 1 \). On the other hand, note that

\[
\frac{1}{k+2}(-xd+k+2)B^{k+2} = B\frac{1}{k+1}(-xd+k+1)B^{k+1},
\]

therefore, multiplying the analogue of \((0.10)\) for \( k+1 \) by \( B \) yields an expression for the same \( \frac{1}{k+2}(-xd+k+2)B^{k+2} \) as linear combination of the \( Bx^{2i} D_{k+1-2i} B \), in which the \( i = 0 \) term, i.e. \( BD_{k+1} B \), appears with coefficient 1. Comparing the two expressions and using \((\ast)_{k'}, k' \leq k \) now yields \((\ast)_{k+1}. \) \( \square \)

It will be convenient to have a 'shifted' version of Proposition 0.2. For any integer \( r \), define a shifted operator \( D_k[r] \) by

\[
D_k[r] = x^r D_k x^{-r}
\]

or in terms of power series,

\[
D_k[r] \sum a_i x^i = \sum \left( \frac{i-1-r}{k} \right) x^i.
\]

The following expansion can be verified easily

\[
D_k[r] = \sum_{j=0}^{k} \left( \frac{r+1-j}{j} \right) D_{k-j}.
\]

Combining this expansion with Proposition 0.2 and eq. (0.5), we conclude
Corollary 0.3. We have

$$C \cdot D_k[r]C = \sum_{i=0}^{k} c_i x^i D_{k+1-i}[r]C - xC.$$

Equivalently,

$$\sum \left( \binom{i - 1 - r}{k} \right) c_i c_{m-i} = -\sum_{i=0}^{k} \left( \binom{m - i - 1 - r}{k + 1 - i} \right) c_i c_{m-i} - c_{m-1}$$

Like (0.8bis), equation (0.12bis) is rather deceptive. Though its two sides look 'roughly' similar, they are in fact completely different in nature, and that they happen to agree is a very special property of the Bernoulli function.

1. Lie atoms: Basic notions

An inclusion $g \to h^+$ of Lie algebras, and more generally a homomorphism of dglas, constitutes in its own right a complex ('mapping cone') endowed with a bracket. Unfortunately, this complex is not usually a dgla (e.g. because the differential is not a derivation with respect to the bracket). Nevertheless, the structure involved is worth encoding, and this is accomplished through the notion of Lie atom.

This section takes up the definition and initial study of Lie atoms. In §1.1 we give the definition, some elementary remarks and constructions, and a basic list of standard examples, drawn mainly from geometry. Typically such examples involve a 'relative' situation, such as the inclusion of a submanifold in an ambient manifold. They will be used as a sort of 'benchmark' as we develop the theory.

For technical reasons it is necessary to define a Lie atom in a slightly different, and finer, manner from the above naive notion, viz. as a $g$-homomorphism $g \to h$ of a Lie algebra $g$ to a $g$-module. From such a homomorphism one can always construct a 'universal hull' $h^+$, which is a Lie algebra receiving a Lie algebra homomorphism $g \to h^+$. This is why our definition is a refinement of the naive one.

In §1.2 we give the construction and basic properties of the Jacobi-Bernoulli complex associated to a Lie atom, which is a (nonobvious) extension of the Jacobi (standard) complex of a Lie algebra or dga. This complex plays a fundamental role in the deformation theory of Lie atoms. The hardest part of the argument is the proof that the construction yields a complex, i.e. that the square of the differential vanishes. This proof requires the Bernoulli identities established in §0.2. From the Jacobi-Bernoulli complex, we construct (see Theorem 1.2.1) the deformation ring $R(g^2)$ of a Lie atom $g^2$, which will later be seen as the base of the universal deformation. We give a version (see Corollary 1.2.3) of the usual 'deformations minus obstructions' estimate on the dimension of the deformation ring, which is important in applications. Then in §1.3 and §1.4 we touch on the fundamental notions of atomic representation and universal enveloping atom, which are natural analogues of the corresponding notions for Lie algebras.

1.1 Definition, examples, remarks. Unless otherwise mentioned, Lie algebras will be understood over an arbitrary commutative unitary ring $S$, which will usually be a $\mathbb{Q}$-algebra.

Definition 1.1.1. By a Lie atom (for 'algebra to module') we shall mean the data $g^2$ consisting of

(i) a Lie algebra $g$;

(ii) a $g$-module $h$;
(iii) a $g$-module homomorphism

$$i : g \to h,$$

where $g$ is viewed as a $g$-module via the adjoint action.

If $i$ is injective, $g^\#$ is said to be a 'pure' Lie atom.

If $h$ is a Lie algebra and $i$ is a Lie homomorphism, $g^\#$ is said to be a 'self-contained' Lie atom or a 'Lie pair'.

Remarks 1.1.2.

(1) Hypothesis (iii) means explicitly that, writing $\langle \ , \ \rangle$ for the $g$-action on $h$,

$$i([a,b]) = \langle a, i(b) \rangle = -\langle b, i(a) \rangle.$$

(2) There is an obvious naive notion of atomic morphism of Lie atoms, hence also of atomic isomorphism and quasi-isomorphism (morphism inducing isomorphism on cohomology). Of course one can also talk about sheaves of Lie atoms, differential graded Lie atoms, etc. Any Lie atom is viewed as a complex in degrees 0,1, and we shall generally consider two atoms to be equivalent if they are atomically quasi-isomorphic. Accordingly, a morphism of Lie atoms would be understood in the sense of the derived category, i.e. a homotopy class of a composition of naive atomic morphisms and inverses of naive atomic quasi-isomorphisms. Thus, for any Lie algebra $g$, the complex $g \to 0$ yields a Lie atom equivalent to (and identified with) $g$; embedding $g$ diagonally in $g \oplus g$ yields a Lie atom equivalent to $g[-1]$. For any Lie atom $(g, h)$, note the 'tautological' morphism $(g, h) \to g$.

(3) Given a Lie atom $g^\#$ as above, note that ker($i$) is an ideal of $g$ and $g^\#$ is an extension of the pure Lie atom ($(g/\ker(i)) \to h$) by the Lie algebra ker($i$). Thus the notion of Lie atom is an amalgamation of those of pure Lie atom and lie algebra.

A basic notion is that of a hull of a Lie atom. Given a Lie atom $g^\# = (g, h, i)$, a hull for $g^\#$ is by definition a Lie algebra $h^+$ with a map $h \to h^+$ such that the composite $g \to h^+$ is a Lie homomorphism and that the given action of $g$ on $h$ extends via $i$ to a 'subalgebra' action of $g$ on $h^+$, i.e. so that

$$\langle a, v \rangle = [i(a), v], \forall a \in g, v \in h^+.$$

Note that any atom admits a universal hull $h^+$, which is simply the quotient of the free Lie algebra on $h$ by the ideal generated by elements of the form

$$[i(a), v] - \langle a, v \rangle, \ a \in g, v \in h$$

(note that the action of $g$ on $h$ extends to an action on $h^+$ by the 'derivation rule'). The basic identity (1.1.1) shows that the map $g \to h^+$ induced by $i$ is a Lie homomorphism.

In what follows, we shall always understand a Lie atom $g^\#$ to come with a choice of hull $h^+$. If $h$ itself is a hull, i.e. if $g^\#$ is a Lie pair, we always choose $h^+ = h$. On the contrary, if no hull is specified, we take $h^+ = h^!$.

Now when $g, h^+$ are nilpotent, $G = \exp(g) \to H^+ = \exp(h^+)$ is a homomorphism of groups, but in general $\exp(h) \subset H^+$ is not $G$-invariant. For this reason we need to consider what is essentially the tangent space to the $G$-orbit of $\exp(h)$. Denote by $g_0 h^i$ the subgroup of $h^+$ generated by all the $a_0 b^i, a \in g, b \in h$, and for $m \in \mathbb{N} \cup \{\infty\}$,

$$h^{[m]} = h + \sum_{i < m+1} g_0 h^i.$$
Also set
\[(1.1.3) \quad g^{[m]} = (g, h^{[m]}) \subset g^+ = (g, h^+).\]
Then it is easy to see that $h^{[m]}$ is a $g$-module. More generally the adjoint action yields a multi-pairing
\[(1.1.4) \quad g \otimes h^{[m_1]} \otimes \ldots \otimes h^{[m_k]} \to h^{[m_1+\ldots+m_k]}\]
Then from the Campbell-Hausdorff formula we conclude

**Lemma 1.1.3.** If $g, h^+$ are nilpotent, then the subset $\exp(h^{[\infty]}) \subset H^+$ is invariant under left or right $G$-multiplication.

**Stock Examples 1.1.4.** We present a list of basic examples to be returned to repeatedly as we develop the theory of Lie atoms.

**A. The general linear atom.** This essentially the universal example, of which every other is a special case. If $j : E_1 \to E_2$ is any linear map of vector spaces, let $g = g(j)$ be the intertwining algebra of $j$, i.e. the Lie subalgebra
\[g = g(j) \subseteq \mathfrak{gl}(E_1) \oplus \mathfrak{gl}(E_2)\]
given by
\[g = \{(a_1, a_2) | j \circ a_1 = a_2 \circ j\}.\]
Thus $g$ is the 'largest' algebra acting on $E_1$ and $E_2$ so that $j$ is a $g$–homomorphism. We define
\[(1.1.5) \quad \mathfrak{gl}(E_1 < E_2) := (g, \mathfrak{gl}(E_2), i_2),\]
with $i_2(a_1, a_2) = a_2$. (and, it goes without saying, choice of hull as $\mathfrak{gl}(E_2)$). Thus when $j$ is injective, $i_2$ is injective.
Next, define
\[(1.1.6) \quad \mathfrak{gl}(E_1 > E_2) := (g, \mathfrak{gl}(E_1), i_1),\]
with $i_1(a_1, a_2) = a_2$. Thus when $j$ is surjective, $i_1$ is injective. These are Lie pairs. The two notions are obviously dual to each other, but since we do not assume $E_1, E_2$ are finite-dimensional, dualising is not necessarily convenient.

Finally, define
\[(1.1.7) \quad \mathfrak{gl}(E_1 \vee E_2) := (g, \mathfrak{gl}(E_1) \oplus \mathfrak{gl}(E_2), i_1 \oplus i_2).\]
In a more global vein, we may consider a vector bundle homomorphism $j : E_1 \to E_2$ and define $\mathfrak{gl}(E_1 < E_2)$ and $\mathfrak{gl}(E_1 > E_2)$ similarly.

The foregoing construction admits a useful generalization to the case of complexes (of vector spaces, locally free sheaves or generally objects in an abelian category). For any complex $(E^i, \partial)$, we denote by $\mathfrak{gl}(E^i)$ the 'internal hom' general linear algebra of $E^i$, that is, the differential graded Lie algebra whose term in degree $i$ is given by
\[\bigoplus_j \text{Hom}(E^j, E^{j+i})\]
and whose differential is given by (signed) commutator with \( \partial \). When \( E \) is an injective resolution, i.e. a complex of injectives, acyclic in a unique degree, note that \( \mathfrak{gl}(E) \) is acyclic in negative degree, hence quasi-isomorphic to a nonnegative complex. Given a morphism
\[
j : E_1 \to E_2
\]
of complexes, there is likewise an intertwining differential graded Lie algebra \( \mathfrak{gl}(j) \); when \( j \) is termwise injective (resp. surjective), \( \mathfrak{gl}(j) \) is a subalgebra of \( \mathfrak{gl}(E_2) \) (resp. \( \mathfrak{gl}(E_1) \)). In any event, there are \( \mathfrak{gl}(j) \)-linear homomorphisms
\[
i_k : \mathfrak{gl}(j) \to \mathfrak{gl}(E_k), k = 1, 2,
\]
and we define differential graded Lie pairs
\[
\mathfrak{gl}(E_1 < E_2) = (\mathfrak{gl}(j), \mathfrak{gl}(E_2), i_2),
\]
\[
\mathfrak{gl}(E_1 > E_2) = (\mathfrak{gl}(j), \mathfrak{gl}(E_1), i_1).
\]
The first (resp. second) definition is especially useful when \( j \) is termwise injective (resp. surjective).

Thus, consider a short exact sequence, say of coherent sheaves on a projective scheme
\[
0 \to A \to B \to C \to 0.
\]
As is well known, this sequence can be resolved into a short exact sequence of complexes of locally free coherent sheaves
\[
0 \to E_1 \to E_2 \to E_3 \to 0.
\]
In fact we may assume that each \( E_i \) is a finite direct sum of line bundles \( O_X(k) \) and that \( E_2 = E_1 \oplus E_3, \forall i \). If \( X \) is smooth (e.g. \( X = \mathbb{P}^n \)), the complexes \( E_i \) may be assumed bounded. It is easy to see that the Lie atoms \( \mathfrak{gl}(E_1 < E_2), \mathfrak{gl}(E_2 > E_3) \) are, up to quasi-isomorphism, independent of the resolution, so we may set
\[
\mathfrak{gl}(A < B) = \mathfrak{gl}(E_1 < E_2), \mathfrak{gl}(B > C) = \mathfrak{gl}(E_2 > E_3).
\]
As we shall see, these Lie atoms control the formal germ at \( B \to C \) of the Quot scheme of \( B \) (a special case of which is the Hilbert scheme).

**B.** If \( i : \mathfrak{g}_1 \to \mathfrak{g}_2 \) is a homomorphism of Lie algebras, then
\[
\mathfrak{g}^\oplus := (\mathfrak{g}_1, \mathfrak{g}_2, i)
\]
is a Lie pair. More generally, if \( \mathfrak{h} \) is any \( \mathfrak{g}_1 \) submodule of \( \mathfrak{g}_2 \) containing \( i(\mathfrak{g}_1) \), then
\[
\mathfrak{g}^\oplus := (\mathfrak{g}_1, \mathfrak{h}, i)
\]
is a Lie atom, whose hull will be taken as the Lie subalgebra of \( \mathfrak{g}_2 \) generated by \( \mathfrak{h} \).

Note that a general Lie atom \( (\mathfrak{g}, \mathfrak{h}, i) \) is essentially of this type, modulo replacing \( \mathfrak{g} \) and \( \mathfrak{h} \) by their images in the hull \( \mathfrak{h}^\oplus \) (though the map \( \mathfrak{h} \to \mathfrak{h}^\oplus \) is not necessarily injective).

**C.** Let \( E \) be an invertible locally free sheaf on a ringed space \( X \) (such as a real or complex manifold), and let \( \mathcal{D}^i(E) \) be the sheaf of \( i \)-th order differential endomorphisms of \( E, i \geq 0 \), and
\[
\mathcal{D}^\infty(E) = \bigcup_{i=0}^{\infty} \mathcal{D}^i(E).
\]
Then $g = \mathcal{O}^1(E)$ and $\mathcal{O}^\infty(E)$ are Lie algebra sheaves and $h = \mathcal{O}^2(E)$ is a $g-$submodule of $\mathcal{O}^\infty(E)$, giving rise to a Lie atom $g^\circ$ with hull $\mathcal{O}^\infty(E)$, which will be called the Heat atom of $E$ and denote by $\mathcal{O}^{1/2}(E)$. Note that if $X$ is a manifold then $g^\circ$ is quasi-isomorphic as a complex to $\text{Sym}^2(T_X)[-1]$ and $h^{[m]} = \mathcal{O}^{m+1}(E)$.

D. Let $Y \subset X$ be an embedding of manifolds (real or complex). Let $T_{X/Y}$ be the sheaf of vector fields on $X$ tangent to $Y$ along $Y$. Then $T_{X/Y}$ is a sheaf of Lie algebras contained in its module $T_X$, giving rise to a Lie pair

$$N_{Y/X}[-1] = (T_{X/Y} \subset T_X),$$

which we call the normal atom to $Y$ in $X$. Notice that $T_{X/Y} \to T_X$ is locally an isomorphism off $Y$, so replacing $T_{X/Y}$ and $T_X$ by their sheaf-theoretic restrictions on $Y$ yields a Lie atom that is quasi-isomorphic to, and identifiable with $N_{Y/X}[-1]$.

More generally, let $f : Y \to X$ be an arbitrary morphism (e.g. a holomorphic map of complex manifolds). Then we have a Lie algebra sheaf on $Y$:

$$T_{X/Y} = \ker(f^{-1}T_X \oplus T_Y \to f^*T_X)$$

$$= \{(u, v) \in \text{Der}(f^{-1}\mathcal{O}_X) \oplus \text{Der}(\mathcal{O}_Y) : f^* \circ u = v \circ f^*\}.$$ 

This sheaf, sometimes called the sheaf of 'f-related vector fields' admits as modules both $f^{-1}T_X$ and $T_Y$, giving rise to Lie pairs on $Y$:

$$N_{f/X}[-1] = (T_{X/Y} \to f^{-1}T_X),$$

$$N_{f/Y}[-1] = (T_{X/Y} \to T_Y).$$

$N_{f/X}[-1]$ is pure whenever $f$ is generically immersive. Note that when $f$ is an inclusion of a submanifold, there is an obvious quasi-isomorphism of Lie atoms $N_{Y/X}[-1] \to T_{f/X}[-1]$. If $f$ is submersive, the Lie atom $N_{f/X}[-1]$ is equivalent to the algebra of vertical vector fields $T_f = \ker(T_Y \xrightarrow{df} f^*T_X)$ while $N_{f/Y}[-1]$ is pure.

E In the situation of the previous example with $Y \subset X$, let $\mathcal{I}_Y$ denote the ideal sheaf of $Y$. Then $\mathcal{I}_Y.T_X$ is also a Lie subalgebra of $T_X$ giving rise to a Lie pair

$$T_X \otimes \mathcal{O}_Y[-1] := (\mathcal{I}_Y.T_X \subset T_X) \sim_{\text{qis}} (T_{X/Y} \subset T_X \otimes T_Y).$$

Note that via the embedding of $Y$ in $Y \times X$ as the graph of the inclusion $Y \subset X$, $T_X \otimes \mathcal{O}_Y[-1]$ is quasi isomorphic as Lie atom to $N_{Y/Y \times X}[-1]$, so this example is essentially a special case of Example D. □

1.2 Jacobi-Bernoulli complex.

Our purpose here is to define the Jacobi-Bernoulli (or JacoBer, for short) complex $J^2(g^\circ)$ associated to a Lie atom $g^\circ = (g \to h)$, which is to play an analogous role in the deformation theory of $g^\circ$ as the Jacobi complex $J(g)$ in the deformation theory of a Lie algebra $g$. We begin with the case where $g^\circ$ is a Lie pair.
For a Lie pair $g^\#$, $J^\sharp(g^\#)$ is a complex in nonpositive degrees whose terms $K^\cdot$ are defined as follows:

$$K^0 = \bigoplus_{j>0} K^{0,j},$$
$$K^i = \bigoplus_{j\geq 0} K^{i,j},$$

where

$$K^{i,j} = -i \Lambda g \otimes \text{Sym}^j \mathfrak{h}.$$  

Next we define the differential $d^i : K^i \to K^{i+1}, i < 0$. This will be a direct sum

$$d^i = \bigoplus d^{i,j,j'} : K^{i,j} \to K^{i+1,j',j'} \leq j + 1.$$  

In particular, $K^\cdot$ will not be a double complex, though it has a natural increasing degree filtration $F. K^\cdot$ defined by

$$(F_r K)^i = \bigoplus_{j \leq i + r} K^{i,j}.$$  

It turns out that the role of $F_r$ will be analogous to that of the 'stupid' filtration on the ordinary Jacob complex. To define the $d^i$, we start with $d^{-1}$. As in §0.2, we denote by $c_t = B_t/t!$ the normalized Bernoulli coefficient. Define $d^{-1,m,m-t+1} : g \otimes \text{Sym}^m \mathfrak{h} \to \text{Sym}^{m-t+1} \mathfrak{h}$, $0 \leq t \leq m$ by

$$(1.2.1) \ a \otimes b^m \mapsto c_t (i(a)_{\otimes b^t}) b^{m-t} = c_t \sum_{j=0}^{m-t} b^j (i(a)_{\otimes b^j}) b^{m-t-j}.$$  

Also define, as in the usual Jacobi complex,

$$(d^{-2,-0,0} : 2 \Lambda g \to g, \ a_1 \wedge a_2 \mapsto [a_1, a_2].)$$  

This then determines the other differentials via the ´derivation rule´:

$$(1.2.2) \ c_t \sum_{i=1}^n (-1)^{i-1} a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots a_n \otimes (i(a_i)_{\otimes b^t}) b^{m-t}$$
$$+ \sum_{i<j} (-1)^{i-j} a_1 \wedge \ldots \wedge [a_i, a_j] \wedge \ldots a_n \otimes b^m$$

It is not obvious that these differentials define a complex, because of the twisting by Bernoulli numbers. We summarize the essential properties of $J^\sharp(g^\#)$ as follows.
Theorem 1.2.1. (i) \( \mathcal{J}^\sharp, F \) is a functor from the category of Lie atoms over \( S \) to that of comultiplicative, cocommutative and coassociative filtered complexes over \( S \).

(ii) The filtration \( F \) is compatible with the comultiplication and has associated graded

\[ F_i/F_{i-1} = \bigwedge^i (g^\sharp). \]

(iii) The quasi-isomorphism class of \( \mathcal{J}^\sharp(g^\sharp) \) depends only on the quasi-isomorphism class of \( g^\sharp \) as Lie atom.

proof. The hard part is proving that \( K^\gamma \) as above is a complex, i.e. that \( d_k d_k+1 = 0 \). Using the definition of \( d_k \), one reduces easily, first to that case \( n = 2 \), then to the vanishing of the component of \( d_{-2} \) going from \( K_{-2, m-1} = \bigwedge g \otimes \text{Sym}^{m-1} h \) to \( K_{0, 1} = h \), that is, to proving that

\[
\sum_{i=0}^m d_{-2, m-1, i} d_{-1, i, 1}(a_1 \wedge a_2 \otimes b^{m-1}) = 0, \forall a_1, a_2 \in g, b \in h.
\]

Plugging into the definitions, (1.2.3) means

\[
\sum_{i+j \leq m-1} c_i c_{m-i} [a_1 \circ a_2 \circ b^i] \circ b^{m-1-i-j} + c_{m-1}[a_1, a_2] \circ b^{m-1} = 0.
\]

We break the big summation in two subsums I and II depending on whether \( j \leq i \) or \( j > i \). Using Lemma 0.1, I can be evaluated as

\[
\sum_{j+2 \alpha \leq i} [a_1 \circ b^j + \alpha, a_2 \circ b^j + \alpha] \circ b^{m-1-2j-2\alpha} \cdot (-1)^{(i-j-1-\alpha)/\alpha} 2(i-j-1-\alpha)/\alpha c_i c_{m-i}.
\]

(\( \text{Note if } m \text{ is even, only even } i \text{'s appear.} \) Setting \( r = j + \alpha \), and using the elementary formula

\[
\sum_{\alpha=0}^x (-1)^\alpha \binom{y}{\alpha} = (-1)^x \binom{y-1}{x}
\]

the coefficient of \( [a_1 \circ r, a_2 \circ r] \circ b^{m-1-2r} \) can be evaluated as

\[
I_{r,m} = \sum_{i} (-1)^r \binom{i-r-2}{r} - 2 \binom{i-r-2}{r-1} c_i c_{m-i}.
\]

Referring to §0.2, the latter is none other than the degree-\( m \) term in

\[
(D_r[r+1]B - 2D_{r-1}[r+1]B)B.
\]

The subsum II can be analyzed in the same way. Now setting \( r = i + \alpha \), we get

\[
II = \sum [a_1 \circ b^r, a_2 \circ b^r] \circ b^{m-1-2r} II_{r,m}
\]
where
\[ II_{r,m} = \sum_i (-1)^{r-i} \left( \binom{m-i-1}{r-i} - 2 \binom{m-i-r-1}{r-i} \right) c_i c_{m-i} \]

As in §0.2, this is just the degree \(-m\) term in
\[- \sum_{i \leq r} c_i x^i D_{r+1-i} [r+1]C + + 2 \sum_{i \leq r-1} c_i x^i D_{r-i} [r+1]C.\]

By Corollary 0.3, we conclude \( I + II + c_{m-1}[a_1,a_2] = 0,\) completing the proof that \( K\) is a complex. [A more conceptual proof of this is much to be desired !!]

Now given that \( J^i(g^i)\) is a complex, functoriality and filtration are obvious, as is the assertion about the graded. As for the comultiplication, its definition is directly analogous to that of the comultiplication in the ordinary Jacobi complex, as developed in [R], based on the natural map
\[ \bigwedge g \otimes \text{Sym}^m \mathfrak{h} \rightarrow \bigoplus_{\substack{i_1+i_2=i \\text{ and } m_1+m_2=m}} (\bigwedge g \otimes \text{Sym}^{m_1} \mathfrak{h}) \otimes (\bigwedge g \otimes \text{Sym}^{m_2} \mathfrak{h}) \]

The proof of the required properties of this comultiplication follows along the same lines as the proof in [R] of the analogous assertions for the Jacobi complex.

Finally, as for (iii), a quasi-isomorphism \( \varphi_1 \rightarrow \varphi_2 \) of Lie atoms induces quasi isomorphisms
\[ \bigwedge (g_1^i) \rightarrow \bigwedge (g_2^i) \]
and the spectral sequence of a filtered complex then shows that the induced map \( J^i(g_1^i) \rightarrow J^i(g_2^i)\) is a quasi-isomorphism. □

**Corollary 1.2.2.** Assume \( \mathfrak{g}^i \) is acyclic in nonpositive degrees. Then there is a second-quadrant spectral sequence with \( E_1 \) term
\[ E_1^{p,q} = \text{Sym}^{q-2p+1} \mathfrak{g}^i \otimes \text{Sym}^{p+2} \mathfrak{g}^i \]
whose abutment has degree 0 part equal to \( \bigoplus E_{\infty}^{p,-p} \) where
\[ E_{\infty}^{p,-p} = \bigoplus g_{A,F}^{p+1} (J^i(g^i)). \]

**proof.** For any complex \((K^i, d)\), its truncation in degree \( r \) is defined by
\[ (K^r)^i = \begin{cases} K^i, i < r, \\ \ker d^r, i = r, \\ 0, i > r. \end{cases} \]

We have
\[ H^i(K^r) = H^i(K), i \leq r, \]
\[ 0, i > r. \]
Then, it is easy to see that
\[ \mathbb{H}^0(J^2((g^2)^2)) = \mathbb{H}^0(J^2(g^4)). \]
Indeed this simply follows from the fact that, for all \( r \),
\[ d_r(E^p_{p-r}) \subset E^p_{p+r-r-1} \]
where, in the case of a JacoBer complex of a Lie atom, the latter group involves only \( \mathbb{H}^1 \) and \( \mathbb{H}^2 \) of the atom.

Therefore, replacing \( g^2 \) by its truncation in degree 2, we may assume \( g^2 \) is acyclic in degrees \( > 2 \).
Then the Corollary becomes simply the spectral sequence associated to the \( F \)-filtration, as in Thorem 1.2.1(ii).

In view of Theorem 1.2.1, there is a natural \( S \)-algebra structure on
\[ R_S(g^4) := S \oplus \text{Hom}_S(\mathbb{H}^0(J^2(g^4)), S) \]
and we call the latter the deformation algebra or ring of the Lie pair \( g^4 \) over \( S \). We now take \( S = C \), and denote \( R_S(g^4) \) simply by \( R(g^4) \).
It is a local ring with maximal ideal \( m = \mathbb{H}^0(J^2(g^4))^* \). Note that the \( m \)-adic filtration on \( R = R(g^4) \) is dual to the \( F \)-filtration on \( J^2(g^4) \). As \( R \) will usually be ‘tested’ by mapping it to a finite-dimensional \( C \)-algebra, our main interest will be in the formal completion
\[ \hat{R} = \lim_{\leftarrow} R/m^e. \]

Now, the spectral sequence of Corollary 1.2.1 yields some information on the associated graded of the \( m \)-adic filtration on \( R \), i.e. the tangent cone \( \text{gr} R \). Set
\[ V = \mathbb{H}^1(g^4)^*, W = \mathbb{H}^2(g^4)^*. \]
Thus, in the above spectral sequence,
\[ E^1_{p, -p} = \text{Sym}^{-p}V^*, E^1_{p, -p+1} = \text{Sym}^{-p-1}V^* \otimes W^*. \]
By construction, the cone \( \text{gr}(R) \) is a quotient of the symmetric algebra \( \text{Sym} V \), and from the spectral sequence, we will obtain a description of the corresponding homogeneous ideal \( I = \bigoplus I^r < \text{Sym} V \).

To this end, note that the spectral sequence endows \( W \) with a decreasing filtration
\[ W = W^2 \supseteq W^3 = \text{im}(d^{2,2}_r)^\perp \supseteq \cdots \supseteq W^r+1 = \text{im}(d^{r,r-1}_r)^\perp \cdots \]
(recall that the image of \( d^{r-r, r-1}_r \) is defined modulo the image of \( d^{r-r, r-1-1}_r \)). The dual of \( d^{r-r, r+1}_r \) yields a map
\[ c_r : W^r/W^{r+1} \rightarrow \text{Sym} V. \]

Now, note that the differential
\[ d^{r-r}_1 : \text{Sym}^{-2}V^* \rightarrow \text{Sym}^{-2}V^* \otimes W^* \]
is compatible with the comultiplication $\text{Sym}^r V^* \to \text{Sym}^{r-2} V^* \otimes \text{Sym}^2 V^*$. Therefore, $d_{-r}^r$ is just the map extended in the obvious way from $d_{-2,2}^r : \text{Sym}^2 V^* \to W^*$. A similar property holds for the other differentials $d_{-r}^r$. Dualizing this fact, we conclude that

$$\text{gr}^r R = (E_{-r}^*)^r = \text{coker} \left( \bigoplus_{i=0}^{r-2} W^{r-i}/W^{r-i+1} \otimes \text{Sym}^i V \to W^r \right)$$

where $\text{gr}$ refers to the $m$-adic filtration and the map $\bar{c}_{r-i}$ on the $i$th summand is the one extended from $c_{r-i}$. In other words, the kernel $I^r$ of the surjection

$$\text{Sym}^r V \to m^r/m^{r+1}$$

is the portion in degree $r$ of of homogeneous ideal in the symmetric algebra $\text{Sym} V$ generated by the images of the $c_{r-i}$, $i = 0, \ldots, r-2$. Thus $I$ has a set of generators corresponding to a $\mathbb{C}$-basis of $W$. In particular, if $V$ and $W$ are finite-dimensional, then $I$ is generated by $\dim W$ many elements and $\text{Sym} V$ is a regular noetherian ring of Krull dimension $\dim V$; by Krull’s theorem, it follows that the Krull dimension of $\text{gr} R$, measured e.g. as $1+ \deg$ of the appropriate Hilbert polynomial

$$p(r) = \dim_{\mathbb{C}}(\text{gr}^r(R)), r >> 0,$$

is at least $\dim V - \dim W$. Since the completion $\hat{R} = \hat{R}(\mathfrak{g}^\sharp) = \varprojlim R/m^r$ is a complete noetherian local ring of the same Krull dimension as the tangent cone, we have proven

**Corollary 1.2.3.** If $\mathfrak{g}^\sharp$ is acyclic in nonpositive degrees and $h^1(\mathfrak{g}^\sharp)$ and $h^2(\mathfrak{g}^\sharp)$ are finite, then

$$\dim \hat{R}(\mathfrak{g}^\sharp) \geq h^1(\mathfrak{g}^\sharp) - h^2(\mathfrak{g}^\sharp). \quad \Box$$

This result is a fundamental 'a priori estimate' on dimension. Given the relation between $R(\mathfrak{g}^\sharp)$ and deformations, to be established in §3 below, it may be viewed as an existence statement, asserting, whenever $h^1(\mathfrak{g}^\sharp) - h^2(\mathfrak{g}^\sharp) > 0$, the existence of a nontrivial deformation. Corollary 1.2.3 unifies and extends a number of 'folklore' results, some of which have had important applications. As just one example, one could mention Mori’s theory of rational curves (see [Kol]).

Next, we will define the JacoBer complex $J^r(\mathfrak{g}^\sharp)$ for a general Lie atom $\mathfrak{g}^\sharp = (\mathfrak{g}, h)$ with hull $\mathfrak{g}^+ = (\mathfrak{g}, h^+)$. $J^r(\mathfrak{g}^\sharp)$ will be a certain subcomplex of $J^r(\mathfrak{g}^\sharp)$, endowed with a sub-stupid filtration that depends on the submodule $\mathfrak{h} \subset \mathfrak{h}^+$. 

Recall the 'adjoint' filtration $(\mathfrak{h}^+)^[\ell]$ on $\mathfrak{h}^+$ (cf. (1.1.2)). It naturally induces a filtration on each $\text{Sym}^i \mathfrak{h}^+$: namely

$$\text{Sym}^i(\mathfrak{h}^+)^[\ell] \subset \text{Sym}^i(\mathfrak{h}^+)$$

is the subgroup generated by all $b_1 \cdots b_i$ such that for some $(j_1, \ldots, j_i)$, we have

$$b_k \in [h^{j_k}], \forall k, \text{ and } \sum j_k = j.$$ 

There is a similar induced filtration on $\mathfrak{g} \otimes \text{Sym}^i \mathfrak{h}^+$, declaring that elements of $\mathfrak{g}$ have filtration level 0. It is evident that this extended bracket filtration is compatible with the differentials on $J^r(\mathfrak{g}^\sharp)$, thus yielding a filtration of $J^r(\mathfrak{g}^\sharp)$ by subcomplexes $J^r(\mathfrak{g}^\sharp)^[\ell]$. We set

$$J^r_m(\mathfrak{g}^\sharp) := F_m(J^r(\mathfrak{g}^\sharp)) := F_m(J^r(\mathfrak{g}^\sharp)) \cap J^r(\mathfrak{g}^\sharp)^[m].$$
Then $J^j_m(\mathfrak{g}^\dagger)$ inherits a comultiplicative structure from $J^j(\mathfrak{g}^\dagger)$, and we set

$$R_m(\mathfrak{g}^\dagger) = S \oplus \text{Hom}_S(\mathbb{H}^0(J^j_m(\mathfrak{g}^\dagger)), S),$$

which is then an $S$-algebra called the $m$th deformation algebra of the Lie atom $\mathfrak{g}^\dagger$.

By construction then, we have a chain of complexes with inclusion maps

$$(1.2.5) \quad J^j_1(\mathfrak{g}^\dagger) = \mathfrak{g}^{[1]} \to \cdots \to J^j_m(\mathfrak{g}^\dagger) \to J^j_{m+1}(\mathfrak{g}^\dagger) \to \cdots \to J^j_{\infty}(\mathfrak{g}^\dagger) \Rightarrow J^j(\mathfrak{g}^\dagger)$$

inducing a chain of rings and homomorphisms

$$(1.2.6) \quad R(\mathfrak{g}^\dagger) \Rightarrow R(\mathfrak{g}^\dagger) \to \cdots \to R_m(\mathfrak{g}^\dagger)^{\oplus} \Rightarrow R_m(\mathfrak{g}^\dagger) \to \cdots .$$

**Lemma 1.2.4.** Suppose $\mathfrak{g}$ is a differential graded Lie algebra with $H^{\leq 0}(\mathfrak{g}) = 0$.

(i) Let $\mathfrak{h}$ be a dg $\mathfrak{g}$ module in nonnegative degrees. Then the map $\eta_m$ in (1.2.6) is surjective and its kernel contains $m_{m+1}(\mathfrak{g}^\dagger)$.

(ii) For any injection $\mathfrak{h}_1 \to \mathfrak{h}_2$ of nonnegative $\mathfrak{g}$-modules, the induced map

$$H^0(J^j_m(\mathfrak{g}, \mathfrak{h}_1)) \to H^0(J^j_m(\mathfrak{g}, \mathfrak{h}_2))$$

is injective.

**proof.** It suffices to prove (ii). This is a standard spectral sequence argument. Note that by choosing a complement to $\partial\mathfrak{g}^0$ in $\mathfrak{g}^1$, we may replace $\mathfrak{g}$ by a sub-dgla $\mathfrak{k}$ in strictly positive degrees that is quasi-isomorphic to $\mathfrak{g}$ and yields a quasi-isomorphic atom $(\mathfrak{k}, \mathfrak{h})$. Then replacing $\mathfrak{g}^\dagger$ by $(\mathfrak{k}, \mathfrak{h})$, the mapping cone of $i_m$ can be represented by a complex in nonnegative degrees, hence has no negative cohomology, which implies our assertion. $\Box$

**Definition 1.2.5.** A dg Lie atom $(\mathfrak{g}, \mathfrak{h})$ is said to be positive if it is isomorphic as Lie atom to one $(\mathfrak{g}', \mathfrak{h}')$ where $\mathfrak{g}'$ (resp. $\mathfrak{h}'$) exists only in positive (resp. nonnegative) degrees.

Clearly, the conclusion of Lemma 1.2.2(i) applies to any positive dg Lie atom.

**Remark 1.2.6.** When $\mathfrak{g}^\dagger$ is a sheaf of (possibly dg) Lie atoms on a topological space $X$, there are sheaf-theoretic analogues of the JacoBer complexes $J^j(\mathfrak{g}^\dagger), J^j_m(\mathfrak{g}^\dagger)$, analogous to the sheaf-theoretic Jacobi complex (cf. [Rcid]). These are complexes defined on certain subset spaces $X\langle m \rangle, X\langle \infty \rangle$, where the ordinary tensor, exterior or symmetric products $\otimes, \wedge, \text{Sym}^i$ are replaced by their ‘external’ analogues $\overset{i}{\otimes}, \overset{i}{\wedge}, \overset{i}{\text{Sym}}$. This construction works well when $X$ is Hausdorff, but not otherwise (e.g. when $X$ is a scheme in the Zariski topology). When $X$ is non-Hausdorff, a version of the subset spaces for Grothendieck topologies still works. However, for any topological space, one can always replace a sheaf of Lie atoms or dg Lie atoms $\mathfrak{g}^\dagger = (\mathfrak{g}, \mathfrak{h})$ by a suitable acyclic (soft or injective) resolution $(\mathfrak{g}, \mathfrak{h}),$ and then $(\Gamma(\mathfrak{g}), \Gamma(\mathfrak{h}))$ is a dg Lie atom whose cohomology is the same as the sheaf cohomology of $\mathfrak{g}^\dagger$. In the sequel, we understand constructions like the JacoBer complex, applied to a Lie atom sheaf $(\mathfrak{g}, \mathfrak{h})$ to mean the non-sheafy version applied to $(\Gamma(\mathfrak{g}), \Gamma(\mathfrak{h})).$

**1.3 Representations.** Now given a Lie atom $\mathfrak{g}^\dagger = (\mathfrak{g}, \mathfrak{h}, i)$, by a left $\mathfrak{g}^\dagger$-module or left $\mathfrak{g}^\dagger$-representation we shall mean the data of a pair $(E_1, E_2)$ of $\mathfrak{g}$-modules with an injective $\mathfrak{g}$-homomorphism $j : E_1 \to E_2$, together with an ‘action rule’

$$\langle \quad \rangle : \mathfrak{h} \times E_2 \to E_2,$$
satisfying the compatibility condition (in which we have written ( ) for all the various action rules):

\[(a, v), x = \langle a, \langle v, x \rangle \rangle - \langle v, \langle a, x \rangle \rangle, \]

\[\forall a \in \mathfrak{g}, v \in \mathfrak{h}, x \in E_2,\]

such that the \(\mathfrak{h}\)-action extends to a Lie action of the hull \(\mathfrak{h}^+\) (if no hull is specified, so \(\mathfrak{h}^+\) is the universal hull, the latter condition is redundant). In other words, a left \(\mathfrak{g}^\mathfrak{h}\)-module is just a homomorphism of Lie atoms

\[\mathfrak{g}^\mathfrak{h} \rightarrow \mathfrak{g}l(E_1 < E_2).\]

Similarly, a right \(\mathfrak{g}^\mathfrak{h}\)-module is defined as a homomorphism of Lie atoms

\[\mathfrak{g}^\mathfrak{h} \rightarrow \mathfrak{g}l(E_1 > E_2).\]

**Examples 1.1.4, cont.** Refer to the previous examples.

**A.** These are the tautological examples: \(\mathfrak{g}l(E_1 < E_2)\) and \(\mathfrak{g}l(E_1 > E_2)\) with \((E_1, E_2)\) as left (resp. right) module in the two cases \(j\) injective (resp. surjective).

**B.** For a Lie atom \(\mathfrak{g}^\mathfrak{h}\) = \((\mathfrak{g}_1, \mathfrak{g}_2, i)\), \(\mathfrak{g}^\mathfrak{h}\) itself is a left \(\mathfrak{g}^\mathfrak{g}\)-module, called the *adjoint representation* while \((\mathfrak{g}^\mathfrak{h})^* = (\mathfrak{g}^*_1, \mathfrak{g}^*_2, i^*), * = dual vector space, is a right \(\mathfrak{g}^\mathfrak{g}\)-module called the *coadjoint representation.*

**C.** In this case \((E, E)\) is a left and right \(\mathfrak{g}^\mathfrak{g}\)-module, called a *Heat module.*

**D.** For an inclusion \(Y \subset X\), the basic left module is \((\mathcal{I}_Y, \mathcal{O}_X)\) which is quasi-isomorphic to \(\mathcal{O}_Y\). Of course we may replace \(\mathcal{I}_Y\) and \(\mathcal{O}_X\) by their topological restrictions on \(Y\). The basic right module of interest is \((\mathcal{O}_X|_Y, \mathcal{O}_Y)\). For a general map \(f : Y \rightarrow X\), the basic left \(\mathcal{N}_{f/X}[1]\)-module of interest is \((f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y)\). For a submersion \(f : Y \rightarrow X\), the basic left \(\mathcal{T}_{f/Y}[1]\)-module is \((f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y)\).

**E.** Realizing \(\mathcal{T}_X \otimes \mathcal{O}_Y[-1] \rightarrow \mathcal{T}_X \otimes \mathcal{T}_Y\), the natural right module of interest is

\[\mathcal{O}_X \rightarrow \mathcal{O}_Y.\]

### 1.4 Universal enveloping atom.

We observe next that there is a natural notion of ‘universal enveloping atom’ associated to a Lie atom \(\mathfrak{g}^\mathfrak{h}\) = \((\mathfrak{g}, \mathfrak{h}, i)\). Indeed, denoting by \(\mathfrak{U}(\mathfrak{g}) = \mathbb{C}1 \oplus \mathfrak{U}^+(\mathfrak{g})\) the usual enveloping algebra, let \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h})\) be the quotient of the \(\mathfrak{U}(\mathfrak{g})\)-bimodule \(\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{h} \otimes \mathfrak{U}(\mathfrak{g})\) by the sub-bimodule generated by elements of the form

\[a \otimes v \otimes 1 - 1 \otimes v \otimes a - 1 \otimes \langle a, v \rangle \otimes 1, 1 \otimes i(a) \otimes b - a \otimes i(b) \otimes 1,\]

\[\forall a, b \in \mathfrak{g}, v \in \mathfrak{h}.\]

**Sorites**

1. \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h})\) is a \(\mathfrak{U}(\mathfrak{g})\)-bimodule.
2. The map \(i\) extends to a bimodule homomorphism

\[i^\mathfrak{g} : \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{g}, \mathfrak{h}).\]

3. \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h})\) is universal with respect to these properties.
4. \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h})\) is generated by \(\mathfrak{h}\) as either right or left \(\mathfrak{U}(\mathfrak{g})\)-module. Moreover the image of \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h})\) in \(\mathfrak{U}(\mathfrak{h}^+)\) is precisely the (left, right or bi-)\(\mathfrak{U}(\mathfrak{g})\)-submodule of \(\mathfrak{U}(\mathfrak{h}^+)\) generated by \(\mathfrak{h}\).
5. An atomic analogue of the Poincaré-Birkhoff-Witt formula holds: it states that, as vector spaces, \(\mathfrak{U}(\mathfrak{g}, \mathfrak{h}) \simeq \text{Sym}^0(\text{im}(i)) \oplus \text{Sym}^1(\mathfrak{g}) \otimes \text{coker}(i)\)
with the two summands corresponding, respectively, to \( \text{im}(i^\sharp) \) and \( \text{coker}(i^\sharp) \). Moreover, under the usual PBW isomorphism

\[
\mathcal{U}(\mathfrak{g}) \cong \text{Sym}(\mathfrak{g}),
\]

the kernel of \( i^\sharp \) corresponds to \( \text{Sym}(\mathfrak{g}) \ker(i) \), i.e. the ideal in \( \text{Sym}(\mathfrak{g}) \) generated by \( \ker(i) \).

Thus

\[
\mathcal{U}(\mathfrak{g}^\sharp) := \left( \mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}, \mathfrak{h}), i \right)
\]

forms an 'associative atom' which we call the universal enveloping atom associated to \( \mathfrak{g}^\sharp \). From (5), it is easy to see that a quasi-isomorphism \( \varphi : \mathfrak{g}_1^\sharp \rightarrow \mathfrak{g}_2^\sharp \) yields a quasi-isomorphism \( \mathcal{U}(\varphi) : \mathcal{U}(\mathfrak{g}_1^\sharp) \rightarrow \mathcal{U}(\mathfrak{g}_2^\sharp) \).

**Examples 1.1.4, cont.**

A. It is elementary that the universal enveloping algebra of the interw ining Lie algebra \( \mathfrak{g} \) is simply the interwining associative algebra

\[ \mathcal{U}(\mathfrak{g}) = \left\{ (a_1, a_2) | j \circ a_1 = a_2 \circ j \right\}, \]

and so the universal enveloping atom of \( \mathfrak{gl}(E_1 < E_2) \) (resp. \( \mathfrak{gl}(E_1 > E_2) \) is just \( \left( \mathcal{U}(\mathfrak{g}), \text{End}(E_2), i \right) \) (resp. \( \left( \mathcal{U}(\mathfrak{g}), \text{End}(E_1), i \right) \).

B. In this case it is clear that \( \mathcal{U}(\mathfrak{g}_1, \mathfrak{h}) \) is just the sub \( \mathcal{U}(\mathfrak{g}_1) - \text{bimodule generated by } \mathfrak{h} \).

### 2. Atomic deformation theory basics

The purpose of this section is to develop the deformation theory associated to a sheaf of Lie atoms \( \mathfrak{g}^\sharp = (\mathfrak{g} \rightarrow \mathfrak{h}^+) \) on a topological space. This is a generalization to a 'relative' setting of the familiar Kodaira-Spencer deformation theory for a sheaf \( \mathfrak{g} \) of Lie algebras, using the same point of view as that as developed in [Rcid]. This theory, which will be reviewed extensively below, is essentially geometric in character. It starts from the notion of a \( \mathfrak{g} \)-deformation, over (or parametrized by) a finite-dimensional local \( \mathbb{C} \)-algebra \( S \) with maximal ideal \( m_S \), as a certain type of torsor \( G^\phi \) with respect to the sheaf of nilpotent groups \( G_S = \exp(\mathfrak{g} \otimes m_S) \). Such a torsor can be 'realized' on any \( \mathfrak{g} \)-module \( E \), yielding a deformation \( E^\phi \) of \( E \).

To a \( \mathfrak{g} \)-deformation one can associate, following the ideas of Kodaira-Spencer, a formal algebraic object known as Kodaira-Spencer class which, in 'raw' form is represented by an element

\[ \phi \in \mathfrak{g}^1 \otimes m_S \]

satisfying the integrability or Maurer-Cartan equation

\[ \partial \phi = -\frac{1}{2} [\phi, \phi]. \]

A more abstract formulation, in terms of the cohomologically-defined 'deformation ring' \( R(\mathfrak{g}) \), can also be given (see below). In this formulation, the element \( \phi \) is replaced by a (multiplicative) homomorphism \( \epsilon = \epsilon(\phi) : R(\mathfrak{g}) \rightarrow S \), called the Kodaira-Spencer homomorphism of the deformation.

To clarify our perspective, we note that some treatments of deformation theory define a deformation formally via the notions of Kodaira-Spencer class and integrability equation. By contrast, our approach deduces these from the geometric definition. While the geometric and formal notions of deformation are closely related in both directions, the exact formulation of this relation, and especially its descent to suitable equivalence or cohomology classes, is somewhat subtle, and depends on appropriate hypotheses of finiteness and automorphism-freeness. Defining a deformation formally via Kodaira-Spencer class sweeps these issues under the rug.
Now extending this deformation theory to the case of a Lie atom $\mathfrak{g}^d = (\mathfrak{g} \to \mathfrak{h}^+)$ means, essentially, considering a $\mathfrak{g}$-deformation, in the form of a torsor $G^\phi$, together with a trivialization of the induced $H^+\text{-torsor (}H^+)^\phi$. For simplicity we ignore in this introduction the role of the $\mathfrak{g}$-submodule $\mathfrak{h} \subset \mathfrak{h}^+$. To a $\mathfrak{g}^d$-deformation one can again associate a Kodaira-Spencer class, which now can be represented, essentially, by a pair

$$(\phi, \psi) \in \mathfrak{g}^1 \otimes m_S \oplus \mathfrak{h}^0 \otimes m_S$$

where $\phi$ satisfies integrability as above, and the pair satisfy compatibility in the form

$$\partial \psi = C(\text{ad}(-\psi))(\phi) = \sum_{n=0}^\infty c_n \text{ad}(-\psi)^n(\phi)$$

where the $c_n$ are the Bernoulli coefficients, defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty c_n x^n.$$ 

These conditions can again be formulated cohomologically via the deformation ring $R(\mathfrak{g}^d)$, which is defined from the cohomology a suitable complex that we call Jacobi-Bernoulli complex, and which generalizes the Jacobi complex of a dgla.

This section is organized as follows. In §2.1 we present basic definitions, properties and examples pertaining to deformations of a Lie atom $\mathfrak{g}^d$, viewed as torsors. The more familiar dgla case is also reviewed. In §2.2 we define the Kodaira-Spencer class of a $\mathfrak{g}^d$-deformation parametrized by $S$, and characterize it in terms of local homomorphisms $R(\mathfrak{g}^d) \to S$, plus some extra data (the latter is not needed if $\mathfrak{g}^d$ is a Lie pair).

2.1 Deformations as torsors.

Our purpose here is to define and study deformations with respect to a Lie atom $\mathfrak{g}^d = (\mathfrak{g}, \mathfrak{h}, i)$. Roughly speaking a $\mathfrak{g}^d$-deformation consists of a $\mathfrak{g}$-deformation $\phi$, plus a ‘trivialization of $\phi$ when viewed as $\mathfrak{h}^+$-deformation’.

We recall first the notion of $\mathfrak{g}$-deformation. Let $\mathfrak{g}$ be a sheaf of Lie algebras over a topological space $X$, let $E$ be a $\mathfrak{g}$-module and $S$ a finite-dimensional local $\mathbb{C}$-algebra with (nilpotent) maximal ideal $m$ of exponent $e$. We set $S_i = S/m^{i+1}$, $m_i = m/m^{i+1}$. Then $\mathfrak{g} \otimes m$ has a natural structure of nilpotent $S$-Lie algebra, with bracket coming from the bracket on $\mathfrak{g}$ and the multiplication $m \times m \to m$. Note that there is a sheaf of groups $G_S$ given by

\begin{equation}
G_S = \exp(\mathfrak{g} \otimes m)
\end{equation}

(where $\exp = \exp_S$ is a finite series by nilpotence of $m$), with multiplication given by the Campbell-Hausdorff formula, where $\exp$, as a map to $\mathfrak{U}(\mathfrak{g} \otimes m)$, is injective because the formal log series gives an inverse.

**Definition 2.1.1.** (i) A $\mathfrak{g}$-deformation over $S$ is a $G_S$-torsor, i.e. sheaf $G^\phi$ of sets with $G_S$-action that is simply transitive, locally over $X$. An equivalence of deformations is a $G_S$-equivariant map $G^\phi \to G^{\phi'}$ over $X$.

(ii) Given the $\mathfrak{g}$-module $E$, a $\mathfrak{g}$-deformation of $E$ over $S$ is a sheaf $E^\phi$ of $S$-modules, together with a maximal atlas of trivialisations

$$\Phi_\alpha : E^\phi|_{U_\alpha} \to E|_{U_\alpha} \otimes S,$$
such that the transition maps

\[ \Psi_{\alpha\beta} := \Phi_{\beta} \circ \Phi_{\alpha}^{-1} \in \tilde{G}_S(U_\alpha \cap U_\beta) \]

where \( \tilde{g} \subset \mathfrak{gl}(E) \) is the image of \( \mathfrak{g} \) and \( \tilde{G}_S \) is the corresponding group sheaf.

*Apology.* The notation \( G^\phi \) for a deformation is somewhat misleading because a deformation does not ’come with’ a \( \phi \) though a suitable \( \phi \) does give rise to a deformation and any deformation comes from a \( \phi \). Hopefully, the context will make our intention clear in each use of this notation.

**Remarks.** (i) Given an abstract \( \mathfrak{g} \)-deformation \( G^\phi \) and a \( \mathfrak{g} \)-module \( E \), a corresponding \( \mathfrak{g} \)-deformation \( E^\phi \) of \( E \) can be defined, either as the deformation with the same transition functions, or as \( E^\phi = (E \times_X G^\phi)/\tilde{G}_S \). Conversely, given a \( \mathfrak{g} \)-deformation \( E^\phi \), it defines a \( \tilde{g} \)-deformation, either as the deformation with the same transition functions or as the sheaf of maps \( E \otimes S \to E^\phi \) (or \( E^\phi \to E \otimes S \)) that are locally in \( \tilde{G}_S \).

(ii) The notion of homomorphism of \( \mathfrak{g} \)-deformations is defined in the obvious way (with local representatives that are \( g \)- and \( S \)-linear). The isomorphism class of a \( \mathfrak{g} \)-deformation is given by the class of \( (\Psi_{\alpha\beta}) \) in the nonabelian Čech cohomology set \( \check{H}^1(X, G_S) \).

(iii) Note that a \( \mathfrak{g} \)-deformation determines a functor from the category of \( \mathfrak{g} \)-modules to that of \( \mathfrak{g} \)-deformations of modules. This functor is exact and linear (in the sense that it induces a \( \mathbb{C} \)-linear map from \( \text{Hom}_R(E, E) \) to \( \text{Hom}_R(E^\phi, E^\phi) \)). Moreover this functor is determined by its value on any *faithful* \( \mathfrak{g} \)-module \( E \). We may call \( E^\phi \) a model of \( \phi \) or \( (\Psi_{\alpha\beta}) \).

(iv) Via the adjoint representation, any \( \mathfrak{g} \)-deformation \( \phi \) over \( S \) determines a \( \mathfrak{g} \)-deformation of \( g \) itself, \( g^\phi \), which is easily seen to be an \( S \)-Lie algebra (\( g^\phi \) also coincides with the algebra of vertical vector fields of \( G^\phi/X \)). In general, \( g^\phi \) need not determine \( \phi \).

(v) If \( (\mathfrak{g}, \partial) \) is a differential graded Lie algebra, one can define an ’operatorial’ notion of deformation as follows. Let \( \mathfrak{g}_\partial \) be the graded Lie algebra \( \mathfrak{g} \oplus \mathbb{C}[-1]|\partial \), split extension of \( \mathbb{C}[-1] \) by \( \mathfrak{g} \) with bracket

\[ [\partial, a] = \partial(a), [\partial, \partial] = 0. \]

Then an operatorial \( \mathfrak{g} \)-deformation is simply a square-zero element of \( \mathfrak{g}_\partial \otimes \mathfrak{m}_S \) congruent to \( \partial \mod \mathfrak{m}_S \), and two such are equivalent if they are in the same orbit under the Adjoint action of \( \exp(\mathfrak{g}^\partial) \). The fact that, if \( \mathfrak{g} \) is a suitable dgla resolution of \( \mathfrak{g} \), then equivalence classes of \( \mathfrak{g} \)-deformations coincide with equivalence classes of operatorial \( \mathfrak{g} \)-deformations is essentially the content of Kodaira-Spencer theory (cf. §2.2 below).

We now turn to the case of Lie atoms and their deformations. Thus let \( \mathfrak{g}^2 = (\mathfrak{g}, \mathfrak{h}, i) \) be a sheaf of \( S \)-Lie atoms on \( X \), and let \( E^2 = (E_1, E_2, J) \) be a sheaf of left \( \mathfrak{g}^2 \)-modules. Note that \( \mathfrak{g}^2 \otimes \mathfrak{m} = (\mathfrak{g} \otimes \mathfrak{m}, \mathfrak{h} \otimes \mathfrak{m}, i \otimes \text{id}) \) is naturally a sheaf of \( S \)-Lie atoms, and we choose for its hull

\[ (2.1.2) \quad \mathfrak{h}^+_m = (\text{Lie closure of } \mathfrak{h} \otimes \mathfrak{m}) \subset \mathfrak{h}^+ \otimes \mathfrak{m}. \]

Note that the inclusion \( \mathfrak{h}^+_m \subset \mathfrak{h}^+ \otimes \mathfrak{m} \) may well be strict, e.g. if \( m^2 = 0 \) then \( \mathfrak{h}^+_m = \mathfrak{h} \otimes \mathfrak{m} \) always. Then \( \mathfrak{h}^+_m \) is endowed with the adjoint filtration as in (1.1.2):

\[ (2.1.3) \quad \mathfrak{h}^+_m = \mathfrak{h} \otimes \mathfrak{m} \subset \ldots \subset \mathfrak{h}^{[i]} = (\mathfrak{h} \otimes \mathfrak{m})^{[i]} \subset \ldots \subset \mathfrak{h}^{[\infty]} \subset \mathfrak{h}^+_m \]

We have

\[ \mathfrak{h}^{[i]} \otimes S_i = \mathfrak{h}^{[i+1]} \otimes S_i = \ldots = \mathfrak{h}^{[\infty]} \otimes S_i. \]

Working with \( \mathfrak{h}^+_m \) and \( \mathfrak{h}^{[i]} \), which depend on \( \mathfrak{h} \), rather than \( \mathfrak{h}^+ \otimes \mathfrak{m} \), is the main purpose of specifying the \( \mathfrak{g} \)-submodule \( \mathfrak{h} \subset \mathfrak{h}^+ \) rather than working solely with \( \mathfrak{h}^+ \). This is the main reason for introducing Lie atoms, as opposed to working just with Lie pairs.
Now set formally

\[ H^+_S = \exp(h^+_m) \]  

(again, the exponential series is finite by nilpotence of \( m \)). By the Campbell-Hausdorff formula, \( H^+_S \) is a group, and there is a group homomorphism \( G_S \to H^+_S \). Moreover, as noted in Lemma 1.1.3, the subset

\[ H^{(\infty)}_S = \exp(h^{(\infty)}_m) \subset H^+_S \]

is (left and right) \( G_S \)-invariant. This notation should be used with care because in general

\[ H^+_S \subset (H^+_S) = \exp(h^+ \otimes m_S). \]

Similarly,

\[ H^+_S = \exp(h^+_m \otimes S_i) \]

Note that we have Lie pairs

\[ g^+_m = (g \otimes m, h^+_m), \]
\[ g^{[i]}_m = (g \otimes m, h^+_m \otimes S_i) \]

and \( g^{[i]}_m = g^+_m \) for \( i \geq e \) where \( e \) is large enough so \( m^{e+1} = 0 \). Also set

\[ g^+_m = (g \otimes m, h^{[\infty]}_m) = (g \otimes m, h^e_m). \]

Thus

\[ g^+_m = (g \otimes m, h^{[i]}_m). \]

There is an obvious action

\[ H^+_S \times E_2 \otimes S_i \to E_2 \otimes S_i, \]

\[ \langle \exp(v), x \rangle = \frac{1}{j!} \sum_{j=0}^i \text{ad}(v)^j(x) \]

where \( \text{ad}(v)(x) = (v, x) \). Such maps are called \( \text{left } h \)-maps of \( E^2 \) if \( v \in h^{[\infty]}_m \). We consider the data of an \( h \)-map to include the element \( v \) (this is of course redundant if the action of \( H_S \) is faithful), and two such maps are considered equivalent if they belong to the same \( G_S \)-orbit. Thus a left \( h \)-map is essentially a \( G_S \)-orbit of an element of \( h \otimes m \) in \( H_S \). Since \( m \) is nilpotent, any left \( h \)-map is an \( S \)-isomorphism. All the above leftmost considerations have obvious rightist analogues.

The notion of \( h \)-map globalizes as follows. Given a \( g \)-deformation \( E^\phi \), a (global) \( \text{left } h \)-map (with respect to \( \phi \)) is a map

\[ A : E_2 \otimes S \to E^\phi_2 \]

such that for any atlas \( \Phi_\alpha \), \( E^\phi_2 \) over an open covering \( U_\alpha \), \( \Phi_\alpha \circ A \) is given over \( U_\alpha \) by a left \( h \)-map.

Note that this condition is independent of the choice of atlas, and is moreover equivalent to the existence of some atlas for which the \( \Phi_\alpha \circ A \) are given by

\[ x \mapsto \langle \exp(v_\alpha), x \rangle, \quad v_\alpha \in h(U_\alpha) \otimes m. \]
We call such an atlas a *good atlas* for $A$. Similarly, an 'abstract' or 'torsor' left $\mathfrak{h}$-map is a map

$$A : (H^+)_S \to (H^+)^\phi$$

from the trivial $H^+$ torsor to the $H^+$ torsor determined by $\phi$, that is locally given by a left $\mathfrak{h}$-map as above. An $\mathfrak{h}$-map as in (2.19) is equivalent to another such,

$$A' : (H^+)_S \to (H^+)^{\phi'}$$

if there exists an equivalence of $\mathfrak{g}$-deformations $\epsilon : G^\phi \to G^{\phi'}$ and an element $\gamma \in G_S$ such that, if we denote by $\epsilon^{H^+}$ the natural extension of $\epsilon$ to an equivalence of $\mathfrak{h}^+$-deformations, then the following commutes

$$
\begin{array}{ccc}
(H^+)_S & \xrightarrow{A} & (H^+)^\phi \\
\downarrow & & \downarrow \epsilon^{H^+} \\
(H^+)_S & \xrightarrow{A'} & (H^+)^{\phi'} \\
\end{array}
$$

The notion of global right $\mathfrak{h}$-map

$$B : F^\phi_1 \to F_1 \otimes S$$

for a right $\mathfrak{g}^\sharp$-module $(F_1, F_2, k)$ is defined similarly, as is that of abstract global $\mathfrak{h}$-maps without specifying a module. A pair $(A, B)$ consisting of a left and right $\mathfrak{h}$-map is said to be a dual pair if there exists a common good atlas with respect to which $A$ has the form (2.2) while $B$ has the form

$$x \mapsto \langle \exp(-v_\alpha), x \rangle$$

with the same $v_\alpha$.

**Definition 2.1.2.** In the above situation, a left $\mathfrak{g}^\sharp$-deformation over $S$ consists of a $\mathfrak{g}$-deformation $G^\phi$ together with a left $\mathfrak{h}$-map from the trivial deformation to the $\mathfrak{h}^+$-deformation corresponding to $\phi$:

$$A : (H^+)_S \to (H^+)^\phi.$$ 

Similarly for right $\mathfrak{g}^\sharp$-deformation. A (2-sided) $\mathfrak{g}^\sharp$-deformation consists of a $\mathfrak{g}$-deformation $\phi$ together with a dual pair $(A, B)$ of $\mathfrak{h}$-maps with respect to $\phi$.

An obvious, yet fundamental observation is that the various notions of $\mathfrak{g}^\sharp$-deformations are functorial with respect to homomorphisms of Lie atoms. In particular, given a Lie algebra $\mathfrak{k}$ and a Lie homomorphism $\mathfrak{k} \to \mathfrak{g}^\sharp$, any $\mathfrak{k}$-deformation over $S$ induces a $\mathfrak{g}^\sharp$-deformation over $S$.

**Examples 1.1.4 cont..**

**A-B** When $E_1 < E_2$ are vector spaces, $\mathfrak{g}(E_1 < E_2)$-deformation theory is just the local geometry of the Grassmannian $G(\dim E_1, \dim E_2)$. When $E_1 < E_2$ are vector bundles, $\mathfrak{g}(E_1 < E_2)$-deformation theory is just the local geometry of the 'Grassmannian' of subbundles of $E_2$ or equivalently, the Quot scheme of $E_2$ localized at the quotient $E_2 \to E_2/E_1$. Generally, if $A \subset B$ are coherent sheaves on a projective scheme, we have the differential graded Lie atoms $\mathfrak{g}(A < B)$ and $\mathfrak{g}(B > B/A)$, and for both of them the associated deformation theory is that of the Quot scheme of $B$ localized at $B/A$.

**C** When $\mathfrak{g}^\sharp = (\mathcal{D}^1(E), \mathcal{D}^2(E))$ is the heat algebra of the invertible sheaf $E$, $\mathfrak{g}^\sharp$-deformations of $E^\sharp = (E, E)$ are called heat deformations. Recall that a $\mathcal{D}^1(E)$-deformation consists of a deformation $\mathcal{O}^\phi$ of the structure sheaf of $X$, together with an invertible $\mathcal{O}^\phi$-module $E^\phi$ that is a deformation
of $E$. Lifting this to a $g^d$–deformation amounts to constructing $S$–linear, globally defined $h$-maps (heat operators)

$$A : E \otimes S \to E^\phi,$$

$$B : E^\phi \to E \otimes S.$$ 

By definition, $h^+_m$ is the subalgebra of $D^\infty(E) \otimes m$ generated by $D^2(E) \otimes m$, so any element of $h^+_m$ is an $S$-linear differential operator $\Box$ that has order $\leq i \mod m^i$ for any $i \geq 2$, and $A$ and $B$ are locally (with respect to an atlas and a trivialisation of $E$) of the form $f \mapsto \sum \Box^i f$.

Note that this operator is of order $\leq 2i - 2$ $\mod m^i$, $i \geq 2$. Indeed writing locally $\Box = \sum \Box_j$, $\Box_j \equiv 0 \mod m^j$ and that the highest-order term in $\Box^{i-1}$ comes from $\Box_2^{i-1}$. Moreover, $A, B$ yield mutually inverse $S$-linear (not $O_X$-linear) isomorphisms. Notice that the heat operator $A$ yields a well-defined lifting of sections (as well as cohomology classes, etc.) of $E$ defined in any open set $U$ of $X$ to sections of $E^\phi$ in $U$ (viz. $s \mapsto A(s \otimes 1)$). In particular, suppose that $X$ is a compact complex manifold. Then $H^*(A) : H^*(X, E) \otimes S \to H^*(X, E^\phi)$ is an $S$-linear isomorphism. Thus for any heat deformation the cohomology module $H^*(E^\phi)$ is not just (relatively) unobstructed but canonically trivialised. Put another way, $H^*(E^\phi)$ is endowed with a canonical flat connection

$$(2.8) \quad \nabla^\phi : H^*(E^\phi) \to H^0(E^\phi) \otimes \Omega_S$$

determined by the requirement that

$$(2.9) \quad \nabla^\phi \circ H^0(A)(H^0(X, E)) = 0,$$

i.e. that the heat lift of sections of $E$ be flat.

**D.** Here $g^d = N_{Y/X}[-1] = (T_{X/Y}, T_X)$. To start with, a $T_{X/Y}$–deformation is simply a deformation in the usual sense of the pair $(Y, X)$, giving rise, e.g. to a deformation $(I_Y^\phi, O_X^\phi)$ of the left $g^d$–module $(I_Y, O_X)$ and to a deformation $(O_X^\phi, O_Y^\phi)$ of the right $g^d$–module $(O_X, O_Y)$. Then a left (resp. right) $g^d$–deformation of $(I_Y, O_X)$ (resp. $(O_X, O_Y)$ consists of a $T_{X/Y}$–deformation, together with a $T_X$–map

$$A : O_X^\phi \to O_X \otimes S,$$

(resp. $B : O_X \otimes S \to O_X^\phi$).

Either $A$ or $B$ yield trivialisations of the deformation $O_X^\phi$. Thus left $g^d$–deformations yield deformations of $Y$ in a fixed $X$, and similarly for right deformations. Conversely, given a deformation of $Y$ in a fixed $X$, let $(x^k_\alpha)$ be local equations for $Y$ in $X$, part of a local coordinate system. Then it is easy to see that we can write equations for the deformation of $Y$ in the form

$$\exp(v_\alpha)(x^k_\alpha), \quad v_\alpha \in T_X \otimes m$$
(\(v_\alpha\) independent of \(k\)), so this comes from a left and a right \(\mathfrak{g}^d\)–deformation of the form \(((\Psi_{\alpha\beta}), (v_\alpha))\) where

\[
\Psi_{\alpha\beta} = \exp(v_\alpha)\exp(-v_\beta) \in \mathfrak{U}(T_X/Y \otimes \mathfrak{m})(U_\alpha \cap U_\beta).
\]

Thus the four notions of left, right and 2-sided \(N_Y/X[-1]\)-deformations and deformations of \(Y\) in a fixed \(X\) all coincide.

Similarly, for a mapping \(f: Y \to X\), an \(N_f/X[-1]\)-deformation is a deformation of \((f,Y)\) fixing \(X\). For \(f\) submersive, a \(T_f/Y[-1]\)-deformation is a deformation of \((Y,f)\) fixing \(Y\).

E. In this case we see similarly that \(T_X \otimes O_Y\)–deformations of \((O_X \to O_Y)\) consist of a deformation of the embedding \(Y \hookrightarrow X\), fixing both \(X\) and \(Y\).

2.2 Kodaira-Spencer class.

Our purpose here is to associate a (higher-order) Kodaira-Spencer class to a \(\mathfrak{g}^d\)-deformation. In the next section we will show that such classes can, under suitable hypotheses, be used to classify \(\mathfrak{g}^d\)-deformations.

The basic idea of Kodaira-Spencer theory can be described thus: let \((\tilde{\mathfrak{g}}, \partial)\) be a suitable acyclic resolution of \(\mathfrak{g}\), let \(G^0_S = \exp(\mathfrak{g}^0 \otimes \mathfrak{m})\). Then a \(\mathfrak{g}\)-deformation, i.e. a \(G_S\)-torsor, may be considered as a subsheaf of the trivial \(G^0_S\)-torsor. This subsheaf can be determined by specifying its tangent space, which can be determined by a suitable operator on \(\tilde{\mathfrak{g}} \otimes \mathfrak{m}\) deforming \(\partial\). The deformation of the operator is essentially the Kodaira-Spencer class \(\phi\) of the \(\mathfrak{g}\)-deformation. Extending this to \(\mathfrak{g}^d\)-deformation is a matter of accounting in terms of \(\phi\) for a trivialization of the \(\mathfrak{h}^+\)-deformation corresponding to \(\phi\). This is trickier than one might expect, in part because the compatibility relation characterizing an \(\mathfrak{h}^+\)-trivialization of \(\phi\) is not a linear or even quadratic condition (as is e.g. the integrability condition on \(\phi\)), but an \(n\)-th degree condition, where \(n\) is the order of the deformation.

Consider then a \(\mathfrak{g}^d\)-deformation as in Definition 2.1.2, consisting of a \(G^d\)-torsor \(G^\phi\), with transitions

\[
\Psi_{\alpha\beta} = \exp(u_{\alpha\beta}), u_{\alpha\beta} \in \mathfrak{g} \otimes \mathfrak{m}
\]

with respect to a suitable open covering \((U_\alpha)\), plus an \(\mathfrak{h}\)-map

\[
A: (H^+)_S \to (H^+)^\phi
\]

locally given as

\[
\exp(v_\alpha), v_{\alpha\beta} \in \mathfrak{h}^\infty_m.
\]

Then the condition that the \(\exp(v_\alpha)\) should glue together to a globally defined map left \(\mathfrak{h}\)-map \(A\) is

\[
(2.2.1) \quad \Psi_{\alpha\beta} \circ \exp(v_\alpha) = \exp(v_\beta).
\]

By analogy with the procedure of [R, p.61], this condition may be analyzed in terms of Kodaira-Spencer cochains, as follows. Let

\[
\tilde{\mathfrak{g}}^d = (\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\partial})
\]

be a suitable acyclic (soft or injective) differential graded Lie algebra resolution of \(\mathfrak{g}\) and set

\[
(\mathfrak{g})^d = (\mathfrak{g}, \mathfrak{h}, \partial) := (\Gamma(\tilde{\mathfrak{g}}), \Gamma(\tilde{\mathfrak{h}}), \partial)
\]

Write

\[
\Psi_{ab} = \exp(-s_a)\exp(s_\beta),
\]
\[ s_\alpha \in \tilde{g}^0(U_\alpha) \otimes m, \]

where the cochain \((\exp(s_\alpha))\) is determined up to left multiplication by an element of the 'soft gauge group'

\[ \exp(\mu) \in G^0_S = \exp(\Gamma(\tilde{g}^0) \otimes m), \]

with \(\mu\) independent of \(\alpha\). Thus

\[ \exp(-s_\alpha) \exp(v_\alpha) = \exp(-s_\beta) \exp(v_\beta) \]

so this element is globally defined and, by the Campbell-Hausdorff formula, can be written as

\[ \exp(\psi(x)), \quad \psi \in (h_{m}^0)^{[\infty]} = (h_{m}^0)^{[e]} \]

(where \(e\) is such that \(m^{e+1} = 0\)). As observed in \([\text{Rcid}]\), the element

\[ \phi := \exp(-s_\alpha) \partial \exp(s_\alpha) \in g^1 \otimes m \]

is globally defined independent of \(\alpha\) and is the Kodaira-Spencer cochain defining the deformation, and we can write formally

\[ \phi = D(-\text{ad}(s))(\partial s) \]

where

\[ D(x) = (e^x - 1)/x \]

is the Deligne function. \(\phi\) is well-defined up to replacing \(\exp(s_\alpha)\) by \(\exp(\mu) \exp(s_\alpha)\) as above (\(\mu\) independent of \(\alpha\)), which is equivalent to conjugating the operator \(\partial + \phi\) by \(\exp(\mu)\). Consider the Bernoulli function

\[ C(x) = 1/D(x) = \sum_{n=0}^{\infty} c_n x^n. \]

Thus \(c_n = B_n/n!\) where \(B_n\) is the \(n\)th Bernoulli number (cf. §0.2). Then a formal calculation in \(\Omega_S(\Gamma(h)^{\pm}_{m})\) shows that

\[ \partial \exp(\psi) = -\phi \exp(\psi), \partial \exp(-\psi) = \exp(-\psi)\phi \]

This implies

\[ i(\phi) = \partial(\exp(\psi)) \exp(-\psi) = D(-\text{ad}(\psi))(\partial \psi) \]

hence

\[ \partial \psi = C(\text{ad}(-\psi)) i(\phi) = \sum_{n=0}^{\infty} c_n \text{ad}(-\psi)^n i(\phi). \quad (2.2.2) \]

In other words, the vector

\[ (-\psi, \phi, -\psi \otimes \phi, \ldots, (-\psi)^n \otimes \phi, \ldots) \in (h_{m}^0)^{[e]} \oplus g^1 \otimes m \oplus \ldots \oplus \text{Sym}^n((h_{m}^0)^{[e]}) \otimes g^1 \otimes m \]

is a cocycle for the JacoBer complex \(J^\flat(\tilde{g}_{m}^0)\). In view of the definition of the JacoBer complex, it follows that for any \(i \geq 1\), the vector

\[ c(\phi, \psi) = (\bigwedge^r \phi \otimes (-\psi)^n) \in \bigoplus_{r+n \leq i} \bigwedge^r (g^1 \otimes m_i) \otimes \text{Sym}^n((h_{m}^0)^{[d]}) \],

\[ r \geq 1, \quad i \geq 1, \quad c \geq 1. \]
which is a priori a 0-cochain for the complex \( J^g_i \), is a cocycle as well. The associated cohomology class

\[
\alpha_i = \alpha_i(\phi, \psi) \in H^0(J^g_i) \subset H^0(J^g_i) \otimes m_i,
\]

is called the \( i \)-th \textit{Kodaira-Spencer class} of the \( g \)-deformation \((G^g, A)\). For all \( i \geq e \), clearly \( \alpha_i \) has the same value, which we denote by \( \alpha(\phi, \psi) \) and call 'the' Kodaira-Spencer class of the deformation.

It is also clear from the definitions that the \( \alpha_i(\phi, \psi) \) are comultiplicative or 'morphic', hence their 'transpose' yields a sequence of ring homomorphisms

\[
\alpha_i^t(\phi, \psi) : R_i(g^g) = C \oplus H^0(J^g_i) \to S_i,
\]

that are mutually compatible via (1.2.6). Obviously, \( \alpha_i(\phi, \psi) \) determines \( \alpha_i(\phi, \psi) \). However, it is not clear a priori that given a homomorphism as in (2.2.5), it necessarily comes from an element of \( H^0(J^g_i) \), as opposed to \( H^0(J^g_i) \otimes m_i \).

Nonetheless, suppose conversely that we have a local artinian \( C \)-algebra \( S \) of exponent \( e \) and a compatible sequence of homomorphisms

\[
\beta_i : R_i(g^g) \to S_i, i \leq e.
\]

Suppose moreover, as in Definition 1.2.3, that \( g^g \) is positive. Then we may further assume that \( g^{\leq 0} = 0, h^{< 0} = 0 \). Clearly each \( \beta_i \) can be written as as \( \alpha_i(\phi_i, \psi_i) \) where

\[
\alpha_i(\phi_i, \psi_i) \in H^0(J^g_i) \otimes m_i.
\]

Thanks to our vanishing hypothesis, the pairs \( \phi_i, \psi_i \) are uniquely determined. Therefore

\[
\psi_i \equiv \psi_e \mod m_i^{i+1}.
\]

Since \( \psi_i \in (h^0) \otimes m_i \), this implies that

\[
\psi_e \in (h^0)^{[\infty]} = (h^0_m)^{[\infty]},
\]

so that

\[
\alpha_e(\phi_e, \psi_e) \in H^0(J^g_e) = H^0(J^g(g \otimes h^{[\infty]})).
\]

and clearly

\[
\beta_i = \alpha_i(\phi_e, \psi_e).
\]

We have established the following

\textbf{Proposition 2.2.1.} Let \( g^g = (g, h) \) be a positive Lie atom, and let \( S \) be a local artinian \( C \)-algebra of exponent \( e \). Then

(i) there is a 1-1 correspondence between compatible sequences of morphic elements

\[
\alpha_i \in H^0(J^g_i) = 1, \ldots, e
\]

and compatible sequences of homomorphisms

\[
\beta_i : R_i(g^g) \to S_i, i \leq e;
\]

(ii) the Kodaira-Spencer class of any \( g^g \)-deformation over \( S \) yields such sequences.

We call a sequence \( \alpha. = (\alpha_1, \ldots, \alpha_e) \) or \( \beta. = (\beta_1, \ldots, \beta_e) \) a Kodaira-Spencer sequence for \( g^g \) and \( S \). The following is obvious
Corollary 2.2.2. (i) A Kodaira-Spencer sequence is uniquely determined by its last element.

(ii) A local homomorphism \( \beta_\infty : R(\mathfrak{g}^i) = R_\infty(\mathfrak{g}^i) \to S \) gives rise to a Kodaira-Spencer sequence \( \beta \). If for each \( i, \beta_\infty \) maps the kernel of \( R(\mathfrak{g}^i) \to R_i(\mathfrak{g}^i) \), an ideal which a priori contains \( \mathfrak{m}_S^{i+1} \), to \( \mathfrak{m}_S^{i+1} \).

(iii) If \( \mathfrak{g}^2 \) is a Lie pair, any local homomorphism \( \beta_\infty : R(\mathfrak{g}^i) \to S \) yields a Kodaira-Spencer sequence.

Remark 2.2.3. When \( \mathfrak{g}^2 \) is not a Lie pair, \( R_i(\mathfrak{g}^i) \) will in general be a proper quotient of \( R_\infty(\mathfrak{g}^i)/\mathfrak{m}_e^{i+1} \), so assertion (iii) above will not hold.

3. Universal deformations

A principal goal of \( \mathfrak{g} \)-deformation theory is the construction of a universal \( \mathfrak{g} \)-deformation, i.e. a \( \mathfrak{g} \)-deformation over \( R(\mathfrak{g}) \), from which any \( \mathfrak{g} \)-deformation over \( S \) is obtained via a (unique) base-change map \( R(\mathfrak{g}) \to S \). [Actually what one seeks is, for each \( e \geq 0 \), an \( e \)-universal deformation, which is one having the above universality property for all \( S \) of exponent \( e \), i.e. such that \( \mathfrak{m}_S^{e+1} = 0 \); then one can take the limit as \( e \to \infty \) to get the (formally) universal deformation. But we will ignore such technicalities in this introductory paragraph.] Proving the existence of a universal deformation typically requires some restrictive hypotheses on \( \mathfrak{g} \), along the lines of nonexistence of automorphisms \( (H^0(\mathfrak{g}) = 0) \), as well as some more technical finiteness hypotheses (‘admissibility’). See [Rcid, Ruvhs] for details. The proof typically proceeds according to the following outline:

(i) ‘inverting’ the Kodaira-Spencer class, e.g. by associating to a local homomorphism
\[
\beta(h) \to S
\]

a \( \mathfrak{g} \)-deformation \( \beta(h) \) over \( S \); in particular, by applying this to the identity map of \( R(\mathfrak{g}) \) we obtain a ‘distinguished’ deformation \( \phi^u \) over \( R(\mathfrak{g}) \) (that one would like to prove is actually universal);

(ii) proving that for any \( \mathfrak{g} \)-deformation \( \phi \) over \( S \), the Kodaira-Spencer homomorphism
\[
\alpha(\phi) : R(\mathfrak{g}) \to S
\]

constructed above depends only on the equivalence (i.e. \( G \)-conjugacy) class of \( \phi \) as deformation;

(iii) proving that for any local homomorphism \( h \) as above, the Kodaira-Spencer homomorphism associated to \( h^*(\phi^u) \) is just \( h \);

(iv) proving that for any \( \mathfrak{g} \)-deformation \( \phi, \phi \) is equivalent to \( \alpha(\phi)^*(\phi^u) \).

The conjunction of (i)-(iv) implies that the assignment
\[
h \mapsto h^*(\phi^u)
\]

yields a bijection
\[
\{ \text{local } \mathbb{C} \text{-algebra homomorphisms } R(\mathfrak{g}) \to S \} \to \{ \mathfrak{g} \text{-deformations over } S \}
\]

which implies the universality of \( \phi^u \).

The main purpose of this section is to construct, under suitable hypotheses, the universal \( \mathfrak{g}^i \)-deformation associated to a sheaf \( \mathfrak{g}^i \) of Lie atoms, which is simultaneously the universal \( \mathfrak{g}^i \)-deformation of any \( \mathfrak{g}^i \)-module \( E^i \) (see Theorem 3.3 below). This universal deformation is the evident analogue of the corresponding notion for \( \mathfrak{g} \)-deformations. We thus extend the main result of [Rcid] to the cases of atoms. We shall also prove a generalization (see Theorem 3.1) of the latter result where the hypothesis of trivial sections of weakened to ‘central sections’, i.e. that \( H^0(\mathfrak{g}) \) maps to the center of \( \mathfrak{g}(U) \) for all open sets \( U \).
3.1 dgla case revisited. We shall assume throughout, without explicit mention, that all sheaves of Lie algebras and modules considered are admissible in the sense of \[\text{Rcid}\]. In addition, unless otherwise stated we shall assume their cohomology is finite-dimensional. We begin by reviewing the main construction of \[\text{Rcid}\] and restating its main theorem in a stronger form. As in §1, we have the Jacobi complex \(J_m(\mathfrak{g})\), which is identified with the Jacobi complex associated to the differential graded Lie algebra \(\mathfrak{g} = (\Gamma(\tilde{\mathfrak{g}}), \partial)\), where \((\tilde{\mathfrak{g}}, \partial)\) is a suitable acyclic (injective, flabby, soft) resolution. This is a complex in degrees \([-m, -1]\) which admits a comultiplicative structure, making

\[R_m(\mathfrak{g}) = \mathbb{C} \oplus H^0(J_m(\mathfrak{g}))^*\]

a \(\mathbb{C}\)-algebra (finite-dimensional by the admissibility hypothesis), and we constructed a certain ‘tautological’ \(\mathfrak{g}\)-deformation \(u_m\) over it (more precisely, \(u_m\) is only defined up to equivalence—more details below). To any \(\mathfrak{g}\)-deformation \(\phi\) over an algebra \((S, m)\) of exponent \(m\) we associated a canonical Kodaira-Spencer homomorphism

\[\alpha = \alpha(\phi) : R_m(\mathfrak{g}) \to S;\]

e.g. \(u_m\) is essentially characterized by the property that \(\alpha(u_m) = \text{id}_{R_m(\mathfrak{g})}\). Although in [Rcid] we made the hypothesis that \(H^0(\mathfrak{g}) = 0\), this is in fact not needed for the foregoing statements, and is only used in the proof that \(u_m\) is an \(m\)-universal deformation.

Now the hypothesis \(H^0(\mathfrak{g}) = 0\) can be relaxed somewhat. Let us say that \(\mathfrak{g}\) has central sections if for each open set \(U \subset X\), the image of the restriction map

\[H^0(\mathfrak{g}) \to \mathfrak{g}(U)\]

is contained in the center of \(\mathfrak{g}(U)\). Equivalently, in terms of a dgla resolution as above \(\mathfrak{g} \to \tilde{\mathfrak{g}}\), the condition is that \(H^0(\mathfrak{g})\) be contained in the center of \(\Gamma(\tilde{\mathfrak{g}})\), i.e. the bracket

\[H^0(\mathfrak{g}) \times \Gamma(\tilde{\mathfrak{g}}) \to \Gamma(\tilde{\mathfrak{g}})\]

should vanish.

**Theorem 3.1.** Let \(\mathfrak{g}\) be an admissible dgla and suppose that \(\mathfrak{g}\) has central sections. Then for any \(\mathfrak{g}\)-deformation \(\phi\) there exists an equivalence of deformations

\[\phi \sim \alpha(\phi)^*(u_m) = u_m \otimes_{R_m(\mathfrak{g})} S;\]

any two such equivalences differ by an element of

\[\text{Aut}(\phi) = H^0(\exp(\mathfrak{g}^\phi \otimes m)).\]

In particular, if \(H^0(\mathfrak{g}) = 0\) then the equivalence is unique. Consequently, for any admissible pair \((\mathfrak{g}, E)\) there are equivalences

\[E^\phi \sim \alpha(\phi)^*(E^{u_m})\]

any two of which differ by an element of \(\text{Aut}(\phi)\).

**proof.** We first prove the isomorphism (3.2). More generally, we will show
Lemma 3.2. For any two deformations \( \phi_1, \phi_2 \), if \( \epsilon(\phi_1) \) and \( \epsilon(\phi_2) \) are cohomologous, then \( \phi_1, \phi_2 \) are equivalent as deformations.

This Lemma yields the isomorphism (3.2) as a special case because \( \epsilon(\phi) \) and \( \epsilon(\alpha(\phi)^*(u_m)) \) are clearly cohomologous. The Lemma is a generalization of [Rcid], Theorem 0.1, Step 4, pp. 63-64. We will give a new, self-contained proof. This proof is by induction on the exponent \( e \) of \( S \). Using induction, we may assume \( \phi_1, \phi_2 \) are equivalent mod \( m^e \). Hence there exist deformations \( \phi'_1, \phi'_2 \in g^1 \otimes m \) equivalent, respectively, to \( \phi_1, \phi_2 \), such that

\[
\phi'_1 \equiv \phi'_2 \mod m^e.
\]

Because equivalent deformations yield cohomologous Kodaira-Spencer classes (Step 2, p.62 of [Rcid]) it follows that

\[
[\epsilon(\phi'_1)] = [\epsilon(\phi_1)] = [\epsilon(\phi_2)] = [\epsilon(\phi'_2)].
\]

Replacing \( \phi_1, \phi_2 \) by \( \phi'_1, \phi'_2 \), we may in fact assume that

\[
\phi_1 \equiv \phi_2 \mod m^e.
\]

This implies that

\[
\phi'_1 = \phi'_2 \in \text{Sym}^i(g^1) \otimes m^i, \forall i > 1.
\]

Our assumption that \( \epsilon(\phi_1) \) and \( \epsilon(\phi_2) \) are cohomologous implies that there exist

\[
\mu_i \in g^0 \otimes \text{Sym}^{i-1}(g^1) \otimes m, \ i = 0, ..., m,
\]

such that \( \epsilon(\phi_1) - \epsilon(\phi_2) \) is the coboundary of \( (\mu_i) \). Recall that the total coboundary of \( J \) consists of a vertical part, induced by \( \partial \), and a horizontal part induced by the bracket. We can write

\[
\mu_i = \mu'_i \otimes \mu''_i, \forall i > 0
\]

with \( \mu''_i \in \text{Sym}^{i-1}(g^1) \otimes m \) linearly independent. Working backwards from the degree-\( m \) component, the fact that the part of the coboundary of \( (\mu_i) \) in \( \text{Sym}^m(\Gamma(g^1)) \) is zero implies that the vertical, \( (\partial\text{-induced}) \) coboundary of \( \mu_m \) vanishes, i.e.

\[
\partial(\mu'_m) = 0.
\]

By Central Sections, it follows that the horizontal (bracket-induced) coboundary of \( \mu_m \) vanishes. Therefore, the vertical coboundary of \( \mu_{m-1} \) vanishes, etc. Continuing backwards in this manner, we conclude eventually that the horizontal coboundary of \( \mu_1 \) is zero and

\[
\partial(\mu'_0) = \phi_1 - \phi_2.
\]

In particular, \( \mu_0 \equiv 0 \mod m^e \), so that \( [\mu_0, \phi_1] = [\mu_0, \phi_2] = 0 \). Therefore clearly \( \phi_1 \) and \( \phi_2 \) are equivalent as deformations, as

\[
\exp(-\mu_0))(\partial + \phi_2) \exp(\mu_0) = \partial + \phi_1.
\]

QED Lemma 3.2.
From Lemma 3.2 we deduce the existence of an isomorphism as in (3.2). Given this, the fact that two such isomorphisms differ by an element of \( \text{Aut}(\phi) \) is obvious. To identify the latter group it suffices to identify its Lie algebra \( \text{aut}(\phi) \), which is given locally by the set of \( g \)-endomorphisms

\[
\text{ad}(v) \in g^0 \otimes m
\]

of the resolution

\[
(g \otimes S, \partial + \text{ad}(\phi)).
\]

It is elementary to check that the condition on \( v \) is precisely

\[
\partial(v) + \text{ad}(\phi)(v) = 0,
\]

i.e. \( v \in g^\phi \). Thus the local endomorphism algebra of \( \phi \) is \( g^\phi \) and the global one is \( \text{aut}(\phi) = H^0(g^\phi) \). Finally, note that \( g^\phi \) admits a Jordan-H"older series with each subquotient isomorphic as a sheaf to \( g \). Therefore if \( H^0(g) = 0 \), we have

\[
H^0(g^\phi) = 0,
\]

hence \( \text{Aut}(\phi) = (1) \). □

Remark. Without the hypothesis of central sections it is still possible to 'classify' \( g \)-deformations over \((S, m)\) in terms of \( H^0(J_m(g), m^\phi) \) but it is not immediately clear how this is related to seminiversal deformations.

3.2 Case of Lie atoms. We now extend these results from Lie algebras to Lie atoms.

**Theorem 3.3.** Let \( g^\sharp = (g, h) \) be an admissible Lie atom such that \( g \) has central sections.

(i) For any \( g^\sharp \)-deformation \((\phi, \psi)\) over an artin local \( \mathbb{C} \)-algebra \( S \), the associated Kodaira-Spencer sequence \( \beta.(\phi, \psi) \) depends only on the equivalence class of \((\phi, \psi)\); conversely, \( \beta.(\phi, \psi) \) determines the equivalence class of \((\phi, \psi)\).

(ii) Assume \( g^\sharp \) is a Lie pair. Then for any natural number \( m \) there exists an equivalence class \( u_m = (\phi_m^u, \psi_m^u) \) of \( g^\sharp \)-deformations over \( R_m(g^\sharp) \) such that for any local artin \( \mathbb{C} \)-algebra of exponent \( \leq m \), the assignments

\[
(\phi, \psi) \mapsto \beta(\phi, \psi),
\]

\[
\beta \mapsto \beta^*(u_m)
\]

establish a bijection between the set of equivalence classes of \( g^\sharp \)-deformation over \( S \) and \( \text{Hom}_{\mathbb{C}-\text{alg}}(R_m(g^\sharp), S) \).

**proof.** (i) The proof of these assertions is the same as in the dglc case, Theorem 3.1 above.

(ii) Set \( S_m = R_m(g^\sharp) \) with its maximal ideal \( m_m \). Then the identity map of \( S_m \) corresponds to a morphic element of \( H^0(J_m(g^\sharp \otimes m_m)) \), which can be written in the form \( \alpha_m(\phi_m^u, \psi_m^u) \), and the fact that \( \alpha_m(\phi_m^u, \psi_m^u) \) comes from a cocycle implies that

\[
u_m = (\phi_m^u, \psi_m^u) \in (\Gamma(g^1) \oplus \Gamma(h^0) \otimes m_m
\]

is a \( g^\sharp \)-deformation, which we call a universal \( m \)-th order \( g^\sharp \) deformation. By (i), the equivalence class of \( u_m \) is independent of the choice of representative for the identity.
Now given an arbitrary $g^\sharp$-deformation $(\phi, \psi)$ over an artin local $\mathbb{C}$-algebra $(S, m)$ of exponent $e$, we get as in §2.2 a Kodaira-Spencer homomorphism

$$\alpha = \alpha_e(\phi, \psi) : R_e(g^\sharp) \to S.$$ 

The proof that $\alpha$ depends only on the equivalence class of $(\phi, \psi)$ is as in the proof of Theorem 3.1, as is the proof that $(\phi, \psi)$ is equivalent to $\alpha^*(u_e)$. □

Remark. For a general, say positive, Lie atom $g^\sharp$, we are not asserting the existence of a universal $g^\sharp$-deformation. If $g^+$ denotes the hull of $g^\sharp$, $g^\sharp$-deformations over $S$ are classified by the set of maps $R_e(g^+) \to S$ that happen to be compatible with a Kodaira-Spencer sequence. But it’s not clear that there is a single algebra $R$ such that $g^\sharp$-deformations over $S$ correspond bijectively with $\text{Hom}_{\mathbb{C} \text{-alg}}(R, S)$.

Examples 1.1.4, conclusion. Applying Theorem 3.3 to our standard examples, we see the following.

A-B When $E_1 < E_2$ are coherent sheaves on a complex projective scheme (or vector bundles on a compact complex manifold) $X$, the universal $\mathfrak{gl}(E_1 < E_2)$-deformation just constructed is the formal completion of the quot scheme $\text{Quot}(E_2)$ at the point $(E_2 \to E_2/E_1)$. The projectivity or compactness hypothesis $X$ is only needed to ensure admissibility. See a forthcoming article [Rsela] for applications of Lie atoms to local moduli and parameter spaces for schemes.

B For heat deformations, we have constructed the universal heat deformation of $E$ (again over a compact complex manifold, to ensure admissibility).

D The universal $N_{Y/X}[-1]$-deformation is the formal germ of the Hilbert scheme or Douady space, first constructed by Kodaira [K]. Note that because $N_{Y/X}[-1]$ is (cohomologically) supported on $Y$, only compactness of $Y$ is needed to ensure admissibility. For a holomorphic map $f : Y \to X$, the universal $T_f$-deformation is the universal deformation of the map $f$, first constructed by other means by Horikawa [Ho]. When $f$ is generically immersive, the universal $N_{f/X}[-1]$-deformation is the universal deformation of $f$ with fixed $X$, and similarly for $N_{f/X}$ when $f$ is generically submersive.

E The universal $T_X \otimes O_Y[-1]$-deformation is the universal deformation of the map $Y \to X$ fixing $X$ and $Y$.

One consequence of applying Theorem 3.3 on these examples is that Corollary 1.2.3 applies to them, and we conclude

Corollary 3.4. In each of the above examples A-E, if the relevant Lie atom $g^\sharp$ satisfies

$$h^1(g^\sharp) - h^2(g^\sharp) > 0,$$

then the corresponding object admits a nontrivial deformation.

As one example, if a submanifold $Y \subset X$ satisfies $(h^0 - h^1)(N_{Y/X}) > 0$, then $Y$ moves in $X$.

Remark. After this was written, the author became aware of a preprint 'L_\infty structures on mapping cones' (arxiv:math/QA/0601312) by D. Fiorenza and M. Manetti which, quoting an earlier version of this paper, considers related problems (in the case of Lie pairs only). Fiorenza and Manetti consider 'deformations' in a purely formal, Lie-theoretic sense, as solutions of the Maurer-Cartan integrability equation, ignoring the connection with the geometric view of deformations as torsors and questions of existence of universal deformations, and therefore also the subtle question of the relation between formal or cohomological equivalence of Maurer-Cartan solutions (measured e.g. via the Jacobi-Bernoulli complex or something similar), and equivalence or conjugacy of deformations, either
viewed as torsors or realized on a particular module (compare the Introduction to §2 above). Also, their results do not apply to general Lie atoms, hence they cannot be applied to Heat deformations. The purely formal results of Fiorenza-Manetti follow easily from our technique of Jacobi-Bernoulli complex. On the other hand the our main results here do focus on geometric deformations, especially universal ones, and for that reason involve, of necessity, some restrictive hypothesis such as finiteness and asymmetry.

As mentioned above, the methods of this paper, such as Jacobi-Bernoulli cohomology, have now been extended from Lie atoms to Semi-simplicial Lie algebras (SELA). It can also be shown that one can associate a SELA $T_X$ to any algebraic scheme $X/C$, so that deformations of $X$ as scheme can be expressed in terms of the Jacobi-Bernoulli cohomology of $T_X$. See [Rsel] for details.

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