Complex complex landscapes

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Spin-glasses have long been considered the paradigm of many variable ‘complex landscapes,’ a subject that includes neural networks and optimization problems, most notably constraint satisfaction. The most tractable family of these are the mean-field spherical $p$-spin models [1] (for a review see [2]) defined by the energy

$$H_0 = \frac{1}{p!} \sum_{i_1 \cdots i_p} J_{i_1 \cdots i_p} z_{i_1} \cdots z_{i_p},$$

where $J$ is a symmetric tensor whose elements are real Gaussian variables and $z \in \mathbb{R}^N$ is constrained to the sphere $z^2 = N$. This problem has been studied in the algebra [3] and probability literature [4, 5]. It has been attacked from several angles: the replica trick to compute the Boltzmann–Gibbs distribution [1], a Kac–Rice [6–8] procedure (similar to the Fadeev–Popov integral) to compute the number of saddle-points of the energy function [9], and the gradient-descent—or more generally Langevin—dynamics starting from a high-energy configuration [10]. Thanks to the simplicity of the energy, all these approaches yield analytic results in the large-$N$ limit.

In this paper we extend the study to complex variables: we shall take $z \in \mathbb{C}^N$ and $J$ to be a symmetric tensor whose elements are complex normal, with $\sum J^2 = p!/2N^{p-1}$ and $\sum J = \kappa |J|^2$ for complex parameter $|\kappa| < 1$. The constraint remains $z^2 = N$.

The motivations for this paper are of two types. On the practical side, there are indeed situations in which complex variables appear naturally in disordered problems: such is the case in which they are phases, as in random laser problems [11]. Quiver Hamiltonians—used to model black hole horizons in the zero-temperature limit—also have a Hamiltonian very close to ours [12].

There is however a more fundamental reason for this study: we know from experience that extending a real problem to the complex plane often uncovers underlying simplicity that is otherwise hidden. Consider, for example, the procedure of starting from a simple, known Hamiltonian $H_{00}$ and studying $\lambda H_{00} + (1-\lambda)H_0$, evolving adiabatically from $\lambda = 0$ to $\lambda = 1$, as is familiar from quantum annealing. The $H_{00}$ is a polynomial of degree $p$ chosen to have simple, known saddles. Because we are working in complex variables, and the saddles are simple all the way (we shall confirm this), we may follow a single one from $\lambda = 0$ to $\lambda = 1$, while with real variables minima of functions appear and disappear, and this procedure is not possible. The same idea may be implemented by performing diffusion in the $J$s and following the roots, in complete analogy with Dyson’s stochastic dynamics [13].

The spherical constraint is enforced using the method of Lagrange multipliers: introducing $\epsilon \in \mathbb{C}$, our energy is

$$H = H_0 + \sum_{i=1}^{N} \frac{p}{2} \epsilon (N - \sum_{i} z_i^2).$$

We choose to constrain our model by $z^2 = N$ rather than $|z|^2 = N$ in order to preserve the analyticity of $H$. The non-holomorphic constraint also has a disturbing lack of critical points nearly everywhere: if $H$ were so constrained, then $0 = \partial^* H = -p\epsilon z$ would only be satisfied for $\epsilon = 0$.

The critical points are of $H$ given by the solutions to the set of equations

$$\sum_{j \neq i} J_{ij_1 \cdots i_{p-1}} z_j z_{i_1} \cdots z_{i_{p-1}} = p\epsilon z_i$$

for all $i = 1, \ldots, N$, which for fixed $\epsilon$ is a set of $N$ equations of degree $p - 1$, to which one must add the constraint. In this sense this study also provides a complement to the work on the distribution of zeroes of random polynomials [14], which are for $N = 1$ and $p \to \infty$. We see from (3) that at any critical point, $\epsilon = H/N$, the average energy.

Since $H$ is holomorphic, any critical point of $\text{Re } H$ is also a critical point of $\text{Im } H$. The number of critical points of $H$ is therefore the same as that of $\text{Re } H$. From each saddle emerge gradient lines of $\text{Re } H$, which are also ones of constant $\text{Im } H$ and therefore constant phase.

Writing $z = x + iy$, $\text{Re } H$ can be considered a real-valued function of $2N$ real variables. Its number of saddle-points is
given by the usual Kac–Rice formula:

\[ N_J(\kappa, \epsilon) = \int dx \, dy \, \delta(\partial_x \Re H) \delta(\partial_y \Re H) \times \left| \det \begin{bmatrix} \partial_x \partial_x \Re H & \partial_x \partial_y \Re H \\ \partial_y \partial_x \Re H & \partial_y \partial_y \Re H \end{bmatrix} \right|. \]  

(4)

The Cauchy–Riemann equations may be used to write this in a manifestly complex way. With the Wirtinger derivative \( \partial = \frac{1}{2} (\partial_x - i \partial_y) \), one can write \( \partial_x \Re H = Re \partial \bar{H} \) and \( \partial_y \Re H = -Im \partial H \). Carrying these transformations through, we have

\[ N_J(\kappa, \epsilon) = \int dx \, dy \, \delta(Re \partial \Re H) \delta(Im \partial H) \times \left| \det \begin{bmatrix} Re \partial \partial H & -Im \partial \partial H \\ -Im \partial \partial H & Re \partial \partial H \end{bmatrix} \right|. \]

(5)

This gives three equivalent expressions for the determinant of the Hessian: as that of a \( 2N \times 2N \) real matrix, that of an \( N \times N \) Hermitian matrix, i.e. the norm squared of that of an \( N \times N \) complex symmetric matrix.

These equivalences belie a deeper connection between the spectra of the corresponding matrices. Each positive eigenvalue of the real matrix has a negative partner. For each such pair \( \pm \lambda, \lambda^2 \) is an eigenvalue of the Hermitian matrix and \( |\lambda| \) is a singular value of the complex symmetric matrix. The distribution of positive eigenvalues of the Hessian is therefore the same as the distribution of singular values of \( \partial \partial H \), or the distribution of square-rooted eigenvalues of \( (\partial \partial H)^T \partial \partial H \).

The expression (5) is to be averaged over \( J \) to give the complexity \( \Sigma \) as \( N \Sigma = \log N = \int dJ \, \log N_J \), a calculation that involves the replica trick. In most the parameter-space that we shall study here, the annealed approximation \( N \Sigma \sim \log N = \int dJ \, N_J \) is exact.

A useful property of the Gaussian \( J \) is that gradient and Hessian at fixed \( \epsilon \) are statistically independent \([8, 15]\), so that the \( \delta \)-functions and the Hessian may be averaged independently. The \( \delta \)-functions are converted to exponentials by the introduction of auxiliary fields \( \hat{z} = \hat{z} + i \hat{y} \). The average of those factors over \( J \) can then be performed. A generalized Hubbard–Stratonovich allows a change of variables from the introduction of auxiliary fields to eight bilinears defined by \( Na = |z|^2 \), \( Na = |\bar{z}|^2 \), \( N\bar{e} = \bar{z}^2 \), \( Nb = z^2 \), and \( Nd = \bar{z} \bar{e} \) (and their conjugates). The result, to leading order in \( N \), is

\[ \overline{N}(\kappa, \epsilon) = \int da \, db \, da \, db^* \, dc \, db^* \, dd \, db^* \, e^{Nf(a, \hat{a}, b, \hat{c}, d)}, \]  

(6)

where the argument of the exponential is

\[ f = 2 + \frac{1}{2} \log \det \left[ \begin{array}{cccc} 1 & a & d & b \\ a & 1 & b^* & d^* \\ d & b^* & \hat{c} & \hat{a} \\ b & d^* & \hat{a} & \hat{c} \end{array} \right] + \int d\lambda \, \rho(\lambda) \log |\lambda|^2 + p \Re \left\{ \frac{1}{8} (\hat{a}a - 1 + (p - 1) d(2a^p - 2 + k(c^* + (p - 1)b^2)) - eb \right\}. \]

(7)

The integral of the distribution \( \rho \) of eigenvalues of \( \partial \partial H \) comes from the Hessian and is dependant on \( a \) alone. This function has an extremum in \( \hat{a}, b, \hat{c}, \) and \( d \) at which its value is

\[ f(a) = 1 + \frac{1}{2} \log \left( \frac{4}{p^2} a^2 - 1 \right) + \int d\lambda \, \rho(\lambda) \log |\lambda|^2 - 2C_+ [\Re(\epsilon \epsilon^{\ast} \lambda)]^2 - 2C_- [\Im(\epsilon \epsilon^{\ast} \lambda)]^2, \]

(8)

where \( \theta = \frac{1}{2} \arg \kappa \) and

\[ C_\kappa = \frac{a^p (1 + p (a^2 - 1)) + a^2 |\kappa|}{a^2 p + (p - 1) a^p (a^2 - 1) |\kappa| - a^2 |\kappa|^2}. \]

(9)

This leaves a single parameter, \( a \), which dictates the magnitude of \( |z|^2 \), or alternatively the magnitude \( y^2 \) of the imaginary part. The latter vanishes as \( a \to 1 \), where (as we shall see) one recovers known results for the real \( p \)-spin.

The Hessian \( \partial \partial H = \partial \partial H_0 - p \epsilon I \) is equal to the unconstrained Hessian with a constant added to its diagonal. The eigenvalue distribution \( \rho \) is therefore related to the unconstrained distribution \( \rho_0 \) by a similar shift: \( \rho(\lambda) = \rho_0(\lambda + p \epsilon) \).
The eigenvalue spectrum of the Hessian of the real part is
\[ \lambda \text{ of } \partial \partial H \] rather than the product over all singular values \[ 18 \]. Therefore, the
density of singular values follows from the jump to the numerator of the Green function \[ 17 \] gives

\[ G(\sigma) = \lim_{n \to 0} \int d\xi \, d\xi^\ast (\xi^{(0)}_i)^* \xi^{(0)}_i \exp \left\{ \frac{1}{2} \sum_{a} \left[ (\xi^{(a)}_i)^* \xi^{(a)}_i - \mathrm{Re} \left( \sum_{j} (\xi^{(a)}_i)^* \xi^{(a)}_j \partial_i \partial_j H \right) \right] \right\}, \quad (12) \]

with sums taken over repeated Latin indices. The average is then made over \( J \) and Hubbard–Stratonovich is used to change variables to the replica matrices \( N\alpha_{ab} = (\xi^{(a)})^* \cdot \xi^{(b)} \) and \( N\chi_{ab} = \xi^{(a)} \cdot \chi^{(b)} \) and a series of replica vectors. The replica-symmetric ansatz leaves all off-diagonal elements and vectors zero, and \( \alpha_{ab} = \alpha_0 \delta_{ab}, \chi_{ab} = \chi_0 \delta_{ab} \). The result is

\[ \overline{G}(\sigma) = N \lim_{n \to 0} \int d\alpha_0 \, d\chi_0 \, d\chi_0^\ast \alpha_0 \exp \left\{ nN \left[ 1 + \frac{p(p-1)}{16} a_p^2 - \frac{\alpha_0 \sigma}{2} + \frac{1}{2} \log(\alpha_0^2 - |\chi_0|^2) + \frac{p}{4} \mathrm{Re} \left( \frac{(p-1)}{8} k^2 \chi_0^2 - \epsilon^2 \chi_0^2 \right) \right] \right\}. \]

The transition from a one-cut to two-cut singular value spectrum naturally corresponds to the origin leaving the support of the eigenvalue spectrum. Weyl’s theorem requires that the product over the norm of all eigenvalues must not be greater than the product over all singular values \[ 18 \]. Therefore, the absence of zero eigenvalues implies the absence of zero singular values. The determination of the threshold energy – the energy at which the distribution of singular values becomes gapped – is then reduced to a geometry problem, and yields

\[ |\epsilon_h|^2 = \frac{p-1}{2p} a_p^2 \left( 1 - |\delta|^2 \right)^2 \frac{1}{1 + |\delta|^2 - 2|\delta| \cos(\arg \kappa + 2 \arg \epsilon)} \]

for \( \delta = \kappa a^{-p} \).

Given \( \rho \), the integral in \( 8 \) may be preformed for arbitrary \( a \). The resulting expression is maximized for \( a = \infty \) for all values of \( \kappa \) and \( \epsilon \). Taking this saddle gives

\[ \log N(\kappa, \epsilon) = N \log(p-1). \]

This is, to this order, precisely the Bézout bound, the maximum number of solutions that \( N \) equations of degree \( p - 1 \) may have \[ 19 \]. That we saturate this bound is perhaps not surprising, since the coefficients of our polynomial equations \( 3 \) are complex and have no symmetries. Reaching Bézout in \( 15 \) is not our main result, but it provides a good check. Analogous asymptotic scaling has been found for the number of pure Higgs states in supersymmetric quiver theories \[ 20 \].

More insight is gained by looking at the count as a function of \( a \), defined by \( N(\kappa, \epsilon, a) = e^{-Nf(a)} \). In the large-\( N \) limit, this is the cumulative number of critical points, or the number of critical points \( z \) for which \( |z|^2 \leq a \). We likewise define the \( a \)-dependent complexity \( \Sigma(\kappa, \epsilon, a) = N \log N(\kappa, \epsilon, a) \)

Everything is analytically tractable for \( \epsilon = 0 \), giving

\[ \Sigma(\kappa, 0, a) = \log(p-1) - 1 \frac{1}{2} \log \left( 1 - |\epsilon|^2 a^{-2(p-1)} \right) \]

(16)
FIG. 2. The complexity of the 3-spin model at \( \epsilon = 0 \) as a function of \( a = |\alpha|^2 = 1 + \nu^2 \) at several values of \( \kappa \). The dashed line shows \( \frac{1}{2} \log(p - 1) \), while the dotted shows \( \log(p - 1) \).

Notice that the limit of this expression as \( a \to \infty \) corresponds with (15), as expected. This is plotted as a function of \( a \) for several values of \( \kappa \) in Fig. 2. For any \( |\kappa| < 1 \), the complexity goes to negative infinity as \( a \to 1 \), i.e., as the spins are restricted to the reals. This is natural, given that the \( y \) contribution to the volume shrinks to zero as that of an \( N \)-dimensional sphere \( \sum_i \lambda_i^2 = N(a-1) \) with volume \( (a-1)^N \). However, when the result is analytically continued to \( \kappa = 1 \) (which corresponds to real \( J \)) something novel occurs: the complexity has a finite value at \( a = 1 \). Since the \( a \)-dependence gives a cumulative count, this implies a \( \delta \)-function density of critical points along the line \( y = 0 \). The number of critical points contained within is

\[
\lim_{a \to 1} \lim_{\kappa \to 1} \log \mathcal{N}(\kappa, 0, a) = \frac{1}{2} N \log(p - 1),
\]

half of (15) and corresponding precisely to the number of critical points of the real \( p \)-spin model (note the role of conjugation symmetry, already underlined in [14]). The full \( \epsilon \)-dependence of the real \( p \)-spin is recovered by this limit as \( \epsilon \) is varied.

These qualitative features carry over to nonzero \( \epsilon \). In Fig. 3 we show that for \( \kappa < 1 \) there is always a gap of \( a \) close to one for which there are no solutions. When \( \kappa = 1 \)—the analytic continuation to the real computation—the situation is more interesting. In the range of energies where there are real solutions this gap closes, which is only possible if the density of solutions diverges at \( a = 1 \). Another remarkable feature of this limit is that there is still a gap without solutions around ‘deep’ real energies where there is no real solution. A moment’s thought tells us that this is a necessity: otherwise a small perturbation of the \( J \)s could produce an unusually deep solution to the real problem, in a region where this should not happen.

The relationship between the threshold and ground, or extremal, state energies is richer than in the real case. In Fig. 4 these are shown in the complex-\( \epsilon \) plane for several examples. Depending on the parameters, the threshold might always come at smaller magnitude than the extremal state, or always come at larger magnitude, or cross as a function of complex argument. For sufficiently large \( a \) the threshold always comes at larger magnitude than the extremal state. If this were to happen in the real case, it would likely imply our replica symmetric computation is unstable, since having a ground state above the threshold implies a ground state Hessian with many negative eigenvalues, a contradiction. However, this is not an obvious contradiction in the complex case. The relationship between the threshold, i.e., where the gap appears, and the dynamics of, e.g., a minimization algorithm or physical dynamics, are a problem we hope to address in future work.

This paper provides a first step for the study of a complex landscape with complex variables. The next obvious one is to study the topology of the critical points and gradient lines of constant phase. We anticipate that the threshold level, where the system develops a mid-spectrum gap, will play a crucial role as it does in the real case.

FIG. 3. The minimum value of \( a \) for which the complexity is positive as a function of (real) energy \( \epsilon \) for the 3-spin model at several values of \( \kappa \).

FIG. 4. Energies at which states exist (green shaded region) and threshold energies (black solid line) for the 3-spin model with \( \kappa = \frac{3}{2} e^{-13\pi/4} \) and (a) \( a = 2 \), (b) \( a = 1.325 \), (c) \( a = 1.125 \), and (d) \( a = 1 \). No shaded region is shown in (d) because no states exist at any energy.

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