SOME LIMIT THEOREMS CONNECTED WITH BROWNIAN LOCAL TIME

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Let \( B = (B_t)_{t \geq 0} \) be a standard Brownian motion and let \((L^x_t; t \geq 0, x \in \mathbb{R})\) be a continuous version of its local time process. We show that the following limit
\[
\lim_{\varepsilon \to 0} \left( \frac{1}{2\varepsilon} \int_0^t \{F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon)\} \, ds \right)
\]
is well defined for a large class of functions \(F(t, x)\), and moreover we connect it with the integration with respect to local time \(L^x_t\). We give an illustrative example of the nonlinearity of the integration with respect to local time in the random case.

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1. Introduction

1.1. The local time of the Brownian motion \( B \) at the point \( a \) is defined as follows:

\[
L^a_t = \mathbb{P} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{|B_s - a| \leq \varepsilon} \, ds,
\]  

which equivalently could be written as follows:

\[
L^a_t = \mathbb{P} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \left( 1_{B_s \leq a - \varepsilon} - 1_{B_s \leq a + \varepsilon} \right) \, ds.
\]

Here we are, more generally, interested in the limit in \( L^1 \):

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon)\} \, ds
\]

for some function \( F \).

Our motivation comes from the desire to connect Chitashvili and Mania results [1] with those of Eisenbaum [2].
Without loss of generality, we restrict our attention to functions defined on $[0,1]$ in particular local time is not a 1-integrator, which is also proved by Eisenbaum [2].

2. Notation and preliminaries

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion and let $(L_t^x; t \geq 0, x \in \mathbb{R})$ be a continuous version of its local time process. Let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration generated by $B$. Without loss of generality, we restrict our attention to functions defined on $[0,1] \times \mathbb{R}$.

For a measurable function $f$ from $[0,1] \times \mathbb{R}$ into $\mathbb{R}$, define the norm $\| \cdot \|$ by

$$
\| f \| = 2 \left( \int_0^1 \int_\mathbb{R} f^2(s,x) e^{-x^2/2s} \frac{ds}{\sqrt{2\pi s}} \right)^{1/2} + \int_0^1 \int_\mathbb{R} |xf(s,x)| e^{-x^2/2s} \frac{ds}{s\sqrt{2\pi s}}. \tag{2.1}
$$

Let $\mathcal{H}$ be the set of functions $f$ such that $\| f \| < \infty$.

In Eisenbaum [2], it is shown that the integration with respect to $L$ is possible in the following sense. Let $f_\Delta$ be an elementary function on $[0,1] \times \mathbb{R}$, meaning that

$$
f_\Delta(t,x) = \sum_{(s_i,x_j) \in \Delta} f_{i,j} 1_{(s_i,s_{i+1})}^1(t) 1_{(x_j,x_{j+1})}^1(x), \tag{2.2}
$$

where $\Delta = \{(s_i,x_j), 1 \leq i \leq n, 1 \leq j \leq m\}$ is an $[0,1] \times \mathbb{R}$ grid, and, for every $(i,j)$, $f_{ij}$ is in $\mathbb{R}$. For such a function, integration with respect to $L$ is defined by

$$
\int_0^1 \int_\mathbb{R} f_\Delta(s,x) dL^x_t = \sum_{(s_i,x_j) \in \Delta} f_{i,j} \left( L^{x_{j+1}}_{t_{i+1}} - L^{x_{j+1}}_{t_{i+1}} - L^{x_j}_{t_{i+1}} + L^{x_j}_{t_i} \right). \tag{2.3}
$$

Let $f$ be an element of $\mathcal{H}$. For any sequence of elementary functions $(f_\Delta)_{k \in \mathbb{N}}$ converging to $f$ in $\mathcal{H}$, the sequence $(\int_0^1 \int_\mathbb{R} f_\Delta(s,x) dL^x_t)_{k \in \mathbb{N}}$ converges in $L^1$. The limit obtained does not depend on the choice of the sequence $(f_\Delta)$ and represents the integral $\int_0^1 \int_\mathbb{R} f(s,x) dL^x_t$.

**Theorem 2.1** (see [2]). Let $(A(x,t); x \in \mathbb{R}, 0 \leq t \leq 1)$ be a continuous random process taking values in $\mathbb{R}$, such that for any $t$ in $[0,1]$ and any $\omega$, $A(\cdot,t)$ is absolutely continuous with respect to $dx$. Note $\partial A/\partial x$ its derivative and ask $\partial A/\partial x$ to be continuous. Then $\int_0^1 \int_\mathbb{R} A(x,s) dL^x_t$ exists and the following hold:

(i) for any couple $(a,b)$ in $\mathbb{R}^2$ with $a < b$

$$
\int_0^t \int_b^a A(x,s) dL^x_t = -\int_0^t \frac{\partial A}{\partial x}(B_s,s) ds + \int_0^t A(b,s) dL^b_t - \int_0^t A(a,s) dL^a_t; \tag{2.4}
$$

(ii)

$$
\int_0^1 \int_\mathbb{R} A(x,s) dL^x_t = -\left. \int_0^1 \frac{\partial A}{\partial x}(B_s,s) ds \right; \tag{2.5}
$$

(iii)

$$
\left( \int_0^t \int_b^a A(x,s) dL^x_t \right)(\omega) = \int_0^t \int_b^a A(x,s)(\omega) dL^x_t(\omega). \tag{2.6}
$$
3. Main results

3.1. Deterministic case

Theorem 3.1. Let $F$ be a bounded element of $\mathcal{H}$. The following equalities hold in $L^1$:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \{ F(s, B_s + \varepsilon) - F(s, B_s - \varepsilon) \} ds = \int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s, \quad \varepsilon > 0,$$

(3.1)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \{ F(s, B_s + \varepsilon) - F(s, B_s) \} ds = \int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s, \quad \varepsilon > 0,$$

(3.2)

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \{ F(s, B_s - \varepsilon) - F(s, B_s + \varepsilon) \} ds = \int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s.$$

(3.3)

Remark 3.2. (1) If we take $F(t, x) = 1_{(x \leq a)}$ in (3.1), we have the very definition of $L^a_t$.

(2) Eisenbaum [2] has shown that for any Borelian function $b(t)$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathbb{I}_{(b(s) - b(s')) \leq \varepsilon} ds = \int_0^t \int_{\mathbb{R}} \mathbb{I}_{(-\infty, b(s))}(x) dL^x_s \quad \text{in } L^1,$$

(3.4)

which corresponds to (3.3) with $F(t, x) = 1_{(x \leq b(t))}$.

Proof. Define $H_\varepsilon(t, x) = (1/\varepsilon) \int_{t-\varepsilon}^t F(t, y) dy$. Then $H_\varepsilon \to F$ in $\mathcal{H}$ as $\varepsilon \downarrow 0$. On the one hand, $(\partial/\partial x)H_\varepsilon(t, x) = (1/\varepsilon) \{ F(t, x) - F(t, x - \varepsilon) \}$. It follows that (see Eisenbaum [2, Theorem 5.1(i)]) $\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL^s_x = - (1/\varepsilon) \int_0^t \{ F(s, B_s) - F(s, B_s - \varepsilon) \} ds$. On the other hand, $\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL^s_x = \int_0^t \int_{\mathbb{R}} F(s, x) dL^s_x$ in $L^1$. \hfill $\square$

Corollary 3.3 (see [3]). The following relation holds in $L^1$:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t g(s) \mathbb{I}_{b(s - \varepsilon) < B_s < b(s) + \varepsilon} ds = \int_0^t g(s) dL^b_s,$$

(3.5)

for a continuous function $g : [0, t] \to \mathbb{R}$ and a continuous curve $b(\cdot)$ with bounded variation on $[0, t]$.

Proof. We apply Theorem 3.1 to the function $F(t, x) = g(t) I(b(t))$. It follows that $(1/2\varepsilon) \int_0^t g(s) I(b(s) - \varepsilon < B_s < b(s) + \varepsilon) ds \to \int_0^t g(s) I(b(s)) dL^b_s$ in $L^1$ as $\varepsilon \downarrow 0$. We conclude using (see [4, Corollary 2.9]) that for the continuous function $g$, we have $\int_0^t g(s) dL^b_s = \int_0^t g(s) dL^b_s$. \hfill $\square$

3.2. Random function case. Let $a, b$ be in $\mathbb{R}$ with $a < b$. Let $\mathcal{M}$ be the set of elementary processes $A$ such that

$$A(s, x) = \sum_{(s_i, x_j) \in \Delta} A_{ij} 1_{s_i \leq s < s_{i+1}}(s) 1_{x_j \leq x < x_{j+1}}(x),$$

(3.6)

where $(s_i)_{1 \leq i \leq m}$ is a subdivision of $(0, 1)$, $(x_j)_{1 \leq j \leq m}$ is a finite sequence of real numbers in $(a, b)$, $\Delta = \{(s_i, x_j), 1 \leq i \leq n, 1 \leq j \leq m\}$, and, is $A_{ij}$ an $\mathcal{F}_{s_i}$-measurable random variable such that $|A_{ij}| \leq 1$ for every $(i, j)$. 

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Eisenbaum [2] asked the following question: does integration with respect to \((L^x_t; 0 \leq t \leq 1, x \in \mathbb{R})\) admit a linear extension to \(\mathcal{P}\) the field generated by \(\mathcal{M}\), verifying the following property?

If \((A_n)_{n \geq 0}\) converges a.e. to \(A(t, x)\), then \(\int_0^1 \int_a^b A_n(s, x) dL^x_s\) converges in \(L^1\) to \(\int_0^1 \int_a^b A(s, x) dL^x_s\).

She only obtained a negative answer to the following weaker question:

Is the set \(\left\{ \int_0^1 \int_a^b A(s, x) dL^x_s, A \in \mathcal{M} \right\}\) bounded in \(L^1\)? (3.7)

Consequently, integration with respect to \((L^x_t; 0 \leq t \leq 1, x \in \mathbb{R})\) does not admit a continuous extension in \(L^1\).

Here we give an illustrative example, thanks to a result obtained by Walsh, which shows the lack of a linear extension.

Let us define \(A_\varepsilon(t, x) = \frac{1}{\varepsilon} \int^x_{x-\varepsilon} L^x_s dy\) and \(\tilde{A}_\varepsilon(t, x) = \frac{1}{\varepsilon} \int^x_{x+\varepsilon} L^x_s dy\). We see easily that \(A_\varepsilon(t, x)\) (resp., \(\tilde{A}_\varepsilon(t, x)\)) converges a.e. to \(L^x_t\), nevertheless we have

\[
\lim_{\varepsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL^x_s \neq \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} \tilde{A}_\varepsilon(s, x) dL^x_s.
\]

(3.8)

Remark 3.4. The integrals \(\int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL^x_s\) and \(\int_0^t \int_{\mathbb{R}} \tilde{A}_\varepsilon(s, x) dL^x_s\) are well defined thanks to Theorem 2.1, however, one does not know whether \(\int_0^t \int_{\mathbb{R}} L^x_s dL^x_s\) is well defined or not.

Let us recall, for the convenience of the reader, Walsh’s theorem about the decomposition of \(A(t, B_t) := \int_0^t 1_{\{B_s \leq B_t\}} ds\).

Theorem 3.5 (see [5]). \(A(t, B_t)\) has the decomposition

\[
A(t, B_t) = \int_0^t L^B_s dB_s + X_t,
\]

where

\[
X_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L^B_s - I^{B,\varepsilon}_s\} ds = t + \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \{L^{B,\varepsilon}_s - I_s^B\} ds.
\]

(3.10)

The limits exist in probability, uniformly for \(t\) in compact sets.

Our example follows by recalling the following property:

\[
\int_0^t \int_{\mathbb{R}} A_\varepsilon(s, x) dL^x_s = -\frac{1}{\varepsilon} \int_0^t \{I^{B,\varepsilon}_s - I^B_s\} ds.
\]

(3.11)

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