Badly approximable infinite products of quadratic polynomials

Dmitry Badziahin and Cameron Eggins

Abstract

We provide a number of conditions on the rational numbers $u$ and $v$ which ensure that the Laurent series $g_{u,v}(x) := \prod_{t=0}^{\infty} (1 + ux^{-3^t} + vx^{-2 \cdot 3^t})$ is badly approximable.

1 Introduction

Given a real number $\xi$, the irrationality exponent of $\xi$ is defined as follows

$$\mu(\xi) := \sup \left\{ \mu \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| < q^{-\mu} \text{ has i.m. solutions } p/q \in \mathbb{Q} \right\}.$$ 

This is one of the most important approximational properties of a number, which indicates, how well it can be approximated by rationals in terms of their denominators.

In recent years there was a lot of interest in understanding the irrationality exponents of Mahler numbers. By Mahler numbers we understand the values of Mahler functions at integer points. The Mahler functions are in turn analytical functions $f \in \mathbb{Q}((z^{-1}))$ which for any $z$ inside their disc of convergence satisfy the equation of the form

$$\sum_{i=0}^{n} P_i(z) f(z^{d^i}) = Q(z)$$

where $n \geq 1, d \geq 2$ are integer, $P_0, \ldots, P_n, Q \in \mathbb{Q}[z]$ and $P_0 P_n \neq 0$.

One of the first results in this direction was achieved in 2011 by Bugeaud [4]. He showed that the irrationality exponent of the Thue-Morse numbers equals
two. These are the numbers of the form $f_{tm}(b) := \sum_{n=0}^{\infty} \frac{t_n}{b^n}$ where $b$ is integer and $t_n$ is the famous Thue-Morse sequence in \{0, 1\} which is recurrently defined as follows: $t_0 := 0, t_{2n} := t_n$ and $t_{2n+1} = 1 - t_{2n}$. One can easily verify that the function $z^{-1}(1 - 2f_{tm}(z))$ satisfies the Mahler equation

$$f_{tm}(z) = (z - 1)f_{tm}(z^2).$$

For more results of this type, see [1], [6], [7], [2].

In [5], the authors provide a non-trivial upper bound for the irrationality exponent of $f(b)$ where the Mahler functions satisfy

$$Q(z) = P_0(z)f(z) + P_1(z)f(z^d).$$

Their bound is quite general and in many cases it is sharp. But that result is often hard to apply because it requires the knowledge about the distribution of non-zero Hankel determinants of $f(z)$, which are not easy to compute. Later Badziahin [3] provided the precise formula for $\mu(f(b))$ for a slightly narrower set of Mahler functions:

**Theorem B.** Let $f(z) \in \mathbb{Q}((z^{-1})) \setminus \mathbb{Q}(z)$ be a solution of the functional equation

$$f(z) = \frac{A(z)}{B(z)}f(z^d), \quad A, B \in \mathbb{Q}[z], \ B \neq 0, \ d \in \mathbb{Z}, \ d \geq 2.$$

Let $b \in \mathbb{Z}, |b| \geq 2$ be inside the disc of convergence of $f$ such that $A(b^m)B(b^m) \neq 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Then

$$\mu(f(b)) = 1 + \limsup_{k \to \infty} \frac{d_{k+1}}{d_k}, \quad (1)$$

Here, $d_k$ is the degree of the denominator of the $k$'th convergent of $f(z)$. We discuss these notions in Section [2].

The upshot is that the irrationality measure of $f(b)$, given by (1) is completely determined by the sequence $d_k$. However determining this sequence for a precise Mahler function $f(z)$ may be problematic. In 2017, the first author [2] verified that $d_k = k$ for all functions $g_u(z)$ which satisfy $g_u(z) = (z + u)g_u(z^2), \ u \in \mathbb{Q}$ and $u \neq 0, 1$. Equivalently, such functions can be written as the infinite products

$$g_u(z) = z^{-1} \prod_{t=0}^{\infty} (1 + uz^{-2t}).$$

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Notice that \( g_0(z) \) and \( g_1(z) \) are rational functions. Therefore we now have a complete understanding of irrationality exponents of \( g_u(b) \).

The next natural case to investigate are the following infinite products:

\[
g_{u,v}(z) = z^{-1} \prod_{t=0}^{\infty} (1 + uz^{-3^t} + vz^{-2 \cdot 3^t}), \quad u, v \in \mathbb{Q}.
\]

In [2], Badziahin started the investigation of sequences \( d_k \) for various pairs \((u, v)\) of integer numbers. It was shown that \( \limsup_{k \to \infty} \frac{d_{k+1}}{d_k} > 1 \) for:

1. \((u, v) = (\pm u, u^2), u \in \mathbb{Q};\)
2. \((u, v) = (\pm s^3, -s^2(s^2 + 1)), s \in \mathbb{Q};\)
3. \((u, v) = (\pm 2, 1).\)

Later, it was shown [3] that in the first two cases the value \( \limsup_{k \to \infty} \frac{d_{k+1}}{d_k} \) is equal to two while in the third case it is \( \frac{7}{5} \). It is shown in the same paper that for functions \( g_{u,v}(z) \) the condition \( \limsup_{k \to \infty} \frac{d_{k+1}}{d_k} = 1 \) is equivalent to \( \forall k \in \mathbb{N}, d_k = k \).

It is conjectured [2, Conjecture A] that \( d_k = k \) for all \( k \in \mathbb{Z} \) for the other pairs \((u, v)\) \in \mathbb{Z}^2 \). This conjecture is verified [2] for large sets of pairs \((u, v)\). In particular it is true if \( u = 0 \) or \( v = 0 \) and also for the region

\[
\{(u, v) \in \mathbb{Q}^2 : u^2 \geq 6, v \geq \max\{3u^2 - 1, 2u^2 + 8\}\}.
\]

Some local conditions on \( u \) and \( v \) modulo 3, ensuring \( d_k = k \) are also provided in [5]. The purpose of this note is to cover as many other pairs \((u, v)\) \in \mathbb{Z}^2 \) as possible and hence make a contribution to the conjecture above.

The main result of this paper provides a number of local conditions on \((u, v)\) modulo any prime \( p \geq 3 \) which ensure that \( d_k = k \) for all \( k \). In particular, for \( p = 3 \) they coincide with those from [5].

**Theorem 1.1.** Let \( p \geq 3 \) be prime and \((u, v) \in \mathbb{Z}^2\) satisfy one of the properties

1. \( u^2 \equiv 3, \quad v \equiv 1 \pmod{p}; \) \( (2) \)
2. \( u^2 \equiv -3, \quad v \equiv -1 \pmod{p}; \) \( (3) \)
3. \( u \equiv \pm \varphi, \quad v \equiv 0 \pmod{p}, \quad \text{where} \quad \varphi^2 + \varphi + 1 \equiv 0 \pmod{p}; \) \( (4) \)
4. \( u \equiv \pm \varphi, \quad v \equiv -1 \pmod{p}, \quad \text{where} \quad \varphi^4 + 4\varphi^2 + 1 \equiv 0 \pmod{p}; \) \( (5) \)
5. \( u \equiv \pm \varphi, \quad v \equiv \delta \pmod{p}, \quad \text{where} \quad \delta^2 - \delta + 1 \equiv 0, \quad \varphi^2 \equiv 2\delta \pmod{p}; \) \( (6) \)
6. \( u \equiv 0, \quad v \equiv \pm \delta \pmod{p}, \quad \text{where} \quad \delta^2 + \delta + 1 \equiv 0 \pmod{p}; \) \( (7) \)
7. \( u = \pm 2\delta^2, \quad v = \delta \pmod{p}, \quad \text{where} \quad \delta^2 + \delta + 1 = 0, \quad p \neq 3. \) \( (8) \)
Then all the partial quotients of the continued fraction for the Mahler function $g_{u,v}(z)$ are linear.

Theorem 1.1 together with (B) imply the following

**Corollary 1.1.** Let $p \geq 3$ be prime and $(u, v) \in \mathbb{Z}^2$ satisfy one of the properties 2–8 of Theorem 1.1. Then for any integer $b$ such that $|b| \geq 2$ and $g_{u,v}(b) \neq 0$ one has
\[
\mu(g_{u,v}(b)) = 2.
\]

## 2 Continued Fractions of Laurent Series

Let $F$ be a field. Consider the set $F[[z^{-1}]]$ of Laurent series together with the valuation: $||\sum_{k=-d}^{\infty} c_k z^{-k}|| = d$, the biggest degree $d$ of $x$ having non-zero coefficient $c_{-d}$. For example, for polynomials $f(z)$ the valuation $||f(z)||$ coincides with their degree. It is well known that in this setting the notion of continued fraction is well defined. In other words, every $f(z) \in F[[z^{-1}]]$ can be written as
\[
f(z) = [a_0(z), a_1(z), a_2(z), \ldots] = a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{a_3(z) + \cdots}}},
\]
where the $a_i(z)$ are non-zero polynomials of degree at least 1, $i \in \mathbb{Z}_{\geq 0}$. We refer the reader to a nice survey [8] for more properties of the continued fractions of Laurent series.

It will be more convenient for us to renormalise this continued fraction to the form
\[
f(z) = a_0(z) + \frac{\beta_1}{a_1(z) + \frac{\beta_2}{a_2(z) + \frac{\beta_3}{a_3(z) + \cdots}}} =: a_0(z) + \frac{1}{\frac{1}{a_1(z)} + \frac{1}{a_2(z)} + \cdots}
\]
where $\beta_i \in F \setminus \{0\}$ are constants and $a_i(z) \in F[z]$ are monic polynomials for $i \geq 1$.

By analogy with the classical continued fractions over $\mathbb{R}$, by $k$’th convergent of $f$ we denote the rational function
\[
\frac{p_k(z)}{q_k(z)} := a_0(z) + \frac{k}{K \sum_{i=1}^{i=K} \frac{\beta_i}{a_i(z)}}.
\]
They satisfy the following recurrent relation
\[
p_0(z) = a_0(z), \quad p_1(z) = a_0(z)a_1(z) + \beta_1, \quad p_n(z) = a_n(z)p_{n-1}(z) + \beta_n p_{n-2}(z),
\]
(10)
q_0(z) = 1, \ q_1(z) = a_1(z), \ q_n(z) = a_n(z)q_{n-1}(z) + \beta_nq_{n-2}(z). \quad (11)

By \( d_k \), we denote the degree (or the valuation) of the denominator \( q_k(z) \) of \( k \)'th convergent of \( f(z) \).

By analogy with the classical case of real numbers, we call a series \( f(z) \in F[[z^{-1}]] \) badly approximable if there exists an absolute constant \( M \) such that \( \forall k \in \mathbb{N}, \|a_k(z)\| \leq M \). Formulae (10), (11) suggest that \( \|a_k(z)\| = d_k-d_k-1 \) therefore an equivalent condition for badly approximable series is \( d_k-d_k-1 \leq M \).

Coming back to the series \( g_{u,v}(z) \), it is known (see [2, Proposition 1]) that \( g_{u,v}(z) \) is badly approximable if and only if \( d_k = k \) for all positive integer \( k \). Now the main tool in the proof of Theorem 1.1 is the following criterion [3, Theorems 2,3] which ensures that condition for \( g_{u,v}(z) \).

**Theorem B2.** Let \( u = (u, v) \in \mathbb{Q}^2 \). Let the sequences \( \alpha_i \) and \( \beta_i \) of rational numbers be computed by the recurrent formulae

\[
\begin{align*}
\alpha_1 &= -u, \quad \alpha_2 = \frac{u(2v - 1 - u^2)}{v - u^2}, \quad \alpha_3 = \frac{-u(v - 1)}{v - u^2}; \\
\beta_1 &= 1, \quad \beta_2 = u^2 - v, \quad \beta_3 = \frac{u^2 + u^4 + v^3 - 3u^2v}{(v - u^2)^2}.
\end{align*}
\]

(12)

and

\[
\begin{align*}
\alpha_{3k+4} &= -u, \quad \beta_{3k+4} = \frac{\beta_{3k+2}}{\beta_{3k+3}}, \\
\beta_{3k+5} &= u^2 - v - \beta_{3k+4}, \quad \alpha_{3k+5} = u - \frac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{\beta_{3k+5}}, \\
\alpha_{3k+6} &= u - \alpha_{3k+5}, \quad \beta_{3k+6} = v - \alpha_{3k+5}^2 \quad (13)
\end{align*}
\]

for any \( k \in \mathbb{Z}_{\geq 0} \). If all algebraic operations in these formulae are valid and \( \beta_i \neq 0 \) for all \( i \in \mathbb{Z} \), then

\[
g_u(z) = a_0(z) + \sum_{i=1}^{\infty} \frac{\beta_i}{a_i(z)}, \quad a_i(z) = z + \alpha_i,
\]

that is, all partial quotients of \( q_u(z) \) are linear.

### 3 Proof of Theorem 1.1

The main idea of the proof is that if a pair \((u, v)\) satisfies one of the conditions 2 – 8 of Theorem 1.1 then the sequences \( \alpha_i \) and \( \beta_i \) from Theorem B2 modulo \( p \) follow a nice pattern and moreover \( \beta_i \) never congruent to zero
modulo $p$. That immediately implies that for all $i \in \mathbb{N}$, $\beta_i \neq 0$ and Theorem B2 implies the required result.

While in each of the cases (2) – (8) the pattern for sequences $\alpha_i$ and $\beta_i$ is different, the proofs are extremely similar. We will provide a detailed proof in the case (2) and only outline the proofs in the remaining cases (3) – (8).

**Lemma 3.1.** If $u^2 \equiv 3 \pmod{p}$ and $v \equiv 1 \pmod{p}$ for odd prime $p$, then the sequences $(\alpha_i)_{i \in \mathbb{N}}$ and $(\beta_i)_{i \in \mathbb{N}}$ are given by the formula for all $k \geq 0$:

\[
\begin{align*}
\alpha_{3k+1} &\equiv -u, \quad \alpha_{3k+2} + \alpha_{3k+3} \equiv u \pmod{p}; \\
\alpha_{9k+2} &\equiv u, \quad \alpha_{9k+5} \equiv \alpha_{3k+3}, \quad \alpha_{9k+8} \equiv 0 \pmod{p}; \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv 2, \quad \beta_{k+3} \equiv 1 \pmod{p}.
\end{align*}
\]

**Proof.** To shorten the notation, in this proof we omit the (mod $p$) as it is implied in every congruence. We use the formulae (12) and (13) to compute the first 9 values of $\alpha_i$ and $\beta_i$:

\[
\begin{align*}
\alpha_1 &\equiv -u, \quad \alpha_2 \equiv u, \quad \alpha_3 \equiv 0, \quad \alpha_4 \equiv -u, \quad \alpha_5 \equiv 0, \\
\alpha_6 &\equiv u, \quad \alpha_7 \equiv -u, \quad \alpha_8 \equiv 0, \quad \alpha_9 \equiv u, \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv 2, \quad \beta_i \equiv 1 \text{ for } 3 \leq i \leq 9.
\end{align*}
\]

Now we prove by induction that for $k \geq 1$:

\[
\begin{align*}
\alpha_{9k+1} &\equiv \alpha_{9k+4} \equiv \alpha_{9k+7} \equiv -u, \\
\alpha_{9k+2} &\equiv u, \quad \alpha_{9k+3} \equiv 0, \quad \alpha_{9k+5} \equiv \alpha_{3k+3}, \quad \alpha_{9k+6} \equiv \alpha_{3k+2}, \quad \alpha_{9k+8} \equiv 0, \quad \alpha_{9k+9} \equiv u \\
\beta_i &\equiv 1, \quad 9k + 1 \leq i \leq 9k + 9
\end{align*}
\]

Which will give the formula we desire. Note that this is the same as equations (18) - (20), as we have just given explicit formulas for the terms defined by $\alpha_{3k+3} = u - \alpha_{3k+2}$.

Suppose that the formulas hold for all $0 \leq k \leq n$. Also note that this implies that up to these values, all every pair $\alpha_{3m+2}, \alpha_{3m+3}$ is either $(0, u)$ or $(u, 0)$ modulo $p$. Now we prove them for $k = n + 1$.

First, it is obvious that $\alpha_{9(n+1)+1} \equiv \alpha_{9(n+1)+4} \equiv \alpha_{9(n+1)+7} \equiv -u$ as they are all of the form $\alpha_{3k+1}$.

Second, by (13) we have $\beta_{9(n+1)+1} \equiv \frac{\beta_{9(n+1)+1}}{\beta_{9(n+1)} - \beta_{9(n+1)-1}} \equiv \frac{1}{1} \equiv 1$ by the induction hypothesis. Then $\beta_{9(n+1)+2} \equiv u^2 - 1 - \beta_{9(n+1)+1} \equiv 3 - 1 - 1 \equiv 1$. 

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Third, we compute:

\[
\begin{align*}
\alpha_{g(n+1)+2} &\equiv u - \frac{\alpha_{3(n+1)+1} + u - \alpha_{g(n+1)-1}\beta_{g(n+1)+1}}{\beta_{g(n+1)+2}} \\
&\equiv u - (\alpha_{3n+4} + u - \alpha_{g(n+1)+1}) \\
&\equiv u - (-u + u - 0) \equiv u
\end{align*}
\]

This then implies \(\alpha_{g(n+1)+3} \equiv u - \alpha_{g(n+1)+2} \equiv 0\). Thus, \(\beta_{g(n+1)+3} \equiv 1 - \alpha_{g(n+1)+2}\alpha_{g(n+1)+3} \equiv 1\).

Fourth, we continue in the same way to compute \(\beta_{g(n+1)+4} \equiv \frac{\beta_{3(n+1)+2}}{\beta_{g(n+1)+3}\beta_{g(n+1)+2}} \equiv \frac{1}{11} \equiv 1\) and \(\beta_{g(n+1)+5} \equiv u^2 - 1 - \beta_{g(n+1)+4} \equiv 1\).

Now this implies:

\[
\begin{align*}
\alpha_{g(n+1)+5} &\equiv u - \frac{\alpha_{3(n+1)+2} + u - \alpha_{g(n+1)+2}\beta_{g(n+1)+4}}{\beta_{g(n+1)+5}} \\
&\equiv u - (\alpha_{3(n+1)+2} + u - \alpha_{g(n+1)+2}) \\
&\equiv u - (\alpha_{3(n+1)+2} + u - u) \\
&\equiv u - \alpha_{3(n+1)+2} \equiv \alpha_{3(n+1)+3}.
\end{align*}
\]

This then implies \(\alpha_{g(n+1)+6} \equiv u - \alpha_{3(n+1)+3} + \alpha_{3(n+1)+2} \equiv 1 - \alpha_{g(n+1)+2}\alpha_{g(n+1)+3} \equiv 1 - \alpha_{3(n+1)+2}\alpha_{3(n+1)+3} \equiv 1\), as by the induction hypothesis one of these are 0 and the other is \(u\).

Finally we have \(\beta_{g(n+1)+7} \equiv \frac{\beta_{3(n+1)+3}}{\beta_{g(n+1)+6}\beta_{g(n+1)+5}} \equiv \frac{1}{11} \equiv 1\) and \(\beta_{g(n+1)+8} \equiv u^2 - 1 - \beta_{g(n+1)+7} \equiv 1\). This implies:

\[
\begin{align*}
\alpha_{g(n+1)+8} &\equiv u - \frac{\alpha_{3(n+1)+3} + u - \alpha_{g(n+1)+5}\beta_{g(n+1)+7}}{\beta_{g(n+1)+8}} \\
&\equiv u - (\alpha_{3(n+1)+3} + u - \alpha_{g(n+1)+5}) \\
&\equiv u - (\alpha_{3(n+1)+3} + u - \alpha_{3(n+1)+3}) \equiv 0.
\end{align*}
\]

This then implies \(\alpha_{g(n+1)+9} \equiv u - \alpha_{g(n+1)+8} \equiv u\) and \(\beta_{g(n+1)+9} \equiv 1 - \alpha_{g(n+1)+8}\alpha_{g(n+1)+9} \equiv 1\).

Thus the formula also holds for \(k = n + 1\), the proof by induction completes.

For the other cases we provide similar lemmata which can be proven in the same way by induction. We leave their proof to the interested reader.
Lemma 3.2. If \( u^2 \equiv -3 \pmod{p} \) and \( v \equiv -1 \pmod{p} \) for odd prime \( p \), then the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) are given by the formula for all \( k \geq 0 \):
\[
\begin{align*}
\alpha_{3k+1} &\equiv -u, \quad \alpha_{3k+2} + \alpha_{3k+3} \equiv u \pmod{p}; \\
\alpha_{9k+2} &\equiv 0, \quad \alpha_{9k+5} \equiv \alpha_{9k+2}, \quad \alpha_{9k+8} \equiv u \pmod{p}; \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv -2, \quad \beta_{k+3} \equiv -1 \pmod{p}.
\end{align*}
\]

Lemma 3.3. If \( u \equiv \varphi \pmod{p} \) and \( v \equiv 0 \pmod{p} \) where \( \varphi \in \mathbb{Z} \) satisfies \( \varphi^2 + \varphi + 1 \equiv 0 \pmod{p} \), then the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) are given by the formula for all \( k \geq 0 \):
\[
\begin{align*}
\alpha_{3k+1} &\equiv -\varphi, \quad \alpha_{3k+2} + \alpha_{3k+3} \equiv \varphi \pmod{p}; \\
\alpha_{9k+2} &\equiv -1, \quad \alpha_{9k+5} \equiv \alpha_{9k+2}, \quad \alpha_{9k+8} \equiv -\varphi^2 \pmod{p}; \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv \varphi^2, \quad \beta_{3k+3} \equiv -\varphi^2, \quad \beta_{3k+4} + \beta_{3k+5} \equiv \varphi^2 \pmod{p}; \\
\beta_{9k+1} &\equiv \beta_{3k+1}, \quad \beta_{9k+4} \equiv -\varphi, \quad \beta_{9k+7} \equiv -1 \pmod{p}.
\end{align*}
\]

One can easily derive from Lemma 3.3 that for \( i \geq 3 \) the value of \( \beta_i \) is congruent to either \(-1, -\varphi\) or \(-\varphi^2\) modulo \( p \), hence it never equals zero.

Lemma 3.4. If \( u \equiv \varphi \pmod{p} \) and \( v \equiv -1 \pmod{p} \) where \( \varphi \in \mathbb{Z} \) satisfies \( \varphi^2 + 4\varphi^2 + 1 \equiv 0 \pmod{p} \) and \( p \) is an odd prime, then the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) are given by the formula for \( k \geq 0 \):
\[
\begin{align*}
\alpha_{3k+1} &\equiv -\varphi, \quad \alpha_{3k+2} + \alpha_{3k+3} \equiv \varphi \pmod{p}; \\
\alpha_{9k+2} &\equiv -\varphi^{-1}, \quad \alpha_{9k+5} \equiv \alpha_{9k+3}, \quad \alpha_{9k+8} \equiv \varphi + \varphi^{-1} \pmod{p}; \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv \varphi^2 + 1, \quad \beta_{3k+3} \equiv -\varphi^{-2}, \quad \beta_{3k+4} + \beta_{3k+5} \equiv \varphi^2 + 1 \pmod{p}; \\
\beta_{9k+1} &\equiv \beta_{3k+1}, \quad \beta_{9k+4} \equiv \varphi^2, \quad \beta_{9k+7} \equiv 1 \pmod{p}.
\end{align*}
\]

One can check that for odd prime \( p \) the values \( \varphi^2 \) and \( \varphi^2 + 1 \) are not congruent to zero modulo \( p \). One can derive from Lemma 3.4 that for \( i \geq 3 \) the value of \( \beta_i \) is congruent to either \(-\varphi^{-2}, \varphi^2\) or 1 modulo \( p \). Hence it never equals zero.

Lemma 3.5. If \( u \equiv \varphi \pmod{p} \) and \( v \equiv \delta \pmod{p} \) where \( \delta, \varphi \in \mathbb{Z} \) satisfy \( \delta^2 - \delta + 1 \equiv 0 \pmod{p} \), \( \varphi^2 \equiv 2\delta \pmod{p} \) and \( p \) is an odd prime, then the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) are given by the formula for \( k \geq 0 \):
\[
\begin{align*}
\alpha_{3k+1} &\equiv -\varphi, \quad \alpha_{3k+2} + \alpha_{3k+3} \equiv \varphi \pmod{p}; \\
\alpha_{9k+2} &\equiv \frac{\varphi}{\delta}, \quad \alpha_{9k+5} \equiv \alpha_{9k+3}, \quad \alpha_{9k+8} \equiv \varphi\delta \pmod{p}; \\
\beta_1 &\equiv 1, \quad \beta_2 \equiv \delta, \quad \beta_{3k+3} \equiv -\delta, \quad \beta_{3k+4} + \beta_{3k+5} \equiv \delta \pmod{p} \\
\beta_{9k+1} &\equiv \beta_{3k+1}, \quad \beta_{9k+4} \equiv -\frac{1}{\delta}, \quad \beta_{9k+7} \equiv 1 \pmod{p}.
\end{align*}
\]
Lemma 3.5 implies that for all \( i \geq 3 \) the value of \( \beta_i \) is congruent to either \(-\delta, -\delta^{-1}\) or 1 modulo \( p \), hence it never equals zero.

**Lemma 3.6.** If \( u \equiv 0 \pmod{p} \) and \( v \equiv \delta \pmod{p} \) where \( \delta \in \mathbb{Z} \) satisfies \( \delta^2 + \delta + 1 \equiv 0 \pmod{p} \), then the sequences \( (\alpha_i)_{i \in \mathbb{N}} \) and \( (\beta_i)_{i \in \mathbb{N}} \) are given by the formula for \( k \geq 0 \):

\[
\alpha_k \equiv 0 \pmod{p};  \\
\beta_1 \equiv 1, \beta_2 \equiv -\delta, \beta_{3k+3} \equiv \delta, \beta_{3k+4} + \beta_{3k+5} \equiv -\delta \pmod{p};  \\
\beta_{9k+1} \equiv \beta_{3k+1}, \beta_{9k+4} \equiv -\delta^{-1}, \beta_{9k+7} \equiv 1 \pmod{p}.
\]

Under conditions of lemma 3.6 for all \( i \geq 3 \) the value of \( \beta_i \) is congruent to either \( \delta, \delta^{-1} \) or 1 modulo \( p \). Hence it never equals zero.

**Lemma 3.7.** If \( u \equiv \pm 2\delta^2 \pmod{p} \) and \( v \equiv \delta \pmod{p} \) where \( \delta \in \mathbb{Z} \) satisfy \( \delta^2 + \delta + 1 \equiv 0 \pmod{p} \) and \( p > 3 \) is an odd prime, then the sequences \( (\alpha_i)_{i \in \mathbb{N}} \) and \( (\beta_i)_{i \in \mathbb{N}} \) are given by the formula for \( k \geq 0 \):

\[
\alpha_{3k+1} \equiv -u, \alpha_{3k+2} + \alpha_{3k+3} \equiv u \pmod{p};  \\
\alpha_{9k+2} \equiv -\frac{2\delta + 4}{3}, \alpha_{9k+5} \equiv \frac{u + \alpha_{3k+2}}{3}, \alpha_{9k+8} \equiv -\frac{4\delta + 2}{3} \pmod{p};  \\
\beta_1 \equiv 1, \beta_2 \equiv 3\delta, \beta_{3k+4} + \beta_{3k+5} \equiv 3\delta \pmod{p}  \\
\beta_{9k+1} \equiv \beta_{3k+1}, \beta_{9k+4} \equiv -\frac{3}{\delta}, \beta_{9k+7} \equiv -3 \pmod{p}.  \\
\beta_{9k+3} \equiv -\frac{\delta}{3}, \beta_{9k+6} \equiv \frac{\beta_{3k+3}}{9}, \beta_{9k+9} \equiv -\frac{\delta}{3} \pmod{p}.
\]

Lemma 3.7 implies that for all \( i \geq 3 \) the value of \( \beta_i \) is congruent to either \( -\delta, -3\delta^{-1}, -3 \) or \( \frac{\beta_i + 1}{9} \) modulo \( p \), the latter of which is inductively never zero. Hence none of \( \beta_i \) equals zero.

Since the proof of Lemma 3.7 involves the most tedious computations, compared to other lemmata, we also outline its proof here.

**Proof.** Again we omit the \( \pmod{p} \) in each congruence for this proof. We’ll also just prove it for the \( u \equiv 2\delta^2 \) case as the proof is essentially the same. We use the formulae (12) and (13) to compute the first 9 values of \( \alpha_i \) and \( \beta_i \):

\[
\alpha_1 \equiv -u, \alpha_2 \equiv -\frac{2\delta + 4}{3}, \alpha_3 \equiv -\frac{4\delta + 2}{3}, \beta_1 \equiv 1, \beta_2 \equiv 3\delta, \beta_3 \equiv -\frac{\delta}{3};  \\
\alpha_4 \equiv -u, \beta_4 \equiv -\frac{3}{\delta}, \beta_5 \equiv -3, \alpha_5 \equiv -\frac{8\delta + 10}{9}, \alpha_6 \equiv -\frac{10\delta + 8}{9}, \beta_6 \equiv -\frac{\delta}{27};
\]

...
\[ \alpha_7 \equiv -u, \ \beta_7 \equiv -3, \ \beta_8 \equiv -\frac{3}{\delta}, \ \alpha_8 \equiv -\frac{4\delta + 2}{3}, \ \alpha_9 \equiv -\frac{2\delta + 4}{3}, \ \beta_9 \equiv -\frac{\delta}{3}. \]

This all clearly satisfy the equations (17) – (21), except for \( \alpha_5 \), so let us check this.

\[ \frac{u + \alpha_2}{3} \equiv \frac{2\delta^2 - \frac{2\delta + 4}{3}}{3} \equiv -\frac{8\delta + 10}{9} \equiv \alpha_5. \]

The base case \( k = 0 \) has been proved.

Now we assume that the equations (17) – (21) are satisfied for \( \alpha_i, \beta_i, \ 1 \leq i \leq 9k \), and verify them for \( 9k + 1 \leq i \leq 9k + 9 \).

First, it is obvious that \( \alpha_{9k+1} \equiv \alpha_{9k+4} \equiv \alpha_{9k+7} \equiv -u \) as they are all of the form \( \alpha_{3k+4} \).

Second, by (13) we have

\[ \beta_{9k+1} = \frac{\beta_{3k+1}}{\beta_{9k} \beta_{k-1}} = \frac{\beta_{3k+1}}{-\frac{3}{\delta} (3\delta - \beta_{9k-2})} = \beta_{3k+1} \]

by the induction hypothesis. Then the equation (13) implies that

\[ \beta_{9k+1} + \beta_{9k+2} \equiv \beta_{9k+4} + \beta_{9k+5} \equiv \beta_{9k+7} + \beta_{9k+8} \equiv u^2 - v \equiv 3\delta. \]

In particular, this together with \( \beta_{9k+1} \equiv \beta_{3k+1} \) implies that \( \beta_{9k+2} \equiv \beta_{3k+2} \).

Third, we compute:

\[ \alpha_{9k+2} \equiv u - \frac{\alpha_{3k+1} + uv - \alpha_{9k-1} \beta_{9k+1}}{\beta_{9k+2}} \]

\[ \equiv 2\delta^2 - \frac{-2\delta^2 + 2 + \frac{4\delta + 2}{3} \beta_{9k+2}}{3\delta - \beta_{9k+1}} \]

\[ \equiv \frac{-2\delta + 4}{3(3\delta - \beta_{9k+1})} \equiv -\frac{2\delta + 4}{3}. \]

Thus we have \( \alpha_{9k+3} \equiv u - \alpha_{9k+2} \equiv -\frac{4\delta + 2}{3} \). Finally, we use the last equation in (13) to compute \( \beta_{9k+3} \):

\[ \beta_{9k+3} \equiv \delta - \alpha_{9k+2} \alpha_{9k+3} \equiv \delta - \frac{(2\delta + 4)(4\delta + 2)}{9} \equiv \frac{\delta}{3}. \]

Fourth, we similarly continue to compute \( \beta_{9k+4} \equiv \frac{\beta_{9k+4}}{\beta_{9k+3} \beta_{9k+2}} \equiv -\frac{\delta}{3} \beta_{9k+2} \equiv -\frac{3}{5} \). This then implies that \( \beta_{9k+5} \equiv 3\delta + \frac{2}{5} \equiv -3. \)
Now we compute:

\[
\alpha_{9k+5} \equiv u - \frac{\alpha_{3k+2} + u\delta - \alpha_{9k+2}\beta_{9k+4}}{\beta_{9k+5}} \\
\equiv 2\delta^2 - \frac{\alpha_{3k+2} + 2 - \frac{2\delta + 4}{3} \cdot \frac{3}{3}}{3} \\
\equiv \frac{2\delta^2 + \alpha_{3(n+1)+2}}{3} \equiv \frac{u + \alpha_{3(n+1)+2}}{3}
\]

This then implies

\[
\alpha_{9k+6} \equiv u - \frac{\alpha_{3(n+1)+2}}{3} \equiv \frac{u + \alpha_{3k+3}}{3}
\]

and then we compute

\[
\beta_{9k+6} \equiv \delta - \alpha_{9k+5}\alpha_{9k+6} \\
\equiv \delta - \frac{2\delta^2 + \alpha_{3k+2}}{3} \cdot \frac{2\delta^2 + \alpha_{3k+3}}{3} \\
\equiv \frac{5\delta - 2\delta^2(\alpha_{3k+3} + \alpha_{3k+2}) - \alpha_{3k+3}\alpha_{3k+2}}{9} \\
\equiv \delta - \frac{\alpha_{3k+3}\alpha_{3k+2}}{9} \equiv \frac{\beta_{3k+3}}{9}.
\]

We finish the proof by computing the last triple of \(\alpha\)'s and \(\beta\)'s. We verify that \(\beta_{9k+7} = \frac{\beta_{3k+3}}{9k+6\beta_{9k+5}} \equiv -3\) and \(\beta_{9k+8} \equiv 3\delta - \beta_{9k+7} \equiv -\frac{3}{2}\). Then we use already known values of \(\alpha_{9k+5}, \beta_{9k+7}, \beta_{9k+8}\) to compute:

\[
\alpha_{9k+8} \equiv u - \frac{\alpha_{3k+3} + u\delta - \alpha_{9k+5}\beta_{9k+7}}{\beta_{9k+8}} \\
\equiv 2\delta^2 - \frac{\alpha_{3k+3} + 2 + 2\delta^2 + \alpha_{3k+2}}{-\frac{3}{2}} \\
\equiv \frac{4\delta^2 + 9}{2}.
\]

This then implies \(\alpha_{9k+9} \equiv u - \alpha_{9k+8} \equiv -\frac{2\delta + 4}{3}\) and

\[
\beta_{9k+9} \equiv \delta - \alpha_{9k+8}\alpha_{9k+9} \equiv -\frac{\delta}{3}.
\]

This finishes the inductions step, thus the proof by inductions completes.

\(\Box\)

All the cases \(\text{(2)} - \text{(8)}\) are now covered and Theorem \(\text{1.1}\) concludes.
4 Further remarks

In view of Theorem 1.1 one can ask a natural question: are (2) – (8) the only local conditions on $u, v$ which guarantee that all the partial quotients of $g_{u,v}(z)$ are linear? In attempt to answer this question, we conduct a computer search of all primes $p$ between 3 and 1000 and all pairs $(u, v) \in \mathbb{Z}_p^2$. The search reveals that every pair that did not seem to ever produce a value of 0 is of the conditions (2) – (8).

These findings, while heuristic, suggest that Theorem 1.1 covers all local conditions which guarantee that the series $g_{u,v}$ is badly approximable.

Also, a quick search reveals that around 82% integer pairs $(u, v) \in [-1000, 1000]^2$ satisfy at least one of the conditions (2) – (8). This indicates that the majority of pairs are covered by Theorem 1.1. However, there are still plenty of pairs for which the conjecture is still to be verified. One of the smallest such pair is $(u, v) = (2, -2)$.

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