DISCRETE ENTROPY OF GENERALIZED JACOBI POLYNOMIALS

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Abstract. Given a sequence of orthonormal polynomials on \( \mathbb{R}, \{p_n\}_{n \geq 0} \), with \( p_n \) of degree \( n \), we define the discrete probability distribution \( \Psi_n(x) = (\Psi_{n,1}(x), \ldots, \Psi_{n,n}(x)) \), with \( \Psi_{n,j}(x) = (\sum_{j=0}^{n-1} p_j^2(x))^{-1} p_{j-1}^2(x) \), \( j = 1, \ldots, n \). In this paper, we study the asymptotic behavior as \( n \to \infty \) of the Shannon entropy \( S(\Psi_n(x)) = -\sum_{j=1}^{n} \Psi_{n,j}(x) \log(\Psi_{n,j}(x)), x \in (-1, 1) \), when the orthogonality weight is \((1-x)^\alpha (1+x)^\beta h(x)\), \( \alpha, \beta > -1 \), and where \( h \) is real, analytic, and positive on \([-1, 1]\). We show that the limit \( \lim_{n \to \infty} (S(\Psi_n(x)) - \log n) \) exists for all \( x \in (-1, 1) \), but its value depends on the rationality of \( \arccos(x)/\pi \).

For the particular case of the Chebyshev polynomials of the first and second kinds, we compare our asymptotic result with the explicit formulas for \( S(\Psi_n(\zeta_j^{(n)})) \), where \( \{\zeta_j^{(n)}\} \) are the zeros of \( p_n \), obtained previously in [2].

1. Introduction

For a discrete probability distribution \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \) with \( \sum_{i=1}^{n} \nu_i = 1 \), we can define its Shannon entropy by

\[
S(\nu) = -\sum_{i=1}^{n} \nu_i \log(\nu_i),
\]

that, by Jensen’s inequality, satisfies

\[
0 \leq S(\nu) \leq \log(n),
\]

and the maximum of \( S(\nu) \) is attained only at the uniform probability distribution

\[
\nu^* = (\nu_1^*, \nu_2^*, \ldots, \nu_n^*) = (1/n, 1/n, \ldots, 1/n).
\]
Thus, along with the Shannon entropy, a natural measure of uncertainty associated with a probability distribution $\nu$ is its “distance” from $\nu^*$, given by the directed or Kullback–Leibler divergence
\[
D(\nu) = D(\nu, \nu^*) = \sum_{i=1}^{n} \nu_i \log \left( \frac{\nu_i}{\nu_i^*} \right) = \log(n) - S(\nu) \geq 0.
\]

Given a probability Borel measure $\mu$ supported on the real line $\mathbb{R}$ with infinite number of points of increase and such that
\[
\int_{\mathbb{R}} x^n d\mu(x) < \infty, \quad n \in \mathbb{N}_0,
\]
we can construct a sequence of orthonormal polynomials
\[
p_n(x) = \kappa_n x^n + \text{lower degree terms}, \quad \kappa_n > 0, \quad n \in \mathbb{N}_0,
\]
such that
\[
\int_{\mathbb{R}} p_n(x) p_m(x) d\mu(x) = \delta_{n,m}, \quad n, m \in \mathbb{N}_0.
\]
The corresponding reproducing kernel is
\[
K_n(x, y) = \sum_{j=0}^{n-1} p_j(x) p_j(y),
\]
that, for $x = y$, becomes the reciprocal of the $n$-th Christoffel function
\[
\lambda_n(x) \overset{\text{def}}{=} \frac{1}{K_n(x, x)}.
\]

For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we can define the discrete probability distribution
\[
\Psi_n(x) = (\Psi_{n,1}(x), \ldots, \Psi_{n,n}(x)), \quad \text{with} \ \Psi_{n,j}(x) = \lambda_n(x)p_{j-1}^2(x), \quad j = 1, \ldots, n.
\]
Observe that this distribution does not depend on the normalization of the measure $\mu$.

Our main goal is to study the asymptotic behavior of the Kullback-Leibler divergence $D(\Psi_n(x))$, or, equivalently, that of the Shannon entropy $S(\Psi_n(x))$, as $n \to \infty$ for $x$ in the bulk of the support of the orthogonality measure $\mu$. We restrict our attention to absolutely continuous measures $\mu$ supported on a bounded interval of $\mathbb{R}$, with $\mu'$ analytic and non-vanishing in the neighborhood of this interval, except for the only possible singularities of a power type at the endpoints of the support. Without loss of generality, we may assume that
\[
d\mu(x) = w(x)dx, \quad w(x) \overset{\text{def}}{=} (1-x)\alpha (1+x)\beta h(x), \quad x \in [-1, 1],
\]
with $\alpha, \beta > -1$, and where $h$ is real, analytic, and positive on $[-1, 1]$. We call such kind of measures and the corresponding orthogonal polynomials “generalized Jacobi”.

In what follows, when we have $x = \cos \theta \in (-1, 1)$, then we also assume that $\theta \in (0, \pi)$.

One of the main results is the following theorem.

\begin{theorem}
For $\mu$ given in (3) and $x = \cos \theta \in (-1, 1)$, the limit
\[
D_\infty(x) \overset{\text{def}}{=} \lim_{n \to \infty} D(\Psi_n(x))
\]
\end{theorem}

1We denote the set of nonnegative integers $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. 

\[\]
exists. Moreover,

\[ D_\infty(x) = \begin{cases} 
1 - \log(2), & \text{if } \frac{\theta}{\pi} \notin \mathbb{Q}, \\
\log(2) + 2\hat{S}_{k,s}, & \text{if } \frac{\theta}{\pi} = \frac{s}{k} \text{ with } s, k \in \mathbb{N}, s < k, \text{ and } \text{GCD}(s, k) = 1,
\end{cases} \]

where

\[ \hat{S}_{k,s} \overset{\text{def}}{=} \frac{1}{k} \sum_{i=0}^{k-1} F \left( \cos \left( (i + 1/2) \frac{\pi s}{k} + \varphi \left( \cos \frac{\pi s}{k} \right) - \pi/4 \right) \right), \]

with

\[ F(x) \overset{\text{def}}{=} \begin{cases} x^2 \log(x^2), & x > 0, \\
0, & x = 0.
\end{cases} \]

and

\[ \varphi(x) \overset{\text{def}}{=} \frac{1}{2} ((\alpha + \beta) \theta - \alpha \pi) + \frac{\sqrt{1-x^2}}{2\pi} \int_{-1}^{1} \frac{\log(h(t))}{\sqrt{1-t^2}} \frac{dt}{t-x}. \]

The integral in the right hand side of (7) is understood in the sense of its principal value, that is,

\[ \int_{-1}^{1} \frac{\log(h(t))}{\sqrt{1-t^2}} \frac{dt}{t-x} \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \left( \int_{-1}^{-1+\varepsilon} \frac{\log(h(t))}{\sqrt{1-t^2}} \frac{dt}{t-x} + \int_{1-\varepsilon}^{1} \frac{\log(h(t))}{\sqrt{1-t^2}} \frac{dt}{t-x} \right), \quad x \in (-1, 1). \]

Remark 1. As formula (10) below shows, the Shannon entropy \( S(\Psi_n(x)) \) (or the Kullback–Leibler divergence \( D(\Psi_n(x)) \)) is closely related to the Christoffel function \( \lambda_n(x) \) or the reproducing kernel \( K_n(x, y) \). The latter exhibits a well-known universal behavior on the support of the orthogonality measure. In its most rudimentary form it is just the first limit in (12) below, while for the more sophisticated “local” version of this universality, leading to the sine kernel, see e.g. [5]. In all cases, the “universal” limit is continuous. This is no longer the case for the Shannon entropy, as Theorem 1 illustrates, since function \( D_\infty \) is discontinuous everywhere in \((-1, 1)\).

Let

\[-1 < \xi_{n}^{(n)} < \cdots < \xi_{1}^{(n)} < 1\]

be the zeros of the \( n \)-th polynomial \( p_n \). In [2], the authors studied the values of

\[ S_{n,j} = S(\Psi_n(\xi_{n}^{(n)})), \quad j = 1, \ldots, n, \]

finding explicit expressions for the case of orthonormal Chebyshev polynomials of the first and second kinds. Recall that the orthonormal Chebyshev polynomials of the first kind are given by the explicit formula

\[ p_n(x) = T_n(x) = \begin{cases} 
\frac{1}{\sqrt{\pi}}, & n = 0, \\
\frac{2}{\sqrt{\pi}} \cos(n\theta), & n \in \mathbb{N}, \quad x = \cos \theta,
\end{cases} \]

Here and in what follows, GCD stands for the greatest common divisor.
for which \( w(x) = (1 - x^2)^{-1/2} \) and
\[
\zeta^{(n)}_j = \cos \left( \frac{(2j - 1)\pi}{2n} \right), \quad j = 1, \ldots, n,
\]
whereas the orthonormal Chebyshev polynomials of the second kind are
\[
p_n(x) = U_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sin ((n + 1)\arccos(x))}{\sqrt{1 - x^2}} = \sqrt{\frac{2}{\pi}} \frac{\sin ((n + 1)\theta)}{\sin(\theta)}, \quad x = \cos \theta, \quad n \in \mathbb{N}_0,
\]
with \( w(x) = (1 - x^2)^{1/2} \) and
\[
\zeta^{(n)}_j = \cos \left( \frac{j\pi}{n + 1} \right), \quad j = 1, \ldots, n.
\]
Thus, it is interesting to study the compatibility of the results from \cite{2} with those stated in Theorem 1. In other words, can we reproduce \cite{4}, “stepping” onto the zeros \( \zeta^{(n)}_j \) only? The answer is yes, but not always.

**Theorem 2.** Consider the orthonormal Chebyshev polynomials of the first or second kind and let \( x = \cos \theta \in (-1, 1) \). If \( \theta/\pi \notin \mathbb{Q} \), then there exists a subsequence \( \Lambda \subset \mathbb{N} \times \mathbb{N} \) such that
\[
\lim_{(n,j) \in \Lambda} \zeta^{(n)}_j = x \quad \& \quad \lim_{(n,j) \in \Lambda} (\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x))) = 0. \tag{8}
\]
If \( \theta/\pi = s/k \) where \( s \in \mathbb{N} \) and \( k \in \mathbb{N} \) with \( s < k \) and \( \text{GCD}(s,k) = 1 \), then \( (8) \) still holds if the polynomials are of the second kind or if \( k \) is even. However, for the Chebyshev polynomials of the first kind and \( k \) odd,
\[
\limsup_n (\mathcal{S}_{n,j_n} - \mathcal{S}(\Psi_n(x))) < 0 \quad \tag{9}
\]
for every subsequence \( \{j_n\} \subset \mathbb{N} \).

**Remark 2.** A stronger statement than \( (9) \) is given in \cite{26} below where \( \mathcal{R} \) is defined in \cite{18}.

## 2. Proof of Theorem 1

Taking into account \cite{11} and \cite{2} we see that
\[
\mathcal{S}(\Psi_n(x)) = -\log(\lambda_n(x)) - \lambda_n(x) \sum_{i=0}^{n-1} p_i^2(x) \log(p_i^2(x)). \tag{10}
\]
A crucial fact about the class of measures given in \cite{3} is that the corresponding orthonormal polynomials satisfy the asymptotic formula, valid uniformly on compact subsets of \((-1,1),
\[
p_n(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{w(x)(1 - x^2)^{1/4}}} \left( \cos ((n + 1/2)\theta + \varphi(x) - \pi/4) + O(1/n) \right), \quad x = \cos \theta, \tag{11}
\]
where the phase function \( \varphi \) is given in \cite{7}; see \cite{3} (1.15) & (1.33)] where this asymptotics was proved using the non-linear steepest descent method based on the Riemann–Hilbert formulation of these polynomials.

Given a generalized Jacobi \( \mu \) as in \cite{3}, it is very well known that
\[
\lim_{n \to \infty} n \lambda_n(x) = \pi w(x) \sqrt{1 - x^2} \quad \& \quad \lim_{n \to \infty} \lambda_n(x)p_n^2(x) = 0, \tag{12}
\]
Assume first that $x \in (-1,1).$ In order to prove Theorem 1, we need to study the behavior of this function. Therefore, we get from (10)–(12) that

$$
S(\Psi_n(x)) = \log \left( \frac{n}{2} \right) - \lambda_n(x) \sum_{i=0}^{n-1} p_i^2(x) \log \left( (\cos ((i + 1/2)\theta + \varphi(x) - \pi/4) + \epsilon_i(x))^2 \right) + o(1),
$$

where $\epsilon_i(x) = o(1)$ as $i \to \infty$ uniformly on compact sets of $(-1,1)$.

We have

$$
\lambda_n(x) \sum_{i=0}^{n-1} p_i^2(x) \log \left( (\cos ((i + 1/2)\theta + \varphi(x) - \pi/4) + \epsilon_i(x))^2 \right) = \frac{2\lambda_n(x)}{\pi w(x) \sqrt{1 - x^2}} \sum_{i=0}^{n-1} \mathcal{F} \left( \cos ((i + 1/2)\theta + \varphi(x) - \pi/4) + \epsilon_i(x) \right),
$$

where $\mathcal{F}$ is the function defined in (6).

Let us denote

$$
y_i(x) \overset{\text{def}}{=} \cos ((i + 1/2)\theta + \varphi(x) - \pi/4), \quad i = 0, 1, \ldots, n - 1,
$$

and consider

$$
\sum_{i=0}^{n-1} \mathcal{F} \left( y_i(x) + \epsilon_i(x) \right) - \sum_{i=0}^{n-1} \mathcal{F} \left( y_i(x) \right) = \sum_{i=0}^{n-1} \mathcal{F}' \left( y_i(x) + \nu_i(x)\epsilon_i(x) \right) \epsilon_i(x), \quad 0 \leq \nu_i(x) \leq 1.
$$

Since $\mathcal{F}'$ is uniformly bounded on compact subsets of $[0, +\infty)$ and $\epsilon_i(x) = o(1)$, we can conclude that

$$
\frac{1}{n} \sum_{i=0}^{n-1} \mathcal{F} \left( y_i(x) + \epsilon_i(x) \right) - \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{F} \left( y_i(x) \right) = o(1), \quad n \to \infty,
$$

uniformly on compact subsets of $(-1,1)$.

Combining (13)–(15), we arrive at the asymptotic expression for the entropy

$$
S(\Psi_n(x)) = \log \left( \frac{n}{2} \right) - \frac{2n\lambda_n(x)}{\pi w(x) \sqrt{1 - x^2}} \hat{S}_n(x) + o(1),
$$

where

$$
\hat{S}_n(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{F} \left( \cos ((i + 1/2)\theta + \varphi(x) - \pi/4) \right).
$$

In order to prove Theorem 1 we need to study the behavior of this function.

Assume first that $x = \cos \theta \in (-1,1)$ with $\theta/\pi \in \mathbb{Q}$. Hence, there exist $s \in \mathbb{N}$ and $k \in \mathbb{N}$ with $s < k$ and $\text{GCD}(s,k) = 1$, such that

$$
\frac{\theta}{\pi} = \frac{s}{k}.
$$

Therefore, there exist non-negative integers $p$ and $q$ with $0 \leq q \leq k - 1$ such that $n - 1 = pk + q$. We use the following straightforward lemma (which is basically the idea behind the FFT algorithm), that can be proved by direct calculation.
Lemma 1. Let \( g(n) \) be periodic with period \( k \), that is, \( g(n + k) = g(n) \) for all \( n \in \mathbb{N} \). Let also \( p \) and \( q \), with \( 0 \leq q \leq k - 1 \), be non-negative integers such that \( n - 1 = pk + q \). Then
\[
\frac{1}{n} \sum_{i=0}^{n-1} g(i) = \frac{1}{k} \sum_{i=0}^{k-1} g(i) + \frac{1}{n} \left( \frac{q + 1}{k} \sum_{i=0}^{k-1} g(i) + \sum_{i=0}^{q} g(i) \right).
\]
In particular, if \( g(n) \) is uniformly bounded, then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(i) = \frac{1}{k} \sum_{i=0}^{k-1} g(i).
\]
Applying Lemma 1 to \( g(n) = \mathcal{F} \left( \cos \left( \frac{(n + 1/2) \pi s}{k} + \varphi(\cos \frac{\pi s}{k}) - \pi/4 \right) \right) \),
we conclude that
\[
\lim_{n \to \infty} \hat{S}_n \left( \cos \frac{\pi s}{k} \right) = \frac{1}{k} \sum_{i=0}^{k-1} \mathcal{F} \left( \cos \left( \frac{(i + 1/2) \pi s}{k} + \varphi(\cos \frac{\pi s}{k}) - \pi/4 \right) \right) = \hat{S}_{k,s}.
\]
Together with (12) and (16), this establishes the assertion of Theorem 1 for the case \( \theta/\pi \in \mathbb{Q} \).

Assume now that \( x = \cos \theta \in (-1, 1) \), but \( \theta/\pi \notin \mathbb{Q} \). By Kronecker’s theorem (also known as Kronecker-Weyl’s theorem), see, e.g., [3, Theorem IV, Chapter III, p. 53], the sequence
\[
\left( \frac{n\theta}{\pi} - \left\lfloor \frac{n\theta}{\pi} \right\rfloor \right)_{n \in \mathbb{N}},
\]
is dense and it is uniformly distributed in [0, 1]. Thus, by (17),
\[
\lim_{n \to \infty} \hat{S}_n(\cos \theta) = \int_0^1 \mathcal{F}(\cos(y\pi + \frac{\theta}{2} + \varphi(\cos \theta) - \pi/4))dy.
\]
Using the periodicity of the cosine function, we get
\[
\lim_{n \to \infty} \hat{S}_n(\cos \theta) = \int_0^1 \mathcal{F}(\cos(y\pi))dy = \frac{1}{2} - \log 2.
\]
Again, combining this with (16) we get the assertion of Theorem 1 for the remaining case \( x = \cos(\theta) \), \( \theta/\pi \notin \mathbb{Q} \).

3. Proof of Theorem 2

1. Let us start with the orthonormal Chebyshev polynomials of the first kind. Recall that \( x = \cos \theta \in (-1, 1) \).

1.1. Assume first that \( \theta/\pi \notin \mathbb{Q} \). Using Theorem 1 and the explicit expression for \( \mathcal{S}_{n,j} \) given in [2, Theorem 1, p. 99], we have
\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = -\mathcal{R} \left( \frac{d_n}{2n} \right) + o(1), \quad n \to \infty,
\]
where \( d_n = \text{GCD}(2j - 1, n) \),
\[
\mathcal{R}(x) = -x(\psi(1 - x) + 2\gamma + \psi(1 + x)),
\]
Here and in what follows, symbol \( \lfloor \cdot \rfloor \) denotes the mathematical integer part.
\( \gamma \) is the Euler-Mascheroni constant \((0.577\ldots)\), and \( \psi(x) \overset{\text{def}}{=} \Gamma'(x)/\Gamma(x) \) is the digamma function. Alternatively, \( R \) can be evaluated using \( \psi \)'s Taylor series expansion that is absolutely convergent for \(|x| < 1\), see, for instance, \( [1, \text{formula (6.3.14), p. 259}] \), and then we obtain

\[
R(x) = 2 \sum_{k=1}^{\infty} \xi(2k+1)x^{2k+1},
\]

where \( \xi(\cdot) \) is the Riemann zeta function.

Denoting by \( P \subset \mathbb{N} \) the set of all prime numbers, we define \( \Lambda \equiv \Lambda_1 \subset P \times \mathbb{N} \subset \mathbb{N} \times \mathbb{N} \) by

\[
\Lambda_1 \overset{\text{def}}{=} \{(p, \lfloor \theta p/\pi \rfloor) : p \in P \}.
\]

Clearly, we can make this sequence linearly ordered by agreeing that \((p_1, j_1) \preceq (p_2, j_2)\) if \( p_1 \leq p_2 \). Thus, when we take limits along \((p, j) \in \Lambda_1\), we understand that \( p \to \infty \).

By the construction, if \((p, j) \in \Lambda_1\), then

\[
\frac{\theta}{\pi} - \frac{3}{2p} \leq \frac{2j-1}{2p} < \frac{\theta}{\pi} - \frac{1}{2p},
\]

so that

\[
\lim_{(p,j) \in \Lambda_1} \frac{2j-1}{2p} = \frac{\theta}{\pi} < 1.
\]

Furthermore, \( \text{GCD}(2j-1, p) \) can take only two values, either \( 1 \) or \( p \). Assuming \( p > 2 \), we must have \( \text{GCD}(2j-1, p) = p \) and, using that \( 2j-1 < 2p \), we have \( 2j-1 = p \), so that \((p, (p+1)/2) \in \Lambda_1 \). If we assume that there is an infinite subsequence of indices from \( \Lambda_1 \) of this form, we would get

\[
\lim_{(p,j) \in \Lambda_1} \frac{2j-1}{2p} = \lim_{(p,j) \in \Lambda_1} \frac{p}{2p} = \frac{1}{2},
\]

that contradicts our assumption that \( \theta/\pi \notin \mathbb{Q} \). Thus, for all sufficiently large \( p \), we actually have \( \text{GCD}(2j-1, p) = 1 \), and, therefore

\[
S_{n,j} - S(\Psi_n(x)) = -R \left( \frac{1}{2p} \right) + o(1) = o(1), \quad n \to \infty,
\]

that proves the assertion when \( \theta/\pi \notin \mathbb{Q} \).

1.2. Assume now that \( \theta/\pi = s/k \) where \( s \in \mathbb{N} \) and \( k \in \mathbb{N} \) with \( s < k \) and \( \text{GCD}(s, k) = 1 \). Using again Theorem \([1, \text{Theorem 1, p. 99}]\) and the explicit expression for \( S_{n,j} \) from \([2, \text{Theorem 1, p. 99}]\) we have

\[
S_{n,j} - S(\Psi_n(x)) = 2\log(2) - 1 - R \left( \frac{d_n}{2n} \right) + 2 \hat{S}_{k,s} + o(1), \quad n \to \infty,
\]

where we used the same notation as before. From the explicit formula \([5]\) for \( \hat{S}_{k,s} \), it is easy to see that in the case of the orthonormal Chebyshev polynomials of the first kind

\[
\hat{S}_{k,s} = \frac{1}{k} \sum_{i=1}^{k-1} \mathcal{F} \left( \cos \left( \frac{\pi is}{k} \right) \right) = \frac{1}{k} \sum_{i=1}^{k-1} \mathcal{F} \left( \cos \left( \frac{\pi i}{k} \right) \right) = \hat{S}_{k,1},
\]

where we have used that

\[
\left\{ \frac{is}{k} \mod 1 : i = 1, 2, \ldots, k \right\} = \left\{ \frac{i}{k} \mod 1 : i = 1, 2, \ldots, k \right\}.
\]

Remark 3. In \([2, \text{formula (27), p. 108}]\), \( \hat{S}_{n,1} \) is normalized in a different way because it lacks the normalizing factor \( 1/n \), so that our formulas will slightly differ from those in \([2]\).
1.2.1. If \( k \) is even, and, thus, \( s \) is odd, we define \( \Lambda \equiv \Lambda_2 \) by
\[
\Lambda_2 = \left\{ \left( \frac{k(2m + 1)}{2}, \frac{s(2m + 1) + 1}{2} \right) : m \in \mathbb{N} \right\} \subset \mathbb{N} \times \mathbb{N}.
\]
Then
\[
(n, j) \in \Lambda_2 \quad \Rightarrow \quad \frac{2j - 1}{2n} = \frac{s}{k} \quad \& \quad \frac{d_n}{2n} = \frac{1}{k}.
\]
Hence,
\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = 2\log(2) - 1 - \mathcal{R}\left(\frac{1}{k}\right) + 2\hat{S}_{k,1} + o(1), \quad n \to \infty. \tag{21}
\]
Observe also that
\[
\hat{S}_{2m,1} = \frac{1}{2m} \sum_{i=1}^{2m-1} F\left(\cos\left(\frac{\pi i}{2(2m + 1)}\right)\right) = \frac{1}{m} \sum_{i=1}^{m-1} F\left(\cos\left(\frac{\pi (2i - 1)}{2(2m + 1)}\right)\right)
\]
for \( m \in \mathbb{N} \), so that we can use formula (40) from [2, Corollary 10, p. 111], by which
\[
2\hat{S}_{k,1} = 1 - 2\log(2) + \mathcal{R}\left(\frac{1}{2k}\right) \tag{22}
\]
when \( k \) is even, and (21) combined with (22) concludes the proof of (8) for this case.

1.2.2. Let us turn to the case when \( k \) is odd. The key identity that holds in this case is
\[
\hat{S}_{2m+1,1} = \frac{1}{2} - \log(2) + \mathcal{R}\left(\frac{1}{2(2m + 1)}\right) - \frac{1}{2}\mathcal{R}\left(\frac{1}{2m + 1}\right), \quad m \in \mathbb{N}. \tag{23}
\]
Indeed,
\[
\sum_{i=1}^{2m} F\left(\cos\left(\frac{\pi i}{2(2m + 1)}\right)\right) = \sum_{i=1}^{m} F\left(\cos\left(\frac{\pi i}{2m + 1}\right)\right) + \sum_{i=1}^{m} F\left(\cos\left(\frac{\pi (2i - 1)}{2(2m + 1)}\right)\right)
\]
\[
= \frac{1}{2} \sum_{i=1}^{2m} F\left(\cos\left(\frac{\pi i}{2m + 1}\right)\right) + \sum_{i=1}^{m} F\left(\sin\left(\frac{\pi i}{2m + 1}\right)\right)
\]
\[
= \frac{2m + 1}{2} \hat{S}_{2m+1,1} + \frac{1}{2} \sum_{i=1}^{2m} F\left(\sin\left(\frac{\pi i}{2m + 1}\right)\right),
\]
so that
\[
\hat{S}_{2m+1,1} = \frac{2}{2m + 1} \sum_{i=1}^{2m} F\left(\cos\left(\frac{\pi i}{2m + 1}\right)\right) - \frac{1}{2m + 1} \sum_{i=1}^{2m} F\left(\sin\left(\frac{\pi i}{2m + 1}\right)\right). \tag{24}
\]
By (22), the first term in (24) is \( 1 - 2\log(2) + \mathcal{R}\left(\frac{1}{2(2m + 1)}\right) \), while, by [2, Proposition 13, p. 114], we have the following identity for the second term
\[
\frac{1}{2m + 1} \sum_{i=1}^{2m} F\left(\sin\left(\frac{\pi i}{2m + 1}\right)\right) = \frac{1}{2} \left( 1 - 2\log(2) + \mathcal{R}\left(\frac{1}{2m + 1}\right) \right), \tag{25}
\]
that yields (23).

By (19) and (23), for \( k \) odd,
\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = 2\log(2) - 1 - \mathcal{R}\left(\frac{d_n}{2n}\right) + 2\hat{S}_{k,1} + o(1)
\]
\[
= 2\mathcal{R}\left(\frac{1}{2k}\right) - \mathcal{R}\left(\frac{1}{k}\right) - \mathcal{R}\left(\frac{d_n}{2n}\right) + o(1), \quad n \to \infty,
\]
where \(d_n = \text{GCD}(2j - 1, n)\).

Observe that the coefficients in the power series expansions \(\mathcal{S}_n(j)\) are all positive, so that \(\mathcal{R}\) is convex on \((0, 1)\). As a consequence, \(\mathcal{R}(x/2) - \mathcal{R}(x)/2\) is decreasing on \((0, 1)\) and, therefore,

\[
\mathcal{R}\left(\frac{x}{2}\right) - \frac{1}{2} \mathcal{R}(x) < 0 \quad \text{for} \quad x \in (0, 1).
\]

In particular, for every choice of \(\Lambda\),

\[
\limsup_{(n,j) \in \Lambda} (\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x))) \leq 2 \mathcal{R}\left(\frac{1}{2k}\right) - \mathcal{R}\left(\frac{1}{k}\right) < 0,
\]

that establishes (26).

2. Now we switch to the orthonormal Chebyshev polynomials of the second kind. Let again \(x = \cos \theta \in (-1, 1)\).

2.1. Assume \(\theta/\pi \notin \mathbb{Q}\). Using Theorem 1 and the explicit expression for \(\mathcal{S}_{n,j}\) given in \([2, \text{Theorem 2, p. 100}]\), we have

\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = \log\left(\frac{n+1}{n}\right) - \mathcal{R}\left(\frac{d_n}{n+1}\right) + o(1), \quad n \to \infty,
\]

where \(d_n = \text{GCD}(j, n + 1)\).

We build \(\Lambda \equiv \Lambda_3 \subset \mathbb{N} \times \mathbb{N}\) and define linear ordering on it similarly as it was done before for \(\Lambda_1\). Namely, for each prime number \(p\) take \(j = \lfloor \theta(p - 1)/\pi \rfloor\) and then we denote all the resulting pairs \((p - 1, j)\) by \(\Lambda_3\).

By the construction,

\[
\lim_{(p,j) \in \Lambda_3} \frac{j}{p} = \frac{\theta}{\pi} < 1,
\]

and GCD\((j, p) = 1\). Thus,

\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = \log\left(\frac{p}{p-1}\right) - \mathcal{R}\left(\frac{1}{p}\right) + o(1) = o(1), \quad n \to \infty.
\]

2.2. Assume now that \(\theta/\pi = s/k\) where \(s \in \mathbb{N}\) and \(k \in \mathbb{N}\) with GCD\((s, k) = 1\). Using again Theorem 1 and the explicit expression for \(\mathcal{S}_{n,j}\) from \([2, \text{Theorem 2, p. 100}]\), we obtain

\[
\mathcal{S}_{n,j} - \mathcal{S}(\Psi_n(x)) = \log\left(\frac{n+1}{n}\right) - 1 + 2 \log(2) - \mathcal{R}\left(\frac{d_n}{n+1}\right) + 2 \mathcal{S}_{k,s} + o(1), \quad n \to \infty,
\]

where \(d_n = \text{GCD}(j, n + 1)\). In the case of the orthonormal Chebyshev polynomials of the second kind, \(\mathcal{S}_{k,s}\) defined in \([5]\), has now the form

\[
\mathcal{S}_{k,s} = \frac{1}{k} \sum_{i=1}^{k} \mathcal{F}\left(\cos\left(\frac{\pi si}{k} - \frac{\pi}{2}\right)\right) = \frac{1}{k} \sum_{i=1}^{k-1} \mathcal{F}\left(\sin\left(\frac{\pi i}{k}\right)\right) = \hat{\mathcal{S}}_{k,1},
\]

where we have used \([20]\) for the second equality.

If we take

\[
\Lambda \equiv \Lambda_4 \overset{\text{def}}{=} \{(mk - 1, sm) : m \in \mathbb{N}\} \subset \mathbb{N} \times \mathbb{N},
\]

then

\[
(n, j) \in \Lambda_4 \quad \Rightarrow \quad \frac{j}{n+1} = \frac{s}{k} \quad \& \quad \frac{d_n}{n+1} = \frac{1}{k}
\]
so that
\[ S_{n,j} - S(\Psi_n(x)) = 2 \log(2) - 1 - R\left(\frac{1}{k}\right) + 2\hat{S}_{k,1} + o(1), \quad n \to \infty. \] (27)

2.2.1 Let \( k \) be even, say, \( k = 2m \). Observe that
\[ \hat{S}_{2m,1} = \frac{1}{2m} \sum_{i=1}^{2m-1} F\left(\sin\left(\frac{\pi i}{2m}\right)\right) = \frac{1}{m} \sum_{i=1}^{m-1} F\left(\sin\left(\frac{\pi i}{2m}\right)\right), \quad m \in \mathbb{N}, \]
so that we can use [2, Proposition 13, p. 114] by which
\[ 2\hat{S}_{2m,1} = 1 - 2 \log(2) + R\left(\frac{1}{2m}\right), \quad m \in \mathbb{N}, \]
and, thus, (8) holds for this case as well.

2.2.2 If \( k \) is odd then we proceed as in 1.2.2 and use (25) in (27) to conclude that (8) holds as well.

The proof of Theorem 2 is complete.

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