Advances in fabrication of sub-micron ferromagnetic elements and their arrays produced a wealth of experiments, many of which are still waiting to be explained by theory. In particular, the first measurement of the four-fold anisotropy of the spin-wave frequencies in square arrays of circular permalloy dots was reported in 1997.

Discovery of the four-fold anisotropy of properties of a square lattice might seem obvious at first. However, a deeper look reveals that, while the distribution of stray fields in a square array of circular magnetic dots depends on orientation of their uniform magnetization in the array plane, the magnetostatic energy of the array is completely isotropic for any in-plane magnetization direction. It directly follows from Maxwell equations that interaction of parallel dipoles (which a uniform magnetization state is) depends only on second powers of direction cosines and can only produce biaxial anisotropy (uniaxial in the array plane), even if the array is rectangular. But, as soon as both periods of the rectangular array are equal, the (in-plane) anisotropy of the magnetic energy of the uniformly magnetized array vanishes.

Thus, it is clear that the effect must come from some kind of non-uniform magnetization distribution within the dots. For example, the two-domain dots (divided by the oppositely-magnetized domains in half) do show the four-fold anisotropy. It can be doubted, however, that such a configuration would survive the high (almost saturating) external field of magnetic resonance experiments.

Based on the approximate analytical approach to micromagnetics of thin flat sub-micron soft ferromagnetic elements, the ansatz for quasi-uniform magnetization distribution in a circular dot was proposed recently. Using this distribution as a starting point, the equilibrium (in the Ritz sense) energy of the square array of the dots in such a state is calculated below, comparing its angular dependence to the experiment.

The magnetization distribution can be expressed in terms of the complex function of complex variable:

$$w(z) = e^{i\varphi_0} \frac{p^2 - e^{-2i\varphi_0} z^2}{p^2 - e^{2i\varphi_0} z^2},$$ (1)

where $z = X/R + iY/R$, $X$, $Y$ are Cartesian coordinates on the dot’s face, $R$ is the dot radius, $i = \sqrt{-1}$, line over a variable denotes the complex conjugation. Components of magnetization unit vector $\vec{m}$ are expressed through $w(z)$ as $m_X + im_Y = 2w/(1 + w\overline{w})$ and $m_Z = (1 - w\overline{w})/(1 + w\overline{w}) = 0$. The dimensionless parameter $p > 1$ describes displacement of skyrmions from the cylinder’s side, their centers (zeros of $w(z, \varphi)$) are located at $X = \pm pR$, $Y = 0$. The value of $p$ is to be found from minimization of the total energy of the array. For $p \to \infty$ the magnetization distribution is uniform, for finite $p$ the corresponding “leaf” quasi-uniform magnetization state is shown in Fig. 1. Compared to the original ansatz in...
magnetization of the dot’s material \((M_s)\). Inside of a dot with magnetization distribution \(\mathbf{M}\) they are:

\[
\sigma_V = \frac{2r \cos(\varphi - \varphi_0)}{\sqrt{r^4 + p^4 - 2p^2 r^2} \cos 2(\varphi - \varphi_0)} \tag{2}
\]

\[
\sigma_S = \frac{p^2 - 1 \cos(\varphi - \varphi_0)}{\sqrt{1 + p^4 - 2 p^2} \cos 2(\varphi - \varphi_0)} \tag{3}
\]

where \(r \leq 1\) and \(0 < \varphi \leq 2\pi\) are the polar coordinates on the dot’s circular face, normalized by the dot radius. These are the rotated charge densities of Ref. 6.

To fully take into account periodicity of the array and all the implied interactions between dots, the density of magnetic charges is represented by its Fourier components:

\[
\tilde{A}_{n,m}(k_z) = \frac{\rho a_{n,m}}{T} \int e^{i \alpha k_z z} dZ = \frac{\rho a_{n,m} \rho \lambda \sin k_z L_Z \pi}{k_z L_Z \pi} \tag{4}
\]

\[
\tilde{a}_{n,m} = \int d\varphi \left( \sigma_F \left|_{r=1} + \int r \, dr \, \sigma_V \right. \right) \tag{5}
\]

\[
F = e^{-2\pi i \rho (n X + m Y)} = e^{-2\pi i \rho r (\cos \varphi + \sin \varphi)} \tag{6}
\]

where \(\rho = R/T\), \(\lambda = L_Z/T\), \(T > 2R\) and \(L_Z\) are the array period and thickness, \(a_{0,0} \equiv 0\) by construction.

Because the Fourier basis functions \(F\) are also the eigenfunctions of the Laplace differential operator, the Poisson equation (resulting from the Maxwell equations under assumptions of the magnetic charges formalism) for the scalar potential becomes algebraic for its Fourier harmonics and can be readily solved. Normalized magnetostatic energy per array cell volume due to the interaction of the stray field with the magnetic charges \(E_M = \int E_M / (\mu_0 \gamma_B M_s^2 T^2 L_Z^2)\) is then

\[
e_M = \frac{\rho^2 \lambda^2}{8 \pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int d\xi \frac{|\tilde{a}_{n,m}|^2 \sin^2 \xi \pi}{\lambda^2 (n^2 + m^2) + \xi^2 (\xi^2)^2} \sin^2 \xi \pi, \tag{7}
\]

where \(|\tilde{a}_{n,m}|^2 = a_{n,m} \overline{a}_{-n,-m}\), \(\xi = k_z L_Z\), \(\mu_0\) is permeability of vacuum in SI units, \(\gamma_B\) is a units-dependent factor equal to 1 in SI (in CGS \(\mu_0 = 1\), \(\gamma_B = 4\pi\)). The integral can be taken, allowing to represent

\[
e_M = \frac{\rho^2}{8 \pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |a_{n,m}|^2 f(2\pi \lambda \sqrt{n^2 + m^2}, \frac{2m \pi}{n^2 + m^2}), \tag{8}
\]

\[
f(x) = 1 + \frac{e^{-x} - 1}{x}. \tag{9}
\]

To evaluate the 2-d Fourier components let us first note that the denominator of both surface and volume charges can be expanded into the power series in \(1/p^4\) (a small parameter in case the magnetization state in the dot is quasi-uniform) using the identity for the generating function of the Legendre polynomials \(P_l(x)\)

\[
\frac{1}{\sqrt{r^4 + p^4 - 2p^2 r^2} \cos \psi} = \frac{1}{p^2} \sum_{i=0}^{\infty} \frac{r^{2i} P_{2i}(\cos \psi)}{p^{2i}}, \tag{10}
\]

where \(\psi = 2(\varphi - \varphi_0)\). Using the identities for the generating functions of Bessel’s functions (formulas 6.521 in Ref. 8), the Fourier basis \(F\) can be represented as

\[
F = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (-i)^k (-1)^l J_k(\alpha n) J_l(\alpha m) e^{i(k+l) \varphi}, \tag{11}
\]

where \(\alpha = 2\pi i p r\) and \(J_k(x)\) are the Bessel’s functions of the first kind.

In such a representation it is easy to evaluate the angular integral, which, for a given \(i\), is non-zero only for \(k + l = \pm (2j + 1)\) with \(j = 1, 2, \ldots, i\). This allows to remove (by making it finite) one infinite summation from a triple \(i, k, l\) sum, resulting from expansions 10, 11. Then, using Bessel’s summation theorem, each of the remaining infinite sums over \(k + l = (2j + 1)\) and \(k + l = -(2j + 1)\) can be evaluated and combined, yielding the representation of \(a_{n,m}\) for a given \(i\) in terms of finite sum of Bessel’s functions of the orders \(2j + 1\) for \(j = 1, 2, 3, \ldots, i\). The resulting terms can be easily integrated in \(r\) for the part of the integral, representing the volume charges. This calculation is quite voluminous in general. Fortunately, to obtain the expansion of the magnetostatic energy up to the \(1/p^4\) (the first neglected term is \(\sim 1/p^6\)) it is sufficient to evaluate the volume charges Fourier components for \(i = 0, 1\) and the surface charges for \(i = 0, 1, 2\) (because of \(p^2\) in Eq. 9).

The final result for the magnetostatic energy can be represented as

\[
e_M = e_0 + \frac{e_1 + \cos(4\varphi_0)e_{14}}{p^2} + \frac{e_2 + \cos(4\varphi_0)e_{2a}}{p^4} + \ldots \tag{12}
\]

\[
e_i = \rho^2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{K_i(\rho k_{m,n} \psi_{m,n})}{m^2 + n^2} f(\lambda k_{m,n}), \tag{13}
\]

where \(k_{m,n} = 2\pi \sqrt{m^2 + n^2}, \psi_{m,n} = \frac{1}{2} \arctan(\frac{2m \pi}{n^2 + m^2})\)

\[
K_0(x) = J_1^2(x), \quad K_1(x) = J_1(x) J_3(x), \quad K_{1a}(x, \psi) = J_1(x) J_3(x) \cos(4\psi), \quad K_2(x) = -2 J_0(x) J_3(x)/x, \quad K_{2a}(x, \psi) = (J_3(x)^2 + 2 J_1(x) J_5(x)) \cos(4\psi)/2. \tag{18}
\]

Functions \(e_i = e_i(\rho, \lambda)\) depend only on geometry of the problem and take into account the self-energy of the volume and surface charges, as well as interaction between them, both in a single dot and across all the dots in the array.

These expressions can be verified by renormalizing them to the unit of magnetic dot volume (as opposed to the volume of lattice cell) and taking the limit \(T \to \infty\). In this case \(e_{1a} = e_{2a} = 0\), and \(e_0, e_1\) and \(e_2\) become identical to the corresponding functions \(e_0^l, e_2\) and \(e_4\) of Ref. 4. Because the same thoroughly verified automatic procedure was used in deriving all the functions \(K_i\), including the angular ones (marked with “a”), it can be
expected with a high degree of confidence that all the expressions for $e_1(\rho, \lambda)$ are correct.

Looking at Eq. (12) it is immediately seen that for uniformly magnetized dots (when $p \to \infty$) the magnetostatic energy shows no angular dependence. The function $e_0$ is, thus, the energy of a square lattice of uniformly magnetized dots, which can be independently verified.

Compared to the above, evaluation of the exchange and Zeeman energy terms is trivial. Both contain no interaction between dots and can be evaluated for each dot separately. Actually, both were already calculated in Ref. 6. Renormalizing (2) from Ref. 6 as $e_E = E_E/(\mu\gamma_B M_S^2 T^2 L_Z)$ and using the expression for the average magnetization (16) therein we get up to $1/p^4$:

$$e_E + e_Z = \frac{\pi}{\gamma_B T^2 p^4} - \pi p^2 h \cos(\phi_0 - \phi_1) \left(1 - \frac{1}{12p^4}\right) + O(1/p^6)$$

where $\tau = T/L_E$, $L_E = \sqrt{C/(\mu_0 M_S^2)}$ is the exchange length of dot’s material, $h = H/(\mu_0 \gamma_B M_S)$ is the normalized field magnitude, and $\phi_1$ is the angle of the in-plane applied field with respect to the lattice.

Now it remains to minimize the sum of (12) and (19) to find the equilibrium values of $p$ and $\phi_0$. This results in two solutions for $\phi_0$ separated by a small angle from the direction of the applied field $\phi_1$, meaning that there will be hysteresis, depending on whether the current direction of the applied field was approached from bigger or smaller angles. Because the typical magnetic resonance measurement is performed in high-saturating magnetic field, this hysteresis can be expected to be small. Let us neglect it here for simplicity by putting $\phi_0 = \phi_1$. Minimizing we get

$$p^2 = \frac{h\pi p^2 + 12\pi/(\gamma_B T^2) + e_2 + e_{2a} \cos 4\phi_1}{6(-e_1 - e_{1a} \cos 4\phi_1)}.$$

Let us assume for now that the denominator here is always positive (that is $e_1 < 0$, $e_{1a} > 0$ and $|e_{1a}| < |e_1|$), which is the case for all the experiments analyzed below. Breaking of this assumption will be discussed at the end. The value of $p$ completely defines magnetic structure of dots in the array. Substituting it back, the equilibrium energy can also be expressed straightforwardly, but let us now concentrate on its angular dependence.

The largest (besides the constant term) Fourier harmonic in the angular dependence of the equilibrium energy is proportional to $\cos 4\phi_1$, which can be clearly classified as the four-fold anisotropy. The higher harmonics, while modifying slightly the dependence itself, do not alter positions of extrema on the angular dependence with minimums located at $\phi_1 = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ and maximums at $\phi_1 = 0, \pi/2, \pi$, and $3\pi/2$. Let us, thus, introduce the fourfold anisotropy constant, representing the equilibrium energy of a single dot (that is, normalized by the dot volume, which is the volume of magnetic material in the array cell) as $E = E_0 + K_4 \cos 4\phi_1$ with $K_4 > 0$. The value of $K_4$ is the half of the equilibrium total energy (renormalized by the dot volume) difference between configurations at $\phi_1 = 0$ and $\phi_1 = \pi/4$. Further, noting that in the resonance experiments the high external field plays a more notable role in stabilization of the “leaf” state than the exchange interaction (stabilizing it in the absence of the applied field), a limit of $\tau \to \infty$ is taken to simplify the expression into

$$K_4 = \frac{\mu_0 \gamma_B M_S^2}{2\pi p^2} \left(\frac{3(e_1 - e_{1a})^2}{12(e_2 - e_{2a}) + \pi h p^2} - \frac{3(e_1 + e_{1a})^2}{12(e_2 + e_{2a}) + \pi h p^2}\right)$$

The full expression, more accurate for the case of the array period comparable to the exchange length, can be easily obtained by the reader (it is also plotted in figures here). Nevertheless, for the subsequent comparison to experiments the precision of (21) is sufficient. At large applied fields $H/(\mu_0 \gamma_B M_S) \gg 1$ it is possible to expand $K_4$ asymptotically as

$$K_4 = -\mu_0 \gamma_B M_S^2 \frac{6e_1 e_{1a}}{\pi^2 p^4 h^4} + O(1/h^6).$$

Let us now apply the obtained knowledge to describe experiments found in literature. The first and foremost is the experiment found in Ref. 1. To interpret it, the authors added the anisotropy of the form $K_4 \sin^2 \psi \cos^2 \psi$ ($\psi$ is the angle between the magnetic moment in the dot and the lattice axis) into their numerical program for finding the resonance frequencies and fitted the angular dependence of the spectra to determine the $K_4$. Because the magnetization of the dot is close to uniform ($\psi = \phi_1$) this term produces the shift in the total dot’s energy, followed by the measured spin-wave mode. Due to $\sin^2 \phi_1 \cos^2 \phi_1$ oscillating between 0 and 1/4, while $\cos 4\phi_1$ between $-1$ and 1 with maximums of one corresponding to minimums of another (and vice versa), there is additional factor $-8$ linking the anisotropy in Ref. 1 to the definition here. Another peculiarity is that the value of $K_4^*$ in Ref. 1 tends to a constant value at large applied fields. This can not be explained by any model of the quasi-uniform state of infinite array, which at $h \to \infty$ must transform into the uniform state, showing no anisotropy. Thus, it seems another factor (probably due to the array shape effect) is present in the experiment. Assuming this factor is field-independent, let us simply add it, expressing $K_4^* = -8K_4 - 0.6 \cdot 10^6$ erg/cm$^3$, plotted in Fig. 2 (taking $M_s = 850$ emu, typical for permalloy, but not specified in Ref. 1) along with the experimental data. The agreement seems to be rather good, noting that the precision of the experiment did not allow to reliably resolve between dependencies of the $K_4^*$ for two considered arrays.

A much more precise recent experiment was done using FMR technique. In FMR it is impossible to change the value of the applied field, which was fixed at 1100 Oe, but the authors tracked the anisotropy field as a function of the array geometry, measuring different arrays with periods from 1100 to 2500 nm. The anisotropy field can be expressed through the anisotropy constant as $H_A = 2K_4/M_S$, which is plotted in Fig. 3 along with
the array parameters and the experimental data from Ref. 11. The agreement is excellent for the case of $K_4$ calculated from (21), whereas the simplified asymptotic expression (22) and its higher order equivalent are clearly not good enough at such a small field.

Let us now look back at (20). It turns out that it is quite possible for the denominator to become zero and negative. This happens for dot geometries approximately defined by the numerical fit $R_{cr}/T = 0.51 - 0.37(L_z/T) + 0.13 \cdot (L_z/T)^2$ in $0 < L_z/T < 1$ with fixing the resulting $R_{cr}/T$ to 0.5 when it exceeds this value. At radii above $R_{cr}$ the parameter $p$ becomes imaginary for some directions of the applied field, which corresponds to the transition to the so-called “flower” state. In this state (as opposed to initially considered “leaf”) the magnetization vectors diverge outwards of the line through the dot center parallel to the applied field.

To conclude, the considered problem belongs to the class, where the leading order energy contribution vanishes due to symmetry, and the system is governed by higher order energy terms. In most other cases details of the quasi-uniform magnetization distribution make only small (and often negligible) corrections to the leading order results, but here they are responsible for the completely new effect, absent in the leading order. Apart from quantitatively describing the four-fold anisotropy in the magnetic dot arrays, the presented theory asks for experiments on thicker dots, which should reveal the new eight-fold (!) anisotropy of the form $K_4 \cos 4 \phi_1 + K_8 \cos 8 \phi_1$ with $K_4$ still given by (21) and $K_8 < 0$. This transition and the resulting anisotropy will be explored in the forthcoming extended paper on the subject.

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