Analysis of Euler beam and bar as negative stiffness mechanism in vibrating structures

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Abstract. Presented here is a new mechanism based on a combination of a bar and Euler beam to illicit negative stiffness in structures. The proposed mechanism consists of an inclined bar, a horizontal Euler beam. The characteristics of the negative stiffness model is presented, and its dynamic features in combination with a vertical spring and damper is investigated under displacement vibration. It was discovered that the design of the mechanical isolator provided a better performance at zero dynamic stiffness during vibration than that of the linear isolator. The proposed mechanism can provide an alternative means of generating negative stiffness within structures.

1. Introduction

In many engineering structures, accuracy is a key performance objective. Hence, equipment needs to be properly stable to meet the accuracy required in measurement. Vibration platform upon which the equipment are placed can adversely affect the accuracy of the measurement. In addition, bridges under the earthquake action may cause damages to adjacent buildings and result in traffic gridlock that will hinder the smooth flow of traffic [1]. There are two central sources of vibration: ground vibration and direct disturbance. Hence, the needs to design equipment holder and our structures against ground vibration and direct disturbance [2].

The use of various forms of non-linear negative stiffness mechanism to isolate equipment has attracted the attention of the engineering field [3-6]. There are numerous studies on the different nonlinear passive isolator which have been documented comprehensively by Ibrahim [7]. This paper aims to present a nonlinear passive vibration isolator using an inclined bar together with an Euler beam.

1.1. Post-buckling of a strut

The relationship between the load and the shortening of the beam is then given by [8-10];

\[ P = P_e \left[1 - \pi \left(\frac{q_0}{L}\right) \left(\frac{\pi q_0}{L}\right)^2 + 4 \left(\frac{y}{L}\right)^2 \right] \left[1 + \frac{\pi^2}{8} \left(\frac{q_0}{L}\right)^2 + \frac{1}{2} \left(\frac{y}{L}\right)^2\right] \]  

(1)

where \( EI, L, P, P_e, y \), and \( q_0 \) are the flexural rigidity, length, axial load, classical Euler critical load, lateral deflection and imperfection, respectively. \( P_e = EI(\pi/L)^2 \). The static displacement of this Euler beam with an inclined bar will be used to illicit negative stiffness mechanism in a car seat.
2. Static features of the proposed negative mechanism

2.1. Force and stiffness relationship

Figure 1. Proposed isolation system containing the negative stiffness from Euler Beam and a bar

The proposed model of the isolator consists of two members eliciting the negative stiffness structures in vertical direction, a damper and a mechanical spring as shown in figure 1. The negative stiffness device is formed by a horizontal Euler beam in series with a bar having the length \( a \), is connected with the vertical spring in parallel. One end of the Euler beam is clamped to the wall while the other end is connected to a slide guide block, which is free to slide in horizontal plane on two parallel rails [11,12]. Other side of the slide block is pinned to one end of the bar through a hinge joint while the end of the bar is connected to the isolating mass.

The principle of virtual work states that in equilibrium the virtual work of the forces applied to a system is zero. Hence, the restoring force displacement relationship equation of the vehicle seat can be derived as:

\[
F = 2P_e \left[ 1 - \pi \left( \frac{q_0}{L} \right)^2 + 4 \left( \frac{L - L_1}{L} \right)^2 \right] \left[ 1 + \frac{\pi^2}{8} \left( \frac{q_0}{L} \right)^2 + \frac{1}{2} \left( \frac{L - L_1}{L} \right) \right] \left( \frac{h - x}{b - L_1} \right)
\]  

(2)

where

\[
h = \sqrt{a^2 - (b - L)^2}
\]  

(3)

\( L_1 \) is the variable length of the beam after the application of force. Dimension force–deflection about \( x \) gives;

\[
\bar{F} = 2 \left[ 1 - \pi \bar{q} \left( \pi^2 \bar{q}^2 + 4 \{1 - \gamma_2 + \Psi\} \right)^{\frac{1}{2}} \left[ 1 + \frac{\pi^2}{8} \bar{q}^2 + \frac{1}{2} \{1 - \gamma_2 + \Psi\} \right] \left( \frac{\bar{h} - \bar{x}}{\Psi} \right)
\]  

(4)

Dimensionless parameters are defined as follows:

\[
\bar{F} = \frac{F}{P_e}, \bar{x} = \frac{x}{L}, \bar{q} = \frac{q}{L}, \gamma_1 = \frac{a}{L}, \gamma_2 = \frac{b}{L}, \bar{h} = \sqrt{\gamma_1^2 - (\gamma_2 - 1)^2}, \Psi = \sqrt{(\gamma_2 - 1)^2 + 2\bar{h} - \bar{x}^2}
\]
Figure 2. Dimensionless force–displacement feature. (a) For $\bar{q} = 0.01$ and $\gamma_1 = 0.8$ and (b) for $\bar{q} = 0.02$ and $\gamma_2 = 1.2$

The dimensionless force–deflection features for various $\gamma_2$ and $\gamma_1$ are calculated by using Eq. (4) and shown in figure 2. It can be seen that this system can lead to some estimates in the design of a negative stiffness device. In figure 2(a), the dependence of $\gamma_2$ on the force displacement is displayed for $\bar{q} = 0.01$ and $\gamma_1 = 0.8$. Increase of $\gamma_2$ more than unity tends to degrade the maximum dimensional force before instability region of negative stiffness is produced. Likewise in figure 2(b), there is a reduction of dimensional force for decrease in $\gamma_1$ values. The relationship of the force and displacement will further be investigated in term of the stiffness and displacement in the next session for the complete system.

2.2. *A quasi-zero-stiffness (QZS) mechanism*

For the system in figure 1, the vertical spring $k$ is in parallel with the inclined bar connected with Euler beam. The restoring force of the vehicle seat can be found by summing the restoring force of the vertical spring to obtain dimensional spring force given by
where
\[
\bar{F}_s = \bar{x} + 2\alpha \left[1 - \pi \bar{q} \left(\pi^2 \bar{q}^2 + 4 \left(1 - \gamma_2 + \Psi\right)\right)^{\frac{1}{2}}\right]\left[1 + \frac{\pi^2}{8} \bar{q}^2 + \frac{1}{2} \left(1 - \gamma_2 + \Psi\right)\right] \left(\frac{\bar{h} - \bar{x}}{\Psi}\right)
\] (5)

Then the non-dimensional stiffness is given as:
\[
\bar{K}_s = \frac{1}{4} \left[4 - \frac{\mu^2 \alpha \Phi \left(1 - \pi \bar{q} / \Gamma\right)}{\Lambda^2} + \frac{4\mu^2 \alpha \left(1 - \pi \bar{q} / \Gamma\right)}{\Lambda^2} + \frac{2\pi \bar{q} \mu^2 \alpha \Phi}{\Gamma^3 \Lambda^3} - \frac{\alpha \Phi \left(1 - \pi \bar{q} / \Gamma\right)}{\Delta}\right]
\] (7)

where
\[
\Delta = \sqrt{\bar{h}^2 - \bar{u}^2} + \left(\gamma_2 - 1\right)^2, \Gamma = \sqrt{4 + \pi^2 \bar{q}^2 + 4\Delta - 4\gamma_2}, \Phi = 12 + \pi^2 \bar{q}^2 + 4\Delta - 4\gamma_2
\]

Figure 3 demonstrates the dimensionless nonlinear stiffness curves for initial imperfection $\bar{q} = 0.01, \alpha = 0.3$ various values $\gamma_1$ and $\gamma_2$. The maximum value of the stiffness occurs at the static equilibrium position. For illustration, in figure 3 (a) when $\gamma_2$ is 1.1 and 1.5, the maximum dimensionless stiffness is -0.025 and 0.28 respectively. Consequently, when $\gamma_2 = 1.5$ the isolator is able to support the load whereas $\gamma_2 = 1.1$ the isolator is unable to carry the imposed weight of the mass.

Substituting $\bar{u} = 0$ in Eq. (7), the dimensionless nonlinear stiffness at the static equilibrium position (SEP) is given as follows:
\[ K_{sep} = \frac{1}{4} \left[ 4 - \frac{\alpha}{\sqrt{\bar{h}^2 + (\gamma_2 - 1)^2}} \left( 1 - \frac{\pi \bar{q}}{4 + \pi^2 \bar{q}^2 + 4\sqrt{\bar{h}^2 + (\gamma_2 - 1)^2} - 4\gamma_2} \right) \times \left( 12 + \pi^2 \bar{q}^2 + 4\sqrt{\bar{h}^2 + (\gamma_2 - 1)^2} - 4\gamma_2 \right) \right] \] (8)

The surface plot of static equilibrium position stiffness \( K_{sep} \) as a function of parameters \( \gamma_1 \) and \( \gamma_2 \) is plotted in Fig. 4. For comparatively huge value of \( \alpha \) with various value of other parameters, the system will negative stiffness values. This implies that the stiffness of the whole system cannot provide any support the mass for large value of \( \alpha \). Therefore, a proper strategy is then expected at this phase to select a configuration that gives quasi-zero-stiffness.

3. Dynamic modelling and solution

The dynamic equations of the isolator are presented in this section in order to analyse its frequency response. The approximation of the exact restoring force given in Eq. (6) can be given as:

\[ \vec{F}_{app} = \vec{h} + \kappa_1 \vec{h} + \kappa_2 \vec{h}^3 \] (9)

where \( \vec{F}_{app} \) is defined as the nonlinear restoring force acting on the driver on the seat. The terms \( \kappa_1 \) and \( \kappa_2 \) characterise the linear, and cubic term, respectively of the Taylor approximation of the restoring force.
Figure 5. Force-deflection characteristics of the zero stiffness system for $\bar{q} = 0, \alpha = 1.2, \gamma_1 = 0.8$

The dynamic equations of motion of the system under displacement excitation $z_0 \cos \omega t$, at the SEP can be expressed as

$$m \frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial y}{\partial t} + k y + k \kappa_1 y + \frac{k \kappa_2}{L^2} y^3 - mg = 0$$

(10)

Let us consider the situation when the car seat is subjected to displacement excitation $z = z_0 \cos \omega t$ at the base, Eq. (10), the approximate dynamic equation of the mass is derived as follows:

$$m \ddot{y} + \eta \dot{y} + k (1 + \kappa_1) y + \frac{k \kappa_2}{L^2} y^3 = -m \ddot{z}$$

(11)

Eq. (11) can be written in non-dimensional form as

$$\ddot{Y} + 2 \xi \dot{Y} + \mu Y + \lambda Y^3 = \Omega^2 \cos \Omega \tau$$

(12)

where

$$Y = \frac{y}{z_0}, \mu = (1 + \kappa_1), \omega_n^2 = \frac{k}{m}, \xi = \frac{\eta}{2 m \omega_n}, \tau = \omega_n t, \Omega = \frac{\omega}{\omega_n}, \lambda = \frac{k \kappa_2}{L^2}$$

The frequency response curves (FRCs) for the displacement excitation systems, can be obtained as:

$$\Omega^2 = \frac{3 \lambda \bar{Y}^4 + 4 \bar{Y}^2 \left( \mu - 2 \xi^2 \right) - \sqrt{\left(3 \lambda \bar{Y}^4 + 4 \mu \bar{Y}^2 \right)^2 - 16 \bar{Y}^4 \left(3 \lambda \bar{Y}^2 + 4 \mu \right) \xi^2 + 64 \bar{Y}^4 \xi^4}}{4 \left( \bar{Y}^2 - 1 \right)}$$

(13)

4. Numerical

In our simulation example, we investigate the suitability and usefulness of using Euler beam in combination with a bar for vehicle vibration isolation. The car seat is modelled with the following properties: mass = 2.14 kg, $\omega_n^2 = 2.162$, $\xi = 0.02$, $z_0 = 10 mm$. Figure 6 shows the transmissivity of the zero stiffness for $\bar{q} = 0$ and $\gamma_1 = 0.75$. It can be observed that as the value of $\alpha$ reduces from 0.5 to 0.21, the natural frequency and transmissivity value also reduce. Therefore, a smaller $\alpha$ gives a better
performance. The parameters of the vehicle seat should be designed in such a way to reduce the transmissivity and reduce the natural frequency so as to be applicable for low frequency vibration.

![Transmissibility curve of the zero stiffness system for $\tilde{q} = 0, \gamma_1 = 0.75$](image)

**Figure 6.** Transmissibility curve of the zero stiffness system for $\tilde{q} = 0, \gamma_1 = 0.75$

5. **Conclusions**

In this paper a new mechanism to achieve negative nonlinear stiffness is presented. The proposed features of the nonlinear isolator consists of three springs; the positive spring acts as the mechanical spring and the two accompanying springs (a bar and Euler beam) act as a negative stiffness. The approximation of the restoring force is achieved by Taylor expansion. Then the equation was solved using harmonic balance method. The design of the isolator provided a better performance at zero dynamic stiffness during vibration than that of the linear isolator.

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