THE MOTIVE OF A SMOOTH THETA DIVISOR

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ABSTRACT. We prove a motivic version of the Lefschetz hyperplane theorem for a smooth ample divisor $\Theta$ on an Abelian variety. We use this to construct a motive $P$ that realizes the primitive cohomology of $\Theta$.

1. Introduction

Let $k$ be an algebraically closed field. Given a smooth projective variety $X$ of dimension $d$ over $k$ and a Weil cohomology $H^*$, there is a decomposition of the diagonal $[\Delta_X] \in H^{2d}(X \times X)$ into its Künneth components:

$$\Delta_X = \Delta_{0,X} + \ldots + \Delta_{2d,X} \in H^{2d}(X \times X) \cong \bigoplus H^j(X) \otimes H^{2d-j}(X)$$

It is one of Grothendieck’s standard conjectures ([7] Section 4) that these Künneth components arise from algebraic cycles; i.e., that there exist correspondences $\pi_{j,X} \in CH^d(X \times X)$ for which $cl(\pi_{j,X}) = \Delta_{j,X}$ under the cycle class map $cl : CH^d(X \times X) \to H^{2d}(X \times X)$. We can state a stronger version of this conjecture as follows:

**Conjecture 1.1** (Chow-Künneth). There exist correspondences $\pi_{j,X} \in CH^d(X \times X)$ satisfying:

(a) $\pi_{j,X}^2 = \pi_{j,X}$, $\pi_{j,X} \circ \pi_{j',X} = 0$ for $j \neq j'$

(b) $\sum \pi_{j,X} = \Delta_X$

(c) $cl(\pi_{j,X}) = \Delta_{j,X}$ for any choice of Weil cohomology.

In this stronger version, the correspondences $\pi_{j,X}$ are actually idempotents, which gives Chow motives $h^j(X) = (X, \pi_{j,X}, 0)$. Moreover, the decomposition of the diagonal into orthogonal components gives a decomposition of the motive of $X$ as $\bigoplus h^j(X)$. An important problem in the theory of motives is to understand these “underlying” objects $h^j(X)$ that represent the various degrees of cohomology (for every choice of cohomology). The Chow-Künneth conjecture is known to hold in some important cases: curves, surfaces ([10] Chapter 6), Abelian varieties ([2]), elliptic modular varieties ([3]).

Suppose that $A$ is an Abelian variety of dimension $g$ and $i : \Theta \hookrightarrow A$ is a smooth ample divisor. The first goal of this note is then to prove the following:

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Theorem 1.1. There exist correspondences \( \pi_{j,\Theta} \in CH^{g-1}(\Theta \times \Theta) \) satisfying conjecture [14].

The Lefschetz hyperplane theorem gives isomorphisms \( i^*: H^j(A) \to H^j(\Theta) \) for \( j < g - 1 \) and \( i_*: H^j(\Theta) \to H^{j+2}(A) \) for \( j > g - 1 \). The proof of Theorem 1.1 gives a particular set of idempotents \( \pi_{j,\Theta} \) and we set \( h^j(\Theta) = (\Theta, \pi_{j,\Theta}, 0) \). We also set \( h^j(A) = (A, \pi_{j,A}, 0) \), where \( \pi_{j,A} \) are the canonical idempotents constructed in [2]. We are then able to prove the following partial result:

\[ \text{Theorem 1.2. } \]

(a) The pull-back \( h^j(i) := \pi_{j,\Theta} \circ t^\Gamma_i \circ \pi_{j,A} : h^j(A) \to h^j(\Theta) \) is an isomorphism for \( j < g - 1 \).

(b) The push-forward \( t^\ast h^j(i) := \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} : h^j(\Theta) \to h^{j+2}(A)(1) \) is an isomorphism for \( j > g - 1 \).

(c) \( \gamma^g-1(i) \) is split-injective and \( t^\ast \gamma^g-1(i) \) is split-surjective.

(d) There is an idempotent \( p \in CH^g(\Theta \times \Theta) \) which is orthogonal to \( \pi_{j,\Theta} \) for \( j \neq g - 1 \) and for which the motive \( P := (\Theta, p, 0) \) satisfies \( H^*(P) = K_{\Theta} := \ker(i_* : H^{g-1}(\Theta) \to H^{g+1}(A)) \).

We can specialize to the case that \( k = \mathbb{C} \) and \( H^* \) is singular cohomology with \( \mathbb{Q} \)-coefficients. The primitive cohomology of \( \Theta \),

\[ K_{\Theta} = \ker(i_* : H^{g-1}(\Theta, \mathbb{Q}) \to H^{g+1}(A, \mathbb{Q})(1)), \]

is the only Hodge substructure of \( H^* \) not coming from \( A \). So, one should expect to encounter difficulty in analyzing the motive \( P \). The simplest nontrivial case is when \( A \) is a principally polarized Abelian fourfold and \([\Theta] \in CH^4(A)\) is its principal polarization. In this case, \( \Theta \) is generally a smooth divisor and \( H^*(P) = K_{\Theta} \) has Hodge level 1. Conjecturally, a motive over \( \mathbb{C} \) whose singular cohomology has Hodge level 1 should correspond to an Abelian variety ([?] Remark 7.12). We have the following partial result:

Proposition 1.1. There exists an Abelian variety \( J \) such that \( h^1(J)(-1) \cong P \Leftrightarrow p_*CH_0(\Theta) = 0 \).

2. Preliminaries

Let \( \mathcal{M}_k \) denote the category of Chow motives over \( k \) whose objects are triples \((X, \pi, n)\), where \( X \) is a smooth projective variety of dimension \( d \), \( \pi \in CH^d(X \times X) \) is an idempotent and \( n \in \mathbb{Z} \). The morphisms are defined as follows:

\[ \text{Hom}_{\mathcal{M}_k}((X, \pi, n), (X', \pi', n')) := \pi' \circ Cor^{n'-n}(X, X') \circ \pi \]

\[ = \pi' \circ CH^{d+n'-n}(X \times X') \circ \pi \]

Here, composition is defined in [3] Chapter 16.1. There is a functor \( \mathcal{Y}_k^{opp} \to \mathcal{M}_k \) from the category of smooth projective varieties over \( k \) with \( \mathcal{Y}(X) = (X, \Delta_X, 0) \) and with \( \mathcal{Y}(g) = t^\Gamma_g \).
for any morphism \( g : X \to X' \). A Weil cohomology theory is a functor \( H^* : Y_k^{opp} \to Vec_K \) (with \( K \) is a field of characteristic 0) satisfying certain axioms (described in [7] Section 4), one of which is the Lefschetz hyperplane isomorphism. Examples include singular, \( \ell \)-adic, crystalline, or de Rham cohomology. This extends to a functor \( H^* : \mathcal{M}_k \to Vec_K \), and for \( M = (X, \pi, m) \), we have
\[
H^j(M) = \pi_* H^{j+2m}(X).
\]
Also, there is the extension of scalars functor \( (\cdot)_L : \mathcal{M}_k \to \mathcal{M}_L \) for any field extension \( k \subset L \).

For \( M = (X, \pi, m) \), we will use the notation \( \mathcal{M}(k, \Delta_k, n) \).

**Lemma 2.1** (Liebermann). Let \( h_X : X' \to X, h_Y : Y \to Y' \) be correspondences of smooth projective varieties. Then, for \( \alpha \in CH^*(X \times Y), \beta \in CH^*(X' \times Y') \), we have
\[
(a) \ (h_X \times h_Y)_* (\alpha) = h_Y \circ \alpha \circ h_X
\]
\[
(b) \ \text{When } f : X \to X' \text{ and } g : Y \to Y' \text{ are morphisms, } (f \times g)_* (\alpha) = \Gamma_g \circ \alpha \circ \Gamma_f.
\]

**Proof.** See [3] Proposition 16.1.1. \( \square \)

**Theorem 2.1** (Shermenev, Deninger-Murre). Let \( A \) be an Abelian variety of dimension \( g \) over \( k \). Then, there is a unique set of idempotents \( \{\pi_{j,A}\} \in CH^g(A \times A) \) satisfying conjecture [7.1] and the following relation for all \( n \in \mathbb{Z} \):
\[
(1) \quad \Gamma_n \circ \pi_{j,A} = n^j \cdot \pi_{j,A} = \pi_{j,A} \circ \Gamma_n
\]

**Proof.** See [2] Theorem 3.1. \( \square \)

Let \( i : \Theta \to A \) be a smooth ample divisor and let \( \mathfrak{h}^j(A) = (A, \pi_{j,A}, 0) \) be the motive for the idempotents in Theorem 2.1. Then, we define the \textit{Lefschetz operator}:
\[
L_{\Theta} := \Delta_* (\Theta) \in CH^{g+1}(A \times A).
\]
The most essential result for the proofs of theorems 1.1 and 1.2 is the following in [8], a motivic version of the Hard Lefschetz theorem:

**Theorem 2.2** (Künemann). Assume that \( [\Theta] = (-1)^g [\Theta] \in CH^1(A) \).

\[
(a) \ (L_{\Theta})_* \alpha = \alpha \cup [\Theta] \text{ for } \alpha \in H^*(A)
\]
\[
(b) \ \text{The operator } \pi_{2g-j,A} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} : \mathfrak{h}^j(A)(g-j) \to \mathfrak{h}^{2g-j}(A) \text{ is an isomorphism of motives for } j \leq g. \text{ That is, there exists a correspondence } \Lambda_{\Theta} \in CH^{g-1}(A \times A) \text{ such that the following relations hold for } j \leq g:
\]
\[
(2) \quad \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} = \pi_{j,A}
\]
\[
\pi_{2g-j,A} \circ L_{\Theta}^{2g-j} \circ \Lambda_{\Theta}^{2g-j} \circ \pi_{2g-j,A} = \pi_{2g-j,A}
\]

\[
(c) \ \text{Set } \pi_{j,A} = 0 \text{ for all } j \notin \{0, 1, \ldots 2g\}. \text{ Then, we have } L_{\Theta} \circ \pi_{j,A} = \pi_{j+2,A} \circ L_{\Theta} \text{ and } \Lambda_{\Theta} \circ \pi_{j,A} = \pi_{j-2,A} \circ \Lambda_{\Theta}.
\]
Proof. See [8] Theorem 4.1. It should be noted that [b] holds more generally for Abelian schemes. It is a technical result that uses properties of the Fourier transform for Chow groups of Abelian schemes.

By Theorem 2.2[a] and the projection formula, we have \( (L_\Theta)_* = \cup \Theta = i_* \circ i^* \). The result below shows that this is true on the level of correspondences:

**Lemma 2.2.** \( L_\Theta = \Gamma_i \circ \Gamma_i \in CH^{g+1}(A \times A) \).

**Proof.** From the obvious commutative diagram:

\[
\begin{array}{ccc}
\Theta & \xrightarrow{\Delta_\Theta} & \Theta \times \Theta \\
i & \downarrow & \downarrow i \times i \\
A & \xrightarrow{\Delta_A} & A \times A
\end{array}
\]

we have \( L_\Theta = (\Delta_A)_*(\Theta) = (\Delta_A)_*(i_* \circ i) = i \times i_*(\Delta_\Theta) = \Gamma_i \circ \Delta_\Theta \circ i_\Gamma_i = \Gamma_i \circ i_\Gamma_i \), where the penultimate step follows from Lemma 2.1(b).

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3. Proofs of Theorems 1.1 and 1.2

Since \( k \) is algebraically closed, it’s possible to find some \( a \in A(k) \) such that \( t_a^* [\Theta] \in CH^1(A) \) is invariant under \((-1)^*\). So, we can assume that \((−1)^*A[Θ] = [Θ]\), so that the results of the previous section are applicable.

**Proof of Theorem 1.1.** For the proof, we will need to exhibit correspondences \( \pi_{j, \Theta} \in CH^{g−1}(\Theta \times \Theta) \) which satisfy conjecture 1.1. These are given as follows:

\[
\begin{align*}
\pi_{j, \Theta} &= t_\Gamma \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \Gamma_i \quad \text{for } j < g − 1, \\
\pi_{g−1, \Theta} &= \Delta_\Theta − \sum_{j \neq g−1} \pi_{j, \Theta}.
\end{align*}
\]

(4)

Since \( \sum \pi_{j, \Theta} = \Delta_\Theta \) holds by definition, it suffices to check conditions [a] and [c] of conjecture 1.1. For \( j < g − 1 \), we have

\[
\begin{align*}
\pi^2_{j, \Theta} &= t_\Gamma \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \Gamma_i \circ t_\Gamma \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \Gamma_i \\
&= t_\Gamma \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \Gamma_i \\
&= t_\Gamma \circ \pi_{j, A} \circ \Lambda_{\Theta}^{g−j} \circ L_{\Theta}^{g−j−1} \circ \Gamma_i = \pi_{j, \Theta}
\end{align*}
\]
Here, the second equality holds by Lemma 2.2, the third holds by Theorem 2.2(b). Similarly, for \( j > g - 1 \) we have:

\[
\pi_{j,\Theta}^2 = I_i \circ \pi_{j,A} \circ L_{\Theta}^{j-g+1} \circ L_{\Theta}^{j-g+1} \circ \pi_{j+2,A} \circ \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \pi_{j+2,A} \circ \Gamma_i
\]

\[
= I_i \circ L_{\Theta}^{j-g+1} \circ L_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ L_{\Theta}^{j-g+1} \circ L_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i
\]

\[
= I_i \circ L_{\Theta}^{j-g+1} \circ \pi_{j+2,A} \circ L_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i
\]

Thus, \( \pi_{j,\Theta} = \pi_{j,\Theta}^2 \) for \( j \neq g - 1 \). Before proving the same for \( j = g - 1 \), we show that the orthogonality condition of (a) (in conjecture 1.1) holds; that is, \( \pi_{j,\Theta} \circ \pi_{j',\Theta} = 0 \) for \( j \neq j' \) and \( j, j' \neq g - 1 \). We do this for the case of \( j = g - 1 \).

\[
\pi_{j,\Theta} \circ \pi_{j',\Theta} = I_i \circ \pi_{j,A} \circ L_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \pi_{j',A} \circ L_{\Theta}^{g-j'} \circ L_{\Theta}^{g-j'-1} \circ \Gamma_i
\]

Again, the second equality holds by Lemma 2.2 and the last equality follows from the orthogonality condition in Theorem 2.1. The third equality holds because we have

\[
\pi_{j,A} \circ L_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} = L_{\Theta}^{g-j} \circ \pi_{j,A}
\]

which follows by repeated application of Theorem 2.2(c). The remaining cases of orthogonality (\( j \neq j' \) and \( j, j' \neq g - 1 \)) are identical to (5).

What remains for the verification of condition (a) is to show that:

(i) \( \pi_{g-1,\Theta} = \pi_{g-1,\Theta}^2 \)

(ii) \( \pi_{g-1,\Theta} \circ \pi_{j,\Theta} = 0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta} \) for \( j \neq g - 1 \)

For (i) let \( \pi = \sum_{k \neq g-1} \pi_{j,\Theta} \). Since the summands are mutually orthogonal idempotents by the preceding verifications, it follows that \( \pi^2 = \pi \). Since \( \pi_{g-1,\Theta} = \Delta_{\Theta} - \pi \) by definition, we have

\[
\pi_{g-1,\Theta}^2 = (\Delta_{\Theta} - \pi)^2 = \Delta_{\Theta} + \pi^2 - 2\pi = \Delta_{\Theta} - \pi = \pi_{g-1,\Theta}
\]

For (ii) let \( j \neq g - 1 \) and note that

\[
\pi_{g-1,\Theta} \circ \pi_{j,\Theta} = (\Delta_{\Theta} - \pi) \circ \pi_{j,\Theta} = \pi_{j,\Theta} - \sum_{k \neq g-1} \pi_{k,\Theta} \circ \pi_{j,\Theta}
\]

\[
= \pi_{j,\Theta} - \pi_{j,\Theta} = 0
\]

where the third equality holds since \( \pi_{k,\Theta} \circ \pi_{j,\Theta} = 0 \) for \( j \neq k \). Similarly, one has \( 0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta} \). This completes the verification of item (a) in conjecture 1.1.

Finally, we prove (c) in conjecture 1.1. It suffices to show that \( \pi_{j,\Theta} \) acts as the identity on \( H^j(\Theta) \) and trivially on \( H^{j'}(\Theta) \) for \( j \neq j' \) and any Weil cohomology \( H^* \). One easily reduces
this to the case that \( j \neq g - 1 \). We will verify this for \( j < g - 1 \). Since \( \pi_{j,A} \) acts as 0 on \( H^j(A) \) for \( j \neq j' \), we need only show that \( \pi_{j,\Theta} \) acts as the identity on \( H^j(\Theta) \). To this end, let \( \phi := \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \) so that

\[
\pi_{j,\Theta} = i^* \Gamma_i \circ \pi_{j,A} \circ \phi
\]

Since \( H^* \) is a Weil cohomology, \( i^* \Gamma_i = i^* : H^j(A) \to H^j(\Theta) \) is an isomorphism (see [7]). Moreover, by Hard Lefschetz, \( (\phi \circ i^* \Gamma_i) = (\Lambda_{\Theta}^{g-j})_* \circ (L_{\Theta}^{g-j})_* \) is the identity on \( H^j(A) \). Thus, \( i^* \) and \( \phi_* \) are inverses, from which it follows that \( (\pi_{j,\Theta})_* \) is the identity on \( H^j(\Theta) \) for \( j < g - 1 \). The case of \( j > g - 1 \) is nearly identical, only that one uses the fact that \( i_* \) is an isomorphism.

**Proof of Theorem 1.2.** The statements of (a) and (b) are that \( h^j(i) \) and \( t h^j(i) \) are isomorphisms for \( j < g - 1 \) and \( j > g - 1 \), respectively. To show this, we need to construct their inverse isomorphisms:

\[
\phi_j := \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \text{ for } j < g - 1
\]

\[
\phi_j := \pi_{j,\Theta} \circ i^* \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \text{ for } j > g - 1
\]

Then, for \( j < g - 1 \), we have

\[
\phi_j \circ h^j(i) = \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \circ i^* \Gamma_i \circ \pi_{j,A}
\]

\[
= \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \circ i^* \Gamma_i \circ \pi_{j,A}
\]

\[
= \pi_{j,\Theta} \circ i^* \Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,A}
\]

\[
= \pi_{j,\Theta} \circ i^* \Gamma_i \circ \pi_{j,A}
\]

where the third and fourth equalities hold by Theorem 2.2(b). Similarly, we have

\[
h^j(i) \circ \phi_j = \pi_{j,\Theta} \circ i^* \Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta}
\]

\[
= \pi_{j,\Theta} = \pi_{j,\Theta}
\]

We conclude that \( h^j(i) \) and \( \phi_j \) are inverses for \( j < g - 1 \), proving (a). For (b) we have

\[
t h^j(i) \circ \phi_j = \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} \circ i^* \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A}
\]

\[
= \pi_{j+2,A} \circ \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A}
\]

\[
= \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j+2,A}
\]

\[
= \pi_{j,A}
\]

Similarly, we have

\[
\phi_j \circ t h^j(i) = \pi_{j,\Theta} \circ i^* \Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta}
\]

\[
= \pi_{j,\Theta} = \pi_{j,\Theta}
\]
As in the proof of Theorem 1.1, one can show that $\mathfrak{h}^{g-1}(i)$ and $\mathfrak{h}^{g-1}(i)$ are split-injective and split-surjective, respectively. Their left and right inverses will be:

$$
\phi_{g-1} = \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \pi_{g-1,\Theta}
$$

$$
\psi_{g-1} = \pi_{g-1,\Theta} \circ \mathfrak{h}^{g-1} \circ \Lambda_{\Theta} \circ \pi_{g+1,A}.
$$

(6)

To this end, we begin by noting that for $j < g - 1$:

$$
\pi_j,\Theta \circ \mathfrak{h}^{g-1} = \mathfrak{h}^{g-1} \circ \Lambda_{\Theta} \circ \mathfrak{h}^{g-1} \circ \Lambda_{\Theta} \circ \pi_{g-1,A}
$$

(7)

Similarly, we have $\Gamma_i \circ \pi_j,\Theta = \pi_{j+2,A} \circ \Gamma_i$ for $j > g - 1$. So, we write $\pi = \sum_{j \neq g-1} \pi_j,\Theta$ as before and obtain:

$$
\Gamma_i \circ \pi \circ \mathfrak{h}^{g-1} \circ \pi_{g-1,A} = \sum_{j < g-1} \Gamma_i \circ \pi_{j,\Theta} \circ \mathfrak{h}^{g-1} \circ \Gamma_i \circ \pi_{g-1,A} + \sum_{j > g-1} \Gamma_i \circ \pi_{j,\Theta} \circ \mathfrak{h}^{g-1} \circ \Gamma_i \circ \pi_{g-1,A}
$$

$$
= \sum_{j < g-1} \Gamma_i \circ \mathfrak{h}^{g-1} \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \pi_{j+2,A} \circ \Gamma_i \circ \mathfrak{h}^{g-1} \circ \Gamma_i \circ \pi_{g-1,A}
$$

(8)

$$
= \sum_{j < g-1} \mathfrak{h}^{g-1} \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \mathfrak{h}^{g-1} \circ \pi_{j,A} \circ \pi_{g-1,A} = 0
$$

where the third equality holds by the mutual orthogonality of $\pi_{j,A}$ and the fourth holds because $L_{\Theta} \circ \pi_{j,A} = \pi_{j+2,A} \circ L_{\Theta}$. Thus, we have:

$$
\phi_{g-1} \circ \mathfrak{h}^{g-1}(i) = \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \pi_{g-1,\Theta} \circ \mathfrak{h}^{g-1}(i)
$$

$$
= \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \mathfrak{h}^{g-1}(i)
$$

$$
= \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \pi \circ \mathfrak{h}^{g-1}(i)
$$

$$
= \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \pi_{g-1,A} \circ \pi_{g-1,\Theta} = \pi_{g-1,\Theta}
$$

Here, the second term on the third line vanishes by (8). So, $\mathfrak{h}^{g-1}(i)$ is split-injective. A similar calculation shows that $\mathfrak{h}^{g-1}(i)$ is split-surjective with right inverse $\psi_j$. The completes the proof of (c).

Finally, for (d) we define:

$$
\pi'_{g-1,\Theta} := \mathfrak{h}^{g-1}(i) \circ \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \in CH^{g-1}(\Theta \times \Theta)
$$

As in the proof of Theorem 1.1 one can show that $\pi'_{g-1,\Theta}$ is an idempotent, is orthogonal to $\pi_{j,\Theta}$ for $j \neq g - 1$. It follows that

$$
\pi'_{g-1,\Theta} \circ \pi_{g-1,\Theta} = \pi'_{g-1,\Theta} - \sum_{j \neq g-1} \pi'_{g-1,\Theta} \circ \pi_{j,\Theta} = \pi'_{g-1,\Theta}
$$
Similarly, one has \( \pi_{g-1, \Theta} \circ \pi'_{g-1, \Theta} = \pi'_{g-1, \Theta} \). Write \( h^{g-1}_1(\Theta) = (\Theta, \pi'_{g-1, \Theta}, 0) \) for the corresponding motive and define:

\[
p := \pi_{g-1, \Theta} - \pi'_{g-1, \Theta} \in CH^{g-1}(\Theta \times \Theta)
\]

We have

\[
p^2 = (\pi_{g-1, \Theta} - \pi'_{g-1, \Theta})^2 = \pi_{g-1, \Theta}^2 + (\pi'_{g-1, \Theta})^2 - 2\pi_{g-1, \Theta} \circ \pi'_{g-1, \Theta}
\]

\[
= \pi_{g-1, \Theta} + \pi'_{g-1, \Theta} - 2\pi'_{g-1, \Theta} = \pi_{g-1, \Theta} - \pi'_{g-1, \Theta} = p
\]

so that \( p \) is an idempotent. Write \( P := (\Theta, p, 0) \) for the corresponding motive. We also have

\[
p \circ \pi'_{g-1, \Theta} = (\pi_{g-1, \Theta} - \pi'_{g-1, \Theta}) \circ \pi'_{g-1, \Theta} = \pi'_{g-1, \Theta} \circ \pi'_{g-1, \Theta} = 0
\]

so that \( p \) and \( \pi'_{g-1, \Theta} \) are orthogonal. This gives a decomposition of motives:

(9) \[
h^{g-1}(\Theta) = P \oplus h^{g-1}_1(\Theta)
\]

The same argument for Theorem 1.1(c) shows that \( H^*(h^{g-1}_1(\Theta)) = i^*H^{g-1}(\Theta) \). Thus, applying \( H^* \) to (9), it follows that \( H^*(P) = K_\Theta \).

\[\square\]

4. The complementary motive \( P \)

Now, let \( k = \mathbb{C} \) and \( H^* \) be singular cohomology with \( \mathbb{Q} \)-coefficients. We consider the case of \( A \) a principally polarized Abelian variety, whose principal polarization is the class of \( i : \Theta \to A \). Since we are interested in the motive \( P \), we need \( \Theta \) to be nonsingular. The simplest nontrivial case is that of \( g = 4 \), where a well-known result of Mumford in [9] is that \( \Theta \) is generally nonsingular. Now, let \( K_{\Theta, \mathbb{Q}} := \ker(i_* : H^{g-1}(\Theta, \mathbb{Q}) \to H^{g+1}(A, \mathbb{Q})(1)) \) be the primitive cohomology. Then, we have the following:

**Lemma 4.1.** \( K_\Theta \) is a rational Hodge structure of level 1 and dimension 10.

**Proof.** Since \( H^3(\Theta) \) and \( H^3(A) \) both have Hodge level 3, we need to show that \( i_* : H^{3,0}(A) \to H^3(\Theta) \) is an isomorphism. Since this map is already injective, it will suffice to show that \( h^{3,0}(\Theta) = h^{3,0}(A) = 4 \). By adjunction, \( \omega_\Theta \cong \mathcal{O}_\Theta(\Theta) \), so \( h^0(\Theta, \mathcal{O}_\Theta(\Theta)) = h^{3,0}(\Theta) \). We can use the long exact sequence to compute \( h^0(\Theta, \mathcal{O}_\Theta(\Theta)) \):

\[
0 \to H^0(A, \mathcal{O}_A) \to H^0(A, \mathcal{O}_A(\Theta)) \xrightarrow{res} H^0(\Theta, \mathcal{O}_\Theta(\Theta)) \to H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A(\Theta)) = 0
\]

Since \( \Theta \) is a principal polarization, \( h^0(A, \mathcal{O}_A(\Theta)) = 1 \) so that the restriction arrow is 0. Moreover, \( h^1(A, \mathcal{O}_A) = 4 \), so it follows that \( h^{3,0}(\Theta) = 4 = h^{3,0}(A) \). Thus, \( i_* : H^{3,0}(A) \to H^{3,0}(\Theta) \) is an isomorphism and \( K_\Theta \) has Hodge level 1. To determine the dimension of \( K_\Theta \), we first compute \( \chi(\Theta) = c_3(T\Theta) \). Applying the Chern polynomial to the adjunction sequence in this case, one obtains that \( c_3(T\Theta) = -c_1(\mathcal{O}(\Theta))^4 = -4! = -24 \). Using the Lefschetz hyperplane theorem, one also computes that \( \chi(\Theta) = 42 - h^3(\Theta) \), so that \( h^3(\Theta) = 66 \). Since \( h^3(A) = \binom{8}{3} = 56 \), it follows that \( K_\Theta \) has dimension 10.\[\square\]
Thus, \( H^*(P, \mathbb{Q}) \) has Hodge level 1 when \( g = 4 \). Now, consider the intermediate Jacobian of \( K_{\Theta} \):

\[
J(K_{\Theta}) = K_{\Theta, \mathbb{C}}/(F^2 K_{\Theta, \mathbb{C}} \oplus K_{\Theta, \mathbb{Z}})
\]

This is a principally polarized Abelian variety of dimension 5, and we have an isomorphism of rational Hodge structures \( H^1(J(K_{\Theta}), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q}) \). The generalized Hodge conjecture predicts that this isomorphism arises from a correspondence \( \Gamma \subset J(K_{\Theta}) \times \Theta \). The existence of \( \Gamma \) was proved in [5]. One may take this a step further and ask whether \( h^1(J(K_{\Theta}))(−1) \) and \( P \) are isomorphic as motives. Proposition 1.1 provides a partial answer to this; i.e., we have 

\[
h^1(J(K_{\Theta}))(−1) \cong P \text{ if } p \text{ acts trivially on } CH^0(\Theta_L) \text{ for all field extensions } \mathbb{C} \subset L \text{ (and conversely).}
\]

We will need the following definition for the proof:

**Definition 4.1.** We say that \( M = (X, \pi, 0) \in M_k \) has representable Chow group in codimension \( i \) if there exists a smooth complete (possible reducible) curve \( C \) and \( \Gamma \in CH^i(C \times X) \) such that \( CH^i_{\text{alg}}(M_L) = \pi_L^* CH^i_{\text{alg}}(X_L) \) lies in \( \Gamma_L \subset CH^i_{\text{alg}}(X_L) \) for every field extension \( k \subset L \).

**Proof of Proposition 1.1.** Suppose that we have some Abelian variety \( J \) for which \( h^1(J)(−1) \cong P \). Then, applying \( CH^3( ) \) to both sides we obtain

\[
p_\ast CH^0(\Theta) = p_\ast CH^3(\Theta) \cong CH^3(h^1(J)(−1)) = CH^2(h^1(J))
\]

From [2] Theorem 2.19, we have \( CH^2(h^1(J)) = 0 \) so that \( p_\ast CH^0(\Theta) = 0 \). For the converse, observe that \( \Theta \) can be defined over some field \( k \) which is the algebraic closure of a finitely generated over \( \mathbb{Q} \). So, let \( \Theta_k \) be a model for \( \Theta \) over \( k \). The operators used in the proof of Theorems 1.1 and 1.2 (\( \Pi \Theta \), \( \Lambda \Theta \), and \( \pi_{j,A} \)) are well-behaved upon passage to an overfield (see [2] and [8]); thus, so is the correspondence \( p_k \in Cor^0(\Theta_k \times \Theta_k) \) constructed above. This means that \( p_k \) coincides with \( p \) (as in the statement of Proposition 1.1), and the assumption that \( p \) acts trivially on \( CH_0 \) becomes the assumption that

\[
p_L \ast CH^3(\Theta_L) = 0
\]

for all overfields \( k \subset L \). Now, let \( P = (\Theta_k, p, 0) \). The task is then to find an Abelian variety \( J \) over \( k \) for which

\[
h^1(J)(−1) \cong P
\]

To this end, we begin with the following lemma:

**Lemma 4.2.** \( P \) has representable Chow group in codimension 2.

**Proof of Lemma.** We will drop the subscript \( k \). We use the same argument as in [1]. There is a localization sequence:

\[
(10) \quad \lim_{D \subset \Theta} CH^2(\Theta \times D) \xrightarrow{(id_\Theta \times j_D)^*} CH^3(\Theta \times \Theta) \xrightarrow{(id_\Theta \times K)^*} CH^3(\Theta_K) \longrightarrow 0
\]
where the limit runs over all (possibly reducible) subvarieties $D$ of codimension 1 and $K = \mathbb{C}(\Theta)$ is the function field of $\Theta$. We have $(id_\Theta \times K)^* \Delta_\Theta = \eta_K$, the generic point of $\Theta$. From Lemma 2.2(a), we have
\[
\eta_K = (p \times id_\Theta)^* \Delta_\Theta = p_K^* (id_\Theta \times K)^* \Delta_\Theta = p_K^* (id_\Theta \times K)^* \eta_K
\]
Since $p_K^* (\eta_K) = 0$ by assumption, the exactness of (10) gives some subvariety $D$ and $\alpha \in CH^2(\Theta \times D)$ for which $p_K^* = (id_\Theta \times j_D)^* \alpha$. After desingularizing, we can assume that $D$ is smooth (although $j_D$ may no longer be an inclusion). By Lemma 2.2(b), we have
\[
p = (id_\Theta \times j_D)^* \alpha = \Gamma_{j_D} \circ \alpha
\]
Thus, we see that the Chow group of $P$ is representable in every codimension. By [12] Theorem 3.4, it follows that the motive of $P$ decomposes as
\[
\bigoplus_i (i)^{\oplus n_i} \oplus h^1(J_i)(-i)
\]
for integers $n_i$ and Abelian varieties $J_i$. Since the cohomology of $P$ is 0 in all but degree 3, this means that $P \cong h^1(J)(-1)$ for some Abelian variety $J$. This gives the proposition. □

Remark 4.1. A more refined version of Proposition 1.1 is that the Abelian variety can be taken to be $J(K_\Theta)$ in the above notation. Indeed, since we have $H^1(J(K_\Theta), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q}) \cong H^1(J, \mathbb{Q})(-1)$ (as rational Hodge structures), it follows that $J$ and $J(K_\Theta)$ are isogenous.

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