Hermitian-Einstein metrics
from noncommutative $U(1)$ instantons

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Abstract
We show that Hermitian-Einstein metrics can be locally constructed by a map from (anti-)
self-dual two-forms on Euclidean $\mathbb{R}^4$ to symmetric two-tensors introduced in [1]. This corre-
spondence is valid not only for a commutative space but also for a noncommutative space. We
choose $U(1)$ instantons on a noncommutative $\mathbb{C}^2$ as the self-dual two-form, from which we derive
a family of Hermitian-Einstein metrics. We also discuss the condition when the metric becomes
Kähler.

1 Introduction
In this article, a linear map from differential two-forms to symmetric two-tensors in two-
dimensional Hermitian manifolds introduced in [1] is studied. The map reveals another as-
pect of Seiberg-Witten map. The original Seiberg-Witten map is a map from noncommutative
gauge fields to commutative gauge fields with a background $B$-field [2]. On the other hand, it
has been interpreted in [1, 3, 4] as a map from a noncommutative gauge field to a Kähler metric.

A purpose of this article is to clarify the map in [1, 3, 4] which locally maps (anti-)self-dual
two-forms on $\mathbb{C}^2$ to Hermitian-Einstein metrics of two-dimensional Kähler manifolds. It might
be worth noting that it is enough for these two-forms to be defined as a symplectic structure
on a commutative manifold, although this map was developed in the context of Seiberg-Witten
map in noncommutative gauge theory. But this correspondence between the self-dual two-form
and Hermitian-Einstein metric can be lifted to noncommutative spaces after (canonical or de-
formation) quantization [5].

The second purpose of this article is to construct explicit examples of Hermitian-Einstein
metrics from noncommutative $U(1)$ instantons. $U(1)$ instantons on noncommutative $\mathbb{C}^2$ were
found by Nekrasov and Schwarz [6]. We will construct the two-form from a multi-instanton
solution given in [7] where the noncommutative $U(1)$ instanton solutions are written in an operator form acting on a Fock space. The Fock space is defined by Heisenberg algebra generated by noncommutative complex coordinates. There is a dictionary between the linear operators acting on the Fock space and usual functions [8]. The dictionary is applicable for arbitrary noncommutative Kähler manifold obtained by deformation quantization with separation of variables [9]. Concrete Hermitian-Einstein metrics are obtained by translating the noncommutative instantons as linear operators into ordinary functions by using the dictionary in [8].

The third purpose is to clarify the Kähler condition for the metrics derived from noncommutative $U(1)$ instantons. Since a Kähler manifold is a symplectic manifold too although the reverse is not necessarily true, one can quantize the Kähler manifold by quantizing a Poisson algebra derived from the underlying symplectic structure of the Kähler geometry, as recently clarified in [5]. We will show that the metric derived from noncommutative $U(1)$ instantons becomes a Kähler metric if the underlying Poisson algebra of $U(1)$ instantons or its quantization is an associative algebra.

Here we mention some studies related with subjects of this article. It has been conjectured in [10, 11] that NC $U(1)$ gauge theory is the fundamental description of Kähler gravity at all scales including the Planck scale and provides a quantum gravity description such as quantum gravitational foams. Recently it was shown in [12, 13, 14] that the electromagnetism in noncommutative spacetime can be realized as a theory of gravity and the symplectization of spacetime geometry is the origin of gravity. Such picture is called emergent gravity and it proposes a candidate of the origin of spacetime. See also related works in Refs. [15, 16, 17, 18, 19, 20, 21, 22, 23] As a bottom-up approach of the emergent gravity formulated in [24], the Eguchi-Hanson metric [25, 26] in four-dimensional Euclidean gravity is used to construct anti-self-dual symplectic $U(1)$ gauge fields, and $U(1)$ gauge fields corresponding to the Nekrasov-Schwarz instanton [6] are reproduced by the reverse process [27]. As a top-down approach of emergent gravity, the $U(1)$ instanton found by Braden and Nekrasov [28] derives a corresponding gravitational metric.

The organization of this paper is as follows. In section 2 some linear algebraic formulas for self-duality are prepared. In section 3 the correspondence between the self-dual two-forms and Hermitian-Einstein metrics is studied. In section 4 Hermitian-Einstein metrics are explicitly constructed from noncommutative $U(1)$ instantons. In section 5 the gauge theory realization of the Kähler condition is studied. In section 6 we discuss an outlook of this subject. Some technical details are left for the appendices.
2 Self-duality

Definition 1 (Hodge star operator). An automorphism $\star$ on the set of $4 \times 4$ alternative matrices is defined as

\[
\star \begin{bmatrix}
0 & \omega_{12} & \omega_{13} & \omega_{14} \\
-\omega_{12} & 0 & \omega_{23} & \omega_{24} \\
-\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\
-\omega_{14} & -\omega_{24} & -\omega_{34} & 0
\end{bmatrix} := \begin{bmatrix}
0 & \omega_{34} & -\omega_{24} & \omega_{23} \\
-\omega_{34} & 0 & \omega_{14} & -\omega_{13} \\
\omega_{24} & -\omega_{14} & 0 & \omega_{12} \\
-\omega_{23} & \omega_{13} & -\omega_{12} & 0
\end{bmatrix},
\]

(i.e., $\omega_{12} \leftrightarrow \omega_{34}$, $\omega_{13} \leftrightarrow \omega_{42}$, $\omega_{14} \leftrightarrow \omega_{23}$).

In other words, $\star \omega_{kl}$ is defined as

\[
\star \omega_{kl} = \frac{1}{2} \sum_{m,n} \epsilon_{klmn} \omega_{mn},
\]

where $\epsilon_{klmn}$ is Levi-Civita symbol. The operator $\star$ is called the Hodge star operation in Euclidean $\mathbb{R}^4$.

Definition 2 (Anti-self-dual matrix). A $4 \times 4$ alternative matrix $\omega^\pm$ is an (anti-)self-dual matrix if

\[
\star \omega^\pm = \pm \omega^\pm. \tag{2.1}
\]

An (anti-)self-dual matrix $\theta^\pm$ is defined as

\[
\theta^\pm := \begin{bmatrix}
0 & -\theta & 0 & 0 \\
\theta & 0 & 0 & 0 \\
0 & 0 & 0 & \mp \theta \\
0 & 0 & \pm \theta & 0
\end{bmatrix} \tag{2.2}
\]

where $\theta$ is a real number. Note that $\omega^\pm$ and $\theta^\mp$ commute each other:

\[
\omega^\pm \theta^\mp = \theta^\mp \omega^\pm. \tag{2.3}
\]

Definition 3 (Matrix $g^\pm$). Let $E_4$ be the $4 \times 4$ unit matrix and $\omega^\pm$ be a $4 \times 4$ (anti-)self-dual matrix. Assume that $\det [E_4 - \omega^\pm \theta^\mp] \neq 0$, then $4 \times 4$ matrix $g^\pm$ is defined as

\[
g^\pm := 2 \left( E_4 - \omega^\pm \theta^\mp \right)^{-1} - E_4.
\]

Remark 1. $g^\pm$ is a symmetric matrix because of (2.3) and it can be inverted to

\[
\omega^\pm = (g^\pm - E_4) \left( g^\pm + E_4 \right)^{-1} (\theta^\mp)^{-1}.
\]
The Remark 1 allows us to regard \( g^\pm \) as a metric tensor since it is symmetric and nondegenerate.

**Lemma 2.1.** For any \( 4 \times 4 \) (anti-)self-dual matrix \( \omega^\pm \);
\[
\ast \omega^\pm = \pm \omega^\pm \implies \det [g^\pm] = 1.
\]

This lemma is proved by a direct calculation.

**Definition 4.** The map \( \iota_{\text{skew}} \) defined as
\[
\iota_{\text{skew}} \left[ \begin{array}{cc}
\omega_{C11} & \omega_{C12} \\
\omega_{C21} & \omega_{C22}
\end{array} \right] = \left( \begin{array}{cccc}
0 & 2i \omega_{C11} & \omega_{C12} - \omega_{C21} & i (\omega_{C12} + \omega_{C21}) \\
-2i \omega_{C11} & 0 & -i (\omega_{C12} + \omega_{C21}) & \omega_{C12} - \omega_{C21} \\
-\omega_{C12} + \omega_{C21} & i (\omega_{C12} + \omega_{C21}) & 0 & 2i \omega_{C22} \\
-i (\omega_{C12} + \omega_{C21}) & -\omega_{C12} + \omega_{C21} & -2i \omega_{C22} & 0
\end{array} \right).
\]

Note that \( \omega_{C11} \) and \( \omega_{C22} \) are pure imaginary.

If the coordinate transformation on the coordinate neighborhood is \( z_1 := x^2 + ix^1, z_2 := x^4 + ix^3 \), then the \( \iota_{\text{skew}} \) is the pull-back of a two-form. This means
\[
\sum_{k,l=1}^2 \omega_{Ckl} dz_k \wedge d\bar{z}_l = \frac{1}{2} \sum_{k,l=1}^4 \omega_{kl} dx^k \wedge dx^l = \frac{1}{2} \sum_{k,l=1}^4 (\iota_{\text{skew}} [\omega_C])_{kl} dx^k \wedge dx^l.
\]

The above \( \iota_{\text{skew}} \) is defined as satisfying this relation.

**Remark 2.** \( \iota_{\text{skew}} \) satisfies the following relation
\[
\det [\iota_{\text{skew}} [\omega_C]] = 16 \left( \det [\omega_C] \right)^2.
\]

Using this result, the following lemma can be deduced.

**Lemma 2.2.** Suppose that the anti-Hermitian matrix \( \omega_C \) satisfies \( \omega_{C22} = -\omega_{C11} \), i.e. \( \text{tr} \omega_C = 0 \). Then the two-form \( \iota_{\text{skew}} [\omega_C] \) is anti-self-dual, i.e.,
\[
\ast \left( \iota_{\text{skew}} \left[ \begin{array}{cc}
\omega_{C11} & \omega_{C12} \\
\omega_{C21} & \omega_{C22}
\end{array} \right] \right) = -\iota_{\text{skew}} \left[ \begin{array}{cc}
\omega_{C11} & \omega_{C12} \\
\omega_{C21} & \omega_{C22}
\end{array} \right].
\]

3 **Hermitian-Einstein metrics and (anti-)self-dual two-forms**

In this section, we discuss how to make a Hermitian-Einstein metric from an anti-self-dual two-form. Let us define a \( u(1) \)-valued two-form on \( \mathbb{R}^4 \) by
\[
\sum_{k,l=1}^4 \omega_{kl} dx^k \wedge dx^l.
\]

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where \( \omega \) is an alternative matrix \((\omega)_{kl} := \omega_{kl}\). If \( \omega \) is an anti-self-dual matrix, then the two-form is called anti-self-dual two-form.

### 3.1 Ricci flat metrics and Hermitian-Einstein metrics

Let \( M \) be a Hermitian manifold and \( h \) be its metric. As a well-known fact, Ricci curvature \( R_{jk} \) for a Hermitian manifold \((M, h, \nabla)\) with the Levi-Civita connection \( \nabla \) takes a simple form

\[
R_{jk} = \partial_j \partial_k \log (\det [h]). \tag{3.1}
\]

See, for example, [29, 30]. Let \( \lambda \) be a cosmological constant. When \( h \) satisfies the Einstein’s equation

\[
R_{kl} = \lambda h_{kl}
\]

then \( M \) is called an Einstein manifold. In this paper we will focus on a Ricci flat manifold (i.e. \( R_{kl} = 0 \) or \( \lambda = 0 \)). We consider \( M \) as a real manifold with local coordinates \( x^\mu (\mu = 1, 2, 3, 4) \).

**Definition 5.** The map \( \iota_{\text{sym}} : \{ h \in M_2[\mathbb{C}] \mid h^\dagger = h \} \longrightarrow M_4[\mathbb{R}] \) is defined as

\[
\iota_{\text{sym}} \left[ \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \right] = \begin{pmatrix} h_{11} & 0 & \frac{1}{2} (h_{12} + h_{21}) & \frac{1}{2} (h_{21} - h_{12}) \\ 0 & h_{11} & -\frac{1}{2} (h_{21} - h_{12}) & \frac{1}{2} (h_{12} + h_{21}) \\ \frac{1}{2} (h_{12} + h_{21}) & -\frac{1}{2} (h_{21} - h_{12}) & h_{22} & 0 \\ \frac{1}{2} (h_{21} - h_{12}) & \frac{1}{2} (h_{12} + h_{21}) & 0 & h_{22} \end{pmatrix},
\]

where \( h \) is a matrix and \((h)_{kl} := h_{lk}\).

**Remark 3.** Assume that \( h \) is a Hermitian metric. If the coordinate transformation on a coordinate neighborhood is \( z^1 := x^2 + ix^1, z^2 := x^4 + ix^3 \), the \( \iota_{\text{sym}} \) is then the pull-back of the Hermitian metric given by

\[
\sum_{k,l=1}^2 h_{kl} dz_k d\bar{z}_l = \sum_{k,l=1}^4 (\iota_{\text{sym}} [h])_{kl} dx^k dx^l.
\]

Hence \( \iota_{\text{sym}} \) squares the determinant:

\[
\det [\iota_{\text{sym}} (h)] = (\det [h])^2.
\]

A Hermitian metric made with \( \iota_{\text{sym}}^{-1} \) will be used below.

**Definition 6.** If \( \tilde{h} \in C^m (U, M_2[\mathbb{C}]) \) and \( \tilde{h}^\dagger = \tilde{h} \), then

\[
\tilde{h} > 0 \text{ in } U \iff \forall u \in U, \quad \tilde{h}(u) > 0
\]

where \( \tilde{h}(u) > 0 \) means that \( \tilde{h} \) is positive definite as a Hermitian matrix.
Lemma 3.1. If $h \in C^\infty(U, M_2[\mathbb{C}])$ is a Hermitian matrix with $\det [h] = 1$ and $h$ is positive (negative) at $\exists p \in U$, then $h$ is positive (negative) in $U$.

Proof. This follows from
\[
\left\{ h \in M_2[\mathbb{C}] \mid h = h^\dagger, \, \det [h] = 1 \right\}
\]
\[
= \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M_2[\mathbb{C}] \mid a, d \in \mathbb{R}, \, a > 0, \, d > 0, \, ad \geq 1, \, |b| = \sqrt{ad - 1} \right\}
\]
\[
\bigcap \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M_2[\mathbb{C}] \mid a, d \in \mathbb{R}, \, a < 0, \, d < 0, \, ad \geq 1, \, |b| = \sqrt{ad - 1} \right\}
\]
which means two spaces are disconnected.

From the above discussions, the following theorem is obtained.

Theorem 3.2. Let $\omega^\pm$ be an (anti-)self-dual two-form on an open neighborhood $U$, i.e. $\star \omega^\pm = \pm \omega^\pm$, and
\[
h^\pm := i^{-1}_{sym} \left[ 2 (E_4 - \omega^\pm \theta^\mp)^{-1} - E_4 \right]. \tag{3.2}
\]
Then $h^\pm$ gives a Ricci-flat Hermitian metric on $U$. So $(U, h^\pm)$ is a local realization of an Einstein manifold.

Proof. Because of Lemma 2.1, if $\star \omega^\pm = \pm \omega^\pm$, then
\[
\det [h^\pm] = 1. \tag{3.3}
\]
Because of Lemma 3.1 and Remark 1, $h^\pm$ is a metric tensor. From equations (3.1) and (3.3),
\[
R_{jk} = \partial_j \partial_k \log (\det [h^\pm]) = 0.
\]
Local complex coordinates can be arranged in such a way that the Jacobians of the transition functions on overlapping charts are one on all the overlaps. In that case, $\det[h^\pm]$ is a globally defined function and the Ricci-flat condition reduces to the Monge-Ampère equation
\[
\det[h^\pm] = \kappa, \tag{3.4}
\]
where the constant $\kappa$ is related to the volume of a Kähler manifold that depends only on the Kähler class. Therefore Theorem 3.2 implies that the self-duality for the two-form $\omega^\pm$ is equivalent to the Ricci-flat condition (3.4) of Kähler manifolds defined by the metric $h^\pm$. 

6
4 Hermitian-Einstein metric from noncommutative instanton on $\mathbb{C}^2$

In the previous section we found the way to construct a Hermitian-Einstein metric from an (anti-)self-dual two-form. To construct the Hermitian-Einstein metric, we will employ the instanton curvature on noncommutative $\mathbb{C}^2$ as the (anti-)self-dual two-form. There are many ways to obtain noncommutative $\mathbb{C}^2$ (see [33] for a review and references therein). We use the Fock representation of noncommutative $\mathbb{C}^2$ given in [8], which is based on the Karabegov’s deformation quantization [9]. There is a simple dictionary between the Fock representation and ordinary functions. Using the dictionary, the Hermitian-Einstein metric is expressed in terms of usual functions.

4.1 Noncommutative $\mathbb{C}^2$

Consider a noncommutative algebra $(C^\infty(\mathbb{C}^2), \{[h]\}, \star)$ led by (A.4) in Appendix A. The star product induces a Heisenberg algebra

$$\left[z^k, \bar{z}^l\right]_\star = -\zeta \delta_{kl}, \quad [z^k, z^l]_\star = 0, \quad \left[\bar{z}^k, \bar{z}^l\right]_\star = 0,$$

(4.1)

where $[x, y]_\star := x \star y - y \star x$. We represent it by creation and annihilation operators given by

$$a_k := \frac{\bar{z}^k}{\sqrt{\zeta_k}}, \quad a_k^\dagger := \frac{z^k}{\sqrt{\zeta_k}},$$

then

$$\left[a_k, a_l^\dagger\right]_\star = \delta_{kl}, \quad \left[a_k, a_l\right]_\star = 0, \quad \left[a_k^\dagger, a_l^\dagger\right]_\star = 0.$$

In the following $\zeta_1 = \zeta_2 = \zeta > 0$ is assumed.

Note that the choice of a noncommutative parameter has the freedom associated with a choice of a background two-form [2]. Here the $\zeta$ in (4.1) is regarded as the only noncommutative parameter. However, in Section 5, we will implicitly assume the identification $\zeta := 2\theta$ since we will work in the background-independent prescription, i.e. $\theta = B^{-1}$.

The algebra $\mathcal{F}$ on $\mathbb{C}$ is defined as follows. The Fock space $\mathcal{H}$ is a linear space spanned by the bases generated by acting $a_i^\dagger$’s on $|0, 0\rangle$:

$$\frac{1}{\sqrt{m_1! m_2!}} \left(a_1^\dagger\right)^{m_1}_\star \left(a_2^\dagger\right)^{m_2}_\star |0, 0\rangle = |m_1, m_2\rangle,$$

(4.2)
where $m_1$ and $m_2$ are positive integers and $(a)^m$ stands for $a \star \cdots \star a$. The ground state $|0, 0\rangle$ satisfies $a_i |0, 0\rangle = 0$, $\forall \ l$. Here, we define the basis of a dual vector space by acting $a_i$’s on $|0, 0\rangle$ as

$$
\frac{1}{\sqrt{n_1!n_2!}} |0, 0\rangle (a_1)^{n_1} (a_2)^{n_2} = \langle n_1, n_2 |,
$$

where $|0, 0\rangle$ satisfies $\langle 0, 0 | a_l^\dagger = 0$, $\forall \ l$. Then we define a set of linear operators as

$$
{\mathcal{F}} := \text{span}_\mathbb{C} (|m_1, m_2\rangle \langle n_1, n_2 | ; m_1, m_2, n_1, n_2 = 0, 1, 2, \cdots )
$$

(4.3)

where $(|m_1, m_2\rangle \langle n_1, n_2 | |k_1, k_2\rangle) = \delta_{k_1n_1} \delta_{k_2n_2} |m_1, m_2\rangle$ and $\langle k_1, k_2 | (|m_1, m_2\rangle \langle n_1, n_2 |) = \delta_{k_1m_1} \delta_{k_2m_2} \langle n_1, n_2 |$. The product on $\mathcal{F}$ is defined as

$$
\langle j_1, j_2 | (k_1, k_2) \circ (|m_1, m_2\rangle \langle n_1, n_2 |) := \delta_{k_1m_1} \delta_{k_2m_2} |j_1, j_2\rangle \langle n_1, n_2 |,
$$

so, $\mathcal{F}$ is an algebra.

There is a one to one correspondence between $\mathcal{F}$ and some subalgebra of $C^\infty (\mathbb{C}^2)$. For arbitrary noncommutative Kähler manifold obtained by deformation quantization with separation of variables [9], we can find the similar correspondence [8]. The following is the simplest example of the correspondence.

**Definition 7.** (Twisted Fock representation). The linear map $\iota : \mathcal{F} \longrightarrow C^\infty (\mathbb{C}^2)$ is defined as

$$
\iota (|m_1, m_2\rangle \langle n_1, n_2 |) = e_{(m_1, m_2, n_1, n_2)} := \frac{z_1^{m_1} z_2^{m_2} e^{-\frac{1}{\xi} z_1^{m_1} z_2^{m_2} z_1^{n_1} z_2^{n_2}}}{\sqrt{m_1!m_2!n_1!n_2!} (\sqrt{\xi})^{m_1+m_2+n_1+n_2}},
$$

(4.4)

especially $\iota (|0, 0\rangle \langle 0, 0 |) = e_{(0, 0, 0, 0)} = e^{-\frac{1}{\xi} z_1^{m_1} z_2^{m_2}}$.

**Proposition 4.1.** Let $\iota (\mathcal{F})$ be defined by

$$
\iota (\mathcal{F}) := \text{span}_\mathbb{C} (e_{(m_1, m_2, n_1, n_2)} ; m_1, m_2, n_1, n_2 = 0, 1, 2, \cdots ) . \tag{4.5}
$$

Then $\{\iota (\mathcal{F}), \star \}$ is an algebra where $\star$ is in (A.4).

**Proof.** After a little algebra, one can deduce the following identity

$$
e_{(k_1, k_2, l_1, l_2)} \star e_{(m_1, m_2, n_1, n_2)} = \delta_{l_1m_1} \delta_{l_2m_2} e_{(k_1, k_2, n_1, n_2)}. \tag{4.6}
$$

Details are given in [8].

The identity (4.6) derives the following fact.

**Proposition 4.2.** The algebras $(\mathcal{F}, \circ)$ and $\{\iota (\mathcal{F}), \star \}$ are isomorphic.
This isomorphism $\iota$ is a “Fock space - function space” dictionary. From this isomorphism, we do not distinguish these two algebras and we only use $\ast$ to represent products in the following.

Here we consider a $U(1)$ gauge theory on noncommutative $\mathbb{C}^2$. $U(1)$ gauge connection in the noncommutative space is defined as follows (see for example [34]).

**Definition 8.** Rescaled coordinates of $\mathbb{C}^2$ are defined as
\[
\hat{\partial}_{z_l} := \frac{\bar{z}_l}{\zeta_l}.
\]
This acts on $\mathcal{H}$ as a linear operator.

Using $\hat{\partial}_{z_l}, \hat{\partial}_{\bar{z}_m}$, let us introduce covariant derivatives and the gauge curvature as follows.

**Definition 9.** Covariant derivatives for a scalar field in fundamental representation $\phi \in \mathcal{F}$ on noncommutative $\mathbb{C}^2$ are defined as
\[
\hat{\nabla}_{z_l} \hat{\phi} := \left[ \hat{\partial}_{z_l}, \hat{\phi} \right] \ast + \hat{A}_{z_l} \ast \hat{\phi} = -\hat{\phi} \ast \hat{\partial}_{z_l} + \hat{D}_{z_l} \ast \hat{\phi}
\]
where we define a local gauge field $\hat{A}_{z_l} \in \mathcal{F}$ and
\[
\hat{D}_{z_l} := \hat{\partial}_{z_l} + \hat{A}_{z_l}.
\]
The gauge curvature is defined as
\[
\hat{F}_{z_l \bar{z}_m} := i \left[ \hat{\nabla}_{z_l}, \hat{\nabla}_{\bar{z}_m} \right] \ast = \frac{i\delta_{lm}}{\zeta_l} + i \left[ \hat{D}_{z_l}, \hat{D}_{\bar{z}_m} \right] \ast, \tag{4.7}
\]
\[
\hat{F}_{z_l z_m} := i \left[ \hat{\nabla}_{z_l}, \hat{\nabla}_{z_m} \right] \ast = i \left[ \hat{D}_{z_l}, \hat{D}_{z_m} \right] \ast,
\]
\[
\hat{F}_{\bar{z}_l \bar{z}_m} := i \left[ \hat{\nabla}_{\bar{z}_l}, \hat{\nabla}_{\bar{z}_m} \right] \ast = i \left[ \hat{D}_{\bar{z}_l}, \hat{D}_{\bar{z}_m} \right] \ast.
\]

**4.2 Ricci-flat metrics from noncommutative $k$-instantons**

In this section, we make Ricci-flat metrics on a local neighborhood from noncommutative instantons on $\mathbb{C}^2$. As we saw in Section 3 (anti)-self-dual two-forms satisfying (2.1) derive Ricci-flat metrics. Nekrasov and Schwarz found in [6] how to construct noncommutative instantons on $\mathbb{C}^2$ by using the ADHM method and the general solutions for the $U(1)$ gauge theory are given in [34]. We introduce the commutation relation of complex coordinates as (1.1). As (anti)-self-dual two-forms in Section 3 we employ noncommutative instantons given in [7].
The general instanton solutions (see [7]) satisfy the (anti)-self-dual relation. An instanton curvature tensor is described by

$$\hat{F}_{\mathcal{C}}^-[k] := \begin{pmatrix} \hat{F}_{z_1 z_1}^- [k] & \hat{F}_{z_2 z_1}^- [k] \\ \hat{F}_{z_2 z_2}^- [k] & -\hat{F}_{z_1 z_1}^- [k] \end{pmatrix},$$

and satisfies (2.1):

$$\star \left( t_{\text{skew}} \left( \hat{F}_{\mathcal{C}}^- [k] \right) \right) = -t_{\text{skew}} \left( \hat{F}_{\mathcal{C}}^- [k] \right).$$

See Lemma 2.2 in Section 2. This fact leads to the following result.

**Proposition 4.3.** If $\hat{F}_{\mathcal{C}}^-$ is a $k$-instanton curvature tensor of $U(1)$ gauge theory on noncommutative $\mathbb{C}^2$, and

$$h[k] := t_{\text{sym}}^{-1} \left\{ 2 \left( E_4 - t_{\text{skew}} \left( \hat{F}_{\mathcal{C}}^- [k] \right) \theta^+ \right)^{-1} - E_4 \right\}$$

$$= \frac{1}{4 \| \hat{F}_{\mathcal{C}}^- [k] \| \theta^2 - 1} \begin{pmatrix} -4i\hat{F}_{z_2 z_1}^- [k] \theta - 2 & -4i\hat{F}_{z_1 z_2}^- [k] \theta \\ -4i\hat{F}_{z_2 z_1}^- [k] \theta & 4i\hat{F}_{z_1 z_1}^- [k] \theta - 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $h[k]$ is an Einstein (Ricci-flat) metric.

A concrete example of $k$-instanton curvature tensors is given in [7] and the curvature is written by using linear operators on a Fock space. It is known from [4] and Proposition 4.2 how to translate the operators into functions. (See also Appendix B.2 and S.) Then the $k$-instanton curvature tensor is expressed by concrete elementary functions as follows:

$$\hat{F}_{z_1 z_1}^- [k] = \frac{i}{\zeta} \sum_{n_2=0}^{\infty} \frac{z_2^{n_2} e^{-1\zeta^{1+2z_2^2}} z_2^{n_2}}{n_2! \zeta^{n_2}} \left( d_1 (0, n_2; k) \right)^2$$

$$- \frac{i}{\zeta} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} e^{-1\zeta^{1+2z_2^2}} z_2^{n_2}}{n_1 n_2! \zeta^{n_1+n_2}} \left\{ \left( d_1 (n_1, n_2; k) \right)^2 - \left( d_1 (n_1 - 1, n_2; k) \right)^2 \right\},$$

$$\hat{F}_{z_1 z_2}^- [k] = -\frac{i}{\zeta} \frac{z_1^{k-1} z_2 e^{-1\zeta^{1+2z_2^2}}}{\sqrt{(k-1)!} \left( \sqrt{\zeta} \right)^k} \left( d_1 (k-1, 1; k) d_2 (0, 0; k) \right)$$

$$- \frac{i}{\zeta} \sum_{n_1=1}^{k-1} \frac{z_1^{n_1+k-1} z_2 e^{-1\zeta^{1+2z_2^2}} z_2^{n_1}}{\sqrt{(n_1 + k-1)! n_1!} \left( \sqrt{\zeta} \right)^{2n_1+k}} \left\{ d_1 (n_1 + k-1, 1; k) d_2 (n_1, 0; k) - d_1 (n_1 - 1, 0; k) d_2 (n_1 - 1, 0; k) \right\}$$

$$- \frac{i}{\zeta} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{z_1^{n_1-1} z_2^{n_2+1} e^{-1\zeta^{1+2z_2^2}} z_2^{n_2}}{\sqrt{(n_1 - 1)!} (n_2 + 1)! n_2! \left( \sqrt{\zeta} \right)^{2n_1+2n_2}} \left\{ d_1 (n_1 - 1, n_2 + 1; k) d_2 (n_1, n_2; k) - d_1 (n_1 - 1, n_2; k) d_2 (n_1 - 1, n_2; k) \right\},$$

$$\times \left\{ d_1 (n_1 - 1, n_2 + 1; k) d_2 (n_1, n_2; k) - d_1 (n_1 - 1, n_2; k) d_2 (n_1 - 1, n_2; k) \right\},$$

$$\times \left\{ d_1 (n_1 - 1, n_2 + 1; k) d_2 (n_1, n_2; k) - d_1 (n_1 - 1, n_2; k) d_2 (n_1 - 1, n_2; k) \right\},$$
\[ F_{z_1 z_2} [k] = - F_{z_2 z_1} [k]^\dagger, \]

where \( n_2 \neq 0 \) and

\[
\begin{align*}
 d_1 (n_1, 0; k) &= \sqrt{n_1 + k + 1} \left( \frac{\Lambda(n_1 + k + 1, 0)}{\Lambda(n_1, k, 0)} \right), \\
 d_1 (n_1, n_2; k) &= \sqrt{n_1 + 1} \left( \frac{\Lambda(n_1 + 1, n_2)}{\Lambda(n_1, n_2)} \right), \\
 d_2 (n_1, 0; k) &= \sqrt{n_2 + 1} \left( \frac{\Lambda(n_1, k, 1)}{\Lambda(n_1 + k, 0)} \right), \\
 d_2 (n_1, n_2; k) &= \sqrt{n_2 + 1} \left( \frac{\Lambda(n_1, n_2 + 1)}{\Lambda(n_1, n_2)} \right). 
\end{align*}
\]

(4.10)

Here

\[ \Lambda [k] (n_1, n_2) = \frac{w_k [k] (n_1, n_2)}{w_k [k] (n_1, n_2) - 2k w_{k-1} [k] (n_1, n_2)}, \]

and

\[ w_n [k] (n_1, n_2) = \sum_{l=0}^{n} \left\{ \frac{n! (n_1 - n_2 + k + l)!}{l! (n_1 - n_2 - k)!} \left( \frac{2(n-l)}{(n-l)!} \right) \frac{(n_2 + (n-l))!}{n_2!} \right\} . \]

Note that some notations are slightly changed from [7] and imaginary unit factor causes here. See also Appendix B.

Using these instanton curvatures, Hermitian-Einstein metrics can be constructed by concrete elementary functions according to the Theorem 3.2.

### 4.3 Einstein metric from finite \( N \)

The full noncommutative \( U(1) \) instanton solution is very complicated. For simplicity, let us consider the \( \zeta \)-expansion.

In the previous subsection, \( \tilde{F}^- \) is represented by an infinite series

\[ \tilde{F}^- = \sum_{n=1}^{\infty} \left( \frac{1}{\zeta} \right)^n \tilde{F}^- (\frac{n}{\zeta}). \]

(4.12)

The anti-self-dual condition \( \star \tilde{F}^- = - \tilde{F}^- \) implies

\[ \star \tilde{F}^- (\frac{n}{\zeta}) = - \tilde{F}^- (\frac{n}{\zeta}) \]

(4.13)
for each $n/2$. Therefore it is possible to employ an arbitrary partial sum of (4.12) determined by a subset $S \subset {\mathbb{Z}}_{>0}$

$$\hat{F}_S^- = \sum_{\frac{n}{2} \in S} \left( \frac{1}{\zeta} \right)^{\frac{n}{2}} \hat{F}^-_{(\frac{n}{2})}$$

(4.14)

for the anti-self-dual two-form to construct a Hermitian-Einstein metric $h$ without losing rigorousness. In the following we consider

$$\hat{F}^-_{(\frac{n}{2})} := \sum_{n=1/2}^{N/2} \left( \frac{1}{\zeta} \right)^{\frac{n}{2}} \hat{F}^-_{(\frac{n}{2})}.$$  

(4.15)

**Example 1.** First let us make the Ricci-flat metric $h\left[{k}\right]_{(1)}$ from $\hat{F}^-_{C} \left[{k}\right]_{(1)}$. The curvature tensor in this case is $\hat{F}^-_{C} \left[{k}\right]_{(1)} = \left( \begin{array}{cc} \frac{i}{\zeta} & 0 \\ 0 & -\frac{i}{\zeta} \end{array} \right)$, and its determinant is $\det \left[ \hat{F}^-_{C} \left[{k}\right]_{(1)} \right] = \frac{1}{\zeta^2}$.

So the metric $h\left[{k}\right]_{(1)}$ is given by

$$h\left[{k}\right]_{(1)} := \frac{1}{4 \det \left[ \hat{F}^-_{C} \left[{k}\right]_{(1)} \right] \theta^2 - 1} \left( \begin{array}{cc} -4i\hat{F}^-_{z1z_1} \left[{k}\right]_{(1)} \theta - 2 & -4i\hat{F}^-_{z2z_2} \left[{k}\right]_{(1)} \theta \\ -4i\hat{F}^-_{z2z_1} \left[{k}\right]_{(1)} \theta & 4i\hat{F}^-_{z1z_1} \left[{k}\right]_{(1)} \theta - 2 \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$$

$$= \frac{1}{1 - 4\zeta^{-2}\theta^2} \left( \begin{array}{cc} 1 - 4\zeta^{-1}\theta + 4\zeta^{-2}\theta^2 & 0 \\ 0 & 1 + 4\zeta^{-1}\theta + 4\zeta^{-2}\theta^2 \end{array} \right) = \left( \begin{array}{cc} \frac{1 - 2\zeta^{-1}\theta}{1 + 2\zeta^{-1}\theta} & 0 \\ 0 & \frac{1 + 2\zeta^{-1}\theta}{1 - 2\zeta^{-1}\theta} \end{array} \right).$$

This corresponds to the Euclidean metric essentially.

**Example 2.** Let us make a Ricci-flat metric $h\left[{k}\right]_{(2)}$ from $\hat{F}^-_{C} \left[{k}\right]_{(2)}$. From (B.16), (B.17),

$$\hat{F}^-_{C} \left[{k}\right]_{(2)} = \frac{i}{\zeta} \left[ 1 - \frac{z_2\bar{z}_2}{\zeta} \left( d_1 \left( 0, 1; k \right) \right)^2 - \frac{z_1\bar{z}_1}{\zeta} \left\{ \left( d_1 \left( 1, 0; k \right) \right)^2 - \left( d_1 \left( 0, 0; k \right) \right)^2 \right\} \right] \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

$$- \frac{id_1 \left( k - 1; 1; k \right) d_2 \left( 0, 0; k \right)}{\zeta^{1+k/2} \sqrt{(k - 1)!}} \left( \begin{array}{cc} 0 & z_1^{-1}z_2 \\ z_1^{k-1}z_2 & 0 \end{array} \right).$$

Then its determinant is

$$\det \left[ \hat{F}^-_{C} \left[{k}\right]_{(2)} \right] = \frac{1}{\zeta^2} \left[ 1 - \frac{z_2\bar{z}_2}{\zeta} \left( d_1 \left( 0, 1; k \right) \right)^2 - \frac{z_1\bar{z}_1}{\zeta} \left\{ \left( d_1 \left( 1, 0; k \right) \right)^2 - \left( d_1 \left( 0, 0; k \right) \right)^2 \right\} \right]^2$$

$$+ \left\{ d_1 \left( k - 1; 1; k \right) \right\}^2 \left\{ d_2 \left( 0, 0; k \right) \right\}^2 \frac{z_1^{k-1}z_2^{k-1}\bar{z}_2}{\zeta^{2+k} \left( k - 1 \right)!}. $$

\footnote{One may choose even more loose condition than (4.14). One can choose a different subset $S$ for each $\hat{F}^-_{z1z_1}, \hat{F}^-_{z2z_2}$ to obtain a Hermitian-Einstein metric.}
So the metric \( h[k]_{\{2\}} \) is given by

\[
h[k]_{\{2\}} := \frac{1}{4 \det \left[ \hat{F}_C^{-} [k]_{\{2\}} \right]} \theta^2 - 1 \left( \begin{array}{cc} -4i \hat{F}_{z_1 z_1}^{-} [k]_{\{2\}} \theta - 2 & -4i \hat{F}_{z_1 z_2}^{-} [k]_{\{2\}} \theta \\ -4i \hat{F}_{z_2 z_1}^{-} [k]_{\{2\}} \theta & 4i \hat{F}_{z_2 z_1}^{-} [k]_{\{2\}} \theta - 2 \end{array} \right) - \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]

which can be calculated concretely though its expression becomes complex. To simplify this we assume \( k > 3 \), then

\[
h[k]_{\{2\}} = \left\{ \begin{array}{c} \frac{2}{1 - 4 \det \left[ \hat{F}_C^{-} [k]_{\{2\}} \right]} \theta^2 - 1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) + \frac{4i \hat{F}_{z_1 z_1}^{-} [k]_{\{2\}} \theta}{1 - 4 \det \left[ \hat{F}_C^{-} [k]_{\{2\}} \right]} \theta^2 \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \\
1 - 4 \theta^2 \zeta^{-2} \left[ 1 - \frac{z_2 \bar{z}_2}{\zeta} (d_1 (0, 1; k))^2 - \frac{z_1 \bar{z}_1}{\zeta} \left\{ (d_1 (1, 0; k))^2 - (d_1 (0, 0; k))^2 \right\} \right]^2 - 1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \\
- \frac{4 \theta}{\zeta} \left[ 1 - \frac{z_2 \bar{z}_2}{\zeta} (d_1 (0, 1; k))^2 - \frac{z_1 \bar{z}_1}{\zeta} \left\{ (d_1 (1, 0; k))^2 - (d_1 (0, 0; k))^2 \right\} \right]^2 \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right\}.
\]

In next subsection, we discuss a Hermitian-Einstein metric obtained from 1-instanton solution.

### 4.4 Hermitian-Einstein metric from a 1-instanton

For the simplest example of the Hermitian-Einstein metric given in the previous discussion, we describe a Hermitian-Einstein metric obtained from a single noncommutative \( U(1) \) instanton. Now we pay attention to low order terms.
For $k = 1$, $\hat{F}_C^{-} [1]$ is

\[
\hat{F}_{z_1 \bar{z}_1}^{-} [1] = \frac{i}{\zeta} - \frac{2i z_2 \bar{z}_2}{3 \zeta^2} (d_1 (0, 1; 1))^2 - \frac{i z_1 \bar{z}_1}{\zeta^2} \left\{ (d_1 (1, 0; 1))^2 - (d_1 (0, 0; 1))^2 \right\} + \mathcal{O} (\zeta^{-3})
\]

\[
= \frac{i}{\zeta} - \frac{2i z_2 \bar{z}_2}{3 \zeta^2} (d_1 (0, 1; 1))^2 + \mathcal{O} (\zeta^{-3}) = \frac{i}{\zeta} - \frac{i}{6\zeta^2} (4z_2 \bar{z}_2 + 7z_1 \bar{z}_1) + \mathcal{O} (\zeta^{-3})
\]

\[
\hat{F}_{z_1 \bar{z}_2}^{-} [1] = -\frac{i z_2}{\zeta} \left( 1 - \frac{z_1 \bar{z}_1}{\zeta} - \frac{z_2 \bar{z}_2}{\zeta} \right) d_1 (0, 1; 1) d_2 (0, 0; 1) + \mathcal{O} (\zeta^{-3})
\]

\[
\hat{F}_{z_2 \bar{z}_1}^{-} [1] = -\frac{i z_2}{\zeta} \left( 1 - \frac{z_1 \bar{z}_1}{\zeta} - \frac{z_2 \bar{z}_2}{\zeta} \right) d_1 (0, 1; 1) d_2 (0, 0; 1) + \mathcal{O} (\zeta^{-3})
\]

from (B.16), (B.18). Then

\[
\det \left[ \hat{F}_C^{-} [1] \right] = \frac{4z_2 \bar{z}_2}{9\zeta^5} (\zeta - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2 - \frac{1}{36\zeta^4} (6\zeta - 7z_1 \bar{z}_1 - 4z_2 \bar{z}_2)^2
\]

(4.16)

From this 1-instaoton curvature, the Hermitian-Einstein metric is given as

\[
h [1]_{(2)} := \frac{1}{4 \det \left[ \hat{F}_C^{-} [1] \right]_{(2)}} \theta^2 - 1 \left( \begin{array}{cc}
-4i \hat{F}_{z_1 \bar{z}_1}^{-} [1] \{2 \} \theta - 2 & -4i \hat{F}_{z_1 \bar{z}_2}^{-} [1] \{2 \} \theta \\
-4i \hat{F}_{z_2 \bar{z}_1}^{-} [1] \{2 \} \theta & 4i \hat{F}_{z_2 \bar{z}_2}^{-} [1] \{2 \} \theta - 2
\end{array} \right) - \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

\[
= \frac{1}{1 - 4 \left\{ \frac{4z_2 \bar{z}_2}{9\zeta^5} (\zeta - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2 - \frac{1}{36\zeta^4} (6\zeta - 7z_1 \bar{z}_1 - 4z_2 \bar{z}_2)^2 \right\} \theta^2}
\]

\[
\times \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) + \frac{\theta}{\zeta} \left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right) + \frac{2\theta}{3\zeta^{5/2}} \left( \begin{array}{cc}
0 & z_2 \\
z_2 & 0
\end{array} \right) + \frac{\theta}{6\zeta^2} \left( \begin{array}{cc}
-4z_2 \bar{z}_2 - 7z_1 \bar{z}_1 & 0 \\
0 & 4z_2 \bar{z}_2 + 7z_1 \bar{z}_1
\end{array} \right)
\]

\[
+ \frac{2\theta}{3\zeta^{5/2}} \left( \begin{array}{cc}
0 & -z_2 (z_1 \bar{z}_1 + z_2 \bar{z}_2) \\
-z_2 (z_1 \bar{z}_1 + z_2 \bar{z}_2) & 0
\end{array} \right) - \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

5 Kähler structure and Bianchi identity

In this section we discuss the Kähler condition on the metric derived from (anti-)self-dual two-forms of noncommutative $U(1)$ instantons. We will clarify this issue by illuminating the duality between the Kähler geometry and $U(1)$ gauge theory claimed in [10].
5.1 Kähler geometry and $U(1)$ gauge theory

Let $M$ be a two-dimensional complex manifold with a Kähler metric

$$ds^2 = h_{ij}(z, \bar{z}) dz^i d\bar{z}^j,$$

(5.1)

where local complex coordinates are given by $z^i = x^{2i} + i x^{2i-1}$, $(i = 1, 2)$. A Kähler manifold is described by a single function $K(z, \bar{z})$, so-called Kähler potential, defined by

$$h_{ij}(z, \bar{z}) = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^j}.$$  

(5.2)

The Kähler potential is not unique but admits a Kähler transformation given by

$$K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})$$

(5.3)

where $f(z)$ and $\bar{f}(\bar{z})$ are arbitrary holomorphic and anti-holomorphic functions. Two Kähler potentials related by the Kähler gauge transformation (5.3) give rise to the same Kähler metric (5.2).

**Definition 10 (Kähler form [30]).** Given a Kähler metric (5.1), the Kähler form is a fundamental closed two-form defined by

$$\Omega = i h_{ij}(z, \bar{z}) dz^i \wedge d\bar{z}^j.$$  

(5.4)

Note that the Kähler form (5.4) can be written as

$$\Omega = dA \quad \text{and} \quad A = \frac{i}{2} (\bar{\partial} - \partial) K(z, \bar{z})$$

(5.5)

where the exterior differential operator is given by $d = \partial + \bar{\partial}$ with $\partial = dz^i \frac{\partial}{\partial z^i}$ and $\bar{\partial} = d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}$. Then the above Kähler transformation (5.3) corresponds to a gauge transformation for the one-form $A$ given by

$$A \rightarrow A + d\lambda$$

(5.6)

where $\lambda = \frac{i}{2} (\bar{f}(\bar{z}) - f(z))$. This implies that the one-form $A$ corresponds to $U(1)$ gauge fields or a connection of holomorphic line bundle. Note that the Kähler form $\Omega$ on a Kähler manifold $M$ is a nondegenerate, closed two-form. Therefore the Kähler form $\Omega$ is a symplectic two-form. This fact leads to the following proposition:

**Proposition 5.1.** A Kähler manifold $(M, \Omega)$ is a symplectic manifold too although the reverse is not necessarily true.

The Kähler condition enforces a specific analytic characterization of Kähler metrics:

**Lemma 5.2.** $ds^2$ is Kähler if and only if it osculates to order 2 to the Euclidean metric everywhere.
The proof of this lemma can be found in [35] (Griffiths-Harris, p. 107). It means that the existence of normal holomorphic coordinates around each point of $M$ is equivalent to that of Kähler metrics.

Let us consider an atlas $\{(U_\alpha, \varphi_\alpha)|\alpha \in I\}$ on the Kähler manifold $M$ and denote the Kähler form $\Omega$ restricted on a chart $(U_\alpha, \varphi_\alpha)$ as $\omega_\alpha \equiv \Omega|_{U_\alpha}$. According to the Lemma 5.2 it is possible to write the local Kähler form as

$$\omega_\alpha = B + F_\alpha, \quad (5.7)$$

where $B$ is the Kähler form of $\mathbb{C}^2$. Since the two-form $F_\alpha$ must be closed due to the Kähler condition, it can be represented by $F_\alpha = dA_\alpha$. Using Eq. (5.5) and $F_\alpha = \omega_\alpha - B$, the one-form $A_\alpha$ on $U_\alpha$ can be written as the form

$$A_\alpha = \frac{i}{2}(\bar{\partial} - \partial)\phi_\alpha(z, \bar{z}) \quad (5.8)$$

where $\phi_\alpha(z, \bar{z}) = K_\alpha(z, \bar{z}) - K_0(z, \bar{z})$ and $K_\alpha(z, \bar{z})$ is the Kähler potential on a local chart $U_\alpha$ and $K_0(z, \bar{z}) = z^\alpha \bar{z}^\beta$ is the Kähler potential of $\mathbb{C}^2$. On an overlap $U_\alpha \cap U_\beta$, two one-forms $A_\alpha$ and $A_\beta$ can be glued using the freedom (5.6) such that

$$A_\beta = A_\alpha + d\lambda_{\alpha\beta} \quad (5.9)$$

where $\lambda_{\alpha\beta}(z, \bar{z})$ is a smooth function on the overlap $U_\alpha \cap U_\beta$. The gluing (5.9) on $U_\alpha \cap U_\beta$ is equal to the Kähler transformation

$$K_\beta(z, \bar{z}) = K_\alpha(z, \bar{z}) + f_{\alpha\beta}(z) + f_{\alpha\beta}(\bar{z}) \quad (5.10)$$

if $\lambda_{\alpha\beta}(z, \bar{z}) = \frac{1}{2}(\bar{f}_{\alpha\beta}(\bar{z}) - f_{\alpha\beta}(z))$.

**Remark 4.** The Kähler transformation (5.10) implies the relation

$$e^{K_\beta} = |e^{f_{\alpha\beta}}|^2 e^{K_\alpha}.$$

So $e^{K(z, \bar{z})}$ is a section of a nontrivial line bundle over $M$.

According to the proposition (5.1), the Kähler manifold $(M, h)$ is also a symplectic manifold $(M, \Omega)$. Therefore one can find a coordinate transformation $\varphi_\alpha \in \text{Diff}(U_\alpha)$ on a local coordinate patch $U_\alpha$ such that $\varphi_\alpha^*(B + F) = B$ according to the famous Darboux theorem or Moser lemma in symplectic geometry [36]. In other words, the electromagnetic fields in the local Kähler form (5.7) can always be eliminated by a local coordinate transformation. To be specific, the Darboux theorem ensures the existence of the local coordinate transformation $\varphi_\alpha : y^\mu \mapsto x^a = x^a(y)$, $\mu, a = 1, \cdots, 4$, obeying [37, 38]

$$\left(B_{ab} + F_{ab}(x)\right)\frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = B_{\mu\nu}. \quad (5.11)$$

Note that $B_{ab}$ and $B_{\mu\nu}$ are constant since they are coming from the Kähler form of $\mathbb{C}^2 \cong \mathbb{R}^4$ according to (5.7) (see also the Lemma 5.2).
Remark 5. So far the coordinates $x^\mu$ have been commonly used for both gravity and field theory descriptions since it does not cause any confusion. However, it is convenient to distinguish two kinds of coordinates $(x^a, y^\mu)$ appearing in the Darboux transformation (5.11). The so-called Darboux coordinates $y^\mu$ will be used for field theory description while the so-called covariant coordinates $x^a$ will be used for gravity description.

Definition 11 (Poisson bracket [36]). Let $	heta := B^{-1} = \frac{1}{2} \theta^{\mu \nu} \frac{\partial}{\partial y^\nu} \wedge \frac{\partial}{\partial y^\mu} \in \Gamma(\Lambda^2 T\mathbb{R}^4)$ be a Poisson bivector. Then the Poisson bracket $\{ \cdot, \cdot \} : C^\infty(\mathbb{R}^4) \times C^\infty(\mathbb{R}^4) \to C^\infty(\mathbb{R}^4)$ is defined by $\{ f, g \} = \theta(df, dg)$ for any smooth functions $f, g \in C^\infty(\mathbb{R}^4)$.

Since both sides of Eq. (5.11) are invertible, one can take its inverse and derive the following relation

$$\Theta^{ab}(x) := \left( \frac{1}{B + F(x)} \right)^{ab} = \theta^{\mu \nu} \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} = \{ x^a(y), x^b(y) \}$$

(5.12)

or

$$- \left( \frac{1}{1 + F\theta} \right)_{ab}(x) = \{ \phi_a(y), \phi_b(y) \}$$

(5.13)

where $\phi_a(y) := B_{ab} x^b(y)$. Recall that we have started with a Kähler manifold with the metric (5.1) and applied the Darboux transformation to the local Kähler form (5.7). Now, in the description (5.12) or (5.13), the curving of the Kähler manifold is described by local fluctuations of $U(1)$ gauge fields on the line bundle $L \to \mathbb{R}^4$. This becomes more manifest by taking the coordinate transformation in Eq. (5.11) as the form

$$\phi_\mu(y) = p_\mu + a_\mu(y)$$

(5.14)

and by calculating the Poisson bracket

$$\{ \phi_\mu(y), \phi_\nu(y) \} = -B_{\mu \nu} + \partial_\mu a_\nu(y) - \partial_\nu a_\mu(y) + \{ a_\mu(y), a_\nu(y) \} \equiv -B_{\mu \nu} + f_{\mu \nu}(y).$$

(5.15)

The functions $a_\mu(y)$ in the Darboux transformation (5.14) will be regarded as gauge fields whose field strength is given by $f_{\mu \nu}(y) = \partial_\mu a_\nu(y) - \partial_\nu a_\mu(y) + \{ a_\mu(y), a_\nu(y) \}$ [2]. Since they respect the non-Abelian structure due to the underlying Poisson structure, they are different from ordinary $U(1)$ gauge fields $A_\mu(x)$ in (5.5), so they will be called “symplectic” $U(1)$ gauge fields. Then Eq. (5.13) leads to the exact Seiberg-Witten map between commutative $U(1)$ gauge fields and symplectic $U(1)$ gauge fields [21, 37, 38]:

$$f_{\mu \nu}(y) = \left( \frac{1}{1 + F\theta} \right)_{\mu \nu}(x) \quad \text{or} \quad F_{\mu \nu}(x) = \left( \frac{1}{1 - f\theta f} \right)_{\mu \nu}(y).$$

(5.16)

Thus the following Lemma is conferred [12, 39, 40]:

\[ \text{Here } a_\mu \text{ is a gauge field of a new } U(1) \text{ gauge symmetry with the Poisson structure rather than the original } U(1) \text{ gauge symmetry. From the original } U(1) \text{ gauge theory point of view, they are local sections of the line bundle } L \to \mathbb{R}^4. \]
Lemma 5.3. The Darboux transformation $\varphi_\alpha \in \text{Diff}(U_\alpha)$ on a local coordinate patch $U_\alpha$ obeying $\varphi_\alpha^*(B + F) = B$ is equivalent to the Seiberg-Witten map between commutative $U(1)$ gauge fields and symplectic $U(1)$ gauge fields.

The gauge theory description of Kähler gravity is realized by viewing a Kähler manifold as a phase space and its Kähler form as the symplectic two-form on the phase space [10]. This viewpoint naturally leads to a Poisson algebra $\mathfrak{P} = \{C^\infty(\mathbb{R}^4), \theta\}$ associated with the Kähler geometry we have started with. The underlying Poisson structure is inherited from the symplectic structure, i.e. $\theta = B^{-1} \in \Gamma(\Lambda^2 T\mathbb{R}^4)$, which is a bivector field called the Poisson tensor.

5.2 Kähler metric and Bianchi identity

Recall that the Seiberg-Witten map (5.16) has been derived from the local Kähler form (5.7). With the identification $\omega^\pm = f^\pm$ and using the map (5.16), the metric $g^\pm$ in the Definition 3 can be written as

$$g^\pm = 2F^\pm \theta^\mp + E_4$$

which can be inverted to yield

$$F^\pm = \frac{1}{2}(g^\pm - E_4)(\theta^\mp)^{-1}. \quad (5.18)$$

Now we will prove the following proposition [27].

Proposition 5.4. Let $F$ be a two-form in (5.18). Then the Kähler condition for the metric $g$ in (5.17) is equivalent to the Bianchi identity for the $U(1)$ curvature $f$.

Proof. First note that the Kähler condition for the metric $g$ in (5.17) is the closedness of the fundamental two-form $\omega = B + F$, which is equal to $dF = 0$. Consider the Jacobi identity

$$\{x^a, \{x^b, x^c\}\} + \{x^b, \{x^c, x^a\}\} + \{x^c, \{x^a, x^b\}\} = 0 \quad (5.19)$$

that is equivalent to the Bianchi identity of symplectic $U(1)$ gauge fields

$$D_a f_{bc} + D_b f_{ca} + D_c f_{ab} = 0, \quad (5.20)$$
where \( D_a f_{bc} = \partial_a f_{bc} + \{ a, f_{bc} \} \). Using Eq. \([5.13]\), let us rewrite the Jacobi identity \([5.19]\) as

\[
0 = \{ x^a, \Theta^{bc}(x) \}_\theta + \{ x^b, \Theta^{ca}(x) \}_\theta + \{ x^c, \Theta^{ab}(x) \}_\theta \\
= \partial_{x^a} \frac{\partial \Theta^{bc}(x)}{\partial y^\mu} + \partial_{x^b} \frac{\partial \Theta^{ca}(x)}{\partial y^\mu} + \partial_{x^c} \frac{\partial \Theta^{ab}(x)}{\partial y^\mu} \\
= \partial_{x^a} \frac{\partial \Theta^{bc}(x)}{\partial x^d} + \partial_{x^b} \frac{\partial \Theta^{ca}(x)}{\partial x^d} + \partial_{x^c} \frac{\partial \Theta^{ab}(x)}{\partial x^d} \\
= \{ x^a, x^d \}_\theta \frac{\partial \Theta^{bc}(x)}{\partial x^d} + \{ x^b, x^d \}_\theta \frac{\partial \Theta^{ca}(x)}{\partial x^d} + \{ x^c, x^d \}_\theta \frac{\partial \Theta^{ab}(x)}{\partial x^d} \\
= \Theta^{ad}(x) \frac{\partial \Theta^{bc}(x)}{\partial x^d} + \Theta^{bd}(x) \frac{\partial \Theta^{ca}(x)}{\partial x^d} + \Theta^{cd}(x) \frac{\partial \Theta^{ab}(x)}{\partial x^d} \\
= -\Theta^{ad} \Theta^{bc} \Theta^{ef} \left( \frac{\partial F_{ef}(x)}{\partial x^d} + \frac{\partial F_{fd}(x)}{\partial x^e} + \frac{\partial F_{de}(x)}{\partial x^f} \right). \tag{5.21}
\]

Since \( \Theta^{ab} \) is invertible, we get from \([5.21]\) the Bianchi identity for the \((1)\) curvature \( F \), i.e.,

\[
\frac{\partial F_{bc}(x)}{\partial x^a} + \frac{\partial F_{ca}(x)}{\partial x^b} + \frac{\partial F_{ab}(x)}{\partial x^c} = 0 \quad \iff \quad dF = 0. \tag{5.22}
\]

The same argument shows that the reverse is also true, i.e., if \( dF = 0 \), the Bianchi identity \([5.20]\) is deduced. This completes the proof. \(\square\)

If one introduces a new bivector \( \Theta = \frac{1}{2} \Theta^{ab}(x) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} \in \Gamma(\Lambda^2 T \mathbb{R}^4) \) using the Poisson tensor in \([5.12]\), Eq. \([5.21]\) shows that the Schouten-Nijenhuis bracket of the bivector \( \Theta \in \Gamma(\Lambda^2 T \mathbb{N}) \) identically vanishes, i.e., \( [\Theta, \Theta]_{SN} = 0 \). This means that the bivector \( \Theta \) defines a new Poisson structure on \( \mathbb{R}^4 \cong \mathbb{C}^2 \). We thus see that the Bianchi identity for symplectic \((1)\) gauge fields leads to the Bianchi identity of commutative \((1)\) gauge fields and vice versa. Since the Bianchi identity \([5.22]\) can be understood as the Kähler condition for the local Kähler form \([5.7]\), the Hermitian-Einstein metrics defined by \( g = \omega \cdot J \) must be Kähler.

Let us quantize the Poisson algebra \( \mathfrak{P} \) to get a noncommutative algebra and a corresponding noncommutative \((1)\) gauge theory. We apply the deformation quantization \( \mathcal{Q} \) in Appendix A and define the quantization map for symplectic \((1)\) gauge fields \([3]\):

\[
\mathcal{Q}(\phi_\mu) := \hat{\phi}_\mu(y) = p_\mu + \hat{A}_\mu(y), \\
\mathcal{Q}([\phi_\mu, \phi_\nu]) := -i[\hat{\phi}_\mu(y), \hat{\phi}_\nu(y)] = -i(-B_{\mu\nu} + \widehat{F}_{\mu\nu}(y)), \tag{5.23}
\]

where \( \mathcal{Q}(f_{\mu\nu}) := \widehat{F}_{\mu\nu}(y) = \partial_\mu \hat{A}_\nu(y) - \partial_\nu \hat{A}_\mu(y) - i[\hat{A}_\mu(y), \hat{A}_\nu(y)] \) is the field strength of noncommutative \((1)\) gauge fields \( \hat{A}_\mu(y) := \mathcal{Q}(a_\mu) \). After quantization, the symplectic \((1)\) gauge fields map to noncommutative \((1)\) gauge fields which contain infinitely many derivative corrections controlled by the noncommutative parameter \( \theta^{\mu\nu} \). For example, the Seiberg-Witten
map (5.16) receives noncommutative corrections and takes a non-local form whose exact form was conjectured in [37]:

\[ F_{\mu\nu}(k) = \int d^4y L^* \left[ \sqrt{1 - \theta \hat{F} (\frac{1}{1 - F \theta} \hat{F})_{\mu\nu}(y) W(y, C)} \right] e^{ik \cdot y}, \] (5.24)

where \( W(x, C) \) is a straight open Wilson line, the determinant and rational function of \( \hat{F} \) should be understood as a power series expansion, and \( L^* \) denotes the integrations together with the path ordering procedure. The conjectured form (5.24) was immediately proved in [58, 59]. In a commutative limit where the derivatives of the field strength can be ignored, the map (5.24) is reduced to the second form in (5.16).

An immediate question arises about the status of Proposition 5.4 after (deformation) quantization. Let us state the result with the following proposition.

**Proposition 5.5.** Let \( F \) be a two-form in (5.24). Then the closedness condition for the commutative \( U(1) \) curvature \( F, dF = 0 \), is equivalent to the Bianchi identity for the noncommutative \( U(1) \) curvature \( \hat{F} \).

This proposition was proved in [58] by proving the conjecture by H. Liu. Theorem 3.2 implies that the Hermitian metric \( h^\pm \) in (3.2) constructed by the identification \( \omega^\pm = \hat{F}^\pm \) still generates a Ricci-flat metric. Therefore Proposition 5.4 may be lifted to noncommutative spaces although we do not have a rigorous proof yet.

## 6 Discussion

We have shown that the Kähler geometry can be described by a \( U(1) \) gauge theory on a symplectic manifold leading to a natural Poisson algebra associated with the Kähler geometry we have started with. Since the Poisson algebra \( \mathfrak{P} \) defined by the Poisson bracket \( \{ f, g \} = \theta(df, dg) \) is mathematically the same as the one in Hamiltonian dynamics of particles, one can quantize the Poisson algebra in the exactly same way as quantum mechanics. Hence we have applied the deformation quantization to the Poisson algebra \( \mathfrak{P} = (C^\infty(\mathbb{R}^4), \{-, -\}) \). The quantization of the underlying Poisson algebra leads to a noncommutative \( U(1) \) gauge theory which arguably describes a quantized Kähler geometry, as claimed in [10] and illuminated in [5]. Then we get a remarkable duality between Kähler gravity and noncommutative \( U(1) \) gauge theory depicted by the following flow chart [5]:

\[
\begin{array}{ccc}
\text{Kähler gravity} & \overset{\text{\(2\gamma^{-1}\)}}{\longrightarrow} & \text{Symplectic \( U(1) \) gauge theory} \\
\mathcal{Q} \downarrow & & \downarrow \mathcal{Q} \\
\text{Quantized Kähler gravity} & \overset{\text{\(\theta\)}}{\longleftarrow} & \text{Noncommutative \( U(1) \) gauge theory}
\end{array}
\] (6.1)
Here $Q : C^\infty(\mathbb{R}^4) \to \mathcal{A}_\theta$ means the quantization and $\mathcal{I}$ means an isomorphism between two theories. In some sense $\mathcal{I}$ corresponds to the gauge-gravity duality. It turns out [5] that it can be interpreted as the large $N$ duality too. Since symplectic $U(1)$ gauge theory is a commutative limit of noncommutative $U(1)$ gauge theory, we understand the classical isomorphism in (6.1) as $\mathcal{I}_\epsilon = \mathcal{I}_{\theta}|_{\epsilon = |\theta| \to 0}$. The duality in (6.1) implies that a quantized Kähler gravity is isomorphically described by a noncommutative $U(1)$ gauge theory. Actually this relation was already observed in [10] in the context of topological strings probing Kähler manifolds where several nontrivial evidences have been analyzed to support the picture. In particular, the authors in [10] argue that noncommutative $U(1)$ gauge theory is the fundamental description of Kähler gravity at all scales including the Planck scale and provides a quantum gravity description such as quantum gravitational foams. The duality in [10] has been further clarified in [42] by showing that it follows from the S-duality of the type IIB superstring.

This duality, if any, suggests an important clue about how to quantize the Kähler gravity. Surprisingly, the correct variables for quantization are not metric fields but dynamical coordinates $x^a(y)$ and their quantization is defined in terms of $\alpha'$ rather than $\hbar$. So far, there is no well-established clue to quantize metric fields directly in terms of $\hbar$ in spite of impressive developments in loop quantum gravity. However, the picture in (6.1) suggests a completely new quantization scheme where quantum gravity is defined by quantizing spacetime itself in terms of $\alpha'$, leading to a dynamical noncommutative spacetime described by a noncommutative $U(1)$ gauge theory [5].

The duality relation in (6.1) may be more accessible with the corresponding relation for solutions of the self-duality equation, i.e., $U(1)$ instantons. Indeed it was shown in [1, 3, 4] that the commutative limit of noncommutative $U(1)$ instantons are equivalent to Calabi-Yau manifolds.

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A Deformation quantizations with separation of variables

We summarize deformation quantization for Poisson manifolds and Kähler manifolds in Appendix A.

**Definition 12** (Deformation quantization of Poisson manifolds). Let $M$ be a Poisson manifold and $C^\infty(M)[[\zeta]]$ be a set of formal power series: $C^\infty(M)[[\zeta]] := \{ f \mid f = \sum f_k \zeta^k, \ f_k \in C^\infty(M) \}$. 

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A star product $\star : C^\infty (M) [[\zeta]] \times C^\infty (M) [[\zeta]] \to C^\infty (M) [[\zeta]]$ is defined as

$$f \star g = \sum_k C_k(f, g) \zeta^k$$

such that the product satisfies the following (i)~(iv) conditions. (i) $(C^\infty (M) [[\zeta]], +, \star)$ is a (noncommutative) algebra. (ii) $C_k(\cdot, \cdot)$ is a bidifferential operator. (iii) $C_0$ and $C_1$ are defined as

$$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = \{f, g\},$$

where $\{f, g\}$ is the Poisson bracket of $M$. (iv) $f \star 1 = 1 \star f = f$.

$(C^\infty (M) [[\zeta]], +, \star)$ is called a deformation quantization of the Poisson manifold $M$.

Karabegov introduced a method to obtain a deformation quantization of a Kähler manifold in [9][43]. His deformation quantization is called deformation quantizations with separation of variables.

**Definition 13** (A star product with separation of variables). Let $\star$ be a star product on a Kähler manifold as a Poisson manifold. The $\star$ is called a star product with separation of variables on a Kähler manifold when

$$a \star f = af$$

for an arbitrary holomorphic function $a$ and

$$f \star b = fb$$

for an arbitrary anti-holomorphic function $b$.

The star product on $\mathbb{C}^2$ constructed by Karabegov’s deformation quantization is given as

$$f \star g = \sum_{n=0}^\infty \frac{\zeta^n}{n!} \delta^{k_1 l_1} \cdots \delta^{k_n l_n} (\partial_{k_1} \cdots \partial_{k_n} f) (\partial_{l_1} \cdots \partial_{l_n} g). \quad (A.4)$$

In this article we made Ricci-flat metrics from (anti-)self-dual two-forms on a noncommutative manifold. A formal power series of symmetric two-form is not defined as a metric in ordinary sense. For this reason we made Ricci-flat metrics from instantons on $\mathcal{F}$ instead of the noncommutative $\mathbb{C}^2$ described as a formal power series, in this article. Here $\mathcal{F}$ is a noncommutative algebra constructed in Section 4 following the method in [8][44]. The product of the algebra $\mathcal{F}$ is given by the star product (A.4). Then we obtain some Ricci flat metrics on $\mathbb{C}^2$.  

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B Noncommutative $U(1)$ instanton in the Fock space

In Appendix B, we make a short review of the method to make a $U(1)$ instanton solution in \[34\] and multi instanton solutions in \[7\].

In noncommutative $\mathbb{R}^4$, Nekrasov and Schwarz found how to construct instanton gauge fields \[6\] by using the ADHM construction \[45\]. Their work has encouraged studies of noncommutative ADHM instantons. (See, for example, \[7, 46, 47\].) Another method to construct noncommutative instantons as smooth deformations of commutative instantons was provided in \[48, 49, 50\]. The correspondence between the smooth deformation and the ADHM construction are discussed in \[51\]. On the other hand, there exist instanton solutions which are not smoothly connected to commutative instantons. The commutative limit of the noncommutative instantons are discussed in \[52, 53\].

Noncommutative instantons are labeled by topological charge called instanton number. The topological number of the noncommutative instanton is studied in \[46, 54, 55, 56, 57\]. It is shown that the topological number coincides with the dimension of a vector space appearing in the ADHM construction. In \[55\], this identification is shown when the noncommutative parameter is self-dual for a $U(N)$ gauge theory. In \[56\], the equivalence is shown with no restrictions on the noncommutative parameters, but a noncommutative version of the Osborn’s identity (Corrigan’s identity) is assumed. In \[53\] final version of the proof was announced.

In Definition 9 a covariant derivative and gauge curvature are given as follows. Covariant derivatives for scalar field $\phi \in \mathcal{F}$ on noncommutative $\mathbb{C}^2$ are defined as $\hat{\nabla}_z \hat{\phi} := \left[ \hat{\partial}_z , \hat{\phi} \right] + \hat{A}_z \hat{\phi} = -\hat{\phi} \hat{\partial}_z + \hat{D}_z \hat{\phi}$ where we define a local gauge field $\hat{A}_z \in \mathcal{F}$ and $\hat{D}_z := \hat{\partial}_z + \hat{A}_z$. The gauge curvature is defined as $\hat{F}_{z\bar{z}} := i \left[ \hat{\nabla}_z , \hat{\nabla}_{\bar{z}} \right] = -\delta_{\mu\nu} \hat{\epsilon} + i \left[ \hat{D}_z , \hat{D}_{\bar{z}} \right]$.

Using this notation, we introduce the ADHM construction in the following.

B.1 Noncommutative ADHM construction

Definition 14. Let $B_1, B_2 \in \mathbb{C}^{k \times k}$, $I \in \mathbb{C}^{k \times N}$, $J \in \mathbb{C}^{N \times k}$ be matrices satisfying

$$
\mu_\mathbb{C} := \left[ B_1, B_2^\dagger \right] + IJ = 0, \quad \mu_\mathbb{R} := \left[ B_1, B_1^\dagger \right] + \left[ B_2, B_2^\dagger \right] + JJ^\dagger - J^\dagger J = (\zeta_1 + \zeta_2) E_k. \quad (B.1)
$$

These equations are called the deformed ADHM equations. Here $\zeta_1$, $\zeta_2$ are noncommutative parameter in \[4.1\].

Let $E_k \in \mathbb{C}^{k \times k}$ be a unit matrix. We put $\beta_1, \beta_2 \in \mathbb{C}^{k \times k}, \tau \in \mathbb{C}^{k \times (2k+1)}, \sigma \in \mathbb{C}^{(2k+1) \times k}, \mathcal{O} \in \mathbb{C}^{(2k+1) \times (2k+1)}).$
\(C^{(2k+1)\times 2k}\) as
\[
\beta_j := \frac{B_j}{\sqrt{\zeta_j}}, \quad \tau := (B_2 - z_2E_k, B_1 - z_1E_k, I), \quad \sigma := (-B_1 + z_1E_k, B_2 - z_2E_k, I)^T
\]
\[
\mathcal{D}^\dagger := \begin{pmatrix} \tau \\ \sigma^\dagger \end{pmatrix} = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \\ -B_1^\dagger + z_1 & B_2^\dagger - z_2 & J^\dagger \end{pmatrix}.
\]

The first step of the ADHM construction is solving the deformed ADHM equations (B.1). The second step of the ADHM construction is solving the equation \(D^\dagger * \Psi = 0, \quad \Psi^\dagger * \Psi = 1\). The third step of the ADHM construction is constructing gauge field \(\hat{A}\) as \(\hat{A}_{z_l} := \Psi^\dagger * \partial_{z_l} \Psi\) where \(\Psi\) is a solution of the equations in the second step.

Then the curvature tensor \(\tilde{F}_{z_l\bar{z}_m}\) constructed from \(\hat{A}_{z_l}, \hat{A}_{\bar{z}_m}\) is self-dual that means \(\tilde{F}_{z_l\bar{z}_m}\) is an instanton curvature tensor.

For the \(U(1)\) case, this construction process can be expressed more explicitly.

Assume
\[
\Psi := \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{\zeta_2} (\beta_2^\dagger - a_2) v \\ \sqrt{\zeta_1} (\beta_1^\dagger - a_1) v \\ \xi \end{pmatrix}
\] \hspace{1cm} (B.2)
\[
\hat{\Delta} := \zeta_1 (\beta_1 - a_1^\dagger) (\beta_1^\dagger - a_1) + \zeta_2 (\beta_2 - a_2^\dagger) (\beta_2^\dagger - a_2),
\] \hspace{1cm} (B.3)

where \(\xi \in \mathcal{F}, \quad v \in \mathbb{C}^k \otimes \mathcal{F}\). \(\mathcal{F}\) is defined in (4.3), and \((\beta^\dagger_a - a_l) v := (\beta^\dagger_a \otimes id - E_k \otimes a_l) v\), where \(id\) is an identity mapping.

A vector space \(\mathcal{H}\) is defined using (4.2) as
\[
\mathcal{H} := \text{span}_\mathbb{C} (|0, 0\rangle, |1, 0\rangle, |0, 1\rangle, |1, 1\rangle, |2, 2\rangle, \cdots).
\]

**Definition 15.** A linear operators \(P\) on \(\mathcal{H}\) is defined as
\[
P := I^\dagger \left( \exp \sum_{\alpha} \beta^\dagger_{\alpha} a_{\alpha} \right) |0, 0\rangle G^{-1} \langle 0, 0| \left( \exp \sum_{\alpha} \beta_{\alpha} a_{\alpha} \right) I,
\]
where
\[
G := \langle 0, 0| \left( \exp \sum_{\alpha} \beta_{\alpha} a_{\alpha} \right) I^\dagger \left( \exp \sum_{\alpha} \beta^\dagger_{\alpha} a_{\alpha} \right) |0, 0\rangle.
\]

**Fact B.1.** This linear operator is a projection operator, i.e., \(PP = P\).

A proposition below is true.
Proposition B.2. Let $\Psi, \hat{\Delta}v, \xi$ be ones given above in (B.2). Then,

$$D^\dagger \Psi = 0, \ \Psi^\dagger * \Psi = 1 \iff \hat{\Delta}v + I\xi = 0, \ v^\dagger \hat{\Delta}v + \xi^\dagger \xi = 1.$$  

Lemma B.3. The operator $S$ which satisfies $SS^\dagger = id, S^\dagger S = id - P$ exists. Let $\Lambda$ be id + $I^\dagger \hat{\Delta}^{-1} I$. If we put

$$\xi = \Lambda^{-1/2} S^\dagger, \ v = -\hat{\Delta}^{-1} I\xi \quad (B.4)$$

then

$$\hat{\Delta}v + I\xi = 0, \ v^\dagger \hat{\Delta}v + \xi^\dagger \xi = 1. \quad (B.5)$$

This lemma means, if we find $\Lambda^{-1/2}$ and $\hat{\Delta}^{-1}$, then we can find a solution.

We define operators $\hat{\partial}_{zi}$ and $\hat{D}_{zi}$ on $\mathcal{H}$ in Section 4 as

$$\hat{\partial}_{zi} := \frac{\bar{z}_i}{\xi_l}, \quad \hat{D}_{zi} := \hat{\partial}_{zi} + \hat{A}_{zi}. \quad (B.5)$$

Noncommutative $U(1)$ instanton curvature in the Fock space is also defined as

$$\tilde{F}_{zi\bar{z}_m} := i \left[ \hat{\partial}_{z_m}, \hat{A}_{zi} \right]_* + i \left[ \hat{\partial}_{zi}, \hat{A}_{z_m} \right]_* + i \left[ \hat{A}_{l}, \hat{A}_{\bar{m}} \right]_*.$$  

Using $\hat{D}_{zi}$, $\tilde{F}$ is rewritten as

$$\tilde{F}_{zi\bar{z}_m} = i \left[ \hat{D}_{zi}, \hat{D}_{z_m} \right]_* + \frac{i\delta_{im}}{\xi_l}. \quad (B.6)$$

Assume $\hat{A}_{zi} := \Psi^\dagger \partial_{zi} \Psi, \ \hat{\partial}_{zi} := \Psi^\dagger \partial_{zi} \Psi$ then

$$\hat{D}_{zi} = -\frac{1}{\xi_l} \Psi^\dagger z_l \Psi, \quad \hat{D}_{zi} = -\frac{1}{\xi_l} \Psi^\dagger z_l \Psi.$$  

Direct calculations derive the following results.

Theorem B.4. If $\Lambda := id + I^\dagger \hat{\Delta}^{-1} I, \xi = \Lambda^{-1/2} S^\dagger, \ v = -\hat{\Delta}^{-1} I\xi$ then

$$\hat{D}_{zi} = -\frac{1}{\sqrt{\xi_l}} S \Lambda^{-1/2} a_l \Lambda^{1/2} S^\dagger, \quad \hat{D}_{zi} = \frac{1}{\sqrt{\xi_l}} S \Lambda^{1/2} a_l^\dagger \Lambda^{-1/2} S^\dagger.$$  

Theorem B.5. If $\tilde{F}_{zi\bar{z}_i}^-$ is given by (B.6) and $\hat{D}_{zi}, \hat{D}_{zi}$ are defined in Theorem B.4 then

$$\tilde{F}_{zi\bar{z}_i}^- [k] = \frac{i}{\xi_1} - i \frac{S \Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^\dagger S \Lambda^{\frac{1}{2}} a_1^\dagger \Lambda^{-\frac{1}{2}} S^\dagger + i \frac{S \Lambda^{\frac{1}{2}} a_1 \Lambda^{-\frac{1}{2}} S^\dagger S \Lambda^{-\frac{1}{2}} a_1^\dagger \Lambda^{\frac{1}{2}} S^\dagger}{\xi_1}}.$$  

$$\tilde{F}_{zi\bar{z}_i}^- [k] = \frac{i}{\xi_2} - i \frac{S \Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^\dagger S \Lambda^{\frac{1}{2}} a_2^\dagger \Lambda^{-\frac{1}{2}} S^\dagger + i \frac{S \Lambda^{\frac{1}{2}} a_2 \Lambda^{-\frac{1}{2}} S^\dagger S \Lambda^{-\frac{1}{2}} a_2^\dagger \Lambda^{\frac{1}{2}} S^\dagger}{\xi_2}}.$$  

$$\tilde{F}_{zi\bar{z}_i}^- [k] = -\frac{i}{\sqrt{\xi_1 \xi_2}} S \Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^\dagger S \Lambda^{\frac{1}{2}} a_2^\dagger \Lambda^{-\frac{1}{2}} S^\dagger + \frac{i}{\sqrt{\xi_1 \xi_2}} S \Lambda^{\frac{1}{2}} a_2 \Lambda^{-\frac{1}{2}} S^\dagger S \Lambda^{-\frac{1}{2}} a_1^\dagger \Lambda^{\frac{1}{2}} S^\dagger.$$  

$$\tilde{F}_{zi\bar{z}_i}^- [k] = -\frac{i}{\sqrt{\xi_1 \xi_2}} S \Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^\dagger S \Lambda^{\frac{1}{2}} a_1^\dagger \Lambda^{-\frac{1}{2}} S^\dagger + \frac{i}{\sqrt{\xi_1 \xi_2}} S \Lambda^{\frac{1}{2}} a_1 \Lambda^{-\frac{1}{2}} S^\dagger S \Lambda^{-\frac{1}{2}} a_2^\dagger \Lambda^{\frac{1}{2}} S^\dagger.$$  

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This curvature is an instanton curvature.

**B.2 $U(1)$ $k$-instanton in the noncommutative $\mathbb{C}^2$**

In this section we summarize $U(1)$ multi-instanton solutions on $\mathbb{C}^2$ in [7]. For simplicity, let us assume $\zeta_1 = \zeta_2 =: \zeta$.

**Definition 16.** Noncommutative instanton curvature in the noncommutative $\mathbb{C}^2$ is defined as

$$\hat{F}^-_C [k] = \begin{pmatrix} \hat{F}^-_{\overline{z}_1 z_1} [k] & \hat{F}^-_{\overline{z}_1 z_2} [k] \\ \hat{F}^-_{\overline{z}_2 z_1} [k] & -\hat{F}^-_{\overline{z}_1 z_1} [k] \end{pmatrix} := \begin{pmatrix} \iota \left( \hat{F}^-_{\overline{z}_1 z_1} [k] \right) & \iota \left( \hat{F}^-_{\overline{z}_1 z_2} [k] \right) \\ \iota \left( \hat{F}^-_{\overline{z}_2 z_1} [k] \right) & \iota \left( -\hat{F}^-_{\overline{z}_1 z_1} [k] \right) \end{pmatrix}$$

where $\iota$ is defined in Definition 7.

We choose

$$B_1 = \sum_{l=1}^{k-1} \sqrt{2l} \zeta e_l e_{l+1}^\dagger, \quad B_2 = 0, \quad I = \sqrt{2k} \zeta e_k, \quad J = 0$$

as a solution of the deformed ADHM equations (B.1). Here

$$e_l^\dagger = \begin{pmatrix} \delta_{1,l} & \delta_{2,l} & \cdots & \delta_{k-1,l} & \delta_{k,l} \end{pmatrix}.$$ 

In this case, the operator $S^\dagger$ in Lemma B.3 is given by

$$S^\dagger = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} |n_1+k,0\rangle \langle n_1,0| + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} |n_1,n_2\rangle \langle n_1,n_2|.$$  \hspace{2cm} (B.7)

From Theorem B.5 and (B.7), a $U(1)$ $k$-instanton curvature in the noncommutative $\mathbb{C}^2$ is
obtained as follows.

\[
\tilde{F}_{\tilde{z}_1}^{-} [k] = \frac{i}{\zeta} - \frac{i}{\zeta} \sum_{n_2=0}^{\infty} |0, n_2\rangle \langle 0, n_2| \left( d_1 (0, n_2; k) \right)^2 \\
- \frac{i}{\zeta} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| \left\{ (d_1 (n_1, n_2; k))^2 - (d_1 (n_1 - 1, n_2; k))^2 \right\}, \\
\]  

(B.8)

\[
\tilde{F}_{\tilde{z}_2}^{-} [k] = - \tilde{F}_{\tilde{z}_1}^{-} [k] \\
\]  

(B.9)

\[
\tilde{F}_{\tilde{z}_1}^{-} [k] = - \frac{i}{\zeta} |k - 1, 1\rangle \langle 0, 0| d_1 (k - 1, 1; k) d_2 (0, 0; k) \\
- \frac{i}{\zeta} \sum_{n_1=1}^{k-1} |n_1 + k - 1, 1\rangle \langle n_1, 0| \left\{ d_1 (n_1 + k - 1, 1; k) d_2 (n_1, 0; k) - d_1 (n_1 - 1, 0; k) d_2 (n_1 - 1, 0; k) \right\} \\
- \frac{i}{\zeta} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} |n_1 - 1, n_2 + 1\rangle \langle n_1, n_2| \\
\times \left\{ d_1 (n_1 - 1, n_2 + 1; k) d_2 (n_1, n_2; k) - d_1 (n_1 - 1, n_2; k) d_2 (n_1 - 1, n_2; k) \right\} 
\]  

(B.10)

\[
\tilde{F}_{\tilde{z}_2}^{-} [k] = \tilde{F}_{\tilde{z}_1}^{-} [k] \\
\]  

(B.11)

where \( d_1 (n_1, n_2; k) \) and \( d_2 (n_1, n_2; k) \) are given by (4.10)-(4.11).

Next we change these curvature operators into functions on \( \mathbb{C}^2 \) using the isomorphism (4.4).

\[
\tilde{F}_{\tilde{z}_1}^{-} [k] = \tau \left( \tilde{F}_{\tilde{z}_1}^{-} [k] \right) \\
= \frac{i}{\zeta} - \frac{i}{\zeta} \sum_{n_2=0}^{\infty} z_2^{n_2} e^{-\frac{z_1 + z_2}{\zeta}} \frac{z_2^{n_2}}{n_2!} (d_1 (0, n_2; k))^2 \\
- \frac{i}{\zeta} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} \frac{z_1^{n_1} \frac{z_2^{n_2}}{n_2!} e^{-\frac{z_1 + z_2}{\zeta}} \frac{z_1^{n_1} z_2^{n_2}}{n_1! n_2! \zeta^{n_1 + n_2}}}{\zeta^{n_1 + n_2}} \left\{ (d_1 (n_1, n_2; k))^2 - (d_1 (n_1 - 1, n_2; k))^2 \right\}. \\
\]  

(B.12)

\[
\tilde{F}_{\tilde{z}_2}^{-} [k] = \tau \left( \tilde{F}_{\tilde{z}_2}^{-} [k] \right) = - \tilde{F}_{\tilde{z}_1}^{-} [k] \\
\]  

(B.13)
\[
\tilde{F}_{z_1 \bar{z}_2} [k] = \iota \left( \tilde{F}_{z_1 \bar{z}_2} [k] \right) \\
= - \left( \frac{i}{z_1} \frac{z_1^{k-1} z_2 e^{-z_1 \bar{z}_1 - z_2 \bar{z}_2}}{\sqrt{(k-1)!} (\sqrt{\zeta})^k} d_1 (k-1, 1; k) d_2 (0, 0; k) \right) \\
- \left( \frac{i}{\zeta} \sum_{n_1=1}^{k-1} \frac{z_1^{n_1+k-1} z_2 e^{-z_1 \bar{z}_1 - z_2 \bar{z}_2}}{\sqrt{(n_1+k-1)! n_1!} (\sqrt{\zeta})^{2n_1+k}} \times \{d_1 \left(n_1+k-1, 1; k) d_2 (n_1, 0; k) - d_1 \left(n_1-1, 0; k) d_2 (n_1-1, 0; k) \right\} \right) \\
\times \{d_1 \left(n_1-1, n_2+1; k) d_2 (n_1, n_2; k) - d_1 \left(n_1-1, n_2; k) d_2 (n_1-1, n_2; k) \right\} \right) \\
\] 
(B.14)

\[
\tilde{F}_{z_2 \bar{z}_1} [k] = \iota \left( \tilde{F}_{z_2 \bar{z}_1} [k] \right) = - \tilde{F}_{z_1 \bar{z}_2} [k] \\
\] 
(B.15)

where \( \bar{a} \) is a complex conjugate of \( a \).

In order to obtain Ricci-flat metrics in Subsection 4.3 and Subsection 4.4, we need the first three terms of the expansion for \( \tilde{F}^- [k] \) in \( \sqrt{1 / \zeta} \).

\[
\tilde{F}_{z_1 \bar{z}_1} [k] = \iota - \frac{i z_1 z_2}{\zeta^2} \left( d_1 (0, 1; k)^2 - \frac{i z_1 \bar{z}_1}{\zeta} \right) \left( d_1 (1, 0; k)^2 - d_1 (0, 0; k)^2 \right) \\
+ \frac{i z_1 \bar{z}_1 z_2^2}{\zeta^3} \left( d_1 (0, 1; k)^2 - \frac{i z_2^2}{\zeta^2} \right) \left( d_1 (0, 1; k)^2 \right) \\
+ \frac{i z_1 \bar{z}_1 z_2^2}{\zeta^3} \left( \{d_1 (1, 0; k)^2 - d_1 (0, 0; k)^2 \} + \bigO \left( \zeta^{-4} \right) \right), \\
\] 
(B.16)

\[
d_1 (0, 0; k) = \sqrt{\frac{(k+1) \Lambda (k+1, 0)}{\Lambda (k, 0)}}, \quad d_1 (1, 0; k) = \sqrt{\frac{(k+2) \Lambda (k+2, 0)}{\Lambda (k+1, 0)}}, \quad d_1 (0, 1; k) = \sqrt{\frac{\Lambda (1, 1)}{\Lambda (0, 1)}}, \\
\]
and

\[
\hat{F}^{-}_{z_{1}, z_{2}} [k] = - \frac{i}{\zeta (\sqrt{\zeta})} \frac{z_{1}^{k-1} z_{2}}{\sqrt{(k-1)!}} \left( 1 - \frac{z_{1}^{2}}{\zeta} - \frac{z_{2}^{2}}{\zeta} + O (\zeta^{-2}) \right) d_{1} (k-1, 1; k) d_{2} (0, 0; k) \\
- \frac{i}{\zeta (\sqrt{\zeta})} \sum_{n_{1}=1}^{k-1} \frac{z_{1}^{n_{1}+k-1} z_{2}^{n_{1}}}{\sqrt{(n_{1}+k-1)! n_{1}!}} \left( 1 - \frac{z_{1}^{2}}{\zeta} - \frac{z_{2}^{2}}{\zeta} + O (\zeta^{-2}) \right) \times \{ d_{1} (n_{1}+k-1, 1; k) d_{2} (n_{1}, 0; k) - d_{1} (n_{1}-1, 0; k) d_{2} (n_{1}-1, 0; k) \} \\
- \frac{i}{\zeta (\sqrt{\zeta})} \sum_{n_{1}=1}^{\infty} \sum_{n_{2}=1}^{\infty} \frac{z_{1}^{n_{1}-1} z_{2}^{n_{2}+1} z_{1}^{n_{1}+n_{2}}}{\sqrt{(n_{1}+n_{2}+1)! (n_{1}+1)! n_{2}! \zeta^{n_{1}+n_{2}}}} \left( 1 - \frac{z_{1}^{2}}{\zeta} - \frac{z_{2}^{2}}{\zeta} + O (\zeta^{-2}) \right) \times \{ d_{1} (n_{1}-1, n_{2}+1; k) d_{2} (n_{1}, n_{2}; k) - d_{1} (n_{1}-1, n_{2}; k) d_{2} (n_{1}-1, n_{2}; k) \}. 
\]

(B.17)

It is useful to distinguish the cases for \( k = 1 \) and \( k > 1 \).

\[
k = 1 \Rightarrow \hat{F}^{-}_{z_{1}, z_{2}} [1] = - \frac{i z_{1}^{2}}{\sqrt{2} \zeta} \left( 1 - \frac{z_{1}^{2}}{\zeta} - \frac{z_{2}^{2}}{\zeta} \right) d_{1} (0, 1; 1) d_{2} (0, 0; 1) + O (\zeta^{-3}). \quad (B.18)
\]

\[
k > 1 \Rightarrow \hat{F}^{-}_{z_{1}, z_{2}} [k] = - \frac{i z_{1}^{k-1} z_{2}}{\zeta (\sqrt{\zeta})} \frac{1}{\sqrt{(k-1)!}} \left( 1 - \frac{z_{1}^{2}}{\zeta} - \frac{z_{2}^{2}}{\zeta} \right) d_{1} (k-1, 1; k) d_{2} (0, 0; k) \\
- \frac{i z_{2}^{2}}{\sqrt{2} \zeta^{3}} k \{ d_{1} (1, 1; k) d_{2} (1, 0; k) - d_{1} (0, 0; k) d_{2} (0, 0; k) \} \\
- \frac{i z_{2}^{z_{1}^{2} z_{2}^{2}}}{\sqrt{2} \zeta^{3}} \{ d_{1} (0, 2; k) d_{2} (1, 1; k) - d_{1} (0, 1; k) d_{2} (0, 1; k) \} + O (\zeta^{-4}).
\]

(B.19)

Functions \( \Lambda, d_{1}, d_{2} \) for \( k = 1 \) are useful for Subsection 4.4:

\[
\Lambda [1] (n_{1}, n_{2}) = \frac{\omega_{1} (n_{1}, n_{2})}{\omega_{1} (n_{1}, n_{2}) - 2 \omega_{0} (n_{1}, n_{2})} = \frac{2 + n_{1} + n_{2}}{n_{1} + n_{2}},
\]

\[
d_{1} (n_{1}, 0; 1) = \sqrt{n_{1} + 2} \frac{\Lambda [1] (n_{1} + 2, 0)}{\Lambda [1] (n_{1} + 1, 0)} = \sqrt{\frac{4 + n_{1} (1 + n_{1})}{(3 + n_{1})}},
\]

\[
d_{1} (n_{1}, n_{2}; 1) = \sqrt{n_{1} + 1} \frac{\Lambda [1] (n_{1} + 1, n_{2})}{\Lambda [1] (n_{1}, n_{2})} = \sqrt{\frac{(n_{1} + 1) (3 + n_{1} + n_{2}) (n_{1} + n_{2})}{(1 + n_{1} + n_{2}) (2 + n_{1} + n_{2})}},
\]

\[
d_{2} (n_{1}, 0; 1) = \left\{ \frac{\Lambda [1] (n_{1} + 1, 1)}{\Lambda [1] (n_{1} + 1, 0)} \right\}^{\frac{1}{2}} = \frac{(n_{1} + 4) (n_{1} + 1)}{(n_{1} + 2) (n_{1} + 3)}.
\]

\[
d_{2} (n_{1}, n_{2}; 1) = \sqrt{\frac{(n_{2} + 1) \Lambda [1] (n_{1}, n_{2} + 1)}{\Lambda [1] (n_{1}, n_{2})}} = \sqrt{\frac{(n_{2} + 1) (n_{1} + n_{2}) (3 + n_{1} + n_{2})}{(n_{1} + n_{2} + 1) (2 + n_{1} + n_{2})}}.
\]

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