Wealth Effect on Portfolio Allocation in Incomplete Markets

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Abstract

We develop a novel five-component decomposition of optimal dynamic portfolio choice, which reveals the simultaneous impacts from market incompleteness and wealth-dependent utilities. Under the HARA utility and a nonrandom interest rate, we can explicitly solve for the optimal policy as a combination of a bond holding scheme and the corresponding simpler CRRA strategy. Under a stochastic volatility model estimated on US equity data, we use closed-form solution to demonstrate the sophisticated impacts from the wealth-dependent utilities, including cycle-dependence and hysteresis effect in optimal portfolio allocation, as well as a risk-return trade-off in investment performance.

Keywords: optimal portfolio choice, stochastic volatility, incomplete market, wealth-dependent utility, closed-form.

JEL Codes: C61, C63, G11.

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1 Introduction

Optimal portfolio choice is a central topic in modern financial economics, drawing continuous attention from both academia and industry. Hedge funds, asset management firms, and pension funds, which manage large positions of portfolios, as well as individual investors, are confronted with this type of decisions frequently. The optimal portfolio choice problem has also drawn long-standing interest in academic research. The celebrated static mean-variance framework of Markowitz (1952) laid a foundation. Following the seminal work by Samuelson (1969) and Merton (1969, 1971), various studies have been developed for the optimal dynamic portfolio choice; see the comprehensive surveys in, e.g., Detemple (2014), Brandt (2010), and Wachter (2010), as well as the references therein. As an optimal stochastic control problem in a continuous-time setting, studying optimal policies usually relies on two approaches. The first one is the well-known dynamic programming method, which employs the highly nonlinear Hamilton-Jacobi-Bellman (HJB hereafter) equation to characterize the optimal policy. The second one is the martingale method pioneered and developed by, e.g., Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), Ocone and Karatzas (1991), Cvitanic and Karatzas (1992), Detemple et al. (2003), and Koijen (2014) among others.

For the purpose of understanding optimal policies from an economic perspective, the decomposition of optimal policies into mean variance and hedging components was initiated by Merton (1971) and matured as a state-of-the-art approach; see, e.g., Detemple et al. (2003), Detemple and Rindisbacher (2005, 2010), and Detemple (2014).\(^1\) For the purpose of implementing and analyzing the behavior of optimal portfolios, existing works largely focus on specific affine models (see, e.g., Duffie et al. (2000)) and wealth-independent utilities, such as the basic constant relative risk aversion (CRRA) utility and its generalization to the recursive utility by separating risk aversion and intertemporal elasticity of substitution (see, e.g., Epstein and Zin (1989) and Duffie and Epstein (1992)). While these specifications can bring mathematical convenience, e.g., closed-form optimal portfolio policies in some rare and limited cases\(^2\), they limit the model capacity to capture empirically flexible market dynamics and investor preferences, e.g., wealth-dependent utilities. As an effective method that can be applied to flexible diffusion models without closed-form policies, Detemple et al. (2003) develop a Monte Carlo simulation approach based on the above-mentioned decomposition of

\[^1\]See also Basak and Chabakauri (2010) for establishing an analytical characterization of optimal portfolios for mean-variance investors.

\[^2\]See, e.g., Kim and Omberg (1996) and Wachter (2002) for modeling stochastic market price of risk of the asset by using an Ornstein-Uhlenbeck model, Liou and Poncet (2001) for considering stochastic interest rates by employing a constant-parameter instantaneous forward rate model, Liu and Pan (2003) for discussing dynamic derivative strategies, Liu et al. (2003) for studying impacts of event risk via affine stochastic volatility models with jumps, Liu (2007) for taking various stochastic environments (e.g., stochastic volatility) into account by modeling the asset return via quadratic affine processes, and Burraschi et al. (2010) for characterizing hedging components against both stochastic volatility and correlation risks under Wishart processes.
the optimal policy. However, this milestone of methods is by far limited to the complete market setting.\(^3\) Now, a notable open question is to obtain an economically insightful decomposition of optimal policies under the more flexible and realistic setting with general incomplete market dynamics and wealth-dependent utilities, e.g., the hyperbolic absolute risk aversion (HARA) utility. We expect such a decomposition to reflect the fundamental impacts from these two effects at the theoretical level, i.e., how market incompleteness and wealth-dependent utility, as well as their interaction, affect the optimal policy. Subsequently, the decomposition facilitates the implementation of the optimal policy via either closed-form solutions or potential numerical methods, and allows us to obtain new economic insights by flexible empirical applications. These challenges are exactly our focus.

In this paper, we develop and implement a decomposition for the optimal portfolio policy under a general class of incomplete market diffusion models and flexible utilities over both intermediate consumption and terminal wealth. In our decomposition, we express each component of the optimal policy as conditional expectation of random variables with sophisticated but explicit dynamics. In an incomplete market, investors cannot fully hedge the risk by investing in the risky assets. As a preparation, we apply and explore the “least favorable completion” principle developed by Karatzas et al. (1991) under general diffusion models. It completes the market by introducing suitable fictitious assets. Then, we establish the equivalence between the optimal policy in the completed market and that in the original market by choosing the appropriate price of risk for these fictitious assets. Such price of risk is endogenously determined by the investor’s utility function and investment horizon, and is thus referred to as the investor-specific price of risk in incomplete market models. It is also known as the “shadow price” of market incompleteness in the literature; see, e.g., Detemple and Rindisbacher (2010).

We begin by revealing and applying the following structure: Under general incomplete market models with wealth-dependent utilities, the appropriate investor-specific price of risk depends not only on the current market state, but also implicitly on investor’s wealth level. The latter dependence is entirely absent in the market price of risk associated with the real assets. Due to this special structure, the state price density, which plays a key role in solving the optimal policy, also becomes wealth-dependent in general incomplete market models. It is fundamentally different from the complete market case, where the state price density only depends on the market state. Thus, the structure of investor-specific price of risk introduces additional channels for the investor’s wealth level to affect the optimal policy. Recognizing such structure is crucial for correctly establishing the decomposition of optimal policy under general incomplete market models with flexible utilities.

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\(^3\)See also Cvitanic et al. (2003) for an alternative simulation approach. Besides, other numerical methods were proposed; see, e.g., the early attempts based on the dynamic programming approach; see, e.g., Fitzpatrick and Fleming (1991), Brennan et al. (1997), Brennan (1998), Chacko and Viceira (2005), Brennan and Xia (2002), and Campbell et al. (2004). We refer to the recent book of Dumas and Luciano (2017) for a survey of different numerical methods available for optimal portfolio choice.
We decompose the optimal policy into four components, all as functions of the current market state variable and investor’s wealth level. The optimal policy includes the mean-variance component, the interest rate hedge component, and two components for hedging the uncertainty in market and investor-specific price of risk respectively. Thus, our results substantially extend the existing complete-market policy decomposition with the mean-variance, interest rate hedge, and market price of risk hedge; see, e.g., Merton (1971) and Detemple et al. (2003). As the investor-specific price of risk depends on investor’s wealth level, the component related to it needs to hedge the uncertainty from both the market state and investor’s wealth. It highlights a key difference from the complete market case. In particular, under the mild assumption that the investor-specific price of risk function is differentiable in its arguments, we show the investor-specific price of risk hedge component can be further decomposed into two parts, which hedge the uncertainty from market state and investor’s wealth respectively. It effectively leads to a five-component decomposition of the optimal policy, and highlights the fundamental differences between our decomposition and the ones in the literature, e.g., Merton (1971), Detemple et al. (2003), Detemple and Rindisbacher (2005, 2010), and Detemple (2014). To our best knowledge, we are the first to identify the last component in the optimal policy, which hedges the uncertainty in investor-specific price of risk due to variation in investor’s wealth. Moreover, we find this component only appears in the optimal policy when the market is incomplete and, at the same time, the utility is wealth-dependent. Thus, it demonstrates the impact of the combination of market incompleteness and wealth-dependent utilities on optimal policies.

Our new decomposition, due to its structural clarity, facilitates the implementation of optimal policy under flexible incomplete-market model with wealth-dependent utility. As an important application, we apply our decomposition to solve the optimal policy under the wealth-dependent HARA utility. Compared with the simple CRRA utility that is commonly used in the literature, the HARA utility offers more flexibility in capturing investor’s preference, and is more realistic in reflecting investment constraints such as investment goal and subsistence level. However, it is much less studied due to its commonly believed mathematical inconvenience; see, e.g., Kim and Omberg (1996) for a rare case with closed-form policy and Duffie et al. (1997) for characterizing the optimal policy by viscosity solution in an incomplete market with constant coefficients for stock and income dynamics. Our decomposition clearly reveals how the investor’s wealth level gets involved in the optimal policy when transiting from the CRRA utility to the HARA utility. In particular, under the special case with nonrandom but possibly time-varying interest rate, we show that we can decompose the optimal policy under HARA utility as a product of its counterpart under CRRA utility and a key multiplier related to investor’s wealth level and bond prices. Our decomposition indicates that the HARA optimal policy is constructed as follows: the investor first buys a series of zero-coupon bonds to exactly satisfy the minimum requirements for terminal wealth and intermediate consumptions, then allocates the remaining wealth just as an investor with CRRA utility. Moreover, this intuitive structure
suggests that the roles of investor wealth and current market state can be explicitly separated in the optimal policy under HARA utility. As such, our decomposition facilitates the closed-form solution of HARA optimal policy under specific models. It also brings potential benefits to the open problem of developing efficient numerical approaches for incomplete market models with wealth-dependent utilities.

To demonstrate the wealth effects in optimal portfolio allocation, we analyze the behaviors of optimal policies for investors with HARA utility under the prototypical stochastic volatility model of Heston (1993) (Heston-SV hereafter). To achieve fresh empirical validity, we calibrate the model using the daily data of SPDR S&P 500 ETF from the recent ten years, i.e., 2010 to 2019. We employ the maximum likelihood estimation approach developed in Aıt-Sahalia et al. (2020), which is an efficient method for estimating continuous-time models with latent factors. With our decomposition results for HARA utility, we can solve the optimal policy in closed-form under the Heston-SV model, which facilitates our subsequent economic analysis. We show that when switching from the simple CRRA utility to the HARA utility, the optimal policy is impacted by not only the investor’s wealth level, but also the interest rate and investment horizon. Thus, the wealth-dependent property of HARA utility can influence the optimal policy via multiple channels beyond investor’s current wealth. Our results provide a potential theoretical explanation to the empirical evidence in the literature that the investment in risky assets increases concavely in investor’s financial wealth; see, e.g., Roussanov (2010), Wachter and Yogo (2010), and Calvet and Sodini (2014).

In addition to the static analysis of the optimal policy at a fixed time point, we analyze the significant wealth impacts from a dynamic perspective, i.e., how the wealth-dependent property of the HARA utility interacts with the complex market dynamics in affecting the optimal allocation strategy and overall investment performance. In particular, we find that the wealth-dependent property of HARA utility leads to sophisticated dependence of the optimal portfolio on the entire paths of the asset price and volatility. As a vivid illustration, we find that the optimal policies of HARA investors with high and low initial wealth levels become more (resp. less) similar to each other in the bull (resp. bear) market regimes. Next, we show the initial wealth level of HARA investors substantially impacts the overall investment performance. It introduces a risk-return trade-off induced by the wealth level: the HARA investor with higher initial wealth allocates more wealth to the risky asset, leading to a higher return but also more risk. We quantify such trade-off by simulating a large number of paths under our estimated model, i.e., accounting for the probabilistic distribution of possible market scenarios. With the simulated paths and the closed-form optimal policy, we compute a priori expectations of performance statistics including excess return mean,

\footnote{Although important, this type of dynamic analysis is rare in the literature. Moreira and Muir (2019) show that, under stochastic volatility, ignoring the hedge component in the optimal policy leads to a substantial utility loss. However, since Moreira and Muir (2019) employ the recursive utility of Duffie and Epstein (1992), the consequent optimal policy is independent of investor’s wealth level, contrasting with our focus on wealth-dependence.}
volatility, Sharpe ratio, and maximum drawdown. We find that as HARA investor’s initial wealth increases by ten times from the subsistence level, the average annual excess return increases from 3.5% to 23.5%, while the volatility increases from 4.2% to 25.4%, and the maximum drawdown jumps from 8.2% to 38.1%. The huge differences in the investment performance exemplify the importance and practical relevance of understanding the wealth effects in delegated portfolio management. Moreover, the average Sharpe ratio increases from 0.74 to 0.82. This surge can be attributed to the additional uncertainty caused by the market cycles, which is more pronounced for low wealth HARA investors. The above cycle-dependence is entirely absent under the simple CRRA utility.

Then, we perform some more in-depth analysis on how the wealth effect impacts the overall investment performance. First, we find that higher interest rate and/or longer investment horizon increase both the average excess return and volatility. It can be interpreted as follows. With higher interest rate or longer horizon, investors can accumulate more wealth during their investment horizon, leading to more allocation on the risky asset. In addition, such effect is more significant for HARA investors with low initial wealth, reflecting the interaction between the wealth effect and model parameters. Finally, we reveal a novel hysteresis effect in the optimal portfolio allocation. That is, the investment performance depends on not only the realization of market states, but also the history of their occurrence. In particular, when we shuffle the path of stock price and move the “good” years with high returns to the beginning of the investment horizon, the average excess return of HARA investors remains almost unchanged, but their volatility increases substantially. Such hysteresis effect essentially stems from the cycle-dependence of optimal policy under the wealth-dependent HARA utility, and entirely vanishes for CRRA investors. These analyses further suggest that the wealth-dependent property can impact the optimal portfolio allocation in sophisticated ways, which must be taken in to account in delegated portfolio management.

The rest of this paper is organized as follows. Section 2 gives the model set-up and recalls the machinery of the fictitious completion method. In Section 3, we establish the decomposition for general incomplete market models with flexible utilities. We reveal the fundamental impact of market incompleteness and wealth-dependent utilities by comparing our decomposition to that under the CRRA utility and/or complete market cases. In Section 4, we apply our decomposition to the wealth-dependent HARA utility and establish a closed-form relation between optimal policies under HARA and CRRA utilities under nonrandom interest rate. We devote Section 5 to the economic analysis of the static and dynamic impacts of wealth-dependence. In particular, we provide explanations for the cycle-dependence and hysteresis effect of optimal policy under wealth-dependent utilities, which is unseen in the previous literature on optimal dynamic portfolio allocation. Section 6 concludes and provides discussions. We collect auxiliary derivations in the appendices. For completeness, we include further results in the online supplementary material.
2 Model set-up and fictitious completion method

We begin by setting up the model, utility function, and optimal dynamic portfolio choice problem under general incomplete market models before recalling the fictitious completion method used to get our new decomposition results.

2.1 Model set-up

Assume that the market consists of \( m \) stocks and one savings account. The stock price \( S_{it} \), for \( i = 1, 2, \ldots, m \), follows the generic SDE:

\[
\frac{dS_{it}}{S_{it}} = (\mu_i(t, Y_t) - \delta_i(t, Y_t)) dt + \sigma_i(t, Y_t)dW_t,
\]

where \( Y_t \) is an \( n \)-dimensional state variable driven by the following generic SDE:

\[
dY_t = \alpha(t, Y_t)dt + \beta(t, Y_t)dW_t.
\]

In (1), \( W_t \) is a standard \( d \)-dimensional Brownian motion; \( \mu_i(t, y) \) is a scalar function for modeling the mean rate of return; \( \delta_i(t, y) \) is a scalar function for modeling the dividend rate; \( \sigma_i(t, y) \) is a \( d \)-dimensional vector-valued function for modeling the volatility. In (2), \( \alpha(t, y) \) is an \( n \)-dimensional vector-valued function for modeling the drift of the state variable \( Y_t \); \( \beta(t, y) \) is an \( n \times d \) dimensional matrix-valued function for modeling the diffusion of the state variable \( Y_t \). We assume the existence and uniqueness of solutions to SDEs (1) and (2).

Besides, we assume that the savings account appreciates at the instantaneous interest rate \( r_t = r(t, Y_t) \) for some scalar-valued function \( r(t, y) \). The state variable \( Y_t \) governs all the investment opportunities in the market through the rate of return, the dividend rate, the volatility, and the instantaneous interest rate.

We mainly focus on the incomplete market case where the number of independent Brownian motions is strictly larger than the number of tradable risky assets, i.e., \( d > m \). In this case, we cannot fully hedge the uncertainty stemming from the Brownian motion by investing in the risky assets. As we will show, due to market incompleteness, the decomposition and subsequent implementation for the optimal portfolio policy become a challenging issue; it enjoys a high-dimensional nature with a sophisticated structure even for one-asset cases. Denote the investor’s wealth process by \( X_t \). Then, it satisfies the following wealth equation:

\[
dX_t = (r_tX_t - c_t)dt + X_t\pi_t^\top [(\mu_t - r_t1_m)dt + \sigma_t dW_t].
\]

\(^5\)Sufficient technical conditions include but are not limited to, e.g., the Lipschitz condition and the polynomial growth condition on the coefficient functions; see, e.g., Sections 5.2 and 5.3 in Karatzas and Shreve (1991). In addition, we assume that the solution to SDE (2) is Malliavin differentiable. A sufficient condition is that the coefficient functions of SDE (2) have bounded derivatives for each order; see, e.g., Section 2.2 of Nualart (2006).
In (3), $\mu_t$ and $\sigma_t$ represent the mean rate of return and volatility of the risky assets, which satisfy $\mu_t = \mu(t, Y_t)$ and $\sigma_t = \sigma(t, Y_t)$ where the $m$–dimensional vector $\mu(t, y)$ and the $m \times d$–dimensional matrix $\sigma(t, y)$ are defined by $\mu(t, y) := \begin{pmatrix} \mu_1(t, y) \\ \mu_2(t, y) \\ \vdots \\ \mu_m(t, y) \end{pmatrix}^\top$ and $\sigma(t, y) := \begin{pmatrix} \sigma_1(t, y) \\ \sigma_2(t, y) \\ \vdots \\ \sigma_m(t, y) \end{pmatrix}^\top$. We assume the volatility function $\sigma(t, y)$ has rank $m$, i.e., its rows are linearly independent. Besides, $c_t$ is the instantaneous consumption rate; $\pi_t$ is an $m$–dimensional vector representing the weights of the risky assets in the portfolio; $1_m$ denotes an $m$–dimensional column vector with all elements equal to 1. The investor maximizes her expected utility over both intermediate consumptions and terminal wealth by dynamically allocating her wealth among the risky assets and the risk-free asset, subject to the non-bankruptcy condition. We can formulate this optimization problem as

$$\sup_{(\pi_t, c_t)} E \left[ \int_0^T u(t, c_t) dt + U(T, X_T) \right], \text{ with } X_t \geq 0 \text{ for all } t \in [0, T],$$

where $u(t, \cdot)$ and $U(T, \cdot)$ are the time-additive utility functions of the intermediate consumptions and the terminal wealth; they are allowed to be time-varying, in order to reflect the time value, e.g., the discount effect, and are assumed to be strictly increasing and concave with $\lim_{x \to \infty} \frac{\partial u(t, x)}{\partial x} = 0$ and $\lim_{x \to \infty} \frac{\partial U(T, x)}{\partial x} = 0$.

Under the incomplete market setting, we aim at developing a useful decomposition and methods of implementation for the optimal policy under general time-additive utility functions. They are sufficiently flexible for capturing the effects of market incompleteness and wealth-dependent property in investor’s risk preference. A representative example of wealth-dependent utilities is the HARA utility. Following the convention (see, e.g., Carroll and Kimball (1996)), we define it by

$$u(t, c) = we^{-\rho t} \frac{(c - \bar{c})^{1-\gamma}}{1-\gamma} \text{ and } U(T, x) = (1 - w)e^{-\rho T} \frac{(x - \bar{x})^{1-\gamma}}{1-\gamma}$$

for $c > \bar{c}$ and $x > \bar{x}$, where $\gamma > 0$ is the risk aversion coefficient; $w \in [0, 1]$ is the weight for balancing the intermediate consumption and the terminal wealth, and $\rho$ is the discount rate. The constants $\bar{c}$ and $\bar{x}$ represent the minimum allowable levels for the intermediate consumption and terminal wealth. The HARA utility allows for imposing lower bound constraints on the intermediate consumption and/or terminal wealth. This realistic feature is particularly suitable for incorporating, e.g., portfolio insurance, investment goal constraints, and subsistence level constraints. Although important, closed-form optimal policies under the HARA utility are rare due to its technical difficulties; see Kim and Omberg (1996) for one such case with the stochastic market price of risk modelled by an Ornstein-Uhlenbeck process. Potential numerical methods include, e.g., the Monte Carlo simulation approaches of Detemple et al. (2003) and Cvitanic et al. (2003), which are, however, developed under complete market settings.

As a simpler and special case, the HARA utility reduces to the widely used CRRA utility when $\bar{c}$ and $\bar{x}$ are set as zero in (5). An alternative generalization of the simple CRRA utility is the
recursive utility that aims at separating risk aversion from elasticity of intertemporal substitution; see, e.g., Epstein and Zin (1989) and Duffie and Epstein (1992). However, most applications with the recursive utility inherit the wealth-independent property of the CRRA utility (see, e.g., Chacko and Viceira (2005) and Moreira and Muir (2019)), thus cannot reveal the fundamental impact of investor’s wealth on the optimal policy. We do not intend to cover the recursive utility in this paper, as the general time-additive utility we consider is flexible enough for our focus on the sophisticated impacts of market incompleteness and wealth dependence on both theoretical and practical aspects. For ease of exposition, we abbreviate time-additive utility as utility in what follows.

2.2 Fictitious completion method

To establish a novel decomposition of the optimal policy, we set as a basic foundation the least favorable completion principle introduced in Karatzas et al. (1991) at the purely theoretical level. For this purpose, we begin by introducing the following notations as necessary preparations, based on which we explore and disentangle the essential structures of the optimal policy in the next section.

We introduce \( d-m \) fictitious assets without dividend payment to complete the market, following Karatzas et al. (1991). Their prices \( F_{it} \), for \( i = 1, 2, \ldots, d-m \), satisfy the following SDE:

\[
\frac{dF_{it}}{F_{it}} = \mu_{it} dt + \sigma_{it}^F(t, Y_t) dW_t,
\]

where the mean rates of returns \( \mu_{it} \) are stochastic processes adaptive to the filtration generated by the Brownian motion \( W_t \). We can choose the volatility function \( \sigma^F(t, y) := (\sigma_1^F(t, y), \ldots, \sigma_{d-m}^F(t, y))^\top \) of the fictitious assets arbitrarily, as long as it has rank \( d-m \) and satisfies the following orthogonal condition with respect to the volatility function \( \sigma(t, y) \) of the real risky assets \( S_t \):

\[
\sigma(t, y)\sigma^F(t, y)^\top \equiv 0_{m \times (d-m)}.
\]

It guarantees that the fictitious and real assets are driven by different Brownian shocks, and thus leads to the success of the market completion.

Combining the \( m \) real risky assets with prices \( S_t \) in (1) and the \( d-m \) fictitious risky assets with prices \( F_t \) in (6), we construct a completed market consisting of \( d \) risky assets and driven formulae of the optimal policy or significant simplifications of the optimization problem under some specific models; see, e.g., Wachter (2002), and Liu (2007) for closed-form optimal policies, as well as Detemple et al. (2003) for a Monte Carlo simulation approach.

\(^7\)As documented in the literature (see, e.g., Karatzas et al. (1991)), we can interpret the terminology “least favorable completion” as follows. Consider all the possible fictitious completions and their associated optimal policies. We naturally say that a completion is more (resp. less) favorable if its corresponding optimal policy results in higher (resp. lower) expected utility. The completion (14) below, which leads to an optimal portfolio with zero weight on the fictitious assets, must be the least favorable one. Indeed, in any other fictitious completion, since this portfolio without the fictitious assets is admissible (i.e., a candidate portfolio strategy), the optimal one must result in a higher expected utility and thus becomes more favorable.
by $d$ independent Brownian motions. In this completed market, we represent the prices of the risky assets, including both the real and fictitious ones, by a $d$-dimensional column vector $S_t = (S_t^T, F_t^T)^T$. According to (1) and (6), $S_t$ is driven by the SDE: $dS_t = \text{diag}(S_t) \left[ \mu_t^S dt + \sigma^S(t, Y_t) dW_t \right]$, with the diagonal matrix $\text{diag}(S_t) = \text{diag}(S_t, F_t)$, the $d$-dimensional column vector $\mu_t^S = ((\mu(t, Y_t) - \delta(t, Y_t))^T, (\mu_F^F)^T)^T$, and the $d \times d$ dimensional matrix $\sigma^S(t, Y_t) = (\sigma(t, Y_t)^T, \sigma^F(t, Y_t)^T)^T$. By linear algebra, the orthogonal condition (7) implies that $\sigma^S(t, y)$ must be nonsingular. Thus, we are now in a complete market, where we can fully hedge the uncertainty stemming from all Brownian motions. The completed market allows for investing in both the real assets $S_t$ and the fictitious assets $F_t$. We denote by $\pi_t$ and $\pi_t^F$ their corresponding weights, which are $m$ and $(d - m)$-dimensional vectors. Similar to (4), we consider the utility maximization problem in this completed market, still with the non-bankruptcy constraint $X_t \geq 0$.

In the completed market, we define the total price of risk as
\[ \theta^S_t := \sigma^S(t, Y_t)^{-1}(\mu_t^S - r(t, Y_t)1_d). \]
By the orthogonal condition (7), we can decompose the total price of risk as:
\[ \theta^S_t = \theta^h(t, Y_t) + \theta^u_t. \quad (8) \]
Here, $\theta^h(t, Y_t)$ and $\theta^u_t$ are the prices of risk associated with the real and fictitious assets, respectively. They are defined by the $d$-dimensional column vectors:
\[ \theta^h(t, Y_t) := \sigma(t, Y_t)^+(\mu(t, Y_t) - r(t, Y_t)1_m) \quad (9a) \]
and
\[ \theta^u_t := \sigma^F(t, Y_t)^+(\mu^F_t - r(t, Y_t)1_{d-m}), \quad (9b) \]
where $A^+ := A^T(AA^T)^{-1}$ denotes the Moore–Penrose inverse (see, e.g., Penrose (1955)) of a general matrix $A$ with linearly independent rows. The term $\theta^h(t, Y_t)$ in (9a) is referred to as the market price of risk, as it is fully determined by the real assets shared by all investors in the market. The term $\theta^u_t$ in (9b), however, is purely associated with the fictitious assets, which are specifically introduced for solving the optimal portfolio choice problem (4) in the incomplete market. As we will show momentarily, $\theta^u_t$ is endogenously determined by the investor’s utility function and the investment horizon. Thus, following the literature, we refer to $\theta^u_t$ as the investor-specific price of risk, since it varies from one investor to another. Our investigation starts from expressing the functional form of $\theta^u_t$. Then, we develop a novel decomposition of the optimal policy based on the structure of $\theta^u_t$.

With the total market price of risk in (8), we introduce the state price density as
\[ \xi^S_t := \exp \left( - \int_0^t r(v, Y_v) dv - \int_0^t (\theta^S_v)^\top dW_v - \frac{1}{2} \int_0^t (\theta^S_v)^\top \theta^S_v dv \right). \quad (10) \]

To guarantee the martingale property of $\xi^S_t \exp(\int_0^t r(v, Y_v) dv)$, we assume that the total price of risk $\theta^u_t$ satisfies the Novikov condition: $E \left[ \exp \left( \frac{1}{2} \int_0^T (\theta^S_v)^\top \theta^S_v dv \right) \right] < \infty.$
For any \( s \geq t \geq 0 \), we define the relative state price density as \( \xi_{t,s}^S = \xi_t^S / \xi_t^S \). By Itô’s formula, it satisfies \( d\xi_{t,s}^S = -\xi_t^S r(s, Y_s) ds + (\theta_s^S)\top dW_s \) with initial value \( \xi_{t,t}^S = 1 \). The above dynamics of \( \xi_t^S \) clearly hinges on the undetermined investor-specific price of risk \( \theta_s^u \).

The martingale approach pioneered by Karatzas et al. (1987) and Cox and Huang (1989) starts by formulating the dynamic problem (4) with information up to time \( t \) as the following equivalent static optimization problem:

\[
\sup_{(c_t, X_T)} E_t \left[ \int_t^T u(s, c_s) ds + U(T, X_T) \right] \quad \text{subject to} \quad E_t \left[ \int_t^T \xi_{t,s}^S c_s ds + \xi_{t,s}^S X_T \right] \leq X_t, \tag{11}
\]

where, throughout the paper, \( E_t \) denotes the expectation condition on the information up to time \( t \) and \( X_t \) is the wealth level assuming that the investor always follows the optimal policy. Then, following the standard method of Lagrangian multiplier, we can represent the optimal intermediate consumption and terminal wealth as \( c_t = I_t^u(t, \lambda_t^\ast) \) and \( X_T = I_T^U(T, \lambda_T^\ast) \), where \( I_t^u(t, \cdot) \) and \( I_T^U(T, \cdot) \) being the inverse marginal utility functions of \( u(t, \cdot) \) and \( U(t, \cdot) \), i.e., the functions satisfying \( \partial u/\partial x(t, I_t^u(t,y)) = y \) and \( \partial U/\partial x(t, I_T^U(t,y)) = y \). The quantity \( \lambda_t^\ast \) denotes the Lagrangian multiplier for the wealth constraint in (11). It is uniquely characterized by

\[
X_t = E_t [G_t, T(\lambda_T^\ast)], \tag{12}
\]

where \( G_t, T(\lambda_T^\ast) \) is defined as \( G_t, T(\lambda_T^\ast) := \Gamma_{t,T}^U(\lambda_T^\ast) + \int_t^T \Gamma_{t,s}^u(\lambda_T^\ast) ds \). Here, \( \Gamma_{t,T}^U(\lambda_T^\ast) \) and \( \Gamma_{t,s}^u(\lambda_T^\ast) \) are given by

\[
\Gamma_{t,T}^U(\lambda_T^\ast) = \xi_{t,T}^S I_T^U(T, \lambda_T^\ast \xi_{t,T}^S) \quad \text{and} \quad \Gamma_{t,s}^u(\lambda_T^\ast) = \xi_{t,s}^S I_t^u(s, \lambda_T^\ast \xi_{t,s}^S). \tag{13a}
\]

By (12), we can determine the multiplier \( \lambda_t^\ast \) with information up to time \( t \). Besides, we introduce the following quantities for expressing the optimal policy in the next section:

\[
\Upsilon_{t,T}^U(\lambda_T^\ast) = \lambda_T^\ast (\xi_{t,T}^S)^2 \frac{\partial I_T^U}{\partial y}(T, \lambda_T^\ast \xi_{t,T}^S) \quad \text{and} \quad \Upsilon_{t,s}^u(\lambda_T^\ast) = \lambda_T^\ast (\xi_{t,s}^S)^2 \frac{\partial I_t^u}{\partial y}(s, \lambda_T^\ast \xi_{t,s}^S). \tag{13b}
\]

Consequently, we can express the optimal policy \( (\pi_t, \pi_T^F) \) for the completed market via the martingale representation theorem (see, e.g., Section 3.4 in Karatzas and Shreve (1991)). With the Clark-Ocone formula (see Ocone and Karatzas (1991)), we can further represent this optimal policy in the form of conditional expectations of suitable random variables. Under a general and flexible complete-market diffusion model, Detemple et al. (2003) propose an explicit conditional expectation form of the optimal policy, and develop a Monte Carlo simulation method for its implementation; see also, e.g., Detemple and Rindisbacher (2010) along this line of contributions and Detemple (2014) for a comprehensive survey of the related developments. We aim at explicitly decomposing the optimal policy for the incomplete market case and reveal the fundamental difference that arises from market incompleteness and wealth-dependent utilities, and more importantly, the particular case where these two situations co-exist.
By the least favorable completion principle proposed in Karatzas et al. (1991), the optimal policy $\pi_t$ for the real assets in the completed market coincides with its counterpart in the original incomplete market, as long as we properly choose the investor-specific price of risk $\theta^u_v$ such that the optimal weights for the fictitious assets are always identically zero, i.e.,

$$\pi^F_v \equiv 0_{d-m}, \text{ for any } 0 \leq v \leq T.$$  \hfill (14)

Given an arbitrary choice of the volatility function $\sigma^F(v, y)$, the least favorable constraint (14) and the orthogonal condition (7) determine the desired $\theta^u_v$ for $0 \leq v \leq T$. Then, the corresponding optimal policy $\pi_t$ of the real assets for the completed market is also optimal for the original incomplete market. In particular, the desired $\theta^u_v$ satisfying (14) and the resulting optimal policy $\pi_t$ are independent of the specific choice of $\sigma^F(v, y)$.

\section{A novel decomposition of optimal dynamic portfolio choice}

In this section, we deliver our new decomposition results based on the previous setup. We substantially develop economic insights regarding the structure of the optimal policy under the incomplete market setting ($d > m$) for models with flexible dynamics (1) – (2) and flexible utility functions.

\subsection{Decomposing optimal policy}

We start by applying the following lemma stating the functional representation of the unknown investor-specific price of risk $\theta^u_v$ that satisfies the least favorable completion constraint (14) in general incomplete market models with flexible utilities.

\begin{lemma}
\label{lemma:theta_u}
The investor-specific price of risk $\theta^u_v$ can be expressed as

$$\theta^u_v = \theta^u(v, Y_v, \lambda^*_v; T),$$  \hfill (15)

for some function $\theta^u(v, y, \lambda; T)$ endogenously determined by the investor’s utility function and investment horizon; it depends on the time $v$, market state $Y_v$, as well as investor’s wealth level $X_v$ via the multiplier $\lambda^*_v$.

\end{lemma}

\begin{proof}
See Section S.1 in the online supplementary material, where we verify such a result by combining the fictitious completion approach in Karatzas et al. (1991) and the minimax local martingale approach in He and Pearson (1991).
\end{proof}

Representation (15) reveals the structure of the investor-specific price of risk $\theta^u_v = \theta^u(v, Y_v, \lambda^*_v; T)$, which is strikingly different from that of the market price of risk $\theta^h(v, Y_v)$ defined in (9a) for real assets. It leads to fundamental differences between incomplete and complete market cases. First,
Theorem 1. Under the incomplete market model (1) – (2), the optimal policy \( \pi_t \) for the real assets with prices \( S_t \) admits the following decomposition:

\[
\pi_t = \pi^{mv}(t, X_t, Y_t) + \pi^r(t, X_t, Y_t) + \pi^\theta(t, X_t, Y_t).
\]  

The terms \( \pi^{mv}(t, X_t, Y_t) \), \( \pi^r(t, X_t, Y_t) \), and \( \pi^\theta(t, X_t, Y_t) \) denote the mean-variance component, the interest rate hedge component, and the price of risk hedge component. The price of risk hedge component \( \pi^\theta(t, X_t, Y_t) \) is further decomposed as

\[
\pi^\theta(t, X_t, Y_t) = \pi^h(t, X_t, Y_t) + \pi^u(t, X_t, Y_t),
\]  

where the first two components \( \pi^h(t, X_t, Y_t) \) and \( \pi^u(t, X_t, Y_t) \) hedge the uncertainties in market and investor-specific price risk. The components can be expressed as conditional expectations on random variables with explicit dynamics\(^9\):

\[
\begin{align*}
\pi^{mv}(t, X_t, Y_t) & = - (\sigma(t, Y_t)^{+})^\top \theta^h(t, Y_t) E_t[Q_{t,T}(\lambda^*_t)]/X_t, \\
\pi^r(t, X_t, Y_t) & = - (\sigma(t, Y_t)^{+})^\top E_t[H_{t,T}^r(\lambda^*_t)]/X_t, \\
\pi^\theta(t, X_t, Y_t) & = - (\sigma(t, Y_t)^{+})^\top E_t[H_{t,T}^\theta(\lambda^*_t)]/X_t,
\end{align*}
\]  

\(^9\)In line with the time–t formulation of the optimization problem (11), we express these components by the time–t state variable \( Y_t \) and the current wealth \( X_t \), rather than the initial wealth \( X_0 \) as in most of the existing literature, e.g., Detemple et al. (2003). We can see that, by solving constraint (12), \( \lambda_t^* \) is a function of \( t, X_t, \) and \( Y_t \).
and

\[ \pi^h(t, X_t, Y_t) = - (\sigma(t, Y_t))^{\top} E_t[H^h_{t,T}]/X_t, \]  
\[ \pi^u(t, X_t, Y_t) = - (\sigma(t, Y_t))^{\top} E_t[H^u_{t,T}(\lambda^*_t)]/X_t. \]

Hereof, \( \lambda^*_t \) is the multiplier uniquely determined by (12), i.e., \( X_t = E_t[\mathcal{G}_{t,T}(\lambda^*_t)] \). It depends on \( X_t \) and satisfies the relation \( \lambda^*_t = \lambda^*_0 \xi^S_t \). The expressions for \( Q_{t,T}(\lambda^*_t), H^r_{t,T}(\lambda^*_t), H^\theta_{t,T}(\lambda^*_t), H^h_{t,T}, H^u_{t,T}(\lambda^*_t) \) and \( H^u_{t,T}(\lambda^*_t) \) are explicitly given in Proposition 1 below. The optimal intermediate consumption \( c_t \) and terminal wealth \( X_T \) are given by \( c_t = I^u(t, \lambda^*_t) \) and \( X_T = I^U(T, \lambda^*_T) \).

**Proof.** See Section S.3 in the online supplementary material.

In Proposition 1 below, we provide the explicit expressions based on standard SDEs for the variables \( Q_{t,T}(\lambda^*_t), H^r_{t,T}(\lambda^*_t), H^\theta_{t,T}(\lambda^*_t), H^h_{t,T}, H^u_{t,T}(\lambda^*_t) \) involved in Theorem 1. For this purpose, we apply the representation of the individual-specific price of risk \( \theta^u \) in (15) to introduce the following \( \lambda^*_t \)-parameterized version of \( \theta^u \):

\[ \theta^u_s(\lambda^*_t) = \theta^u(s, Y_s, \lambda^*_s; T) = \theta^u(s, Y_s, \lambda^*_s \xi^S_{t,s}(\lambda^*_t); T), \]  
where the second equality follows from the definition \( \xi^S_{t,s} = \xi^S_s/\xi^S_t \) and the relation \( \lambda^*_t = \lambda^*_0 \xi^S_t \), i.e., \( \lambda^*_s = \lambda^*_0 \xi^S_t = \lambda^*_t \xi^S_s \); \( \xi^S_{t,s}(\lambda^*_t) \) in (20) denotes the \( \lambda^*_t \)-parameterized version of the state price density, which evolves according to

\[ d\xi^S_{t,s}(\lambda^*_t) = - \xi^S_{t,s}(\lambda^*_t) [r(s, Y_s)ds + \theta^S_s(\lambda^*_t)^{\top}dW_s], \]  
with

\[ \theta^S_s(\lambda^*_t) = \theta^h_s(t, Y_t) + \theta^u(s, Y_s, \lambda^*_s \xi^S_{t,s}(\lambda^*_t); T). \]  

We introduce these \( \lambda^*_t \)-dependent versions (20)–(22) to highlight the impact from the investor’s wealth, as \( \lambda^*_t \) depends on the current wealth level \( X_t \) via (12), i.e., \( X_t = E_t[\mathcal{G}_{t,T}(\lambda^*_t)] \). In addition, we see that \( \lambda^*_t \) can be fully determined by the information at time \( t \). Thus, these \( \lambda^*_t \)-dependent versions clearly reveal the temporal structure of the optimal policy by isolating the information available at time \( t \).

**Proposition 1.** The quantities \( Q_{t,T}(\lambda^*_t), H^r_{t,T}(\lambda^*_t), \) and \( H^\theta_{t,T}(\lambda^*_t) \) in (18a) – (18c) are given by

\[ Q_{t,T}(\lambda^*_t) = \Upsilon^U_{t,T}(\lambda^*_t) + \int_t^T \Upsilon^u_{t,s}(\lambda^*_t)ds, \]  
\[ H^r_{t,T}(\lambda^*_t) = (\Gamma^U_{t,T}(\lambda^*_t) + \Upsilon^U_{t,T}(\lambda^*_t)) H^r_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^*_t) + \Upsilon^u_{t,s}(\lambda^*_t)) H^r_{t,s}ds, \]  
\[ H^\theta_{t,T}(\lambda^*_t) = (\Gamma^U_{t,T}(\lambda^*_t) + \Upsilon^U_{t,T}(\lambda^*_t)) H^\theta_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^*_t) + \Upsilon^u_{t,s}(\lambda^*_t)) H^\theta_{t,s}(\lambda^*_t)ds, \]  

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where $\Gamma^{U}_{t,T}(\lambda^*_t)$, $\Gamma^{u}_{t,s}(\lambda^*_t)$, $\Upsilon^{U}_{t,T}(\lambda^*_t)$, and $\Upsilon^{u}_{t,s}(\lambda^*_t)$ are defined in (13a) and (13b) except for replacing the relative state price density $\xi^S_{t,s}$ by the $\lambda^*_t$–dependent version $\xi^S_{t,s}(\lambda^*_t)$. $H^{r}_{t,s}$ in (23b) is a $d$–dimensional vector-valued processes evolving according to SDEs:

$$dH^{r}_{t,s} = (D_{t}Y_{s}) \nabla r(s,Y_{s})ds,$$

for $t \leq s \leq T$, with initial value $H^{r}_{t,t} = 0_d$. Here and throughout this paper, $\nabla$ denotes the gradient of functions with respect to the arguments in the place of $Y_{s}$\textsuperscript{10}, and $D_{t}$ denotes the Malliavin derivative at time $t$. Specifically, $D_{t}Y_{s}$ is a $d \times n$ matrix with each element given by $\nabla_{s}$ component the usual complete-market decomposition with only three components; see, e.g., Merton (1971) the same way as that for $X_{t}$. Besides, $H^{\theta}_{t,s}(\lambda^*_t)$ in (23c) follows the SDE\textsuperscript{11}

$$dH^{\theta}_{t,s}(\lambda^*_t) = \bigg[(D_{t}Y_{s}) \nabla \theta^h(s,Y_{s}) + D_{t}\theta^u_{s}(\lambda^*_t)\bigg]\bigg(\theta^S_{s}(\lambda^*_t)ds + dW_{s}\bigg),$$

with initial value $H^{\theta}_{t,t}(\lambda^*_t) = 0_d$. Finally, the terms $H^{h}_{t,T}$ and $H^{u}_{t,T}(\lambda^*_t)$ in (19a)–(19b) are defined in the same way as that for $H^{\theta}_{t,T}(\lambda^*_t)$ in (23c) except for replacing $H^{\theta}_{t,s}(\lambda^*_t)$ by $H^{h}_{t,s}$ and $H^{u}_{t,s}(\lambda^*_t)$ for $t \leq s \leq T$, which are both $d$–dimensional vector-valued processes evolving according to SDEs:

$$dH^{h}_{t,s} = (D_{t}Y_{s}) \nabla \theta^h(s,Y_{s})\theta^h(s,Y_{s})ds + dW_{s},$$

$$dH^{u}_{t,s}(\lambda^*_t) = D_{t}\theta^u_{s}(\lambda^*_t)\theta^u_{s}(\lambda^*_t)ds + dW_{s},$$

with initial values $H^{h}_{t,t} = H^{u}_{t,t}(\lambda^*_t) = 0_d$.

**Proof.** See Section S.1 in the online supplementary material. 

The explicit structure of the optimal policy, with four components given in (16) and (17) extends the usual complete-market decomposition with only three components; see, e.g., Merton (1971) and Detemple et al. (2003). We now analyze the economic implication of each component. The component $\pi^{mu}(t,X_{t},Y_{t})$ in (18a) is the mean-variance component, as reflected by the market price of risk $\theta^h(t,Y_{t})$ defined in (9a) – a mean-variance trade-off for the risky assets. The second component $\pi^{r}(t,X_{t},Y_{t})$ in (18b) is for hedging the uncertainty in the interest rate, as seen from the gradient of interest rate $\nabla r$ in (24). Finally, the component $\pi^{\theta}(t,X_{t},Y_{t})$ given by (18c) hedges the uncertainty in the price of risk. By (17), it can be further decomposed into two components $\pi^{h}(t,X_{t},Y_{t})$ and $\pi^{u}(t,X_{t},Y_{t})$.

\textsuperscript{10}For an $m$–dimensional vector-valued function $f(t,y) = (f_{1}(t,y),f_{2}(t,y),\cdots,f_{m}(t,y))$, its gradient is an $n \times m$ matrix with each element given by $[\nabla f(t,y)]_{ij} = \partial f_{j}/\partial y_{i}(t,y)$, for $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,m$.

\textsuperscript{11}Here, $D_{t}\theta^u_{s}(\lambda^*_t)$ is a $d \times d$ matrix with $D_{t}\theta^u_{s}(\lambda^*_t) = ((D_{t1}\theta^u_{s}(\lambda^*_t))^{\top},(D_{t2}\theta^u_{s}(\lambda^*_t))^{\top},\cdots,(D_{td}\theta^u_{s}(\lambda^*_t))^{\top})^{\top}$, where each $D_{t}\theta^u_{s}(\lambda^*_t)$ is a $d$–dimensional column vector.

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\( \pi^u(t, X_t, Y_t) \). The two components hedge the uncertainty in market price of risk \( \theta^h \) and investor-specific price of risk \( \theta^u \), as seen by the gradient \( \nabla \theta^h \) in (27a) and the Malliavin term \( \mathcal{D}_t \theta^u_s(\lambda^*_s) \) in (27b). As we will discuss in Section 3.4, the last component \( \pi^u(t, X_t, Y_t) \) is absent under complete market models.

In particular, as a natural analogue to a classical derivative, we can intuitively understand the Malliavin derivative \( \mathcal{D}_t \) as the sensitivity to the underlying Brownian motion \( W_t \); see Appendix D of Detemple et al. (2003) for an accessible survey of Malliavin calculus in finance.\(^\text{12}\) Indeed, throughout this paper, the Malliavin derivative on the state variable \( \mathcal{D}_t Y_s \) can be viewed as a standard multidimensional diffusion process with dynamics given explicitly in (25). On the other hand, the Malliavin derivative on the investor-specific price of risk \( \mathcal{D}_t \theta^u_s(\lambda^*_s) \) clearly depends on the structure of \( \theta^u_s(\lambda^*_s) \) given in (20). Specifically, the underlying Brownian motion \( W_t \) impacts the value of \( \theta^u_s(\lambda^*_s) = \theta^u(s, Y_s, \lambda^*_s; T) \) via both the state variable \( Y_s \) and the multiplier \( \lambda^*_s \). This structure introduces additional variation in the investor-specific price of risk \( \theta^u(s, Y_s, \lambda^*_s; T) \), which is absent in the market price of risk \( \theta^h(s, Y_s) \).

Our decompositions in Theorem 1 and Proposition 1 provide an explicit representation of the optimal policy in general incomplete market models. In addition, they clearly reveal how investor’s wealth \( X_t \) impacts the optimal policy, i.e., the channels for the wealth-dependent effect. It can be seen by checking how the time-\( t \) multiplier \( \lambda^*_t \) gets involved in the optimal policy, as it is determined by \( X_t \) according to (12). First, the time-\( t \) multiplier \( \lambda^*_t \) directly appears in the optimal policies (18a) – (18c) through the functions \( \Gamma_{i,t}^{\mathcal{D}}(\lambda^*_t) \) and \( \Upsilon_{i,t}^{\mathcal{V}}(\lambda^*_t) \) for \( i \in \{u, U\} \), which are given by (13a) and (13b). Second, as shown by (21) and (26), the multiplier \( \lambda^*_t \) affects the dynamics of the building blocks \( \xi_{t,s}^S(\lambda^*_t) \) and \( H^{\theta}_{t,s}(\lambda^*_t) \). It introduces an implicit impact of the wealth level on the optimal policy. By (22), such implicit impact essentially stems from the special structure of the investor-specific price of risk \( \theta^u_s(\lambda^*_s) \), which only appears in incomplete market models. Thus, the market incompleteness leads to additional channels for the wealth-dependent effect, i.e., via the structure of the investor-specific price of risk.

Now, we still face one remaining difficulty in solving for the optimal policy – the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) in (15) is still undetermined. For this task, we establish a novel equation for characterizing \( \theta^u(v, Y_v, \lambda^*_v; T) \) in Theorem 2 below. To obtain \( \theta^u(v, Y_v, \lambda^*_v; T) \), we first express the optimal policy for the fictitious assets \( \pi^F_v \) using representations similar to those in Theorem 1. Then, we combine the least favorable completion principle (14) and the orthogonal condition (7). The derivation of the proper \( \theta^u(v, Y_v, \lambda^*_v; T) \) again shows that it is essential to understand the correct structure of investor-specific price of risk in (15) and establish the explicit decomposition of optimal

\(^\text{12}\)We can view Malliavin calculus as the stochastic calculus of variation in the space of sample paths. Malliavin calculus has proven its important role in financial economics through its merit in solving portfolio choice problems, see, e.g., Ocone and Karatzas (1991), Detemple et al. (2003), Detemple and Rindisbacher (2005), and Detemple and Rindisbacher (2010). See, for example, Nualart (2006) for a book-length discussion of the theory of Malliavin calculus.
policy as in Theorem 1.

**Theorem 2.** The proper investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) for the least favorable completion satisfies the following \( d \)-dimensional equation:

\[
\theta^u(v, Y_v, \lambda^*_v; T) = \frac{\sigma(v, Y_v)^+ \sigma(v, Y_v) - I_d}{E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]} \times (E[H^r_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]),
\]

for \( 0 \leq v \leq T \). Here \( I_d \) denotes the \( d \)-dimensional identity matrix; \( \sigma(v, Y_v)^+ \) is given by \( \sigma(v, Y_v)^+ = \sigma(v, Y_v)^\top (\sigma(v, Y_v)\sigma(v, Y_v)^\top)^{-1} \).

**Proof.** See Section S.2 in the online supplementary material.

Theorem 2 shows that the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) is determined by a complex multidimensional equation system, which consists of equation (28) and the SDEs of \( Y_s, \xi^S_{v,s}(\lambda^*_v), H^r_{v,s}(\lambda^*_v) \), and \( D_{v,s}Y_s \) given in (2), (21), (24), (26), and (25). In the literature, other types of differential equations are employed for characterizing optimal portfolios in incomplete markets. For example, He and Pearson (1991) indirectly relate the optimal policy and its equivalent local martingale measure to the solution of a quasi-linear PDE. Detemple and Rindisbacher (2005) derive and solve a forward-backward SDE governing the shadow price under a model of partially hedgeable Gaussian interest rate and CRRA utility (see also Detemple and Rindisbacher (2010) for generalizing such results).

The contribution of our characterization (28) lies in that it reveals the explicit structure of the investor-specific price of risk using the representation (15). In particular, it clearly shows the wealth level \( X_v \) impacts the value of \( \theta^u(v, Y_v, \lambda^*_v; T) \) via the multiplier \( \lambda^*_v \), as shown by the SDEs of \( \xi^S_{v,s}(\lambda^*_v) \) and \( H^\theta_{v,s}(\lambda^*_v) \) in (21) and (26).

In addition, equation (28) implies its terminal condition:

\[
\theta^u(T, Y_T, \lambda^*_T; T) \equiv 0_d,
\]

which corresponds to the investor-specific price of risk with the investment horizon shrinking to zero. It suggests that, under a vanishing investment horizon, the returns of the fictitious assets \( \mu^F_t \) employed in the least favorable completion converge to the risk-free return \( r(t, Y_t) \). It follows from \( \theta^u(v, Y_v, \lambda^*_v; T) = \sigma^F(v, Y_v)^+ (\mu^F_t - r(v, Y_v)1_{d-m}) \) by (9b) and (15). The terminal condition (29) plays an important role in potential numerical methods for solving the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \). For example, in a simulation approach, it can serve as the boundary condition for backward induction. It verifies that the investor-specific price of risk indeed endogenously depends on the investment horizon of each investor.

### 3.2 Revealing wealth-related component under differentiability

In this section, we reveal a novel structural impact of the wealth-dependent property on the optimal policy, under the additional assumption that the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \)
is differentiable in its arguments. As we will show in Section 4.2, this natural assumption holds for
the classic Heston-SV model.

**Proposition 2.** Under the assumption that the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) is differentiable in its arguments, we can decompose the component \( \pi^u(t, X_t, Y_t) \) as

\[
\pi^u(t, X_t, Y_t) = \pi^{u,Y}(t, X_t, Y_t) + \pi^{u,\lambda}(t, X_t, Y_t),
\]

where

\[
\pi^{u,Y}(t, X_t, Y_t) = -(\sigma(t, Y_t)^\top E_t[\mathcal{H}^{u,Y}_{t,T}(\lambda^*_t)])/X_t,
\]

\[
\pi^{u,\lambda}(t, X_t, Y_t) = -(\sigma(t, Y_t)^\top E_t[\mathcal{H}^{u,\lambda}_{t,T}(\lambda^*_t)])/X_t.
\]

The terms \( \mathcal{H}^{u,Y}_{t,T}(\lambda^*_t) \) and \( \mathcal{H}^{u,\lambda}_{t,T}(\lambda^*_t) \) in (31a)–(31b) are defined in the same way as that for \( \mathcal{H}^\theta_{t,T}(\lambda^*_t) \) in (23c) except for replacing \( H^\theta_{t,s}(\lambda^*_t) \) by \( H^{u,Y}_{t,s}(\lambda^*_t) \) and \( H^{u,\lambda}_{t,s}(\lambda^*_t) \) for \( t \leq s \leq T \), which follow SDEs:

\[
dH^{u,Y}_{t,s}(\lambda^*_t) = (\mathcal{D}_t Y_s) \nabla \theta^u(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T)(\theta^u_s(\lambda^*_t)ds + dW_s),
\]

\[
dH^{u,\lambda}_{t,s}(\lambda^*_t) = -\lambda^*_s \xi^S_{t,s}(\lambda^*_t)(\theta^S(\lambda^*_t) + H^\theta_{t,s} + H^{u,\lambda}_{t,s}(\lambda^*_t))
\cdot \partial \theta^u / \partial \lambda(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T)(\theta^u_s(\lambda^*_t)ds + dW_s).
\]

**Proof.** See Section S.3 in the online supplementary material. \( \square \)

As one of our contributions, we show in (30) that we can further decompose the investor-specific price of risk hedge component \( \pi^u(t, X_t, Y_t) \) into two interpretable parts under the differentiability assumption. By combining (30) with (16) – (17) in Theorem 1, we obtain a final decomposition of the optimal policy with five interpretable components. Specifically, the first term \( \pi^{u,Y}(t, X_t, Y_t) \) hedges the fluctuation in the investor-specific price of risk that arises from the state variable \( Y_t \), as explicitly reflected by the gradient \( \nabla \theta^u(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T) \) in (27b). Its structure resembles that of \( \pi^h(t, X_t, Y_t) \), except for replacing \( \theta^h \) by \( \theta^u \) in (27a). The second term \( \pi^{u,\lambda}(t, X_t, Y_t) \), however, is introduced via a fundamentally different channel: it essentially hedges the uncertainty in the investor-specific price of risk due to the variation in investor’s wealth level. To see this, we note that \( \pi^{u,\lambda}(t, X_t, Y_t) \) hinges on the partial derivative \( \partial \theta^u / \partial \lambda(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T) \) in (32b). It captures the sensitivity of \( \theta^u \) with respect to the multiplier \( \lambda^*_s = \lambda^*_t \xi^S_{t,s}(\lambda^*_t) \), which is directly related to investor’s wealth level \( X_t \) via \( X_t = E_s[G_{s,T}(\lambda^*_s)] \) by (12).

By the above analysis, we see that investors need to hedge the uncertainty in investor-specific price of risk from both the state variable and investor’s wealth level. The two channels of uncertainty stems from the structure of investor-specific price of risk in (15) — it depends on both the state variable \( Y_s \) and investor’s wealth level \( X_s \) via the multiplier \( \lambda^*_s \). For the market price of risk \( \theta^h(s, Y_s) \), however, only the uncertainty from the state variable needs to be hedged via the component \( \pi^h(t, X_t, Y_t) \). In
addition, as we will show in the next two sections, the term \( \pi^{u,\lambda}(t, X_t, Y_t) \) can only possibly appear when the market is incomplete and, at the same time, the utility is wealth-dependent. That is, it vanishes under the CRRA utility, and also under complete market models. Thus, it highlights the impact on the optimal policy coming from the interaction of market incompleteness and wealth-dependent utilities, i.e., the proper way for the investor to complete the market depends on her current wealth level. To our best knowledge, we are the first to show this additional term in optimal dynamic portfolio allocation.

3.3 Decomposing optimal policy under the CRRA utility

To reveal the structural impact of wealth-dependent utility on the optimal policy, we now investigate how the optimal policy degenerates under the wealth-independent CRRA utility, i.e., with \( \bar{c} = 0 \) and \( \bar{x} = 0 \) in (5). The results are obtained by applying our general decomposition in Theorems 1 and 2, and then simplifying the results using the special structure of the CRRA utility. We provide the main results here and discuss the key insights. We refer to Appendix A for a further detailed discussion.

First, we can show that the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) introduced in (15) is independent of the multiplier \( \lambda^*_v \) and thus can be reduced to \( \theta^u(v, Y_v; T) \) under the CRRA utility. Thus, it is no longer affected by investor’s wealth level \( X_v \) via the equation \( X_v = E_v[G_{v,T}(\lambda^*_v)] \). It means that under incomplete market models, the proper way for a CRRA investor to complete the market only depends on the market state and its horizon. Consequently, the relative state price density \( \xi^S_{t,s}(\lambda^*_t) \) and Malliavin term \( H^\theta_{t,s}(\lambda^*_t) \), as given in general by dynamics (21) and (26), become independent of the multiplier \( \lambda^*_t \). Thus, they can be written as \( \xi^S_{t,s} \) and \( H^\theta_{t,s} \) with dynamics

\[
\begin{align*}
    d\xi^S_{t,s} &= -\xi^S_{t,s}[r(s, Y_s)ds + (\theta^h(s, Y_s) + \theta^u(s, Y_s; T))^\top dW_s], \\
    dH^\theta_{t,s} &= (D_t Y_s) (\nabla \theta^h(s, Y_s) + \nabla \theta^u(s, Y_s; T))[\theta^S(s, Y_s; T)ds + dW_s],
\end{align*}
\]

where \( \theta^S(s, Y_s; T) = \theta^h(s, Y_s) + \theta^u(s, Y_s; T) \).

By the above analysis, we see the wealth-independent property of the CRRA utility is embodied by multiple building blocks of the optimal policy: the investor-specific price of risk \( \theta^u(v, Y_v; T) \), the relative state price density \( \xi^S_{t,s} \), and the Malliavin term \( H^\theta_{t,s} \) are all independent of investor’s current wealth level, in contrast to their counterparts under general utilities. With these preparations, we provide the decomposition of optimal policy under the CRRA utility in the following proposition.

**Proposition 3.** Under the incomplete market model (1) – (2) and the CRRA utility function given in (5) with \( \bar{c} = \bar{c} = 0 \), the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) does not depend on the multiplier \( \lambda^*_v \), and can be simplified as \( \theta^u(v, Y_v; T) \). Consequently, the relative state price density
\( \xi^S_{t,s}(\lambda^*_t) \) and Malliavin term \( \mathcal{H}_{t,s}(\lambda^*_t) \) become independent of \( \lambda^*_t \), with dynamics given in (33a) – (33b). The optimal policy is wealth-independent, and allows the decomposition:

\[
\pi_t = \pi^{mv}(t, Y_t) + \pi^r(t, Y_t) + \pi^\theta(t, Y_t)
\]

and \( \pi^\theta(t, Y_t) = \pi^h(t, Y_t) + \pi^u(t, Y_t) \), where the mean-variance component \( \pi^{mv}(t, Y_t) \) can be explicitly solved as

\[
\pi^{mv}(t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t)^\top \theta^h(t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t)\sigma(t, Y_t)^\top)^{-1} (\mu(t, Y_t) - r(t, Y_t)1_m) ;
\]

the other components and the equation characterizing \( \theta^u(v, Y_v; T) \) are explicitly given in the appendix. Under the differentiability assumption for \( \theta^u(v, y; T) \), we have \( \pi^u(t, Y_t) = \pi^{u,Y}(t, Y_t) \) in (30), i.e., the last component \( \pi^{u,\lambda}(t, X_t, Y_t) \) vanishes under the CRRA utility.

Proof. See Section S.4 in the online supplementary material.

Comparing the decomposition results in Proposition 3 with those in Theorem 1, we find that the optimal policy under the CRRA utility differs from that under general utilities in the following aspects. First, the optimal policy under the CRRA utility is independent of investor’s wealth level \( X_t \), reflecting the wealth-independent property of the CRRA utility. It is guaranteed as follows. First, the time–t multiplier \( \lambda^*_t \) does not appear in the optimal policy (34) under the CRRA utility. Second, with the simplified structure of \( \theta^u(s, Y_s; T) \), the building blocks \( \xi^S_{t,s} \) and \( \mathcal{H}_{t,s} \) in (33a) – (33b) are also independent of \( \lambda^*_t \). It ensures that the wealth level \( X_t \) is not implicitly involved in the optimal policy.

Next, the mean-variance component \( \pi^{mv}(t, Y_t) \) can be explicitly solved as (35) under the CRRA utility. It is given as the product of the inverse covariance matrix \( (\sigma(t, Y_t)^{-1}) \) and the excess return \( \mu(t, Y_t) - r(t, Y_t)1_m \), and then further divided by the investor’s risk aversion level \( \gamma \). Moreover, the mean-variance component is “myopic” under the CRRA utility, as it is independent of the investor’s horizon. Finally, the last component \( \pi^{u,\lambda}(t, X_t, Y_t) \) in (30) vanishes in the CRRA policy, as \( \theta^u(s, Y_s; T) \) no longer depends on the multiplier \( \lambda^*_s \). It suggests investors do not need to hedge the uncertainty in investor-specific price of risk due to variation in their wealth level. This again reflects the impact of the wealth-independent property of the CRRA utility.

3.4 Discussing the complete market case

We now briefly describe the decomposition of optimal policy under complete market models. By comparing it with its counterpart under incomplete market models, we can reveal the fundamental impact of market incompleteness on optimal portfolio allocation. The complete market model is set under the general model (1) and (2) with the number of risky assets being equal to the number of driving Brownian motions, i.e., \( m = d \). In addition, we assume that the resulting square matrix
\( \sigma(t, y) \) is non-singular. Thus, we can fully hedge the risk by investing in the risky assets, and do not need the completion procedure with fictitious assets. In this case, the market price of risk in (9a) is defined as \( \theta(t, Y_t) := \sigma(t, Y_t)^{-1} (\mu(t, Y_t) - r(t, Y_t)1_m) \), while the investor-specific price of risk \( \theta_t^u \) vanishes as the investor does not need to complete the market by fictitious assets. Then, the decomposition of the optimal policy follows as a special case of the incomplete market results in Theorem 1, via replacing the components by their counterparts in the complete market.

The market completeness introduces the following two simplifications of optimal policy decomposition. First, as the undetermined investor-specific price of risk \( \theta^u \) is not involved, the dynamics of the relative state price density \( \xi_{t,s} \) and Malliavin term \( H_{t,s}^q \), as given in (21) and (26) for general incomplete markets, are now explicitly given. Furthermore, they no longer depend on the multiplier \( \lambda^* \), thus are independent of investor’s wealth level. Second, it is straightforward to verify that the last component, i.e., \( \pi^u(t, X_t, Y_t) \), in the optimal policy decomposition (16) vanishes under the complete market model, as the investors do not need to hedge the uncertainty in the investor-specific price of risk. We omit further details, and refer to, e.g., Detemple et al. (2003).

4 Closed-form optimal policies for wealth-dependent HARA utility

In the previous section, we have revealed the fundamental impact of wealth-dependent utility on optimal policy in general incomplete market models. As discussed in Section 3.1, such impact essentially stems from the special structure of the investor-specific price of risk, which implicitly depends on the investor’s wealth level \( X_v \) via the multiplier \( \lambda^*_v \). In this and subsequent sections, we show that the wealth impact on investor’s portfolio allocation goes far beyond this. In particular, we study the optimal policy under the HARA utility in (5), which is a representative wealth-dependent utility. First, under nonrandom but possibly time-varying interest rate, we show that the optimal policy under the HARA utility can be decomposed into a bond holding scheme and the corresponding CRRA portfolio. Next, we solve the optimal policies in closed-form under the Heston-SV model for investors with HARA utility. We further calibrate the model using the S&P 500 ETF data in the recent ten years, which is then used in Section 5 to illustrate the static and dynamic impacts of wealth-dependence and get further economic insights.

4.1 Connection between optimal policies under HARA and CRRA utilities

We first apply our decomposition results in Theorems 1 and 2 to explicitly characterize the optimal policy under HARA utility. To save space, we provide and discuss the results in Corollary 4 of Appendix B. Comparing with the CRRA policy in Section 3.3, our decomposition clearly reveals the channels for investor’s wealth to affect the optimal policy under the HARA utility. These channels are essentially introduced by the wealth constraints \( \bar{c} \) and \( \bar{x} \) in (5) for HARA investors.
Next, we study the optimal policy under HARA utility with nonrandom but possibly time-varying interest rate. In this special case, we show the investor-specific price of risk under HARA utility is indeed identical to that under the corresponding CRRA utility. Furthermore, the optimal policies under HARA and CRRA utilities are connected to each other by a simple multiplier related to current wealth level and bond prices. This new relationship sheds light on the construction of the optimal policy under the HARA utility. The results are summarized in Proposition 4 below.

**Proposition 4.** With nonrandom but possibly time-varying interest rate \( r_t \), the investor-specific price of risk \( \theta^u \) under HARA utility (5) coincides with its counterpart under CRRA utility. It does not depend on the multiplier \( \lambda^u \) and allows the representation \( \theta^u = \theta^u (v, Y_V; T) \). The optimal policy under HARA utility satisfies the following simple ratio relationship with its counterpart under CRRA utility:

\[
\pi^u(t, X_t, Y_t) = \pi^u_C(t, X_t, Y_t) \frac{X_t}{\bar{X}_t},
\]

as well as \( \bar{X}_t = X_t - \bar{X}_{t,T} \) due to the deterministic nature of interest rate. Here, the subscripts \( H \) and \( C \) represent for the HARA and CRRA utilities. Besides, \( \bar{X}_t \) in (36) is given by

\[
\bar{X}_t = X_t - \bar{X}_{t,T} - \overline{\int_t^T B_{t,s} ds}, \quad \text{for } w > 0,
\]

and

\[
\bar{X}_t = X_t - \bar{X}_{t,T}, \quad \text{for } w = 0,
\]

where \( B_{t,s} := \exp(-\int_t^s r_v dv) \) is the price at time \( t \) for a zero-coupon bond with face value one that matures at time \( s \).

**Proof.** See Appendix C.1. \( \square \)

We now discuss the main findings in Proposition 4. First, the investor-specific prices of risk under the two utilities agree with each other. That is, when there is no uncertainty in the interest rate, the investor completes the market in exactly the same way under the two utilities, and the impact of the current wealth level entirely vanishes in the investor-specific price of risk for HARA investors. Second, the ratio relationship (36) connects the optimal policy under HARA utility to its counterpart under the CRRA utility, which is much easier to solve in closed form (or implement via, e.g., simulation methods) due to its wealth-independent nature. The relationship (36) provides a convenient way to compute the optimal policy under HARA utility. Although we assume a deterministic interest rate in Proposition 4, no assumptions are imposed on the state variable. Thus, the relationship (36) can be applied to various models with sophisticated state variables and complex dynamics. In Section 4.2, we explicitly illustrate such an application using the Heston-SV model used in Liu (2007). Extension of Proposition 4 to the case with random interest rates can be regarded as an open research topic,
for which the change of numeraire techniques in Detemple and Rindisbacher (2010) may render a useful tool.

The relationship (36) allows for the following intuitive economic interpretation. With a deterministic interest rate, \( B_{t,s} \) represents the time-\( t \) price of a zero-coupon bond with face value one maturing at time \( s \). Thus, \( \bar{X}_t \) given in (37a), i.e., \( \bar{X}_t = X_t - \bar{\pi}B_{t,T} - \bar{\pi}\int_t^T B_{t,s} ds \) is the remaining wealth after the investor buys \( \bar{\pi} \) zero-coupon bonds maturing at \( T \) and a continuum of \( \bar{\pi} \) zero-coupon bonds maturing at \( s \) for all \( s \in [t, T] \). The continuous payments from this bonds holding position exactly render the minimum terminal wealth \( \bar{\pi} \) and intermediate consumption \( \bar{\pi} \) required by the HARA utility (5), i.e., \( \bar{\pi} \) at time \( T \) and \( \bar{\pi} \) at each \( s \in [t, T] \). After purchasing the bonds, the HARA investor allocates the remaining wealth following the optimal policy under the CRRA utility, i.e., \( \pi_{mv}^C(t, Y_t) \bar{X}_t \) and \( \pi_{\theta}^C(t, Y_t) \bar{X}_t \) for the mean-variance and price of risk hedge components. It leads to the optimal policy under HARA utility given in (36). To summarize these insights, the HARA investor first buys a series of zero-coupon bonds to satisfy the minimum requirements for terminal wealth and intermediate consumptions over the entire investment horizon, then allocates her remaining wealth just as a pure CRRA investor.\(^{13}\) This simple but important relationship demonstrates again the application potential of our decomposition results for general incomplete market models in Theorems 1 and 2. Such a structure explains how HARA investors maximize their expected utility while fulfilling the minimum requirements for terminal wealth and intermediate consumptions.

From an economic aspect, we can view relationship (36) as a decomposition of the HARA policy that separates the roles of the state variable and the investor’s wealth level: the state variable \( Y_t \) impacts the optimal policy only via the CRRA policies \( \pi_{mv}^C(t, Y_t) \) and \( \pi_{\theta}^C(t, Y_t) \), while the current wealth level \( X_t \) impacts the optimal policy only via the ratio \( \bar{X}_t/X_t \). Moreover, the decomposition (36) implies the following behavior of the optimal policy. First, the HARA investors allocate more on risky assets as their wealth level increases, since the ratio \( \bar{X}_t/X_t \) monotonically increases in the current wealth level \( X_t \). In the limit as \( X_t \) approaches infinity, the optimal policies \( \pi_{mv}^H(t, X_t, Y_t) \) and \( \pi_{\theta}^H(t, X_t, Y_t) \) converge to their CRRA counterparts, as the multiplier \( \bar{X}_t/X_t \) converges to one. These findings reconcile the analysis from the Arrow-Pratt relative risk aversion coefficient defined by \( \gamma^U(x) := -(\partial U(t, x)/\partial x)^{-1}x\partial U^2(t, x)/\partial x^2 \). Under the HARA utility for terminal wealth, it is given by \( \gamma^U(X_t) = \gamma X_t/(X_t - \pi) \), which decreases in \( X_t \) and converges to its CRRA counterpart \( \gamma \) as \( X_t \) approaches infinity. Thus, the decrease in risk aversion degree motivates HARA investors to invest more in risky assets as their wealth level increases.

\(^{13}\) A similar intuition is developed in Detemple and Zapatero (1992). They show that under a complete market with deterministic coefficients, investors with habit formation will first invest in a perfectly safe portfolio that finances habit consumption and then invest as a standard CRRA investor.
4.2 Closed-form HARA policy under Heston-SV model

In what follows, we solve the optimal policy in closed form for HARA investors under the Heston-SV model, applying the closed-form relationship of Proposition 4. We begin by setting up the model. The asset price \( S_t \) follows
\[
dS_t/S_t = (r + \lambda V_t)dt + \sqrt{V_t}dW_{1t}, \tag{38a}
\]
and the variance \( V_t \) follows
\[
dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t}(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}), \tag{38b}
\]
where \( W_{1t} \) and \( W_{2t} \) are two independent standard one-dimensional Brownian motions. Here, the parameter \( r \) denotes the risk-free interest rate; the parameter \( \lambda \) controls the market price of risk; the positive parameters \( \kappa, \theta, \) and \( \sigma \) represent the rate of mean-reversion, the long-run mean, and the proportional volatility of the variance process \( V_t \). We assume the Feller’s condition holds: \( 2\kappa \theta > \sigma^2 \).

The leverage effect parameter \( \rho \in [-1, 1] \) measures the instantaneous correlation between the asset return and the change in its variance.

The Heston-SV model belongs to the class of affine models (Duffie et al., 2000). This model and its generalizations can match a number of stylized facts in stock return and variance dynamics, and are thus widely applied in the empirical literature (see, e.g., Poteshman (2001) and Pan (2002)). For example, when \( \rho < 0 \), the Heston-SV model captures the “leverage effect”, i.e., the stock return and variance are negatively correlated. Besides, the market price of risk (Sharpe ratio) of the risky asset, \( \lambda \sqrt{V_t} \), is proportional to \( \sqrt{V_t} \) and thus increases in the volatility. As noted in Liu (2007), this feature of the Heston-SV model is empirically supported by Campbell and Cochrane (1999). In what follows, we solve the optimal policy under the Heston-SV model for investors with HARA utility over the terminal wealth, i.e.,
\[
U(T, x) = (x - \bar{x})^{1-\gamma}/(1-\gamma) \tag{39}
\]
for some risk aversion coefficient \( \gamma > 1 \). Note that in the case of \( \bar{x} = 0 \), it reduces to the CRRA utility, i.e.,
\[
U(T, x) = x^{1-\gamma}/(1-\gamma). \tag{40}
\]

While the optimal policy for CRRA investors under the Heston-SV model is available in closed form (see Liu (2007)), its counterpart for HARA investors, to our best knowledge, is absent in the literature. It is probably due to the common consensus that HARA utility usually causes mathematical inconvenience. We circumvent this challenge by applying Proposition 4 on decomposing the optimal policy for HARA investors, and deduce the next corollary.

Corollary 1. Under the Heston-SV model in (38a) – (38b) and the HARA utility (39) over terminal wealth, the optimal policy \( \pi_H(t, X_t, V_t) \) can be solved in closed-form as
\[
\pi_H(t, X_t, V_t) = \frac{\bar{X}_t}{X_t}\pi_C(t, V_t), \tag{41}
\]
where $\bar{X}_t$ follows
\[
\bar{X}_t = X_t - \pi \exp(-r(T - t)),
\]
and $\pi_C(t, V_t)$ is the optimal policy under the corresponding CRRA utility (40). It admits the following closed-form decomposition
\[
\pi_C(t, V_t) = \pi_C^{mv}(t, V_t) + \pi_C^\theta(t, V_t),
\]
where the mean-variance component and the price of risk hedge component are given by $\pi_C^{mv}(t, V_t) = \lambda / \gamma$ and $\pi_C^\theta(t, V_t) = -\rho \delta \phi(T - t)$. The function $\phi(\tau)$ is defined by
\[
\phi(\tau) = \frac{2[\exp(\varsigma \tau) - 1]}{(\bar{\kappa} + \varsigma) [\exp(\varsigma \tau) - 1] + 2\varsigma},
\]
with $\bar{\kappa} = \kappa - (1 - \gamma) \lambda \rho \sigma / \gamma$, $\delta = -(1 - \gamma) \lambda^2 / (2 \gamma^2)$, and $\varsigma = \sqrt{\bar{\kappa}^2 + 2\delta \sigma^2 (\rho^2 + \gamma^2(1 - \rho^2))}$. Finally, the investor-specific price of risk, i.e., the shadow price of market incompleteness, can be solved in closed form as
\[
\theta^u(t, V_t; T) = (0, \theta^u_2(t, V_t; T))^\top,
\]
where
\[
\theta^u_2(t, V_t; T) = -\frac{1}{\rho} \gamma \sqrt{1 - \rho^2} \sqrt{V_t} \pi_C^\theta(t, V_t) = \gamma \sqrt{1 - \rho^2} \sigma \delta \phi(T - t) \sqrt{V_t}.
\]

Proof. See Appendix C.2. \qed

The above explicit results illustrate again how our decomposition can be potentially applied to obtain closed-form optimal policies under sophisticated settings. They will be further applied in Section 5 for conducting economically insightful comparative studies. Closed-form solution (44) for the investor-specific price of risk $\theta^u$ provides a concrete example for solving the complex equation system developed in Theorem 2, demonstrating the existence, uniqueness, and differentiability of the solution for that particular model. Besides, straightforward calculation based on (9b) renders $\theta^u_2(t, V_t; T) = (\mu^F_t - r) / \sigma^F_2(t, V_t)$, i.e., $\theta^u_2(t, V_t; T)$ also equals the Sharpe ratio of the fictitious asset. Similar to the market price of risk $\lambda \sqrt{V_t}$, the investor-specific price of risk increases in variance via the square-root term $\sqrt{V_t}$. However, it also additionally depends on the remaining investment horizon $T - t$, as revealed in representation (15).

### 4.3 Empirical model estimation

To achieve empirical validity up to date in the subsequent analysis, we estimate the parameters of Heston-SV model based on the daily data of SPDR S&P 500 ETF from the recent ten years, i.e., 2010/1/3 to 2019/12/31. We employ the maximum likelihood estimation approach developed by Aït-Sahalia et al. (2020), which is an efficient method for estimating continuous-time models with latent factors. The annualized parameter estimates are obtained as
\[
\kappa = 12.5850, \ \rho = -0.8141, \ \lambda = 6.6992, \ \theta = 0.0193, \ r = 0.0051, \ \text{and} \ \sigma = 0.5385.
\]
The large value of $\kappa$ indicates that the variance process is highly mean-reverting. The large negative value of $\rho$ implies a strong leverage effect, i.e., the price changes of the risky asset and its variance process are negatively correlated. Moreover, the market is characterized by a low long-run volatility $\sqrt{\theta} \approx 0.1389$, a low risk-free rate $r$, and a high market price of risk coefficient $\lambda$. These features reflect the behavior of the US financial market between 2010 and 2020, which is the longest bull market in the history, associated with remarkably low volatility and interest rate, as well as significantly high returns (e.g., the S&P 500 Index surges by 185% during this period).

5 Static and dynamic impacts of wealth-dependence

In this section, we further reveal the impact of wealth-dependent utility on optimal portfolio allocation using the representative Heston-SV model with HARA utility. First, we investigate static impacts from wealth-dependent utility. Second, we examine dynamic aspects, namely cycle-dependence of optimal policy. Then, we quantify the dynamic impacts of the wealth-dependent utility on investment performance for delegated portfolio management, i.e., how the investment performance varies with the investor’s initial wealth level and its interaction with other model parameters. It induces a risk-return trade-off that is unseen under wealth-independent utilities. Next, we demonstrate a novel hysteresis effect in the portfolio allocation, which stems from cycle-dependence of optimal policy under the wealth-dependent HARA utility. We show that the optimal policy and investment performance depend not only on the composition of market states, but also on their sequence of occurrence. Thanks to the closed-form optimal policy under the estimated Heston-SV model of Section 4.2, we can use path simulations to quantify hysteresis effect by taking different market scenarios into account, instead of using just one stock price path. Our analysis provides novel valuable understandings for delegated portfolio management.

5.1 Static impacts from wealth-dependence

We now apply the closed-form formulae (41) – (43) under the Heston-SV model to explicitly examine how the optimal policy $\pi_H(t, X_t, V_t)$ is impacted by the wealth-dependent property of the HARA utility. In particular, we examine static impacts via three channels: the wealth level $X_t$, the interest rate $r$, and the investment horizon $T - t$. For the subsequent impact analysis, it helps that the corresponding optimal policy $\pi_C(t, V_t)$ under CRRA utility is independent of the wealth level $X_t$ and the interest rate $r$. In the three panels of Figure 1, we plot how the optimal policies vary with the current wealth level $X_t$ (left), interest rate $r$ (middle), and investment horizon $T - t$ (right). The red square and blue circle curves denote the policy under CRRA and HARA utilities. By (41), we see that the optimal policy $\pi_H(t, X_t, V_t)$ under HARA utility is impacted by the
wealth level \(X_t\) via the ratio \(\tilde{X}_t/X_t\). By (42), we can express this ratio as

\[
\frac{\tilde{X}_t}{X_t} = 1 - \left(\frac{X_t}{\bar{X}}\right)^{-1} \exp(-r(T-t)).
\] (46)

Here, \(X_t/\bar{X}\) measures the current wealth level \(X_t\) relative to the minimum requirement \(\bar{X}\). This formula explicitly shows that \(\tilde{X}_t/X_t\) increases concavely in \(X_t\) with \(\lim_{X_t\to\infty} \tilde{X}_t/X_t = 1\). The left panel of Figure 1 shows that the impact of wealth level can be substantial. As \(X_t\) increases from \(2\bar{X}\) to \(10\bar{X}\), the allocation on the risky asset from HARA investors increases by approximately 70% from 97% to 167%. But such an impact diminishes as the wealth level becomes higher, i.e., the minimum requirement constraint becomes less binding. Such a concave increase in the allocation for risky asset with respect to investor’s wealth is empirically observed in household finance literature; see, e.g., Roussanov (2010), Wachter and Yogo (2010), and Calvet and Sodini (2014).

![Figure 1: Behaviors of optimal policy in the Heston-SV model under the HARA and CRRA utilities.](image)

The three panels plot the optimal policies in the Heston-SV model under the HARA (blue circle) and CRRA (red square) utilities at different wealth level (left), interest rate (middle), and investment horizon (right). These figures are generated using our estimated parameters in (45) with the risk aversion parameter \(\gamma = 4\). Besides, we set \(T - t = 10\) years in the left and middle panels, and \(X_t/\bar{X} = 3\) in the middle and right panels.

We next analyze how the optimal policy changes with the interest rate \(r\). With a higher interest rate, the risk-free asset delivers a higher return. Thus, we might expect that investors should increase their allocation on the risk-free asset, and accordingly decrease their allocation on the risky asset. However, thanks to our closed-form optimal policy, we can show that the investors with HARA utility will actually behave in the opposite way, i.e., they will increase their allocation on the risky asset under a higher interest rate. Indeed, by the closed-form formulae (41) and (42), we find that the optimal policy \(\pi_H(t, X_t, V_t)\) depends on the interest rate \(r\) via the ratio \(\tilde{X}_t/X_t\), which increases concavely in \(r\) according to (46). We can interpret this behavior according to the optimal investment...
strategy described after Proposition 4. While a higher risk-free rate does not impact the optimal policy under CRRA utility, it lowers the bond price \( B_{t,T} = \exp(-r(T-t)) \). So, it costs less for HARA investors to ensure their minimum requirement \( \bar{x} \) by investing in bonds, and thus increases their remaining wealth \( \bar{X}_t \) for investing in the risky asset following the CRRA policy. The above analysis is demonstrated by the middle panel of Figure 1, where we observe that the optimal policy \( \pi_H(t, X_t, V_t) \) increases concavely with \( r \). The magnitude of such an impact is indeed sizeable: the optimal policy \( \pi_H(t, X_t, V_t) \) increases by approximately 30% as the interest rate increases from 0 to 0.08.

Finally, by the closed-form formulae (43), the investment horizon \( T-t \) affects the optimal policy \( \pi_C(t, V_t) \) for CRRA investors only through the price of risk hedge component \( \pi^\theta_C(t, V_t) = -\rho \sigma \phi(T-t) \). When \( \rho < 0 \), this component increases monotonically in \( T-t \). However, for HARA investors, in addition to the dependence through the corresponding CRRA policy as mentioned above, their optimal policy is impacted by \( T-t \) also via the ratio \( X_t/\bar{x} \) according to (41) and (46), which increases concavely in \( T-t \). The right panel of Figure 1 shows that both the optimal policies \( \pi_C(t, V_t) \) and \( \pi_H(t, X_t, V_t) \) increase with the investment horizon \( T-t \). For the CRRA case, the sharp increase due to the price of risk hedge component mainly occurs when the investment horizon is short; under longer investment horizons, the optimal policy becomes almost insensitive to \( T-t \). For the HARA case, however, beside the similar sharp increase for short horizons, \( \pi^\theta_H(t, X_t, V_t) \) keeps increasing in \( T-t \) even for longer investment horizons. We can interpret these behaviors as follows. Under CRRA utility, a longer investment horizon increases the uncertainty in the price of risk. It leads to a larger hedging demand for risk-averse investors, and results in a larger hedge component in the optimal policy. On the other hand, the increase in \( \pi^\theta_H(t, X_t, V_t) \) under the HARA utility is generated by a combination of two effects: first, the decrease of the bond price \( B_{t,T} = \exp(-r(T-t)) \) and thus the increase of the remaining wealth \( \bar{X}_t \) in (42), and second, the increase of the corresponding CRRA hedging demand as discussed above. The first effect is significant even for long investment horizons, leading to a more lasting impact from the investment horizon.

The above comparative analysis illustrates how we can apply our theoretical decompositions to understand the behavior of the optimal policy under incomplete market models with wealth-dependent utilities. In particular, we have explicitly shown that the wealth-dependent property of the HARA utility should not be taken only “literally”. Precisely speaking, by Proposition 4, the optimal policy of HARA investors is determined by the remaining wealth \( \bar{X}_t \), which depends on the current wealth level \( X_t \), interest rate \( r \), and investment horizon \( T-t \) according to (42). It gives further understanding of what is induced by the wealth-dependent property of the HARA utility.
5.2 Cycle-dependence of optimal policy

In addition to the above static analysis of optimal policies at an arbitrary fixed time $t$, we now employ the Heston-SV model as representative model to analyze the significant wealth impact of HARA utility from a dynamic perspective, i.e., how the wealth-dependent property of HARA utility interacts with the complex market dynamics in determining the optimal allocation strategy and overall investment performance. In the literature, the vast majority of studies on optimal dynamic portfolio allocation focuses on solving optimal policies and then conducting static impact analysis from model specifications under wealth-independent utilities; see e.g., Wachter (2002), Chacko and Viceira (2005), and Liu (2007), among others. As an important issue in our setting, the dynamic impacts on investment performance over different asset allocation strategies and/or investor profiles remain to be investigated. Here, we aim to shed light on this under an environment with stochastic volatility and wealth-dependent utility. A recent study on the dynamic impacts from different policies can be found in Moreira and Muir (2019). It relies on a stochastic volatility model and shows that ignoring the hedge component in the optimal policy would lead to a substantial utility loss. However, Moreira and Muir (2019) assume that the investor shares the recursive utility of Duffie and Epstein (1992), which generalizes the wealth-independent CRRA utility. Then, the derived optimal policy is independent of the investor’s wealth level.

In contrast, the wealth-dependent property of HARA utility leads to a sophisticated dependence of the optimal portfolio policy on the entire path of asset price and its volatility. This feature is totally absent under the simple CRRA utility. The ratio $\bar{X}_t / X_t$ given in (46) is determined by the investor’s wealth level $X_t$, which evolves according to the following explicit dynamics

$$dX_t = rX_t dt + [X_t - \bar{X}\exp(-r(T - t))]\left[\frac{\lambda}{\gamma} - \rho \sigma \delta \phi(T - t)\right] V_t dt + \sqrt{V_t} dW_1.$$

(47)

It is obtained by plugging in the HARA policy (41), specifying $\mu_t = r + \lambda V_t$, $\sigma_t = \sqrt{V_t}$, as well as $r_t = r$ in (3) according to the specification of Heston-SV model in (38a), and setting $c_t \equiv 0$ since intermediate consumption is not included in investor’s utility. The wealth dynamics (47) obviously depends on the entire path of the stock price $S_t$ and variance $V_t$, due to their common Brownian shock $W_{1t}$. Thus, the optimal policy of HARA investors is also influenced by the historical path of market dynamics through the ratio $\bar{X}_t / X_t$ in (41). On the contrary, under CRRA utility, we see from (43) that the optimal policy $\pi_C(t, V_t)$ is simply independent of both current wealth level $X_t$ and variance $V_t$, and only depends on the remaining investment horizon $T - t$. Thus, the optimal policy of CRRA investor is entirely deterministic and does not depend on the realization of the model dynamics (38a) – (38b). This stark contrast has important implications on delegated portfolio management. The delegated portfolio manager for HARA investors must adjust the optimal policy in a stochastic way according to the realization of market dynamics, while, in contrast, portfolio managers for CRRA investors can simply determine the optimal policy for the entire investment horizon by (43) at the
beginning of the period.

Let us illustrate the dynamic impact of the HARA utility, as discussed above in theory, by examining how the initial wealth level of HARA investors affects their optimal portfolio allocation over the entire investment horizon. Consider a market where the stock price $S_t$ and its variance $V_t$ follow the Heston-SV model with parameters given in (45). Without loss of generality, the initial price and variance are set as $S_0 = 100$ and $V_0 = \theta = 0.0195$. We consider two investors with HARA utilities over terminal wealth for an investment horizon of $T = 10$ years. Their risk aversion coefficient and minimum requirement for terminal wealth are set to $\gamma = 4$ and $\pi = 10^6$, i.e., one million. The two investors only differ in their initial wealth levels: the high-wealth investor has an initial wealth of $X_0^H = 5 \times 10^6$, while the low-wealth investor has an initial wealth of $X_0^L = 2 \times 10^6$. Thus, their ratios of initial wealth over the minimum requirement are equal to $X_0^H / \pi = 5$ and $X_0^L / \pi = 2$. Denote the optimal policies of the two investors by $\pi_t^H$ and $\pi_t^L$. Then, the ratio $\pi_t^H / \pi_t^L$ measures how the optimal policy of the high-wealth investor differs from that of the low-wealth investor. We simulate market scenarios and the corresponding dynamics of optimal policies $\pi_t^H$ and $\pi_t^L$ for the entire investment period. We conduct the simulations using a standard Euler scheme on the Heston-SV model (38a) – (38b). Along the simulated path, we evaluate the optimal policies $\pi_t^H$ and $\pi_t^L$ via (41), and the investor’s wealth evolves according to equation (47).

Figure 2 shows a representative market scenario, i.e., a simulated path of the policy ratio $\pi_t^H / \pi_t^L$ (in red dashdotted with the right $y$-axis in both panels), and the corresponding paths of stock price $S_t$ (in blue solid with the left $y$-axis in the upper panel) and its realized variance $RV_t$ (in black solid with the left $y$-axis in the lower panel). Given the simulated path of the spot variance $V_t$, we approximate the realized variance by averaging the daily spot variance over the past one-month time window, i.e., $RV_t = \Sigma_{i=0}^{n-1} V_{i-\Delta}/n$ with $\Delta = 1/252$ and $n = 22$. We classify the market regimes following the method proposed by Lunde and Timmermann (2004): a transition from a bear to bull (resp. bull to bear) market is triggered when the stock price increases by 20% (resp. drops by 15%) from its lowest (resp. highest) level in the current bear (resp. bull) market. We represent the bear markets by the shaded areas in Figure 2. Not surprisingly, the realized variance spikes during bear markets due to the leverage effect, i.e., the changes of stock prices and variance are negatively correlated.

Recall that the optimal policy under CRRA utility, as given in (43), is independent of the variance level $V_t$, stock price $S_t$, and wealth level $X_t$. Thus, we simply have $\pi_t^H / \pi_t^L = 1$ for all $t$ under CRRA utility, i.e., the optimal policies from the high-wealth and low-wealth investors always coincide. However, under HARA utility, we observe that the ratio $\pi_t^H / \pi_t^L$ varies markedly during the investment period. The relative difference in $\pi_t^H$ and $\pi_t^L$ can be as large as 50%, i.e., the ratio $\pi_t^H / \pi_t^L$ can exceed 1.5. In particular, it is negatively correlated with the stock price, but positively correlated with the realized volatility. It is not an incidental result out of a specific path. With $10^4$ trials of simulations, we find that the average correlation between $\pi_t^H / \pi_t^L$ and $S_t$, resp. $RV_t$, is approximately
−0.90, resp. 0.10; both are statistically significant at 0.1% level. We can interpret such a pattern as follows. Under our estimated Heston-SV model, both HARA investors hold positive positions in the risky asset, so their wealth levels decrease when the stock price drops. Moreover, according to our closed-form formulae (41) and (46), the impact of the wealth level on the optimal policy is more significant when the wealth level is low, i.e., the minimum constraint is more binding, as shown by the left panel of Figure 1. Thus, when stock price drops (and realized variance increases from the leverage effect), we expect to see a larger difference in the optimal policies of high- and low-wealth HARA investors, resulting in a higher ratio $\pi_t^H / \pi_t^L$. In contrast, during the bull market when stock price increases, both investors become wealthier and behave more similarly, it leads to a smaller ratio $\pi_t^H / \pi_t^L$.

This example illustrates the impact of HARA utility from a dynamic perspective. As opposed to being fully determined by their different initial wealth levels, the investor’s optimal investment decisions indeed depend on the historical path of market performance. In particular, cycles matter for HARA investors. As we can see from Figure 2, the optimal policies of high- and low-wealth HARA investors tend to diverge, as shown by a larger ratio $\pi_t^H / \pi_t^L$, during the bear markets (marked by the shaded areas), where stock price drops and volatility spikes. However, this important market cycle dependence is totally absent under the wealth-independent CRRA utility. Such a dependence is important for delegated portfolio management. In particular, investment advices can be considerably erroneous and lead to completely suboptimal strategies if we ignore the wealth-dependence property of investor’s HARA utility.
This figure plots a simulated path of stock price $S_t$ in blue solid with left $y$-axis in the upper panel, realized variance $RV_t$ in black solid with left $y$-axis in the lower panel, and that of the corresponding optimal policy ratio between the high– and low–wealth investors under HARA utility, i.e., $\pi^H_t/\pi^L_t$, in red dashdotted with right $y$-axis in both two panels. The ratios are calculated by the closed-form formulae (41) and (43). The shaded areas depict the bear market regimes. The parameters for the Heston SV model are chosen as our estimated ones in (45).

5.3 Impacts on investment performance for delegated portfolio management

The previous analysis documents that the initial wealth of HARA investor induces a substantial impact on their portfolio allocation. In particular, the HARA investor with a higher initial wealth will allocate more of her wealth to the risky asset, as the ratio $\pi^H_t/\pi^L_t$ is greater than 1 over the investment horizon. Thus, it is intuitive that a high-wealth HARA investor enjoys higher returns but is exposed to more risk. However, precisely quantifying this intuition on investment performance remains an open problem with practical relevance for delegated portfolio management. For
example, how do the return mean and Sharpe ratio from the optimal investment strategy vary for HARA investors with different initial wealth levels? While the impact analysis as shown in Figure 2 partly sheds light on such issues, it cannot give reliable answers owing to the limitation of capturing only one specific market scenario among others. This type of limitation is common in backtesting investment strategies. In contrast, thanks to our closed-form optimal policy (41), we can precisely and quantitatively analyze the over-all impacts on investment performance by taking various market scenarios into account under our estimated Heston-SV model. Specifically, for the purpose of covering different market scenarios, we simulate a large number of paths for the investment problem for HARA investors with different levels of initial wealth.14

For each simulated path, we compute the excess return mean, volatility, and Sharpe ratio over the entire investment horizon \([0, T]\) using the annualized daily excess returns of the wealth, i.e.,

\[
R_i = \ln \left( \frac{X_i}{X_{i-1}} \right) / \Delta - r \quad \text{with} \quad \Delta = 1/252 \quad \text{and} \quad i = 1, 2, ..., N,
\]

where \(N\) denotes the total number of trading days in the entire investment horizon, i.e., \(T = N\Delta\). The excess return mean and volatility are calculated as the sample mean and standard deviation of daily excess returns, i.e.,

\[
\bar{R} = \frac{1}{N} \sum_{i=1}^{N} R_i \quad \text{and} \quad SD = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (R_i - \bar{R})^2} / (N - 1).
\]

Then, the Sharpe ratio is simply computed as \(SR = \bar{R} / SD\). Besides, we also compute the maximum drawdown of wealth, that measures the risk of extreme losses in the investment strategy. It is calculated as \(MD = \max_{0 \leq n \leq N} (1 - \frac{M_{\min}^n}{M_{\max}^n})\), where \(M_{\max}^n\) and \(M_{\min}^n\) denote the running maximum and minimum of investor's wealth until day \(n\), i.e.,

\[
M_{\max}^n = \max_{0 \leq k \leq n} X_k \Delta \quad \text{and} \quad M_{\min}^n = \min_{0 \leq k \leq n} X_k \Delta.
\]

Then, to evaluate the over-all performance across different market scenarios, we compute the average of these time–series performance statistics across all the simulated paths. Those averages estimate the expectations \(E[\bar{R}], E[SD], E[SR],\) and \(E[MD]\) over different realizations of market dynamics. In this implementation, we employ a large sample of \(10^4\) simulated paths for each initial wealth level. The left panel of Figure 3 plots the estimates for average excess return means (red square) and volatilities (blue circle) for different initial wealth levels; the middle and right panel plot the estimates for average Sharpe ratio and maximum drawdown. The dashed flat lines in each panel show the corresponding performance statistics of a CRRA investor, whose policy is independent of the initial wealth level.

Figure 3 reveals that the initial wealth level greatly influences the investment performance as expected from the results of the previous section. As shown in Figure 2, HARA investors with a higher initial wealth invest more in the risky asset, i.e., \(\pi_t^H / \pi_t^L > 1\). In Figure 3, we see that, on average, it leads to a higher mean and Sharpe ratio, but also a larger volatility and maximum drawdown. From this perspective, we can interpret the impact on investment performance as a risk-return trade-off induced by different initial wealth levels of HARA investors. Such a trade-off is sizeable: as the initial wealth \(X_0\) increases from \(\overline{\pi}\) to \(10\overline{\pi}\), the average excess return mean increases

\[14\text{Without the closed-form optimal policy (41), this task would become computationally intensive due to the nested simulations for evaluating the optimal policies during the investment period.}\]
from 3.5% to 23.5%, while the average excess return volatility increases from 4.2% to 25.4% and the average maximum drawdown of wealth increases from 8.2% to 38.1%. Thus, a high-wealth HARA investor with initial wealth $X_0 = 10\bar{x}$ expects to earn an annual excess return of more than 20%, but also to experience a more than 38% loss of the wealth due to potential market crashes. On the other hand, a HARA investor with relatively low initial wealth $X_0 = \bar{x}$ will allocate only a tiny portion of her wealth to the risky asset, because, by the investment strategy described after Proposition 4 for HARA investors, most of her wealth will be allocated to the risk-free asset in order to ensure that the minimum requirement for terminal wealth can always be satisfied. Such a conservative strategy leads to a low average annual excess return mean of 3.5%, but also a low average volatility (4.2%) and maximum drawdown (8.2%). The above-mentioned sharp contrast between the investment performance of high- and low-wealth HARA investors shows again that it is crucial to understand the wealth-dependence property of the utility in delegated portfolio management. Besides, the impact of the initial wealth is more significant at low wealth levels, and quickly decays as $X_0/\bar{x}$ further increases.

Let us now provide an analysis for the Sharpe Ratio. In the above, we know that HARA investors with higher initial wealth will allocate more to the risky asset, and it naturally leads to higher average excess return mean and volatility, as well as larger maximum drawdown. However, the increase of average Sharpe ratio with the initial wealth of HARA investors, as shown in the middle panel of

Figure 3: Performance statistics of dynamic optimal portfolio allocation with different initial wealth levels under the HARA utility.

This figure the estimates of average excess return mean and volatility (left), Sharpe ratio (middle), and maximum drawdown (right) for the investment problem under HARA utility and our empirically estimated Heston-SV model given in (45), given that investors dynamically follow the optimal policy. The corresponding levels under CRRA utility are represented by the dashed lines. The averages are computed over $10^4$ simulated paths. We set the risk aversion and investment horizon as $\gamma = 4$ and $T - t = 10$ years.

Let us now provide an analysis for the Sharpe Ratio. In the above, we know that HARA investors with higher initial wealth will allocate more to the risky asset, and it naturally leads to higher average excess return mean and volatility, as well as larger maximum drawdown. However, the increase of average Sharpe ratio with the initial wealth of HARA investors, as shown in the middle panel of

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Figure 3, appears to be puzzling at first sight. In the following, we show that it can be attributed to the cycle-dependence of optimal policy, which essentially originates from the wealth-dependent property of investor’s utility.

We begin by introducing the following notations to ease exposition. Since the daily time increment $\Delta = 1/252$ is small, we can approximate the annualized daily excess return as $R_i = \frac{\ln(X_i/\Delta) - r}{\Delta}$, according to the wealth dynamics (3) under the Heston-SV model, where $Z_i = (W_i - W_i)/\Delta$ is a normal random variable with zero mean and variance $1/\Delta$. Thus, the excess return $R_i$ is approximately linear in the optimal policy $\pi_i$.

As we mentioned, the HARA investors with higher initial wealth will allocate more wealth on the risky asset, leading to larger $\pi_i$. To eliminate the impact from the overall levels of $\pi_i$, we normalize the excess returns $R_i$ by the average optimal policy over the entire investment horizon, i.e., $\bar{\pi} = \frac{1}{N} \sum_{i=1}^{N} \pi_i$. That is, we define the scaled excess return $R'_i := R_i/\bar{\pi}$ as $\pi'_i = \pi_i/\bar{\pi}$ denotes the scale optimal policy. We then compute the mean and volatility of the scaled excess return $R'_i$ as $\bar{R}' = \frac{1}{N} \sum_{i=1}^{N} R'_i$ and $SD' = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (R'_i - \bar{R'})^2}$. For each simulated path, it is easy to verify the Shape ratio coincides with the one calculated based on the scaled excess returns, i.e., $SR = \bar{R}/SD = \bar{R}'/SD'$ since we have $\bar{R}' = \bar{R}/\bar{\pi}$ and $SD' = SD/\bar{\pi}$. Thus, while the average optimal policy $\bar{\pi}$ significantly impacts both the return mean $\bar{R}$ and volatility $SD$, it does not directly affect the Sharpe ratio.

Those computations show that the average optimal policy $\bar{\pi}$, which increases in the initial wealth level of HARA investors, cannot explain the variation in Sharpe ratio observed in the middle panel of Figure 3. Instead, we can explain the increasing pattern of Sharpe ratio by the cycle-dependence property of optimal policy under HARA utility. It can be interpreted as follows. As we analyzed in Section 5.2, the optimal policy under HARA utility depends on the historical path of market performance. Moreover, due to the wealth constraint, such cycle-dependence is more pronounced for low-wealth HARA investors, i.e., the portfolio allocation of low-wealth HARA investors is more sensitive to the path of market performance. This cycle-dependence introduces additional uncertainty in the scaled optimal policy $\pi'_i$ for HARA investors with low initial wealth, even after we normalize it with the average level $\bar{\pi}$. It leads to more volatility in the scaled excess return $R'_i$ and thus a lower Sharpe ratio for low-wealth HARA investors. On the other hand, the optimal policy of high-wealth HARA investors is less sensitive to the market performance; it reduces the uncertainty in their scaled optimal policy $\pi'_i$ and thus produces a higher Sharpe ratio.

We use simulations to illustrate the additional uncertainty in the scaled optimal policy $\pi'_i$ for low-wealth HARA investors. Figure 4 exhibits the representative quantiles of the scaled optimal policy $\pi'_i$ for HARA investors with initial wealth $X_0 = 1$ (top) and $X_0 = 4$ (bottom). For ease of comparison, we use the same vertical axis in the two panels. We show the quantiles at the beginning of each quarter in the investment horizon, which are computed based on $10^4$ simulation
trials. The drop near the end is due to the decrease in the price of risk hedge component when investment horizon shrinks to zero, as analyzed in Section 5.1. Comparisons between the two panels, i.e., the low- vs. high-wealth investors, clearly support our interpretations. The distributions of \( \pi'_i \Delta \) spread out over much wider ranges for the low-wealth investor, suggesting more uncertainty in her optimal policy. It explains the increasing pattern of the expected Sharpe ratio in the middle panel of Figure 3.

Figure 4: Representative quantiles of the scaled optimal policy \( \pi'_i \Delta \) for HARA investors

This figure summarizes the distribution of the scaled optimal policy \( \pi'_i \Delta \) at every quarter for HARA investors with initial wealth \( X_0/\tau = 1 \) (top) and \( X_0/\tau = 4 \) (bottom), based on the simulation trials under the estimated Heston-SV model given in (45). Besides, we set the risk aversion and investment horizon as \( \gamma = 4 \) and \( T-t = 10 \) years. The box plots gather the 2.5th and 97.5th percentiles, the first and third quartiles, as well as the median, by using the short bars at the ends of two whiskers, the upper and lower edges of the blue box, as well as the red bar inside the box.
5.4 Interaction between wealth level and model parameters

In Section 5.1, we have shown that the wealth-dependent property of HARA utility can affect the optimal policy via other channels besides the current wealth level, i.e., the interest rate \( r \) and investment horizon \( T \). In this section, we study such effects from a dynamic perspective, i.e., the impact of interest rate and investment horizon on the investment performance for HARA investors. Moreover, we analyze how these impacts vary for HARA investors with different initial wealth levels. Such variation reflects the dynamic interaction between model parameters and investor’s wealth level.

![Figure 5: Performance statistics of dynamic optimal portfolio allocation with different interest rate under the HARA utility.](image)

This figure plots the average excess return mean (left panel), volatility (middle panel), and maximum drawdown (right panel) at different levels of interest rate \( r \) for the investment problem under HARA utility and our empirically estimated Heston-SV model given in (45). We consider three investors: low-wealth HARA investor with \( X_0/\bar{x} = 1 \) (blue circle), high-wealth HARA investor with \( X_0/\bar{x} = 4 \) (red square), and a CRRA investor (black dashed). The averages are computed over \( 10^4 \) simulated paths. We set the risk aversion and investment horizon as \( \gamma = 4 \) and \( T - t = 10 \) years.

In Figure 5, we plot the average excess return mean (left panel), volatility (middle panel), and maximum drawdown (right panel) at different levels of interest rate \( r \). The other model parameters are given in (45). We set the investment horizon \( T = 10 \) years and the risk aversion level \( \gamma = 4 \). Same as other studies, we estimate the investment performance by \( 10^4 \) simulation trials. In each panel, we show the corresponding statistics for three investors: low-wealth HARA investor with \( X_0/\bar{x} = 1 \) (blue circle), high-wealth HARA investor with \( X_0/\bar{x} = 4 \) (red square), and a CRRA investor (black dashed). The optimal policy of the CRRA investor corresponds to the limit case of \( X_0/\bar{x} \to \infty \) for a HARA investor, as suggested by the relationship (41).
We now examine how the investment performance changes with interest rate \( r \) for the three investors. By the left and middle panels of Figure 5, we see the average excess return mean and volatility increase in \( r \) for the two HARA investors, but remain almost unchanged for the CRRA investor. Moreover, the increase is much more significant for the low-wealth HARA investor than that for the high-wealth one. It can be interpreted as follows. First, by the discussion in Section 5.3, a higher interest rate decreases the prices of bonds held by HARA investors. Second, with a higher interest rate, the investor’s wealth increases faster as her savings account enjoys a higher return. Both effects lead to more allocation on the risky asset for the HARA investors, which produces higher average excess return mean and volatility. Moreover, by Figure 1, the optimal policy of the low-wealth HARA investor is more sensitive to the wealth level, as her wealth constraint is more binding. Thus, the impact of \( r \) is larger for the low-wealth HARA investor than that for the high-wealth one. On the other hand, the optimal policy of the CRRA investor is independent of both the wealth level and interest rate. So the corresponding average excess return mean and volatility are almost unaffected by \( r \).

In addition, by the right panel of Figure 5, we find that the average maximum drawdown can react differently to interest rate \( r \) for the three investors. In particular, the average maximum drawdown decreases slightly in \( r \) for the CRRA and high-wealth HARA investors, but increases significantly for the low-wealth HARA investor. We can explain such different responses by the dual role played by interest rate in affecting the maximum drawdown. On one hand, a higher \( r \) increases the return of investor’s portfolio,\(^{15}\) leading to a smaller maximum drawdown. On the other hand, as \( r \) increases, the wealth of HARA investors accumulates at a faster speed. Thus, they would invest more aggressively on the risky asset. It introduces more risk and thus increases the maximum drawdown, as discussed in Section 5.3. For the low-wealth HARA investor, the second effect dominates as her optimal policy is more sensitive to the wealth level. It produces the increasing pattern of average maximum drawdown in the interest rate. However, for the CRRA and high-wealth HARA investors, the first effect outweighs the second as their optimal policy is less impacted by the wealth level. It explains the mild decreasing pattern of average maximum drawdown in the right panel of Figure 5.

\(^{15}\) To see this, note that the instantaneous expected return of portfolio is given by \( r + \lambda \pi_t V_t \).
Figure 6: Performance statistics of dynamic optimal portfolio allocation with different investment horizon under the HARA utility.

This figure plots the average excess return mean (left), volatility (middle), and maximum drawdown (right) at different investment horizon $T$ for the investment problem under HARA utility and our empirically estimated Heston-SV model given in (45). We consider three investors: low-wealth HARA investor with $X_0/\bar{x} = 1$ (blue circle), high-wealth HARA investor with $X_0/\bar{x} = 4$ (red square), and a CRRA investor (black dashed). The averages are computed over $10^4$ simulated paths. We set the risk aversion and investment horizon as $\gamma = 4$ and $T - t = 10$ years.

Next, Figure 6 shows how the investment horizon impacts the investment performance for the three investors. The average excess return mean, volatility, and maximum drawdown are plotted in the left, middle, and right panel. The model parameters are set as (45) and the risk aversion is $\gamma = 4$. By the left and middle panels, we see the average excess return mean and volatility increase in the investment horizon $T$ for the two HARA investors, but stay almost unchanged for the CRRA investor.\(^{16}\) In addition, the magnitudes are much larger for the low-wealth HARA investor. We can explain these patterns by the wealth effect as discussed for the impact of interest rate. With longer investment horizons, investors can accumulate more wealth as time goes by, leading to higher wealth levels. Besides, by the analysis in Secion 5.1, a longer investment horizon makes the bond prices lower. Both effects increase the allocation on the risky asset for the HARA investors, which translates to higher average excess return mean and volatility. Again, these effects are more significant for the low-wealth HARA investor, leading to a larger increase in the average excess return mean and volatility.

Finally, the right panel of Figure 6 shows that the average maximum drawdown increases in the investment horizon $T$ for all the three investors, and the magnitude of increase is the largest for

\(^{16}\)The very small increase in the average excess return mean for the CRRA investor is due to the increase in the price of risk hedge component when investment horizon becomes longer, see the right panel of Figure 1.
the low-wealth HARA investor. Indeed, increasing $T$ leads to a larger average maximum drawdown for two reasons. First, the maximum drawdown naturally increases in $T$ as investors are likely to experience bigger loss when they have a longer investment horizon. Second, as analyzed before, a longer $T$ increases the allocation on the risky asset from HARA investors. It translates to more uncertainty that also contributes to a larger maximum drawdown. The second effect is absent for the CRRA investor, but more significant for the low-wealth HARA investor. Combining these two effects explain the patterns in the right panel of Figure 6 for the average maximum drawdown.

5.5 Hysteresis effect in optimal portfolio allocation

In this section, we demonstrate a novel additional impact from the wealth-dependent utility: the hysteresis effect in optimal portfolio allocation. That is, the optimal policy and investment performance depend not only on the composition of market states (e.g., bear and bull markets) as exemplified in Figure 2, but also the sequence of their occurrences. This effect is fully absent for wealth-independent utilities, and has not yet been investigated in the literature.

To illustrate this hysteresis effect, we perform the following experiments. For a given market path of stock price $S_t$ and variance $V_t$, we shuffle it by moving the “good” periods to the beginning of the investment horizon. It is done as follows. Denote the increments in the Brownian motions by $\Delta_{W,i}^{(k)} = W_{k,i} \Delta - W_{k,(i-1)} \Delta$, for $i = 1, 2, ..., T/\Delta$ and $k \in \{1, 2\}$. The increments $\{\Delta_{W,i}^{(k)}\}$ can be viewed as the realization of random market states. It determines the path of the two Brownian motions, and thus the stock price and variance processes via (38a) – (38b). For each year, we compute the annual return of the stock as $\ln(S_t/S_{t-1})$. We select the three years with the highest annual returns. Then, we move the increments $\Delta_{W,i}^{(k)}$ in these three years to the beginning of the investment horizon for both the stock price and its volatility process. That is, we rearrange the sequence of $\{\Delta_{W,i}^{(k)}\}$, $k \in \{1, 2\}$, such that for $j = 1, 2, 3$, the increments in the year with the $j$-th highest annual return is now moved to the $j$-th year of the new path. The sequence in the other years remain unchanged. With the shuffled increments, we construct the new stock price and variance processes $S_t'$ and $V_t'$ following (38a) – (38b). Thus, the shuffling does not alter the values of the increments in the two processes, but only changes the sequence of their occurrences by moving the three years with the best stock performance to the beginning of the horizon.
Let us analyze how the shuffling of path impacts the optimal policy and investment performance. In Figure 7, we plot a representative path of stock price (upper panel) and its corresponding optimal policy (lower panel). The paths before (resp. after) the shuffling are represented by the blue solid (resp. red dashdotted) lines. The model parameters are set as (45) with the risk aversion level $\gamma = 4$. By the upper panel, we see that the stock price after shuffling increases faster at the start of the investment horizon. It is not surprising as our shuffling is to move the “good” periods ahead. On the other hand, the terminal price $S'_T$ remains almost the same after the shuffling since the values of the increments are not changed. The rearrangement of the sequence of market states has notable impact on the optimal policy. By the lower panel, we see that the optimal policy from the new path, i.e., with “good” years ahead, is generally larger than that from the original path, and the difference is especially significant in the first several years. The explanation is the wealth effect of HARA utility.
In the shuffled path with “good” periods ahead, investors will accumulate more wealth at the start of their horizon compared with the original path. It leads to larger optimal policy under the HARA utility, as we show in Section 5.1.

The comparison in Figure 7 highlights the hysteresis effect in asset allocation under the HARA utility, i.e., the sequence of market states matters. This effect essentially stems from the wealth-dependent property of the HARA utility. The hysteresis effect can also affect the investment performance. For example, with more allocation on the risky asset, the HARA investor is expected to bear more risk along the shuffled path in Figure 7. In the following, we quantify such impact using simulations with a large number of trials. In Figure 8, we plot the average excess return mean (left panel) and volatility (right panel) for HARA investors with different initial wealth levels under three scenarios. The black dashed line shows the results from original paths without any shuffling, i.e., same as the results in the left panel of Figure 8. The blue circle (resp. red square) line reports the corresponding results after we shuffle each path by moving the increments in the three years with the highest (resp. lowest) annual stock returns to the beginning of the horizon. The points on the far right in each panel represent the estimates for a CRRA investor. It coincides with the limit case as the initial wealth level goes to infinity. Same as the other studies, each average point in the figure is estimated by $10^4$ simulation trials.

By the left panel of Figure 8, we find that the average excess return means are almost identical in the three scenarios, i.e., unaffected by the shuffling. The reason for this seemingly surprising result is the following. We take the scenario with “good” periods ahead for illustration (blue circle line). In this scenario, HARA investors tend to allocate more on the risky asset as shown by the lower panel of Figure 7, leading to higher excess return means. However, in the shuffled path with “good” periods ahead, investors are more likely to experience negative shocks in the subsequent years of their investment horizon. When the negative shocks happen, the larger weights on the risky asset magnify the loss and decrease the excess return mean. These two effects offset each other and the average excess return mean remains almost unchanged. A similar discussion also applies to the scenario where we move the “bad” periods ahead. Thus, while the sequence of market states matters for the optimal policy, it almost does not impact the average excess return mean.
Figure 8: Performance statistics of dynamic optimal portfolio allocation with different initial wealth levels under the HARA utility.

This figure plots the average excess return mean (left) and volatility (right) for the investment problem under HARA utility and our empirically estimated Heston-SV model given in (45). The corresponding levels under CRRA utility are represented by point at the right ends. Three scenarios are considered: “good” periods ahead (blue circle), “bad” periods ahead (red square), and without shuffling (black dashed). The averages are computed over $10^4$ simulated paths. We set the risk aversion and investment horizon as $\gamma = 4$ and $T - t = 10$ years.

On the contrary, by the right panel of Figure 8, we see that the sequence of market states does impact the average excess return volatility. In particular, for a given initial wealth level, the scenario with “good” (resp. “bad”) periods ahead leads to the largest (resp. smallest) volatility. Such variation highlights the hysteresis effect in optimal portfolio allocation under the HARA utility and its impact on investment performance. This pattern can be interpreted following our discussion for Figure 7.

When the “good” (resp. “bad”) periods are moved ahead, investors will accumulate more (resp. less) wealth at the beginning of their horizon; it leads to more (resp. less) allocation on the risky asset under the HARA utility, and produces a larger (resp. smaller) volatility. On the other hand, the average volatilities are almost the same under the three scenarios for a CRRA investor, as shown by the points on the right end of the panel. That is, the hysteresis effect entirely vanishes under the CRRA utility. It is because the optimal policy under the CRRA utility is independent of the market path.
6 Conclusions and discussions

This paper establishes and implements a novel decomposition of the optimal policy under general incomplete-market diffusion models with flexible wealth-dependent utilities. The decomposition, as an indispensable generalization of the existing complete-market policy decomposition in, e.g., Merton (1971) and Detemple et al. (2003), contains four components: the mean-variance component, the interest rate hedge component, and two components for hedging uncertainties in market and investor-specific price of risk. The structural clarity of our decomposition reveals how investor’s wealth level impacts the optimal policy under incomplete market models. It facilitates the implementation of optimal policy via closed-form solutions or potential numerical approaches.

We apply our decomposition under the wealth-dependent HARA utility. We show that with non-random interest rate, the optimal policy under HARA utility can be decomposed as a bond holding scheme and the corresponding CRRA strategy. It further leads to closed-form solution under specific models. To demonstrate the wealth effects in optimal portfolio allocation, we analyze the behavior of optimal policy for HARA investors in a typical incomplete market featuring stochastic volatility. With the model parameters calibrated from US data, we find the wealth-dependent property leads to sophisticated cycle-dependence for optimal policy, as well as a substantial risk-return trade-off and a novel hysteresis effect in investment performance.

We can adapt or generalize our decomposition for optimal portfolio policies to other settings, e.g., the forward measure based representation considered in Detemple and Rindisbacher (2010). Moreover, it is interesting, among other possible extensions, to consider other (exotic) types of market incompleteness, e.g., the short-selling constraint or the “rectangular” constraint considered in Cvitanic and Karatzas (1992) and/or Detemple and Rindisbacher (2005), as well as the presence of jumps considered in, e.g., Aït-Sahalia et al. (2009) and Jin and Zhang (2012). Another important direction is to develop decomposition results for the optimal policy under incomplete market models and general recursive utilities, e.g., those generalizing the wealth-independent recursive utilities employed in the literature and thus leading to wealth-dependent optimal policies.
References

Aït-Sahalia, Y., Cacho-Diaz, J., Hurd, T., 2009. Portfolio choice with a jumps: A closed form solution. Annals of Applied Probability 19, 556–584.

Aït-Sahalia, Y., Li, C., Li, C. X., 2020. Maximum likelihood estimation of latent markov models using closed-form approximations. Journal of Econometrics (Forthcoming).

Basak, S., Chabakauri, G., 2010. Dynamic mean-variance asset allocation. Review of Financial Studies 23 (8), 2970–3016.

Brandt, M. W., 2010. Portfolio choice problems. In: Aït-Sahalia, Y., Hansen, L. P. (Eds.), Handbook of Financial Econometrics, 1st Edition. Vol. 1. North-Holland, San Diego, pp. 269 – 336.

Brennan, M., Schwartz, E., Lagnado, R., 1997. Strategic asset allocation. Journal of Economic Dynamics and Control 21, 1377–1403.

Brennan, M. J., 1998. The role of learning in dynamic portfolio decisions. European Finance Review 1 (3), 295–306.

Brennan, M. J., Xia, Y., 2002. Dynamic asset allocation under inflation. Journal of Finance 57 (3), 1201–1238.

Burrauchi, A., Porchia, P., Trojani, F., 2010. Correlation risk and optimal portfolio choice. Journal of Finance 65 (1), 393–420.

Calvet, L. E., Sodini, P., 2014. Twin picks: Disentangling the determinants of risk-taking in household portfolios. Journal of Finance 69 (2), 867–906.

Campbell, J. Y., Chacko, G., Rodriguez, J., Viceira, L. M., 2004. Strategic asset allocation in a continuous-time VAR model. Journal of Economic Dynamics and Control 28 (11), 2195–2214.

Campbell, J. Y., Cochrane, J. H., 1999. By force of habit: A consumption-based explanation of aggregate stock market behavior. Journal of Political Economy 107 (2), 205–251.

Carroll, C. D., Kimball, M. S., 1996. On the concavity of the consumption function. Econometrica 64 (4), 981–992.

Chacko, G., Viceira, L. M., 2005. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. Review of Financial Studies 18 (4), 1369–1402.

Cox, J. C., Huang, C.-f., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49 (1), 33–83.
Cvitanic, J., Goukasian, L., Zapatero, F., 2003. Monte Carlo computation of optimal portfolios in complete markets. Journal of Economic Dynamics and Control 27, 971–986.

Cvitanic, J., Karatzas, I., 1992. Convex duality in constrained portfolio optimization. Annals of Applied Probability 2 (4), 767–818.

Detemple, J., 2014. Portfolio selection: A review. Journal of Optimization Theory and Applications 161 (1), 1–21.

Detemple, J., Garcia, R., Rindisbacher, M., 2003. A Monte Carlo method for optimal portfolios. Journal of Finance 58 (1), 401–446.

Detemple, J., Rindisbacher, M., 2005. Closed-form solutions for optimal portfolio selection with stochastic interest rate and investment constraints. Mathematical Finance 15 (4), 539–568.

Detemple, J., Rindisbacher, M., 2010. Dynamic asset allocation: Portfolio decomposition formula and applications. Review of Financial Studies 23, 25–100.

Detemple, J. B., Zapatero, F., 1992. Optimal consumption-portfolio policies with habit formation. Mathematical Finance 2 (4), 251–274.

Duffie, D., Epstein, L. G., 1992. Stochastic differential utility. Econometrica, 353–394.

Duffie, D., Fleming, W., Soner, H. M., Zariphopoulou, T., 1997. Hedging in incomplete markets with HARA utility. Journal of Economic Dynamics and Control 21 (4-5), 753–782.

Duffie, D., Pan, J., Singleton, K. J., 2000. Transform analysis and asset pricing for affine jump-diffusions. Econometrica 68, 1343–1376.

Dumas, B., Luciano, E., 2017. The Economics of Continuous-Time Finance. MIT Press, Cambridge.

Epstein, L. G., Zin, S. E. V., 1989. Risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework. Econometrica 57 (4), 937 – 969.

Fitzpatrick, B. G., Fleming, W. H., 1991. Numerical methods for an optimal investment-consumption model. Mathematics of Operations Research 16 (4), 823–841.

He, H., Pearson, N. D., 1991. Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. Journal of Economic Theory 54, 259–304.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bonds and currency options. Review of Financial Studies 6, 327–343.
Jin, X., Zhang, A. X., 2012. Decomposition of optimal portfolio weight in a jump-diffusion model and its applications. Review of Financial Studies 25, 2877–2919.

Karatzas, I., Lehoczky, J. P., Shreve, S. E., 1987. Optimal portfolio and consumption decisions for a small investor on a finite horizon. SIAM Journal on Control and Optimization 25, 1557–1586.

Karatzas, I., Lehoczky, J. P., Shreve, S. E., Xu, G.-L., 1991. Martingale and duality methods for utility maximization in an incomplete market. SIAM Journal on Control and Optimization 29 (3), 702–730.

Karatzas, I., Shreve, S. E., 1991. Brownian Motion and Stochastic Calculus, 2nd Edition. Vol. 113 of Graduate Texts in Mathematics. Springer-Verlag, New York.

Kim, T., Omberg, E., 1996. Dynamic nonmyopic portfolio behavior. Review of Financial Studies 9, 141–161.

Koijen, R., 2014. The cross-section of managerial ability, incentives, and risk preferences. Journal of Finance 69 (3), 1051–1098.

Lioui, A., Poncet, P., 2001. On optimal portfolio choice under stochastic interest rates. Journal of Economic Dynamics and Control 25, 1141–1865.

Liu, J., 2007. Portfolio selection in stochastic environments. Review of Financial Studies 20 (1), 1–39.

Liu, J., Longstaff, F., Pan, J., 2003. Dynamic asset allocation with event risk. Journal of Finance 58, 231–259.

Liu, J., Pan, J., 2003. Dynamic derivative strategies. Journal of Financial Economics 69, 401–430.

Lunde, A., Timmermann, A., 2004. Duration dependence in stock prices: An analysis of bull and bear markets. Journal of Business & Economic Statistics 22 (3), 253–273.

Markowitz, H., 1952. Portfolio selection. Journal of Finance 7 (1), 77–91.

Merton, R. C., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. Review of Economics and Statistics 51, 247–257.

Merton, R. C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373–413.

Moreira, A., Muir, T., 2019. Should long-term investors time volatility? Journal of Financial Economics 131 (3), 507–527.
Nualart, D., 2006. The Malliavin Calculus and Related Topics, 2nd Edition. Probability and Its Applications. Springer, Berlin.

Ocone, D., Karatzas, I., 1991. A generalized Clark representation formula, with application to optimal portfolios. Stochastics and Stochastics Reports 34, 187–220.

Pan, J., 2002. The jump-risk premia implicit in options: Evidence from an integrated time-series study. Journal of Financial Economics 63 (1), 3–50.

Penrose, R., 1955. A generalized inverse for matrices. Mathematical Proceedings of the Cambridge Philosophical Society 51, 406–413.

Pliska, S. R., 1986. A stochastic calculus model of continuous trading: Optimal portfolios. Mathematics of Operations Research 11, 239–246.

Poteshman, A. M., 2001. Underreaction, overreaction, and increasing misreaction to information in the options market. Journal of Finance 56 (3), 851–876.

Roussanov, N., 2010. Diversification and its discontents: Idiosyncratic and entrepreneurial risk in the quest for social status. Journal of Finance 65 (5), 1755–1788.

Samuelson, P., 1969. Lifetime portfolio selection by dynamic stochastic programming. Review of Economics and Statistics 51, 239–246.

Wachter, J. A., 2002. Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. Journal of Financial and Quantitative Analysis 37, 63–91.

Wachter, J. A., 2010. Asset allocation. Annual Review of Financial Economics 2 (1), 175–206.

Wachter, J. A., Yogo, M., 2010. Why do household portfolio shares rise in wealth? Review of Financial Studies 23 (11), 3929–3965.
Appendix A  Degeneration of decomposition under wealth-independent utility

As briefly discussed in Section 3.3, we further document in Corollaries 2 and 3 below on how our general policy decomposition in Theorems 1 and 2 degenerates under CRRA utility. The comparison demonstrates the structural impact of wealth-dependent utility on optimal policy in general models. We first provide the explicit result for optimal policy under CRRA utility, then discuss in detail its fundamental difference with the optimal policy under general wealth-dependent utilities.

Corollary 2. Under CRRA utility, the interest rate hedge and price of risk hedge components are given by

\[ \pi^r(t, Y_t) = -\left(\sigma(t, Y_t)\right)^\top \frac{E_t[\hat{H}^r_{t,T}]}{E_t[\hat{G}_{t,T}]} \quad \text{and} \quad \pi^\theta(t, Y_t) = -\left(\sigma(t, Y_t)\right)^\top \frac{E_t[\hat{H}^\theta_{t,T}]}{E_t[\hat{G}_{t,T}]} \]  

(A.1)

The functions \( \hat{H}^\theta_{t,T}, \hat{H}^r_{t,T}, \) and \( \hat{G}_{t,T} \) are defined as

\[ \hat{H}^\theta_{t,T} := \left(1 - \frac{1}{\gamma}\right) \left(1 - w\right) \frac{\frac{1}{\gamma} e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,T})^{1-\frac{1}{\gamma}} H^\theta_{t,T} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,s})^{1-\frac{1}{\gamma}} H^\theta_{t,s} ds \right), \]  

(A.2a)

\[ \hat{H}^r_{t,T} := \left(1 - \frac{1}{\gamma}\right) \left(1 - w\right) \frac{\frac{1}{\gamma} e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,T})^{1-\frac{1}{\gamma}} H^r_{t,T} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,s})^{1-\frac{1}{\gamma}} H^r_{t,s} ds \right), \]  

(A.2b)

and

\[ \hat{G}_{t,T} := (1 - w) \frac{\frac{1}{\gamma} e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,T})^{1-\frac{1}{\gamma}} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\sigma^T}{\xi}} (\xi^S_{t,s})^{1-\frac{1}{\gamma}} ds}, \]  

(A.2c)

with \( \xi^S_{t,s}, H^\theta_{t,s}, \) and \( H^r_{t,s} \) evolving according to (33a), (33b), and (24). The investor-specific price of risk \( \theta^u(v, Y_v; T) \) satisfies the following \( d \)-dimensional equation

\[ \theta^u(v, Y_v; T) = \frac{\sigma(v, Y_v)^\top \sigma(v, Y_v) - I_d}{E[\hat{Q}_{v,T}|Y_v]} \times (E[\hat{H}^r_{v,T}|Y_v] + E[\hat{H}^\theta_{v,T}|Y_v]), \]  

(A.3)

where \( \hat{Q}_{v,T} = -\hat{G}_{v,T}/\gamma, \) as defined by (A.2c), i.e.,

\[ \hat{Q}_{v,T} := -\frac{1}{\gamma} \left(1 - w\right) \frac{\frac{1}{\gamma} e^{-\frac{\sigma^T}{\xi}} (\xi^S_{v,T})^{1-\frac{1}{\gamma}} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\sigma^T}{\xi}} (\xi^S_{v,s})^{1-\frac{1}{\gamma}} ds}. \]  

(A.4)

The investor-specific price of risk \( \theta^u(v, Y_v; T) \) is determined by a multidimensional equation system consisting of equation (A.3), as well as the SDEs of \( Y_s, \xi^S_{t,s}, H^r_{t,s}, H^\theta_{t,s}, \) and \( D_i Y_s \) given in (2), (33a), (24), (33b), and (25), which are all independent of the multiplier \( \lambda_i \).

Proof. See Section S.3 in the online supplementary material.

Though as a special case, the explicit decomposition in Corollary 2 is new relative to the existing analysis on the structure of the optimal policy under CRRA utility by means of, e.g., HJB equations (see, e.g., Liu (2007)). By comparing the decomposition in Corollary 2 and that in Theorems 1 and
2, we explicitly observe how the structure of the optimal policy under wealth-independent CRRA utility differs from that under general wealth-dependent utilities, as well as how the specific structure of CRRA utility allows for significant simplification of the decomposition. This comparative study demonstrates again the importance of our explicit decomposition results in Theorems 1 and 2.

First, the building blocks employed in these two decompositions obviously have different structures. Since investor-specific price of risk does not depend on the multiplier \( \lambda^*_s \) for the CRRA case, it takes the form \( \theta^u_s = \theta^u(s, Y^*_s; T) \). By the analysis similar to those immediately prior to Theorem 1, \( \theta^u_s \) here is consequently independent of the wealth level \( X_s \). Thus, the market completion under CRRA utility enjoys a simpler mechanism. We now compare the dynamics of \( \xi^S_{t,s}(\lambda^*_t) \) in (21) with that of \( \xi^S_{t,s} \) in (33a), as well as the dynamics of \( H^\theta_{t,s}(\lambda^*_t) \) in (26) with that of \( H^\theta_{t,s} \) in (33b). Owing to the absence of multiplier \( \lambda^*_s \) from \( \theta^u(v, Y^*_v, \lambda^*_v; T) \), dynamics (33a) and (33b) are obviously simpler than (21) and (26). In particular, they are all independent of multiplier \( \lambda^*_t \).

Next, our decomposition results illustrate how the current wealth level impacts the optimal policy under general wealth-dependent utilities, but not under CRRA utility.\(^{17}\) By Theorems 1 and 2 for general wealth-dependent utilities, the current wealth level \( X_t \) impacts the optimal policy through two channels. First, it directly appears in the optimal policy as the denominator in (18a) – (18c). Second, due to the wealth equation \( X_t = E_{t}[G_{t,T}(\lambda^*_t)] \), \( X_t \) is implicitly involved in the optimal policy through the time-\( t \) multiplier \( \lambda^*_t \) in the functions \( \Theta_{t,T}(\lambda^*_t), H^\theta_{t,T}(\lambda^*_t) \), and \( H^\theta_{t,T}(\lambda^*_t) \) with the building blocks \( \xi^S_{t,s}(\lambda^*_t) \) and \( H^\theta_{t,s}(\lambda^*_t) \). However, both channels are absent under CRRA utility, thanks to its special structure. As shown in (35) and (A.1), both \( X_t \) and \( \lambda^*_t \) vanish in the components of the optimal policy. Furthermore, by (33a) and (33b), the building blocks \( \xi^S_{t,s} \) and \( H^\theta_{t,s} \) are also independent of \( \lambda^*_t \). Such an independence guarantees that \( X_t \) is not implicitly involved in the optimal policy through \( \lambda^*_t \) as in the case with general wealth-dependent utilities. In essence, it is again because the investor-specific price of risk \( \theta^u(s, Y^*_s; T) \) does not depend on \( \lambda^*_s \) under CRRA utility.

In addition, the decomposition of the price of risk hedge component, as given in (17) for general cases, can be simplified under CRRA utility, as shown in Corollary 3 below.

**Corollary 3.** Under the incomplete market model (1) – (2) and the CRRA utility function given in (5) with \( \bar{\pi} = \overline{v} = 0 \), the price of risk hedge component \( \pi^\theta(t,Y_t) \) in (34) can be further decomposed as

\[
\pi^\theta(t,Y_t) = \pi^{h,Y}(t,Y_t) + \pi^{u,Y}(t,Y_t); \quad \text{(A.5)}
\]

where the first and second components hedge the uncertainty in market and investor-specific price of risk, due to variation of the state variable \( Y_t \). They are given by

\[
\pi^{h,Y}(t,Y_t) = -(\sigma(t,Y_t)^+)^\top \frac{E_t[H^h_{t,T}(\lambda^*_t)]}{E_t[G_{t,T}]} \quad \text{and} \quad \pi^{u,Y}(t,Y_t) = -(\sigma(t,Y_t)^+)^\top \frac{E_t[H^u_{t,T}(\lambda^*_t)]}{E_t[G_{t,T}]}.
\]

\(^{17}\)This wealth-independent property of CRRA utility reconciles the conclusions in Detemple et al. (2003) and Ocone and Karatzas (1991) for complete market models.
Here, $\tilde{H}_{t,T}^{h,Y}$ and $\tilde{H}_{t,T}^{u,Y}$ are defined in the same way as $\tilde{H}_{t,T}^0$ in (A.2a), except for replacing $H_{t,s}^0$ for $t \leq s \leq T$ by $H_{t,s}^{h,Y}$ and $H_{t,s}^{u,Y}$, which follow dynamics in (27a) and
\begin{equation}
\frac{d}{dt}H_{t,s}^{u,Y} = \mathcal{D}_t \theta^u(s, Y_s; T)(\theta^u(s, Y_s; T)ds + dW_s).
\end{equation}
Under the assumption that $\theta^u(v, y; T)$ is differentiable in its argument, the above dynamics can be further expressed as
\begin{equation}
\frac{d}{dt}H_{t,s}^{u,Y} = (\mathcal{D}_t Y_s) \nabla \theta^u(s, Y_s; T)(\theta^u(s, Y_s; T)ds + dW_s).
\end{equation}

Proof. The proof follows directly by combining the representation of the investor-specific price of risk $\theta^u(s, Y_s; T)$ under CRRA utility and decomposition (17).

That is, the component $\pi^{u, \lambda}(t, X_t, Y_t)$ in (30) of Proposition 2 vanishes under CRRA utility. As discussed in Section 3.3, this is because the investor-specific price of risk $\theta^u(s, Y_s; T)$ does not depend on the wealth level via the multiplier $\lambda^*_s$ under CRRA utility.

**Appendix B  Decomposition of optimal policy under HARA utility**

In this section, we establish the decomposition of optimal policy under general incomplete market model (1) – (2) with the wealth-dependent HARA utility (5). The result is summarized in the corollary below. It is obtained by directly particularizing Theorems 1 and 2 under the HARA utility.

**Corollary 4.** Under the HARA utility (5) with $w > 0$, the investor-specific price of risk $\theta^u_v$ in (15) satisfies the following $d$-dimensional equation
\begin{equation}
\theta^u(v, Y_v, \lambda^*_v; T) = \frac{\sigma(v, Y_v) - I_d}{E[\tilde{Q}_{v,T}(\lambda^*_v)]} \times (E[\tilde{H}_{v,T}^0(\lambda^*_v)|Y_v, \lambda^*_v] + E[\tilde{H}_{v,T}^0(\lambda^*_v)|Y_v, \lambda^*_v] + (\lambda^*_v)^\frac{1}{w} E[\tilde{Q}_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]).
\end{equation}

Here, $\tilde{H}_{v,T}^r(\lambda^*_v)$, $\tilde{H}_{v,T}^0(\lambda^*_v)$, and $\tilde{Q}_{v,T}(\lambda^*_v)$ are defined as (A.2b), (A.2a), and (A.4), except for replacing $\xi^S_{v,s}$ and $H^\theta_{v,s}$ by the $\lambda^*_v$-dependent version $\xi^S_{v,s}(\lambda^*_v)$ and $H^\theta_{v,s}(\lambda^*_v)$, which evolve according to SDEs (21) and (26). Besides, $\xi_{v,T}(\lambda^*_v)$ is a $d$-dimensional column vector given by
\begin{equation}
\xi_{v,T}(\lambda^*_v) = \xi_{v,T}^r(\lambda^*_v) + \xi_{v,T}^\theta(\lambda^*_v),
\end{equation}
where
\begin{equation}
\xi_{v,T}^r(\lambda^*_v) := \bar{\xi}_{v,T}^r(\lambda^*_v) H_{v,T}^r + \bar{c} \int_v^T \xi_{v,s}^S(\lambda^*_v) H_{v,s}^r ds,
\end{equation}
\begin{equation}
\xi_{v,T}^\theta(\lambda^*_v) := \bar{\xi}_{v,T}^\theta(\lambda^*_v) H_{v,T}^\theta(\lambda^*_v) + \bar{c} \int_v^T \xi_{v,s}^S(\lambda^*_v) H_{v,s}^\theta(\lambda^*_v) ds,
\end{equation}
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with \( \bar{x} \) and \( \bar{c} \) being the minimum allowable levels for terminal wealth and intermediate consumption under the HARA utility (5). The optimal policy under HARA utility follows by 
\[
\pi_t = \pi^{mv}(t, X_t, Y_t) + \pi^r(t, X_t, Y_t) + \pi^0(t, X_t, Y_t),
\]
where
\[
\pi^{mv}(t, X_t, Y_t) = -\frac{1}{X_t} (\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) (\lambda^*_t)^{-\frac{1}{n}} E_t[\hat{Q}_{t,T}(\lambda^*_t)],
\]
and the hedge components given by:
\[
\begin{align*}
\pi^r(t, X_t, Y_t) &= -\frac{1}{X_t} (\sigma(t, Y_t)^+)^\top \left( (\lambda^*_t)^{-\frac{1}{n}} E_t[\hat{H}^r_{t,T}(\lambda^*_t)] + E_t [\zeta^r_{t,T}(\lambda^*_t)] \right), \\
\pi^0(t, X_t, Y_t) &= -\frac{1}{X_t} (\sigma(t, Y_t)^+)^\top \left( (\lambda^*_t)^{-\frac{1}{n}} E_t[\hat{H}^\theta_{t,T}(\lambda^*_t)] + E_t [\zeta^\theta_{t,T}(\lambda^*_t)] \right). 
\end{align*}
\]

The multiplier \( \lambda^*_t \) is characterized as the unique solution for the wealth constraint:
\[
(\lambda^*_t)^{-\frac{1}{n}} E_t[\hat{G}_{t,T}(\lambda^*_t)] + \pi E_t \left[ \xi^S_{t,s}(\lambda^*_t) \right] + \tau E_t \left[ \int_t^T \xi^S_{t,s}(\lambda^*_t) ds \right] = X_t,
\]
where \( \hat{G}_{t,T}(\lambda^*_t) = -\gamma \hat{Q}_{t,T}(\lambda^*_t) \), defined by (A.2c) with \( \xi^S_{t,s} \) replaced by the \( \lambda^*_t \)-dependent version \( \xi^S_{t,s}(\lambda^*_t) \). For the case of \( w = 0 \) in utility (5), the above representation still holds except for dropping the terms related to \( \bar{c} \) in (B.9b), (B.9c), and (B.11).

Corollary 4 shows how investor’s wealth level gets involved in the optimal policy when switching from the wealth-independent CRRA utility to the HARA utility. First, by (B.8), we can see that the multiplier \( \lambda^*_v \) directly appears in the investor-specific price of risk under the HARA utility via the term \( (\lambda^*_v)^{\frac{1}{n}} E [\zeta_{v,T}(\lambda^*_v)] \), which is fully absent under the CRRA utility by (A.3). It introduces the dependence of \( \theta^\mu(v, Y_v, \lambda^*_v; T) \) on the multiplier \( \lambda^*_v \) under the HARA utility. As the multiplier \( \lambda^*_v \) is determined by the wealth level \( X_v \) via (12), the investor-specific price of risk is wealth-dependent under the HARA utility. By the dynamics in (21) and (26), the components \( \xi^S_{v,s}(\lambda^*_v) \) and \( H^\theta_{v,s}(\lambda^*_v) \), also depend on the multiplier \( \lambda^*_v \). It further introduces \( \lambda^*_v \) into the functions \( \hat{H}^r_{v,T}(\lambda^*_v), \hat{H}^\theta_{v,T}(\lambda^*_v) \), \( \hat{G}_{v,T}(\lambda^*_v) \), and \( \hat{Q}_{v,T}(\lambda^*_v) \) by (A.2a) – (A.2c) and (A.4).

The above comparison shows that the term \( (\lambda^*_v)^{\frac{1}{n}} E [\zeta_{v,T}(\lambda^*_v)] \) in (B.8) plays an essential role in differentiating the optimal policy under the HARA utility from that under the CRRA utility. By definitions (B.9a) – (B.9c), we see that \( \zeta_{v,T}(\lambda^*_v) \) is essentially introduced by the coefficients \( \bar{c} \) and \( x \), which denote the lower bounds on intermediate consumption and terminal wealth in (5) for HARA investors. When \( \bar{x} = \bar{c} = 0 \), the term \( \zeta_{v,T}(\lambda^*_v) \) vanishes in (B.8) as the HARA utility reduces to the CRRA utility, under which the optimal policy becomes independent of wealth. Our decomposition suggests that the wealth-dependent property of the HARA utility is indeed introduced by the wealth constraints \( \bar{c} \) and \( \bar{x} \).
Appendix C  Proofs of Proposition 4 and Corollary 1

Appendix C.1  Proof of Proposition 4

As a supplement to Proposition 4, we first characterize the investor-specific price of risk \( \theta^u_v \) under HARA utility with nonrandom interest rate \( r_t \). By Proposition 4, it coincides with the counterpart under CRRA utility, and thus can be simplified as \( \theta^u_v = \theta^u(v,Y_v;T) \). It satisfies the following \( d \)-dimensional equation:

\[
\theta^u(v,Y_v;T) = \frac{\sigma(v,Y_v) + \sigma(v,Y_v)}{E[\tilde{Q}_{v,T}|Y_v]} \times E[\tilde{H}_{v,T}|Y_v],
\]

where \( \tilde{H}_{v,T} \) and \( \tilde{Q}_{v,T} \) are given by (A.2a) and (A.4). In the following, we prove Proposition 4 together with the representation (C.12).

Next, as a technical preparation, we prove the following lemma.

**Lemma 2.** With deterministic interest rate \( r_s \), the following relationship holds for general utility functions:

\[
E_t[\xi^S_{t,s}(\lambda^*_t) H^\theta_{t,s}(\lambda^*_t)] \equiv 0_d, \text{ for any } s \geq t.
\]

Here, \( \xi^S_{t,s}(\lambda^*_t) \) is the relative state price density, and \( H^\theta_{t,s}(\lambda^*_t) \) is the Malliavin term related to the uncertainty in the total price of risk, with dynamics explicitly given in (21) and (26).

**Proof.** By (10), we get \( \xi^S_t = \exp(-\int_0^t r_v dv - \int_0^t (\theta^S_v) ^\top dW_v - \frac{1}{2} \int_0^t (\theta^S_v) ^\top \theta^S_v dv) \) for the state price density in incomplete markets. We can decompose it into two parts related to the interest rate and total price of risk, i.e., \( \xi^S_t = B_t \eta_t \), where

\[
B_t = \exp \left( -\int_0^t r_v dv \right) \quad \text{and} \quad \eta_t = \exp \left( -\int_0^t (\theta^S_v) ^\top dW_v - \frac{1}{2} \int_0^t (\theta^S_v) ^\top \theta^S_v dv \right).
\]

With a deterministic interest rate \( r_s \), the discount term \( B_t \) is also deterministic. A straightforward application of Ito’s formula leads to the SDE of \( \eta_t \) as

\[
d\eta_t = -\eta_t (\theta^S_v) ^\top dW_t.
\]

The martingale property of \( \eta_t \) leads to

\[
E_t[\eta_s] = \eta_t, \text{ for any } s \geq t.
\]

Next, we prove

\[
E_t[\eta_s H^\theta_{t,s}(\lambda^*_t)] \equiv 0_d,
\]

using standard Ito calculus. By the dynamics of \( \eta_s \) in (C.15) and that of \( H^\theta_{t,s}(\lambda^*_t) \) in (26), i.e.,

\[
dH^\theta_{t,s}(\lambda^*_t) = M_{t,s}(\lambda^*_t) [\theta^S_{s}(\lambda^*_t) ds + dW_s]
\]

with

\[
M_{t,s}(\lambda^*_t) = (D_t Y_s) \nabla \theta^b(s,Y_s) + D_t \theta^u_s(\lambda^*_t),
\]

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a straightforward application of the Ito’s formula leads to

\[ d(\eta_s H^\theta_{t,s}(\lambda^*_v)) = (-\eta_s \theta^S_{t,s}(\lambda^*_v)^T H^\theta_{t,s}(\lambda^*_v) + \eta_s M_{t,s}(\lambda^*_v))dW_s. \]

Then, (C.17) follows by \( E_t[\eta_s H^\theta_{t,s}(\lambda^*_v)] = \eta_s H^\theta_{t,t}(\lambda^*_v) \equiv 0_d \) according to initial condition \( H^\theta_{t,t}(\lambda^*_v) \equiv 0_d \).

Finally, relationship (C.13) comes from \( E_t[\xi^S_{t,s}(\lambda^*_v) H^\theta_{t,s}(\lambda^*_v)] = E_t[\xi^S_{t,s} H^\theta_{t,s}(\lambda^*_v)]/\xi^S_{t,t} = B_s E_t[\eta_s H^\theta_{t,s}(\lambda^*_v)]/\xi^S_{t,t} = 0_d \), where the second equality follows from the deterministic nature of the discount term \( B_s \) as well as (C.17).

Now, we are ready to prove Proposition 4 for the optimal policy under HARA utility with a deterministic interest rate. Without loss of generality, we assume \( w > 0 \) in utility (5), as the case of \( w = 0 \) follows in a similar fashion.\(^{18}\)

**Proof. Part 1:** First, we show that with a deterministic interest rate, the investor-specific price of risk \( \theta^u(v, Y_v, \lambda^*_v; T) \) under HARA utility coincides with its counterpart under CRRA utility given in Corollary 2, and is thus independent of multiplier \( \lambda^*_v \). To begin with, we employ the dual problem technique introduced in He and Pearson (1991) to characterize the investor-specific price of risk \( \theta^u \) by

\[ \inf_{\theta^u \in \text{Ker}(\sigma)} E \left[ \int_0^T \tilde{u}(v, \lambda^*_v) dv + \tilde{U}(T, \lambda^*_T) \right], \quad (C.18) \]

where \( \tilde{u}(t, y) := \sup_{x \geq 0} (u(t, x) - yx) \) and \( \tilde{U}(t, y) := \sup_{x \geq 0} (U(t, x) - yx) \); the constraint \( \theta^u \in \text{Ker}(\sigma) \) abbreviates for \( \theta^u \in \text{Ker}(\sigma(v, Y_v)) \) for \( 0 \leq v \leq T \), with \( \text{Ker}(\sigma(v, Y_v)) := \{ w \in \mathbb{R}^d : \sigma(v, Y_v)w \equiv 0_m \} \) denoting the kernel of \( \sigma(v, Y_v) \). Using the explicit forms of functions \( I^u(t, y) \) and \( I^U(t, y) \) under HARA utility, we can explicitly specify the dual problem (C.18) conditional on information up to time \( v \) as

\[ \inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w)^{\gamma} e^{-\eta S_{v,T}^\gamma} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} \int_v^T e^{-\eta S_{s,T}^\gamma} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds + \frac{\gamma - 1}{\gamma} A^*_{v,T} \right], \quad (C.19) \]

where \( A^*_{v,T} = \bar{c} \lambda^*_T + \bar{c} \int_v^T \lambda^*_s ds \). Here, \( \bar{c} \) and \( \bar{c} \) are the minimum requirements for terminal wealth and intermediate consumption. On the other hand, under CRRA utility, the dual problem (C.18) specifies to

\[ \inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w)^{\gamma} e^{-\eta S_{v,T}^\gamma} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w^{\frac{1}{\gamma}} \int_v^T e^{-\eta S_{s,T}^\gamma} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds \right], \quad (C.20) \]

conditioning on information up to time \( v \). Comparing (C.19) and (C.20), we see that the term \( A^*_{v,T} \) in (C.19) distinguishes the dual problem under HARA utility from that under CRRA utility.

With a deterministic interest rate, we then verify that \( E_v[A^*_{v,T}] \) does not depend on the control process \( \theta^u_v \) for \( v \in [v, T] \) and thus can be dropped from the dual problem (C.19) to simplify it as the CRRA counterpart (C.20). To see this, we use the relationship \( \lambda^*_v = \lambda^*_v \xi^S_{v,s} = \lambda^*_v \xi^S_{v,v,s} \) to derive that

\[ E_v[A^*_{v,T}] = \bar{c} E_v[\lambda^*_T] + \bar{c} \int_v^T E_v[\lambda^*_s] ds = \lambda^*_v \left[ \bar{c} E_v[\xi^S_{v,T}] + \bar{c} \int_v^T E_v[\xi^S_{s,v,s}] ds \right]. \quad (C.21) \]

\(^{18}\)For the proof under the case of \( w = 0 \), we just need to drop all the terms related to \( \bar{w} \).
We express the conditional expectation $E_v[\xi^S_{v,s}]$ as $E_v[\xi^S_{v,s}] = E_v[B_{v,s}\eta_{v,s}]$, where $B_{v,s} := B_s/B_v$ and $\eta_{v,s} := \eta_s/\eta_v$ following (C.14). A straightforward application of Ito’s formula leads to the SDE of $\eta_{v,s}$ as $d\eta_{v,s} = -\eta_{v,s} (\theta^S_v)^T dW_v$. As we assume a deterministic interest rate, $B_{v,s}$ is also deterministic. Thus, we have

$$E_v[\xi^S_{v,s}] = E_v[B_{v,s}\eta_{v,s}] = B_{v,s}E_v[\eta_{v,s}] = B_{v,s},$$

(C.22)

where the last equality follows from the martingale property of $\eta_{v,s}$ as a process in $s$ and the fact that $\eta_{v,v} = 1$. Plugging $E_v[\xi^S_{v,s}]$ into (C.21), we obtain that $E_v[A_v,T] = \lambda_v^*[\bar{x}B_v,T + \bar{c}^T B_v dW_s]$, which obviously does not depend on the control process $\theta^S_v$. Thus, we can drop the term $A_v,T$ from (C.19).

By the above arguments, we show that with a deterministic interest rate, the investor-specific price of risk $\theta^u_v$ under HARA utility is uniquely characterized as the control process for the dual problem (C.20) with the underlying Markov process $(Y_v, \xi^S_{v,s})$ for $v \leq s \leq T$. Thus, we can verify that the dual problems, as well as the underlying Markov process, are actually the same under the HARA and CRRA utilities. They directly lead to the same optimal control process $\theta^u_v$. It proves that with a deterministic interest rate, the investor-specific price of risk $\theta^u(v, Y_v, \lambda^*_v; T)$ under HARA utility coincides with its counterpart under CRRA utility, and is thus independent of the multiplier $\lambda^*_v$. So, we can express it as $\theta^u_v = \theta^u(v, Y_v; T)$ with the same function $\theta^u(v, Y_v; T)$ that satisfies equation (B.8) under CRRA utility. Consequently, the quantities $\tilde{H}_{v,T}^c(\lambda^*_v)$, $\tilde{H}_{v,T}^\theta(\lambda^*_v)$, $\bar{Q}_{v,T}(\lambda^*_v)$, and $\tilde{Q}_{v,T}(\lambda^*_v)$ for HARA utility are also independent of the multiplier $\lambda^*_v$, and coincide with their counterparts under CRRA utility, which are given in (A.2b), (A.2a), (A.4), and (A.2c).

Next, we establish equation (C.12) that governs the investor-specific price of risk $\theta^u(v, Y_v; T)$. It follows from equation (B.8) that, whether the interest rate is deterministic or not, $\theta^u(v, Y_v, \lambda^*_v; T)$ under HARA utility is characterized by

$$\theta^u(v, Y_v, \lambda^*_v; T) = \sigma(v, Y_v)^+\sigma(v, Y_v) - I_d \frac{E[H_{v,s}^c(\lambda^*_v)|Y_v, \lambda^*_v] + E[H_{v,s}^\theta(\lambda^*_v)|Y_v, \lambda^*_v] + (\lambda^*_v)^\frac{1}{2} E[\zeta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[\bar{Q}_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]},$$

(C.23)

where $\zeta_{v,T}(\lambda^*_v) = \zeta^c_{v,T}(\lambda^*_v) + \zeta^\theta_{v,T}(\lambda^*_v)$ according to (B.9a). With a deterministic interest rate, we have $H_{v,s}^c \equiv 0_d$ due to (24) and $\nabla r(s, Y_s) \equiv 0_n$. Thus, it follows from (A.2b) and (B.9b) that $\tilde{H}_{v,T}^c(\lambda^*_v) \equiv 0_d$ and $\zeta^c_{v,T}(\lambda^*_v) \equiv 0_d$. Also recall that $\theta^u(v, Y_v, \lambda^*_v; T)$ is independent of $\lambda^*_v$ and thus simplifies to $\theta^u(v, Y_v; T)$. Then, we simplify equation (C.23) to

$$\theta^u(v, Y_v; T) = \sigma(v, Y_v)^+\sigma(v, Y_v) - I_d \frac{E[H_{v,T}^c(\lambda^*_v)|Y_v, \lambda^*_v] + E[H_{v,T}^\theta(\lambda^*_v)|Y_v, \lambda^*_v] + (\lambda^*_v)^\frac{1}{2} E[\zeta^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[\bar{Q}_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]},$$

(C.24)

where $\zeta^\theta_{v,T}(\lambda^*_v)$ is defined by (B.9c) as $\zeta^\theta_{v,T}(\lambda^*_v) = \bar{x}\xi^S_{v,T}(\lambda^*_v)H_{v,T}^\theta(\lambda^*_v) + \bar{c}^T \xi^S_{v,s}(\lambda^*_v)H_{v,s}^\theta(\lambda^*_v)ds$. By Lemma 2, its expectation is always zero under a deterministic interest rate, i.e.,

$$E_v[\xi^\theta_{v,T}(\lambda^*_v)] = \bar{x}E_v[\xi^S_{v,T}(\lambda^*_v)H_{v,T}^\theta(\lambda^*_v)] + \bar{c} \int_0^T E_v[\xi^S_{v,s}(\lambda^*_v)H_{v,s}^\theta(\lambda^*_v)]ds \equiv 0_d.$$

(C.25)
Thus, the last term \((\lambda_v^*)^{-\frac{1}{2}} E[\zeta_{v,T}^\theta(\lambda_v^*)|Y_v, \lambda_v^*]\) vanishes in (C.24), and the equation further simplifies to
\[
\theta^u (v, Y_v; T) = \frac{\sigma (v, Y_v)^{\top} \sigma (v, Y_v) - I_d}{E[\bar{Q}_{v,T}(\lambda_v^*)|Y_v, \lambda_v^*]} \times E[\bar{\mathcal{H}}_{v,T}^\theta(\lambda_v^*)|Y_v, \lambda_v^*].
\]
By examining the definitions of \(\bar{Q}_{v,T}(\lambda_v^*)\) and \(\bar{Q}_{v,T}(\lambda_v^*)\) as well as the SDEs of \(\xi_{t,s}^\theta(\lambda_v^*)\) and \(H_{v,s}^\theta(\lambda_v^*)\) in (21) and (26), we confirm that \(\bar{Q}_{v,T}(\lambda_v^*)\) and \(\bar{Q}_{v,T}(\lambda_v^*)\) reduce to \(\bar{Q}_{v,T}\) and \(\bar{Q}_{v,T}\) given in (A.2a) and (A.4). Hence, the multiplier \(\lambda_v^*\) does not show up in either the above equation system or its solution \(\theta^u (v, Y_v; T)\). It proves equation (C.12) for \(\theta^u (v, Y_v; T)\).

**Part 2:** Next, we look into the optimal policy under HARA utility with a deterministic interest rate. Under this circumstance, we have \(H_{t,s}^\gamma \equiv 0 \) by (24). Thus, it follows from (A.2b), (B.9b), and (B.10b) that the interest hedge component \(\pi_H^\gamma (t, X_t, Y_t) = 0\), i.e., there is no need to hedge interest rate uncertainty. So, we only need to focus on the mean-variance and price of risk hedge components. First, we solve for the multiplier \(\lambda_v^*\) from the wealth equation (B.11), i.e., \((\lambda_v^*)^{-\frac{1}{2}} E_t[\bar{G}_{t,T}] + \bar{\gamma} E_t[\xi_{t,s}^\theta] + \bar{\gamma} E_t[\int_t^T \xi_{t,s}^\theta ds] = X_t\). Here, we drop the dependence on \(\lambda_v^*\) from \(\bar{G}_{t,T}\) and \(\xi_{t,s}^\theta\). It is because we have shown in Part 1 that the investor-specific price of risk \(\theta^u (t, Y_t; T)\) does not depend on \(\lambda_v^*\), and neither do \(\bar{G}_{t,T}\) and \(\xi_{t,s}^\theta\) according to (A.2c) and (21). By (C.22), we have \(E_t[\xi_{t,s}^\theta] = B_{t,s}\). Plugging it to the above equation, we solve \((\lambda_v^*)^{-\frac{1}{2}}\) as
\[
(\lambda_v^*)^{-\frac{1}{2}} = \frac{X_t}{E_t[\bar{G}_{t,T}]}, \tag{C.26}
\]
where \(X_t\) is defined in (37a), i.e., \(X_t = X_t - \bar{\gamma} B_{t,T} - \bar{\gamma} \int_t^T B_{t,s} ds\); \(\bar{G}_{t,T}\) is defined in (A.2c). Plugging (C.26) into the mean-variance component in (B.10a) and invoking the relationship \(\bar{G}_{t,T} = -\bar{\gamma} \bar{Q}_{t,T}\), we can derive
\[
\pi_{H}^{mv} (t, X_t, Y_t) = -\frac{1}{X_t} (\sigma (t, Y_t)^{\top})^{\top} \theta^b (t, Y_t) (\lambda_v^*)^{-\frac{1}{2}} E_t[\bar{Q}_{t,T}] = -\frac{1}{X_t} (\sigma (t, Y_t)^{\top})^{\top} \theta^b (t, Y_t) \frac{X_t}{E_t[\bar{G}_{t,T}]} E_t[\bar{Q}_{t,T}] = \frac{X_t}{\gamma X_t} (\sigma (t, Y_t)^{\top})^{\top} \theta^b (t, Y_t). \tag{C.27}
\]
Next, in (B.10c), we have
\[
\pi_{H}^\theta (t, X_t, Y_t) = -\frac{1}{X_t} (\sigma (t, Y_t)^{\top})^{\top} \left((\lambda_v^*)^{-\frac{1}{2}} E_t[\bar{H}_{t,T}^\theta] + E_t[\xi_{t,s}^\theta] \right) = -\frac{1}{X_t} (\sigma (t, Y_t)^{\top})^{\top} (\lambda_v^*)^{-\frac{1}{2}} E_t[\bar{H}_{t,T}^\theta]
\]
for the price of risk hedge component, where the second equality follows from (C.25). Plugging (C.26) into the right-hand side, we obtain
\[
\pi_{H}^\theta (t, X_t, Y_t) = -(\sigma (t, Y_t)^{\top})^{\top} \frac{X_t}{X_t} E_t[\bar{H}_{t,T}^\theta] = (\sigma (t, Y_t)^{\top})^{\top} \frac{X_t}{X_t} \frac{E_t[\bar{H}_{t,T}^\theta]}{X_t \gamma E_t[\bar{Q}_{t,T}]}, \tag{C.28}
\]
where the last equality follows from the relation \(\bar{Q}_{t,T} = -\bar{G}_{t,T}/\bar{\gamma}\) by (A.2c) and (A.4). Finally, we can obtain relationships (36) by comparing the optimal components in (C.27) and (C.28) with their counterparts under CRRA utility. □
Appendix C.2 Proof of Corollary 1

Proof. The optimal policy (41) directly follows from the HARA policy decomposition (36) established in Proposition 4 as well as the simple fact that the interest rate hedge component is zero for both CRRA and HARA utilities as the interest rate is constant in the Heston-SV model. In particular, $\bar{X}_t$ in (42) follows from (37b); the decomposition of the policy under CRRA utility in (43) follows from Theorems 1 and 2 as well as the results in Liu (2007). The form of (44) for the investor-specific price of risk is derived as follows. By the definition (9b) and the orthogonal condition (7), we have $\sigma(t, Y_t) \theta^u_t \equiv 0_m$. Under Heston-SV model with HARA utility, it implies $\sigma(t, V_t) \theta^u(t, V_t; T) \equiv 0$ with $\sigma(t, V_t) = (\sqrt{V_t}, 0)$. Combining (C.12) and (A.1) as well as plugging in $\pi_c^\theta(t, V_t) = -\rho \sigma \delta \phi(T - t)$, we obtain $\theta^u_2(t, V_t; T)$ in closed-form as (44). 

$\square$
Wealth Effect on Portfolio Allocation in Incomplete Markets*

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Abstract

This online supplementary material for Wealth Effect on Portfolio Allocation in Incomplete Markets documents: (1) Proof of Lemma 1, Theorem 1, and Proposition 1 (Section S.1), (2) Proof of Theorem 2 (Section S.2), (3) Proof of Proposition 2 (Section S.3), and (4) Proof of Corollary 2 (Section S.4).

Keywords: optimal portfolio choice, stochastic volatility, incomplete market, wealth-dependent utility, closed-form.

JEL Codes: C61, C63, G11.

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S.1 Proof of Lemma 1, Theorem 1, and Proposition 1

We first provide the following lemma that represents the optimal policy under the completed market with both real and fictitious assets, assuming the investor-specific price of risk process $\theta^u_s$ were known. Recall that in the completed market, the assets $S_t = (S^T_t, F^T_t)^T$ follow the dynamics:

$$dS_t = \text{diag}(S_t) \left[ \mu^S_t dt + \sigma^S(t, Y_t) dW_t \right],$$

(S.1)

with $\mu^S_t = ((\mu(t, Y_t) - \delta(t, Y_t))^T, (\mu^F)^T)^T$ and $\sigma^S(t, Y_t) = (\sigma(t, Y_t)^T, \sigma^F(t, Y_t)^T)^T$.

**Lemma S.1.** In the completed market with dynamics (S.1) and (2), the optimal policy $(\pi_t, \pi^F_t)^T$ for both the real and fictitious assets admits the following representation

$$(\pi_t, \pi^F_t)^T = -\frac{1}{X_t} (\sigma^S(t, Y_t)^T)^{-1} \left( \theta^S_t E_t[Q_{t,T}(\lambda^S_0)] + E_t[H^\theta_{t,T}(\lambda^S_0)] \right),$$

(S.2)

where $\theta^S_t$ is the total price of risk defined in (8); $E_t$ denotes the expectation conditional on the information up to time $t$; $\xi^S_t$ is the state price density defined in (10); $\lambda^*_0$ is the multiplier uniquely determined by the wealth equation

$$X_0 = E[\xi^*_0],$$

(S.3)

where $X_0$ is the initial wealth and the function $\xi^*_{t,T}(\cdot)$ is defined as

$$\xi^*_{t,T}(\lambda^*_0) := \Gamma^U_{t,T}(\lambda^*_0) + \int_t^T \Gamma^u_{t,s}(\lambda^*_0) ds,$$

(S.4a)

the components $Q_{t,T}(\lambda^S_0), H^\theta_{t,T}(\lambda^S_0)$, and $H^\theta_{t,T}(\lambda^S_0)$ are given by

$$Q_{t,T}(\lambda^S_0) = \Gamma^U_{t,T}(\lambda^S_0) + \int_t^T \Gamma^u_{t,s}(\lambda^S_0) ds,$$

(S.4b)

$$H^\theta_{t,T}(\lambda^S_0) = (\Gamma^U_{t,T}(\lambda^S_0) + \Gamma^U_{t,T}(\lambda^S_0))H^\theta_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^S_0) + \Gamma^u_{t,s}(\lambda^S_0))H^\theta_{t,s} ds,$$

(S.4c)

$$H^\theta_{t,T}(\lambda^S_0) = (\Gamma^U_{t,T}(\lambda^S_0) + \Gamma^U_{t,T}(\lambda^S_0))H^\theta_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^S_0) + \Gamma^u_{t,s}(\lambda^S_0))H^\theta_{t,s} ds,$$

(S.4d)

with functions $\Gamma^U_{t,T}(\cdot), \Gamma^u_{t,s}(\cdot), \Gamma^U_{t,T}(\cdot)$, and $\Gamma^u_{t,s}(\cdot)$ defined in (13a) and (13b). Here, the terms $H^r_{t,s}$ and $H^\theta_{t,s}$ in (S.4c) and (S.4d) satisfy

$$dH^r_{t,s} = D_r r(s, Y_s) ds + dW_s,$$

(S.5)

with initial values $H^r_{t,t} = H^\theta_{t,t} = 0$, where $D_r r(s, Y_s)$ and $D_r \theta^S_s$ denote the Malliavin derivatives of the interest rate $r(s, Y_s)$ and total price of risk $\theta^S_s$, respectively.

**Proof.** The statement follows from the martingale approach arguments that lead to Theorem 1 in Detemple et al. (2003) (see also, e.g., Karatzas et al. (1987) and Cox and Huang (1989)). □
In what follows, we prove Theorem 1 and Proposition 1.

Proof. This proof consists of three parts consecutively. In the first part, we prove the relationship \( \lambda_t^* = \theta_0^* \xi_t^S \) and apply it to verify the representation of the investor-specific price of risk in (15). In the second part, we start to apply Lemma S.1 and focus on deriving the explicit dynamics of \( \xi_t^S(\lambda_t^*) \), \( H_t^{\vartheta} \), and \( H_t^{\vartheta}(\lambda_t^*) \) as (21), (24), and (26), respectively, based on the representation (15) of \( \vartheta_v^u \) and the dynamics in (10) and (S.5). In the third part, we consequently establish the decompositions of the optimal policy (16) and (17) with components (18a) – (18c) and (19a) – (19b), respectively.

Part 1: We first briefly prove the relationship

\[
\lambda_t^* = \theta_0^* \xi_t^S. \tag{S.6}
\]

As a foundation, the existence and uniqueness of \( \lambda_t^* \), as the solution to equation (12), follow from standard calculus: the utilities \( u(t, \cdot) \) and \( U(t, \cdot) \) are strictly increasing and concave with \( \lim_{x \to \infty} \partial u(t, x) / \partial x = 0 \) and \( \lim_{x \to \infty} \partial U(T, x) / \partial x = 0 \) (see similar discussions in Cox and Huang (1989)). We now proceed to show the relationship (S.6). Assuming the investor follows the optimal policy in the completed market, we follow Karatzas et al. (1987) and Cox and Huang (1989) to derive that the time–t optimal wealth satisfies

\[
\xi_t^S X_t = E_t \left[ \xi_t^S I_U(T, \lambda_t^* \xi_t^S) + \int_t^T \xi_s^S I_u(s, \lambda_0^* \xi_s^S) ds \right],
\]

where \( \lambda_0^* \) is characterized by (S.3). By dividing \( \xi_t^S \) on both sides of the above equation and using the relation \( \xi_t^S = \xi_t^S \xi_t^S \xi_t^S \) for any \( s \geq t \), we obtain \( X_t = E_t \left[ \xi_t^S I_U(T, \lambda_t^* \xi_t^S) + \int_t^T \xi_t^S I_u(s, \lambda_0^* \xi_s^S) ds \right]. \)

By the definition of \( G_{t,T}(\cdot) \) in (S.4a), the above equation is equivalent to \( X_t = E_t[G_{t,T}(\lambda_0^* \xi_t^S)]. \) By the uniqueness of solution to equation (12), we establish the relationship (S.6), i.e., \( \lambda_t^* = \theta_0^* \xi_t^S. \) Then, we prove Lemma 1 by verifying the representation of the investor-specific price of risk in (15), i.e., \( \vartheta_v^u = \vartheta_v^u(v, Y_v, \lambda_v^*; T) \) for some function \( \vartheta_v^u(v, y, \lambda; T) \). This verification hinges on linking the least favorable completion approach of Karatzas et al. (1991) and the minimax local martingale approach of He and Pearson (1991), two independently developed martingale approaches for solving optimal portfolios under incomplete market settings.

By Theorem 9.3 of Karatzas et al. (1991), the investor-specific price of risk \( \vartheta_v^u \) satisfying (14) must lead to the smallest utility among all possible completions, i.e., the least favorable completion. More precisely, the desired \( \vartheta_v^u \) satisfying (14) serves as the optimizer for the following dual problem

\[
\inf_{\vartheta^u \in \text{Ker}(\sigma)} \left\{ \sup_{(c_t, X_t) \in A_{\vartheta^u}} E \left[ \int_0^T u(t, c_t) dt + U(T, X_T) \right] \right\}, \tag{S.7}
\]

where \( A_{\vartheta^u} = \{(c_t, X_t) : E[\int_0^T \xi_t^S c_t dt + \xi_t^S X_T] \leq X_0 + X_t \geq 0 \text{ for all } t \in [0, T]\}. \) Here, corresponding to the orthogonal condition in (S.22), we use \( \vartheta^u \in \text{Ker}(\sigma) \) to abbreviate \( \vartheta_v^{\text{opt}} \in \text{Ker}(\sigma(v, Y_v)) \) for any \( 0 \leq v \leq T, \) with \( \text{Ker}(\sigma(v, Y_v)) := \{w \in R^d : \sigma(v, Y_v)w \equiv 0_m\} \) denoting the kernel of \( \sigma(v, Y_v). \)
Problem (S.7) is also discussed in He and Pearson (1991) for the same goal of characterizing the optimal portfolio in the incomplete market case, though the language of He and Pearson (1991) hinges on the class of arbitrage-free state prices, which indeed correspond to the state price density $\xi_t^S$ of the completed market defined by (10). According to Theorem 2 and the discussion prior to Theorem 7 of He and Pearson (1991), the solution of problem (S.7) also solves the following optimization problem:

$$\inf_{\theta^u \in \text{Ker}(\sigma)} E \left[ \int_0^T \tilde{u}(v, \lambda^*_v) dv + \tilde{U}(T, \lambda^*_T) \right],$$  \hspace{1cm} (S.8)

where $\lambda^*_v$ is the time-$v$ multiplier characterized by the equation $X_v = E_w[G_w(T(\lambda^*_v))]$. By (S.6), i.e., $\lambda^*_v = \lambda^*_0 \xi^S_t$, an application of Ito formula leads to $d\lambda^*_v = -\lambda^*_v (r(v, Y_v) dv + (\theta^h_v(v, Y_v) + \theta^u_v) dW_v)$, in which $\theta^u_v$ serves as a control process. Besides, $\tilde{u}(t, x)$ and $\tilde{U}(t, x)$ in (S.8) denote the conjugates of utility functions $u(t, x)$ and $U(t, x)$, defined by $\tilde{u}(t, y) := \sup_{x \geq 0} (u(t, x) - xy)$ and $\tilde{U}(t, y) := \sup_{x \geq 0} (U(t, x) - xy)$, respectively. We can check that $\sup_{x \geq 0} (u(t, x) - xy)$ and $\sup_{x \geq 0} (u(t, x) - xy)$ take their maximum at $x = \I^u(t, y)$ and $x = \I^U(t, y)$ respectively.

To temporarily summarize, by linking the problems (S.7) and (S.8), we verify that the desired investor-specific price of risk $\theta^u_v$ satisfying the least favorable completion (14) is also the solution of the optimization problem (S.8). Next, we proceed to obtain the functional representation of $\theta^u_v$ by looking into the optimization problem (S.8). Since $(Y_v, \lambda^*_v) = (Y_v, \lambda^*_0 \xi^S_t)$ forms a Markov process, an application of the feedback law (see, e.g., Theorem 9.1 of Touzi (2012)) to the control problem (S.8) implies that the control process $\theta^u_v$ must be a measurable function of the time $v$, the state variable $Y_v$, and the multiplier $\lambda^*_v$. Besides, $\theta^u_v$ also depends on the investment horizon $T$, since it is involved in the objective function of the optimization problem (S.8). Although $T$ is a fixed parameter, it is economically and technically important for revealing the structural differences of optimal policies between complete and incomplete market settings. Thus, we arrive to the representation (15) of the control process $\theta^u_v$, i.e., $\theta^u_v = \theta^u(v, Y_v, \lambda^*_v; T)$ for some investor-specific price of risk function $\theta^u(v, y, \lambda; T)$. It completes the proof for Lemma 1.

Part 2: In this part, we prove the dynamics of $\xi^S_{t,s}(\lambda^*_t)$, $H^r_{t,s}$, and $H^\theta_{t,s}(\lambda^*_t)$ given in (21), (24), and (26), respectively. To begin with, by applying the representation of $\theta^u_v$ in (15) and the relationship (S.6), we obtain the following $\lambda^*_t$-dependent version of the total price of risk $\theta^S_s$ introduced in (8):

$$\theta^S_s(\lambda^*_t) = \theta^h(s, Y_s) + \theta^u(s, Y_s, \lambda^*_s \xi^S_{t,s}(\lambda^*_t); T),$$  \hspace{1cm} (S.9)

which plays an important role in explicitly deriving the desired dynamics in what follows. We now apply Lemma S.1 to the completed market (S.1) with the total price of risk $\theta^S_s$ taking the specific form given in (S.9). By definition (10), the relative state price density $\xi^S_{t,s} = \xi^S_s / \xi^S_t$ satisfies the dynamics

$$d\xi^S_{t,s} = -\xi^S_{t,s} [r(s, Y_s) ds + (\theta^S_s)^T dW_s].$$  \hspace{1cm} (S.10)
Accordingly, by the generic dynamics of $\xi_{t,s}^S$ in (S.10) and that of $H_{t,s}^\theta$ in (S.5), it is easy to verify that $\lambda_t^*$ gets involved in them through the term $\theta_s^S(\lambda_t^*)$. Thus, similar to the spirit of creating the $\lambda_t^*$-dependent notation $\theta_s^S(\lambda_t^*)$, we express $\xi_{t,s}^S$ and $H_{t,s}^\theta$ by their $\lambda_t^*$-dependent versions $\xi_{t,s}^S(\lambda_t^*)$ and $H_{t,s}^\theta(\lambda_t^*)$, respectively, for emphasizing their dependences on $\lambda_t^*$. Applying the representation (S.9) to the generic dynamics of $\xi_{t,s}^S(\lambda_t^*)$ given in (S.10), we obtain the explicit dynamics of $\xi_{t,s}^S(\lambda_t^*)$ in (21), i.e., $d\xi_{t,s}^S(\lambda_t^*) = -\xi_{t,s}^S(\lambda_t^*)[r(s,Y_s)ds + \theta_s^S(\lambda_t^*)]dt + dW_s$. For $H_{t,s}^\theta$ from the first SDE in (S.5), it is straightforward to apply the chain rule of Malliavin derivative to obtain (24), i.e., $dH_{t,s}^\theta = (D_tY_s)r(s,Y_s)ds$. Here, $D_tY_s$ denotes the Malliavin derivative of the state variable, which satisfies SDE (25). Next, to derive the explicit dynamics of $H_{t,s}^\theta(\lambda_t^*)$, we plug the representation of $\theta_s^S(\lambda_t^*)$ in (S.9) into the second SDE in (S.5) to obtain

$$dH_{t,s}^\theta(\lambda_t^*) = D_t\theta_s^S(\lambda_t^*)[\theta_s^S(\lambda_t^*)]ds + dW_s. \quad \text{(S.11)}$$

By (S.9) and the chain rule of Malliavin derivative, we have

$$D_t\theta_s^S(\lambda_t^*) = D_t\theta^h(s,Y_s) + D_t\theta^a(\lambda_t^*) = (D_tY_s)\nabla \theta^h(s,Y_s) + D_t\theta^a(\lambda_t^*). \quad \text{(S.12)}$$

Then, the explicit dynamics of $H_{t,s}^\theta(\lambda_t^*)$ in (26) follows by plugging the above equation into into SDE (S.11).

**Part 3:** We now proceed to prove the decomposition of the optimal policy given in (18a) – (19a). Since we apply Lemma S.1 to the completed market (S.1) with the total price of risk $\theta^S$ taking the specific form given in (S.9), the components $Q_{t,T}(\lambda_t^*\xi_t^S)$, $H_{t,T}(\lambda_t^*\xi_t^S)$, and $H_{t,T}^\theta(\lambda_t^*\xi_t^S)$ in (S.12) – (S.4d) of Lemma S.1 exactly coincide with the components $Q_{t,T}(\lambda_t^*)$, $H_{t,T}(\lambda_t^*)$, and $H_{t,T}^\theta(\lambda_t^*)$ in (23a) – (23c) of Proposition 1. Indeed, this correspondence hinges on the following two reasons. First, by the relationship (S.6), i.e., $\lambda_t^* = \lambda_t^*\xi_t^S$, we can substitute $\lambda_t^*\xi_t^S$ in $Q_{t,T}(\lambda_t^*\xi_t^S)$, $H_{t,T}^\theta(\lambda_t^*\xi_t^S)$, and $H_{t,T}^\theta(\lambda_t^*\xi_t^S)$ by the time–$t$ multiplier $\lambda_t^*$. Second, as proved in **Part 2** above, the building blocks $\xi_{t,s}^S$ and $H_{t,s}^\theta$ in Lemma S.1 are realized by their $\lambda_t^*$–dependent versions $\xi_{t,s}^S(\lambda_t^*)$ and $H_{t,s}^\theta(\lambda_t^*)$, with explicit dynamics (21) and (26), while the dynamics of $H_{t,s}^\theta$ is explicitly computed as (24). Following the above discussions, we can represent the optimal policy $(\pi_t, \pi_t^F)$ in (S.2) for the completed market as

$$(\pi_t, \pi_t^F) = -\frac{1}{\lambda_t^*}(\sigma^S(t,Y_t)\nabla) - \left(\theta_t^S(\lambda_t^*)E_t[Q_{t,T}(\lambda_t^*)] + E_t[H_{t,T}^\theta(\lambda_t^*)] + E_t[H_{t,T}^\theta(\lambda_t^*)]\right), \quad \text{(S.13)}$$

where $\theta^S(\lambda_t^*) = \theta^h(t,Y_t) + \theta^a(t,Y_t, \lambda_t^*; T)$ according to the $\lambda_t^*$-dependent representation in (S.9) and the fact that $\xi_{t,s}^S(\lambda_t^*) = 1$. Here, in (S.13), the components $Q_{t,T}(\lambda_t^*)$, $H_{t,T}(\lambda_t^*)$, and $H_{t,T}^\theta(\lambda_t^*)$ are given by (23a), (23b), and (23c), respectively; besides, the building blocks $\xi_{t,s}^S(\lambda_t^*)$, $H_{t,s}^\theta$, and $H_{t,s}^\theta(\lambda_t^*)$ now follow the dynamics in (21), (24), and (26), respectively.
Next, combining (S.13) with the following algebraic fact:
\[
(\sigma^S(t, Y_t)^{\top})^{-1} = (\sigma^S(t, Y_t)^{-1})^{\top} = ((\sigma(t, Y_t)^+)^{\top}, (\sigma^F(t, Y_t)^+)^{\top});
\]  
(S.14)
the second equality follows from
\[
\sigma^S(t, Y_t)^{-1} = (\sigma(t, Y_t)^+, \sigma^F(t, Y_t)^+),
\]
which can be obtained by the orthogonal condition (7). We explicitly represent the optimal policy for real assets as
\[
\pi_t = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^{\top} \left( \theta^S_t(\lambda^*_t)E_t[Q_{t,T}(\lambda^*_t)] + E_t[H^u_{t,T}(\lambda^*_t)] + E_t[H^\theta_{t,T}(\lambda^*_t)] \right).
\]  
(S.15)
We can further simplify this expression using the following algebraic fact
\[
(\sigma(t, Y_t)^+)^{\top} \theta^S_t(\lambda^*_t) = (\sigma(t, Y_t)^+)^{\top} \theta^h(t, Y_t)
\]  
(S.16)
with \(\theta^h(t, Y_t)\) defined in (9a). To verify this, we use definition of Moore-Penrose inverse for \(\sigma(t, Y_t)^+\), the orthogonal condition in (S.22), as well as representation (15) to deduce that
\[
(\sigma(t, Y_t)^+)^{\top} \theta^u(t, Y_t, \lambda^*_t; T) = (\sigma(t, Y_t)^+)^{\top} (\theta^h(t, Y_t) + \theta^u(t, Y_t, \lambda^*_t; T)) = (\sigma(t, Y_t)^+)^{\top} \theta^h(t, Y_t).
\]
By (8), we can compute the terms \((\sigma(t, Y_t)^+)^{\top} \theta^S_t(\lambda^*_t)\) in (S.15) as
\[
(\sigma(t, Y_t)^+)^{\top} \theta^S_t(\lambda^*_t) = (\sigma(t, Y_t)^+)^{\top} (\theta^h(t, Y_t) + \theta^u(t, Y_t, \lambda^*_t; T)) = (\sigma(t, Y_t)^+)^{\top} \theta^h(t, Y_t).
\]
Hence, by (S.16), we can further simplify the representations (S.15) as
\[
\pi_t = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^{\top} \left( \theta^h(t, Y_t)E_t[Q_{t,T}(\lambda^*_t)] + E_t[H^u_{t,T}(\lambda^*_t)] + E_t[H^\theta_{t,T}(\lambda^*_t)] \right).
\]  
(S.17)
The decomposition (16) given by (18a), (18b), and (18c) of the optimal policy \(\pi_t\) for real assets directly follows the representation (S.17).

Finally, we prove the decomposition of the price of risk hedge component \(\pi^\theta(t, X_t, Y_t)\) in (17). By definition (9a) and orthogonal condition (S.22), we can verify that the market price of risk \(\theta^h(t, Y_t)\) and investor-specific price of risk \(\theta^S_t\) are orthogonal, i.e.,
\[
\theta^h(t, Y_t)^{\top} \theta^u = (\mu(t, Y_t) - r(t, Y_t)1_m)^{\top} (\sigma(t, Y_t)^+) = 0.
\]  
(S.18)
Combining this and (S.9) leads to
\[
\theta^S_t(\lambda^*_t)^{\top} \theta^S_t(\lambda^*_t) = \theta^h(t, Y_t)^{\top} \theta^h(t, Y_t) + \theta^u_t(\lambda^*_t)^{\top} \theta^u(\lambda^*_t).
\]
Taking the Malliavin derivatives on both sides and using the chain rule, we can derive
\[
\mathcal{D}_t \theta^S_t(\lambda^*_t)^{\top} \theta^S_t(\lambda^*_t) = \mathcal{D}_t \theta^h(s, Y_s)^{\top} \theta^h(s, Y_s) + \mathcal{D}_t \theta^u_t(\lambda^*_t)^{\top} \theta^u(\lambda^*_t)
\]
\[(D_tY_t) \nabla \theta^h(s, Y_s) \theta^h(s, Y_s) + D_t \theta^u_s(\lambda_t^*) \theta^u_s(\lambda_t^*). \quad (S.19)\]

Plugging (S.19) into the dynamics of $H^0_{t,s}(\lambda_t^*)$ in (S.11), we verify that $H^0_{t,s}(\lambda_t^*)$ can be decomposed into two parts as

$$H^0_{t,s}(\lambda_t^*) = H^h_{t,s} + H^u_{t,s}(\lambda_t^*),$$

where $H^h_{t,s}$ and $H^u_{t,s}(\lambda_t^*)$ follow the dynamics in (27a) and (27b), respectively. The decomposition of price of risk hedge component $\pi^\theta(t, X_t, Y_t)$ in (17) then follows from (18c), (23c), and the above decomposition.

\[\square\]

### S.2 Proof of Theorem 2

**Proof.** To develop equation (28) that governs the investor-specific price risk, we decompose the optimal policy for the fictitious assets and then invoke the least favorable completion condition (14). Similar to Theorem 1 and Proposition 1, we can decompose the optimal portfolio policy for the fictitious assets as $\pi^F(t, X_t, Y_t) = \pi^{mv,F}(t, X_t, Y_t) + \pi^{r,F}(t, X_t, Y_t) + \pi^\theta,F(t, X_t, Y_t)$, where the mean-variance component, interest rate hedge component, and price of risk hedge component for fictitious assets can be expressed as:

$$\pi^{mv,F}(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma^F(t, Y_t) + \theta^u(t, Y_t, \lambda_t^*; T)E_t[Q_{t,T}(\lambda_t^*)]), \quad (S.20a)$$
$$\pi^{r,F}(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma^F(t, Y_t) + E_t[H^r_{t,T}(\lambda_t^*)]), \quad (S.20b)$$
$$\pi^\theta,F(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma^F(t, Y_t) + E_t[H^\theta_{t,T}(\lambda_t^*)]); \quad (S.20c)$$

the terms $Q_{t,T}(\lambda_t^*)$, $H^r_{t,T}(\lambda_t^*)$, and $H^\theta_{t,T}(\lambda_t^*)$ are defined in (23a), (23b), (23c), respectively. The mean-variance component $\pi^{mv,F}(t, X_t, Y_t)$ captures the risk-return trade-off in the fictitious assets, as shown by the definition of $\theta^u(t, Y_t, \lambda_t^*; T)$ in (9b).

By the least favorable completion (14), the optimal policy for the fictitious assets should be equal to zero, i.e., $\pi^{mv,F}(t, X_t, Y_t) + \pi^{r,F}(t, X_t, Y_t) + \pi^\theta,F(t, X_t, Y_t) = 0_{d \times m}$. Plugging in the components in (S.20a), (S.20b), and (S.20c), we can characterize the investor-specific price of risk $\theta^u(t, Y_t, \lambda_t^*; T)$ by

$$\begin{align*}
(\sigma^F(t, Y_t)^+)^\top \theta^u(t, Y_t, \lambda_t^*; T)E_t[Q_{t,T}(\lambda_t^*)] &= -(\sigma^F(t, Y_t)^+)^\top (E_t[H^r_{t,T}(\lambda_t^*)] + E_t[H^\theta_{t,T}(\lambda_t^*)]). \quad (S.21)
\end{align*}$$

Then, equation (28) can be obtained by above equation and the orthogonal constraint

$$\sigma(t, Y_t) \theta^u_t \equiv 0_m, \quad (S.22)$$

which follows from the definition (9b) and the orthogonal condition (7). We next verify the simple fact that

$$E_v[Q_{v,T}(\lambda_v^*)] = E[Q_{v,T}(\lambda_v^*)|Y_v, \lambda_v^*], \quad E_v[H^r_{v,T}(\lambda_v^*)] = E[H^r_{v,T}(\lambda_v^*)|Y_v, \lambda_v^*], \quad E_v[H^\theta_{v,T}(\lambda_v^*)] = E[H^\theta_{v,T}(\lambda_v^*)|Y_v, \lambda_v^*]. \quad (S.23)$$
Without loss of generality, we take \( E_v[Q_{v,T}(\lambda^*_v)] \) as an example to verify this fact. Indeed, it follows from (2), (21), and (23a) that the joint process \( (Y_s, \xi^S_{s,v}(\lambda^*_v), Q_{v,s}(\lambda^*_v)) \) in the time variable \( s \geq v \) is Markovian with the starting point given by \( (Y_v, \xi^S_{v,v}(\lambda^*_v), Q_{v,v}(\lambda^*_v)) \equiv (Y_v, 1, \lambda^*_v \partial I^U/\partial y(v, \lambda^*_v)) \). That is, the conditioning in \( E_v[Q_{v,T}(\lambda^*_v)] \) is reduced to \( Y_v \) and \( \lambda^*_v \). Thus, we can write \( E_v[Q_{v,T}(\lambda^*_v)] = E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] \).

By representation (S.23), equation (S.21) leads to

\[
(\sigma^F(v, Y_v)^+)^\top \theta^u (v, Y_v, \lambda^*_v; T) E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] = -(\sigma^F(v, Y_v)^+)^\top (E[H^T_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v])
\]  

\( \text{ (S.24) } \)

It is straightforward to derive from above that

\[
(\sigma^F(v, Y_v)^+)^\top \theta^u (v, Y_v, \lambda^*_v; T) = -(\sigma^F(v, Y_v)^+)^\top \frac{E[H^T_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}.
\]  

\( \text{ (S.25a) } \)

Since \( (\sigma^F(v, Y_v)^+)^\top \) is a \((d - m) \times d\) matrix, (S.25a) provides \((d - m)\) equations governing the \(d\)-dimensional column vector \( \theta^u (v, Y_v, \lambda^*_v; T) \). We get the other \( m \) equations for governing \( \theta^u (v, Y_v, \lambda^*_v; T) \) out of the orthogonal condition in (S.22), i.e. \( \sigma(v, Y_v)\theta^u (v, Y_v, \lambda^*_v; T) = 0_m \). Thus, it follows that

\[
(\sigma(v, Y_v)^+)^\top \theta^u (v, Y_v, \lambda^*_v; T) = (\sigma(v, Y_v)\sigma(v, Y_v)^+)^{-1}\sigma(v, Y_v)\theta^u (v, Y_v, \lambda^*_v; T) = 0_m.
\]  

\( \text{ (S.25b) } \)

By combining (S.25a) and (S.25b), the function \( \theta^u (v, Y_v, \lambda^*_v; T) \) solves

\[
\theta^u (v, Y_v, \lambda^*_v; T) = -\left( (\sigma(v, Y_v)^+)^\top \right)^{-1} \frac{0_{m \times d}}{(\sigma^F(v, Y_v)^+)^\top} \frac{E[H^T_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}.
\]  

\( \text{ (S.26) } \)

We now further simplify the above equation. By (S.14), we have

\[
\left( (\sigma(v, Y_v)^+)^\top \right)^{-1} (\sigma^F(v, Y_v)^+)^\top = \sigma^S(v, Y_v)^\top = (\sigma(v, Y_v)^+ \sigma^F(v, Y_v)^\top).
\]  

\( \text{ (S.27) } \)

Thus, equation (S.26) can be further deduced as

\[
\theta^u (v, Y_v, \lambda^*_v; T) = -\sigma(v, Y_v)^\top \sigma^F(v, Y_v)^\top \frac{0_{m \times d}}{(\sigma^F(v, Y_v)^+)^\top} \frac{E[H^T_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}
\equiv -\sigma^F(v, Y_v)^\top (\sigma^F(v, Y_v)^+)^\top \frac{E[H^T_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v] + E[H^\theta_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}{E[Q_{v,T}(\lambda^*_v)|Y_v, \lambda^*_v]}.
\]  

\( \text{ (S.28) } \)

By the definition of Moore-Penrose inverse, we can simplify the coefficient in the above equation as

\[
\sigma^F(v, Y_v)^\top (\sigma^F(v, Y_v)^+)^\top = \sigma^F(v, Y_v)^\top (\sigma^F(v, Y_v)\sigma^F(v, Y_v)^+)^{-1}\sigma^F(v, Y_v) = \sigma^F(v, Y_v)^+ \sigma^F(v, Y_v).
\]  

\( \text{ (S.29) } \)

Besides, by (S.27), we note that

\[
I_d = (\sigma(v, Y_v)^+ \sigma^F(v, Y_v)^+)^\top (\sigma(v, Y_v)^+ \sigma^F(v, Y_v)^+) \equiv \sigma(v, Y_v)^+ \sigma(v, Y_v) + \sigma^F(v, Y_v)^+ \sigma^F(v, Y_v).
\]  

\( \text{ (S.30) } \)
Combining (S.30) with (S.29), we get
\[ \sigma^F(v, Y_v)^\top (\sigma^F(v, Y_v)^+)^\top = \sigma^F(v, Y_v)^+ \sigma(v, Y_v). \] (S.31)

Then, (28) follows by plugging (S.31) into (S.28).

\[ \square \]

**S.3 Proof of Proposition 2**

**Proof.** Under the assumption that the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) is differentiable in its arguments, we can apply the chain rule to the Malliavin derivative \( D_t \theta^u_s(\lambda^*_t) \) to express it as

\[
D_t \theta^u_s(\lambda^*_t) = \langle D_t Y_s \rangle \nabla \theta^u(s, Y_s, \lambda^*_t \xi^{S}_{t,s}(\lambda^*_t); T) \]
\[ + \lambda^*_t (D_t \xi^{S}_{t,s}(\lambda^*_t)) \partial \theta^u / \partial \lambda(s, Y_s, \lambda^*_t \xi^{S}_{t,s}(\lambda^*_t); T), \] (S.32)

where we must take into account the dependence of \( \theta^u_s(\lambda^*_t) \) on both state variable \( Y_s \) and relative state price density \( \xi^{S}_{t,s}(\lambda^*_t) \). Besides, by the property of Malliavin derivative on Ito integrals (see, e.g., the survey in Appendix D of Detemple et al. (2003)), we have

\[
D_t \xi^{S}_{t,s}(\lambda^*_t) = -\xi^{S}_{t,s}(\lambda^*_t) (\theta^{S}_{t} (\lambda^*_t) + H^{\theta}_{t,s}(\lambda^*_t)) \].

Further applying this to (S.32), we obtain

\[
D_t \theta^u_s(\lambda^*_t) = \langle D_t Y_s \rangle \nabla \theta^u(s, Y_s, \lambda^*_t \xi^{S}_{t,s}(\lambda^*_t); T) \]
\[ - \lambda^*_t \xi^{S}_{t,s}(\lambda^*_t) (\theta^{S}_{t} (\lambda^*_t) + H^{\theta}_{t,s}(\lambda^*_t)) \partial \theta^u / \partial \lambda(s, Y_s, \lambda^*_t \xi^{S}_{t,s}(\lambda^*_t); T). \] (S.33)

Plugging this into the dynamics of \( H^{u}_{t,s}(\lambda^*_t) \) in (27b), i.e.,

\[ dH^{u}_{t,s} = D_t \theta^u_s(\lambda^*_t)(\theta^u_s(\lambda^*_t)ds + dW_s) \]

we verify we can decompose \( H^{u}_{t,s}(\lambda^*_t) \) as

\[ H^{u}_{t,s}(\lambda^*_t) = H^{u,y}_{t,s}(\lambda^*_t) + H^{u,\lambda}_{t,s}(\lambda^*_t), \]

where \( H^{u,y}_{t,s}(\lambda^*_t) \) and \( H^{u,\lambda}_{t,s}(\lambda^*_t) \) follow the dynamics in (32a) and (32b), respectively. Then, the decomposition of the investor-specific price of risk hedge component \( \pi^u(t, X_t, Y_t) \) in (30) follows from (19b) and (31a) – (31b).

\[ \square \]

**S.4 Proof of Corollary 2**

**Proof.** For the incomplete market model with CRRA utility, i.e.,

\[ u(t, c) = we^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} \text{ and } U(T, x) = (1 - w)e^{-\rho T} \frac{x^{1-\gamma}}{1-\gamma}, \] (S.34)
we follow the general decomposition established in Theorems 1 and 2, and then develop substantial structural simplifications of the results based on the special properties of CRRA utility.

First, we prove that the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) is independent of the parameter \( \lambda \) for any \( 0 \leq v \leq T \) under CRRA utility. Equivalently, it leads to that the investor-specific price of risk \( \theta^u \), which ought to be \( \theta^u(v, Y_v, \lambda^*_v; T) \) under general utilities according to representation (15), is independent of the multiplier \( \lambda^*_v \). To begin with, thanks to the explicit forms of functions \( I^u(t, y) \) and \( I^U(t, y) \) under CRRA utility (S.34), we can specify the dual problem in (S.8) for characterizing the investor-specific price of risk function as:

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\sigma T}{\gamma}} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w \frac{1}{\gamma} \int_0^T e^{-\frac{\sigma s}{\gamma}} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds \right].
\]  
(S.35)

According to the principle of dynamic programming, we can solve the optimal \( \theta^u \) at arbitrary time \( v \) from the following time-v version of problem (S.35):

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\sigma T}{\gamma}} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w \frac{1}{\gamma} \int_v^T e^{-\frac{\sigma s}{\gamma}} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds \right].
\]  
(S.36)

Using the relationship \( \lambda^*_s = \lambda^*_0 \xi^S_s = \lambda^*_v \xi^S_v \) as well as the fact that the multiplier \( \lambda^*_v \) is known with information available up to time \( v \), we can extract the factor \( (\lambda^*_v)^{1-\frac{1}{\gamma}} = \) from the conditional expectation in (S.36) to get \( \inf_{\theta^u \in \text{Ker}(\sigma)} (\lambda^*_v)^{1-\frac{1}{\gamma}} E_v \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\sigma T}{\gamma}} (\xi^S_{v,T})^{1-\frac{1}{\gamma}} + w \frac{1}{\gamma} \int_v^T e^{-\frac{\sigma s}{\gamma}} (\xi^S_{v,s})^{1-\frac{1}{\gamma}} ds \right] \). According to He and Pearson (1991), as the Lagrangian multiplier of the static optimization problem (11), \( \lambda^*_0 \) must be positive. It implies that \( \lambda^*_v = \lambda^*_0 \xi^S_v \) is also positive. Thus, we can drop the factor \( (\lambda^*_v)^{1-\frac{1}{\gamma}} \) in this optimization problem. Besides, the process \( (Y_s, \xi^S_{v,s}) \) for \( v \leq s \leq T \) is Markovian with the initial value \( (Y_v, 1) \). Thus, we use the feedback control law to conclude that \( \theta^u \) admits the representation \( \theta^u = \theta^u(v, Y_v; T) \) for some function \( \theta^u(v, y; T) \). In other words, the function \( \theta^u(v, y, \lambda; T) \) introduced in (15) is independent of the parameter \( \lambda \) under CRRA utility. Then, by (8), the total price of risk under CRRA utility can be parameterized and represented as \( \theta^S(s, Y_s; T) = \theta^0(s, Y_s) + \theta^u(s, Y_s; T) \). Plugging this representation into (21) and (26), we can prove that \( \xi^S_{t,s} \) and \( H^0_{t,s} \), under CRRA utility, satisfy the dynamics in (33a) and (33b), respectively. In particular, as both \( \xi^S_{t,s} \) and \( H^0_{t,s} \) are independent of the time-t multiplier \( \lambda^*_t \) under CRRA utility, we drop \( \lambda^*_t \) as opposed to writing their general expressions \( \xi^S_{t,s}(\lambda^*_t) \) and \( H^0_{t,s}(\lambda^*_t) \).

We now establish the representations of optimal policy in (35) and (A.1) as well as the equation governing \( \theta^u(v, Y_v; T) \) in (A.3). First, we note the following algebraic fact: with the specification of the CRRA utility function given in (S.34), the functions \( Q_{t,T}(\lambda^*_t), H^0_{t,T}(\lambda^*_t), H^0_{t,T}(\lambda^*_t) \), and \( G_{t,T}(\lambda^*_t) \) defined in (23a) – (23c) and (S.4a) are simplified to the following separable forms:

\[
Q_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \tilde{Q}_{t,T}, \quad H^0_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \tilde{H}^0_{t,T}, \quad H^0_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \tilde{H}^0_{t,T}, 
\]  
(S.37)

and \( G_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \tilde{G}_{t,T} \), where \( \tilde{H}^0_{t,T}, \tilde{H}^0_{t,T}, \) and \( \tilde{G}_{t,T} \) are introduced in (A.2a), (A.2b), and (A.2c), respectively.
respectively, and the function $\tilde{Q}_{t,T}$ is given by

$$\tilde{Q}_{t,T} = -\frac{1}{\gamma} \tilde{G}_{t,T}.$$  \hfill (S.38)

With the separable forms, the wealth equation in (12), i.e., $X_t = E_t[G_{t,T}(\lambda_t^*)]$, is equivalent to

$$X_t = (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{G}_{t,T}].$$  \hfill (S.39)

For the mean-variance component $\pi_{\text{mv}}(t, X_t, Y_t) = -(\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) E_t[Q_{t,T}(\lambda_t^*)]/X_t$ in (18a), we use the relationships (S.37) and (S.38) to get $E_t[Q_{t,T}(\lambda_t^*)] = (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{G}_{t,T}]/\gamma$. Then, plugging it into (18a) yields

$$\pi_{\text{mv}}(t, X_t, Y_t) = (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{G}_{t,T}](\sigma(t, Y_t)^+/\gamma X_t) = (\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t)/\gamma,$$

where the second equality follows by (S.39). Finally, by the definition of $\theta^h(t, Y_t)$ in (9a), we obtain the representation in (35) as

$$\pi_{\text{mv}}(t, X_t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t)\sigma(t, Y_t)^\top)^{-1} (\mu(t, Y_t) - r(t, Y_t)1_m).$$

Similarly, for the interest rate and price of risk hedge components given by (18b) and (18c), their representations under CRRA utility (A.1) follow by combining the separable forms in (S.37) and the constraint (S.39).

Finally, we derive equation (A.3) for the investor-specific price of risk function $\theta^u(v, Y_v; T)$ under CRRA utility. With the separable forms given in (S.37) and the relationship (S.38), we can express the conditional expectations in the general equation (28) as $E[H_v,T(\lambda_v^*)|Y_v, \lambda_v^*] = (\lambda_v^*)^{-\frac{1}{2}} E[H_v,T|Y_v], E[H_v,T(\lambda_v^*)|Y_v, \lambda_v^*] = (\lambda_v^*)^{-\frac{1}{2}} E[H_v,T|Y_v],$ and $E[Q_v,T(\lambda_v^*)|Y_v, \lambda_v^*] = (\lambda_v^*)^{-\frac{1}{2}} E[\tilde{Q}_v,T|Y_v] = - (\lambda_v^*)^{-\frac{1}{2}} E[\tilde{G}_v,T|Y_v]/\gamma$. Then, (A.3) follows directly by plugging them in (28). The term $(\lambda_v^*)^{-\frac{1}{2}}$ cancels out in both the nominator and denominator, which reconciles with the investor-specific price of risk function $\theta^u(v, Y_v; T)$ being indeed independent of multiplier $\lambda_v^*$.
References

Cox, J. C., Huang, C.-f., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49 (1), 33–83.

Detemple, J., Garcia, R., Rindisbacher, M., 2003. A Monte Carlo method for optimal portfolios. Journal of Finance 58 (1), 401–446.

He, H., Pearson, N. D., 1991. Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. Journal of Economic Theory 54, 259–304.

Karatzas, I., Lehoczky, J. P., Shreve, S. E., 1987. Optimal portfolio and consumption decisions for a small investor on a finite horizon. SIAM Journal on Control and Optimization 25, 1557–1586.

Karatzas, I., Lehoczky, J. P., Shreve, S. E., Xu, G.-L., 1991. Martingale and duality methods for utility maximization in an incomplete market. SIAM Journal on Control and Optimization 29 (3), 702–730.

Touzi, N., 2012. Optimal stochastic control, stochastic target problems, and backward SDE. Vol. 29. Springer Science and Business Media, New York.
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