Abstract. As in [5], we study holomorphic maps of positive degree between compact complex manifolds, and prove that any holomorphic map of degree one from a compact complex manifold to itself is biholomorphic. This conclusion confirms that under a mild restriction the holomorphic Gromov relation “≥” is indeed a partial order.

Keywords: holomorphic map, degree, biholomorphic, Gromov’s partial order.

AMSC: 32Q, 32H.

1. Introduction

In 1978, Gromov introduced a notion of domination between smooth manifolds in a lecture at the Graduate Center CUNY as follows:

Let $X$ and $Y$ be $n$-dimensional closed smooth manifolds. We say that $X \geq Y$ if there is a smooth map of positive degree from $X$ to $Y$.

Gromov asked whether the relation “$\geq$” is a partial order in the context of real hyperbolic manifolds (i.e., the manifolds with a constant negative curvature metric). Here, as in [5], we consider this problem for general complex manifolds. We introduce the following notions.

Definition 1.1. Let $X$ and $Y$ be connected compact complex manifolds of the same dimension. Then $X \geq_1 Y$ (resp. $X \geq Y$) means that from $X$ to $Y$ there exists a holomorphic map of degree one (resp. positive degree).

Then we rephrase the question of Gromov’s partial order as follows:

Question 1.2. Let $X$ and $Y$ be connected compact complex manifolds of the same dimension.

(a) If $X \geq_1 Y$ and $Y \geq_1 X$, are $X$ and $Y$ biholomorphic?
(b) If $X \geq Y$ and $Y \geq X$, are $X$ and $Y$ biholomorphic?

In this note, we study holomorphic maps of degree one between connected compact complex manifolds and obtain the following.

Theorem 1.3. Let $X$ and $Y$ be connected compact complex manifolds of the same dimension and with the same second Betti number. If $f : X \to Y$ is a holomorphic map of degree one, then $f$ is biholomorphic.

Using this theorem, we can answer Question 1.2 partly.

Theorem 1.4. Let $X$ and $Y$ be connected compact complex manifolds of the same dimension.

(a) If $X \geq_1 Y$ and $Y \geq_1 X$, then $X$ and $Y$ are biholomorphic.
(b) Suppose that $X \geq Y$ and $Y \geq X$. If $X$ and $Y$ are not biholomorphic, then $X$ and $Y$ both admit a holomorphic self-map of degree greater than one.

Note that in [5] G. Bharali, I. Biswas and M. Mahan proved the above two theorems under the extra condition that $X$ or $Y$ belongs to the Fujiki class $C$. Hence they can prove that the answer to Question 1.2 (a) is positive if $X$ or $Y$ belongs to one of the following four classes: (i) projective manifolds of general type; (ii) Kähler manifolds which are Kobayashi hyperbolic; (iii) rational homogeneous projective manifolds with Picard number one which are not biholomorphic to $\mathbb{C}P^n$; or (iv) smooth projective hypersurfaces of dimension greater than one and of degree greater than two.

The second part of Theorem 1.4 reminds us to consider which manifold does not admit a holomorphic self-map of degree greater than one. Certainly, the Kobayashi hyperbolic manifolds have this property. In fact, the measure hyperbolic manifolds, see section 2.4 in [13], which are the generalization of Kobayashi hyperbolic manifolds, also admit this property. For more results on this question, we refer to [1, 2, 3, 9, 11]. Here, we consider a special class of compact complex manifolds, i.e., the Calabi-Yau manifolds. A Calabi-Yau manifold $X$ in this note means a compact complex manifold $X$ with finite fundamental group and with $K_X^\otimes m = O_X$ for some positive integer $m$. Note that we do not require a Calabi-Yau manifold being Kählerian.

**Theorem 1.5.** If $X$ is a Calabi-Yau manifold, then every surjective holomorphic self-map $f : X \to X$ is biholomorphic.

Combining the above discussions, we immediately get the following result.

**Corollary 1.6.** Let $X$ and $Y$ be connected compact complex manifolds of the same dimension. Suppose that $X \geq Y$ and $Y \geq X$. If $X$ or $Y$ is a measure hyperbolic manifold or a Calabi-Yau manifold, then $X$ and $Y$ are biholomorphic.

The paper is organized as follows. Theorem 1.3 and 1.4 will be proved in Section 2, and Theorem 1.5 will be proved in Section 3.

2. HOLOMORPHIC MAPS OF DEGREE ONE

Let $M$ and $N$ be $n$-dimensional compact oriented smooth manifolds, and let $f : M \to N$ be a smooth map. If $N$ is connected, then the degree of $f$ is defined as

$$\deg f := \sum_{x \in f^{-1}(y)} \text{sgn}(f)(x),$$

where $y \in N$ is a regular value and $\text{sgn}(f)(x)$ is the sign of the Jacobi $J(f)(x)$ of $f$ at $x$. The above definition does not depend on the regular value $y$ (cf. [14]).

**Remark 2.1.** If $f : X \to Y$ is a holomorphic map between $n$-dimensional complex manifolds $X$ and $Y$, then $f$ has a real Jacobi $J_R(f)(x)$, which is a real determinant of order $2n$, and a complex Jacobi $J_C(f)(x)$, which is a complex determinant of order $n$. In the definition of degree of $f$, $f$ is regarded as a smooth map between $2n$-dimensional smooth manifolds $X$ and $Y$, so $\text{sgn}(f)(x)$ refer to the sign of the real Jacobi $J_R(f)(x)$ in this situation.
Remark 2.2. If \( f : X \to Y \) is a holomorphic map between \( X \) and \( Y \) which are both \( n \)-dimensional connected compact complex manifolds, then \( f \) is surjective if and only if \( \deg f \neq 0 \). Moreover, in this case, we have \( \deg f > 0 \). Indeed, suppose that \( f \) is surjective. Let \( S' \) be the set of its critical values. Then \( Y - S' \) is the set of its regular values. For any \( y \in Y - S' \) and \( x \in f^{-1}(y) \), we have \( \text{sgn}(f)(x) = 1 \) as the real Jacobi \( J_\mathbb{R}(f)(x) \) is nonnegative. Therefore, \( \deg f > 0 \). The other direction is obvious by the definition of degree.

In this section, we consider holomorphic maps of degree one. We first recall the definition of a modification. A homomorphic map \( f : X \to Y \) between connected compact complex manifolds is called a \textit{modification}, if there is a nowhere dense analytic subset \( F \subset Y \), such that \( f^{-1}(F) \subset X \) is nowhere dense and \( f : X - f^{-1}(F) \to Y - F \) is biholomorphic. If \( F \) is the minimal analytic subset satisfying the above condition, then we call \( E = f^{-1}(F) \) the \textit{exceptional set} of the modification \( f \). We need the following basic result on the modifications.

**Theorem 2.3** \([10]\), page 215 and \([6]\), page 170. If \( f : X \to Y \) is a modification between connected compact complex manifolds and the exceptional set \( E \) is not empty, then \( E \) has pure codimension one in \( X \) and \( \text{codim}_Y(f(E)) \geq 2 \).

Recall a proposition of A. Fujiki in \([8]\) about modifications, whose original proof uses the method of local cohomology. For a convenience, we give a simpler proof as in \([7]\).

**Proposition 2.4** \([8]\), Proposition 1.1. If \( f : X \to Y \) is a modification of \( n \)-dimensional compact complex manifolds with exceptional set \( E \), then there is an exact sequence

\[
0 \to H_{2n-2}(E, \mathbb{R}) \xrightarrow{i_*} H_{2n-2}(X, \mathbb{R}) \xrightarrow{f_*} H_{2n-2}(Y, \mathbb{R}) \to 0,
\]

where \( i : E \to X \) is the inclusion. Moreover, if \( E_1, \ldots, E_r \) are the irreducible components of \( E \), then \( H_{2n-2}(E, \mathbb{R}) = \bigoplus_{j=1}^{r} \mathbb{R}[E_j] \), where \( [E_j] \in H_{2n-2}(E, \mathbb{R}) \) is the fundamental class of \( E_j \) in \( E \) for \( j = 1, \ldots, r \).

**Proof.** By the projection formula

\[
f_* f^* = \text{id} : H_r(Y, \mathbb{R}) \to H_r(Y, \mathbb{R}),
\]

\( f_* : H_r(X, \mathbb{R}) \to H_r(Y, \mathbb{R}) \) is surjective for \( r = 0, \ldots, 2n \). Here \( f^* : H_r(Y, \mathbb{R}) \to H_r(X, \mathbb{R}) \) is induced by the pull back as follows:

\[
\begin{array}{ccc}
H_r(Y, \mathbb{R}) & \xrightarrow{f^*} & H_r(X, \mathbb{R}) \\
\downarrow_{PD_Y} & & \downarrow_{PD_X} \\
H^{2n-r}(Y, \mathbb{R}) & \xrightarrow{f^*} & H^{2n-r}(X, \mathbb{R})
\end{array}
\]

where \( PD_X \) (resp. \( PD_Y \)) is the Poincaré duality of \( X \) (resp. \( Y \)).

Let \( F = f(E), U = X - E, \) and \( V = Y - F \). Then \( f \mid_U : U \to V \) is a biholomorphic map. Consider the following commutative diagram of exact sequences of Borel-Moore homology

\[
\begin{array}{cccccccc}
H_{2n-1}(X, \mathbb{R}) & \xrightarrow{f_*} & H_{2n-1}(U, \mathbb{R}) & \xrightarrow{=} & H_{2n-2}(E, \mathbb{R}) & \xrightarrow{i_*} & H_{2n-2}(X, \mathbb{R}) & \xrightarrow{f_*} & H_{2n-2}(U, \mathbb{R}) \\
\downarrow & & & & \downarrow \cong & & \downarrow & & \\
H_{2n-1}(Y, \mathbb{R}) & \xrightarrow{f_*} & H_{2n-1}(V, \mathbb{R}) & \xrightarrow{=} & H_{2n-2}(F, \mathbb{R}) & \xrightarrow{=} & H_{2n-2}(Y, \mathbb{R}) & \xrightarrow{f_*} & H_{2n-2}(V, \mathbb{R})
\end{array}
\]
By Theorem 2.3, \( \dim F \geq 2 \), so \( H_{2n-2}(F, \mathbb{R}) = 0 \). By the second long exact sequence, we know that \( H_{2n-1}(Y, \mathbb{R}) \to H_{2n-1}^{BM}(V, \mathbb{R}) \) is surjective. Since \( f_* : H_{2n-1}(X, \mathbb{R}) \to H_{2n-1}(Y, \mathbb{R}) \) is surjective, \( H_{2n-1}(X, \mathbb{R}) \to H_{2n-1}^{BM}(U, \mathbb{R}) \) is surjective. Hence \( i_* \) is injective by the first long exact sequence.

If \( \alpha \in H_{2n-2}(X, \mathbb{R}) \) and \( f_*(\alpha) = 0 \), then the image of \( \alpha \) in \( H_{2n-2}^{BM}(U, \mathbb{R}) \cong H_{2n-2}^{BM}(V, \mathbb{R}) \) is zero. Hence, \( \alpha \) is in the image of \( i_* \) by the first long exact sequence. Thus, \( \text{Ker} f_* \subseteq \text{Im} i_* \). Form the fact \( H_{2n-2}(F, \mathbb{R}) = 0 \), we also have \( f_* i_* = 0 \), i.e., \( \text{Im} i_* \subseteq \text{Ker} f_* \). Therefore \( \text{Im} i_* = \text{Ker} f_* \). Combining the above discussions, we get the short exact sequence (1).

By Theorem 2.3, \( E_1, \ldots, E_r \) all have dimension \((n - 1)\). Set

\[
A := \bigcup_{i \neq j} (E_i \cap E_j),
E_i' := E_i - A \cap E_i.
\]

Then, all \( E_i' \) for \( i = 1, \ldots, r \) do not intersect with one another and \( E - A = \bigcup_i E_i' \). We consider the exact sequence of Borel-Moore homology for \((E, A)\)

\[
H_{2n-2}(A, \mathbb{R}) \longrightarrow H_{2n-2}(E, \mathbb{R}) \longrightarrow H_{2n-2}^{BM}(E - A, \mathbb{R}) \longrightarrow H_{2n-3}(A, \mathbb{R}).
\]

By the definition of \( A \), \( \dim A \leq n - 2 \), so \( H_{2n-2}(A, \mathbb{R}) = H_{2n-3}(A, \mathbb{R}) = 0 \). Hence

\[
H_{2n-2}(E, \mathbb{R}) = H_{2n-2}^{BM}(E - A, \mathbb{R}) = \oplus_i H_{2n-2}^{BM}(E_i', \mathbb{R}).
\]

Considering the long exact sequence of Borel-Moore homology for \((E_i, A \cap E_i)\), we obtain

\[
H_{2n-2}^{BM}(E_i', \mathbb{R}) = H_{2n-2}(E_i, \mathbb{R}) = \mathbb{R}[E_i].
\]

Hence, \( H_{2n-2}(E, \mathbb{R}) = \oplus_i \mathbb{R}[E_i] \). □

Having made the above preparations, we can prove Theorem 1.3.

**Proof.** Denote \( J_C(f)(x) \) the complex Jacobi of \( f \) at \( x \). Define

\[
S := \{ x \in X \mid J_C(f)(x) = 0 \},
\]

and \( S' = f(S) \). That is, \( S \) is the set of critical points, and \( S' \) is the set of critical values. Then \( Y - S' \) is the set of regular values.

By Remark 2.2, \( f : X \to Y \) is surjective and for any \( y \in Y - S' \) and \( x \in f^{-1}(y) \), \( \text{sgn}(f)(x) = 1 \). Since

\[
\sum_{x \in f^{-1}(y)} \text{sgn}(f)(x) = \deg f = 1,
\]

\( f^{-1}(y) \) contains only one point. Hence, \( f : X - f^{-1}(S') \to Y - S' \) is injective. Therefore it is biholomorphic.

By [17], Corollary 1.7, \( S \) is a nowhere dense analytic subset of \( X \). Then by the proper mapping theorem, \( S' = f(S) \) is an analytic subset. Since

\[
\dim S' = \dim f(S) \leq \dim S < \dim X = \dim Y,
\]

\( S' \) is also nowhere dense in \( Y \). We claim that the analytic subset \( f^{-1}(S') \) is also nowhere dense in \( X \). Indeed, if \( f^{-1}(S') \) is dense in \( X \), then \( f^{-1}(S') = X \) since \( X \) is connected. Hence, \( S' = f(f^{-1}(S')) = f(X) = Y \), which contradicts that \( S' \) is nowhere dense in \( Y \). Therefore, \( f \) is a modification.
Suppose the exceptional set \( E \subseteq X \) of \( f \) is not empty. By Theorem 2.3, \( E \) has pure codimension one, \( \text{codim}_Y f(E) \geq 2 \) and \( f : X - E \to Y - f(E) \) is biholomorphic. Assume that \( r \) is the number of irreducible components of \( E \). By Theorem 2.4, \( b_2(X) = b_2(Y) + r \). Then by the hypothesis of theorem, \( r = 0 \), i.e., \( E = \emptyset \), which contradicts to the previous assumption. Therefore, \( f \) is a biholomorphic map.

\[ \square \]

**Corollary 2.5.** (a) Any holomorphic self-map of degree one of a connected compact complex manifold must be biholomorphic.

(b) Let \( f : X \to Y \) be a holomorphic map of degree one. If \( X \) and \( Y \) are both K3 surfaces, Enriques surfaces or complex tori of the same dimension, then \( f \) is biholomorphic.

**Proof.** We get part (a) and (b) immediately by Theorem 1.3 if we note that in all cases \( b_2(X) = b_2(Y) \).

\[ \square \]

**Remark 2.6.** We can also obtain Corollary 2.5, (b) by Proposition 3.4, (b) immediately.

Now, we prove Theorem 1.4 as follows.

**Proof.** (a) If \( f : X \to Y \) and \( g : Y \to X \) are both holomorphic maps of degree one, then the composition \( g \circ f : X \to X \) is a self-endomorphism of degree one. By Corollary 2.5, (a), \( g \circ f \) is biholomorphic. Hence, \( f \) is injective. Since \( f \) is also surjective, it is bijective. Therefore, \( f \) is a biholomorphic map.

(b) By Definition 1.1 there exist positive degree holomorphic maps \( f : X \to Y \) and \( g : Y \to X \). Hence the push out \( f_* : H^2_2(X, \mathbb{R}) \to H^2_2(Y, \mathbb{R}) \) and \( g_* : H^2_2(Y, \mathbb{R}) \to H^2_2(X, \mathbb{R}) \) are surjective. Thus, \( b_2(X) = b_2(Y) \). Now we have the conclusions that \( \deg f > 1 \) and \( \deg g > 1 \). Otherwise by Theorem 1.3 \( f \) or \( g \) is biholomorphic, which contradicts the hypothesis of theorem. Hence, \( \deg(f \circ g) > 1 \) and \( \deg(g \circ f) > 1 \).

\[ \square \]

Next we give another sufficient condition for a degree-one holomorphic map to be biholomorphic. We first recall some notation. For a compact complex manifold \( X \), define its Neron-Severi group

\[
\text{NS}(X) = \text{Im}(c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z})).
\]

Denote \( \text{NS}(X)_\mathbb{R} := \text{NS}(X) \otimes \mathbb{Z} \mathbb{R} \) and \( \rho(X) := \dim \text{NS}(X)_\mathbb{R} \). The number \( \rho(X) \) is called the Picard number of \( X \).

**Proposition 2.7.** Let \( X \) be a connected projective manifold and \( Y \) a connected compact complex manifold. If \( f : X \to Y \) is a modification and \( E_1, \ldots, E_r \) are the irreducible components of its exceptional set \( E \), then there is an exact sequence

\[
0 \to \bigoplus_{j=1}^r \mathbb{R}[E_j] \xrightarrow{i_*} \text{NS}(X)_\mathbb{R} \xrightarrow{f_*} \text{NS}(Y)_\mathbb{R} \to 0,
\]

where \( i : E \to X \) is the inclusion and \( [E_j] \) is the fundamental class of \( E_j \) in \( X \) for \( j = 1, \ldots, r \).

**Proof.** For any \( L \in \text{Pic}(X) \), there is a divisor \( D \) on \( X \) such that \( L = \mathcal{O}(D) \) and hence, \( f_*(c_1(L)) = c_1(\mathcal{O}(f_*(D))) \). Therefore, \( f_*(\text{NS}(X)_\mathbb{R}) \subseteq \text{NS}(Y)_\mathbb{R} \). On the other hand, since \( f^*(\text{NS}(Y)_\mathbb{R}) \subseteq \text{NS}(X)_\mathbb{R} \), by the projection formula \( f_* f^* = \text{id} \), we have

\[
\text{NS}(Y)_\mathbb{R} = f_* f^*(\text{NS}(Y)_\mathbb{R}) \subseteq f_*(\text{NS}(X)_\mathbb{R}).
\]
Hence, \( f_* \) is surjective. Then since \([E_z] \in NS(X)_{\mathbb{R}}, \) sequence 2 is exact subsequently by Proposition 2.4.

**Theorem 2.8.** Let \( X \) be a connected projective manifold and \( Y \) a connected compact complex manifold with \( \dim X = \dim Y \). Suppose that the Picard numbers \( \rho(X) = \rho(Y) \). If \( f : X \to Y \) is a holomorphic map of degree one, then \( f \) is a biholomorphic map.

**Proof.** As the proof of Theorem 1.3 we know that \( f \) is a modification. Then by Proposition 2.7, the number of irreducible components of its exceptional set is zero. Hence, \( f \) is a biholomorphic map.

We will give an application of Theorem 1.3. G. Bharali and I. Biswas considered the rigidity of a holomorphic self-map of a fiber space in [4], where Theorem 1.2 is the following under the extra condition that \( \dim H^1(X_s, \mathcal{O}_{X_s}) \) is independent on \( s \in S \).

**Theorem 2.9 ([4]).** Suppose that \( S \) is a connected compact complex manifold and \( p : X \to S \) is a family of connected compact complex manifolds (i.e., \( p \) is a proper holomorphic submersion with connected compact fibers). Let \( F : X \to X \) be a holomorphic map such that there exist two points \( a, b \in S \) satisfying \( F(X_a) \subseteq X_b \). Then

(a) \( F \) is a holomorphic map of fiber spaces, i.e., there exists a holomorphic map \( f : S \to S \) such that \( p \circ F = f \circ p \); and

(b) If \( F \vert_{X_a} : X_a \to X_b \) has degree one, then \( F \) is a fiberwise biholomorphism.

**Proof.** Part (a) is same as Theorem 1.2, a) in [4]. From the original proof of Theorem 1.2, b) in [4], we know that if \( F \vert_{X_s} : X_a \to X_b \) has degree one, then for any \( s \in S \), \( F \vert_{X_s} : X_s \to X_{f(s)} \) has also degree one. Since all fibers of \( p : X \to S \) are diffeomorphic, \( b_2(X_s) = b_2(X_{f(s)}) \). Hence, by Theorem 1.3 we get part (b).

### 3. Holomorphic Maps with Positive Degree

In this section we consider surjective holomorphic maps. A holomorphic map \( f : X \to Y \) of complex manifolds is called finite, if \( f \) is proper and for any point \( y \in Y \), \( f^{-1}(y) \) is a finite set.

**Proposition 3.1.** Let \( X \) be a compact Kähler manifold and \( Y \) a compact complex manifold. Suppose that the betti numbers of \( X \) and \( Y \) are equal. If \( f : X \to Y \) is a surjective holomorphic map, then \( f \) is finite. Therefore, \( \dim X = \dim Y \) and \( Y \) is a Kähler manifold.

**Proof.** If \( f : X \to Y \) is finite, then \( \dim X = \dim Y \) and \( Y \) is a Kähler manifold by Theorem 2 in [18]. So, we only need to prove that \( f \) is finite. Assume that \( f \) is not finite. Then there exists a point \( y \in Y \) such that \( \dim f^{-1}(y) \geq 1 \). We choose an irreducible analytic set \( Z \subseteq f^{-1}(y) \) such that \( \dim Z = r \geq 1 \). If we set \( \dim X = n \) and \( \dim Y = m \), since \( f(Z) = \{y\} \), then \( f_*([Z]) = 0 \) in \( H^{2m-2r}(Y, \mathbb{R}) \). Here \( [Z] \in H_{2r}(X, \mathbb{R}) \) is the fundamental class of \( Z \) on \( X \), and \( f_* : H^{2n-2r}(X, \mathbb{R}) \to H^{2m-2r}(Y, \mathbb{R}) \) is the \( 2r \)-th Gysin map.

Since \( X \) is a compact Kähler manifold, \( [Z] \neq 0 \) in \( H^{2m-2r}(X, \mathbb{R}) \). Hence we can choose a class \( \gamma \in H^{2r}(X, \mathbb{R}) \) such that \( [Z] \cup \gamma \neq 0 \) in \( H^{2n}(X, \mathbb{R}) \). By Lemma 7.28 in [19], \( f^* : H^{2r}(Y, \mathbb{R}) \to H^{2r}(X, \mathbb{R}) \) is injective. Since \( b_{2r}(X) = b_{2r}(Y) \), \( f^* \) is bijective. Hence, there is \( \beta \in H^{2r}(Y, \mathbb{R}) \) such that \( \gamma = f^*(\beta) \). So \( [Z] \cup \gamma = [Z] \cup f^*(\beta) = f_*[Z] \cup \beta = 0 \). It contradicts the choice of \( \gamma \).
Consequently, we have the following.

**Corollary 3.2.** Let $X$ be a compact Kähler manifold. If $f : X \to X$ is a surjective holomorphic map, then $f$ is finite.

The following proposition is essentially proved by A. Fujimoto in [19], where $X$ and $Y$ are both projective manifolds with $\dim X = \dim Y$ and $\rho(X) = \rho(Y)$.

**Proposition 3.3.** Let $X$ be a projective manifold and $Y$ a compact complex manifold such that $\rho(X) = \rho(Y)$ or $b_2(X) = b_2(Y)$. If $f : X \to Y$ is a surjective holomorphic map, then $f$ is finite and $\dim X = \dim Y$.

**Proof.** Obviously, we only need to prove that $f$ is finite. Assume that $f$ is not finite. Then there exists a point $y \in Y$ such that $\dim f^{-1}(y) \geq 1$. Since $X$ is a projective manifold, $f^{-1}(y)$ is a projective variety, hence contains a projective curve $C$. Suppose $n = \dim X \geq \dim Y = m$. Since $f(C) = \{y\}$, $f_\ast[C] = 0$ in $H^{2m-2}(Y, \mathbb{R})$, where $f_\ast : H^{2n-2}(X, \mathbb{R}) \to H^{2m-2}(Y, \mathbb{R})$ is the second Gysin map. The $r$-th Gysin map is defined by pushing out through the Poincaré dualities as follows (see [19], page 178):

$$
\begin{align*}
H^{2n-r}(X, \mathbb{R}) & \xrightarrow{f_\ast} H^{2m-r}(Y, \mathbb{R}) \\
PD_X & \downarrow \quad \downarrow PD_Y \\
H_r(X, \mathbb{R}) & \xrightarrow{f_\ast} H_r(Y, \mathbb{R})
\end{align*}
$$

By [19], Lemma 7.28, $f_\ast : H^2(Y, \mathbb{R}) \to H^2(X, \mathbb{R})$ is injective. When $\rho(X) = \rho(Y)$, $f_\ast : NS(Y)_{\mathbb{R}} \to NS(X)_{\mathbb{R}}$ is an isomorphism. Let $L$ be an ample line bundle on $X$. Then there is an $\alpha \in NS(Y)_{\mathbb{R}}$ such that $c_1(L) = f^\ast \alpha$. So $c_1(L) \cup [C] = f^\ast \alpha \cup [C] = \alpha \cup f_\ast[C] = 0$. It contradicts the ampleness of $L$. When $b_2(X) = b_2(Y)$, we can obtain the contradiction as in the proof of Proposition 3.1 when we choose $Z = C$ here. \qed

Now, we consider when a surjective holomorphic map is an unramified covering map. A surjective holomorphic map is called an unramified covering map if it is a finite covering map in the topological sense.

**Proposition 3.4.** (a) Let $X$ be a compact complex manifold with non-negative Kodaira dimension. If $f : X \to X$ is a surjective holomorphic map, then $f$ is an unramified covering map.

(b) Let $X$ and $Y$ be $n$-dimensional compact complex manifolds with $K^k_X = \mathcal{O}_X$ and $K^l_Y = \mathcal{O}_Y$ for some positive integers $k$ and $l$ respectively. If $f : X \to Y$ is a surjective holomorphic map, then $f$ is an unramified covering map.

(c) Let $f : X \to Y$ be a surjective finite map of complex manifolds (which may be non-compact). If the number $|f^{-1}(y)|$ of the points contained in $f^{-1}(y)$ is independent with $y \in Y$, then $f$ is an unramified covering map.

**Proof.** (a) is proved in [13], Theorem 7.6.11, or in [15].

(b) Let $f_\ast : T_X \to f^\ast T_Y$ be the tangent map. Then

$$
\bigwedge^n f_\ast : \bigwedge^n T_X \to f^\ast(\bigwedge^n T_Y)
$$
defines a global section $\sigma \in \Gamma(X, K_X \otimes f^*K_Y^{-1})$. If $D(\sigma)$ is the divisor defined by $\sigma$, then $K_X = f^*K_Y \otimes O(D(\sigma))$. Since $K_X^{\otimes k} = O_X$ and $K_Y^{\otimes l} = O_Y$, $O(D(\sigma^{kl})) = O_X$. Hence, the divisor $D(\sigma^{-kl}) = 0$, which implies the divisor $D(\sigma) = 0$. Therefore $f$ is a holomorphic submersion map. Since $\dim X = \dim Y$, $f$ is a surjective local isomorphism. By [12], Lemma 2, $f$ is an unramified covering map.

(c) For any $y_0 \in Y$, set $f^{-1}(y_0) = \{x_1, \ldots, x_d\}$. By [6], Theorem in page 145, $f$ is an open map. So we can choose an open neighbourhood $W_i$ of $x_i$ for $i = 1, \ldots, d$ such that $W_i \cap W_j = \emptyset$ for $i \neq j$. Define $H_i := f(W_i)$ for $i = 1, \ldots, d$. Let $V := \bigcap_{i=1}^d H_i$ and $U_i := W_i \cap f^{-1}(V)$. Then, $f|_{U_i}: U_i \to V$ for $i = 1, \ldots, d$ is surjective. For any $y \in V$, $(f|_{U_i})^{-1}(y)$ is not empty and

$$\sum_{i=1}^d |(f|_{U_i})^{-1}(y)| \leq |f^{-1}(y)| = d.$$  

So for each $i$, $|(f|_{U_i})^{-1}(y)| = 1$ and $f^{-1}(y) = \bigcup_{i=1}^d (f|_{U_i})^{-1}(y)$, i.e., $f|_{U_i}$ is bijective and $f^{-1}(V) = \bigcup_{i=1}^d U_i$. Hence, $f|_{U_i}$ is biholomorphic. We have proved that $f$ is an unramified covering map. $\square$

**Proposition 3.5.** Let $X$ be an $n$-dimensional connected compact complex manifold and $f: X \to Y$ an unramified holomorphic covering map. Suppose that $X$ satisfies one of the following conditions:

(a) There exists a Chern number $P(X) := \int_X P(c_1(X), \ldots, c_n(X)) \neq 0$, where

$$P(T_1, \ldots, T_n) = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n}$$

is a polynomial on $T_1, \ldots, T_n$ over $\mathbb{Q}$ satisfying $i_j \in \mathbb{N}$ for $j = 1, \ldots, n$ and $i_1 + 2i_2 + \cdots + ni_n = n$. Especially, the Euler characteristic $\chi(O_X) \neq 0$ of $O_X$, or the topological Euler characteristic $\chi^{top}(X) \neq 0$ of $X$, or the signature $\sigma(X) \neq 0$ of $X$ when $n$ is even; or

(b) The fundamental group $\pi_1(X)$ has no proper subgroup isomorphic to itself.

Then $f$ is biholomorphic.

**Proof.** (a) Since $f$ is an unramified covering map, $T_X = f^*T_X$ and for each $i$, $c_i(X) = f^*c_i(X)$. Hence, we have

$$P(X) = \int_X P(f^*c_1(X), \ldots, f^*c_n(X)) = \int_X f^*(P(c_1(X), \ldots, c_n(X))) = \deg f \cdot \int_X P(c_1(X), \ldots, c_n(X)) = \deg f \cdot P(X).$$

Since $P(X) \neq 0$, $\deg f = 1$. Then by Theorem 1.3, $f$ is biholomorphic.

(b) Let $x_0$ be any point in $X$ and $F = f^{-1}(x_0)$ the fiber of $f$ at $x_0$. Consider the exact sequence

$$\cdots \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(X) \xrightarrow{f_*} \pi_1(X) \longrightarrow \pi_0(F) \longrightarrow \pi_0(X) \longrightarrow \cdots$$
where $i_*$ is induced by the inclusion $i : F \rightarrow X$. Clearly, $\pi_1(F) = 0$. Since $X$ is a connected manifold, it is path-connected, hence $\pi_0(X) = 0$. Since $\pi_1(X)$ does not contain any proper subgroup isomorphic to itself, $f_*$ is an isomorphism. Hence, $\pi_0(F) = 0$, i.e., $F$ contains only one point. So $f$ is biholomorphic. □

**Remark 3.6.** Y. Fujimoto in [9] proved Theorem 3.5, (a) for $\chi(O_X) \neq 0$ in the case of projective manifolds.

Now we can give a proof of Theorem 1.5.

**Proof.** By Proposition 3.4, (b), $f$ is an unramified map. Then by Proposition 3.5, (b), $f$ is a biholomorphic map. □

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