A full set of Casimir operators for the Lie superalgebra $gl(m/\infty)$ is constructed and shown to be well defined in the category $O_{FS}$ generated by the highest weight irreducible representations with only a finite number of non-zero weight components. The eigenvalues of these Casimir operators are determined explicitly in terms of the highest weight. Characteristic identities satisfied by certain (infinite) matrices with entries from $gl(m/\infty)$ are also determined.
I. INTRODUCTION

During the last years the infinite dimensional Lie algebras and Lie superalgebras play an important role in several areas of theoretical and mathematical physics\(^{1-9}\). They have applications in the theory of integrable field equations, string theory, two-dimensional statistical models. In addition these algebras are of interest as examples of Kac-Moody Lie (super-)algebras of infinite type.

However, for these algebras such a fundamental concept as Casimir invariants has not yet been determined. The present paper is a step in solving this problem.

We construct a full set of Casimir operators for the infinite dimensional general linear Lie superalgebra \(gl(m/\infty)\) corresponding to the natural matrix realization, namely

\[
\text{gl}(m/\infty) = \{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_{m \times m}, B \in M_{m \times \infty}, C \in M_{\infty \times m}, D \in M_{\infty \times \infty}, \text{all but a finite number of } X_{ij} \in \mathbb{C} \text{ are zero} \},
\]

where \(M_{p \times q}\) is the space of all \(p \times q\) complex matrices. The even subalgebra \(\text{gl}(m/\infty)_0\) has \(B = 0\) and \(C = 0\); the odd subspace \(\text{gl}(m/\infty)_1\) has \(A = 0\) and \(D = 0\).

A basis for the Lie superalgebra \(\text{gl}(m/\infty)\) is given by the Weyl generators \(E_{ij}\), \(i, j = -m + 1, -m + 2, \ldots, 0, 1, \ldots\). Assign to each index \(i\) a degree \(\langle i \rangle\), which is zero for \(i \in \mathbb{Z}_+\) and 1 for \(i \in \mathbb{N}\) (see the notation at the end of the Introduction). Then the generator \(E_{ij}\) is even (resp. odd), if \(\langle i \rangle + \langle j \rangle\) is an even (resp. odd) number. The multiplication (\(\equiv\)the supercommutator) \([\ , \ ]\) of \(\text{gl}(m/\infty)\) is given by the linear extension of the relations:

\[
[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{\langle(i)\rangle+\langle(j)\rangle} \delta_{il}E_{kj}.
\]

We will consider the category \(O_{FS}\) generated by all highest weight irreducible \(\text{gl}(m/\infty)\) modules \(V(\Lambda)\) with a finite number of non-zero highest weight components \(\Lambda_i\) of the highest weight

\[
\Lambda \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k, 0, 0, \ldots) \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k, \hat{0}).
\]

The highest weight \(\Lambda\) of \(V(\Lambda)\) uniquely characterized the module and satisfies the conditions:

\[
\Lambda_i - \Lambda_{i+1} \in \mathbb{Z}_+, \quad \forall i \neq 0.
\]

Denote by \(H\) the Cartan subalgebra of \(\text{gl}(m/\infty)\). The dual space \(H^*\) of \(H\) is described by the forms \(\varepsilon_i, i = -m + 1, -m + 1, \ldots\), where \(\varepsilon_i : X \to A_{ii}\), for \(-m + 1 \leq i \leq 0\) and \(\varepsilon_i : X \to D_{ii}\), \(\forall i \in \mathbb{N}\), and \(X\) is given by (1) only for diagonal \(X\). On \(H^*\) there is a bilinear form \((\ , \ )\) defined by

\[
(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad \text{for } -m + 1 \leq i, j \leq 0;
\]

\[
(\varepsilon_i, \varepsilon_j) = 0, \quad \text{for } -m + 1 \leq i \leq 0 \text{ and } j \in \mathbb{N};
\]

\[
(\varepsilon_i, \varepsilon_j) = -\delta_{ij}, \quad \text{for } i, j \in \mathbb{N}.
\]
The roots $\epsilon_i - \epsilon_j$ ($i \neq j$) of $gl(m/\infty)$ are the non-zero weights of the adjoint representation. The positive roots are those given by the set:

$$\Phi^+ = \{\epsilon_i - \epsilon_j | i < j, \ i, j = -m + 1, -m + 2, \ldots\}. \quad (6)$$

Define

$$\rho = \frac{1}{2} \sum_{i=-m+1}^{0} (1 - 2i - 2m)\epsilon_i + \frac{1}{2} \sum_{i=1}^{\infty} (1 - 2i + 2m)\epsilon_i. \quad (7)$$

Let $D_n$ be the set of $gl(m/\infty)$ weights:

$$D_n = \{\nu | \nu = (\nu_{-m+1}, \ldots, \nu_0, \nu_1, \ldots, \nu_n, 0), \ \nu_i \in \mathbb{Z}_+, \ i = -m + 1, -m + 2, \ldots, n - 1, \ \nu_n \in \mathbb{N}\}, \quad (8)$$

and let $D^+_n \subset D_n$ be the subset of integral dominant weights in $D_n$:

$$D^+_n = \{\nu | \nu \in D_n, \ \nu_i - \nu_{i+1} \in \mathbb{Z}_+, \ \forall i \neq 0\}. \quad (9)$$

Note that if $\nu$ is a weight in $V(\Lambda)$, $\Lambda \in D^+_k$, then $\nu \in D_n$, for some $n \in \mathbb{Z}_+$.

In Section II we construct a full set of Casimir operators convergent on each module $V(\Lambda)$. The eigenvalues of these Casimir invariants for all modules from the category $O_{FS}$ are computed in Section III.

In Section IV we present a derivation of the polynomial identities satisfied by certain matrices with entries from $gl(m/\infty)$.

Throughout the paper we use the following notation:

irrep(s) - irreducible representation(s);

$\mathbb{C}$ - the complex numbers;

$\mathbb{Z}_+$ - all non-negative integers;

$\mathbb{N}$ - all positive integers;

$U(A)$ - the universal enveloping algebra of $A$;

$\langle i \rangle = \begin{cases} 0 & \text{for } i \in -\mathbb{Z}_+ \\ 1 & \text{for } i \in \mathbb{N}. \end{cases}$

II. CONSTRUCTION OF CASIMIR OPERATORS

An obvious invariant for $gl(m/\infty)$ is the first order invariant

$$I_1 = \sum_{i=-m+1}^{\infty} E_{ii}. \quad (10)$$

It is not clear, however, how to construct appropriate higher order Casimir operators for $gl(m/\infty)$. Let us first consider the second order invariant $I^{(m,n)}_2$ of $gl(m/n)$:

$$I^{(m,n)}_2 = \sum_{i,j=-m+1}^{n} (-1)^{\langle j \rangle} E_{ij} E_{ji} = \sum_{i,j=-m+1}^{0} E_{ij} E_{ji} - \sum_{i,j=1}^{n} E_{ij} E_{ji} + \sum_{i=1}^{n} \sum_{j=-m+1}^{0} E_{ij} E_{ji}$$
\[- \sum_{i=-m+1}^{0} \sum_{j=1}^{n} E_{ij} E_{ji} = \sum_{i=-m+1}^{0} \sum_{j<i=-m+1}^{0} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} \sum_{j>i=-m+1}^{0} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} E_{ii}^2 \]

\[- \sum_{i=1}^{n} \sum_{j<i=1}^{n} E_{ij} E_{ji} - \sum_{i=1}^{n} \sum_{j>i=1}^{n} E_{ij} E_{ji} - \sum_{i=1}^{n} E_{ii}^2 + 2 \sum_{i=1}^{n} \sum_{j=-m+1}^{0} E_{ij} E_{ji} - \sum_{i=-m+1}^{0} \sum_{j=1}^{n} E_{ij} E_{ji} (E_{ii} + E_{jj}) \]

\[= 2 \sum_{i=-m+1}^{0} \sum_{j<i=-m+1}^{0} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} \sum_{j=j<i=-m+1}^{0} (E_{ii} - E_{jj}) + \sum_{i=-m+1}^{0} \sum_{j=i=-m+1}^{0} E_{ij} E_{ji} - 2 \sum_{i=-m+1}^{0} \sum_{j=i<j=-m+1}^{0} E_{ij} E_{ji} \]

\[= 2 \sum_{i=-m+1}^{n} \sum_{j<i=-m+1}^{n} (-1)^{j} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii} + 1 - m - 2i) - \sum_{i=1}^{n} E_{ii}(E_{ii} + 1 + n - 2i) \]

\[- n \sum_{i=-m+1}^{0} E_{ii} - m \sum_{i=1}^{n} E_{ii} \]

\[= 2 \sum_{i=-m+1}^{n} \sum_{j<i=-m+1}^{n} (-1)^{j} E_{ij} E_{ji} + \sum_{i=-m+1}^{n} (-1)^{i} E_{ii}(E_{ii} + 1 - 2i) - (m + n)I_1^{(m,n)} \]

\[= 2 \sum_{i=-m+1}^{n} \sum_{j<i=-m+1}^{n} (-1)^{j} E_{ij} E_{ji} + \sum_{i=-m+1}^{n} (-1)^{i} E_{ii}(E_{ii} + 1 - 2i) - 2mI_1^{(m,n)} + (m - n)I_1^{(m,n)}, \]

(11)

where $I_1^{(m,n)} \equiv \sum_{i=-m+1}^{n} E_{ii}$ is the first order invariant of $gl(m/n)$. Due to the last term in (11) the $gl(m/n)$ second order invariant diverges as $n \to \infty$. Eliminating the last term in (11) (the rest of the expression is also an invariant) and taking the limit $n \to \infty$ one obtains the following quadratic Casimir for $gl(m/\infty)$:

\[I_2 = 2 \sum_{i=-m+1}^{\infty} \sum_{j<i=-m+1}^{\infty} (-1)^{j} E_{ij} E_{ji} + \sum_{i=-m+1}^{\infty} (-1)^{i} E_{ii}(E_{ii} + 1 - 2i) - 2mI_1, \]

(12)

which is convergent (see formula (21)) on the category $O_{FS}$ of irreps considered. On $V(\Lambda)$, $\Lambda \in D_k^+$, $I_2$ takes constant value

\[\chi_{\Lambda}(I_2) = \sum_{i=-m+1}^{k} \left( (-1)^{(i)} \Lambda_i(\Lambda_i + 1 - 2i) - 2m\Lambda_i \right) = (\Lambda, \Lambda + 2\rho). \]

(13)

This consideration shows how to construct the higher order Casimir operators of $gl(m/\infty)$.

Introduce to this end the characteristic matrix

\[A^j = (-1)^{(j)} E_{ji}. \]

(14)

Define the powers of the matrix $A$ recursively by

\[(A^q)^j = \sum_{k=-m+1}^{\infty} A^k(A^{q-1})^j, \quad [(A^0)^j \equiv \delta_{ij}]. \]

(15)
Using induction and the $gl(m/\infty)$ commutation relations (2) one obtains:

**Proposition 1:**

\[
[E_{kl}, (A^q)^{ij}] = (-1)^{(i)+1} \left( \delta_{lj}(A^q)^{ik} - \delta_{lk}(A^q)^{ij} \right).
\]

(16)

Therefore the matrix supertraces

\[
str(A^q) \equiv \sum_{i=-m+1}^{\infty} (-1)^{(i)} (A^q)^i
\]

(17)

are formally Casimir operators. They are, however, divergent except for $q = 1$ in which case we obtain the first order invariant (10). Our purpose is to construct a full set of Casimir invariants which are well defined and convergent on the category $O_{FS}$.

**Theorem 1:** The Casimir operators defined recursively by

\[
I_1 = \sum_{i=-m+1}^{\infty} (-1)^{(i)} A_i^i = str(A);
\]

\[
I_q = \sum_{i=-m+1}^{\infty} (-1)^{(i)} [(A^q)^i I_{q-1}] = str[A^q - I_{q-1}]
\]

(18)

form a full set of convergent $gl(m/\infty)$ Casimir operators on each module $V(\Lambda) \in O_{FS}$.

Observe that the operators $I_q$ are indeed Casimir invariants (see *Proposition 1*). Then it remains to prove they are convergent on the category $O_{FS}$. We will do this by induction. Consider first the case $q = 2$:

\[
I_2 \equiv \sum_{j=-m+1}^{\infty} (-1)^{(j)} (A^2)^{ij} - I_1 = \sum_{j=-m+1}^{0} \left[ \sum_{i=-m+1}^{\infty} E_{ij}E_{ji} - I_1 \right]
\]

\[
- \sum_{j=1}^{\infty} \left[ \sum_{i=-m+1}^{\infty} E_{ij}E_{ji} - I_1 \right] = \sum_{j=-m+1}^{0} \sum_{i=-m+1}^{\infty} E_{ij}E_{ji} + \sum_{j=-m+1}^{0} \sum_{i=1}^{\infty} E_{ij}E_{ji} - mI_1
\]

\[
- \sum_{j=1}^{\infty} \sum_{i=-m+1}^{0} E_{ij}E_{ji} - \sum_{j=1}^{\infty} \left[ \sum_{i=1}^{\infty} E_{ij}E_{ji} - I_1 \right]
\]

\[
= 2 \sum_{j=-m+1}^{0} \sum_{i=-m+1}^{0} E_{ij}E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii} + 1 - m - 2i) - mI_1 + 2 \sum_{j=-m+1}^{0} \sum_{i=1}^{\infty} E_{ij}E_{ji}
\]

\[
- \sum_{i=-m+1}^{\infty} \sum_{j=1}^{0} (E_{ii} + E_{jj}) - \sum_{j=1}^{\infty} \left[ 2 \sum_{i=j+1}^{\infty} E_{ij}E_{ji} + \sum_{i<j}^{\infty} (E_{ii} - E_{jj}) + E_{jj}^2 - I_1 \right]
\]

\[
= 2 \sum_{i=-m+1}^{\infty} \sum_{j=-m+1}^{0} (-1)^{(j)} E_{ij}E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii} + 1 - m - 2i) - mI_1 - m \sum_{i=1}^{\infty} E_{ii}
\]

\[
- \sum_{j=1}^{\infty} \left[ \sum_{i<j}^{\infty} (E_{ii} - E_{jj}) + E_{jj}^2 - \sum_{i=1}^{\infty} E_{ii} \right]
\]
\[
\begin{align*}
= 2 & \sum_{i=-m+1}^{\infty} \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} E_{ii} (E_{ii} + 1 - 2i) - 2mI_1 - \sum_{i=1}^{\infty} E_{ii} (E_{ii} + 1 - 2i), \\
\end{align*}
\]

which agrees with the definition (12).

Now let \( v \in V(\Lambda), \Lambda \in D_+^r \), be an arbitrary weight vector. Then the weight of \( v \) has the form

\[
\nu = (\nu_{-m+1}, \nu_{-m+2}, \ldots, \nu_0, \ldots, \nu_r, 0).
\]

Since

\[
A_i^j v = (-1)^{(i,j)} E_{ji} v = 0, \quad \forall i > r,
\]

the second order invariant \( I_2 \) is convergent on each \( V(\Lambda) \in O_{FS} \) [c.f. formula (13)]

Applying Proposition 1 and (21), for \( i > r \) one obtains

\[
(A^q)^i v = \sum_{i=-m+1}^{\infty} A_i^j (A^{q-1})_j^i v = \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} E_{ji} (A^{q-1})^i_j v
\]

\[
= \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} \left\{ (-1)^{(i,j)} A^{q-1}_j - (A^{q-1})_j^i \right\} v + (-1)^{(i,j)} (A^{q-1})_j^i E_{ji} v
\]

\[
= \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} \left[ (A^{q-1})_j^i - (A^{q-1})_j^i \right] v. \quad (22)
\]

For the case \( q = 2 \) we have

\[
(A^2)^i v = \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} \left[ A_j^i - A_i^j \right] v = \sum_{j=-m+1}^{\infty} E_{ji} v = I_1 v, \quad \forall i > r \quad (23)
\]

so that

\[
((A^2)^i - I_1) v = 0, \quad \forall i > r, \quad (24)
\]

which is another proof for the convergence of \( I_2 \). More generally

**Proposition 2:** For any weight vector \( v \in V(\Lambda) \), and \( q \in \mathbb{N} \) there exist \( r \in \mathbb{N} \) such that

\[
((A^q)^i - I_{q-1}) v = 0, \quad \forall i > r. \quad (25)
\]

**Proof:** We proceed by induction. Assume \( v \) has weight \( \nu \) as in (20). Formula (25) is valid for \( q = 2 \) (24). Let the result be true for a given \( q \), i.e.

\[
(A^q)^i v = I_{q-1} v, \quad \forall i > r.
\]

Then (see (22))

\[
(A^{q+1})^i v = \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} \left[ (A^q)_j^i - (A^q)^i_j \right] v = \sum_{j=-m+1}^{\infty} (-1)^{(i,j)} \left[ (A^q)_j^i - I_{q-1} \right] v = I_q v, \quad \forall i > r, \quad (26)
\]

which proves (25). \[\square\]
$I_q$ (18) is convergent on each $V(\Lambda)$ for $q = 2$. Assume it is well defined and convergent on $V(\Lambda)$ for a given $q$. Then, with $v$ as in (25), we have

$$I_{q+1}v = \sum_{i=-m+1}^{\infty} (-1)^i \left[(A^{q+1})_i^j - I_q \right] v = \sum_{i=-m+1}^{r} (-1)^i \left[(A^{q+1})_i^j - I_q \right] v$$

$$= \sum_{i=-m+1}^{r} (-1)^i (A^{q+1})_i^j v + (r-m)I_q v. \quad (27)$$

Therefore $I_{q+1}$ is convergent and well defined on $V(\Lambda)$.

This completes the (inductive) proof of Theorem 1.

III. EIGENVALUE FORMULA FOR CASIMIR OPERATORS

In this section we apply our previous results to evaluate the spectrum of the operators (18).

Let $v \in V(\Lambda)$, be an arbitrary vector of weight $\nu = (-m+1, \nu_{m+2}, \ldots, \nu_0, \nu_1, \ldots, \nu_r, \hat{0})$. Then, keeping in mind Proposition 1, the fact that $(A^{q-1})_k^j$ has weight $\varepsilon_k - \varepsilon_j$ under the adjoint representation of $gl(m/\infty)$ and that all vectors of $V(\Lambda)$ have weight components $\nu_i$ in $\mathbb{Z}_+$, we must have for $j \leq r$

$$(A^{q-1})_k^j v = 0, \quad \forall k > r. \quad (28)$$

Therefore

$$(A^q)_i^j v = \sum_{k=-m+1}^{\infty} A_k^j (A^{q-1})_k^j v = \sum_{k=-m+1}^{r} A_k^j (A^{q-1})_k^j v. \quad (29)$$

Proceeding recursively we may therefore write

$$(A^q)_i^j v = (\bar{A}^q)_i^j v, \quad \forall i, j = -m + 1, -m + 2, \ldots, r, \quad (30)$$

where $(\bar{A})_i^j = (-1)^{(i)(j)} E_{ji}$, $\forall i, j = -m + 1, \ldots, r$, is the $gl(m/r)$ characteristic matrix, and the powers of the matrix $\bar{A}$ are defined by (15) with $i, j, k = -m + 1, \ldots, r$ and $\bar{A}$ instead of $A$. It follows then that the formula (27) can be written as:

$$I_q v = \sum_{i=-m+1}^{r} (-1)^i \left[(\bar{A}^q)_i^j - I_{q-1} \right] v = \left[I_q^{(m,r)} - (m-r)I_{q-1} \right] v, \quad (31)$$

with

$$I_q^{(m,r)} = \sum_{i=-m+1}^{r} (-1)^i (\bar{A}^q)_i^j, \quad (32)$$

being the $q^{th}$ order invariant of $gl(m/r)$. Formula (31) is valid $\forall q \in \mathbb{N}$, which gives a recursion relation for the $I_q$ with initial condition

$$I_1 v = \chi\Lambda(I_1) v. \quad (33)$$

In particular it follows from (31) that the invariants $I_q$ are certainly convergent on all weight vectors $v \in V(\Lambda)$.

To determine the eigenvalues of $I_q$ let $v = v^+_\Lambda$ be the highest weight vector of the $V(\Lambda)$ module and let

$$\Lambda = (\bar{\Lambda}, \hat{0}) \in D^+_k, \quad \bar{\Lambda} = (\Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k). \quad (34)$$
Then for the eigenvalues of the $I_q$ one obtains the recursion relation (see (31)):

$$\chi(\Lambda(I_q)) = \chi(\Lambda(I_{q-1})) - (m-k)\chi(\Lambda(I_{q-1})), \quad \chi(I_1) = \sum_{i=-m+1}^{k} \Lambda_i,$$

where $\chi(\Lambda(I_{q-1}))$ is the eigenvalue of the $q^{th}$ order invariant (32) of $gl(m/k)$ on the irreducible $gl(m/k)$ module with highest weight $\Lambda$; the latter is given explicitly by

$$\chi(\Lambda(I_q)) = \sum_{i=-m+1}^{k} (-1)^i \alpha_i^q \prod_{j \neq i=-m+1}^{k} \left( \frac{\alpha_i - \alpha_j + (-1)^j}{\alpha_i - \alpha_j} \right),$$

where

$$\alpha_i = (-1)^i (\Lambda_i - i + 1) - m.$$ 

Therefore we obtain for the eigenvalues of the Casimir operators $I_q$

$$\chi(\Lambda(I_q)) = \sum_{i=-m+1}^{k} (-1)^i P_q(\alpha_i) \prod_{j \neq i=-m+1}^{k} \left( \frac{\alpha_i - \alpha_j + (-1)^j}{\alpha_i - \alpha_j} \right),$$

for suitable polynomials $P_q(x)$ which, from Eq. (35), satisfy the recursion relation

$$P_q(x) = x^q - (m-k)P_{q-1}(x), \quad P_1(x) = x.$$

In particular

$$P_2(x) = x^2 - (m-k)x = \frac{x^2 - (m-k)^2}{x + (m-k)}, \quad (39a)$$

$$P_3(x) = x^3 - (m-k)(x^2 - (m-k)x) = \frac{x^3 + (m-k)^3}{x + (m-k)}, \quad (39b)$$

and more generally, it is easily established by induction that

$$P_q(x) = x \frac{x^q - (-1)^q(m-k)^q}{x + (m-k)}, \quad (40)$$

Thus we have

**Theorem 2:** The eigenvalues of the Casimir operators $I_q$ (18), on the irreducible $gl(m/\infty)$ module $V(\Lambda), \Lambda \in D_k^+$ are given by

$$\chi(\Lambda(I_q)) = \sum_{i=-m+1}^{k} (-1)^i \alpha_i \left( \frac{\alpha_i^q - (-1)^q(m-k)^q}{\alpha_i + (m-k)} \right) \prod_{j \neq i=-m+1}^{k} \left( \frac{\alpha_i - \alpha_j + (-1)^j}{\alpha_i - \alpha_j} \right),$$

where $\alpha_i = (-1)^i (\Lambda_i - i + 1) - m.$




IV. POLYNOMIAL IDENTITIES

Let $\Delta$ be the comultiplication on the enveloping algebra $U[gl(m/\infty)]$ of $gl(m/\infty)$ ($\Delta(E_{ij}) = E_{ij} \otimes 1 + 1 \otimes E_{ij}$, $i, j = -m + 1, -m + 2, \ldots$ with 1 being the unit in $U[gl(m/\infty)]$). Applying $\Delta$ to the second order Casimir operator (12) of $gl(m/\infty)$ we obtain:

$$
\Delta(I_2) = I_2 \otimes 1 + 1 \otimes I_2 + 2 \sum_{i,j=-m+1}^{\infty} (-1)^{i+j} E_{ij} \otimes E_{ji}.
$$

Therefore

$$
\sum_{i,j=-m+1}^{\infty} (-1)^{i+j} E_{ij} \otimes E_{ji} = \frac{1}{2} [\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2].
$$

Denote by $\pi_{e_{-m+1}}$ the irrep of $gl(m/\infty)$ afforded by $V(e_{-m+1})$. The weight spectrum for the vector module $V(e_{-m+1})$ consists of all weights $\epsilon_i$, $i = -m + 1, -m + 2, \ldots$, each occurring exactly once. Denote by $e_{ij}$, $i, j = -m + 1, -m + 2, \ldots$ the generators on this space

$$
\pi_{e_{-m+1}}(E_{ij}) = e_{ij},
$$

with $e_{ij}$ an elementary matrix.

Introduce the characteristic matrix

$$
A = \frac{1}{2} (\pi_{e_{-m+1}} \otimes 1) [\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2].
$$

Therefore

$$
A_k^l = \sum_{i,j=-m+1}^{\infty} (-1)^{(i+j)(i+j)} \pi_{e_{-m+1}}(E_{ij}) l_{ik} (1 - 1) E_{ji} = (-1)^{k+l} E_{lk}.
$$

The matrix $A$ is the infinite matrix introduced in Sec. II (see (14)) and the entries of the matrix powers $A^q$ are given recursively by (15). We will see that the characteristic matrix satisfies a polynomial identity acting on the $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_1^+$. Let $\pi_{\Lambda}$ be the representation afforded by $V(\Lambda)$. From Eq. (45), acting on $V(\Lambda)$ we may interpret $A$ as an invariant operator on the tensor product module $V(e_{-m+1}) \otimes V(\Lambda)$:

$$
A \equiv \frac{1}{2} (\pi_{e_{-m+1}} \otimes \pi_{\Lambda}) [\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2].
$$

Following Ref. 11 it is easy to see that the tensor product space admits a filtration of submodules

$$
V(e_{-m+1}) \otimes V(\Lambda) = V_{k+1} \supseteq V_k \supseteq \ldots \supseteq V_0 \supseteq \ldots \supseteq V_{-m+1} \supseteq \{0\},
$$

where each factor module $M_i = V_i/V_{i+1}$, if non-zero, is indecomposable and cyclically generated by a highest weight vector of weight $\Lambda + \epsilon_i$. We emphasize that $M_i$ is only non-zero when $\Lambda + \epsilon_i$ is integral dominant.

Then it follows that the generalized eigenvalues of $A$ on the tensor product space are given by

$$
\frac{1}{2} [(\Lambda + \epsilon_i, \Lambda + \epsilon_i + 2\rho) - (e_{-m+1}, e_{-m+1} + 2\rho) - (\Lambda, \Lambda + 2\rho)] = (-1)^{(i)} (A_i + 1 - i) - m,
$$

(see Theorem 2). Thus we have
**Theorem 3:** On each $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D^+_k$ the characteristic matrix satisfies the polynomial identity
\[
\prod_{i=-m+1}^{k+1} (A - \alpha_i) = 0,
\]
with $\alpha_i = (-1)^{i} (\Lambda_i + 1 - i) - m$ the characteristic roots.

Note that the characteristic identities (50) are the $gl(m/\infty)$ counterpart of the polynomial identities encountered for $gl(m/n)$ by Jarvis and Green \(^\text{12}\) (more precisely their adjoint identities).

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