Abstract

We use the expansion of superalgebras procedure (summarized in the text) to derive Chern-Simons (CS) actions for the \((p,q)\)-Poincaré supergravities in three-dimensional spacetimes. After deriving the action for the \((p,0)\)-Poincaré supergravity as a CS theory for the expansion \(osp(p|2;\mathbb{R})(2,1)\) of \(osp(p|2;\mathbb{R})\), we find the general \((p,q)\)-Poincaré superalgebras and their associated \(D=3\) supergravity actions as CS gauge theories from an expansion of the simple \(osp(p+q|2;\mathbb{R})\) superalgebras, namely \(osp(p+q|2;\mathbb{R})(2,1,2)\).
1 Introduction and results

Some important limits in physics can be described in terms of Lie algebra contractions [1-4]. For instance, the simple de Sitter so(4, 1) and anti-de Sitter so(3, 2) algebras lead by contraction to the D=4 Poincaré algebra. The contraction parameter may be related to the AdS$_4$ constant curvature when SO(3, 2) is interpreted as the isometry group of four-dimensional spacetime (it is $1/R$ where $R$ is the radius of the universe), or to the square root of the cosmological constant $\Lambda$ when the algebra is taken as the starting point for the construction of gravity via gauging (for so(4, 1) the cosmological constant changes sign). The most familiar example -which in fact motivated the idea [1, 2]- is the Galilei algebra as a $c\to\infty$ İnönü-Wigner (I-W) contraction of the Poincaré one. Of course, the procedure also applies to superalgebras: for instance, the (N=1) $D=4$ superPoincaré algebra is an I-W contraction of the fourteen-dimensional $osp(1|4)$ superalgebra, the even part of which is the adS$_4$ algebra so(3, 2) $\sim sp(4, \mathbb{R})$.

Contractions can take Lie algebras that are direct sums of two Lie algebras into others that no longer have a direct sum structure or that in general alter the structural relation among the original algebras. The oldest example is the centrally extended eleven-dimensional Galilei algebra, which is a $c\to\infty$ I-W contraction of the direct sum of the Poincaré algebra and $u(1)$. This quantum-mechanical Galilei algebra arises because the $c\to\infty$ limit involves both terms of the direct sum [3,5]; the supersymmetric case was discussed in [6]. Another example, in $D=2$, is the centrally extended Poincaré algebra which is used to construct the $D=1+1$ CGHS model [7] of gravity as a gauge theory. As noted in [8], the relevant centrally (magnetic-like) extended $D=1+1$ Poincaré algebra, of dimension four, may be obtained by a contraction. Such a contraction was called ‘unconventional’ in [8] but it is, in fact, an ordinary I-W contraction involving the direct sum of the three-dimensional adS$_2$ = so(1, 2) and a one-dimensional algebra. As with the contraction of the trivially extended $D=4$ Poincaré algebra to the centrally extended Galilei one, this I-W contraction involves both algebras in the direct sum, and the dimensions $(3+1)$ of the original and contacted algebras are the same (the term ‘extension’ is used throughout in its mathematical sense).

The supersymmetric version of the CGHS model was shown in [10] to be the gauge theory of an algebra obtained by a similar trick, which in this case corresponds to performing an I-W contraction of a semidirect extension of the five-dimensional super de Sitter algebra $osp(1|2; \mathbb{R})$ (which is the supersymmetric version of so(1, 2)) by a four-dimensional abelian algebra which contains two bosonic charges and a pair of fermionic generators. The resulting $(5+4)$-dimensional contracted algebra is the superalgebra suitable for two-dimensional dilaton supergravity [10], rather than the five-dimensional $N=1$ $D=1+1$ superPoincaré algebra which is obtained by an I-W contraction of $osp(1|2; \mathbb{R})$ by rescaling the two bosonic ‘translations’ and the two fermionic generators, respectively, by $\mu$ and $\mu^{1/2}$. In fact, the superalgebra in [10] is an extension of the $N=1$, $D=2$ superPoincaré one by the four-dimensional ideal of the additional generators, in which one of the bosonic generators is central. For other examples of this type of I-W contractions (in planar physics) see [11].

The anti-de Sitter algebra for $D$-dimensional spacetime is so($D-1$, 2). For $D=3$, it is adS$_3$ =

---

1 The name ‘unconventional contraction’ is often used to denote standard I-W contractions where the generators of the original algebra are rescaled at the same time that new, additional generators are introduced. As a result, the contraction being performed is in fact an ordinary I-W contraction of a larger algebra, not of the original one. The re-scaling of the generators is singular in the contraction limit, but allows for the contraction in the algebra commutators (see [5,6] for the cohomological meaning of the procedure and [3] for the earliest work).
so(2, 2) = so(1, 2) ⊕ so(1, 2), which is not simple; its supersymmetric generalization is osp(1|2; \mathbb{R}) ⊕ osp(1|2; \mathbb{R}). There are actually two non-isomorphic osp(1|2; \mathbb{R}) algebras, osp(1|2; \mathbb{R})\pm (see Sec. 2 for the ± signs). In [12] a set of D=3 AdS-type supergravities was given as Chern-Simons (CS) models based on the superalgebras osp(p|2; \mathbb{R})_+ ⊕ osp(q|2; \mathbb{R})_- = AdS(p, q) in the terminology of [13]. A problem appeared when taking the Poincaré limit of these AdS(p, q) CS supergravities for p ≥ 2 or q ≥ 2 (the action does not have a limit). The only consistent Poincaré limit yields [13] the Marcus-Schwarz D = 3 N-extended Poincaré theories [14], for which the superalgebra does not include the gauge so(p) and so(q) generators. The difficulty was overcome for a class of (p, q)-Poincaré supergravities, related to the AdS(p, q) anti-de Sitter ones (the (1, 0) case being the old three-dimensional example in [15]). Since the problem to obtain a Poincaré limit of the AdS(p, q) CS theories was due to the so(p) and so(q) gauge fields terms in the CS action in [12], the authors of [16] started from a larger algebra, the trivial extension osp(p|2; \mathbb{R})_+ ⊕ osp(q|2; \mathbb{R})_- ⊕ so(p) ⊕ so(q) of AdS(p, q) by so(p) ⊕ so(q). This superalgebra was used to take a particular I-W contraction that gave a well defined Poincaré limit, here denoted sP(p, q), to obtain the associated (p, q)-Poincaré N-supergravities, N = p + q. Clearly\footnote{\text{dim \textit{osp}(m|q; \mathbb{R}) = \frac{1}{2}m(m - 1) + \frac{1}{2}q(q + 1) + mq, q = 2n, where \textit{mq} is the dimension of the odd part; the even subalgebra is \textit{so}(m) \oplus \textit{sp}(q). For q=2, \text{dim \textit{osp}(m|2; \mathbb{R}) = \frac{1}{2}m(m - 1) + 3 + 2m.}}\text{2\textit{dim \textit{osp}(m|q; \mathbb{R}) = \frac{1}{2}m(m - 1) + \frac{1}{2}q(q + 1) + mq, q = 2n, where \textit{mq} is the dimension of the odd part; the even subalgebra is \textit{so}(m) \oplus \textit{sp}(q). For q=2, \text{dim \textit{osp}(m|2; \mathbb{R}) = \frac{1}{2}m(m - 1) + 3 + 2m.}}\text{2}}\text{.}

\begin{equation}
\text{dim \textit{sP}(p, q) = p(p - 1) + q(q - 1) + 6 + 2(p + q),}
\end{equation}
in which 2\text{(p + q) = 2N is the dimension of the odd sector. These D=3 (p, q)-Poincaré supergravity models [16] constitute the subject of this paper[2].}

We provide here an alternative derivation of both the sP(p, q) superalgebras and of the associated D=3 (p, q)-Poincaré CS supergravities by means of a construction directly based on the osp(N|2; \mathbb{R}) superalgebra. Rather than extending trivially the non-simple adS(p, q) superalgebra to then perform a I-W contraction, we will use a comparatively novel procedure, the \textit{expansion of algebras}, which leads to new algebras and superalgebras by ‘expanding’ the original ones. This technique was used first in [19] and studied in general in [20, 21], where it was called the \textit{expansion method} (see also [22] for further developments). By series expanding the Maurer Cartan forms in the MC equations of the original algebra and then looking at the equations that result by equating equal powers in the expansion parameter, it is possible to retain a number of the coefficient one-forms in the different series in such a way that the resulting equations become the MC ones of a new Lie (super)algebra, the expanded one. Clearly, the dimension of the expanded algebra is \textit{larger} in general than that of the original algebra. It may be seen [20, 21], nevertheless, that the expansion procedure also includes the general Weimar-Woods contractions [1] (which in turn include the I-W ones) as a particular case, for which the expansion is of course dimension-preserving. In [19, 20] various sets of conditions were provided to cut the series expansions of the MC forms in such a way that the retained one-form coefficients satisfy the MC equations of a new, expanded Lie (super)algebra, the structure constants of which follow from those of the original algebra.

The expansion procedure is convenient to generate new, larger (super)algebras from a given one[4]. Besides, it presents computational advantages, particularly for model building. For instance, starting from a CS model based on a simple Lie algebra and then expanding the gauge

\footnote{\text{It may be worth mentioning that, in another context, the analysis of (2+1)-dimensional gravity with higher-spin fields (or HS AdS gravity) has also led to the appearance of larger algebras, namely W-algebras [17, 13].}}\text{.}
field one-forms exactly in the same way as the MC forms, every power in the expanded CS model provides a CS action for a certain expanded Lie algebra. This fact was used to obtain the $D = 3 N=1$ Poincaré supergravity action as a CS gauge theory by expanding an $osp(1|2; \mathbb{R})$-based CS model in a parameter $\lambda$ ($|\lambda| = L^{-\frac{1}{2}}$) and by looking at the term in $\lambda^2$. This $D=1+2$ Poincaré superalgebra or $s\mathcal{P}(1,0)$ (which is the expansion $osp(1|2; \mathbb{R})(2,1)$) can also be obtained as an I-W contraction of the eight-dimensional direct sum $osp(1|2; \mathbb{R}) \oplus sp(2)$, but the expansion method leads directly to the $s\mathcal{P}(1,0)$ superalgebra and to the CS action of the $(1,0)$-Poincaré supergravity.

It is clear that one cannot expect that every contraction of an extended group be given by an expansion, since the expansion procedure has much less freedom than the combination of extensions (which involve pairs of algebras) and contractions. So it makes sense to investigate whether the $s\mathcal{P}(p, q)$ superalgebra and the $(p, q)$-Poincaré supergravities can be obtained from expansions of a certain CS model based on a simple superalgebra when $p + q > 1$, as we already know to be the case for $(p, q) = (1, 0)$ (or $(0, 1)$). We shall show that the $D = 3 (p, q)$-superPoincaré algebra $s\mathcal{P}(p,q)$, obtained in [16] as an I-W contraction of $adS(p, q) \oplus (so(p) \oplus so(q))$ which mixes the $so(p)$ and $so(q)$ generators present in both parts of the direct sum (and that cannot be obtained by contracting a simple superalgebra), can be derived from the expansion $osp(p + q|2; \mathbb{R})(2,1,2)$ of $osp(p + q|2; \mathbb{R})$. The process will involve taking the quotient of the expansion by an ideal. The gauge fields associated with the generators of this ideal will not appear in the expanded CS lagrangian, which coincides with that of the $D = 3 (p, q)$-Poincaré CS supergravity models given in [16]. This is the result of this paper: the gauge superalgebra $s\mathcal{P}(q, p)$ underlying the $D=2+1 (p, q)$-Poincaré supergravities as well as their corresponding CS actions can be obtained from the expansion $osp(p|2; \mathbb{R})(2,1)$ when $q$ (say) is zero and, when $p, q \neq 0$, from $osp(p + q|2; \mathbb{R})(2,1,2)$, in this case by virtue of the physical irrelevance in the action of a bosonic ideal of this superalgebra.

The plan of the paper is as follows. In Sec. 2 we recall the properties of $osp(p|2; \mathbb{R})$ through its MC equations to fix the notation and write the corresponding gauge free differential algebra (FDA); similarly, Sec. 3 reviews the ingredients of the expansion procedure relevant here. Sec. 4.1 considers the case $(p, 0)$-Poincaré supergravity, where it is shown that the expanded algebra $osp(p|2; \mathbb{R})(2, 1)$ is the $(p, 0)$-superPoincaré one, $s\mathcal{P}(p, 0)$. In Sec. 4.2 the general $(p, q)$ case is examined, and in particular the $(p, q)$-Poincaré supergravity CS lagrangian is obtained from the expansion $osp(p + q|2; \mathbb{R})(2,1,2)$. Sec. 5 contains some final remarks.

2 The superalgebra $osp(p|2; \mathbb{R})$

$OSp(p|2; \mathbb{R})$ is the supergroup of transformations preserving the $(p+2) \times (p+2)$ orthosymplectic metric $C$, $g^t C g = C$, where $g$ is a $(p, 2)$-type supermatrix, $t$ indicates supertranspose, and $C$ is given by

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & -iI_{p \times p} \end{pmatrix} \quad (2.1)$$

in which $\epsilon_{\alpha\beta}$ is the $2 \times 2$ symplectic metric. Therefore, the superalgebra elements $X \in osp(p|2; \mathbb{R})$ satisfy $X^t C + CX = 0$. Let then $a$ be a $osp(p|2; \mathbb{R})$-valued MC one-form. The previous condition means that $a$ will have the block form structure

$$a = \begin{pmatrix} \ell^{\alpha \beta} & iu^{i\alpha} \\ \nu_{\beta} & h^{ij} \end{pmatrix} \quad , \quad \alpha, \beta = 1, 2 ; \quad i, j = 1, \ldots, p \quad , \quad (2.2)$$
or one where the $i$ at the top right box is removed and a $-i$ is added to the bottom left box. In (2.2), $\nu^i_\alpha = \epsilon_{\alpha\beta} \nu^\beta_i$, $(\nu^\alpha i)^* = \nu^{\alpha i}$, $h^{ij} = -h^{ji}$ (the euclidean indices $i,j$ will be always up), and the symplectic subalgebra matrix $l^{\alpha\beta}$ has the property $l_{\alpha\beta} = l_{\beta\alpha}$.

The MC equations for the supermatrix form (2.2) read

$$da = -a \wedge a \quad i.e.,$$

$$
\left( \begin{array}{ccc}
 d\nu^\alpha_\beta & i\nu^{\alpha i} & i\nu^{i\gamma} \\
 d\nu^i_\beta & dh^{ij} & -i\nu^{j\gamma} \\
 dh^{ij} & -i\nu^{i\gamma} & h^{kj}
\end{array} \right) = -
\left( \begin{array}{ccc}
 l^\alpha_\gamma & i\nu^{k\alpha} & i\nu^{j\gamma} \\
 l^i_\gamma & h^{ik} & -i\nu^{j\gamma} \\
 -i\nu^{i\gamma} & h^{kj} & -i\nu^{k\alpha} & \nu^{i\beta}
\end{array} \right)
\right).$$

(2.4)

Explicitly, the $osp(p|2; \mathbb{R})$ MC equations are

$$dh^{ij} = -h^{ik} \wedge h^{kj} - i\nu^{i\gamma} \wedge \nu^{j\gamma},$$

$$d\nu^{i\alpha} = -h^{jk} \wedge \nu^{k\alpha} - l^{i\gamma} \wedge \nu^{j\gamma},$$

$$dl^{\alpha\beta} = -l^{\alpha_\gamma} \wedge l^{\beta_\gamma} - i\nu^{k\alpha} \wedge \nu^{i\beta}.$$  

(2.5)

For the other possible choice of the matrices $a$, the MC equations have a plus sign in the two fermion bilinears above. These two $osp$ algebras, inequivalent as real algebras and denoted $osp(p|2; \mathbb{R})_+$ and $osp(p|2; \mathbb{R})_-$ respectively, correspond to taking a different sign for the $\{Q_\alpha, Q_\beta\}$ anticommutator [13]; clearly, their complexified versions coincide. Unless otherwise stated, $osp(p|2; \mathbb{R})$ will mean $osp(p|2; \mathbb{R})_+$ as given by eq. (2.5).

The curvatures $F$ of the gauge fields $A$ associated with the different $osp(p|2; \mathbb{R})$ generators, needed for constructing a CS model, are now directly obtained from the MC equation (2.3) by writing

$$F = dA + A \wedge A$$

(2.6)

in terms of the ‘soft’ forms $A$, which satisfy eqs. (2.5) when the curvatures vanish. Writing the matrix blocks of $A$ and $F$ respectively as

$$A = \left( \begin{array}{ccc}
 f^{\alpha\beta} & i\xi^{j\alpha} & i\xi^{i\gamma} \\
 \xi^{j}_\beta & A^{ij} & \xi^{i}_\beta
\end{array} \right), \quad F = \left( \begin{array}{ccc}
 \Omega^{\alpha\beta} & i\Psi^{j\alpha} & i\Psi^{i\gamma} \\
 \Psi^{j}_\beta & F^{ij} & \Psi^{i}_\beta
\end{array} \right);$$

(2.7)

eq. (2.6) gives

$$F^{ij} = dA^{ij} + A^{ik} \wedge A^{kj} + i\xi^{i}_\alpha \wedge \xi^{j\gamma},$$

$$\Psi^{j\alpha} = d\xi^{j\alpha} + f^{\alpha}_\gamma \wedge \xi^{j\gamma} + A^{jk} \wedge \xi^{k\alpha},$$

$$\Omega^{\alpha\beta} = df^{\alpha\beta} + f^{\alpha}_\gamma \wedge f^{\gamma}_\beta + i\xi^{k\alpha} \wedge \xi^{k}_\beta.$$  

(2.8)

Similarly, the infinitesimal gauge variations characterized by a matrix $\phi$

$$\phi = \left( \begin{array}{ccc}
 i\tilde{\nu}^{\alpha\beta} & i\tilde{\zeta}^{j\alpha} & i\tilde{\zeta}^{i\gamma} \\
 \tilde{\zeta}^{j}_\beta & \tilde{\alpha}^{ij} & \tilde{\alpha}^{i\beta}
\end{array} \right)$$

(2.9)

and are given by

$$\delta_\phi A = d\phi + A \phi - \phi A, \quad \delta_\phi F = F \phi - \phi F.$$  

(2.10)

In components, the above equations read

$$\delta_\phi A^{ij} = d\tilde{a}^{ij} + A^{ik} \tilde{a}^{kj} - \tilde{a}^{ik} A^{kj} + i\xi^{i}_\gamma \tilde{\zeta}^{j\gamma} - i\tilde{\zeta}^{i}_\gamma \xi^{j\gamma},$$

(2.11)
Then, it is consistent to expand the MC forms of $V_i$ those of $i.e.$ Expansions of a Lie algebra require that each power in $\lambda$ vanishes separately. Various sets of conditions that guarantee that it is possible to accompany the MC equations of the new expanded (super)algebra were given in [19, 20]. The result that is relevant in this paper is the following.

Finite Lie algebras are obtained by cutting the expansions of the MC one-forms in such a way that the coefficient one-forms $\omega$ $(\omega, \lambda)$ be a Lie superalgebra, $\dim \mathcal{G} = 2$ be a Lie superalgebra, $\delta^{\alpha} = 0$. Let the MC equations of $G$ and $V$ satisfy one of the three conditions below

$$\phi^2 = 0, \phi = 0, \phi = 2, b$$

$$\frac{1}{2} \epsilon^{ks}_{ps} \omega^{sp} \wedge \omega^{hq} \quad p, q, s = 0, 1, 2. \quad (3.1)$$

Then, it is consistent to expand the MC forms of $V_0 \oplus V_2$ in terms of even powers of $\lambda$ and those of $V_1$ in terms of odd powers of $\lambda$, up to the orders below, as (see [20], Sec. 5)

$$\omega^{i_0} = \sum_{a_0=0, \, a_0 \, \text{even}}^{N_0} \lambda^{a_0} \omega^{i_0, a_0}, \quad (\omega^{i_1}, \omega^{i_2}, \omega^{i_3}, N_3), \quad \text{providing that the even} \quad N_0, N_2 \, \text{and odd} \quad N_1 \, \text{integers satisfy one of the three conditions below} \quad (3.3) \quad (N_0 = N_1 + 1 = N_2) \; ; \quad N_0 = N_1 - 1 = N_2 \; ; \quad N_0 = N_1 - 1 = N_2 - 2. \quad (3.3)$$

3 Chern-Simons actions from the expansion procedure

Expansions of a Lie algebra $\mathcal{G}$ are obtained by series expanding some of its MC forms $\omega^i, \, i = 1, \ldots, \dim \mathcal{G}$, in terms of a parameter $\lambda$. These series follow by re-scaling by a power of $\lambda$ some group coordinates $g^i$ in the expression of the MC forms as given, say, by the left invariant one-forms on the associated group manifold (see [20]). When the expansions of the MC forms are introduced in the MC equations, an infinite-dimensional algebra is formally obtained by requiring that each power in $\lambda$ vanishes separately.

Finite Lie algebras are obtained by cutting the expansions of the MC one-forms in such a way that the coefficient one-forms $\omega^{i_0}$ accompanying $\lambda^{i_0}$ in $\omega^i(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{i_0, \alpha}$ satisfy equations (obtained by equating equal powers in $\lambda$) that become the MC equations of a new algebra, the expanded algebra. Various sets of conditions that guarantee that it is possible to retain consistently a finite number of $\omega^{i_0}$ satisfying the MC equations of the new expanded (super)algebra were given in [19, 20]. The result that is relevant in this paper is the following.

Let $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ be a Lie superalgebra, $V_1$ its Grassmann odd part, $V_0 \oplus V_2$ the even one, and $V_0$ a subalgebra of $\mathcal{G}$. Let $\omega^{i_0}, \omega^{i_1}$ and $\omega^{i_2}$ the MC forms dual to the algebra basis generators in the vector subspaces $V_0, V_1$ and $V_2 \subset \mathcal{G}$ respectively; obviously, $i_0 = 1, \ldots, \dim V_0, i_1 = 1, \ldots, \dim V_1, i_2 = 1, \ldots, \dim V_2$. Let the MC equations of $\mathcal{G}$ be written in the form

$$d \omega^{k_s} = -\frac{1}{2} \epsilon^{ks}_{ps} \omega^{i_p} \wedge \omega^{j_q} \quad p, q, s = 0, 1, 2. \quad (3.1)$$

Then, it is consistent to expand the MC forms of $V_0 \oplus V_2$ in terms of even powers of $\lambda$ and those of $V_1$ in terms of odd powers of $\lambda$, up to the orders below, as (see [20], Sec. 5)
This means that the $\omega^{i_1,\alpha_1}, \omega^{i_2,\alpha_2}, \omega^{i_3,\alpha_3}$ one-forms, where the $i_s = 1, \ldots, \dim V_s$ \((s = 1, 2, 3)\) and the $\alpha_s$ have the ranges in the sums above, define formally the MC forms of finite-dimensional superalgebras. These expanded superalgebras, subordinated to the splitting of the vector space of the original superalgebra $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$, will be denoted by $\mathcal{G}(N_0, N_1, N_2)$. Their dimension is given by

$$\dim \mathcal{G}(N_0, N_1, N_2) = \left(\frac{N_0 + 2}{2}\right) \dim V_0 + \left(\frac{N_1 + 1}{2}\right) \dim V_1 + \left(\frac{N_2}{2}\right) \dim V_2, \quad (3.4)$$

and are determined by the following structure constants

$$C^{k_s,\alpha_s}_{i_p,\beta_p, j_q, \gamma_q} = \begin{cases} 0 & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ \lambda_{i_p j_q}^{k_s} & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases}, \quad (3.5)$$

which are expressed in terms of those of the original algebra $\mathcal{G}$. As stated, the above result is obtained by substituting \((3.2)\) in \((3.1)\), identifying equal powers of $\lambda$, and taking $N_0$, $N_1$ and $N_2$ in such a way that the resulting MC equations describe a superalgebra.

From the above expanded Lie superalgebras it is possible to obtain, by moving to the corresponding generic ‘soft’ forms, $\omega^{i_s,\alpha_s} \to A^{i_s,\alpha_s}$, their gauge free differential algebra (FDA) and gauge variations (of infinitesimal parameters $\varphi^{i_s,\alpha_s}$)

$$F^{k_s,\alpha_s} = dA^{k_s,\alpha_s} + \frac{1}{2} C^{k_s,\alpha_s}_{i_p,\beta_p, j_q, \gamma_q} A^{i_p,\beta_p} \wedge A^{j_q,\gamma_q},$$

$$dF^{k_s,\alpha_s} = C^{k_s,\alpha_s}_{i_p,\beta_p, j_q, \gamma_q} F^{i_p,\beta_p} \wedge A^{j_q,\gamma_q},$$

$$\delta A^{k_s,\alpha_s} = d\varphi^{k_s,\alpha_s} - C^{k_s,\alpha_s}_{i_p,\beta_p, j_q, \gamma_q} A^{i_p,\beta_p} \wedge A^{j_q,\gamma_q}. \quad (3.6)$$

One important point, which will be used in what follows, is that eqs. \((3.6)\) may also be obtained by inserting in the equations of the gauge FDA associated with $\mathcal{G}$,

$$F^{k_s} = dA^{k_s} + \frac{1}{2} C^{k_s}_{i_p j_q} A^{i_p} \wedge A^{j_q},$$

$$dF^{k_s} = C^{k_s}_{i_p j_q} F^{i_p} \wedge A^{j_q},$$

$$\delta A^{k_s} = d\varphi^{k_s} - C^{k_s}_{i_p j_q} \varphi^{i_p} \wedge A^{j_q}. \quad (3.7)$$

the formal expansions of $A^{k_s}$, $F^{k_s}$ and $\varphi^{k_s}$ in terms of $A^{k_s,\alpha_s}$, $F^{k_s,\alpha_s}$ and $\varphi^{k_s,\alpha_s}$ with exactly the same structure as that of the MC forms $\omega^{k_s}$ in \((3.2)\), and then identifying equal powers of $\lambda$.

**Expansions and CS models**

Lie algebra expansions may be used to obtain, from a given gauge Chern-Simons (CS) model based on a Lie (super)algebra $\mathcal{G}$, new CS ones associated with the different expansions of $\mathcal{G}$ (see \([20,21]\) for details). Let $\mathcal{G}$ be a Lie superalgebra, for instance of the type described above, and let $k_{i_1, \ldots, i_l}$ be the coordinates of a symmetric invariant $l$-tensor of $\mathcal{G}$, where $i = 1, \ldots, \dim \mathcal{G}$. Then, it follows that the $2l$-form $H$ constructed out of the curvatures $F$,

$$H = k_{i_1, \ldots, i_l} F^{i_1} \wedge \cdots \wedge F^{i_l}, \quad (3.8)$$

is closed and gauge invariant and may be taken as the starting point in a Chern-Weil construction. Since gauge FDAs are contractible \([23,24]\), they have trivial de Rham cohomology so that
$H$ defines a $(2l - 1)$-form (the CS form) $B$, such that $dB = H$. Then, integrating $B$ over a $(2l - 1)$-dimensional manifold $M^{2l-1}$, a CS model is obtained through the action

$$I[A] = \int_{M^{2l-1}} B(A).$$  \hfill (3.9)

It is possible now to perform an expansion of the $A^i$ and $F^i$ fields of the form \(A^i = \sum_{\alpha} \lambda^\alpha p A^i_{\alpha p}\) and similarly for $F^i$. This leads to an expansion of the action \(I[A]\) given by

$$I[A, \lambda] = \int_{M^{2l-1}} B(A, \lambda) = \int_{M^{2l-1}} \sum_{N=0}^\infty \lambda^N B_N(A) = \sum_{N=0}^\infty \lambda^N I_N[A],$$  \hfill (3.10)

where in the sums $N$ is even. The same expansion, when applied to \(I[A]\), leads to

$$H(F, \lambda) = \sum_{N=0}^\infty \lambda^N H_N,$$  \hfill (3.11)

where, necessarily,

$$H_N = dB_N(A).$$  \hfill (3.12)

This means that, for each $N$, the actions

$$I_N = \int_{M^{2l-1}} B_N(A)$$  \hfill (3.13)

define CS models obtained by the expansion of the original, $G$-based one. Given $N$, the Lie algebra corresponding to the CS action $I_N$ is the one that contains all the gauge curvatures (field strengths) included in $H_N$. The most common situation is that all the curvatures $F^{k_0, \alpha_0}$ (for $\alpha_0 = 0, 2, \ldots, N$), $F^{k_2, \alpha_2}$ (for $\alpha_2 = 2, 4, \ldots, N$) and $F^{k_1, \alpha_1}$ (for $\alpha_1 = 1, 3, \ldots, N - 1$) are present in $H_N$, so that the Lie superalgebra corresponding to $I_N$ is $G(N, N - 1, N)$, which is the first case in \(3.3\). It may be, however, that not all the field strengths actually appear in the CS action \(3.13\). This will be the case for the expansion of $osp(p + q|2; \mathbb{R})$ of interest here, as will be shown at the end of Sec. 4.2.

### 4 The $(p, q)$-Poincaré supergravities from an expansion of \(osp(p + q|2; \mathbb{R})\)

Consider first a natural CS model associated with $osp(p|2; \mathbb{R})$. It is constructed from the closed, invariant four form

$$H = Tr(F \wedge F),$$  \hfill (4.1)

where $F$ is given in eq. \(2.7\), $Tr$ stands for the supertrace and we have moved back to the notation of Sec. 2. $H$ is exact, $H = dB$, and the potential CS form $B$ is gauge invariant up to the exterior derivative of a two-form. As a result, the CS action functional

$$I[A] = \int_{M^3} B = \int_{M^3} Tr \left( F \wedge A - \frac{1}{3} A \wedge A \wedge A \right)$$  \hfill (4.2)

is gauge invariant up to boundary terms that depend on the topology of $M^3$ and on the type of gauge transformations considered (‘large’ vs. ‘small’). Since the field equations are not affected
by these boundary terms, they are gauge invariant, and it is easy to check that they are given by the vanishing of the curvatures, $F = 0$.

Using the $osp(p|2; \mathbb{R})$ components of the $\mathbb{F}$ and $\mathbb{A}$ matrices given in (2.7), the above four-form $H$ and CS action $I[A]$ turn out to be

$$
H = \Omega^\gamma \wedge \Omega^\alpha - F^{ik} \wedge F^{ki} - 2i\Psi^i_\gamma \wedge \Psi^{i\gamma},
$$

$$
I[A] = \int_{M^3} \Omega^\alpha \wedge J_\alpha - \frac{1}{3} f^{\alpha \gamma \delta \delta} A^{\alpha \gamma \delta} - f^{\alpha \gamma} \wedge F^\gamma - F^{ik} \wedge A^{ki} + \frac{1}{3} A^{ik} \wedge A^{kl} \wedge A^{li} - i\xi^{k\alpha} \wedge A^{kl} \wedge \xi^i_l
$$

$$
+ 2i\Psi^{k\alpha} \xi^k_i, \quad i, l, k = 1, \ldots, p .
$$

(4.3)

Taking the above $OSp(q|2; \mathbb{R})_+$ CS action minus a copy of that corresponding to $OSp(q|2; \mathbb{R})_-$ with indices $i', k' = 1, \ldots, q$, and identifying appropriately the gauge fields, the CS actions of the $AdS(p, q)$ supergravities [12, 13] are obtained. As already mentioned, the limit of these $AdS(p, q)$ supergravities when the $AdS$ dimensionful parameter is taken to zero is not straightforward. A way out was found in [16] using a CS model based on a larger superalgebra, $osp(p|2; \mathbb{R})_+ \oplus osp(q|2; \mathbb{R})_- \oplus so(p) \oplus so(q)$. The I-W contraction limit was then taken after making suitable linear combinations of the $so(p)$ generators in $osp(p|2; \mathbb{R})_+ \oplus so(p)$ and $so(q)$ in $osp(q|2; \mathbb{R})_- \oplus so(q)$. In this way, the I-W contraction limit gives the $sp(p, q)$ superalgebra, the gauge $so(p) \oplus so(q)$ fields remain coupled to the gravity fields, and the action of the $(p, q)$-Poincaré supergravities is obtained. Clearly, $adS(p, q) \sim adS(q, p)$ but, as we shall see, $sp(p, q) \sim sp(q, p)$. It is not difficult to see why this is so: the difference in sign for the $\{Q_\alpha, Q_\beta\}$ anticommutators in $adS$ cannot be absorbed by redefining the generators in their r.h.s., something that can be done for superalgebras of Poincaré type.

We derive now the $D=3$ $(p, q)$-Poincaré supergravities by means of the expansion procedure. Since the cases $(p, 0)$ (or $(0, q)$) and $(p, q), p \neq 0, q \neq 0$, turn out to be slightly different, they will be treated separately.

4.1 $D=3$ $(p, 0)$-Poincaré supergravity as a CS theory of the expansion $osp(p|2; \mathbb{R})(2, 1)$

We begin with a $osp(p|2; \mathbb{R})$-based model\footnote{It is not difficult to check that the same $(p, 0)$-Poincaré supergravity would follow from $osp(p|2; \mathbb{R})_-$, since the sign differences in the $osp(p|2; \mathbb{R})_{\pm}$ algebras are reabsorbed and disappear at the Poincaré level.} and perform an expansion of the $\mathbb{F}$ and $\mathbb{A}$ fields in the action, eq. (1.2) or (1.3). It is possible to assign physical dimensions to the expansion parameter $\lambda$ so that the coefficient one-forms in the expansion are dimensionful too (the original $osp(p|2; \mathbb{R})$ gauge fields are dimensionless, as it has to be the case for a simple algebra with dimensionless structure constants). In order to obtain gravity actions, it is convenient to take $[\lambda] = L^{-\frac{4}{2}}$ in geometrized units. The next step is to identify the vector spaces $V_0$, $V_1$ and $V_2$ of Sec. 3. The space $V_1$ has to be associated with the odd generators, so its dual is generated by the fermionic one-form fields $\xi^{i\alpha}$. Here it is convenient to choose $V_2 = 0$ and $V_0$ as the whole $so(p) \oplus sp(2) \subset osp(p|2; \mathbb{R})$ bosonic subalgebra, to which the bosonic $A^{ij}$ and $f^{\alpha \beta}$ one-form fields are associated.
The expansion $osp(p|2;\mathbb{R})(2,1)$ is determined by that of the MC forms in (2.5) up to the orders $N_0=2$ (bosonic part) and $N_1=1$ (fermionic part). This means that the gauge forms and field strengths associated with $osp(p|2;\mathbb{R})(2,1)$ are those that correspond to $osp(p|2;\mathbb{R})$, eqs. (2.7), expanded up to these orders, namely:

$$f^a_{\beta} = \omega^a_{\beta} + \lambda^2 e^a_{\beta} + o(\lambda^4)$$
$$A_{ij} = A_{ij}^{0} + \lambda^2 A_{ij}^{2} + o(\lambda^4)$$
$$\xi^{ia} = \lambda \psi^{ja} + o(\lambda^3)$$
$$\Omega^a_{\beta} = R^a_{\beta} + \lambda^2 T^a_{\beta} + o(\lambda^4)$$
$$F^{ij} = F^{ij}_{0} + \lambda^2 F^{ij}_{2} + o(\lambda^4)$$
$$\Psi^{ia} = \lambda \mathcal{D} \psi^{ja} + o(\lambda^3) \quad i,j=1,\ldots,p \ , \quad (4.4)$$

where we have put $f^{a\beta,0} = \omega^{a\beta}, f^{a\beta,2} = e^{a\beta}, \xi^{i0} = \psi^{ia}$ and $\Omega^{a\beta,0} = R^{a\beta}$ and $\mathcal{D} \psi^{ia}$ is defined in the next formula. The relations among the $osp(p|2;\mathbb{R})(2,1)$ gauge fields and field strengths above may be found from the general expressions in eqs. (3.6) and (3.5), or directly by expanding eqs. (2.8). They are given by

$$R^a_{\beta} = d \omega^a_{\beta} + \omega^a_{\gamma} \wedge \omega^\gamma_{\beta} \ , \quad T^a_{\beta} = d e^a_{\beta} + \omega^a_{\gamma} \wedge e^\gamma_{\beta} + e^a_{\gamma} \wedge \omega^\gamma_{\beta} + i \psi^{ja} \wedge \psi^k_{\beta} \equiv D e^a_{\beta} + i \psi^{ja} \wedge \psi^k_{\beta} \ ,$$
$$F^{ij}_{0} = d A^{ij} + A^{ik} \wedge A_{kj} \ , \quad F^{ij}_{2} = d A^{ij2} + A^{ik} \wedge A_{kj2} + A^{ik2} \wedge A_{kj} + i \psi^{ja} \wedge \psi^{j\gamma} \ ,$$
$$\mathcal{D} \psi^{ia} = d \psi^{ja} + \omega^a_{\gamma} \wedge \psi^{j\gamma} + A^{jk} \wedge \psi^{ka} \equiv D \psi^{ja} + A^{jk} \wedge \psi^{ka} \ . \quad (4.5)$$

In view of the above, it will be natural to introduce $D=3$ imaginary gamma matrices and make the following identifications, $\omega^a_{\beta} \propto \omega_{ab}(\gamma^{ab})^a_{\beta}$, which is dimensionless $[\omega^a_{\beta}] = L^0$, is the Lorentz spin connection in three spacetime dimensions; $R^a_{\beta} \propto R_{ab}(\gamma^{ab})^a_{\beta}$ is its curvature; $e^a_{\beta} \propto e_a(\gamma^a)^a_{\beta}$, $[e^a_{\beta}] = L^1$, may be identified with the dreibein and its curvature, $T^a_{\beta} \propto T_a(\gamma^a)^a_{\beta}$, with torsion; $\psi$ and $\mathcal{D} \psi$, with length dimensions $[\psi] = [\mathcal{D} \psi] = L^2$, are the gravitino one-form field and its complete (Lorentz plus gauge) covariant derivative; $A^{ij0}$ are the dimensionless gauge one-form fields, $[A^{ij0}] = L^0$, and $F^{ij0}$ their curvatures. Finally, the $SO(p)$ one- and two-forms $A^{ij2}, F^{ij2}$ have length dimensions $L^1$.

The above dreibein, spin connection and gravitino identifications are justified by the form of the action (3.13) for $N=2$ that follows from the expansion of (4.3), which becomes the supergravity action. The choice $N=2$ is selected because then the action $I_2$ has length dimensions $[I_2] = L^1$, which are the dimensions of a $D=3$ action in geometrized units ($L^{D-2}$ for an action on general $D$-dimensional spacetimes). Indeed, $I_2$ is given by

$$I_2 = \int_{\mathcal{M}^3} \left( 2 R^a_{\beta} \wedge e^\gamma_{\alpha} - 2 F^{ik0} \wedge A^{kj2} - 2 i \mathcal{D} \psi^{ja} \wedge \psi^{j\gamma} \right) , \quad (4.6)$$

which can be seen to coincide with the $N=2$ term of the expansion of the second line in (4.3) except for the integral of an exact form on $\mathcal{M}^3$. The integrand in (4.6) is a potential form of

\footnote{In terms of Pauli matrices, $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^3$ (mostly minus metric); the antisymmetrized product $\gamma^{ab}$ is taken with weight one.}
the $\lambda^2$ term in the expansion of the first line of (4.3) as in (3.11), which is given by
\begin{equation}
H_2 = 2F^\alpha_\gamma \wedge T_\gamma^\alpha - 2F^{ik,0} \wedge F^{ki,2} - 2i\mathcal{D}_{i}^j \wedge \mathcal{D}_{j}^\gamma .
\end{equation}

The field equations obtained by varying $e^\gamma_\alpha$, $\omega^\gamma_\alpha$, $A^{ik,0}$, $A^{ik,2}$ and $\psi^j_\gamma$ are, respectively, $R^\alpha_\gamma = 0$, $T^\alpha_\gamma = 0$, $F^{ki,2} = 0$, $F^{ki,0} = 0$ and $\mathcal{D}_{i}^j \wedge \mathcal{D}_{j}^\gamma = 0$, and express the vanishing of the curvatures.

Since $H_2$ in (4.7) and $I_2$ in (4.6) contain all the gauge fields in the expansion up to order $\lambda^2$, they determine a CS model based on the Lie superalgebra $osp(p|2; \mathbb{R})$ when $q = 0$, as will be seen in Sec. 4.2 below. Therefore, the three-dimensional $(p, 0)$-Poincaré gravity is a CS theory constructed from a specific expansion of a simple superalgebra.

The next step is to split the underlying vector space of $osp(p+q|2; \mathbb{R})$ as $osp(p+q|2; \mathbb{R}) = V_0 \oplus V_1 \oplus V_2$, where $V_1$ is its odd part, and $V_0$ is a subalgebra. The latter will be the $so(p) \oplus so(q) \oplus sp(2)$ subalgebra, the basis of which corresponds to the gauge fields $\{ A^{ij}, A^{i'j'}, f^\alpha_\beta \}$. The remaining

4.2 The general case: $D=3$ $s\mathcal{P}(p,q)$ or $(p,q)$-Poincaré supergravity from an expansion of $osp(p+q|2; \mathbb{R})$

The first question to address here is what should be the starting superalgebra that, by expansion, leads to the general $(p,q)$-Poincaré supergravities. One might think of the direct sum $adS(p,q)$ superalgebra itself as a possible candidate. However, besides not being simple, for $q = 0$ it reduces to $osp(p|2; \mathbb{R}) \oplus sp(2)$, which is larger than $osp(p|2; \mathbb{R})$, the expansion $osp(p|2; \mathbb{R}) (2, 1)$ of which is behind the $(p, 0)$-Poincaré supergravity models as shown in Sec. 4.1.

Additionally, it may be checked by direct computation that an expansion of the $adS(p,q)$ algebra will not produce a superalgebra leading to a CS action for $\mathcal{P}(p,q)$-supergravities. These, however, may still be seen to be CS theories constructed from a specific expansion of a simple superalgebra.

To show that the superalgebra $s\mathcal{P}(p,q)$ of the $(p,q)$-Poincaré supergravities and their CS actions follow from an expansion of $osp(p+q|2; \mathbb{R})$, we write first the $osp(p+q|2; \mathbb{R})$ gauge FDA in a convenient form using eq. (2.8), where the $i = 1, \ldots, p + q$ index is split into two indices $(i, i')$, $i = 1, \ldots, p$ and $i' = 1, \ldots, q$. In this way, the $osp(p+q|2; \mathbb{R})$ curvatures read
\begin{align}
F^{ij} &= dA^{ij} + A^{ik} \wedge A^{kj} + A^{ij} \wedge A^{ik} + i\xi^i_\gamma \wedge \xi^j_\gamma \\
F^{i'j'} &= dA^{i'j'} + A^{i'k} \wedge A^{j'k} + A^{i'k} \wedge A^{j'k} + i\xi^{i'}_\gamma \wedge \xi^{j'}_\gamma \\
F^{ij'} &= -F^{ij'} = dA^{ij} + A^{ik} \wedge A^{kj'} + A^{ij} \wedge A^{ik'} + i\xi^i_\gamma \wedge \xi^{j'}_\gamma \\
\Psi^{ja} &= d\xi^a_\gamma \wedge \xi^j_\gamma + A^{jk} \wedge \xi^{k_\alpha} + A^{jk} \wedge \xi^{k_\alpha} \\
\Psi^{j'a} &= d\xi^{j'a} \wedge \xi^a_\gamma + A^{jk} \wedge \xi^{k_\alpha} + A^{jk} \wedge \xi^{k_\alpha} \\
\Omega^{\alpha_\beta} &= df^{\alpha_\beta} + f^{\alpha_\gamma} \wedge \xi^\gamma_\beta + i\xi^{j_\alpha} \wedge \xi^i_\beta + i\xi^{j'\alpha} \wedge \xi^{i'}_\beta.
\end{align}

note the mixed indices terms $A^{ij'}$, $F^{ij'}$. The equations for the exterior derivatives of the curvatures can be deduced from (4.8) and will not be needed here. By simply setting the above curvatures equal to zero, the MC equations of $osp(p+q|2; \mathbb{R})$ are recovered in the present language.

The next step is to split the underlying vector space of $osp(p+q|2; \mathbb{R})$ as $osp(p+q|2; \mathbb{R}) = V_0 \oplus V_1 \oplus V_2$, where $V_1$ is its odd part, and $V_0$ is a subalgebra. The latter will be the $so(p) \oplus so(q) \oplus sp(2)$ subalgebra, the basis of which corresponds to the gauge fields $\{ A^{ij}, A^{i'j'}, f^\alpha_\beta \}$. The remaining
bosonic subspace, $V_2$, corresponds to the $p \times q$ gauge fields \( \{ A_i^{\alpha} = -A_i^{\alpha'} \} \). The expansions of the gauge forms and the curvatures of \( \text{osp}(p+q|2; \mathbb{R}) \), subordinated to the above splitting and up to order \( \lambda^2 \), are

\[
\begin{align*}
 f^{\alpha}_{\beta} &= \omega^{\alpha}_{\beta} + \lambda^2 c^{\alpha}_{\beta} + o(\lambda^4) \\
 A^{ij} &= A^{ij,0} + \lambda^2 A^{ij,2} + o(\lambda^4) \\
 A^{ij'} &= A^{ij',0} + \lambda^2 A^{ij',2} + o(\lambda^4) \\
 A^{ij''} &= -A^{ij''} = \lambda^2 A^{ij''} + o(\lambda^4) \\
 \zeta^{i\alpha} &= \lambda \psi^{i\alpha} + o(\lambda^3) \\
 \xi^{i\alpha} &= \lambda \psi^{i\alpha} + o(\lambda^3),
\end{align*}
\]

where \( \zeta^{i\alpha,1} \equiv \psi^{i\alpha} \), \( \zeta^{i\alpha,1} \equiv \psi^{i\alpha} \), and

\[
\begin{align*}
 \Omega^{\alpha}_{\beta} &= R^{\alpha}_{\beta} + \lambda^2 T^{\alpha}_{\beta} + o(\lambda^4) \\
 F^{i}_{j} &= F^{ij,0} + \lambda^2 F^{ij,2} + o(\lambda^4), \quad i, j = 1, \ldots, p, \\
 F^{i}_{j'} &= F^{ij',0} + \lambda^2 F^{ij',2} + o(\lambda^4), \quad i', j' = 1, \ldots, q, \\
 F^{i}_{j''} &= \lambda^2 F^{ij''} + o(\lambda^4) \\
 \Psi^{i\alpha} &= \lambda \varphi^{i\alpha} + o(\lambda^3) \\
 \xi^{i\alpha} &= \lambda \varphi^{i\alpha} + o(\lambda^3),
\end{align*}
\]

where \( \Omega^{\alpha\beta} \equiv T^{\alpha\beta} \); note that \( F^{ij,0} \equiv 0 \) since the expansions in the \( V_2 \) subspace begin already with \( \lambda^2 \).

Similarly, the curvatures are related with the gauge fields by

\[
\begin{align*}
 F^{ij,0} &= dA^{ij,0} + A^{ik,0} \wedge A^{kj,0} \\
 F^{i0j'} &= dA^{i0j'} + A^{i0k} \wedge A^{k0j'} + i\psi^{i0j}_{\gamma} \wedge \psi^{j\gamma} \\
 F^{ij'} &= dA^{ij'} + A^{i0k} \wedge A^{k0j'} + A^{ik,0} \wedge A^{k0j'} + i\psi^{ij'}_{\gamma} \wedge \psi^{j\gamma} \\
 F^{ij''} &= dA^{ij''} + A^{ik,0} \wedge A^{k0j''} + A^{ik,0} \wedge A^{k0j''} + i\psi^{ij''}_{\gamma} \wedge \psi^{j\gamma} \\
 \mathcal{D}\psi^{i\alpha} &= d\psi^{i\alpha} + \omega^{\alpha}_{\beta} \wedge \psi^{i\beta} + A^{ik,0} \wedge \psi^{k\alpha} \equiv D\psi^{i\alpha} + A^{ik,0} \wedge \psi^{k\alpha} \\
 \mathcal{D}\psi^{i\alpha} &= d\psi^{i\alpha} + \omega^{\alpha}_{\beta} \wedge \psi^{i\beta} + A^{ik,0} \wedge \psi^{k\alpha} \equiv D\psi^{i\alpha} + A^{ik,0} \wedge \psi^{k\alpha} \\
 R^{\alpha}_{\beta} &= d\omega^{\alpha}_{\beta} + \omega^{\alpha}_{\gamma} \wedge \omega^{\gamma}_{\beta} \\
 T^{\alpha}_{\beta} &= d\psi^{i\alpha} + e^{\alpha}_{\beta} \wedge \psi^{i\beta} + e^{\alpha}_{\gamma} \wedge \psi^{i\beta} + \omega^{\alpha}_{\beta} \wedge \psi^{i\beta} + \omega^{\alpha}_{\beta} \wedge \psi^{i\beta} \\
 &= D\psi^{i\alpha} + i\psi^{k\alpha} \wedge \psi^{k\beta} + i\psi^{k\alpha} \wedge \psi^{k\beta}.\tag{4.11}
\end{align*}
\]

For zero curvatures (and recalling that we took \( N_0 = 2, N_1 = 1 \) and \( N_2 = 2 \)), eqs. (4.11) reproduce the MC equations of \( \text{osp}(p+q|2; \mathbb{R}) \), \( (2,1,2) \), the \( N_1 = 1, N_0 = 2 = N_2 \) expansion of \( \text{osp}(p+q|2; \mathbb{R}) \).

The various fields appearing in (4.10) and (4.11) can be interpreted as in the \( (p,0) \) case, but now there are also additional \( SO(q) \) \( A^{ij',0} \), \( A^{ij',2} \) gauge fields. Further, (4.11) contains a set of \( p \times q \) one-forms \( A^{ij',2} = -A^{ij',2} \), which transform under the \( SO(p) \) and \( SO(q) \) groups acting on the corresponding indices.

The MC equations of \( \text{osp}(p+q|2; \mathbb{R})(2,1,2) \), obtained by taking vanishing curvatures in (4.11), are not yet those of the superalgebra \( s\mathcal{P}(p,q) \) of the \( (p,q) \)-Poincaré supergravity models.
we are interested in. The MC equations for \( s\mathcal{P}(p, q) \) follow by setting the curvatures equal to zero in eq. (7.7) of ref. [16], which in its notation is

\[
\begin{align*}
F^{ij} &= dA^{ij} + A^{ik} \wedge A^{kj} \\
F^{ij'} &= dA^{ij'} + A^{ik'} \wedge A^{kj'} \\
G^{ij} &= dC^{ij} + A^{ik} \wedge C^{kj} + C^{ik} \wedge A^{kj} - \psi^i_j \wedge \psi_j^i \\
G^{ij'} &= dC^{ij'} + A^{ik'} \wedge C^{kj'} + C^{ik'} \wedge A^{kj'} + \psi^i_{j'} \wedge \psi_j^{i'}
\end{align*}
\]

\[
\mathcal{D}\psi^{ja} = d\psi^{ja} + \frac{i}{2} \omega^a(\gamma^a) \wedge \psi^{ja} + A^{jk} \wedge \psi^{ka} = D\psi^{ja} + A^{jk} \wedge \psi^{ka}
\]

\[
\mathcal{D}\psi^{j'a} = d\psi^{j'a} + \frac{i}{2} \omega^a(\gamma^a) \wedge \psi^{j'a} + A^{j'k'} \wedge \psi^{k'a} = D\psi^{j'a} + A^{j'k'} \wedge \psi^{k'a}
\]

\[
F^a(\omega) = d\omega - \frac{1}{2} e_{bc} \omega^b \wedge \omega^c
\]

\[
T^a = de^a - e^a_{bc} \omega^b \wedge e^c - \frac{1}{4} \tilde{\psi}^{i\gamma} \gamma^a \psi^i - \frac{1}{4} \tilde{\psi}^{i'}_{\gamma'} \gamma^a \psi^{i'}
\]

\[
\equiv D e^a - \frac{1}{4} \tilde{\psi}^{i\gamma} \gamma^a \psi^i - \frac{1}{4} \tilde{\psi}^{i'\gamma'} \gamma^a \psi^{i'}. \quad (4.12)
\]

Reverting these expressions to the present notation \( i.e., \) making the replacements \( A^{ij} \rightarrow A^{ij}, F^{ij} \rightarrow F^{ij} \) (and similarly for the primed indices), \( G^{ij} \rightarrow -F^{ij}, C^{ij} \rightarrow -A^{ij}, G^{ij'} \rightarrow F^{ij'}, C^{ij'} \rightarrow A^{ij'}, \) setting \( \omega_{\alpha\beta} = \frac{1}{2} \omega^a(\gamma^a)_{\alpha\beta} \) (where \( \omega^a \propto e_{abc} \omega^c \)), \( R^a_{\alpha\beta} = \frac{1}{2} F^a_{\gamma\alpha\beta}, e^a_{\alpha\beta} = -2ie^a(\gamma^a)_{\alpha\beta} \) and \( T^a_{\beta} = -2iT^a_{\gamma\alpha\beta} \) in terms of the \( D=3 \) gamma matrices, and identifying \( \tilde{\psi} \) with the index down (so that \( \tilde{\psi}^{i\gamma} \psi^i = \psi_i^a(\gamma^a)_{\alpha\beta} \psi^j \)), we see that eq. (4.12) reproduces eq. (4.11) but for the fifth line in \( F^{ij},2 \). In fact, the algebra generators corresponding to the \( A^{ij},2 \) fields, missing in (4.12), generate a \( (p \times q) \)-dimensional abelian subalgebra of \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2) which is, in fact, an ideal (clearly trivial if either \( p \) or \( q \) are zero, which explains why it was absent in the \( (p, 0) \) case). Denoting this abelian ideal by \( \mathcal{C} \), it is seen that \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2)/\( \mathcal{C} = s\mathcal{P}(p, q) \) i.e., \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2) is an extension of the \( D = 3 \) \( (p, q) \)-supersymmetric algebra \( s\mathcal{P}(p, q) \) by \( \mathcal{C} \), which is neither central nor semidirect. It is not semidirect because \( s\mathcal{P}(p, q) \) is not a subalgebra of \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2): the commutator of the algebra generators dual to \( \tilde{\psi}^i \) and \( \tilde{\psi}^{i'} \) have a component in the subspace corresponding to \( A^{ij},2 \), which is not in \( s\mathcal{P}(p, q) \). It is not central either since the abelian ideal \( \mathcal{C} \) does not belong to the centre of \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2).

Again, we can make an easy dimensional check using (3.3): \( \dim osp(p+q|2; \mathbb{R}) (2, 1, 2)/\mathcal{C} = \left( 2 \left( \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + 3 \right) + 2(p+q) + pq \right) - pq = \dim s\mathcal{P}(p, q) \) by eq. (4.11). In fact, it turns out that the \( N = 2 \) term in the expansion (3.10) of the action in (4.13) for \( osp(p+q|2; \mathbb{R}) \) actually selects the Lie superalgebra \( s\mathcal{P}(p, q) \) rather than \( osp(p+q|2; \mathbb{R}) \) (2, 1, 2). Indeed, the first equation in (4.3) leads, by expanding in \( \lambda \) and selecting the \( \lambda^2 \) term, to

\[
H_2 = 2R^a_{\gamma} \gamma^a \wedge T^{\gamma\alpha} - 2F^{ik,0} \wedge F^{ki,2} - 2F^{ik',0} \wedge F^{ki',2} - 2iD\psi^i_{\gamma} \wedge D\psi^j_{\gamma} - 2iD\psi^{i'}_{\gamma'} \wedge D\psi^{j'}_{\gamma'} \quad (4.13)
\]

(the first term containing \( F^{ij,2} - F^{ij',2} \wedge F^{ij',2} \), is of order \( \lambda^2 \)). Nevertheless, \( osp(p+q|2; \mathbb{R}) (2, 1, 2) \) remains a symmetry of the action by construction. This may be rephrased by noticing that the Casimir of \( osp(p+q|2; \mathbb{R}) \), expanded up to order \( \lambda^2 \), is degenerate on \( \mathcal{C} \), which is its radical as seen eq. (4.13). Thus, once \( \mathcal{C} \) is quotiented out, the resulting metric is no longer degenerate.

Therefore, the action of the \( (p, q) \)-Poincaré supergravity is found to be

\[
I_2 = \frac{1}{2} \int_{M^3} \left( \ deriv^a \gamma^a - F^{ik,0} \wedge A^{ki,2} - F^{ik',0} \wedge A^{ki',2} - iD\psi^i_{\alpha} \wedge \psi^{i\alpha} - iD\psi^{i'}_{\alpha} \wedge \psi^{i'\alpha} \right). \quad (4.14)
\]
Note that $C^{ij'}$ is absent from both (4.13) and (4.14); of course, for $q=0$, both expressions reduce to eqs (1.7) and (1.6) respectively. Again, the field equations are given by the vanishing of the curvatures, and the resulting models coincide with those of [16] (eqs. (7.11) there).

Therefore, the expansion procedure applied to $osp(p + q|2; \mathbb{R})$ determines the $s\mathcal{P}(p,q) = osp(p + q|2; \mathbb{R})(2, 1, 2)/\mathcal{E}$ algebras associated with the $(p,q)$-Poincaré supergravities as well as their CS actions.

5 Concluding remarks

We conclude with some remarks on the expansion method and the gauge structure of (super)gravities in general. The search for a gauge structure of (super)gravities in various space-time dimensions for some underlying group is an old one. On general odd-dimensional space-times, gauge theories with a CS structure are natural candidates; see [25], [26, 27] and [28] for a review. In fact, for $D=3$, the CS structure of $N=1$ supergravity has been known for quite a long time [29, 30]. The expansion method was immediately used for lower dimensional supergravities [20, 21]. It has also been applied to $D=5$ CS AdS gravity in an attempt to obtain the Einstein-Hilbert lagrangian [31] from it, albeit the dynamics turns out to be very different from that of general relativity.

Even when there is not a clearly singularized initial (super)group to start with, the expansion procedure may be useful. It was already pointed out in the original CJS $D=11$ supergravity paper [32] that the $OSp(1|32)$ symmetry might play a relevant role in eleven-dimensional supergravity. It was shown in [33] that it is indeed possible to associate a Lie algebra to the $D=11$ CJS supergravity FDA by considering its three-form field as a composite of one-form fields, which are identified with the soft MC forms of a superlagebra; this algebra may be said to trivialize the original FDA one. The general solution that trivializes the FDA structure of $D=11$ supergravity was found in [34]. It was shown there that the underlying gauge group structure of CJS supergravity is described by the members of a one-parameter family of deformations of a specific expansion of the $osp(1|32)$ algebra, namely of $osp(1|32)(2, 3, 2)$, the expansion itself being excluded. On its part, the M-theory full superlagebra (i.e., including the Lorentz automorphism algebra) may be seen to be [20, 21] the expansion $osp(1|32)(2, 1, 2)$; also, the usefulness of expansions in obtaining the flat limit for the WZ term of a superstring in anti-de Sitter space has been shown in [19, 35].

For the moment, however, it may be said that a clear understanding of the symmetry structure of $D=11$ supergravity and of M-theory remains elusive. At a more modest level, the analysis of the $D=3$ $(p,q)$-Poincaré supergravities presented here does exhibit the expansion character of the various $\mathcal{P}(p,q)$ Poincaré superalgebras behind them and provides further examples of the usefulness of the expansion procedure to construct the corresponding CS actions.

Acknowledgements. This work has been partially supported by research grants from the Spanish Ministry of Science and Innovation (FIS2008-01980, FIS2005-03989), the Junta de Castilla y León (VA013C05) and EU FEDER funds.
References

[1] I.E. Segal, A class of operator algebras which are determined by groups, Duke Math. J. 18, 221-265 (1951)

[2] E. İnönü and E.P. Wigner, On the contraction of groups and their representations, Proc. Nat. Acad. Sci. U.S.A. 39, 510-524 (1953); E. İnönü, contractions of Lie groups and their representations, in Group theoretical concepts in elementary particle physics, F.Gürsey ed., Gordon and Breach, pp. 391-402 (1964)

[3] E.J. Saletan, Constructions of Lie groups, J.Math. Phys. 2, 1-21 (1961)

[4] E. Weimar-Woods, Constructions of Lie algebras: generalized İnönü-Wigner contractions versus graded contractions, J. Math. Phys. 36, 4519-4548 (1995); Constructions, generalized İnönü and Wigner contractions and deformations of finite-dimensional Lie algebras, Rev. Math. Phys. 12, 1505-1529 (2000)

[5] V. Aldaya and J.A. de Azcárraga, Cohomology, central extensions and dynamical groups, Int. J. Theor. Phys. 24, 141-154 (1985)

[6] J.A. de Azcárraga and D. Ginestar, Non-relativistic limit of supersymmetric theories, J. Math. Phys, 32, 2500-3508 (1991)

[7] C. G. . Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Evanescent black holes, Phys. Rev. D45, 1005-1009 (1992) [arXiv:hep-th/9111056].

[8] D. Cangemi and R. Jackiw, Gauge invariant formulations of lineal gravities, Phys. Rev. Lett. 69, 233-236 (1992) [arXiv:hep-th/9203056].
R. Jackiw, Higher symmetries in lower dimensional models, in Proc. of the XXIII GIFT Seminar (Salamanca, 1992), L. A. Ibort and M. A. Rodríguez eds., NATO ASI Series 409, 289-316 (1993).

[9] J.A. de Azcárraga and J.M. Izquierdo, Lie groups, Lie algebras, cohomology and some applications in physics, Camb. Univ. Press., 1995,

[10] V. O. Rivelles, Topological two-dimensional dilaton supergravity Phys. Lett. B321 189-192 (1992) [arXiv:hep-th/9301029].

[11] P. A. Horváthy, M. S. Plyushchay, M. Valenzuela, Supersymmetry of the planar Dirac-Deser-Jackiw-Templeton system, and of its non-relativistic limit, J. Math. Phys. 51, 092108 (2010) [arXiv:1002.4729 [hep-th]].

[12] A. Achúcarro and P.K. Townsend, A Chern-Simons action for three-dimensional anti-De Sitter supergravity theories, Phys. Lett. B 180 (1987) 89-92 (1986);

[13] A. Achúcarro and P.K. Townsend, Extended supergravities in d=2+1 as Chern-Simons theories, Phys. Lett. B 229, 383-387 (1989).

[14] N. Marcus and J.H. Schwarz, Three-dimensional supergravity theories, Nucl. Phys. B228, 145-162 (1983)
[15] S.J. Gates, M.T. Grisaru M. Roček and W. Siegel, *Superspace: 1001 lessons in supersymmetry*, Benjamin/Cummings (1983)

[16] P. S. Howe, J. M. Izquierdo, G. Papadopoulos, P. K. Townsend, *New supergravities with central charges and Killing spinors in (2+1)-dimensions*, Nucl. Phys. B467, 183-214 (1996) [hep-th/9505032].

[17] M. Henneaux, S.-J. Rey, *Nonlinear $W_\infty$ as asymptotic symmetry of three-dimensional higher spin AdS gravity*, JHEP 1012, 007 (2010) [arXiv:1008.4579 [hep-th]].

[18] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, JHEP 1011, 007 (2010) [arXiv:1008.4744 [hep-th]].

[19] M. Hatsuda and M. Sakaguchi, *Wess-Zumino term for the AdS superstring and generalized İnönü-Wigner contraction*, Prog. Theor. Phys. 109, 853-867 (2003) [arXiv:hep-th/0106114].

[20] J. A. de Azcárraga, J. M. Izquierdo, M. Picón and O. Varela, *Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity*, Nucl. Phys. B662, 185-219 (2003) [arXiv:hep-th/0212347].

[21] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, *Extensions, expansions, Lie algebra cohomology and enlarged superspaces*, Class. Quant. Grav. 21, S1375-S1384 (2004) [arXiv:hep-th/0401033]; *Expansions of algebras and superalgebras and some applications*, Int. J. Theor. Phys. 46, 2738-2752 (2007) [arXiv:hep-th/0703017].

[22] F. Izaurieta, E. Rodríguez and P. Salgado, *Expanding Lie (super)algebras through abelian semigroups*, J. Math. Phys. 47, 123512 (2006) [arXiv:hep-th/0606215].

[23] D. Sullivan, *Infinitesimal computations in topology*, Inst. des Haut. Étud. Sci., Pub. Math 47, 269-331 (1977)

[24] P. van Nieuwenhuizen, *Free graded differential superalgebras*, in *Group theoretical methods in physics*, M. Serdaroglu and E. İnönü Eds., Lect. Notes in Phys. 180, 228-247 (1983)

[25] A. H. Chamseddine, *Topological gravity and supergravity in various dimensions*, Nucl. Phys. B346, 213-234 (1990).

[26] M. Bañados, R. Troncoso, J. Zanelli, *Higher dimensional Chern-Simons supergravity*, Phys. Rev. D54, 2605-2611 (1996) [gr-qc/9601003].

[27] F. Izaurieta, E. Rodríguez, P. Minning, P. Salgado and A. Pérez, *Standard general relativity from Chern-Simons gravity*, Phys. Lett. B678, 213-217 (2009) [arXiv:0905.2187 [hep-th]].

[28] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008*, arXiv:hep-th/0502193.

[29] P. van Nieuwenhuizen, *Three-dimensional conformal supergravity and Chern-Simons terms*, Phys. Rev. D32, 872-878 (1985)
[30] E. Witten, \textit{(2+1)-dimensional gravity as an exactly soluble system}, Nucl. Phys. \textbf{B311}, 46-78 (1988)

[31] J. D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, \textit{Lie algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian}, Phys. Lett. \textbf{B640}, 278-284 (2006) [arXiv:hep-th/0605174].

[32] E. Cremmer, B. Julia and J. Scherk, \textit{Supergravity Theory in Eleven-Dimensions}, Phys. Lett. \textbf{B76}, 409-412 (1978).

[33] R. D’Auria and P. Fré, \textit{Geometric supergravity in d = 11 and its hidden supergroup}, Nucl. Phys. \textbf{B201}, 101-140 (1982) [Erratum-ibid. \textbf{B206}, 496 (1982)].

[34] I. A. Bandos, J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, \textit{On the underlying gauge group structure of D=11 supergravity}, Phys. Lett. \textbf{B596}, 145-155 (2004) [hep-th/0406020];
I. A. Bandos, J. A. de Azcárraga, M. Picón, O. Varela, \textit{On the formulation of D = 11 supergravity and the composite nature of its three-form gauge field}, Annals Phys. \textbf{317}, 238-279 (2005) [hep-th/0409100].

[35] M. Hatsuda and M. Sakaguchi, \textit{Wess-Zumino term for AdS superstring}, Phys. Rev. \textbf{D66}, 045020 (2002) [arXiv:hep-th/0205092].