One-dimensional model of freely decaying two-dimensional turbulence

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Abstract
We construct a discrete shell model for two-dimensional turbulence that takes into account local and nonlocal interactions between velocity modes in Fourier space. In real space, its continuous limit is described by the one-dimensional Burgers equation. We find a novel approximate scaling solution of such an equation and show that it well describes the main characteristics of the energy spectrum in fully developed, freely decaying two-dimensional turbulence.

Keywords Two-dimensional turbulence · Shell models · Analytical methods

1 Introduction
Freely decaying, two-dimensional hydrodynamic (2HD) turbulence has been the object of intensive studies in the last few decades (for a review of two-dimensional turbulence see, e.g., [1]). This is because, some three-dimensional turbulent systems in nature, such as large-scale motions in the atmosphere and oceans, are well approximated by two-dimensional hydrodynamical models.

The main features of statistically homogeneous and isotropic turbulence are encoded in the so-called kinetic energy spectrum \( E(k, t) \), which defines the energy contained in a given (Fourier) velocity mode of wavenumber \( k \) at the time \( t \). Such a spectrum exhibits, in both direct numerical simulations and laboratory experiments, peculiar “universal” properties that completely characterize the evolution of fully developed turbulence. In particular, the energy spectrum unveils the presence of three distinct regions, or ranges, of turbulence with different characteristics: the large-scale range, the inertial range, and the dissipative range.

The aim of this paper is to introduce and discuss a reduced dimensional model of Navier-Stokes equation that governs two-dimensional turbulence. Such a model is, under plausible assumptions, analytically solvable and seems to describe well the main properties of freely decaying two-dimensional turbulence, namely when external sources are not present in the turbulent state.

The plan of the paper is as follows. In the following section, we review the main features of the energy spectrum in freely decaying 2HD turbulence. In Sect. 3, we apply particular types of scaling arguments, first introduced by Olesen in the study of freely decaying, three-dimensional magnetohydrodynamic turbulence, to the hydrodynamical case in two dimensions. In Sect. 4, we introduce our one-dimensional model for two-dimensional turbulence. In Sect. 5, we find and discuss the solution of the equation describing such a reduced dimensional model. In Sect. 6, we draw our conclusion. Finally, in the Appendix, we derive the Saffman spectrum from the Burgers equation.

2 Hydrodynamics in two dimensions
The evolution of an incompressible fluid in two dimensions is described by the two-dimensional Navier-Stokes equation,

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \tag{1}
\]

and the incompressibility condition, \( \nabla \cdot \mathbf{v} = 0 \), where \( \mathbf{v}(\mathbf{x}, t) = (v_1, v_2) \) is the velocity of bulk fluid motion, \( p(\mathbf{x}, t) \) is the pressure, and \( \nu \) is the kinematic viscosity (see, e.g., [2, 3]). One can define the kinematic Reynolds number, \( Re = vl/\nu \), where \( v \) and \( l \) are the typical velocity and typical length scale of the fluid motion. Hydrodynamic turbulence occurs when \( Re \gg 1 \).
In Fourier space, the two-dimensional hydrodynamic equations take the form
\[
\left( \frac{\partial}{\partial t} + v k^2 \right) u_\alpha(k) = i P_{\alpha\beta}(k) \int d^2 \rho u_\rho(p) u_\beta(k-p)
\]  
(2)
and \( k_\alpha u_\alpha(k, t) = 0 \), where \( u(k, t) \) is the velocity field in Fourier space,
\[
P_{\alpha\beta}(k) = \frac{1}{2\pi} \left( \frac{1}{k^2} k_\alpha k_\beta - \frac{1}{2} k_\gamma \delta_{\alpha\beta} - \frac{1}{2} k_\beta \delta_{\alpha\gamma} \right).
\]  
(3)
and \( k = |k| \). The pressure \( p \) has been eliminated by use of the incompressibility condition. Reality of the velocity field in real space imposes that \( u(-k) = u^*(k) \). (Greek subscripts range from 1 to 2 and summation over repeated indices is implied.)

For isotropic fluids, two-dimensional hydrodynamics admits two invariants of motion in the limit of null kinematic viscosity (the inviscid case): the kinetic energy density (in a unitary two-dimensional volume),
\[
E(t) = \left\langle \frac{1}{2} \int d^2 x \mathbf{v}^2 \right\rangle = \int_0^\infty dk E(k, t),
\]  
(4)
and the enstrophy,
\[
\Omega(t) = \left\langle \int d^2 x \mathbf{\omega}^2 \right\rangle = \int_0^\infty dk k^2 E(k, t),
\]  
(5)
where
\[
E(k, t) = \pi k \left\langle |u(k)|^2 \right\rangle
\]  
(6)
is the kinetic energy density spectrum, and \( \omega = \epsilon_{\alpha\beta}\partial_\gamma v^\gamma = \partial_1 v_2 - \partial_2 v_1 \) is the “scalar vorticity” field. Since we are interested in the evolution of statistically homogeneous and isotropic velocity fields, an ensemble average denoted by \( \left\langle \ldots \right\rangle \) has been introduced in the above definitions of energy and enstrophy.

The main features of locally homogeneous and isotropic turbulence are described by the kinetic energy spectrum to which we now turn our attention. In freely decaying turbulence, the energy spectrum displays some universal characteristics at both large and small scales. In particular, the results of direct numerical simulations and laboratory experiments clearly show the presence of three distinct regions or ranges: (i) the large-eddies range, \( k \ll k_i \), (ii) the enstrophy inertial range, \( k_i \ll k \ll k_{\text{diss}} \), and the dissipation range, \( k \gg k_{\text{diss}} \). A sketch of the energy spectrum in these ranges is shown in Fig. 1. The existence of such three regions is not surprising. Indeed, in freely decaying 2HD there are only two independent length scales: the dissipation length \( L_{\text{diss}}(t) = \sqrt{\nu t} \) and the initial scale \( L_i \), which we assume to be much greater than \( L_{\text{diss}} \), \( L_i \gg L_{\text{diss}} \). The wavenumber corresponding to \( L_i \) is \( k_i = 1/L_i \), while the wavenumber corresponding to \( L_{\text{diss}} \) is \( k_{\text{diss}} = 1/L_{\text{diss}} \). These two length scales define the three ranges introduced above.

**Enstrophy inertial range** A physical homogeneous and isotropic field not interacting with external systems is dependent only upon the initial conditions which introduces a mean velocity \( u_i \) and a length scale \( L_i \). This is the case, for example, of turbulence behind a grid (with \( u_i \) being the velocity field at the grid and \( L_i \) being the spacing of the bars of the grid perpendicular to the direction of the flow). In numerical simulations, instead, it is assumed that the initial velocity field is homogeneous and isotropic. In this case, the mean value of the velocity is zero and \( u_i \) in the following equations must be regarded as a root-mean-square (r.m.s) value of the field. More generally, \( L_i \) and \( u_i \) can be considered as typical scale values at the onset of fully developed turbulence.

By definition, an inertial range is a range where dissipation is negligible and the dynamics is independent on the initial conditions. Dimensional analysis, then, completely gives the shape of the energy spectrum in such a range (see, e.g., [4]),
\[
E(k, t) = ct^{-2} k^{-3}, \quad k_i \ll k \ll k_{\text{diss}},
\]  
(7)
where \( c \) is a dimensionless constant. In forced turbulence, a time-independent spectrum of the form \( E(k, t) \propto k^{-3} \) is known as Batchelor spectrum [5].

**Dissipative range** At small scales, \( k \gg k_{\text{diss}} \), the 2HD equations can be exactly solved if one assumes that the rate of energy transfer due to nonlinear interactions is completely negligible. Indeed, it can be shown that the energy spectrum
changes in time as $\partial E(k, t)/\partial t = T(k, t) - v k^2 E(k, t)$, where $T(k, t)$ is the energy change caused by nonlinear interactions (see, e.g., [1]). When this term is negligible with respect to the viscous term, one obtains

$$E(k, t) = c_\infty \nu^2 k e^{-vk^2}, \quad k \gg k_{\text{diss}},$$

(8)

where $c_\infty$ is a dimensionless constant.

The hypothesis that the rate of nonlinear energy transfer is negligible is certainly true in the “mathematical” limit $k \to \infty$. However, such a hypothesis is not justified in a region around the dissipation wavenumber. Indeed, Tatsumi and Yanase [6], using the so-called “modified zero fourth-order cumulant approximation”, found an analytic expression of the energy spectrum in the dissipation range of the form

$$E(k, t) \propto e^{-bk \sqrt{t}}, \quad k \gtrsim k_{\text{diss}}.$$  

(9)

where $b$ is a dimensionless constant. The same authors found, by direct numerical integration of 2HD equations, that $E(k, t) \propto \exp \left[ -b(k \sqrt{t} \nu)^s \right]$ with $s$ in the range $[1.3, 1.4]$. The discrepancy with Eq. (9) indicates, according to the authors, that either the asymptotic behaviour $s = 1$ is realized beyond the numerical coverage or the numerical results are not accurate enough. In either case, the asymptotic form of the spectrum is different from the purely viscous spectrum (8) for $k \gtrsim k_{\text{diss}}$.

**Large-eddies range** In the limit $k \to 0$, a Maclaurin expansion of the energy spectrum gives (see, e.g. [3])

$$E(k, t) = \begin{cases} 4\pi \mathcal{L} + O(k^3), & \mathcal{L} \neq 0, \\ I(t)k^3 + O(k^5), & \mathcal{L} = 0. \end{cases}$$

(10)

Here, $\mathcal{L}$ is known as the two-dimensional Saffman integral, and its time-independence is a consequence of the conservation of linear momentum [3]. Indeed, Davidson [7] has shown that the Saffman integral in two dimensions can be written as

$$\mathcal{L} = \left\langle \frac{1}{V} \left( \int \nu d^2x \mathbf{v} \right)^2 \right\rangle,$$

(11)

where $V$ is some large two-dimensional volume. Accordingly, $\mathcal{L}$ is nonzero whenever the turbulence has sufficient linear momentum. In this case, then, the energy spectrum at large scales is linear in the wavenumber, a result known as Saffman spectrum. If the total linear momentum is instead negligible, a spectrum of the form $I(t)k^3$ develops at large scales where $I(t)$, the so-called Loitsiansky integral in two dimensions, is generally expected to be an increasing function of time [3].

Both cases of turbulence have been generated in computer simulations, while the experimental situation as to whether the grid turbulence is of the Saffman or $k^3$ type is still unclear (see [7] for a brief discussion about this point).

### 3 Dimensional scaling

**Olesen’s scaling** The two-dimensional HD equations, under the scaling transformations

$$x \to \ell x, \quad t \to \ell^{1-h} t,$$

(12)

admit solutions of the type

$$v(\ell x, \ell^{1-h} t) = \ell^h \psi(x, t),$$

(13)

provided that the dissipative parameter $v$ scales as

$$v(\ell^{1-h} t) = \ell^{1+h} \psi(t).$$

(14)

Here $\ell > 0$ is the “scaling factor” and $h$ is a arbitrary real parameter. Starting from the scaling relations, and following Olesen’s analysis [8], we obtain the energy spectrum

$$E(k, t) = k t^p \psi(k t^{1+p/2}),$$

(15)

where $p = (1 + h)/(1 - h)$ and $\psi$ is a arbitrary scaling-invariant function of its argument. We have two cases:

(i) If dissipation is negligible, as it happens at large scales and, by definition, in the inertial range, then $h$ (and in turns $p$) is completely arbitrary, to wit, it is not fixed by scaling arguments.

(ii) If dissipation is important, as in the dissipative range (at very small scales), then $h$ is fixed by the scaling properties of $v$. Indeed, if we differentiate Eq. (14) with respect to $\ell$, and put $\ell = 1$ afterwards, we get $v \propto \ell^p$. In particular, as we will assume in this paper, if $v$ is constant then we must take to $p = 0$ and, accordingly,

$$E(k, t) = k \psi(k t^{1/2}).$$

(16)

If dissipation is not negligible and $v$ does not evolve in time as a power law, then Eq. (13) is not a solution of the 2HD equations and in turn the energy spectrum does not exhibit a scaling of the type given by Eq. (15).

**Dimensional analysis** For the sake of later convenience let us re-write Eqs. (16) and (15) in a form such that the scaling functions $\Psi$ and $\psi$ are dimensionless. It is easy to check that they must be written as

$$E(k, t) = \nu^2 k \psi(k \sqrt{vt})$$

(17)

and

$$E(k, t) = E_0 k \tau^{2p} \psi(\kappa \tau^{1+p/2}),$$

(18)
respectively. Here, \( E_t = u^2 L_t \), \( \kappa = k/k_1 \), \( \tau = t/t_1 \), and \( t_1 = L_1/u_1 \) is the so-called initial “eddy turnover time”.

It is interesting to observe that form of the energy spectrum in the inertial range is completely fixed by scaling considerations only. Indeed, since in this range the dynamics is completely independent on dissipation (and on the initial conditions), the only possible form for the scaling function \( \psi \) such that the energy spectrum does not depend on the kinematic viscosity is \( \psi(x) = cx^{-4} \). This gives the Batchelor spectrum in Eq. (7) with \( c \) being a “universal” dimensionless constant.

4 One-dimensional model

In this section, we propose a one-dimensional, continuous (toy) model for two-dimensional hydrodynamics. We start by generalizing the well-known discrete shell model [9] for HD turbulence by including all possible local and nonlocal interaction terms in Fourier space. Next, we consider the continuous limit of such a discrete shell model. The resulting model will be analyzed in the next section.

Shell model with nonlocal interactions Let us consider a shell model constructed in a discrete wavenumber space which is approximated by \( 2N \) shells,

\[
\{k_n | k_n = Kn, \ n = -N, -N + 1, \ldots, 0, \ldots, N - 1, N \}, \tag{19}
\]

where \( K \) gives the spacing between two consecutive shells. The velocity is represented by a set of complex variables \{\( u_n \)\}, where \( u_n \) stands for the velocity components whose scalar wavenumber \( k \in \mathbb{R} \) satisfies \( k_n \leq k \leq k_{n+1} \). We now construct the Navier-Stokes equation in this scalar model in the following way,

\[
\left( \frac{d}{dt} + v k^2 \right) u_n = iP_n \sum_{j=-N}^{N} u_j u_{n-j}, \tag{20}
\]

where \( P_n \) is a real vector and \( u_{-n} = u^*_n \). The sum on the left-hand side of (20) includes all possible local and nonlocal interaction terms between different Fourier modes.

In the shell model, the kinetic energy density is \( E(t) = \sum_{n=0}^{N} E_n(t) \), where \( E_n(t) = \pi k_n^2 |u_n|^2 \) is the energy spectrum (for a justification of the form of the energy spectrum, see below.) One can check that a sufficient condition for the conservation of energy in the inviscid limit is \( P_n = -P_{-n} \). Mimicking Eq. (3), we can assume assume that \( P_n \) is linear in \( n \), \( P_n \propto k_n \). With this \( P_n \), it is easy to check that the quantity \( \sum_{n=0}^{N} \sum_{j=-N}^{N} f(|k_n|) |u_n|^2 \), with \( f(x) \) any function such that \( f(0) = 0 \), is conserved in the limit of vanishing viscosity. In particular, for \( f(x) = x^3 \) the above expression is the analogue of the enstrophy.

Continuous limit The continuous limit of the above shell model corresponds to take \( N \to \infty \) and \( K \to 0 \). In this case, \( k_n \) is replaced by \( \tilde{k} \), the discrete set \( u_n(t) \) is replaced by a continuous complex variable \( u(\tilde{k}, t) \), the real vector \( P_n \) by a real function of \( P(\tilde{k}) \), and sums are replaced by integrals, e.g., \( K \sum_{n=-N}^{N} \to \int_{-\infty}^{+\infty} d\tilde{k} \). The continuous limit of the discrete shell model (20) is then given by

\[
\left( \frac{d}{dt} + v k^2 \right) u(\tilde{k}) = iP(\tilde{k}) \int_{-\infty}^{+\infty} d\tilde{q} u(\tilde{q}) u(\tilde{k} - \tilde{q}), \tag{21}
\]

where \( k = |\tilde{k}|, P(\tilde{k}) \propto \tilde{k} \), and \( u(\tilde{k}) = u^*(\tilde{k}) \). The energy spectrum is

\[
E(k, t) = \pi k |u(\tilde{k})|^2 \tag{22}
\]

and the energy is \( E(t) = \int_{0}^{\infty} dk E(k, t) \). The energy, and in the general the quantity \( \int_{0}^{\infty} dg(k) |u(\tilde{k})|^2 \) with \( g(k) \) being any real function such that the previous integral exists, is conserved in the limit of vanishing viscosity.

In the following, we take

\[
P(\tilde{k}) = -\tilde{k}/2\sqrt{2\pi}. \tag{23}
\]

A possible multiplicative constant in the above definition of \( P(\tilde{k}) \) can be reabsorbed in the definition of time and kinematic viscosity, and then will be neglected in the following.

Burgers equation The factor \(-1/2\sqrt{2\pi} \) in Eq. (23) has been introduced for convenience. In this case, indeed, the inverse Fourier transform of Eq. (21) is the well-known viscous Burgers equation (see [10] and references therein)

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \xi} = \nu \frac{\partial^2 v}{\partial \xi^2}, \tag{24}
\]

where \( v(\xi, t) = \int \frac{dk}{\sqrt{2\pi}} e^{ik\xi} u(k, t) \).

It is assumed in the literature that the viscous Burgers equation cannot describe the main features of hydrodynamic turbulence [10]. This is because an exact solution of such an equation can be given that does not share an important property of the Navier–Stokes equation, namely the sensitivity to small changes in the initial conditions (at sufficiently high Reynolds numbers) that trigger the spontaneous arise of randomness by chaotic dynamics. Such an exact solution is found by applying a Cole-Hopf transformation defined by introducing the function \( f(\xi, t) \) as

\[
v = -2\nu \frac{\partial f}{\partial \xi}, \tag{25}
\]

Inserting the above equation in the viscous Burgers equation, one gets

\[
H \frac{\partial f}{\partial \xi} = \frac{\partial H}{\partial \xi}, \tag{26}
\]
where

\[
H = \frac{\partial f}{\partial t} - \nu \frac{\partial^2 f}{\partial \xi^2}.
\]  

(27)

Consequently, if \( f \) solves the “heat equation” \( H = 0 \) then \( v \), given the Cole-Hopf transformation, solves the viscous Burgers equation. Since the heat equation can be exactly solved, the exact form of \( f \) is known. An inverse Cole-Hopf transformation then gives the analytical expression for \( v \) [10]. Moreover, it can be shown that such a solution for \( v \) is unique if the initial condition

\[
v(\xi, 0) = v_0(\xi)
\]

(28)

is given. The existence of such a solution, then, excludes the possibility that the Burger equation could describe the transition from an initial state to a state of fully developed hydrodynamic turbulence.

However, the above arguments do not exclude the possibility that the Burger equation could describe the evolution of the system after a turbulent state has been reached. Indeed, our claim is that this is the case, in the sense that Burgers equation does describe the main features of the energy spectrum in fully developed, freely decaying two-dimensional turbulence. In order for this to be possible one has to assume that the Burger equation is a model of turbulence only at times \( t \gg t_0 \), namely only when turbulence is already fully developed. In particular, the initial condition (24) is inapplicable in this case, which means that the velocity field must be not defined for \( t = 0 \) (or, in general, for \( t = t_0 \), where \( t_0 \) is the initial time).

A possible solution of the Burger equations that satisfies the above condition and that will correspond, as discussed in Sect. 5, to the case of large wavenumbers can be obtained by writing \( v(\xi, t) \) as

\[
v(\xi, t) = \sqrt{\frac{\nu}{t}} \hat{f}\left(-\frac{\zeta}{\sqrt{vt}}\right).
\]

(29)

with \( \hat{f}(\alpha) \) a “scaling function” to be determined. Observe that this solution is not defined for \( t = 0 \).

Inserting into Eq. (24) we get

\[
2\hat{f}'' + 2i\hat{f}' + \omega^2 \hat{f} + \hat{f} = 0,
\]

(30)

where a prime indicates a derivative with respect to \( \omega = -\zeta/\sqrt{vt} \). The solution of the above equation is

\[
\hat{f}(\omega) = \frac{2c_1c_2H_{c_1 - \left(\frac{\omega}{2}\right)} - \omega c_2H_{c_1 - \left(\frac{\omega}{2}\right)} + F\left(\frac{-c_1}{2}, \frac{1}{2}, \frac{\omega^2}{4}\right) + c_1 \frac{1}{2} F\left(1 - \frac{c_1}{2}, \frac{3}{2}, \frac{\omega^2}{4}\right)}{c_2H_{c_1 - \left(\frac{\omega}{2}\right)} + F\left(\frac{-c_1}{2}, \frac{1}{2}, \frac{\omega^2}{4}\right)}
\]

where \( c_1 \) and \( c_2 \) are real constants of integration, \( H_n(x) \) is the Hermite polynomial of degree \( n \), and \( F_1(a, b; x) \) is the Kummer confluent hypergeometric function [11].

As we will discuss later, the solution (24) describes the system in a state of fully developed HD turbulence. The constants \( c_1 \) and \( c_2 \) are then to be considered “universal” and cannot be determined by initial conditions. Indeed, they will correspond to the constants \( C \) and \( c_{\infty} \) defined in Sect. 5.

Let us finally observe that the function \( f(\xi, t) \) defined by the Cole-Hopf transformation does not satisfy the heat equation. Indeed, writing

\[
f(\xi, t) = \sqrt{\frac{\nu}{t}} \phi\left(-\frac{\zeta}{\sqrt{vt}}\right)
\]

(32)

where \( \phi(\omega) \) is an arbitrary function of its argument, and inserting into the Cole-Hopf transformation (25), we get

\[
\phi(\omega) = e^{\frac{1}{2}\int d\omega \hat{f} \hat{g}},
\]

(33)

with \( \hat{g} \) given by Eq. (31). It is easy to check now, by direct inspection, that \( f(\xi, t) \) as given by Eqs. (32) and (33) does not satisfy the heat equation.

In the next paragraph, we show that, under particular conditions, the Burger equation describes the dynamics of a two-dimensional fluid. Moreover, the results of this analysis will provide a justification of the definition of energy spectrum introduced above. In the next section, instead, we will discuss in some detail the above “scaling solution” of Burger equation. For the sake of convenience, we will work in Fourier space and then consider Eq. (21).

**Compressible fluid with negligible pressure** Let us consider a compressible fluid with negligible pressure. We introduce the “complex velocity field” \( V(x, y) \) as

\[
V(x, y) = v_x + iv_y.
\]

(34)

Using the Navier-Stokes equation, we find that the \( V \) satisfies the equation

\[
\frac{\partial V}{\partial t} + \left(\text{Re } [V] \frac{\partial}{\partial x} + \text{Im } [V] \frac{\partial}{\partial y}\right)V = \nu \nabla^2 V.
\]

(35)

Taking the complex conjugate of the above equation, it is easy to see that a possible solution of it is such that \( V^* = e^{-i\varphi} V \), with \( \varphi \in \mathbb{R} \).
Let us introduce the new real quantity \( v(x, y) \) as

\[
u(x, y) = e^{-ip/2}V = \frac{1}{2}[\sec(\varphi/2)v_x + \csc(\varphi/2)v_y].\]

(36)

It satisfies the equation

\[
\frac{\partial v}{\partial t} + v \left[ \cos(\varphi/2) \frac{\partial}{\partial x} + \sin(\varphi/2) \frac{\partial}{\partial y} \right] v = v \nabla^2 v.
\]

(37)

We can diagonalize the operator in square parenthesis in the above equation by performing a rotation of an angle \( \varphi \) of the coordinate system,

\[
\xi = \cos(\varphi/2)x + \sin(\varphi/2)y, \quad \eta = -\sin(\varphi/2)x + \cos(\varphi/2)y.
\]

(38)

(39)

The function \( v \) in the new coordinate system, \( v(\xi, \eta) \), satisfies the equation

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \xi} = v \nabla^2 v,
\]

(40)

where \( \nabla^2 = \partial^2_\xi + \partial^2_\eta \). The r.m.s. value of the velocity field, \( v_{\text{rms}} \), is given by

\[
v^2_{\text{rms}}(t) = \int d^2x v^2(x) = \int d^2\xi v^2(\xi) = \int dk w^2_{\text{rms}}(k),
\]

(41)

where \( \xi = (\xi, \eta) \) and

\[
w^2_{\text{rms}}(k, t) = 2\pi k w(k) v^*(k),
\]

(42)

with \( w_{\text{rms}} \) being the spectrum of the r.m.s. velocity. Here, \( w(k) \) is the Fourier transform of \( v(\xi, \eta) \). The equation (40) reduces then to the Burgers equation. The r.m.s. velocity is still given by Eq. (41), while its spectrum reads

\[
w^2_{\text{rms}}(k, t) = 2\pi k |w(k, t)|^2,
\]

(43)

where \( k = |k_\xi| \). For constant density (case that is not realized in compressible, pressurless 2HD turbulence), we have \( E(t) = v^2_{\text{rms}}/2 \) (with density equal to one), so that the energy spectrum is \( E(k, t) = w^2_{\text{rms}}(k, t)/2 = \pi k |w(k_\xi, t)|^2 \). This result justifies the definition of energy spectrum introduced above.

5 Results

As discussed in Sect. 2, the state of a turbulent isotropic fluid depends entirely on the initial conditions, \( L_i \) and \( u_i \), and on the dissipation parameter \( \nu \). The energy contained in a given wavenumber \( k \) evolves in time with different characteristics in the three different ranges defined by \( k \ll k_i, k \ll k_{\text{diss}}, \) and \( k \gg k_{\text{diss}} \). Indeed, while in the large-scale range the energy spectrum generally depends on the initial conditions but not on dissipation, in the dissipative range it depends on \( \nu \) but not on \( L_i \) and \( u_i \). The inertial range, that in wavenumber space is placed in between the two above ranges, is instead characterized by a universal energy spectrum not dependent on either initial conditions or viscosity. Based on this peculiarities of the energy spectrum, it is plausible to assume that the wave-numbers defined by \( k \ll k_i \) and \( k \gg k_i \) “do not communicate”, in the sense that nonlocal interactions between modes in this different ranges are negligible. We will assume, in the following, that this is the case. Accordingly, and roughly speaking, the integral in Eq. (21) can be split in two parts

\[
\int_{-\infty}^{+\infty} d\bar{q} \ u(\bar{q}) u \left( \bar{k} - \bar{q} \right) = \theta(k_i - k)
\]

\[
\int_{q<k_i} d\bar{q} \ u(\bar{q}) u \left( \bar{k} - \bar{q} \right) + \theta(k - k_i)
\]

\[
\int_{q>k_i} d\bar{q} \ u(\bar{q}) u \left( \bar{k} - \bar{q} \right).
\]

(44)

where \( \theta(x) \) is the Heaviside step function.

Enstrophy inertial range In this range, the energy spectrum has the form (17). Accordingly, we look for solutions of Eq. (21) of the form

\[
u(\bar{k}, t) = \nu \phi(\bar{k} \sqrt{\nu t}).
\]

(45)

Inserting into Eq. (21) and taking into account Eq. (44), we find that a possible solution is defined through the scaling function \( \phi(x) \) that, in turns, satisfies the equation

\[
x \frac{d\phi(x)}{dx} + 2x^2 \phi(x) = -\frac{i}{\sqrt{2\pi}} x \int_{|y|>k/k_{\text{diss}}} dy \phi(y) \phi(x - y),
\]

(46)

where \( x = \bar{k} \sqrt{\nu t} = \bar{k}/k_{\text{diss}} \). In the inertial range, the viscous term in not effective and can be neglected. Also, and for the sake of simplicity, we will take the limit \( k_i/k_{\text{diss}} \to 0 \) in the bound of the integral in (46). In this case, the Fourier transform of Eq. (46) gives

\[
2\dot{\phi} \ddot{\phi} + \phi' \dot{\phi} + \dot{\phi} = 0,
\]

(47)

where \( \dot{\phi}(\omega) \) is the Fourier transform of \( \phi(x) \) and a prime indicates the derivative with respect to \( \omega \). Observe that \( \dot{\phi}(\omega) \) is exactly the scaling function introduced in Sect. 4 [see Eq. (29)] and Eq. (47) corresponds to Eq. (30) when the dissipative term (proportional to \( \dot{\phi}' \)) is neglected.

Equation. (47) admits two solutions

\[
\dot{\phi}(\omega) = -\frac{1}{2} \left( \omega \pm \sqrt{\omega^2 + C^2} \right)
\]

(48)
specified by the ± sign, where \( C \in \mathbb{R} \) is a constant of integration. We can neglect the first term in Eq. (48) since its inverse Fourier transforms is proportional to the derivative of the \( \delta(x) \) function and we are assuming that \( |x| = k/k_{\text{dis}} > k_{i}/k_{\text{dis}} \neq 0 \). In this case, taking the inverse Fourier transform of Eq. (48) gives

\[
\phi(x) = \pm \frac{C}{\sqrt{2\pi|x|}} K_1(C|x|),
\]

(49)

where \( K_1(x) \) is the modified Bessel function of the second kind \([11]\).

For small values of |x|, \( C|x| \ll 1 \), the asymptotic expansion of \( \phi(x) \) is

\[
\phi(x) = \pm \frac{1}{\sqrt{2\pi x^2}}
\]

(50)

at the leading order in \( x \). This gives the Batchelor spectrum (7) with \( c = 1/2 \).

**Dissipative range** This range is defined by the condition \( k \geq k_{\text{dis}} \). We can distinguish two “sub-dissipative ranges”: In the “pre-viscous damping range”, the rate of energy transfer due to nonlinear interactions is not negligible. In this case, we expect a “slower” decay of the energy than the “fast” decay in the purely viscous case. In the “viscous damping range”, mathematically defined by \( k/k_{\text{dis}} \rightarrow \infty \), instead, the rate of energy transfer due to nonlinear interactions is absent and the dynamics is dominated by viscosity.

**Pre-viscous damping range.** – At the leading order, the asymptotic expansion of \( \phi(x) \) for large |x|, \( C|x| \gg 1 \), is

\[
\phi(x) = \pm \frac{\sqrt{C}}{2|x|^{3/2}} e^{-C|x|}.
\]

(51)

This gives the energy spectrum

\[
E(k, t) = \frac{\pi C k^{1/2}}{4k^2 T^{1/2}/2} e^{-2ck^{-1/3}}.
\]

(52)

The above result is compatible with the results of Tatsumi and Yanase discussed in Sect. 2.

**Viscous damping range.** – In this range, we can neglect the right-hand side of Eq. (46). Accordingly, we get

\[
\phi(x) = \sqrt{c_{\infty}} \frac{\pi}{e^{-x}} e^{-x}, \quad \text{where} \quad c_{\infty} > 0 \quad \text{is a constant.}
\]

This gives the energy spectrum (8) in the purely viscous case.

The viscous term is negligible when the second term in Eq. (46) is negligible with respect to the first one, \( |d \ln \phi/k^2| \gg 1 \). Inserting Eq. (49), this is the case when \( K_1(C|x|)/K_2(C|x|) \ll C/2|x| \) or, using the asymptotic expansion of \( K_1(x) \) for large and small arguments, when \( |x| \leq 1 \) if \( C|x| \leq 1 \) or \( |x| \leq C/2 \) if \( C|x| \geq 1 \). This in turns gives two possibilities for the scaling function \( \psi(|x|) = \pi(|\phi(x)|)^2 \):

\[
\psi(|x|) = \begin{cases} \frac{1}{2\pi} e^{-|x|}, & |x| \leq \min (1/C, 1), \\ c_{\infty} e^{-2|x|}, & |x| \geq \min (1/C, 1), \end{cases}
\]

(53)

if \( C \leq \sqrt{2} \) or

\[
\psi(|x|) = \begin{cases} \frac{1}{2\pi} e^{-|x|}, & |x| \leq 1/C, \\ \frac{C}{4|x|^2} e^{-2|x|}, & 1/C \leq |x| \leq C/2, \\ c_{\infty} e^{-2|x|}, & |x| \geq C/2, \end{cases}
\]

(54)

if \( C \geq \sqrt{2} \). We can get an estimate of \( c_{\infty} \) if we impose the continuity of \( \psi(|x|) \) at \( |x| = \min (1/C, 1) \) for the first case \( (C \leq \sqrt{2}) \) and at \( |x| = C/2 \) for the second case \((C \geq \sqrt{2})\).

Roughly speaking, we get that \( c_{\infty} \) is of order unity if \( C \ll \sqrt{2} \) and exponentially suppressed, \( c_{\infty} \sim e^{-C^2/2} \), if \( C \gg \sqrt{2} \). Accordingly, the exponential tail \( c_{\infty} e^{-2|x|} \) in the spectrum (54) is exponentially small when \( C \) is large, and then difficult to see both in experiments and direct numerical simulations of two-dimensional turbulence.

**Large scales** In the limit \( k \rightarrow 0 \), dissipation is not effective. Taking into account the Olesen’s arguments and Eq. (44), we can search for solutions of Eq. (21) of the form

\[
u(\kappa, t) = u_0\sqrt{T} \Phi(\kappa^2\tau^p),
\]

(55)

where \( \kappa = \kappa/k_i, q = (1 + p)/2, \) and \( \Phi(x) \) is a dimensionless function of its argument. Inserting the above equation in Eq. (21) and neglecting the term proportional to \( \nu \), we find

\[
q \frac{d^2\Phi(z)}{dz^2} + p\Phi(z) = -\frac{i}{2\sqrt{2\pi}} z \int_{-\infty}^{+\infty} dz' \Phi(z')\Phi(z - z'),
\]

(56)

where \( z = \kappa^2\tau^p \). In the following, and for the sake of simplicity, we will take the limit \( \tau = T/T_i \rightarrow \infty \) in the bounds of the integral in Eq. (56) and we will assume that \( q > 0 \), namely

\[
p > -1.
\]

(57)

For \( z \rightarrow 0 \) (corresponding to \( k \rightarrow 0 \)), we can write \( \Phi(z - z') \approx \Phi(-z') \). The last assumption is true if \( |\Phi(z)|^2 \) is integrable at \( z = 0 \). Accordingly, to the leading order in \( z \rightarrow 0 \), the right-hand side of Eq. (56) can be approximated by \( i\Gamma_0 z \), where \( \Gamma_0 = -(1/2\sqrt{2\pi}) \int_{-\infty}^{+\infty} dz|\Phi(z)|^2 \) (assuming that the integral exists). We then find, to the leading order in \( z \rightarrow 0 \),

\[
\Phi(z) = \begin{cases} 3\Gamma_0 (1/3 - i\frac{z}{2} + \ln z), & p = -1/3, \\ \frac{i\Gamma_0}{p+q} + C_p e^{-p/q}, & p \neq -1/3, \end{cases}
\]

(58)
where $C_{-1/3}$ and $C_p$ are and dimensionless constants of integration. The condition $\Phi(\tau) = \Phi^*(\tau)$ implies that $C_{-1/3} \in \mathbb{R}$, and that $C_p \in \mathbb{R}$ if $p/2q \in \mathbb{Z}$, $C_p$ is purely imaginary if $(p/q - 1)/2 \in \mathbb{Z}$, and Im $[C_p]/\text{Re} [C_p] = \tan(p/2q)$, otherwise.

For $p = -1/3$, the energy spectrum is given, at the leading order, by

$$E(k, t) = c_{-1/3} k^3 \ln^2 \left[ \frac{k}{k_i t_i} \right]^{1/3}, \quad (59)$$

where $c_{-1/3} = 9 \pi u^2 L_i^2 \Gamma_0^2$. For $p \neq -1/3$, we have two cases:

(i) If $C_p = 0$, the energy spectrum is given by

$$E(k, t) = c_p t^{1+3p} k^3, \quad (60)$$

where $c_p = \pi u^3 L_i^{1+3p} C_p/(1 + 3p)^2$. The cases $E(k, t) \propto t^3 k^3$ with $p = 1, 2, 4$ found in the literature (see [3]), then correspond to $p = 0, 1/3, 1/2, 1$, respectively. Observe, also, that the energy spectrum at large scales is increasing in time for $p > -1/3$ and decreasing in time for $p < -1/3$. Finally, observe that for $p = 1$ the energy spectrum does not depend on $L_i$. In this case $E(k, t) \propto t^3 k^3$. For $p = -1$, instead, the energy spectrum does not depend on $u_i$.

In this case $E(k, t) \propto t^{-2} k^3$.

(ii) If $C_p \neq 0$, the integrability condition of $\Phi(z)^2$ at $z = 0$ implies that $-1 < p < 1$. This in turns gives two cases:

(a) For $-1 < p < -1/3$, we have $-p/q > 1$, so that $\Phi(z) = i t^q \sqrt{z}/(p + q)$ at the leading order. The energy spectrum is then given by Eq. (60).

(b) For $-1/3 < p < 1/3$, instead, $\Phi(z) = C_p z^{-3p/4}$ at the leading order. The energy spectrum is then given by

$$E(k, t) = c'_p k^{6p}, \quad (61)$$

where $c'_p = \pi u^2 L_i^{2(1-p)/(1+3p)}|C_p|^2$ and $a_p = (1 - 3p)/(1 + p)$. Imposing that the Saffman integral is finite we get $a_p \geq 1$, corresponding to $p \leq 0$. Accordingly, $1 \leq a_p < 3$.

Particularly important is the case $p = 0$ (see the Appendix for a different derivation of the expression of the energy spectrum in this case). This is the only case where $\Phi(0)$ is finite and different form zero. In this case, we have the Saffman spectrum

$$E(k, t) = c_0 k, \quad (62)$$

where $c_0 = \pi u^2 L_i^2 |\Phi(0)|^2$.

6 Conclusions

In this paper, we have proposed a (continuous) one-dimensional, toy model for two-dimensional turbulence. Starting by the well-known (discrete) shell model of hydrodynamics, we have constructed such a model by adding all possible nonlocal interaction terms between velocity modes in Fourier space, and then by taking its continuous limit. In real space, such a model corresponds to the one-dimensional Burgers equation.

As it is well-known in the literature, Burgers equation cannot describe the transition from an initial state to a state of fully developed hydrodynamic turbulence. This is because the solution of Burgers equation with given Dirichlet boundary conditions is unable to describe the spontaneous arise of randomness by chaotic dynamics.

Nevertheless, we have shown that particular asymptotic scaling solutions of Burgers equation, which do not satisfy Dirichlet boundary conditions, do indeed well describe the behaviour of a two-dimensional fluid in a turbulent state, after the transition to chaotic dynamics has happened. Such scaling solutions successfully explain the main characteristic of the kinetic energy spectrum at both small, intermediate, and large scales, as seen in both direct numerical simulations and laboratory experiments. In particular:

(i) If the velocity field admits a Maclaurin expansion at small wavenumbers, then the energy spectrum is that

Analytical case. – If one assumes that $\Phi(z)$ is analytical at $z = 0$, namely it admits a Maclaurin expansion, then it is easy to see that the form of energy spectrum is that given in Eq. (10) with $L = c_0 /4 \pi$ and $I(t) = c_p \Gamma_0$, where $p \neq -1/3$, and $c_p$ and $c_0$ are the same as those previously defined.

Absolute state Let us conclude this section by observing that, in the inviscid limit, Eq. (21) admits a static solution of the form

$$u(k, t) = \frac{e^{ik} a}{\sqrt{a + bk^2}}, \quad (63)$$

where $\theta \in \mathbb{R}$, $a > 0$, and $b > 0$ are constants. This solution corresponds to the static energy spectrum

$$E(k, t) = \frac{\pi k}{a + bk^2}. \quad (64)$$

The above spectrum has, correctly, the form of the “canonical” distribution for two-dimensional turbulence first discussed by Kraichnan [12] (see, also, [2]), with $a$ and $b$ related to the inviscid invariants $E$ and $\Omega$.
given in Eq. (10) as theoretically predicted, under particular assumptions, in the exact two-dimensional theory. However, our model also admits the possible existence of a time-independent, power-law spectrum of the form given in Eq. (61).

(ii) The model predicts the existence of an inertial range, where the dynamics in independent on initial conditions and viscosity. Moreover, the energy spectrum is correctly given by the Batchelor spectrum (7).

(iii) At small scales, corresponding to the dissipative range, the model admits the existence of two different sub-dissipative ranges. In the “pre-viscous damping range” [see Eq. (52)], where we expect the rate of energy transfer due to nonlinear interactions to be not negligible, the energy decays slower than in the “viscous damping range” [see Eq. (8)], where dynamics is dominated by viscosity.

In conclusion, the results of our analysis of freely-decaying two-dimensional turbulence by means of a simple and solvable, one-dimensional toy model, are in agreement with numerical simulations and laboratory experiments, open the theoretical possibility of novel power-law type energy spectra at large scales, and indicate the existence of a pre-viscous damping range. To our knowledge, the existence of such a range has been discussed in the literature only once, in 1981 by Tatsumi and Yanase [6]. Needless to say, further investigations are needed to confirm the existence and characteristics of this peculiar range in freely-decaying two-dimensional turbulence, as well as the possibility of different power laws for the energy spectra at large scales.

Declarations

Conflict of interest None.

Appendix: Saffman spectrum from Burgers equation

For $p = 0 \ (q = 1/2)$ Eq. (55) reads

$$u(\vec{k}, t) = u_i L_i \Phi(\vec{k}/u_i L_i t)$$

and, in turns, Eq. (56) becomes

$$z \frac{d\Phi(z)}{dz} = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz' \Phi(z') \Phi(z - z'),$$

where $z = \vec{k}/u_i L_i t$. Taking the Fourier transform of the above equation, we find

$$2\Phi \Phi' + \sigma \Phi' + \Phi = 0,$$

where $\Phi(\sigma)$ is the Fourier transform of $\Phi(z)$ and a prime indicates the derivative with respect to $\sigma$. Equation (67) admits two solutions

$$\Phi(\sigma) = -\frac{1}{2} \left[ \sigma \pm \sqrt{\sigma^2 + c^2} \right]$$

specified by the ± sign, where $c \in \mathbb{R}$ is a constant of integration. Since the limit $\vec{k} \to 0$ corresponds to $\sigma \to \pm \infty$ (large scales), we must take the minus sign in Eq. (68) in order to have a finite velocity field and then a finite energy spectrum. In this case, we have

$$\Phi(\sigma) = \frac{c^2}{4\sigma} + O(\sigma^{-3}).$$

The inverse Fourier transform of the above equation reads

$$\Phi(z) = -\frac{i\sqrt{\pi}c^2}{4\sqrt{2}} \operatorname{sgn}(z) + O(z^2),$$

where $\operatorname{sgn}(\lambda)$ is the sign function. The leading term of the energy spectrum at large scales is then given by Eq. (62) with $c_0 = \pi^2 c^4 u_i L_i^2/32$. Comparing this expression for $c_0$ with the one after Eq. (62), we find that $c$ is related to $\Phi(0)$ by $c = 2(2/\pi)^{1/4} |\Phi(0)|^{1/2}$.

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