MULTIPHOTON PRODUCTION AT HIGH ENERGIES IN THE STANDARD MODEL II

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ABSTRACT

We examine multiphoton production in the electroweak sector of the Standard Model in the high energy limit using the equivalence theorem in combination with spinor helicity techniques. We utilize currents consisting of a charged scalar, spinor, or vector line that radiates $n$ photons. Only one end of the charged line is off shell in these currents, which are known for the cases of like-helicity and one unlike-helicity photons. We obtain a wide variety of helicity amplitudes for processes involving two pairs of charged particles by considering combinations of four currents. We examine the situation with respect to currents which have both ends of the charged line off-shell, and present solutions for the case of like-helicity photons. These new currents may be combined with two of the original currents to produce additional amplitudes involving Higgs, longitudinal $Z$ or neutrino pairs.

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I. INTRODUCTION

In this paper we will conclude a study of the high energy scatterings involving many vector bosons and Higgs bosons in a spontaneously symmetry-broken gauge theory begun in references [1] and [2]. This program is a generalization of the work on QCD by Berends and Giele [3]. In particular, we will consider the Weinberg-Salam-Glashow model [4], with a focus upon processes involving an arbitrary number of photons, plus one or two pairs of charged particles ($W^+W^−$, $\ell^\pm W^\mp$, or $\ell^+\ell^-$), and possibly one or two $Z$’s, Higgs bosons, or neutrinos.

Our work is based upon three main ideas: the equivalence theorem [5], the multispinor [6,7,8, 9] representation of a vector field, and the use of recursion relations [1,2,3]. The equivalence theorem allows us to identify the longitudinal degrees of freedom of the $W^\pm$ and $Z$ bosons with the corresponding would-be Goldstone bosons $\phi^\pm$ and $\phi_2$, up to corrections of the order of the vector boson mass divided by the center-of-mass energy. The multispinor representation of a vector field allows us to treat fermions and vector bosons on an equal footing by replacing the conventional Lorentz 4-vector with a second rank spinor which may be thought of as a combination of two spin-$\frac{1}{2}$ objects. We will use Weyl-van der Waerden spinors in this work (see Appendix A for a summary of our conventions).

Reference [2] develops recursion relations for currents consisting of a single charged line, with $n$ on-shell photons attached. This line could be scalar, spinor, or vector in nature, and has one end off shell. It is possible to solve the recursion relations in the cases where either all, or all but one, of the photons have the same helicity. From the solutions for the currents, we may obtain amplitudes for various processes. Those amplitudes which are calculable from the currents
taken either singly or in pairs are discussed in reference [2]. In this paper, we will

discuss processes involving combinations of four currents, such as

\[ e^+ e^- \rightarrow W^+ W^- \gamma \ldots \gamma. \]

By choosing various combinations of the four currents, a wide range of processes
is covered. All of these amplitudes involve diagrams which contain some type of
neutral particle propagating between the charged lines. The next logical question
to ask concerns processes for which the propagating particle itself carries a charge.
In that case, charge conservation dictates that we combine three currents, one of
which has two off-shell particles. The complications involved in this case is the
topic of the remainder of the paper.

We have organized our presentation as follows. Section 2 is a brief review
of the electroweak recursion relations and their solutions. All of the ingredients
necessary to compute quadruple current amplitudes like (1.1) are presented there.
Section 3 describes, using an explicit example, how to perform calculations in-
volving four currents. In addition, the results for the helicity combinations we are
able to solve for are collected there. Next, we proceed to discuss the prospects
for obtaining double-off-shell currents. As we see in Section 4, although we can-
not always obtain expressions for the currents themselves, we are able to obtain
enough information to compute the desired amplitudes. Section 5 presents an
example of such a computation, along with the rest of the results involving three
currents. The final section contains a few concluding remarks.
II. THE ELECTROWEAK RECURRENCE RELATIONS AND THEIR SOLUTIONS

In this section we will review the recursion relations for currents containing a charged line plus \( n \) photons presented in reference [2]. All of the photons will be on shell, and one end of the charged line will be off shell. Within the framework of the Weinberg-Salam-Glashow model, the charged line can have spin 0, spin \( \frac{1}{2} \), or spin 1. We will consider each possibility in turn.

2.1 The longitudinal \( W \) current

We define the quantity \( \Phi(P; 1, 2, \ldots, n) \) to represent the sum of all tree-level diagrams consisting of an unbroken scalar line (the would-be Goldstone bosons \( \phi^\pm \)) with \( n \) photons attached in all possible ways. By the equivalence theorem, this represents a longitudinally polarized \( W \equiv W_L \) that radiates \( n \) photons. Choose the convention that all momenta flow into the diagram. Then, the on-shell \( \phi^+ \) has momentum \( P \), the \( n \) on-shell photons have momenta \( k_1, k_2, \ldots, k_n \), and the off-shell \( \phi^- \) has momentum \( Q = -[P + k_1 + \ldots + k_n] \equiv -[P + \kappa(1,n)] \).

In reference [2], we show that \( \Phi(P; 1, 2, \ldots, n) \) satisfies the recursion relation

\[
\Phi(P; 1, 2, \ldots, n) = \frac{-e \sqrt{2}}{[P + \kappa(1,n)]^2} \left[ \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \epsilon^{\hat{\alpha}\alpha}(n)[P + \kappa(1,n)]_{\alpha\hat{\alpha}} \Phi(P; 1, 2, \ldots, n-1) \right.
\]

\[
+ e \sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{2! (n-2)!} \epsilon^{\hat{\alpha}\alpha}(n-1) \epsilon_{\alpha\hat{\alpha}}(n) \Phi(P; 1, 2, \ldots, n-2) \right]
\]

(2.1)

where the symbol \( \mathcal{P}(1\ldots n) \) denotes the set of permutations of the momenta \( k_1, k_2, \ldots, k_n \). This form of the recursion relation is permitted by the Bose symmetry enjoyed by the photons. The sources of the two terms in (2.1) are obvious:
the first term is from the three-point vertex, while the second term is generated by the seagull vertex. By definition

\[ \Phi(P) \equiv 1. \quad (2.2) \]

We denote by \( \Phi(1, 2, \ldots, n; Q) \) the same current, but with the \( \phi^+ \) off shell instead. The two currents are connected in the expected manner:

\[ \Phi(1, 2, \ldots, n; Q) = (-1)^n \Phi(Q; 1, 2, \ldots, n). \quad (2.3) \]

Closed form solutions to the recursion relation (2.1) are known for two special helicity configurations. If all of the photons have the same helicity, or all but one of the photons have the same helicity, then it is possible to choose the gauge momenta such that

\[ \bar{\epsilon}^{\dot{\alpha}}(j)\epsilon_{\alpha\dot{\alpha}}(\ell) = 0 \quad (2.4) \]

for any pair of polarization spinors. The advantage of having (2.4) hold is the vanishing of all of the seagull contributions to (2.1), leaving what is effectively a single-term recursion relation. Thus, to obtain \( \Phi(P; 1^+, \ldots, n^+) \), we choose

\[ \epsilon_{\alpha\dot{\alpha}}(j^+) \equiv \frac{u_\alpha(g)\bar{u}_{\dot{\alpha}}(k_j)}{\langle j\ g \rangle}, \quad (2.5) \]

with the same gauge spinor \( g \) for each of the photons. In this case, the solution to (2.1) is [2]

\[ \Phi(P; 1^+, \ldots, n^+) = (-e\sqrt{2})^n \sum_{P(1\ldots n)} \frac{\langle P\ g \rangle}{\langle P|1, \ldots, n|g \rangle}, \quad (2.6) \]

where

\[ \langle P|1, 2, \ldots, n|g \rangle \equiv \langle P\ 1 \rangle\langle 1\ 2 \rangle \cdots \langle n\ g \rangle. \quad (2.7) \]

Some useful properties of this “string” of spinor inner products are given in Appendix A. This expression has the proper \( n = 0 \) limit to match smoothly onto (2.2).
Next, consider the case of one photon with differing helicity. For concreteness, let us choose that photon to carry momentum $k_1$. If we choose $g = k_1$ in (2.5) for $j = 2, \ldots, n$, and set

$$\epsilon_{a\dot{a}}(1^-) \equiv \frac{u_\alpha(k_1)\bar{u}_{\dot{a}}(h)}{\langle 1'h \rangle^*},$$

(2.8)

with $h$ an arbitrary null-momentum, it is not hard to see that (2.4) still holds. The solution to (2.1) for this case reads [2]

$$\Phi(P; 1^-, 2^+, \ldots, n^+) =
\begin{equation}
= -(-e\sqrt{2})^n \sum_{P(2\ldots n)} \frac{\langle P 1 \rangle}{\langle P|2, \ldots, n|1 \rangle!} \left\{ \frac{\langle h P \rangle^*}{\langle h|1 P \rangle^*}
\right. \\
+ (1 - \delta_{n1}) \sum_{j=2}^{n} a^\alpha(k_1)\Pi_{\alpha\beta}(P, 1, 2, \ldots, j)u_{\beta}(k_1) \left\}
\right.
\end{equation}$$

(2.9)

Notice that since the first photon is distinguishable by its helicity, it has been explicitly removed from the permutation sum. The quantity $\Pi$ appearing in (2.9) is defined by

$$\Pi_{\alpha\beta}(P, 1, 2, \ldots, j) \equiv \frac{(k_j)_{a\dot{a}}[\bar{P} + \kappa(1, j)]^{\dot{a}\beta}}{[P + \kappa(1, j-1)]^2[P + \kappa(1, j)]^2}.$$  

(2.10)

Some useful properties of $\Pi$ are listed in Appendix A. Equation (2.9) is valid for all $n \geq 1$.

### 2.2 The fermion currents

Next, we consider the spin-$\frac{1}{2}$ case. We define the quantity $\bar{\psi}(p; 1, 2, \ldots, n)$ to represent the sum of all tree-level diagrams consisting of an unbroken fermion line with $n$ photons attached in all possible ways. Again, all momenta flow into the diagram. The on-shell positron has momentum $p$, the $n$ on-shell photons have momenta $k_1, k_2, \ldots, k_n$, and the off-shell electron has momentum $q = -[p + \ldots$. 

κ(1, n)]. Berends and Giele [3] obtain recursion relations for \( \bar{\psi} \), which may be cast in the form [2]

\[
\bar{\psi}_\alpha(p^+; 1, 2, \ldots, n) = -e\sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \bar{\psi}_\beta(p^+; 1, 2, \ldots, n-1) \bar{\epsilon}^{\beta\beta}(n) \frac{[p + \kappa(1, n)]\beta\dot{\alpha}}{|p + \kappa(1, n)|^2}, \tag{2.11a}
\]

\[
\bar{\psi}_\alpha(p^-; 1, 2, \ldots, n) = -e\sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \bar{\psi}_\beta(p^-; 1, 2, \ldots, n-1) \bar{\epsilon}^{\beta\beta}(n) \frac{[\bar{p} + \bar{\kappa}(1, n)]\beta\dot{\alpha}}{|\bar{p} + \bar{\kappa}(1, n)|^2}. \tag{2.11b}
\]

In the massless limit considered here, the helicities of all fermions are conserved; hence the pair of recursion relations. The zero-photon currents are

\[
\bar{\psi}_\dot{\alpha}(p^+) \equiv \bar{u}_{\dot{\alpha}}(p) \tag{2.12a}
\]

and

\[
\bar{\psi}_\alpha(p^-) \equiv u^\alpha(p). \tag{2.12b}
\]

We may also define currents in which the positron is off shell and the electron is on shell. These will be denoted by \( \psi(1, 2, \ldots, n; q) \) and are related to the \( \bar{\psi} \)'s by

\[
\psi^{\dot{\alpha}}(1, 2, \ldots, n; q^+) = (-1)^n \bar{\psi}^{\dot{\alpha}}(q^+; 1, 2, \ldots, n), \tag{2.13a}
\]

\[
\psi_\alpha(1, 2, \ldots, n; q^-) = (-1)^n \bar{\psi}_\alpha(q^-; 1, 2, \ldots, n), \tag{2.13b}
\]

as required by charge-conjugation symmetry.

The gauge choice (2.5) allows us to solve the recursion relations (2.11) for like-helicity photons, with the results [2]

\[
\bar{\psi}_\dot{\alpha}(p^+; 1^+, \ldots, n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{-u^\beta(g)[p + \kappa(1, n)]\beta\dot{\alpha}}{\langle p|1, 2, \ldots, n|g \rangle}, \tag{2.14a}
\]

\[
\bar{\psi}_\alpha(p^-; 1^+, \ldots, n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{u^\alpha(p) \langle p g \rangle}{\langle p|1, 2, \ldots, n|g \rangle}. \tag{2.14b}
\]
Notice that except for the required spinor structure, the solutions (2.14) contain the same functional form as the solution (2.6) for $\Phi(P; 1^+, \ldots, n^+)$. That is, currents of differing spins are proportional to each other, a SUSY-like relation. The $n = 0$ forms of (2.14) match smoothly onto (2.12).

To obtain solutions when the first photon has negative helicity, set $g = k_1$ in (2.5), and take (2.8) for $\epsilon_{\alpha \dot{\alpha}}(1^-)$. Then, the following solutions result:

\[
\bar{\psi}_\alpha(p^+, 1^-, 2^+, \ldots, n^+) = \left( -e \sqrt{2} \right)^n \sum_{P(2 \ldots n)} \frac{u^\beta(k_1)p + \kappa(1, n)]_{\beta \dot{\alpha}}}{\langle p|2, \ldots, n|1 \rangle} \left\{ \frac{\langle h | p \rangle^*}{\langle h|1|p \rangle^*} \right\} + (1 - \delta_{n1}) \sum_{j=2}^n u^\alpha(k_1) \Pi^\beta_\alpha(p, 1, 2, \ldots, j) u_\beta(k_1),
\]

\[
\bar{\psi}_\alpha(p^-, 1^-, 2^+, \ldots, n^+) = \left( -e \sqrt{2} \right)^n \sum_{P(2 \ldots n)} \frac{-\langle p | 1 \rangle}{\langle p|2, \ldots, n|1 \rangle} \left\{ \frac{\langle h | p \rangle^*}{\langle h|1|p \rangle^*} u^\alpha(p) - \frac{u^\alpha(k_1)}{\langle p | 1 \rangle^*} \right\} + (1 - \delta_{n1}) \langle p | 1 \rangle \sum_{j=2}^n u^\beta(k_1) \Pi^\alpha_\beta(p, 1, 2, \ldots, j) \right\}.
\]

Even though the fermionic case has no seagulls to dispose of, and hence a simpler recursion relation than the scalar case, it is still not possible to solve for currents containing more than one unlike helicity.

### 2.3 The transverse $W$ currents

Finally, we consider spin-1 currents. We define $W(P; 1, 2, \ldots, n)$ to represent the sum of all tree-level diagrams consisting of an unbroken vector line with $n$ photons attached in all possible ways. As usual, all momenta flow into the diagram. The on-shell $W^+$ has momentum $P$, the $n$ on-shell photons have
momenta \( k_1, k_2, \ldots, k_n \), and the off-shell \( W^- \) has momentum \( Q = -[P + \kappa(1, n)] \).

In Lorentz 4-vector notation the recursion relation for \( W \) reads [2]
\[
W^\mu(P; 1, 2, \ldots, n) = -e \left[ \frac{1}{[P + \kappa(1, n)]^2} \sum_{P(1 \ldots n)} \frac{1}{(n-1)!} \left[ \epsilon(n), W(P; 1, 2, \ldots, n-1) \right]^\mu \right. \\
+ e \left. \sum_{P(1 \ldots n)} \frac{1}{2! (n-2)!} \left\{ \epsilon(n-1), W(P; 1, 2, \ldots, n-2) \epsilon(n) \right\}^\mu \right],
\]
where
\[
\left[ \epsilon(n), W(P; 1, 2, \ldots, n-1) \right]^\mu \equiv \\
= 2[P + \kappa(1, n-1)] \cdot \epsilon(n) W^\mu(P; 1, 2, \ldots, n-1) \\
- 2k_n \cdot W(P; 1, 2, \ldots, n-1) \epsilon^\mu(n) \\
+ \epsilon(n) \cdot W(P; 1, 2, \ldots, n-1) \left[ k_n - [P + \kappa(1, n-1)] \right]^\mu,
\]
and
\[
\left\{ \epsilon(n-1), W(P; 1, 2, \ldots, n-2), \epsilon(n) \right\}^\mu \equiv \\
= \epsilon(n-1) \left[ \epsilon(n)W^\mu(P; 1, 2, \ldots, n-2) - W(P; 1, 2, \ldots, n-2)\epsilon^\mu(n) \right] \\
- \epsilon(n) \left[ W(P; 1, 2, \ldots, n-2)\epsilon^\mu(n-1) - \epsilon(n-1)W^\mu(P; 1, 2, \ldots, n-2) \right].
\]

The current \( W \) is a conserved quantity, satisfying [2]
\[
[P + \kappa(1, n)] \cdot W(P; 1, 2, \ldots, n) = 0.
\]

The current \( W(1, 2, \ldots, n; Q) \), with an on-shell \( W^- \) of momentum \( Q \) and an off-shell \( W^+ \) is related to \( W(Q; 1, 2, \ldots, n) \) by
\[
W(1, 2, \ldots, n; Q) = (-1)^n W(Q; 1, 2, \ldots, n).
\]

The recursion relation is soluble if all of the particles have the same helicity or only one helicity differs. If all of the particles have positive helicity, we choose (2.5) for the photons and
\[
W_{\alpha\dot{\alpha}}(P^+) = \frac{u_{\alpha}(g)\bar{u}_{\dot{\alpha}}(P)}{\langle P\ g \rangle}.
\]
It is not hard to show that with this choice of gauge spinors, not only does (2.4) hold, but also

\[ \bar{\epsilon}^{\dot{\alpha}}(j^+) W_{\alpha\dot{\alpha}}(P^+; 1^+, \ldots, n^+) = 0, \]  

(2.22)

for all \( j \) and \( n \). Because of (2.4) and (2.22), the seagull contributions represented by the curly bracket function (2.18) vanish, and the square bracket function (2.17) reduces to

\[ \left[ \epsilon(n), W(P; 1, 2, \ldots, n-1) \right]^\mu = 
\]

\[ = 2[P + \kappa(1, n-1)] \cdot \epsilon(n) W^\mu(P; 1, 2, \ldots, n-1) \]

(2.23)

\[ - 2k_n \cdot W(P; 1, 2, \ldots, n-1) \epsilon^\mu(n). \]

These simplifications are sufficient to allow solution of the recursion relation (2.16), with the result [2]

\[ W_{\alpha\dot{\alpha}}(P^+; 1^+, \ldots, n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1 \ldots n)} \frac{-u_\alpha(g)u^\beta(g)[P + \kappa(1, n)]_{\beta\dot{\alpha}}}{\langle P \mid g \rangle \langle P[1, 2, \ldots, n|g} \]  

(2.24)

Once more, the same functional form as in the scalar case appears, with the added spinor structure required to describe a spin-1 particle. Equation (2.24) reduces to (2.21) for \( n = 0 \).

If we set \( g = k_1 \) in (2.5) and (2.21), and use the choice (2.8) for the first photon, then we are able to obtain \( W(P^+; 1^-, 2^+, \ldots, n^+) \). Because all of the polarization spinors are proportional to \( u_\alpha(k_1) \), the key properties (2.4) and (2.22) still hold, and we may eliminate the seagull contributions and use (2.23) when solving the recursion relation. The result is not surprising [2]:

\[ W_{\alpha\dot{\alpha}}(P^+; 1^-, 2^+, \ldots, n^+) = 
\]

\[ = \left( -e\sqrt{2} \right)^n \sum_{\mathcal{P}(2 \ldots n)} \frac{u_\alpha(k_1)u^\beta(k_1)[P + \kappa(1, n)]_{\beta\dot{\alpha}}}{\langle P \mid k_1 \rangle \langle P[2, \ldots, n|k_1} \left\{ \frac{\langle h \mid P \rangle^*}{\langle h \mid 1 \rangle \langle P \rangle^*} \right\} 
\]

(2.25)

\[ + (1 - \delta_{n1}) \sum_{j=2}^{n} u^\gamma(k_1)\Pi_\gamma^\delta(P, 1, 2, \ldots, j)u_\delta(k_1) \right\}, \]
valid for \( n \geq 1 \).

Instead of allowing one of the photons to have negative helicity, we may choose to let the \( W \) have negative helicity. In this case, we set \( g = P \) in (2.5) and write

\[
W_{\alpha \dot{\alpha}}(P^-) = \frac{u_\alpha(P) \bar{u}_{\dot{\alpha}}(h)}{\langle P \ h \rangle^*}.
\] (2.26)

The recursion relation simplifies as before, and we easily obtain [2]

\[
W_{\alpha \dot{\alpha}}(P^-; 1^+) = -e \sqrt{2} \frac{u_\alpha(P) \bar{u}_{\dot{\alpha}}(k_1) \langle h \ 1 \rangle^*}{[P + k_1]^2 \langle P \ h \rangle^*},
\] (2.27a)

\[
W_{\alpha \dot{\alpha}}(P^-; 1^+, \ldots, n^+) = (-e \sqrt{2})^n \sum_{P(1, \ldots, n)} \frac{\langle h \ 1 \rangle^* u_\alpha(P) u^{\dot{\beta}}(P) [P + \kappa(1, n)]_{\beta \dot{\alpha}}}{\langle P \ h \rangle^* \langle P \ 1 \rangle \langle P \ 2, \ldots, n \rangle \langle P \rangle} \times \sum_{\ell=2}^{n} u^{\gamma}(P) \Pi^{\gamma \delta}(P, 1, \ldots, \ell) u_{\delta}(P).
\] (2.27b)

Note that (2.27b) holds only for \( n \geq 2 \). For \( n = 1 \) we use (2.27a) and for \( n = 0 \) we use (2.26).

III. QUADRUPLE CURRENT AMPLITUDES

Since we have already discussed those amplitudes which may be obtained from the currents taken either singly or in pairs [2], we begin with an examination of those processes which may be computed from the combination of four currents. Each process involves a pair of charged lines, and the corresponding Feynman diagrams have the basic topology of Figure 1. Depending upon the identities of the four currents, variations upon this basic structure may be possible. In particular, note that the \( \phi^4 \) vertex enters directly when all four currents are scalars, and the strength of the coupling \( \lambda \) as compared to \( e \) becomes an issue when deciding which contributions are the most important.
The layout of this section is as follows. First, we will illustrate the techniques required to compute quadruple current amplitudes by discussing the process

$$e^+e^- \rightarrow W^+W^-\gamma\gamma\gamma\gamma.$$  

(3.1)

Then, we will tabulate the various results which may be obtained using the currents reviewed in the previous section.

### 3.1 The process $e^+e^- \rightarrow W^+W^-\gamma\gamma\gamma\gamma$

As illustrated in Figure 1, there are three basic contributions to the process (3.1). The first two contributions have unbroken $W$ lines with either a photon or a $Z$-boson connecting the $W$ line to the fermion line, as illustrated in the upper half of Figure 1. We will refer to these as the photon and $Z$ contributions respectively. The third diagram, which is only present when the fermion line is left-handed, consists of a broken $W$ line, with a neutrino propagating across the gap, as pictured in the lower half of Figure 1. This will be referred to as the neutrino contribution.

We will choose the convention that all of the momenta flow into the diagram. Hence, variations on the basic process (3.1) are easily obtained by crossing. The positron will have momentum $p$, the electron momentum $q$, the $W^+$ momentum $P$ and the $W^-$ momentum $Q$. The photons will have momenta labelled by $k_1, k_2, \ldots, k_n$. The first photon will be the one which is selected to carry negative helicity; the remaining $n - 1$ photons will all have positive helicity.
3.1.1 The photon contribution

Denote by $\mathcal{M}_\gamma$ those contributions to (3.1) that involve the exchange of a photon between the spinor and vector lines. From Figure 1, we see that we have

$$\mathcal{M}_\gamma(p, q; P, Q; 1, \ldots, n) =$$

$$= \sum_{\mathcal{P}(1\ldots n)} \sum_{r=0}^{n} \sum_{s=r}^{n} \sum_{t=s}^{n} \frac{W_\mu(P; 1, \ldots, r)}{r!} \frac{W_\nu(r+1, \ldots, s; Q)}{(s-r)!} \times (-ie)V_{\mu\nu\lambda}[P + \kappa(1, r), \kappa(r+1, s) + Q, -K]$$

$$\times \frac{-ig_{\lambda\sigma}}{K^2} \bar{\psi}(p; s+1, \ldots, t) \psi(t+1, \ldots, n; q)$$

$$+ \sum_{\mathcal{P}(1\ldots n)} \sum_{r=1}^{n} \sum_{s=r}^{n} \sum_{t=s}^{n} \frac{W_\mu(P; 1, \ldots, r-1)}{(r-1)!} \frac{W_\nu(r+1, \ldots, s; Q)}{(s-r)!} \times (-ie^2)S^{\mu\nu\lambda\rho}$$

$$\times \frac{-ig_{\lambda\sigma}}{K^2} \bar{\psi}(p; s+1, \ldots, t) \psi(t+1, \ldots, n; q)$$

$$\frac{\epsilon_\rho(r)}{(t-s)!} \frac{-ie_\sigma}{(n-t)!}$$

(3.2)

where we have used the notations

$$V^{\mu\nu\lambda}(k_1, k_2, k_3) = g^{\mu\nu}(k_1 - k_2)^\lambda + g^{\nu\lambda}(k_2 - k_3)^\mu + g^{\lambda\mu}(k_3 - k_1)^\nu$$

(3.3)

to designate the three-point vertex function and

$$S^{\mu\nu\lambda\rho} = 2g^{\mu\nu}g^{\lambda\rho} - g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho}$$

(3.4)

for the seagull vertex function. The momentum carried by the virtual photon is

$$K \equiv P + \kappa(1, s) + Q$$

$$= -[p + \kappa(s+1, n) + q].$$

(3.5)

For the helicity combination of interest, namely all vector bosons having positive helicity with the exception of a single photon, we know that each of the currents $W_{\alpha\dot{\alpha}}$ are proportional to $u_\alpha(k_1)$. Thus, all products of the form $W \cdot W'$ vanish.

Furthermore, since all of the polarization spinors are also proportional to $u_\alpha(k_1)$ as well, their dot products with any of the $W$’s is also zero. Since the metric tensor combinations appearing in (3.4) inevitably leading to the forms $W \cdot W'$ or $\epsilon \cdot W$,
these two important properties tell us that there are no seagull contributions to the amplitude. In addition, only two of the three terms appearing in (3.3) actually contribute. Thus, (3.2) reduces to

\[ M_\gamma(p,q;P,Q;1,\ldots,n) = 2ie^2 \sum_{P(1\ldots n)} \sum_{r=0}^{n} \sum_{s=r}^{n} \sum_{t=s}^{n} \frac{1}{r!(s-r)!(t-s)!(n-t)!} \frac{1}{K^2} \]

\[ \times \left\{ W_\mu(P;1,\ldots,r)W_\nu(r+1,\ldots,s;Q) - W_\nu(P;1,\ldots,r)W_\mu(r+1,\ldots,s;Q) \right\} \]

\[ \times \bar{\psi}(p; s+1,\ldots,t)\gamma^{\nu}\psi(t+1,\ldots,n;q)K^{\mu} \]

(3.6)

after applying current conservation and rearranging a bit. Use of the multispinor replacement rules (A.22) and (A.23) plus the Schouten identity (A.8) allows us to simplify further, yielding

\[ M_\gamma(p^+,q^-;P,Q;1,\ldots,n) = \]

\[ = -i(-e\sqrt{2})^2 \sum_{P(1\ldots n)} \sum_{s=0}^{n} \sum_{r=0}^{s} \sum_{t=s}^{n} \frac{1}{r!(s-r)!(t-s)!(n-t)!} \bar{\psi}_\beta(p^+; s+1,\ldots,t) \]

\[ \times \frac{\bar{K}^{\dot{\beta}\alpha}}{K^2} W_{\alpha\dot{\gamma}}(r+1,\ldots,s;Q)\bar{W}^{\dot{\gamma}\beta}(P;1,\ldots,r)\psi_\beta(t+1,\ldots,n;q^-), \]

(3.7)

where we have specialized to a left-handed fermion line.

In order to actually work with (3.7), it is necessary to explicitly write out the part of the permutation sum involving the first photon, since for the amplitude we wish to compute that photon is distinguishable by its negative helicity. The result is a sequence of four terms, corresponding to the possibility that this photon was radiated by any one of the four charged particles. At the same time, it is convenient to use the permutation sum so that the remaining photons radiated by that particle are numbered \( k_2,\ldots,k_r \). Thus, we find the
following four contributions to $\mathcal{M}_\gamma(p^+, q^-; P^+, Q^+; 1^-, 2^+, \ldots, n^+)$:

$$\mathcal{M}_{\gamma 1} \equiv -i(-e\sqrt{2})^2 \sum_{P(2\ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{s} \sum_{t=s}^{n} \frac{1}{(r-1)!(s-r)!(t-s)!(n-t)!}$$

$$\times \tilde{\psi}_\beta(p^+; (s+1)^+, \ldots, t^+) \frac{K^\beta\alpha}{K^2} W_{\alpha\bar{\alpha}}((r+1)^+, \ldots, s^+, Q^+)$$

$$\times W^{\bar{\alpha}\beta}(P^+; 1^-, 2^+, \ldots, r^+) \psi_\beta((t+1)^+, \ldots, n^+, q^-),$$

(3.8)

$$\mathcal{M}_{\gamma 2} \equiv -i(-e\sqrt{2})^2 \sum_{P(2\ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{s} \sum_{t=s}^{n} \frac{1}{(r-1)!(s-r)!(t-s)!(n-t)!}$$

$$\times \tilde{\psi}_\beta(p^+; (s+1)^+, \ldots, t^+) \frac{K^\beta\alpha}{K^2} W_{\alpha\bar{\alpha}}(1^-, 2^+, \ldots, r^+, Q^+)$$

$$\times W^{\bar{\alpha}\beta}(P^+; (r+1)^+, \ldots, s^+) \psi_\beta((t+1)^+, \ldots, n^+, q^-),$$

(3.9)

$$\mathcal{M}_{\gamma 3} \equiv i(-e\sqrt{2})^2 \sum_{P(2\ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{s} \sum_{t=s}^{n} \frac{1}{(r-1)!(s-r)!(t-s)!(n-t)!}$$

$$\times \tilde{\psi}_\beta(p^+; 1^-, 2^+, \ldots, r^+) \frac{K^\beta\alpha}{K^2} W_{\alpha\bar{\alpha}}((t+1)^+, \ldots, n^+, Q^+)$$

$$\times W^{\bar{\alpha}\beta}(P^+; (s+1)^+, \ldots, t^+) \psi_\beta((r+1)^+, \ldots, s^+, q^-),$$

(3.10)

$$\mathcal{M}_{\gamma 4} \equiv i(-e\sqrt{2})^2 \sum_{P(2\ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{s} \sum_{t=s}^{n} \frac{1}{(r-1)!(s-r)!(t-s)!(n-t)!}$$

$$\times \tilde{\psi}_\beta(p^+; (r+1)^+, \ldots, s^+) \frac{K^\beta\alpha}{K^2} W_{\alpha\bar{\alpha}}((t+1)^+, \ldots, n^+, Q^+)$$

$$\times W^{\bar{\alpha}\beta}(P^+; (s+1)^+, \ldots, t^+) \psi_\beta(1^-, 2^+, \ldots, r^+, q^-).$$

(3.11)

The last two contributions contain the new notation

$$K \equiv p + \kappa(1, s) + q,$$

(3.12)

which arises because of the different momentum routing used in this pair of terms. The actual evaluation of each of the four terms follows essentially the same procedure. We will illustrate the method using $\mathcal{M}_{\gamma 1}$, and then simply quote the results for the remaining three contributions.
When we insert (2.13), (2.14), (2.20), (2.24), and (2.25) for the currents appearing in (3.8) we obtain

\[ \mathcal{M}_{\gamma 1} = i (-e \sqrt{2})^{n+2} \frac{\langle 1 \mid q \rangle^2}{\langle P \mid 1 \mid Q \rangle} \sum_{P(2 \ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{t=s}^{n} \frac{1}{K^2} \]

\[ \times \frac{\delta \kappa [P + \bar{\kappa}(1, r)]_{\alpha \epsilon} u_{\epsilon}(k_1)}{\langle P\mid 2, \ldots, r \mid 1 \rangle \langle 1 \mid r+1, \ldots, s \mid Q \rangle} \]

\[ \times \frac{\gamma \kappa \kappa (k_1)[p + \kappa(s+1, t)]_{\gamma \beta} \tilde{K}^{\tilde{\beta} \alpha} u_{\alpha}(k_1)}{\langle p \mid s+1, \ldots, t \mid 1 \rangle \langle 1 \mid t+1, \ldots, n \mid q \rangle} \]

\[ \times \left\{ \frac{\langle h P \rangle^*}{\langle h^* \mid P \rangle} + (1 - \delta_r) \sum_{j=2}^{r} u^{\omega}(k_1) \Pi^{\theta} (P, 1, 2, \ldots, j) u_{\theta}(k_1) \right\}. \]

We begin by performing the sum on \( t \). In order to write

\[ \frac{1}{\langle p \mid s+1, \ldots, t \mid 1 \rangle \langle 1 \mid t+1, \ldots, n \mid q \rangle} = \frac{\langle t \mid t+1 \rangle}{\langle t+1 \mid t \rangle} \frac{1}{\langle p \mid s+1, \ldots, n \mid q \rangle} \]

for all \( t \) in the indicated summation range, we adopt the device that when \( t = s \), we write \( p \) and when \( t = n + 1 \), we write \( q \). These assignments mesh nicely with the momentum sum in the numerator, which becomes

\[ [p + \kappa(s+1, t)]_{\gamma \beta} = \sum_{v=s}^{t} (k_v)_{\gamma \beta}. \]

Thus, we write the relevant factors for the sum on \( t \) as

\[ \sigma_{\gamma 1 t} \equiv \sum_{t=s}^{n} \sum_{v=s}^{t} \frac{\langle t \mid t+1 \rangle}{\langle t \mid 1 \rangle \langle 1 \mid t+1 \rangle} \langle 1 \mid v \rangle \bar{u}_{\beta}(k_v) \tilde{K}^{\beta \alpha} u_{\alpha}(k_1). \]

Interchanging the order of the sums and applying (A.16) to do the sum on \( t \) yields

\[ \sigma_{\gamma 1 t} = \sum_{v=s}^{n} \frac{\langle v \mid q \rangle}{\langle v \mid 1 \rangle \langle 1 \mid q \rangle} \langle 1 \mid v \rangle \bar{u}_{\beta}(k_v) \tilde{K}^{\beta \alpha} u_{\alpha}(k_1) \]

\[ = \frac{1}{\langle 1 \mid q \rangle} u^{\gamma}(q)[p + \kappa(s+1, n)]_{\gamma \beta} \tilde{K}^{\beta \alpha} u_{\alpha}(k_1). \]

We may use the Weyl equation to extend the sum in the first factor of (3.17) to read \( p + \kappa(s+1, n) + q = -K \):

\[ \sigma_{\gamma 1 t} = \frac{-1}{\langle 1 \mid q \rangle} u^{\gamma}(q) K_{\gamma \beta} \tilde{K}^{\beta \alpha} u_{\alpha}(k_1) \]

\[ = K^2, \]
where we have applied (A.6) and (A.13a) to obtain the last line. Returning to (3.13) and using (A.6) to extend one of the momentum factors from $P + \kappa(1, r)$ to $K$, we have

$$\mathcal{M}_{\gamma 1} = i(-e \sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \sum_{r=1}^{s} u^{(1)(k_1)\{[\kappa(r+1, s) + Q]_{\delta \delta} \hat{K}^{\bar{\alpha} e} u_{\epsilon}(k_1) \}}{\langle P|2, \ldots, r|1\rangle \langle 1|r+1, \ldots, s|Q \rangle} \times \frac{1}{\langle p|s+1, \ldots, n|q \rangle \langle h|1|P \rangle^*}

+ i(-e \sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=2}^{n} \sum_{r=2}^{s} u^{(1)(k_1)\{[\kappa(r+1, s) + Q]_{\delta \delta} \hat{K}^{\bar{\alpha} e} u_{\epsilon}(k_1) \}}{\langle P|2, \ldots, r|1\rangle \langle 1|r+1, \ldots, s|Q \rangle} \times \frac{1}{\langle p|s+1, \ldots, n|q \rangle} \sum_{j=2}^{r} u^{(\omega)(1)(k_1)\Pi_{\omega} \theta(P, 1, 2, \ldots, j)u_{\theta}(k_1)}. \quad (3.19)

Denote the first of the two terms in (3.19) by $\mathcal{M}_{\gamma 1a}$. We may do the sum on $r$ by defining $r = 1$ to mean $P$ and $r = s + 1$ to mean $Q$. Notice that $k_1$ is thus naturally absent from any momentum sums that result. Isolating the $r$-dependent factors of $\mathcal{M}_{\gamma 1a}$ yields

$$\sigma_{\gamma 1a} \equiv \sum_{r=1}^{s} \sum_{w=r+1}^{s+1} \frac{\langle r r+1 \rangle}{(r|1|r+1)} \langle 1 w \rangle u^{(1)(k_1)\bar{\alpha} \delta \bar{\alpha} \bar{e} \bar{e} \bar{u}_{\delta}(k_w) \bar{K}^{\bar{\alpha} e} u_{\epsilon}(k_1) \rangle. \quad (3.20)

Interchanging the order of summation and proceeding as before we find that

$$\sigma_{\gamma 1a} = \sum_{w=2}^{s+1} \frac{\langle P w \rangle}{\langle P|1|w \rangle} \langle 1 w \rangle u^{(\omega)(1)(k_1)\Pi_{\omega} \theta(P, 1, 2, \ldots, j)u_{\theta}(k_1)} \rangle \quad (3.21)

\begin{align*}
= \frac{1}{\langle P 1 \rangle} u^{(\delta)(P)\{[\kappa(2, s) + Q]_{\delta \delta} \hat{K}^{\bar{\alpha} e} u_{\epsilon}(k_1) \}}

= [P + \kappa(2, s) + Q]^2.
\end{align*}

Hence,

$$\mathcal{M}_{\gamma 1a} = i(-e \sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \times \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{\langle P + \kappa(2, s) + Q \rangle^2}{\langle P|2, \ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle \langle h|1|P \rangle^*}. \quad (3.22)$$
The additional summation appearing in the second term of (3.19) makes the evaluation of the sums there somewhat more complex. We may join the denominator strings involving $r$ using the same ground rules outlined in the previous paragraph. Thus, we are led to consider

$$
\sigma_{1b} \equiv \sum_{r=j}^{s} \sum_{w=r+1}^{s+1} \frac{\langle r \, r+1 \rangle}{\langle r | 1 | r+1 \rangle} \langle 1 \, w \rangle \bar{u}_{\alpha}(k_w) \bar{K}^\alpha \epsilon \, u_{\epsilon}(k_1).
$$

(3.23)

The (unwritten) sum on $j$ has been moved to the left of the sum on $r$ and now ranges from 2 to $s$. We begin the evaluation as before:

$$
\begin{align*}
\sigma_{1b} &= \sum_{w=j+1}^{s+1} \frac{\langle j \, w \rangle}{\langle j | 1 | w \rangle} \langle 1 \, w \rangle \bar{u}_{\alpha}(k_w) \bar{K}^\alpha \epsilon \, u_{\epsilon}(k_1) \\
&= \frac{-1}{\langle 1 \, j \rangle} u^\delta(k_j) [\kappa(j+1, s) + Q]_{\delta \alpha} \bar{K}^\alpha \epsilon \, u_{\epsilon}(k_1). \\
\end{align*}
$$

(3.24)

At this stage, we transpose the order of multiplication, the shifting of three contractions producing a net sign change. In addition, we write $\kappa(j+1, s) + Q = K - [P + \kappa(1, j)]$ to obtain

$$
\sigma_{1b} = K^2 - \frac{1}{\langle 1 \, j \rangle} u^\epsilon(k_1) K_{\epsilon \alpha}[\bar{P} + \bar{\kappa}(1, j)]^{\alpha \delta} \bar{u}_{\delta}(k_j).
$$

(3.25)

Insertion of (3.25) into the second term of (3.19) produces

$$
\mathcal{M}_{1b} = i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 | Q \rangle} \sum_{P(2 \ldots n)}^{n} \sum_{s=2}^{n} \frac{K^2}{\langle P | 2, \ldots, s | Q \rangle \langle p | s+1, \ldots, n | q \rangle} \times \sum_{j=2}^{s} u^\omega(k_1) \Pi_{\omega \theta}(P, 1, 2, \ldots, j) u_{\theta}(k_1)
$$

$$
- i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 | Q \rangle} \sum_{P(2 \ldots n)}^{n} \sum_{s=2}^{n} \frac{1}{\langle P | 2, \ldots, s | Q \rangle \langle p | s+1, \ldots, n | q \rangle} \times \sum_{j=2}^{s} u^\epsilon(k_1) K_{\epsilon \alpha}[\bar{P} + \bar{\kappa}(1, j)]^{\alpha \delta} \Pi_{\delta \theta}(P, 1, 2, \ldots, j) u_{\theta}(k_1)
$$

(3.26)

where we have used the fact that $u^\omega(k_1) \Pi_{\omega \theta}(P, 1, 2, \ldots, j) u_{\theta}(k_1)$ contains a factor of $\langle 1 \, j \rangle$ to obtain the second term.
It is possible to do the sum on $j$ appearing in the second term ($\equiv \mathcal{M}_{\gamma_{1b2}}$) of (3.26) with the help of (A.27). Thus, we write

$$\mathcal{M}_{\gamma_{1b2}} = -i(-e\sqrt{2})^{n+2} \frac{1}{\langle P|1|Q \rangle} \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{u^\epsilon(k_1)K_{\epsilon\alpha}}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle}$$

$$\times \left[ \frac{[P + \bar{k}_1]^{\alpha\theta}u_\theta(k_1)}{[P + k_1]^2} - \frac{[P + \bar{k}(1,s)]^{\alpha\theta}u_\theta(k_1)}{[P + \kappa(1,s)]^2} \right],$$

(3.27)

where we have used the fact that the quantity in square brackets vanishes for $s = 1$ to extend the range of the summation. We may use the Weyl equation and (A.6) to rewrite (3.27) as

$$\mathcal{M}_{\gamma_{1b2}} = -i(-e\sqrt{2})^{n+2} \frac{1}{\langle P|1|Q \rangle}$$

$$\times \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{u^\epsilon(k_1)K_{\epsilon\alpha}}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle \langle P|1 \rangle^*}$$

$$+ i(-e\sqrt{2})^{n+2} \frac{1}{\langle P|1|Q \rangle} \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{1}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle}$$

$$\times \frac{u^\epsilon(k_1)Q_{\epsilon\alpha}}{[P + \kappa(1,s)]^2} \langle \bar{P} + \bar{k}(1,s) \rangle^{\alpha\theta}u_\theta(k_1).$$

(3.28)

Notice that the last factor in (3.28) is just $K^2u^\epsilon(k_1)\Pi^\theta_\epsilon(P,1,2,\ldots,s,Q)u_\theta(k_1)$. Hence, we may combine (3.28) with (3.26) to form

$$\mathcal{M}_{\gamma_{1b}} = i(-e\sqrt{2})^{n+2} \frac{1}{\langle P|1|Q \rangle} \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1,s) + Q]^2}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle}$$

$$\times \sum_{j=2}^{s+1} \frac{u^\omega(k_1)\Pi^\theta_\omega(P,1,2,\ldots,j)u_\theta(k_1)}{\langle j = s+1 \equiv Q \rangle}$$

$$- i(-e\sqrt{2})^{n+2} \frac{1}{\langle P|1|Q \rangle} \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{\bar{u}_{\delta}(P)[P + \bar{k}(1,s) + Q]^{\alpha\epsilon}u_\epsilon(k_1)}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle \langle P|1 \rangle^*},$$

(3.29)

where the notation $j = s + 1 \equiv Q$ is used to remind us to write $Q$ for $s + 1$ in the sum on $j$. 
Applying the same procedure to $M_{\gamma 2}$ yields the contributions

$$
M_{\gamma 2a} = -i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 \rangle \langle Q \rangle} \times \sum_{P(2...n)} \sum_{s=1}^{n} \frac{[P + \kappa(2, s) + Q]^2}{\langle P | P \rangle \langle 2 | s \rangle \langle Q | q \rangle} \langle h \mid Q \rangle^*.
$$

and

$$
M_{\gamma 2b} = -i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 \rangle \langle Q \rangle} \sum_{P(2...n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + Q]^2}{\langle P | s, s-1, \ldots, 2 \rangle \langle q | s+1, \ldots, n \rangle} \\
\times \sum_{j=2}^{s+1} u^\omega(k_1) P_\omega \theta(Q, 1, 2, \ldots, j) u_\theta(k_1) \bigg|_{j=s+1=P} \\
+ i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 \rangle \langle Q \rangle} \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\bar{\bar{u}}_\bar{\alpha}(Q)[\bar{P} + \bar{\kappa}(1, s) + \bar{Q}]^{\bar{\alpha} \bar{\epsilon}} u_\bar{\epsilon}(k_1)}{\langle P | 2, \ldots, s \rangle \langle Q | q \rangle} \langle h \rangle^*. 
$$

(3.31)

The structure $\langle P | s, s-1, \ldots, 2 \rangle \langle Q \rangle$ appearing in (3.31) is a consequence of (2.20) and the desire to write the current in a form containing $\Pi(Q, 1, 2, \ldots, j)$. Other forms of this term are possible, but this is the most convenient.

The remaining contributions, $M_{\gamma 3}$ and $M_{\gamma 4}$, turn out to be so nearly identical to $M_{\gamma 1}$ and $M_{\gamma 2}$ respectively that it would be redundant to write them down. These terms may be obtained by exchanging $p \leftrightarrow P$ and $q \leftrightarrow Q$ in (3.22), (3.29), (3.30), and (3.31) inside the permutation sums only. The factor $\langle 1 \, q \rangle^2 \langle P | 1 \rangle^{-1}$ appearing outside the permutation sums remains unchanged. This close relation, which becomes apparent after the explicit expressions for the currents are inserted into (3.10) and (3.11), is a reflection of the SUSY-like relationships between the currents.

At this stage, we may combine some of the fragments that do not depend on $\Pi$. Application of (A.15) to the sum of (3.22) and (3.30) gives

$$
M_{\gamma 12a} = -i(-e\sqrt{2})^{n+2} \frac{\langle 1 \, q \rangle^2}{\langle P | 1 \rangle \langle Q \rangle} \times \sum_{P(2...n)} \sum_{s=1}^{n} \frac{[P + \kappa(2, s) + Q]^2}{\langle P | 2, \ldots, s \rangle \langle Q | q \rangle} \langle P \rangle^*.
$$

(3.32)
Likewise, the second term of (3.29) may be added to the second term of (3.31) to produce

\[ M_{\gamma 12b2} \equiv -i(-e\sqrt{2})^{n+2} \left( \frac{\langle 1, q \rangle^2}{\langle P|1|Q \rangle} \right) \sum_{P(2\ldots n)} \sum_{s=1}^{n} \left[ \bar{u}_{\dot{\alpha}}(P) \langle P 1 \rangle^s - \bar{u}_{\dot{\alpha}}(Q) \langle Q 1 \rangle^s \right] \]

\[ \times \frac{[\bar{P} + \bar{\kappa}(1, s) + \bar{Q}]^s \hat{\alpha}_s u_{\epsilon}(k_1)}{\langle P|2\ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle} \]

\[ = -i(-e\sqrt{2})^{n+2} \left( \frac{\langle 1, q \rangle^2}{\langle P|1|Q \rangle} \right) \sum_{P(2\ldots n)} \sum_{s=1}^{n} \left( \frac{\langle P Q \rangle^*}{\langle P|1|Q \rangle^*} \right) \]

\[ \times \bar{u}_{\dot{\alpha}}(k_1) \left[ \bar{P} + \bar{\kappa}(1, s) + \bar{Q} \right]^s \hat{\alpha}_s u_{\epsilon}(k_1) \]

\[ \frac{\langle P|2\ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle}{\langle P|1|Q \rangle} \]

\[ (3.33) \]

Since

\[ \bar{u}_{\dot{\alpha}}(k_1) \left[ \bar{P} + \bar{\kappa}(1, s) + \bar{Q} \right]^s \hat{\alpha}_s u_{\epsilon}(k_1) = 2k_1 \cdot [P + \kappa(1, s) + Q], \]

\[ (3.34) \]

we see that (3.33) is precisely what must be added to (3.32) to extend its numerator from \([P + \kappa(2, s) + Q]^2\) to \([P + \kappa(1, s) + Q]^2\). Thus,

\[ M_{\gamma 12a} + M_{\gamma 12b2} = -i(-e\sqrt{2})^{n+2} \left( \frac{\langle 1, q \rangle^2}{\langle P|1|Q \rangle} \right) \]

\[ \times \sum_{P(2\ldots n)} \sum_{s=1}^{n} \left( \frac{\langle P + \kappa(1, s) + Q \rangle^2}{\langle P|2\ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle} \langle P|1|Q \rangle^*. \right. \]

\[ (3.35) \]

We now present the final result for the photon exchange graphs. There are a total of six terms: (3.35), the first term of (3.29), the first term of (3.31), and the
counterparts to these three terms generated from \( M_3 \) and \( M_4 \). Hence,

\[
M_\gamma(p^+, q^-; P^+, Q^+; 1^-, 2^+, \ldots, n^+) = \\
= i(-e\sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + Q]^2}{\langle P|2, \ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle} \\
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi_{\omega}^\theta (P, 1, 2, \ldots, j) u_\theta(k_1) |j = s+1 \equiv Q \rangle \\
- i(-e\sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + Q]^2}{\langle P|s, s-1, \ldots, 2|Q \rangle \langle p|s+1, \ldots, n|q \rangle} \\
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi_{\omega}^\theta (Q, 1, 2, \ldots, j) u_\theta(k_1) |j = s+1 \equiv P \rangle \\
+ i(-e\sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + q]^2}{\langle p|2, \ldots, s|Q \rangle \langle p|s+1, \ldots, n|Q \rangle} \\
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi_{\omega}^\theta (p, 1, 2, \ldots, j) u_\theta(k_1) |j = s+1 \equiv Q \rangle \\
- i(-e\sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + q]^2}{\langle p|s, s-1, \ldots, 2|Q \rangle \langle p|s+1, \ldots, n|Q \rangle} \\
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi_{\omega}^\theta (q, 1, 2, \ldots, j) u_\theta(k_1) |j = s+1 \equiv P \rangle \\
- i(-e\sqrt{2})^{n+2} \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \\
\times \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + Q]^2}{\langle P|2, \ldots, s|Q \rangle \langle p|s+1, \ldots, n|q \rangle \langle P|Q \rangle^*} \\
\times \sum_{p(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + q]^2}{\langle p|2, \ldots, s|Q \rangle \langle p|s+1, \ldots, n|Q \rangle \langle p|1|q \rangle^*}.
\]

(3.36)
3.1.2 The \( Z \) contribution

Given that, in the massless limit, the \( Z \)-boson is almost a photon with “odd” couplings, it is not hard to see how the inclusion of \( Z \)-exchange modifies (3.36). Emission of a photon from the \( W \) line occurs at a vertex \( ig \sin \theta_W V^\mu\nu\lambda \), while \( Z \)-emission involves \( ig \cos \theta_W V^\mu\nu\lambda \). The same ratio of coupling constants holds between the corresponding pair of seagull vertices. The fermion-photon vertex is

\[
-ie \gamma^\mu = -ig \gamma^\mu \left[ \frac{1}{2} (1 - \gamma_5) \sin \theta_W + \frac{1}{2} (1 + \gamma_5) \sin \theta_W \right],
\]

while the fermion-\( Z \) vertex is

\[
\frac{ig}{4 \cos \theta_W} \gamma^\mu (-1 + 4 \sin^2 \theta_W + \gamma_5) = -ig \gamma^\mu \left[ \frac{1}{2} (1 - \gamma_5) \frac{\cos 2 \theta_W}{2 \cos \theta_W} + \frac{1}{2} (1 + \gamma_5) \frac{-\sin^2 \theta_W}{\cos \theta_W} \right].
\]

Thus, we see that if we replace the exchanged photon by an exchanged \( Z \), we simply replace

\[
(-e \sqrt{2})^{n+2} = 2g^2 \sin^2 \theta_W (-e \sqrt{2})^n \rightarrow -2g^2 \sin^2 \theta_W (-e \sqrt{2})^n
\]

if the electron is right-handed and

\[
2g^2 \sin^2 \theta_W (-e \sqrt{2})^n \rightarrow g^2 \cos 2 \theta_W (-e \sqrt{2})^n
\]

if the electron is left-handed. The consequence of (3.39) is the exact cancellation of the tree-level photon-exchange diagrams with the \( Z \)-exchange diagrams for \( e^+ \bar{e}^- \rightarrow W^+ W^- \gamma \cdots \gamma \) in the high-energy limit. Since the right-handed neutrinos which would be required for the third type of diagram do not exist within the Standard Model, this means that

\[
\mathcal{M}(p^-, q^+; P, Q; 1, \ldots, n) = 0
\]

to tree level in the high-energy limit independent of the helicity combination of the (transverse) vector bosons. This is a reflection of a number of properties
of the high-energy limit of the Standard Model. First, the $W$-boson has only left-handed couplings to fermions. Thus, a right-handed fermion cannot radiate the $W'$s directly. Second, the photon and $Z$ exchanges between a $W$ line and a right-handed fermion line conspire to cancel. Finally, helicity conservation for what are effectively massless fermions means that the emission of any number of photons in any helicity configuration cannot change a right-handed fermion line into a left-handed one. Hence, we have (3.41).

For the left-handed fermions we have been considering in detail, the net effect when the photon and $Z$ contributions are summed is to make the replacement

$$(-e\sqrt{2})^{n+2} \rightarrow g^2(-e\sqrt{2})^n$$

in (3.36).

### 3.1.3 The neutrino contribution

For left-handed fermions, there is also a contribution in which the $W$'s are radiated directly by the fermions, with a neutrino propagating in the gap. From Figure 1, we see that this contribution is

$$\mathcal{M}_\nu(p^+, q^-; P, Q; 1, \ldots, n) =$$

$$= \sum_{\mathcal{P}(1\ldots n)} \sum_{r=0}^{n} \sum_{s=r}^{n} \sum_{t=s}^{n} \tilde{\psi}(p^+; s+1, \ldots, t) \frac{ig}{(t-s)!} \frac{W(t+1, \ldots, n; Q)}{\sqrt{2}} \frac{1}{(n-t)!} \frac{1}{2}(1-\gamma_5)$$

$$\times \frac{\sqrt{2}}{[P + \kappa(1, s) + q]^2} \frac{ig}{r!} \frac{W(P; 1, \ldots, r)}{2} \psi(r+1, \ldots, s; q^-)$$

$$= -ig^2 \sum_{\mathcal{P}(1\ldots n)} \sum_{r=0}^{n} \sum_{s=r}^{n} \sum_{t=s}^{n} \frac{1}{r!(s-r)!(t-s)!(n-t)!} \tilde{\psi}_\alpha(p^+; s+1, \ldots, t)$$

$$\times \overline{\psi}_\beta(P + \kappa(1, s) + q) [P + \kappa(1, s) + q]^2 \overline{\nabla}^{i\alpha} \frac{W^{i\alpha}(t+1, \ldots, n; Q)}{[P + \kappa(1, s) + q]^2}$$

$$\times \psi_\beta(r+1, \ldots, s; q^-)$$

(3.44)
Evaluation of (3.44) for the case of all like helicities except for the first photon is exactly analogous to the evaluation of the photon-exchange case already discussed. Hence, we will only present the result:

\[
\mathcal{M}_\nu(p^+, q^-; p^+, Q^+; 1^-, 2^+, \ldots, n^+) =
\]
\[
= ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^2}{\langle P|2, \ldots, s|q \rangle \langle p|s+1, \ldots, n|Q \rangle}
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi^\theta(P, 1, 2, \ldots, j) u_\theta(k_1) |_{j=s+1=q}
\]
\[
- ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^2}{\langle p|2, \ldots, s|Q \rangle \langle P|s+1, \ldots, n|q \rangle}
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi^\theta(q, 1, 2, \ldots, j) u_\theta(k_1) |_{j=s+1=p}
\]
\[
+ ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + Q]^2}{\langle p|s, s-1, \ldots, 2|Q \rangle \langle P|s+1, \ldots, n|q \rangle}
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi^\theta(p, 1, 2, \ldots, j) u_\theta(k_1) |_{j=s+1=q}
\]
\[
- ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + Q]^2}{\langle p|s, s-1, \ldots, 2|Q \rangle \langle P|s+1, \ldots, n|q \rangle}
\times \sum_{j=2}^{s+1} u^\omega(k_1) \Pi^\theta(Q, 1, 2, \ldots, j) u_\theta(k_1) |_{j=s+1=p}
\]
\[
- ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + q]^2}{\langle P|2, \ldots, s|q \rangle \langle p|s+1, \ldots, n|Q \rangle \langle P|1|q \rangle^*}
\]
\[
- ig^2(-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(1, s) + Q]^2}{\langle p|2, \ldots, s|Q \rangle \langle P|s+1, \ldots, n|q \rangle \langle p|1|Q \rangle^*}.
\]

(3.45)

### 3.1.4 Cross-channel identities

The contributions represented by (3.36) and (3.45) look quite similar to each other. Since they contain the same combination of coupling constants (recall (3.42)!), one is led to investigate whether or not they may be re-written in such
a way so as to be combined into a simpler form. This is indeed the case, as is shown in Appendix B. The results relevant to the present discussion are (see (B.3), (B.16), (B.17), and (B.41)):

\[
\sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^2}{\langle P|2, \ldots, s|q\rangle \langle p|s+1, \ldots, n|Q\rangle} = - \sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + Q]^2}{\langle P|2, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle} + \sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{\langle P\,1\rangle \bar{u}_\alpha(k_1)\bar{P} + \bar{\kappa}(1, s) + \bar{Q}\bar{\alpha}\beta u_\beta(q)}{\langle P\,q\rangle \langle P|2, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle}\tag{3.46}
\]

and

\[
\sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^2}{\langle P|2, \ldots, s|q\rangle \langle p|s+1, \ldots, n|Q\rangle} \times \sum_{j=2}^{s+1} u^\gamma(k_1)\Pi_{\gamma} \delta(P, 1, 2, \ldots, j)u_\delta(k_1)\big|_{j=s+1 \equiv q} =
\]

\[
\sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^2}{\langle P|2, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle} \times \sum_{j=2}^{s+1} u^\alpha(k_1)\Pi_{\alpha} \delta(P, 1, 2, \ldots, j)u_\delta(k_1)\big|_{j=s+1 \equiv Q}\tag{3.47}
\]

\[
\sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{\langle q|1|Q\rangle}{\langle q\,Q\rangle} \left[ \frac{1}{\langle P|2, \ldots, s|q\rangle \langle p|s+1, \ldots, n|Q\rangle} - \frac{1}{\langle P|2, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle} \right] + \sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \frac{\langle P\,1\rangle \bar{u}_\alpha(P)\bar{P} + \bar{\kappa}(1, s) + \bar{Q}\bar{\alpha}\beta u_\beta(q)}{\langle P\,q\rangle \langle P|2, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle \langle P\,1\rangle}\tag{3.48}
\]

In addition to (3.46) and (3.47), we require variants of these identities that are easily obtained by permuting among \(P, Q, p,\) and \(q\). We apply the appropriate form of (3.46) or (3.47) to each of the six terms appearing in (3.45), and combine them with (3.36). Every one of the double-sum terms in (3.45) (i.e. those containing \(\Pi\)) produce a contribution that cancels one of the double-sum terms in
In addition, each of the single-sum terms in (3.45) produces a piece that combines logically with a corresponding piece in (3.36) via (A.15). The result of (carefully) doing all of this algebra may be organized into the following ten contributions:

\[ \mathcal{M}_A = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle q|1|Q \rangle}{\langle q Q \rangle} \times \left[ \frac{1}{\langle P|2,...,s|q \rangle \langle p|s+1,...,n|Q \rangle} - \frac{1}{\langle P|2,...,s|q \rangle \langle p|s+1,...,n|Q \rangle} \right. \\
- \left. \frac{1}{\langle P|s+1,...,n|q \rangle \langle p|2,...,s|Q \rangle} + \frac{1}{\langle P|s+1,...,n|q \rangle \langle p|2,...,s|Q \rangle} \right], \tag{3.48} \]

\[ \mathcal{M}_B = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle P|1|p \rangle}{\langle P p \rangle} \times \left[ \frac{1}{\langle P|2,...,s|q \rangle \langle p|s+1,...,n|Q \rangle} - \frac{1}{\langle P|2,...,s|q \rangle \langle p|s+1,...,n|Q \rangle} \right. \\
- \left. \frac{1}{\langle P|s+1,...,n|q \rangle \langle p|2,...,s|Q \rangle} + \frac{1}{\langle P|s+1,...,n|q \rangle \langle p|2,...,s|Q \rangle} \right], \tag{3.49} \]

\[ \mathcal{M}_C = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle P|1 \rangle}{\langle P Q \rangle} \frac{\bar{u}_{\dot{\alpha}}(P)K^{\dot{\alpha}\beta}u_{\beta}(q)}{\langle P|2,...,s|Q \rangle \langle p|s+1,...,n|Q \rangle \langle P|1 \rangle^*}, \tag{3.50} \]

\[ \mathcal{M}_D = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle P|1 \rangle}{\langle P Q \rangle} \frac{\bar{u}_{\dot{\alpha}}(p)K^{\dot{\alpha}\beta}u_{\beta}(Q)}{\langle P|2,...,s|Q \rangle \langle p|s+1,...,n|Q \rangle \langle P|1 \rangle^*}, \tag{3.51} \]

\[ \mathcal{M}_E = -C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle Q|1 \rangle}{\langle P Q \rangle} \frac{\bar{u}_{\dot{\alpha}}(Q)K^{\dot{\alpha}\beta}u_{\beta}(p)}{\langle P|2,...,s|Q \rangle \langle p|s+1,...,n|Q \rangle \langle 1 Q \rangle^*}, \tag{3.52} \]

\[ \mathcal{M}_F = -C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle q|1 \rangle}{\langle P q \rangle} \frac{\bar{u}_{\dot{\alpha}}(q)K^{\dot{\alpha}\beta}u_{\beta}(P)}{\langle P|2,...,s|q \rangle \langle p|s+1,...,n|Q \rangle \langle 1 q \rangle^*}, \tag{3.53} \]

\[ \mathcal{M}_G = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{K^2}{\langle P|2,...,s|Q \rangle \langle p|s+1,...,n|Q \rangle \langle Q|1|q \rangle^*}, \tag{3.54} \]

\[ \mathcal{M}_H = C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{K^2}{\langle p|2,...,s|Q \rangle \langle P|s+1,...,n|Q \rangle \langle q|1|Q \rangle^*}, \tag{3.55} \]

\[ \mathcal{M}_I = -C \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle P|1 \rangle}{\langle P q \rangle} \frac{\bar{u}_{\dot{\alpha}}(k_1)K^{\dot{\alpha}\beta}u_{\beta}(q)}{\langle P|2,...,s|Q \rangle \langle p|s+1,...,n|q \rangle \langle P|1|q \rangle^*}, \tag{3.56} \]
\[ \mathcal{M}_J = -C \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{\langle p \ 1 \rangle}{\langle P \ 1 \rangle} \frac{\bar{u}_{\dot{\alpha}}(k_1) \bar{K}^{\dot{\alpha} \dot{\beta}} u_{\beta}(Q)}{\langle P \ q \rangle^* \langle 1 \ q \rangle^*} \langle p \ q \rangle^* \langle P|s+1,\ldots,n|Q \rangle \langle p|1|Q \rangle^*, \] (3.57)

where we use the abbreviation

\[ C \equiv ig^2 (-e\sqrt{2})^n \frac{\langle 1 \ q \rangle^2}{\langle P|1|Q \rangle^*}. \] (3.58)

Recall that \( K \) and \( \mathcal{K} \) are as given in (3.5) and (3.12) respectively. At first glance, it seems like the trade of (3.36) and (3.45) for (3.48)–(3.57) has produced more terms than it has cancelled. We shall now demonstrate how to combine this bewildering array of terms into a simple result.

The first observation to make is that \( \mathcal{M}_A \) and \( \mathcal{M}_B \) both vanish. This is readily seen by noting that the momenta of the second two terms in the square brackets may be relabelled using

\[ \{s+1,\ldots,n\} \rightarrow \{2,\ldots,n-s+1\} \]

\[ \{2,\ldots,s\} \rightarrow \{n-s+2,\ldots,n\}, \] (3.59)

followed by the variable change \( s' = n - s + 1 \). If we subsequently drop the primes, the net effect is

\[ \{2,\ldots,s\} \leftrightarrow \{s+1,\ldots,n\}. \] (3.60)

Upon applying (3.60) to the appropriate parts of (3.48) and (3.49), we see that both contributions are identically zero.

The next step is to combine \( \mathcal{M}_C \) with \( \mathcal{M}_I \). The factors which differ between these two terms read

\[ \frac{\bar{u}_{\dot{\alpha}}(P)}{\langle P \ 1 \rangle^*} - \frac{\langle P \ q \rangle^* \bar{u}_{\dot{\alpha}}(k_1)}{\langle P \ 1 \rangle^* \langle 1 \ q \rangle^*} = \frac{\bar{u}_{\dot{\alpha}}(q)}{\langle 1 \ q \rangle^*}, \] (3.61)

where we have applied (A.14). Thus, the sum of (3.50) and (3.56) is

\[ \mathcal{M}_{CI} = -C \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{\langle P \ 1 \rangle}{\langle P \ q \rangle} \frac{\bar{u}_{\dot{\alpha}}(q) \bar{K}^{\dot{\alpha} \dot{\beta}} u_{\beta}(q)}{\langle P|2,\ldots,s|Q \rangle \langle p|s+1,\ldots,n|q \rangle \langle 1 \ q \rangle^*}. \] (3.62)
The combination of $\mathcal{M}_D$ with $\mathcal{M}_J$ proceeds in analogous fashion to produce

$$\mathcal{M}_{DJ} = -\mathcal{C} \sum_{P(2...n)} \sum_{s=1}^{n} \frac{\langle p \, 1 \rangle}{\langle p \, Q \rangle} \overline{u}_{\dot{\alpha}} (Q) \overline{K}^{\dot{\alpha} \beta} u_\beta (Q) \frac{1}{\langle 1 \, Q \rangle^*}.$$  \hspace{1cm} (3.63)

In order to obtain the same denominator strings in all of the contributions, we apply the momentum relabelling given by (3.60). The action of (3.60) on $K$ is

$$K = p + \kappa(1,s) + q \rightarrow p + k_1 + \kappa(s+1,n) + q$$

$$= k_1 - P - \kappa(1,s) - Q$$  \hspace{1cm} (3.64)

$$= k_1 - K.$$  

If we relabel $\mathcal{M}_H$ in this manner and combine it with $\mathcal{M}_G$, the result is

$$\mathcal{M}_{GH} = \mathcal{C} \sum_{P(2...n)} \sum_{s=1}^{n} \frac{2k_1 \cdot K}{\langle p \, 2, \ldots, s \, |Q \rangle \langle p \, s+1, \ldots, n \, |q \rangle \langle Q \, 1 \, |q \rangle}.$$  \hspace{1cm} (3.65)

We will set this contribution aside for cancellation.

The result of using (3.60) to get all of the remaining contributions on a common denominator is

$$\mathcal{M}_{CDEFIJ} = \mathcal{C} \sum_{P(2...n)} \sum_{s=1}^{n} \frac{1}{\langle p \, 2, \ldots, s \, |Q \rangle \langle p \, s+1, \ldots, n \, |q \rangle}
\times \left\{- \frac{1}{\langle 1 \, q \rangle^* \langle P \, q \rangle} \overline{u}_{\dot{\alpha}} (q) \overline{K}^{\dot{\alpha} \alpha} u_\alpha (q) + \frac{1}{\langle 1 \, Q \rangle^* \langle p \, Q \rangle} \overline{u}_{\dot{\alpha}} (Q) \overline{K}^{\dot{\alpha} \alpha} u_\alpha (Q) - \frac{1}{\langle 1 \, Q \rangle^* \langle P \, q \rangle} \overline{u}_{\dot{\alpha}} (q) \overline{K}^{\dot{\alpha} \alpha} u_\alpha (P) \right\}.$$  \hspace{1cm} (3.66)

Equation (3.66) consists of the contributions from (3.52), (3.53), (3.62), and (3.63). Denoting the portion of (3.66) in curly brackets by $\mathcal{R}$, we have, after grouping the terms appropriately,

$$\mathcal{R} = \frac{1}{\langle 1 \, q \rangle^* \langle P \, q \rangle} \left[ -u_\alpha (q) \langle P \, 1 \rangle + u_\alpha (P) \langle q \, 1 \rangle \right] + \frac{1}{\langle 1 \, Q \rangle^* \langle P \, q \rangle} \left[ u_\alpha (Q) \langle p \, 1 \rangle - u_\alpha (p) \langle Q \, 1 \rangle \right] - \frac{\langle p \, 1 \rangle}{\langle p \, Q \rangle} \langle 1 \, Q \rangle - \frac{\langle q \, 1 \rangle}{\langle P \, q \rangle} \langle 1 \, P \rangle.$$  \hspace{1cm} (3.67)
The terms in square brackets are simplified using (A.14), yielding
\[
\mathcal{R} = \left[ -\frac{\bar{u}_\alpha(q) + \bar{u}_\alpha(Q)}{\langle 1 q \rangle} + \bar{u}_\alpha(Q) \right] K^{\alpha\dot{\alpha}} u_\alpha(k_1) - \frac{\langle p|1|Q \rangle}{\langle p Q \rangle} - \frac{\langle q|1|P \rangle}{\langle q P \rangle},
\]
where we have again regrouped. One final application of (A.14) produces
\[
\mathcal{R} = -2k_1 \cdot K \frac{\langle Q q \rangle^*}{\langle Q|1|q \rangle^*} + \frac{\langle P|1|q \rangle}{\langle P q \rangle} - \frac{\langle p|1|Q \rangle}{\langle p Q \rangle}.
\]
Notice that the first term in (3.69), when inserted back into (3.66), exactly cancels (3.65). Thus, the only surviving contributions to the total amplitude come from the second two terms of (3.69), leading to the result
\[
\mathcal{M}(p^+, q^-; P^+, Q^+; 1^-, 2^+, \ldots, n^+) = ig^2 (-e\sqrt{2})^n \frac{\langle 1 q \rangle^2}{\langle P|1|Q \rangle} \sum_{p(2\ldots n)} \sum_{s=1}^n \frac{1}{\langle P|2\ldots,s|Q \rangle \langle p|s+1\ldots,n|q \rangle}.
\]

The only technique required to do the remaining quadruple current amplitudes which was not illustrated by this example is the disposal of non-vanishing seagull terms. However, the procedure is very much like the seagull-disposal method described in reference [2], and so we will not repeat it here.

### 3.2 Summary of quadruple current amplitudes

We now summarize the results of those processes we are able to compute from four currents. They fall naturally into two groupings, as described below. In order to list such a large variety of processes in as compact a form as possible, we adopt a “standard” process, namely
\[
P Q p q \longrightarrow \gamma \gamma \cdots \gamma.
\]

In (3.71), \( P \) and \( p \) denote positively-charged particles of momenta \( P \) and \( p \) respectively, while \( Q \) and \( q \) are negatively-charged particles of momenta \( Q \) and \( q \).
The $n$ photons have momenta $k_1, \ldots, k_n$. All momenta are directed inward, so each amplitude listed below can be made to describe many different processes via the crossing relations. The helicity labels appearing in the tables of results in this section reflect this convention.

The first group of amplitudes all involve $n$ like-helicity photons. They have the generic form

$$M(P, Q; p, q; 1^+, \ldots, n^+) = i g^2 (-e \sqrt{2})^n \sum_{P(1\ldots n)} \sum_{s=0}^n f(P, Q, p, q) \langle P|1, \ldots, s|Q\rangle \langle p|s+1, \ldots, n|q\rangle,$$

(3.72)

where $f(P, Q, p, q)$ is a scalar function that depends upon the spins and helicities of the charged particles. Table 1 lists the values of $f$ appropriate for the various combinations of four currents in this grouping.

Certain aspects of the entries in Table 1 deserve special mention. Notice that although the amplitudes for transverse $W$ production from “wrong”-helicity fermions vanish in accordance with (3.41), the same is not true for longitudinal $W$ production. Both types of fermion pairs may produce longitudinal $W$’s in the high-energy limit.

Many of the entries contain functions of the weak-mixing angle $\theta_W$. This quantity enters in because of interference between photon and $Z$ exchanges. The entry involving four longitudinal $W$’s contains two terms, one of which was generated from the $\phi^4$ coupling of the Higgs sector. Obviously, the (still unknown) size of $\lambda$ is crucial in determining the importance of this contribution.
Table 1: Group one quadruple current amplitude helicity functions

|   |   |   |   |   |
|---|---|---|---|---|
| $P$ | $Q$ | $p$ | $q$ | $f(P, Q, p, q)$ |
| $W^+_\uparrow$ | $W^-_\downarrow$ | $\bar{e}_\downarrow$ | $e_\downarrow$ | $-\langle Q q \rangle^3 \langle P q \rangle^{-1}$ |
| $W^+_\uparrow$ | $W^-_\downarrow$ | $\bar{e}_\uparrow$ | $e_\downarrow$ | $\langle P q \rangle^2 \langle P p \rangle \langle p Q \rangle^{-1}$ |
| $W^+_\uparrow$ | $W^-_\downarrow$ | $\bar{e}_\downarrow$ | $e_\uparrow$ | $0$ |
| $W^+_\downarrow$ | $W^-_\uparrow$ | $\bar{e}_\downarrow$ | $e_\uparrow$ | $0$ |
| $W^+_\uparrow$ | $W^-_\downarrow$ | $W^+_L$ | $W^-_L$ | $-\langle p \rangle \langle Q q \rangle^2 \langle P q \rangle^{-1}$ |
| $W^-_\uparrow$ | $W^-_\downarrow$ | $W^+_L$ | $W^-_L$ | $-\langle P p \rangle \langle p Q \rangle \langle p Q \rangle^{-1}$ |
| $\bar{e}_\uparrow$ | $e_\downarrow$ | $\bar{\mu}_\downarrow$ | $\mu_\uparrow$ | $-\frac{1}{2} \sec^2 \theta_W \langle Q q \rangle^2$ |
| $\bar{e}_\downarrow$ | $e_\uparrow$ | $\bar{e}_\downarrow$ | $e_\downarrow$ | $\frac{1}{2} \sec^2 \theta_W \langle P p \rangle \langle Q q \rangle^3 \langle P q \rangle^{-1} \langle p Q \rangle^{-1}$ |
| $\bar{e}_\uparrow$ | $e_\downarrow$ | $\bar{\mu}_\downarrow$ | $\mu_\uparrow$ | $\tan^2 \theta_W \langle p Q \rangle^2$ |
| $\bar{e}_\downarrow$ | $e_\uparrow$ | $\bar{\mu}_\downarrow$ | $\mu_\uparrow$ | $-2 \tan^2 \theta_W \langle P p \rangle^2$ |
| $\bar{e}_\downarrow$ | $e_\uparrow$ | $\bar{e}_\downarrow$ | $e_\uparrow$ | $2 \tan^2 \theta_W \langle P p \rangle^3 \langle Q q \rangle \langle P q \rangle^{-1} \langle p Q \rangle^{-1}$ |
| $W^+_L$ | $W^-_L$ | $\bar{e}_\uparrow$ | $e_\downarrow$ | $\frac{1}{2} \sec^2 \theta_W \langle P q \rangle \langle Q q \rangle$ |
| $W^+_L$ | $W^-_L$ | $\bar{e}_\downarrow$ | $e_\uparrow$ | $-\tan^2 \theta_W \langle P p \rangle \langle p Q \rangle$ |
| $W^+_L$ | $W^-_L$ | $W^+_L$ | $W^-_L$ | $-\frac{1}{2} \sec^2 \theta_W \langle P p \rangle \langle Q q \rangle - 4 \lambda g^{-2} \langle P Q \rangle \langle p q \rangle$ |

For processes involving two fermion lines, the question of indistinguishability is especially interesting, since there is more than one type of charged fermion in the standard model. Extra graphs are possible when both fermion lines have both the same helicity and the same identity: the two fermions (or the two antifermions) may be exchanged. When these graphs are included (with the relative
minus sign dictated by Fermi statistics), the entries for $e\bar{e}e\bar{e}$ listed result, and differ from the entries for $e\bar{e}\mu\bar{\mu}$. On the other hand, if the two fermion lines have different helicities, the additional graphs do not exist. In this case, $e\bar{e}e\bar{e}$ and $e\bar{e}\mu\bar{\mu}$ give the same results.

We now turn to the results for the second group of amplitudes, all of which have a single negative-helicity photon and $n-1$ positive-helicity photons. Most of the amplitudes in this group have a very complicated structure, as evidenced by the general form

$$M(P, Q; p, q; 1^-, 2^+, \ldots, n^+) = ig^2(-e\sqrt{2})^n \times \left\{ \sum_{P(2n)} \sum_{s=1}^{n} \sum_{j=2}^{s+1} \frac{F_1^{\alpha\beta}(P, Q, p, q, 1)\Pi_{\alpha}^{\beta}(P, 1, 2, \ldots, j)}{\langle P|2, \ldots, s|Q\rangle\langle p|s+1, \ldots, n|q\rangle}_{j=s+1\equiv Q} + \sum_{P(2n)} \sum_{s=1}^{n} \sum_{j=2}^{s+1} \frac{F_2^{\alpha\beta}(P, Q, p, q, 1)\Pi_{\alpha}^{\beta}(Q, 1, 2, \ldots, j)}{\langle P|s, s-1, \ldots, 2|Q\rangle\langle p|s+1, \ldots, n|q\rangle}_{j=s+1\equiv P} + \sum_{P(2n)} \sum_{s=1}^{n} \sum_{j=2}^{s+1} \frac{F_3^{\alpha\beta}(P, Q, p, q, 1)\Pi_{\alpha}^{\beta}(p, 1, 2, \ldots, j)}{\langle p|2, \ldots, s|Q\rangle\langle P|s+1, \ldots, n|Q\rangle}_{j=s+1\equiv q} + \sum_{P(2n)} \sum_{s=1}^{n} \sum_{j=2}^{s+1} \frac{F_4^{\alpha\beta}(P, Q, p, q, 1)\Pi_{\alpha}^{\beta}(q, 1, 2, \ldots, j)}{\langle p|s, s-1, \ldots, 2|q\rangle\langle P|s+1, \ldots, n|Q\rangle}_{j=s+1\equiv p} \right\}$$

Most of the structure in (3.73) is simply the reflection that the negative helicity photon can be radiated from by one of the four charged particles. The functions and momenta appropriate to various processes are summarized in Table 2.
Table 2 (continued): Group two quadruple current amplitude helicity functions

| $P$ | $Q$ | $p$ | $q$ | $f_S(P,Q,p,q,1)$ |
|-----|-----|-----|-----|------------------|
| $W^+_t$ | $W^-_t$ | $\bar{e}_t$ | $e_i$ | $(1 \, q)^2 \left[ \langle 1 \, q \rangle \langle P \, q \rangle^{-1} \langle 1 \, Q \rangle^{-1} - \langle 1 \, 1 \rangle \langle P \, Q \rangle^{-1} \langle P \, 1 \rangle^{-1} \right]$ |
| $W^+_t$ | $W^-_t$ | $W^+_L$ | $W^-_L$ | $\langle p|1|q \rangle \left[ \langle 1 \, q \rangle \langle P \, q \rangle^{-1} \langle 1 \, Q \rangle^{-1} - \langle 1 \, 1 \rangle \langle P \, Q \rangle^{-1} \langle P \, 1 \rangle^{-1} \right]$ |
| $\bar{e}_t$ | $e_i$ | $\bar{\mu}_t$ | $\mu_i$ | $-\frac{1}{2} \sec^2 \theta_W \langle Q \, q \rangle \langle 1 \, Q \rangle \langle 1 \, q \rangle \left[ (k_1 + q)^{-2} - (k_1 + Q)^{-2} \right]$ |
| $\bar{e}_t$ | $e_i$ | $\bar{\mu}_t$ | $\mu_i$ | $2 \tan^2 \theta_W \langle P \, p \rangle \langle P \, 1 \rangle \langle p \, 1 \rangle \left[ (P + k_1)^2 - (p + k_1)^2 \right]$ |
| $W^+_L$ | $W^-_L$ | $\bar{e}_t$ | $e_i$ | $\frac{1}{2} \sec^2 \theta_W \langle P|q,1|Q \rangle (k_1 + q)^{-2}$ |
| $W^+_L$ | $W^-_L$ | $\bar{e}_t$ | $e_i$ | $\tan^2 \theta_W \langle P|p,1|Q \rangle (p + k_1)^{-2}$ |
| $W^+_L$ | $W^-_L$ | $W^+_L$ | $W^-_L$ | $0$ |
| $W^+_L$ | $W^-_L$ | $W^+_L$ | $W^-_L$ | $0$ |

Table 2 requires a few comments. First, the only four-fermion amplitudes given involve $e\bar{e}\mu\bar{\mu}$, because no additional simplifications were found in the $e\bar{e}e\bar{e}$ case. Hence, to obtain the result for two identical fermion lines with the same helicity, proceed as follows. First, write down the contribution implied by the appropriate entry in Table 2. Then, subtract the same quantity with $Q \leftrightarrow q$, to account for the indistinguishability. The amplitude involving two fermion lines of opposing helicities receives no such additional contribution.
The process with four $W_L$’s has been given two entries in Table 2. The first one is the sum of photon and $Z$ exchanges. To it should be added another contribution obtained by exchanging $Q$ and $q$, since the $W_L$’s are indistinguishable. The other entry is the result from the $\phi^4$ coupling. Since the form of this vertex automatically accounts for the indistinguishability of the $W_L$’s, this contribution is complete as it stands and requires no $Q \leftrightarrow q$ addition.

IV. CURRENTS WITH TWO OFF-SHELL PARTICLES

The quadruple current amplitudes discussed in the previous section all share a common feature. The particle which is propagating between the two charged lines is neutral. It cannot radiate photons. Suppose we change the situation, and consider processes where the virtual particle has a non-zero electric charge. Examples include:

\[ e \bar{\nu} \longrightarrow W_L^- H \gamma \ldots \gamma, \quad (4.1) \]

which contains a propagating $\phi^\pm$,

\[ W^+ W^- \longrightarrow \nu \bar{\nu} \gamma \ldots \gamma, \quad (4.2) \]

which contains a propagating $e^\pm$, and

\[ e \bar{e} \longrightarrow \nu \bar{\nu} \gamma \ldots \gamma, \quad (4.3) \]

which contains a propagating $W^\pm$. In every case, we require a current which has both ends of the charged line off shell. In this section we will examine the situation with respect to the computability of such amplitudes. We will find that, although we can obtain the actual double-off-shell current only for a scalar line,
we can obtain the combination of factors required to compute all three of the above types of amplitudes.

4.1 The double-off-shell scalar current

We define the current $\Theta(\mathcal{P}; 1, \ldots, n; \mathcal{Q})$ as consisting of a charged scalar line with $n$ photons attached all possible ways. All momenta are directed inward. The off-shell $\phi^+$ has momentum $\mathcal{P}$, $\mathcal{P}^2 \neq 0$. The off-shell $\phi^-$ has momentum $\mathcal{Q}$, $\mathcal{Q}^2 \neq 0$. The argument list of $\Theta$ is overspecified in that

$$\mathcal{P} + \kappa(1, n) + \mathcal{Q} = 0,$$

(4.4)

allowing us to always eliminate one of the momenta from the result. We define the zero-photon current by

$$\Theta(\mathcal{P}; \mathcal{Q}) = \frac{i}{\mathcal{P}^2} = \frac{i}{\mathcal{Q}^2},$$

(4.5)

that is, just a propagator for the scalar particle. A moment’s reflection upon the derivation[2] of the recursion relation for the scalar current $\Phi(\mathcal{P}; 1, \ldots, n)$ will reveal that nowhere was the fact that $\mathcal{P}^2 = 0$ used in the development. Hence, we have the same recursion formula as (2.1), but with $\Phi$ replaced by $\Theta$, and seeded by (4.5) instead of (2.2).

Furthermore, if we make the gauge choice indicated by (2.5), we are able to solve this recursion relation in the case of all like-helicity photons. Since the seagull contributions all vanish in this gauge (cf. equation (2.4)), the recursion relation for $\Theta$ becomes

$$\Theta(\mathcal{P}; 1^+; \ldots, n^+; \mathcal{Q}) =
\frac{-e\sqrt{2}}{[\mathcal{P} + \kappa(1, n)]^2} \sum_{\mathcal{P}(1 \ldots n)} \frac{1}{(n-1)!} \epsilon^{\dot{\alpha} \alpha}(n)[\mathcal{P} + \kappa(1, n)]_{\dot{\alpha} \alpha} \Theta(\mathcal{P}; 1^+; \ldots, (n-1)^+; \mathcal{Q})$$

$$= \frac{-e\sqrt{2}}{(n-1)!} \frac{\bar{u}_\alpha(k_n)[\mathcal{P} + \bar{\kappa}(1, n)]^{\dot{\alpha} \alpha} u_\alpha(g)}{\langle n \mid g \rangle [\mathcal{P} + \kappa(1, n)]^2} \Theta(\mathcal{P}; 1^+; \ldots, (n-1)^+; \mathcal{Q})$$

(4.6)
We may iterate (4.6) until we reach \( \Theta(\mathcal{P}; \mathcal{Q}) \). The result is

\[
\Theta(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q}) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \Theta(\mathcal{P}; \mathcal{Q}) \prod_{\ell=1}^{n} \frac{\bar{u}_\alpha(k_\ell)(\bar{P} + \bar{Q}(1, \ell))^\alpha}{\langle \ell \mid g \rangle \langle \mathcal{P} + \kappa(1, \ell) \rangle^2} \Theta(\mathcal{P}; \mathcal{Q}) ^n \prod_{\ell=1}^{n} \bar{u}_\alpha(k_\ell)(\bar{P} + \bar{Q}(1, \ell))^\alpha u_\alpha(g).
\]

(4.7)

We recognize that (4.7) contains a factor of \( \Xi(1, n) \), as defined in Appendix A. Thus, we apply (A.25) to obtain

\[
\Theta(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q}) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \Theta(\mathcal{P}; \mathcal{Q}) \prod_{\ell=1}^{n} \frac{\bar{u}_\alpha(k_\ell)(\bar{P} + \bar{Q}(1, \ell))^\alpha}{\langle \ell \mid g \rangle \langle \mathcal{P} + \kappa(1, \ell) \rangle^2} u_\alpha(g) \Pi_{\alpha\beta}(\mathcal{P}, 1, 2, \ldots, \ell) u_\beta(g).
\]

(4.8)

Inserting (4.5), we find that

\[
\Theta(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q}) =
\]

\[
= i(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{1}{\langle \mathcal{P} \rangle^{1\ldots n | g \rangle}} \prod_{\ell=1}^{n} u_\alpha(g) \Pi_{\alpha\beta}(\mathcal{P}, 1, 2, \ldots, \ell) u_\beta(g).
\]

(4.9)

Equation (4.9) is valid for \( n \geq 1 \). The limit onto \( n = 0 \) is not smooth, and we must treat that case separately.

We note that the derivation for \( \Theta(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q}) \) matches the derivation of \( \Phi(\mathcal{P}; 1^+, \ldots, n^+) \) until the point at which the zero particle current is introduced. It is at this stage where \( P^2 = 0 \) was used to eliminate the sum over the various \( \Pi \)'s in the latter case. One might hope, in light of this similarity, that it would be possible to compute \( \Theta(\mathcal{P}; 1^-, 2^+, \ldots, n^+; \mathcal{Q}) \), a current with one unlike helicity. Unfortunately, because of the extra complications introduced by the form of (4.9) as compared to (2.6), we have been unable to obtain a simplified expression in this case.

### 4.2 The double-off-shell spinor current

We define the current \( \Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n; \mathcal{Q}) \) as consisting of a charged spinor line with \( n \) photons attached all possible ways. All momenta are directed inward.
The off-shell $e^+$ has momentum $\mathcal{P}$, $\mathcal{P}^2 \neq 0$. The off-shell $e^-$ has momentum $\mathcal{Q}$, $\mathcal{Q}^2 \neq 0$. We choose the line to be left-handed, as we are interested in diagrams that couple this current to the $W$ of the standard model. A right-handed current could also be defined analogously.

The zero-photon current is given by a propagator for the fermion:

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; \mathcal{Q}) = -\frac{i\mathcal{P}_{\alpha\dot{\alpha}}}{\mathcal{P}^2} = \frac{i\mathcal{Q}_{\alpha\dot{\alpha}}}{\mathcal{Q}^2}.$$ (4.10)

The only place in which the on-shell condition was used in the derivation of the recursion for $\bar{\psi}\dot{\alpha}(p^+; 1, 2, \ldots, n)$ is in the form of the zero photon current—that is, a massless spinor was used for the on-shell particle. If we remove it from (2.11a), we obtain the recursion relation for $\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n; \mathcal{Q})$:

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n; \mathcal{Q}) =$$

$$= -e\sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n-1; \mathcal{Q}) e^{\beta\dot{\beta}(n)} \frac{[\mathcal{P} + \kappa(1,n)]_{\beta\dot{\alpha}}}{[\mathcal{P} + \kappa(1,n)]^2}.$$ (4.11)

In the case of all like-helicity photons, we are able to solve (4.11) for the combination $u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q})$, where $g$ is the gauge momentum of the photons. Fortunately, for the processes under consideration in this paper, this is sufficient. We see this by noting that typically, one of the transverse $W$ currents is contracted into the undotted index of $\Psi$. Since all of the $W$’s are proportional to $u(g)$ (we may have to choose a specific value for $g$, however), the above combination is all that is required.

Contracting $u^\alpha(g)$ into both sides of (4.11) and employing (2.5) for the polarization spinors, we have

$$u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1^+, \ldots, n^+; \mathcal{Q}) =$$

$$= -e\sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1^+, \ldots, (n-1)^+; \mathcal{Q})$$

$$\times \bar{u}^\alpha(k_n) \frac{u^{\beta}(g)[\mathcal{P} + \kappa(1,n)]_{\beta\dot{\alpha}}}{\langle n\ g \rangle [\mathcal{P} + \kappa(1,n)]^2}.$$ (4.12)
From the form of (4.12), it is obvious that $u^\alpha(g)\Psi_{\alpha\dot{\alpha}}$ satisfies the following ansatz:

$$u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(P; 1^+, \ldots, n^+; Q) = u^\alpha(g)[P + \kappa(1, n)]_{\alpha\dot{\alpha}} \mathcal{Y}(P; 1^+, \ldots, n^+; Q), \quad (4.13)$$

where, according to (4.10), we have

$$\mathcal{Y}(P; Q) = -\frac{i}{P^2}. \quad (4.14)$$

If we insert (4.13) into (4.12), we obtain

$$\mathcal{Y}(P; 1^+, \ldots, n^+; Q) =$$

$$= -e\sqrt{2} \sum_{P(1\ldots n)} \frac{1}{(n-1)!} \bar{u}_\dot{\alpha}(k_n) [\bar{P} + \bar{\kappa}(1, n)]^{\dot{\alpha}\beta} u_\beta(g) \mathcal{Y}(P; 1^+, \ldots, (n-1)^+; Q). \quad (4.15)$$

A comparison of (4.15) with (4.6) reveals that $\Theta$ and $\mathcal{Y}$ satisfy the same recursion relation. Hence, the two solutions are proportional, differing only by the ratio of the zero-photon currents. Thus, we find another example of a SUSY-like relationship between currents of differing spins. Hence, we have

$$u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(P; 1^+, \ldots, n^+; Q) =$$

$$= -i(-e\sqrt{2})^n \sum_{P(1\ldots n)} \frac{u^\alpha(g)[P + \kappa(1, n)]_{\alpha\dot{\alpha}}}{\langle g|1, \ldots, n|g\rangle} \sum_{\ell=1}^n u^\alpha(g)\Pi_{\alpha\beta}(P, 1, 2, \ldots, \ell) u_\beta(g), \quad (4.16)$$

valid for $n \geq 1$. As in the scalar case, $n = 0$ remains separate.

### 4.3 The modified vector current

The case of a vector line with both ends off shell is much more difficult to solve in general. This is because the vector current with one off-shell $W$ is a conserved current, while the vector current with two off-shell $W$’s is not. Let us denote this current by $I_{\mu\nu}(P; 1, \ldots, n; Q)$. The $\mu$ index is associated with the incoming $W^+$ of momentum $P$, while the $\nu$ index belongs to the incoming $W^-$.
of momentum $Q$. Since the zero photon current is just a propagator, we have simply

$$I_{\mu\nu}(\mathcal{P}; Q) = -\frac{ig_{\mu\nu}}{p^2} = -\frac{ig_{\mu\nu}}{Q^2}. \quad (4.17)$$

It is straightforward to repeat the derivation of the transverse $W$ current given in reference [2] using (4.17) as the starting point, and avoiding the use of current conservation. Hence, we shall immediately present the result:

$$I_{\mu\nu}(\mathcal{P}; 1, 2, \ldots, n; Q) =$$

$$= \frac{-e}{[\mathcal{P} + \kappa(1, n)]^2} \left[ \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \left[ \epsilon(n), I(\mathcal{P}, 1, \ldots, n-1; Q) \right]_{\mu\nu} \right]$$

$$+ e \sum_{\mathcal{P}(1\ldots n)} \frac{1}{2!(n-2)!} \left\{ \epsilon(n-1), I(\mathcal{P}, 1, \ldots, n-2; Q), \epsilon(n) \right\}_{\mu\nu}, \quad (4.18)$$

where

$$\left[ \epsilon(n), I(\mathcal{P}, 1, \ldots, n-1; Q) \right]_{\mu\nu} \equiv$$

$$= I_{\mu\nu}(\mathcal{P}, 1, \ldots, n-1; Q) \epsilon(n) \cdot \left\{ 2[\mathcal{P} + \kappa(1, n-1)] + k_n \right\}$$

$$- I_{\mu\xi}(\mathcal{P}; 1, \ldots, n-1; Q) \left\{ [\mathcal{P} + \kappa(1, n-1)] + 2k_n \right\}^{\xi} \epsilon_{\nu}(n)$$

$$+ I_{\mu\xi}(\mathcal{P}; 1, \ldots, n-1; Q) \epsilon^{\xi}(n) \left\{ k_n - [\mathcal{P} + \kappa(1, n-1)] \right\}^{\nu}, \quad (4.19)$$

and

$$\left\{ \epsilon(n-1), I(\mathcal{P}, 1, \ldots, n-2; Q), \epsilon(n) \right\}_{\mu\nu} \equiv$$

$$= \epsilon^{\xi}(n-1) [\epsilon_{\xi}(n) I_{\mu\nu}(\mathcal{P}; 1, \ldots, n-2; Q) - I_{\mu\xi}(\mathcal{P}; 1, \ldots, n-2; Q) \epsilon_{\nu}(n)]$$

$$- \epsilon^{\xi}(n) [I_{\mu\xi}(\mathcal{P}; 1, \ldots, n-2; Q) \epsilon_{\nu}(n-1) - \epsilon_{\xi}(n-1) I_{\mu\nu}(\mathcal{P}; 1, \ldots, n-2; Q)]. \quad (4.20)$$

Equations (4.19) and (4.20) reduce to (2.17) and (2.18) respectively if we contract in a transverse polarization vector $W^\mu(\mathcal{P})$, multiply by $iP^2$ to remove the propagator for the $W^+$, and let $P^2 \to 0$.

As written, the recursion relation for the double off-shell $W$ current is prohibitively difficult to solve. Instead, let us define a “modified” transverse $W$
current, \(\mathcal{V}(P^*;1,\ldots,n)\), in the spirit of reference [10]. This current differs from the usual transverse \(W\) current in that \(P^2 \neq 0\), and

\[
\mathcal{V}_{\alpha\dot{\alpha}}(P^*) \equiv u_\alpha(g)u_\beta(g)P_{\beta\dot{\alpha}}
\]  

(4.21)

replaces \(W_{\alpha\dot{\alpha}}(P)\). The form of (4.21) is such that

\[
\bar{P}^{\dot{\alpha}}_{\alpha} \mathcal{V}_{\alpha\dot{\alpha}}(P^*) = 0,
\]  

(4.22)

just as if \(\mathcal{V}(P^*)\) were a real polarization spinor. Consequently, \(\mathcal{V}(P^*;1,\ldots,n)\) is a conserved current, just like \(W(P;1,\ldots,n)\). Furthermore, since \(\mathcal{V}(P^*)\) is proportional to \(u(g)\),

\[
\bar{\epsilon}^{\dot{\alpha}}_{\alpha}(j^+)\mathcal{V}_{\alpha\dot{\alpha}}(P^*) = 0
\]  

(4.23)

for the gauge choice (2.5). As a result of (4.22) and (4.23), \(\mathcal{V}(P^*;1^+,\ldots,n^+)\) satisfies the same simplified form of the recursion relation as \(W(P^+;1^+,\ldots,n^+)\), namely [2]

\[
\mathcal{V}_{\alpha\dot{\alpha}}(P^*;1^+,\ldots,n^+) = -e\sqrt{2} \sum_{P(1\ldots n)} \frac{1}{(n-1)!} \frac{[P + \kappa(1,n)]_{\beta\dot{\beta}}}{[P + \kappa(1,n)]^2}
\]

\[
\times \left[ \bar{\epsilon}^{\dot{\beta}}_{\beta}(n^+)\mathcal{V}_{\alpha\dot{\alpha}}(P^*;1^+,\ldots,(n-1)^+) - \bar{\nabla}^{\dot{\beta}}_{\beta}(P^*;1^+,\ldots,(n-1)^+)\epsilon_{\alpha\dot{\alpha}}(n^+) \right].
\]  

(4.24)

The form of (4.21) is consistent with the following ansatz for the spinor structure of \(\mathcal{V}(P^*;1^+,\ldots,n^+)\):

\[
\mathcal{V}(P^*;1^+,\ldots,n^+) = u_\alpha(g)u_\beta(g)[P + \kappa(1,n)]_{\beta\dot{\alpha}} \mathcal{Z}(P^*;1^+,\ldots,n^+),
\]  

(4.25)

with

\[
\mathcal{Z}(P^*) = 1.
\]  

(4.26)
The correctness of this ansatz is not immediately obvious, but it is easily proven inductively. Assume that the ansatz is true for the \((n-1)\)-particle current. Then, the \(n\)-particle current is given by

\[
\mathcal{V}_{\alpha \bar{\alpha}}(P^*; 1^+, \ldots, n^+) = -e \sqrt{2} \sum_{P(1 \ldots n)} \frac{1}{(n-1)!} \frac{\mathcal{Z}(P^*; 1^+, \ldots, (n-1)^+)}{\langle n \ g \rangle [P + \kappa(1, n)]^2} \times \left\{ u^\beta(g)[P + \kappa(1, n)]_{\beta \beta} \bar{u}^\gamma(k_n) u^\gamma(g)[P + \kappa(1, n-1)]_{\gamma \bar{\gamma}} + u^\beta(g)[P + \kappa(1, n)]_{\beta \beta} \bar{P} + \bar{\kappa}(1, n-1) \bar{\beta} \bar{\gamma} \ u_\gamma(g) u^\alpha(g) \bar{u}_\alpha(k_n) \right\}.
\]

We simplify the quantity in curly brackets may be simplified using (A.6), obtaining

\[
u_\alpha(g) \left\{ u^\beta(g)[P + \kappa(1, n)]_{\beta \beta} \bar{u}^\gamma(k_n) u^\gamma(g)[P + \kappa(1, n-1)]_{\gamma \bar{\gamma}} + u^\beta(g)[P + \kappa(1, n)]_{\beta \beta} \bar{P} + \bar{\kappa}(1, n-1) \bar{\beta} \bar{\gamma} \ u_\gamma(g) \right\} = u_\alpha(g) \bar{u}_\beta(k_n) \bar{P} + \bar{\kappa}(1, n-1) \bar{\beta} \bar{\gamma} u_\gamma(g)
\]

If we insert (4.28) into (4.27), we see that the following recursion relation for \(\mathcal{Z}(P^*; 1^+, \ldots, n^+)\) must hold:

\[
\mathcal{Z}(P^*; 1^+, \ldots, n^+) = -e \sqrt{2} \sum_{P(1 \ldots n)} \frac{1}{(n-1)!} \frac{\bar{u}_\beta(k_n)[P + \kappa(1, n)]_{\beta \beta} \bar{u}^\gamma(g)}{\langle n \ g \rangle [P + \kappa(1, n)]^2} \mathcal{Z}(P^*; 1^+, \ldots, (n-1)^+).
\]

When (4.29) is compared to (4.6), we see that \(\mathcal{Z}\) also satisfies the same recursion relation as \(\Theta\). Thus, we immediately write down

\[
\mathcal{V}(P^*; 1^+, \ldots, n^+) = (-e \sqrt{2})^n \sum_{P(1 \ldots n)} \frac{P^2}{\langle g \rangle [1, \ldots, n \ g]} \sum_{\ell=1}^n u^\alpha(g) \Pi^\beta_{\alpha} (P, 1, 2, \ldots, \ell) u_\beta(g).
\]
V. TRIPLE CURRENT AMPLITUDES

In this section we will examine the amplitudes which may be computed from a combination of three currents. Because one of these currents must have two off-shell particles, we are limited to those cases with like-helicity photons. The process we will illustrate the computational methods with is

\[ \bar{e}_\uparrow \nu_\downarrow \rightarrow W_L H \gamma_\uparrow \cdots \gamma_\uparrow. \]  

(5.1)

We have selected (5.1) since it demonstrates how the modified vector current \( V \) fits into the picture. It is straightforward to apply \( \Theta \) and \( \Psi \) to the amplitudes in which they appear.

5.1 The process \( \bar{e}_\uparrow \nu_\downarrow \rightarrow W_L H \gamma_\uparrow \cdots \gamma_\uparrow \)

Figure 2 illustrates the Feynman diagrams describing the process (5.1) and indicates the momentum routing that has been chosen in order to evaluate them. According to Figure 2, we have

\[ \mathcal{M}(p, q; H, \nu; 1, \ldots, n) = \]

\[ = -\frac{g^2}{2\sqrt{2}} \sum_{\mathcal{P}(1 \ldots n)} \sum_{s=0}^{n} \sum_{t=s}^{n} \frac{1}{s!(t-s)!(n-t)!} \bar{\psi}(p; 1, \ldots, s) \gamma_\lambda \frac{1}{2} (1 - \gamma_5) \ u(\nu) \]

\[ \times \Phi(t+1, \ldots, n; q) \left[ H - \{ q + \kappa(t+1, n) \} \right] I^\mu\lambda(\mathcal{P}; s+1, \ldots, t; \mathcal{Q}) \]

\[ - \frac{g^2e}{2\sqrt{2}} \sum_{\mathcal{P}(1 \ldots n)} \sum_{s=0}^{n-1} \sum_{t=s}^{n-1} \frac{1}{s!(t-s)!(n-t-1)!} \bar{\psi}(p; 1, \ldots, s) \gamma_\lambda \frac{1}{2} (1 - \gamma_5) \ u(\nu) \]

\[ \times \Phi(t+2, \ldots, n; q) \epsilon_\mu(t+1) I^\mu\lambda(\mathcal{P}; s+1, \ldots, t; \mathcal{Q}). \]  

(5.2)

In (5.2) the positron has momentum \( p \), the \( W_L^- \) has momentum \( q \), the Higgs boson has momentum \( H \), and the neutrino has momentum \( \nu \). All of these momenta are directed into the diagram. We also have defined

\[ \mathcal{P} \equiv p + \kappa(1, s) + \nu \]  

(5.3)
and

$$Q \equiv H + \kappa(t+1, n) + q.$$  \hspace{1cm} (5.4)

Note the presence of $I^\mu\lambda(\mathcal{P}; s+1, \ldots, t; Q)$, the vector current with both $W$’s off shell. The momenta $\mathcal{P}$ and $Q$ are directed towards the center of this current. Thus, $\mathcal{P}$ is the momentum of the off-shell $W^+$, while $Q$ is the momentum of the off-shell $W^−$.

We now demonstrate how to connect $I^\mu\lambda$ to $\mathcal{V}_{\alpha\dot{\alpha}}$. One way to obtain the current with two off-shell $W$’s is to begin with a current with just one off-shell $W$, remove the polarization vector of the other $W$ by differentiation, and supply a propagator for the newly off-shell particle. Hence, we formally write

$$I^\mu\lambda(\mathcal{P}; s+1, \ldots, t; Q) = -\frac{i}{\mathcal{P}^2} \frac{\partial}{\partial \epsilon_{\lambda}(\mathcal{P})} W^\mu(\mathcal{P}; s+1, \ldots, t). \hspace{1cm} (5.5)$$

Applying (5.5) to (5.2) and translating to multispinor form via (A.22) and (A.23), we have

$$\mathcal{M}(p^+, q^0; H^0, \nu^−; 1^+, \ldots, n^+) =$$

$$= \frac{ig^2}{2\sqrt{2}} \sum_{\mathcal{P}(1\ldots n)} \sum_{s=0}^{n-1} \sum_{t=s}^{n} \frac{1}{s!(t-s)!(n-t)!} \bar{\psi}_{\dot{\beta}}(p^+; 1^+, \ldots, s^+) u_{\beta}(\nu)$$

$$\times \Phi((t+1)^+, \ldots, n^+; q) \left[ H - \{q + \kappa(t+1, n)\} \right]_{\alpha\dot{\alpha}}$$

$$\times \frac{1}{\mathcal{P}^2} \frac{\partial}{\partial \epsilon_{\beta}(\mathcal{P})} \bar{\mathcal{W}}^{\dot{\alpha}}(\mathcal{P}; (s+1)^+, \ldots, t^+)$$

$$+ \frac{ig^2 e}{2} \sum_{\mathcal{P}(1\ldots n)} \sum_{s=0}^{n-1} \sum_{t=s}^{n-1} \frac{1}{s!(t-s)!(n-t-1)!} \bar{\psi}_{\dot{\beta}}(p^+; 1^+, \ldots, s^+) u_{\beta}(\nu)$$

$$\times \Phi((t+2)^+, \ldots, n^+; q) \epsilon_{\alpha\dot{\alpha}}((t+1)^+)$$

$$\times \frac{1}{\mathcal{P}^2} \frac{\partial}{\partial \epsilon_{\beta}(\mathcal{P})} \bar{\mathcal{W}}^{\dot{\alpha}}(\mathcal{P}; (s+1)^+, \ldots, t^+) \hspace{1cm} (5.6)$$

In (5.6) we have specialized to the helicity configuration we are able to compute. In both terms of (5.6) we are instructed to remove the polarization spinor for
the $W^+$ and replace it by $\bar{\psi}_\beta(p^+; 1^+, \ldots, s^+) u_\nu(\nu)$. But, according to (2.14a), we may write

$$\bar{\psi}_\beta(p^+; 1^+, \ldots, s^+) = u^\gamma(g)[p + \kappa(1, s)]\gamma_\beta Y(p^+; 1^+, \ldots, s^+)$$

where

$$Y(p^+; 1^+, \ldots, s^+) \equiv \sum_{\mathcal{P}(1...s)} \frac{-(-e\sqrt{2})^s}{\langle p|1, \ldots, s|g\rangle}.$$  \tag{5.8}$$

Thus, the spinor structure of what we replace the polarization spinor with is

$$u^\gamma(\nu)[p + \kappa(1, s)]\gamma_\beta u_\nu(\nu) = u_\beta(\nu)u^\gamma(\nu)\mathcal{P}\gamma_\beta,$$  \tag{5.9}$$

where we have chosen $g = \nu$ and applied the Weyl equation along with (5.3) to obtain a form that matches (4.21). Consequently, (5.6) becomes

$$\mathcal{M}(p^+, q^0; H^0, \nu^-, 1^+, \ldots, n^+) =$$

$$= \frac{ig^2}{2\sqrt{2}} \sum_{\mathcal{P}(1...n)} \sum_{s=0}^{n} \sum_{t=s}^{n} \frac{1}{s!(t-s)!(n-t)!} \frac{1}{\mathcal{P}^2} Y(p^+; 1^+, \ldots, s^+)$$

$$\times [H - \{q + \kappa(t+1, n)\}]\bar{\mathcal{V}}^{\alpha\alpha}(\mathcal{P}^*; (s+1)^+, \ldots, t^+)$$

$$\times \Phi((t+1)^+, \ldots, n^+; q)$$

$$+ \frac{ig^2e}{2} \sum_{\mathcal{P}(1...n)} \sum_{s=0}^{n-1} \sum_{t=s}^{n-1} \frac{1}{s!(t-s)!(n-t-1)!} \frac{1}{\mathcal{P}^2} Y(p^+; 1^+, \ldots, s^+)$$

$$\times \bar{\mathcal{V}}^{\dot{\alpha}\dot{\alpha}}(\mathcal{P}^*; (s+1)^+, \ldots, t^+)\epsilon_{\dot{\alpha}\dot{\alpha}}((t+1)^+)$$

$$\times \Phi((t+2)^+, \ldots, n^+; q).$$  \tag{5.10}$$

Let us insert (4.25) to incorporate the spinor structure of $\bar{\mathcal{V}}^{\dot{\alpha}\dot{\alpha}}$. Because of (2.5), the second term in (5.10) vanishes. This leaves just

$$\mathcal{M}(p^+, q^0; H^0, \nu^-, 1^+, \ldots, n^+) =$$

$$= \frac{ig^2}{2\sqrt{2}} \sum_{\mathcal{P}(1...n)} \sum_{s=0}^{n} \sum_{t=s}^{n} \frac{1}{s!(t-s)!(n-t)!} \frac{1}{\mathcal{P}^2} Y(p^+; 1^+, \ldots, s^+)$$

$$\times \mathcal{Z}(\mathcal{P}^*; (s+1)^+, \ldots, t^+)\Phi((t+1)^+, \ldots, n^+; q)$$

$$\times [\vec{H} - \{\vec{q} + \vec{\kappa}(t+1, n)\}]^{\alpha\dot{\alpha}} u_\alpha(\nu)u^\beta(\nu)[\mathcal{P} + \kappa(s+1, t)]_{\beta\dot{\alpha}}.$$  \tag{5.11}$$
Let us examine the last line of factors appearing in (5.11), which we will call $\chi$.

Applying (5.3), we find that

$$
\chi = u^\beta(\nu)[p + \kappa(1, t) + \nu]_{a} [2\bar{H} - \{\bar{H} + \bar{\kappa}(t+1, n) + \bar{q}\}]^{\dot{a}} \alpha u_\alpha(\nu)
$$

(5.12)

where we have used momentum conservation in the second line. Because of (A.6) and the antisymmetry of the spinor product, this reduces to

$$
\chi = -2u^\beta(\nu)[H + \kappa(t+1, n) + q]_{a} [\bar{H} - \{\bar{H} + \bar{\kappa}(s+1, n) + \bar{q}\}]^{\dot{a}} \alpha u_\alpha(\nu).
$$

(5.13)

At this stage, we insert (2.3) and (2.6) for $\Phi$, (4.26) and (4.30) for $Z$, (5.8) for $Y$, and (5.13) for $\chi$ back into (5.11). Because of the special form of $Z$ when $t = s$, we obtain two separate contributions. After collecting related factors, these two terms are

$$
\mathcal{M}_1 \equiv \frac{i g^2}{\sqrt{2}} \langle -e \sqrt{2} \rangle \sum_{\mathcal{P}(1...n)} \sum_{s=0}^{n} \frac{\langle H \nu \rangle \langle \nu q \rangle \bar{u}_{\dot{a}}(H)[\bar{H} + \bar{\kappa}(s+1, n) + \bar{q}]^{\dot{a}} \alpha u_\alpha(\nu)}{(p|1, \ldots, s|\nu \rangle \langle \nu |s+1, \ldots, n|q)}[p + \kappa(1, s) + \nu]^2
$$

(5.14)

and

$$
\mathcal{M}_2 \equiv \frac{i g^2}{\sqrt{2}} \langle -e \sqrt{2} \rangle \sum_{\mathcal{P}(1...n)} \sum_{\ell=1}^{n} \sum_{t=\ell}^{n} \sum_{s=0}^{\ell-1} \frac{\bar{u}_{\dot{a}}(H)[\bar{\kappa}(t+1, n) + \bar{q}]^{\dot{a}} \alpha u_\alpha(\nu)}{(p|1, \ldots, s|\nu \rangle \langle \nu |s+1, \ldots, t|\nu)} \times \frac{\langle H \nu \rangle \langle \nu q \rangle}{\langle \nu |t+1, \ldots, n|q \rangle} u^\gamma(\nu) \Pi^\delta(\mathcal{P}, s+1, \ldots, \ell) u_\delta(\nu).
$$

(5.15)

Note that we have reordered the sums appearing in (5.15) so that the sum on $\ell$, which came from $Z$, is to be done last.

Since it is not immediately obvious how to simplify $\mathcal{M}_1$, we begin with $\mathcal{M}_2$. Since $\mathcal{P}$ is never the last argument of $\Pi$, we have

$$
\Pi(\mathcal{P}, s+1, \ldots, \ell) = \Pi(p, \nu, 1, \ldots, \ell),
$$

(5.16)
where we have applied (5.3) and exploited the definition (2.10) for $\Pi$. Thus, the $s$-dependence of (5.15) may be summarized in

$$
\sum_{s=0}^{\ell-1} \langle s \, s+1 \rangle = \frac{\langle p \, \ell \rangle}{\langle p | \nu | \ell \rangle}.
$$

(5.17)

Application of (5.17) to (5.15) produces

$$
\mathcal{M}_2 = \frac{-i g^2}{\sqrt{2}} (-\epsilon \sqrt{2})^n \sum_{\mathcal{P}(1...n)} \sum_{t=1}^{n} \frac{\langle H \, \nu \rangle \langle \nu \, q \rangle}{\langle p \, \nu \rangle} \frac{\bar{u}_\alpha(H)[\bar{\kappa}(t+1, n) + \bar{q}]^{\bar{\alpha}} u_\alpha(\nu)}{\langle p|1, \ldots, t|\nu\rangle \langle \nu|t+1, \ldots, n|q \rangle}
\times \frac{\langle H \, \nu \rangle \langle \nu \, q \rangle}{\langle p \, \nu \rangle} u_\gamma(p) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(\nu).
$$

(5.18)

The sum on $t$ appearing in (5.18) is not much harder, although we do have to deal with the $t$-dependence of the numerator:

$$
\sum_{t=t}^{n+1} \frac{\langle t \, t+1 \rangle}{\langle t | \nu | t+1 \rangle} (u \, H)^* \langle u \, \nu \rangle = \sum_{t=t}^{n+1} \frac{\langle \ell \, u \rangle}{\langle \ell | \nu | u \rangle} (u \, H)^* \langle u \, \nu \rangle
\times \frac{\langle \ell \, u \rangle}{\langle \ell | \nu | u \rangle} u_\gamma(p) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(\nu).
$$

(5.19)

Thus,

$$
\mathcal{M}_2 = \frac{-i g^2}{\sqrt{2}} (-\epsilon \sqrt{2})^n \sum_{\mathcal{P}(1...n)} \sum_{t=1}^{n} \frac{\langle H \, \nu \rangle \langle \nu \, q \rangle}{\langle p \, \nu \rangle} \frac{\bar{u}_\alpha(H)[\bar{\kappa}(t+1, n) + \bar{q}]^{\bar{\alpha}} u_\alpha(\nu)}{\langle p|1, \ldots, t|\nu\rangle \langle \nu|t+1, \ldots, n|q \rangle}
\times \frac{\langle \ell \, u \rangle}{\langle \ell | \nu | u \rangle} u_\gamma(p) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(\nu).
$$

(5.20)

At this stage, we apply (A.28) to turn $u_\gamma(p) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(\nu)$ into $u_\gamma(\nu) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(p)$. The result of this manipulation is

$$
\mathcal{M}_{2A} \equiv \frac{-i g^2}{\sqrt{2}} (-\epsilon \sqrt{2})^n \sum_{\mathcal{P}(1...n)} \sum_{t=1}^{n} \frac{\langle H \, \nu \rangle \langle \nu \, q \rangle}{\langle p \, \nu \rangle} \frac{1}{\langle p|1, \ldots, n|q \rangle}
\times \bar{u}_\alpha(H)[\bar{H} + \bar{\kappa}(\ell+1, n) + \bar{q}]^{\bar{\alpha}} u_\alpha(\nu) \Pi_\gamma^{\delta}(p, \nu, 1, \ldots, \ell) u_\delta(p),
$$

(5.21)

$$
\mathcal{M}_{2B} \equiv \frac{-i g^2}{\sqrt{2}} (-\epsilon \sqrt{2})^n \sum_{\mathcal{P}(1...n)} \sum_{t=1}^{n} \frac{\langle H \, \nu \rangle \langle \nu \, q \rangle}{\langle p \, \nu \rangle} \frac{\bar{u}_\alpha(H)[\bar{\kappa}(t+1, n) + \bar{q}]^{\bar{\alpha}} u_\alpha(\nu)}{\langle p|1, \ldots, n|q \rangle}
\times \frac{1}{\langle \nu |\ell \rangle} \left[ \frac{1}{|p + \nu + \kappa(1, \ell-1)|^2} - \frac{1}{|p + \nu + \kappa(1, \ell)|^2} \right].
$$

(5.22)
We have written the contributions from the three terms of (A.28) as two separate pieces since we will deal with them differently.

We begin by considering $M_{2A}$. Momentum conservation allows us to replace $[\bar{H} + \bar{\kappa}(\ell+1, n) + \bar{q}]\hat{\alpha}\alpha$ by $-[\bar{p} + \bar{\nu} + \bar{\kappa}(1, \ell)]\hat{\alpha}\alpha$. Hence, we may apply (A.27) and perform the sum on $\ell$, giving

$$M_{2A} = \frac{ig^2}{\sqrt{2}}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle\langle \nu q \rangle}{\langle p \nu \rangle\langle p|1,\ldots,n|q \rangle} \frac{1}{\langle p|1,\ldots,n|q \rangle} \times \bar{u}_\alpha(H) \left[ \frac{[\bar{p} + \bar{\nu}]\hat{\alpha}\delta}{|p + \nu|^2} - \frac{[\bar{p} + \bar{\nu} + \bar{\kappa}(1, n)]\hat{\alpha}\delta}{|p + \nu + \kappa(1, n)|^2} \right] \bar{u}(p).$$

(Momentum conservation plus straightforward spinor algebra produces

$$M_{2A} = \frac{ig^2}{\sqrt{2}}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle\langle \nu q \rangle}{\langle p \nu \rangle\langle p|1,\ldots,n|q \rangle} \frac{1}{\langle p|1,\ldots,n|q \rangle} \times \bar{u}_\alpha(H) \left[ \frac{[\bar{p} + \bar{\nu}]\hat{\alpha}\delta}{|p + \nu|^2} - \frac{[\bar{p} + \bar{\nu} + \bar{\kappa}(1, n)]\hat{\alpha}\delta}{|p + \nu + \kappa(1, n)|^2} \right] \bar{u}(p).$$

(5.24)

We now return to $M_{2B}$. The form of this contribution is such that we hope to combine it with $M_1$ in some manner. To this end, we shift the sum on $\ell$ by 1 in the first of the two contributions to (5.22):

$$M_{2B} = \frac{-ig^2}{\sqrt{2}}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \sum_{\ell=0}^{n-1} \frac{\langle H \nu \rangle\langle \nu q \rangle}{\langle p|1,\ldots,n|q \rangle} \frac{\bar{u}_\alpha(H)[\bar{\kappa}(\ell+2, n) + \bar{q}]\hat{\alpha}\alpha}{|p + \nu + \kappa(1, \ell)|^2} \bar{u}_\alpha(k_{\ell+1})$$

$$+ \frac{ig^2}{\sqrt{2}}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \sum_{\ell=1}^{n} \frac{\langle H \nu \rangle\langle \nu q \rangle}{\langle p|1,\ldots,n|q \rangle} \frac{\bar{u}_\alpha(H)[\bar{\kappa}(\ell+1, n) + \bar{q}]\hat{\alpha}\alpha}{|p + \nu + \kappa(1, \ell)|^2} \bar{u}_\alpha(k_{\ell})$$

(5.25)

The Weyl equation allows us to extend the momentum sum appearing in the first term of (5.25) to $\kappa(\ell+1, n)$, matching the second term. We now extend both sums to the range $\ell \in [0, n]$, and compensate. Note that we choose to let $k_0 \equiv p$
and \( k_{n+1} \equiv q \) respectively. Thus, we obtain

\[
\mathcal{M}_{2B} = \frac{-ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \sum_{\ell=0}^{n} \frac{\langle H \nu \rangle \langle \nu \rangle q}{\langle p|1,\ldots,n|q \rangle} \times \frac{\bar{u}_\alpha (H) \bar{\kappa}(\ell+1, n) + \bar{q}^\dagger \alpha}{[p + \nu + \kappa(1, \ell)]^2} \left[ \frac{u_\alpha (k_{\ell+1})}{\langle \nu \ell+1 \rangle} - \frac{u_\alpha (k_\ell)}{\langle \nu \ell \rangle} \right] \\
+ \frac{ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle \langle \nu \rangle q}{\langle p|1,\ldots,n|q \rangle} \bar{u}_\alpha (H) \bar{q}^\dagger \alpha \frac{u_\alpha (p)}{[p + \nu + \kappa(1, n)]^2} \langle \nu q \rangle \\
- \frac{ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \langle H \nu \rangle \langle \nu \rangle q \bar{u}_\alpha (H) \bar{\kappa}(1, n) + \bar{q}^\dagger \alpha \frac{u_\alpha (p)}{[p + \nu]^2} \langle \nu p \rangle. \\
\tag{5.26}
\]

The first term of (5.26) contains

\[
\frac{u_\alpha (k_{\ell+1})}{\langle \nu \ell+1 \rangle} - \frac{u_\alpha (k_\ell)}{\langle \nu \ell \rangle} = u_\alpha (\nu) \frac{\langle \ell \ell+1 \rangle}{\langle \ell|\nu|\ell+1 \rangle}, \\
\tag{5.27}
\]

where we have applied (A.14). The second term vanishes because of the Weyl equation. We use momentum conservation and a little bit of spinor algebra to simplify the third term. The result of these manipulations is

\[
\mathcal{M}_{2B} = \frac{-ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \sum_{\ell=0}^{n} \frac{\langle H \nu \rangle \langle \nu \rangle q \bar{u}_\alpha (H) \bar{\kappa}(\ell+1, n) + \bar{q}^\dagger \alpha u_\alpha (\nu)}{\langle p|1,\ldots,\ell|\nu\ell+1,\ldots,n|q \rangle [p + \kappa(1, \ell) + \nu]^2} \\
- \frac{ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle \langle \nu \rangle q}{\langle p|\nu \rangle} \frac{\langle H \nu \rangle}{\langle p|\nu \rangle} \frac{1}{\langle p|1,\ldots,n|q \rangle}. \\
\tag{5.28}
\]

The first term of (5.28) exactly cancels the contribution from \( \mathcal{M}_1 \) given in (5.14). The second term of (5.28) disposes of the first term of (5.24), leaving the second term from \( \mathcal{M}_{2A} \) as the sole surviving contribution. Therefore, we have

\[
\mathcal{M}(p, q; H, \nu; 1, \ldots, n) = \frac{ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle \langle \nu \rangle q}{\langle p|\nu \rangle} \frac{\langle p|q \rangle}{\langle p|H q \rangle} \frac{\langle p|1,\ldots,n|q \rangle}{\langle p|1,\ldots,n|q \rangle}. \\
\tag{5.29}
\]

We may apply (A.19) to write this in the form

\[
\mathcal{M}(p, q; H, \nu; 1, \ldots, n) = \frac{ig^2}{\sqrt{2}} (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\langle H \nu \rangle \langle \nu \rangle q}{\langle p|\nu \rangle} \frac{\langle p|q \rangle}{\langle p|H q \rangle} \frac{n}{\prod_{j=1}^{n} \langle p|j|q \rangle}. \\
\tag{5.30}
\]
5.2 Summary of triple current amplitudes

We now summarize the results of those processes we are able to compute from three currents. These are limited to the production of like-helicity photons since the double-off-shell currents are limited to that case. We adopt the “standard” process to be

\[ PQ N_1 N_2 \rightarrow \gamma \gamma \cdots \gamma. \]  

(5.31)

In (5.31), \( P \) is a positively-charged particle of momentum \( P \), \( Q \) is a negatively-charged particle of momentum \( Q \), \( N_1 \) and \( N_2 \) are neutral particles of momenta \( N_1 \) and \( N_2 \). The \( n \) photons have momenta \( k_1, \ldots, k_n \). The results presented below have all momenta directed into the diagrams. The helicity labels applied to quantities in this section reflect this convention. Amplitudes for variants on the above process are easily obtained from crossing symmetry.

It should be noted that not every diagram contributing to the processes listed in this section contains three currents. In particular, the amplitudes involving a pair of transversely-polarized \( W \)’s have at least two types of contributions. First, there is the three-current type of diagram where the \( W_T \) line is broken, with a double-off-shell fermion or scalar current in the intervening space. The \( W \)’s can annihilate into a \( Z \), however, and the \( Z \) can then pair-produce the fermion-antifermion or scalar pair. Such diagrams contain only two currents. In the neutral scalar case, there is also the possibility of a seagull vertex that joins two \( W \)’s and two \( \phi \)’s directly. For the cases considered here, this vertex always vanishes, since it contains \( W_{\alpha \dot{\alpha}} \bar{W}^{\dot{\alpha} \alpha} = 0 \).
The triple current amplitudes may be cast into the form

$$\mathcal{M}(P, Q; N_1, N_2; 1^+, \ldots, n^+) = \frac{ig^2}{2}(-e\sqrt{2})^n F(P, Q, N_1, N_2) \langle P Q \rangle^{n-1} \prod_{j=1}^{n} \langle P|j|Q \rangle,$$

(5.32)

where the scalar function $F(P, Q, N_1, N_2)$ depends on the identities and helicities of the four particles. The values of these helicity functions are listed in Table 3.

Table 3: Triple current amplitude helicity functions

| $P$ | $Q$ | $N_1$ | $N_2$ | $F(P, Q, N_1, N_2)$ |
|-----|-----|-------|-------|---------------------|
| $W^+_{\uparrow}$ | $W^-_{\downarrow}$ | $\bar{\nu}^\uparrow$ | $\nu_{\downarrow}$ | $2\langle N_1 Q \rangle^2\langle N_2 Q \rangle\langle N_1 N_2 \rangle^{-1}\langle P N_2 \rangle^{-1}$ |
| $W^+_{\downarrow}$ | $W^-_{\downarrow}$ | $\bar{\nu}^\uparrow$ | $\nu_{\downarrow}$ | $-2\langle N_1 P \rangle^3\langle N_1 N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
| $W^+_{\uparrow}$ | $W^-_{\downarrow}$ | $H$ | $H$ | $\langle P Q \rangle\langle N_1 N_2 \rangle\langle N_1 P N_2 \rangle^{-1}$ |
| $W^+_{\uparrow}$ | $W^-_{\downarrow}$ | $Z_L$ | $H$ | $-i\langle N_1 Q N_2 \rangle\langle N_1 N_2 \rangle^{-1}\sum_{\ell=1}^{2}\langle N_\ell Q \rangle\langle P N_\ell \rangle^{-1}$ |
| $W^+_{\uparrow}$ | $W^-_{\downarrow}$ | $Z_L$ | $Z_L$ | $\langle P Q \rangle\langle N_1 N_2 \rangle\langle N_1 P N_2 \rangle^{-1}$ |
| $W^+_{\downarrow}$ | $W^-_{\downarrow}$ | $H$ | $H$ | $\langle P Q \rangle\langle N_1 N_2 \rangle\langle N_1 P N_2 \rangle^{-1}$ |
| $W^+_{\downarrow}$ | $W^-_{\downarrow}$ | $Z_L$ | $H$ | $-i\langle N_1 P N_2 \rangle\langle N_1 N_2 \rangle^{-1}\sum_{\ell=1}^{2}\langle P N_\ell \rangle\langle N_\ell Q \rangle^{-1}$ |
| $W^+_{\downarrow}$ | $W^-_{\downarrow}$ | $Z_L$ | $Z_L$ | $\langle P Q \rangle\langle N_1 N_2 \rangle\langle N_1 P N_2 \rangle^{-1}$ |
| $\bar{e}^\uparrow$ | $e_{\downarrow}$ | $\bar{\nu}^\uparrow$ | $\nu_{\downarrow}$ | $-2\langle N_2 Q \rangle^2\langle P Q \rangle\langle P N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
| $\bar{e}^\uparrow$ | $W_{L\downarrow}$ | $H$ | $\nu_{\downarrow}$ | $\sqrt{2}\langle N_1 N_2 \rangle\langle N_2 Q \rangle\langle P Q \rangle\langle P N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
| $\bar{e}^\uparrow$ | $W_{L\downarrow}$ | $Z_L$ | $\nu_{\downarrow}$ | $i\sqrt{2}\langle N_1 N_2 \rangle\langle N_2 Q \rangle\langle P Q \rangle\langle P N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
| $W_{L\uparrow}$ | $e_{\downarrow}$ | $\bar{\nu}^\uparrow$ | $H$ | $\sqrt{2}\langle P Q \rangle^2\langle Q N_2 \rangle\langle P N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
| $W_{L\uparrow}$ | $e_{\downarrow}$ | $\nu_{\downarrow}$ | $Z_L$ | $-i\sqrt{2}\langle P Q \rangle^2\langle Q N_2 \rangle\langle P N_2 \rangle^{-1}\langle N_1 Q \rangle^{-1}$ |
Not listed in Table 3 are the combinations involving two positive-helicity $W$’s. These amplitudes vanish for all-positive-helicity photon production. Although the amplitudes involving two negative-helicity $W$’s do not vanish, we are unable to compute them from the available currents. Such amplitudes would require the simultaneous use of $W(P^-; 1^+, \ldots, s^+)$ and $W((t+1)^+, \ldots, n^+; Q^-)$. The known solution for the first of these two quantities requires that the gauge momentum be $g \equiv P$, while the latter requires that $g \equiv Q$.

The amplitudes involving a pair of neutral scalars forms an interesting series. While Bose symmetry dictates that the amplitude for the production of two $H$’s or two $Z_L$’s be symmetric under the interchange of the momenta of the two scalars, no such requirement holds for the production of a one $H$ plus one $Z_L$. Instead, the form of the Feynman rules forces the amplitude to be antisymmetric under the interchange of $H$ and $Z_L$. This may be traced back to the identifications of the Higgs with the $\phi_1$ and the $Z_L$ with the $\phi_2$ of the unbroken theory. Only the combinations $\phi_1 \pm i\phi_2$ appear as part of the original complex scalar doublet of hypercharge $+1$. The extra $i$ in front of the $\phi_2$ appears in the vertices, and leads to the antisymmetry between $H$ and $Z_L$ just mentioned. It is also the source of the differing relative signs between the last two pairs of amplitudes.

Conspicuously absent from Table 3 is

$$ W_L^+ W_L^- H H \rightarrow \gamma \gamma \cdots \gamma $$

(5.33)

as well as its relatives involving $Z_L$’s. The reason for this is the presence of diagrams containing the double-off-shell vector current, but contracted into scalar vertices at both ends. In the mixed case ($W_L H e \nu$), it was always possible to avoid the complications caused by the seagulls on the scalar line by making the fermion line into the effective polarization spinor, and using the form of the modified $W$
current to eliminate the seagulls. If both lines are scalars, however, it is not apparent where to begin. In order to show that the modified $W$ current even appears in the amplitude, it seems that one must eliminate the seagulls first. But eliminating the seagulls requires knowledge of the current tying the two scalar lines together! Hence, we have been unable to obtain expressions for the process (5.33).

**VI. CONCLUSION**

We have seen how to combine the currents of reference [2] in groups of four to produce helicity amplitudes for processes containing two charged lines. An important example of this type of process involves the production of a $W^+W^-$ pair from an $e\bar{e}$ pair. We are able to obtain expressions for amplitudes containing photons of all the same helicity or one differing helicity. The latter type of amplitude cannot be easily obtained from within the $U(N)$ framework of reference [1]. We have considered currents which have both ends of the charged line off shell. We have found that except for the case of the double off-shell scalar current, we cannot solve for these quantities directly. We are able, however, to find expressions for the combination of the spinor and vector double off-shell currents with some other suitable factors. These combinations occur naturally in amplitudes containing a pair of Higgs particles, transversely-polarized $Z$-bosons, or neutrinos. We are limited to all like-helicity photons in this case.

All of the cautions mentioned in the conclusion to reference [2] apply to the results obtained here. In particular, it is potentially difficult to square most of the amplitudes in this paper. A notable exception is the set of triple current amplitudes, which may be squared trivially. A second potential difficulty involves
the finite masses of the particles. In those special regions of phase space where some of the invariants formed from pairs of momenta are of the same order as the neglected masses, the corrections to the amplitudes presented here are potentially large. A further problem lies in the fact that all of the amplitudes presented here are infrared divergent. In principle, the satisfactory treatment of the divergences involves a knowledge of loop diagrams. Finally, current experimental capabilities preclude the measurement of helicity-projected amplitudes. Thus, it would be desirable to obtain a complete set of helicity amplitudes, in order to be able to sum over helicities. In spite of the difficulties, however, from the large variety of processes for which we have obtained amplitudes, it is clear that the combination of multispinors, the equivalence theorem, and recursion relations forms a powerful tool in the study of the high energy limit of the Standard Model.

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A. MULTISPINOR CONVENTIONS

Below we list the important results of application of Weyl-van der Waerden spinor calculus to gauge theories. Readers interested in the details should refer to references [1] and [11].

We use the Weyl basis

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \]  

(A.1)

for the Dirac matrices. In (A.1), \( \sigma^\mu \) and \( \bar{\sigma}^\mu \) refer to the convenient Lorentz-covariant grouping of the \( 2 \times 2 \) Pauli matrices plus the unit matrix:

\[ \sigma^\mu \equiv (1, \vec{\sigma}), \]  

(A.2a)

\[ \bar{\sigma}^\mu \equiv (1, -\vec{\sigma}), \]  

(A.2b)

and satisfy the anticommutators

\[ (\bar{\sigma}^\mu)^{\dot{\alpha} \dot{\beta}}(\sigma^\nu)_{\beta \dot{\beta}} + (\sigma^\nu)_{\dot{\alpha} \dot{\beta}}(\sigma^\mu)_{\beta \dot{\beta}} = 2g^{\mu \nu} \delta^\dot{\alpha}_{\dot{\beta}}, \]  

(A.3a)

\[ (\sigma^\mu)_{\alpha \beta}(\bar{\sigma}^\nu)^{\dot{\alpha} \dot{\beta}} + (\sigma^\nu)_{\dot{\alpha} \dot{\beta}}(\bar{\sigma}^\mu)^{\alpha \beta} = 2g^{\mu \nu} \delta^\beta_\alpha. \]  

(A.3b)

To each Lorentz 4-vector there corresponds a rank two multispinor, formed from the contraction of the 4-vector with \( \sigma^\mu \):

\[ W_{\alpha \dot{\beta}} = \frac{1}{\sqrt{2}} \sigma^\mu_{\alpha \dot{\beta}} W\mu, \]  

(A.4a)

\[ \bar{W}^{\dot{\alpha} \dot{\beta}} = \frac{1}{\sqrt{2}} \bar{\sigma}^{\dot{\alpha} \dot{\beta}} W^\mu. \]  

(A.4b)

For the purposes of normalization, it is convenient to use a different convention when converting momenta:

\[ k_{\alpha \dot{\beta}} = \sigma^\mu_{\alpha \dot{\beta}} k_\mu, \]  

(A.5a)
Useful consequences of (A.5) and (A.3) are

\[ \bar{k}^{\dot{\alpha} \beta} k_{\dot{\beta} \beta} = k^2 \delta_{\beta}^{\dot{\beta}}, \]  
(A.6a)

\[ k_{\alpha \dot{\beta}} \bar{k}^{\dot{\beta} \beta} = k^2 \delta_{\alpha}^{\beta}. \]  
(A.6b)

The spinor indices may be raised and lowered using the 2-component antisymmetric tensor:

\[ u^{\alpha} = \varepsilon^{\alpha \beta} u_{\beta}, \]  
(A.7a)

\[ \bar{u}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha} \dot{\beta}} \bar{u}_{\dot{\beta}}, \]  
(A.7b)

\[ \varepsilon^{\alpha \beta} = \varepsilon_{\alpha \beta}, \]  
(A.7c)

\[ \varepsilon^{\dot{\alpha} \dot{\beta}} = \varepsilon_{\dot{\alpha} \dot{\beta}}, \]  
(A.7d)

\[ \varepsilon_{12} = \varepsilon_{\dot{1} \dot{2}} = 1. \]  
(A.7e)

Many useful relations may be easily proven from the Schouten identity

\[ \delta^{\alpha}_{\gamma} \delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta} \delta^{\beta}_{\gamma} + \varepsilon^{\alpha \beta} \varepsilon_{\gamma \delta} = 0, \]  
(A.8)

the generator of 2-component Fierz transformations.

We denote by \( u(k) \) and \( \bar{u}(k) \) the solutions to the 2-component Weyl equations:

\[ \bar{k}^{\dot{\alpha} \beta} u_{\beta}(k) = 0, \]  
(A.9a)

\[ \bar{u}_{\dot{\beta}}(k) \bar{k}^{\dot{\beta} \alpha} = 0. \]  
(A.9b)

These two spinors are related by complex conjugation

\[ \bar{u}_{\alpha}(k) = [u_{\alpha}(k)]^*, \]  
(A.10)
and have the normalization

$$u_{\alpha}(k) \bar{u}_{\dot{\alpha}}(k) = k_{\alpha\dot{\alpha}}. \quad (A.11)$$

It is useful to define a scalar product

$$\langle 1 \ 2 \rangle \equiv u^{\alpha}(k_1) u_{\alpha}(k_2), \quad (A.12)$$

which has two elementary properties

$$\langle 1 \ 2 \rangle = -\langle 2 \ 1 \rangle, \quad (A.13a)$$

$$\langle 1 \ 2 \rangle^* \langle 1 \ 2 \rangle = 2 k_1 \cdot k_2. \quad (A.13b)$$

Contraction of $u_{\alpha}(k_1) u_{\beta}(k_2) u_{\gamma}(k_3) u_{\delta}(k_4)$ into (A.8) produces the extremely useful relation

$$\langle 1 \ 2 \rangle \langle 3 \ 4 \rangle + \langle 1 \ 3 \rangle \langle 4 \ 2 \rangle + \langle 1 \ 4 \rangle \langle 2 \ 3 \rangle = 0. \quad (A.14)$$

A second relation of great utility may be derived from (A.14):

$$\frac{\langle 1 \ 2 \rangle}{\langle 1 \ P \rangle \langle P \ 2 \rangle} + \frac{\langle 2 \ 3 \rangle}{\langle 2 \ P \rangle \langle P \ 3 \rangle} = \frac{\langle 1 \ 3 \rangle}{\langle 1 \ P \rangle \langle P \ 3 \rangle}. \quad (A.15)$$

Equation (A.15) may be used to demonstrate that

$$\sum_{j=\ell}^{m-1} \frac{\langle j \ j+1 \rangle}{\langle j \ P \rangle \langle P \ j+1 \rangle} = \frac{\langle \ell \ m \rangle}{\langle \ell \ P \rangle \langle P \ m \rangle}. \quad (A.16)$$

An important structure built from the inner products is

$$\langle p \vert 1, 2, \ldots, n \rangle {\equiv} \langle p \ 1 \rangle \langle 1 \ 2 \rangle \cdots \langle n \ q \rangle. \quad (A.17)$$

We note the following basic properties of $\langle p \vert 1, 2, \ldots, n \rangle$:

$$\langle p \vert \ q \rangle \equiv \langle p \ q \rangle \quad (A.18a)$$

$$\langle p \vert 1, 2, \ldots, j-1 \rangle \langle j \ j+1, j+2, \ldots, n \rangle = \langle p \vert 1, 2, \ldots, n \rangle \quad (A.18b)$$
\[ \langle q|n, n-1, \ldots, 1|p \rangle = (-1)^{n-1} \langle p|1, 2, \ldots, n|q \rangle. \quad (A.18c) \]

The inverse of this “string” of spinor inner products is easy to sum over permutations: [12]
\[
\sum_{\mathcal{P}(1 \ldots n)} \frac{1}{\langle p|1, 2, \ldots, n|q \rangle} = \frac{\langle p \rangle^{n-1} \langle q \rangle}{n \prod_{j=1}^{n} \langle j|q \rangle}. \quad (A.19) 
\]

Helicities ±1 for massless vector bosons may be described by
\[
\epsilon_{\alpha \dot{\alpha}}(k^+) \equiv \frac{u_{\alpha}(q) \bar{u}_{\dot{\alpha}}(k)}{\langle k \rangle \langle q \rangle}, \quad (A.20a) 
\]
\[
\epsilon_{\alpha \dot{\alpha}}(k^-) \equiv \frac{u_{\alpha}(k) \bar{u}_{\dot{\alpha}}(q)}{\langle k \rangle^* \langle q \rangle}, \quad (A.20b) 
\]
where q is any null-vector such that \( k \cdot q \neq 0 \). As the choice of q does not affect any physics result, we will refer to \( u(q) \) and \( \bar{u}(q) \) as gauge spinors. The corresponding polarization vectors \( \epsilon^\mu(k) \) defined through (A.4) differ from the “standard” polarization vectors
\[
\epsilon^\mu_0(k^\pm) = \left(0, \mp \frac{1}{\sqrt{2}}, \mp i \frac{1}{\sqrt{2}}, 0 \right), \quad (A.21a) 
\]
\[
k^\mu = (k, 0, 0, k), \quad (A.21b) 
\]
by a \( q \)-dependent phase and gauge transformation [1].

To save accounting for a large number of indices, an efficient method is to initially write quantities in the usual formalism and then convert to multispinor notation at a later stage using the substitutions
\[
k \cdot k' = \frac{1}{2} \bar{k}^{\dot{\alpha}} \epsilon_{\alpha \dot{\alpha}}(k') = \frac{1}{2} k_{\alpha \dot{\alpha}} \bar{k}^{\alpha \dot{\alpha}}, \quad (A.22a) 
\]
\[
k \cdot \epsilon(k') = \frac{1}{\sqrt{2}} \bar{k}^{\dot{\alpha}} \epsilon_{\alpha \dot{\alpha}}(k') = \frac{1}{\sqrt{2}} k_{\alpha \dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} \epsilon^\alpha(k'), \quad (A.22b) 
\]
\[
\epsilon(k) \cdot \epsilon(k') = \bar{\epsilon}^{\dot{\alpha}}(k) \epsilon_{\alpha \dot{\alpha}}(k') = \epsilon_{\alpha \dot{\alpha}}(k) \bar{\epsilon}^{\dot{\alpha}}(k'), \quad (A.22c) 
\]
for Lorentz dot products and

\[
\frac{1}{2}(1 - \gamma_5)\psi \rightarrow \psi_\alpha, \quad (A.23a)
\]

\[
\frac{1}{2}(1 + \gamma_5)\psi \rightarrow \psi^{\dot{\alpha}}, \quad (A.23b)
\]

\[
\frac{1}{2}(1 - \gamma_5)W\frac{1}{2}(1 + \gamma_5) \rightarrow \sqrt{2} W_{\alpha\dot{\alpha}}, \quad (A.23c)
\]

\[
\frac{1}{2}(1 + \gamma_5)W\frac{1}{2}(1 - \gamma_5) \rightarrow \sqrt{2} W^{\dot{\beta}\alpha}, \quad (A.23d)
\]

\[
\frac{1}{2}(1 - \gamma_5)\tilde{k}\frac{1}{2}(1 + \gamma_5) \rightarrow k_{\alpha\dot{\alpha}}, \quad (A.23e)
\]

\[
\frac{1}{2}(1 + \gamma_5)\tilde{k}\frac{1}{2}(1 - \gamma_5) \rightarrow \sqrt{2} W_{\alpha\dot{\alpha}}, \quad (A.23f)
\]

in strings of Dirac matrices. Note the unequal treatments of momenta versus other 4-vectors caused by the conventions (A.4) and (A.5).

The following combination of factors forms the key building block in the solutions to the recursion relations:

\[
\Xi(j, n) \equiv \sum_{\mathcal{P}(j \ldots n)} \prod_{\ell = j}^{n} \bar{u}_\beta(k_\ell)[\bar{P} + \bar{\kappa}(1, \ell)]^{\dot{\beta}\beta}u_\beta(g)\frac{\langle \ell g \rangle}{\langle \ell g \rangle} - \left[\frac{1}{\mathcal{P} + \kappa(1, \ell-1)}\frac{1}{[P + \kappa(1, \ell)]^2}\right]. (A.24)
\]

In reference 2, we show that this expression reduces to

\[
\Xi(j, n) = \sum_{\mathcal{P}(j \ldots n)} \frac{[P + \kappa(1, j-1)]^2}{\langle g|j, \ldots, n|g \rangle} \sum_{\ell = j}^{n} u^\alpha(g)\Pi_\alpha^\beta(P, 1, 2, \ldots, \ell)u_\beta(g), \quad (A.25)
\]

where

\[
\Pi_\alpha^\beta(P, 1, 2, \ldots, \ell) \equiv \frac{(k_\ell)_{\alpha\dot{\alpha}}[\bar{P} + \bar{\kappa}(1, \ell)]^{\dot{\beta}\beta}}{[P + \kappa(1, \ell-1)]^2[P + \kappa(1, \ell)]^2}. (A.26)
\]

The factor \(\Pi\) has a pair of useful properties:

\[
[\bar{P} + \bar{\kappa}(1, j)]^{\dot{\beta}\alpha}\Pi_\alpha^\beta(P, 1, \ldots, j) = \frac{[\bar{P} + \bar{\kappa}(1, j-1)]^{\dot{\beta}\beta}}{\mathcal{P} + \kappa(1, j-1)} - \frac{[\bar{P} + \bar{\kappa}(1, j)]^{\dot{\beta}\beta}}{\mathcal{P} + \kappa(1, j)} (A.27)
\]

\[
u^\alpha(g)\Pi_\alpha^\beta(P, 1, 2, \ldots, j)u_\beta(h) = \nu^\alpha(h)\Pi_\alpha^\beta(P, 1, \ldots, j)u_\beta(g)
\]

\[+ \langle g h \rangle \left[\frac{1}{\mathcal{P} + \kappa(1, j-1)} - \frac{1}{\mathcal{P} + \kappa(1, j)}\right]. \quad (A.28)
\]

Both (A.27) and (A.28) contain combinations of terms which are trivially summed over \(j\).
B. CROSS-CHANNEL IDENTITIES

In many of the quadruple current amplitudes, there are structures which result from particles propagating in two different channels. They are characterized by denominators that look like

\[ D_1 = \langle P|2,\ldots,s|q\rangle\langle p|s+1,\ldots,n|Q \rangle \]  

(B.1)

as opposed to

\[ D_2 = \langle P|2,\ldots,s|Q\rangle\langle p|s+1,\ldots,n|q \rangle. \]  

(B.2)

The key to the generation of identities that these two forms is the observation that the way in which we labelled the photon momenta on the diagrams was dictated by convenience. Any other labelling scheme is physically equivalent. Hence, we may turn one type of term into the other simply by “undoing” one of the sums tying the legs together, relabelling the momenta, and redoing the sum. Unfortunately, it does not seem possible to start with the diagrams written in a form which leads to the same denominators in both cases. The gauge momentum \( g \) appears naturally as the momentum that “links” the denominator strings. Unfortunately, the linking momentum required to derive these identities is different. It is this conflict that motivated the approach outlined below.

A simple example of this type of identity may be generated from the fifth term of (3.45). Let us define

\[
\Lambda \equiv \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1,s) + q]^2}{\langle P|2,\ldots,s|q\rangle\langle p|s+1,\ldots,n|Q \rangle}
\]

\[ = \sum_{P(2\ldots n)} \sum_{s=1}^{n} \frac{[p + \kappa(s+1,n) + Q]^2}{\langle P|2,\ldots,s|q\rangle\langle p|s+1,\ldots,n|Q \rangle}. \]  

(B.3)
Since we wish to split one of the denominator strings, making it look like a pair of unsummed currents, we use (A.6) and the Weyl equation to write

$$
\frac{[p + \kappa(s+1, n) + Q]^2}{\langle p|s+1, \ldots, n|Q \rangle} = \frac{u^\alpha(q)[p + \kappa(s+1, n) + Q]|_{\alpha\dot{\alpha}}[\bar{p} + \bar{\kappa}(s+1, n)]^\dot{\alpha}\beta u_\beta(q)}{\langle p|s+1, \ldots, n|Q \rangle \langle q|Q \rangle}.
$$

(B.4)

Our goal is to break up $\langle p|s+1, \ldots, n|Q \rangle$. To this end, we note that the proper endpoint momentum assignments are “$p$” for “$s$” and “$Q$” for “$n+1$”. Hence

$$
\frac{[p + \kappa(s+1, n) + Q]^2}{\langle p|s+1, \ldots, n|Q \rangle} = \frac{u^\alpha(q)[p + \kappa(s+1, n) + Q]|_{\alpha\dot{\alpha}} n \sum_{v=s}^{n} k_{v}^\dot{\alpha}\beta u_\beta(q) \frac{1}{\langle q|Q \rangle}}{\langle p|s+1, \ldots, n|Q \rangle}.
$$

(B.5)

Supplying a factor of $\langle v q \rangle$ to numerator and denominator and rearranging the factors in a suggestive manner produces

$$
\frac{[p + \kappa(s+1, n) + Q]^2}{\langle p|s+1, \ldots, n|Q \rangle} = \frac{u^\alpha(q)[p + \kappa(s+1, n) + Q]|_{\alpha\dot{\alpha}} n \sum_{v=s}^{n} \bar{u}_{v}^\dot{\alpha}(k_{v})\langle v q \rangle \frac{\langle v q |Q \rangle}{\langle v q |Q \rangle}}{\langle p|s+1, \ldots, n|Q \rangle}.
$$

(B.6)

We may convert $\langle v q \rangle\langle v q |Q \rangle^{-1}$ into the factor required to split the denominator by using (A.16) in reverse:

$$
\frac{[p + \kappa(s+1, n) + Q]^2}{\langle p|s+1, \ldots, n|Q \rangle} = \frac{u^\alpha(q)[p + \kappa(s+1, n) + Q]|_{\alpha\dot{\alpha}} n \sum_{v=s}^{n} \sum_{t=v}^{n} k_{v}^\dot{\alpha}\beta u_\beta(q) \langle t t+1 \rangle}{\langle p|s+1, \ldots, n|Q \rangle \langle t|q|t+1 \rangle}.
$$

(B.7)

Interchanging the order of the sums, we see that

$$
\frac{[p + \kappa(s+1, n) + Q]^2}{\langle p|s+1, \ldots, n|Q \rangle} = \sum_{t=s}^{n} \frac{u^\alpha(q)[\kappa(t+1, n) + Q]|_{\alpha\dot{\alpha}}[\bar{p} + \bar{\kappa}(s+1, t)]^\dot{\alpha}\beta u_\beta(q)}{\langle p|s+1, \ldots, t|q \rangle \langle q|t+1, \ldots, n|Q \rangle},
$$

(B.8)

where we have used (A.6) to shorten the momentum sum in the first factor.

Hence, we may write

$$
\Lambda = \sum_{\mathcal{P}(2..n)} \sum_{s=1}^{n} \sum_{t=s}^{n} \frac{u^\alpha(q)[\kappa(t+1, n) + Q]|_{\alpha\dot{\alpha}}[\bar{p} + \bar{\kappa}(s+1, t)]^\dot{\alpha}\beta u_\beta(q)}{\langle P|2, \ldots, s|q \rangle \langle p|s+1, \ldots, t|q \rangle \langle q|t+1, \ldots, n|Q \rangle}.
$$

(B.9)
Since any set of labels for the dummy momenta appearing inside the permutation sum is equivalent to any other, we choose to relabel the momenta as follows:

\[
\begin{align*}
\{t+1, \ldots, n\} & \rightarrow \{s+1, \ldots, s+n-t\}, \\
\{s+1, \ldots, t\} & \rightarrow \{s+n-t+1, \ldots, n\},
\end{align*}
\]

producing

\[
\Lambda = \sum_{\mathcal{P}(2 \ldots n)} \sum_{s=1}^{n} \sum_{t=s}^{n} \frac{u^\alpha(q)[\kappa(s+1, s+n-t) + Q][\bar{p} + \bar{\kappa}(s+n-t+1, n)]^{\dot{\alpha} \beta} u_\beta(q)}{\langle P|2, \ldots, s|q\rangle \langle q|s+1, \ldots, s+n-t|Q\rangle \langle p|s+n-t+1, \ldots, t|q\rangle}.
\]

\[
\left(\text{B.10}\right)
\]

The replacement of \( t \) by \( r \equiv s + n - t \) immediately suggests itself:

\[
\Lambda = \sum_{\mathcal{P}(2 \ldots n)} \sum_{r=1}^{n} \sum_{s=1}^{r} \frac{u^\alpha(q)[\kappa(s+1, r) + Q][\bar{p} + \bar{\kappa}(r+1, n)]^{\dot{\alpha} \beta} u_\beta(q)}{\langle P|2, \ldots, s|q\rangle \langle q|s+1, \ldots, r|Q\rangle \langle p|r+1, \ldots, n|q\rangle}.
\]

\[
\left(\text{B.11}\right)
\]

At this stage, we use the Weyl to add \( \bar{q}^{\dot{\alpha} \beta} \) to the second momentum factor in (B.12), causing it to read

\[
\bar{\bar{p}} + \bar{\bar{\kappa}}(r+1, n) + \bar{q}^{\dot{\alpha} \beta} = -[\bar{P} + \bar{\kappa}(1, r) + \bar{Q}]^{\dot{\alpha} \beta}.
\]

\[
\left(\text{B.13}\right)
\]

Next, we perform the sum on \( s \) appearing in (B.12). Extracting the relevant factors, we have

\[
\sigma_s \equiv \sum_{s=1}^{r} \sum_{w=s+1}^{r+1} \frac{\langle s s+1 \rangle u^\alpha(q)(k_w)_{\alpha \dot{\alpha}}}{\langle s|q|s+1 \rangle}.
\]

\[
\left(\text{B.14}\right)
\]

where “1” corresponds to “\( P \)” and “\( r+1 \)” corresponds to “\( Q \)”. The sum is easily performed using (A.16), yielding

\[
\sigma_s = \frac{1}{\langle P q \rangle} u^\alpha(P)[P + \kappa(2, r) + Q].
\]

\[
\left(\text{B.15}\right)
\]

To get the final form, we add and subtract \( (k_1)_{\alpha \dot{\alpha}} \) in (B.15). Inserting the result into (B.12) produces

\[
\Lambda = -\sum_{\mathcal{P}(2 \ldots n)} \sum_{r=1}^{n} \frac{[P + \kappa(1, r) + Q]^2}{\langle P|2, \ldots, r|Q\rangle \langle p|r+1, \ldots, n|q\rangle} + \sum_{\mathcal{P}(2 \ldots n)} \sum_{r=1}^{n} \frac{\langle P 1 \rangle \bar{u}_{\dot{\alpha}}(k_1)[P + \bar{\kappa}(1, r) + \bar{Q}]^{\dot{\alpha} \beta} u_\beta(q)}{\langle P q \rangle \langle P|2, \ldots, r|Q\rangle \langle p|r+1, \ldots, n|q\rangle},
\]

\[
\left(\text{B.16}\right)
\]
which is the desired identity.

It is somewhat more difficult to derive identities of this ilk when the terms being targeted contain sums over $\Pi$. Simply put, the sum involving the various $\Pi$’s gets tangled up with the sum introduced for the purpose of splitting the denominator. To see how this works, consider

$$Z = \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^{2}}{[P|2, \ldots, s|q]\langle p|s+1, \ldots, n|Q\rangle} \times \sum_{j=2}^{s+1} u^{\gamma}(k_{1})\Pi^{\delta}(P, 1, 2, \ldots, j)u_{\delta}(k_{1})\bigg|_{j=s+1=q}.$$  \hspace{1cm} (B.17)

We begin isolating the $j = s + 1$ contribution to $Z$ and setting it aside:

$$Z_1 = \sum_{\mathcal{P}(2\ldots n)} \sum_{s=1}^{n} \frac{[P + \kappa(1, s) + q]^{2} u^{\gamma}(k_{1})\Pi^{\delta}(P, 1, 2, \ldots, s, q)u_{\delta}(k_{1})}{[P|2, \ldots, s|q]\langle p|s+1, \ldots, n|Q\rangle}.$$  \hspace{1cm} (B.18)

We apply momentum conservation to the remainder to obtain $[p + \kappa(s+1, n) + q]^{2}$ in the numerator and use (B.8) to write

$$Z_2 = \sum_{\mathcal{P}(2\ldots n)} \sum_{s=2}^{n} \sum_{t=s}^{n} \frac{u^{\alpha}(q)[\kappa(t+1, n) + Q]_{\alpha\dot{\alpha}}[\bar{p} + \bar{\kappa}(s+1, t)]^{\dot{\alpha}\beta}u_{\beta}(q)}{[P|2, \ldots, s|q]\langle p|s+1, \ldots, t|q\rangle\langle t|q+1, \ldots, n|q\rangle} \times \sum_{j=2}^{s} u^{\gamma}(k_{1})\Pi^{\delta}(P, 1, 2, \ldots, j)u_{\delta}(k_{1}).$$  \hspace{1cm} (B.19)

We now use (B.10) to relabel the momenta appearing in (B.19) and make the variable change, $r = s + n - t$:

$$Z_2 = \sum_{\mathcal{P}(2\ldots n)} \sum_{s=2}^{n} \sum_{r=s}^{n} \frac{u^{\alpha}(q)[\kappa(s+1, r) + Q]_{\alpha\dot{\alpha}}[\bar{p} + \bar{\kappa}(r+1, n) + q]^{\dot{\alpha}\beta}u_{\beta}(q)}{[P|2, \ldots, s|q]\langle q|s+1, \ldots, r|Q\rangle\langle r|q+1, \ldots, n|q\rangle} \times \sum_{j=2}^{s} u^{\gamma}(k_{1})\Pi^{\delta}(P, 1, 2, \ldots, j)u_{\delta}(k_{1}).$$  \hspace{1cm} (B.20)

Note that since for the range of $j$ and $s$ considered, $\{2, \ldots, j\} \subseteq \{2, \ldots, s\}$, there has been no change in the arguments of $\Pi$. In preparation for doing the sum on $s$, we transpose the order of the matrix multiplication and re-order the sums:

$$Z_2 = \sum_{\mathcal{P}(2\ldots n)} \sum_{r=2}^{n} \sum_{j=2}^{r} \sum_{s=j}^{r} \frac{u^{\alpha}(q)[p + \kappa(r+1, n) + q]_{\beta\dot{\alpha}}[\bar{\kappa}(s+1, r) + \bar{Q}]_{\dot{\alpha}\beta}u_{\beta}(q)}{[P|2, \ldots, s|q]\langle q|s+1, \ldots, r|Q\rangle\langle r|q+1, \ldots, n|q\rangle} \times u^{\gamma}(k_{1})\Pi^{\delta}(P, 1, 2, \ldots, j)u_{\delta}(k_{1}).$$  \hspace{1cm} (B.21)
At this stage we see clearly that even though we tried to keep $\Pi$ out of this process, the appearance of $j$ as one of the limits in the sum on $s$ will force it to become involved. The sum required by (B.21) is easily performed using (A.16):

$$\sum_{s=j}^{r+1} \sum_{w=j+1}^{r+1} \frac{s s+1}{s q s+1} \bar{\epsilon}_{w}^{\alpha \dot{\alpha}} u_{\alpha}(q) = \frac{-1}{q j} [\bar{k}(j+1, r) + \bar{Q}]_{\alpha \dot{\alpha}} u^{\alpha}(k_{j}). \quad (B.22)$$

Combining (B.22) with (B.21) results in

$$Z_{2} = Z_{2A} < P|2, \ldots, r|Q \rangle \langle p|r+1, \ldots, n|q \rangle \times \prod_{\alpha} \delta(P, 1, 2, \ldots, j) u_{\delta}(k_{1}) \frac{\langle 1 j \rangle}{\langle q j \rangle}, \quad (B.23)$$

where we have chosen to extract the implicit factor of $\langle 1 j \rangle$ from the $\Pi$-structure and contract the momentum sum into $\Pi$ instead.

At this stage, we break $Z_{2}$ into two contributions by writing

$$u^{\delta}(q)[P + \kappa(r+1, n) + Q]_{\beta \dot{\alpha}}[\bar{k}(j+1, r) + \bar{Q}]_{\dot{\alpha} \alpha} =$$

$$= u^{\delta}(q)[P + \kappa(r+1, n) + Q]_{\beta \dot{\alpha}}[\bar{k}(1, r) + \bar{Q} - \bar{k}(1, j)]_{\dot{\alpha} \alpha}$$

$$= -[P + \kappa(1, r) + Q]^{2} u^{\alpha}(q)$$

$$+ u^{\beta}(q)[P + \kappa(1, r) + Q]_{\beta \dot{\alpha}}[\bar{P} + \bar{k}(1, j)]_{\dot{\alpha} \alpha}, \quad (B.24)$$

where we have used momentum conservation and (A.6). The first term in (B.24) produces

$$Z_{2A} \equiv - \sum_{\alpha} \sum_{P(2 \ldots n)}^{n} \frac{\langle P + \kappa(1, r) + Q \rangle^{2}}{\langle p|2, \ldots, r|Q \rangle \langle p|r+1, \ldots, n|q \rangle} \times \sum_{j=2}^{r} u^{\alpha}(k_{1}) \prod_{\alpha} \delta(P, 1, 2, \ldots, j) u_{\delta}(k_{1}), \quad (B.25)$$

which we set aside for the moment. The rest of (B.24) gives

$$Z_{2B} \equiv \sum_{\alpha} \sum_{P(2 \ldots n)}^{n} \frac{\langle 1 j \rangle}{\langle q j \rangle} \frac{u^{\delta}(q)[P + \kappa(1, r) + Q]_{\beta \dot{\alpha}}}{\langle p|2, \ldots, r|Q \rangle \langle p|r+1, \ldots, n|q \rangle} \times \langle p + \bar{k}(1, j) \rangle_{\dot{\alpha} \alpha} \prod_{\alpha} \delta(P, 1, 2, \ldots, j) u_{\delta}(k_{1}). \quad (B.26)$$
We apply (A.27) to (B.26). We cannot do the sum over \( j \) in this case because of the factor \( \langle 1 \ j \rangle / \langle q \ j \rangle \) appearing in (B.26). Instead, we must write

\[
Z_{2B} = \sum_{P(2...n)} \sum_{r=2}^{n} \sum_{j=1}^{r-1} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle} = \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle}.
\]

(B.27)

Let us shift the sum over \( j \) by one unit in the first of the two terms in (B.27):

\[
Z_{2B} = \sum_{P(2...n)} \sum_{r=2}^{n} \sum_{j=1}^{r-1} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle} = \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle}.
\]

(B.28)

It is easy to show from (A.14) that

\[
\frac{\langle 1 \ j + 1 \rangle}{\langle q \ j + 1 \rangle} - \frac{\langle 1 \ j \rangle}{\langle q \ j \rangle} = \frac{\langle 1 \ q \rangle}{\langle j | q | j + 1 \rangle}.
\]

(B.29)

This combination will appear in (B.28) if we make the ranges of the summations be identical. In addition to extending the range of \( j \) in both terms, we also add \( r = 1 \):

\[
Z_{2B} = \sum_{P(2...n)} \sum_{r=1}^{n} \sum_{j=1}^{r} \frac{\langle 1 \ q \rangle}{\langle q | j + 1 \rangle} \frac{\langle 1 \ j \rangle}{\langle q | j \rangle} \frac{\langle 1 \ j \rangle}{\langle q | j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q | j + 1 \rangle} = \frac{\langle 1 \ j \rangle}{\langle q | j \rangle} \frac{\langle 1 \ j \rangle}{\langle q | j \rangle} \frac{\langle 1 \ j + 1 \rangle}{\langle q | j + 1 \rangle}.
\]

(B.30)

Notice that we have chosen the values of the endpoint terms to be “\( P \)” for \( j = 1 \) and “\( Q \)” for \( j = r + 1 \). This allowed us to write

\[
\frac{1}{\langle P | 2, \ldots, r | Q \rangle} \frac{\langle j | j + 1 \rangle}{\langle j | q | j + 1, \ldots, r | Q \rangle} = \frac{1}{\langle P | 2, \ldots, j | q \rangle} \frac{\langle q | q + 1, \ldots, r | Q \rangle}{\langle q | j + 1, \ldots, r | Q \rangle}.
\]

(B.31)
throughout the entire summation range in the first term of (B.30).

We now concentrate on the first term of (B.30), which we will refer to as $Z_{2B1}$. Momentum conservation plus the Weyl equation permits us to write

$$u^\beta(q)[P + \kappa(1, r) + Q]_{\beta \dot{\alpha}} = -u^\beta(q)[p + \kappa(r+1, n)]_{\beta \dot{\alpha}} \quad (B.32)$$

in the numerator. At this stage, we relabel the momenta once more:

$$\{r+1, \ldots, n\} \rightarrow \{j+1, \ldots, j+n-r\}, \quad (B.33)$$

This has the effect of restoring the original ordering of the denominator strings, as is readily apparent after eliminating $r$ in favor of $i \equiv j + n - r$:

$$Z_{2B1} = \sum_{\mathcal{P}(2\ldots n)} \sum_{j=1}^n \sum_{i=j}^n \frac{-\langle 1 \rangle u^\beta(q)[p + \kappa(j+1, i)]_{\beta \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)}{\langle P|2, \ldots, j|q\rangle \langle p|j+1, \ldots, i|q \rangle \langle q|i+1, \ldots, n|Q \rangle [P + \kappa(1, j)]^2}.$$  \quad (B.34)

Letting $i = j$ stand for $p$ and $i = n + 1$ stand for $Q$ gives us the following for the sum over $i$:

$$\sum_{i=j}^n \sum_{\ell=j}^i \frac{\langle i \ell | q \rangle}{\langle i | q \rangle} u^\beta(q)(k_{\ell})_{\beta \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)$$

$$= \sum_{\ell=j}^n \frac{\langle \ell | q \rangle}{\langle \ell | q \rangle} u^\beta(Q)[p + \kappa(j+1, n)]_{\beta \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)$$

$$= \frac{1}{\langle q | Q \rangle} u^\beta(Q)[p + \kappa(j+1, n)]_{\beta \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)$$

$$= \frac{-1}{\langle q | Q \rangle} u^\beta(Q)[P + \kappa(1, j) + q]_{\beta \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)$$

$$= -\frac{\langle Q | 1 \rangle}{\langle q | Q \rangle}[P + \kappa(1, j)]^2 - \frac{1}{\langle q | Q \rangle} u^\beta(Q)q_{\dot{\beta} \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)$$

where we have made free use of momentum conservation, the Weyl equation, and (A.6). Applying (B.35) to (B.34) gives us

$$Z_{2B1} = \sum_{\mathcal{P}(2\ldots n)} \sum_{j=1}^n \frac{\langle q | 1 | Q \rangle}{\langle q | Q \rangle} \frac{1}{\langle P|2, \ldots, j|q\rangle \langle p|j+1, \ldots, n|Q \rangle}$$

$$- \sum_{\mathcal{P}(2\ldots n)} \sum_{j=1}^n \frac{u^\beta(k_1)q_{\dot{\beta} \dot{\alpha}}[\tilde{P} + \bar{\kappa}(1, j)]^{\dot{\alpha} \delta} u_\delta(k_1)}{\langle P|2, \ldots, j|q\rangle \langle p|j+1, \ldots, n|Q \rangle [P + \kappa(1, j)]^2}.$$  \quad (B.36)
Supplying a factor of $[P + \kappa(1, j) + q]^2$ to the numerator and denominator of the second term in (B.36) allows us to recognize the presence of a factor of $\Pi$:

$$Z_{2B1} = \sum_{\mathcal{P}(2\ldots n)} \sum_{j=1}^{n} \frac{\langle q|1|Q \rangle}{\langle q|Q \rangle} \frac{\langle P|2,\ldots,j|q \rangle\langle p|j+1,\ldots,n|Q \rangle}{\langle P|2,\ldots,j|q \rangle\langle p|j+1,\ldots,n|Q \rangle}$$

$$- \sum_{\mathcal{P}(2\ldots n)} \sum_{j=1}^{n} \frac{[P + \kappa(1, j) + q]^2 u^\gamma(k_1)\Pi^\delta(P, 1, 2,\ldots,j, q)u_\delta(k_1)}{\langle P|2,\ldots,j|q \rangle\langle p|j+1,\ldots,n|Q \rangle}. $$

(B.37)

Thus, the point of the manipulations on $Z_{2B1}$ becomes apparent, for the second term of (B.37) exactly cancels the contribution from $Z_1$, as a comparison with (B.18) readily attests.

We now return to the two remaining terms of (B.30):

$$Z_{2B2} \equiv - \sum_{\mathcal{P}(2\ldots n)} \sum_{r=1}^{n} \frac{\langle 1|Q \rangle u^\beta(q)[P + \kappa(1, r) + Q]_{\gamma\delta}[(\bar{P} + \bar{\kappa}(1, r))]^{\gamma\delta}u_\delta(k_1)}{\langle q|Q \rangle\langle P|2,\ldots,r|Q \rangle\langle p|r+1,\ldots,n|Q \rangle [P + \kappa(1, r)]^2}$$

$$+ \sum_{\mathcal{P}(2\ldots n)} \sum_{r=1}^{n} \frac{\langle 1|P \rangle u^\beta(q)[P + \kappa(1, r) + Q]_{\gamma\delta}\bar{P}^{\gamma\delta}u_\delta(k_1)}{\langle q|P \rangle\langle P|2,\ldots,r|Q \rangle\langle p|r+1,\ldots,n|Q \rangle [P + k_1]^2}.$$  

(B.38)

Application of (A.6) to the first term of (B.38) yields

$$Z_{2B2} = - \sum_{\mathcal{P}(2\ldots n)} \sum_{r=1}^{n} \frac{\langle q|1|Q \rangle}{\langle q|Q \rangle\langle P|2,\ldots,r|Q \rangle\langle p|r+1,\ldots,n|Q \rangle} \frac{1}{\langle p|j+1,\ldots,n|Q \rangle}$$

$$- \sum_{\mathcal{P}(2\ldots n)} \sum_{r=1}^{n} \frac{\langle 1|Q \rangle u^\beta(q)Q_{\gamma\delta}[P + \kappa(1, r)]^{\gamma\delta}u_\delta(k_1)}{\langle q|Q \rangle\langle P|2,\ldots,r|Q \rangle\langle p|r+1,\ldots,n|Q \rangle [P + \kappa(1, r)]^2}$$

$$+ \sum_{\mathcal{P}(2\ldots n)} \sum_{r=1}^{n} \frac{\langle 1|P \rangle \bar{u}_\delta(P)[\bar{P} + \bar{\kappa}(1, r) + \bar{Q}]^{\gamma\delta}u_\beta(q)}{\langle q|P \rangle\langle P|2,\ldots,r|Q \rangle\langle p|r+1,\ldots,n|Q \rangle} \frac{1}{\langle P|1 \rangle}. $$

(B.39)
A factor of $[P + \kappa(1, r) + Q]^2$ supplied to the numerator and denominator of the second term of (B.39) converts this to

$$Z_{2B2} = - \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \frac{\langle q|1|Q \rangle}{\langle q\ Q \rangle} \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle}$$

$$- \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \frac{[P + \kappa(1, r) + Q]^2 u^\gamma(k_1) \Pi_{\gamma \delta}^\delta(P, 1, 2,...,j, Q)u_\delta(k_1)}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle}$$

$$+ \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \langle 1\ P \rangle \bar{u}_{\dot{\alpha}}(P) [\bar{P} + \bar{\kappa}(1, r) + \bar{Q}]^{\dot{\alpha} \beta} u_\beta(q) \frac{1}{\langle q\ P \rangle} \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle}$$

$$\times \sum_{j=2}^{r+1} u^\alpha(k_1) \Pi_{\alpha \delta}^\delta(P, 1, 2,...,j)u_\delta(k_1) \bigg|_{j=r+1 \equiv Q}$$

$$+ \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \frac{\langle q|1|Q \rangle}{\langle q\ Q \rangle} \left[ \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle} - \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle} \right]$$

$$+ \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \langle P\ 1 | \bar{u}_{\dot{\alpha}}(P) [\bar{P} + \bar{\kappa}(1, r) + \bar{Q}]^{\dot{\alpha} \beta} u_\beta(q) \frac{1}{\langle p\ r+1,...,n|q \rangle} \langle P|1 \rangle^\ast \rangle.$$

Comparing to (B.25), we see that the second term of (B.40) is precisely the contribution required to add a $j = r + 1 \equiv Q$ term to $Z_{2A}$.

We now combine (B.18), (B.25), (B.37) and (B.40) to obtain the final result:

$$Z = - \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \frac{[P + \kappa(1, r) + Q]^2}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle}$$

$$\times \sum_{j=2}^{r+1} u^\alpha(k_1) \Pi_{\alpha \delta}^\delta(P, 1, 2,...,j)u_\delta(k_1) \bigg|_{j=r+1 \equiv Q}$$

$$+ \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \langle q|1|Q \rangle \left[ \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle} - \frac{1}{\langle P|2,...,r|Q\rangle\langle p|r+1,...,n|q \rangle} \right]$$

$$+ \sum_{\mathcal{P}(2...n)} \sum_{r=1}^{n} \langle P\ 1 | \bar{u}_{\dot{\alpha}}(P) [\bar{P} + \bar{\kappa}(1, r) + \bar{Q}]^{\dot{\alpha} \beta} u_\beta(q) \frac{1}{\langle p\ r+1,...,n|q \rangle} \langle P|1 \rangle^\ast \rangle.$$

Although (B.41) looks unwieldy, large reductions result when it is applied to actual amplitudes. Typically, the first term will cancel a corresponding contribution from the uncrossed graphs. The remaining “fragments” may be neatly reduced when all of the pieces of the amplitude are combined. Unfortunately, it is not always possible to produce a suitable identity when $\Pi$ is present: it is very important to have the “right” spinors contracted into $\Pi$. 
REFERENCES

1. C. Dunn and T.–M. Yan, Nucl. Phys. B352, 402 (1991).

2. G. Mahlon and T.–M. Yan, Cornell preprint CLNS 91/1119 (1992).

3. F. A. Berends and W. T. Giele, Nucl. Phys. B306, 759 (1988).

4. S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam in Proc. 8th Nobel Symposium, Aspenäsgarden, edited by N. Svartholm, (Almqvist and Wiksell, Stockholm, 1968), p. 367.

5. J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. D10, 1145 (1974); B. W. Lee, C. Quigg, and H. Thacker, Phys. Rev. D16, 1519 (1977); M. S. Chanowitz and M. K. Gaillard, Nucl. Phys. B261, 379 (1985); G. J. Gounaris, R. Kogerler, and H. Neufeld, Phys. Rev. D34, 3257 (1986).

6. J. Schwinger, Particles, Sources and Fields, (Addison-Wesley, Redwood City, 1970), Vol. I; Ann. Phys. 119, 192 (1979).

7. The spinor technique was first introduced by the CALCUL collaboration, in the context of massless Abelian gauge theory: P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Phys. Lett. 105B, 215 (1981); P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Nucl. Phys. B206, 53 (1982); F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Nucl. Phys. B206, 61 (1982); F.A. Berends, P. De Causmaecker, R. Gastmans, R. Kleiss, W. Troost, and T.T. Wu, Nucl. Phys. B239, 382 (1984); B239, 395 (1984); B264, 243 (1986); B264, 265 (1986).

By now, many papers have been published on the subject. A partial list of references follows.

P. De Causmaecker, thesis, Leuven University, 1983; R. Farrar and F. Neri, Phys. Lett. 130B, 109 (1983); R. Kleiss, Nucl. Phys. B241, 61 (1984); Z. Xu, D.H. Zhang and Z. Chang, Tsingua University preprint TUTP-84/3, 84/4, and 84/5a (1984); Nucl. Phys. B291, 392 (1987); J.F. Gunion and Z. Kunszt, Phys. Lett. 161B, 333 (1985); F.A. Berends, P.H. Davereldt, and R. Kleiss, Nucl. Phys. B253, 441 (1985); R. Kleiss and W.J. Stirling, Nucl. Phys. B262 235 (1985); J.F. Gunion and Z. Kunszt, Phys. Lett. 159B, 167 (1985); 161B, 333 (1985); S.J. Parke and T.R. Taylor, Phys. Rev. Lett. 56, 2459 (1986); Z. Kunszt, Nucl. Phys. B271, 333 (1986); J.F. Gunion and J. Kalinowski, Phys. Rev. D34, 2119 (1986); R. Kleiss and W.J. Stirling, Phys. Lett. 179B, 159 (1986); M. Mangano and S.J. Parke, Nucl. Phys. B299, 673 (1988); M. Mangano, S.J. Parke, and Z. Xu, Nucl. Phys. B298, 653 (1988); D.A. Kosower, B.–H. Lee, and V.P. Nair, Phys. Lett. 201B, 85 (1988); M.
Mangano and S.J. Parke, Nucl. Phys. B299, 673 (1988); F.A. Berends and W.T. Giele, Nucl. Phys. B313, 595 (1989); M. Mangano, Nucl. Phys. B315, 391 (1989); D.A. Kosower, Nucl. Phys. B335, 23 (1990); Phys. Lett. B254, 439 (1991); Z. Bern and D.A. Kosower, Nucl. Phys. B379, 451 (1992); C.S. Lam, McGill preprint McGill/92-32, 1992.

8. Many of the results for processes containing six or fewer particles are collected in R. Gastmans and T.T. Wu, The Ubiquitous Photon: Helicity Method for QED and QCD (Oxford University Press, New York, 1990).

9. The excellent review by Mangano and Parke provides a guide to the various approaches to and extensive literature on the subject: M. Mangano and S.J. Parke, Phys. Reports 200, 301 (1991).

10. G. Mahlon, T.–M. Yan and C. Dunn, Cornell preprint CLNS 91/1120, 1992.

11. For a brief introduction to properties of two-component Weyl-van der Waerden spinors, see, for example, M. F. Sohnius, Phys. Reports 128, 39 (1985).

12. M. Mangano, Nucl. Phys. B309, 461 (1988).