Localization of Energy and Momentum in an Asymptotically Reissner-Nordström Non-singular Black Hole Space-time Geometry

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The space-time geometry exterior to a new four-dimensional, spherically symmetric and charged black hole solution that, through a coupling of general relativity with a non-linear electrodynamics, is everywhere non-singular, for small $r$ it behaves as a de Sitter metric, and asymptotically it behaves as the Reissner-Nordström metric, is considered in order to study the energy-momentum localization. For the calculation of the energy and momentum distributions, the Einstein, Landau-Lifshitz, Weinberg and Møller energy-momentum complexes have been applied. The results obtained show that in all prescriptions the energy depends on the mass $M$ of the black hole, the charge $q$, two parameters $a \in \mathbb{Z}^+$ and $\gamma \in \mathbb{R}^+$, and on the radial coordinate $r$. The calculations performed in each prescription show that all the momenta vanish. Additionally, some limiting and particular cases for $r$ and $q$ are studied, and a possible connection with strong gravitational lensing and microlensing is attempted.

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I. INTRODUCTION

The problem of energy-momentum localization, being one of the most challenging problems in classical general relativity in the search of a physically meaningful expression for the energy-momentum of the gravitational field, has triggered a lot of interesting research work, but still remains open and rather not fully understood. As a result, there is not a generally accepted definition for the notion of the localized energy-momentum associated with the gravitational field. Nevertheless, several and often different approaches have been tried in an attempt to achieve the aforementioned energy-momentum localization. Among the mathematical tools utilized, the most notable are superenergy tensors [1]-[3], quasi-local energy definitions [4]-[8] and a number of the so called energy-momentum complexes [9]-[15]. In particular, the energy-momentum complexes of Einstein [9]-[10], Landau-Lifshitz [11], Papapetrou [12], Bergmann-Thomson [13] and Weinberg [15] are coordinate-dependent pseudo-tensorial quantities which can be used in Cartesian and quasi-Cartesian coordinates, more precisely in Schwarzschild Cartesian coordinates and Kerr-Schild Cartesian coordinates, and have yielded so far many physically meaningful as well as interesting results [16]-[24]. In fact, it has been found that different pseudo-tensor complexes lead to the same energy for any metric of the Kerr-Schild class and even for space-times more general than those described by this class (see, e.g., [25], [26] and [27] for some reviews and references therein). Furthermore, the Møller energy-momentum complex [14] allows the calculation of energy and momenta in any coordinate system, including the aforesaid Schwarzschild Cartesian coordinates and Kerr-Schild Cartesian coordinates, and it has also provided a number of physically interesting results for many space-time geometries, in particular for four-dimensional, three-dimensional, two-dimensional, and one-dimensional space-times [28]-[39].

More recently, the relevant research has also turned to the teleparallel theory of gravitation whereby a number of similar results for the energy of the gravitational field has been obtained [40]-[48], while one should also underline the notion of the quasi-local mass introduced by Penrose [49] and further developed by Tod [50] for various gravitating systems. It must be stressed that the Einstein, Landau-Lifshitz, Papapetrou, Bergmann-Thomson, Weinberg and Møller prescriptions agree with this quasi-local mass definition. Furthermore, some rather pertinent and modern approaches concern the quasi-local energy-momentum associated with a closed 2-surface, and the concept of the
Wang-Yau quasi-local energy [51-52]. Indeed, the effort to rehabilitate the value of the energy-momentum complexes has led to the study of pseudo-tensors and quasi-local approaches in the context of a Hamiltonian formulation with a choice of a four-dimensional isometric Minkowski reference geometry on the boundary. It was found that for any closed 2-surface there exists a common value for the quasi-local energy for all expressions that agree (to linear order) with the Freud superpotential. In other words, all the quasi-local expressions in a large class yield the same energy-momentum [53], [54].

The present paper is organized as follows. In Section 2 the new four-dimensional, non-singular and charged black hole space-time that asymptotically behaves as the Reissner-Nordström solution is described. Then, the Einstein pseudo-tensorial complex applied for the calculation of the energy and momentum distributions is introduced in Section 3, together with the results obtained. In Section 4, the Landau-Lifshitz energy-momentum complex and the calculated expressions for energy-momentum are presented. Section 5 is devoted to the depiction of the Weinberg prescription and the presentation of the evaluated expressions for the energy and momentum. In Section 6 we introduce the Møller energy-momentum complex and show the results obtained. Section 7 contains a discussion of the main results and some limiting and particular cases. Finally, in Section 8 we present the implied conclusions of our study. In our calculations we have used geometrized units ($c = G = 1$) and for the signature of the metric we have chosen $(+,−,−,−)$. We have used the Schwarzschild Cartesian coordinates $\{t, x, y, z\}$ in the case of the Einstein, Landau-Lifshitz and Weinberg prescriptions, and the Schwarzschild coordinates $\{t, r, \theta, \phi\}$ for the Møller prescription, respectively. Finally, Greek indices take values from 0 to 3, while Latin indices run from 1 to 3.

II. THE ASYMPTOTICALLY REISSNER-NORDSTRÖM NON-SINGULAR BLACK HOLE SPACE-TIME GEOMETRY

In this section we introduce the asymptotically Reissner-Nordström non-singular black hole space-time geometry. This new spherically symmetric and charged non-singular black hole is built based on the metric function $f(r) = 1 - \frac{2M}{r} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^a} \right]^{3/a}$ given by eq. (56) in the L. Balart and E. C. Vagenas paper [55]. The method consists in adding a new term which will make the metric behave asymptotically as the Reissner–Nordström metric. For the new term, the Dagum distribution function that contains a factor $\frac{q^2}{r^2}$ was employed, so that the metric function is given by

$$f(r) = 1 - \frac{2M}{r} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^a} \right]^{3/a} + \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^a} \right]^{4/a},$$

(1)

with $a \geq 2$ representing an integer and a constant $\gamma \in \mathbb{R}^+$. By setting $\gamma \geq (2/3)^a$ it is seen that the black hole solution satisfies the weak energy condition. The associated electric field is expressed as

$$E(r) = \frac{q}{r^2} \left( \frac{3\gamma(3+a)(\frac{q^2}{Mr})^{a-1}}{2 \left[ 1 + \gamma \left( \frac{q^2}{Mr} \right)^a \right]^{2+3/a}} + \frac{1 - \gamma(3+a) \left( \frac{q^2}{Mr} \right)^a}{\left[ 1 + \gamma \left( \frac{q^2}{Mr} \right)^a \right]^{2+3/a}} \right).$$

(2)

For small values of the $r$ coordinate, the black hole metrics obtained from eq. (1) show a de Sitter black hole behaviour with

$$f(r) \approx 1 - M^4 \frac{1}{\gamma^4 q^6} (2\gamma^2 - 1)r^2.$$

(3)

Notice that the factor in front of the term $r^2$ cannot become zero because the values of $\gamma$ are restricted at $\gamma \geq (2/3)^a$. For $(1/2)^a \leq \gamma < (2/3)^a$, these black hole metrics remain non-singular, without satisfying the weak energy condition, but they exhibit a de Sitter center. The conclusion is that if a black hole metric is non-singular and satisfies the weak energy condition, then it possesses a de Sitter center, but if the metric has a de Sitter behavior when approaching the center, this does not necessarily imply that it satisfies the weak energy condition. For $\gamma < (1/2)^a$, the black hole metric is singular. A special case of eq. (1) arises when we choose $a = 2$ and $\gamma = M^2/q^2$, corresponding to, as it was pointed out in [55], the black hole metric presented in [56].

Thus, the new spherically symmetric, static and charged asymptotically Reissner-Nordström non-singular black hole metric considered is described by a line element of the form

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(4)

with $B(r) = f(r)$, $A(r) = \frac{1}{f(r)}$, and the metric function is given by eq. (1).
III. EINSTEIN PRESCRIPTION AND THE ENERGY DISTRIBUTION OF THE ASYMPTOTICALLY
REISSNER-NORDSTRÖM NON-SINGULAR BLACK HOLE

The Einstein energy-momentum complex \([9]-[10]\) for a \((3 + 1)\)-dimensional space-time has the expression

\[
\theta^\mu_{,\nu} = \frac{1}{16\pi} h^\mu_{\nu,\lambda} \lambda.
\]

(5)

The superpotentials \(h^\mu_{\nu,\lambda}\) are given by

\[
h^\mu_{\nu,\lambda} = \frac{1}{\sqrt{-g}} g_{\nu\sigma} \left[-g(\theta^\mu_{,\sigma} g^{\lambda\kappa} - g^{\lambda\kappa} g_{\mu\kappa})\right]_{,\kappa}
\]

(6)

and exhibit the property of antisymmetry

\[
h^\mu_{\nu,\lambda} = -h^\lambda_{\nu,\mu}.
\]

(7)

In the Einstein prescription the local conservation law is written:

\[
\theta^\mu_{,\nu,\mu} = 0
\]

(8)

The energy and momentum are evaluated by

\[
P^\mu = \int\int\int \theta^0_\mu dx^1 dx^2 dx^3,
\]

(9)

where \(\theta^0_\mu\) and \(\theta^0_i\) represent the energy and momentum density components, respectively.

Applying Gauss' theorem, the energy-momentum has the expression

\[
P^\mu = \frac{1}{16\pi} \int\int h^{0i}_{\mu} n_i dS,
\]

(10)

with \(n_i\) the outward unit normal vector over the surface element \(dS\). In eq. (10) \(P^0\) is the energy.

To calculate the energy distribution and momentum, the line element given by eq. (1) is converted to Schwarzschild Cartesian coordinates \(t, x, y, z\) using the coordinate transformation

\[
x = r \sin \theta \cos \phi,
\]

\[
y = r \sin \theta \sin \phi,
\]

\[
z = r \cos \theta.
\]

(11)

The line element given by eq. (1) and eq. (4) reads now

\[
ds^2 = f(r)dt^2 - \left(dx^2 + dy^2 + dz^2\right) - \frac{1}{f(r)} \left(\frac{1}{r^2} - \frac{1}{r^4}(xdx + ydy + zdz)^2\right),
\]

(12)

where

\[
r = \sqrt{x^2 + y^2 + z^2}.
\]

(13)

For \(\mu = 0, 1, 2, 3\) and \(i = 1, 2, 3\) we conclude that the following components of the superpotential \(h^{0i}_{\mu}\) in quasi-Cartesian coordinates vanish:

\[
h^{01}_{1} = h^{02}_{1} = h^{03}_{1} = 0,
\]

\[
h^{01}_{2} = h^{02}_{2} = h^{03}_{2} = 0,
\]

\[
h^{01}_{3} = h^{02}_{3} = h^{03}_{3} = 0
\]

(14)

while the non-vanishing components of the superpotential are given by

\[
h^{01}_{0} = \frac{2x}{r^2} \left\{ \frac{2M}{r} \left[ \frac{1}{1 + \gamma \left(\frac{q^2}{3Mr}\right)^a}\right]^{3/a} - \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left(\frac{q^2}{3Mr}\right)^a}\right]^{4/a} \right\},
\]

(15)

\[
h^{02}_{0} = \frac{2y}{r^2} \left\{ \frac{2M}{r} \left[ \frac{1}{1 + \gamma \left(\frac{q^2}{3Mr}\right)^a}\right]^{3/a} - \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left(\frac{q^2}{3Mr}\right)^a}\right]^{4/a} \right\},
\]

(16)
Combining the expression for the energy-momentum distribution given by eq. (10) and eqs. (15)-(17) the expression for the energy distribution in the Einstein prescription for the non-singular and charged black hole space-time that asymptotically behaves as the Reissner-Nordström solution is given by

\[
E_E = M \left\{ \frac{1}{1 + \gamma \left( \frac{q^2}{M} \right)^a} \right\}^{3/a} - \frac{q^2}{r^2} \left\{ \frac{1}{1 + \gamma \left( \frac{q^2}{M} \right)^a} \right\}^{4/a}.
\]  

(18)

From eq. (10) and eq. (14), we obtain that all the momentum components vanish:

\[
P_x = P_y = P_z = 0.
\]  

(19)

In Fig. 1 we plot the energy in the Einstein prescription for the choice of parameters \(a = 2\) and \(\gamma = \frac{4}{9}\), and with \(M = 1\) and \(q = 0.1\).
IV. LANDAU-LIFSHITZ ENERGY-MOMENTUM COMPLEX AND THE ENERGY DISTRIBUTION OF THE ASYMPTOTICALLY REISSNER-NORDSTROM NON-SINGULAR BLACK HOLE

The Landau-Lifshitz energy-momentum complex is given by [11]

\[ L_{\mu\nu} = \frac{1}{16\pi} S_{\mu\rho\sigma}^{\nu}, \]

(20)

whith the Landau-Lifshitz superpotentials

\[ S_{\mu\rho\sigma}^{\nu} = -g(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}). \]

(21)

The \( L_{00} \) and \( L_{0i} \) components represent the energy and the momentum densities, respectively. For the Landau-Lifshitz prescription the local conservation is respected

\[ L_{\mu\nu}^{\mu} = 0. \]

(22)

By integrating \( L^{\mu\nu} \) over the 3-space, one obtains the expressions for the energy and momentum:

\[ P_{\mu} = \int \int \int L_{0}^{\mu} d^{3}x. \]

(23)

With the aid of Gauss’ theorem we get

\[ P_{\mu} = \frac{1}{16\pi} \int \int S_{\nu}^{\mu0} n_{i} dS = \frac{1}{16\pi} \int \int U_{\nu}^{\mu0} n_{i} dS. \]

(24)

In the Landau-Lifshitz prescription, the calculations have to be performed using the line element (12). The non-vanishing components of the the Landau-Lifshitz superpotentials are

\[ U_{001} = \frac{2\pi}{r^2} \left\{ \frac{2M}{r} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{4\pi^2} \right)^3} \right]^{\frac{3}{a}} - \frac{q^2}{r^2} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{4\pi^2} \right)^3} \right]^{\frac{4}{a}} \right\}. \]

(25)
\[ U^{002} = \frac{2y}{r^2} \frac{2M}{r} \left[ \frac{\gamma}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{3/a} - \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{4/a}, \]  
(26)

\[ U^{003} = \frac{2z}{r^2} \frac{2M}{r} \left[ \frac{\gamma}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{3/a} - \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{4/a}. \]  
(27)

Using (25)-(27) and (24) we obtain the energy

\[ E_{LL} = \frac{r}{2} \frac{2M}{r} \left[ \frac{\gamma}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{3/a} - \frac{q^2}{r^2} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr^4} \right)} \right]^{4/a}. \]  
(28)

The energy in the Landau-Lifshitz prescription is plotted in Fig. 3 for \( a = 2, \gamma = \frac{4}{5}, M = 1 \) and \( q = 0.1 \).

\('M=1', 'q=0.1', '\gamma=\frac{4}{5}, a=2'\)

![Graph of Landau-Lifshitz energy versus the radial distance r.](image)

FIG. 3: Landau-Lifshitz energy versus the radial distance r.

In Fig. 4 we present the graph of the energy near the origin for the same values of \( a, M, q \) and \( \gamma \).
V. WEINBERG PRESCRIPTION AND THE ENERGY DISTRIBUTION OF THE ASYMPTOTICALLY REISSNER-NORDSTRÖM NON-SINGULAR BLACK HOLE

The Weinberg energy-momentum complex [15] is given by the expression

$$ W^{\mu \nu} = \frac{1}{16\pi} D^{\lambda \mu \nu}, $$ (29)

where the corresponding superpotentials are

$$ D^{\lambda \mu \nu} = \frac{\partial h_\kappa}{\partial x_\lambda} \eta^{\mu \nu} - \frac{\partial h_\kappa}{\partial x_\mu} \eta^{\lambda \nu} - \frac{\partial h_\kappa}{\partial x_\nu} \eta^{\lambda \mu} + \frac{\partial h^{\lambda \mu}}{\partial x_\kappa} \eta^{\nu} + \frac{\partial h^{\lambda \nu}}{\partial x_\kappa} \eta^{\mu} + \frac{\partial h^{\mu \nu}}{\partial x_\kappa} \eta^{\lambda} , $$ (30)

and

$$ h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu}. $$ (31)

The $W^{00}$ and $W^{0i}$ components represent the energy and the momentum densities, respectively. In the Weinberg prescription the local conservation law is respected:

$$ W^{\mu \nu}_{\nu} = 0 . $$ (32)

By integrating $W^{\mu \nu}$ over the 3-space, one gets the expression for the energy-momentum

$$ P^\mu = \int \int \int W^{\mu 0} dx^1 dx^2 dx^3 . $$ (33)

Using Gauss’ theorem and integrating over the surface of a sphere of radius $r$, the energy-momentum distribution takes the form:

$$ P^\mu = \frac{1}{16\pi} \int \int D^{0\mu} n_i dS . $$ (34)
The nonvanishing superpotential components are as follows:

\[
D^{100} = \frac{2x}{r^2} \frac{2M}{r} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{3/a} - \frac{q^2}{r^2} \frac{1}{\gamma} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{4/a},
\]

(35)

\[
D^{200} = \frac{2y}{r^2} \frac{2M}{r} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{3/a} - \frac{q^2}{r^2} \frac{1}{\gamma} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{4/a},
\]

(36)

\[
D^{300} = \frac{2z}{r^2} \frac{2M}{r} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{3/a} - \frac{q^2}{r^2} \frac{1}{\gamma} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{4/a},
\]

(37)

Substituting these expressions into (34), we obtain for the energy distribution inside a 2-sphere of radius \( r \) the expression

\[
E_W = \frac{r}{2} \frac{2M}{r} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{3/a} - \frac{q^2}{r^2} \frac{1}{\gamma} \left[ \frac{1}{1+\gamma \left( \frac{q^2}{M_r} \right)^a} \right]^{4/a}.
\]

(38)

The energy is plotted in Fig. 5 for \( a = 2 \), \( \gamma = \frac{4}{9} \), \( M = 1 \) and \( q = 0.1 \).

\('M=1'q=0.1'\gamma=\frac{4}{9},a=2'\)

FIG. 5: Weinberg energy versus the radial distance \( r \).
Fig. 6 shows the behaviour of the energy near the origin for the same values of $a$, $M$, $q$ and $\gamma$.

\[
\mathcal{M} = 1', q = 0.1', \gamma = \frac{4}{9}, a = 2'
\]

**FIG. 6:** Weinberg energy versus the radial distance $r$ near the origin.

**VI. MÖLLER PRESCRIPTION AND THE ENERGY DISTRIBUTION OF THE ASYMPTOTICALLY REISSNER-NORDSTRÖM NON-SINGULAR BLACK HOLE**

The expression for the Møller energy-momentum complex [14] is

\[
\mathcal{J}_\nu^\mu = \frac{1}{8\pi} M_\nu^{\mu\lambda},
\]

where $M_\nu^{\mu\lambda}$ represent the Møller superpotentials:

\[
M_\nu^{\mu\lambda} = \sqrt{-g} \left( \frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma}.
\]

The Møller superpotentials $M_\nu^{\mu\lambda}$ are also antisymmetric

\[
M_\nu^{\mu\lambda} = -M_\lambda^{\mu\nu}.
\]

Møller’s energy-momentum complex satifies the local conservation law

\[
\frac{\partial \mathcal{J}_\nu^\mu}{\partial x^\mu} = 0,
\]

with $\mathcal{J}_0^\mu$ representing the energy density and $\mathcal{J}_i^0$ the momentum density components, respectively.

For the Møller prescription, the energy and momentum distributions are given by

\[
P_\mu = \iiint \mathcal{J}_\nu^0 dx^1 dx^2 dx^3
\]

The energy distribution is calculated by using

\[
E = \iiint \mathcal{J}_0^0 dx^1 dx^2 dx^3.
\]
Applying Gauss’ theorem one gets
\[ P_\mu = \frac{1}{8\pi} \int \int M_{\mu i} n_i dS. \] (45)

In the Møller prescription we use Schwarzschild coordinates \( \{ t, r, \theta, \phi \} \) for the line element (4) and the metric function given by eq. (1). We found that the only non-vanishing component of the Møller superpotential (40) has the expression
\[
M_{01}^{01} = \frac{2M}{r^2} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right]^{3/a} \left[ 2 - \frac{6\gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right] - 2q^2 \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right]^{4/a} \left[ 1 - \frac{2\gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right] \sin \theta. \] (46)

Combining eqs. (45) and (46) and after a few groupings of the terms, we obtain the expression for the energy distribution:
\[
E_M = M \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right]^{3/a} \left[ 1 - \frac{3\gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right] - \frac{q^2}{r} \left[ \frac{1}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right]^{4/a} \left[ 1 - \frac{2\gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}}{1 + \gamma \left( \frac{q^2}{Mr} \right)^{\frac{a}{2}}} \right]. \] (47)

Furthermore, because of the vanishing of the spatial components of the Møller superpotential, all the momentum components are vanishing everywhere:
\[ P_r = P_\theta = P_\phi = 0. \] (48)

In Fig. 7 we plot the energy in the Møller prescription for the parameter \( a = 2 \) with \( M = 1, q = 0.1, \) and \( \gamma = \frac{4}{9} \).

**FIG. 7: Møller energy versus the radial distance \( r \).**
Fig. 8 shows the behaviour of the Möller energy near the origin for the same values of $a$, $M$, $q$ and $\gamma$.

In Fig. 9 we present a comparison of the energy distributions in the Einstein, Landau-Lifshitz, Weinberg and Möller prescriptions for $a = 2$, $M = 1$, $q = 0.1$ and $\gamma = \frac{4}{9}$.

FIG. 8: Möller energy versus the radial distance $r$ near the origin.

FIG. 9: Comparison of energy in the Einstein, Landau-Lifshitz, Weinberg, and Möller prescriptions versus the radial distance $r$. 
In this paper we have calculated the energy and momentum distributions for a new spherically symmetric and charged asymptotically Reissner-Nordström non-singular black hole space-time geometry.

The Einstein, Landau-Lifshitz, Weinberg and Møller prescriptions have been used, and we have found that these four prescriptions give the same result regarding the momentum components, namely that all the momenta vanish. The expressions of the energy are well-defined and physically meaningful showing a dependence on the mass $M$, the charge $q$, two parameters $\gamma$ and $a$, and the radial coordinate $r$. Notice that the Landau-Lifshitz and Weinberg energy-momentum complexes yield exactly the same expression for the energy distribution. Also, for both prescriptions the energy is equal to the ADM mass also in the Landau-Lifshitz and Weinberg prescriptions. This result is in agreement with the result obtained by Virbhadra for the energy distribution of the Schwarzschild black hole solution [27]. Indeed, in the particular case $q = 0$, the energy in the Landau-Lifshitz and Weinberg prescriptions is given by the expression $\frac{M}{1 - \frac{2M}{r}}$ that is obtained for the energy of the Schwarzschild black hole solution in Schwarzschild Cartesian coordinates.

Table 1 shows the limiting behavior of the energy for $r \to 0$ and $r \to \infty$, and in the particular case $q = 0$.

| Case             | $r \to 0$ | $r \to \infty$ | $q = 0$ |
|------------------|-----------|-----------------|---------|
| Einstein         | 0         | $M$             | $M$     |
| Landau-Lifshitz  | 0         | $\frac{M}{1 - \frac{2M}{r}}$ | $M$     |
| Weinberg         | 0         | $M$             | $M$     |
| Møller           | 0         | $\frac{M}{1 - \frac{2M}{r}}$ | $M$     |

Table 1: Limiting behaviour

For $r \to 0$ the new spherically symmetric and charged non-singular black hole solution considered exhibits a de Sitter behaviour near zero, but this does not necessarily imply that it satisfies the weak energy condition.

Fig. 10 shows the energy distributions in the Einstein, Landau-Lifshitz, Weinberg and Møller prescriptions near the origin, as a function of $r$, for the particular case $a = 2$ and $\gamma = \frac{4}{9}$, with $M = 1$, $q = 0.1$.

The behaviour of the energy near the origin, that is for $r \to 0$, is a special limiting case of particular interest. For some spacetime geometries the metric “goes infinite” and, as a consequence, a singularity appears, while the energy and momentum take extreme values. This particular behavior is related to the specific nature of the space-time geometry. In the case of the Einstein, Landau-Lifshitz, Weinberg and Møller prescriptions, we notice that for $r \to 0$ the energy tends to zero. Carrying out a more detailed investigation of the behavior of energy near the origin we found that the Einstein energy tends to zero from positive values, as expected from eq. (18), being an increasing function of $r$ which tends to zero from the maximum value $M$ that is the ADM mass. The Landau-Lifshitz and Weinberg energy presents points of divergence and takes both positive and negative values. Analyzing Fig. 11, we deduce that for $r \lesssim 0.00077$ the energy in these two prescriptions becomes positive and tends to zero. Also, for $r \gtrsim 1.995$ the Landau-Lifshitz and Weinberg energy takes positive values and, finally, for large values of the radial coordinate $r$, it becomes equal to the ADM mass $M$. The values of the $r$ coordinate $r = 0.00077$ and $r = 1.995$ represent the two points of divergence of the Landau-Lifshitz and Weinberg energy. The Møller energy tends also to zero and, according to Fig. 12, close enough to zero, i.e. for $r \lesssim 0.0135$, it acquires negative values. For values of $r$ greater than 0.0135, the Møller energy is positive and an increasing function of $r$ acquiring the value $M$, which is the ADM mass, for $r \to \infty$. 

VII. RESULTS AND DISCUSSION
FIG. 10: Comparison of energy in the Einstein, Landau-Lifshitz, Weinberg, and Møller prescriptions versus the radial distance $r$ near the origin.

$'M = 1', q = 0.1', \gamma = \frac{4}{9}, a = 2'$

FIG. 11: Landau-Lifshitz energy versus the radial distance $r$ near the origin.
As we noticed, from Fig. 11, we observe that the Landau-Lifshitz and Weinberg energy exhibits two points of divergence (singularities) whose values depend on the values of the parameters $a$ and $\gamma$, and on the mass $M$ and the charge $q$, jumping from negative to positive values, and finally reaches the value of the ADM mass $M$ for $r \to \infty$.

These results come to support the use of the Einstein, Landau-Lifshitz, Weinberg and Møller energy-momentum complexes for the evaluation of the energy of a space-time geometry, while keeping in mind that the positive energy region serves as a convergent lens and the negative one as a divergent lens [57]. Also, the negativity of the expressions of energy in the case of the Landau-Lifshitz, Weinberg and Møller prescriptions, for a range of values of $a$, $\gamma$, $M$, $q$ and $r$, highlights the existence of some difficulty in the physically meaningful interpretation of the energy in certain regions of space-time.

VIII. CONCLUSIONS

The study of the asymptotically Reissner-Nordström non-singular black hole space-time geometry could be of great importance for black hole physics as it would allow to test this particular black hole, the best approach being gravitational lensing. In this light, the space-time described by the metric given by eq. (4) with the metric function (1) allows obtaining useful information concerning effects in strong gravitational lensing. As we noted before, the positive and negative regions of the energy serve as convergent and divergent lenses, [58]-[59] respectively, and the study of the behaviour of the energy in the Einstein, Landau-Lifshitz, Weinberg and Møller prescriptions near the event horizon also indicates what type of microlensing can occur within each pseudotensorial prescription. The behaviour of the energy near the event horizon could be analyzed by performing a Taylor expansion of $E_E(r)$, $E_{LL}(r)$, $E_W(r)$ and $E_M(r)$ as a function of $r = 0.00077$ and $r = 1.995$ in the particular case $a = 2$, $\gamma = \frac{4}{9}$, $M = 1$ and $q = 0.1$. We choose these two values for the radial coordinate $r$ because in the case of the event horizon the equation $f(r) = 0$, with $f(r)$ given by eq. (1) has two real roots, $r = 0.00077$ and $r = 1.995$, and, also, two complex roots. The energy in the Einstein prescription near the event horizon $E_{HE}$ has only positive values playing the role of a convergent lens. In the case of the Landau-Lifshitz, Weinberg and Møller prescriptions, the expressions of the energy near the event horizon $E_{HLL}$, $E_{HW}$ and $E_{HM}$ take both positive and negative values and serve as a convergent and divergent lens, respectively. To obtain more information about the microlensing, a detailed analysis of the effects of dark energy on the strong gravitational lensing in the case of the asymptotically Reissner-Nordström non-singular black hole is needed.

As a conclusion, the energy in the Einstein prescription takes only positive values, while in the case of the Landau-Lifshitz, Weinberg and Møller prescriptions the energy takes both positive and negative values depending on the
values of the radial coordinate \( r \) and of \( a, \gamma, M \) and \( q \). The apparent weakness of the Landau-Lifshitz, Weinberg and Møller prescriptions could be justified by the properties of the particular metric describing the asymptotically Reissner-Nordström non-singular black hole. Note that a similar behaviour of the Møller energy-momentum complex was described in [60] and [61]. In the case of the asymptotically Reissner-Nordström non-singular black hole under study, this strange behaviour is due, as in the case of the metrics presented in [60] and [61], to the special properties of these black hole solutions that originate in the coupling of the gravitational field to non-linear electrodynamics.

The pseudotensorial prescriptions used in this work constitute instructive and useful tools for the energy-momentum localization. We consider a challenging future issue to employ other pseudotensorial prescriptions, as well as the teleparallel equivalent theory of general relativity, for further investigation of the energy-momentum localization in the context of the asymptotically Reissner-Nordström non-singular black hole.

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