A CHARACTERIZATION OF CAUCHY SEQUENCES IN FUZZY METRIC SPACES AND ITS APPLICATION TO FUZZY FIXED POINT THEORY

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Abstract. We introduce the concept of fuzzy Ćirić-Matkowski contractive mappings as a generalization of fuzzy Meir-Keeler type contractions. We also introduce a class $\Psi_1$ of gauge functions $\psi : (0, 1] \to (0, 1]$ in the sense that, for any $r \in (0, 1)$, there exists $\rho \in (r, 1)$ such that $1 - r > \tau > 1 - \rho$ implies $\psi(\tau) \geq 1 - r$. We show that fuzzy $\psi$-contractive mappings ($\psi \in \Psi_1$) are fuzzy Ćirić-Matkowski contractive mappings. Then, we present a characterization of $M$-Cauchy sequences in fuzzy metric spaces. This characterization is used to establish new fuzzy fixed point theorems. Our results include those of Mihet (Fuzzy $\psi$-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets Syst. 159(2008) 739–744.), Wardowski (Fuzzy contractive mappings and fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 222(2013) 108–114) and others. Examples are given to support the results.

1. Introduction

Fuzzy metric spaces were initiated by Kramosil and Michálek [11]. Later, in order to obtain a Hausdorff topology in fuzzy metric spaces, George and Veeramani [6] modified the conditions formulated in [11]. The study of fixed point theory in fuzzy metric spaces started with the work of Grabiec [7], by extending the well-known fixed point theorems of Banach [2] and Edelstein [5] to fuzzy metric spaces. Many authors followed this concept by introducing and investigating different types of fuzzy contractive mappings. The concept of fuzzy contractive mapping, initiated by Gregori and Sapena [9], has become of interest for many authors; see, e.g., [14, 15, 16, 17, 21, 22, 23, 24, 25]. They reconsidered the Banach contraction principle by initiating a new concept of fuzzy contractive mapping in fuzzy metric spaces in the sense of George and Veeramani [6] and also in the sense of Kramosil and Michálek [11]. However, in their results, there were used strong conditions for completeness, namely $G$-completeness [7], of a fuzzy metric space. Being aware of this problem, they raised the question whether the fuzzy contractive sequences are Cauchy in the usual sense, namely $M$-Cauchy [6]. Very recently many papers have appeared concerning this subject; see, for example, the interesting results of Mihet [14, 15, 16].

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In a recent paper, Wardowski [24] introduced the concept of fuzzy $H$-contractive mappings and formulated conditions guaranteeing the convergence of fuzzy $H$-contractive sequences to a unique fixed point in a complete fuzzy metric space. This established notion of contraction turns out to be a generalization of the fuzzy contractive condition of Gregori and Sapena [9]. The paper includes a comprehensive set of examples showing the generality of the results and demonstrating that the formulated conditions are significant and cannot be omitted. However, in [8], it is shown that some assertions made in [24] are not true. In [17], Mihet considered a larger class of $H$-contractive mappings in the setting of fuzzy metric spaces in the sense of Kramosil and Michalek as well as in the setting of strong fuzzy metric spaces.

In this paper, we are concerned with contractive mappings of Meir-Keeler type in fuzzy metric spaces, which are analogous with Meir-Keeler type contractions in metric spaces; [11] [4] [10] [12] [13]. After gathering together some preliminaries to fuzzy metric spaces in Section 2, we introduce, in Section 3, the concept of fuzzy Ćirić-Matkowski contractive mappings (which are generalizations of fuzzy Meir-Keeler contractive mappings). We introduce a class $\Psi_1$ of gauge functions that contains the class $\Psi$ introduced in [16], and establish a relation between fuzzy Ćirić-Matkowski contractive mappings and fuzzy $\psi$-contractive mappings ($\psi \in \Psi_1$).

In Section 4, adopting the same method used in [1], we present a characterization of $M$-Cauchy sequences in fuzzy metric spaces. Using this characterization, it is then easily proved, in Section 5 that fuzzy Ćirić-Matkowski contractive mappings on complete fuzzy metric spaces have unique fixed points. The fixed point theorems presented in Section 5 extend some recent results in this area (e.g., [8] [16] [17] [24]).

2. Preliminaries

A binary operation $*$ on $[0,1]$ is called a triangular norm or a t-norm [20] if it is associative, commutative, and satisfies the following properties:

1. $a * 1 = a$, for all $a \in [0,1]$;
2. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

A t-norm $*$ is called positive if $a * b > 0$ whenever $a > 0$ and $b > 0$. Some typical examples of t-norms are the following:

$$a * b = ab, \quad \text{(product)}$$
$$a * b = \min\{a,b\}, \quad \text{(minimum)}$$
$$a * b = \max\{0,a + b - 1\}, \quad \text{(Lukasiewicz)}$$
$$a * b = \frac{ab}{a + b - ab}, \quad \text{(Hamacher)}.$$

Definition 2.1 (Kramosil and Michalek, [11]). A fuzzy metric space is a triple $(X,M,*)$, where $X$ is a nonempty set, $*$ is a continuous t-norm, and $M : X^2 \times [0,\infty) \to [0,1]$ is a mapping with the following properties:

1. $M(x,y,0) = 0$, for all $x, y \in X$;
2. $x = y$ if and only if $M(x,y,t) = 1$, for all $t > 0$;
If, in the above definition, the triangular inequality (5) is replaced by
\[(5') M(x,z,t) \geq M(x,y,t) * M(y,z,t), \text{ for all } x,y,z \in X \text{ and } t > 0,\]
then the triple \((X,M,\cdot)\) is called a strong fuzzy metric space. It is easy to check that the triangle inequality \((5')\) implies \((5)\), that is, every strong fuzzy metric space is itself a fuzzy metric space.

In [6], the authors modified the above definition in order to introduce a Hausdorff topology on the fuzzy metric space.

Definition 2.2 (George and Veeramani, [6]). A fuzzy metric space is a triple \((X,M,\cdot)\), where \(X\) is a nonempty set, \(\cdot\) is a continuous t-norm, and \(M : X^2 \times (0,\infty) \to [0,1]\) is a mapping satisfying the following properties, for all \(x,y \in X\) and \(t > 0\):

(a) \(M(x,y,t) > 0\);
(b) \(x = y\) if and only if \(M(x,y,t) = 1\);
(c) \(M(x,y,t) = M(y,x,t)\);
(d) \(M(x,y,\cdot) : (0,\infty) \to [0,1]\) is continuous;
(e) \(M(x,z,s+t) \geq M(x,y,s) * M(y,z,t)\).

Example 2.3 ([6]). Let \((X,d)\) be a metric space. Define a t-norm by \(a \cdot b = ab\), and set \(M_d(x,y,t) = \frac{t}{t + d(x,y)}, \quad (x,y \in X, t > 0)\).

Then \((X,M_d,\cdot)\) is a (strong) fuzzy metric space; \(M_d\) is called the standard fuzzy metric induced by \(d\). It is interesting to note that the topologies induced by the standard fuzzy metric \(M_d\) and the corresponding metric \(d\) coincide.

Example 2.4 ([6]). Let \((X,d)\) be a metric space. Define \(a \cdot b = ab\), and
\[M(x,y,t) = \exp\left[-\frac{d(x,y)}{t}\right],\]
for all \(x,y \in X\) and \(t > 0\). Then \((X,M,\cdot)\) is a (strong) fuzzy metric space.

A sequence \((x_n)\) in a fuzzy metric space \((X,M,\cdot)\) converges to a point \(x \in X\), if
\[\forall t > 0, \quad \lim_{n \to \infty} M(x_n, x, t) = 1.\]

In [7], notions of Cauchy sequences and complete fuzzy metric spaces are defined as follows: The sequence \((x_n)\) in \(X\) is called \(G\)-Cauchy if
\[\forall n \in \mathbb{N}, \forall t > 0, \quad \lim_{n \to \infty} M(x_n, x_{n+m}, t) = 1.\]

A fuzzy metric space in which every \(G\)-Cauchy sequence is convergent is called a \(G\)-complete fuzzy metric space. With this definition of completeness, even \((\mathbb{R},M_d,\cdot)\), where \(d\) is the Euclidean metric on \(\mathbb{R}\) and \(a \cdot b = ab\), fails to be complete. Hence, in [6], the authors redefined Cauchy sequence as follows.
Definition 2.5 (George and Veeramani, [6]). A sequence \((x_n)\) in a fuzzy metric space \((X, M, \ast)\) is said to be \textit{Cauchy} (or \textit{M-Cauchy}) if, for each \(r \in (0,1)\) and each \(t > 0\), there exists \(N \in \mathbb{N}\) such that
\[
\forall m, n \geq N, \quad M(x_n, x_m, t) > 1 - r.
\]
A fuzzy metric space in which every Cauchy sequence is convergent, is called a \textit{complete} (or \textit{M-complete}) fuzzy metric space.

Note that a metric space \((X, d)\) is complete if and only if the induced fuzzy metric space \((X, M_d, \ast)\) is complete.

In this paper, we work in the setting of fuzzy metric spaces in the sense of George and Veeramani, in which by a Cauchy sequence we always mean an \(M\)-Cauchy sequence, and by a complete space we always mean an \(M\)-complete space.

3. Fuzzy \(\psi\)-contractive Mappings

Before discussing the main subject of the section, for the reader’s convenience, we briefly recall some known facts and results about \(\phi\)-contractive mappings on metric spaces. Many authors have extended the Banach contraction principle in various types. Two types of such generalizations are as follows. One type involves the Meir-Keeler type conditions, and the other involves contractive gauge functions. Theorems for equivalence between the two types of contractive definitions are established in [18].

Definition 3.1. A self-map \(T\) of a metric space \(X\) is said to be a \textit{Meir-Keeler contraction} if, for every \(\epsilon > 0\), there exists \(\delta > \epsilon\) such that
\[
\forall x, y \in X, \quad \epsilon \leq d(x, y) < \delta \implies d(Tx, Ty) < \epsilon.
\]

Theorem 3.2 (Meir and Keeler [13]). If \(T\) is a Meir-Keeler contraction on a complete metric space \(X\), then \(T\) has a unique fixed point \(z\), and \(T^n x \to z\), for every \(x \in X\).

Čirić [4] and Matkowski [12, Theorem 1.5.1] generalizes the above Meir-Keeler fixed point theorem as follows.

Definition 3.3. A self-map \(T\) of a metric space \(X\) is said to be a \textit{Čirić-Matkowski contraction} if \(d(Tx, Ty) < d(x, y)\) for \(x \neq y\), and, for every \(\epsilon > 0\), there exists \(\delta > \epsilon\) such that
\[
\forall x, y \in X, \quad \epsilon < d(x, y) < \delta \implies d(Tx, Ty) \leq \epsilon.
\]

As it is mentioned in [10, Proposition 1], condition (3.2) in Definition 3.3 can be replaced by the following:
\[
\forall x, y \in X, \quad d(x, y) < \delta \implies d(Tx, Ty) \leq \epsilon.
\]

Theorem 3.4 (Čirić [4], Matkowski [12]). If \(T\) is a Čirić-Matkowski contraction on a complete metric space \(X\), then \(T\) has a unique fixed point \(z\), and \(T^n x \to z\), for every \(x \in X\).
In 1995, Jachymski [10] presented a fixed point theorem involving gauge functions as follows.

**Definition 3.5.** Let \( \Phi_1 \) denote the class of all functions \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with the property that, for any \( \epsilon > 0 \), there exists \( \delta > \epsilon \) such that \( \epsilon < s < \delta \) implies \( \phi(s) \leq \epsilon \). Given \( \phi \in \Phi_1 \), a self-map \( T \) of a metric space \((X, d)\) is said to be \( \phi \)-contractive if

1. \( d(Tx, Ty) < d(x, y) \) for \( x \neq y \),
2. \( d(Tx, Ty) \leq \phi(d(x, y)) \), for all \( x, y \).

**Theorem 3.6** (Jachymski [10]). For every \( \phi \in \Phi_1 \), every \( \phi \)-contractive mapping \( T \) on a complete metric space \( X \) has a unique fixed point \( z \), and \( T^n x \to z \), for every \( x \in X \).

The following result shows that Theorems 3.4 and 3.6 are equivalent. (An examination of the lemma and its proof in [18], shows that the self-map \( T \) there, has no role.)

**Lemma 3.7** ([18] Lemma 3.1). Let \( E \) and \( F \) be two nonnegative functions on a nonempty set \( X \) such that \( E(x) \leq F(x) \), for all \( x \in X \). Then the following statements are equivalent:

(i) There is a function \( \phi \in \Phi_1 \) such that \( E(x) \leq \phi(F(x)) \), for all \( x \in X \).
(ii) For any \( \epsilon > 0 \) there is \( \delta > \epsilon \) such that \( \epsilon < F(x) < \delta \) implies \( E(x) \leq \epsilon \).

We now turn to the class of \( \psi \)-contractive mappings on fuzzy metric spaces.

**Definition 3.8.** Let \( \Psi_1 \) denote the class of all functions \( \psi : (0, 1] \rightarrow (0, 1] \) with the property that, for any \( r \in (0, 1) \), there exists \( \rho \in (r, 1) \) such that \( 1 - r > \tau > 1 - \rho \) implies \( \psi(\tau) \geq 1 - r \). Given \( \psi \in \Psi_1 \), a self-map \( T \) of a fuzzy metric space \((X, M, *)\) is said to be \( \psi \)-contractive if

1. \( M(Tx, Ty, t) > M(x, y, t) \) for \( x \neq y \) and \( t > 0 \),
2. \( M(Tx, Ty, t) \geq \psi(M(x, y, t)) \), for all \( x, y \) and \( t > 0 \).

In [16] (see also [14]) Mihet defined a class \( \Psi \) consisting of all continuous, non-decreasing functions \( \psi : (0, 1] \rightarrow (0, 1] \) with the property that \( \psi(\tau) > \tau \), for all \( \tau \in (0, 1) \). Then, they proved a fixed point theorem for \( \psi \)-contractive mappings \( (\psi \in \Psi) \) in complete strong fuzzy metric spaces. We will prove fixed point theorems for \( \psi \)-contractive mappings \( (\psi \in \Psi_1) \) in complete fuzzy metric spaces (Theorems 5.3 and 5.4). Since the class \( \Psi_1 \) properly contains the class \( \Psi \) (Proposition 3.9 and Example 6.1), and the class of fuzzy metric spaces contains the class of strong fuzzy metric spaces, we see that our theorem extends Mihet’s result (see also Example 6.2).

**Proposition 3.9.** \( \Psi \subseteq \Psi_1 \).

**Proof.** Take \( \psi \in \Psi \). For every \( r \in (0, 1) \), since \( \psi(1 - r) > 1 - r \), the continuity of \( \psi \) at \( 1 - r \) implies the existence of some \( \rho \in (r, 1) \) such that \( 1 - r > \tau > 1 - \rho \) implies \( \psi(\tau) \geq 1 - r \). This shows that \( \psi \in \Psi_1 \), and thus \( \Psi \subseteq \Psi_1 \).

**Remark 1.** We will present, in Example 6.1 some function \( \psi \in \Psi_1 \) which is not continuous. This confirms that \( \Psi \not\subseteq \Psi_1 \).
As in [24], we let $\mathcal{H}$ be the family of all strictly decreasing bijections $\eta : (0, 1] \to [0, \infty)$. In a recent paper, Wardowski [24] introduced the notion of fuzzy $\mathcal{H}$-contractive mappings, and proved a fixed point theorem for such mappings. In [8], it is shown that fuzzy $\mathcal{H}$-contractive mappings are included in the class of fuzzy $\psi$-contractive mappings ($\psi \in \Psi$). One might define new fuzzy contractive mappings by composing functions $\phi \in \Phi_1$. Given $\eta \in \mathcal{H}$ and $\phi \in \Phi_1$, let us call a self-map $T$ of a fuzzy metric space $(X, M, \ast)$ a fuzzy $\phi \cdot \mathcal{H}$-contractive mapping, if

$$\forall t > 0, \forall x, y \in X, \eta(M(Tx, Ty, t)) \leq \phi(\eta(M(x, y, t))).$$

We show that these kinds of fuzzy contractive mappings are included in the class of fuzzy $\psi$-contractive mappings ($\psi \in \Psi_1$). First, we have the following result.

**Proposition 3.10.** For every $\eta \in \mathcal{H}$, the mapping $\phi \mapsto \eta^{-1} \circ \phi \circ \eta$, is a one-to-one correspondence from $\Phi_1$ onto $\Psi_1$.

**Proof.** Take $\phi \in \Phi_1$, and let $\psi = \eta^{-1} \circ \phi \circ \eta$. Given $r \in (0, 1)$, let $\epsilon = \eta(1 - r)$. Since $\phi \in \Phi_1$, there is $\delta > \epsilon$ such that $\epsilon < s < \delta$ implies $\phi(s) \leq \epsilon$. Let $\rho = 1 - \eta^{-1}(\delta)$. Now, if $1 - r > \tau > 1 - \rho$ then $\epsilon < \eta(\tau) < \delta$, and thus $\phi(\eta(\tau)) \leq \epsilon$. Therefore, $\psi(\tau) = \eta^{-1}(\phi(\eta(\tau))) \geq \eta^{-1}(\epsilon) = 1 - r$.

Similarly, if $\psi \in \Psi_1$, then $\phi = \eta \circ \phi \circ \eta^{-1}$ belongs to $\Phi_1$. \qed

**Corollary 3.11.** The class of fuzzy $\phi \cdot \mathcal{H}$-contractive mappings is included in the class of fuzzy $\psi$-contractive mappings, for $\psi \in \Psi_1$.

We conclude this section by introducing the concept of fuzzy Ćirić-Matkowski contractive mappings.

**Definition 3.12.** A self-map $T$ of a fuzzy metric space $(X, M, \ast)$ is said to be a fuzzy Ćirić-Matkowski contractive mapping if

1. $M(Tx, Ty, t) > M(x, y, t)$ for $x \neq y$ and $t > 0$,
2. for every $t > 0$ and $r \in (0, 1)$, there exists $\rho \in (r, 1)$ such that

$$\forall x, y \in X, \quad 1 - r > M(x, y, t) > 1 - \rho \implies M(Tx, Ty, t) \geq 1 - r.$$  

**Lemma 3.13.** Let $E$ and $F$ be two nonnegative functions on a nonempty set $X$ such that $E(x) \geq F(x)$, for all $x \in X$. Then the following statements are equivalent:

1. There is a function $\psi \in \Psi_1$ such that $E(x) \geq \psi(F(x))$, for all $x \in X$.
2. For any $r \in (0, 1)$ there is $\rho \in (r, 1)$ such that

$$\forall x \in X, \quad 1 - r > F(x) > 1 - \rho \implies E(x) \geq 1 - r.$$
Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (i) Take a function $\eta \in \mathcal{H}$, and set $E_1(x) = \eta(E(x))$ and $F_1(x) = \eta(F(x))$. It is a matter of calculation to see that, for any $\epsilon > 0$, there is $\delta > \epsilon$ such that $\epsilon < E_1(x) < \delta$ implies $E_1(x) \leq \epsilon$. Hence, by Lemma 5.7 there is $\phi \in \Phi_1$ such that $E_1(x) \leq \phi(F_1(x))$, for all $x \in X$. Therefore,

$$E(x) = \eta^{-1}(E_1(x)) \geq \eta^{-1}(\phi(F_1(x))) = \eta^{-1}(\phi(\eta(F(x)))).$$  

If we take $\psi = \eta^{-1} \circ \phi \circ \eta$, then $\psi \in \Psi_1$ (by Proposition 5.10) and $E(x) \geq \psi(F(x))$, for all $x \in X$.

Obviously, the class of fuzzy Ćirić-Matkowski contractive mappings contains the class of $\psi$-contractive mappings ($\psi \in \Psi_1$). We can say more:

**Theorem 3.14.** Let $T$ be a self-map of a fuzzy metric space $(X, M, \ast)$ such that 

$$\forall t > 0, \forall x, y \in X, \ M(Tx, Ty, t) \geq M(x, y, t).$$

Among the following statements, the implications (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) hold.

(i) There exists $\psi \in \Psi_1$ such that 

$$\forall t > 0, \forall x, y \in X, \ M(Tx, Ty, t) \geq \psi(M(x, y, t)).$$

(ii) For every $r \in (0, 1)$, there exists $\rho \in (r, 1)$ such that 

$$\forall t > 0, \forall x, y \in X, \ 1 - r > M(x, y, t) > 1 - \rho \Rightarrow M(Tx, Ty, t) \geq 1 - r.$$  

(iii) For every $t > 0$ and $r \in (0, 1)$, there exists $\rho \in (r, 1)$ such that 

$$\forall x, y \in X, \ 1 - r > M(x, y, t) > 1 - \rho \Rightarrow M(Tx, Ty, t) \geq 1 - r.$$  

(iv) For every $t > 0$, there exists $\psi_t \in \Psi_1$ such that 

$$\forall x, y \in X, \ M(Tx, Ty, t) \geq \psi_t(M(x, y, t)).$$

**Proof.** (i) ⇔ (ii) follows from Lemma 3.13 by considering the nonnegative functions 

$$E(x, y, t) = M(Tx, Ty, t), \quad F(x, y, t) = M(x, y, t),$$

on $X^2 \times (0, \infty)$. The implication (ii) ⇒ (iii) is obvious, and (iii) ⇔ (iv) also follows from Lemma 3.13 by considering, for each $t > 0$, the nonnegative functions 

$$E_t(x, y) = M(Tx, Ty, t), \quad F_t(x, y) = M(x, y, t),$$

on $X \times X$.  

4. A Characterization of Cauchy Sequences

In this section, a characterization of Cauchy sequences in fuzzy metric spaces is presented. This characterization then will be used, in Section 5, to give new fixed point theorems. First, we need a couple of definitions.

**Definition 4.1.** Let $(x_n)$ be a sequence in a fuzzy metric space $(X, M, \ast)$.

1. $(x_n)$ is called asymptotically regular if $M(x_n, x_{n+1}, t) \to 1$, for every $t > 0$.
2. $(x_n)$ is called uniformly asymptotically regular if, for any sequence $E = \{t_i : i \in \mathbb{N}\}$ of positive numbers with $t_i \to 0$, we have $M(x_n, x_{n+1}, t) \to 1$, uniformly on $t \in E$.  

The following is the main result of the section (it is analogous with [1, Lemma 2.1]).

**Lemma 4.2.** Let \((x_n)\) be a sequence in \((X,M,\ast)\). Suppose, for every \(t > 0\) and \(r \in (0,1)\), for any two subsequence \((x_{p_n})\) and \((x_{q_n})\), if \(\lim \inf M(x_{p_n}, x_{q_n}, t) \geq 1 - r\), then, for some \(N\),

\[
M(x_{p_n+1}, x_{q_n+1}, t) \geq 1 - r, \quad (n \geq N).
\]

Then \((x_n)\) is Cauchy, provided it is uniformly asymptotically regular. In case \((X,M,\ast)\) is a strong fuzzy metric space, we only need \((x_n)\) be asymptotically regular.

**Proof.** To get a contradiction, assume that \((x_n)\) is not Cauchy. Then, there exist \(t > 0\) and \(r \in (0,1)\) such that

\[
\forall k \in \mathbb{N}, \exists p,q \geq k, \quad M(x_p, x_q, t) < 1 - r.
\]

Set \(a_i = 1/i\), and \(E = \{a_i : i \in \mathbb{N}\}\). In case \((x_n)\) is uniformly asymptotically regular, we have \(M(x_n, x_{n+1}, a_it) \to 1\), uniformly on \(E\). Hence, there exist positive integers \(k_1 < k_2 < \cdots\) such that

\[
M(x_{m}, x_{m+1}, a_it) > 1 - \frac{r}{n}, \quad (m \geq k_n, i \in \mathbb{N}).
\]

For each \(k_n\), by (4.2), there exist integers \(p_n\) and \(q_n\) such that

\[
q_n > p_n \geq k_n \quad \text{and} \quad M(x_{p_n+1}, x_{q_n+1}, t) < 1 - r.
\]

We choose \(q_n\) be the smallest such integer so that \(M(x_{p_n+1}, x_{q_n}, t) \geq 1 - r\). Now, for every \(n\) and \(i\), we have

\[
M(x_{p_n}, x_{q_n}, t) \geq M(x_{p_n}, x_{p_n+1}, a_it) \ast M(x_{p_n+1}, x_{q_n}, (1 - a_i)t)
\]

\[
> \left(1 - \frac{r}{n}\right) \ast M(x_{p_n+1}, x_{q_n}, (1 - a_i)t).
\]

This is true for every \(i \in \mathbb{N}\). If \(i \to \infty\) then \((1 - a_i)t \to t\), and the continuity of \(M\) gives

\[
M(x_{p_n}, x_{q_n}, t) \geq \left(1 - \frac{r}{n}\right) \ast M(x_{p_n+1}, x_{q_n}, t) \geq \left(1 - \frac{r}{n}\right) \ast (1 - r).
\]

This implies that \(\lim \inf M(x_{p_n}, x_{q_n}, t) \geq 1 - r\). However, we have

\[
M(x_{p_n+1}, x_{q_n+1}, t) < 1 - r, \quad (n \in \mathbb{N}).
\]

This is a contradiction.

In case \(X\) is a strong fuzzy metric space, we choose \(k_1 < k_2 < \cdots\) such that (4.3) holds only for \(i = 1\), that is,

\[
M(x_{m}, x_{m+1}, t) > 1 - \frac{r}{n}, \quad (m \geq k_n).
\]

Then, instead of (4.5), we have the following

\[
M(x_{p_n}, x_{q_n}, t) \geq M(x_{p_n}, x_{p_n+1}, t) \ast M(x_{p_n+1}, x_{q_n}, t)
\]

\[
> \left(1 - \frac{r}{n}\right) \ast M(x_{p_n+1}, x_{q_n}, t).
\]

The rest of the proof is similar. \(\square\)

The following result follows directly from the above lemma.
\textbf{Theorem 4.3.} Let \((x_n)\) be a sequence in \((X,M,\ast)\), and let \(\mathfrak{M}(x,y,t)\) be a non-negative function on \(X^2 \times (0,\infty)\) such that, for any two subsequences \((x_{p_n})\) and \((x_{q_n})\),

\begin{equation}
\forall t > 0, \liminf_{n \to \infty} \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq \liminf_{n \to \infty} M(x_{p_n}, x_{q_n}, t).
\end{equation}

Suppose, for every \(t > 0\) and \(r \in (0,1)\), for any two subsequences \((x_{p_n})\) and \((x_{q_n})\), condition

\begin{equation}
\liminf_{n \to \infty} \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq 1 - r,
\end{equation}

implies that, for some \(N \in \mathbb{N}\),

\begin{equation}
M(x_{p_n+1}, x_{q_n+1}, t) \geq 1 - r, \quad (n \geq N).
\end{equation}

Then \((x_n)\) is Cauchy, provided it is uniformly asymptotically regular. In case \((X,M,\ast)\) is a strong fuzzy metric space, we only need \((x_n)\) to be asymptotically regular.

Proof. Using Lemma 4.2 let \(t > 0\) and \(r \in (0,1)\), and let \((x_{p_n})\) and \((x_{q_n})\) be two subsequences of \((x_n)\) with

\[ \liminf M(x_{p_n}, x_{q_n}, t) \geq 1 - r. \]

Then \(\liminf \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq 1 - r\), and thus (4.9) holds. All conditions in Lemma 4.2 are fulfilled and so the sequence is Cauchy.

The following result helps us apply Lemma 4.2 and Theorem 4.3 to fuzzy Ćirić-Matkowski contractive mappings. In fact, if \(T\) is such a contraction, then the Picard iterates \(x_n = T^n x, n \in \mathbb{N}\), satisfy condition (i) in the following lemma.

\textbf{Lemma 4.4.} Let \((x_n)\) be a sequence in \((X,M,\ast)\). For a nonnegative function \(\mathfrak{M}(x,y,t)\) on \(X^2 \times (0,\infty)\), the following statements are equivalent:

(i) for every \(t > 0\) and \(r \in (0,1)\), there exists \(\rho \in (r,1)\) and \(N \in \mathbb{Z}^+\) such that

\begin{equation}
\forall p,q \geq N, \quad \mathfrak{M}(x_{p,q}, t) > 1 - \rho \implies M(x_{p+1}, x_{q+1}, t) \geq 1 - r.
\end{equation}

(ii) for every \(t > 0\) and \(r \in (0,1)\), for any two subsequences \((x_{p_n})\) and \((x_{q_n})\), if

\[ \liminf \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq 1 - r \]

then, for some \(N\),

\[ M(x_{p_n+1}, x_{q_n+1}, t) \geq 1 - r, \quad (n \geq N). \]

Proof. (i) \(\Rightarrow\) (ii) Take \(t > 0\) and \(r \in (0,1)\). Assume, for subsequences \((x_{p_n})\) and \((x_{q_n})\), we have \(\liminf \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq 1 - r\). By (i), there exists \(\rho \in (r,1)\) and \(N_1 \in \mathbb{Z}^+\) such that (4.10) holds. Take \(N_2 \in \mathbb{Z}^+\) such that \(\mathfrak{M}(x_{p_n}, x_{q_n}, t) > 1 - \rho\) for \(n \geq N_2\). Then

\[ M(x_{p_n+1}, x_{q_n+1}, t) \geq 1 - r, \quad (n \geq \max\{N_1, N_2\}). \]

(ii) \(\Rightarrow\) (i) Assume, to get a contradiction, that (i) fails to hold. Then there exist \(t > 0\), \(r \in (0,1)\), and subsequences \((x_{p_n})\) and \((x_{q_n})\) such that

\[ \mathfrak{M}(x_{p_n}, x_{q_n}, t) > (1 - r) \ast \left(1 - \frac{1}{n}\right) \quad \text{and} \quad 1 - r > M(x_{p_n+1}, x_{q_n+1}, t). \]

This contradicts (i) because \(\liminf \mathfrak{M}(x_{p_n}, x_{q_n}, t) \geq 1 - r\). \qed
5. Fixed Point Theorems

In this section, new fixed point theorems for Ćirić-Matkowski contractive mappings on complete fuzzy metric spaces are presented. These theorems include many recent results in fuzzy fixed point theory including those in [16], [24], and others. In proving the theorems, we use the characterization of Cauchy sequences given in previous section.

**Definition 5.1.** Let $T$ be a self-map of a fuzzy metric space $(X, M, *)$ and $x \in X$. We say that $T$ is (uniformly) asymptotically regular at $x$, if the sequence $(T^n x)$ is (uniformly) asymptotically regular (Definition 4.1).

**Lemma 5.2.** If $T$ is a fuzzy Ćirić-Matkowski contractive mapping, then $T$ is asymptotically regular at each $x \in X$.

**Proof.** Let $x \in X$ and set $x_n = T^n x$, $n \in \mathbb{N}$. If $x_m = x_n$, for some $m$, then $x_m = x_n$ for all $n \geq m$, and there is nothing to prove. Suppose $x_n \neq x_{n+1}$, for all $n$. Then, by induction, we have $M(x_{n+1}, x_{n+2}, t) > M(x_n, x_{n+1}, t) > 0$, for all $t > 0$. Therefore, for every $t > 0$, the sequence $M(x_n, x_{n+1}, t)$ converges to some number $L(t) \in (0, 1]$, and $M(x_n, x_{n+1}, t) < L(t)$. We show that $L(t) = 1$. If $L(t) < 1$, take $r = 1 - L(t)$. There is $\rho \in (r, 1)$ such that (5.3) holds. Therefore,

$$L(t) > M(x_n, x_{n+1}, t) > 1 - \rho \Rightarrow M(x_{n+1}, x_{n+2}, t) \geq L(t).$$

This is a contradiction, and thus $L(t) = 1$. □

The following two fixed point theorems follow directly from Lemma 4.2, Lemma 4.4, and Lemma 5.2; they are extensions of [16, Theorem 3.1].

**Theorem 5.3.** Let $(X, M, *)$ be a complete strong fuzzy metric space. Then every fuzzy Ćirić-Matkowski contractive mapping (in particular, every $\psi$-contractive mapping for $\psi \in \Psi_1$) on $X$ has a unique fixed point.

**Theorem 5.4.** Let $(X, M, *)$ be a complete fuzzy metric space. Then every fuzzy Ćirić-Matkowski contractive mapping (in particular, every $\psi$-contractive mapping for $\psi \in \Psi_1$) on $X$ has a unique fixed point provided $T$ is uniformly asymptotically regular at some point $x_0$.

Before proceeding to next result, a remark is in order.

**Remark 2.** One crucial condition in the main theorem in [24] is the following.

- $\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\}$ is bounded, for all $x \in X$ and any sequence $(t_i)$ of positive numbers with $t_i \searrow 0$.

It is interesting to note that this condition implies that $T$ is uniformly asymptotically regular at each point $x \in X$. We see that Theorem 5.4 also generalizes [24, Theorem 3.2].

We now present our final fixed point theorem. For a self-map $T$ of $(X, M, *)$ and nonnegative real numbers $\alpha, \beta$, define a function $\mathfrak{M}$ on $X^2 \times (0, \infty)$ by

$$\mathfrak{M}(x, y, t) = M(x, y, t) \ast M(x, Tx, t)^\alpha \ast M(y, Ty, t)\beta.$$ (5.1)
**Theorem 5.5.** Let \((X, M, \ast)\) be a complete fuzzy metric space, and \(T\) be a continuous self-map of \(X\). Define \(M\) by (5.1), and suppose

\(i)\ M(Tx, Ty, t) > M(x, y, t)\), for \(x \neq y\) and \(t > 0\);

\(ii)\ For every \(t > 0\) and \(r \in (0, 1)\), there exist \(p \in (r, 1)\) and \(N \in \mathbb{Z}^+\) such that, for every \(x, y \in X\),

\[
M(T^N x, T^N y, t) > 1 - \rho \Rightarrow M(T^{N+1} x, T^{N+1} y, t) \geq 1 - r.
\]

Then \(T\) has a unique fixed point, provided \(T\) is uniformly asymptotically regular at some \(x_0 \in X\). In case \((X, M, \ast)\) is a strong fuzzy metric space, we only need \(T\) be asymptotically regular at \(x_0\).

**Proof.** First, let us show that \(T\) has at most one fixed point. Suppose \(x = Tx\) and \(y = Ty\). By (5.1), we have \(M(x, y, t) = M(x, y, t) = M(Tx, Ty, t)\), for all \(t > 0\). Now, condition (i) implies that \(x = y\).

Suppose \(T\) is (uniformly) asymptotically regular at \(x_0\), and set \(x_n = T(x_{n-1})\), for \(n \geq 1\). Then \(M(x_n, x_{n+1}, t) \to 1\), for all \(t > 0\). Therefore, for any two subsequences \((x_{p_n})\) and \((x_{q_n})\), we have

\[
\lim \inf M(x_{p_n}, x_{q_n}, t) = \lim \inf M(x_{p_n}, x_{p_n}, t) \ast M(x_{p_n}, x_{p_n+1}, t)^\alpha \ast M(x_{q_n}, x_{q_n+1}, t)^\beta = \lim \inf M(x_{p_n}, x_{q_n}, t),
\]

which means that condition (4.8) in Theorem 4.3 holds. Moreover, (5.2) implies (4.10) in Lemma 4.4. All conditions in Theorem 4.3 are fulfilled, and hence the Picard iterates \(T^n x_0\) form a Cauchy sequence. Since \(X\) is complete, there is \(z \in X\) with \(T^n x_0 \to z\). Since \(T\) is continuous, we get \(Tz = z\). \(\square\)

**Corollary 5.6.** Let \((X, M, \ast)\) be a complete fuzzy metric space, and \(T\) be a continuous self-map of \(X\). Define \(M\) by (5.1), and suppose

\(i)\ M(Tx, Ty, t) > M(x, y, t)\), for \(x \neq y\) and \(t > 0\);

\(ii)\ For every \(t > 0\), there exists \(\psi_t \in \Psi_1\) such that, for every \(x, y \in X\),

\[
M(Tx, Ty, t) \geq \psi_t(M(x, y, t)).
\]

Then \(T\) has a unique fixed point, provided \(T\) is uniformly asymptotically regular at some \(x_0 \in X\). In case \((X, M, \ast)\) is a strong fuzzy metric space, we only need \(T\) be asymptotically regular at \(x_0\).

**Corollary 5.7.** Let \((X, M, \ast)\) be a complete fuzzy metric space, and \(T\) be a continuous self-map of \(X\). Define \(M\) by (5.1), and suppose there exists \(\psi \in \Psi\) such that

\[
\forall t > 0, \forall x, y \in X, \quad M(Tx, Ty, t) \geq \psi(M(x, y, t)).
\]

Then \(T\) has a unique fixed point, provided \(T\) is uniformly asymptotically regular at some \(x_0 \in X\). In case \((X, M, \ast)\) is a strong fuzzy metric space, we only need \(T\) be asymptotically regular at \(x_0\).
6. Examples

To justify our results in previous sections, we devote this section to some examples.

**Example 6.1.** This example stems from [10, Example 1]. We present a gauge function \( \psi \in \Psi_1 \) that is not continuous. Beside Proposition 3.9, this shows that \( \Psi_1 \) properly contains \( \Psi \). Define \( \psi : (0, 1] \to (0, 1] \) by

\[
(6.1) \quad \psi(\tau) = \begin{cases} 
\frac{1}{2}, & \tau < \frac{1}{2}; \\
\frac{1}{n+2}, & \frac{n}{1+n} \leq \tau < \frac{n+1}{n+2}; \\
1, & \tau = 1.
\end{cases}
\]

Obviously, \( \psi \) is not continuous and thus \( \psi \notin \Psi \). To show that \( \psi \in \Psi_1 \), first we see that \( \psi \) is nondecreasing and satisfies the following conditions:

\[
(6.2) \quad \psi(\tau) > \tau \quad \text{and} \quad \lim_{n \to \infty} \psi^n(\tau) = 1, \quad (0 < \tau < 1).
\]

Towards a contradiction, suppose \( \psi \notin \Psi_1 \). Then, there exist \( r \in (0, 1) \) and a sequence \( \{\tau_n\} \) in \((0, 1 - r)\) such that \( \tau_n \to 1 - r \) and \( \psi(\tau_n) < 1 - r \). Take \( \tau \in (0, 1 - r) \). Then \( \tau < \tau_n < 1 - r \), for some \( \tau_n \). Since \( \psi \) is nondecreasing, we get \( \psi(\tau) \leq \psi(\tau_n) < 1 - r \). Replacing \( \tau \) by \( \psi(\tau) \), we get \( \psi^2(\tau) < 1 - r \). By induction, we have \( \psi^n(\tau) < 1 - r \), for all \( \tau \in (0, 1 - r) \) and \( n \in \mathbb{N} \). This contradicts the fact that \( \psi^n(\tau) \to 1 \), for all \( \tau \in (0, 1) \), and so \( \psi \in \Psi_1 \).

**Remark 3.** We bring the above argument for the reader’s convenience. In fact, \( \psi = \eta^{-1} \circ \phi \circ \eta \), where \( \eta(\tau) = 1/\tau - 1 \), and \( \phi \in \Phi_1 \) is given by [10]

\[
(6.3) \quad \phi(s) = \begin{cases} 
0, & s = 0; \\
\frac{1}{n+1}, & \frac{1}{n+1} < s \leq \frac{1}{n}; \\
1, & s > 1.
\end{cases}
\]

The following example shows that Theorems 5.3 and 5.4 are genuine extensions of [16, Theorem 3.1].

**Example 6.2.** Let \( X = [0, \infty) \) and define \( d \) as follows [10]

\[
d(x, y) = \begin{cases} 
\max\{x, y\}, & x \neq y; \\
0, & x = y.
\end{cases}
\]

That \( (X, d) \) is a complete metric space is easy to verify. Let \( \ast \) be the product \( t \)-norm, and \( M_d \) be the fuzzy metric on \( X \) induced by \( d \); that is

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad (x, y \in X, \ t > 0).
\]

Then \( (X, M_d, \ast) \) is a complete strong fuzzy metric space. Define \( \phi : X \to X \) as in (6.3). Then \( \phi \) is a continuous self-map of the fuzzy metric space \( (X, M_d, \ast) \). In fact, if \( M_d(x_n, x, t) \to 1 \), for \( t > 0 \), then \( d(x_n, x) \to 0 \) and thus \( x = 0 \) and \( x_n \to 0 \). It is then obvious that \( \phi(x_n) \to 0 \) in \( (X, M_d, \ast) \) which shows that \( \phi \) is continuous.
We show that $\phi$ is a fuzzy Ćirić-Matkowski contractive mapping. Using Theorem 3.14 it suffices to show that there exists, for every $t > 0$, a function $\psi_t \in \Psi_1$ such that

$$\forall x, y \in X, \quad M_d(\phi(x), \phi(y), t) \geq \psi_t(M_d(x, y, t)).$$

For $t > 0$, define $\eta_t(\tau) = t/\tau - t$ and $\psi_t = \eta_t^{-1} \circ \phi \circ \eta_t$. Since $\phi \in \Phi_1$ and $\eta_t \in \mathcal{H}$, Proposition 3.10 shows that $\psi_t \in \Psi_1$. Now, a simple calculation shows that the following statements are equivalent:

1. $M_d(\phi(x), \phi(y), t) \geq \psi_t(M_d(x, y, t))$, for all $x, y \in X$ and $t > 0$,
2. $d(\phi(x), \phi(y)) \leq \phi(d(x, y))$, for all $x, y \in X$,

and the latter is obvious. Hence $\phi$ is a fuzzy Ćirić-Matkowski contractive mapping.

Now, we show that $\phi$ fails to be a $\psi$-contractive mapping for any $\psi \in \Psi$. To get a contradiction, assume $\phi$ is a $\psi$-contractive mapping for some $\psi \in \Psi$. Since $\psi$ is continuous and $\psi(1/2) > 1/2$, there exists $\rho \in (0, 1/2)$ such that $\rho < \tau \Rightarrow \psi(\tau) > 1/2$.

Take $\delta > 0$ such that $\rho = 1/(2 + \delta)$. Then take $x = 1$, $y = 1 + \delta/2$ and $t = 1$. We have

$$M_d(x, y, 1) = \frac{1}{1 + d(1, 1 + \delta/2)} = \frac{1}{2 + \delta/2}, \quad M_d(\phi(x), \phi(y), 1) = \frac{1}{1 + d(1/2, 1)} = \frac{1}{2}.$$

We see that $\rho < M_d(x, y, 1) < 1/2$ and thus we must have

$$\frac{1}{2} = M_d(\phi(x), \phi(y), 1) \geq \psi(M_d(x, y, 1)) > \frac{1}{2},$$

which is absurd.

**Example 6.3.** This example shows that Theorem 5.5 is a real extension of Theorems 5.3 and 5.4. Take $X = \{0, 1, 2, 5\}$, and define $T : X \to X$ as follows:

$$T_0 = 0, \quad T_1 = 5, \quad T_2 = 0, \quad T_5 = 2.$$

Let $*$ be the product t-norm, and set

$$M(x, y, t) = \exp\left[\frac{(|x - y|)}{t}\right].$$

Then $(X, M, \ast)$ is a complete strong fuzzy metric space. It is easy to see that $T$ is continuous and asymptotically regular at each point of $X$. Define

$$\mathfrak{M}(x, y, t) = M(x, y, t) \ast M(x, T x, t)^2 \ast M(y, T y, t)^2.$$

Then $M(T x, T y, t) \geq \mathfrak{M}(x, y, t)^{5/7}$; that is $M(T x, T y, t) \geq \psi(\mathfrak{M}(x, y, t))$, where $\psi(\tau) = \tau^{5/7}$. Hence, all conditions in Theorem 5.3 (Corollary 5.7) are fulfilled. However, for $x = 0$ and $y = 1$, we see that $M(T x, T y, t) < M(x, y, t)$, for all $t > 0$, and thus Theorems 5.3 and 5.4 and, in particular, Theorem 3.1 in [16] do not apply to $T$. 

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