UNFOLDING A BYKOV ATTRACTOR:
FROM AN ATTRACTING TORUS TO STRANGE ATTRACTORS

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Abstract. In this paper we present a comprehensive mechanism for the emergence of strange attractors in a two-parametric family of differential equations acting on a three-dimensional sphere. When both parameters are zero, its flow exhibits an attracting heteroclinic network (Bykov network) made by two 1-dimensional and one 2-dimensional separatrices between two hyperbolic saddles-foci with different Morse indices. After slightly increasing both parameters, while keeping the one-dimensional connections unaltered, we focus our attention in the case where the two-dimensional invariant manifolds of the equilibria do not intersect.

Under some conditions on the parameters and on the eigenvalues of the linearisation of the vector field at the saddle-foci, we prove the existence of many complicated dynamical objects, ranging from an attracting quasi-periodic torus to Hénon-like strange attractors, as a consequence of the Torus Bifurcation Theory. The mechanism for the creation of horseshoes and strange attractors is also discussed. Theoretical results are applied to show the occurrence of strange attractors in analytic unfoldings of some Hopf-zero singularities.

1. Introduction

1.1. Strange attractors. Many aspects contribute to the richness and complexity of a dynamical system. One of them is the existence of strange attractors. Before going further, we introduce the following notion:

Definition 1. A (Hénon type) strange attractor of a two-dimensional dissipative diffeomorphism $R$, defined in a compact and riemannian manifold, is a compact invariant set $\Lambda$ with the following properties:

• $\Lambda$ equals the closure of the unstable manifold of a hyperbolic periodic point;
• the basin of attraction of $\Lambda$ contains an open set (and thus has positive Lebesgue measure);
• there is a dense orbit in $\Lambda$ with a positive Lyapounov exponent (exponential growth of the derivative along its orbit);
• $\Lambda$ is not hyperbolic.

A vector field possesses a strange attractor if the first return map to a cross section does.

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The rigorous proof of the strange character of an invariant set is a great challenge and the proof of the persistence (in measure) of such attractors is a very involving task. Based on [12], Mora and Viana [37] proved the emergence of strange attractors in the process of creation or destruction of the Smale horseshoes that appear through a bifurcation of a tangential homoclinic point. In the present paper, rather than exhibit the existence of strange attractors, we explore a mechanism to obtain them in the unfolding of a Bykov attractor, an expected phenomenon close to a $SO(2)$-equivariant system. In the present paper, the abundance of strange attractors is a consequence of the Torus Bifurcation developed in [1, 3, 4, 8]. See also [49].

1.2. The object of study. Our starting point is a two-parametric differential equation \( \dot{x} = f_{(A, \lambda)}(x) \) in the three-dimensional sphere \( S^3 \) with two saddle-foci whose organizing center \( (A = \lambda = 0) \) share all the invariant manifolds, of dimensions one and two, forming an attracting heteroclinic network \( \Gamma \) with a non-empty basin of attraction \( U \). We study the global transition of the dynamics from \( \dot{x} = f_{(0,0)}(x) \) to a smooth two-parameter family \( \dot{x} = f_{(A, \lambda)}(x) \) that breaks the network (or part of it). Note that, for small perturbations, the set \( U \) is still positively invariant. When \( A, \lambda \neq 0 \), we assume that the one-dimensional connections persist, and the two dimensional invariant manifolds are generically transverse (either intersecting or not).

When \( \lambda > A \geq 0 \), this gives rise to a complex network, that consists of a union of Bykov cycles [15], contained in \( U \). The dynamics in the maximal invariant set contained in \( U \), contains, but does not coincide with, the suspension of horseshoes accumulating on the heteroclinic network described in [6, 29, 33, 43, 44]. In addition, close to the organizing center, the flow contains infinitely many heteroclinic tangencies and attracting limit cycles with long periods, coexisting with sets with positive entropy, giving rise the so called quasi-stochastic attractors. All dynamical models with quasi-stochastic attractors were found, either analytically or by computer simulations, to have tangencies of invariant manifolds [22]. Recently, there has been a renewal of interest of this type of heteroclinic bifurcation in the reversible [19, 29, 36], equivariant [33, 43, 44] and conservative [13] contexts.

The novelty: The case \( A > \lambda \geq 0 \) corresponds to the case where the two-dimensional invariant manifolds do not intersect and has not been studied yet. Although the cycle associated to the equilibria is destroyed, complex dynamics appears near the ghost of the cycle. In the present article, it is shown that the perturbed system may manifest regular behaviour corresponding to the existence of a smooth invariant torus, and may also have chaotic regimes.

In the region of transition from regular behaviour (attracting torus) to chaotic dynamics (suspended horseshoes), using known results about Arnold tongues, we prove the existence of lines with homoclinic tangencies to dissipative periodic solutions, responsible for the existence of persistent strange attractors nearby. This phenomenon has already been observed by [49] in the context of non-autonomous differential equations. Our theoretical results may be applied in generic unfoldings of the Hopf-zero singularity of a given type (Case III of [23]).

1.3. This article. We study the dynamics arising near a differential equation with a specific attracting heteroclinic network (Bykov attractor). We show that, when a two-dimensional connection is broken with a prescribed configuration, the dynamics undergoes a global transition from regular to chaotic dynamics.

We discuss the global bifurcations that occur as the parameters \((A, \lambda)\) vary. We complete our results by reducing our problem to that of a first return map having an attracting
(non-contractible) curve and we study their general bifurcations. Chirality is an essential information in this problem. As we will see in Section 9, the mechanism of the transition from regular dynamics to chaotic dynamics is not so standard as in [41,51], where just saddle-node, periodic doubling and Newhouse bifurcations were involved in the annihilation of hyperbolic horseshoes.

This article is organised as follows. In Section 2 after some basic definitions, we describe precisely the object of study and in Section 3 we review some literature related to it. In Section 4 we state the main results of the present article. The coordinates and other notation used in the rest of the article are presented in Section 5. In Sections 6, 7 and 8 we prove the main results of the article. In Section 9 we describe a mechanism of breakdown of a two-dimensional torus due to the onset of homoclinic tangencies produced by the stable and unstable manifolds of a dissipative saddle. These tangencies are the origin of the persistence strange attractors. This mechanism is discussed in Section 10 for the Hopf-zero singularity (Case III of [23]). For the reader’s convenience, we have compiled at the end of the paper a list of definitions in a short glossary.

Throughout this paper, we have endeavoured to make a self contained exposition bringing together all topics related to the proofs. We have stated short lemmas and we have drawn illustrative figures to make the paper easily readable.

2. Setting

We will enumerate the main assumptions concerning the configuration of the cycle and the intersection of the invariant manifolds of the equilibria. We refer the reader to the Appendix A for precise definitions.

2.1. The organising center. For $\varepsilon > 0$ small enough, consider the two-parameter family of $C^3$-smooth differential equations

$$\dot{x} = f_{(A,\lambda)}(x) \quad x \in S^3 \quad A, \lambda \in [-\varepsilon, \varepsilon] \quad (2.1)$$

endowed with the usual topology, where $S^3$ denotes the unit three-sphere. Denote by $\varphi_{(A,\lambda)}(t, x)$, $t \in \mathbb{R}$, the associated flow$^1$, satisfying the following hypotheses for $A = 0$ and $\lambda = 0$:

(P1) There are two different equilibria, say $O_1$ and $O_2$.

(P2) The spectrum of $df_X$ is:

(P2a) $E_1$ and $-C_1 \pm \omega_1 i$ where $C_1 > E_1 > 0$, $\omega_1 > 0$, for $X = O_1$;

(P2b) $-C_2$ and $E_2 \pm \omega_2 i$ where $C_2 > E_2 > 0$, $\omega_2 > 0$, for $X = O_2$.

Thus the equilibrium $O_1$ possesses a 2-dimensional stable and 1-dimensional unstable manifold and the equilibrium $O_2$ possesses a 1-dimensional stable and 2-dimensional unstable manifold. We also assume that:

(P3) The manifolds $W^u(O_2) \cap W^s(O_1)$ coincide and this intersection consists of two-sphere (also called the $2D$-connection).

and

$^1$Since $S^4$ is a compact set without boundary, the local solutions of (2.1) could be extended to $\mathbb{R}$.}
(P4) There are two trajectories, say $\gamma_1, \gamma_2$, contained in $W^u(O_1) \cap W^s(O_2)$, each one in each connected component of $S^3 \backslash W^u(O_2)$ (also called the 1D-connections).

![Figure 1. Sketch of the Bykov attractor $\Gamma$ satisfying (P1)–(P4).](image)

The two equilibria $O_1$ and $O_2$, the two-dimensional heteroclinic connection from $O_2$ to $O_1$ refered in (P3) the two trajectories listed in (P4) build a heteroclinic network we will denote hereafter by $\Gamma$. This network has two cycles. See Figure 1 for an illustration. This set has an attracting character [34], this is why it will be called a Bykov attractor.

**Lemma 2.1** [34]. The set $\Gamma$ is asymptotically stable.

Therefore, we may find an open neighborhood $\mathcal{U}$ of the heteroclinic network $\Gamma$ having its boundary transverse to the flow of $\dot{x} = f_{(0,0)}(x)$ and such that every solution starting in $\mathcal{U}$ remains in it for all positive time and is forward asymptotic to $\Gamma$.

2.2. **Chirality.** There are two different possibilities for the geometry of the flow around $\Gamma$, depending on the direction trajectories turn around the one-dimensional heteroclinic connection from $O_1$ to $O_2$. To make this rigorous, we need some new concepts.

Let $V_1$ and $V_2$ be small disjoint neighbourhoods of $O_1$ and $O_2$ with disjoint boundaries $\partial V_1$ and $\partial V_2$, respectively. These neighbourhoods will be constructed with detail in Section 5. Trajectories starting at $\partial V_1 \backslash W^s(O_1)$ near $W^s(O_1)$ go into the interior of $V_1$ in positive time, then follow the connection from $O_1$ to $O_2$, go inside $V_2$, and then come out at $\partial V_2$ as in Figure 2. Let $\Phi$ be a piece of trajectory like this from $\partial V_1$ to $\partial V_2$. Now join its starting point to its end point by a line segment as in Figure 2, forming a closed curve, that we call the loop of $\Phi$. The loop of $\Phi$ and the cycle $\Gamma$ are disjoint closed sets.

**Definition 2.** [34] We say that the two saddle-foci $O_1$ and $O_2$ in $\Gamma$ have the same chirality if the loop of every trajectory is linked to $\Gamma$ in the sense that the two closed sets cannot be disconnected by an isotopy. Otherwise, we say that $O_1$ and $O_2$ have different chirality.

Then our assumption is:
(P5) The saddle-foci $O_1$ and $O_2$ have the same chirality.

For $r \geq 3$, denote by $\mathcal{X}^r(S^3)$, the set of $C^r$-vector fields on $S^3$ endowed with the $C^r$-Whitney topology, satisfying Properties (P1)–(P5).

2.3. Perturbing terms. With respect to the effect of the two parameters $A$ and $\lambda$ on the dynamics, we assume that:

(P6) For $A, \lambda > 0$, the manifolds $W^u(O_1)$ and $W^s(O_2)$ persists.

By Kupka-Smale Theorem, generically the invariant two-dimensional manifolds $W^u(O_2)$ and $W^s(O_1)$ are transverse (intersecting or not). Throughout this article, we assume that:

(P7a) For $A, \lambda > 0$, the two-dimensional manifolds $W^u(O_2)$ and $W^s(O_1)$ do not intersect.

and

(P8) The transitions along the connections $[O_1 \rightarrow O_2]$ and $[O_2 \rightarrow O_1]$ are given, in local coordinates, by the Identity map and by

$$(x, y) \mapsto (x, y + A + \lambda \Phi(x))$$

respectively, where $\Phi : S^1 \rightarrow S^1$ is a smooth map ($x \in S^1 = \mathbb{R} \text{ (mod } 2\pi)$). This assumption will be detailed later in Section 5.

All critical points of $\Phi : S^1 \rightarrow S^1$ are non-degenerate and the graph of $y = \Phi(x)$ in $\text{Out}(O_2)$ is transversal to the $x$–axis. For the moment, let us also assume that $\Phi(x) = \sin x$, $x \in [0, 2\pi]$, which has exactly two critical points. To simplify the notation, in what follows we will sometimes drop the subscript $(A, \lambda)$, unless there is some risk of misunderstanding.
2.4. Constants. For future use, we settle the following notation:

\[\delta_1 = \frac{C_1}{E_1} > 1, \quad \delta_2 = \frac{C_2}{E_2} > 1, \quad \delta = \delta_1 \delta_2 > 1 \quad (2.2)\]

and

\[K = \frac{E_2 + C_1}{E_1 E_2} > 0, \quad K_\omega = \frac{E_2 \omega_1 + C_1 \omega_2}{E_1 E_2} > 0 \quad \text{and} \quad a = \frac{\lambda}{A}. \quad (2.3)\]

3. Overview

In this section, we review some known results about the bifurcation of codimension 2 under consideration, which are summarized in Table 1.

3.1. Case 1: \( A = \lambda = 0 \). The network \( \Gamma \) is asymptotically stable (see Lemma 2.1). The two-dimensional manifolds \( W^u(O_2) \) and \( W^s(O_1) \) coincide and the global attractor of \( f(0,0) \) is made of the equilibria \( O_1, O_2 \), and the two connections \( [O_1 \rightarrow O_2] \), together with a sphere which is both the stable manifold of \( O_1 \) and the unstable manifold of \( O_2 \). See Figure 1.

This attractor, the so called Bykov attractor, has a finite number of moduli of stability and the points of its proper basin of attraction have historic behavior [16]. The coincidence of the two-dimensional invariant manifolds of \( O_1 \) and \( O_2 \) prevents visits to both cycles.

| Case(s) | Parameters | \( \lambda = 0 \) | \( \lambda > 0 \) |
|---------|------------|-----------------|-----------------|
| 1 and 2 | \( A = 0 \) | Attracting network | Horseshoes + strange attractors [35] + homoclinic tangencies [35, 45] |
| 3       | \( 0 < A < \lambda \) | —               | Horseshoes + strange attractors [35] + homoclinic tangencies [35, 45] |
| 4       | \( A > \lambda \) | Attracting torus | Torus or horseshoes + strange attractors (additional parameter: \( K_\omega \)) |

Table 1. Overview of the results.

3.2. Case 2: \( A = 0 \) and \( \lambda > 0 \). Let \((f_{(A,\lambda)})_{(A,\lambda)}\) be a two-parameter family of vector fields in \( X^3(S^3) \) satisfying (P1)–(P6) and where Property (P7a) does not hold, meaning that the connection \( W^u_{(0,\lambda)}(O_1) \cap W^s_{(0,\lambda)}(O_2) \) is kept but the manifolds \( W^u_{(0,\lambda)}(O_2) \) and \( W^s_{(0,\lambda)}(O_1) \) rearrange themselves and create either transversal or tangential intersections. This upheaval causes an explosion of the non-wandering set of the flow, bringing forth a countable union of suspended horseshoes inside the set of trajectories that remain for all positive times in \( U \). These horseshoes accumulate at the stable/unstable manifolds of the equilibria (cf. [15, 33]). Moreover, for a sequence of positive parameters arbitrarily close to zero, say \((\lambda_j)_j\), the flow
associated to $f_{(0,\lambda)}$ exhibits homo and heteroclinic tangencies, sinks with long periods and strange attractors (cf. [35, 37, 38]).

3.3. **Case 3:** $\lambda > A > 0$. In this case, the manifolds $W^u_{(A,\lambda)}(O_2)$ and $W^s_{(A,\lambda)}(O_1)$ still meet transversely along at least two connections and thus the results of [33, 35, 44] still hold. The transverse intersection of the two-dimensional invariant manifolds of the two equilibria implies that the set of trajectories that remain for all time in a small neighbourhood of the Bykov cycle contains a locally-maximal hyperbolic set admitting a complete description in terms of symbolic dynamics, reminiscent of the results of [22, 47]. An obstacle to the global symbolic description of these trajectories is the existence of tangencies that lead to the birth of stable periodic solutions, as described in [21, 22, 39].

3.4. **Case 4:** $A > \lambda \geq 0$. As far as we know, the cases $A > \lambda = 0$ and $A > \lambda > 0$ have not been studied. The goal of this article is to explore this case where there are no cycles associated to $O_1$ and $O_2$. In contrast to Cases 2 and 3, where the involved bifurcations are related to horseshoe destruction, tangencies and Newhouse phenomena [35, 41], in Case 4, the bifurcations are connected with the torus breakdown [1, 3, 4, 8]. Hence, hereafter, we also assume that:

($P7b$) $A > \lambda \geq 0$ (or, equivalently, $a = \frac{\lambda}{A} \in ]0,1[$).

Note that under hypothesis ($P8$), we get ($P7a$) $\iff$ ($P7b$). When we refer to ($P7$) we refer one of the above.

The set of vector fields $f_{(0,0)} \in X^{3}_{Byk}(S^3) \subset X^r(S^3)$, $r \geq 3$, which satisfy the conditions ($P1$)–($P8$) contains a non-empty open subset of families $(f_{(A,\lambda)})_{(A,\lambda)}$ (cf. §4.1.3.2 of [5]).

4. **Main results**

Let $T$ be a neighborhood of the Bykov attractor $\Gamma$ that exists for $A = \lambda = 0$. For $\varepsilon > 0$ small enough, let $(f_{(A,\lambda)})_{(A,\lambda)\in [-\varepsilon,\varepsilon]^2}$ be a two-parameter family of vector fields in $X^3_{Byk}(S^3)$ satisfying conditions ($P1$)–($P8$).

**Theorem A.** For $\varepsilon > 0$ small, let $(f_{(A,\lambda)})_{(\lambda,A)\in \mathbb{R}^2} \in X^3_{Byk}(S^3)$. Then, the first return map to a given cross section to $\Gamma$ may be approximated by a map of the form:

$$F_{(A,\lambda)}(x,y) = \left[ x - K_\omega \ln(y + A + \lambda \sin x) \pmod{2\pi}, \; (y + A + \lambda \sin x)\delta \right].$$

where $(x,y) \in D = \{x \in \mathbb{R} \pmod{2\pi} \; \text{ and } \; y \in [-\varepsilon,\varepsilon]\}$.

The proof of Theorem A is done in Section 6 by composing local and transition maps. Since $\delta > 1$, for $A$ small enough, the second component of $F_{(A,\lambda)}$ is contracting and, under an additional hypothesis, the dynamics of $F_{(A,\lambda)}$ is dominated by the family of circle maps.

**Theorem B.** Let $(f_{(A,\lambda)})_{(\lambda,A)\in \mathbb{R}^2} \in X^3_{Byk}(S^3)$. If $\frac{\lambda}{A} < \frac{1}{\sqrt{1 + K^2}}$, then there is an invariant closed curve $C \subset D$ as the maximal attractor for $F_{(A,\lambda)}$. This closed curve is not contractible on $D$. 
The proof of Theorem is performed in Section 7 using the Afraimovich’s Annulus Principle. The curve $C$ is globally attracting in the sense that, for every $p \in D$, there exist a point $p_0 \in C$ such that
\[
\lim_{n \to +\infty} \left| \mathcal{F}_n^{(A,\lambda)}(p) - \mathcal{F}_n^{(A,\lambda)}(p_0) \right| = 0.
\]
The attracting invariant curve for $\mathcal{F}^{(A,\lambda)}$ is the graph of a smooth map and corresponds to an attracting two-torus for the flow of (2.1). Thus, in particular:

**Corollary C.** Let $(f^{(A,\lambda)})_{(\lambda,A) \in \mathbb{R}^2} \in \mathcal{X}_{B\gamma k}^3(S^3)$. If either $K_\omega > 0$ or $\lambda > 0$ are sufficiently small, then there is a two-dimensional torus which is globally attracting. This attracting torus persists for $(A, \lambda) \approx (0, 0)$.

**Proof.** The existence of an invariant torus is a direct corollary of Theorem B. Since the torus is normally hyperbolic, its persistence follows from the theory for normally hyperbolic manifolds developed in Hirsch et al. [26]. □

The following result says that there is a set positive Lebesgue measure such that the torus of Corollary C has a dense orbit (the whole torus is a minimal attractor).

**Corollary D.** Let $(f^{(A,\lambda)})_{(\lambda,A) \in \mathbb{R}^2} \in \mathcal{X}_{B\gamma k}^3(S^3)$. For either $K_\omega > 0$ or $\lambda > 0$ sufficiently small, the dynamics of $\mathcal{F}^{(A,\lambda)}$ induces on $C$ a circle diffeomorphism. In this case, for any given interval of unit length $I$, there is a positive measure set $\Delta \subset I$ so that the rotation number of $\mathcal{F}^{(A,\lambda)}|_C$ is irrational if and only if $a = \lambda/A \in \Delta$.

**Proof.** The first part of this result follows from Corollary C; the second part, concerning rotation number, may be found in Herman [25] on circle diffeomorphisms. □

**Theorem E.** Let $(f^{(A,\lambda)})_{(\lambda,A) \in \mathbb{R}^2} \in \mathcal{X}_{B\gamma k}^3(S^3)$. If $1 > \frac{\lambda}{A} > \frac{\exp\left(\frac{4\pi}{K_\omega}\right)^{-1}}{\exp\left(\frac{4\pi}{K_\omega}\right)^{-1/4}}$, then there exists a hyperbolic invariant closed subset $\Lambda$ in a cross section to $\Gamma$ such that $\mathcal{F}^{(A,\lambda)}|_\Lambda$ is topologically conjugate to the Bernoulli shift of two symbols.

**Remark 4.1.** Under a stronger (but reasonable) hypothesis, the conclusion of Theorem E may be rephrased as: $\mathcal{F}^{(A,\lambda)}|_\Lambda$ is topologically conjugate to the Bernoulli shift of $m$ symbols, with $2 < m \in \mathbb{N}$. See Lemma S.2.

As a corollary of Theorems E and E in the bifurcation diagram $(a, K_\omega) = \left(\frac{\lambda}{A}, K_\omega\right)$, we may draw, in the first quadrant, two smooth curves, the graphs of $g$ and $f$ in Figure 3 (left), such that:

1. $g(K_\omega) = \frac{1}{\sqrt{1+K_\omega^2}}$ and $f(K_\omega) = \frac{\exp\left(\frac{4\pi}{K_\omega}\right)^{-1}}{\exp\left(\frac{4\pi}{K_\omega}\right)^{-1/4}}$;
2. the region below the graph of $g$ correspond to flows having an invariant and attracting torus with zero topological entropy (regular dynamics);
3. the region above the graph of $f$ correspond to vector fields whose flows exhibit suspended horseshoes (chaotic dynamics).
Figure 3. Left: Invariant attracting 2-torus and chaotic regions for the map $F_{(A,\lambda)}$ with respect to the parameters $\frac{A}{\lambda}$ and $K_\omega$. (Right): For $K_\omega > 0$ fixed (red dashed line on left), the transition from the graph of $g$ to that of $f$ may be explained by the Arnold tongues associated to the torus bifurcations. $B_1^k$ or $B_2^k$: saddle-node bifurcation. $M_1^k, M_2^k$: above these points, in $B_1^k$ or $B_2^k$, the unstable manifold of the saddle-node is not homeomorphic to a circle. $\text{Hom}_1^k, \text{Hom}_2^k$: the flow has a homoclinic tangency to a dissipative point. $D^k$ above this line the torus is destroyed. $S^k$: Transition to chaos. Source: [1, 7, 8].

For $A, K_\omega > 0$ fixed, as $\lambda$ increases, one observes the “breaking of the wave” which accompanies the break of the invariant circle $C$. In the chaotic region, a neighbourhood of $y = 0$ is folded and mapped into itself, leading to the formation of horseshoes and strange attractors.

The behavior of $F_{(A,\lambda)}$ is unknown for the parameter range between the graphs of $f$ and $g$. Nevertheless, for $A, K_\omega > 0$ fixed (red dashed line in Figure 3(left)), the transition from the graph of $g$ to that of $f$ may be explained via Arnold tongues associated to the torus bifurcations [1, 7, 8]. In particular, in the bifurcation diagram $(A, \frac{A}{\lambda})$, there is a set with positive Lebesgue measure for which the family (2.1) exhibits strange attractors.

**Theorem F.** Fix $K_\omega^0 > 0$. In the bifurcation diagram $(A, \frac{A}{\lambda})$, where $(A, \lambda)$ is such that $g(K_\omega^0) < \frac{A}{\lambda} < f(K_\omega^0)$, there exists a positive measure set $\Delta$ of parameter values, so that for every $a \in \Delta$, $F_{(A,\lambda)}$ admits a strange attractor (of Hénon-type) with an ergodic SRB measure.

The proof of Theorem F is a consequence of the Torus Bifurcation Theory [11, 33, 44, 8] combined with the results by Newhouse [38] and Mora and Viana [37]. A discussion of these results will be performed in Section 9.

**An application: strange attractors in the unfolding of Hopf-zero singularities.** By applying the previous theory, we may show the occurrence of strange attractors in a specific case of unfoldings of a Hopf-zero singularity (Type I of [11], Case III of [23, 3], the case which

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2Care is needed to compare both works because the constants choice and signs are different.
has not been dealt in \[11\]: \( \sigma \gg O(\varepsilon) \). Since the hypotheses are very technical, we decide to postpone the precise result to Section 10.

5. Linearisation and transition maps

In this section we will analyze the dynamics near the Bykov attractor \( \Gamma \) through local maps, after selecting appropriate coordinates in neighborhoods of the saddle-foci \( O_1 \) and \( O_2 \) (see Figure 4), as done in \[40\].

![Figure 4. Local cylindrical coordinates inside \( V_2 \) and \( V_1 \) and near \( O_2 \) and \( O_1 \), respectively.](image)

5.1. Local coordinates. In order to describe the dynamics around the Bykov cycles of \( \Gamma \), we use the local coordinates near the equilibria \( O_1 \) and \( O_2 \) introduced in \[35\]. See also Ovsyannikov and Shilnikov \[40\].

In these coordinates, we consider cylindrical neighbourhoods \( V_1 \) and \( V_2 \) in \( \mathbb{R}^3 \) of \( O_1 \) and \( O_2 \), respectively, of radius \( \rho = \varepsilon > 0 \) and height \( z = 2\varepsilon \) — see Figure 4. After a linear rescaling of the variables, we may also assume that \( \varepsilon = 1 \). Their boundaries consist of three components: the cylinder wall parametrised by \( x \in \mathbb{R} \) (mod \( 2\pi \)) and \(|y| \leq 1\) with the usual cover

\[
(x, y) \mapsto (1, x, y) = (\rho, \theta, z)
\]

and two discs, the top and bottom of the cylinder. We take polar coverings of these disks

\[
(r, \varphi) \mapsto (r, \varphi, \pm 1) = (\rho, \theta, z)
\]

where \( 0 \leq r \leq 1 \) and \( \varphi \in \mathbb{R} \) (mod \( 2\pi \)). The local stable manifold of \( O_1 \), \( W^s(O_1) \), corresponds to the circle parametrised by \( y = 0 \). In \( V_1 \) we use the following terminology suggested in Figure 4:

- \( \text{In}(O_1) \), the cylinder wall of \( V_1 \), consisting of points that go inside \( V_1 \) in positive time;
- \( \text{Out}(O_1) \), the top and bottom of \( V_1 \), consisting of points that go outside \( V_2 \) in positive time.
We denote by $\text{In}^+(O_1)$ the upper part of the cylinder, parametrised by $(x, y)$, $y \in [0, 1]$ and by $\text{In}^-(O_1)$ its lower part.

The cross-sections obtained for the linearisation around $O_2$ are dual to these. The set $W^s(O_2)$ is the $z$-axis intersecting the top and bottom of the cylinder $V_2$ at the origin of its coordinates. The set $W^u(O_2)$ is parametrised by $z = 0$, and we use:

- $\text{In}(O_2)$, the top and bottom of $V_2$, consisting of points that go inside $V_2$ in positive time;
- $\text{Out}(O_1)$, the cylinder wall of $V_2$, consisting of points that go inside $V_2$ in negative time, with $\text{Out}^+(O_2)$ denoting its upper part, parametrised by $(x, y)$, $y \in [0, 1]$ and $\text{Out}^-(O_2)$ its lower part.

We will denote by $W^u_{\text{loc}}(O_2)$ the portion of $W^u(O_2)$ that goes from $O_2$ up to $\text{In}(O_1)$ not intersecting the interior of $V$ and by $W^s_{\text{loc}}(O_1)$ the portion of $W^s(O_1)$ outside $V_2$ that goes directly from $\text{Out}(O_2)$ into $O_1$. The flow is transverse to these cross-sections and the boundaries of $V_1$ and of $V_2$ may be written as the closure of $\text{In}(O_1) \cup \text{Out}(O_1)$ and $\text{In}(O_2) \cup \text{Out}(O_2)$, respectively. The orientation of the angular coordinate near $O_2$ is chosen to be compatible with that the direction induced along $V$.

Observe that the flow is transverse to these cross-sections and that the boundaries of $V_1$ and of $V_2$ are just the closures of $\text{In}(O_1) \cup \text{Out}(O_1)$ and $\text{In}(O_2) \cup \text{Out}(O_2)$, respectively. The local stable manifold of $O_1$, say $W^s_{\text{loc}}(O_1)$, corresponds precisely to the disk in $V_1$ parameterized by $y = 0$. The local stable manifold $W^s_{\text{loc}}(O_2)$ is the $z$-axis, intersecting the top and bottom of the cylinder $V_2$ at the origin of its coordinates. The manifold $W^u(O_2)$ is parameterized by $z = 0$. in $O_1$.

5.2. Local maps near the saddle-foci. Following [18] [40], the trajectory of a point $(x, y)$ with $y > 0$ in $\text{In}^+(O_1)$, leaves $V_1$ at $\text{Out}(O_1)$ at

$$
\Phi_1(x, y) = \left( y^{\delta_1} + S_1(x, y; A, \lambda), -\frac{\omega_1 \ln y}{E_1} + x + S_2(x, y; A, \lambda) \right) = (r, \phi) \quad \text{where} \quad \delta_1 = \frac{C_1}{E_1} > 1,
$$

(5.1)

where $S_1$ and $S_2$ are smooth functions which depend on $\lambda$ and satisfy:

$$
\left| \frac{\partial^{k+l+m}}{\partial x^k \partial r^l \partial \lambda^m} S_i(x, y; A, \lambda) \right| \leq C y^{\delta_1 + \sigma - 1},
$$

(5.2)

and $C$ and $\sigma$ are positive constants and $k, l, m$ are non-negative integers. Similarly, a point $(r, \phi)$ in $\text{In}(O_2) \setminus W^s_{\text{loc}}(O_2)$, leaves $V_2$ at $\text{Out}(O_2)$ at

$$
\Phi_2(r, \varphi) = \left( -\frac{\omega_2 \ln r}{E_2} + \varphi + R_1(r, \varphi; A, \lambda), r^{\delta_2} + R_2(r, \varphi; A, \lambda) \right) = (x, y) \quad \text{where} \quad \delta_2 = \frac{C_2}{E_2} > 1
$$

(5.3)

and $R_1$ and $R_2$ satisfy a condition similar to (5.1). The terms $S_1$, $S_2$, $R_1$, $R_2$ correspond to asymptotically small terms that vanish when $y$ and $r$ go to zero. A better estimate under a stronger eigenvalue condition has been obtained in [27] Prop. 2.4].
5.3. **The transitions.** The coordinates on $V_1$ and $V_2$ are chosen so that $[O_1 \rightarrow O_2]$ connects points with $z > 0$ (resp. $z < 0$) in $V_1$ to points with $z > 0$ (resp. $z < 0$) in $V_2$. Points in $\text{Out}(O_1) \setminus W_{\text{loc}}^u(O_1)$ near $W^u(O_1)$ are mapped into $\text{In}(O_2)$ along a flow-box around each of the connections $[O_1 \rightarrow O_2]$. We will assume that the transition

$$\Psi_{1 \rightarrow 2} : \text{Out}(O_1) \rightarrow \text{In}(O_2)$$

does not depend neither on $\lambda$ nor $A$ and is the Identity map, a choice compatible with hypothesis (P4) and (P5). Denote by $\eta$ the map

$$\eta = \Phi_2 \circ \Psi_{1 \rightarrow 2} \circ \Phi_1 : \text{In}(O_1) \rightarrow \text{Out}(O_2).$$

From (5.1) and (5.3), omitting high order terms in $y$ and $r$, we infer that, in local coordinates, for $y > 0$ we have

$$\eta(x, y) = \left( x - K_\omega \log y \ (\text{mod } 2\pi), \ y^\delta \right)$$

with

$$\delta = \delta_1 \delta_2 > 1 \quad \text{and} \quad K_\omega = \frac{C_1 \omega_2 + E_2 \omega_1}{E_1 E_2} > 0.$$  (5.5)

A similar expression is valid for $y < 0$, after suitable changes in the sign of $y$. The chirality hypothesis (P5) is implicitly used when the terms in the numerator of $K_\omega$ have the same sign. Using (P7) and (P8), for $A > \lambda \geq 0$, we still have a well defined transition map

$$\Psi^{(A, \lambda)}_{2 \rightarrow 1} : \text{Out}(O_2) \rightarrow \text{In}(O_1)$$

that depends on the parameters $\lambda$ and $A$, given by

$$\Psi^{(A, \lambda)}_{2 \rightarrow 1}(x, y) = (x, y + A + \lambda \sin x) \quad \text{where} \quad A > \lambda.$$  (5.6)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Geometry of the global map $\Psi^{(A, \lambda)}_{2 \rightarrow 1}$. For $A > \lambda \geq 0$, $W^s(O_1)$ intersects the wall $\text{Out}(O_2)$ of the cylinder $V_2$ and $W^u(O_2)$ intersects the wall $\text{In}(O_1)$ of the cylinder $V_1$ on closed curves given, in local coordinates, by the graph of a $2\pi$–periodic function of sinusoidal type. Compare the right side with Figure 4 of [20].}
\end{figure}
6. Proof of Theorem A

The proof of Theorem A is straightforward by composing the local and transition maps constructed in Section 5. Let

\[ F_{(A, \lambda)}(x, y) = \eta \circ \Psi_{2 \rightarrow 1}^{(A, \lambda)} =: D \subset \text{Out}(O_2) \rightarrow D \subset \text{Out}(O_2). \]  

be the first return map to \( \text{Out}(O_2) \), where \( D \) is the set of initial conditions \((x, y) \in \text{Out}(O_2)\) whose solution returns to \( \text{Out}(O_2) \). Composing \( \eta \) with \( \Psi_{2 \rightarrow 1}^{(A, \lambda)} \), the analytic expression of \( F_{(A, \lambda)} \) is given by:

\[
F_{(A, \lambda)}(x, y) = \left[-K_\alpha \log \left(y + A + \lambda \sin x\right) + x \pmod{2\pi}, (y + A + \lambda \sin x)^\delta\right] = \left(F_1^{(A, \lambda)}(x, y), F_2^{(A, \lambda)}(x, y)\right)
\]

The following technical result will be useful in the sequel.

**Lemma 6.1.** For small enough \( A \) and \( y \in D \), the map \( F_2^{(A, \lambda)} \) is a contraction in \( y \).

**Proof.** Since \( \delta > 1 \), we may write:

\[
\left|\frac{\partial F_2^{(A, \lambda)}(x, y)}{\partial y}\right| = |\delta(y + A + \lambda \sin x)^\delta| = O(A^{\delta-1}) < 1,
\]

and we get the result. \( \square \)

**Remark 6.2.** For \( y \in D \), we have:

\[
(y + A + \lambda \sin x)^\delta = A^\delta \left(\frac{y}{A} + 1 + \frac{\lambda}{A} \sin x\right)^\delta = A^\delta \left(1 + \frac{\lambda}{A} \sin x\right)^\delta + o(y^\delta) < 1,
\]

for \( A > 0 \) small enough.

7. Proof of Theorem B

To prove the existence of an invariant and attracting smooth closed curve for \( F_{(A, \lambda)} \), we make use if the *Afraimovich Annulus Principle* \([3, 4]\). We start by proving the existence of a flow-invariant annulus in \( \text{Out}^+(O_2) \) in Subsection 7.1, then we check the conditions required by the Annulus Principle (one by one) in Subsection 7.2.

7.1. The existence of an annulus. Let

\[
\mathcal{B} = \left\{(x, y) : A^\delta \left(1 - \frac{\lambda}{A}\right)^\delta \leq y \leq 2A^\delta \left(1 + \frac{\lambda}{A}\right)^\delta \text{ and } x \in \mathbb{R} \pmod{2\pi}\right\} \subset \text{Out}^+(O_2)
\]

In what follows, if \( \mathcal{X} \subset \mathbb{S}^3 \), let \( \overset{o}{\mathcal{X}} \) denote the topological interior of \( \mathcal{X} \).

**Lemma 7.1.** If \( A > 0 \) small enough, then \( F_{(A, \lambda)}(\mathcal{B}) \subseteq \overset{o}{\mathcal{B}} \).

**Proof.** If \((x, y) \in \mathcal{B}\), we have:
\[ (y + A + \lambda \sin x)^\delta = A^\delta \left( \frac{y}{A} + 1 + \frac{\lambda}{A} \sin x \right)^\delta \]
\[ \leq A^\delta \left( 2A^\delta - 1 \left( 1 + \frac{\lambda}{A} \right)^\delta + 1 + \frac{A}{\lambda} \right)^\delta + o(y^\delta) \]
\[ \leq A^\delta \left( \left( 1 + \frac{\lambda}{A} \right) \left( 2A^\delta - 1 \left( 1 + \frac{\lambda}{A} \right)^{\delta - 1} + 1 \right) \right)^\delta \]
\[ \leq A^\delta \left( 1 + \frac{\lambda}{A} \right)^\delta \left( O \left( A^\delta \right) + 1 \right) \]
\[ < 2A^\delta \left( 1 + \frac{\lambda}{A} \right)^\delta \]

Analogously, if \((x, y) \in \mathcal{B}\), we may write:
\[ (y + A + \lambda \sin x)^\delta = A^\delta \left( \frac{y}{A} + 1 + \frac{\lambda}{A} \sin x \right)^\delta \]
\[ \geq A^\delta \left( A^\delta - 1 \left( 1 - \frac{\lambda}{A} \right)^\delta + 1 - \frac{A}{\lambda} \right)^\delta \]
\[ \geq A^\delta \left( \left( 1 - \frac{\lambda}{A} \right) \left( A^\delta - 1 \left( 1 - \frac{\lambda}{A} \right)^{\delta - 1} + 1 \right) \right)^\delta \]
\[ \geq A^\delta \left( 1 - \frac{\lambda}{A} \right)^\delta \left( O \left( A^\delta \right) + 1 \right)^\delta \]
\[ > A^\delta \left( 1 - \frac{\lambda}{A} \right)^\delta \]

Hence, the region \(\mathcal{B}\) is forward flow-invariant. \(\square\)

We recall the annulus principle (version [4]), adapted to our purposes. Let us introduce the following notation: for a vector-valued or matrix-valued function \(\Phi(x, y)\), define
\[ \|\Phi\| = \sup_{(x, y) \in \mathcal{D}} \|\Phi(x, y)\|, \]
where \(\|\ast\|\) is the standard Euclidian norm.

**Theorem 7.2 ([4])**. Let \(G(x, y) = (x + g_1(x, y), g_2(x, y))\) be a C\(^1\) map, with \(x \in \mathbb{R} \pmod{2\pi}\), defined on the annulus
\[ \mathcal{B} = \{(x, y) : a \leq y \leq b \text{ and } x \in \mathbb{R} \pmod{2\pi}\}, \]
where \(0 < a < b\), and satisfying:

1. \(g_1\) and \(g_2\) are \(2\pi\)-periodic in \(x\) and smooth;
2. \(G(\mathcal{B}) \subseteq \mathcal{B}\);
(3) $0 < \left\| 1 + \frac{\partial g_1}{\partial x} \right\| < 1$;

(4) $\left\| \frac{\partial g_2}{\partial y} \right\| < 1$ (i.e., the map $g_1$ is a contraction in $y$);

(5) $2 \sqrt{\left\| 1 + \frac{\partial g_1}{\partial x} \right\|^2 \left\| \frac{\partial g_2}{\partial y} \right\| \left\| \frac{\partial g_1}{\partial y} \right\| < 1 - \left\| 1 + \frac{\partial g_1}{\partial x} \right\|^{-1} \left\| \frac{\partial g_2}{\partial y} \right\|}$;

(6) $\left\| 1 + \frac{\partial g_1}{\partial x} \right\| + \left\| \frac{\partial g_2}{\partial y} \right\| < 2$,

then the maximal attractor in $B$ is an invariant closed curve, the graph of a $2\pi$-periodic, $C^1$ function $y = h(x)$.

Figure 6. Illustration of Theorem B if $\lambda < \sqrt{\frac{A}{1+K^2}}$, then there is an invariant closed curve $C \subset \text{Out}^+(O_2)$ as the maximal attractor of $F_{(A,\lambda)}$.

7.2. Proof of Theorem B In this section, we prove Theorem B as an application of Theorem 7.2 Let us define:

$g_1(x, y) = F^{(A,\lambda)}_1(x, y) - x = -K_\omega \log(y + A + \lambda \sin x)$

$g_2(x, y) = F^{(A,\lambda)}_2(x, y) = (y + A + \lambda \sin x)^\delta$

where

$K_\omega = \frac{C_1\omega_2 + E_2\omega_1}{E_1E_2}$,

whose derivatives may be written as:

$\frac{\partial g_1(x, y)}{\partial x} = -K_\omega \frac{\lambda \cos x}{y + A + \lambda \sin x}$,

$\frac{\partial g_1(x, y)}{\partial y} = -K_\omega \frac{\lambda \cos x}{y + A + \lambda \sin x}$,

$\frac{\partial g_2(x, y)}{\partial x} = \delta \lambda (y + A + \lambda \sin x)^{\delta-1} \cos x$,

$\frac{\partial g_2(x, y)}{\partial y} = \delta (y + A + \lambda \sin x)^{\delta-1}$. 

We state and prove an auxiliary result that will be used in the sequel.

Lemma 7.3. If $0 < \lambda < \frac{A}{\sqrt{1+K^2}}$, the range of $\frac{\partial F^{(A,\lambda)}}{\partial x}$ is $]0,1[.$

Proof. First note that

$$\frac{\partial F^{(A,\lambda)}}{\partial x} = 1 + \frac{\partial g_1(x,y)}{\partial x} = 1 - K_\omega \frac{\lambda A \cos x}{1 + \frac{\lambda A}{3} \sin x} + o(y) \quad (7.1)$$

and that, for $A$ small enough, the expression (7.1) is non-negative. Taking $a = \frac{\lambda A}{3}$, we get:

$$\frac{\partial^2 F^{(A,\lambda)}}{\partial^2 x} = \frac{\partial}{\partial x} \left( 1 - K_\omega \frac{a \cos x}{1 + a \sin x} \right)$$

$$= K_\omega \frac{a \sin x(1 + a \sin(x)) + a^2 \cos^2(x)}{(1 + a \sin(x))^2}$$

$$= K_\omega \frac{a \sin x + a^2}{(1 + a \sin(x))^2}$$

Therefore $\frac{\partial^2 F^{(A,\lambda)}(x,y)}{\partial^2 x} = 0$ if and only if $a = 0$ or $\sin(x) = -a$. Let us define $x^* \in [\pi, 3\pi/2]$ such that $\sin(x^*) = -a$. Therefore:

$$0 < 1 + \frac{\partial g_1(x^*,y)}{\partial x} < 1 \iff 0 < 1 - K_\omega \frac{a \cos x^*}{1 + a \sin x^*} < 1$$

$$\iff 0 < 1 - \frac{K_\omega a \sqrt{1 - a^2}}{1 - a^2} < 1$$

$$\iff 0 < \frac{K_\omega a}{\sqrt{1 - a^2}} < 1$$

$$\iff \frac{1 - a^2}{a^2} > K_\omega$$

$$\iff a < \frac{1}{\sqrt{K^2_\omega + 1}}$$

$$\iff \lambda < \frac{A}{\sqrt{K^2_\omega + 1}}$$

\[
\square
\]

In order to prove Theorem 13, we check one by one the hypotheses of Theorem 7.2 putting all pieces together.

(1) It is easy to see to see that $g_1(x,y) = F_1^{(A,\lambda)}(x,y) - x$ and $g_2(x,y) = F_2^{(A,\lambda)}(x,y)$ are $2\pi$-periodic in the variable $x$.

(2) This item follows from Lemma 7.1 where $a = A^\delta (1 - \frac{A}{3})^\delta$ and $b = 2A^\delta (1 + \frac{A}{3})^\delta$.

(3) Using Lemma 7.3 we know that $0 < \|1 + \frac{\partial g_1}{\partial x}\| < 1$ if $a = \frac{\lambda A}{3} \in \left[0, \frac{1}{\sqrt{1+K^2}}\right]$. In particular, under the same condition, we get $\|1 + \frac{\partial g_1}{\partial x}\|^{-1} < \infty$. 
(4) The proof of this item follows from Lemma 6.1. Indeed,
\[ \left| \frac{\partial g_2(x, y)}{\partial y} \right| = \sup_{(x, y) \in D} |\delta(y + A + \lambda \sin x)^{\delta-1}| = O(A^{\delta-1}) < 1. \]

(5) Since
\[ \frac{\partial g_1(x, y)}{\partial y} = -K\omega \approx O \left( \frac{1}{A} \right) \quad \text{and} \]
\[ \frac{\partial g_2(x, y)}{\partial x} = \delta A(y + A + \lambda \sin x)^{\delta-1} \cos x \leq \delta A(y + A + \lambda \sin x)^{\delta-1} = O(A^{\delta}), \]

does not imply that:
\[ \sqrt{1 + \left| \frac{\partial g_1}{\partial x} \right|^2 \left| \frac{\partial g_2}{\partial y} \right|^2} = 1 + O(A^{\delta/2}) O(A^{-1/2}) = O\left( A^{\frac{\delta+1}{2}} \right) \]

and thus:
\[ 1 - \left| 1 + \frac{\partial g_1}{\partial x} \right|^{-1} \left| \frac{\partial g_2}{\partial y} \right| = 1 - O(A^{\delta-1}) > O\left( A^{\frac{\delta-1}{2}} \right) = \sqrt{1 + \left| \frac{\partial g_1}{\partial x} \right|^2 \left| \frac{\partial g_2}{\partial y} \right|^2 \left| \frac{\partial g_1}{\partial y} \right|^2}. \]

(6) Using again Lemmas 6.1 and 7.3 we get:
\[ \left| 1 + \frac{\partial g_1}{\partial x} \right| + \left| \frac{\partial g_2}{\partial y} \right| < 1 + O(A^{\delta-1}) < 2. \]

This ends the proof of the existence of a globally attracting 2-torus for the equation (2.1).

Remark 7.4. For $\lambda = 0$ and any $A > 0$, the model $F_{(A, \lambda)}$ has an invariant closed curve which is attracting. This curve is not contractible because it is the graph of a $C^1$-smooth map $h$.

Remark 7.5. For $A > \lambda \geq 0$, the existence of an attracting invariant torus depends on the interaction between the value of $a = \frac{\lambda}{A}$ and $K\omega$.

Geometric interpretation. Condition $\lambda < \frac{4}{K^2} \cdot \sqrt{1 + K^2}$ leads that the image of the annulus $B$ under $F_{(A, \lambda)}$ is also an annulus bounded by two curves without folds in the angular coordinate. The subsequent image of this annulus is self-alike too, and so on. As a result, we obtain a sequence of embedded annuli (as suggested in Figure 6); moreover, the contraction in the $y$-variable guarantees that these annuli intersect in a single and smooth closed curve. This curve is invariant and attracting.

8. Proof of Theorem E

In this section, under an appropriate hypothesis, we prove the existence of two rectangles whose image under $F_{(A, \lambda)}$ overlaps with $\text{Out}(O_2)$ at least twice. Therefore, we obtain a construction similar to the hyperbolic Smale horseshoe: a small tubular neighbourhood of $y = 0$ is folded and mapped into itself.
8.1. Stretching the angular component. The first technical result says that the image under $F_1^{(A,\lambda)}(x, y)$ of the curve $W^u(O_2) \cap \text{Out}(O_2)$ with $x \in [\pi/2, 3\pi/2]$ is monotonic, meaning that there are no folds (according to Section 5 of [34], the graph does not have reversals of orientation).

**Lemma 8.1.** The angular map $F_1^{(A,\lambda)}(x, y)$ is an increasing map for $x \in [\pi/2, 3\pi/2]$ and $y \in D$.

**Proof.** One knows that:

$$\frac{\partial F_1^{(A,\lambda)}}{\partial x}(x, y) = 1 - K_\omega \frac{\lambda \cos x}{y + A + \lambda \sin x}$$

$$= 1 - K_\omega \frac{\frac{A}{\lambda} \cos x}{\frac{y}{A} + \frac{1}{A} \sin x}$$

$$= 1 - K_\omega \frac{\frac{\lambda}{A} \cos x}{1 + \frac{1}{A} \sin x} + o(y)$$

Since $\frac{A}{\lambda} \ll 1$ and $\frac{\lambda}{A} \cos x < 0$ in $[\pi/2, 3\pi/2]$, it follows that

$$\forall x \in [\pi/2, 3\pi/2], \quad \forall y \in D, \quad \frac{\partial F_1^{(A,\lambda)}}{\partial x}(x, y) > 0$$

and the result follows. \(\square\)

For all $c > 0$ sufficiently small, define $\theta_0 = \pi + \arcsin(c)$. It is easy to see that $\theta_0 \in [\pi, 3\pi/2]$ and $\sin(\theta_0) = \sin(\pi + \arcsin(c)) = -c < 0$. Therefore, if $\delta \in [0, 3\pi/2 - \theta_0 [\, \text{then} \, \delta \in [0, \pi/2 [\, \text{and therefore}$

$$F_1^{(A,\lambda)}(y, 3\pi/2 - \delta) - F_1^{(A,\lambda)}(y, \theta_0) =$$

$$= -K_\omega \log[y + A + \lambda \sin(3\pi/2 - \delta)] + (3\pi/2 - \delta) + K_\omega \log(y + A + \lambda \sin \theta_0 - \theta_0)$$

$$= (3\pi/2 - \delta - \theta_0) + K_\omega \log \left[ \frac{y + A + \lambda \sin \theta_0}{y + A + \lambda \sin(3\pi/2 - \delta)} \right]$$

$$= (3\pi/2 - \delta - \theta_0) + K_\omega \log \left[ \frac{\frac{1}{A}y + 1 + \frac{\lambda}{A} \sin \theta_0}{\frac{1}{A}y + 1 + \frac{1}{A} \sin(3\pi/2 - \delta)} \right]$$

$$= (3\pi/2 - \delta - \theta_0) + K_\omega \log \left[ \frac{1 - \frac{\lambda c}{A}}{1 + \frac{1}{A} \sin(3\pi/2 - \delta)} \right] + O(y)$$

$$= (3\pi/2 - \delta - \theta_0) + K_\omega \log \left[ \frac{1 - \frac{\lambda c}{A}}{1 - \frac{1}{A} \cos(\delta)} \right] + O(y)$$

Motivated by [3], we define now the map:
\[ P\left( \delta, \frac{\lambda}{A}\right) \mapsto K_\omega \ln \left[ \frac{1 - \frac{\lambda}{A} \cos(\delta)}{1 - \frac{\lambda}{A}} \right] \]

**Lemma 8.2.** If \( \frac{\lambda}{A} > \exp\left( \frac{\pi}{K_\omega c}\right)^{-1} - \frac{c}{\exp(\frac{\pi}{K_\omega c}) - c} \), then \( P(0, \frac{\lambda}{A}) > \frac{\pi}{c} \). In particular, given \( n \in \mathbb{N} \), there exists \( c_n > 0 \) small enough, such that \( \frac{\lambda}{A} > \exp\left( \frac{\pi}{K_\omega c_n}\right)^{-1} - c_n \), then \( P(0, \frac{\lambda}{A}) > 2(n + 1)\pi \).

**Proof.** Taking \( a = \frac{\lambda}{A} \), we get:

\[
P(0, a) > \frac{\pi}{c} \iff K_\omega \log \left( \frac{1 - ac}{1 - a} \right) > \frac{\pi}{c}
\]

\[
\iff \log \left( \frac{1 - ac}{1 - a} \right) > \frac{\pi}{K_\omega c}
\]

\[
\iff \frac{1 - ac}{1 - a} > \exp \left( \frac{\pi}{K_\omega c} \right)
\]

\[
\iff 1 - ac > (1 - a) \exp \left( \frac{\pi}{K_\omega c} \right)
\]

\[
\iff 1 - ac > \exp \left( \frac{\pi}{K_\omega c} \right) - a \exp \left( \frac{\pi}{K_\omega c} \right)
\]

\[
\iff a \left( \exp \left( \frac{\pi}{K_\omega c} \right) - c \right) > \exp \left( \frac{\pi}{K_\omega c} \right) - 1
\]

\[
\iff a > \frac{\exp \left( \frac{\pi}{K_\omega c} \right) - 1}{\exp \left( \frac{\pi}{K_\omega c} \right) - c}
\]

Note that, for \( c > 0 \) fixed, we get:

\[
\lim_{K_\omega \to 0} \frac{\exp \left( \frac{\pi}{K_\omega c} \right) - 1}{\exp \left( \frac{\pi}{K_\omega c} \right) - c} = \lim_{X \to +\infty} \frac{\exp(X) - 1}{\exp(X) - c} = 1.
\]

By continuity with respect to \( \delta \), there exists \( \delta_0 < 3\pi/2 - \theta_0 \) such that \( P(\delta_0, a) > \frac{\pi}{c} \) if \( a > \frac{\exp \left( \frac{\pi}{K_\omega c} \right)^{-1}}{\exp \left( \frac{\pi}{K_\omega c} \right) - c} \). Therefore there are two disjoint intervals \( I_1 = [x_1, x_2] \) and \( I_2 = [x_3, x_4] \) such that

\[
\pi < \theta_0 < x_1 < x_2 < x_3 < x_4 < \frac{3\pi}{2} - \delta_0 \tag{8.1}
\]

such that we may define two horizontal rectangles:

\[ H_1 = \{(x, y) \in D : x \in I_1\}, \quad \text{and} \quad H_2 = \{(x, y) \in D : x \in I_2\}. \]

on which the map \( F_{(A, \lambda)}|_{H_1 \cup H_2} \) is topologically conjugate to a Bernoulli shift with two symbols. The construction of this horseshoe is the goal of the next subsection.
8.2. The construction of the topological horseshoe. We now recall from [44] the main steps of the construction of the horseshoes which are invariant by the two-dimensional map $F_{(A,\lambda)}|_{\text{Out}(O_2)}$. The argument uses the generalized Conley-Moser conditions [23, 31, 50] to ensure the existence for $\frac{\lambda}{\lambda_0} > \frac{\exp\left(\frac{\pi}{\omega_0}\right)-1}{\exp\left(\frac{\pi}{\omega_0}\right)-e}$ of an invariant set $\Lambda \subset \text{Out}^+(O_2)$ topologically conjugated to a Bernoulli shift with (at least) 2 symbols.

Given a rectangular region in $\text{Out}(O_2)$, parameterized by a rectangle $[w_1, w_2] \times [z_1, z_2]$, a horizontal strip in $\mathcal{R}$ is a set

$$\mathcal{H} = \{(y, x) : x \in [u_1(y), u_2(y)] \quad y \in [w_1, w_2]\}$$

where $u_1, u_2 : [w_1, w_2] \to [z_1, z_2]$ are Lipschitz functions such that $u_1(y) < u_2(y)$. The horizontal boundaries of a horizontal strip are the graphs of the maps $u_i$; the vertical boundaries are the lines $\{w_i\} \times [u_1(w_i), u_2(w_i)]$. In an analogous way we define a vertical strip across $\text{Out}(O_2)$, a vertical rectangle, with the roles of $x$ and $y$ reversed.
We may deduce (by construction) that:

1. For $i = 1, 2$, as suggested in Figure 8, the horizontal strip $H_i$ is mapped (homeomorphically) by $F_{(A, \lambda)}$ into a vertical strip across $\text{Out}(O_2)$.

2. $F_{(A, \lambda)}(H_1) \cap F_{(A, \lambda)}(H_2) = \emptyset$ because $F_{(A, \lambda)}$ is a diffeomorphism, where it is well defined.

3. $F_{(A, \lambda)}(H_1)$ and $F_{(A, \lambda)}(H_2)$ have full intersections (see Definition 2.3.4 of [50]) because of Lemma 8.2 and subsequent remark;

4. For $i, j = 1, 2$, defining $V_{ji} := F_{(A, \lambda)}(H_i) \cap H_j$, $H_{ij} := F_{(A, \lambda)}^{-1}(V_{ji}) = H_i \cap F_{(A, \lambda)}^{-1}(H_j)$ and denoting by $\partial_v V_{ji}$ the vertical boundaries of $V_{ji}$, we get, by construction, that:

   (a) $\partial_v V_{ji} \subset \partial_v F_{(A, \lambda)}(H_i)$;
   
   (b) the map $F_{(A, \lambda)}$ maps $H_{ij}$ homomorphically onto $V_{ji}$;
   
   (c) $F_{(A, \lambda)}^{-1}(\partial_v V_{ji}) \subset \partial_v H_i$.

\[ \text{Figure 8. If } \lambda > \exp\left(\frac{\pi - 1}{\exp\left(\frac{\pi - 1}{\exp\left(\frac{\pi - 1}{\cdots}ight)}\right)}\right), \text{ there are at least two rectangles whose image under } F_{(A, \lambda)} \text{ overlaps with } \text{Out}(O_2) \text{ at least twice, giving rise to a Smale horseshoe – see Figure 7(D).} \]

Using [50], we may conclude that there exists a $F_{(A, \lambda)}$-invariant set of initial conditions

\[ \Lambda = \bigcap_{n \in \mathbb{Z}} F_{(A, \lambda)}^n(H_1 \cup H_2) \]
on which the map $F_{(A,\lambda)}|\Lambda$ is topologically conjugate to a Bernoulli shift with two symbols.

8.3. **Hyperbolicity.** To prove the hyperbolicity of $\Lambda$ with respect to the map $F_{(A,\lambda)}$, we apply the following result due to Afraimovich, Bykov and Shilnikov [2].

**Theorem 8.3 [2].** Let $H : U \to \mathbb{R}^2$ be a $C^1$ map where $U$ is an open convex subset of $\mathbb{R}^2$ such that $H(x, y) := (F_1(x, y), F_2(x, y))$, where $x, y \in \mathbb{R}$. If:

1. $\| \frac{\partial F_2}{\partial y} \| < 1$
2. $\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \| < 1$;
3. $1 - \| \frac{\partial F_3}{\partial y} \| \left\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right\| > 2 \sqrt{\| \frac{\partial F_2}{\partial x} \| \| \frac{\partial F_1}{\partial y} \| \left\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right\|}$
4. $(1 - \| \frac{\partial F_3}{\partial y} \|) \left( 1 - \| \frac{\partial F_3}{\partial y} \|^{-1} \right) > \| \frac{\partial F_2}{\partial x} \| \left\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right\| \| \frac{\partial F_1}{\partial y} \|$, 

then any compact invariant set $\Lambda \subset U$ is hyperbolic.

To finish the proof of Theorem 8.3, we check one by one the hypotheses of Theorem 8.3 when applied to the compact set $\Lambda \subset \text{Out}^+(O_2)$, where $F_1 = F_{1;(A,\lambda)}$ and $F_2 = F_{2;(A,\lambda)}$.

1. The proof follows from Lemma 6.1.
2. One knows that:
   $$\left\| \frac{\partial F_1}{\partial x} \right\| = \sup_{(x,y) \in \Lambda} \left\| \frac{\partial F_1}{\partial x} \right\| = \sup_{(x,y) \in \Lambda} \left| 1 - K_A \frac{A \cos x + o(y)}{1 + \frac{A}{4} \sin x} \right| > 1$$

The last inequality follows from the fact that $\pi < x < \frac{3\pi}{2}$ (see (8.1)) in $\Lambda \subset H_1 \cup H_2$.

3. We know that:
   $$1 - \left( \frac{\partial F_1}{\partial y} \right)^{-1} = 1 - O(A^{-1}) \left( \frac{\partial F_1}{\partial x} \right)^{-1}$$

On the other hand, we may write:

$$2 \sqrt{\left\| \frac{\partial F_2}{\partial x} \right\| \left\| \frac{\partial F_1}{\partial y} \right\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \left\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right\|} = 2 \left\| \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right\| O(A^{\frac{\delta-1}{2}})$$

4. Similarly, we write:
   
   $$(1 - \left( \frac{\partial F_1}{\partial y} \right)^{-1}) (1 - \left( \frac{\partial F_1}{\partial x} \right)^{-1}) = (1 - O(A^\delta)) \left( 1 - \left( \frac{\partial F_1}{\partial x} \right)^{-1} \right)$$

and

$$\left( \frac{\partial F_2}{\partial x} \right) \left( \frac{\partial F_1}{\partial y} \right)^{-1} \left\| \frac{\partial F_1}{\partial y} \right\| = O(A^\delta) O(A^{-1}) \left( \frac{\partial F_1}{\partial x} \right)^{-1}$$
**Geometric interpretation.** Condition $a = \frac{\lambda}{A} > \exp\left(\frac{\pi}{K\omega c}\right) - 1$ provides enough expansion in the $x$-variable (angular coordinate) within the region Out($O_2$). It means that there are at least two rectangles whose image under $F_{(A,\lambda)}$ overlaps with Out($O_2$) at least twice. Therefore, we obtain a construction similar to the Smale horseshoe. Hyperbolicity is obtained by considering rectangles whose vertical boundaries do not have reversals of orientation. Since $c > 0$ may be taken small enough, instead of two horizontal rectangles, we may perform a construction with $n \in \mathbb{N}$ rectangles, provided $P\left(0, \frac{A}{\lambda}\right) > 2\pi(n + 1)$.

**Remark 8.4.** The dynamics of $\Lambda$ is mainly governed by the geometric configuration of the global invariant manifold $W^u(O_2)$. The topological horseshoe is a *rotational horseshoe* in the terminology of [42].

9. **Proof of Theorem F via torus-bifurcation**

As noticed in Section 4 in the bifurcation diagram $(\frac{\lambda}{A}, K\omega)$, we may draw, in the first quadrant, two smooth curves, the graphs of $g$ and $f$, such that:

1. $g(K\omega) = \frac{1}{\sqrt{1 + K^2\omega}}$ and $f(K\omega) = \frac{\exp\left(\frac{4\pi}{K\omega}\right) - 1}{\exp\left(\frac{4\pi}{K\omega}\right) - 1/4}$;
2. the region below the graph of $g$ correspond to flows having an invariant and attracting torus with zero topological entropy (regular dynamics);
3. the region above the graph of $f$ correspond to vector fields whose flows exhibit chaos (chaotic dynamics).

In this section, we describe a mechanism of breakdown of a two-dimensional torus due to the onset of homoclinic tangencies produced by the stable and unstable manifolds of a dissipative saddle (note that, for small $A$, the first return map $F_{(A,\lambda)}$ is still contracting because $\delta > 1$; cf Remark 6.2). These tangencies are the source of strange attractors. We address the reader to [1, 3, 4, 8] for more information on the subject.

9.1. **Arnold tongue.** The choice of parameters in Subsection 4 lets us build the bifurcation diagram of Figure 3 (right), in the plane of the parameters $(A, \lambda/A)$ in the domain

$$\{0 \leq A < 1, \quad 0 < \lambda/A < \mu_0\},$$

for $0 < \mu_0 < 1$ sufficiently small. See Figure 7.

Within this region, we may define an Arnold tongue, denoted by $T_k$, adjoining the horizontal axis at a point

$$A_k = (\exp(-2\pi k), 0),$$

where $k$ is a large enough integer. Inside, there coexist at least a pair of fixed points of the Poincaré map $F_{(A,\lambda)}$. Their images are the periodic orbits of period $2\pi k$. The borders of $T_k$ are the bifurcation curves $B_1^k$ and $B_2^k$ on which the fixed points merge into a saddle-node. The curve $B_2^k$ continue up to the line $\lambda/A = 1$, while the curve $B_1^k$ bend to the left staying below $\lambda/A = 1$. Eventually these curves may touch the corresponding curves of other tongue (meaning that there are parameter values for which the periodic points of periods $2\pi k$ and $2\pi m$ coexist), $m, k \in \mathbb{N}$. 
Below the line $D^k$ (see Figure 9), the closed invariant curve which exists for $\lambda_1 < f_1(K_0^0)$ is the unstable manifold of the saddle fixed point $Q_k$ which closes on the fixed point $P_k$. After crossing the curve $D_k$ the invariant curve no longer exists.

The borders of the Arnold tongue $T_k$ are the bifurcation curves $B^k_1$ and $B^k_2$ on which the fixed points merge to a saddle-node. Inside the tongue $T_k$, one of the fixed points of the map is always of saddle-type (say $Q_k$). The other point is a sink, $P_k$. The points $M^k_1$ and $M^k_2$ correspond to pre-wiggles: below these points, in $B^k_1$ and $B^k_2$, the limit set of $W^u(Q_k)$ is the saddle-node itself and the is homeomorphic to a circle. Above this point, the set $W^u(Q_k)$ is not homeomorphic to a circle (loses smoothness).

![Figure 9](image)

**Figure 9.** In the Arnold tongue $T_k$, there is a line corresponding to the emergence of a homoclinic orbit is due to the tangency of the stable and unstable manifolds of the saddle (dissipative) periodic point.

### 9.2. Strange attractors

A mechanism of breakdown of the invariant circle is due to the onset of homoclinic tangencies produced by the stable and unstable manifolds associated to a dissipative point. The tangencies occur along the bifurcation curves $\text{Hom}^k_1$ and $\text{Hom}^k_2$ where each line corresponds to a homoclinic contact of the components $W^s(Q_k)$ and $W^u(Q_k)$, where $Q_k$ is a dissipative saddle (as illustrated in Figure 9). The curves $\text{Hom}^k_1$ and $\text{Hom}^k_2$ divide the region above $D^k$ into two regions with simple and complex dynamics. In the zone above the curves $\text{Hom}^k_1$ and $\text{Hom}^k_2$, there is a fixed point $Q_k$ exhibiting a transverse homoclinic intersection, and thus the corresponding map $F_{(A, \lambda)}$ exhibits nontrivial hyperbolic chaotic sets. Other stable points of large period exist in the region above the curves $\text{Hom}^k_1$ and $\text{Hom}^k_2$ since the homoclinic tangencies arising in these lines are quadratic. Therefore, it follows from Newhouse [38, 39] that above these curves, in the parameter space $(A, \frac{\lambda}{A})$, there is a dense set of parameters for which the map $F_{(A, \lambda)}$ has infinitely many sinks [17].

Homoclinic tangencies are a route of access to new non-hyperbolic dynamics. In this case, they are the gateway to the existence of strange attractors. The existence of $\text{Hom}^k_1$ and $\text{Hom}^k_2$ shows that there are parameters $(\frac{\lambda}{A}, K_0)$ for which the corresponding first return map has
Newhouse phenomena. In particular, using now, there exists a positive measure set $\Delta$ of parameter values, so that for every $a \in \Delta$, $\mathcal{F}_{(A,\lambda)}$ admit a strange attractor of Hénon-type with an ergodic SRB measure.

**Remark 9.1.** In this type of result, the number of connected components with which the strange attractors intersect the section $\text{Out}(O_2)$ is not specified nor is the size of their basins of attraction.

**Remark 9.2.** For $k$ large enough, the lines $\text{Hom}_k^1$ and $\text{Hom}_k^2$ within the Arnold tongues $T_k$ will approach the vertical axis and the points $M_k^1, M_k^2$ will approach the origin.

10. An application: Hopf-zero singularity unfolds strange attractors

In this section, we prove the existence of strange attractors in analytic unfoldings of a Hopf-zero singularity for a particular case which has not been considered in [11]. In order to improve the readability of the paper, we recall to the reader the most important steps about generic unfoldings of a Hopf-zero singularity.

From now on, we consider Hopf-Zero singularities, that is, three-dimensional vector fields $f^*$ in $\mathbb{R}^3$ such that:

- $O \equiv (0,0,0)$ is an equilibrium of $f^*$;
- the spectrum of $df^*(0,0,0)$ is $\{\pm i\omega, 0\}$, with $\omega > 0$.

Without loss of generality, we can assume that:

$$Df^*(0,0,0) = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (10.1)$$

Observe that the lowest codimension singularities in $\mathbb{R}^3$ with a three-dimensional center manifold are the ones whose linear part is linearly conjugated to (10.1).

10.1. The normal form. The normal form of a degenerate jet with linear part given by

$$\begin{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

may be written in cylindrical coordinates $(r, \theta, z)$ by:

$$\begin{cases} \dot{r} = a_1 rz + a_2 r^3 + a_3 r z^2 + O(|r, z|^4) \\ \dot{\theta} = \omega + O(|r, z|^2) \\ \dot{z} = b_1 r^2 + b_2 z^2 + b_3 r^2 z + b_4 z^3 + O(|r, z|^4) \end{cases} \quad (10.2)$$

where $\omega > 0$ and $a_1, a_2, a_3, b_1, b_2, b_3, b_4 \in \mathbb{R}\{0\}$. The vector field associated to (10.2) contains no $\theta$-dependent terms; this is why the angular component $\theta$ may be decoupled. Truncating (10.2) at order 2 and removing the angular coordinate $\theta$, we obtain:

$$\begin{cases} \dot{r} = a_1 rz \\ \dot{z} = b_1 r^2 + b_2 z^2. \end{cases} \quad (10.3)$$
FIGURE 10. (a) Phase diagram of (10.4) for $a > 0$ and $b = -1$. (b) Stable heteroclinic cycle associated to $O_1$ and $O_2$ for the differential equation (10.5) with $\mu_1 = 0$.

Setting $r^{\text{new}} = -\sqrt{|b_1b_2|} r$, $z^{\text{new}} = -b_2 z$, and dropping the superscripts "new", we get the differential equation:

$$\begin{align*}
\dot{r} &= arz \\
\dot{z} &= br^2 - z^2
\end{align*}$$

(10.4)

where $a = -a_1/b_2$ and $b = \pm 1$. The phase diagram of (10.4) for $a > 0$ and $b = -1$ is shown in Figure 10(a).

Takens [48] proved that there are six topological types for the normal form (10.4), but from now on, we are only interested in the one characterized by the conditions $b = -1$ and $a > 0$ (Type I of [11]; Case III of [23]). The line defined by $r = 0$ ($z$-axis) is flow-invariant. According to [23], any generic unfolding of (10.4) may be written as:

$$\begin{align*}
\dot{r} &= \mu_1 r + arz \\
\dot{z} &= \mu_2 r^2 - z^2
\end{align*}$$

(10.5)

whose flow satisfy the following properties (for $\mu_2 \geq 0$ and $\mu_2 > \frac{\mu_1^2}{a^2}$):

- there are two equilibria of saddle-type, say $p_1 = (0, \sqrt{\mu_2})$, $p_2 = (0, -\sqrt{\mu_2})$, whose eigenvalues of $df^*$ at the equilibria are $\mu_1 \pm a\sqrt{\mu_2}$ and $\mp 2\sqrt{\mu_2}$;
- there is another equilibrium given by $\left(\sqrt{\mu_2 - \frac{\mu_1^2}{a^2}}, -\frac{\mu_1}{a}\right)$ which is a center;
- for $\mu_1 = 0$, there is a heteroclinic cycle associated to $p_1$ and $p_2$;
• for $\mu_1 = 0$, if $G(r, z) = 2r \frac{2}{a} \left( \mu_2 - \frac{r^2}{1+a} - z^2 \right)$ then the Lie derivative of $G$ with respect to the vector field associated to (10.5) satisfies the inequality:

$$\mathcal{L}_v G \equiv 0,$$

meaning that there is a family of non-trivial periodic solutions limiting the inner part of the planar heteroclinic cycle associated to $p_1$ and $p_2$. This cycle is Lyapunov stable – see Figure 10(b).

**Figure 11.** Bifurcation diagram for the differential equation (10.6) when $3c+e > d+3f$ and $3(3c+e) + d + 3f < 0$. (H): Hopf bifurcation; (AHC): attracting heteroclinic cycle.

The truncated normal form of order 2 is not enough to our purposes because the heteroclinic cycle is not asymptotically stable.

10.2. **Truncating at order 3.** Truncating at third order any generic unfolding of (10.4) with $a > 0$ and $b = -1$, we may write:

$$\begin{cases}
\dot{r} = \mu_1 r + arz + cr^3 + drz^2 \\
\dot{z} = \mu_2 - r^2 - z^2 + er^2z + fz^3
\end{cases} \quad (10.6)$$

where the new parameters $c, d, e, f \in \mathbb{R}$ satisfy the open conditions:

$$3c + e > d + 3f \quad \text{and} \quad 3(3c + e) + d + 3f < 0. \quad (10.7)$$
Following [24, 30], the flow of (10.6) exhibits a heteroclinic cycle associated to
\[ \tilde{p}_1 \approx (0, \sqrt{\mu_2} + f/2) \quad \text{and} \quad \tilde{p}_2 \approx (0, -\sqrt{\mu_2} + f/2) \]
at all points \((\mu_1, \mu_2)\) lying at the line \((AHC)\) defined by
\[ (AHC) : \quad \mu_2 = \frac{-4\mu_1}{3(3c + e) + d + 3f} + O(\sqrt{\mu_2}). \]
This cycle is asymptotically stable cycle provided conditions (10.7) are valid. Line \((AHC)\) is depicted in Figure 11.

10.3. **General perturbations.** Adding the angular coordinate \(\theta\) to equation (10.6), we define a \(SO(2)\)-equivariant vector field, say \(f(\mu_1, \mu_2)\). Its flow has an attracting heteroclinic cycle associated to the lift of \(\tilde{p}_1\) and \(\tilde{p}_2\), say \(O_1\) and \(O_2\). This cycle is made by one 1D and one 2D heteroclinic connections associated to two hyperbolic saddles-foci with different Morse indices. The coincidence of the invariant manifolds of the hyperbolic saddle-foci is exceptional and they are expected to split.

Generic unfoldings of the Hopf-zero singularities were considered in [19]; the authors introduced an extra parameter \(\varepsilon = \sqrt{\mu_2}\) and obtained a singular perturbation problem with a pure rotation when \(\varepsilon = 0\) or a family with rotation speed tending to \(+\infty\) as \(\varepsilon \to 0\). Since the imaginary part of the eigenvalues of the vector field at the equilibria has the form \(O(1/\varepsilon)\), this means that \(K_\omega \to 0\); details in Section 3 of [19]. In this paper, we are interested in a particular case of unfoldings of (10.4).

**Definition 3.** A Gaspard-type unfolding of (10.4) admits (10.6) as a truncated normal form of order 3 and its high order terms are of the type \(O(\varepsilon)\) (i.e. they have the form of equations (2.15) and (2.16) of Gaspard [20]).

**Proposition 10.1.** Let \(f^*\) be a Hopf-zero singularity. If \(f_{(\mu_1, \mu_2)}(\mu_1, \mu_2)\) is a Gaspard-type unfolding of \(f^*\), then there exists an analytic curve \(C\) in the parameter space \((\mu_1, \mu_2)\) and a domain \(W\) contained in a wedge shaped neighbourhood of \(C\) such that: if \((\mu_1, \mu_2) \in W\) then the flow of \(f_{(\mu_1, \mu_2)}\) has a strange attractor (of Hénon-type) with an ergodic SRB measure.

**Proof.** We apply Theorem [ ] to prove Proposition [10.1]. Indeed, if \((\mu_1, \mu_2) \in (AHC)\), then the flow of \(f_{(\mu_1, \mu_2)}\) satisfies:

- there are two hyperbolic saddle-foci \(O_1\) and \(O_2\) satisfying (P1)–(P2);
- the manifolds \(W^u(O_2)\) and \(W^s(O_1)\) coincide and one branch of \(W^u(O_1)\) coincide with \(W^s(O_2)\) – see Remark [10.4]. In particular, there is an attracting heteroclinic cycle associated to \(O_1\) and \(O_2\) meaning that (P3)–(P4) are satisfied;
- by construction on the way the angular coordinate is acting on (10.6), the saddle-foci have the same chirality – (P5) is valid;
- by hypothesis, we perturb \(f_{(\mu_1, \mu_2)}\) in such a way that the manifolds \(W^u(O_2)\) and \(W^s(O_1)\) do not intersect and the one-dimensional manifolds of the equilibria are preserved, emerging an attracting 2-torus. This perturbation correspond to (P6)–(P7).

Breaking the 2D-connection, a smooth torus is expected to persist. The dynamics on it exhibits intervals of frequency locking and irrational flow as the rotation number varies. Generic perturbations (without symmetry) will break the 2-torus and, by Theorem [ ], the result follows, where the parameter \(\varepsilon\) plays the role of \(\lambda\).
10.4. A conjecture. Following Bonckaert and Fontich [14], when the scaling parameters defined in [19] tend to 0, the invariant manifolds have a limit position given by the invariant manifolds of the equilibria at the 2-jet (when $z > 0$). Therefore, for any generic unfolding of the Hopf-zero singularity (Type I of [11]), the splitting distance is well defined for the 1D and 2D connections. The case of 1D was obtained in [9] – the distance $S^1$ between the 1D invariant manifolds is exponentially small with respect to $\varepsilon > 0$ and the coefficient in front of the dominant term depends on the full jet of the singularity. The splitting function for the 2D invariant manifolds, say $S^2$, has been obtained in [10].

When $\sigma = \mu_1/\varepsilon = O(\varepsilon)$, conclusive results are given in [11]: the manifolds $W^u(O_2)$ intersects transversely $W^s(O_1)$, giving rise to infinitely many Bykov cycles. Any analytic unfolding of a Hopf-zero singularity exhibits Shilnikov bifurcations and strange attractors [27]. When the parameter $\sigma = \mu_1/\varepsilon \gg O(\varepsilon)$, the 2-dimensional manifolds of the saddle-foci do not intersect and the distance between them is of order $\sigma$ (see [10]). In this case, there exist trapping regions which prevent the existence of Shilnikov homoclinic cycles; more details in [11, 19]. The existence of suspended horseshoes may be ensured by our Theorem [E] provided its hypotheses are satisfied.

The problem about the existence of strange attractors in generic unfoldings of Hopf-zero singularities is still open. However, if we introduce the distances $S^1$ ([9]) and $S^2$ ([10]) in the definition of $\Psi_{1 \to 2}$ and $\Psi_{2 \to 1}$ (see Section 5), we conjecture that Proposition [10] may be restated for all generic unfoldings of the Hopf-zero singularities, not just for the Gaspard-type unfoldings. We defer this task for a future work.

**Final remark**

We finish this article with two important remarks that should pointed out. First, note that our study is based on a model satisfying Hypothesis (P8), the transition map along the connections. This assumption is necessary to make precise computations in Sections 6, 7 and 8. It condensates all “good properties” of a generic breaking of invariant manifolds and has already been used (in a different context) by [49]. See also §3 of [20].

Finally, observe that, in (P1)–(P5) we have asked for the existence of a heteroclinic network with two 1D-connections. Nevertheless the existence of an attracting torus and its bifurcations do not rely on the existence of such a network, provided the map $\Psi_{2 \to 1}$ sends $W^u_{\text{loc}}(O_2)$ into $\text{In}^+(O_1)$ (see Section 5). This remark is important since the flow associated to the lift of the differential equation (10.6) does not have a network, just a cycle.

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Appendix A. Glossary

For $\varepsilon > 0$ small enough, consider the two-parameter family of $C^3$-smooth autonomous differential equations

$$\dot{x} = f(A,\lambda)(x) \quad x \in S^3 \quad A, \lambda \in [-\varepsilon, \varepsilon]$$

Denote by $\varphi(A,\lambda)(t,x), t \in \mathbb{R}$, the associated flow.

A.1. Symmetry. Given a group $G$ of endomorphisms of $S^3$, we will consider two-parameter families of vector fields $(f(A,\lambda))$ under the equivariance assumption $f(A,\lambda)(\gamma x) = \gamma f(A,\lambda)(x)$ for all $x \in S^3$, $\gamma \in G$ and $(A,\lambda) \in [-\varepsilon, \varepsilon]^2$. For an isotropy subgroup $\tilde{G} < G$, we will write $\text{Fix}(\tilde{G})$ for the vector subspace of points that are fixed by the elements of $\tilde{G}$. Observe that, for $G$-equivariant differential equations, the subspace $\text{Fix}(\tilde{G})$ is flow-invariant.

A.2. Attracting set. A subset $A$ of a topological space $\mathcal{M}$ for which there exists a neighborhood $U \subset \mathcal{M}$ satisfying $\varphi(t,U) \subset U$ for all $t \geq 0$ and $\bigcap_{t \in \mathbb{R}^+} \varphi(t,U) = A$ is called an attracting set by the flow $\varphi$, not necessarily connected. Its basin of attraction, denoted by $\mathcal{B}(A)$, is the set of points in $\mathcal{M}$ whose orbits have $\omega$-limit in $A$. We say that $A$ is asymptotically stable (or that $A$ is a global attractor) if $\mathcal{B}(A) = \mathcal{M}$. An attracting set is said to
be quasi-stochastic if it encloses periodic solutions with different Morse indices, structurally unstable cycles, sinks and saddle-type invariant sets.

A.3. Heteroclinic phenomenon. Suppose that $O_1$ and $O_2$ are two hyperbolic saddle-foci of $f_{(A,\lambda)}$ with different Morse indices (dimension of the unstable manifold). There is a heteroclinic cycle associated to $O_1$ and $O_2$ if $W^u(O_1) \cap W^s(O_2) \neq \emptyset$ and $W^u(O_2) \cap W^s(O_1) \neq \emptyset$. For $i, j \in \{1, 2\}$, the non-empty intersection of $W^u(O_i)$ with $W^s(O_j)$ is called a heteroclinic connection between $O_i$ and $O_j$, and will be denoted by $[O_i \to O_j]$. Although heteroclinic cycles involving equilibria are not a generic feature within differential equations, they may be structurally stable within families of systems which are equivariant under the action of a compact Lie group $G \subset O(n)$, due to the existence of flow-invariant subspaces [23].

A.4. Bykov cycle. A heteroclinic cycle between two hyperbolic saddle-foci of different Morse indices, where one of the connections is transverse (and so stable under small perturbations) while the other is structurally unstable, is called a Bykov cycle. A Bykov network is a connected union of heteroclinic cycles, not necessarily in finite number. We refer to [28] for an overview of heteroclinic bifurcations and substantial information on the dynamics near different kinds of heteroclinic cycles and networks.

A.5. Suspended horseshoe. Given $(A, \lambda) \in [-\epsilon, \epsilon]^2$, suppose that there is a cross-section $S_\lambda$ to the flow $\varphi_{(A,\lambda)}$ such that $S_{(A,\lambda)}$ contains a compact set $K_{(A,\lambda)}$ invariant by the first return map $F_{(A,\lambda)}$ to $S_{(A,\lambda)}$. Assume also that $F_{(A,\lambda)}$ restricted to $K_{(A,\lambda)}$ is conjugate to a full shift on a finite alphabet. Then the suspended horseshoe associated to $K_{(A,\lambda)}$ is the flow-invariant set $\overline{K_{(A,\lambda)}} = \{ \varphi_{\lambda}(t, x) : t \in \mathbb{R}, x \in K_{(A,\lambda)} \}$.

A.6. SRB measure. Given an attracting set $A$ for a continuous map $R : \mathcal{M} \to \mathcal{M}$ of a compact manifold $\mathcal{M}$, consider the Birkhoff average with respect to the continuous function $T : \mathcal{M} \to \mathbb{R}$ on the $R$-orbit starting at $x \in \mathcal{M}$:

$$L(T, x) = \lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=0}^{n-1} T \circ R^i(x). \quad \text{(A.2)}$$

Suppose that, for Lebesgue almost all points $x \in B(A)$, the limit (A.2) exists and is independent on $x$. Then $L$ is a continuous linear functional in the set of continuous maps from $\mathcal{M}$ to $\mathbb{R}$ ($C(\mathcal{M}, \mathbb{R})$). By the Riesz Representation Theorem, it defines a unique probability measure $\mu$ such that:

$$\lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=0}^{n-1} T \circ R^i(x) = \int_A T \, d\mu \quad \text{(A.3)}$$

for all $T \in C(\mathcal{M}, \mathbb{R})$ and for Lebesgue almost all points $x \in B(A)$. If there exists an ergodic measure $\mu$ supported in $A$ such that (A.3) is satisfied for all continuous maps $T \in C(\mathcal{M}, \mathbb{R})$ for Lebesgue almost all points $x \in B(A)$, where $B(A)$ has positive Lebesgue measure, then $\mu$ is called a SRB measure and $A$ is a SRB attractor.