THE SPECIALIZATIONS IN A SCHEME

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ABSTRACT. In this paper we will obtain some further properties for specializations in a scheme. Using these results, we can take a picture for a scheme and a picture for a morphism of schemes. In particular, we will prove that every morphism of schemes is specialization-preserving and of norm not greater than one (under some condition); a necessary and sufficient condition will be given for an injective morphism between irreducible schemes.

INTRODUCTION

Specializations are concrete and intuitive for one to study classical varieties\textsuperscript{[6]}. The results on this topic relating to schemes are mainly presented in Grothendieck’s EGA. In this paper we try to obtain some properties for the specializations in a scheme such as the lengths of specializations.

Together with specializations, a scheme can be regarded as a partially ordered set. In §1 we will prove that every morphism of schemes is specialization-preserving (Proposition 1.3) and that every specialization in a scheme is contained in an affine open subset (Proposition 1.9). Using those results, we can take a picture for a scheme (Remark 1.10):

A scheme can be described to be a number of trees standing on the ground such that

- each irreducible component is a tree;
- the generic point of an irreducible component is the root of the corresponding tree;
- the closed points of an irreducible component are the top leaves of the corresponding tree;
- each specialization in an irreducible component are the branches of the corresponding tree.

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In §2 we will discuss the lengths of specializations, where we will notice that the length and the dimension of a subset in a scheme are not equal in general (Remarks 2.2-3).

Using the lengths, in §3 we will define the norm of a morphism of schemes and demonstrate that any morphism of schemes is of norm not greater than one under some condition. Then we will obtain a picture of morphisms of schemes (Remarks 3.6-7): As schemes are trees, morphisms exactly scale down the trees under that condition. A necessary and sufficient condition will be given for an injective morphism between irreducible schemes.

In §4, last section, we will present an application of specializations.

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1. Preliminaries

In the section we will fix the notations and then obtain the basic facts for specializations in a scheme. To start with, we will discuss the specializations in a topological space (which are not Hausdorff in general) since a scheme itself is a space.

Let $E$ be a topological space. Given any $x, y \in E$. Then $y$ is a specialization of $x$ (or $x$ is a generalization of $y$) in $E$ if $y$ is in the closure $\{x\}$, and we denote it by $x \rightarrow y$ (in $E$). For $x \in E$, we put

$$Sp(x) = \{y \in E \mid x \rightarrow y\}$$

and

$$Gen(x) = \{y \in E \mid y \rightarrow x\}.$$  

If $x \rightarrow y$ and $y \rightarrow x$ both hold in $E$, $y$ is called a generic specialization of $x$ in $E$, and we denote it by $x \leftrightarrow y$ (in $E$). The point $x$ is initial if we have $x \leftrightarrow z$ for any $z \in E$ such that $z \rightarrow x$; $x$ is final if we have $x \leftrightarrow z$ for any $z \in E$ such that $x \rightarrow z$.

Let $x \rightarrow y$ in $E$. Then $y$ is said to be a closest specialization of $x$ in $E$ if we have either $z = x$ or $z = y$ for any $z \in E$ such that $x \rightarrow z$ and $z \rightarrow y$ in $E$.

Obviously, we have the following statements:
(i) Let $X$ be a scheme, and $x \in X$. Then we have

$$Sp(x) = \{x\}$$

and

$$Gen(x) \cong Spec(O_x).$$

(ii) Let $E$ be a topological space. Take any $x, y \in E$. Then $x \to y$ in $E$ if and only if $Sp(x) \supseteq Sp(y)$; $x \leftrightarrow y$ in $E$ if and only if $Sp(x) = Sp(y)$; $x$ is final if $x$ is a closed point in $E$; $x \in E$ is initial if and only if $x$ is a generic point of an irreducible component of $E$. In particular, $Sp(x)$ is an irreducible closed subset in $E$.

**Example 1.1.** Let $X$ be an Artinian scheme. Then every point $x \in X$ is both initial and final.

**Proof.** As every $x \in X$ is closed, we have $Sp(x) = \{x\}$ for any $x \in X$. □

**Definition 1.2.** Let $f : E \to F$ be a mapping of topological spaces.

(i) $f$ is said to be IP—preserving if the condition is satisfied:

Given any closed subset $U$ of $E$. Then $f(x_0)$ is an initial point of $f(U)$ if $x_0$ is initial in $U$.

(ii) $f$ is specialization-preserving if we have $f(x) \to f(y)$ in $F$ for any $x \to y$ in $E$.

Now we obtain the main result in the section.

**Proposition 1.3.** (i) Every morphism of schemes is IP—preserving.

(ii) Every morphism of schemes is specialization-preserving.

**Proof.** It is immediate from Lemmas 1.6-7. □

**Lemma 1.4.** Given any scheme $X$.

(i) Let $X = Spec(A)$ be affine. Then we have $x \to y$ in $X$ for any $x, y \in X$ if and only if $j_x \subseteq j_y$ in $A$, where $j_x$ and $j_y$ denote the prime ideals in $A$ corresponding to $x$ and $y$, respectively.

(ii) Take any $x, y \in X$. Then we have $x \leftrightarrow y$ in $X$ if and only if $x = y$.

**Proof.** (i) Let $x \to y$ in $X$. We have $Sp(x) = V(j_x)$ and $Sp(y) = V(j_y)$; then $Sp(x) \supseteq Sp(y)$, and hence $j_x \subseteq j_y$. Evidently, the converse is true.

(ii) It suffices to prove $\Rightarrow$. Let $x \leftrightarrow y$. Take an affine open subset $U$ of $X$ such that $x \in U$. As $x \leftrightarrow y$, there is the identity $Sp(x) = Sp(y)$; then

$$Sp(x) \cap U = Sp(y) \cap U,$$
which are open subsets in $Sp(x)$; as $Sp(x)$ and $Sp(y)$ are irreducible, we have $x, y \in U$, and hence $x \leftrightarrow y$ in $U$. By (i) we have $x = y$ in $U$. □

Let $X_0$ be an irreducible closed subset of a scheme $X$. Take an affine open subset $U$ of $X$ such that $U \cap X_0 \neq \emptyset$, where $U = Spec(A)$. Then

$$U_0 = U \cap X_0$$

is closed in $U$ and is open in $X_0$. Let

$$\Sigma = \{ j_z | z \in U_0 \}.$$ 

With inclusion $\subseteq$, $\Sigma$ is a partially ordered set. There exist minimal elements in $\Sigma$, and we denote by $\Sigma_0$ the set of such minimal elements in $\Sigma$. Let $z_0 \in U_0$ with $j_{z_0} \in \Sigma_0$. Then $z_0$ is an initial point in $U_0$. As $X_0$ is irreducible, $U_0$ is irreducible; hence, $z_0$ is an initial point in $X_0$.

If $z'_0$ is another initial point in $X_0$, we have $z_0 \leftrightarrow z'_0$ in $X_0$, and then $z_0 = z'_0$. This proves (Lemma 1.6) that any irreducible closed subset of a scheme has one and only one initial point (i.e., generic point). In general, there is the following definition.

**Definition 1.5.** Assume $E$ is a topological space satisfying the condition:

There exists one and only one initial point $x_U$ in every irreducible closed subset $U$ of $E$, and we have $x_U \neq x_V$ for any irreducible closed subset $V$ of $E$ such that $U \neq V$.

Then the space $E$ is said to have the ($UIP$)−property.

Obviously, there are many spaces which are of the ($UIP$)−property such as Hausdorff spaces.

**Lemma 1.6.** An irreducible $T_0$−space which has an initial point has the ($UIP$)−property. In particular, every scheme is of the ($UIP$)−property.

**Lemma 1.7.** Let $f : E \to F$ be a mapping of topological spaces.

(i) $f$ is specialization-preserving if and only if $f$ is IP−preserving.  

(ii) Let $F$ be of the ($UIP$)−property. Then $f$ is specialization-preserving if $f$ is continuous.

**Proof.** (i) It is immediate from definition.

(ii) Let $f$ be continuous. Take any $x \to y$ in $E$. It is clear that $f(\text{Sp}(x))$ is irreducible, and then $\bar{f(\text{Sp}(x))}$ is irreducible in $F$; as $f(x) \in f(\text{Sp}(x))$, there is

$$\text{Sp}(f(x)) \subseteq \overline{f(\text{Sp}(x))}.$$
as $F$ has the $(UIP)$—property, there is
\[ f(\text{Sp}(x)) = \text{Sp}(f(x)). \]

Hence, we have
\[ f(\text{Sp}(y)) = \text{Sp}(f(y)). \]

As $\text{Sp}(x) \supseteq \text{Sp}(y)$, we have
\[ f(\text{Sp}(x)) \supseteq f(\text{Sp}(y)); \]
then there is
\[ f(\text{Sp}(x)) \supseteq f(\text{Sp}(y)), \]
and it follows that
\[ \text{Sp}(f(x)) \supseteq \text{Sp}(f(y)) \]
holds. Hence, there is the specialization $f(x) \to f(y)$ in $F$. $\square$

**Lemma 1.8.** Let $X$ be a scheme. For every specialization $x \to y$ in $X$, there is an affine open subset $U$ of $X$ such that $x, y \in U$.

**Proof.** Let $x \to y$ in $X$. Hypothesize that there is no affine open set $W$ such that $x, y \in W$. Let $V$ be an affine open set such that $y \in V$ but $x \not\in V$. Then $y$ is not a limit point of the set $\{x\}$, and we will obtain a contradiction. $\square$

**Proposition 1.9.** Let $X$ be a scheme, and $x, y \in X$ such that $x \to y$. Then there is an affine open set $U = \text{Spec}(A)$ in $X$ such that $x, y \in U$ and $j_x \subseteq j_y$ in $A$,

where $j_x$ and $j_y$ are the prime ideals in $A$ corresponding to $x$ and $y$, respectively.

**Proof.** It is immediate from Lemmas 1.4 and 1.8. $\square$

Now we have got the following remark.

**Remark 1.10. (The Picture of a Scheme).** From the point of view of specializations, a scheme can be regarded as a number of trees standing on the ground such that

(i) each irreducible component is a tree;

(ii) the initial point of an irreducible component is the root of the corresponding tree;

(iii) the final points of an irreducible component are the top leaves of the corresponding tree;

(iv) each specialization in an irreducible component are the branches of the corresponding tree.
2. Definition for Lengths of Specializations

In this section we will define the lengths of specializations, which will be served to define the norm of a morphism of schemes in Section 3. There exist differences between the dimensions and the lengths of subsets in a scheme (Remarks 2.2-3).

Let $E$ be a topological space, and $x, y \in E$ with $x \rightarrow y$. By a **restrict series of specializations** from $x$ to $y$ in $E$, denoted by $\Gamma (x, y)$, we understand a series of specializations

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y$$

in $E$ satisfying

(i) $y = x$ if $x$ is a specialization of $y$ in $E$;

(ii) each $x_i$ is not a specialization of $x_{i+1}$ for $i = 0, 1, \ldots, n-1$ if $x$ is not a specialization of $y$ in $E$.

The **length** of the restrict series $\Gamma (x, y)$ is defined to be $n$. The **length** from $x$ to $y$ (or the **length of the specialization** $x \rightarrow y$), denoted by $l(x, y)$, is defined to be the supremum among all the lengths of restrict series of specializations from $x$ to $y$.

Set

$$l(E) = \sup \{ l(x, y) \mid x, y \in E \text{ such that } x \rightarrow y \}.$$ 

Then $l(E)$ is said to be the **length of the topological space** $E$.

A restrict series $\Gamma$ of specializations in $E$ is called a **presentation for the length** of $E$ if the length of $\Gamma$ is equal to $l(E)$. The **length of a point** $x \in E$, denoted by $l(x)$, is defined to be the length of the subspace $Sp(x)$ in $E$.

Obviously, $l(E_0) \leq l(E)$ holds for any subspace $E_0$ of $E$ since a restrict series of specializations in $E_0$ must be in $E$.

**Lemma 2.1.** Let $E$ be a topological space such that $\dim E < \infty$. The following statements are true.

(i) Let $\Gamma (x_0, x_n)$ be a presentation for the length of $E$. Then $x_0$ is initial and $x_n$ is final in $E$.

(ii) Let $E$ be of the (UIP)−property. Then $l(E) = \dim E$.

**Proof.** (i) It is immediate from definition.

(ii) Hypothesize that $l(E) = \infty$. That is, for any $n \in \mathbb{N}$ there is a restrict series of specializations in $E$

$$x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n.$$ 

Then we have a chain of closed subsets in $E$

$$Sp(x_0) \supsetneq Sp(x_1) \supsetneq \cdots \supsetneq Sp(x_n).$$
As $Sp(x_0), Sp(x_1), \ldots, Sp(x_n)$ are irreducible closed subsets, we will get $\dim E = \infty$, which is in contradiction with the assumption that $\dim E < \infty$. Hence, we must have $l(E) < \infty$. This also proves that $l(E) \leq \dim E$.

Let $\dim E = n_0$. We have a chain of irreducible closed subsets in $E$

$$F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_{n_0}.$$ 

As $E$ is of the (UIP)-property, each subset $F_j$ has the unique initial point $x_{F_j}$. There is a restrict series of specializations in $E$

$$x_{F_0} \rightarrow x_{F_1} \rightarrow \cdots \rightarrow x_{F_{n_0}}.$$ 

Then $l(E) \geq \dim E$ holds. This completes the proof. □

**Remark 2.2.** Let $E$ be a topological space of the (UIP)—property. Take a subspace $E_0$ of $E$. In general, it is not true that $l(E_0) = \dim E_0$; but for the whole space $E$, $l(E)$ and $\dim E$ coincide with each other. That is due to the fact

(i) $\dim E_0$ is determined by $E_0$ itself;
(ii) $l(E_0)$ is defined both by $E_0$ and by $E$ externally.

**Remark 2.3.** Let $X$ be a scheme, and $X_0$ a subscheme of $X$.

(i) $l(X_0) \geq \dim X_0$.
(ii) Let $X_0$ be closed in $X$. Then $\dim X_0 < \infty$ if and only if $l(X_0) < \infty$; moreover, $\dim X_0 = l(X_0)$ if $\dim X_0 < \infty$.

3. Main Results

Let $x, y$ be two points in a topological space $E$. Then $x$ and $y$ are said to be $Sp$—connected if either $x \rightarrow y$ in $E$ or $y \rightarrow x$ in $E$ holds; $x$ and $y$ are said to be $Sp$—disconnected if they are not $Sp$—connected.

**Definition 3.1.** Let $f : X \rightarrow Y$ be a morphism of schemes. (i) $f$ is said to be bounded if there exists a constant $\beta \in R$ such that

$$l(f(x_1), f(x_2)) \leq \beta l(x_1, x_2)$$

holds for any $x_1 \rightarrow x_2$ in $X$ with $l(x_1, x_2) < \infty$.

(ii) Let $f$ be bounded. If $\dim X = 0$, define $\|f\| = 0$; if $\dim X > 0$, define

$$\|f\| = \sup \left\{ \frac{l(f(x_1), f(x_2))}{l(x_1, x_2)} : x_1 \rightarrow x_2 \text{ in } X, \ 0 < l(x_1, x_2) < \infty \right\}.$$ 

Then the number $0 \leq \|f\| \leq \beta$ is said to be the norm of $f$.7
Example 3.2.

(i) The $k$–rational points of a $k$–variety are morphisms of norm zero.

(ii) Let $s, t$ be variables over a field $k$. Suppose

$$f : \text{Spec} (k [s, t]) \to \text{Spec} (k [t])$$

is induced from the embedding of the $k$–algebras. Then $\| f \| = 1$.

(iii) Let $t$ be a variable over $\mathbb{Q}$. Suppose

$$f : \text{Spec} (k [t]) \to \text{Spec} (\mathbb{Z} [t])$$

is induced from the evident embedding. Then $\| f \| = 2$.

Here are the main results of the paper.

Theorem 3.3. Let $f : X \to Y$ be a morphism of schemes. Then $f$ is bounded and $0 \leq \| f \| \leq 1$ if $f$ satisfies

**Condition (**) :** For any $x \in X$, either

$$f (\text{Sp} (x)) = \text{Sp} (f (x))$$

holds or

$$f^{-1} (\text{Sp} (f (x))) \neq \emptyset$$

is $\text{Sp}$–connected.

**Proof.** Without loss of generality, assume dim $X > 0$ and dim $Y > 0$. Let $X$ and $Y$ be both irreducible. Take any affine open subset $U$ of $X$. We will prove

$$\| f | _U \| \leq 1.$$  

Take any $x_1 \to x_2$ in $U$ such

$$0 < (x_1, x_2) < \infty \text{ and } l (f (x_1), f (x_2)) > 0.$$  

We will proceed in two steps.

(i) Let $x_1 \to x_2$ in $U$ be closest. Hypothesize that $f (x_1) \to f (x_2)$ in $Y$ is not closest. That is, assume

$$l (f (x_1), f (x_2)) \geq 2$$

in $Y$. We have

$$\text{Sp} (f (x_1)) \supseteq \text{Sp} (f (x_2));$$

then dim $\text{Sp} (f (x_1)) \geq 2$.

Take $y_0 \in Y$ such that

$$f (x_1) \to y_0 \to f (x_2) \text{ in } Y$$

and

$$f (x_1) \neq y_0 \neq f (x_2).$$
We have 
\[ \text{Sp}(f(x_1)) \supsetneq \text{Sp}(y_0) \supsetneq \text{Sp}(f(x_2)) \]
in $Y$. As
\[ f^{-1}(\text{Sp}(f(x_1))) \supsetneq f^{-1}(\text{Sp}(y_0)) \supsetneq f^{-1}(\text{Sp}(f(x_2))) \]
hold in $X$, we obtain
\[ \dim U \cap f^{-1}(\text{Sp}(f(x_1))) \geq 2 \]
from Condition (*); then
\[ l(x_1, x_2) \geq 2, \]
and hence $x_1 \rightarrow x_2$ in $U$ is not closest, where there will be a contradiction. This proves $f(x_1) \rightarrow f(x_2)$ in $Y$ is closest.

(ii) Assume that $x_1 \rightarrow x_2$ in $U$ is not closest. Let
\[ l(x_1, x_2) = n \]
in $U$. Then there are the closest specializations in $U$
\[ z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_{n+1} \]
where $z_1 = x_1$ and $z_{n+1} = x_2$.

Obviously, we have either
\[ f(z_i) = f(z_{i+1}) \text{ for some } 1 \leq i \leq n \]
or
a restrict series of specializations
\[ f(z_1) \rightarrow f(z_2) \rightarrow \cdots \rightarrow f(z_{n+1}) \]
in $Y$.

For the latter case, by (i) it is seen that these specializations are closest in $Y$.

Hence,
\[ l(f(x_1), f(x_2)) > l(x_1, x_2) \]
never holds in $U$. This proves
\[ \|f\|_{U} \leq 1, \]
and it follows that
\[ \|f\| \leq 1 \]
holds in $X$ since any two $x_1, x_2 \in X$ with $x_1 \rightarrow x_2$ are contained in an affine open subset of $X$. \qed

Remark 3.4. (The Picture of a Morphism of Schemes). Theorem 3.3 affords us a longitudinal classification of morphisms of schemes. Let $f : X \rightarrow Y$ be a morphism of schemes.
(i) \( f \) is **length-preserving** if we have
\[
l(x, y) = l(f(x), f(y))
\]
for any \( x, y \in X \) such that \( x \to y \).
(ii) \( f \) is **asymptotic** if \( \|f\| = 1 \).
(iii) \( f \) is **null** if \( \|f\| = 0 \).

**Remark 3.5. (The Picture of a Morphism of Schemes).**
There exists a latitudinal classification of morphisms of schemes. That is, let \( f : X \to Y \) be a morphism of schemes.
(i) \( f \) is **level-separated** if \( f(x) \) and \( f(y) \) are \( \text{Sp-} \)disconnected in \( Y \) for any \( x, y \in X \) which are \( \text{Sp-} \)disconnected and of the same lengths in \( X \).
(ii) \( f \) is **level-reduced** if \( f(x) \) and \( f(y) \) are \( \text{Sp-} \)connected in \( Y \) for any \( x, y \in X \) which are \( \text{Sp-} \)disconnected and of the same lengths in \( X \).
(iii) \( f \) is **level-mixed** if \( f \) is neither level-separated nor level-reduced.

**Remark 3.6.** Let \( f : X \to Y \) be a morphism of schemes.
(i) Let \( \dim X > 0 \). Then \( \|f\| = 1 \) if \( f \) is length-preserving.
(ii) Let \( \dim X = \dim Y < \infty \). Then \( \|f\| \geq 1 \) if \( f \) is surjective.
(iii) Let \( \|f\| = 1 \). In general, it is not true that \( f \) is injective since there exists a scheme \( X \) which can not be totally ordered by specializations.

**Theorem 3.7.** Let \( X \) and \( Y \) be irreducible schemes, and \( f : X \to Y \) be a morphism satisfying Condition \((*)\). Suppose \( \dim Y < \infty \). Then \( f \) is injective if and only if \( f \) is length-preserving and level-separated.

**Proof.** Prove \( \implies \). Assume that \( f \) is injective. As \( \dim Y < \infty \), we have \( \dim X < \infty \) by Proposition 1.3.

Show \( f \) is length-preserving. Take any restrict series of specializations in \( X \)
\[
z_1 \to z_2 \to \cdots \to z_n.
\]
We have
\[
f(z_1) \to f(z_2) \to \cdots \to f(z_n)
\]
in \( Y \). As \( f \) is injective, we get \( f(z_i) \neq f(z_j) \) for all \( i \neq j \); then
\[
l(z_1, z_n) = l(f(z_1), f(z_n))
\]
holds for any specialization \( z_1 \to z_n \) in \( X \) which is of finite length. As \( \dim X = l(X) < \infty \),
it is seen that
\[
l(x, y) = l(f(x), f(y))
\]
holds for any $x \to y$ in $X$.

Show $f$ is level-separated. Take any $x_1, x_2 \in X$ which are $Sp$–disconnected and of the same lengths, that is, $x_1 \neq x_2$ and $l(x_1) = l(x_2)$; then $f(x_1)$ and $f(x_2)$ are $Sp$–disconnected; otherwise, if $f(x_1) \to f(x_2)$ in $Y$, we have

$$Sp(f(x_1)) = Sp(f(x_2))$$

since

$$l(f(x_1)) = l(f(x_2)) \leq l(Y) = \dim Y < \infty;$$

it follows that $f(x_1) = f(x_2)$ holds, which is in contradiction with the assumption.

Prove $\iff$ . Conversely, suppose that $f$ is length-preserving and level-separated. We have $\dim X < \infty$. In deed, if $\dim X = \infty$, we will obtain $\dim Y = \infty$ since $f$ is length-preserving.

Take any $x, y \in X$. We will prove $f(x) \neq f(y)$ if $x \neq y$.

Let $\xi$ be the generic point of $X$. There are three cases.

Case (i) : Let $\dim X = 0$. Then $x = y$ and $f$ is injective.

Case (ii) : Let $\dim X > 0$ and $x = \xi$.

We have $y \neq \xi$ and $x \to y$; then $l(x, y) > 0$; as $f$ is length-preserving, it is seen that

$$l(f(x), f(y)) = l(x, y) > 0.$$ 

Hence, $f(x) \neq f(y)$.

Case (iii) : Let $\dim X > 0$, $x \neq \xi$, and $y \neq \xi$. As $\dim X < \infty$, we have $l(z) \leq l(X) < \infty$ for any $z \in X$. There are several subcases.

If $l(x) = l(y)$ and $y \in Sp(x)$ (or $x \in Sp(y)$, respectively), we have $x = y$ and then $f(x) = f(y)$.

If $l(x) = l(y)$, $y \not\in Sp(x)$, and $x \not\in Sp(y)$ hold, we have $f(y) \not\in Sp(f(x))$ and then $f(x) \neq f(y)$ since $f$ is level-separated.

If $l(x) > l(y)$ and $y \in Sp(x)$ hold, we have

$$l(f(x), f(y)) = l(x, y) > 0$$

since $f$ is length-preserving; hence, $f(x) \neq f(y)$.

Now suppose $l(x) > l(y)$ and $y \not\in Sp(x)$ without loss of generality. It is seen that $x, y$ are $Sp$–disconnected. Taking a presentation for the length $l(x)$, we have $x_0 \in Sp(x)$ such that $l(x_0) = l(y) < \infty$. Then $l(x, x_0) > 0$ and $x_0 \neq y$. As $f$ is level-separated, we have $f(x_0) \neq f(y)$.

As $l(x_0) = l(y)$, we have $l(\xi, x_0) = l(\xi, y)$ by taking presentations for the lengths; as

$$l(\xi, x_0) = l(\xi, x) + l(x, x_0)$$

holds, we have

$$l(\xi, y) = l(\xi, x) + l(x, x_0).$$
Hence, 
\[ l(f(x), f(x_0)) = l(x, x_0) > 0 \]
and 
\[ l(f(\xi), f(y)) = l(f(\xi), f(x_0)) = l(f(\xi), f(x)) + l(f(x), f(x_0)) < \infty. \]

We must have \( f(x) \neq f(y) \); otherwise, if \( f(x) = f(y) \), there is 
\[ l(f(\xi), f(x)) = l(f(\xi), f(x)) + l(f(x), f(x_0)); \]
then \( l(f(x), f(x_0)) = 0 \), where there will be a contradiction. \( \square \)

**Corollary 3.8.** Let \( f : X \to Y \) be a morphism of schemes. Then \( \|f\| = 1 \) if \( f \) is injective and satisfies Condition (\( \ast \)).

**Proof.** As every ideal is contained in a maximal ideal in a commutative ring, we can take an irreducible open subspace \( X_0 \) of \( X \) such that \( l(X_0) < \infty \). Then we have \( \|f|_{X_0}\| = 1 \); as \( \|f\| \leq 1 \) by Theorem 3.3, we get \( \|f\| = 1 \). \( \square \)

4. **An Application of Specializations**

**Definition 4.1.** (i) Let \( R, S \) be commutative rings with 1. A homomorphism \( \tau : R \to S \) is said to be of \( J- \) type if \( \tau^{-1}(\tau(I)S) = I \) holds for every prime ideal \( I \) in \( R \).

(ii) Let \( X, Y \) be schemes. A morphism \( f : X \to Y \) is said to be of **finite \( J- \) type** if \( f \) is of finite type and the induced homomorphism 
\[ f^\# \mid_V : O_Y(V) \to f_*O_X(U) \]
is of \( J- \) type for any affine open sets \( V \) of \( Y \) and \( U \) of \( f^{-1}(V) \).

**Proposition 4.2.** Let \( X \) and \( Y \) be irreducible schemes, and \( f : X \to Y \) be a morphism. Then we have \( \dim X = \dim Y \) if \( f \) is length-preserving and of finite \( J- \) type.

**Proof.** Let \( f \) be length-preserving and of finite \( J- \) type. It follows that 
\[ l(X) = l(f(X)) \leq l(Y) \]
hold. Then we have \( \dim X \leq \dim Y \) since 
\[ \dim X = l(X) \text{ and } l(Y) = \dim Y. \]

Take any \( x \in X \) and \( y = f(x) \in Y \). As \( f \) is of finite \( J- \) type, there are affine open subsets \( V \) of \( Y \) and \( U \) of \( f^{-1}(V) \) such that 
\[ f^\# \mid_V : O_Y(V) \to f_*O_X(U) \]
is a homomorphism of \( J- \) type, where \( x \in U \) and \( y \in V \).
Set $V = \text{Spec}(R)$ and $U = \text{Spec}(S)$. As $X$ and $Y$ are irreducible, we have

$$\dim U = \dim X \text{ and } \dim V = \dim Y.$$  

Take any restrict series of specializations

$$y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n$$

in $V$. Then we obtain a chain of prime ideals

$$j_{y_0} \subsetneq j_{y_1} \subsetneq \cdots \subsetneq j_{y_n}$$

in $R$, where each $j_{y_i}$ is the prime ideal in $R$ corresponding to $y_i$ in $V$. By Corollary 2.3 of [5] there are prime ideals

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n$$

in $S$ such that $f^{#-1}(I_i) = j_{y_i}$.

Hence, we obtain a restrict series of specializations

$$x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$$

in $U$ such that $f(x_i) = y_i$ and $j_{x_i} = I_i$. This proves $l(U) \geq l(V)$.

As

$$\dim X = l(X) \geq l(U)$$

and

$$l(V) \geq \dim V = \dim Y;$$

we have

$$\dim X \geq \dim Y.$$

This completes the proof. $\square$
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