QCD$_4$ Glueball Masses from AdS$_6$
Black Hole Description

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Abstract

By using the generalized version of gauge/gravity correspondence, we study the mass spectra of several typical QCD$_4$ glueballs in the framework of AdS$_6$ black hole metric of Einstein gravity theory. The obtained glueball mass spectra are numerically in agreement with those from the AdS$_7 \times S^4$ black hole metric of the 11-dimensional supergravity.

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The idea that the non-perturbative aspects of 4-dimensional QCD may have a dual description in terms of supergravity limit of string theories is perhaps among the most appealing concepts in particle physics. With the advent of D-branes[1], the AdS/CFT correspondence[2] and then the thermal mechanism for breaking supersymmetry and conformal symmetry[3], studies have been intensified for phenomenologies of this expanded holographic principle in recent years. An approach to QCD from the gauge/gravity correspondence is to estimate glueball masses of QCD (in the large \( N \) limit) from supergravity models in higher dimensions. Although the supergravity approximation is not satisfactory for a serious study of large \( N \) QCD, it is surprising that the mass spectra of QCD glueballs obtained from the supergravity dual descriptions have qualitatively coincided with those from lattice QCD calculations[4, 5, 6, 7, 8, 9].

There have basically been two kinds of approaches for calculating QCD\(_4\) glueball masses from the conjectured gauge/gravity correspondence. The first one consists of starting from the dilaton-free truncations of the 11-dimensional supergravity and breaking supersymmetries with different compactifications for bosonic and fermionic degrees of freedom[3, 6]. This involves at least one compact space-like dimension (the so-called thermal circle \( S^1 \)) where anti-periodic boundary conditions are assumed for the fermionic fields while the bosonic fields remain periodic. It leads to a AdS\(_7\) × S\(^4\) AdS-Schwarzschild black hole metric that can be related to QCD\(_4\). The second one begins with a slice of the AdS\(_5\) metric of 5-dimensional Einstein gravity theory[7]. The brane on which QCD\(_4\) fields and interactions are constrained is assumed to be a 4-dimensional boundary of this AdS\(_5\) slice.

With the perspective of the first kind of approaches, it appears more natural to conjecture that the dual description of the QCD\(_4\) should be such a gravity theory that is only involved six dimensions of spacetime. Besides having the 4-dimensional spacetime on which the QCD\(_4\) interactions lie, it has to have a thermal circle and an extra dimension. Based on this understanding, in this paper we intend to calculate the large \( N \) QCD\(_4\) glueball masses from the AdS\(_6\) AdS-Schwarzschild black hole metric. The AdS\(_6\) AdS-Schwarzschild black hole metric is difficult to be identified as a non-BPS brane solution of Type II supergravity, however, it is an exact solution of 6-dimensional Einstein gravity theory with non-vanishing cosmological constant[8, 10]. In view of the general holographic principle between gauge theories and gravity[11], at least in estimating the glueball masses of QCD\(_4\), we conjecture that AdS\(_6\) AdS-Schwarzschild black hole metric provides a probable dual description of...
QCD\(_4\) gauge theory.

Let us begin with the near-horizon AdS\(_6\) AdS-Schwarzschild black hole metric,

\[ ds^2 = (r^2 - \frac{1}{r^3})d\tau^2 + (r^2 - \frac{1}{r^3})^{-1}dr^2 + r^2 \sum_{i=1}^{4} dx_i^2. \]

The 6-dimensional bulk is described by a radial coordinate \( r \) and a 4-dimensional Euclidean space-time coordinates \( x_i \) \((i = 1, 2, 3, 4)\), a radial coordinate \( r \), and the thermal coordinate \( \tau \). The 4-dimensional Euclidean space-time is assumed to be an effective 3-brane on which the QCD\(_4\) glueballs live (we regard \( x_4 \) as the time coordinate). This metric defines a horizon at \( r = 1 \) so that the physically relevant region is within the region \( 1 \leq r < \infty \). The thermal coordinate \( \tau \) is necessary to be periodic in order to avoid a conical singularity at the horizon and break conformal symmetries. In writing Eq. (1) we have adopted the units \( R_{AdS} = 1 \) so that the period of \( \tau \) is \( \beta = 4\pi/5 \) [3, 10]. If the theory is embedded into some supergravity/superstring framework, we have to impose further the anti-periodic boundary conditions for fermionic degrees of freedom to break the probable Supersymmetries. The gravitational fluctuations \( h_{MN} \) are defined by,

\[ \bar{g}_{MN} = g_{MN} + h_{MN} \]

where \( g_{MN} \) denotes the AdS\(_6\) background metric [11] which is a solution of six dimensional Einstein field equation with a negative cosmological constant:

\[ R_{MN} = -5g_{MN} \]

By the generalized gauge/gravity conjecture, the gravitational fluctuations in AdS\(_6\) bulk are nothing but the glueballs of the large \( N \) QCD\(_4\) gauge theory on the 4-dimensional boundary \( x_i \) \((i = 1, \cdots, 4)\). For simplicity these metric perturbations have no dependence on the spatial coordinates \( x_i \) \((i = 1, 2, 3)\) and compactified thermal coordinate \( \tau \). Because glueballs are free particles at \( N \to \infty \), we make an ansatz \( h_{MN} = H_{MN}(r)e^{-mx_4} \) where \( H_{MN}(r) \) is the radial profile tensor and \( m \) is the mass of the corresponding QCD\(_4\) glueball. It follows from Einstein equation \( \bar{R}_{MN} = -5\bar{g}_{MN} \) that the metric perturbations \( h_{MN} \) satisfy the equations of motion:

\[ \frac{1}{2} \nabla_M \nabla_N h_L^L + \frac{1}{2} \nabla^2 h_{MN} - \nabla^L \nabla_{(M} h_{N)L} - 5h_{MN} = 0 . \]
They will be solved as an eigenvalue problem for determining the mass $m$ of QCD$_4$ glueballs.

As a tensor in 6-dimensional bulk the metric perturbation $h_{MN}$ has a variety of polarizations (with total number 21). However, not all of these polarizations are independent. Relying on the diffeomorphism invariance of the gravity theory under the infinitesimal transformation of the coordinates $x^M \rightarrow x'^M = x^M + \epsilon^M(x)$, the equations of motion of the perturbations are symmetric under the following "gauge transformations":

$$h'_{MN}(x) = h_{MN}(x) - g_{ML}(x)\partial_N\epsilon^L(x) - g_{NL}(x)\partial_M\epsilon^L(x) - \epsilon^L(x)\partial_L g_{MN}(x).$$

Consequently, there are 12 components non-independent in the metric perturbation tensor. As a tensor in 6-dimensional bulk the metric perturbation $h_{MN}$ is symmetric under the following "gauge transformations":

$$h'_{MN}(x) = h_{MN}(x), \quad \forall M.$$ (5)

The other components are expressed as,

$$H_{rr}(r) = S_1(r), \quad H_{rr}(r) = A(r), \quad H_{ri}(r) = \sqrt{r^4 - \frac{1}{r}V_i(r)}, \quad H_{ii}(r) = [r^2 T_1(r) + S_3(r)], \quad H_{12}(r) = r^2 T_3(r), \quad H_{13}(r) = r^2 T_4(r), \quad H_{22}(r) = [r^2 T_2(r) + S_3(r)], \quad H_{23}(r) = r^2 T_5(r), \quad H_{33}(r) = [S_3(r) - r^2 T_1(r) - r^2 T_2(r)].$$ (7)

Substitution of these ansatz into Eq.(5) leads to $A(r) = B_i(r) (i = 1, 2, 3) = 0$. Besides,

$$r(r^5 - 1)T''_i(r) + (6r^5 - 1)T'_i(r) + m^2 r^2 T_i(r) = 0,$n$$r(r^5 - 1)V''_j(r) + (6r^5 - 1)V'_j(r) + [m^2 r^2 - 25/4r(r^5 - 1)]V_j(r) = 0.$$(8)

for $(i = 1, 2, 3, 4, 5)$ and $(j = 1, 2, 3)$ and where $T'_i(r) = \partial_r T_i(r)$, etc. From the perspective of the boundary QCD$_4$, the former five independent perturbations $T_i(r) (i = 1, 2, \cdots, 5)$ form the spin-2 representation of algebra.
SO(3) and the latter three perturbations \( V_j(r) \) \((j = 1, 2, 3)\) form the spin-1 representation. These perturbations do respectively represent a QCD 4 tensor glueball and a vector glueball. The remaining perturbations are expected to form the spin-0 representation of \( SO(3) \) so that a dual description of the QCD 4 scalar glueball arises. As expected, only \( S_3(r) \) in the remaining fluctuations \( S_i(r) \) \((i = 1, 2, 3)\) is independent, which satisfies a third order differential equation:

\[
\begin{align*}
    &r^3(r^5 - 1)^2[4(r^5 - 1) + m^2 r^3]S''_3(r) + r^2(r^5 - 1)[8(r^5 - 1)(2r^5 + 3) \\
    &+ m^2 r^3(2r^5 - m^2 r^3 + 13)]S''_3(r) - r[16(r^5 + 4)(r^5 - 1)^2 \\
    &+ m^2 r^3(4r^{10} + m^2 r^8 - 3r^5 + 4m^2 r^3 - 26)]S'_3(r) + [80(r^5 - 1)^2 \\
    &+ m^2 r^3(4r^{10} + 2m^2 r^8 - m^4 r^6 - 28r^5 - 2m^2 r^3 - 26)]S_3(r) = 0 .
\end{align*}
\]

(9)

Once \( S_3(r) \) is known, the other two perturbations can be determined by the following equations:

\[
\begin{align*}
    &4r^3(2r^5 + 3)(r^5 - 1)^2S'''_3(r) + r(r^5 - 1)[4(2r^5 + 3)^2 - 15m^2 r^3]S'_3(r) \\
    &- 2(r^5 - 1)[4(3r^5 + 2)(2r^5 + 3) + m^2 r^3(8r^5 + 3m^2 r^3 - 3)]S_3(r) \\
    &- 2m^2 r^8[4(r^5 - 1) + m^2 r^3]S_1(r) = 0 ,
\end{align*}
\]

(10)

and

\[
\begin{align*}
    &15r^2(r^5 - 1)S''_3(r) + 6r(r^5 - 1)(m^2 r^3 - 5)S_3(r) \\
    &- 2r^6(10 - m^2 r^3)S_1(r) - 4(2r^5 + 3)(r^5 - 1)^2S_2(r) = 0 .
\end{align*}
\]

(11)

It is difficult in general to solve a differential equation of the order higher than two. However, in the considered case, the “gauge symmetry” of Eq.(9) enable us to reduce the order of this equation for determining the QCD 4 scalar glueball masses to two. Our strategy is to find another gauge in which \( H_{44}(r) \) does still remain vanishing. Choosing the (nonzero) gauge transformation parameters as follows:

\[
\begin{align*}
    &\epsilon'(r) = S_3(r)/2r , \quad \epsilon^i(r) = S_3(r)/2mr^2 ,
\end{align*}
\]

(12)

we get from ansatz (7) the expressions of the metric perturbations in the new gauge:

\[
\begin{align*}
    &H'_{11}(r) = S_1(r) - (1 + 3/2r^5)S_3(r) , \\
    &H'_{12}(r) = S_2(r) + r[4(r^5 + 1)S_3(r) - 2r(r^5 - 1)S'_3(r)]/2(r^5 - 1)^2 , \\
    &H'_{14}(r) = -S'_3(r)/2m + [2(r^5 - 1) + m^2 r^3]S_3(3)/2mr(r^5 - 1) , \\
    &H'_{22}(r) = r^2T_2(r) , \\
    &H'_{33}(r) = -r^2T_1(r) - r^2T_2(r) .
\end{align*}
\]

(13)
The other perturbations have the same expressions as those in the transverse
gauge. These perturbations in the new gauge are in fact some special com-
binations of the old ones and their derivatives (with respect to coordinate $r$)
in transverse gauge. We expect that they may satisfy the conventional two
order equations. By a tedious but straightforward calculation, we find that
$H_{\tau\tau}'(r)$ is indeed subject to a two order differential equation:

$$
r(r^5 - 1)\frac{d^2}{dr^2} S(r) + (6r^5 - 1)\frac{d}{dr} S(r) + \left[ m^2 r^2 + \frac{750r^4}{(8r^5 - 3)^2} \right] S(r) = 0 \quad (14)
$$

where,

$$
S(r) := r^3 H_{\tau\tau}'(r)/(8r^5 - 3) = r^3 [S_1(r) - (1 + 3/2r^5)S_3(r)]/(8r^5 - 3) . \quad (15)
$$

We now calculate the discrete mass spectrum for these three kinds of
 glueballs. It follows from Eqs. (8) and (14) that the asymptotic behavior of
their linear independent solutions at $r \to \infty$ are

$$
T_i(r), V_j(r), S(r) \sim r^{-5}, \quad \forall i, j. \quad (16)
$$

and at $r = 1$ are

$$
\begin{align*}
& T_i(r), \quad S(r) \sim 1, \quad \log(r - 1), \\
& V_j(r) \sim \sqrt{r - 1}, \quad \sqrt{r - 1}\log(r - 1), \quad \forall i, j.
\end{align*} \quad (17)
$$

In all cases the reasonable boundary conditions at $r = 1$ are the ones without
the logarithmic singularity. At $r \to \infty$ the singular asymptotic behavior
is necessary for having a normalizable eigenstate. The expected boundary
conditions which are also compatible with the field equations read

$$
\begin{align*}
& T_i(r_1) = 1, \quad T_i'(r_1) = -m^2/5 ; \\
& V_j(r_1) = 0, \quad V_j'(r_1) \to \infty ; \\
& S(r_1) = 1, \quad S'(r_1) = -6 - m^2/5 .
\end{align*} \quad (18)
$$

at $r_1 = 1$ and

$$
\lim_{r \to \infty} T_i(r) = \lim_{r \to \infty} V_j(r) = \lim_{r \to \infty} S(r) = 0, \quad \forall i, j . \quad (19)
$$

at $r_\infty \simeq \infty$. Generically there are no solutions to Eqs. (8) and (15) that not
only satisfy these boundary conditions but can also be represented as series
expansions convergent throughout the whole physical region $1 \leq r < \infty$. In fact, the series expansions of solutions

$$
T^{(\infty)}_i(r) = \frac{1}{r^5} + \sum_{n=1}^{\infty} a^{(\infty)}_n \frac{1}{r^{5+n}}, \quad V^{(\infty)}_i(r) = \frac{1}{r^5} + \sum_{n=1}^{\infty} b^{(\infty)}_n \frac{1}{r^{5+n}},
$$

$$
S^{(\infty)}(r) = \frac{1}{r^5} + \sum_{n=1}^{\infty} c^{(\infty)}_n \frac{1}{r^{5+n}},
$$

(20)

do converge in the region $I(\infty) = \{ r \mid 1 < r < \infty \}$, while the series expansions

$$
T^{(1)}_i(r) = 1 + \sum_{n=1}^{\infty} a^{(1)}_n (r - 1)^n, \quad V^{(1)}_i(r) = (r - 1)^{\frac{3}{2}} + \sum_{n=1}^{\infty} b^{(1)}_n (r - 1)^{n+\frac{1}{2}},
$$

$$
S^{(1)}(r) = 1 + \sum_{n=1}^{\infty} c^{(1)}_n (r - 1)^n,
$$

(21)

are convergent only in the neighborhoods of the horizon $I_T(1) = \{ r \mid 0 \leq (r - 1) < 1 \}$, $I_V(1) = \{ r \mid 0 \leq (r - 1) < 1 \}$ and $I_S(1) = \{ r \mid 0 \leq (r - 1) < 1 \}$ and $I_S(1) \approx 0.177$. The expansion coefficients are associated with the squared masses of glueballs. Therefore, coincidence of these solutions in the overlap, see $I(\infty) \cap I_T(1)$, yields a discrete set of eigenvalues $m^2_n$, where $n$ is the number of zeros of the following vanishing Wronski determinant:

$$
\begin{vmatrix}
T^{(\infty)}_i(r_0) & (T^{(\infty)}_i)'(r_0) \\
T^{(1)}_i(r_0) & (T^{(1)}_i)'(r_0)
\end{vmatrix} = 0,
$$

(22)

for an arbitrary point $r_0 \in I(\infty) \cap I_T(1)$. In numerical calculations this $r_0$ should be chosen far away from the endpoints of the regions $I_T(1)$, $I_V(1)$ and $I_S(1)$ because we have to truncate the series expansions of (20) and (21) into some polynomials. As a check to the above calculation, we have also solved these boundary problems by an independent Runge-Kutta method in which the boundary conditions play crucial roles. It turns out that the calculation results from these two schemes coincide. The obtained mass spectrum of the QCD$_4$ glueballs (in units $R_{AdS} = 1$) are presented in Table 1.

The spectrum of these glueball masses is quantitative in agreement with those from the AdS$_7$-BH dual theory which is presented in Table 2. In particular, both approaches indicate that the mass gap obeys an inequality $m(0^{++}) < m(2^{++}) < m(1^{-+})$, and the lowest mass scalar comes from the gravitational multiplet. By consulting with the available lattice spectrum
|   | $S$ ($0^{++}$) | $V$ ($1^{-+}$) | $T$ ($2^{++}$) |
|---|---|---|---|
| $n=0$ | 2.52 | 5.00 | 4.06 |
| $n=1$ | 6.20 | 7.73 | 6.69 |
| $n=2$ | 8.93 | 10.34 | 9.25 |
| $n=3$ | 11.54 | 12.91 | 11.79 |
| $n=4$ | 14.11 | 15.45 | 14.31 |

Table 1: QCD$_4$ Glueball Mass Spectrum $m_n$ from AdS$_6$-BH Approach

|   | $S$ ($0^{++}$) | $V$ ($1^{-+}$) | $T$ ($2^{++}$) |
|---|---|---|---|
| $n=0$ | 7.308 | 31.985 | 22.097 |
| $n=1$ | 46.986 | 72.489 | 55.584 |
| $n=2$ | 94.485 | 126.174 | 102.456 |
| $n=3$ | 154.981 | 193.287 | 162.722 |
| $n=4$ | 228.777 | 273.575 | 236.400 |

Table 2: QCD$_4$ Glueball Mass-squared $m^2_n$ from AdS$_7$-BH Approach

for pure SU(3) QCD$_4$ [9], the qualitative agreement is also good. Here we have to emphasize that it is such a qualitative agreement that is important. Because the supergravity approximation to the gauge/gravity duality is in a “wrong phase” [3, 10], a reasonable estimation based on it should not be expected to give reliable quantitative results. Consequently, the conjectured duality between QCD$_4$ and AdS$_6$ supergravity works well in estimating the mass gap of nonperturbative QCD$_4$.

In conclusion, we have seen that the AdS$_6$ black hole metric of a 6-dimensional Einstein gravitational theory can also be applied to estimate QCD$_4$ glueball mass spectrum, even though it is difficult to be identified as a nonextremal bane solution of the Type II supergravity. It would be very important to understand whether this agreement is purely a numerical coincidence or whether there is a deeper mechanism behind it. It is also important to know whether this conjectured AdS$_6$ black hole dual description can be used to describe other non-perturbative properties of QCD$_4$. 
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