On star edge colorings of bipartite and subcubic graphs

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A B S T R A C T

A star edge coloring of a graph is a proper edge coloring with no 2-colored path or cycle of length four. The star chromatic index $\chi'_s(G)$ of $G$ is the minimum number $t$ for which $G$ has a star edge coloring with $t$ colors. Star edge coloring was recently introduced by Liu and Deng [14], motivated by the vertex coloring version, see e.g. [1,5]. This notion is intermediate between acyclic edge coloring, where every two-colored subgraph must be acyclic, and strong edge coloring, where every color class is an induced matching.

Dvořák et al. [4] studied star edge colorings of complete graphs and obtained the currently best upper and lower bounds for the star chromatic index of such graphs. A fundamental open question here is to determine whether $\chi'_s(K_n)$ is linear in $n$. Bezegová et al. [2] investigated star edge colorings of trees and outerplanar graphs. Lei et al. [11] proved that it is NP-complete to determine if a graph $G$ satisfies that $\chi'_s(G) \leq 3$. Wang et al. [19,20] quite recently obtained some upper bounds on the star chromatic index of graphs with maximum degree four, and also for some families of planar and related classes of graphs. Some further results on star edge colorings of subcubic (i.e. with maximum degree at most three) and sparse graphs appear in [6,8–10,12,15]. Besides these results, very little is known about star edge colorings.

In this paper, we primarily consider star edge colorings of bipartite graphs. As for complete graphs, a fundamental problem for complete bipartite graphs is to determine whether the star chromatic index is a linear function on the number of vertices. We determine the star chromatic index of complete bipartite graphs where one part has size at most 3, and obtain some bounds on the star chromatic index for larger complete bipartite graphs. Note that the complete bipartite graph $K_{3,4}$ requires exactly $rs$ colors for a strong edge coloring; indeed it has been conjectured [3] that any bipartite graph where the parts have maximum degrees $\Delta_1$ and $\Delta_2$, respectively, has a strong edge coloring with $\Delta_1 \Delta_2$ colors. As we shall see, for star edge colorings the situation is quite different.

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Furthermore, we study star chromatic index of bipartite graphs where the vertices in one part all have small degrees. Nakprasit [17] proved that if \( G \) is a bipartite graph where the maximum degree of one part is 2, then \( G \) has a strong edge coloring with \( 2\Delta(G) \) colors. Here we obtain analogous results for star edge colorings: we obtain a sharp upper bound for the star chromatic index of a bipartite graph where one part has maximum degree two.

Finally, we consider the following conjecture first posed in [4].

**Conjecture 1.1.** If \( G \) has maximum degree at most 3, then \( \chi''_s(G) \leq 6 \).

Dvořák et al. [4] proved a slightly weaker version of Conjecture 1.1, namely that \( \chi''_s(G) \leq 7 \) if \( G \) is subcubic; Bezegová et al. established [2] that Conjecture 1.1 holds for all trees and outerplanar graphs, while it is still open for e.g. planar graphs. In this paper we verify that the conjecture holds for some families of graphs with maximum degree three, namely bipartite graphs where one part has maximum degree 2, cubic Halin graphs and another family of planar graphs.

2. Bipartite graphs

In this section we consider star edge colorings of bipartite graphs. For an edge coloring \( f \) of a graph \( G \) and a vertex \( u \) of \( G \), we shall denote by \( f(u) \) the set of colors of all edges incident with \( u \).

We first consider complete bipartite graphs. Trivially \( \chi''_s(K_{1,d}) = d \), and it is straightforward that \( \chi''_s(K_{2,2}) = 3 \). For general complete bipartite graphs where one part has size 2, we have the following easy observation.

**Proposition 2.1.** For the complete bipartite graph \( K_{2,d}, d \geq 2 \), we have \( \chi''_s(K_{2,d}) = 2d - \left\lceil \frac{d}{2} \right\rceil \).

**Proof.** Suppose \( K_{2,d} \) has parts \( X \) and \( Y \), where \( X = \{x_1, x_2\} \) and \( Y = \{y_1, \ldots, y_d\} \).

If \( x_1 \) and \( x_2 \) have at least \( \lceil d/2 \rceil + 1 \) common colors on their incident edges, say \( 1, \ldots, \lceil d/2 \rceil + 1 \), then there is at least one vertex in \( Y \) which is incident with two edges both of which have colors from \( \{1, \ldots, \lceil d/2 \rceil + 1\} \); this implies that there is a 2-colored \( P_4 \) or \( C_4 \) in \( K_{2,d} \). Hence, there are at least \( 2d - \lfloor d/2 \rfloor \) distinct colors in a star edge coloring of \( K_{2,d} \).

To prove the upper bound, we give an explicit star edge coloring \( f \) of \( K_{2,d} \). We set \( f(x_i y_i) = i \), for \( i = 1, \ldots, d \), and

\[
f(x_3 y_i) = d - i + 1, \quad i = 1, \ldots, \lfloor d/2 \rfloor, \quad f(x_2 y_i) = d + 1, \quad i = \lceil d/2 \rceil, \ldots, d.
\]

The coloring \( f \) is a star edge coloring using exactly \( 2d - \left\lceil \frac{d}{2} \right\rceil \) colors. \( \square \)

Wang et al. [20] proved that \( \chi''_s(K_{3,4}) = 7 \), and it is known that \( \chi''_s(K_{3,3}) = 6 \) [4]. Using Proposition 2.1, we can prove the following.

**Theorem 2.2.** For the complete bipartite graph \( K_{3,d}, d \geq 5 \), it holds that \( \chi''_s(K_{3,d}) = 3 \left\lceil \frac{d}{2} \right\rceil \).

**Proof.** Let us first consider the case when \( d \) is even. The lower bound \( \frac{3d}{2} \) follows immediately from Proposition 2.1, so let us turn to the proof of the upper bound. We shall give an explicit star edge coloring of \( K_{3,d} \) using \( \frac{3d}{2} \) colors.

Let \( X \) and \( Y \) be the parts of \( K_{3,d} \), where \( X = \{x_1, x_2, x_3\}, Y = U \cup V, U = \{u_1, \ldots, u_k\}, V = \{v_1, \ldots, v_k\} \), and \( d = 2k \).

We define a star edge coloring \( f \) by setting

\[
f(x_1 u_i) = i, \quad f(x_2 u_i) = i + k, \quad f(x_3 u_i) = i + 2k, \quad i = 1, \ldots, k,
\]

and

- \( f(x_1 v_i) = i + k, \quad i = 1, \ldots, k, \)
- \( f(x_2 v_i) = i + 2k + 1, \quad i = 1, \ldots, k - 1, \quad f(x_2 v_k) = 2k + 1, \)
- \( f(x_3 v_i) = i + 2, \quad i = 1, \ldots, k - 2, \quad f(x_3 v_{k-1}) = 1, \quad f(x_3 v_k) = 2. \)

Clearly, \( f \) is a proper edge coloring of \( K_{3,2k} \) with \( 3k \) colors. Suppose that \( K_{3,2k} \) contains a 2-edge-colored path or cycle \( F \) with four edges. Let us prove that \( F \) does not contain two edges \( e_1 \) and \( e_2 \) incident with the same vertex from \( X \), say \( x_1 \), and two other edges \( e_3 \) and \( e_4 \) incident with another vertex from \( X \), say \( x_2 \). Then, since the restriction \( f' \) of \( f \) to the subgraph induced by \( X \cup U \) satisfies \( f'(x_1) \cap f'(x_2) = \emptyset \) (and similarly for the subgraph induced by \( X \cup V \)), we may assume that \( f(e_1) \in \{1, \ldots, k\} \) and \( f(e_2) \in \{k+1, \ldots, 2k\} \). However, no edge incident with \( x_2 \) is colored by a color from \( \{1, \ldots, k\} \), which contradicts that \( F \) is 2-edge-colored.

Suppose now that there is a 2-edge-colored path \( F \) on 4 edges, where exactly two edges are incident to the same vertex from \( X \), say \( x_2 \). Then, as before, we may assume that the two edges \( e_2 \) and \( e_3 \) of \( F \) that are incident with \( x_2 \) satisfy that \( f(e_2) \in \{2k+1, \ldots, 3k\} \) and \( f(e_3) \in \{k+1, \ldots, 2k\} \). This means that the edge \( e_1 \) of \( F \) that is incident with \( x_1 \) must satisfy \( f(e_1) \in \{k+1, \ldots, 2k\} \), and so \( f(e_1) = f(e_3) \). Thus, for the edge \( e_4 \) of \( F \) incident with \( x_1 \) it holds that \( f(e_4) \in \{2k+1, \ldots, 3k\} \) and \( f(e_4) = f(e_2) \). However, by the construction of \( f \) we have that \( f(e_4) = f(e_2) - 1 \) (except if \( f(e_2) = 2k+1 \), which implies that \( f(e_4) = 3k \)); this contradicts that \( F \) is 2-edge-colored.

Let us now consider the case when \( d \) is odd; suppose \( d = 2k+1 \). The upper bound follows immediately from the even case, since \( K_{3,2k+1} \) is a subgraph of \( K_{3,2k+2} \). Let us prove the lower bound.

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Let $X$ and $Y$ be the parts of $K_{3,d}$, where $X = \{x_1, x_2, x_3\}$ and consider a star edge coloring of $K_{3,d}$. Since the subgraph induced by $\{x_1, x_2\} \cup Y$ is isomorphic to $K_{2,d}$, there are at most $k$ colors which can appear at both $x_1$ and $x_2$. Since the same holds for $x_1$ and $x_3$, and $x_2$ and $x_3$, it follows that we need at least $3k + 3$ colors for a star edge coloring of $K_{3,d}$. □

Next, we consider complete bipartite graphs where the parts have size at least 4. Let us first establish a lower bound on the star chromatic index of such graphs.

**Proposition 2.3.** For the complete bipartite graph $K_{4,d}$ ($d \geq 4$) it holds that $\chi_{st}'(K_{4,d}) \geq \frac{5d}{2}$.

**Proof.** Let $X$ and $Y$ be the parts of $K_{4,d}$, where $X = \{x_1, x_2, x_3, x_4\}$, and let $C_i$ be the set of colors used on the edges incident with the vertex $x_i$. From the argument in the proof of Proposition 2.1, we can conclude that the sets $C_i$ may overlap in at most $d/2$ colors. Thus the optimization problem

$$
\text{minimize } \left| \bigcup_{i=1}^{4} C_i \right|
$$

subject to $|C_i| = d$, $i = 1, 2, 3, 4$

$$
|C_i \cap C_j| \leq \frac{d}{2}, \quad i, j = 1, 2, 3, 4, \quad i \neq j
$$

gives a lower bound on $\chi_{st}'(K_{4,d})$. A linear integer program description of this problem is given in Appendix A. Solving the linear relaxation of this problem gives the desired lower bound. □

For $d \leq 12$, the values of $\chi_{st}'(K_{4,d})$ are given in Table 1. Explicit colorings realizing these values are given in Appendix B, and for all values of $d$ except $d = 6$ they can be proved optimal by solving the optimization problem in the proof of Proposition 2.3. For the case $d = 6$, a computer search showing that the value in Table 1 is optimal has been conducted.

The fact that $\chi_{st}'(K_{4,12}) = 20$ can be used to derive a general upper bound on the star chromatic index of complete bipartite graphs with four vertices in one part.

**Proposition 2.4.** For the complete bipartite graph $K_{4,d}$ it holds that $\chi_{st}'(K_{4,d}) \leq 20 \left\lceil \frac{d}{12} \right\rceil$.

**Proof.** Set $s = \left\lceil \frac{d}{12} \right\rceil$. The proposition follows easily by decomposing $K_{4,d}$ into $s$ copies $H_1, \ldots, H_i$ of $K_{4,12}$ and possibly one copy $J$ of a complete bipartite graph $K_{4,a}$, where $1 \leq a \leq 11$, and then using disjoint sets of colors for star edge colorings of each of the graphs $H_1, \ldots, H_i$ and $J$; these star edge colorings together form a star edge coloring of $K_{4,d}$. □

Note that it follows from Proposition 2.3 that the upper bound in the preceding proposition is in fact sharp for an infinite number of values of $d$.

Using computer searches we have also determined the star chromatic index for some additional complete bipartite graphs; see Tables 2 and 3. Again, explicit colorings appear in Appendix B.
Moreover, using a similar linear program technique as in the proof of Proposition 2.3, one can prove lower bounds on the star chromatic index for further families of complete bipartite graphs. Let us here just list a few cases corresponding to the values in the tables above:

- \( \chi'_s(K_{r,d}) \geq \frac{10d}{r} \) if \( d \geq 5 \);
- \( \chi'_s(K_{6,6}) \geq \frac{7d}{4} \) if \( d \geq 6 \);
- \( \chi'_s(K_{7,7}) \geq \frac{7d}{4} \) if \( d \geq 7 \);
- \( \chi'_s(K_{8,8}) \geq \frac{10d}{10} \) if \( d \geq 8 \).

Finally, let us note some further consequences of the above results for general complete bipartite graphs. By decomposing a general complete bipartite graph \( K_{r,d} \) into complete bipartite graphs where one part has size e.g. at most 3, and using disjoint sets of colors for star edge colorings of distinct complete bipartite subgraphs, we deduce, using Theorem 2.2, that \( \chi'_s(K_{r,d}) \leq 3 \left\lceil \frac{d}{2} \right\rceil \left\lceil \frac{r}{2} \right\rceil \). However, using the values of star chromatic indices in Tables 2 and 3, it is possible to deduce an upper bound on \( \chi'_s(K_{r,d}) \) which is better for large values of \( r \) and \( d \).

**Corollary 2.5.** For any \( r, d \geq 1 \) it holds that \( \chi'_s(K_{r,d}) \leq 15 \left\lceil \frac{d}{3} \right\rceil \left\lceil \frac{r}{3} \right\rceil \).

Note that Dvořák et al. [4] obtained an asymptotically better bound: it follows from their results that for every \( \varepsilon > 0 \), there is a constant \( C > 0 \), such that for every \( n \geq 1 \), \( \chi'_s(K_{n,n}) \leq Cn^{1+\varepsilon} \).

Next, we turn to general bipartite graphs with restrictions on the vertex degrees. Our first task is to generalize Proposition 2.1 to general bipartite graphs with even maximum degree.

In the following we use the notation \( G = (X, Y; E) \) for a bipartite graph \( G \) with parts \( X \) and \( Y \) and edge set \( E = E(G) \). We denote by \( \Delta(X) \) and \( \Delta(Y) \) the maximum degrees of the vertices in the parts \( X \) and \( Y \), respectively. A bipartite graph \( G = (X, Y; E) \) where all vertices in \( X \) have degree 2 and all vertices in \( Y \) have degree \( d \) is called \((2, d)\)-biregular.

If the vertices in one part of a bipartite graph \( G \) has maximum degree 1, then trivially \( \chi'_s(G) = \Delta(G) \). For the case when the vertices in one of the parts have maximum degree two, we have the following.

**Theorem 2.6.** If \( G = (X, Y; E) \) is a bipartite graph, where \( \Delta(X) = 2 \) and \( \Delta(Y) = 2k \), then \( \chi'_s(G) \leq 3k \).

Note that the upper bound in Theorem 2.6 is sharp, as follows from Proposition 2.1.

For the proof of this theorem we shall use the following lemma.

**Lemma 2.7.** If \( G \) is \((2, 2k)\)-biregular with parts \( X \) and \( Y \), then it decomposes into subgraphs \( F_i \) such that \( d_{F_i}(x) \in \{0, 2\} \) for every \( x \in X \) and \( d_{F_i}(y) = 2 \) for every \( y \in Y \).

**Proof.** From a \((2, 2k)\)-biregular graph \( G \), construct a \( 2k \)-regular multigraph \( H \) by replacing every path of length 2 with an internal vertex of degree two by a single edge. By Petersen’s 2-factor theorem [18], \( H \) has a decomposition into 2-factors; these 2-factors induce the required subgraphs of \( G \). \( \Box \)

We shall also use the simple fact that every even cycle has a star edge coloring with three colors, the proof of which is left to the reader.

**Proof of Theorem 2.6.** If \( G = (X, Y; E) \) is not \((2, 2k)\)-biregular, then it is a subgraph of such a graph, so it suffices to consider the case when \( G \) is \((2, 2k)\)-biregular.

Assume, consequently, that \( G \) is a \((2, 2k)\)-biregular graph. By the preceding lemma, \( G \) decomposes into subgraphs \( F_1, \ldots, F_k \) such that \( d_{F_i}(x) \in \{0, 2\} \) for every \( x \in X \) and \( d_{F_i}(y) = 2 \) for every \( y \in Y \). Since each \( F_i \) is a collection of even cycles, it has a star edge coloring with three colors.

For \( i = 1, \ldots, k \), we color each \( F_i \) with colors \( 3i-2, 3i-1, 3i \) so that each \( F_i \) gets a star edge coloring with 3 colors. This yields a star edge coloring of \( G \); indeed, for every vertex \( x \) of \( X \), all colors on edges incident with \( x \) is in \{3i−2, 3i−1, 3i\} for some \( i \). Hence, if there is a 2-colored cycle or path \( J \) with four edges in \( G \), then since \( J \) contains at least two vertices

| \( r \) | \( d \) | \( \chi'_s(K_{r,d}) \) |
|---|---|---|
| 6 | 6 | 13 |
| 7 | 7 | 14 |
| 8 | 8 | 15 |

---

Table 3
The values of \( \chi'_s(K_{r,d}) \) for \( r = 6, 7, 8 \) and \( d \leq 8 \).
from $X$, it must be colored by two colors from $\{3i−2, 3i−1, 3i\}$ for some $i$. This implies that all edges of $f$ are in $F_i$, which contradicts that the restriction of $f$ to each $F_i$ is a star edge coloring. We conclude that $f$ is in fact a star edge coloring of $G$. □

For the case $d = 3$, we can generalize Proposition 2.1 as follows.

**Theorem 2.8.** If $G = (X, Y; E)$ is a bipartite graph with $\Delta(X) \leq 2$ and $\Delta(Y) \leq 3$, then $\chi_{st}'(G) \leq 5$.

To prove the theorem, we shall use the following easy lemma from [15], and the notion of a list star edge coloring. A list assignment $L$ for a graph $G$ is a map which assigns to each edge $e$ of $G$ a set $L(e)$ of colors. If each of the lists has size $k$, we call $L$ a $k$-list assignment. If $G$ admits a star edge coloring $\phi$ such that $\phi(e) \in L(e)$ for every edge $e$ of $G$, then $G$ is star $L$-edge-colorable; $\phi$ is a star $L$-edge coloring of $G$. The graph $G$ is star $k$-edge-choosable if it is star $L$-edge-colorable for every list assignment $L$, where $|L(e)| \geq k$ for every $e \in E(G)$.

**Lemma 2.9.** If $C$ is any cycle distinct from $C_d$, then $C$ is star $3$-edge-choosable.

By the distance between two edges $e$ and $e'$ of a graph, we mean the smallest number of edges in a path from an endpoint of $e$ to an endpoint of $e'$.

Before proving Theorem 2.8, we have to notice that for $C_4$-free graphs satisfying the condition in Theorem 2.8, this result can be deduced from the result of [16]. In [16], it is proved that the incidence chromatic number of a subcubic graph is at most 5; the incidence chromatic number of a graph $G$ is equal to the strong chromatic index of $G^*$, where $G^*$ is the graph obtained from $G$ by subdividing each edge of $G$. When $G$ is a cubic graph (with no multiple edges), the graph $G^*$ is a $(2, 3)$-biregular graph with no cycles of length four. Now, since the strong chromatic index of a graph is an upper bound for its star chromatic index, the result follows.

Since the result of [16] only applies to $C_4$-free $(2, 3)$-biregular graphs, and to have a self-contained paper, we give our short proof of Theorem 2.8.

**Proof of Theorem 2.8.** Since any cycle of even length has a star 3-edge coloring, we may assume that $G$ has maximum degree 3.

Assume that $G$ is a counterexample to the theorem which minimizes $|V(G)| + |E(G)|$. Then $G$ satisfies the following:

(i) $G$ is connected;
(ii) $G$ does not contain any vertex of degree 1;
(iii) no two vertices of degree 3 are adjacent;
(iv) $G$ does not contain two vertices of degree 3 that are linked by a path $P$ of length at least four, where all internal vertices of $P$ have degree 2. Thus any vertex of degree 2 has two neighbors of degree 3, so $G$ is $(2, 3)$-biregular.

Statements (i)-(iii) are straightforward. To see (iv), assume that $P$ is such a path, and let $u_1 u_2$ and $u_2 u_3$ be two adjacent edges of $P$, where $u_1$, $u_2$ and $u_3$ all have degree 2. By assumption $G - u_2$ has a star edge coloring with 5 colors. Now we can color $u_1 u_2$ with a color not appearing on an edge of distance at most 1 from $u_1 u_2$ in $G$; there are at most four such edges, so this is possible. Next, we can color $u_2 u_3$ by a color not appearing on $u_1 u_2$ or on any edge at distance at most 1 from $u_2 u_3$ in $G - u_1 u_2$; there are at most four such edges, so we can pick a color from $\{1, 2, 3, 4, 5\}$ for $u_2 u_3$. This yields a 5-edge coloring of $G$; a contradiction, and so (iv) holds.

Let $C_{2k} = u_1 u_2 \ldots u_{2k} u_1$ be a shortest cycle of $G$; if $G$ does not have a cycle, then it has a vertex of degree 1 and thus violates condition (ii), a contradiction. Then the graph $H = G - E(C_{2k})$ has a star 5-edge coloring.

We define a new star edge coloring $f$ of $H$ by recoloring every pendant edge $e$ of $H$ by a color from $\{1, 2, 3, 4, 5\}$ not appearing on any edge of distance at most 1 from $e$; there are at most three such edges, so this is possible.

Next, we define a list assignment $L$ for $C_{2k}$ with colors from $\{1, 2, 3, 4, 5\}$ by for each edge $u_i u_{i+1}$ of $C_{2k}$ forbidding the colors on the two pendant edges of $H$ with smallest distance to $u_i u_{i+1}$ in $G$. Then every edge of $C_{2k}$ receives a list of size at least 3; so by Lemma 2.9 it has star $L$-edge coloring. This coloring along with the edge coloring $f$ of $H$ form a star edge coloring of $G$ with 5 colors. □

Note that the preceding theorem settles a particular case of Conjecture 1.1.

Let us briefly remark that there are $(2, 3)$-biregular graphs with $\chi_{st}'(G) = 4$; while such examples with $\chi_{st}'(G) = 3$ trivially do not exist. Take two copies of $P_5$, and denote these copies by $H_1 = u_1 u_2 u_3 u_4 u_5$ and $H_2 = v_1 v_2 v_3 v_4 v_5$. Next, we add the edges $E' = \{u_1 v_2, u_3 v_4, u_2 v_1, u_4 v_5\}$ to $H_1 \cup H_2$; the resulting graph $G$ is $(2, 3)$-biregular. We define a proper edge coloring $f$ of this graph by setting

$$f(u_1 u_2) = f(u_3 u_4) = f(v_3 v_4) = 3, \quad f(u_2 u_3) = f(v_2 v_3) = f(v_4 v_5) = 2,$$

and

$$f(u_4 u_5) = f(v_1 v_2) = 1,$$

and by coloring all edges in $E'$ by color 4. The coloring $f$ is a star edge coloring, because all edges of $E'$ are adjacent to edges of three distinct colors.
For bipartite graphs $G = (X, Y; E)$ with $\Delta(X) = 2$ and odd maximum degree at least 5, we can use Theorem 2.8 for proving the following.

**Theorem 2.10.** If $G = (X, Y; E)$ is a bipartite graph with $\Delta(X) = 2$ and $\Delta(Y) = 2k + 1$, then $\chi'_c(G) \leq 3k + 2$.

Note that by Proposition 2.1, Theorem 2.10 is sharp.

For the proof of Theorem 2.10 we shall need the following theorem due to Bäbler, see e.g. [7].

**Theorem 2.11.** Let $G$ be a $(2k + 1)$-regular multigraph. If $G$ has at most $2k$ bridges, then $G$ has a 2-factor.

If the graph $G$ is obtained from $H$ by subdividing every edge of $H$, then we say that $H$ is the condensed version of $G$.

Note that if $G$ is $(2, 2k + 1)$-biregular, then $H$ is $(2k + 1)$-regular.

The proof of Theorem 2.10 is similar to the proof of the main result of [7]; hence we omit some details.

**Proof of Theorem 2.10.** Since every graph satisfying the conditions in the theorem is a subgraph of a $(2, 2k + 1)$-biregular graph, it suffices to prove the theorem for $(2, 2k + 1)$-biregular graphs. The proof is by induction on $k$. The case $k = 0$ is trivial, and Theorem 2.8 settles the case $k = 1$.

Now assume that $k \geq 2$ and that $G$ is a $(2, 2k + 1)$-biregular graph. Let $H$ be the condensed version of $G$; then $H$ is $(2k + 1)$-regular.

If $H$ has at most one bridge, then by Theorem 2.11, $H$ has a 2-factor. In $G$, this 2-factor corresponds to a subgraph $F$, where all vertices of $Y$ have degree 2, and every vertex of $X$ has degree 2 or 0. Note that the graph $G$ obtained from $G - E(F)$ by removing all isolated vertices is a $(2, 2k - 1)$-biregular graph. By the induction hypothesis, $G$ has a star edge coloring with $3(k - 1) + 2$ colors. By star edge coloring all cycles of $F$ with 3 additional colors, we obtain a star edge coloring with $3k + 2$ colors of $G$.

Now assume that $H$ has at least two bridges. We proceed as in [7]: Let $B_1, \ldots, B_r$ be the maximal bridgeless connected subgraphs obtained from $H$ by removing all bridges. For each subgraph $B_i$ we construct a $(2k + 1)$-regular multigraph containing $B_i$ by proceeding as follows: If there is an even number of bridges in $H$ with endpoints in $B_i$, then we add a number of copies of the graph $A$ consisting of $2k$ parallel edges, the endpoints of which we join to endpoints in $B_i$ of removed bridges by a single edge, respectively; if there is an odd number of bridges with endpoints in $B_i$, then we also add a subgraph $T$ consisting of a triangle $xxyz$ where $x$ and $y$ are joined by $k + 1$ parallel edges, $x$ and $z$ are joined by $k$ parallel edges, and $y$ and $z$ are joined by $k$ parallel edges, and $z$ is joined by an edge to one endpoint in $B_i$ of a removed bridge. This yields a $(2k + 1)$-regular multigraph $J_i$ containing $B_i$. We set $J = J_1 \cup \cdots \cup J_r$; so $J$ is a $(2k + 1)$-regular multigraph containing $B = B_1 \cup \cdots \cup B_r$. Note that in $J$ a bridge $b$ of $H$ is replaced by two edges joining the endpoints of $b$ with vertices of subgraphs isomorphic to $A$ or $T$.

By Theorem 2.11, $J$ has a 2-factor. Thus, by proceeding as in the preceding case, we may construct a star edge coloring $f$ with $3k + 2$ colors of the corresponding $(2, 2k + 1)$-biregular graph $D$ obtained from $J$ by subdividing all edges of $J$. Now, let $K$ be the graph obtained from $D$ by removing all edges of $D$ that are in subgraphs that correspond to the added subgraphs in $J$ that are isomorphic to $A$ or $T$, and thereafter removing all isolated vertices. The obtained graph $K$ is identical to the graph obtained from $B = B_1 \cup \cdots \cup B_r$ by

(i) subdividing all edges of $B$, and

(ii) for every bridge $uv$ of $H$ adding a path $u_2u_3$ with origin at $u$, where $u_2, u_3$ are new vertices and $u_3$ has degree 1, and adding a path $v_2v_3$ with origin at $v$, where $v_2, v_3$ are new vertices and $v_3$ has degree 1.

Thus each path of length 2 in $G$ that corresponds to a bridge in $H$ is represented by two distinct paths of length 2 in $K$; moreover, if we identify every pair of such paths corresponding to the same bridge in $H$, then we obtain a graph isomorphic to $G$.

Let $f'_c$ be the restriction of $f$ to $K$; this is a star edge coloring of $K$ with $3k + 2$ colors. We recolor every pendant edge of $K$ by a color from $\{1, \ldots, 3k + 2\}$ which does not appear on an edge of distance at most 1 from the pendant edge; the obtained coloring $f''_c$ is a star edge coloring with the property that no pendant edge is in a bicolored path of length at least 3. Now, to obtain a star edge coloring of $G$ from $f''_c$ we may successively “paste” together components of $K$ by identifying paths that correspond to the same bridge in $H$ and permuting colors in one of the components so that the colorings agree on the identified paths; this “pasting process” can be done exactly as in [7] (e.g. by doing a Depth-First-Search in the tree with vertices for the subgraphs $B_i$ and edges for the bridges of $H$); so we omit the exact details here. Since in $K$, any bicolored path with a pendant edge has length at most 2, this yields a star edge coloring of $G$. □

3. Planar cubic graphs

As mentioned above, Conjecture 1.1 has been verified for outerplanar graphs. A particularly interesting special case of Conjecture 1.1 is planar graphs; this particular case is still wide open. In this section we provide two results in this direction.

Recall that a Halin graph is a planar graph constructed from a planar drawing of a tree with at least four vertices and with no vertices of degree two by connecting its leaves by a cycle that crosses none of its edges. We shall first prove that
cubic Halin graphs have star chromatic index at most 6. Note that this upper bound is sharp, since the complement of a 6-cycle is a cubic Halin graph attaining this bound (see e.g. [15]). Our proof is similar to the proof in [13] of the fact that cubic Halin graphs have strong chromatic index at most 7.

**Theorem 3.1.** If G is a cubic Halin graph, then \( \chi'_s(G) \leq 6 \).

**Proof.** Let \( G = T \cup C \), where \( T \) is a tree and \( C \) is an adjacent cycle containing all pendant vertices of \( T \). Our proof proceeds by induction on the length \( m \) of the cycle \( C \). It is straightforward that every cubic Halin graph with \( m \leq 5 \) has star chromatic index at most 6; indeed in such a graph \( G = T \cup C \), the tree \( T \) consists of a path \( P = u_0u_1 \ldots u_k \) along with the vertices \( u'_1, \ldots, u'_{k-1} \), where the unique neighbor of \( u'_i \) in \( P \) is \( u_i \), \( i = 1, \ldots, k - 1 \). So let us assume that \( m \geq 6 \).

Let \( P = u_0u_1 \ldots u_k \) be a path of maximum length in \( T \). Since \( \Delta(T) \leq 3 \) and \( m \geq 6, l \geq 5 \). Moreover, since \( P \) is maximum, all neighbors of \( u_1 \), except \( u_2 \), are leaves. We set \( w = u_3, u = u_2, v = u_1 \). Moreover, let \( v_1 \) and \( v_2 \) be the neighbors of \( v \) on \( C \) and label some other vertices in \( G \) according to Fig. 1.

Since \( d_G(u) = 3 \), there is a path \( Q \) from \( u \) to \( x_1 \) or \( y_1 \), with \( V(P) \cap V(Q) = \{u\} \). Suppose without loss of generality that there is such a path from \( u \) to \( y_1 \). Then, since \( P \) is a path of maximum length in \( T \), \( Q \) has length at most two; that is, \( uy_3 \in E(T) \) or \( u = y_3 \). If the former holds, then \( y_3y_2 \in E(T) \), and in the latter case, \( uy_1 \in E(T) \).

**Case 1.** \( uy_3 \in E(T) \):

Let \( z \in V(C) \) be the vertex distinct from \( y_1 \) that is adjacent to \( y_2 \). Let \( G' \) be the graph obtained from \( G \) by removing vertices \( v, v_1, v_2, y_1, y_2, y_3 \), and adding two new edges \( uw_1 \) and \( uz \). By the induction hypothesis, there is a star edge coloring \( f' \) with colors \( 1, \ldots, 6 \) of \( G' \). Without loss of generality, we assume that \( f'(uw) = 1, f'(ux_1) = 2 \) and \( f'(uz) = 3 \). Let \( f'(x_1) = \{s_1, s_2\} \) and \( f'(z) = \{t_1, t_2\} \). Note that if \( 3 \in f'(x_1) \), then \( 2 \notin f'(z) \), and vice versa. From \( f' \), we shall define a star 6-edge coloring \( f \) of \( G \); we begin by setting \( f(e) = f'(e) \) for all edges \( e \in E(G') \cap E(G) \), \( f(x_1v_1) = f(uw) = 2 \) and \( f(yz) = f(uy_3) = 3 \). We extend \( f \) to the remaining uncolored edges of \( G \) by considering some different cases. For brevity, we shall in the following use the notation \( \{2, 3, s_1, s_2, t_1, t_2\} \) for a set containing all distinct colors from the sequence \( 2, 3, s_1, s_2, t_1, t_2 \).

**Subcase 1.1.** \( |\{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\}| \geq 2 \):

Let \( \{c_1, c_2\} \subseteq \{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\} \). Without loss of generality, we assume that \( c_2 \neq 1 \) and set \( f(v_1v) = f(y_2y_3) = c_2 \) and \( f(v_1v_2) = f(y_1y_2) = c_1 \). To obtain a star edge coloring of \( G \) we now properly color the edges \( vv_2, v_2y_1, y_1y_3 \) by two colors in \( \{1, \ldots, 6\} \setminus \{2, 3, c_1, c_2\} \) so that neither of \( vv_2 \) and \( y_1y_3 \) is colored 1.

**Subcase 1.2.** \( |\{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\}| = 1 \):

Let \( c_1 \in \{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\} \).

**Subcase 1.2.1.** \( \{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset \):

If \( \{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset \), then \( 2 \in f'(z) \) or \( 3 \in f'(x_1) \). Without loss of generality we assume that the former holds; so \( t_1 = 2 \) and thus \( 3 \notin \{s_1, s_2\} \).

If \( c_1 = 1 \), then we set

\[
    f(v_1v_2) = f(y_1y_2) = c_1, f(vv_1) = f(y_1y_3) = t_2, f(vv_2) = f(y_2y_3) = s_1, \quad \text{and} \quad f(v_2y_1) = s_2.
\]

This yields a star edge coloring of \( G \).

If, on the other hand \( c_1 \neq 1 \), then we set \( f(v_1v_1) = f(y_2y_3) = c_1 \) and \( f(v_1v_2) = 3 \). Without loss of generality, we further assume that \( s_1 \neq 1 \) and set \( f(vv_2) = f(y_1y_3) = s_1, f(v_2y_1) = t_2 \) and \( f(y_1y_2) = s_2 \).
Subcase 1.2.2. $|\{s_1, s_2\} \cap \{t_1, t_2\}| = 1$:

Suppose that $s_1 = t_1$ and note that the conditions imply that $2 \not\in f'(z)$ and $3 \not\in f'(x_1)$.

If $c_1 = 1$, then we set
\[
f(v_1v_2) = f(y_1y_2) = c_1, f(vv_1) = f(y_1y_3) = t_2, f(vv_2) = f(y_2y_3) = s_2, \text{ and } f(v_2y_1) = s_1.
\]

If, on the other hand $c_1 \neq 1$, then we set $f(vv_1) = f(y_2y_3) = c_1, f(v_1v_2) = 3$ and $f(y_1y_2) = 2$. We then color the edges of the path $vv_2y_1y_3$ properly by colors $s_2$ and $t_2$ so that neither of $vv_2$ and $y_1y_3$ is colored 1.

Subcase 1.3. $\{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\} = \emptyset$:

Without loss of generality, we assume that $s_1 = 1$. We obtain a star edge coloring of $G$ by setting $f(vv_1) = f(y_1y_3) = t_2, f(vv_2) = f(y_2y_3) = s_2, f(v_1v_2) = t_1, f(y_1y_2) = s_1$, and $f(y_1v_2) = 3$.

Case 2. $u = y_3$:

Let $G'$ be the graph obtained from $G$ by removing $v, v_1, v_2, y_1$, and adding the new edges $ux_1$ and $uy_2$. By the induction hypothesis, there is a star 6-edge coloring $f'$ of $G'$. Without loss of generality we assume that $f'(ux_1) = 1, f'(uy_2) = 2$ and $f'(y_3) = 3$. Let $f'(x_1) = \{2, s_1, s_2\}$ and $f'(y_2) = \{3, t_1, t_2\}$. From $f'$, we shall define a star 6-edge coloring $f$ of $G$; we begin by setting $f(e) = f'(e)$ for all edges $e \in E(G') \cap E(G), f(x_1v_1) = 2$ and $f(y_1v_2) = 3$. We extend $f$ to the remaining uncolored edges of $G$ by considering some different cases.

Subcase 2.1. $2 \not\in \{t_1, t_2\}$:

We set $f(uv) = 3$ and $f(uv_1) = 2$ and consider three different subcases.

Subcase 2.1.1. $|\{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\}| \geq 2$:

Let $\{c_1, c_2\} \subseteq \{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\}$. Without loss of generality, we assume that $c_2 \neq 1$ and set $f(vv_1) = f(v_2y_1) = c_2$ and $f(v_1v_2) = c_1$. We then color $vv_2$ by a color from $\{1, \ldots, 6\} \setminus \{2, 3, c_1, c_2\}$ to obtain a star edge coloring of $G$.

Subcase 2.1.2. $|\{1, \ldots, 6\} \setminus \{2, 3, s_1, s_2, t_1, t_2\}| = 1$:

Let $c_1 \in \{1, \ldots, 6\} \setminus \{2, 3, t_1, t_2, s_2\}$. Suppose first that $3 \in \{s_1, s_2\}$, e.g. that $s_1 = 3$; then the colors 2, $c_1, s_1, s_2, t_1, t_2$ are distinct and we set $f(vv_2) = c_1, f(vv_1) = t_2, f(vv_2) = t_1$ and $f(v_2y_1) = s_2$ to obtain a star edge coloring of $G$.

Suppose now that $\{s_1, s_2\} \cap \{t_1, t_2\} \neq \emptyset$, e.g. $t_1 = s_1$; then the colors $c_1, t_1, t_2$ are all distinct and not equal to 2 or 3. We set $f(vv_2) = t_1, f(v_2y_1) = s_2$, and color $vv_1, v_1v_2$ by the colors $c_1, t_2$ so that $f(vv_1) \neq 1$.

Subcase 2.1.3. $\{1, \ldots, 6\} \not\subseteq \{2, 3, s_1, s_2, t_1, t_2\} = \emptyset$:

Without loss of generality, we assume that $s_2 \neq 1$, and set
\[
f(vv_1) = t_2, f(v_1v_2) = t_1, f(vv_2) = s_2, f(v_2y_1) = s_1.
\]

Subcase 2.2. $2 \in \{t_1, t_2\}$:

We assume $t_1 = 2$; note that this implies that $3 \not\in f'(x_1)$. We set $f(uv_1) = 2, f(vv_1) = 3$ and color $uv$ by a color $c_1 \leq 6$ satisfying that $c_1 \not\in f'(w) \cup \{2, 3\}$.

Assume first that $1 \not\in \{s_1, s_2\}$. Then we set $f(v_1v_2) = 1$, color $v_2y_1$ by a color $c_2 \in \{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$, and thereafter color $vv_1$ by a color from $\{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$.

Assume now that $1 \in \{s_1, s_2\}$, e.g. that $s_1 = 1$. If $c_1 \neq t_2$, then we color $v_2y_1$ by $c_1$, $v_1v_2$ by a color $c_2 \in \{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$, and $vv_2$ by a color $c_3 \in \{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$.

If, on the other hand $c_1 = t_2$, then we color $v_1v_2$ by a color $c_2 \in \{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$, $vv_2$ by color 2, and $v_2y_1$ by a color $c_3 \in \{1, \ldots, 6\} \setminus \{1, 2, 3, c_1, c_2\}$.

Finally, we have the following for planar graphs. The proof uses a similar approach as in [19,20].

**Proposition 3.2.** Let $G$ be a subcubic planar graph with girth at least 7. If $G$ has a perfect matching, then $\chi'_6(G) \leq 6$.

**Proof.** Let $G$ be a subcubic planar graph with girth at least 7, and suppose that $M$ is a perfect matching in $G$. Let $H$ be the graph obtained from $G$ by contracting all edges of $M$. Since $G$ is planar and has girth at least 7, $H$ is a planar triangle-free graph. Thus, by Grötzsch’s theorem, $H$ has a proper vertex coloring $\varphi$ with colors $1, 2, 3$. We obtain a partial strong edge coloring $f$ of $G$ by coloring every edge of $M$ by the color of the corresponding vertex in $H$.

Now, by Lemma 2.9, $G - M$ has a star edge coloring with three colors; use colors $4, 5, 6$ for such a coloring $g$ of $G - M$. By combining the colorings $g$ and $f$, we obtain a star edge coloring of $G$ with colors $1, \ldots, 6$. Indeed, there is no 2-colored path or cycle of length four with colors only in $\{1, 2, 3\}$ or $\{4, 5, 6\}$, because both $f$ and $g$ are star edge colorings. Moreover, there is no 2-colored path or cycle of length four with one color from $\{1, 2, 3\}$ and one color from $\{4, 5, 6\}$, because $f$ is a strong edge coloring of $M$ with respect to $G$. □
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Appendix A. The optimization problem in the proof of Proposition 2.3

The optimization problem in the proof of Proposition 2.3 can be formulated as a linear integer program in the following way.

\[
\text{minimize } \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} x_{ijk\ell}
\]

subject to

\[
\sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} x_{1jk\ell} = d
\]

\[
\sum_{i=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} x_{1ik\ell} = d
\]

\[
\sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{\ell=0}^{1} x_{ij1\ell} = d
\]

\[
\sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} x_{ijk1} = d
\]

\[
\sum_{k=0}^{1} \sum_{\ell=0}^{1} \sum_{i=0}^{1} x_{11k\ell} \leq \frac{d}{2}
\]

\[
\sum_{j=0}^{1} \sum_{\ell=0}^{1} \sum_{i=0}^{1} x_{1j1\ell} \leq \frac{d}{2}
\]

\[
\sum_{j=0}^{1} \sum_{k=0}^{1} x_{1jk1} \leq \frac{d}{2}
\]

\[
\sum_{i=0}^{1} \sum_{\ell=0}^{1} \sum_{k=0}^{1} x_{i11\ell} \leq \frac{d}{2}
\]

\[
\sum_{i=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} x_{ik11} \leq \frac{d}{2}
\]

\[
x_{ijk\ell} \geq 0 \text{ and integer } \forall i, j, k, \ell \in \{0, 1\}
\]

The variables \(x_{ijk\ell}\) count the number of colors in various sets from the proof of Proposition 2.3; for instance \(x_{1010} = r\) means that there are exactly \(r\) colors contained in the set \(C_1 \cap \overline{C}_2 \cap \overline{C}_3 \cap \overline{C}_4\).

The optimal solution to the linear relaxation of this problem is \(\frac{10d}{6}\), which is attained by

\[
x_{0011} = x_{0010} = x_{0111} = x_{0110} = x_{1001} = x_{1010} = x_{1011} = x_{1100} = x_{1101} = x_{1110} = \frac{d}{6},
\]

\[
x_{0000} = x_{0001} = x_{0010} = x_{0100} = x_{1000} = x_{1111} = 0.
\]

Appendix B. Star edge colorings of small complete bipartite graphs

In this appendix we give explicit colorings of some small complete bipartite graphs. The colorings are given in the form of an array where rows and columns correspond to vertices, cells correspond to edges and the contents of the cells correspond to colors (see Figs. 2–23).
Fig. 2. A star edge coloring of $K_{4,4}$ with 7 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 1 & 2 & 8 \\
2 & 8 & 4 & 9 & 10 \\
4 & 5 & 7 & 10 & 6 \\
\end{array}
\]

Fig. 3. A star edge coloring of $K_{4,5}$ with 10 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 1 & 2 & 3 \\
2 & 9 & 4 & 10 & 6 & 11 \\
5 & 11 & 2 & 8 & 10 & 7 \\
\end{array}
\]

Fig. 4. A star edge coloring of $K_{4,6}$ with 11 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 1 & 2 & 3 & 11 \\
2 & 10 & 4 & 11 & 6 & 12 & 13 \\
5 & 3 & 12 & 10 & 13 & 7 & 8 \\
\end{array}
\]

Fig. 5. A star edge coloring of $K_{4,7}$ with 13 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\
2 & 11 & 5 & 6 & 12 & 13 & 8 & 14 \\
14 & 7 & 13 & 11 & 8 & 1 & 9 & 10 \\
\end{array}
\]

Fig. 6. A star edge coloring of $K_{4,8}$ with 14 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 1 & 2 & 3 & 4 & 14 \\
2 & 12 & 5 & 6 & 13 & 15 & 8 & 14 & 16 \\
8 & 9 & 16 & 12 & 4 & 7 & 11 & 15 & 10 \\
\end{array}
\]

Fig. 7. A star edge coloring of $K_{4,9}$ with 16 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 \\
2 & 13 & 6 & 7 & 16 & 14 & 17 & 5 & 15 & 9 \\
8 & 16 & 11 & 6 & 3 & 12 & 10 & 17 & 2 & 14 \\
\end{array}
\]

Fig. 8. A star edge coloring of $K_{4,10}$ with 17 colors.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
12 & 13 & 14 & 15 & 16 & 1 & 2 & 3 & 4 & 5 & 17 \\
2 & 14 & 6 & 5 & 18 & 16 & 8 & 17 & 10 & 19 & 20 \\
3 & 9 & 15 & 20 & 11 & 4 & 16 & 19 & 18 & 7 & 13 \\
\end{array}
\]

Fig. 9. A star edge coloring of $K_{4,11}$ with 20 colors.
Fig. 10. A star edge coloring of $K_{4,12}$ with 20 colors.

\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 1 & 2 \\
2 & 8 & 4 & 9 & 10 \\
3 & 6 & 10 & 2 & 11 \\
5 & 11 & 2 & 7 & 9 \\
\end{array}

Fig. 11. A star edge coloring of $K_{5,5}$ with 11 colors.

\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 1 & 2 & 3 \\
2 & 9 & 4 & 10 & 6 & 11 \\
5 & 11 & 2 & 8 & 10 & 7 \\
8 & 10 & 1 & 12 & 4 & 2 \\
\end{array}

Fig. 12. A star edge coloring of $K_{5,6}$ with 12 colors.

\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 1 & 2 & 3 \\
2 & 10 & 4 & 12 & 6 & 13 & 11 \\
4 & 7 & 5 & 9 & 14 & 12 & 8 \\
11 & 3 & 13 & 5 & 10 & 7 & 14 \\
\end{array}

Fig. 13. A star edge coloring of $K_{5,7}$ with 14 colors.

\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\
2 & 11 & 5 & 13 & 12 & 7 & 14 & 15 \\
6 & 13 & 9 & 3 & 2 & 15 & 8 & 10 \\
13 & 7 & 1 & 10 & 14 & 4 & 9 & 5 \\
\end{array}

Fig. 14. A star edge coloring of $K_{5,8}$ with 15 colors.

\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14 & 1 & 2 & 3 & 4 \\
2 & 12 & 5 & 6 & 13 & 15 & 16 & 17 & 8 \\
5 & 9 & 7 & 10 & 12 & 11 & 17 & 4 & 15 \\
17 & 6 & 9 & 1 & 8 & 16 & 14 & 11 & 10 \\
\end{array}

Fig. 15. A star edge coloring of $K_{5,9}$ with 17 colors.

\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 \\
2 & 13 & 6 & 7 & 8 & 15 & 16 & 17 & 10 & 14 \\
7 & 9 & 18 & 3 & 6 & 12 & 15 & 10 & 11 & 13 \\
9 & 8 & 14 & 17 & 11 & 4 & 3 & 18 & 16 & 12 \\
\end{array}

Fig. 16. A star edge coloring of $K_{5,10}$ with 18 colors.
Fig. 17. A star edge coloring of $K_{5,11}$ with 20 colors.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|-----|-----|
| 12|   |   |   |   |   |   | 7 |   |   |     |     |
| 13|   |   |   |   |   |   | 8 |   | 9 | 1 | 2   |
| 14|   | 2 |   | 9 | 10 | 3 | 11 | 12 |
| 15|   | 3 | 4 | 12 | 13 | 9 | 5  |    |
| 16| 6  | 12| 7 | 8  | 3  | 13 |    |    |
| 17| 10 | 3  | 13| 6  | 1  | 11 |    |    |

Fig. 18. A star edge coloring of $K_{6,6}$ with 13 colors.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|
| 18|   |   |   |   |   |   | 8 |   | 9 | 10 | 11 | 1  |
| 19| 2  |   | 10| 4  | 12 | 11 | 5  | 13 |    |    |    |    |
| 20| 4  | 8  | 14| 13 | 3  | 10 | 6  |    |    |    |    |    |
| 21| 6  | 12 | 8 | 7  | 2  | 13 | 9  |    |    |    |    |    |
| 22| 7  | 13 | 11| 5  | 9  | 3  | 14 |    |    |    |    |    |

Fig. 19. A star edge coloring of $K_{6,7}$ with 14 colors.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|
| 23|   |   |   |   |   |   | 9 |   | 10| 11 | 12 | 1  | 2   |
| 24| 2  | 13 | 4 | 14 | 10 | 5 | 12 | 7  |    |    |    |    |    |
| 25| 3  | 15 | 13| 2  | 11 | 8 | 5  | 10 |    |    |    |    |    |
| 26| 6  | 9  | 12| 7  | 2  | 13 | 15 | 3  |    |    |    |    |    |
| 27| 8  | 11 | 6 | 9  | 4  | 14 | 2  | 13 |    |    |    |    |    |

Fig. 20. A star edge coloring of $K_{6,8}$ with 15 colors.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|
| 28|   |   |   |   |   |   | 8 |   | 9 | 10 | 11 | 1  | 2   | 3   |
| 29| 2  | 10 | 4 | 12 | 11 | 5 | 13 |    |    |    |    |    |    |    |
| 30| 4  | 8  | 14| 13 | 3  | 10 | 6  |    |    |    |    |    |    |    |
| 31| 6  | 12 | 8 | 7  | 2  | 13 | 9  |    |    |    |    |    |    |    |
| 32| 7  | 13 | 11| 5  | 9  | 3  | 14 |    |    |    |    |    |    |    |
| 33| 12 | 11 | 1 | 14 | 7 | 4  | 8  |    |    |    |    |    |    |    |

Fig. 21. A star edge coloring of $K_{7,7}$ with 14 colors.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|
| 34|   |   |   |   |   |   | 9 |   | 10| 11 | 12 | 1  | 2   | 3   |
| 35| 2  | 13 | 4 | 14 | 10 | 5 | 12 | 7  |    |    |    |    |    |    |
| 36| 3  | 15 | 13| 2  | 11 | 8 | 5  | 10 |    |    |    |    |    |    |
| 37| 6  | 9  | 12| 7  | 2  | 13 | 15 | 3  |    |    |    |    |    |    |
| 38| 8  | 11 | 6 | 9  | 4  | 14 | 2  | 13 |    |    |    |    |    |    |
| 39| 11 | 12 | 5 | 6  | 14 | 15 | 1  | 2  |    |    |    |    |    |    |

Fig. 22. A star edge coloring of $K_{7,8}$ with 15 colors.
Fig. 23. A star edge coloring of $K_{8,8}$ with 15 colors.

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