DICKSON INVARIANTS IN THE IMAGE OF THE STEENROD SQUARE

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Abstract. Let $D_n$ be the Dickson invariant ring of $\mathbb{F}_2[X_1, \ldots, X_n]$ acted by the general linear group $\text{GL}(n, \mathbb{F}_2)$. In this paper, we provide an elementary proof of the conjecture by [3]: each element in $D_n$ is in the image of the Steenrod square in $\mathbb{F}_2[X_1, \ldots, X_n]$, where $n > 3$.

1. Introduction

A polynomial in $\mathbb{F}_2[X_1, X_2, \ldots, X_n]$ is hit if it is in the image of the summation of the Steenrod square: $\sum_{i \geq 1} \text{Sq}^i$. Let $D_n$ be the Dickson invariant algebra of $n$-variables. In this paper, we will prove the following,

Theorem 1.1. When $n > 3$, each polynomial in the Dickson invariant ring $D_n$ is hit.

In [3], Hung studies the Dickson invariants in the image of the Steenrod square. Since it is trivial that $D_1$ and $D_2$ are not hit, the problem starts interesting from $n = 3$. In the same paper, Hung shows that each element in $D_3$ is hit and conjectured that it is true for $D_{n>3}$. So our result provides a positive answer to the conjecture, which supports to the positive answer of the conjecture on the spherical classes: there are no spherical classes in $Q_0S^0$, except the Hopf invariant one and Kervaire invariant one elements. We refer to [3] and an excellent expository paper [5], p501 for more background regarding to this conjecture.

Remark 1.2. Recently, K. F. Tan and the author [4] has obtained an elementary proof of the case $n = 3$.

2. Proof of Theorem 1.1

We first recall some basic properties regarding the Dickson algebra. Write $V_n$ for the product

$$\prod_{\alpha_i \in \{0,1\}, i=1,\ldots,n-1} (\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + x_n).$$

Then we have the following theorem.

Theorem 2.1 (Hung [3]).

$$\text{Sq}^i V_n = \begin{cases} 
V_n & \text{if } i = 0 \\
V_n Q_{n-1, s} & \text{if } i = 2^{n-1} - 2^s, \ 0 \leq s \leq n - 1 \\
V_n^2 & \text{if } i = 2^{n-1} \\
0 & \text{otherwise.}
\end{cases}$$

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Using the previous observation, the last polynomial is hit, since the order of each time.

We use the induction on $n$ to prove Theorem 1.1. Suppose that the statement is true for $n$. Then we will prove that each polynomial in $D_{n+1}$ is hit.

Recall that

$$Q_{n+1,k} = Q_{n,k-1}^2 + V_{n+1}Q_{n,k}$$ for $1 \leq k \leq n$.

So any monomial in $F_2[Q_{n+1,0}, Q_{n+1,1}, ..., Q_{n+1,n}]$ can be written as the summation of the following form:

$$A := V_{n+1}^{a.n}Q_{n,0}^{n_0}Q_{n,1}^{n_1}Q_{n,2}^{n_2} \cdots Q_{n,n-1}^{n_{n-1}}.$$

Hence by the hypothesis of the induction, it is sufficient to show that $A$ is hit for any $a > 0$. Notice that

$$V_{n+1} = \sum_{s=1}^{n} \text{Sq}^1(Q_{n,s}X_{n+1}^{2^s-1}). \quad (1)$$

When $n_1$ is even, we have the hit polynomial

$$A = \text{Sq}^1(\left(\sum_{s=1}^{n} Q_{n,s}x_{n+1}^{2^s-1}\right) V_{n+1}^{a-1} Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \cdots Q_{n,n-1}^{n_{n-1}}).$$

If $n_1$ is odd and $n_2$ is even, then $A$ can be written as the hit polynomial:

$$\text{Sq}^2(\sum_{s=1}^{n} Q_{n,s}x_{n+1}^{2^s-1}) V_{n+1}^{a-1} Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \cdots Q_{n,n-1}^{n_{n-1}} + \text{Sq}^1(\left(\sum_{s=1}^{n} Q_{n,s}x_{n+1}^{2^s-1}\right) V_{n+1}^{a-1} Q_{n,0}^{n_0} (\text{Sq}^1 Q_{n,1}^{n_1-1})^2 Q_{n,2}^{n_2+1} \cdots Q_{n,n-1}^{n_{n-1}}).$$

In the following, we will always assume that $n_1$ and $n_2$ are both odd. When $n = 3$, $n_0$ is even and $a$ is odd, we have

$$A = (V_4^{a-1} \text{Sq}^4 V_4)Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2-1} = V_4 \chi(\text{Sq}^4)[V_4^{a-1} Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2-1}] \pmod{\text{the hits}} = V_4^n Q_{3,1} \left(\text{Sq}^2[Q_{3,0}^{n_0} Q_{3,1}^{n_1-1} Q_{3,2}^{n_2-1}]\right)^2 \pmod{\text{the hits}}.$$ 

Using the previous observation, the last polynomial is hit, since the order of $Q_{3,2}$ is even.

When $n = 3$, $n_0$ is even and $a$ is even, notice that

$$Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2} = Q_{3,0}^{n_0} Q_{3,1}^{n_1-1} Q_{3,2}^{n_2-1} \text{Sq}^4 Q_{3,1}.$$
Then using the \( \chi \)-trick and doing some basic computation, we can see that the monomial \( Q_{3,1}^{n_1}Q_{3,2}^{n_2}Q_{3,3}^{n_3} \) is in the image of \( \sum_{i=1}^{4} \text{Sq}^i \). In fact,

\[
Q_{3,1}^{n_1}Q_{3,2}^{n_2}Q_{3,3}^{n_3} = \text{Sq}^i \text{Sq}^{2i} \text{Sq}^{3i} \text{Sq}^{4i} (
\text{mod the hits})
\]

Using the previous discussion, we can conclude that all these polynomials are hit, except for those when \( n_i \) is odd. Then \( \chi\text{Sq}^{i}\text{Sq}^{2i}\text{Sq}^{3i}\text{Sq}^{4i} = 0 \) for \( i = 1, 2, 3 \) and 4. Therefore using the \( \chi \)-trick, we know that \( A \) is hit.

When \( n = 3 \), \( n_0 \) is odd and \( a \) is even, \( \nu \) is the integer such that \( a = 2^\nu b \) where \( b \) is odd. Then

\[
V_4^{n_0} = \text{Sq}^{4a} \text{Sq}^{2a} \ldots \text{Sq}^{8b} V_4^b.
\]

Hence

\[
A = (\text{Sq}^{4a} \text{Sq}^{2a} \ldots \text{Sq}^{8b} V_4^b)(Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2})
\]

\[
\equiv V_4^{n_0} \chi(\text{Sq}^{2b}) \ldots \chi(\text{Sq}^{4a}) \chi(Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2}) \quad (\text{mod the hits}).
\]

After expanding the last polynomial using Theorem 2, it is easy to see that each resulting term belongs to one of the previous cases. Therefore \( A \) is hit.

When \( n \geq 4 \), the polynomial \( A \) takes the following form,

\[
V_4^{n_0} (\text{Sq}^{2n-4} Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \ldots Q_{n,n-1}^{n_{n-1}}).
\]

Using a result of Don Davis, Theorem 2. of [1] and the \( \chi \)-trick, we know that it is sufficient to show the polynomial:

\[
Q_{n,1}^{n_1} \text{Sq}^{2n-1} \ldots \text{Sq}^{n_0} \chi(\text{Sq}^{4}) \{ V_{n+1}^{a} Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \ldots Q_{n,n-1}^{n_{n-1}} \}
\]

is hit.

After expansion using the Steenrod operation, the above polynomial can be written as the summation of the form:

\[
V_{n+1}^{a} Q_{n,0}^{k_0} Q_{n,1}^{k_1} Q_{n,2}^{k_2} \ldots Q_{n,n-1}^{k_{n-1}}.
\]

Using the previous discussion, we can conclude that all these polynomials are hit, except for those when \( k_1 \) and \( k_2 \) are both odd. But in this case, we can replace \( n_i \) by \( k_i \) for all \( i \) in [1] and carry out the above process again. After using this process sufficiently many times with modulo the hits, we can conclude that the new \( k_0, k_1 \)
and \( k_2 \) are independent of the process. To keep \( k_0, k_1 \) and \( k_2 \) unchanged with the process, we must require that

\[
\text{Sq}^{2^{n-1}} \cdots \text{Sq}^8 \chi(\text{Sq}^4) \left\{ V_{n+1}^a Q_n^k \cdots Q_n^{k_{n-1}} \right\} \mod \text{the hits}
\]

contributes \( Q_{n,2} \) after each process is done, since for \( j \leq 2^{n-1} \) and \( t < n \), \( \text{Sq}^j Q_{n,0} = Q_{n,0} Q_{n,t} \) only if \( j = t = n - 1(> 2) \). Finally because all \( k_i \) \( (0 \leq i < n) \) are finite, we conclude that \( A \) is hit after carrying on the process further for enough many times.

**References**

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