Free oscillations of a dissipative oscillator with double quadratic nonlinearity

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Abstract. Approximate formulas for calculating the amplitudes of free damped oscillations of the oscillator with a quadratic nonlinear elastic characteristic under the action of a resistance force that is proportional to the square of the velocity are derived by the energy balance method. Two variants of this method have been implemented. In the first, an approximate differential equation of the envelope graph of oscillations is composed and its analytical solution is constructed. As a result, for the calculation of amplitudes, iterative relations were obtained using the Lambert W function. The argument of this special function is positive for the hard power characteristic and negative for the soft one. Asymptotic approximations of the Lambert W function, which simplify the practical implementation of analytical solutions, are proposed, and the possibility of using known tables of this special function is indicated. In presenting the second variant of the energy balance method, the recurrent relation between the amplitudes of oscillations related to the analytical solution of the cubic equation is derived. Unlike the first option, it does not require iterations. It has been made a comparison of numerical results, which these methods of calculating the amplitudes lead to, and numerical computer integration of the differential equation of oscillations of the oscillator. Satisfactory consistency of the results obtained in different ways confirmed the suitability of the derived approximate formulas for engineering calculations. The main advantage of this method is that it does not involve the construction and use of an exact solution of the double nonlinear differential equation of motion of the oscillator. In addition to the direct problem, the inverse problem of determining the coefficient of quadratic resistance of the medium based on the results of measuring the amplitudes of free oscillations on the oscillogram is also analytically solved here. The obtained solution of the direct dynamics problem was used to check the accuracy of the coefficient identification.

1. Introduction

Significant mathematical difficulties arise in the analytical solution of the equations of oscillations of nonlinear dissipative oscillators.
In order to overcome them the energy balance method is used. This method has long been used in the calculations of oscillations of dissipative linear elastic systems. Here the task is to apply this method for nonlinearly elastic dissipative systems. Free oscillations were chosen for the study. Analytical solutions of the problem of free damped oscillations should be used to identify the value of the coefficient of viscous resistance and verify the correctness of its value. This coefficient is needed to calculate the resonant forced oscillations of the system because the use of identification results improves the adequacy of the forced oscillation model. Therefore, the calculation of free oscillations usually precedes the calculation of forced oscillations, which the efficiency and service life of the structure depend on [1-3].

2. Review of related literature and problem setting
Free undamped oscillations with a pure quadratic force characteristic were considered in [4].

The exact analytical solution of the oscillation equation is expressed in two forms. In the first solution of the problem - periodic Ateb-functions, and in the second - Jacobi elliptic functions. In [5] free oscillations without damping are described for an asymmetric quadratic-nonlinear force characteristic. Asymptotic methods of nonlinear mechanics are used to construct the solution of the dynamic problem. This paper uses the energy balance method, which is described in [6-10]. It discusses different types of dissipative resistance from turbulent viscous to dry friction. The joint action of different resistance forces during oscillations of a linearly elastic system was studied by this method in [11-14]. Here the method is extended to the case of free oscillations of nonlinearly elastic dissipative oscillators.

3. Objectives
The objective of the paper is to derive and test approximate formulas for calculating the amplitudes of free damped oscillations of an oscillator with a symmetric quadratically nonlinear force characteristic, in the presence of a quadratic viscous resistance.

In this case, the oscillations are described by a differential equation with two nonlinear items.

4. Results and discussions
The motion of the oscillator is described by the differential equation:

\[ m\ddot{x} + kx^2 \sign(x) + c_1x + c_2x^3 \sign(x) = 0, \]

where \( m \) – a mass of an oscillator; \( k \) – the square resistance; \( c_1, c_2 \) – characteristics of the elasticity of the system; \( x(t) \) – deviation of the oscillator from the equilibrium position as a function of time \( t \); the point above \( x \) means the derivative of \( t \).

The reason for the oscillations is the initial deviation of the system by \( a_0 \) from the equilibrium position \( x=0 \). Therefore, equation (1) is supplemented by the initial conditions:

\[ x(0) = -a_0; \dot{x}(0) = 0. \]

In order not to construct a solution of this nonlinear Cauchy problem based on the energy balance method (EBM), we consider the iterative half-cycle of oscillations, which begins with the amplitude deviation \( x = a_{-1} \) and ends with the amplitude deviation \( x = a_i \). The change in potential energy \( \Delta I \) at this stage of the motion is:

\[ \Delta I = \frac{1}{2} c_1 (a_i^2 - a_{-1}^2) + \frac{1}{3} c_2 (a_i^3 - a_{-1}^3). \]

The work of the force of quadratic resistance \( A_f \) is approximately represented by the integral:
\[ A_r \approx -k \int_{-a}^{a} \hat{x}^2 \, dx \]  

(4)

taking analogically [8]:

\[ \alpha = \frac{1}{2} (a_{i+1} + a_i); \quad x = -a \cos(\omega_i t), \quad dx = a\omega_i \sin(\omega_i t) \, dt. \]  

(5)

Then:

\[ A_r \approx -ka^3 \omega_i^3 \int_0^{\pi/a_i} \sin^3(\omega_i t) \, dt = -\frac{4}{3} ka^3 \omega_i^3. \]  

(6)

Here \( \omega_i \) – average value of the oscillation frequency in the iterative half-cycle.

Equation of the envelope graph of oscillations and its solution. To obtain this equation, instead of (3), we assume approximately:

\[ \Delta \Pi \approx (c_1 a + c_2 a^2) (a_i - a_{i-1}), a_i - a_{i-1} = \frac{\pi}{\omega_i} \frac{da}{a_i \, dt}. \]  

(7)

According to the energy balance method \( \Delta \Pi = A_r \) and, taking into account the accepted approximations, we obtain the following differential equation of the envelope:

\[ \left( c_1 a + c_2 a^2 \right) \frac{da}{dt} = -\frac{4}{3} k \omega_i^3 a^3. \]  

(8)

Dividing the variables here, we obtain:

\[ \left( \frac{c_1}{a^2} + \frac{c_2}{a} \right) da = -\frac{4}{3} k \omega_i^3 dt. \]  

(9)

By further integration we find the constant \( c \) with accuracy:

\[ c_2 \ln a - \frac{c_1}{a} = -\frac{4}{3} \omega_i^3 t + c. \]  

(10)

To determine the constant \( c \) we use the condition \( a = a_{i-1} \) at \( t=0 \).

Then (10) takes the form:

\[ \ln \left( \frac{1}{a_i} + \frac{c_1}{c_2 a_i} \right) = \ln \left( \frac{1}{a_{i-1}} + \frac{c_1}{c_2 a_{i-1}} + \frac{4}{3} \omega_i^3 t \right). \]  

(11)

By substituting in (11) \( t = \pi / \omega_i \) we obtain the relation:

\[ \ln \left( \frac{1}{a_i} + \frac{c_1}{c_2 a_i} \right) = \ln \left( \frac{1}{a_{i-1}} + \frac{c_1}{c_2 a_{i-1}} + \frac{4}{3} \omega_i^3 \frac{\pi}{c_2} \right). \]  

(12)

Next, we will distinguish between cases of hard and soft power characteristics. In the case of \( c_1 > 0 \) we present (12) in the form:

\[ \ln \left( \frac{c_1}{c_2 a_i} \right) + \frac{c_1}{c_2 a_i} = y_i, \]  

(13)

where
\begin{equation}
y_i = \ln \left( \frac{c_i}{c_2 a_{i-1}} \right) + \frac{c_i}{c_2 a_{i-1}} + \frac{4}{3} k \omega_i^2.
\end{equation}

The solution of the equation (13) is expressed in terms of the Lambert \( W(\zeta) \) function of a positive argument. Consequently:

\begin{equation}
a_i = \frac{c_i}{c_2 W[\exp(y_i)]}.
\end{equation}

For an oscillator with a hard power characteristic we may take:

\begin{equation}
\omega_i^2 = \frac{c_i}{m} + \frac{4c_2}{3\pi m} (a_{i-1} + a_i).
\end{equation}

Using (14), (15) and (16), the calculation of the amplitude is reduced to iterative relations:

\begin{equation}
a_i^{(j+1)} = \frac{c_i}{c_2 W[\exp(\gamma_i^{(j)})]}, j = 0, 1, 2, \ldots;
\end{equation}

\begin{equation}
\gamma_i^{(j)} = \ln \left( \frac{c_i}{c_2 a_{i-1}} \right) + \frac{c_i}{c_2 a_{i-1}} + \frac{4}{3} k \frac{c_i}{m} \left[ \frac{4c_2}{3\pi m} (a_{i-1} + a_i) \right],
\end{equation}

To calculate \( a_i \) you need to perform several iterations, assuming a zero approximation:

\begin{equation}
a_i^{(0)} = \frac{a_{i-1}}{1 + \frac{4}{3} k, a_{i-1}}.
\end{equation}

The values of the \( W(\zeta) \) function can be found in the tables printed in [15], and for large \( \zeta \) it is advisable to use asymptotics [16]:

\begin{equation}
W(\zeta) \approx P - Q + \frac{Q}{P} + \frac{Q(Q - 2)}{2P^2} + \frac{Q(2Q^2 - 9Q + 6)}{6P^3} + \frac{Q(3Q^3 - 22Q^2 + 36Q - 12)}{12P^4},
\end{equation}

where \( P = \ln \zeta, Q = \ln P \).

There are other variants of approximations \( W(\zeta) \) for large \( \zeta \), one of which is:

\begin{equation}
W(\zeta) \approx P \left[ 1 - \frac{3 + 3P + 2Q}{4P + 1} + \sqrt{\left( \frac{3 + 3P + 2Q}{4P + 1} \right)^2 - \frac{6Q}{4P + 1}} \right].
\end{equation}

The errors of approximations (20) and (21) are written down in Table 1, which also shows the values of \( W(\zeta) \), calculated on a computer in the environment "Maple".

\begin{table}[h]
\centering
\caption{The values of \( W(\zeta) \), calculated in three ways}
\begin{tabular}{cccccccc}
\hline
\( \zeta \) & \( f.(20) \) & \( f.(21) \) & «Maple» & \( \zeta \) & \( f.(20) \) & \( f.(21) \) & «Maple» \\
\hline
Values \( W(\zeta) \) & Values \( W(\zeta) \) \\
3 & 1.0297 & 1.0499 & 1.0499 & 50 & 2.8604 & 2.8607 & 2.8609 \\
4 & 1.2017 & 1.2022 & 1.2022 & 100 & 3.3852 & 3.3854 & 3.3056 \\
5 & 1.3333 & 1.3267 & 1.3267 & 200 & 3.9294 & 3.9296 & 3.9297 \\
10 & 1.7448 & 1.7454 & 1.7455 & 1000 & 5.2495 & 5.2495 & 5.2496 \\
\hline
\end{tabular}
\end{table}
Calculations show that the errors of the approximation (21) are smaller than the approximation (20).

Example 1. Using formula (17) we calculate ten oscillation amplitudes. For calculations we take: \( m = 2 \text{ kg}; \ k = 10 \text{ N s}^2/\text{m}^2; \ c_1 = 800 \text{ N/m}; \ c_2 = 4000 \text{ N/m}^2; \ a_0 = 0.05 \text{ m}. \) The results of the calculations are written down in Table 2.

The differences in the values of the amplitudes of the oscillations obtained in two ways are insignificant. The error of the approximate method is less than 1%.

| Table 2. The values of the amplitudes at \( c_2 > 0 \) | \( \varepsilon \) | \( f(17) \) | \( \times 10^2 a_i, \text{m} \) | \( \varepsilon \) | \( f(17) \) | \( \times 10^2 a_i, \text{m} \) |
|---|---|---|---|---|---|---|
| 1 | 3.7736 | 3.7851 | 6 | 1.6864 | 1.6969 |
| 2 | 3.0278 | 3.0417 | 7 | 1.5176 | 1.5272 |
| 3 | 2.5268 | 2.5406 | 8 | 1.3795 | 1.3881 |
| 4 | 2.1674 | 2.1802 | 9 | 1.2643 | 1.2721 |
| 5 | 1.8970 | 1.9087 | 10 | 1.1668 | 1.1740 |

Here are the calculation formulas for the soft elasticity characteristic. In the case of \( c_2 < 0 \) the equation (12) will have the following form:

\[
\ln \left( \frac{c_1}{c_2 |a_i|} \right) - \frac{c_1}{c_2 |a_{i-1}|} = z_i, \tag{22}
\]

where

\[
z_i = \ln \left( \frac{c_1}{c_2 |a_{i-1}|} \right) - \frac{c_1}{c_2 |a_{i-1}|} - \frac{4}{3} \frac{k}{c_2} \omega_i^2; \tag{23}
\]

\[
\omega_i^2 = \frac{c_1}{m} - \frac{4}{3} c_2 \left( a_{i-1} + a_i \right). \tag{24}
\]

The solution of the equation (22) is expressed through the second branch \( W_2(-\zeta) \) of the Lambert W function of the negative argument, namely:

\[
a_i = -\frac{c_1}{|c_2| W_2(-\exp(z_i))}. \tag{25}
\]

Therefore, the calculation of amplitudes is reduced to the use of an iterative formula:

\[
a_i^{(j+1)} = -\frac{c_1}{|c_2| W_2(-\exp(z_i^{(j)}))}, \quad j = 0, 1, 2, \ldots, \tag{26}
\]

in which:

\[
z_i^{(j)} = \ln \left( \frac{c_1}{|c_2| a_{i-1}} \right) - \frac{c_1}{|c_2| a_{i-1}} - \frac{4}{3} \frac{k}{c_2} \left[ \frac{c_1}{m} - \frac{4}{3} \frac{k}{c_2} \left( a_{i-1} + a_i^{(j)} \right) \right]. \tag{27}
\]

The zero approximation \( a_i^{(0)} \), is still determined by formula (19).
The table of the function $W_2(-\zeta)$ is printed in [15]. It is possible to use approximation for calculations of values of the special function along with it in the interval $\zeta \in \left[ \frac{1}{e}; 0 \right]$: 

$$W_2(-\zeta) \approx -P \left[ 1 + \frac{3 - 3P + 2Q}{4P - 1} + \sqrt{\frac{3 - 3P + 2Q}{4P - 1}} + \frac{6Q}{4P - 1} \right],$$

(28)

where $P = -\ln(\zeta)$; $Q = \ln(P)$.

The accuracy of this approximation is provided in Table 3, which also indicates the values $W_2(-\zeta)$, borrowed from [15].

| Value: $-W_2(-\zeta)$ | $f(28)$ | $f(15)$ | Error % |
|------------------------|---------|---------|--------|
| 0.001                  | 9.1182  | 9.1180  | 0.002  |
| 0.010                  | 6.4732  | 6.4728  | 0.006  |
| 0.100                  | 3.5787  | 3.5772  | 0.041  |
| 0.200                  | 2.5448  | 2.5426  | 0.087  |
| 0.300                  | 1.7829  | 1.7813  | 0.090  |

As you can see, formula (28) provides a fairly high accuracy of calculations.

Example 2. Table 4 shows the values of the amplitudes of free oscillations of the oscillator with a soft characteristic of elasticity calculated by formula (17), when $m=2$ kg; $k=10$ N s$^2$/m$^2$; $c_1=800$ N/m; $c_2=-4000$ N/m$^2$; $a_0=0.05$m.

In the same place, for comparison, the values of $a_i$ are written down, which are caused by the numerical computer integration of equation (1), under the initial conditions (2).

The differences between the results of the calculation by two methods are insignificant, which confirms the probability of the approximate formula (26).

| $i$ | $f(26)$ | numerical integration value $10^2 a_i, m$ | $f(26)$ | numerical integration value $10^2 a_i, m$ |
|-----|---------|------------------------------------------|---------|------------------------------------------|
| 1   | 3.7143  | 3.6784                                   | 4       | 2.1099                                   | 2.0809 |
| 2   | 2.9602  | 2.9227                                   | 5       | 1.8460                                   | 1.8211 |
| 3   | 2.4629  | 2.4295                                   | 6       | 1.6411                                   | 1.6195 |

Let us consider further the implementation of the second variant of the energy balance method without using the equation of the envelope graph of oscillations. To do this, the change in potential energy in the iterative half-cycle is approximately given by the expression:

$$\Delta I \approx (c_s a + c_{2s} a^2)(a_i - a_{i-1}).$$

(29)

Then the equality $\Delta I = A_f$, taking into account (6) and (29), is reduced to the cubic equation:

$$\xi_i^3 + a_i \xi_i^2 + a_i \xi_i + c_i = 0,$$

(30)

in which $\xi_i = a_i + a_{i-1}; a_i = \frac{2k}{3m} + \frac{c_1}{c_i}; a_i = \frac{c_2 a_{i-1}}{c_i} - \frac{mc_i}{16kc_2}$. 

6
Let us introduce the notation: \( p = \sigma - \frac{1}{3}a_{+}^2 \); \( q = c_{+} - \frac{1}{3}a_{+} \alpha_{+} + 2 \left( \frac{a_{+}}{3} \right)^3 \).

The solution of the equation (30), at \( c_{+} > 0 \), has the form:

\[
\xi_i = 2\sqrt{\frac{p}{3}} \cos \frac{\alpha - \frac{a_{+}}{3}}{2},
\]

where \( \alpha = \arccos \left( -\frac{q}{2\sqrt{\left( p / 3 \right)^3}} \right) \).

If \( c_{+} < 0 \), then:

\[
\xi_i = -2\sqrt{\frac{p}{3}} \cos \frac{\alpha + \pi}{3} - \frac{a_{+}}{3},
\]

Then we can calculate \( a_{+} \), because:

\[
a_{+} = 2\xi_i - a_{-+}.
\]

Using formulas (31), (32) and (33) the results of calculations of amplitudes of oscillations are written down in Table 5. There are also errors in these formulas, if the results of numerical computer integration, listed in Table 2 and Table 4, are taken for the exact values of the amplitudes.

| Table 5. The results of calculations by formulas (31) - (33) |
|-------------------------------------------------------------|
| \( i \) | \( 100a_{+}, \text{m} \) | \( \text{error,\%} \) | \( 100a_{+}, \text{m} \) | \( \text{error,\%} \) |
|---|---|---|---|---|
| 1 | 3.7567 | 0.75 | 3.6889 | 0.29 |
| 2 | 3.0102 | 1.04 | 2.9356 | 0.44 |
| 3 | 2.5114 | 1.15 | 2.4422 | 0.52 |
| 4 | 2.1543 | 1.19 | 2.0928 | 0.57 |
| 5 | 1.8860 | 1.19 | 1.8319 | 0.59 |
| 6 | 1.6770 | 1.17 | 1.6294 | 0.61 |

Calculations show that the errors of the energy balance method are small when using the cubic equation (30).

Identification of the value of the quadratic resistance coefficient. Using dependences (6) and (29), the equality \( \Delta II = A_f \) will have the following form:

\[
(c_{ii} + c_{+}a_{+}^2)(a_{+} - a_{-+}) = -\frac{4}{3}ka_{+}^3 \alpha_{+}^2.
\]

From which the formula for calculating follows:

\[
k = \frac{3\left( c_{ii} + c_{+}a_{+}\right)(a_{+} - a_{-+})}{4a_{+}^2 \alpha_{+}^2}
\]

Here, \( a = 0.5(a_{-+} + a_{+}) \), \( \alpha_{+}^2 \) – are defined by expressions (16) and (22).

Example 3. Using (35), we calculate the values \( k \) at: \( m = 2 \text{kg} \); \( c_{i} = 800 \text{N/m} \); \( c_{+} = 4000 \text{N/m}^2 \); \( a_0 = 0.04 \text{ m} \); \( a_{i} = 0.03 \text{ m} \). The values of \( a_{0} \) and \( a_{i} \) are considered measured on the vibrogram of free oscillations. Substituting the numerical data in (35) we obtain: \( k \approx 12.527 \text{ N s}^2/\text{m} \). To check the correctness of the identification we use formulas (15) and (21). With the above calculated data and
а₀=0.04 m it gives a₁ ≈ 0.03014 m, which is close to the value of a₁ involved in the identification. Thus, the solution of the direct problem of dynamics confirmed the correctness of the identification.

5. Conclusions

Our studies have confirmed that the calculation of the amplitudes of free damped oscillations of an oscillator with double quadratic nonlinearity can be performed approximately by the energy balance method without constructing an analytical solution of the nonlinear differential equation of motion.

The implementation of this method is possible in two forms: using the Lambert W function and without it. Both forms of the method lead to small errors in determining the amplitudes of damped oscillations, which makes it possible to identify the coefficient of viscous resistance by measuring the amplitudes of free oscillations on the vibrogram.

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