ON THE LIE GROUP STRUCTURE
OF PSEUDO-FINSLER ISOMETRIES

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ABSTRACT. Using an extension to isometries of the associated Sasaki structure,
we establish a Lie transformation group structure for the set of isometries of a
pseudo-Finsler conical metric.

1. INTRODUCTION

A very classical problem in mathematics is to establish which topological groups
have a Lie group structure (Hilbert’s fifth problem). A complete answer to this
question has been given by several authors in the fifties, see for instance [4, 13].
According to Gleason and Yamabe’s result, a topological group has a Lie group
structure compatible with its topology if and only if it does not contain small sub-
groups, i.e., if there is some neighborhood of the identity that does not contain any
proper subgroup. A natural and important extension of this problem in Geometry,
is to establish when the action of some group \( G \) on a differentiable manifold \( M \) is
a Lie transformation group. Recall that a Lie transformation group consists of a
Lie group \( G \) and a smooth action of \( G \) on a differentiable manifold \( M \) by diffeo-
morphisms. By a result of Kuranishi (see [7]), an effective action (by diffeomor-
phisms) of a locally compact group on a smooth manifold is a Lie transformation
group. This result is particularly useful in order to establish the existence of a Lie
transformation group structure for groups of distance preserving maps of metric
spaces. Namely, the isometry group of a locally compact metric space is a locally
compact topological group, endowed with the compact–open topology. By a well
known result of Myers and Steenrod (see [10]), isometries of a Riemannian mani-
ifold coincide with isometries of the underlying metric structure; in particular, the
natural action of the isometry group of a Riemannian manifold is a Lie transfor-
mation group. A similar argument has been employed more recently by Deng and
Hou to show that the group of isometries of a (non necessarily reversible) Finsler
manifold is a Lie transformation group, see [3]. We will see here that a natural aver-
aging procedure allows to reduce the Finsler case to the standard Riemannian case
(Theorem B). Myers and Steenrod’s result has been further developed by Palais
(see [11]), who showed that the differentiable structure of a Riemannian manifold
can be recovered merely from its metric space structure.

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1The action of a group \( G \) on a set \( X \) is effective if the unique element of \( G \) that fixes all the
elements of \( X \) is the identity.
When it gets to isometries of metrics with signature, like Lorentzian metrics, or more generally pseudo-Riemannian metrics, there is no naturally associated metric space structure, and thus Myers and Steenrod’s techniques do not apply. A beautiful general theory, developed mostly in [12] and [6], studies the question of establishing a differentiable structure for the set of automorphisms of a $G$-structure on a smooth manifold $M$. Such a theory allows to reduce to a relatively simple algebraic problem the question of establishing for which Lie group $G \subset GL(n)$, given any $G$-structure $P$ on any $n$-manifold $M$, the group of automorphisms of $P$ is a (finite dimensional) Lie subgroup of the group of diffeomorphisms of $M$. Curiously enough, such algebraic problem only involves the Lie algebra of $G$. The result applies, in particular, to all orthogonal groups $O(n, k)$, as well as conformal groups; thus, the set of isometries or the set of conformal diffeomorphisms of any pseudo-Riemannian manifold is a Lie transformation group. An interesting issue of the theory is the question of regularity of the automorphisms, and the corresponding topology in the automorphism group. Thanks to Myers–Steenrod’s (or Palais’) result, for Riemannian isometries continuity is equivalent to smoothness, and all $C^k$-topologies coincide in the isometry group, $k = 0, \ldots, +\infty$. In particular, the group of Riemannian isometries endowed with the compact–open topology is a Lie transformation group. By the $G$-structure automorphism theory, for pseudo-Riemannian isometries one must consider the $C^1$-topology, while for conformal diffeomorphisms one has to consider the $C^2$-topology. As a matter of facts, an elementary argument using the exponential map shows that, also in the isometry group of a general pseudo-Riemannian manifold the $C^1$-topology coincides with the compact-open topology. Interestingly enough, the differentiability class of automorphisms of a $G$-structure coincides with the so-called order of the $G$-structure, which is roughly speaking the minimal order of derivatives at a fixed point needed to determine uniquely an automorphism of the given structure. Finiteness of the order of a $G$-structure is the key property for the development of the theory.

It is an important question to study automorphisms of pseudo-Finsler structures, which arise naturally in General Relativity. A pseudo-Finsler structure on a (connected) manifold $M$ consists of an open subset $\mathcal{T} \subset T_0M$, where $T_0M$ denotes the tangent bundle with its zero section removed, and a smooth function $F : \mathcal{T} \rightarrow \mathbb{R}^+$ satisfying the following properties:

- for all $p \in M$, the intersection $\mathcal{T}_p = \mathcal{T} \cap T_pM$ is a non empty open cone of the tangent space $T_pM$;
- $F(tv) = tF(v)$ for all $v \in \mathcal{T}$ and all $t > 0$;
- for all $v \in \mathcal{T}$, the second derivative $g_v = \left( \frac{\partial^2 F^2}{\partial y_i \partial y_j}(v) \right)_{ij}$ in the vertical directions is nondegenerate.

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2 By $C^k$-topology on the group of diffeomorphisms of a manifold, we mean the weak Whitney $C^k$-topology, i.e., the topology of uniform convergence on compacta of all derivatives up to order $k$.

3 Although in the Riemannian case, by a somewhat involved argument, the compact–open topology coincides with all other $C^k$-topologies in the conformal group.

4 There are several notions of pseudo-Finsler structures in the literature. Here we use a quite general notion, sometimes called conic pseudo-Finsler structure. A somewhat different notion is given in [2]. See [5] for a discussion on the several notions of pseudo-Finsler manifolds.
By continuity, the fundamental tensor $g_v$ has constant index, which is called the index of the pseudo-Finsler structure. The case when $\mathcal{T} = T_0 M$ and the index of $g_v$ is zero, i.e., $g_v$ is positive definite for all $v$, is the standard Finsler structure. When $g_v$ does not depend on $v$, then we have a standard pseudo-Riemannian manifold. An automorphism (or isometry) of the pseudo-Finsler structure $(M, \mathcal{T}, F)$ is a diffeomorphism $f$ of $M$, with $df(\mathcal{T}) = \mathcal{T}$ and $F \circ df = F$. Clearly, the set $\text{Iso}(M, \mathcal{T}, F)$ of such automorphisms is a group with respect to composition, and one has a natural action of $\text{Iso}(M, \mathcal{T}, F)$ on $M$. In order to establish a Lie transformation group structure for this set, which is the purpose of the paper, one cannot apply metric space techniques, nor $G$-structure techniques. Namely, it is not hard to show that the $G$-structure corresponding to Finsler or pseudo-Finsler metrics has never finite order. Similarly, also the averaging technique mentioned above for the standard Finsler example does not work in the pseudo-Finsler case, due to the fact that:

- sums (or even convex combinations) of non positive definite nondegenerate symmetric bilinear forms may fail to be nondegenerate;
- at every point $p \in M$, the indicatrix $\Sigma_p = F^{-1}(1) \cap T_p$ is never compact.

In this paper we will use general techniques from calculus with non linear connections in vector bundles and sprays to prove the following results.

**Theorem A.** The group of isometries of a pseudo-Finsler structure $(M, F)$, endowed with the $C^1$-topology, is a Lie transformation group of $M$.

The same proof of Theorem A will also yield the following:

**Corollary.** An isometry of a pseudo-Finsler structure $(M, F)$ is a $C^\infty$-map, and it is completely determined by its second jet at any point.

We will also discuss briefly the averaging technique mentioned above, that allows to reduce Deng–Hou’s result to the standard Riemannian case, proving:

**Theorem B (S. Deng and Z. Hou).** The group of isometries of a Finsler structure $(M, F)$, endowed with the compact–open topology, is a Lie transformation group of $M$. Finsler isometries are smooth, and they are uniquely determined by their first jet at any point of $M$.

The proof of our results will make it clear that totally analogous results hold in the case of different notions of pseudo-Finsler structure. More precisely, a Lie transformation group structure exists for any group of diffeomorphisms of a manifold $M$ that preserve a geodesic spray defined in suitable open subsets of $TM$, see next section for details.

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## 2. Proofs

**Quasi-tangent structure of $TM$.** In order to define a (non linear) connection associated to a pseudo-Finsler structure, we will follow Grifone’s terminology, see [9]. Let $\pi : TM \to M$ be the canonical projection; for $v \in TM$, denote by $\text{Ver}_v = \text{Ker}(d\pi_v)$ the vertical subspace of $T_v(\mathcal{T})$; $\text{Ver}$ will denote the vertical
distribution on $T M$. First, one defines the quasi-tangent structure of $T M$ as the $(1, 1)$ tensor $J$ in $T M$ by:

$$J(X) = i_v (d\pi (X)),$$

where $v \in T_p M$, $X \in T_v (T M)$, and $i_v : T_p M \rightarrow \text{Ver}_v$ is the canonical identification ($i_v$ is the differential at $v$ of the inclusion $T_p M \hookrightarrow T M$).

**Lemma 1.** If $f : M \rightarrow M$ is a diffeomorphism (of class $C^2$); the quasi-tangent structure $J$ of $T M$ is invariant by the diffeomorphism $d f : T M \rightarrow T M$, i.e., the pull-back $(d f)^* (J)$ equals $J$.

**Proof.** Since $d f$ send fibers of $T M$ into fibers, then clearly $d f$ preserves the vertical distribution, which is the tangent distribution to the fibers. For $p \in M$ and $v \in T_p M$ (in fact, for $v \in T_p M$), one has the following commutative diagrams:

$$\begin{align*}
T_v (T M) & \xrightarrow{d^2 f} T_{d f(v)} (T M) \\
\downarrow d\pi & \quad \downarrow d\pi \\
T_p M & \xrightarrow{d f} T_{f(p)} M
\end{align*}$$

The commutativity of the first diagram is obvious. For the second, it suffices to differentiate the commutative diagram:

$$\begin{align*}
T_p M & \xrightarrow{d f} T_{f(p)} M \\
\downarrow \text{inclusion} & \quad \downarrow \text{inclusion} \\
T M & \xrightarrow{d f} T M.
\end{align*}$$

The equality $(d f)^* (J) = J$ follows readily from (2.1). \qed

**Orthogonal distribution associated to a pseudo-Finsler structure.** Consider now a pseudo-Finsler structure $(M, T, F)$, and let $S$ denote the vector field in $T$ given by the geodesic spray of $F$. There is a complement to this space associated to $S$, the horizontal space, which is defined as follows. The spray $S$ satisfies the identity $J(S) = C$, where $C$ is the tautological vertical field of $T M$, or Liouville field, (i.e., $C_v = i_v (v)$), and the identity $[C, S] = S$, where $[,]$ are the Lie brackets of $T M$. Moreover, the Lie derivative $\Gamma_S = -L_S (J)$ of the quasi-tangent structure $J$ is a $(1, 1)$ tensor on $T$ that satisfies (see [9]):

$$\begin{align*}
(\Gamma_S)^2 & = \text{Id}, \\
\text{Ker} (\Gamma_S + I) & = \text{Ver}.
\end{align*}$$

By (2.2), $\text{Hor}^S := \text{Ker} (\Gamma_S - \text{Id})$ is a distribution in $T$ which is complementary to $\text{Ver}$, and it will be called the orthogonal distribution associated to the pseudo-Finsler structure $(M, T, F)$.

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5The identity $J(S) = C$ means that the integral curves of $S$ are of the form $t \mapsto \gamma'(t) \in T$, for some curve $t \mapsto \gamma (t) \in M$. Such curves $\gamma$ are precisely the geodesics of $S$. The identity $[C, S] = S$ means that affine reparameterizations of geodesics of $S$ are geodesics.
The Sasaki metric. Denote by $k$ the index of the fundamental tensor $g_v$ of the pseudo-Finsler structure $(M, T, F)$; we will now define a pseudo-Riemannian metric $g^F$ on $T$ having index $2k$. For $v \in T$, the spaces $\text{Ver}_v$ and $\text{Hor}_v^S$ are $g^F$-orthogonal. The restriction of $g^F$ to $\text{Ver}_v$ is the push-forward of $g_v$ by the isomorphism $i_v : T_pM \to \text{Ver}_v$, and the restriction of $g^F$ to $\text{Hor}_v^S$ is the pull-back of $g_v$ by the isomorphism $d\pi_v : \text{Hor}_v^S \to T_pM$. Clearly, $g^F$ is a smooth $(0, 2)$-tensor field on $T$ which is everywhere nondegenerate and of index $2k$; the tensor $g^F$ is the Sasaki metric of the pseudo-Finsler structure $(M, T, F)$.

The central result is the following:

**Proposition 2.** Let $f : M \to M$ be an isometry of $(M, T, F)$, i.e., $f : M \to M$ is a diffeomorphism of class $C^2$, with $df(T) = T$, and $(df)^s(F) = F$. Then, $df : T \to T$ is an isometry of the Sasaki metric $g^F$.

**Proof.** We have already observed in Lemma 1 that $df$ preserves the vertical distribution $\text{Ver}$ and the quasi-tangent structure $\mathcal{F}$. Since $f$ is an isometry of $(M, T, F)$, then $df$ preserves the geodesic spray $S$. Hence, by construction, $df$ preserves also the orthogonal distribution $\text{Hor}^S$. The commutativity of the diagram on the left of (2.1) (when $d^2f$ is restricted to $\text{Hor}^S$) shows that $d^2f$ preserves the restriction of $g^F$ to the horizontal distribution. The commutativity of the diagram on the right of (2.1) shows that $d^2f$ preserves the restriction of $g^F$ to the vertical distribution. In conclusion, $df$ is an isometry of the pseudo-Riemannian manifold $(T, g^F)$. □

**Remark.** It is also immediate to prove that, conversely, if $f : M \to M$ is a $C^2$-map such that $df(T) = T$ and such that $df|_T : T \to T$ is a $g^F$-isometry, then in fact $f$ is a diffeomorphism of $M$ and it is an isometry of $(M, T, F)$. For this, one uses the fact that $T_p = T \cap T_pM$ is a nonempty open subset of $T_pM$ for all $p \in M$.

**Final argument.** Let us denote by $\text{Iso}(M, T, F)$ the group of $C^2$-isometries of the pseudo-Finsler structure $(M, T, F)$, and by $\text{Iso}(T, g^F)$ the isometry group of the pseudo-Riemannian manifold $(T, g^F)$. It is well known (see for instance [3]) that $\text{Iso}(T, g^F)$ is a Lie group, and that the natural action of $\text{Iso}(T, g^F)$ on $T$ is smooth. Moreover, every element of $\text{Iso}(T, g^F)$ is determined by its first jet at any point of $T$.

The proof of Theorem A and its Corollary will be obtained directly from the following two results.

**Proposition 3.** The map $\text{Iso}(M, T, F) \ni f \mapsto df \in \text{Iso}(T, g^F)$ is an injective group homomorphism, whose image is closed in the $C^1$-topology.

**Proof.** The given map is a group homomorphism, by the chain rule; it is obviously injective. In order to prove that its image is closed in the $C^1$-topology, assume that $f_n$ is a sequence of $C^2$-diffeomorphisms of $M$, with $df_n(T) = T$ such that $(df_n)|_T$ converges as $n \to \infty$ in the $C^1$-topology to a $C^1$-diffeomorphism $\Psi : T \to T$, then, by elementary arguments:

(a) $f_n$ is $C^1$-convergent to some diffeomorphism $f_\infty$ of $M$ (namely, if $s$ is a local section of $TM$ taking values in $T$, then locally $f_n = \pi \circ (df_n) \circ s$);

(b) $\Psi = df_\infty$, and thus $f_\infty$ is of class $C^2$.

This concludes the proof. □
The statements in Theorem A and its Corollary follow almost entirely from Proposition \[3\]. As to the action of \(\text{Iso}(M, \mathcal{T}, F)\), what Proposition \[3\] says is that the map \(\text{Iso}(M, \mathcal{T}, F) \times \mathcal{T} \ni (f, v) \mapsto df(v) \in \mathcal{T}\) is smooth, and it is a Lie transformation group of \(\mathcal{T}\). From this, it follows easily (see Lemma \[4\] below) that the natural action of \(\text{Iso}(M, \mathcal{T}, F)\) on \(M\) is a Lie transformation group of \(M\).

**Lie transformation groups of submersions.** Assume that \(q : E \to B\) is a smooth surjective submersion, and let \(f : E \to E\) be a diffeomorphism that carries fibers of \(q\) (diffeomorphically) onto fibers. Then one has an induced map \(\tilde{f} : B \to B\), which is again a diffeomorphism. If \(G\) is a Lie transformation group of the total space \(E\) such that the action of every element \(g \in G, E \ni x \mapsto g \cdot x \in E\), carries fibers onto fibers, then one has an induced action of \(G\) on the base \(B\). We state the following elementary result, which may have some interest of its own.

**Lemma 4.** Let \(q : E \to B\) be a smooth surjective submersion, and let \(G\) be a Lie transformation group of the total space \(E\). Assume that the action of each element of \(G\) carries fibers of \(q\) onto fibers. Then, the induced action of \(G\) on the base \(B\) makes \(G\) into a Lie transformation group of \(B\).

**Proof.** The smoothness of the induced action of \(G\) on \(B\) follows easily from the existence of local sections of \(q\). \(\square\)

The proof of Theorem A is concluded by applying Lemma \[4\] to the surjective submersion \(\pi|_\mathcal{T} : \mathcal{T} \to M\) and to the Lie transformation group \(\text{Iso}(M, \mathcal{T}, F) \times \mathcal{T} \ni (f, v) \mapsto df(v) \in \mathcal{T}\).

**The Finsler case: proof of Theorem B.** Given a Finsler structure \((M, F)\), one can define a Riemannian metric \(h_F\) on \(M\) obtained as the average of the fundamental tensor. More precisely, for all \(p \in M\), let \(\Sigma_p\) be the indicatrix of \(F\) at \(p\):

\[
\Sigma_p = \{ v \in T_p M : F(v) = 1 \}.
\]

Then, \(h_F\) is defined by:

\[
h_F(v, w) = \int_{\Sigma_p} g_u(v, w) \, d\Omega_p(u),
\]

where \(v, w \in T_p M\), and \(d\Omega_p\) is the volume associated to the Riemannian metric in \(\Sigma_p\) given by the restriction of the fundamental tensor. This averaged metric was first defined in \[8\].

The proof of Theorem B is obtained readily from the following:

**Proposition 5.** The group of isometries of \((M, F)\) is contained as a closed subgroup of \(\text{Iso}(M, h_F)\) (in the compact–open topology).

**Proof.** If \(f : M \to M\) is a diffeomorphism that preserves \(F\), then \(df\) carries indicatrices onto indicatrices, and it also preserves the fundamental tensor of \(F\), as well as the volume forms on the indicatrices associated to the fundamental tensor. Thus, \(f\) preserves \(h_F\). The condition \(f^*(F) = F\) is closed in the \(C^1\)-topology, hence \(\text{Iso}(M, F)\) is closed in \(\text{Iso}(M, h_F)\) with respect to the \(C^1\)-topology. On the other hand, the compact–open topology and the \(C^1\)-topology coincide on \(\text{Iso}(M, h_F)\). This concludes the proof. \(\square\)
REFERENCES

[1] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann–Finsler geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[2] J. K. Beem, *Indefinite Finsler spaces and timelike spaces*, Canad. J. Math. 22 (1970), 1035–1039.

[3] S. Deng and Z. Hou, *The group of isometries of a Finsler space*, Pacific J. Math., 207 (2002), pp. 149–155.

[4] A. M. Gleason, *Groups without small subgroups*, Ann. of Math. (2) 56, (1952), 193–212.

[5] M. A. Javaloyes, M. Sánchez, *On the definition and examples of Finsler metrics*, arXiv:1111.5066, to appear in Ann. Sc. Norm. Sup. Pisa, DOI Number: 10.2422/2036-2145.201203_002.

[6] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70. Springer-Verlag, New York–Heidelberg, 1972.

[7] M. Kuranishi, *On conditions of differentiability of locally compact groups*, Nagoya Math. J. 1 (1950), 71–81.

[8] R. Gallego Torromé, *Averaged structures associated with a Finsler structure*, arXiv:math/0501058v9 [math.DG].

[9] J. Grifone, *Structure presque-tangente et connexions I, II*, Ann. Inst. Fourier (Grenoble) 22 (1972), no. 1, 287–334, no. 3, 291–338.

[10] S. B. Myers and N. E. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2), 40 (1939), pp. 400–416.

[11] R. S. Palais, *On the differentiability of isometries*, Proc. Amer. Math. Soc. 8 (1957), 805–807.

[12] S. Sternberg, *Lectures on differential geometry*, Second edition. Chelsea Publishing Co., New York, 1983.

[13] H. Yamabe, *A generalization of a theorem of Gleason*, Ann. of Math. (2) 58, (1953), 351–365.

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