THE ENERGETICS OF A FLARING SOLAR ACTIVE REGION AND OBSERVED FLARE STATISTICS

M. S. WHEATLAND

School of Physics, University of Sydney, NSW 2006, Australia

Received 2008 January 20; accepted 2008 February 26

ABSTRACT

A stochastic model for the energy of a flaring solar active region is presented, generalizing and extending the approach of Wheatland and Glukhov. The probability distribution for the free energy of an active region is described by the solution to a master equation involving deterministic energy input and random jump transitions downward in energy (solar flares). It is shown how two observable distributions, the flare frequency-energy distribution and the flare waiting-time distribution, may be derived from the steady state solution to the master equation, for given choices for the energy input and for the rates of flare transitions. An efficient method of numerical solution of the steady state master equation is presented. Solutions appropriate for flaring, involving a constant rate of energy input and power-law distributed jump transition rates, are numerically investigated. The flarelike solutions exhibit power-law flare frequency-energy distributions below a high-energy rollover, set by the largest energy the active region is likely to have. The solutions also exhibit approximately exponential (i.e., Poisson) waiting-time distributions, despite the rate of flaring depending on the free energy of the system.

Subject headings: methods: statistical — Sun: activity — Sun: corona — Sun: flares

1. INTRODUCTION

Solar active regions are the sites of occurrence of most solar flares. Large active regions may persist for several solar rotations and produce dozens of significant flares during a transit of the disk (Richardson 1951). The largest flares are believed to involve the release of more than $10^{27}$ J of stored magnetic energy, the energy appearing in accelerated particles, heating, bulk motion of material, and radiation (Hudson 1991). The process underlying flares is accepted to be magnetic reconnection, although the details of the process remain a subject of study (Priest & Forbes 2002).

The dynamical energy balance of solar active regions presents challenges to our understanding. The energy liberated in flares is thought to be excess or “free” magnetic energy associated with electric current systems in the solar atmosphere, but the origin of the currents, and hence the source of the energy, is not well understood (McClymont & Fisher 1989; Tandberg-Hanssen & Emslie 1988; Leka et al. 1996; Wheatland 2000a; Schrijver et al. 2005). Two popular pictures for energy supply are first that existing coronal magnetic structures are twisted and sheared by photospheric motions, producing currents, and second that new current-carrying magnetic flux emerges through the photosphere. These mechanisms suggest that energy supply to an active region may be described as a continuous process, driven by slow photospheric and subphotospheric motions. In contrast, flare energy release is rapid and unexpected. An important point is that the size of the downward jump in energy associated with a flare may be very large, by comparison with the amount of stored active region energy.

The understanding of active region energy balance is hampered by the inability to calculate coronal magnetic energy. In principle, the magnetohydrodynamic (MHD) virial theorem permits the calculation of magnetic energy from vector field values inferred in the chromosphere subject to the assumption that the field is everywhere force-free (e.g., Metcalf et al. 2005). However, few chromospheric vector field determinations are made, and the reliability of the method is unknown. Methods for modeling coronal magnetic fields from photospheric or chromospheric boundary conditions are being developed (Metcalf et al. 2008; Schrijver et al. 2008), and these may permit estimation of coronal magnetic energy. There has also been recent progress in methods for estimating the rate of supply of energy to an active region directly from observations (e.g., Welsch et al. 2007). For example, in ideal MHD the Poynting flux is $S = (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{B}/\rho_s$, where $\mathbf{u}$ is the fluid velocity and $\mathbf{B}$ is the magnetic field, and in principle, this quantity may be estimated at the photosphere from observations.

Although we lack detailed quantitative information about the rate of energy supplied to and the energy stored in active regions, we do have detailed information about solar flare occurrence. The energy released in flares may be estimated (albeit subject to some error), and the rate of occurrence of flares is observed. Two related statistical properties of flares—the frequency-energy distribution and the distribution of times between events—have been studied in some detail, as summarized below.

Studies of the frequency-energy distribution show that it is a power law over many decades in energy (Hudson 1991; Crosby et al. 1993; Aschwanden et al. 1998). Specifically, the distribution may be written as

$$N(E) = AE^{-\gamma},$$

where $N(E)$ is the number of flares per unit time and per unit energy $E$, the factor $A$ is a (time-dependent) measure of the total flaring rate, and $\gamma \approx 1.5$. Typically, this distribution is determined for flares from all active regions on the Sun over some period of time, but it also appears to apply to individual active regions (Wheatland 2000b), which suggests that the power law is intrinsic to the flare mechanism. A popular model explaining the power law is the avalanche model (Lu & Hamilton 1991; Charbonneau et al. 2001), in which the magnetic field in the corona is assumed to be in a self-organized critical state and subject to avalanches of small-scale reconnection events.

There has been considerable interest in the flare waiting-time distribution and, more generally, in waiting-time distributions for what they reveal about underlying physics in a variety of systems (Sánchez et al. 2002). Determinations of flare waiting-time distributions have given varied results (Pearce et al. 1993; Biesecker 1994; Wheatland et al. 1998; Boffetta et al. 1999; Moon et al. 2001,
2002; Wheatland 2001). The results suggest that the observed distribution depends on the particular active region and on time, and that it is also influenced by event definition and selection procedures (Wheatland 2001; Buchlin et al. 2005; Paczuski et al. 2005; Baiesi et al. 2006). For some active regions, the distribution appears to be consistent with a simple Poisson process, i.e., independent events occurring at a constant mean rate (Moon et al. 2001). The corresponding waiting-time distribution is exponential. Other active regions show time variation in the flaring process, and flare occurrence may be approximated by a piecewise constant or, more generally, by a time-varying Poisson process (Wheatland 2001). On longer timescales, a power-law tail is observed for events from the whole Sun (Boffetta et al. 1999). This may be accounted for in terms of a time-dependent Poisson model (Wheatland & Litvinenko 2002), although some authors have argued that the power law has fundamental significance (Lepreti et al. 2001). We note that the waiting-time distribution has often been considered in isolation from other flare statistics, in particular the frequency-energy distribution.

To understand the statistics of flare occurrence, it is desirable to have a general theory for the energetics of an active region, relating the free energy of the active region to the observed frequency-energy and waiting-time distributions. Rosner & Vaiana (1978) presented the first model of this kind, which was analogous to Fermi acceleration. In this model, active regions experience exponential growth in free energy between flares which occur as a Poisson process in time, depleting all accumulated energy. This gives a power-law flare frequency-energy distribution for large energies and presupposes Poisson occurrence. Litvinenko (1994, 1996) generalized the model to incorporate different rates of supply of energy to the system, but the Litvinenko models retained the feature that each flare releases all of the free energy. This aspect of the Rosner & Vaiana (1978) model was criticized by Lu (1995). Wheatland & Glukhov (1998) introduced a model permitting arbitrary changes in free energy at each flare. The model assumes the free energy \( E \) of an active region increases secularly between jump transitions downward in energy (flares), which occur at a rate \( \alpha(E, E') \) for jumps from \( E \) to \( E' \), per unit energy. A master equation describes the steady state energy balance, and the solution of this equation is the probability distribution \( P(E) \) for the free energy. The flare frequency-energy distribution is given by the convolution

\[
N(E) = \int_{E}^{\infty} P(E') \alpha(E', E - E') dE'.
\]

Wheatland & Glukhov (1998) investigated solutions to the master equation for constant energy input and for a choice of transition rates \( \alpha(E, E') \sim (E - E')^{-\gamma} \), which leads to power-law behavior \( N(E) \sim E^{-\gamma} \) (below a high-energy rollover set by the highest energy an active region is likely to have). Wheatland & Glukhov (1998) did not determine a waiting-time distribution for the model. It was argued that the derived power-law solutions are consistent with an avalanche-type model, and avalanche models have simple Poisson waiting-time statistics (Wheatland et al. 1998; Sánchez et al. 2002). However, this presents a puzzle: the total rate of flaring in the Wheatland & Glukhov (1998) model is given by

\[
\lambda(E) = \int_0^E \alpha(E, E') dE',
\]

which depends on the energy of the system. Hence, it is expected that the occurrence of flares is not strictly Poisson, since the occurrence of a flare changes the energy of the system and, hence, the instantaneous total rate of flaring. Non-Poisson waiting-time statistics might then be expected.

Recently, Daly & Porporato (2007) demonstrated how to determine steady state waiting-time distributions for continuous time processes with arbitrary jump transitions. The Daly & Porporato (2007) theory is quite general, applying to any system described by a single time-dependent stochastic variable \( x(t) \) following a deterministic trajectory interrupted by positive or negative jumps of random timing and size. The probability distribution \( P(x, t) \) for \( x(t) \) is described by a master equation, and the waiting-time distribution for jumps in the steady state may be obtained from the solution to the steady state master equation. Daly & Porporato (2007) demonstrated the theory in application to simple models for human attention, for voltage across a nerve membrane, and for soil moisture content associated with rainfall events. In each case, the models were analytic, involving simple solutions to the master equation. In this paper, the Daly & Porporato (2007) theory is applied to the Wheatland & Glukhov (1998) model for active region free energy, and waiting-time distributions are derived for solutions of relevance to flares. The application of the theory is relatively straightforward: the Wheatland & Glukhov (1998) model is in the class of models considered by Daly & Porporato (2007), the time-dependent stochastic variable being the active region energy \( E(t) \). Minor modifications to the theory are required, because \( E \) is a positive definite quantity and because the jump transitions in \( E \) are always negative. A specific difficulty in applying the theory is that for the power-law form for rate transitions relevant for flares, the master equation is not amenable to analytic solution. In this paper, an efficient numerical method of solution of the steady state master equation is presented and applied. This paper also considers a more general form for flare transition rates than considered in Wheatland & Glukhov (1998). The results resolve the puzzle outlined above concerning whether the model produces an exponential waiting-time distribution.

The sections of the paper are as follows. Section 2 presents the model, starting with the time-dependent master equation (§ 2.1). Section 2.2 shows how the steady state waiting-time distribution may be obtained, § 2.3 presents simple analytic solutions illustrating the theory, and § 2.4 considers the information provided by moments of the master equation. Section 3 presents flarelike solutions to the master equation, starting with a justification of appropriate choices for the rates of transitions and for the energy supply rate (§ 3.1). Section 3.2 describes the numerical method, and § 3.3 presents the results. Section 4 discusses the results and their significance for understanding solar flares.

2. MODEL

2.1. Master Equation

An active region is modeled as a system with free energy \( E(t) \) which evolves in time due to secular energy input and jumps downward in energy at random times and of random sizes. The system is described by the time-dependent master equation (van Kampen 1992; Gardiner 2004) for the probability distribution \( P(E, t) \) of the free energy,

\[
\frac{\partial P(E, t)}{\partial t} = -\frac{\partial}{\partial E} \left[ \beta(E, t) P(E, t) \right] - \lambda(E) P(E, t) + \int_E^{\infty} P(E', t) \alpha(E', E) dE',
\]

where \( \beta(E, t) \) describes the energy input rate at time \( t \), \( \alpha(E, E') \) describes the rate of flare jumps from \( E \) to \( E' \), and \( \lambda(E) \) is the total rate of flaring, given by equation (3). The terms on the right-hand
side of equation (4) describe the system gradually increasing in energy due to energy input, falling to a lower energy due to a flare, and falling from a higher energy due to a flare, respectively. This is the time-dependent version of the master equation given in Wheatland & Glukhov (1998).

Following Daly & Porporato (2007), we note that the system is also described by the stochastic differential equation

\[ \frac{dE}{dt} = \beta(E, t) - \lambda(E, t), \tag{5} \]

where \( \lambda(E, t) = \sum_{i=1}^{N(t)} \Delta E_i \delta(t - t_i) \) describes accumulated losses in energy due to flaring, \( \delta(x) \) is the delta function, and the times \( t_i \) are given by a state-dependent Poisson process with occurrence rate \( \lambda(E(t)) \). The jump amplitudes \( \Delta E \) are distributed according to the (state-dependent) distribution \( h(\Delta E, E) \), defined by

\[ \alpha(E, E - \Delta E) = \lambda(E)h(\Delta E, E), \tag{6} \]

so that \( \int_0^E h(\Delta E, E)d(\Delta E) = 1 \).

2.2. Steady State Waiting-Time Distribution

Daly & Porporato (2007) showed how—assuming a steady state—the waiting-time distribution for the jump transitions may be derived. In this section we briefly reiterate the theory, as it applies to the present model.

Consider a deterministic trajectory described by equation (5), starting at energy \( E \), and ending at a higher energy \( E_t \), the instant before a jump occurs. The distribution \( p_s(E) \) of final energies \( E_t \) is given by the rate of jumping at a given energy divided by the mean total rate of jumping, i.e.,

\[ p_s(E) = \frac{\lambda(E)P(E)}{\langle \lambda \rangle}, \tag{7} \]

where \( P(E) \) is the steady state solution to the master equation (4) and

\[ \langle \lambda \rangle = \int_0^\infty \lambda(E)P(E)dE \tag{8} \]

is the mean total rate. The distribution \( p_s(E) \) of starting energies \( E_s \) is then given by \( p_s(E) \) together with the distribution of jumps \( h(\Delta E, E) \),

\[ p_s(E) = \int_E^\infty p_s(E')h(E' - E, E')dE', \tag{9} \]

which using equations (6) and (7) may be rewritten as

\[ p_s(E) = \frac{1}{\langle \lambda \rangle} \int_E^\infty P(E')\alpha(E', E)dE'. \tag{10} \]

The waiting-time distribution is given by

\[ p_r(\tau) = -\frac{d\mathcal{F}}{d\tau} \tag{11} \]

with

\[ \mathcal{F}(\tau) = \int_0^\infty p_s(E, \tau)dE, \tag{12} \]

where \( p_s(E, \tau) \) is the solution to

\[ \frac{\partial p_s(E, \tau)}{\partial \tau} = -\frac{\partial}{\partial E} \left[ \beta(E)p_s(E, \tau) - \lambda(E)p_s(E, \tau) \right] \tag{13} \]

with the initial condition \( p_s(E, 0) = p_s(E) \). Equation (13) describes the evolution of the system before a flaring jump occurs, i.e., over the deterministic trajectory starting at energy \( E_t \) and ending at energy \( E_e \).

A simpler form for the waiting-time distribution may be obtained when \( \beta(E) = \beta_0 \), a constant. Solution of equation (13) by characteristics then gives

\[ p_s(E, \tau) = p_s(E - \beta_0 t) \exp \left\{ -\int_0^\tau \lambda(E - \beta_0(t - s))ds \right\}, \tag{14} \]

assuming \( E \geq \beta_0 t \), and \( p_s(E, \tau) = 0 \) otherwise. In this case,

\[ \mathcal{F}(\tau) = \int_0^\infty p_s(u)f(u, \tau)du, \tag{15} \]

where

\[ f(u, \tau) = \exp \left[ -\int_0^\tau \lambda(\beta_0 s + u)ds \right], \tag{16} \]

so the waiting-time distribution is

\[ p_r(\tau) = \int_0^\infty p_s(u)\lambda(\beta_0 \tau + u)f(u, \tau)du. \tag{17} \]

2.3. Steady State Analytic Solutions

Two analytic examples illustrate the application of the theory. The examples are not relevant for flares, because they do not produce power-law frequency-energy distributions, but they show how Poisson and non-Poisson waiting-time distributions may be obtained.

The case \( \beta(E) = \beta_0 \) and \( \alpha(E, E') = \alpha_0 \) (where \( \alpha_0 \) and \( \beta_0 \) are constants) was considered by Wheatland & Glukhov (1998). In this case, equation (3) gives \( \lambda(E) = \alpha_0 E \), so the total rate of jumps is energy dependent and the waiting-time distribution will not correspond to a simple Poisson process. The analytic solution to the steady state master equation is

\[ P(E) = aEe^{-(1/2)aE^2}, \tag{18} \]

with \( a = \alpha_0/\beta_0 \), and from equation (2), the frequency-energy distribution for jumps is Gaussian,

\[ \mathcal{N}(E) = \alpha_0 e^{-(1/2)aE^2}. \tag{19} \]

From equation (8) the mean total rate is \( \langle \lambda \rangle = (\alpha_0/\beta_0)^{1/2} \), and using equations (7) and (10), the distributions of end and start energies for deterministic trajectories are

\[ p_s(E) = \left( \frac{2}{\pi} \right)^{1/2} a^{3/2} E^{2} e^{-(1/2)aE^2}, \tag{20} \]

\[ p_s(E) = \left( \frac{2}{\pi} \right)^{1/2} a^{1/2} e^{-(1/2)aE^2}, \tag{21} \]
respectively. Using equations (17) and (21), it follows that the waiting-time distribution is also Gaussian,

\[ p_r(t) = \left( \frac{2\alpha_0 \beta_0}{\pi} \right)^{1/2} e^{-2\alpha_0 \beta_0 \beta_0 \sigma^2}. \] (22)

As a second example, we consider the case \( \beta(E) = \beta_0 \) and \( \lambda(E) = \lambda_0 \) (where \( \beta_0 \) and \( \lambda_0 \) are constants). Since the total rate is constant, the waiting-time distribution must be

\[ p_r(t) = \lambda_0 e^{-\lambda_0 t}, \] (23)

i.e., jumps occur in time as a simple Poisson process. From equation (3) this case requires \( \alpha(E, E') = \lambda_0/E \). The corresponding solution to the steady state master equation is

\[ P(E) = b^2 E e^{-bE}, \] (24)

with \( b = \lambda_0/\beta_0 \), and from equation (2), the frequency-energy distribution of jumps is exponential,

\[ N(E) = b \lambda_0 e^{-bE}. \] (25)

The mean total rate of jumps given by equation (8) is \( \langle \lambda \rangle = \lambda_0 \), and using equations (7) and (10), we have \( p_r(E) = b^2 E e^{-bE} \) and \( p_r(E) = b e^{-bE} \). From equation (17) it follows that the waiting-time distribution is indeed given by equation (23).

2.4. Moments of the Master Equation

Moments of the master equation give useful information about the global behavior of solutions (Litvinenko & Wheatland 2001). The zeroth moment, obtained by integrating equation (4) over all energies, gives the simple result

\[ \frac{d}{dt} \int_0^\infty P(E, t)dE = 0, \] (26)

i.e., normalization is preserved, provided \( \beta(E, t)P(E, t) \) goes to zero as \( E \to 0 \) and \( E \to \infty \).

The first moment, obtained by multiplying equation (4) by \( E \) and integrating over all energies, gives the simple statement of global energy balance,

\[ \frac{d}{dt} \langle E \rangle = \langle \beta \rangle - \langle r \rangle, \] (27)

where for any quantity \( f = f(E, t) \), the mean \( \langle f \rangle \) is defined by

\[ \langle f \rangle = \int_0^\infty f(E, t)P(E, t)dE, \] (28)

and where

\[ r(E) = \int_0^E (E - E')\alpha(E, E')dE' \] (29)

is the total rate of energy loss at energy \( E \). Equation (27) requires \( \beta(E, t)P(E, t) \) to go to zero as \( E \to 0 \) and \( E \to \infty \). In the steady state, equation (27) gives

\[ \langle \beta \rangle = \langle r \rangle. \] (30)

3. FLARELIKE SOLUTIONS

In the following we consider solutions to the steady state master equation which may be of relevance to solar flares.

3.1. Choices Appropriate for Flares

We restrict attention to the case \( \beta(E) = \beta_0 \), a constant. The motivation is that active regions are externally driven, i.e., the energy is supplied from the subphotosphere by external processes. In the absence of a back-reaction, it is then expected that the energy supply rate does not depend on the state of the system. In passing we note that in general the energy supply rate may depend on time. However, in this section we consider only steady state solutions to the master equation. We return to this point in \( \S \) 4.

We consider the form

\[ \alpha(E, E') = \alpha_0 E^\delta (E - E')^{-\gamma}(E - E' - E_c) \] (31)

for the flare transition rate, where \( E_c \) is a low-energy cutoff and \( \theta(x) \) is the step function. The case \( \delta = 0 \) was considered in Wheatland & Glukhov (1998). The motivation for equation (31) is that it may describe an avalanche-type system, in which energy transitions are intrinsically power-law distributed. The power law is assumed to originate in the microphysics of the flare process and must be assumed at the level of this model. (This is in contrast to models which attempt to account for the power law, e.g., Rosner & Vaiana 1978.) The low-energy cutoff \( E_c \) is needed to ensure \( \lambda(E) \) is finite. The \( E^\delta \) factor represents a possible dependence of the transition rate on the energy of the system. It is plausible that an avalanche-type system with more energy is more likely to contain unstable sites and, hence, will flare at a higher rate. In the following we take \( \gamma = 1.5 \) in every instance and consider two cases: \( \delta = 0 \), following Wheatland & Glukhov (1998); and \( \delta = 1 \). The choice of transition rates from equation (31) leads to the total flaring rate

\[ \dot{\lambda}(E) = \frac{\alpha_0}{\gamma - 1} E^\delta (E_c - \frac{1}{\gamma} E)^{1/((\gamma + 1))}. \] (32)

Hence, the rate of occurrence of flares is energy dependent, and the waiting-time distribution will not correspond to a simple Poisson process, as pointed out in \( \S \) 1.

Application of the first moment with the choice of transition rates from equation (31) leads to a simple estimate for the mean energy of the system (Wheatland & Litvinenko 2002). Specifically, from equation (29) we have

\[ r(E) \approx \frac{\alpha_0}{2 - \gamma} E^{\delta + 2 - \gamma}, \] (33)

for \( E \gg E_c \). Taking averages and making the approximation \( (E^{\delta + 2 - \gamma}) \approx (E^{\delta + 2 - \gamma})_0 \) together with \( \langle \beta \rangle = \beta_0 \) in equation (30) gives

\[ (E) \approx \left( \frac{2 - \gamma}{\alpha_0/\beta_0} \right)^{1/(\delta + 2 - \gamma)}. \] (34)

Substituting equation (31) into equation (2) leads to the form for the flare frequency-energy distribution

\[ N(E) = \alpha_0 E^{-\gamma} \int_E^\infty (E')^\delta P(E')dE' \] (35)

for \( E \geq E_c \). Hence, it follows that the frequency-energy distribution will be a power law with index \( \gamma \) up to energies \( E \) at which
P(\varepsilon) becomes very small. Equation (34) provides a crude estimate (a lower bound) for the energy at which the frequency-energy distribution is expected to depart from power-law behavior.

3.2. Numerical Method

The steady state master equation may be nondimensionalized by introducing new variables $\bar{\varepsilon} = \varepsilon/E_0$, $\bar{P} = P/E_0$, $\bar{\varepsilon} = \varepsilon/\varepsilon_\text{max}$, and $\bar{\varepsilon} = E_0^2\alpha/\beta_0$, where $E_0$ is a chosen scale for energy. (For the solutions corresponding to eq. [31] we take $E_0 = E_e$.) This procedure gives

$$
\frac{d\bar{P}}{d\bar{\varepsilon}} + \lambda\bar{P} - \int_{\varepsilon_\text{min}}^{\varepsilon_\text{max}} \bar{P}(\varepsilon') \bar{\sigma}(\varepsilon', \varepsilon) d\varepsilon' = 0,
$$

(36)

where

$$
\lambda = \int_{\varepsilon_\text{min}}^{\varepsilon_\text{max}} \bar{\sigma}(\varepsilon', \varepsilon) d\varepsilon'.
$$

(37)

Hereafter, we assume nondimensional equations, but omit the bars.

Equation (36) is linear in $P(\varepsilon)$ and, hence, may be solved by discretizing in energy and solving a coupled system of linear equations. These equations must be supplemented by the normalization condition on $P(\varepsilon)$. Direct back-substitution provides an efficient method of solution, and the details of the procedure are given in the Appendix. In Wheatland & Glukhov (1998), the steady state master equation was solved by a relaxation procedure, but the approach given here is more numerically efficient.

The flare frequency-energy distribution is obtained from the solution for $P(\varepsilon)$ via numerical evaluation of equation (2). The waiting-time distribution is similarly determined via numerical evaluation of equation (17), using an analytic form for $f(u, \tau)$ obtained from equation (16). All numerical integrations use the extended trapezoidal rule. The numerical solution was tested on the analytic cases given in § 3.2.

3.3. Results

First, we consider the case $\delta = 0$, following Wheatland & Glukhov (1998). Figure 1 illustrates the numerical solution of the steady state master equation (36) for the case $\alpha_0 = 0.1$, which is one of the two cases considered in Wheatland & Glukhov (1998). The top panel shows the probability distribution $P(\varepsilon)$ for active region energy (as a linear-log plot), the middle panel shows the flare frequency-energy distribution $\lambda(\varepsilon)$ (as a log-log plot), and the bottom panel shows the waiting-time distribution $\phi(\tau)$ (as a log-linear plot). As explained in § 3.1, the frequency-energy distribution is expected to be a power law with index $\gamma = 1.5$ below energies at which $P(\varepsilon)$ becomes small, and equation (34) provides a lower bound for the departure from power-law behavior. The lower bound is shown in the top and middle panels by a vertical line. The top and middle panels confirm the results of Wheatland & Glukhov (1998). The bottom panel shows the waiting-time distribution $\phi(\tau)$ (solid curve) as well as the Poisson distribution $\lambda(\varepsilon)e^{-(\lambda/\varepsilon\varepsilon_\text{max})\tau}$ (dotted line) corresponding to the mean rate of flaring implied by the form of $\lambda(\varepsilon)$ and the solution for $P(\varepsilon)$ (see eq. [8]). Note that the units for time in the bottom panel are $\varepsilon_\text{max}/\beta_0$, following the nondimensionalization in § 3.2.

The waiting-time distribution for the model is approximately Poisson, although there is a slight deficiency of long waiting times by comparison with the Poisson distribution.

Figure 2 shows the case $\delta = 0$ and $\alpha_0 = 0.02$, which is the other case considered by Wheatland & Glukhov (1998) in the same format as Figure 1. For a lower value of $\alpha$, the system flares less often and, hence, is more likely to have larger energy. Hence, the distribution $P(\varepsilon)$ shown in the top panel is shifted to higher energy and has a higher mean. The flare frequency-energy distribution (middle panel) is a power law over more decades in energy than for the case $\alpha_0 = 0.1$. These results are consistent with the findings of Wheatland & Glukhov (1998). The bottom panel shows the waiting-time distribution (solid curve) as well as the Poisson
distribution corresponding to $\langle \lambda \rangle$ (dotted curve), although the two curves are almost indistinguishable.

The results in Figures 1 and 2 suggest that the $\delta = 0$ model has a waiting-time distribution which is close to being strictly Poisson (exponential), and that the approximation becomes better for smaller values of $\alpha_0$. This may be understood in terms of equation (32). For $E \gg E_c$, the total rate is $\lambda(E) \approx \alpha_0 E^{-\gamma} + 1 / (\gamma - 1)$, which is constant, in which case a Poisson waiting-time distribution is expected. For smaller values of $\alpha_0$ the system is more likely to have larger energy, and hence, the approximation $E \gg E_c$ will be better. Figure 3 illustrates this explanation for the case $\delta = 0$ and $\alpha_0 = 0.02$. The solid curve shows the total rate as a function of energy, and the dashed line shows the mean total rate $\langle \lambda \rangle$. The energy distribution $P(E)$ is shown, with arbitrary normalization, by the dotted curve. We see that, over the range of energy for which the distribution $P(E)$ is substantial, the total rate is approximately constant and equal to the mean total rate. These results resolve the puzzle identified in § 1: the waiting-time distribution for the model is not strictly Poisson, but is a good approximation to an exponential.

Next, we consider the case $\delta = 1$, to examine what happens when the rate of flare transitions increases with the energy of the system. Figures 4 and 5 show the cases $\alpha_0 = 10^{-3}$ and $10^{-5}$ respectively, in the same format as Figures 1 and 2. First consider Figure 4. The energy distribution $P(E)$ shown in the top panel is qualitatively similar to the $\delta = 0$ case, although the distribution declines more rapidly at higher energies, so that it is more skewed in a linear-log representation. Since the rate of transitions increases with energy, the system is less likely to be found at very large energies, and this explains the rapid decline. The middle panel shows the flare frequency-energy distribution $N(E)$, which is a power law with index $\gamma$ below a high-energy rollover. The estimate from equation (34) for the mean of the distribution (vertical line) provides a lower bound for the departure from power-law behavior. The bottom panel shows the waiting-time distribution $p_w(T)$ and the Poisson distribution implied by $\langle \lambda \rangle$. The waiting-time distribution is approximately Poisson, but has a deficit of long waiting times. Figure 5 illustrates the case with reduced flare transition rates. The distribution $P(E)$ (top panel) is shifted to higher energies and is again quite skewed in the linear-log representation. The frequency-energy distribution $N(E)$ (middle panel) is a power law over more decades in energy. The waiting-time distribution $p_w(T)$ (bottom panel) is again approximately exponential, but departs somewhat from the Poisson model, including showing an excess of long waiting times.

The results in the bottom panels of Figures 4 and 5 suggest that, for the $\delta = 1$ model, the waiting-time distribution is approximately exponential but shows some departure from the Poisson case depending on the parameters of the model. The approximate Poisson behavior is perhaps surprising, because in this case, the total rate of flaring (given by eq. [32]) varies approximately linearly with $E$ (for $E \gg E_c$). Figure 6 illustrates this for the case $\delta = 1$ and $\alpha_0 = 10^{-5}$, using the same format as Figure 3. The total rate (solid curve) increases substantially over the range of energies the system is likely to have [the dotted curve shows $P(E)$], and may be substantially different than $\langle \lambda \rangle$ (dashed line).
A stochastic model is presented for the free magnetic energy of a flaring solar active region. The energy of an active region is assumed to grow deterministically between random flare events at which the energy jumps downward by an amount equal to the flare energy. Flare jumps occur from energy $E$ to $E'$ with a rate $\alpha(E, E')$ per unit time and per unit energy, and energy input occurs at a rate $\beta(E)$. Active region energy is then described by a distribution $P(E)$ which is the steady state solution to a master equation. This distribution determines two observable distributions, namely, the flare frequency-energy distribution and the waiting-time distribution. The model generalizes and extends the approach of Wheatland & Glukhov (1998). Novel aspects of the work presented here include the determination of waiting-time distributions (following general theory presented by Daly & Porporato 2007), consideration of a more general form for the rate of flare transitions, and introduction of an efficient method of numerical solution of the steady state master equation.

The form $\alpha(E, E') = \alpha_0 E^\delta (E - E')^{1.5}$ for flare transitions is investigated, for the cases $\delta = 0$ and 1. The case $\delta = 0$ was considered by Wheatland & Glukhov (1998) motivated by the avalanche model. For both cases, the model is shown to produce power-law flare frequency-energy distributions below a rollover at high energies, due to the active region having a finite energy. For both cases, the waiting-time distribution is approximately exponential (Poisson). For the case $\delta = 0$, this may be understood in that the total rate $\lambda(E)$ is approximately constant for $E \gg E_c$, which becomes a good approximation for small $\alpha_0$, when the system is more likely to be found at large energies. This result is consistent with the interpretation of this model as avalanche-like, since avalanche models have simple Poisson statistics. For the case $\delta = 1$, the interpretation is more complicated, because the total rate varies approximately linearly with energy. However, the waiting-time distribution is determined by an average of rates over the possible energies of the active region, and numerical evaluation shows that the result is approximately exponential.

The general model introduced in § 2.1 includes time dependence in the driving rate, but we have focused on the steady state throughout this paper. Many active regions exhibit large variations in the rate of flaring during a transit of the disk (Wheatland 2001). This behavior is often linked, e.g., with the emergence of new magnetic flux (Romano & Zuccarello 2007), which suggests that it is a response to an increased rate of driving. Hence, we have neglected an important aspect of active region energetics. Time-dependent driving will influence the observed waiting-time distribution. In the simplest case, time variation might be represented by a piecewise constant variation in the driving rate. If the system adjusts suitably quickly to changes in driving, the steady state solution applies to each piece. The waiting-time distribution is then a weighted sum over the steady state distributions applying to each piece (Litvinenko & Wheatland 2001). Based on the results presented in this paper, the waiting-time distribution for active regions is expected to be approximately exponential provided the rate of driving of the system is constant. If the rate of driving is time varying, the distribution will depart from exponential. A time-dependent model will be investigated in future work.

One shortcoming of the model is that it does not describe energy loss from the system by mechanisms other than flaring, for example, loss due to flux submergence or quasi-steady background dissipation. However, a simple generalization of the master equation permits this. Specifically, if the secular energy “input” is replaced by small jumps in energy (which may be positive or negative), then the energy gains and losses may be represented by Fokker-Planck terms in a generalized master or Chapman-Kolmogorov equation. Specifically, the secular energy increase term $-\partial [\beta(E, t)P(E, t)]/\partial E$ on the right-hand side of equation (4) may be replaced by a pair of terms $-\partial [\beta_i(E, t)P(E, t)]/\partial E + \frac{1}{2} \partial^2 [\beta_i(E, t)P(E, t)]/\partial E^2$, where the coefficients $\beta_i(E, t), i = 1, 2$ represent first and second moments of energy changes associated with the small jump transitions (van Kampen 1992; Gardiner 2004). It is straightforward to solve the resulting generalized master equation by discretization and solution of the resulting linear system, extending the approach presented in the Appendix. However, in this case the Daly & Porporato (2007) method for determining the steady state waiting-time distribution needs to be modified. This model will be investigated more completely in future work.

The results presented here show how it is possible to construct a model for active region energetics which directly predicts observable flare statistics, namely, the flare frequency-energy and waiting-time distributions. In principle, observations may be used to determine the energy supply and flare energy release terms in the model, i.e., $\beta(E, t)$ and $\alpha(E, E')$. However, the observations are not really precise or unambiguous enough to identify these terms with certainty. In particular, the interpretation of the waiting-time distribution is complicated by the time dependence of the energy supply. The situation would be improved by an ability to estimate coronal free energy and the rate of supply of energy to active regions from observations. Reliable methods for estimating these quantities are the subject of current research (Welsch et al. 2007; Schrijver et al. 2008). If such methods are developed, the theory developed here will be of greater significance. It may provide valuable insight into the flare mechanism, as well as being of practical benefit for flare prediction.

The author thanks Ian Craig for pointing out that the master equation can be solved as a linear system by back-substitution and a referee for helpful comments which improved the presentation.
NUMERICAL SOLUTION OF THE STEADY STATE MASTER EQUATION

Discretizing equation (36) at energies $E_i = i\Delta$ gives

$$\frac{P_{i+1} - P_{i-1}}{2\Delta} + \lambda_i P_i - \frac{1}{2} \Delta \sum_{j=0}^{N-2} (P_j \alpha_{j,i} + P_{j+1} \alpha_{j+1,i}) = 0, \quad (A1)$$

where

$$\lambda_i = \frac{1}{2} \Delta \sum_{j=0}^{i-1} (\alpha_{i,j} + \alpha_{i,j+1}). \quad (A2)$$

$P_i = P(E_i)$, and $\alpha_{i,j} = \alpha(E_i, E_j)$. Centered differencing is used for the derivative, and the extended trapezoidal rule is used for the integrals. Equation (A1) with $i = 1, 2, \ldots, N - 1$ (and the assumption $P_N = 0$) may be supplemented by the normalization condition

$$\frac{1}{2} \Delta \sum_{i=0}^{N-2} (P_i + P_{i+1}) = 1 \quad (A3)$$

to give $N$ linear equations in the $N$ unknowns $P_0, P_1, P_2, \ldots, P_{N-1}$. The resulting linear system may be solved efficiently by back-substitution as follows. Equation (A1) may be rewritten as

$$P_{i-1} = P_{i+1} + 2\Delta \left[ \lambda_i P_i - \frac{1}{2} \Delta \sum_{j=0}^{N-2} (P_j \alpha_{j,i} + P_{j+1} \alpha_{j+1,i}) \right], \quad (A4)$$

which expresses $P_{i-1}$ in terms of $P_i, P_{i+1}, P_{i+2}, \ldots, P_{N-2}$. Hence, if we assume a value for $P_{N-1}$, we can apply equation (A4) to solve for $P_{N-2}$ and then apply it again to solve for $P_{N-3}$, etc. In this way we can determine $P_{N-1}, P_{N-2}, P_{N-3}, \ldots, P_0$ up to an unknown normalization factor. The factor is determined by applying equation (A3). Specifically, the solution is given by $P_i'$ (with $i = 0, 1, 2, \ldots, N - 1$), where

$$P_i' = \frac{P_i}{(1/2) \Delta \sum_{j=0}^{N-2} (P_j + P_{j+1})}. \quad (A5)$$

REFERENCES

Aschwanden, M. J., Dennis, B. R., & Benz, A. O. 1998, ApJ, 497, 972
Baiesi, M., Paczuski, M., & Stella, A. L. 2006, Phys. Rev. Lett., 96, 051103
Biesecker, D. 1994, Ph.D. thesis, Univ. New Hampshire
Boffetta, G., Carbone, V., Giuliani, P., Veltri, P., & Vulpiani, A. 1999, Phys. Rev. Lett., 83, 4662
Buchlin, E., Galtier, S., & Velli, M. 2005, A&A, 436, 355
Charbonneau, P., McIntosh, S. W., Liu, H.-L., & Bogdan, T. J. 2001, Sol. Phys., 203, 321
Crospy, N. B., Aschwanden, M. J., & Dennis, B. R. 1993, Sol. Phys., 143, 275
Daly, E., & Porporato, A. 2007, Phys. Rev. E, 75, 011119
Gardiner, C. W. 2004, Handbook of Stochastic Methods (3rd ed.; Berlin: Springer)
Hudson, H. S. 1991, Sol. Phys., 133, 357
Leka, K. D., Canfield, R. C., McLennan, M. N., & van Driel-Gesztelyi, L. 1996, ApJ, 462, 547
Lepreti, F., Carbone, P., & Veltri, P. 2001, ApJ, 555, L133
Litvinenko, Y. E. 1994, Sol. Phys., 151, 195
———. 1996, Sol. Phys., 167, 321
Litvinenko, Y. E., & Wheatland, M. S. 2001, ApJ, 550, L109
Lu, E. T. 1995, ApJ, 447, 416
Lu, E. T., & Hamilton, R. J. 1991, ApJ, 380, L89
McClintock, A. N., & Fisher, G. H. 1989, in Yosemite Conf. Outstanding Problems in Solar System Plasma Physics: Theory and Instrumentation, ed. J. Waite, J. Burch, & R. Moore (Geophys. Monogr. 54; Washington: AGU), 219
Metcalf, T. R., Leka, K. D., & Mickey, D. L. 2005, ApJ, 623, L53
Metcalf, T. R., et al. 2008, Sol. Phys., 247, 269
Moon, Y.-J., Choe, G. S., Park, Y. D., Wang, H., Gallagher, P. T., Chae, J., Yun, H. S., & Goode, P. R. 2002, ApJ, 574, 434
Moon, Y.-J., Choe, G. S., Yun, H. S., & Park, Y. D. 2001, J. Geophys. Res. Space Phys., 106, 29951
Paczuski, M., Boettcher, S., & Baiesi, M. 2005, Phys. Rev. Lett., 95, 181102
Pearce, G., Rowe, A. & Yeung, J. 1993, Ap&SS, 208, 99
Priest, E. R., & Forbes, T. G. 2002, Astron. Astrophys. Rev., 10, 313
Richardson, R. S. 1951, ApJ, 114, 356
Romano, P., & Zuccarello, F. 2007, A&A, 474, 633
Rosner, R., & Vaiana, G. 1978, ApJ, 222, 1104
Sánchez, R., Newman, D. E., & Carreras, B. A. 2002, Phys. Rev. Lett., 88, 068302
Schrijver, C. J., DeRosa, M. L., Title, A. M., & Metcalf, T. R. 2005, ApJ, 628, 501
Schrijver, C. J., et al. 2008, ApJ, 675, 1637
Tandberg-Hanssen, E., & Emslie, A. G. 1988, The Physics of Solar Flares (Cambridge: Cambridge Univ. Press)
van Kampen, N. G. 1992, Stochastic Processes in Physics and Chemistry (Amsterdam: Elsevier)
Welsch, B. T., et al. 2007, ApJ, 670, 1434
Wheatland, M. S. 2000a, ApJ, 532, 616
———. 2000b, ApJ, 532, 1209
———. 2001, Sol. Phys., 203, 87
Wheatland, M. S., & Glukhov, S. 1998, ApJ, 494, 858
Wheatland, M. S., & Litvinenko, Y. E. 2002, Sol. Phys., 211, 255
Wheatland, M. S., Sturrock, P. A., & McTiernan, J. M. 1998, ApJ, 509, 448