Rollage: Efficient Rolling Average Algorithm to Estimate ARMA Models for Big Time Series Data
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Abstract
We develop a new efficient algorithm for the analysis of large-scale time series data. We firstly define rolling averages, derive their analytical properties, and establish their asymptotic distribution. These theoretical results are subsequently exploited to develop an efficient algorithm, called Rollage, for fitting an appropriate AR model to big time series data. When used in conjunction with the Durbin’s algorithm, we show that the Rollage algorithm can be used as a criterion to optimally fit ARMA models to big time series data. Empirical experiments on large-scale synthetic time series data support the theoretical results and reveal the efficacy of this new approach, especially when compared to existing methodology.

1 Introduction
A time series is a collection of random variables indexed according to the order they are obtained in time. The primary objective of time series analysis is to develop statistical models to forecast the future behavior of the system. These models have proved their effectiveness and advantages in modeling and analyzing stochastic dynamic systems, and continue to gain in popularity for modeling a wide range of applications spanning from supply chains and energy systems to epidemiology and engineering problems [1,10,11,12,16].

The autoregressive moving average (ARMA) model is a widely applied model to achieve this objective. The model was popularized by Box and Jenkins [3] for analyzing stationary time series data. Extensions of this model were subsequently introduced, such as the autoregressive integrated moving average (ARIMA) model for analyzing non-stationary time series data which posses a trend in mean, and the seasonal ARIMA (SARIMA) model to deal with time series data displaying seasonal effects [16]. All in all, each SARIMA model is, at its core, an ARMA model for a linearly transformed time series constructed by differencing the original time series at proper lags.

Broadly speaking, the Box-Jenkins method involves three steps, consisting of identification, estimation and diagnosis of an ARMA model. The first step, model identification, involves making an initial guess for an appropriate order of the model. Throughout the literature, there have been numerous procedures proposed to estimate the orders of an ARMA model [7]. Commonly the orders are chosen through use of the autocorrelation function (ACF) in conjunction with the partial autocorrelation function (PACF), and observing their respective plots.

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This method is not without its drawbacks as modelers require a high level of expertise to interpret the ACF and PACF plots manually [17]. In addition, the PACF only uses the last over-fitted coefficient to estimate the order of the autoregressive (AR) component which may not truly incorporate all information from the sample. The main goal of this paper is to establish a new method to estimate the order of an AR model that utilizes all over-fitted coefficients in big data regime.

The second step of the Box-Jenkins method involves estimating the parameters for the chosen ARMA model. For pure AR models, this can be achieved analytically by performing a variant of maximum likelihood estimation (MLE) known as conditional maximum likelihood estimation (CMLE) [14]. However, for pure MA and combined ARMA models, both the likelihood and conditional likelihood are complex non-linear functions. Hence, finding a solution to either the MLE or CMLE is intractable. Consequently, numerical optimization methods are often employed to estimate the parameters of MA and ARMA models.

Durbin [8] worked to overcome the intractability of the maximum likelihood equations of MA models by exploiting the asymptotic equivalence between AR models of infinite order and MA models of finite order. The solution was to fit a large order AR model to represent the infinite order AR model using CMLE. The residuals of that model can then be used as estimates of the unobservable white noise of the MA process. Standard linear regression techniques could then be used to estimate the parameters of the MA model. Durbin [9] then extended this initial work to estimate the parameters for a full ARMA model using a similar approach. Durbin’s method is promising as it replaces a non-linear estimation problem with two stages of linear estimation [5]. However, a good question to ask is, how large should the order of the AR model be in Durbin’s methodology to provide both accurate estimates and optimal efficiency?

Through practice and simulation, Broersen [5,6] identified that increasing the order of the large AR model does not necessarily result in more accurate estimate of an ARMA model and that the optimal order is in fact finite. As a result, an appropriately large order is essential in Durbin’s algorithm to ensure parameter accuracy and optimal forecasts, as well as improving computational complexity. The use of model selection criteria, such as the Akaike Information Criteria (AIC) [2], has been suggested to aid in the selection of an appropriately large order AR model [5,15].

With these developments serving as motivation, this paper will look to develop a new algorithm to appropriately estimate the order of an AR model, and then use this algorithm as a criterion in Durbin’s methodology to optimally fit an ARMA model to big time series data. In particular, our contributions can be summarized as follows:

(i) We introduce the concept of a rolling average and derive its theoretical properties,

(ii) Using the rolling average properties, we develop a highly-efficient algorithm, called Rollage, for fitting an appropriate AR model to big time series data,

(iii) We use the Rollage algorithm as a model selection criterion in Durbin’s methodology to optimally fit an ARMA model,

(iv) We empirically demonstrate the effectiveness of the Rollage algorithm to estimate ARMA models on large-scale synthetic time series data, when compared to existing criteria including BIC and GIC.

The structure of this paper is as follows: Section 2 introduces three times series models utilized in this paper, namely AR, MA and ARMA models, and covers their important properties and estimation techniques. Section 3 introduces the concept of a rolling average, derives its theoretical results, and develops a new methodology to estimate AR, MA, and ARMA models, appropriately called Rollage.
algorithm. Section 4 illustrates the efficacy of the new methodology by implementing it on several synthetic big time series data and comparing it to existing methodology. Section 5 concludes the paper and addresses future work.

Notation

Throughout the paper, vectors and matrices are denoted by bold lower-case, and upper-case letters respectively (e.g., \(v\) and \(M\)). All vectors are assumed to be column vectors. We use regular lower-case to denote scalar constants (e.g., \(c\)). Random variables are denoted by regular upper-case letters (e.g., \(X\)). For a real vector, \(v\), its transpose is denoted by \(v^\top\). For a vector \(v\) and a matrix \(M\), \(\|v\|\) and \(\|M\|\) denote vector \(\ell_2\) norm and matrix spectral norm, respectively. The determinant and adjugate of a square matrix \(M\) are denoted by \(\text{det}(M)\) (or \(|M|\), used interchangeably) and \(\text{adj}(M)\), respectively. Adopting MatLab notation, we use \(A(i,:)\) and \(A(:,j)\) to refer to the \(i\)th row and \(j\)th column of the matrix \(A\), respectively, and consider them as a column vector.

2 Background

In this section, we present a brief overview of the three time series models considered in this paper, namely autoregressive models (Section 2.1), moving average models (Section 2.1), and autoregressive moving average models (Section 2.3).

2.1 Autoregressive Models

A time series \(\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}\) is called (weakly) stationary, if the mean \(\mathbb{E}[Y_t]\) is independent of time \(t\), and the auto-covariance \(\text{Cov}(Y_t, Y_{t+h})\), denoted by \(\gamma_h\), depends only on the lag \(h\) for any integer values \(t\) and \(h\). A stationary time series \(\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}\) with the constant mean \(\mathbb{E}[Y_t] = 0\) is an AR model with the order \(p\), denoted by \(\text{AR}(p)\), if we have

\[
Y_t = \phi_1^{(p)} Y_{t-1} + \cdots + \phi_p^{(p)} Y_{t-p} + W_t,
\]

where \(\phi_p^{(p)} \neq 0\) and the time series \(\{W_t; t = 0, \pm 1, \pm 2, \ldots\}\) is a Gaussian white noise with the mean \(\mathbb{E}[W_t] = 0\) and variance \(\text{Var}(W_t) = \sigma_W^2\). Gaussian white noise is a stationary time series in which each individual random variable \(W_t\) has a normal distribution and any pair of random variables \(W_{t_1}\) and \(W_{t_2}\) for distinct values of \(t_1, t_2 \in \mathbb{Z}\) are uncorrelated. For the sake of simplicity, we assume that \(\mathbb{E}[Y_t] = 0\)

It is readily seen that each \(\text{AR}(p)\) model has \(p+2\) unknown parameters consisting of the order \(p\), the coefficients \(\phi_i^{(p)}\) and the variance of white noises \(\sigma_W^2\). Following is a brief explanation of a common method in the literature for estimating the unknown order \(p\).

**Estimating the order \(p\).** A common method to estimate the order of an \(\text{AR}(p)\) model is to use the partial autocorrelation function (PACF) [16, Chapter 3]. The PACF of a stationary time series \(\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}\) at lag \(m\) is defined by

\[
\text{PACF}_m := \begin{cases} 
\rho(Y_t, Y_{t+1}) & \text{for } m = 1, \\
\rho(Y_{t+m} - \hat{Y}_{t+m}, Y_t - \hat{Y}_{t,m}) & \text{for } m \geq 2,
\end{cases}
\]
where \( \rho \) denotes the correlation function, and where \( \hat{Y}_{t,m} \) and \( \hat{Y}_{t+m,-m} \) denote the linear regression, in the population sense, of \( Y_t \) and \( Y_{t+m} \) on \( \{Y_{t+1}, \ldots, Y_{t+m-1}\} \), respectively. In order to apply the PACF values to estimate the order of an AR model, we first need to introduce the concept of “causality”, as given in Definition 1.

**Definition 1 (Causal AR Model).** An AR\((p)\) model is said to be “causal”, if the time series \( \{Y_t; t = 0, \pm 1, \pm 2, \ldots\} \) can be written as

\[
Y_t = \sum_{i=0}^{\infty} \psi_i W_{t-i},
\]

where \( \psi_0 = 1 \) and the constant coefficients \( \psi_i \) satisfy \( \sum_{i=0}^{\infty} |\psi_i| < \infty \).

It can be shown that for a causal AR\((p)\) model, while the theoretical PACF (2) at lags \( m = 1, \ldots, p-1 \) may be non-zero and at lag \( m = p \) is strictly non-zero, at lag \( m = p + 1 \) it drops to zero and then remains at zero henceforth [16, Chapter 3]. Theorem 1 indicates the statistical properties of the parameter estimates using PACF and the PACF estimates. These can be used to select the model order in practice by plotting the sample PACF versus lag \( m \) along with a 95\% zero-confidence boundary, that is two horizontal lines at \( \pm 1.96/\sqrt{n} \), are plotted. Then, the largest lag \( m \) in which the sample PACF lies out of the zero-confidence boundary for PACF is used as an estimation of the order \( p \).

Theorem 1 plays a crucial role in developing theoretical results in Section 3.

**Theorem 1 (Asymptotic Distribution of Estimated Coefficients [4]).** Suppose the time series \( \{Y_1, \ldots, Y_n\} \) be a stationary casual AR\((p)\) model as given in (1) and fit an AR\((m)\) model \((m > p)\) to the time series data, that is

\[
Y_t = \phi_1^{(m)} Y_{t-1} + \cdots + \phi_m^{(m)} Y_{t-m} + W_t.
\]

The maximum likelihood estimate of the coefficient vector, denoted by \( \hat{\phi}_{p,m} = \left[ \hat{\phi}_1^{(m)} \ldots \hat{\phi}_m^{(m)} \right]^\top \), asymptotically, has a multivariate normal distribution

\[
\sqrt{n}(\hat{\phi}_{p,m} - \phi_{p,m}) \sim \mathcal{MN}(0, \Sigma_{p,m}),
\]

where \( \phi_{p,m} := \left[ \phi_1^{(p)} \ldots \phi_p^{(p)} 0 \ldots 0 \right]^\top \), the covariance matrix \( \Sigma_{p,m} = \sigma^2 W^2 \Gamma_{p,m}^{-1} \), and

\[
\Gamma_{p,m} = \begin{pmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} \\
\gamma_1 & \gamma_0 & \cdots & \gamma_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{m-1} & \gamma_{m-2} & \cdots & \gamma_0
\end{pmatrix}.
\]
is the autocovariance matrix of the given time series. Furthermore, $\hat{\phi}_m^{(m)}$ is an unbiased estimate for $\text{PACF}_m$ with the limit distribution

$$\sqrt{n}\hat{\phi}_m^{(m)} \sim N[0, 1].$$

2.2 Moving Average Models

A stationary time series $\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}$ with the constant mean $E[Y_t] = 0$ is an MA model with the order $q$, denoted by $\text{MA}(q)$, if we have

$$Y_t = \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q} + W_t,$$

where $\theta_q \neq 0$ and the time series $\{W_t; t = 0, \pm 1, \pm 2, \ldots\}$ is a Gaussian white noise with the mean $E[W_t] = 0$ and variance $\text{Var}(W_t) = \sigma^2_W$.

Similar to an AR($p$) model, each MA($q$) model has $q + 2$ unknown parameters consisting of the order $q$, the coefficients $\theta_i$ and the variance of white noises $\sigma^2_W$. Here, we briefly explain the common methods in the literature to estimate each of these unknown parameters.

**Estimating the order $q$.** A common method to estimate the order of an MA($q$) model is to use the autocorrelation function (ACF) [16, Chapter 3]. The ACF of a stationary time series $\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}$ at lag $m$ is defined by

$$\text{ACF}_m := \begin{cases} \gamma_m & \text{for } m = 0, 1, \ldots, \\ \gamma_0 & \text{for } m = -1, -2, \ldots. \end{cases}$$

In order to apply the ACF values to estimate the order of an MA model, we first need to introduce the concept of “invertibility”, as given in Definition 2.

**Definition 2 (Invertible MA Model).** An $\text{MA}(q)$ model is said to be “invertible”, if embedded white noises of the time series $\{Y_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as

$$W_t = \sum_{i=0}^{\infty} \pi_i Y_{t-i},$$

where $\pi_0 = 1$ and the constant coefficients $\pi_i$ satisfy $\sum_{i=0}^{\infty} |\pi_i| < \infty$.

It can be shown that for an invertible MA($q$) model, while the theoretical ACF (4) at lags $m = 1, \ldots, q - 1$ may be non-zero and at lag $m = q$ is strictly non-zero, at lag $m = q + 1$ it drops to zero and then remains at zero henceforth [16, Chapter 3]. Furthermore, it can be shown that if $y_1, \ldots, y_n$ is a time series realization of an invertible MA($q$) model, the (estimated) sample ACF values, scaled by $\sqrt{n}$, at lags greater than $q$ has a standard normal distribution, in limit. Thus, in practice, the (estimated) sample ACF versus lag $m$ along with a 95% zero-confidence boundary, that is two horizontal lines at $\pm 1.96/\sqrt{n}$, are plotted. Then, the largest lag $m$ in which the sample ACF lies out of the zero-confidence boundary for ACF is used as an estimation of the order $q$. 

5
Maximum likelihood estimation of the coefficients $\theta_i$ and variance $\sigma^2_W$. Unlike an AR model, for a MA($q$) model both the log-likelihood function and the conditional log-likelihood function are complicated non-convex functions and cannot be maximized analytically \cite[Chapter 5]{14}. So, one approach in estimating the parameters of a MA model is maximizing the corresponding (log-)likelihood function approximately by applying some numerical optimization algorithms, such as the gradient descent method.

The main difficulty in dealing with the likelihood function of a MA model is that the lagged values of white noises are not known before fitting a model to the data. \cite{8} overcame this problem by developing a new method for MA model fitting. Motivated from Definition 2, it is readily seen that an “invertible” MA($q$) model can be represented by

$$Y_t = \sum_{i=1}^{\infty} (-\pi_i)Y_{t-1} + W_t,$$

implying that an invertible MA($q$) model is equivalent to an AR($\infty$) model. Durbin exploited this equivalence to estimate the parameters of a MA model and later extended this for ARMA models. As a result, Durbin’s Algorithm first fits an AR model with a sufficiently large order, say $\tilde{p}$, to the data and approximates the values of white noises $W_t$ by finding the residuals of the fitted AR model, that is,

$$\tilde{w}_t = y_t - \sum_{i=1}^{\tilde{p}} (-\hat{\pi}_i)y_{t-i}. \tag{5}$$

In the next step, the algorithm approximates estimates the coefficients $\theta_i$ by regressing the time series over the $q$ lagged values of the estimated residuals in (5), that is

$$Y_t = \tilde{\theta}_1\tilde{w}_{t-1} + \cdots + \tilde{\theta}_q\tilde{w}_{t-q} + W_t. \tag{6}$$

The steps of Durbin’s Algorithm are depicted in Algorithm 1. \cite{15} extended this algorithm with trimming steps to improve the initial parameter estimates, as well as suggesting the use the Bayesian Information Criterion (BIC) as a model selection criterion. Details regarding the BIC can be seen in Definition 3. Both algorithms are readily applicable to ARMA models with the inclusion of the appropriate lags of the primary time series in Step 3.

**Definition 3** (Bayesian Information Criterion \cite{16}). Consider a regression model with $k$ coefficients and denote the maximum likelihood estimator for the variance as

$$\hat{\sigma}^2_k = \frac{SSE(k)}{n}$$

where $SSE(k)$ is the residual sum of squares under the model with $k$ regression coefficients.

The Bayesian Information Criterion, denoted $BIC$, is

$$BIC = \log(\hat{\sigma}^2_k) + \frac{k \log(n)}{n},$$

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with the value of $k$ yielding the minimum BIC specifying the best regression model.

**Algorithm 1 Durbin’s Algorithm: An Algorithm for MA Fitting**

**Input:**
- Time series data $\{y_1, \ldots, y_n\}$;
- The estimated order $q$;

**Step 1.** Choose a sufficiently high order $\hat{p} > q$;

**Step 2.** Find the CMLE of the coefficient vector, $(\hat{\pi}_1, \ldots, \hat{\pi}_{\hat{p}})$ and compute the white noise approximations $\tilde{w}_t$ as in (5);

**Step 3.** Regress the primary time series over $q$ lagged values of $\tilde{w}_t$ to estimate the coefficient vector, $(\tilde{\theta}_1, \ldots, \tilde{\theta}_q)$ as in (6);

**Output:** Estimated coefficients $(\hat{\theta}_1, \ldots, \hat{\theta}_q)$.

[5, 6] identified, through practice and simulation, that simply increasing the size of $\hat{p}$ in the intermediate AR model does not necessarily improve the quality of estimates of the associated MA model. Broerson [5, 6] also identified that the optimal $\hat{p}$ value is in fact finite and advocated the use of model selection criteria, such as the AIC [2], in order to choose an appropriate $\hat{p}$. Instead, Broerson [5, 6] developed a new selection criterion for optimally choosing $\hat{p}$ by generalizing the AIC (GIC), details of which can be seen in Definition 4.

**Definition 4 (Generalized Information Criterion [5]).** Consider an $\text{MA}(q)$ process as defined in (3) with variance

$$
\sigma^2_Y = \sigma^2_W \left( 1 + \sum_{i=1}^{q} \theta^2_i \right) = \frac{\sigma^2_W}{\prod_{i=1}^{\infty} (1 - k_i^2)}
$$

where $\theta_i$ are the coefficients of an MA model and $k_i$ are the reflection coefficients of a long AR model. Then the residual sum of squares for $N$ observations of an $\text{MA}(q)$ process, denoted $\text{RSS}_p$, is given by

$$
\text{RSS}_p = \sum_{n=1}^{N} \sigma^2_Y \prod_{i=1}^{p} (1 - k^2_i) = \text{RSS}_{p-1}(1 - k^2_p).
$$

The Generalized Information Criterion, denoted $\text{GIC}(p, \alpha)$, with penalty factor 1 for $\alpha$, is

$$
\text{GIC}(p, \alpha) = \log \left( \frac{\text{RSS}_p}{N} \right) + \frac{\alpha p}{N}
$$

where the optimal order $k$ is the order with minimum $\text{GIC}(p, 1), p = 0, 1, \ldots, \infty$. 

7
2.3 Autoregressive Moving Average Models

A stationary time series \( \{ Y_t; t = 0, \pm 1, \pm 2, \ldots \} \) with the constant mean \( \mathbb{E}[Y_t] = 0 \) is an ARMA model with orders \( p \) and \( q \), denoted by \( \text{ARMA}(p, q) \), if we have

\[
Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},
\]

where \( \phi_p \neq 0, \theta_q \neq 0 \) and \( W_t \) is Gaussian white noise with the mean \( \mathbb{E}[W_t] = 0 \) and variance \( \text{Var}(W_t) = \sigma_W^2 \). It is readily seen that an ARMA model is a combination of the AR and MA model seen previously in Section 2.1 and Section 2.2 respectively. Consequently, each \( \text{ARMA}(p, q) \) model has \( p + q + 3 \) unknown parameters consisting of orders \( p \) and \( q \), the coefficients \( \phi_i \) and \( \theta_i \) and the variance of the white noises \( \sigma_W^2 \). Here, we briefly explain the common methods in the literature to estimate each of these unknown parameters.

**Estimating the orders \( p \) and \( q \)**

Since each ARMA model contains both AR and MA components, the order of an \( \text{ARMA}(p, q) \) model is commonly estimated by using the PACF seen in Section 2.1 in conjunction with the ACF seen in Section 2.2. Table 1 [16] summarises the behaviour of the ACF and PACF for ARMA models and its derivatives. For pure AR and MA models, the behaviour of the PACF and ACF is clear and estimation of \( p \) or \( q \) is straightforward as previously discussed in Section 2.1 and Section 2.2 respectively. However, the behaviour of the PACF and ACF for an ARMA model is ambiguous with no clear cut off after a particular lag to estimate the orders \( p \) and \( q \). As a result, a more precise inspection of the PACF and ACF is required, and often multiple models, consisting of different combinations of orders \( p \) and \( q \), are made as an initial guess of what the true orders may be. Steps 2 and 3 of the Box-Jenkins method, estimation and diagnosis, are then performed to help distinguish the best modelling combination of \( p \) and \( q \).

| ACF | AR(p) | MA(q) | ARMA(p, q) |
|-----|-------|-------|------------|
| PACF | Cuts off after lag \( p \) | Tails off | Tails off |

Table 1: This table highlights the behaviour of the ACF and PACF for AR(p), MA(q) and ARMA(p, q) processes.

**Maximum likelihood estimation of the coefficients \( \phi_i \) and \( \theta_i \) and variance \( \sigma_W^2 \):** Due to the MA component, both the log-likelihood and conditional log-likelihood function of an ARMA\((p, q)\) model are complicated non-convex functions and cannot be maximized analytically. Numerical optimization techniques are commonly employed to deal with estimating the coefficients of ARMA models. Durbin’s methodology, Algorithm 1, of exploiting the asymptotic equivalence between AR\((\infty)\) and MA\((q)\) models can similarly be used to estimate the coefficients of ARMA models. The optimal order \( \hat{p} \) can also be chosen using model selection criteria such as the BIC utilised by Hannan and Rissanen [15] and the GIC utilised by [5, 6] as seen in Section 2.2. Durbin’s methodology, Algorithm 1, along with the BIC and GIC criteria will be used as the estimation method for ARMA models for the remainder of this paper, and will be compared empirically in Section 4 to the new algorithm, called Rollage, developed in Section 3.
3 Theoretical Results

In this section, we introduce the concept of rolling average and develop its theoretical properties. These results are utilized to construct an efficient algorithm to estimate an appropriate AR model for big time series data. It should be noted that all proofs of this section as well as technical lemmas/propositions/theorems used in the proofs are presented in the Supplementary Materials document.

3.1 Rolling Average

We start this section by introducing the rolling average which is plays a central role in this work.

**Definition 5 (Rolling Average).** Suppose the time series \( \{Y_1, \cdots, Y_n\} \) is a causal AR\( (p) \) process. Fit an AR\((m)\) model \((m > p)\) to the data and find the MLE of the coefficient vector, \( \hat{\phi}_{p,m} = (\hat{\phi}_1^{(m)}, \cdots, \hat{\phi}_m^{(m)})^T \). The “rolling average” of this estimation is denoted by \( \bar{\phi}_{p,m} \) and defined as follows:

\[
\bar{\phi}_{p,m} := \frac{1}{m - p} \sum_{j=p+1}^{m} \hat{\phi}_j^{(m)}.
\]

The main motivation of introducing rolling averages is the convergence result on the asymptotic distribution of the MLEs of an AR model coefficients fitted to (large enough) time series data, as expressed in Theorem 1. More precisely, Theorem 1 states that if the underlying model of the time series data is truly an AR\( (p) \) model, but an AR\((m)\) model \((m > p)\) is (over)fitted to the data, the MLE of the (over-fitting) coefficient vector, asymptotically, has a multivariate normal distribution, that is,

\[
\sqrt{n} \left[ \hat{\phi}_{p+1}^{(m)} \cdots \hat{\phi}_m^{(m)} \right]^T \sim \text{MVN}(0, \Sigma_{p,m}(p + 1, m : p + 1 : m)).
\]

Accordingly, the rolling average \( \bar{\phi}_{p,m} \) which is the sample mean of the estimated (over-fitting) coefficients \( \hat{\phi}_1^{(m)}, \cdots, \hat{\phi}_m^{(m)} \) should also have a Normal distribution with a mean zero, in limit. This implies that, similar to the PACF values, the rolling averages can be also considered as a tool to estimate the order of an AR model. Furthermore, while the former approach looks only at the last (over-fitting) coefficient \( \hat{\phi}_m^{(m)} \) to estimate the order of the model, rolling averages consider all (over-fitting) coefficients together. Hence, one could expect that the latter exploits more information from the sample, and consequently may provide more efficient and accurate estimates. This is the motivation behind developing a new algorithm for estimating an AR model based on rolling averages.

Hence, we should derive the variance of the rolling averages to be able to utilize them in an algorithmic way to estimate the order of an AR model. For this purpose, we first define nested lower right corner matrix in Definition 6 and then obtain its structure in Theorem 2. Finally, Theorem 3 applies all these results to establish the asymptotic distribution of the rolling average \( \bar{\phi}_{p,m} \).
Definition 6 (Nested Lower Right Corner Matrix (NLRC)). Suppose the time series \( \{Y_1, \cdots, Y_n\} \) is a causal AR\((p)\) process. Fit an AR\((m)\) model \((m > p)\) to the data and find the covariance matrix \(\Sigma_{p,m}\). The \((m - p) \times (m - p)\) square matrix \(\Sigma_{p,m}(p+1:m, p+1:m)\) extracted from the lower right corner of the covariance matrix is called “nested lower right corner” matrix and denoted by \(\text{NLRC}_{p,m}\).

Theorem 2 (Closed Form for \(\text{NLRC}_{p,m}\)). Let the time series \(\{Y_1, \cdots, Y_n\}\) be a causal AR\((p)\) model. The nested lower right corner matrix \(\text{NLRC}_{p,m}\) for a fixed \(p \geq 1\) and \(m = p+1, p+2, \ldots\), is a symmetric positive definite matrix with the lower triangular coordinates satisfying the following “nested” equations for \(m = p+2, p+3, \ldots\),

\[
\text{NLRC}_{p,m}(i, j) =
\begin{cases}
  \phi_0^{(p)} & \text{if } m \leq 2p + 1, \\
  \phi_{m-p-i}^{(p)} \phi_{m-p-1}^{(p)} + \text{NLRC}_{p,m-1}(i, 1) & \text{for } j = 1, i = 1, \ldots, m - p - 1, \\
  \phi_0^{(p)} \phi_{m-p-1}^{(p)} & \text{for } j = 1, i = m - p, \\
  \text{NLRC}_{p,m-1}(i, 1) & \text{for } j = 1, i = 1, \ldots, m - p - 1, \\
  0 & \text{for } j = 1, i = m - p, \\
  \text{NLRC}_{p,m-1}(i-1, j-1) & \text{for } 2 \leq j \leq i \leq m - p,
\end{cases}
\]

where \(\phi_0^{(p)} := -1\) and the initial condition is \(\text{NLRC}_{p,p+1}(1, 1) = (\phi_0^{(p)})^2\). Moreover, the lower
triangular coordinates equal

\[
\begin{align*}
\text{NLRC}_{p,m}(i,j) &= \left\{ \begin{array}{ll}
\min\{m-p-i,p\} \sum_{k=0}^{\min\{m-p-i,p\}} (\phi_k^{(p)})^2 & \text{for } j = i, \ i = 1, \ldots, m - p \\
\min\{m-p-i,p-1\} \sum_{k=0}^{\min\{m-p-i,p-1\}} \phi_k^{(p)} \phi_{k+1}^{(p)} & \text{for } j = i - 1, \ i = 2, \ldots, m - p \\
\vdots & \vdots \\
\min\{m-p-i,\ell\} \sum_{k=0}^{\min\{m-p-i,\ell\}} \phi_k^{(p)} \phi_{k+\ell}^{(p)} & \text{for } j = i - \ell, \ i = \ell + 1, \ldots, m - p \\
\vdots & \vdots \\
\phi_0^{(p)} \phi_{\ell}^{(p)} & \text{for } j = i - p, \ i = p + 1, \ldots, m - p \\
0 & \text{elsewhere,}
\end{array} \right.
\end{align*}
\]

and above the diagonal coordinates are calculated through

\[
\text{NLRC}_{p,m}(i,j) = \text{NLRC}_{p,m}(j,i) \quad \text{for } 1 \leq i < j \leq m - p.
\]

Note that those coordinates where the corresponding range for \(i\) has a lower bound greater than the upper bound should be disregarded.

**Example 1.** For an \(\text{AR}(2)\) model, the lower triangular coordinates of the symmetric matrix \(\text{NLRC}_{2,7}\) are illustrated below (for sake of simplicity, the superscript (2) has been omitted):

\[
\text{NLRC}_{2,7} = \begin{pmatrix}
\phi_0^2 + \phi_1^2 + \phi_2^2 \\
\phi_0\phi_1 + \phi_1\phi_2 \\
\phi_0\phi_2 \\
0
\end{pmatrix}.
\]

**Theorem 3** (Asymptotic Distribution of Rolling Average). Let the time series \(\{Y_1, \ldots, Y_n\}\) be a causal \(\text{AR}(p)\) process. If an \(\text{AR}(m)\) model \((m > p)\) is fitted to the data, asymptotically, we have

\[
\sqrt{n} \tilde{\phi}_{p,m} \sim N[0, \sigma_{p,m}^2],
\]
where $\sigma^2_{p,m}$ satisfies the recursion,

\[
(m-p)^2\sigma^2_{p,m} = \begin{cases} 
(m - p - 1)^2\sigma^2_{p,m-1} + (\phi_0^{(p)} + \cdots + \phi_{m-1}^{(p)})^2 & \text{for } m = 2p + 1, 2p + 2, \ldots, \\
(m - p - 1)^2\sigma^2_{p,m-1} + (\phi_0^{(p)} + \cdots + \phi_{m-p-1}^{(p)})^2 & \text{for } m = p + 2, \ldots, 2p, \\
(\phi_0^{(p)})^2 & \text{for } m = p + 1,
\end{cases}
\]

with the general solution for $\sigma^2_{p,m}$,

\[
\begin{cases} 
(\phi_0^{(p)})^2 & \text{for } m = p + 1, \\
\frac{1}{2^2} (\phi_0^{(p)})^2 + (\phi_0^{(p)} + \phi_1^{(p)})^2 & \text{for } m = p + 2, \\
\vdots & \vdots \\
\frac{1}{\ell^2} (\phi_0^{(p)})^2 + (\phi_0^{(p)} + \phi_1^{(p)})^2 + \cdots + (\phi_0^{(p)} + \cdots + \phi_{\ell-1}^{(p)})^2 & \text{for } m = p + \ell, \\
\vdots & \vdots \\
\frac{1}{p^2} (\phi_0^{(p)})^2 + (\phi_0^{(p)} + \phi_1^{(p)})^2 + \cdots + (\phi_0^{(p)} + \cdots + \phi_{p-1}^{(p)})^2 & \text{for } m = 2p, \\
\frac{1}{(p+k)^2} (p^2\sigma^2_{p,2p} + k(\phi_0^{(p)} + \cdots + \phi_p^{(p)})^2) & \text{for } m = 2p + k, k = 1, 2, \ldots.
\end{cases}
\]

### 3.2 Rollage Algorithm for Fitting AR Models

Based on theoretical results developed in Section 3.1, we introduce the Rollage algorithm, depicted in Algorithm 2 to fit an appropriate AR model to a given time series data.

**Typical procedure for calculating rolling average.** Tables 2 and 3 illustrate how the rolling averages are calculated in the Rollage algorithm.

| Model | Parameters |
|-------|------------|
| AR(1) | $\phi_1^{(1)}$ |
| AR(2) | $\phi_1^{(2)}, \phi_2^{(2)}$ |
| AR(3) | $\phi_1^{(3)}, \phi_2^{(3)}, \phi_3^{(3)}$ |
| $\vdots$ | $\vdots$ |
| AR($\bar{p}$) | $\phi_1^{(p)}, \cdots, \phi_{\bar{p}}^{(p)}$ |

Table 2: Fitting $\bar{p}$ models AR(1), $\ldots$, AR($\bar{p}$) to the data and finding the MLE of their parameters

### 3.3 Rollage Algorithm for Fitting ARMA Models

In order to fit ARMA (and MA) models using the theory developed in Section 3.2, we must use the Rollage algorithm in conjunction with Durbin’s Algorithm 1. To achieve this, we must first
Algorithm 2: Rollage: A Novel Algorithm for AR Fitting

**Input:**
- Time series data \{y_1, \ldots, y_n\};
- A relatively large value \(\bar{p} \ll n\);

**Step 0.** Set \(\ell = 1\);

**while** \(\ell < \bar{p}\) **do**

1. \(\ell \leftarrow \ell + 1\);
2. Find the CMLE of the coefficient vector, \((\hat{\phi}_1^{(\ell)}, \ldots, \hat{\phi}_\ell^{(\ell)})\);
3. Compute the values of rolling averages, \(\bar{\phi}_{h,\ell}\), for \(h = 1, \ldots, \ell - 1\) as in Definition 5;
4. Compute the variance of rolling averages, \(\hat{\sigma}^2_{\ell,m}\), for \(m = \ell + 1, \ldots, \bar{p}\) as in Theorem 3;

**end while**

**Step 5.** Estimate \(p\) as the largest \(\ell\) such that at least 5% of inequalities \(|\bar{\phi}_{\ell,m}| \geq 1.96\sigma_{\ell,m}/\sqrt{n - \bar{p}}\) for \(m = \ell + 1, \ldots, \bar{p}\) are held;

**Output:** Estimated order \(p\) and coefficients \((\hat{\phi}_1^{(p)}, \ldots, \hat{\phi}_p^{(p)})\).

| Model | Rolling average |
|-------|-----------------|
| AR(1) | \(\hat{\phi}_{1,2} = \hat{\phi}_2^{(2)}\) |
| AR(2) | \(\hat{\phi}_{1,3} = \frac{1}{2}(\hat{\phi}_2^{(3)} + \hat{\phi}_3^{(3)})\), \(\hat{\phi}_{2,3} = \hat{\phi}_3^{(3)}\) |
| : | : |
| AR(\(\bar{p}\)) | \(\hat{\phi}_{1,\bar{p}} = \frac{1}{\bar{p} - 1}(\hat{\phi}_2^{(p)} + \cdots + \hat{\phi}_p^{(p)})\), \(\ldots, \hat{\phi}_{\bar{p}-1,\bar{p}} = \hat{\phi}_\bar{p}^{(p)}\) |

Table 3: Calculating the rolling averages

introduce a threshold hyper-parameter to Rollage so that an appropriately large \(\bar{p}\) value is chosen for the AR model, in the intermediate step of Algorithm 1. In this scenario, Rollage will act as a stopping criterion to choose an appropriately large \(\bar{p}\) value.

We define the threshold hyper-parameter, denoted \(\delta\), as the following inequality:

\[
\delta < \max \left( \frac{|\bar{\phi}_{\ell,m}|}{1.96\sigma_{\ell,m}/\sqrt{n - \bar{p}}} \right),
\]

whereby the estimate of the large \(\bar{p}\) value for the intermediate AR model is the first value of \(\ell\) such that the inequality is violated. This extension to Durbin’s Algorithm 1 can be seen in Algorithm 3, and is called Rollage*. 

13
Algorithm 3 Rollage*: An Algorithm for ARMA (and MA) Fitting

Input:
- Time series data \( \{y_1, \ldots, y_n\} \);
- The estimated order \( q \);
- Threshold parameter \( \delta \)

Step 1. Using Rollage (2) choose the sufficiently high order \( \tilde{p} > q \) to be the first value of \( l \) such that the inequality \( \delta < \max \left( \frac{|\hat{\phi}_{l,m}|}{1.96\sigma_{l,m}/\sqrt{n-p}} \right) \) is violated in the Rollage algorithm;

Step 2. Find the CMLE of the coefficient vector, \( (\hat{\pi}_1, \ldots, \hat{\pi}_{\tilde{p}}) \) and compute the white noise approximations \( \tilde{w}_t \) as in (5);

Step 3. Regress the primary time series over \( q \) lagged values of \( \tilde{w}_t \) to estimate the coefficient vector, \( (\hat{\theta}_1, \ldots, \hat{\theta}_q) \) as in (6);

Output: Estimated coefficients \( (\hat{\theta}_1, \ldots, \hat{\theta}_q) \).

4 Empirical Results

In this section, we present the performance of the Rollage algorithm on several synthetic time series data. The data were simulated using models with randomly generated sets of coefficient parameters. The variance of the error parameter was 1 for each case. The resulting models were confirmed to be causal and invertible using MATLAB’s arima() function. Parameters were randomly generated for AR\((p)\) and MA\((q)\) models with \( p = 5, 10, \ldots, 100 \) and \( q = 5, 10, \ldots, 100 \). All possible parameter combinations were then utilised to produce 400 ARMA\((p,q)\) models.

Numerical analysis is presented in three subsequent sections. In Section 4.1, the Rollage algorithm is applied to the AR synthetic data where its overall efficacy is analysed. In Section 4.2, the Rollage algorithm is applied to the MA data where it is used as a criterion in Durbin’s Algorithm 1 and compared to the BIC and GIC criteria for estimating an optimal \( \tilde{p} \) value then subsequently fitting an MA model. The relative errors of parameter estimation are also analysed amongst the Rollage, BIC and GIC criteria. Analogously, Section 4.3 compares the Rollage, BIC and GIC criteria in estimating ARMA models.

4.1 Autoregressive Models

From each of the 20 AR\((p)\) models, 500,000 synthetic time series realizations were generated. The Rollage algorithm was applied to each with results being the average of 20 replications.

For each of the 20 models, the Rollage algorithm correctly identifies the order \( p \), showing its efficacy in estimating this parameter. Figure 1 displays the rolling average graphs produced by the Rollage algorithm at various lags for the data generated from the AR\((20)\) model. In general, we notice that for lags less than or equal to the true order \( p \), at least one rolling average lies significantly outside the 95% confidence boundary. However, for all subsequent lags greater than the true order \( p \) all rolling averages lie within the 95% confidence boundary. Specifically, Figure 1(a-e) displays lags less than or equal to \( p = 20 \), where it can be seen that the rolling averages lie outside the 95% confidence bounds. Figure 1(f-i) then shows lags greater than \( p = 20 \) where the rolling averages
lie within the 95% confidence bounds. This feature is dictated by the inequalities in Step 5 of the Rollage algorithm. It should be noted that the confidence boundaries in Figure 1(a-f) display similar curvature to those in Figure 1(g-i), also dictated by Step 5, however cannot be distinguished due to the larger scale in their respective graphs. In this sense, the Rollage graphs can be used as a tool, analogous to the PACF, in choosing the order of an AR model.

![Images of rolling averages for different lags](images)

(a) $p = 5$
(b) $p = 10$
(c) $p = 15$
(d) $p = 19$
(e) $p = 20$
(f) $p = 21$
(g) $p = 25$
(h) $p = 30$
(i) $p = 35$

Figure 1: This figure illustrates the rolling averages at different lags for nine AR($p$) models where the underlying data is generated from an AR(20) model. It is readily seen that up to $p = 20$, there exists at least one rolling average which is significantly out of the 95% confidence boundaries. However, from $p = 21$ onwards, all rolling averages at all lags lie within the 95% confidence bounds.

4.2 Moving Average Models

For each of the 20 MA($q$) models, synthetic time series data was generated for sample sizes $n = 10,000$, $20,000$, $50,000$, $100,000$, $200,000$, $500,000$ and $1,000,000$. For each dataset, the Rollage, BIC and GIC criteria are used to estimate an optimal $\tilde{p}$ value for fitting an AR model before subsequently fitting an MA model. The threshold hyper-parameter for the Rollage algorithm was varied from 2.5
to 4 in 0.25 increments throughout the experiments with threshold equal to 3 providing the best
trade off between estimating an optimal $\hat{p}$ and producing the lowest relative error of parameter
estimates, on average. As a result, all subsequent empirical results utilise a threshold parameter of
3 in the Rollage algorithm.

Figure 2 displays the estimated $\hat{p}$ values for each $q$ (i.e. $\hat{p}$ vs $q$) from the 20 MA models at the
various sample sizes $n$ (excluding $n = 10,000$). More precisely, the blue, red and magenta graphs
are associated with the Rollage, BIC and GIC criteria, respectively. It can be observed that for
all sample sizes, the Rollage algorithm produces a smaller than or equal to $\hat{p}$ value compared to
both the BIC and GIC criteria, with this becoming more noticeable as $q$ and $n$ increases. Table 4
highlights this, comparing the average $\hat{p}$ and associated relative error produced by the Rollage,
BIC and GIC criteria for each of the respective sample sizes $n$. On average, the Rollage algorithm
provides smaller estimates for $\hat{p}$ than both the BIC and GIC criteria, with the difference between
Rollage and the other criteria increasing as $n$ increases. The bottom 2 rows of Table 4 quantify
this relative difference between Rollage and the BIC and GIC criteria. The relative difference of
$\hat{p}$ is calculated as

$$\frac{|\hat{p}_R - \hat{p}_A|}{\hat{p}_R}$$

where $\hat{p}_R$ is given by the Rollage algorithm and $\hat{p}_A$ is suggested by the mentioned alternative.
The total average column of Table 4 summarises this trend for all sample sizes, showing that the
Rollage criterion provides $\hat{p}$ values that are 18.06% smaller than the BIC and 32.26% smaller than
the GIC criteria, on average. The results of Figure 2 and Table 4 provide empirical evidence that
the Rollage algorithm is computationally less expensive than both the BIC and GIC criteria, on
average.

Similarly, Figure 3 displays the estimated $\hat{p}$ values for each sample size $n$ (i.e. $\hat{p}$ vs $n$) for various
MA($q$) models where a similar conclusion, that the Rollage criterion provides smaller estimates for $\hat{p}$
than the BIC and GIC criteria, is reached. Additionally, Figure 2 suggests that $\hat{p}$ increases linearly
as $q$ increases, and becomes more evident as $n$ increases, whereas Figure 3 suggests that $\hat{p}$ increases
logarithmically as $n$ increases. We have fitted linear models to predict the optimal $\hat{p}$, which find $q$,
log($n$) and $q\log(n)$ to be statistically significant predictors for $\hat{p}$ across all three criteria. Table 5
provides the coefficients for linear models to predict the optimal $\hat{p}$ for each of the Rollage, BIC
and GIC criteria. For the Rollage algorithm, the rearranged linear model is

$$\hat{p} = q(0.81\log(n) - 7.12) + 3.19\log(n)$$

highlighting that $\hat{p}$ is heavily dependent on $q$, however the sample size $n$ is less impactful as log($n$)
stunts this impact. The linear models can’t be applied generally to prescribe a $\hat{p}$, even within the
range of $q$ and $n$ considered, as the set of 20 MA model coefficients may not be representative of
the entire population. The models do however provide a good initial approximation for $\hat{p}$ and offer
insight into variable relationships.

Analogous to Figure 2, Figure 4 shows the corresponding relative error associated with estimating
the parameters of an MA($q$) model, in percentage. Here we define the relative error of parameter
estimates by

$$\frac{||\hat{\theta} - \theta||}{||\theta||}$$

where $\hat{\theta}$ is the parameter estimates and $\theta$ is the true parameter values for an MA($q$) model. It
is observed that for all sample sizes, the Rollage criterion produces similar relative errors to the
Figure 2: Each graph illustrates the $\hat{p}$ values estimated by the Rollage, BIC and GIC criteria to initially fit an AR($p$) model and then subsequently fit an associated MA($q$) model. The Rollage, BIC and GIC criteria are represented by the blue, red and magenta graphs respectively. The six graphs represent the different sample sizes $n$ that were used to fit an MA($q$) model. It is readily seen that the Rollage criteria regularly produces the smallest estimate for $\hat{p}$.

BIC and GIC criteria over the whole range of MA($q$) models. Table 4 summarises this by showing the average relative errors below the average estimate of $\hat{p}$. For all criteria, the relative error of parameter estimates becomes smaller as $n$ increases and for big data regimes (i.e. $n = 1,000,000$) all criteria produce relative errors of approximately 1.5%. As a result, the Rollage algorithm provides a great trade off between computational runtime and algorithmic accuracy for fitting an MA model, when compared to the BIC and GIC criteria.

In addition, Figure 4 suggests that the relative error increases linearly as a function of $q$, however as $n$ increases the linear relationship appears to become more constant across all values of $q$ (particularly $n = 1,000,000$). In contrast, Figure 5 displays the relative error of parameter estimates as a function of the sample size $n$ for various MA($q$) models, where we see the relative errors decrease exponentially. This is further evidence that the Rollage algorithm may be appropriate for fitting MA models to big time series data when compared to current alternatives.

4.3 Autoregressive Moving Average Models

For each of the 400 ARMA($p$, $q$) models, synthetic time series data was generated for sample sizes $n = 10,000, 20,000, 50,000, 100,000, 200,000, 500,000$ and $1,000,000$. For each dataset, the Rollage, BIC and GIC criteria produced estimates of the optimal $\hat{p}$ value for fitting a large AR model before subsequently fitting the full ARMA model. The threshold hyper-parameter for the Rollage algorithm was varied from 2.5 to 4 in 0.25 increments throughout the experiments and 3 was found to provide
| Rollage | BIC    | GIC    | Relative to BIC | Relative GIC |
|---------|--------|--------|----------------|--------------|
|         | 10k    | 20k    | 50k            | 100k         | 200k         | 500k         | 1M            | Total Average |
|         | 71     | 86     | 115            | 134          | 168          | 216          | 297           | 155           |
|         | (14.09%) | (10.26%) | (7.14%)          | (5.76%)         | (3.93%)        | (2.99%)        | (1.78%)        | (6.56%)        |
|         | 76     | 94     | 129            | 155          | 205          | 263          | 363           | 183           |
|         | (13.69%) | (9.75%) | (6.66%)         | (5.22%)         | (3.47%)        | (2.55%)        | (1.48%)        | (6.12%)        |
|         | 100    | 113    | 151            | 174          | 217          | 276          | 403           | 205           |
|         | (12.79%) | (9.30%) | (6.34%)         | (5.07%)         | (3.40%)        | (2.53%)        | (1.42%)        | (5.84%)        |
| Relative to BIC | 7.04% | 9.30% | 12.17% | 15.67% | 22.02% | 21.76% | 22.22% | 18.06% |
| Relative GIC | 40.85% | 31.40% | 31.30% | 29.85% | 29.17% | 27.78% | 35.69% | 32.26% |

Table 4: Each cell provides the average $\tilde{p}$ (rounded to the nearest integer) and average associated relative error (rounded to 2 d.p.) of the subsequently fitted MA models for the given sample size and estimation method.

| Rollage | BIC    | GIC    | $R^2$ |
|---------|--------|--------|-------|
| $q$     | log(n) | $q\log(n)$ | $q\log(n)$ | $R^2$ |
| -7.12   | 3.19   | 0.81   | 0.95  |
| -9.39   | 3.27   | 1.06   | 0.95  |
| -9.43   | 5.76   | 1.05   | 0.94  |

Table 5: Coefficients for the corresponding terms of linear models fitted to predict the optimal $\tilde{p}$ chosen by each criterion based on all MA simulated time series data.

the lowest average relative errors for the parameter estimates over the various $n < 1,000,000$ and $3.5$ for $n = 1,000,000$.

Similar to the case with MA models, Figure 6 provides examples of the Rollage algorithm consistently providing smaller values of $\tilde{p}$ than the other criteria. The total average column of Table 6 summarises this trend for all considered sample sizes, showing also that GIC consistently provides the largest values. Rollage begins suggesting significantly smaller $\tilde{p}$ than BIC for $n > 50,000$, and for all considered sample sizes for GIC. The bottom 2 rows of Table 6 quantify this relative difference, which is calculated as per (9). The relative difference becomes significant as $n$ gets large. However, the Rollage algorithm provides relative errors that closely matches those for BIC and GIC for all considered sample sizes. When $n = 1,000,000$, we see that BIC provides a relative error only $1.5\%$ lower than Rollage, but the $\tilde{p}$ prescribed by BIC is almost $20\%$ larger than that for Rollage. This is evidence that the Rollage algorithm could provide significant computational savings within the context of big data with a limited reduction in parameter accuracy.

Figures 6 and 7 suggest that $\tilde{p}$ grows linearly as a function of $q$ and logarithmically as a function of $n$. The relationship with $p$ is less obvious, but linear models find $q$, $p$, $p\log(n)$, $q\log(n)$ and $\log(n)$ to be significant predictors for $\tilde{p}$ across all three criterion. The coefficients of the three linear models are provided in Table 7. The dependence of $\tilde{p}$ suggested by Rollage on $p$ and $q$ is clarified if we consider the rearranged model

$$\hat{p} = p(2.08 - 0.14 \log(n)) + q(0.76 \log(n) - 6.29) + 4.19 \log(n).$$

Here we can see that for the range of sample sizes considered for this study, the coefficients of $p$ and $q$ are both positive, but as $n$ increases the coefficient of $p$ decreases while the coefficient of $q$
Figure 3: This figure illustrates $\tilde{p}$ as a function of the sample size $n$ for various MA($q$) models. The Rollage criteria is shown to consistently produce the smallest estimate for $\tilde{p}$. A logarithmic relationship can be observed between $\tilde{p}$ and the sample size $n$.

5 Conclusion

In this paper, we have developed a new efficient algorithm to estimate an AR model. Motivated by Theorem 1, we utilise the concept of a rolling average to develop an algorithm, called Rollage, to estimate the order of an AR model and subsequently fit the model. When used in conjunction
Figure 4: Each graph illustrates the Relative Error % of estimating the parameters of an MA(q) model for each of the Rollage, BIC and GIC criteria. The Rollage, BIC and GIC criteria are represented by the blue, red and magenta graphs respectively. The six graphs represent the different sample sizes $n$ that were used to fit an MA(q) model. All three criteria produce similar relative errors in estimating the parameters of an associated MA(q) model. Relative errors reduce as sample size $n$ increases for all criteria.

with existing methodology, specifically Durbin’s, the Rollage algorithm can be used as a criterion to optimally fit ARMA models to big time series data. Empirical results on large-scale synthetic time series data show the efficacy of Rollage and when compared to existing criteria, the Rollage algorithm provides a great trade off between computational complexity and algorithmic accuracy.
Figure 5: This figure illustrates Relative Error % of estimating the parameters of an MA(q) model as a function of the sample size $n$. The Rollage criterion is shown to produce similar relative errors when compared to the BIC and GIC criteria. A negative exponential relationship can be observed between Relative Error % and the sample size $n$. 
Table 6: Each cell provides the average $\hat{p}$ (rounded to the nearest integer) and average associated relative error (rounded to 2 d.p.) of the subsequently fitted ARMA models for the given sample size and estimation method.

|       | 10k | 20k | 50k | 100k | 200k | 500k | 1M    | Total Average |
|-------|-----|-----|-----|------|------|------|-------|---------------|
| Rollage | 130 | 142 | 168 | 172  | 205  | 269  | 301   | 195 (23.88%)  |
| BIC    | 131 | 143 | 169 | 182  | 222  | 292  | 358   | 211 (22.84%)  |
| GIC    | 170 | 186 | 220 | 219  | 263  | 344  | 425   | 257 (22.69%)  |
| Relative to BIC | 0.77% | 0.70% | 0.60% | 5.81% | 8.29% | 8.55% | 18.94% | 8.21% |
| Relative GIC | 30.77% | 30.99% | 30.95% | 27.33% | 28.29% | 27.88% | 41.20% | 31.79% |

Table 7: Coefficients for the corresponding terms of linear models fitted to predict the optimal $\hat{p}$ chosen by each criterion based on all ARMA simulated time series data.

|       | $p$  | $q$  | $\log(n)$ | $p\log(n)$ | $q\log(n)$ | $R^2_a$ |
|-------|------|------|------------|------------|------------|---------|
| Rollage | 2.08 | -6.29 | 4.19 | -0.14 | 0.76 | 0.92 |
| BIC    | 1.87 | -8.12 | 3.55 | -0.11 | 0.94 | 0.94 |
| GIC    | 2.54 | -8.86 | 7.09 | -0.19 | 1.03 | 0.90 |
Figure 6: The first two plots (left to right) show the values of $\tilde{p}$ suggested by the 3 criterion plotted against $n$ for two ARMA models. The last two plots show the Relative Error % of the full ARMA parameter estimates for the same models when the optimal $\tilde{p}$ according to each criterion was used in Durbin’s Algorithm (1).
Figure 7: The values of $\hat{p}$ suggested by the three criterion are plotted against $p$ and $q$ for $n = 1,000,000$. The dependence of $\hat{p}$ linearly on $q$ is clearly more significant than the dependence on $p$. 
A Technical Lemmas/Propositions/Theorems and Proofs

A.1 Proof of Theorem 2

We first present Lemma 1 [13], Theorem 4 [4], and Theorem 5 [13] and then prove Propositions 1 and 2 which are used in the proof of Theorem 2.

**Lemma 1** (Block Matrix Inversion [13]). Consider the $2 \times 2$ block matrix

\[
M = \begin{pmatrix} c & b^\top \\ b & A \end{pmatrix},
\]

where $A$, $b$, and $c$ are an $m \times m$ matrix, an $m \times 1$ vector and a scalar, respectively. If $A$ is invariable, the inverse of matrix $M$ exists and can be calculated as follows

\[
M^{-1} = \frac{1}{k} \begin{pmatrix} 1 & -b^\top A^{-1} \\ -A^{-1}b & kA^{-1} + A^{-1}bb^\top A^{-1} \end{pmatrix},
\]

where $k = c - b^\top A^{-1}b$.

**Theorem 4** (Recursion for Autocovariance Function [4]). For a causal AR($p$) model with an induced Gaussian white noise series with mean 0 and variance $\sigma^2_w$, the autocovariance function at lag $j$ is given by

\[
\gamma_j = \begin{cases} 
\phi_1^{(p)} \gamma_1 + \cdots + \phi_p^{(p)} \gamma_p + \sigma^2_w & \text{for } j = 0, \\
\phi_1^{(p)} \gamma_{j-1} + \cdots + \phi_p^{(p)} \gamma_{j-p} & \text{for } j = 1, 2, \ldots, \\
\gamma_{-j} & \text{for } j = -1, -2, \ldots.
\end{cases}
\]

**Theorem 5** (Inverse and Adjugate of a Square Matrix [13]). If $A$ is an invertible square matrix, its inverse can be represented by

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A),
\]

where $\text{adj}(A)$ is the adjugate of matrix $A$, that is, the transpose of its cofactor matrix.

**Proposition 1** (Recursion For Matrix $\Sigma_{p,m}$). Let the time series $\{Y_1, \cdots, Y_n\}$ be a causal AR($p$) model. The covariance matrix $\Sigma_{p,m}$ for a fixed $p \geq 1$ and $m = p + 2, p + 3, \ldots$, satisfies
the following recursion

\[
\Sigma_{p,m} = \begin{pmatrix} 1 & \psi_{p,m-1}^\top \\ \psi_{p,m-1} & \Sigma_{p,m-1} + V_{p,m-1} \end{pmatrix},
\]

where \( \psi \) is an \((m - 1) \times 1 \) vector given by

\[
\psi_{p,m-1}^\top := \begin{bmatrix} \phi_1(p) & \cdots & \phi_p(p) & 0 & \cdots & 0 \end{bmatrix}
\]

and \( V_{p,m-1} \) is a symmetric \((m - 1) \times (m - 1) \) matrix defined by

\[
V_{p,m-1} := \psi_{p,m-1} \psi_{p,m-1}^\top.
\]

Proof. From Theorem 1, we know that \( \Sigma_{p,m} = \sigma_W^2 \Gamma_{p,m}^{-1} \). Furthermore, we have

\[
\Gamma_{p,m} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m-1} & \gamma_{m-2} & \cdots & \gamma_0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} \\ \gamma_1 & \Gamma_{p,m-1} & \cdots & \gamma_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m-1} & \gamma_{m-2} & \cdots & \gamma_0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_{p,m-1}^\top \\ \gamma_{p,m-1} & \Gamma_{p,m-1} \end{pmatrix},
\]

where \( \gamma_{p,m-1} := [\gamma_1 \cdots \gamma_{m-1}] \). So, Lemma 1 implies that

\[
\Gamma_{p,m}^{-1} = \frac{1}{k_{p,m-1}} \begin{pmatrix} 1 & -\gamma_{p,m-1} \Gamma_{p,m-1}^{-1} \\ -\gamma_{p,m-1}^\top \Gamma_{p,m-1}^{-1} & k_{p,m-1} \Gamma_{p,m-1}^{-1} + \Gamma_{p,m-1}^{-1} \gamma_{p,m-1} \Gamma_{p,m-1}^{-1} \end{pmatrix},
\]

where \( k_{p,m-1} = \gamma_0 - \gamma_{p,m-1}^\top \Gamma_{p,m-1}^{-1} \gamma_{p,m-1} \). Firstly, we simplify \( \Gamma_{p,m-1}^{-1} \gamma_{p,m-1} \) appearing in all blocks of the inverse matrix \( \Gamma_{p,m-1}^{-1} \). For this purpose, let define \( \psi_{p,m-1} := \Gamma_{p,m-1}^{-1} \gamma_{p,m-1} \) or equivalently, \( \Gamma_{p,m-1} \psi_{p,m-1} = \gamma_{p,m-1} \). More precisely,

\[
\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-2} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m-2} & \gamma_{m-3} & \cdots & \gamma_0 \end{pmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{m-1} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{m-1} \end{bmatrix}.
\]

Following 4 as well as the uniqueness of the solution of the above system of linear equations (due to the invertibility of matrix \( \Gamma_{p,m-1} \)), we can conclude that for any \( m \geq p + 2 \), we have

\[
\psi_{p,m-1}^\top = \begin{bmatrix} \phi_1(p) & \cdots & \phi_p(p) & 0 & \cdots & 0 \end{bmatrix}.
\]

Therefore,

\[
k_{p,m-1} = \gamma_0 - \gamma_{p,m-1}^\top \Gamma_{p,m-1}^{-1} \gamma_{p,m-1}
\]
\begin{align*}
\gamma_0 - \gamma_{p,m-1}^T v_{p,m-1} \\
= \gamma_0 - \phi_1^{(p)} \gamma_1 - \cdots - \phi_1^{(p)} \gamma_p
\end{align*}

(Theorem 4) = \sigma_W^2.

Thus, all these results conclude that

\begin{align*}
\Gamma_p^{-1} = \frac{1}{\sigma_W^2} \begin{pmatrix}
1 & -v_{p,m-1}^T \\
-v_{p,m-1} & \sigma_W^2 \Gamma_{p,m-1}^{-1} + v_{p,m-1} v_{p,m-1}^T
\end{pmatrix},
\end{align*}

and, equivalently,

\begin{align*}
\Sigma_p = \begin{pmatrix}
1 & -v_{p,m-1}^T \\
-v_{p,m-1} & \Sigma_{p,m-1} + V_{p,m-1}
\end{pmatrix}.
\end{align*}

\begin{proposition}[Recursion for Determinant of Autocovariance Matrix] Let the time series \( \{Y_1, \cdots, Y_n\} \) be a causal AR\((p)\) model with an induced Gaussian white noise series with mean 0 and variance \( \sigma_W^2 \). We have the following recursion for the determinant of the autocovariance matrix:

\[
\text{det}(\Gamma_p) = \sigma_W^2 \text{det}(\Gamma_{p,m-1}), \quad \text{for } p \geq 1, \ m \geq p + 1.
\]

\end{proposition}

\begin{proof}
The autocovariance matrix \( \Gamma_{p,m} \) for some \( m > p \) is given by

\[
\Gamma_{p,m} = \begin{pmatrix}
\gamma_0 & \cdots & \gamma_{m-2} & \gamma_{m-1} \\
\vdots & \ddots & \vdots & \vdots \\
\gamma_{m-2} & \cdots & \gamma_0 & \gamma_1 \\
\gamma_{m-1} & \cdots & \gamma_1 & \gamma_0
\end{pmatrix} = \begin{pmatrix}
\Gamma_{p,m-1} & \gamma_{m-1} \\
\vdots & \ddots & \vdots & \vdots \\
\gamma_{m-1} & \cdots & \gamma_1 & \gamma_0
\end{pmatrix}.
\]

Let \( \Gamma_{p,m}(\cdot, j) \) for \( j = 1, \cdots, m \) denote the \( j \)th column of the autocovariance matrix \( \Gamma_{p,m} \). Motivated from Theorem 4, performing the following column operation on the last column of \( \Gamma_{p,m} \) does not change the determinant:

\[
\Gamma_{p,m}(\cdot, m) = \Gamma_{p,m}(\cdot, m) - \phi_1 \Gamma_{p,m}(\cdot, m-1) - \cdots - \phi_p \Gamma_{p,m}(\cdot, m-p).
\]

This results in

\[
\text{det}(\Gamma_{p,m}) = \begin{vmatrix}
\gamma_{m-1} - \phi_1 \gamma_{m-2} - \cdots - \phi_p \gamma_{m-p-1} \\
\Gamma_{p,m-1} \\
\gamma_{m-1} - \phi_1 \gamma_0 - \cdots - \phi_p \gamma_{p-1} \\
\gamma_1 - \phi_1 \gamma_1 - \cdots - \phi_p \gamma_p
\end{vmatrix}.
\]

\end{proof}
Applying Theorem 4 to column \( m \) implies that,

\[
\det(\Gamma_{p,m}) = \begin{vmatrix} \Gamma_{p,m-1} & 0 \\ \vdots & \ddots & \ddots \\ \gamma_{m-1} & \cdots & \gamma_1 & \sigma_W^2 \end{vmatrix} = (-1)^{m+m} \sigma_W^2 \det(\Gamma_{p,m-1}) = \sigma_W^2 \det(\Gamma_{p,m-1}).
\]

By considering all these results together, we can now prove Theorem 2.

**Proof of Theorem 2**

*Proof.* We prove the nested equation for the matrix \( \text{NLRC}_{p,m} \) by fixing \( p \) and varying \( m = p + 1, p + 2, \ldots \). For the initial condition at \( m = p + 1 \), we have

\[
\text{NLRC}_{p,p+1}(1,1) = \text{Var}(\sqrt{n}\hat{\phi}_{p+1}^{(p+1)}) = \text{Var}(\sqrt{n}\hat{\text{PACF}}_{1}),
\]

where \( \hat{\text{PACF}}_{1} \) is the sample PACF at lag 1. Since the variance of the sample PACF, scaled by \( \sqrt{n} \), at any lag is equal to 1 (cf. Theorem 1), it results in

\[
\text{NLRC}_{p,p+1}(1,1) = (\phi_p^{(p)})^2.
\]

Now, consider a fixed value of \( m > p + 1 \). According to Definition 6 and Proposition 1, we have

\[
\text{NLRC}_{p,m}(2:m-p,2:m-p) = \Sigma_{p,m}(p+2:m,p+2:m) = \Sigma_{p,m-1}(p+1:m-1,p+1:m-1)
\]

\[
+ V_{p,m-1}(p+1:m-1,p+1:m-1) = \Sigma_{p,m-1}(p+1:m-1,p+1:m-1) = \text{NLRC}_{p,m-1}.
\]

Note that the second last equality is due to the structure of matrix \( V_{p,m-1} \) as defined in Proposition 1. More precisely, except the \( p \times p \) block on the top left corner of this matrix, all the other coordinates are equal to zero, including the \((m-p-1) \times (m-p-1)\) block on the lower right corner, that is \( V_{p,m-1}(p+1:m-1,p+1:m-1) \).

To complete this proof, we only need to show that the nested equations provided for the first column of \( \text{NLRC}_{p,m} \) hold. For this purpose, we utilize Theorems 1 and 5 in the proof. For a fixed \( i \in \{1, \ldots, m-p\} \), we have

\[
\text{NLRC}_{p,m}(i,1) = \Sigma_{p,m}(p+i,p+1) = \sigma_W^2 \Gamma_{p,m}^{-1}(p+i,p+1) \]

\[
= \sigma_W^2 \frac{\text{adj}(\Gamma_{p,m})(p+i,p+1)}{\det(\Gamma_{p,m})}. \quad (13)
\]
The value of $\text{adj}(\Gamma_{p,m})(p+i,p+1)$ is equal to the coordinate $(p+1,p+i)$ of the cofactor matrix of $\Gamma_{p,m}$. However, as the latter matrix is symmetric, its cofactor matrix is symmetric as well, implying that both coordinates $(p+1,p+i)$ and $(p+i,p+1)$ of the cofactor matrix are identical. This results in

$$\text{adj}(\Gamma_{p,m})(p+i,p+1) = (-1)^{2p+i+1}$$

$$\begin{bmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & \gamma_{m-1} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\gamma_{p+i-2} & \cdots & \gamma_{i-1} & \gamma_{i-3} & \cdots & \gamma_{m-p+i+1} \\
\gamma_{p+i} & \cdots & \gamma_{i+1} & \gamma_{i-1} & \cdots & \gamma_{m-p+i-1} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\gamma_{m-1} & \cdots & \gamma_{m-p} & \gamma_{m-p-2} & \cdots & \gamma_0
\end{bmatrix}.$$ 

We find this determinant in four cases and consequently derive $\text{NLRC}_{p,m}(i,1)$ for each case:

- **Case 1:** $1 \leq i \leq m-p-1$ and $m \leq 2p+1$. Similar to the proof of Proposition 2, we can add a weighted sum of some columns to the last column and then utilize Theorem 4 to simplify the result while keeping the determinant unchanged. However, since $m \leq 2p+1$ and the $(p+1)^{st}$ column of the matrix $\Sigma_{p,m}$ has been already removed, implementing such operations on the last column will result in the coordinates of the removed column multiplied by its corresponding coefficient as in Theorem 4. More precisely, by adding to the last column the $p-1$ preceding columns multiplied by $-\phi_1^{(p)}, \cdots, -\phi_{m-p-2}^{(p)}, -\phi_{m-p}^{(p)}, \cdots, -\phi_p^{(p)}$, respectively, and applying Theorem 4, we yield

$$\text{adj}(\Gamma_{p,m})(p+i,p+1) = (-1)^{i+1}$$

$$\begin{bmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & \phi_{m-p-1}^{(p)} \gamma_p \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\gamma_{p+i-2} & \cdots & \gamma_{i-1} & \gamma_{i-3} & \cdots & \phi_{m-p-1}^{(p)} \gamma_{i-2} \\
\gamma_{p+i} & \cdots & \gamma_{i+1} & \gamma_{i-1} & \cdots & \phi_{m-p-1}^{(p)} \gamma_i \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\gamma_{m-1} & \cdots & \gamma_{m-p} & \gamma_{m-p-2} & \cdots & \phi_{m-p-1}^{(p)} \gamma_{m-p-1} + \sigma_W^2
\end{bmatrix}.$$ 

In the last equality, the determinant is multiplied by $(-1)^{m-p-2}$ due to the $m-p-2$ column exchange operations to bring the last column to the $(p+1)^{st}$ column. Now, by implementing a similar operation on the last row we obtain

$$\text{adj}(\Gamma_{p,m})(p+i,p+1) = (-1)^{m-p+i-1}\phi_{m-p-1}^{(p)}$$
This result along with (13) and Proposition 2 imply that,

\[ \text{NLRC}_{p,m}(i,1) = \frac{\phi^{(p)}_{m-p-i} \phi^{(p)}_{m-p-i} \det(\Gamma_{p,m-1}) + \sigma^2_W \text{adj}(\Gamma_{p,m-1})(p + i, p + 1)}{\det(\Gamma_{p,m})} \]

\[ (\text{Proposition 2}) = \phi^{(p)}_{m-p-i} \phi^{(p)}_{m-p-i} + \sigma^2_W \frac{\text{adj}(\Gamma_{p,m-1})(p + i, p + 1)}{\det(\Gamma_{p,m-1})} \]

\[ = \phi^{(p)}_{m-p-i} \phi^{(p)}_{m-p-i} + \text{NLRC}_{p,m-1}(i,1) \quad \text{for} \ i = 1, \ldots, m - p - 1. \]
• Case 2: $i = m - p$ and $m \leq 2p + 1$. Analogous to Case 1, we have

$$\text{adj}(\Gamma_{p,m})(m, p + 1) = (-1)^{m-p+1} \begin{vmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & \gamma_{m-1} \\
\gamma_1 & \cdots & \gamma_{p-2} & \gamma_{p} & \cdots & \gamma_{m-2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{m-2} & \cdots & \gamma_{m-p-1} & \gamma_{m-p} & \cdots & \gamma_1
\end{vmatrix}$$

$$= (-1)^{m-p+1} \begin{vmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & \gamma_{m-1} \\
\gamma_1 & \cdots & \gamma_{p-2} & \gamma_{p} & \cdots & \gamma_{m-2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{m-2} & \cdots & \gamma_{m-p-1} & \gamma_{m-p} & \cdots & \gamma_1
\end{vmatrix}$$

$$= (-1)^{(m-p+1)+(m-p-2)\phi_{m-p-1}^{(p)}} \phi_0^{(p)} \phi_{m-p-1}^{(p)} \det(\Gamma_{p,m-1}).$$

This result along with (13) and Proposition 2 imply that,

$$\text{NLRC}_{p,m}(m-p,1) = \sigma_W^2 \frac{\phi_0^{(p)} \phi_{m-p-1}^{(p)} \det(\Gamma_{p,m-1})}{\det(\Gamma_{p,m})}$$

(Proposition 2) = $\phi_0^{(p)} \phi_{m-p-1}^{(p)}$.

• Case 3: $1 \leq i \leq m - p - 1$ and $m > 2p + 1$. Unlike Case 1, in this case all preceding $p$ columns of the last column exist in the determinant. So, analogous to the proof of Proposition 2, we add to the last column the $p$ preceding columns multiplied by $-\phi_1, \ldots, -\phi_p$, respectively, and apply Theorem 4 to yield

$$\text{adj}(\Gamma_{p,m})(p+i,p+1) = (-1)^{i+1} \begin{vmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{p+i-2} & \cdots & \gamma_{i-1} & \gamma_i & \cdots & 0 \\
\gamma_{p+i} & \cdots & \gamma_{i+1} & \gamma_{i-1} & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{m-1} & \cdots & \gamma_{m-p} & \gamma_{m-p-2} & \cdots & \sigma_W^2
\end{vmatrix}$$

$$= (-1)^{(i+1)+(2(m-i)-1)} \sigma_W^2 \begin{vmatrix}
\gamma_0 & \cdots & \gamma_{p-1} & \gamma_{p+1} & \cdots & \gamma_{m-2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{p+i-2} & \cdots & \gamma_{i-1} & \gamma_i & \cdots & \gamma_{m-p-i} \\
\gamma_{p+i} & \cdots & \gamma_{i+1} & \gamma_{i-1} & \cdots & \gamma_{m-p-i-2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\gamma_{m-2} & \cdots & \gamma_{m-p-1} & \gamma_{m-p-3} & \cdots & \gamma_0
\end{vmatrix}$$

$$= \sigma_W^2 \text{adj}(\Gamma_{p,m-1})(p+i,p+1).$$
This result along with (13), Proposition 2, and Theorem 1 imply that,

\[
\text{NLRC}_{p,m}(i, 1) = \sigma_W^2 \frac{\text{adj}(\Gamma_{p,m-1})(p+i,p+1)}{\text{det}(\Gamma_{p,m})}.
\]

(Proposition 2) = \sigma_W^2 \frac{\text{adj}(\Gamma_{p,m-1})(p+i,p+1)}{\text{det}(\Gamma_{p,m})}.

(Theorem 1) = \text{NLRC}_{p,m-1}(i, 1) \text{ for } i = p + 2, \ldots, m - p - 1.

• Case 4: \(i = m - p\) and \(m > 2p + 1\). Analogous to Cases 2 and 3, we obtain

\[
\text{adj}(\Gamma_{p,m})(m, p + 1) = (-1)^{m-p+1} \begin{vmatrix}
\gamma_0 & \gamma_{p-1} & \gamma_{p+1} & \cdots & \gamma_{m-1} \\
\gamma_1 & \gamma_{p-2} & \gamma_{p} & \cdots & \gamma_{m-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{m-2} & \gamma_{m-p-1} & \gamma_{m-p-3} & \cdots & \gamma_1 \\
\gamma_0 & \gamma_{p-1} & \gamma_{p+1} & \cdots & 0 \\
\gamma_1 & \gamma_{p-2} & \gamma_{p} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{m-2} & \gamma_{m-p-1} & \gamma_{m-p-3} & \cdots & 0
\end{vmatrix} = (-1)^{m-p+1} \text{adj}(\Gamma_{p,m})(m, p + 1) = 0.
\]

This result along with (13) imply that,

\[
\text{NLRC}_{p,m}(m - p, 1) = 0.
\]

Now, by starting with the scaler \(\text{NLRC}_{p,p+1}(1, 1) = 1\) and implementing the nested equations iteratively on \(m\), the coordinates of the matrix \(\text{NLRC}_{p,m}\) are accordingly derived. □

A.2 Proof of Theorem 3

Proof. We prove this theorem by fixing \(p\) and utilizing Definition 5 and Theorem 2 in three cases:

• Case 1: \(m = p + 1\).

\[
\sigma_{p,p+1}^2 = \text{Var}(\sqrt{n}\phi_{p+1}(p+1)) = \text{Var}(\sqrt{n} \phi_{p+1}(p+1)) = \text{NLRC}_{p,p+1}(1, 1) = (\phi_{00}^{(p)})^2.
\]

• Case 2: \(p + 1 < m \leq 2p\).

\[
(m - p)^2 \sigma_{p,m}^2 = (m - p)^2 \text{Var}(\sqrt{n} \phi_{p,m}) = (m - p)^2 \text{Var}(\sqrt{n} \phi_{p,m}) = (m - p)^2 \text{Var}(\frac{1}{m - p} \sum_{i=p+1}^{m} \sqrt{n} \phi_{i}^{(m)})
\]

(Definition 5) = \((m - p)^2 \text{Var}(\sum_{i=p+2}^{m} \sqrt{n} \phi_{i}^{(m)} + 2 \text{Cov}(\sum_{i=p+2}^{m} \sqrt{n} \phi_{i}^{(m)})\)

32
(Theorem 2) = \( \text{NLRC}_{p,m}(1, 1) + \left( \sum_{i=2}^{m-p} \text{NLRC}_{p,m}(i, i) + 2 \sum_{i=3}^{m-p} \sum_{j=2}^{i-1} \text{NLRC}_{p,m}(i, j) \right) \)

\[ + 2 \sum_{i=2}^{m-p} \text{NLRC}_{p,m}(i, 1) \]

(Theorem 2) = \( \left( \sum_{i=2}^{m-p} \text{NLRC}_{p,m-1}(i-1, i-1) + 2 \sum_{i=3}^{m-p} \sum_{j=2}^{i-1} \text{NLRC}_{p,m-1}(i-1, j-1) \right) \)

\[ + \left( \text{NLRC}_{p,m}(1, 1) + 2 \sum_{i=2}^{m-p} \text{NLRC}_{p,m}(i, 1) \right) \]

\[ = \left( \sum_{i=1}^{m-1} \text{NLRC}_{p,m-1}(i, i) + 2 \sum_{i=2}^{m-p} \sum_{j=1}^{i-1} \text{NLRC}_{p,m-1}(i, j) \right) \]

\[ + \left( \text{NLRC}_{p,m}(1, 1) + 2 \sum_{i=2}^{m-p} \text{NLRC}_{p,m}(i, 1) \right) \]

(Theorem 2) = \( \left( \sum_{i=p+1}^{m} \text{Var}(\sqrt{n\hat{\phi}_{i}^{(m)}}) + 2 \sum_{i=p+2}^{m-1} \sum_{j=p+1}^{i-1} \text{Cov}(\sqrt{n\hat{\phi}_{i}^{(m)}}, \sqrt{n\hat{\phi}_{j}^{(m)}}) \right) \)

\[ + \left( \sum_{k=0}^{m-p-1} (\phi_{k}^{(p)})^2 + 2 \sum_{i=2}^{m-p-m-1} \sum_{k=0}^{m-p-i} \phi_{k}^{(p)} \phi_{k+i}^{(p)} \right) \]

\[ = \left( \text{Var}(\sum_{i=p+1}^{m-1} \sqrt{n\hat{\phi}_{i}^{(m-1)}}) + \left( \sum_{k=0}^{m-p-1} (\phi_{k}^{(p)})^2 + 2 \sum_{i=1}^{m-p-1} \sum_{k=0}^{m-p-i} \phi_{k}^{(p)} \phi_{k+i}^{(p)} \right) \right) \]

\[ = (m - p - 1)^2 \sigma_{p,m-1}^2 + \left( \phi_{0}^{(p)} + \cdots + \phi_{m-p-1}^{(p)} \right)^2. \]

• Case 3: \( m \geq 2p + 1 \). The proof of this case is analogous to Case 2. To avoid duplication, it is omitted.

Now, it is readily seen that by starting from Case 1 and implementing the recursion iteratively, the provided general solution is derived.

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