Disagreement and Polarization in Two-Party Social Networks*

Yuhao Yi*  Stacy Patterson*

* Department of Computer Science, Rensselaer Polytechnic Institute, Troy, NY 12180 (e-mail: yiy3@rpi.edu, sep@cs.rpi.edu).

Abstract: We investigate disagreement and polarization in a social network with two polarizing sources of information. First, we define disagreement and polarization indices in two-party leader-follower models of opinion dynamics. We then give expressions for the indices in terms of a graph Laplacian. The expressions show a relationship between these quantities and the concepts of resistance distance and biharmonic distance. We next study the problem of designing the network so as to minimize disagreement and polarization. We give conditions for optimal disagreement and polarization, and further, we show that a linear combination of disagreement and polarization of the follower nodes is a convex function of the edge weights between followers. We propose algorithms to address some related continuous and discrete optimization problems and also present analytic results for some interesting examples.

Keywords: Consensus, Control of Networks, Multi-agent Systems

1. INTRODUCTION

Social networks play an important role in shaping people’s opinions. As such, a significant amount of study has been devoted to analyzing how opinions are formed in social groups and how these opinions are impacted by both external and internal influence. Problems of interest include how quickly members of a social network converge to agreement French Jr (1956); DeGroot (1974); Friedkin and Johnsen (1990), how to select individuals to maximize influence within a network Vassio et al. (2014), and the impact of competing external sources of opinion Acemo˘glu et al. (2013); Como and Fagnani (2016); Mai and Abed (2019).

A popular model for opinion formation in social networks is consensus dynamics French Jr (1956); DeGroot (1974); Friedkin and Johnsen (1990). In a consensus network, every node (representing an individual) seeks to minimize the difference between its opinion and the opinions of its neighbors. The French-DeGroot model French Jr (1956); DeGroot (1974) and the Freidkin-Johnson model Friedkin and Johnsen (1990) are among the most popular opinion consensus models. In the French-DeGroot model, individuals update their opinions based only on the differences between their own opinions and those of their neighbors in the network. The Friedkin-Johnsen model is similar in that an individual’s opinion is influenced by its neighbors, but individuals also display some stubbornness against changing their opinions.

In this work, we investigate the impact of two sources of polarizing opinions on the opinions of a social network. We model these sources differently for each opinion model. In the French-DeGroot model, the node set (set of individuals) is partitioned into a set of opinion leaders and a set of followers. The opinion leaders have one of two polarized opinions, and they never change their opinions, while follower nodes update their opinions by comparing with their neighbors. In the Friedkin-Johnsen model, every node has some susceptibility of persuasion to each source of polarizing opinion, and it also has preferences for each source. Each node updates its state according to its own state, the states of its neighbors, and these sources of opinions. The differences in the models lead to differences in the resulting steady-state opinion of the network, as well as in how external sources of impact these steady-state opinions.

We are interested, in particular, in how the the polarizing opinions effect the disagreement and polarization in the network. The disagreement is defined as the sum of differences between all neighbors Musco et al. (2018). In an opinion network with small disagreement, most of the nodes share similar opinions with their neighbors. Polarization is defined as the sum of the deviation of every node’s state from the (weighted) system average Musco et al. (2018). Polarization describes the extent of macroscopic deviations of opinions in the network. In a network with a community structure Girvan and Newman (2002), small disagreement means that consensus is nearly achieved within each community. We call this phenomenon group consensus. However, if the communities are dominated by different sources of polarizing opinions, and the connections between communities are weak, the enhancement of group consensus may lead to large polarization between different communities. Thus, there can be a trade-off between disagreement and polarization.

We analyze the relationship between the network topology, the locations of the opinion sources (i.e., which nodes act as inputs), and the resulting polarization and disagreement in the social network. We first give expressions for disagreement and polarization in terms of the Laplacian...
matrix of the network. In the French-DeGroot model, the expressions show a connection between these indices and distance metrics over the graph topology. Namely, these indices depend on the resistance distance and biharmonic distance between opinion leaders from opposing parties.

We then pose four network design problems to minimize a performance index that is a convex combination of disagreement and polarization. Our analysis leads to efficient algorithms with optimality guarantees for the first three problems. For the fourth, we show that the performance index is a convex function of the node preferences. We then give a scalable heuristic solution using convex optimization with $t_1$ norm regularization. Finally, we further explore the tradeoff between disagreement and polarization through analytical and numerical examples.

**Related Work**: Several works have addressed related optimization problems in the French-DeGroot model and the Friedkin-Johnsen model. In Vassio et al. (2014) the authors investigated the problem of single leader placement in a French-DeGroot network to maximize influence of an opinion. A few works studied the problem of influence maximization via leader selection in French-DeGroot Mai and Abed (2019); Hunter and Zaman (2018) or Friedkin-Johnsen Gionis et al. (2013) networks. Abebe et al. (2018) investigated influence maximization through modifying the susceptibility to persuasion of nodes in the network. A recent paper Musco et al. (2018) studied the problem of minimizing disagreement and polarization in a simpler Friedkin-Johnsen model with uniform susceptibility to persuasion. Our work is motivated by this work, while we discuss more general models, present new analytic results, and investigate a larger set of optimization problems.

**Paper Outline**: The remainder of the paper is organized as follows. In Section 2, we introduce notation and related concepts used in the paper. In Section 3, we describe the opinion dynamics models, give formal definitions of polarization, and disagreement, and pose several network design problems related to these performance measures. In Section 4, we give expressions for disagreement and polarization, and in Section 5 we present solutions to the design problems. We give some analytical examples in Section 7 and numerical results in Section 7. Finally, we conclude in Section 8.

2. PRELIMINARIES

2.1 Notation

We consider a connected undirected graph $G = (V, E, w)$, where $V = [n]$ is the node set and $E$ is the edge set. The function $w : E \to \mathbb{R}^+$ is the edge weight function. Every individual in a social network is represented by a node in the graph, and the social links are modeled by the edges. We let $|E| = m$ and let $N_i$ be the neighbors of node $i$. Since the graph is undirected, for any $(u, v) \in E$, $(u, v) = (v, u)$; $i \in N_j$ if and only if $j \in N_i$. Let $w_{\text{min}} = \min_{(u, v) \in E} w(u, v)$ and $w_{\text{max}} = \max_{(u, v) \in E} w(u, v)$.

A Laplacian matrix of a graph $G$ is defined as $L^G = D - A$ where $D$ is the diagonal degree matrix and $A$ is the adjacency matrix of the graph. Specifically, $A_{i,j} = w(i, j)$ if $(i, j) \in E$ and $A_{i,j} = 0$ otherwise. $D_{i,i} = \sum_{j=1}^{n} A_{i,j}$ for all $i$ and $D_{i,j} = 0$ for any $i \neq j$. We also use $L$ instead of $L^G$ when context is clear. We define $e_i$ as the $i$-th canonical basis of $\mathbb{R}^n$. And we define $b_{u,v} := e_u - e_v$. Then the Laplacian matrix $L$ of graph $G$ can be written as $L = \sum_{(u,v) \in E} w(u,v)b_{u,v}^Tb_{u,v}$. In addition, we use $1$ (or $0$) to represent all one (or all zero) vector with appropriate length, and use $1$ (or $0$) to represent scalar values. We also denote $I$ as the identity matrix, $e_i$ as the $i$-th canonical basis of $\mathbb{R}^n$, and denote $L^T$ as the Moore-Penrose pseudoinverse of $L$.

2.2 Resistance Distance and Biharmonic Distance

For an undirected graph $G = (V, E, w)$, an electrical graph $G$ is defined by replacing each edge in $G$ by a resistor with resistance $r(e) = 1/w(e)$. The resistance distance between two nodes $u$ and $v$ is defined as the voltage difference between these two nodes when unit current is injected into $u$ and extracted from $v$. We recall the the following expression for resistance distance Klein and Randić (1993).

**Lemma 2.1.** The resistance distance between nodes $u$ and $v$ in an undirected graph $G$ is given by

$$r_{u,v} = b_{u,v}^T L b_{u,v}.$$  

A standard result Klein and Randić (1993), following from Ohm’s law and Kirchhoff’s law is that the vector $L^Tb_{u,v}$ gives the voltage of all nodes in the electrical network when unit current is injected to $u$ and extracted from $v$, and the average voltage of all nodes is restricted to be 0.

We further recall the definition of biharmonic distance Lipman et al. (2010); Yi et al. (2018).

**Definition 2.1.** The biharmonic distance between nodes $u$ and $v$ in an undirected graph $G$ is defined by

$$d_B(u, v) = \sqrt{b_{u,v}^T L^2 b_{u,v}}.$$  

3. PROBLEM FORMULATION

3.1 System Model

We first introduce the considered opinion models. In the models, the opinion of a node is also called the state of the node. The states of all nodes are denoted by the vector $x \in \mathbb{R}^n$. The entry $x_v$ is the state of vertex $v$.

**French-DeGroot Model** We consider a leader-follower French-DeGroot opinion network. The node set $V$ is partitioned as $V = \{S, F\}$, where $S = \{s_0, s_1\}$ and $F = V \setminus S$. The node $s_0$ is the leader for opinion 0, and $s_1$ is the leader set for opinion 1. The nodes in $F$ are followers.

Leader $s_0$ has initial state $x_{s_0}(0) = 0$, and leader $s_1$ has initial state $x_{s_1}(0) = 1$. A leader node never changes its state, so the leader states states satisfy $\dot{x}_v(t) = 0, \forall v \in S$. A follower node $v$ starts with any initial state $x_v(0) = x_v^0$ and updates its state by the dynamics

$$\dot{x}_v(t) = -\sum_{u \in N_v} w(u, v)(x_v(t) - x_u(t)).$$  

We write the state vector as $x = (x_S^T, x_F^T)$, where $x_S$ and $x_F$ are the state vectors for the leaders and followers, respectively. Similarly, we write $L$ as a block matrix with...
the first two rows and columns associated with the leader nodes
\[ L = \begin{pmatrix} L_{SS} & L_{SF} \\ L_{FS} & L_{FF} \end{pmatrix}. \]

Then, the dynamics of the system is given by the following compact form
\[ \dot{x}_S = 0, \]
\[ \dot{x}_F = -L_{FF}x_F - L_{FS}x_S. \]

Since \( S \neq \emptyset \), \(-L_{FS}\) is Hurwitz, and therefore, the system converges to an unique steady state Mesbahi and Egerstedt (2010)
\[ \dot{x} = -(L_{FF})^{-1}L_{FS}x_S = -(L_{FF})^{-1}L_{FS}, \]
where \( L_{FS} \) is the column of \( L_{FS} \) that corresponds to \( s_1 \).

**Friedkin-Johnsen Model** We next consider a similar opinion dynamics model, in which the sources of polarizing opinions come from outside the network, and these opinion sources affect every node in the network. This model is a variation of the Friedkin-Johnsen model Friedkin and Johnsen (1990). We assume every node in the network is affected by both opinion 1 and opinion 0. Each node \( v \) starts with any initial state \( x_v(0) = x_v^0 \). Node \( v \) updates its state according to
\[ \dot{x}_v(t) = \beta_v \kappa_v \cdot (1 - x_v(t)) + (1 - \beta_v) \kappa_v \cdot (0 - x_v(t)) + \sum_{u \in N_v} (x_u(t) - x_v(t)), \]
in which \( \kappa_v > 0 \) is node \( v \)'s susceptibility to persuasion Abebe et al. (2018), and \( 0 \leq \beta_v \leq 1 \) is the preference of node \( v \) to opinion 1 over opinion 0.

The dynamics of the system can be written as
\[ \dot{x}(t) = -(L + K)x(t) + BK1, \]
where \( B \) is the diagonal matrix with \( B_{vv} = \beta_v \), and \( K \) is the diagonal matrix with \( K_{vv} = \kappa_v \). The matrix \(-(L + K)\) is Hurwitz, so the system converges to a unique steady state
\[ \dot{x} = (L + K)^{-1}BK1. \]

### 3.2 Disagreement and Polarization

We study the disagreement and polarization in the social network using definitions given in Musco et al. (2018). The disagreement between neighbors \( u \) and \( v \) is defined as
\[ d(u, v) := w(u, v) (\dot{x}_u - \dot{x}_v)^2. \]
The disagreement of the network is then defined as
\[ D := \sum_{(u, v) \in E} d(u, v). \]

Disagreement characterizes the sum of squared differences between all neighbors in the network. By Markov's inequality, in a social network with small disagreement, most of the neighbors share similar opinions.

Polarization measures how opinions at steady state deviate from the average. We first define the steady state deviation from average vector as
\[ \bar{x} = \dot{x} - \frac{1}{n} 11^T \dot{x}. \]
Then, the total polarization of the network is given by
\[ P := \sum_{u \in V} \bar{x}_u^2 = \bar{x}^T \bar{x}. \]

We further define a **Polarization-Disagreement index** as
\[ I = \rho D + (1 - \rho) P, \]
for \( \rho \in [0, 1] \). Minimizing \( D \) alone may lead to group consensus. If different communities in the graph are dominated by different opinions, group consensus indicates that the network has a large \( P \). On the other hand, minimizing \( P \) does not necessarily make \( D \) small. We explore the relationship between disagreement and polarization in the next two sections.

For the Friedkin-Johnsen Model with non-uniform \( \kappa \), we define a variation of polarization called the **weighted polarization**. We first define a weighted average \( \bar{x} \) of \( \dot{x} \)
\[ \alpha = \frac{\sum_{v \in V} \kappa_v \dot{x}_v}{\sum_{v \in V} \kappa_v}. \]
Then the deviation from the weighted average is given by
\[ \dot{x} = \dot{x} - \alpha 1. \]
And the weighted polarization is defined as
\[ \tilde{P} = \sum_{v \in V} \kappa_v \dot{x}_v^2. \]
The weighted average \( \alpha \) has the following interpretation. For a Friedkin-Johnsen model, one can intuitively minimize both disagreement and polarization by strengthening the connections between all neighbors. Let's consider a graph \( G' \) in which all edge weights are strengthened such that \( L_{G'} = a L \). Let's consider a system in which the graph \( G \) is replaced by \( G' \), while \( \kappa_v \) and \( \beta_v \) remain the same for each \( v \in V \). Then the new equilibrium is given by
\[ \dot{x}' = (aL + K)^{-1} BK1 = (aL + K)^{-1} (L + K) \dot{x}. \]
Then, we consider the limit \( a \to +\infty \),
\[ \lim_{a \to +\infty} (aL + K)^{-1} = \frac{1}{\sum_{v \in V} \kappa_v} 11^T, \]
because all other eigenvalues of \((aL + K)^{-1}\) approach 0 at the limit \( a \to +\infty \), and we obtain
\[ \lim_{a \to +\infty} \dot{x}' = \frac{11^T K \dot{x}}{\sum_{v \in V} \kappa_v} = \alpha 1. \]
This means that \( \alpha \) is the consensus value of the system at the limit \( a \to +\infty \).

In order to study the trade off between \( D \) and \( \tilde{P} \), we define the weighted Polarization-Disagreement index \( \tilde{I} \) as follows
\[ \tilde{I} = \rho D + (1 - \rho) \tilde{P}. \]

The following propositions give expressions for the disagreement and polarization of the steady-state opinions.

**Proposition 3.1.** The disagreement of the opinion network is given by
\[ D = \dot{x}^T L \dot{x}. \]

**Proposition 3.2.** The polarization of the opinion network is given by
\[ \mathcal{P} = \dot{x}^T J \dot{x}, \]
where \( J = I - \frac{1}{n} 11^T \).

**Proposition 3.3.** The weighted polarization of the opinion network is given by
\[ \mathcal{P} = \dot{x}^T P^T K \dot{x}, \]
where \( P = I - \frac{1}{\sum_{v \in V} \kappa_v} \).
3.3 Problem Formulation

We study two network design problems for the French-DeGroot model. The first relates to identifying opinion leaders to optimize the disagreement and polarization of the network.

Problem 3.1. In a French-DeGroot model, given the graph \( G = (V, E, w) \), and a opinion leader \( s_0 \) for opinion 0, choose a single opinion leader \( s_1 \) for opinion 1 such that \( I \) is minimized.

The second problem addresses how to design a network topology so as to minimize network disagreement and polarization for any choice of \( s_0 \) and \( s_1 \), subject to a budget constraint edges in the network.

Problem 3.2. In a French-DeGroot model, if the vertex set \( V \) is given, design the edge set \( E \) and weight function \( w \) of the graph \( G = (V, E, w) \) with a cardinality constraint \( |E| \leq k \) and a budget on total weight \( \sum_{e \in E} w(e) \leq W \), such that \( \max_{s_0, s_1} \hat{I}(s_0, s_1) \) is minimized.

We also study two network design problems in the context of Friedkin-Johnsen opinion dynamics. The first problem addresses how to optimize select weights to minimize the weighted Polarization-Dissagreement index.

Problem 3.3. In the Friedkin-Johnsen Model, given the node set \( V \) and edge set \( E \), design the edge weights \( w(e) \in [\ell, p] \) with budget \( \sum_{e \in E} w(e) \leq W \) such that the quantity \( \hat{I} \) is minimized.

In the second problem, we study the disagreement and polarization in a network where nodes have binary preferences for opinion 0 or 1. The problem addresses how to flip the preferences of a small number of nodes to minimize disagreement and polarization of the opinion network.

Problem 3.4. In the Friedkin-Jonson Model, given \( G = (V, E, w) \), an integer \( k \), diagonal matrices \( K \) and \( B \), in which \( K_{v,v} = \kappa_v \) and \( B_{v,v} = \beta_v \). Assume the preference of node \( v \) satisfies \( \beta_v \in \{0,1\} \) for all \( v \in V \), we flip \( \beta_v \) of at most \( \eta \) nodes (from 0 to 1 or from 1 to 0), such that \( \hat{I} \) is minimized.

4. ANALYSIS

In this section, we give expressions for disagreement and polarization in the French-DeGroot model. The analytic results in this section also give insight into the aforementioned network design problems.

4.1 Expressions for \( D \) and \( P \) in French-DeGroot model

From Yi et al. (2019), we have the following result.

Lemma 4.1. In a two-party French-DeGroot model, let \( s_0 \) and \( s_1 \) be the leaders for opinion 0 and 1. The steady state \( \hat{x}_v \) of node \( v \) is given by

\[
\hat{x}_v = \frac{\beta_v^{T} \xi_{s_1}^{T} L \rho^{T} b_{s_1, s_0}}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}.
\]

Given this lemma, we express the steady state vector \( \hat{x} \) as

\[
\hat{x} = \frac{(I - \xi_{s_0}^{T} \xi_{s_1}^{T} L \rho^{T} b_{s_1, s_0})}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}.
\]

Next we express \( D \) in terms of the resistance distance between \( s_0 \) and \( s_1 \).

Theorem 4.2. In the French-DeGroot model, the disagreement \( D \) in the considered opinion network is

\[
D = \frac{1}{r_{s_1, s_0}}.
\]

Proof. From Proposition 3.1 and (21), we obtain

\[
D = \frac{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}} = \frac{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}.
\]

Therefore, we obtain that \( D = \frac{1}{r_{s_1, s_0}} \).

Next, we give an expression for \( P \) based on the resistance distance and the biharmonic distance.

Theorem 4.3. The polarization \( P \) in the French-DeGroot opinion network is

\[
P = \frac{\lambda_{s_0}(D_{s_1, s_0})}{\lambda_{s_0}(2D_{s_1, s_0})}.
\]

Proof. From Proposition 3.2 and (21) we attain

\[
P = \frac{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}} = \frac{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}{b_{s_1, s_0}^{T} L \rho^{T} b_{s_1, s_0}}.
\]

From Lemma 2.1 and Definition 2.1, we obtain the result of Theorem 4.3.

Theorem 4.3 shows that the value of \( P \) is determined by the ratio of biharmonic distance to effective resistance between \( s_0 \) and \( s_1 \).

5. OPTIMIZING DISAGREEMENT AND POLARIZATION

We now use the analysis in the previous section to derive solutions for the network design problems posed in Section 3.3 for each of the two opinion models.

5.1 Network Design for French-DeGroot Dynamics

We derive solutions for network design in the French-DeGroot model with opinion leaders from opposing parties.

Placing a Single Leader We first consider Problem 3.1 to minimize \( \hat{I} \) when \( \rho = 1 \), i.e., to minimize disagreement. This result follows directly from Theorem 4.2.

Theorem 5.1. Given \( G \) and \( s_0 \), when \( \rho = 1 \), \( \hat{I} \) is minimized when we select \( s_1 \) such that \( r(s_0, s_1) \) is maximized.

We next consider the same problem when \( \rho = 0 \). We derive conditions under which a node \( v \) is the node, that when selected to be the 1 opinion leader, will minimize the polarization.
Theorem 5.2. Given $G$ and $s_0$, when $\rho = 0$, if there exists a vertex $v \neq s_0$, and $N_v = N_{s_0}$, $w(u, v) = w(u, s_0)$ for all $u \in (N_v \setminus \{s_0\} \cup \{s_1\})$, then $v$ is a minimizer of $I$.

Before we prove Theorem 5.2, we present a lower bound for $P$.

Lemma 5.3. The polarization $P$ in an French-DeGroot opinion network satisfies $P \geq \frac{1}{2}$.

Proof. From Cauchy-Schwartz inequality,
$$b_{s_1, s_0}^T L b_{s_1, s_0} \leq \|b_{s_1, s_0}\|_2 \cdot \|L b_{s_1, s_0}\|_2 = \sqrt{2} b_{s_1, s_0}^T L^2 b_{s_1, s_0}$$

From (25) we obtain
$$P = \frac{b_{s_1, s_0}^T L^2 b_{s_1, s_0}}{2 \cdot b_{s_1, s_0}^T L b_{s_1, s_0}} = \frac{1}{2}.$$

This completes the proof.

Then we proceed to prove Theorem 5.2.

Proof. In the proof of Lemma 5.3, we have shown that $P \geq \frac{1}{2}$, and the equality holds only when $b_{s_1, s_0}$ is multiple of $L_{s_1, s_0} b_{s_1, s_0}$. This only happens if $b_{s_1, s_0}$ is an eigenvector of matrix $L$. And this implies that $b_{s_1, s_0}$ is an eigenvector of matrix $L$. We let
$$h := \begin{cases} \sum_{u \in N_{s_0}} w(u, s_1) & s_1 \notin N_{s_0} \\ w(s_1, s_0) + \sum_{u \in N_{s_0}} w(u, s_1) & s_1 \in N_{s_0}. \end{cases}$$

When $s_0$ and $s_1$ have the same neighbors and the same weight to every neighbor, then $L b_{s_1, s_0} = h \cdot b_{s_1, s_0}$, and $L^2 b_{s_1, s_0} = h^{-1} \cdot b_{s_1, s_0}$. It is easy to verify from Theorem 4.3 that $P = \frac{1}{2}$.

In general, $D$ and $P$ are not minimized simultaneously. When the condition of Theorem 5.2 is satisfied, $D = \frac{1}{2}$, where $h$ is as defined in (27). This means $D = \frac{|N_{s_0} + |N_{s_1} \cap \{s_1\}|}{2}$. From Theorem 4.2 we know that $D$ is determined by the distance between $s_0$ and $s_1$. However the condition given in Theorem 5.2 does not guarantee a large $r_{s_0, s_1}$.

To solve Problem 3.1 for any graph $G$ and general $\rho$, we can use the expressions (23) and (24) and calculate $D$, $P$, and $T$ for each candidate of $s_1$. Then, we find the best choice of $s_1$ by simply comparing the results. Calculating $L^1$ and $L^{2^i}$ takes $O(n^3)$ running time, and it takes $O(n)$ running time to calculate $d_0(s_0, s_1)$ and $r_{s_0, s_1}$ for each candidate $s_1$ by computing a vector inner product. Therefore, the total running time is $O(n^3 + n^2) = O(n^3)$.

Designing a Robust Structure Problem 3.2 seeks to find a graph structure and an edge weight function $w$ such that $T$ is minimized. We begin by considering minimizing $P$.

Without the cardinality constraint and the total weight constraint, an unweighted complete graph guarantees $P = \frac{1}{2}$ for any $s_0$ and $s_1$.

Lemma 5.4. For an unweighted complete graph $G = K_n$, $P$ is minimized for any pairs of $s_0$ and $s_1$.

Proof. For any $s_1$, $s_0$,
$$P = \frac{b_{s_1, s_0}^T (nI - 11^T) b_{s_1, s_0}}{(b_{s_1, s_0}^T (nI - 11^T) b_{s_1, s_0})^2} = \frac{2n^{-2}}{4n^{-2}} = \frac{1}{2}.$$

By Corollary 5.3, $P \geq \frac{1}{2}$. Therefore $P$ is minimized for any $s_0$ and $s_1$ in an unweighted complete graph.

To solve Problem 3.2 with cardinality constraint, we show that a sparsification of a complete graph also guarantees that $P$ is small for any $s_0$ and $s_1$. The following lemma states the existence of a spectral sparsifier.

Lemma 5.5. (Batson et al. (2012), Theorem 1.1). For every $d > 1$, every undirected graph $G = (V, E, w)$ on $n$ nodes contains a weighted subgraph $H = (V, F, \tilde{w})$ with $|d(n - 1)|$ edges (i.e., average degree at most $2d$) that satisfies:
$$L_G \preceq L_H \preceq \left(\frac{d + 1 + 2\sqrt{d}}{d + 1 - 2\sqrt{d}}\right) L_G.$$

Further, there exists such an algorithm to find such a sparsifier $H$ for any $G$.

The algorithm is given in Batson et al. (2012). We note that other sparsification algorithms can also be applied to get a sparsifier for the unweighted complete graph.

Then, we show that a sparsifier of an unweighted complete graph $K_n$ guarantees that $P$ is small for any $s_0$ and $s_1$.

Lemma 5.6. For a graph $H$ whose Laplacian matrix $L_H$ satisfies
$$L_C \preceq L_H \preceq (1 + \epsilon)L_C$$

where $L_C$ is the Laplacian matrix of an unweighted complete graph, the following result holds
$$1/2 \leq P \leq (1 + \epsilon^2)/2.$$

Proof. From Corollary 5.3 we know that $P \geq \frac{1}{2}$.

From (30) we know that all eigenvalue of $L_H$ are in $[n, (1 + \epsilon)n]$; therefore
$$\frac{1}{(1 + \epsilon)^2} (L_C)^{2^i} \preceq (L_H)^{2^i} \preceq (L_C)^{2^i}$$

Then, we obtain
$$P = \frac{b_{s_1, s_0}^T L_H^{2^i} b_{s_1, s_0}}{(b_{s_1, s_0}^T L_H^{2^i} b_{s_1, s_0})^2} \leq \frac{b_{s_1, s_0}^T L_C^{2^i} b_{s_1, s_0}}{(b_{s_1, s_0}^T L_H^{2^i} b_{s_1, s_0})^2} \leq \frac{(1 + \epsilon)^2}{2}.$$

Lemma 5.6 and 5.5 lead to the following lemma.

Lemma 5.7. In a French-DeGroot model, there exists a graph $H = (V, F, \tilde{w})$ (and an algorithm to find $H$) with $O(\frac{n}{\epsilon^2})$ edges, such that for any leaders $s_0$, $s_1$, the polarization $P$ satisfies $P \in \left(\frac{1}{2}, (1 + \epsilon)\frac{1}{2}\right)$.

Lemma 5.7 shows that we can find a $(1 + \epsilon)$ approximation for optimum $P$ if $k = \Omega(\frac{n}{\epsilon^2})$. Next we will show how to get a solution that satisfies the constraint for total weight of edges $\sum_{e \in E} w(e) \leq W$.

Proposition 5.8. For any graph $G$ whose Laplacian matrix is $L$, and a constant $a$, we construct a new graph $G'$ with the Laplacian matrix $L' = aL$ by uniformly scaling the edge weights by a factor of $a$. Then for given $s_0$ and $s_1$, the value of $P$ is same in $G$ and $G'$.

Proof. From (24) we get $(2 \cdot P)^{\frac{1}{2}} = \frac{b_{s_1, s_0}^T L b_{s_1, s_0}}{\|b_{s_1, s_0}\|_2 \|L b_{s_1, s_0}\|_2}$.

This is the cosine value of the angle between vectors
Finally, we obtain the following theorem.

**Theorem 5.9.** In a French-DeGroot model, there exists a graph \( H' = (V, F, \bar{w}) \) (and an algorithm to find \( H' \)) with \( O(2) \) edges that satisfies \( \sum_{e \in E} w(e) \leq W \), such that for any leaders \( s_0, s_1, \mathcal{P} \in \{\mathbf{1}, (1 + \epsilon)\mathbf{1}\} \).

**Proof.** From Lemma 5.7 we obtain a graph \( H = (V, F, \bar{w}) \) with \( O(2) \) edges that guarantees \( \mathcal{P} \in \{1/2, (1 + \epsilon)/2\} \) for any \( s_0 \) and \( s_1 \). Then we let \( \bar{W} = \sum_{e \in F} \bar{w}(e) \). If \( \bar{W} \leq W \), then the constraint is satisfied, we let \( H' = H \). Otherwise, let \( \bar{w}(e) = \frac{W}{W} \bar{w}(e) \) for all \( e \in F \), and let \( H' = (V, F, \bar{w}) \).

Next, we consider minimizing \( D \). Minimizing \( D \) turns out to be easy in this case. For a graph \( G \) with the Laplacian matrix \( L \), we let the disagreement of the system be \( D(G, s_0, s_1) \) for leaders \( s_0 \) and \( s_1 \). And for a graph \( G' \) with a Laplacian matrix \( L' = aL \), we let disagreement of the system be \( D(G', s_0, s_1) \) for the same \( s_0 \) and \( s_1 \). Then from (23) we know that \( D(G', s_0, s_1) = a D(G, s_0, s_1) \). So it suffices to let \( a \) be a small number to get arbitrarily small \( D \) without changing the value of \( \mathcal{P} \).

To minimize \( I \) one only need to first minimize \( P \) according to Theorem 5.9. Then, one can make \( D \) arbitrarily small without changing the value of \( \mathcal{P} \).

### 5.2 Network Design for Friedkin-Johnsen Dynamics

We now provide solutions to the network design problems for Friedkin-Johnsen opinion dynamics.

**Designing Weights** We first study Problem 3.3. In Musco et al. (2018), the authors showed that if \( \kappa = 1 \) for all nodes, when \( \rho = \frac{1}{2} \), the index \( I = \frac{1}{2}D + \frac{1}{2}P \) is a convex function of the edge weights \( w \) of the graph. Below, we show that with nonuniform \( \kappa \), when \( \rho = \frac{1}{2} \), the index \( \tilde{I} = \frac{1}{2}D + \frac{1}{2}P \) is a convex function of all edge weights in the graph. As a result, Problem 3.3 can be formulated as an semidefinite programming (SDP) problem, which can be solved in polynomial time using standard SDP solvers Boyd and Vandenberghe (2004).

**Theorem 5.10.** In the Friedkin-Johnsen Model, in a graph \( G = (V, E, w) \), with susceptibility to persuasion matrix \( K \) and preference matrix \( B \), the weighted Polarization-Disagreement Index \( \tilde{I} = \frac{1}{2}D + \frac{1}{2}P \) is a convex function of the edge weights \( w \) of the graph. \( w_e = (1) e \in E \).

To simplify the notation, we let \( s := BK1 \). In addition we let \( \tilde{s} = PTs \), in other words,

\[
\tilde{s} = s - \frac{\kappa^{1T}}{\sum_{v \in V} K_v} s.
\]

To prove Theorem 5.10, we require the following three lemmas.

**Lemma 5.11.** The following results hold

\[
J(L + K)^{-1}s = J(L + K)^{-1}\tilde{s},
\]

\[
P(L + K)^{-1} = (L + K)^{-1}\tilde{s}.
\]

**Proof.** We start from the right hand side of (34),

\[
J(L + K)^{-1}s = \left(s - \frac{\kappa^{1T}}{\sum_{v \in V} K_v} s\right)
\]

\[
= J(L + K)^{-1}s - J(L + K)^{-1} \frac{1^{T}s}{\sum_{v \in V} K_v}
\]

where the first equality follows from (33). Since \( (L + K)1 = \kappa \), we have \( (L + K)^{-1}1 = 1 \). Then,

\[
J(L + K)^{-1}1 = J(L + K)^{-1}1 = J1 \cdot \left(1^{T}s/\sum_{v \in V} K_v\right)
\]

Thus (34) holds. Next we prove (35):

\[
P(L + K)^{-1}s = (L + K)^{-1}s - \frac{1\kappa^{T}}{\sum_{v \in V} K_v} (L + K)^{-1}s
\]

\[
= (L + K)^{-1}s - \frac{1\kappa^{T}}{\sum_{v \in V} K_v} (L + K)^{-1}s
\]

\[
= (L + K)^{-1} \left(1 - \frac{1\kappa^{T}}{\sum_{v \in V} K_v}\right)s = (L + K)^{-1}\tilde{s}.
\]

**Lemma 5.12.**

\[
\tilde{D} = \tilde{s}T(L + K)^{-1}L(L + K)^{-1}\tilde{s},
\]

\[
\tilde{P} = \tilde{s}T(L + K)^{-1}K(L + K)^{-1}\tilde{s}.
\]

**Proof.** From (17), we obtain

\[
\tilde{D} = \tilde{s}T(L + K)^{-1}L(L + K)^{-1}s
\]

\[
= (L + K)^{-1}s - \frac{1\kappa^{T}}{\sum_{v \in V} K_v} (L + K)^{-1}s
\]

\[
= (L + K)^{-1} \left(1 - \frac{1\kappa^{T}}{\sum_{v \in V} K_v}\right)s = (L + K)^{-1}\tilde{s}.
\]

**Lemma 5.13.**

\[
\tilde{I} = \frac{1}{2}\tilde{s}T(L + K)^{-1}\tilde{s},
\]

\[
\tilde{I} \succ 0, Y \succ 0
\]

**Proof.** By combining the results in Lemma 5.12, \( s = BK1 \), and (33), we obtain the result in Lemma 5.13.

Now we give the proof for Theorem 5.10.

**Proof.** Similar to (Boyd and Vandenberghe, 2004, Section 3.1.7), we investigate the epigraph of \( f(Y, \tilde{s}) = \tilde{s}TY^{-1}\tilde{s} \), where \( Y = (L + K) \). A function is convex if and only if its epigraph is a convex set.

\[
\text{epi} f = \{(Y, t)|Y \succ 0, \tilde{s}TY^{-1}\tilde{s} \leq t\}
\]

\[
= \{(Y, t)|Y \succ 0, \tilde{s}(\tilde{s}Tt) \geq 0, Y \succ 0\}
\]

Since the matrix inequality

\[
(Y \tilde{s}(\tilde{s}Tt) \geq 0
\]

References

Boyd, S., and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press.
defines a convex set, \( f(Y, \tilde{s}) \) is a convex function of both \( Y \) and \( \tilde{s} \). Since \( Y = g(w) \) is an affine function of \( w \), we know that \( f(g(w)) \) is a convex function of \( w \).

Following from Theorem 5.10, Problem 3.3 can be formulated as an SDP problem as shown in (Boyd and Vandenberghe, 2004, Section 7.5.2), which can be solved in polynomial time by standard SDP solvers.

Designing Preferences Finally, we consider Problem 3.4. Since Problem 3.4 itself is combinatorial, we instead solve a convex relaxation of the problem. We first show that the objective \( \tilde{I} \) is a convex function of the node susceptibilities.

**Theorem 5.14.** In the considered Friedkin- johnsen model, in a graph \( G = (V, E, w) \), with susceptibility to persuasion matrix \( K \) and preference matrix \( E \), the indices \( D, \tilde{P}, \tilde{I} \) are all convex functions of the vector \( \beta \), where the entries of the vector are defined as \( \beta_v = \beta_i \) for all \( v \in V \).

**Proof.** Because \( \tilde{s} \) is an affine function of \( \beta \), we obtain the convexity of \( \tilde{I} \) (for \( \rho = \frac{1}{2} \)) as a function of \( \beta \) from the proof of Lemma 5.13. In addition, we know from (36) and (37) that \( D \) and \( \tilde{P} \) are quadratic forms in \( \tilde{s} \), and the matrices \( (L+K)^{-1}L(L+K)^{-1} \) and \( (L+K)^{-1}K(L+K)^{-1} \) are positive semidefinite. Therefore \( D, \tilde{P}, \tilde{I} \) are all convex functions of \( \beta \), for any \( \rho \in [0, 1] \).

We next give a formal definition of Problem 3.4. We define an flip vector \( d \), \( d_v \in \{0, 1\} \). The flip vector flips the preference of every node whose preferences are flipped. We aim to find a sparse \( d \), \( \|d\|_0 \leq k \), such that the new preference vector is \( \theta = \beta + 2 \cdot \text{diag}(d) \cdot p \), in which the preferences of nodes chosen by \( d \) is flipped. The objective is to minimize \( \tilde{I} \).

In other words,

\[
d^* \in \arg \min_{d} \tilde{I} \quad \text{subject to} \quad \|d\|_0 \leq k.
\]

The problem given by (43) is a convex optimization with cardinality constraint, which is intractable in general. So we instead study a convex relaxation of the problem and propose a heuristic for (43). In particular, we solve an \( \ell_1 \)-regularized relaxation of the problem

\[
d^* \in \arg \min_{d} \tilde{I} + \lambda \|d\|_1,
\]

where \( d \in \mathbb{R}^n \) and \( d_v \in \{0, 1\} \) for all \( v \in V \). \( \lambda \) is a parameter for the regularization term. We need to tune this parameter to get a sparse \( d^* \) and further round all none zero entries to 1.

6. EXAMPLES

In this section, we give analytic results for \( D \) and \( \tilde{P} \) in the French-DeGroot model for some graphs with special structures to give additional insight into the relationship between disagreement and polarization. In particular, we study the scaling of \( D \) and \( \tilde{P} \). We note that similar examples can be constructed for the Friedkin-Johnsen model in a similar manner.

In a French-DeGroot model, Theorem 5.2 shows that \( \tilde{P} \) is minimized when \( \{s_1\} \cup N_{s_1} = \{s_0\} \cup N_{s_0} \). Next, we give an example where \( \tilde{P} \) does not obtain this minimum but is still constant in the number of nodes in the network.

**Proposition 6.1.** Given a node set \( V \), leaders \( s_0 \) and \( s_1 \), we construct a complete graph supported on nodes \( V \setminus \{s_1, s_0\} \). And we let \( s_0 \) connect to a node \( l_0 \) in \( V \setminus \{s_1, s_0\} \) and \( s_1 \) connect to a node \( l_1 \) in \( V \setminus \{s_1, s_0, l_0\} \). All edges have the same weight 1. Then \( \tilde{P} = O(1) \) and \( D = O(1) \).

**Proof.** \( L^1b_{s_1, s_0} \) gives the voltage of all nodes while unit current is injected to \( s_1 \) and extracted from \( s_0 \), and the average voltage is shifted to 0. By symmetry we know that \( n - 4 \) nodes have the voltage 0, 1 node has the voltage \( \frac{1}{n} \), 1 node has the voltage \( -\frac{1}{n} \), and the leaders have voltages \( \pm \left(1 + \frac{1}{n-4}\right)\). Further, the resistance distance between \( s_0 \) and \( s_1 \) is \( \left(2 + \frac{2}{n-4}\right) \). It is easy to verify from the definitions that \( \tilde{P} = \frac{1}{2(n-4)^2+1} \). When \( n \to \infty \), \( \tilde{P} \to \frac{1}{2} \). In addition, \( D = \frac{n-4}{2(n-3)} \); therefore, when \( n \to \infty \), we obtain \( D \to \frac{1}{2} \).

This example shows that in a graph where followers are well connected and leaders are only connected to a small number of nodes, both \( D \) and \( \tilde{P} \) are both small and independent of the size of the network. Thus, \( \tilde{I} = O(1) \) for any \( \rho \in [0, 1] \).

We next present an example that shows a tradeoff between disagreement and polarization that varies depending on the locations of the opinion leaders.

**Proposition 6.2.** Given an unweighted \( q \)-Barbell graph \( G \) which consists of two \( q \)-cliques and a single edge connecting two clique \( (n = 2q \) and \( m = q(q-1)+1) \), if \( s_0 \) and \( s_1 \) are in the same clique and neither of them are end vertices of the bridge, then \( \tilde{P} = O(1) \) and \( D = O(n) \). If \( s_0 \) and \( s_1 \) are in different cliques and none of them are end vertices of the bridge, then \( \tilde{P} = O(n) \) and \( D = O(1) \).

**Proof.** For the case where \( s_0 \) and \( s_1 \) are in the same clique and neither are end vertices of the bridge, then according to Lemma 5.2 we know that \( \tilde{P} = \frac{n^2}{2q^2} \). The effective resistance between \( s_0 \) and \( s_1 \) is \( \frac{4}{q} \), therefore \( D = \frac{2}{q} \). If \( s_0 \) and \( s_1 \) are in different cliques and neither of them are end vertices of the bridge, then we consider \( v := L^1b_{s_0, s_1} \) again. One node has the voltage \( \frac{1}{2} \); another node has the voltage \( -(\frac{1}{2} + \frac{2}{q}) \); \( q-1 \) nodes have the voltage \( \frac{1}{2} \) and another \( q-1 \) nodes have the voltage \( -(\frac{1}{2} + \frac{1}{q}) \). Then, we find the expression for \( \tilde{P} \) is \( \tilde{P} = \frac{2a}{(2q+1)^2} + \frac{2a}{(2q+1)^2} \). Because \( q = O(n) \), we obtain \( \tilde{P} = O(n) \). Furthermore, the resistance distance between \( s_0 \) and \( s_1 \) is \( \frac{4}{q} \); therefore, \( D = \frac{2}{2q} \), i.e. \( D = O(1) \).

This example shows that if leaders with different opinions locate in different cliques, then polarization grows linearly in the number of nodes, while disagreement is constant in the number of nodes. If the opinion leaders are located in the same clique, then disagreement grows linearly, while polarization is constant in the number of nodes.

7. EXPERIMENTS

In this section, we show the effectiveness of the heuristic algorithm proposed in Section 5.2.2 for Problem 3.4, i.e., solving the \( \ell_1 \) regularized problem (44).
To compare with the proposed $\ell_1$ regularized heuristic, we consider another heuristic algorithm also based on the convex relaxation of (43). We consider the solution of the following optimizing problem

$$d^* \in \arg \min_{d} \tilde{F}, \text{ subject to } 1^T d \leq k,$$

(45)

where $d \in \mathbb{R}^n$ and $d_v \in [0, 1]$ also holds for all $v \in V$. We round the largest $k$ entries of $d^*$ to 1 and otherwise to 0 to get a selecting vector.

We evaluate the effectiveness of the heuristic algorithm by conducting an experiment on the Haggle social network Chaintreau et al. (2007). We take the largest connected component (LCC) of the network and remove all duplicate edges. The LCC of the network has 274 nodes and 2124 edges. We let all edge weights be 1, and let $\beta_i$ be 0 with probability 0.35 and otherwise take the value of 1.

In Figure 1 we show the result of the $\ell_1$ regularized heuristic algorithm. We search the value of $\lambda$ in the range $[0.45, 1.0]$. The $\ell_1$ regularized heuristic algorithm returns a vector $d^*$. We compare the result of flipping opinions of nodes $v$ with nonzero $d_v^*$ and the result of flipping opinions of nodes with $k$ largest values in $d_v^*$. We also compare the results with the result of flipping the opinions of $k$ random nodes. The results show that the $\ell_1$ regularized relaxation heuristic returns best result compared to the other two methods. We notice that worse results are obtained when $\lambda$ is small and therefore the algorithm returns a $d^*$ with more nonzero entries. Thus, our method is most effective when the goal is to identify a small number of preferences to flip.

8. CONCLUSION

We have studied two models of opinion dynamics in social networks with two sources of polarizing opinions, namely French-DeGroot and Friedkin-Johnsen opinion dynamics. We have defined the quantities of disagreement and polarization in these opinion network models, and we have given expressions for these quantities using the Laplacian matrix of the graph. We have also proposed several network design problems that minimize disagreement and polarization through different methods of graph construction and augmentation, and we have presented algorithms for these problems. Finally, we have presented examples to further explore the relationship between network design, disagreement, and polarization. In future work, we will consider additional network design problems including tuning individual susceptibility to persuasion.

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