WELL-POSEDNESS FOR A SYSTEM OF QUADRATIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS IN ALMOST CRITICAL SPACES

HIROYUKI HIRAYAMA, SHINYA KINOSHITA, AND MAMORU OKAMOTO

Abstract. In this paper, we consider the Cauchy problem of the system of quadratic derivative nonlinear Schrödinger equations introduced by Colin and Colin (2004). We determine an almost optimal Sobolev regularity where the smooth flow map of the Cauchy problem exists, except for the scaling critical case. This result covers a gap left open in papers of the first and second authors (2014, 2019).

1. Introduction

We consider the Cauchy problem of the system of nonlinear Schrödinger equations:

\[
\begin{aligned}
(i\partial_t + \alpha \Delta) u &= - (\nabla \cdot w) v, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \beta \Delta) v &= - (\nabla \cdot w) u, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \gamma \Delta) w &= \nabla (u \cdot v), \quad t > 0, \quad x \in \mathbb{R}^d, \\
(u, v, w)|_{t=0} &= (u_0, v_0, w_0) \in \mathcal{H}^s(\mathbb{R}^d),
\end{aligned}
\]

where \( \alpha, \beta, \gamma \in \mathbb{R}\{0\} \), the unknown functions \( u, v, w \) are \( d \)-dimensional complex vector valued. Moreover, \( \mathcal{H}^s(\mathbb{R}^d) \) denotes the \( L^2 \)-based Sobolev space, and we set

\[
\mathcal{H}^s(\mathbb{R}^d) := (\mathcal{H}^s(\mathbb{R}^d))^d \times (\mathcal{H}^s(\mathbb{R}^d))^d \times (\mathcal{H}^s(\mathbb{R}^d))^d.
\]

Our aim in this paper is to determine regularities where the smooth flow map of (1.1) exists.

The system (1.1) was introduced by Colin and Colin in [7] as a model of laser-plasma interaction. (See also [8], [9].) The local existence of the solution of (1.1) in \( \mathcal{H}^s(\mathbb{R}^d) \) for \( s > \frac{d}{2} + 3 \) is obtained in [7]. The system (1.1) is invariant under the following scaling transformation:

\[
A_\lambda(t, x) = \lambda^{-1} A(\lambda^{-2} t, \lambda^{-1} x) \quad (A = (u, v, w)),
\]

Key words and phrases. Schrödinger equation; Well-posedness; Cauchy problem; Bilinear estimate.
and the scaling critical regularity is $s_c = \frac{d}{2} - 1$. We set

$$\mu := \alpha \beta \gamma \left( \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma} \right), \quad \kappa := (\alpha - \beta)(\alpha - \gamma)(\beta + \gamma). \quad (1.2)$$

We note that $\kappa = 0$ does not occur when $\mu \geq 0$ for $\alpha, \beta, \gamma \in \mathbb{R}\setminus\{0\}$.

First, we mention some known results for related problems. Since the system (1.1) has quadratic nonlinear terms which contain a derivative, a derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata [23] considered the Schrödinger equation

$$\begin{cases}
i \partial_t u - \Delta u = (b_1(x) \cdot \nabla)u, & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\u(0, x) = u_0(x), & x \in \mathbb{R}^d\end{cases}$$

and proved that the uniform bound

$$\sup_{x \in \mathbb{R}^d, \omega \in S^{d-1}, R > 0} \left| \Re \int_0^R b_1(x + r \omega) \cdot \omega dr \right| < \infty$$

is a necessary condition for the $L^2(\mathbb{R}^d)$ well-posedness. Furthermore, Christ [6] proved that the flow map of the nonlinear Schrödinger equation

$$\begin{cases}
i \partial_t u - \partial_x^2 u = u \partial_x u, & t \in \mathbb{R}, \ x \in \mathbb{R}, \\u(0, x) = u_0(x), & x \in \mathbb{R}\end{cases}$$

is not continuous on $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. From these results, it is difficult to obtain the well-posedness for quadratic derivative nonlinear Schrödinger equation in general. See [25] and references therein for derivative nonlinear Schrödinger equations with cubic or higher order nonlinearities.

Next, we introduce the previous results for (1.1). In [16] and [17], the first and second authors proved the well-posedness of (1.1) in $H^s(\mathbb{R}^d)$ under the condition $\kappa \neq 0$, where $s$ is given in Table 1 below.

| $\mu$ | $d = 1$ | $d = 2$ or 3 | $d \geq 4$ |
|-------|--------|-------------|----------|
| $\mu > 0$ | $s \geq 0$ | $s \geq s_c$ | $s \geq s_c$ |
| $\mu = 0$ | $s \geq 1$ | $s \geq 1$ | $s \geq s_c$ |
| $\mu < 0$ | $s \geq \frac{1}{2}$ | $s \geq \frac{1}{2}$, $s > s_c$ | $s \geq s_c$ |

**Table 1.** Regularities to be well-posed in [16] and [17].

In [16], the first author also considered the case $\kappa = 0$ and proved the well-posedness of (1.1) in $H^s(\mathbb{R}^d)$ for $s \geq \frac{1}{2}$ if $d = 1$, $s > 1$ if $d = 2$ or 3, and $s > s_c$ if $d \geq 4$ under the condition $\alpha = \beta$ and $(\beta + \gamma)(\gamma - \alpha) = 0$. On the other hand, the
first author proved that the flow map is not $C^2$ for $s < 1$ if $\mu = 0$, for $s < \frac{1}{2}$ if $\mu < 0$ and $(\beta + \gamma)(\gamma - \alpha) \neq 0$, and for any $s \in \mathbb{R}$ if $(\beta + \gamma)(\gamma - \alpha) = 0$. Therefore, the well-posedness obtained in [16] and [17] are optimal except for the following cases (A1)–(A4) as far as we use the iteration argument:

(A1) $d = 1$, $\mu > 0$, and $s < 0$,
(A2) $d = 2$ or $3$, $\alpha = \beta$, $(\beta + \gamma)(\gamma - \alpha) \neq 0$, and $\frac{1}{2} \leq s \leq 1$,
(A3) $d = 3$, $\kappa \neq 0$, $\mu < 0$ and $s = \frac{1}{2}$ (which is scaling critical),
(A4) $d \geq 4$, $\alpha = \beta$, $(\beta + \gamma)(\gamma - \alpha) = 0$, and $s = s_c$.

The radial settings are also considered in [18].

We point out that the results in [16] and [17] do not contain the asymptotic behavior of the solution for $d \leq 3$ under the condition $\mu = 0$ (and also $\mu < 0$). In [19], Ikeda, Katayama, and Sunagawa considered the system of quadratic nonlinear Schrödinger equations

$$
\left( i\partial_t + \frac{1}{2m_j} \Delta \right) u_j = F_j(u, \partial_x u), \quad t > 0, \ x \in \mathbb{R}^d, \ j = 1, 2, 3, \tag{1.3}
$$

under the mass resonance condition $m_1 + m_2 = m_3$ (which corresponds to the condition $\mu = 0$ for (1.1)), where $u = (u_1, u_2, u_3)$ is $\mathbb{C}^3$-valued, $m_1, m_2, m_3 \in \mathbb{R}\setminus\{0\}$, and $F_j$ is defined by

$$
\begin{align*}
F_1(u, \partial_x u) &= \sum_{|\alpha|,|\beta| \leq 1} C_{1,\alpha,\beta} (\partial^\alpha u_2)(\partial^\beta u_3), \\
F_2(u, \partial_x u) &= \sum_{|\alpha|,|\beta| \leq 1} C_{2,\alpha,\beta} (\partial^\alpha u_3)(\partial^\beta u_1), \\
F_3(u, \partial_x u) &= \sum_{|\alpha|,|\beta| \leq 1} C_{3,\alpha,\beta} (\partial^\alpha u_1)(\partial^\beta u_2)
\end{align*}
$$

(1.4)

with some constants $C_{1,\alpha,\beta}, C_{2,\alpha,\beta}, C_{3,\alpha,\beta} \in \mathbb{C}$. They obtained the small data global existence and the scattering of the solution to (1.3) in the weighted Sobolev space for $d = 2$ if the nonlinear terms $F_j$ in (1.4) satisfy the null condition. They also proved the same result for $d \geq 3$ without the null condition. In [20], Ikeda, Kishimoto, and third author proved the small data global well-posedness and the scattering of the solution to (1.3) in $\mathcal{H}^s(\mathbb{R}^d)$ for $d \geq 3$ and $s \geq s_c$ under the null condition. They also proved the local well-posedness in $\mathcal{H}^s(\mathbb{R}^d)$ for $d = 1$ and $s \geq 0$, $d = 2$ and $s > s_c$, and $d = 3$ and $s \geq s_c$ under the same conditions. (The results in [16] for $d \leq 3$ and $\mu = 0$ say that if the nonlinear terms do not have null condition, then $s = 1$ is optimal regularity to obtain the well-posedness by using the iteration argument.)

In [26], Sakoda and Sunagawa considered (1.3) for $d = 2$ and $j = 1, \ldots, N$ with

$$
F_j(u, \partial_x u) = \sum_{|\alpha|,|\beta| \leq 1} \sum_{1 \leq k,l \leq 2N} C_{j,k,l}^{\alpha,\beta} (\partial^\alpha u_k^\#)(\partial^\beta u_l^\#), \tag{1.5}
$$
where $u_k^* = u_k$ if $k = 1, \ldots, N$, and $u_k^* = u_{N+k}$ if $k = N+1, \ldots, 2N$. They obtained the small data global existence and the time decay estimate for the solution under some conditions for $m_1, \ldots, m_N$ and the nonlinear terms (1.5). Note that their argument covered (1.1) with $\mu = 0$. In [21], Katayama and Sakoda considered (1.3) for $d = 1$ and 2 with more general nonlinearity than (1.5), and obtained asymptotic behavior of solution for small initial data. In particular, they gave the examples of non-scattering solutions to (1.1) for small initial data under the condition $\mu = 0$.

Moreover, it is known that the existence of the blow up solutions for the system of nonlinear Schrödinger equations. Ozawa and Sunagawa ([24]) gave the examples of the derivative nonlinearity which causes the small data blow up for a system of Schrödinger equations. See [13], [14], [15] and references therein for a system of nonlinear Schrödinger equations without derivative nonlinearity.

Now, we give our main results. The first result is that the flow map fails to be $C^3$ for the case (A1).

**Theorem 1.1.** Let $d = 1$, $\mu > 0$, and $s < 0$. Then, the flow map of (1.1) is not $C^3$ in $\mathcal{H}^s(\mathbb{R}^d)$.

It is known that the flow map is smooth if we prove the well-posedness by using the contraction mapping theorem (or the iteration argument). While there is a gap between the failure of the smoothness of the flow map and ill-posedness, Theorem 1.1 says that the contraction mapping theorem does not work to prove the well-posedness of (1.1) for $s < 0$.

While the nonlinear term in (1.1) is quadratic, to show the existence of an irregular flow map, we need to consider the third iteration term as in the KdV equation (see Section 6 in [5]).

Next, we consider the case (A2).

**Theorem 1.2.** Let $d = 2$ or 3. Assume that $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ satisfy

$$\alpha = \beta \neq 0, \quad (\beta + \gamma)(\gamma - \alpha) \neq 0.$$ 

Then, (1.1) is locally well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ for $s \geq \frac{1}{2}$ and $s > s_c$.

We mention the difference of Theorem 1.2 and the previous results in [17]. In [17], they used the fact that the sizes of the frequencies of $u$, $v$, and $w$ are almost the same when the oscillation is small. On the other hand, when $\alpha = \beta$, the frequency of $w$ may be smaller than that of $u$ and $v$ even if the oscillation is small. See Lemma 2.7 below. However, since the derivative only hits $w$ in (1.1), we can treat this case.
For the proof of Theorem 1.2, we use the Fourier restriction norm method introduced by Bourgain in [4]. Namely, we rely on the contraction mapping theorem in the Fourier restriction norm space. A bilinear estimate in the Fourier restriction norm space plays a key role in the proof. See Proposition 2.2 below. Moreover, the flow map obtained in Theorem 1.2 is smooth. From Theorems 1.1 and 1.2 with the previous results in [16] and [17], we obtain a classification of the regularity where the smooth flow map exists except for the scaling critical cases (A3) and (A4).

Notation. We denote the spatial Fourier transform by \( \hat{\cdot} \) or \( \mathcal{F}_x \), the Fourier transform in time by \( \mathcal{F}_t \) and the Fourier transform in all variables by \( \hat{\cdot} \) or \( \mathcal{F}_{tx} \). For \( \sigma \in \mathbb{R} \), the free evolution \( e^{it\sigma\Delta} \) on \( L^2 \) is given as a Fourier multiplier

\[
\mathcal{F}_x [e^{it\sigma\Delta} f](\xi) = e^{-it|\xi|^2} \hat{f}(\xi).
\]

We will use \( A \lesssim B \) to denote an estimate of the form \( A \leq CB \) for some constant \( C \) and write \( A \sim B \) to mean \( A \lesssim B \) and \( B \lesssim A \). We will use the convention that capital letters denote dyadic numbers, e.g. \( N = 2^n \) for \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and for a dyadic summation we write \( \sum_N a_N := \sum_{n \in \mathbb{N}_0} a_{2^n} \) and \( \sum_{N \geq M} a_N := \sum_{n \in \mathbb{N}_0, 2^n \geq M} a_{2^n} \) for brevity. Let \( \chi \in C^\infty_0((-2,2)) \) be an even, non-negative function such that \( \chi(t) = 1 \) for \( |t| \leq 1 \). We define \( \psi(t) := \chi(t) - \chi(2t) \), \( \psi_1(t) := \chi(t) \), and \( \psi_N(t) := \psi(N^{-1}t) \) for \( N \geq 2 \). Then, \( \sum_N \psi_N(t) = 1 \). We define frequency and modulation projections by

\[
\mathcal{P}_N u(\xi) := \psi_N(\xi) \hat{u}(\xi), \quad \mathcal{Q}^\sigma_L u(\tau, \xi) := \psi_L(\tau + \sigma|\xi|^2) \hat{u}(\tau, \xi).
\]

Furthermore, we define \( Q^\sigma_{\geq M} := \sum_{L \geq M} Q^\sigma_L \) and \( Q^\sigma_{< M} := Id - Q^\sigma_{\geq M} \).

The rest of this paper is planned as follows. In Section 2, we restate Theorem 1.2 and collect some results on the linear and bilinear Strichartz estimates and the property of low modulation. In Section 3, we prove Proposition 2.2 for \( d = 2 \). In Section 4, we show Proposition 2.2 for \( d = 3 \). In Section 5, we prove Theorem 1.1.

2. Preliminary

In this section, we state some preliminary results used in the proof of Theorem 1.2. In Subsection 2.1, by using the condition for coefficients, we restate Theorem 1.2. We then define the Fourier restriction norm space. In Subsection 2.2, we collect some useful lemmas.

2.1. Fourier restriction norm spaces. First, we rewrite Theorem 1.2 by using a change of variables. Set \( \sigma = \alpha^{-1} \gamma \) and

\[
U(t, x) = \alpha^{-1} u(\alpha^{-1} t, x), \quad V(t, x) = \alpha^{-1} v(\alpha^{-1} t, x), \quad W(t, x) = \alpha^{-1} w(\alpha^{-1} t, x).
\]
Then, (1.1) can be rewritten
\[
\begin{aligned}
(i\partial_t + \Delta)U &= -(\nabla \cdot W)V, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \Delta)V &= -(\nabla \cdot W)U, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \sigma\Delta)W &= \nabla(U \cdot V), \quad t > 0, \quad x \in \mathbb{R}^d, \\
(U, V, W)|_{t=0} &= (U_0, V_0, W_0) \in H^s(\mathbb{R}^d),
\end{aligned}
\]
(2.1)

and the condition \((\beta + \gamma)(\gamma - \alpha) \neq 0\) is equivalent to \(\sigma \neq \pm 1\). Hence, Theorem 1.2 is equivalent to the following.

**Theorem 2.1.** Let \(d = 2\) or \(3\) and \(\sigma \in \mathbb{R} \setminus \{0, \pm 1\}\). Then, (2.1) is locally well-posed in \(H^s(\mathbb{R}^d)\) for \(s \geq \frac{1}{2}\) and \(s > s_c\).

Now, we define the Fourier restriction norm, which was introduced by Bourgain in [4].

**Definition 1.** Let \(s \in \mathbb{R}, \ b \in \mathbb{R}, \ \sigma \in \mathbb{R} \setminus \{0\}\). We define \(X^{s,b}_\sigma := \{u \in S'(\mathbb{R}_t \times \mathbb{R}^d_x) \mid \|u\|_{X^{s,b}_\sigma} < \infty\}\), where
\[
\|u\|_{X^{s,b}_\sigma} := \|\langle \xi \rangle^{s}(\tau + \sigma|\xi|^2)^b\tilde{u}(\tau, \xi)\|_{L^2_{\tau,\xi}}
\sim \left(\sum_{N \geq 1} \sum_{L \geq 1} N^{2s}L^{2b}\|Q^s_L P_N u\|^2_{L^2}ight)^{\frac{1}{2}},
\]
where \(\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}\), \(P_N\) and \(Q^s_L\) are defined in Notation at the end of Section 1.

The key estimates to obtain Theorem 2.1 are the following.

**Proposition 2.2.** Let \(d = 2, 3, \ \sigma \in \mathbb{R} \setminus \{0, \pm 1\}, \ s \geq \frac{1}{2}, \ s > s_c, \ and \ j \in \{1, \ldots, d\}\). Then there exist \(b' \in (0, \frac{1}{2})\) and \(C > 0\) such that
\[
\|\partial_j W\|_{X^{s-b'}_1} \leq C\|W\|_{X^{s'}_1} \|V\|_{X^{b'}_1},
\]
\[
\|\partial_j W\|_{X^{s-b'}_1} \leq C\|W\|_{X^{s'}_1} \|U\|_{X^{b'}_1},
\]
\[
\|\partial_j (U \nabla W)\|_{X^{s-b'}_1} \leq C\|U\|_{X^{s'}_2} \|V\|_{X^{b'}_1},
\]
hold for any \(U, V \in X^{s,b'}_1\) and \(W \in X^{s,b'}_\sigma\), where \(\partial_j = \frac{\partial}{\partial x_j}\).

**Remark 2.1.** Note that \(\|\nabla\|_{X^{s',b'}_1} = \|V\|_{X^{s',b'}_1}\). By the duality argument, to obtain Proposition 2.2 it suffices to show that
\[
\left|\int_{\mathbb{R}} \int_{\mathbb{R}^d} U(t, x)V(t, x)\partial_j W(t, x)dxdt\right| \lesssim \|U\|_{X^{s_1,b'}_1} \|V\|_{X^{s_2,b'}_1} \|W\|_{X^{-s_3,b'}_\sigma}
\]
(2.2)
for \((s_1, s_2, s_3) \in \{(s, s, -s), (s, -s, s), (-s, s, s)\}\) with \(s \geq \frac{1}{2}\) and \(s > s_c\). Moreover, Plancherel’s theorem yields that the left hand side of (2.2) is written as follows:

\[
\left| \int_\mathbb{R} \int_\mathbb{R}^d U(t,x) V(t,x) \partial_j W(t,x) \, dx \, dt \right|
\]

\[
= \left| \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \left( \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R}^d \tilde{U}(\tau_1, \xi_1) \tilde{V}(\tau_2, \xi_2) (\xi_1^{(j)} + \xi_2^{(j)}) \tilde{W}(-\tau_1 - \tau_2, -\xi_1 - \xi_2) \partial \tau_1 \partial \xi_1 \partial \tau_2 \partial \xi_2 \right) \right|
\]

where \(\xi_1^{(j)}\) and \(\xi_2^{(j)}\) are the \(j\)-th components of \(\xi_1\) and \(\xi_2\), respectively. Hence, (2.2) is equivalent to the following:

\[
\left| \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R}^d (\xi_1^{(j)} + \xi_2^{(j)}) f_1(\tau_1, \xi_1) f_2(\tau_2, \xi_2) f_3(\tau_1 + \tau_2, +\xi_1 + \xi_2) \partial \tau_1 \partial \xi_1 \partial \tau_2 \partial \xi_2 \right|
\]

\[
\lesssim \| \mathcal{F}_{\tau, \xi}^{-1}[f_1]\|_{X^{s_1, \nu}_1} \| \mathcal{F}_{\tau, \xi}^{-1}[f_2]\|_{X^{s_2, \nu}_2} \| \mathcal{F}_{\tau, \xi}^{-1}[f_3]\|_{X^{s_3, \nu}_3}.
\]

We note that

\[
\| u \|_{X^{s, \nu}(T)} \lesssim T^{b'-b} \| u \|_{X^{s, b}_2(T)}
\]

holds for any \(s \in \mathbb{R}\), \(\sigma \in \mathbb{R}\setminus\{0\}\), \(\frac{1}{b} < b \leq 1\), \(0 \leq b' \leq 1 - b\), and \(0 < T \leq 1\), where \(X^{s, b}(T)\) denotes the time restricted space of \(X^{s, b}\). Hence, by using the fixed point argument with Proposition 2.2, we can obtain Theorem 2.1. Since this argument is standard by now, we omit the details in this paper. See [16] and [17] for example.

2.2. Linear and bilinear estimates. In this subsection, we collect some propositions used in the proof of Proposition 2.2. First, we state the Strichartz and bilinear Strichartz estimates. We say that \((p,q)\) is an admissible pair if \(p\) and \(q\) satisfy \(2 \leq p, q \leq \infty\), \(\frac{2}{p} + \frac{d}{q} = \frac{d}{2}\), and \((p,q,d) \neq (2, \infty, 2)\).

**Proposition 2.3** (Strichartz estimate (cf. [12], [22])). Let \(\sigma \in \mathbb{R}\setminus\{0\}\) and \((p,q)\) be admissible. Then, we have

\[
\| e^{it\sigma \Delta} \varphi \|_{L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| \varphi \|_{L^2(\mathbb{R}^d)}
\]

for any \(\varphi \in L^2(\mathbb{R}^d)\).

The Strichartz estimate implies the following. See the proof of Lemma 2.3 in [11].

**Corollary 2.4.** Let \(L \in 2^\mathbb{N}_0\), \(\sigma \in \mathbb{R}\setminus\{0\}\), and \((p,q)\) be admissible. Then, we have

\[
\| Q^\sigma_L u \|_{L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim L^\frac{d}{2} \| Q^\sigma_L u \|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^d)}
\]

for any \(u \in L^2(\mathbb{R} \times \mathbb{R}^d)\).

We have the following bilinear Strichartz estimates.
Proposition 2.5. Let $d \geq 2$, $\sigma_1$, $\sigma_2 \in \mathbb{R}\{0\}$. For any dyadic numbers $N_1$, $N_2$, $N_3 \in 2^{N_0}$ and $L_1$, $L_2 \in 2^{N_0}$, we have

\[
\| P_{N_3}(Q_{L_1}^{\sigma_1}P_{N_1}u_1 \cdot Q_{L_2}^{\sigma_2}P_{N_2}u_2) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim (N_{\min}^{12})^{\frac{d}{2}-1} \left( \frac{N_{\min}^{12}}{N_{\max}^{12}} \right)^{\frac{1}{2}} \| Q_{L_1}^{\sigma_1}P_{N_1}u_1 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \| Q_{L_2}^{\sigma_2}P_{N_2}u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)},
\]

where $N_{\max}^{12} = N_1 \lor N_2$ and $N_{\min}^{12} = N_1 \land N_2$. Furthermore, if $\sigma_1 + \sigma_2 \neq 0$, then we have

\[
\| P_{N_3}(Q_{L_1}^{\sigma_1}P_{N_1}u_1 \cdot Q_{L_2}^{\sigma_2}P_{N_2}u_2) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N_{\min}^{\frac{d}{2}-1} \left( \frac{N_{\min}}{N_{\max}} \right)^{\frac{1}{2}} \| Q_{L_1}^{\sigma_1}P_{N_1}u_1 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \| Q_{L_2}^{\sigma_2}P_{N_2}u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)},
\]

where $N_{\max} = \max_{1 \leq j \leq 3} N_j$, $N_{\min} = \min_{1 \leq j \leq 3} N_j$.

Proposition 2.5 can be obtained by the same way as in the proof of Lemma 1 in [10]. See also Lemma 3.1 in [16].

An interpolation argument yields the following. Since the proof is the same as that of Corollary 2.5 in [17], we omit the details here.

Corollary 2.6. Let $d \geq 2$, $b' \in (\frac{1}{4}, \frac{1}{2})$, and $\sigma_1$, $\sigma_2 \in \mathbb{R}\{0\}$. We set $\delta = \frac{1}{2} - b'$. For any dyadic numbers $N_1$, $N_2$, $N_3 \in 2^{N_0}$ and $L_1$, $L_2 \in 2^{N_0}$, we have

\[
\| P_{N_3}(Q_{L_1}^{\sigma_1}P_{N_1}u_1 \cdot Q_{L_2}^{\sigma_2}P_{N_2}u_2) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim (N_{\min}^{12})^{\frac{d}{2}-1+4\delta} \left( \frac{N_{\min}^{12}}{N_{\max}^{12}} \right)^{\frac{1}{2}-2\delta} L_1^b L_2^b \| Q_{L_1}^{\sigma_1}P_{N_1}u_1 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \| Q_{L_2}^{\sigma_2}P_{N_2}u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)}.
\]

Furthermore, if $\sigma_1 + \sigma_2 \neq 0$, then we have

\[
\| P_{N_3}(Q_{L_1}^{\sigma_1}P_{N_1}u_1 \cdot Q_{L_2}^{\sigma_2}P_{N_2}u_2) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N_{\min}^{\frac{d}{2}-1+4\delta} \left( \frac{N_{\min}}{N_{\max}} \right)^{\frac{1}{2}-2\delta} L_1^b L_2^b \| Q_{L_1}^{\sigma_1}P_{N_1}u_1 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)} \| Q_{L_2}^{\sigma_2}P_{N_2}u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^d)}.
\]

Next, we consider the low modulation case $L_{\max} := \max_{1 \leq j \leq 3} L_j \ll N_{\max}^2$.

Lemma 2.7. Let $\sigma \in \mathbb{R}\{0, \pm 1\}$. We assume that $(\tau_1, \xi_1)$, $(\tau_2, \xi_2)$, $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^d$ satisfy $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$. If it holds that

\[
\max\{|\tau_1 + |\xi_1|^2|, |\tau_2 - |\xi_2|^2|, |\tau_3 - \sigma| |\xi_3|^2|\} \ll \max_{1 \leq j \leq 3} |\xi_j|^2,
\]

then we have

\[
|\xi_1| \sim |\xi_2| \gtrsim |\xi_3|.
\]
Proof. We set
\[ \Phi(\xi_1, \xi_2, \xi_3) := |\xi_1|^2 - |\xi_2|^2 - \sigma |\xi_3|^2. \]
Then, we have
\[ \Phi(\xi_1, \xi_2, \xi_3) = (1 - \sigma)|\xi_1|^2 - 2\sigma \xi_1 \cdot \xi_2 - (1 + \sigma)|\xi_2|^2 \]
\[ = (1 - \sigma)\left| \xi_1 - \frac{\sigma}{1 - \sigma}\xi_2 \right|^2 - \frac{1}{1 - \sigma}|\xi_2|^2 \]
\[ = - (1 + \sigma)\left| \xi_2 + \frac{\sigma}{1 + \sigma}\xi_1 \right|^2 + \frac{1}{1 + \sigma}|\xi_1|^2 \]
and
\[ \Phi(\xi_1, \xi_2, \xi_3) = 2\xi_2 \cdot \xi_3 + (1 - \sigma)|\xi_3|^2 \]
\[ = (1 - \sigma)\left| \xi_3 + \frac{1}{1 - \sigma}\xi_2 \right|^2 - \frac{1}{1 - \sigma}|\xi_2|^2. \]
Therefore, if it holds \( |\xi_1| \gg |\xi_2| \) or \( |\xi_1| \ll |\xi_2| \) or \( |\xi_3| \gg |\xi_2| \), then we have
\[ \max\{ |\tau_1 + |\xi_1|^2|, |\tau_2 - |\xi_2|^2|, |\tau_3 - c|\xi_3|^2| \} \geq |\Phi(\xi_1, \xi_2, \xi_3)| \sim \max_{1 \leq j \leq 3} |\xi_j|^2. \]
It implies the conclusion. \( \square \)

Lemma 2.7 is different from the case \( \kappa \neq 0 \), where \( \kappa \) is as in (1.2). More precisely, \( |\xi_2| \gg |\xi_3| \) occurs in this case, while we have \( |\xi_1| \sim |\xi_2| \sim |\xi_3| \) if \( \kappa \neq 0 \) (see Lemma 4.1 in [16]).

3. Proof of bilinear estimates for \( d = 2 \)

In this section, we prove Proposition 2.2 for \( d = 2 \). To treat the low modulation interaction, we first introduce the angular frequency localization operators which were utilized in [2].

Definition 2 ([2]). We define the angular decomposition of \( \mathbb{R}^2 \) in frequency. We define a partition of unity in \( \mathbb{R} \),
\[ 1 = \sum_{j \in \mathbb{Z}} \omega_j, \quad \omega_j(s) = \psi(s - j) \left( \sum_{k \in \mathbb{Z}} \psi(s - k) \right)^{-1}. \]
For a dyadic number \( A \geq 64 \), we also define a partition of unity on the unit circle,
\[ 1 = \sum_{j=0}^{A-1} \omega_j^A, \quad \omega_j^A(\vartheta) = \omega_j \left( \frac{A\vartheta}{\pi} \right) + \omega_{j-A} \left( \frac{A\vartheta}{\pi} \right). \]
We observe that \( \omega_j^A \) is supported in
\[ \Theta_j^A = \left[ \frac{\pi}{A} (j - 2), \frac{\pi}{A} (j + 2) \right] \cup \left[ -\pi + \frac{\pi}{A} (j - 2), -\pi + \frac{\pi}{A} (j + 2) \right]. \]
We now define the angular frequency localization operators $R^A_j$,

$$F_x(R^A_j f)(\xi) = \omega^A_j(\vartheta)F_x f(\xi), \quad \text{where} \ \xi = |\xi|(\cos \vartheta, \sin \vartheta).$$

For any function $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, we set $(R^A_j u)(t, x) = (R^A_j u(t, \cdot))(x)$. This operator localizes function in frequency to the set

$$\mathcal{D}^A_j = \{(\tau, |\xi| \cos \vartheta, |\xi| \sin \vartheta) \in \mathbb{R} \times \mathbb{R}^2 \mid \vartheta \in \Theta^A_j\}. \quad (3.1)$$

Immediately, we can see

$$u = \sum_{j=0}^{A-1} R^A_j u.$$

The following propositions play an important role in the proof of Proposition 2.2.

**Proposition 3.1.** Let $N_1, N_2, N_3, L_1, L_2, L_3, A \in 2^{N_0}$, and $j_1, j_2 \in \{0, 1, \ldots, A - 1\}$. We assume $A \geq 64$, $|j_1 - j_2| \lesssim 1$, and $N_3 \lesssim N_1 \sim N_2$. Then, we have the following estimate:

$$\|P_{N_3}(R^A_{j_1} Q^1_{L_1} P_{N_1} u_1 \cdot R^A_{j_2} Q^{-1}_{L_2} P_{N_2} u_2)\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \lesssim \left( \frac{N_1}{N_3, A} \right)^{\frac{1}{2}} L_1^{\frac{3}{2}} L_2^{\frac{1}{2}} \|R^A_{j_1} Q^1_{L_1} P_{N_1} u_1\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \|R^A_{j_2} Q^{-1}_{L_2} P_{N_2} u_2\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)}. \quad (3.2)$$

**Proposition 3.2.** Let $\sigma \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $N_1, N_2, N_3, L_1, L_2, L_3, A \in 2^{N_0}$, and $j_1, j_2 \in \{0, 1, \ldots, A - 1\}$. We assume $L_{\max} \ll N_{\max}^2$, $A \geq 64$, and $|j_1 - j_2| \leq 32$. Then, we have the following estimate:

$$\|P_{N_3}(R^\sigma_{j_1} Q^1_{L_1} P_{N_1}(R^\sigma_{j_2} Q^{-\sigma}_{L_2} P_{N_2} u_2) \cdot Q^{-\sigma}_{L_3} u_3)\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \lesssim A^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|R^\sigma_{j_2} Q^{-\sigma}_{L_2} P_{N_2} u_2\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \|Q^{-\sigma}_{L_3} u_3\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)}. \quad (3.3)$$

**Proposition 3.3.** Let $\sigma \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $N_1, N_2, N_3, L_1, L_2, L_3, A \in 2^{N_0}$, and $j_1, j_2 \in \{0, 1, \ldots, A - 1\}$. We assume $L_{\max} \ll N_{\max}^2$, $A \geq 64$, and $16 \leq |j_1 - j_2| \leq 32$. Then the following estimate holds:

$$\|Q^\sigma_{L_3} P_{N_3}(R^\sigma_{j_1} Q^1_{L_1} P_{N_1} u_1 \cdot R^\sigma_{j_2} Q^{-\sigma}_{L_2} P_{N_2} u_2)\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \lesssim A^{\frac{1}{2}} N_{\max}^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|R^\sigma_{j_1} Q^1_{L_1} P_{N_1} u_1\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)} \|R^\sigma_{j_2} Q^{-\sigma}_{L_2} P_{N_2} u_2\|_{L^2_t(\mathbb{R} \times \mathbb{R}^2)}. \quad (3.4)$$

The bound such as in $(3.2)$ does not appear for $\kappa \neq 0$ in $(1.6)$ and $(1.7)$, where $\kappa$ is as in $(1.2)$. On the other hand, similar bounds as in Propositions 3.2 and 3.3 also appear for $\kappa \neq 0$. See Theorem 2.8 and Proposition 2.9 in $(1.7)$. However, we need to treat the case $N_3 \ll N_1 \sim N_3$ in Propositions 3.2 and 3.3. Hence, while a part of proof of Propositions 3.2 and 3.3 is similar to that in the previous results, we need a more careful calculation. See Lemma 3.6 below for example.
We postpone the proof of Propositions 3.1, 3.2, and 3.3 to the next subsection. Assuming Propositions 3.1, 3.2, and 3.3, we here prove the bilinear estimates.

**Proof of Proposition 2.2 for** \(d = 2\). Let \( s \geq \frac{1}{2} \) and 

\[
(s_1, s_2, s_3) \in \{(s, s, -s), (s, -s, s), (-s, s, s)\}.
\]

We prove (2.2). We set

\[
 u_{N_1, L_1} := Q_{L_1}^1 P_{N_1} U, \ v_{N_2, L_2} := Q_{L_2}^{-1} P_{N_2} V, \ w_{N_3, L_3} := Q_{L_3}^{-\sigma} P_{N_3} W.
\]

Then, we have

\[
\left| \int_{\mathbb{R} \times \mathbb{R}^2} U(t, x) V(t, x) \partial_j W(t, x) dx dt \right| \leq \sum_{N_1, N_2, N_3 \geq 1} \sum_{L_1, L_2, L_3 \geq 1} N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1, L_1} v_{N_2, L_2} w_{N_3, L_3} dx dt \right|.
\]

It suffices to show that

\[
N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1, L_1} v_{N_2, L_2} w_{N_3, L_3} dx dt \right| \lesssim N_3^{s\varepsilon} (L_1 L_2 L_3)^{\varepsilon} \left( \frac{N_{\min}}{N_{\max}} \right)^{\varepsilon} \| u_{N_1, L_1} \|_{L^2_{tx}} \| v_{N_2, L_2} \|_{L^2_{tx}} \| w_{N_3, L_3} \|_{L^2_{tx}} \quad (3.5)
\]

for some \( b' \in (0, \frac{1}{2}) \), \( c \in (0, b') \), and \( \varepsilon > 0 \). Indeed, from (3.5) and the Cauchy-Schwarz inequality, we obtain

\[
\sum_{N_1, N_2, N_3 \geq 1} \sum_{L_1, L_2, L_3 \geq 1} N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1, L_1} v_{N_2, L_2} w_{N_3, L_3} dx dt \right|
\]

\[
\lesssim \sum_{N_1, N_2, N_3 \geq 1} \sum_{L_1, L_2, L_3 \geq 1} N_1^{s\varepsilon} \left( \frac{N_{\min}}{N_{\max}} \right)^{\varepsilon} (L_1 L_2 L_3)^{\varepsilon} \| u_{N_1, L_1} \|_{L^2_{tx}} \| v_{N_2, L_2} \|_{L^2_{tx}} \| w_{N_3, L_3} \|_{L^2_{tx}}
\]

\[
\lesssim \sum_{N_1 \geq 1} \sum_{N_2 \geq 1} N_2^{-(s_2 + s_3) - \varepsilon} \left( \sum_{N_1 \geq 1} N_1^{s_{11} + \varepsilon} \sum_{L_1 \geq 1} L_1^{-(b' - c)} L_1^b \| u_{N_1, L_1} \|_{L^2_{tx}} \right)
\]

\[
\times \left( \sum_{L_2 \geq 1} L_2^{-(b' - c)} L_2^b \| v_{N_2, L_2} \|_{L^2_{tx}} \right) \left( \sum_{L_3 \geq 1} L_3^{-(b' - c)} L_3^b \| w_{N_3, L_3} \|_{L^2_{tx}} \right)
\]

\[
\lesssim \| u \|_{X_{s_1, b'}^1} \| v \|_{X_{s_2, b'}^2} \| w \|_{X_{s_3, b'}^3},
\]

since \( s - s_1 \geq 0, s - s_1 - s_2 - s_3 = 0, \) and \( b' - c > 0 \). The summations for \( N_2 \lesssim N_1 \sim N_3 \) and \( N_3 \lesssim N_1 \sim N_2 \) are similarly handled.
Now, we prove (3.5).

Case 1: High modulation, $L_{\text{max}} \gtrsim N_{\text{max}}^2 N_{\text{min}}^{-\frac{2}{3}}$

We first assume $L_3 = L_{\text{max}}$. By the symmetry, we can assume $N_1 \leq N_2$. Then, by the Cauchy-Schwarz inequality and (2.3), we have

$$\begin{aligned} N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \, dx \, dt \right| & \lesssim N_3 \| P_{N_3} (u_{N_1,L_1} v_{N_2,L_2}) \|_{L^{6}_t} \| w_{N_3,L_3} \|_{L^2_x} \\
 & \lesssim N_3 N_{1}^{4\delta} \left( \frac{N_1}{N_2} \right)^{\frac{1}{2} - 2\delta} L_3^c L_1^c \| u_{N_1,L_1} \|_{L^2_t} \| v_{N_2,L_2} \|_{L^2_t} \| w_{N_3,L_3} \|_{L^2_x} \\
 & \lesssim N_3 N_{1}^{4\delta} \left( \frac{N_1}{N_2} \right)^{\frac{1}{2} - 2\delta} \left( N_{\text{max}}^2 N_{\text{min}}^{-\frac{2}{3}} \right)^{-c} (L_1 L_2 L_3)^c \\
 & \quad \times \| u_{N_1,L_1} \|_{L^2_t} \| v_{N_2,L_2} \|_{L^2_t} \| w_{N_3,L_3} \|_{L^2_x},
\end{aligned}$$

where $\delta = \frac{1}{2} - c$. If $N_3 \lesssim N_1 \sim N_2$, then we obtain

$$N_3 N_{1}^{4\delta} \left( \frac{N_1}{N_2} \right)^{\frac{1}{2} - 2\delta} \left( N_{\text{max}}^2 N_{\text{min}}^{-\frac{2}{3}} \right)^{-c} \sim N_3^{3 - \frac{16}{3} c - s} N_1^{4c - \frac{4}{3}}.$$

If $N_1 \lesssim N_2 \sim N_3$, then we obtain

$$N_3 N_{1}^{4\delta} \left( \frac{N_1}{N_2} \right)^{\frac{1}{2} - 2\delta} \left( N_{\text{max}}^2 N_{\text{min}}^{-\frac{2}{3}} \right)^{-c} \sim N_1^{3 - \frac{16}{3} c - s} N_3^{4c - \frac{4}{3}}.$$

Therefore, by choosing $b'$ and $c$ as max$\{\frac{3(3-s)}{16}, \frac{3}{8}, \frac{1}{3}\} < c < b' < \frac{1}{2}$ for $s > \frac{1}{3}$, we get (3.5). The case $L_1 = L_{\text{max}}$ and $L_2 = L_{\text{max}}$ is similarly treated, but we use (2.4) instead of (2.3).

Case 2: Low modulation, $L_{\text{max}} \ll N_{\text{max}}^2 N_{\text{min}}^{-\frac{2}{3}} (\ll N_{\text{max}}^2)$

By Lemma 2.7, we can assume $N_3 \lesssim N_1 \sim N_2$. We set

$$M := L_{\text{max}}^{-\frac{3}{4}} N_1^{\frac{3}{4}} N_3^{-\frac{1}{4}} \gg 1$$

and decompose $\mathbb{R}^3 \times \mathbb{R}^3$ as follows:

$$\mathbb{R}^3 \times \mathbb{R}^3 = \left( \bigcup_{0 \leq j_1, j_2 \leq M - 1} \mathcal{D}^M_{j_1} \times \mathcal{D}^M_{j_2} \right) \cup \left( \bigcup_{64 \leq A \leq M} \bigcup_{0 \leq j_1, j_2 \leq A - 1} \bigcup_{16 \leq |j_1 - j_2| \leq 32} \mathcal{D}^A_{j_1} \times \mathcal{D}^A_{j_2} \right).$$
where $\mathcal{D}_j^A$ is as in (3.1). We can write

\[
\left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} dxdt \right| \\
\leq \sum_{0 \leq j_1,j_2 \leq M-1 \atop |j_1 - j_2| \leq 16} \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1,L_1,j_1}^M v_{N_2,L_2,j_2}^M w_{N_3,L_3} dxdt \right| \\
+ \sum_{64 \leq A \leq M} \sum_{0 \leq j_1,j_2 \leq A-1 \atop 16 \leq |j_1 - j_2| \leq 32} \left| \int_{\mathbb{R} \times \mathbb{R}^2} u_{N_1,L_1,j_1}^A v_{N_2,L_2,j_2}^A w_{N_3,L_3} dxdt \right| \\
=: \text{I} + \text{II},
\]

where

\[
u_{N_1,L_1,j_1}^A = R_{j_1}^A u_{N_1,L_1}, \quad \nu_{N_2,L_2,j_2}^A = R_{j_2}^A v_{N_2,L_2}.
\]

For the contribution from the first term I in (3.7), we first assume $L_{\text{max}} = L_3$. By (3.7), the Hölder inequality, (3.2), and (3.6), we get

\[
N_3 \cdot \text{I} \leq \sum_{0 \leq j_1,j_2 \leq M-1 \atop |j_1 - j_2| \leq 16} N_3 \left( \frac{N_1}{N_3 M} \right)^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left\| u_{N_1,L_1,j_1}^M \right\|_{L_2^t} \left\| v_{N_2,L_2,j_2}^M \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t} \\
\leq N_3^{\frac{1}{2}} \left( \frac{N_3}{N_1} \right)^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{3}{4}} \left\| u_{N_1,L_1,j_1} \right\|_{L_2^t} \left\| v_{N_2,L_2,j_2} \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t} \\
\leq N_3^{\frac{1}{2}} \left( \frac{N_3}{N_1} \right)^{\frac{1}{4}} (L_1 L_2 L_3)^{\frac{3}{4}} \left\| u_{N_1,L_1,j_1} \right\|_{L_2^t} \left\| v_{N_2,L_2,j_2} \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t},
\]

which shows (3.5). If $L_{\text{max}} = L_1$ or $L_{\text{max}} = L_2$, then we can use a better estimate (3.3) instead of (3.2). Hence, we obtain (3.5) in this case.

For the second term II in (3.7), by Proposition 3.3 and (3.6), we get

\[
N_3 \cdot \text{II} \leq \sum_{64 \leq A \leq M} \sum_{0 \leq j_1,j_2 \leq A-1 \atop 16 \leq |j_1 - j_2| \leq 32} N_3 \left( \frac{N_3}{N_1} \right)^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{3}{4}} \left\| u_{N_1,L_1,j_1}^A \right\|_{L_2^t} \left\| v_{N_2,L_2,j_2}^A \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t} \\
\leq N_3^{\frac{1}{2}} \left( \frac{N_3}{N_1} \right)^{\frac{1}{4}} L_{\text{max}}^{\frac{3}{4}} (L_1 L_2 L_3)^{\frac{1}{2}} \left\| u_{N_1,L_1} \right\|_{L_2^t} \left\| v_{N_2,L_2} \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t} \\
\leq N_3^{\frac{1}{2}} \left( \frac{N_3}{N_1} \right)^{\frac{1}{4}} (L_1 L_2 L_3)^{\frac{1}{2}} \left\| u_{N_1,L_1} \right\|_{L_2^t} \left\| v_{N_2,L_2} \right\|_{L_2^t} \left\| w_{N_3,L_3} \right\|_{L_2^t},
\]
which shows (3.5). This completes the proof of Proposition 2.2 for \( d = 2 \). \( \square \)

### 3.1. Proof of key propositions

In this subsection, we prove Propositions 3.1, 3.2, and 3.3. First, we show Proposition 3.1.

**Proof of key propositions.** In this subsection, we prove Propositions 3.1, 3.2, and 3.3. First, we show Proposition 3.1.

**Proof of Proposition 3.1.** If \( A \lesssim \frac{N_1}{N_3} \), then we obtain (3.2) by the Hölder inequality and Corollary 2.4 with \( p = q = 4 \), since \( 1 \lesssim \frac{N_1}{N_3} \). Therefore, we can assume

\[
A \gg \frac{N_1}{N_3} \gtrsim 1. \quad (3.8)
\]

We set \( f_1 = \mathcal{F}[R_{j_1}^A Q_{L_1} P_{N_1} u_1] \) and \( f_2 = \mathcal{F}[R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2] \). By the duality argument, it suffices to show that

\[
\left| \int_{\Omega} f_1(\tau_1, \xi_1) f_2(\tau_2, \xi_2) f(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\]

\[
\lesssim \left( \frac{N_1}{N_3 A} \right)^{\frac{1}{2}} L_{1/2}^{1/2} \| f_1 \|_{L_{1/4}^2} \| f_2 \|_{L_{1/4}^2} \| f \|_{L_{1/4}^2}
\]

(3.9)

for any \( f \in L^2(\mathbb{R} \times \mathbb{R}^2) \), where \( \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}) = |\xi_i| (\cos \theta_i, \sin \theta_i), \ (i = 1, 2) \) and

\[
\Omega = \left\{ (\tau_1, \tau_2, \xi_1, \xi_2) \bigg| \begin{array}{l}
|\xi_1| \sim N_1, \ |\xi_2| \sim N_2, \ |\xi_1 + \xi_2| \sim N_3, \\
\theta_1 \in \Theta_{j_1}^A, \ \theta_2 \in \Theta_{j_2}^A, \\
|\tau_1 + |\xi_1|^2| \sim L_1, \ |\tau_2 - |\xi_2|^2| \sim L_2
\end{array} \right\}.
\]

By the Cauchy-Schwarz inequality, we have

\[
\left| \int_{\Omega} f_1(\tau_1, \xi_1) f_2(\tau_2, \xi_2) f(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\]

\[
\lesssim \| f_1 \|_{L_{1/4}^2} \| f_2 \|_{L_{1/4}^2} \left( \int_{\Omega} |f(\tau_1 + \tau_2, \xi_1 + \xi_2)|^2 d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right)^{\frac{1}{2}}
\]

(3.10)

By changing variables \((\xi_1, \xi_2) \mapsto (\tilde{\xi}_1, \tilde{\xi}_2)\) as

\[
(\tilde{\xi}_1, \tilde{\xi}_2) = \left( \xi_1 R \left( -\frac{\pi}{A} j_2 \right), \xi_2 R \left( -\frac{\pi}{A} j_2 \right) \right), \ R(\theta) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right).
\]

we have

\[
\int_{\Omega} |f(\tau_1 + \tau_2, \xi_1 + \xi_2)|^2 d\tau_1 d\tau_2 d\xi_1 d\xi_2
\]

\[
= \int_{\tilde{\Omega}} \left| f \left( \tau_1 + \tau_2, (\tilde{\xi}_1 + \tilde{\xi}_2) R \left( \frac{\pi}{A} j_2 \right) \right) \right|^2 d\tau_1 d\tau_2 d\tilde{\xi}_1 d\tilde{\xi}_2,
\]

(3.11)
where \( \tilde{\xi}_i = (\tilde{\xi}_i^{(1)}, \tilde{\xi}_i^{(2)}) = |\tilde{\xi}_i| (\cos \tilde{\theta}_i, \sin \tilde{\theta}_i), \ (i = 1, 2) \) and

\[
\tilde{\Omega} = \left\{ (\tau_1, \tau_2, \tilde{\xi}_1, \tilde{\xi}_2) \mid \begin{array}{l}
|\tilde{\xi}_1| \sim N_1, \ |\tilde{\xi}_2| \sim N_2, \ |\tilde{\xi}_1 + \tilde{\xi}_2| \sim N_3, \\
\tilde{\theta}_1 \in \Theta^A_{j_1-j_2}, \ \tilde{\theta}_2 \in \Theta^A_0, \\
|\tau_1 + |\tilde{\xi}_1|^2| \sim L_1, \ |\tau_2 - |\tilde{\xi}_2|^2| \sim L_2
\end{array} \right\}.
\]

Since \(|j_1 - j_2| \lesssim 1\), we have

\[
\min\{|\tilde{\theta}_1|, |\pi - \tilde{\theta}_1|\} \lesssim A^{-1}, \ \min\{|\tilde{\theta}_2|, |\pi - \tilde{\theta}_2|\} \lesssim A^{-1}
\]

for \( \tilde{\xi}_1 \in \Theta^A_{j_1-j_2} \) and \( \tilde{\xi}_2 \in \Theta^A_0 \). Therefore, it follows from (3.8) that

\[
|\tilde{\xi}_1^{(2)} + \tilde{\xi}_2^{(2)}| \leq |\tilde{\xi}_1| \sin \tilde{\theta}_1 + |\tilde{\xi}_2| \sin \tilde{\theta}_2 \lesssim (N_1 + N_2)A^{-1} \ll N_3.
\]

It says that \( |\tilde{\xi}_1^{(1)} + \tilde{\xi}_2^{(1)}| \sim N_3 \), since \( |\tilde{\xi}_1 + \tilde{\xi}_2| \sim N_3 \) in \( \tilde{\Omega} \). By changing variables \((\tau_1, \tau_2) \mapsto (c_1, c_2)\) and \((\tilde{\xi}_1, \tilde{\xi}_2) \mapsto (\mu, w, z)\) as

\[
c_1 = \tau_1 + |\tilde{\xi}_1|^2, \ c_2 = \tau_2 - |\tilde{\xi}_2|^2, \\
\mu = c_1 + c_2 - |\tilde{\xi}_1|^2 - |\tilde{\xi}_2|^2, \\
w = \tilde{\xi}_1 + \tilde{\xi}_2, \ z = \tilde{\xi}_2^{(2)},
\]

we have

\[
\int_{\tilde{\Omega}} \left| f \left( \tau_1 + \tau_2, (\tilde{\xi}_1 + \tilde{\xi}_2)R \left( \frac{\pi}{A}j_2 \right) \right) \right|^2 d\tau_1 d\tau_2 d\tilde{\xi}_1 d\tilde{\xi}_2 \lesssim \left( \int_{|c_1| \lesssim L_1} dc_1 dc_2 \right) \left( \int_{|c_2| \lesssim L_2} dz \right)
\]

\[
\times \left( \int_{\mathbb{R} \times \mathbb{R}^2} \left| f \left( \mu, wR \left( \frac{\pi}{A}j_2 \right) \right) \right|^2 1_{|\tilde{\xi}_1 + \tilde{\xi}_2| \sim N_3} d\mu dw \right),
\]

where

\[
J(\tilde{\xi}_1, \tilde{\xi}_2) = \left| \det \frac{\partial(\mu, w, z)}{\partial(\tilde{\xi}_1, \tilde{\xi}_2)} \right| = |\tilde{\xi}_1^{(1)} + \tilde{\xi}_2^{(1)}| \sim N_3.
\]

Therefore, we obtain

\[
\int_{\tilde{\Omega}} \left| f \left( \tau_1 + \tau_2, (\tilde{\xi}_1 + \tilde{\xi}_2)R \left( \frac{\pi}{A}j_2 \right) \right) \right|^2 d\tau_1 d\tau_2 d\tilde{\xi}_1 d\tilde{\xi}_2 \lesssim \frac{N_2 A^{-1}}{N_3} L_1 L_2 \|f\|_{L^2}^2
\] (3.12)

since

\[
\int_{\mathbb{R} \times \mathbb{R}^2} \left| f \left( \mu, wR \left( \frac{\pi}{A}j_2 \right) \right) \right|^2 d\mu dw = \int_{\mathbb{R} \times \mathbb{R}^2} |f(\mu, w)|^2 d\mu dw = \|f\|_{L^2}^2.
\]

As a result, we get (3.9) from (3.10), (3.11), and (3.12). \( \square \)

Before the proof of Proposition 3.2, we state an elementary lemma.
Lemma 3.4. Let $L, M, N \in 2^{\mathbb{N}_0}$. Assume that $N^2 \gg L$. Then, we have

$$\left| \{ x \in \mathbb{R} \mid |(x-M)^2 - N^2| \lesssim L \} \right| \lesssim \frac{L}{N}.$$

Proof. When $|(x-M)^2 - N^2| \lesssim L$, a direct calculation shows that $x$ is in

$$\left[ M - \sqrt{N^2 - CL}, M - \sqrt{N^2 + CL} \right] \cup \left[ M + \sqrt{N^2 - CL}, M + \sqrt{N^2 + CL} \right]$$

for some constant $C > 0$. Hence, from $N^2 \gg L$, we have

$$\left| \{ x \in \mathbb{R} \mid |(x-M)^2 - N^2| \lesssim L \} \right| \lesssim \sqrt{N^2 + CL} - \sqrt{N^2 - CL}$$

$$= \frac{2CL}{\sqrt{N^2 + CL} + \sqrt{N^2 - CL}} \sim \frac{L}{N},$$

which concludes the proof. \hfill \Box

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 2.7 and $L_{\max} \ll N_{\max}^2$, we can assume $N_3 \lesssim N_1 \sim N_2$. By Plancherel’s theorem as in Remark 2.1, it suffices to show that

$$\left\| \psi^1_{N_1, L_1, j_1} (\tau_1, \xi_1) \int_{\mathbb{R} \times \mathbb{R}^2} f_2(\tau_2, \xi_2) f_3(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_2 d\xi_2 \right\|_{L^2_{\tau_1, \xi_1}} \lesssim A^{-\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}} \| f_2 \|_{L^2_{\tau_2}} \| f_3 \|_{L^2_{\tau_2}},$$

where $f_2 = F[R^2_{j_2} Q_{L_2}^{-1} P_{N_1} u_2]$, $f_3 = F[Q_{L_3}^* P_{N_3} u_3]$, and $\psi^1_{N_1, L_1, j_1} (\tau_1, \xi_1) = \omega^A_{j_1}(\theta_1) \psi_{L_1} (\tau_1 + |\xi_1|^2) \psi_{N_1} (\xi_1)$ for $(\tau_1, \xi_1) = (\tau_1, |\xi_1| \cos \theta_1, |\xi_1| \sin \theta_1)$. If $A \sim 1$, (3.13) follows from Corollary 2.4 with $p = q = 4$. We hence assume that $A \gg 1$.

By the Cauchy-Schwarz inequality, we have

$$\left\| \psi^1_{N_1, L_1, j_1} (\tau_1, \xi_1) \int_{\mathbb{R} \times \mathbb{R}^2} f_2(\tau_2, \xi_2) f_3(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_2 d\xi_2 \right\|_{L^2_{\tau_1, \xi_1}} \lesssim \| \psi^1_{N_1, L_1, j_1} (\tau_1, \xi_1) \|

\times \left( \int_{\mathbb{R} \times \mathbb{R}^2} |f_2(\tau_2, \xi_2)|^2 |f_3(\tau_1 + \tau_2, \xi_1 + \xi_2)|^2 d\tau_2 d\xi_2 \right)^{\frac{1}{2}} \| E(\tau_1, \xi_1) \|_{L^2_{\tau_1, \xi_1}}^{\frac{1}{2}} \| f_2 \|_{L^2_{\tau_2}} \| f_3 \|_{L^2_{\tau_2}},$$

where

$$E(\tau_1, \xi_1) = \left\{ (\tau_2, \xi_2) \in D^A_{j_2} \mid \langle \tau_2 - |\xi_2|^2 \rangle \sim L_2, \langle \tau_1 + \tau_2 + |\xi_1 + \xi_2|^2 \rangle \sim L_3 \right\}.$$
We set \( \tilde{E}(\tau_1, \xi_1) = \{ \xi_2 \mid (\tau_2, \xi_2) \in E(\tau_1, \xi_1) \} \) for some \( \tau_1 \in \mathbb{R} \). Then, it holds that
\[
|E(\tau_1, \xi_1)| \lesssim \min\{L_2, L_3\} |\tilde{E}(\tau_1, \xi_1)|.
\]
(3.15)

For \( \xi_2 = (|\xi_2| \cos \theta_2, |\xi_2| \sin \theta_2) \in \tilde{E}(\tau_1, \xi_1) \), we obtain
\[
- (\tau_1 + |\xi_1|^2) - (\tau_2 - |\xi_2|^2) + (\tau_1 + \tau_2 + \sigma|\xi_1 + \xi_2|^2)
\]
\[
= -|\xi_1|^2 + |\xi_2|^2 + \sigma|\xi_1 + \xi_2|^2
\]
\[
= \frac{((1 + \sigma)|\xi_2| + \sigma|\xi_1| \cos \angle(\xi_1, \xi_2))^2 - (1 - \sigma^2 \sin^2 \angle(\xi_1, \xi_2))|\xi_1|^2}{1 + \sigma}.
\]
(3.16)

From \( (\tau_1, \xi_1) \in \text{supp}_N \phi_{N_1, L_1, j_1} \) and \( \xi_2 \in \tilde{E}(\tau_1, \xi_1) \), it says that
\[
((1 + \sigma)|\xi_2| + \sigma|\xi_1| \cos \angle(\xi_1, \xi_2))^2
\]
\[
=(1 - \sigma^2 \sin^2 \angle(\xi_1, \xi_2))|\xi_1|^2 - (1 + \sigma)(\tau_1 + |\xi_1|^2) + O(\max\{L_2, L_3\}).
\]

Here, a simple calculation shows that
\[
\frac{x}{2} \leq \sin x \leq x
\]
(3.17)
for \( 0 \leq x \leq \frac{\pi}{2} \). It follows from \( \theta_1 \in \Theta_{j_1}^A, \theta_2 \in \Theta_{j_2}^A, |j_1 - j_2| \leq 32, \) and \( A \gg 1 \) that
\[
|\sigma \sin \angle(\xi_1, \xi_2)| \leq |\sigma \angle(\xi_1, \xi_2)| \lesssim A^{-1} \ll 1.
\]
(3.18)

Since \( \xi_2 = (|\xi_2| \cos \theta_2, |\xi_2| \sin \theta_2) \), we may regard that \( |\xi_2| \) and \( \theta_2 \) are independent variables. Then, Lemma 3.4 with (3.15) and (3.18) yields that \( |\xi_2| \) is restricted to a set \( \Omega \) of measure at most \( O\left( \frac{\max\{L_2, L_3\}}{N_1} \right) \). As a result, we obtain
\[
|\tilde{E}(\tau, \xi)| \lesssim \int_{\theta_2 \in \Theta_{j_2}^A} \left( \int_{|\xi_2| \in \Omega} |\xi_2| \text{d}|\xi_2| \right) \text{d}\theta_2
\]
\[
\lesssim A^{-1} \max\{L_2, L_3\},
\]
(3.19)
since \( |\xi_2| \sim N_2 \sim N_1 \).

Therefore, (3.13) follows from (3.11), (3.15), and (3.19). This concludes the proof of Proposition 3.2.

To prove Proposition 3.3, we use a nonlinear version of the Loomis-Whitney inequality.

**Proposition 3.5 (3 Corollary 1.5).** Assume that the surface \( S_i \) \((i = 1, 2, 3)\) is an open and bounded subset of \( S_i^* \) which satisfies the following conditions (Assumption 1.1 in [3]).

(i) \( S_i^* \) is defined as
\[
S_i^* = \{ \lambda_i \in U_i \mid \Phi_i(\lambda_i) = 0, \nabla \Phi_i \neq 0 \}
\]
for a convex $U_i \subset \mathbb{R}^3$ such that $\text{dist}(S_i, U_i^c) \geq \text{diam}(S_i)$ and $\Phi_i \in \mathcal{C}^{1,1}(U_i)$;

(ii) the unit normal vector field $n_i$ on $S_i^*$ satisfies the Hölder condition

$$\sup_{\lambda, \lambda' \in S_i^*} \left( \frac{|n_i(\lambda) - n_i(\lambda')|}{|\lambda - \lambda'|} + \frac{|n_i(\lambda) \cdot (\lambda - \lambda')|}{|\lambda - \lambda'|^2} \right) \lesssim 1;$$

(iii) there exists $a > 0$ such that the matrix

$$N(\lambda_1, \lambda_2, \lambda_3) = (n_1(\lambda_1), n_2(\lambda_2), n_3(\lambda_3))$$

satisfies the transversality condition

$$a \leq \det N(\lambda_1, \lambda_2, \lambda_3) \leq 1$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in S_1^* \times S_2^* \times S_3^*$.

We also assume $\text{diam}(S_i) \lesssim a$. Then for functions $f \in L^2(S_1)$ and $g \in L^2(S_2)$, the restriction of the convolution $f \ast g$ to $S_3$ is a well-defined $L^2(S_3)$-function which satisfies

$$\|f \ast g\|_{L^2(S_3)} \lesssim \frac{1}{\sqrt{a}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}.$$

We first claim the following.

**Lemma 3.6.** Let $N_1, N_2, N_3, A \in 2^{\mathbb{N}_0}$, $A \geq 64$, and $16 \leq |j_1 - j_2| \leq 32$. Assume that $\xi_1 = (|\xi_1| \cos \theta_1, |\xi_1| \sin \theta_1)$ and $\xi_2 = (|\xi_2| \cos \theta_2, |\xi_2| \sin \theta_2)$ satisfy $|\xi_1| \sim N_1$, $|\xi_2| \sim N_2$, $|\xi_1 + \xi_2| \sim N_3$, $\theta_1 \in \Theta^A_{j_1}$, and $\theta_2 \in \Theta^A_{j_2}$. If $N_3 \lesssim N_1 \sim N_2$, then $A \geq \frac{N_1}{N_3}$.

**Proof.** If $N_3 \sim N_1$, then the claim is clearly true, since $\frac{N_1}{N_3} \sim 1$. So, we assume $N_3 \ll N_1$. By using the rotation, we can assume $\theta_1 \in \Theta^A_{j_1 - j_2}$ and $\theta_2 \in \Theta^A_0$. Then, it follows from $16 \leq |j_1 - j_2| \leq 32$, and (3.17) that

$$|\sin \theta_1| = |\sin(\pi + \theta_1)| \geq \sin \left( \frac{|j_1 - j_2| - 2}{A} \pi \right) \geq \sin \left( \frac{14}{A} \pi \right) \geq \frac{7}{A} \pi$$

and

$$|\sin \theta_2| = |\sin(\pi + \theta_2)| \leq \sin \frac{2}{A} \pi \leq \frac{2}{A} \pi.$$

We therefore obtain

$$|\xi_1 + \xi_2| \geq ||\xi_1| \sin \theta_1 + |\xi_2| \sin \theta_2|$$

$$\geq |\xi_1| (|\sin \theta_1| - |\sin \theta_2|) - |\xi_1 + \xi_2| |\sin \theta_2|$$

$$\geq |\xi_1| \left( \frac{7}{A} \pi - \frac{2}{A} \pi \right) - |\xi_1 + \xi_2| \frac{2}{A} \pi,$$

which yields that

$$N_3 \sim |\xi_1 + \xi_2| \gtrsim \frac{|\xi_1|}{A} \sim \frac{N_1}{A}.$$

This shows the desired bound. □
We then present the proof of Proposition 3.3.

**Proof of Proposition 3.3.** By Lemma 2.7 and $L_{\text{max}} \ll N_{\text{max}}^2$, we can assume $N_3 \lesssim N_1 \sim N_2$. We divide the proof into the following two cases:

(I) $L_{\text{max}} \geq A^{-2}N_1^{-1}N_3^{-1}$,  
(II) $L_{\text{max}} \leq A^{-2}N_1^{-1}N_3^{-1}$.

We first consider the case (I). For the case $L_{\text{max}} = L_3$, by using the (3.2), we have

$$
\|Q_{L_3}^{-\sigma} P_{N_3} (R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1 \cdot R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2)\|_{L_{\xi t}^2} \\
\lesssim \left( \frac{N_1}{N_3 A} \right)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1\|_{L_{\xi t}^2} \|R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2\|_{L_{\xi t}^2} \\
\lesssim A^{\frac{1}{2}} N_1^{-1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}} \|R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1\|_{L_{\xi t}^2} \|R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2\|_{L_{\xi t}^2},
$$

which shows (3.4). For $L_{\text{max}} = L_1$, by the duality argument, Hölder inequality, and (3.3), we have

$$
\|Q_{L_1}^{\sigma} P_{N_3} (R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1 \cdot R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2)\|_{L_{\xi t}^2} \\
\sim \sup_{\|u_3\|_{L_2} = 1} \left| \int_{\mathbb{R} \times \mathbb{R}^2} (R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1) (R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2) (Q_{L_3}^{-\sigma} P_{N_3} u_3) \, dx \, dt \right| \\
\lesssim \|R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1\|_{L_2} \\
\times \sup_{\|u_3\|_{L_2} = 1} \|R_{j_1}^A Q_{L_1}^{-1} P_{N_1} (R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2 \cdot Q_{L_3}^{-\sigma} P_{N_3} u_3)\|_{L_{\xi t}^2} \\
\lesssim A^{\frac{1}{2}} N_1^{-1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}} \|R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1\|_{L_{\xi t}^2} \|R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2\|_{L_{\xi t}^2}.
$$

The case $L_{\text{max}} = L_2$ can be treated similarly.

For (II), by Plancherel’s theorem and the duality argument, (3.4) is verified by the following estimate:

$$
\left\| \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}} f_1 (\tau_1, \xi_1) f_2 (\tau_2, \xi_2) f_3 (\tau_1 + \tau_2, \xi_1 + \xi_2) \, d\tau_1 \, d\tau_2 \, d\xi_1 \, d\xi_2 \right\| \\
\lesssim A^{\frac{1}{2}} N_1^{-1} (L_1 L_2 L_3)^{\frac{1}{2}} \|f_1\|_{L_{\tau \xi}^2} \|f_2\|_{L_{\tau \xi}^2} \|f_3\|_{L_{\tau \xi}^2}
$$

where $f_1 = \mathcal{F}_{\tau x} [R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1]$, $f_2 = \mathcal{F}_{\tau x} [R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2]$, and $f_3 = \mathcal{F}_{\tau x} [Q_{L_3}^{-\sigma} P_{N_3} u_3]$.

Let $(\tau_i, \xi_i) \in \text{supp} f_i$ for $i = 1, 2$ and $(\tau_1 + \tau_2, \xi_1 + \xi_2) \in \text{supp} f_3$. We write $\xi_i$ as

$$
\xi_i = (|\xi_i| \cos \theta_i, |\xi_i| \sin \theta_i).
$$

The assumption $16 \leq |j_1 - j_2| \leq 32$ yields that $|\angle (\xi_1, \xi_2)|$ is confined to a set of measure $\sim A^{-1}$. If $A \gg 1$, from Lemma 3.4 with (3.16) and (3.18), the range of
$|\xi_2|$ is restricted to a set of measure $\sim \frac{L_{\text{max}}}{N_1}$ for any $(\tau_1, \xi_1) \in \text{supp} f_1$. Here, the assumption (II) and Lemma 3.6 yield that

$$\frac{L_{\text{max}}}{N_1} \leq A^{-2} N_1^2 N_3^{-1} \lesssim A^{-1} N_1 =: \delta.$$ 

Namely, for any fixed $(\tau_1, \xi_1) \in \text{supp} f_1$, $|\xi_2|$ is restricted to a set of measure $\sim \delta$. If $A \sim 1$, we set $\delta := N_1$. From $(\tau_2, \xi_2) \in \text{supp} f_2$ and $N_1 \sim N_2$, $|\xi_2|$ is trivially restricted to a set of measure $\sim \delta$.

Next, we decompose $f_1$ by thickened circular localization characteristic functions. Namely, by setting

$$\mathcal{S}_\delta^R = \{ (\tau, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid R \leq \langle \xi \rangle \leq R + \delta \}$$

for $R > 0$, we have

$$f_1 = \sum_{k=0}^{[\frac{N_1}{\delta}]+1} \mathbf{1}_{\mathcal{S}^{N_1+k\delta}_\delta} f_1,$$

where $[s]$ denotes the maximal integer which is not greater than $s \in \mathbb{R}$. For $k = 0, 1, \ldots, [\frac{N_1}{\delta}] + 1$, we may assume that $\text{supp} f_2$ is confined to $\mathcal{S}_\delta^{0(k)}$ with some fixed $\xi^0(k) \sim N_2$.

Let

$$C_\delta(\xi) := \{ (\tau, \xi) \in \mathbb{R}^3 \mid |\xi - \xi'| \leq \delta \}.$$ 

We apply a harmless decomposition to $f_1, f_2, f_3$ and assume that there exist $\xi_{f_i}^0 \in \mathbb{R}^2$ such that $\text{supp} f_i \subset C_\delta(\xi_{f_i}^0)$ for $i = 1, 2, 3$.

We apply the same strategy as that of the proof of Proposition 4.4 in [2]. Applying the transformation $\tau_1 = -|\xi_1|^2 + c_1$ and $\tau_2 = |\xi_2|^2 + c_2$ and Fubini’s theorem, we find that it suffices to prove

$$\left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_1(\phi^-_{c_1}(\xi_1)) f_2(\phi^+_{c_2}(\xi_2)) f_3(\phi^-_{c_1}(\xi_1) + \phi^+_{c_2}(\xi_2)) d\xi_1 d\xi_2 \right| \lesssim A^{\frac{1}{2}} N_1^{-1} \|f_1 \circ \phi^-_{c_1}\|_{L_{\xi}^1} \|f_2 \circ \phi^+_{c_2}\|_{L_{\xi}^2} \|f_3\|_{L_{\xi}^2},$$

where $f_3(\tau, \xi)$ is supported in $c_0 \leq \tau - \sigma|\xi|^2 \leq c_0 + 1$ and

$$\phi^-_{c_1}(\xi_1) = (-|\xi_1|^2 + c_1, \xi_1), \quad \phi^+_{c_2}(\xi_2) = (|\xi_2|^2 + c_2, \xi_2).$$

We use the scaling $(\tau, \xi) \to (N_1^2 \tau, N_1 \xi)$ to define

$$g_1(\tau_1, \xi_1) = f_1(N_1^2 \tau_1, N_1 \xi_1),$$
$$g_2(\tau_2, \xi_2) = f_2(N_1^2 \tau_2, N_1 \xi_2),$$
$$g_3(\tau, \xi) = f_3(N_1^2 \tau, N_1 \xi).$$
If we set $\tilde{c}_k = N_1^{-2}c_k$, inequality (3.20) reduces to
\[
\left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} g_1(\phi_{c_1}^-(\xi_1))g_2(\phi_{c_2}^+(\xi_2))g_3(\phi_{c_1}^-(\xi_1) + \phi_{c_2}^+(\xi_2))d\xi_1d\xi_2 \right| \\
\lesssim A^\frac{1}{2}N_1^{-1}\|g_1 \circ \phi_{c_1}^-\|L^2_\xi \|g_2 \circ \phi_{c_2}^+\|L^2_\xi \|g_3\|L^2_\xi .
\]
Note that $g_3$ is supported in $S_3(N_1^{-2})$, where
\[
S_3(N_1^{-2}) = \left\{ (\tau, \xi) \in C_{N_1^{-1}\delta}(N_1^{-1}\xi_{f_3}) : |\xi| + \frac{c_0}{N_1^2} \leq \tau \leq |\xi| + \frac{c_0 + 1}{N_1^2} \right\}.
\]
By density and duality arguments, it suffices to show for continuous $g_1$ and $g_2$ that
\[
\|g_1|_{S_1} \ast g_2|_{S_2}\|_{L^2(S_3(N_1^{-2}))} \lesssim A^\frac{1}{2}N_1^{-1}\|g_1\|_{L^2(S_1)}\|g_2\|_{L^2(S_2)},
\] (3.22)
where $S_1$ and $S_2$ denote the following surfaces
\[
S_1 = \{ \phi_{c_1}^-(\xi_1) : \xi_1 \in \mathbb{R}^2 \} \cap C_{N_1^{-1}\delta}(N_1^{-1}\xi_{f_1}),
\]
\[
S_2 = \{ \phi_{c_2}^+(\xi_2) : \xi_2 \in \mathbb{R}^2 \} \cap C_{N_1^{-1}\delta}(N_1^{-1}\xi_{f_2}).
\]
Then, (3.22) is immediately obtained by
\[
\|g_1|_{S_1} \ast g_2|_{S_2}\|_{L^2(S_3)} \lesssim A^\frac{1}{2}\|g_1\|_{L^2(S_1)}\|g_2\|_{L^2(S_2)},
\] (3.23)
where
\[
S_3 = \left\{ (\psi(\xi), \xi) \in C_{N_1^{-1}\delta}(N_1^{-1}\xi_{f_3}) : |\psi(\xi)| = \sigma|\xi|^2 + \frac{c_0}{N_1^2} \right\}.
\]
Since $|N_1^{-1}\xi_{f_1}| \sim |N_1^{-1}\xi_{f_2}| \sim 1$, $|N_1^{-1}\xi_{f_3}| \sim N_1^{-1}N_3 \lesssim 1$, and $N^{-1}\delta \sim A^{-1}$, we have
\[
\text{diam}(S_i) \lesssim A^{-1}
\]
for $i = 1, 2, 3$. By applying a harmless decomposition, we can assume
\[
\text{diam}(S_i) \leq 2^{-10}(\sigma)^{-1}A^{-1}
\] (3.24)
for $i = 1, 2, 3$.

For any $\lambda_i \in S_i$, $i = 1, 2, 3$, there exist $\xi_1$, $\xi_2$, $\xi$ such that
\[
\lambda_1 = \phi_{c_1}^-(\xi_1), \quad \lambda_2 = \phi_{c_2}^+(\xi_2), \quad \lambda_3 = (\psi(\xi), \xi).
\]
By (3.21), the unit normals $n_i$ on $\lambda_i$ are written as
\[
n_1(\lambda_1) = \frac{1}{\sqrt{1 + 4|\xi_1|^2}} \left( 1, \ 2\xi_1^{(1)}, \ 2\xi_1^{(2)} \right),
\]
\[
n_2(\lambda_2) = \frac{1}{\sqrt{1 + 4|\xi_2|^2}} \left( 1, \ -2\xi_2^{(1)}, \ -2\xi_2^{(2)} \right),
\]
\[
n_3(\lambda_3) = \frac{1}{\sqrt{1 + 4\sigma^2|\xi|^2}} \left( -1, \ 2\sigma\xi^{(1)}, \ 2\sigma\xi^{(2)} \right),
\]
where $\xi^{(i)}$ ($i = 1, 2$) denotes the $i$-th component of $\xi$. Clearly, the surfaces $S_1$, $S_2$, $S_3$ satisfy the following Hölder condition.

$$
\sup_{\lambda_1, \lambda'_1 \in S_i} \left( \frac{|n_i(\lambda_1) - n_i(\lambda'_1)|}{|\lambda_1 - \lambda'_1|} + \frac{|n_i(\lambda_1) \cdot (\lambda_1 - \lambda'_1)|}{|\lambda_1 - \lambda'_1|^2} \right) \leq 2^3. \quad (3.25)
$$

We may assume that there exist $\xi'_1, \xi'_2, \xi'_3 \in \mathbb{R}^2$ such that

$$\xi'_1 + \xi'_2 = \xi',\quad \phi_{e_1}(\xi'_1) \in S_1, \quad \phi_{e_2}^+(\xi'_2) \in S_2, \quad (\psi(\xi'), \xi') \in S_3,$$

otherwise the left-hand side of (3.23) vanishes. Let $\lambda'_1 = \phi_{e_1}(\xi'_1)$, $\lambda'_2 = \phi_{e_2}^+(\xi'_2)$, $\lambda'_3 = (\psi(\xi'), \xi')$. For any $\lambda_1 = \phi_{e_1}(\xi_1) \in S_1$, we deduce from $\lambda_1, \lambda'_1 \in S_1$, (3.24), and (3.25) that

$$|n_1(\lambda_1) - n_1(\lambda'_1)| \leq 2^{-7} \langle \sigma \rangle^{-1} A^{-1}. \quad (3.26)$$

Similarly, for any $\lambda_2 \in S_2$ and $\lambda_3 \in S_3$, we have

$$|n_2(\lambda_2) - n_2(\lambda'_2)| \leq 2^{-7} \langle \sigma \rangle^{-1} A^{-1}, \quad (3.27)$$

$$|n_3(\lambda_3) - n_3(\lambda'_3)| \leq 2^{-7} \langle \sigma \rangle^{-1} A^{-1}. \quad (3.28)$$

From (3.24) and (3.25), once the following transversality condition

$$\langle \sigma \rangle^{-1} A^{-1} \leq |\det N(\lambda_1, \lambda_2, \lambda_3)| \quad \text{for any } \lambda_i \in S_i \quad (3.29)$$

is verified, we obtain the desired estimate (3.23) by applying Proposition 3.3 with $a = \langle \sigma \rangle^{-1} A^{-1/32}$.

Finally, we show (3.29). For $\lambda'_1 = \phi_{e_1}^-(\xi'_1)$, $\lambda'_2 = \phi_{e_2}^+(\xi'_2)$, $\lambda'_3 = (\psi(\xi'), \xi')$ and $\xi'_1 + \xi'_2 = \xi'$, a direct calculation shows that

$$|\det N(\lambda'_1, \lambda'_2, \lambda'_3)| \geq \langle \sigma \rangle^{-1} \frac{1}{\langle 2|\xi'_1| \rangle \langle 2|\xi'_2| \rangle} \left| \operatorname{det} \begin{pmatrix} 1 & 1 & -1 \\ \xi''_1(1) & -\xi''_2(1) & \sigma \xi''_1(1) \\ \xi''_1(2) & -\xi''_2(2) & \sigma \xi''_1(2) \end{pmatrix} \right|$$

$$\geq \frac{1}{8} \langle \sigma \rangle^{-1} \left| \frac{\xi''_1(1) - \xi''_1(2)}{\xi''_1(1) \xi''_1(2)} \right| \geq \frac{\langle \sigma \rangle^{-1} A^{-1}}{16}.$$ 

Combining this bound with (3.26)–(3.28), we obtain (3.29). This concludes the proof of Proposition 3.3. \(\square\)

1Strictly speaking, we need to construct a larger set $S_i^*$ and replace $S_i$ by $S_i^*$. However, since $S_i$ is a graph in this setting, this is a sight modification. Indeed, by setting $U_1 = C_{2N_1^{-1}}(N_1^{-1} \xi''_i)$ and $S_i^* = \{ \phi_{e_1}(\xi_1) \mid \xi_1 \in \mathbb{R}^2 \} \cap U_1$, we have $\operatorname{dist}(S_1, U_1^*) \geq 2N_1^{-1} \delta = 2A^{-1}$. Similarly, we can set $S_2^*$ and $S_3^*$. Moreover, since estimates with $S_i$ replaced by $S_i^*$ are similarly obtained, we omit the details.
4. Proof of bilinear estimates for \( d = 3 \)

In this section, we prove Proposition 2.2 for \( d = 3 \). We first give the operators with respect to angular variables introduced in [1].

Definition 3 ([1]). For each \( A \in \mathbb{N} \), \( \{\omega_A^j\}_{j \in \Omega_A} \) denotes a set of spherical caps of \( S^2 \) with the following properties:

(i) The angle \( \angle(x, y) \) between any two vectors in \( x, y \in \omega_A^j \) satisfies

\[
|\angle(x, y)| \leq A^{-1}.
\]

(ii) Characteristic functions \( \{1_{\omega_A^j}\} \) satisfy

\[
1 \leq \sum_{j \in \Omega_A} 1_{\omega_A^j}(x) \leq 3
\]

for any \( x \in S^2 \).

We define the function

\[
\alpha(j_1, j_2) = \inf \{|\angle(\pm x, y)| : x \in \omega_A^{j_1}, y \in \omega_A^{j_2}\},
\]

which measures the minimal angle between any two straight lines through the spherical caps \( \omega_A^{j_1} \) and \( \omega_A^{j_2} \), respectively. It is easily observed that for any fixed \( j_1 \in \Omega_A \) there exist only a finite number of \( j_2 \in \Omega_A \) which satisfies \( \alpha(j_1, j_2) \sim A^{-1} \).

Based on the above construction, for each \( j \in \Omega_A \), we define

\[
\mathcal{D}_j^A = \left\{(\tau, \xi) \in \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) : \frac{\xi}{|\xi|} \in \omega_A^j\right\}
\]

and the corresponding localization operator

\[
\mathcal{F}(R_j^A u)(\tau, \xi) = \frac{\chi_{\omega_A^j}(\frac{\xi}{|\xi|})}{\chi(\frac{\xi}{|\xi|})} \mathcal{F}u(\tau, \xi).
\]

The following estimates are three dimensional version of Propositions 3.1, 3.2, and 3.3.

**Proposition 4.1.** Let \( N_1, N_2, N_3, L_1, L_2, L_3, A \in 2^\mathbb{N}_0 \), and \( j_1, j_2 \in \Omega_A \). We assume \( A \geq 64 \), \( \alpha(j_1, j_2) \lesssim A^{-1} \), and \( N_3 \lesssim N_1 \sim N_2 \). Then, we have the following estimate:

\[
\|P_{N_3}(R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1 \cdot R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2)\|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \frac{N_1}{N_3^A} L_1^\frac{3}{2} L_2^\frac{1}{2} \|R_{j_1}^A Q_{L_1}^1 P_{N_1} u_1\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} \|R_{j_2}^A Q_{L_2}^{-1} P_{N_2} u_2\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)}. \tag{4.1}
\]
Proposition 4.2. Let $\sigma \in \mathbb{R}\setminus \{0, \pm 1\}$. Let $N_1$, $N_2$, $N_3$, $L_1$, $L_2$, $L_3$, $A \in 2^{\mathbb{N}_0}$, and $j_1, j_2 \in \Omega_A$. We assume $L_{\max} \ll N_{\max}^2$, $A \geq 64$, and $\alpha(j_1, j_2) \lesssim A^{-1}$. Then, we have the following estimate:

$$
\| R_{j_1L_1}^A Q_{L_1}^{-1} P_{N_1} (R_{j_2L_2}^A Q_{L_2}^{-1} P_{N_2} u_2 \cdot Q_{L_3}^{-1} P_{N_3} u_3) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \\
\lesssim A^{-\frac{3}{2}} N_{\max}^2 \frac{1}{L_2^3} \frac{1}{L_3^3} \| R_{j_2L_2}^A Q_{L_2}^{-1} P_{N_2} u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \| Q_{L_3}^{-1} P_{N_3} u_3 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)}.
$$

(4.2)

Proposition 4.3. Let $\sigma \in \mathbb{R}\setminus \{0, \pm 1\}$. Let $N_1$, $N_2$, $N_3$, $L_1$, $L_2$, $L_3$, $A \in 2^{\mathbb{N}_0}$, and $j_1, j_2 \in \Omega_A$. We assume $L_{\max} \ll N_{\max}^2$, $A \geq 64$, and $\alpha(j_1, j_2) \sim A^{-1}$. Then the following estimate holds:

$$
\| Q_{L_3}^\sigma P_{N_3} (R_{j_1L_1}^A Q_{L_1}^1 P_{N_1} u_1 \cdot R_{j_2L_2}^A Q_{L_2}^{-1} P_{N_2} u_2) \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \\
\lesssim N_{\max}^\frac{1}{2} L_1^\frac{1}{2} L_2^\frac{1}{2} L_3^\frac{1}{2} \| R_{j_1L_1}^A Q_{L_1}^1 P_{N_1} u_1 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \| R_{j_2L_2}^A Q_{L_2}^{-1} P_{N_2} u_2 \|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)}.
$$

If supp $f_i \subset \mathcal{D}_{j_i}^A$ $(i = 1, 2)$, after applying rotation in space and suitable decomposition, we may assume that the supports of $f_1$ and $f_2$ are both contained in the following slab

$$
\Sigma_3 (N_1 A^{-1}) := \{ (\tau, \tilde{\xi}, \xi^{(3)}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \mid |\xi^{(3)}| \leq N_1 A^{-1} \}.
$$

Set

$$
\psi_{N_3,L_3}^{\sigma, \tau} (\tau, \xi) = \psi_{L_3} (\tau - \sigma |\xi|^2) \psi_{N_3} (\xi),
$$

where $\psi$ is as in the notation at the end of Section IV. We claim that if

$$
\left\| \psi_{N_3,L_3}^{\sigma, \tau} (\tau, \tilde{\xi}, \xi^{(3)}) \int_{\mathbb{R} \times \mathbb{R}^2} f_1 (\tau_1, \tilde{\xi}_1, \xi^{(3)}_1) f_2 (\tau - \tau_1, \tilde{\xi} - \tilde{\xi}_1, \xi^{(3)} - \xi^{(3)}_1) d\tau_1 d\tilde{\xi}_1 \right\|_{L_t^2(\mathbb{R} \times \mathbb{R}^2)} \\
\lesssim K \| f_1 (\xi^{(3)}_1) \|_{L_{\tau\xi}^2} \| f_2 (\xi^{(3)} - \xi^{(3)}_1) \|_{L_{\tau\xi}^2}
$$

(4.3)

holds uniformly for $\xi^{(3)}$ and $\xi^{(3)}_1$ with $|\xi^{(3)} - \xi^{(3)}_1| \leq N_1 A^{-1}$ and $|\xi^{(3)}_1| \leq N_1 A^{-1}$, then we can obtain

$$
\left\| \psi_{N_3,L_3}^{\sigma, \tau} (\tau, \xi) \int_{\mathbb{R} \times \mathbb{R}^3} f_1 (\tau_1, \xi_1) f_2 (\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1 \right\|_{L_t^2(\mathbb{R} \times \mathbb{R}^3)} \\
\lesssim A^{-\frac{3}{2}} N_1^\frac{1}{2} K \| f_1 \|_{L_{\tau\xi}^2} \| f_2 \|_{L_{\tau\xi}^2}.
$$
Indeed, once (4.3) holds, from Minkowski’s inequality and Young’s inequality, we have

\[
\| \psi_{N_3, L_3}^p (\tau, \xi) \int_{\mathbb{R}^2} f_1 (\tau_1, \xi_1) f_2 (\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1 \|_{L^2_{\tau \xi}} 
\]

\[
= \| \psi_{N_3, L_3}^p (\tau, \tilde{\xi}, \xi^{(3)}) \times \int_{\mathbb{R}^2} f_1 (\tau_1, \tilde{\xi}_1, \xi^{(3)}) f_2 (\tau - \tau_1, \tilde{\xi} - \tilde{\xi}_1, \xi^{(3)} - \xi_1^{(3)}) d\tau_1 d\tilde{\xi}_1 \|_{L^2_{\tau \xi}} 
\]

\[
\lesssim K \left[ \int_{\mathbb{R}} \| f_1 (\xi_1^{(3)}) \|_{L^2_{\tau \xi}} \| f_2 (\xi^{(3)} - \xi_1^{(3)}) \|_{L^2_{\tau \xi}} d\xi_1^{(3)} \right] 
\]

\[
\lesssim K \left( \sup_{(\xi^{(3)})} \int_{\mathbb{R}} \| f_1 (\xi_1^{(3)}) \|_{L^2_{\tau \xi}} \| f_2 (\xi^{(3)} - \xi_1^{(3)}) \|_{L^2_{\tau \xi}} d\xi_1^{(3)} \right) 
\]

\[
\lesssim A^{-\frac{1}{2}} N_2^\frac{1}{2} K \| f_1 \|_{L^2_{\tau \xi}} \| f_2 \|_{L^2_{\tau \xi}}. 
\]

Therefore, to show Propositions 4.1, 4.2, 4.3 it suffices to prove (4.3) for

\[
K = \left( \frac{N_1}{N_3 A} \right)^\frac{1}{2} L_1^\frac{1}{2} L_2^\frac{1}{2}, \quad A^{-\frac{1}{2}} L_2^\frac{1}{2} L_3^\frac{1}{2}, \quad \text{and} \quad A^\frac{1}{2} N^{-1}_{\text{max}} L_1^\frac{1}{2} L_2^\frac{1}{2} L_3^\frac{1}{2}; 
\]

respectively. Since we can get these estimates by similar argument as the proof of Proposition 3.1, 3.2, and 3.3 (See, also [17]), we omit its proof.

Now, we give the proof of the bilinear estimates.

**Proof of Proposition 2.2 for d = 3.** Let $1 > s > \frac{1}{2}$ and

\[
(s_1, s_2, s_3) \in \{(s, s, -s), (s, -s, s), (-s, s, s)\}. 
\]

We prove (2.2). We set

\[
u_{N_1 L_1} := Q_{L_1}^1 P_{N_1} U, \quad v_{N_2 L_2} := Q_{L_2}^{-1} P_{N_2} V, \quad w_{N_3 L_3} := Q_{L_3}^{-\sigma} P_{N_3} W. 
\]
Then, we have
\[
\left| \int_{\mathbb{R} \times \mathbb{R}^3} U(t, x)V(t, x) \partial_j W(t, x) \, dx \, dt \right| \leq \sum_{N_1, N_2, N_3 \geq 1} \sum_{L_1, L_2, L_3 \geq 1} N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} \, dx \, dt \right|.
\]

By the same reason for \( d = 2 \), it suffices to show that
\[
N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} \, dx \, dt \right| \leq N_{\min}^s (L_1 L_2 L_3)^c \left( \frac{N_{\min}}{N_{\max}} \right)^\varepsilon \| u_{N_1, L_1} \|_{L_t^2}^2 \| u_{N_2, L_2} \|_{L_t^2}^2 \| u_{N_3, L_3} \|_{L_t^2}
\]
for some \( b' \in (0, \frac{1}{2}) \), \( c \in (0, b') \), and \( \varepsilon > 0 \).

Now, we prove (4.4).

Case 1: High modulation, \( L_{\max} \gtrsim N_{\max}^2 \)

We first assume \( L_3 = L_{\max} \). By the symmetry, we can assume \( N_1 \leq N_2 \). Then, by the Cauchy-Schwarz inequality and (2.3), we have
\[
N_3 \left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} \, dx \, dt \right| \leq N_3 \| P_{N_3} (u_{N_1, L_1} u_{N_2, L_2}) \|_{L_t^2} \| u_{N_3, L_3} \|_{L_t^2}
\]
\[
\quad \leq N_3 \frac{1}{N_1^{1/4} + \delta} \left( \frac{N_1}{N_2} \right)^\delta L_2^5 L_5^2 \| u_{N_1, L_1} \|_{L_t^2} \| u_{N_2, L_2} \|_{L_t^2} \| u_{N_3, L_3} \|_{L_t^2}
\]
\[
\quad \leq N_3 \frac{1}{N_1^{1/4} + \delta} \left( \frac{N_1}{N_2} \right)^\delta N_{\max}^{-2c} (L_1 L_2 L_3)^c \times \| u_{N_1, L_1} \|_{L_t^2} \| u_{N_2, L_2} \|_{L_t^2} \| u_{N_3, L_3} \|_{L_t^2},
\]
where \( \delta = \frac{1}{2} - c \). If \( N_3 \lesssim N_1 \sim N_2 \), then we obtain
\[
N_3 \frac{1}{N_1^{1/4} + \delta} \left( \frac{N_1}{N_2} \right)^\delta \sim N_{\max}^{-2c} N_3^{\frac{1}{2} - 6c - s} N_3^{s} \left( \frac{N_2}{N_1} \right)^{6c - s}. \]
If \( N_1 \lesssim N_2 \sim N_3 \), then we obtain
\[
N_3 \frac{1}{N_1^{1/4} + \delta} \left( \frac{N_1}{N_2} \right)^\delta \sim N_{\max}^{-2c} N_3^{\frac{1}{2} - 6c - s} N_3^{s} \left( \frac{N_1}{N_3} \right)^{4c - s}. \]
Therefore, by choosing \( b' \) and \( c \) as \( \max \{ \frac{1}{2} (\frac{7}{2} - s), \frac{5}{12}, \frac{3}{8} \} < c < b' < \frac{1}{2} \) for \( s > \frac{1}{2} \), we get (4.4). The case \( L_1 = L_{\max} \) and \( L_2 = L_{\max} \) is similarly handled, but we use (2.4) instead of (2.3).
Case 2: Low modulation, $L_{\text{max}} \ll N_{\text{max}}^2$

By Lemma 2.7, we can assume $N_3 \lesssim N_1 \sim N_2$. We set

$$M = N_1^3 N_3^{-s} \quad (4.5)$$

and decompose $\mathbb{R}^4 \times \mathbb{R}^4$ as follows:

$$\mathbb{R}^4 \times \mathbb{R}^4 = \left( \bigcup_{j_1, j_2 \in \Omega_M} \mathcal{D}_j^{M} \times \mathcal{D}_j^{M} \right) \cup \left( \bigcup_{64 \leq A \leq M} \bigcup_{j_1, j_2 \in \Omega_A} \mathcal{D}_j^{A} \times \mathcal{D}_j^{A} \right).$$

We can write

$$\left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1} v_{N_2, L_2} w_{N_3, L_3} dx \right|$$

$$\leq \sum_{j_1, j_2 \in \Omega_M} \left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1, j_1} v_{N_2, L_2, j_2} w_{N_3, L_3} dx \right|$$

$$+ \sum_{64 \leq A \leq M} \sum_{j_1, j_2 \in \Omega_A} \left| \int_{\mathbb{R} \times \mathbb{R}^3} u_{N_1, L_1, j_1} v_{N_2, L_2, j_2} w_{N_3, L_3} dx \right|$$

$$=: I + II, \quad (4.6)$$

where

$$u_{N_1, L_1, j_1} = R_{j_1} u_{N_1, L_1}, \quad v_{N_2, L_2, j_2} = R_{j_2} v_{N_2, L_2}. \quad (4.6)$$

For the first term $I$ in (4.6), we first assume $L_{\text{max}} = L_3$. By (4.6), the Hölder inequality, (4.1), and (4.5), we get

$$N_3 \cdot I \lesssim \sum_{j_1, j_2 \in \Omega_M} \frac{N_3 \| P_N (u_{N_1, L_1, j_1} v_{N_2, L_2, j_2}) \|_{L_{t,x}}^2 \| w_{N_3, L_3} \|_{L_{t,x}^2}}{N_1^2 M}$$

$$\lesssim \sum_{j_1, j_2 \in \Omega_M} \frac{N_3}{N_1^2 M} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \| u_{N_1, L_1, j_1} \|_{L_{t,x}^2} \| v_{N_2, L_2, j_2} \|_{L_{t,x}^2} \| w_{N_3, L_3} \|_{L_{t,x}^2}$$

$$\lesssim N_3 \left( \frac{N_3}{N_1} \right)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \| u_{N_1, L_1} \|_{L_{t,x}^2} \| v_{N_2, L_2} \|_{L_{t,x}^2} \| w_{N_3, L_3} \|_{L_{t,x}^2}$$

$$\lesssim N_3 \left( \frac{N_3}{N_1} \right)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{2}} \| u_{N_1, L_1} \|_{L_{t,x}^2} \| v_{N_2, L_2} \|_{L_{t,x}^2} \| w_{N_3, L_3} \|_{L_{t,x}^2},$$

which shows (4.4). If $L_{\text{max}} = L_1$ or $L_{\text{max}} = L_2$, then we use (4.2) instead of (4.1).
We also define the iteration terms for (1.1) as follows:

\[ k \text{ and } s < \]

\[ \text{Proposition 5.1.} \]

\[ \text{Let } \]

\[ \text{conclude the proof of Proposition 2.2 for } \]

\[ d = 28 \text{ H. HIRAYAMA, S. KINOSHITA, AND M. OKAMOTO} \]

\[ \text{get } \]

\[ M \]

\[ \text{here we used } \log \]

\[ \sigma \]

\[ \text{The following proposition implies Theorem 1.1.} \]

\[ \text{The lack of the third differentiability of the flow map} \]

\[ C > \]

\[ N \]

\[ \text{it} \]

\[ \text{∂} \]

\[ \text{∂} \]

\[ \sigma \]

\[ \text{define the Duhamel integral operator } \mathcal{I}_\sigma \text{ as} \]

\[ \mathcal{I}_\sigma(f)(t) := \int_0^t e^{i(t-t')\sigma \partial_x^2} f(t') dt'. \]  

\[ (5.1) \]

\[ \text{We also define the iteration terms for (1.1) as follows:} \]

\[ u^{(1)}(t) := e^{it\alpha \partial_x^2} u_0, \quad v^{(1)}(t) := e^{it\beta \partial_x^2} v_0, \quad w^{(1)}(t) := e^{it\gamma \partial_x^2} w_0, \]

\[ u^{(k)}(t) := i \sum_{k_1, k_2 \in \mathbb{N}} \mathcal{I}_\alpha (\partial_x u^{(k_1)} \cdot v^{(k_2)})(t), \]

\[ v^{(k)}(t) := i \sum_{k_1, k_2 \in \mathbb{N}} \mathcal{I}_\beta (\partial_x \overline{u^{(k_1)} \cdot u^{(k_2)}})(t), \]

\[ w^{(k)}(t) := -i \sum_{k_1, k_2 \in \mathbb{N}} \mathcal{I}_\gamma (\partial_x (u^{(k_1)} \overline{v^{(k_2)}}))(t) \]

\[ (5.2) \]

\[ \text{for any integer } k \text{ greater than 1.} \]

\[ \text{The following proposition implies Theorem 1.1.} \]

\[ \text{Proposition 5.1. Let } d = 1, 0 < T \ll 1, \text{ and let } \mu \text{ be as in (1.2). Assume } \mu > 0 \]

\[ \text{and } s < 0. \text{ For every } C > 0, \text{ there exist } u_0, v_0, w_0 \in H^s(\mathbb{R}) \text{ such that} \]

\[ \sup_{0 \leq t \leq T} \| u^{(3)}(t) \|_{H^s} \geq C(\| u_0 \|_{H^s} \| w_0 \|_{H^s}^2 + \| u_0 \|_{H^s} \| v_0 \|_{H^s}^2). \]

\[ (5.3) \]
Proof. We set $v_0 = 0$. Then, it follows from (5.2) that

$$v^{(1)} = u^{(2)} = w^{(2)} = v^{(3)} = 0,$$

$$v^{(2)} = i\bar{\mathcal{I}}_\beta(\partial_x w^{(1)} \cdot u^{(1)}),$$

$$w^{(3)} = i\bar{\mathcal{I}}_\alpha(\partial_x w^{(1)} \cdot v^{(2)}), \quad w^{(3)} = -i\bar{\mathcal{I}}_\gamma(\partial_x (u^{(1)} \overline{v^{(2)}})).$$

(5.4)

For $\xi, \eta \in \mathbb{R}$, we set

$$\Phi(\xi, \eta) := \alpha \xi^2 - \beta (\xi - \eta)^2 - \gamma \eta^2.$$

By (5.1) and (5.4), we have

$$\widehat{v^{(2)}}(t', \eta) = i \int_0^t e^{-i(t' - t'')\beta \eta^2} \int_\mathbb{R} i\eta_1 \overline{w^{(1)}}(t'', \eta_1) \widehat{u^{(1)}}(t'', \eta - \eta_1) d\eta_1 dt''$$

$$= -e^{-it' \beta \eta^2} \int_\mathbb{R} \left( \int_0^{t'} e^{-it'' \Phi(\eta - \eta_1 - \eta_2)} dt'' \right) \eta_1 \overline{u_0}(\eta - \eta_1) \overline{w_0}(\eta - \eta_1) d\eta_1$$

$$= e^{-it' \beta \eta^2} \int_\mathbb{R} \left( \int_0^{t'} e^{-it'' \Phi(\eta + \xi_2, \xi_2)} dt'' \right) \xi_2 \overline{u_0}(\eta + \xi_2) \overline{w_0}(\eta) d\xi_2.$$

Moreover, we obtain

$$\widehat{u^{(3)}}(t, \xi) = i \int_0^t e^{-i(t' - t')\alpha \xi^2} \int_\mathbb{R} i\xi_1 \overline{w^{(1)}}(t', \xi_1) \widehat{v^{(2)}}(t', \xi - \xi_1) d\xi_1 dt'$$

$$= -e^{-it\alpha \xi^2} \int_\mathbb{R} \left( \int_0^t e^{-it'' \Phi(\xi, \xi_1)} \int_0^{t'} e^{-it'' \Phi(\xi - \xi_1 + \xi_2, \xi_2)} dt'' dt' \right)$$

$$\times \xi_1 \xi_2 \overline{u_0}(\xi_1) \overline{w_0}(\xi_2) \overline{u_0}(\xi - \xi_1 + \xi_2) d\xi_1 d\xi_2.$$

(5.5)

Now, we give the initial data $u_0$ and $w_0$ as

$$\widehat{u_0}(\xi) = N^{-s} 1_{[kN, kN + 1]}(\xi), \quad \widehat{w_0}(\xi) = N^{-s} 1_{[N, N + 1]}(\xi),$$

(5.6)

where $k$ is a constant will be chosen later. Then, we have

$$\|u_0\|_{H^s} \sim \|w_0\|_{H^s} \sim 1.$$  

(5.7)

By choosing $k$ appropriately, we have

$$|\Phi(\xi - \xi_1 + \xi_2, \xi_2)| \sim N^2, \quad |\Phi(\xi, \xi_1)| \sim N^2, \quad |\Psi(\xi, \xi_1, \xi_2)| \lesssim 1$$

(5.8)

for $\xi_1, \xi_2 \in [N, N + 1]$ and $\xi - \xi_1 + \xi_2 \in [kN, kN + 1]$, where

$$\Psi(\xi, \xi_1, \xi_2) := \Phi(\xi, \xi_1) - \Phi(\xi - \xi_1 + \xi_2, \xi_2).$$

(5.9)
Here, we prove (5.3) by assuming (5.8). It follows from (5.5) that
\[ |\hat{w}_3(t, \xi)| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t e^{it'\Phi(\xi,\xi_1,\xi_2)} \Phi(\xi - \xi_1 + \xi_2, \xi_2) dt' \xi_1 \xi_2 \right| \\
\times \hat{w}_0(\xi_1)\hat{w}_0(\xi_2) \hat{u}_0(\xi - \xi_1 + \xi_2) d\xi d\xi_2 \\
\geq N^{2-3s} \mathbf{1}_{[kN, kN+\frac{1}{2}]}(\xi). \]
Hence, (5.8) yields that
\[ \|I\|_{H^s} \gtrsim t N^{-2} N^{2-3s} \|\langle \xi \rangle^{s} \mathbf{1}_{[kN, kN+\frac{1}{2}]}(\xi)\|_{L^2} \sim t N^{-2s} \]
for $0 < t \ll 1$. On the other hand, by (5.8), Young's inequality, and (5.6), we have
\[ \|II\|_{H^s} \leq N^s N^{-2} \|\hat{w}_0 \ast \hat{w}_0 \ast \hat{u}_0\|_{L^2} \lesssim N^s N^{-2} \|\hat{w}_0\|_{L^1}^2 \|u_0\|_{L^2} \sim N^{-2s-2}. \]
Therefore, we obtain
\[ \|u_3(t)\|_{H^s} \gtrsim t N^{-2s} - N^{-2s-2} \sim t N^{-2s} \] (5.10)
provided that $t \gg N^{-2}$. Then, (5.3) follows from (5.7), (5.10), and $s < 0$.

It remains to prove (5.8) by choosing $k$ appropriately. If $\xi_1 = N + \varepsilon_1, \xi_2 = N + \varepsilon_2,$ and $\xi = kN + \varepsilon$ for $0 \leq \varepsilon, \varepsilon_1, \varepsilon_2 \leq 1$, then (5.9) yields that
\[ \Psi(\xi, \xi_1, \xi_2) = \alpha \xi^2 - \gamma \xi^2 - \alpha (\xi - \xi_1 + \xi_2)^2 + \gamma \xi_2^2 \\
= (\xi_1 - \xi_2) \{2\alpha \xi - (\alpha + \gamma)\xi_1 + (\alpha - \gamma)\xi_2\} \\
= (\varepsilon_1 - \varepsilon_2) \{2(\alpha k - \gamma)N + 2\alpha \varepsilon - (\alpha + \gamma)\varepsilon_1 + (\alpha - \gamma)\varepsilon_2\} \] (5.11)
Therefore, if we choose $k$ as
\[ k = \frac{\gamma}{\alpha}, \]

\[ k \]
then the third condition (5.8) holds. Furthermore, we have

$$|\Phi(\xi - \xi_1 + \xi_2, \xi_2)| = |\alpha(\xi - \xi_1 + \xi_2)^2 - \beta(\xi - \xi_1)^2 - \gamma \xi_2^2|$$

$$\sim |\alpha k^2 - \beta(k - 1)^2 - \gamma|N^2$$

$$= \left|\frac{(\alpha - \gamma)\mu}{\alpha^2}\right|N^2,$$

where $\mu$ is as in (1.2). Since the assumption $\mu > 0$ implies $\alpha \neq \gamma$, (5.12) shows the first condition in (5.8).

Finally, it follows from (5.9), (5.11), and (5.12) that

$$|\Phi(\xi, \xi_1)| = |\Psi(\xi, \xi_1, \xi_2) + \Phi(\xi - \xi_1 + \xi_2, \xi_2)| \sim N^2.$$

We therefore obtain (5.8). This concludes the proof of Proposition 5.1.

ACKNOWLEDGEMENTS

This work was supported by JSPS KAKENHI Grant Numbers JP17K14220 and JP20K14342, Program to Disseminate Tenure Tracking System from the Ministry of Education, Culture, Sports, Science and Technology, and the DFG through the CRC 1283 ”Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications”.

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(H. Hirayama) Institute for Tenure Track Promotion, University of Miyazaki, 1-1, Gakuenkibanadai-nishi, Miyazaki, 889-2192 Japan
E-mail address, H. Hirayama: h.hirayama@cc.miyazaki-u.ac.jp

(S. Kinoshita) Universität Bielefeld, Fakultät für Mathematik, Postfach 10 01 31 33501, Bielefeld, Germany
E-mail address, S. Kinoshita: kinoshita@math.uni-bielefeld.de

(M. Okamoto) Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address, M. Okamoto: okamoto@math.sci.osaka-u.ac.jp