SET-THEORETICAL PROBLEMS IN ASYMPTOLOGY

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Dedicated to the memory of Kenneth Kunen

Abstract. In this paper we collect some open set-theoretic problems that appear in the large-scale topology (called also Asymptology). In particular, we ask problems on: (i) critical cardinalities of some special (large, indiscrete, inseparable) coarse structures on $\omega$, (ii) the interplay between properties of a coarse space and its Higson corona, (iii) some special ultrafilters ($T$-points and cellular $T$-points) related to finitary coarse structures on $\omega$, (iv) partitions of coarse spaces into thin pieces, and (v) coarse groups having various extremal properties.

Introduction

In this paper we collect selected open problems in the large-scale topology that have set-theoretic flavor. The large-scale topology (or else Asymptology \cite{37}) studies properties of coarse spaces. Coarse spaces defined in term of balls were introduced under name balleans in \cite{29} independently and simultaneously with \cite{40}. The necessary preliminary information on coarse spaces is collected in Section 1. More information related to large-scale topology can be found in the monographs \cite{29}, \cite{37}, \cite{40}. In Section 2 we pose some open problems on the critical cardinalities of some special (large, inseparable, indiscrete) coarse structures on $\omega$ and in Section 3 we investigate the interplay between properties of a coarse space and its Higson corona. Section 4 is devoted to some special ultrafilters on $\omega$, and Section 5 to the interplay between coarse properties of an action of a group $G \subseteq S_\omega$ on $\omega$ and the induced action of $G$ on the compact Hausdorff space $\omega^* = \beta\omega \setminus \omega$. In Section 6 we study partitions of coarse spaces into finitely many thin pieces and in the final Section 7 we collect some questions about coarse groups with various extremal properties.

1. Coarse structures and coarse spaces

Coarse structures can be introduced via balls as in \cite{29}, \cite{37} or via entourages as in \cite{40}. An entourage on a set $X$ is any subset $E$ of the square $X \times X$ that contains the diagonal $\Delta_X := \{(x, x) : x \in X\}$ of $X \times X$. For two entourages $E, F$, the sets

$E^{-1} = \{(y, x) : (x, y) \in E\}$ and $EF = \{(x, z) : \exists y \in X \ (x, y) \in E \land (y, z) \in F\}$

are entourages on $X$.

For an entourage $E$ on $X$, a point $x \in X$ and a set $A \subseteq X$, the sets

$E(x) := \{y \in X : (x, y) \in E\}$ and $E[A] = \bigcup_{a \in A} E(a)$

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are called the $E$-balls around $x$ and $A$, respectively.

An entourage $E$ is called

- **symmetric** if $E = E^{-1}$;
- **cellular** if $E^{-1} = E = EE$ (i.e., $E$ is an equivalence relation on $X$);
- **locally finite** if for any $x \in X$ the set $E^\pm(x) := E(x) \cup E^{-1}(x)$ is finite;
- **finitary** if the cardinal $\sup_{x \in X} |E^\pm(x)|$ is finite;
- **trivial** if the set $E \setminus \Delta_X$ is finite.

It is clear that each cellular entourage is symmetric, each trivial entourage is finitary, and each finitary entourage is locally finite.

A coarse structure on a set $X$ is any family $\mathcal{E}$ of entourages on $X$ satisfying the following axioms:

(A1) for any $E, F \in \mathcal{E}$, the entourage $EF^{-1}$ is contained in some entourage $G \in \mathcal{E}$;
(A2) $\bigcup \mathcal{E} = X \times X$;
(A3) for any entourages $E \subseteq F$ on $X$ the inclusion $F \in \mathcal{E}$ implies $E \in \mathcal{E}$.

A subfamily $\mathcal{B} \subseteq \mathcal{E}$ is called a base of the coarse structure $\mathcal{E}$ if each entourage $E \in \mathcal{E}$ is contained in some entourage $B \in \mathcal{B}$. Any base of a coarse structure satisfies the axioms (A1), (A2). On the other hand, any family of entourages $\mathcal{B}$ satisfying the axioms (A1), (A2) is a base of the unique coarse structure $\downarrow \mathcal{B} = \bigcup_{B \in \mathcal{B}} \{E \subseteq X \times X : \Delta_X \subseteq E \subseteq B\}$.

For a coarse structure $\mathcal{E}$ on a set $X$, its weight $w(\mathcal{E})$ is defined as the smallest cardinality of a base of $\mathcal{E}$.

A coarse structure $\mathcal{E}$ on a set $X$ is called

- **bounded** if $X \times X \in \mathcal{E}$;
- **locally finite** if each entourage $E \in \mathcal{E}$ is locally finite;
- **finitary** if each entourage $E \in \mathcal{E}$ is finitary;
- **cellular** if $\mathcal{E}$ has a base consisting of cellular entourages.

Each set $X$ carries the smallest coarse structure consisting of all trivial entourages on $X$. The smallest coarse structure is finitary and cellular. The family $\mathcal{E}_{<\omega}(X)$ (resp. $\mathcal{E}_{\omega}(X)$) of all finitary (resp. locally finite) entourages on $X$ is the largest finitary (resp. locally finite) coarse structure on $X$.

Each action $\alpha : G \times X \to X$ of a group $G$ on a set $X$ induces the finitary coarse structure $\mathcal{E}_G$ on $X$, which is generated by the base consisting of the entourages

$$(L \times L) \cup \{(x, y) \in X \times X : y \in Fx\}$$

where $L$ is a finite subset of $X$ and $F$ is a finite subset of the group $G$ that contain the identity $1_G$ of $G$. The action of the permutation group $S_X$ on $X$ induces the largest finitary coarse structure $\mathcal{E}_{S_X} = \mathcal{E}_{<\omega}(X)$ on a set $X$. The trivial subgroup $\{1_G\}$ induces the smallest finitary coarse structure on $X$.

The following fundamental fact was proved by Protasov in [21] (see also [31] and [27]).

**Theorem 1.1.** Every finitary coarse structure on a set $X$ coincides with the finitary coarse structure $\mathcal{E}_G$ induced by the action of some subgroup $G \subseteq S_X$. 
Every metric space \((X, d)\) carries the canonical coarse structure \(\mathcal{E}_d\), which is generated by the base

\[
\{ \{(x, y) \in X \times X : d(x, y) < \varepsilon \} : 0 < \varepsilon < \infty \}.
\]

A coarse structure \(\mathcal{E}\) on a set \(X\) is called \textit{metrizable} if \(\mathcal{E} = \mathcal{E}_d\) for some metric \(d\) on \(X\).

The following metrizability theorem can be found in [37, 2.1.1].

**Theorem 1.2.** A coarse structure on a set is metrizable if and only if it has a countable base.

A \textit{coarse space} is a pair \((X, \mathcal{E})\) consisting of a set \(X\) and a coarse structure \(\mathcal{E}\) on \(X\). A coarse space is called \textit{metrizable} (resp. \textit{bounded}, \textit{cellular}, \textit{locally finite}, \textit{finitary}) if so is its coarse structure \(\mathcal{E}\).

Let \((X, \mathcal{E})\) be a coarse space. A subset \(B \subseteq X\) is called \textit{bounded} if \(B \subseteq E(x)\) for some \(E \in \mathcal{E}\) and \(x \in X\). The family of all bounded subsets in a coarse space is called the \textit{bornology} of the coarse space.

For two coarse spaces \((X, \mathcal{E}_X)\) and \((Y, \mathcal{E}_Y)\), a function \(f : X \to Y\) is called

- \textit{macro-uniform} if for any entourage \(E_X \in \mathcal{E}_X\) there exists an antourage \(E_Y \in \mathcal{E}_Y\) such that \(\{(f(x), f(x')) : (x, x') \in E_X\} \subseteq E_Y\);
- an \textit{asymorphism} if \(f\) is bijective and both maps \(f\) and \(f^{-1}\) are macro-uniform;
- a \textit{coarse equivalence} if \(f\) is macro-uniform and there exists a macro-uniform map \(g : Y \to X\) such that \(\{(g \circ f(x), x) : x \in X\} \subseteq E_X\) and \(\{(f \circ g(y), y) : y \in Y\} \subseteq E_Y\) for some entourages \(E_X \in \mathcal{E}_X, E_Y \in \mathcal{E}_Y\).

**Theorem 1.3** ([5]). For any countable locally finite groups \(G, H\), the finitary coarse spaces \((G, \mathcal{E}_G)\) and \((H, \mathcal{E}_H)\) are coarsely equivalent.

2. **Critical cardinalities related to some special coarse structures on \(\omega\)**

In this section we consider some uncountable cardinals that appear as smallest weights of coarse structures on \(\omega\) that possess some (pathological) properties. First we recall the necessary information on cardinal characteristics of the continuum (see [7, 10, 44] for more details).

For a set \(X\), its cardinality is denoted by \(|X|\). Cardinals are identified with the smallest ordinals of a given cardinality. We denote by \(\omega\) and \(\omega_1\) the smallest infinite and the smallest uncountable cardinals, respectively. By \(c\) we denote the cardinality of the real line. For a set \(X\), let \([X]^{<\omega}\) be the family of all finite subsets of \(X\), and \([X]^{\omega}\) be the family of all countably infinite subsets of \(X\).

For two sets \(A, B\), we write \(A \subseteq^* B\) if \(A \setminus B\) is finite. For two functions \(f, g : \omega \to \omega\) we write \(f \leq g\) if \(f(n) \leq g(n)\) for all \(n \in \omega\), and \(f \leq^* g\) if the set \(\{n \in \omega : f(n) \not\leq g(n)\}\) is finite.

Now we recall the definitions of some small uncountable cardinals, namely:

- \(b = \min\{|B| : B \subseteq \omega^\omega \land \forall f \in \omega^\omega \exists g \in B \ g \not\leq^* f\}\);
- \(d = \min\{|D| : D \subseteq \omega^\omega \land \forall f \in \omega^\omega \exists g \in D \ f \leq g\}\);
- \(r = \min\{|R| : R \subseteq [\omega]^\omega \land \forall a \in [\omega]^\omega \exists r \in R \ (r \subseteq^* a \lor r \subseteq^* \omega \setminus a)\}\);
- \(s = \min\{|S| : S \subseteq [\omega]^\omega \land \forall a \in [\omega]^\omega \exists s \in S \ (a \not\subseteq^* s \land a \not\subseteq^* \omega \setminus s)\}\);
- \(t = \min\{|T| : T \subseteq [\omega]^\omega \land (\forall s, t \in T \ s \subseteq^* t \lor t \subseteq^* s) \text{ and } (\forall s \in [\omega]^\omega \exists t \in T \ s \not\subseteq^* t)\}\);
- \(u\) is the smallest cardinality of a base of a free ultrafilter on \(\omega\).

The order relations between these cardinals are described by the following diagram (see [7, 10, 44]), in which for two cardinals \(\kappa, \lambda\) the arrow \(\kappa \to \lambda\) indicates that \(\kappa \leq \lambda\) in ZFC.
Each family of sets $\mathcal{I}$ with $\bigcup \mathcal{I} / \notin \mathcal{I}$ has four basic cardinal characteristics:

- $\text{add}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \land \bigcup A / \notin \mathcal{I}\}$;
- $\text{cov}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \land \bigcup A = \bigcup \mathcal{I}\}$;
- $\text{non}(\mathcal{I}) = \min\{|A| : A \subseteq \bigcup \mathcal{I} \land A / \notin \mathcal{I}\}$;
- $\text{cof}(\mathcal{I}) = \min\{|J| : J \subseteq \mathcal{I} \land \forall I \in \mathcal{I} \exists J \in J (I \subseteq J)\}$.

These cardinal characteristics are usually considered for the $\sigma$-ideals $\mathcal{M}$ and $\mathcal{N}$ of meager sets and Lebesgue null sets on the real line, respectively. The cardinal characteristics of the $\sigma$-ideals $\mathcal{M}$ and $\mathcal{N}$ are described by the famous Cichoń diagram:

Now let us return to coarse spaces and introduce some properties of coarse structures that will lead us to some new cardinal characteristics of the continuum.

Let $\mathcal{E}$ be a coarse structure on a set $X$. A subset $A \subseteq X$ is called
- $\mathcal{E}$-bounded if $A \subseteq E(x)$ for some $E \in \mathcal{E}$ and $x \in X$;
- $\mathcal{E}$-unbounded if $A$ is not $\mathcal{E}$-bounded;
- $\mathcal{E}$-large if $E[A] = X$ for some $E \in \mathcal{E}$;
- $\mathcal{E}$-discrete (or else $\mathcal{E}$-thin) if for every $E \in \mathcal{E}$ there exists an $\mathcal{E}$-bounded set $B \subseteq A$ such that $A \cap E(a) = \{a\}$ for every $a \in A \setminus B$.

Two subsets $A, B \subseteq X$ are called
- $\mathcal{E}$-close if there exists an entourage $E \in \mathcal{E}$ such that $A \subseteq E[B]$ and $B \subseteq E[A]$;
- $\mathcal{E}$-separated if for every entourage $E \in \mathcal{E}$ the intersection $E[A] \cap E[B]$ is $\mathcal{E}$-bounded.

A coarse structure $\mathcal{E}$ on a set $X$ is called
- indiscrete if each $\mathcal{E}$-discrete set in $X$ is $\mathcal{E}$-bounded;
- inseparated if any $\mathcal{E}$-unbounded sets $A, B \subseteq X$ are not $\mathcal{E}$-separated;
- large if any $\mathcal{E}$-unbounded set in $X$ is $\mathcal{E}$-large.

A coarse space is called indiscrete (resp. inseparated, large) if so is its coarse structure.

For any infinite locally finite coarse space we have the implications:

large $\Rightarrow$ inseparated $\Rightarrow$ indiscrete $\Rightarrow$ nonmetrizable.

The largest finitary coarse structure on $\omega$ is large and hence inseparated and indiscrete. Using indiscrete, inseparated or large coarse structures we can characterize certain known cardinal characteristics of the continuum.

Theorem 2.1 ([1]). The cardinal

1. $b$ is equal to the smallest weight of an indiscrete locally finite coarse structure on $\omega$;
(2) \( b \) is equal to the smallest weight of an inseparated locally finite coarse structure on \( \omega \);
(3) \( \vartheta \) is equal to the smallest weight of a large locally finite coarse structure on \( \omega \);
(4) \( \varepsilon \) is equal to the smallest weight of a large finitary coarse structure on \( \omega \).

The smallest weights of indiscrete (or inseparated) finitary coarse structures on \( \omega \) determine new cardinal characteristics of the continuum, which were introduced in [1]:

- \( \Delta \) is the smallest weight of an indiscrete finitary coarse structure on \( \omega \);
- \( \Sigma \) is the smallest weight of an inseparated finitary coarse structure on \( \omega \).

The relation of these two cardinals to other characteristics of the continuum are described in the following theorem, proved in [1]:

**Theorem 2.2** ([1]). \( \max\{b, s, \text{cov}(N)\} \leq \Delta \leq \Sigma \leq \text{non}(M) \).

**Problem 2.3** ([1]). Are the strict inequalities \( \max\{b, s, \text{cov}(N)\} < \Delta < \Sigma < \text{non}(M) \) consistent?

The cardinal \( \Delta \) and \( \Sigma \) have dual versions defined as

\[
\hat{\Delta} = \min\{|A| : A \subseteq [\omega]^\omega \land \forall E \in E_{<\omega} \exists A \in A \ (A \text{ is } \{E\}-\text{discrete})\};
\]

\[
\hat{\Sigma} = \min\{|A| : A \subseteq [\omega]^\omega \land \forall E \in E_{<\omega} \exists A, B \in A \ (A, B \text{ are } \{E\}-\text{separated})\}.
\]

For the cardinals \( \Delta \) and \( \Sigma \) we have lower and upper bounds, which are dual to those from Theorem 2.2.

**Theorem 2.4** ([1]). \( \text{cov}(M) \leq \hat{\Sigma} \leq \hat{\Delta} \leq \min\{\vartheta, \tau, \text{non}(N)\} \).

**Problem 2.5** ([1]). Are the strict inequalities \( \text{cov}(M) < \hat{\Sigma} < \hat{\Delta} \leq \min\{\vartheta, \tau, \text{non}(N)\} \) consistent?

The cardinal characteristics \( \Delta, \Sigma, \hat{\Delta}, \hat{\Sigma} \) nicely fit into the following enriched version of the famous Cichoń diagram, see [1].

\[
\begin{array}{cccc}
\Sigma & \text{non}(M) & \text{cov}(M) & \text{cov}(N) \\
\Delta & s & \vartheta & c \\
\text{cov}(N) & \omega_1 & b \\
\text{add}(N) & \text{add}(M) & \text{cov}(M) & \hat{\Sigma} \\
\end{array}
\]

Considering critical cardinalities of cellular coarse structures on \( \omega \), we obtain six cardinals:

\[
\begin{align*}
\Delta_{\omega}^c &= \min\left(\{c^+\} \cup \{w(\mathcal{E}) : \mathcal{E} \text{ is an indiscrete cellular finitary coarse structure on } \omega\}\right); \\
\Sigma_{\omega}^c &= \min\left(\{c^+\} \cup \{w(\mathcal{E}) : \mathcal{E} \text{ is an inseparated cellular finitary coarse structure on } \omega\}\right); \\
\Lambda_{\omega}^c &= \min\left(\{c^+\} \cup \{w(\mathcal{E}) : \mathcal{E} \text{ is a large cellular finitary coarse structure on } \omega\}\right); \\
\Delta_{\omega} &= \min\left(\{c^+\} \cup \{w(\mathcal{E}) : \mathcal{E} \text{ is an indiscrete cellular locally finite coarse structure on } \omega\}\right);
\end{align*}
\]
\[\Sigma^o_\omega = \min \left( \{ c^+ \} \cup \{ w(E) : E \text{ is an inseparable cellular locally finite coarse structure on } \omega \} \right) ;
\]
\[\Lambda^o_\omega = \min \left( \{ c^+ \} \cup \{ w(E) : E \text{ is a large cellular locally finite coarse structure on } \omega \} \right) ;\]

The cardinal \( c^+ \) appears in the definitions of those cardinals in order to make these cardinals well-defined (this indeed is necessary for the cardinal \( \Lambda^o_\omega \), which is equal to \( c^+ \) in ZFC).

The following diagram (taken from \[1\]) describes all known order relations between the cardinals \( \Delta^o_{<\omega}, \Sigma^o_{<\omega}, \Lambda^o_{<\omega}, \Delta^o_\omega, \Sigma^o_\omega, \Lambda^o_\omega \) and the cardinals \( t, b, d, \Delta, \Sigma, c \). For two cardinals \( \alpha, \beta \) an arrow \( \alpha \to \beta \) (without label) indicates that \( \alpha \leq \beta \) in ZFC. A label at an arrow indicates the assumption under which the corresponding inequality holds.

Non-trivial arrows at this diagram follow from Theorem 2.2 and the following theorem taken from \[1\].

**Theorem 2.6** \([1]\).

1. \( \Lambda^o_{<\omega} = c^+ \).
2. \( \Delta^o_{<\omega} \leq c \).
3. \( \Delta^o_{<\omega} = c \) implies \( \Sigma^o_\omega \leq \Sigma^o_\omega = c \).
4. \( t = b \) implies \( \Delta^o_\omega = \Sigma^o_\omega = b \).
5. \( t = d \) implies \( \Lambda^o_\omega = d \).

This theorem raises many set-theoretic questions.

**Problem 2.7** \([1]\). Is \( \Sigma^o_{<\omega} \leq c \) in ZFC? This is equivalent to asking if there exists an inseparable cellular finitary coarse structure on \( \omega \) in ZFC?

Problem 2.7 has an affirmative answer if the following problem has a negative answer.

**Problem 2.8** \([1]\). Is \( \Delta^o_{<\omega} < c \) consistent?

**Problem 2.9** \([1]\). Does there exist a large cellular locally finite coarse structure on \( \omega \) in ZFC? If yes, is \( d \) equal to the smallest weight of a large cellular locally finite coarse structure on \( \omega \)?

**Problem 2.10** \([1]\). Is \( b \) equal to the smallest weight of an indiscrete cellular locally finite coarse structure on \( \omega \)?

Two coarse structures \( E, E' \) on a set \( X \) are called

- \( \delta \)-equivalent if two subsets \( A, B \subseteq X \) are \( E \)-close if and only if they are \( E' \)-close;
- \( \lambda \)-equivalent if two subsets \( A, B \subseteq X \) are \( E \)-separated if and only if they are \( E' \)-separated.
Remark 2.13. By [26], any metrizable unbounded locally finite coarse structure on \( \omega \) structures on \( \omega \) structures on \( \omega \). All of them are \( \delta \)-equivalent, which means that Problem 2.11 has negative answer under \( b = c \). So we are actually interested in the ZFC-answer to Problem 2.11.

Remark 2.14. By [28], any metrizable unbounded locally finite coarse structure \( E \) on \( \omega \) is \( \lambda \)-equivalent to some finitary cellular finitary coarse structure \( E_0 \subseteq E \).

It can be shown that two metrizable coarse structures on the same set are equal if and only if they are \( \delta \)-equivalent.

Problem 2.14 ([28, 33]). Let \( E, E' \) be two \( \delta \)-equivalent finitary coarse structures on \( \omega \). Is \( E \) metrizable if \( E' \) is metrizable?

Finally we discuss some problems related to cardinal characteristics of the family \( E^{\bullet}_{<\omega} \) of nontrivial cellular finitary entourages on \( \omega \). We are interested in two cardinal characteristics of the family \( E^{\bullet}_{<\omega} \):

\[
\uparrow(E^{\bullet}_{<\omega}) := \min \{|A| : A \subseteq E^{\bullet}_{<\omega}, \forall E \in E^{\bullet}_{<\omega} \exists E' \in E^{\bullet}_{<\omega} \exists A \in A (E' \subseteq E \cap A)\};
\]

\[
\downarrow(E^{\bullet}_{<\omega}) := \min \{|A| : A \subseteq E^{\bullet}_{<\omega}, \forall E \in E^{\bullet}_{<\omega} \exists E' \in E^{\bullet}_{<\omega} \exists A \in A (E \cup A \subseteq E')\}.
\]

The cardinal characteristics \( \uparrow(E^{\bullet}_{<\omega}) \) and \( \downarrow(E^{\bullet}_{<\omega}) \) are introduced and studied in [1, §7], where the following theorem is proved.

Theorem 2.15 ([1, 7.1]). \( \Sigma \leq \uparrow(E^{\bullet}_{<\omega}) \leq \text{non}(M) \) and \( \text{cov}(M) \leq \downarrow(E^{\bullet}_{<\omega}) \leq \hat{\Delta} \).

This theorem raises the following set-theoretic problems.

Problem 2.16. Is \( \downarrow(E^{\bullet}_{<\omega}) \leq \hat{\Sigma} \)?

Problem 2.17. Which of the following strict inequalities is consistent?

1. \( \Sigma < \uparrow(E^{\bullet}_{<\omega}) \);  
2. \( \uparrow(E^{\bullet}_{<\omega}) < \text{non}(M) \);  
3. \( \text{cov}(M) < \downarrow(E^{\bullet}_{<\omega}) \);  
4. \( \downarrow(E^{\bullet}_{<\omega}) < \hat{\Sigma} \).

3. THE HIGSON CORONA OF A COARSE SPACE

Let \((X, \mathcal{E})\) be a coarse space. A complex-valued function \( f : X \to \mathbb{C} \) is called slowly oscillating if for any \( \varepsilon \in (0, \infty) \) and \( E \in \mathcal{E} \), there exists an \( \mathcal{E} \)-bounded set \( B \subseteq X \) such that \( \text{diam} f[E(x)] < \varepsilon \) for any \( x \in X \setminus B \).

Let \( \text{so}(X, \mathcal{E}) \) be the algebra of bounded slowly oscillating complex-valued functions on \( X \) and

\[
\delta : X \to \mathbb{C}^{\text{so}(X, \mathcal{E})}, \quad \delta : x \mapsto (f(x))_{f \in \text{so}(X, \mathcal{E})},
\]

be the canonical map. Since each function \( f \in \text{so}(X, \mathcal{E}) \) is bounded, the image \( \delta(X) \) has compact closure \( h(X, \mathcal{E}) \), called the Higson compactification of the coarse space \((X, \mathcal{E})\). Since each function \( f : X \to \mathbb{C} \) with finite support \( \text{supp}(f) = \{x \in X : f(x) \neq 0\} \) is slowly oscillating, the map \( \delta \) is injective and \( \delta(X) \) is an open discrete subspace of \( h(X, \mathcal{E}) \).

Since every function \( f : X \to \mathbb{C} \) with \( \mathcal{E} \)-bounded support is slowly oscillating, for every \( \mathcal{E} \)-bounded set \( B \subseteq X \) the closure \( \overline{\delta(B)} \) is an open subset of \( h(X, \mathcal{E}) \). Then the complement

\[
h(X, \mathcal{E}) = h(X, \mathcal{E}) \setminus \bigcup \{\overline{\delta(B)} : B \subseteq X \text{ is } \mathcal{E} \text{-bounded in } X\}
\]
is a compact subset of $h(X, E)$, called the Higson corona of $(X, E)$. If each bounded set in $(X, E)$ is finite, then the Higson corona $h(X, E)$ coincides with the remainder $h(X, E) \setminus \delta(X)$ of the Higson compactification of $X$.

By the Higson corona of a coarse structure $E$ on a set $X$ we will understand the Higson corona of $(X, E)$. More information on Higson (and other coronas) can be found in [20], [4], and [40].

Each macro-uniform map $f : X \to Y$ between coarse spaces $X, Y$ induces a continuous map

$$hf : h(X) \to h(Y), \quad hf : (x_\phi)_{\phi \in s_0(X)} \mapsto (x_{\psi \circ f})_{\psi \in s_0(Y)},$$

between their Higson compactifications. If $f$ is a coarse equivalence, then the induced map $hf = hf \restriction h(X) : h(X) \to h(Y)$ is a homeomorphism of the Higson coronas.

The Higson corona reflects many properties of the coarse space. For example,

- for any inseparated coarse space its Higson corona is a singleton;
- for any indiscrete coarse space its Higson corona is finite.

Let us recall that a compact Hausdorff space $X$ is perfectly normal if each closed subset of $X$ is of type $G_\delta$ in $X$. In [4] we studied the problem of reconstruction of a coarse space from its Higson corona and proved the following theorem.

**Theorem 3.1 ([4]).**

1. Every perfectly normal compact Hausdorff space is homeomorphic to the Higson corona of some finitary coarse structure on $\omega$.
2. Under $\Delta^c_{<\omega} = \mathfrak{c}$, every perfectly normal compact Hausdorff space is homeomorphic to the Higson corona of some cellular finitary coarse structure on $\omega$.
3. Under $\omega_1 = \mathfrak{c}$, every compact Hausdorff space of weight $\leq \omega_1$ is homeomorphic to the Higson corona of some cellular finitary coarse structure on $\omega$.

This theorem raises two open set-theoretic problems.

**Problem 3.2.** Is each compact Hausdorff space of weight $\leq \omega_1$ homeomorphic to the Higson corona of a finitary coarse structure on $\omega$?

**Problem 3.3.** Is each compact metrizable space homeomorphic to the Higson corona of some cellular finitary coarse structure on $\omega$?

By Theorem 4 in [19] (see also Theorem 11 in [9]), for any cellular metrizable finitary coarse space $(X, E)$ its Higson corona $h(X, E)$ is a zero-dimensional compact Hausdorff space. Under CH, we have a much more precise result.

**Theorem 3.4 ([23]).** Under CH the Higson corona of any unbounded separable ultrametric space is homeomorphic to $\beta\omega \setminus \omega$.

On the other hand, under $\mathfrak{u} < \mathfrak{d}$ we have an opposite result.

**Theorem 3.5 ([3]).** Let $G$ be any countable locally finite group. Under $\mathfrak{u} < \mathfrak{d}$, for a metrizable finitary coarse space $(X, E)$ the following conditions are equivalent:

1. $(X, E)$ is coarsely equivalent to the coarse space $(G, E_G)$;
2. the Higson coronas $h(X, E)$ and $h(G, E_G)$ are homeomorphic;
3. $(X, E)$ is cellular and the character of each point in $h(X, E)$ is $\geq \mathfrak{d}$. 

We recall that the character of a point of a topological space is the smallest cardinality of a neighborhood base at the point.

By [43] (see also [45], [11]), under Proper Forcing Axiom (briefly, PFA), each homeomorphism of the Stone-Čech remainder $\beta\omega \setminus \omega$ is induced by a bijection between subsets with finite complement in $\omega$. This result can be reformulated as follows.

**Theorem 3.6.** Let $E$ be the smallest coarse structure on the set $X = \omega$. Under PFA, each homeomorphism of the corona $h(X, E)$ is induced by some coarse equivalence of $(X, E)$.

This theorem suggests the following problem.

**Problem 3.7.** Assume PFA. Let $X, Y$ be two metrizable (cellular) finitary coarse spaces.

1. Are the coarse spaces $X, Y$ coarsely equivalent if their Higson coronas are homeomorphic?
2. Is any homeomorphism between the Higson coronas of $X, Y$ induced by a coarse equivalence between $X$ and $Y$?

**Remark 3.8.** Recent results of Braga, Farah, Vignati [8] suggest that the answer to Problem 3.7 can be affirmative for metrizable finitary coarse spaces $X, Y$ of finite asymptotic dimension (more generally, for metrizable finitary coarse spaces having Yu’s property A). On the other hand, by an old result of Rudin [41], under CH the Stone-Čech remainder has $2^\omega$ homeomorphisms, which implies that under CH the answer to Problem 3.7(2) is negative even for discrete coarse spaces.

### 4. Special ultrafilters on coarse spaces

First we recall the definitions of some special types of free ultrafilters on $\omega$.

A free ultrafilter $U$ on $\omega$ is called

- a **P-point** if for any sequence of sets $\{U_n\}_{n \in \omega} \subseteq U$ there exist a set $U \in U$ such that $U \subseteq^* U_n$ for every $n \in \omega$;
- a **Q-point** if for any locally finite cellular entourage $E$ on $\omega$ there exists a set $U \in U$ such that $|U \cap E(x)| \leq 1$ for any $x \in \omega$;
- a **Ramsey** if for any map $f: \omega \to \omega$ there exists a set $U \in U$ such that $f|U$ is either constant or injective;
- a **weak P-point**, if for any sequence $(U_n)_{n \in \omega}$ of free ultrafilters on $\omega$ that are not equal to $U$ there exists a set $U \in U \setminus \bigcup_{n \in \omega} U_n$;
- an **OK$_\kappa$-point** for a cardinal $\kappa$ if for any sequence of sets $\{U_n\}_{n \in \omega} \subseteq U$ there exists a family $(V_\alpha)_{\alpha \in \kappa} \subseteq U$ such that for any ordinals $\alpha_1 < \cdots < \alpha_n$ in $\kappa$ we have $\bigcap_{i=1}^n V_{\alpha_i} \subseteq^* U_n$;
- a **rapid** if for any function $f \in \omega^\omega$ there exists a function $g \in \omega^\omega$ such that $f \leq g$ and $\{g(n) : n \in \omega\} \in U$;
- a **discrete** if for any injective function $\varphi: \omega \to \mathbb{R}$ there exists a set $U \in U$ whose image $\varphi(U)$ is a discrete subspace of $\mathbb{R}$;
- **nowhere dense** if for any injective function $\varphi: \omega \to \mathbb{R}$ there exists a set $U \in U$ whose image $\varphi(U)$ is nowhere dense in $\mathbb{R}$;
- a **(cellular) T-point** if for any increasing sequence $(E_n)_{n \in \omega}$ of finitary (and cellular) entourages on $\omega$ there exists a set $U \in U$ such that for every $n \in \omega$ the set $\{x \in U : \{x\} \neq U \cap E_n(x)\}$ is finite;
• **dynamically discrete** if for any action of a countable group $G$ on $\omega$ the subspace $\{gU\}_{g \in G}$ is discrete in $\beta\omega$.

It is known that a free ultrafilter on $\omega$ is Ramsey if and only if it is both a $P$-point and a $Q$-point. More information on $Q$-points, $P$-points, weak $P$-points, $OK_\kappa$-points can be found in the survey [15]; discrete and nowhere dense ultrafilters were considered in [2] and [42]; (cellular) $T$-points were introduced and studied by Petrenko and Protasov in [30], [31].

The following diagram describes the implications between various properties of free ultrafilters on $\omega$.

According to a famous result of Kunen [12], [13] (see also [15 4.5.2]), $OK_\kappa$-points exist in ZFC. On the other hand, there are models of ZFC containing no nowhere dense ultrafilters [42] and there are models of ZFC containing no rapid ultrafilters [16]. A well-known open problem [15, p.563] asks whether there exists a model of ZFC containing no $P$-points and no $Q$-points.

This information motivates the following open problems.

**Problem 4.1** ([30]). Do $T$-points exist in ZFC?

**Problem 4.2** ([30]). Is each weak $P$-point (or $OK_\kappa$-point) a $T$-point?

**Problem 4.3** ([31]). Is each discrete ultrafilter a $T$-point?

**Problem 4.4** ([31]). Is each cellular $T$-point a $T$-point?

**Remark 4.5.** By [30 Proposition 4], under CH there exists a $T$-point which is neither a weak $P$-point, nor a $Q$-point nor a nowhere dense ultrafilter. Jana Flašková noticed that a rapid ultrafilter needs not be a $T$-point, see [31, p.350].

It is easy to see that weak $P$-points are dynamically discrete ultrafilters. The converse is not true because of the following result. Two ultrafilters $U, V$ on $\omega$ are called *isomorphic* if there exists a bijection $f$ of $\omega$ such that $U = \{f(V) : V \in V\}$. By [15 4.5.2], there exists $2^\omega$ pairwise non-isomorphic weak $P$-points.

**Theorem 4.6.** If $(U_n)_{n \in \omega}$ is a sequence of pairwise non-isomorphic weak $P$-points, then each ultrafilter $W$ in the closure $\overline{\{U_n : n \in \omega\}} \subseteq \beta\omega \setminus \omega$ is dynamically discrete.

**Proof.** Since $W \in \overline{\{U_n : n \in \omega\}}$, there exists an ultrafilter $V$ on $\omega$ such that the family

$$\big\{ \bigcup_{n \in V} U_n : V \in V \land (U_n)_{n \in V} \in \prod_{n \in V} U_n \big\}$$

is a base of the ultrafilter $W$. If the ultrafilter $V$ is principal, then $W = U_n$ for some $n \in \omega$ and $W = U_n$ is dynamically discrete, being a weak $P$-point. So, we assume that $V$ is a free ultrafilter. Let $G$ be any countable subgroup of the permutation group $S_\omega$ of $\omega$. Write $G$ as
the union $G = \bigcup_{n \in \omega} G_n$ of a sequence $(G_n)_{n \in \omega}$ of finite sets such that $G_n^{-1} = G_n \subseteq G_{n+1}$ for every $n \in \omega$.

For every $n \in \omega$ the weak $P$-point $U_n$ does not belong to the closure of the countable set 
\{$gU_i : g \in G, i \in \omega, U_n \neq gU_i \}$. Consequently, there exists a set $U_n \in \mathcal{U}_n$ such that $U_n \notin gU_i$ for every $g \in G$ and $i \in \omega$ with $U_n \neq gU_i$. It follows that for every $n \in \omega$ the set
\[ U_n' = U_n \setminus \bigcup \{gU_i : g \in G_n, i \leq n, U_n \neq gU_i \} \]
belongs to the ultrafilter $\mathcal{U}_n$.

To see $W$ is dynamically discrete, it suffices to show that the ultrafilter $W$ is an isolated point of its orbit \{$gW\}_{g \in G}$. This will follow as soon as we check that for every $g \in G$ with $gW \neq W$, the set $W = \bigcup_{n \in \omega} U_n' \in W$ does not belong to the ultrafilter $gW$. Find a number $m \in \omega$ such that $g \in G_m$. Since $W \neq gW$, we can chose a set $O_g \in W$ such that $O_g \cap gO_g = \emptyset$. The choice of $\mathcal{V}$ ensures that the set $V_g = \{n \in \omega : O_g \in U_n, n \geq m\}$ belongs to the ultrafilter $\mathcal{V}$. We claim that the set $W_g = \bigcup_{n \in \omega} gU_n' \in gW$ is disjoint with the set $W$. Assuming that $W \cap W_g \neq \emptyset$, we could find two numbers $i \in \omega$ and $j \in V_g$ such that $U_i' \cap gU_j' \neq \emptyset$. If $i = j$, then the inclusion $i \in V_g$ implies $U_i \neq gU_i$ and then the definition of the set $U_i'$ ensures that $U_i' \cap gU_i' = \emptyset$, which contradicts the choice of the numbers $j, i$. Therefore, $i \neq j$.

The non-isomorphness of the ultrafilters $\mathcal{U}_i, \mathcal{U}_j$ guarantees that $gU_i \neq U_i \neq g^{-1}U_j$. If $j < i$, then the definition of the set $U_i'$ ensures that $U_i' \cap gU_j' = \emptyset$. If $i < j$, then the definition of the set $U_i'$ ensures that $U_i' \cap g^{-1}U_i' = \emptyset$. In both cases we obtain $U_i' \cap gU_j' = \emptyset$, which contradicts the choice of the numbers $i, j$. This completes the proof of dynamical discreteness of $W$. $\square$

5. Coarse structures and topological dynamics

By Theorem 1.1, every finitary coarse structure on a set $X$ is equal to the finitary coarse structure $\mathcal{E}_G$ induced by the action of suitable subgroup $G \subseteq S_X$ of the permutation group of $X$. The action of the group $G$ extends to a continuous action of $G$ on the Stone-Čech compactification $\beta X$ of the discrete space $X$. The remainder $X^* = \beta X \setminus X$ is an invariant subset of the action, so we obtain a dynamical system $(X^*, G)$ and can study the interplay between properties of the coarse structure $\mathcal{E}_G$ and the properties of the dynamical system $(X^*, G)$, see the papers [22, 28, 38] for more information on this topic.

The following theorem (proved in [38, Theorem 15]) shows that the orbit structure of the dynamical system $(\omega^*, G)$ uniquely determines the coarse structure $\mathcal{E}_G$.

**Theorem 5.1.** For two subgroups $G, H \subseteq S_\omega$ the coarse structures $\mathcal{E}_G$ and $\mathcal{E}_H$ on $\omega$ coincide if and only if \{Gr : r \in \omega^*\} = \{Hr : r \in \omega^*\}.

The following fact is proved in [28, Theorem 3.15].

**Theorem 5.2.** If for a group $G$ of permutations of a set $X$, the finitary coarse space $(X, \mathcal{E}_G)$ is indiscrete, then there exists an ultrafilter $p \in X^*$ whose orbit $Gp$ is not discrete.

Under $t = c$ we can prove a bit more.

**Theorem 5.3.** Let $G$ be a group of permutations of a countable set $X$ such that the coarse space $(X, \mathcal{E}_G)$ is indiscrete. Then there exists an ultrafilter $p \in X^*$ whose orbit $Gp$ contains at least $t$ pairwise disjoint nonempty open sets and hence $|Gp| \geq t$. If $t = c$, then $p$ is a $P$-point and the space $Gp$ is not discrete but every subspace $A \subseteq Gp$ of cardinality $|A| < c$ is discrete.
Proof. Write the family of all subsets of $X$ as $\{X_\alpha\}_{\alpha \in \mathfrak{t}}$.

For every ordinal $\alpha \in \mathfrak{t}$ we shall inductively choose an infinite set $U_\alpha \subseteq X$ and an element $g_\alpha \in G$ such that the following conditions are satisfied:

(i) $U_\alpha \cup g_\alpha U_\alpha \subseteq^* U_\beta$ for all $\beta \in \alpha$;
(ii) $U_\alpha \subseteq X_\alpha$ or $U_\alpha \subseteq X \setminus X_\alpha$;
(iii) $U_\alpha \cap g_\alpha U_\alpha = \emptyset$.

Assume that for some ordinal $\alpha \in \mathfrak{t}$ we have constructed a family of infinite sets $(U_\beta)_{\beta < \alpha}$ satisfying the condition (i). Since $\alpha < \mathfrak{t}$, there exists an infinite set $W_\alpha \subseteq X$ such that $W_\alpha \subseteq^* U_\beta$ for all $\beta \in \alpha$. Since one of the sets $W_\alpha \cap X_\alpha$ or $W_\alpha \setminus X_\alpha$ is infinite, we can replace $W_\alpha$ by a smaller infinite set and additionally assume that $W_\alpha \subseteq X_\alpha$ or $W_\alpha \subseteq X \setminus X_\alpha$.

Since the coarse space $(\omega, \mathcal{E}_G)$ is indiscrete, the subset $W_\alpha$ is not $\mathcal{E}_G$-discrete. Consequently, there exists an element $g_\alpha \in G$ such that the set $V_\alpha = \{x \in W_\alpha : x \neq g_\alpha x \in W_\alpha\}$ is infinite. Choose an infinite subset $U_\alpha \subseteq V_\alpha$ such that $U_\alpha \cap g_\alpha U_\alpha = \emptyset$. It is easy to see that the set $U_\alpha$ satisfies the conditions (i)–(iii).

After completing the inductive construction, take any ultrafilter $p \in X^*$ containing the family $\{U_\alpha\}_{\alpha \in \mathfrak{t}}$. The inductive conditions (i) and (iii) imply that for every ordinals $\beta < \alpha < \mathfrak{t}$ we have

$$g_\alpha U_\alpha \cap g_\beta U_\beta \subseteq^* U_\beta \cap (X \setminus U_\beta) = \emptyset,$$

which implies that the family $(Gp \cap g_\alpha U_\alpha)_{\alpha \in \mathfrak{t}}$ consists of pairwise disjoint nonempty open sets in $Gp \subseteq X^* \subset \beta X$.

Now assume that $\mathfrak{t} = \mathfrak{c}$. In this case the condition (ii) implies that $\{U_\alpha\}_{\alpha \in \mathfrak{t}}$ is a base of the ultrafilter $p$. The condition (i) implies that the ultrafilter $p$ is a $P$-point (even a $P_{<\mathfrak{c}}$-point).

To see that the orbit $Gp$ is not discrete, take any neighborhood $O_p$ of $p$ in $\beta X$ and find $\alpha \in \mathfrak{t}$ such that $\overline{U_\alpha} \subseteq O_p$. The inductive condition (i) implies that $g_{\alpha+1} U_{\alpha+1} \subseteq^* U_\alpha$ and hence $g_{\alpha+1} p \in \overline{U_\alpha} \subseteq O_p$, witnessing that the orbit $Gp$ is not discrete.

It remains to prove that every subspace $A \subseteq Gp$ of cardinality $|A| < \mathfrak{c}$ is discrete. Given any point $gp \in A$, for every $a \in A \setminus \{gp\}$, find an ordinal $\alpha_a \in \mathfrak{c}$ such that $g U_{\alpha_a} \not\subseteq a$. By Proposition 6.4 in [7], the cardinal $\mathfrak{t}$ is regular. Consequently, there exists an ordinal $\alpha \in \mathfrak{c} = \mathfrak{t}$ such that $\sup_{a \in A} \alpha_a < \alpha$. The inductive condition (i) guarantees that $g U_\alpha \subseteq^* g U_{\alpha_a}$ for all $a \in A \setminus \{gp\}$. Then $g U_\alpha$ is a neighborhood of $gp$ in $\beta X$, which is disjoint with the set $A \setminus \{gp\}$ and witnesses that the space $A$ is discrete.

By a dynamical system we understand a compact Hausdorff space $K$ endowed with the continuous action of some group $G$. A dynamical system $(K, G)$ is called

- minimal if the orbit $Gx$ of any point $x \in K$ is dense in $K$;
- topologically transitive if the orbit $GU$ of any nonempty open set $U \subseteq K$ is dense in $K$.

The following theorem proved in [25] characterizes large and inseparated coarse structures in dynamical terms.

**Theorem 5.4 ([25]).** For an infinite set $X$ and a subgroup $G \subseteq S_X$ of the permutation group, the coarse structure $\mathcal{E}_G$ is

1. large if and only if the dynamical system $(X^*, G)$ is minimal;
2. inseparated if and only if $(X^*, G)$ is topologically transitive.
It is well-known that a dynamical system $(K, G)$ of a (metrizable) compact Hausdorff space $K$ is topologically transitive if (and only if) the orbit $Gx$ of some point $x \in K$ is dense in $K$. In this case we say that the dynamical system has a dense orbit.

**Example 5.5.** Under $\Sigma < \mathfrak{c}$ there exists a subgroup $G \subseteq S_\omega$ of cardinality $|G| = \Sigma < \mathfrak{c}$ such that the coarse structure $E_G$ is inseparated and hence the dynamical system $(\omega^*, G)$ is topologically transitive but has no dense orbits (as the space $\omega^*$ has density $\mathfrak{c} > |G|$).

**Proof.** By the definition of the cardinal $\Sigma$, there exists an inseparated finitary coarse structure $E$ of weight $w(E) = \Sigma$ on $\omega$. By Theorem 5.6 the coarse structure $E$ coincides with the coarse structure $E_G$ generated by some subgroup $G \subseteq S_\omega$ that has cardinality $|G| = w(E) = \Sigma$. Since the coarse space $E = E_G$ is inseparated, we can apply Theorem 5.4(2) and conclude that the dynamical system $(\omega^*, G)$ is topologically transitive. Since the Stone-Čech remainder $\omega^*$ contains continuum many pairwise disjoint open sets, its density is equal to the cardinality of continuous. Since $|G| = \Sigma < \mathfrak{c}$, no orbit $G\alpha$ is dense in $\omega^*$. $\square$

Example 5.5 cannot be proved in ZFC because of the following theorem.

**Theorem 5.6.** Under $t = \mathfrak{c}$, for every subgroup $H$ of the homeomorphism group of $\omega^*$, the dynamical system $(\omega^*, H)$ is topologically transitive if and only if it has a dense orbit.

**Proof.** The “if” part is trivial and holds without any set-theoretic assumptions. To prove the “only if” part, assume that $t = \mathfrak{c}$ and the dynamical system $(\omega^*, H)$ is topologically transitive.

Let $(A_\alpha)_{\alpha \in \mathfrak{c}}$ be an enumeration of all infinite subsets of $\omega$. By transfinite induction we shall construct a transfinite sequence of infinite subsets $(U_\alpha)_{\alpha \in \mathfrak{c}}$ of $\omega$ and a transfinite sequence $(g_\alpha)_{\alpha \in \mathfrak{c}}$ of elements of the group $H$ such that for every $\alpha \in \mathfrak{c}$ the following conditions are satisfied:

(a) $U_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha$;

(b) $g_\alpha(U_\alpha) \subseteq^* A_\alpha$.

To start the inductive construction, put $U_0 = A_0$ and $g_0$ be the identity of the group $H$. Assume that for some ordinal $\alpha \in \mathfrak{c}$, a transfinite sequence $(U_\beta)_{\beta < \alpha}$ satisfying the condition (a) has been constructed. By the definition of the tower number $t$ and the equality $t = \mathfrak{c} > \alpha$, there exists an infinite subset $V_\alpha \subseteq \omega$ such that $V_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha$. The infinite sets $V_\alpha$ and $A_\alpha$ determine basic open sets $V_\alpha^*$ and $A_\alpha^*$ in the space $\omega^*$. Since the action of the group $H$ on $\omega^*$ is topologically transitive, there exist $g_\alpha \in H$ and an infinite subset $U_{\alpha_0} \subseteq V_{\alpha_0}$ such that $g_\alpha(U_{\alpha_0}) \subseteq A_{\alpha_0}^*$ and hence $g_\alpha(U_{\alpha_0}) \subseteq^* A_{\alpha_0}$. This completes the inductive step.

After completing the inductive construction, extend the family $(U_\alpha)_{\alpha \in \mathfrak{c}}$ to a free ultrafilter $U$ and observe that its orbit intersects each basic open set $A_\alpha^*$, $\alpha \in \mathfrak{c}$ and hence is dense in $\omega^*$. $\square$

Theorem 5.4 and the (original) definition of the cardinal $\Sigma$ given in [1] imply the following dynamical characterization of $\Sigma$.

**Theorem 5.7.** The cardinal $\Sigma$ is equal to the smallest cardinality of a subgroup $G \subseteq S_\omega$ such that the dynamical system $(\omega^*, G)$ is topologically transitive.

Theorems 5.6 and 5.7 suggest to consider the cardinal $\Sigma^*$ defined as the smallest cardinality of a subgroup $H \subseteq \text{Homeo}(\omega^*)$ such that the dynamical system $(\omega^*, H)$ is topologically transitive.

**Theorem 5.8.** $t \leq \Sigma^* \leq \Sigma$. 

Proof. The inequality $\Sigma^* \leq \Sigma$ follows from Theorem 5.47. To prove that $t \leq \Sigma^*$, we need to show that a subgroup $H \subseteq \text{Homeo}(\omega^*)$ has cardinality $|H| \geq t$ if the dynamical system $(\omega^*, H)$ is topologically transitive.

By [11, 3.6], there exists a family $(A_\alpha)_{\alpha \in t}$ of infinite subsets of $\omega$ such that for any distinct ordinals $\alpha, \beta \in t$ the intersection $A_\alpha \cap A_\beta$ is finite. Repeating the argument from the proof of Theorem 5.46 we can use the topological transitivity of the dynamical system $(\omega^*, H)$ and construct a transfinite sequence of infinite subsets $(U_\alpha)_{\alpha \in t}$ of $\omega$ and a transfinite sequence $(g_\alpha)_{\alpha \in t}$ of elements of the group $H$ such that for every $\alpha \in t$ the following conditions are satisfied:

(a) $U_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha$;
(b) $g_\alpha(U_\alpha) \subseteq^* A_\alpha$.

We claim that the sequence $(g_\alpha)_{\alpha \in t}$ consists of pairwise distinct elements of the group $H$. To derive a contradiction, assume that $g_\alpha = g_\beta$ for some ordinals $\alpha < \beta$ in $t$. Then

$$g_\beta(U_\beta) = g_\alpha(U_\beta) \cap g_\beta(U_\beta) \subseteq^* g_\alpha(U_\alpha) \cap g_\beta(U_\beta) \subseteq A_\alpha \cap A_\beta$$

and hence the set $g_\beta(U_\beta)$ is finite and so is the set $U_\beta$, which contradicts the choice of $U_\beta$. This contradiction shows that $|H| \geq |(g_\alpha)_{\alpha \in t}| = t$ and hence $\Sigma^* \geq t$. \qed

Problem 5.9. Is the strict inequality $\Sigma^* < \Sigma$ consistent?

A point $x$ of a topological space $X$ is called

- a $P$-point if every $G_\delta$-set $G \subseteq X$ that contains $x$ is a neighborhood of $x$;
- a weak $P$-point if $x \notin \overline{C}$ for any countable set $C \subseteq X \setminus \{x\}$.

For any coarse space $(X, E)$, the canonical map $\delta : X \to h(X, E)$ from $X$ to its Higson compactification has a unique continuous extension $\bar{\delta} : \beta X \to h(X, E)$ to the Stone-Cech compactification $\beta X$ of $X$ endowed with the discrete topology. If the coarse structure $E$ is locally finite, then $\bar{\delta}(\beta X \setminus X) = h(X, E)$. For a free ultrafilter $p$ on $X$ let $\bar{p}$ be the set $\bar{\delta}^{-1}(\delta(p))$. For two free ultrafilters $p, q$ on $X$ we have $\bar{p} = \bar{q}$ if and only if for any bounded slowly oscillating function $f : X \to \mathbb{C}$ and its continuous extension $\bar{f} : \beta X \to \mathbb{C}$ we have $\bar{f}(p) = \bar{f}(q)$.

Our next questions ask about the interplay between properties of points in $\beta X \setminus X$ and $h(X, E)$. By Theorem 3 [24], for a metrizable finitary coarse structure $E$ on $\omega$ the Higson corona $h(\omega, E)$ contains a weak $P$-point. Moreover, $h(\omega, E)$ contains a $P$-point if $\beta \omega \setminus \omega$ contains a $P$-point.

**Remark.** Let $E$ be a finitary metrizable coarse structure on $\omega$ and $p$ be a free ultrafilter on $\omega$.

(i) Is $\bar{\delta}(p)$ a weak $P$-point in $h(\omega, E)$ if $p$ is a weak $P$-point?

(ii) Does the set $\bar{p}$ contain a weak $P$-point in $\beta \omega \setminus \omega$ if $\bar{\delta}(p)$ is a weak $P$-point in $h(\omega, E)$?

Given a countable subgroup $G \subseteq S_\omega$, consider the finitary coarse structure $E_G$ on $\omega$ induced by the action of the group $G$. By analogy with Theorem 4.1 [22], it can be shown that for any $P$-point $p \in \omega^*$ the set $\bar{p}$ coincides with the closure of the orbit of $p$ under the action of the group $G$ on $\omega^*$.

**Problem 5.11** [22]. Is a free ultrafilter $p$ on $\omega$ a $P$-point if $\bar{p} = \overline{Gp}$ for any countable subgroup $G \subseteq S_\omega$?

If an answer to this question is negative, then another question is of interest.
Problem 5.12 (22). In ZFC, does there exist a free ultrafilter \( p \) on \( \omega \) such that \( \bar{p} = Gp \) for any countable subgroup \( G \subseteq S_\omega \)?

6. PARTITIONS OF COARSE SPACES INTO THIN SUBSETS

Let \((X, \mathcal{E})\) be a coarse space and \( m \) be a natural number. A subset \( T \subseteq X \) is called \( m \)-thin if for any entourage \( E \in \mathcal{E} \) there exists an \( \mathcal{E} \)-bounded set \( B \subseteq X \) such that \( |E(x) \cap T| \leq m \) for every \( x \in T \setminus B \). It is clear that \( T \subseteq X \) is 1-thin if and only if \( T \) is \( \mathcal{E} \)-discrete; 1-thin sets are called thin.

The following dynamical characterization of \( m \)-thin sets was proved in [36, Proposition 4] (for \( m = 1 \) this characterization was proved in Theorem 3.2 in [35]).

**Theorem 6.1.** Let \( m \in \mathbb{N} \) and \( G \subseteq S_\omega \) be a subgroup. A subset \( T \subseteq \omega \) of the coarse space \((\omega, \mathcal{E}_G)\) is \( m \)-thin if and only if \(|T \cap Gp| \leq m\) for any ultrafilter \( p \in \omega^* \). Here \( \overline{T} \) is the closure of the set \( T \) in \( \beta \omega \).

This theorem implies the following characterization of decomposability into finitely many thin pieces.

**Theorem 6.2.** For a subgroup \( G \subseteq S_\omega \), a subset \( T \subseteq \omega \) of the coarse space \((\omega, \mathcal{E}_G)\) can be covered by finitely many thin sets if and only if any ultrafilter \( p \in \mathcal{T} \cap \omega^* \) contains a set \( P \in p \) such that \(|\overline{P} \cap Gq| \leq 1\) for every \( q \in \omega^* \).

By [34], for every \( n, m \in \mathbb{N} \) and Abelian group \( G \) of cardinality \( \aleph_n \), every \( m \)-thin subset \( T \subseteq G \) in the coarse space \((G, \mathcal{E}_G)\) is the union of \( m^{n+1} \) thin subsets. On the other hand, there exists a group \( G \) of cardinality \( \aleph_\omega \) such that the finitary coarse space \((G, \mathcal{E}_G)\) contains a 2-thin subset which cannot be covered by finitely many thin subsets. In fact, similar examples can be found among finitary coarse spaces of countable cardinality.

A group \( G \) is called Boolean if each element \( x \in G \) has order \( \leq 2 \). A subgroup \( G \subseteq S_\omega \) is defined to be almost Boolean if the quotient group \( G/(G \cap S_{<\omega}) \) is Boolean. In this definition \( S_{<\omega} \) denotes the normal subgroup of \( S_\omega \) consisting of permutations \( f \in S_\omega \) that have finite support \( \text{supp}(f) = \{ x \in \omega : f(x) \neq x \} \).

**Theorem 6.3.** There exists an almost Boolean subgroup \( G \subseteq S_\omega \) of cardinality \(|G| = \tau \) such that the finitary coarse space \((\omega, \mathcal{E}_G)\) is cellular and 2-thin but \( \omega \) cannot be covered by finitely many 1-thin subspaces.

**Proof.** We divide the proof of this theorem into three lemmas.

**Lemma 6.4.** There exists an almost Boolean subgroup \( G \subseteq S_\omega \) of cardinality \(|G| = \aleph \) such that the finitary coarse space \((\omega, \mathcal{E}_G)\) is cellular and 2-thin but \( \omega \) cannot be covered by finitely many 1-thin subspaces.

**Proof.** By [7, 8.1], there exists a family \( \mathcal{A} \subseteq [\omega]^\omega \) of cardinality \(|\mathcal{A}| = \aleph \), which is almost disjoint in the sense that \( A \cap B \) if finite for any distinct sets \( A, B \in \mathcal{A} \).

Fix any ultrafilter \( \mathcal{U} \) on \( \omega \) containing the filter
\[
\mathcal{F} = \{ X \in [\omega]^\omega : |\{ A \in \mathcal{A} : A \nsubseteq X \}| < \aleph \}.
\]

Let \( \{ U_\alpha \}_{\alpha \in \mathcal{A}} \) be an enumeration of the ultrafilter \( \mathcal{U} \). By transfinite induction, for every \( \alpha \in \aleph \) choose a set \( A_\alpha \in \mathcal{A} \setminus \{ A_\beta \}_{\beta < \alpha} \) such that the set \( A_\alpha \cap U_\alpha \) is infinite. The choice of \( A_\alpha \) is always possible since the set \( \{ A \in \mathcal{A} : |A \cap U_\alpha| = \omega \} \) has cardinality of continuum (in the
opposite case, the set $\omega \setminus U_\alpha$ belongs to the filter $\mathcal{F} \subseteq \mathcal{U}$ but has empty intersection with $U_\alpha$, which is a desired contradiction.

After completing the inductive construction, for every $\alpha \in \mathfrak{c}$, choose an involution $f_\alpha$ of $\omega$ whose support $\text{supp}(f_\alpha) := \{x \in \omega : f(x) \neq x\}$ is an infinite subsets of $U_\alpha \cap A_\alpha$. Let $G \subseteq S_\omega$ be the subgroup, generated by the set $\{f_\alpha\}_{\alpha \in \mathfrak{c}}$. Taking into account that the family $(\text{supp}(f_\alpha))_{\alpha \in \mathfrak{c}}$ is almost disjoint, one can show that the group $G$ is almost Boolean and the coarse space $(\omega, \mathcal{E}_G)$ is cellular and 2-thin.

To see that $(\omega, \mathcal{E}_G)$ cannot be covered by finitely many thin sets, it suffices to observe that for every finite partition of $\omega$ one of the cells of the partition belongs to the ultrafilter $\mathcal{U}$, and that for every $\alpha \in \mathfrak{c}$ the set $U_\alpha \in \mathcal{U}$ is not 1-thin since the set $\{x \in U_\alpha : f_\alpha(x) \neq x\}$ is infinite.

For treating the case $\mathfrak{r} < \mathfrak{c}$, we shall exploit the following lemma, which is a special case of Theorem 2.1 in [6].

Lemma 6.5. For any cardinal $\kappa < \mathfrak{c}$ and family of infinite sets $\{I_\alpha\}_{\alpha \in \kappa} \subseteq [\omega]^\omega$, there exists an almost disjoint family $\{A_\alpha\}_{\alpha \in \kappa} \subseteq [\omega]^\omega$ such that $A_\alpha \subseteq I_\alpha$ for every $\alpha \in \kappa$.

Lemma 6.6. If $\mathfrak{r} < \mathfrak{c}$, then there exists an almost Boolean subgroup $G \subseteq S_\omega$ of cardinality $|G| = \mathfrak{r}$ such that the finitary coarse space $(\omega, \mathcal{E}_G)$ is cellular and 2-thin but $\omega$ cannot be covered by finitely many 1-thin subspaces.

Proof. By definition of the cardinal $\mathfrak{r}$, there exists a family $\{R_\alpha\}_{\alpha \in \mathfrak{r}} \subseteq [\omega]^\omega$ such that for any finite partition of $\omega$ one of the cells of the partition contains the set $R_\alpha$ for some $\alpha \in \mathfrak{r}$.

By Lemma 6.5, there exists an almost disjoint family of infinite sets $(A_\alpha)_{\alpha \in \mathfrak{r}}$ in $\omega$ such that $A_\alpha \subseteq R_\alpha$ for every $\alpha \in \mathfrak{r}$. For every $\alpha \in \mathfrak{r}$ choose a bijective function $f_\alpha \in S_\omega$ such that $f_\alpha \circ f_\alpha$ is the identity map of $\omega$ whose support $\text{supp}(f_\alpha) = \{x \in \omega : f_\alpha(x) \neq x\}$ coincides with the infinite set $A_\alpha$.

Let $G$ be the subgroup of $S_\omega$, generated by the set of involutions $\{f_\alpha\}_{\alpha \in \mathfrak{r}}$. Taking into account that the family $(\text{supp}(f_\alpha))_{\alpha \in \mathfrak{r}}$ is almost disjoint, we can show that the group $G$ is almost Boolean and the coarse space $(G, \mathcal{E}_G)$ is cellular and 2-thin.

Next, we show that $\omega$ cannot be covered by finitely many 1-thin sets. Assuming that such a finite cover exists, we can find $\alpha \in \mathfrak{r}$ such that the set $R_\alpha$ is contained in one of the cells of the cover and hence $R_\alpha$ is 1-thin. On the other hand, for the entourage $E_\alpha = \Delta_\omega \cup \{(x, f_\alpha(x)) : x \in \omega\}$ the set $\{x \in \omega : R_\alpha \cap E_\alpha(x) \neq \{x\}\}$ contains the infinite set $A_\alpha$, which means that the set $R_\alpha$ is not 1-thin. □

Problem 6.7. Is there a ZFC-example of a subgroup $G \subseteq S_\omega$ of cardinality $|G| = \omega_1$ such that the finitary coarse space $(\omega, \mathcal{E}_G)$ is 2-thin but cannot be covered by finitely many 1-thin subspaces.

Theorem 6.8. For every number $n \in \mathbb{N}$ there exists an almost Boolean subgroup $G \subseteq S_\omega$ of cardinality $|G| = \omega$ such that

1. the finitary coarse space $(\omega, \mathcal{E}_G)$ is cellular and 2-thin;
2. $(\omega, \mathcal{E}_G)$ can be covered by $n$ thin sets;
3. $(\omega, \mathcal{E}_G)$ cannot be covered by $(n - 1)$ thin sets.

Proof. Write the set $\omega$ as the union $\omega = \bigcup_{i \in \mathbb{N}} \Omega_i$ where $\Omega_0 = \{nk : k \in \omega\}$ and $\Omega_i = \Omega_0 + i$ for $i \in \mathbb{N}$. 
By Proposition 8.1 in [7], there exists an almost disjoint family \( \mathcal{A} \subseteq [\Omega_0]^\omega \) of cardinality \( |\mathcal{A}| = \aleph_0 \). Consider the filter

\[
\mathcal{F} = \{X \subseteq [\Omega_0]^\omega : |\{A \subseteq [\Omega_0]^\omega : A \not\subseteq X\}| < \aleph_0\}.
\]

Fix any free ultrafilter \( \mathcal{U} \) on \( \Omega_0 \) possessing a base \( \{U_\alpha\}_{\alpha \in \mathcal{U}} \) of cardinality \( \aleph_0 \). If \( \aleph_0 = \aleph_1 \), then we can additionally assume that \( \mathcal{F} \subseteq \mathcal{U} \).

**Lemma 6.9.** There exists an almost disjoint family \( \{A_\alpha\}_{\alpha \in \mathcal{U}} \subseteq [\Omega_0]^\omega \) such that \( A_\alpha \subseteq U_\alpha \) for all \( \alpha \in \mathcal{U} \).

**Proof.** If \( \aleph_0 < \aleph_1 \), then the existence of the family \( (A_\alpha)_{\alpha \in \mathcal{U}} \) follows from Lemma 6.8. So, we assume that \( \aleph_0 = \aleph_1 \). In this case the ultrafilter \( \mathcal{U} \) contains the filter \( \mathcal{F} \).

By transfinite induction, for every \( \alpha \in \mathcal{U} \) choose a set \( A'_\alpha \in \mathcal{A} \setminus \{A'_\beta\}_{\beta < \alpha} \) such that the set \( A_\alpha = A'_\alpha \cup U_\alpha \) is infinite. The choice of the set \( A'_\alpha \) is always possible since the family \( \{A \in \mathcal{A} : |A \cup U_\alpha| = \omega\} \) has cardinality of continuum (in the opposite case, the set \( \omega \setminus U_\alpha \) belongs to the filter \( \mathcal{F} \subseteq \mathcal{U} \)).

Consider the finite family

\[
[n]^2 = \{(i, j) \in n \times n : i < j\},
\]

which can be identified with the family of 2-element subsets of the ordinal \( n = \{0, \ldots, n - 1\} \).

For every \( \alpha \in \mathcal{U} \) and \( p \in [n]^2 \) choose an infinite set \( A_{\alpha, p} \subseteq A_\alpha \subseteq \Omega_0 \) such that the family \( (A_{\alpha, p})_{p \in [n]^2} \) is disjoint. Consider the involution \( f_\alpha : \omega \rightarrow \omega \) defined by the formula

\[
f_\alpha(x) = \begin{cases} x + (j - i) & \text{if } x - i \in A_{\alpha, p} \text{ for some } p = (i, j) \in [n]^2; \\ x - (j - i) & \text{if } x - j \in A_{\alpha, p} \text{ for some } p = (i, j) \in [n]^2; \\ x & \text{otherwise.} \end{cases}
\]

Let \( G \subseteq S_\omega \) be the subgroup generated by the set of involutions \( \{f_\alpha\}_{\alpha \in \mathcal{U}} \).

Taking into account that the family \( (\text{supp}(f_\alpha))_{\alpha \in \mathcal{U}} \) is almost disjoint, one can show that the group \( G \) is almost Boolean and the coarse space \( (\omega, \mathcal{E}_G) \) is cellular and 2-thin. The choice of the involutions \( f_\alpha \) guarantees that for every \( i \in n \) the set \( \Omega_i \) is thin. Therefore, \( \omega = \bigcup_{i \in n} \Omega_i \) is the union of \( n \) thin sets \( \Omega_0, \ldots, \Omega_{n-1} \).

Now take any partition \( \{P_1, \ldots, P_{n-1}\} \) of \( \omega \) into \( n - 1 \) pieces. For every \( i \in n \) consider the ultrafilter \( \mathcal{U}_i = \{V \subseteq \omega : \exists U \in \mathcal{U}_i \text{ such that } U + i \subseteq V\} \) on \( \omega \), and observe that for some number \( k_i \in \{1, \ldots, n\} \) the set \( P_{k_i} \) belongs to \( \mathcal{U}_i \). By the Pigeonhole principle, there exists an index \( k \in \{1, \ldots, n - 1\} \) such that the set \( \{i \in n : k_i = k\} \) contains two numbers \( i < j \). Hence \( P_k \in \mathcal{U}_i \cap \mathcal{U}_j \) and there exists \( \alpha \in \mathcal{U} \) such that \( U_\alpha \cup \{i, j\} \subseteq P_k \). Consider the pair \( p = (i, j) \in [n]^2 \) and observe that for the entourage \( E_\alpha = \Delta_\omega \cup \{(x, f_\alpha(x)) : x \in \omega\} \) the set \( \{x \in P_k : P_k \cap E_\alpha(x) \neq \{x\}\} \) contains the set \( A_{\alpha, p} \cup \{i, j\} \) and hence is infinite, witnessing that \( P_k \) is not thin.

**Problem 6.10.** Let \( G \) be a group of cardinality \( \aleph_1 \) endowed with the finitary coarse group structure \( \mathcal{E}_G \). Can every \( n \)-thin subset of \( G \) be partitioned into \( n \) thin subsets? This is so if \( G \) is Abelian, see [36].

## 7. Coarse groups

A topological space \( X \) with no isolated points is called **maximal** if \( X \) has an isolated point in every stronger topology. Under CH, there exists a maximal topological group \( [14] \). On the other hand, if there exists a maximal topological group, then there is a \( P \)-point in \( \beta \omega \setminus \omega \) [17].
If a topological group $G$ contains an infinite totally bounded subset, then there exists a non-closed discrete subset of $G$ \[18\]. Every discrete subset of a maximal group is closed. Answering a question from \[18\], Reznichenko and Sipacheva \[39\] proved that if there exists a countable non-discrete topological group in which every discrete subset is closed, then there is a rapid ultrafilter on $\omega$.

For a group $G$, a family $\mathcal{I}$ of subset of $G$ is called \[32\] a group ideal if $\bigcup \mathcal{I} = G$, $\mathcal{I}$ is closed under taking subsets and finite unions, and $A, B \in \mathcal{I}$ implies $AB^{-1} \in \mathcal{I}$. Every group ideal $\mathcal{I}$ defines a coarse structure on $G$ with the base $\{(x, Fx) : x \in G \} : 1_G \in F \in \mathcal{I}$ and $G$ endowed with this coarse structure is called a coarse group.

A coarse group $(G, \mathcal{E})$ is called maximal $(G, \mathcal{E})$ is unbounded but $G$ is bounded in every (not necessary group) coarse structure on $G$, which is strictly larger that $\mathcal{E}$.

Under CH, there exists a maximal coarse Boolean group \[38\].

Problem 7.1 \[38\]. Does there exist a maximal coarse group in ZFC?

Every maximal coarse group $(G, \mathcal{E})$ is large \[38\] and hence indiscrete.

Problem 7.2. In ZFC, does there exist an indiscrete coarse group $(G, \mathcal{E})$?

Let $G$ be a totally bounded topological group endowed with the finitary coarse group structure $\mathcal{E}_G$. Then there exist $\mathcal{E}_G$-discrete subsets $A, B \subseteq G$ such that $A$ is dense and $B$ has the unique limit point \[25\]. For open problems concerning $\mathcal{E}_G$-discrete subsets of topological groups, see \[25\], where the following problem is posed.

Problem 7.3. In ZFC, does there exist a countable non-discrete topological group $G$ in which every $\mathcal{E}_G$-discrete subset is closed?

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