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A NOTE ON MEAN CURVATURE, MASLOV CLASS AND SYMPLECTIC AREA OF LAGRANGIAN IMMERSIONS

Abstract. In this note we prove a simple relation between the mean curvature form, symplectic area, and the Maslov class of a Lagrangian immersion in a Kähler-Einstein manifold. An immediate consequence is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.

1. Introduction

Let $(M, \omega)$ be a Kähler-Einstein manifold whose Ricci curvature is a multiple of the metric by a real number $\lambda$. Hence the Kähler form $\omega$ and the first Chern class $c_1(M)$ are related by $c_1(M) = \lambda[\omega]$. Let $L$ be an immersed Lagrangian submanifold of $M$ and let $\sigma_L$ be the mean curvature form of $L$ (which is a closed 1-form on $L$). Let $F : \Sigma \to M$ be a smooth map from a compact connected surface to $M$ whose boundary $\partial F$ is contained in $L$. Let $\mu(F)$ be the Maslov class of $F$ and $\omega(F)$ its symplectic area. The goal of this note is to prove the following simple relation between these quantities:

\begin{equation}
\mu(F) - 2\lambda\omega(F) = \frac{\sigma_L(\partial F)}{\pi}.
\end{equation}

This relation was given in \cite{M} for $\mathbb{C}^n$ and in \cite{Ar} for Calabi-Yau manifolds. Dazord \cite{D} showed that the differential of the mean curvature form is the Ricci form, so in the Kähler-Einstein case $\sigma_L$ is closed. Y.G. Oh \cite{Oh2} investigated the symplectic area in the case that the mean curvature form is exact.

In the case $\lambda > 0$, Lagrangian submanifolds for which the left-hand side vanishes on all disks $F$ are called monotone in the symplectic geometry literature, cf. \cite{Oh1}. An immediate consequence of (1) is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.

In view of the condition $c_1(M) = \lambda[\omega]$, the left-hand side of (1) depends only on the boundary of $F$. Thus if the map $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$ is trivial it defines a cohomology class $\delta_L \in H^1(L; \mathbb{R})$ via $\delta_L(\gamma) := \mu(F) - 2\lambda\omega(F)$ for some 2-cycle $F$ with $\partial F = \gamma$. It follows that in this case the cohomology class of the mean curvature form $\sigma_L$ is invariant under symplectomorphisms of $M$. This generalizes Oh’s observation \cite{Oh2} that the cohomology class is invariant under Hamiltonian deformations. One consequence is the following:

Let $(M, \omega)$ be a Kähler manifold with $c_1(M) = \lambda[\omega] \in H_2(M; \mathbb{R})$. Let $L$ be an immersed Lagrangian submanifold of $M$ such that the map $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$ is trivial and $\delta_L \neq 0$. Suppose there is a Kähler-Einstein metric $\omega_{KE}$ in the same cohomology class as $\omega$ and $\phi : (M, \omega) \to (M, \omega_{KE})$ is a symplectomorphism (e.g. the one provided by Moser’s theorem). Then $\phi(L)$ is Lagrangian but not minimal.

Note that for $\lambda \neq 0$ most Lagrangian submanifolds $L$ with nontrivial first Betti number such that $H_1(L) \to H_1(M)$ vanishes have $\delta_L \neq 0$: For any such $L$, pick a

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normal vector field \( v \) to \( L \) such that \( i_v \omega \) is closed on \( L \) and non-trivial cohomologically. Then small time variations of \( L \) through \( v \) produce Lagrangian submanifolds with nontrivial \( \delta_L \).

2. Notation

We first recall the definition of the Maslov index that is suitable for our purposes. Let \( V \) be a Hermitian vector space of complex dimension \( n \). Let \( \Lambda^{(n,0)}V \) be the (one-dimensional) space of holomorphic \((n,0)\)-forms on \( V \) and set

\[
K^2(V) := \Lambda^{(n,0)}V \otimes \Lambda^{(n,0)}V.
\]

Let \( L \) be a Lagrangian subspace of \( V \). We can associate to \( L \) an element \( \kappa(L) \) in \( \Lambda^{(n,0)}V \) of unit length which restricts to a real volume form on \( L \). This element is unique up to sign and therefore defines a unique element of unit length

\[
\kappa^2(L) := \kappa(L) \otimes \kappa(L) \in K^2(V).
\]

Thus we get a map \( \kappa^2 \) from the Grassmanian \( \text{Gr}_{\text{Lag}}(V) \) of Lagrangian planes to the unit circle in \( K^2(V) \). This map induces a homomorphism \( \kappa_2^* \) of fundamental groups

\[
\kappa_2^* : \pi_1(\text{Gr}_{\text{Lag}}(V)) \to \mathbb{Z}.
\]

To understand the map \( \kappa_2^* \), let \( L \) be a Lagrangian subspace and let \( v_1, \ldots, v_n \) be an orthonormal basis for \( L \). For \( 0 \leq t \leq 1 \) consider the subspace

\[
L_t = \text{span}\{v_1, \ldots, v_{n-1}, e^{\pi it}v_n\}.
\]

This loop \( \{L_t\} \) is the standard generator of \( \pi_1(\text{Gr}_{\text{Lag}}(V)) \). The induced elements in \( \Lambda^{(n,0)}V \) are related by \( \kappa(L_t) = \pm e^{-\pi it}\kappa(L) \), so \( \kappa^2(L_t) = e^{-2\pi it}\kappa^2(L) \) and \( \kappa^2(\{L_t\}) = -1 \). Thus we see that the homomorphism \( \kappa_2^* \) is related to the Maslov index \( \mu \) (as defined, e.g., in [ALP]) by

\[
\kappa_2^* = -\mu : \pi_1(\text{Gr}_{\text{Lag}}(V)) \to \mathbb{Z}.
\]

Now let \((M, \omega)\) be a symplectic manifold of dimension \( 2n \). Pick a compatible almost complex structure \( J \) on \( M \) and let \( K(M) \) be the canonical bundle of \( M \), i.e., \( K(M) := \Lambda^{(n,0)}T^*M \) is the bundle of \((n,0)\)-forms on \( M \). Note that \( c_1(K(M)) = -c_1(M) \). Let \( K^2(M) := K(M) \otimes K(M) \) be the square of the canonical bundle.

Let \( L \) be an immersed Lagrangian submanifold of \( M \). For any point \( l \in L \) there is an element of unit length \( \kappa(l) \) of \( K(M) \) over \( l \), unique up to sign, which restricts to a real volume form on the tangent space \( T_lL \). The squares of these elements give rise to a section of unit length

\[
\kappa_2^L : L \to K^2(M).
\]

Now let \( F : \Sigma \to M \) be a smooth map with boundary \( \partial F \) on \( L \). The \textit{symplectic area} of \( F \) is

\[
\omega(F) = \int_{\Sigma} F^*\omega.
\]

This defines a map from the relative second homology group to \( \mathbb{R} \),

\[
\left[ \omega \right] : H_2(M, L; \mathbb{Z}) \to \mathbb{R}.
\]

To define the Maslov class \( \mu(F) \), choose a unitary frame for the tangent bundle \( TM \) along \( F \). Consider the dual frame and wedge all its elements. Thus we get a
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unit length section \( \kappa_F \) of \( K(M) \) over \( F \). Now on the boundary \( \partial F = F(\partial \Sigma) \) we also have the section \( \kappa_L^2 \) defined above. We can uniquely write

\[
\kappa_L^2 = e^{i\theta} \kappa_F^2
\]

for a function \( e^{i\theta} : \partial \Sigma \to S^1 \) to the unit circle. The Maslov class \( \mu(F) \) is minus its winding number,

\[
\mu(F) := -\frac{1}{2\pi} \int_{\partial F} d\theta.
\]

This defines a map

\[
\mu : H_2(M, L; \mathbb{Z}) \to \mathbb{Z}.
\]

In view of the discussion above, this definition agrees with the usual definition of the Maslov class, cf. \[ALP\].

Now suppose that \( c_1(M) = \lambda[\omega] \in H_2(M; \mathbb{R}) \). It is well-known that if \( \partial F \) is trivial in \( H_1(L; \mathbb{R}) \), then \( F \) represents an element in \([F] \in H_2(M; \mathbb{R})\) and

\[
\mu(F) = 2c_1(M)([F]) = 2\lambda \omega(F).
\]

So in this case \( \mu(F) - 2\lambda \omega(F) \) depends only on the boundary \( \partial F \in H_1(L; \mathbb{R}) \). If, moreover, the map \( H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R}) \) is trivial, this expression defines a cohomology class \( \delta_L \in H^1(L; \mathbb{R}) \) via

\[
\delta_L(\gamma) := \mu(F) - 2\lambda \omega(F)
\]

for some 2-cycle \( F \) with \( \partial F = \gamma \).

3. PROOF

Now assume that \((M, \omega)\) is Kähler-Einstein, i.e., \( M \) carries a Kähler metric whose Ricci curvature is a multiple of the metric by a constant \( \lambda \in \mathbb{R} \). This is equivalent to saying that the curvature form of the canonical bundle \( K(M) \) equals \(-\frac{2}{\pi} \lambda \omega\). We denote the connections on \( K(M) \) and \( K^2(M) \) (induced by the Levi-Civita connection) by \( \nabla \).

Let \( L \) be an immersed Lagrangian submanifold of \( M \) and let \( \kappa_L^2 \) be the canonical section of \( K^2(M) \) over \( L \) as above. The section \( \kappa_L^2 \) defines a connection 1-form \( \xi_L \) for \( K^2(M) \) over \( L \) by the condition \( \nabla \kappa_L^2 = \xi_L \otimes \kappa_L^2 \). Since \( \kappa_L^2 \) has constant length \( 1 \), \( \xi_L \) is an imaginary valued 1-form on \( L \). From the Einstein condition and the fact that \( L \) is Lagrangian we get \( d(i\xi_L) = -4\pi \lambda \omega|_L = 0 \), so the form \( i\xi_L \) is closed.

Let \( H \) be the trace of the second fundamental form of \( L \) (the mean curvature vector field of \( L \)). Thus \( H \) is a section of the normal bundle to \( L \) in \( M \) and we have a corresponding 1-form \( \sigma_L := i_H \omega \) on \( L \). The following fact goes back to \[Oh2\] (see also \[Gold1\] for a proof):

\[
\sigma_L = i\xi_L/2.
\]

(Here the factor \( 1/2 \) is due to the fact that \( \xi_L \) is a connection 1-form for \( K^2(M) \) rather than \( K(M) \).) Thus \( \sigma_L \) is a closed 1-form on \( L \), called the mean curvature form on \( L \).

Having explained all the terms in formula \[1\], we now turn to its proof. Let \( F : \Sigma \to M \) be a smooth map from a compact surface with boundary on \( L \). Define the section \( \kappa_F \) of \( K(M) \) over \( F \) as above, using a unitary trivialization of \( TM \) over
Let $\xi_F$ be the connection 1-form along $F$ defined by $\nabla \kappa^2_F = \xi_F \otimes \kappa^2_F$. The Einstein condition tells us that $d(i\xi_F) = -4\pi \lambda F^* \omega$. Thus by Stokes’ theorem,

$$2\lambda \omega(F) = \int_{\partial F} -i \xi_F.$$ 

Recall that along $\partial F$ we have $\kappa^2_L = e^{i\theta} \kappa^2_F$ for a function $e^{i\theta} : \partial \Sigma \to S^1$, and the Maslov class is given by

$$\mu(F) = -\frac{1}{2\pi} \int_{\partial F} d\theta.$$ 

The connection 1-forms $\xi_F$ and $\xi_L$ are related by $\xi_L = \xi_F + i \, d\theta$.

Thus

$$\frac{\sigma_L(\partial F)}{\pi} = \int_{\partial F} \frac{i \xi_L}{2\pi} = \int_{\partial F} \frac{i \xi_F}{2\pi} - \int_{\partial F} \frac{d\theta}{2\pi} = \mu(F) - 2\lambda \omega(F).$$

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