Towards a Covariant Loop Quantum Gravity

Etera R. Livine*
Perimeter Institute, 31 Caroline St N., Waterloo, ON, Canada N2L 2Y5 and
Laboratoire de Physique, ENS Lyon, CNRS UMR 5672, 46 Allée d’Italie, 69364 Lyon Cedex 07

We review the canonical analysis of the Palatini action without going to the time gauge as in the standard derivation of Loop Quantum Gravity. This allows to keep track of the Lorentz gauge symmetry and leads to a theory of Covariant Loop Quantum Gravity. This new formulation does not suffer from the Immirzi ambiguity, it has a continuous area spectrum and uses spin networks for the Lorentz group. Finally, its dynamics can easily be related to Barrett-Crane like spin foam models.

I. INTRODUCTION

Over the recent years, Loop Quantum Gravity (LQG) has become a promising approach to quantum gravity (see e.g. [1, 2] for reviews). It has produced concrete results such as a rigorous derivation of the kinematical Hilbert space with discrete spectra for areas and volumes, the resulting finite isolated horizon entropy counting and regularization of black hole singularities, a well-defined framework for a (loop) quantum cosmology, and so on. Nevertheless, the model still has to face several key issues: a well-defined dynamics with a semi-classical regime described by Newton gravity law and general relativity, the existence of a physical semi-classical state corresponding to an approximately flat space-time, a proof that the no-gravity limit of LQG coupled to matter is standard quantum field theory, the Immirzi ambiguity,... Here, we address a fundamental issue at the root of LQG, which is necessarily related to these questions: why the SU(2) gauge group of Loop Quantum Gravity? Indeed, the compactness of the SU(2) gauge group is directly responsible for the discrete spectra of areas and volumes, and therefore is at the origin of most of the successes of LQG: what happens if we drop this assumption?

As we will see, this leads to a theory of Covariant Loop Quantum Gravity [8, 11, 13], which uses the same techniques and tools as LQG but whose gauge group is the Lorentz group SL(2, C) instead of SU(2).

Let us start by reviewing the general structure of LQG and how the SU(2) gauge group arises. In a first order formalism, General Relativity (GR) is formulated in term of tetrad $e$ which indicates the local Lorentz frame and a Lorentz connection $\omega$ which describes the parallel transport. The theory is invariant under local Lorentz transformations and (space-time) diffeomorphisms.

The complex formulation of LQG is equivalent to that first order formalism. It is a canonical formulation based on a splitting of the space-time as a spatial slice evolving in time. The canonical variables are the Ashtekar variables: a self-dual complex connection $A^{\text{Ash}}$ and its conjugate triad field $E$. The theory is invariant under the Lorentz group SL(2, C) (seen as the complexified SU(2) group) and under space-time diffeomorphisms. In these variables, GR truly looks like a SU(2) gauge theory. The difficulty comes from reality constraints expressing that the imaginary part of the triad field $E$ vanishes and that the real part of the connection $A^{\text{Ash}}$ is actually a function of the $E$. More precisely, on one hand, keeping the metric real under the Hamiltonian flow requires that $\text{Re} \ E \nabla [E, E] = 0$ and, on the other hand, the real part of $A^{\text{Ash}} = \Gamma(E) + iK$ is the spin-connection $\Gamma(E)$ while its imaginary part is the extrinsic curvature. Such constraints must be taken into account by the measure of the space of connection and render the quantization complicated.

The real formulation of LQG came later as a way to avoid the reality constraint issue and has now become the standard formulation of LQG. It uses the real Ashtekar-Barbero connection $A^\gamma = \Gamma(E) + \gamma K$ and its conjugate triad field $E$. $\gamma$ is called the Immirzi parameter and is an arbitrary real parameter. The theory is derived from the original first order GR formulation in a particular (partial) gauge fixing, the time gauge, which breaks the local Lorentz invariance down to a local SU(2) gauge invariance. The theory then has a compact gauge invariance, is free from complicated reality conditions and its Hamiltonian (constraint) can be regularized and quantized. Nevertheless, it appears as the result of a gauge fixing. The natural question is whether this affects the quantization or not: can we

* elivine@perimeterinstitute.ca
trust all the results of the real LQG formulation? As we will see, considering SU(2) as the gauge group of GR instead of the non-compact Lorentz group is related to several issues faced by the standard formulation of LQG.

- Since we have chosen a particular gauge fixing, should not we take it into account in the measure on the phase space through a Faddeev-Popov determinant? Would it not change the spectrum of the observables of the theory? Moreover, does choosing the time gauge constrain us to a specific class of measurements?

- The Ashtekar-Barbero connection $A^\gamma$, on the spatial slice, is not the pull-back of a space-time connection [3], since one can show that its holonomy on the spatial slice depends on the embedding on that slice in space-time. This is true unless the Immirzi parameter is taken equal to the purely imaginary values $\gamma = \pm i$ corresponding to the original self-dual Ashtekar connection. From that point of view, the real connection $A^\gamma$ cannot be considered as a genuine gauge field and SU(2) cannot be viewed as the gauge group of gravity.

- The Complex LQG formalism has a simple polynomial Hamiltonian constraint. On the other hand, the real LQG formulation has an extra non-polynomial term. In fact, it seems we trade the reality condition problem with the issue of a more complicated Hamiltonian.

- There is a discrepancy with the standard spin foam models for GR. Spin foam models have been introduced as discretization of the GR path integral seen as a constrained topological theory [4]. They naturally appear as the space-time formalism describing the evolution and dynamics of the LQG canonical theory. Nevertheless, they use the Lorentz group as gauge group and therefore the quantum states of quantum geometry are spin networks for the Lorentz group [5] instead of the standard SU(2) spin networks of LQG.

- In three space-time dimension, the standard Loop Gravity quantization of 3d gravity has as gauge group the full Lorentz group and not only the little group of spatial rotations. Indeed, in three space-time dimensions, the gauge group is always the Lorentz group, SO(3) in the Riemannian version [6] and SO(2, 1) in the Lorentzian theory [7]. This allows a precise matching between the LQG framework and the spin foam quantization for 3d gravity.

- Finally, the real LQG formulation faces the issue of the Immirzi ambiguity: $\gamma$ is an arbitrary unfixed parameter. It enters the spectrum of geometrical observables such as areas and volumes (at the kinematical level). It is usually believed that black hole entropy calculations should fix this ambiguity by requiring a precise match with the semi-classical area-entropy law. More recently, $\gamma$ has been argued to be related to parity violation when coupling fermions to gravity. Nevertheless, at the level of pure gravity, there still lacks a clear understanding of the physical meaning of $\gamma$: it does not change the classical phase space and canonical structure but leads to unitarily inequivalent quantization (at the kinematical level). We can not forget the possibility that this dependence on $\gamma$ might only be due to the choice of the time gauge.

Here, we review a Lorentz covariant approach to Loop Quantum Gravity, which has been coined Covariant Loop Quantum Gravity. It is based on an explicit canonical analysis of the original Palatini action for GR without any time gauge first performed by Alexandrov [8]. The canonical variables are a Lorentz connection and its conjugate triad (a 1-form valued in the Lorentz algebra). The states of quantum geometry are Lorentz spin networks which reduce in a particular case to the standard SU(2) spin networks.

The main difference with the standard LQG is a continuous spectrum for areas at the kinematical level. The main advantages of the formalism is that the Immirzi ambiguity disappears and it becomes possible to make contact between the canonical theory and spin foam models. The main drawbacks of the approach are a non-compact gauge group and a non-commutative connection. Finally, there is still a lot of work left in order to precisely define the framework: rigourously define and study the Hilbert space (the problem is to deal with the non-commutativity of the connection) and derive the dynamics of the theory (quantize the Hamiltonian constraint and compare to the standard spin foam models).

II. LORENTZ COVARIANT CANONICAL ANALYSIS

In a first order formalism, GR is formulated in terms of the space-time connection $\omega = \omega^I \epsilon_I J_{I\mu} dx^\mu$, defined as a $\mathfrak{so}(3,1)$-valued 1-form, and the tetrad field $e^\mu = e^\mu_\nu dx^\nu$. The space-time is a four-dimensional Lorentzian manifold $\mathcal{M}$.
We decompose the tetrad field \( e \) and then quantize the theory. Here, we follow the canonical analysis of [8].

\[
S[\omega, e] = \int_M \left[ \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) - \frac{1}{\gamma} e^I \wedge e^J \wedge F_{IJ}(\omega) \right],
\]

where \( F(\omega) = d\omega + \omega \wedge \omega \) is the curvature tensor of the connection \( \omega \). The metric is defined from the tetrad field as \( g_{\mu\nu} = e^I \eta_{IJ} e^J \). The first term of the previous action is the standard Palatini action. Its equations of motion are equivalent to the usual Einstein equations when the tetrad is non-degenerate. The second term actually has no effect on the equations of motion and thus does not matter at the classical level. The coupling constant \( \gamma \) is the Immirzi parameter.

The difficulty in the canonical analysis comes from the second class constraints. Indeed, the canonically conjugate variable to the connection \( \omega^I_J \) is \( \pi^a_{IJ} = e^{a\delta} \epsilon_{IJKL} e^b_{\delta} \). These variables are unfortunately not independent and they satisfy the simplicity constraints:

\[
\forall a, b, \epsilon^{IJKL} \pi^a_{IJ} \pi^b_{KL} = 0.
\]

These constraints are the non-trivial part of the canonical structure. Nevertheless, Holst showed in [9] that, in the time gauge \( e^a_0 = 0 \), the tetrad \( e \) reduces to a triad field \( E \), the simplicity constraints do not appear and we recover the canonical phase space and constraints of the real formulation of LQG with the Ashtekar-Barbero connection \( A^{(\gamma)} \) conjugated to \( E \) and the Immirzi parameter \( \gamma \).

The natural question is: how did the simplicity constraints go away? Barros pushed Holst’s analysis further and showed it is possible to solve these constraints explicitly [10]. The phase space is parameterized by two couples of conjugate variables \( (A, E) \) and \( (\chi, \zeta) \). The first couple of canonical variables is a generalization of the Ashtekar-Barbero connection and triad. The new variable \( \chi \) is the time normal (or internal time direction) defined as the normalised space component (in the internal indices) of the time component of the tetrad field: \( \chi^i = -e^{0i}/e^{00} \). Finally, it is possible to gauge fix the boost part of the Lorentz gauge symmetry by fixing \( \chi \). This is the time gauge. In this frame, we exactly retrieve the variables and constraints of LQG. However, the price is the loss of the explicit Lorentz covariance of the theory.

A. Second class constraints and the Dirac bracket

The strategy of Covariant Loop Gravity is to compute the whole set of second class constraints, derive the associated Dirac bracket and then quantize the theory. Here, we follow the canonical analysis of [8].

We start with a space-time \( \mathcal{M} \sim \mathbb{R} \times \Sigma \) where we distinguish the time direction from the three space dimensions. We decompose the tetrad field \( e^I \) as:

\[
e^0 = N dt + \chi_a E^a_0 dx^a
\]

\[
e^i = E^a_i N^a dt + E^a_i dx^a
\]

where \( i = 1, 3 \) is an internal index (space components of \( I \)) and \( a \) is the space index labelling the coordinates \( x^a \). \( N \) and \( N^a \) are respectively the lapse and the shift. \( \chi^i \) indicates the deviation of the normal to the canonical hypersurface from the time direction: the time normal is defined as the normalised time-like 4-vector \( \chi = (1, \chi_i)/\sqrt{1 - |\chi|^2} \).

Let’s call \( X, Y, \ldots = 1, 6 \) \( \mathfrak{sl}(2, \mathbb{C}) \)-indices labelling antisymmetric couples \( [IJ] \). We define new connection/triad variables valued in \( \mathfrak{sl}(2, \mathbb{C}) \) instead of the standard \( \mathfrak{su}(2) \) of LQG. The Lorentz connection \( A^X_a \) is:

\[
A^X_a = \left( \frac{1}{2} \omega^{0i} - \frac{1}{2} \epsilon^{i}{}_{jk} \omega^j \chi^k \right).
\]

Then we define a “rotational” triad and a boost triad,

\[
R^X_X = (-\epsilon^{ik} E^i_a \chi^k, E^0_a), \quad B^X_X = (\ast R^X) X = (E^0_a, \epsilon^{ik} E^i_a \chi^k),
\]

where \( \ast \) is the Hodge operator on \( \mathfrak{sl}(2, \mathbb{C}) \) switching the boost and rotation part of the algebra. We further define the actual projectors on the boost and rotation sectors of \( \mathfrak{sl}(2, \mathbb{C}) \), \( (P_R)_Y^X = R_X^a R^a_Y, (P_B)_Y^X = B_X^a B^a_Y \):

\[
P_R = \begin{pmatrix}
\frac{\sqrt{\chi^2}}{1 - \chi^2} & \frac{\epsilon^{ik} \chi^k}{1 - \chi^2} \\
\frac{\epsilon^{ik} \chi^k}{1 - \chi^2} & \frac{\sqrt{\chi^2}}{1 - \chi^2}
\end{pmatrix}, \quad P_B = \text{Id} - P_R, \quad P_B P_R = 0.
\]
\( P_R \) projects on the subspace \( \mathfrak{su}(2)_\chi \) generating the rotations leaving the vector \( \chi \) invariant, while \( P_B \) projects on the complementary subspace. The action then reads:

\[
S = \int dt d^3x \left( B_a^x - \frac{1}{\gamma} R_a^x \right) \partial_t A_a^x + \Lambda^X \mathcal{G}_X + N^a \mathcal{H}_a + N \mathcal{H}.
\]  

(6)

The phase space is thus defined with the Poisson bracket,

\[
\{ A_a^x(x), \left( B_b^y - \frac{1}{\gamma} R_b^y \right)(y) \} = \delta_a^x \delta_b^y \delta^{(3)}(x, y).
\]  

(7)

\( \Lambda^X, N^a, N \) are Lagrange multipliers enforcing the first class constraints:

\[
\mathcal{G}_X = \mathcal{D}_A \left( B_x - \frac{1}{\gamma} R_x \right), \quad \mathcal{H}_a = -\left( B_a^b - \frac{1}{\gamma} R_a^b \right) F_{ab}(A), \quad \mathcal{H} = \frac{1}{1 + \frac{1}{\gamma}} \left( B - \frac{1}{\gamma} R \right) F(A).
\]  

(8)

However, in contrast to the usual LQG framework, we also have second class constraints:

\[
\phi^{ab} = (\ast R^a)^X R_b^X = 0, \quad \psi^{ab} \approx RRD_A R.
\]  

(9)

The constraint \( \phi = 0 \) is the simplicity constraint. The constraint \( \psi = 0 \) comes from the Poisson bracket \( \{ \mathcal{H}, \phi \} \) and is required in order that the constraint \( \phi = 0 \) is preserved under gauge transformations (generated by \( \mathcal{G}, \mathcal{H}_a, \mathcal{H} \)) and in particular under time evolution. \( \psi \) corresponds to the reality constraint \( \text{Re}[E \nabla[E, E]] = 0 \) of Complex LQG.

To solve the second class constraints, we define the Dirac bracket \( \{ f, g \}_D = \{ f, g \} - \{ f, \varphi \}_D^{-1} \{ \varphi, g \} \) where the Dirac matrix \( \Delta_{rs} = \{ \varphi_r, \varphi_s \} \) is made of the Poisson brackets of the constraints \( \varphi = \{ \phi, \psi \} \). Following \([8, 11]\), one then checks that the algebra of the first class constraints is not modified. Defining smeared constraints, we find the following Dirac brackets:

\[
\mathcal{G}(\Lambda) = \int \Sigma \Lambda^X \mathcal{G}_X, \quad \mathcal{H}(N) = \int \partial_t N \mathcal{H}, \quad \mathcal{D}(\tilde{N}) = \int \Sigma N^a(\mathcal{H}_a + A_a^X \mathcal{G}_X),
\]

\[
\{ \mathcal{G}(\Lambda_1), \mathcal{G}(\Lambda_2) \}_D = \mathcal{G}(\{ \Lambda_1, \Lambda_2 \}), \quad \{ \mathcal{D}(\tilde{N}), \mathcal{D}(\tilde{M}) \}_D = -\mathcal{D}(\{ \tilde{N}, \tilde{M} \}),
\]

\[
\{ \mathcal{D}(\tilde{N}), \mathcal{G}(\Lambda) \}_D = -\mathcal{G}(N^a \partial_a \Lambda), \quad \{ \mathcal{H}(\tilde{N}), \mathcal{H}(N) \}_D = -\mathcal{H}(\partial N),
\]

\[
\mathcal{H}(N), \mathcal{G}(\Lambda) \}_D = 0, \quad \{ \mathcal{H}(\tilde{N}), \mathcal{H}(M) \}_D = \mathcal{D}(\tilde{K}) - \mathcal{G}(K^b A_b),
\]

\[
[\Lambda_1, \Lambda_2]^X = f_{YZ}^X \Lambda_Y^Z, \quad [\tilde{N}, \tilde{M}]^a = N^b \partial_b M^a - M^b \partial_b N^a, \quad K^b = (N \partial_b M - M \partial_b N) R^a \partial_b g^{XY},
\]

where \( f_{YZ}^X \) are the structure constant of the algebra \( \mathfrak{su}(2, \mathbb{C}) \). With \( A \in \{ 1, 2, 3 \} \) boost indices and \( B \in \{ 4, 5, 6 \} \sim \{ 1, 2, 3 \} \) rotation indices, we have \( f_{AA}^A = f_{BB}^B = f_{AB}^B = 0 \) and \( f_{AA}^A = -f_{AB}^B = -f_{BB}^B \) given by the antisymmetric tensor \( \epsilon \).

The \( \mathcal{G}'s \) generate \( \text{SL}(2, \mathbb{C}) \) gauge transformations. The vector constraint \( \mathcal{H}_a \) generates spatial diffeomorphisms on the canonical hypersurface invariant \( \Sigma \). Finally, the scalar constraint \( \mathcal{H} \) is called the Hamiltonian constraint and generates the (time) evolution of the canonical variables.
B. The choice of connection and the Area spectrum

As shown in [8, 11, 12], although the triad field \( R \) is still commutative for the Dirac bracket, the properties of the connection \( A \) change drastically: it is not canonically conjugated to the triad and it does not commute with itself. Nevertheless, one should keep in mind that when using the Dirac bracket the original canonical variables lose their preferred status and we should feel free to identify better suited variables. Following [12], we do not modify the triad \( R \) but we look for a new connection \( A \) satisfying the following natural criteria:

- \( A \) must be a Lorentz connection i.e. it should behave correctly under the Gauss law \( \mathcal{G} \):

\[
\{ \mathcal{G}(\Lambda), A^X_a \}_D = \partial_a \Lambda^X - [\Lambda, A_a]^X = \partial_a \Lambda^X - f_{YZ}^X \Lambda^Y A^Z_a.
\]  

(10)

- \( A \) must be a 1-form et therefore properly transform under spatial diffeomorphisms:

\[
\{ D(\vec{N}), A \}_D = A^X_b \partial_a N^b - N^b \partial_a A^X_b.
\]  

(11)

- \( A \) must be conjugated to the triad \( R \). This is required in order that the area operators \( \text{Area}_S \sim \int_S d^2x \sqrt{n_a n_b R_a^X R_b^X} \) (with \( n_a \) the normal to the surface \( S \)) be diagonalized in the spin network basis resulting from a loop quantization. This condition reads:

\[
\{ A^X_a(x), R^Y_b(y) \}_D \propto \delta^X_a \delta^{(3)}(x, y).
\]  

(12)

We obtain a 2-parameter family of such connections \( A(\lambda, \mu) \) [12]:

\[
A^X_a(\lambda, \mu) = A^X_a + \frac{1}{2} (\gamma + \lambda - \mu \ast) P_R \left[ \frac{\gamma - \ast}{1 + \gamma^2} [B_\lambda, \mathcal{G}]^X \right.
+ \left( \lambda + (1 - \mu \ast) (P_R \ast A^X_a + \Theta^X_a(R)) \right),
\]

(13)

with

\[
\Theta^X_a(R) = \Theta^X_a(\chi) = \left( -\frac{\epsilon^{ijk} \partial_i \chi_j}{1 - \chi^2}, \frac{\partial_a \chi^i}{1 - \chi^2} \right).
\]

Their commutation relation with the triad are very simple:

\[
\{ A^X_a(\lambda, \mu), B^Y_b \}_D = \delta^X_a \left[ (\mu - \lambda \ast) P_B \right]_Y^X
\]

(14)

\[
\{ A^X_a(\lambda, \mu), P_B \}_D = \{ A^Y_b(\lambda, \mu), \chi \}_D = 0.
\]

(15)

Despite this, the bracket \( \{ A, A \}_D \) remains complicated. From there, the loop quantization chooses functions of \( A \) (Wilson loops and spin networks) as wave functions and raises the triads \( B, R \) to derivation operators. Each connection \( A(\lambda, \mu) \) will lead to a non-equivalent quantization. We can then compute the action of an area operator on a \( A(\lambda, \mu) \) Wilson line and we find [12, 13]:

\[
\text{Area}_S \sim l_P^2 \sqrt{\lambda^2 + \mu^2} C(\text{su}(2), \chi) - \mu^2 C_1(\text{sl}(2, \mathbb{C})) + \lambda \mu C_2(\text{sl}(2, \mathbb{C})).
\]

where \( C(\text{su}(2), \chi) = \vec{J} \cdot \vec{J} \) is the Casimir operator of \( \text{su}(2) \) (stabilizing the vector \( \chi \)), \( C_1(\text{sl}(2, \mathbb{C})) = T_X T_X = \vec{J}^2 - \vec{K}^2 \) and \( C_2(\text{sl}(2, \mathbb{C})) = (\ast T)^X T_Y = \vec{J} \cdot \vec{K} \) are the two (quadratic) Casimirs of \( \text{sl}(2, \mathbb{C}) \). Since the algebra \( \text{su}(2) \) enters the formula, one could think at first that this area spectrum is not Lorentz invariant. However, one must not forget that \( \chi \) enters the formula and gets rotated under Lorentz transformations. Thus we see two alternatives:

1. Either we work with functionals of the connection \( A \). Then a basis of quantum states is provided by spin networks for the Lorentz group. These are labelled by unitary representations of \( \text{sl}(2, \mathbb{C}) \), they diagonalize \( C_1(\text{sl}) \) and \( C_2(\text{sl}) \), but they do not diagonalize \( C(\text{su}) \). Therefore they do not diagonalize the area operator.
2. Or we work with functionals of both the connection $\mathcal{A}$ and the time normal field $\chi$. This is possible with $\mathcal{A}$ and $\chi$ commute (see (15)). It is possible to introduce \textit{projected spin networks}, which project on given eigenvalues of $C(\mathfrak{su})$ and therefore diagonalize the area operator. We will discuss the details of these states later.

In the following, we will work with the latter alternative. Then the irreducible unitary representations (of the principal series) of $\mathfrak{sl}(2, \mathbb{C})$ are labelled by a couple of numbers $(n \in \mathbb{N}, \rho \geq 0)$. The Casimir’s values are then:

$$C_1 = n^2 - \rho^2 - 1, \quad C_2 = 2n\rho, \quad C = j(j+1), \quad \text{with } j \geq n. \quad (16)$$

The restriction $j \geq n$ comes from the decomposition of the $\mathfrak{sl}(2, \mathbb{C})$ representations on $\mathfrak{su}(2)$ irreducible representations. Moreover this condition ensures that the area eigenvalues are all real (and positive) for any value of $(\lambda, \mu)$. This is a nice consistency check. Note however that, since the formula involves the real parameter $\rho$, we lose the discreteness of the spectrum, which was a key result of LQG!

Now, it seems that we do not have any preferred choice of connection, and therefore no rigorous prediction on the area spectrum. This would be an extra ambiguity besides the choice of the Immirzi parameter $\gamma$. Instead, we choose to impose further constraints on the connection $\mathcal{A}(\lambda, \mu)$ and two criteria naturally appear:

1. We require that the connection behaves properly under \textit{space-time} diffeomorphisms, generated by $\mathcal{H}_a$ and $\mathcal{H}$.

2. We require that the connection be commutative, i.e that $\{\mathcal{A}, \mathcal{A}\}_D$ vanishes.

Unfortunately, these two conditions are not compatible. As we will in the next sections, the first choice corresponds to the only unique choice of a \textit{covariant connection} and is the one used by the proposed Covariant LQG. Very interestingly, the area spectrum for this covariant connection does not depend on the Immirzi parameter $\gamma$. While this resolves the Immirzi ambiguity, it is still complicated to quantize the theory due to the non-commutativity of the connection.

On the other hand, the second criteria leads to a unique commutative Lorentz extension of the Ashtekar-Barbero connection. It allows to recover the $\mathfrak{su}(2)$ structure and area spectrum and Immirzi ambiguity of the real formulation of LQG.

This raises the issue of the space-time covariance of the standard formulation of LQG based on the Ashtekar-Barbero connection. Although there is no doubt that $\mathcal{H}_a$ and $\mathcal{H}$ satisfy the same algebra as the generators of the space-time diffeomorphisms, the action of $\mathcal{H}$ on the connection is not the usual one. This means that this connection is not a space-time 1-form and thus does not have a clear geometric interpretation. Although it is not clear to which extent this is a problem, we expect this to be an obstacle when studying the quantum dynamics of the theory.

### III. THE COVARIANT CONNECTION AND PROJECTED SPIN NETWORKS

#### A. A continuous area spectrum

As shown in [12, 13], there is a unique space-time connection, i.e which transforms as a 1-form under space-time diffeomorphism generated by the constraints $\mathcal{H}_a, \mathcal{H}$. It is actually the unique connection which is equal to the original connection $A$ on the constrained surface $\mathcal{G}^X = \mathcal{H}_a = \mathcal{H} = 0$. It corresponds to the choice $(\lambda, \mu) = (0, 1)$ and we will simply write $\mathcal{A}$ for $\mathcal{A}(0,1)$ in the following sections. Its brackets with the triad is:

$$\{A^X_a, B^b_Y\}_D = \delta^b_a(P_B)^X_Y, \quad \{A^X_a, (P_B)^Y_2\}_D = 0. \quad (17)$$

The first bracket says that only the boost part of the connection seems to matter. The second relation is also very important and states that the field $\chi$ commutes with both the connection and can thus be treated as an independent variable. Then, following the results of the previous section, it turns out that the area spectrum does not depend on the Immirzi parameter at all and is given by the boost part of the $\mathfrak{sl}(2, \mathbb{C})$ Casimir:

$$\text{Area} \sim l_p^2 \sqrt{C(\mathfrak{su}(2))} - C_1(\mathfrak{sl}(2, \mathbb{C})) = l_p^2 \sqrt{j(j+1) - n^2 + \rho^2 + 1}.$$  

Interestingly, this spectrum is \textit{not} the standard $\sqrt{j(j+1)} \mathfrak{su}(2)$-Casimir area spectrum, but it contains a term coming from the Lorentz symmetry which makes it \textit{continuous}. 

The problem with this connection is that it is non-commutative. Indeed, the bracket \( \{A^X, A^Y\}_D \) does not vanish and turns out to be complicated. At least, it is possible to prove that it does not depend on the Immirzi parameter. Actually it was shown [13] that this complicated bracket was due to the fact that the rotational part of \( A \) was not independent from the triad field but equal to the spin-connection:

\[
P_R A^X = \Gamma(R)_A^X \sim [R, \partial R] + RR[R, \partial R].
\]

The explicit expression can be found in [13–15]. This relation is reminiscent of the reality constraint of the complex LQG formulation where the real part of the self-dual connection is a function of the triad \( E \) and is constrained to be the spin-connection \( \Gamma(E) \). Moreover, it turns out that both the rotation and the boost parts of the connection are commutative:

\[
\{ (P_B A)^X, (P_B A)^Y \}_D = \{ (P_B A)^X, (P_B A)^Y \}_D = 0. \tag{18}
\]

At the end of the day, the non-commutativity of the connection comes from the facts that \( P_B A \) is canonically conjugate to the (boost) triad \( B = \ast R \) and that the other half of the connection \( P_R A \) is a function of the triad. It thus seems as this non-commutativity comes from taking into account the reality constraints.

**B. Projected spin networks**

In order to talk about the quantum theory and the area spectrum, we should precisely define the Hilbert space and our quantum states of space(-time) geometry. Since geometric observables (such as the area) involve \( \chi \) and that \( \chi \) commutes with \( A \), it is natural to consider functionals \( f(A, \chi) \) as wave functions for the quantum geometry. Then requiring gauge invariance under the Lorentz group SL(2, \( \mathbb{C} \)) reads:

\[
\forall g \in \text{SL}(2, \mathbb{C}) \ f(A, \chi) = f(\varphi A = gA^{-1} + g\partial g^{-1}, g.) \chi).
\]

Assuming that \( \chi \) is time-like everywhere (i.e. the canonical hypersurface is space-like everywhere) and that all the fields are smooth, we can do a smooth gauge transformation to fix \( \chi \) to \( \chi_0 = (1, 0, 0, 0) \) everywhere. Thus the wave function is entirely defined by its section \( f_{\chi_0}(A) = f(A, \chi_0) \) at \( \chi = \chi_0 \) constant:

\[
f(A, \chi) = f_{\chi_0}(\varphi A) \quad \text{for all } g \text{ such that } g.A = \chi_0.
\]

Then \( f_{\chi_0} \) has a residual gauge invariance under \( \text{SU}(2)_{\chi_0} \). We are actually considering functionals of the Lorentz connection \( A \) which are not invariant under the full Lorentz group \( \text{SL}(2, \mathbb{C}) \) but only under the compact group of spatial rotations (defined through the field \( \chi \)).

To proceed to a loop quantization, we introduce **cylindrical functionals** which depend on the fields \( A, \chi \) through a finite number of variables. More precisely, given a fixed oriented graph \( \Gamma \) with \( E \) links and \( V \) vertices, a cylindrical function depends on the holonomies \( U_1, \ldots, U_E \in \text{SL}(2, \mathbb{C}) \) of \( A \) along the edges of \( \Gamma \) and on the values \( \chi_1, \ldots, \chi_V \) of \( \chi \) at the vertices of the graph. The gauge invariance then reads:

\[
\forall k_v \in \text{SL}(2, \mathbb{C}), \ \phi(U_{e_1}, \ldots, \chi_{v_1}, \ldots) = \phi(k_{s(e)}U_{k_{t(e)}^{-1}}, \ldots, k_v, \chi_v, \ldots), \tag{20}
\]

where \( s(e), t(e) \) denote the source and target vertices of an edge \( e \). As previously, such an invariant function is fully defined by its section \( \phi_{\chi_0}(U_1, \ldots, U_E) \) at constant \( \chi_1 = \ldots = \chi_V = \chi_0 \). The resulting function \( \phi_{\chi_0} \) is invariant under \( \text{SU}(2)_{\chi_0} \): we effectively reduced the gauge invariance from the non-compact \( \text{SL}(2, \mathbb{C}) \) to the compact \( \text{SU}(2)_{\chi_0} \).

Physically, the field \( \chi \) describe the embedding of the hypersurface \( \Sigma \) in the space-time \( \mathcal{M} \). From the point of view of the cylindrical functionals, the embedding is defined only at a finite number of points (the graph’s vertices) and is left fuzzy everywhere else. At these points, the normal to the hypersurface is fixed to the value \( \chi_v \) and the symmetry thus reduced from \( \text{SL}(2, \mathbb{C}) \) to \( \text{SU}(2)_{\chi_v} \).

Since the gauge symmetry is compact, we can use the Haar measure on \( \text{SL}(2, \mathbb{C}) \) to define the scalar product on the space of wave functions:

\[
\langle \phi|\psi \rangle = \int_{\text{SL}(2, \mathbb{C})^E} \prod_e dg_e \tilde{\phi}(g_e, \chi_e)\psi(g_e, \chi_e) = \int_{\text{SL}(2, \mathbb{C})^E} \prod_e dg_e \tilde{\phi}_{\chi_0}(g_e)\psi_{\chi_0}(g_e). \tag{21}
\]
The Hilbert space $H_{\Gamma}$ is finally defined as the space of $L^2$ cylindrical functions with respect to this measure. A basis of this space is provided by the projected spin networks [14, 16]. Following the standard construction of spin networks, we choose one (irreducible unitary) SL$(2, \mathbb{C})$ representation $\mathcal{I}_e = (n_e, \rho_e)$ for each edge $e \in \Gamma$. However, we also choose one SU$(2)$ representation $j^{(e)}_v$ for each link $e$ at each of its extremities $v$. Moreover, we choose a SU$(2)$ intertwiner $i_v$ for each vertex instead of an SL$(2, \mathbb{C})$ intertwiner. This reflects that the gauge invariance of the cylindrical function is SU$(2)^V$.

Let’s call $R^{(n, \rho)}$ the Hilbert space of the SL$(2, \mathbb{C})$ representation $\mathcal{I} = (n, \rho)$ and $V^j$ the space of the SU$(2)$ representation $j$. If we choose a (time) normal $x \in$ SL$(2, \mathbb{C})$/SU$(2)$ and consider the subgroup SU$(2)_x$ stabilizing $x$, we can decompose $R^x$ onto the irreducible representations of SU$(2)_x$:

$$R^{(n, \rho)} = \bigoplus_{j \geq n} V^j_{(x)}.$$  \hspace{1cm} (22)

Let’s call $P^j_{(x)}$ the projector from $R^{(n, \rho)}$ onto $V^j_{(x)}$:

$$P^j_{(x)} = \Delta_j \int_{SU(2)_x} dg \, \overline{\mathcal{C}_x^{(j)}}(g) D^{(n, \rho)}(g),$$  \hspace{1cm} (23)

where $\Delta_j = (2j + 1)$ is the dimension of $V^j$, the integration is over SU$(2)_x$, $D^x(g)$ is the matrix representing the group element $g$ acting on $R^x$ and $\mathcal{C}^j$ the character of the $j$-representation. To construct a projected spin network, we insert this projector at the end vertices of every link which allows to glue the Lorentz holonomies to the SU$(2)$ intertwiners. The resulting functional is:

$$\phi^{(\mathcal{I} + \mathcal{J} + \mathcal{L})}(U_e, \chi_v) = \prod_v i_v \left[ \bigotimes_{e \leftarrow v} |\mathcal{I}_e \chi_v j^{(e)}_v(m_e)\rangle \right] \prod_e (|\mathcal{I}_e \chi_{s(e)} j^{(s(e))}_e(m_e)\rangle |\mathcal{I}_e \chi_{t(e)} j^{(t(e))}_e(m_e)\rangle),$$  \hspace{1cm} (24)

with an implicit sum over the $m$’s. $|\mathcal{I} x m\rangle$ is the standard basis of $V^j_{(x)} \hookrightarrow R^x$ with $m$ running from $-j$ to $j$. In short, compared to the usual spin networks, we trace over the subspaces $V^j_{(x)}$ instead of the full spaces $R^x$.

Using these projected spin networks allows to project the Lorentz structures on specific fixed SU$(2)$ representations. This allows to diagonalize the area operators. Considering a surface $\Sigma$ intersecting the graph $\Gamma$ only on one edge $e$ at the level of a (possibly bivalent) vertex, its area operator $\text{Area}_\Sigma$ will be diagonalized by the projected spin network basis with the eigenvalues given above:

$$\text{Area}_\Sigma |\phi^{(\mathcal{I} + \mathcal{J} + \mathcal{L})}_{e}(\mathcal{I}_e j^{(e)}_e, \chi_v)\rangle = \ell_P^2 \sqrt{j_e(j_e + 1)} - n_e^2 + \rho_e^2 + 1 |\phi^{(\mathcal{I} + \mathcal{J} + \mathcal{L})}_{e}(\mathcal{I}_e j^{(e)}_e, \chi_v)\rangle.$$

The procedure is now simple. Given a graph $\Gamma$ and a set of surfaces, in order to have a spin network state diagonalizing the area operators associated to all these surfaces, we simply need to project that spin network state at all the intersections of the surfaces with $\Gamma$. If we want to obtain quantum geometry states diagonalizing the areas of all the surfaces in the hypersurface $\Sigma$, we would need to consider an “infinite refinement limit” where we project the spin network state at all points of its graph $\Gamma$. Such a procedure is described in more details in [13, 14]. However, from the point of view that space-time is fundamentally discrete at microscopic scales, it sounds reasonable to be satisfied with quantum geometry states that diagonalize the areas of a discrete number of surfaces (intersecting the graph at the points where we have projected the spin network states). This is consistent with the picture that considering a projected spin network state, the embedding of the hypersurface $\Sigma$ into the space-time is only well-defined at the vertices of the graph, where we know the time normal $\chi$: at all other points, $\Sigma$ remains fuzzy and so must be the surfaces embedded in $\Sigma$.

C. Simple spin networks

Up to now, we have described quantum states $f(A)$ and the action of triad-based operators on them. We should also define the action of connection-based operators. We normally expect that $A$ would act by simple multiplication of a
wave function \( f(\mathcal{A}) \). Unfortunately, in our framework, \( \mathcal{A} \) does not commute with itself, so this naïve prescription does not work. The point is that, due to the second class constraint, the rotation part of the connection \( P_B \mathcal{A} \) is constrained and must be equal to the spin connection \( \Gamma[R] \) defined by the triad \( R \). This reflects the reality constraints of LQG. The natural way out is that we would like wave functions which do not correspond to the choice \( \Gamma[R] \) and must be equal to the spin connection \( \Gamma^R \). Intuitively, while \( \hat{\Gamma} \) does not work. The point is that, due to the second class constraint, the rotation part of the connection \( \mathcal{A} \) is not an independent variable. Actually, in the time gauge where the field \( \chi \) is taken constant equal to \( \chi_0 \), \( \mathcal{A} \) reduces to the SU(2)-connection of the real Ashtekar-Barbero formalism. Then \( \mathcal{A} \) is the natural Lorentz extension of that SU(2)-connection [13]. Finally, the area spectrum for this connection reproduces exactly the standard spectrum:

\[
\text{Area}_S \sim l_p^2 \sqrt{C(\text{su}(2))} = l_p^2 \sqrt{j(j+1)}
\]
In order to completely recover LQG, we still need to take care of the second class constraints. To faithfully represent the Dirac bracket, we would indeed like wave functions which do not depend on the boost part of the connection $P_B A$. We take this into account in the scalar product. Instead of using the SL(2, C)-Haar measure, we can restrict ourselves to the SU(2) subgroup. More precisely, we define the scalar product using the $\chi = \chi_0$ section of the wave functions:

$$\langle f | g \rangle = \int_{\text{SU}(2)^2} dU e^{\chi_0(U)^{1/2}} g_{\chi_0}(U).$$

(30)

We are considering the usual scalar product of LQG, and a basis is given by the standard SU(2) spin networks.

V. SPIN FOAMS AND THE BARRETT-CRANE MODEL

Up to now, we have described the kinematical structure of Covariant Loop (Quantum) Gravity. We still need to tackle the issue of defining the dynamics of the theory. On one hand, one can try to regularize and quantize à la Thiemann the action of the Hamiltonian constraint either on the covariant connection $A$ or the commutative connection $a$. In this case, we will naturally have to study the volume operator and face the usual ambiguities of LQG. On the other hand, one can turn to the spin foam formalism. Spin foams have evolved independently but parallelly to LQG. Inspired from state sum models, they provide well-defined path integrals for “almost topological” theories, which include gravity-like theories. Moreover, they use the same algebraic and combinatorial structures as LQG. In particular, spin networks naturally appear as the kinematical states of the theory. From this perspective, spin foams allow a covariant implementation of the LQG dynamics and a rigorous definition of the physical inner product of the theory.

In three space-time dimensions, pure gravity is described by a BF theory and is purely topological. The spin foam quantization is given by the Ponzano-Regge model [17]. Its partition function defines the projector onto the gravity physical states i.e wave functions on the moduli space of flat Lorentz connections.

In four space-time dimensions, it turns out that general relativity can be recast as a constrained BF theory. One can quantize the topological BF theory as a spin foam model and then impose the extra constraints directly on the partition function at the quantum level (e.g [18]). For 4d gravity, this leads to the Barrett-Crane model [19]. There are of course ambiguities in the implementation of the constraints, which lead to different versions of this model. We show below that the Barrett-Crane model provide a dynamical framework for Covariant LQG.

A. Gravity as a constrained topological theory

Let us start with the Plebanski action:

$$S[\omega, B, \phi] = \int_M \left[ B^{IJ} \wedge F_{IJ}[\omega] - \frac{1}{2} \phi_{IJKL} B^{KL} \wedge B^{IJ} \right],$$

(31)

where $\omega$ is the $so(3, 1)$ connection, $F[\omega] = d_\omega \omega$ its curvature, $B$ a $so(3, 1)$-valued 2-form and $\phi$ a Lagrange multiplier satisfying $\phi_{IJKL} = -\phi_{KJIL} = -\phi_{IJLK} = \phi_{KLIJ}$ and $\phi_{IJKL} \epsilon_{JKL} = 0$. The equations of motion are:

$$d_B + [\omega, B] = 0, \quad F_{IJ}(\omega) = \phi_{IJKL} B_{KL}, \quad B^{IJ} \wedge B^{KL} = e \epsilon_{JKL},$$

with $e = \frac{1}{4} \epsilon_{IJKL} B^{IJ} \wedge B^{KL}$. When $e \neq 0$, the constraint on $B$ is equivalent to the simplicity constraint, $\epsilon_{IJKL} B^{IJ} B_{KL} = \epsilon_{abcd} e$.

This constraint is satisfied if and only if there exists a real tetrad field $e' = e_d^a dx^a$ such that either $B = \pm e \wedge e$ (sector $I_{\pm}$) or $B = \pm * (e \wedge e)$ (sector $II_{\pm}$). These four sectors are due to the symmetry of the constraints under $B \rightarrow (sB)$. The $*$ operation allows to switch the sectors: $I_+ \rightarrow I_+ \rightarrow I_- \rightarrow II_- \rightarrow I_+$. Restricting ourselves to the $I_+$ sector, the action reduces to $S = \int * (e \wedge e) \wedge F$ and we recover general relativity in the first order formalism.

A first remark is that we still have to get rid of the $I_{\pm}$ and $II_{\pm}$ sectors in the path integral at the quantum level.
These are respectively related to the chirality of the 3-volume and to the issue of time orientation [21, 22]. A second remark is that taking a more general constraint on the $\phi$ field, for instance $a\phi_{IJ} I^J + b\phi_{IJKL} I^J K^L = 0$, we recover the Palatini-Holst action for general relativity with Immirzi parameter [20].

B. Simple spin networks again

The spin foam strategy is to first discrete and quantize the topological BF theory as a state sum model, then to impose the $B$-constraints on the discretized partition function.

In order to discretize the path integral, we choose a triangulation (or more generally a cellular decomposition) of the 4d space-time gluing 4-simplices together. We then associate the $B$ field to triangles, $B^{IJ}(t) = \int_t B^{IJ}$, and the connection curvature to the dual surfaces. The simplicity constraint of the 2-form, $\epsilon_{IJKL} B_{ab}^{IJ} B_{cd}^{KL} = e\epsilon_{abcd}$, is then translated to the discrete setting. For any two triangles $t, t'$, we have:

$$\epsilon_{IJKL} B^{IJ}(t) B^{KL}(t') = \int_{t, t'} c d^2\sigma \wedge d^2\sigma' = V(t, t'),$$

where $V(t, t')$ is the 4-volume spanned by the two triangles. In particular, for any two triangles who share an edge, we have:

$$\epsilon_{IJKL} B^{IJ}(t) B^{KL}(t') = 0. \tag{32}$$

These are the Barrett-Crane constraints which are implemented at the quantum level. More precisely, we associate a copy of the $\mathfrak{sl}(2, \mathbb{C})$-algebra to each triangle $t$ and we quantize the $B^{IJ}(t)$’s as the Lorentz generators $J_i^{IJ}$. For a given triangle $t$, the previous constraint for $t = t'$ becomes $\epsilon_{IJKL} J_i^{IJ} J_i^{KL} = 0$, which is the vanishing of the second Casimir $C_2(\mathfrak{sl}(2, \mathbb{C})) = 0$. This means that the representation $I_t$ associated to a triangle $t$ must be simple: either $(n_t, 0)$ or $(0, \rho_t)$. The first Casimir $C_1 = J_{iJ} J_i^{J'}$ gives the (squared) area of the triangle. For the discrete series, $C_1(n, 0) = -n^2 + 1$ is negative and the triangle is time-like. For the continuous series, $C_1(0, \rho) = \rho^2 + 1$ is positive and the triangle is space-like. Thus we recover the same simplicity of the $\mathfrak{sl}(2, \mathbb{C})$ representations as in Covariant LQG. The only difference is that we only consider space-like triangles in the canonical framework, and therefore only obtain the $(0, \rho)$ representations. The time-like representations would naturally appear in the canonical setting if considering a time-like normal $\chi$ (e.g. [15]). In the following, we will restrict ourselves to the $(0, \rho)$ representations.

Coupling between different triangles happens at the level of tetrahedra: to each tetrahedron is associated an intertwiner between the representations attached to its four triangles. Solving the constraints $\epsilon_{IJKL} J_i^{IJ} J_i^{KL} = 0$ for every couple of triangles $(t, t')$ of the tetrahedron leads to a unique intertwiner. This Barrett-Crane intertwiner $I_{BC} : \otimes_{t=1}^4 R^{(0, \rho_t)} \to \mathbb{C}$ is the only $SU(2)$-invariant intertwiner:

$$I_{BC} = \int_{\text{SL}(2, \mathbb{C})/SU(2)} d\chi \bigotimes_{t=1}^4 \langle (0, \rho_t), \chi, j = 0 |. \tag{33}$$

We recover the intertwiner structure of the simple spin networks introduced for Covariant LQG. More precisely, the quantum geometry states associated to any space-like slice of the triangulation in the Barrett-Crane model are simple spin networks [13, 22].

This makes the link between the kinematical states of the canonical theory and the spin foam states. Then the transition amplitudes of the Barrett-Crane model can be translated to the canonical context and considered as defining the dynamics of Covariant LQG.

C. The issue of the second class constraints

In the previous spin foam quantization, we discretize and quantize the path integral for general relativity. We have dealt with the simplicity constraint $B.(\ast B) = 0$ by imposing on the path integral. A priori, this corresponds to the simplicity constraint $(9), \phi = R.(\ast R) = 0$ of the canonical analysis. However, it seems as we are missing the other second class constraint $\psi \sim \epsilon RR D A R$. The $\psi$ constraints are essential to the computation of the Dirac bracket: shouldn’t we discretize them too and include them in the spin foam model?
The spin foam point of view is that we have already taken them into account. Indeed, the $\psi$ are secondary constraints, coming from the Poisson bracket $H, \phi$: at first, $\phi = 0$ is only imposed on the initial hypersurface and we need $\psi = 0$ to ensure we keep $\phi = 0$ under the Hamiltonian evolution. On the other hand, the Barrett-Crane model is fully covariant and $\phi = 0$ is directly imposed on all the space-time structures: we have projected on $\phi = 0$ at all stages of the evolution (i.e. on all hypersurfaces). The Barrett-Crane construction ensures that a simple spin network will remain a simple spin network under evolution. In this sense, we do not need the secondary constraints $\psi$. It would nevertheless be interesting to check that a discretized version of $\psi$ vanishes on the Barrett-Crane partition function.

VI. CONCLUDING REMARKS

Starting with the canonical analysis of the Palatini-Holst action, we have shown how the second class constraints are taken into account by the Dirac bracket. Requiring a good behavior of the Lorentz connection under Lorentz gauge transformations and space diffeomorphisms, we obtain a two-parameter family of possible connection variables. Requiring that the connection further behaves as a 1-form under space-time diffeomorphisms, we obtain a unique covariant connection. This leads to a Covariant LQG with “simple spin networks” (for the Lorentz group), a continuous area spectrum and an evolution dictated by the Barrett-Crane spin foam model. The theory turns out independent from the Immirzi parameter. The main obstacle to a full quantization is that the non-commutativity of this connection. This can be understood as reflecting the reality conditions of the complex formulation of LQG. On the other hand, there exists a unique commutative connection. It turns out to be a generalization of the Ashtekar-Barbero connection of the real formulation of LQG. We further recover the SU(2) spin networks, the standard discrete area spectrum and the usual Immirzi ambiguity.

It seems that Covariant LQG could help address some long-standing problems of the standard formulation of LQG, such as the Immirzi ambiguity, the issue of the Lorentz symmetry, the quantization of the Hamiltonian constraint and how to recover the space-time diffeomorphisms at the quantum level.

Finally, a couple of issues which should be addressed within the Covariant LQG theory to ground it more solidly are:

- a study of the 3-volume operator acting on simple spin networks.
- a derivation of the spin foam amplitudes from the Covariant LQG Hamiltonian constraint, possibly following the previous work in 3d gravity [23].

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