Gravity in the dynamical approach to the cosmological constant

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One of the most disturbing difficulties in thinking about the cosmological constant is that it is not stable under radiative corrections. The feedback mechanism proposed in ref. [1] is a dynamical way to protect a zero or small cosmological constant against radiative corrections. Hence, while this by itself does not solve the cosmological constant problem, it can help solving the problem. In the present paper we investigate stability and gravity in this approach and show that the feedback mechanism is both classically and quantum mechanically stable and has self-consistent, stable dynamics. 

I. INTRODUCTION

The cosmological constant is one of the most outstanding mysteries in contemporary physics [2,3]. There are actually two cosmological constant problems. One is the “old” problem: “why is the cosmological constant vanishingly small compared to the energy scale of, say, the Planck scale?” The ratio of the critical density of the universe today, than which energy density of the cosmological constant must be smaller, to the Planckian energy density is as small as $10^{-120}$. Since the cosmological constant is not protected against quantum corrections, there is no manifest reason why it should be so tiny. We now have the second problem: “why is the dark energy roughly of the same order as matter energy density today?” The observation of high-redshift, Type Ia supernovae is strongly inconsistent with a universe with vanishing dark energy, whose typical form is a cosmological constant [4–6]. This implies that there should be a small but non-vanishing dark energy and that it should be comparable to matter energy density today. However, since dark energy and matter scale differently as the universe expands, there seems to be no manifest reason why they should be of the same order today. This problem is often called the coincidence problem. 

Although there are many approaches to the coincidence problem [7–10], all of them seem to have a common weakness: they can be spoiled by other contributions to the cosmological constant. This is because it is a priori assumed that the old problem can be solved. A tiny correction to the unknown solution to the old problem can produce a relatively huge cosmological constant. For example, in the quintessence scenario [7] it is a priori assumed that the asymptotic value of the quintessence potential exactly vanishes. If we added a non-vanishing cosmological constant to the potential then the scenario would not work. Moreover, if there are many proposals to solve the coincidence problem and if they can coexist then we will end up with a too much cosmological constant, provided that each contribution is additive. Hence, a good solution to the coincidence problem must include a mechanism which almost completely cancels other contributions to the cosmological constant. Unfortunately, no phenomenologically acceptable solution with such a mechanism is known.

There are also many approaches to the old problem. Among them are quantum cosmology [11–13], membrane creation [14–17], a scalar field with a very small mass [18,19], backreaction models [20,21], eternal inflation [22], extra dimensions [23], and so on [24,25].

We do not consider the quantum cosmology approach based on the Euclidean quantum gravity partly because it is not well understood and may be ill-defined. (For example, see refs. [26,27] for challenging arguments.) There is another reason why we do not consider this approach. Quantum cosmology is, in a sense, a theory of the beginning, or the initial condition, of the universe. For example, the Hartle-Hawking boundary condition [28] predicts a sharp peak at a vanishing cosmological constant, but this value should be considered as the effective cosmological constant at the beginning of the universe, including all contributions from many fields and quantum corrections. Hence, if this prediction is correct then the universe cannot experience inflation nor any phase transitions after its birth. If the effective cosmological constant were zero at the beginning of the universe and if a phase transition occurred after that then the cosmological constant now would be negative. If we try to avoid this disagreeable conclusion by modifying or changing the proposal then the peak must be moved to a non-vanishing cosmological constant. For example, the proposals of Linde [29] and Vilenkin [30] prefer a large cosmological constant. This indicates the onset of a Planck scale inflation, but does not seem to provide a solution to the cosmological constant problem without anthropic consideration [31].

A proposal by Rubakov [24] might relax the cosmological constant in the Brans-Dicke frame to a small value if the cosmological constant in the Einstein frame is exactly zero. However, for his proposal to work, it is essential to

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include an inflaton sector in the Einstein frame. Hence, we anyway need to explain why the cosmological constant in the Einstein frame is exactly zero today. A similar proposal by Hebecker and Wetterich [25] not only has a similar problem but also predicts a too much deviation from Einstein theory for weak gravity and, thus, is phenomenologically unacceptable.

Dolgov’s backreaction model [20] and its variants [21] use the kinetic energy of a running scalar field to cancel the bare cosmological constant. In these models the effective gravitational coupling vanishes and, thus, the deviation from Einstein gravity is too large.

As far as the author knows, all other previously known approaches to the old cosmological constant problem are either dependent of the anthropic principle or phenomenologically unacceptable (e.g. a non-standard Friedmann equation at low energy, a non-Einstein weak gravity, a naked singularity, etc.). This situation has led to an increase in speculation for the necessity of the anthropic principle [32]. Before resorting to this, it is worthwhile to ask whether anything could possibly do what the cosmological constant data requires [3]. It seems likely that the correct way to interpret the tiny value of the cosmological constant is that conventional quantum field theory is not the whole story, so it is worth seeking acceptable modifications.

In ref. [1] a new dynamical approach to the cosmological constant problem was proposed. Although this idea by itself does not explain why the cosmological constant is so small, it does provide a way to protect a zero or small cosmological constant against radiative corrections (without using supersymmetry) and thus can help solving the cosmological constant problem. In this approach the true value of the vacuum energy is negative, but the dynamics is such that the true minimum is never attained, and the universe settles down to a near zero energy state.

The purpose of the present paper is to investigate stability and gravity in this dynamical approach. In particular, we show that (i) the effective cosmological constant decreases in time and asymptotically approaches zero from above; (ii) the evolution of a homogeneous, isotropic universe is described by the standard Friedmann equation at low energy; that (iii) classical, linearized gravity in Minkowski background is described by Einstein gravity at distances longer than \( l_p = \sqrt{\alpha \kappa} \) and at energies lower than \( l^{-1}_\kappa \), where \( \kappa \) is the Planck length and \( \alpha \) is a dimensionless, positive parameter of the model; that (iv) the mechanism is stable under radiative corrections and thus a zero or small cosmological constant is protected against radiative corrections; and that (v) quantum fluctuation at low energy is so small that the effective cosmological constant at low energy does not overshoot zero even quantum mechanically.

This paper is organized as follows. In Sec. II we explain the basic idea in analogy with a simple system of a charged particle. In Sec. III we review the feedback mechanism proposed in ref. [1] in a more general setting. In Sec. IV we derive equations of motion in detail and show that the standard Friedmann equation is recovered at low energy. The result in Sec. IV is confirmed numerically in Sec. V. In Sec. VI we consider reheating in this model. In Sec. VII we investigate weak gravity in Minkowski background and show the recovery of the linearized Einstein theory. In Secs. VIII we investigate stabilities against radiative corrections and quantum fluctuations. Sec. IX is devoted to a summary of this paper.

II. BASIC IDEA

One of the most disturbing difficulties in thinking about the cosmological constant is that it is not protected against radiative corrections, which usually generate enormous vacuum energies compared to what we observe. The feedback mechanism proposed in ref. [1] can be considered as a dynamical way to protect a zero or small cosmological constant against radiative corrections. Hence, although the feedback mechanism by itself does not solve the cosmological constant problem, it can help solving the problem. In this section we explain the basic idea underlying the feedback mechanism.

Let us consider a simple system of a non-relativistic, charged particle in a background electrostatic potential. The action of this simple system is

\[
I = \int dt \left[ \frac{M}{2} \dot{x}^2 - eV(x) \right],
\]

where \( V(x) \) is the electrostatic potential. Let us suppose that \( eV(x) \) is an increasing function of \( x \) so that the particle moves towards the negative \( x \) region. Suppose, however, that for some reason, we would like to stop the particle just before its crossing \( x = 0 \). How can we stop it? One obvious solution is to control the electrostatic potential so that it has a minimum near \( x = 0 \), and introduce a friction. By controlling the potential and the friction carefully, we can achieve our purpose. A difficulty in this solution is that there might be forces (e.g. strong wind, gravity, someone who may kick the particle, etc.) other than the electrostatic force and the friction. These forces make the control of the potential extremely subtle and difficult.
A different, rather radical solution would be available if we could control the mass $M$ of the particle. In this case we make $M$ to diverge when $x$ is sufficiently close to $x = 0$. To make this successful we of course need to watch the value of $x$ in real time. This case does not involve a subtle control of anything: $x$ completely stops when $M$ diverges, irrespective of the value of $eV'(x)$ and other forces. One might worry that the kinetic term in the action might diverge. Actually, this is not the case, and the kinetic term does not diverge but vanishes. Since the equation of motion of the position $x$ of the charged particle is

$$ (M\dot{x}) + eV'(x) = 0, $$

the momentum $\pi \equiv M\dot{x}$ remains finite in finite time as far as $eV'(x)$ is finite. (This is the reason why $x$ stops when $M$ diverges.) Hence, the kinetic term actually vanishes when $M$ diverges:

$$ \frac{M}{2} \dot{x}^2 = \frac{\pi^2}{2M} \to 0 \quad (M \to \infty). $$

Indeed, the more singular the mass is, the more quickly the kinetic term vanishes.

We apply this second idea to a scalar field, whose potential determines the effective cosmological constant but is not fine-tuned. The reason we do not consider a fine-tuned potential, or the analogue of the first idea above, is that it is not stable under radiative corrections, which add vacuum energies to the potential and also change the form of the potential. We still want to stop the scalar field at or near zero cosmological constant. Let us suppose that a scalar field $\phi$ is described by the action

$$ I = \int d^4x \sqrt{-g} \left[ -\frac{\partial^\mu \phi \partial_\mu \phi}{2f} - V(\phi) \right], $$

where $f$ is chosen so that it vanishes at or near zero cosmological constant. A simple choice might be to make $f$ dependent of the value of $\phi$ itself so that $f$ vanishes at a root of the potential. In this case, $\phi$ stops when the potential vanishes. However, this choice is too naive and would not be stable under radiative corrections: radiative corrections add vacuum energies to the potential and, hence, change the potential value at which $\phi$ stops. Indeed, if $f$ is a function of $\phi$ only then we can always make a change of variable $\phi \to \bar{\phi} \equiv \int d\phi/\sqrt{f}$ so that the kinetic term of the new variable $\bar{\phi}$ is canonically normalized. In this language, the choice of $f$ is nothing but a fine tuning of the potential of $\phi$ and, thus, this is never stable against radiative corrections. Hence, we cannot protect zero or small cosmological constant against radiative corrections if $f$ depends only on $\phi$. In other words, $\phi$ itself does not know where to stop.

On the other hand, the spacetime curvature seems to know where to stop $\phi$ since it reflects the value of the effective cosmological constant via the Friedmann equation. Hence, instead of considering a $\phi$-dependent $f$, we postulate that $f$ depends on the spacetime curvature so that the curvature can tell when $\phi$ should stop. For simplicity, we restrict our discussion to the case where $f$ depends on the Ricci scalar $R$ only and vanishes at $R = 0$. In this case, $\phi$ stops rolling when $R = 0$. We do not provide a reason why $f$ vanishes at $R = 0$. However, we shall show that this choice of $f$ is stable under radiative corrections and leads to a stable dynamics. In other words, instead of providing a solution to the cosmological constant problem, we propose a dynamical mechanism to protect a zero or small cosmological constant against radiative corrections (without using supersymmetry) as a first step. The singular kinetic term makes the scalar field stop at zero curvature, even without a fine-tuned potential.

With this choice of $f$, we need to make it sure that $R = 0$ really corresponds to zero cosmological constant. If the kinetic term dominates over the potential term then the value of the potential determined by $R = 0$ does not need to be zero and, thus, this model does not seem to work. Hence, we are left with the following two possibilities. One possibility is that there is a scaling solution to scalar dynamics in which the kinetic and potential energies of the scalar field decrease in tandem, so that the field is slowed down when small potential energy is achieved. Unfortunately, in our specific realizations, there always seems an instability invalidating the scaling solutions we tried.

The second possibility, which we shall focus on in the rest of this paper, is that the potential term dominates over the kinetic term so that $R$ is determined solely by the contribution from the potential. In this case, the scalar field indeed stops at zero cosmological constant automatically. We could make $f$ dependent of $\phi$ as well, but such a dependence can be removed from the behavior of the kinetic term near $R = 0$ by redefinition of $\phi$ without loss of generality. We suppose that the kinetic term in (4) represents the term which is the most singular-looking at $R = 0$ among many possible terms in the kinetic part and we did not include less singular-looking terms since they are not important at low energy. We also omitted all other dynamical fields since, as we shall see in the following sections, the dynamics of $\phi$ is so slow that any dynamical fields other than $\phi$ will settle into their ground state before the universe approaches a sufficiently low energy state. The potential $V(\phi)$ includes the ground state energies of all such fields.

Since $f$ is in the denominator and vanishes in the $R \to 0$ limit, the kinetic term threatens to be singular at low energy. Moreover, since $f$ is a function of the Ricci scalar $R$, it introduces corrections to Einstein equation and
possibly destabilize the system. On the other hand, from the charged-particle analogue above, we have learned that a singular-looking kinetic term is actually regular. The same is true here! The more singular $1/f$ is, the smaller the kinetic term and corrections to Einstein equation are! This rather counter-intuitive but very simple fact means that by making $1/f$ (for the most singular-looking term) sufficiently singular at $R = 0$, we can make corrections to Einstein equation as small as we like. In particular, it is indeed possible to make the corrections due to the kinetic term much smaller than those from usual higher derivative terms like $R^2$ at low energy so that higher derivative corrections and stability are completely controlled by the usual higher derivative terms. Note that if there are several (or many) terms in the kinetic part, we do not need to require the singular behavior for all coefficients: just one term is enough.

III. FEEDBACK MECHANISM

In the previous section we have learned from a simple analogous system that a scalar field stops in a completely regular way when the coefficient of a kinetic term diverges. In this section we consider a scalar field with such a singular-looking kinetic term.

The field Lagrangian we consider is

\[
I = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \alpha R^2 + L_{\text{kin}} - V(\phi) \right],
\]

\[
L_{\text{kin}} = \frac{\kappa^{-4}K^q}{2qf^{q-1}},
\]

where $f$ is a function of the Ricci scalar $R$, $\kappa$ is the Planck length, $\alpha$ and $q$ are constants, and $K = -\kappa^4 \partial^\mu \phi \partial_\mu \phi$. Our sign convention for the metric is $(- + + +)$. In a word, the idea of this model is to assume a non-standard kinetic term. The dependence on curvature is what stalls the field at small cosmological constant.

We can absorb any nonzero cosmological term into $V(\phi)$ without loss of generality. We assume that (i) the minimum of the potential $V(\phi)$ is negative; that (ii) $q > 1/2$; that (iii) $\alpha > 0$; and that (iv) the function $f(R)$ is non-negative and behaves near $R = 0$ as

\[
f(R) \sim (\kappa^2 R)^{2m},
\]

where $m$ is an integer satisfying

\[
2(m - 1) > \frac{q}{2q - 1}.
\]

In the $m \to \infty$ limit, this choice of $f(R)$ may be replaced by $f(R) \sim \exp[-\kappa^{-4}R^{-2}]$ or similar functions. We suppose that the kinetic term in (5) is the most singular-looking one among many possible terms in the kinetic part. We assume that the most singular-looking term behaves like (6) with the condition (7), but other terms do not need to behave like that. If there are several terms in the kinetic part then the most singular-looking term is dominant and other terms are not important at low energy. Thus, eg. adding the standard canonical kinetic term does not affect anything.

Since $f$ is in the denominator of the kinetic term and vanishes in the $\kappa^2 R \to 0$ limit, the kinetic term threatens to be singular at low energy. We shall see below that the large $m$ makes $\phi$ evolve slowly, that the numerator $K^q$ vanishes more quickly than the denominator and that the kinetic term is actually regular.

Now let us explain the motivations for the assumptions (i)-(iv). (i) We would like to propose a mechanism in which the scalar field stops rolling at or near zero vacuum energy. For this purpose, the minimum of the potential $V(\phi)$ should be negative so that $V(\phi)$ has a root. (ii) For the stability of inhomogeneous perturbations, it is necessary that the sound velocity squared $c_s^2 = L_{\text{kin},K}/(2K L_{\text{kin},KK} + L_{\text{kin},K})$ is positive [33]. In our model this condition is reduced to $q > 1/2$. (iii) The $R$-dependence of the kinetic term $L_{\text{kin}}$ produces higher derivative corrections to Einstein equation which might destabilize gravity. As we shall explain below, the term $\alpha R^2$ can stabilize gravity at low energy if $\alpha$ is positive. (iv) For the term $\alpha R^2$ to control the stability, we need to make it sure that the term $\alpha R^2$ is dominant over $L_{\text{kin}}$ at low energy. As shown below, this is the case if and only if the condition (7) is satisfied.

We do not know a parent theory that will provide our Lagrangian as the low-energy effective theory. In particular, we do not give a reason why $f$ vanishes at $R = 0$. In this sense, this model by itself does not solve the cosmological constant problem. However, we shall show below that this choice of $f$ is radiatively stable, leads to a stable dynamics and could help solve the cosmological constant problem. We treat this model as a purely phenomenological suggestion that might motivate further research into the possible parent theory, which is presumably not based entirely on conventional four-dimensional field theory.
We could make $f$ dependent of $\phi$, but such a dependence can be removed from the behavior of $L_{\text{kin}}$ near $R = 0$ by redefinition of $\phi$ without loss of generality. Note that $L_{\text{kin}}$ above represents the term which is the most singular-looking at $R = 0$ among many possible terms in the kinetic part and that we did not include less singular-looking terms since they are not important at low energy. We also omitted all other dynamical fields since, as we shall see below, the dynamics of $\phi$ is so slow that any dynamical fields other than $\phi$ will settle into their ground state before the universe approaches a sufficiently low energy state.

We now argue that the Lagrangian (5) gives a feedback mechanism that makes the field stall at or near zero vacuum energy. The argument for the $q = 1$ case was already given in ref. [1]. Here, let us simply generalize it to a general $q (> 1/2)$.

In the flat Friedmann-Robertson-Walker (FRW) background

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

the equation of motion for a homogeneous $\phi$ is

$$\ddot{\pi} + 3H\pi + V'(\phi) = 0, \quad \pi \equiv \frac{\dot{\phi}}{f} \cdot \left(\frac{\kappa^4 f^2}{f^2}\right)^{q-1},$$

where $H = \dot{a}/a$, a dot denotes the time derivative and a prime applied to $V(\phi)$ denotes the derivative with respect to $\phi$. Near $V = 0$, $V$ can be approximated by a linear function as

$$V \simeq c\kappa^{-3}(\phi - \phi_0),$$

where $c$ and $\phi_0$ are constants. This approximation is extremely good since, as we shall see, $\phi$ rolls very slowly near $V = 0$ and does not probe the global shape of the potential.

Without any fine-tuning, the dimensionless constant $c$ should be of order unity. The equation of motion (9) can be rewritten as

$$\frac{d\mathcal{B}}{dx} + \left(3 - \frac{\dot{H}}{H^2}\right)\mathcal{B} + 1 = 0,$$

where $\mathcal{B} \equiv \kappa^3 H\pi/c$ and $x \equiv \ln a(t)$. This implies that $\mathcal{B}$ approaches $-(3 - \dot{H}/H^2)^{-1}$, if $\dot{H}/H^2$ changes slowly compared to $a$ and is smaller than 3. See Fig. 1 for confirmation of this statement by numerical calculation. Hence, the asymptotic behavior of $\pi$ is

$$\kappa^2 \pi \sim -c\kappa^{-1} H^{-1}.$$  (12)

If the kinetic term is small compared to the potential term then the Friedmann equation implies that

$$3H^2 \simeq \kappa^2 V,$$

and (12) can be rewritten as

$$\kappa\partial_t(\kappa^4 V) \sim c\kappa^2 \dot{\phi} = cf \cdot \kappa^2 \pi \cdot \left(\frac{\kappa^4 V}{\kappa^4 V + \kappa^2 H^2}\right)^{q-1} \frac{1}{\kappa} \sim -(c^2)^{\frac{q}{q+1}} \frac{1}{\kappa^4 V} \sim -(c^2)^{\frac{q}{q+1}} (\kappa^4 V)^{2m - \frac{4q - 1}{2q - 1}}.$$  (14)

Hence, we obtain

$$(\kappa^4 V)^{-\gamma/2} \sim (c^2)^{\frac{q}{q+1}} \frac{t - t_0}{\kappa}, \quad \gamma = 4m - \frac{4q - 1}{2q - 1}.$$  (15)

where $t_0$ is a constant. From the condition (7), $\gamma > 3$ and

$$\kappa^4 V \to +0 \quad (t/\kappa \to \infty).$$  (16)

This result means that the field stalls near $V = 0$. See Figs. 4 and 5 for confirmation of this behavior by numerical calculation.
We have assumed that the kinetic energy is small compared to the potential energy. This assumption is easily verified. At low energy ($\kappa H \ll 1$),

$$L_{\text{kin}} = \frac{\kappa^{-4} f}{2q} (\kappa^4 \pi^2) \frac{\pi^2}{\kappa^4 (\kappa H)^4 \sim \kappa^{-4} \left( \frac{c^2}{\kappa^2 H^2} \right)^4 \sim \kappa^{-4} (\kappa H)^4 \sim \kappa^{-4} (\kappa H)^2 \sim V},$$

provided that $c = O(1)$. Here, we have used the condition (7) to obtain the first inequality. We have implicitly assumed that the standard Friedmann equation is valid at low energy. Let us justify this assumption in the next section.

IV. CLASSICAL STABILITY I - RECOVERY OF FRIEDMANN EQUATION

We can also show the essential, and somewhat surprising result, that the standard Friedmann equation is recovered at low energy, starting from the action (5). There are higher derivative corrections to the Einstein equation due to the $R$-dependence of $L_{\text{kin}}$ and $\alpha R^2$.

Since there are higher-derivative terms, the stability of the system is a non-trivial question. In order to see the non-triviality, let us consider the Klein-Gordon equation $(\Box - M^2) \varphi = 0$ as a standard equation and add $\epsilon(\varphi) \Box^2 \varphi / M^2$ to the right hand side. One might expect that the standard equation should be recovered whenever $\epsilon \to 0$. Actually, this is not true. The standard equation is certainly recovered in the $\epsilon \to 0$ limit if $\epsilon > 0$. On the other hand, if $\epsilon < 0$ then the system has a tachyonic degree and is unstable. Moreover, if $\varphi$ crosses a root of $\epsilon$ then the system experiences a singularity ($\Box^2 \varphi$ diverges) and the low energy effective theory cannot be trusted unless a miracle cancellation occurs.

In our system, from the estimate of $L_{\text{kin}}$ in (17) with (7), $L_{\text{kin}}$ is much smaller than $\alpha R^2$ ($\sim H^4$) at low energy if $\alpha \neq 0$. Hence, $\alpha R^2$ should dominate the higher derivative corrections and control the stability. For the theory $R/2\kappa^2 + \alpha R^2$, roughly speaking, the parameter $\alpha$ plays the role of $\epsilon$ above (including the sign) and it is known that the low energy dynamics is stable if and only if $\alpha \geq 0$ [34]. Here, stability means that as the universe expands, the system keeps away from unphysical spurious solutions and approaches the standard low energy evolution asymptotically. If we did not include the term $\alpha R^2$ then, as we shall see below, $L_{\text{kin}}$ would make the quantity corresponding to $\epsilon$ above to be negative essentially because $f(R)$ is in the denominator. Therefore our system should be stable and the standard Friedmann equation should be recovered at low energy if and only if $\alpha > 0$ and (7) are satisfied.

The equation of motion derived from the action (5) includes up to fourth order derivatives of the metric. However, since the Friedmann equation is a constraint equation, it does not include the highest order derivatives of the metric. This means that the generalized Friedmann equation describing a homogeneous, isotropic universe in our model includes derivatives of the metric only up to third order. Hence, it should be of the following form.

$$\frac{\epsilon(t)}{H} \partial_t \left( \frac{\dot{H}}{H^2} \right) = \kappa^2 (\rho_{\text{other}} + \rho_{\phi,0} + \Delta \rho_{\phi}) - 3H^2,$$

where the dimensionless coefficient $\epsilon(t)$ is written in terms of $(\dot{H} / H^2, \kappa H, \kappa^2 \pi)$, $\rho_{\text{other}}$ is the energy density of fields other than $\phi$, $\rho_{\phi,0} = 2K L_{\text{kin,k}} - L_{\text{kin}} + V$ is a part of the energy density of $\phi$ which would be obtained by neglecting the $R$-dependence of $L_{\text{kin}}$, $\Delta \rho_{\phi}$ is corrections to $\rho_{\phi,0}$ which depends on $H$ and $\dot{H}$ but not on $H$. We shall see the detailed form of the generalized Friedmann equation soon.

In this form of the generalized Friedmann equation, $\epsilon$ and $\Delta \rho_{\phi}$ characterize corrections to the standard Friedmann equation. Indeed, if $\epsilon \equiv 0$ and if $\Delta \rho_{\phi} \equiv 0$ then this equation reduces to the standard Friedmann equation $3H^2 = \kappa^2 (\rho_{\text{other}} + \rho_{\phi,0})$. However, this does NOT necessarily guarantees the recovery of the standard Friedmann equation in the limit $\epsilon \to 0$, $\Delta \rho_{\phi} \to 0$. As in the above example of a Klein-Gordon equation with a higher derivative correction, what makes the limit rather subtle is the sign of the coefficient $\epsilon$ of the highest order derivative term. As we shall see soon, if $\epsilon$ approaches zero from the positive side then the standard Friedmann equation is indeed recovered in the limit. This situation is somehow similar to having an infinitely massive extra degree of freedom. On the other hand, if $\epsilon$ approaches zero from the negative side then the situation is similar to having an infinitely unstable tachyonic degree and, thus, the system is completely unstable. Moreover, if $\epsilon$ reaches zero in finite time then the system encounters a singularity where $\partial_t (\dot{H} / H^2)$ diverges. Hence, both the stability and the recovery of the standard Friedmann equation require $\epsilon$ to be positive.

In our model there are two contributions to $\epsilon$: one from $L_{\text{kin}}$ and the other from $\alpha R^2$. We shall see below that the first contribution is negative. This implies that the system would be unstable in the absence of the term $\alpha R^2$ and that we really need this term (or other ordinary higher derivative terms). On the other hand, we shall see below that
the sign of the second contribution is the same as the sign of $\alpha$. Moreover, as we have already seen in the previous section, $L_{kin}$ is much smaller than $H^4$ and, thus, than $\alpha R^2$ at low energy. This means that the contribution from the $\alpha R^2$ term determines the sign of $\epsilon$ at low energy and that $\epsilon$ is positive if and only if $\alpha$ is positive. Therefore, if and only if $\alpha$ is positive, the system is stable and the standard Friedmann equation is recovered at low energy. In the following, we shall show this statement more explicitly.

For the stability analysis and numerical works, it is sometimes more convenient to work with a set of first order differential equations than a set of higher order differential equations. In the end of this section, we shall use the set of first order differential equations to show the recovery of the standard Friedmann equation analytically. Some numerical results are shown in Sec. V.

A set of first order differential equations is obtained in Appendix A for a general action of the form (A1) with (A24). Restricting it to our model action (5), we obtain

$$
\dot{\varphi} = f \pi \cdot (\kappa^4 \pi^2) \frac{1}{2\kappa^2} - \frac{1}{2},
$$

$$
\dot{\pi} = -3H \pi - V'(\varphi),
$$

$$
\dot{H} = H^2 \Omega,
$$

$$
\epsilon \Omega = H \mathcal{F},
$$

$$
\kappa^2 \dot{\varphi} = -H^2 (G_0 + \Delta G),
$$

$$
\dot{\rho}_{\text{other}} = -3H (\rho_{\text{other}} + p_{\text{other}}),
$$

(19)

where

$$
\Omega \equiv \frac{\dot{H}}{H^2},
$$

$$
\varphi \equiv 2(L_R) - 2HL_R,
$$

$$
L_R = 2\alpha R + L_{\text{kin},R},
$$

(20)

and

$$
\epsilon \equiv 18\kappa^2 H^2 \left[ 4\alpha - \frac{(2q - 1)f''}{q\kappa^4} (\kappa^4 \pi^2)^{\frac{q - 1}{2}} \right],
$$

(21)

$$
\mathcal{F} \equiv \frac{3\kappa^2 \varphi}{H} + \frac{3(2q - 1)}{q\kappa^2} \left[ 12\Omega (\Omega + 2) H^2 f'' - f' \right] (\kappa^4 \pi^2)^{\frac{q - 1}{2}}
$$

$$
- \frac{6(3H \pi + V')}{H} f' \kappa^2 \pi \cdot (\kappa^4 \pi^2)^{-\frac{q - 1}{2\kappa^2}} + 72\alpha \kappa^2 H^2 (1 - 2\Omega)(\Omega + 2),
$$

$$
G_0 \equiv 2\Omega + \frac{1}{\kappa^2 H^2} \left[ \kappa^4 (\rho_{\text{other}} + p_{\text{other}}) + f \cdot (\kappa^4 \pi^2)^{\frac{q - 1}{2}} \right],
$$

$$
\Delta G \equiv -\frac{3(2q - 1)\Omega f'}{q\kappa^2} (\kappa^4 \pi^2)^{\frac{q - 1}{2\kappa^2}} + 72\alpha \kappa^2 H^2 (\Omega + 2),
$$

$$
R = 6H^2 (\Omega + 2).
$$

Here, $\rho_{\text{other}}$ and $p_{\text{other}}$ are energy density and pressure of fields and/or matter other than $\phi$. Besides the above dynamical equations, there is a constraint equation of the form

$$
\frac{3\kappa^2 \varphi}{H} = \mathcal{F}_0 + \Delta G - 36\alpha \kappa^2 H^2 (\Omega + 2)^2,
$$

(22)

where

$$
\mathcal{F}_0 \equiv \frac{\kappa^2 (\rho_{\text{other}} + \rho_{\phi,0}) - 3H^2}{H^2},
$$

$$
\rho_{\phi,0} \equiv 2KL_{\text{kin},K} - L_{\text{kin}} + V = \frac{2q - 1}{2q\kappa^4} f \cdot (\kappa^4 \pi^2)^{\frac{q - 1}{2\kappa^2}} + V.
$$

(23)

Note that the standard Friedmann equation and the standard dynamical equation are $\mathcal{F}_0 = 0$ and $G_0 = 0$, respectively. When the system is analyzed numerically, we can use the constraint equation to set the initial condition for $\varphi$ and also to check numerical accuracy.
It is also possible to use the constraint equation to eliminate \( \varphi \) from the set of first-order equations: \( \mathcal{F} \) in the forth equation in (19) is rewritten as

\[
\mathcal{F} = \mathcal{F}_0 + \Delta \mathcal{F},
\]

where

\[
\Delta \mathcal{F} = \frac{3(2q - 1)}{q\kappa^2} \left[ 12\Omega(\Omega + 2)H^2 f'' - (\Omega + 1)f' \right] \left( \kappa^4 \pi^2 \right)^{\frac{3}{q}} - 6(3H\pi + V')f' \kappa^2 \pi \cdot \left( \kappa^4 \pi^2 \right)^{\frac{1-q}{q}} - 108\alpha^2 H^2 \Omega(\Omega + 2).
\]

Hence, the fourth equation in (19) is reduced to (18) with \( \Delta \rho_\phi = \kappa^{-2} H^2 \Delta \mathcal{F} \).

Now let us show the recovery of the standard Friedmann equation at low energy by using the above set of first-order equations.

From the inequalities in (17), both \( L_{\text{kin}} \) and \( \alpha R^2 \sim H^4 \) are small compared to the Einstein-Hilbert term \( R/2\kappa^2 \) \( \sim H^2/\kappa^2 \) at low energy \( \kappa H \ll 1 \). Thus, even without any calculations we can conclude that

\[
e(t) \to 0, \quad \frac{\kappa^2}{H^2} (\rho_{\phi,0} - V) \to 0, \quad \frac{\kappa^2 \Delta \rho_\phi}{H^2} = \Delta \mathcal{F} \to 0,
\]

in the low energy limit \( \kappa H \to 0 \). Of course, it is straightforward to confirm this by using the explicit expressions presented here. More specifically, the second term in the expression (21) of \( \epsilon(t) \) is small compared to the first term at low energy \( \kappa H \ll 1 \):

\[
\kappa^{-4} f''(R) \cdot \left( \kappa^4 \pi^2 \right)^{3/q} \sim (\kappa^4 \pi^2)^{3/q} (\kappa^2 H^2)^2(m-1) \ll \alpha,
\]

where we have used the condition (7). Note that the second term in (21) is negative if \( f''(R) \) is positive. Therefore, if and only if \( \alpha \) is positive and the condition (7) is satisfied, \( \epsilon(t) \) is positive at low energy. In the next paragraph we shall see that the positivity of \( \epsilon(t) \) and, thus, the positivity of \( \alpha \) and the condition (7) are essential for the recovery of the standard Friedmann equation.

Having the behavior (26) and the positivity of \( \epsilon(t) \), it is easy to see that

\[
[k^2(\rho_{\text{other}} + V) - 3H^2]/H^2 \to 0 \quad (t \to \infty),
\]

provided that \( H > 0 \) and \( \dot{H}/H^2 < 0 \). This is equivalent to \( \mathcal{F}_0 \to 0 \quad (t \to \infty) \) since the kinetic part in \( \rho_{\phi,0} \) in (23) goes to zero much faster than the potential \( V \). Actually, if the right hand side of (18) is positive (or negative) then \( \dot{H}/H^2 \) will increase (or decrease, respectively). Hence, \( \dot{H} \) will decrease less (or more, respectively) rapidly. Moreover, \( \rho_{\text{other}} \) decreases more (or less, respectively) rapidly if it satisfies the weak energy condition. Thus, as a result, the right hand side of (18) will decrease (or increase, respectively). The decrease (or increase, respectively) should continue until the right hand side of (18) becomes almost zero, and we expect the behavior (28), which is nothing but the recovery of the standard Friedmann equation at low energy. This behavior, namely the recovery of the standard Friedmann equation, can be confirmed numerically. See Figs. 10-13 in the next section for numerical confirmation of the recovery of the standard Friedmann equation \( (\mathcal{F}_0 \to 0) \) and the standard dynamical equation \( (\mathcal{G}_0 \to 0) \). In the low energy Friedmann equation, \( \kappa^2 V \) plays the role of the cosmological constant \( \Lambda_{\text{eff}} \). 

V. NUMERICAL RESULTS

It is straightforward to integrate the set of first order equations (19) numerically. In this section we show some results. In the following we set

\[
q = 1, \quad m = 2, \quad c = 1, \quad \alpha = 1; \quad \rho_{\text{other}} \equiv 0, \quad p_{\text{other}} \equiv 0,
\]

for simplicity. In order to set initial values we need four independent conditions at \( t = 0 \) since we have five variables and one constraint equation. In the following we show numerical results for four different sets of initial values. We shall see that the system exhibits an attractor behavior. Because of the attractor behavior, in figures showing results of long numerical integration, plots for those four different initial values are degenerate.

First, let us confirm the behavior \( \mathcal{E} \equiv \kappa^2 H \pi/c \to -(3 - \dot{H}/H^2)^{-1} \) stated just after (11). Note that the feedback mechanism and the stability analysis are based on this important behavior. Fig. 1 shows that \((3 - \dot{H}/H^2)\mathcal{E} \) indeed
converges to 1 rather quickly. Thus, the evolution of $\pi$ is determined by this attractor behavior indecently of the initial condition. Figs 2 and 3 show the evolution of $\pi$.

**FIG. 1.** The behavior of $B \equiv \kappa^3 H\pi/c$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. This figure indeed shows that $-(3 - \dot{H}/H^2)B$ quickly converges to 1.

**FIG. 2.** The evolution of $\pi$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. 
FIG. 3. The asymptotic evolution of $\pi$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa (\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.

Second, let us see that the feedback mechanism indeed works and the effective cosmological constant stalls near zero. Figs. 4 and 5 show that. It is worth while stressing again that $V(\phi)$ includes the ground state energies of all fields and that adding finite terms to the potential does never spoil the feedback mechanism as far as the minimum of $V(\phi)$ remains negative. Although the additional finite terms change the form of $V(\phi)$ but the behavior of $V(\phi)$ near its root is always characterized by two parameters $c$ and $\phi_0$ (i.e. the first order Taylor expansion). The feedback mechanism works irrespective of the values of $c$ and $\phi_0$.

FIG. 4. The evolution of $\kappa^2 \Lambda_{eff} = \kappa^4 V(\phi) = c\kappa (\phi - \phi_0)$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa (\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. 

FIG. 5. The asymptotic evolution of $\kappa^2 \Lambda_{\text{eff}} = \kappa^4 V(\phi) = c \kappa (\phi - \phi_0)$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c \kappa (\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.

Third, because of the behavior of $\Lambda_{\text{eff}}$ governed by the feedback mechanism, the Hubble parameter $H$ also approaches zero very slowly. Figs 6 and 7 show this behavior of $H$. Figs 8 and 9 show that the dimensionless evolution rate $H/H^2$ of $H$ indeed approaches zero. Thus, asymptotically $H$ does not change in cosmological time scale. These again confirm the statement that $\Lambda_{\text{eff}}$ stalls near zero and that it does more slowly than matter or radiation.

FIG. 6. The evolution of the Hubble parameter $H$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c \kappa (\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. 

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FIG. 7. The asymptotic evolution of the Hubble parameter $H$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.

FIG. 8. The evolution of the dimensionless evolution rate $\dot{H}/H^2$ of $H$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$.
FIG. 9. The asymptotic evolution of the dimensionless evolution rate $\dot{H}/H^2$ of $H$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t=0)$. The initial values of $\phi$ and $\dot{H}/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $\dot{H}/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)|_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.

Fourth, we can also see that the standard Friedmann equation is recovered ($F_0 \to 0$) and that the standard dynamical equation is also recovered ($G_0 \to 0$). What actually happens is that the standard dynamical equation is recovered earlier than the standard Friedmann equation and that the latter is gradually recovered after that. Figs. 10, 11 and 12 show the early stage of the recovery process. Fig. 13 shows the late stage the recovery of the standard Friedmann equation.

FIG. 10. The early stage of the recovery of the standard Friedmann equation ($F_0 \to 0$) for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t=0)$. The initial values of $\phi$ and $\dot{H}/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $\dot{H}/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)|_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. 


FIG. 11. Recovery of the standard dynamical equation \((G_0 \to 0)\) for four different sets of initial values. In the figure, \(a\) is the scale factor and \(a_0 = a(t = 0)\). The initial values of \(\phi\) and \(H/H^2\) are \(c\chi(\phi - \phi_0)|_{t=0} = 0.01\) and \(H/H^2|_{t=0} = -0.02\). Two more independent conditions at \(t = 0\) are given by setting \(F_0\) and \(G_0\) in four different ways as \((F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)\).

FIG. 12. The early stage of the recovery process of the standard Friedmann and dynamical equations \((F_0 \to 0, G_0 \to 0)\) for four different sets of initial values. In the figure, \(a\) is the scale factor and \(a_0 = a(t = 0)\). The initial values of \(\phi\) and \(H/H^2\) are \(c\chi(\phi - \phi_0)|_{t=0} = 0.01\) and \(H/H^2|_{t=0} = -0.02\). Two more independent conditions at \(t = 0\) are given by setting \(F_0\) and \(G_0\) in four different ways as \((F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)\).
FIG. 13. The late stage of the recovery of the standard Friedmann equation ($F_0 \to 0$) for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.

Finally, we can check that the extra degree of freedom $\varphi$ goes to zero. Figs 14 and 15 confirm that behavior.

FIG. 14. The evolution of the extra degree $\varphi$ for four different sets of initial values. In the figure, $a$ is the scale factor and $a_0 = a(t = 0)$. The initial values of $\phi$ and $H/H^2$ are $c\kappa(\phi - \phi_0)|_{t=0} = 0.01$ and $H/H^2|_{t=0} = -0.02$. Two more independent conditions at $t = 0$ are given by setting $F_0$ and $G_0$ in four different ways as $(F_0, G_0)_{t=0} = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)$. Four lines are degenerate because of the attractor behavior.
FIG. 15. The asymptotic evolution of the extra degree \( \varphi \) for four different sets of initial values. In the figure, \( a \) is the scale factor and \( a_0 = a(t = 0) \). The initial values of \( \phi \) and \( H/H^2 \) are \( \exp(\phi - \phi_0)|\varphi| = 0.01 \) and \( H/H^2|\varphi| = -0.02 \). Two more independent conditions at \( t = 0 \) are given by setting \( F_0 \) and \( G_0 \) in four different ways as \((F_0, G_0)|\varphi| = (0.1, 0.1), (-0.1, 0.1), (0.1, -0.01), (-0.1, -0.01)\). Four lines are degenerate because of the attractor behavior.

VI. REHEATING

We have achieved the vanishing cosmological constant in a way that is stable under radiative corrections and that has self-consistent, stable dynamics. However, although the cosmological constant approaches zero, it does so more slowly than matter or radiation so that without additional structure, the universe would be empty. It is not entirely clear that a dynamical model where this is not the case could be successful since it is this property that makes it possible for all fields other than \( \phi \) to settle into their ground state before \( \phi \) stalls at zero curvature so that the zero curvature really corresponds to the vanishing cosmological constant. Moreover, the singular behavior of a kinetic term coefficient and, thus, the slow evolution of \( \varphi \) are required for stability. This does imply, however, that should this mechanism be responsible for a low cosmological constant, reheating would be required to thermally populate the universe after the cosmological constant has decreased to a small value. Nonetheless, as we shall see below, this model can at the very least reduce fine-tuning by 60 orders of magnitude or provide a new mechanism for sampling possible cosmological constants and implementing the anthropic principle.

The reheating process requires further speculation, and is a subject for future research. A couple of possibilities are:

(I) Low-energy inflation. One can consider an extra scalar field \( \chi \) with mass \( m_\chi \sim 10^{-3} eV \) and a term like \(-R\chi^2\). When \( R \sim m_\chi^2 \), a phase transition would occur (as in hybrid inflation [35]) and the universe would be reheated up to temperature \( \sim TeV \). This phase transition happens when the energy stored in \( \chi \) plus the energy stored in \( \phi \) yields a Hubble constant of approximately \( m_\chi \). The energy in \( \chi \) will decrease during the phase transition; the energy after the phase transition must be very small. In this case the cosmological constant problem is reduced from \((M_{Pl}/10^{-3}eV)^4 \sim 10^{120} \) to \((TeV/10^{-3}eV)^4 \sim 10^{60} \). For smaller \( m_\chi \), the reheat temperature would be lower and the tuning of the cosmological constant would presumably be smaller.

(II) Energy inflow from extra dimensions. For example, in a non-elastic scattering of branes, a part of the kinetic energy due to the relative motion can be converted to radiation on our brane without changing the brane tension and the cosmological constant. For this to work, branes should be sufficiently flat and parallel.

After reheating, the conventional standard cosmology with a vanishingly small cosmological constant can be restored. This is because the scalar field \( \phi \) is just frozen at low energy. Namely,

\[
|\kappa^2 \dot{\phi}| \sim (\kappa H)^{4m} \times |\kappa^2 \pi| \frac{1}{\kappa^2 \pi}
\]

(30)
can be made arbitrarily small at low energy \((\kappa H \ll 1)\) by considering a sufficiently large \( m \), where \( \kappa^2 \pi \) is estimated by (12). In the scenario (I), both \( \pi \) and \( H \) evolve continuously through the reheating epoch. Hence we obtain

\[
|\kappa^5 \dot{V}| \sim (\kappa H)^{4m-\frac{3}{2}}
\]

(31)
and the 4d cosmological constant does not overshoot zero within the present age of the universe if \( m \) is large enough to ensure that 
\[
(k^2 R_{\text{reh}})^{2m - \frac{3}{m + 2}} < \kappa H_{\text{today}}^{4m} / (k H_{\text{before}})^{\frac{3}{m + 2}},
\]
where we have used the Friedmann equation \( 3H^2_{\text{before}} \sim k^2 V \). Here, \( H_{\text{before}} \) and \( H_{\text{after}} \) are the Hubble parameter before and after the energy inflow from extra dimensions, respectively. By using (11) it is easily shown that \( k^3 \pi H + c/5 \propto a^{-5} \) during the radiation-dominated epoch \( (H / H^2 = -2) \). Hence, \( k^2 \pi H \) decays from \( \sim H_{\text{after}} / H_{\text{before}} \) to \( O(1) \) in the time scale \( \Delta t_{\text{relax}} \sim (H_{\text{after}} / H_{\text{before}})^{3/5} H^{-1}_{\text{after}} \). Thus, if \( m \) is large enough to ensure that 
\[
(k H_{\text{after}})^{4m - \frac{3}{m + 2}} < (k H_{\text{before}})^{\frac{3}{m + 2}} + \frac{c}{5}
\]
then \( V \) does not overshoot zero in the time scale \( \Delta t_{\text{relax}} \), provided that \( k H_{\text{after}} \ll 1 \). After that, we can use the estimate (31) with \( H \) being the realtime value, and \( V \) does not overshoot zero if \( m \) is sufficiently large. Let us recall that a choice like \( f(R) \sim \exp(-k^{-4} R^{-2}) \) corresponds to \( m \to \infty \).

One might worry about symmetry restoration and phase transitions that occur after reheating. This is not a problem in both examples if \( m \) is sufficiently large. The reason is that \( R_{\text{reh}} / 4 - k^2 (U(\chi^+)|_{R=R_{\text{reh}}} - U(\chi^-)) \) (or \( k^2 V \), respectively) is the cosmological constant at zero temperature since the temperature before the reheating is zero. The large \( m \) ensures that \( \Lambda_{\text{today}} \) is still positive and small since \( \phi \) continues to be almost frozen all the way down to the present epoch including the time when the symmetry is restored and during the time the phase transition takes place.

Even if all else fails, although not our initial subjective, the existence of \( \phi \) can at the very least provide a natural framework in which to implement the anthropic principle. If we assume eternal inflation, there would be many inflationary universes. In each universe, the cosmological constant is determined by how much \( \phi \) has rolled when inflation ends. That in turn depends on the number of \( e \)-foldings that occurred before inflation stopped. In an eternal inflation scenario, different numbers of \( e \)-foldings would occur in different domains and therefore different \( \phi \) values, and hence different values of the cosmological constant would occur in different regions.

Our model in general predicts \( w_\phi \equiv p_\phi / \rho_\phi \sim -1 \) today because of (31) since pre- and post-reheating behavior requires a large \( m \). In the scenario (I), \( |\dot{V}| \) continuously decreases. In the scenario (II), \( |\dot{V}| \) increases suddenly at reheating but is adjusted to the behavior (31) in the time scale \( \Delta t_{\text{relax}} \). Note that \( H_{\text{today}} \Delta t_{\text{relax}} < (H_{\text{before}} / H_{\text{after}})^{3/5} \ll 1 \) since \( H^2_{\text{after}} \gg H^2_{\text{before}} \approx \Lambda_{\text{before}} / 3 > \Lambda_{\text{today}} / 3 \approx H^2_{\text{today}} \). This means that the behavior (31) and \( w_\phi \simeq -1 \) are realized before the present stage of the universe.

VII. CLASSICAL STABILITY II - LINEARIZED GRAVITY

In this section, restricting to the \( q = 1 \) case for simplicity, we show that linearized Einstein gravity in a Minkowski background is recovered at distances larger than the length scale \( l_\ast = \sqrt{\alpha k} \) and at energies lower than \( l_\ast^{-1} \). The extension to general \( q \) \( (> 1/2) \) should be straightforward.

A possible source of instability which would disturb weak gravity in a Minkowski background could be an inhomogeneous fluctuation of \( \phi \), which could possibly generate violent breakdown of linearized Einstein gravity. Quite surprisingly, this is not the case and the seemingly most dangerous part, the kinetic term, is not as dangerous as it looks. Essential to this conclusion is the constraint equation, which prevents \( \phi \) from fluctuating freely and forces the denominator and the numerator to fluctuate in a strongly correlated way so that the contributions of the kinetic term to the equation of motion are regular and much smaller than those of the \( \alpha R^2 \) term. Namely, the scalar field and metric cannot fluctuate independently. In fact, it is a well-known fact in the standard scalar field cosmology that there is only one physical degree of freedom out of scalar field and metric degrees for scalar-type perturbations. In the longitudinal gauge (see (40) or (D11)), one of the constraint equations is the \((0i)\)-component of the perturbed Einstein equation, which is roughly of the form

\[
\pi \partial_i (\delta \phi) \simeq \frac{\delta G_{0i}}{\kappa^2} - \delta T^{(other)}_{0i},
\]

where \( \pi \) is the momentum conjugate to the homogeneous background \( \phi \) defined in (9), \( \delta \phi \) is the inhomogeneous perturbation of \( \phi \), \( \delta T^{(other)}_{0i} \) is the \((0i)\)-component of the stress-energy tensor of fields and matter other than \( \phi \). Note that homogeneous perturbations can be absorbed into the background and that, without loss of generality, we can
restrict our analysis to inhomogeneous perturbations, for which \( \partial_t(\delta \phi) \neq 0 \). The right hand side of this equation is manifestly regular and we already know that \( \pi \) behaves like (12). Thus, we conclude that

\[
\delta \phi \simeq \kappa H \times (\text{regular expression}).
\]  

(34)

This means that \( \delta \phi \) is indeed suppressed at low energy (\( \kappa H \ll 1 \)). Similarly, the other constraint equation, namely the traceless part of the \((ij)\)-components of the perturbed Einstein equation, leads to

\[
(\delta \phi) = (\kappa H)^3 \times (\text{regular expression}).
\]  

(35)

This again means that inhomogeneous perturbation of \( \phi \) is suppressed at low energy. Hence, from the constraint equations, we expect that the stress energy tensor of the scalar field should be small enough. More rigorous treatment of the constraint equations is included in the complete analysis shown in Appendix D.

Since contributions of the singular-looking kinetic term are expected to be small enough, the only possibly important correction to Einstein gravity is again due to the \( \alpha R^2 \) term. This tells us that the linearized gravity in our model should be similar to that in the theory \( R/2\kappa^2 + \alpha R^2 \). Hence, we should be able to recover the linearized Einstein gravity in Minkowski background at distances longer than the length scale \( l_* = \sqrt{\kappa \alpha} \) and at energies lower than \( l_*^{-1} \) [36].

A complete analysis of the weak gravity in Minkowski background requires much more carefulness. One of the reasons is that the action for the scalar field is not manifestly well defined in Minkowski background since the denominator of the kinetic term vanishes at \( R = 0 \). Moreover, it is not totally clear how to treat the perturbations of the denominator. This situation requires a sort of regularization for the field equation. In the end of the calculation we of course need to take the limit where the regularization is turned off. Unlike calculations in quantum field theories, we must not renormalize anything since we are dealing with classical dynamics. Namely, before turning off the regularization, we must not subtract anything from the regularized field equation and must treat it as it is.

For the purpose of the rigorous treatment of weak gravity, we investigate in detail perturbations around the flat FRW background in the longitudinal gauge and take the \( \kappa H \to +0 \) limit in the end of the calculation. This strategy makes it possible to analyze linear perturbations around Minkowski background, on which the action of the scalar field perturbation is not apparently well-defined. For simplicity, we assume that the FRW background is driven by the scalar field \( \phi \). Of course, we include a general matter stress-energy tensor into perturbations and carefully investigate how they couple to gravity. A critical point for this analysis is that constraint equations ((0\(i\))-components and the traceless part of \((ij)\)-components of Einstein equation) prevent \( \phi \) from fluctuating freely, and it turns out that the \( \phi \) perturbation is of sufficiently high order in the \( \kappa H \) expansion that the perturbative analysis is under control.

The result of the analysis is simple. The linearized gravity in our model is similar to that in the theory \( R/2\kappa^2 + \alpha R^2 \) in the sense that the only difference is the existence of an extra scalar-type massless mode in our model. The extra massless mode is decoupled from the matter stress-energy tensor at the linearized level. Hence, as far as we are concerned with classical, linear perturbations generated by matter sources, these two theories give exactly the same prediction. Therefore, in our model the linearized Einstein gravity in Minkowski background is recovered at distances longer than the length scale \( l_* = \sqrt{\alpha \kappa} \) and at energies lower than \( l_*^{-1} \).

Let us begin by defining the metric perturbation \( \delta g_{\mu \nu} \) by

\[
g_{\mu \nu} = g_{\mu \nu}^{(0)} + \delta g_{\mu \nu},
\]  

(36)

where

\[
g_{\mu \nu}^{(0)} dx^\mu dx^\nu = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j
\]  

(37)

\( (i = 1, 2, 3) \). We shall investigate the system of coupled equations for the scalar field perturbation \( \delta \phi \), the metric perturbation \( \delta g_{\mu \nu} \) and the stress energy tensor contributions from other fields and matter, taking into account all higher derivative corrections. In order to investigate linearized gravity in Minkowski background, in the end of calculations, we shall take the limit

\[
\kappa H \to +0,
\]  

(38)

keeping \( \partial_t^n H/H^{n+1} (n = 1, 2, 3) \) finite, where \( H = \partial_t a/a \). In the following, we shall denote \( \partial_t \) by an overdot.

In order to take the full advantage of the symmetry of the background spacetime (with \( H \neq 0 \)), we expand the scalar field perturbation, the metric perturbation and the stress energy tensor contributions from other matter and fields by harmonics on the 3-plane with each expansion coefficient being a function of the time \( t \) only. Now it is well-known and easily shown that there are three distinct types of perturbations and that each type is decoupled
from the others in the linearized level. They are called scalar-, vector- and tensor-type perturbations and each of them is an irreducible representation of the symmetry of the background FRW spacetime: spin-0, spin-1 and spin-2 representations, respectively. Since scalar-, vector-, and tensor-type perturbations are decoupled from each other in the linearized level, we can analyze perturbations of each type separately.

The vector- and tensor-type perturbations are decoupled from scalar-type perturbations and, thus, from the perturbation of the scalar field $\phi$. Moreover, the linear perturbation of the Ricci scalar vanishes for vector- and tensor-type perturbations. Thus, the $\kappa H \rightarrow +0$ limit of perturbation equations is manifestly well-defined. Actually, in the $\kappa H \rightarrow +0$ limit, the linear perturbation of the stress energy tensor $T_{\phi \mu \nu}$ of $\phi$ vanishes. Therefore, we obtain

$$\delta G_{\mu \nu} = \kappa^2 T_{\mu \nu}^\text{other}. \quad (39)$$

This is nothing but the linearized Einstein equation. In other words, possible corrections to the linearized Einstein equation appear only for the scalar-type perturbations.

So, let us concentrate on the scalar-type perturbations. As is well-known, in linear perturbations we have not only physical degrees of freedom but also gauge degrees of freedom so we need to fix gauge. (See subsection D 1 of Appendix D for the formula of infinitesimal gauge transformation and a gauge choice.) In the longitudinal gauge, the perturbed metric is written as

$$g_{\mu \nu}dx^\mu dx^\nu = -(1 + 2\Phi Y)dt^2 + (1 - 2\Psi Y)a^2 \delta_{ij}dx^i dx^j, \quad (40)$$

where $\Phi$ and $\Psi$ are functions of $t$, and $Y$ is the scalar harmonics on the 3-plane with 3-momentum $k_i$. Here, we omit to write $k_i$ and the integration over all possible values of $k_i$.

As already stated, we shall analyze the system of coupled equations for the scalar field perturbation $\delta \phi$, the metric perturbation $\delta g_{\mu \nu}$ and the stress energy tensor contributions from other fields and matter in the FRW background, taking into account all higher derivative corrections. In the end of calculations, we shall take the limit $\kappa H \rightarrow +0$, keeping $\partial_i^0 H/H^{n+1}$ ($n = 1, 2, 3$) finite. For this purpose, we introduce a small dimensionless parameter $\epsilon$ so that

$$\kappa H = O(\epsilon), \quad \frac{\kappa^2 k^2}{a^2} = O(\epsilon^0). \quad (41)$$

and expand all quantities and equations. We shall keep the expanded equations up to the order $O(\epsilon^2)$ to obtain equations governing the quantities of order $O(\epsilon^0)$. In the end of the calculation, we shall take the limit $\epsilon \rightarrow 0$.

Complete analysis of the scalar-type perturbations is given in subsection D 2 of Appendix D. The equations governing the $O(\epsilon^0)$ order metric perturbations are obtained after going through the procedure described above and eliminating the scalar field perturbation. The result is summarized as the following equations for linearized gravity in Minkowski background.

$$\kappa^2 k^2 \Psi_+ + \kappa^4 \left( \tau_{00} + 2k^2 \tau_{(LL)} \right) = 0,$$

$$\Box \left[ (1 - 12\alpha k^2 \Box) \Psi_- + 4\alpha \kappa^2 \left( 3\Psi_+ + k^2 \Psi_+ \right) - 2\kappa^2 \tau_{(LL)} \right] = 0, \quad (42)$$

where $\tau$'s are scalar-type perturbations of the stress energy tensor contributions from other fields and matter defined by (D20). Here, we have redefined the spatial coordinates as $a_0 x^i \rightarrow x^i$ and

$$\Box \equiv -\partial_i^0 - k^2. \quad (43)$$

Before taking the $\kappa H \rightarrow 0$ limit, $a_0$ is defined as the value of the scale factor at the time when a weak gravity experiment is performed, assuming that the duration of the experiment is much shorter than the cosmological time scale $H^{-1}$. After taking the $\kappa H \rightarrow 0$ limit, the scale factor is a constant and, thus, the above rescaling of the spatial coordinates makes perfect sense at any time. The perturbed metric, after the $\kappa H \rightarrow +0$ limit and the above redefinition of the spatial coordinates, is

$$ds^2 = -(1 + 2\Phi Y)dt^2 + (1 - 2\Psi Y)\delta_{ij}dx^i dx^j, \quad (44)$$

and $\Psi_{\pm} = \Psi \pm \Phi$. The conservation equation is

$$\tau_{00} + k^2 \tau_{(L)0} = 0,$$

$$\tau_{(L)0} + \frac{4}{3} k^2 \tau_{(LL)} - \tau_{(Y)} = 0. \quad (45)$$
Now it is time to show that the linearized Einstein gravity in Minkowski background is recovered at distances longer than the length scale \( l_\alpha = \sqrt{\alpha \kappa} \) and at energies lower than \( l_\alpha^{-1} \). (Note that we have assumed that \( \alpha \) is positive.) For this purpose and to make the arguments qualitative, let us compare the result (42) for scalar-type perturbations in our model with the corresponding equations in the higher-curvature theory whose gravitational action is

\[
I_{HD} = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \alpha R^2 \right).
\]

For scalar perturbations given by (44) and (D20), linearized gravity equations in the theory (46) are

\[
\kappa^2 k^2 \Psi_+ + \kappa^4 \left( \tau_{00} + 2k^2 \tau_{(LL)} \right) = 0, \\
(1 - 12\alpha \kappa^2 \Box) \Psi_- + 4\alpha \kappa^2 \left( 3\Psi_+ + k^2 \Psi_+ \right) - 2\kappa^2 \tau_{(LL)} = 0.
\]

Therefore, for linearized gravity the only difference between our model and the higher-curvature theory (46) is that there is an extra massless mode for \( \Psi_- \) in our model. (Notice the extra \( \Box \) in the second equation in (42).) However, this zero mode does not couple to the matter stress energy \( T_{\mu\nu}^{\text{other}} \) directly since the extra \( \Box \) in the second equation in (42) is applied to the whole expression. Hence, as far as we are concerned with linear perturbations generated by matter sources, these two theories give exactly the same prediction. Thus, it is concluded that the scalar-type linearized gravity in Minkowski background in our model is effectively described by the theory (46).

In the theory (46), the linearized Einstein gravity is recovered at distances longer than the length scale \( l_\alpha = \sqrt{\alpha \kappa} \) and at energies lower than \( l_\alpha^{-1} \). To see this, let us quote the equations governing scalar perturbations in Einstein gravity:

\[
\kappa^2 k^2 \Psi_+ + \kappa^4 \left( \tau_{00} + 2k^2 \tau_{(LL)} \right) = 0, \\
\Psi_- - 2\kappa^2 \tau_{(LL)} = 0.
\]

It is easy to see that these equations in Einstein theory are recovered from the corresponding equations (47) in the theory (46) if \( \alpha \kappa^2 \Psi_\perp / \Psi_\pm \) and \( \alpha \kappa^2 k^2 \) are sufficiently small. For the recovery, it is fairly important that we had assumed that \( \alpha \) is positive. Actually, the higher derivative terms in the second equation in (47) do not introduce extra instabilities if and only if \( \alpha \) is non-negative. Note that the stability of homogeneous, isotropic evolution of the universe has also required \( \alpha > 0 \).

**VIII. QUANTUM MECHANICAL STABILITY**

Now let us think about quantum mechanical stabilities.

The stability under radiative corrections is an important feature of our model. Radiative corrections will produce additional regular terms in the action, but the field will stall whether or not these are present. Adding finite terms to the potential part does not change anything since the curvature \( R \) feels the corrected potential via the gravity equation. As for the kinetic part, we would like to stress again that the singular behavior of the kinetic term coefficient is imposed only on the most singular-looking term among many possible terms in the kinetic part and, thus, adding any kinetic terms which are less singular-looking at \( R = 0 \) does not change anything. Of course, adding a more singular-looking kinetic term just makes the condition more robust. The more singular a kinetic term looks, the more stable it is under radiative corrections.

Since \( f(R) \) is in the denominator of a kinetic term and vanishes at \( R = 0 \), one might also worry about additional singular potential terms like \( 1/f(R) \) being generated through radiative corrections. However, this does not happen. In order to see this, it is convenient to normalize quantum fluctuation of the scalar field \( \phi \) around the homogeneous classical background. For simplicity, we restrict our discussion to the \( q = 1 \) case and work in the unit where \( \kappa = 1 \). The kinetic term is expanded as

\[
-\frac{\partial^\mu \phi \partial_\mu \phi}{2f} = -\frac{1}{2} \left[ \frac{1}{f_0} + \left( \frac{1}{f} \right)'_0 \left( \delta R + \delta_2 R \right) + \left( \frac{1}{f} \right)''_0 \left( \delta R \right)^2 \right] \times (g^\mu_\nu - h^\mu_\nu + h^{\mu\lambda} h_\lambda^\nu) \partial_\mu \left( \phi_0 + \sqrt{f_0} \delta \phi_c \right) \partial_\nu \left( \phi_0 + \sqrt{f_0} \delta \phi_c \right)
\]

up to the quadratic order in perturbations, where a quantity with the subscript 0 represents the value on the FRW background considered in Sec. IV, \( h_{\mu\nu} \) is metric perturbation, \( \delta R \) and \( \delta_2 R \) are the \( O(h) \) and \( O(h^2) \) parts of the Ricci scalar perturbation, respectively, and
\[ \delta \phi_c \equiv \frac{\delta \phi}{\sqrt{f_0}} \]  

(50)

is the normalized quantum fluctuation of the scalar field. Hence, the contributions of the kinetic term to the full quadratic action multiplied by \(-2\) are

\[ g'^\mu \partial_\mu (\delta \phi_c) \partial_\nu (\delta \phi_c) \]

\[ -R_0 f_0 \frac{\dot{R}_0}{R_0} \delta \phi_c (\delta \phi_c) - \frac{1}{4} \left( R_0 f_0 \frac{\dot{R}_0}{R_0} \right)^2 (\delta \phi_c)^2 - 2 \sqrt{f_0} R_0 f_0 \left( \frac{1}{f} \right) \delta R (\delta \phi_c) - \sqrt{f_0} R_0 h^\mu \partial_\mu (\delta \phi_c) \]

\[ -\sqrt{f_0} R_0 f_0 \left( \frac{1}{f} \right) (\delta \phi_c)^2 - R_0 f_0 \frac{\dot{R}_0}{R_0} \delta R \delta \phi_c - \sqrt{f_0} R_0 f_0 \frac{\dot{R}_0}{R_0} R_0 h^\mu \delta \phi_c - R_0 f_0 \frac{\dot{R}_0}{R_0} R_0 h^\mu h^\nu \delta \phi_c \]

\[ \left( \frac{1}{f} \right)^2 R_0 f_0 \left( \frac{1}{f} \right) (\delta R)^2 - R_0 f_0 \left( \frac{1}{f} \right) R_0 \delta_2 R - \left( \frac{\sqrt{f_0} R_0}{R_0} \right)^2 R_0 f_0 \left( \frac{1}{f} \right) R_0 h^\mu \delta R. \]

(51)

Actually, all except for the first term vanish in the low energy limit since

\[ R_0 \propto H^2 \to 0, \]

\[ \frac{\dot{R}_0}{R_0} \propto H \cdot \frac{\dot{H}}{H^2} \to 0, \]

\[ \frac{\sqrt{f_0} R_0}{R_0} \propto H^{2m-3} \to 0 \]

(52)

in the low energy limit \(H \to 0\) and

\[ \frac{R_0 f'_0}{f_0} \sim R_0 f_0 \left( \frac{1}{f} \right) \sim R_0^2 f_0 \left( \frac{1}{f} \right)^2 \sim 1. \]

(53)

This means that \(\delta \phi_c\) asymptotically approaches a canonically normalized fluctuation in the low energy limit. In terms of \(\delta \phi_c\), each term in the potential part can include a positive power of \(\sqrt{f_0}\) but not negative power. Therefore, loop contributions of \(\delta \phi_c\) to the radiatively corrected potential part can include positive powers of \(\sqrt{f_0}\) but not negative power.

More rigorous treatment requires careful consideration of constraint equations among the scalar field perturbation and the metric perturbation. Namely, we need to introduce a variable analogous to the Mukhanov variable [37] in the standard field theory cosmology. Hence, rigorous treatment seems much more complicated than the above. Nonetheless, the above argument is convincing enough and we do not expect terms with negative power of \(\sqrt{f_0}\) to appear in the potential part of the proper perturbation variable.

In the above, it has been shown that the feedback mechanism is stable under radiative corrections and thus a zero or small cosmological constant is protected against radiative corrections. In the following, we show that quantum fluctuation at low energy is so small that the effective cosmological constant at low energy does not overshoot zero even quantum mechanically.

In the \(\kappa^2 R \to 0\) limit the scalar field becomes completely weakly coupled. Hence, one might think that the scalar field could overshoot zero curvature by quantum fluctuation. This does not happen, as far as the (effective) cosmological constant is a substantial component of the energy density of the universe. The reason is as follows. (For simplicity we consider the \(q = 1\) case only, but it is easy to generalize it to a general \(q > 1/2\). We shall again work in the unit where \(\kappa = 1\), and assume that the stability condition \(m > 3/2\) is satisfied.)

The fluctuation \(\delta \phi\) of the scalar field \(\phi\) around a homogeneous background does not have a canonically normalized kinetic term. As we showed in the above, the normalized fluctuation \(\delta \phi_c \equiv \delta \phi/\sqrt{f_0}\) has a canonically normalized kinetic term plus additional terms which vanish in the low energy limit \(H \to 0\). Hence, by dimensionality, amplitude of the quantum fluctuation \(\delta \phi_c\) should be \(|\delta \phi_c| \sim H\). Hence, the quantum fluctuation of the cosmological constant is estimated as

\[ |\delta \Lambda_{\text{eff}}| \sim |c \delta \phi| \propto H^{2m+1}, \]

(54)

where we have used the behavior \(f_0 \propto H^{4m}\) near \(H = 0\). On the other hand, if the (effective) cosmological constant is a substantial component of the cosmological energy density then

\[ \Lambda_{\text{eff}} \sim H^2. \]

(55)
Note that under the stability condition \( m > 3/2 \) (see (7) with \( q = 1 \)), the cosmological constant asymptotically dominates the cosmological energy density. Since \( 2m + 1 > 2 \) from the stability condition, the estimates (54) and (55) imply that
\[
|\delta \Lambda_{\text{eff}}| \ll \Lambda_{\text{eff}} \tag{56}
\]
at low energy \((H \ll 1)\). This means that \( \Lambda_{\text{eff}} \) does not jump to a negative value by quantum fluctuation.

**IX. SUMMARY**

In the present paper, we have investigated gravity in the recently proposed dynamical approach to the cosmological constant. We have shown that (i) the effective cosmological constant decreases in time and asymptotically approaches zero from above; (ii) the evolution of a homogeneous, isotropic universe is described by the standard Friedmann equation at low energy; that (iii) classical, linearized gravity in Minkowski background is described by Einstein gravity at distances longer than \( l^* = \sqrt{\alpha \kappa} \) and at energies lower than \( l^{-1} \), where \( \kappa \) is the Planck length and \( \alpha \) is a dimensionless, positive parameter of the model; that (iv) the mechanism is stable under radiative corrections and thus a zero or small cosmological constant is protected against radiative corrections; and that (v) quantum fluctuation at low energy is so small that the effective cosmological constant at low energy does not overshoot zero even quantum mechanically.

One of the most disturbing difficulties in thinking about the cosmological constant is that it is not protected against radiative corrections, which usually generate enormous vacuum energies compared to what we observe. Because of the above properties (i)-(v), the feedback mechanism can be considered as a dynamical way to protect a zero or small cosmological constant against radiative corrections. Hence, although the feedback mechanism by itself does not solve the cosmological constant problem, it can help solving the problem.

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**APPENDIX A: DERIVATION OF EQUATIONS OF MOTION**

In this appendix we consider a more general action of the form
\[
I[g_{\mu\nu}, \phi] = \int d^Dx \sqrt{-g} \left[ \frac{R}{2\kappa^2} + L(R, X, K, \phi) \right],
\tag{A1}
\]
where \( X = R^{\mu\nu} R_{\mu\nu} \), \( K = -\kappa^4 \partial^\mu \phi \partial^\mu \phi \), and \( R \) and \( R_{\mu\nu} \) are the Ricci scalar and the Ricci tensor of the metric \( g_{\mu\nu} \). (See Appendix B for an alternative description.) Our sign convention for the metric is \((-+++\))

1. **Variational formulas**

In this subsection we derive some general formulas for variations of geometrical objects in \( D \)-dimension. Let us decompose the metric \( g_{\mu\nu} \) into the background \( g^{(0)}_{\mu\nu} \) and perturbation \( \delta g_{\mu\nu} \):
\[
g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \tag{A2}
\]
The indices of any linear-order quantities are lowered by \( g^{(0)}_{\mu\nu} \) and raised by the inverse \( g^{(0)\mu\nu} \) of \( g^{(0)}_{\mu\nu} \). For example,
\[
\delta g^{\mu\nu} = g^{(0)\mu\rho} g^{(0)\nu\sigma} \delta g_{\rho\sigma}. \tag{A3}
\]
Hence,
\[ g^{\mu\nu} = g^{(0)\mu\nu} - \delta g^{\mu\nu} \] (A4)

up to the linear order.

Next, we can easily expand \( \sqrt{-g} \) and the Christoffel symbol \( \Gamma^\rho_{\mu\nu} \) as follows, where \( g \) is the determinant of \( g_{\mu\nu} \).

\[ \sqrt{-g} = \sqrt{-g^{(0)}} \left( 1 + \frac{1}{2} \delta g \right), \]
\[ \Gamma^\rho_{\mu\nu} = \Gamma^{(0)}_{\mu\nu} + \delta \Gamma^\rho_{\mu\nu} \] (A5)

up to the linear order, where \( \delta g = g^{(0)\mu\nu} \delta g_{\mu\nu} \), \( \Gamma^{(0)}_{\mu\nu} \) is the Christoffel symbol for the background metric \( g^{(0)}_{\mu\nu} \), and

\[ \delta \Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{(0)\rho\sigma} \left( \delta g_{\sigma\mu;\nu} + \delta g_{\sigma\nu;\mu} - \delta g_{\mu\nu;\sigma} \right). \] (A6)

Here, a semicolon denotes the covariant derivative compatible with the background metric \( g^{(0)}_{\mu\nu} \).

Thirdly, the Ricci tensor is expanded as follows.

\[ R_{\mu\nu} = R^{(0)}_{\mu\nu} + \delta R_{\mu\nu} \]
(A7)

up to the linear order, where \( R^{(0)}_{\mu\nu} \) is the Ricci tensor for the background metric \( g^{(0)}_{\mu\nu} \), and

\[ \delta R_{\mu\nu} = \frac{1}{2} \left( \delta g^{\rho\nu}_{\mu;\rho} + \delta g^{\rho\mu}_{\nu;\rho} - \delta g^{\rho}_{\mu;\nu;\rho} \right). \] (A8)

Therefore, we obtain

\[ R = R^{(0)} + \delta R, \]
\[ X = R^{(0)\mu\nu} R^{(0)}_{\mu\nu} + \delta X, \] (A9)

where

\[ \delta R = -R^{(0)\mu\nu} \delta g_{\mu\nu} + (\delta g^{\mu\nu}_{\nu;\mu} - \delta g^{\mu\nu})_{;\mu}, \]
\[ \delta X = 2R^{(0)\mu\nu} \delta R_{\mu\nu} - 2R^{(0)\mu\rho} R^{(0)\nu}_{\rho} \delta g_{\mu\nu}. \] (A10)

2. Equations of motion

By using the variational formulas presented in the previous subsection, the variation of the action \( \delta I \) is calculated as

\[ \delta I = \int d^Dx \sqrt{-g} \left[ -\frac{1}{2\kappa^2} \left( G^{\mu\nu} - \kappa^2 T^{\mu\nu}_{\phi} \right) \delta g_{\mu\nu} + E_\phi \delta \phi \right], \] (A11)

where

\[ T^{\mu\nu}_{\phi} = 2\kappa^4 \mathbf{L}_K \partial^\mu \phi \partial^\nu \phi + \mathbf{L}_K g^{\mu\nu} - 2\mathbf{L}_R R^{\mu\nu} + 2(\mathbf{L}_R^\rho R^{\mu\nu})_{;\rho} - 2(\mathbf{L}_R^\rho R^{\mu\nu})_{;\rho} g^{\mu\nu} \\
+ 2(\mathbf{L}_X R^{\mu\rho})_{;\rho} + 2(\mathbf{L}_X R^{\rho\nu}) g^{\mu\nu} - 2(\mathbf{L}_X R^{\mu\rho})_{;\rho} g^{\mu\nu} - 4\mathbf{L}_X R^{\mu\rho} R^{\nu}_{\rho}, \]
\[ E_\phi = 2\kappa^4 (\mathbf{L}_K \partial^\mu \phi)_{;\mu} + \mathbf{L}_\phi. \] (A12)

Hence we obtain the following set of equations of motion.

\[ G^{\mu\nu} = \kappa^2 \left( T^{\mu\nu}_{\phi} + T^{\mu\nu}_{\text{other}} \right), \]
\[ E_\phi = 0, \] (A13)

where \( T^{\mu\nu}_{\text{other}} \) is the stress energy tensor of other fields whose action should be added to the action (A1).
Let us consider a homogeneous $\phi = \phi(t)$ in the $D = 4$ flat FRW background (8). With this ansatz, the stress energy tensor is

$$T^\mu_{\phi\nu} = \begin{pmatrix} -\rho_\phi & 0 & 0 & 0 \\ 0 & p_\phi & 0 & 0 \\ 0 & 0 & p_\phi & 0 \\ 0 & 0 & 0 & p_\phi \end{pmatrix},$$  \tag{A14}$$

where

$$\rho_\phi = 2\kappa^4 L_{,K} \dot{\phi}^2 - L - 6H \left[ L_{,R} + 2(3H^2 + 2\dot{H})L_{,X} \right] + 6 \left[ (H^2 + \dot{H}) L_{,R} + 2(3H^4 + 3H^2\dot{H} + 2\dot{H}^2)L_{,X} \right],$$

$$\rho_\phi + p_\phi = 2\kappa^4 L_{,K} \dot{\phi}^2 + 2 \left[ L_{,R} + 2(3H^2 + 2\dot{H})L_{,X} \right]^2 - 2H \left[ L_{,R} + 6H^2L_{,X} \right] - 4\dot{H} \left[ L_{,R} + 6(H^2 + \dot{H})L_{,X} \right],$$ \tag{A15}

and

$$R = 6(\dot{H} + 2H^2),$$

$$X = 12(\dot{H}^2 + 3H^2\dot{H} + 3H^4).$$ \tag{A16}$$

Here, a dot represents the derivative with respect to the proper time $t$. The equations of motion are

$$H^2 = \frac{\kappa^2}{3} (\rho_\phi + \rho_{other}),$$ \tag{A17}$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ (\rho_\phi + p_\phi) + (\rho_{other} + p_{other}) \right],$$ \tag{A18}$$

$$0 = \dot{\pi} + 3H\pi - L\phi,$$ \tag{A19}$$

and the equation of motion for other fields, where

$$\pi = 2\kappa^4 L_{,K} \dot{\phi},$$ \tag{A20}$$

and $\rho_{other}(t)$ and $p_{other}(t)$ are energy density and pressure of other fields:

$$T^\mu_{other\nu} = \begin{pmatrix} -\rho_{other} & 0 & 0 & 0 \\ 0 & p_{other} & 0 & 0 \\ 0 & 0 & p_{other} & 0 \\ 0 & 0 & 0 & p_{other} \end{pmatrix}.$$ \tag{A21}$$

The conservation of $T^\mu_{other\nu}$ is expressed as

$$\dot{\rho}_{other} + 3H (\rho_{other} + p_{other}) = 0.$$ \tag{A22}$$

Note that (A17), (A18) and (A19) are not independent. Actually, there is an identity

$$\dot{\rho}_\phi + 3H (\rho_\phi + p_\phi) + \dot{\phi}E_\phi = 0,$$ \tag{A23}$$

which can be checked explicitly and corresponds to the conservation equation of $T^{\phi\mu\nu}$.

**3. First order equations**

Hereafter in this appendix, we assume that $L$ is of the form

$$L = \frac{\kappa^{-4}}{2q} F(R, X) K^q - V(\phi) + G(R, X),$$ \tag{A24}$$

where $F$ and $G$ are functions of $R$ and $X$, and $q$ is a constant. In this case, $\rho_\phi$ and $p_\phi$ are expressed as
where

\[ \kappa^4 \rho_\phi = \frac{2q-1}{2q} F^{-\frac{1}{2q}} (\kappa^4 \pi^2)^{\frac{1}{2q-1}} + \kappa^4 (V - G) + \frac{3H^2 \Omega}{q} \left[ F_{,R} + 2H^2 (2\Omega + 3)F_{,X} \right] \left( \frac{\kappa^4 \pi^2}{F} \right)^{\frac{1}{2q-1}} + 6\kappa^4 H^2 \Omega \left[ G_{,R} + 2H^2 (2\Omega + 3)G_{,X} \right] - 3\kappa^4 H \phi, \]

\[ \kappa^4 (\rho_\phi + p_\phi) = F^{-\frac{1}{2q}} (\kappa^4 \pi^2)^{\frac{1}{2q-1}} + \frac{3H^2 \Omega}{q} \left[ F_{,R} + 2H^2 (2\Omega + 3)F_{,X} \right] \left( \frac{\kappa^4 \pi^2}{F} \right)^{\frac{1}{2q-1}} + 6\kappa^4 H^2 \Omega \left[ G_{,R} + 2H^2 (2\Omega + 3)G_{,X} \right] + \kappa^4 \dot{\phi}, \] (A25)

where

\[ \pi \equiv FK^{q-1} \dot{\phi}, \]

\[ \Omega \equiv \frac{\dot{H}}{H^2}, \]

\[ \varphi \equiv 2[\mathbf{L}_{,R} + 2(3H^2 + 2\dot{H})\mathbf{L}_{,X}] - 2H(\mathbf{L}_{,R} + 6H^2 \mathbf{L}_{,X}). \] (A26)

Hence, the equations of motion (A18), (A19) and (A22) are rewritten as the following set of first-order equations:

\[ \dot{\phi} = f \cdot (\kappa^4 \pi^2)^{\frac{1}{2q-1}}, \]

\[ \dot{\pi} = -3H \pi - V'(\phi), \]

\[ \dot{H} = H^2 \Omega, \]

\[ \epsilon \equiv 2\mathbf{H} \cdot \mathbf{H}, \]

\[ \kappa^2 \dot{\varphi} = -H^2 (\mathbf{G} + \Delta \mathbf{G}), \]

\[ \dot{\rho}_{\text{other}} = -3H (\rho_{\text{other}} + p_{\text{other}}), \] (A27)

where

\[ f \equiv F^{-\frac{1}{2q-1}}, \]

\[ \epsilon \equiv 12\kappa^2 H^2 \left[ 3G_{,RR} + 12H^2 (2\Omega + 3)G_{,RX} + 12H^4 (2\Omega + 3)^2 G_{,XX} + 2G_{,X} \right] \]

\[ - \frac{6(2q-1)}{q^2} H^2 \left[ 3f_{,RR} + 12H^2 (2\Omega + 3)f_{,RX} + 12H^4 (2\Omega + 3)^2 f_{,XX} + 2f_{,X} \right] (\kappa^4 \pi)^{\frac{1}{2q-1}}, \]

\[ \mathcal{F} \equiv \frac{3\kappa^2 \varphi}{H} - \frac{6(3H^2 + 2\dot{H})}{H} \left[ f_{,R} + 2H^2 (2\Omega + 3) f_{,X} \right] \kappa^2 \pi \cdot (\kappa^4 \pi^2)^{-\frac{1}{2q-1}} + \frac{6(2q-1)}{q^2} \left\{ -f_{,R} + 2(4\Omega^2 + 6\Omega - 3)H^2 f_{,X} \]

\[ + 12\Omega \left[ (\Omega + 2)H^2 f_{,RR} + 2(4\Omega^2 + 13\Omega + 12)H^4 f_{,RX} + 8(2\Omega + 3)(\Omega^2 + 3\Omega + 3)H^6 f_{,XX} \right] \right\} \left( \kappa^4 \pi^2 \right)^{\frac{1}{2q-1}} \]

\[ - 6\kappa \left\{ -G_{,R} + 2(4\Omega^2 + 6\Omega - 3)H^2 G_{,X} \]

\[ + 12 \Omega \left[ (\Omega + 2)H^2 G_{,RR} + 2(4\Omega^2 + 13\Omega + 12)H^4 G_{,RX} + 8(2\Omega + 3)(\Omega^2 + 3\Omega + 3)H^6 G_{,XX} \right] \right\}, \]

\[ \mathcal{G}_0 \equiv 2\mathbf{G} + \frac{1}{\kappa^2 H^2} \left[ (\kappa^4 (\rho_{\text{other}} + p_{\text{other}})) + f \cdot (\kappa^4 \pi^2)^{\frac{1}{2q-1}} \right], \]

\[ \Delta \mathcal{G} \equiv \frac{3(2q-1)}{q^2} \left[ f_{,R} + 2H^2 (2\Omega + 3) f_{,X} \right] (\kappa^4 \pi^2)^{\frac{1}{2q-1}} + 6\kappa^2 \Omega \left[ G_{,R} + 2H^2 (2\Omega + 3) G_{,X} \right], \]

\[ R \equiv 6H^2 (\Omega + 2), \]

\[ X \equiv 12H^4 (\Omega^2 + 3\Omega + 3). \] (A28)

The constraint equation (A17) is now written as

\[ \frac{3\kappa^2 \varphi}{H} = \mathcal{F}_0 + \Delta \mathcal{G} - \frac{\kappa^2 \mathbf{G}}{H^2}, \] (A29)

where

\[ \mathcal{F}_0 \equiv \frac{\kappa^2 (\rho_{\text{other}} + \rho_{\phi,0}) - 3H^2}{H^2}, \]

\[ \rho_{\phi,0} \equiv 2KL_{\text{kin},K} - L_{\text{kin}} + V = \frac{2q-1}{2q\kappa^2} f \cdot (\kappa^4 \pi^2)^{\frac{1}{2q-1}} + V. \] (A30)
When the system is analyzed numerically, we can use the constraint equation to set the initial condition for $\varphi$ and also to check numerical accuracy.

It is also possible to use the constraint equation to eliminate $\varphi$ from the set of first-order equations: $F$ in the forth equation in (A27) is rewritten as

$$\Delta F = \frac{6(3H^2 + V^\prime)}{H} \left[ f_{,R} + 2H^2(2\Omega + 3)f_{,X} \right] \kappa^2 \pi \cdot (\kappa^4 \pi^2)^{-\frac{q-1}{q}}$$

$$+ \frac{3(2q - 1)}{q \kappa^2} \left\{ -(\Omega + 1)f_{,R} + 2(2\Omega^2 + 3\Omega - 3)H^2f_{,X} \\ + 12\Omega \left[ (\Omega + 2)H^2f_{,RR} + 2(4\Omega^2 + 13\Omega + 12)H^4f_{,RX} + 8(2\Omega + 3)(\Omega^2 + 3\Omega + 3)H^6f_{,XX} \right] \right\} (\kappa^4 \pi^2)^{-\frac{q}{q-1}}$$

$$- 6\kappa^2 \left\{ -(\Omega + 1)G_{,R} + 2(2\Omega^2 + 3\Omega - 3)H^2G_{,X} \\ + 12\Omega \left[ (\Omega + 2)H^2G_{,RR} + 2(4\Omega^2 + 13\Omega + 12)H^4G_{,RX} + 8(2\Omega + 3)(\Omega^2 + 3\Omega + 3)H^6G_{,XX} \right] \right\} - \frac{\kappa^2 G}{H^2}. \quad (A32)$$

**APPENDIX B: EQUIV ALENT ACTION**

In this appendix let us consider an action of the form

$$I[g_{\mu\nu}, \phi] = \int d^Dx \sqrt{-g} F(R, K, \phi), \quad (B1)$$

where $R$ is the Ricci scalar of the metric $g_{\mu\nu}$, $\phi$ is a scalar field and $K = -\kappa^4 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. We shall show that this action is equivalent to the following action.

$$\tilde{I}[\tilde{g}_{\mu\nu}, \psi, \phi] = \int d^Dx \sqrt{-\tilde{g}} \tilde{F}(\tilde{R}, \tilde{K}, \tilde{F}, \psi, \phi),$$

where the new field $\psi$ and the function $R(\psi, K, \phi)$ are defined by

$$e^{\gamma \kappa \psi} = 2\kappa^2 F_R(R, K, \phi) \leftrightarrow R = R(\psi, K, \phi), \quad (B3)$$

the metric $\tilde{g}_{\mu\nu}$ is defined by

$$\tilde{g}_{\mu\nu} = e^{\beta \kappa \psi} g_{\mu\nu}, \quad (B4)$$

and $\tilde{R}$ is the Ricci scalar for the new metric $\tilde{g}_{\mu\nu}$. Here, $\kappa$ is an arbitrary positive constant, $F_R \equiv (\partial F/\partial R)_{K, \phi}$, $\tilde{K} \equiv -\kappa^4 \tilde{g}_{\mu\nu} \partial_\mu \phi \partial_\nu \phi$, $\tilde{g}^{\mu\nu} \equiv (\tilde{g})^{-1}_{\mu\nu}$, and constants $\alpha$, $\beta$ and $\gamma$ are

$$\alpha = \frac{D}{\sqrt{(D - 1)(D - 2)}},$$

$$\beta = \frac{2}{\sqrt{(D - 1)(D - 2)}},$$

$$\gamma = \frac{\sqrt{D - 2}}{\sqrt{D - 1}}. \quad (B5)$$

In (B3) the function $R(\psi, K, \phi)$ is defined by solving the left equation with respect to $R$ and, thus, we have implicitly assumed that $\partial^2 F/\partial R^2 \neq 0$. The action (B2) actually describes the Einstein gravity plus two scalar fields $\phi$ and $\psi$. Equations of motion derived from the two actions are the same and are
\[ G_{\mu\nu} - \kappa^2 \left( \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} \bar{g} \partial_\mu \psi \partial_\nu \psi \bar{g}_{\mu\nu} \right) - \kappa^2 \left( e^{-\alpha \kappa \psi} F \bar{g}_{\mu\nu} + 2\kappa^4 e^{-\gamma \kappa \psi} F K \partial_\mu \phi \partial_\nu \phi \right) + \frac{1}{2} e^{-\beta \kappa \psi} R \bar{g}_{\mu\nu} = 0, \]

\[ \kappa^4 \frac{\partial}{\sqrt{-\bar{g}}} \left[ \sqrt{-\bar{g}} \bar{g}_{\mu\nu} e^{-\gamma \kappa \psi} F K \partial_\mu \phi \partial_\nu \phi \right] + \frac{1}{2} e^{-\alpha \kappa \psi} F \phi = 0, \quad (B6) \]

where it is understood that \( R = R(\psi, e^{\beta \kappa \psi} \tilde{K}, \phi) \) and \( K = e^{\beta \kappa \psi} \tilde{K} \) are substituted into \( R, F, F_K \equiv \left( \partial F / \partial K \right)_{R,\phi} \) and \( F_\phi \equiv \left( \partial F / \partial \phi \right)_{R,K} \). The equation of motion for \( \psi \) can be derived from the second equation and the divergence of the first equation, and is

\[ \frac{1}{\sqrt{-\bar{g}}} \frac{\partial}{\partial \psi} \left[ \sqrt{-\bar{g}} \bar{g}_{\mu\nu} \partial_{\nu} \psi \right] - \alpha e^{-\alpha \kappa \psi} \bar{F} - \beta \kappa^5 e^{-\gamma \kappa \psi} F K \bar{g}_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\beta}{2\kappa} e^{-\beta \kappa \psi} R = 0. \quad (B7) \]

Again, it is understood that \( R = R(\psi, e^{\beta \kappa \psi} \tilde{K}, \phi) \) and \( K = e^{\beta \kappa \psi} \tilde{K} \) are substituted into \( R, F \) and \( F_K \).

Now let us show that the actions (B1) and (B2) are classically equivalent. It is an extension of ref. [38], in which the scalar field \( \phi \) has a usual canonical kinetic term.

First, it is easy to show that

\[ \frac{1}{\sqrt{-g}} \delta \left( \sqrt{-g} \text{F}(R, K, \phi) \right) = \frac{1}{2} F g^{\mu\nu} \delta g_{\mu\nu} + F_R \delta R + F_K \delta K + F_\phi \delta \phi = E^{\mu\nu} \delta g_{\mu\nu} + E \delta \phi + X^\mu_{\mu}, \quad (B8) \]

where

\[ E^{\mu\nu} = \frac{1}{2} F g^{\mu\nu} - F_R R^{\mu\nu} + F_R^{;\nu} F_R^{;\mu} + \kappa^4 F_K \phi^{;\mu} \phi^{;\nu}, \]

\[ E = 2\kappa^4 (F_K \partial_\mu \phi) \delta \mu + F_\phi, \]

\[ X^\mu = F_R (\delta g^{\mu\nu} - \delta g^{;\nu} \phi^{;\mu}) - F_R^{;\nu} \delta g^{\mu\nu} + F_R^{;\mu} \delta g^{\nu}_{\nu} - 2\kappa^2 F_K \phi^{;\mu} \delta \phi, \quad (B9) \]

and a semicolon represents the covariant derivative compatible with the metric \( g_{\mu\nu} \). Equations of motion derived from \( \delta I = 0 \) are \( E^{\mu\nu} = 0 \) and \( E = 0 \). These equations, supplemented by suitable initial conditions, govern dynamics of the system. Evidently, unless \( F_R \) is independent of \( R, E^{\mu\nu} \) includes up to the forth order derivatives of the metric \( g_{\mu\nu} \). In the following, we shall introduce an auxiliary field so that the resulting action includes only up to the second derivatives of fields. Moreover, it will be shown that after a conformal transformation, the gravitational part of the resulting action is of the form of the Einstein-Hilbert action.

Second, let us perform a conformal transformation

\[ \tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \quad (B10) \]

where \( \omega \) is a function to be determined below. The relation between the Ricci tensor \( R_{\mu\nu} \) of the original metric \( g_{\mu\nu} \) and the Ricci tensor \( \tilde{R}_{\mu\nu} \) of the conformally transformed metric \( \tilde{g}_{\mu\nu} \) can be found in textbooks in general relativity [39].

\[ \tilde{R}_{\mu\nu} = R_{\mu\nu} - (D - 2) \omega_{;\mu} \omega_{;\nu} + (D - 2) \omega^{;\mu} \omega_{;\nu} - 2 \omega_{;\mu ;\nu} + 2 \omega_{;\nu ;\mu} - \omega^{;\mu ;\nu} + \omega^{;\nu ;\mu} g_{\mu\nu}. \quad (B11) \]

Accordingly, the relation between Einstein tensors \( G_{\mu\nu} \) and \( \tilde{G}_{\mu\nu} \) for \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \), respectively, is

\[ G_{\mu\nu} = \tilde{G}_{\mu\nu} - (D - 2) \left[ \omega_{;\mu} \omega_{;\nu} + \frac{1}{2} (D - 3) \omega^{;\mu} \omega_{;\nu} + (D - 2) \omega_{;\mu ;\nu} - \omega^{;\mu ;\nu} + \omega^{;\nu ;\mu} \right]. \quad (B12) \]

Thus, we obtain the following expression of \( E_{\mu\nu} \).

\[ -E_{\mu\nu} = F_R \tilde{G}_{\mu\nu} - \frac{1}{2} (F - F_R R) \delta g_{\mu\nu} - (D - 2) F_R \left[ \omega_{;\mu} \omega_{;\nu} + \frac{1}{2} (D - 3) \omega^{;\mu} \omega_{;\nu} \right] - \kappa^4 F_K \phi^{;\mu} \phi^{;\nu}, \]

\[ + (D - 2) F_R (\omega_{;\mu ;\nu} - \omega^{;\nu ;\mu} \delta g_{\mu\nu}) - (F_R ;\mu \delta g_{\mu\nu}) \quad (B13) \]

The higher derivative terms in the last line can be canceled up to lower derivative terms if we chose

\[ \omega = \frac{1}{D - 2} \ln(2\kappa^2 F_R), \quad (B14) \]
Thus, the set of equations of motion derived from the action \( \tilde{\mathcal{I}} \) is the same as that from \( I \).

Finally, by using the relations

\[
R = e^{\beta \kappa \psi} \tilde{R} + 2 \gamma^{-1} \kappa \psi_{;\mu}^{;\mu} + \kappa^2 \psi^{;\mu} \psi_{;\mu},
\]

\[
\psi_{;\mu}^{;\mu} = e^{\beta \kappa \psi} \tilde{\nabla}_{\mu} \psi - \gamma \kappa e^{\beta \kappa \psi} \tilde{g}_{\mu \nu} \psi_{;\mu ;\nu},
\]

it is shown that \( \tilde{I}[\tilde{g}_{\mu \nu}, \psi, \phi] = I[\tilde{g}_{\mu \nu}, \phi] \).

**APPENDIX C: HARMONICS ON A PLANE**

In this Appendix we give definitions of scalar, vector and tensor harmonics on an \( n \)-dimensional plane. For definitions and properties of more general harmonics, see Appendix B of ref. [40] and Appendix A of ref. [41].
1. Scalar harmonics

The scalar harmonics are given by

\[ Y = \exp(-ik_jx^j), \]  

(C1)

by which any function \( f \) can be expanded as

\[ f = \int d^n k \; cY, \]  

(C2)

where \( c \) is a constant depending on \( k \). Hereafter, we write \( k_i \) as \( k \) in most cases, and sometimes we omit it.

2. Vector harmonics

In general, any vector field \( v_i \) can be decomposed as

\[ v_i = v_{(T)i} + \partial_i f, \]  

(C3)

where \( f \) is a function and \( v_{(T)i} \) is a transverse vector field:

\[ \delta^{ij} \partial_i v_{(T)j} = 0. \]  

(C4)

Thus, the vector field \( v_i \) can be expanded by using the scalar harmonics \( Y \) and transverse vector harmonics \( V_{(T)i} \) as

\[ v_i = \int d^n k \left[ c_{(T)} V_{(T)i} + c_{(L)} \partial_i Y \right]. \]  

(C5)

Here, \( c_{(T)} \) and \( c_{(L)} \) are constants depending on \( k \), and the transverse vector harmonics \( V_{(T)i} \) is given by

\[ V_{(T)i} = u_i \exp(-ik_jx^j), \]  

(C6)

where the constant vector \( u_i \) satisfies the condition

\[ \delta^{ij} k_i u_j = 0. \]  

(C7)

Because of the expansion (C5), it is convenient to define longitudinal vector harmonics \( V_{(L)i} \) by

\[ V_{(L)i} \equiv \partial_i Y = -ik_i Y. \]  

(C8)

3. Tensor harmonics

In general, a symmetric second-rank tensor field \( t_{ij} \) can be decomposed as

\[ t_{ij} = t_{(T)ij} + \partial_i v_j + \partial_j v_i + f \delta_{ij}, \]  

(C9)

where \( f \) is a function, \( v_i \) is a vector field and \( t_{(T)ij} \) is a transverse traceless symmetric tensor field:

\[ \delta^{ij} t_{(T)ij} = 0, \]

\[ \delta^{ij} \partial_i t_{(T)j} = 0. \]  

(C10)

Thus, the tensor field \( t_{ij} \) can be expanded by using the scalar harmonics \( Y \), the vector harmonics \( V_{(T)i} \) and \( V_{(L)i} \), and transverse traceless tensor harmonics \( T_{(T)ij} \) as

\[ t_{ij} = \int d^n k \left[ c_{(T)} T_{(T)ij} + c_{(LT)} (\partial_i V_{(T)j} + \partial_j V_{(T)i}) 
+ c_{(L)} (\partial_i V_{(L)j} + \partial_j V_{(L)i}) + \tilde{c}_i Y \delta_{ij} \right]. \]  

(C11)
Here, $c(T)$, $c(LT)$, $c(LL)$, and $\tilde{c}(Y)$ are constants depending on $k$, and the transverse traceless tensor harmonics $T_{(T)ij}$ is given by

$$T_{(T)ij} = s_{ij} \exp(-ik_j x^i), \quad \text{(C12)}$$

where the constant symmetric second-rank tensor $s_{ij}$ satisfies the condition

$$\delta^{ii} k_i s_{i'j} = 0, \quad \delta^{ij} s_{ij} = 0. \quad \text{(C13)}$$

Because of the expansion (C11), it is convenient to define tensor harmonics $T_{(LT)ij}$, $T_{(LL)ij}$, and $T_{(Y)ij}$ by

$$T_{(LT)ij} \equiv \partial_i V_{(T)j} + \partial_j V_{(T)i},$$
$$T_{(LT)ij} = -i(u_k k_j + u_j k_k) Y,$$
$$T_{(LL)ij} \equiv \partial_i V_{(L)j} + \partial_j V_{(L)i} - \frac{2}{n} \delta_{ij} \delta^{i'j'} \partial_{i'} V_{(L)j'}$$
$$= \left( -2k_i k_j + \frac{2}{n} \delta^{i'j'} k_i k_j \delta_{ij} \right) Y,$$
$$T_{(Y)ij} \equiv \delta_{ij} Y. \quad \text{(C14)}$$

APPENDIX D: DETAILED ANALYSIS OF LINEARIZED GRAVITY

In this appendix we show that linearized Einstein gravity in Minkowski background is recovered at distances longer than the length scale $l_* = \sqrt{\alpha \kappa}$ and at energies lower than $l_*^{-1}$. For simplicity we consider the $q = 1$ case, but extension to a general $q (> 1/2)$ should be straightforward.

For this purpose we investigate in detail perturbations around the flat FRW background in the longitudinal gauge and take the $\kappa H \rightarrow +0$ limit in the end of the calculation. This strategy makes it possible to analyze linear perturbations around Minkowski background, on which the action of the scalar field perturbation is not apparently well-defined. For simplicity, we assume that the FRW background is driven by the scalar field $\phi$. Of course, we include a general matter stress-energy tensor into perturbations and carefully investigate how they couple to gravity.

A critical point for this analysis is that constraint equations ($\{0i\}$-components and the traceless part of $\{ij\}$-components of Einstein equation) prevent $\phi$ from fluctuating freely, and it turns out that the $\phi$ perturbation is of sufficiently high order in the $\kappa H$ expansion that the perturbative analysis is under control.

As stated in Sec. VII the result of the analysis is simple. The linearized gravity in our model is similar to that in the theory $R/2\kappa^2 + \alpha R^2$ in the sense that the only difference is the existence of an extra scalar-type massless mode in our model. The extra massless mode is decoupled from the matter stress-energy tensor at the linearized level. Hence, as far as we are concerned with classical, linear perturbations generated by matter sources, these two theories give exactly the same prediction. Therefore, in our model the linearized Einstein gravity in Minkowski background is recovered at distances longer than the length scale $l_* = \sqrt{\alpha \kappa}$ and at energies lower than $l_*^{-1}$.

Let us begin by defining the metric perturbation $\delta g_{\mu\nu}$ by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \quad \text{(D1)}$$

where

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \quad \text{(D2)}$$

$(i = 1, 2, 3)$. In order to investigate linearized gravity in Minkowski background, in the end of calculations, we shall take the limit

$$\kappa H \rightarrow +0, \quad \text{(D3)}$$

keeping $\partial^n H/H^{n+1}$ $(n = 1, 2, 3)$ finite, where $H = \partial_t a/a$. In the following, we shall denote $\partial_t$ by an overdot.
1. Gauge choice and three types of perturbations

In order to take the full advantage of the symmetry of the background spacetime, we expand the metric perturbation by harmonics on the 3-plane:

\[
\delta g_{\mu\nu}dx^\mu dx^\nu = h_{00}Y dt^2 + 2(h_{(T)0}V_{(T)i} + h_{(L)0}V_{(L)i})dtdx^i + (h_{(T)}T_{(T)ij} + h_{(LT)}T_{(LT)ij} + h_{(LL)}T_{(LL)ij} + h_{(Y)}T_{(Y)ij})dx^i dx^j,
\]

(D4)

where \( Y, V_{(T,L)} \) and \( T_{(T,L,LL,Y)} \) are scalar, vector and tensor harmonics, respectively, and the coefficients \( h_{00}, h_{(T)0}, h_{(T,L,LL,Y)} \) depend only on the time \( t \). In this expression and hereafter, we omit the 3-momentum \( k_i \) and the integration over 3-momenta. See Appendix C for definitions of the harmonics.

We also expand the perturbation of the scalar field \( \phi \) by harmonics as

\[
\phi = \phi^{(0)}(t) + \delta \phi Y,
\]

(D5)

where \( \phi^{(0)}(t) \) is the background depending only on the time \( t \) and the coefficient \( \delta \phi \) of the perturbation also depends only on the time \( t \). In the following we shall derive the equation of motion linearized with respect to \( h \)'s and \( \delta \phi \).

Now, it is easily shown from the orthogonality among different kinds of harmonics that each of the following sets of variables form a closed set of equations: (S) \( h_{00}, h_{(L)0}, h_{(LL)}, h_{Y} \) and \( \phi \); (V) \( h_{(T)0}, h_{(LT)} \); and (T) \( h_{T} \). Perturbations in different sets are completely decoupled from each other at the linearized level. The perturbations in the category (S) form a spin-0 representation and are called \textit{scalar perturbations}. The perturbations in the category (V) form a spin-1 representation and are called \textit{vector perturbations}. Finally, the perturbations in the category (T) form a spin-2 representation and are called \textit{tensor perturbations}.

The metric perturbation \( \delta g_{\mu\nu} \) includes not only physical degrees of freedom but also gauge freedom. An infinitesimal gauge transformation is given by

\[
\begin{align*}
\delta g_{\mu\nu} &\rightarrow \delta g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}, \\
\delta \phi Y &\rightarrow \delta \phi Y - \xi^\mu \partial_\mu \phi^{(0)},
\end{align*}
\]

(D6)

where \( \xi_\mu \) is an arbitrary vector field and a semicolon denotes the covariant derivative compatible with the background metric \( g^{(0)}_{\mu\nu} \). Hence, by expanding the vector \( \xi_\mu \) in terms of harmonics as

\[
\tilde{\xi}_\mu dx^\mu = \xi_0 Y dt + (\xi_{(T)}V_{(T)i} + \xi_{(L)}V_{(L)i})dx^i,
\]

(D7)

we obtain the following infinitesimal gauge transformation for the expansion coefficients in (D4) and (D5) with \( k_i \neq 0 \).

\[
\begin{align*}
h_{00} &\rightarrow h_{00} - 2\xi_0, \\
h_{(T)0} &\rightarrow h_{(T)0} - a^2(\xi_{(T)})', \\
h_{(L)0} &\rightarrow h_{(L)0} - \xi_0 - a^2(\xi_{(L)})', \\
h_{(T)} &\rightarrow h_{(T)}, \\
h_{(LT)} &\rightarrow h_{(LT)} - \xi_{(T)}, \\
h_{(LL)} &\rightarrow h_{(LL)} - \xi_{(L)}, \\
h_{(Y)} &\rightarrow h_{(Y)} + 2a^2 H\xi_0 + \frac{2}{3}k^2\xi_{(L)}, \\
\delta \phi &\rightarrow \delta \phi + \xi_0 \phi^{(0)}.
\end{align*}
\]

(D8)

From the gauge transformation (D8) it is evident that for modes with \( k_i \neq 0 \), we can choose the gauge so that

\[
h_{(L)0} = h_{(LT)} = h_{(LL)} = 0.
\]

(D9)

With this condition, the gauge degrees of freedom represented by \( \xi_0, \xi_{(T)} \) and \( \xi_{(L)} \) are completely fixed for modes with \( k_i \neq 0 \). On the other hand, we do not need to consider modes with \( k_i = 0 \) since they can be absorbed into the background without loss of generality.

We also take into account the stress-energy tensor \( T_{\mu\nu} \) of other fields, consider itself as a perturbation around \( T_{\mu\nu} = 0 \) and expand it by harmonics as
\[
T_{\mu\nu}dx^\mu dx^\nu = \tau_{00}Ydt^2 + 2(\tau(T)0\nu(V(T))_i + \tau(L)_0V(L)_i)dx^i \\
+ (\tau(T)T(T)_{ij} + \tau(L)T(L)_{ij} + \tau(LL)T(LL)_{ij} + \tau(Y)T(Y)_{ij})dx^i dx^j. \tag{D10}
\]

Since gauge freedom is already completely fixed, all the coefficients \(\tau\)'s represent gauge-invariant degrees of freedom.

Hence, in the gauge (D9), all gauge-inequivalent degrees of freedom are represented by the following variables: (S) \(h_{00}\), \(h_{(Y)}\), \(\delta\phi\), \(\tau_{00}\), \(\tau_{(L)0}\), \(\tau_{(L)L}\) and \(\tau_{(Y)}\) for scalar perturbations; (V) \(h_{(T)0}\), \(\tau_{(T)0}\) and \(\tau_{(LT)}\) for vector perturbations; and (T) \(h_{(T)}\) and \(\tau_{(T)}\) for tensor perturbations. As already mentioned, scalar-, vector- and tensor-type perturbations are completely decoupled from each other at the linearized level. Hence, in the following, we shall analyze each type separately.

For vector- and tensor-type perturbations, as shown in Sec. VII, there is no difference between weak gravity in our model and that in Einstein theory. Hence, in the following we consider scalar-type perturbations only.

2. Scalar perturbations

For scalar-type metric perturbations, we have two gauge-invariant degrees \(h_{00}\) and \(h_{(Y)}\). By introducing \(\Phi(t)\) and \(\Psi(t)\) by \(\Phi = -h_{00}/2\) and \(\Psi = -a^{-2}h_{(Y)}/2\), the perturbed metric is written as

\[
g_{\mu\nu}dx^\mu dx^\nu = -(1 + 2\Phi Y)dt^2 + (1 - 2\Psi Y)a^2\delta_{ij}dx^i dx^j, \tag{D11}
\]

For this metric, the perturbed Ricci tensor is

\[
R_{\mu\nu}dx^\mu dx^\nu = R^{(0)}_{\mu\nu}dx^\mu dx^\nu + \delta R_{00}Y dt^2 + 2\delta R_{(L)0}V(L)_i dt dx^i + (\delta R_{(LL)}T(LL)_{ij} + \delta R_{(Y)}T(Y)_{ij})dx^i dx^j. \tag{D12}
\]

up to the linear order in \(\Phi\) and \(\Psi\), where \(R^{(0)}_{\mu\nu}\) is the background Ricci tensor

\[
R^{(0)}_{00} = -3(\dot{H} + H^2),
R^{(0)}_{ij} = (\dot{H} + 3H^2)a^2\delta_{ij}, \tag{D13}
\]

and

\[
\delta R_{00} = 3\dot{\Psi} + 6H\dot{\Psi} + 3H\bar{\Phi} - \frac{k^2}{a^2}\Phi,
\delta R_{(L)0} = 2\dot{\Psi} + 2H\Phi,
\delta R_{(LL)} = \frac{1}{2} (\Psi - \Phi),
\delta R_{(Y)} = -a^2 \left[ \dot{\Psi} + 6H\dot{\Psi} + H\bar{\Phi} + 2(\dot{H} + 3H^2)(\Psi + \Phi) + \frac{4k^2}{3a^2}\Psi - \frac{1}{3} \frac{k^2}{a^2} \Phi \right]. \tag{D14}
\]

Hence, the perturbed Ricci scalar and the perturbed Einstein tensor are

\[
R = R^{(0)} + \delta RY,
G_{\mu\nu}dx^\mu dx^\nu = G^{(0)}_{\mu\nu}dx^\mu dx^\nu + \delta G_{00}Y dt^2 + 2\delta G_{(L)0}V(L)_i dt dx^i + (\delta G_{(LL)}T(LL)_{ij} + \delta G_{(Y)}T(Y)_{ij})dx^i dx^j \tag{D15}
\]

up to the linear order, where \(R^{(0)}\) and \(G^{(0)}_{\mu\nu}\) are the background Ricci scalar and the background Einstein tensor

\[
R^{(0)} = 6(\dot{H} + 2H^2),
G^{(0)}_{00} = 3H^2,
G^{(0)}_{ij} = -(2\dot{H} + 3H^2)a^2\delta_{ij}, \tag{D16}
\]

and

\[
\delta R = -2 \left[ 3\dot{\Psi} + 12H\dot{\Psi} + 3H\bar{\Phi} + 6(\dot{H} + 2H^2)\Phi + 2\frac{k^2}{a^2}\Psi - \frac{k^2}{a^2}\Phi \right],
\delta G_{00} = -2 \left[ 3H\dot{\Psi} + \frac{k^2}{a^2}\Psi \right],
\delta G_{(L)0} = \delta R_{(L)0},
\delta G_{(LL)} = \delta R_{(LL)},
\delta G_{(Y)} = 2a^2 \left[ \dot{\Psi} + 3H\dot{\Psi} + H\bar{\Phi} + (2\dot{H} + 3H^2)(\Psi + \Phi) + \frac{1}{3} \frac{k^2}{a^2}(\Psi - \Phi) \right]. \tag{D17}
\]
The Bianchi identity

\[ \nabla^\mu G_{\mu \nu} = 0, \]  

where \( \nabla \) is the covariant derivative compatible with the perturbed metric \( g_{\mu \nu} \), is reduced to

\[
\delta G_{00} + 3H \delta G_{00} + 3H \frac{\delta G(Y)}{a^2} + k^2 \frac{\delta G(L)_{0}}{a^2} \delta G_{(L)0} = \left( 3G_{00}^{(0)} + a^{-2} \delta^i j G_{ij}^{(0)} \right) \dot{\Psi} + 2 \dot{G}_{00}^{(0)} \dot{\Phi} - 2a^{-2} \delta^i j G_{ij}^{(0)} H(\Psi + \Phi),
\]

\[
\delta G_{(L)0} + 3H \delta G_{(L)0} + 4 \frac{k^2}{a^2} G_{(LL)} - \frac{\delta G(Y)}{a^2} \delta G_{(Y)} = \frac{1}{3} \left( 3G_{00}^{(0)} + a^{-2} \delta^i j G_{ij}^{(0)} \right) \Phi + \frac{2}{3} a^{-2} \delta^i j G_{ij}^{(0)} \Psi.
\]  

These two equations are, of course, satisfied by the components shown in (D17).

As for fields other than \( \phi \), we have the stress energy tensor

\[
T^\mu_{\nu} \  \  \  dx^\mu dx^\nu = \tau_{00} Y dt^2 + 2 \tau_{(L)0} V_{(L)i} dt dx^i + (\tau_{(LL)} T_{(LL)ij} + \tau_{(Y)} T_{(Y)ij}) dx^i dx^j.
\]  

The conservation equation

\[ \nabla^\mu T^\mu_{\nu} = 0, \]  

where \( \nabla \) is the covariant derivative compatible with the perturbed metric \( g_{\mu \nu} \), is reduced to

\[
\dot{\tau}_{00} + 3H \tau_{00} + 3H \frac{\tau(Y)}{a^2} + \frac{k^2}{a^2} \tau_{(L)0} = 0,
\]

\[
\dot{\tau}_{(L)0} + 3H \tau_{(L)0} + \frac{4}{3} \frac{k^2}{a^2} \tau_{(LL)} - \frac{\tau(Y)}{a^2} = 0.
\]  

On the other hand, the stress energy tensor of \( \phi \) (with \( q = 1 \)) is given by

\[
T^\mu_{\nu} \equiv \frac{1}{f} \partial^\mu \phi \partial^\nu \phi + L \delta^\mu \nu - 2L R^\mu \nu + 2 L^\mu_{\rho \sigma} \delta^\nu \sigma - 2L^i_{\mu \rho} \delta^\nu \sigma,
\]  

where

\[
L \equiv \alpha R^2 + L_{kin} - V,
\]

\[
L_R \equiv 2\alpha R - \frac{\kappa^{-4} f^2}{2f^2},
\]  

and a prime applied to \( f \) denotes derivative with respect to \( R \). We now expand \( T_{\phi \mu \nu} \) up to the linear order:

\[
T_{\phi \mu \nu} \  \  \  dx^\mu dx^\nu = T^{(0)}_{\phi \mu \nu} \  \  \  dx^\mu dx^\nu + \tau_{000} Y dt^2 + 2 \tau_{(L)0} V_{(L)i} dt dx^i + (\tau_{(LL)} T_{(LL)ij} + \tau_{(Y)} T_{(Y)ij}) dx^i dx^j.
\]  

The background \( T^{(0)}_{\phi \mu \nu} \) is given by

\[
T^{(0)}_{\phi \mu \nu} = \begin{pmatrix}
-\rho_{\phi} & 0 & 0 & 0 \\
0 & \rho_{\phi} & 0 & 0 \\
0 & 0 & \rho_{\phi} & 0 \\
0 & 0 & 0 & \rho_{\phi}
\end{pmatrix},
\]  

where

\[
\rho_{\phi} = \pi \dot{\phi}^{(0)} - L - 6H(L_R) + 6(H^2 + \dot{H})L_R,
\]

\[
\rho_{\phi} + \rho_{\phi} = \pi \ddot{\phi}^{(0)} + 2(L_R) - 2H(L_R) + 4H L_R,
\]  

\[
\pi \equiv \frac{\dot{\phi}}{f},
\]

\[
L = \alpha R^2 + \frac{\dot{\phi}^{(0)2}}{2f} - V,
\]

\[
L_R = 2\alpha R - \frac{f \dot{\phi}^{(0)2}}{2f^2},
\]

\[
R = 6(\dot{H} + 2H^2).
\]
and a dot represents the derivative with respect to the proper time $t$. The linear part is

$$
\tau_{(00)} = F^{(0)}(\phi^{(0)})(\dot{\phi} + \frac{1}{2} F_{,R}^{(0)} \phi^{(0)} 2 \delta R + 2(V^{(0)} - \alpha R^{(0)} 2)\Phi + V^{(0)}' \delta \phi - 2 \alpha \kappa^2 R^{(0)} \delta R

- 3(\dot{H} + H^2) \delta W + W^{(0)} \delta R_{(00)} + 3 H \delta \dot{W} + \frac{k^2}{a^2} \delta W - 3 \ddot{W}^{(0)} \phi ;
$$

$$
\tau_{(L)0} = F^{(0)}(\phi^{(0)})(\dot{\phi} + W^{(0)} \delta R_{(L)0} - \delta \dot{W} + H \delta W + \dddot{W}^{(0)} \Phi ,
$$

$$
\tau_{(LL)} = - \frac{1}{2} \delta W + W^{(0)} \delta R_{(LL)};
$$

$$
\tau_{(Y)} = a^2 \left[ F^{(0)}(\phi^{(0)})(\dot{\phi} + \frac{1}{2} F_{,R}^{(0)} \phi^{(0)} 2 \delta R - F^{(0)}(\phi^{(0)2}(\Psi + \Phi) + 2(V^{(0)} - \alpha R^{(0)} 2)\Psi - V^{(0)}' \delta \phi

+ 2 \alpha R^{(0)} \delta R + (\ddot{H} + H^2) \delta W + W^{(0)} a^{-2} \delta R_{(Y)} - \delta \dot{W} - 2 \dot{H} \delta W - \frac{2 k^2}{3 a^2} \delta W

+ 2 \dddot{W}^{(0)}(\Psi + \Phi) + 4 H \dddot{W}^{(0)}(\Psi + \Phi) + \dddot{W}^{(0)}(2 \Psi + \Phi) \right] ,
$$

(D29)

where

$$
V^{(0)} = V(\phi^{(0)}),
$$

$$
V^{(0)'} = V'(\phi^{(0)}),
$$

$$
F^{(0)} = \frac{1}{f(R^{(0)})},
$$

$$
F_{,R}^{(0)} = - \left[ f'(R^{(0)}) \right]^2 f(R^{(0)}),
$$

$$
F_{,RR}^{(0)} = \frac{2 f'(R^{(0)})^2 - f'(R^{(0)}) f''(R^{(0)})}{f(R^{(0)})^3},
$$

$$
W^{(0)} = - F_{,R}^{(0)} \phi^{(0)2} - 4 \alpha R^{(0)} ,
$$

$$
\delta W = - F_{,RR}^{(0)} \phi^{(0)2} \delta R + 2 F_{,R}^{(0)} \phi^{(0)2} \phi - 2 F_{,R}^{(0)} \phi^{(0)2} \delta \phi - 4 \alpha \delta R .
$$

(D30)

Note that there is an identity

$$
\tau_{(00)} + 3 H \tau_{(L)0} + 4 \frac{k^2}{3 a^2} \tau_{(LL)} - \frac{\tau_{(Y)}}{a^2} = \frac{1}{3} \left[ 3 T_{(00)}^{(0)} + a^{-2} \delta^{ij} T_{(0)}^{(0)} \right] \phi + \frac{2}{3 a^2} \delta^{ij} T_{(0)}^{(0)} \Psi - E_{(0)}^{(0)} \delta \phi ,
$$

(D31)

where $E_{(0)}^{(0)}$ is given by

$$
E_{(0)}^{(0)} = - \left[ F^{(0)}(\phi^{(0)})(\dot{\phi} + \frac{1}{2} F_{,R}^{(0)} \phi^{(0)} 2 \delta R) - 3 H F^{(0)}(\phi^{(0)} - V'(\phi) ,
$$

(D32)

and gives the field equation $E_{(0)}^{(0)} = 0$ for the background $\phi$. The identity (D31) is equivalent to the $(L)$-component of the linearized conservation equation of $T_{\phi\mu\nu}$. Hence, combining this identity with the Bianchi identity (D19) and the conservation equation (D22) for other fields, it is concluded that the $(Y)$-component $\delta G_{(Y)} = \kappa^2 (\tau_{(Y)} + \tau_{(L)}^{(Y)})$ of the linearized equation of motion follows from other components. On the other hand, the 00-, $(L)0$- and $(LL)$-components give all independent equations.

Actually, the $(L)0$- and $(LL)$-components of the equation of motion can be, in a sense, considered as constraint equations since we have imposed the gauge condition $h_{(L)0} = h_{(LL)} = 0$. The $(LL)$-component $\delta G_{(LL)} = \kappa^2 (\tau_{(LL)} + \tau_{(LL)})$ is

$$
\kappa^2 \delta W = - \left( 1 - \kappa^2 W^{(0)} \right) \Psi + 2 \kappa^2 \tau_{(LL)} .
$$

(D33)

The $(L)0$-component $\delta G_{(L)0} = \kappa^2 (\tau_{(L)0} + \tau_{(L)0})$ is, by using the $(LL)$-component, reduced to

$$
\kappa^2 F^{(0)}(\phi^{(0)})(\dot{\phi} = \left( 1 - \kappa^2 W^{(0)} \right) \left( \dot{\Psi} + H \dot{\Psi} + \kappa^2 \frac{2}{2} W^{(0)}(-\dot{\Psi} + 3 \dot{\Psi}) + \kappa^2 (2 \tau_{(LL)} - 2 H \tau_{(LL)} - \tau_{(L)0}) .
$$

(D34)
To be precise, this equation is \( \delta G_{(L)0} - 2a(\delta G_{(LL)})/a \) is \( \kappa^2(\tau_{(L)L} + \tau_{(L)0}) - 2\kappa^2a[(\tau_{(LL)} + \tau_{(L)})]/a \). Here, we have defined

\[
\Psi_\pm \equiv \Psi \pm \Phi.
\]  

(D35)

The remaining 00-component \( \delta G_{00} = \kappa^2(\tau_{00} + \tau_{0}) \) is much more complicated than the above two components since it is a dynamical equation while the above two are constraint equations. By using the (LL)- and (L)0-components, the 00-component is reduced to

\[
\left( C_2 + \kappa^2 \Phi_+ + C_1 + \kappa \Phi_+ + C_0 + \Psi_+ \right) + \left( C_2 - \kappa^2 \Phi_- + C_1 - \kappa \Phi_- + C_0 - \Psi_- \right) + S_{00} = 0,
\]

(D36)

where

\[
C_{2+} = -1 + 2\kappa^2 F_{,R}^{(0)} \phi^{(0)2} - 4\alpha \kappa^2 R^{(0)},
\]

\[
C_{1+} = 3\kappa^2 \dot{W}^{(0)} + \left( \frac{\kappa F^{(0)}}{\partial \kappa F^{(0)}} - \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} \right) \left( 1 - \kappa^2 W^{(0)} \right) - \kappa H \left( 4 + \frac{7}{2} \kappa^2 W^{(0)} - \frac{15}{2} \kappa^2 F_{,R}^{(0)} \phi^{(0)2} + 30 \alpha \kappa^2 R^{(0)} \right),
\]

\[
C_{0+} = -\kappa^4 \left( V^{(0)} - \alpha R^{(0)2} \right) - \kappa^2 \left( H(1 - \kappa^2 W^{(0)}) \right) - \frac{\kappa^2 \kappa^2}{a^2} \left( 1 - \frac{1}{2} \kappa^2 W^{(0)} - \frac{1}{2} \kappa^2 F_{,R}^{(0)} \phi^{(0)2} + 2 \alpha \kappa^2 R^{(0)} \right) + \frac{1}{2} \kappa^4 \dot{W}^{(0)}
\]

\[
+ \left( \frac{\kappa F^{(0)}}{F^{(0)}} + \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} - \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} \right) \left( \kappa H(1 - \kappa^2 W^{(0)}) - \frac{1}{2} \kappa^3 \dot{W}^{(0)} \right) + 3 \kappa^2 (\dot{H} + 2H^2) \left( \kappa^2 F_{,R}^{(0)} \phi^{(0)2} - 4 \alpha \kappa^2 R^{(0)} \right),
\]

\[
C_{2-} = 3\kappa^2 F_{,R}^{(0)} \phi^{(0)2},
\]

\[
C_{1-} = 9\kappa^3 H F_{,R}^{(0)} \phi^{(0)2},
\]

\[
C_{0-} = \kappa^4 \left( V^{(0)} - \alpha R^{(0)2} \right) - 3\kappa^2 \left( H + H^2 \right) \left( 1 - \kappa^2 W^{(0)} \right) + 3\kappa^2 F_{,R}^{(0)} \phi^{(0)2} \left( \kappa^2 \kappa^2 \right) \left( \kappa^2 \kappa^2 \right) - \frac{3}{2} \kappa^4 \dot{W}^{(0)}
\]

\[
+ \frac{3}{2} \left( -2 \kappa H + \frac{\kappa F^{(0)}}{F^{(0)}} + \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} - \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} \right) \left( \kappa^3 \dot{W}^{(0)} - 3 \kappa^2 (\dot{H} + 2H^2) \left( \kappa^2 F_{,R}^{(0)} \phi^{(0)2} - 4 \alpha \kappa^2 R^{(0)} \right) \right),
\]

\[
S_{00} = -\kappa^4 \tau_{00} + \kappa^4 \tau_{(L)L} - 2\kappa^4 \left[ \tau_{(LL)} + 2\dot{H} \tau_{(LL)} - (4 \dot{H} + 3H^2) \tau_{(LL)} + \frac{k^2}{a^2} \tau_{(LL)} \right]
\]

\[
- \left( \frac{\kappa F^{(0)}}{F^{(0)}} + \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} - \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} \right) \left( \kappa^3 \tau_{(L)0} - 2 \kappa^3 \tau_{(L)L} + 2 \kappa^3 H \tau_{(L)L} \right).
\]

(D37)

The (LL)-component (D33) can also be rewritten as a differential equation for \( \Psi_\pm \) by using the (L)0-component (D34) as follows.

\[
\left( D_{2+} + \kappa^2 \Phi_+ + D_{1+} + D_{0+} \right) + \left( D_{2-} - \kappa^2 \Phi_- + D_{1-} - D_{0-} \right) + S_{(LL)} = 0,
\]

(D38)

where

\[
D_{2+} = -(1 - \kappa^2 W^{(0)}) + \frac{3 \kappa^2 F^{(0)}}{2 F_{,R}^{(0)}} \left( F_{,R}^{(0)} \phi^{(0)2} + 4 \alpha \right),
\]

\[
D_{1+} = \left( \frac{\kappa F^{(0)}}{F^{(0)}} + \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} - \kappa H \right) \left( 1 - \kappa^2 W^{(0)} \right) + \frac{3}{2} \kappa^3 \dot{W}^{(0)} + \frac{15 \kappa^2 F^{(0)}}{2 F_{,R}^{(0)}} \left( F_{,R}^{(0)} \phi^{(0)2} + 4 \alpha \right) \kappa H,
\]

\[
D_{0+} = \frac{1}{2} \kappa^4 F_{,R}^{(0)} \phi^{(0)2} + \left( \frac{\kappa F^{(0)}}{F^{(0)}} + \frac{\kappa V^{(0)}}{\partial \kappa V^{(0)}} \right) \left[ \kappa H \left( 1 - \kappa^2 W^{(0)} \right) - \frac{1}{2} \kappa^3 \dot{W}^{(0)} \right] - \kappa \left[ \kappa H \left( 1 - \kappa^2 W^{(0)} \right) \right] + \frac{1}{2} \kappa^4 \dot{W}^{(0)} + \frac{1}{2} \kappa^2 F_{,R}^{(0)} \left( F_{,R}^{(0)} \phi^{(0)2} + 4 \alpha \right) \left[ \frac{\kappa^2 \kappa^2}{a^2} + 6 \kappa^2 (\dot{H} + 2H^2) \right],
\]

\[
D_{2-} = \frac{3 \kappa^2 F^{(0)}}{2 F_{,R}^{(0)}} \left( F_{,R}^{(0)} \phi^{(0)2} + 4 \alpha \right),
\]
For this purpose, as we shall see, we actually need to keep terms up to the second order in $\kappa H$. This means that the duration $t_0$ of the experiment is much shorter than the cosmological time scale $H^{-1}$. Of course, this assumption is justified since we shall take the $\kappa H \rightarrow +0$ limit in the end of calculation.

Now let us solve the equations (D36) and (D38) with respect to $\Psi_\pm$ up to the zeroth order in the $\kappa H$ expansion. For this purpose, as we shall see, we actually need to keep terms up to the second order in $\kappa H$ in intermediate steps.

First, let us introduce a small dimensionless parameter $\epsilon$, assume that

$$\kappa H = O(\epsilon),$$
$$\frac{\kappa^2 k^2}{a^2} = O(\epsilon^0),$$

and estimate orders of other quantities as

$$F^{(0)} = O(\epsilon^{-4m}),$$
$$\kappa^{-2} F^{(0)}_{,R} = O(\epsilon^{-4m-2}),$$
$$\kappa^{-4} F^{(0)}_{,RR} = O(\epsilon^{-4m-4}),$$
$$\kappa^2 R^{(0)} = O(\epsilon^2),$$
$$W^{(0)} = O(\epsilon^2),$$
$$\kappa^3 \mathcal{V}^{(0)} = c = O(\epsilon^0),$$
$$\kappa^2 F^{(0)} \phi^{(0)} = O(\epsilon^{-1}).$$

Next, $\kappa^2 \dot{H}$ is shown to be of order $\epsilon^4$ by using the equation of motion.

$$\kappa^2 \dot{H} = \frac{-\kappa^4 F^{(0)} \phi^{(0)2} + \kappa^4 \dot{W}^{(0)} - \kappa^4 H \ddot{W}^{(0)}}{2(1 - \kappa^2 \dot{W}^{(0)})} = O(\epsilon^4).$$

This means that $\kappa H$ can be considered as a constant up to the order $O(\epsilon^2)$, which is sufficient for our purpose. As a consequence, we can regard $F^{(0)}$, $F^{(0)}_{,R}$, $F^{(0)}_{,RR}$ and $R^{(0)}$ as constants up to the order $O(\epsilon^2)$. The estimate (D42) leads to the expansion of $1/a^2$ as

$$\frac{1}{a^2} = \frac{1}{a_0^2} \left[ 1 - 2H(t - t_0) + 2H^2(t - t_0)^2 + O(\epsilon^2) \right],$$

around the time $t = t_0$ when an experiment is performed, where $a_0 = a(t_0)$ is a constant. Here, we have assumed that the duration $t - t_0$ of the experiment is much shorter than the cosmological time scale $H^{-1}$. Of course, this assumption is justified since we shall take the $\kappa H \rightarrow +0$ limit in the end of calculation.

Thirdly, from (11) and the statement after that, we obtain the estimate that

$$\frac{\kappa^2 H F^{(0)} \phi^{(0)}}{c} + \frac{1}{3} = B + \frac{1}{3} = O(\dot{H}/H^2) = O(\epsilon^2),$$

where we have used (D42). With this estimation and the background equation of motion, we obtain

$$\frac{\kappa \dot{F}^{(0)}}{F^{(0)}} + \frac{\kappa \dot{\phi}^{(0)}}{\phi^{(0)}} = -3\kappa H - \frac{c}{\kappa^2 F^{(0)} \phi^{(0)}} = O(\epsilon^3).$$

By using the background equations of motion, we also obtain
\[ \kappa^4 \left( V^{(0)} - \alpha R^{(0)2} \right) = 3\kappa^2 H^2 \left( 1 + \kappa^2 W^{(0)} \right) + \kappa^2 H \left( 1 + 2\kappa^2 W^{(0)} \right) - \frac{1}{2} \kappa^4 \dot{W}^{(0)} - \frac{5}{2} \kappa^4 H W^{(0)} = 3\kappa^2 H^2 + O(\epsilon^4). \]  

Fourthly, let us expand all other relevant quantities as

\[
\Psi_\pm = \sum_{i=0}^{\infty} \Psi_{\pm[i]},
\]

\[
\tau_{00} = \sum_{i=0}^{\infty} \tau_{00[i]},
\]

\[
\tau_{(L)0} = \sum_{i=0}^{\infty} \tau_{(L)0[i]},
\]

\[
\tau_{(Y)} = \sum_{i=0}^{\infty} \tau_{(Y)[i]},
\]

\[
\tau_{(LL)} = \sum_{i=0}^{\infty} \tau_{(LL)[i]}, \tag{D47}
\]

where terms with the subscript \([i]\) is of order \(O(\epsilon^i)\). According to the expansion of \(\tau\)'s, we obtain the following expansion of the conservation equation (D22):

\[
\dot{\tau}_{00[0]} + \frac{k^2}{a^2_0} \tau_{(L)0[0]} = 0,
\]

\[
\dot{\tau}_{(L)0[0]} + \frac{4 k^2}{3 a^2_0} \tau_{(LL)[0]} - \frac{\tau_{(Y)[0]}}{a^2_0} = 0 \tag{D48}
\]

in the order \(O(\epsilon^0)\),

\[
\dot{\tau}_{00[1]} + \frac{k^2}{a^2_0} \tau_{(L)0[1]} + 3 H \tau_{00[0]} + 3 H \frac{\tau_{(Y)[0]}}{a^2_0} - 2 \frac{k^2}{a^2_0} \tau_{(L)0[0]} H(t - t_0) = 0,
\]

\[
\dot{\tau}_{(L)0[1]} + \frac{4 k^2}{3 a^2_0} \tau_{(LL)[1]} - \frac{\tau_{(Y)[1]}}{a^2_0} + 3 H \tau_{(L)0[0]} H(t - t_0) + 2 \frac{\tau_{(Y)[0]}}{a^2_0} H(t - t_0) = 0. \tag{D49}
\]

in the order \(O(\epsilon^1)\), and

\[
\dot{\tau}_{00[2]} + \frac{k^2}{a^2_0} \tau_{(L)0[2]} + 3 H \tau_{00[1]} + 3 H \frac{\tau_{(Y)[1]}}{a^2_0} - 2 \frac{k^2}{a^2_0} \tau_{(L)0[1]} H(t - t_0)
\]

\[
-6 \frac{\tau_{(Y)[0]}}{a^2_0} H^2(t - t_0) + 2 \frac{k^2}{a^2_0} \tau_{(L)0[0]} H^2(t - t_0)^2 = 0,
\]

\[
\dot{\tau}_{(L)0[2]} + \frac{4 k^2}{3 a^2_0} \tau_{(LL)[2]} - \frac{\tau_{(Y)[2]}}{a^2_0} + 3 H \tau_{(L)0[1]} - 8 \frac{k^2}{3 a^2_0} \tau_{(LL)[1]} H(t - t_0) + 2 \frac{\tau_{(Y)[1]}}{a^2_0} H(t - t_0)
\]

\[
+ \frac{8 k^2}{3 a^2_0} \tau_{(LL)[0]} H^2(t - t_0)^2 - 2 \frac{\tau_{(Y)[0]}}{a^2_0} H^2(t - t_0)^2 = 0. \tag{D50}
\]

in the order \(O(\epsilon^2)\). These conservation equations are used throughout the forthcoming calculations.

Fifthly, the coefficients of the equations (D36) and (D38) are correspondingly expanded as

\[
C_{I\pm} = \sum_{i=0}^{\infty} C_{I\pm[i]},
\]

\[
S_{00} = \sum_{i=0}^{\infty} S_{00[i]},
\]

\[
D_{I\pm} = \sum_{i=0}^{\infty} D_{I\pm[i]},
\]

\[
S_{(LL)} = \sum_{i=0}^{\infty} S_{(LL)[i]}, \tag{D51}
\]

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where
\[
C_{2+0} = -1, \quad C_{2+1} = 0, \quad C_{2+2} = -4\alpha\kappa^2 R^{(0)}, \\
C_{1+0} = 0, \quad C_{1+1} = -\kappa H, \quad C_{1+2} = 0, \\
C_{0+0} = -\frac{\kappa^2 k^2}{a_0^2}, \quad C_{0+1} = 2\frac{\kappa^2 k^2}{a_0^2} H(t - t_0), \quad C_{0+2} = -2\frac{\kappa^2 k^2}{a_0^2} H^2(t - t_0)^2 - 4\alpha\frac{\kappa^2 k^2}{a_0^2} \kappa^2 R^{(0)}, \\
C_{2-0} = C_{2-1} = C_{2-2} = C_{1-0} = C_{1-1} = C_{1-2} = C_{0-0} = C_{0-1} = C_{0-2} = 0, \\
S_{00[0]} = -\kappa^4 \tau_{00[0]} + \kappa^4 \hat{\tau}_{(LL)0[0]} - 2\kappa^4 \tau_{(LL)0[0]} - 2\frac{\kappa^2 k^2}{a_0^2} \kappa^2 \tau_{(LL)[0]} \\
+ 2\kappa^4 H \hat{\tau}_{(LL)0[0]} - 3\kappa^4 H \tau_{(LL)0[0]} + 4\kappa^2 \frac{\kappa^2}{a_0^2} \kappa^2 \tau_{(LL)[0]} H(t - t_0), \\
S_{00[1]} = -\kappa^4 \tau_{00[1]} + \kappa^4 \hat{\tau}_{(LL)0[1]} - 2\kappa^4 \tau_{(LL)0[1]} - 2\frac{\kappa^2 k^2}{a_0^2} \kappa^2 \tau_{(LL)[1]} \\
+ 2\kappa^4 H \hat{\tau}_{(LL)0[1]} - 3\kappa^4 H \tau_{(LL)0[1]} + 4\kappa^2 \frac{\kappa^2}{a_0^2} \kappa^2 \tau_{(LL)[1]} H(t - t_0) - 4\frac{\kappa^2 k^2}{a_0^2} \kappa^2 \tau_{(LL)0[0]} H^2(t - t_0)^2,
\]
\[
D_{2+0} = -1, \quad D_{2+1} = 0, \quad D_{2+2} = -4\alpha\kappa^2 R^{(0)} + 6\alpha\frac{\kappa^2 F^{(0)}}{F_R^{(0)}}, \\
D_{1+0} = 0, \quad D_{1+1} = -\kappa H, \quad D_{1+2} = 0, \\
D_{0+0} = D_{0+1} = 0, \quad D_{0+2} = 2\alpha \frac{\kappa^2 k^2}{a_0^2} \frac{\kappa^2 F^{(0)}}{F_R^{(0)}}, \\
D_{2-0} = D_{2-1} = 0, \quad D_{2-2} = 6\alpha \frac{\kappa^2 F^{(0)}}{F_R^{(0)}}, \\
D_{1-0} = D_{1-1} = D_{1-2} = 0, \\
D_{0-0} = D_{0-1} = 0, \quad D_{0-2} = \frac{\kappa^2 F^{(0)}}{F_R^{(0)}} \left( \frac{1}{2} + 6\alpha \frac{\kappa^2 k^2}{a_0^2} \right), \\
S_{(LL)[0]} = \kappa^4 \hat{\tau}_{(LL)0[0]} - 2\kappa^4 \tau_{(LL)0[0]}, \\
S_{(LL)[1]} = \kappa^4 \hat{\tau}_{(LL)0[1]} - 2\kappa^4 \tau_{(LL)0[1]} + 2\kappa^4 H \tau_{(LL)0[0]}, \\
S_{(LL)[2]} = \kappa^4 \hat{\tau}_{(LL)0[2]} - 2\kappa^4 \tau_{(LL)0[2]} + 2\kappa^4 H \tau_{(LL)0[1]} - \frac{\kappa^2 F^{(0)}}{F_R^{(0)}} \kappa^2 \tau_{(LL)[0]}. \tag{D52}
\]

Now it is time to expand the equations (D36) and (D38). For this purpose we shall use the conservation equations (D48), (D49) and (D50). In the order \(O(\epsilon^5)\), by subtracting (D38) from (D36) we obtain
\[
\frac{\kappa^2 k^2}{a_0^2} \Psi_{+[0]} = -\kappa^4 \left( \tau_{00[0]} + 2\frac{\kappa^2 k^2}{a_0^2} \tau_{(LL)[0]} \right). \tag{D53}
\]

It is easily shown by using the conservation equations that both (D36) and (D38) are simultaneously satisfied by this solution in the order \(O(\epsilon^5)\).

In the order \(O(\epsilon^4)\), by subtracting (D38) from (D36), substituting the solution (D53), and using the conservation equations, we obtain
\[
\frac{\kappa^2 k^2}{a_0^2} \Psi_{+[1]} = -\kappa^4 \left( \tau_{00[1]} + 2\frac{\kappa^2 k^2}{a_0^2} \tau_{(LL)[1]} + 3H \tau_{(LL)0[0]} + 2\tau_{00[0]} H(t - t_0) \right). \tag{D54}
\]

It is easily shown by using the conservation equations that both (D36) and (D38) are simultaneously satisfied by this solution in the order \(O(\epsilon^4)\).

In the order \(O(\epsilon^3)\), by subtracting (D38) from (D36), substituting the solutions (D53) and (D54), and using the conservation equations, we obtain
The perturbed metric, after the $\kappa H$ limit, where we have redefined the spatial coordinates as $a_0 x^i \rightarrow x^i$, is

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j,$$

and $\Psi_\pm = \Psi \pm \Phi$. The conservation equation is reduced to

$$\dot{\tau}_{00} + k^2 \tau_{(LL)} = 0,$$

$$\dot{\tau}_{(L)L} + \frac{4}{3} k^2 \tau_{(LL)} - \tau_{(Y)} = 0,$$

where $\tau$'s are defined by (D20).

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