Towards the Gravity Dual of Quarkonium in the Strongly Coupled QCD Plasma

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We build a “bottom-up” holographic model of charmonium by matching the essential spectral data. We argue that this data must include not only the masses but also the decay constants of the \( J/\psi \) and \( \psi' \) mesons. Relative to the “soft-wall” models for light mesons, such a matching requires two new features in the holographic potential: an overall upward shift as well as a narrow “dip” near the holographic boundary. We calculate the spectral function as well as the position of the complex singularities (quasinormal frequencies) of the retarded correlator of the charm current at finite temperatures. We further extend this analysis by showing that the residues associated with these singularities are given by the boundary derivative of the appropriately normalized quasinormal mode. We find that the “melting” of the \( J/\psi \) spectral peak occurs at a temperature \( T \approx 540 \text{ MeV} \), or \( 2.8 T_c \), in good agreement with lattice results.

I. INTRODUCTION

The fate of quarkonium states such as \( J/\psi \) in the quark-gluon plasma has been a subject of continuous interest for both theorists and experimentalists since the famous proposal [1] to use the suppression of \( J/\psi \) production in heavy-ion collisions as a signature of the quark-gluon plasma formation. The screening of color forces should weaken or prevent \( c\bar{c} \) pair binding, manifesting in \( J/\psi \) suppression in heavy-ion collisions capable of creating quark-gluon plasma. Making detailed predictions for this effect is a formidable challenge, requiring understanding of such factors as \( c\bar{c} \) pair production, binding, and survival of charmonium states in medium, as well as their interplay in the dynamically evolving high energy density medium (see, e.g., Ref. [2] for review).

A field-theoretical quantity which encodes much, but certainly not all, information necessary in this regard is the spectral function of the operator of charm current, \( J^\mu = \bar{c}\gamma^\mu c \). This operator, acting on the vacuum state, produces charmonium \( 1^{--} \) states such as \( J/\psi, \psi' \), etc. By studying the response of the thermal medium to the action of such an operator we can learn about the fate of the charmonium states in this medium. The dissociation, or “melting”, of the charmonium states manifests itself in the gradual decrease with temperature of the strength of the response at the frequencies at which the vacuum state would “resonate” by creating a charmonium bound state.

The current-current correlator and its spectral function are field-theoretically well-defined quantities, interesting in their own right. They can be, in principle, calculated using \textit{ab initio} lattice Monte Carlo approach. The task of extracting the charmonium spectral functions from discrete Euclidean correlators measured on the lattice is difficult, but significant progress has been made recently using the Maximum Entropy Method (MEM) [3–10]. The spectral peak corresponding to the \( J/\psi \) state appears to be still prominent up to temperatures as high as almost \( 2.5 T_c \), where \( T_c \approx 190 \text{ MeV} \) (see, e.g, Ref. [11] for a review) is the critical/crossover temperature at which the thermal QCD medium undergoes deconfining transition. To understand lattice results better, it is very desirable to have analytically controllable models which could allow us to study the “melting” of the charmonium spectral peak.

The models based on a non-relativistic Schrödinger potential have been traditionally used for this purpose. In the pioneering work of Ref. [12], the effective temperature-dependent potential was modeled by introducing Debye screening into the linearly confining potential. The recently studied models have used a non-relativistic Schrödinger potential, whose dependence on temperature is introduced either by input from the lattice [13–21], or resummation-improved perturbation theory [22–24].

The non-relativistic thermal potential models typically predict charmonium dissociation at temperatures approximately 1.2–1.5 \( T_c \) [18], which appear to be too low compared to the dissociation temperatures inferred from the lattice. Several possible resolutions of this puzzle have been proposed (as reviewed, e.g., in Refs. [18, 24, 26]).

Gauge-gravity holographic correspondence [27–29] provides a new field-theoretically consistent framework for modeling the properties of thermal strongly-coupled medium (see, e.g., [30, 31] for recent reviews). Although \textit{ab initio} calculations in QCD-proper are still not possible within this approach, many properties of QCD, both in vacuum and at finite temperature can be modeled. “Top-down” reduction of string theory on suitably chosen backgrounds allows one to study QCD-like field theories, such as theories with flavored matter in fundamental color representation [32, 33]. Alternatively, reversing the rules of the holographic correspondence one can build “bottom-up” models by matching the relevant features of QCD, such as chiral symmetry breaking, operator product expansion constraints on correlators and linear con-
The fate of meson states in strongly coupled thermal medium has been a subject of many studies using a “top-down” approach to implementing massive flavor degrees of freedom introduced in Refs. 33–53. Spectral functions and quasinormal modes were studied in Refs. 44, 45, 50, 52, 53. Structure of quarkonium in these theories was also studied in Ref. 55.

It is reasonable to expect that reproducing the charmonium spectrum correctly is an important prerequisite for predicting its finite temperature properties. One important way in which the spectrum of heavy quarkonium excitations in “top-down” models differs from QCD is that in these models the lowest meson mass \( m_1 \) emerges as the only scale determining the masses of both the ground state, and the excited mesons \( m_n \sim n m_1, n \gg 1 \). In contrast, in the heavy quark limit of QCD, the mass of the ground state, such as \( J/\psi \), is controlled by a parameter (heavy quark mass) different from the parameter (string tension, or \( \Lambda_{\text{QCD}} \)) controlling the level spacing of excited states: \( m_n^2 \sim m_1^2 + n^2 \Lambda_{\text{QCD}}^2 \). Refs. 57, 58. It would be interesting to find a holographic model which could represent these features of charmonium spectrum correctly.

### II. APPROACH AND OUTLINE

In this paper, we shall approach this problem from the complementary side of the “top-down” models. Taking the quarkonium properties at zero temperature as input, we shall construct the holographic dual which matches these properties in the spirit of the “bottom-up” approach of Refs. 40–42. Then we shall study how such a model would evolve with temperature.

The pioneering calculation of the \( J/\psi \) spectral functions within such an approach has been performed in Ref. 59–61. Building on the success of the “bottom-up” models for light mesons 40–42 the authors of Ref. 59 changed the scale in the “soft-wall” model of Ref. 42 to match the mass of \( J/\psi \) meson, in place of the \( \rho \) meson.\(^1\) However, such an approach suffers from the drawbacks already discussed above for the “top-down” models: the scale that sets the level spacing of quarkonium excited states is the same as the scale that sets the mass of the ground state, in effect \( \Lambda_{\text{QCD}} \sim m_{J/\psi} \) in the models of Refs. 59–62. What we would like instead is a spectrum which has a gap between the vacuum and the ground state whose scale is set by a parameter independent from the spacing of higher excited levels. Experimentally, the slope of the radial excitation trajectory for \( J/\psi \) is similar to that of the \( \rho \) mesons 57, 58, and we want the model to match that property of QCD.

The matching of the quarkonium mass spectrum is not the only new ingredient which we need to introduce. We point out that another important quantity to be matched is the decay constant of the quarkonium. Simply put, in order for the spectral function to be correct at high temperature, we should begin with a function correct at zero temperature, which requires matching not only the position, but also the strength of the \( J/\psi \) resonance, i.e. the decay constant.

One of the most striking qualitative consequences of such a more realistic model of quarkonium is a much more robust spectral peak of \( J/\psi \), which persists up to temperature 540 MeV, i.e., 2.8 \( T_c \), in agreement with lattice studies. This is in contrast to Ref. 59 which finds charmonium peak “melting” at around 1.2 \( T_c \). We take better agreement of our results with the lattice calculations as evidence that the model we introduce captures essential features of quarkonium.

In Section III to make the paper more self-contained, we lay out the general setup of the holographic model. Section IV discusses the features of the holographic potential we introduce to model charmonium. Section V applies the methods of holographic QCD at finite temperature to our model and, similar to Section III should be familiar to practitioners. The finite temperature spectral functions obtained using the proposed new holographic potential are presented in Section VI. In Section VII we further our study of the thermal properties of charmonium by considering quasinormal modes. To facilitate this analysis, in Section VIII we generalize the quantitative relation between the holographic wave-function and the residue at the corresponding singularity of the 2-point correlator (i.e., decay constant) to the case of quasinormal modes at nonzero temperature. Interestingly, this relationship is the same as the known one in vacuum, after the quasinormal mode is appropriately normalized. We derive the expression for such a norm in Section VIII. In Section IX we use the quasinormal modes and the residues to analyze the charmonium spectral function more quantitatively. We conclude in Section X with a summary and a discussion of the results.

### III. SETUP

We shall focus on the two-point correlation function of the heavy quark (charm) current, \( J^{\mu} = \bar{c} i \gamma^\mu c \), and define the retarded Green’s function:

\[
G_R(q) = -i \int d^4x e^{iqx} \theta(x_0) \left\langle [J^\perp(x), J^\perp(0)] \right\rangle, \tag{1}
\]

where \( J^\perp \) is a component of the current orthogonal to the 4-vector \( q^\mu \): \( q_\mu J^\mu = 0 \) and \( \langle \ldots \rangle \) denotes thermal average. For simplicity, we shall consider only the case \( q = 0 \), i.e., \( q = (\omega, 0) \). Thus \( J^\perp \) will denote any spatial component of \( J^\mu \), e.g., \( J^x \). We shall also define \( G_R(\omega) = G_R(\omega, 0) \).

\(^1\) Similar “rescaling” has been used in earlier papers in order to calculate the charmonium dissociation temperature, Refs. 61–62, but not the spectral functions.
The spectral function is defined, for real values of $\omega$, as
\[
\rho(\omega) \equiv \rho(\omega, 0) = -\text{Im} \, G_R(\omega).
\] (2)

Among standard properties of $\rho(\omega)$, which can be shown using spectral representation, is that $\rho(\omega, q)$ equals the Fourier transform of $\langle [J^I(x), J^I(0)] \rangle$, and that $\rho(\omega)/\omega > 0$ for all $\omega$.

In the spirit of the holographic approach, we shall assume that the generating functional of the heavy quark vector current $J^\mu$ can be represented by the effective action obtained by integrating over a bulk 5D gauge field $V_M$ (dual to the current) at given fixed boundary values (equal to the source of the current). The action for the 5D gauge field is given by
\[
S = -\frac{1}{4g_5^2} \int d^5x \sqrt{g} e^{-\Phi} V_{MN} V^{MN},
\] (3)
where $g_5^2$ is the 5D gauge coupling and $V_{MN} = \partial_M V_N - \partial_N V_M$. The 2-point current correlator is given by the linear response of the field $V_M$ to an infinitesimal perturbation of its boundary condition.

We choose the conformally flat representation for the 5D background metric $g_{MN}$ with 4D Lorentz-isometry:
\[
ds^2 = g_{MN} dx^M dx^N = e^{2A(z)} \left[ \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right],
\] (4)
where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric tensor. The effect of confinement is represented by the non-trivial background profile of the scalar field $\Phi$ in Eq. (3) in the same way as it is done in the soft-wall model with dilaton background in Ref. [42]. We shall assume that the physics associated with the mass of the quark and chiral condensate operator $\bar{c}c$ can be also represented as a contribution to the background field $\Phi$. Such a contribution is essential if the model is to be extended to describe different flavors, which must have different holographic backgrounds, depending on their respective masses. In a putative top-down approach this contribution could arise through the interaction of the gauge field with the scalar (“tachyon”) field dual to the quark mass operator $\bar{c}c$, e.g., by a mechanism explored in Refs. [63, 64].

Following the rules of the holographic correspondence, we shall calculate the generating functional for correlation functions of the heavy-quark current by evaluating the action at its extremum for given boundary conditions. The extremum is given by the solution of the equations of motion, which in $V_5 = 0$ gauge read:
\[
\partial_z [e^{B(z)} \partial_z V] + q^2 e^{B(z)} V = 0,
\] (5)
where $V$ is any of the three components $V_\mu$ of $V_\mu(q, z)$ transverse to 4-vector $q^\mu$ ($q^4 = 0$) and
\[
B = A - \Phi.
\] (6)

Discrete values of $q^2 = m_n^2$, for which the Eq. (5) possesses a normalizable solution $V = v_n(z)$ satisfying the boundary condition $V|_{z=0} = 0$, correspond to the masses $m_n$ of the charmonium states, $n = 1, 2, \ldots = J/\psi, \psi', \ldots$.

We normalize such solutions as
\[
\int_0^\infty dz \, e^{B(z)} v_n(z)^2 = 1.
\] (7)

Back on the field theory side, the decay constants $f_n$ are defined via matrix elements of the charmonium current between vacuum and a given vector charmonium state $n$ with polarization $\epsilon_\mu$ as
\[
\langle 0 | J_\mu(0) | n \rangle = f_n m_n \epsilon_\mu.
\] (8)

Holographic correspondence relates these constants to the (second) derivative of the normal mode $v_n$ (see Refs. [37, 40]):
\[
f_n = \frac{1}{g_5 m_n} \left. v_n'(z) e^{B(z)} \right|_{z \to 0}.
\] (9)

As one can see from Eqs. (5–9), the masses and decay constants of the charmonium states are determined by the combination $B$ of the gravity ($A$) and dilaton/tachyon ($\Phi$) backgrounds [41]. In the spirit of the bottom-up approach, we shall choose the function $B$ so as to satisfy a number of phenomenological and field-theoretical QCD constraints. We assume that such a background arises dynamically, but do not attempt to model the corresponding dynamics.

We require that the ultraviolet behavior of the current-current correlator is conformal, viz. $G_R(\omega) \sim \omega^2 \log(-\omega^2)$ as $\omega^2 \to -\infty$, which translates into:
\[
e^{B(z)} \quad \xrightarrow{z \to 0} \quad z^{-1},
\] (10)
and matches that of QCD, which fixes $g_5^2 = 12\pi^2/N_c$.

By performing a Liouville transformation,
\[
\Psi = e^{B(z)/2} V,
\] (12)
we can bring Eq. (5) to the canonical Schrödinger-like form
\[
-d^2 \Psi/dz^2 + U(z) \Psi = q^2 \Psi,
\] (13)
with the holographic potential given by
\[
U(z) = \frac{B''(z)}{2} + \frac{\left(B'(z)\right)^2}{2}.
\] (14)

We shall now choose $B$ or, equivalently $U$, to match not only the mass of $J/\psi$, but also the mass of $\psi'$ as well as (and more importantly) the decay constants of $J/\psi$ and $\psi'$. 
IV. HOLOGRAPHIC POTENTIAL

In order to explain the role and motivation for the features of the holographic Schrödinger potential which we are led to introduce in this paper, we shall add those features one at a time.

To begin with, it is easy to implement the two-scale excitation spectrum, correcting the drawback of Ref. [59], by adding a constant, $c^2$, to the potential. Taking the potential from the soft-wall model, $U_{(a)}$, which reproduces the equidistant mass-squared spectrum controlled by parameter $a$, we thus consider

$$U_{(a,c)} = U_{(a)} + c^2 = 3/(4z^2) + (a^2z)^2 + c^2.$$  \hfill (15)

For this potential, the mass of the ground state is controlled by $c$, independent from $a$.\footnote{The first term is the potential for a conformally invariant theory. It can be, curiously, viewed as a centrifugal term for a radial Schrödinger equation in 2d with the radial quantum number $m = 1$ (coinciding with the spin of the mesons we study [42]).}

The shifted potential in Eq. (15), however, has an important drawback, which is significant as far as the finite temperature properties of the quarkonium are concerned. Since those properties are encoded in the spectral function of quarkonium at finite temperature, it is clear that to get those properties right on a quantitative level, one must begin with correct spectral function at zero temperature. The spectral function at zero temperature consists of a series of peaks, representing the quarkonium states. These peaks are characterized not only by their position (masses) but also by their strengths, i.e., the decay constants. There are 4 parameters in this potential, and we use the simplest approximation for a narrow dip: a negative Dirac delta function.

What should be done to the potential (15) to increase the value of the $J/\psi$ decay constant? To understand the solution to this problem, one should recall the relationship between the decay constant and the rate of change of the holographic wave-function at the boundary $z \to 0$, Eq. (10):

$$f_n = \frac{1}{g_5 m_n} \psi_n''(0) = \frac{1}{g_3 m_n} (\sqrt{z} \psi_n)'|_{z=0},$$  \hfill (16)

where we used Eq. (10) and the normalizable solution $\psi_n$ to Eq. (15), $\psi_n = e^{B/2} \psi_n$. That means the steeper the wave-function $\psi_n$ is near the boundary $z = 0$, the larger is the decay constant. For example, increasing parameter $a$ would make the function steeper, by squeezing its support. But that will unacceptably increase the radial trajectory slope, $dm_n^2/dn$, which we want to preserve. A simple and natural solution is to provide an “incentive” for the wave function to be larger only in a narrow region close to the boundary, thus forcing it to approach the boundary value $\psi_n(0) = 0$ steeper. This can be easily achieved by creating a “dip” in the potential at small $z \sim 3d$.

What should be the shape of the dip at $z_d$? In this paper we shall consider a minimalistic example of the potential necessary to match the quarkonium spectral data. We shall use the simplest approximation for a narrow dip: a negative Dirac delta function.\footnote{Replacing the delta-function with a narrow square well produces similar results.}

Rather than superimposing the delta function onto the potential $U_{(a,c)}$ in Eq. (15), we first observe that, since the potential undergoes a qualitative change in behavior at $z = z_d$, manifested in the delta function, there is no reason to expect that it is described by a function continuous across $z_d$. A more natural, and convenient, choice is to consider the following piecewise analytic function:

$$U(z) = \frac{3}{4z^2} \theta(z_d - z) + ((a^2z)^2 + c^2) \theta(z - z_d) - a \delta(z - z_d),$$  \hfill (17)

which represents the necessary large and small $z$ behavior. There are 4 parameters in this potential, and we use them to fit 4 experimental data points: the masses and the decay constants of $J/\psi$ and $\psi'$ in Table I. We find (see also Fig. 1)

$$a = 0.970 \text{ GeV}, \quad c = 2.781 \text{ GeV},$$
$$\alpha = 1.876 \text{ GeV}, \quad z_d = 2.211 \text{ GeV}. $$  \hfill (18)

One can see the effect of the delta function on the holographic wave functions explicitly as shown in Fig. 2. As expected, the wave functions, varying relatively smoothly for $z > z_d$, “dive” steeply towards their boundary value $\psi_n(0) = 0$ once $z < z_d$.\footnote{The first term is the potential for a conformally invariant theory. It can be, curiously, viewed as a centrifugal term for a radial Schrödinger equation in 2d with the radial quantum number $m = 1$ (coinciding with the spin of the mesons we study [42]).}

| Observable | Experiment | $U_{(a)}$ (MeV) | $U_{(a,c)}$ (MeV) |
|------------|------------|-----------------|-------------------|
| $m_{J/\psi}$ | 3096       | 3096*           | 3096*             |
| $m_{\psi'}$ | 3685       | 4378            | 3685*             |
| $f_{J/\psi}$ | 416        | 348             | 145               |
| $f_{\psi'}$ | 296        | 348             | 173               |

TABLE I: Comparison of the masses and decay constants obtained using the scaled soft-wall (as in Ref. [59]) and shifted soft-wall potentials in Eq. (15) with experimental values, Ref. [67]. Analytic formulas from Ref. [42] are used: $m_n^2 = 4a^2n + c^2$ and $f_n = \sqrt{8n^2a^2/(g_3m_n)}$, $n = 1,2$. The observables which are fitted to determine parameter $a$ in $U_{(a)}$ and $a, c$ in $U_{(a,c)}$ are marked by asterisks. Both models significantly underpredict the decay constant of $J/\psi$. The shifted potential $U_{(a,c)}$ is worse because the parameter $a$ needed for the fit is smaller.

For the shifted potential in Eq. (15), we first observe that, since $\psi$ is overpredicted by 20% and that of $\psi'$ is underpredicted by 15% – see Table I. For the shifted potential in Eq. (15), the situation is much worse: the decay constants are overpredicted by 20% and that of $J/\psi$ is overpredicted by 15% – see Table I. For the shifted potential in Eq. (15), we first observe that, since $\psi$ is overpredicted by 20% and that of $\psi'$ is underpredicted by 15% – see Table I. For the shifted potential in Eq. (15), we first observe that, since $\psi$ is overpredicted by 20% and that of $\psi'$ is underpredicted by 15% – see Table I.
The potential in Eq. (17) can be, certainly, improved by applying further constraints, e.g., fitting the masses and decay constants of the higher excited states, and also by applying constraints which follow from the operator product expansion of the heavy quark current-current correlator beyond the leading order.

We shall defer such improvements to future work, and concentrate here on the consequences of the two major properties of the potential we introduced: the shift, providing the two-scale level pattern, and the dip at small \(z\), supporting the necessary size of the decay constants.

From the point of view of the nonrelativistic model of quarkonium, the effect of increasing the decay constant can be understood as the effect of decreasing the size of the bound state: the decay constant is proportional to the probability of finding quark and antiquark at the same point, which is larger for a more compact state. It is reasonable to expect that a more compact bound state survives up to higher temperatures. This is indeed the result we find (see Section VI).

V. FINITE TEMPERATURE

Once the zero-temperature parameters of the model are fixed, we wish to determine the spectral function of the charm current at finite temperature, as defined on the field theory side by Eqs. (1) and (2).

On the holographic side, finite temperature correlation functions of heavy quark current are determined by a similar action to Eq. (1), but on a different, temperature dependent, background \(\Phi\) and \(g_{MN}\), whose main feature is the presence of a black-hole horizon at some value of \(z = z_h\). In a “top-down” approach that background would itself be an extremum of the same gravity/dilaton/tachyon action. We shall not attempt to model the dynamics of these background fields here, leaving it to further work. Lacking the equations of motion, which would determine the background for each temperature, we shall instead make a minimalistic assumption about the behavior of this background.

We choose the following representation for the metric consistent with the 3d Euclidean spatial isometry, corresponding to finite temperature:

\[
\begin{align*}
  ds^2 &= e^{2A(z)} \left[ h dt^2 - dx^2 - h^{-1} dz^2 \right].
\end{align*}
\]

If the function \(h(z)\) has a simple zero, viz. \(h(z_h) = 0\), the space described by Eq. (21) possesses an event horizon at \(z = z_h\). The temperature \(T\) corresponding to this background is related to \(z_h\) as

\[
T = \frac{1}{4\pi} |h'(z_h)|. \tag{20}
\]

We assume the simplest ansatz for \(h\), interpolating between \(h(0) = 1\) and \(h(z_h) = 0\) with the power \(z^4\) dictated by the dimension of the operator of the energy-momentum:

\[
h = 1 - (z/z_h)^4, \tag{21}
\]

which turns out to be the same as in the familiar \(AdS_5\) black hole solution to Einstein equations with negative cosmological constant. The temperature \(T\) corresponding to this background is related to \(z_h\) as

\[
z_h = (\pi T)^{-1}. \tag{22}
\]

As already noted, in principle, we should expect the function \(A(z)\) as well as the background \(\Phi(z)\) to depend on the temperature. Since we do not attempt to model the background dynamically, we have little choice but to neglect the dependence of the background \(B\) in Eq. (9) on...
Our purpose here is to describe the effect of the temperature semiquantitatively. The presence of the black-hole horizon in the metric is the main property that our ansatz is aimed to capture.

Our approach to introducing temperature dependence here is similar to that of Ref. [59]. In addition to making our model simpler, this will allow us to see the effect of the features of the potential we introduce (specifically, the “dip”), in comparison with Ref. [59], more clearly.

Therefore, the equation we solve to determine the correlator of the heavy quark current at finite temperature is given by

$$\partial_z (he^B \partial_z V) + \omega^2 h^{-1} e^B V = 0 ,$$

where function $B(z)$ is determined by solving Eq. (14) for given $U(z)$. The retarded correlation function is given, according to the well-known prescription of Ref. [68, 69], by

$$G_R(\omega) = -\frac{1}{g_5} \frac{e^{B(z_\omega)} V'(z, \omega)}{z = \epsilon} = -\frac{1}{g_5} \frac{V'(\epsilon, \omega)}{\epsilon} ,$$

where $\epsilon \to 0$ is an ultraviolet regulator and $V(z, \omega)$ is the solution of Eq. (24) with boundary conditions:

$$V(\omega, \epsilon) = 1 ;$$

$$V(\omega, z) \xrightarrow{z \to z_h} C(\omega)(1 - z/z_h)^{-\omega/(4\pi T)} ,$$

where, for each $\omega$, $C(\omega)$ is a free constant determined, if needed, by solving equation (23). The spectral function can be calculated as the imaginary part of $G_R(\omega)$ given by Eq. (24). Alternatively, one can observe that $\text{Im}(he^B V' \partial_z V)$ is $z$-independent and evaluate it at $z = z_h$, instead of $z = \epsilon$:

$$g_5^2 \rho(\omega) = \text{Im} e^{-1} V'(\epsilon, \omega) = |C(\omega)|^2 e^{B(z_h)} \omega ,$$

which demonstrates explicitly that $\rho(\omega)/\omega > 0$ condition is a built-in property of the model.

The Green’s function obtained using Eq. (24) is logarithmically divergent as $\epsilon \to 0$. This logarithmic dependence on the UV regulator $\epsilon$ matches the expectation on the field theory side. It does not concern us here because the imaginary part of Eq. (24), i.e., the spectral function given by Eq. (20), is finite, as it should be in the corresponding field theory.

The solutions of Eq. (23) are easier to study by applying the Liouville transformation

$$\Psi(\zeta(z)) = e^{B(z)/2} V(z); \quad \zeta(z) = \int_0^z dz'/h(z') ,$$

which yields the equivalent Schrödinger equation for the function $\Psi(\zeta)$:

$$-d^2 \Psi/d\zeta^2 + U_T(\zeta) \Psi = \omega^2 \Psi ,$$

where the potential is given by

$$U_T(\zeta(z)) = \frac{d^2 B/d\zeta^2}{2} + \frac{(dB/d\zeta)^2}{2}$$

$$= \left[ \frac{B''(z)}{2} + \frac{(B'(z))^2}{2} + \frac{B'(z)h'(z)}{2h(z)} \right] h(z)^2 .$$

At $T = 0$ this equation becomes Eq. (14) with $\zeta = z$. We use Eq. (14) with $U(z)$ given by Eq. (17) to find $B(z)$, which then, upon substitution into Eq. (29) gives us $U_T(\zeta(z))$.

VI. RESULTS

For the purpose of presenting the results, it is useful to observe that in a theory with conformal behavior at short distances, one can expect that

$$\rho(\omega)/\omega^2 \xrightarrow{\omega \to \infty} \text{dimensionless const}.$$  

This large $\omega$ behavior is dominated by the contribution of the unit operator in the OPE of the product of the currents. In QCD, the dimensionless constant can be calculated, due to the asymptotic freedom, $\rho(\omega)/\omega^2 \xrightarrow{\omega \to \infty} N_c/(24\pi)$, and it is matched in the holographic model by Eqs. (10), (11):

$$\rho(\omega)/\omega^2 \xrightarrow{\omega \to \infty} \frac{\pi}{2g_5^2} .$$

Equation (31) motivates the use of the rescaled spectral function:

$$\tilde{\rho}(\omega) = \frac{\rho(\omega)/\omega^2}{\left(\rho(\omega)/\omega^2\right)_{\omega \to \infty}} = \frac{2g_5^2 \rho}{\pi \omega^2} ,$$

such that $\tilde{\rho}(\infty) = 1$. The spectral functions of the charm quark current obtained in our model at three representative temperatures are shown in Fig. 3.

We observe that a pronounced $J/\psi$ spectral peak survives at least beyond $T = 400$ MeV $\approx 2 T_c$. This is in agreement with lattice results discussed in the introduction. In comparison, the model in Ref. [59], predicts the disappearance of the peak already above $T \sim 1.2 T_c$. We can also see that the $J/\psi$ peak has mostly disappeared at $T = 600$ MeV.

Another interesting observation is that the peak at zero $\omega$ grows with temperatures. This peak is related to DC conductivity $\sigma$ by Kubo formula

$$\sigma = \lim_{\omega \to 0} \rho(\omega)/\omega .$$

It is easy to show, using Eq. (25) and the fact that $V = 1$ is the solution to Eq. (23) at $\omega = 0$, that (c.f. Ref. [70])

$$\sigma = g_5^{-2} e^{B(z_h)} .$$

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5 Similar approach was used earlier in Refs. [68, 69] to study light flavor mesons at finite temperature.
On the field theory side, the growth of the charm conductivity is a natural consequence of the dissociation of quarkonium bound states.

It is also instructive to look at the temperature evolution of the holographic potential. Comparing the potentials at $T = 200$ and 400 MeV in Fig. 4, one can see that the barrier protecting the (quasi-)bound state from decay becomes thinner, and the depth of the delta function dip becomes smaller with increasing temperature.

In order to obtain a more objective and quantitative description of the evolution of the size and shape of the spectral peak, we shall now take a look at the quasinormal modes defined by equation (23) with outgoing wave boundary condition from Eq. 25.

VII. QUASINORMAL MODES

In order to study the properties of the peak in the spectral function, corresponding to $J/\psi$, we shall use its relation to one of the poles of the function $G_R(\omega)$ in the complex plane. Such poles correspond to solutions $V = v_\nu$ of the equation 25 satisfying boundary conditions

$$v_\nu(\epsilon) = 0; \quad v_\nu(z) \xrightarrow{z \to z_h} c_n(1 - z/z_h)^{-\omega/(4\pi T)},$$

similar to the b.c. in Eq. (25) at the horizon $z = z_h$, but not at the boundary $z = \epsilon$. Unlike the b.c. in Eq. (25), the b.c. in Eq. (35) are homogeneous and specify the solution only up to an overall normalization constant. We describe the most suitable normalization condition for $v_\nu$ in Section VIII.

Such solutions $v_\nu$ are called quasinormal modes due to their similarity to the normal modes of the $T = 0$ equations. Similarly to normal modes, the quasinormal modes exist only for a discrete set of frequencies $\omega_n$. However, due to the nature of the b.c. in Eq. (35), these frequencies are complex.

Quasinormal modes in the context of gauge/gravity duality have been studied extensively, see, e.g., Refs. 71–77, and in particular, in application to the fate of mesons at finite temperature in Refs. 44, 45, 50, 52, 53. We follow a standard numerical method to obtain the values of the quasinormal frequencies as well as the corresponding eigenfunctions of the mode equation. In summary, the main challenge arises due to the fact that, since $\text{Im} \omega_n < 0$, the boundary condition selects the solution dominant as $z \to z_h$. Thus, numerical integration of the equation must be carried out with precision much better than $(1 - z/z_h)^{-\text{Im} \omega_n/(2\pi T)}$, to make sure the “wrong” solution is not mixed in. This effectively prevents the numerical integration from reaching $z = z_h$. The standard solution to this problem [76], and the one we adopt, is to build the solution in a small but finite interval $[z_h(1 - \delta), z_h]$ using a truncated Taylor (Frobenius) expansion. The numerical integration can then be carried out starting from the point $z_h(1 - \delta)$ away from $z_h$.

The trajectories of the first two quasinormal frequencies on the complex plane are shown in Fig. 5. One can see that the imaginary part of the second mode $\omega_2$ increases much faster with temperature than that of $\omega_1$. This corresponds to the observation that the $J/\psi$ peak “melts” very early, in accordance with the sequential dissociation scenario 77.

We can demonstrate the relationship between the quasinormal modes and the peak more clearly by calculating the residue $r_n$ of $G_R$ in the pole corresponding to the mode $n$

$$r_n \equiv \lim_{\omega \to \omega_n} (\omega - \omega_n)G_R(\omega).$$

It is easy to see that at $T = 0$, the residue is related to
the $T = 0$ decay constant in Eq. (9) as
\begin{equation}
   r_n = \frac{f_n^2 m_n^2}{2\omega_n} \quad (T = 0) .
\end{equation}

We shall define the contribution of the pole to the rescaled spectral function (32) as
\begin{equation}
   \tilde{\rho}_n(\omega) \equiv \text{Im} \left( \frac{\tilde{r}_n}{\omega - \omega_n} \right) , \quad \text{where} \quad \tilde{r}_n = \frac{2g_n^2}{\pi} \frac{r_n}{\omega_n} .
\end{equation}

Due to the well-known property of the Green’s function, $G_R(\omega)^* = G_R(-\omega^*)$, the poles must come in pairs, $\omega_n$ and $\omega_{-n} \equiv -\omega_n^*$, with residues $r_n$ and $r_{-n} \equiv -r_n^*$. In Section VIII we show how the quasinormal mode could in fact be normalized and derive a useful expression for the residue $r_n$ in terms of the corresponding quasinormal mode. The result is an extension of the $T = 0$ equation (9).

To demonstrate the usefulness of the expression for $r_n$, given by Eqs. (33, 54) we derive below, we show in Fig. 6 the spectral function $\tilde{\rho}$, together the contribution of the pole and its “mirror,” $\tilde{\rho}_1 + \tilde{\rho}_{-1}$, as well as the result of the subtraction $\tilde{\rho} - \tilde{\rho}_1 - \tilde{\rho}_{-1}$ which, as expected, leaves only monotonic background around the location of the peak, $\omega \approx \text{Re} \omega_1$.  

\section{VIII. Normalization of the Modes and the Residues}

In this Section, we describe how the residue $r_n$ for a pole at quasinormal frequency $\omega_n$ can be calculated given only the corresponding quasinormal mode $v_n(z)$. We shall show that the expression is similar to the zero temperature result, Eq. (9), for the decay constants $f_n$. In order to do that, we would need to be able to generalize the normalization condition Eq. (7) at $T = 0$ to the case of quasinormal modes at finite $T$. We shall indeed obtain this generalization as a byproduct of our derivation of $r_n$.

Similar to the derivation of the $T = 0$ result in Ref. 40 we shall introduce the Green’s function $G(\omega; z, z')$ for the mode equation (23):
\begin{equation}
   \partial_z \left( e^{\beta \omega} G \right) + \omega^2 e^{\beta \omega} G = \delta(z - z') ,
\end{equation}
with (homogeneous) boundary conditions similar to Eq. (35)
\begin{align}
   G(\omega; \epsilon, z') &= 0 ; \\
   G(\omega; z, z') &\xrightarrow{z \to z_h} C(\omega, z') (1 - z/z_h)^{-i\omega/(4\pi T)} ,
\end{align}
where, $C(\omega, z')$ is an arbitrary function of $\omega$ and $z'$. It can be easily checked that
\begin{equation}
   V(\omega, z') = -z^{-1} \partial_z G(\omega; z, z')|_{z = \epsilon} \quad (41)
\end{equation}
satisfies defining equations (23) and (25).

The Green’s function $G$ can be constructed as
\begin{equation}
   G(\omega; z, z') = \frac{u(\omega, z) v(\omega, z')}{he^{\beta |u|}} ,
\end{equation}
where $z(\epsilon, z') = \{\min, \max\}(z, z')$; $u$ and $v$ are two solutions of equation (23) each obeying separately one of the two boundary conditions in Eq. (40):
\begin{align}
   u(\omega, \epsilon) &= 0 ; \quad (43a) \\
   v(\omega, z) &\sim (1 - z/z_h)^{-i\omega/(4\pi T)} \text{ for } z \to z_h . \quad (43b)
\end{align}
Both boundary conditions (13) are homogeneous and thus define $u$ and $v$ only up to overall normalization constants. However, the normalization of either $u$ or $v$ is canceled out from Eq. (12) due to the Wronskian
\begin{equation}
   [u, v] \equiv u \partial_z v - v \partial_z u ,
\end{equation}
\begin{equation}
   \text{FIG. 5: Quasinormal modes $\omega_1$ (J/$\psi$) and $\omega_2$ ($\psi'$) for a sequence of temperatures: } T = 100, 200, 300, 400, 500, 600 \text{ MeV for } \omega_1 \text{ and } T = 100, 200, 300 \text{ MeV for } \omega_2 . \quad \text{Colors (online) correspond to temperatures.}
\end{equation}

\begin{equation}
   \text{FIG. 6: The scaled spectral function at } T = 200 \text{ MeV. The dashed line is the contribution of the first quasinormal mode, } \tilde{\rho}_1 + \tilde{\rho}_{-1} . \quad \text{The dotted line is the result of subtraction of that contribution from the spectral function.}
\end{equation}
in the denominator. Note also that the denominator in Eq. (12) is independent of $z$: $\partial_z (he^B [u,v]) = 0$.

The zeros of the Wronskian $[u,v]$ as a function of $\omega$ are the poles of the Green’s function (12). The corresponding complex value of $\omega$ is the quasinormal frequency, $\omega_n$. The residue associated with such a pole is given by

$$\lim_{\omega \to \omega_n} (\omega - \omega_n) G(\omega; z, z') = \frac{u(\omega, z') v(\omega, z)}{he^B \partial_\omega [u,v]} \bigg|_{\omega=\omega_n}. \quad (45)$$

Since $he^B \partial_\omega [u,v]$ is independent of $z$, we can calculate it by setting $z$ to a convenient value, which happens to be $z = \epsilon$. At that point $u(\omega, \epsilon) = 0$ and $\partial_\omega u(\omega, \epsilon) = 0$. Furthermore, since at $\omega = \omega_n$ the function $v$ is a constant multiple of $u$, also $v(\omega_n, \epsilon) = 0$. This means,

$$he^B \partial_\omega [u,v] = he^B [u, \partial_\omega v] \bigg|_{z=\epsilon} \quad (at \ \omega = \omega_n). \quad (46)$$

Differentiating Eq. (23) obeyed by $v$ with respect to $\omega$, we obtain the equation for $\partial_\omega v$:

$$\partial_\omega \left( he^B \partial_\omega [\partial_\omega v] \right) + i\omega^2 \delta v e^B (\partial_\omega v) + 2\omega h^{-1} e^B v = 0. \quad (47)$$

We multiply Eq. (47) by $u$, then take Eq. (23) obeyed by $u$ and multiply it by $\partial_\omega v$, and then subtract the two resulting equations to find

$$\partial_\omega (he^B [u, \partial_\omega v]) = -2\omega h^{-1} e^B uv. \quad (48)$$

Integrating Eq. (45) over $z$ from $z = \epsilon$ to $z = z_h(1 - \delta)$ and using Eq. (46) we find (at $\omega = \omega_n$)

$$he^B \partial_\omega [u,v] = 2\omega \int_{\epsilon}^{z_h(1 - \delta)} dx \frac{e^B \partial_\omega [u,v]}{h} e^B uv + he^B [u, \partial_\omega v] \bigg|_{z=z_h(1 - \delta)}. \quad (49)$$

As $\delta \to 0$, we can (provided $- \text{Im} \omega_n < 2\pi T$, see below) replace $v$ in the last term by its asymptotics (33). Thus

$$\partial_\omega v \bigg|_{z \to z_h} \to -\frac{i}{4\pi T} \log(1 - z/z_h) v, \quad (50)$$

and therefore

$$[u, \partial_\omega v] \bigg|_{z=(1-\delta)z_h} \to -\frac{i}{4\delta} \log \delta \frac{\delta}{4}[u, v] + \frac{i}{4\delta} uv. \quad (51)$$

At $\omega = \omega_n$, $u \sim v$ and $[u, v] = 0$. Thus we can write, for $\delta \to 0$, Eq. (49) as

$$he^B \partial_\omega [u,v] = 2\omega \int_{\epsilon}^{(1-\delta)z_h} dx \frac{e^B uv + ie^B v }{h} \bigg|_{z=(1-\delta)z_h}. \quad (52)$$

If we choose the normalization of $u$ and $v$ at $\omega = \omega_n$ in such a way that $u = v = v_n$ and the r.h.s. of Eq. (52) equals $2\omega_n$, we find from Eqs. (45), (41), (30) and (24):

$$r_n = \frac{1}{g^2} \frac{(v_n'(e)/e)^2}{2\omega_n}, \quad (53)$$

where $\omega_n$ is the quasinormal frequency and $v_n$ is the corresponding quasinormal mode normalized as

$$\lim_{\delta \to 0} \left[ \int_{\epsilon}^{(1-\delta)z_h} dz \frac{e^{Bv_n^2}}{h} + \frac{i}{2\omega_n} e^{Bv_n^2} \right]_{z=(1-\delta)z_h} = 1. \quad (54)$$

Note that, at $T = 0$, the normalization condition in Eq. (54) reduces to the normalization in Eq. (7) for the normal mode. The expression for the residue in Eq. (53) also reduces to (9) via Eq. (37).

It is important to realize, however, that $v_n^2$ is involved in Eq. (54), not $|v_n^2|$, and that both terms in Eq. (54) are complex. In particular, that means that complex condition in Eq. (54) fixes both the magnitude and the phase of $v_n$.

One could also note that $\int (dz/h) e^{Bv^2} = \int d\xi \psi^2$, where $\zeta$ and $\psi$ are the Schrödinger variables defined in Eq. (27). Written in terms of $\zeta$ and $\psi$, the normalization condition in Eq. (7) is equivalent to the one introduced by Zeldovich in the study of the quasi-discrete levels in quantum-mechanical decays [78] and used in the theory of quasinormal modes of cavities [79].

Equation (54) has a drawback, which becomes important when $-\text{Im} \omega_n/T$ is sufficiently large. Since $\text{Im} \omega_n < 0$, $v_n \sim (1 - z/z_h)^{-i\omega/(4\pi T)} \to \infty$ as $z \to z_h$. The last term in Eq. (54) serves to cancel the divergence in the integral. However, the remainder is finite only if $-\text{Im} \omega_n < 2\pi T$. This follows from the Frobenius expansion of the solution $v$ near $z = z_h$:

$$v(\omega, (1 - \delta)z_h) \sim \delta^{-\omega/(4\pi T)} (1 + O(\delta)).$$

Substituting into Eq. (54), we find that after cancellation of the divergent terms $\delta^{-\omega/(2\pi T)}$, the remaining terms are of order $\delta^{-\omega/(2\pi T)+1}$ and vanish as $\delta \to 0$ for $-\text{Im} \omega_n < 2\pi T$. However, for $-\text{Im} \omega_n > 2\pi T$, the remaining terms are still divergent as $\delta \to 0$. This drawback can be cured by returning to Eq. (49) and keeping more terms in the asymptotics of $\partial_\omega v$ in Eq. (54). In practice, we do use Eq. (49), and calculate the last term on the r.h.s. using the truncated Frobenius expansion (which we have to generate anyway in order to obtain the solution of the mode equation near $z_h$, as explained in Section VII).

IX. RESULTS OF QUASINORMAL MODE ANALYSIS

The analysis of quasinormal modes allows us to study the fate of the $J'/\psi$ more quantitatively. Rather than fitting the spectral function $\rho$, as done in Ref. [80], we can read off the peak’s height and width directly from the values of the residue and the imaginary part of the quasinormal frequency obtained by solving Eq. (23), with b.c. (35) and using Eq. (30) and (54). We define the peak height $H_n$ as

$$H_n \equiv \bar{\rho}_n(\omega) \bigg|_{\omega=\text{Re} \omega_n} = \frac{\text{Re} \bar{r}_n}{\text{Im} \omega_n}, \quad (55)$$
where $\bar{\rho}_n(\omega)$ and $\bar{r}_n$ are given by Eq. (35). The width of the peak can be defined, as usual, as

$$\Gamma_n \equiv -\text{Im} \, \omega_n \, .$$

(56)

It is instructive to compare our results with those obtained by applying the same quasinormal mode analysis to the scaled soft-wall model [59].

The dependence of the $J/\psi$ peak height $H_1$ on $T$ is shown in Fig. 7. Using $H_1$, one can define a useful criterion for identifying the “melting” temperature, which quantifies reasonably well the qualitative perception. We can define the disappearance of the peak as a point at which the height of the peak as defined by Eq. (56) becomes smaller than the asymptotic value of the background, i.e., $\bar{\rho}(\infty) = 1$.

According to this criterion, the peak survives until temperatures about 540 MeV, i.e., about 2.8 $T_c$. In contrast, in the rescaled soft-wall model of Ref. [59] the peak height drops quickly and disappears at about 230 MeV, or 1.2 $T_c$. The peak is more robust in our model because of the delta function in the holographic potential $U$ in Eq. (17), which can additionally be checked by reducing $\alpha$ to 0.

The dependence of the $J/\psi$ peak width $\Gamma_1$ (in units of $2\pi T$) is shown in Fig. 8. Interestingly, the peak begins to broaden somewhat earlier in our model ($T \sim 100$ MeV vs $T \sim 150$ MeV). This can be understood as the consequence of the fact that our potential has a “softer wall” at large $z$, and thus is sensitive to lower temperatures. However, even after the soft wall is “melted” at about 200 MeV, the dip at small $z_d$ continues to support the $J/\psi$ peak. On the contrary, in the rescaled soft model Ref. [59], once the wall has “melted”, no feature remains in the potential to support the narrow quasi-discrete $J/\psi$ state.

It is also interesting to note that for $T \to \infty$ the width $\Gamma_1/(2\pi T) \to 1$, which corresponds to the value of the $n = 1$ quasinormal frequency, $\omega_n = 2\pi T n(\pm 1 - i)$ (see, e.g., Ref. [76]) in the AdS black-hole background. This agrees with the expectation that at asymptotically large temperatures the theory should appear almost conformal.

X. SUMMARY, DISCUSSION AND OUTLOOK

In this paper, we attempted to build a holographic model of charmonium using the “bottom-up” approach. As was done in Ref. [10], we required the matching of the UV behavior of the charm current correlator to that of QCD, which is known due to asymptotic freedom. This fixed the small $z$ behavior of the holographic potential. Similarly, the large $z$ behavior was fixed using the condition that the meson spectrum is asymptotically equidistant in mass squared, as in the soft wall model of Ref. [12].

We then observed that, in QCD, the asymptotic spacing of levels is controlled by a parameter, $\Lambda_{QCD}$, theoretically independent from the mass of the ground state, $m_{1/\psi}$. We incorporated this feature by a shift in the soft wall model holographic potential. At this point our model became different from that of Ref. [59], where both scales were controlled by one parameter.

We then required that not only the spectrum of the meson masses is matched to QCD phenomenology, but also the values of the decay constants. This requirement led us to introduce a “dip” in the holographic potential. We modeled this feature minimalistically, by a delta function.

A possible way one could anticipate such a feature from a putative “top-down” construction is by observing that the existing models describing heavy mesons do indeed have potentials with the characteristic scale of the width of the well (flavor brane extension in the $z$ coordinate) of order of the inverse meson mass $z_d \sim m^{-1}$. From that point of view, the potential we need differs from these models by a slower rising “soft wall” at large $z$, permitting the wave-function to “leak” significantly beyond scales of order inverse meson mass $m^{-1}$, and reaching
up to the confinement scale, such as confinement radius, \( R_{\text{conf}} \sim \Lambda_{QCD}^{-1} \), as illustrated in Fig. 9.

![Graph](image)

**FIG. 9:** A sketch emphasizing the features of the holographic Schrödinger potentials. Left – the “top-down” model potential (see, e.g., Eq. (3.2) in Ref. [51]) has one scale, \( m \), characterizing both the ground state mass and the level spacing. Right – a potential with separate scales for the mass gap, \( m \), and for the level spacing, \( \Lambda_{QCD} \). Due to slower rising wall of the potential at large \( z \), the holographic wave function is allowed to “leak” beyond the scale of \( m^{-1} \). In this paper the “dip” at small \( z \) is approximated by a negative delta function.

It is easy to see that such a modification of the potential is what accounts for the broadening of the \( J/\psi \) peak below the dissociation temperature in our model. In contrast, the “top-down” models would predict no broadening below dissociation temperature, though this can change if instanton-like tunneling effects are considered [80].

Having thus constructed a potential with 4 free parameters, we fit those parameters to the 2 masses and 2 decay constants of \( J/\psi \) and \( \psi' \).

Obviously, the “ascetic” potential which we introduced can be further improved by matching more than just the 2 states in the charmonium spectrum. The problem of reconstructing the potential from the spectral data is a standard inverse problem. The fact that the spectrum together with the derivatives of the eigenfunctions (initial velocities, or normalizing constants) uniquely specifies the potential for a Dirichlet problem [51], suggests that matching decay constants is a natural part of such “bottom-up” approach.

An interesting related observation is that the same spectral data can be obtained, in principle, from a non-relativistic model of quarkonium. It would be interesting to know if a more direct relationship exists between the holographic potential and the potential in the non-relativistic Schrödinger description of quarkonia in the heavy quark limit.

It might be also interesting to consider the constraints on the potential arising from the operator product expansion of the current correlator [65]. By quark-hadron duality these constraints can substitute the spectral data. It would be interesting to explore the finite temperature consequences of these constraints and compare to the studies of Ref. [82 84].

Having specified the model at zero temperature, we consider the evolution of the spectral function of the charm current with temperature. Here too, we make a minimalistic assumption about the temperature dependence of the background, similar to the previous study of Ref. [54]. The necessity for the assumption comes from the fact that we do not model the dynamics of the background itself. This can be remedied in a more comprehensive model which we leave to further work. We believe that the minimalistic assumption about \( z \)-dependence of the temperature factor \( h \) and the scale factor \( e^{B} \) is sufficient to make semiquantitative predictions at the level of precision generally expected from such a model approach.

One of the new tools we introduce to study properties of the finite temperature Green’s function is based on the relationship between the residue in the pole and the holographic wave function, which we derive, Eqs. (53), (54). This relationship is a nontrivial generalization of the zero temperature expression for the decay constants [40]. This allows us to study not only the position of the pole in the complex plane, corresponding to the quasinormal mode, but also the strength of the contribution of this pole to the spectral function. In particular, this allows us to devise a simple quantitative criterion for the disappearance of the peak based on its height relative to the asymptotic level of the background. Since the dissociation of quarkonium and disappearance of the peak is not a sharp transition, a criterion of this type might find its use in comparing results of different approaches in a more uniform and objective way.

Finally, we find that the new features of the holographic potential, such as the “dip”, which we introduce to model quarkonium more realistically, strengthen the robustness of the \( J/\psi \) peak, allowing it to persist out to temperatures of about 540 MeV, i.e., 2.8 \( T_{c} \), according to our criterion. This is in good agreement with lattice studies, but not with most non-relativistic Schrödinger models of the quarkonium or the holographic model of Ref. [59], which typically predict dissociation at about 1.2 \( T_{c} \). The comparison with the lattice results can be made even more direct by calculating the Euclidean correlators of the current in the holographic model. We defer this to future work.

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