ON THE CLOSE INTERACTION BETWEEN ALGORITHMIC RANDOMNESS AND CONSTRUCTIVE/COMPUTABLE MEASURE THEORY

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Abstract. This is a survey of constructive and computable measure theory with an emphasis on the close connections with algorithmic randomness. We give a brief history of constructive measure theory from Brouwer to the present, emphasizing how Schnorr randomness is the randomness notion implicit in the work of Brouwer, Bishop, Demuth, and others. We survey a number of recent results showing that classical almost everywhere convergence theorems can be used to characterize many of the common randomness notions including Schnorr randomness, computable randomness, and Martin-Löf randomness. Last, we go into more detail about computable measure theory, showing how all the major approaches are basically equivalent (even though the definitions can vary greatly).

1. Introduction

Starting with the work of Turing in 1936 on the computability of real numbers, it has been understood that many of the basic concepts of analysis — e.g. continuous functions, metric spaces, and open sets — have computable analogues. This “computable interpretation” of analysis has been developed through many interrelated mathematical traditions, including the Russian and American constructivist traditions, computable analysis, and reverse mathematics.

One sub-branch of analysis, measure theory, has presented one of the largest challenges to this program, as the American constructivist Bishop observed.

Any constructive approach to mathematics will find a crucial test in the ability to assimilate the intricate body of mathematical thought called measure theory. [...] It was recognized by Lebesgue, Borel, and other pioneers in abstract function theory that the mathematics they were creating relied, in a way almost unique at the time, on set-theoretic methods, leading to results whose constructive content was problematical. [Bis67 p. 154]

In the 1960s, Bishop [Bis67] — in addition to the Russian school of constructivists Šanin [San68], Kosovsky [Kos69a, Kos69b, Kos69c, Kos69d], and Demuth [Dem65, Dem67, Dem67a, Dem67b, Dem67c, Dem67d, Dem69a, Dem69b, Dem69c, Dem69d, Dem70, Dem73] — overcame these difficulties to develop constructive theories of measurable sets, measurable functions, integrable functions, null sets, and almost everywhere convergence (drawing on earlier work of Brouwer [Bro19]). Their work was later incorporated into computable analysis and reverse mathematics.
Also in the 1960s, Martin-Löf [ML66] developed his own notion of constructive null set, providing one of the most successful definitions of randomness. Namely, a point is (Martin-Löf) random if it is not in any (Martin-Löf) constructive null set. Around 1970, Schnorr [Sch70a, Sch71a] felt that Martin-Löf’s notion of constructive null set was too inclusive. He developed two other randomness notions, now known as Schnorr randomness and computable randomness, each having their own corresponding notion of constructive null set.

This article will show there is a deep connection between computable measure theory and algorithmic randomness. At the heart of this discussion is the notion of an effective (or constructive) null set.

After a short introduction to algorithmic randomness in Section 2, we will give a survey of constructive measure theory in Section 3. The purpose of this survey is twofold: to highlight common approaches to measure theory among constructivists such as Brouwer, Demuth, Bishop, Martin-Löf and others, and to show that algorithmic randomness naturally arises out of these approaches. This will provide motivation for some of the more technical results in the rest of the paper.

In Section 4, we survey a number of recent results characterizing Schnorr randomness, computable randomness, and Martin-Löf randomness using theorems from classical analysis. For example, we will see that a real \( x \in [0,1] \) is Martin-Löf random if and only if \( f \) is differentiable at \( x \) for every computable function \( f: [0,1] \to \mathbb{R} \) of bounded variation. Theorems of this type provide a useful-but-informal measure of the “naturalness” of a randomness notion. We also show how the characterization results are connected to results in constructive analysis and reverse mathematics. For example, the results that are constructively provable (or provable in \( \text{RCA}_0 \)) are those most connected to Schnorr randomness.

Lastly, in Section 5, we turn to the foundations of computable measure theory. We systematically organize the various definitions in the computable and constructive mathematics literature of effectively measurable set, measurable function, integrable function, and almost uniform convergence. Although there is a number of definitions of these notions, they are basically equivalent. Once again randomness arises naturally.

Some [Das11, Del11] have argued that Martin-Löf randomness is the correct randomness notion — just as Church-Turing computability is the correct computability notion. Others, have argued the same for other different randomness notions. Porter [Por16], on the other hand, has argued against any one correct randomness notion. This survey, especially Section 4, supports this latter viewpoint. A variety of randomness notions have been naturally characterized by a.e. convergence theorems in analysis.

Nonetheless, there is one randomness notion that stands out in this survey, especially given its limited treatment in the literature. We will repeatedly see that Schnorr randomness, while much weaker than Martin-Löf randomness, has very strong connections to constructive and computable measure theory.

We hope this paper serves as a talking point between those from the constructive analysis, the computable analysis, and the algorithmic randomness communities. We also hope that others, who may not be interested in randomness for its own sake, will still find this survey to be a good starting point to learn about past and recent developments in constructive and computable measure theory.
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2. A Quick Introduction to Effective Null Sets and Algorithmic Randomness

Before getting into constructive measure theory in more depth, let us introduce the concept of an effective null set. This notion is at the heart of computable and constructive measure theory (especially from a point-set view), and it is the starting point of algorithmic randomness.

2.1. Computable analysis on \( \mathbb{R} \). We assume the reader has some basic understanding of what it means for a function \( f : \mathbb{N} \to \mathbb{N} \) to be computable. See, for example, Coo04, Son16, Od189, Od99. A real number \( r \) is computable if there is a computable function \( f : \mathbb{N} \to \mathbb{Q} \) such that \( |f(n) - r| \leq 2^{-n} \) for all \( n \in \mathbb{N} \). We will denote the set of computable reals as \( \mathbb{R}_{\text{comp}} \).

An effectively open set \( U \subseteq [0, 1] \) is a set of the form \( \bigcup_n I_n \) where \( (I_n)_{n \in \mathbb{N}} \) is a computable listing of open intervals with rational endpoints. (Under the usual topology of \([0, 1]\), the interval \([0, 1/2]\) is an open interval since it is the intersection of the open interval \((-1/2, 1/2)\) and \([0, 1]\).) An effectively closed set is the complement of an effectively open set. If \( D \subseteq [0, 1] \), a computable function \( f : D \to \mathbb{R} \) is a function such that for every effectively open set \( U \subseteq [0, 1] \), we can (uniformly in the code for \( U \)) compute an effectively open set \( V \) such that \( V \cap D = f^{-1}(U) \cap D \). (This is one of many equivalent definitions.) If \( f : \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}} \) is computable, then we say that \( f \) is Markov computable.

These definitions also extend naturally to Cantor space \( \{0, 1\}^\mathbb{N} \), the space of infinite binary sequences. Let \( \{0, 1\}^{<\mathbb{N}} \) denote the space of finite binary sequences. Instead of rationals, use sequences containing finitely many 1s. Instead of rational intervals, use cylinder sets \( \{\sigma\} \) which is the set of all \( x \in \{0, 1\}^\mathbb{N} \) of which \( \sigma \in \{0, 1\}^{<\mathbb{N}} \) is a prefix. For more background on computable analysis, see BW99, BC06, Grz57, Lac55a, Lac55b, PER89, Wei00.

2.2. Martin-Löf randomness and Schnorr randomness. For now, let \( \mu \) be the usual Lebesgue measure on \([0, 1]\) or the fair-coin measure on \( \{0, 1\}^\mathbb{N} \) given by \( \mu([\sigma]) = 2^{-|\sigma|} \). While there are many notions of algorithmic randomness and effective null set, we start with the most important two. The first is due to Martin-Löf [ML66].

**Definition 1.** A Martin-Löf test is a computable sequence of effectively open sets \( U_n \) such that \( \mu(U_n) \leq 2^{-n} \) for all \( n \in \mathbb{N} \). A Martin-Löf null set \( E \) is any set covered by this test, that is \( E \subseteq \bigcap_n U_n \). A point \( x \) is called Martin-Löf random if it is not in any Martin-Löf null set.

The second definition of effective null set has roots in the constructive measure theory of Brouwer, but was first introduced in a computability theory setting by Schnorr [Sch70a, Sch71a].

**Definition 2.** A Schnorr test is a computable sequence of effectively open sets \( U_n \) such that \( \mu(U_n) \leq 2^{-n} \) for all \( n \in \mathbb{N} \) and \( \mu(U_n) \) is computable uniformly in \( n \). A
Schnorr null set $E$ is any set covered by this test, that is $E \subseteq \bigcap_n U_n$. A point $x$ is called Schnorr random if it is not in any Schnorr null set.

Both of these definitions are effectivizations of outer regularity, the result that any null set can be covered by an arbitrarily small open set.

By definition, every Schnorr null set is a Martin-Löf null set. Therefore, every Martin-Löf random is Schnorr random. Moreover, every computable real number $r$ is covered by a Schnorr null test. For example, $1/2$ is covered by the Schnorr test $U_n = (1/2 - 2^{-(n+1)}, 1/2 + 2^{-(n+1)})$. Therefore, no computable real is Schnorr random or Martin-Löf random. However, consider the set $\mathbb{R}_{\text{comp}} \cap [0,1]$ of all computable reals in the unit interval. This is where Schnorr null sets and Martin-Löf null sets differ.

**Proposition 3** (Martin-Löf [ML66]). There is a universal Martin-Löf null set which contains all other Martin-Löf null sets. Therefore, $\mathbb{R}_{\text{comp}} \cap [0,1]$ is a Martin-Löf null set.

**Proposition 4** (Schnorr [Sch71b, Sch71a]). Given (a code for) a Schnorr null set $E$, one can compute (uniformly in the code) a computable point $x \notin E$. Therefore, $\mathbb{R}_{\text{comp}} \cap [0,1]$ is not a Schnorr null set.

This distinction has led many to assume Martin-Löf randomness is more natural but note that Proposition 4 is an effectivization of what is arguably the most fundamental principle in point-set measure theory.

**Proposition 5.** Any property which holds almost everywhere, holds somewhere.

To see the connection, say that a property $P$ holds effectively almost everywhere (in the sense of Schnorr) if the set of points not satisfying $P$ form a Schnorr null set. Proposition 4 says for every effectively such property $P$ we can effectively compute some $x$ for which the property $P$ holds.

2.3. **Other algorithmic randomness notions.** Besides Schnorr and Martin-Löf randomness, there is a whole zoo of randomness notions. We will need some of them at certain points, and we list them here for reference. For more information the reader is directed to the survey [DHNT06] or the books by Downey-Hirschfeldt [DH10] and Nies [Nie09]. (The reader may wish to skip this subsection and refer back to it as needed.)

The third most important randomness concept we will need is computable randomness. Also defined by Schnorr [Sch71a], it arises naturally in certain convergence theorems in analysis. We use an equivalent definition due to Merkle, Mihailović, and Slaman [MMS06]. A computable probability measure on $[0,1]$ is a Borel probability measure $\nu$ on $[0,1]$ such that $p \mapsto \int_0^1 p(x) \, d\mu(x)$ is a computable map from polynomials $p$ with rational coefficients to their integrals. (For Cantor space, $\{0,1\}^\mathbb{N}$, a computable probability measure is a Borel probability measure on $\{0,1\}^\mathbb{N}$ for which

1“Despite Schnorr’s critique, [Martin-Löf randomness] has remained the paradigmatic notion of algorithmic randomness, and has received considerably more attention than Schnorr randomness. One reason may simply be that Martin-Löf’s definition came first, and is perfectly adequate for many results. Another important reason, however, is that the mathematical theory of Schnorr randomness is not as well behaved as that of [Martin-Löf randomness]. For example, the existence of universal Martin-Löf tests (and corresponding universal objects such as universal c.e. martingales and prefix-free complexity) is a powerful tool in the study of [Martin-Löf randomness] that is not available in the case of Schnorr randomness.” [DH10] §7.1.2
\( \sigma \mapsto \nu([\sigma]) \) is computable for \( \sigma \in \{0,1\}^\mathbb{N} \). See Subsection 5.3 for a uniform definition.

**Definition 6.** A bounded Martin-Löf test is a computable sequence of effectively open sets \( U_n \) such that there is a computable probability measure \( \nu \) for which 
\[
\mu(U_n \cap A) \leq 2^{-n} \nu(U_n \cap A) \quad \text{for any measurable set } A.
\] 
(It suffices that \( A \) ranges over rational intervals \([a,b]\) for \([0,1]\) and cylinder sets \([\sigma]\) for \(\{0,1\}^\mathbb{N}\).) A computably null set \( E \) is any set covered by this test, that is \( E \subseteq \bigcap_n U_n \). A point \( x \) is called computably random if it is not in any computably null set.

The remainder of the randomness notions are defined via complexity of sets. The \( \Pi^0_n \) and \( \Sigma^0_n \) sets are respectively the effectively closed and effectively open sets. A \( \Sigma^0_n \) set, also known as an effective \( F_\sigma \) set, is a computable union of \( \Pi^0_n \) sets. Similarly, a \( \Pi^0_n \) set, also known as an effective \( G_\delta \) set, is a computable intersection of \( \Sigma^0_n \) sets. By recursion, one can define \( \Sigma^0_0 \) and \( \Pi^0_n \) for all \( n \).

**Definition 7.** A weak \( n \)-null set is any subset of a null \( \Sigma^0_{n+1} \) set. A point \( x \) is called weak \( n \)-random if it is not in any weak \( n \)-null set (or equivalently is not in any null \( \Pi^0_n \) set). Weak 1-randomness is known as Kurtz randomness.

Many do not consider Kurtz randomness to be a true randomness notion. One reason is that there is a Kurtz random real \( x \in [0,1] \) whose binary digits \( (x_n) \) do not satisfy the strong law of large numbers, \( \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} x_k = \frac{1}{2} \) [Nie09, 3.5.3, 3.5.4].

**Definition 8.** An \( n \)-Martin-Löf test is a computable sequence of \( \Sigma^0_n \) sets \( A_n \) such that \( \mu(A_n) \leq 2^{-n} \). An \( n \)-Martin-Löf null set \( E \) is any set covered by this test, that is \( E \subseteq \bigcap_n A_n \). A point \( x \) is called \( n \)-random if it is not in any \( n \)-Martin-Löf null set.

Notice that 1-randomness is Martin-Löf randomness. It also turns out that 2-randomness is equivalent to Martin-Löf randomness relative to the halting problem \( \emptyset' \), and \( n \)-randomness is equivalent to Martin-Löf randomness relative to \( \emptyset^{(n-1)} \).

In summary, the randomness notions are as follows listed in order of strength (the weakest notions, which give rise to the largest set of randomness, are listed first): Kurtz random, Schnorr random, computable random, Martin-Löf random, weak \( n \)-random \( (n \geq 2) \), \( n \)-random, weak \( (n+1) \)-random, ...

## 3. Randomness and Constructive Mathematics

Constructive mathematics arose out of the desire to ensure that proofs have computational meaning. While the early constructivist work of Brouwer and others pre-dates Turing’s work, it is largely recognized that constructivism has a computational interpretation (the Brouwer-Heyting-Kolmogorov interpretation). A constructive proof of “there exists a function \( f \ldots \),” provides a construction of a computable function \( f \).

A consequence of this computable interpretation is that constructive mathematics is consistent with Church’s thesis: all functions are computable functions, and in particular, all reals are computable reals. Nonetheless, it is still constructively

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2This computable interpretation can be formalized via realizability or Hyland’s effective topos.

3Church’s thesis in constructive mathematics is stronger than the similarly named Church-Turing thesis (also called Church’s thesis), which only says that all intuitively computable functions are computable in the sense of Church and Turing. See the discussion in Beeson [Bees05, III.8]. (Although our version is closer to what Beeson calls the False Church’s thesis.)
provable, using Cantor’s diagonalization argument, that the set of real numbers is not countable. On the other hand, it is more subtle to constructively prove that the unit interval is not a null set. This comes down to the definition of a null set. Classically, a set \( A \subseteq [0,1] \) is null if for any \( \varepsilon > 0 \), the set \( A \) can be covered by a sequence of intervals \( (I_n)_{n \in \mathbb{N}} \) such that the sum of the lengths of the intervals \( \sum_{n \in \mathbb{N}} |I_n| \) is less than \( \varepsilon \). Under the computable interpretation, this covering corresponds to a Martin-Löf test. However, Kreisel and Lacombe [KL57] and Zaslavskiš and Ceštin [ZC62] explicitly constructed coverings of the computable reals which have arbitrarily small size. Zaslavskiš and Ceštin call these singular coverings.

Therefore, one quickly runs into the following paradox of singular coverings:

**Theorem 9** (Paradox of singular coverings, first version). The following set of statements is constructively inconsistent for any definition of “null set.”

1. The set of computable reals \( \mathbb{R}_{comp} \cap [0,1] \) is a null set.
2. (Church’s thesis) All reals are computable. (Hence \( [0,1] \subseteq \mathbb{R}_{comp} \).
3. If \( A \subseteq B \) and \( B \) is null, then so is \( A \).
4. The unit interval \( [0,1] \) is not null.

For, (1)–(3) imply the negation of (4). Statements (3) and (4) are basic facts of measure theory that one needs to develop a consistent notion of measurable set and measure. That means in order to develop measure theory, we need to reject (1) or (2). Some, for example Martin-Löf, have used this argument to reject Church’s thesis. The negation of Church’s thesis, not all reals are computable, does not actually imply (constructively) that there is a noncomputable real. For example, Brouwer’s intuitionism—in particular his fan principle—is incompatible with Church’s thesis, but still compatible with weak Church’s thesis: there does not exist a nonconstructive real.

Nonetheless, there are still issues with adopting the above “covering” definition of a null set.

**Theorem 10** (Paradox of singular coverings, second version). The following set of statements is constructively inconsistent for any definition of “null set.”

1. The unit interval \( [0,1] \) is a measurable set with measure one.
2. The set of computable reals \( \mathbb{R}_{comp} \cap [0,1] \) is a null set.
3. If \( A \) has positive measure and \( B \) is null then \( A \setminus B \) has positive measure.
4. Every measure one set contains a point.
5. (Weak Church’s thesis) There does not exist a noncomputable real.

For, (1)–(4) imply the existence of a noncomputable real, contradicting (5). Again, (1), (3), and (4) are basic properties of measure theory that would be nice to have in any constructive development of point-set measure theory. Again, one is left with the choice of denying weak Church’s thesis or using a different definition of null set in which one can’t constructively prove that the real numbers are null. Most

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4To say that \( [0,1] \) is not countable is to say there does not exist an enumeration \( \{r_n\}_{n \in \mathbb{N}} \) of \( [0,1] \). Under the computable interpretation this is saying that there is no computable enumeration of \( \mathbb{R}_{comp} \cap [0,1] \).

5That is, to constructively prove that a specific set \( A \) is null, we would for each (code of) \( \varepsilon > 0 \), explicitly construct a cover \( (I_n)_{n \in \mathbb{N}} \) such that \( \sum_{n \in \mathbb{N}} |I_n| \leq \varepsilon \). Letting \( \varepsilon = 2^{-k} \), we have that \( U_k = \bigcup_n I_n \) is an effectively open set uniformly in \( k \) and that \( \mu(U_k) \leq \varepsilon = 2^{-k} \) for all \( k \).

6See Beeson [Bee05] for a more in-depth discussion on this paradox, including a work-around not mentioned here.
constructivists, starting with Brouwer, opted to go with the latter, defining null sets via regular coverings, that is coverings where $\sum_n |I_n|$ constructively exists.  

Regular coverings, under the computable interpretation, correspond to Schnorr tests. Indeed, Schnorr \cite{Sch70a, Sch71b} referred to his null sets as “total recursive null sets in the sense of Brouwer”.

What follows is a short survey on constructive measure theory and related subjects, emphasizing the deep connections with effective null sets and, in some cases, algorithmic randomness.

3.1. Brouwerian intuitionism. In 1919, Brouwer \cite{Bro19} developed a constructive measure theory on the unit square. (See the presentation in Heyting’s book \cite[Ch. VI]{Hey56}.) In Brouwer’s measure theory, a set is null if it is enclosed in a measurable open set of arbitrarily small measure. Here a measurable open set is an open set in which the measure constructively exists, and arbitrarily small means that given a natural number $n$, one can construct a measurable open set enclosing $A$ with measure less than $2^{-n}$. In the computable interpretation, a measurable open set corresponds to an effectively open set of computable measure, and therefore the Brouwerian null sets correspond to Schnorr null sets.

Further, in Brouwer’s measure theory, a set $Q$ is measurable if for each $n$, there is a measurable open set $U_n$ of measure less than $2^{-n}$ and a finite union of rational rectangles $V_n$ such that $Q = V_n$ outside of $U_n$ (that is $Q \Delta V_n \subseteq U_n$ where $\Delta$ is symmetric difference) \cite[p. 29]{Bro19}. Then $\mu(Q)$ is defined as $\lim_n \mu(V_n)$, where the measure $\mu(V_n)$ is the geometric area of $V_n$. Brouwer gave definitions of measurable functions and integrable functions as well. In general, Brouwer’s approach is the one followed by many later constructivists, insofar as their approaches are equivalent.

Brouwer and his students developed a large amount of measure theory constructively, including fundamental results about measurable functions and sets, the monotone convergence theorem, the dominated convergence theorem, and Egoroff’s theorem \cite[Ch. VI]{Hey56}. However, it should be noted that Brouwer’s intuitionism is incompatible with classical logic. For example, it is a Brouwerian theorem that every function on the unit interval is uniformly continuous. As a corollary, every bounded function defined almost everywhere is measurable \cite[§§6.2.2, Thm. 1]{Hey56}. Also, Brouwer adopted the fan principle, which later constructivists deemed non-constructive. Using this theorem, one can prove the dominated convergence theorem and Egoroff’s theorem \cite[§6.5.4]{Hey56}. The latter says that (on a probability space) a.e. convergence implies almost uniform convergence.

3.2. The Russian school of constructive mathematics. The Russian school of constructive mathematics — led by Markov and his students Shanin, Zaslavskiĭ, and Ceitin — combined the ideas of Turing and Brouwer. In particular Church’s thesis

\footnote{In constructive mathematics, one cannot in general prove that a bounded monotone sequence converges. There are examples of bounded monotone computable sequences whose limit is not computable.}

\footnote{One slight difference with later constructivists is that in Brouwer’s measure theory, a measurable function need not be defined on a set of full measure. In this case the function is assumed to be zero on almost all of those undefined points. However, it is shown that such partial functions can be extended to a full domain \cite[§6.2.2]{Hey56}. In that case, Brouwer’s definition is compatible with the later constructivists.}
— that every function is (Markov) computable — was explicitly assumed. Therefore, Russian recursive constructivism is very similar to modern computable analysis (except that the Russian constructivists avoided most nonconstructive principles such as the law of the excluded middle\(^9\) and avoided reference to non-computable object.) See the surveys [Kus99, DK79] and the books [Kus84, BR87] for more on Russian constructive mathematics.

In 1962, Šanin wrote a book on constructive analysis, emphasizing constructive metric spaces, which appeared in English translation in 1968 [San68]. Formally, a constructive metric space is identified with a metric \(\rho\) on the natural numbers, and the constructive points in this constructive metric space are identified with constructive sequences \((n_k)\) of natural numbers such that \(\rho(n_k, n_\ell) \leq 2^{-k}\) for all \(k \leq \ell\). The idea is to encode a metric on a countable set, e.g. the Euclidean distance on \(\mathbb{Q}\), and the constructive metric space is the completion of this metric, e.g. \(\mathbb{R}\). (However, by Šanin’s use of Church’s thesis, this constructive completion only consists of computable points.)

Šanin used computable metric spaces to give constructive definitions of measurable sets, measurable functions, and integrable functions. For example, consider the \(L^1\)-metric \(\rho(f, g) = \int_0^1 |f(x) - g(x)|\, dx\) on rational step functions. This describes a constructive metric space, and the corresponding constructive points are the constructive integrable functions — the integrable FR-constructs in Šanin’s terminology. Similarly, Šanin defined measurable sets and measurable functions in a similar manner (see Subsection 5.3). Kosovski [Kos69a, Kos69b, Kos70, Kos73a, Kos73b] further extended Šanin’s work to probability theory, proving constructive versions of the strong law of large numbers, developing a theory of constructive stochastic processes, and extending Šanin’s ideas to arbitrary spaces given by normed Boolean algebras of sets.

Šanin’s and Kosovskiĭ’s approach is different from Brouwer’s in that it is point-free. Each integrable FR-construct is not a true function, but instead a point in a metric space of function-like objects. (Recall that, classically, the metric space \(L^1([0, 1])\) is the space of equivalence classes of integrable functions modulo a.e. equivalence.) Unlike Brouwer’s integrable functions, the statement \(f(0) = 1\) is not meaningful for an integrable FR-construct \(f\). We will return to this point-free theme in Subsections 3.8 and 5.5.

Also in 1962, Zaslavskiĭ and Čeitin [ZC62] wrote about the singular coverings mentioned at the beginning of this section. While their focus was on the pathological case of singular coverings, they added the following note.

We call a covering \(\Phi\) regular if the sequence of numbers \(\sum_{k=0}^n |\Phi_k|\) is constructively convergent as \(n \to \infty\). The set \(\mathcal{M}\) of [constructive real numbers] will be said to be a set of measure zero if for arbitrary \(\varepsilon\) there can be realized a regular \(\varepsilon\)-bounded covering by intervals of the set. [...] Consequently, in spite of the existence of constructive singular coverings, it is possible to give a reasonable definition of the constructive concept of a set of measure zero. Other concepts of the constructive theory of measure can be defined in a similar way. [ZC62, p. 58 in English translation] (Emphasis in original.)

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\(^9\)They did however adopt Markov’s principle, which states that for each binary sequence \((a_n)\), if every no term \(a_n\) equals 0, then there exists a term equal to 1. This is a weak form of the law of excluded middle.
While Zaslavskii and Cȩi̧tin do not define such “other concepts”, Demuth \cite{Dem65,Dem67a,Dem67b,Dem68a,Dem68b,Dem69a,Dem69c,Dem70,Dem73} does take up this work, giving constructive definitions of integrable functions and measurable sets. (A detailed survey of Demuth’s work on constructive measure theory can be found in Demuth and Kučera \cite{DK79}. Also see the surveys by Slaman and Kučera \cite{KS01, Bnk} and Kučera, Nies, and Porter \cite{KNP}.) Demuth’s work is particularly relevant because he, independently of Martin-Löf and Schnorr, defined the same randomness notions (or at least considered the corresponding null sets). Kučera, Nies, and Porter comment on Demuth’s path to randomness.

Demuth considered a number of different notions of effective null set. They are equivalent to several major randomness notions that have been introduced independently.

It is striking that Demuth never actually referred to random or non-random sequences. Instead, he characterized these classes in terms of non-approximability in measure and approximability in measure, respectively. This reflects the fact that Demuth’s motivation in introducing these classes differed significantly from the motivation of the recognized “fathers” of algorithmic randomness. Whereas the various randomness notions were introduced and developed by Martin-Löf, Kolmogorov, Levin, Schnorr, Chaitin, and others in the context of classical probability, statistics, and information theory, Demuth developed these notions in the context of and for application in constructive analysis, where the notion of approximability plays a central role \cite{KNP} §4.

Demuth’s measure theory takes place entirely on the constructive reals. A property \( P \) of the constructive real numbers is said to hold for almost every constructive real number if (in modern terminology) it holds outside of a Schnorr null set \cite{DK79}. While these “step functions” are only defined \( f: \subseteq \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}} \), we get a set-point interpretation of Šanin’s point-free approach (see the remark in \cite{Dem68a} as follows. A partial function \( f: \subseteq \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}} \) is integrable if there is a computable sequence of rational step functions \( s_n \) such that for all \( n \geq m, \| s_m - s_n \|_{L^1} \leq 2^{-m} \) and \( f(x) = \lim_n s_n(x) \) for almost every \( x \in \mathbb{R}_{\text{comp}} \). The integral \( \int_0^1 f(x) \, dx \) is equal to \( \lim_n \int_0^1 s_n(x) \, dx \) (where the integral of the step function \( s_n \) is defined in the usual way). Demuth similarly defines a measurable function using the metric \( \rho(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx \). A set \( A \subseteq \mathbb{R}_{\text{comp}} \) is measurable if there is an integrable function \( f: \subseteq \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}} \) such that \( 1_A(x) = f(x) \) for almost every \( x \in \mathbb{R}_{\text{comp}} \). Then \( \mu(A) \) is defined as \( \int_0^1 f(x) \, dx \) \cite{DK79} §4.

While Demuth’s measurable sets are restricted to the constructive real numbers, this is just the computable interpretation of constructive mathematics at play. His definitions work equally well on the whole unit interval, and if taken as such, they are constructively equivalent to those of Brouwer.\footnote{When Demuth considers an “integrable function” \( f: \subseteq \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}} \) he is defining \( f \) as a constructive limit of rational “step functions” \( s_n \). While these “step functions” are only defined on \( \mathbb{R}_{\text{comp}} \), they have natural extensions \( \hat{s}_n \) defined on \( [0,1] \). The classical limit \( \lim_n \hat{s}_n \) of these step functions converges almost everywhere to a function \( f: [0,1] \to \mathbb{R} \). Then \( f = f \mid \mathbb{R}_{\text{comp}} \) and the classical integral of \( f \) is the same as Demuth’s “integral” of \( f \). Moreover, a “measurable set” \( A \subseteq \mathbb{R}_{\text{comp}} \) in Demuth’s terminology can be identified with a \( \{0,1\} \)-valued “integrable function” \( f \). By extending \( f \) to its classical counterpart \( \hat{f} \), we get a set \( A = \{ x : f(x) = 1 \} \) such that}
Demuth proved constructive versions of a number of differentiability results in measure theory including the Lebesgue differentiation theorem \[DK79\] Thm. 4.14. Demuth was particularly interested in the differentiability of functions of bounded variation. He showed that for every constructively absolutely continuous function \(f: \mathbb{R}_{\text{comp}} \to \mathbb{R}_{\text{comp}}\), the set of non-differentiable points can be covered by a (not necessarily regular) constructive covering. Translated into a modern perspective, Demuth’s result shows that absolutely continuous Markov computable functions are differentiable at Martin-Löf randoms (cf. Theorem 14). To avoid the paradox of singular coverings, Demuth (slightly) abandoned Church’s thesis, enlarging the constructive interval to contain “pseudo-reals”, that is reals computable in the halting problem, \(\emptyset'\) (see, for instance, [Dem75a] and [Dem75b]).

3.3. Bishop’s constructive mathematics. In 1967, Bishop published a book on constructive mathematics [Bis67], showing that a large amount of mathematical analysis could be proved constructively. A major portion of his work was on measure theory. Whereas Brouwer’s intuitionism and the constructive mathematics of the Russian school allows one to prove nonclassical results (such as all functions are uniformly continuous or all functions are computable) Bishop’s constructivism is compatible with classical mathematics [Bee85]. Therefore, any result proved in Bishop’s book is classically valid, but also constructive — and therefore has a computable interpretation.

Bishop’s measure theory progressed through a number of revisions. His first development [Bis67, Ch. 6] was for probability measures on locally compact metric spaces. (See Bridges and Demuth [BD91] or Beeson [Bee85 §I.13] for short presentations.) Later Bishop and Cheng [BC72] extended this framework to arbitrary integration spaces via the Daniell integral. (Also see Bishop and Bridges [BB83, Ch. 6].) In both cases, measures are defined via a linear integration functional. We will briefly explain how Bishop’s approach applies to the space \([0, 1]\) with the Lebesgue measure \(\mu\). This measure \(\mu\) can be defined via the Riemann integral \(\int_0^1 f(x) \, dx\) on uniformly continuous functions \(f: [0, 1] \to \mathbb{R}\). An integrable function is a partial function \(f: \subseteq [0, 1] \to \mathbb{R}\) constructed as follows. Take a sequence of uniformly continuous functions \(f_n\) such that \(\sum_n \int_0^1 |f_n(x)| \, dx\) constructively converges. Set the domain of \(f\) to be the set of all \(x \in [0, 1]\) such that \(\sum_n |f_n(x)|\) constructively converges. For such \(x\), set \(f(x) = \sum_n f_n(x)\). A set is full if it contains the domain of some integrable \(f\).

If the sequence \(f_n\) is a computable sequence of uniformly continuous functions, then the corresponding full set \(\{x: \sum_n |f_n(x)|\text{ converges}\}\) is the complement of a Schnorr null set. Conversely, every Schnorr null set is of this form (Theorem 13). Moreover, Bishop’s definitions and theorems largely agree with those of Brouwer.

\[A = \bar{A} \cap \mathbb{R}_{\text{comp}}\text{ for “almost every constructive real” } x \text{ in the sense of Demuth, and Demuth’s “measure” of } A \text{ is the same as the classical measure of } \bar{A}.\]

\[11\text{ Technically, this is a notion of “non-pseudo-differentiability” since Markov computable functions are only defined on constructive reals. See [DK79] or [KNP] for more details.}\]

\[12\text{ Unlike Brouwer, Bishop does not adopt the fan principle. Therefore, he cannot prove Ergorov’s theorem that almost everywhere convergence is the same as almost uniform convergence. Instead his definition of almost everywhere convergence is closer to almost uniform convergence. In particular, his dominated convergence theorem is weaker than Brouwer’s, and therefore weaker than the classical version.}\]
A noteworthy constructive theorem of Bishop is that every measurable set of positive measure contains a point \([\text{Bis67}, \text{Ch. 6, Prop. 2}]\) (compare with Proposition 4).

Bishop-style constructivism continues to receive a lot of attention. There have been a number of results in Bishop-style constructive measure theory and probability theory \([\text{Cha69}, \text{Cha72a}, \text{Cha74b}, \text{Cha74a}, \text{Cha75}, \text{Bri77}, \text{Bri79}, \text{Cha81}]\), including on advanced topics such as ergodic theory \([\text{Bis67}, \text{Bis68}, \text{Nub72}, \text{Spi02}, \text{Spi06a}, \text{Spi06c}]\), stochastic processes \([\text{Cha72a}, \text{Cha76}, \text{Cha81}]\), potential theory \([\text{Cha77}, \text{Cha81}]\), and quantum mechanics \([\text{Hel93}, \text{Hel97}, \text{BS00}]\). It also influenced some of the later Russian constructivists, such as Kreinovich’s work on constructive Wiener measure \([\text{Kre74a, Kre74b}]\).

3.4. Martin Löf’s constructive mathematics. In 1966, Martin-Löf [ML66] introduced his definition of constructive null set and Martin-Löf randomness. Later, he turned his focus to constructive type theory. In 1970, during this transitional period, Martin-Löf wrote a book on constructive analysis [ML70a], including a chapter devoted to measure theory.

His style is similar to that of the Russian school, mentioning computable objects explicitly, but he does not work explicitly in the constructive real numbers. Indeed, Martin-Löf rejects the idea that the continuum is made up only of computable points. He invokes the existence of singular coverings — which is a stronger form of Kreisel and Lacombe’s theorem [KL57] that there is an effective open set not equal to the reals which contains all computable reals. Of this result Martin-Löf writes,

In classical mathematics the continuum is conceived as the totality of its points. One might therefore, like Markov and his school, try to constructivize the continuum by looking upon it as the totality of its constructive points. This leads, as shown by Kreisel and Lacombe’s theorem [KL57] that there is an effective open set not equal to the reals which contains all computable reals. Of this result Martin-Löf writes,

Martin-Löf’s definition of measurable set is as follows.

A Borel set \(A\) is **measurable** if for every computable real number \(\varepsilon > 0\) \([\ldots]\) we can find a simple set \(P\) \([that is, a finite union of disjoint basic open sets]\) and an open set \(U\) such that

\[
A \triangle P \subseteq U
\]

and \(U\) is bounded by \(\varepsilon\) \([that is, \(\mu(Q) \leq \varepsilon\) for every simple set \(Q \subseteq U\)]\). [ML70a, p. 92]

Notice that unlike Brouwer’s definition before, \(\mu(U)\) need not (constructively) exist.

Martin-Löf was aware of the difference.

Secondly, the fact that our definition allows the construction of an inner limit set of measure zero which contains all constructive
points, although troublesome to those whose continuum consists of constructive points only, is in full agreement with the intuitionistic concept of the continuum as a medium of free choice. [ML70a, p. 101]

Last, he ends his defense of his definition of measurable set by referring to his notion of randomness and his theorem that there is a universal Martin-Löf constructive null set.

Thirdly, the definition we have adopted enables us to prove a new theorem which may serve as a justification of the notion of a random sequence conceived by von Mises and elaborated by Wald and Church 1940. [ML70a, p. 101]

3.5. Reverse mathematics. Constructive mathematics gets its computable interpretation from restricting itself to a subset of classical logic. There is, however, another way of doing mathematics, which both has a computational interpretation and uses classical logic. That is RCA₀, a subsystem of second order arithmetic, which forms the basis for the reverse mathematics program of Friedman and Simpson [Sim09b].

While RCA₀ and BISH (Bishop’s constructive system) are similar, there are also key differences. RCA₀ uses classical logic, whereas BISH does not. Conversely, various versions of the axiom of choice hold in BISH which do not in RCA₀. There are also differences in methodology between reverse mathematics and Bishop style constructivism. While a constructivist desires to move much of mathematics under a constructive lens, the goal of reverse mathematics is to determine exactly which set existence axioms (added to RCA₀) are required to prove a theorem of mathematics. It turns out that a large number of theorems in mathematics are equivalent (over RCA₀) to one of the following five systems of reverse mathematics (listed in increasing proof-theoretic strength), RCA₀, WKL₀, ACA₀, ATR₀, and Π₁¹-CA₀. (For an introduction to reverse mathematics, see [Sim09b].)

However, when Yu and Simpson [YS90] looked at the reverse mathematics of measure theory, another system WWKL₀ arose, strictly between RCA₀ and WKL₀. The axiom weak weak König’s lemma (WWKL) states that if T is a subtree of \( \{0,1\}^\mathbb{N} \) with no infinite path, then

\[
\lim_{n \to \infty} \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} = 0.
\]

The system WWKL₀ is RCA₀ + WWKL. Yu and Simpson [Yu87, YS90, Yu90, Yu93] showed that a large amount of measure theory can be developed in the system WWKL₀. Moreover, the axiom WWKL is equivalent over RCA₀ to a number of basic principles of measure theory (see [Sim09b, §X.1]):

- Every closed set of positive measure contains a point.
- Every sequence of intervals \((a_n, b_n)\) covering \([0, 1]\) satisfies \(\sum_{n=0}^{\infty} (b_n - a_n) \geq 1\).
- If \(U, V \subseteq \{0,1\}^\mathbb{N}\) are disjoint open sets such that \(U \cup V = \{0,1\}^\mathbb{N}\) then \(\mu(U) + \mu(V) = 1\).

In short (using the terminology from earlier), WWKL prevents the pathologies of singular coverings. WWKL is also closely related to Martin-Löf randomness. Indeed WWKL is equivalent (over RCA₀) to the existence of a Martin-Löf random relative to each \(x \in \{0,1\}^\mathbb{N}\) [ADR12, Thm 3.1].
The reverse mathematics of measure theory relies on both point-free definitions of integrable functions and sets (using the $L^1$ metric space), as well as pointwise versions. Yu [Yu94], Brown, Giusto, and Simpson [BGS02], Simic [Sim04], and Avigad, Dean, and Rute [ADR12] define the pointwise version of an integrable function $f$ as the pointwise limit of a sequence $(p_n)$ of certain continuous functions which approximate $f$ in the $L^1$-norm. Using WWKL, they show that these $(p_n)$ converge outside of a (relativized) Martin-Löf null set. By the later work of Pathak, Rojas, and Simpson [PRS14] and Rute [Rut13], as well as the constructivists already mentioned, it is likely provable in RCA$_0$ that this convergence happens outside of a (relativized) Schnorr null set. Indeed, it seems that a large amount of measure theory can be developed in RCA$_0$ — including many of the results proved using WWKL in Yu [Yu87, Yu90, Yu94], Brown, Giusto, and Simpson [BGS02], and Simic [Sim04].

Nonetheless, there are a number of theorems not provable in RCA$_0$. For example, over RCA$_0$, both (a certain version of) the monotone convergence theorem [Yu94] and the Vitali covering theorem [BGS02] are equivalent to WWKL. Yu showed that Borel regularity is provable in ATR$_0$ [Yu93], and that many theorems of measure theory are equivalent (over RCA$_0$) to ACA$_0$ [Yu87, Yu90, Yu96]. Simic [Sim04, Sim07] showed that the pointwise ergodic theorem is equivalent to ACA. Avigad and Simic [AS06] showed the same for the mean ergodic theorem. Avigad, Dean, and Rute [ADR12] showed that the following are all equivalent (over RCA$_0$) to an axiom called 2WWKL:

- Egoroff’s theorem
- the Cauchy version of the dominated convergence theorem
- every $G_δ$ set of positive measure contains a point
- collection axiom $BΣ_2$ plus the existence of a 2-random (Definition 8) relative to each $x \in \{0, 1\}^N$.

We also remark that reverse mathematics has inspired a similar program called constructive reverse mathematics which replaces the base theory RCA$_0$ with BISH (or some other suitable constructive base theory). Nemoto [Nem10] has investigated WWKL in constructive reverse mathematics, and Beeson [Bee06] has investigated the constructive strength of the statement the every sequence of intervals $(a_n, b_n)$ covering $[0, 1]$ satisfies $\sum_{n=0}^{\infty} (b_n - a_n) \geq 1$.

3.6. **Computable analysis.** Computable analysis, like constructive analysis, studies the computable content of theorems in mathematical analysis. Unlike constructive mathematics or RCA$_0$, computable analysis does not rely on any restricted framework of logic or mathematics. Instead, it explicitly refers to computable functions, computable reals, etc. Also like constructive analysis, computable analysis
developed in many separate but interrelated traditions (see Avigad and Brattka [AB14] for a historical survey).

Early work combining the measure-theoretic and computability theoretic can be found in Kreisel and Lacombe’s [KL57] result that there is a $\Sigma^0_1$ set of arbitrarily small measure covering all the computable reals, as well as Jockusch and Soare’s [JS72] work showing that the complete extensions of Peano arithmetic have measure zero.

Later Friedman and Ko [KF82, Ko91] studied the polynomial-time complexity of measurable functions and sets, via approximability. Ko [Ko91, Ch. 5] showed that by replacing “polynomial-time computable” with “computable”, the approximable sets and functions are equivalent to the measurable sets and functions of Šanin. Pour-El and Richards [PER89] developed computable analysis on Banach spaces, focusing significantly on $L^p$ spaces, again using a point-free treatment similar to Šanin.

Starting around the turn of the millennium, there have been a large number of papers on computable measure theory. Many of these papers follow the type-2 effectiveness approach [Wei00, BHW08] or the domain theory approach [AJ94]. Most of these papers have been concerned with computable representations of measures or probability distributions [Wei99, Mul99, WW06, SSS06, Sch07, Eda09, HR09a, MTY13, Col]. While most of these representations are equivalent, the generality of the underlying spaces vary. Other papers have been about computable representations of measurable sets, integrable functions, and measurable functions or their properties [WD05, WD06, Eda09, HR09a, HR09c, Bos08, Wu12, WT14, Wei17, Col]. Again, these representations are basically equivalent, but the details are a bit more complicated. As we will see in Section 3, the various representations can be broken up into three categories corresponding to those that are point-free, those that are defined outside of a Martin-Löf null set, and those that are defined outside of a Schnorr null set.

Yet others are interested in computable stochastic processes, including Brownian motion [DF13, FM13, BE17, Col] and Lévy and Feller processes [Ma15].

There have also been a number of papers about the computability of various theorems in measure theory, e.g. the ergodic theorem [AGT10, Hoy13], the Riesz representation theorems [LW08, LW07, JW13], various decomposition theorems [JW14, HRW12], as well as other results [PF17]. Additional works on computable probability theory are motivated by probabilistic programming [FR12, AFR17, Misb, Misa, VKS, AAF+, HMS], and others still, as we will see, are motivated by work in algorithmic randomness.

3.7. Algorithmic randomness. Algorithmic randomness is closely tied to computable analysis, and many researchers have focused on exploring these connections.

In the 1960s and 1970s, Solomonoff, Kolmogorov, Martin-Löf, Levin, Schnorr, Chaitin, and others grappled with the relationship between information theory, probability theory, dynamical systems, and computability. (See Schnorr [Sch77] for a survey of that time period.) Besides the already mentioned characterizations of Martin-Löf and Schnorr randomness via measures and effectively open sets, there are also characterizations of randomness via algorithmic complexity (see [DHNT06, LV08, Nic09, DH10]). This is closely connected to the work on effective Hausdorff dimension by Lutz, Mayordomo, and others [Lut00, Lut03, Lut05, May02, Rei08].
It also led to fruitful research by V'yugin and others connecting algorithmic complexity, entropy, dimension, and ergodic theory [V'y98, Hoc09, Hoy12, Sim15].

While most work in algorithmic randomness has taken place on Cantor space \( \{0,1\}^\mathbb{N} \) or the unit interval with the Lebesgue measure, there have been extensions of the theory to other spaces. Martin-Löf [ML66 §V] considered Martin-Löf randomness for other Bernoulli measures, and Schnorr [Sch71c Ch. 5] did the same for Schnorr randomness. Levin [Lev73, Lev76, Lev84] generalized Martin-Löf randomness to noncomputable probability measures on Cantor space.

Asarin and Prokrovskii [APS86] extended Martin-Löf randomness to Brownian motion, and this work has been taken up by Fouché and others [Fou00a, Fou00b, KH07, Fou08, Fon09, HR09c, KHN08, KHS11, Fou13, FMD14, ABS14]. Hertling and Weihrauch [HW03], Gács [Gács05], and Hoyrup and Rojas [HR09a, HR09d] extended Martin-Löf's and Levin's ideas to other computable metric spaces. Hoyrup and Rojas [HR09a] also realized that the effectively measurable functions and sets of Edalat [Eda09] could be characterized in terms of Martin-Löf randomness. This approach is called layerwise computability, and Hoyrup and Rojas’s ideas have been extended to Schnorr randomness by Pathak, Rojas, and Simpson [PRS14], Miyabe [Miy13], and Rute [Rut13].

In Section 4 we survey more results showing that Schnorr randomness, computable randomness, and Martin-Löf randomness can all be characterized via classical convergence theorems in analysis, and we will highlight the powerful tools which make it easy to translate analytic theorems into results about randomness.

3.8. **Point-free measure theory: measure algebras, locales, forcing, and category theory.** Measure theory is usually presented in a point-set-theoretic manner: One first develops a theory of points, sets, and functions. Then certain sets and functions are deemed to be “measurable”. This is, more or less, the approach of many of the early constructivists, including Brouwer, Demuth, Bishop, and Martin-Löf. In classical practice, one often goes a step further, considering equivalence classes modulo almost everywhere equivalence. For example, let \( \mu \) be a measure on \( \{0,1\}^\mathbb{N} \). Then one has the vector space \( L^0(\mu) \) of measurable functions modulo \( \mu \)-a.e. equivalence, the Banach space \( L^1(\mu) \) of \( \mu \)-integrable functions modulo \( \mu \)-a.e. equivalence, and the complete Boolean algebra of measurable sets modulo \( \mu \)-a.e. equivalence. These spaces are all complete separable metric spaces.

The point-free approach to measure theory proceeds differently. In it, one formally defines “measurable functions” and “measurable sets” directly as objects in the above metric spaces, without explicitly mentioning the underlying functions, sets, and points. The “functions” and “sets” in these spaces are merely formal objects, not actual functions or sets.

Indeed, we already saw that Šanin [San68] and Kosovskii [Kos69a, Kos69b, Kos70, Kos73a, Kos73b] used this approach to reason about a large subset of probability theory. An equivalent approach is given by Coquand and Palmgren [CP02], who construct a space of measurable sets as the metric completion of a countable Boolean ring with a measure on it. Using this approach, they give constructive proofs of Kolmogorov’s 0-1 law, the first Borel-Cantelli lemma, and the strong law of large numbers. Spitters [Spi06a] extended this approach to include integrable and measurable functions.

This all ties in to point-free topology, a field which has close connections to constructive mathematics (see Section 5 of [BPT10]). One type of point-free space,
generalizing topological spaces, is a locale. A locale is given by a partial order which behaves like the partial order of open sets in a topological space under the subset relationship — this partial order has top and bottom elements, is closed under arbitrary joins \( \bigcup \) and finite meets \( \cap \), and satisfies the distributive law \( (\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (U \cap V_i) \). A morphism \( f : X \to Y \) between locales \( X \) and \( Y \) behaves like a continuous function between topological spaces; formally it is given by a map from the “open sets” of \( Y \) to the “open sets” of \( X \) which preserves finite meets, and arbitrary joins. If \( \mu \) is a Borel probability measure on \([0, 1] \), the measurable sets modulo \( \mu \)-a.e. equivalence form a locale, the \( \mu \)-measurable locale. If we denote the \( \mu \)-measurable locale as \( (\{0, 1\}^\mathbb{N}, \mu) \), then the morphisms \( f : (\{0, 1\}^\mathbb{N}, \mu) \to \mathbb{R} \) (where \( \mathbb{R} \) has the standard topology/locale) are exactly the measurable functions modulo \( \mu \)-a.e. equivalence (Notice, that if \( \mu \) is the Lebesgue measure, the \( \mu \)-measurable locale is not homeomorphic to any topological space necessitating the use of point-free methods.)

Not only can one reason about measure theory in the locale of \( \mu \)-measurable sets, but one can also use the measurable locale to give a rigorous formulation of randomness. One can naively view probability theory as the study of random events, whereby a random event is one satisfying every probability one property. While such “random events” do not actually exist, the measurable locale can be viewed as the space of random points.

This ties in closely with set-theoretic forcing. In forcing one has two mathematical universes \( \mathcal{U} \subseteq \mathcal{V} \), the smaller of which is known as the ground model. If one takes a locale \( L \) in the ground model, forcing allows one to construct objects \( g \) in the larger universe, called generics, which behave as if they are “points” in the “space” \( L \). In Solovay forcing [Jec03, Ch. 26], one forces with the \( \mu \)-measurable locale (also known as the measure algebra of \( \mu \)-measurable sets). The resulting generics are known as Solovay randoms. Being a Solovay random is equivalent to being in every \( \mu \)-measure one set in the ground model. We now have an analogy to Schnorr randomness, which is equivalent to being in every constructive \( \mu \)-measure one set. In Subsection 5.10 we strengthen this analogy by giving an effective version of Solovay forcing, where the generics are the Schnorr randoms.

Simpson [Sim12] has proposed another locale as a model for randomness. The locale of random sequences is the locale of open sets of \( \{0, 1\}^\mathbb{N} \) modulo a.e. equivalence. This locale is analogous to Kurtz randomness. (Recall, a point is Kurtz

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14Recall that the Boolean algebra of measure sets modulo a.e. equivalence is complete, and therefore closed under arbitrary joins, not just countable joins.

15While we are not aware of a fully constructive treatment of the \( \mu \)-measurable locale, we note that none of the constructive definitions of measurable set given so far are constructively closed under infinite countable unions. Nonetheless, we suggest as a candidate the locale whose “open sets” are given by the representation \( \delta_k \) in [Wei14, Wei17] of point-free measurable sets computable from below. Computably, this has the closure properties of a \( \sigma \)-locale ([Wei17, Thm. 4.1]) and the computable morphisms \( f : (\{0, 1\}^\mathbb{N}, \mu) \to \mathbb{R} \) are exactly the point-free measurable functions of Šanin and others (see the representation \( \delta_{inf} \) in [Wei17]).

16Assume the locale \( (\{0, 1\}^\mathbb{N}, \mu) \) is homeomorphic to a topological space \( X \). For each measurable set \( B \) of \( (\{0, 1\}^\mathbb{N}, \mu) \), let \( \bar{B} \) be the corresponding open set in \( X \). Consider a point \( x \in X \). For each \( k \), there is exactly one \( \sigma \in \{0, 1\}^k \) such that \( x \in \bar{\sigma} \). Existence follows from \( \bigcup \{ \bar{\sigma} : \sigma \in \{0, 1\}^k \} = \{0, 1\}^\mathbb{N} = X \). Uniqueness follows from \( \bar{\sigma} \cap \bar{\tau} = \emptyset \). Let \( U_k = \bigcup \{ \bar{\sigma} : \sigma \in \{0, 1\}^k, x \notin \bar{\sigma} \} \). Then \( \mu(U_k) = 1 - 2^{-k} \). Since, \( \mu(\bigcup_k U_k) = 1 \), we have \( x \in \bigcup_k \bar{U}_k \) contradicting the definition of \( U_k \).
random if it is in every measure one effectively open set.) Like Kurtz randomness, the locale of random sequences does not always satisfy the strong law of large numbers [Sim09a]. This analogy can also be made formal with forcing.

Locales and forcing are part of a larger categorical framework, including sheaves, toposes, type theory, and other tools important to modern constructive mathematics. There is new work approaching measure theory and probability from this perspective [Jac06, Rod09, Vic11, Sim17, Sim, FS, Cla, nLa18], much of it building on the work of Giry [Gir82]. While this work is in progress, we conjecture that in these settings, questions about randomness will once again naturally arise, both implicitly and explicitly. To the extent that these categorical models are reasoned about constructively or computably, we will again find connections and analogies with algorithmic randomness.

4. Characterizing algorithmic randomness via theorems in analysis

One of the most important characteristics of algorithmic randomness is that it satisfies many of the almost everywhere theorems of mathematics. For example, every Schnorr random (and therefore every Martin-Löf random) satisfies the strong law of large numbers — that is the sequence of binary digits \((x_n)\) of \(x \in \{0, 1\}^\mathbb{N}\) satisfies \(\lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} x_k = \frac{1}{2}\). However, the strong law of large numbers, or even the more advanced law of the iterated logarithm, does not characterize Schnorr randomness. This is simply because one can construct a computable sequence \(x \in \{0, 1\}^\mathbb{N}\) for which both theorems hold [PS12].

However, it turns out that many of the more general theorems in analysis and probability, usually involving a free parameter, do characterize the standard algorithmic randomness notions. These characterization results show that Martin-Löf randomness, computable randomness, and Schnorr randomness are all natural randomness notions. What follows is a survey of some of these results.

4.1. Monotone convergence. A variation of the monotone convergence theorem in measure theory states that given an increasing sequence of continuous nonnegative functions \(g_n : [0, 1] \to [0, \infty]\), if \(\sup_n \int_0^1 g_n(x) dx\) is finite, then \(\sup_n g_n(x) < \infty\) for almost every \(x\). This can be used to characterize Schnorr randomness and Martin-Löf randomness.

Theorem 11 (Levin [Lev76]). The following are equivalent for a real \(x \in [0, 1]\).

1. The real \(x\) is Martin-Löf random.
2. The supremum \(\sup_n g_n(x)\) is finite for every increasing computable sequence of continuous functions \(g_n : [0, 1] \to [0, \infty]\) such that \(\sup_n \int_0^1 g_n(x) dx\) is finite.

Moreover, a set \(E\) is a Martin-Löf null set if and only if \(E \subseteq \{x : \lim_n g_n(x) = \infty\}\) for some such sequence \((g_n)\).

Theorem 12 (Rute [Rut16a]). The following are equivalent for a real \(x \in [0, 1]\).

1. The real \(x\) is Schnorr random.
2. The supremum \(\sup_n g_n(x)\) is finite for every increasing computable sequence of continuous functions \(g_n : [0, 1] \to [0, \infty]\) such that there is some computable probability measure \(\mu\) such that \(\int_A g_n(x) dx \leq \mu(A)\) for all Borel sets \(A \subseteq [0, 1]\).
Moreover, a set $E$ is a Schnorr null set if and only if $E \subseteq \{ x : \lim_n g_n(x) = \infty \}$ for some such sequence $(g_n)$.

**Theorem 13** (Miyabe \cite{Miy13}). The following are equivalent for a real $x \in [0, 1]$.

1. The real $x$ is Schnorr random.
2. The supremum $\sup_n g_n(x)$ is finite for every increasing computable sequence of continuous functions $g_n : [0, 1] \to [0, \infty)$ such that $\sup_n \int_0^1 g_n(x) \, dx$ is finite and computable.

Moreover, a set $E$ is a Schnorr null set if and only if $E \subseteq \{ x : \lim_n g_n(x) = \infty \}$ for some such sequence $(g_n)$.

4.2. Differentiability. A theorem of Lebesgue states that every function of bounded variation is differentiable almost everywhere. A function $f$ of bounded variation with effectively integrable derivative is differentiable almost everywhere. A function $f$ is differentiable at $x$ if there is a constant $C > 0$ such that for all $a \leq x_0 < \ldots < x_n \leq b$, one has $\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \leq c$. The minimum such bound is the variation $\Var^1(f)$.

**Theorem 14** (\(\Rightarrow\) Demuth \cite{Dem75b}, \(\Leftarrow\) Brattka, Miller, Nies \cite{BMN16}). The following are equivalent for a real $x \in [0, 1]$.

1. The real $x$ is Martin-Löf random.
2. The function $f$ is differentiable at $x$ for every computable function $f : [0, 1] \to \mathbb{R}$ of bounded variation.

**Theorem 15.** The following are equivalent for a real $x \in [0, 1]$.

1. The real $x$ is computably random.
2. The function $f$ is differentiable at $x$ for every computable function $f : [0, 1] \to \mathbb{R}$ of bounded variation with a computable variation $\Var^1(f)$.

**Proof.** Brattka, Miller, and Nies \cite{BMN16} Cor. 4.3 proved this theorem for nondecreasing $f$. Therefore it is sufficient to find two nondecreasing computable functions $f^+$ and $f^-$ such that $f = f^+ - f^-$. Let $f^+ = \Var_0^1(f)$ and $f^- = \Var_0^1(f) - f$. Both are non-decreasing. Since $\Var_0^1(f)$ is computable, so is $\Var_0^1(f)$. (Indeed, $\Var_0^1(f)$ is both computable from below, and computable from above by the calculation $\Var_0^1(f) = \Var_0^1(f) - \Var_1^1(f)$.)

In this next result, a function $f : [0, 1] \to \mathbb{R}$ is effectively integrable if there is a computable sequence of rational polynomials $p_n$ such that $\|f - p_n\|_{L^1} = \int (f - p_n) \, d\mu \leq 2^{-n}$.

**Theorem 16** (Rute \cite{Rut13} Cor. 4.17, p. 48, Cor. 12.5, p. 67). The following are equivalent for a real $x \in [0, 1]$.

1. The real $x$ is Schnorr random.
2. The function $f$ is differentiable at $x$ for every computable function $f : [0, 1] \to \mathbb{R}$ of bounded variation with effectively integrable derivative $f'$.

Now, let us consider Rademacher’s theorem that says that every Lipschitz function is almost everywhere differentiable. Recall, a function $f : [0, 1] \to \mathbb{R}$ is Lipschitz if there is a constant $C > 0$ such that for all $x, y \in [0, 1]$, $|f(x) - f(y)| \leq C|x - y|$.

**Theorem 17** (Freer, Kjos-Hanssen, Nies, Stephan \cite{FKHNS14}). The following are equivalent for a real $x \in [0, 1]$.

\footnote{In Section \ref{sec:effective-integrability} we provide three different definitions of “effectively integrable function”. This is the point-free version. Many authors refer to these as $L^1$-computable functions.}
(1) The real $x$ is computably random.
(2) Every computable Lipschitz function $f : [0, 1] \to \mathbb{R}$ is differentiable at $x$.

Lebesgue’s differentiation theorem states that if $f : [0, 1] \to \mathbb{R}$ is integrable, then
\[ \frac{1}{r} \int_{x-r}^{x+r} f(y) \, dy \text{ converges to } f(x) \text{ as } r \to 0 \text{ for almost every } x. \]

**Theorem 18** (Pathak, Rojas, Simpson [PRS14], Rute [Rut13]). The following are equivalent for a real $x \in [0, 1]$.

(1) The real $x$ is Schnorr random.
(2) The averages $\frac{1}{r} \int_{x-r}^{x+r} f(y) \, dy$ converge as $r \to 0$ for every effectively integrable function $f$.

This version of the Lebesgue’s differentiation theorem also holds in multiple dimensions. In Theorem 37 we will address the question, “To which value does $\frac{1}{r} \int_{x-r}^{x+r} f(y) \, dy$ converge?”

### 4.3. Martingale theory

In this subsection, we will work in the fair-coin measure on Cantor space $\{0,1\}^\mathbb{N}$ for convenience. If $f$ is an integrable function and $\mathcal{F}$ is a $\sigma$-algebra, then the conditional expectation $\mathbb{E}[f \mid \mathcal{F}]$ is the unique (up to a.e. equivalence) integrable function $g$ such that $\int_A g \, d\mu = \int_A f \, d\mu$ for all $A \in \mathcal{F}$. If $\mathcal{F}$ is the least $\sigma$-algebra for which the functions $h_0, \ldots, h_{n-1}$ are $\mathcal{F}$-measurable, then we write $\mathbb{E}[f \mid h_0, \ldots, h_{n-1}] = \mathbb{E}[f \mid \mathcal{F}]$. A martingale is a sequence of integrable functions $(f_n)_{n \in \mathbb{N}}$ such that for all $n \geq 1$,
\[ \mathbb{E}[f_n \mid f_0, \ldots, f_{n-1}] = f_{n-1} \text{ $\mu$-a.e.} \]

Doob’s martingale convergence theorem states that if $(f_n)$ is a martingale such that $\sup_n \|f_n\|_{L^2} < \infty$, then $f_n(x)$ converges for almost every $x$. We will say that a martingale $(f_n)$ is computable if $(f_n)$ is a computable sequence of computable functions. (The next two theorems can be strengthened to include martingales on arbitrary computable probability measures as in Subsection 5.4 where the functions $f_n$ are Brouwer/Schnorr effectively measurable as in Subsection 6.5. See footnote 10 (p. 33) and Theorem 7.11 (p. 55) in Rute [Rut13].

**Theorem 19** (Takahashi [Tak05], Merkle, Mihalović, Slaman [MMS06]). The following are equivalent for a sequence $x \in \{0,1\}^\mathbb{N}$.

(1) The sequence $x$ is Martin-Löf random.
(2) The sequence $f_n(x)$ converges for every computable martingale $(f_n)$ such that $\sup_n \|f_n\|_{L^2}$ is finite.

**Theorem 20** (Rute [Rut13 Thm. 7.11, p. 55, Thm. 12.9, p. 68]). The following are equivalent for a sequence $x \in \{0,1\}^\mathbb{N}$.

(1) The sequence $x$ is Schnorr random.
(2) The sequence $f_n(x)$ converges for every computable martingale $(f_n)$ such that $\sup_n \|f_n\|_{L^2}$ is finite and computable and such that $\lim_n f_n$ is effectively integrable.

Most of the martingale work in algorithmic randomness, however, has been focused on computable dyadic martingales (often just called computable martingales), that is, martingales $f_n$ of the form $f_n(x) = g(x \mid n)$ for some computable function $g$. This provides another convenient characterization of computable randomness.
Theorem 21 (Folklore [DH10, Theorem 7.1.3], following Schnorr [Sch71a]). The following are equivalent for a sequence $x \in \{0, 1\}^\mathbb{N}$.

1. The sequence $x$ is computably random.
2. The sequence $f_n(x)$ converges for every nonnegative computable dyadic martingale $(f_n)$.

4.4. Ergodic theory. Again, we work in the fair-coin measure on Cantor space. A measure preserving transformation $T : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N}$ is a measurable map such that $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A$. The pointwise ergodic theorem states that for any integrable function $f : \{0, 1\}^\mathbb{N} \to \mathbb{R}$, the following average converges for almost every $x$.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

In this next theorem, an almost everywhere computable map is one which is computable on a $\Pi^0_2$ set of measure one. (Every result in this subsection concerning almost everywhere computable maps also holds for the more general Brouwer/Schnorr effectively measurable maps that we describe in Subsection 5.5. The results also extend to computable probability measures on computable metric spaces as discussed in Subsection 5.4. For full generalizations of the next two theorems, see Hoyrup and Rojas [HR09a, Thm. 8] and Rute [Rut13, p. 72], respectively.)

Theorem 22 (V’yugin [V’y98], Franklin, Towsner [FT14]). The following are equivalent for a sequence $x \in \{0, 1\}^\mathbb{N}$.

1. The sequence $x$ is Martin-Löf random.
2. The ergodic averages

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

converge for every integrable, a.e. computable $f : \{0, 1\}^\mathbb{N} \to \mathbb{R}$ and for every a.e. computable measure-preserving $T$.

A measure preserving transformation is ergodic if and only if $T^{-1}(A) = A$ implies that $\mu(A)$ is 0 or 1. For ergodic $T$, the ergodic theorem states that almost surely

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f \, d\mu.$$

Theorem 23 (Gács, Hoyrup, Rojas [GHR11]). The following are equivalent for a sequence $x \in [0, 1]$.

1. The sequence $x$ is Schnorr random.
2. The ergodic averages

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

converge for every a.e. computable $f : \{0, 1\}^\mathbb{N} \to \mathbb{R}$ which is effectively integrable and for every a.e. computable ergodic measure-preserving $T$.

A special case of the ergodic theorem is the strong law of large numbers (SLLN). As mentioned above, SLLN alone does not characterize any algorithmic randomness notions. Nonetheless, starting with Von Mises [vM19], there have been attempts to
define randomness by requiring that a sequence $x$ not only satisfy SLLN, but that certain transformations $T(x)$ of that sequence do as well. The transformations that Von Mises considered were subsequences of $x$ given by a selection rule; that is, one has to choose whether to select the bit $x_i$ based only on the values of the former bits $x_0,...x_{i-1}$. A Church stochastic sequence is the formulation of this notion where the selection rules are computable, that is given by a computable function $f : \{0,1\}^N \rightarrow \{\text{yes},\text{no}\}$ [DH10 Def. 7.4.1]. While this stochasticity notion and its generalizations are not as useful as the established notions of randomness, Schnorr realized that Von Mises's ideas can be used to define Schnorr randomness if one uses the correct class of transformations $T(x)$.

**Theorem 24** (Schnorr [Sch71c Thm. 12.1]). The following are equivalent for a sequence $x \in \{0,1\}^N$.

1. The sequence $x$ is Schnorr random.
2. The frequency of 1s in $T(x)$ converges to $1/2$, i.e. 
   $$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (T(x))_k = \frac{1}{2}$$
   for every a.e. computable measure-preserving map $T : [0,1] \rightarrow [0,1]$ where $(T(x))_k$ is the $k$th bit of $T(x)$.

Note, not every Church selection rule corresponds to an a.e. computable measure-preserving map. While, the Church selection rules are total functions $f : \{0,1\}^N \rightarrow \{\text{yes},\text{no}\}$, the corresponding transformation $T : \{0,1\}^N \rightarrow \{0,1\}^N$ may be partial, and the measure of the domain of $T$ may be less than one. Indeed, Schnorr randomness and Church stochasticity are incomparable notions [DH10 §8.4].

4.5. **Some additional remarks.** The above results show that each of Schnorr randomness, computable randomness, and Martin-Löf randomness can be characterized naturally via theorems from analysis. However, if one looks at the proofs, for the most part these results can be rewritten in terms of effective null sets. For example, Theorem 14 can be adapted as follows.

**Theorem 25.** For each computable function $f : [0,1] \rightarrow \mathbb{R}$ of bounded variation, the set $\{x \in [0,1] : f \text{ is not differentiable at } x\}$ is a Martin-Löf null set. Conversely, for each Martin-Löf null set $A$, there is a computable function $f : [0,1] \rightarrow \mathbb{R}$ of bounded variation such that $A \subseteq \{x \in [0,1] : f \text{ is not differentiable at } x\}$.

By relativizing the second part of this theorem, one gets the following corollary.

**Corollary 26.** For each null set $A \subseteq [0,1]$, there is a continuous function $f : [0,1] \rightarrow \mathbb{R}$ of bounded variation such that $A \subseteq \{x \in [0,1] : f \text{ is not differentiable at } x\}$.

What this logic tells us is that if we have a result which characterizes all Martin-Löf randoms (e.g. Theorem 14), it should relativize to a result (e.g. Corollary 26) which characterizes all null sets. The same holds for Schnorr and computable randomness, or any other notion which deserves to be called a “randomness notion”.

This allows us to instantly rule out some theorems as those which characterize randomness notions. For example, the strong law of large numbers only characterizes a single null set, namely the set of numbers which are not simply normal in base 2. Therefore, there is no algorithmic randomness notion characterized by the strong law of large numbers.
A more interesting example is a theorem of Weyl. Given a sequence of distinct integers \((a_n)\), the set \(\{a_nx\}_n\) is uniformly distributed modulo one for almost every \(x \in [0, 1]\). Avigad \cite{Avi13} defined a real \(x \in [0, 1]\) to be UD-random if \(\{a_nx\}_n\) is uniformly distributed modulo one for all computable sequences \((a_n)\) of distinct integers. However, Avigad noticed that there is a specific null set \(C\) such that for every sequence \((a_n)\) of distinct integers (not necessarily computable), there is a real \(x \in C\) where \(\{a_nx\}_n\) is uniformly distributed modulo one. Hence it is impossible to use Weyl’s theorem to characterize null sets, and “UD randomness” is not a true notion of randomness (for the Lebesgue measure).

So far we have been talking about randomness relative to the Lebesgue measure. It is possible that Weyl’s theorem characterizes null sets for a different measure \(\mu\) on \([0, 1]\). It is also possible that UD randomness is not associated with null sets for a single measure, but instead sets which are null for all measures in a family of measures (see, for example, \cite{Rei08, BGH+11}). Indeed, every type of “exceptional set” in mathematics has its own notion of effectively random-like objects. For example, effective Cohen genericity corresponds to meager sets. Kurtz randomness (which does not behave like a typical randomness notion) corresponds to subsets of null \(F_\sigma\) sets (that is, a countable union of closed sets). Such sets are both null and meager.

4.6. Connections with constructive and reverse mathematics. These characterization theorems have a close connection to constructive mathematics and reverse mathematics. (Some even call this approach “reverse randomness” because of the similarities.) For example, consider the Lebesgue differentiation theorem. It is constructive, as shown by Bishop \cite[Ch. 8, Thm. 5]{Bis67} and Demuth \cite[Thm. 4.4]{DK79}, and it also holds of Schnorr randomness (Theorem 18). Conversely, the nonconstructive theorems such as the ergodic theorem and the martingale convergence theorem do not hold for all Schnorr randoms (Theorems 22 and 19). This is not a coincidence, but instead a fundamental connection between Schnorr randomness and constructive mathematics.

**Informal Principle 1.** Consider an a.e. theorem \(T\) of the form

\[
\text{for all objects } a, \text{ for almost every } x, \text{ it holds that } P(x, a)
\]

where “almost every \(x\)” is defined using the constructive null sets of Brouwer, Demuth, or Bishop (recall that, under a computable interpretation, these are basically Schnorr null sets). If \(T\) is constructively provable, then \(P(x, a)\) holds for all Schnorr randoms \(x\) and all computable objects \(a\).

**Informal justification.** Assume \(T\) is constructively provable. Fix a computable \(a\). From a constructive proof of \(T\) one can explicitly construct a Schnorr null set \(N\), for which if \(P(x, a)\) does not hold then \(x \in N\). Therefore, \(P(x, a)\) holds for all Schnorr randoms \(x\).

A common special case of the above principle is a.e. convergence, which by Ergoroff’s theorem is classically equivalent to almost uniform convergence (when working in a probability space). Most constructive proofs of a.e. convergence proceed through almost uniform convergence.

**Informal Principle 2.** Consider an almost uniform convergence theorem \(T\) of the form
given a sequence \((f_n)\) of uniformly continuous functions \(f_n : [0, 1] \to \mathbb{R}\), satisfying some property \(P((f_n))\), then the sequence \((f_n)\) converges almost uniformly

where “almost uniformly” is defined using the constructive definitions of Brouwer, Demuth, Bishop, or Šanin (see Definitions 32 and 36). If \(T\) is constructively provable, then \(f_n(x)\) converges for all Schnorr randoms \(x\) and all computable sequences of computable functions \(f_n\) such that \(P((f_n))\) holds and is effectively realized.

**Informal justification.** Bishop constructively observed that if \((f_n)\) converges almost uniformly then \(f_n(x)\) convergences for almost every \(x\) [Bis67, p. 196]. The rest follows from Informal Principle 1.

An alternate justification is as follows. From the constructive proof of \(T\) and a realizer of \(P((f_n))\) we can extract a computable rate of almost uniform convergence. From this, we can apply the result that an effective rate of almost uniform convergence is sufficient to show convergence on Schnorr randomness. (This is due to Hoyrup, Rojas, Galatolo [CHR10, Theorem 1] and Rute [Rut13, Lemma 3.19, p. 41]. Also, see Lemma 39(4).) □

**Remark 27.** We would like to regard these previous two results as informal recipes for translating a constructive result into one about Schnorr randomness, rather than true meta-theorems. The constructive systems of Bishop and others are not given by formal axioms, making it difficult to truly formalize this result. Also, there are small subtleties, such as what it means for \(a\) to be computable or \(P((f_n))\) to be effectively realized, that are not worth considering here.

Also while it is convenient that the definition of null set used in, say, Bishop’s work is equivalent to that used by Schnorr, it is not strictly necessary for the above results to hold. Even Martin-Löf’s constructive theorems or the point-free theorems of Šanin can be used to extract computable results about Schnorr randomness. See Section 5 for details on how to effectively convert between the different definitions.

The converse of our informal principle is technically not true. For example, consider the theorem that for every monotone sequence of bounded continuous functions \(g_n : [0, 1] \to [0, 1]\), the sequence \(g_n(x)\) converges for almost every \(x\). This theorem is not constructive, but it is true that for every computable sequence of computable functions \((g_n)\), the sequence \(g_n(x)\) converges for every Schnorr random \(x\) (and indeed every \(x \in [0, 1]\)).

Nonetheless, the converse of our informal principle seems to be “true is spirit”. The natural a.e. theorems holding for Schnorr randomness — the law of large numbers, the Lebesgue differentiation theorem, the ergodic theorem for ergodic measures, etc. — are provable in constructive mathematics.

Moreover, reverse-mathematics-type results seem to shed light on the connections between Schnorr randomness and a.e. convergence theorems. For example, the following are constructively equivalent for an increasing sequence of nonnegative uniformly continuous functions \(g_n : [0, 1] \to [0, \infty)\) such that \(\int_0^1 g_n(x) \, dx\) is bounded (Bishop [Bis67, Ch. 7, Thm. 5]).

\[
\begin{align*}
(1) & \quad \int_0^1 g_n(x) \, dx \text{ converges.} \\
(2) & \quad g_n \text{ converges almost uniformly.}
\end{align*}
\]

What is interesting about this result is the following connections to Theorem 13.
(1) Using the second Informal Principle, one can use the forward direction of Bishop’s result to get the forward direction of Theorem 13.

(2) Bishop’s result suggests (but does not alone prove!) that one cannot remove the condition that \(\sup_n \int_0^1 g_n(x) \, dx\) is computable from Theorem 13. (We know this is true by Theorem 11 along with the fact that Schnorr randomness and Martin-Löf randomness are different.)

(3) Bishop’s result suggests (but does not prove!) that there is no other stronger “reasonable hypotheses” one can place on \(\int_0^1 g_n(x) \, dx\) in Theorem 13. (This is too vague to be provable, but it agrees with experience. While there are examples, as above, where \(g_n(x)\) converges on all Schnorr randoms and \(\int_0^1 g_n(x) \, dx\) is not computable, these seem contrived and unnatural.)

It would be interesting to explore this connection more. For another example, Spitters [Spi06] showed a constructive characterization for when ergodic averages converge. This characterization aligns well with experience about Schnorr randomness and the ergodic decomposition (Rute [Rut13], Thm. 10.2, p. 72), also see Hoyrup [Hoy13].

As for the theorems which characterize Martin-Löf randomness — Lebesgue’s theorem for functions of bounded variation, the martingale convergence theorem, and the ergodic theorem — these results are all nonconstructive. (See Problems 3, 9, and 11 in Bishop [Bis67, pp. 242–243] as well as various computability theoretic counterexamples [V’y01, BMN16, AGT10].) Nonetheless, Bishop [Bis67, Ch. 8, §3] showed that these three theorems can be made constructive by weakening the conclusion but not the hypothesis. For these “equal hypothesis results”, Bishop used upcrossings.

Pick two rationals \(a < b\). A sequence of real numbers \((x_n)\) has at least \(k\) \((a,b)\)-upcrossings if there are indices \(m_1 < n_1 < \cdots < m_k < n_k\) such that \(x_{m_j} < a < b < x_{n_j}\) for all \(j \in \{1, \ldots, k\}\). Classically, a bounded sequence converges exactly if for each pair of rationals \(a < b\), the number of \((a,b)\)-upcrossings is bounded. Therefore, a sequence of measurable functions \((f_n)\) converges almost everywhere if there is an upper bound on both \(\int |f_n| \, d\mu\) and \(\int U_{a,b} \, d\mu\) where \(U_{a,b}(x)\) is the number of \((a,b)\)-upcrossings of \((f_n(x))\).

Doob’s nonconstructive proof of martingale convergence proceeded via a constructive proof of an upcrossing inequality bounding \(\int U_{a,b} \, d\mu\) [Bis66]. Bishop [Bis67] §8.3, in turn, gave constructive upcrossing inequalities for both the ergodic theorem and Lebesgue’s theorem concerning the differentiability of bounded variation functions. Indeed, V’yugin [V’y98] used the former to prove that the ergodic theorem holds for Martin-Löf randomness (Theorem 22). Similarly, the latter can be used to give an alternate proof of Demuth’s result (Theorem 14) that Lebesgue’s theorem holds for Martin-Löf randomness.

As for reverse mathematics, given the close connection between \(\text{WWKL}\) and Martin-Löf randomness, one may expect that theorems such as the pointwise ergodic theorem are equivalent to \(\text{WWKL}\) over \(\text{RCA}_0\). However, this depends on how one formalizes the theorem. In \(\text{RCA}_0\), there are two nonequivalent ways to say that a sequence \((x_n)\) converges. One way is to say that \(\lim_n x_n\) exists. Using this limit characterization of convergence, Simic [Sim07] showed that the pointwise ergodic theorem is equivalent to \(\text{ACA}_0\). (The main idea is that there is a computable ergodic system whose limit is Turing equivalent to \(\psi\).) The other characterization of convergence is to say that \((x_n)\) is Cauchy. Using this Cauchy
characterization of convergence (in the definition of differentiable), Nies, Triplet, and Yokoyama [NTY17] showed that Lebesgue’s theorem about the differentiability of functions of bounded variation is equivalent to WKL over RCA₀. It is natural to conjecture that for each of the pointwise ergodic theorem, the martingale convergence theorem, and Lebesgue’s theorem about the differentiability of functions of bounded variation, that the “limit” version is equivalent to ACA and the Cauchy version is equivalent to WKL over RCA₀.

Last, another way that constructive mathematics sheds light on algorithmic randomness is via relativization. A real x is Martin-Löf random relative to a real y, if x is not contained in any Martin-Löf null set computable from y. While, at first, this may seem natural, it does not necessarily agree with constructive mathematics. In constructive mathematics, when one says that an object A exists given another object B, one constructs a uniformly computable function which takes (a code for) any such object B and returns (a code for) a corresponding object A. This suggests, an alternative definition: A real x is Martin-Löf random uniformly relative to a real y if x is not contained in any Martin-Löf null set uniformly computable from y (see [MR13, Rut18] for formal definitions). While these two definitions agree for Martin-Löf randomness, they disagree for Schnorr randomness [MR13, Rut18]. One needs to be cautious of this when relativizing a result. For example, the following is the correct (and most general) way to relativize Theorem 13. (This is actually the definition of relative Schnorr randomness given in Rute [Rut18].)

**Theorem 28.** For every oracle \( a \in \mathbb{N} \), the following are equivalent for a real \( x \in [0,1] \).

1. The real x is Schnorr random uniformly relative to a.
2. The supremum \( \sup_n g_n^a(x) \) is finite for every increasing computable sequence of continuous functions \( g_n^a : [0,1] \rightarrow [0,\infty) \) uniformly computable in a where \( \int_0^1 g_n^a(x) \, dx \) converges with a rate of convergence uniformly computable in a.

Just as with Theorem 13 we could apply the second Informal Principle to construct a proof of this theorem. Using Bishop’s proof, we can extract an algorithm which takes as input an oracle a, a function \( a \mapsto (g_n^a) \), and a function which maps a to a rate of convergence for \( \int_0^1 g_n^a(x) \, dx \). The output of this algorithm is (a code for) a null set \( E^a \) for which \( \{ x : \lim_n g_n^a(x) = \infty \} \subseteq E^a \).

Moreover, Schnorr randomness behaves much better under uniform computability [FS10, Miy11, MR13, Rut18], solving many of the perceived flaws of Schnorr randomness. (For example, Porter [Por12, §§10.4.2] cataloged the four main objections to Schnorr randomness. Each of these can be fixed by replacing “computable” with “uniformly computable”.)

### 4.7. Further investigations in randomness and analysis.

Besides the aforementioned topics, there are questions that only make sense in the context of randomness. Fouche [Fou08] showed that if \( \phi : [0,1] \rightarrow \mathbb{R} \) is a Martin-Löf random Brownian motion path (also called a complex oscillation) then \( \phi(1) \) is Martin-Löf random. Hoyrup and Rojas [HR09a, §§5.3] showed the converse also holds in the sense that if \( x \) is Martin-Löf random, then \( \phi(1) = x \) for some Martin-Löf random complex oscillation \( \phi \). Rute [Rut18, Ex. 9.6] showed that this is true of Schnorr randomness as well. There are many more such results in randomness, e.g. [DKH12, PC15]. This is especially true in probability theory, where one quickly
passes between multiple representations of the same object. A random sequence of independent fair coin tosses can be used to construct a uniform random variable on $[0,1]$, a random walk on the integers, a random graph, a random percolation model, and a number of other random objects. It is important to know that randomness on one space is (in some sense) equivalent to randomness on another.

Four basic tools have been developed for this purpose. With the correct definitions, these theorems hold for both Schnorr and Martin-Löf randomness (and often, but not always, hold for computable randomness). Since the details are a bit technical, we state them here vaguely with citations to the full theorems.

1. **Randomness conservation** If $f : (\Omega, \mathbb{P}) \to X$ is "sufficiently effectively measurable" and $\omega \in \Omega$ is $\mathbb{P}$-random, then $f(\omega)$ is random for the push-forward measure $\mathbb{P}_f$ (given by $\mathbb{P}_f(A) = \mathbb{P}(f^{-1}(A))$) [BP12, Thms. 3.2, 4.1], [HR09b, Prop. 5], [Rut16b], [Rut18, Prop. 9.2], [BHS17, Thm. 2].

2. **No randomness from nothing** If $f : (\Omega, \mathbb{P}) \to X$ is "sufficiently effectively measurable," and $x \in X$ is $\mathbb{P}_f$-random, then $x = f(\omega)$ for some $\mathbb{P}$-random $\omega \in \Omega$ [BP12, Thm. 3.5], [HR09b, Prop. 5], [Rut16b, Thm. 7], [Rut18, Cor. 9.5], [BHS17].

3. **Van Lambalgen’s theorem and its generalizations** Given $(\Omega_1 \times \Omega_2, \mathbb{P})$, the pair $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ is $\mathbb{P}$-random if and only if $\omega_1$ is $\mathbb{P}_1$-random and $\omega_2$ is $\mathbb{P}(\cdot \mid \omega_1)$-random relative to $\omega_1$, assuming that $\mathbb{P}$ can be "effectively decomposed" into the projection measure $\mathbb{P}_1$ on $\Omega_1$ and the family of conditional probabilities $\omega_1 \mapsto \mathbb{P}(\cdot \mid \omega_1)$ on $\Omega_2$ [Tak08, Tak11], [Bau17, Thm. 4], [BST, Thm. 5], [Rut18, Thm. 8.2].

A special case of this is where $\mathbb{P}$ is the product of two independent measures $\mathbb{P}_1 \otimes \mathbb{P}_2$. In this case, $(\omega_1, \omega_2)$ is $\mathbb{P}$-random if and only if $\omega_1$ is $\mathbb{P}_1$-random and $\omega_2$ is $\mathbb{P}_2$-random relative to $\omega_1$. [DH10, Thm. 6.9.1], [Nie09, Thm. 3.4.6], [MK13], [Myi11].

For those interested in learning more about these new directions in algorithmic randomness (at least with respect to Martin-Löf randomness), we recommend Gács [Gács], Bienvenu, Gács, Hoyrup, Rojas, and Shen [BGH+11], Hoyrup and Rojas [HR09c], [HR09a], [HR09b], and Allen, Bienvenu, and Slaman [ABS14]. For computable randomness, see Rute [Rut16a, Rut16b]. For Schnorr randomness, see [Rut18, Rut13].

5. **Randomness and the foundations of computable measure theory**

We saw in Section 3 that constructive measure theory has been developed through a number of different constructive and computable traditions — each tradition using slightly different definitions, terminology, and techniques. This nonlinear development, unfortunately, gives the outsider (and even the insider) the appearance

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18 For the reader wishing to connect these results with Section 5, we remark that when we say "sufficiently effectively measurable" it is sometimes sufficient for the map $f : (\Omega, \mathbb{P}) \to X$ to be Brouwer/Schnorr effectively measurable as in Definition 34. Other times one must also require that the conditional probability map $x \mapsto \mathbb{P}(\cdot \mid f = x)$ be Brouwer/Schnorr effectively measurable as well.

19 The measure $Q$ is absolutely continuous with respect to $P$ if every $P$-null set is $Q$-null.
that “a systematic general framework for computability in measure and integration theory still remains in its infancy” [Eda09]. This is far from the case.

In this section, we give a short presentation on the foundations of computable measure theory. Our presentation shows that, while there are many approaches to constructive/computable measure theory, they are basically equivalent. One piece of evidence for this is that the definitions of measurable set, measurable function, integrable function, and almost uniform convergence in the computable and constructive literature basically agree. Specifically, most definitions fall into three categories:

1. Point-free definitions.
2. Definitions which are well-defined outside of a Martin-Löf null set.
3. Definitions which are well-defined outside of a Schnorr null set.

Moreover, all three categories are equi-computable, in the sense that, given a computable object of one type, one can uniformly compute an equivalent object of another type. As the descriptions of these categories suggest, Martin-Löf and Schnorr randomness naturally arise out of these definitions (although, in most cases there was no mention of randomness in the original definitions). (For simplicity, we only focus on whether our definitions are computably equivalent, ignoring whether they are constructively equivalent.)

5.1. Computable metric spaces and computable topology. To do computable analysis, one needs a good notion of a computable space. The early constructivists restricted their work to Euclidean space $\mathbb{R}^d$ or Cantor space $\{0,1\}^\mathbb{N}$. Later work gradually incorporated compact and locally compact metric spaces, separable Banach spaces, complete separable metric spaces, and finally a wide variety of topological and abstract spaces.

For this presentation, we will use complete separable metric spaces (also known as Polish spaces). These spaces are sufficiently rich, but still easy to work with. (Most random variables in probability theory, for example, takes values in a complete separable metric space.)

**Definition 29.** A computable metric space $X$ is a triple $(X, \rho, A)$ where $(X, \rho)$ is a complete separable metric space, and $A \subseteq X$ is a dense indexed set $\{a_i\} \subseteq X$ (possibly with repetition) such that $i, j \mapsto \rho(a_i, a_j)$ is computable. A point $x \in X$ is computable if there is a computable sequence $(i_n)$ such that for all $m < n$, $\rho(a_{i_n}, a_{i_m}) < 2^{-m}$ and $x = \lim_n a_{i_n}$. The sequence $(i_n)$ is called the Cauchy name of $x$.

The effectively open sets of $X$ are computable sets of the form $U = \bigcup_i B(x_i, r_i)$ where $(x_i)_i$ is a computable sequence of points in $A$, $(r_i)_i$ is a computable sequence of positive rationals, and $B(x_i, r_i) = \{x \in X : \rho(x, x_i) < r_i\}$. The effectively closed sets are the complements of effectively open sets.

A partial map $f: D \subseteq X \to Y$ (where $Y$ is a computable metric space) is computable if there is a partial computable map $\Phi: \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ which takes every $X$-Cauchy name for every $x \in D$ to a $Y$-Cauchy name of $f(x)$.

5.2. Computable measure spaces. The set theoretic concept of a measure is so general that it is difficult to distill it down to a computably representable form. There are a few generally accepted approaches to do this. One approach, which is simple and elegant, is to divorce the measure from the underlying topological structure of the space. Any measure whose $\sigma$-algebra is countably generated can
be represented with this approach. Recall, that a ring of sets is a collection $\mathcal{R}$ of subsets of $X$ closed under union, intersection, empty set, and set difference. If $X \in \mathcal{R}$, then $\mathcal{R}$ is a Boolean algebra. We say that a countable ring $\mathcal{R} = \{ R_i \}$ is computable if the index of $R_i \cap R_j$ is uniformly computable from $i$ and $j$ for $\square \in \{ \cup, \cap, \setminus \}$.

**Definition 30.** A computable $\sigma$-finite measure space is a tuple $(X, \mathcal{A}, \mathcal{R}, \mu)$ where $\mathcal{R} = \{ R_i \}$ is a computable ring of $X$ which generates the $\sigma$-algebra $\mathcal{A}$ on $X$ and $i \mapsto \mu(R_i)$ is computable. A computable finite measure space is a computable $\sigma$-finite measure space $(X, \mathcal{A}, \mathcal{R}, \mu)$ where $\mathcal{R}$ is a Boolean algebra of $X$. A computable probability space is a computable finite measure space where $\mu(X) = 1$.

This is the definition of Wu and Weihrauch [WW06]. Also, there is no loss in loosening the Boolean operations on $\mathcal{R}$ up to $\mu$-a.e. equivalence. For example, if $R, S \in \mathcal{R}$, then we only require that there is a set $T \in \mathcal{R}$ such that $S \cup R = T$ $\mu$-a.e. The fair-coin probability measure on $\{0,1\}$ is computable with the Boolean algebra of cylinder sets. The Lebesgue measure on $\mathbb{R}$ is similarly computable with the ring of half-open rational intervals $(a, b]$.

Following Coquand and Palmgren [CP02], one can make this definition completely point-free by replacing the ring of sets $\mathcal{R}$ with any countable algebraic Boolean ring (without a unit) and the measure $\mu$ on sets with a measure on the ring $\mathbb{R}$ Similarly, a Boolean algebra of sets is replaced with an algebraic Boolean algebra, that is a Boolean ring with a unit 1. Another formal, point-free approach, based on the Danielle integral, was used by Bishop and Cheng [BC72, BB85]. Coquand and Palmgren [CP02] and Wu and Weihrauch [WW06] showed that one can effectively translate between the Danielle integral approach and the ring approach.

For simplicity, we will focus only on probability measure spaces $(X, \mathcal{A}, \mathcal{R}, \mu)$, with an occasional footnote on finite and $\sigma$-finite measures. Also, because these spaces do not have a topology, we cannot define algorithmic randomness in the usual way. We will show in Subsection 5.10 that one can still define Schnorr randomness for computable measure spaces via “effectively generic ultrafilters.”

### 5.3. The point-free approach to computable measure theory.

Assume that $(X, \mathcal{A}, \mathcal{R}, \mu)$ is a computable probability space and $Y$ is a computable metric space. Many of the objects of measure theory can be described in a point-free way as follows:

1. There are two senses in which the sigma-algebra $\mathcal{A}$ of a measure $\mu$ is generated by a countable family of sets $\mathcal{R}$. In a set theoretic sense, $\mathcal{A}$ is the minimum $\sigma$-algebra extending $\mathcal{R}$. In a measure theoretic sense, $\mathcal{A}$ is the minimum $\mu$-complete sigma-algebra extending $\mathcal{R}$. That is $\mathcal{A}$ contains all $\mu$-null sets. Since measure theory is normally “up to a null set” the differences are negligible. However, for concreteness, when we say $\mathcal{R}$ generates $\mathcal{A}$, we mean the latter. When we speak later about Borel measures, we will mean the completion of a Borel measure.

2. A Boolean ring is a commutative ring where $x^2 = x$. Ring multiplication and addition correspond to intersection and symmetric difference. Union $x \cup y$ corresponds to $x + y + xy$. A measure $\mu$ on a ring $\mathcal{R}$ is a nonnegative function $\mu : \mathcal{R} \to [0, \infty)$ satisfying $\mu(x \cup y) = \mu(x) + \mu(y) - \mu(x \cap y)$ and $\mu(\emptyset) = 0$.

3. The key observation of computable finite measures is that, with the exception of the zero measure, they are computable probability measures scaled by a computable real. The key observation of computable $\sigma$-finite measures is that there is a computable partition $X_n$ of disjoint ring elements such that $X = \bigcup_{n=0}^{\infty} X_n$ $\mu$-a.e., $\mu(X_n) > 0$, and the map $i \mapsto m$ such that $R_i \subseteq \bigcup_{n=0}^{m-1} X_n$ $\mu$-a.e. is computable [WT14]. Therefore a computable $\sigma$-finite measure space is just a disjoint union of uniformly computable finite measure spaces $(X_n, \mathcal{A}_n, \mathcal{R}_n, \mu_n)$. Write $\mu = \sum_n \mu_n$. 
points in a computable metric space. As we discussed in Subsections 3.2 and 3.8, this approach goes back to Šanín [San68] and has been developed by many others.

- The space $\text{MSet}(X, \mu)$ of $A$-measurable sets (modulo a.e. equivalence) is a computable metric space under the metric $\rho(A, B) = \mu(A \triangle B)$. Call the computable points in this space point-free effectively measurable sets.

- The space $L^0(X, \mu)$ of measurable functions $f : (X, \mu) \to \mathbb{R}$ (modulo a.e. equivalence) is a computable metric under the following metric (where $d_Y$ is the metric of $Y$).

$$\rho(f, g) = \int \frac{|f-g|}{1 + |f-g|} \, d\mu$$

Call the computable points in this space point-free effectively measurable functions.

- Similarly, the space $L^0(X, \mu; Y)$ of measurable functions $f : (X, \mu) \to Y$ (modulo a.e. equivalence) is a computable metric under the following metric (where $d_Y$ is the metric of $Y$).

$$\rho(f, g) = \int \frac{d_Y(f, g)}{1 + d_Y(f, g)} \, d\mu$$

Call the computable points in this space point-free effectively measurable functions from $(X, \mu)$ to $Y$.

- The space, $L^p(X, \mu)$ of $p$-integrable functions (modulo a.e. equivalence) for computable $1 \leq p < \infty$ is a computable metric space under the metric

$$\rho(f, g) = \|f - g\|_{L^p} = \left( \int \|f - g\|^p \, d\mu \right)^{1/p}.$$  

We will call computable points in this space point-free effective $L^p$ functions or point-free effectively integrable functions when $p = 1$.

**Remark 31.** We have not yet mentioned which countable dense set to use for each metric space. For measurable sets, use the Boolean algebra $\mathcal{R}$. For $L^0$ and $L^p$, use the set of rational step functions $\sum_{i=0}^{n-1} q_i 1_{R_i}$ where $R_0, \ldots, R_{n-1} \in \mathcal{R}$ is a partition of $X$ and $q_i \in \mathbb{Q}$. For the $Y$-valued measurable functions, use the same idea with the dense set $A$ of the computable metric space $Y$ taking the place of the rationals.

These above point-free definitions are equal to many others in the literature. We list a few which are easily deducible from the definitions.

- A set $A$ is point-free effectively measurable if and only if the characteristic function $1_A$ is point-free effectively measurable (or point-free effectively $L^p$ or any computable $p$) [Rut13, Prop. 3.24, p. 41].

- A measurable function $f : (X, \mu) \to Y$ is point-free effectively measurable if and only if for every effectively open set $U \subseteq X$, there is a sequence of effectively measurable sets $A_0, A_1, \ldots$ (computable uniformly from the index

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23For a $\sigma$-finite measure space, $\rho(A, B) = \mu(A \triangle B)$ is a metric for the space of finitely measurable sets. The space of all measurable sets is given by the metric $\rho(A, B) = \sum_n 2^{-n} \min\{1, \mu_n(A \triangle B)\}$ where $\mu = \sum_n \mu_n$ as in the previous footnote.

24This is just one of many computable equivalent metrics, also including the metric $\rho(f, g) = \int \min\{|f-g|, 1\} \, d\mu$ and the Ky-Fan metric. Again, for $\sigma$-finite measurable spaces, use the metric $\rho(f, g) = \sum_n 2^{-n} \min\{1, \mu_n(f \triangle g)\}$.

25For $\sigma$-finite measures $\mu = \sum_n \mu_n$, there is also a space of locally $p$-integrable functions given by the metric $\rho(f, g) = \sum_n 2^{-n} \min\{1, \|f-g\|_{L^p(\mu_n)}\}$. 
of $U$) such that $f^{-1}(U) = \bigcup A_i \mu$-a.e. (This is basically the representation 
\delta_{nfo}$ of [Wei17, Thm. 5.4]. Also see Subsection [5.4])

- An $L^p(X, \mu)$ function $f$ (for computable $p \geq 1$) is point-free effectively $L^p$
if and only if $f$ is point-free effectively measurable and $\|f\|_{L^p}$ is finite and computable [Rut13, Prop. 3.20, p. 41].

- A bounded measurable function $f : (X, \mu) \to [0, 1]$ is point-free effectively measurable if and only if $f$ is point-free effectively measurable for any (and hence all) computable $p \geq 1$ [Rut13, Prop. 3.20, p. 41].

See Spitters [Spi02, Ch. 3][Spi06a] for a modern constructive treatment of this metric
approach. Moreover, Ko [Ko91, §5.1] has given descriptions of these classes via “recursively
approximable sets” and “recursively approximable functions”. He also gave a characterization of the point-free measurable functions via effective convergence in measure [Ko91, Cor. 5.13] (see also Rute [Rut13, Prop. 3.15]). Edalat
[Eda09] gave a slightly different, but equivalent, characterization of bounded measurable functions via interval-valued functions.

Notice that in our point-free framework there is only one null set, namely the
equivalence class of the empty set. Even with such a limited definition of “null set,” many almost everywhere results can still be described in this framework. For example, two sets $A$ and $B$ are a.e.

\begin{equation}
\forall x \in X : \forall m \exists k > K(m, n) \forall Y (f_k(x), f(x)) > 2^{-m} \leq 2^{-n}.
\end{equation}

This definition was considered a constructive or effective version of "almost
everywhere (or almost sure) convergence" by Kosovski [Kos73b], V’yugin [V’y97],
Coquand and Palmgren [CP02] as well as many others.\footnote{Some authors use different but effectively equivalent definitions, e.g. replacing \(\delta_{nfo}\) with $\forall m \mu(x \in X : \exists k > K(m, n) d_Y(f_k(x), f(x)) > 2^{-m}) \leq 2^{-n}$.}

Recall, that if a sequence of functions converges almost uniformly, then it converges almost everywhere. Egoroff’s theorem says that the converse holds for a probability space. However, Egoroff’s theorem is not constructive.\footnote{Egoroff’s theorem also fails, in general, for \(\sigma\)-finite measures. However, one can easily develop a notion of “local almost uniform convergence” (and its effective analogue) which is classically equivalent to a.e. convergence.}

For that reason (and also the reason that Egoroff’s theorem fails for the convergence of continuously indexed families of functions $\{f_t\}_{t \in [0, \infty)}$ as $t \to \infty$), we choose to call this almost uniform convergence \footnote{Egoroff’s theorem holds in Brouwer’s measure theory because of the fan principle [Heyd, §6.5.4], Bishop [Bis67, Ch. 7, Theorem 4] on the other hand, modified the definition of almost everywhere convergence, making Egoroff’s theorem trivial. Kosovski [Kos70, 2.5.1] gave a constructive counterexample to Egoroff’s theorem, and Avigad, Dean, and Rute [ADR12] show that Egoroff’s theorem is equivalent to 2-WWKL over RCA0.}

Also note that the above definition of almost uniform convergence is “point-free”
in the sense that $\{x \in X : \forall m \exists k > K(m, n) d_Y(f_k(x), f(x)) > 2^{-m}\}$ is $\mu$-almost everywhere equal to $\{x \in X : \forall m \exists k > K(m, n) d_Y(g_k(x), g(x)) > 2^{-m}\}$ for any sequence $\{g_k\}$ which is $\mu$-a.e. equal to $(f_k)$ and any $g$ which is $\mu$-a.e. equal to $f$.}
5.4. Computable measures on computable metric spaces. While the definition of a computable measure space in Definition 30 is both general and elegant, it requires imposing an arbitrary ring structure on the space, effectively treating the space as zero-dimensional. Now we will consider an alternative definition which preserves the topological and metric structure of computable metric space $X$, while also inducing a computable metric structure on the space of probability measures on $X$. For the majority of probability theory it is sufficient to work with Borel probability measures on a Polish space. For analysis, it is also common to work with locally finite Borel measures on locally compact Polish spaces. Again, we will focus on the probability measure case, with an occasional footnote about locally finite measures.

If $X$ is a computable metric space, then the space $M_1(X)$ of Borel probability measures on $X$ is a computable metric space under the Levy-Prokhorov metric or the Wasserstein metric. (For the Wasserstein metric, one must first modify $X$ to be a bounded metric space.) The computable probability measures $\mu \in M_1(X)$ are the computable points in this metric space.

Equivalently, the computable probability measures can be described as follows.

1. By an effective version of the Reisz representation theorem, $\mu \in M_1(X)$ is computable if and only if $f \mapsto \int f \, d\mu$ is a computable operator on bounded computable functions $f : X \to [0,1]$ [HR09d, Cor. 4.3.1, 4.3.2].

2. Using valuation theory, $\mu \in M_1(X)$ is computable if and only if $U \mapsto \mu(U)$ is a lower semicomputable operator on effectively open sets [HR09d, Thm. 4.2.1] (For Cantor space, this is equivalent to the map $\sigma \mapsto \mu([\sigma])$ being computable where $\sigma \in \{0,1\}^\mathbb{N}$.)

All of these approaches give the space $M_1(X)$ the topology of weak convergence. For more on computable measures, see Schröder [Sch07] and Hoyrup and Rojas [HR09d]. For a constructive, point-free treatment of integral operators and valuations, see Coquand and Spitters [CS09].

Every computable probability space $(X, \mathcal{A}, \mathcal{R}, \mu)$ is isomorphic to the computable measure $\nu$ on Cantor space $\{0,1\}^\mathbb{N}$ given by

$$\nu([\sigma]) = \mu \left( \bigcap_{i < |\sigma|} R_i \cap \bigcap_{i < |\sigma|} R_i^c \right)$$

where $\mathcal{R} = \{R_i\}$.

For locally finite measures $\mu \in \mathcal{M}_{\text{loc}}(X)$, one also needs $\mu(X)$ to be computable. For locally finite measures $\mu \in \mathcal{M}_{\text{loc}}(X)$, see, for instance, Edalat [Eda09].

Similarly, every computable $\sigma$-finite measure space is isomorphic to a measure on the locally compact space $\mathbb{N} \times \{0,1\}^\mathbb{N}$. 

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Footnotes:

29 Recall that a locally compact Polish space is the same as a locally compact second-countable Hausdorff space. A locally finite measure is one in which every point is contained in a neighborhood of finite measure. For Borel measures on locally compact Polish spaces, locally-finite measures are equivalent to $\sigma$-finite measures. These are also called Radon measures.

30 There are also metrics one can use for the space $\mathcal{M}_{\text{loc}}(X)$ of locally finite measures, e.g. Kallenberg [Kal83, §15.7].

31 For locally finite measures $\mu \in \mathcal{M}_{\text{loc}}(X)$, $(f, K) \mapsto \int f \, d\mu$ is a computable operator on pairs of computable functions $f : X \to [0,1]$ and effectively compact sets $K$ such that $\text{supp} f \subseteq K$. See Bishop [Bis67, Ch. 6].

32 For finite measures $\mu \in \mathcal{M}(X)$, one also needs $\mu(X)$ to be computable. For locally finite measures $\mu \in \mathcal{M}_{\text{loc}}(X)$, see, for instance, Edalat [Eda09].

33 Similarly, every computable $\sigma$-finite measure space is isomorphic to a measure on the locally compact space $\mathbb{N} \times \{0,1\}^\mathbb{N}$. 

Conversely, given a computable probability measure $\mu$ on a computable metric space $X$, there is a computable sequence of radii $r_i > 0$, dense in $[0, 1]$, such that $\mu\{d_X(x, a_i) = r_i\} = 0$ for the dense set $A = \{a_i\}$ used to generate $X$. In this way, the balls $B(a_i, r_i)$ form a basis of $X$ and $i, j \mapsto \mu(B(a_i, r_i))$ is computable. The space $(X, A, R, \mu)$ is a computable measure space where $R$ is the free Boolean algebra generated by these balls and $A$ is the ($\mu$-completion of the) Borel sigma-algebra of $X$.\footnote{A similar construction can be done for the locally finite measures on effectively locally compact computable metric spaces. This is basically the idea of Bishop’s theory of profiles\cite[Ch. 6]{BB85}. In this way we can construct a ring $R$ of open sets of computable measure which generates the corresponding $\sigma$-finite measure space.} In this way we can extend all the point-free definitions of the previous subsection to computable metric spaces with computable probability measures.

We can also now talk about the pushforward measure $\mu_f \in M_1(Y)$ of a measurable function $f : (X, \mu) \to Y$ given by $\mu_f(A) = \mu(f^{-1}(A))$. This provides yet another characterization of point-free effectively measurable functions. A function $f : (X, \mu) \to Y$ is point-free effectively measurable if and only if $\mu_f$ is computable and the map $A \mapsto f^{-1}(A)$ is a computable map of type $\text{MSet}(Y, \mu_f) \to \text{MSet}(X, \mu)$.\footnote{Since each set is open, by “complement” in $R$ we mean the interior of the complement. This is acceptable, since, by construction, the boundary of each set in $R$ is null.}

5.5. Two pointwise approaches. While the point-free approach is elegant, it is noticeably different from classical measure theory, where a measurable function is actually a function and a measurable set is actually a set. Also, there is a certain conceptual advantage to thinking about functions as algorithms which take a point in one space and assign it to a value in another space.

There are two similar, but different pointwise variants of measure theory in the constructive/computable literature. The first we will call the Brouwer/Schnorr variant, because it was the approach used by Brouwer\cite{Bro19} and it implicitly uses Schnorr null sets. This variant is equivalent to approaches used by Demuth\cite{DK79}, Bishop\cite{Bis67}, BC\cite{BC72}, BB\cite{BB85}, Pathak, Rojas, and Simpson\cite{PRS14}, Rute\cite{Rut13}, and Miyabe\cite{Miy13}. The second variant we will call the Martin-Löf variant since it was used by Martin-Löf\cite{ML70a} and implicitly uses Martin-Löf null sets. This variant is equivalent to approaches given by Edalat\cite{Eda09}, Yu\cite{Yu94}, Brown, Giusto, and Simpson\cite{BCS02}, Pathak\cite{Pat09}, and Hoyrup and Rojas\cite{HR09a}.

Assume $X$ and $Y$ are computable metric spaces and $\mu \in M_1(X)$ is a computable measure. In the previous subsection, we saw there is a countable Boolean algebra $R$ of effectively open sets of computable measure which generates this measure space.\footnote{The basic sets are the elements of this Boolean algebra, where as the basic functions $g : (X, \mu) \to Y$ are the step functions of the form $g(x) = a_i$ if $x \in R_i$ where $R_0, \ldots, R_{n-1} \in R$ is a finite partition of $R$ and each $a_i$ is from the dense set generating $Y$. (The basic functions are partial computable since we don’t include the boundaries of the sets $R_i$.)} The basic sets are the elements of this Boolean algebra, where as the basic functions $g : (X, \mu) \to Y$ are the step functions of the form $g(x) = a_i$ if $x \in R_i$ where $R_0, \ldots, R_{n-1} \in R$ is a finite partition of $R$ and each $a_i$ is from the dense set generating $Y$. (The basic functions are partial computable since we don’t include the boundaries of the sets $R_i$.)

**Definition 33.** For a set $Q \subseteq X$,

- $Q$ is Martin-Löf effectively measurable if there is a computable sequence of basic sets $(R_n)$ and a computable sequence $(U_n)$ of effectively open sets
such that \( \mu(U_n) \leq 2^{-n} \) and
\[
Q \triangle R_n \subseteq U_n.
\]
- \( Q \) is \textit{Brouwer/Schnorr effectively measurable} if, moreover, \( \mu(U_n) \) is computable from \( n \).

Notice that a measure zero Martin-Löf effectively measurable set is exactly a Martin-Löf null set, and a measure zero Brouwer/Schnorr effectively measurable set is exactly a Schnorr null set.

In the following, let \( f(x)^\uparrow \) denote that \( x \) is not in the domain of \( f \).

**Definition 34.** For a partial function \( f : X \to Y \), where the metric of \( Y \) is \( d_Y \),
- \( f \) is \textit{Martin-Löf effectively measurable} if there is a computable sequence of basic functions \((g_n)\) and a computable sequence \((U_n)\) of effectively open sets such that \( \mu(U_n) \leq 2^{-n} \), and
\[
\{ x : f(x)^\uparrow \lor g_n(x)^\uparrow \lor d_Y(f(x), g_n(x)) > 2^{-n} \} \subseteq U_n.
\]
- \( f \) is \textit{Brouwer/Schnorr effectively measurable} if, moreover, \( \mu(U_n) \) is computable from \( n \).

**Definition 35.** For a partial function \( f : X \to \mathbb{R} \),
- \( f \) is \textit{Martin-Löf effectively integrable} if there is a computable sequence of basic functions \((g_n)\) and a computable sequence \((U_n)\) of effectively open sets such that \( \mu(U_n) \leq 2^{-n} \), and
\[
\{ x : f(x)^\uparrow \lor g_n(x)^\uparrow \lor |f(x) - g_n(x)| > 2^{-n} \} \subseteq U_n,
\]
and
\[
\int |f - g_n| \, d\mu \leq 2^{-n}.
\]
- \( f \) is \textit{Brouwer/Schnorr effectively integrable} if, moreover, \( \mu(U_n) \) is computable from \( n \).

The Martin-Löf and Brouwer/Schnorr effective \( L^p \) functions are defined analogously.

**Definition 36.** Given a sequence of Martin-Löf effectively measurable functions \( f_k : X \to Y \) and a Martin-Löf effectively measurable function \( f : X \to Y \),
- \( f_k \) converges to \( f \) \textit{Martin-Löf effectively almost uniformly} if there is a computable rate of almost uniform convergence \( K : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) and a computable sequence of effectively open sets \( U_n \) where for all \( n \), \( \mu(U_n) \leq 2^{-n} \) and
\[
\{ x \in X : \forall m \exists k \geq K(m, n) (f(x)^\uparrow \lor f_k(x)^\uparrow \lor d_Y(f_k(x), f(x)) > 2^{-m}) \} \subseteq U_n.
\]
- \( f_n \) converges to \( f \) \textit{Brouwer/Schnorr effectively almost uniformly} if, moreover, \( \mu(U_n) \) is computable from \( n \).

These pointwise versions allow us to treat measurable functions as true functions taking values \( x \) and providing values \( f(x) \). For example, the Schnorr randomness version of the Lebesgue differentiation theorem (Theorem 18) can be strengthened to include a limit.

**Theorem 37** (Pathak, Rojas, Simpson [PRS14], Rute [Rut13, Thm. 4.10, p. 46, Thm. 12.3, p. 56]). \textit{The following are equivalent for a real \( x \in [0, 1] \).}
The real \( x \) is Schnorr random.

For every Brouwer/Schnorr effectively integrable function \( f : [0, 1] \to \mathbb{R} \),

\[
\lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(y) \, dy = f(x).
\]

5.6. The equivalence of the three approaches and the connection with randomness. The three approaches — point-free, Martin-Löf, and Brouwer/Schnorr — are all essentially equivalent. This next theorem is stated for effectively measurable functions, but also holds for effectively measurable sets, effective \( L^p \) functions, and effective almost uniform convergence.

Theorem 38.

1. A Brouwer/Schnorr effectively measurable function is a Martin-Löf effectively measurable function.

2. The equivalence class of a Martin-Löf effectively measurable function is a point-free effectively measurable function.

3. If \( f : (X, \mu) \to Y \) is a point-free effectively measurable function given by a sequence \( (f_n) \) of basic functions such that

\[
\int \frac{d_Y(f, f_n)}{1 + d_Y(f, f_n)} \, d\mu \leq 2^{-n},
\]

then the partial function \( \tilde{f} : X \to Y \) given by \( \tilde{f}(x) := \lim_n f_n(x) \) (where the limit exists) is a Brouwer/Schnorr effectively measurable function.

Moreover the exceptional set \( \{ x : f_n(x) \text{ diverges} \} \) is a Schnorr null set. Also, if \( f'_n \) is an alternate sequence of basic functions satisfying (5.2), then \( \{ x : \lim_n f_n(x) \neq \lim_n f'_n(x) \} \) is a Schnorr null set. (See Pathak, Rojas, Simpson [PRS14, Thm. 3.9], Rute [Rut13, Prop. 3.18], Demuth and Kučera [DK79, Thm. 4.1], and Bishop and Bridges [BB85, Props. 8.2, 8.3].)

In particular, this above theorem implies that for every point-free effectively measurable function \( f \) and for every Schnorr random \( x \), there is a unique canonical value \( \tilde{f}(x) := \lim_n f_n(x) \). This also shows that most theorems about the pointfree and Martin-Löf effectively measurable functions naturally generalize to the Brouwer/Schnorr effectively measurable functions.

Schnorr randomness is the weakest randomness notion for this purpose. Rute [Rut13, Thm. 12.19, p. 70] showed that there is no weaker randomness notion for which Theorem 38 holds. This again demonstrates how Schnorr randomness naturally arises out of computable and constructive analysis (and that it is more than a coincidence that the Brouwer/Schnorr definition is the pointwise definition adopted by most of the early constructivists).

5.7. Other equivalent representations. Many of the constructive definitions in the literature are equivalent to the Brouwer/Schnorr approach, including the definitions of Brouwer, Demuth, and Bishop. However, there are a few caveats. First, Brouwer’s and Bishop’s definitions are not computable, so one first needs to give them a computable interpretation. Although Brouwer’s definition of measurable set is not defined on a measure one set, it can be extended to one [Hey56, §6.2.2]. This extension is equivalent to the Brouwer/Schnorr approach. Demuth’s definitions are
restricted to the computable reals, but these definitions naturally extend to the set of real numbers.

While verifying all these equivalences would take us too far afield, much of the work can be done via the following lemma. Extending the definition from Section 4 if $X$ and $Y$ are computable metric spaces and $\mu \in \mathcal{M}_1(X)$ is computable, then a partial function $f : (X, \mu) \to Y$ is almost everywhere computable if there is a $\Pi^0_2$ subset $A \subseteq X$ of $\mu$-full measure such that $f : A \to Y$ is computable. (These are just the functions that are computable almost surely. A definition in this regard, avoiding mention of $\Pi^0_2$ sets, can be found in Rute [Rut16a, Defs. 7.1, 7.4].)

**Lemma 39** (See Rute [Rut13] §3, p. 36). Let $X$ and $Y$ be a computable metric spaces and $\mu \in \mathcal{M}_1(X)$ be a computable probability measure.

1. Every computable function $f : X \to Y$ is Brouwer/Schnorr effectively measurable.
2. Every almost everywhere computable function $f : X \to Y$ is Brouwer/Schnorr effectively measurable.
3. If $(f_n)$ is a computable sequence of Brouwer/Schnorr effectively measurable functions such that (the equivalence classes of) $(f_n)$ converge point-free effectively almost uniformly (Definition 33), then $f_n$ converges Brouwer/Schnorr effectively almost uniformly and the pointwise limit $f = \lim_n f_n$ is Brouwer/Schnorr effectively measurable.
4. For computable $p$, the Brouwer/Schnorr effectively $L^p$ functions $f : (X, \mu) \to \mathbb{R}$ are exactly the Brouwer/Schnorr effectively measurable functions such that $\|f\|_{L^p}$ is computable.
5. The Brouwer/Schnorr effectively measurable sets $A$ are exactly the sets such that $1_A$ is Brouwer/Schnorr effectively measurable.

Also, inner and outer regularity along with Luzin’s theorem provide convenient representations which are, respectively, equivalent to the Brouwer/Schnorr effectively measurable sets and the Brouwer/Schnorr effectively measurable functions.

- **(Inner and outer regularity, Schnorr layerwise decidability)** The Brouwer/Schnorr effectively measurable sets $A$ are exactly those with a computable sequence of effectively closed sets $C_n$ and effectively open sets $U_n$ such that $C_n \subseteq A \subseteq U_n$, $\mu(C_n)$ is computable in $n$, $\mu(U_n)$ is computable in $n$, and $\mu(U_n - C_n) \leq 2^{-n}$. (The sequence $C_n$ can also be modified to be compact — in constructive analysis terminology these are called effectively located sets, in computable analysis these are called computable sets.) [Rut13] Prop. 3.22, p. 41

- **(Luzin’s theorem, Schnorr layerwise computability)** The Brouwer/Schnorr effectively measurable functions $f : (X, \mu) \to Y$ are exactly those with a computable sequence of closed (or even effectively located/computable) sets

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36 Also, to be pedantic, in our definition of, say, Brouwer/Schnorr integrable function there are $2^{2^{\aleph_0}}$ Brouwer/Schnorr effectively integrable functions. For example, any function $f$ a.e. equal to 0 is Brouwer/Schnorr effectively integrable if $\{x : f(x) \neq 0\}$ is a Schnorr null set. (In this way, our Brouwer/Schnorr representation is a multi-representation, whereby each name corresponds to a set of objects.) Whereas, some otherwise equivalent definitions of effectively integrable functions require that $f(x) = \lim_n g_n(x)$ for a computable sequence of simple functions and that the domain of $f$ is exactly the set of $x$ for which that limit converges. In this case, there would only be countably many Brouwer/Schnorr integrable functions, and every Brouwer/Schnorr integrable function would be Borel-measurable.
such that \( \mu(K_n) \leq 1 - 2^{-n} \) and \( \mu(K_n) \) is computable in \( n \) and there is a sequence of computable functions \( f_n : K_n \to Y \) such that \( f \upharpoonright K_n = f_n. \)

For the constructive version of these results, see Spitters [Spi05]. These above definitions can be modified so that they are equivalent to the Martin-Löf effectively measurable sets and functions by removing the restriction that \( \mu(C_n) \), \( \mu(U_n) \), and \( \mu(K_n) \) are computable. These notions are called layerwise decidable sets and layerwise computable functions [HR09a, HR09c]. The idea is that if \( K_n \) is the complement of the universal Martin-Löf test, then to compute \( f(x) \) for a Martin-Löf random \( x \), one only needs to know \( x \) and an upper bound on the least \( n \) such that \( x \in K_n \). This least \( n \) is known as the randomness deficiency of \( x \). Layerwise computability is a useful notion, because it gives a very quick and intuitive method for showing that a function \( f \) is Martin-Löf effectively measurable.

5.8. Other non-equivalent representations. While most of the definitions in the literature align with the ones given above, it should be mentioned that there are other useful representations. For example, just as there are computable reals and reals computable from below, there are natural representations of what it means for a measure, a real-valued measurable function, and a measurable set to be “computable from below” (or “from above”). In particular, for measurable sets this captures the notion that measurable sets form a locale (as mentioned in Subsection 3.8). See Weihrauch [Wei17]. Also, often it is sufficient to represent a random variable, not as a measurable function, but only as a distribution (probability measure) [Mül99]. Rute’s work [Rut16b, Rut18] shows that it is convenient in randomness to represent a measurable function \( f : (X, \mu) \to Y \) by both a name for \( f \) (as above) and also a name for the conditional probability map \( y \mapsto \mu(\cdot | f = y) \) which is a measurable function of type \( (Y, \mu_f) \to M_1(X) \). It appears that all natural examples of measurable functions are computable in this stronger sense.

Alternatively, in effective descriptive set theory [Mos09] one follows Borel’s transfinite inductive definition to get effectively Borel measurable sets whose measures are hyperarithmetic reals. As Martin-Löf [ML70b] first showed, this leads to its own notion of randomness. This “higher randomness” has since become its own area of study [CY15, Ch. 14]. Coquand [Coq01] showed it is possible to reason constructively about measure theory in the Borel hierarchy using a hyperarithmetic definition of the reals.

5.9. Computing effectively measurable functions. So far our discussion of effectively measurable functions has been a bit abstract. However, those interested in the foundations of computable probability — including probabilistic algorithms and simulating probabilistic processes — are right to ask the question, “Can all this be implemented on a computer?” The answer is yes!

Just as the effectively continuous functions \( f : X \to Y \) are the same as the computable functions from \( X \) to \( Y \) (which can be implemented on a computer — in theory), effectively measurable functions \( f : (X, \mu) \to Y \) are the same as recursively approximable functions (which can be implemented on a computer). The definition goes back to Friedman and Ko [KF82].

Returning to the continuous case, assume \( f : \{0, 1\}^N \to \mathbb{R} \) is computable. Then there is an algorithm \( g : \mathbb{N} \times \{0, 1\}^N \to \mathbb{Q} \) which takes \( x \in \{0, 1\}^N \) and \( n \in \mathbb{N} \), and
returns an approximation \( g(n, x) \) such that \( |g(n, x) - f(x)| \leq 2^{-n} \). In short, this algorithm approximates \( f \) in distance.

For a measurable function, we want an algorithm which approximates \( f \) both in distance and in probability. (The following definitions naturally generalize to any measurable function \( f: (X, \mu) \to Y \). See Bosserhoff [Bos08].)

**Definition 40.** A measurable function \( f: \{\{0, 1\}^N, \mu\} \to \mathbb{R} \) is recursively approximable if there is an algorithm \( g: \mathbb{N} \times \{\{0, 1\}^N \to \mathbb{Q} \) which takes in \( x \in \{0, 1\}^N \) and \( n \in \mathbb{N} \), and outputs an approximation \( g(n, x) \) such that for all \( n \in \mathbb{N} \),

\[
\mu \{ x : |g(n, x) - f(x)| > 2^{-n} \} \leq 2^{-n}.
\]

That is to say, for each \( n \), there is a small probability \( \leq 2^{-n} \) that the algorithm will return a bad approximation. (To be clear, the algorithm need not know the approximation is bad.)

Notice that this definition is point-free in that it is invariant under almost everywhere equivalence. Also, we could modify this definition to allow \( g \) to be partial. Assume \( g \) is the same as above, except that it is partial and

\[
\mu \{ x : g(n, x) \uparrow \lor |g(n, x) - f(x)| > 2^{-n} \} \leq 2^{-n}
\]

where \( g(n, x) \uparrow \) means \( g \) does not halt with those inputs. Then let \( h(n, x) \) be the same as \( g(n + 1, x) \) except that after \( g(n + 1, x) \) has halted for at least \( 1 - 2^{n+1} \) \( \mu \)-measure of the \( x \), we set \( h(n, x) = 0 \) for the rest.

**Theorem 41** (Ko, Thm. 5.12 [Ko91]). The recursively approximable functions are the same as the point-free effectively measurable functions.

**Theorem 42.** A measurable function \( f: \{\{0, 1\}^N, \mu\} \to \mathbb{R} \) is Brouwer/Schnorr effectively measurable if and only if \( f \) is recursively approximable with algorithm \( g \) and

\[
f(x) = \lim_{n \to \infty} g(n, x) \quad \text{on all } x \text{ where } g(n, x) \text{ converges.}
\]

**Proof.** If \( f \) is Brouwer/Schnorr effectively measurable and given by a sequence of basic functions \( (g_n) \), and a Schnorr test \( (U_n) \), then set \( g(n, x) = g_n(x) \) and we have

\[
\mu \{ x : |g(n, x) - f(x)| > 2^{-n} \} \leq \mu(U_n) \leq 2^{-n}.
\]

Hence it \( f \) is recursively approximable. Moreover, for all \( x \notin \bigcap_n U_n \), \( \lim_n g(x, n) = f(x) \). Finally, one can slightly modify \( g(n, x) \) so that it does not converge for any \( x \in \bigcap_n U_n \).

Conversely, if \( f \) is recursively approximable given as the limit of \( g(n, x) \), then the sequence \( g(n, x) \) converges point-free effectively almost uniformly. By Lemma 39, \( f \) is Brouwer/Schnorr effectively measurable. \( \square \)

Now, we give a concrete (and interesting) example of a recursively approximable function which is not just almost everywhere computable.

**Example 43.** Let \( \lambda \) denote the fair-coin measure on \( \{0, 1\}^N \). Let \( \bar{x}_\ell \) denote the frequency of 1’s in the first \( \ell \) bits of \( x \), i.e. \( \frac{1}{\ell} \sum_{k=0}^{\ell-1} x_k \). Let \( f: \{0, 1\}^N \to \mathbb{R} \) be \( f(x) = \sup \bar{x}_\ell \). This function \( f \) is recursively approximable as follows. Given \( n \), by standard probability estimates (e.g. martingale inequalities), there is some \( m \) computable from \( n \) such that

\[
\lambda \left\{ x : \sup_{\ell < m} \bar{x}_\ell = \sup_{\ell} \bar{x}_\ell \right\} > 1 - 2^{-n}.
\]
Therefore, we can estimate $f(x)$ with $g(n, x) = \sup_{\ell < n(m_n)} \bar{x}_\ell$. This estimate is correct (even exact) with probability at least $1 - 2^{-n}$. Since $f$ is recursively approximate and $f(x) = \lim_n g(n, x)$, it is also Brouwer/Schnorr effectively measurable.

Notice, however, that $f$ cannot be almost everywhere computable. If it were, then for almost all strings $x$ such that $f(x) < 2/3$, one could read finitely many bits and be sure that $f(x) < 2/3$. However, it is impossible to know this almost surely from finitely bits of $x$. (Specifically, $\{x : f(x) < 2/3\}$ is a nowhere dense set of positive measure.)

5.10. Obtaining Schnorr randomness through effective Solovay forcing. In Subsection 3.8 we saw that the Boolean algebra of measurable sets modulo a.e. equivalence seems to capture the intuitive notion of randomness. In particular, by forcing with this poset (Solovay random forcing) the resulting generics are exactly those reals which are in every measure one set in the ground model.

Now, we will consider the effective analogue of Solovay’s forcing construction. (Compare to the presentation in Jech [Jec03 pp. 511–515].) Let $B$ be the Boolean algebra of point-free effectively measurable sets. A set $A \in F$ if it is upward-closed and closed under finite meets (that is $A \cap B \in F$ whenever $A \in F$ and $B \in F$). Moreover, $F$ is an ultrafilter if for each $A \in B$ either $A$ or its complement is in $F$. Say that an ultrafilter $G$ is effectively generic if for every computable sequence $A = (A_n)$ of elements in $G$, if $\bigcap_n A_n$ is in $B$, then $\bigcap_n A_n$ is in $G$. For a topological space, let $G_{cpt}$ (resp. $G_{cl}$) be the collection of all compact (resp. closed) sets whose equivalence class is in $G$.

If we are working in a computable probability measure on a computable metric space, then for every point-free effectively measurable set $A \in B$ and every Schnorr random $x$, by Theorem 44, there is a canonical value $\overline{1}_A(x)$ which is either 0 or 1. We write $x \in A$ if $\overline{1}_A(x) = 1$ and $x \notin A$ if $\overline{1}_A(x) = 0$.

Proposition 44. Fix a computable probability measure $\mu$ on a computable metric space $X$ and let $B$ be the Boolean algebra of point-free effectively measurable sets. The following are equivalent for any collection $G \subseteq B$.

1. $G$ is an effectively generic ultrafilter.
2. $G$ is an ultrafilter and $\bigcap G_{cl} = \bigcap G_{cpt} = \{x\}$ for some Schnorr random $x$.
3. $G = \{A \in B : x \in A\}$ for some Schnorr random $x$.

The Schnorr randoms $x$ in (2) and (3) are the same.

Proof: (1) $\rightarrow$ (2): Since $G$ is an effectively generic ultrafilter, $G_{cpt}$ is nonempty and contains subsets of arbitrarily small diameter. (Indeed, one can effectively compute a countable cover of closed balls of computable measure less than $\varepsilon$ [HR09d Lemma 5.1.1].) Then one can find compact subsets of these balls of measure arbitrarily close to the measure of the ball. This gives a computable sequence of compact sets $K_n$ of computable measure such that $\mu(\bigcup_n K_n) = 1$. If $K_n \notin G_{cpt}$ for all $n$, then since $G$ is an ultrafilter, the complement $U_n$ of $K_n$ is in $G$ for all $n$. But $\bigcap_n U_n = \emptyset \mu$-a.e. violating the fact that $G$ is effectively generic.) Since $G$ is a filter, $G_{cpt}$ is closed under finite intersections, and by compactness the intersection $G = \bigcap G_{cpt}$ is nonempty. Since the sets of $G_{cpt}$ have arbitrarily small diameter, the set $G$ is a singleton set $\{g\}$.

We also have that $G_{cl} = \{g\}$ since any effectively measurable, closed set $C \in B$ which does not contain $g$ must be disjoint from some ball around $g$. Therefore, $C$ is
also disjoint from some effectively measurable, compact set $K \in \mathcal{G}_{cpt}$ of arbitrarily small diameter containing $x$. Hence, $\mathcal{G}_{cl}$ is made up precisely of the effectively measurable, closed sets containing $x$.

To see that $g$ is Schnorr random, consider a Schnorr test $(U_n)$ and let $(C_n)$ be the complementary sequence of closed sets. To show $g \notin \bigcap_n U_n$, it suffices to show $C_n \in \mathcal{G}_{cl}$ for some $n$, because then, $g \notin U_n$. Assume for a contraction that $C_n \notin \mathcal{G}_{cl}$ for all $n$. Then $U_n \in \mathcal{G}$ for all $n$. Since $(U_n)$ is a Schnorr test and $\mathcal{G}$ is effectively generic, $\emptyset = \bigcap_n U_n \in \mathcal{G}$ violating that $\mathcal{G}$ is a filter.

(2) $\to$ (3): Assume $\mathcal{G}$ is an ultrafilter and $\bigcap \mathcal{G}_{cl} = \{x\}$. Since $\mathcal{G}$ is an ultrafilter, $\mathcal{G}_{cl}$ is precisely the collection of all effectively measurable, closed sets $C$ which contain $x$. The rest follows from the following two regularity facts for an arbitrary effectively measurable set $A$ and a Schnorr random $x$ (which can be found in Rute [Rut13, Prop. 3.22, p. 41]).

- If $x \notin \tilde{A}$, then there exists a closed, effectively measurable set $C \subseteq A$ $\mu$-a.e. such that $x \in C$.
- If $x \notin \tilde{A}$, then there exists an open, effectively measurable set $U \supseteq A$ $\mu$-a.e. such that $x \notin U$.

(3) $\to$ (1): Assume that $x$ is a Schnorr random and let $\mathcal{G}_x = \{A \in \mathcal{B} : x \in \tilde{A}\}$. To see that $\mathcal{G}_x$ is an effectively generic ultrafilter it is enough to show the following.

- $x \notin \tilde{\emptyset}$.
- For all $A, B \in \mathcal{B}$, if $x \notin \tilde{A}$ and $A \subseteq B$ $\mu$-a.e. then $x \in \tilde{B}$.
- For all $A \in \mathcal{B}$, if $x \in A$ then $x \notin A^c$.
- For any computable sequence $(A_n)$ from $\mathcal{B}$, if $x \in \tilde{A_n}$ for all $n$ and $\bigcap_n A_n$ is effectively measurable, then $x \in \bigcap_n A_n$.

These results can all be found in Rute [Rut13 Prop. 3.28, p. 42].

This result shows that we can consistently extend Schnorr randomness to any arbitrary computable probability space as in Definition 30, even when there is not an underlying computable metric space.

**Definition 45.** For a computable probability space $(X, \mathcal{B}, R, \mu)$, define a Schnorr random to be an effectively generic ultrafilter $\mathcal{G}$ in the Boolean algebra of effectively measurable sets $\mathcal{B}$.

Forcing is important in computability theory and proof theory. See Shore [Sho10] for a survey, Downey and Hirschfeld [DH10] for examples of forcing in computably theory and randomness, and Avigad [Avi04] for examples of forcing in reverse and constructive mathematics. An alternative interpretation of “effective Solovay forcing” is due to Kautz [Kau91] [DH10 §§7.2.5]. It is also known that forcing with effectively closed sets of computable measure can be used to construct Schnorr randoms with pathological properties; see, for example, Yu [Yu11].

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