Generalized Berry phase for a bosonic Bogoliubov system with exceptional points

Yuki Kawaguchi, Nagoya university

arXiv:1904.08724

Collaborators

Terumichi Ohashi (D2, Nagoya Univ.)

Shingo Kobayashi (RIKEN)
Topological Insulators/Superconductors

Bulk: insulator/SC (gapped spectrum)
Surface: metal

Topologically nontrivial structure of the electric wave function in the momentum space

\[ \text{Bi}_{1-x}\text{Sb}_x \]

\[ x = 0.12 \]

→ resulting phenomena are robust

protected by the topology of the system
Cold-atomic Gases

- gaseous clouds of neutral atoms
  confined in a magnetic or optical trap
- isolated in vacuum
  → total energy and number of particles are conserved
  unitary time evolution
- w/ and w/o periodic potential, gauge field, internal degrees of freedom
Topological Phenomena in Cold-atomic Systems

**Chern # of Hofstadter bands**

M. Aidelsburger, et al., Nat. Phys. **11**, 162 ('15)

**Aharonov-Bohm phase in the reciprocal space**

L. Duca, et al., Science **347**, 288 ('15)

**Chiral edge states in synthetic dimension**

M. Mancini, et al., Science **349**, 1510 ('15)
B. K. Stuhl, et al., Science **349**, 6255 ('15)
Bose-Einstein condensates of cold atoms

- Bose-Einstein condensate (BEC)
  macroscopic number of particles are condensed in a single-particle state

- Excitation spectrum from a BEC
  BEC works as a particle bath + Bose-Einstein statistics
  → non-Hermitian Bogoliubov equation
  → complex eigenvalues may appear
Motivation

- $\mathbb{Z}_2$ topological invariant

\[ \nu_{\text{PHS}} = \sum_{n: E_n < 0} \int dk \, i \langle u_{k,n} | \partial_k | u_{k,n} \rangle \quad \text{mod} \ 2\pi \]
\[ = 0 \quad \text{or} \quad \pi \in \mathbb{Z}_2 \]

cannot be defined when particle and hole bands touch

- Bosonic Bogoliubov Spectrum

It often happens that particle and hole bands cross.

We cannot label "particle" or "hole" for eigenstates with complex eigenvalues
Question:
Can we define a topo. charge in the presence of exceptional points?

- Issues in non-Hermitian systems
- Generic properties of bosonic Bogoliubov Hamiltonian
  - particle-hole symmetry & pseudo Hermiticity
  - EPs appear when two real eigenvalue modes change into a pair of complex eigenvalue modes
- Exceptional points can be avoided by introducing complex momentum space
  - Define the $Z_2$ invariant in a inversion symmetric 1D BEC
  - Confirm bulk-edge correspondence in two toy models
Topology of Bogoliubov bands

- Excitation from BEC in a periodic potential
  \[ \text{Brillouin Zone is well-defined} \]

- 2D Haldane model  
  \[ \mathcal{N}_{ch} = \sum_n \int_{BZ} dk \ i \epsilon_{\ell m} \left\langle \frac{\partial}{\partial k_{\ell}} u_n(k) | \tau_3 | \frac{\partial}{\partial k_m} u_n(k) \right\rangle \]
  
  Furukawa & Ueda, New. J. Phys. (2015)

- 1D Inversion symmetric BEC  
  \[ \nu_{IS} = \sum_n \int_{BZ} dk \ i \langle u_n(k) | \tau_3 | \frac{\partial}{\partial k} u_n(k) \rangle \mod 2\pi \]
  
  Engelhardt & Brandes, Phys. Rev. A (2015)

  \[ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

  \[ \rightarrow \text{Bulk-edge correspondence is confirmed} \]

But, ... discussed only the case when all eigenvalues are real. How is the case when complex eigenvalues appear?
Issues in non-Hermitian systems

- bi-orthogonal basis $|u^R_n\rangle \neq |u^L_n\rangle$
  - right eigenvector $H|u^R_n\rangle = E_n|u^R_n\rangle$
  - left eigenvector $\langle u^L_n|H = \langle u^L_n|E_n \iff H^\dagger|u^L_n\rangle = E_n^*|u^L_n\rangle$

- orthonormal relation
  \[
  \langle u^R_m|u^L_n\rangle = \langle u^L_m|u^R_n\rangle = \delta_{mn}
  \]
  \[
  \sum_n |u^R_n\rangle\langle u^L_n| = \sum_n |u^L_n\rangle\langle u^R_n| = 1
  \]
Excitation from BEC in a periodic potential

$\rightarrow$ Brillouin Zone is well-defined

- 2D Haldane model $\text{Furukawa \& Ueda, New. J. Phys. (2015)}$

$$N_{ch} = \sum_n \int_{\text{BZ}} dk \ i\epsilon_{\ell m} \langle \frac{\partial}{\partial k_{\ell}} u_n(k) | \tau_3 | \frac{\partial}{\partial k_m} u_n(k) \rangle \cdot \langle \frac{\partial}{\partial k_{\ell}} u^L_n(k) | \frac{\partial}{\partial k_m} u^R_n(k) \rangle$$

- 1D Inversion symmetric BEC $\text{Engelhardt \& Brandes, Phys. Rev. A (2015)}$

$$\nu_{IS} = \sum_n \int_{\text{BZ}} dk \ i \langle u_n(k) | \tau_3 | \frac{\partial}{\partial k} u_n(k) \rangle \mod 2\pi \cdot \langle u^L_n(k) | \frac{\partial}{\partial k} u^R_n(k) \rangle$$

$\rightarrow$ Bulk-edge correspondence is confirmed

But, ... discussed only the case when all eigenvalues are real. How is the case when complex eigenvalues appear?
Issues in non-Hermitian systems

- bi-orthogonal basis $|u_n^R\rangle \neq |u_n^L\rangle$
  - right eigenvector $H|u_n^R\rangle = E_n|u_n^R\rangle$
  - left eigenvector $\langle u_n^L|H = \langle u_n^L|E_n \iff H^\dagger|u_n^L\rangle = E_n^*|u_n^L\rangle$

- orthonormal relation
  $$\langle u_m^R|u_n^L\rangle = \langle u_m^L|u_n^R\rangle = \delta_{mn}$$
  $$\sum_n |u_n^R\rangle\langle u_n^L| = \sum_n |u_n^L\rangle\langle u_n^R| = 1$$  ※except for exceptional points

- exceptional point: the point where $H$ cannot be diagonalized
  - Jordan normal form
    $$\begin{pmatrix}
    \ddots & 0 & 0 & 0 \\
    0 & E & 1 & 0 \\
    0 & 0 & E & 0 \\
    0 & 0 & 0 & \ddots
    \end{pmatrix}$$
  - reduction of the number of linearly independent eigenvectors $|u_n^R\rangle = |u_{n+1}^R\rangle$
  - left and right eigenvectors become orthogonal: $\langle u_n^L|u_n^R\rangle = 0$
  → Berry connection is ill-defined

  topological charge of exceptional point
  Shen, et al., PRL(18'), Kawabata, et al., arXiv(19')
non-Hermitian Bogoliubov Equation

Hamiltonian (single component BEC in a uniform system)

\[ \hat{\mathcal{H}} = \sum_k (\xi_k - \mu) \hat{a}_k^\dagger \hat{a}_k + \frac{g}{2} \sum_{k_1 k_2 k_3 k_4} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3} \hat{a}_{k_4} \delta_{k_1 + k_2, k_3 + k_4} \]

\[ \approx \frac{1}{2} \sum_k (\hat{a}_k^\dagger \hat{a}_k - \hat{a}_{-k}^\dagger \hat{a}_{-k}) \begin{pmatrix} \xi_k - \mu + g |\psi|^2 & g \psi^2 \\ g (\psi^*)^2 & \xi_{-k} - \mu + g |\psi|^2 \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{a}_{-k}^\dagger \end{pmatrix} \]

\[ \equiv \tilde{H}(k) : \text{エルミート} \]

Bogoliubov transformation

\[ \begin{pmatrix} \hat{a}_k \\ \hat{a}_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_{-k}^* \\ v_k & u_{-k}^* \end{pmatrix} \begin{pmatrix} \hat{b}_k \\ \hat{b}_{-k}^\dagger \end{pmatrix} =: T(k) \begin{pmatrix} \hat{b}_k \\ \hat{b}_{-k}^\dagger \end{pmatrix} \]

para-unitary transformation

\[ T(k)^\dagger \tau_3 T(k) = \tau_3 \]

bosonic commutation relation

\[ [\hat{a}_k, \hat{a}_{k'}^\dagger] = [\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta_{k, k'} \]

Bogoliubov equation

\[ \tau_3 \tilde{H}(k) \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} \]

\[ \equiv H(k) \text{ non-Hermitian "Bogoliubov Hamiltonian"} \]

mean field approx.

\[ \hat{a}_0 \sim \hat{a}_0^\dagger \sim \psi \]
Generic properties of Bogoliubov Eq.

- **Bogoliubov equation**

\[ H(k) |u_n^R(k)\rangle = E_n(k) |u_n^R(k)\rangle \]

\[ H(k) = \begin{pmatrix} H^{(1)}(k) & H^{(2)}(k) \\ -[H^{(2)}(-k)]^* & -[H^{(1)}(-k)]^* \end{pmatrix} \]

- **Pseudo-Hermiticity**

\[ \tau_3 H(k) \tau_3^{-1} = H^\dagger(k) \]

- **Particle-hole symmetry**

\[ \mathcal{C} H(k) \mathcal{C}^{-1} = -H(-k), \quad \mathcal{C} = \tau_1 K \]
Generic properties of Bogoliubov Eq.

**Pseudo-Hermiticity**

\[ \tau_3 H(k) \tau_3^{-1} = H^\dagger(k) \]

- \[ H(k) |u_n^R\rangle = E |u_n^R\rangle \]
- \[ H(k) \tau_3 |u_n^L\rangle = E^* \tau_3 |u_n^L\rangle \]

**Particle-hole symmetry**

\[ \mathcal{C} H(k) \mathcal{C}^{-1} = -H(-k), \quad \mathcal{C} = \tau_1 K \]

- \[ H(-k) \mathcal{C} |u_n^R\rangle = -E^* \mathcal{C} |u_n^R\rangle \]
- \[ H(-k) \tau_3 \mathcal{C} |u_n^L\rangle = -E \tau_3 \mathcal{C} |u_n^L\rangle \]

**Real eigenvalue**

- \[ \tau_3 |u_n^L\rangle = + |u_n^R\rangle \]

**Particle band**

- **Complex eigenvalue**

  - a pair of complex conjugate eigenvalues
  
  \[ |u_n^R\rangle, \quad \tau_3 |u_n^L\rangle = |u_{n'}^R\rangle \]

**Hole band**

- Particle and hole bands are distinguished

- \[ \tau_3 \mathcal{C} |u_n^L\rangle = -\mathcal{C} \tau_3 |u_n^L\rangle = -\mathcal{C} |u_n^R\rangle \]
Emergence mechanism for complex eigenvalues

Complex eigenvalue appears when the particle band and hole band cross

- Total excitation energy is zero
- Instability grows with conserving the total energy of the system
Appearance of the exceptional point

Exchange of the linear dependent partners

\[ |u_1^R(k_0) \rangle \propto |u_2^R(k_0) \rangle \]

\[ E_1 - E_2 \approx 2a \sqrt{k_0 - k} \]

\[ E_1 - E_2: \text{real} \]

\[ E_1 - E_2: \text{pure imaginary} \]
Avoiding exceptional points

\[ H(k) = \begin{pmatrix} H^{(1)}(k) & H^{(2)}(k) \\ -[H^{(2)}(-k)]^* & -[H^{(1)}(-k)]^* \end{pmatrix} \]

\[ \begin{pmatrix} H^{(1)}(k) & H^{(2)}(k) \\ -[H^{(2)}(-k^*)]^* & -[H^{(1)}(-k^*)]^* \end{pmatrix} \]

complex \( k \)

pseudo-Hermiticity holds only on real axis

- extended particle-hole symmetry
  \[ C H(k) C^{-1} = -H(-k^*) \]
  \[ C H(k_x + i k_y) C^{-1} = -H(-k_x + i k_y) \]

\[ \leftarrow \text{sign of } k_y \text{ does not change} \]

- different from 2D
- the imaginary part of \( k \) is introduced just for a theoretical procedure
Avoiding exceptional points

\[ H(k) = \begin{pmatrix} H^{(1)}(k) & H^{(2)}(k) \\ -[H^{(2)}(-k)]^* & -[H^{(1)}(-k)]^* \end{pmatrix} \]

two branches of \( a(k_0 - k)^{1/2} \)

\[ \pm a \sqrt{k_0 - k + g(k)} \quad g(k) \in \mathbb{R} \]

- EP is isolated in complex k plane
- can be avoided by taking a path with nonzero Im k

\( \text{pseudo-Hermiticity holds only on real axis} \)

\( \text{analytic continuation} \)
Choosing integration path

- The choice of the integration path (upper or lower half plane) determines how the eigen-spectrum are continued.
- Choose the integration path such that the bands are continuously labeled, being consistent with the symmetry of the system.

\[
\Lambda(\kappa) = \sqrt{(\kappa - \kappa_1)(\kappa - \kappa_2)}
\]
Choosing integration path

- The choice of the integration path (upper or lower half plane) determines how the eigen-spectrum are continued.

- Choose the integration path such that the bands are continuously labeled, being consistent with the symmetry of the system.

- $\mathbb{Z}_2$ invariant associated with inversion symmetry in 1D is always well-defined

- $\mathbb{Z}_2$ invariant associated with particle-hole symmetry in 1D is well-defined for some special cases.
$\mathbb{Z}_2$ invariant in a 1D inversion symmetric system

- Space inversion: $\mathcal{P}H(k)\mathcal{P}^{-1} = H(-k)$

- Labeling rule:
  A pair of inversion symmetric states have the same sign

| index          | $n > 0$                           | $n < 0$                           |
|----------------|-----------------------------------|-----------------------------------|
| Real eigenvalue| $|u_n^R\rangle = +\tau_3|u_n^L\rangle$ | $|u_n^R\rangle = -\tau_3|u_n^L\rangle$ |
| Complex eigenvalue | Im $E_n > 0$ | Im $E_n < 0$ |

$|u^R\rangle$: eigenstate of $H(k)$
$\mathcal{P}|u^R\rangle$: eigenstate of $H(-k)$

- The above labeling rule results in a symmetric path with respect to $k=0$. 
\( \mathbb{Z}_2 \) invariant in a 1D inversion symmetric system

- **Berry connection matrix**
  \[
  \mathcal{A}_{mn}(k) = \frac{i}{2} \left[ \langle u_m^L(k) | \partial_k u_n^R(k) \rangle + \langle u_m^R(k) | \partial_k u_n^L(k) \rangle \right]
  \]

- **\( \mathbb{Z}_2 \) invariant**
  \[
  (-1)^{\nu_{\text{IS}}} = \exp \left( i \int_{-\pi}^{\pi} dk \sum_{n=1}^{n_x} [\mathcal{A}(k)]_{nn} \right) = \prod_{n=1}^{n_x} \frac{\xi_n(\pi)}{\xi_n(0)}
  \]

**Derivation:**

- Inversion symmetry
  \[
  \sum_{n=1}^{n_x} [\mathcal{A}(-k)]_{nn} + \sum_{n=1}^{n_x} [\mathcal{A}(k)]_{nn} = -\frac{i}{2} \partial_k \left[ \log \det[U^\mathcal{P}_L(k) U^\mathcal{P}_R(k)] \right]
  \]

- Integration \( 0 \leq k \leq \pi \) \rightarrow \[
  \int_{-\pi}^{\pi} dk \sum_{n=1}^{n_x} [\mathcal{A}(k)]_{nn}
  \]

  \[
  [U^\mathcal{P}_L(k)]_{mn} = \langle u_m^L(k) | \mathcal{P} | u_n^R(-k) \rangle
  \]
  \[
  [U^\mathcal{P}_R/RL(0, \pi)]_{nn} = \xi_n(0, \pi)
  \]

- **same as the former work** [Engelhardt & Brandes, PRA(15)]
- **applicable in the presence of exceptional points**
Toy Model 1: spin-orbit coupled 1D BEC

- Hamiltonian:
  \[ \hat{H} = \sum_j \left[ -t(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j+1,\uparrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j+1,\downarrow} + \text{H.c.}) + h(\hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\downarrow} - \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\uparrow}) + \lambda(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j+1,\downarrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j-1,\downarrow} + \text{H.c.}) + \frac{g}{2}(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow}^\dagger + \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\uparrow}^\dagger) \right] \]

- Condensed state: \( \uparrow \) atoms are uniformly distributed in all site.

- Phase diagram:
Toy Model 1: spin-orbit coupled 1D BEC

- Hamiltonian:
  \[ \hat{H} = \sum_j \left[ -t(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j+1,\uparrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j+1,\downarrow}) + \hbar(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\downarrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\uparrow}) + \lambda(\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j+1,\downarrow} - \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j+1,\uparrow}) + \frac{g}{2} (\hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\uparrow}^\dagger \hat{a}_{j,\uparrow} \hat{a}_{j,\uparrow} + \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\downarrow}^\dagger \hat{a}_{j,\downarrow} \hat{a}_{j,\downarrow}) \right] + \text{H.c.} \]

- Condensed state: \( \uparrow \) atoms are uniformly distributed in all sites.

- Phase diagram:
  
  ![](phase_diagram.png)
  
  - Periodic boundary
  - Open boundary
  
  \( \nu_{IS} = 0 \) (l)
  \( \nu_{IS} = 1 \)
  Unstable

Nagoya University, Sep. 25, 2019
Toy Model 2: 1D superlattice (SSH model)

- Hamiltonian:
  \[ \hat{\mathcal{H}} = \sum_j \left[ -t_1 (\hat{a}_{j,A}^\dagger \hat{a}_{j,B} + \hat{a}_{j,B}^\dagger \hat{a}_{j,A}) 
  - t_2 (\hat{a}_{j+1,A}^\dagger \hat{a}_{j,B} + \hat{a}_{j,B}^\dagger \hat{a}_{j+1,A}) 
  + \frac{g}{2} (\hat{a}_{j,A}^\dagger \hat{a}_{j,A}^\dagger \hat{a}_{j,A} \hat{a}_{j,A} + \hat{a}_{j,B}^\dagger \hat{a}_{j,B}^\dagger \hat{a}_{j,B} \hat{a}_{j,B}) \right] \]

- Condensed state: all sites are equivalently occupied

- Phase diagram:
  - \( t_1/t_2 \)
  - \( \nu_{\text{IS}} = 1 \) (unstable)
  - \( \nu_{\text{IS}} = 0 \) (stable)
  - \( t_1, t_2 > 0 \)
Toy Model2: 1D superlattice (SSH model)

- Hamiltonian:
  \[
  \hat{\mathcal{H}} = \sum_j \left[ -t_1 (\hat{a}_{j,A} \hat{a}_{j,B} + \hat{a}_{j,B} \hat{a}_{j,A}) 
  - t_2 (\hat{a}_{j+1,A} \hat{a}_{j,B} + \hat{a}_{j,B} \hat{a}_{j+1,A}) 
  + \frac{g}{2} (\hat{a}_{j,A} \hat{a}_{j,A} \hat{a}_{j,A} \hat{a}_{j,A} + \hat{a}_{j,B} \hat{a}_{j,B} \hat{a}_{j,B} \hat{a}_{j,B}) \right]
  \]

- Condensed state: all sites are equivalently occupied

- Phase diagram:
  \[ \frac{t_1}{t_2} \]
  - \[ \nu_{IS} = 1 \] (a)
  - \[ \nu_{IS} = 0 \] (c)

  - Open boundary
  - Periodic boundary

  \( t_1, t_2 > 0 \)
Summary

Question:
Can we define a topo. charge in the presence of exceptional points?

→ YES

- Exceptional points can be avoided by introducing complex momentum space
- We have defined the $\mathbb{Z}_2$ invariant in a inversion symmetric 1D BEC
- We have confirmed bulk-edge correspondence in two toy models

T. Ohashi, S. Kobayashi, YK, arXiv:1904.08724