ARITHMETIC PROPERTIES OF TRACES OF SINGULAR MODULI ON CONGRUENCE SUBGROUPS

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Abstract. After Zagier proved that the traces of singular moduli \( j(z) \) are Fourier coefficients of a weakly holomorphic modular form, various properties of the traces of the singular values of modular functions mostly on the full modular group \( \text{PSL}_2(\mathbb{Z}) \) have been investigated such as their exact formulas, limiting distribution, duality, and congruences. The purpose of this paper is to generalize these arithmetic properties of traces of singular values of a weakly holomorphic modular function on the full modular group to those on a congruence subgroup \( \Gamma_0(N) \).

1. Introduction

Singular moduli are special values of the classical modular invariant \( j(z) \) at imaginary quadratic arguments in the upper half plane \( \mathbb{H} \). These important algebraic numbers have a long history in number theory. Recently, the work of Borcherds [3] and Zagier [21] have inspired many works on connecting the traces of singular moduli to the Fourier coefficients of weakly holomorphic modular forms or Harmonic Maass forms of half-integral weight. (See [19] for the list of references.)

To describe the context of the work of Borcherds and Zagier, we first define the modular trace of a weakly holomorphic modular function. In this paper, \( D \) is always a positive integer congruent to 0 or 3 modulo 4 unless otherwise specified. For each non-square \( D \), we let \( Q_{D,N} \) denote the set of positive definite integral binary quadratic forms

\[
Q(x, y) = [Na, b, c] = Nax^2 + bxy + cy^2
\]

with discriminant \(-D = b^2 - 4Nac\). The congruence subgroup \( \Gamma_0(N) \subseteq \Gamma(1) := \text{PSL}_2(\mathbb{Z}) \) acts on the set \( Q_{D,N} \). For a fixed solution \( \beta \pmod{2N} \) of \( \beta^2 \equiv -D \pmod{4N} \), a smaller set \( Q_{D,N,\beta} = \{[Na, b, c] \in Q_{D,N} \mid b \equiv \beta \pmod{2N}\} \) is still invariant under the action of \( \Gamma_0(N) \).

A significance of these forms is that there is a canonical bijection between \( Q_{D,N,\beta}/\Gamma_0(N) \) and \( Q_{D,1}/\Gamma(1) \) when the discriminant \( D \) is not divisible by the square of any prime divisor of \( N \) [16]. For each quadratic form \( Q \) in \( Q_{D,N,\beta} \), the corresponding Heegner point on the modular curve \( X_0(N) \) is the unique root of \( Q(x, 1) \),

\[
z_Q = \frac{-b + i\sqrt{D}}{2Na} \in \mathbb{H}.
\]

Denoting the stabilizers of \( Q \) in \( \Gamma_0(N) \) by \( \Gamma_0(N)_Q \), we define the trace of a weakly holomorphic modular function \( f \) on \( \Gamma_0(N) \) by

\[
t_f(D) = \sum_{Q \in Q_{D,N,\beta}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)_Q|} f(z_Q).
\]
In particular, the class number $H_N(D)$ is given by
\begin{equation}
H_N(D) = t_1(D) = \sum_{Q \in Q_{D,N,\beta}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)Q|}
\end{equation}
which is the Hurwitz-Kronecker class number $H(D)$ when the level $N = 1$.

We consider a generalized Hilbert class polynomial $H_D$ by
\begin{equation}
H_D(X) = \prod_{Q \in Q_{D,1}/\Gamma(1)} (X - j(z_Q))^{1/|\Gamma(1)Q|}
\end{equation}
that reduces to the Hilbert class polynomial when $-D < 0$ is a fundamental discriminant. If $q = e^{2\pi iz}$ and $j_1(z) = j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots$ denotes the normalized Hauptmodul for $\Gamma(1)$, then the $q$-expansion of the polynomial is given by [21, Eq. (11)]
\begin{equation}
H_D(j(z)) = q^{-H(D)}(1 - t_1(D)q + O(q^2)).
\end{equation}
In [3, p. 204], [21, Theorem 3], Borcherds proved that
\begin{equation}
H_D(j(z)) = q^{-H(D)} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2,D)},
\end{equation}
where $A(d,D)$ are the Fourier coefficients of a weight $1/2$ weakly holomorphic modular form. More precisely, for $D \geq 0$, the set of the functions
$$f_D(z) = q^{-D} + \sum_{d \equiv 0,3 (\text{mod} 4)} A(d,D)q^d$$
form a unique basis of the space of weakly holomorphic modular forms of weight $1/2$. Comparing equations (1.4) and (1.5), we see that $t_1(D) = A(1,D)$ for all $D > 0$. On the other hand, Zagier showed that
$$g_d(z) = q^{-d} - \sum_{D \equiv 0,3 (\text{mod} 4)} A(d,D)q^D$$
is a weakly holomorphic modular form of weight $3/2$ and the set \{$g_d|d > 0$\} form a basis of the space of weakly holomorphic modular forms of weight $3/2$. Accordingly, $t_1(D)$ is the coefficient of $q^D$ in the Fourier expansion of $g_1(z)$.

This type of duality relation involving traces of the values of Niebur-Poincaré series on $\Gamma(1)$ has been found by Bringmann and Ono [7, Theorems 1.1 and 1.2]. For a non-negative integer $m$ and complex numbers $s$ and $z = x + iy$ with $y > 0$, we define the weight zero $m$th Niebur-Poincaré series on a congruence subgroup $\Gamma$ by
\begin{equation}
\mathcal{F}_m(z,s) = \sum_{M \in \Gamma_\infty \setminus \Gamma} e(-m\text{Re}Mz)(\text{Im}Mz)^{1/2}I_{s-1/2}(2\pi m\text{Im}Mz),
\end{equation}
where $\Gamma_\infty \subset \Gamma$ is the subgroup of translations, $e(z) = e^{2\pi iz}$, and $I_{s-1/2}$ is the modified Bessel function of the first kind. Niebur observed that every modular function on $\Gamma(1)$ that is holomorphic away from the cusp at infinity can be written as a linear sum of these Poincaré series and Duke [10] utilized this property of the Poincaré series to obtain explicit formulas for the traces of CM values of Hecke type Faber polynomials $j_m(z) = P_m(j(z))$ in terms of Kloosterman sums.
and the class number $H(D)$. Extending their ideas to a modular function of prime level $p$, the authors and Choi and Jeon [9] derived exact formulas for the traces of singular values on $\Gamma_0(p)$.

Moving on to divisibility properties of the modular trace, Ahlgren and Ono [1] showed that $t_j(p^2D) \equiv 0 \pmod{p}$ for an odd prime that splits in $\mathbb{Q}(\sqrt{-D})$ and Osburn [20] generalized the congruence relation to that of the traces of CM values of Hauptmodul with prime level. In [8, Theorem 1.4], the authors with Choi and Jeon established Tramer type divisibility property of the traces of singular moduli with arbitrary level.

Although, Bruinier and Funke [5] generalized Zagier’s result by showing that the modular traces of any weakly holomorphic modular functions are Fourier coefficients in the holomorphic part of a Harmonic weak Maass form, most results on traces of singular moduli are on level 1 case. In this paper, we generalize the properties of traces of singular moduli discussed in this introduction such as duality, exact formulas, and congruences to the traces of singular values of any modular function that is completely determined by its principal part at infinity. In the next section, we will state our results with interesting examples. Then in the three sections that follow we give the proofs of our theorems. In the last section, we discuss very briefly a property of the modular trace that we don’t treat in this paper.

2. statements of results and Examples

2.1. Exact formulas for traces of singular moduli on congruence subgroups. Throughout the rest of the paper, $\Gamma$ always denotes the group $\Gamma^*_0(N)$ generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions. Then the set $\mathcal{Q}_{D,N}$ is invariant under the action of $\Gamma$. We can define the trace of a weakly holomorphic modular function with respect to $\Gamma$ as

$$t^*_f(D) = \sum_{Q \in \mathcal{Q}_{D,N}/\Gamma} \frac{1}{|\Gamma_Q|} f(z_Q).$$

We denote $H_N^*(D)$ by the corresponding class number, i.e., $t^*_f(D)$. It is easy to see that if $a$ is the number of prime divisors $p$ of $(\beta, N)$ such that $p \nmid N/(\beta, N)$, then

$$t^*_f(D) = \frac{1}{2^a} t_f(D).$$

For this reason, we may study $t^*_f(D)$ for $t_f(D)$ avoiding the inconvenience of dealing with $\beta$ in $\mathcal{Q}_{D,N,\beta}$. Furthermore, using a theta lift, we establish the following relation between class numbers that will be needed later.

**Theorem 2.1.** For every positive integer $D$ congruent to 0 or 3 modulo 4,

$$(2.2) \quad H_N^*(4D) = 3H(D).$$

Recall the Niebur-Poincaré series $F_m(z, s)$ on $\Gamma$ in (1.6). When $m = 0$, the series is the non-holomorphic Eisenstein series for $\Gamma$ of weight zero. The Poincaré series $F_m(z, s)$ converges absolutely for $\text{Re} \, s > 1$ and can be analytically continued to the entire $s$ plane and it has no poles at $\text{Re}(s) = 1$ [18, Theorem 5]. Moreover, it is an eigenfunction for the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ with eigenvalue $s(1-s)$ so that $F_m(z, 1)$ is annihilated by the Laplacian $\Delta$ and thus almost holomorphic on $\mathbb{H}$. Niebur showed that any modular function on $\Gamma(1)$ that is holomorphic away from the cusp at infinity is a linear combination of the $F_m(z, 1)$ on $\Gamma(1)$ [18, Theorem 6]. Using the similar argument in [9, p. 4], we can generalize the result to $F_m(z, 1)$ on $\Gamma$. 
Proposition 2.2. Suppose $f$ is a modular function on $\Gamma$ whose poles are supported only at infinity and the principal part is given by $\sum_{m=1}^\ell a_m e(-mz)$. If we define

$$F_m^*(z, s) = (2\pi \sqrt{m}) F_m(z, s) + c_m,$$

where $-c_m$ is the constant term in $(2\pi \sqrt{m}) F_m(z, 1)$, then

$$f(z) = \sum_{m=1}^\ell a_m F_m^*(z, 1).$$

For example, a Faber polynomial $j_m(z) = m(j - 744)\mid_{T_m}$ on $\Gamma(1)$ satisfies

$$j_m(z) = F_m^*(z, 1) = 2\pi \sqrt{m} F_m(z, 1) - 24 \sigma(m),$$

where $|T_m$ is the weight 0 Hecke operator and $\sigma(m)$ is the divisor function [6, 10]. If $N$ is a prime $p$ and $p^\alpha \mid m$, then for a modular function $f$ satisfying the conditions in Proposition 2.2, we have [9]

$$(2.4) \quad f(z) = \sum_{m=1}^\ell a_m \left( 2\pi \sqrt{m} F_m(z, 1) - 24 \left( \frac{-p^{\alpha+1}}{p+1} \sigma(m/p^\alpha) + \sigma(m) \right) \right).$$

The constant $c_m$ for arbitrary level $N$ can be determined explicitly using properties of Ramanujan sum.

Theorem 2.3. Let $F_m(z, s)$ be the Poincaré series defined in (1.6). Then the constant term $-c_m$ in $(2\pi \sqrt{m}) F_m(z, 1)$ is given by

$$(2.5) \quad -c_m = 24 \frac{m}{m_N} \sigma(m_N) \prod_{p \mid N} (1 - p^{-2})^{-1} \sum_{e \mid N} \frac{1}{e} \prod_{p \mid \frac{N}{e}} \delta_e(p) p^{1 - \beta_{e,p}} \left( (1 - p^{\beta_{e,p} - \alpha - 2})(1 + p^{-1}) - 1 \right).$$

Here $\alpha_p := \text{ord}_p(m)$, $\beta_{e,p} = \text{ord}_p(N/e)$, $m_N$ denotes the largest exact divisor of $m$ satisfying $(m_N, N) = 1$, and $\delta_e(p)$ is defined by

$$\delta_e(p) = \begin{cases} 1, & \text{if } \beta_{e,p} \leq \alpha_p + 1, \\ 0, & \text{otherwise}. \end{cases}$$

The constant $c_m = -24 \sigma(m)$ when $N = 1$ and $c_m = -24 \left( \frac{-p^{\alpha+1}}{p+1} \sigma(m/p^\alpha) + \sigma(m) \right)$ when $N = p$ and $p^\alpha \mid m$. (See (2.3) and (2.4), respectively.) If $N = p^t$, a prime power, then

$$c_m = -24 \frac{p^\alpha \sigma(m/p^\alpha)}{1 - p^{-2}} \times \begin{cases} p^{1-t}(1 - p^{t-\alpha-2})(1 + p^{-1}) - 1 + p^{-t}, & \text{if } t \leq \alpha + 1, \\ p^{-t}, & \text{otherwise}. \end{cases}$$

Moreover, if $m$ is relatively prime to $N$, then

$$c_m = -24 \sigma(m) \frac{\sum_{e \mid N} \mu(N/e) e}{N^2 \prod_{p | N} (1 - p^{-2})},$$

where $\mu(n)$ is the Möbius function. This implies that

$$c_m = \sigma(m)c_1 \quad \text{whenever} \quad (m, N) = 1.$$
By Proposition 2.2, one may write the traces of the singular values of \( f \) in terms of the traces of Poincaré series \( \mathcal{F}_m^*(z, 1) \) that lead to exact formulas for the traces of the values of \( f \) at Heegner points. We first find explicit formulas for the traces of the values of \( \mathcal{F}_m^* \) at Heegner points.

**Lemma 2.4.** Let \( \mathcal{F}_m^*(z, s) = (2\pi \sqrt{m}) \mathcal{F}_m(z, s) + c_m \), where \( \mathcal{F}_m(z, s) \) is defined in (1.6) and \(-c_m\) is the constant term in \((2\pi \sqrt{m}) \mathcal{F}_m(z, 1)\) which is explicitly given in (2.5). Then the trace of the values of \( \mathcal{F}_m^*(z, s) \) at Heegner points is given by

\[
t_m(D) := \sum_{Q \in \mathcal{Q}_{D,N} / \Gamma} \frac{1}{|\Gamma_Q|} \mathcal{F}_m^*(z_Q, 1) = c_m H_N^*(D) + \sum_{c > 0} S_D(m, c) \sinh \left( \frac{4\pi m \sqrt{D}}{c} \right),
\]

where

\[
(2.11) \quad S_D(m, c) = \sum_{x^2 \equiv -D \mod c} e(2mx/c) \quad \text{for any positive integers } m \text{ and } c.
\]

It follows from Proposition 2.2 and Lemma 2.4 that

**Corollary 2.5.** Suppose that \( f \) is a modular function for \( \Gamma \) whose poles are supported only at \( \infty \) and the principal part is given by \( \sum_{m=1} a_m e(-mz) \). Then

\[
t_f^*(D) = \sum_{m=1} \sum_{c > 0} c_m H_N^*(D) + \sum_{c > 0} S_D(m, c) \sinh \left( \frac{4\pi m \sqrt{D}}{c} \right),
\]

where \(-c_m\) is given in (2.5).

**Example 1.** Consider

\[
j_{45}^* = -1 + \left( \frac{\eta(3z)^2 \eta(15z)^2}{\eta(z) \eta(5z) \eta(9z) \eta(45z)} \right),
\]

where \( \eta(z) \) is the Dedekind eta function defined by \( \eta(z) = q^{1/24} \prod_{n=1}^\infty (1 - q^n) \). Then \( j_{45}^* \) is the Hauptmodul for \( \Gamma_0^*(45) \) which is of genus 0 and has a Fourier expansion of the form \( q^{-1} + 0 + O(q) \). Let \( D = 20 \). Since the representatives for \( \mathcal{Q}_{20,45,40}/\Gamma_0(45) \) are given by \([405,40,1]\) and \([90,-50,7]\), we find from equations (2.1), (2.8) and (2.12) that

\[
j_{45}^*(z_{[405,40,1]}) + j_{45}^*(z_{[90,-50,7]}) = -1 + 2 \sum_{c > 0} S_{20}(1, c) \sinh \left( \frac{8\pi \sqrt{5}}{c} \right).
\]

The left hand side of the equation is known to be \(-3\), and hence the exponential sum on the right hand side has the value \(-1\).

**Example 2.** Let \( P_m(j_4^*) \) denote the polynomial of the Hauptmodul \( j_4^*(z) \) on \( \Gamma_0^*(4) \) with Fourier development of the form \( P_m(j_4^*)(z) = q^{-m} + O(q) \). For an odd integer \( m \), we obtain from
Corollary 2.5 and (2.8) that
\[
(2.13) \quad t^*_m(j^*_4)(4D) = -8\sigma(m)H_4^*(4D) + \sum_{c > 0 \atop c \equiv 0 \pmod{16}} S_{4D}(m, c) \sinh \left( \frac{4\pi m \sqrt{4D}}{c} \right).
\]
But by an analogous proof to [8, Theorem 1.2], we find that
\[
(2.14) \quad t^*_m(j^*_4)(4D) = -24\sigma(m)H(D).
\]
Comparing two equations (2.13) and (2.14) and applying Theorem 2.1, we find that
\[
(2.15) \quad \sum_{c > 0 \atop c \equiv 0 \pmod{16}} S_{4D}(m, c) \sinh \left( \frac{4\pi m \sqrt{4D}}{c} \right) = 0
\]
for all positive integer $D \equiv 0, 3 \pmod{4}$.

2.2. Duality and Divisibility. For more arithmetic properties of traces of the values of Niebur-Poincaré series and modular functions holomorphic away from the cusp at infinity, we make the Kloosterman sum representation of the trace of the Niebur-Poincaré series. In order to define Kloosterman sum, we need the extended Kronecker symbol $\left( \frac{c}{d} \right)$ and
\[
\varepsilon_d := \begin{cases} 
1, & \text{if } d \equiv 1 \pmod{4}, \\
 i, & \text{if } d \equiv 3 \pmod{4}
\end{cases}
\]
that is defined for odd $d$. For $c, m, n, \lambda \in \mathbb{Z}$ with $c \equiv 0 \pmod{4}$, the weight $k := \lambda + 1/2$ Kloosterman sum $K_\lambda(m, n, c)$ is defined by
\[
(2.16) \quad K_\lambda(m, n, c) := \sum_{v \pmod{c}} \left( \frac{c}{v} \right) \varepsilon_v^{2\lambda + 1} e \left( \frac{mv + nv}{c} \right),
\]
where the sum runs through the primitive residue classes modulo $c$ and $v \bar{v} \equiv 1 \pmod{c}$.

Following the method developed in [13], [4], [6], [7], and [17], we construct a half integral weight Maass-Poincaré series for arbitrary level $4N$ whose holomorphic coefficients are represented by the Kloosterman sums: For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we define
\[
\mathcal{M}_s(y) := |y|^{-k/2} M_{\frac{3}{2} \text{sgn}(y), s - \frac{1}{2}}(|y|),
\]
where $M_{\nu, \mu}$ is the usual M-Whittaker function. And for $m \geq 1$ with $(-1)^{\lambda+1} m \equiv 0, 1 \pmod{4}$, we define
\[
\varphi_{-m, s}(z) := \mathcal{M}_s(-4\pi my)e(-mx).
\]
With these notations, we define the Poincaré series for $\text{Re}(s) > 1$ by
\[
(2.17) \quad \mathfrak{P}_{\lambda, n}(-m, s; z) := \sum_{M \in \Gamma_{\infty}\backslash \Gamma_0(4N)} (\varphi_{-m, s}\chi_M)(z)
\]
where $|_k$ is the usual weight $k$ slash operator.

Now as in [7] and [17], we apply Kohnen’s projection operator [15, p. 250] $\text{pr}_\lambda$ to (2.17) to obtain a new family of weak Maass forms.
\begin{equation}
\mathcal{P}_{\lambda,N}(-m, z) := \begin{cases} \frac{3}{2} \mathcal{P}_{\lambda,n}(-m, \frac{k}{2}; z) |pr_\lambda, & \text{if } \lambda \geq 1; \\ \frac{3}{2(1-k)} \mathcal{P}_{\lambda,n}(-m, 1 - \frac{k}{2}; z) |pr_\lambda, & \text{if } \lambda \leq 0. \end{cases}
\end{equation}

The series $\mathcal{P}_{\lambda,N}(-m, z)$ is a weakly holomorphic modular form of weight $\lambda + 1/2$ and level $4N$ satisfying Kohnen plus-condition if $\lambda > 1$ and it is a weak Maass form if $\lambda \leq 1$ that has Fourier expansion
\begin{equation}
\mathcal{P}_{\lambda,N}(-m, z) = q^{-m} + \sum_{n \geq 0} b_{\lambda,N}(-m, n) q^n + \mathcal{P}_{\lambda,N}^-(m, z),
\end{equation}
where $\mathcal{P}_{\lambda,N}^-(m, z)$ is the non-holomorphic part.

It follows from [17, Theorem 2.1] that if $m, n$ are positive integers such that $(-1)^{\lambda+1} m, (-1)^\lambda n \equiv 0, 1 \pmod{4}$ and $N$ is odd, then the Fourier coefficients $b_{\lambda,N}(-m, n)$ of the weak Maass form $\mathcal{P}_{\lambda,N}(-m, z)$ is given by
\begin{equation}
b_{\lambda,N}(-m, n) := (-1)^{\frac{\lambda+1}{2}} \sqrt{2} (n/m)^{\frac{\lambda+1}{2}} (1 - (-1)^\lambda i) \sum_{\substack{c > 0 \\text{n} | 4N | c}} \frac{K_{\lambda}(m, n, c)}{c} \delta_1(c/4) I_{\lambda - 1/2} \left( \frac{4\pi \sqrt{nm}}{c} \right),
\end{equation}
where $\delta_1(d) = 2$ if $d$ is odd and 1 otherwise.

Now, the trace of the values of Niebur-Poincaré series at Heegner points is a linear sum of these coefficients and class numbers.

**Lemma 2.6.** Let $t_m(D)$ be the trace of the Poincaré series of odd level $N$ at Heegner points in (2.10). If $(m, N) = 1$, then
\begin{equation}
t_m(D) = -\sum_{\nu | m} \nu B(-\nu^2, D),
\end{equation}
where $B(-m, n)$ is given by
\begin{equation}
B(-m, n) = -c_1 \delta_1(m) H_N^*(n) + b_{1,N}(-m, n).
\end{equation}
Here $-c_1$ is the constant given in (2.8) with $m = 1$, the function $\delta_1(m) = 1$ if $m$ is a square and zero otherwise, and $b_{1,N}(-m, n)$ is given in (2.20) with $\lambda = 1$.

Generalizing the duality relation $b_{\lambda,1}(-m, n) = -b_{1,\lambda,1}(-n, m)$ by Bringmann and Ono [7, Theorem 1.1], we can establish a duality relation for $B(-m, n)$ in (2.21). Using the Bruinier-Funke theta lift as in [8], one can construct a weight $3/2$ Harmonic weak Maass form $G_N(z)$ whose holomorphic part is $\sum_D H_N^*(D) q^D$. Let
\begin{equation}
\mathcal{P}_{1,N}^*(m, z) := \mathcal{P}_{1,N}(-m, z) + (-c_1) \delta_1(m) G_N(z).
\end{equation}
In addition, we define
\begin{equation}
\mathcal{P}_{0,N}^*(m, z) := \mathcal{P}_{0,N}(-m, z) + c_1 H_N^*(m) \theta(z)/2,
\end{equation}
where $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ is the Jacobi theta series. Then we have the following duality relation between coefficients of $\mathcal{P}_{1,N}^*(m, z)$ and $\mathcal{P}_{0,N}^*(m, z)$.
**Theorem 2.7.** Let \( P^*_1,N(-m,z) \) and \( P^*_{0,N}(-m,z) \) be weak Maass forms defined in (2.22) and (2.23), respectively, with \( N \) odd. Assume they have Fourier coefficients of \( q^n \) for \( n \geq 0 \), \( B_{1,N}(-m,n) \) and \( B_{0,N}(-m,n) \), respectively. If \( m \) is a positive integer that is a square modulo \( 4N \), then for every positive integer \( n \) with \( -n \equiv \emptyset \pmod{4N} \), we have

\[
B_{1,N}(-m,n) = -B_{0,N}(-n,m).
\]

Note that \( B(-m,n) \) in (2.21) is the coefficient \( B_{1,N}(-m,n) \) in the holomorphic part of \( P^*_1,N(-m,z) \).

Lastly, we examine a divisibility property of the traces of singular moduli that is a generalization of [1] and [20].

**Theorem 2.8.** Suppose that \( f \) is a modular function for \( \Gamma \) with odd level \( N \) which is holomorphic away from the cusp at infinity. If \( f \) has the principal part \( \sum_{m=1}^{\ell} a_m e(-mz) \) at \( \infty \) with \( a_m \in \mathbb{Z} \) such that for all non-zero \( a_m \), \( (m,N) = 1 \), then for every prime \( p \) for which \( (p,N) = 1 \) and \( \left( \frac{-D}{p} \right) = 1 \),

\[
(2.24) \quad t_f(p^2D) = -p \sum_{m=1}^{\ell} a_m \left( \sum_{\nu | m} \nu B(-p^2\nu^2,D) + \sum_{\nu | m/p} \nu B(-\nu^2,D) \right).
\]

If we assume a modular function \( f \in \mathbb{Q}(q) \) satisfies the condition in Theorem 2.8 and both trace of \( f \) and the sum \( \sum_{m} a_m (\sum_{\nu | m} \nu B(-p^2\nu^2,D) + \sum_{\nu | m/p} \nu B(-\nu^2,D)) \) are integers, we have the congruence

\[
t_f(p^2D) \equiv 0 \pmod{p}.
\]

**Example 3.** Consider

\[
f = \left( \frac{\eta(z)}{\eta(37z)} \right)^2 - 2 + 37 \left( \frac{\eta(37z)}{\eta(z)} \right)^2.
\]

Then \( f \) is a modular function for \( \Gamma_0(37) \) which is of genus 1 and has a Fourier expansion of the form \( q^{-3} - 2q^{-2} - q^{-1} + 0 + O(q) \). When \( D = 11 \), the computation from Theorem 2.3, Lemma 2.4 and Corollary 2.5 shows that \( t_f(11) = t_3(11) - 2t_2(11) - t_1(11) = 5 \) is an integer while \( t_1(11) \), \( t_2(11) \), \( t_3(11) \) are not integers. Furthermore, \( t_f(p^2 \cdot 11) \equiv 0 \pmod{p} \) for a prime \( p \) for which \( \left( \frac{-11}{p} \right) = 1 \). For example, for the first three of such primes 3, 5 and 23, we have \( t_f(99) = t_f(3^2 \cdot 11) = -6 \equiv 0 \pmod{3} \), \( t_f(275) = t_f(5^2 \cdot 11) = -75 \equiv 0 \pmod{5} \), and \( t_f(5819) = t_f(23^2 \cdot 11) = 246920364 \equiv 0 \pmod{23} \).

3. **Exact Formulas for traces**

In this section, we will give the proofs of theorems listed in Section 2.1.

**Proof of Proposition 2.2.** First, we recall that the Niebur-Poincaré series \( F_m(z,s) \) in (1.6) has the following Fourier expansion \([18, \text{Theorem 1}]; \text{for } \Re s > 1,\)

\[
F_m(z,s) = e(-mx) y^{1/2} I_{s-1/2}(2\pi my) + \sum_{n=-\infty}^{\infty} b_n(y,s; -m) e(nx),
\]
where \( b_n(y, s; -m) \rightarrow 0 \) \((n \neq 0)\) exponentially as \( y \rightarrow \infty \). Hence the pole of \( F_m(z, 1) \) at infinity may occur only in \( e(\pi x) y^{1/2} I_{1/2}(2\pi m y) \), which is equal to
\[
(3.2) \quad \frac{1}{\pi y^{1/2} m^{1/2}} \sinh(2\pi m y)^{1/2} e(-m x) = \frac{1}{2\pi \sqrt{m}} (e(-m z) - e(-m \bar{z})).
\]
We normalize \( F_m(z, 1) \) by multiplying with \( 2\pi \sqrt{m} \) so that the coefficient of \( e(-m z) \) is normalized. It is easy to check that
\[
F^*_m(z, 1) = (2\pi \sqrt{m}) F_m(z, 1) + c_m,
\]
where \(-c_m\) is the constant term in \((2\pi \sqrt{m}) F_m(z, 1)\), is a \( \Gamma \)-invariant harmonic function and \( F^*_m(z, 1) - e(-m z) \) has a zero at \( \infty \). Hence Proposition 2.2 follows from [18, Theorem 6]. □

As the explicit formulas for \( t_m(D) \) given in Lemma 2.4 can be derived in a very similar way to Lemma 3 in [9], we omit the proof of Lemma 2.4. Hence in order to complete the proof of exact formulas for the traces of the values of a weakly holomorphic modular function on \( \Gamma \), it remains to evaluate the constant term \( c_m \) given in Theorem 2.3.

Proof of Theorem 2.3. It follows from (3.1) and [18, Theorem 1] that the constant term in \((2\pi \sqrt{m}) F_m(z, 1)\) is
\[
(3.3) \quad \frac{-c_m}{s-1} = \lim_{s \rightarrow 1} 2\pi \sqrt{m} b_0(y, s, -m) = \lim_{s \rightarrow 1} 2\pi \sqrt{m} (2\pi m s^{s-1/2} \phi_m(s)/\Gamma(s)) y^{1-s}/(2s-1) = 4\pi^2 m \lim_{s \rightarrow 1} \phi_m(s).
\]
Here \( \phi_m(s) = \sum_{c>0} S(m, 0; c) e^{-2s} \) and \( S(m, n; c) \) is the general Kloosterman sum \( \sum_{0 \leq d < |c|} e((ma + nd)/c) \) for \((a, c) \in \Gamma\). As \( M \in \Gamma^*(N) \) if and only if \( M \) is of the form \( \left( \frac{\sqrt{e} a}{N c/\sqrt{e}}, \frac{b/\sqrt{e}}{N c/\sqrt{e}} \right)_{det=1} \) for some \( e \mid N \) with \( a, b, c, d \in \mathbb{Z} \) and \((a, N c/e) = 1\), we can identify the sum \( S(m, 0; c) = \sum_{0 \leq d < |c|} e((ma)/c) \) with the sum of \( m \)-th powers of primitive \( N c/e \)-th roots of unity. Hence if we denote the Ramanujan sum by \( u_n(q) \), that is the sum of \( n \)-th powers of primitive \( q \)-th roots of unity, we find that
\[
\phi_m(s) = \sum_{c>0} S(m, 0; c) e^{-2s} = \sum_{e \mid N} \sum_{c \geq 1} u_m(N c/e) e^{-2s} N^{2s} e^{-s} c^{-2s}
\]
\[
= \sum_{e \mid N} e^{-s} \sum_{c \geq 1} u_m(N c/e) (N c/e)^{-2s} \quad \text{if } (e, c) = 1
\]
\[
= \sum_{e \mid N} e^{-s} \prod_{p \mid N} \prod_{k \geq \text{ord}_p(e)} \frac{u_m(p^k) p^{-2sk} \prod_{p \mid N} \sum_{k \geq 0} u_m(p^k) p^{-2sk}}{\phi(p^k)},
\]
where the last equality holds due to the multiplicative property of \( u_n(q) \) as a function of \( q \) and the fact \((e, c) = 1\). Recall the following known fact on the Ramanujan sum:
\[
(3.5) \quad u_n(p^k) = \begin{cases} 0, & \text{if } p^{k-1} \nmid n, \\ -p^{k-1}, & \text{if } p^{k-1} \mid n, \\ \varphi(p^k), & \text{if } p^k \mid n, \end{cases}
\]
where \( \varphi(n) \) is Euler’s totient function. Letting \( \alpha = \alpha_p = \text{ord}_p(m) \) and \( m_N \) be the largest exact divisor of \( m \) satisfying \( (m_N, N) = 1 \), we deduce from (3.5) that

\[
\prod_{p \mid N} \sum_{k \geq 0} u_m(p^k)p^{-2sk} = \prod_{p \mid N} (1 + \varphi(p)p^{-2s} + \varphi(p^2)p^{-4s} + \cdots + \varphi(p^n)p^{-n2s} - p^\alpha p^{-2(\alpha+1)s})
\]

(3.6)

\[
= \prod_{p \mid N} (1 - p^{-2s})(1 + p^{1-2s} + \cdots + (p^{1-2s})^\alpha)
\]

\[
= \frac{\zeta(2s)^{-1}}{\prod_{p \mid N}(1 - p^{-2s})} \sigma_{1-2s}(m_N).
\]

Also, if \( \beta = \beta_{v,p} = \text{ord}_p(\frac{N}{e}) \) and \( \delta_v(p) \) is defined as in (2.6), then we obtain from (3.5) that

\[
\prod_{p \mid \frac{N}{e}} \sum_{k \geq \beta} u_m(p^k)p^{-2sk} = \prod_{p \mid \frac{N}{e}} \delta_v(p) \left( \varphi(p^\beta)p^{-2\beta s} + \varphi(p^{\beta+1})p^{-2(\beta+1)s} + \cdots + \varphi(p^n)p^{-n2s} - p^\alpha p^{-2(\alpha+1)s} \right)
\]

(3.7)

\[
= \prod_{p \mid \frac{N}{e}} \delta_v(p)(p^{1-2s})^{\beta-1} \left[ (1 + p^{1-2s} + \cdots + (p^{1-2s})^{\alpha-\beta+1})(1 - p^{-2s}) - 1 \right]
\]

\[
= \prod_{p \mid \frac{N}{e}} \delta_v(p)(p^{1-2s})^{\beta-1} \left( \frac{1 - (p^{1-2s})^{\alpha-\beta+2}}{1 - p^{1-2s}}(1 - p^{-2s}) - 1 \right).
\]

Therefore, the theorem follows from (3.3), (3.4), (3.6), and (3.7). \( \square \)

We close this section with the proof of Theorem 2.1.

Proof of Theorem 2.1. Following the notations in [5, 8] we denote the theta lifting of \( f \) by \( I(\tau, f) \) and Zagier’s Eisenstein series of weight 3/2 by \( F(\tau) \) which is given by

\[
F(\tau) = \sum_D H(D)q^D + \frac{1}{16\pi \sqrt{v}} \sum_{m \in \mathbb{Z}} \beta(4\pi vm^2)q^{-m^2},
\]

where \( \tau = u + iv \) with \( v > 0 \) and \( \beta(s) = \int_1^\infty t^{-3/2}e^{-st}dt \). Bruinier and Funke [5] showed that the holomorphic part of \( I(\tau, f) \) is the generating function of traces of the values at CM points of \( f \). A little careful application of this result as in [8, eq. (4.3)] shows that the holomorphic part of \( I(\tau, 1)|_{U_4} \) is the generating function of class numbers \( 4H^*_4(4D) \), whereas the non-holomorphic part of \( I(\tau, 1)|_{U_4} \) is given from [8, (3.5) and Section 4] by

\[
\frac{3 \cdot 1 \cdot 1}{4\pi \sqrt{v}/4} \sum_{m \in \mathbb{Z}} \beta(4\pi vm^2)q^{-m^2}.
\]

This implies that the difference between \( I(\tau, 1)|_{U_4} \) and \( 12F(\tau) \) lies in the space of holomorphic modular forms of weight 3/2 on \( \Gamma_0(4) \) satisfying Kohnen’s plus condition, which should be zero. Thus we have the identity \( I(\tau, 1)|_{U_4} = 12F(\tau) \). Comparing the coefficients in \( q^D \), the assertion follows. \( \square \)
4. KLOOSTERMAN SUM REPRESENTATION AND DUALITY

We first prove the Kloosterman sum representation of the trace of the values of Niebur-Poincaré series at Heegner points in Lemma 2.6 and then the duality relation given in Theorem 2.7.

Proof of Lemma 2.6. From Kohnen’s result on the relation between Kloosterman sum and Salicé sum given in [15, Proposition 5] and [17, Proposition 2.2], we can write the sum in the far right side in (2.10) as

$$\sum_{c>0 \atop c \equiv 0 \mod 4N} S_D(m, c) \sinh \left( \frac{4\pi m \sqrt{D}}{c} \right) = \sum_{c>0 \atop d \mid (\frac{c}{m})} \sum_{4N \mid c} (1 + i)(c/d)^{-1/2} \delta_o(c/4d) \times K_1(-\frac{m^2}{d^2}, D, c/d) \left( \frac{2m^2 \sqrt{D}}{c} \delta_o(c/4d) \right)^{1/2} I_1/2 \left( \frac{4\pi m \sqrt{D}}{c} \right).$$

(4.1)

Since $(m, N) = 1$ and $c_m = \sigma(m)c_1$ when $(m, N) = 1$, we deduce from (2.10) and (4.1) that

$$t_m(D) = c_1H^*_N(D) \sum_{\nu \mid m} \sum_{c>0 \atop 4N \mid c} (1 + i)(c/\nu)^{-1/2} \delta_o(c/4\nu) \times K_1(-\frac{m^2}{\nu^2}, D, c/\nu) \left( \frac{2\nu \sqrt{D}}{c/\nu} \right)^{1/2} I_1/2 \left( \frac{4\pi m \sqrt{D}}{c/\nu} \right).$$

(4.2)

Replacing $c/\nu$ by $c$ and $m/\nu$ by $\nu$, we complete the proof.

Next, we prove the duality between the coefficients of $P^*_1, N(-m, z)$ and $P^*_0, N(-m, z)$ in (2.22) and (2.23).

Proof of Theorem 2.7. According to the Fourier development of $P^*_\lambda, N(-m, z)$ computed in [17, Theorem 2.1], the coefficient $b_{1, N}(-m, n)$ in the holomorphic part of $P^*_1, N(-m, z)$ is given by

$$b_{1, N}(-m, n) = -\pi \sqrt{2}(n/m)^{1/4}(1 + i) \sum_{c>0 \atop 4N \mid c} \delta_o(c/4) \frac{K_1(-m, n, c)}{c} I_1/2 \left( \frac{4\pi \sqrt{nm}}{c} \right).$$

Applying [7, Proposition 3.1], we may write the right-hand side as

$$-\pi \sqrt{2}(n/m)^{1/4}(1 + i) \sum_{c>0 \atop 4N \mid c} \delta_o(c/4) \frac{K_0(-m, n, c)}{c} I_1/2 \left( \frac{4\pi \sqrt{nm}}{c} \right),$$

which is the Fourier coefficients $-b_{0, N}(-n, m)$ in the holomorphic part of $P^*_0, N(-m, z)$ by [17, Theorem 2.1]. Thus

$$b_{1, N}(-m, n) = -b_{0, N}(-n, m).$$

(4.3)

By the definition of $B^*_\lambda, N(-m, n)$, we have

$$B_{1, N}(-m, n) = -c_1 \delta_{\square, n} H^*_N(n) + b_{1, N}(-m, n)$$

(4.4)
and

\begin{equation}
B_{0,N}(-m,n) = c_1 \delta_{m,n} H_N^*(m) + b_{0,N}(-m,n).
\end{equation}

Now the theorem follows from (4.3), (4.4) and (4.5). \hfill \Box

5. Congruences of traces

We start the proof of Theorem 2.8 by showing that the Niebur-Poincaré series $F_m(z,1)$ is generated by the action of Hecke operator on $F_1(z,1)$.

Lemma 5.1. For every positive integer $m$ relatively prime to $N$, we have

\begin{equation}
F_m^*(z,1) \big| T_m = F_m^*(z,1),
\end{equation}

where $|T_m$ denotes the usual action of the $m$th Hecke operator $T_m$ on the space of weakly holomorphic modular functions. And for a prime $p$ not dividing $N$,

\begin{equation}
F_m^*(z,1) \big| T_p = F_{pm}^*(z,1) + p F_m^{*p}(z,1),
\end{equation}

where $F_m^{*p}(z,1)$ is defined to be zero unless $m/p \in \mathbb{Z}$.

Proof. As we can easily deduce (5.1) from (5.2), we only prove (5.2). By [2, Lemma 6] and [14, Lemmas 2.5 and 2.6], $F_m^*(z,1)$ is on $\Gamma_0(N)$ and has a pole only at $\infty$. Since $F_m^*(z,1)$ is harmonic, so is $F_m^*(z,1) \big| T_p$. Now if we compare the principal parts of the functions in both sides of (5.2) and apply [18, Theorem 6], we obtain the result immediately. \hfill \Box

By Theorem 2.6, for $m$ and $p$ satisfying $(m, N) = 1$ and $(p, N) = 1$, we have

\begin{equation}
t_{pm}(D) = - \left( \sum_{\nu|m} \nu B(-\nu^2, D) + \sum_{\nu|m} \nu B(-p^2 \nu^2, D) \right).
\end{equation}

On the other hand, by the definition of $t_m(D)$ and (5.2) (cf. [21, Section 6]),

\begin{equation}
t_{pm}(D) = \left( t_m(p^2D) + \left( \frac{-D}{p} \right) t_m(D) + pt_m(D/p^2) \right) - pt_{m/p}(D).
\end{equation}

Since $t_f(D) = \sum_m a_m t_m$, it follows from (5.3) and (5.4) that

\begin{equation}
t_f(p^2D) + \left( \frac{-D}{p} \right) t_f(D) + pt_f(D/p^2) = t_f(D) - pC(D)
\end{equation}

where

\begin{equation}
C(D) = \sum_m a_m \left( \sum_{\nu|m} \nu B(-\nu^2, D) + \sum_{\nu|m/p} \nu B(-\nu^2, D) \right).
\end{equation}

Therefore, for a prime $p$ for which $\left( \frac{-D}{p} \right) = 1$, we find that

\begin{equation}
t_f(p^2D) = -pC(D)
\end{equation}

which proves (2.24).
6. The Limiting Distribution of the Modular Trace

Using his result on uniform distribution of CM points on the modular curve \( X_0(1) \) [11], Duke [10, Theorem 1] proved a conjecture on the limiting distribution of traces of singular moduli \( j_1(z) \) made by Bruinier, Jenkins, and Ono [6, Theorem 1.1]. The limiting distribution of traces of singular moduli, so-called “24 Theorem”, is equivalent to

\[
\sum_{c > \sqrt{D}/3 \atop c \equiv 0 \pmod{4}} S_D(c) \sinh(4\pi\sqrt{D}/c) = o(H(D)),
\]

where the Salié sum \( S_D(c) \) is defined for any positive integer \( c \) by

\[
S_D(c) := \sum_{x^2 \equiv -D \pmod{c}} e(2x/c).
\]

This has been recently generalized to the limiting distribution of the twisted trace of Maass-Poincaré series \( F_m(z, s) \) for \( \Gamma(1) \) by Folsom and Masri [12, Theorem 1.1]. Although we could evaluate exact values of \( \sum_c S_D(m, c) \sinh \left( \frac{4\pi \sqrt{D}}{c} \right) \) for many cases as in Examples 1 and 2 in Section 2, we were not able to establish the analogue of the equation (6.1) due to the difficulty of determining a proper bound in the fundamental domain of \( \Gamma_0(N) \) which contains enough of Heegner points in \( X_0(N) \). From our computational experiment in level 4 case, though, we find that

\[
\sum_{c > 16 \sqrt{D}/3 \atop c \equiv 0 \pmod{4}} S_D(c) \sinh(4\pi\sqrt{D}/c) = o(H(D)).
\]

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