Static and rotating electrically charged black holes in three-dimensional Brans-Dicke gravity theories

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Abstract

We obtain static and rotating electrically charged black holes of a Einstein-Maxwell-Dilaton theory of the Brans-Dicke type in (2+1)-dimensions. The theory is specified by three fields, the dilaton $\phi$, the graviton $g_{\mu\nu}$, and the electromagnetic field $F^{\mu\nu}$, and two parameters, the cosmological constant $\Lambda$ and the Brans-Dicke parameter $\omega$. It contains eight different cases, of which one distinguishes as special cases, string theory, general relativity and a theory equivalent to four dimensional general relativity with one Killing vector. We find the ADM mass, angular momentum, electric charge and dilaton charge and compute the Hawking temperature of the solutions. Causal structure and geodesic motion of null and timelike particles in the black hole geometries are studied in detail.

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1. Introduction

The discovery, by Bañados, Teitelboim and Zanelli [1, 2], of the BTZ black hole solution in (2+1)-dimensional Einstein theory with a negative cosmological constant has attracted much attention to gravity in (2+1) dimensions [3]. Since then, the inclusion of dilaton or electromagnetic fields has been made and the corresponding theories have been studied.

Einstein-Maxwell theories in (2+1) dimensions have been discussed by several authors [4]-[8] who have noticed that if an electric charge is included, the BTZ solution presented in the original paper [1] is valid only in the absence of rotation. Rotating electrically charged solutions in the Einstein-Maxwell theory have been obtained by Clément [4], and Martínez, Teitelboim and Zanelli [9] using a coordinate transformation. Assuming self-dual or anti-self-dual conditions between the electromagnetic fields, Kamata and Koikawa [5] and Cataldo and Salgado [7] have also obtained solutions in the Einstein-Maxwell theory. However, as noticed by Chan [10], the mass and angular momentum of solutions discussed in [5] diverge at spatial infinity. To overcome this divergence a boundary contribution has been taken into account [11]. The presence of divergences in the mass and angular momentum is a usual feature present in charged solutions. One way to regularize them is by the introduction of a topological Chern-Simons term. This procedure has been done by Clément [12], Fernando and Mansouri [6] and Dereli and Obukhov [13] who have analyzed the self-dual solutions for the Einstein-Maxwell-Chern-Simons theory in (2+1) dimensions.

Einstein-Maxwell-Dilaton theories in (2+1) dimensions have also been studied. Chan and Mann [14] have found static electrically charged solutions and Fernando [15] has analyzed the Einstein-Maxwell-Dilaton theory when the self-dual condition is imposed. Chen [16] has applied T-duality to obtain new rotating solutions of Einstein-Maxwell and Einstein-Maxwell-Dilaton theories in (2+1) dimensions.

In this paper we find and study in detail the static and rotating electrically charged solutions of a Einstein-Maxwell-Dilaton action of the Brans-Dicke type. The static uncharged solutions were found and analyzed by Sá, Kleber and Lemos [20] and the angular momentum has been added by Sá and Lemos [21]. The uncharged theory is specified by two fields, the dilaton \( \phi \) and the graviton \( g_{\mu\nu} \), and two parameters, the cosmological constant \( \Lambda \) and the Brans-Dicke parameter \( \omega \). It contains seven different cases and each \( \omega \) can
be viewed as yielding a different dilaton gravity theory. For instance, for \( \omega = -1 \) one gets the simplest low-energy string action \([25]\), for \( \omega = 0 \) one gets a theory related (through dimensional reduction) to four dimensional General Relativity with one Killing vector \([22]\) and for \( \omega = \pm\infty \) one obtains three dimensional General Relativity analyzed in \([1, 2]\).

The electrically charged theory that we are going to study is specified by the extra electromagnetic field \( F^{\mu\nu} \). It contains eight different cases. For \( \omega = 0 \) one gets a theory related (through dimensional reduction) to electrically charged four dimensional General Relativity with one Killing vector \([23]\) and for \( \omega = \pm\infty \) one obtains electrically charged three dimensional General Relativity \([4, 9]\).

Since magnetically charged solutions in (2+1) dimensions have totally different properties from the electrically charged ones, we do not study them here (see \([17]-[19]\) for magnetically charged solutions of Einstein-Maxwell and Einstein-Maxwell-Dilaton theories).

The plan of this article is the following. In Section 2 we set up the action and the field equations. The static general solution of the field equations are found in section 3 and we write the scalar \( R^{\mu\nu}R_{\mu\nu} \) which, in (2+1) dimensions, signals the presence of singularities. The angular momentum is added in section 4. In section 5 we use an extension of the formalism of Regge and Teitelboim to derive the mass, angular momentum, electric charge and dilaton charge of the black holes. In section 6 we study the properties of the different cases that appear naturally from the solutions. We work out in detail the causal structure and the geodesic motion of null and timelike particles for typical values of \( \omega \) that belong to the different ranges. The Hawking temperature is computed in section 7. Finally, in section 8 we present the concluding remarks.

2. Field equations

We are going to work with an action of the Maxwell-Brans-Dicke type in three-dimensions written in the string frame as

\[
S = \frac{1}{2\pi} \int d^3x \sqrt{-g} e^{-2\phi} [R - 4\omega(\partial\phi)^2 + \Lambda + F^{\mu\nu}F_{\mu\nu}],
\]

where \( g \) is the determinant of the 3D metric, \( R \) is the curvature scalar, \( \phi \) is a scalar field called dilaton, \( \Lambda \) is the cosmological constant, \( \omega \) is the three-
dimensional Brans-Dicke parameter and $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the Maxwell tensor, with $A_\mu$ being the vector potential. Varying this action with respect to $g_{\mu\nu}$, $F_{\mu\nu}$ and $\phi$ one gets the Einstein, Maxwell and ditaton equations, respectively

$$\frac{1}{2} G_{\mu\nu} - 2(\omega + 1) \nabla_\mu \phi \nabla_\nu \phi + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla_\gamma \nabla^\gamma \phi + (\omega + 2) g_{\mu\nu} \nabla_\gamma \phi \nabla^\gamma \phi - \frac{1}{4} g_{\mu\nu} \Lambda = \frac{\pi}{2} T_{\mu\nu}, \quad (2)$$

$$\nabla_\nu (e^{-2\phi} F_{\mu\nu}) = 0, \quad (3)$$

$$R - 4\omega \nabla_\gamma \nabla^\gamma \phi + 4\omega \nabla_\gamma \phi \nabla^\gamma \phi + \Lambda = -F^{\gamma\sigma} F_{\gamma\sigma}, \quad (4)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $\nabla$ represents the covariant derivative and $T_{\mu\nu} = \frac{4}{\pi} (g^{\gamma\sigma} F_{\mu\gamma} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^{\gamma\sigma} F_{\gamma\sigma})$ is the Maxwell energy-momentum tensor.

We want to consider now a spacetime which is both static and rotationally symmetric, implying the existence of a timelike Killing vector $\partial/\partial t$ and a spacelike Killing vector $\partial/\partial \phi$. The most general static metric with a Killing vector $\partial/\partial \phi$ with closed orbits in three dimensions can be written as

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{-2\nu(r)} dr^2 + r^2 d\phi^2, \quad (5)$$

with $0 \leq \phi \leq 2\pi$. Each different $\omega$ has a very rich and non-trivial structure of solutions which could be considered on its own. As in [20, 21] we work in the Schwarzschild gauge, $\mu(r) = -\nu(r)$, and compare different black hole solutions in different theories. For this ansatz the metric is written as

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{-2\nu(r)} dr^2 + r^2 d\phi^2. \quad (5)$$

We also assume that the only non-vanishing components of the vector potential are $A_t(r)$ and $A_\phi(r)$, i.e.,

$$A = A_t dt + A_\phi d\phi. \quad (6)$$

This implies that the non-vanishing components of the symmetric Maxwell tensor are $F_{tr}$ and $F_{t\phi}$.

Inserting the metric (5) into equation (2) one obtains the following set of equations

$$\phi_{,rr} + \phi_{,r} \nu_{,r} + \frac{\phi_{,r}}{r} - (\omega + 2)(\phi_{,r})^2 - \frac{\nu_{,r}}{2r} + \frac{1}{4} \Lambda e^{-2\nu} = \frac{\pi}{2} e^{-4\nu} T_{tt}, \quad (7)$$

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\[-\phi_{,r} \nu_{,r} - \frac{\phi_{,r}}{r} - \omega (\phi_{,r})^2 + \frac{\nu_{,r}}{2r} - \frac{1}{4} \Lambda e^{-2\nu} = \frac{\pi}{2} T_{rr}, \quad (8)\]
\[\phi_{,rr} + 2 \phi_{,r} \nu_{,r} - (\omega + 2) (\phi_{,r})^2 - \frac{\nu_{,rr}}{2} - (\nu_{,r})^2 + \frac{1}{4} \Lambda e^{-2\nu} = -\frac{\pi}{2} e^{-2\nu} \frac{F_{\gamma\sigma} F_{\gamma\sigma}}{r^2} T_{\varphi\varphi}, \quad (9)\]
\[0 = \frac{\pi}{2} T_{t\varphi} = e^{-2\nu} F_{tr} F_{\varphi r}, \quad (10)\]

where \(,r\) denotes a derivative with respect to \(r\). In addition, inserting the metric (5) into equations (3) and (4) yields
\[\partial_r [e^{-2\phi_r} (F_{tr} + F_{\varphi r})] = 0, \quad (11)\]
\[\omega \phi_{,rr} + 2 \omega \phi_{,r} \nu_{,r} + \omega \frac{\phi_{,r}}{r} - \omega (\phi_{,r})^2 + \frac{\nu_{,r}}{r} + \frac{\nu_{,rr}}{2} + (\nu_{,r})^2 - \frac{1}{4} \Lambda e^{-2\nu} = \frac{1}{4} e^{-2\nu} F_{\gamma\sigma} F_{\gamma\sigma}. \quad (12)\]

3. The general static solution

From the above equations valid for a static and rotationally symmetric spacetime one sees that equation (10) implies that the electric and magnetic fields cannot be simultaneously non-zero, i.e., there is no static dyonic solution. In this work we will consider the electrically charged case alone \((A_{\varphi} = 0, A_t \neq 0)\).

So, assuming vanishing magnetic field, one has from Maxwell equation (11) that
\[F_{tr} = -\frac{\chi}{4r} e^{2\phi}, \quad (13)\]
where \(\chi\) is an integration constant which, as we shall see in (14), is the electric charge. One then has that
\[F_{\gamma\sigma} F_{\gamma\sigma} = \frac{\chi^2}{8r^2} e^{4\phi}, \quad T_{tt} = \frac{\chi^2}{16\pi r^2} e^{2\nu} e^{4\phi}, \quad T_{rr} = -\frac{\chi^2}{16\pi r^2} e^{-2\nu} e^{4\phi}, \quad T_{\varphi\varphi} = \frac{\chi^2}{16\pi} e^{4\phi}. \quad (14)\]

To proceed we shall first consider the case \(\omega \neq -1\). Adding equations (11) and (8) one obtains \(\phi_{,rr} = 2(\omega + 1)(\phi_{,r})^2\), yielding for the dilaton field the
following solution

\[ \phi = -\frac{1}{2(\omega + 1)} \ln[2(\omega + 1)r + a_1] + a_2, \quad w \neq -1 \quad (15) \]

where \( a_1 \) and \( a_2 \) are constants of integration. One can, without loss of generality, choose \( a_1 = 0 \). Then, equation (15) can be written as

\[ e^{-2\phi} = a(\alpha r)^{\frac{1}{\omega + 1}}, \quad w \neq -1, \quad (16) \]

where \( \alpha \) is an appropriate constant that is proportional to the cosmological constant [see equation (21)]. The dimensionless constant \( a \) can be viewed as a normalization to the action (1). Since it has no influence in our calculations, apart a possible redefinition of the mass, we set \( a = 1 \). The vector potential \( A = A_\mu(r)dx^\mu = A_t(r)dt \) with \( A_t(r) = \int F_t dr \) is then

\[ A = \frac{1}{4} \chi(\omega + 1)(\alpha r)^{\frac{1}{\omega + 1}} dt, \quad w \neq -1. \quad (17) \]

Inserting the solutions (13)-(16) in equations (7)-(12), we obtain for the metric

\[ ds^2 = -\left[(\alpha r)^2 - \frac{b}{(\alpha r)^{\omega + 1}} + \frac{k\chi^2}{(\alpha r)^{\omega + 1}}\right] dt^2 + \frac{dr^2}{(\alpha r)^{\frac{1}{\omega + 1}}} + \frac{k\chi^2}{(\alpha r)^{\frac{4}{\omega + 1}}} + r^2 d\varphi^2, \quad \omega \neq -2, -\frac{3}{2}, -1, \quad (18) \]

\[ ds^2 = -\left[(1 + \frac{\chi^2}{4} \ln r)r^2 - br\right] dt^2 + \frac{dr^2}{(1 + \frac{\chi^2}{4} \ln r)r^2 - br} + r^2 d\varphi^2, \quad \omega = -2, \quad (19) \]

\[ ds^2 = -r^2[-\Lambda \ln(br) + \chi^2 r^2] dt^2 + \frac{dr^2}{r^2[-\Lambda \ln(br) + \chi^2 r^2]} + r^2 d\varphi^2, \quad \omega = -\frac{3}{2}, \quad (20) \]

where \( b \) is a constant of integration related with the mass of the solutions, as will be shown, and \( k = \frac{(\omega + 1)^2}{8(\omega + 2)} \). For \( \omega \neq -2, -\frac{3}{2}, -1 \) \( \alpha \) is defined as

\[ \alpha = \sqrt{\frac{(\omega + 1)\Lambda}{(\omega + 2)(2\omega + 3)}}. \quad (21) \]
For $\omega = -2, -\frac{3}{2}$ we set $\alpha = 1$. For $\omega = -2$ equations (7) and (8) imply $\Lambda = \chi^2/8$ so, in contrast with the uncharged case [20, 21], the cosmological constant is not null.

Now, we consider the case $\omega = -1$. From equations (6)-(12) it follows that $\nu = C_1$, $\phi = C_2$, where $C_1$ and $C_2$ are constants of integration, and that the cosmological constant and electric charge are both null, $\Lambda = \chi = 0$. So, for $\omega = -1$ the metric gives simply the three-dimensional Minkowski spacetime and the dilaton is constant, as occurred in the uncharged case [20, 21].

In (2+1) dimensions, the presence of a curvature singularity is revealed by the scalar $R_{\mu \nu} R^{\mu \nu}$

\[
R_{\mu \nu} R^{\mu \nu} = 12\alpha^4 + \frac{4\omega}{(\omega + 1)^2} \frac{b\alpha^4}{(\chi r)^{(2\omega+3)/(\omega+1)}} + \frac{2(\omega^2 + 4\omega + 3)}{2(\omega + 1)^4} \frac{b^2\alpha^4}{(\chi r)^{(2\omega+3)/(\omega+1)}} - \frac{(\omega - 1)}{(\omega + 1)^2} \frac{k\chi^2\alpha^4}{(\chi r)^{(2\omega+3)/(\omega+1)}} - \frac{(\omega^2 + 2\omega + 2)}{(\omega + 1)^4} \frac{k\chi^2b\alpha^4}{(\chi r)^{(2\omega+3)/(\omega+1)}}
\]

\[(22) \quad \omega \neq -2, -\frac{3}{2}, -1,
\]

\[
R_{\mu \nu} R^{\mu \nu} = 8 + \frac{32}{r} + \frac{6}{r^2} + \chi^2 \left[ 6 \ln r + \frac{4 \ln r}{r} + \frac{3}{r} + 5 \right] + \chi^4 \left[ \frac{3}{4} \ln^2 r + \frac{5}{4} \ln r + 9 \right], \quad \omega = -2,
\]

\[(23) \quad \omega = -\frac{3}{2},
\]

\[
R_{\mu \nu} R^{\mu \nu} = \Lambda^2 \left[ 12 \ln^2 (br) + 20 \ln (br) + 9 \right] + \Lambda \chi^2 r^2 \left[ 5 \ln (br) + \frac{9}{2} \right] + \frac{9}{16} \chi^4 r^4
\]

\[(24) \quad \omega = -\frac{3}{2}.
\]

An inspection of these scalars in (22)-(24) reveals that for $\omega < -2$ and $\omega > -1$ the curvature singularity is located at $r = 0$ and for $-\frac{3}{2} < \omega < -1$ the curvature singularity is at $r = +\infty$. For $-2 \leq \omega \leq -\frac{3}{2}$ both $r = 0$ and $r = +\infty$ are singular. For $\omega = \pm \infty$ spacetime has no curvature singularities.

Note that in the uncharged case [20, 21], for $-2 \leq \omega < -\frac{3}{2}$ the curvature singularity is located only at $r = 0$. 
4. The general rotating solution

In order to add angular momentum to the spacetime we perform the following coordinate transformations (see e.g. [21]-[24])

\[ t \rightarrow \gamma t - \frac{\theta}{\alpha^2} \varphi, \]
\[ \varphi \rightarrow \gamma \varphi - \theta t, \]  
(25)

where \( \gamma \) and \( \theta \) are constant parameters. Substituting (25) into (18)-(20) we obtain

\[
ds^2 = -\left[ (\gamma^2 - \frac{\theta^2}{\alpha^2}) \right. \\
\left. \frac{b}{(\alpha r)^{\frac{1}{\omega+1}}} - \frac{k\chi^2}{(\alpha r)^{\frac{1}{\omega+1}}} \right] dt^2 - \frac{\gamma \theta}{\alpha^2 \left[ \frac{b}{(\alpha r)^{\frac{1}{\omega+1}}} - \frac{k\chi^2}{(\alpha r)^{\frac{1}{\omega+1}}} \right]} 2 dtd\varphi + \left[ \frac{dr^2}{(\alpha r)^{\frac{1}{\omega+1}}} - \frac{b}{(\alpha r)^{\frac{1}{\omega+1}}} + \frac{k\chi^2}{(\alpha r)^{\frac{1}{\omega+1}}} \right] d\varphi^2, \]
\[
\omega \neq -2, -\frac{3}{2}, -1, \]  
(26)

\[
ds^2 = -\left[ \left( \gamma^2 - \theta^2 \right) + \frac{\chi^2 \ln r}{4} \right] r^2 - \gamma^2 br \right] dt^2 + \gamma \theta \left[ \frac{\chi^2}{4} r^2 \ln r - br \right] 2 dtd\varphi + \left[ \left( \gamma^2 - \theta^2 \right) - \frac{\chi^2}{4} \ln r \right] r^2 + \theta^2 br \right] d\varphi^2, \]
\[
\omega = -2, \]  
(27)

\[
ds^2 = -r^2 \left[ -\gamma^2 A \ln (br) - \theta^2 + \gamma^2 \chi^2 r^2 \right] dt^2 - \gamma \theta r^2 \left[ A \ln (br) + 1 - \chi^2 r^2 \right] 2 dtd\varphi + \frac{dr^2}{(1 + \frac{\chi^2}{4} \ln r)r^2 - br} + \left[ \left( \gamma^2 - \theta^2 \right) - \frac{\chi^2}{4} \ln r \right] r^2 + \theta^2 br \right] d\varphi^2, \]
\[
\omega = -\frac{3}{2}. \]  
(28)

Introducing transformations (25) into (17) we obtain that the vector potential \( A = A_\mu(r)dx^\mu \) is now given by

\[
A = \gamma A(r) dt - \frac{\theta}{\alpha^2} A(r) d\varphi, \quad w \neq -1, \]  
(29)

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where \( A(r) = \frac{1}{4} \chi (\omega + 1)(\alpha r)^{-1} \). Solutions (26)-(29) represent electrically charged stationary spacetimes and also solve (4). Analyzing the Einstein-Rosen bridge of the static solution one concludes that spacetime is not simply connected which implies that the first Betti number of the manifold is one, i.e., closed curves encircling the horizon cannot be shrunk to a point. So, transformations (25) generate a new metric because they are not permitted global coordinate transformations [26]. Transformations (25) can be done locally, but not globally. Therefore metrics (18)-(20) and (26)-(28) can be locally mapped into each other but not globally, and such they are distinct.

5. Mass, angular momentum and electric charge of the solutions

In this section we will calculate the mass, angular momentum, electric charge and dilaton charge of the static and rotating electrically charged black hole solutions. To obtain these quantities we apply the formalism of Regge and Teitelboim [27] (see also [9, 20, 21, 22]).

We first write the metrics (26)-(28) in the canonical form involving the lapse function \( N^0(r) \) and the shift function \( N^\varphi(r) \)

\[
ds^2 = -(N^0)^2 dt^2 + \frac{dr^2}{f^2} + H^2 (d\varphi + N^\varphi dt)^2,
\]

where \( f^{-2} = g_{rr}, \ H^2 = g_{\varphi\varphi}, \ H^2 N^\varphi = g_{t\varphi} \) and \((N^0)^2 - H^2 (N^\varphi)^2 = g_{tt}\). Then, the action can be written in the Hamiltonian form as a function of the energy constraint \( \mathcal{H} \), momentum constraint \( \mathcal{H}_{\varphi} \) and Gauss constraint \( G \)

\[
S = -\int dt d^2 x [N^0 \mathcal{H} + N^\varphi \mathcal{H}_{\varphi} + A_t G] + B
\]

\[
= -\Delta t \int dr N \left[ \frac{2\pi^2}{H^3} e^{-2\phi} - 4f^2 (H \phi_r e^{-2\phi})_r - 2H \phi_r (f^2)_r e^{-2\phi} + 2f (fH_r)_r e^{-2\phi} + 4\omega H f^2 (\phi_r)^2 e^{-2\phi} - \Lambda H e^{-2\phi} + \frac{2H}{f} e^{-2\phi} (E^2 + B^2) \right]
+ \Delta t \int dr N^\varphi \left[ (2\pi e^{-2\phi})_r + \frac{4H}{f} e^{-2\phi} E^\varphi B \right]
+ \Delta t \int dr A_t \left[ -\frac{4H}{f} e^{-2\phi} \partial_r E^\varphi \right] + B,
\]
where \( N = \frac{N^0}{r} \), \( \pi \equiv \pi_\varphi = -\frac{fH^3(N^\varphi)}{N} \) (with \( \pi_\varphi \) being the momentum conjugate to \( g_{r\varphi} \)), \( E^r \) and \( B \) are the electric and magnetic fields and \( B \) is a boundary term. Upon varying the action with respect to \( f(r) \), \( H(r) \), \( \pi(r) \), \( \phi(r) \) and \( E^r(r) \) one picks up additional surface terms. Indeed,

\[
\delta S = -\Delta tN \left[ (H_r - 2H\phi_r)e^{2\phi}\delta f^2 - (f^2)_r e^{2\phi}\delta H - 4f^2He^{-2\phi}\delta(\phi, r) \right] \\
+ 2H[(f^2)_r + 4(\omega + 1)f^2\phi_r]e^{-2\phi}\delta\phi + 2f^2 e^{-2\phi}\delta(H, r) \\
+ \Delta tN^\varphi \left[ 2e^{-2\phi}\delta\pi - 4\pi e^{-2\phi}\delta\phi \right] + \Delta tA_t \left[ -\frac{4H}{f}e^{-2\phi}\delta E^r \right] + \delta B \\
+ \text{(terms vanishing when the equations of motion hold).} \tag{32}
\]

In order that the Hamilton’s equations are satisfied, the boundary term \( B \) has to be adjusted so that it cancels the above additional surface terms. More specifically one has

\[
\delta B = -\Delta tN\delta M + \Delta tN^\varphi \delta J + \Delta tA_t \delta Q , \tag{33}
\]

where one identifies \( M \) as the mass, \( J \) as the angular momentum and \( Q \) as the electric charge since they are the terms conjugate to the asymptotic values of \( N \), \( N^\varphi \) and \( A_t \), respectively.

To determine the \( M \), \( J \) and \( Q \) of the black hole one must take the black hole spacetime and subtract the background reference spacetime contribution, i.e., we choose the energy zero point in such a way that the mass, angular momentum and charge vanish when the black hole is not present.

Now, note that for \( \omega < -2 \), \( \omega > -3/2 \) and \( \omega \neq -1 \), spacetime \([20, 21]\) has an asymptotic metric given by

\[
- \left( \gamma^2 - \frac{\theta^2}{\alpha^2} \right) \alpha^2 r^2 dt^2 + \frac{dr^2}{\alpha^2 r^2} + \left( \gamma^2 - \frac{\theta^2}{\alpha^2} \right) r^2 d\varphi^2 , \tag{34}
\]

i.e., it is asymptotically an anti-de Sitter spacetime. In order to have the usual form of the anti-de Sitter metric we choose \( \gamma^2 - \theta^2/\alpha^2 = 1 \). For the cases \(-2 \leq \omega \leq -3/2 \) we shall also choose \( \gamma^2 - \theta^2/\alpha^2 = 1 \), as has been done for the uncharged case \([20, 21]\). For \( \omega \neq -3/2, -1 \) the anti-de Sitter spacetime is also the background reference spacetime, since the metrics \([20] \) and \([27] \) reduce to \([34] \) if the black hole is not present \((b = 0 \text{ and } \varepsilon = 0)\). For \( \omega = -3/2 \) the above described procedure of choosing the energy zero
point does not apply since for any value of $b$ and $\varepsilon$ one still has a black hole solution. Thus, for $\omega = -3/2$ the energy zero point is chosen arbitrarily to correspond to the black hole solution with $b = 1$ and $\varepsilon = 0$.

Taking the subtraction of the background reference spacetime into account and noting that $\phi - \phi_{\text{ref}} = 0$ and that $\phi_{r} - \phi_{r,\text{ref}} = 0$ we have that the mass, angular momentum and electric charge are given by

\[
M = (2H\phi_{,r} - H_{,r})e^{-2\phi}(f^{2} - f_{\text{ref}}^{2}) + (f^{2})_{,r}e^{-2\phi}(H - H_{\text{ref}}) - 2f^{2}e^{-2\phi}(H_{,r} - H_{r,\text{ref}}),
J = -2e^{-2\phi}(\pi - \pi_{\text{ref}}),
Q = \frac{4H}{f}e^{-2\phi}(E_{r} - E_{r,\text{ref}}),
\]

where $\phi_{,r}$ and $\phi_{r,\text{ref}}$ are the energy zero point is chosen arbitrarily to correspond to the black hole solution with $b = 1$ and $\varepsilon = 0$.

Then, for $\omega > -3/2$ and $\omega \neq -1$, we finally have that the mass and angular momentum are (after taking the appropriate asymptotic limit: $r \to +\infty$ for $\omega > -1$ and $r \to 0$ for $-3/2 < \omega < -1$, see the Penrose diagrams on section 6.3 to understand the reason for these limits)

\[
M = b\left[\frac{\omega + 2}{\omega + 1}\gamma^{2} + \frac{\theta^{2}}{\alpha^{2}}\right] = M_{Q=0},
J = \frac{\gamma\theta b^{2}}{\alpha^{2}}\frac{2\omega + 3}{\omega + 1} = J_{Q=0}, \quad \omega > -3/2, \omega \neq -1, \quad (36)
\]

where $M_{Q=0}$ and $J_{Q=0}$ are the mass and angular momentum of the uncharged black hole. For $\omega < -3/2$, the mass and angular momentum are (after taking the appropriate asymptotic limit, $r \to +\infty$)

\[
M = b\left[\frac{\omega + 2}{\omega + 1}\gamma^{2} + \frac{\theta^{2}}{\alpha^{2}}\right] + \text{Div}_{M}(\chi, r) = M_{Q=0} + \text{Div}_{M}(\chi, r), \quad (37)
J = \frac{\gamma\theta b^{2}}{\alpha^{2}}\frac{2\omega + 3}{\omega + 1} + \text{Div}_{J}(\chi, r) = J_{Q=0} + \text{Div}_{J}(\chi, r), \quad \omega < -3/2, \quad (38)
\]

where $\text{Div}_{M}(\chi, r)$ and $\text{Div}_{J}(\chi, r)$ are terms proportional to the charge $\chi$ that diverge at the asymptotic limit. These kind of divergent terms are present even in the case $\omega = \pm \infty$ which gives the electrically charged BTZ black hole [9]. Following [9] these divergences can be treated as follows. One considers
a boundary of large radius \( r_0 \) involving the black hole. Then, one sums and subtracts \( \text{Div}_M(\chi, r_{0}) \) to (37) so that the mass (37) is now written as

\[
M = M(r_0) + [\text{Div}_M(\chi, r) - \text{Div}_M(\chi, r_{0})],
\]

where \( M(r_0) = M_{Q=0} + \text{Div}_M(\chi, r_0) \), i.e.,

\[
M_{Q=0} = M(r_0) - \text{Div}_M(\chi, r_{0}).
\]

(40)

The term between brackets in (39) vanishes when \( r \to r_{0} \). Then \( M(r_0) \) is the energy within the radius \( r_{0} \). The difference between \( M(r_0) \) and \( M_{Q=0} \) is \(-\text{Div}_M(\chi, r_{0})\) which is interpreted as the electromagnetic energy outside \( r_{0} \) apart from an infinite constant which is absorbed in \( M(r_0) \). The sum (40) is then independent of \( r_{0} \), finite and equal to the total mass.

To handle the angular momentum divergence, one first notice that the asymptotic limit of the angular momentum per unit mass \( (J/M) \) is either zero or one, so the angular momentum diverges at a rate slower or equal to the rate of the mass divergence. The divergence on the angular momentum can then be treated in a similar way as the mass divergence. So, the divergent term \(-\text{Div}_J(\chi, r_{0})\) can be interpreted as the electromagnetic angular momentum outside \( r_{0} \) up to an infinite constant that is absorbed in \( J(r_{0}) \).

In practice the treatment of the mass and angular divergences amounts to forgetting about \( r_{0} \) and take as zero the asymptotic limits: \( \lim \text{Div}_M(\chi, r) = 0 \) and \( \lim \text{Div}_J(\chi, r) = 0 \). So, for \( \omega < -3/2 \) the mass and angular momentum are also given by (36).

Interesting enough, as has been noticed in [9], is the fact that in four spacetime dimensions there occurs a similar situation. For example, the \( g_{tt} \) component of Reissner-Nordström solution can be written as \( 1 - \frac{2M(r_0)}{r} + Q^2(\frac{1}{r} - \frac{1}{r_{0}}) \). The total mass \( M = M(r_0) + \frac{Q^2}{2r_0} \) is independent of \( r_0 \) and \( \frac{Q^2}{2r_0} \) is the electrostatic energy outside a sphere of radius \( r_0 \). In this case, since \( \frac{Q^2}{2r_0} \) vanishes when \( r_0 \to \infty \), one does not need to include an infinite constant in \( M(r_0) \). Thus, in this general Brans-Dicke theory in 3D we conclude that both situations can occur depending on the value of \( \omega \). The \( \omega > -3/2, \omega \neq -1 \) case is analogous to the the Reissner-Nordström black hole in the sense that it is not necessary to include an infinite constant in \( M(r_0) \), while the case \( \omega < -3/2 \) is similar to the electrically charged BTZ black hole [9] since an infinite constant must be included in \( M(r_0) \).
For $\omega = -3/2$ the mass and angular momentum are ill defined since the boundaries $r \to \infty$ and $r \to 0$ have logarithmic singularities in the mass term even in the absence of the electric charge.

Now, we calculate the electric charge of the black holes. To determine the electric field we must consider the projections of the Maxwell field on spatial hypersurfaces. The normal to such hypersurfaces is $n^\nu = (1/N^0, 0, -N^\varphi/N^0)$ so the electric field is $E^\mu = g^{\mu\sigma} F_{\sigma\nu} n^\nu$. Then, from (35), the electric charge is

$$Q = -\frac{4Hf}{N^0} e^{-2\phi} (\partial_r A_t - N^\varphi \partial_r A_\varphi) = \gamma \chi, \quad \omega \neq -1.$$  \hfill (41)

One can also define a dilaton charge as the flux of the dilaton field over an asymptotic one-sphere

$$Q_{\text{Dil}} \equiv \frac{2}{\pi} \int d\varphi \sqrt{\sigma} u^\mu \partial_\mu (\phi - \phi_{\text{ref}}),$$  \hfill (42)

where $\sqrt{\sigma} = \sqrt{g_{\varphi\varphi}}$ and $u^\mu = (0, \sqrt{g^{rr}}, 0)$ is the normal vector to the one-sphere boundary. As the background reference spacetime has the same dilaton solution as the black hole spacetime one conclude that the dilaton charge is zero.

The mass, angular momentum and electric charge of the static black holes can be obtained by putting $\gamma = 1$ and $\theta = 0$ on the above expressions [see (25)].

Now, we want to cast the metric in terms of $M$, $J$ and $Q$. For $\omega \neq -2, -3/2, -1$, we can use (36) to solve a quadratic equation for $\gamma^2$ and $\theta^2/\alpha^2$. It gives two distinct sets of solutions

$$\gamma^2 = \frac{\omega + 1}{2(\omega + 2)} \frac{M(2 - \Omega)}{b}, \quad \frac{\theta^2}{\alpha^2} = \frac{M\Omega}{2b},$$  \hfill (43)

$$\gamma^2 = \frac{\omega + 1}{2(\omega + 2)} \frac{M\Omega}{b}, \quad \frac{\theta^2}{\alpha^2} = \frac{M(2 - \Omega)}{2b},$$  \hfill (44)

where we have defined a rotating parameter $\Omega$ as

$$\Omega \equiv 1 - \sqrt{1 - \frac{4(\omega + 1)(\omega + 2) J^2 \alpha^2}{(2\omega + 3)^2 M^2}}, \quad \omega \neq -2, -3/2, -1.$$  \hfill (45)
When we take $J = 0$ (which implies $\Omega = 0$), (43) gives $\gamma \neq 0$ and $\theta = 0$ while (44) gives the nonphysical solution $\gamma = 0$ and $\theta \neq 0$ which does not reduce to the static original metric. Therefore we will study the solutions found from (43). For $\omega = -2$ we have $\gamma^2 = J^2/Mb$ and $\theta^2 = M/b$.

The condition that $\Omega$ remains real imposes for $-2 > \omega > -1$ a restriction on the allowed values of the angular momentum: $|\alpha J| \leq (2\omega + 3)M/(2\sqrt{(\omega + 1)(\omega + 2)})$. For $-2 > \omega > -1$ we have $0 \leq \Omega \leq 1$. In the range $-2 < \omega < -3/2$ and $-3/2 < \omega < -1$ we have $\Omega < 0$. The condition $\gamma^2 - \theta^2/\alpha^2 = 1$ fixes the value of $b$ and from (41) we can write $k\chi^2$ as a function of $b, M, \Omega, Q$. Thus,

$$ b = \frac{M}{2(\omega + 2)}\left[2(\omega + 1) - (2\omega + 3)\Omega\right], \quad (46) $$

$$ k\chi^2 = \frac{b}{4(\omega + 1)}\frac{Q^2}{M(2 - \Omega)}, \quad \omega \neq -2, -3/2, -1, \quad (47) $$

and

$$ b = \frac{J^2 - M^2}{M}, \quad \chi^2 = \frac{Q^2Mb}{J^2}, \quad \omega = -2. \quad (48) $$

The metrics (26) and (27) may now be cast in the form

$$ ds^2 = -\left[(\alpha r)^2 - \frac{(\omega + 1)}{2(\omega + 2)} \frac{M(2 - \Omega)}{(\alpha r)^{\frac{2}{\omega + 1}}} + \frac{(\omega + 1)^2}{8(\omega + 2)} \frac{Q^2}{(\alpha r)^{\frac{2}{\omega + 1}}}ight]dt^2 $$

$$ -\frac{\omega + 1}{2\omega + 3}J\left[(\alpha r)^{-\frac{2}{\omega + 1}} - \frac{(\omega + 1)Q^2}{4M(2 - \Omega)}(\alpha r)^{-\frac{2}{\omega + 1}}\right]dtd\phi $$

$$ + \left[(\alpha r)^2 - \frac{M[2(\omega + 1) - (2\omega + 3)\Omega]}{2(\omega + 2)(\alpha r)^{\frac{2}{\omega + 1}}} + \frac{(\omega + 1)Q^2[2(\omega + 1) - (2\omega + 3)\Omega]}{8(\omega + 2)(2 - \Omega)(\alpha r)^{\frac{2}{\omega + 1}}}\right]^{-1}dr^2 $$

$$ + \frac{1}{\alpha^2}\left[(\alpha r)^2 + \frac{M\Omega}{2(\alpha r)^{\frac{2}{\omega + 1}}} - \frac{(\omega + 1)\Omega Q^2}{8(2 - \Omega)(\alpha r)^{\frac{2}{\omega + 1}}}\right]d\phi^2, \quad \omega \neq -2, -3/2, -1, \quad (49) $$

$$ ds^2 = -\left[\left(1 + \frac{Q^2}{4} \ln r\right)r^2 - \frac{J^2}{M}r\right]dt^2 + J\left[\frac{Q^2M}{4J^2}r^2 \ln r - r\right]dtd\phi. $$

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Analyzing the function $\Delta = g_{rr}^{-1}$ in (26), (27), and (46)-(48) we can set the conditions imposed on the mass and angular momentum of the solution obtained for the different values of $\omega \neq -3/2, -1$, in order that the black holes might exist. These conditions are summarized on Table 1.

**Table 1.** Values of the angular momentum for which black holes with positive and negative masses might exist.

| Range of $\omega$ | Black holes with $M > 0$ | Black holes with $M < 0$ |
|-------------------|--------------------------|--------------------------|
| $-\infty < \omega < -2$ | $|\alpha J| \leq \frac{|(2\omega+3)M|}{2\sqrt{(\omega+1)(\omega+2)}}$ | $|\alpha J| \leq \frac{|(2\omega+3)M|}{2\sqrt{(\omega+1)(\omega+2)}}$ |
| $\omega = -2$ | might exist for any $J$ | $|J| < |M|$ |
| $-2 < \omega < -\frac{3}{2}$ | do not exist for any $J$ | might exist for any $J$ |
| $-\frac{3}{2} < \omega < -1$ | $|\alpha J| > M$ | might exist for any $J$ |
| $-1 < \omega < +\infty$ | $|\alpha J| \leq \frac{(2\omega+3)M}{2\sqrt{(\omega+1)(\omega+2)}}$ | $|\alpha J| \leq \frac{(2\omega+3)M}{2\sqrt{(\omega+1)(\omega+2)}}$ |

We can mention the principal differences between the charged and uncharged theory. The charged theory has black holes which are not present in the uncharged theory for the following range of parameters: (i) $-\infty < \omega < -2$, $M < 0$; (ii) $\omega = -2$, $M > 0$, $|J| < M$; (iii) $-3/2 < \omega < -1$, $M < 0$, $|\alpha J| > M$; (iv) $-1 < \omega < +\infty$, $M > 0$, $M < |\alpha J| < (2\omega + 3)M/2\sqrt{(\omega+1)(\omega+2)}$ and (v) $-1 < \omega < +\infty$, $M < 0$, $|\alpha J| < |M|$.

6. Causal and geodesic structure of the charged black holes

6.1. Analysis of the causal structure

In order to study the causal structure we follow the procedure of Boyer and Lindquist [28] and Carter [29] and write the metrics (26)-(28) in the form [see (23)]

$$ds^2 = -\Delta \left( \gamma dt - \frac{\theta}{\alpha^2} d\varphi \right)^2 + \frac{dr^2}{\Delta} + r^2 \left( \gamma d\varphi - \theta dt \right)^2,$$  \hspace{1cm} (51)
where

\[ \Delta = (\alpha r)^2 - b(\alpha r)^{-\frac{1}{\omega+1}} + k\chi^2(\alpha r)^{-\frac{1}{\omega+1}}, \quad \omega \neq -2, -3/2, -1, \quad (52) \]

\[ \Delta = (1 + \frac{\chi^2}{4} \ln r)r^2 - br, \quad \omega = -2, \quad (53) \]

\[ \Delta = r^2[-\Lambda \ln(br) + \chi^2 r^2], \quad \omega = -\frac{3}{2} \quad (54) \]

and in (52) \( b \) and \( k\chi^2 \) are given by (46) and (47).

Below we describe the general procedure to draw the Penrose diagrams. Following Boyer and Lindquist [28], we choose a new angular coordinate which straightens out the spiraling null geodesics that pile up around the event horizon. A good choice is

\[ \bar{\varphi} = \gamma \varphi - \theta t. \quad (55) \]

Then (51) can be written as

\[ ds^2 = -\Delta \left( \frac{1}{\gamma} dt - \frac{\theta}{\alpha^2 \gamma} d\bar{\varphi} \right)^2 + \frac{dr^2}{\Delta} + r^2 d\bar{\varphi}^2. \quad (56) \]

Now the null radial geodesics are straight lines at 45°. The advanced and retarded null coordinates are defined by

\[ u = \gamma t - r_* \quad \text{and} \quad v = \gamma t + r_*, \quad (57) \]

where \( r_* = \int \Delta^{-1} dr \) is the tortoise coordinate. In general, the integral defining the tortoise coordinate cannot be solved explicitly for the solutions (52)-(54). Moreover, the maximal analytical extension depends critically on the values of \( \omega \). There are seven cases which have to be treated separately: \( \omega < -2, \omega = -2, -2 < \omega < -3/2, \omega = -3/2, -3/2 < \omega < -1, \omega > -1 \) and \( \omega = \pm \infty \). As we shall see, on some of the cases the \( \Delta \) function has only one zero and so the black hole has one event horizon, for other cases \( \Delta \) has two zeros and consequently two horizons are present. If \( \Delta \) has one zero, \( r = r_+ \), we proceed as follows. In the region where \( \Delta < 0 \) we introduce the Kruskal coordinates \( U = +e^{-ku} \) and \( V = +e^{+kv} \) and so \( UV = +e^{k(v-u)} \). In the region where \( \Delta > 0 \) we define the Kruskal coordinates as \( U = -e^{-ku} \) and \( V = +e^{+kv} \) in order that \( UV = -e^{k(v-u)} \). The signal of the product \( UV \) is chosen so that the factor \( \Delta/UV \), that appears in the metric coefficient \( g_{UV} \),
is negative. The constant $k$ is introduced in order that the limit of $\Delta/UV$ as $r \to r_+$ stays finite.

If $\Delta$ has two zeros, $r = r_-$ and $r = r_+$ (with $r_- < r_+$), one has to introduce a Kruskal coordinate patch around each of the zeros of $\Delta$. The first patch constructed around $r_-$ is valid for $0 < r < r_+$. For this patch, in the region where $\Delta < 0$ we introduce the Kruskal coordinates $U = +e^{+k_-u}$ and $V = +e^{-k_-v}$ and so $UV = +e^{k_-(u-v)}$. In the region where $\Delta > 0$ we define the Kruskal coordinates as $U = -e^{+k_-u}$ and $V = +e^{-k_-v}$ in order that $UV = -e^{k_-(u-v)}$. The metric defined by this Kruskal coordinates is regular in the patch $0 < r < r_+$ and, in particular, is regular at $r_-$. However, it is singular at $r_+$. To have a metric non singular at $r_+$ one has to define new Kruskal coordinates for the second patch which is constructed around $r_+$ and is valid for $r_- < r < \infty$. For this patch, in the region where $\Delta < 0$ we introduce the Kruskal coordinates $U = +e^{-k_+u}$ and $V = +e^{+k_+v}$ and so $UV = +e^{k_+(v-u)}$. In the region where $\Delta > 0$ we define the Kruskal coordinates as $U = -e^{-k_+u}$ and $V = +e^{+k_+v}$ in order that $UV = -e^{k_+(v-u)}$. $k_-$ and $k_+$ are constants obeying the same condition defined above for $k$ and the sign of $UV$ is also chosen in order to have metric coefficient $g_{UV} \propto \Delta/UV$ negative. Now, these two different patches have to be joined together. Finally, to construct the Penrose diagram one has to define the Penrose coordinates by the usual arctangent functions of $U$ and $V$: $U = \arctan U$ and $V = \arctan V$.

The horizon is mapped into two mutual perpendicular straight null lines at $45^\circ$. In general, to find what kind of curve describes the lines $r = 0$ or $r = \infty$ one has to take the limit of $UV$ as $r \to 0$, in the case of $r = 0$, and the limit of $UV$ as $r \to \infty$, in the case of $r = \infty$. If this limit is $\infty$ the corresponding line is mapped into a curved null line. If the limit is $-1$ the corresponding line is mapped into a curved timelike line and finally, when the limit is $+1$ the line is mapped into a curved spacelike line. The asymptotic lines are drawn as straight lines although in the coordinates $U$ and $V$ they should be curved outwards, bulged. It is always possible to change coordinates so that the asymptotic lines are indeed straight lines.

The lines of infinite redshift, $r = r_{rs}$, are given by the vanishing of the $g_{tt}$ metric component. There are closed timelike curves, $r_{CTC}$, whenever $g_{\varphi\varphi} < 0$.

The Penrose diagram for the static charged black hole is similar to the one drawn for the corresponding rotating charged black hole. The only difference is that the infinite redshift lines coincide with the horizons and there are no closed timelike surfaces. This similarity is due to the fact that the
rotating black hole is obtained from the static one by applying the coordinate transformations (25).

In practice, to find the curve that describes the asymptotic limits \(r = 0\) and \(r = \infty\), we can use a trick. We study the behavior of \(\Delta\) at the asymptotic limits, i.e. we find which term of \(\Delta\) dominates as \(r \to 0\) and \(r \to \infty\). Then, in the asymptotic region, we take \(\Delta \sim \Delta_0\) in the vicinity of \(r = 0\) and \(\Delta \sim \Delta_\infty\) in the vicinity of \(r = \infty\). The above procedure of finding the Kruskal coordinates is then applied to the asymptotic regions, e.g. in the vicinity of \(r = \infty\) we can take \(r_* \sim r_*^{\infty} = \int (\Delta_\infty)^{-1} dr\) and find the character of the \(r = \infty\) curve from the limit of UV as \(r \to \infty\).

6.2. Analysis of the geodesic structure

Let us now consider the geodesic motion. The equations governing the geodesics can be derived from the Lagrangian

\[
\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\frac{\delta}{2},
\]

where \(\tau\) is an affine parameter along the geodesic which, for a timelike geodesic, can be identified with the proper time of the particle along the geodesic. For a null geodesic one has \(\delta = 0\) and for a timelike geodesic \(\delta = +1\). From the Euler-Lagrange equations one gets that the generalized momentums associated with the time coordinate and angular coordinate are constants: \(p_t = E\) and \(p_\phi = L\). The constant \(E\) is related to the time-like Killing vector \((\partial/\partial t)^\mu\) which reflects the time translation invariance of the metric, while the constant \(L\) is associated to the spacelike Killing vector \((\partial/\partial \phi)^\mu\) which reflects the invariance of the metric under rotation. Note that since the spacetime is not asymptotically flat, the constants \(E\) and \(L\) cannot be interpreted as the local energy and angular momentum at infinity.

From the geodesics equations we can derive two equations which will be specially useful since they describe the behavior of geodesic motion along the radial coordinate. For \(\omega < -2\) and \(\omega > -1\) these are

\[
\begin{align*}
    r^2 \dot{r}^2 &= \left[-r^2 \delta + \frac{(\omega + 2)c_0^2}{2(\omega + 1) - (2\omega + 3)\Omega}\right] \Delta + \frac{2(\omega + 1)c_1^2}{2(\omega + 1) - (2\omega + 3)\Omega} r^2, \\
    r^2 \dot{r}^2 &= (E^2 - \alpha^2 L^2) r^2 + \frac{Mc_0^2}{2(\alpha r)^{\frac{\omega}{2}}} - \frac{\sqrt{2}(\omega + 1)}{16\sqrt{2} - 1\Omega} \frac{c_0^2 Q^2}{(\alpha r)^{\frac{\omega}{2}}} - r^2 \delta \Delta.
\end{align*}
\]
For $-2 < \omega < -3/2$ and $-3/2 < \omega < -1$ the two useful equations are

$$r^2 \dot{r}^2 = -\left[r^2 \delta - \frac{(\omega + 2)c_0^2}{2(\omega + 1) - (2\omega + 3)\Omega}\right] \Delta + \frac{2(\omega + 1)c_1^2}{2(\omega + 1) - (2\omega + 3)\Omega} r^2$$

$$r^2 \dot{r}^2 = (E^2 - \alpha^2 L^2)r^2 - \frac{Mc_0^2}{2(\alpha r)^{\omega+1}} + \sqrt{2}(\omega + 1) \frac{c_0^2 Q^2}{16\sqrt{2} - 2(\alpha r)^{\omega+1}} - r^2 \delta \Delta. \quad (62)$$

In the above equations $\Delta$ is the inverse of the metric component $g_{rr}$ and we have introduced the definitions

$$c_0^2 = \left[\frac{\sqrt{|\Omega|}}{\alpha} E \mp \left[\omega + 1 \sqrt{\frac{\omega + 2}{2(\omega + 1)}} \sqrt{2 - \frac{\Omega}{\alpha L}}\right]^2, \quad \text{and} \quad (63)$$

$$c_1^2 = \left[\sqrt{\frac{2 - \Omega}{2}} E - \left[\omega + 2 \sqrt{\frac{\omega + 2}{2(\omega + 1)}} \sqrt{\frac{\Omega}{\alpha L}}\right]^2, \quad (64)$$

where in (63) the minus sign is valid for $\omega < -2$ and $\omega > -1$ and the plus sign is applied when $-2 < \omega < -3/2$ or $-3/2 < \omega < -1$. There are turning points, $r_{tp}$, whenever $\dot{r} = 0$. If this is the case, equations (59) and (61) will allow us to make considerations about the position of the turning point relatively to the position of the horizon. For this purpose it will be important to note that for $\omega < -2$ and $\omega > -1$ one has $0 \leq \Omega \leq 1$, and for $-2 < \omega < -3/2$ or $-3/2 < \omega < -1$ we have that $\Omega < 0$. As a first and general example of the interest of equations (59) and (61) note that the turning points coincide with the horizons when the energy and the angular momentum are such that $c_1 = 0$. From equations (60), (62) and after some graphic computation we can reach interesting conclusions.

6.3. Penrose diagrams and geodesics for each range of $\omega$

We are now in position to draw the Penrose diagrams and study the geodesic motion. Besides the cases $\omega = -2$ and $\omega = -3/2$, for each different range of $\omega$ ($\omega < -2$, $-2 < \omega < -3/2$, $-3/2 < \omega < -1$, $\omega > -1$ and $\omega = \pm \infty$) we will consider a particular value of $\omega$. These will be precisely the ones that have been analyzed in the uncharged study of action (1) [20, 21].

We will study the solutions with positive mass, $M > 0$, and describe briefly the Penrose diagrams of the solutions with negative mass.
6.3.1 $\omega < -2$

For this range of $\omega$ there is the possibility of having black holes with positive mass whenever $|\alpha J| \leq \frac{|2\omega + 3M|}{2\sqrt{(\omega + 1)(\omega + 2)}}$. We are going to analyze the typical case $\omega = -3$. The $\Delta$ function, (52), is

$$\Delta = (\alpha r)^2 - b\sqrt{\alpha r} + c(\alpha r),$$

where from (44) and (47) one has that $b > 0$ and $c \equiv k\chi^2 < 0$ (if $M > 0$).

For $|\alpha J| \leq \frac{3M}{2\sqrt{2}}$, $\Delta$ has always one and only one zero given by

$$r_+ = \frac{1}{3\alpha} \left[-2c + \frac{c^2}{s} + s \right],$$

where

$$s = \left[\frac{1}{2} \left(27b^2 + 2c^3 + 3\sqrt{3}b\sqrt{27b^2 + 4c^3}\right)\right]^\frac{1}{3},$$

$\Delta > 0$ for $r > r_+$ and $\Delta < 0$ for $r < r_+$. The curvature singularity at $r = 0$ is a spacelike line in the Penrose diagram while $r = +\infty$ is a timelike line. The Penrose diagram is drawn in figure 1. There is no extreme black hole for this case.

![Penrose diagram](image)

**Figure 1:** Penrose diagram for the $\omega = -3$, $M > 0$, $|\alpha J| \leq 3M/2\sqrt{2}$ black hole.

When we consider the solutions with negative mass we conclude that, for $|\alpha J| \leq \frac{3M}{2\sqrt{2}}$, one has black holes with two horizons, with one (extreme case) or a spacetime without black holes. For the black hole with two horizons the Penrose diagram is shown in figure 2.(a) of and the extreme black hole has a Penrose diagram which is drawn in figure 2.(b).
Figure 2: (a) Penrose diagram for the $\omega = -3$, $M < 0$, $|\alpha J| \leq 3|M|/2\sqrt{2}$ black hole with two horizons. (b) Penrose diagram for the $\omega = -3$, $M < 0$, $|\alpha J| \leq 3|M|/2\sqrt{2}$ extreme black hole.

Let us now consider the geodesic motion. Analyzing (59) and noting that $0 \leq \Omega \leq 1$ we see that for the null and timelike geodesics the coefficient of $\Delta$ is always negative so, for $0 < r < r_+^+$, the first term of (59) is positive. Since the second term is positive or null we conclude that whenever there are turning points, they are $r_{tp}^1 = 0$ and $r_{tp}^2 \geq r_+$. From (60) we conclude the following about the geodesic motion. (i) If $E^2 - \alpha^2 L^2 > 0$ null particles produced at $r = 0$ escape to $r = +\infty$ and null particles coming in from infinity are scattered at $r_{tp}^1 = 0$ and spiral back to infinity. (ii) Null geodesics with $E^2 - \alpha^2 L^2 < 0$ are bounded between the singularity $r_{tp}^1 = 0$ and a maximum ($r_{tp}^2 \geq r_+$) radial distance. The turning point $r_{tp}^2$ is exactly at the horizon $r_+$ if and only if the energy and the angular momentum are such that $c_1 = 0$. (iii) Null geodesics with energy and angular momentum such that $c_0 = 0$ can reach and “stay” at the curvature singularity $r = 0$. (iv) All the timelike geodesics present the same features as the null geodesics with $E^2 - \alpha^2 L^2 < 0$. So, any timelike geodesic is bounded within the region $r_{tp}^1 \leq r \leq r_{tp}^2$ (with $r_{tp}^1 = 0$ and $r_{tp}^2 \geq r_+$), and no timelike particle can either escape to infinity or reach and “stay” at $r = 0$. (v) Neither null or timelike geodesics have stable or unstable circular orbits.
6.3.2 \( \omega = -2 \)

For \( \omega = -2 \), \( \Delta \) is given by (53) and, in the case \( M > 0 \), \( \Delta \) has one zero given by

\[
r_+ = b \left[ 4 \chi^2 \text{ProdLog} \left( \frac{be^{4\chi^2}}{4\chi^2} \right) \right]^{-1},
\]

with \( \text{ProdLog}(x) = z \) being such that \( x = ze^z \). The scalar \( R_{\mu\nu}R^{\mu\nu} \) (23) diverge at \( r = 0 \) and \( r = +\infty \). For \( |J| > |M| \), \( \Delta \) is positive for \( r > r_+ \) and negative for \( r < r_+ \). At \( r = +\infty \) the curvature singularity is timelike while \( r = 0 \) is a null curvature singularity. The Penrose diagram is represented in figure 3.(a). For \( |J| < |M| \), unlike the uncharged case, the theory also has a black hole with a horizon located at \( r_+ \) given by (67). \( \Delta \) is negative for \( r > r_+ \) and positive for \( r < r_+ \). The \( r = +\infty \) curvature singularity is spacelike and \( r = 0 \) is a null curvature singularity. The Penrose diagram is drawn in figure 3.(b). For \( |J| = |M| \) one has \( b = \chi = 0 \). Then, \( r = 0 \) is a naked null singularity and the boundary \( r = \infty \) changes character and has no singularity, being a timelike line in the Penrose diagram drawn in figure 3.(c).

\[\text{Figure 3: (a)}\] Penrose diagram for the black hole of: i) \( \omega = -2 \), \( M > 0 \), \( |J| > M \); ii) \( \omega = -9/5 \), \( M < 0 \). \( \text{(b)} \) Penrose diagram for the \( \omega = -2 \), \( M > 0 \), \( |J| < M \) naked singularity. \( \text{(c)} \) Penrose diagram for the \( \omega = -2 \), \( M > 0 \), \( |J| = M \) naked singularity.

Now, we study the geodesic motion for \( M > 0 \). The behavior of geodesic motion along the radial coordinate can be obtained from the following two equations

\[
r^2\dot{r}^2 = -[r^2\delta + c_0^2]\Delta + c_1^2r^2 \tag{68}
\]
\[ r^2 \dot{r}^2 = (E^2 - L^2) r^2 - \frac{\chi^2 c_0^2}{4} r^2 \ln r + \frac{J^2 - M^2}{M} c_0^2 r - r^2 \delta \Delta, \quad (69) \]

where
\[
c_0^2 = \left[ \left( \frac{J^2}{M^2} - 1 \right)^{-1/2} E - \left( 1 - \frac{M^2}{J^2} \right)^{-1/2} L \right]^2, \quad (70)
\]
\[
c_1^2 = \left[ \left( 1 - \frac{M^2}{J^2} \right)^{-1/2} E - \left( \frac{J^2}{M^2} - 1 \right)^{-1/2} L \right]^2. \quad (71)
\]

We first consider the case \(|J| > M\). From equation (68) we conclude that whenever there are turning points, they are given by \(r_{tp}^1 = 0\) and \(r_{tp}^2 \geq r_+\). The turning point \(r_{tp}^2\) is exactly at the horizon \(r_+\) if and only if the energy and the angular momentum are such that \(c_1 = 0\), i.e., \(E = ML/J\). From the graphic computation of (69) we conclude that: (i) the only particles that can escape to \(r = +\infty\) or \(r = 0\) are null particles that satisfy \(c_0 = 0\) which implies \(E = JL/M\); (ii) all other null geodesics and all timelike geodesics are bounded between \(r_{tp}^1 = 0\) and a maximum \((r_{tp}^2 \geq r_+)\) radial distance.

For \(|J| < M\) one has that: (i) timelike and null spiraling particles with \(E \neq JL/M\) start at \(r_{tp} \geq r_+\) and reach infinity radial distances or timelike and null geodesics start at \(r = +\infty\) and spiral toward \(r_{tp}\) and then return back to infinity; (ii) null particles can escape to \(r = +\infty\) or \(r = 0\) if \(c_0 = 0\), i.e., \(E = JL/M\); (iii) timelike geodesics with \(E = JL/M\) can be bounded between \(r_{tp}^1 = 0\) and a maximum \((r_{tp}^2 \geq r_+)\) radial distance, or start at \(r_{tp}^3 \geq r_+\) and reach infinity radial distances.

### 6.3.3 \(-2 < \omega < -3/2\)

In this range of \(\omega\), spacetime has no black holes since from (46) and (47) one has \(b < 0\) and \(k\chi^2 > 0\) so the \(\Delta\) function (52) is always positive. The curvature singularity \(r = 0\) is a naked null singularity and the curvature singularity \(r = +\infty\) is a naked timelike singularity. The Penrose diagram is drawn in figure 4.

When we consider the solutions with negative mass we conclude that one might have black holes with one horizon. If this is the case, the Penrose diagram is exactly equal to the one shown in figure 3.(a) (which represents the typical case \(\omega = -9/5\)).
Figure 4: Penrose diagram for the spacetime of: i) \( \omega = -2, M > 0, |J| = M \); ii) \(-2 < \omega < -3/2, M > 0\); iii) \( \omega = -3/2 \) (large \( \Lambda/\chi^2 \)); iv) \( \omega = -4/3, M < 0, |\alpha J| = M \) (the only difference is that \( r = 0 \) is a topological singularity rather than a curvature singularity).

Although there are no black holes with positive mass, it is interesting to study the geodesics that we might expect for this spacetime. From graphic computation of (62) we conclude the following for null and timelike geodesics.

(i) For \( E^2 - \alpha^2 L^2 \leq 0 \) there is no possible motion. (ii) This situation also occurs for small positive values of \( E^2 - \alpha^2 L^2 \). (iii) However, when we increase the positive value of \( E^2 - \alpha^2 L^2 \) there is a critical value for which a stable circular orbit is allowed. (iv) And for positive values of \( E^2 - \alpha^2 L^2 \) above the critical value, null and timelike geodesics are bounded between a minimum \( (r_{tp}^1) \) and a maximum \( (r_{tp}^2) \) radial distance. (v) Null particles with energy and angular momentum satisfying \( c_0 = 0 \) that are produced at \( r = 0 \) escape to \( r = +\infty \) and null particles coming in from infinity are scattered at \( r_{tp} = 0 \) and spiral back to infinity. (vi) Timelike particles with energy and angular momentum satisfying \( c_0 = 0 \) are bounded within the region \( 0 \leq r \leq r_{tp} \), with \( r_{tp} \) finite.

6.3.4 \( \omega = -3/2 \)

In this case, \( \Delta \) is given by (54). Depending on the value of \( \frac{\Lambda}{\chi^2} \) one has black holes with two horizons (small \( \frac{\Lambda}{\chi^2} \)), with one (extreme case) or a spacetime without black holes (large \( \frac{\Lambda}{\chi^2} \)).

For the black hole with two horizons, we have \( \Delta > 0 \) in \( r < r_- \) and \( r > r_+ \). For the patch \( r_- < r < \infty \), the curvature singularity \( r = \infty \) is mapped into two symmetrical timelike lines and the horizon \( r = r_+ \) is mapped into two mutual perpendicular straight lines at \( 45^\circ \). For the patch \( 0 < r < r_+ \), the curvature singularity \( r = 0 \) is mapped into a pair of null lines and the horizon \( r = r_- \) is mapped into two mutual perpendicular straight lines at \( 45^\circ \). One has to join these two different patches and then repeat them over in the vertical. The resulting Penrose diagram is shown in figure 5.(a). For the
extreme black hole the two curvature singularities \( r = 0 \) and \( r = \infty \) are still null and timelike lines (respectively), but the event and inner horizon join together in a single horizon \( r_+ \). The Penrose diagram is like the one drawn in figure 5.(b). The spacetime with no black hole present has a Penrose diagram like the one represented in figure 4.

![Penrose diagrams](image)

**(Figure 5)**: (a) Penrose diagram for the black hole with two horizons of: i) the \( \omega = -3/2 \), (small \( \Lambda/\chi^2 \)); ii) \( \omega = -4/3 \), \( M < 0 \), \( |\alpha J| < |M| \) (the only difference is that \( r = 0 \) is a topological singularity rather than a curvature singularity).

(b) Penrose diagram for the extreme black hole of: i) \( \omega = -3/2 \); ii) \( \omega = -4/3 \), \( M < 0 \), \( |\alpha J| < |M| \) (the only difference is that \( r = 0 \) is a topological singularity rather than a curvature singularity).

Now, we study the geodesic motion.

The behavior of geodesic motion along the radial coordinate can be obtained from the following two equations

\[
\begin{align*}
 r^2 \dot{r}^2 &= -r^2 \delta + c_0^2 \Delta + c_0^2 r^2 \\
 \dot{r}^2 &= c_0^2 [\Lambda \ln(b r) - \chi^2 r^2 + 1] - \delta \Delta
\end{align*}
\]  

where \( c_0 = (\theta E - \gamma L) \). From equation (72) we conclude that whenever there are turning points, they are given by \( r_{tp}^1 \leq r_- \) and \( r_{tp}^2 \geq r_+ \). The turning points coincide with the horizons if and only if the energy and the angular momentum are such that \( c_0 = 0 \). From the graphic computation of (73) we conclude that: (i) Whenever \( c_0 = 0 \) null particles describe stable circular orbits wherever they are located; (ii) Null geodesics with \( c_0 \neq 0 \) and all timelike geodesics describe a bound orbit between \( r_{tp}^1 \) and \( r_{tp}^2 \).
For the extreme black hole the two horizons coincide so all null geodesics and all timelike geodesics describe a stable circular orbit.

6.3.5 $-3/2 < \omega < -1$

For this range of $\omega$ there is the possibility of having black holes with positive mass whenever $|\alpha J| > M$ (Table 1). We are going to analyze the typical case $\omega = -4/3$. The $\Delta$ function, (52), is

$$\Delta = (\alpha r)^2 - b(\alpha r)^3 + c(\alpha r)^6,$$

where from (46) and (47) one has that $b > 0$ and $c \equiv k\chi^2 < 0$ if $M > 0$ and $|\alpha J| > M$. $\Delta$ is negative for $r > r_+$ and positive for $r < r_+$, where $r_+$ is the only zero of $\Delta$ given by

$$r_+ = \frac{1}{2\alpha} \left( \frac{b}{c} \right)^{1/3} \left[ \sqrt{s} - \frac{2}{\sqrt{s}} - s \right],$$

where

$$s = \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4^4c^{-4}}{3^3b^4}} \right]^{1/3} + \left[ \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4^4c^{-4}}{3^3b^4}} \right]^{1/3}. \quad (75)$$

The physical curvature singularity is located inside the horizon at $r = +\infty$ and is a spacelike line in the Penrose diagram. At $r = 0$ there is a null topological singularity. The Penrose diagram is sketched in figure 6.

![Penrose diagram](image)

**Figure 6:** Penrose diagram for the black hole of: i) $\omega = -4/3$, $M > 0$, $|\alpha J| > M$; ii) $\omega = -4/3$, $M < 0$, $|\alpha J| > |M|$.

When we consider the solutions with negative mass we conclude that, for $|\alpha J| > |M|$, one has black holes with a Penrose diagram equal to the one drawn in figure 6. For $|\alpha J| < |M|$ one has black holes with two horizons, with
one (extreme case) or a spacetime without black holes. For the black hole with two horizons the Penrose diagram is similar to the one shown in figure 5.(a) and the extreme black hole has a Penrose diagram which is similar to the one drawn in figure 5.(b). For $|\alpha J| = |M|$ the Penrose diagram is similar to figure 4. The only difference is the fact that $r = 0$ is now a topological singularity rather than a curvature singularity.

Let us now consider the geodesic motion for positive mass. Analyzing (11) and noting that from (15) the condition $|\alpha J| > M$ implies $\Omega < -2$ we conclude that, for null geodesics, whenever there are turning points, they are $r_{1\text{tp}} = 0$ and $r_{2\text{tp}}^2 \geq r_+$. From (12) we conclude the following about the null geodesic motion. (i) For $E^2 - \alpha^2 L^2 \leq 0$ there is no possible motion. (ii) Null geodesics with $E^2 - \alpha^2 L^2 > 0$ are bounded between the singularity $r_{1\text{tp}} = 0$ and a maximum ($r_{2\text{tp}}^2 \geq r_+$) radial distance. The turning point $r_{2\text{tp}}^2$ is exactly at the horizon $r_+$ if and only if the energy and the angular momentum are such that $c_1 = 0$.

The timelike geodesic motion is radically different. (i) For $E^2 - \alpha^2 L^2 \leq 0$ we have timelike spiraling particles that start at $r_{\text{tp}}$ and reach infinity radial distances or timelike geodesics that start at $r = +\infty$ and spiral toward $r_{\text{tp}}$ and then return back to infinity. (ii) For small positive values of $E^2 - \alpha^2 L^2$, timelike particles that are produced at $r = 0$ escape to $r = +\infty$ and timelike particles coming in from infinity are scattered at $r_{\text{tp}} = 0$ and spiral back to infinity. (iii) When we increase the positive value of $E^2 - \alpha^2 L^2$ there is a critical value for which an unstable circular orbit is allowed. (iv) And for positive values of $E^2 - \alpha^2 L^2$ above the critical value, timelike particles are allowed to be bounded between $r_{1\text{tp}} = 0$ and a maximum ($r_{2\text{tp}}^2$) radial distance or to start at $r_{3\text{tp}}^3 > r_{2\text{tp}}^2$ and escape to infinity.

6.3.6 $\omega > -1$

The range $\omega > -1$ is not discussed here since the properties of the typical case $\omega = 0$ have been presented in [23], where the three-dimensional gravity theory of $\omega = 0$ was obtained through dimensional reduction from four-dimensional General Relativity with one Killing vector field.

6.3.7 $\omega = \pm \infty$

The case $\omega = \pm \infty$ is also not discussed here since this case reduces to the electrically charged BTZ black hole which has been studied in detail in
7. Hawking temperature of the charged black holes

To compute the Hawking temperature of the rotating black holes, one starts by writing the metric in the canonical form (30). To proceed it is necessary to first perform the coordinate transformation to coordinates $t, \tilde{\varphi}$ which corotate with the black hole. In other words, the angular coordinate $\varphi$ must be changed to $\tilde{\varphi} = \varphi - \Omega_H t$, where $\Omega_H = -\frac{\partial t}{\partial \varphi} \big|_{r=r_+} = -N^\varphi (r_+)$ is the angular velocity of the black hole. With this transformation the metric (30) becomes

$$ds^2 = -(N^0)^2 dt^2 + \frac{dr^2}{f^2} + H^2 \left[ d\tilde{\varphi} + \left( N^\varphi (r) - N^\varphi (r_+) \right) dt \right]^2 ,$$  

(76)

Then, one applies the Wick rotation $t \to -i \tau$ in order to obtain the euclidean counterpart of (76). Now, one studies the behavior of the euclidean metric in the vicinity of the event horizon, $r_+$. In this vicinity, one can write $N^\varphi (r) - N^\varphi (r_+) \sim 0$ and take the expansion $\Delta (r) \sim \frac{d\Delta (r)}{dr} (r - r_+) + \cdots$. One proceeds applying the variable change $\frac{1}{\Delta (r)} dr^2 = d\rho^2$ so that one has

$$\rho = 2 \sqrt{\frac{r-r_+}{d\Delta (r_+) / dr}} \quad \text{and} \quad \Delta (\rho) \sim [d\Delta (r_+) / dr]^{\rho^2 / 4} .$$

With this procedure the euclidean metric in the vicinity of the event horizon can be cast in the form

$$ds^2 \sim (2\pi / \beta_H)^2 \rho^2 d\tau^2 + d\rho^2 + H^2 d\tilde{\varphi}^2 .$$

Applying a final variable change, $\bar{\tau} = (2\pi / \beta_H) \tau$, the metric becomes $ds^2 \sim \rho^2 d\bar{\tau}^2 + d\rho^2 + H^2 d\tilde{\varphi}^2$. To avoid the canonical singularity at the event horizon one must demand that the period of $\bar{\tau}$ is $2\pi$ which implies $0 \leq \tau \leq \beta_H$. Finally, the Hawking temperature is defined as $T_H = (\beta_H)^{-1}$.

Applying the above procedure, one finds for the Hawking temperature of the rotating black holes the following expressions

$$T_H = \frac{1}{4\pi} \frac{r_+^2 \left[ 2\alpha r_+ + \frac{b^2}{\omega_{+1}} (\alpha r_+) - \frac{2k\chi_1^2 (\alpha r_+)}{\omega_{+1}} \right]^2}{r_+^2 - \frac{\alpha^2}{\omega_{+1}} \left[ -b(\alpha r_+) - \frac{1}{\omega_{+1}} + k\chi_2^2 (\alpha r_+) \right]} ,$$  

(77)

$\omega \neq -2, -\frac{3}{2}, -1$. 

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where in (77), $b$, $k\chi^2$ and $\theta^2$ are given by (46), (47) and (43), respectively.

The Hawking temperature of the static charged black holes can be obtained from (77)-(79) by taking $\gamma = 1$ and $\theta = 0$ [see (25)] (see also [22] for the $\omega = 0$ uncharged black hole).

8. Conclusions

We have added the Maxwell action to the action of a generalized 3D dilaton gravity specified by the Brans-Dicke parameter $\omega$ introduced in [20, 21]. We have concluded that for the static spacetime the electric and magnetic fields cannot be simultaneously non-zero, i.e. there is no static dyonic solution. In this work we have considered the electrically charged case alone. We have found the static and rotating black hole solutions of this theory. It contains eight different cases that appear naturally from the solutions. For $\omega = 0$ one gets a theory related (through dimensional reduction) to electrically charged four dimensional general relativity with one Killing vector [23] and for $\omega = \pm \infty$ one obtains electrically charged three dimensional general relativity [4, 8]. For $\omega = -1$ the Maxwell term does not alter the uncharged solution, i.e. one still gets the two-dimensional string theory black hole cross $S^1$ (For a discussion see [20, 41, 42]). The uncharged theory imposed, in the case $\omega = -2$, that the cosmological constant had to be null. This no longer happens in the charged extension of the theory which imposes, for $\omega = -2$, a relation between the charge and the cosmological constant.

For $\omega > -3/2$ the ADM mass and angular momentum of the solutions are finite, well-behaved and equal to the ADM masses of the uncharged solutions. However, for $\omega < -3/2$ the ADM mass and angular momentum
of the solutions have terms proportional to the charge that diverge at the asymptotic limit, as frequently occurs in the extended theories including a Maxwell field (see, e.g. \[4, 9, 11\]). We have shown how to treat this problem. For each range of $\omega$ we have determined what conditions must be imposed on the ADM masses of the solutions in order to be possible the existence of black holes. Our results show that there is no upper bound on the electric charge.

The causal and geodesic structure of the charged solutions is, in some of the cases, quite different from the ones of the uncharged case. The Hawking temperature has been computed.

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