Almost Periodicity of All $L^2$-bounded Solutions of a Functional Heat Equation

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Abstract. In this paper, we continue the investigations done in the literature about the so called Bohr-Neugebauer property for almost periodic differential equations. More specifically, for a class of functional heat equations, we prove that each $L^2$-bounded solution is almost periodic. This extends a result in [5] to the delay case.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary, $\tau$ be a positive constant and $\mathcal{C} = C([-\tau,0], L^2(\Omega, \mathbb{R}))$ denote the space of continuous functions $\varphi: [-\tau,0] \to L^2(\Omega, \mathbb{R})$ with the norm defined by $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \|\varphi(\theta)\|_{L^2}$, here $\|\varphi(\theta)\|_{L^2} = \left( \int_{\Omega} \varphi^2(\theta, x) \, dx \right)^{1/2}$ for $\theta \in [-\tau,0]$.

In this paper, we consider the boundary problem of partial functional differential equation

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= \Delta u + f(t, x, u_t) & \text{if } (t, x) \in \mathbb{R} \times \Omega, \\
u(t, x) &= 0 & \text{if } (t, x) \in \mathbb{R} \times \partial \Omega,
\end{aligned}
\end{equation}

where $\Delta$ is the Laplace operator acting on the variable $x \in \Omega$, $f: \mathbb{R} \times \Omega \times \mathcal{C} \to \mathbb{R}$ is continuous, and the time delay function $u_t \in \mathcal{C}$ defined by $u_t(\theta)(\cdot) = u(t+\theta, \cdot) \in L^2(\Omega, \mathbb{R})$ for $\theta \in [-\tau,0]$.

There have been much research activity for the qualitative behavior of partial differential equations with or without delays, see, e.g., the references [1–3, 6, 8, 9, 13, 14]. It is worth mentioning that the authors in [4, 7, 11, 12, 15] studied the Bohr-Neugebauer property for some special abstract differential equations. A differential equation is said to have Bohr-Neugebauer property if its any bounded solution is almost periodic. This issue also

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occurred in Corduneanu’s monograph [5, Chapter 7], where the author considered the following heat equation

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) = \Delta u + \tilde{f}(t, x, u) & \text{if } (t, x) \in \mathbb{R} \times \Omega, \\
u(t, x) = 0 & \text{if } (t, x) \in \mathbb{R} \times \partial\Omega
\end{cases}
\]

and, under the assumption that \( u(t, x) \) was a solution of (1.2) with the property

\[
\sup_{t \in \mathbb{R}} \int_{\Omega} u^2(t, x) \, dx < \infty,
\]

obtained a conclusion that this \( L^2 \)-bounded solution \( u(t, x) \) was almost periodic.

The main objective of this paper is to extend the conclusion of (1.2) to (1.1). For this purpose, we assume that

(H1) \( b \in C(\mathbb{R} \times \bar{\Omega}, \mathbb{R}) \) with \( b(t, x) \geq 0 \) for \( (t, x) \in \mathbb{R} \times \bar{\Omega} \);

(H2) for any \( \varphi_1, \varphi_2 \in C \) and \( (t, x) \in \mathbb{R} \times \bar{\Omega} \),

\[
|f(t, x, \varphi_1) - f(t, x, \varphi_2)| \leq b(t, x) \|\varphi_1 - \varphi_2\|
\]

(H3) \( \lambda > 0 \) is the smallest eigenvalue of the boundary-value problem [9,10]

(1.3) \[
\begin{cases}
\Delta w + \lambda w = 0 & \text{if } x \in \Omega, \\
w = 0 & \text{if } x \in \partial\Omega;
\end{cases}
\]

(H4) \( (\int_{\Omega} b^2(t, x) \, dx)^{1/2} \leq b_0 < \lambda \) for all \( t \in \mathbb{R} \).

We remark that the existence of \( L^2 \)-solutions of functional heat equations had been studied in monograph [13]. Next we consider only the almost periodicity of \( L^2 \)-solutions of (1.1).

2. Main results

As usual, by \( C([-\tau, 0], \mathbb{R}) \) we denote the Banach space of real-valued functions on \([-\tau, 0]\) with supremum norm. In what follows, we will require an important conclusion, which extends the result in [5, Proposition 6.5].

**Lemma 2.1.** Let \( \psi: \mathbb{R} \to \mathbb{R}_+ \) be bounded, differential and satisfy

\[
\psi'(t) \leq \omega(\psi_t), \quad t \in \mathbb{R},
\]

where \( \psi_t \in C([-\tau, 0], \mathbb{R}) \) is defined by \( \psi_t(\theta) = \psi(t+\theta) \) for \( \theta \in [-\tau, 0] \), \( \omega: C([-\tau, 0], \mathbb{R}) \to \mathbb{R} \) is continuous, and \( \omega(\psi) < 0 \) for \( \|\psi\| > \mu > 0 \). Then

\[
\psi(t) \leq \mu, \quad t \in \mathbb{R}.
\]
Proof. We first consider the case that
\[ \psi(t) \leq \psi(t_M), \quad t \in \mathbb{R}, \]
where \( t_M \) is some point in \( \mathbb{R} \). That is, \( \psi \) obtains its maximum value at the point \( t_M \).
Then, we have
\[ 0 = \psi'(t_M) \leq \omega(\psi_{t_M}), \]
which, together with the assumption \( \omega(\psi) < 0 \) for \( \|\psi\| > \mu > 0 \), results in
\[ \mu \geq \|\psi_{t_M}\| = \psi(t_M) \geq \psi(t), \quad t \in \mathbb{R}. \]

In the case that \( \limsup_{t \to \infty} \psi(t) = \sup \{ \psi(t) : t \in \mathbb{R} \} \), we can choose a sequence \( \{t_n\} \) with \( t_n \to \infty \) as \( n \to \infty \), such that \( \lim_{n \to \infty} \psi(t_n) = \sup \{ \psi(t) : t \in \mathbb{R} \} \), and
\[ \psi'(t_n) \geq 0 \quad \text{for sufficiently large } n. \]

Similarly, by
\[ 0 \leq \psi'(t_n) \leq \omega(\psi_{t_n}) \quad \text{for sufficiently large } n, \]
we obtain that
\[ \mu \geq \|\psi_{t_n}\| \geq \psi(t_n) \quad \text{for sufficiently large } n, \]
which means
\[ \mu \geq \sup \{ \psi(t) : t \in \mathbb{R} \}. \]

In case \( \limsup_{t \to -\infty} \psi(t) = \sup \{ \psi(t) : t \in \mathbb{R} \} \), we assert that
\[ (2.1) \quad \sup \{ \psi(t) : t \in \mathbb{R} \} \leq \mu \quad \text{for } t \in \mathbb{R}. \]

Otherwise, there exists a \( t_N < 0 \) such that \( \psi(t_N) > \mu \), which yields
\[ \psi'(t_N) \leq \omega(\psi_{t_N}) < 0 \]
and leads to
\[ \psi(t_N) \leq \psi(t) \leq \sup \{ \psi(t) : t \in \mathbb{R} \} \quad \text{for } t \leq t_N. \]

Now by the assumption on \( \omega \), we have \( \omega(\psi_{t}) \leq -m < 0 \) for \( t \leq t_N \), and this, in combination with the assumption \( \psi'(t) \leq \omega(\psi_{t}) \), induces \( \sup \{ \psi(t) : t \in \mathbb{R} \} = \infty \), which conflicts with our assumption on \( \psi \). In other words, the assertion \( (2.1) \) is true. The proof is complete. \( \Box \)

Referring to \([5\text{, Chapter 7]}\), by an \( L^2(\Omega) \)-almost periodic function \( f(t, x, \varphi) \) in \( t \) uniformly with respect to \( \varphi \in \mathcal{C} \) we mean that, for each \( \varepsilon > 0 \), there exists a number \( l = l(\varepsilon) > 0 \) such that any interval \( [\mu, \mu + l] \subset \mathbb{R} \) contains a point \( \sigma \) with the property
\[ (2.2) \quad \int_{\Omega} |f(t + \sigma, x, \varphi) - f(t, x, \varphi)|^2 \, dx < \varepsilon^2 \quad \text{for all } (t, \varphi) \in \mathbb{R} \times \mathcal{C}. \]
Theorem 2.2. Suppose that \( f(t, x, \varphi) \) is \( L^2(\Omega) \)-almost periodic in \( t \) uniformly with respect to \( \varphi \in C \). Then, under the assumptions (H1)–(H4), each \( L^2 \)-bounded solution \( u(t, x) \) of (1.1) is almost periodic in the sense of mapping \( t \in \mathbb{R} \to u(t, \cdot) \in L^2(\Omega, \mathbb{R}) \).

Proof. The proof is similar to that in [5, Theorem 7.5]. By the assumption on \( f(t, x, \varphi) \), for each \( \varepsilon > 0 \), there exists an \( l = l(\varepsilon) > 0 \) such that any interval \( [\mu, \mu + l] \subset \mathbb{R} \) contains a point \( \sigma \) with the property (2.2). For the fixed \( \sigma \in \mathbb{R} \) we define

\[
v(t, x) = u(t + \sigma, x) - u(t, x).
\]

Then we have

\[
\begin{aligned}
\frac{\partial}{\partial t} v(t, x) &= \Delta v + f(t + \sigma, x, u_{t+\sigma}) - f(t, x, u_t) & \text{if } (t, x) \in \mathbb{R} \times \Omega, \\
v(t, x) &= 0 & \text{if } (t, x) \in \mathbb{R} \times \partial \Omega.
\end{aligned}
\]

Let

\[
V(t) = \int_\Omega v^2(t, x) \, dx, \quad t \in \mathbb{R}
\]

and

\[
\|V_t\| = \sup_{-\tau \leq \theta \leq 0} \int_\Omega |u(t + \sigma + \theta, x) - u(t + \theta, x)|^2 \, dx, \quad t \in \mathbb{R}.
\]

Then

\[
\sqrt{\|V_t\|} = \|u_{t+\sigma} - u_t\|,
\]

and \( V(t) \) is bounded on \( \mathbb{R} \).

Now invoking (2.3), we get

\[
\frac{1}{2} \frac{dV}{dt} = \int_\Omega v \frac{\partial v}{\partial t} \, dx
\]

\[
= \int_\Omega v \Delta v \, dx + \int_\Omega v(f(t + \sigma, x, u_{t+\sigma}) - f(t, x, u_t)) \, dx.
\]

Note that, from Green’s formula and Poincaré’s inequality, it follows that

\[
\lambda \int_\Omega v^2(t, x) \, dx \leq \int_\Omega |\text{grad } v(t, x)|^2 \, dx = -\int_\Omega v \Delta v \, dx,
\]

where \( \lambda \) is the smallest eigenvalue of (1.3). Consequently, from (2.4) we derive

\[
\frac{1}{2} \frac{dV}{dt} \leq -\lambda \int_\Omega v^2(t, x) \, dx
\]

\[
+ \int_\Omega v(f(t + \sigma, x, u_{t+\sigma}) - f(t + \sigma, x, u_t)) \, dx
\]

\[
+ \int_\Omega v(f(t + \sigma, x, u_t) - f(t, x, u_t)) \, dx.
\]
In addition, the Hölder inequality leads us to
\[
\int_{\Omega} v(f(t + \sigma, x, u_{t+\sigma}) - f(t + \sigma, x, u_{t})) \, dx \\
\leq \left( \int_{\Omega} v^2 \, dx \right)^{1/2} \left( \int_{\Omega} b^2(t + \sigma, x) \, dx \right)^{1/2} \|u_{t+\sigma} - u_t\|
\]
\[
\leq b_0 \left( \int_{\Omega} v^2 \, dx \right)^{1/2} \|u_{t+\sigma} - u_t\|
\]
and
\[
\int_{\Omega} v(f(t + \sigma, x, u_{t}) - f(t, x, u_{t})) \, dx \\
\leq \left( \int_{\Omega} v^2 \, dx \right)^{1/2} \left( \int_{\Omega} |f(t + \sigma, x, u_{t}) - f(t, x, u_{t})|^2 \, dx \right)^{1/2},
\]
where for the first two inequalities we have imposed the assumptions (H2) and (H4), respectively. Hence, from (2.5) we obtain
\[
\frac{1}{2} \frac{dV}{dt} \leq -\lambda V + b_0 \sqrt{V} \|u_{t+\sigma} - u_t\| + \varepsilon \sqrt{V}
\]
and then
\[
\frac{1}{2} \frac{dV}{dt} \leq -\lambda V + b_0 \|V_t\| + \varepsilon \sqrt{\|V_t\|},
\]
where we have used (2.2) for the first inequality and \(\sqrt{V(t)} \leq \|V_t\| = \|u_{t+\sigma} - u_t\|\) for the second one. Since
\[
-\lambda V + b_0 \|V_t\| + \varepsilon \sqrt{\|V_t\|} \geq -\lambda \|V_t\| + b_0 \|V_t\| + \varepsilon \sqrt{\|V_t\|},
\]
we first consider
\[
-\lambda \|V_t\| + b_0 \|V_t\| + \varepsilon \sqrt{\|V_t\|} \geq 0
\]
and get
\[
\sqrt{\|V_t\|} \leq \frac{\varepsilon}{\lambda - b_0}.
\]
On the other hand, by the boundedness of \(V(t)\) we have
\[
V(t_M) = \sup \{V(t) : t \in \mathbb{R}\} \quad \text{for some } t_M \in \mathbb{R},
\]
or
\[
\limsup_{t \to \infty} V(t) = \sup \{V(t) : t \in \mathbb{R}\}, \quad \text{or} \quad \limsup_{t \to -\infty} V(t) = \sup \{V(t) : t \in \mathbb{R}\},
\]
which induce
\[ \{ V : -\lambda V + b_0 \| V_t \| + \varepsilon \sqrt{\| V_t \|} \geq 0 \} = \{ V : -\lambda \| V_t \| + b_0 \| V_t \| + \varepsilon \sqrt{\| V_t \|} \geq 0 \} . \]

Hence, by Lemma 2.1, (2.6) and (2.7), we learn that
\[ V(t) \leq \left( \frac{\varepsilon}{\lambda - b_0} \right)^2, \quad t \in \mathbb{R}, \]
namely,
\[ \int_{\Omega} |u(t + \sigma, x) - u(t, x)|^2 \, dx \leq \left( \frac{\varepsilon}{\lambda - b_0} \right)^2, \quad t \in \mathbb{R}, \]
which shows that the $L^2$-bounded solution $u(t, x)$ of (1.1) is almost periodic. The proof is complete.

\[ \square \]

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