Emulating a many-body topological transition with an aberrated optical cavity

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The Ince-Gauss modes of a cavity with aberrated mirrors are shown to constitute a system that is mathematically analogous to the Bose-Hubbard dimer (Bosonic Josephson junction). The latter involves interacting quantum particles while the former involves purely linear optics. The roles of particle interactions and hopping are played by spherical aberration and astigmatism, respectively, and particle number difference corresponds to net optical orbital angular momentum. Both systems feature a topological transition which in the ray-optical/mean-field limit is represented by a Viviani curve on a generalized Poincaré/Bloch sphere. The possibility of emulating the squeezing of a coherent state input is also demonstrated.

Introduction. The rich dynamical behavior of the bosonic Josephson junction (BJJ) has been observed in many systems such as coupled reservoirs of superfluid helium [1, 2], coupled atomic Bose-Einstein condensates (BECs) [3–9], and coupled polariton BECs in semiconductor microcavities [10, 11]. In the mean-field theory (Gross-Pitaevskii equation) the BJJ is analogous to a nonrigid pendulum whose angular displacement ϕ corresponds to the relative phase between the BECs. The dynamics of this system includes plasma oscillations (librational motion), Josephson ac oscillations (rotational motion), and π-oscillations (stable oscillations about the inverted position) [12–14]. Nonlinear effects are particularly evident in the phenomenon of macroscopic quantum self-trapping (MQST): as the ratio of coupling to interactions drops below a critical value, the inverted stable point bifurcates into two new stable points, locking the condensate populations at unequal values [13, 15]. This bifurcation is an example of a dynamical phase transition [16–19] and corresponds to orbits with different topologies on a Poincaré/Bloch sphere (see Figs. 1 and 2).

The two-site Bose-Hubbard (BH) dimer provides a minimal many-particle model for all these effects, and is described by the Hamiltonian

\[ \hat{H}_{\text{BH}} = \frac{E_c}{8} \left( \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right)^2 - \frac{J}{2} \left( \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} \right), \tag{1} \]

where \( \hat{a} \) and \( \hat{b} \) represent the annihilation operators for each site. The operator \( \hat{H}_{\text{BH}} \) is related to the Lipkin-Meshkov-Glick model [20] in nuclear physics and to Ising models with long-range interactions [21–24]. It is able to describe processes not captured by the mean-field approximation, such as squeezing of the number difference [25–28], Bloch oscillations and Bragg resonances [29, 30], quantum revivals [31], and caustics in Fock space [32–34]. The first term in \( \hat{H}_{\text{BH}} \) accounts for on-site particle–particle interactions characterized by the “charging energy” \( E_c \), while the second describes hopping between sites at frequency \( J/\hbar \). The ratio \( \varepsilon \equiv 4J/E_c \) with respect to the particle number \( N \) defines three regimes: Rabi \( N \ll \varepsilon \), Josephson \( 1/N \ll \varepsilon \ll N \) and Fock \( \varepsilon \ll 1/N \) [35, 36]. The topological transition occurs at \( \varepsilon_{\text{crit}} = 2N \).

Many condensed matter systems can be simulated using optical fields for which laser sources give easy access to high levels of coherence [37–41]. The lack of direct light-light interaction is usually addressed by using nonlinear media such as two-level atoms, artificial atoms, or semiconductors [10, 11, 42–47]. In particular, nonlinear propagation of light in a birefringent fiber was shown to obey the two-mode Gross-Pitaevskii equation [48]. Optical implementations that behave like a BH dimer and that do not require a nonlinear medium have also been proposed, such as light propagation in an appropriate lattice of \( N \) waveguides [49–52].

In this work we show that a simple standard optical system, namely a resonant cavity composed of two slightly aberrated curved mirrors, is described by an equation analogous to the BH dimer model, and hence presents the same rich dynamics. Further, the modes of such cavities correspond to beams with closed-form expressions: the Ince-Gauss (IG) modes [53–62]. While aberrations in cavities are typically considered as defects that cause deformation of the resulting modes, here we show that moderate amounts of the two most basic aberrations, astigmatism and spherical aberration, play analogous roles to those of hopping and on-site particle–particle interactions, respectively. The ratio between these aberrations is then the analogue of \( \varepsilon \) and defines two limiting cases: the Hermite-Gauss-like regime where astigmatism dominates, and the Laguerre-Gauss-like regime where spherical aberration dominates, which
correspond to the Rabi and Fock regimes of the BH dimer.

Resonant-cavity. In the paraxial regime, a resonant cavity composed of two identical unaberrated curved mirrors is mathematically analogous to an isotropic two-dimensional harmonic oscillator (2DHO) [55, 56, 63–67]. As shown in the Supplemental Material (SM), a single pass through the cavity is described by the action of the operator \( \hat{C} = \exp \left(-2i\chi \hat{T}_0\right) \) on the beam’s field profile, where \( \chi \) is determined by the cavity length and focusing power of the mirrors, and \( \hat{T}_0 \) plays the role of a Hamiltonian and takes the form

\[
\hat{T}_0 = \frac{1}{2w^2}(\hat{q}_x^2 + \hat{q}_y^2) + \frac{k^2w^2}{8}(\hat{p}_x^2 + \hat{p}_y^2),
\]

with \( k \) and \( w \) being the wavenumber and waist of the fundamental Gaussian mode, respectively. The position representation is obtained by substituting \((\hat{q}_x, \hat{q}_y) \rightarrow (x, y) \) and \((\hat{p}_x, \hat{p}_y) \rightarrow -ik^{-1}(\partial_x, \partial_y) \). Cavity eigenmodes, known as structured Gaussian modes [64–66, 68, 69], separate into families with the same eigenvalue \((N + 1)/2\), where \( N \) is referred to as the total order of the mode. This eigenvalue is proportional to the energy for the 2DHO, and to the rate of phase accumulation of the mode under propagation near its waist plane.

Schwinger’s coupled oscillator model [70] can be used to map the degenerate family of modes onto a collective spin with total angular momentum \( N/2 \) [64–66, 71] by introducing the Fradkin-Stokes operators \( \hat{T} = (\hat{T}_1, \hat{T}_2, \hat{T}_3) \):

\[
\hat{T}_1 = \frac{1}{2w^2}(\hat{q}_x^2 - \hat{q}_y^2) + \frac{k^2w^2}{8}(\hat{p}_x^2 - \hat{p}_y^2),
\]

\[
\hat{T}_2 = \frac{1}{w^2}\hat{q}_x\hat{q}_y + \frac{k^2w^2}{4}\hat{p}_x\hat{p}_y, \quad \hat{T}_3 = \frac{k}{2}(\hat{q}_x\hat{p}_y - \hat{q}_y\hat{p}_x),
\]

which satisfy the \( su(2) \) commutation relations \([\hat{T}_0, \hat{T}_j] = 0 \) and \([\hat{T}_i, \hat{T}_j] = i\sum_k \epsilon_{ijk}\hat{T}_k\), with \( \epsilon_{ijk} \) being the Levi-Civita tensor. Here, different spin axes correspond to different modes [64, 66, 71]: the Hermite-Gauss (HG) and vortex Laguerre-Gauss (LG) modes are eigenstates of \( \hat{T}_1 \) and \( \hat{T}_3 \), respectively. Note that vortex LG modes carry orbital angular momentum (OAM) [72, 73], while real LG modes do not and are eigenmodes of \( \hat{T}_3^2 \).

Aberrated cavity and IG modes. The effect of aberrations is to deform the wavefront after reflection according to the operator \( exp(i\vec{W}) \) where \( \vec{W} = \epsilon_1(\hat{q}_y - \hat{q}_x) + \epsilon_2\hat{q}_y^2 \), the first term corresponding to astigmatism and the second to spherical aberration [see Fig. 1(a)]. Assuming that the effect of these aberrations is perturbative and that \( \chi \neq 0, \pi/2, \) or \( \pi \), it is shown in the SM that the operator of the aberrated cavity after \( M \gg 1 \) passes is given by

\[
\hat{C}_a^M \approx \exp\left[\frac{iM\epsilon_2w^4}{2} - \frac{4\chi}{\epsilon_2w^4}\hat{T}_0 + 3\hat{T}_0^2 + \frac{1}{4} - \hat{T}_1\right]
\]

where \( \hat{T}_1 \) is the “Ince operator” defined as

\[
\hat{T}_1 = \hat{T}_3^2 + \frac{\epsilon}{2}\hat{T}_1
\]

Each of these modes is identified by its total order \( N \), parity \( p \), and index \( \mu \), ordered such that the corresponding eigenvalues satisfy \( a_{N,\mu}^{(p)} < a_{N,\mu+2}^{(p)} \) and \( a_{N,\mu}^{(o)} < a_{N,\mu}^{(e)} \). The HG and real LG modes, are obtained for \( \epsilon = 0 \), respectively.

When written in terms of annihilation and creation operators, \( \hat{T} \) takes the same form as \( \hat{H}_{BH} \) for \( N \) particles with the ratio \( \epsilon \) between aberrations playing the role of the ratio between hopping and on-site interactions [36]. The two operators differ only in the sign in front of the
second term, but this could be changed by reversing the sign of one aberration, causing a change of orientation of the major axis of the elliptical coordinate system to be along \( y \) instead of \( x \); both attractive and repulsive interactions can be emulated [74]. We hence can consider only \( \varepsilon > 0 \) without loss of generality.

It is remarkable that the BH operator leads to a closed-form 2D solution separable in elliptical coordinates, and the \( N + 1 \) Ince polynomials provide an analogical analogue to the \( N + 1 \) eigenstates of the \( N \) particle BH Hamiltonian. The values of \( N \) and \( \varepsilon \) can be controlled by simply tuning the cavity. Note also that the eigenstates of \( \hat{T}_3 = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2 \) are Fock states giving half the population difference between sites. Thus, the vortex LG modes play the role of the Fock basis and the net OAM is the analogue of the particle difference.

**Ray structure.** The analogue of the mean-field limit for the BH model is the ray-optical limit for the cavity. For the particular case of IG modes the ray structure is obtained by considering the scalar counterparts of the eigenvalue relations for \( \hat{T}_0 \) and \( \hat{\ell} \). The one for \( \hat{T}_0 \) leads to the equation \( t_1^2 + t_2^2 + t_3^2 = 1 \), where \( t_j = 2T_j/(N+1) \) and \( \hat{T}_j \) is the ray-optical version of the operator \( \hat{T}_j \), which defines a Poincaré/Bloch sphere. Points on the surface of this sphere represent rays that follow the same elliptic orbit of the 2DHO in both space and momentum and are referred to here as elliptical-families of rays (EFR). Each cavity mode is represented by a different closed curve on the surface of the sphere known as a Poincaré Path (PP) as illustrated in Figs. 1 and 2, and is thus described by a continuous set of EFR [68]. Modes are therefore described by two periodic parameters: one labels the ellipse (position around the PP) and the other labels the position around the ellipse (a specific ray), and this implies that each mode is topologically equivalent to a torus [64]. For IG modes, the PP can be found as contours over the sphere for the scalar version of the Ince operator. Such curves correspond to the intersection of the Poincaré sphere with the cylinder \( (t_1 - c)^2 + t_2^2 = R^2 \), aligned along the \( t_3 \) axis, centered at \( c = \varepsilon/2(N+1) \) and of radius \( R = [1+c^2-a/(N+1)^2]^{1/2} \) as shown in Fig. 2. This type of curve is referred to as a generalized Viviani curve [75]. The cylinder means that the projection of the PP onto the sphere’s equatorial plane is always circular.

As shown in Figs. 1 and 2, for \( c + R > 1 \) the cavity is dominated by astigmatism and the modes resemble deformed HG modes. On the other hand, for \( c + R < 1 \) the cavity is dominated by spherical aberration and the modes resemble deformed LG modes. The boundary \( c + R = 1 \) defines the Viviani separatrix, a figure of eight-shaped curve (like the original Viviani curve), where the topological transition takes place. (Due to the choice \( \varepsilon > 0 \), the merging point is located at \( t_1 = 1 \) instead of \( -1 \), the usual location in most condensed-matter works [15].) This topological transition is shown in Fig. 2 for a projection of the complete 4D ray phase-space structure onto the \( P_y = 0 \) subspace, where the torus corresponding to a single loop in the HG regime splits into two at the Viviani separatrix. By further projecting onto \( P_x = 0 \), the EFRs at the waist plane are obtained, showing the transition from an HG-like to an LG-like mode. These are plotted together with the corresponding field distribution, showing that the ray structure mimics the wave solution.

The ray structure also allows estimating the eigenvalue in Eq. (6) through the self-consistency condition in wave estimates [32, 65, 68, 76–82]; the solid angle enclosed by each loop of the PP must be an odd multiple of \( 2\pi/(N+1) \) [68]. This quantization can be shown to be related to a geometric phase through the Pancharatnam-Berry connection [65, 69, 83, 84]. In the HG-like case, where the PP consists of only one loop, the total subtended solid angle must be an odd multiple of \( 2\pi/(N+1) \), but in the LG-like case, where the PP is composed of two
loops, each loop must satisfy this condition so the total solid angle must be an even multiple of \(2\pi/(N+1)\). This discrepancy creates a discontinuity at the Viviani separatrix where the semiclassical estimate fails, as shown in Fig. 3. Nonetheless, away from the separatrix, the semiclassical estimate for the eigenvalue in Eq. (6) is nearly indistinguishable from the exact eigenvalue, even for small \(N\).

Ray and wave evolution. In the ray/mean-field limit \(1/N \to 0\), the wave evolution inside the cavity can be understood in terms of the ray—optic equations. After each reflection the aberrations lead to small changes in the ray directions proportional to the gradient of the wavefront \(W\). As shown in the SM, the evolution of the rays can then be separated into two parts acting on different scales in time (or number of bounces \(M\)): that of the evolution of the rays within an EFR, and that of the evolution of the EFR. The aberrations change slightly the rate at which rays cycle within an EFR. This change can be interpreted as a Hannay angle, the classical analogue of a geometric phase [85]. On the other hand, the evolution of the EFR as a whole causes the point that represents it on the Poincaré sphere to move along the PP (a generalized Viviani curve), following a discrete analogue of the evolution equations for a BH dimer. The update equations take a particularly telling form when expressed in terms of the angle \(\varphi\) defined in Fig. 2 (not to be confused with the azimuthal angle \(\phi\)):

\[
\Delta^2 \varphi_n = \frac{(N+1)^3}{2} \epsilon_{2\xi_1} w^6 R \sin \varphi_n,
\]

where \(\Delta^2 \varphi_n = \Delta \varphi_{n+1} - \Delta \varphi_n\) and \(\Delta \varphi_n = \varphi_{n+1} - \varphi_n\). Equation (7) is a discretized version of the simple (rigid) pendulum [75] which reveals that the ray spends more time in the EFRs where the pendulum slows down, namely EFRs represented by points closer to the equator. This simple pendulum representation is more intuitive than the nonrigid pendulum, where the length depends on the angular momentum, obtained when writing the equation in terms of the azimuthal angle [13, 14].

The limit of high \(N\) and small \(\xi_3\), for which the PP is close to the equator, is of particular interest because the dynamics mimic the traditional Josephson junction between bulk superconductors. The dynamical equations then reduce to those of a simple pendulum even in the azimuthal angle \(\phi\) [35]. In the optical analogy this limit corresponds to cavities with nearly flat mirrors, which have propagation-invariant modes known as Mathieu beams [86], also separable in elliptical coordinates.

Finally, consider the evolution of an incident coherent mode (corresponding to modes with elliptical shapes known as the generalized Hermite-Laguerre-Gauss modes [65, 66, 71, 87]) whose Q function corresponds to a circular blob on the sphere (see SM for details). For \(N \to \infty\) this blob reduces to a point and can be represented by a single EFR. As the beam bounces back and forth in the cavity, the blob follows a generalized Viviani curve whose projection onto the equatorial plane evolves like a simple pendulum in the angle \(\varphi\) [75]. Depending on the value of \(\xi\) and the initial state, the evolution corresponds to Rabi oscillations (plasma or \(\pi\) oscillations, which are topologically identical) or MQST. For smaller \(N\) or paths close to the separatrix, wave effects become manifest, simulating quantum effects in the BJJ. Figure 4 shows the evolution of the Q function of a coherent mode initially placed at the merging of the separatrix. This evolution was computed with the exact operator for the aberrated cavity using a Fox-Li algorithm [88], showing that the state is squeezed in an analogous fashion to that seen in atomic BECs [25–28]. This squeezing can be quantified by computing the squeezing parameter [89, 90] which gives a minimum value of 0.44 < 1 after 9600 passes.

Concluding remarks. It is shown that an optical cavity without nonlinear optical effects can emulate interacting quantum particles, including a topological transition and squeezing (often considered a quintessentially quantum effect that requires nonlinearity). At the heart of this treatment are the IG modes and the Ince polynomials associated with them: they provide the wave solutions in the aberrated cavity. These special functions have connections yet to be explored to several other physical systems, including the planar quantum pendulum and the Razavy potential [91, 92], as well as the rotational dynamics of celestial bodies [93, 94]. Note that the model presented here does not consider the quantization of the optical field itself, the quantum nature of the BH model corresponds to the wave nature of light, with the mean-
field approximation being well represented by the ray picture. Including the photon picture for the field in the form of a discretization of the mode amplitudes would correspond to a “third quantization”.

The results presented here are more general than the optical cavity: any system equivalent to a 2DHO with quartic and anisotropic perturbations, such as levitated nanoparticles [95, 96], can be used to implement versions of BJJ and superconducting qubits [97–99]. A classical nonlinear optical system analogous to the aberrated cavity would be a gradient-index waveguide with an appropriate refractive index distribution. Finally, we note that this coupling would not map onto the standard multisite BH of BJJ and superconducting qubits [97–99]. A classical nonparticles [95, 96], can be used to implement versions of BJJ and superconducting qubits [97–99]. A classical

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Supplemental material: Emulating a many-body topological transition with an aberrated optical cavity

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This document presents supporting material and proofs for the results presented in the main text. Section I presents a brief summary of the key results for structured-Gaussian beams emphasizing the operator formalism through the use of Dirac’s notation. Section II presents the derivation of the operator for the aberrated cavity. Section III presents the main properties of the Ince polynomials and in position representation they take the form \( \mathbf{H}_{\pm} \) and \( \mathbf{L}_{\pm} \) for the \( 2DHO \) and \( 2DHO \) analogies to that of energy in the \( 2DHO \) analogy, and is proportional to the rate of phase accumulation near the waist of the beam under propagation. The position and momentum operators, \( \hat{q}_i \) and \( \hat{p}_i \) for \( i = x, y \), satisfy the canonical commutation relation

\[
\{ \hat{q}_i, \hat{p}_j \} = \frac{i}{\hbar} \delta_{ij},
\]

and in position representation they take the form \( \hat{q}_x = x \) and \( \hat{p}_x = -i\hbar^{-1} \partial_x \) (and similarly for \( y \)).

The Schwinger oscillator model provides the underlying structure of the degenerate space of SG modes with the same total order. This is achieved by introducing the operators \( [2–6] \)

\[
\hat{T}_1 = \frac{1}{2w^2}(\hat{q}_x^2 - \hat{q}_y^2) + \frac{k^2w^2}{8}(\hat{p}_x^2 - \hat{p}_y^2),
\]

\[
\hat{T}_2 = \frac{1}{w^2}\hat{q}_x\hat{q}_y + \frac{k^2w^2}{4}\hat{p}_x\hat{p}_y,
\]

\[
\hat{T}_3 = \frac{k}{2}(\hat{q}_x\hat{p}_y - \hat{q}_y\hat{p}_x),
\]

which satisfy the commutation relations of \( su(2) \),

\[
[\hat{T}^2, \hat{T}_j] = 0, \quad [\hat{T}_i, \hat{T}_j] = i \sum_k \epsilon_{ijk} \hat{T}_k,
\]

with \( \hat{T} = (\hat{T}_1, \hat{T}_2, \hat{T}_3) \) and \( \epsilon_{ijk} \) being the Levi-Civita tensor. Using this set of operators it is possible to study SG modes through the spin algebra. The most common choices of basis for this space are the HG and the LG modes, which satisfy particularly simple eigenrelations

\[
\hat{T}_1 |HG_{N,\ell}\rangle = \frac{\ell}{2} |HG_{N,\ell}\rangle, \quad \hat{T}_3 |LG_{N,\ell}\rangle = \frac{\ell}{2} |LG_{N,\ell}\rangle
\]

I. STRUCTURED-GAUSSIAN MODES

Summary. Structured Gaussian (SG) beams are self-similar (up to a scaling factor) solutions to the paraxial wave equation and eigenmodes of cavities with curved mirrors. They are mathematically analogous to the quantum two-dimensional harmonic oscillator 2DHO and can thus be studied through an operator formalism analogous to the one used in quantum mechanics. This section presents some key identities, emphasizing the role of the Schwinger oscillator model, and the Hermite-Gauss (HG) and Laguerre-Gauss (LG) modes.

SG modes, denoted here by the ket \( |U\rangle \), are modes of a resonant cavity with curved mirrors. This system is mathematically analogous to the 2DHO [1]. SG modes are completely determined by their initial profile, which is given as the product of a Gaussian and a polynomial. All SG modes satisfy the eigenvalue relation

\[
\hat{T}_0 |U\rangle = \frac{N + 1}{2} |U\rangle,
\]

where

\[
\hat{T}_0 = \frac{1}{2w^2}(\hat{q}_x^2 + \hat{q}_y^2) + \frac{k^2w^2}{8}(\hat{p}_x^2 + \hat{p}_y^2),
\]

with \( k \) being the wavenumber and \( w \) the waist of the Gaussian. \( N \) is referred to as the total order and plays a

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with their position representation at \( z = 0 \) being
\[
\langle r | H_{G,\ell} \rangle = \frac{1}{w \sqrt{2^{N-1} \pi}} \left( \frac{N + \ell}{2} \right)! \left( \frac{N - \ell}{2} \right)! e^{-\frac{r^2}{w^2}} \times H_{N,\ell} \left( \sqrt{2} r / w \right),
\]
\[
\langle r | \ell \rangle = \frac{i^{\ell - N}}{w} \sqrt{\frac{2^{(\ell + 1)} \left( \frac{N - \ell}{2} \right)! \left( \frac{N + \ell}{2} \right)!}{\pi}} e^{-\frac{x^2}{w^2}} \times (r / w)^\ell e^{i \phi \ell} L_{\ell - \ell}^{|\ell|} \left( \frac{2 r^2}{w^2} \right),
\]
where \( H_n \) and \( L_{\ell}^{[m]} \) are the Hermite and Laguerre polynomials, respectively. Note, however, that not all SG beams satisfy such simple eigenvalue relations in terms of the operators \( \hat{T}_i \) (with \( i = 1, 2, 3 \)).

Given this operator formalism it is possible to define the equivalent of spin coherent states [7–9]. Choosing the LG modes as a basis and denoting them simply as \( |N, \ell\rangle \), the coherent modes are defined as
\[
|N; \theta, \phi\rangle = \hat{D}(\phi, \theta, 0) |N, N\rangle,
\]
where
\[
\hat{D}(\phi, \theta, \chi) = e^{-i \hat{T}_3 \phi} e^{-i \hat{T}_2 \theta} e^{-i \hat{T}_3 \chi}.
\]
is an SU(2) group transformation parametrized in terms of Euler angles. The coherent modes can be identified as the generalized Gaussian modes, obtained experimentally by performing astigmatic transformations (i.e. with a pair of cylindrical lenses) on the LG modes, that have elliptic spatial distributions outlining the classical orbits of a 2DHO [10, 11]. These modes allow us to define the Q function on the Poincaré/Bloch sphere through
\[
Q(\theta, \phi) = \frac{N + 1}{4 \pi} |\langle N; \theta, \phi | U \rangle|^2,
\]
where \( \theta \) and \( \phi \) are the sphere’s polar and azimuthal angles, respectively [12, 13].

### II. ABERRATED CAVITY: THE INCE OPERATOR

**Summary.** Using the operator formalism introduced in the previous section and the matrix representation of linear canonical transformations, the operators of the unaberrated and aberrated cavities are derived. In particular, we show how the Ince operator emerges from considering small amounts of astigmatism and spherical aberration.

#### A. Resonant cavity with spherical mirrors

Let us start by considering the operators for the two types of elements of a cavity. The first is the Fresnel operator which describes paraxial free propagation:
\[
\hat{F}(z) = \exp \left( -\frac{i k z \varphi^2}{2} \right),
\]
while the second is the operator for a paraxial focusing element, such as a curved mirror:
\[
\hat{R}(\Phi) = \exp \left( -\frac{i k \Phi^2}{2} \right).
\]

In these equations, \( k \) is the wavenumber, \( z \) is the propagation distance and \( \Phi \) is the focusing power determined by the radius of curvature \( \Phi = 2 / R_c \). Here, the position and momentum operators were introduced through their norm squared \( \hat{q} \hat{q} = \hat{q}^2 + Q_0^2 \) and \( \hat{p} \hat{p} = \hat{p}^2 + P_0^2 \).

Let us assume for simplicity a cavity where the two mirrors are identical. A single pass through this cavity can be written as free propagation followed by reflection and focusing by the mirror, or equivalently by reflection and focusing followed by free propagation. It is convenient, though, to use a symmetric form, so the focusing is split into two equal parts at the beginning and end of the trip:
\[
\hat{C} = \hat{R}(\Phi_m / 2) \hat{F}(l_c) \hat{R}(\Phi_m / 2),
\]
where \( l_c \) is the cavity length and \( \Phi_m \) the focusing power of each mirror. The operator in Eq. (S5) can be interpreted as the propagator for going from the curved surface at the first mirror to the curved surface at the second.

This transformation can be represented by what is known as an ABCD matrix generating the corresponding linear canonical transformation [14, 15]. This ABCD matrix is the same one that is used to model the effect of first-order optics on rays represented by a four-dimensional vector \((Q, P)\) where \( Q \) and \( P \) are the transverse position and direction, respectively. The resulting ABCD matrix for the cavity can be easily obtained by concatenating the three elementary operations
\[
C = \left( \begin{array}{cc} 1 - l_c \Phi_m / 2 & l_c l_2 \\ -\Phi_m (1 - l_c \Phi_m / 4) l_2 & 1 - l_c \Phi_m / 2 \end{array} \right),
\]
where \( l_2 \) is the 2 by 2 identity matrix. Defining \( \cos \chi = (1 - l_c \Phi_m / 2), \sin \chi = l_c / g \) and \( g = |\Phi_m(l_c^{-1} - \Phi_m / 4)|^{-1/2} \) the matrix for the cavity can be written as
\[
C = \left( \begin{array}{cc} \cos \chi l_2 & g \sin \chi l_2 \\ -g^{-1} \sin \chi l_2 & \cos \chi l_2 \end{array} \right),
\]
which is just a scaled rotation matrix. Note that the conditions 0 < \( \Phi_m < l_c^{-1} \) for cavity stability are implied, so that 0 < \( \chi < \pi \).

Using Collins’ formula [15] the kernel of the integral transformation performed by the cavity operator is given by
\[
C(\chi) = \langle q | \hat{C} | q' \rangle = \frac{1}{\sqrt{2 \pi N(N + 1)}} \left( \frac{N + 1}{2} \right)! \left( \frac{N - 1}{2} \right)! \left( \frac{N}{2} \right)! e^{-\frac{q^2}{2}},
\]
\[
\frac{k}{2\pi i g \sin \chi} \exp \left\{ \frac{ik}{2g \sin \chi} \left( [q^2 + q'^2] \cos \chi - 2q' \cdot q \right) \right\}.
\]
It can be shown by direct computation that this kernel satisfies the following equation
\[
\frac{i}{k} \frac{\partial \mathcal{C}}{\partial \chi} = \frac{1}{2g} (q^2 - \frac{g^2}{k^2} \nabla^2_q) \mathcal{C} = \hat{H} \mathcal{C}.
\]
This is the equation of a 2DHO so that the cavity operator takes the following form
\[
\hat{C} = \exp \left( -ik\chi \hat{H} \right)
\]
where
\[
\hat{H} = \frac{1}{2g} q^2 + \frac{g^2}{2} \theta^2.
\]
By identifying \( g = kw^2/2 \) we can relate this operator to \( \hat{T}_0 \) used in the main text as
\[
\hat{H} = \frac{2}{k} \hat{T}_0.
\]
Thus, the cavity operator can be written as
\[
\hat{C} = \exp \left( -2i\chi \hat{T}_0 \right)
\]
which clearly shows that any SG beam is an eigenmode of the cavity, provided the phase consistency, i.e. that the field forms a longitudinal mode inside the cavity, is satisfied [1].

**B. Aberrated cavity**

Let us now consider an aberrated cavity. Considering only astigmatism and spherical aberration the wavefront deviation can be written as
\[
W = \epsilon_1 (q_y^2 - q_z^2) + \epsilon_2 q^4,
\]
where \( \epsilon_1 \) and \( \epsilon_2 \) control the amount of astigmatism and spherical aberration, respectively. Since the aberrations take place at the mirrors, the operator describing a single pass through the aberrated cavity can be written as
\[
\hat{C}_a = \hat{C} \exp (i\hat{W}),
\]
where we promoted the wavefront aberration to the operator
\[
\hat{W} = \epsilon_1 (q_y^2 - q_z^2) + \epsilon_2 q^4.
\]
Let us assume that the aberrations are small, so that
\[
\exp (i\hat{W}) \approx 1 + i\hat{W}
\]
and
\[
\hat{C}_a \approx \hat{C} (1 + i\hat{W}).
\]
Given that the mode traverses the cavity many times, let us consider instead
\[
\hat{C}_a^M \approx \prod_{n=1}^M \hat{C} (1 + i\hat{W}) \approx \hat{C}^M + i \sum_{n=0}^{M-1} \hat{C}^{M-n} \hat{W} \hat{C}^n,
\]
where in the last equality we removed all terms that are second order or higher in \( \hat{W} \).

From the commutation relation of the position and momentum operators it is then easy to show that the following relations hold:
\[
[\hat{q}_i, \hat{F}(z)] = \hat{F}(z) z \hat{p}_i, \quad [\hat{p}_i, \hat{F}(z)] = 0, \quad [\hat{p}_i, \hat{R}(\Phi)] = -\hat{R}(\Phi) \Phi \hat{q}_i, \quad [\hat{q}_i, \hat{R}(\Phi)] = 0.
\]
From here it can be seen that, for any constant coefficients \( a, b \),
\[
(a\hat{p}_i + b\hat{q}_i) \hat{C} = \hat{C} (a'\hat{p}_i + b'\hat{q}_i)
\]
where
\[
\begin{pmatrix}
  a' \\
  b'
\end{pmatrix} =
\begin{pmatrix}
  \cos \chi & g \sin \chi \\
  -g^{-1} \sin \chi & \cos \chi
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix}
\]
and more generally,
\[
(a\hat{p}_i + b\hat{q}_i)^m \hat{C}^m = \hat{C}^m (a_n \hat{p}_i + b_n \hat{q}_i)^m
\]
where
\[
\begin{pmatrix}
  a_n \\
  b_n
\end{pmatrix} =
\begin{pmatrix}
  \cos n\chi & g \sin n\chi \\
  -g^{-1} \sin n\chi & \cos n\chi
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix}.
\]
Note that here we used a two-dimensional version of the ABCD matrix since the full 4D matrix is separable in \( x \) and \( y \). With this, the aberrated cavity operator can be written as
\[
\hat{C}_a^M \approx \hat{C}^M (1 + i \sum_{n=0}^{M-1} \hat{W}_n)
\]
with
\[
\hat{W}_n = \epsilon_1 [(c_n \hat{q}_y + g s_n \hat{p}_y)^2 - (c_n \hat{q}_x + g s_n \hat{p}_x)^2]
\]
where we used the shorthand \( c_n = \cos n\chi \) and \( s_n = \sin n\chi \).

Assuming that \( \chi \neq 0 \) (short cavity with flat mirrors), \( \pi/2 \) (confocal cavity) or \( \pi \) (concentric cavity) and that the number of passes \( M \) is large we have that

\[
\sum_{n=0}^{M-1} c_n^2 \approx \sum_{n=0}^{M-1} s_n^2 \approx \frac{M}{2},
\]

\[
\sum_{n=0}^{M-1} c_n^4 \approx \sum_{n=0}^{M-1} s_n^4 \approx \frac{3M}{8}, \quad \sum_{n=0}^{M-1} c_n^2s_n^2 \approx \frac{M}{8},
\]

\[
\sum_{n=0}^{M-1} c_n s_n \approx \sum_{n=0}^{M-1} c_n^3s_n \approx \sum_{n=0}^{M-1} c_n^3n_s^3 \approx 0.
\]

Thus

\[
\sum_{n=0}^{M-1} \hat{W}_n \approx \epsilon_1 \frac{M}{2} \left( q_n^2 + g_n^2p_n^2 - \hat{q}_n^2 - \hat{g}_n^2\hat{p}_n^2 \right)
\]

\[
+ \epsilon_2 \frac{M}{8} \left\{ 3(q_n^4 + g_n^4p_n^4) + g_n^2\hat{q}_n^2 + \hat{p}_n^2\hat{q}_n^2 + (\hat{q}_n \cdot \hat{p} + \hat{\mathbf{p}} \cdot \hat{q})^2 \right\}.
\]

Noting that

\[
3(q_n^4 + g_n^4p_n^4) + g_n^2\hat{q}_n^2 + \hat{p}_n^2\hat{q}_n^2 + (\hat{q}_n \cdot \hat{p} + \hat{\mathbf{p}} \cdot \hat{q})^2
\]

\[
= w^4 \left( 12\hat{T}_0^2 - 4\hat{T}_1^2 + 1 \right)
\]

and

\[
\hat{q}_n^2 + g_n^2\hat{p}_n^2 - \hat{q}_n^2 - \hat{g}_n^2\hat{p}_n^2 = 2w^2\hat{T}_1
\]

we can write

\[
\sum_{n=0}^{M-1} \hat{W}_n \approx -\epsilon_1 w^2 M\hat{T}_1 + \epsilon_2 w^4 \frac{M}{8} (12\hat{T}_0^2 - 4\hat{T}_1^2 + 1).
\]

Given that all the operators that appear in this expression commute with \( \hat{T}_0^2 \), we can readily write the operator of the aberrated cavity as

\[
\hat{C}_a^M \approx \exp \left[ iM \left( 2\hat{T}_0^2 + \frac{3\epsilon_2 w^4 M}{8} \hat{T}_0^2 - \frac{\epsilon_2 w^4 M}{2} \hat{T}_1 \right) \right]
\]

(S6)

where we introduced the Ince operator

\[
\hat{I} = \hat{T}_0^2 + \frac{\epsilon}{2} \hat{T}_1
\]

with \( \epsilon = 4\epsilon_1/\epsilon_2 w^2 \) indicating the ratio between the two aberrations. It is tempting to write the single pass operator by simply removing the \( N \) in the previous expression but it is the effect of several passes that leads to this expression. For a single pass there are other terms that come into play that are of the same order as the ones that appear in Eq. (S6) but tend to cancel out after several passes.

### III. The Ince-Gauss Modes

**Summary.** The Ince equation and its solutions, the trigonometric Ince polynomials, are presented along with their main properties. These are then used, along with their hyperbolic counterparts, to define the Ince-Gauss (IG) modes which are shown to be eigenfunction of the Ince operator by writing it in elliptic coordinates.

**A. The Ince equation and its solutions**

The Ince polynomials, \( K_{N,\mu}^{(p)} \), are the solutions to the second-order differential equation

\[
d^2K \over d\nu^2 + \epsilon \sin 2\nu dK \over d\nu + (a - N\epsilon \cos 2\nu)K = 0,
\]

(S8)

where \( N \) is a non-negative integer, \( \epsilon \) is a positive real parameter and \( a \) is the eigenvalue of the equation. Equation (S8) is known as the Ince equation and corresponds to a particular case of the Whittaker-Hill equation [16, 17]. It accepts two independent families of solutions based on their parity, which can be expanded as finite Fourier series giving trigonometric polynomials

\[
C_{N,\mu}(\nu) = \sum_{n=0}^{N/2} A_n^{(N,\mu)} \cos 2n\nu, \quad \text{if } N \text{ is even}
\]

\[
+ \sum_{n=0}^{(N-1)/2} A_n^{(N,\mu)} \cos (2n+1)\nu, \quad \text{if } N \text{ is odd}
\]

\[
S_{N,\mu}(\nu) = \sum_{n=0}^{N/2} B_n^{(N,\mu)} \sin 2n\nu, \quad \text{if } N \text{ is even}
\]

\[
+ \sum_{n=0}^{(N-1)/2} B_n^{(N,\mu)} \sin (2n+1)\nu, \quad \text{if } N \text{ is odd}
\]

Here, we use \( K_{N,\mu}^{(p)} \) to denote both the even \( C_{N,\mu} \) and odd \( S_{N,\mu} \) Ince polynomials when \( p = e, o \), respectively. By substituting these expressions into the Ince equation, an eigenvalue equation with tridiagonal matrices is obtained for each type of solution. The eigenvectors provide the coefficients \( A_n^{(N,\mu)} \) and \( B_n^{(N,\mu)} \), while the eigenvalues can be used to order the solutions. The eigenvalues are labeled such that \( a_{N,\mu}^{(e)} > a_{N,\mu}^{(o)} > a_{N,\mu}^{(e)} \), where the index \( \mu \) ranges from 0 (1) to \( N \) in steps of two for even (odd) solutions. Therefore, the index \( \mu \) has the same parity as \( N \) and is directly related to the number of zeros in the solution. The normalization of the Ince polynomials is given by

\[
\int_0^{2\pi} [K_{N,\mu}^{(p)}(\nu)]^2 d\nu = \pi,
\]

and they satisfy the orthogonality relation for \( \mu_1 \neq \mu_2 \)

\[
\int_0^{2\pi} K_{N,\mu_1}^{(p)}(\nu)K_{N,\mu_2}^{(p)}(\nu)e^{-\nu \cos 2\nu} d\nu = 0.
\]

The Ince polynomials have hyperbolic counterparts which are obtained by the simple change of variable \( \nu \rightarrow i\xi \). These are solutions of the hyperbolic Ince equation given by

\[
d^2K_h \over d\xi^2 - \epsilon \sinh 2\xi dK_h \over d\xi - (a - N\epsilon \cosh 2\xi)K_h = 0.
\]
As was already mentioned, \( K_\nu(\xi) = K(i\xi) \) and its functional form is obtained from the trigonometric solutions by replacing sines and cosines with their hyperbolic counterparts.

### B. The Ince-Gauss modes and their eigenvalue relations

The Ince equation results from solving the 2DHO or the paraxial wave equation in elliptical coordinates \([18, 19]\). Its solutions are the IG modes, a sub-family of the SG modes. Their profile at a plane orthogonal to their main propagation direction is sufficient to completely define them:

\[
\text{IG}_{N,\mu}^{(p)}(r, \varepsilon) = k_{N,\mu}^{(p)} K_{N,\mu}^{(p)}(i\xi, \varepsilon) K_{N,\mu}^{(p)}(\nu, \varepsilon) e^{-r^2/w^2},
\]

where \( x = f \cosh \xi \cos \nu \) and \( y = f \sinh \xi \sin \nu \), with \( \xi \geq 0 \) and \( 0 \leq \nu < 2\pi \) being the radial and angular elliptic variables, respectively, \( f \) being the semi-focal distance, \( \varepsilon = 2f^2/w^2 \), and \( k_{N,\mu} \) being a normalization constant. Note that the variable in the Gaussian factor is \( r^2 = x^2 + y^2 = f^2(\cosh 2\xi + \cos 2\nu)/2 \). Figure S1 shows the amplitude distribution for the first IG modes and their dependence on the parity and indices \( N \) and \( \mu \). Given that IG beams are the separable Gaussian solutions in elliptical coordinates, they interpolate between HG and LG beams. However, the LG beams in question are not the eigenstates of \( \mathbf{\hat{T}}_3 \) (those carrying orbital angular momentum) but their real and imaginary parts which are eigenstates of \( \mathbf{\hat{T}}_3^2 \).

Since the IG beams are SG beams they satisfy the defining relation

\[
\mathbf{\hat{T}}_0 \text{IG}_{N,\mu}^{(p)}(r, \varepsilon) = \frac{N+1}{2} \text{IG}_{N,\mu}^{(p)}(r, \varepsilon).
\]

It is also possible to write an eigenvalue relation in terms of the Schwinger operators, by using the Ince operator defined in Eq. S7. In elliptical coordinates, this operator takes the form

\[
\mathbf{\hat{T}} = -\frac{1}{4(\cosh 2\xi - \cos 2\nu)}(\cos 2\nu \partial_\xi^2 + \cosh 2\xi \partial_\nu^2)
+ \frac{\varepsilon^2}{16}(1 + \cos 2\nu \cosh 2\xi).
\]

One can show that the IG modes then satisfy the eigenvalue equation

\[
\mathbf{\hat{T}} \text{IG}_{N,\mu}^{(p)}(r; \varepsilon) = \frac{a_{N,\mu}^{(p)}}{4} \text{IG}_{N,\mu}^{(c,o)}(r; \varepsilon),
\]

where \( a_{N,\mu}^{(p)} \) is the corresponding eigenvalue of the Ince equation.

### IV. RAY STRUCTURE OF SG MODES

**Summary.** This section presents the parametrization of the ray structure of SG beams in terms of a Jones vector and how it can be represented as a path on the surface of the Poincaré sphere. It also shows how the ray formalism is related to the operator formalism presented in Sec. I.

Following [20], in the ray optical limit, SG beams can be described by a two-parameter (\( \eta \) and \( \tau \)) family of rays given by

\[
\begin{align*}
Q(\eta, \tau) &= w_0 \sqrt{N+1} \Re \{ \mathbf{v} [\theta(\eta), \phi(\eta)] e^{-ir} \}, \\
P(\eta, \tau) &= \frac{2}{kw_0} \sqrt{N+1} \Im \{ \mathbf{v} [\theta(\eta), \phi(\eta)] e^{-ir} \},
\end{align*}
\]

where \( Q \) and \( P \) denote the transverse position and momentum (the transverse direction) of the corresponding ray, and

\[
\mathbf{v}(\theta, \phi) = \cos(\theta/2) e^{-i\phi} \epsilon_+ + \sin(\theta/2) e^{i\phi} \epsilon_-.
\]

is a mathematical analog of the Jones vector (applied here to ray families, not to polarization), with \( \epsilon_\pm = (\hat{x} \pm i\hat{y})/2^{1/2} \). For fixed \( \eta \) and \( \tau \) ranging from 0 to \( 2\pi \), upon propagation, the rays describe a ruled hyperboloid with an elliptic cross section. The angles \( \theta \) and \( \phi \), parametrized by \( \eta \), determine the shape of the ray families of ruled hyperboloids [or elliptic families of rays (EFR)] and can be interpreted as coordinates over the surface of a ray-optical Poincaré sphere. An example is shown in Fig. S2. These EFR correspond to the coherent states defined in Eq. (S4) in the ray-optical limit. In particular, the vectors \( \epsilon_\pm \) represent LG modes with \( \ell = N \).

A SG beam is composed of a one-parameter continuous set of EFRs, represented by a Poincaré path (PP) on the surface of the RPS, as shown in Fig. S3. The periodicity of the two parameters \( \eta \) and \( \tau \) makes the parametrization
of SG beams topologically equivalent to that of a torus [5].

This parametrization satisfies the scalar (or ray-optical) equivalent of Eq. (S1):

$$\frac{1}{2w_0} \|Q\|^2 + \frac{k^2w_0^2}{8} \|P\|^2 = \frac{N + 1}{2},$$

(S12)

where $\|F\|^2 = F^* \cdot F$ for a complex vector function $F$. That is, the four-dimensional phase space $(Q, P)$ for a ray is confined to the surface of a 3-sphere (assuming an appropriate scaling of the parameters). Fortuitously, this space can be mapped onto the 2-sphere through the Hopf fibration [5]. This leads to

$$t_1^2 + t_2^2 + t_3^2 = 1,$$

which is Eq. (S12) written in terms of the normalized scalar quantities $t_j = 2T_j/(N + 1)$ where $T_j$ is the scalar equivalent of the operator $\mathcal{T}_j$. This spherical representation stems from the underlying SU(2) structure of SG beams and is simply the scalar version of the Schwinger oscillator model introduced in Eq. S2.

V. RAY EVOLUTION INSIDE THE CAVITY

Summary. This section outlines the proof leading to Eq. (7) of the main text. Using results from Sec. II, it is shown that the EFRs follow the evolution of a simple pendulum inside the aberrated cavity.

Knowing the ray structure of SG modes, it is possible to compute their evolution inside the cavity. The unaberrated cavity causes a rotation in phase space, where the scaling between $Q$ and $P$ is determined by $g$. Given a ray determined by Eq. (S10), this rotation causes a shift $\tau \to \tau + \chi$. After this phase space rotation, the aberrations change the direction of the rays according to the gradient of $W$. These compounded effects of a pass through the cavity are simpler to write in terms of the complex vector $(N + 1)^{1/2}z = Q/w_0 + ikw_0P/2,

\begin{align*}
\zeta_{n+1} &= \zeta_n e^{-i\chi} + \frac{1}{2} \left[ -\delta_1 \sigma_z \cdot (\zeta_n + \zeta_n^*) \right. \\
&\left. + \delta_2 \left| (\zeta_n e^{-i\chi} + \zeta_n^* e^{i\chi}) \right|^2 (\zeta_n e^{-i\chi} + \zeta_n^* e^{i\chi}) \right],
\end{align*}

where the real and imaginary parts of $\zeta_n$ represent the position and direction coordinates of the ray after $n$ passes, and $\delta_1$ and $\delta_2$ are proportional to $\epsilon_2$ and $\epsilon_2$, respectively. To simplify the notation, the Pauli matrix $\sigma_z$ was introduced, where the Pauli matrices are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The dominant effect on the rays traversing the cavity is the phase space rotation induced by the unaberrated cavity, which gives a global phase to $z$. Factoring out this phase by introducing $\zeta_n = z_n \exp(i\chi)$, the evolution equation of the rays can be written as

$$\zeta_{n+1} \approx \zeta_n + i\left\{ -\delta_1 \sigma_z \cdot \zeta_n + \delta_2 \left[ (\zeta_n \cdot \zeta_n) \zeta_n^* + 2(\zeta_n \cdot \zeta_n^*) \zeta_n + 2(\zeta_n \cdot \zeta_n^*)^2 \right] \right\}.$$

After several passes through the cavity, the terms with oscillatory dependence in $\chi$ average to zero (provided $\chi \neq 0, \pi/2, \text{or} \pi$), and are therefore dropped. This rotating-wave approximation leads to

$$\zeta_{n+1} \approx \zeta_n + i\left\{ -\delta_1 \sigma_z \cdot \zeta_n + \frac{\delta_2}{2} \left[ (\zeta_n \cdot \zeta_n) \zeta_n^* + 2(\zeta_n \cdot \zeta_n^*) \zeta_n \right] \right\}.$$

Using the identity

$$\left( \zeta_n \cdot \zeta_n^* \right) \zeta_n^* + 2(\zeta_n \cdot \zeta_n^*) \zeta_n = 3(\zeta_n \cdot \zeta_n^*)\zeta_n - (\zeta_n \cdot \sigma_y \cdot \zeta_n)\sigma_y \cdot \zeta_n,$$

the previous equation can be written as
\[ \zeta_{n+1} \approx \zeta_n + i \left\{ - \delta_1 \sigma_z \cdot \zeta_n + \frac{\delta_2}{2} \left[ 3 (\zeta_n \cdot \zeta_n^*) \zeta_n - (\zeta_n^* \cdot \sigma_y \cdot \zeta_n) \sigma_y \cdot \zeta_n \right] \right\}. \]

Assuming the aberrations are small, a ray initially belonging to an EFR will still belong to an EFR, generally different from the initial one, after several passes. This means that \( z_n \cdot z_n^* = \zeta_n \cdot \zeta_n^* = 1 \) throughout the evolution. This simplifies the evolution equation to

\[ \zeta_{n+1} \approx \zeta_n \left( 1 + i \frac{3}{2} \delta_2 \right) - i \left\{ \delta_1 \sigma_z \cdot \zeta_n + \frac{\delta_2}{2} \left[ (\zeta_n^* \cdot \sigma_y \cdot \zeta_n) \sigma_y \cdot \zeta_n \right] \right\}. \]

Note that \( 1 + i3\delta_2/2 \approx \exp(i3\delta_2/2) \), which shows that the linear term arising from the spherical aberration contributes to a global phase, which corresponds to the one arising from the \( T_0^2 \) term in the aberrated cavity operator in Eq. (S6). This phase-space rotation is a Hannay angle [21], the classical analogue of a geometric phase, accumulated when a ray moves between EFRs.

The effect of the other terms is best understood by considering the coordinates of the EFR on the Poincaré sphere given by

\[ t_1 = \zeta^* \cdot \sigma_z \cdot \zeta, \quad t_2 = \zeta^* \cdot \sigma_x \cdot \zeta, \quad t_3 = \zeta^* \cdot \sigma_y \cdot \zeta. \]

These relations remain valid if \( \zeta \) is substituted with \( z \). Neglecting terms that are second-order or higher in \( \delta_1 \) and \( \delta_2 \) leads to

\[ t_1^{(n+1)} = t_1^{(n)} - \delta_2 t_3^{(n)} t_2^{(n)}, \]
\[ t_2^{(n+1)} = t_2^{(n)} - 2 \delta_1 t_3^{(n)} + \delta_2 t_3^{(n)} t_1^{(n)}, \]
\[ t_3^{(n+1)} = t_3^{(n)} + 2 \delta_1 t_2^{(n)}. \]

These equations describe how EFRs evolve after several traversals of the aberrated cavity. They are the discrete analogs of the evolution equations for a BH dimer on the surface of the Bloch sphere.

It can be shown that the quantity \( h = \frac{t_3^{(n+1)} - 4 \delta_1 t_1^{(n)} / \delta_2 \right\} \) remains constant to first order in \( \delta_1 \) and \( \delta_2 \) indicating that the evolution of the EFR traces a generalizd Viviani curve on the surface of the Poincaré sphere. Therefore, the trajectory on the equatorial plane follows a circle of radius \( R = (1 - h + 4 \delta_1^2 / \delta_2) \) translated by \( c = 2\delta_1 / \delta_2 \) along the \( t_1 \) axis. Writing \( t_1^{(n)} = c + R \cos \varphi_n \) and \( t_2^{(n)} = R \sin \varphi_n \), and dropping higher-order terms in \( \delta_1 \) and \( \delta_2 \) leads to

\[ \varphi_{n+1} = \varphi_n + \delta_2 t_3^{(n)}. \]

This equation can be written solely in terms of \( \varphi \) as

\[ \varphi_{n+1} = \varphi_n \pm \delta_2 \sqrt{1 - R^2 - c^2 - 2 R c \cos \varphi_n}. \]

This equation tells us that the change in \( \varphi \), \( \Delta \varphi_n = \varphi_{n+1} - \varphi_n \), is determined by the angular velocity of a simple pendulum. The relation to the simple pendulum can be written more explicitly by considering the change of the change in \( \varphi \),

\[ \Delta^2 \varphi_n = \Delta \varphi_{n+1} - \Delta \varphi_n = \delta_2 \Delta t_3^{(n)} = 2 \delta_2 \delta_1 t_2^{(n)} = 2 \delta_2 \delta_1 R \sin \varphi_n. \]

This last equality is a discretized version of the evolution equation for the simple pendulum. Writing it in terms of the wavefront parameters \( \epsilon_1 \) and \( \epsilon_2 \), this equation reads

\[ \Delta^2 \varphi_n = \frac{(N + 1)^3}{2} \epsilon_2 \epsilon_1 w^6 R \sin \varphi_n. \]

VI. Modeling the Dynamical Evolution

Summary. This section explains how the evolution of a beam inside the aberrated cavity is computed. The exact evolution inside the aberrated cavity is computed through a Fox-Li algorithm. These results are then compared with those obtained with a Schrödinger-like equation by identifying the operator of the aberrated cavity as an evolution operator, thus verifying the validity of the results obtained in Sec. II. As an example, the squeezing of an initial coherent state is considered.

In order to simulate the Bose-Hubbard (BH) dynamics in the aberrated cavity, it is necessary to consider the evolution of the mode as it bounces back and forth without taking into account the superposition with previous round-trips. This is equivalent to considering the unfolded system of a series of aberrated lenses separated by free space. Both situations can be modeled using a Fox-Li algorithm [22], where the focusing and aberrations of the mirrors (or lenses) are taken into account in configuration space, by simply multiplying by the appropriate phase factor, and free propagation is computed through multiplication by a quadratic phase in Fourier space. This allows computing the exact evolution of a mode in the aberrated cavity and comparing it with the evolution of the corresponding BH dimer. Given that the Ince operator was obtained under the assumption that
FIG. S5. Q function of the mode evolution in the aberrated cavity for (a) $M = 0$, (b) 3200, (c) 6400 and (d) 9600 passes. The parameters are the same as those in Fig. S4.

FIG. S6. Modulus of the coefficients $c_{N\ell}$ of the expansion in terms of LG modes after (a) $M = 0$, (b) 3200, (c) 6400 and (d) 9600 passes inside the aberrated cavity. The parameters are the same as those in Fig. S4.

the effect of the aberrations is small, we expect to see some leakage into modes of different total order in the exact Fox-Li propagation method. However, the smaller the aberrations the smaller this leakage will be.

In the simulations, the transverse and longitudinal scales are chosen so that $w = 1$ and $k = 1$ which fixes $g = 1/2$. Then, by choosing a value for the parameter $\chi$ all the parameters of the unaberrated cavity are set. The input mode is chosen as the coherent mode with $\theta = \pi/2$ and $\phi = 0$, which corresponds to the HG mode where all the lobes lie on a horizontal line. Figures S4 and S5 show the spatial distribution and Q function of this mode. This mode was chosen because its Q function corresponds to the smallest spot size for a given $N$ and lies on the separatrix, which leads to squeezing and entanglement in a BH dimer [23, 24]. Figures S4 and S5 show the results of running the simulation for $N = 21$, $\chi = 0.2$, $\epsilon_1 = 10^{-4}$, and $\epsilon_2 = 2.5 \times 10^{-5}$. Figure S4 shows the mode distribution at various stages of the evolution and Fig. S5 the corresponding Q function. To compute the Q function, the mode was decomposed in terms of the LG basis of same $N$. Figure S6 quantifies the leakage into neighboring modes by computing the mode decomposition with respect to a larger set of LG modes encompassing several total orders. As long the aberrations are sufficiently small, the leakage will be negligible so that it can be safely ignored. Placing a coherent state at the merging point of the separatrix leads to squeezing [24] which can be appreciated by the stretching of the Q function in Fig. S5. Squeezing in spin systems can be quantified by several parameters [25], such as the variance computed along perpendicular directions to the mean-spin direction [26, 27]. In the example presented here, the mean direction is $t_1$ so that the spin-squeezing parameter is given by

$$\xi^2_{S}(\vartheta) = \frac{4\Delta \hat{T}^2(\vartheta)_{\perp}}{N},$$

with

$$\hat{T}_{\perp} = \hat{T}_2 \cos \vartheta + \hat{T}_3 \sin \vartheta.$$  (S14)

Figure S7 shows the plot of the parameter as a function of $\vartheta$ for the same four cases shown in Figs. S1, S5 and S6. For $M = 9600$ the squeezing parameter attains a minimum value of 0.44.

FIG. S7. Spin squeezing parameter as a function of $\vartheta$ for $M = 0$, 3200, 6400 and 9600 passes.

FIG. S8. Evolution of a coherent SG beam with $N = 21$ in an aberrated cavity according to the operator found in Eq. (S6) with the same parameters as those in used for Figs. S4 and S5 for (a) 3200, (b) 6400 and (c) 9600 passes. The inset shows the field amplitude and phase (coded as hue) for the corresponding Husimi distribution.
To further confirm the validity of the approximation, the mode evolution according to the Ince operator is computed. This done through the equation

$$\frac{d}{d\xi} |U; \xi\rangle = -i\hat{\mathcal{I}} |U; \xi\rangle,$$

(S15)

where $\xi = M_\epsilon w^4/2$. This equation is analogous to the time dependent Schrödinger equation where $\xi$, through the number of passes, plays the role of time. By decomposing the SG mode in terms of the LG basis, this equation leads to a set of $N + 1$ coupled first-order linear differential equations, known as the Raman-Nath equations [28], which can be easily solved by diagonalizing $\hat{\mathcal{I}}$. The extra phase terms arising from the operator $\hat{T}_0$ and the identity in the full expression of the aberrated cavity operator are also taken into account. Figure S8 shows the results obtained by choosing the same parameters as in the exact simulation of the aberrated cavity. The strong resemblance of Figs. S4, S5 and S8 confirms the agreement between the two calculations.

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