Singlet structure function $F_1$ in double-logarithmic approximation

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Received: 25 September 2017 / Accepted: 22 February 2018 / Published online: 12 March 2018
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Abstract The conventional ways to calculate the perturbative component of the DIS singlet structure function $F_1$ involve approaches based on BFKL which account for the single-logarithmic contributions accompanying the Born factor $1/x$. In contrast, we account for the double-logarithmic (DL) contributions unrelated to $1/x$ and because of that they were disregarded as negligibly small. We calculate the singlet $F_1$ in the double-logarithmic approximation (DLA) and account at the same time for the running $\alpha_s$ effects. We start with a total resummation of both quark and gluon DL contributions and obtain the explicit expression for $F_1$ in DLA. Then, applying the saddle-point method, we calculate the small-$x$ asymptotics of $F_1$, which proves to be of the Regge form with the leading singularity $\omega_0 = 1.066$. Its large value compensates for the lack of the factor $1/x$ in the DLA contributions. Therefore, this Reggeon can be identified as a new Pomeron, which can be quite important for the description of all QCD processes involving the vacuum (Pomeron) exchanges at very high energies. We prove that the expression for the small-$x$ asymptotics of $F_1$ scales: it depends on a single variable $Q^2/x^2$ only instead of $x$ and $Q^2$ separately. Finally, we show that the small-$x$ asymptotics reliably represent $F_1$ at $x \leq 10^{-6}$.

1 Introduction

A description of the singlet structure function $F_1$ in the framework of Collinear Factorization usually involves DGLAP [1–4] to calculate the perturbative contributions. In this case $F_1$ is represented in the form of two convolutions:

$$F_1 = C_q(x/y) \otimes \Delta q(y, Q^2) + C_g(x/y) \otimes \Delta g(y, Q^2),$$

(1)

where $C_q$ and $C_g$ are the evolved (with respect to $Q^2$) quark and gluon distributions, respectively. These distributions are solutions to the DGLAP equations which govern the $Q^2$-evolution of the initial quark and gluon distributions $\delta q(x, \mu^2)$ and $\delta g(x, \mu^2)$, evolving them from the scale $\mu^2$ to $Q^2$. Both $\delta q$ and $\delta g$ are defined at $x \sim 1$ and $Q^2 = \mu^2 \sim 1\text{GeV}^2$. The parameter $\mu$ is also called the factorization scale. The $x$-dependence of $F_1$ is described by the coefficient functions $C_{q,g}$ as well as by the phenomenological factors in $\delta q, \delta g$. In the framework of DGLAP the evolution in the $k_\perp$-space is separated from the evolution with respect to $x$. Such a separation takes place at $x \sim 1$ only and breaks at small $x$ as was shown in Refs. [5–7]. It is the theoretical reason not to use DGLAP at small $x$. A practical reason is that DGLAP, by its design, accounts for the total resummation of $\ln^n Q^2$, while contributions $\sim \ln^n x$ are present in the DGLAP expressions in few first orders in $\alpha_s$ only (through the coefficient functions in NLO, NNLO, etc.).

On the other hand, such contributions are very important at small $x$, so it would be appropriate to substitute the DGLAP expressions for the DIS structure functions by new ones which include the total resummation of all double-logarithmic (DL) contributions. In the first place there are DL terms $\sim (\alpha_s \ln^2(1/x))^n$, then the terms $\sim (\alpha_s \ln(1/x) \ln Q^2)^n$, etc. Expressions accounting for resummation of DL contributions and for the running $\alpha_s$ effects were obtained for several structure functions with non-vacuum exchanges in the $t$-channel: the spin structure function $g_1$ (the singlet and non-singlet components) and the non-singlet component of $F_1$ (see the overview of [8] and the references therein). Besides, the expressions were obtained for $g_1$ and non-singlet $F_1$ combining the DGLAP results and resummation of the DL contributions, which made it possible to apply these expressions at arbitrary $x$ and $Q^2$.

However, a similar generalization of DGLAP was not obtained for the singlet $F_1$. The point is that by that time $F_1$ in the small-$x$ region has been intensively investigated in terms of approaches based on BFKL [9–13] and this looked as the only way to study $F_1$ at small $x$. Indeed, the lead-
ing $x$-dependent contributions to $F_1$ proved to be the single-logarithmic (SL) terms accompanying the “Born” factor $1/x$:

$$(1/x)[1 + c_1 \alpha_s \ln(1/x) + c_2(\alpha_s \ln(1/x))^2 + \cdots],$$

while the DL contributions proportional to $1/x$, i.e. the terms

$$(1/x)[1 + c_1^{\text{DL}} \alpha_s \ln^2(1/x) + c_2^{\text{DL}}(\alpha_s \ln(1/x))^4 + \cdots],$$

cancel each other (i.e. $c_k^{\text{DL}} = 0$ for $k = 1, 2, \ldots$) as was found first in Ref. [14]. As a result, the common strategy for investigating the QCD processes with vacuum exchanges in the $t$-channel was based on the use of the BFKL results. In particular, SL contributions to the structure functions $F_{1,2}$ was presented in Refs. [15–17]; SL contributions to $F_2$ in Refs. [18,19] were calculated with inclusion of the resummed anomalous dimensions in the renormalization group equation, while $F_2$ in Ref. [20] was calculated with direct unification of DGLAP and FFKL.

The solution to the BFKL equation is expressed through the series of the high-energy asymptotics of the Regge form, with the leading asymptotics commonly addressed as the BFKL Pomeron, so at $x \to 0$

$$F_1 \sim x^{-(1+\Delta P)},$$

where $\Delta P$ is the Pomeron intercept. As $\Delta P > 0$ for both the LO and the NLO BFKL Pomerons, they are called supercritical. As we are not going to use BFKL or its modifications like [21,22] in the present paper, we just mention that extensive literature on this issue can be found in Ref. [23].

Instead of using the BFKL results or trying to increase the accuracy of the method of Refs. [16,17], in the present paper we account for total resummation of the double-logarithmic contributions to $F_1$. In the first place we account for the $x$-dependent contributions

$$1 + c_1^{\text{DL}} \alpha_s \ln^2(1/x) + c_2^{\text{DL}}(\alpha_s \ln(1/x))^2 + \cdots$$

and then for DL terms combining logs of $x$ and $Q^2$. These DL contributions do not involve the large factor $1/x$ and for this reason they have been neglected in the BFKL approach. We calculate the singlet structure function $F_1$ in DLA, summing DL contributions coming from virtual gluon and quark exchanges. As a result, our expressions for coefficient functions and anomalous dimensions contain total resumptions of appropriate DL terms. To calculate $F_1$ we compose and solve Infra-Red Evolution Equations (IREEs) in the same way as we did for calculating the DIS singlet structure function $g_1$ (see Ref. [8]), investigating the cases of fixed and running $\alpha_s$. We recall that the IREE method was suggested by Lipatov in Refs. [24–27]. It is based on factorization of DL contributions of the partons with minimal transverse momenta first noticed by Gribov in Ref. [28] in the context of QED of hadrons. The technology of implementation of this method to DIS is described in detail in Ref. [8]. In contrast to DGLAP and BFKL equations, we compose the two-dimensional evolution equations: They control the evolutions in both $x$ and $Q^2$. We obtain the explicit expression for $F_1$ and then, applying the saddle-point method, we calculate the small-$x$ asymptotics of $F_1$ automatically complemented by the asymptotic $Q^2$-dependence. The asymptotics proves to be of the Regge form. The large value of the intercept compensates for the lack of the factor $1/x$ in the DL contributions and thereby renders the DLA asymptotics of the same order as the BFKL one. This proves that the DL contributions to $F_1$ at small $x$ are, at least, no less important than the contributions coming from the BFKL Pomeron.

Our paper is outlined as follows: in Sect. 2 we compose and solve IREE for the Compton amplitudes $A_{q,g}$ related to $F_1$ by the optical theorem. In this section we express $A_{q,g}$ through the amplitudes of the $2 \to 2$ scattering of partons. Those amplitudes are calculated in Sect. 3. In Sect. 4 we apply the saddle-point method to obtain an explicit expression for the small-$x$ asymptotics of $F_1$ and prove that this asymptotics depends on the single variable $Q^2/x^2$ instead of depending separately on $Q^2$ and $x$. In Sect. 5 we consider in detail the intercept of the Pomeron in DLA, embracing the cases of fixed and running $\alpha_s$. We also fix the region where the small-$x$ asymptotics can reliably represent $F_1$. Finally, Sect. 6 is for our concluding remarks.

## 2 IREE for the amplitudes of Compton scattering off partons

Following the DGLAP pattern, we consider $F_1$ in the framework of Collinear factorization and represent $F_1$ through the convolutions of the perturbative components $T_q$ and $T_g$ with non-perturbative initial quark and gluon distributions $\phi_{q,g}$, respectively:

$$F_1 = F_{1q} \otimes \phi_q + F_{1g} \otimes \phi_g.$$  

(6)

Throughout the paper we will consider the perturbative objects $F_{1q} \otimes \phi_q$ only. It is convenient to consider the Compton amplitudes $A_q$ and $A_g$ related to $T_{1q} \otimes \phi_q$ by the optical theorem:

$$F_{1q} \otimes \phi_q(x, Q^2/\mu^2) = -\frac{1}{2\pi} 3 A_{q,g}(x, Q^2/\mu^2),$$  

(7)

where we have introduced the factorization scale $\mu$ and used the standard notation $x = Q^2/w$, with $w = 2pq$ and $Q^2 = -q^2$. The next step is to represent $A_{q,g}$ in terms of the Mellin transform:
\[ A_{q,g}(w/\mu^2, Q^2/\mu^2) = \int_{-\infty}^{\infty} \frac{dw}{2\pi i} (w/\mu^2)^{\xi_1^{(+)}(\omega)} F_{q,g}(\omega, Q^2/\mu^2) \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega \rho} F_{q,g}(\omega, y), \]  

where we have introduced the signature factor \( \xi^{(+)}(\omega) = (1 + e^{-\omega})/2 \approx 1 \) and the logarithmic variables \( \rho, y \) (using the standard notation \( w = 2pq \)):  

\[ \rho = \ln(w/\mu^2), \quad y = \ln(Q^2/\mu^2). \]  

In what follows we will call \( F_q, F_g \) Mellin amplitudes and will use the same form of the Mellin transform for the other amplitudes as well. For instance, the Mellin transform for the color singlet amplitude \( A_{gg} \) of the elastic gluon–gluon scattering in the forward kinematics is  

\[ A_{gg}(w/\mu^2, Q^2/\mu^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} (w/\mu^2)^{\xi_1^{(-)}(\omega)} f_{gg}(\omega) \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega \rho} f_{gg}(\omega). \]  

We have presumed in Eq. (10) that the virtualities of all external gluons are \( \sim \mu^2 \). Let us notice that the only difference between the Mellin representation for the Compton amplitudes \( A_{q,g} \) and similar amplitudes related to the singlet \( g_1 \) is in the signature factors: the signature factor for \( g_1 \) is \( \xi^{(-)}(\omega) = (-1 + e^{-\omega})/2 \). Otherwise, the technology of composing and solving IREE for \( A_{gg} \) and singlet \( g_1 \) is the same. Because of that, we present IREE for \( A_q, F_g \) (and for auxiliary amplitudes as well) with short comments only. The full-length derivation of all involved IREE can be found in Ref. [8]. Now all is set up to construct IREEs for \( F_{q,g} \). In the kinematics where  

\[ w \gg Q^2 \gg \mu^2, \]  

the amplitudes \( F_q, F_g \) obey the partial differential equations:  

\[ \left[ \hat{\partial}/\hat{\partial} y + \omega \right] F_q(\omega, y) = F_q(\omega, y) h_{qq}(\omega) + F_g(\omega, y) h_{qq}(\omega), \]

\[ \left[ \hat{\partial}/\hat{\partial} y + \omega \right] F_g(\omega, y) = F_g(\omega, y) h_{gg}(\omega) + F_q(\omega, y) h_{gg}(\omega), \]  

where we have used the following convenient notation:  

\[ h_{rr'} = \frac{1}{8\pi i} f_{rr'}, \]  

with \( r, r' = q, g \) and \( f_{rr'} \) being the parton–parton amplitudes. We will calculate \( h_{rr'} \) in the next section. Actually, the equations in (12) manifest a strong resemblance with the DGLAP equations. Indeed, the first factor in brackets in the l.h.s. of (12) exists in DGLAP too. The second term vanishes when the Mellin factor \( (s/\mu^2)^{\omega} \) is replaced by the factor \( x^{-\omega} \), which is used in the DGLAP equations. When the parton amplitudes \( f_{rr'} \) are in the Born approximation, Eq. (12) coincides with the DGLAP equations. A general solution to Eq. (12) is  

\[ F_q(\omega, y) = e^{-\omega y} \left[ C_+(\omega)e^{\Omega_+ y} + C_-(\omega)e^{\Omega_- y} \right], \]

\[ F_g(\omega, y) = e^{-\omega y} \left[ C_+(\omega) h_{gg} - h_{qq} + \sqrt{R} e^{\Omega_+ y} \right] + C_-(\omega) h_{gg} - h_{qq} + \sqrt{R} e^{\Omega_- y}, \]  

where \( C_+(\omega) \) and \( C_-(\omega) \) are arbitrary factors whereas  

\[ \Omega_\pm = \frac{1}{2} \left[ h_{gg} + h_{qq} \pm \sqrt{R} \right]. \]  

and  

\[ R = (h_{gg} + h_{qq})^2 - 4(h_{gq} h_{gg} - h_{gg} h_{gq}) = (h_{gg} - h_{qq})^2 + 4 h_{gq} h_{gg}. \]  

We specify the factors \( C_\pm(\omega) \) by the matching with the Compton amplitudes \( f_q, f_g \) calculated in the kinematics \( Q^2 \approx \mu^2 \), i.e. at \( y = 0 \). The matching condition is  

\[ F_q(\omega, y)|_{y=0} = f_q(\omega), \quad F_g(\omega, y)|_{y=0} = f_g(\omega), \]  

which leads to the following expressions:  

\[ C_+ = \frac{h_{gg} f_g(\omega) - (h_{gg} - h_{qq} - \sqrt{R}) f_q(\omega)}{2\sqrt{R}}, \]

\[ C_- = \frac{-h_{gq} f_g(\omega) + (h_{gg} - h_{qq} + \sqrt{R}) f_q(\omega)}{2\sqrt{R}}. \]  

Now let us express \( f_q, f_g \) through the parton–parton amplitudes \( h_{rr'} \). To this end, we construct IREE for them. As \( f_q, f_g \) do not depend on \( Q^2 \), the IREE for them are algebraic:  

\[ \alpha f_q(\omega) = a_{\gamma q} + f_q(\omega) h_{qq}(\omega) + f_g(\omega) h_{gq}(\omega), \]

\[ \alpha f_g(\omega) = f_q(\omega) h_{gg}(\omega) + f_g(\omega) h_{gq}(\omega), \]  

where \( a_{\gamma q} = e^2 \), with \( e^2 \) being the total electric charge of the involved quacks, so that \( a_{\gamma q}/\omega \) is the Born value of amplitude \( f_q(\omega) \). There is no similar term in the equation for \( f_g(\omega) \). The only difference between the r.h.s. of (19) and (12) is the factor \( a_{\gamma q} \) in Eq. (19). The solution to Eq. (19) is  

\[ f_q(\omega) = a_{\gamma q} \frac{(\omega - h_{gg})}{G(\omega)}, \]

\[ f_g(\omega) = a_{\gamma q} \frac{h_{gg}}{G(\omega)}, \]
with \( G(\omega) \) being the determinant of the system (19):

\[
G = (\omega - h_{qq})(\omega - h_{gg}) - h_{gg}h_{qq}.
\]

(21)

Combining Eqs. (18) and (20), we express \( C_\pm \) through the parton–parton amplitudes:

\[
C_+ = a_{\gamma q} q_m \frac{h_{qq} h_{qq} - (\omega - h_{gg})(h_{gg} - h_{qq} - \sqrt{\Delta})}{2G\sqrt{\Delta}},
\]

\[
C_- = a_{\gamma q} q_m \frac{-h_{qq} h_{qq} + (\omega - h_{gg})(h_{gg} - h_{qq} + \sqrt{\Delta})}{2G\sqrt{\Delta}}.
\]

(22)

Combining Eqs. (22, 15) and (14), we can easily express \( F_{qq, gg} \) in terms of the parton–parton amplitudes \( h_{rr} \):

\[
\omega h_{qq} = b_{qq} + h_{qq} h_{qq} + h_{gg} h_{gg},
\]

\[
\omega h_{gg} = b_{gg} + h_{qq} h_{qq} + h_{gg} h_{gg},
\]

\[
\omega h_{gg} = b_{gg} + h_{gg} h_{gg} + h_{gg} h_{gg},
\]

\[
\omega h_{gg} = b_{gg} + h_{gg} h_{gg} + h_{gg} h_{gg}.
\]

(23)

where the terms \( b_{rr} \) include the Born factors \( a_{rr} \) and contributions of non-ladder graphs \( V_{rr} \):

\[
b_{rr} = a_{rr} + V_{rr}.
\]

(24)

The Born factors are (see Ref. [8] for details):

\[
a_{qq} = \frac{A(\omega)C_F}{2\pi}, \quad a_{gg} = \frac{A'(\omega)C_F}{\pi}, \quad a_{gq} = \frac{2N A(\omega)}{\pi},
\]

\[
a_{gg} = \frac{2N A(\omega)}{\pi}.
\]

(25)

with \( \eta = \ln(\mu^2/\Lambda^2_{QCD}) \) and \( b \) being the first coefficient of the Gell-Mann–Low function. When the running effects for the QCD coupling are neglected, \( A(\omega) \) and \( A'(\omega) \) are replaced by \( \alpha_s \). The terms \( V_{rr} \) are represented in a similar albeit more involved way (see Ref. [8] for details):

\[
V_{rr} = \frac{m_{rr}}{\pi} D(\omega),
\]

(27)

with

\[
m_{rr} = \frac{C_F}{2N}, \quad m_{gg} = -2N^2, \quad m_{gq} = n_f \frac{C_F}{2},
\]

\[
m_{gg} = -NC_F,
\]

and

\[
D(\omega) = \frac{1}{2b^2} \int_0^\infty dz e^{-\omega z} \ln \left( \frac{z + \eta}{(z + \eta)^2 + \pi^2} \right) - \frac{1}{z + \eta}.
\]

(29)

Let us note that \( D = 0 \) when the running coupling effects are neglected. It corresponds to the total compensation of DL contributions of non-ladder Feynman graphs to scattering amplitudes with the positive signature as was first noticed in Ref. [29]. When \( \alpha_s \) is running, such a compensation is only partial. A solution to Eq. (23) is

\[
h_{qq} = \frac{1}{2} \left[ \omega - Z - \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{qq} = \frac{b_{gg}}{Z},
\]

\[
h_{gg} = \frac{1}{2} \left[ \omega - Z + \frac{b_{gg} - b_{qq}}{Z} \right], \quad h_{gg} = \frac{b_{gg}}{Z},
\]

(30)

where

\[
Z = \frac{1}{\sqrt{2}} \sqrt{Y + W},
\]

(31)

with

\[
Y = \omega^2 - 2(b_{qq} + b_{gg})
\]

(32)

and

\[
W = \sqrt{\left(\omega^2 - 2(b_{qq} + b_{gg})\right)^2 - 4(b_{qq} - b_{gg})^2 - 16b_{gg}b_{gg}}.
\]

(33)

The algebraic equations (23) are non-linear, so they yield four expressions for \( Z \). We selected in Eq. (31) the solution obeying the matching with the Born amplitudes \( h_{rr}^{\text{Born}} \); at large \( \omega \)

\[
h_{rr} \to h_{rr}^{\text{Born}} = a_{rr}/\omega.
\]

(34)

Substituting the expressions of Eq. (30) in (20), we obtain explicit expressions for the amplitudes \( f_q, f_g \). Combining them with Eqs. (22, 15) and (14), we obtain explicit expressions for \( f_q \) and \( f_g \). Substituting them in Eq. (8), we arrive at the explicit expressions for the Compton amplitudes \( A_q \) and \( A_g \). Finally, applying the optical theorem (7) to \( A_q \) and \( A_g \), we arrive at the singlet structure function \( F_1 \).
4 Small-x asymptotics of the structure function $F_1$

The regular way to obtain the small-x asymptotics of $A_q$ and $A_g$ is to write explicit expressions for $F_q$ and $F_g$ in Eq. (8), then push $x \to 0$ and apply the saddle-point method. However, before doing this, let us consider in detail how to calculate the asymptotics of the gluon–gluon scattering amplitude $A_{gg}$, presuming virtualities of all external gluons $\sim \mu^2$.

4.1 Asymptotics of $F_1$

The small-x asymptotics of $A_q$ and $A_g$ can be obtained with applying the saddle-point method to Eq. (8). As $\Omega_+(\omega) > \Omega_-(\omega)$, we neglect the terms $C(-)$ in (14) and represent Eq. (8) in the following form:

$$A_q \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega \xi} \tilde{F}_q(\omega) e^{\Omega_+\omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\psi_q} \tilde{F}_q(\omega) e^{\Omega_+\omega},$$

$$A_g \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\omega \xi} \tilde{F}_g(\omega) e^{\Omega_+\omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{\psi_g} \tilde{F}_g(\omega) e^{\Omega_+\omega},$$

(35)

with $\xi = \ln(1/x)$ and

$$\tilde{F}_q = C_+, \quad \tilde{F}_g = C_+ \frac{(h_{gg} - h_{qq} + \sqrt{R})}{2h_{gg}}$$

(36)

and

$$\psi_q = \omega \xi + \ln \tilde{F}_q, \quad \psi_q = \omega \xi + \ln \tilde{F}_q.$$ 

(37)

The stationary point at $x \to 0$ of $\psi_q$ is given by the rightmost root $\omega_0$ of the following equation:

$$d\psi_q/d\omega = \xi + \frac{\partial \tilde{F}_q}{\tilde{F}_q}(\omega_0) = 0.$$ 

(38)

When $\xi \to \infty$, it must be equated by some negative singular contribution in the second term of Eq. (38). Using the explicit formulas for $\tilde{F}_{g,q}$, one can conclude that such a contribution comes from the factor $1/W$. So, the stationary point $\omega_0$ is the rightmost root of the equation

$$(\omega^2 - 2b_{qq} - 2b_{gg})^2 - 4(b_{qq} - b_{gg})^2 - 16b_{qq}b_{gg} = 0.$$ 

(39)

We consider in detail solutions to Eq. (39) at fixed and running $\alpha_s$ in the next section. In the vicinity of $\omega_0$ we can represent Eq. (38) as

$$\psi'_q = \xi + \frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{dW}{d\omega} = \xi + \frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{\lambda}{W} = \xi - \frac{\varphi \lambda}{W} = 0,$$ 

(40)

with

$$\varphi_q = -\partial \ln \tilde{F}_q/\partial W$$ 

(41)

and

$$\lambda = 2\omega(\omega^2 - 2(b_+ + b_-)).$$ 

(42)

so in vicinity of the singularity $\omega_0$

$$W \approx W_0 = \varphi \frac{\lambda}{\xi}.$$ 

(43)

Expanding $\Psi_q$ in a series, we obtain

$$\Psi_q(\omega) \approx \Psi_q(\omega_0) + (1/2)\Psi''(\omega_0)(\omega - \omega_0)^2.$$ 

(44)

In order to calculate $\Psi''$ we notice that the most singular contributions come from differentiation of the numerator in Eq. (40), so

$$\Psi'' \approx -\frac{\partial \tilde{F}_q}{\tilde{F}_q} \frac{\lambda dW^{-1}}{d\omega} = \varphi \frac{\lambda^2}{W} = \frac{\xi^3}{\lambda \varphi_q^2}$$ 

(45)

and therefore the asymptotics of $A_q$ at $x \to 0$ is

$$A_q(x, Q^2/\mu^2) \sim A_q^{as}(x, Q^2/\mu^2)$$

$$= C_+(\omega_0)\varphi_q(\omega_0) \sqrt{\frac{\lambda}{2\pi \xi^3}} \left(\frac{1}{x}\right)^{\omega_0} \left(\frac{Q^2}{\mu^2}\right)^{\Omega_+(\omega_0)}.$$ 

(46)

Repeating the reasoning above for $A_g$ and applying to them the optical theorem, we conclude that the small-x asymptotics of $F_1$ is

$$F_1 \sim \Pi(\omega_0, \xi) \left(\frac{1}{x}\right)^{\omega_0} \left(\frac{Q^2}{\mu^2}\right)^{\Omega_+(\omega_0)},$$ 

(47)

where the factor $\Pi(\omega_0, \xi)$ is

$$\Pi(\omega_0, \xi) \approx C_+ \sqrt{\frac{\lambda}{2\pi \xi^3}} \left[\varphi_q \delta q + \varphi_g \left(h_{gg}(\omega_0) - h_{gg}(\omega_0) + \sqrt{R(\omega_0)}\right) \delta q \right].$$ 

(48)

with $\delta q$ and $\delta q$ being the initial quark and gluon densities. They do not include singular factors $\sim x^{-a}$, with positive $a$. The Regge form of the asymptotics is brought about entirely by the perturbative contributions. Let us notice that $\Pi \sim \ln^{-3/2}(1/x)$. Equation (47) exhibits that the total resummation of DL contributions leads to the Regge behavior of $F_1$ at small $x$.  

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4.2 Asymptotic scaling

Substituting the explicit expressions for \( h_{rr'} \) of Eq. (30) in Eq. (15) and using Eq. (39), we obtain \( \Omega_{\omega_0}(\omega_0) = \omega_0/2 \). This allows us to write the asymptotics of \( F_1 \) of Eq. (47) in the following way:

\[
F_1 \sim \Pi(\omega_0, \xi) \left( \frac{1}{x} \right)^{\omega_0} \left( \frac{Q^2}{\mu^2} \right)^{\omega_0/2}
= \Pi(\omega_0, \xi) \left( \frac{Q^2}{x^2 \mu^2} \right)^{\omega_0/2}.
\] (49)

Equation (49) manifests that \( F_1(x, Q^2) \) at asymptotically high energies depends on the single variable \( Q^2/x^2 \) only. We name such a confluence of the \( x \) and \( Q^2 \) dependence an asymptotic scaling. The same form of asymptotic scaling was obtained earlier for the structure function in Eq. (30) in Eq. (47) in in Ref. [8] for details. We stress that the asymptotic scaling for \( F_1 \) can be checked by an analysis of available experimental data. Moreover, \( F_2 = 2(x F_1) \) at very small \( x \), which proves the asymptotic scaling for \( F_2 \). Finally, let us notice that the leading singularity \( \omega_0 \) in Eq. (49) does not depend on \( Q^2 \).

5 Anatomy of the leading singularity \( \omega_0 \)

In this section we consider in detail the leading singularity \( \omega_0 \), which is the rightmost root of Eq. (39). In order to make the asymptotics of \( F_1 \) look similar to Eq. (4), we denote

\[
\omega_0 = 1 + \Delta,
\] (50)

so that \( \omega_0 \) could look similar to the BFKL leading singularity; see Eq. (4). Now let us discuss different scenarios for calculating \( \omega_0 \). In what follows we will call \( \Delta \) the DL Pomeron intercept. We recall that in the straightforward Reggeology concept \( \Delta = 0 \) and the Pomeron with \( \Delta > 0 \) is called the supercritical Pomeron.

5.1 Intercept under approximation of fixed QCD coupling

In the first place let us estimate \( \omega_0 \) for the case of fixed \( \alpha_s \). In this case DL contributions of non-ladder graphs totally cancel each other, so that \( D = 0 \) and \( b_{rr'} = a_{rr'} \), with \( a_{rr'} \) defined in Eq. (25), where \( A(\omega) \) and \( A'(\omega) \) should be replaced by \( \alpha_s \). Then the solution to Eq. (39) is

\[
\omega_0^{\text{fix}} = \left( \frac{\alpha_s^{\text{fix}}}{\pi} \right)^{1/2} \left[ 4N + CF \right]^{1/2} + \sqrt{(4N - CF)^2 - 8n_f CF} \approx 2.63 \sqrt{\alpha_s},
\] (51)

which fairly coincides with the well-known LO BFKL intercept \( \Delta_{LO} \). However, \( \Delta_{LO} \) corresponds to accounting for gluon contributions only, while \( \Delta^{fix} \) accommodates both gluon and quark contributions. When the quark contributions in Eq. (51) are dropped, the purely gluonic intercept \( \Delta^{fix}_g \) becomes somewhat greater:

\[
\Delta^{fix}_g = \omega_0^{fix} - 1 = 0.29,
\] (52)

which again bears a strong resemblance to the LO BFKL intercept. However, we are positive that the approximation of fixed \( \alpha_s \) can be used for rough estimating only, so we will not pursue this approximation any longer.

5.2 Intercept for the case of running coupling

Now we account for the running coupling effects in Eq. (39). Because of that, Eq. (39) can be solved only numerically. As the couplings \( A \) and \( A' \) included in the factors \( b_{rr'} \) depend on \( \mu \) through \( \eta = \ln(\mu^2/\Lambda^2) \), the solution, \( \omega_0^{fix} \), is also \( \mu \)-dependent. Numerical calculations yield the plot of the \( \eta \)-dependence of \( \omega_0 \) presented in Fig. 1.

The curve in Fig. 1 has the maximum \( \omega_0^{DLA} = 1.066 \) at \( \mu/\Lambda = 13.8 \). We call

\[
\mu_0 = 13.8 \Lambda
\] (54)
the optimal mass scale and call
\[ \Delta = \omega_0^{DLA} - 1 = 0.066 \]  \hspace{1cm} (55)

the intercept of the Pomeron in DLA. It is interesting to notice that \( \Delta \) is close to the NLO BFKL intercept. In contrast, when the quark contributions are neglected, the purely gluonic intercept \( \Delta_g^{DLA} \) is much greater:
\[ \Delta_g = 0.254 . \]  \hspace{1cm} (56)

Confronting Eqs. (53)–(52) and (56)–(55) demonstrates that accounting for the quark contributions decreases the intercept. Similarly, confronting Eqs. (52)–(55) exhibits that accounting for the quark contributions decreases the purely gluonic intercept \( \alpha \) corresponds to the case of BFKL sums the single-logarithmic terms. Moreover, Eq. (55) obtained from resummation of DL contributions, while the DL contributions, the impact of their effects essentially decreases the intercept value. We also would like to stress that despite our values of \( \Delta \) in Eqs. (53) and (50) being close to the values of the LO BFKL and NLO BFKL intercepts, respectively, this similarity if just a coincidence: our intercepts are values of the LO BFKL and NLO BFKL intercepts, respectively. This can be seen in Fig. 3.

5.3 Applicability region of the small-\( x \) asymptotics

It is obvious that the small-\( x \) asymptotic expressions, like Eq. (47), are always much simpler than non-asymptotic expressions. However, it is important to find at which values of \( x \) the asymptotics can reliably be used. To answer this question we numerically investigate \( R_{as} \), defined as follows:
\[ R_{as}(x, Q^2) = \frac{A_q^{as}(x, Q^2)}{A_q(x, Q^2)} . \]  \hspace{1cm} (57)

The \( x \)-dependence of \( R_{as} \) at fixed \( Q^2 \) is shown in Fig. 2 for the case when \( Q^2 \approx \mu^2 \).

![Fig. 2](image)

**Fig. 2** Approach of \( A_q \) to its asymptotics \( A_q^{as} \) at fixed \( Q^2 = \mu^2 \)

Figure 2 demonstrates that \( R_{as} = 0.9 \) at \( x \approx 8.10^{-5} \), while the curve in Fig. 3, where \( Q^2 = 100\mu^2 \), grows slower and reaches the value \( R_{as} = 0.9 \) much later, at \( x \approx 3.10^{-7} \).

Therefore, the applicability region of the small-\( x \) asymptotics essentially depends on the \( Q^2 \) value. The plots in Figs. 2 and 3 lead us to conclude that the small-\( x \) asymptotics reliably represent \( F_1 \) in the wide range of \( Q^2 \) when \( x < x_{max} \), with \( x_{max} \approx 10^{-6} \).

6 Summary and outlook

In this paper we have calculated the perturbative contributions \( F_1^{d,x} \) and \( F_1^{s} \) to the structure function \( F_1 \) in the Double-Logarithmic Approximation, by collecting the DL contributions and at the same time accounting for the running \( \alpha_s \) effects. We obtained the explicit expressions for \( F_1^{d,x} \) and then, applying the saddle-point method, calculated the small-\( x \) asymptotics of \( F_1 \), arriving at a new DL contribution to the QCD Pomeron. We demonstrated that despite the lack of the factor \( 1/x \) in the DL contributions, the impact of their total resummation makes this Pomeron supercritical, albeit the value of the intercept strongly depends on the accuracy of the calculations. The maximal value of the intercept corresponds to the roughest approximation where the quark contributions are neglected and \( \alpha_s \) is fixed. Then the value of the intercept decreases when the accuracy of the calculations increases: first, when the quark contributions are accounted for and then, notably, when the running \( \alpha_s \) effects are taken into account. Nevertheless, the Pomeron remains supercritical as \( \omega_0 = 1.066 \). Such a monotonic decrease allows us to suggest that further accounting for sub-leading contributions can decrease the value of the intercept down to zero, so that eventually the intercept will satisfy the Froissart bound. We proved that the \( x \)- and \( Q^2 \)-dependencies of \( F_1 \) converge at small \( x \) in dependence on the single variable \( Q^2/x^2 \). We call this convergence asymptotic scaling. We stress that this prediction of asymptotic scaling can be confirmed by an analysis of the available experimental data. As asymptotically
\[ F_2 \sim 2x F_1, \]

the asymptotic scaling should also take place for \( F_2 \). Investigating the applicability region for the asymptotics, we found that \( F_1 \) can reliably be represented by its asymptotics at \( x \leq x_{\text{max}} \), with \( x_{\text{max}} \approx 10^{-6} \).

Although we have discussed the structure function \( F_1 \), we would like to notice that the experimental data available in the literature are mostly on the structure functions \( F_2 \) and \( F_L \). As such, we would be interesting to apply our approach to calculating \( F_2 \) and \( F_L \) as well. Calculating \( F_2 \) in DLA can be done in a way quite similar to that we have used for \( F_1 \). As a result, we see that \( F_2 \) in DLA can be represented through \( F_1 \):

\[ F_2 = 2x F_1, \tag{58} \]

which coincides with the well-known Born relation between \( F_1 \) and \( F_2 \). Equation (58) entails that in DLA \( F_L = 0 \). In order to estimate the deviation of \( F_L \) from zero, one should account for sub-leading contributions to both \( F_1 \) and \( F_2 \). In the first place, such contributions are the single-logarithmic (SL) ones. In this regard we recall that the SL contributions to \( F_2 \) following from emission of gluons with momenta widely separated in rapidity and not ordered in transverse momenta were accounted for in Refs. [15–19], which involved dealing with the BFKL characteristic function. However, there are the SL contributions unrelated to BFKL, i.e., for the case of \( F_1 \), the SL terms unaccompanied by the factor \( 1/x \), similar to the DL terms in Eq. (5). In contrast to the DL contributions (5), there is not a general technology in the literature for resummations of such non-BFKL SL terms. On the other hand, we were able to modify the IREE method for the spin structure function \( g_1 \) (see Ref. [8] and the references therein) to account for the SL contributions which are complementary to the ones calculated in Refs. [15–19], namely, the SL following from emission of the partons with momenta ordered in the \( k_\perp \)-space and disordered in the longitudinal space. We plan to adapt this approach to calculating the SL contributions to \( F_{1,2} \).

Finally, we stress that in contrast to DGLAP we do not need singular factors \( \sim x^{-d} \) in fits for the initial parton distributions for \( F_1 \). Such factors cause a steep rise of the structure functions at small \( x \) and lead to the Regge asymptotics of \( F_1 \). However, we have shown in Sect. 4 that the resummation of the DL contributions to \( F_1 \) automatically leads to the Regge asymptotics, which makes an inclusion unnecessary of the singular terms into the fits. This result agrees with our earlier results (see Ref. [8]) for the non-singlet structure functions \( g_1 \) and \( F_1 \) and also with the results of Refs. [18, 19] obtained for the small-\( x \) behavior of the structure function \( F_2 \). The latter agreement is especially interesting because the approaches used in Refs. [18, 19] and in the present paper are totally different.

Acknowledgements We are grateful to Mario Greco for interesting discussions of the running \( \alpha_s \) effects. Work of B.I. Ermolaev was supported in part by the RFBR Grant no. 16-02-00790-a.

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