Collision of water wave solitons

Nora Fenyvesi*, Gyula Bene†

Institute of Physics, Loránd Eötvös University
Pázmány Péter sétány 1/A H-1117 Budapest, Hungary

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Abstract: A classification of the time evolution of the two-soliton solutions of the Boussinesq equation is given, based on the number of extrema of the wave. For solitons moving in the same directions, three different scenarios are found, while it is shown that only one of these scenarios exists in case of oppositely moving solitons.

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1. Introduction

Interest in environmental flows [1–18] both utilize and motivate theoretical work in nonlinear wave phenomena. Among these are solitons [19–26], which appear in many different situations and have applications in several other branches of physics [27, 28]. Although solitons are well known and much studied, we believe that a simple classification scheme of the evolving wave shapes of two colliding water wave solitons may not be in vain.

Following the first observation of a soliton in a narrow channel (J.Scott Russell [29]), both Boussinesq [30] and Lord Rayleigh [31] explained some of its basic properties theoretically by assuming that the wavelength is much larger than the depth of water which in turn is much larger than the height of the wave.

Boussinesq later derived his equation [32] in this approximation for the wave amplitude $\eta(x, t)$ which can be written in dimensionless form as

\[ \eta_{tt} - \eta_{xx} - 3 \left( \eta^3 \right)_{xx} - \eta_{xxxx} = 0 \]  

Boussinesq [33] and later (independently) Korteweg and de Vries [34] derived the equation now called Korteweg - de Vries equation for waves travelling to a single direction, in dimensionless form it is

\[ \eta_t + \eta_x + \frac{1}{2} \left( \eta^2 \right)_x + \eta_{xxx} = 0 \]  

Solitons in other nonlinear equations and via several techniques were found since. Especially, several recent results are available for generalized Boussinesq systems. The solitary wave solution of a modified Boussinesq equation, with power law nonlinearity, was obtained by using the solitary wave Ansatz method [35]. A generalized form of the Boussinesq equation was studied with $\left( u^{2n} \right)_{xx}$ type nonlinearity in [36] and [37]. By applying the Ansatz method, the mapping method, the exponential function...
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2. General properties of the two-soliton solution of the Boussinesq equation

The Boussinesq equation in dimensionless form is given by

\[ n_{tt} - n_{xx} - 3 \left( n^2 \right)_{xx} - n_{xxxx} = 0 \]  

(3)

Two-soliton solutions are generated by the Zakharov-Shabat scheme [19], and may be written as

\[ \eta = -a \frac{d}{dx} \left\{ \frac{k_1(1 + q) + k_2(1 + p) - (k_1 + k_2)a}{(1 + q)(1 + p) - a} \right\}, \]

(4)

where the new variables \( p \) and \( q \) are defined by

\[ p = e^{2k_1x - 2\omega_1t}, \quad q = e^{2k_2x - 2\omega_2t} \]

(5)

and the parameter \( a \) is given by

\[ a = \frac{4k_1k_2}{(k_1 + k_2)^2 + \frac{1}{12} \left( \frac{\omega_1}{k_1} - \frac{\omega_2}{k_2} \right)^2}. \]

(6)
In these expressions \( k_1 \) and \( k_2 \) are arbitrary positive parameters characterizing the individual solitons and

\[
\omega_j = \pm k_j \sqrt{1 + 4k_j^2}, \quad (j = 1, 2)
\] (7)

with positive sign for a soliton propagating from left to right, and negative otherwise. Note that if \( \omega_1 \neq \omega_2 \),

\[
0 < a < 1.
\] (8)

On the other hand, if \( \omega_1 = \omega_2 \), we also have \( k_1 = k_2 \) that corresponds to a one-soliton solution, not being of interest here.

An important observation is that in the extreme cases when \( k_1, k_2 \ll 1/2 \) or \( k_1, k_2 \gg 1/2 \) the parameter \( a \) depends only on the ratio

\[
\kappa = \frac{k_2}{k_1}.
\]

Explicitly, we get for solitons moving in the same direction

\[
a = \begin{cases} 
\frac{4k}{(1 + \kappa)^2 + \frac{1}{2}(1 - \kappa)^2} & \text{if } k_1, k_2 \gg \frac{1}{2} \\
\frac{4k}{(1 + \kappa)^2} & \text{if } k_1, k_2 \ll \frac{1}{2}
\end{cases}
\] (9)

and for solitons moving oppositely

\[
a = \begin{cases} 
\frac{4k}{(1 + \kappa)^2 + \frac{1}{2}(1 - \kappa)^2} & \text{if } k_1, k_2 \gg \frac{1}{2} \\
0 & \text{if } k_1, k_2 \ll \frac{1}{2}
\end{cases}
\] (10)

For the spatial derivative of the wave we have

\[
\frac{\partial \eta}{\partial x} = \frac{8}{((1 + q)(1 + p) - a)^3} \sum_{n=1}^{5} \sum_{j=0}^{n} c_{n,j} p^j q^{n-j},
\] (11)

where the coefficients are given by

\[
\begin{align*}
c_{10} &= -k_1^2(a - 1)^2 \\
c_{11} &= -k_1^3(a - 1)^2 \\
c_{20} &= -k_1^2(a - 1) \\
c_{21} &= -(k_1 + k_2) \left[ a(k_1 + k_2)^2 - 3k_1^2 + 3k_1k_2 - 3k_2^2 \right] (a - 1) \\
c_{22} &= -k_1^3(a - 1) \\
c_{31} &= (k_1 - k_2) \left[ a(2k_1^2 + 5k_1k_2 + 2k_2^2) - 3k_1^2 - 3k_1k_2 - 3k_2^2 \right] \\
c_{32} &= (k_2 - k_1) \left[ a(2k_1^2 + 5k_1k_2 + 2k_2^2) - 3k_1^2 - 3k_1k_2 - 3k_2^2 \right] \\
c_{33} &= -k_1^3 \\
c_{41} &= -(k_1 + k_2) \left[ a(k_1 + k_2)^2 - 3k_1^2 + 3k_1k_2 - 3k_2^2 \right] \\
c_{42} &= -(k_1 + k_2) \left[ a(k_1 + k_2)^2 - 3k_1^2 + 3k_1k_2 - 3k_2^2 \right] \\
c_{43} &= -k_1^3 \\
c_{51} &= k_1^3 \\
c_{52} &= k_1^3 \\
c_{53} &= k_1^3 \\
\end{align*}
\] (12)

All the other coefficients are zero. Again, in the extremes \( k_1, k_2 \ll 1/2 \) or \( k_1, k_2 \gg 1/2 \) a common factor \( k_1^3 \) can be pulled out of all the coefficients and the rest depends only on the ratio \( k_2/k_1 \). Accordingly, for both very small and very large wave numbers the behavior of the solitons is determined by this ratio, up to a scaling.

For minima and maxima of the amplitude \( \eta(x, t) \) at a given time \( t \) we have

\[
\sum_{n=1}^{5} \sum_{j=0}^{n} c_{n,j} p^j q^{n-j} = 0
\] (13)

expressing the vanishing of the first spatial derivative. Additionally, due to the definitions (7) we have

\[
q^{k_1}p^{-k_2} = e^{2\omega_1 t - \omega_2 t}.
\] (14)

Thus, extrema are given by the intersections of the graphs of Eqs.(13) and (14) on the \( q-p \) plane. The graph of Eq.(14) is a power function with a time-dependent coefficient. In fact, time appears only here. As for the graph of Eq.(13), we may prove some general properties on the basis of Eqs.(12), namely

1. Exchanging \( k_1 \) with \( k_2 \) and \( \omega_1 \) with \( \omega_2 \) is equivalent with exchanging \( p \) with \( q \). This follows directly from Eqs.(4)-(6).

2. The curves can have at most three intersections with a line \( p = \text{const. or } q = \text{const} \). Indeed, for a fixed \( p \) Eq. (13) is a third order polynomial of \( q \) and vice versa.

3. The transformation

\[
p \rightarrow (1 - a)\frac{1}{p}, \quad q \rightarrow (1 - a)\frac{1}{q}
\] (15)

leaves Eq.(13) invariant.

Indeed, direct substitution shows that under transformation (15) the expression

\[
\sum_{n=1}^{5} \sum_{j=0}^{n} c_{n,j} p^j q^{n-j}
\] goes over into

\[
- \left(1 - a\right)^3 \sum_{n=1}^{5} \sum_{j=0}^{n} c_{n,j} p^j q^{n-j}.
\] (16)

In view of the definition of \( p \) and \( q \), the symmetry (15) means that the two-soliton solution is invariant with respect to a simultaneous spatial and temporal reflection, with respect to suitably chosen origins. Explicitly, transformation (15) is equivalent with

\[
x \rightarrow 2x_0 - x, \quad t \rightarrow 2t_0 - t,
\] (17)
where
\[
\begin{align*}
x_0 &= \frac{1}{4} \frac{\omega_1 - \omega_2}{k_1 \omega_2 - k_2 \omega_1} \ln(1 - a), \\
f_0 &= \frac{1}{4} \frac{k_2 - k_1}{k_1 \omega_2 - k_2 \omega_1} \ln(1 - a).
\end{align*}
\] (18)

The symmetry manifests itself in the log-log plots of Fig. 1, Fig. 3, Fig. 5, Fig. 7, Fig. 9 as an inversion symmetry with respect to the symmetry point
\[ p = q = \sqrt{1 - a}. \] (19)

4. Although \( p = 0, q = 0 \) satisfies Eq. (13), in the first quadrant no curve starts from the origin. This is because for small \( p \) and \( q \) the first order terms dominate, and (for positive \( p \) and \( q \)) they are both negative.

5. For \( p \to \infty \) two asymptotes exist, namely, one at \( q = 0 \) and another one at \( q = 1 \). Similarly, for \( q \to \infty \) we have asymptotes at \( p = 0 \) and \( p = 1 \). This result can easily be obtained since for large values of \( p \) the terms containing the highest (third) power of \( p \) dominate, i.e.,
\[
k^3 p^3 q^3 - k^3 p^3 q = 0.
\] (20)

6. Near the above asymptotes the curve may be approximated by
\[
\begin{align*}
q &= \frac{k_3}{k_2^3} (1 - a) \frac{1}{a} \frac{1}{p} \\
1 - q &= \frac{k_3}{k_2} \left[ 8 - a \left( 4 + 6 \frac{k_1}{k_2} - \left( \frac{k_2}{k_1} \right)^3 \right) \right] \frac{1}{p} \\
p &= \frac{k_3}{k_2} (1 - a) \frac{1}{q} \\
1 - p &= \frac{k_3}{k_2} \left[ 8 - a \left( 4 + 6 \frac{k_1}{k_2} - \left( \frac{k_2}{k_1} \right)^3 \right) \right] \frac{1}{q}
\end{align*}
\] (21, 22, 23, 24)

This can be readily shown by taking into account the next-to-highest (second) power of the large variable.

The right hand sides of Eqs. (21), (23) are positive, thus the curve approaches the asymptote from above and from the right, respectively.\(^1\) The same is true for Eqs. (22), (24) in case of oppositely moving solitons \((\omega_1, \omega_2 < 0)\). In contrast, for solitons moving in the same direction \((\omega_1, \omega_2 > 0)\) the right hand sides of Eqs. (22), (24) can be both negative or positive, depending on the parameters \(k_1, k_2\).

The above properties allow us to explain the possible topologies of the curve (13).

![Figure 1](image1.png)

**Figure 1.** The zeros of Eq. (13) for solitons moving in the same direction. Left panel: Type I. case (at \( k_1 = 1, k_2 = 1.5 \)), right panel: Type II. case (at \( k_1 = 1, k_2 = 1.7 \)).

Under transformation (15) the point \( p = 1, q = \infty \) goes over into \( p = 1 - a, q = 0 \), similarly, the point \( p = \infty, q = 1 \) goes over into \( p = 0, q = 1 - a \). Finally, the point \( p = 0, q = \infty \) goes over into \( p = \infty, q = 0 \). This allows only three possible topologies, according to the fact that the point \( p = 1 - a, q = 0 \) should be continuously connected with either \( p = 0, q = 1 - a \) (Fig. 1 left panel), or \( p = \infty, q = 0 \) (Fig. 1 right panel), or \( p = 1, q = \infty \). This last possibility is visualized again by the right panel of Fig. 1 if \( p \) is exchanged with \( q \), which, according to property 1 above, is equivalent with exchanging \( k_1 \) with \( k_2 \). Since these curves are intersections of a smooth surface with a plain, other possibilities are ruled out. We shall

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\(^1\) Hence they lie in the first quadrant of the \((p, q)\) coordinate system.
call the topology shown in the left panel of Fig. 1 the Type I case, and the topology shown in the right panel of Fig. 1 the Type II case. The topology obtained from the Type II case via exchanging the axes will be called the Type $\Pi$ case. Note that the same transformation does not change the topology Type I.

While all the three situations do occur for solitons moving in the same direction, the Type I case never occurs for oppositely moving solitons. This can be shown by considering the intersections of the curve (13) with the $p = q$ line, i.e., the zeros of

$$\xi \left[ \xi^2 - (1 - a) \right] \left[ \xi^2 + b \xi + (1 - a) \right] .$$

where $\xi = p = q$ and

$$b = 2 - a \frac{(k_1 + k_2)^3}{k_1^2 + k_2^2} .$$

Polynomial (25) always has zeros at $\xi = 0$ and $\xi = \pm \sqrt{1 - a}$. The discriminant of the last quadratic factor is

$$D = b^2 - 4(1 - a) = \frac{a(k_1 + k_2)^6}{k_1^2 + k_2^2} \left[ a - 12k_1k_2(k_1^3 + k_2^3) \right] .$$

It is a simple exercise to show that the last factor on the right hand side of Eq.(27) is always negative in the case of oppositely moving solitons (cf. Eq.(6)). Hence, in that case only a single positive root exists, while the case Type I shown in the left panel of Fig. 1 requires three positive roots.

When changing the parameters $k_1, k_2$, the curves with different topologies go over into each other. If one considers solitons moving into the same direction, it is sufficient to consider the case $k_1 < k_2$, since the parameters must not coincide and the opposite case (i.e. $k_1 > k_2$) simply corresponds to the exchange of the axes. While the relative difference of parameters is sufficiently large, we have the topology Type II. If the relative difference is diminished, we get the topology Type I. The crossover between the two topologies if shown in left panel of Fig. 2.

One might wonder whether the middle branch of the curve touches both the other branches indeed at the same parameters. The answer is affirmative, and follows again from the symmetry (15). At the crossings of the branches one gets zero first derivatives in two independent directions, hence, one has simultaneously

$$f(p, q, k_1, k_2) = 0 ,$$
$$\frac{\partial f(p, q, k_1, k_2)}{\partial p} = 0 ,$$
$$\frac{\partial f(p, q, k_1, k_2)}{\partial q} = 0 .$$

where

$$f(p, q, k_1, k_2) = \sum_{n=1}^{5} \sum_{j=0}^{n} c_n p^n q^{n-j} .$$

Applying now the transformation (15) to a crossing point, it is straightforward to show that at the transformed point

$^2$ This is actually an intersection of a saddle with the tangent plane.
Eqs.(28)-(30) are also satisfied. Indeed, the two crossings go over into each other under transformation (15)\(^3\).

In case of oppositely moving solitons, \(k_1 = k_2\) is certainly possible, and a transition from Type II. to Type II. occurs exactly when the two parameters coincide (cf. right panel of Fig.2).

### 3. Time evolution of the extrema

Extrema of the waves are the intersection points of the graphs of Eqs.(13) and (14). As time goes on, the coefficient of the power function (14) changes from zero to infinity, and the corresponding curve\(^4\) “sweeps through” the curve (13). According to the possible shapes of curve (13) we observe different scenarios, according to the number of maxima. As shown in Figs.3 and 4, in the Type I. case we always get three intersections, i.e., there are always two maxima and a minimum between them.

\footnote{3} The transformation is obviously an involution, i.e., it is equal to its own inverse.

\footnote{4} On the log-log plots the graph of Eq.(14) is a straight line.

In contrast, in the Type II. and Type II. cases there are periods when only a single maximum exists. However, one has to distinguish here two subcases:

a) If the middle branch is steep enough, namely, if

\[
\frac{\partial \ln q}{\partial \ln p} > \frac{k_2}{k_1}
\]

at the symmetry point (19), then at certain times the two maxima reappear (Figs.5 and 6). This situation will be called Type II.a (or Type II.a).

\[\text{Figure 4. Type I. topology as seen in wave pattern for solitons moving in the same direction. Parameters and time instants identified by the colors are the same as in Fig.3.}\]

\[\text{Figure 5. Type II.a topology for intersections of the curve (13) and (14) at parameters } k_1 = 1.0 \text{ and } k_2 = 1.6 \text{ for solitons moving in the same direction. Different colors of the straight lines correspond to different time instants.}\]

In such a case the initially well separated solitons (two maxima) coalesce, the remnant of the smaller soliton being only a “drooping shoulder” at the front side. Further on, the “shoulder” moves towards the maximum, it rises and develops a second maximum. Thus a shallow valley is created on the top of the wave. At later times these events take place in reversed order: the rear bank of the valley goes down, the corresponding maximum disappears and becomes a drooping shoulder at the rear side, then it develops a maximum again behind the taller wave and the two solitons are again separated.

b) If the steepness of middle branch is smaller than \(k_2/k_1\), no valley is created on the top of the wave, as shown in
Figure 6. Type II.a topology as seen in wave pattern for solitons moving in the same direction. Parameters and time instants identified by the colors are the same as in Fig. 5.

Both Type II.a and Type II.b can be observed in case of solitons moving in the same direction. In contrast, for oppositely moving solitons, only the case Type II.b (or Type II.b) can exist. This can be proven on the basis of Eqs. (32), (19). Indeed, we have at the symmetry point

\[
\frac{\partial \ln q}{\partial \ln p} = \frac{4k_2^3(1-a) - a(k_2 - k_1)^3 + \sqrt{1-a}(4k_1^3 - a(k_1 + k_2)^3)}{4k_2^3(1-a) + a(k_2 - k_1)^3 + \sqrt{1-a}(4k_2^3 - a(k_1 + k_2)^3)},
\]

which is smaller than \(k_2/k_1\) for oppositely moving solitons, if \(k_2 \geq k_1\).

Figure 7. Type II.b topology for intersections of the curve (13) and (14) at parameters \(k_1 = 1.0\) and \(k_2 = 1.8\) for solitons moving in the same direction. Different colors of the straight lines correspond to different time instants.

Figure 9. Type II.b topology for intersections of the curve (13) and (14) at parameters \(k_1 = 1.0\) and \(k_2 = 1.5\) for solitons moving in opposite directions. Different colors of the straight lines correspond to different time instants.

Figure 10. Type II.b topology as seen in wave pattern for solitons moving in opposite directions. Parameters and time instants identified by the colors are the same as in Fig. 9.
4. Parameter space

The scenarios described above are summarized in parameter space in Figs. 11 and 12.

For solitons moving in the same direction, at the border between Type II.a and Type II.b in parameter space (see Fig. 11) the expression (33) is equal to $k_2/k_1$. This condition defines the border. As noted after Eq.(12), for both $k_1 \ll 1$ and $k_2 \gg 1$ the expression (33) depends only on the ratio $k_2/k_1$, hence the border looks linear. In fact, its slope slightly differs for small and large $k$ values.

As discussed above, the border between Type I. and Type II.a in parameter space is given by Eqs.(28)-(31). Again, the border is not exactly a straight line, its slope is slightly different for large and small $k$ values.

The shaded and the unshaded regions in Fig. 11 are obtained by exchanging $k_1$ and $k_2$.

As noted before, the $k_1 = k_2$ line is not allowed. For solitons moving in opposite directions, the parameter space is even simpler (see Fig. 12). All parameter values are allowed, and the crossover from Type II.b to Type II.b occurs at $k_1 = k_2$.

5. Nearly identical solitons

Let us consider now the situation when two nearly identical solitons interact, i.e., two solitons moving in the same direction with almost the same speed and thus having almost the same amplitude, but which are spatially separated. If they were strictly identical (i.e., $k_2 = k_1$ and $\omega_2 = \omega_1$), we would obtain a single soliton solution rather than a two-soliton solution. Hence approaching this limit can be interesting. This situation corresponds to case I.

Maxima correspond to intersections of the line (14) with the two outer segments of the graph of (13). Obviously, as time goes on, the initially well separated solitons, while propagating in the same direction, get closer to each other, i.e., the distance between the two maxima decreases, then, without coalescing, their distance grows again. The minimal distance between the maxima may be estimated (cf. Fig.(13)) as

$$\delta x_{\text{min}} \geq -\frac{1}{k_1 + k_2} \ln(1 - \alpha) = \frac{1}{k_1 + k_2} \ln \left( \frac{(k_1 + k_2)^2 + \frac{1}{12} \left(\frac{\omega_1}{k_1} - \frac{\omega_2}{k_2}\right)^2}{(k_1 - k_2)^2 + \frac{1}{12} \left(\frac{\omega_1}{k_1} - \frac{\omega_2}{k_2}\right)^2} \right).$$

If one considers now the corresponding waves (Fig.14), initially one sees two very similar solitons, the (slightly) taller one chasing the smaller one. When the taller soliton gets closer to the smaller one, the taller soliton gradually

Evidently, $\delta x_{\text{min}}$ diverges logarithmically as $k_1 \to k_2$. Hence, in that limit one soliton is always staying at the infinity, and one observes only the other one.
loses its height and speed, at the same time, the smaller soliton gains in height and speed, the distance between them starts increasing, and eventually, we see the original solitons again, but this time the smaller one chasing and the taller one escaping.

Finally, let us note that here we present only a phenomenological description. The understanding of the underlying mechanism, if it is possible to find one in physical terms beyond the backward scattering transform, could be very interesting.

6. Summary and discussion

A simple classification scheme of the two soliton solutions of the Boussinesq equation have been presented. The scheme is based on the behavior of local maxima of the wave. We have shown that for solitons moving in the same direction there can be three different scenarios. In the Type I. case (see Figs. 3 and 4) there are two maxima all the time, separated by a minimum. In the Type IIa. case (see Figs. 5 and 6) initially, when the solitons are still separated, there are two maxima with a minimum in between. During the collision the two solitons merge and only one maximum remains, the remnant of the other shows up only as a shoulder. Later on, however, the second maximum reappears and grows. Then the first maximum disappears for a while, but as the solitons become separated, it reappears and we have the initial solitons in a reversed ordering along the line. In the Type IIb. case (see Figs. 7 and 8) the separated solitons merge to a wave having a single maximum, and later on a second maximum reappears and the solitons separate again. For solitons moving in opposite directions only the Type IIb. case exists (see Figs. 9 and 10). In that case the wave numbers \( k_1 \) and \( k_2 \) may coincide (see Figs. 15 and 16). In contrast, for solitons moving in the same direction \( k_1 \) and \( k_2 \) must be different. If \( k_1 \to k_2 \) (see Figs. 13 and 14) we have an extreme Type I. case, namely, there remains a large minimal distance between the solitons all the time, and the chasing soliton loses in height, while the escaping soliton gains in height during the collision. As a result, eventually the two solitons change their ordering along the line, without any close contact.

Since our result are based on the small amplitude approximation, we expect that they should be observable in that limit. Also, the results for solitons moving in the same
direction should follow from the Korteweg-de Vries equation as well, since that equation is obtained in the same approximation. The derivation of both the Korteweg-de Vries equation and the Boussinesq equation assumes irrotational flow of an incompressible, ideal fluid. The flow is two-dimensional, with a vertical and one horizontal direction (i.e., with no dependence on the other horizontal coordinate). The fluid layer has infinite extension in the horizontal direction, and constant equilibrium depth. One considers surface gravitational waves with small amplitude compared to the fluid depth (meaning $\epsilon \ll 1$) and with long characteristic horizontal length scale compared to the depth (meaning $\delta \ll 1$). Note that this latter assumption may be relaxed and also, that the effect of surface tension may be included [19]. In this setup, one considers the next-to-leading order in $\epsilon \propto \delta^2$ of the governing equations. If one considers waves travelling in only one direction one obtains a first order wave equation $u_t + cu_x = 0$ in zeroth order and the Korteweg-de Vries equation in first order. Without this additional assumption, one gets in first order the Boussinesq equation. Hence, if applied to waves travelling in only one direction both equations should lead to the same physical results. Indeed, as mentioned in the introduction, Lax [22] got the same classification as we did for solitons travelling in the same direction. The condition of the different cases are, however, different. In [22], the condition was expressed solely in terms of the ratio of the initially well separated soliton velocities. In our case, such a simple expression was not available, except in the long wavelength limit, when the borders between the different cases depends only on $k_2/k_1 = \sqrt{(c_2^2 - 1)/(c_1^2 - 1)} \approx \sqrt{(c_2 - 1)/(c_1 - 1)}$. Note that Lax used a standard form of the Korteweg-de Vries equation, namely, $u_t + uu_x + u_{xxx} = 0$. This corresponds to a coordinate system which moves with the velocity of the linear shallow water waves, hence, his soliton velocities correspond to ours minus unity.

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