ON PROPER HOLOMORPHIC MAPPINGS BETWEEN TWO EQUIDIMENSIONAL FBH-TYPE DOMAINS

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Abstract. We introduce a new class of domains $D_{n,m}(\mu, p)$, called FBH-type domains, in $\mathbb{C}^n \times \mathbb{C}^m$, where $0 < \mu \in \mathbb{R}$ and $p \in \mathbb{N}$. In the special case of $p = 1$, these domains are just the Fock–Bargmann–Hartogs domains $D_{n,m}(\mu)$ in $\mathbb{C}^n \times \mathbb{C}^m$ introduced by Yamamori. In this paper we obtain a complete description of an arbitrarily given proper holomorphic mapping between two equidimensional FBH-type domains. In particular, we prove that the holomorphic automorphism group $\text{Aut}(D_{n,m}(\mu, p))$ of any FBH-type domain $D_{n,m}(\mu, p)$ with $p \neq 1$ is a Lie group isomorphic to the compact connected Lie group $U(n) \times U(m)$. This tells us that the structure of $\text{Aut}(D_{n,m}(\mu, p))$ with $p \neq 1$ is essentially different from that of $\text{Aut}(D_{n,m}(\mu))$.

1. Introduction

For any positive integers $n, m$ and $p$ and any positive real number $\mu$, we define a domain $D_{n,m}(\mu, p)$ in $\mathbb{C}^N$ by

$$D_{n,m}(\mu, p) = \{(z, w) \in \mathbb{C}^N \mid \|w\|^2 < e^{-\mu\|z\|^{2p}}\},$$

where $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ with $N = n + m$. This is an unbounded pseudoconvex domain in $\mathbb{C}^N$ with real analytic boundary. Since the complex Euclidean space $\mathbb{C}^n$ is now embedded in $D_{n,m}(\mu, p)$ in the canonical manner, it is not hyperbolic in the sense of Kobayashi [9]. In the special case where $p = 1$, this domain reduces to the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ introduced by Yamamori [24]. For this reason, we would like to call $D_{n,m}(\mu, p)$ an FBH-type domain in $\mathbb{C}^N$.

The Fock–Bargmann–Hartogs domains $D_{n,m}(\mu) = D_{n,m}(\mu, 1)$ have been studied from various points of view. For example, Kim, Ninh and Yamamori [8] studied the structure of the holomorphic automorphism group $\text{Aut}(D_{n,m}(\mu))$ of $D_{n,m}(\mu)$ and showed that $\text{Aut}(D_{n,m}(\mu))$ has the structure of a non-compact connected Lie group of dimension $n^2 + m^2 + 2n$. For two equidimensional Fock–Bargmann–Hartogs domains $D_{n,m}(\mu)$ and $D_{k,\ell}(\nu)$, Tu and Wang [23] proved that every proper holomorphic mapping $f : D_{n,m}(\mu) \to D_{k,\ell}(\nu)$ is necessarily a biholomorphic mapping, provided that $m \geq 2$. After that, Kodama [14, Theorem 2] gave an

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alternative proof of this fact. Moreover, in a recent paper [16] we determined completely the structure of the space consisting of all proper holomorphic mappings from \( D_{n,m}(\mu) \) to \( D_{k,e}(v) \). In view of these, in this paper we shall restrict ourselves to the case where \((p, q) \neq (1, 1)\) and we would like to study the following.

**Questions.** For two equidimensional FBH-type domains \( D_{n,m}(\mu, p) \) and \( D_{k,e}(v, q) \), under what conditions does there exist a proper holomorphic mapping from \( D_{n,m}(\mu, p) \) to \( D_{k,e}(v, q) \)? And further, if there exists such a mapping, can we give a complete description of all such mappings? In particular, can we determine the structure of the holomorphic automorphism group \( \text{Aut}(D_{n,m}(\mu, p)) \)?

Anyway we would like to ask: What happens when the Fock–Bargmann–Hartogs domains \( D_{n,m}(\mu) \) are replaced by the FBH-type domains \( D_{n,m}(\mu, p) \) with \( p \neq 1 \) in [16, Theorem 1]? The answer is something remarkable, as stated below.

The main purpose of this paper is to give affirmative answers to the questions above. First of all, we can verify the following fundamental fact on the existence of proper holomorphic automorphism group \( \text{Aut}(D_{n,m}(\mu, p)) \).

**Theorem 1.** Let \( D_1 = D_{n_1,m_1}(\mu_1, p_1) \) and \( D_2 = D_{n_2,m_2}(\mu_2, p_2) \) be two equidimensional FBH-type domains in \( \mathbb{C}^N \). Assume that there exists a proper holomorphic mapping \( f : D_1 \to D_2 \). Then we have

\[
(n_1, m_1) = (n_2, m_2), \quad f(\Delta_{D_1}) = \Delta_{D_2} \quad \text{and} \quad f(D_j^*) = D_j^*,
\]

where \( \Delta_{D_j} = \{ (z_j, w_j) \in D_j \mid w_j = 0 \} \cong \mathbb{C}^{n_j} \) and \( D_j^* = D_j \setminus \Delta_{D_j} \) for \( j = 1, 2 \).

Moreover, both the restrictions

\[
f|_{\Delta_{D_1}} : \Delta_{D_1} \to \Delta_{D_2} \quad \text{and} \quad f|_{D_j^*} : D_j^* \to D_j^*
\]

are also proper holomorphic mappings.

By making use of this, we first prove the following theorem, which tells us that the structure of the holomorphic automorphism group of the FBH-type domain \( D_{n,m}(\mu, p) \) with \( p \neq 1 \) is essentially different from that of the Fock–Bargmann–Hartogs domain \( D_{n,m}(\mu) = D_{n,m}(\mu, 1) \).

**Theorem 2.** Let \( D_{n,m}(\mu, p) \) be an FBH-type domain in \( \mathbb{C}^N \) with \( p \neq 1 \). Then every holomorphic automorphism \( f \) of \( D_{n,m}(\mu, p) \) can be written in the form

\[
f(z, w) = (Az, Bw) \quad \text{for} \quad (z, w) \in D_{n,m}(\mu, p),
\]

where \( A \in U(n) \) and \( B \in U(m) \). In particular, \( \text{Aut}(D_{n,m}(\mu, p)) \) is a Lie group isomorphic to the compact connected Lie group \( U(n) \times U(m) \).

Together with a result in [8] (see Fact D in the next section), Theorem 2 gives us an explicit expression of any holomorphic automorphism of the FBH-type domains \( D_{n,m}(\mu, p) \).

Since the holomorphic automorphism group of every Fock–Bargmann–Hartogs domain is non-compact, we obtain the following characterization of the Fock–Bargmann–Hartogs domains among all the FBH-type domains.

**Corollary.** Let \( D \) be an FBH-type domain in \( \mathbb{C}^N \). Then \( D \) is a Fock–Bargmann–Hartogs domain if and only if \( \text{Aut}(D) \) is non-compact.
Now let us consider the questions concerning proper holomorphic mappings between two equidimensional FBH-type domains $D_{n,m}(\mu, p)$ and $D_{k,\ell}(v, q)$ in $\mathbb{C}^N$. Then, thanks to Theorem 1, we may assume that $(n, m) = (k, \ell)$ from the beginning. Under this assumption, we can establish the following theorems. For the convenience of the reader, we shall split our result into two theorems according to $m = 1$ or $m \geq 2$.

**Theorem 3.** Let $D_{n,1}(\mu, p)$ and $D_{n,1}(v, q)$ be two equidimensional FBH-type domains in $\mathbb{C}^{n+1}$ with $(p, q) \neq (1, 1)$. Then we have the following cases.

**Case I:** $n = 1$. In this case, there exists a proper holomorphic mapping from $D_{1,1}(\mu, p)$ to $D_{1,1}(v, q)$ if and only if $p = q$.

Moreover, by putting $r = p/q$, every proper holomorphic mapping $f : D_{1,1}(\mu, p) \to D_{1,1}(v, q)$ can be expressed as follows:

If $q = 1$, then $r = p \geq 2$ and

$$f(z, w) = (\sqrt{k}\sqrt{\mu/v}Az^p + \gamma/\sqrt{v}, e^{-v(\sqrt{k}\sqrt{\mu/v}Az^p, v/\sqrt{v})-(v/2)|\gamma/\sqrt{v}|^2}Bw^k)$$

for $(z, w) \in D_{1,1}(\mu, p)$, where $k \in \mathbb{N}, \gamma \in \mathbb{C}$ and $A \in U(1), B \in U(1)$. In particular, $f$ is not a biholomorphic mapping.

If $q \geq 2$, then $p \geq 2$ and

$$f(z, w) = (\sqrt{k}\sqrt{\mu/v}Az^p, Bw^k)$$

for $(z, w) \in D_{1,1}(\mu, p)$, where $k \in \mathbb{N}$ and $A \in U(1), B \in U(1)$. In particular, $f$ is a biholomorphic mapping if and only if $p = q$ and $k = 1$.

**Case II:** $n \geq 2$. In this case, there exists a proper holomorphic mapping from $D_{n,1}(\mu, p)$ to $D_{n,1}(v, q)$ if and only if $p = q$.

Moreover, every proper holomorphic mapping $f : D_{n,1}(\mu, p) \to D_{n,1}(v, q)$ has the form

$$f(z, w) = (\sqrt{k}\sqrt{\mu/v}Az^p, Bw^k)$$

for $(z, w) \in D_{n,1}(\mu, p)$, where $k \in \mathbb{N}$ and $A \in U(n), B \in U(1)$. In particular, $f$ is a biholomorphic mapping if and only if $k = 1$.

**Theorem 4.** Let $D_{n,m}(\mu, p)$ and $D_{n,m}(v, q)$ be two equidimensional FBH-type domains in $\mathbb{C}^N$ with $m \geq 2$ and $(p, q) \neq (1, 1)$. Then we have the following cases.

**Case I:** $n = 1$. In this case, there exists a proper holomorphic mapping from $D_{1,m}(\mu, p)$ to $D_{1,m}(v, q)$ if and only if $p/q \in \mathbb{N}$.

Moreover, by putting $r = p/q$, every proper holomorphic mapping $f : D_{1,m}(\mu, p) \to D_{1,m}(v, q)$ can be expressed as follows:

If $q = 1$, then $r = p \geq 2$ and

$$f(z, w) = (\sqrt{\mu/v}Az^p + \gamma/\sqrt{v}, e^{-v(\sqrt{\mu/v}Az^p, v/\sqrt{v})-(v/2)|\gamma/\sqrt{v}|^2}Bw)$$

for $(z, w) \in D_{1,m}(\mu, p)$, where $\gamma \in \mathbb{C}$ and $A \in U(1), B \in U(m)$. In particular, $f$ is never a biholomorphic mapping.

If $q \geq 2$, then $p \geq 2$ and

$$f(z, w) = (\sqrt{\mu/v}Az^p, Bw)$$

for $(z, w) \in D_{1,m}(\mu, p)$.
where \( A \in U(n) \) and \( B \in U(m) \). In particular, \( f \) is a biholomorphic mapping if and only if \( p = q \).

Case II: \( n \geq 2 \). In this case, there exists a proper holomorphic mapping from \( D_{n,m}(\mu, p) \) to \( D_{n,m}(\nu, q) \) if and only if \( p = q \).

Moreover, every proper holomorphic mapping \( f : D_{n,m}(\mu, p) \to D_{n,m}(\nu, q) \) has the form

\[
  f(z, w) = \left( \sqrt[2q]{\mu/\nu} A z, B w \right)
\]

for \((z, w) \in D_{n,m}(\mu, p)\), where \( A \in U(n) \) and \( B \in U(m) \). In particular, \( f \) is always a biholomorphic mapping.

Combining these with a result in [16, Theorem 1] (in which the structure of all proper holomorphic mappings between two equidimensional Fock–Bargmann–Hartogs domains was clarified), we obtain a complete description of an arbitrarily given proper holomorphic mapping between two equidimensional FBH-type domains. Therefore we obtain affirmative answers to our questions.

Finally, the following fact should be mentioned. Namely, in the previous paper [15] we obtained the following characterization of the Fock–Bargmann–Hartogs domain \( D_{n,m}(\mu) \) from the viewpoint of the holomorphic automorphism group: Let \( M \) be a connected Stein manifold of dimension \( N \) and let \( D_{n,m}(\mu) \) be a Fock–Bargmann–Hartogs domain in \( \mathbb{C}^N \) with \( N = n + m \). Assume that \( m \geq 2 \) and the identity component of \( \text{Aut}(M) \) is isomorphic to \( \text{Aut}(D_{n,m}(\mu)) \) as topological groups. Then \( M \) is biholomorphically equivalent to \( D_{n,m}(\mu) \). However, there is no analogue to this for the FBH-type domains, in general. In fact, by our theorems we obtain many examples of pairs \((D, D')\) consisting of FBH-type domains \( D \) and \( D' \) in \( \mathbb{C}^n \times \mathbb{C}^m \) for any \( n, m \geq 1 \) such that \( D \) is not biholomorphically equivalent to \( D' \), and further, both the holomorphic automorphism groups \( \text{Aut}(D) \) and \( \text{Aut}(D') \) are isomorphic to \( U(n) \times U(m) \) as Lie groups. Of course, being pseudoconvex domains with real analytic boundaries, the FBH-type domains are Stein manifolds.

After some preparations in Section 2, we prove our Theorems 1, 2, 3 and 4 in Sections 3, 4, 5 and 6, respectively.

**Notation.** Throughout this paper we use the following notation. For a given \( n \in \mathbb{N} \) and a subset \( S \) of \( \mathbb{C}^n \), we denote by

- \( U(n) \) the unitary group of degree \( n \),
- \( \langle \cdot, \cdot \rangle \) (respectively \( \| \cdot \| \)) the standard Hermitian inner product (respectively its associated Euclidean norm) on \( \mathbb{C}^n \),
- \( B^n = \{ \zeta \in \mathbb{C}^n \mid \| \zeta \| < 1 \} \) the open unit ball in \( \mathbb{C}^n \), and
- \( \partial S \) (respectively \( \overline{S} \)) the boundary (respectively closure) of \( S \) in \( \mathbb{C}^n \).

Let \( D \) be a domain in \( \mathbb{C}^n \) and \( f : D \to \mathbb{C}^n \) a holomorphic mapping. Then we denote by

- \( \text{Aut}(D) \) the group of all holomorphic automorphisms of \( D \) equipped with the compact-open topology, and thus \( \text{Aut}(D) \) is a Hausdorff space satisfying the second axiom of countability,
- \( f|_S : S \to \mathbb{C}^n \) the restriction of \( f \) to \( S \), where \( S \) is a subset of \( D \).
• \( J_f(\zeta) \) the complex Jacobian determinant of \( f \) at \( \zeta \in D \), and
• \( V_f = \{ \zeta \in D \mid J_f(\zeta) = 0 \} \).

2. Preliminaries

Given the FBH-type domain

\[
D_{n,m}(\mu, s) = \{(z, w) \in \mathbb{C}^N \mid \|w\|^2 < e^{-\mu \|z\|^2s}\}
\]

in \( \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m \), let us define a non-singular linear transformation \( L_{\mu,s} \) of \( \mathbb{C}^N \) by setting

\[
L_{\mu,s}(z, w) = (\sqrt[2]{\mu}z, w) \quad \text{for} \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N.
\]

(2.1)

Then it is obvious that \( L_{\mu,s} \) induces a linear equivalence between the FBH-type domains \( D_{n,m}(\mu, s) \) and \( D_{n,m}(1, s) \). Taking this into account, in this section we study the structure of \( D_{n,m}(1, s) \) more closely. We set for a while

\[
D = D_{n,m}(1, s), \quad \Delta_D = \{(z, w) \in D \mid w = 0\} = \mathbb{C}^n \quad \text{and} \quad D^* = D \setminus \Delta_D.
\]

Also, for the given points \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), \( w = (w_1, \ldots, w_m) \in \mathbb{C}^m \), we set

\[
\zeta = (\zeta_1, \ldots, \zeta_N) = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N
\]

and write as usual

\[
\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N} \quad \text{for} \quad \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N, \quad \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_{\geq 0}^N.
\]

where \( \mathbb{Z}_{\geq 0} \) denotes the set of all non-negative integers.

Let \( A^2(D) \) be the Hilbert space consisting of all square integrable holomorphic functions on \( D \) with the inner product

\[
\langle f, g \rangle = \int_D f(\zeta) \overline{g(\zeta)} \, dV(\zeta) \quad \text{for} \quad f, g \in A^2(D),
\]

where \( dV(\zeta) \) is the Lebesgue measure on \( \mathbb{C}^N \). Then, by direct calculations, we can see that

\[
\mathfrak{M} := \{ \zeta^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^N \} \subset A^2(D) \quad \text{and} \quad \langle \zeta^\alpha, \zeta^\beta \rangle = 0 \text{ if } \alpha \neq \beta.
\]

(2.2)

Moreover, since \( D \) is a Reinhardt domain in \( \mathbb{C}^N \) containing the origin 0, every holomorphic function on \( D \) can be expanded uniquely as a power series around 0, which converges absolutely and uniformly on compact subsets of \( D \). Hence, the set \( \mathfrak{M} \) of all monomials \( \zeta^\alpha \) forms a complete orthogonal system for \( A^2(D) \); consequently, the Bergman kernel function \( K_D \) for \( D \) can be expressed as

\[
K_D(\zeta, \eta) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^N} c_\alpha \zeta^\alpha \overline{\eta^\alpha} \quad \text{for} \quad \zeta, \eta \in D
\]

(2.3)

with \( c_\alpha = \|\zeta^\alpha\|^{-2} > 0 \) for each \( \alpha \in \mathbb{Z}_{\geq 0}^N \). For the given \( r = (r_1, \ldots, r_N) \in \mathbb{R}_{>0}^N \) and \( \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N \), we set

\[
r \cdot \zeta := (r_1 \zeta_1, \ldots, r_N \zeta_N), \quad 1/r := (1/r_1, \ldots, 1/r_N).
\]
It then follows from (2.3) that, for \( r, t \in \mathbb{R}^N_{>0} \) and \( \zeta, \eta \in \mathbb{C}^N \),

\[
K_D(r \cdot \zeta, (1/r) \cdot \eta) = K_D(t \cdot \zeta, (1/t) \cdot \eta)
\]

whenever \( r \cdot \zeta, t \cdot \zeta, (1/r) \cdot \eta, (1/t) \cdot \eta \in D \); hence, for any points \( \zeta, \eta \in D \),

\[
K_D(r \cdot \zeta, (1/r) \cdot \eta) = K_D(\zeta, \eta) \quad \text{if} \quad r \cdot \zeta, (1/r) \cdot \eta \in D.
\]

Therefore, the same procedure as used in the proofs of Tu and Wang [23, Theorems 2.3 and 2.5] can be applied to obtain the following (see also [2, 4, 13, 18, 22]): For each compact subset \( E \) of \( D \), there exists an open neighborhood \( O = O(E) \) of \( \overline{D} \) such that \( K_D(\zeta, \eta) \) extends to be holomorphic on \( O \) as a function of \( \zeta \) for each \( \eta \in E \). Thanks to this fact together with Bell’s transformation rule for Bergman kernels under proper holomorphic mappings, we obtain the following fact.

**Fact A.** Let \( D' \) be another FBH-type domain in \( \mathbb{C}^N \) and let \( f : D \rightarrow D' \) be a proper holomorphic mapping. Then \( f \) extends holomorphically to an open neighborhood \( W \) of \( \overline{D} \).

Notice that the mapping \( f \) (extended to \( W \)) in Fact A is locally biholomorphic near almost all points of \( \partial D \), since \( \{ \zeta \in W \mid J_f(\zeta) = 0 \} \cap \partial D \) is at most a nowhere dense closed subset of \( \partial D \). This fact will be used later.

Next we define a real analytic function \( \rho \) on \( \mathbb{C}^N \) by

\[
\rho(\zeta) = -1 + \|w\|^2 e^{\|z\|^2} \quad \text{for} \quad \zeta = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N,
\]

where \( 2 \leq s \in \mathbb{N} \). It then follows that \( \rho \) gives a global defining function for the FBH-type domain \( D \). Consider here the complex Hessian form

\[
H_\rho(\zeta; t) = \sum_{i,j=1}^N \frac{\partial^2 \rho(\zeta)}{\partial \zeta_i \partial \zeta_j} t_i t_j \quad \text{for} \quad t = (t_1, \ldots, t_N) \in \mathbb{C}^N
\]

of \( \rho \) at \( \zeta \in \mathbb{C}^N \). Then, for any point \( \zeta_0 = (a, b) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N \), we have

\[
H_\rho(\zeta_0; t) = e^{\|a\|^2 s} \left\{ s^2 \|a\|^{2(s-1)} \|b\|^2 \|\langle a, u \rangle\|^2 + s(s - 1)\|a\|^{2(s-2)}\|b\|^2 \|\langle a, u \rangle\|^2 \right.
\]

\[
+ s\|a\|^{2(s-1)}\|b\|^2 \|\langle a, u \rangle\| v + \|v\|^2
\]

\[
\left. - 2s\|a\|^{2(s-1)}\|b\| \|\langle a, u \rangle\| v + \|v\|^2
\right.
\]

\[
\geq e^{\|a\|^2 s} \left\{ s\|a\|^{2(s-1)} \|b\|^2 \|\langle a, u \rangle\| v + \|v\|^2
\right.
\]

\[
+ s(s - 1)\|a\|^{2(s-2)}\|b\|^2 \|\langle a, u \rangle\| v + \|v\|^2
\]

\[
\leq e^{\|a\|^2 s} \left\{ s\|a\|^{2(s-1)} \|b\|^2 \|\langle a, u \rangle\| v + \|v\|^2
\right.
\]

\[
+ s(s - 1)\|a\|^{2(s-2)}\|b\|^2 \|\langle a, u \rangle\| v + \|v\|^2
\]

\[
\geq 0
\]

for all \( t = (u, v) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N \) by Schwarz’s inequality. Thus \( \rho \) is plurisubharmonic on \( \mathbb{C}^N \) and moreover it is strongly plurisubharmonic on \( \{ (z, w) \in \mathbb{C}^N \mid \|z\| \|w\| \neq 0 \} \). On the other hand, the complex tangent space \( T^c_\zeta(\partial D) \) to \( \partial D \) at \( \zeta = (a, b) \in \partial D \) is given by

\[
T^c_\zeta(\partial D) = \{ t = (u, v) \in \mathbb{C}^n \times \mathbb{C}^m \mid s\|a\|^{2(s-1)}\|b\|^2 \|\langle a, u \rangle\| + \langle v, b \rangle = 0 \}.
\]

Thus, for every point \( \zeta_0 \in \partial D \) of the form \( \zeta_0 = (0, b) \in \mathbb{C}^n \times \mathbb{C}^m \), we have

\[
H_\rho(\zeta_0; t) = 0 \quad \text{for} \quad t = (u, 0) \in \mathbb{C}^n \times \{0\} \subset T^c_\zeta(\partial D).
\]

Consequently, \( \zeta_0 \) is a weakly (not strictly) pseudoconvex boundary point of \( D \). Summarizing the above, we obtain the following fact.
Fact B. Let \( D = D_{n,m}(1, s) \) be the FBH-type domain in \( \mathbb{C}^N \) with \( 2 \leq s \leq \mathbb{N} \). Let \( S(D) \) be the set of all strictly pseudoconvex boundary points of \( D \) and \( \mathcal{W}(D) \) the set of all weakly (not strictly) pseudoconvex boundary points of \( D \). Then we have

\[
S(D) = \{ (z, w) \in \partial D \mid z \neq 0 \} \quad \text{and} \quad \mathcal{W}(D) = \{ (z, w) \in \partial D \mid z = 0 \}.
\]

For a given \( s \in \mathbb{N} \), let us define the domains \( E_s \) and \( \mathcal{E}_s \) in \( \mathbb{C}^{n+1} \) by setting

\[
E_s = \{ (u, v) \in \mathbb{C} \times \mathbb{C}^n \mid \text{Im} \, u - \| v \|^2 > 0 \},
\]

\[
\mathcal{E}_s = \{ (\xi, \eta) \in \mathbb{C} \times \mathbb{C}^n \mid ||\xi||^2 + \| \eta \|^2 < 1 \}.
\]

(2.4)

For later purpose, here we wish to give the precise description of automorphisms \( \phi \) of \( \mathcal{E}_s \). To this end, notice that the correspondence \( \phi_s : \mathcal{E}_s \rightarrow \mathbb{C}^{n+1} \) given by

\[
\phi_s(u, v) = \left( \frac{u - i}{u + i}, \frac{2}{u + i} \right)^{1/s} \quad \text{for} \quad (u, v) \in \mathcal{E}_s
\]

induces a biholomorphic mapping from \( \mathcal{E}_s \) onto \( E_s \) with the inverse

\[
\phi_s^{-1}(\xi, \eta) = \left( \frac{1 + \xi}{1 - \xi}, \frac{1}{1 - \xi} \right)^{1/s} \quad \text{for} \quad (\xi, \eta) \in E_s.
\]

Moreover, if \( s = 1 \), it is well known that every automorphism of \( \mathcal{E}_1 = B^{n+1} \) is given as a linear fractional transformation described in terms of some element of \( SU(n + 1, 1) \). And, if \( s \neq 1 \), we know by [12] that every element \( \widehat{\phi} \) in \( \text{Auto}(E_s) \) can be written in the form

\[
\widehat{\phi}(\xi, \eta) = \left( \frac{\lambda \xi - \alpha}{1 - \alpha \xi}, \rho(\xi)^{1/2} U \eta \right) \quad \text{for} \quad (\xi, \eta) \in E_s,
\]

(2.6)

where \( |\lambda| = 1, |\alpha| < 1, U \in U(n) \) and \( \rho(\xi) \) is a nowhere vanishing holomorphic function on the unit disc \( \Delta \) in \( \mathbb{C} \) defined by

\[
\rho(\xi) = \frac{1 - |\alpha|^2}{(1 - \alpha \xi)^2} \quad \text{for} \quad \xi \in \Delta.
\]

Thus one may obtain the explicit form of any element \( \phi \) in \( \text{Auto}(\mathcal{E}_s) = \phi_s^{-1} \text{Auto}(E_s) \phi_s \). In fact, by expressing \( \phi = (\phi_0, \phi_1, \ldots, \phi_n) \) with respect to the coordinate system \( (u, v) = (u, v_1, \ldots, v_n) \) in \( \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1} \), the following fact can be verified.

Fact C. Every automorphism \( \phi \) of \( \mathcal{E}_s \) can be written in the following form.

1. If \( s = 1 \), then

\[
\phi_j(u, v) = \frac{\alpha_{0j}u + \sum_{i=1}^{n} \alpha_{ij} v_j + \beta_i}{\gamma_{0j}u + \sum_{i=1}^{n} \gamma_{ij} v_j + \delta}, \quad 0 \leq i \leq n,
\]

(2.7)

where all the coefficients \( \alpha_{ij}, \beta_i, \gamma_i \) \( (0 \leq i, j \leq n) \) and \( \delta \) are suitable complex constants. Moreover, if \( \phi \) is an affine automorphism of \( \mathcal{E}_1 \), that is, a non-singular affine transformation of \( \mathbb{C}^{n+1} \) leaving \( \mathcal{E}_1 \) invariant, then

\[
\phi(u, v) = (ku + a + 2i \langle Bv, b \rangle + i\|b\|^2, Bv + b)
\]

(2.8)

for \( (u, v) \in \mathcal{E}_1 \), where \( a \in \mathbb{R}, b \in \mathbb{C}^n \) and \( 0 < k \in \mathbb{R}, B \in \text{GL}(n, \mathbb{C}) \) with \( k\|v\|^2 = \|Bv\|^2 \) for all \( v \in \mathbb{C}^n \) or \( (1/\sqrt{k})B \in U(n) \).
Proof. Let us consider the plurisubharmonic function \( \zeta \) so assume that there exists a point \( h \). Once it is shown that \( h \) gives rise to a continuous plurisubharmonic function on \( V \) and

\[
\varphi(u, v) = \left( \frac{i A_{\lambda, \alpha} u + i B_{\lambda, \alpha}}{C_{\lambda, \alpha} u + i D_{\lambda, \alpha}}, \left( \frac{4(|\alpha|^2 - 1)}{(C_{\lambda, \alpha} u + i D_{\lambda, \alpha})^2} \right)^{1/2} \right) U v
\]  

for \( (u, v) \in E, \) where \(|\lambda| = 1, |\alpha| < 1, U \in U(n)\) and \( A_{\lambda, \alpha}, B_{\lambda, \alpha}, C_{\lambda, \alpha}, D_{\lambda, \alpha} \) are complex constants given by

\[
\begin{pmatrix}
A_{\lambda, \alpha} & B_{\lambda, \alpha} \\
C_{\lambda, \alpha} & D_{\lambda, \alpha}
\end{pmatrix}
= \begin{pmatrix}
1 - \lambda \alpha + \lambda - \bar{\alpha} & 1 - \lambda \alpha - \lambda + \bar{\alpha} \\
1 + \lambda \alpha - \lambda - \bar{\alpha} & 1 + \lambda \alpha + \lambda + \bar{\alpha}
\end{pmatrix}.
\]

(For the precise description of automorphisms of \( \mathcal{E}_1 \), see [10, Section 3]. Also, for the affine automorphisms of \( \mathcal{E}_1 \), see [20, Section 2].)

We finish this section by the following fact on the holomorphic automorphism group of the Fock–Bargmann–Hartogs domains \( D_{n,m}(\mu) = D_{n,m}(\mu, 1) \) due to Kim, Ninh and Yamamori [8, Theorem 10].

Fact D. Let \( D = D_{n,m}(\mu) \) be a Fock–Bargmann–Hartogs domain in \( \mathbb{C}^N \). Then the automorphism group \( \text{Aut}(D) \) is generated by the following mappings:

\[
\begin{align*}
\varphi_A : (z, w) &\mapsto (Az, w), \quad A \in U(n), \\
\varphi_B : (z, w) &\mapsto (z, Bw), \quad B \in U(m), \\
\varphi_\gamma : (z, w) &\mapsto (z + \gamma, e^{-\mu(z, \gamma) - (\mu/2)\|\gamma\|^2} w), \quad \gamma \in \mathbb{C}^n.
\end{align*}
\]

More precisely, every automorphism \( \varphi \) of \( D \) can be written as the composite mapping \( \varphi = \varphi_\gamma \circ \varphi_B \circ \varphi_A \) of automorphisms \( \varphi_A, \varphi_B \) and \( \varphi_\gamma \) of the above type.

3. Proof of Theorem 1

We begin with the following lemma, which will be used in the proofs of our theorems. Although, in the proof below of this lemma, there is some overlap with our previous paper [16], we carry out the proof in detail for the sake of completeness and self-containedness.

Lemma 1. Let \( D = D_{n,m}(\mu, s) \) be an FBH-type domain in \( \mathbb{C}^N \) and let \( V \) be an irreducible complex analytic subvariety of \( D \) with \( \dim_{\mathbb{C}} V > 0 \). Assume that \( \nabla \cap \partial D = \emptyset \). Then \( V \) is contained in \( \Delta_D = \{ (z, w) \in D \mid w = 0 \} \equiv \mathbb{C}^n \).

Proof. Let us consider the plurisubharmonic function \( h \) on \( \mathbb{C}^N \) given by

\[
h(\xi) = \|w\|^2 \quad \text{for} \quad \xi = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N.
\]

Then \( h \) gives rise to a continuous plurisubharmonic function on \( V \) and

\[
h(\xi) = \|w\|^2 < e^{-\mu \|\xi\|^2} \leq 1 \quad \text{for all} \quad \xi = (z, w) \in V \subset D.
\]

Once it is shown that \( h(\xi) \equiv 0 \) on \( V \), we conclude that \( V \subset \Delta_D \). We argue by contradiction, so assume that there exists a point \( \xi_0 = (z_0, w_0) \in V \) such that \( h(\xi_0) = \|w_0\|^2 \neq 0 \). Then

\[
0 < \|w_0\|^2 = h(\xi_0) \leq \sup\{h(\xi) \mid \xi \in V\} =: L \leq 1,
\]
and hence, there is a sequence \( \zeta_v = (z_v, w_v) \in V, \; v = 1, 2, \ldots \), such that
\[
\|w_v\|^2 / 2 \leq \|w_v\|^2 = h(\zeta_v) \leq L, \quad v = 1, 2, \ldots, \text{ and } \lim_{v \to \infty} h(\zeta_v) = L.
\]
Passing to a subsequence, if necessary, we may assume that \( \{w_v\}_{v=1}^\infty \) converges to some point \( w^* \in \mathbb{C}^m \) with \( \|w^*\|^2 = L \). Moreover, we have
\[
\|z_v\|^2 < -\frac{1}{2} (1/\mu) \log(\|w_v\|^2/2) < \infty, \quad v = 1, 2, \ldots,
\]
because \( \zeta_v = (z_v, w_v) \in V \subset D \). Thus, passing again to a subsequence, we may further assume that \( \{\zeta_v\}_{v=1}^\infty \) itself converges to a point \( \zeta^* = (z^*, w^*) \in \overline{V} \). Here, by our assumption, \( V \) may be regarded as a closed complex analytic subvariety of \( \mathbb{C}^N \) contained in \( D \). Accordingly, we have
\[
\zeta^* \in V \subset D \quad \text{and} \quad h(\zeta^*) = L.
\]
Hence, \( h(\zeta) \equiv h(\zeta_v) > 0 \) on \( V \) by the maximum principle for plurisubharmonic functions on a closed connected complex analytic subvariety of \( \mathbb{C}^N \) (cf. [6, Ch. IX]). Thus \( V \) is contained in the bounded subset \( \{ (z, w) \in D \mid \|w\| = \|w_v\| \} \) of \( \mathbb{C}^N \). Therefore, being a compact, irreducible complex analytic subvariety of \( \mathbb{C}^N \) contained in \( D \), \( V \) reduces to a singleton \( \{ \zeta_v \} \) and \( \dim V = 0 \), which contradicts our assumption. As a result, we have shown that \( h(\zeta) \equiv 0 \) on \( V \) or \( V \subset \Delta_D \), proving the lemma. \( \Box \)

**Proof of Theorem 1.** Let \( f \) be a proper holomorphic mapping between two FBH-type domains \( D_1 = D_{n_1, m_1}(\mu_1, p_1) \) and \( D_2 = D_{n_2, m_2}(\mu_2, p_2) \) in \( \mathbb{C}^N \) as in Theorem 1.

We first assert that \( f(D_1) \subset \Delta_{D_2} \). For this, write \( f = (g, h) \) with respect to the coordinate system \( (z_2, w_2) \in C^{m_2} \times C^{m_2} = C^N \) and recall that \( (z_1, 0, 1) \in D_1 \) for all \( z_1 \in C^{n_1} \). Then one can define holomorphic mappings \( \tilde{g} : C^{n_1} \to C^{n_2} \) and \( \tilde{h} : C^{n_1} \to C^{n_2} \) by setting
\[
\tilde{g}(z_1) = g(z_1, 0), \quad \tilde{h}(z_1) = h(z_1, 0) \quad \text{for } z_1 \in C^{n_1}.
\] (3.1)
Since \( |\tilde{h}(z_1)| \leq 1 \) on \( C^{n_1} \), it then follows from Liouville’s theorem that \( \tilde{h} \) is a constant mapping; hence, \( \tilde{h}(z_1) \equiv w_2^0 \) on \( C^{n_1} \) for some point \( w_2^0 \in C^{m_2} \) with \( \|w_2^0\| < 1 \). To prove our first assertion, it suffices to show that \( w_2^0 = 0 \), the origin of \( C^{m_2} \). Assume not. Then, since
\[
\|\tilde{g}(z_1)\|^2 \leq -(1/\mu_2) \log \|w_2^0\|^2 < \infty \quad \text{for all } z_1 \in C^{n_1},
\]
Liouville’s theorem again tells us that \( \tilde{g}(z_1) \equiv w_2^0 \) on \( C^{n_1} \) for some point \( z_2^0 \in C^{m_2} \); consequently, \( f(z_1, 0) = (z_2^0, w_2^0) \in D_2 \) for all \( (z_1, 0) \in D_1 \). However, this contradicts the fact that \( f : D_1 \to D_2 \) is a proper holomorphic mapping. Thus we have shown that \( w_2^0 = 0 \) and \( f(\Delta_{D_1}) \subset \Delta_{D_2} \), as asserted.

We next prove that \( n_1 = n_2 \) and so \( m_1 = m_2 \). To this end, consider the holomorphic mapping \( \tilde{g} : C^{n_1} \to C^{n_2} \) appearing in (3.1) and let \( k = \max\{\text{rank } M_{\tilde{g}}(z_1) \mid z_1 \in C^{n_1}\} \), where \( M_{\tilde{g}}(z_1) \) denotes the complex Jacobian matrix of \( \tilde{g} \) at \( z_1 \). Choose a point \( z_1^0 \in C^{n_1} \) with rank \( M_{\tilde{g}}(z_1^0) = k \). If \( n_1 > n_2 \), then we have \( k \leq n_2 < n_1 \); and hence, by the rank theorem, the point \( z_1^0 \) is never isolated in \( \tilde{g}^{-1}(\tilde{g}(z_1^0)) \). Obviously this implies that \( f^{-1}(f(z_1^0)) \) is an infinite subset of \( D_1 \) for the point \( \zeta_0 := (z_1^0, 0) \in D_1 \). Since \( f : D_1 \to D_2 \) is a proper holomorphic mapping, this is absurd. Thus \( n_1 \leq n_2 \). For the reverse inequality, it suffices to verify the following:
\[
f^{-1}(\Delta_{D_2}) \subset \Delta_{D_1}; \quad \text{accordingly,} \quad f^{-1}(\Delta_{D_2}) = \Delta_{D_1} \quad \text{and} \quad f(\Delta_{D_1}) = \Delta_{D_2}. \quad (3.2)
\]
For this purpose, taking an arbitrary irreducible component $V$ of the complex analytic subvariety $f^{-1}(\Delta_{D_2})$ of $D_1$, we claim that $V \subset \Delta_D$. Once it is shown that $\overline{V} \cap \partial D = \emptyset$, then this comes from Lemma 1, since $\dim C V > 0$. Now, assuming that there exists a point $\zeta_0 \in \overline{V} \cap \partial D_1$, we choose a sequence $\{\zeta_i\}_{i=1}^{\infty}$ in $V$ converging to $\zeta_0$. Then, since our proper holomorphic mapping $f : D_1 \rightarrow D_2$ can be regarded as a mapping defined on some open neighborhood of $\overline{D}_1$ by Fact A, we obtain a contradiction:

$$\partial D_2 \ni f(\zeta_0) = \lim_{i \rightarrow \infty} f(\zeta_i) \in \Delta_{D_2} \subset D_2.$$ 

Thus we have shown the assertion (3.2) and hence $(n_1, m_1) = (n_2, m_2)$, as required.

Moreover, since $f : D_1 \rightarrow D_2$ is a proper holomorphic mapping between two equidimensional domains $D_1$ and $D_2$ in $\mathbb{C}^N$, we have $f(D_1) = D_2$ (cf. [21, p. 301]). Hence, this combined with (3.2) assures us that $f|_{D_1^*} = D_2^*$ and both the restrictions $f|_{D_1^*} : D_1^* \rightarrow D_2^*$ and $f|_{\Delta_{D_1}} : \Delta_{D_1} \rightarrow \Delta_{D_2}$ are proper holomorphic mappings. Therefore, the proof of Theorem 1 is completed.

\[\square\]

4. Proof of Theorem 2

For simplicity, we denote by $D$ the FBH-type domain $D_{r,m}(\mu, p)$ in $\mathbb{C}^N$ as in Theorem 2. Also, choosing an element $f$ of Aut($D$) arbitrarily, we write $f = (g, h)$ by coordinates.

We first assert that $f$ is a linear automorphism of $D$, that is, a non-singular linear transformation of $\mathbb{C}^N$ leaving $D$ invariant. Indeed, we know by (2.2) that the set $\mathfrak{M}$ of all monomials $\zeta^\alpha$ is contained in $A^2(D)$. Therefore, once it is shown that $f(0) = 0$, the linearity of $f$ is a direct consequence of Yamamori [25, Theorem 4.1]. Now, in order to check that $f(0) = 0$, notice that $f$ induces a CR-diffeomorphism of the pseudoconvex real analytic hypersurface $\partial D$, because $f$ as well as $f^{-1}$ extends holomorphically beyond $\partial D$ by Fact A. Hence, $f$ has to map $\mathcal{W}(D)$ onto itself, where $\mathcal{W}(D)$ is the set of all weakly (not strictly) pseudoconvex boundary points of $D$. Thus $g(0, w) = 0$ for all $(0, w) \in \partial D$ by Fact B. Then we have

$$g(0, w) = 0 \quad \text{for all } (0, w) \in D,$$

and, in particular, $g(0) = 0$. Indeed, to prove (4.1), note that the set $\{ (z, w) \in D \mid z = 0 \}$, the cross-section of $D$ by the $w$-coordinate space in $\mathbb{C}^N$, can be naturally identified with the unit ball $B^m$ in $\mathbb{C}^m$. Under this identification, let us define the function $k$ by

$$k(w) = \|g(0, w)\|^2 \quad \text{for } w \in \overline{B^m}.$$ 

Note that $k$ is defined on some open neighborhood $O$ of $\overline{B^m}$ by Fact A and is plurisubharmonic on $O$ with $k(w) \equiv 0$ on $\partial B^m$. Consequently, we have $k(w) \equiv 0$ on $B^m$ by the maximum principle for plurisubharmonic functions; proving the assertion (4.1). Moreover, we know by Theorem 1 that $f(\Delta_D) = \Delta_D$; and hence, $h(0) = 0$. Therefore we conclude that $f(0) = 0$, as asserted.

We next assert that our linear automorphism $f$ of $D$ can be written in the form as in Theorem 2. Indeed, since $f(\Delta_D) = \Delta_D$ and $g$ satisfies the condition (4.1), $f$ has to be of the form

$$f(z, w) = (Az, Bw) \quad \text{with some } A \in \text{GL}(n; \mathbb{C}), \; B \in \text{GL}(m; \mathbb{C}).$$
Since $f$ maps $\partial D$ onto itself, it then follows that
\[
\|Bw\|^2 = e^{-\mu\|Az\|^{2\rho}} \quad \text{whenever} \quad \|w\|^2 = e^{-\mu\|z\|^{2\rho}}.
\tag{4.2}
\]
Taking the points $(0, w) \in \mathbb{C}^n \times \mathbb{C}^m$ with $\|w\| = 1$, we therefore have
\[
\|Bw\| = 1, \quad \text{accordingly,} \quad B \in U(m).
\]
Together with (4.2), this implies that
\[
\|Az\|^{2\rho} = \|z\|^{2\rho} \quad \text{for all} \quad (z, w) \in \partial D.
\]
Notice that this equation holds for any $z \in \mathbb{C}^n$ because one can find an element $w \in \mathbb{C}^m$ in such a way that $(z, w) \in \partial D$. We thus conclude that $A \in U(n)$. As a result, we have shown that $f$ has the form required in Theorem 2; completing the proof. \hfill \Box

5. Proof of Theorem 3

Before undertaking the proof of Theorem 3, we need some preparation.

Let $D_1$ and $D_2$ be two domains in $\mathbb{C}^N$. Let $O$ be an open subset of $\mathbb{C}^N$ containing a point $p \in \partial D_1$ and $F : O \cap D_1 \to \mathbb{C}^N$ a holomorphic mapping. Under this situation, we consider the following condition.

Condition $(\dagger)$. There exists an open neighborhood $U_p \subset O$ of $p$ such that $F$ extends to a biholomorphic mapping, denoted by the same letter $F$, from $U_p$ into $\mathbb{C}^N$ satisfying
\[
F(U_p \cap D_1) \subset D_2 \quad \text{and} \quad F(U_p \cap \partial D_1) \subset \partial D_2.
\]
If this condition is satisfied, we say that $F$ satisfies the condition $(\dagger)$ at $p$.

Let $L_{\mu, s}$ be the non-singular linear transformation of $\mathbb{C}^N$ defined in (2.1). As mentioned before, $L_{\mu, s}$ induces a linear equivalence between the FBH-type domains $D_{n,m}(\mu, s)$ and $D_{n,m}(1, s)$. Therefore, we need only prove Theorems 3 and 4 for the FBH-type domains $D_{n,m}(\mu, s)$ with $\mu = 1$. Taking this into account, we shall deal primarily with the FBH-type domains $D_{n,m}(\mu, s)$ with $\mu = 1$ and set
\[
D_s = D_{n,m}(1, s), \quad \Delta_{D_s} = \{(z, w) \in D_s \mid w = 0\} \cong \mathbb{C}^n \quad \text{and} \quad D_s^* = D_s \setminus \Delta_{D_s} \tag{5.1}
\]
in the following part of this paper. Also, for the given $s \in \mathbb{N}$, we shall use the following holomorphic self-mapping $\pi_s$ of $\mathbb{C}^2$ given by
\[
\pi_s(x, y) = (x, y^s) \quad \text{for} \quad (x, y) \in \mathbb{C}^2.
\]
Clearly $\pi_s$ induces a proper holomorphic mapping from $E_s$ (respectively $\mathcal{E}_s$) onto $E_1 = B^2$ (respectively $E_1$), where $E_s$ and $\mathcal{E}_s$ are the domains in $\mathbb{C}^2$ appearing in (2.4) with $n = 1$. Moreover, $\pi_s$ is injective on some open neighborhood of any point $(\xi, \eta) \in \partial E_s$ (respectively $(u, v) \in \partial \mathcal{E}_s$) with $\eta \neq 0$ (respectively $v \neq 0$).

We need the following lemma in the proof of Theorem 3.

**Lemma 2.** Let $\mathcal{E}_p$ and $\mathcal{E}_q$ be the domains in $\mathbb{C}^2$ given in (2.4) with $n = 1$ and assume that $(p, q) \neq (1, 1)$. Let $F : \mathcal{E}_p \to \mathcal{E}_q$ be a holomorphic mapping satisfying the condition $(\dagger)$ at some point $\zeta_o \in \partial \mathcal{E}_p$. Then $F : \mathcal{E}_p \to \mathcal{E}_q$ is a proper holomorphic mapping and $p/q \in \mathbb{N}$. Moreover, we have the following:
(1) If \( q = 1 \), then \( F \) has the form \( F = \varphi \circ \pi_p \) with some \( \varphi \in \text{Aut}(E_1) \).

(2) If \( q \geq 2 \), then \( p \geq 2 \) and \( F \) has the form \( F = \varphi \circ \pi_{p/q} \) with some \( \varphi \in \text{Aut}(E_q) \).

**Proof.** Denoting by \( \phi_s : E_s \to E_s \) the biholomorphic mapping given in (2.5), we here consider the holomorphic mapping
\[
\hat{F} := \phi_q \circ F \circ \phi_p^{-1} : E_p \to E_q.
\]
By our assumption, there exists an open neighborhood \( U_{\zeta_0} \) of \( \zeta_0 \in \partial E_p \) such that \( F \) satisfies the condition (†) at \( \zeta_0 \). Put
\[
x_0 = \phi_p(\zeta_0) \in \partial E_p, \quad V_{x_0} = \phi_p(U_{\zeta_0}) \quad \text{and} \quad y_0 = \hat{F}(x_0) \in \partial E_q, \quad V_{y_0} = \hat{F}(V_{x_0}).
\]
After shrinking \( U_{\zeta_0} \) sufficiently small, if necessary, we may assume that \( \hat{F} \) defines a biholomorphic mapping from \( V_{x_0} \) onto \( V_{y_0} \). Taking a suitable nearby point of \( \zeta_0 \), if necessary, we may further assume that
\[
\pi_p|_{V_{x_0}} : V_{x_0} \to \pi_p(V_{x_0}) \quad \text{and} \quad \pi_q|_{V_{y_0}} : V_{y_0} \to \pi_q(V_{y_0})
\]
are biholomorphic mappings. Thus, putting \( W_{x_0} = \pi_p(V_{x_0}) \) and \( W_{y_0} = \pi_q(V_{y_0}) \), we obtain a biholomorphic mapping
\[
\tilde{F} := \pi_q|_{V_{y_0}} \circ \hat{F} \circ (\pi_p|_{V_{x_0}})^{-1} : W_{x_0} \to W_{y_0}
\]
satisfying \( \tilde{F}(W_{x_0} \cap B^2) \subset B^2 \) and \( \tilde{F}(W_{y_0} \cap \partial B^2) \subset \partial B^2 \). Consequently, \( \tilde{F} \) extends to a holomorphic automorphism, say again \( \tilde{F} \), of \( B^2 \) by Alexander [1]. Therefore
\[
\pi_q(\tilde{F}(\xi, \eta)) = \tilde{F}(\pi_p(\xi, \eta)) \quad \text{for} \quad (\xi, \eta) \in E_p
\]
by analytic continuation. In particular, this implies that \( \tilde{F} : E_p \to E_q \) is a proper holomorphic mapping; and hence, it follows from a result due to Landucci [17] that \( r := p/q \in \mathbb{N} \) and \( \tilde{F} \) can be described as \( \tilde{F} = \hat{\varphi} \circ \pi_r \) with some \( \hat{\varphi} \in \text{Aut}(E_q) \).

If \( q = 1 \), then \( r = p \geq 2 \) by our assumption \((p, q) \neq (1, 1)\) and \( \tilde{F} = \hat{\varphi} \circ \pi_p \) with some \( \hat{\varphi} \in \text{Aut}(E_1) \). Note that \( \pi_p \circ \phi_p = \phi_1 \circ \pi_p \) on \( E_p \). Then, putting \( \varphi := \phi_1^{-1} \circ \hat{\varphi} \circ \phi_1 \in \text{Aut}(E_1) \), we have
\[
F = \phi_1^{-1} \circ \hat{F} \circ \phi_p = \phi_1^{-1} \circ \hat{\varphi} \circ \pi_p \circ \phi_p = \phi_1^{-1} \circ \hat{\varphi} \circ \phi_1 \circ \pi_p = \varphi \circ \pi_p \quad \text{on} \quad E_p.
\]
Thus we obtain the assertion (1) in the lemma.

If \( q \geq 2 \), then \( p \geq 2 \) and the automorphism \( \hat{\varphi} \) of \( E_q \) has the form as in (2.6) with \( s = q \). Note that \( \pi_r \circ \phi_p = \phi_q \circ \pi_r \) on \( E_p \). Then, by putting \( \varphi := \phi_q^{-1} \circ \hat{\varphi} \circ \phi_q \in \text{Aut}(E_q) \) and by repeating the same computation as above, it can be seen that \( F \) has the form \( F = \varphi \circ \pi_r \), as required in (2) of the lemma. Thereby the proof of the lemma is completed. \( \square \)

**Proof of Theorem 3.** By routine computations, it can be seen that the mapping \( f \) described in Theorem 3 gives a proper holomorphic mapping from \( D_{n,1}(\mu, p) \) onto \( D_{n,1}(v, q) \) in each case.

Conversely, assuming the existence of a proper holomorphic mapping from \( D_{n,1}(\mu, p) \) to \( D_{n,1}(v, q) \), we would like to show that every proper holomorphic mapping \( f : D_{n,1}(\mu, p) \to D_{n,1}(v, q) \) has the form written in Theorem 3 in each case. To this end, we may assume that \((\mu, v) = (1, 1)\), as we have already mentioned before. In the following part
of this section, we denote by \( D_s \) and \( D_s^* \) the domains in \( \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1} \) introduced in (5.1) with \( m = 1 \); and put

\[
\partial^* D_s^* = \{(z, w) \in \partial D_s^* | w \neq 0\} = \partial D_s \cup \partial D_s^*
\]

for \( s = p, q \).

Let \( \mathcal{E}_s \) be the domain in \( \mathbb{C}^{n+1} \) appearing in (2.4) and consider the holomorphic mapping \( \varpi \) from \( \mathcal{E}_s \) into \( \mathbb{C}^n \times \mathbb{C}^n \) defined by

\[
\varpi(u, v) = (v, e^{iu/2}) \quad \text{for} \ (u, v) \in \mathcal{E}_s \subset \mathbb{C} \times \mathbb{C}^n.
\]

Then it is easily seen that \( \varpi(\mathcal{E}_s) = D_s^* \) and \( \mathcal{E}_s \) is the universal covering of \( D_s^* \) with the covering projection \( \varpi \). Clearly, \( \varpi \) is, in fact, defined on \( \mathbb{C} \times \mathbb{C}^n \) and \( \varpi(\partial \mathcal{E}_s) = \partial^* D_s^* \).

Let \( f : D_p \rightarrow D_q^* \) be the given proper holomorphic mapping. It then follows from Theorem 1 that \( f(D_p^*) = D_q^* \) and \( f \) can be regarded as a proper holomorphic mapping from \( D_p^* \) onto \( D_q^* \). Hence we can lift \( f \) to a holomorphic mapping \( F : \mathcal{E}_p \rightarrow \mathcal{E}_q \) in such a way that

\[
\varpi(F(u, v)) = f(\varpi(u, v)) \quad \text{for all} \ (u, v) \in \mathcal{E}_p,
\]

(5.2)

because \( f \circ \varpi : \mathcal{E}_p \rightarrow D_q^* \) is a holomorphic mapping from the simply connected domain \( \mathcal{E}_p \) to the domain \( D_q^* \) and \( (\mathcal{E}_q, \varpi) \) is the universal covering space of \( D_q^* \). Here, since \( f \) extends holomorphically to an open neighborhood \( W \) of \( \overline{D_p} \) by Fact A and since \( f \) is locally biholomorphic near almost all points of \( \partial^* D_p^* = \partial D_p \), we may assume that \( F \) is defined on some open neighborhood of \( \overline{\mathcal{E}_p} \) and there exists an open neighborhood \( U_{\zeta_0} \) of some point \( \zeta_0 \in \partial \mathcal{E}_p \) such that \( F \) satisfies the condition ( \( \dagger \) ) at \( \zeta_0 \). Moreover, note that

\[
\varpi^{-1}(\varpi(u, v)) = \{(u + 4 \pi v, v) \mid v \in \mathbb{Z} \} \quad \text{for any} \ (u, v) \in \mathcal{E}_p.
\]

Then equation (5.2) tells us the following: Represent \( F = (F_0, F_1, \ldots, F_n) \) with respect to the coordinate system \( (u, v) = (u, v_1, \ldots, v_n) \) in \( \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1} \). Then, for any point \( x = (u, v) \in \mathcal{E}_p \) and any integer \( v \), there exists an integer \( n(x, v) \) such that

\[
F_0(u + 4 \pi v, v) = F_0(u, v) + 4 \pi n(x, v),
\]

and

\[
F_i(u + 4 \pi v, v) = F_i(u, v), \quad 1 \leq i \leq n.
\]

(5.3)

Since \( F \) is holomorphic on \( \mathcal{E}_p \), the integer \( n(x, v) \) depends continuously on \( x \in \mathcal{E}_p \) for each fixed \( v \in \mathbb{Z} \). Consequently, \( n(x, v) \) is independent of \( x \); so that we may write \( n(x, v) = n(v) \).

The proof of Theorem 3 is now divided into two cases as follows.

**Case I:** \( n = 1 \). Since the holomorphic mapping \( F : \mathcal{E}_p \rightarrow \mathcal{E}_q \) satisfies the condition ( \( \dagger \) ) at some point \( \zeta_0 \in \partial \mathcal{E}_p \) as we have already seen above, it follows from Lemma 2 that \( F \) is a proper holomorphic mapping and \( p/q \in \mathbb{N} \).

If \( q = 1 \), then it follows from the assertion (1) of Lemma 2 that \( F : \mathcal{E}_p \rightarrow \mathcal{E}_1 \) has the form \( F = \varphi \circ \pi_p \) with some \( \varphi \in \text{Aut}(\mathcal{E}_1) \) or \( F \) can be written in the form

\[
F(u, v) = \left( \frac{\alpha_0 u + \alpha_1 v^p + \beta_0}{\gamma_0 u + \gamma_1 v^p + \delta}, \frac{\alpha_1 u + \alpha_{11} v^p + \beta_1}{\gamma_0 u + \gamma_1 v^p + \delta} \right)
\]
for \((u, v) \in \mathcal{E}_p\) by (2.7). We here assert that \(\varphi\) is affine. Indeed, it follows from the equation in (5.3) for \(i = 1\) that

\[
\frac{4\pi v \cdot \alpha_{10} + \alpha_{10} u + \alpha_{11} v^p + \beta_1}{4\pi v \cdot \gamma_0 + \gamma_0 u + \gamma_1 v^p + \delta} = \frac{\alpha_{10} u + \alpha_{11} v^p + \beta_1}{\gamma_0 u + \gamma_1 v^p + \delta}
\]

for all \((u, v) \in \mathcal{E}_p\) and for all \(v \in \mathbb{Z}\). If \(\gamma_0 \neq 0\), we then have

\[
\alpha_{10}/\gamma_0 = F_1(u, v) \quad \text{for all } (u, v) \in \mathcal{E}_p,
\]

in contradiction to the fact that \(F : \mathcal{E}_p \to \mathcal{E}_1\) is a proper holomorphic mapping. Thus \(\gamma_0 = 0\).

In such a case, it follows at once that \(\alpha_{10} = 0\). Hence \(F_1\) does not depend on the variable \(u\), so \(F_1\) has the form \(F_1(u, v) = F_1(v)\). Next consider the first equation in (5.3). If \(\alpha_{00} = 0\), then \(F\) itself does not depend on \(u\). Clearly this is absurd, because \(F : \mathcal{E}_p \to \mathcal{E}_1\) is a proper holomorphic mapping. Thus \(\alpha_{00} \neq 0\) and

\[
\gamma_1 v^p + \delta = \alpha_{00} \cdot v/n(v) \quad \text{for all } (u, v) \in \mathcal{E}_p,
\]

where \(v\) is any integer with \(n(v) \neq 0\). This can only happen when \(\gamma_1 = 0\); so that

\[
\varphi(u, v) = ((\alpha_{00} u + \alpha_{01} v + \beta_0)/\delta, (\alpha_{11} v + \beta_1)/\delta)
\]

is actually an affine automorphism of \(\mathcal{E}_1\), as asserted. Thus \(\varphi\) can now be written in the following form by (2.8):

\[
\varphi(u, v) = (ku + a + 2i(Bv, b) + i|b|^2, Bv + b) \quad \text{for } (u, v) \in \mathcal{E}_1,
\]

where \(a \in \mathbb{R}, b \in \mathbb{C}\) and \(0 < k \in \mathbb{R}, B \in \text{GL}(1, \mathbb{C})\) with \(|v|^2 = |Bv|^2\) for all \(v \in \mathbb{C}\) or \((1/\sqrt{k})B \in U(1)\). So, if we write \(B = \sqrt{k}B\) with \(B \in U(1)\), then by (5.2)

\[
f(v, e^{iu/2}) = (\sqrt{k}Bv^p + b, e^{-2\sqrt{k}Bv^p} - (1/2)|b|^2 e^{(a/2)i} (e^{iu/2}^2))
\]

for all \((u, v) \in \mathcal{E}_p\). Moreover, since \(f\) is a single-valued holomorphic mapping defined on \(D_p\), the positive real number \(k\) has to be an integer. Hence, replacing \(\tilde{B}, b\) and \(e^{(a/2)i}\) by \(A, \gamma\) and \(B\), respectively, we conclude that \(f\) has the form

\[
f(z, w) = (\sqrt{k}Az^p + \gamma, e^{-2\sqrt{k}Az^p} - (1/2)|\gamma|^2 Bw^k) \quad \text{for } (z, w) \in D_p
\]

by analytic continuation, where \(k \in \mathbb{N}, \gamma \in \mathbb{C}\) and \(A, B \in U(1)\); completing the proof of Theorem 3 in the case where \(n = m = 1\) and \(q = 1\).

If \(q \geq 2\), then we have \(p \geq 2\). Hence \(F\) has the form

\[
F(u, v) = \left( i A_{\lambda,\alpha} u + i B_{\lambda,\alpha} C_{\lambda,\alpha} u + i D_{\lambda,\alpha} \right) \left( \frac{4(|\alpha|^2 - 1)}{(C_{\lambda,\alpha} u + i D_{\lambda,\alpha})^2} \right)^{1/2q} \delta^{p/q}
\]

for \((u, v) \in \mathcal{E}_p\) by Lemma 2 (2) and Fact C (2), where \(\delta \in \mathbb{C}\) with \(|\delta| = 1\) and \(\lambda, \alpha, A_{\lambda,\alpha}, \ldots\) are the same objects as in (2.9). By the same reasoning as in the case of \(q = 1\), it then follows that \(C_{\lambda,\alpha} = 0\); consequently

\[
A_{\lambda,\alpha} = 2(1 - \tilde{\alpha}), \quad B_{\lambda,\alpha} = 2(\alpha - \tilde{\alpha})/(\alpha - 1) \quad \text{and} \quad D_{\lambda,\alpha} = 2(|\alpha|^2 - 1)/(\alpha - 1).
\]

Thus, if we put \(k = |1 - \alpha|^2/(1 - |\alpha|^2)\) and \(\theta = (\text{Im } \alpha)/(1 - |\alpha|^2)\), then \(F\) can be written in the form

\[
F(u, v) = (ku + 2\theta, \sqrt{k}8^q)^{2q/\delta^{p/q}} \quad \text{for } (u, v) \in \mathcal{E}_p,
\]

for \((u, v) \in \mathcal{E}_p\) by (2.7). We here assert that \(\varphi\) is affine. Indeed, it follows from the equation in (5.3) for \(i = 1\) that
where $\widehat{\delta} \in \mathbb{C}$ with $|\widehat{\delta}| = 1$. Together with (5.2), this says that

$$f(v, e^{iu/2}) = (\sqrt[2k]{k\delta}v/pq, e^{iu}(e^{iu/2})^k) \quad \text{for } (u, v) \in \mathcal{E}_p.$$ 

Moreover, $k$ must be a positive integer, as in the case of $q = 1$. Hence $f$ has the form

$$f(z, w) = (2\sqrt[2k]{kAz^p/q}, Bw^k) \quad \text{for } (z, w) \in D_p$$
by analytic continuation, where $k \in \mathbb{N}$ and $A, B \in U(1)$. (Note that, for any positive real number $r$, there exist infinitely many points $\alpha \in \Delta$ such that $|1 - \alpha|^2/(1 - |\alpha|^2) = r$.) This completes the proof of Theorem 3 in the case where $n = m = 1$ and $q \geq 2$. Therefore we have proved Theorem 3 in Case I.

**Case II:** $n \geq 2$. Let $F : \mathcal{E}_p \to \mathcal{E}_q$ be the lifting of $f : D^*_p \to D^*_q$ satisfying the condition (5.2) and consider again the holomorphic mapping $\widehat{F} := \phi_q \circ F \circ \phi_p^{-1} : E_p \to E_q$, where $\phi_s : \mathcal{E}_s \to \mathcal{E}_s$ is the biholomorphic mapping defined in (2.5) for $s = p, q$.

First we wish to show that $F$ is a biholomorphic mapping from $\mathcal{E}_p$ onto $\mathcal{E}_q$ and $p = q$. For this, it suffices to prove that $\widehat{F}$ is a biholomorphic mapping from $E_p$ onto $E_q$. Since $F$ satisfies the condition $(\dagger)$ at some point $\xi_o \in \partial \mathcal{E}_p$, there exists an open neighborhood $V_{\xi_o}$ of a point $x_o \in \partial E_p$ such that $\widehat{F}$ extends to a biholomorphic mapping, say again $\widehat{F}$, from $V_{\xi_o}$ into $\mathbb{C}^{n+1}$ satisfying

$$\widehat{F}(V_{\xi_o} \cap E_p) \subset E_q \quad \text{and} \quad \widehat{F}(V_{\xi_o} \cap \partial E_p) \subset \partial E_q.$$ 

Notice here that the sets of all weakly (not strictly) pseudoconvex boundary points of $E_p$ and $E_q$, respectively, are contained in the coordinate subspace $\{(\zeta, \eta) \in \mathbb{C} \times \mathbb{C}^n \mid \eta = 0\}$ of $\mathbb{C}^{n+1}$ with codimension $n \geq 2$. Hence, by a result on the localization principle for biholomorphic mappings between generalized complex ellipsoids due to Dini and Primicerio [5], Kodama [11] or Monti and Morbidelli [19], we conclude that $\widehat{F} : E_p \to E_q$ is a biholomorphic mapping and $p = q$, as desired.

Since $F$ is now a holomorphic automorphism of $\mathcal{E}_p = \mathcal{E}_q$, it has the form as in (2.9) with $s = p = q$. Moreover, in exactly the same way as in Case I, it can be seen that $C_{\lambda, \alpha} = 0$. Thus, by putting again $k = |1 - \alpha|^2/(1 - |\alpha|^2)$ and $\theta = (\text{Im } \alpha)/(1 - |\alpha|^2)$, $F$ can be described as

$$F(u, v) = (ku + 2\theta, 2\sqrt{k}Uv) \quad \text{for } (u, v) \in \mathcal{E}_p,$$

where $U \in U(n)$. Hence, by the same reasoning as in Case I, we conclude that $f$ has the form

$$f(z, w) = (2\sqrt{k}Az, Bw^k) \quad \text{for } (z, w) \in D_p$$
with some $k \in \mathbb{N}$, $A \in U(n)$ and $B \in U(1)$; proving Theorem 3 in Case II.

Therefore, the proof of Theorem 3 is completed.

6. **Proof of Theorem 4**

It is easily seen that the mapping $f$ described in Theorem 4 gives a proper holomorphic mapping from $D_{n,m}(\mu, p)$ onto $D_{n,m}(v, q)$ in each case.

Conversely, assuming the existence of a proper holomorphic mapping from $D_{n,m}(\mu, p)$ to $D_{n,m}(v, q)$, we shall show that any such mapping $f$ has the form written in Theorem 4 in each case. Without loss of generality, we may again assume that $(\mu, v) = (1, 1)$. Also, as in (5.1), we set $D_s = D_{n,m}(1, s)$. 

$\square$
Case I: \( n = 1 \). In this case, we shall employ the holomorphic self-mapping \( \Pi_s \) of \( \mathbb{C}^{m+1} \) given by

\[
\Pi_s(x, y) = (x^s, y) \quad \text{for } (x, y) \in \mathbb{C} \times \mathbb{C} = \mathbb{C}^{m+1}.
\]

Clearly \( \Pi_s \) induces a proper holomorphic mapping from the FBH-type domain \( D_s = D_{1,m}(1, s) \) onto the Fock–Bargmann–Hartogs domain \( D_1 = D_{1,m}(1, 1) \).

Thanks to Fact A, we can now choose a point \( \zeta_0 \in \partial D_p \) and an open neighborhood \( U_{\zeta_0} \) of \( \zeta_0 \) such that \( f \) gives a biholomorphic mapping from \( U_{\zeta_0} \) onto \( U'_{\zeta_0} := f(U_{\zeta_0}) \). By taking a suitable nearby point of \( \zeta_0 \) and by shrinking \( U_{\zeta_0} \) sufficiently small, if necessary, we may assume that the restrictions

\[
\Pi_p |_{U_{\zeta_0}} : U_{\zeta_0} \to \Pi_p(U_{\zeta_0}) \quad \text{and} \quad \Pi_q |_{U'_{\zeta_0}} : U'_{\zeta_0} \to \Pi_q(U'_{\zeta_0})
\]

are biholomorphic mappings. Thus, putting \( V_{\zeta_0} = \Pi_p(U_{\zeta_0}) \) and \( V'_{\zeta_0} = \Pi_q(U'_{\zeta_0}) \), we obtain a biholomorphic mapping

\[
\widehat{f} := \Pi_q |_{U'_{\zeta_0}} \circ f \circ (\Pi_p |_{U_{\zeta_0}})^{-1} : V_{\zeta_0} \to V'_{\zeta_0}
\]

satisfying \( \widehat{f}(V_{\zeta_0} \cap D_1) \subseteq D_1 \) and \( \widehat{f}(V_{\zeta_0} \cap \partial D_1) \subseteq \partial D_1 \). Recall that \( D_1 = D_{1,m}(1, 1) \) is the Fock–Bargmann–Hartogs domain in \( \mathbb{C} \times \mathbb{C}^m \) with \( m \geq 2 \). Then, an extension theorem due to Kodama [14, Theorem 1] can now be applied to obtain an element \( F \in \text{Aut}(D_1) \) such that \( F(\zeta) = \widehat{f}(\zeta) \) for \( \zeta \in V_{\zeta_0} \cap D_1 \); and hence,

\[
\Pi_q(f(\zeta)) = F(\Pi_p(\zeta)) \quad \text{for all } \zeta \in D_p \tag{6.1}
\]

by analytic continuation. Recall that \( F \in \text{Aut}(D_1) \) has the form

\[
F(z, w) = (Az + \gamma, e^{-(Az,\gamma)-(1/2)|\gamma|^2}Bw) \quad \text{for } (z, w) \in D_1
\]

by Fact D, where \( \gamma \in \mathbb{C}, A \in U(1) \) and \( B \in U(m) \). Represent \( f = (g, h) \) with respect to the coordinate system \((z, w) \in \mathbb{C} \times \mathbb{C}^m \). It then follows from (6.1) that

\[
(g(z, w))^q = Az^p + \gamma \quad \text{and} \quad h(z, w) = e^{-(Az^p,\gamma)-(1/2)|\gamma|^2}Bw \tag{6.2}
\]

for all \((z, w) \in D_p \). Consider the entire function \( \widehat{g}(z) := g(z, 0) \) for \( z \in \mathbb{C} \). Then the first equation in (6.2) says that \( \widehat{g} \) is a polynomial of degree \( p/q \); and hence, \( p/q \in \mathbb{N} \).

If \( q = 1 \), then \( p \geq 2 \) by our assumption \((p, q) \neq (1, 1) \) and

\[
f(z, w) = (Az^p + \gamma, e^{-(Az^p,\gamma)-(1/2)|\gamma|^2}Bw) \quad \text{for } (z, w) \in D_p.
\]

If \( q \geq 2 \), then we have \( p \geq 2 \) and \( (\widehat{g}(z))^q = Az^p + \gamma \) on \( \mathbb{C} \) by (6.2). Thus \( \gamma = 0 \) and \( g(z, w) = \widetilde{A}z^{p/q} \) with \(|\widetilde{A}| = 1 \), because the algebraic equation \( Az^p + \gamma = 0 \) admits a multiple root only when \( \gamma = 0 \). Therefore, we have shown that \( f \) has the form described in Case I of Theorem 4.

Case II: \( n \geq 2 \). We first assert that the given proper holomorphic mapping \( f : D_p \to D_q \) is a biholomorphic mapping. For this, we may assume by Fact A that \( f \) is a holomorphic mapping defined on some connected open neighborhood \( W \) of \( D_p \). We put

\[
\widehat{V}_f = \{ \zeta \in W \mid J_f(\zeta) = 0 \} \quad \text{and} \quad V_f = \widehat{V}_f \cap D_p.
\]
Once it is shown that $V_f = \emptyset$, then $f : D_p \to D_q$ is an unramified finite covering. Hence it must be a biholomorphic mapping, since $D_q$ is a simply connected domain in $\mathbb{C}^N$. Therefore, we have only to show that $V_f = \emptyset$. So, assuming to the contrary that $V_f \neq \emptyset$, we shall derive a contradiction. To this end, choose an arbitrary irreducible component $V$ of $V_f$ and fix an irreducible component $\hat{V}$ of $\hat{V}_f$ such that $V \subset \hat{V}$. Then $V$ is a complex analytic subvariety of $D_p$ with $\dim_{\mathbb{C}} V = N - 1 > n$, since $m \geq 2$. If $\hat{V} \cap \partial D_p = \emptyset$, then $V \subset \Delta_{D_p} \cong \mathbb{C}^n$ by Lemma 1, which is of course impossible. Thus the verification of our first assertion is now reduced to showing that $\hat{V} \cap \partial D_p = \emptyset$, because $\hat{V} \cap \partial D_p \subset \hat{V} \cap \partial D_p$.

Assuming that $\hat{V} \cap \partial D_p \neq \emptyset$, we want to obtain a contradiction. For this, we first assert that $\hat{V} \cap \partial D_p \subset W(D_p)$. Indeed, let $\rho_s(\zeta) := -1 + \|w\|^2 e^{|z|^2}$ be the plurisubharmonic function on $\mathbb{C}^N$ appearing in Section 2 and put $r(\zeta) = \rho_q(f(\zeta))$ for $\zeta \in W$. It then follows from the Hopf lemma that $r(\zeta)$ is a real analytic defining function for $D_p$ as well as $\rho_p(\zeta)$. Therefore, by using the same method as in the last paragraph of the proof of Bell [3, Theorem 2], it can be seen that $J_f(\zeta_o) \neq 0$ for any point $\zeta_o \in S(D_p)$. Thus $\hat{V} \cap S(D_p) = \emptyset$ or $\hat{V} \cap \partial D_p \subset W(D_p)$, as asserted. On the other hand, we can prove that $\hat{V} \cap \partial D_p \nsubseteq W(D_p)$ contradictory to the assertion above. To this end, consider the set $M$ consisting of all regular points of $\hat{V}$. Then $M$ is a connected complex submanifold of $\mathbb{C}^N$ of dim$_{\mathbb{C}} M = N - 1$ and $M$ is an open dense subset of $\hat{V}$. In particular, we have $M \cap D_p \neq \emptyset$. Moreover, $M \cap \partial D_p \neq \emptyset$ and $M$ intersects $\partial D_p$ transversely. Indeed, if $M \cap \partial D_p = \emptyset$, then $M \subset D_p$ and $\hat{V} = M \subset D_p \subset W$. Thus $\hat{V}$ can be regarded as a closed connected complex analytic subvariety of $\mathbb{C}^N$ contained in $W$. Define now the plurisubharmonic function $\hat{h}$ on $\hat{V}$ by

$$\hat{h}(\zeta) = \|w\|^2$$

for $\zeta = (z, w) \in \hat{V} \subset D_p$.

Then we have

$$\hat{h}(\zeta) \leq 1 \quad \text{for all } \zeta \in \hat{V} \quad \text{and} \quad \hat{h}(\zeta) = 1 \quad \text{for all } \zeta \notin \hat{V} \cap \partial D_p,$$

since

$$\hat{V} \cap \partial D_p \subset W(D_p) = \{(z, w) \in \partial D_p \mid z = 0\} = \{(0, w) \in \mathbb{C}^n \times \mathbb{C}^m \mid \|w\| = 1\}$$

by Fact B. Thus $\hat{h}(\zeta) \equiv 1$ on $\hat{V}$ by the maximum principle for plurisubharmonic functions and so $\hat{V}$ is contained in the compact subset $W(D_p)$ of $\mathbb{C}^N$. Then $\hat{V}$ reduces to a singleton and hence dim$_{\mathbb{C}} \hat{V} = 0$, which is a contradiction. Therefore, we have $M \cap \partial D_p \neq \emptyset$. Assume next that $M$ does not intersect $\partial D_p$ transversely. Then there exist a point $\zeta_o \in M \cap \partial D_p$ and a connected open neighborhood $O$ of $\zeta_o$ in $M$ such that $O \subset \overline{D_p}$. Without loss of generality, we may assume that $O$ is contained in some local holomorphic coordinate system $(U, \psi)$ on $M$ such that $\psi(U) = B^{N-1}$ and $\psi(\zeta_o) = 0$, the origin of $\mathbb{C}^{N-1}$. Let $\rho_p(\zeta)$ be the global defining function for $D_p$ as above and consider the plurisubharmonic function $\rho$ on $B^{N-1}$ given by $\rho(x) := \rho_p(\psi^{-1}(x))$ for $x \in B^{N-1}$. Then we have

$$\rho(x) \leq 0 \quad \text{for all } x \in \psi(O) \quad \text{and} \quad \rho(0) = 0;$$

from which it follows that $\rho(x) \equiv 0$ on $\psi(O)$ and $O \subset \partial D_p$. Then $\hat{V} = M \subset D_p$ by analytic continuation, and hence we have $V \subset \hat{V} \cap \partial D_p = \emptyset$, which is a contradiction. Therefore, $M$ intersects $\partial D_p$ transversely, as asserted. In particular, we see that dim$_{\mathbb{R}}(M \cap \partial D_p) = 2N - 3$. On the other hand, we have dim$_{\mathbb{R}} W(D_p) = 2m - 1 \leq 2N - 5$, because $n \geq 2$
by our assumption. Then $M \cap \partial D_p \not\subset \mathcal{W}(D_p)$ and hence $\hat{V} \cap \partial D_p \not\subset \mathcal{W}(D_p)$, as required. Eventually we have proved that $V_f = \emptyset$ and $f : D_p \to D_q$ is, in fact, a biholomorphic mapping.

Our next task is to show that $f : D_p \to D_q$ is linear, that is, $f$ is a non-singular linear transformation of $\mathbb{C}^N$ such that $f(D_p) = D_q$. This can be shown along the same lines as in the proof of Yamamori [25, Theorem 4.1]. Indeed, the inclusion $\mathfrak{M} \subset A^2(D)$ in (2.2) guarantees that $K_{D_p}(0, 0) > 0$ and $T_{D_p}(0, 0)$ is positive definite for $s = p, q$, where $K_{D_p}$ is the Bergman kernel for $D_p$ and $T_{D_p}$ is an $N \times N$ matrix defined by

$$
T_{D_p}(\zeta, \eta) = \begin{pmatrix}
\partial^2 \log K_{D_p}(\zeta, \eta)/\partial \overline{\eta^1} \partial \zeta_1 & \cdots & \partial^2 \log K_{D_p}(\zeta, \eta)/\partial \overline{\eta^1} \partial \zeta_N \\
\vdots & \ddots & \vdots \\
\partial^2 \log K_{D_p}(\zeta, \eta)/\partial \overline{\eta^N} \partial \zeta_1 & \cdots & \partial^2 \log K_{D_p}(\zeta, \eta)/\partial \overline{\eta^N} \partial \zeta_N
\end{pmatrix}.
$$

Moreover, since our $f : D_p \to D_q$ is now a biholomorphic mapping, $f$ induces a CR-diffeomorphism between the real analytic hypersurfaces $\partial D_p$ and $\partial D_q$; accordingly,

$$
f(\mathfrak{S}(D_p)) = \mathfrak{S}(D_q) \quad \text{and} \quad f(\mathcal{W}(D_p)) = \mathcal{W}(D_q).
$$

Thus, representing $f = (g, h)$ by coordinates and repeating exactly the same argument as in the proof of assertion (4.1), we can see that $g(0, w) = 0$ for all $(0, w) \in D_p$. This together with $f(\Delta_{D_p}) = \Delta_{D_q}$ from Theorem 1 yields that $f(0) = 0$. Thus, by the same method as in the proof of Yamamori [25, Theorem 4.1] based on the results in Ishi and Kai [7, Propositions 2.1 and 2.6] it can be shown that $f$ is linear, as desired.

Finally, express $f(z, w) = (Az, Bw)$ by some $A \in \text{GL}(n; \mathbb{C})$ and $B \in \text{GL}(m; \mathbb{C})$. Then, by a slight modification of the proof of Theorem 2, one can easily check that $A \in U(n)$, $B \in U(m)$ and $p = q$; proving Theorem 4 in Case II.

Therefore the proof of Theorem 4 is now completed. \qed

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