It is surprising but true that if \( A \) is a nice enough ring, then any \( A \)-left-module of finite flat dimension also has finite projective dimension. A result to this effect was proved in \([3, \text{prop. 6}]\) under the condition that the number

\[
\text{FPD}(A) = \sup \left\{ \text{pd} \ M \ \bigg| \ M \text{ is an } A\text{-left-module with } \text{pd} \ M < \infty \right\},
\]

known as the left big finitistic projective dimension of \( A \), is finite. Here \( \text{pd} \ M \) denotes the projective dimension of \( M \).

Unfortunately, this number is not known to be finite even if \( A \) is a finite dimensional algebra over a field, where, indeed, its finiteness is a celebrated conjecture. On the other hand, for such an algebra, finite flat certainly implies finite projective dimension, simply because each flat module is projective.

So it seems that there might be results based on other conditions than finiteness of \( \text{FPD}(A) \). The germ of such a result is in \([2, \text{cor. 3.4}]\) which shows that finite flat implies finite projective dimension for a ring which is a homomorphic image of a noetherian commutative Gorenstein ring with finite Krull dimension.

Now, for such a ring, \( \text{FPD}(A) \) is in fact finite, and so formally, \([2, \text{cor. 3.4}]\) follows from \([3, \text{prop. 6}]\). However, the method of proof of \([2, \text{cor. 3.4}]\) lends itself to generalization, and in this note I will use it to show that finite flat implies finite projective dimension for any right-noetherian algebra which admits a dualizing complex.

This includes finite dimensional algebras, but also more interesting cases such as noetherian complete semi-local PI algebras, and filtered algebras whose associated graded algebras are connected and noetherian, and either PI, graded FBN, or with enough normal elements.

**Setup.** \( k \) is a field, and \( A \) is a right-noetherian \( k \)-algebra for which there exists a left-noetherian \( k \)-algebra \( B \) and a dualizing complex \( BD_A \).
To make sense of this setup, here is the definition of dualizing complexes, reproduced from [6, def. 1.1]. Note that the definition uses derived categories; for notation relating to these, see for instance [5, sec. 1] or [6, sec. 1].

**Definition.** The complex $BDA$ in $D(B \otimes_k A^{\text{op}})$ is called a dualizing complex if it satisfies the following.

(i) The cohomology of $D$ is bounded and finitely generated both over $B$ and over $A^{\text{op}}$.
(ii) The injective dimensions $\text{id}_B D$ and $\text{id}_{A^{\text{op}}} D$ are finite.
(iii) The canonical morphisms $A \rightarrow \text{RHom}_B(D, D)$ in $D(A^e)$ and $B \rightarrow \text{RHom}_{A^{\text{op}}}(D, D)$ in $D(B^e)$ are isomorphisms.

The dualizing complex $D$ is a sort of co-tilting object, so it is not surprising that the functors $DL \otimes -$ and $\text{RHom}(D, -)$ induce certain equivalences of categories. These were noted in [1, sec. 2] in the commutative case and in [4, sec. 2] in the non-commutative graded case. The methods carry over to the present setup with the following result:

Let $D \text{id} (A)$ denote the full subcategory of $D(A)$ consisting of complexes which are isomorphic to a bounded complex of flat modules, and let $D \text{id} (B)$ denote the full subcategory of $D(B)$ consisting of complexes which are isomorphic to a bounded complex of injective modules. Then there are quasi-inverse equivalences of categories

$$D \text{id} (A) \xleftrightarrow{DL \otimes A} \xrightarrow{\text{RHom}_B(D, -)} D \text{id} (B). \tag{1}$$

Using this, I can prove the main result that finite flat implies finite projective dimension.

**Theorem.** Let $M$ be an $A$-left-module. Then $\text{fd} M < \infty$ implies $\text{pd} M < \infty$.

**Proof.** The case $M = 0$ is trivial, so let me assume $M \not\cong 0$. I will then prove more than claimed, namely, $\text{fd} M < \infty$ implies

$$\text{pd}_A M \leq - \inf \{ i \mid H^i(D \otimes_A M) \not\cong 0 \} + \text{id}_B D. \tag{2}$$

This implies the theorem because both terms on the right hand side are $< \infty$, the first one since $D$ has bounded cohomology while $M$ has finite flat dimension, and the second one by assumption on $D$.

The right hand side of (2) does not change if $D$ is replaced by some suspension $\Sigma^j D$, so I can suppose

$$\text{id}_B D = 0.$$
If I now set
\[ n = -\inf\{ i \mid H^i(D \otimes_A M) \neq 0 \} \]
then the inequality \([\text{2}]) amounts to
\[ \text{pd}_A M \leq n. \]

To prove this, let me start by showing \( n \geq 0 \). The condition \( \text{id}_B D = 0 \) implies that the cohomology of \( D \) is concentrated in cohomological degrees \( \leq 0 \), and since \( M \) is a complex concentrated in degree 0 and so has cohomology concentrated in degree 0, it follows that the cohomology \( H(D \otimes_A M) \) is concentrated in degrees \( \leq 0 \). Provided \( H(D \otimes_A M) \neq 0 \) holds, this proves \( \inf\{ i \mid H^i(D \otimes_A M) \neq 0 \} \leq 0 \) and hence \( n \geq 0 \). To see \( H(D \otimes_A M) \neq 0 \), observe that as \( M \) has finite flat dimension, the equivalences \([\text{1}]\) give the isomorphism \( M \cong \text{RHom}_B(D, D \otimes_A M) \). So \( M \neq 0 \) implies \( D \otimes_A M \neq 0 \), hence \( H(D \otimes_A M) \neq 0 \).

Now consider a projective resolution \( P \) of \( M \). There is a short exact sequence
\[ 0 \to \Omega^{n+1} M \to P_n \to \Omega^n M \to 0 \]
involving two of the syzygies of \( M \), as defined by means of \( P \). I shall prove that this sequence splits, whence \( \Omega^n M \) is projective so \( \text{pd}_A M \leq n \) as desired.

To see that the sequence splits, I will in fact prove
\[ \text{Ext}_A^1(\Omega^n M, \Omega^{n+1} M) = 0. \]

For this, consider the following chain of isomorphisms,
\[
\begin{align*}
\text{Ext}_A^1(\Omega^n M, \Omega^{n+1} M) &\cong \text{Ext}_A^{n+1}(M, \Omega^{n+1} M) \\
&\cong \text{Hom}_{D(A)}(M, \Sigma^{n+1} \Omega^{n+1} M) \\
&\cong \text{Hom}_{D(B)}(D \otimes_A M, D \otimes_A (\Sigma^{n+1} \Omega^{n+1} M)) \\
&= (\ast),
\end{align*}
\]
where \( \Sigma^{n+1} \) denotes \((n + 1)\)st suspension, and where \((\ast)\) is by the equivalences in equation \([\text{1}]\) which apply because both \( M \) and \( \Omega^{n+1} M \) have finite flat dimension.

Let \( T \) be a truncation of \( D \otimes_A M \) at cohomological degree
\[ -n = \inf\{ i \mid H^i(D \otimes_A M) \neq 0 \}, \]
so $T$ is quasi-isomorphic to $D \otimes_A M$ and concentrated in degrees $\geq -n$. Then

$$(*) \cong \text{Hom}_{D(B)}(T, D \otimes_A (\Sigma^{n+1} \Omega^{n+1} M)) = (**) .$$

To continue the computation, let $BI_A$ be a left-bounded injective resolution of $BD_A$ over $B \otimes_k A^{op}$. Forgetting the $A$-structure, $BI$ is an injective resolution of $BD$ over $B$. The condition $\text{id}_B D = 0$ implies that truncating $BI$ at cohomological degree 0 gives a quasi-isomorphic complex which is concentrated in degrees $\leq 0$ and consists of injective $B$-left-modules. So truncating $BI_A$ at degree 0 gives a quasi-isomorphic complex $BJ_A$ which is concentrated in degrees $\leq 0$ and consists of $B$-left-$A$-right-modules which are injective when viewed as $B$-left-modules. As $I$ is left-bounded, the truncation $J$ is bounded.

Also, let $F$ be a bounded flat resolution of $\Sigma^{n+1} \Omega^{n+1} M$. As the complex $\Sigma^{n+1} \Omega^{n+1} M$ is just the $(n+1)$'st suspension of a module, $F$ can be taken to be concentrated in cohomological degrees $\leq -n - 1$, and I have

$$D \otimes_A (\Sigma^{n+1} \Omega^{n+1} M) \cong D \otimes_A F \simeq J \otimes_A F.$$

As $J$ is a bounded complex of $B$-left-$A$-right-modules which are injective when viewed as $B$-left-modules while $F$ is a bounded complex of flat $A$-left-modules, $J \otimes_A F$ is a bounded complex of injective $B$-left-modules. Hence $J \otimes_A F$ is an injective resolution which can be used to compute homomorphism groups in $D(B)$, so

$$(**) \cong \text{Hom}_{K(B)}(T, J \otimes_A F) = (***) .$$

But $J$ is concentrated in cohomological degrees $\leq 0$ and $F$ is concentrated in degrees $\leq -n - 1$, so $J \otimes_A F$ is concentrated in degrees $\leq -n - 1$. On the other hand, $T$ is concentrated in degrees $\geq -n$. This implies

$$(***) = 0$$

as desired. \hfill \Box

**Corollary.** Let $R$ be a noetherian $k$-algebra and suppose that one of the following holds.

(i) $R$ is a complete semi-local PI algebra.

(ii) $R$ has a filtration $F$ so that the associated graded algebra $\text{gr}^F R$ is connected and noetherian, and either PI, graded FBN, or with enough normal elements.

Let $M$ be an $R$-left-module. Then $\text{fd} M < \infty$ implies $\text{pd} M < \infty$. 
Proof. The algebra $R$ can be used as $A$ in the theorem because $B$ and $D$ exist. In case (i) this is by [5, cor. 0.2], and in case (ii) by [6, cor. 6.9]. □

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