A study on the vibration dissipation mechanism of the rotating blade with dovetail joint

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Abstract
The vibration dissipation mechanism of the rotating blade with a dovetail joint is studied in this paper. Dry friction damping plays an indispensable role in the direction of vibration reduction. The vibration level is reduced by consuming the total energy of the turbine blade with the dry friction device on the blade-root in the paper. The mechanism of the vibration reduction is revealed by the variation of the friction force and the energy dissipation ratio of dry friction. In this paper, the flexible blade with a dovetail interface feature is discretized by using the spatial beam element based on the finite element theory. Then the classical Coulomb-spring friction model is introduced to obtain the dry friction model on the contact interfaces of the tenon-mortise structure. What is more, the effects of the system parameters (such as the rotating speed, the friction coefficient, the installation angle of the tenon) and the excitation level on blade damping characteristics are discussed, respectively. The results show that the variation of the system parameters leads to a significant change of damping characteristics of the blade. The variation of the tangential stiffness and the position of the external excitation will affect the nonlinear characteristics and vibration damping characteristics.

Keywords
Dry friction damping, rotating blade, dovetail joint, vibration dissipation mechanism, nonlinear characteristics

Introduction
As a rotating and thin-walled structure, the blade will be subjected to high cycle fatigue damage because of the complex external excitation, which will reduce the reliability of the engine and even lead to catastrophic accidents. In order to reduce the vibration strain of the blade, it is necessary to avoid the resonance frequency and increase the damping of the blades when they work. Many scholars have researched the modal characteristics of the blade. Yoo et al. and Lim et al.¹–³ presented a modeling method for the vibration analysis of rotating pre-twisted blades with a concentrated mass. Moreover, the effects of the dimensionless parameters on the modal characteristics of the rotating blade were investigated through numerical analysis. Shukla and Harsha⁴ were concerned with a comparative study of the modal analysis of the steam turbine blade with the analytical method and experimental method. The variation in natural frequency and mode shapes for the cracked and un-cracked blade was studied. Li et al.⁵,⁶ developed the rotor-blade coupling system model to study the effect of shaft bending on the coupling system and analyze the nonlinear properties of the continuous model.
Dry friction damping plays an essential role in the field of vibration reduction. Vibration amplitude is reduced by dissipating the total energy of the blade with the dry friction device. Therefore, it is necessary to develop a reasonable dry friction mechanical model which is used to calculate the complex contact force between the contact interfaces. The researchers have done much work on dry friction damping and established a series of dry friction mechanical models. Denhartog developed a rational dry friction model that accurately analyzes the impact of dry friction damping. Iwan developed a model with bilinear hysteretic force, and the dynamic response is solved by introducing it in the single-degree-of-freedom system and the two-degree-of-freedom system. Yang and Menq studied the double interface friction model, and the mathematical expression of friction force on the friction interface is presented. These models can be well used to simulate the nonlinear force between the contact interfaces and lay a foundation for future research.

Nowadays, the wedge-shaped damper and the shrouded structure are the typical dry friction damping structure in the blade system. There are many pieces of research on the wedge-shaped damper. He et al. researched the dynamic response of the turbine blade with the under-platform damper by using the finite element method and modal superposition method. Moreover, the vibration reduction effect of the under-platform damper was well verified. Pesaresi et al. presented an experimental damper rig and found that the localized damper motion has a strong effect on the nonlinear response of the blade. The finite element model was established, and the threedimensional friction contact element was used to simulate the contact interface. The results were compared with the experiment. Sever et al. presented an experimental method of the rotating blade with under-platform damper and investigated the forced response of the blade with and without mechanical mistuning. Firrone et al. firstly analyzed the influence of static load on the dynamic characteristics of the blade with friction damping, then coupled the blade displacement with the damper displacement, established the finite element model of blade system, and analyzed the blade response characteristics.

Meanwhile, there are many numerical analyses and experimental researches on the shrouded structure, which have been done by the researchers at home and abroad. Afzal et al. gave an analytical expression to efficiently evaluate the matrix while computing the friction contact forces in the time domain employing the alternate frequency–time domain approach. Siewert et al. presented a reduced order model of a shrouded turbine blade, including a contact model to determine the nonlinear contact force to predict the resulting vibration amplitudes. Gu et al. developed a method to predict the nonlinear vibratory response of the bladed disc system with shrouded dampers based on the sector model, numerical 3D friction contact model, and the hybrid frequency–time domain method. Cao et al. proposed an approximate approach for analyzing the two-dimensional friction contact problem to compute the dynamic response of a structure constrained by friction interfaces due to tip-rub. The nonlinear vibration characteristics of the blade were investigated in terms of the Poincare graph, and the frequency spectrum of the responses and the amplitude–frequency curves. Pust et al. investigated the coupled vibration of blades damped by dry friction contact in a shroud by using the method of friction waveform and time domain response. Byrtus and Zeman used a friction element placed between two blades to imitate the friction effects in blade shrouding. The methodology was applied to two actual blade models and is extended to the whole blade disc.

In recent years, the integrally bladed disk structure is used in the engine with more and more requirements on the reliability and the thrust-weight ratio of the engine. Besides, some researchers have studied the effect of the ring damper and the damping coating, which are usually used in the integrally bladed disk structure on the vibration characteristics of the blades. Laxalde et al. investigated the damping characteristics of the blisk system with a friction ring. The nonlinear response was obtained by using the multi-harmonic balance method. Tang and Epureanu presented a reduced-order modeling method to research the nonlinear dynamics of mistuned bladed disks with ring dampers. Moreover, the results were well validated. Chen et al. investigated the natural characteristics and the damping effect of the blisk with hard coating on blades. The effect of the NiCrAlY coating on the natural frequencies, modal loss factors, and frequency responses was developed.

To sum up, many scholars have proved the significance of the dry friction damping by numerical analysis and experimental research. However, these studies do not explain the vibration dissipation mechanism. Also, there are some shortcomings about the integrally bladed disk structure. For example, the rotor blade damage will result in the abandonment of the integrally bladed disk structure. In the high-pressure compressor, the blades are short and thin, the weight reduction effect brought by adopting the integrally bladed disk structure is not apparent. Therefore, it is also necessary to conduct the vibration reduction research on the blade-disk system assembled by the tenon-mortise structure. Besides, fewer studies have considered the problem of the friction damping between blade and disk. Therefore, taking the rotating blade with a dovetail joint as the research object in this paper. Firstly, the dynamic model of the rotating blade is established, and the blade is discretized by using the
spatial beam element based on the finite element theory. Then the classical Coulomb-spring friction model is introduced to obtain the dry friction model. The vibration dissipation mechanism of the blade with a dovetail interface joint is then discussed.

**The establishment of the dynamic model**

**The establishment of the element matrix**

In the rotating machines, such as gas turbine and air-compressor machine, the blade as a critical component should have enough rigidity and excellent damping effect. In engineering applications, the blades are usually assembled with the disk in the form of tenon-mortise structure. The dovetail joints are widely used because of its simple structure and easy manufacture. The blade with a dovetail joint is shown in Figure 1.

The contact interfaces of the tenon-mortise structure will be fitted to each other as a result of the centrifugal force of the blade when the blade is in rotating condition. The schematic diagram of a rotating blade system is shown in Figure 2. The blade is assembled on the disk through the tenon-mortise structure, where $OXYZ$ is the
global coordinate system and the coordinate origin $O$ is in correspondence with the center of the disk. $oxyz$ is the local coordinate system of the blade, and the coordinate origin is located at the blade root.

Meanwhile, the direction of the axis $x$ is in correspondence with that of axis $X$. The displacements of the blade in the direction of $x$, $y$, and $z$ are $u$, $v$, $w$, respectively. $\beta_i$ is the installation angle of the blade, $R$ is the radius of the disk, $\Omega$ is the rotating speed, $P$ is an arbitrary point on the blade.

It is necessary to select the appropriate element type and number when discretizing the blade. In this paper, the spatial beam element is applied to discretize the blade into 30 elements. The point $P$ exists the bending deformation in the $oxy$ plane and the $oxz$ plane in Figure 2. The torsional deformation around the $x$-axis is not considered. Therefore, the stiffness and mass matrix can be derived from the viewpoint of combined deformation for any spatial beam element. The deformation of any element can be decomposed into the following three groups:

1. The bending deformation in the $oxy$ plane.
2. The bending deformation in the $oxz$ plane.
3. The axial deformation along the $x$-axis.

Taking the bending deformation in the $oxy$ plane as an example and taking into account the effect of shearing deformation, the strain energy of the element is calculated as follows

$$U_e^c = U_b^c + U_s^c = \frac{1}{2} \int_{(i-1)l}^{il} EI \left( \frac{\partial^4 v_b^c}{\partial x^4} \right)^2 + G \frac{\partial v_s^c}{\partial x} \frac{\partial v_s^c}{\partial x} \, dx \quad (1)$$

where $U_b^c$ and $U_s^c$ are the bending and shearing energy. $E$ denotes Young’s Modulus, $G$ is the modulus of shearing, $I$ represents the moment of inertia, $A$ is the cross-sectional area of the blade, $k$ is the shear correction factor, $l$ is the element length, $i$ is the serial number of elements and $i = 1, 2, 3, \ldots, n$. In this paper, $n = 30$. The bending stiffness matrix $k_b$ and the shearing stiffness matrix $k_s$ are derived from equation (1) in the $oxy$ plane. Each node has three displacement components: $v_b$, $v_s$, and $\theta$, and two independent variables $v$ and $\theta$ are obtained by using the equilibrium equation and geometric relation of the element. The stiffness matrix $k_i^c$ of the spatial beam element in the $oxy$ plane is obtained through integrating $k_b$ and $k_s$.

The centrifugal force acts on the blade when it rotates. In addition, the strain energy of any element in the blade under the action of centrifugal force in the $oxy$ plane can be shown as

$$U_e^c = \frac{1}{2} \int_{(i-1)l}^{il} N_i^c(x_j) \cos \beta_i v^c(x_j) \, dx_j \quad (2)$$

where $x_j$ is the distance between any point in an element and the front end of the element; $v^c(x_j)$ is the transverse displacement of this point; $N_i^c(x_j)$ is the centrifugal force of this point and can be written as follows

$$N_i^c(x_j) = \int_{(i-1)l+x_j}^{L} \rho A \Omega^2 [R + (i - 1)l + x_j] \, dx \quad (3)$$

where $L$ is the length of the blade. Therefore, the centrifugal stiffness matrix $k_i^c$ in the $oxy$ plane can be obtained through the centrifugal strain energy of an element.

The kinetic energy of a spatial beam element in the $oxy$ plane can be expressed as given below

$$T_i^c = T_b^c + T_s^c = \frac{1}{2} \rho A \int_{(i-1)l}^{il} \left( \frac{\partial v_b^c}{\partial t} \right)^2 + \left( \frac{\partial v_s^c}{\partial t} \right)^2 \, dt \quad (4)$$

The bending mass matrix $m_b$ and the shearing mass matrix $m_s$ are obtained according to the kinetic energy of an element. Therefore, the mass matrix $m_i^c$ of the spatial beam element in the $oxy$ plane can be obtained by integrating $m_b$ and $m_s$.

The derivation of the element matrix in the $oxz$ plane is the same as that in the $oxy$ plane. The axial deformation of the blade along the $x$-axis can be regarded as a one-dimensional link element to derive the element
matrix. The stiffness matrix, the centrifugal stiffness matrix, and the mass matrix derived from the above three kinds of deformations are combined to form the whole stiffness, mass, and centrifugal stiffness matrixes of the spatial beam element. They can be written as $k^e$, $m^e$, and $k^ec$, respectively.

There is an angle between the local coordinate system and the global coordinate system. Therefore, the coordinate transformation needs to be performed so that the element matrixes can be converted from the local coordinate system to the global coordinate system. The key is to get the conversion matrix $T$. It is known that the $x$-axis of the local coordinate system coincides with the $X$ axis of the global coordinate system, while the angle between the $y$-axis and the $Y$-axis is $\beta_1$. The angle between the $z$-axis and the $Z$-axis is $\beta_1$ too. The coordinate conversion matrix $T$ can be obtained according to the angle between the coordinate axis. The element stiffness matrix $k^e$, the element centrifugal stiffness matrix $k^ec$, and the element mass matrix $m^e$ can be expressed as follows

$$
\begin{align*}
\tilde{k}^e &= T^T k^e T \\
\tilde{k}^ec &= T^T k^ec T \\
\tilde{m}^e &= T^T m^e T
\end{align*}
$$

where the matrix $k^e$, $k^ec$, $m^e$, and $T$ are provided in Appendix 1.

**The establishment of dry friction mechanics model**

The loading situation of the blade is complicated when it is in a rotating condition. There will be a complex nonlinear force between the contact interfaces of the tenon-mortise joint. The dry friction model is introduced to simulate the loading situation of the blade based on the Coulomb friction law. The dry friction model is shown in Figure 3. Where $N_2$ is the centrifugal force of the blade; $A_1$ and $A_2$ represent the contact interfaces; $\gamma$ is the angle of tenon; $\phi$ and $\eta$ are the angles between the contact interface and the vertical direction. $Y(t)$ is the horizontal displacement of the tenon; $d_1$ and $d_2$ are the displacements of the contact interface and the vertical direction. $N_1$ and $N_2$ are the normal pressure on $A_1$ and $A_2$; $k_1$, $f_1$, and $\mu_1$ are the tangential stiffness, the friction force and the friction coefficient of $A_1$, respectively. The meaning of $k_2$, $f_2$, and $\mu_2$ are the same as $k_1$, $f_1$, and $\mu_1$. $w_1$ and $w_2$ are the displacements of the dry friction damper. Due to the existence of the installation angle $\beta_1$, when the external excitation $Q$ acts on the blade, there are two components along the $y$-direction and the $z$-direction of the local coordinate, $Q_1$ and $Q_2$, respectively. $\beta_2$ is the angle between the installation direction of the tenon and the axis of the disk, which is called the angle of the tenon installation. $\beta_2$ is zero in Figure 3. In this paper, we call it a straight tenon when $\beta_2$ is zero, or it is called a diagonal tenon.

Here, the following simplifications are made:

1. The effect of the friction force on the normal pressure on the contact interface is neglected.
2. The contact interfaces of the tenon-mortise structure are always in the contact state, and there is no separation phenomenon.
3. The displacement limit of the tenon is neglected.

In this paper, it is assumed that the tenon and dovetail are always in contact with each other and the dovetail is a rigid body, so the tenon in $x$ direction is completely constrained. However, due to the need for deducing the
friction force, we added the virtual displacement in the \( x \) direction into the force analysis. In addition, this paper does not pay attention to the modes in the centrifugal direction of the blade. In order to simplify the calculation, we constrain the degree of freedom in \( x \) direction of node 1 (where the tenon is located), and only use \( Y(t) \) which is made up of the displacement of \( y \) direction and \( z \) direction to calculate the friction.

It can be seen from the geometric relation in Figure 3

\[
\begin{align*}
    d_1 &= Y(t)/\cos \gamma \\
    d_2 &= Y(t)/\cos \gamma
\end{align*}
\]  

(6)

According to the dry friction model in Figure 3, the mathematical expression of friction force can be deduced as follows

\[
f_1 = \begin{cases} 
    k_1(d_1 - w_1) & k_1|d_1 - w_1| < \mu_1N_1 \\
    \mu_1N_1 \text{sgn}(\dot{w}_1) & k_1|d_1 - w_1| \geq \mu_1N_1
\end{cases} \\

f_2 = \begin{cases} 
    k_2(d_2 - w_2) & k_2|d_2 - w_2| < \mu_2N_2 \\
    \mu_2N_2 \text{sgn}(\dot{w}_2) & k_2|d_2 - w_2| \geq \mu_2N_2
\end{cases}
\]

(7)

While ignoring the effect of friction, the normal pressure acting on the \( A_1 \) and \( A_2 \) surfaces can be deduced according to the balance of force that can be expressed as

\[
N_1 = N_2 = \frac{N_\Omega}{2\cos \gamma}
\]

(8)

The centrifugal force of the blade is as follows

\[
N_\Omega = \int_R^{R+L} \rho A\Omega^2xdx
\]

(9)

where \( \Omega \) is the rotating speed.

As can be seen from equation (7), the friction force is segmented during the vibration period. The following describes the transformation relationship of the motion state on the contact surface \( A_1 \). The situation of \( A_2 \) is similar to \( A_1 \). When contact surface \( A_1 \) is in a sticking state, the displacement of the damper remains unchanged, \( \dot{w}_1 = 0 \); when contact surface \( A_1 \) is in a slipping state, the damper maintains the same velocity with the tenon contact surface, \( \dot{w}_1 = d_1 \), i.e.

\[
\dot{w}_1 = \begin{cases} 
    0 & \text{sticking state} \\
    d_1 & \text{slipping state}
\end{cases}
\]

(10)

The integral of the above equation is obtained

\[
w_1 = \begin{cases} 
    c_1 & \text{sticking state} \\
    d_1 + c_2 & \text{slipping state}
\end{cases}
\]

(11)

where \( c_1, c_2 \) are constant. The values of \( c_1, c_2 \) are unknown. \( c_1 \) and \( c_2 \) can be eliminated by subtracting the displacement in the previous period from the displacement in the later period of the damper. The results are as follows

\[
\Delta w_{1s} = w_{1s} - w_{1(s-1)} = \begin{cases} 
    0 & \Delta d_{1s} \text{ sticking state} \\
    \Delta d_{1s} & \text{slipping state}
\end{cases}
\]

(12)

where \( s \) is the order of iterations.

The friction forces \( f_1 \) and \( f_2 \) on the two contact surfaces are decomposed in the vertical and horizontal direction, i.e.

\[
\begin{align*}
    f_v &= f_1 \sin \gamma - f_2 \sin \gamma \\
    f_h &= f_1 \cos \gamma + f_2 \cos \gamma
\end{align*}
\]

(13)
where \( f_v = 0 \) because of \( f_1 = f_2 \). There is only the component of dry friction force \( f_h \) acting on the tenon. In this paper, because of the existence of the installation angle of the tenon \( \beta_2 \), the dry friction force acting on the tenon can be written as

\[
 f = f_h \cos \beta_2 \\
\]

Therefore, the friction force vector \( F(x,t) \) is obtained by putting \( f \) on the corresponding degree of freedom.

**Matrices assembly of the rotating blade system**

The blade is divided into a number of elements by using the finite element theory. Then assemble the element matrix into the global stiffness matrix \( K \), the global centrifugal stiffness matrix \( K_c \), and the global mass matrix \( M \), i.e.

\[
\begin{align*}
K &= \sum_{e=1}^{n} G_e^T k_e G_e \\
K_c &= \sum_{e=1}^{n} G_e^T k_{ce} G_e \\
M &= \sum_{e=1}^{n} G_e^T m_e G_e
\end{align*}
\]

where \( G_e \) is the element transformation matrix which is provided in Appendix 1. The damping of the system is expressed in terms of Rayleigh damping

\[
 C = \alpha M + \beta K
\]

\[
\alpha = 2(\xi_2/\omega_2 - \xi_1/\omega_1)/(1/\omega_2^2 - 1/\omega_1^2) \\
\beta = 2(\xi_2\omega_2 - \xi_1\omega_1)/(\omega_2^2 - \omega_1^2)
\]

where \( \alpha \) and \( \beta \) are the coefficients of the Rayleigh damping. \( \omega_1 \) and \( \omega_2 \) are the first order and the second order natural angular frequency of the cantilevered blade, \( \omega_1 = 1591.3 \text{ rad/s} \) and \( \omega_2 = 9876.1 \text{ rad/s} \); \( \xi_1 \) and \( \xi_2 \) are the damping coefficients.

The dynamic equation of the rotating blade system is written in the following form

\[
M\ddot{X} + C\dot{X} + (K + K_c)X = Q - F(x,t)
\]

where \( Q \) represents the external excitation vector.

In order to visualize the actual finite element model that is being developed, we draw a schematic of the actual finite element model which is shown in Figure 4. Where \( OXYZ \) is the global coordinate system and the coordinate origin \( O \) is in correspondence with the center of the disk. \( oxyz \) is the local coordinate system of the blade, and the coordinate origin is located at the blade root. \( OXYZ \) and \( oxyz \) have the same meaning as those in Figure 2.

**The model verification**

It can be known from equations (7), (8), and (9) that the friction force is proportional to the square of the rotational speed \( f \propto \Omega^2 \). Therefore, the response of the blade changes linearly when the rotating speed is high enough. The tenon-mortise structure is in a fixed state at this moment. According to equation (7), the friction generated on the contact interfaces is static friction when the contact interfaces are in sticking state. According to the friction model given in this paper, the boundary condition of the blade at this moment is linear elastic support.
boundary. So the verification of the natural characteristic is conducted by using ANSYS software when the rotating speed is 600 rad/s and 800 rad/s. The blade is divided into 30 elements by using BEAM188 element, and the COMBINE14 element is used to establish the constraint of the blade. The centrifugal stiffening effect is considered.

The excitation point and response point are shown in Figure 4, and we use the response amplitude in the $y$ direction of the response point as the $y$-axis of the subsequent amplitude–frequency curves. In Figure 5, the green and brown solid lines correspond to the resonance frequencies 310.4 Hz and 349.3 Hz of the blade, respectively, when the rotating speeds are 600 rad/s and 800 rad/s. The green and brown dashed lines in the figure are the natural frequencies of the blade obtained by ANSYS, which are 303.5 Hz and 337.4 Hz, respectively. The results calculated by ANSYS software are smaller than those of the present. The relative errors are $-2.2\%$ and $-3.4\%$, respectively, which are within the allowable range. Therefore, it can be proved that the results of this paper are correct.

**Numerical cases and discussion**

In this paper, the parameters of the rotating blade system are shown in Table 1.

The simple harmonic excitation $Q = P \cdot \sin(\omega t + \varphi)$ is applied on the middle of the blade along the tangent direction of the disk circumference. $P$ is the excitation amplitude and $P = 1.5$ kN; $\omega$ is the excitation angular frequency; $\varphi$ is the initial phase. The friction force on the contact interface is directly related to the pressure and the friction coefficient of that: the friction force is proportional to the square of the rotating speed ($f \propto \Omega^2$); the friction force is also proportional to the friction coefficient of the contact interface ($f \propto \mu$). Moreover, the damping characteristics of the system are changed by the change of the constraint state of the blade-root, which is related to the installation angle of the tenon and the tangential stiffness. Therefore, the response of the blade under the simple harmonic excitation $Q$ is solved. In other words, we did a harmonic analysis. The excitation
frequency ranges from 500 to 3000 rad/s, and the interval of excitation frequency is 10 rad/s. So we obtain results with a resolution of 10 rad/s. The influence of the rotating speed, the friction coefficient, the installation angle of the tenon, the tangential stiffness, the excitation amplitude and the excitation position on the damping reduction characteristics of the system are discussed as follows.

The rotating speed

The constraint state of the blade is affected by the change of the rotating speed. Moreover, the natural characteristics of the blade will change with the rotating speed. The amplitude–frequency curves of the blade under different rotating speeds are shown in Figure 6(a). It can be known that there is a difference in the amplitude–frequency curves when the rotating speeds are different. Because of the nonlinear friction at the blade root, the amplitude–frequency curve presents a nonlinear phenomenon. The response amplitude of the blade decreases, and the resonance peak shows a tendency to move right. Nevertheless, as the speed increases, the nonlinear

| Nos | Parameters                  | Value     |
|-----|-----------------------------|-----------|
| 1   | Disk radius (R)             | 350 mm    |
| 2   | Blade length (L)            | 150 mm    |
| 3   | Blade width (b)             | 60 mm     |
| 4   | Blade thickness (h)         | 7 mm      |
| 5   | Young's modulus (E)         | 200 GPa   |
| 6   | Poisson ratio (ν)           | 0.3       |
| 7   | Density (ρ)                 | 7850 kg/m³|
| 8   | Element number (n)          | 30        |
| 9   | Blade installation angle (β1) | π/6      |
phenomenon of amplitude–frequency curve gradually weakens. Therefore, the nonlinear phenomenon disappears when the rotating speed reaches 600 rad/s. However, when the rotating speed continues to increase, the stiffness of the blade system increases because of the centrifugal rigidity effect. When the rotation speed continues to increase, the increase of the system stiffness makes the natural blade’s frequency increase and its resonance peak decrease.

The nonlinear dry friction force on the tenon-mortise structure causes the nonlinear phenomenon shown by the response of the blade. The changes of the maximum friction force on blade root with the excitation angular frequency under different rotating speeds are shown in Figure 6(b). Take \( \Omega = 400 \text{ rad/s} \) as an example, when the excitation angular frequency is less than 1660 rad/s and greater than 1810 rad/s, the vibration amplitude of the blade does not reach the critical friction transition conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency is between 1660 rad/s and 1810 rad/s, the vibration amplitude of the blade reaches the critical friction transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has apparent segmental linear behavior, as well as the sticking state. As a result, the nonlinearity of the amplitude–frequency curve gradually weakens, and the response amplitude of the blade increases. Taking rotating speed 400 rad/s as an example, the maximum friction force keeps constant when the excitation frequency is within the range of 1660–1810 rad/s, which indicates that there is a relative slipping phenomenon between the contact interfaces, and it can be seen from Figure 6 that the nonlinear phenomenon in the frequency range is visible. The tenon-mortise structure will always stick when the rotating speed reaches high enough. Therefore, the energy-consuming ratio of the friction force is zero, and there is no nonlinearity in Figure 6. For example, the rotating speeds of 600 rad/s and 800 rad/s.

**The friction coefficient**

The change of the friction coefficient will affect the sliding friction force, thus affecting the constraint state and changing the blade characteristics. When the rotating speed is 300 rad/s, the amplitude–frequency curves of the blade under different friction coefficient are shown in Figure 7(a). It is known that the amplitude–frequency curves are nonlinear, and the vibration amplitude of the blade decreases due to the existence of the friction. However, when the friction coefficient increases, it is difficult for the contact interfaces to occur in the relative slipping state. As a result, the nonlinearity of the amplitude–frequency curve gradually weakens, and the response amplitude of the blade increases in the resonance region.

The variation of the maximum friction force with the excitation frequency is shown in Figure 7(b). Take \( \beta_2 = 20^\circ \) as an example, when the excitation angular frequency is less than 1450 rad/s and greater than 1780 rad/s, the vibration amplitude of the blade does not reach the critical friction transition conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency is between 1450 rad/s and 1780 rad/s, the blade’s vibration amplitude reaches the critical friction transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has apparent segmental linear behavior, as well as the overall nonlinear behavior. In addition, it is known that the change of the friction coefficient will not only affect the sliding friction force but also affect the range of the excitation frequency when the contact interfaces are in the relative slipping. The maximum friction force remains constant in the resonance region, which indicates that the relative slipping phenomenon of the contact interfaces occurs at the excitation frequency.
However, when the friction coefficient increases, the range of the excitation frequency where the maximum friction force remains constant decreases, which results in the nonlinearity of the response weakens.

The energy dissipation ratios of dry friction under different friction coefficients are shown in Figure 7(c). It can be known that it is difficult for the contact interfaces to be in a relative slipping state with the increase of the friction coefficient. Meanwhile, the energy dissipation ratio of the dry friction in the resonance region decreases gradually, which leads to a more significant response amplitude at the excitation frequency.

Sliding friction force varies with the coefficient of friction. Therefore, it is difficult for the contact interfaces to be in a relative slipping state when the coefficient of friction increases. This will lead to the decrease of the energy dissipation ratio of the dry friction and weaken the nonlinearity response, and the increase of the response amplitude under blade resonance.

As an example, we take a friction coefficient is 0.3 In Figure 7, when the coefficient of friction is 0.3. The maximum friction force will keep constant when the excitation frequency is within the range of 1560–1750 rad/s, which indicates that there is a relative slipping phenomenon between the contact interfaces, and the vibration dissipation phenomenon occurs at this moment. As a result, the response amplitude of the blade begins to decrease, and the amplitude–frequency curve shows a nonlinear phenomenon. On the contrary, the contact interfaces will remain in the sticking state, and the energy dissipation ratio is zero.

The installation angle of the tenon

The response of the blade under different installation angle of the tenon is shown in Figure 8(a). It can be known that the response of the blade has a noticeable difference when the installation angle of the tenon (β2) is different. The blade response amplitude will decrease gradually, and the resonance frequency has the trend of right shift with the increase of the installation angle of the tenon.

The variation of the maximum friction force on the contact interface with the excitation frequency is shown in Figure 8(b). Take $k = 0.5 \times 10^6$ N/m as an example, when the excitation angular frequency is less than 950 rad/s and greater than 1360 rad/s, the vibration amplitude of the blade does not reach the critical friction transition.
conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency is between 950 rad/s and 1360 rad/s, the blade’s vibration amplitude reaches the critical friction of transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has apparent segmental linear behavior, as well as the overall nonlinear behavior. In addition, it can be known from Figure 8(b) that the maximum friction force remains constant in the resonance region, which indicates that the contact interfaces exist in the relative slipping state. What is more, the excitation frequency range of the maximum friction force keeps constantly increasing when the installation angle of the tenon increases.

Figure 8(c) shows the energy-consuming ratios of dry friction under diverse installation angles. It is known that the energy-consuming ratio of dry friction in the resonance region of the blade increases with the installation angle of the tenon. There is the existence of the energy dissipation ratio when the maximum friction force keeps constant. On the contrary, the energy dissipation ratio is zero.

It can be known from equation (14) that the components of the friction force at the blade-root will change with the variation of the installation angle of the tenon, which changes the constraint state and the blade damping characteristics. The energy dissipation by dry friction begins to increase, and the vibration amplitude of the blade becomes smaller with the increase of the installation angle of the tenon. While taking the installation angle of 20° as an example, the maximum friction force remains constant when the excitation frequency is within the range of 1450 rad/s–1780 rad/s, which indicates that there is a relative slipping phenomenon between the contact interfaces, and the vibration dissipation phenomenon occurs at this moment. The response amplitude of the blade decreases.

**The tangential stiffness**

In Figure 3, the variation of the tangential stiffness \(k_1\) and \(k_2\) of the contact interfaces will affect the constraint force of the blade, thereby change the blade’s natural characteristics. The response of the blade is shown in Figure 9(a); it can be known that the response of the blade shows the soft nonlinearity when the tangential stiffness is small. However, the nonlinear phenomenon of the blade gradually transforms from soft
to hard when the tangential stiffness increases. The resonance peak decreases, and the resonance frequency increases than before. The amplitude–frequency curve has shown evident hard nonlinear phenomenon when the tangential stiffness reaches $2 \times 10^6$ N/m.

The slip-stick state of the contact interfaces is affected by the tangential stiffness. The variation of the maximum friction force with the excitation frequency under different tangential stiffness is shown in Figure 9(b). Take $k = 0.5 \times 10^6$ N/m as an example, when the excitation angular frequency is less than 950 rad/s and greater than 1360 rad/s, the vibration amplitude of the blade does not reach the critical friction transition conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency is between 950 rad/s and 1360 rad/s, the vibration amplitude of the blade reaches the critical friction transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has obvious segmental linear behavior, as well as the overall nonlinear behavior. In addition, the range of the excitation frequency where the maximum friction force remains constant varies significantly when the tangential stiffness is different. Moreover, when the tangential stiffness is more significant, the maximum friction force will keep constant firstly with the variation of the excitation frequency, then change with that, and lastly keep constant again.

The energy dissipation ratio of the dry friction with the variation of the excitation frequency under different tangential stiffness is shown in Figure 9(c). It can be known that the constraint state of the blade changes with the variation of the tangential stiffness. When the tangential stiffness increases, the stiffness of the blade system increases, which increases the blade’s natural frequency. Therefore, the maximum energy consuming ratio has a right shift trend and gradually decreases.

It can be known from equation (7) that the friction force is segmented. The variation of the tangential stiffness will cause the constraint state of the blade to change, which will affect the damping characteristics and the nonlinear characteristics of the blade. By taking the tangential stiffness of $0.5 \times 10^6$ N/m as an example, the maximum friction force keeps constant when the excitation frequency is within the range of 950–1360 rad/s, which indicates that there is a relative slipping phenomenon between the contact interfaces, and the vibration dissipation

Figure 9. The effect of the tangential stiffness. (a) Amplitude-frequency curve. (b) Maximum friction force-frequency curve. (c) Energy dissipation ratio-frequency curve.
phenomenon occurs at this moment. Then the blade’s vibration amplitude decreases. The tangential stiffness is smaller at this moment, which makes the blade’s response show a soft nonlinearity.

**The excitation amplitude**

The vibration displacement and the total energy of the blade are affected by the variation of the excitation amplitude. The blade’s response under different excitation amplitudes is shown in Figure 10(a). It can be known that the nonlinearity of the response becomes stronger, and the blade’s vibration amplitude increases with the increase of the excitation frequency. In Figure 10(a), a more significant vibration level at low excitation frequency when external excitation amplitude increases to 4 kN, which caused by the jumping phenomenon of the whole blade.

It is known that the variation of excitation amplitude does not affect the magnitude of sliding friction. However, the vibration displacement of the blade is affected by the excitation amplitude, which will affect the transformation of the stick-slip state between the contact interfaces. The variation of the maximum value of the friction force with the excitation frequency under different excitation amplitudes is shown in Figure 10(b). Take $P = 4.0 \text{kN}$ as an example, when the excitation angular frequency is greater than 1850 rad/s, the vibration amplitude of the blade does not reach the critical friction transition conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency is less than 1850 rad/s, the blade’s vibration amplitude reaches the critical friction transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has obvious segmental linear behavior, as well as the overall nonlinear behavior. In addition, it is found from the figure that the range of the excitation frequency where the maximum friction force remains constant increases with the excitation amplitude, which indicates that it is easy for the contact interfaces to be in the relative slipping state. Meanwhile, the maximum friction force also keeps constant when the excitation frequency is near the blade’s jumping frequency, which is caused by the blade’s jumping phenomenon.

![Figure 10](image-url). The effect of the excitation amplitude. (a) Amplitude-frequency curve. (b) Maximum friction force-frequency curve. (c) Energy dissipation ratio-frequency curve.
The energy dissipation ratio of the dry friction with the variation of the excitation frequency under different excitation amplitudes is shown in Figure 10(c). It can be known that the energy dissipation ratio of dry friction increases with the excitation amplitude. When the excitation amplitude is 4 kN, the frequency range where the energy dissipation phenomenon occurs is the largest because of the jumping phenomenon of the blade. The energy dissipation phenomenon exists near the jumping frequency of the blade, but the ratio is relatively smaller when the small excitation amplitude is acting.

The total vibration energy of the blade varies with the excitation amplitude acting on the blade. The larger the excitation amplitude, the greater the blade’s total energy, so that the blade will have a more significant vibration displacement. It can be known from equation (7) that the contact interfaces are accessible in the relative slipping when the vibration displacement is massive, which makes the blade’s response show nonlinearity. Taking the excitation amplitude of 3 kN as an example, the maximum friction force remains constant when the excitation frequency is less than 880 rad/s and within the range of 1000 rad/s–1850 rad/s. At this moment, the energy dissipation phenomenon occurs on the contact interfaces, and the response amplitude of the blade decreases.

It can be found from Figure 10(a) that there is a peak when the excitation amplitude is 4 kN, and the excitation frequency is 700 rad/s. This is caused by the jumping phenomenon of the blade. In order to verify the jumping phenomenon of the blade, the responses of the blade-tip and blade-root are introduced when the excitation amplitude is 4 kN, respectively. The results are shown in Figure 11.

It can be found from Figure 11(b) that the vibration amplitude of the blade-tip is about 7.5 mm and that of the blade-root is about 5.8 mm. This can prove that the blade does have a jumping phenomenon when the excitation frequency is about 700 rad/s. The jumping displacement of the blade is the primary vibration displacement. Therefore, there is a more significant response amplitude of the blade at this moment.

**The excitation position**

The effect of the excitation position on the amplitude-frequency curve of the blade is shown in Figure 12(a). It can be known that the position of the excitation has a significant effect on the amplitude–frequency curve of the blade. The nonlinear characteristics showed by the amplitude–frequency curve gradually transform from the soft type to

![Figure 11. The response of the blade. (a) The time-domain response. (b) The frequency spectrum.](image)
the hard type when the excitation position is transformed from the blade-tip to the blade root. The amplitude–frequency curve has shown a visible nonlinearity while the excitation acts on the middle of the blade.

The variation of the maximum friction force with the excitation frequency under different positions of the excitation is shown in Figure 12(b). Take 1/2 L as an example, when the excitation angular frequency ranges from 870 rad/s to 1480 rad/s and greater than 1780 rad/s, the vibration amplitude of the blade does not reach the critical friction transition conditions. The maximum friction force increases with the increase of vibration amplitude, which is consistent with the sticking state of equation (7). When the excitation angular frequency ranges from 1480 rad/s to 1780 rad/s and less than 870 rad/s, the blade's vibration amplitude reaches the critical friction transition conditions. At this time, the maximum friction force of the vibration amplitude changes and remains unchanged, which is consistent with the slipping state in equation (7). Thus, it can be found that the dry friction has apparent segmental linear behavior, as well as the overall nonlinear behavior. In addition, the range of the excitation frequency where the maximum friction force remains constant decreases when the excitation position gradually transforms from the blade-tip to the blade-root. What is more, no matter where the excitation is located, there is a sudden change in the friction force when the excitation frequency is 870 rad/s, which is caused by the jumping phenomenon of the blade.

The energy dissipation ratio of dry friction is shown in Figure 12(c). It can be known that the ratio gradually decreases when the excitation position transforms from the blade-tip to the blade-root. Meanwhile, the energy dissipation phenomenon also occurs at a low frequency because of the jumping phenomenon of the blade.

The bending moment at the blade-root is affected by the position of the excitation, which will affect the contact state of the tenon-mortise structure and the nonlinear characteristics of the blade. The energy dissipation ratio of the dry friction gradually decreases when the excitation position transforms from the blade-tip to the blade-root, but the response amplitude also decreases. This is because as the excitation position gradually approaches the blade-root, it is more difficult for the blade to vibrate, and the total energy of the blade decreases. Taking the excitation position at 1/2 L from the blade-tip as an example, the maximum of the friction force keeps in a constant value when the excitation frequency is less than 870 rad/s and within the range of 1480–1780 rad/s, which indicates that there is a relative slipping phenomenon between the contact interfaces, and the vibration dissipation phenomenon occurs at this moment. The amplitude–frequency curve shows a visible nonlinear phenomenon.

Figure 12. The effect of the excitation position. (a) Amplitude-frequency curve. (b) Maximum friction force-frequency curve. (c) Energy dissipation ratio-frequency curve.
Conclusions
This paper established a dynamic model of the flexible rotating blade, and the blade is discretized by using the spatial beam element. The dry friction mechanics model and the finite element theory are introduced to obtain the dynamic equation of the rotating blade system. The vibration dissipation mechanism of a flexible rotating blade with a dovetail joint is discussed. The results as follows:

The energy dissipation ratio of dry friction gradually decreases when the rotating speed increases. The contact interfaces will always be in sticking state, and there is no energy dissipation at this moment when the rotating speed is high enough. Besides, the tenon-mortise structure has a better damping effect, while the friction coefficient is small. Therefore, it is necessary to ensure the roughness and the wear resistance of the blade joints. Besides, the damping reduction effect of the blade increases with the installation angle of the tenon, which makes the response amplitude of the blade decrease. Therefore, it should be used with a reasonable installation angle when the blade is assembled. At last, the variation of the tangential stiffness and the excitation position will affect the nonlinear characteristics of the blade, which causes the change of the constraint state of the blade and affects the vibration reduction characteristics of the blade.

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Appendix

**Notation**

- **A**: cross-sectional area of the blade
- **A₁, A₂**: contact interfaces
- **B**: blade width
- **c₁, c₂**: constant
- **C**: system damping
- **d₁, d₂**: displacements of the contact interface a₁ and a₂
- **E**: Young’s modulus
- **fᵥ, fₓ**: two components of friction forces in the vertical and horizontal direction
- **f**: dry friction force acting on the tenon
- **F(x,t)**: friction force vector
- **G**: shearing modulus
- **Ge**: element transformation matrix
- **H**: blade thickness
- **I**: moment of inertia
- **I**: serial number of elements
- **k_b**: element bending stiffness matrix in the oxy plane
- **k_s**: element shearing stiffness matrix in the oxy plane
- **k_e**: total stiffness matrix of element in the oxy plane
- **k_c**: total centrifugal stiffness matrix of element in the oxy plane
- **k′_e**: total stiffness matrixes of element in local coordinate system
- **k_e**: total centrifugal stiffness matrixes of element in local coordinate system
- **k_c**: element centrifugal stiffness matrix in global coordinate system
- **k₁, f₁, μ₁**: tangential stiffness, friction force and friction coefficient of a₁
- **k₂, f₂, μ₂**: tangential stiffness, friction force and friction coefficient of a₂
- **K**: shear correction factor
- **K**: global stiffness matrix
- **K_c**: global centrifugal stiffness matrix
- **L**: element length
- **L**: blade length
- **m_b**: bending mass matrix of element in the oxy plane
$m_s$, shearing mass matrix of element in the $oxy$ plane

$m^e_1$, total mass matrix of element in the $oxy$ plane

$m^e$, total mass matrices of element in local coordinate system

$m^e_1$, element mass matrix in global coordinate system

$M$, global mass matrix

$N$, element number

$N^e_r(x_j)$, centrifugal force of any point

$N^e_\Omega$, centrifugal force of the blade

$N_1, N_2$, normal pressure on $a_1$ and $a_2$

$OXYZ$, global coordinate system of the blade disk

$Oxyz$, local coordinate system of the blade

$P$, excitation amplitude

$P$, an arbitrary point on the blade

$Q$, external excitation

$Q$, external excitation vector

$Q_1, Q_2$, two components of external excitation along the $y$-direction and the $z$-direction

$R$, disk radius

$T$, coordinate conversion matrix

$T^e_1$, element kinetic energy in the $oxy$ plane

$\Gamma$, tenon angle

$U^e_1$, element strain energy in the $oxy$ plane

$U^e_b$, element bending energy in the $oxy$ plane

$U^e_s$, element shearing energy in the $oxy$ plane

$U^e_c$, centrifugal strain energy of element in the $oxy$ plane

$u, v, w$, displacements of the blade in the direction of $x$, $y$ and $z$

$V$, Poisson ratio

$v^e(x_j)$, transverse displacement of any point

$w_1, w_2$, displacements of the dry friction damper

$x_j$, distance between any point in an element and the front end of the element

$X$, displacement vector

$Y(t)$, horizontal displacement of the tenon

$\gamma, \beta$, coefficients of the Rayleigh damping

$\beta_1$, blade installation angle

$\beta_2$, tenon installation angle

$\Omega$, rotating speed

$\phi, \eta$, angles between the contact interface and the vertical direction

$\rho$, density

$\omega_1, \omega_2$, first order and second order natural angular frequency of the cantilevered blade

$\zeta_1, \zeta_2$, damping coefficients correspond to $\omega_1$ and $\omega_2$

$\omega$, excitation angular frequency

$\Phi$, initial phase

### Appendix I

The matrix $m^e$, $k^e$, and $k^e_c$ can be given as

$$ m^e = \begin{bmatrix}
m_{1,1} & m_{1,2} & \cdots & m_{1,9} & m_{1,10} \\
m_{2,2} & \cdots & m_{2,9} & m_{2,10} \\
\vdots & \vdots & \vdots & \ddots \\
SyM & m_{9,9} & m_{9,10} \\
m_{10,10} & \end{bmatrix} $$
where the unspecified elements of matrix are 0

\[ l = L/n, \ k = 3/2, \ L_y = hb^3/12, \ L_y = hb^3/12, \ G = E/(2(1 + v)) \]

\[ A = b h, \ b_c = (12kEL_z)/(GAl^2), \ b_z = (12kEL_y)/(GAl^2) \]

\[ m_{11} = A\rho/3, \ m_{16} = A\rho/6 \]

\[ m_{22} = \left( A\rho\left(70b_y^3 + 147b_y + 78\right)/210(b_y + 1)^2 \right) \]

\[ m_{25} = \left( A\rho\left(35b_y^2 + 77b_y + 44\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{27} = \left( A\rho\left(35b_y^2 + 63b_y + 27\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{210} = -\left( A\rho\left(35b_y^2 + 63b_y + 26\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{3,8} = \left( A\rho\left(35b_y^2 + 77b_y + 44\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{3,9} = \left( A\rho\left(35b_y^2 + 63b_y + 26\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{4,8} = -\left( A\rho\left(35b_y^2 + 63b_y + 26\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{5,5} = A\rho\left(840(b_y + 1)^2 + \hat{P}/120\right) \]

\[ m_{5,10} = A\rho\left(840(b_y + 1)^2 - \hat{P}/120\right) \]

\[ m_{6,6} = A\rho/3 \]

\[ m_{7,7} = \left( A\rho\left(70b_y^2 + 147b_y + 78\right)/\left(210(b_y + 1)^2\right) \right) \]

\[ m_{7,10} = -\left( A\rho\left(35b_y^2 + 77b_y + 44\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{8,8} = \left( A\rho\left(70b_y^2 + 147b_y + 78\right)/\left(210(b_y + 1)^2\right) \right) \]

\[ m_{8,9} = \left( A\rho\left(35b_y^2 + 77b_y + 44\right)/\left(840(b_y + 1)^2\right) \right) \]

\[ m_{9,9} = A\rho\left(840(b_y + 1)^2 + \hat{P}/120\right) \]

\[ m_{9,10} = A\rho\left(840(b_y + 1)^2 + \hat{P}/120\right) \]

\[ \mathbf{k}^* = \begin{bmatrix} 
  k_{1,1} & k_{1,2} & \cdots & k_{1,9} & k_{1,10} \\
  k_{2,2} & \cdots & k_{2,9} & k_{2,10} \\
  \vdots & \vdots & \vdots & \vdots \\
  \text{SYM} & \text{SYM} & \cdots & \text{SYM} \\
  k_{9,9} & k_{9,10} & \cdots & k_{10,10} 
\end{bmatrix} \]

where the unspecified elements of matrix are 0

\[ k_{1,1} = AE/l, \ k_{1,6} = -AE/l, \ k_{2,2} = (12EL_z)/(\hat{P}(b_y + 1)) \]

\[ k_{2,7} = -\left(12EL_z\right)/(\hat{P}(b_y + 1)), \ k_{2,10} = (6EL_z)/(\hat{P}(b_y + 1)) \]

\[ k_{3,4} = -\left(6EL_z\right)/(\hat{P}(b_y + 1)), \ k_{3,8} = -\left(12EL_z\right)/(\hat{P}(b_y + 1)) \]

\[ k_{4,4} = EL_z/l + \left(EL_z(4b_y + 4) - EI_y(b_y + 1)\right)/(l(b_y + 1)^2) \]

\[ k_{4,8} = (6EL_z)/(\hat{P}(b_y + 1)) \]

\[ k_{4,9} = \left(EL_z(2b_y + 2) + EI_y(b_y + 1)\right)/(l(b_y + 1)^2) - EL_z/l \]

\[ k_{5,5} = \left(EL_z(4b_y + 4) - EI_y(b_y + 1)\right)/(l(b_y + 1)^2) \]

\[ k_{5,7} = -\left(6EL_z\right)/(\hat{P}(b_y + 1)) \]

\[ k_{5,10} = \left(EL_z(2b_y + 2) + EI_y(b_y + 1)\right)/(l(b_y + 1)^2) - EL_z/l, \ k_{6,6} = AE/l \]

\[ k_{7,7} = (12EL_z)/(\hat{P}(b_y + 1)), \ k_{7,10} = -\left(6EL_z\right)/(\hat{P}(b_y + 1)) \]

\[ k_{8,8} = \left(6EL_z\right)/(\hat{P}(b_y + 1)) \]

\[ k_{9,9} = EI_y/l + \left(EL_z(4b_y + 4) - EI_y(b_y + 1)\right)/(l(b_y + 1)^2) \]

\[ k_{10,10} = EL_z/l + \left(EL_z(4b_y + 4) - EI_y(b_y + 1)\right)/(l(b_y + 1)^2) \]

\[ \mathbf{k}^c = \begin{bmatrix} 
  k_{1,1}^c & k_{1,2}^c & \cdots & k_{1,9}^c & k_{1,10}^c \\
  k_{2,2}^c & \cdots & k_{2,9}^c & k_{2,10}^c \\
  \vdots & \vdots & \vdots & \vdots \\
  \text{SYM} & \text{SYM} & \cdots & \text{SYM} \\
  k_{9,9}^c & k_{9,10}^c & \cdots & k_{10,10}^c 
\end{bmatrix} \]
where the unspecified elements of matrix are 0

\[
\begin{bmatrix}
k_{x,2} & k_{x,5} & k_{x,7} & k_{x,10} \\
k_{y,2} & k_{y,5} & k_{y,7} & k_{y,10} \\
k_{z,2} & k_{z,5} & k_{z,7} & k_{z,10} \\
k_{oxz,2} & k_{oxz,5} & k_{oxz,7} & k_{oxz,10}
\end{bmatrix}
= \mathbf{k}_{\text{oxy}},
\quad
\begin{bmatrix}
k_{3,3} & k_{3,4} & k_{3,8} & k_{3,9} \\
k_{4,3} & k_{4,4} & k_{4,8} & k_{4,9} \\
k_{5,3} & k_{5,4} & k_{5,8} & k_{5,9} \\
k_{6,3} & k_{6,4} & k_{6,8} & k_{6,9}
\end{bmatrix}
= \mathbf{k}_{\text{oxz}}
\]

\(\mathbf{k}_{\text{oxy}}\) and \(\mathbf{k}_{\text{oxz}}\) can be derived as follows:

\[
\alpha_0 = -\frac{3}{2} \hat{F}^2 + \frac{3}{2} \hat{F}^2 i - \frac{3}{2} \hat{F}^2 n + \hat{F}^{-1} n + \hat{F}^{-1} \hat{R} - li \hat{R} + \ln \hat{R} \quad \alpha_1 = 2l - 2li - R, \quad \alpha_2 = -1/2
\]

\[
F_{\text{oxy}} = \rho A (\Omega \cos \beta_1)^2 (\alpha_2 x^2 + \alpha_1 x + \alpha_0)
\quad
F_{\text{oxz}} = \rho A (\Omega \sin \beta_1)^2 (\alpha_2 x^2 + \alpha_1 x + \alpha_0)
\]

\[
\mathbf{N}_b = \begin{bmatrix} 1 - 3x^2/\hat{F}^2 + 2x^3/\hat{F}^3 & x - 2x^2/\hat{F} + x^3/\hat{F}^2 & 3x^2/\hat{F}^2 - 2x^3/\hat{F}^3 & x^3/\hat{F}^2 - x^2/\hat{F} \end{bmatrix}
\]

\[
\mathbf{N}_a = \begin{bmatrix} 1 - x/l & 0 & x/l & 0 \end{bmatrix}
\]

\[
\mathbf{B}_{\text{oxy--bend}} =
\begin{bmatrix}
1/(1 + b_y) & -b_y / (2(1 + b_y)) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/(1 + b_y) & b_y / (2(1 + b_y)) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{B}_{\text{oxz--bend}} =
\begin{bmatrix}
1/(1 + b_z) & -b_z / (2(1 + b_z)) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/(1 + b_z) & b_z / (2(1 + b_z)) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{S}_{\text{oxy--bend}} =
\begin{bmatrix}
b_y / (1 + b_y) & b_y / (2(1 + b_y)) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b_y / (1 + b_y) & -b_y / (2(1 + b_y)) \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{S}_{\text{oxz--bend}} =
\begin{bmatrix}
b_z / (1 + b_z) & b_z / (2(1 + b_z)) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b_z / (1 + b_z) & -b_z / (2(1 + b_z)) \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{k}_{\text{oxy--bend}} = F_{\text{oxy}} \left( \mathbf{B}_{\text{oxy--bend}}^T (\mathbf{dN}_b / \mathbf{d}x)^T + \mathbf{S}_{\text{oxy--bend}}^T (\mathbf{dN}_a / \mathbf{d}x)^T \right) \left( (\mathbf{dN}_b / \mathbf{d}x) \mathbf{B}_{\text{oxy--bend}} + (\mathbf{dN}_a / \mathbf{d}x) \mathbf{S}_{\text{oxy--bend}} \right)
\]

\[
\mathbf{k}_{\text{oxz--bend}} = F_{\text{oxz}} \left( \mathbf{B}_{\text{oxz--bend}}^T (\mathbf{dN}_b / \mathbf{d}x)^T + \mathbf{S}_{\text{oxz--bend}}^T (\mathbf{dN}_a / \mathbf{d}x)^T \right) \left( (\mathbf{dN}_b / \mathbf{d}x) \mathbf{B}_{\text{oxz--bend}} + (\mathbf{dN}_a / \mathbf{d}x) \mathbf{S}_{\text{oxz--bend}} \right)
\]

\[
\mathbf{k}_{\text{oxy}} = \int_0^l \mathbf{k}_{\text{oxy--bend}} \mathbf{d}x, \quad \mathbf{k}_{\text{oxz}} = \int_0^l \mathbf{k}_{\text{oxz--bend}} \mathbf{d}x
\]
The matrix $T$ and $G^e$ can be given as

$$
T = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \beta_1 & -\sin \beta_1 \\
0 & \sin \beta_1 & \cos \beta_1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \beta_1 & -\sin \beta_1 \\
\sin \beta_1 & \cos \beta_1 \\
1 & 0 & 0 \\
0 & \cos \beta_1 & -\sin \beta_1 \\
0 & \sin \beta_1 & \cos \beta_1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \beta_1 & -\sin \beta_1 \\
\sin \beta_1 & \cos \beta_1 \\
\end{bmatrix}
$$

where the unspecified elements of matrix are 0

$$
G^e = \begin{bmatrix}
G_{1,1} & G_{1,2} & \cdots & G_{1,5(1+n)-1} & G_{1,5(1+n)} \\
G_{2,1} & G_{2,2} & \cdots & G_{2,5(1+n)-1} & G_{2,5(1+n)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
G_{9,1} & G_{9,2} & \cdots & G_{9,5(1+n)-1} & G_{9,5(1+n)} \\
G_{10,1} & G_{10,2} & \cdots & G_{10,5(1+n)-1} & G_{10,5(1+n)} \\
\end{bmatrix}
$$

where the unspecified elements of matrix are 0

$$
\begin{bmatrix}
G_{1,5i-4} & G_{1,5i-3} & \cdots & G_{1,5i+4} & G_{1,5i+5} \\
G_{2,5i-4} & G_{2,5i-3} & \cdots & G_{2,5i+4} & G_{2,5i+5} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
G_{9,5i-4} & G_{9,5i-3} & \cdots & G_{9,5i+4} & G_{9,5i+5} \\
G_{10,5i-4} & G_{10,5i-3} & \cdots & G_{10,5i+4} & G_{10,5i+5} \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix}
$$