Non Linear Diffusion and Wave Damped Propagation: Weak Solutions and Statistical Turbulence Behavior

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Abstract

We present Mathematical arguments on the existence and uniqueness of weak solutions for a special class of non-linear parabolic and hyperbolic equations of the Mathematical-Physics subject to random initial conditions sampled in $L^2(\Omega)$ space (homogeneous statistical turbulence). In order to achieve such goal, we use the results of compactness on $L^p$ spaces with weak topologies, instead of the somewhat elaborated results from the methods of parabolic variational inequalities often utilized in such mathematical studies.

1 Introduction

One of the most important problems in the Mathematical-Physics of the nonlinear diffusion and wave damped propagation is to establish the existence and uniqueness of weak solutions in some convenient Banach Spaces for the associated non-linear evolution equation (see ref. [1], [3], [4], [5]). Another important class of initial-value problems for nonlinear diffusion – damping came from statistical turbulence as modeled by nonlinear diffusion or damped hyperbolic partial differential equations with random initial conditions associated to Gaussian processes sampled in certain Hilbert spaces ([2], [5]) and simulating the Turbulence physical
phenomena [6].

The purpose of this note is to contribute for such mathematical-physicist studies by using functional spaces compacity arguments in order to produce proofs for the existence and uniqueness of weak solutions for a class of special non-linear diffusion equations on a “smooth” $C^\infty$ domain with compact closure $\Omega \subset \mathbb{R}^3$ with Dirichlet boundary condition and initial conditions belonging to the space $L^2(\Omega)$. This study is presented in Section 2.

In Section 3, we present similar analysis for to the Wave equation with a non-linear damping analogous in its form to the non linear term studied on Section 2 for the diffusion equation.

In Section 4, we present in some details the solution of the associated problem for random initial conditions in terms of our proposed cylindrical functional measures representations previously proposed on the literature ([6]) and defined by a cylindrical measure on the Banach space $L^\infty((0, T), L^2(\Omega))$, a new result on the subject.

Finally in Section 5, we present a complementary semi-group analyzes on the very important problem of anomalous diffusion for random conditions in the path-integral framework.

2 The Theorem for Parabolic Non-Linear Diffusion

Let us consider the following non-linear diffusion equation in some strip $\Omega \times [0, T] \subset \mathbb{R}^4$

$$\frac{\partial U(x,t)}{\partial t} = (-AU)(x,t) + \Delta (F(U(x,t))) + f(x,t)$$

with the initial and Dirichlet boundary condition

$$U(x,0) = g(x) \in L^2(\Omega) \subset L^1(\Omega)$$

$$U(x,t)|_{\partial \Omega} \equiv 0$$

where $A$ denotes a second order self-adjoint uniform elliptic positive differential operator, $F(x)$ is a real function continuously differentiable on the extended real line $(-\infty, +\infty)$ with its derivative $F'(x)$ strictly positive on $(-\infty, +\infty)$. The external source $f(x,t)$ is supposed to be on the space $L^\infty([0, T] \times L^2(\Omega)) = L^\infty([0, T], L^2(\Omega))$.

We now state the existence and uniqueness theorem of ours.
Theorem 1. In the initial-value non linear diffusion eq.(1) – eq.(3), for any \( g(x) \in L^2(\Omega) \), and \( A = -\Delta \) (minus Laplacean) there exists an unique solution \( \bar{U}(x, t) \) on \( L^\infty([0, T] \times L^2(\Omega)) \) satisfying this problem in a certain weak sense with a test functional space given by \( C_0^\infty([0, T], H^2(\Omega) \cap H^1_0(\Omega)) \).

Proof. The existence proof will be given for a general \( A \) as stated below eq.(3). Let \( \{ \varphi_i(x) \} \) be the spectral eigenfunctions associated to the operator \( A \). Note that each \( \varphi_i(x) \in H^2(\Omega) \cap H^1_0(\Omega) \) and this set is complete on \( L^2(\Omega) \) as a result of the Gelfand Generalized Spectral theorem applied for \( A \) ([1]) since \( H^2(\Omega) \cap H^1_0(\Omega) \) is compactly immersed in \( L^2(\Omega) \).

Consider the following system of non-linear ordinary differential equations associated to eq.(1)–eq.(3) – the well-known Galerkin system ([1]).

\[
\left( \frac{\partial U^{(n)}(x, t)}{\partial t}, \varphi_j(x) \right)_{L^2(\Omega)} + (AU^{(n)}(x, t), \varphi_j(x))_{L^2(\Omega)} \\
= (\nabla \cdot [(F'(U^{(n)}(x, t)))\nabla U^{(n)}(x, t)], \varphi_j(x))_{L^2(\Omega)} \\
+ (f^{(n)}(x, t), \varphi_j(x))_{L^2(\Omega)}
\]

subject to the initial-condition

\[
U^{(n)}(x, 0) = \sum_{i=1}^{n} (g, \varphi_i)_{L^2(\Omega)} \cdot \varphi_i(x)
\]

where the Finite-dimensional Galerkin approximants are given exactly in terms of the spectral basis \( \{ \varphi_i(x) \} \) as

\[
U^{(n)}(x, t) = \sum_{i=1}^{n} U_i^{(n)}(t) \varphi_i(x)
\]

\[
f^{(n)}(x, t) = \sum_{i=1}^{n} (f(x, t), \varphi_i(x))_{L^2(\Omega)} \varphi_i(x)
\]

and \( (, )_{L^2(\Omega)} \) is the usual inner product on \( L^2(\Omega) \). Note that

\[
U^{(n)}(x, t)\big|_{\partial\Omega} = \sum_{i=1}^{n} U_i^{(n)}(t)(\varphi_i(x))\big|_{\partial\Omega} = 0.
\]

Let us introduce the short notations

\[
U^{(n)}(x, t) \equiv U^{(n)}
\]
\[(g, \varphi_i)_{L^2(\Omega)} \equiv g_i^{(n)} \quad (10)\]

\[(f(x, t), \varphi_i(x))_{L^2(\Omega)} = f_i^{(n)}. \quad (11)\]

By multiplying the Galerking system eq.(4) by \(U^{(n)}\) as usual ([3]) we get the a priori identity

\[
\frac{1}{2} \frac{d}{dt} ||U^{(n)}||^2_{L^2(\Omega)} + (AU^{(n)}, U^{(n)})_{L^2(\Omega)}
+ \int_{\Omega} dx F'(U^{(n)})(\nabla U^{(n)} \nabla U^{(n)}) = (f, U^{(n)})_{L^2(\Omega)}
\quad (12)
\]

By a direct application of the G\"ardeing-Poincaré inequality to the quadratic form associated to the operator \(A\), one has that there is a positive constant \(\gamma(\Omega)\) such that ([3])

\[(AU^{(n)}, U^{(n)})_{L^2(\Omega)} \geq \gamma(\Omega)||U^{(n)}||^2_{L^2(\Omega)}. \quad (13)\]

This yields the following estimate for any integer positive \(p\)

\[
\frac{1}{2} \frac{d}{dt} (||U^{(n)}||^2_{L^2(\Omega)}) + \gamma(\Omega)||U^{(n)}||^2_{L^2(\Omega)} + \|(F'(U^{(n)}))^{1/2} \nabla U^{(n)}||^2_{L^2(\Omega)}
\leq \frac{1}{2} \left\{ p||f(x, t)||^2_{L^2(\Omega)} + \frac{1}{p}||U^{(n)}||^2_{L^2(\Omega)} \right\}. \quad (14)
\]

At this point we observe that the square root of the function \(F'(x)\) makes sense since it is always positive \(F'(x) > 0\).

By choosing on eq.(14) \(p\) big enough and such that \(a_p \equiv \gamma(\Omega) - \frac{1}{2p} > 0\), we have that

\[
\frac{1}{2} \frac{d}{dt} (||U^{(n)}||^2_{L^2(\Omega)}) + a_p||U^{(n)}||^2_{L^2(\Omega)} \leq \frac{1}{2} p||f||^2_{L^2(\Omega)}(t), \quad (15)
\]

or by the Gronwall lemma, there is a uniform constant \(M\) \([even for T = +\infty if f \in L^2([0, \infty], L^2(\Omega))]\)

such that

\[
\sup_{0 \leq t \leq T} (||U^{(n)}(t)||^2_{L^2(\Omega)}) \leq \sup_{0 \leq t \leq T} \left\{ e^{-a_p t} \left[ \int_0^T ds ||f||^2_{L^2(\Omega)}(s) e^{+a_p s} + ||g||^2_{L^2(\Omega)} \right] \right\} = M. \quad (16)
\]

Note that the Galerkin system of (non-linear) ordinary differential equations for \(\{U_i^{(n)}(t)\}\)

has unique global solution on the interval \([0, T]\) as consequence of eq.(16) – which means that

\[
\sum_{i=1}^n (U_i^{(n)}(t))^2 \leq M \quad \text{and the Peano-Caratheodory theorem since } f(t, \cdot) \in L^\infty([0, T]).
\]

As a consequence of the Banach-Alaoglu theorem applied to the bounded set \(\{U^{(n)}\}\) on \(L^\infty([0, T], L^2(\Omega))\), there is a sub-sequence weak-star convergent to an element (function) \(\overline{U}(t, x) \in L^\infty([0, T], L^2(\Omega))\).
\( L^\infty([0, T], L^2(\Omega)) \) and this subsequence will still be denoted by \( \{U^{(n)}(x, t)\} \) in the analysis that follows.

A important remark is at this point. Since \( F(x) \) is a Lipschtizian function on the closed interval where it is defined we have that the set of functions \( \{F(U^{(n)})\} \) is a bounded set on \( L^\infty([0, T], L^2(\Omega)) \) if the set \( \{U^{(n)}\} \) has this property.

As a consequence of this remark and of a priori inequalite given in details on Appendix A, we can apply the famous Aubin-Lion theorem ([1]) to insures the strong convergence of the sequence \( \{U^{(n)}(x, t)\} \) to the function \( \overline{U}(x, t) \). As a straightforward consequence of our special non-linearity on eq.(1), namely \( F(x) \) is a Lipschitizian function, we obtain, thus, the strong convergence of the sequence \( \{F(U^{(n)}(x, t))\} \) to the \( L^2(\Omega) \)-function \( F(\overline{U}(x, t)) \).

As a consequence of the above made comments and by noting that \( L^2(\Omega) \) is continuously immersed on \( H^{-2}(\Omega) \) and the weak star convergence definition, we have the weak equation below in the test space function of \( v(x, t) \in C_c^\infty([0, T], H^2(\Omega) \cap H^1_0(\Omega)) \)

\[
\lim_{n \to \infty} \int_0^T dt \left[ \left( -U^{(n)}, \frac{dv}{dt} \right)_{L^2(\Omega)} + (AU^{(n)}, v)_{L^2(\Omega)} - (F(U^{(n)}), \Delta v)_{L^2(\Omega)} \right] = \lim_{n \to \infty} \int_0^T dt (f^{(n)}, v)_{L^2(\Omega)} \tag{17}
\]

or by passing the weak-star limit

\[
\int_0^T dt \left[ \left( -\overline{U}(x, t), \frac{dv(x, t)}{dt} \right)_{L^2(\Omega)} + (\overline{U}(x, t), (Av)(x, t))_{L^2(\Omega)} - (F(\overline{U}(x, t)), \Delta v(x, t))_{L^2(\Omega)} \right] = \int_0^T dt (f(x, t), v(x, t))_{L^2(\Omega)} \tag{18}
\]

This concludes the existence proof or our Theorem 1, since \( v(0, x) = v(T, x) = 0 \).

Let us apply it for a concrete problem of random exponential non-linear diffusion on a cube \( [0, L]^3 \subset R^3 \), mainly for explanation of the above written abstract results

\[
\frac{\partial U(x, t)}{\partial t} = \nabla \left\{ \left( \frac{k_0}{2} + e^{-U(x, t)} \right) \nabla U(x, t) \right\} + \frac{k_0}{2} \Delta U(x, t) \tag{19}
\]

with

\[
U(x, 0) = g(x) \in L^2(\Omega) \tag{20}
\]
where \( g(x) \) are the samples of a stochastic Gaussian process on \( \Omega \) with correlation function defining a operator of Trace class on \( L^2(\Omega) \). Namely

\[
E\{g(x)g(y)\} = K(x, y) \in \int_1^\infty (L^2(\Omega))
\]

or in terms of the spectral set associated to the Laplacian, i.e.,

\[
g(x) = \lim_{n \to \infty} \sum_{i=0}^{n} g_i^{(n)} \varphi_i(x)
\]

with

\[
E\{g_i g_k\} = K_{ik} = \int_{\Omega \times \Omega} dxdy K(x, y) \varphi_i(x) \varphi_k(y).
\]

As a consequence of the Theorem 1, an explicitly solution of the equation (19) must be taken in \( L^\infty([0, \infty), L^2(\Omega)) \) in the weak sense of eq.(18) and given thus by the Trigonometrical series sequence of functions:

\[
U(x, t, [g]) = \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \left\{ \sum_{i,j,k=1}^{n_1,n_2,n_3} d^{i,j,k}(t, [g_0]) \left( \frac{i\pi}{L} x \right) \left( \frac{j\pi}{L} y \right) \left( \frac{k\pi}{L} z \right) \right\}
\]

where the set of absolutely continuous function \( \{d^{i,j,k}(t, [g_0])\} \) on \([0, T]\) satisfy the Galerkin set of ordinary differential equations with “random” Gaussian initial conditions

\[
d^{i,j,k}(0, [g_0]) = \left( g(x), \left( \frac{i\pi}{L} x \right), \left( \frac{j\pi}{L} y \right), \left( \frac{k\pi}{L} z \right) \right)_{L^2(\Omega)} = g_0.
\]

Let us now comment on the uniqueness solution problem for the non-linear initial-value diffusion eq.(1) - eq.(3).

In the case under study, the uniqueness result comes from the following technical lemma

**Lemma 1.** If \( \overline{U}_1 \) and \( \overline{U}_2 \) in \( L^\infty([0, T] \times L^2(\Omega)) \) are two functions satisfying the weak relationship, for any \( v \in C_0^\infty([0, T], H^2(\Omega) \cap H^1_0(\Omega)) \)

\[
\int_0^T dt \left\{ \left( \overline{U}_1 - \overline{U}_2, -\frac{\partial v}{\partial t} \right)_{L^2(\Omega)} + (\overline{U}_1 - \overline{U}_2, Av)_{L^2(\Omega)} - (F(\overline{U}_1) - F(\overline{U}_2); \Delta v)_{L^2(\Omega)} \right\} \equiv 0
\]

(27-a)
then \( \overline{U}_1 = \overline{U}_2 \) a.e in \( L^\infty([0, T] \times L^2(\Omega)) \) ([5]).

Let us notice that eq.(27-a) is a weak statement on the assumption of vanishing initial values and it is a consequence of

\[
\lim_{t \to 0^+} \text{sup} \text{ ess} \left\{ \int_{\Omega} d^2x \left| \overline{U}_1(x, t) - \overline{U}_2(x, t) \right|^2 \right\} = 0. \tag{27-b}
\]

The proof of eq.(27) is a direct consequence of the fact that \( F(x) \) is a Lipschitz function satisfying an inequalite of the form

\[
|F(\overline{U}_1) - F(\overline{U}_2)|(x, t) \leq (\sup_{-\infty < x < +\infty} F'(x))|\overline{U}_1 - \overline{U}_2|(x, t); \tag{27-c}
\]

(for a technical proof see the Avner Friedmon book on ref. [5]). However in the case of \( A = -\Delta \) (minus Laplacian), as stated in our theorem, the idendity eq.(27-a) means that for functions of the form \( v(x, t) = e^{-\lambda t}v_\lambda(x) \in C_0^\infty([0, T], H^2(\Omega) \cap H^1_0(\Omega)) \) with \( \Delta v_\lambda(x) = -\lambda v_\lambda(x) \), the following identity must holds true

\[
\int_0^T dt (F(\overline{U}_1) - F(\overline{U}_2), v_\lambda)_{L^2(\Omega)} \equiv 0 \tag{28}
\]

which means that \( F(\overline{U}_1) = F(\overline{U}_2) \) a.e. and thus, \( \overline{U}_1 = \overline{U}_2 \) a.e., since \( F(x) \) satisfies the lower bound estimative

\[
|F(x) - F(y)| \geq (\inf_{-\infty < x < +\infty} |F'(x)|)|x - y|.
\]

This result produces a rigorous uniqueness result for \( A = -\Delta \).

Another important example of non-linear diffusion equation somewhat related to eq.(1) is the density equation for a gas in a porous medium with physical saturation. In this case the law of conservation of mass

\[
\nabla \cdot (\rho \vec{V}) + \frac{\partial \rho}{\partial t} = 0 \tag{29}
\]

where \( \vec{V} \) is the velocity of gas, \( \rho \) the density of the gas and \( P \) is the pressure, together with the Darcy’s law and the isothermic equation of state, namely

\[
P = c\rho^\gamma \tag{30}
\]

\[
\vec{V} = -k\nabla P
\]
leads to the following non-linear diffusion equation

\[
\begin{cases}
\frac{\partial \rho(x,t)}{\partial t} = -\gamma ck \nabla (\rho \nabla \rho)(x, t) \\
\rho(x, 0) = f(x) \in L^2(\Omega)
\end{cases}
\tag{31}
\]

Since it is showed that for finite volume open, convex with \(C^\infty\)-boundary domain \(\Omega\) as in our study, the function \(\rho(x, t)\) which satisfies eq.(31) should be a bounded function for any \(t \geq t_0 > 0\) (with \(t_0\) fixed) [5], it is clear that one can replace the above written mathematical Gas Porous medium equation by a more physical equation of the form taking into account the physical phenomena of saturation

\[
\begin{cases}
\frac{\partial \rho(x,t)}{\partial t} = -\gamma c \nabla (F(\varepsilon)(\rho) \nabla \rho)(x, t) \\
\rho(x, 0) = f(x) \in L^2(\Omega)
\end{cases}
\tag{32}
\]

with \(F(\varepsilon)(x)\) denoting a differentiable regularizing function of the function \(x^\gamma\) on the interval \([0, M]\), where \(M\) is the global upper-bound of the function \(\rho(x, t)\).

One can thus apply our Theorem 1 to obtain an explicitly set of functions \(\{\rho^{(n)}(\varepsilon)(x, t)\}\) converging in the weak star topology of \(L^\infty([t_0, \infty), L^2(\Omega))\) to the solution \(\rho(\varepsilon)(x, t)\) of the problem eq.(32). It is a reasonable conjecture that in the non-saturation case, one should takes the \(\varepsilon \to 0\) limit on eq.(32), namely: \(\rho(x, t) = \lim_{\varepsilon \to 0} \rho(\varepsilon)(x, t)\); since \(F(\varepsilon)(x)\) converge to \(x^\gamma\) in the \(C^\infty\)-topology of \(C([0, M])\); As a consequence it is expected that a solution for the physical equation eq.(31) should be produced. A full technical description of this limiting process will appears in a extended mathematical oriented paper.

3 The Hyperbolic non-linear damping

We aim on this third section state a theorem analogous to the Theorem 1 of paragraph 2 on diffusion but now for the important case of existence of non-linear damping on the Hyperbolic initial-value problem on \(\Omega \times [0, T]\) with imposed Dirichlet boundary conditions, including the new global case of \(T = +\infty\) (see Section 2) and a damping positive constant \(\nu\).

\[
\frac{\partial^2 U(x,t)}{\partial t^2} + (AU)(x,t) = -\nu \frac{\partial U(x,t)}{\partial t} + \Delta \left(F\left(\frac{\partial U}{\partial t}(x,t)\right)\right) + f(x,t)
\tag{33}
\]
The $L^2(\Omega)$-initial conditions are given by

$$U(x, 0) = g(x) \in L^2(\Omega)$$  \hspace{1cm} (34-a)$$

$$U_t(x, 0) = h(x) \in L^2(\Omega)$$  \hspace{1cm} (34-b)$$

$$U(x, t)|_{\partial \Omega} = 0$$  \hspace{1cm} (34-c)$$

and now the non-homogenous term $f(x, t)$ is considered to be a function belonging to the functional space

$$L^2([0, T] \times \Omega) \cap L^\infty([0, T], L^2(\Omega)).$$  \hspace{1cm} (35)$$

We have thus the following theorem of existence (without uniqueness)

**Theorem 2.** There exists a solution $\bar{U}(x, t)$ on $L^\infty([0, T] \times L^2(\Omega))$ for eq.(33)–eq.(35) in the weak sense with a test functional space as given by $C^\infty_0([0, T], H^2(\Omega) \cap H^1_0(\Omega))$. In order to arrive at such theorem, let us consider the analogous of the estimate eq.(12) for eq.(33) with $\dot{U}^{(n)} \equiv \frac{\partial}{\partial t}(U^{(n)}(x, t))$. Namely:

$$\frac{1}{2} \frac{d}{dt} ||\dot{U}^{(n)}||^2_{L^2(\Omega)} + \nu ||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (A\dot{U}^{(n)}, U^{(n)})_{L^2(\Omega)} + ||(\dot{F}'(\dot{U}^{(n)}))^{1/2} \nabla \dot{U}^{(n)}||^2_{L^2(\Omega)}$$

$$= (f, \dot{U}^{(n)})_{L^2(\Omega)}$$  \hspace{1cm} (36)$$

or equivalently

$$\frac{1}{2} \frac{d}{dt} \left( ||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (A\dot{U}^{(n)}, U^{(n)})_{L^2(\Omega)} \right) + \nu \left( ||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (A\dot{U}^{(n)}, U^{(n)})_{L^2(\Omega)} \right) + ||(\dot{F}'(\dot{U}^{(n)}))^{1/2} \nabla \dot{U}^{(n)}||^2_{L^2(\Omega)} \leq \nu (A\dot{U}^{(n)}, U^{(n)})$$

$$+ \frac{1}{2} \left( p||f||^2_{L^2(\Omega)} + \frac{1}{p} ||\dot{U}^{(n)}||^2_{L^2(\Omega)} \right)$$  \hspace{1cm} (37)$$

If one chooses here the integer $p$ such that $\frac{1}{2p} = \nu$, we have the simple bound

$$\frac{1}{2} \frac{d}{dt} \left( ||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (A\dot{U}^{(n)}, U^{(n)})_{L^2(\Omega)} \right) \leq \frac{1}{4\nu} ||f||^2_{L^2(\Omega)}.$$  \hspace{1cm} (38)
As a consequence of eq.(38), there is a constant $M$ such that the uniform bounds holds true (even if for the case $T = +\infty$ for the case of $f \in L^2([0, \infty), L^2(\Omega))$).

\[
\sup_{0 \leq t \leq T} \text{ess sup}_{t \leq T} \|\dot{U}(t)\|^2_{L^2(\Omega)} \leq M \tag{39}
\]

\[
\sup_{0 \leq t \leq T} \|U(t)\|^2_{L^2(\Omega)} \leq M. \tag{40}
\]

As a consequence of the bounds eq.(39) – eq.(40), there are two functions $\overline{U}(x, t)$ and $\overline{P}(x, t)$ such that we have the weak star convergence on $L^\infty([0, T] \times L^2(\Omega))$

\[
\text{weak-star} \quad \lim_{n \to \infty} (U^{(n)}(x, t)) = \overline{U}(x, t) \tag{41}
\]

\[
\text{weak-star} \quad \lim_{n \to \infty} (\frac{\partial}{\partial t} U^{(n)}(x, t)) = \overline{P}(x, t). \tag{42}
\]

We have thus that the relationship below hold true for any test function $v(x, t) \in C^\infty_c([0, T] \times H^2(\Omega) \cap H_0^1(\Omega))$ obviously satisfying the relations $v(x, 0) = v(x, T) = \Delta v(x, 0) = \Delta v(x, T) = v_t(x, 0) = v_t(x, T) = v_{tt}(x, 0) = v_{tt}(x, T) \equiv 0$ as a consequence of applying the Aubin-Lion theorem in a similar way it was used on eq.(17).

\[
\int_0^T dt \left[ \left( \overline{U}, \frac{d^2v}{dt^2} \right)_{L^2(\Omega)} + (\overline{U}, Av)_{L^2(\Omega)} + v \left( \overline{U}, -\frac{dv}{dt} \right)_{L^2(\Omega)} - (F(\overline{P}), \Delta v)_{L^2(\Omega)} \right]
\]

\[
= \int_0^T dt (f, v)_{L^2(\Omega)} \tag{43}
\]

with the initial conditions

\[
\overline{U}(x, 0) = g(x) \in L^2(\Omega) \tag{44}
\]

\[
\overline{P}(x, 0) = h(x) \in L^2(\Omega). \tag{45}
\]

Let us show now that

\[
\overline{U}(x, t) = \int_0^t ds \overline{P}(x, s). \tag{46}
\]
Firstly, let us remark that integrating on the interval \(0 \leq t \leq T\) the relationship eq.(36), one obtains the following estimate

\[
\frac{1}{2} \left( \|\dot{U}^n(t)\|_{L^2(\Omega)}^2 - \|\dot{U}^n(0)\|_{L^2(\Omega)}^2 \right) \\
+ \left( \nu \int_0^t ds \|\dot{U}_n\|_{L^2(\Omega)}^2(s) \right) \\
+ \frac{1}{2}(AU^n(t), U^n(t))_{L^2(\Omega)} \\
- \frac{1}{2}(AU^n(0), U^n(0)) + \int_0^t ds \| (F'(\dot{U}^n))^\frac{1}{2} \nabla \dot{U}^n \|_{L^2(\Omega)}^2 \\
\leq \frac{p'}{2} \int_0^t ds \|f\|_{L^2(\Omega)}^2 + \frac{1}{2p'} \int_0^t \|\dot{U}_n(s)\|^2 ds
\]

Since the operator \(A\) satisfies the Gårding-Poincaré inequality on \(L^2(\Omega)\)

\[
(AU^n, U^n)_{L^2(\Omega)}(t) \geq \gamma(\Omega)\|U^n\|_{L^2(\Omega)}^2(t),
\]

one can see straightforwardly from eq.(47) by choosing \(2p' > \frac{1}{\nu}\) and the previous bounds eq.(39) that there is a positive Constant \(B\) such that

\[
\int_0^T ds \|\dot{U}^n(s)\|_{L^2(\Omega)}^2 \leq B = MT
\]

which by its turn yields that \(\frac{dU(t,x)}{dt}\) is weakly convergent to \(\overline{P}(x, t)\) in \(L^2([0, T], L^2(\Omega))\).

As a consequence of general theorems of Function convergence on space of integrable functions (Aubin-Lion theorem again ([1])), one has that \(\overline{P}(x, t)\) is the time-derivative of the function \(\overline{U}(x, t)\) almost everywhere on \(\Omega\), since it is expected that \(U^n(x, t)\) should be a strongly convergent sequence to \(\overline{U}(x, t)\) on the separable and reflexive Banach Space \(L^\infty([0, T] \times L^2(\Omega))\) ([1]).

(See Appendix B for mathematical details).

4 A Path-Integral Solution for the Parabolic Non-Linear Diffusion

Let us start the physicist oriented section of our note by writing the abstract scalar Parabolic equation eq.(1) – eq.(3) in the integral (weak) form

\[
U(t) = e^{-At}U(0) + \int_0^t ds \; e^{-(t-s)A} \Delta F(U(s))
\]

where \(F(x)\) is a general non-linear scalar functional such that \(F: L^2(\Omega) \rightarrow H^2(\Omega)\).
Let us suppose that the initial conditions for eq.(1) are samples of a Gaussian stochastic processes belonging to the $L^2(\Omega)$ space. For instance, this mathematical fact is always true when the processes correlation function defines an integral operator of the trace class on $L^2(\Omega)$

$$E\{U(0,x)U(0,y)\} = K(x,y)$$

(51)

where

$$\int_{\Omega} dx K(x,x) < \infty$$

(52)

At a formal path-integral method, one aims to compute the initial condition average of arbitrary product of the function $U(x,t) \equiv U_x(t)$ for arbitrary space-time points. This task is achieved by considering the associated characteristic functional for eq.(51), namely

$$Z[j(t)] = \frac{1}{Z(0)} E\left\{ \exp \left( i \int_0^T dt \langle j(t), U(t) \rangle_{L^2(\Omega)} \right) \right\}. $$

(53)

In order to write a path-integral representation for eq.(53), we follow our previous studies on the subject ([6]) by realizing the random initial conditions $U(0) = U_0$ average as a Gaussian functional integral on $L^2(\Omega)$, namely:

$$Z[f(t)] = \frac{1}{Z(0)} \int_{L^2(\Omega)} d\mu[U_0] \times \int_{L^2([0,T], L^2(\Omega))} \left( \prod_{x \in \Omega; t \in [0,T]} dU(x,t) \right) \times \delta^{(F)} \left[ U(t) - \left( e^{-At} U_0 + \int_0^t ds \ e^{-\left( t-s \right)A} \Delta F(U(s)) \right) \right] \times \exp \left( i \int_0^\infty dt \langle j(t), U(t) \rangle_{L^2(\Omega)} \right) \right\}$$

(54)

where formally

$$d\mu[U_0] = \left( \prod_{x \in \Omega} (dU_0(x)) \right) \times \exp \left\{ -\frac{1}{2} \int_{\Omega} dx dy U_0(x) K^{-1}(x,y) U_0(y) \right\}. $$

(55)

By using the well-known Fourier functional integral representation for the delta-functional inside the identity eq.(54) ([6]) (see this old idea on Monin A.S. and Yaglom A.M. – “Statistical Fluid Mechanics” – Mit Press, Cambridge, 1971, vol. 2)) ([6]), one can re-write eq.(54) in the
following form

\[ Z[j(t)] = \frac{1}{Z(0)} \left\{ \int_{L^\infty([0,T],L^2(\Omega))} \prod (dU(x,t)) \times \int_{L^2([0,T],L^2(\Omega))} d\lambda(t) \right. \\
\times \exp \left\{ i \int_0^t \left\langle \lambda(t), U(t) + \int_0^t ds e^{-(t-s)A} \Delta F(U)(s) \right\rangle_{L^2(\Omega)} \right. \\
\times \exp \left\{ -\frac{1}{2} \int_0^t ds \left\langle \lambda(s) ((e^{-As}K^{-1}e^{+As})\lambda)(s) \right\rangle_{L^2(\Omega)} \right\} \right. \\
\times \exp \left\{ i \int_0^T ds \int_\Omega dx j(s,x)U(s,x) \right\} \}. \] (56)

After evaluating the Gaussian cylindrical measure associated to the Lagrange multiplier fields \(\lambda(x,t)\), one obtains our formal Path-integral representation, however with a weight well-defined mathematically as showed in Section 2, as the main conclusion of Sections 2, 3, 4

\[ Z[j(t)] = \frac{1}{Z(0)} \left\{ \int_{L^\infty((0,T),L^2(\Omega))} \prod dU(x,t) \right. \\
\times \exp \left\{ -\frac{1}{2} \int_0^T ds \int_0^T ds' \int_\Omega dx \left( \frac{\partial U}{\partial t} + AU - \Delta F(U) \right)(x,s) \right. \\
\times \left\langle \left( \frac{\partial U}{\partial s} + AU - \Delta F(U) \right)(s'), \right. \\
\times \exp \left\{ i \int_0^T ds \int_\Omega dx j(s,x)U(s,x) \right\} \}. \] (57)

It is worth re-write the result eq.(57) in the usual Physicist notation of Feynman path integrals ([6])

\[ Z[f(x,t)] = \frac{1}{Z(0)} \left\{ \int_{L^\infty((0,T),L^2(\Omega))} D^F[U(x,s)] \right. \\
\times \exp \left\{ -\frac{1}{2} \int_0^T ds \int_0^T ds' \int_\Omega dx \left( \frac{\partial U}{\partial t} + AU - \Delta F(U) \right)(x,s) \right. \\
\times \left\langle \left( \frac{\partial U}{\partial t} + AU - \Delta F(U) \right)(x,s'), \right. \\
\times \exp \left\{ i \int_0^T ds \int_\Omega dx j(s,x)U(s,x) \right\} \}. \] (58)

5 Random Anomalous Diffusion, a semi-group approach
Recently, it has been an important issue on mathematical-physics of diffusion on random porous medium, the study of the anomalous diffusion equation on the full space $R^D$ but under the presence of a positive random potential as modeled by the parabolic quasi-linear equation $(0 < t < \infty)$ as written below ([7]).

\[
\frac{\partial U(x, t)}{\partial t} = -D_0(-\Delta)^\alpha U(x, t) + V(x)U(x, t) + F(U(x, t)) \quad (59-a)
\]

\[
U(x, 0) = f(x) \quad (59-b)
\]

where $U(x, t)$ is the diffusion field, $D_0$ the medium diffusibility constant, $V(x)$ denotes the stochastic samples (positive functions) associated to a general random field processes with realizations on the space of square integrable functions (by the Minlo’s theorem – see Appendix D), $F(x)$ denotes the problem’s non-linearity represented by a Lipschitzian function as on the previous sections and, finally, $(-\Delta)^\alpha$ represents the effect of the anomalous diffusion of the field $U(x, t)$ on the ambient $R^D$ where the diffusion takes place and it is represented here by a fractional power of the Laplacean operator. The constant $\alpha$ is called here the anomalous diffusion exponent.

In order to give a precise mathematical meaning we will present a different mathematical scheme of the previous sections for eq.(59). Let us, thus, proceed for a moment by re-writing it formally in the weak-integral form for those initial-values $f(x) \in L^2(R^D)$ as in eq.(54). Namely

\[
U(t, [V]) = \exp[-t(D_0(-\Delta)^\alpha + V)]f \\
+ g \int_0^t ds \{ \exp[-(t-s)(D_0(-\Delta)^\alpha + V)]\} F(U(s)) \quad (60)
\]

where we have used a notation emphasising the stochastic variable nature of the diffusion field $U(x, t, [V])$ as a functional of the samples $V(x) \in L^2(R^D)$ associated to our random diffusion potential.

Physical quantities are functionals of the diffusion field $U(t, [V])$ and after its determination one should average over these random potentials samples. The whole averaging information is contained in the space-time characteristic functional as pointed out on section 4

\[
Z[j(x, t)] = \int_{L^2(R^D)} d\mu[V] \times \exp \left\{ \int_{R^D} dx \int_0^\infty dt j(x, t) \cdot U(x, t, [V]) \right\}. \quad (61)
\]
Here $d\mu[V]$ means the random potentials cylindrical measure associated the $V(x)$-characteristic functional

$$Z[h(x)] = \int_{L^2(R^D)} d\mu[V] \exp i\langle h, v \rangle_{L^2(R^D)}.$$  \hspace{1cm} (62)

Note that we do not suppose the Gaussianity of the field statistics on eq.(62).

In order to write a functional integral representation for the characteristic functional eq.(61) as much as similar representation obtained on section 4, let us re-write eq.(61) into the weak-integral form as done in eq.(56)

$$Z[j(x,t)] = \int_{L^2(R^D)} d\mu[V] \int_{L^2(R^D)} D^F[\lambda] \int_{L^2(R^D)} D^F[U]$$

$$\times \exp i\{g(\lambda, F(U) + e^{-tA})_{L^2(R^D)}\}$$  \hspace{1cm} (63)

where $I(F(U))$ and $e^{-tA}$ denotes the objects on the right hand-side of eq.(60) and the (formal at this point of our study) contractive generator semi-group below

$$A = D_0(-\Delta)^\alpha + V.$$  \hspace{1cm} (64)

At this point one could proceed in a physicist way by re-writing eq.(63) as a path-integral associated to a dynamics of three fields, (the well-known Martin-Siggin-Rose component path-integral – ref. [6]).

$$Z[j(x,t)] = \int_{L^2(R^D)} d\mu[V] \int_{L^2(R^D)} D^F[\lambda] \int_{L^2(R^D)} D^F[U]$$

$$\times \exp i \left( \begin{bmatrix} 0 & \frac{1}{2} (\frac{\partial}{\partial x} - D_0(-\Delta)^\alpha) \\ \frac{1}{2} (-\frac{\partial}{\partial x} - D_0(-\Delta)^\alpha) & 0 \end{bmatrix} \right)_{L^2(R^D)}$$

$$\times \exp i \left( \begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix} \right)_{L^2(R^D)} \left( \begin{bmatrix} \lambda \\ U \end{bmatrix} \right)_{L^2(R^D)}$$  \hspace{1cm} (65)

It is worth remark that in the usual case of the cylindrical measures $d\mu[V]$ be a purely Gaussian measure, formally written in the Physicist notation as

$$d\mu[V] = D^F[V] \exp \left\{-\frac{1}{2} \int_{L^2(R^D)} dx \int_{L^2(R^D)} dy V(x) K^{-1}(x,y)V(y) \right\}$$  \hspace{1cm} (66)
with \( K(x,y) \) defining an integral operator of the trace class on \( L^2(R^D) \), one can easily make a further simplification by integrating out exactly the \( V(x) \)-Gaussian functional integral and producing the effective quartic interaction term as written below

\[
\int_{L^2(R^D)} d\mu[V] \exp i \left( [\lambda, U], \begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix} \right) \right)_{L^2(R^D)} \\
= \exp \left\{ -\frac{1}{2} \int_{R^D} dx \int_0^\infty dt \int_{R^D} dy \int_0^\infty ds (\lambda(x,t)U(x,t))K(x,y)(\lambda(y,s)U(y,s)) \right\} \tag{67}
\]

Let us call the reader attention that at this point one can straightforwardly implement the usual Feynman-Wild-Martin-Siggia-Rosen diagramatics with the free-propagator given explicitly in momentum space by (\([6]\))

\[
[\partial_t - D_0(-\Delta)^\alpha]^{-1} \leftrightarrow \frac{1}{iw - D_0(|\vec{k}|^{2\alpha}).} \tag{68}
\]

It is important remark that all the above analysis is still somewhat formal at this point of our study since it is based strongly on the hypothesis that the operator \( A \) as given by eq.(64) is a \( C_0 \)-generator of a contractive semi-group on \( L^2(R^D) \) which straightforwardly leads to the existence and uniqueness of global solution for eq.(59). Let us give a rigorous proof of ours of such self-adjointness result, which by its turn will provides a strong connection between the parameter \( \alpha \) of the Laplacean power, related to the anomalous diffusion coefficient, and the underline dimension \( D \) of the space where the anomalous diffusion is taking place.

Let us first recall the famous Kato-Rellich theorem on self-adjointness perturbative of self-adjoint operators.

**Theorem of Kato-Rellich.** Suppose that \( A \) is an self-adjoint operator on \( L^2(R^D) \), \( B \) is a symmetric operator with \( \text{Dom}(B) \supset \text{Dom}(A) \) and such that \( 0 \leq a < 1 \)

\[
||B\varphi||_{L^2(R^D)} \leq a||A\varphi||_{L^2(R^D)} + b||\varphi||_{L^2(R^D)}. \tag{69}
\]

Then, \( A + B \) is self-adjoint on \( \text{Dom}(A) \) and essentially self-adjoint on any core of \( A \).

In order to apply this powerfull theorem to our general case under study, let us recall that the fractional power of the Laplacean operator, \((-\Delta)^\alpha \), has as an operator domain the
(Sobolev) space of all square integrable functions \( \varphi(x) \) such its Fourier Transform \( \hat{\varphi}(k) \) satisfies the condition \(|k|^\alpha \hat{\varphi}(k) \in L^2(R^D)\) and as a core domain the Schwartz space \( S(R^D) \) ([8]).

As a consequence, we have the straightforward estimative for \( \varphi \in S(R^D) \)

\[
\|\hat{\varphi}\|_{L^2(R^D)} \leq \left\| \frac{1}{1 + k^{2\alpha}} \|\hat{\varphi}\|_{L^2(R^D)} \right\| (k^{2\alpha} + 1) \|\hat{\varphi}\|_{L^2(R^D)} \\
\leq C(\alpha, D)(\|k^{2\alpha} \hat{\varphi}\|_{L^2(R^D)} + \|\hat{\varphi}\|_{L^2(R^D)}) \tag{70}
\]

here the finite constant

\[
C(\alpha, D) = \int_{R^D} \frac{dk}{(1 + k^{2\alpha})^2} < \infty \tag{71}
\]

if \( \alpha > \frac{D}{4} \), a condition relating the “anomalous” diffusion coefficient \( \alpha \) and the intrinsic space-time dimensionality \( D \) as said before on the introduction. Note that for the physical case of \( D = 3 \), we have that \( \alpha = \frac{3}{4} + \varepsilon \) with \( \varepsilon > 0 \).

Let us re-scale the function \( \hat{\varphi}(k) \) \((r > 0)\) (for \( \beta > 0 \) arbitrary)

\[
\hat{\varphi}_r(k) = r^\beta \hat{\varphi}(rk) \tag{72}
\]

We have thus

\[
\|\hat{\varphi}_r\|_{L^1(R^D)} = \int_{R^D} dr k^\beta \hat{\varphi}(rk) = r^{\beta-D} \|\hat{\varphi}\|_{L^1(R^D)} \tag{73}
\]

and

\[
\|\hat{\varphi}_r\|_{L^2(R^D)} = \sqrt{r}^{(2\beta-D)/2} \|\hat{\varphi}\|_{L^2(R^D)} \tag{74}
\]

together with

\[
\|K^{2\alpha} \hat{\varphi}_r\|_{L^2(R^D)} = r^{(2\beta-D-4\alpha)/2} \|K^{2\alpha} \hat{\varphi}\|_{L^2(R^D)} \tag{75}
\]

Let us substitute eqs.(73)–(75) into eq.(70). We arrive at the estimate

\[
\|\hat{\varphi}_r\|_{L^1(R^D)} \leq C(\alpha, D) \left[ r^{(2\beta-D-4\alpha)/2} \|K^{2\alpha} \hat{D}\|_{L^2(R^D)} \\
+ r^{(2\beta-D)/2} \|\hat{\varphi}\|_{L^2(R^D)} \right] \tag{76}
\]
or equivalently

\[ ||\tilde{\varphi}_1||_{L^1(R^D)} \leq C(\alpha, D) \left( r^{\frac{D}{2} - 2\alpha} ||K^{2\alpha}\tilde{\varphi}||_{L^2(R^D)} + r^{\frac{D}{2}} ||\tilde{\varphi}||_{L^2(R^D)} \right). \]  

(77)

Let us now estimate the function \( V\varphi \) on the space \( L^2(R^D) \)

\[ ||V\varphi||_{L^2(R^D)} \leq ||V||_{L^2(R^D)} ||\varphi||_{L^\infty(R^D)} \]
\[ \leq ||V||_{L^2(R^D)} ||\varphi||_{L^1(R^D)} \]
\[ \leq (||V||_{L^2(R^D)} C(\alpha, D) \cdot r^{\frac{D}{2} - 2\alpha}) ||K^{2\alpha}\tilde{\varphi}||_{L^2(R^D)} \]
\[ + (||V||_{L^2(R^D)} C(\alpha, D) \cdot r^{\frac{D}{2}}) ||\tilde{\varphi}||_{L^2(R^D)} \]  

(78)

Now one just choose \( r \) such that

\[ ||V||_{L^2(R^D)} \times C(\alpha, D) \times r^{(D-4\alpha)/2} < 1 \]  

(79)

in order to obtain the validity of the bound \( a < 1 \) on eq.(69) and, thus, concluding that \((-\Delta)^\alpha + V\) is an essentially self-adjoint operator on \( S(R^D) \). So, its closure on \( L^2(R^D) \) produces the operator used on the above exposed semi-group \( C_0 \)-construction for \( V(x) \) being a positive function almost everywhere. This result of ours is a substantial generalization of Theorem X.15 presented in ref. [8]. Note that the \( V(x) \) samples positivity is always the physical case of random porosity in Porous medium. In such case, the random potential is of the exponential form \( V(x) = V_0(\exp\{ga(x)\}) \); with \( g \) a positive small parameter and \( V_0 \) is a back-ground porosity term. The effective cylindrical measure on the random porosity parameters is written as (see eq.(66))

\[ d\mu[V] \overset{g<<1}{\approx} D^F[a(x)] \exp \left\{ -\frac{1}{2} \int_{R^D} dx \int_{R^D} dy e^{ga(x)} K^{-1}(x,y) e^{ga(y)} \right\} \]  

(80)

The above exposed \( C_0 \)-semi group study complements the study on non-linear purely diffusion made on the previous sections by purely compacity arguments.

Finally, let us briefly sketch on the important case of wave propagation \( U(x,t) \) in a random medium described by a small stochastic damping positive \( L^2(R^D) \) function \( \nu(x) \), such that \( \frac{\nu^2(x)}{4} \)
is a Gaussian process sampled on the $L^2(R^D)$ space. The hyperbolic linear equation governing such physical wave propagation is given by (see Section 3)

$$\frac{\partial^2(U(x,t))}{\partial^2 t} + \nu(x)\frac{\partial U(x,t)}{\partial t} = -(-\Delta)^\alpha U(x,t)$$  

$$U(x,0) = f(x) \in L^2(R^D)$$  

$$U_t(x,0) = g(x) \in L^2(R^D)$$  

where

$$E \left( \frac{\nu^2(x) \nu^2(y)}{4} \right) = K(x,y) \in \int_1(\mathcal{L}^2(R^D))$$  

On can see that after the variable change

$$U(x,t,[\nu]) = e^{-\frac{\nu(x)}{2} t} \cdot \Phi(x,t,[\nu]),$$  

the damped-stochastic wave equation takes the suitable form of a wave in the presence of a random potential

$$\frac{\partial^2 \Phi(x,t)}{\partial^2 t} = -(-\Delta)^\alpha \Phi(x,t) + \left( \frac{\nu^2(x)}{4} \right) \Phi(x,t)$$

$$\Phi(x,0) = f(x)$$

$$\Phi_t(x,0) = \frac{\nu(x)}{2} f(x) + g(x) = \mathcal{G}(x).$$  

Equation (86) has an operatorial scalar weak-integral solution for $t > 0$ of the following form

$$\Phi(t) = \cos(t\sqrt{A}) f + \frac{\sin(t\sqrt{A})}{\sqrt{A}} \mathcal{G}$$  

here, the non-positive self-adjoint operator $A$ is given explicitly by

$$A = (-\Delta)^\alpha - \frac{\nu^2(x)}{4}.$$  

As a conclusion, one can see that anomalous diffusion can be handled mathematically in the framework of our previously technique of path integrals with constraints.
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In this mathematical appendix, we intend to give a rigorous mathematical proof of the result used on Section 2 about Parabolic non-linear diffusion in relation to the weak continuity of the non-linearity $\Delta F(U)$ used on the passage of eq.(17) for eq.(18).

Firstly we consider the physical hypothesis on the non-linearity of the parabolic eq.(1) that the Laplacean operator has a cut-off in its spectral range, namely

$$\Delta F(U(x,t)) \to \Delta^{(A)} F(U(x,t)) \quad (A-1)$$

where the regularized Laplacean $\Delta^{(A)}$ means the bounded operator of norm $\Lambda$, i.e., in terms of the spectral theorem of the Laplacean

$$\Delta = \int_{-\infty}^{+\infty} \lambda dE_s(\lambda) \quad (A-2)$$

it is given by

$$\Delta^{(A)} = \int_{|\lambda|<\Lambda} \lambda dE_s(\lambda) \quad (A-3)$$

Let us impose either the well-known Path-integral Sturm-Liouville time boundary conditions imposed on the elements of the path-integral domain (eq.(58)), when one is defining the path integral by means of fluctuations around classical configurations

$$\overline{U}(x,T) = \overline{U}(x,0) = 0 \quad (A-4)$$

It is straightforward to see the validity of the following chain of inequalities with $a > 1$

$$\int_0^T dt \left\| \frac{dU_n(t)}{dt} \right\|^2_{L^2(\Omega)} \leq \text{Real}((AU_n(t),U_n(t)) - (AU_n(0),U_n(0))) + \int_0^T dt \left\| \Delta^{(A)} F(U_n(t)) \cdot \frac{dU_n}{dt} \right\|_{L^2(\Omega)}$$

$$\leq 0 + 0 + a \left( \int_0^T \left\| \Delta F(U_n(t)) \right\|^2_{L^2(\Omega)} dt \right) + \frac{1}{a} \left( \int_0^T dt \left\| \frac{dU_n}{dt} \right\|^2_{L^2(\Omega)} \right) \quad (A-5)$$
In other words

\[
\left(1 - \frac{1}{a}\right) \left(\int_{0}^{T} dt \left\| \frac{dU_n}{dt} \right\|_{L^2(\Omega)}^2\right) \\
\leq c a \Lambda^2 \cdot \int_{0}^{T} \left\| U_n(t) \right\|_{L^2(\Omega)}^2 \\
\leq C a \Lambda^2 T M
\]  

(A-6)

where \(c = \sup_{a \leq x \leq b} |F'(x)|\) and \(M\) is the bound given by eq.(16).

Now by a direct application of the well-known Aubin-Lion theorem ([1]), one has that the sequence \(\{U_n\}\) is a compact set on \(L^2(\Omega)\). So its converges strongly to \(\overline{U}\) on \(L^2(\Omega)\) as a consequence of the Lipschitzian property of \(F(x)\).

We have either the strong convergence of \(F(U_n)\) to \(F(\overline{U})\), since for each \(t\) we have that the inquality below holds true

\[
\int_{\Omega} dx |F(U_n) - F(\overline{U})|_{L^2(\Omega)}^2(t) \leq \left( \sup_{-\infty < x < +\infty} (F'(x)) \right) \left( \int_{\Omega} dx |U_n - \overline{U}|_{L^2(\Omega)}^2(t) \right) \to 0. \quad \text{(A-7)}
\]

It is, thus, an immediate result the validity of the weak convergence on \(L^2(\Omega)\) used on eq.(18)

\[
\lim_{n \to \infty} \int_{0}^{T} dt (F(U_n), \Delta v)_{L^2(\Omega)} = \int_{0}^{T} dt (F(\overline{U}), \Delta v)_{L^2(\Omega)}.
\]  

(A-8)
APPENDIX B

In this appendix, we show that the function \( U(x, t) \in L^\infty([0, T], L^2(\Omega)) \) the weak solution of eq.(1) in the test space \( D((0, T), L^2(\Omega)) \) (the Schwart space of \( L^3(\Omega) \)-valued test functions), satisfies the initial condition eq.(2). In order to show such result, let us consider a test function possessing the following form for each \( \varepsilon > 0 \),

\[
 v_i^{(\varepsilon)}(x, t) = \varphi_i(x) \left( \chi_{[0,\varepsilon]}(t)/\varepsilon \right),
\]

where \( \varphi_i(x) \) is a member of the spectral set associated to the self-adjoint operator \( A: H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega) \).

The notation \( \chi_{[a,b]}(t) \) denotes the characteristic function of the interval \( [a, b] \).

Since \( U_m(x, t) \) converges to \( U(x, t) \) in the weak-star topology of \( L^\infty((0, T), L^2(\Omega)) \) and \( D((0, T), L^2(\Omega)) \subset L^1((0, T), L^2(\Omega)) \), we have the relation below for each \( v_i^{(\varepsilon)}(x, t) \) holds true

\[
\int_0^T dt \langle U_m(x, t), v_i^{(\varepsilon)}(x, t) \rangle_{L^2(\Omega)} \longrightarrow \int_0^T dt \langle U(x, t), v_i^{(\varepsilon)}(x, t) \rangle_{L^2(\Omega)}
\]

or equivalently:

\[
\frac{1}{\varepsilon} \int_0^\varepsilon dt \langle U_m(x, t), \varphi_i(x) \rangle_{L^2(\Omega)} \longrightarrow \frac{1}{\varepsilon} \int_0^\varepsilon dt \langle U(x, t), \varphi_i(x) \rangle_{L^2(\Omega)}.
\]

By means of the mean-value theorem applied to both sides of eq.(2-A) and taking the limit \( \varepsilon \to 0 \), we have for each \( i \in \mathbb{Z} \) that:

\[
\langle U_m(x, 0), \varphi_i(x) \rangle_{L^2(\Omega)} \longrightarrow \langle U(x, 0), \varphi_i(x) \rangle_{L^2(\Omega)}
\]

as a direct consequence of our choice of the Galerkin elements eq.(6) whose functional form do not changes on function of the order \( m \in \mathbb{Z} \).

As a consequence of eq.(5) and eq.(6), it yields the result

\[
\langle g(x), \varphi_i \rangle_{L^2(\Omega)} = \langle U(x, 0), \varphi_i(x) \rangle_{L^2(\Omega)}
\]

or by means of Parsedval-Fourier theorem

\[
\lim_{t \to 0} ||U(x, t) - g(x)||_{L^2(\Omega)} = 0
\]

result, which by its turn, means that the function \( U(x, t) \) obtained by means of our compacity technique satisfies the initial-condition eq.(2) in the \( L^2(\Omega) \)-mean sense.
APPENDIX C

Le us show that the functions $U(x, t) \in L^\infty([0, T], L^2(\Omega))$ given by eq.(41) and $P(x, t) \in L^\infty([0, T], L^2(\Omega))$ – eq.(42) are coincident as elements of the above written functional space of $L^2(\Omega)$ – valued essential bounded functions on $(0, T)$.

To verify such result, let us call the reader attention that since $U_m(x, t)$ is wearly convergent to $U(x, t)$ in $L^\infty([0, T], L^2(\Omega))$, we have that the set $\{U_m(x, t)\}$ is convergent to the function $U(x, t)$ as a Schwartz distribution on $L^2(\Omega)$ since $D([0, T], L^2(\Omega)) \subset L^1([0, T], L^2(\Omega))$.

This means that

$$U_m(x, t) \to U(x, t) \quad \text{in} \quad D'([0, T], L^2(\Omega)). \quad (1-C)$$

Analogous result hold true for the time-derivative of the above written equation as a result of $U(x, t)$ be a function

$$\frac{\partial}{\partial t} U_m(x, t) \to \frac{\partial}{\partial t} U(x, t) \quad \text{in} \quad D'([0, T], L^2(\Omega)) \quad (2-C)$$

By the other side, the set $\left\{\frac{\partial U_m(x, t)}{\partial t}\right\}$ converges weakly star in $L^1([0, T], L^2(\Omega))$ to $P(x, t) \in L^\infty([0, T], L^2(\Omega))$ which, by its turn, means that

$$\frac{\partial U_m(x, t)}{\partial t} \to P(x, t) \quad \text{in} \quad D'([0, T], L^2(\Omega)). \quad (3-C)$$

By the uniqueness of the limit on the distributional space $D'([0, T], L^2(\Omega))$, we have the coincidence of $\frac{\partial U(x, t)}{\partial t}$ and $P(x, t)$ as elements of $D'([0, T], L^2(\Omega))$. However, $P(x, t)$ is a function, so by general theorems on Schwartz distribution theory $\left\{\frac{\partial U(x, t)}{\partial t}\right\}$ must be a function either since $L^2(\Omega)$ is a separable Hilbert Sapce. As a consequence we have that $\frac{\partial U(x, t)}{\partial t} = P(x, t)$ as elements of $L^\infty([0, T], L^2(\Omega))$, which is the result searched

$$\frac{\partial U(x, t)}{\partial t} = P(x, t) \quad \text{a.e. in} \quad ([0, T] \times \Omega). \quad (4-C)$$
Appendix D

Probability Theory in terms of Functional Integrals
and the Minlos Theorem – an overview

In this complementary appendix, we discuss briefly and for the mathematical oriented reader
the mathematical basis and concepts of functional integrals formulation for stochastic process
and the important Minlos theorem on Hilbert space support of probabilistic measures.

The first basic notion of Kolmogorov’s probability theory framework is to postulate a con-
venient topological space (Polish spaces) $\Omega$ formed by the phenomena random events. The
chosen topology of $\Omega$ should posses a rich set of non-trivial compact subsets. After that,
we consider the $\sigma$-algebra generated by all opens sets of this – so called – topological Sample
space and denoted by $\mathcal{F}_\Omega$. Thirdly, one introduces a regular measure $d\mu$ on this set algebra $\mathcal{F}_\Omega$,
assigning values in $[0, 1]$ to each member of $\mathcal{F}_\Omega$.

The (abstract) triplet $(\Omega, \mathcal{F}_\Omega, d\mu)$ is called thus a probability space in the Kolmogorov-
Schwartz Scheme ([10]).

Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a set of measurables real functions on $\Omega$, which may be taken as continuous
injutive functions of compact support on $\Omega$ (without loss of generality).

It is a standard result that one can “immerse” (represent) the abstract space $\Omega$ on the
“concrete” infinite product compact space $R^\infty = \prod_{\alpha \in \Lambda} (\hat{R}_\alpha)$, where $\hat{R}$ is a compactified copy of
the real line. In order to achieve such result we consider the following injection of $\Omega$ in $R^\infty$
defined by the family $\{X_\alpha\}_{\alpha \in \Lambda}$

$$I: \Omega \to R^\infty$$

$$\omega \to \{X_\alpha(\omega)\}_{\alpha \in \Lambda}$$

(D-1)

A new $\sigma$-algebra of events on $\Omega$ can be induced on $\Omega$ and affiliated to the family $\{X_\alpha\}_{\alpha \in \Lambda}$.

It is the $\sigma$-algebra generated by all the “finite-dimensional cylinders” sub-sets, which are
explicitly given in its generic formulas by

\[ C_{\Lambda_{\text{fin}}} = \left\{ \omega \in \Omega \mid \Lambda_{\text{fin}} \text{ is a finite subset of } \Lambda \right\} \]

\[ \text{and } [(X_{\alpha})_{\alpha \in \Lambda_{\text{fin}}} (\Omega)] \subset \prod_{\alpha \in \Lambda_{\text{fin}}} [a_{\alpha}, b_{\alpha}] \]

The measure restriction of the initial measure \( \mu \) on this cylinder sets of \( \Omega \) will be still denoted by \( \mu \) on what follows. Now it is a basic theorem of Probability theory that \( \mu \) induces a measure \( \nu^{(\infty)} \) on \( \prod_{\alpha \in \Lambda} (\hat{R})_{\alpha} \), which can be identified with the space of all functions of the index set \( \Lambda \) to the real compactified line \( \hat{R}; F(\Lambda, R) \) with the topology of pontual convergence.

At this point, it is straightforward see that the average of any Borelian (mensurable) function \( G(\omega) \) (a Random variable) is given by the following functional integral on the functional space \( F(\Lambda, R) \)

\[ \int_{\Omega} d\mu(\omega) G(\omega) = \int_{F(\Lambda, R)} d\nu^{(\infty)}(f) \cdot G(I^{-1}(f)) \quad (D-2) \]

It is worth recall that the real support of the measure \( d\nu^{(\infty)}(f) \) in most practical cases is not the whole space \( F(\Lambda, R) \) but only a functional sub-space of \( F(\Lambda, R) \). For instance, in the most practical cases \( \Lambda \) is the index set of a algebraic vectorial base of a vector space \( E \) and \( \prod_{\alpha \in \Lambda} (R)_{\alpha} \) (without the compactification) is the space of all sequences with only a finite number of non-zero entries as it is necessary to consider in the case of the famous extension theorem of Kolmogorov. It turns out that one can consider the support of the probability measure as the set of all linear functionals on \( E \), the so called Algebraic Dual of \( E \). This result is a direct consequence of considering the set of Random variables given by the family \( \{e_{\alpha}^*\}_{\alpha \in \Lambda} \). For applications one is naturally leads to consider the probabilistic object called characteristic functional associated to the measure \( d\nu^{(\infty)}(f) \) when \( F(\Lambda, R) \) is identified with the vector space of all algebraic linear functionals \( E_{\text{alg}} \) of a given vector space \( E \) as pointed out above, namely

\[ Z[j] = \int_{E_{\text{alg}}} d\nu^{(\infty)}(f) \exp(if(j)) \quad (D-3) \]

where \( j \) are elements of \( E \) written as \( j = \sum_{\alpha \in \Lambda_{\text{fin}}} x_{\alpha} e_{\alpha} \) with \( \{e_{\alpha}\}_{\alpha \in \Lambda} \) denoting a given Hammel (vectorial) basis of \( E \).
One can show that if there is a sub-space $H$ of $E$ with a norm $|| \cdot ||_H$ coming from a inner product $(\cdot , \cdot)_H$ such that
\[
\int_{E \cap H} d\nu^{(\infty)}(f) \cdot ||f||_H^2 < \infty
\] (D-4)
one can show that the support of the measure of exactly the Hilbert space $(H, (\cdot , \cdot)_H)$ – the famous Minlos theorem.

Another important theorem (the famous Wiener theorem) is when we have the following condition for the family of random variables with the index set $\Lambda$ being a real interval of the form $\Lambda = [a, b]$
\[
\int_E d\nu^{(\infty)}(f) |X_{t+h}(f) - X_t(f)|^p \leq c|h|^{1+\varepsilon}
\] (D-5)
for $p, \varepsilon, c$ positive constants. We have that it is possible to choose as the support of the functional measure $d\nu^{(\infty)}(\cdot)$ the space of the real continuous function on $\Lambda$ the well-known famous Brownian path space
\[
\int_{\Omega} d\mu(\omega) G(\omega) = \int_{C([a,b],R)} d\nu^{(\infty)}(f(x)) \tilde{G}(f(x))
\] (D-6)

For rigorous proofs and precise formulations of the above sketched deep mathematical results, the interested reader should consult the basic mathematical book on the subject: L. Schwartz – “Random Measures on Arbitrary Topological Spaces and Cylindrical Measures” – Tata Institute – Oxford University Press, 1973 ([10]).

Let us finally now give a concrete example of such results consider a generating functional $Z[J(x)]$ of the exponential quadratic functional form on $L^2(R^N)$
\[
Z[J(x)] = \exp \left\{ -\frac{1}{2} \int_{R^N} d^N x \int_{R^N} d^N y j(x)K(x,y)j(y) \right\}
\] (D-7)
and associated to a Gaussian Random field with two-point correlation function given by the kernel of eq.(D-7)
\[
E\{v(x)v(y)\} = K(x,y).
\] (D-8)

In the case of the above written kernel be a class-trace operator on $L^2(R^N)$ ($\int_{R^N} d^N x K(x,x) < \infty$), on can represent eq.(B-7) by means of a Gaussian measure with support on the Hilbert space $L^2(R^N)$. This result was called the reader attention for on the bulk of this paper.