Projected Semi-Stochastic Gradient Descent Method with Mini-Batch Scheme under Weak Strong Convexity Assumption

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Abstract. We propose a projected semi-stochastic gradient descent method with mini-batch for improving both the theoretical complexity and practical performance of the general stochastic gradient descent method (SGD). We are able to prove linear convergence under weak strong convexity assumption. This requires no strong convexity assumption for minimizing the sum of smooth convex functions subject to a compact polyhedral set, which remains popular across machine learning community. Our PS2GD preserves the low-cost per iteration and high optimization accuracy via stochastic gradient variance-reduced technique, and admits a simple parallel implementation with mini-batches. Moreover, PS2GD is also applicable to dual problem of SVM with hinge loss.

Keywords: stochastic gradient, variance reduction, support vector machine (SVM), linear convergence, weak strong convexity

1 Introduction

The problem we are interested in is to minimize a constrained convex problem,

\[
\min_{w\in W} \left\{ F(w) := g(Aw) + q^T w \right\}.
\]

where \( w \in \mathbb{W} \subseteq \mathbb{R}^d \), \( A \in \mathbb{R}^{n \times d} \), and assume that \( F \) can be further written as

\[
F(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w).
\]

This type of problem is prevalent through machine learning community. Specifically, applications which benefit from efficiently solving this kind of problems include face detection, fingerprint detection, fraud detection for banking systems, image processing, medical image recognition, and automatic cars etc. To exploit the problem, we further make the following assumptions:

Assumption 1. The functions \( f_i : \mathbb{R}^d \rightarrow \mathbb{R} \) are convex, differentiable and have Lipschitz continuous gradients with constant \( L > 0 \). That is,

\[
\|\nabla f_i(w_1) - \nabla f_i(w_2)\| \leq L\|w_1 - w_2\|,
\]

for all \( w_1, w_2 \in \mathbb{R}^d \), where \( \| \cdot \| \) is \( L2 \) norm.
Assumption 2. The function $g : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and strongly convex with parameter $\mu > 0$ on its effective domain that is assumed to be open and non-empty, i.e., $\forall z_1, z_2 \in \text{dom}(g) \subseteq \mathbb{R}^n$,
\[
 g(z_1) \geq g(z_2) + \nabla g(z_2)^T (z_1 - z_2) + \frac{\mu}{2} \|z_1 - z_2\|^2.
\]  

Assumption 3. The constraint set is a compact polyhedral set, i.e.,
\[
 \mathcal{W} = \{ w \in \mathbb{R}^d : Cw \leq c \}, \quad \text{where } C \in \mathbb{R}^{m \times d}, c \in \mathbb{R}^m.
\]

Remark 1. Problem (1) usually appears in machine learning problems, where $A$ is usually constructed by a sequence of training examples $\{a_i\}_{i=1}^n \subseteq \mathbb{R}^d$. Note that $n$ is the number of data points and $d$ is the number of features. Problem (2) arises as a special form as problem (1) which is a general form in a finite sum structure, which covers empirical risk minimization problems. As indicated in the problem setting, there are two formulations of the problem with different pairs of $A$ and $\mathcal{W}$ given a sequence of labeled training examples $\{\{a_i, b_i\}\}_{i=1}^n$ where $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$. Define the set $[m] \stackrel{\text{def}}{=} \{1, 2, \ldots, m\}$ for any positive integer $m$.

**Type I Primal Setting** A commonly recognized structure for this type of problem is to apply (1) to primal problem of finite sum structured problems and to represent $g$ as $g(Aw) = \frac{1}{n} \sum_{i=1}^n g_i(a_i^T w)$ where $g_i$ are $\mathbb{R} \to \mathbb{R}$. In this way, $f_i$ in (1) can be defined as $f_i(w) \stackrel{\text{def}}{=} g_i(a_i^T w) + q^T w$. We need $g_i$ to have Lipschitz continuous gradients with constants $L/\|a_i\|^2$ to fulfill Assumption 1, i.e.,
\[
 \|\nabla f_i(w_1) - \nabla f_i(w_2)\| = \|(a_i \nabla g_i(a_i^T w_1) + q) - (a_i \nabla g_i(a_i^T w_2) + q)\|
 = \|a_i\| \|\nabla g_i(a_i^T w_1) - \nabla g_i(a_i^T w_2)\| \leq \|a_i\| (L/\|a_i\|^2) \|a_i^T w_1 - a_i^T w_2\|
 = (L/\|a_i\|) \|a_i^T (w_1 - w_2)\| \leq (L/\|a_i\|) \|a_i\| \|w_1 - w_2\| = L \|w_1 - w_2\|,
\]
where the last inequality follows from Cauchy Schwartz inequality.

Popular problems in this type from machine learning community are logistic regression and least-squares problems by letting $q = 0$, i.e., $f_i(w) = g_i(a_i^T w) = \log(1 + \exp(-b_i a_i^T w))$ and $f_i(w) = g_i(a_i^T w) = \frac{1}{2} (a_i^T w - b_i)^2$, respectively. These problems are widely used in both regression and classification problems. Our results and analyses are also valid for any convex loss function with Lipschitz continuous gradient.

To deal with overfitting and enforce sparsity to the weights $w$ in real problems, a widely used technique is to either add a regularized term to the minimization problem or enforce constraints to $w$. A well-known fact is that regularized optimization problem can be equivalent to some constrained optimization problem under proper conditions. The problem of our interest is formulated to solve constrained optimization problem. Under Assumption 3 several popular choices of polyhedral constraints exist, such as $\mathcal{W} = \{ w \in \mathbb{R}^d : \|w\|_1 \leq \zeta \}$ and $\mathcal{W} = \{ w \in \mathbb{R}^d : \|w\|_{\infty} \leq \zeta \}$. 


Type II Dual Setting We can also apply (1) to dual form of some special SVM problems. With the same sequence of labeled training examples \(\{(a_i, b_i)\}_{i=1}^n\), let us denote \(a_i \equiv (a_{i1}, \ldots, a_{id})^T \in \mathbb{R}^d\), then an example is the dual problem of SVM with hinge loss, which has the objective function:

\[
g(A\alpha) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n b_i b_j a_i^T a_j \alpha_i \alpha_j = \frac{1}{2} \alpha^T A^T A \alpha = \frac{1}{2} \|A\alpha\|^2
\]

where the \(i^{th}\) column of \(A\) is \(b_i a_i\) so that \([A^T A]_{ij} = (b_i a_i)^T (b_j a_j) = b_i b_j a_i^T a_j\) and we should also know that \(A \in \mathbb{R}^{d \times n}\).

By defining \(a_s^{(c)} \equiv (b_1 a_{s1}, b_2 a_{s2}, \ldots, b_n a_{sn})^T \in \mathbb{R}^n\), \(\forall s \in [d]\), then \(a_s^{(c)}\) is the \(s^{th}\) row vector of \(A\) which is also called the feature vector. By deleting unnecessary \(a_s^{(c)}\) corresponds to feature \(s\), we can guarantee that \(\|a_s^{(c)}\| \neq 0, \forall s \in [d]\) and easily scale \(a_s^{(c)}\); so similar to Type I, Type II problem can also satisfy Assumption 1. Under this type, \(\forall i \in [d]\), \(f_i\) can be written as

\[
f_i(\alpha) = g_i \left((a_s^{(c)})^T \alpha\right) + q^T \alpha
\]

with

\[
g_i \left((a_s^{(c)})^T \alpha\right) = \frac{d}{2} \| (a_s^{(c)})^T \alpha \|^2 \text{ and } q = (-1, \ldots, -1)^T \in \mathbb{R}^n,
\]

and \(F(\alpha) = \frac{1}{d} \sum_{i=1}^d f_i(\alpha)\).

The dual formulation of SVM with hinge loss is

\[
\min_{\alpha \in W} g(A\alpha) + q^T \alpha
\]

with \(g\) defined in (5), \(A \in \mathbb{R}^{d \times n}\), \(q = (-1, \ldots, -1)^T \in \mathbb{R}^n\) and \(W = \{\alpha : \alpha_i \in [0, \lambda n], \forall i \in [n]\} \subset \mathbb{R}^n\), where \(\lambda\) is regularization parameter. This problem satisfies Assumptions 1 and 2, which is within our problem setting.

To be consistent, let us use notation \(a_i\) for \(a_i^{(c)}\), and \(w\) for \(\alpha\) to describe the algorithm and analyses in the following sections.

Remark 2. Assumption 2 covers a wide range of problems. Note that this is not a strong convexity assumption for the original problem \(F(w)\) since the convexity of \(F\) is dependent on the data \(A\); nevertheless, the choice of \(g\) is independent of \(A\). Popular choices for \(g(z)\) have been mentioned in Remark 1, i.e., \(\frac{1}{2}\|z-b\|^2\), \(\frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-b_i z_i))\) in Type I and \(\frac{1}{2}\|z\|^2\) in Type II.

Related Work A great number of methods have been delivered to solve problem (1) during the past years. One of the most efficient algorithms that have been extensively used is FISTA. However, this is considered a full gradient algorithm, and is impractical in large-scale settings with big \(n\) since \(n\) gradient evaluations are needed per iteration. Two frameworks are imposed to reduce the cost per iteration—stochastic gradient algorithms [26, 29, 34, 18, 8, 30] and randomized coordinate descent methods [20, 23, 25, 17, 28, 14, 16, 24, 113, 113]. However,
even under strong convexity assumption, the convergence rates in expectation is only sub-linear, while full gradient methods can achieve linear convergence rates \[21,32\]. It has been widely accepted that the slow convergence in standard stochastic gradient algorithms arises from its unstable variance of the stochastic gradient estimates. To deal with this issue, various variance-reduced techniques have been applied to stochastic gradient algorithms \[12,27,32,11,3,10,22\]. These algorithms are proved to achieve linear convergence rate under strong convexity condition, and remain low-cost in gradient evaluations per iteration.

The topic whether an algorithm can achieve linear convergence without strong convexity assumptions remains desired in machine learning community. Recently, the concept of \textit{weak strong convexity property} has been proposed and developed based on \textit{Hoffman bound} \[7,31,13,6,33\]. In particular, Ji and Wright \[13\] first proposed the concept as \textit{optimally strong convexity} in March 2014 \footnote{Even though the concept was first proposed by Liu and Wright in \[13\] as \textit{optimally strong convexity}, to emphasize it as an extended version of strong convexity, we use the term \textit{weak strong convexity} as in \[6\] throughout our paper.}. Necoara \[15\] established a general framework for weak non-degeneracy assumptions which cover the weak strong convexity. Hui \[33\] provides a complete of summary on weak strong convexity, including their connections. This kind of methodology could help to improve the theoretical analyses for series of fast convergent algorithms and to apply those algorithms to a broader class of problems.

\textbf{Our contributions} In this paper, we combine the stochastic gradient variance-reduced technique and weak strong convexity property based on Hoffman bound to derive a projected semi-stochastic gradient descent method (PS2GD). This algorithm enjoys three benefits. First, PS2GD promotes the best convergence rate for solving (1) without strong convexity assumption from sub-linear convergence to linear convergence in theory. Second, stochastic gradient variance-reduced technique in PS2GD helps to maintain the low-cost per iteration of the standard stochastic gradient method. Last, PS2GD comes with a mini-batch scheme, which admits a parallel implementation, suggesting probably speedup in clocktime in an HPC environment.

Moreover, we have shown in Remark 1 that our framework covers the dual form of SVM problem with hinge loss. Instead of applying SDCA \[27,28\], we can also apply PS2GD as a stochastic dual gradient method.

\section{Projected Algorithms and PS2GD}

A common approach to solve (1) is to use \textit{gradient projection methods} \[21,31,6\] by forming a sequence \(\{y_k\}\) via

\[
y_{k+1} = \arg\min_{w \in \mathcal{W}} \left[ U_k(w) \right] \text{def} = F(y_k) + \nabla F(y_k)^T (w - y_k) + \frac{1}{2h} \| w - y_k \|^2,
\]

where \(U_k\) is an upper bound on \(F\) if \(h > 0\) is a stepsize parameter satisfying \(h \leq \frac{1}{L}\). This procedure can be equivalently written using the \textit{projection operator}
Projected Semi-Stochastic Gradient Descent (PS2GD)

as follows:

\[ y_{k+1} = \text{proj}_W(y_k - h \nabla F(y_k)) \]

where

\[ \text{proj}_W(z) \overset{\text{def}}{=} \arg \min_{w \in W} \left\{ \frac{1}{2} \| w - z \|^2 \right\} \]

In large-scale setting, instead of updating the gradient by evaluating \( n \) component gradients, it is more efficient to consider the projected stochastic gradient descent approach, in which the proximal operator is applied to a stochastic gradient step:

\[ y_{k+1} = \text{proj}_W(y_k - h G_{k,t}), \quad (6) \]

where \( G_{k,t} \) is a stochastic estimate of the gradient \( \nabla F(y_k) \). Of particular relevance to our work are the SVRG [9,32] and S2GD [11] methods where the stochastic estimate of \( \nabla F(y_k) \) is of the form

\[ G_{k,t} = \nabla F(w) + (\nabla f_i(y_k) - \nabla f_i(w)), \quad (7) \]

where \( w \) is an “old” reference point for which the gradient \( \nabla F(w) \) was already computed in the past, and \( i \in [n] \) is picked uniformly at random. A mini-batch version of similar form is introduced as mS2GD [10] with

\[ G_{k,t} = \nabla F(w) + \frac{1}{b} \sum_{i \in A_{k,t}} (\nabla f_i(y_k) - \nabla f_i(w)), \quad (8) \]

where the mini-batch \( A_{k,t} \subset [n] \) of size \( b \) is chosen uniformly at random. Apparently, the gradient estimate (7) is a special case of (8) with \( b = 1 \). Notice that \( G_{k,t} \) is an unbiased estimate of the gradient:

\[ \mathbb{E}_i[G_{k,t}] \overset{\text{S}}{=} \nabla F(w) + \frac{1}{b} \frac{b}{n} \sum_{i=1}^{n} (\nabla f_i(y_k) - \nabla f_i(w)) \overset{\text{E}}{=} \nabla F(y_k). \]

Methods such as SVRG [9,32], S2GD [11] and mS2GD [10] update the points \( y_k \) in an inner loop, and the reference point \( x \) in an outer loop. This ensures that \( G_{k,t} \) has low variance, which ultimately leads to extremely fast convergence.

We now describe the PS2GD method in mini-batch scheme (Algorithm 1).
Algorithm 1 PS2GD

1: \textbf{Input:} $M$ (max # of stochastic steps per epoch); $h > 0$ (stepsize); $w_0 \in \mathbb{R}^d$ (starting point); linear coefficients $q \in \mathbb{R}^d$; mini-batch size $b \in [n]

2: \textbf{for} $k = 0, 1, 2, \ldots$ \textbf{do}

3: Compute and store $v_k \leftarrow \nabla F(w_k) = \frac{1}{n} \sum_i \nabla f_i(w_k) = \frac{1}{n} \sum_i a_i \nabla g_i(a_i^T w_k) + q$ 

4: Initialize the inner loop: $y_{k,0} \leftarrow w_k$

5: Let $t_k \leftarrow t \in \{1, 2, \ldots, M\}$ uniformly at random

6: \textbf{for} $t = 0$ to $t_k - 1$ \textbf{do}

7: Choose mini-batch $A_{kt} \subset [n]$ of size $b$ uniformly at random

8: Compute a stochastic estimate of $\nabla F(y_{k,t})$: 

9: $G_{k,t} \leftarrow v_k + \frac{1}{b} \sum_{i \in A_{kt}} [\nabla g_i(a_i^T y_{k,t}) - \nabla g_i(a_i^T w_k)] a_i$

10: $y_{k,t+1} \leftarrow \text{proj}_W (y_{k,t} - hG_{k,t})$

11: \textbf{end for}

12: Set $w_{k+1} \leftarrow y_{k,t_k}$

13: \textbf{end for}

The algorithm includes both outer loops indexed by epoch counter $k$ and inner loops indexed by $t$. To begin with, the algorithm runs each epoch by evaluating $v_k$, which is the full gradient of $F$ at $w_k$, then it proceeds to produce $t_k$ — the number of inner loops, where $t_k = t \in \{1, 2, \ldots, M\}$ is chosen uniformly at random.

Subsequently, we run $t_k$ iterations in the inner loop — the main step of our method (Step 8). Each new iterate is given by the projected update (6); however, with the stochastic estimate of the gradient $G_{k,t}$ in (8), which is formed by using a mini-batch $A_{kt} \subset [n]$ of size $|A_{kt}| = b$. Each inner iteration takes 2b component gradient evaluations.²

3 Complexity Result

In this section, we state our main complexity results and comment on how to optimally choose the parameters of the method. Denote $\mathcal{W}^* \subseteq \mathcal{W}$ as the set of optimal solutions. Then following ideas from the proof of Theorem 1 in [10], we conclude the following theorem. In Appendix B.4, we provide the complete proof.

**Theorem 1.** Let Assumptions 1, 2 and 3 be satisfied and let $w_* \in \mathcal{W}^*$ be any optimal solution to (1). In addition, assume that the stepsize satisfies $0 < h \leq \min \left\{ \frac{1}{4L\alpha(b)}, \frac{1}{4hL\alpha(b)} \right\}$ and that $M$ is sufficiently large so that

$$\rho \overset{\text{def}}{=} \frac{\beta + 4\mu h^2 L\alpha(b)(M + 1)}{\mu h (1 - 4hL\alpha(b)) M} < 1,$$

² It is possible to finish each iteration with only $b$ evaluations for component gradients, namely $\{\nabla f_i(y_{k,t})\}_{i \in A_{kt}}$, at the cost of having to store $\{\nabla f_i(x_k)\}_{i \in [n]}$, which is exactly the way that SAG [12] works. This speeds up the algorithm; nevertheless, it is impractical for big $n$. 

where \( \alpha(b) = \frac{m-b}{b(m-1)} \) and \( \beta \) is some finite positive number dependent on the structure of \( A \) in (1) and \( C \) in (4). Then PS2GD has linear convergence in expectation:

\[
E(F(w_k) - F(w_*)) \leq \rho^k (F(w_0) - F(w_*)).
\]

Remark 3. Consider the special case of strong convexity, when \( F \) is strongly convex with parameter \( \mu_F \),

\[
F(w) - F(w_*) \geq \frac{\mu_F}{2} \|w - w_*\|^2,
\]

then we have

\[
\rho = \frac{1}{h \mu_F(1 - 4h \alpha(b)) M} + \frac{4h L \alpha(b)(M+1)}{(1 - 4h \alpha(b)) M},
\]

which recovers the convergence rate from [10] and it is better than [32] computationally since their algorithm requires computation of an average over \( M \) points, while we continue with the last point, which is computationally more efficient.

In the special case when \( b = 1 \) we get \( \alpha(b) = 1 \), and the rate given by (9) exactly recovers the rate achieved by VRPSG [6] (in the case when the Lipschitz constants of \( \nabla f_i \) are all equal).

From Theorem 4 it is not difficult to conclude the following corollary, which aims to detect the effects of mini-batch on PS2GD. The proof of the corollary follows from the proof of Theorem 2 in [10], and thus is omitted.

**Corollary 2.** Fix target decrease \( \rho_* \geq \rho \), where \( \rho \) is given by (9) and \( \rho_* \in (0, 1) \).

If we consider the mini-batch size \( b \) to be fixed and define the following quantity,

\[
\tilde{h} \overset{\text{def}}{=} \sqrt{\beta^2 \left( \frac{1 + \rho}{\rho \mu} \right)^2 + \frac{\beta}{4 \mu \alpha(b) L} - \frac{\beta (1 + \rho)}{\rho \mu}},
\]

then the choice of stepsizes \( h^*_b \) and size of inner loops \( m^*_b \), which minimizes the work done — the number of gradients evaluated — while having \( \rho \leq \rho_* \), is given by the following statements.

If \( \tilde{h} \leq \frac{1}{\rho} \), then \( h^*_b = \tilde{h} \) and

\[
m^*_b = \frac{2 \kappa}{\rho} \left\{ \left( 1 + \frac{1}{\rho} \right) 4 \alpha(b) + \sqrt{\frac{4 \alpha(b)}{\kappa} + \left( 1 + \frac{1}{\rho} \right)^2 [4 \alpha(b)]^2} \right\},
\]

where \( \kappa \overset{\text{def}}{=} \frac{\beta L}{\mu} \) is the condition number; otherwise, \( h^*_b = \frac{1}{\rho} \) and

\[
m^*_b = \frac{\kappa + 4 \alpha(b)}{\rho - 4 \alpha(b)(1 + \rho)}.
\]

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3 We only need to prove the existence of \( \beta \) and do not need to evaluate its value in practice. Lemma 4 provides the existence of \( \beta \).
If $m^*_b < m^*_1/b$ for some $b > 1$, then mini-batching can help us reach the target decrease $\rho_*$ with fewer component gradient evaluations. Equation (11) suggests that as long as the condition $\tilde{h}^b \leq 1/L$ holds, $m^*_b$ is decreasing at a rate roughly faster than $1/b$. Hence, we can attain the same decrease with no more work, compared to the case when $b = 1$.

4 Numerical Experiments

In this section, we deliver preliminary numerical experiments to substantiate the effectiveness and efficiency of PS2GD. We experiment mainly on constrained logistic regression problems introduced in Remark 1 (Type I), i.e.,

$$\min_{w \in \mathcal{W}} \{F(w) := \frac{1}{n} \sum_{i=1}^{n} \log[1 + \exp(-b_ia_i^T w)]\}, \text{ with } \mathcal{W} = \{w \in \mathbb{R}^d : \|w\|_{\infty} \leq \zeta\}$$

(13)

where $\{(a_i, b_i)\}_{i=1}^{n}$ is a set of training data points with $a_i \in \mathbb{R}^d$ and $b_i \in \{+1, -1\}$ for binary classification problems.

We performed experiments on three publicly available binary classification datasets, namely rcv1, news20 and astro-ph. In a logistic regression problem, the Lipschitz constant of function $f_i$ can be derived as $L_i = \|a_i\|^2/4$. We assume (Assumption 1) the same constant $L$ for all functions since all data points can be scaled to have proper Lipschitz constants. We set the bound of the norm $\zeta = 0.1$ in our experiments. A summary of the three datasets is given in Table 1 including the sizes $n$, dimensions $d$, their sparsity as proportion of nonzero elements and Lipschitz constants $L$.

| Dataset   | n    | d    | Sparsity  | L      |
|-----------|------|------|-----------|--------|
| rcv1      | 20,242 | 47,236 | 0.1568%   | 0.2500 |
| news20    | 19,996 | 1,355,191 | 0.0336%  | 0.2500 |
| astro-ph  | 62,369 | 99,757 | 0.0767%   | 0.2500 |

Table 1: Summary of datasets used for experiments.

We implemented the following prevalent algorithms. SGD, SGD+ and FISTA are only enough to demonstrate sub-linear convergence without any strong convexity assumption.

4 Although both L1 norm and infinity norm are applicable in our problem setting, infinity norm enforces an implementation for data with sparse structure, hence, our experiments are conducted with infinity norm in $\mathcal{W}$.
5 rcv1 and news20 are available at [http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/)
6 Available at [http://users.cecs.anu.edu.au/~xzhang/data/](http://users.cecs.anu.edu.au/~xzhang/data/)
1. **PS2GD b=1**: the PS2GD algorithm without mini-batch, i.e., with mini-batch size $b = 1$. Although a safe step-size is given in our theoretical analyses in Theorem 1, we ignored the bound, experimented with various step-sizes and used the constant step-size that gave the best performance.

2. **PS2GD b=4**: the PS2GD algorithm with mini-batch size $b = 4$. We used the constant step-size that gave the best performance.

3. **SGD**: the proximal stochastic gradient descent method with the constant step-size which gave the best performance in hindsight.

4. **SGD+**: the proximal stochastic gradient descent with adaptive step-size $h = h_0/(k + 1)$, where $k$ is the number of effective passes and $h_0$ is some initial constant step-size. We used $h_0$ which gave the best performance in hindsight.

5. **FISTA**: fast iterative shrinkage-thresholding algorithm proposed in [1]. This is considered as the full gradient descent method in our experiments.

![Fig. 1: Comparison of different algorithms on rcv1 (left), news20 (middle) and astroph(right).](image)

In Figure 1 each effective pass is considered as $n$ component gradient evaluations, where each $f_i$ in (2) is named as a component function, and each full gradient evaluation counts as one effective pass. The $y$-axis is the distance from the current function value to the optimum, i.e., $F(w) - F(w^*)$. The nature of SGD suggests unstable positive variance for stochastic gradient estimates, which induces SGD to oscillate around some threshold after a certain number of iterations with constant step-sizes. Even with decreasing step-sizes over iterations, SGD are still not able to achieve high accuracy (shown as SGD+ in Figure 1). However, by incorporating a variance-reduced technique for stochastic gradient estimate, PS2GD maintains a reducing variance over iterations and can achieve higher accuracy with fewer iterations. FISTA is worse than PS2GD due to large numbers of component gradient evaluations per iteration.

Meantime, increase of mini-batch size up to some threshold does not hurt the performance of PS2GD and PS2GD can be accelerated in the benefit of...
simple parallelism with mini-batches. Figure 2 compares the best performances of PS2GD with different mini-batch sizes on datasets rcv1 and news20. Numerical results on rcv1 with no parallelism imply that, PS2GD with $b = 2, 4, 8, 16, 32$ are comparable or sometimes even better than PS2GD without any mini-batch ($b = 1$); while on news20, PS2GD with $b = 4, 32$ are better than and the others are worse but comparable to PS2GD with $b = 1$. Moreover, with parallelism, the results are promising. The bottom row shows results of ideal speedup by parallelism, which would be achievable if and only if we could always evaluate the $b$ gradients efficiently in parallel.\footnote{In practice, it is impossible to ensure that evaluating different component gradients takes the same time; however, Figure 2 implies the potential and advantage of applying mini-batch scheme with parallelism.}

![Fig. 2: Comparison of PS2GD with different mini-batch sizes on rcv1 (left) and news20 (right).](image)

5 Conclusion

In this paper, we have proposed a mini-batch projected semi-stochastic gradient descent method, for minimizing the sum of smooth convex functions subject to a compact polyhedral set. This kind of constrained optimization problems arise in inverse problems in signal processing and modern statistics, and is popular among the machine learning community. Our PS2GD algorithm combines the variance-reduced technique for stochastic gradient estimates and the mini-batch scheme, which ensure a high accuracy for PS2GD and speedup the algorithm. Mini-batch technique applied to PS2GD also admits a simple implementation for parallelism in HPC environment. Furthermore, in theory, PS2GD has a great
improvement that it requires no strong convexity assumption of either data or objective function but maintains linear convergence; while prevalent methods under non-strongly convex assumption only achieves sub-linear convergence. PS2GD, belonging to the gradient descent algorithms, has also been shown applicable to dual problem of SVM with hinge loss, which is usually efficiently solved by dual coordinate ascent methods. Comparisons to state-of-the-art algorithms suggest PS2GD is competitive in theory and faster in practice even without parallelism. Possible implementation in parallel and adaptiveness for sparse data imply its potential in industry.

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References

1. Amir Beck and Marc Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sciences*, 2(1):183–202, 2009.
2. Paul H. Calamai and Jorge J. Moré. Projected gradient methods for linearly constrained problems. *Mathematical Programming*, 39:93–116, 1987.
3. Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In *NIPS*, 2014.
4. Olivier Fercoq, Zheng Qu, Peter Richtárik, and Martin Takáč. Fast distributed coordinate descent for non-strongly convex losses. In *IEEE Workshop on Machine Learning for Signal Processing*, 2014.
5. Olivier Fercoq and Peter Richtárik. Accelerated, parallel and proximal coordinate descent. *SIAM J. Optimization*, 25(4):1997–2023, 2013.
6. Pinghua Gong and Jieping Ye. Linear convergence of variance-reduced projected stochastic gradient without strong convexity. *arXiv:1406.1102*, 2014.
7. Alan J. Hoffman. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards*, 49(4):263–265, 1952.
8. Martin Jaggi, Virginia Smith, Martin Takáč, Jonathan Terhorst, Thomas Hofmann, and Michael I Jordan. Communication-efficient distributed dual coordinate ascent. In *NIPS*, pages 3068–3076, 2014.
9. Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *NIPS*, pages 315–323, 2013.
10. Jakub Konečný, Jie Liu, Peter Richtárik, and Martin Takáč. Mini-batch semi-stochastic gradient descent in the proximal setting. *IEEE Journal of Selected Topics in Signal Processing*, 10:242–255, 2016.
11. Jakub Konečný and Peter Richtárik. Semi-stochastic gradient descent methods. *arXiv:1312.1666*, 2013.
12. Nicolas Le Roux, Mark Schmidt, and Francis Bach. A stochastic gradient method with an exponential convergence rate for finite training sets. In *NIPS*, pages 2672–2680, 2012.
13. Ji Liu and Stephen J. Wright. Asynchronous stochastic coordinate descent: Parallelism and convergence properties. *SIAM J. Optimization*, 25(1):351–376, 2015.
14. Jakub Mareček, Peter Richtárik, and Martin Takáč. Distributed block coordinate descent for minimizing partially separable functions. *Numerical Analysis and Optimization 2014, Springer Proceedings in Mathematics and Statistics*, pages 261–286, 2014.
15. Ion Necoara. Linear convergence of first order methods under weak nondegeneracy assumptions for convex programming. arXiv:1504.06298, 2015.
16. Ion Necoara and Dragos Clipici. Parallel random coordinate descent method for composite minimization: Convergence analysis and error bounds. *SIAM J. Optimization*, 26(1):197–226, 2016.
17. Ion Necoara and Andrei Patrascu. A random coordinate descent algorithm for optimization problems with composite objective function and linear coupled constraints. *Computational Optimization and Applications*, 57(2):307–337, 2014.
18. Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM J. Optimization*, 19(4):1574–1609, 2009.
19. Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Kluwer, Boston, 2004.
20. Yurii Nesterov. Efficiency of coordinate descent methods on huge-scale optimization problems. *SIAM J. Optimization*, 22:341–362, 2012.
21. Yurii Nesterov. Gradient methods for minimizing composite functions. *Mathematical Programming*, 140(1):125–161, 2013.
22. Lam M. Nguyen, Jie Liu, Katya Scheinberg, and Takáč. SARAII: A novel method for machine learning problems using stochastic recursive gradient. arXiv:1703.00102, 2017.
23. Peter Richtárik and Martin Takáč. Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function. *Mathematical Programming*, 144(1-2):1–38, 2014.
24. Peter Richtárik and Martin Takáč. Distributed coordinate descent method for learning with big data. *Journal of Machine Learning Research*, 17:1-25, 2016.
25. Peter Richtárik and Martin Takáč. Parallel coordinate descent methods for big data optimization. *Mathematical Programming*, Series A:1–52, 2015.
26. Shai Shalev-Shwartz, Yoram Singer, Nathan Srebro, and Andrew Cotter. Pegasos: Primal estimated sub-gradient solver for svm. *Mathematical Programming: Series A and B- Special Issue on Optimization and Machine Learning*, pages 3-30, 2011.
27. Shai Shalev-Shwartz and Tong Zhang. Accelerated mini-batch stochastic dual coordinate ascent. In *NIPS*, pages 378–385, 2013.
28. Shai Shalev-Shwartz and Tong Zhang. Stochastic dual coordinate ascent methods for regularized loss. *Journal of Machine Learning Research*, 14(1):567–599, 2013.
29. Ohad Shamir and Tong Zhang. Stochastic gradient descent for non-smooth optimization: Convergence results and optimal averaging schemes. In *ICML*, pages 71–79. Springer, 2013.
30. Martin Takáč, Avleen Singh Bijral, Peter Richtárik, and Nathan Srebro. Minibatch primal and dual methods for SVMs. In *ICML*, pages 537–552. Springer, 2013.
31. Po-Wei Wang and Chih-Jen Lin. Iteration complexity of feasible descent methods for convex optimization. *Journal of Machine Learning Research*, 15:1523–1548, 2014.
32. Lin Xiao and Tong Zhang. A Proximal Stochastic Gradient Method with Progressive Variance Reduction. *SIAM Journal on Optimization*, 24(4):2057–2075, 2014.

33. Hui Zhang. The restricted strong convexity revisited: analysis of equivalence to error bound and quadratic growth. *Optimization Letters*, pages 1–17, 2016.

34. Tong Zhang. Solving large scale linear prediction using stochastic gradient descent algorithms. In *ICML*, pages 919–926. Springer, 2004.
A Technical Results

Lemma 1. Let set $W \subseteq \mathbb{R}^d$ be nonempty, closed, and convex, then for any $x, y \in \mathbb{R}^d$,
$$\|\text{proj}_W(x) - \text{proj}_W(y)\| \leq \|x - y\|.$$  

Note that the above contractiveness of projection operator is a standard result in optimization literature. We provide proof for completeness.

Inspired by Lemma 1 in [32], we derive the following lemma for projected algorithms.

Lemma 2 (Modified Lemma 1 in [32]). Let Assumption 4 hold and let $w_* \in W^*$ be any optimal solution to Problem (1). Then for any feasible solution $w \in W$, the following holds,
$$\frac{1}{n} \sum_{i=1}^{n} \|a_i[\nabla g_i(a_i^T w) - \nabla g_i(a_i^T w_*)]\| = \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(w) - \nabla f_i(w_*)\| \leq 2L[F(w) - F(w_*)].$$

(14)

Lemma 3 and Lemma 4 come from [10] and [31], respectively. Please refer to the corresponding references for complete proofs.

Lemma 3 (Lemma 4 in [10]). Let $\{\xi_i\}_{i=1}^{n}$ is a collection of vectors in $\mathbb{R}^d$ and $\mu \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} \xi_i \in \mathbb{R}^d$. Let $\hat{S}$ be a $\tau$-nice sampling. Then
$$\mathbb{E} \left[\left\|\frac{1}{\tau} \sum_{i \in \hat{S}} \xi_i - \mu\right\|^2\right] = \frac{1}{n\tau} \frac{n - \tau}{(n - 1)} \sum_{i=1}^{n} \|\xi_i\|^2.$$  

(15)

Combining the proof of Corollary 3 in [32], by applying Lemma 3 with $\xi_i := \nabla f_i(y_{k,t-1}) - \nabla f_i(w_k) = a_i[\nabla g_i(a_i^T y_{k,t-1}) - \nabla g_i(a_i^T w_k)]$ and Lemma 2 we have the bound for variance as follows.

Theorem 3 (Bounding Variance). Considering the definition of $G_{k,t}$ in Algorithm 4 conditioned on $y_{k,t}$, we have $\mathbb{E}[G_{k,t}] = \frac{1}{n} \sum_{i=1}^{n} \nabla g_i(y_{k,t}) + q = \nabla F(y_{k,t})$ and the variance satisfies,
$$\mathbb{E}\left[\|G_{k,t} - \nabla F(y_{k,t})\|^2\right] \leq \frac{n - b}{b(n - 1)} \frac{4L[F(y_{k,t}) - F(w_*) + F(w_k) - F(w_*)]}{\alpha(b)}.$$  

(16)

Lemma 4 (Hoffman Bound, Lemma 15 in [31]). Consider a non-empty polyhedron
$$\{w_* \in \mathbb{R}^d|Cw_* \leq c, Aw_* = r\}.$$  

For any $w$, there is a feasible point $w_*$ such that
$$\|w - w_*\| \leq \theta(A, C) \left\|\frac{|Cw - c|}{Aw - r}\right\|,$$
where $\theta(A, C)$ is independent of $x$,
$$\theta(A, C) = \sup_{u, v} \left\{ \frac{\|u\|}{\|v\|} \left\|C^Tu + A^Tv\right\| = 1, u \geq 0, \text{ The corresponding rows of } C, A \text{ to } u, v \text{'s non-zero elements are linearly independent.} \right\}.$$  

(17)
Lemma 5 (Weak Strong Convexity). Let $w \in W := \{w \in \mathbb{R}^d : Cw \leq c\}$ be any feasible solution (Assumption [3]) and $w^* = \text{proj}_W(w)$ which is an optimal solution for Problem [1]. Then under Assumptions [2-3], there exists a constant $\beta > 0$ such that for all $w \in W$, the following holds,

$$F(w) - F(w^*) \geq \frac{\mu}{2\beta} \|w - w^*\|^2,$$

where $\mu$ is defined in Assumption [2]. $\beta$ can be evaluated by $\beta = \theta^2$ where $\theta$ is defined in (17).

B Proofs

B.1 Proof of Lemma [1]

For any $x, y \in \mathbb{R}^d$, by Projection Theorem, the following holds,

$$[y - \text{proj}_W(y)]^T[\text{proj}_W(x) - \text{proj}_W(y)] \leq 0,$$

similarly, by symmetry, we have

$$[x - \text{proj}_W(x)]^T[\text{proj}_W(y) - \text{proj}_W(x)] \leq 0.$$

Then (18) + (19) gives

$$\|\text{proj}_W(x) - \text{proj}_W(y)\|^2 \leq (x - y)^T[\text{proj}_W(x) - \text{proj}_W(y)],$$

and by Cauchy-Schwarz inequality, we have

$$\|\text{proj}_W(y) - \text{proj}_W(x)\| \leq \|x - y\|.$$

when $\text{proj}_W(x) = \text{proj}_W(y)$ are distinct; in addition, when $\text{proj}_W(x) = \text{proj}_W(y)$, the above inequality also holds. Hence, for any $x, y \in \mathbb{R}^d$, which is the same to

$$\|\text{proj}_W(x) - \text{proj}_W(y)\| \leq \|x - y\|.$$

B.2 Proof of Lemma [2]

For any $i \in \{1, \ldots, n\}$, consider the function

$$\phi_i(w) = f_i(w) - f_i(w^*) - \nabla f_i(w^*)^T(w - w^*),$$

then it should be obvious that $\nabla \phi_i(w^*) = \nabla f_i(w^*) - \nabla f_i(w^*) = 0$, hence $\min_{w \in \mathbb{R}^d} \phi_i(w) = \phi_i(w^*)$ because of the convexity of $f_i$. By Assumption [1] and Remark [1] $\nabla \phi_i(w)$ is Lipschitz continuous with constant $L$, hence by Theorem 2.1.5 from [19] we have

$$\frac{1}{2L}\|\nabla \phi_i(w)\|^2 \leq \phi_i(w) - \min_{w \in \mathbb{R}^d} \phi_i(w) = \phi_i(w) - \phi_i(w^*) = \phi_i(w).$$
which, by \cite{24}, suggests that
\[
\|\nabla f_i(w) - \nabla f_i(w_*)\|^2 \leq 2L[f_i(w) - f_i(w_*) - \nabla f_i(w_*)^T(w - w*)].
\]
By averaging the above equation over \(i = 1, \ldots, n\) and using the fact that \(F(w) = \frac{1}{n}\sum_{i=1}^{n} f_i(w)\), we have
\[
\frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(w) - \nabla f_i(w_*)\|^2 \leq 2L[F(w) - F(w_*) - \nabla F(w_*)^T(w - w*)],
\]
which, together with \(\nabla F(w_*)^T(w - w*) \geq 0\) indicated by the optimality of \(w_*\) for Problem (1), completes the proof for Lemma 2.

B.3 Proof of Lemma 5

First, we will prove by contradiction that there exists a unique \(r\) such that \(W^* = \{w \in \mathbb{R}^d : Cw \leq c, Aw = r\}\) which is non-empty. Assume that there exist distinct \(w_1, w_2 \in W^*\) such that \(Aw_1 \neq Aw_2\). Let us define the optimal value to be \(F^*\) which suggests that \(F^* = F(w_1) = F(w_2)\). Moreover, convexity of function \(F\) and feasible set \(W\) suggests the convexity of \(W^*\), then \(\frac{1}{2}(w_1 + w_2) \in W^*\). Therefore,
\[
F^* = F\left(\frac{1}{2}(w_1 + w_2)\right) = g\left(A\frac{1}{2}(w_1 + w_2)\right) + \frac{1}{2}g^T(w_1 + w_2)
= g\left(\frac{1}{2}Aw_1 + \frac{1}{2}Aw_2\right) + \frac{1}{2}q^T(w_1 + w_2).
\]

Strong convexity indicated in Assumption 2 suggests that
\[
F^* = \frac{1}{2}(F(w_1) + F(w_2)) \geq \frac{1}{2}[g(Aw_1) + q^T w_1] + \frac{1}{2}[g(Aw_2) + q^T w_2]
= \left(\frac{1}{2}g(Aw_1) + \frac{1}{2}g(Aw_2)\right) + \frac{1}{2}q^T(w_1 + w_2)
> g\left(\frac{1}{2}Aw_1 + \frac{1}{2}Aw_2\right) + \frac{1}{2}q^T(w_1 + w_2) \overset{\text{23}}{=} F^*,
\]
which is a contradiction, so there exists a unique \(r\) such that \(W^*\) can be represented by \(\{w \in \mathbb{R}^d : Cw \leq c, Aw = r\}\).

For any \(w \in W = \{x \in \mathbb{R}^d : Cx \leq c\}, [Cw - c]^+ = 0\), then by Hoffman’s bound in Lemma 3 for any \(w \in W\), there exists \(w' \in W^*\) and a constant \(\theta > 0\) defined in (17), dependent on \(A\) and \(C\), such that
\[
\|w - w'\| \leq \theta \left\|\frac{[Cw - c]^+}{Aw - r}\right\| = \theta \|Aw - r\| = \theta \|Aw - Aw_*\|, \forall w \in W^*.
\]

Being aware of that by choosing \(w_* = \text{proj}_{W^*}(w)\), we have that \(\|w - w_*\| \leq \|w - w'\|\), which suggests that
\[
\|w - w_*\| \leq \|w - w'\| \overset{\text{22}}{\leq} \theta \|Aw - Aw_*\|,
\]
or equivalently,
\[
\|Aw - Aw_*\|^2 \geq \frac{1}{\theta^2}\|w - w_*\|^2, \forall w_* \in W^*.
\]

(23)
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where $\beta = \theta^2 > 0$.

Optimality of $w^*$ for Problem 1 suggests that

$$\nabla F(w^*)^T (w - w^*) \geq 0; \quad (24)$$

then we can conclude the following,

$$g(Aw) \geq g(Aw^*) + \nabla g(Aw^*)^T (Aw^* - Aw) + \frac{\mu}{2} ||Aw - Aw^*||^2, \quad (25)$$

which, by considering $F(w) = g(Aw) + q^T w$ in Problem 1, is equivalent to

$$F(w) - F(w^*) \geq g(Aw) - g(Aw^*) + q^T (w - w^*) \geq \frac{\mu}{2} ||Aw - Aw^*||^2 \quad (26)$$

B.4 Proof of Theorem 1

The proof is following the steps in [10,32]. For convenience, let us define the stochastic gradient mapping

$$d_{k,t} = \frac{1}{h}(y_{k,t} - y_{k,t+1}) = \frac{1}{h}(y_{k,t} - \text{proj}_W(y_{k,t} - hG_{k,t})), \quad (27)$$

then the iterate update can be written as

$$y_{k,t+1} = y_{k,t} - hd_{k,t}. \quad (28)$$

Let us estimate the change of $||y_{k,t+1} - w^*||$. It holds that

$$||y_{k,t+1} - w^*||^2 = ||y_{k,t} - hd_{k,t} - w^*||^2 = ||y_{k,t} - w^*||^2 - 2hd_{k,t}^T(y_{k,t} - w^*) + h^2||d_{k,t}||^2. \quad (29)$$

By the optimality condition of $y_{k,t+1} = \text{proj}_W(y_{k,t} - hG_{k,t}) = \arg\min_{w \in W} \{\frac{1}{2}||w - (y_{k,t} - hG_{k,t})||^2\}$, we have

$$[y_{k,t+1} - (y_k - hG_{k,t})]^T (w^* - y_{k,t+1}) \geq 0,$$

then the update $y_{k,t+1} = y_k - hd_{k,t}$ suggests that

$$G_{k,t}^T (w^* - y_{k,t+1}) \geq d_{k,t}^T (w^* - y_{k,t+1}). \quad (30)$$

Moreover, Lipschitz continuity of the gradient of $F$ implies that

$$F(y_{k,t}) \geq F(y_{k,t+1}) - \nabla F(y_{k,t})^T (y_{k,t+1} - y_{k,t}) - \frac{L}{2} ||y_{k,t+1} - y_{k,t}||^2. \quad (31)$$
Let us define the operator \( \Delta_{k,t} = G_{k,t} - \nabla F(y_{k,t}) \), so

\[
\nabla F(y_{k,t}) = G_{k,t} - \Delta_{k,t}
\]

(Convexity of \( F \) suggests that

\[
F(w^*) \geq F(y_{k,t}) + \nabla F(y_{k,t})^T (w^* - y_{k,t})
\]

\[\geq F(y_{k,t+1}) - \nabla F(y_{k,t})^T (y_{k,t+1} - y_{k,t}) - \frac{L}{2} \|y_{k,t+1} - y_{k,t}\|^2 + \nabla F(y_{k,t})^T (w^* - y_{k,t+1}) \]

\[= F(y_{k,t+1}) - \frac{L}{2} \|y_{k,t+1} - y_{k,t}\|^2 + \nabla F(y_{k,t})^T (w^* - y_{k,t+1}) \]

\[ \geq \left( \frac{L}{2} - \frac{Lh^2}{2} \right) \|d_{k,t}\|^2 + \frac{Lh^2}{2} (w^* - y_{k,t} + y_{k,t+1}) - \Delta_{k,t}^T (w^* - y_{k,t+1}) \]

\[= F(y_{k,t+1}) + \frac{Lh^2}{2} \|d_{k,t}\|^2 + \frac{Lh^2}{2} (w^* - y_{k,t}) - \Delta_{k,t}^T (w^* - y_{k,t+1}) \]

\[h \leq 1^L \]

\[ \geq F(y_{k,t+1}) + \frac{h}{2} \|d_{k,t}\|^2 + \frac{h}{2} (w^* - y_{k,t}) - \Delta_{k,t}^T (w^* - y_{k,t+1}), \]

then equivalently,

\[ - d_{k,t}^T (y_{k,t} - w_*) + \frac{h}{2} \|d_{k,t}\|^2 \leq F(w_*) - F(y_{k,t+1}) - \Delta_{k,t}^T (y_{k,t+1} - w_*). \]

Therefore,

\[ \|y_{k,t+1} - w_*\|^2 \leq \|y_{k,t} - w_*\|^2 + 2h \left( F(w_*) - F(y_{k,t+1}) - \Delta_{k,t}^T (y_{k,t+1} - w_*) \right) \]

\[= \|y_{k,t} - w_*\|^2 - 2h \Delta_{k,t}^T (y_{k,t+1} - w_*) - 2h [F(y_{k,t+1}) - F(w_*)]. \]

In order to bound \(-\Delta_{k,t}^T (y_{k,t+1} - w_*)\), let us define the proximal full gradient update

\[ \bar{y}_{k,t+1} = \text{proj}_y(y_{k,t} - h\nabla F(y_{k,t})), \]

with which, by using Cauchy-Schwartz inequality and Lemma 4, we can conclude that

\[ -\Delta_{k,t}^T (y_{k,t+1} - w_*) = -\Delta_{k,t}^T (y_{k,t+1} - \bar{y}_{k,t+1}) - \Delta_{k,t}^T (\bar{y}_{k,t+1} - w_*) \]

\[= -\Delta_{k,t}^T \left[ \text{proj}_y(y_{k,t} - hG_{k,t}) - \text{proj}_y(y_{k,t} - h\nabla F(y_{k,t})) \right] - \Delta_{k,t}^T (\bar{y}_{k,t+1} - w_*) \]

\[\leq \|\Delta_{k,t}\| \|y_{k,t} - hG_{k,t}\| - (y_{k,t} - h\nabla F(y_{k,t})) \| - \Delta_{k,t}^T (\bar{y}_{k,t+1} - w_*), \]

\[= h \|\Delta_{k,t}\|^2 - \Delta_{k,t}^T (\bar{y}_{k,t+1} - w_*). \]

So we have

\[ \|y_{k,t+1} - w_*\|^2 \]

\[\leq \|y_{k,t} - w_*\|^2 + 2h \left( h \|\Delta_{k,t}\|^2 - \Delta_{k,t}^T (\bar{y}_{k,t+1} - w_*) - [F(y_{k,t+1}) - F(w_*)] \right). \]

\[\text{Note that this quantity is never computed during the algorithm. We can use it in the analysis nevertheless.}\]
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By taking expectation, conditioned on \( y_{k,t} \) we obtain

\[
E[\|y_{k,t+1} - w_\ast\|^2] \leq \|y_{k,t} - w_\ast\|^2 + 2h \left(h E[\|\Delta_k,t\|^2] - E[F(y_{k,t+1}) - F(w_\ast)]\right),
\]

(34)

where we have used that \( E[\Delta_k,t] = E[G_k,t] - \nabla F(y_{k,t}) = 0 \) and hence \( E[-\Delta_k,t(y_{k,t+1} - w_\ast)] = 0 \). Now, if we put (16) into (34) we obtain

\[
E[\|y_{k,t+1} - w_\ast\|^2] \leq \|y_{k,t} - w_\ast\|^2 + 2h (4Lh\alpha(b)(F(y_{k,t}) - F(w_\ast)) + F(w_k) - F(w_\ast) - E[F(y_{k,t+1}) - F(w_\ast)]),
\]

(35)

where \( \alpha(b) = \frac{m-b}{b(m-1)} \).

Now, if we consider that we have just an lower-bounds \( \nu_F \geq 0 \) of the true strong convexity parameter \( \mu_F \), then we obtain from (35) that

\[
E[\|y_{k,t} - w_\ast\|^2] \leq \|y_{k,t-1} - w_\ast\|^2 + 2h (4Lh\alpha(b)(F(y_{k,t}) - F(w_\ast)) + F(w_k) - F(w_\ast) - E[F(y_{k,t+1}) - F(w_\ast)]),
\]

(36)

which, by decreasing the index \( t \) by 1, is equivalent to

\[
E[\|y_{k,t} - w_\ast\|^2] \leq \|y_{k,t-1} - w_\ast\|^2 + 2h (4Lh\alpha(b)(F(y_{k,t-1}) - F(w_\ast)) + F(w_k) - F(w_\ast)) + 8h^2L\alpha(b)(F(y_{k,t}) - F(w_\ast)) + F(w_k) - F(w_\ast)).
\]

Now, by the definition of \( w_k \) we have that

\[
E[F(w_{k+1})] = \frac{1}{M} \sum_{t=1}^{M} E[F(y_{k,t})].
\]

(37)

By summing (36) multiplied by \((1 - h\nu_F)^{M-t}\) for \( t = 1, \ldots, M \), we can obtain the left hand side

\[
LHS = \sum_{t=1}^{M} E[\|y_{k,t} - w_\ast\|^2] + 2h \sum_{t=1}^{M} E[F(y_{k,t}) - F(w_\ast)]
\]

(38)

and the right hand side

\[
\begin{align*}
RHS &= \sum_{t=1}^{M} E[\|y_{k,t-1} - w_\ast\|^2] + 8h^2L\alpha(b) \sum_{t=1}^{M} E[F(y_{k,t-1}) - F(w_\ast) + F(w_k) - F(w_\ast)] \\
&= \sum_{t=0}^{M-1} E[\|y_{k,t} - w_\ast\|^2] + 8h^2L\alpha(b) \left( \sum_{t=0}^{M-1} E[F(y_{k,t}) - F(w_\ast)] \right) \\
&\quad + 8h^2L\alpha(b)M E[F(w_k) - F(w_\ast)] \\
&\leq \sum_{t=0}^{M-1} E[\|y_{k,t} - w_\ast\|^2] + 8h^2L\alpha(b) \left( \sum_{t=0}^{M} E[F(y_{k,t}) - F(w_\ast)] \right) \\
&\quad + 8Mh^2L\alpha(b) E[F(w_k) - F(w_\ast)].
\end{align*}
\]

(39)

For simplicity, we omit the \( E[\cdot | y_{k,t}] \) notation in further analysis.\(^9\)

\(\bar{y}_{k,t+1}\) is constant, conditioned on \( y_{k,t}\)\(^10\).
Applying the above linear convergence relation recursively with chained expectations $F$ and realizing that

$$\text{E}[\|y_{k,M} - w_*\|^2] + 2h \sum_{t=1}^{M} \text{E}[F(y_{k,t}) - F(w_*)]$$

$$\leq \text{E}[\|y_{k,0} - w_*\|^2] + 8Mh^2 L_o(b) \text{E}[F(w_k) - F(w_*)]$$

$$+ 8h^2 L_o(b) \left( \sum_{t=1}^{M} \text{E}[F(y_{k,t}) - F(w_*)] \right)$$

$$+ 8h^2 L_o(b) \text{E}[F(y_{k,0}) - F(w_*)].$$

Now, using (37) we obtain

$$\text{E}[\|y_{k,M} - w_*\|^2] + 2Mh(\text{E}[F(w_{k+1})] - F(w_*))$$

$$\leq \text{E}[\|y_{k,0} - w_*\|^2] + 8Mh^2 L_o(b) \text{E}[F(w_k) - F(w_*)]$$

$$+ 8Mh^2 L_o(b) \left( \text{E}[F(w_{k+1})] - F(w_*) \right)$$

$$+ 8h^2 L_o(b) \text{E}[F(y_{k,0}) - F(w_*)].$$

(40)

Note that all the above results hold for any optimal solution $w_* \in W^*$; therefore, they also hold for $w'_* = \text{proj}_{W^*}(w_k)$, and Lemma 5 implies that, under weak strong convexity of $F$, i.e., $\nu_F = 0$,

$$\|w_k - w'_*\|^2 \leq \frac{2\beta}{\mu} [F(w_k) - F(w'_*)].$$

(41)

Considering $\text{E}[\|y_{k,M} - w'_*\|^2] \geq 0$, $y_{k,0} = w_k$, and using (41), the inequality (40) with $w_*$ replaced by $w'_*$ gives us

$$2Mh \left\{ 1 - 4hL_o(b) \right\} \text{E}[F(w_{k+1})] - F(w'_*)]$$

$$\leq \left\{ \frac{2\beta}{\mu} + 8Mh^2 L_o(b) + 8h^2 L_o(b) \right\} [F(w_k) - F(w'_*)],$$

or equivalently,

$$\text{E}[F(w_{k+1})] - F(w'_*)] \leq \rho [F(w_k) - F(w'_*)],$$

when $1 - 4hL_o(b) > 0$ (which is equivalent to $h \leq \frac{1}{4L_o(b)}$), and when $\rho$ is defined as

$$\rho = \frac{\beta/\mu + 4h^2 L_o(b)(M + 1)}{h(1 - 4hL_o(b))M}.$$

The above statement, together with assumptions of $h \leq 1/L$, implies

$$0 < h \leq \min \left\{ \frac{1}{4L_o(b)}, \frac{1}{L} \right\}.$$

Applying the above linear convergence relation recursively with chained expectations and realizing that $F(w'_*) = F(w_*)$ for any $w_* \in W^*$ since $w_*, w'_* \in W^*$, we have

$$\text{E}[F(w_k) - F(w_*)] \leq \rho^k [F(w_0) - F(w_*)],$$
