Non-axisymmetric Torsional Oscillations of Relativistic Stars

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Abstract. We examine the non-axisymmetric Alfvén oscillations of relativistic stars with a strong dipole magnetic fields, where we have done the 2D simulation. We find that the spectrum of non-axisymmetric axial-type Alfvén oscillations is discrete as well as the axisymmetric polar Alfvén oscillations, in contrast to the spectrum of axisymmetric axial Alfvén oscillations which is continuum. We also show the dependence of Alfvén frequency on the strength of magnetic field and find that such frequencies are smaller than those of the axisymmetric axial Alfvén oscillations. That is, to explain the observed evidence of quasi-periodic oscillations in the soft gamma repeaters, the non-axisymmetric axial oscillations could also play an important role.

1. Introduction

One of the most promising candidates of magnetars is the soft gamma repeaters (SGRs), which rarely emit quite strong gamma-rays called as “giant flares”. Up to now, at least three giant flares have been detected in three different SGRs. Through the timing analysis of the decaying tail followed after the initial strong sharp burst in the giant flares, the existence of quasi-periodic oscillations (QPOs) is revealed. The specific frequencies of these QPOs are in the range from tens Hz up to a few kHz \cite{1}. The many researchers are considering that these QPOs come from the oscillations of magnetars and the observations of QPOs in SGRs might be first evidences to detect the oscillations of neutron stars directly. To explain these QPO frequencies in SGRs theoretically, many attempts have been done. However, most of them are based on the axisymmetric oscillations, e.g., the oscillations in the crust region \cite{2, 3} and the Alfvén oscillations \cite{4, 5, 6, 7}. On the other hand, the non-axisymmetric oscillations on the relativistic magnetar are not really acknowledged, in spite of the fact that consideration of non-axisymmetric oscillations is more realistic and more natural. There are a few studies about such oscillations, but those are mainly based on the Newtonian framework \cite{8, 9}. In practice, considering the non-axisymmetric oscillations is more complicated because the axial type oscillations are coupled with the polar type oscillations due to the presence of magnetic field even for a non-rotating magnetar. This coupling could be important for the observation of the gravitational waves, if one stands on the statement that the polar oscillations would be induced from the axial oscillation. Thus in this article we consider only the axial type non-axisymmetric oscillations as a first step, where we neglect the coupling with polar type ones. In order to know the feature of axial type non-axisymmetric oscillations, we will perform the 2D numerical simulations about the perturbation equations of relativistic magnetar models, which have a global dipole magnetic field.
2. Equilibrium Models
We neglect the stellar deformation from spherical symmetry due to the magnetic pressure, because the magnetic energy is much smaller than the gravitational binding energy. Thus the equilibrium of the non-rotating neutron star becomes static spherical symmetry. The magnetic field inside the star is superimposed on this stellar model, where we focus only on the dipole magnetic field with ideal MHD approximation. This magnetic field is given as [10].

3. Perturbation Equations
We restrict attention to non-axisymmetric axial-type perturbations with relativistic Cowling approximation, for which the perturbed matter quantities should be zero except for the \( \theta \) and \( \phi \) components of perturbed four-velocity, \( \delta u^\theta \) and \( \delta u^\phi \). The perturbation equations describing \( \delta u^\theta \) and \( \delta u^\phi \) can be derived from the \( \theta \) and \( \phi \) components of the linearized equation of motion, which corresponds to the equation (35) in [3]. On the other hand, the perturbations of magnetic fields are determined from the linearized induction equation, which is corresponding to the equation (37) in [3]. Now, we can replace the variables \( \delta u^\theta \) and \( \delta u^\phi \) with

\[
\delta u^\theta = im \mathcal{U}_1(t, r, \theta)e^{im\phi} / \sin \theta, \quad \text{and} \quad \delta u^\phi = \mathcal{U}_2(t, r, \theta)e^{im\phi} / \sin \theta.
\]

Then the perturbation equations to solve can be written as

\[
\begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix}
\begin{bmatrix}
\partial_t^2 \mathcal{U}_1 \\
\partial_t^2 \mathcal{U}_2
\end{bmatrix}
= \begin{bmatrix}
\mathcal{F}_1 \\
\mathcal{F}_2
\end{bmatrix},
\]

where the coefficients \( A_{00}, A_{01}, A_{10}, \) and \( A_{11} \), and the source terms \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are some functions of the background properties.

Since the angular dependences for the axial perturbations can be generally described as

\[
\delta u^\theta \sim \partial_\theta Y_{\ell m} / \sin \theta \quad \text{and} \quad \delta u^\phi \sim (\partial_\theta Y_{\ell m}) / \sin \theta,
\]

the \( \theta \) dependences of the variables \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) become \( \mathcal{U}_1 \sim P_{\ell m}^m(\cos \theta) \) and \( \mathcal{U}_2 \sim \partial_\theta P_{\ell m}^m(\cos \theta) \), where \( P_{\ell m}^m(\cos \theta) \) is the associated Legendre polynomial of order \( \ell \) and \( m \). Thus the boundary conditions for \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are

\[
\mathcal{U}_1 = 0 \quad \text{and} \quad \partial_\theta \mathcal{U}_2 = 0 \quad \text{for} \quad m = 1, \quad \text{(3)}
\]

\[
\mathcal{U}_1 = 0 \quad \text{and} \quad \mathcal{U}_2 = 0 \quad \text{for} \quad m \neq 1, \quad \text{(4)}
\]

on the axis (\( \theta = 0 \)), while

\[
\mathcal{U}_1 = 0 \quad \text{and} \quad \partial_\theta \mathcal{U}_2 = 0 \quad \text{as} \quad (\ell + m) \quad \text{is odd}, \quad \text{(5)}
\]

\[
\partial_\theta \mathcal{U}_1 = 0 \quad \text{and} \quad \mathcal{U}_2 = 0 \quad \text{as} \quad (\ell + m) \quad \text{is even}, \quad \text{(6)}
\]

on the equatorial plane (\( \theta = \pi/2 \)). On the other hand, with respect to the radial direction, since in the vicinity of stellar center the variables \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are expanded as \( \mathcal{U}_1 \sim C_1 r^{\ell-1} + \cdots \) and \( \mathcal{U}_2 \sim C_2 r^{\ell-1} + \cdots \), where \( C_1 \) and \( C_2 \) are some constant, we can impose the boundary conditions at \( r = 0 \) as \( \mathcal{U}_1 = \mathcal{U}_2 = 0 \). The boundary conditions at the stellar surface are the zero traction conditions, which lead to \( \partial_r \mathcal{U}_1 = \partial_r \mathcal{U}_2 = 0 \).

4. Numerical Results
In order to construct magnetar models, we adopt a polytropic equation of state (EOS) called EOS A (see [11] for more details). Especially, we will focus on the stellar model with \( M = 1.4M_\odot \). For simplicity, we consider only the pure poloidal magnetic component in this article. It should be noticed that the perturbed variables \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) should be the real variables if the stellar magnetic field has only poloidal component. To evolve the perturbation equations (2), we adopt
the iterated Crank-Nicholson scheme together with a fourth-order Kreiss-Oliger dissipation. The region to calculate in this article is restricted in the two-dimensional region of \((r, \theta)\) with \(100 \times 40\) grid points, where \(r\) and \(\theta\) are in the range of \(0 \leq r \leq R\) and \(0 \leq \theta \leq \pi/2\), respectively. After obtaining the evolutions of perturbed variables, one can extract the specific frequencies for each spacial position by performing a fast Fourier transformation (FFT) of the time-depending perturbed variables.

First, to see the spectrum feature for the non-axisymmetric axial oscillations, we check the position of peak frequencies in FFT with different spacial position in the star. Now, figure 1 shows the FFT of perturbation variables \(U_2\) at various points inside the star for a stellar model with \(B = 5 \times 10^{16}\) Gauss, where the value of \(m\) is fixed to be \(m = 2\) in the evolution. In this figure, three lines are corresponding to the different angular positions as \(\theta \approx 0\), \(\pi/4\), and \(\pi/2\), while three different panels correspond to the different radial positions as \(r \approx 0\), \(R/2\), and \(R\). From this figure, one can observe two peaks, whose frequencies are 98 and 180 Hz, and one can say that the positions of peak frequencies in FFT are independent of the observer inside the star. The FFT for \(U_1\) is also similar picture to figure 1. That is, the spectrum of non-axisymmetric axial Alfven oscillation could not be continuum but discrete.

Figure 2 shows the dependence of the frequencies of non-axisymmetric axial Alfven oscillations on the magnetic field strength, where the labels of \((2, 2)\) and \((4, 2)\) denote the values of \((\ell, m)\) and \(B_{16}\) is defined as \(B_{16} = B/10^{16}\). These Alfven modes are corresponding to the peak frequencies in the above FFTs. In the figure, the marks are shown our numerical results while the lines are corresponding to the fitting formula as mentioned the below. Obviously one can see that the frequencies are proportional to the magnetic field strength. With our numerical results, we can obtain the fitting formula as

\[
f_{22}/B_{16} = 19.31 \quad \text{and} \quad f_{42}/B_{16} = 35.99, \tag{7}\]

where \(f_{22}\) and \(f_{42}\) denote the frequencies of Alfven modes with \((\ell, m) = (2, 2)\) and \((4, 2)\), respectively. With these fitting formulae, the oscillation frequencies for the stellar model with \(B = 4 \times 10^{15}\) Gauss become \(f_{22} = 7.7\) Hz and \(f_{42} = 14.4\) Hz. On the other hand, the minimum frequency of the axisymmetric axial Alfven oscillation for the same stellar model is around 15 Hz [4]. That is, the non-axisymmetric axial oscillations can be realized the lower oscillation rather than the axisymmetric axial oscillations. This result could be quite important to explain theoretically the observed evidence of quasi-periodic oscillations in the SGR, because it has been difficult to explain the lower part of frequencies (see for the discussion in [3, 4]). We are considering that the non-axisymmetric axial Alfven oscillations could play an important role in the oscillations of magnetar and probably it might be possible to explain clearly the observed evidence with these type of oscillations. To fit the possible stellar model with the observations in SGRs, however, it is necessary to produce more oscillation frequencies with different value of \((\ell, m)\) for the stellar models constructed with different EOSs. Since now we find that the non-axisymmetric axial oscillations become discrete spectrum, one can safely study the calculation

![Figure 1. The FFT of the perturbation function \(U_2\).](image-url)
Figure 2. Frequencies of the Alfvén modes as functions of magnetic field strength.

to determine the specific frequencies by using modal decomposition as the study by [2, 8] and this method could be better to determine the frequencies of stellar oscillations. As a future work, we will do this and find the most possible magnetar model to explain the observations.

5. Summary and Discussion
To examine the non-axisymmetric axial-type oscillations on magnetar, we derive the perturbation equations for an arbitrary magnetic distribution inside the star, where we neglect the effect of the solid crust near the stellar surface. For simplicity, we consider the magnetar models with the pure poloidal magnetic field and with this stellar model the perturbation variables can reduce the real ones. Performing the time-evolution of two-dimensional perturbation equations, we estimate the Alfvén oscillations. In order to know the spectrum feature for these type oscillations, we check the dependence of positions of peak frequencies in FFTs upon the observer position inside the star. Then we observe that the positions of peak frequencies in FFTs are independent of the observers, i.e., we can conclude that these type oscillations become discrete spectrum as well as the axisymmetric polar Alfvén oscillations. Additionally we see the dependence of frequencies on the magnetic field strength and find that the frequencies are obviously proportional to the magnetic field strength. The frequencies of the non-axisymmetric axial oscillation become smaller than those of the axisymmetric axial oscillations. In order to explain the observation of specific frequencies of the quasi-periodic oscillations in SGRs, these smaller frequencies could be play an important role and as a future work we will find the most possible magnetar model by determining systematically more frequencies with different values of $\ell, m$ and with different EOSs. At the end, we make a comment about the introduction of the solid crust. As shown in the previous articles such as [3], the oscillation of solid curst region becomes discrete spectrum. Thus even if considering this region in the magnetar model, we can expect that there exist both frequencies of non-axisymmetric axial Alfvén oscillations and the crustal oscillations. Anyway taking into account the existence of solid crust region is also important future work.

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