LETTER TO THE EDITOR

Quantum fields and ‘big rip’ expansion singularities

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Abstract
The effects of quantized conformally invariant massless fields on the evolution of cosmological models containing a ‘big rip’ future expansion singularity are examined. Quantized scalar, spinor and vector fields are found to strengthen the accelerating expansion of such models as they approach the expansion singularity.

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Studies of distant type Ia supernovae [1, 2] strongly indicate that the expansion of the universe is actually accelerating. Combined with the analysis of the WMAP cosmic microwave background [3], it appears that the universe is filled with a ‘dark energy’ amounting to about 70% of the closure density. This dark energy is characterized by an effective equation of state parameter, \( w = p/\rho \), the ratio of the pressure to the energy density.

The simplest model for the dark energy is that it is a cosmological constant, \( \Lambda \), for which \( w = -1 \). Quintessence models of classical scalar fields, which have attracted much attention [4, 5], yield values of \( w > -1 \). Models of the dark energy with \( w < -1 \) are dubbed ‘phantom energy’. Phantom energy has the odd property that its energy density increases as the universe expands; it also violates the dominant energy condition. A recent analysis [6] finds \( w = -1.02^{+0.13}_{-0.19} \). Clearly, the possibility that our universe contains a phantom energy with \( w < -1 \) cannot be ruled out at present.

In addition to the odd material properties of phantom energy, cosmological models in which phantom energy becomes dominant undergo divergent expansion, leading to a future spacelike singularity at a finite cosmic time, termed a ‘big rip’ [7]. The neighbourhood of cosmological singularities is widely recognized as one of the very few locations where quantum gravitational effects may play an important dynamical role. The possible existence of a future big rip allows us to consider a case in which quantum gravitational effects can be examined in a predictive fashion, rather than merely studying the consistency of quantum gravitational models applied to observations of the early universe. In this letter, we begin
such a study by examining the form of the vacuum stress–energy for quantized conformally invariant scalar, spinor and vector fields in big rip spacetimes. We seek to determine whether such fields, at the semiclassical level, will weaken or strengthen the development of a big rip singularity.

For a spatially flat Friedmann–Robertson–Walker (FRW) spacetime with metric
\[ ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \] (1)
the Einstein equation for cosmology containing a mass fraction \( \Omega_m \) of ordinary and dark matter, and \( 1 - \Omega_m \) of dark energy, is [7]
\[ \left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ \frac{\Omega_m}{a^3} + (1 - \Omega_m)a^{-3(1+w)} \right], \] (2)
where \( H_0 \) is the Hubble constant. This may be solved implicitly for the scale factor
\[ \frac{1}{H_0} \int_0^{a(t)} \xi^{1/2} \left[ \Omega_m + (1 - \Omega_m)\xi^{-3w} \right]^{1/2} d\xi = t, \] (3)
where the lower boundary of the integral has been chosen so that the big bang occurred when \( t = 0 \). The fact that the integral is convergent for any value \( w < -1 \) with \( a \to \infty \) demonstrates that the big rip occurs at a finite time for such values of \( w \).

We are interested in the behaviour near the big rip, where quantum effects are expected to become significant, and so we can approximate equation (3) by
\[ T - t = \frac{1}{H_0(1 - \Omega_m)^{1/2}} \int_{a(t)}^\infty \xi^{(1+3w)/2} d\xi, \] (4)
where \( T \) is the time of the big rip, i.e., \( a(T) \to \infty \). This yields, as an approximate description of the scale factor near the big rip,
\[ a(t) \simeq \left[ \left( \frac{1}{2} \right) H_0 (1 - \Omega_m)^{1/2} \right] \left( 1 + w \right) (T - t)^{1/2}. \] (5)

It is well known that the vacuum stress–energy of quantized conformally invariant fields in an FRW spacetime depends only on the trace anomaly and the choice of appropriate vacuum state, determined by the mapping of the FRW spacetime’s Cauchy surface to the conformally related Minkowski or Rindler space. A good summary of these points is given by Candelas and Dowker [8]. Spatially flat FRW spacetimes are conformally related to Minkowski space, and hence the vacuum stress–energy tensor of a quantized conformally invariant field is given simply by
\[ \langle T_{\mu\nu} \rangle = \alpha \left( g_{\mu\nu}R_{\sigma\tau} - R_{\mu\sigma\nu\tau} + RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2 \right)
+ \beta \left( \frac{2}{3} RR_{\mu\nu} - R_{\mu\sigma}^{\sigma\nu} R_{\sigma\tau} + \frac{1}{2}g_{\mu\nu}R_{\sigma\tau} R^\sigma\tau - \frac{1}{4}g_{\mu\nu}R^2 \right), \] (6)
where \( R_{\mu\nu} \) is the Ricci tensor, and \( R \) is the Ricci scalar. The constants \( \alpha \) and \( \beta \) are related to the number and spins of the fields present (see table 1).

Evaluating \( \langle T_{\mu\nu} \rangle \) for the metric of equation (1) with the scale factor given by equation (5), we find that the vacuum energy densities of the scalar, spinor and vector fields \( \langle \rho_0 \rangle, \langle \rho_{1/2} \rangle \) and \( \langle \rho_1 \rangle \) respectively) can be written in the form
\[ \langle \rho_a \rangle = \langle T_{00} \rangle |_{\text{spin}=a} = \frac{P_a}{19440\pi^2(T - t)^4(1 + w)^4}, \] (7)
Table 1. Spin coefficients.

| Spin | α   | β   |
|------|-----|-----|
| 0    | 1   | 1   |
| 1/2  | 3/200π² | 11 | 5600π² |
| 1    | -9/1400π² | 31 | 1400π² |

where \( a = 0, 1/2, 1 \), and \( P_a \) is a second-degree polynomial in \( w \),

\[
P_0 = -5 + 18w + 27w^2 \\
P_{1/2} = -5 + 54w + 81w^2 \\
P_1 = 205 - 162w - 243w^2.
\]

Of these three polynomials, \( P_0 \) and \( P_{1/2} \) are strictly positive for all \( w < -1 \); \( P_1 \) is positive for \( w_0 < w < -1 \) and negative for \( w < w_0 \), where \( w_0 = -\frac{1}{3} - \frac{2}{9} \sqrt{174} \simeq -1.31 \). The signs of the energy densities, by equation (7), agree with the signs of the polynomials as functions of \( w \).

For all three spins, the ratio of the expected value of the pressure to the expected value of the energy density is a constant

\[
\langle w \rangle = \frac{\langle p_a \rangle}{\langle \rho_a \rangle} = 1 + 2w,
\]

which is independent of the spin of the field.

In the absence of a complete self-consistent theory of quantum gravity, calculated expectation values of the vacuum stress–energy of quantized fields in a classical background may be used to indicate, in a perturbative sense, in which direction inclusion of quantum effects will change the classical solution. We accomplish this by examining the effective change in the total stress–energy tensor when the quantized vacuum energies are included. The qualitative effect of such inclusion on the dynamics of an FRW cosmology, i.e., whether the evolution towards a big rip singularity is strengthened or weakened, is simple to discern, due to the high degree of symmetry of the spacetime metric.

In de Sitter spacetime, the expectation value of the stress–energy tensor, \( \langle T_{\mu\nu} \rangle \), and the cosmological constant term, \( \Lambda g_{\mu\nu} \), have the same form and are usually dealt with as one term (the first renormalizing the value of the second). However, in the FRW cosmology with phantom energy \( (w < -1) \), the pressure to energy density ratio of the quantized fields’ vacuum stress–energy, \( \langle w \rangle \), is not equal to the background ratio, \( w \). For the combination of the phantom energy plus the quantized fields’ vacuum stress–energy we define an effective equation of state parameter by

\[
w_{\text{eff}} \equiv \frac{P_{\text{total}}}{\rho_{\text{total}}} = \frac{P_{\text{background}} + \langle p \rangle}{\rho_{\text{background}} + \langle \rho \rangle}.
\]

Using equation (11), this can be rewritten as

\[
w_{\text{eff}} = w + (1 + w) \frac{\langle \rho \rangle}{\rho + \langle \rho \rangle}.
\]

If \( w_0 < w < -1 \), then \( \langle \rho \rangle > 0 \) for all of the conformally invariant quantized fields, which ensures that

\[
w_{\text{eff}} < w.
\]
In this case, the effect of the vacuum energy density of the quantized conformally invariant fields is to strengthen the accelerated expansion that leads to the big rip singularity.

If the only conformally invariant quantized fields present were vector fields, and $w < w_0$, then $\langle \rho \rangle < 0$, and, so long as the vacuum energy is considered a small perturbation ($\langle \rho \rangle \ll \rho$), the effect of the quantized vector fields would be to weaken the evolution towards the big rip singularity, since in this case $w_{\text{eff}} > w$. However, if $w_0 < w < -1$, then the quantized vector fields strengthen the big rip. The effect of quantized conformally invariant vector fields is thus to ‘push’ $w_{\text{eff}}$ towards the value $w_0$, in which case there is still a big rip singularity caused by divergent expansion. Even quantized conformally invariant vector fields cannot, within the semiclassical approximation, weaken a big rip singularity to the point of rendering it regular.

The above calculations were done within the context of the big rip cosmological models constructed by Caldwell et al [7], which assume that the parameter, $w$, in the dark-energy equation of state, $w = p/\rho$, remains constant near the big rip, and in which the scale factor, $a(t)$, diverges at a finite cosmic time. In contrast, the models constructed by Barrow [9], in which $w \sim (T_{\text{rip}} - t)^{-1-\alpha}$ (with $\alpha = -1 + \epsilon$) possess a milder form of expansion singularity, in which the scale factor remains finite (although its second derivative diverges, with a related divergence in the pressure) at the singularity. The effects of quantized conformally invariant fields in such models have been examined by Nojiri and Odinstov [10], who have found that, in those cases, the quantum effects cause the singularity to become milder.

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