MIXING TIME OF METROPOLIS CHAIN BASED ON RANDOM TRANSPOSITION WALK CONVERGING TO MULTIVARIATE EWENS DISTRIBUTION

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We prove sharp rates of convergence to the Ewens equilibrium distribution for a family of Metropolis algorithms based on the random transposition shuffle on the symmetric group, with starting point at the identity. The proofs rely heavily on the theory of symmetric Jack polynomials, developed initially by Jack [Proc. Roy. Soc. Edinburgh Sect. A 69 (1970/1971) 1–18], Macdonald [Symmetric Functions and Hall Polynomials (1995) New York] and Stanley [Adv. Math. 77 (1989) 76–115]. This completes the analysis started by Diaconis and Hanlon in [Contemp. Math. 138 (1992) 99–117]. In the end we also explore other integrable Markov chains that can be obtained from symmetric function theory.

1. Introduction. There is a well-known bijection between the set of partitions of $n$ and the conjugacy classes of the symmetric group $S_n$. The partition that a permutation $\sigma \in S_n$ corresponds to is simply given by its cycle structure. In fact this connection is the basis for the classical representation theory of $S_n$ (see, e.g., [9]): the set of irreducible representations of $S_n$ is indexed by the set $\mathcal{P}_n$ of partitions of $n$. Since $S_n$ is finite, it can also be endowed with a probability space structure. The most natural measure on $S_n$ is thus the uniform measure, with each permutation getting a weight of $1/n!$. Sampling from this uniform measure is important for many statistical applications [5], such as testing independence of $n$ i.i.d. uniform random variables on an ordered set. Its intimate connection with card shuffling models has also generated a wonderful array of mathematics, most notably the determination of their mixing times; see, for instance, [2, 7] and [19].

One of the most natural generalizations of the uniform measure on $S_n$ is a 1-parameter family of so-called multivariate Ewens distributions, named after Warren Ewens, who derived the partition function of this probability measure. It is defined by giving each permutation $\sigma$ a weight of $\alpha^{\ell(\sigma)}$, $\alpha > 0$, where $\ell(\sigma)$ is the number of cycles in $\sigma$; hence it can be viewed as an exponentially tilted family based on the uniform measure. The distribution was first applied to population genetics, in which it describes the distribution of frequencies of alleles in a sample of genes ([13], Chapter 41).

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Many important properties of the uniform measure on $S_n$ continue to hold for the multivariate Ewens distribution [1]. For instance, the two perfect sampling schemes of the uniform measure, Feller coupling and the Chinese restaurant process, both generalize to the Ewens case. In this article, we describe a much more subtle property of the uniform measure that has been successfully generalized to the $\alpha$ deformed setting. In a nutshell, the characters of the symmetric group $S_n$ form a basis in the Fourier space of class functions on $S_n$ under the uniform measure [5]. When the underlying measure is $\alpha$-deformed from the uniform, one can only make sense of Fourier transforms of a particular type of class functions, namely the ones supported on transpositions and the identity class. In that case, the basis in the Fourier space becomes the matrix coefficients of the transition from Jack symmetric polynomials basis to the power sum symmetric polynomials basis. These generalize the characters of $S_n$, which happen to be the transition coefficients from Schur polynomial basis to the power-sum polynomial basis.

This Fourier analytic property was first obtained by Stanley in [20]. Later Hanlon [11] applied it to the study of the Metropolis Markov chain based on random transposition walk on $S_n$ that converges to the multivariate Ewens distribution. Diaconis and Hanlon [21] further initiated the investigation of total variation mixing time of this chain.

In light of the sharp result in [7] for the uniform case ($\alpha = 1$), it is natural to wonder what’s the exact mixing time for the Diaconis–Hanlon Metropolis walk ($\alpha \neq 1$). In this paper, we prove a pair of matching upper and lower bound for the mixing time that applies to all $\alpha > 0$, which together imply the cut-off phenomenon. Previously Diaconis and Hanlon [21] outlined a proof of the upper bound in the case $\alpha > 1$ and conjectured that it was tight.

In the Appendices, we include some preliminary attempts to generalize the walk studied here in various directions. These were motivated by questions of Diaconis on whether other nontrivial Markov chains can be constructed from symmetric function theory. First we look at the action of the Sekiguchi–Debiard operator on other classical bases of symmetric polynomials. We also consider higher order operators, as given by the operator valued generating function in [17], page 317. These turn out to give new local move Markov chains converging to MED($\theta$), albeit without simple group theoretic interpretations. Finally we look at Laplace–Beltrami operators associated with other root systems. Recall the Schur–Weyl duality between the simple Lie groups $SU(n)$ and the finite groups $S_n$. This leads to an interpretation of the Sekiguchi–Debiard operator (as well as their quantized version given by Macdonald [17]) as associated with root system of type $A_n$. The appropriate generalizations were first discovered by Heckman and Opdam [12] in the context of Hamiltonian systems of particles on a circle, and later extended to the Macdonald case in [18]; see also [15] and references therein for a 5-parameter generalization.
Jack polynomials, which form the backbone of the argument presented here, turn out to be special cases of Macdonald polynomials of type $A_n$. Diaconis and Ram [6] interpreted them as eigenfunctions of an auxiliary variable algorithm on the space of partitions, which can be viewed as a quantized version of the local walk studied here.

2. Metropolis walk starting at the identity class. The multivariate Ewens distribution with parameter $\alpha$ is defined on $S_n$ with $P(\sigma)$ proportional to $\alpha^{\ell(\sigma)}$ where $\ell(\sigma)$ is the number of cycles of $\sigma$. The normalization constant $z_n(\alpha) = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$.

Consider now the random transposition walk on $S_n$, defined by picking a pair of numbers $i \neq j$ at random, and multiplying the current state in $S_n$ by the transposition $(ij)$. In this form, the walk is periodic and does not converge. But if we make it lazy, then it converges to the uniform measure on $S_n$. By metropolizing the nonlazy walk to the multivariate Ewens distribution with parameter $\alpha$, we create a new Markov chain that converges to MED($\alpha$); see [21] for details on the Metropolis algorithm. The walk behaves differently for $\alpha > 1$ and $\alpha < 1$. And as long as $\alpha \neq 1$, it converges, because it always has positive holding probability.

Warning. The $\alpha$ parameter here will be the reciprocal of $\theta$ below.

If we start the Metropolis walk described in the introduction from the identity element, then we can view it as a walk either on the symmetric group or on the set of partitions. It is the latter interpretation that allows for sharp analysis with other starting points.

**Theorem 2.1.** For $\theta \in (0, \infty) \setminus \{1\}$, let $P_\theta$ be the discrete time Metropolis chain based on random transposition walk starting from id, converging to the multivariate Ewens distribution $\pi$ with parameter $\theta^{-1}$ (so identity has the largest mass when $\theta < 1$). Explicitly, let $\lambda(\pi)$ be the cycle structure of the permutation $\pi$, $\lambda^t$ be the transposition of $\lambda$ as a Ferrers diagram and denote $n(\lambda) = \sum_{i=1}^{n} \binom{\lambda^t_i}{2}$. Then the transition rule is given by

$$P_\theta(\pi, \sigma) = \begin{cases} 
1 - 1 \land \theta + \frac{n(\lambda(\pi)^t)}{\binom{n}{2}} (1 \land \theta - 1 \land \theta^{-1}), & \text{if } \sigma = \pi, \\
\frac{1}{\binom{n}{2}} (1 \land \theta), & \text{if } \sigma = \pi(i, j) \text{ and } \ell(\sigma) = \ell(\pi) - 1, \\
\frac{1}{\binom{n}{2}} (1 \land \theta^{-1}), & \text{if } \sigma = \pi(i, j) \text{ and } \ell(\sigma) = \ell(\pi) + 1, \\
0, & \text{otherwise.} 
\end{cases}$$
Here $1 \leq i < j \leq n$. The chain $P_\theta$ has a total variation cut-off at $t = \frac{1}{2}(\frac{1}{\theta} \lor 1)n \log n$. This means

$$\lim_{c \to \infty} \limsup_{n \to \infty} \|P_{id}^t - \pi\|_{TV} = 0,$$

$$\lim_{c \to -\infty} \liminf_{n \to \infty} \|P_{id}^t - \pi\|_{TV} = 1$$

for $t(c) := \frac{1}{2}(\frac{1}{\theta} \lor 1)n(\log n + c)$.

**Remark 1.** (1) Notice the chain has no intrinsic holding: when $\theta = 1$ it corresponds to the completely industrious random transposition walk with probability $\frac{1}{(\frac{1}{\theta})}$ to go to a neighboring permutation. If one inserts a holding of $1/n$, that is, $P \mapsto \frac{1}{n}I + (1 - \frac{1}{n})P$, then the asymptotic cut-off profile stays the same and has no removable discontinuity at $\theta = 1$.

(2) The Metropolis chain defined above can be projected to conjugacy classes of $S_n$, namely partitions, provided we start at the identity element. The transition matrix takes the following form:

$$P_\theta(\lambda, \mu) = \begin{cases} 
1 - 1 \wedge \theta + \frac{n(\lambda^i)}{(\frac{1}{\theta})}(1 \wedge \theta - 1 \wedge \theta^{-1}), & \text{if } \mu = \lambda, \\
\frac{\lambda_i \lambda_j}{(\frac{1}{\theta})}(1 \wedge \theta), & \text{if } \mu_k = \lambda_i + \lambda_j, \\
\frac{\lambda_k}{(\frac{1}{\theta})}(1 \wedge \theta^{-1}), & \text{if } \mu_i + \mu_j = \lambda_k \text{ and } \mu_i \neq \mu_j, \\
\frac{\lambda_k}{2(\frac{1}{\theta})}(1 \wedge \theta^{-1}), & \text{if } \mu_i + \mu_j = \lambda_k \text{ and } \mu_i = \mu_j, \\
0, & \text{otherwise.}
\end{cases}$$

Here $1 \leq i < j \leq \ell(\mu)$ and $1 \leq k \leq \ell(\lambda)$. Furthermore in the second line, $\mu \setminus \mu_k = \lambda \setminus \{\lambda_i, \lambda_j\}$. In other words, $\mu$ is obtained from $\lambda$ by joining $\lambda_i$ and $\lambda_j$ into a single part $\mu_k$. Similarly, for the third and fourth lines of the formula above, $\mu \setminus \{\mu_i, \mu_j\} = \lambda \setminus \lambda_k$, that is, $\mu$ is obtained by breaking a part $\lambda_k$ in $\lambda$ into two parts, $\mu_i$ and $\mu_j$.

(3) The first order phase transition at $\theta = 1$ for the cut-off value is not surprising, because the Metropolis chain has different forms for $\theta < 1$ and for $\theta > 1$.

(4) $\theta$ denotes the inverse of the Ewens sampling parameter the chain $P_\theta$ converges to. This choice of convention is justified by the fact that the left eigenfunctions of the chain $P_\theta$ are the transition coefficients from the Jack polynomials with parameter $\theta$ to the power sum polynomials, as derived in [11].

(5) It will be interesting to see what happens when the $\theta$ value in the transition probability is allowed to be state dependent, but satisfying some uniform bound
c < θ(λ) < c^{-1} for c independent of n. My conjecture is that it will always take at least \( \frac{1}{2} n \log n \) steps to converge to its stationarity distribution, which is no longer in the Ewens family.

The next four sections will be devoted to the proof of Theorem 2.1.

3. Preliminaries on \( L^2 \) mixing time.

**Lemma 3.1.** Given a reversible ergodic Markov chain \( P \) on a finite state space \( X \), let \( f_j \) be the right eigenfunctions, normalized so that
\[
\sum_x f_j(x)^2 \pi(x) = 1,
\]
with corresponding eigenvalues \( \beta_j \). Then \( g_j(x) := f_j(x)\pi(x) \) are left eigenfunctions of \( P \), with the same eigenvalues, satisfying
\[
\sum_x g_j(x)^2 \frac{1}{\pi(x)} = 1.
\]
Furthermore,
\[
\frac{1}{\pi(x)} = \sum_j f_j^2(x),
\]
\[
\pi(x) = \sum_j g_j^2(x).
\]

**Proof.** By reversibility, we have \( \pi(x) P(x, y) = \pi(y) P(y, x) \). Therefore,
\[
\beta_j f_j(x) = \sum_y P(x, y) f_j(y) = \sum_y \frac{\pi(x)}{\pi(x)} P(x, y) f_j(y)
\]
\[
= \sum_y \frac{\pi(y)}{\pi(x)} P(y, x) f_j(y).
\]
Now multiplying both sides by \( \pi(x) \), we get
\[
\beta_j \pi(x) f_j(x) = \sum_y \pi(y) f_j(y) P(y, x).
\]
This proves the first part.

The last two identities are nothing but a restatement of the fact that the matrix \( \Pi(x, y) := \pi(x) P(x, y) / \pi(y) \) is doubly stochastic; that is, each row and column sums to 1. Here is a formal proof. Since \( \{ f_j \} \) forms a basis, we can decompose the function \( z \mapsto 1_x(z) \) in it,
\[
1_x(z) = \sum_j c_j f_j,
\]
where the coefficients $c_j$ are given by

$$c_j = \langle f_j, 1_x \rangle_{L^2(\pi)} = \sum_z 1_x(z) f_j(z) \pi(z) = f_j(x) \pi(x).$$

The first equality follows immediately. The second is similar. □

**Lemma 3.2.** Under the same notation as the previous lemma, one can bound the total variation distance to stationarity at time $k$ starting at state $x$ by

$$4 \| P^k_x - \pi \|_{TV}^2 \leq \left\| \frac{P^k_x}{\pi(x)} - 1 \right\|_{L^2(\pi)}^2 \leq \frac{1}{\pi^2(x)} \sum_j \beta_j^{2k} g_j^2(x) - 1.$$

**Proof.** The first inequality (3) follows directly from the Cauchy–Schwarz inequality. To prove the second formula (4), first write

$$\left\| \frac{P^k_x}{\pi} - 1 \right\|_2^2 = \sum_y \pi(y) \left[ \left( \frac{P^k_x(y)}{\pi(y)} \right)^2 - 1 \right] = \sum_y \frac{(P^k_x(y))^2}{\pi(y)} - \pi(y).$$

Using reversibility again [in the extended form $\pi(x) P^k(x, y) = \pi(y) P^k(y, x)$], we can write

$$\frac{(P^k_x(y))^2}{\pi(y)} = \frac{P^k_x(y) P^k_y(x)}{\pi(x)}.$$

Thus summing over $y \in X$, we get

$$\left\| \frac{P^k_x}{\pi} - 1 \right\|_2^2 = \frac{P^{2k}(x, x)}{\pi(x)} - 1.$$

Next write the function $y \mapsto P^{2k}_x(y)$ as the result of a row vector multiplied by a matrix,

$$P^{2k}_x(y) = \sum_z 1_x(z) P^{2k}(z, y).$$

By the previous lemma, we have

$$1_x(y) = \sum_j c_j g_j(y),$$

where $c_j = \sum_z \frac{1}{\pi(z)} 1_x(z) g_j(z) = \frac{g_j(x)}{\pi(x)}$.

Finally evaluating at $y = x$ in (5), we obtain

$$\frac{P^{2k}(x, x)}{\pi(x)} = \sum_j \frac{g_j(x)^2}{\pi(x)^2} \beta_j^{2k}. \quad \square$$
4. Results from symmetric function theory. First we recall from Hanlon [11] that the eigenvalues for the chain \( P_\theta \) projected onto conjugacy classes of \( S_n \), with \( \theta > 1 \), are given by

\[
\beta_\lambda = \frac{n(\lambda^t) - \theta^{-1}n(\lambda)}{(\frac{n}{2})}.
\]

For an independent proof with pointers to literature, see the proof of Theorem 4.2.

Next we derive the eigenvalues of the chain \( P_\theta \), for \( \theta \in (0, 1) \). Notice this is not the same chain as that studied in [11]. Here the identity element gets the biggest mass, whereas in [11], identity has the smallest mass [Ewens sampling with parameter \( \in (0, 1) \)]. But the same result of Macdonald can be used here to derive eigenvalues. Indeed, consider the following matrix \( T_\theta \) defined by

\[
T_\theta(\pi, \sigma) = \begin{cases} 
(\theta - 1)n(\pi) \theta^{n(\pi)}, & \text{if } \sigma = \pi, \\
1 \theta^{n(\pi)}, & \text{if } \sigma = \pi(i, j) \text{ and } \ell(\sigma) = \ell(\pi) - 1, \\
1 \theta^{n(\pi)}, & \text{if } \sigma = \pi(i, j) \text{ and } \ell(\sigma) = \ell(\pi) + 1, \\
0, & \text{otherwise},
\end{cases}
\]

where \( n(\pi) = \sum_i \binom{n_i}{2} \) and \( \{n_i\} \) is the partition structure of \( \pi \). This quantity gives the number of ways to break a part in the partition structure of \( \pi \) into two parts, using multiplication by a transposition.

Hanlon considered the case \( \theta \geq 1 \) (his \( \alpha \) is our \( \theta \)), here we extend to \( \theta \in (0, 1) \), which is no longer a Markov matrix because the diagonal entries are no longer nonnegative. Nevertheless The rows still sum to 1. Then his Theorem 3.5 continues to hold because the proof never uses \( \theta \geq 1 \). Likewise, Theorem 3.9 holds for \( \theta < 1 \). To get \( P_\theta \), we simply need to rescale \( T_\theta \) by \( \theta \) and add a constant multiple \( cI \) of identity matrix. \( c \) can be obtained by looking at the top eigenvalue. By Theorems 3.5 and 3.9 of [11], \( \theta T_\theta \) has eigenvalues \( \theta n(\lambda^t) - n(\lambda) \). Thus we need to add \( 1 - \theta \) in order for \( \beta(n) \) to equal 1.

Combining the two cases, we have the following formula for eigenvalues of \( P_\theta \):

\[
\beta_\lambda(\theta) = 1 - \theta \land 1 + \frac{\theta n(\lambda^t) - n(\lambda)}{(\theta \lor 1)(\frac{n}{2})}.
\]

Denote \( r(\lambda) = \frac{n(\lambda^t) - n(\lambda)}{(\frac{n}{2})} \). The following lemma collects a bunch of estimates about \( \beta_\lambda \):

**Lemma 4.1.** Let \( \geq \) be the natural partial order on the set of partitions defined as follows: given two partitions represented by Ferrers diagrams \( \lambda \) and \( \lambda' \), say \( \lambda \geq \lambda' \) if \( \lambda \) can be obtained by successive up and right moves of blocks of \( \lambda' \).
(1) \(n(\lambda')\) is monotone, and \(n(\lambda)\) is anti-monotone in the above partial order; that is, for \(\lambda \geq \lambda'\),
\[ n(\lambda') > n(\lambda''), \quad n(\lambda) < n(\lambda'). \]

(2) \(\beta_\lambda\) is monotone with respect to the natural partial order on \(\lambda\). Thus \(\beta_{(n)} \geq \beta_\lambda\) for all \(\lambda \vdash n\), \(\beta_\lambda \leq \beta_{(\lambda_1, n - \lambda_1)}\) and \(\beta_\lambda \geq \beta_{(\lambda_1, 1^{n-\lambda_1})}\).

(3) Furthermore, for \(\lambda_1 \geq \frac{\theta}{\lambda_1} \geq \lambda_1\),
\[
\beta_\lambda \leq 1 - (\theta \land 1) - \frac{2\lambda_1(n - \lambda_1)}{n(n - 1)},
\]
and in general
\[
\beta_\lambda \leq 1 - (\theta \land 1) \left(1 - \frac{\lambda_1 - 1}{n - 1}\right). \tag{9}
\]
In particular, if \(\beta_\lambda(\theta) \geq 0\), the above two inequalities hold with \(|\beta_\lambda|\).

(4) Finally, if \(\beta_\lambda(\theta) < 0\), then \(\beta_{\lambda_1}(\theta) > 0\) and \(|\beta_\lambda| \leq \beta_{\lambda_1}\).

PROOF. (1) It suffices to check the first assertion for \(\lambda\) and \(\lambda'\) that differ by one block, that is, \(\lambda_i = \lambda_i' + 1\), \(\lambda_j = \lambda_j' - 1\), \(i < j\). Then
\[
n(\lambda') - n(\lambda'') = \frac{1}{2} [\lambda_i(\lambda_i - 1) + \lambda_j(\lambda_j - 1) - \lambda_i'(\lambda_i' - 1) - \lambda_j'(\lambda_j' - 1)] = \lambda_i' - \lambda_j' \geq 0,
\]
using the fact \(\lambda_i \geq \lambda_j\) and \(\lambda_i' \geq \lambda_j'\) by definition of Ferrers diagram. The antimonotonicity of \(n(\lambda)\) follows by taking transpose.

(2) This follows directly from the previous assertion and formula (7) for \(\beta_\lambda\) in terms of \(n(\lambda)\) and \(n(\lambda')\).

(3) Equation (8) follows from \(\lambda \leq (\lambda_1, n - \lambda_1)\) and monotonicity, after throwing away the term \(-\frac{n(\lambda)}{\theta(\lor 1)(\frac{n}{2})}\).

For (9), we again throw away the term \(-\frac{n(\lambda)}{\theta(\lor 1)(\frac{n}{2})}\) in \(\beta_\lambda\) to obtain
\[
\beta_\lambda \leq 1 - \theta \land 1 + \frac{\theta n(\lambda')}{(\theta \lor 1)(\frac{n}{2})} = 1 - (\theta \land 1) \left(1 - \frac{n(\lambda')}{\frac{n}{2}}\right)
\]
\[
= 1 - (\theta \land 1) \left(1 - \sum_j \frac{\lambda_j(\lambda_j - 1)}{n(n - 1)}\right)
\]
\[
\leq 1 - (\theta \land 1) \left(1 - \frac{(\lambda_1 - 1) \sum_j \lambda_j}{n(n - 1)}\right) = 1 - (\theta \land 1) \left(1 - \frac{\lambda_1 - 1}{n - 1}\right). \tag{4}
\]

(4) Here we consider \(\theta \geq 1\) and \(\theta < 1\) separately. When \(\theta \geq 1\),
\[
\beta_\lambda = \frac{1}{\theta(\frac{n}{2})}(\theta n(\lambda') - n(\lambda)).
\]
If $\beta_\lambda \leq 0$, then $\theta n(\lambda^t) - n(\lambda) \leq 0$. Since $\theta \geq 1$, $n(\lambda) \geq n(\lambda^t)$. So

$$\theta n(\lambda) - n(\lambda^t) \geq n(\lambda) - \theta n(\lambda^t) \geq 0,$$

which implies $\beta_{\lambda^t} \geq |\beta_\lambda| \geq 0$.

Next let $\theta < 1$. Then we can write

$$\beta_\lambda = 1 - \theta + \theta \frac{n(\lambda^t)}{n(\lambda)} - \frac{n(\lambda^t)}{n(\lambda)} = \left[1 - \frac{n(\lambda^t)}{n(\lambda)}\right] - \theta \left[1 - \frac{n(\lambda^t)}{n(\lambda)}\right].$$

If $\beta_\lambda \leq 0$, then since $\theta < 1$,

$$1 - \frac{n(\lambda^t)}{n(\lambda)} \geq 1 - \frac{n(\lambda)}{n(\lambda^t)}.$$

Switching $\lambda$ and $\lambda^t$ we again get

$$\beta_{\lambda^t} \geq |\beta_\lambda| \geq 0.$$

We also need some definitions and results from Diaconis and Hanlon [21]:

**Definition 1.** We collect some notation to be used in the main proof below, some of which will be repeated; they are, for the most part, taken from [21]:

1. Given a partition $\lambda \vdash n$, and a position $s = (i, j)$ in its Ferrers diagram (i.e., $j \leq \lambda_i$), define

   $$h^*(s) = h^*_{\lambda}(s) := (a + 1)\theta + \ell,$$

   $$h^*_*(s) = h^*_{\lambda}(s) := a\theta + (\ell + 1),$$

   where $a = \lambda_i - j$ denotes the number of positions in the same row and strictly to the right of $s$ (the arm length), and $\ell = \lambda^t_j - i$ denotes the number of positions in the same column and strictly below $s$ (the leg length).

2. Define the generalization of hooklength product,

   $$j_\lambda = j_\lambda(\theta) := \prod_{s \in \lambda} h^*_*(s)h^*(s).$$

   When $\theta = 1$, this becomes the product of the hooklengths of all the blocks in the diagram of $\lambda$.

3. Define $c_{\lambda, \rho} = c_{\lambda, \rho}(\theta)$ to be the change of basis coefficients from Jack symmetric polynomials $J_\lambda(\theta)$ (not to be confused with $j_\lambda$ above) to power sum polynomials, that is,

   $$J_\lambda(\theta) = \sum_{\rho \vdash n} c_{\lambda, \rho}(\theta) p_\rho.$$

   See [20] for extensive development of properties of Jack polynomials. When $\theta = 1$, $J_\lambda(1) = H_\lambda s_\lambda$, where $H_\lambda = j_\lambda(1)$ is the hooklength product, and $s_\lambda$ is the Schur polynomial indexed by $\lambda$. 
(4) Denote by $\pi = \pi_\theta$ the Ewens sampling measure with parameter $\theta^{-1}$; recall $\pi_\theta(\sigma) = \theta^{-\ell(\sigma)}/z_n(\theta^{-1})$, where $z_n(\theta^{-1}) = \prod_{i=1}^n (\theta^{-1} + i - 1)$ is the Ewens sampling formula. Also let $\Pi = \Pi_{n,\theta} := \pi_\theta(1^n)^{-1} = \prod_{i=1}^n (1 + \theta(i-1))$.

Note that when $\theta = 1$, $j_\lambda(1)$ is exactly the square of the product of hook lengths of all positions in $\lambda$, which is well known to be $(n!/\dim\pi\lambda)^2$ by the hooklength formula. By Wedderburn’s structure theorem (see [8], Chapter 18, Theorem 10), we also have

$$n! = \sum_\lambda \dim\pi_\lambda^2,$$

therefore $\sum_\lambda \frac{1}{j_\lambda(1)} = \frac{1}{n!}$.

**Theorem 4.2.** The left eigenfunctions of the Metropolis chain $P_\theta$ defined on partitions, normalized in $L^2(\mathcal{P}_n, 1/\pi_\theta)$, are given by

$$g_\lambda(\rho) = \frac{c_{\lambda,\rho}}{(f_\lambda\Pi/(\theta^n n!))^{1/2}}$$

with corresponding eigenvalues stated in (7).

**Proof.** We synthesize the arguments found in [11], Definition 3.8 to Definition 3.12 and [21], Theorem 1. The result from [17], Chapter VI, Section 4, shows that the Macdonald polynomials are simultaneous eigenfunctions of the Macdonald operators $D_{q,t}^r$, $r = 0, \ldots, n$. Specializing to the limit $q = t^\theta$, $t \to 1$ and after some affine linear transformation, the same results hold for Jack polynomials and the associated Sekiguchi–Debiard operators (37), $D_\theta(X)$. The $X^2$ coefficient of this operator valued generating function turns out to be the following Laplace–Beltrami-type operator (our notation differs slightly from [17], page 320): let $f$ be a homogeneous polynomial of degree $N$ in $n$ variables, then

$$D_\theta^2 f = \left(-\frac{\theta^2}{2} U_n - \theta V_n + c_n\right) f,$$

where $U_n = \sum_{i=1}^n (x_i \partial_i)^2 - x_i \partial_i = \sum_{i=1}^n x_i^2 \partial_i^2$, $V_n = \frac{1}{2} \sum_{i \neq j} \frac{x_i^2 \partial_i - x_j^2 \partial_j}{x_i - x_j}$, and $c_n = \theta^2 \binom{N}{2} + \theta N \binom{n}{2} + \frac{1}{4} \binom{3}{2} (3n - 1)$; see (40) for a proof.

After an affine transform, we arrive at the following cleaner operator:

$$(12) \quad L_\theta^2 := \frac{1}{\binom{2}{2}} \left(\frac{1}{2} U_n + \frac{1}{\theta} (V_n - (n - 1)N)\right),$$

which readily admits a Markov chain interpretation, when acting on power sum polynomials. Combining (39), (38) and (40), we have

$$L_\theta^2 p_\lambda = \frac{P_\lambda}{\binom{N}{2}} \left(\sum_{s < t} \lambda_s \lambda_t \frac{P_{\lambda_s + \lambda_t}}{p_{\lambda_t} p_{\lambda_s}} + (1 - \theta^{-1}) n(\lambda') + \theta^{-1} \sum_s \frac{\lambda_s^{-1}}{2} \sum_{r=1}^{\lambda_s} \frac{P_r p_{\lambda_s-r}}{p_{\lambda_s}}\right).$$
Observe that for \( \theta > 1 \), the first and third terms above correspond to joining two cycles into one and splitting a cycle into two cycles, respectively, whereas the middle term gives the holding probability at \( \lambda \). In other words, the probability of going from \( \lambda \) to \( \mu \) in one step under the Jack–Metropolis walk is given by the \( p_\mu \) coefficient of \( L^2_\theta p_\lambda \). This translates to

\[
L^2_\theta p_\lambda = \sum_{\mu \vdash N} T_\theta(\lambda, \mu) p_\mu.
\]

Next we show that \( L^2_\theta J_\lambda = \beta_\lambda J_\lambda \), with \( \beta \) given by (6) (i.e., when \( \theta > 1 \)); the general case follows by an appropriate affine transform. In [17], page 317, it is shown that for the Macdonald operator-valued generating function \( D_n(X; q, t) \), eigenvalues are given by \( \beta_\lambda(X; q, t) = \prod_{i=1}^n (1 + Xtn_i - iq\lambda_i) \). Now using Example 3(c) on page 320, one can derive the eigenvalues for \( D_n(X; \alpha) \), by considering the limiting operator \( \lim_{t \to 1} (t^{-1} - n) Y^n D_n(Y^{-1}; q, t) \) where \( Y = (t - 1)X - 1 \). Extracting the \( X^{n-2} \) term gives \( \beta_\lambda \), which is stated in Example 3(b) of page 327 as \( \alpha \binom{N}{2} e_\lambda(\alpha) \) (\( \alpha \) is the same as \( \theta \) in our notation), since \( \square_n^\alpha = \alpha \binom{N}{2} L^2_\alpha \).

Finally we prove the formula for the left eigenfunctions. Define the inner product \( \langle \cdot \, , \cdot \rangle_\theta \) by

\[
\langle p_\lambda, p_\mu \rangle_\theta = \delta_{\lambda\mu} z_\lambda \theta^{\ell(\lambda)}.
\]

In [20] (see also Lemma 3.11 of [11]), it is shown that \( \langle J_\lambda, J_\lambda \rangle_\theta = j_\lambda(\theta) \) as defined before. Here recall the normalization of \( J_\lambda \) is fixed by requiring that in the monomial symmetric function basis, its \( m_1 \) coefficient be 1. Therefore expressing \( J_\lambda \) in terms of \( p_\mu \)'s, we have

\[
\sum_{\rho \vdash N} c^2_{\lambda, \rho} z_\rho \theta^{\ell(\rho)} = j_\lambda.
\]

On the other hand, the normalization constant for the MED(\( \theta^{-1} \)) distribution is

\[
z_n(\theta^{-1}) = \theta^{-1}(\theta^{-1} + 1) \cdots (\theta^{-1} + n - 1) = \Pi \theta^{-n},
\]

hence \( \pi_\theta(\rho) = \theta^{-\ell(\rho)} \frac{n!}{z_\rho} z_n(\theta)^{-1} \), and with \( g_\lambda(\rho) \) given in (11) we have

\[
\sum_{\rho \vdash N} g_\lambda(\rho)^2 \pi_\theta(\rho)^{-1} = \sum_{\rho} \frac{c^2_{\lambda, \rho}}{j_\lambda / (\theta^n n!)} \Pi \theta^{\ell(\rho)} z_\rho \frac{1}{n!} = \sum_{\rho} \frac{c^2_{\lambda, \rho} \theta^{\ell(\rho)} z_\rho}{j_\lambda} = 1,
\]

by the previous equation. This shows \( g_\lambda \) are indeed left eigenfunctions by Lemma 3.1. \( \square \)

**Corollary 4.3.** The right eigenfunctions of \( P_\theta \) are proportional to

\[
f_\lambda(\rho) = g_\lambda(\rho) \theta^{\ell(\rho)} z_\rho.
\]

**Proof.** Since \( \pi_\theta(\rho) \propto \theta^{-\ell(\rho)} \frac{n!}{z_\rho} \), this follows from Lemma 3.1 and the previous theorem. \( \square \)

**Lemma 4.4.** For any \( \lambda \vdash n \),

\[
c_{\lambda, 1^n} = 1.
\]
PROOF. This follows from the following formula in [20]:
\[ J_\lambda(1^n; \theta) = \prod_{(i,j) \in \lambda} (n - (i - 1) + \theta(j - 1)), \]
ture for all \( n \in \mathbb{N} \), by reading coefficients of powers of \( n \); See [21], Section 4, Theorem 1. □

**Lemma 4.5.** \( j_\lambda \) admits the following inductive bound on the parts of \( \lambda \):
\[ j_\lambda \geq \lambda_1! \theta^{2\lambda_1 - 1} \lambda_1^{-1} - 1 e^{-\pi^2/12\theta^2} j(\lambda_2, \ldots, \lambda_n). \]
Note that the constant \( e^{-\pi^2/12\theta^2} \) is not important.

**Proof of Lemma 4.5.** From the definition, we have
\[
\begin{align*}
\hat{j}_\lambda \geq & \left[ \prod_{i=1}^{\lambda_1} \prod_{i=1}^{\lambda_1 - 1} (i\theta) (i\theta + 1) \right] j(\lambda_2, \ldots, \lambda_n) \\
= & \left[ \prod_{i=1}^{\lambda_1} \prod_{i=1}^{\lambda_1 - 1} (i\theta)(1 + (i\theta)^{-1}) \right] j(\lambda_2, \ldots, \lambda_n) \\
\geq & \lambda_1! \theta^{\lambda_1}(\lambda_1 - 1)! \theta^{\lambda_1 - 1} \exp\left( \frac{1}{\theta} \log \lambda_1 - \frac{1}{\theta^2} \frac{\pi^2}{12} \right) j(\lambda_2, \ldots, \lambda_n) \\
= & \lambda_1! \theta^{\lambda_1}(\lambda_1 - 1)! \theta^{\lambda_1 - 1} \lambda_1^{-1/\theta} \exp\left( -\frac{1}{\theta^2} \frac{\pi^2}{12} \right) j(\lambda_2, \ldots, \lambda_n),
\end{align*}
\]
where we used the fact that \( 1 + x \geq e^{x-x^2/2} \) for \( x \geq 0 \), applied to \( x = (i\theta)^{-1} \), and the zeta sum,
\[ \sum_i \frac{1}{2i^2} \leq \frac{\pi^2}{12}. \] □

**5. Mixing time upper bound.** By Theorem 4.2, under the same notation there, *four times* the total variation distance of \( P_{1^n}^k \) from \( \pi \) can be bounded by
\[ \| P_{1^n}^k - \pi \|_2^2 \leq \frac{1}{\pi^2(x)} \sum_\lambda \beta_{\lambda}^{2k} g_{\lambda}^2(1^n) - 1, \]
where we use the sloppy (but standard) notation \( \| P_{1^n}^k - \pi \|_2 \) to mean \( \| P_{1^n}^k - \pi \|_{L^2(\pi)} \).

For \( \lambda = (n) \), corresponding to the trivial representation on \( S_n \), and starting point \( x = (1^n) \), the summand exactly cancels \(-1\): \( \beta(n) = 1 \), \( j(n) = \Pi \theta^n n! \) and \( c(n), 1^n = 1 \) (by Lemma 4.4), whereas \( \pi(1^n) = \Pi^{-1} \), so
\[ \frac{1}{\pi(1^n)^2} \beta_{(n)}^{2k} g_{(n)}^2(1^n) = 1. \]
Thus using the explicit formula for $g_{\lambda}$, we immediately have

\begin{equation}
\|P^k_x - \pi\|^2_2 = \theta^n n! \sum_{\lambda \vdash n, \lambda \neq (n)} \frac{\beta_{\lambda}^{2k}}{j_{\lambda}}.
\end{equation}

We now break the sum according to the sign of $\beta_{\lambda}$,

\begin{equation}
\sum_{\lambda \vdash n, \lambda \neq (n)} \frac{\beta_{\lambda}^{2k}}{j_{\lambda}} = \sum_{\beta_{\lambda} \geq 0} \frac{\beta_{\lambda}^{2k}}{j_{\lambda}} + \sum_{\beta_{\lambda} < 0} \frac{\beta_{\lambda}^{2k}}{j_{\lambda}},
\end{equation}

where $\sum^*$ denotes summation skipping the top eigenvalue indexed by $\lambda = (n)$.

Next we can rewrite the first summand on the right according to the size of $\lambda_1$, and obtain the following bound:

\begin{equation}
\theta^n n! \sum_{\lambda \vdash n, \lambda \neq (n)} \frac{\beta_{\lambda}^{2k}}{j_{\lambda}} \leq \sum_{s=1}^{n-1} \theta^n n! \sum_{\lambda: \lambda_1 = s, \beta_{\lambda} \geq 0} \frac{\theta^n n!}{j_{\lambda}}.
\end{equation}

Splitting $\Pi = \Pi_n, \theta$ into two subproducts, and using Lemma 4.5, we have

\begin{equation}
\frac{\Pi_n! \theta^n}{j_{\lambda}} \leq \frac{(\Pi_n / (\Pi_{n-\lambda_1})) (n! / ((n-\lambda_1)!) \theta^{\lambda_1} \Pi_{n-\lambda_1} (n-\lambda_1)!) e^{-\pi^2/12\theta^2}}{j_{(\lambda_2, \ldots, \lambda_n)}},
\end{equation}

where the second quotient factor happens to be $\frac{1}{\pi(n-s)} g_{(\lambda_2, \ldots, \lambda_n)}^2 (1^n - (n-\lambda_1))$; see Theorem 4.2 and Lemma 4.4. Also denote the first factor by $q_{n, \lambda_1}$.

By (2), the definition of $\Pi_{n-s} := \pi((n-s)-1)$ (see Definition 1), and the fact $\Pi_n = \theta^n (n-1)! e^{\theta^{-1} \sum_{i=2}^n 1/(i-1)}$, we can bound the summand of the right-hand side of (16) for a fixed $s$ as

\begin{equation}
\sum_{\lambda: \lambda_1 = s} \frac{\Pi_n n! \theta^n}{j_{\lambda}} \leq q_{n,s} \sum_{\mu=1}^{n-s} \frac{1}{\pi((n-s)!)^2 \theta^{\lambda_1} \Pi_{n-\lambda_1} (n-\lambda_1)!) e^{-\pi^2/12\theta^2}} \leq \left(\frac{n}{s}\right)^{\theta^{-1} - 1} \theta^{n-s+1} \frac{n!}{s!} \theta^{n-s+1} \frac{e^{\pi^2/12\theta^2}}{(n-s)!}.
\end{equation}

We will now reduce the $L^2$ bound (14) to bounding the following quantity:

\begin{equation}
b_{n,+} := \sum_{s=n-1}^{1} \beta_{s,+}^{2k} \left(\frac{n}{s}\right)^{\theta^{-1} - 1} \theta^{n-s+1} \frac{n!}{s!} \theta^{n-s+1} \frac{e^{\pi^2/12\theta^2}}{(n-s)!},
\end{equation}

where $\beta_{s,+} := \max\{\beta_{\lambda}: \beta_{\lambda} \geq 0, \lambda_1 = s\}$.
For the second summand of (15), we obtain

\[
\sum_{\beta_k < 0} \frac{\beta_k^2}{j_k} \leq \sum_{s=n-1}^1 \max\{\beta_k^2 : \beta_k < 0, \lambda'_1 = s\} \sum_{\lambda : \lambda'_1 = s} \frac{\theta^n n! \Pi_n}{j_k}
\]

(18)

\[+ \beta_1^n \frac{\theta^n n! \Pi_n}{j_1^n}.
\]

Using the explicit formula (7) for \(\beta_\lambda\), we get

\[\beta_1^n = 1 - (\theta \land 1) - (\theta^{-1} \land 1) \in (-1, 1),\]

for \(\theta \neq 1\). On the other hand,

\[
\Pi_n n! \theta^n / j_1^n = \prod_{i=1}^{n-1} (1 + i \theta) n! \theta^n / \prod_{i=1}^{n} (i \theta)(1 + (i - 1) \theta) = 1,
\]

since \(j_1^n = \prod_{i=1}^{n} (i \theta)(1 + (i - 1) \theta)\) by definition. Thus

\[\beta_1^\Omega(n) \Pi_n n! \theta^n / j_1^n = o(1),\]

which has negligible contribution in (14).

For the remaining terms in (18), first observe that

\[j_{\lambda'} \geq \prod_{i=1}^{\lambda_1} (i + \theta) j_{(\lambda_2, \ldots, \lambda_n)'},\]

Hence when \(\theta < 1\),

\[j_{\lambda'} \geq \prod_{i=1}^{\lambda_1} (i \theta) \prod_{i=1}^{\lambda_1-1} (i \theta + 1) j_{(\lambda_2, \ldots, \lambda_n)'},\]

and using the bound \(|\beta_\lambda| \leq \beta_{\lambda'}\), for \(\beta_k < 0\), we get

\[
\sum_{\beta_k < 0} \frac{\beta_k^2}{j_k} \leq \sum_{\beta_k \geq 0} \frac{\beta_k^2}{j_k} + o(1).
\]

If \(\theta > 1\), the \(j_{\lambda'}\) is comparable to \(j_\lambda\) within an exponential factor

\[j_{\lambda'} \geq j_\lambda \theta^{2n}.
\]

Furthermore by the explicit formula of \(\beta_\lambda\), we have

\[\beta_\lambda(\theta) = \frac{n(\lambda')}{(n)} - \theta^{-1} n(\lambda) \left(\frac{\theta}{(2)}\right) \leq \theta^{-1} n(\lambda) \left(\frac{n}{2}\right) - n(\lambda) \left(\frac{\theta}{2}\right) \leq \theta^{-1} n(\lambda) \left(\frac{\theta}{2}\right) \leq -\theta^{-1} \beta_\lambda(1).\]
So since $k := \frac{1}{2(\theta \wedge 1)} n(c + \log n) = \Theta(n \log n)$, $\theta^{-2k} \theta^{2n} = o(1)$. Thus we can still compare the negative $\beta_\lambda$ sum to the positive one,

$$\sum_{\beta_\lambda < 0} \beta_\lambda^{2k} \leq b_{n,+} + o(1).$$

It remains to bound $2b_{n,+}$. First note that the factor 2 in front is immaterial, since for $\lambda \neq (n)$,

$$\beta_\lambda^{cn} < \beta_{(n-1,1)}^{cn} \leq e^{-\Omega_0(c)},$$

thanks to the monotonicity of $\beta_\lambda$’s. So by increasing $c$ in $k = \frac{1}{2(\theta \wedge 1)} n(c + \log n)$, we can decrease $b_{n,+}$ by a factor of 2. The factor $c_\theta = e^{\pi^2/2\theta^2}$ can be ignored similarly. We can also get rid of the factor $(\frac{n}{s})^{\theta^{-1} - 1}$ in (17) as follows.

For $s \geq n/2$, $(\frac{n}{s})^{\theta^{-1} - 1} = o(1)$, so again increasing $c$ annihilates it. For $s < n/2$, recall the second bound on $\beta_\lambda$ (9), which implies that for $\lambda_1 < n/2$, $\beta_\lambda$ is bounded away from 1 uniformly in $n$. Now in the definition of $b_{n,+}$, $\beta_\lambda$ is assumed to be nonnegative (alternatively, $\beta_{1^n}$ is bounded uniformly away from $-1$), hence raising $\beta_\lambda$ to the power $\Omega(n)$ easily cancels any power of $n$, that is,

$$n^{\theta^{-1} - 1} \beta_\lambda^{cn} = o(1).$$

So together, by increasing $c$, we can reduce the problem to bounding the following quantity:

$$B_{n,+} := \sum_{s=n-1}^{1} \frac{\theta^{n-s+1} (n!)^2}{(n-s)! s!^2} \beta_\lambda^{2k}.$$

The only estimates we rely on now are (8) and (9) from Lemma 4.1; the idea will be similar to [7]; see also [5]. First note that it suffices to show

$$\frac{\theta^{n-s+1} (n!)^2}{(n-s)!} \beta_\lambda^{2k} = O(1),$$

uniformly for all $s \in [1, n-1]$ and $c$ sufficiently large. Indeed using (9), we have

$$\beta_{s,+} \leq e^{-x-x^2/2} \leq e^{-x}$$

for $x = (\theta \wedge 1) \frac{n-s}{n}$. Therefore

$$O(1) \sum_{s=n-1}^{1} \beta_{s,+}^{(cn)/(2(1+\theta))} \leq O(1) \sum_{t=1}^{n} e^{-tc} = o_c(1),$$

by geometric summation; in fact, using (8) we can get a better bound, but that’s not necessary.
Next recall (8) as well as the estimates (no Stirling formula needed)

\[
\frac{n!}{s!} \leq e^{\int_{s}^{n} \log x \, dx + \log n - \log s} = e^{n \log n - s \log s - (n-s) + \log n - \log s},
\]

\[
(n-s)! \geq e^{\int_{n-s}^{n} \log x \, dx} = e^{(n-s) \log(n-s) - (n-s-1)}.
\]

Taking logarithm, and letting \( s = \alpha n \), we can bound the left-hand side of (19) by

\[
\log \left[ \frac{\theta^{n-s+1}}{(n-s)! \left( \frac{n}{s}! \right)^2} \beta_{s+}^{2k} \right] \leq (1 - 2\alpha)(1 - \alpha)n \log n - (1 - \alpha)n \log(1 - \alpha) - 2\alpha n \log \alpha - 2\log \alpha + (C_1(\theta) - 2c\alpha)(1 - \alpha)n + C_2(\theta),
\]

where \( C_1(\theta), C_2(\theta) \) are constants that depend only on \( \theta \).

For \( \alpha \geq \alpha_0 \in (1/2, 1) \), the right-hand side of (20) can be further simplified to

\[
(1 - \alpha)n \left[ (1 - 2\alpha) \log n - \log(1 - \alpha) + C'_1(\theta) - c \right] + C'_2(\theta).
\]

Since \((1 - \alpha)n \geq 1\), and we can choose \( c \) as large as we want, it suffices to show the expression inside the square brackets above is \( O(1) \). But when \( \alpha = 1 - 1/n \) or \( 1/2 \), this is clearly true. Furthermore, the derivative

\[
\frac{d}{d\alpha} \left[ (1 - 2\alpha) \log n - \log(1 - \alpha) \right] = -2 \log n + \frac{1}{1 - \alpha}
\]

is monotone increasing, showing that the function \( \alpha \mapsto (1 - 2\alpha) \log n - \log(1 - \alpha) \) is convex, and its value for any \( \alpha \in [1/2, 1 - 1/n] \) is bounded above by the values at the boundary points.

Next let \( \alpha < \alpha_0 \). Using the second bound for \( \beta_{\lambda} \), (9), we have

\[
\beta_{\lambda} \leq e^{-(\theta \land 1)(1 - (s/n))}.
\]

Then the logarithm of the left-hand side of (19) is bounded by

\[
(1 - \alpha)n \log n + (1 - \alpha)(\log \theta - \log(1 - \alpha) + 1)n - 2\alpha n \log \alpha + 2 \log \alpha - (1 - \alpha)n \log n - (1 - \alpha)cn \leq (1 - \alpha)(C(\theta) - c)n + 2(1 - \alpha n) \log \alpha.
\]

Clearly \((1 - \alpha n) \log \alpha = O(n) \) for \( \alpha \in [\frac{1}{n}, \alpha_0] \). So for sufficiently large \( c \), the right-hand side above goes to \( -\infty \). Together this shows (19) is true for all \( s \in [\frac{1}{n}, 1 - \frac{1}{n}] \), and concludes the upper bound for the mixing time.

6. Mixing time lower bound. We rely heavily on results from [20]. Again we collect some notation needed in the analysis below:

**Definition 2.** For \( \lambda, \rho \vdash n \), let:

- \( H_\lambda \) be the product of all hook-lengths of the Ferrers diagram for \( \lambda \);
- \( z_\rho := \prod_{i=1}^{n} i^{m_i} m_i! \) and \( m_i = m_i(\rho) \) is the number of parts in \( \rho \) of length \( i \);
• \( \chi_\lambda(\rho) \) be the character of \( S_n \) indexed by \( \lambda \) evaluated at an element of conjugacy class \( \rho \); alternatively, they can be defined by the system

\[
p_\rho = \sum_\lambda \chi_\lambda(\rho)s_\lambda,
\]

where \( s_\rho \) are the Schur polynomials.

**Warning.** Note the Schur polynomials are not direct specializations of the Jack polynomials; they differ by a factor

\[
s_\lambda = H^{-1}_\lambda J_\lambda(1).
\]

Thus the matrix \( \chi_\lambda(\rho) \) is the inverse of \( H^{-1}_\lambda c_{\lambda,\rho}(1) \).

**Lemma 6.1** ([17], Chapter I, equation (7.5); see also [20], equation (50)). The relation between \( c_{\lambda,\rho}(1) \) and \( \chi_\lambda(\rho) \) is given by

\[
c_{\lambda,\rho}(1) = H_\lambda z_\rho^{-1} \chi_\lambda(\rho).
\]

**Corollary 6.2.** The inverse matrix to \( \chi_\lambda(\rho) \) is given by \( \chi_\lambda(\rho) z_\rho^{-1} \), that is,

\[
s_\lambda = \sum_\rho \chi_\lambda(\rho) z_\rho^{-1} p_\rho.
\]

**Proof.** By relation (22) and the lemma above, we have

\[
s_\lambda = H^{-1}_\lambda J_\lambda(1)
\]

\[
= H^{-1}_\lambda \sum_\rho H_\lambda z_\rho^{-1} \chi_\lambda(\rho)p_\rho.
\]

Comparing the coefficients with (10) in Definition 1 yields the result. 

As in the \( \theta = 1 \) case studied by Diaconis and Shahshahani, the strategy will be to use a certain eigenfunction \( f \) of the chain as test function and compare the probabilities of the event \( \{ f < \eta \} \) for some \( \eta \in \mathbb{R} \) under the stationary distribution and the distribution at time slightly before the mixing time, which in our case is

\[
k(c) := \frac{1}{\Sigma(\theta+1)} n(\log n - c).
\]

In the case where \( \theta = 1 \), \( \rho \mapsto \chi_{(n-1,1)}(\rho) = m_1(\rho) - 1 \) is the desired eigenfunction. So it is natural to guess that a suitable affine transformation of the fixed-point (aka 1-cycle) counting function \( \rho \mapsto m_1(\rho) \) is the desired eigenfunction.

Lemma 6.1 shows that the following normalized version of \( c_{\lambda,\rho} \) is the right analogue of characters of the symmetric group

\[
d_{\lambda,\rho}(\theta) := c_{\lambda,\rho}(\theta) z_\rho^{-(n-\ell(\rho))} H_\lambda^{-1}.
\]

Thus our candidate test function will be \( d_\lambda(\rho) = d_{\lambda,\rho} \).
It is straightforward to compute $\mathbb{P}_\infty(d_\lambda < \eta)$ where $\mathbb{P}_\infty$ denotes the stationary measure; the number of cycles are asymptotically independent and Poisson distributed. To estimate $\mathbb{P}_k(d_\lambda < \eta)$, one uses the second moment method. The first moment of $d_{(n-1,1)}$ is easily computed since it is proportional to the right eigenfunctions of the chain $P_0$; see Corollary 4.3. For second moments, we need to decompose $d_{(n-1,1)}^2$ as linear combinations of other $d_\lambda$’s. This is accomplished by first expressing $d_{(n-1,1)}$ and other $d_\lambda$’s in terms of powers of $m_\alpha$’s, the number of $i$-cycles (not to be confused with monomial symmetric functions), then deducing the relationship by solving the appropriate system of linear equations. The analysis below will be an elaboration of this strategy.

First we need:

**Proposition 6.3 ([20], Proposition 7.5).** Let $d_{\lambda,\rho} = d_{\lambda,\rho}(\theta)$ be defined as above. Then

$$\theta^{k+1}(k+n)d_{(n,1^k),\rho}(\theta) = \sum_{j=0}^{k} (-1)^{k-j} (j + (n+k-j)\theta) \prod_{v+j} \left( \frac{m_i(\rho)}{m_i(\nu)} \right) (-1)^{j-\ell(\nu)} \theta^{\ell(\nu)}.$$ 

Note that the partitions in the proposition above is for $n+k$, rather than $n$.

From this, we easily obtain

$$d_{(n-1,1),\rho}(\theta) = -\frac{1}{\theta} + \frac{1 + (n-1)\theta}{\theta n} m_1(\rho)$$

and

$$d_{(n-2,1^2),\rho}(\theta) = \frac{1}{\theta^2} - \left( \frac{1 + (n-1)\theta}{n \theta^2} + \frac{2 + (n-2)\theta}{2 \theta n} \right) m_1(\rho)$$

$$+ \frac{2 + (n-2)\theta}{2 \theta n} m_1(\rho)^2 - \frac{2 + (n-2)\theta}{\theta^2 n} m_2(\rho).$$

Note in particular,

$$\chi_{(n-1,1)}(\rho) = m_1(\rho) - 1,$$

$$\chi_{(n-2,1^2)}(\rho) = 1 - \frac{3}{2} m_1(\rho) + \frac{1}{2} m_1(\rho)^2 - m_2(\rho),$$

as expected.

Using the Schur–Weyl relation

$$\chi_{n-1,1}^2 = \chi_n + \chi_{n-1,1} + \chi_{n-2,2} + \chi_{n-2,1^2},$$

and we also obtain

$$\chi_{(n-2,2)} = \frac{1}{2} m_1^2 - \frac{3}{2} m_1 + m_2.$$

To get $J_{(n-2,2)}$, we need the conjecture right after Proposition 7.2 as well as Corollary 3.5 from [20]. The conjecture has been proved in [14]. Notice the parameter $\alpha$ is the same as our parameter $\theta$. 
**Proposition 6.4** ([20], Proposition 7.2). The Jack polynomials corresponding to the partition \((2^i, 1^j)\) have the following expansion in terms of the monomial symmetric basis \(m_\lambda\):

\[
J_{(2^i, 1^j)} = \sum_{r=0}^{i} (i)_r (\theta + i + j)_r (2(i - r) + j)! m_{(2^{r+1}, 1^{i-r+j})},
\]

where \((i)_r := i(i-1) \cdots (i-r+1)\).

**Proposition 6.5** ([14], Theorem 1.1). In terms of Schur polynomial basis, we have

\[
J_{(2^i, 1^j)} = \sum_{r=0}^{i} (i)_r (\theta + i + j)_r (i - r - \theta)_r (i + j - r)! s_{(2^{r+1}, 1^{i-r+j})}.
\]

The next result relates Jack polynomials corresponding to conjugate partitions, when expressed in terms of the power sum basis.

**Proposition 6.6** ([20], Corollary 3.5). Let \(J_\lambda = \sum_\mu c_{\lambda \mu}(\theta) p_\mu\), then

\[
J_\lambda^t = \sum_\mu (-\theta)^{n-\ell(\mu)} c_{\lambda \mu} \left(\frac{1}{\theta}\right) p_\mu.
\]

We also recall from Corollary 6.2 that \(s_\lambda = \sum_\rho \chi_\lambda(\rho) z_\rho^{-1} p_\rho\), where \(\chi_\lambda(\rho)\) is the character of \(\lambda\) evaluated at \(\rho\).

Combining the previous results, we easily get

\[
J_{n-2,2}(\theta) = \sum_\rho (-\theta)^{n-\ell(\rho)} \left[\left(2 - \frac{1}{\theta}\right) \left(1 - \frac{1}{\theta}\right)(n-2)\chi_\rho^{1^n} + 2\left(n - 2 + \frac{1}{\theta}\right) \left(1 - \frac{1}{\theta}\right)(n-3)!\chi_{2^1, 1^{n-2}}(\rho) + 2\left(n - 2 + \frac{1}{\theta}\right) \left(n - 3 + \frac{1}{\theta}\right)(n-4)!\chi_{2^2, 1^{n-4}}(\rho)\right] z_\rho^{-1} p_\rho,
\]

where \(\chi_\lambda(\rho)\) is the irreducible character \(\lambda\) evaluated at \(\rho\). Therefore we can read off the coefficients

\[
c_{(n-2,2), \rho}(\theta) = \frac{(-\theta)^{n-\ell(\rho)}}{z_\rho} \left[\left(2 - \frac{1}{\theta}\right) \left(1 - \frac{1}{\theta}\right)(n-2)!\chi_{1^n}(\rho) + 2\left(n - 2 + \frac{1}{\theta}\right) \left(1 - \frac{1}{\theta}\right)(n-3)!\chi_{2^1, 1^{n-2}}(\rho) + 2\left(n - 2 + \frac{1}{\theta}\right) \left(n - 3 + \frac{1}{\theta}\right)(n-4)!\chi_{2^2, 1^{n-4}}(\rho)\right],
\]

(28)
and using the relation $\chi_{\ell}(\rho) = \chi_{\ell}(\rho) \text{sgn}(\rho)$, as well as the formula for $\chi_n$, $\chi_{n-1,1}$, $\chi_{n-2,2}$ and $\chi_{n-2,12}$ derived above, we also get

$$d_{(n-2,2),\rho}(\theta) = m_1(n-2 + (1/\theta)\bigg) \frac{(n-3)\left(1 - \frac{1}{\theta}\right) - \frac{3}{2}\left(n - 3 + \frac{1}{\theta}\right)}{(n-1)(n-2)}$$

$$+ m_1^2 \frac{2(n-2 + (1/\theta))(n-3 + (1/\theta))}{2(n-1)(n-2)}$$

$$+ m_2 \frac{(n-2 + (1/\theta))(n-3 + (1/\theta))}{(n-1)(n-2)}.$$  \hfill (29)

Using $i = 0$ and $j = n$, one gets

$$J(n)(\theta) = \sum_{\rho} \theta^{n-\ell(\rho)} n! \rho^{-1} \rho.$$  \hfill (30)

Hence

$$c(n,\rho)(\theta) = \theta^{n-\ell(\rho)} n! \rho^{-1},$$

and

$$d(n,\rho)(\theta) = 1.$$  \hfill (31)

To mimic the case of $\theta = 1$, we need to express $d_{(n-1,1),\rho}(\theta)^2$ in terms of the other $d_{\ell}$’s. First using (24), we get

$$d_{(n-1,1),\rho}(\theta)^2 = \frac{1}{\theta^2} + \left(\frac{1}{\theta n} + \frac{n-1}{n}\right)^2 m_1^2 - \frac{2}{\theta} \left(\frac{1}{\theta n} + \frac{n-1}{n}\right) m_1.$$  \hfill (33)

Using $m_1, m_1^2, m_2$ and 1 as a basis, we can write

$$d_{(n-1,1),\rho}(\theta)^2 = u + v d_{(n-1,1),\rho}(\theta) + w d_{(n-2,12),\rho}(\theta) + x d_{(n-2,2),\rho}(\theta)^2,$$  \hfill (32)

for some indeterminates $u, v, w, x$.

Comparing coefficients of the $m_i^k$’s, we get the following four equations:

$$u - v + \frac{w}{\theta^2} = \frac{1}{\theta^2},$$  \hfill (33)

$$v\left(\frac{1}{\theta n} + \frac{n-1}{n}\right) - w\left(\frac{1 + (n-1)\theta}{n\theta^2} + \frac{2 + (n-2)\theta}{2\theta n}\right)$$

$$+ x \frac{n-2 + (1/\theta)}{(n-1)(n-2)} \left[\left(n - 3\right)\left(1 - \frac{1}{\theta}\right) - \frac{3}{2}\left(n - 3 + \frac{1}{\theta}\right)\right]$$

$$= -\frac{2}{\theta} \left(\frac{1}{\theta n} + \frac{n-1}{n}\right),$$  \hfill (34)

$$w \frac{2 + (n-2)\theta}{2\theta n} + x \frac{(n-2 + (1/\theta))(n-3 + (1/\theta))}{2(n-1)(n-2)} = \left(\frac{1}{\theta n} + \frac{n-1}{n}\right)^2,$$  \hfill (35)

$$-w \frac{2 + (n-2)\theta}{\theta^3 n} + x \frac{(n-2 + (1/\theta))(n-3 + (1/\theta))}{(n-1)(n-2)} = 0.$$  \hfill (36)
Solving, we get
\[ u = \frac{(n^2 - 4n + 3)\theta^3 + (n - 1)\theta^2 + (n - 1)^2\theta + n - 3}{\theta^2(\theta + 1)(\theta(n - 3) + 1)n} \]
\[ = \frac{1}{\theta^2(\theta + 1)} + \frac{1}{\theta + 1} + \mathcal{O}\left(\frac{1}{n}\right), \]
\[ v = \frac{(n^3 - 6n^2 + 11n - 6)\theta^3 + 2(2n^2 - 7n + 4)\theta^2 + (3n + 2)\theta - 4}{\theta n((n^2 - 5n + 6)\theta^2 + (3n - 8)\theta + 2)} \]
\[ = 1 + \mathcal{O}\left(\frac{1}{n}\right), \]
\[ w = \frac{2(1 + \theta(n - 1))^2}{(1 + \theta)(2 + \theta(n - 2))n} = \frac{2\theta}{1 + \theta} + \mathcal{O}\left(\frac{1}{n}\right), \]
\[ x = \frac{2(1 + \theta(n - 1))^2(n - 1)(n - 2)}{(1 + \theta)n^2(1 + \theta(2n - 5) + \theta^2(n - 2)(n - 3))} = \frac{2}{1 + \theta} + \mathcal{O}\left(\frac{1}{n}\right). \]

Notice \( x + w = 2 + \mathcal{O}\left(\frac{1}{n}\right). \)

Also by (24), (25), (29) and (31), we get
\[ d_{n,1^n} = 1, \]
\[ d_{(n-1,1)^n} = n - 1, \]
\[ d_{(n-2,1^2)^n} = \frac{(n - 1)(n - 2)}{2}, \]
\[ d_{(n-2,2)^n} = \frac{n(n - 3)}{2}, \]
which are independent of \( \theta \) because of the normalization chosen for Jack polynomials.

Finally we recall \( c_\lambda(\theta) \) are eigenfunctions of \( P_\theta \), hence so are \( d_\lambda(\theta) \), with eigenvalue \( \beta_\lambda \). We list the relevant eigenvalues here:

\[ \beta_{(n)} = 1, \]
\[ \beta_{(n-1,1)} = 1 - (\theta \wedge 1) + \frac{\theta(n-1)^2 - 1}{(\theta \vee 1)\binom{n}{2}} = 1 - (1 \wedge \theta)\frac{2}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \]
\[ \beta_{(n-2,1^2)} = 1 - (\theta \wedge 1) + \frac{\theta(n-2)^2 - 3}{(\theta \vee 1)\binom{n}{2}} = 1 - (\theta \wedge 1)\frac{4}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \]
\[ \beta_{(n-2,2)} = 1 - (\theta \wedge 1) + \frac{\theta(n-2)^2 - 2}{(\theta \vee 1)\binom{n}{2}} = 1 - (\theta \wedge 1)\frac{4}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \]
Notice
\[
\lim_{n \to \infty} \frac{1 - \beta_{(n-1,1)}}{1 - \beta_{(n-2,1^2)}} = \lim_{n \to \infty} \frac{1 - \beta_{(n-1,1)}}{1 - \beta_{(n-2,2)}} = 2.
\]

Now we are fully equipped to prove the lower bound. First observe
\[ L_\infty(d_{(n-1,1)}(\theta)) \prec \text{Poi}(\theta^{-1}) + 1, \]
which comes from Feller coupling. Here \( d_{(n-1,1)} \) stands for the random variable \( d_{(n-1,1),\rho} \) where \( \rho \) has Ewens sampling distribution with parameter \( \theta^{-1} \), as indicated by the subscript \( \infty \). Therefore,
\[ \lim_{n \to \infty} \mathbb{P}_\infty(d_{(n-1,1)}(\theta) \leq \eta) = 1. \]

Furthermore,
\[ \mathbb{P}_k(d_{(n-1,1)}(\theta) \leq \eta) \leq \frac{\text{var}_k(d_{(n-1,1)})}{(\eta - \mathbb{E}_k(d_{(n-1,1)}))^2}. \]

Let \( k = \frac{1}{2}(\theta^{-1} \vee 1)n(\log n - c) \) for any \( c > 0 \).
Since \( d_\lambda \) are eigenfunctions, we can compute the mean and variance at time \( k \),
\[
\mathbb{E}_kd_{(n-1,1)} = (n-1)\left(1 - (\theta \wedge 1)\frac{2}{n}\right)^k + O(1) = e^c + O(1),
\]
\[
\text{var}_kd_{(n-1,1)} = \mathbb{E}_kd_{(n-1,1)}^2 - (\mathbb{E}_kd_{(n-1,1)})^2
\]
\[
= u + (n-1)ve^{-(\theta \wedge 1)((2k)/n)(1+O(1/n))}
\]
\[
+ \left(\frac{(n-1)(n-2)}{2}w + \frac{n(n-3)}{2}x\right)e^{-(\theta \wedge 1)((4k)/n)(1+O(1/n))}
\]
\[
- (n-1)^2e^{-(\theta \wedge 1)((4k)/n)(1+O(1/n))}
\]
\[
\leq \frac{1}{1 + \theta} + \frac{1}{\theta^2(1 + \theta)} + (n-1)e^{-(\theta \wedge 1)((2k)/n)} + O\left(\frac{1}{n}\right) \leq O(e^c).
\]

Therefore if we let \( \eta = \frac{1}{2}e^c \), then
\[
\lim_{c \to \infty} \lim_{n \to \infty} \mathbb{P}_k(c)[d_{(n-1,1)} < \eta] \leq \lim_{c \to \infty} \lim_{n \to \infty} \mathbb{P}(O(1)\frac{e^c}{((1/2)e^c + O(1))^2}) = 0.
\]

Thus
\[
\lim_{c \to \infty} \lim_{n \to \infty} \|\delta_1 P^k(c) - \pi\|_{TV}
\geq \lim_{c \to \infty} \lim_{n \to \infty} \left|\mathbb{P}_k(c)\left[d_{(n-1,1)} < \frac{1}{2}e^c\right] - \mathbb{P}_\infty\left[d_{(n-1,1)} < \frac{1}{2}e^c\right]\right| = 1.
\]

**Remark 2.** Wilson’s method gives a suboptimal lower bound, once we know the “geometric” information that \( d_{(n-1,1),\rho} = -\frac{1}{\theta} + \frac{1 + (n-1)\theta}{\theta n}m_1(\rho) \): let \( X_1 \) be the
random variable distributed as $\delta_x P$. We have
\[
R := \sup_{x \in S_n} \mathbb{E}(d(n-1,1),X_1 - d(n-1,1),x)^2 \leq 1,
\]
\[
\log \frac{1}{\beta(n-1,1)} = (\theta \wedge 1) \frac{2}{n} + O\left(\frac{1}{n^2}\right)
\]
and $d(n-1,1)^n = n - 1$. Hence by Wilson [16],
\[
t_{mix}(\varepsilon) \geq \frac{1}{2\log(1/\beta(n-1,1))} \left[ \log \left( \frac{(1 - \beta(n-1,1))d(n-1,1)^n}{2R} \right) + \log \frac{1 - \varepsilon}{\varepsilon} \right]
\]
\[
\geq \frac{n}{4(1 \wedge \theta)} \log n + \log \varepsilon^{-1} + O(1).
\]
This misses a factor of 2 from the lower bound obtained by second moment method. The discrepancy is possibly due to the nonlocal nature of the random transposition walk.

**APPENDIX A: SEKIGUCHI–DEBIARD OPERATOR OVER OTHER BASES**

Having seen the probabilistic interpretation of the second order differential operator (12) expressed in the power sum symmetric basis $p_{\lambda}$, it is natural to consider the following:

**QUESTION 1.** *Are there other bases of the symmetric polynomials $\Lambda_n$ over which $L^2_{\theta}$ has natural probabilistic interpretation?*

Here we examine the remaining four fundamental bases: monomial, elementary and complete. The action of $L^2_{\theta}$ on the monomial basis $m_{\lambda}$ is well known to be strictly upper triangular ([17], page 317), when the rows and columns of the Markov matrix are arranged in a total order compatible with the natural partial order on the set of partitions $\mathcal{P}_n$ of $n$: $\mu < \lambda$ if $\mu_1 + \cdots + \mu_r \leq \lambda_1 + \cdots + \lambda_r$ for all $r$. In particular, if $L^2_{\theta}$ does define a Markov matrix (meaning the entries are nonnegative), it has a single absorbing state at $(1^n)$.

Next consider the action of $L^2_{\theta}$ on $e_{\lambda}$, the elementary symmetric polynomials. This has been studied in detail in [3]. Here we give a quick development avoiding lengthy computations. The action of $U$ is easy to describe. For any simple elementary polynomial $e_r$, the operators $x_i \partial_i$ and $(x_i \partial_i)^2$ simply collect all the terms in $e_r$ that contains the factor $x_i$. So after summing over $i \in [n]$, this results in a constant multiple of the identity. Thus to get a nontrivial action, one must consider a composite $e_{r_1,r_2} := e_{r_1} e_{r_2}$. In this case, one can show that for $r_1 \leq r_2$,
\[
U(e_{r_1,r_2}) = (1 + r_1)e_{r_1,r_2} - \sum_{j=0}^{r_1-1} 2(r_1 + r_2 - 2j)e_{r_1+r_2-j,j}.
\]
Thus \( U \) is strictly lower triangular with respect to the partial order \( \preceq \). It turns out that the action of \( V \) on the \( e_\lambda \) is diagonal: first of all \( V \) satisfies a product rule on \( e_\lambda = \prod_{j=1}^{\ell(\lambda)} e_{\lambda_j} \); by pairing up \( j \neq k \), one also sees that \( Ve_r \) consists of monomials with no repeated factors, hence by symmetry must be a multiple of \( e_r \). Thus the following linear combination yields a legitimate Markov matrix:

\[
M_e(c_1, c_2) := c_1 I + c_2 \left( \frac{\theta}{2} U + V \right).
\]

Notice that we need to add the multiple of \( I \) to make sure that the diagonal entries of \( M_e \) are nonnegative. Also observe that the Jack parameter \( \theta \) needs to be nonpositive for the off-diagonal entries to be nonnegative. It is clear from the description of \( U \) and \( V \) that this Markov chain is absorbing at \((n)\), because the next step either stays in the current state or goes to a state corresponding to a partition bigger than that of the current state.

Next we look at the complete symmetric polynomials \( h_\lambda \). First consider the action of \( \langle X, \nabla \rangle \) on \( h_r \), one of the generators. Since \( h_r = s_{(r)} \) is a degree \( r \) homogeneous polynomial, the action of \( \langle X, D \rangle = \sum_{i=1}^{n} x_i \partial_i \) is simply multiplication by \( r \); that is, any homogeneous polynomials are eigenfunctions \( \langle X, D \rangle \). However, the operator \( (\langle X, D \rangle)^2 \) acts nontrivially on \( h_r \),

\[
\sum_{i=1}^{n} (x_i \partial_i)^2 h_4 = -2h_{14} + 10h_{2,1,1} - 8h_{2,2} - 12h_{3,1} + 28h_4.
\]

For partitions of more than one part, the computation gets unwieldy, and I have not tried to express \( U(h_r h_r) \) and \( V(h_r h_r) \) in terms of \( h_\lambda \) explicitly because of the following numerical observation: for \( \lambda = (3, 2, 1) \), we have

\[
Uh_\lambda = 2h_{2,1,4} - 8h_{2,2,1} - 2h_{3,1^3} + 14h_{3,2,1} + 6h_{3^2} + 6h_{4,1^2} + 8h_{4,2} + 10h_{5,1},
\]

\[
Vh_\lambda = -h_{2,1,4} + 4h_{2,2,1} + h_{3,1^3} + 32h_{3,2,1}.
\]

The only linear combination of the above two expressions that yields nonnegative coefficients is \( \frac{1}{2} U + V \), which corresponds to \( \theta = 1 \). But in that case, the Markov chain is again absorbing at \((6)\). So we arrive at the following result:

**PROPOSITION A.1.** The operator \( L_\theta^2 \) gives a Markov matrix under the complete symmetric polynomial basis \( h_\lambda \) for all \( n \) if and only if \( \theta = 1 \). In this case, the Markov chain never goes toward partitions of fewer or equal parts, hence is absorbing at \((n)\).

**PROOF.** When \( \theta = 1 \), \( h_\lambda \) are dual to \( e_\lambda \) with respect to the Jacobi–Trudy identity. Hence the walk defined by \( D_1^2 \) on \( h_\lambda \) can be obtained from the upper-triangular walk on \( e_\lambda \) under the map \( \lambda \mapsto \lambda^t \); in particular the walk is absorbing at \((r)\). For \( \theta \neq 1 \), the numerical example above suffices to show the associated walk is not positive. \( \square \)
For $\theta \neq 1$, the resulting signed Markov matrix seems to have nontrivial left eigenvector corresponding to the eigenvalue 1. I have not checked if this corresponds to some nice stationary distribution on $\mathcal{P}_n$; presumably it will define a signed measure.

**APPENDIX B: HIGHER ORDER SEKIGUCHI–DEBIARD OPERATORS**

Throughout this section $N$ will denote the number of underlying variables in the symmetric functions, and $n$ will denote the weight of partitions, consistent with previous sections. It is possible to study higher order differential operators on $\Lambda_N$ from the Sekiguchi–Debiard operator-valued generating function (see [17], page 328),

$$D_{\theta}(X) := a_\delta(x)^{-1} \sum_{w \in S_N} \varepsilon(w)x^{w_\delta} \prod_{j=1}^{N}(X + (w_\delta)_j + \theta x_j \partial_j)$$

(37)

$$= \sum_{k=0}^{N} D_{\theta}^k X^{N-k}.$$  

Since the seminal work of Diaconis and Ram [6] interpreting $D_2^2$ above as the generator of an auxiliary variable Markov chain, it has been tempting to consider the following:

**QUESTION 2.** Does any of the higher order $D_{\theta}^k$’s admit natural probabilistic interpretation?

Below we give complete analysis of $D_{\theta}^3$, and show that the answer is not nearly as nice as for $D_{\theta}^2$. To begin, it suffices to understand the following two-parameter family of operators:

$$D(\lambda, \mu; h) = a_\delta(x)^{-1} \sum_{w \in S_N} \varepsilon(w) \prod_{i=1}^{\ell} \left( \sum_{j=1}^{N} (w_\delta)^{\lambda_i} (x_j \partial_j)^{\mu_i} \right) \sum_{j=1}^{N} (x_j \partial_j)^h$$

$$= a_\delta(x)^{-1} \sum_{j_1, \ldots, j_\ell} \prod_{i=1}^{\ell} (x_{j_i} \partial_{j_i})^{\lambda_i} a_\delta(x)(x_{j_i} \partial_{j_i})^{\mu_i} \sum_{j=1}^{N} (x_j \partial_j)^h,$$

where $\lambda, \mu$ are positive integer compositions and $\ell = \ell(\mu) = \ell(\lambda)$. Indeed, it is easy to see that [denoting by $(j_1, \ldots, j_k)$ all distinct indices]

$$D_{\theta}^k = a_\delta(x)^{-1} \sum_{w \in S_N} \sum_{(j_1, \ldots, j_k)} \sum_{u=0}^{k} \frac{\theta^u}{u!(k-u)!} \prod_{i=1}^{k-u} (w_\delta)_i \prod_{i=k-u+1}^{k} (x_{j_i} \partial_{j_i}),$$

which can be expressed as a linear combination of $D(\lambda, \mu, h)$’s with $|\lambda| + |\mu| + h = k$ and $h \neq 1$; the factors of the form $\sum_{j=1}^{N} (w_\delta)_j^k$ or $\sum_{j=1}^{N} (x_j \partial_j)^k$ all evaluate to constant by symmetry, and the operator $\sum_{j=1}^{N} (x_j \partial_j)$ acts on $p_\lambda$ by constant multiplication.
For \( k = 3 \), we thus have three operators to consider: \( D((1), (2); 0) \), \( D((2), (1); 0) \) and \( D(\emptyset, \emptyset; 3) \). We compute the action of each on \( p_\lambda \) below. We need the following three computations:

- \( a_\delta(x)^{-1}x_i \partial_i a_\delta(x) = \sum_{j \neq i} \frac{x_i}{x_i - x_j} \);

- \( (x_i \partial_i)^2 p_\lambda = x_i \partial_i \sum_{s=1}^{\ell(\lambda)} \frac{p_\lambda}{p_{\lambda_s}} (x_i \partial_i) p_{\lambda_s} \)
  \[= \sum_{s=1}^{\ell(\lambda)} \frac{p_\lambda}{p_{\lambda_s}} (x_i \partial_i)^2 p_{\lambda_s} + \sum_{t \neq s} \frac{p_\lambda}{p_{\lambda_s} p_{\lambda_t}} [(x_i \partial_i) p_{\lambda_s}] [(x_i \partial_i) p_{\lambda_t}] \]
  \[= \sum_{s=1}^{\ell(\lambda)} \lambda_s^2 x_i^{\lambda_s} \frac{p_\lambda}{p_{\lambda_s}} + \sum_{s \neq t} \lambda_s \lambda_t x_i^{\lambda_s + \lambda_t} \frac{p_\lambda}{p_{\lambda_s} p_{\lambda_t}}; \]

- \( \sum_{i=1}^N \sum_{j \neq i} x_i^{r+1} x_j^{r+1} \equiv \left( \frac{1}{2} \sum_{i \neq j} x_i^{r+1} x_j^{r+1} \right) \equiv \frac{1}{2} \sum_{i \neq j} \sum_{u=0}^{r} x_i^u x_j^{r-u} . \)

From this we have

\[ D((1), (2); 0) p_\lambda = a_\delta(x)^{-1} \sum_{i=1}^N \left[ x_i \partial_i a_\delta(x) (x_i \partial_i)^2 p_\lambda \right] \]

\[= \lambda_s \frac{\lambda_t^2}{2} \left[ (2N - \lambda_s - 1) + \sum_{u=1}^{\lambda_s-1} \frac{p_u p_{\lambda_s-u}}{p_{\lambda_s}} \right] \]

\[+ \sum_{s \neq t} \frac{\lambda_s \lambda_t}{2} \left[ (2N - \lambda_s - \lambda_t - 1) \frac{p_{\lambda_s+\lambda_t}}{p_{\lambda_s} p_{\lambda_t}} + \sum_{u=1}^{\lambda_s+\lambda_t-1} \frac{p_u p_{\lambda_s+\lambda_t-u}}{p_{\lambda_s} p_{\lambda_t}} \right]. \]

It is worth pointing out that the sum \( \sum_{u=1}^{\lambda_s+\lambda_t-1} \frac{p_u p_{\lambda_s+\lambda_t-u}}{p_{\lambda_s} p_{\lambda_t}} \) contains a constant term (when \( u \in \{\lambda_s, \lambda_t\} \)).

Next we consider \( D((2), (1); 0) \). Again we collect some computations below:

- \( x_i \partial_i p_\lambda = \sum_{s=1}^{\ell(\lambda)} \lambda_s \lambda_i x_i^{\lambda_s} \frac{p_\lambda}{p_{\lambda_s}}. \)
$a_\delta(x)^{-1}(x_i \partial_i)^2 a_\delta(x)$

\[
= (x_i \partial_i)^2 \log a_\delta(x) + x_i^2 (\partial_i \log a_\delta(x))^2
\]

\[
= \sum_{j \neq i} \frac{-x_i x_j}{(x_i - x_j)^2} + \sum_{j \neq i} \sum_{k \neq i, j} \frac{x_i^2}{(x_i - x_j)(x_i - x_k)} + \sum_{j \neq i} \frac{x_i^2}{(x_i - x_j)^2}
\]

\[
= \sum_{j \neq i} \frac{x_i}{x_i - x_j} + \sum_{j \neq k: \ j, k \neq i} \frac{x_i^2}{(x_i - x_j)(x_i - x_k)}.
\]

\[
\sum_{i=1}^N \sum_{j \neq k: \ j, k \neq i} x_i^{r+2} \frac{x_i}{x_i - x_j}(x_i - x_k)
\]

\[
= \sum_{T \subset [N]: \ |T|=3} \sum_{i \in T} \sum_{j \neq k: \ j, k \in T \setminus \{i\}} \frac{x_i^{r+2}}{(x_i - x_j)(x_i - x_k)}
\]

\[
= 2 \sum_{T \subset [N]: \ |T|=3} s_r(T) = 2 \sum_{T \subset [N]: \ \lambda=r} \sum_{\lambda \vdash r} m_{\lambda}(T)
\]

\[
= 2 \sum_{T \subset [N]: \ |T|=3} \sum_{\lambda \vdash r: \ \ell(\lambda) \leq 3} m_{\lambda}(T)
\]

\[
= 2 \left[ \left( \begin{array}{c} N-3 \end{array} \right) \sum_{\lambda \vdash r: \ \ell(\lambda)=3} m_{\lambda} + \left( \begin{array}{c} N-2 \end{array} \right) \sum_{\lambda \vdash r: \ \ell(\lambda)=2} m_{\lambda} + \left( \begin{array}{c} N-1 \end{array} \right) \sum_{\lambda \vdash r: \ \ell(\lambda)=1} m_{\lambda} \right],
\]

where $s_{\lambda}(T)$ denotes the Schur polynomial over the variables indexed by $T$, and similarly for $m_{\lambda}$.

\[
\sum_{\lambda \vdash r: \ \ell(\lambda)=2} m_{\lambda} = \sum_{\lambda \vdash r: \ i \neq j} x_i^{\lambda_1} x_j^{\lambda_2} I(\lambda_1 \neq \lambda_2) + \frac{1}{2} x_i^{\lambda_1} x_j^{\lambda_2} I(\lambda_1 = \lambda_2)
\]

\[
= \frac{1}{2} \sum_{r_1+r_2=r: \ r_1 \geq 1} (p_{r_1} p_{r_2} - p_r) = \frac{1}{2} \sum_{r_1+r_2=r: \ r_1 \geq 1} p_{r_1} p_{r_2} - \frac{r-1}{2} p_r.
\]

\[
\sum_{\lambda \vdash r: \ \ell(\lambda)=3} m_{\lambda} = \frac{1}{6} \left[ \sum_{r_1+r_2+r_3=r: \ r_i \geq 1} p_{r_1} p_{r_2} p_{r_3} - 3 \sum_{r_1+r_2=r: \ r_i \geq 1} (r_1 - 1) p_{r_1} p_{r_2} + (r-1)(r-2) p_r \right]
\]
\[
D((2), (1); 0) p_\lambda = a_\delta(x)^{-1} \sum_{w \in S_N} \varepsilon(w) x^w \sum_{i=1}^N (w_1 \partial_i) p_\lambda
\]

(38)
and

\[ D(\varnothing, \varnothing; 2)p_\lambda = a_\delta(x)^{-1} \sum_{w \in S_N} \varepsilon(w)x^{\omega}\sum_{i=1}^N (x_i \partial_i)^2 p_\lambda \]

\[ = p_\lambda \left( \sum_{s=1}^{\ell(\lambda)} \frac{\lambda_s^2}{s^2} + \sum_{s \neq t} \frac{\lambda_s \lambda_t}{\lambda_s \lambda_t} \right). \]

We can now compute the action of \( D_2^2 \) on power sum polynomials; see [17], Example VI.3.3(e).

\[ D_2^2 p_\lambda = a_\lambda(x)^{-1} \sum_{w \in S_N} \varepsilon(w)x^{\omega}\sum_{i \neq j} \left( \frac{1}{2}(w_\omega)_i (w_\omega)_j + \theta (w_\omega)_i (x_j \partial_j) \right) p_\lambda \]

\[ = a_\delta(x)^{-1} \sum_{w} \varepsilon(w)x^{\omega}\left( \frac{1}{2} \left[ \left( \sum_i (w_\omega)_i \right)^2 - \sum_i (w_\omega)^2_i \right] \\
+ \theta \left[ \left( \sum_i (w_\omega)_i \right) \left( \sum_k x_i \partial_k \right) - \sum_i (w_\omega)_i(x_i \partial_i) \right] \\
+ \frac{\theta^2}{2} \left[ \left( \sum_i x_i \partial_i \right)^2 - \sum_i (x_i \partial_i)^2 \right] \right) \]

\[ = \frac{1}{2} \left[ \left( \frac{N(N+1)}{2} \right)^2 - \frac{N(N+1)(2N+1)}{6} \right] \\
+ \theta \left[ \frac{N(N+1)n}{2} - D((1), (1); 0) \right] + \frac{\theta^2}{2} \left[ n^2 - D(\varnothing, \varnothing; 2) \right] p_\lambda \]

\[ = \left( \frac{1}{2} \left[ \theta^2 n^2 + \theta n N(N-1) + \frac{N(N-1)(N-2)(3N-1)}{12} \right] \\
- \theta D((1), (1); 0) - \frac{\theta^2}{2} D(\varnothing, \varnothing; 2) \right) p_\lambda. \]

Similarly we can compute \( D_3^3 \) using the inclusion-exclusion principle,

\[ D_3^3 p_\lambda = a_\delta(x)^{-1} \]

\[ \times \sum_{w} \varepsilon(w)x^{\omega} \sum_{(i,j,k)} \left[ \frac{1}{6} (w_\omega)_i (w_\omega)_j (w_\omega)_k + \frac{\theta}{3} (w_\omega)_i (w_\omega)_j x_k \partial_k \right. \]

\[ + \frac{\theta^2}{3} (w_\omega)_i (x_j \partial_j) (x_k \partial_k) + \frac{\theta^3}{6} (x_i \partial_i) (x_j \partial_j) (x_k \partial_k) \]
\[
\begin{align*}
&= \left( c_N(3, \theta) + \frac{\theta^3}{6} [2D(\emptyset, \emptyset; 3) - 3nD(\emptyset, \emptyset; 2)] \\
&\quad + \frac{\theta^2}{3} \left[ 2D((1), (1); 0) - 2nD((1), (1); 0) - \binom{N}{2} D(\emptyset, \emptyset; 2) \right] \\
&\quad + \frac{2\theta}{3} \left[ D((2), (1); 0) - \binom{N}{2} D((1), (1); 0) \right] \right)p_\lambda,
\end{align*}
\]

where \( c_N(3, \theta) = \frac{1}{6} \left[ \binom{N}{3} - \frac{3}{4} \binom{N}{2} \binom{2N}{2} + 2 \binom{N}{2} \binom{N}{2} \right] + \frac{\theta}{2} \left[ \binom{N}{2} n - \frac{1}{4} \binom{2N}{2} n + \frac{\theta^2}{3} \binom{N}{2} n^2 \right] + \frac{\theta^3 n^3}{6} \). Unfortunately I cannot extract any natural interpretation of Markov chains from the right-hand side. This is not so surprising since the composite walk \( P_\theta^k \) for \( k \geq 2 \) does not correspond to some affine transformation of the Metropolization of \( P_1^k \) with respect to the measure \( \text{MED}(\theta) \). Nonetheless this gives a new Markov chain that converges to the multivariate Ewens distribution with parameter \( \theta^{-1} \), since the operators \( D_\theta^i \) are simultaneously diagonalized and the left eigenfunction corresponding to the eigenvalue 1 is simply the stationary distribution.

I also computed a numerical example using the symmetric reduction function in Mathematica and the SF package in Maple. We take the power sum polynomial \( p_\lambda \) with \( \lambda = (3, 1^2) \):

\[
\begin{align*}
D_\theta^2 p_3 p_1^2 &= -3\theta p_2 p_1^3 + (33\theta + 35 + 7\theta^2) p_3 p_1^2 - 6\theta^2 p_4 p_1 - p_2 p_3^2, \\
D_\theta^3 p_3 p_1^2 &= (2/3)\theta p_1^5 + (-4\theta^2 - 8\theta) p_2 p_1^3 \\
&\quad + (22\theta^2 + (73/6)\theta^3 + 50 + (307/3)\theta) p_3 p_1^2 + 4\theta^2 p_1 p_2^2 \\
&\quad + (4\theta^3 - 20\theta^2) p_4 p_1 + (-4\theta^2 + 2\theta^2) p_3 p_2 + 6\theta^3 p_5, \\
D_\theta^5 \circ D_\theta^2 p_3 p_1^3 &= 3\theta^2 p_5^5 + (-21\theta - 4\theta^3 - 19\theta^2) p_2 p_1^3 \\
&\quad + (505\theta^3 + 1225 + 2310\theta + 4\theta^4 + 1579\theta^2) p_3 p_2^2 + 24\theta^3 p_1 p_2^2 \\
&\quad + (41\theta^3 - 6\theta^4 - 42\theta^2) p_4 p_1 \\
&\quad + (-6\theta^3 - \theta^4 - 7\theta^2) p_3 p_2 + 30 p_6.
\end{align*}
\]

This example shows that \( D_\theta^2, D_\theta^3, D_\theta^2 \circ D_\theta^2 \) and id are independent operators on \( \Lambda_N \).

Notice also that \( D_\theta^3 p_\lambda \) has positivity issues: the partitions of Cayley distance 2 from the starting partition \( \lambda \) are always positive, whereas the ones that differ from \( \lambda \) by one transposition might become negative. So in order to make \( D_\theta^3 \) into a Markov matrix, one needs to add a sufficiently negative multiple of \( D_\theta^2 \). We have not tried to compute the optimal multiple here since we are unable to glean any nice pattern from the numerical example above; in particular, the coefficients cannot be made into simple powers of \( \theta \). Observe that \( D_\theta^2 \circ D_\theta^2 p_\lambda \) also has positivity problem, but it is much easier to fix, since one can simply add a multiple of the identity to \( D_\theta^2 \) as in the case treated by [21].
APPENDIX C: COMPOSITIONS OF $D_\theta^2$ FOR DIFFERENT $\theta$ VALUES

In general the eigenvalues and eigenfunctions of a Markov chain can be highly intractable, due to the need to solve for high degree polynomials. For instance, the Metropolis chain based on 3-cycle shuffle on $\mathcal{P}_n$ already requires taking square roots for $n = 4$:

$$M_4^{(3)}(\theta) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1/8 & 3/4 & 0 & 1/8 \\ 0 & 1 & 0 & 0 & 0 \\ \theta^2 & 0 & 0 & 1 - \theta^2 & 0 \\ 0 & \theta^2 & 0 & 0 & 1 - \theta^2 \end{pmatrix}. $$

The eigenvalues are

$$\{1, 1, -\theta^2, \frac{1}{16}(1 - 8\theta^2 - \sqrt{193 - 208\theta^2 + 64\theta^4}), \frac{1}{16}(1 - 8\theta^2 + \sqrt{193 - 208\theta^2 + 64\theta^4})\}. $$

The following result was discovered in numerical experiments:

**Proposition C.1.** For any Laurent polynomial $p$ in $m$ variables, $p(D_{\theta_1}^2, \ldots, D_{\theta_m}^2)$ gives rise to a Markov chain on the set of partitions $\mathcal{P}_n$, with eigenvalues, and left and right eigenvectors given by rational functions of $\theta_1, \ldots, \theta_m$. In particular, the stationary distribution is given by rational functions of $\theta_i$’s also.

**Proof.** When expressed in the monomial symmetric basis, $D_{\theta}^2$ is unipotent (upper triangular with 1’s on the diagonal), with respect to any total ordering on $\mathcal{P}_n$ compatible with the natural partial ordering $\preceq$, whereby $\lambda \preceq \mu$ if $\lambda_1 + \cdots + \lambda_r \leq \mu_1 + \cdots + \mu_r$ for all $r$; see [17], page 317, equation (3.7). Thus fixing this total-ordering, any Laurent polynomial of $D_{\theta_i}$’s is clearly still unipotent. The eigenvalues are simply the diagonal entries, and the eigenvectors can be computed using simple row reduction, which also results in rational components.

The above result is clearly also true for $D_{\theta}^k$ in general and even Macdonald operators. Thus in principle, one can compute the $L^2$ mixing time of Markov chain of the form $D_{\theta_1}^2 D_{\theta_2}^2$, whose stationary distribution can be quite complicated: for $n = 3$, the stationary probabilities are

$$\left\{ \begin{array}{l} \frac{\theta_2(\theta_1(3 - 4\theta_2) + \theta_1^2(-1 + \theta_2) + \theta_2)}{8 + \theta_2 + \theta_1^2(-1 + \theta_2)\theta_2 - \theta_1(-1 + 9\theta_2 + \theta_2^2)}, \\
\frac{3(-(-3 + \theta_2)\theta_2 + \theta_1(1 - 4\theta_2 + \theta_2^2))}{8 + \theta_2 + \theta_1^2(-1 + \theta_2)\theta_2 - \theta_1(-1 + 9\theta_2 + \theta_2^2)}, \\
\frac{-2\theta_1 + 2(-2 + \theta_2)^2}{8 + \theta_2 + \theta_1^2(-1 + \theta_2)\theta_2 - \theta_1(-1 + 9\theta_2 + \theta_2^2)} \end{array} \right\}. $$
The appropriate generalization of the Laplace–Beltrami operator to root systems other than $A_N$ is given by the Heckman–Opdam operator (see [12] and [3]),

$$L_N(\kappa, R) = \Delta + \kappa V_N := \sum_{i=1}^{N} \partial_{t_i}^2 + \sum_{\alpha \in R_+} \kappa_\alpha \coth(\alpha/2) \partial_\alpha,$$

where $R$ denotes an arbitrary root system, $R_+$ a designated set of positive roots and $\kappa_\alpha$ is called a multiplication function, invariant under the action of the Weyl group on $R_+$. For more on root systems and Weyl groups, consult the first 3 chapters of [10] as well as Chapters 19 and 20 of [4].

Fascinated by the success of the $A_N$ root system, Diaconis raised the following:

**Question 3.** Are there other root systems beside those of type $A_N$ whose associated Heckman–Opdam operators give rise to nontrivial Markov chains with algebraically tractable spectral decomposition?

We study root system $D_N$ in detail here; $B_N$ and $C_N$ are similar. These come from the irreducible decomposition of the adjoint representation of the maximal torus in the compact Lie groups $SO(2N, \mathbb{R})$. The positive roots can be chosen as the set $\{x_i x_j^{-1}, x_i x_j : 1 \leq i < j \leq N\}$ on the maximal torus; in the associated Cartan subalgebra (the Lie subalgebra corresponding to the maximal torus), they become $\{t_i - t_j, t_i + t_j : 1 \leq i < j \leq N\}$. The appropriate analogue of the power sum polynomials appears to be the following power sum symmetric Laurent polynomials:

$$p_a = \sum_{i=1}^{N} \cosh(at_i) = \sum_{i=1}^{N} [x_i^a + x_i^{-a}] / 2,$$

where $x_i = e^{t_i}$. And as in the case of $A_N$, $p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}$. By direct computation we have

$$\Delta p_\lambda = p_\lambda \left\{ \sum_{i=1}^{\ell(\lambda)} \lambda_i^2 + \sum_{1 \leq i < j \leq \ell(\lambda)} \lambda_i \lambda_j \left[ \frac{p_{\lambda_i + \lambda_j}}{p_{\lambda_i} p_{\lambda_j}} - \frac{p_{\lambda_i - \lambda_j}}{p_{\lambda_i} p_{\lambda_j}} \right] \right\},$$

$$\sum_{\theta \in R_+} \coth(\theta/2) \partial_\alpha p_a = 2a \sum_{i \neq j}^{a-1} \sum_{\ell=0} \cosh(\ell t_i) \cosh((a-\ell)t_j)$$

$$= (2an - a^2 - a) p_a + 2a \sum_{\ell=1}^{a-1} p_{\ell} p_{a-\ell} - a \sum_{\ell=1}^{a-1} p_{a-2\ell},$$
\[
\frac{a-1}{\ell=1} \sum p_{a-2\ell} = \begin{cases} 
\frac{a}{2} 
& \text{if } a \text{ is odd,} \\
N + 2 \sum_{\ell=1}^{(a-2)/2} p_{a-2\ell}, 
& \text{if } a \text{ is even} 
\end{cases}
\]

if we define \( p_0 := \frac{N}{2} \).

Therefore for \( n = |\lambda| = \sum_i \lambda_i \),

\[
\sum_{\alpha \in R_+} \coth(\alpha/2) \partial_\alpha p_{\lambda} 
= \left( (2N - 1)n - \sum_{i=1}^{\ell(\lambda)} \lambda_i^2 \right) p_{\lambda} + \sum_{i=1}^{\ell(\lambda)} p_{\lambda} \frac{2\lambda_i}{p_{\lambda_i}} \left[ \sum_{\ell=1}^{\lambda_i-1} p_{\lambda} p_{\lambda_i-\ell} - \sum_{\ell=1}^{[\lambda_i/2]} p_{\lambda_i-2\ell} \right].
\]

Restricting to partitions of the top grading, \( n \), clearly the transition coefficients are affine transformation of those in the \( A_N \) case, and hence nothing new is obtained this way. There are several pathological features regarding the action of \( L_N(D, \kappa) := \Delta + \sum_{\alpha \in R_+} \kappa_\alpha \coth(\alpha/2) \partial_\alpha \) on the power sum analogues of symmetric Laurent polynomials (see the toy example below):

(1) there are no easy ways to make the entries all positive;
(2) the row sums are not the same.

Thus it remains difficult to interpret the full transition matrix as a Markov kernel. For root system \( D_N \), there is only one Weyl orbit, hence \( \kappa_\alpha \equiv \kappa \). The Heckman–Opdam functions have rational transition coefficients to this power sum Laurent basis, as illustrated by the following numerical example (\( P_k \) denotes the set of partitions of \( k \)):

\[
M_\kappa(N)|_{P_3 \cup P_1} := 
\begin{pmatrix} 
9 + \kappa(-12 + 6N) & 12\kappa & 0 & -6\kappa \\
2 & 5 + \kappa(-8 + 6N) & 4\kappa & -2 - 2\kappa N \\
0 & 3 & 3 + \kappa(-6 + 6N) & -3 \\
0 & 0 & 0 & 1 + \kappa(-2 + 2N)
\end{pmatrix},
\]

where the columns and rows are indexed by \((3), (21), (1^3), (1)\). The eigenvalues are very clean,

\[
3 + 6\kappa(-2 + N), \quad 9 + 6\kappa(-1 + N), \\
5 + \kappa(-8 + 6N), \quad 1 + 2\kappa(-1 + N).
\]
The left eigenvectors are rational functions of $\kappa$, which we display as rows of the following matrix:

$$
\begin{pmatrix}
\frac{-5 + \theta + 2N}{3(-1 + N)} & -\frac{-5 + \theta + 2N}{1 + N} & \frac{2(-5 + \theta + 2N)}{3(-1 + N)} \\
0 & 0 & 1 \\
\frac{-2\theta(-1 + 2\theta + N)}{3(1 + 2\theta + N)} & -\frac{-2(-1 + 2\theta + N)}{1 + 2\theta + N} & \frac{-4(-1 + 2\theta + N)}{3\theta(1 + 2\theta + N)} \\
\frac{\theta(-3 + 2\theta + 2N)}{\theta + 2\theta^2 - 2N + 2\theta N} & -\frac{2(-1 + \theta)(-3 + 2\theta + 2N)}{\theta + 2\theta^2 - 2N + 2\theta N} & \frac{12 - 8\theta - 8N}{\theta + 2\theta^2 - 2N + 2\theta N}
\end{pmatrix}.
$$

Here $\theta = \kappa^{-1}$ corresponds to the parameter in the $A_N$ case.

We have also tried to adjust the diagonal entries to make the row sum equal to 1; the resulting matrix however does not have rational eigenvalues in the entries.

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