Conformal Ricci and Matter Collineations for Two Perfect Fluids

M. Sharif * and Naghmana Tehseen
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

Conformal Ricci and conformal matter collineations for the combination of two perfect fluids in General Relativity are investigated. We study the existence of timelike and spacelike conformal Ricci and matter collineations by introducing the kinematical and dynamical properties of such fluids and using the Einstein field equations. Some recent studies on conformal collineations are extended and new results are found. It is worth mentioning that we recover all the previous results as special cases.

Keywords: Conformal collineations, Two perfect fluids.
PACS: 04.20.Jb Exact solutions.

1 Introduction

The Einstein field equations (EFEs), whose fundamental constituent is the spacetime metric $g_{ab}$, are highly nonlinear partial differential equations, and therefore it is very difficult to obtain their exact solutions. Symmetries of the geometrical/physical relevant quantities of the General Relativity (GR) theory are known as collineations. In general, these can be represented as $\mathcal{L}_\xi A = \mathcal{B}$, where $A$ and $\mathcal{B}$ are the geometric/physical objects, $\xi$ is the vector field generating the symmetry, and $\mathcal{L}_\xi$ signifies the Lie derivative operator along the vector field $\xi$.

A one-parameter group of conformal motions generated by a conformal Killing vector (CKV) $\xi$ is defined as

$$\mathcal{L}_\xi g_{ab} = 2\psi g_{ab},$$

where $\mathcal{L}$ is the Lie derivative operator, $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{,ab} \neq 0$, the CKV is said to be proper. Otherwise, $\xi$ reduces to the special conformal Killing vector (SCKV) if $\psi_{,ab} = 0$, but $\psi_{,a} \neq 0$. Other subcases are a homothetic vector (HV) if $\psi_{,a} = 0$ and a Killing vector (KV) if $\psi = 0$.

Duggal introduced a new symmetry called a Ricci inheritance collineation defined by

$$\mathcal{L}_\xi R_{ab} = \alpha R_{ab},$$

where $\alpha = \alpha(x^a)$ is a scalar function. We shall use the term conformal Ricci collineation (CRC) onward which reduces to Ricci collineation (RC) for $\alpha = 0$. Similarly, we can define a matter inheritance collineation or conformal matter collineation (CMC) by

$$\mathcal{L}_\xi T_{ab} = \alpha T_{ab},$$

where $T_{ab}$ is the energy-momentum tensor, and this becomes a matter collineation (MC) when $\alpha = 0$. The function $\alpha$ is called the inheriting or the conformal factor.

Recently, there has been a keen interest in the study of the CKVs and affine conformal vectors (ACVs) in a class of fluid spacetimes. Herrera et al. [3] investigated CKVs with anisotropic fluids. Duggal and Sharma [4] extended this work to the more general case of special ACV. Mason and Maartens [5] studied the kinematics and dynamics of conformal collineations with the general class of anisotropic fluids and no energy flux. Duggal [2, 6] discussed curvature inheritance symmetry and timelike CRCs in a perfect fluid spacetime. Yavuz and Yilmaz

* msharif@math.pu.edu.pk
et al. [7, 8] considered inheriting conformal and SCKVs and worked on the curvature inheritance symmetry in string cosmology. Yilmaz [9] also considered timelike and spacelike collisions in a string cloud. Baysal et al. [10] worked on spacelike CRCs in models of a string cloud and string fluid stress tensor. Mason and Tsamparlis [11] investigated spacelike CKVs in a spacelike congruence. Saridakis and Tsamparlis [12] studied the applications for spacelike CKVs and matter described either by a perfect fluid or by an anisotropic fluid. Tsamparlis [13] has discussed the general symmetries of string fluid spacetime. Sharif and Umber [14] have investigated timelike and spacelike CMCs for specific forms of the energy-momentum tensor. In a very recent paper, Sharif and Naghmana [15] have explored the spacetimes admitting CRCs and CMCs in anisotropic fluids.

In this paper, we shall use the following notation: \( a, b, c \) are the four velocities of the two perfect fluids and \( a, b \) indicates derivative w.r.t. \( y \). Similarly, for the fluid 2, when we use \( \dot{v}_a = v_{a;b}b^b \), the symbol \( v_{a;b} \) indicates derivative w.r.t. \( y \) and for \( \dot{v}_a = v_{a;\hat{b}}\hat{b}^\hat{b}, v_{a;\hat{b}} \) means derivative w.r.t. \( x \).

The two pair of vectors \( u^a, x^a \), \( v^a, y^a \) define projection operators as

\[
H_{1ab} = h_{1ab} - x_a x_b, \quad H_{2ab} = h_{2ab} - y_a y_b
\]

which are normal to both \( u^a, x^a \) and \( v^a, y^a \). These projection operators satisfy the following properties

\[
\begin{align*}
H_{1ab} u^a &= 0 = H_{1ab} v^a, & H_{1ab} h_{1c}^b &= H_{1ac}, \\
H_{1a}^a &= 2 = H_{2a}^a, & H_{2ab} v^a &= 0 = H_{2ab} y^a, \\
H_{2ab} h_{2c}^b &= H_{2ac}.
\end{align*}
\]

\section{Notations and Some General Results}

We consider the timelike vectors \( u^a, v^a \) and spacelike vectors \( x^a, y^a \) corresponding to the combination of two perfect fluids satisfying the following relations:

\[
\begin{align*}
u^a u_a &= -1 = v^a v_a, & x^a x_a &= 1 = y^a y_a, \\
u^a v_a &\neq 0 = x^a y_a, & u^a x_a &= 0 = v^a x_a, \\
u^a y_a &= 0 = v^a y_a.
\end{align*}
\]

The projection tensors corresponding to the vectors \( u^a \) and \( v^a \) are defined as follows:

\[
h_{1ab} = g_{ab} + u_a u_b, \quad h_{2ab} = g_{ab} + v_a v_b
\]

which project normal to \( u^a, v^a \) and produce the well-known \( 1 + 3 \) decomposition of the tensor algebra along \( u^a, v^a \) respectively. The two well-known examples of the \( 1 + 3 \) decomposition [10] are:

\[
\begin{align*}
u_{a;b} &= \sigma_{1ab} + \omega_{1ab} + \frac{1}{3} \theta h_{1ab} - \dot{u}_a u_b, \\
v_{a;b} &= \sigma_{2ab} + \omega_{2ab} + \frac{1}{3} \theta h_{2ab} - \dot{v}_a v_b, \\
T_{ab} &= \mu_1 u_a u_b + \mu_2 v_a v_b + F_1 h_{1ab} + F_2 h_{2ab} \\
&\quad + 2q_1(u_b) + \pi_{1ab} + 2q_2(v_b) + \pi_{2ab}.
\end{align*}
\]

In this paper we shall use the following notation:

For the fluid 1, when \( \dot{u}_a = u_{a;b} b^b \), the notation \( u_{a;b} \) means derivative w.r.t. \( x \) but for \( \dot{u}_a = u_{a;b} b^b \), \( u_{a;b} \) denotes derivative w.r.t. \( y \). Similarly, for the fluid 2, when we use \( \dot{v}_a = v_{a;b} b^b \), the symbol \( v_{a;b} \) indicates derivative w.r.t. \( y \) and for \( \dot{v}_a = v_{a;\hat{b}}\hat{b}^\hat{b}, v_{a;\hat{b}} \) means derivative w.r.t. \( x \).

The two pair of vectors \( u^a, x^a \) and \( v^a, y^a \) define projection operators as

\[
H_{1ab} = h_{1ab} - x_a x_b, \quad H_{2ab} = h_{2ab} - y_a y_b
\]

which are normal to both \( u^a, x^a \) and \( v^a, y^a \). These projection operators satisfy the following properties

\[
\begin{align*}
H_{1ab} u^a &= 0 = H_{1ab} v^a, & H_{1ab} h_{1c}^b &= H_{1ac}, \\
H_{1a}^a &= 2 = H_{2a}^a, & H_{2ab} v^a &= 0 = H_{2ab} y^a, \\
H_{2ab} h_{2c}^b &= H_{2ac}.
\end{align*}
\]
The vectors $x_{a;b}$ and $y_{a;b}$ can be decomposed using the same procedure as given in [17]. This gives the following

\[ x_{a;\beta} = A_{1ab} + x_{a}^{\gamma} x_{\gamma} - \dot{x}_{a} u_{\beta} + u_{a} [f^{\beta} u_{f,\beta} + (f^{\beta} \dot{u}_{f}) u_{\beta} - (f^{\beta} u_{f}^{\gamma}) x_{\gamma}], \]

\[ y_{a;\beta} = A_{2ab} + y_{a}^{\gamma} y_{\gamma} - \dot{y}_{a} v_{\beta} + v_{a} [f^{\beta} v_{f,\beta} + (f^{\beta} \dot{v}_{f}) v_{\beta} - (f^{\beta} v_{f}^{\gamma}) y_{\gamma}], \]

where

\[ A_{1ab} = S_{1ab} + R_{1ab} + \frac{1}{2} \varepsilon_{1} H_{1ab}, \]

\[ A_{2ab} = S_{2ab} + R_{2ab} + \frac{1}{2} \varepsilon_{2} H_{2ab}, \]

and $s^{\alpha} = s_{\cdots ;\alpha} x^{\alpha}$. The notations $x_{\beta}$ and $y_{\beta}$ in Eqs. (11) and (12) mean derivative w.r.t. $x$ and $y$ respectively. We note that $S_{1ab} = S_{1ba}$ and $S_{b}^{a} = 0$ is the traceless part (shear tensor). Also, $R_{1ab} = -R_{1ba}$ is antisymmetric part (rotation tensor) and $\varepsilon_{1}, \varepsilon_{2}$ are the traces (expansion). The following relations can be found [17]

\[ S_{1ab} = H_{1a}^{c} H_{1b}^{d} x_{(c;d)} - \frac{1}{2} \varepsilon_{1} H_{1ab}, \]

\[ R_{1ab} = H_{1a}^{c} H_{1b}^{d} x_{(c;d)}, \quad \varepsilon_{1} = H_{1}^{ab} x_{a;b}, \]

\[ S_{2ab} = H_{2a}^{c} H_{2b}^{d} y_{(c;d)} - \frac{1}{2} \varepsilon_{2} H_{2ab}, \]

\[ R_{2ab} = H_{2a}^{c} H_{2b}^{d} y_{(c;d)}, \quad \varepsilon_{2} = H_{2}^{ab} y_{a;b}. \]

The bracket terms of $u^{a}$ and $v^{a}$ in Eqs. (11) and (12) can be written as

\[ -N_{1b} + 2w_{1f} u_{f}^{\beta} + H_{1b}^{c} x_{c}^{\beta}, \]

\[ -N_{2b} + 2w_{2f} v_{f}^{\beta} + H_{2b}^{c} y_{c}^{\beta}, \]

where $N_{1b}$, $N_{2b}$ are given by

\[ N_{1b} = H_{1b}^{a} (\dot{x}_{a}^{\gamma} - u_{a}^{\gamma}), \quad N_{2b} = H_{2b}^{a} (\dot{y}_{a}^{\gamma} - v_{a}^{\gamma}). \]

They are called Greenberg vectors [18]. This vector vanishes if and only if the vector fields $u^{a}$, $v^{a}$, $x^{a}$, $y^{a}$ are surface forming, i.e., if and only if $\mathcal{L}_{\xi} x^{a} = A u^{a} + B x^{a}$. From the kinematical point of view, the vector $N_{1a}$ vanishes if and only if the vector field $x^{a}$ is frozen along the observer.

Substituting Eq. (15) in Eqs. (11) and (12), it follows that

\[ x_{a;\beta} = A_{1ab} + x_{a}^{\gamma} x_{\gamma} - \dot{x}_{a} u_{\beta} + H_{1b}^{c} x_{c} u_{a} + (2w_{1c} x_{c}^{t} - N_{1c}) u_{a}, \]

\[ y_{a;\beta} = A_{2ab} + y_{a}^{\gamma} y_{\gamma} - \dot{y}_{a} v_{\beta} + H_{2b}^{c} y_{c} v_{a} + (2w_{2c} y_{c}^{t} - N_{2c}) v_{a}. \]

Notice that $\mathcal{L}_{\xi}$ means Lie derivative with respect to the vector field $\xi$, otherwise $\xi$ is used as a scalar.

### 3 The Two Perfect Fluids

The energy-momentum tensor for a non-interacting combination of two (non-zero) perfect fluids is [19]

\[ T_{ab} = (\rho_{1} + p_{1}) u_{a} u_{b} + (\rho_{2} + p_{2}) v_{a} v_{b} + (p_{1} + p_{2}) g_{ab}, \]

where $p_{1}$, $p_{2}$ are pressures, $\rho_{1}$, $\rho_{2}$ are densities, $u^{a}$, $v^{a}$ are the four-velocities ($u^{a} u_{a} = -1 = v^{a} v_{a}$, $u^{a} v_{a} \neq 0$) and $x^{a}$, $y^{a}$ are unit spacelike vectors normal to the four velocities ($u^{a} x_{a} = v^{a} y_{a} = 0 = u^{a} y_{a} = v^{a} x_{a}$). The energy-momentum tensor of the two fluids can also be written as

\[ T_{ab} = \rho_{1} u_{a} u_{b} + \rho_{2} v_{a} v_{b} + p_{1} h_{1ab} + p_{2} h_{2ab}. \]
Comparing Eqs. (8) and (23), we obtain

\[ \mu_1 = \mu_1, \quad \mu_2 = \mu_2, \quad F_1 = p_1, \quad F_2 = p_2, \]
\[ q_1^a = 0, \quad q_2^a = 0, \quad \pi_{1ab} = 0, \quad \pi_{2ab} = 0. \] (24)

This implies that the two perfect fluids is an anisotropic fluid for which heat flux and traceless anisotropic stress tensor are zero. The EFEs can be written as

\[ R_{ab} = T_{ab} + (\Lambda - \frac{1}{2} T)g_{ab}, \] (25)

where \( \Lambda \) is the cosmological constant. For the two perfect fluids, it takes the form

\[ R_{ab} = (\rho_1 + p_1)u_a u_b + (\rho_2 + p_2)v_a v_b + \frac{1}{2}(\rho_1 + p_2 - p_1 - p_2 + 2\Lambda)g_{ab}. \] (26)

The 1 + 3 decomposition of \( R_{ab} \) takes the form

\[ R_{ab} = \frac{1}{2}(\rho_1 + 3p_1 - \Lambda)u_a u_b + \frac{1}{2}(\rho_2 + 3p_2 - \Lambda)v_a v_b + \frac{1}{2}(\rho_1 - p_1 + \Lambda)h_{1ab} \]
\[ + \frac{1}{2}(\rho_2 - p_2 + \Lambda)h_{2ab}. \] (27)

This gives the field equations in terms of the combination of two perfect fluid variables. Using Eq. (27), we find \( \mathcal{L}_\xi R_{ab} \) in terms of the two perfect fluids.

\[ \frac{1}{\xi} \mathcal{L}_\xi R_{ab} = \left[ \frac{1}{2}(\rho_1 - \rho_2 + 3p_1 + p_2) + (\rho_1 - \rho_2 + 3p_1 + p_2 - 2\Lambda)(\ln \xi) \right] u_a u_b + \left[ (\rho_1 - p_1 + \Lambda) \right] \]
\[ - \rho_2 + 3p_1 + p_2 - 2\Lambda)(\xi_u - (\ln \xi)_u) u_{(a} h_{b)}^c \]
\[ + (\rho_1 + p_1 - p_2)(\xi_u - (\ln \xi)_u) h_{1cd} \]
\[ + (\rho_1 + p_1 - p_2)(\xi_u - (\ln \xi)_u) h_{2ab} \] (28)

Similarly, the Lie derivative of the Ricci tensor along the spacelike vector \( \xi^a = \xi^a x^a \) can be written in terms of the 1 + 1 + 2 dynamic quantities.

\[ \frac{1}{\xi} \mathcal{L}_\xi R_{ab} = \left[ \frac{1}{2}(\rho_1 - \rho_2 + 3p_1 + p_2) \right] \]
\[ + (\rho_1 - p_1 + \Lambda) \left[ (\rho_1 + p_1 - p_2) \right] \]
\[ - \rho_2 + 3p_1 + p_2 - 2\Lambda)(\xi_u - (\ln \xi)_u) u_{(a} h_{b)}^c \] (29)

Expressions (28) and (29) are general and hold for all collineations and any two perfect fluids. Similarly, we can write expressions for the second timelike and spacelike vectors.

The conservation equations for the two perfect fluids are (there arise the following two cases)

**Case (1):**

\[ \dot{\rho}_1 + (\rho_1 + p_1)\dot{\theta}_1 - (\rho_2 + p_2)\dot{u}_a v^a - \dot{p}_2 - (\rho_2 + p_2)u_a v^a - (\rho_2 + p_2)u_a v^a \tau_1 = 0, \] (30)

\[ (\rho_1 + p_1)\dot{v}^a + (\rho_2 + p_2)(\dot{v}^c + u^a v_a u_c) + (\rho_2 + p_2)(\dot{v}^c + v^f u_f u^c + u_a v^a \tau_1 u_c) \]
\[ + (\rho_2 + p_2)\tau_2 v^c + \dot{h}_1^f (p_1 + p_2) = 0. \] (31)
Project the second equation along $x^c$ and with $H^a_{1c}$, we get the two equations

$$p_1^a + p_2^a + (\rho_1 + p_1)x_f u_f^j + (\rho_2 + p_2) \times x_f \hat{v}^f = 0,$$

$$H^a_{1c}[\rho_1(\rho_1 + p_1)\hat{u}_c + ((\rho_2 + p_2)v + (\rho_2 + p_2)\tau_2)v_c + (\rho_2 + p_2)\hat{u}_c + h^a_{1c}(\rho_1 + p_2)\tau_2] = 0. \quad (33)$$

**Case (ii):**

$$\rho_2 + (\rho_2 + p_2)\theta_2 - (\rho_1 + p_1)v^a u_a - (\rho_1 + p_1)u_a v^a - \rho_1 + p_1)u_a v^a \tau_2 = \hat{p}_1 = 0,$$

$$(\rho_2 + p_2)v^c + (\rho_1 + p_1)(u^c + u^a v_a v_c) + (\rho_1 + p_1)(\hat{u}^c + \hat{u}^f v_f v^c + u_a v^a \tau_2 v_c)$$

$$+ (\rho_1 + p_1)\tau_1 u^c + h^a_{2f}(\rho_1 + p_2)\tau_2 = 0. \quad (35)$$

Project the second equation along $y^c$ and with $H^a_{2c}$, the equations are

$$p_1^a + p_2^a + (\rho_2 + p_2)y_f \hat{v}^f + (\rho_1 + p_1)y_f \hat{v}^f = 0,$$

$$H^a_{2c}(\rho_2 + p_2)\hat{u}_c + ((\rho_1 + p_1)u_c + (\rho_1 + p_1)\hat{u}_c + (\rho_1 + p_1)\tau_1)u_c + (\rho_1 + p_1)\hat{u}_c + h^a_{2c}(\rho_1 + p_2)\tau_2] = 0. \quad (37)$$

### 4 Kinematic Conditions for the Two Perfect Fluids

Kinematics and dynamics in GR are clearly defined by considering the kinematical and dynamical variables, and the identities and the constraints they have to satisfy. Symmetries are an important form of constraints which restrict a physical system. These restrictions are expressed as relations among the parameters specifying the state of the system. Collineations restrict the system by the two levels, i.e., at the kinematical level (relations among the kinematical and geometric variables) and at the dynamical level (relations among the kinematical and the dynamical variables). Here we use kinematic restrictions coming from a general collineation, in particular, from a CRC. Since collineations are defined in terms of the Lie derivative of the metric tensor $g_{ab}$ and its derivatives, all types of collineations can be expressed by the quantity $L_{\xi}g_{ab}$. We define the decomposition as

$$L_{\xi}g_{ab} = 2\psi g_{ab} + 2P_{ab}, \quad (38)$$

where $\psi(x^a)$ is a function (the conformal factor) and $P_{ab}(x^a)$ is a symmetric traceless tensor. This implies that every collineation can be expressed in terms of $\psi(x^a)$, $P_{ab}(x^a)$ and their derivatives. For $P_{ab} = 0$, this gives a CKV; $\psi_{ab} = 0 = P_{ab,c}$ yields an affine collineation, etc. The kinematic conditions of a general collineation are relations among the kinematic quantities (shear, rotation, expansion) of the vector field involved and the parameters $\psi(x^a)$, $P_{ab}(x^a)$.

#### 4.1 Timelike Collineation

There arise the following two cases:

(i) $\xi^a = \xi u^a$,  
(ii) $\xi^a = \xi v^a$, ($\xi \neq 0$).

**Case (i):** For this case, Eq. (38) can be written in the form

$$\xi_{ab} + \xi_{kb} = 2\psi g_{ab} + 2P_{ab}, \quad (39)$$

where

$$\psi = \frac{\xi}{4}[\ln(\xi) + \theta_1],$$

$$P_{ab} = \xi[\sigma_{1ab} + \frac{1}{3}\theta_1 h_{1ab} - \hat{u}_{(a} u_{b)} + (\ln(\xi))_{(a} u_{b)} - \frac{1}{4}(\ln(\xi) + \theta_1)g_{ab}]. \quad (41)$$

We $1 + 3$ decompose $P_{ab}$ w.r.t. the timelike vector $u^a$ in the following

$$P_{ab} = \frac{\xi}{4}[\theta_1 - 3(\ln(\xi))]u_a u_b - \xi[u_c - h^c_{1e}(\ln(\xi),d) \times h^e_{1c}(u_b) + \frac{1}{12}\xi[\theta_1 - 3(\ln(\xi))]h_{1ab} + \xi\sigma_{1ab}. \quad (42)$$
If we take
\[ \mu_1 P = \frac{\xi}{4}[\theta_1 - 3(\ln \xi)], \]
\[ L_1 P = \frac{\xi}{12}[\theta_1 - 3(\ln \xi)], \]
\[ \Upsilon_1 P = \xi[\dot{u}_c - h^d_1c(\ln \xi),d], \]
\[ M_{P_{1ab}} = \sigma_{1ab} \]
then Eq. (42) can take the following form
\[ P_{ab} = \mu_1 P u_a u_b + L_1 P h_{1ab} - 2\Upsilon_1 P (a u_b) + \xi M_{P_{1ab}}. \]

We express these equations as conditions among the kinematic variables of the timelike congruence \( u^a \), which gives a system of equations called the kinematic conditions of the collineations.

**Case (ii):** In this case, we decompose the traceless tensor \( P_{ab} \) w.r.t. the timelike vector \( v^a \). Using the same procedure as above, it follows that
\[ P_{ab} = \mu_2 P v_a v_b + L_2 P h_{2ab} - 2\Upsilon_2 P (a v_b) + \xi M_{P_{2ab}}, \]
where
\[ \mu_2 P = \frac{\xi}{4}[\theta_2 - 3(\ln \xi)], \]
\[ L_2 P = \frac{\xi}{12}[\theta_2 - 3(\ln \xi)], \]
\[ \Upsilon_2 P = \xi[\dot{v}_c - h^d_2c(\ln \xi),d], \]
\[ M_{P_{2ab}} = \sigma_{2ab}. \]

The kinematic conditions for a CRC involve the second derivatives of the quantities \( \psi, P_{ab} \). We evaluate these conditions by taking a general vector field \( \xi^a \) with the identities
\[ \mathcal{L}_\xi R_{ab} = (\mathcal{L}_\xi \Gamma^c_{ab}),c - (\mathcal{L}_\xi \Gamma^c_{ac}),b, \]
\[ \mathcal{L}_\xi \Gamma^c_{bc} = \frac{1}{2} g^{ad}(\mathcal{L}_\xi g_{bd}) + (\mathcal{L}_\xi g_{cd}) - (\mathcal{L}_\xi g_{bc})d. \]

Using Eqs. (53), (54), the following general result for any vector \( \xi^a \) can be verified.

**Proposition 1:** A fluid spacetime \( u^a \) admits a CRC \( \xi^a \) if and only if
\[ \Delta \psi = \frac{1}{3}(P_{ab} - aR), \]
\[ \Delta P_{ab} = 2K_{ab} - 2A_{ab} - 2Z_{ab}, \]
where \( R \) is the Ricci scalar and
\[ K_{ab} = P_{(a},b)c - \frac{1}{4} g_{ab} P_{cd}, \quad A_{ab} = \psi_{;ab} - \frac{1}{4} g_{ab} \Delta \psi, \]
\[ Z_{ab} = R_{ab} - \frac{1}{4} g_{ab} R, \]
which is a geometric result. The kinematic conditions can be obtained by replacing \( \psi, P_{ab} \) from Eqs. (40), (41) in terms of the kinematic variables. The resulting expressions will become very tedious and hence will not be given here. These are the constraints satisfied by any solution.
4.2 Spacelike Collineation

The kinematic restrictions in this case involve all the nine quantities, i.e.,

\[
\begin{align*}
\sigma_{1ab}, & \quad \omega_{1ab}, \quad \theta_1, \quad \dot{u}_a, \quad S_{1ab}, \quad R_{1ab}, \quad \epsilon_1, \quad \dot{x}_a, \quad u^a_0, \\
\sigma_{2ab}, & \quad \omega_{2ab}, \quad \theta_2, \quad \dot{v}_a, \quad S_{2ab}, \quad R_{2ab}, \quad \epsilon_2, \quad \dot{y}_a, \quad v^a_0,
\end{align*}
\]

plus the parameters \(\psi, P_{ab}\) and their derivatives. Again we have two cases according to

(i) \(\xi^a = \xi x^a\),  
(ii) \(\xi^a = \xi y^a\).

**Case (i):** We make the \(1 + 1 + 2\) decomposition of \(P_{ab}\) by considering Eq. (38) and contract with 

\[
\begin{align*}
u^a u^b, & \quad u^a x^b, \quad x^a x^b, \quad H_{1c}^b u^a, \quad H_{1c}^b x^a, \quad H_{1ab}, \\
H_{1c}^a H_{1d}^b = & \quad \frac{1}{2} H_{1b}^a H_{1cd},
\end{align*}
\]

it turns out that

\[
\begin{align*}
\psi = & \quad \frac{\xi}{4}[\epsilon_1 + (\ln \xi)^* - \dot{x}^c u_c], \\
\lambda_{1P} = & \quad P_{ab} u^a u^b = \frac{\xi}{4}[\epsilon_1 + (\ln \xi)^* + 3\dot{x}^c u_c], \\
2K_{1P} = & \quad -2P_{ab} u^a u^b = -\xi[(\ln \xi) + x^c u_c], \\
\gamma_{1P} = & \quad P_{ab} x^a x^b = -\frac{\xi}{4}[\epsilon_1 - 3(\ln \xi)^* - \dot{x}^c u_c], \\
2S_{1Pc} = & \quad -2P_{ab} H_{1c}^b u^a = -\xi(N_{1c} - 2w_{1c} x^c), \\
2\theta_{1Pc} = & \quad 2P_{ab} H_{1d}^a u^b = \xi(H_{1c}^b (\ln \xi)_b + x^c), \\
a_{1P} = & \quad P_{ab} H_{1}^a = \frac{\xi}{2}[\epsilon_1 - (\ln \xi)^* + \dot{x}^c u_c], \\
D_{1Pab} = & \quad \xi S_{1ab} = (H_{1c}^a H_{1d}^b - \frac{1}{2} H_{1b}^a H_{1cd}) P_{ab}.
\end{align*}
\]

Thus the \(1 + 1 + 2\) decomposition of \(P_{ab}\) is given by

\[
P_{ab} = \lambda_{1P} u_a u_b + 2K_{1P} u_a x_b + 2S_{1P} u_a u_b + \gamma_{1P} x_a x_b + 2\theta_{1P} x_a x_b + \frac{1}{2} a_{1P} H_{1ab} + D_{1Pab}.
\]

The property \(P_{ab} = 0\) implies that

\[
a_{1P} - \lambda_{1P} + \gamma_{1P} = 0.
\]

**Case (ii):** The \(1 + 1 + 2\) decomposition in this case can be obtained by contracting Eq. (38) with 

\[
\begin{align*}
v^a v^b, & \quad v^a y^b, \quad y^a y^b, \quad H_{2c}^b v^a, \quad H_{2c}^b y^a, \\
H_{2a}^b H_{2d}^b = & \quad \frac{1}{2} H_{2b}^a H_{2cd}.
\end{align*}
\]

It follows that

\[
\begin{align*}
\psi = & \quad \frac{\xi}{4}[\epsilon_2 + (\ln \xi)^* - \dot{y}^c v_c], \\
\lambda_{2P} = & \quad P_{ab} v^a v^b = \frac{\xi}{4}[\epsilon_2 + (\ln \xi)^* + 3\dot{y}^c v_c], \\
2K_{2P} = & \quad 2P_{ab} v^a v^b = \xi[(\ln \xi) + y^c v_c], \\
\gamma_{2P} = & \quad P_{ab} y^a y^b = -\frac{\xi}{4}[\epsilon_2 - 3(\ln \xi)^* - \dot{y}^c v_c], \\
2S_{2Pc} = & \quad -2P_{ab} H_{2c}^b v^a = -\xi(N_{2c} - 2w_{2c} y^c), \\
2\theta_{2Pc} = & \quad 2P_{ab} H_{2d}^a v^b = \xi(H_{2c}^b (\ln \xi)_b + y^c), \\
a_{2P} = & \quad P_{ab} H_{2}^a = \frac{\xi}{2}[\epsilon_2 - (\ln \xi)^* + \dot{y}^c v_c], \\
D_{2Pab} = & \quad \xi S_{2ab} = (H_{2c}^a H_{2d}^b - \frac{1}{2} H_{2b}^a H_{2cd}) P_{ab}.
\end{align*}
\]
The $1 + 1 + 2$ decomposition of $P_{ab}$ turns out to be

$$P_{ab} = \lambda_{2P} v_a v_b + 2K_{2P} v_{(a} v_{b)} + 2S_{2P(a} v_{b)} + \gamma_{2P} y_a y_b + 2\theta P_{2(a} y_{b)} + \frac{1}{2} a_{2P} H_{ab} + D_{2P}.$$

(76)

From $P^a_a = 0$, we have

$$a_{2P} - \lambda_{2P} + \gamma_{2P} = 0. \tag{77}$$

## 5 Conformal Ricci Collineations for the Two Perfect Fluids

In this section we shall discuss the existence of timelike and spacelike CRCs for the two perfect fluids.

### 5.1 Timelike Conformal Ricci Collineations

We shall give the necessary and sufficient conditions for the timelike CRCs of the two perfect fluids for the following two cases:

(i) $\xi^a = \xi u^a$, (ii) $\xi^a = \xi v^a$.

From Eq. (26), we have the $1 + 3$ decomposition of the Ricci tensor and hence CRC gives the condition

$$\mathcal{L}_\xi R_{ab} = \alpha \left( \frac{1}{2}(\rho_1 + 3\rho_1 - \lambda)u_au_b + \frac{1}{2}(\rho_2 + 3\rho_2 - \lambda)v_av_b + \frac{1}{2}(\rho_1 - \rho_1 + \lambda)h_{ab} ight)$$

$$+ \frac{1}{3}(\rho_2 - \rho_2 + \lambda)\ell_{2ab}, \tag{78}$$

and the corresponding $1 + 3$ expression of $\mathcal{L}_\xi R_{ab}$ is given in Eq. (28). Equating these two expressions and after some calculation we find the following results.

**Proposition 2:** A two perfect fluid spacetime admits a CRC $\xi^a = \xi u^a$ if and only if

$$\dot{\rho}_1 = \dot{\rho}_2 + (\rho_2 + \rho_2)(\dot{v}^a u_a + u_a v^a \tau_2) - (\rho_1 + \rho_1)\dot{\theta}_1 + (\rho_2 + \rho_2)u_a u^a,$$

$$\dot{\rho}_1 = \frac{3}{2}\dot{\rho}_2 - \frac{3}{2}\dot{\rho}_2 + (\rho_2 + \rho_2)(\dot{v}^a u_a + v_a u^a \tau_2) + (\rho_2 + \rho_2)u_a v_a + \frac{1}{3}(\rho_2 - \rho_2 - 5\rho_1)$$

$$-2\rho_2 + 4\Lambda \dot{\theta}_1 + \frac{2}{3}(\rho_2 + \rho_2)u_a u_b + \frac{4}{3}\rho_2 \times (\dot{v}^a u^b u^a + u^a v_a \ln \xi + u^a v^b (\ln \xi) + v_a \dot{u}^a + v_a \dot{v}^a v^b) - \frac{3}{3}(\rho_1 + \rho_1 - 3\rho_1 - 5\rho_2 + 6\Lambda + 2(\rho_2 + \rho_2)u^a v^a b^b), \tag{79}$$

$$\rho_1 - \rho_2 + 3\rho_1 + \rho_2 + 2\Lambda)(\dot{u}_a - (\ln \xi)_a - (\ln \xi)_a + 2(\rho_2 + \rho_2)(u^a v_a + v^a u^a u^a u^a)$$

$$+ (\rho_2 + \rho_2)(\dot{v}^a u_a + v^a u^a u^a u^a + v_a u^a u_a + u_a v^a u^a u^a + \dot{v}_a u^a u^a u^a + v_a u^a u^a u^a u^a + v_a u^a u^a u^a u^a) = \beta(\rho_2 + \rho_2)(v^a u^a u_a + v^a u^a u^a u^a + v^a u^a u^a u^a), \tag{80}$$

$$\rho_2 - \rho_1 - 3\rho_1 + \rho_2 + 2\Lambda)(\dot{\theta}_1 = (\ln \xi)_a) + 2(\rho_2 + \rho_2)(\dot{v}_a u^a v^a + (\rho_2 + \rho_2)(\dot{v}^a u^a + v^a u^a u^a u^a)$$

$$+ v_a \dot{u}^a + v_a \dot{v}^a (\ln \xi) + 2(\rho_2 + \rho_2)(\dot{v}_a u^a v^a + v^a u^a u^a u^a), \tag{81}$$

$$+ 2\beta(\rho_2 + \rho_2)(v^a u^a u_a + v^a u^a u^a u^a + v^a u^a u^a u^a) = \beta(\rho_2 + \rho_2 + \Lambda + \rho_2 + \rho_2)u^a v^a b^b), \tag{82}$$

$$\rho_2 + 2\Lambda)(\dot{v}_a v_b + 2v^a u^a (u_b v_b) + v^a u^a u^a u^a u^a u_b + \frac{1}{3} h_{1ab} + \frac{1}{3} h_{1ab} v^a u^a u^a u^a + (\rho_2 + \rho_2 - \rho_1)$$

$$= - \rho_2 + 2\Lambda)(\dot{\theta}_1 = (\ln \xi)_a) + 2(\rho_2 + \rho_2)(\dot{v}_a v_b + v^a u^a u^a u^a u^a u_b + \frac{1}{3} h_{1ab} + \frac{1}{3} h_{1ab} v^a u^a u^a u^a u^a u_b$$

$$+ v^a u^a u^a u^a u^a u_b + v^a u^a u^a u^a u^a u_b + v^a u^a u^a u^a u^a u_b + \frac{1}{3} h_{1ab} + \frac{1}{3} h_{1ab} v^a u^a u^a u^a u^a u_b$$

$$+ u^a v^a (\ln \xi)_a = \beta(\rho_2 + \rho_2)(v^a v^a + 2v^a u^a v^a + v^a u^a u^a u^a), \tag{83}$$

$$+ \frac{1}{3} h_{1ab} = \frac{1}{3} h_{1ab} v^a u^a u^a u^a). \tag{84}$$
If we take \( \alpha = 0 \) in the above relations we find the results for a RC. Using 1 + 3 decomposition of the tensor \( P_{ab} \), given in Eqs. (85-88), the above system of equations can be written as follows:

\[
\begin{align*}
\dot{\rho}_1 = \dot{\rho}_2 + (\rho_1 + p_2)(\hat{u}^a v_a + u_a u^a) - (\rho_1 + \rho_2)\theta_1 + (\rho_1 + \rho_2)\hat{u}^a v_a, \\
\dot{\rho}_1 = \frac{1}{3} \rho_2 - \frac{2}{3} \rho_2 + (\rho_1 + p_2)(\hat{u}^a v_a + u_a u^a) - (\rho_2 + p_2)\theta_1 + (\rho_2 + p_2)\hat{u}^a v_a + \frac{1}{3} (2\rho_2 - \rho_1 - 5p_2 - 2p_2) + 2\Lambda \theta_1 + u_a u^a v^ib + 4 \frac{1}{3} (\rho_2 - \rho_1 - 3p_2 - 2p_2 - 2\Lambda) \theta_1 + (\rho_2 + p_2)u^a v^ib + u^a v^ib (\ln \dot{\xi}) + u^a v^ib (\frac{1}{3} \theta_1)
\end{align*}
\]

\[
\begin{align*}
\frac{4}{3 \xi} \hat{P}^{(1)}_{1P} + u_a \hat{u}^a + v_a \hat{u}^a v^ib - \frac{\beta}{3} (3\rho_1 + \rho_2 - 3p_1 - 5p_2 + 6\Lambda + 2(\rho_2 + p_2)u_a u^a v^ib),
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\xi} (p_1 - \rho_2 + 3p_1 + p_2 - 2\Lambda) \Pi_{1Pa} + 2(\rho_2 + \rho_2) \times (u_a u^a v^ib + u^a v^ib u^a u^a + 2(\rho_2 + p_2)u_a u^a v^ib + v_a u_a v^ib u^a),
\end{align*}
\]

\[
\begin{align*}
2(\rho_2 + \rho_2)u_a u^a v^ib + v_a u_a v^ib u^a + v_a u_a v^ib = \beta(\rho_2 + p_2)(v_a u_a v^ib + v_a u_a v^ib u^a),
\end{align*}
\]

\[
\begin{align*}
\frac{1}{3} (\rho_1 - p_2 - 3p_1 - p_2 + 2\Lambda) (\theta_1 + \frac{4}{3 \xi} \hat{P} (1P)) + (\rho_2 + p_2) \times (u_a u^a v^ib + v_a u_a v^ib + v_a u_a v^ib u^a),
\end{align*}
\]

\[
\begin{align*}
\frac{1}{3} (\rho_1 - p_2 - 3p_1 - p_2 + 2\Lambda) (\theta_1 + \frac{4}{3 \xi} \hat{P} (1P)) + (\rho_2 + p_2) \times (u_a u^a v^ib + v_a u_a v^ib + v_a u_a v^ib u^a),
\end{align*}
\]

\[
\begin{align*}
\frac{1}{3} \rho_1 - 5p_2 - 5p_2 - 2\Lambda) (\theta_1 + \frac{4}{3 \xi} \hat{P} (1P)) + (\rho_2 + p_2) \times (u_a u^a v^ib + v_a u_a v^ib + v_a u_a v^ib u^a),
\end{align*}
\]

These are the necessary and sufficient conditions for a CRC in the form of kinematical quantities. 

**Proposition 3:** A two perfect fluid spacetime admits a CRC \( \xi^a = \xi v^a \) if and only if

\[
\begin{align*}
\dot{\rho}_2 = \dot{\rho}_1 + (\rho_1 + p_1)(\hat{u}^a v_a + u_a u^a) - (\rho_2 + p_2)\theta_2 + (\rho_1 + p_1)\hat{u}^a v_a, \\
\dot{\rho}_2 = \frac{1}{3} \rho_1 - \frac{2}{3} \rho_1 + (\rho_1 + p_1) \times (u_a u^a v^ib + v_a u_a v^ib),
\end{align*}
\]

\[
\begin{align*}
-2p_1 - 5p_2 + 4\Lambda) \theta_2 + \frac{1}{3} (p_1 + p_1) (u_a u^a v^ib + 4 \frac{1}{3} (\rho_1 + p_1) (u_a u^a v^ib + u_a u^a) (\ln \dot{\xi}) + u_a u^a v^ib + u_a u^a + u_a u^a v^ib - \frac{\beta}{3} (\rho_1 + 3p_2 - 5p_1 - 3p_2 + 6\Lambda + 2(\rho_2 + p_2) v_a u_a v^ib),
\end{align*}
\]

\[
\begin{align*}
(\rho_2 - p_1 + p_1 + 3p_2 - 2\Lambda) (\theta_2 - (\ln \dot{\xi})), a - (\ln \dot{\xi})u_a + 2(\rho_1 + p_1) (u_a u^a v^ib + u_a u^a) (\ln \dot{\xi}) + u_a u^a v^ib + u_a u^a + u_a u^a v^ib - \frac{\beta}{3} (\rho_1 + 3p_2 - 5p_1 - 3p_2 + 6\Lambda + 2(\rho_2 + p_2) v_a u_a v^ib),
\end{align*}
\]

\[
\begin{align*}
(\rho_2 - p_1 + p_1 + 3p_2 - 2\Lambda) (\theta_2 - (\ln \dot{\xi})), a - (\ln \dot{\xi})u_a + 2(\rho_1 + p_1) (u_a u^a v^ib + u_a u^a) (\ln \dot{\xi}) + u_a u^a v^ib + u_a u^a + u_a u^a v^ib - \frac{\beta}{3} (\rho_1 + 3p_2 - 5p_1 - 3p_2 + 6\Lambda + 2(\rho_2 + p_2) v_a u_a v^ib),
\end{align*}
\]

\[
\begin{align*}
(\rho_2 - p_1 + p_1 + 3p_2 - 2\Lambda) (\theta_2 - (\ln \dot{\xi})), a - (\ln \dot{\xi})u_a + 2(\rho_1 + p_1) (u_a u^a v^ib + u_a u^a) (\ln \dot{\xi}) + u_a u^a v^ib + u_a u^a + u_a u^a v^ib - \frac{\beta}{3} (\rho_1 + 3p_2 - 5p_1 - 3p_2 + 6\Lambda + 2(\rho_2 + p_2) v_a u_a v^ib),
\end{align*}
\]
Using the kinematical conditions for the timelike case, i.e., Eqs. (49)–(52) in Eqs. (89)–(92), we can obtain an expression of the above equations in terms of $P_{ab}$.

5.2 Spacelike Conformal Ricci Collineations

In this case we use the $1 + 1 + 2$ decomposition of the Ricci tensor given in Eq. (20) and the corresponding $1 + 1 + 2$ expression of $\mathcal{L}_\xi R_{ab}$ given in Eq. (29). Equating these two expressions we find the following results corresponding to the following two cases:

(i) $\xi^a = \xi x^a$,  
(ii) $\xi^a = \xi y^a$.

**Proposition 4:** A two perfect fluid spacetime admits CRC $\xi^a = \xi x^a$ if and only if

\[
2\rho_1^* - \rho_2^* + (p_1 - p_2 + 3p_1 + p_2 + 2\Lambda)\rho_2^* x_a = 3(\rho_1 + \rho_2 - p_1 - p_2 + 2\Lambda)(\ln x)^* \\
+ (\rho_2 + p_2)^* u_a u^a v_b v_b + 2(p_2 + p_2)(v_a^* u^a v_a v_b) - x_a v^a v_b v_b = \beta(2p_1 + p_2 - p_2 + 2\Lambda) \\
+ (\rho_1 + \rho_2 - p_1 - p_2 + 2\Lambda)[x^a + (\ln x)_a - (\ln x)^* x_a] = 0, \\
(\rho_1 + \rho_2 - p_1 - p_2 + 2\Lambda)(2u^a x_a + \epsilon_1) + p_2^* + p_2^* + 2(p_2 + p_2)v^a x_a = (\rho_2 + p_2)^* v^a u^a v_b v_b \\
- 2(p_2 + p_2)^* v^a u^a v_b v_b + 2(p_2 + p_2)x_a v^a v_b = \beta(2p_1 - 4p_1 - 3p_2 + 4\Lambda - p_2 + p_2)^* v^a u^a v_b v_b, \\
(\rho_2 + p_2)^* v^a u^a v_b v_b + 2(p_2 + p_2)(c^a u^a v_a v_b) = \beta(c^a u^a v_a v_b), \\
+ 2(p_1 - p_2 + 3p_1 + p_2 - 2\Lambda)\omega_{1+1} x_b f + (p_1 - p_2 - p_2 + 2\Lambda)\Lambda_1 + (\rho_2 + p_2)^* (u^a v_a v_a + v^a u^a v_a) \\
- (p_2 + p_2)^* (v_a^* u^a v_b v_b - 2(p_2 + p_2)^* u^a v_a v_b) = \beta(p_2 + p_2 - p_2), \\
(\rho_2 + p_2)^* (v_a v_b + 2v^a u^a v_a v_b + v^a v^a v^a v_b - \frac{1}{2} H_{1ab} - \frac{1}{2} H_{1ab} v^a v_b d^a d^b) + 2(p_1 + p_2 - p_1 - p_2 + 2\Lambda) S_{1ab} + 2(p_2 + p_2) (v^a v_b) \\
- v^a v^a v_b v_b - 2(p_2 + p_2)^* v^a v^a v_b v_b) = \beta(p_2 + p_2 - p_2) \\
= \beta(p_2 + p_2)^* (v_a v_b + 2v^a u^a v_a v_b + v^a v^a v_b v_b - \frac{1}{2} H_{1ab} - \frac{1}{2} H_{1ab} v^a v_b d^a d^b), \\
(\rho_1 + p_1)\beta_1 - (p_2 + p_2)(v^a u^a + v^a u^a u^a) - (p_2 + p_2)^* v_a u^a - \rho_2 = 0, \\
p_1^2 + p_2^2 + (p_1 + p_1)^2 \xi^2 + (p_2 + p_2)^2 = 0, \\
(\rho_1 + p_1)^2 (p_1 + p_1)^2 \xi^2 + (p_2 + p_2)^2 = 0, \\
H_{1ab} [(p_1 + p_2)^2 (p_1 + p_2)^2) v_a + (p_2 + p_2)^2 = 0]. \\
\]
Proposition 5: A two perfect fluid spacetime admits a CRC $\xi^a = \xi y^a$ if and only if
\begin{align*}
2p^2 - p_1^2 + p_2^2 + (p_2 - p_1 + p_1 + 3p_2 + 2\Lambda)(\Delta y^c y_c + 3(\rho + p_2 - p_1 - p_2 + 2\Lambda)(\Delta y^c y_c) + (p_1 + p_1)u^a v^a u^b v^b + 2(p_1 + p_1)(u^a v^a u^b v^b - y_a u^a y_b) &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\rho_1 + p_2 - p_1 - p_2 + 2\Lambda)(\Delta y^c y_c) - (\Delta y^c y_c) y_a = 0, \\
(\rho_1 + p_2 - p_1 - p_2 + 2\Lambda)\Delta y^c y_c + p_1^2 + p_1^2 + 2(p_1 + p_1)u^a v^a u^b v^b - 2(p_1 + p_1)^2 u^a v^a u^b v^b + 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 - 4p_2 - 3p_1 + 4\Lambda - (p_1 + p_1)u^a v^a u^b v^b), \\
(\rho_1 + p_1)(u^a v^a u^b v^b + 2u^a v^b v^b u^a v^a + u^a v^a u^a) = (p_1 + p_1)(y_a u^a + 2y_a u^a v^a v^a + 3u^a v^a y_a), \\
+2(p_1 - p_1 + p_1 + 3p_2 - 2\Lambda)\omega_{2a} y^f y^f + (p_1 + p_2 - p_1 - p_2 + 2\Lambda)\Delta y^c y_c + (p_1 + p_1)^2 (u^a v^a u^b), \\
+u^a v^b v^c u^d v^e v^f = \beta(p_1 + p_1)(y_a u^a + v^a u^b v^a u^b), \\
(\rho_1 + p_1)u^a v^a u^b v^b - 2(p_1 + p_1)u^a v^a u^b v^b + 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
(\rho_1 + p_1)u^a v^a u^b v^b + 2u^a v^b v^b u^a v^a + \frac{1}{2} H_{2ab} - \frac{1}{2} H_{2ab} u^a v^a u^b v^b &= 2(p_1 + p_2 - p_1 - p_2 + 2\Lambda)\Delta y^c y_c + 2(p_1 + p_1)u^a v^a u^b v^b + 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
2v_a^a v^a v^b v^e v^f v^b - 2v_a^a v^a v^b v^a v^b v^b - 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
(\rho_1 + p_1)u^a v^a u^b v^b + 2u^a v^b v^b u^a v^a + \frac{1}{2} H_{2ab} - \frac{1}{2} H_{2ab} u^a v^a u^b v^b &= 2(p_1 + p_2 - p_1 - p_2 + 2\Lambda)\Delta y^c y_c + 2(p_1 + p_1)u^a v^a u^b v^b + 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
\rho_2 + (p_2 + p_2)\theta - (p_1 + p_1)(u^a v^a u^b v^b - (p_1 + p_1)u^a v^a u^b v^b - \rho_1 = 0, \\
p_2^2 + p_1^2 + (p_2 + p_2)\theta - (p_1 + p_1)(u^a v^a u^b v^b - (p_1 + p_1)u^a v^a u^b v^b - \rho_1 = 0, \\
H_{2ab}[(\rho_1 + p_2 + p_2)\theta + (p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
\rho_2 + (p_2 + p_2)\theta - (p_1 + p_1)(u^a v^a u^b v^b - (p_1 + p_1)u^a v^a u^b v^b - \rho_1 = 0, \\
H_{2ab}[(\rho_1 + p_2 + p_2)\theta + (p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 + p_1 - (p_1 + p_1)u^a v^a u^b v^b), \\
\rho_2 - (p_2 - p_1)\theta - (p_1 + p_1)u^a v^a u^b v^b - (p_1 + p_1)u^a v^a u^b v^b - \rho_1 = 0, \\
\rho_2 - (p_2 - p_1)\theta - (p_1 + p_1)u^a v^a u^b v^b - (p_1 + p_1)u^a v^a u^b v^b - \rho_1 = 0,
\end{align*}

We can use the $1 + 1 + 2$ decomposition of the tensor $P_{ab}$ given in Eqs. (68–72) to write these equations in terms of the irreducible parts of $P_{ab}$.

6 Conformal Matter Collineations

6.1 Timelike Conformal Matter Collineations

Here we give the necessary and sufficient conditions for the timelike CMCs of the two perfect fluids for the following cases:

(i) $\xi^a = \xi u^a$, (ii) $\xi^a = \xi v^a$.

Proposition 6: A two perfect fluid admits a CMC $\xi^a = \xi u^a$ if and only if
\begin{align*}
(p_2 - p_1)[\rho_1 - (\Delta y^c y_c - u_a (\Delta y^c y_c)) + (p_2 + p_2)(u^a v^a u^b v^b + 2u^a v^a u^b v^b + 2u^a v^a u^b v^b) &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^e v^f v^b - 2v_a^a v^a v^b v^a v^b v^b + 2(p_1 + p_1)^2 u^a v^a u^b v^b + 2(p_1 + p_1)u^a v^a u^b v^b &= \beta(p_1 - 4p_2 - 3p_1 + 4\Lambda - (p_1 + p_1)u^a v^a u^b v^b), \\
+2v_a^a v^a v^b v^b + u^a v^a u^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b + u^a v^a u^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b + u^a v^a u^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0, \\
+2v_a^a v^a v^b v^b &= \beta(p_1 + 2p_2 - p_1 + 2\Lambda) + (\Delta y^c y_c) y_a = 0.
\end{align*}
\[ (\rho_2 + p_2) \hat{\vartheta}_a - (\rho_1 + p_1) \dot{\vartheta}_a - \dot{p}_2 - \rho_2 \dot{\vartheta}_a - (\rho_2 + p_2) \dot{v}^a u_a - (\rho_2 + p_2) u_\alpha \dot{v}^\alpha - \rho_2 = 0, \]

(117)

We take \( \alpha = 0 \) in the above relations we get result for the matter collineation. We can use the \( 1+3 \) decomposition of the tensor \( P_{ab} \) given in Eq. (119) to write these equations in terms of irreducible parts of \( P_{ab} \).

**Proposition 7:** A two perfect fluid solution admits a CMC \( \xi^a = \xi v^a \) if and only if

\[ \hat{\rho}_2 + (p_2 + p_2) \dot{\theta}_2 - (\rho_1 + p_1) \dot{v}^a \dot{u}_a - \dot{p}_1 - (\rho_1 + p_1) \dot{v}^a u_a - (\rho_1 + p_1) u_\alpha \dot{v}^\alpha \dot{\tau}_2 = 0, \]

(118)

\[ + v_\alpha u^\alpha v_d u^a \dot{v}_b + 2(\rho_1 + p_1) u^a \dot{v}_b + \frac{2}{3}(p_1 + p_1) \dot{v}_a u^a u^b + \frac{2}{3}(p_1 + p_1) u_\alpha \dot{v}^\alpha u^b, \]

(119)

\[ + 2(\rho_2 + p_2) \hat{\xi} v^a u_a - (\rho_2 + p_2) \dot{v}^a u_a - (\rho_2 + p_2) \dot{v}^a u_a - \rho_2 = 0. \]

(120)

\[ + 2(\rho_1 + p_1) \hat{\xi} v^a u_a - (\rho_1 + p_1) \dot{v}^a u_a - (\rho_1 + p_1) \dot{v}^a u_a - \rho_1 \dot{\tau}_2 = 0, \]

(121)

\[ + v_\alpha u^\alpha v_d u^a \dot{v}_b + 2(\rho_1 + p_1) u^a \dot{v}_b + \frac{2}{3}(p_1 + p_1) \dot{v}_a u^a u^b + \frac{2}{3}(p_1 + p_1) u_\alpha \dot{v}^\alpha u^b, \]

We can write these equations by using the \( 1+3 \) decomposition of the tensor \( P_{ab} \) given in Eqs. (119)–(121).

### 6.2 Spacelike Conformal Matter Collineations

The necessary and sufficient conditions for the timelike CMCs of the two perfect fluids are given for the following two cases:

(i) \( \xi^a = \xi x^a \),

(ii) \( \xi^a = \xi y^a \).

**Proposition 8:** A two perfect fluid spacetime admits the CMC \( \xi^a = \xi x^a \) iff

\[ (\rho_1 - p_1)^a + (\rho_2 + p_2)^a u^a v_b u^b + 2(\rho_2 + p_2) v^a u^b v_b + 2(\rho_1 - p_1) \dot{u}^a x_a - (\rho_2 + p_2) \dot{v}^a v_b v_b = \beta[(\rho_1 - p_2) + (\rho_2 + p_2) u^a v_b v_b], \]

(122)
(p_1 + p_2)[x^*_a + (\ln \xi)_a - (\ln \xi)^*_a] = 0,  
(123)
(p_2 - p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b 
= \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],  
(131)
(p_1 + p_2)_a = \beta(p_2 + p_2)\hat{v}^a y_a v_b u^b,  
(132)
(p_1 + p_1)^* - 2(p_1 + p_2)\hat{v}^a y_a + 4(p_1 + p_1)y_a \hat{u}^a - (p_1 + p_1)\hat{v}^a y_a v_b u^b 
- 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_1 + p_2)v_a u^a v_b u^b y_a = \beta(p_1 - p_1 - 2p_2 - (p_1 + p_1)v_a u^a v_b u^b),  
(133)
(p_1 + p_1, y_a, u^a, v_b u^b) = \beta(p_1 + p_1)(u^a v_a u_a + u^a v_a u^a v^a u^d v_a),  
(134)
(p_1 + p_1)^* + 4(p_1 + p_2)(\ln \xi)^* + 2(p_1 + p_1)y_a \hat{u}^a - (p_1 + p_1)\hat{v}^a y_a v_b u^b - 2(p_1 + p_2)\hat{v}^a y_a v_b u^b 
- 2(p_1 + p_1)v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b y_a = \beta(p_1 - p_1 - (p_1 + p_1)v_a u^a v_b u^b),  
(135)
(p_1 + p_1)^*(u_a u_b + 2v_a v^a v^d u^d v_a + v_a v^d v_a v_b u^d v_b v_a) + \frac{1}{2}H_{2ab} - \frac{1}{2}H_{2ab}v_c v^d v^d v_a u^b + 2(p_1 + p_2)S_{2ab} 
+ 2(p_1 + p_1)(v^a u_a) + v^a v^d u^d a b + v^a v^d v^a v^a v^a v^a v^d v_a + v^a v^d v^a v^a v^d v_a 
- y_a \hat{v}^a v^a u_a v^a u^b v_b + \frac{1}{2}H_{2ab}(y_a \hat{v}^a + y_a \hat{v}^a v^d v^d v_b) = \beta(p_1 + p_1)(u_a u_b + 2v_a v^a v^d u^d v_a) 
+ v_a v^d v^a u_a v^b + \frac{1}{2}H_{2ab} - \frac{1}{2}H_{2ab}v_c v^d v^d v_a u^b,  
(136)
\hat{v}^a u_a (1 + 1 + 2) decomposition of the CMC \(\xi^a = \xi^a y^a\) iff 
(p_2 - p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b + 2(p_1 + p_1)u^a v^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b 
= \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],  
(129)
(p_2 - p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b 
= \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],  
(130)
These equations can be written in terms of the irreducible parts of \(P_{ab}\) using the 1 + 1 + 2 decomposition of the tensor \(P_{ab}\).

**Proposition 9:** A perfect fluid spacetime admits the CMC \(\xi^a = \xi^a y^a\) iff

\[
(p_2 - p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b + 2(p_1 + p_1)u^a v^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b = \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],
\]

\[
(p_1 + p_1, y_a, u^a, v_b u^b) = \beta(p_1 + p_1)(u^a v_a u_a + u^a v_a u^a v^a u^d v_a),
\]

\[
(p_1 + p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b = \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],
\]

\[
(p_2 - p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b = \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],
\]

\[
(p_1 + p_1)^* + 4(p_1 + p_2)(\ln \xi)^* + 2(p_1 + p_1)y_a \hat{u}^a - (p_1 + p_1)\hat{v}^a y_a v_b u^b - 2(p_1 + p_2)\hat{v}^a y_a v_b u^b
\]

\[
(p_1 + p_1)^*(u_a u_b + 2v_a v^a v^d u^d v_a) + \frac{1}{2}H_{2ab} - \frac{1}{2}H_{2ab}v_c v^d v^d v_a u^b + 2(p_1 + p_2)S_{2ab}
\]

\[
(p_1 + p_1)^* + (p_1 + p_1)^* v_a u^a v_b u^b - 2(p_1 + p_1)v_a u^a v_b u^b + 2(p_2 - p_1)\hat{v}^a y_a - (p_1 + p_1)\hat{v}^a y_a v_b u^b = \beta[(p_2 - p_1) + (p_1 + p_1)v_a u^a v_b u^b],
\]

\[
(p_1 + p_1)^* + 4(p_1 + p_2)(\ln \xi)^* + 2(p_1 + p_1)y_a \hat{u}^a - (p_1 + p_1)\hat{v}^a y_a v_b u^b - 2(p_1 + p_2)\hat{v}^a y_a v_b u^b
\]

\[
(p_1 + p_1)^*(u_a u_b + 2v_a v^a v^d u^d v_a) = \beta(p_1 + p_1)(u_a u_b + 2v_a v^a v^d u^d v_a) + \frac{1}{2}H_{2ab} - \frac{1}{2}H_{2ab}v_c v^d v^d v_a u^b,
\]

We can use the 1 + 1 + 2 decomposition of the tensor \(P_{ab}\) given in Eqs. (128–125) to write these equations in terms of the irreducible parts of \(P_{ab}\).

### 7 Outlook

This section contains a summary and discussion of the results obtained. We have studied two types of collineations for the combination of the two perfect fluids. We have derived conditions for the existence of
CRCs and CMCs. In terms of the kinematic quantities of the vector fields, these conditions are used to calculate kinematical effects. Using the kinematic and the dynamic equations, we have obtained the set of equations under the symmetry assumption. For the physical system, the full set of equations is the set of these equations plus the Einstein field equations and any other equations for the extra fields or other geometrical identities. In the case of timelike CRCs and CMCs, we have found five conditions for the two perfect fluids. In the case of spacelike CRCs and CMCs, we have obtained nine conditions. In the following, we discuss some special cases of the fluid spacetimes.

1. When either \( \rho_1 = 0 = p_1 \) or \( \rho_2 = 0 = p_2 \), the conditions of timelike CRC become

\[
\begin{align*}
((\rho_1 + 3p_1 - 2\Lambda)\xi u^a)_{;a} &= -\beta(\rho_1 - 3p_1 + 4\Lambda), \\
(\rho_1 + 3p_1 - 2\Lambda)[\dot{u}_a - (\ln \xi)_{,a} - \theta_1 u_a] &= 2\beta(\rho_1 + \Lambda)u_a, \\
(\rho_1 - p_1 + 2\Lambda)\sigma_{1ab} &= 0.
\end{align*}
\]

(140) (141) (142)

These are the same conditions as for the timelike CRC perfect fluid [17].

2. If \( p_1 = 0 = p_2 \), this reduces to the case of two dusts.

3. If we substitute \( \rho_1 = p_1 \) and \( \rho_2 = p_2 \) in the two fluids, the respective conditions reduce to the conditions of stiff matter.

4. For \( \rho_1 = 3p_1, \rho_2 = 3p_2 \) we have the radiation case.

5. When \( \rho_1 = -\rho_1, \rho_2 = -p_2 \), it gives a dark energy component.

Similarly, we can classify spacelike CRCs.

If we take either \( \rho_1 = 0 = p_1 \) or \( \rho_2 = 0 = p_2 \), the conditions for the existence of timelike CMCs become

\[
\begin{align*}
\dot{\rho}_1 + (\rho_1 + p_1)\theta &= 0, \\
\rho_1(\dot{u}_a - (\ln \xi)_{,a} - u_a(\ln \xi)) &= 0, \\
3\dot{p}_1 + 2p_1\theta &= 3\beta p_1, \\
2p_1\sigma_{1ab} &= 0, \\
2\rho_1(\ln \xi) - p_1\theta &= \beta p_1.
\end{align*}
\]

(143) (144) (145) (146) (147)

These are the conditions for a perfect fluid already available in the literature [14]. Similarly, we can write down the conditions for the timelike and spacelike CMCs two dust, stiff matter, radiation, and dark energy component. It is interesting to note that if we replace \( \frac{1}{2}(\rho_1 + 3p_1 - \Lambda) \) by \( \rho_1, \frac{1}{2}(\rho_2 + 3p_2 - \Lambda) \) by \( \rho_2 \) and \( \frac{1}{2}(\rho_1 - p_1 + \Lambda) \) by \( p_1, \frac{1}{2}(\rho_2 - p_2 + \Lambda) \) by \( p_2 \) and vice versa, then we can obtain the conditions of CMCs from CRCs and vice versa. Also, it is important to note that the results obtained in the case (i) for the two perfect fluids (in terms of \( u_a \) and \( x_a \)) can be modified for the case (ii) by substituting \( v_a \) and \( y_a \) instead of \( u_a \) and \( x_a \). It is worth mentioning that when \( \alpha = 0 \), the conditions for the existence of CRCs and CMCs in all the cases of the two perfect fluids reduce to the conditions for the existence of RCs and MCs, respectively [14] [17].

References

[1] G. H. Katzin, J. Levine, and W. R. Davis, J. Math. Phys. 10, 617 (1969).
[2] K. L. Duggal, J. Math. Phys. 33, 2989 (1992).
[3] L. Herrera and J. Ponce de León, J. Math. Phys. 26, 778, 2018, 2647 (1985).
[4] K. L. Duggal and R. Sharma, J. Math. Phys. 27, 2511 (1986).
[5] D. P. Mason and R. Maartens, J. Math. Phys. 28, 2182 (1987).
[6] K. L. Duggal, Acta Appl. Math. 31, 225 (1993).

[7] I. Yavuz and I. Yilmaz, Gen. Rel. Grav. 29, 1295 (1997).

[8] I. Yilmaz, I. Tarhan, I. Yavuz, H. Baysal, and U. Camci, Int. J. Mod. Phys. D8, 659 (1999).

[9] I. Yilmaz, Int. J. Mod. Phys. D10, 681 (2001).

[10] H. Baysal, U. Camci, I. Yilmaz, I. Tarhan, and I. Yavuz, Int. J. Mod. Phys. D11, 463 (2002).

[11] D. P. Mason and M. Tsamparlis, J. Math. Phys. 26, 2881 (1985).

[12] E. Saridakis and M. Tsamparlis, J. Math. Phys. 32, 1541 (1991).

[13] M. Tsamparlis, Gen. Rel. Grav. 38, 279 (2006).

[14] M. Sharif and U. Sheikh, Int. J. Mod. Phys. A21, 3213 (2006).

[15] M. Sharif and N. Tehseen, Chinese J. Phys. 45, 592 (2007).

[16] H. Stephani, General Relativity: An Introduction to the Theory of the Gravitational Field (Cambridge University Press, 1990).

[17] K. L. Duggal and R. Sharma, Symmetries of Spacetimes and Riemann Manifolds (Kluwer Academic Publishers, 1999).

[18] P. S. Greenberg, J. Math. Anal. Appl. 30, 128 (1970).

[19] G. S. Hall and D. A. Negm, Int. J. Theor. Phys. 25, 405 (1986).