Convexity of effective Lagrangian in nonlinear electrodynamics as derived from causality

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Abstract

In nonlinear electrodynamics, by implementing the causality principle as the requirement that the group velocity of elementary excitations over a background field should not exceed unity, and the unitarity principle as the requirement that the residue of the propagator should be nonnegative, we find restrictions on the behavior of massive and massless dispersion curves and establish the convexity of the effective Lagrangian on the class of constant fields, also the positivity of all characteristic dielectric and magnetic permittivity constants. Violation of the general principles by the one-loop approximation in QED at exponentially large magnetic field is analyzed resulting in complex energy tachyons and super-luminal ghosts that signal the instability of the magnetized vacuum. General grounds for kinematical selection rules in the process of photon splitting/merging are discussed.

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I. INTRODUCTION

The effective action that is defined as the Legendre transform of the generating functional of the Green functions [1] and, in its turn, is itself a generating functional of the (one-particle-irreducible) vertices makes a basic quantity in quantum field theory. This is a c-numerical functional of fields and their derivatives, a knowledge of which is meant to supply one with the final solution to the theory. For this reason it seems important to see, how the most fundamental principles manifest themselves as some general properties of the effective action to be respected by model- or approximation-dependent calculations, and whose violation might signal important inconsistencies in the theory underlying these calculations. Such inconsistencies may show themselves first of all as ghosts and tachyons, that play an important role [2] in cosmological speculations about forming the Λ-term and dark energy using a scalar (Higgs) field yet to be discovered in the coming experiments on the Large Hadronic Collider.

It is stated [1] basing on a formal continual integral representation for the propagator that, when the effective action \( \Gamma(\phi) \) of a scalar field with mass \( m \) is considered, its second variational derivative \( \Sigma(x - y|\phi_0) = \delta^2 \Gamma/\delta \phi(x) \delta \phi(y)|_{\phi=\phi_0} \) calculated at the constant background value of this field, \( \phi(x) = \phi_0 \), i.e. the mass operator against this background, is a nonpositive quantity, \( \Sigma \leq 0 \). In other words, the effective Lagrangian is expected -- to the extent that that formal property survives perturbative or other calculations -- to be a nonconvex (while the effective potential to be a convex) function of a constant scalar field. However, the same statement may be considered as the one directly prescribed by the causality principle. Indeed, the spectral curve of small excitations over the constant field background, \( k_0 = \sqrt{k^2 + m^2 - \Sigma(k)} \), where \( k = (k_0, k) \) is the (4-momentum) variable, Fourier-conjugate to the 4-coordinate difference \( x - y \), satisfies the causal propagation condition reading that its group velocity should not exceed unity, the absolute speed limit for any signal, \( |\partial k_0/\partial k| = |k|/k_0 \leq 1 \) for any nonnegative mass squared \( m^2 \geq 0 \), provided, again, that \( \Sigma \leq 0 \).

The case under our consideration here is much less trivial as we deal not with a massive scalar, but with a massless vector gauge field. The results apply, first of all, to electromagnetic field, but also -- in a restrictive way -- to nonabelian gluon fields. Nonlinear electrodynamical models are also considered for cosmological purposes [3] with the advantage...
that instead of the scalar field, uncertain to be physically identified, only well established electromagnetic field is involved.

We are going to demonstrate that the requirement of the causal propagation of elementary excitations over the vacuum occupied by a background field with a constant and homogeneous field strength, supplemented by the requirements of translation-, Lorentz-, gauge-, P- and C-invariances and unitarity has a direct impact on the effective Lagrangian. For the case - which is general for electromagnetic field, but special for a nonabelian field - where the Lagrangian depends on gauge-invariant combinations (field strengths) \( F_{\alpha\beta}(z) = \partial_\alpha A_\beta(z) - \partial_\beta A_\alpha(z) \) of the background field potentials we make sure that the above requirements are expressed as certain inequalities to be obeyed by the effective Lagrangian and its first and second derivatives with respect to the two field invariants

\[
\mathfrak{F} = \frac{1}{4} F_{\rho\sigma} F_{\rho\sigma} = \frac{1}{2} (B^2 - E^2) \quad \text{and} \quad \mathfrak{G} = \frac{1}{4} F_{\rho\sigma} \tilde{F}_{\rho\sigma} = (EB),
\]

where \( \mathbf{E} \) and \( \mathbf{B} \) are background electric and magnetic fields, respectively, and the dual field tensor is defined as \( \tilde{F}_{\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\lambda\kappa} F_{\lambda\kappa} \), where the completely antisymmetric unit tensor is defined in such a way that \( \epsilon_{1230} = 1 \). More specifically, we demonstrate that it is a convex function with respect to the both variables \( \mathfrak{F}, \mathfrak{G} \) for any constant value of \( \mathfrak{F} \gtrless 0 \) and \( \mathfrak{G} = 0 \).

In Section II model- and approximation-independent study is undertaken.

In Subsection A we are basing on the general diagonal representation of the polarization operator and photon Green function in terms of its eigenvectors and eigenvalues, obtained for arbitrary values of the momentum \( k \) and for nonzero constant field invariants \( \mathfrak{F}, \mathfrak{G} \) in [4], to find limitations on the location of dispersion curves, imposed by demanding that the group velocity of the vacuum excitations be less than/or equal to unity. We find that the massless branches of these curves ("photons"), whose existence is always guarantee by the gauge invariance, for every polarization mode are outside the light cone (or on it) in the momentum space \( k^2 = 0 \), whereas the massive branches all should pass below a certain curve in the plane \((k_0^2 - k_3^2, k_\perp^2)\), where \( k_3 \) and \( k_\perp \) are the excitation momentum components along and across the direction of the background magnetic and electric fields in the special frame, where these are mutually parallel.

In Subsection B we confine ourselves to the infrared asymptotic behavior \( k_\mu \rightarrow 0 \) of the polarization operator, in which case its eigenvalues can be expressed in terms of first and second derivatives of the effective Lagrangian with respect to the field invariants \( \mathfrak{F}, \mathfrak{G} \). Massless dispersion curves are explicitly found in terms of these derivatives for the "magnetic-like"
case \( \mathcal{F} > 0, \mathcal{G} = 0 \). The restrictions of Subsection A, now supplemented with the unitarity requirement that the residue of the Green function in the pole, corresponding to the mass shell of the elementary excitation, be nonnegative (completeness of the set of states with nonnegative norm), actualize as a number of inequalities, to be satisfied by these derivatives. They mean, in particular, that the effective Lagrangian is a convex function of the field invariants in the point \( \mathcal{G} = 0 \). Basing on the study made in Appendix we reveal the physical sense of the quantities subject to these inequalities as dielectric and magnetic permeabilities responsible for polarizing small static charges and currents of special configurations (There is no universal linear response function able to cover every configuration). In Subsection C the inequalities of Subsection B are extended to include also the ”electric-like” background field \( \mathcal{F} < 0, \mathcal{G} = 0 \), so in the end the whole axis of the variable \( \mathcal{F} \) is included into result.

In Subsection D we find the contribution of the polarization operator into effective Lagrangian, which is local in the infrared limit and presents the Lagrangian for small, slow, long-wave perturbations of the background field. This gives us the possibility to define their energy-momentum tensor via the Noether theorem. By imposing the conditions of the positivity of the energy density and of non-spacelikeness of the energy-momentum flux vector – the Weak Energy Condition and Dominant Energy Condition of Hawking and Ellis – that might be considered as an alternative to the requirements used in Subsections A and B, we find some inequalities that do not contradict to those found in the previous subsection, but are weaker. This urges us to make an important admission that those popular conditions may be in a certain respect insufficient.

In Section III we test the restrictions obtained in Section II for the Euler-Heisenberg one-loop effective Lagrangian of Quantum Electrodynamics and for the Lagrangian of Born-Infeld to establish that the latter perfectly satisfies all of them. On the contrary, some of them are violated by the Euler-Heisenberg Lagrangian at exponentially large magnetic field, leading to appearance of ghosts and tachyons, signifying the instability of the magnetized vacuum due to the lack of asymptotic freedom in QED. (The instability of the electrified vacuum in this approximation thanks to Schwinger’s electron-positron pair creation goes without saying). It is a surprise that the convexity itself is not violated at any value of the magnetic field.

In Section IV we decided to use the opportunity, presented by the fact that all the appropriate circumstances have been exposed, to discuss a somewhat different matter about
S. Adler’s kinematical selection rule that is established by appealing to one-loop approximation and forbids some transitions between photon modes in the cubic process of the photon splitting in a strong magnetic field, what is important for formation of radiation in the pulsar magnetosphere. Within our context this rule is an inequality between derivatives of the effective Lagrangian, involved in the previous analysis. We propose arguments that may rule out a violation of Adler’s kinematical selection rule basing on dual symmetry consideration.

In concluding Section V we perform an attempt of comparative discussion of our approach with other ways of introducing causality into consideration.

II. UNITARITY AND CAUSALITY

A. Configuration of exact dispersion curves

Let $\mathcal{L}(z)$ be the nonlinear part of the effective Lagrangian as a function of the two electromagnetic field invariants $\mathcal{F}$ and $\mathcal{G}$ and, generally, of other Lorentz scalars that can be formed by the electromagnetic field tensor $F_{\mu\nu}$ and its space-time derivatives. The total action is $S_{\text{tot}} = \int L_{\text{tot}}(z)d^4z$, where $L_{\text{tot}}(z) = -\mathcal{F}(z) + \mathcal{L}(z)$. It is assumed that

$$\left. \frac{\delta \Gamma}{\delta \mathcal{F}} \right|_{\mathcal{F}=\mathcal{G}=0} = 0,$$

where $\Gamma = \int \mathcal{L}(z)d^4z$, according to the correspondence principle, since $-\mathcal{F}$ is the classical Lagrangian.

We consider the background field, which is constant in time and space and has only one nonvanishing invariant: $\mathcal{F} \neq 0, \mathcal{G} = 0$ (although $\mathcal{G}$ may be involved in intermediate equations). This field is purely magnetic in a special Lorentz frame, if $\mathcal{F} > 0$, and purely electric in the opposite case, $\mathcal{F} < 0$. Such fields will be called magnetic- or electric-like, respectively.

Polarization operator is responsible for small perturbations above the constant-field background. In accordance with the role of the effective action as the generating functional of vertex functions, the polarization operator is defined as the second variational derivative with respect to the vector potentials $A_\mu$

$$\Pi_{\mu\tau}(x, y) = \left. \frac{\delta^2 S}{\delta A_\mu(x) \delta A_\tau(y)} \right|_{\mathcal{G}=0, \mathcal{F}=\text{const}}.$$

5
The action $S$ here is meant to be - prior to the two differentiations over $A_\mu, A_\tau$ - a functional containing field derivatives of arbitrary order, but the fields are set constant after the differentiations. Nevertheless, their derivatives do contribute into the polarization operator leading to its complicated dependence on the momentum $k$, the variable, Fourier conjugated to $(x-y)$.

It follows from the translation- Lorentz-, gauge-, P- and charge-invariance that the Fourier transform of the tensor is diagonal

$$\Pi_{\mu\tau}(k, p) = \delta(k-p)\Pi_{\mu\tau}(k), \quad \Pi_{\mu\tau}(k) = \sum_{a=1}^{3} \kappa_a(k) \frac{\varphi^{(a)}_{\mu} \varphi^{(a)}_{\tau}}{(\gamma^{(a)})^2}$$

in the following basis:

$$\varphi^{(1)}_{\mu} = (F^2 k)_{\mu} k^2 - k_{\mu}(k F^2 k), \quad \varphi^{(2)}_{\mu} = (\bar{F} k)_{\mu}, \quad \varphi^{(3)}_{\mu} = (F k)_{\mu}, \quad \varphi^{(4)}_{\mu} = k_{\mu},$$

where $(\bar{F} k)_{\mu} \equiv \bar{F}_{\mu\tau} k_{\tau}$, $(F k)_{\mu} \equiv F_{\mu\tau} k_{\tau}$, $(F^2 k)_{\mu} \equiv F_{\mu\tau}^2 k_{\tau}$, $k F^2 k \equiv k_{\mu} F_{\mu\tau}^2 k_{\tau}$, formed by the eigenvectors of the polarization operator

$$\Pi_{\mu\tau} \varphi^{(a)}_{\tau} = \kappa_a(k) \varphi^{(a)}_{\mu}.$$ 

We are working in Euclidian metrics with the results analytically continued to Minkowsky space, hence we do not distinguish between co- and contravariant indices. All eigenvectors are mutually orthogonal, $\varphi^{(a)}_{\mu} \varphi^{(b)}_{\mu} \sim \delta_{ab}$, this means that the first three ones are 4-transversal, $\varphi^{(a)}_{\mu} k_{\mu} = 0$; correspondingly $\kappa_4 = 0$ as a consequence of the 4-transversality of the polarization operator. The unit matrix is decomposed as

$$\delta_{\mu\tau} = \sum_{a=1}^{4} \frac{\varphi^{(a)}_{\mu} \varphi^{(a)}_{\tau}}{(\gamma^{(a)})^2} \quad \text{or} \quad \delta_{\mu\tau} - \frac{k_{\mu} k_{\tau}}{k^2} = \sum_{a=1}^{3} \frac{\varphi^{(a)}_{\mu} \varphi^{(a)}_{\tau}}{(\gamma^{(a)})^2}.$$

The eigenvalues $\kappa_a(k)$ of the polarization operator are scalars and depend on $\mathfrak{F}$ and on any two of the three momentum-containing Lorentz invariants $k^2 = k^2 - k_{0}^2$, $k F^2 k$, $k \bar{F}^2 k$, subject to one relation $\frac{k \bar{F}^2 k}{2\mathfrak{F}} - k^2 = \frac{k^2 F^2 k}{2\mathfrak{F}}$. The squares of the eigenvectors are

$$(\varphi^{(1)})^2 = -(k F^2 k)(k F^2 k) + 2\mathfrak{F} k^2 = k^2 k^2 (2\mathfrak{F} k^2 (k^2 - k_{0}^2)),$$

$$(\varphi^{(2)})^2 = -(k F^2 k), \quad (\varphi^{(3)})^2 = -(k \bar{F}^2 k)$$

The diagonal representation of the photon Green function as an exact solution to the Schwinger-Dyson equation with the polarization operator taken for the kernel is (up to
The dispersion equations that define the mass shells of the three eigen-modes are

\[ \kappa_a(k^2, \frac{k F^2 k}{2\mathcal{F}}, \mathcal{F}) = k^2, \quad a = 1, 2, 3. \] (9)

All the equations above are valid both for magnetic- and electric-like cases, \( \mathcal{F} \leq 0, \mathcal{G} = 0 \). If, specifically, the magnetic-like background field \( \mathcal{F} > 0, \mathcal{G} = 0 \) is considered, in the special frame the field-containing invariants become

\[ \frac{k F^2 k}{2\mathcal{F}} = k_3^2 - k_0^2, \quad \frac{k F^2 k}{2\mathcal{F}} = -k_\perp^2, \quad \mathcal{F} = \frac{B^2}{2}, \] (10)

where we directed the magnetic field \( \mathbf{B} \) along the axis 3, and the two-dimensional vector \( \mathbf{k}_\perp \) is the photon momentum projection onto the plane orthogonal to it. On the contrary, if we deal with the electric-like background field \( \mathcal{F} < 0, \mathcal{G} = 0 \), in the special frame, where only electric field \( \mathbf{E} \) exists and is directed along axis 3, we have, instead of (10), the following relations for the background-field- and momentum-containing invariants

\[ \frac{k F^2 k}{2\mathcal{F}} = k_\perp^2, \quad \frac{k F^2 k}{2\mathcal{F}} = k_0^2 - k_3^2, \quad \mathcal{F} = \frac{-E^2}{2}, \] (11)

where the two-dimensional vector \( \mathbf{k}_\perp \) now is the photon momentum projection onto the plane orthogonal to \( \mathbf{E} \). In the both cases the dispersion equations (9) can be represented in the same form

\[ \kappa_a(k^2, k_\perp^2, \mathcal{F}) = k^2, \quad a = 1, 2, 3 \] (12)

and their solutions have the following general structure, provided by relativistic invariance

\[ k_0^2 = k_3^2 + f_a(k_\perp^2), \quad a = 1, 2, 3. \] (13)

It is notable that the structure (13) retains when the second invariant is also nonzero, \( \mathcal{G} \neq 0 \), this time the direction 3 being the common direction of the background electric and magnetic fields in the special reference frame, where these are mutually parallel. Hence,
the restrictions on the way the dispersion curves pass to be obtained below in the present subsection will remain valid in this general case, too. The only specific feature of the general case is that the eigenvectors \( b_\mu \) are no longer given by the final expressions (14), but are now linear combinations of the vectors (14) with generally unknown coefficients depending on the scalar combinations of the background field and momentum [4], [6].

The causality principle requires that the modulus of the group velocity, calculated on each mass shell (13), be less or equal to the speed of light in the free vacuum \( c = 1 \):

\[
|v_{gr}|^2 = \left( \frac{\partial k_0}{\partial k_3} \right)^2 + \left| \frac{\partial k_0}{\partial k_\perp} \right|^2 = \frac{k_3^2}{k_0^2} + \left| \frac{k_\perp}{k_0} \cdot f'_a \right|^2 = \frac{k_3^2 + k_\perp^2 \cdot (f'_a)^2}{k_3^2 + f_a(k_\perp^2)} \leq 1,
\]

(14)

where \( f'_a = df_a(k_\perp^2)/dk_\perp^2 \). This imposes the obligatory condition on the form and location of the dispersion curves (13), i.e. on the function \( f_a(k_\perp^2) \), to be fulfilled within every reasonable approximation (remind that \( k_3^2 + f_a(k_\perp^2) \geq 0 \) due to (13)):

\[
k_\perp^2 \left( \frac{df_a(k_\perp^2)}{dk_\perp^2} \right)^2 \leq f_a(k_\perp^2).
\]

(15)

This inequality requires first of all that \( f_a(k_\perp^2) \geq 0 \), hence no branch of any dispersion curve may get into the region \( k_0^2 - k_\perp^2 < 0 \). If it might, the photon energy \( k_0 \) would have an imaginary part within the momentum interval \( 0 < k_\perp^2 < -f_a(k_\perp^2) \), corresponding to the vacuum excitation exponentially growing in time. This sort of tachyon would signal the instability of the magnetized vacuum. Inequality (15) further requires that

\[
\frac{df_a(k_\perp^2)}{dk_\perp^2} \leq 1, \quad \text{or} \quad f_a(k_\perp^2) \leq \text{const} + k_\perp.
\]

(16)

The unitarity imposes the limitations that the residues of the photon propagator (8) in the poles corresponding to every photon mass shell (9) be nonnegative - the positive definiteness of the norm of every elementary excitation of the vacuum. This requirement implies:

\[
1 - \frac{\partial \kappa_a(k_0^2, k_\perp^2, \vec{3})}{\partial k_0^2} \bigg|_{k_0^2 - k_\perp^2 = f_a(k_\perp^2)} \geq 0.
\]

(17)

We shall prove somewhat later (see eq. (24) below) that

\[
\kappa_a(0, 0, \vec{3}) = 0, \quad a = 1, 2, 3.
\]

(18)

This property implies that for each mode there always exists a dispersion curve with \( f_a(0) = 0 \), which passes through the origin in the \((k_0^2 - k_\perp^2, k_\perp^2)\)-plane. It is such branches that are
called photons, since they are massless in the sense that the energy \( k_0 \) turns to zero for the particle at rest, \( k_3 = k_\perp = 0 \) (although, generally, \( k^2 \neq 0 \) where \( \mathbf{k} \neq \mathbf{0} \)). Other branches for each mode may also appear provided that a dynamical model includes an existence of a massive excitation of the vacuum with quantum numbers of a photon, for instance the positronium atom \([7]\) or a massive axion. Thus, for photons, the integration constant \( \text{const} \) in (16) should be chosen as zero. We conclude that the causality requires that in the plane \((\sqrt{k_0^2 - k_3^2}, k_\perp)\) the photon dispersion curves are located outside the light cone: \( k^2 \geq 0 \). (Remind that the light cone \( k^2 = 0 \) is the mass shell of a photon in the vacuum without an external field.) However, unlike the previous case, a violation of this ban would not lead to a complex-energy tachyon or directly signalize the vacuum instability.

The refraction index squared \( n_a^2 \) is defined for photons of mode \( a \) on the mass shell (13) as

\[
    n_a^2 \equiv \frac{|\mathbf{k}|^2}{k_0^2} = 1 + \frac{k_\perp^2 - f_a(k_\perp^2)}{k_0^2}.
\]

(19)

It follows from (16) with \( \text{const} = 0 \) that the refraction index is greater than unity - the statement common in standard optics of media (this is not, certainly, true for (massive) plasmon branches). Consequently, the modulus of the phase velocity in each mode \( v^\text{ph}_a = \frac{k_0}{|\mathbf{k}|} \) equal to \( 1/n_a \) is, for the photon proper, also smaller than the velocity of light in the vacuum \( c = 1 \). This is not the case for a massive – e.g. positronium – branch of the photon dispersion curve, where \( |v^\text{ph}_a| > 1 \) without any importance for causality.

Now that we established that for photons one has \( k^2 \geq k_0^2 \), or \( k^2 \geq 0 \), we see from the dispersion equation (9) that the eigenvalues \( \kappa_a \) are nonnegative in the momentum region, where the photon dispersion curves lie, i.e. the polarization operator is nonnegatively defined matrix there.

**B. Infrared limit: properties of the Lagrangian as a function of constant fields**

Hitherto, we were dealing with the elementary excitation of arbitrary 4-momentum \( k_\mu \). To get the (infrared) behavior of the polarization operator at \( k_\mu \sim 0 \) it is sufficient to have at one’s disposal the effective Lagrangian as a function of constant field strengths, since their derivatives, if included in the Lagrangian, would supply extra powers of the momentum \( k \) in the expression (2) for the polarization operator. We shall restrict ourselves to the infrared
asymptotic below. Our goal is to establish some inequalities imposed on the derivatives of the effective Lagrangian \( \mathcal{L} \) over the constant fields by the requirement \((15)\) that any elementary excitation of the vacuum should not propagate with the group velocity larger than unity and the requirement \((17)\) that the residue of the Green function is positive in the photon pole. To proceed beyond this limit we had to include the space and time derivatives of the fields into the Lagrangian. Then, utilizing the same requirements \((15), (17)\) the results concerning the convexity of the effective Lagrangian with respect to the constant fields to be obtained below, might be, perhaps, extended to convexities with respect to the derivative-containing field variables.

Aiming at the infrared limit we do not include time- and space-derivatives of the field strengthes in the equations that follow. Using the definition

\[
\mathcal{F}_\alpha\beta(z) = \partial_\alpha A_\beta(z) - \partial_\beta A_\alpha(z)
\]

we find

\[
\frac{\delta}{\delta A_\mu(x)} \int \mathcal{F}(z) d^4z = \int F_{\alpha\mu}(z) \frac{\partial}{\partial z_\alpha} \delta^4(x - z) d^4z,
\]

\[
\frac{\delta}{\delta A_\mu(x)} \int \mathcal{G}(z) d^4z = \int \tilde{F}_{\alpha\mu}(z) \frac{\partial}{\partial z_\alpha} \delta^4(x - z) d^4z.
\]

(20)

Then, for the first variational derivative of the action one has

\[
\frac{\delta \Gamma}{\delta A_\mu(x)} = \int \left[ \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} F_{\alpha\mu}(z) + \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \tilde{F}_{\alpha\mu}(z) \right] \frac{\partial}{\partial z_\alpha} \delta^4(x - z) d^4z. \tag{21}
\]

By repeatedly applying eq. \((21)\) we get for the infrared (IR) limit of the polarization operator in a constant external field

\[
\Pi_{\mu\tau}^{IR}(x, y) = \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\tau(y)} \bigg|_{\mathcal{F}, \mathcal{G} = \text{const}} = \left\{ \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left( \frac{\partial^2}{\partial x_\alpha \partial x_\mu} - \Box \delta_{\mu\tau} \right) - \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z) \partial \mathcal{G}(z)} \left( \frac{\partial}{\partial x_\alpha} \right) \left( \frac{\partial}{\partial x_\beta} \right) \right\} F_{\text{const}} \delta^4(x - y). \tag{22}
\]

The P-invariance requires that the effective Lagrangian should be an even function of the pseudoscalar \( \mathcal{G} \). Hence all the terms in the third line of eq. \((22)\) vanish for the "single-invariant" fields with \( \mathcal{G} = 0 \) under consideration.

Thus, we find for the infrared limit of the polarization operator in the magnetic- or
electric-like field in the momentum representation, \( \Pi_{\mu \nu}^{IR}(k, p) = \delta(k - p) \Pi_{\mu \nu}^{IR}(k) \),

\[
\Pi_{\mu \nu}^{IR}(k) = \left( \frac{d \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}} \right)_k \delta(\mu k^2 - k_{\mu} k_{\tau}) + \frac{d^2 \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}^2} (F_{\mu \alpha} k_{\alpha})(F_{\tau \beta} k_{\beta}) + \frac{\partial^2 \mathcal{L}(\mathbf{F}, \mathbf{G})}{\partial \mathbf{G}^2} \bigg|_{\mathbf{G} = 0} (\tilde{F}_{\mu \alpha} k_{\alpha})(\tilde{F}_{\tau \beta} k_{\beta}).
\]

(23)

Here the scalar \( \mathbf{F} \) and the tensors \( F, \tilde{F} \) are already set to be space- and time-independent. By comparing this with (3) we identify the eigenvalues of the polarization operator in the infrared limit as

\[
\kappa_1(k^2, k F^2 k, \mathbf{F}) \big|_{k \to 0} = k^2 \frac{d \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}},
\]

\[
\kappa_2(k^2, k F^2 k, \mathbf{F}) \big|_{k \to 0} = k^2 \frac{d \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}} - (k F^2 k) \frac{\partial^2 \mathcal{L}(\mathbf{F}, \mathbf{G})}{\partial \mathbf{G}^2} \bigg|_{\mathbf{G} = 0},
\]

\[
\kappa_3(k^2, k F^2 k, \mathbf{F}) \big|_{k \to 0} = k^2 \frac{d \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}} - (k F^2 k) \frac{d^2 \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}^2}.
\]

(24)

This is the leading behavior of the polarization operator in the magnetic-like field near zero-momentum point \( k_{\mu} = 0 \). Every eigenvalue \( \kappa_a \) turns into zero quadratically when all the momentum components disappear. Thereby, eq. (18) is proved.

For the sake of completeness, we give the same eqs. (24) also in terms of the invariant variables

\[
H = \sqrt{\mathbf{F} + \sqrt{\mathbf{F}^2 + \mathbf{G}^2}} \quad E = \sqrt{-\mathbf{F} + \sqrt{\mathbf{F}^2 + \mathbf{G}^2}}
\]

(25)

that are, respectively, the magnetic and electric fields in the Lorentz frame, where these are parallel. Then, with the notation \( L(H, E) = \mathcal{L}(\mathbf{F}, \mathbf{G}) \) the coefficients in (24) are:

\[
\frac{d \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}} = \frac{1}{H} \frac{d L(H, 0)}{d H},
\]

\[
\frac{d^2 \mathcal{L}(\mathbf{F}, 0)}{d \mathbf{F}^2} = \frac{1}{2\mathbf{F}} \left( \frac{d^2 L(H, 0)}{d H^2} - \frac{d L(H, 0)}{d H} \right),
\]

\[
\frac{\partial^2 \mathcal{L}(\mathbf{F}, \mathbf{G})}{\partial \mathbf{G}^2} \bigg|_{\mathbf{G} = 0} = \frac{1}{2\mathbf{F}} \left( \frac{1}{E} \frac{\partial L(H, E)}{\partial E} \right) \bigg|_{E = 0} + \frac{1}{2\mathbf{F}} \frac{d L(H, 0)}{d H}.
\]

(26)

At this step we turn to the special case of magnetic-like background and shall be sticking to it until the end of the present Subsection, keeping the extension of some results to the electric-like case \( \mathbf{F} < 0 \) to the next Subsection C.
The dispersion curves $f_{\alpha}(k^2_\perp)$ near the origin may be found by solving equations (9) in the special frame with the right-hand sides taken as (24) and with eqs. (10) taken into account. This gives for the photons of modes 2 and 3

$$f_2(k^2_\perp) = k^2_\perp \left( \frac{1 - \mathcal{L}_\delta}{1 - \mathcal{L}_\delta + 2\mathcal{F}\mathcal{L}_\phi} \right),$$

(27)

$$f_3(k^2_\perp) = k^2_\perp \left( 1 - 2\mathcal{F}_\delta \mathcal{L}_\delta \right),$$

(28)

where we are using the notations $\mathcal{L}_\delta = \frac{d^2\mathcal{L}(\delta,0)}{d\delta^2}$, $\mathcal{L}_\delta = \frac{d\mathcal{L}(\delta,\omega)}{d\delta}$, $\mathcal{L}_\phi = \frac{\partial^2\mathcal{L}(\delta,\phi)}{\partial\phi^2} \bigg|\phi=0$.

As for mode 1, the dispersion equation in the present approximation has only the trivial solution $k^2 = 0$ that makes the vector potential $\mathcal{A}_\mu^{(1)}$ corresponding to it purely longitudinal, with no electromagnetic field carried by the mode. This is a nonpropagating mode in the infrared limit (it is also nonpropagating within the one-loop approximation beyond this limit; however, massive-positronium solutions in mode 1 do propagate [7]).

The unitarity condition (17), as applied to mode 2, gives via the second equation in (24)

$$1 - \mathcal{L}_\delta + 2\mathcal{F}\mathcal{L}_\phi \geq 0.$$  

(29)

Then, from the behavior of the dispersion curve (27) and the causality (15) it follows that

$$1 - \mathcal{L}_\delta \geq 0$$

(30)

and

$$\mathcal{L}_\phi \geq 0.$$  

(31)

(Remind that for the magnetic-like case under consideration one has $\mathcal{F} > 0$.)

Analogously, the unitarity condition (17), as applied to mode 3, gives via the third equation in (24) again the result (30). (This inequality also provides the positiveness of the norm of the non-propagating mode 1.) Then from the behavior of the dispersion curve (28) and the causality (15) it follows that

$$1 - \mathcal{L}_\delta - 2\mathcal{F}\mathcal{L}_\delta \geq 0$$

(32)

and

$$\mathcal{L}_\delta \geq 0.$$  

(33)
Inequalities eq. (30), eq. (32) together provide that all the three residues of the photon Green function in the complex plane of $k_2$, the same as in the complex plane of $(k_3^2 - k_0^2)$, eq. (17), are also nonnegative

$$1 - \left. \frac{\partial \kappa_a(k^2, k_\perp^2, \mathfrak{f})}{\partial k_\perp^2} \right|_{k_0^2 - k_3^2 = f_a(k_\perp^2)} \geq 0,$$  \hspace{1cm} (34)

at least in the infrared limit. We do not know whether this statement is prescribed by general principles and therefore might be expected to hold beyond this limit.

Relations (31), (33) indicate that the extremum of the effective action at $\mathfrak{g} = 0$ (note that $(\partial \mathfrak{L})/\partial \mathfrak{g})|_{\mathfrak{g}=0} = 0$ due to P-invariance) is a minimum for any $\mathfrak{f}$ and that the Lagrangian is a convex function of $\mathfrak{f}$ for any $\mathfrak{f} > 0$ and of $\mathfrak{g}$ for $\mathfrak{g} = 0$.

Relations (29), (30), (32) indicate positiveness of various dielectric and magnetic permittivity constants that control electro- and magneto-statics of charges and currents of certain configurations. Eqs. (24) imply that the quantities that are subject to the inequalities (29), (29) and (32) are expressed in terms of different infra-red limits of the polarization operator eigenvalues as

$$1 - \mathcal{L}_{\mathfrak{f}} = \lim_{k_\perp^2 \to 0} \left( 1 - \frac{\kappa_2|_{k_0 = k_3 = 0}}{k_\perp^2} \right) \equiv \varepsilon_{\text{tr}}(0),$$

$$1 - \mathcal{L}_{\mathfrak{f}} = \lim_{k_\perp^2 \to 0} \left( 1 - \frac{\kappa_1|_{k_0 = k_3 = 0}}{k_\perp^2} \right) \equiv (\mu^w_{\text{tr}}(0))^{-1},$$

$$1 - \mathcal{L}_{\mathfrak{f}} = \lim_{k_3^2 \to 0} \left( 1 - \frac{\kappa_3|_{k_0 = k_3 = 0}}{k_3^2} \right) \equiv (\mu^l_{\text{long}}(0))^{-1},$$

$$1 - \mathcal{L}_{\mathfrak{f}} + 2 \mathfrak{f} \mathcal{L}_{\mathfrak{f}} \mathfrak{g} = \lim_{k_\perp^2 \to 0} \left( 1 - \frac{\kappa_2|_{k_0 = k_3 = 0}}{k_3^2} \right) \equiv \varepsilon_{\text{long}}(0),$$

$$1 - \mathcal{L}_{\mathfrak{f}} - 2 \mathfrak{f} \mathcal{L}_{\mathfrak{f}} \mathfrak{g} = \lim_{k_\perp^2 \to 0} \left( 1 - \frac{\kappa_3|_{k_0 = k_3 = 0}}{k_\perp^2} \right) \equiv (\mu^l_{\text{tr}}(0))^{-1}.$$  \hspace{1cm} (35)

It is demonstrated in Appendix that $\varepsilon_{\text{long}}$ and $\varepsilon_{\text{tr}}$ are dielectric constants responsible for polarizing the homogeneous electric fields parallel and orthogonal to the external magnetic field, which are produced, respectively, by uniformly charged planes (sufficiently far from them), oriented across the external magnetic field and parallel to it, see eqs. (123) and (125). These are determined by the eigenvalue $\kappa_2$, the virtual photons of the mode 2 being carriers of electrostatic force.
The quantity $\mu_{\text{tr}}^w(0)$ is the magnetic permittivity constant responsible for attenuation of the magnetic field produced by a constant current concentrated on a line, parallel to the external magnetic field, sufficiently far from the current-carrying line, see eq. (110) with $\mu(0)$ replaced by $\mu_{\text{tr}}^w(0)$ in it. The same quantity $\mu_{\text{tr}}^w(0)$ governs the constant magnetic field of a plane current flowing along the external field. This magnetic permittivity is determined by the mode 1. The other two magnetic permittivities, $\mu_{\text{pl}}^{\text{long}}(0)$ and $\mu_{\text{tr}}^{\text{pl}}(0)$ are determined by the mode 3. The permittivity $\mu_{\text{tr}}^{\text{pl}}(0)$ is responsible for remote attenuation of the magnetic field produced by a constant current, homogeneously concentrated on a plane, parallel to the external magnetic field, and flowing in the direction transverse to it, see eq. (135). This magnetic field is homogeneous and parallel to the external field. Finally, permittivity $\mu_{\text{pl}}^{\text{long}}(0)$ is responsible for remote attenuation of the magnetic field produced by a constant straight current, homogeneously concentrated on a plane, transverse to the external magnetic field, see eq. (138). This field is also homogeneous. Virtual photons of the modes 1 and 3 are carriers of magneto-static force.

By using the wordings ”sufficiently far” and ”remote” we mean distances from the corresponding sources that essentially exceed a characteristic length of an underlying microscopic theory, wherein the linear response is formed. In a material medium that may be an inter-atomic distance; in perturbative QED this is the electron Compton length.

Relations (35), (36), (37) mean that the inequalities (29), (30) and (32) signify the positiveness of all the characteristic permittivities of the magnetized vacuum, which was derived above on general basis. Besides, thanks to (35), there exists the equality between one dielectric and two (inverse) magnetic permittivities

$$\varepsilon_{\text{tr}}(0) = (\mu_{\text{tr}}^w(0))^{-1} = (\mu_{\text{pl}}^{\text{long}}(0))^{-1}. \quad (38)$$

The first equality here is a direct consequence of the invariance under the Lorentz boost along the magnetic field in the special frame (see eq. (74) in Section IV) and can be extended to the permittivity functions as defined in Appendix by (128) and the right equation (121),

$$\varepsilon_{\text{tr}}(k_{\perp}^2) = (\mu_{\text{tr}}^w(k_{\perp}^2))^{-1}. \quad (38)$$
C. Electric-like background field

In this Subsection we shall see how the inequalities (29)–(33) derived in the previous Subsection are extended to the negative domain of the invariant \( \mathfrak{f} \).

Bearing in mind eqs. (11) we may solve again dispersion equations (12) using eqs. (24) to get the photon dispersion curves in the electric-like background field in the infrared approximation. For mode 2 this results in

\[
 k_0^2 - k_3^2 = k_\perp^2 \left( 1 + \frac{2 \mathfrak{f}_I G G}{1 - \mathfrak{f}_G} \right),
\]

while for mode 3 in

\[
 k_0^2 - k_3^2 = k_\perp^2 \left( \frac{1}{1 - \mathfrak{f}_G - 2 \mathfrak{f}_I \mathfrak{f}_G} \right)
\]

(compare this with (27), (28)). The unitarity relation (17) applied to mode 2 leads to the inequality (30). The causality condition (15), when applied to (39) requires that

\[
 \left( 1 + \frac{2 \mathfrak{f}_I G G}{1 - \mathfrak{f}_G} \right)^2 \leq \left( 1 + \frac{2 \mathfrak{f}_I G G}{1 - \mathfrak{f}_G} \right).
\]

This implies that the right-hand side of the inequality (41) be positive and thus the both sides can be divided on it. Then the inequality (41) becomes the inequality (29)

\[
 \left( 1 + \frac{2 \mathfrak{f}_I G G}{1 - \mathfrak{f}_G} \right) < 1.
\]

In view of (30) this means that \( 2 \mathfrak{f}_I G G < 0 \). Once \( \mathfrak{f} \) is negative for the electric-like case under consideration now, we come again to the convexity condition (31), now in the domain of negative \( \mathfrak{f} \). By applying the same procedure to mode 3 we quite analogously reproduce eqs. (32) and (33).

D. Energy-momentum conditions

Apart from the relations derived above, there is an extra relation that does not include derivatives of the Lagrangian and follows from the positiveness of the energy density of the background field. The standard expression for the energy-momentum tensor

\[
 T_{\mu\nu}(x) = - \frac{\partial L_{\text{tot}}}{\partial (\partial A_\alpha/\partial x_\nu)} \frac{\partial A_\alpha}{\partial x_\mu} + \delta_{\mu\nu} L_{\text{tot}}^{\text{sqr}},
\]

\( \text{(43)} \)
\( g_{00} = -1 \), leads in the constant magnetic-like background, after symmetrization, to

\[
T_{\mu\nu} = - F^2 \, \mu\nu (1 - \mathcal{L} \delta) + \delta_{\mu\nu} L_{\text{tot}}. \tag{44}
\]

The trace of this tensor is \( 4(\mathcal{L} - \mathcal{F} \mathcal{L}) \). The energy density in the special frame is

\[
T_{00} = -L_{\text{tot}} = \mathcal{F} - \mathcal{L}. \tag{45}
\]

Therefore, the condition

\[
\mathcal{L} < \mathcal{F} \tag{46}
\]

should hold. Once \( T_{i0} = 0 \) the spacial part of the energy-momentum vector is zero, hence the latter is directed along the time.

Now we proceed by describing general restrictions imposed by the physical requirement that the energy density of elementary excitations of the magnetic-like background (magnetized vacuum) be nonnegative ("weak energy condition" in terms of Ref. [8])

\[
t_{00} \geq 0 \tag{47}
\]

and that their energy-momentum flux density be non-spacelike ("dominant energy condition" of Ref. [8])

\[
t_{0\nu}^2 \leq 0. \tag{48}
\]

To this end we have to define the energy-momentum tensor \( t_{\mu\nu}(x) \) of small perturbations of the background field by first defining their Lagrangian.

The total effective Lagrangian \( L_{\text{tot}} = -\mathcal{F} + \mathcal{L} \) expanded near the background constant magnetic field contributes into the total action – in view of the definition (2) – the following correction, quadratic in the small perturbation \( a_\mu(x) \) above the background:

\[
S_{\text{tot}}^{\text{sqf}} = \frac{1}{2} \int a_\mu(x) \left\{ - \left( \delta_{\mu\nu} \partial_\alpha^2 - \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \right) \delta(x - y) + \Pi_{\mu\nu}(x,y) \right\} a_\nu(y) d^4 x d^4 y. \tag{49}
\]

The field intensity of the perturbation will be denoted as \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). Using the diagonal form of the polarization operator [3] we get in the momentum representation

\[
L_{\text{tot}}^{\text{sqf}}(k) = \frac{1}{4} f^2 + \frac{1}{4} \left( -\frac{\kappa_1}{k^2} f^2 + \frac{\kappa_1 - \kappa_2}{2kF^2k} (f \tilde{F})^2 + \frac{\kappa_1 - \kappa_3}{2kF^2k} (f F)^2 \right). \tag{50}
\]

16
Here the notations are used: \((fF)_{\mu\nu} = f_{\mu\alpha}F_{\alpha\nu} = (Ff)_{\nu\mu}\), \((fF)_{\mu\mu} = (FF)\), \(f_{\mu\nu}^2 = f_{\mu\nu}f_{\nu\mu}\). \(f^2 = f_{\mu\nu}^2 = -(f_{\mu\nu})^2\), and we have exploited the relations \(f^2 = -2a_\mu(k^2\delta_{\mu\nu} - k_\mu k_\nu)a_\nu\), \((fF) = 2(aFk)\). This Lagrangian is nonlocal, since it depends on momenta in a complicated way, in other words, it depends highly nonlinearly on the derivatives with respect to coordinates. It becomes local if we restrict ourselves to the infrared limit by substituting eqs.(24) into it. Then the quadratic Lagrangian acquires the very compact form

\[
L_{\text{tot}}^{\text{sq}} = \frac{1}{4}f^2(1 - \mathcal{L}_8) + \frac{1}{8}(\mathcal{L}_{\phi\phi}(f\tilde{F})^2 + \mathcal{L}_{\tilde{F}\tilde{F}}(fF)^2).
\]

(51)

This Lagrangian, quadratic in the field \(f_{\mu\nu}(x)\), does not contain its derivatives, \(F_{\mu\nu}, \tilde{F}_{\mu\nu}, \mathcal{L}_8, \mathcal{L}_{\phi\phi}\) and \(\mathcal{L}_{\tilde{F}\tilde{F}}\) being constants depending upon the background field alone. It governs small-amplitude low-frequency and low-momentum perturbations of the magnetized vacuum, free of/ or created by small sources. It might be obtained also directly by calculating the second derivative \((2)\) of the Lagrangian defined on constant fields.

Once the background is translation-invariant, there is a conserved energy-momentum tensor \(t_{\mu\nu}(x)\) of the field \(f_{\mu\nu}\) provided by the Noether theorem by considering variations of this field. Applying the definition \((52)\) to the field of small perturbation \(a_\mu\) and to its Lagrangian \((51)\) we get

\[
t_{\mu\nu}(x) = -\frac{\partial L_{\text{tot}}^{\text{sq}}}{\partial(\partial a_\alpha/\partial x_\nu)} \frac{\partial a_\alpha}{\partial x_\mu} + \delta_{\mu\nu}L_{\text{tot}}^{\text{sq}} = -\frac{\partial a_\alpha}{\partial x_\mu} \left( f_{\alpha\nu}(1 - \mathcal{L}_8) + \frac{1}{2}(f\tilde{F})\mathcal{L}_{\phi\phi}\tilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathcal{L}_{\tilde{F}\tilde{F}}F_{\alpha\nu} \right) + \delta_{\mu\nu}L_{\text{tot}}^{\text{sq}}.
\]

(52)

The Maxwell equations for small sourceless perturbations of the magnetized vacuum are

\[
\delta L_{\text{tot}}^{\text{sq}} / \delta a_\alpha = \partial / \partial x_\nu (\partial L_{\text{tot}}^{\text{sq}} / \partial x_\nu) / (\partial a_\alpha / \partial x_\nu) = -\partial / \partial x_\nu \left( f_{\alpha\nu}(1 - \mathcal{L}_8) + \frac{1}{2}(f\tilde{F})\mathcal{L}_{\phi\phi}\tilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathcal{L}_{\tilde{F}\tilde{F}}F_{\alpha\nu} \right) = 0.(53)
\]

We are going to use the standard indeterminacy in the definition of the energy-momentum tensor to let it depend only on the field strength \(f_{\mu\nu}\), and not on its potential. To this end we add the quantity (the designation \(\doteq\) below means ”equal up to full derivative”)

\[
\frac{\partial L_{\text{tot}}^{\text{sq}}}{\partial(\partial a_\alpha / \partial x_\nu)} \frac{\partial a_\mu}{\partial x_\alpha} + \frac{\partial L_{\text{tot}}^{\text{sq}}}{\partial(\partial a_\alpha / \partial x_\nu)} = \frac{\partial a_\mu}{\partial x_\alpha} \left\{ (f_{\alpha\nu}(1 - \mathcal{L}_8) + \frac{1}{2}(f\tilde{F})\mathcal{L}_{\phi\phi}\tilde{F}_{\alpha\nu} + \frac{1}{2}(fF)\mathcal{L}_{\tilde{F}\tilde{F}}F_{\alpha\nu} \right\}
\]

(54)
to [52], that disappears due to the Maxwell equations [53], taking into account the anti-symmetry of the expression inside the braces. Hence the energy-momentum tensor may be equivalently written as

\[ t_{\mu\nu}(x) = -f_{\mu\nu}^2(1 - \mathcal{L}_g) - \frac{1}{2}(f \tilde{F})\mathcal{L}_{\phi\phi}(f \tilde{F})_{\mu\nu} - \frac{1}{2}(f F)\mathcal{L}_{\delta\delta}(f F)_{\mu\nu} + \frac{\delta_{\mu\nu}}{4}\left(f^2(1 - \mathcal{L}_g) + \frac{1}{2}\mathcal{L}_{\phi\phi}(f \tilde{F})^2 + \frac{1}{2}\mathcal{L}_{\delta\delta}(f F)^2\right). \]  

(55)

This tensor is traceless, \( t_{\mu\mu} = 0 \). It obeys the continuity equation with respect to the second index

\[ \frac{\partial t_{\mu\nu}}{\partial x^\nu} = 0 \]  

(56)

owing to the Maxwell equations [53]. Hence, the 4-momentum vector obtained by integrating \( t_{0\mu} \) over the spatial volume \( d^3x \) conserves in time.

Let us take [55], first, on the monochromatic – with 4-momentum \( k_\mu \) – real solution of the Maxwell equations [53] that belongs to the eigen-mode 3: \( f_{\mu\nu}^{(3)} = k_\mu\nu^{(3)} - k_\nu\mu^{(3)}. \) One has \( (f^{(3)}F)_{\mu\nu} = \gamma_\nu^{(3)} - k_\mu(F^2k)_\nu, (f^{(3)}F) = -2(kF^2k), (f^{(3)})^2_{\mu\nu} = -k^2\gamma_\nu^{(3)} + k_\mu k_\nu(kF^2k), (f^{(3)})^2 = 2k^2(F^2k), (f^{(3)}\tilde{F}) = 0. \) With the substitution \( f_{\mu\nu} = f_{\mu\nu}^{(3)} \) the Maxwell equation [53] is satisfied, when

\[ \gamma^\alpha_\mu \{k^2(1 - \mathcal{L}_g) + (kF^2k)\mathcal{L}_{\delta\delta}\} = 0, \]  

(57)

i.e., naturally, on the dispersion curve [28] for mode 3. It is seen that the Lagrangian [51] disappears on the mass shell of mode 3, \( L^{tot(3)}_{\mathcal{S}} = 0. \) Then, the reduction of the energy momentum tensor [55] onto this mode, \( t_{\nu\mu}^{(3)}(x), \) should be written with its \( \delta_{\mu\nu} \) part dropped:

\[ t_{\mu\nu}^{(3)}(x) = (1 - \mathcal{L}_g)(k^2\gamma^{(3)}_{\mu\nu} - k_\mu k_\nu(kF^2k)) + (kF^2k)\mathcal{L}_{\delta\delta}(\gamma^{(3)}_{\mu\nu} - k_\mu(kF^2k)_\nu). \]  

(58)

Although we referred to the magnetic-like background above in this Subsection, all the equations written in it up to now remain, as a matter of fact, valid also for the electric-like case. In the rest of this Subsection we actually specialize to the magnetized vacuum, although the conclusions may be readily extended to cover the electrified vacuum, as well. When \( \mathcal{F} > 0, \) in the special frame (see eqs. [114] in Appendix), it holds \( \gamma_0^{(3)} = 0, (F^2k)_{0,3} = 0, (F^2k)_{1,2} = -2\mathcal{F}k_{1,2}. \) Then, after omitting the positive common factor \(-(kF^2k) = 2\mathcal{F}k^2_1, \) we get for energy-momentum density vector

\[ t^{(3)}_{0\nu}(x) = k_0 \{(1 - \mathcal{L}_g)k_\nu + \mathcal{L}_{\delta\delta}(F^2k)_\nu\} \]  

(59)
It is convenient to write it in components (counted as 0, 1, 2, 3 downwards)

\[
\begin{pmatrix}
  k_0 (1 - \mathcal{L}_\delta) \\
  k_1 (1 - \mathcal{L}_\delta - 2 \delta \mathcal{L}_{\delta\delta}) \\
  k_2 (1 - \mathcal{L}_\delta) \\
  k_3 (1 - \mathcal{L}_\delta)
\end{pmatrix}_\nu = k_0
\]

\[
(60)
\]

The positive definiteness of the energy density (47) results again in the requirement that the inequality (30) be satisfied. The causality in the form of the dominant energy condition (48) makes us expect that vector (60) should be non-spacelike. Now, from (60) with the use of the dispersion law (28) this condition becomes

\[
t^{(3)2}_{0,\nu} = k^2_0 \left( (k^2_3 - k^2_0)(1 - \mathcal{L}_\delta)^2 + k^2_\perp (1 - \mathcal{L}_\delta - 2 \delta \mathcal{L}_{\delta\delta})^2 \right) = \\
-2 \delta \mathcal{L}_{\delta\delta} k^2_0 k^2_\perp (1 - \mathcal{L}_\delta - 2 \delta \mathcal{L}_{\delta\delta}) \leq 0.
\]

\[
(61)
\]

The same operations, performed over the energy-momentum tensor (55) taken on mode 2 result in

\[
t^{(2)2}_{0,\nu} = -2 \delta \mathcal{L}_{\epsilon\epsilon} k^2_0 k^2_\perp (1 - \mathcal{L}_\delta + 2 \delta \mathcal{L}_{\epsilon\epsilon}) \leq 0.
\]

\[
(62)
\]

The fulfillment of (61), (62) is guaranteed by the inequalities (29), (31)–(32) established in Subsection B. However, the inverse statement would be wrong: the inequalities (61), (62), derived in the present Subsection do not yet lead to (29), (31)–(32). This may indicate that pair of conditions (17) (unitarity as the positivity of the residue) and (14) (causality as the boundedness of the group velocity), used to derive the limitations (29)–(32) of Subsection B, are together more restrictive than the two principles (47) (energy positiveness) and (48) (causality as non-spacelikeness of the energy-momentum density), although the latter provide the fact that when solving the Cauchy problem initial data have no influence on what occurs outside their light cone. (This is proved in [8] within General Relativity context. In this connection it is interesting to mention the observation in Refs. [9] that the background field may be represented by an equivalent effective metric tensor at least as far as dispersion equations are concerned. That metric tensor may be apparently used for representing the Lagrangian (51) in geometric form.)
III. TESTING EULER-HEISENBERG AND BORN-INFELD LAGRANGIANS

In the one-loop approximation of QED the quantities involved can be calculated either using the Euler-Heisenberg effective Lagrangian \( \mathcal{L} = \mathcal{L}^{(1)} \) as long as the infrared limit is concerned or, alternatively, the one-loop polarization operator calculated in [4] for off-shell photons – within and beyond this limit. In the infrared limit the photon-momentum-independent coefficients in (24) within one loop are the following functions of the dimensionless magnetic field \( b = eB/m^2 \), where \( e \) and \( m \) are the electron charge and mass:

\[
\mathcal{L}^{(1)}_{\mathfrak{F}} = \frac{\alpha}{2\pi} \int_0^\infty \frac{dt}{t} \exp \left( -\frac{t}{b} \right) \left( -\coth \frac{t}{2} + \frac{1}{\sinh^2 t} + \frac{2}{3} \right), \tag{63}
\]

\[
2\mathfrak{F} \mathcal{L}^{(1)}_{\mathfrak{G}_G} = \frac{\alpha}{3\pi} \int_0^\infty \frac{dt}{t} \exp \left( -\frac{t}{b} \right) \left( -3 \coth \frac{t}{2} + \frac{3}{2 \sinh^2 t} + t \coth t \right), \tag{64}
\]

\[
2\mathfrak{F} \mathcal{L}^{(1)}_{\mathfrak{G}_F} = \frac{\alpha}{3\pi} \int_0^\infty \frac{dt}{t} \exp \left( -\frac{t}{b} \right) \left( 3 \coth \frac{t}{2} - \frac{t \coth t}{\sinh^2 t} + \frac{3}{2 \sinh^2 t} \right). \tag{65}
\]

Here \( \alpha = e^2/4\pi = 1/137 \) is the fine-structure constant. Eq. (63) turns to zero as \( \mathfrak{F} \sim b^2 \), since the divergent linear in \( \mathfrak{F} \) part of the one-loop diagram was absorbed in the course of renormalization into \( \mathcal{L}_{\text{cl}} \). It can be verified that the general relations (29)–(33) ordained by unitarity (17) and causality (15) to the infrared limit are obeyed by the one-loop approximation within the vast range of the magnetic field values. However, due to the known lack of asymptotic freedom in QED [10], some of them are violated for the exponentially strong fields. One can establish the asymptotic behavior of (63)–(65) in the limit \( b = eB/m^2 \to \infty \)

\[
\mathcal{L}^{(1)}_{\mathfrak{F}} \simeq \frac{\alpha}{3\pi} (\ln b - 1.79), \quad 2\mathfrak{F} \mathcal{L}^{(1)}_{\mathfrak{G}_G} \simeq \frac{\alpha}{3\pi} (b - 1.90), \quad 2\mathfrak{F} \mathcal{L}^{(1)}_{\mathfrak{G}_F} \simeq \frac{\alpha}{3\pi}. \tag{66}
\]

Thanks to the linearly growing [11] term in \( \mathcal{L}^{(1)}_{\mathfrak{G}_G} \), for mode 2 the positive-norm condition (the left relation in (29)) is fulfilled for any \( b \), and also the dispersion curve (27) goes outside the light cone, as it is prescribed by the causality in the form of eq. (16) with \( \text{const} = 0 \). However, the bracket in (27) becomes negative for \( b > b_2^{\mathfrak{F}} = \exp\{1.79 + 3\pi/\alpha\} \), and mode 2 becomes a complex energy tachyon. For mode 3, the positive norm condition ( relation (30)) is fulfilled for \( b < b_2^{\mathfrak{F}} \). However, within the range \( \exp\{0.79 + 3\pi/\alpha\} = b_3^{\mathfrak{F}} < b < b_2^{\mathfrak{F}} \) the bracket in (28) is negative, and mode 3 is a complex energy tachyon. For \( b > b_2^{\mathfrak{F}} \) the dispersion curve (28) for mode-3 photon gets inside the light cone and becomes a superluminal ghost with real energy and negative norm. An instability of the magnetized vacuum...
with respect to production of a constant field is associated with the imaginary energy at zero momentum. The elementary excitation with this property appears in mode 3 at a smaller threshold value, $b_{cr}^3$, than in mode 2, $b_{cr}^2$. The instability associated with mode-2 tachyons may lead to gaining the constant field with $G \neq 0$, since the (pseudo)vector-potential $\gamma_\mu^{(2)}$ carries an electric field component, parallel to the background magnetic field, whereas in $\gamma_\mu^{(3)}$ this component is perpendicular to $B$. It is interesting to note that, in spite of the instabilities and appearance of super-luminal excitations pointed above, the convexity properties (31), (33) are left intact under any magnetic field within one loop.

The borders of stability of the magnetic field found here by analyzing the one-loop approximation are characterized by the large exponential $\exp\{1/\alpha\}$. It is much larger than the border found earlier [12] as the value where the mass defect of the bound electron-positron pair completely compensates the $2m$ energy gap between the electron and positron, which is of the order of $\exp\{1/\sqrt{\alpha}\}$. These values are of the Planck scale.

The situation is quite different for the Born-Infeld electrodynamics with its Lagrangian

$$L_{tot} = L^{BI} = a^2 \left( 1 - \sqrt{1 + \frac{2\delta}{a^2} - \frac{\mathcal{G}^2}{a^4}} \right)$$

(67)

viewed upon as final, not subject to further quantization. Here $a$ is an arbitrarily large parameter with the dimensionality of mass squared. The correspondence principle (11) is respected by eq. (67). It does not contain field derivatives, hence all the infra-red limits encountered in this paper should be understood as exact values, for instance, going to the limit is unnecessary in (35), (36), (37). The Lagrangian (67) was derived long ago [13] basing on very general geometrical principles of reparametrization-invariance, and besides it attracted much attention in recent decades thanks to the fact that it appears responsible for the electromagnetic sector of a string theory [14] and thus is expected not to suffer from the lack of asymptotic freedom. For this reason our statement to follow that all the fundamental requirements established in Section 2 are obeyed in the Born-Infeld electrodynamics (67) is instructive. We assume again that there is the constant and homogeneous magnetic-like external background and set $\mathcal{G} = 0$ after differentiation. Then, we get from (67)

$$1 - \mathcal{L}^{BI}_3 = \left( 1 + \frac{2\delta}{a^2} \right)^{-\frac{1}{2}} \geq 0,$$

$$\mathcal{L}^{BI}_{\mathcal{G}\mathcal{G}} = a^{-2} \left( 1 + \frac{2\delta}{a^2} \right)^{-\frac{3}{2}} \geq 0,$$

$$\mathcal{L}^{BI}_{\mathcal{G}} = a^{-2} \left( 1 + \frac{2\delta}{a^2} \right)^{-\frac{1}{2}} \geq 0,$$

$$1 - \mathcal{L}^{BI}_3 + 2\delta \mathcal{L}^{BI}_{\mathcal{G}\mathcal{G}} = \left( 1 + \frac{2\delta}{a^2} \right)^{\frac{1}{2}} \geq 0,$$

$$1 - \mathcal{L}^{BI}_3 - 2\delta \mathcal{L}^{BI}_{\mathcal{G}} = \left( 1 + \frac{2\delta}{a^2} \right)^{-\frac{1}{2}} \geq 0$$

(68)
where $\mathcal{L}^{BI} = L^{BI} + 2\Phi$. Thus, relations (29)–(33) are all satisfied, hence there are neither ghosts, nor tachyons. The mode 1 remains nonpropagating. As for modes 2 and 3, their dispersion curves coincide, since $f_2(k_1^2) = f_3(k_1^2)$ in (27), (28) due eqs. (68). This reflects the known absence of birefringence in the Born-Infeld electrodynamics [15]. Still, beyond the mass shell one has $\kappa_2 \neq \kappa_3$, consequently the corresponding permeabilities (35), (36), (37) are different. The same as in the one-loop QED, in the limit of large external field there is a linearly growing contribution in $\kappa_2$, so mode 2 dominates, the dielectric permeability (36) behaving like the middle equation in (66)

$$\varepsilon^{BI}_{\text{long}}(0) \simeq 2\Phi \mathcal{L}^{BI}_{\Phi} \simeq \frac{B}{a},$$

with the identification $a = (3\pi/\alpha)B_0$, where $B = m^2/e = 4.4 \times 10^{14}$ Gauss is the characteristic field strength in QED.

If we include the electric-like case we shall see that eqs. (68) are all fulfilled within the interval $-(2\Phi/a) < \Phi < \infty$, at the border of which the Lagrangian (67) becomes imaginary (recall that $\Phi = 0$.)

**IV. GENERAL BASIS FOR ADLER’S SELECTION RULE**

There is an important statement that the dispersion in mode-2 photon is stronger than that in mode 3 throughout the range of continuity of the dispersion curves, i.e. $f_2(k_1^2) < f_3(k_1^2)$ there. This statement holds within the one-loop approximation, where this range is $0 < k_0^2 - k_3^2 < 4m^2$, for all external fields and is crucial for establishing the kinematical selection rules for the photon splitting process [16]. In approximation-independent way this statement in the infrared limit might be expressed, following eqs. (27), (28) as

$$\mathcal{L}_{\Phi \Phi} - \mathcal{L}_{\Phi \Phi} \geq 2\Phi \frac{\mathcal{L}_{\Phi \Phi} \mathcal{L}_{\Phi \Phi}}{1 - \mathcal{L}_{\Phi}}. \quad (70)$$

Bearing in mind that the quantities $\mathcal{L}_{\Phi}, \mathcal{L}_{\Phi \Phi}, \mathcal{L}_{\Phi \Phi}$ are of the order of the fine structure constant, this may be simplified just to

$$\mathcal{L}_{\Phi \Phi} > \mathcal{L}_{\Phi \Phi}. \quad (71)$$

We, however, do not know whether this statement, simple as it is, can be deduced from any fundamental principle. We can argue, nevertheless, that the inequality $f_2(k_1^2) < f_3(k_1^2)$,
once fulfilled in the one-loop approximation (small $\alpha$), or at least in the infrared limit, or 
at least for small magnetic field, will remain valid for any $\alpha$, any momentum and any field. In other 
words, dispersion curves of modes 2 and 3, considered as functions of any of these 
parameters, cannot intersect, except in the point $k_\mu = 0$. Indeed, if they did, i.e. if the 
equality $f_2(k_\perp^2) = f_3(k_\perp^2)$ might hold for a given choice of $\alpha$, $B$, and $k_\mu \neq 0$, it would follow 
from (9) and (13) that also 

$$\kappa_2 = \kappa_3$$

would be true on the mass shell for the same choice. This degeneracy does take place at zero 
momentum due to the property (18), also in the free case $\alpha = 0$, where all $\kappa_a$’s are zero, and 
in the no-external-field case, where the isotropy of the vacuum is expressed as $\kappa_1 = \kappa_2 = \kappa_3$, 
but would imply an extra symmetry in the case of nonzero momentum.

Before discussing what sort of symmetry this might be, we dwell on other degeneracies 
of the polarization operator - the ones that are due to the residual Lorentz invariance left 
after the magnetic field is imposed. These are the invariance under rotations about the 
magnetic field direction (when $k_\perp = 0$) and under Lorentz boosts along the magnetic field 
(when $k_3 = k_0 = 0$). In the limit $k_\perp = 0$ the eigenvectors $\nu^{(1,3)}_\mu (4)$, when normalized, turn 
into two unit 2-vectors lying in the plane orthogonal to the magnetic field and orthogonal 
to each other. They transform through each other under rotations in this plane, while $\nu^{(2)}_\mu$ 
remains intact (see [17] or eq. (114) in Appendix for the explicit form of the eigenvectors 
in the special frame to make sure of this fact). Hence, referring to the representation (3), 
the isotropy of the polarization tensor in this plane is expressed as 

$$\kappa_1|_{k_\perp=0} = \kappa_3|_{k_\perp=0}.$$  

This degeneracy provides that the virtual longitudinally directed photons of modes 2 and 3, 
whose electric fields are lying in the plane orthogonal to the magnetic field and are transverse 
to each other, may be linearly combined to form two counter- and clockwise transversely 
polarized eigenmodes. In the meanwhile the mode 2 is a longitudinally polarized virtual 
eigenwave directed along the magnetic field that corresponds to the quite different eigenvalue 
$\kappa_2$.

In the other limiting case of $k_3 = k_0 = 0$, quite analogously, the eigenvectors $\nu^{(1,2)}_\mu (4)$ 
turn after normalization into two unit mutually orthogonal 2-vectors lying in the hyperplane
$(k_3, k_0)$. They transform through each other under Lorentz boosts along the magnetic field, while $\gamma^{(3)}_\mu$ is unchanged. Hence, the isotropy of the polarization tensor $(3)$ in this hyperplane is expressed as

$$\kappa_1|_{k_{0,3}=0} = \kappa_2|_{k_{0,3}=0}.$$  \hspace{1cm} (74)

Eqs. (74) and (73) are certainly obeyed within the one-loop approximation.

We now come back to the wouldbe equality (72). Noting that $\gamma^{(2)}_\mu$ in (4) is a pseudovector, whereas $\gamma^{(3)}_\mu$ is a vector, we see that (72), if true, would imply the on-shell degeneracy with respect to parity. The transformation that interchanges the vectors $\gamma^{(2,3)}_\mu$ is the discrete duality transformation $B \rightarrow iE$, $E \rightarrow -iB$, $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu}$ – not to be confused with continual duality. If we complete the definition of the duality transformation by requiring that on-shell the photon momenta do not change under it we find that eq. (72) would express the invariance of the polarization operator in the form (3) under the duality transformation. No such invariance holds in QED already because there is no magnetic charge carrier in it. (The effective Lagrangian on the class of constant fields is still dual-invariant, since the scalars $\mathfrak{F}$ and $\mathfrak{G}$ on which it depends are. The Born-Infeld Lagrangian above shared the same property, but it was completed by the on-shell invariance (72) of the polarization operator as well, expressed as the absence of birefringence, since the asymmetry between virtual magnetic and electric charges (electrons) does not lie in the basis of Born-Infeld theory). Eq. (72) is not fulfilled in any known approximation, except for the trivial situations listed above. We conclude that an intersection of dispersion curves of modes 2 and 3 should be ruled out as a completely unbelievable event.

V. DISCUSSION

In the present paper, for establishing obligatory properties of the effective Lagrangian we exploited two general principles – unitarity and causality – taken in the special form of the requirements of nonnegativity of the residue (17) and of boundedness of the group velocity (14). We feel it necessary to confront this way of action with other approaches.

Usually, consequences of causality are discussed referring to holomorphic properties of the polarization operator (dielectric permittivity tensor) that follow from the retardation of the linear response and are expressed – after being supplemented by certain statements con-
cerning the high-frequency asymptotic conditions – as the Kramers-Kronig (once-subtracted) dispersion relations. Although the general proof of an analog of the Kramers-Kronig relation in a background field is lacking from the literature, for the magnetized vacuum the holomorphity of the polarization operator eigenvalues $\kappa_a$ in a cut complex plane of the variable $(k_0^2 - k_3^2)$ was established within the one-loop approximation \cite{5}, \cite{6}, the probability of electron-positron pair creation by a photon making the cut discontinuity. Nevertheless, as we could see in Section 3, this approximation contradicts some consequences of the causality. Thus, the knowledge of the holomorphic properties is not enough to exploit the causality requirement at full.

More specifically the causality is approached by studying what is called "causal propagation". Here the Hadamard’s method of characteristic surface (the wave front), across which the first derivative of the propagating solution may undergo a discontinuity is used. The propagation is causal if the normal vector to the characteristic surface is time- or light-like. (This criterion looks very close to the group velocity criterion \cite{14} appealed to by us.) Certain conditions obtained in this way that should be obeyed by the "structural function $H$", the knowing of which is equivalent to the effective Lagrangian, may be found among numerous relations in a scrupulous study of Jerzy Plebański. It seems, however, that inequalities (9.176) derived in his Lectures \cite{15}, relating to the general case $\mathfrak{F} \neq 0$, $\mathfrak{G} \neq 0$, and the subsequent formulae, relating to the null-field subcase, $\mathfrak{F} = \mathfrak{G} = 0$, need to be supplemented by consequences of some requirements intended to substitute for unitarity or positiveness of the energy, not exploited in \cite{15}, before/in-order-that a comparison with our conclusions might become possible.

On the other hand, when considering the causal propagation the implementation \cite{8} of Dominant Energy Condition (DEC) \cite{18} completed by Weak Energy Condition (WEC) \cite{17} is also popular. The first one implies that the causality is reassured while solving the Cauchy boundary problem. One might expect that these two conditions are equivalent: first one to the group-velocity boundedness, and second one, at least partially, to the unitarity as the completeness of the set positive-energy states. The implementation of DEC and WEC to the problem of elementary excitations over the magnetized vacuum undertaken in Subsection C of Section II has indicated, however, as we already discussed it in that Subsection, that these two conditions lead to somewhat weaker conclusions than the ones that followed in Subsection B from imposing the conditions \cite{17}, \cite{14}.
We conclude by the remark that previously the appeal to the group velocity has shown its fruitfulness in establishing the phenomenon of canalization of the photon energy along the external magnetic field [23], [3] and the capture of gamma-quanta by a strong nonhomogeneous magnetic field of a pulsar [24], [4].

Appendix

Here we are going to reveal direct physical meanings to the constants involved in eqs. (29), (32) in terms of various long-wave limits of the dielectric and magnetic permeability of the vacuum in external magnetic field. Before doing it we have to define these notions within the technique of eigenvalues of the polarization operator [4], [6] used throughout the present paper. We shall stress that the proportionality relation between the electric induction and electric field strength for electrostatic case common in homogeneous isotropic medium cannot be naively extended to the vacuum with magnetic field even though only one polarization mode is dealt with. On the contrary, different small-momentum limits of one and the same scalar dielectric function serve polarization of external electric charges of different configurations.

For the sake of direct comparison with the case of electrodynamics of homogeneous isotropic medium we shall first consider this case using the corresponding version of the technique of eigenvalues of the polarization operator [18].

In any homogeneous background the (second pair of) Maxwell equations can be written in the form

\[
(k^2 g_{\mu\rho} - k_\mu k_\rho) A^\rho(k) - \Pi_{\mu\rho}(k) A^\rho(k) = -j_\mu(k), \quad (j_\mu(k)k^\mu) = 0, \tag{75}
\]

where \(j_\mu(k)\) is the conserved (4-transversal) external current and \(\Pi_{\mu\rho}(k)\) is the 4-transversal polarization operator, \(k_\mu \Pi_{\mu\rho}(k) = \Pi_{\mu\rho}(k) k_\rho = 0\). Define the electric induction \(d\) as

\[
d_n = \varepsilon_{nj} e_j \quad n, j = 1, 2, 3, \tag{76}
\]

where \(e\) is the electric field

\[
e_n = -i(k_0 A_n - k_n A_0) \tag{77}
\]

and

\[
\varepsilon_{nj} = \delta_{nj} + \frac{\Pi_{nj}}{k_0^2}. \tag{78}
\]
With the definition of the magnetic induction
\[ b_n = -i \epsilon_{nji} k_j A_i \] (79)
the first pair of the Maxwell equations \( \mathbf{k} \mathbf{b} = 0 \) and \( \epsilon_{nji} k_j \epsilon_i - k_0 b_n = 0 \) is satisfied as a consequence of the definitions (77), (79), while (75) becomes the second pair of the (linearized near the external field) Maxwell equations for \( \mathbf{d} \) and \( \mathbf{b} \) with external source
\[ -i k \mathbf{d} = j_0, \quad -i(\epsilon_{nji} k_j b_i + k_0 d_n) = j_n \] (80)
with no polarization charges and currents explicitly involved. These equations are valid in the regions, where the fields produced by the sources \( j \) are small as compared to the external field. This form of the Maxwell equations, wherein the magnetic field strength and induction are not distinguished may be found in [19]. We reserve the letter \( h \) for the magnetic field produced by the same currents, but with all the magnetization effects disregarded. The magnetic permeability will be defined with respect to those fields.

We shall also need below eq.(76) in the form
\[ d_n = \epsilon_n + i \frac{\Pi_{n\mu} A^\mu}{k_0}, \] (81)
which follows from (77), (78) and the 4-transverseness of the polarization tensor. The Fourier transform is defined as
\[ D_{\mu\nu}(x) = \frac{1}{(2\pi)^4} \int \exp(ikx) D_{\mu\nu}(k) \, d^4k, \quad \mu, \nu = 0, 1, 2, 3. \] (82)
The equation giving the 4-potential in terms of the external 4-current to be used throughout this Appendix is
\[ A_\mu(x) = \int D_{\mu\nu}(x - y) j^\nu(y) d^4y, \quad \mu, \nu = 0, 1, 2, 3. \] (83)
Here \( x \) and \( y \) are 4-coordinates, and \( D_{\mu\nu}(x - y) \) is the photon Green function in a magnetic field in the coordinate representation.

A. Isotropic medium

The most general covariant 4-transversal polarization tensor of an isotropic homogeneous medium [20] formed with the use of the 4-velocity \( u_\mu \) of the medium is diagonal [18] in the
basis \( a_\mu, c_\mu^{(n)}, n = 1, 2 \)

\[
\Pi_{\mu\nu}(k) = \kappa_1(k^2, (uk)^2) \sum_{n=1,2} \frac{c_\mu^{(n)} c_\nu^{(n)}}{(c^{(n)})^2} + \kappa_2(k^2, (uk)^2) \frac{a_\mu a_\nu}{a^2}
\]

(84)

with

\[
a_\mu = u_\mu k^2 - k_\mu (uk), \quad a^2 = k^2(k^2 - (uk)^2), \quad (au) = 0,
\]

(85)

and \( c_\mu^{(1)} \) defined as any 4-vector orthogonal to the hyperplane, where the two vectors \( k_\mu \) and \( a_\mu \) lie, whereas \( c_\mu^{(2)} \equiv \varepsilon_{\mu\nu\rho\lambda} c_\nu^{(1)} a_\rho k_\lambda \) is also orthogonal to this hyperplane and, besides, orthogonal to \( c_\mu^{(1)} \). Thus the four vectors \( k_\mu, a_\mu, \) and \( c_\mu^{(1,2)} \) make an orthogonal basis in the Minkowski space. They are four eigenvectors of the polarization operator

\[
\Pi_{\mu\nu}(k) c_\nu^{(1,2)} = \kappa_1 c_\mu^{(1,2)}, \quad \Pi_{\mu\nu}(k) a_\nu = \kappa_2 a_\mu, \quad \Pi_{\mu\nu}(k) k_\nu = 0.
\]

(86)

Only three basis vectors appear in the decomposition (84) because one eigenvalue is zero, according to (86). The four basis vectors \( k_\mu, a_\mu, \) and \( c_\mu^{(1,2)} \) are 4-vector potentials of the electromagnetic eigen-waves. In the rest-frame of the medium \( (u_\mu = \delta_{\mu0}) \) (and arbitrary normalization) these may be taken in the form – the components are counted downwards as \( \nu = 0, 1, 2, 3 \) –

\[
c_\nu^{(1)} = \begin{pmatrix} 0 \\ k_3 \\ 0 \\ -k_1 \end{pmatrix}, \quad c_\nu^{(2)} = \begin{pmatrix} 0 \\ k_1 k_2 \\ -(k_3^2 + k_1^2) \\ k_2 k_3 \end{pmatrix},
\]

\[
a_\nu = k^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - k_0 \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad k_\nu = \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_\nu.
\]

(87)

The orientations of the corresponding electric and magnetic fields, calculated basing on these vector-potentials, are described in detail in [18]. In the Lorentz frame, where the medium is at rest, mode 1 is transversely-polarized electromagnetic wave, while mode 2 is purely longitudinal electric wave, its magnetic field being equal to zero. The degeneracy
corresponding to the fact that there is a common eigenvalue $\kappa_1$ for the two eigenvectors $c^{(1,2)}_\mu$ reflects the axial symmetry of the problem, which in the rest frame reduces to the symmetry under rotations around the direction of the photon 3-momentum $k$. If the kinematical condition $k^2 = (u k)^2$ is fulfilled, additional degeneracy

$$
\kappa_1(k^2, k^2) = \kappa_2(k^2, k^2), \quad \text{if} \quad (u k) \neq 0 \quad (88)
$$

appears that reflects a symmetry, which in the rest frame is spherical symmetry due to the disappearance of the direction specialized by the photon 3-momentum: in this frame the above kinematic condition becomes just $k^2 = 0$. The above-said relates to real (on-shell) photons of the eigenmodes and to virtual (off-shell) photons, as well. The latter are subject to two, generally different, dispersion equations

$$
k^2 = \kappa_{1,2}(k^2, (u k)^2). \quad (89)
$$

To see this consider the Schwinger-Dyson equation for the photon Green function $D_{\mu\nu}(k)$ in momentum space:

$$(k^2 g_{\mu\rho} - k_\mu k_\rho) D^\rho_\sigma(k) - \Pi_{\mu\rho}(k) D^\rho_\nu(k) = g_{\mu\nu} - k_\mu k_\nu / k^2. \quad (90)$$

After the substitution of (84) its solution is readily found to be

$$
D_{\mu\nu}(k) = \frac{1}{k^2 - \kappa_1} \sum_{n=1,2} c^{(n)}_\mu c^{(n)}_\nu \left( \frac{a_\mu a_\nu}{a^2} + k_\mu k_\nu D^L(k) \right). \quad (91)
$$

Here $D^L(k)$ is an arbitrary function, not determined by the Schwinger-Dyson equation.

1. Electrostatics in isotropic medium.

Consider the electrostatic problem with a source comprised of charges that are at rest in the rest frame of the medium:

$$
j_\nu(k) = \delta_\nu \delta(k_0) q(k), \quad (j k) = 0. \quad (92)
$$

Then the field produced by this static source is given by the vector-potential

$$
A_\mu(x) = \frac{1}{(2\pi)^3} \int D_{\mu\rho}(0, k) \exp(-i k x) q(k) d^3k. \quad (93)
$$
Here zero stands for the $k_0$-argument of the photon propagator. Among the eigenvectors there is only one, whose zeroth component survives the substitution $k_0 = 0$. It is $a_\nu$. For this reason only the second term in (91) contributes to (93). Spatial components of $a_\nu$ are zero in this limit. Therefore,

$$A_0(x) = \frac{1}{(2\pi)^3} \int \frac{e^{-ikx}q(k)d^3k}{k^2 - \kappa^2(k^2,0)}, \quad A_{1,2,3}(x) = 0. \quad (94)$$

Certainly, the static potential has only its zeroth component different from zero and carries no magnetic field. Using this fact in the definition of the induction (81) we get for the induction (76) corresponding to the potential (94) (note that $a_n/k_0 = -k_n$ and that $a^2 = k^2k^2$)

$$d = i k A_0(k) \left( 1 - \frac{\kappa^2(k^2,0)}{k^2} \right) = e\varepsilon(k^2), \quad (95)$$

where

$$\varepsilon(k^2) = 1 - \frac{\kappa^2(k^2,0)}{k^2} \quad (96)$$

is the static dielectric permittivity with spacial dispersion, equal to the inverse refraction index squared, $\varepsilon(k^2) = n^2 - 1$, defined on the mass shell as (19). The field strength and induction are parallel in the momentum space, but in the configuration space,

$$e(x) = \frac{i}{(2\pi)^3} \int k e^{-ikx}q(k)d^3k, \quad (97)$$

$$d(x) = \frac{i}{(2\pi)^3} \int \frac{k e^{-ikx}q(k)d^3k}{k^2} \quad (98)$$

they, generally, are not, except, for instance, spherical charge distribution, $q(k) = q(k^2)$ or when considered far from the charges – where one may take $q(k) \approx q(0)$, – and some other special cases.

Let us examine, next, a homogeneously charged, infinitely extended plane, say the $(1,2)$-coordinate plane. This corresponds to the choice $q(k) = (2\pi)^2 \rho \delta^2(k_\perp)$ with the 2-vector $k_\perp$ lying in the chosen plane and $\rho$ being a constant surface charge density. Then

$$e_3(x_3) = \frac{i\rho}{2\pi} \int \frac{k_3 e^{-ik_3x_3}d^3k_3}{k_3^2 \varepsilon(k_3^2)}, \quad e_{1,2}(x) = 0. \quad (99)$$
If $|x_3|$ is large, only small $|k_3|$ contribute. Then we get for the electric field the asymptotic expression

$$e_3(x_3) \approx \frac{i \rho}{2 \pi \varepsilon(0)} \int \frac{e^{-ik_3x_3}k_3dk_3}{k_3^2}$$

that is $1/\varepsilon$ multiplied by the field of the charged plane without the polarization taken into account. The latter in the present case coincides with the induction $\mathcal{E}$.

To define the integral in the infra-red region one may introduce a regularizing mass parameter $m > 0$ and let it tend to zero afterwards. (This gives the same result as the causal shift of the pole $k_0^2 - k^2 + i0$ in the photon Green function). Then the field is

$$e_3(x_3) \approx \lim_{m \to 0} -\rho \frac{1}{2 \pi \varepsilon(0)} \frac{d}{dx_3} \int \frac{e^{-ik_3x_3}dk_3}{k_3^2 + m^2} = \lim_{m \to 0} -\rho \frac{1}{2 \pi \varepsilon(0)} \frac{d}{dx_3} \exp(-|x_3|m).$$

Finally, in the isotropic medium the electric field of a charged plane parallel to the coordinate plane $(1,2)$ at large distance from this plane is a constant field pointing to the plane:

$$e_3(x_3) \approx \frac{\rho}{2 \varepsilon(0)} \text{sgn}(x_3),$$

where $\text{sgn}(x) = \pm 1$ for $x \geq 0$. We have reproduced this well-known result to stress that it is direction-independent: it gives the electric field, orthogonal to the chosen charged plane, as a function of the distance from that plane by a universal formula, independent of the orientation of the plane. We shall see in the next subsection, how this result will be modified in the magnetized vacuum.

2. **Magneto-statics of isotropic medium.**

Now consider the magneto-static problem with the source corresponding to a constant current flowing in the special frame along the direction 3.

$$j_\mu = \delta_{\mu 3}j(k_\perp)\delta(k_0)\delta(k_3), \quad (kj) = 0,$$

where $k_\perp$ is the two-component momentum in the $(1,2)$-plane. It produces the 4-vector potential

$$A_\mu(x_\perp) = \frac{1}{(2\pi)^3} \int D_{\mu 3}(0, 0, k_\perp) \exp(-ik_\perp x_\perp)j(k_\perp)d^2k_\perp,$$
where the zeros stand for the \( k_0 \) and \( k_3 \)-arguments of the photon propagator. Among the eigenvectors (87) there is only one, whose third component survives the substitution \( k_0 = 0 \). It is \( c^{(1)}_\nu \). For this reason only the first term in (91) with \( n = 1 \) contributes to (104). The \( (\nu \neq 3) \)-components of \( c^{(1)}_\nu \) are zero in this limit. Therefore,

\[
A_3(x_\perp) = \frac{1}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp} j(k_\perp) d^2k_\perp}{k_\perp^2 - \kappa_1(k_\perp^2, 0)}, \quad A_{0,1,2}(x_\perp) = 0.
\]

(105)

This 4-potential has only its third component different from zero and carries no electric field. The magnetic induction, formed with the use of this 4-potential according to eq. (79)

\[
b_n(x_\perp) = -\frac{i\epsilon_{nm3}}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp} j(k_\perp) k_m d^2k_\perp}{k_\perp^2 - \kappa_1(k_\perp^2, 0)}
\]

(106)
differs from the magnetic field \( h_n^{\text{non}}(x_\perp) \) produced by the same current (103) in the absence of the medium (i.e. when \( \kappa_1 = 0 \))

\[
h_n^{\text{non}}(x_\perp) = -\frac{i\epsilon_{nm3}}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp} j(k_\perp) k_m d^2k_\perp}{k_\perp^2}
\]

(107)
by the factor in the integrand

\[
\mu(k_\perp^2) = \left(1 - \frac{\kappa_1(k_\perp^2, 0)}{k_\perp^2}\right)^{-1}
\]

(108)
to be identified as magnetic permeability. Its long-wave limit \( \mu(0) = (1 - (\kappa_1(k^2, 0)/k^2)|_{k^2=0})^{-1} \) serves the asymptotic behavior of magnetic field \( b(x_\perp) \), \( |x_\perp| \to \infty \) produced by the current, which flows along the axis \( x_3 \) and whose density decreases in the orthogonal plane \( (1,2) \) away from the origin sufficiently fast, \( j(0) \neq \infty \) in (103) - otherwise the integral might be infra-red-divergent. In this case

\[
b_n(x_\perp) = -\frac{i\epsilon_{nm3}}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp} \mu(k_\perp^2) j(k_\perp) k_m d^2k_\perp}{k_\perp^2} \approx -\frac{i\epsilon_{nm3} j(0) \mu(0)}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp} k_m d^2k_\perp}{k_\perp^2}
\]

(109)
Eq. (109) also covers the case of the current, flowing along an infinitely thin cylindric rectilinear wire, with \( j(k_\perp) \) being taken as \( j(k_\perp) = 2\pi J \), where \( J \) is the total constant current. Then

\[
b_n(x_\perp) \approx -\frac{iJ \mu(0) \epsilon_{nm3}}{(2\pi)^2} \int \frac{e^{-ik_\perp x_\perp} k_m d^2k_\perp}{k_\perp^2} = \frac{-iJ \mu(0) \epsilon_{nm3}}{(2\pi)^2 |x_\perp|} \int \frac{e^{-ik_\perp x_\perp} k_m d^2k_\perp}{k_\perp^2} \]

(110)
where \( \hat{\mathbf{x}}_\perp \) is the unit vector along the transverse coordinate \( \hat{\mathbf{x}}_\perp = \frac{\mathbf{x}_\perp}{|\mathbf{x}_\perp|} \). To make sure that the last integration is correct note that the projection of the integral onto the direction orthogonal to \( \mathbf{x}_\perp \)

\[
\int_0^\infty dk \int_0^{2\pi} d\phi \sin \phi e^{-ik \cos \phi} = 0
\]

(111)
disappears due to the angle integration. It could not be otherwise, since \( \mathbf{x}_\perp \) is the only vector in the integrand, hence the integral, which is a vector, and not a pseudovector, cannot help being parallel to it. The projection on \( \mathbf{x}_\perp \) is

\[
\int_0^{2\pi} d\phi \int_0^\infty dk \cos \phi e^{-ik \cos \phi} = -i2\pi
\]

(112)

Certainly, the well-known relation (110) does not, in the present case of isotropic medium, depend on the specific choice of the direction of the current along the axis \( x_3 \) made above.

**B. Magnetized vacuum**

In the present subsection we refer to the special frame and find it more convenient to list the arguments in the eigenvalues \( \kappa_a \) in (8) in a different order, also without indicating the dependence on the magnetic field:

\[
\kappa_a(k^2, -kF^2k, \vec{\mathbf{k}}) = \bar{\kappa}_a(k^2, k_3^2, k_\perp^2)
\]

(113)
The first three (meaningful) 4-eigenvectors \( \{\vec{\mathbf{b}}\} \) of the polarization tensor \( \Pi_{\mu\nu} \) take in the special frame (up to the normalization, which we chose differently here) the form - the components are counted downwards as \( \nu = 0, 1, 2, 3 \) -

\[
\vec{\mathbf{b}}^{(1)}_\nu = k^2 \begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \end{pmatrix}_\nu, \quad \vec{\mathbf{b}}^{(2)}_\nu = \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_\nu, \quad \vec{\mathbf{b}}^{(3)}_\nu = \begin{pmatrix} 0 \\ k_0 \\ -k_1 \\ k_2 \end{pmatrix}_\nu.
\]

(114)

Their lengths are

\[
(\vec{\mathbf{b}}^{(1)})^2 = -k^2 k_\perp^2 (k_0^2 - k_3^2), \quad (\vec{\mathbf{b}}^{(2)})^2 = -(k_0^2 - k_3^2), \quad (\vec{\mathbf{b}}^{(3)})^2 = -k_\perp^2.
\]

(115)
1. Electrostatics of magnetized vacuum.

Consider the electrostatic problem with source \( (92) \) comprised of charges that are at rest in the special frame. Then \( (83) \) results again in \( (93) \).

Among the eigenvectors \( (114) \) there is only one, whose zeroth component survives the substitution \( k_0 = 0 \). It is \( \flat(2) \). This implies that out of the four ingredients of the general decomposition of the photon propagator \( (8) \) only the term with \( a = 2 \), \( D_2(k)\flat(2)\flat(2)/(\flat(2))^2 \), participates in \( (93) \), i.e. only mode-2 (virtual) photon may be a carrier of electro-static interaction, and not photons of modes 1,2, nor the purely gauge mode 4. Bearing in mind that \( (\flat(2))^2 = k_3^2 - k_0^2 \), we have

\[
A_0(x) = \frac{1}{(2\pi)^3} \int \frac{e^{-i \mathbf{k} \cdot \mathbf{x}} q(\mathbf{k}) d^3k}{\mathbf{k}^2 - \bar{\kappa}_2(0, k_3^2, k_\perp^2)}, \quad A_{1,2,3}(x) = 0. \tag{116}
\]

Here \( k_\perp^2 = k_1^2 + k_2^2 \). Thus, the static charge gives rise to electric field only, as it might be expected. Equation for electric field \( (77) \) corresponding to the potential \( (116) \) can be represented as

\[
e(x) = \frac{1}{(2\pi)^3} \int \frac{i k e^{-i \mathbf{k} \cdot \mathbf{x}} q(\mathbf{k}) d^3k}{\mathbf{k}^2 - \bar{\kappa}_2(0, k_3^2, k_\perp^2)} = \frac{1}{(2\pi)^3} \int \frac{i k e^{-i \mathbf{k} \cdot \mathbf{x}} q(\mathbf{k}) d^3k}{\varepsilon(k_3^2, k_\perp^2) \mathbf{k}^2}, \tag{117}
\]

where

\[
\varepsilon(k_3^2, k_\perp^2) = 1 - \frac{\bar{\kappa}_2(0, k_3^2, k_\perp^2)}{k^2} \tag{118}
\]

is the coefficient – to be understood as the dielectric function – of proportionality between the (Fourier transforms of) electric field in the magnetized vacuum and that with the vacuum polarization disregarded

\[
e_{\text{non}}(x) = \frac{1}{(2\pi)^3} \int \frac{i k e^{-i \mathbf{k} \cdot \mathbf{x}} q(\mathbf{k}) d^3k}{\mathbf{k}^2}. \tag{119}
\]

It is equal to the inverse of the refraction index squared \( (19) \). Eq. \( (119) \) does not coincide with the induction, \( e_{\text{non}}(x) \neq d(x) \), defined as \( (76) \) or \( (81) \), which is not, generally, parallel with the electric field already in the momentum space

\[
d_n = i k_n A_0(\mathbf{k}) \left( 1 - \delta_{n3} \frac{\bar{\kappa}_2(0, k_3^2, k_\perp^2)}{k_3^2} \right) = i k_n q(\mathbf{k}) \frac{1 - \delta_{n3} \bar{\kappa}_2(0, k_3^2, k_\perp^2)}{(2\pi)^3 \frac{k^2 - \bar{\kappa}_2(0, k_3^2, k_\perp^2)}}. \tag{120}
\]
Once, unlike the isotropic case (96), $\kappa_2$ depends separately on the two momentum squared components $k_3^2, k_\perp^2$, there is no universal, direction-independent static dielectric permeability. On the contrary, depending on the character of the external charge distribution, one may speak of different dielectric functions, which are, for instance, the two different long-wave limits of (118)

$$
\varepsilon_{\text{long}}(k_3^2) = 1 - \frac{\kappa_2(0, k_3^2, 0)}{k_3^2}, \quad \varepsilon_{\text{tr}}(k_\perp^2) = 1 - \frac{\kappa_2(0, 0, k_\perp^2)}{k_\perp^2}.
$$

(121)

These two dielectric functions control the potential far from the region where the charges are located, in the directions across, $\varepsilon_{\text{long}}(k_3^2)$, and along the magnetic field, $\varepsilon_{\text{tr}}(k_\perp^2)$. One of situations of that sort, namely the field of a point-like charge that decreases with different speeds along different directions following an anisotropic Coulomb law, was studied in detail in [21], [17] (see also [22]) in the limit of large magnetic field. If, on the contrary, the charge is not localized, but distributed homogeneously along or across the magnetic field the role of these two dielectric functions may be described more definitely.

The first one, $\varepsilon_{\text{long}}(k_3^2)$, is responsible for polarization caused by the charge distribution, homogenous in the direction orthogonal to the magnetic field, $q(k) = \delta^2(k_\perp)\tilde{q}(k_3)$. In particular, the electric field strength of a plane $(1,2)$ charged with a constant surface density $\rho$, $\tilde{q}(k_3) = (2\pi)^2\rho$, - oriented transversally to the external magnetic field - is obtained from (117) as

$$
e_3(x_3) = i\rho \int \frac{k_3 e^{-ik_3 x_3} dk_3}{k_3^2 - \kappa_2(0, k_3^2, 0)}, \quad e_\perp(x) = 0.
$$

(122)

If $|x_3|$ is large, only small $|k_3|$ contribute. Then, keeping the lowest term in the power series expansion of $\kappa_2(0, k_3^2, 0)$ with respect to $k_3^2$, we get – in the same way as in the previous subsection – that in the vacuum, magnetized along the axis $x_3$, the electric field of a charged plane parallel to the co-ordinate plane $(1,2)$ at large distance from this plane is a constant field pointing to the plane:

$$
e_3(x_3) \approx \frac{\rho}{2\varepsilon_{\text{long}}(0) \text{sgn}(x_3)}.
$$

(123)

In this case the induction is the same as the electric field without the vacuum polarization, $\varepsilon_{\text{long}}(0)e(x) = e_{\text{non}}(x) = d(x)$.

The second dielectric function (121), $\varepsilon_{\text{tr}}(k_\perp^2)$, is responsible for polarization caused by the charge distribution, homogenous in the direction parallel to the magnetic field, $q(k) = \delta^2(k_\perp)$.
\[ \delta(k_3) \tilde{q}(k_\perp). \] In particular, the electric field of the coordinate plane \((1, \beta)\) charged with the constant surface density \(\rho\), \(\tilde{q}(k_\perp) = (2\pi)^2 \rho \delta(k_1)\) is obtained from (117) as

\[ e_2(x_2) = \frac{\rho}{2\pi} \int \frac{ik_2 e^{-ik_2 x_2} dk_2}{k_2^2 - \overline{k}_2(0, 0, k_2^2)}, \quad e_{1,3}(x) = 0. \tag{124} \]

Keeping this time the lowest term in the power series expansion of \(\varepsilon(0, k_2^2)\) with respect to \(k_2^2\), we get far from the surface

\[ e_2(x_2) \approx \frac{\rho}{2\varepsilon_{\text{tr}}(0)} \text{sgn}(x_2). \tag{125} \]

Certainly, due to the axial symmetry of the problem this result is basically the same for any charged plane containing the vacuum magnetization direction \(\beta\), but differs from (123) in that it contains the different dielectric constant. In the present case the induction is not \(\varepsilon_{\text{tr}}(0)e(x) = e_{\text{non}}(x)\), but, on the contrary, coincides with the electric field \(d(x) = e(x)\). We, nevertheless, define the dielectric permeability with respect to \(e_{\text{non}}(x)\), and not with respect to \(d(x)\), the latter being only introduced to give the Schwinger-Dyson equation the form of the Maxwell equations in a medium.

By confronting eqs. (121) with (24) we establish the connection between the dielectric constants \(\varepsilon_{\text{long}}(0), \varepsilon_{\text{tr}}(0)\) with the quantities (29), and thus the nonnegativity of the former

\[ \varepsilon_{\text{tr}}(0) = 1 - \mathcal{L}_\delta \geq 0, \quad \varepsilon_{\text{long}}(0) = 1 - \mathcal{L}_\delta + 2\mathcal{L}\mathcal{E}_\theta \geq 0. \tag{126} \]

2. Magneto-statics of magnetized vacuum.

Now consider the magneto-static problem in the magnetized vacuum with the source (103) corresponding to a constant current flowing in the special frame along the direction of the external magnetic field \(\beta\). It produces the same 4-vector potential as (104), but with the photon propagator given as (8). Among the three meaningful eigenvectors (114) with \(a = 1, 2, 3\) there is only one, whose third component survives – after normalization – the substitution \(k_0 = 0\). It is \(b_\nu^{(1)}\). Indeed, \((\tilde{z}^{(1)})^2 = k^2 k_\perp^2 (k_3^2 - k_0^2) = k^2 k_\perp^2 k_3^2\), hence \(b_3^{(1)}/\sqrt{(b^{(1)})^2} = 1\) after \(k_3 = 0\) is substituted. For this reason only the first term in (8) with \(a = 1\) contributes to (104). The \((\nu \neq 3)\)-components of \(b_\nu^{(1)}\) are zero in this limit. Therefore, for the vector potential we have the equation

\[ A_3(x_\perp) = \frac{1}{(2\pi)^3} \int \frac{e^{-ik_\perp x_\perp}}{k_\perp^2 - \overline{k}_1(0, 0, k_\perp^2)} j(k_\perp) dk_\perp, \quad A_{0,1,2}(x_\perp) = 0, \tag{127} \]
very similar to (105), but with the external-magnetic-field-dependent eigenvalue ω1, from (5), (8). The magnetic induction, formed with the use of this 4-potential according to eq. (79) differs from the magnetic field produced classically by the same current (103) in the non-magnetized vacuum (i.e. when ω1 = 0) by the factor in the momentum space

\[ \mu^\text{w}_{1\text{r}}(k^2) = \left(1 - \frac{\bar{\omega}_1(0,0,k^2)}{k^2}\right)^{-1} \]  

(128)
to be identified as magnetic permeability. Its long-wave limit \( \mu^\text{w}_{1\text{r}}(0) = \left(1 - \bar{\omega}_1(0,0,k^2)/k^2\right)_{k^2=0}^{-1} \) serves the asymptotic behavior of magnetic induction \( b(x) \), \(|x| \to \infty \) produced by the current, which flows along the external magnetic field and whose density decreases in the orthogonal plane (1,2) away from the origin sufficiently fast, \( j(0) \neq \infty \) in (103). (This case includes the straight-linear current of an infinitely thin wire.) Now, eqs. (109) and (110) for the magnetic induction of the current oriented along the axis 3, remain valid, but with \( \mu(k^2) \) and \( \mu(0) \) replaced, respectively, by \( \mu^\text{w}_{1\text{r}}(k^2) \) and \( \mu^\text{w}_{1\text{r}}(0) \) in them.

The same quantity (109) controls the magnetic induction of a current also flowing parallel to the axis 3, but homogeneously concentrated on the plane that contains the external magnetic field, say the (1,3)-plane, \( j(k^2) = (2\pi)^2 j\delta(k_1) \), where \( j \) is a finite density of current per unit length along the axis 1. Now, the potential (127) becomes

\[ A_3(x) = \frac{j}{(2\pi)} \int \frac{e^{-ik_2x_2}dk_2}{k_2^2\mu^\text{w}_{1\text{r}}(k^2)} \]  

(129)
and the corresponding magnetic induction far from the surface, behaves (the infra-red issue to be treated in the same way as in the electrostatic problem of a charged plane (100) above) as

\[ b_1(x) \approx \frac{j\mu^\text{w}_{1\text{r}}(0)}{2\text{sgn}(x_2)}. \]  

(130)
The formulae hitherto obtained in this Subsubsection, however, are not applicable to other directions of the current.

Let us, then, examine a constant current flowing in the magnetized vacuum across its magnetic field, say along the axis 1

\[ j_\nu(k) = \delta_\nu_1\delta(k_0)\delta(k_1)j(k_3, k_2), \quad (jk) = 0. \]  

(131)
The only vector among (114) that has nonzero component 1, when \( k_1 = k_0 = 0 \), and thus exclusively contributes into the photon Green function (8) is \( \phi_\nu^{(3)} \). All the other components
of $\lambda^{(3)}_n$ disappear in this limit. Therefore, the potential produced by the current (131) is

$$A_1(x_2, x_3) = \frac{1}{(2\pi)^3} \int \frac{e^{-i(k_2 x_2 + k_3 x_3)} j(k_2, k_3) dk_2 dk_3}{k_2^2 + k_3^2 - \bar{\kappa}_3(0, k_3^2, k_2^2)}, \quad A_{0,2,3}(x_2, x_3) = 0. \quad (132)$$

An essential difference of this expression from (127) or (105) is that the axis 1 is not a symmetry axis. This is reflected in the fact that $\bar{\kappa}_3(0, k_3^2, k_2^2)$ in (132) does not depend on the combination $k_3^2 + k_2^2$, but contains the variables $k_2^2$ and $k_3^2$ separately. For this reason we have to further specify two different current configurations.

Let, first, the current, flowing transverse to the external magnetic field, in the direction 1, is homogeneously distributed along the direction 3, i.e. along the external magnetic field, $j(k_2, k_3) = \delta(k_3) \tilde{j}(k_2)$. Then, the magnetic induction produced by this current is parallel to direction 3, orthogonal to the current, and parallel to the external magnetic field, and depends only upon the coordinate $x_2$ across the external field:

$$b_3(x_2) = \frac{i}{(2\pi)^3} \int \frac{k_2 e^{-ik_2 x_2} \tilde{j}(k_2) dk_2}{k_2^2 - \bar{\kappa}_3(0, 0, k_2^2)} = \frac{i}{(2\pi)^3} \int \frac{k_2 e^{-ik_2 x_2} \tilde{j}(k_2) \mu_{pl}^{tr}(k_2^2) dk_2}{k_2^2}, \quad (133)$$

where

$$\mu_{pl}^{tr}(k_2^2) = \left(1 - \frac{\bar{\kappa}_3(0, 0, k_2^2)}{k_2^2}\right)^{-1}. \quad (134)$$

If the current, besides, is totally concentrated on the infinitely thin surface coinciding with the coordinate plane $(1,3)$, $\tilde{j}(k_2) = (2\pi)^2 j$, where $j$ is a finite, constant linear current density, defined as the ratio of the total current to the length unit of the axis 3, its magnetic induction far from the surface, behaves as

$$b_3(x_2) \approx \frac{j \mu_{pl}^{tr}(0)}{2} sgn(x_2). \quad (135)$$

Let, second, the current, flowing transverse to the external magnetic field, in the direction 1, is homogeneously distributed along the direction 2, orthogonal to the external magnetic field, $j(k_2, k_3) = \delta(k_2) \tilde{j}(k_3)$. Then, the magnetic induction produced by this current is parallel to direction 2, orthogonal to the current and to the external magnetic field, and depends only upon the coordinate $x_3$ along the external field:

$$b_2(x_3) = -\frac{i}{(2\pi)^3} \int \frac{k_3 e^{-ik_3 x_3} \tilde{j}(k_3) dk_3}{k_3^2 - \bar{\kappa}_3(0, k_3^2, 0)} = -\frac{i}{(2\pi)^3} \int \frac{k_3 e^{-ik_3 x_3} \tilde{j}(k_3) \mu_{pl}^{long}(k_3^2) dk_3}{k_3^2}, \quad (136)$$

where

$$\mu_{pl}^{long}(k_3^2) = \left(1 - \frac{\bar{\kappa}_3(0, k_3^2, 0)}{k_3^2}\right)^{-1}. \quad (137)$$
If the current, besides, is totally concentrated on the infinitely thin surface coinciding with the coordinate plane \((1,2)\), \(\tilde{j}(k_3) = (2\pi)^2 j\), where \(j\) is a finite, constant current density per unit length along the axis 2, its magnetic induction far from the surface, behaves as

\[
b_2(x_3) \approx -\frac{j\mu_{\text{pl}}(0)}{2} \text{sgn}(x_3).
\]

According to (24), (137)

\[
(\mu_{\text{tr}}(0))^{-1} = \left(\mu_{\text{long}}(0)\right)^{-1} = 1 - L_{\delta} \geq 0
\]

and to (24), (134)

\[
(\mu_{\text{tr}}(0))^{-1} = 1 - L_{\delta} + 2F_{\delta} \geq 0.
\]

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