On a conjecture about enumerating \((2+2)\)-free posets

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Abstract. Recently, Kitaev and Remmel posed a conjecture concerning the generating function for the number of unlabeled \((2+2)\)-free posets with respect to number of elements and number of minimal elements. In this paper, we present a combinatorial proof of this conjecture.

Key words: \((2+2)\)-free poset, minimal element.

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1 Introduction

A poset is said to be \((2+2)\)-free if it does not contain an induced subposet that is isomorphic to \(2+2\), the union of two disjoint 2-element chains. In a poset, let \(D(x)\) be the set of predecessors of an element \(x\) (the strict down-set of \(x\)). Formally, \(D(x) = \{ y : y < x \}\). A poset \(P\) is \((2+2)\)-free if and only if its sets of predecessors, \(D(P) = \{ D(x) : x \in P \}\) can be written as

\[ D(P) = \{ D_0, D_1, \ldots, D_k \} \]

where \(\emptyset = D_0 \subset D_1 \subset \ldots \subset D_k\), see [1, 2]. In such context, we say that \(x \in P\) has level \(i\) if \(D(x) = D_i\). An element \(x\) is said to be a minimal element if \(x\) has level 0.

Let \(p_n\) be the number of unlabeled \((2+2)\)-free posets on \(n\) elements. El-Zahar [4] and Khamis [5] used a recursive description of \((2+2)\)-free posets to derive a pair of functional equations that define the generating function for the number \(p_n\). But they did not solve these equations. Recently, using functional equations and the Kernel method, Bousquet-Mélou et al. [2] showed that the generating function for the number \(p_n\) of unlabeled \((2+2)\)-free posets on \(n\) elements is given by

\[ P(t) = \sum_{n \geq 0} p_n t^n = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1-t)^i). \quad (1.1) \]

Note that throughout this paper, the empty product as usual is taken to be 1. In fact, they studied a more general function of unlabeled \((2+2)\)-free posets.
according to number of elements, number of levels and level of minimum maximal elements. Zagier [8] proved that Formula (1.1) is also the generating function for certain involutions introduced by Stoimenow [7].

Given a sequence of integers \( x = (x_1, x_2, \ldots, x_n) \), we say that the sequence \( x \) has an ascent at position \( i \) if \( x_i < x_{i+1} \). The number of ascents of \( x \) is denoted by \( \text{asc}(x) \). A sequence \( x = (x_1, x_2, \ldots, x_n) \) is said to be an ascent sequence of length \( n \) if it satisfies \( x_1 = 0 \) and \( 0 \leq x_i \leq \text{asc}(x_1, x_2, \ldots, x_{i-1}) + 1 \) for all \( 2 \leq i \leq n \). Ascent sequences were introduce by Bousquet-Mélou et al. [2] to unify three combinatorial structures. Bousquet-Mélou et al. [2] constructed bijections between unlabeled \((2+2)\)-free posets and ascent sequences, between ascent sequences and permutations avoiding a certain pattern, between unlabeled \((2+2)\)-free posets and a class of involutions introduced by Stoimenow [7].

Recently, Kitaev and Remmel [6] extended the work of Bousquet-Mélou et al. [2]. They found generating function for unlabeled \((2+2)\)-free posets when four statistics are taken into account, one of which is the number of minimal elements in a poset. The key strategy used by Bousquet-Mélou et al. [2] and Kitaev and Remmel [6] is to translate statistics on \((2+2)\)-free posets to statistics on ascent sequences using the bijection between unlabeled \((2+2)\)-free posets and ascent sequences given by Bousquet-Mélou et al. [2]. Let \( p_{n,k} \) be the number of \((2+2)\)-free posets on \( n \) elements with \( k \) minimal elements, with the assumption \( p_{0,0} = 1 \). Under the bijection between unlabeled \((2+2)\)-free posets and ascent sequences, the number of unlabeled \((2+2)\)-free posets on \( n \) elements with \( k \) minimal elements is equal to that of ascent sequences of length \( n \) with \( k \) zeros. Kitaev and Remmel [6] derived that the generating function for the number \( p_{n,k} \) is given by

\[
P(t, z) = \sum_{n \geq 0, k \geq 0} p_{n,k} z^k t^n = 1 + \sum_{n \geq 0} \frac{zt}{(1-tz)^{n+1}} \prod_{i=1}^{n} (1 - (1-t)^{i-1}(1-zt)),
\]

by counting ascent sequences with respect to length and number of zeros. Moreover, they conjectured the function \( P(t, z) \) can be written in a simpler form.

**Conjecture 1.1**

\[
P(t, z) = \sum_{n \geq 0, k \geq 0} p_{n,k} z^k t^n = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1-t)^{i-1}(1-zt)). \tag{1.2}
\]

The objective of this paper is to give a combinatorial proof of Conjecture 1.1. In order to prove the conjecture, we need two more combinatorial structures: upper triangular matrices with non-negative integer entries such that all rows and
columns contain at least one non-zero entry, which was introduced by Dukes and Parviainen [3], and upper triangular (0, 1)-matrices in which all columns contain at least one non-zero entry.

Let $A_n$ be the collection of upper triangular matrices with non-negative integer entries which sum to $n$. A (0, 1)-matrix is a matrix in which each entry is either 0 or 1. Let $\mathcal{M}_n$ be the set of (0, 1)-matrices in $A_n$ in which all columns contain at least one non-zero entry. Denote by $I_n$ the set of matrices in $A_n$ in which all rows and columns contain at least one non-zero entry. Given a matrix $A$, denoted by $A_{i,j}$ the entry in row $i$ and column $j$. Let $\text{dim}(A)$ be the number of rows in the matrix $A$. The sum of all entries in row $i$ is called the row sum of row $i$, denoted by $\text{rsum}_i(A)$. The column sum of column $i$, denoted by $\text{csum}_i(A)$, can be defined similarly. A row is said to be zero if its row sum is zero.

Let $A$ be a matrix in $M_n$, define $\text{min}_i(A)$ to be the least value of $j$ such $A_{j,i}$ is non-zero. A column $i$ of $A$ is said to be improper if it satisfies one of the following two cases: (1) $\text{csum}_i(A) \geq 2$; (2) for $1 < i \leq \text{dim}(A)$, we have $\text{csum}_i(A) = 1$, $\text{rsum}_i(A) = 0$, and $\text{min}_i(A) < \text{min}_{i-1}(A)$. Otherwise, column $i$ is said to be proper. The matrix $A$ is said to be improper if there is at least one improper column in $A$; otherwise, the matrix $A$ is said to be proper. Given an improper matrix $A \in \mathcal{M}_n$, define $\text{index}(A)$ to be the largest value $i$ such that column $i$ is improper. Denote by $\mathcal{P}\mathcal{M}_n$ the set of proper matrices in $\mathcal{M}_n$.

**Example 1.2** Consider the following matrix $A \in \mathcal{M}_8$:

$$
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

We have $\text{dim}(A) = 6$, $\text{min}_1(A) = 1$, $\text{min}_2(A) = 1$, $\text{min}_3(A) = 1$, $\text{min}_4(A) = 2$, $\text{min}_5(A) = 2$, $\text{min}_6(A) = 1$. There are two improper columns, that is, columns 3 and 6. Hence, we have $\text{index}(A) = 6$.

Denote by $\mathcal{P}\mathcal{M}_{n,k}$ the set of matrices $A \in \mathcal{P}\mathcal{M}_n$ with $\text{rsum}_1(A) = k$ and $\mathcal{I}_{n,k}$ the set of matrices $A \in \mathcal{I}_n$ with $\text{rsum}_1(A) = k$. Dukes and Parviainen [3] constructed a recursive bijection between the set $\mathcal{I}_n$ and the set of ascent sequences of length $n$. Under their bijection, they showed that the number of upper triangular matrices $A \in \mathcal{I}_n$ with $\text{rsum}_1(A) = k$ is equal to the number of ascent sequences of length $n$ with $k$ zeros, which implies that the cardinality of
In this paper, we will prove Conjecture 1.1 by showing that the generating function for the number of matrices in $\mathcal{I}_{n,k}$ is given by the right-hand side of Formula (1.2).

In Section 2, we present a parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$. In Section 3, we prove that the right-hand side of Formula (1.2) is the generating function for the number of matrices in $\mathcal{P}\mathcal{M}_{n,k}$. Moreover, we show that there is a bijection between the set $\mathcal{P}\mathcal{M}_{n,k}$ and the set $\mathcal{I}_{n,k}$ in answer to Conjecture 1.1.

2 A parity reversing and weight preserving involution

In this section, we will construct a parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$. Before constructing the involution, we need some definitions.

Given a matrix $A \in \mathcal{M}_n$, the weight of the matrix $A$ is assigned by $z^{r\sum_1(A)}$. Given a subset $S$ of the set $\mathcal{M}_n$, the weight of $S$, denoted by $W(S)$, is the sum of the weights of all matrices in $S$. We define the parity of the matrix $A$ to be the parity of the number $n - \dim(A)$. Denote by $\mathcal{E}\mathcal{M}_n$ (resp. $\mathcal{O}\mathcal{M}_n$) the set of matrices in $\mathcal{M}_n$ whose parity are even (resp. odd).

**Theorem 2.1** There is a parity reversing and weight preserving involution $\Phi$ on the set $\mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$. Furthermore, we have $W(\mathcal{E}\mathcal{M}_n) - W(\mathcal{O}\mathcal{M}_n) = W(\mathcal{P}\mathcal{M}_n)$.

**Proof.** Given a matrix $A \in \mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$, suppose that $\text{index}(A) = i$. We now have two cases. (1) We have $c\sum_i(A) \geq 2$. (2) We have $1 < i \leq \dim(A)$, $c\sum_i(A) = 1$, $r\sum_i(A) = 0$, and $\min_i(A) < \min_{i-1}(A)$.

For Case (1), we obtain a new matrix $\Phi(A)$ from the matrix $A$ in the following way. In $A$, replace the entry in row $\min_i(A)$ of column $i$ with zero. Then, insert a new zero row between row $i$ and row $i+1$ and insert a new column between column $i$ and $i+1$. Let the new column be filled with all zeros except that the entry in row $\min_i(A)$ is filled with 1. In this case, we have $\Phi(A) \in \mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$ with $\text{index}(\Phi(A)) = i + 1$, $\dim(\Phi(A)) = \dim(A) + 1$ and $r\sum_1(\Phi(A)) = r\sum_1(A)$.

For Case (2), we may obtain a new matrix $\Phi(A)$ by reversing the construction for Case (1) as follows. In $A$, replace the entry in row $\min_i(A)$ of column $i - 1$ with
Then remove column $i$ and row $i$. In this case, we have $\Phi(A) \in \mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$ with $\text{index}(\Phi(A)) = i - 1$, $\text{dim}(\Phi(A)) = \text{dim}(A) - 1$ and $\text{rsum}_1(\Phi(A)) = \text{rsum}_1(A)$.

In both cases, the map $\Phi$ reverse the parities and preserve the the weights of the matrices. Hence, we obtain a desired parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{P}\mathcal{M}_n$. Note that if a matrix $A \in \mathcal{M}_n$ is proper, then there is exactly one 1 in each column. Hence for each $A \in \mathcal{P}\mathcal{M}_n$, the parity of $A$ is even. By applying the involution, we can deduce that

$$W(\mathcal{E}\mathcal{M}_n) - W(\mathcal{O}\mathcal{M}_n) = W(\mathcal{P}\mathcal{M}_n).$$

\[ \square \]

**Example 2.2** Consider the following two matrices in $\mathcal{M}_6$:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

For matrix $A$, we have $\text{index}(A) = 3$. Thus we have

$$\Phi(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the new inserted row and column are illustrated in bold.

For matrix $B$, we have $\text{index}(B) = 4$. Thus we have

$$\Phi(B) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

In fact, we have $\Phi(A) = B$ and $\Phi(B) = A$.

### 3 Proof of Conjecture 1.1

In this section, we will show that the right-hand side of Formula 1.2 is the generating function for the number of matrices in $\mathcal{P}\mathcal{M}_{n,k}$. Furthermore, we prove
that there is a bijection between the set $PM_{n,k}$ and the set $I_{n,k}$, which implies Conjecture 1.1.

Let

$$A(t, z) = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1 - t)^{i-1}(1 - zt)).$$

With the assumption that the empty product is as usual taken to be 1, we have

$$A(t, z) = 1 + \sum_{n \geq 1} \prod_{i=1}^{n} \left( (i-1) + z \left( \binom{i-1}{j-1} \right) \right) (-1)^{j-1} p^j.$$

Define $A_n(z)$ to be the coefficient of $t^n$ in $A(t, z)$ for $n \geq 1$, that is

$$A(t, z) = 1 + \sum_{n \geq 1} A_n(z) t^n. \quad (3.1)$$

Thus we have

$$A_n(z) = \sum_{d=1}^{n} \sum_{n_1 + n_2 + \ldots + n_d = n} (-1)^{n-d} \prod_{j=1}^{d} \left( \binom{j-1}{n_j} + z \left( \binom{j-1}{n_j-1} \right) \right),$$

where the second summation is over all compositions $n_1 + n_2 + \ldots + n_d = n$ such that $n_j \geq 1$ for $j = 1, 2, \ldots, d$.

**Lemma 3.1** For $n \geq 1$, we have

$$A_n(z) = W(\mathcal{EM}_n) - W(\mathcal{OM}_n).$$

**Proof.** Let $\mathcal{M}(n_1, n_2, \ldots, n_d)$ be the set of matrices in $\mathcal{M}_n$ with $d$ columns in which the column sum of column $j$ is equal to $n_j$ for all $1 \leq j \leq d$. In order to get a matrix $A \in \mathcal{M}(n_1, n_2, \ldots, n_d)$, we should choose $n_j$ places in column $j$ form $j$ places to arrange 1’s for all $1 \leq j \leq d$. we have two cases. (1) If $A_{1,j} = 0$, then we have $\binom{j-1}{n_j}$ ways to arrange 1’s in column $j$. (2) If $A_{1,j} = 1$, then we have $\binom{j-1}{n_j-1}$ ways to arrange the remaining 1’s in column $j$. In the former case, column $j$ contributes 1 to the weight of $A$. While in the latter case, column $j$ contributes $z$ to the weight of $A$. Altogether, column $j$ contributes $\binom{j-1}{n_j} + z \binom{j-1}{n_j-1}$ to the weight of $\mathcal{M}(n_1, n_2, \ldots, n_d)$, which implies that

$$W(\mathcal{M}(n_1, n_2, \ldots, n_d)) = \prod_{j=1}^{d} \left( \binom{j-1}{n_j} + z \binom{j-1}{n_j-1} \right).$$
It is clear that the parity of each matrix in $\mathcal{M}(n_1, n_2, \ldots, n_d)$ is the parity of the number $n - d$. When $d$ ranges from 1 to $n$ and $n_1, n_2, \ldots, n_d$ range over all compositions $n_1 + n_2 + \ldots + n_d = n$ such that $n_j \geq 1$ for all $1 \leq j \leq d$, we get the desired result.

Denote by $a_{n,k}$ the cardinality of the set $\mathcal{PM}_{n,k}$. Assume that $a(0,0) = 1$.

**Theorem 3.2** We have

$$A(t, z) = \sum_{n \geq 0, k \geq 0} a_{n,k}z^k t^n = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1 - t)^{i-1}(1 - zt)).$$

*Proof.* Combining Theorem 2.1 and Lemma 3.1 we deduce that $A_n(z) = W(\mathcal{PM}_n)$ for $n \geq 1$. Note that $W(\mathcal{PM}_n) = \sum_{k=1}^{n} a_{n,k}z^k$ for $n \geq 1$. Hence we have

$$A(t, z) = 1 + \sum_{n \geq 1} A_n(z)t^n = \sum_{n \geq 0, k \geq 0} a_{n,k}z^k t^n,$$

which implies the desired result.  

From Theorem 3.2, in order to prove Conjecture 1.1 it suffices to prove that $a_{n,k} = p_{n,k}$. In a matrix $A$, the operation of adding column $i$ to column $j$ is defined by increasing $A_{k,j}$ by $A_{k,i}$ for each $k = 1, 2, \ldots, \dim(A)$. Note that a matrix $A \in \mathcal{M}_n$ is proper if and only if it satisfies

- each column has exactly one 1;
- if $rsum_i(A) = 0$, then we have $min_i(A) \geq min_{i-1}(A)$ for $2 \leq i \leq \dim(A)$.

This observation will be essential in the construction of the bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$.

**Theorem 3.3** There is a bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$.

*Proof.* Let $A$ be a matrix in the $\mathcal{PM}_{n,k}$, we can construct a matrix $A'$ in $\mathcal{I}_{n,k}$. If there is no zero rows in $A$, then we do nothing for $A$ and let $A' = A$. In this case, the resulting matrix $A'$ is contained in $\mathcal{I}_{n,k}$. Otherwise, we can construct a new upper triangular matrix $A'$ by the following removal algorithm.

- Find the least value $i$ such that row $i$ is a zero row. Then we obtain a new upper triangular matrix by adding column $i$ to column $i - 1$ and remove column $i$ and row $i$.
• Repeat the above procedure for the resulting matrix until there is no zero row in the resulting matrix.

Clearly, the obtained matrix $A'$ is a matrix in $I_n$. Since the algorithm preserves the sums of entries in each non-zero rows of $A$, we have $rsum_1(A') = rsum_1(A)$. Hence, the resulting matrix $A'$ is in $I_{n,k}$.

Conversely, we can construct a matrix in $PM_{n,k}$ from a matrix in $I_{n,k}$. Let $B$ be a matrix in the $I_{n,k}$. If the sum of entries in each column is equal to 1, then we do nothing for $B$ and let $B' = B$. Otherwise, we can construct a new upper triangular matrix $B'$ by the following addition algorithm.

• Find the largest value $i$ such that $csum_i(B) \geq 2$. Then we obtain a new upper triangular matrix by decreasing the entry in row $max_i(B)$ of column $i$ by 1, where $max_i(B)$ is defined to be the largest value $j$ such that $B_{j,i}$ is non-zero. Since $B$ is upper triangular, we have $max_i(B) \leq i$.

• Insert one column between column $i$ and column $i + 1$ and one zero row between row $i$ and row $i + 1$ such that the new inserted column is filled with all zeros except that the entry on row $max_i(B)$ is filled with 1.

• Repeat the above procedure for the resulting matrix until there is no column whose column sum is larger than 1.

Clearly, the obtained matrix $B'$ is a matrix in $M_n$. From the construction of the above algorithm we know that the column sum of each column in $B'$ is equal to 1. Furthermore, if row $j$ is a zero row, then we must have $min_j(B') \geq min_{j-1}(B')$. Thus, the resulting matrix $B'$ is proper. Since the algorithm preserves the sums of entries in each non-zero row of $B$, we have $rsum_1(B') = rsum_1(B)$. Hence, the resulting matrix $B'$ is in $PM_{n,k}$. This completes the proof.

Example 3.4 Consider a matrix $A \in PM_{6,3}$. By applying the removal algorithm, we get

\[
A = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \leftrightarrow \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \leftrightarrow \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 \\
\end{bmatrix} \leftrightarrow A' = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]
where the removed rows and columns are illustrated in bold at each step of the removal algorithm. Conversely, given \( A' \in \mathcal{I}_{6,3} \), by applying addition algorithm, we can get \( A \in \mathcal{P}\mathcal{M}_{6,3} \), where the inserted new rows and columns are illustrated in bold at each step of the addition algorithm.

Combining Theorems 2.1, 3.2 and 3.3 we obtain a combinatorial proof of Conjecture 1.1. Note that specializing \( z = 1 \) implies a combinatorial proof of Formula (1.1), which was proved by Bousquet-Mélou et al. [2] by using functional equations and the Kernel method.

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