Twisted Alice Loops as Monopoles

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Abstract
Symmetry breaking can produce “Alice” strings, which alter scattered charges and carry monopole number and charge when twisted into loops. We apply recent topological results, fixing Alice strings’ stability and prescribing their twisting into loops with monopole charge, to several models. We show that Alice strings of condensed matter systems (nematic liquid crystals, $^3$He-A, and related systems of non-chiral Bose condensates and amorphous chiral superconductors) are topologically Alice, and carry fundamental monopole charge when twisted into loops. They might thus be observed indirectly, not as strings, but as loop-like point defects. Other models yield Alice loops that carry only deposited, and not fundamental, charge.
INTRODUCTION

Among the defects created in spontaneous symmetry breakdown are Alice strings.\(^1\), \(^2\), \(^3\) Like monopoles,\(^4\) Alice strings obstruct the global extension of unbroken symmetries, making them multivalued when parallel transported around the string. This algebraic obstruction causes nonconservation of associated charges, when Aharonov-Bohm scattered around the string; it also induces monopoles, as twisted loops of Alice string. Alice strings arise in both particle physics and condensed matter models, with condensed matter systems offering the most likely prospects for their observation.\(^5\) We show here that known condensed matter Alice strings form twisted loops with fundamental monopole charge, suggesting a second avenue for their potential observation: Alice strings might be observed, not as strings, but as looplike point defects, when twisted loops comprise the energetically favored solution of fundamental monopole charge.

We recently established in\(^6\) a topological criterion for Alice behavior, stating when Alice strings must form, and when strings’ Alice features may be deformed away. Consider the symmetry breakdown of Lie group \(G \rightarrow H\), taking for \(G\) the simply connected cover of the initial Lie symmetry. A topological string then has homotopy \(\pi_0(H)\); that is, its flux \(U(2\pi)\) lies in a disconnected component of the unbroken symmetry group \(H\). Monopoles have homotopy \(\pi_1(H)\), describing loops \(h(\alpha)\) of different winding in \(H\). Our criterion labels strings with flux \(U(2\pi)\) topologically Alice if they alter the topological charge of monopoles circumnavigating them:

\[
h(\alpha) \rightarrow \tilde{h}(\alpha) = U(2\pi) h(\alpha) U^{-1}(2\pi) \not\sim h(\alpha).
\]

This is a topological criterion, corresponding to a nontrivial action of \(\pi_0(H)\) on \(\pi_1(H)\), where \(h_o = U(2\pi) \in \pi_0(H)\) alters the topological winding of loop \(h(\alpha) \in \pi_1(H)\):

\[
\tilde{h}(\alpha) = h_o h(\alpha) h_o^{-1} \not\sim h(\alpha).
\]  

(0.1)

Using this criterion we constructed a prescription for twisting Alice strings into loops carrying monopole charge. We showed that the twisted Wilson line

\[
U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha) h_o^{-1},
\]

(0.2)

with \(h(\alpha)\) and \(h_o\) as in our criterion above, generates a single-valued twisted Alice loop. \(U(\varphi, \alpha)\) interpolates between the point \(h_o^{-1}\) at \(\varphi = 0\) and the loop \(h^{-1}(\alpha) \tilde{h}(\alpha)\) at \(\varphi = 2\pi\).
The twisted Alice loop thus carries a nontrivial monopole charge $h^{-1}(\alpha) \tilde{h}(\alpha)$ if and only if the Alice string obeys our topological criterion. This monopole charge corresponds to that deposited on the Alice loop in the monopole circumnavigation $h^{-1}(\alpha) \rightarrow \tilde{h}^{-1}(\alpha)$. \[6\]

Two points are key. First, our topological arguments only indicate that deposited monopole charge can be carried by a twisted Alice loop. Monopole charge is typically not deposited in single fundamental units, leaving open the question of whether twisted Alice loops can carry fundamental monopole charge. Second, we took as twisting function $h(\alpha)$ for the twisted Wilson line $U(\varphi, \alpha)$, the loop in $H$ representing a fundamental monopole (call this the fully-twisted Alice loop). We remain free to choose a different twisting function $h(\alpha)$ in $H$, so long as it renders $U(\varphi, \alpha)$ singlevalued in the angle $\alpha$. Propitious choice of $h(\alpha)$, in some models, allows construction of twisted Alice loops with fundamental monopole charge.

We exploit both points here, in examining the monopole charge carried by model twisted Alice loops. We study primarily those models which offer the best condensed matter candidates for Alice strings: the original Alice string of Schwarz, coinciding with the Alice string of liquid crystals and of non-chiral Bose condensates \[1, 7\]; and the Alice string of $^3$He-A \[8\], arising anew for unconventional spin-triplet superconductors \[9\]. In both models we find strings which are topologically Alice. We also find twisted Alice loops supporting fundamental monopole charge, even though monopole charge deposits onto string loops in even increments. Thus, for the Alice strings of interest to condensed matter, even the most fundamental singular point defect, or monopole, can take the form of a twisted Alice loop.

We show, in other models, different outcomes for the topology of Alice strings and their twisted loops. When Alice behavior is nontopological, the fully-twisted Alice loop has trivial monopole charge. However, an Alice loop with different twist $h(\alpha)$ may still exist, carrying nontrivial monopole charge. We examine these possibilities in the context of a nontopologically Alice string discussed in \[2\] and \[6\]. Finally, string loops which are topologically Alice may admit only the full twisting invoked in our topological argument. In this case, twisted Alice strings carry only deposited, and not fundamental, monopole charge. We realize this possibility for a new topological Alice string, formed in the symmetry breakdown $SU(3) \rightarrow O(2)$. Here twisted Alice loops support only even deposited monopole charge, and cannot form fundamental monopoles.

Through these models we show that whether twisted Alice loops can support fundamental monopole charge depends closely on the symmetry-breaking pattern. Specifically, it depends
on the initial symmetry group $G$, through the identification of loops in $H$ with monopoles via the exact sequence for $\pi_2(G/H)$. When fundamental monopoles correspond to nonminimal-winding loops in $H$, alternative choices for the loop twisting $h(\alpha)$ exist; and these can induce a twisted Alice loop with fundamental monopole charge. Algebraic commutations in the model may also allow alternative twistings $h(\alpha)$, yielding twisted Alice loops with fundamental monopole charge.

**THE SCHWARZ, OR NEMATIC, ALICE STRING**

We start with the simplest example, the canonical Schwarz Alice string, whose symmetry-breaking pattern coincides with Alice strings in nematic liquid crystals and in non-chiral Bose condensates.

Here $G$ is $SO(3)$, with Higgs $\phi$ transforming in the adjoint representation. When $\phi$ develops the vev $\langle \phi \rangle = \text{diag} (1,1,-2)$, $SO(3)$ breaks to the residual symmetry $H = O(2)$, containing $z$-rotations $R_z(\alpha)$ and the discrete symmetry element $h_o = R_x(\pi) = \text{diag} (1,-1,-1)$. Here $\pi_0(H) = Z_2$ and $\pi_1(H) = Z$ so we have topological strings and monopoles. The canonical Alice string has Wilson line $U(\varphi) = R_x(\varphi/2)$ with $U(2\pi) = h_o$. This string is Alice, as $U(2\pi)$ fails to commute with the unbroken symmetry generator $T_z$; in fact, on parallel transport around the string,

$$T_z \to U(2\pi) \ T_z \ U^{-1}(2\pi) = -T_z \ .$$

(0.3)

This canonical Alice string meets our topological criterion, of changing topological monopole charge upon circumnavigation. By the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $O(2)$ which can be unwound in $SO(3)$. Since only even winding loops in $O(2)$ can be unwound in $SO(3)$, the fundamental monopole in this canonical Alice model has a loop in $O(2)$ of winding 2.

In applying our topological criterion, we choose $h_o$ as our representative of the string, a nontrivial element of $\pi_0(H)$, and $h(\alpha) = R_z(2\alpha)$ as our representative of the fundamental Alice monopole, a winding 2 element of $\pi_1(H)$. This gives

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h^{-1}(\alpha) \ ,$$

from equation (0.3). Note that $h^{-1}(\alpha)$ has $O(2)$ winding -2, topologically distinct from $h(\alpha)$ of $O(2)$ winding 2. Thus $\tilde{h}(\alpha) \not\sim h(\alpha)$ and our topological criterion is fulfilled.
We now construct a monopole as a twisted Alice loop. From Eq. (0.2), the twisted Wilson line
\[ U(\varphi, \alpha) = h^{-1}(\alpha/2) \, U(\varphi) \, h(\alpha/2) \, h_0^{-1} \]
generates an Alice loop with single-valued condensate. (We take \( h(\alpha/2) \) because we need only for \( h \) to be single-valued in \( \alpha \), and \( h(\alpha/2) \), the winding 1 loop in \( O(2) \), first fulfills that requirement.) \( U(\varphi, \alpha) \) interpolates between \( h_0^{-1} \) at \( \varphi = 0 \) and \( h^{-1}(\alpha) \) at \( \varphi = 2\pi \). It is thus the fundamental antimonopole in the model. Note that the inverse twisted Alice loop, with twisted Wilson line \( U^{-1}(\varphi, \alpha) \), generates the fundamental monopole.

**THE ALICE STRING OF \(^3\text{He}-\text{A}\)**

A more complicated global symmetry-breaking pattern describes the Alice string expected in \(^3\text{He}-\text{A}\) [5, 8], and more recently predicted in amorphous chiral superconductors with p-wave pairing, such as \( \text{Sr}_2\text{RuO}_4 \) [9].

Here \( G \) is \( SO(3)_L \times SO(3)_S \times U(1)_N \), describing spatial rotations, spin rotations, and a \( U(1) \) phase symmetry associated with number conservation of helium atoms. (The \( U(1) \) symmetry is approximate, as is independence of spin and orbital rotations due to minimal spin-orbit coupling, but both describe \(^3\text{He}-\text{A}\) well.) The matrix order parameter \( A \) transforms under symmetry transformations as \( A \rightarrow e^{2i\theta} \, R_S \, A \, R_L^{-1} \), where \( R_S \) and \( R_L \) are spin and orbital rotations, respectively.

The order parameter develops the form
\[ A_{ij} = \Delta_A \, \hat{d}_i \, (\hat{m}_j + i \hat{n}_j) \]
where \( \hat{m} \) and \( \hat{n} \) are perpendicular, determining \( \hat{l} = \hat{m} \times \hat{n} \), the direction of the condensate’s angular momentum vector. This breaks \( G \) to the residual symmetry \( H = U(1)_{S_d} \times U(1)_{L_{l-N/2}} \times Z_2 \), consisting of spin rotations about the \( \hat{d} \) axis; spatial rotations about the \( \hat{l} \) axis when compensated by a matching \( U(1)_N \) phase rotation, and the discrete \( Z_2 \) transformation \( h_o \), with \( h_o : \hat{d}, \quad \hat{m} + i \hat{n} \rightarrow -\hat{d}, \quad -(\hat{m} + i \hat{n}) \).

Identifying the defect topology requires care in this setting, as the exact sequences relating \( \pi_2(G/H) \) and \( \pi_1(G/H) \), the monopole and string homotopy groups, to \( \pi_1(H) \) and \( \pi_0(H) \) are highly nontrivial. Note that \( \pi_2(G/H) = Z \), corresponding to the loops \( \pi_1(U(1)_{S_d}) \), which can be unwound in \( G \). (Loops of the other \( U(1) \) factor cannot be unwound in \( G \), as they
contain unshrinkable $U(1)_N$ loops.) $\pi_1(G/H) = Z_4$, which describes strings of two different origins. First, the Alice strings, called half-quantum vortices, have Wilson lines ending in a disconnected component of $H$, getting topological stability from $\pi_0(H)$. Second, a $Z_2$ winding one vortex, nontrivial in $SO(3)_L$ in $G$, induces as its image a $Z_2$ winding one vortex in $G/H$, with topological stability inherited from $\pi_1(G)$. These two classes of vortices are not independent: instead winding twice about a half-quantum vortex is equivalent to once around a winding one vortex, and the full string homotopy is $\pi_1(G/H) = Z_4$, or windings $0, \pm 1/2, \text{and } 1 \text{ modulo } 2$, with Alice strings corresponding to windings $\pm 1/2$.

The Volovik-Mineev Alice string, of winding $\pm 1/2$, has order parameter $A_{ij}$ with $\hat{d} = \hat{x}$ in spin space, and $\{\hat{l}, \hat{m}, \hat{n}\} = \{\hat{x}, \hat{y}, \hat{z}\}$ in ordinary space. This is acted on by Wilson line $U(\varphi) = e^{\pm i\varphi/2} R_{S_{\hat{x}}}(\varphi/2)$ to give, asymptotically in $r$,

$$A_{ij}(\varphi) = \Delta_A e^{\pm i\varphi/2} (\cos(\varphi/2) \hat{x}_j + \sin(\varphi/2) \hat{y}_j)_S (\hat{x}_j + i\hat{y}_j)_L ,$$

single-valued in $\varphi$. Note that $U(2\pi) = -R_{S_{\hat{x}}}(\pi)$ lies in the same homotopy class as $h_o$. This string is Alice, making unbroken symmetry generator $T_{S_{\hat{x}}}$ double-valued. Physically, this means that a particle flips its spin, and hence its magnetization, on circumnavigating the Alice string.

This long-studied Alice string meets our topological criterion, of changing topological monopole charge upon circumnavigation. By the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $U(1)_{S_{\hat{x}}}$ which can be unwound in $SO(3)_S$. As in the nematic case, only even winding loops in $U(1)_{S_{\hat{x}}}$ can be unwound in $SO(3)_S$. Thus the fundamental monopole in $^3$He-A corresponds to a loop in $U(1)_{S_{\hat{x}}}$ of winding 2.

In applying our topological criterion, we choose $U(2\pi)$ as our representative of the string, a nontrivial element of $\pi_o(H)$, and $h(\alpha) = R_{S_{\hat{x}}}(2\alpha)$ as our representative of the fundamental monopole, a winding 2 element of $\pi_1(U(1)_{S_{\hat{x}}})$. This gives

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h^{-1}(\alpha) ,$$

since $T_{S_{\hat{x}}} \to -T_{S_{\hat{x}}}$. Note that $h^{-1}(\alpha)$ has $U(1)_{S_{\hat{x}}}$ winding -2, topologically distinct from $h(\alpha)$ of $U(1)_{S_{\hat{x}}}$ with winding 2. Thus $\tilde{h}(\alpha) \not\sim h(\alpha)$, meeting our topological criterion.

As in the nematic case, we construct a monopole as a twisted Alice loop. From Eq. (0.2),

$$U(\varphi, \alpha) = h^{-1}(\alpha/2) \ U(\varphi) \ h(\alpha/2) \ h_o^{-1}$$

generates an Alice loop with single-valued condensate. (Again \( h(\alpha/2) \) appears, the winding 1 loop in \( U(1)_{S_2} \), as our minimal single-valued choice in constructing \( U(\varphi, \alpha) \).) \( U(\varphi, \alpha) \) interpolates between \( h_0^{-1} \) at \( \varphi = 0 \) and \( h^{-1}(\alpha) \) at \( \varphi = 2\pi \). It is thus the fundamental antimonopole in the model, which contains monopoles and antimonopoles of even winding in \( U(1)_{S_2} \) only. Note that the inverse twisted Alice loop, with twisted Wilson line \( U^{-1}(\varphi, \alpha) \), again generates the fundamental monopole.

A NONTOPOLOGICALLY ALICE STRING

A nontopologically Alice string arises when a Higgs \( \phi \), transforming in the adjoint representation under \( G = SO(6) \), acquires the vev \( \langle \phi \rangle = \text{diag}(1^3, -1^3) \). As discussed in \cite{2}, this condensate leaves unbroken an \( SO(3) \times SO(3) \) subgroup of \( SO(6) \) and a discrete \( Z_2 \) transformation \( h_1 = -1 \), so \( H = SO(3) \times SO(3) \times Z_2 \). Here \( \pi_o(H) = Z_2 \) and \( \pi_1(H) = Z_2 \times Z_2 \), so topological strings and monopoles form, with monopoles and antimonopoles topologically identified. As noted in \cite{6}, strings in this model can have algebraic Alice behavior, for \( U(2\pi) = h_o = \text{diag}(1^2, (-1)^4) = -R_{12}(\pi) \), which makes generators \( T_{13} \) and \( T_{23} \) of rotations in the 13- and 23-planes double-valued. Yet that Alice behavior fails our topological criterion. Taking as our nontrivial monopole loop \( h(\alpha) = R_{13} \), with monopole charge \((1,0)\), we find \( \tilde{h}(\alpha) = h_0 h(\alpha) h_0^{-1} = h^{-1}(\alpha) \), since \( T_{13} \rightarrow -T_{13} \). Thus the \((1,0)\) monopole transforms into its antimonopole on traversing the string. However, as monopoles and antimonopoles are identified, that transformation is nontopological. This string’s Alice behavior is thus nontopological; it can be deformed away by deforming \( U(2\pi) \) to the topologically equivalent value \( h_1 \), with no algebraic Alice behavior.

We might still hope to construct a \((1,0)\) monopole as a twisted string loop, taking for our string the algebraic, but nontopologically Alice string \( U(\varphi) = R_{34}(\varphi/2) R_{56}(\varphi/2) \), with algebraic Alice flux \( U(2\pi) = h_o = -R_{12}(\pi) \) as above. From Eq. (0.2), the twisted Wilson line

\[
U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha) h_0^{-1}
\]

generates an Alice loop with single-valued condensate. \( U(\varphi, \alpha) \) interpolates between \( h_0^{-1} \) at \( \varphi = 0 \) and \( h^{-2}(\alpha) \) at \( \varphi = 2\pi \). This is a loop in \( H \) of winding \((2,0)\); however, winding \((2,0)\) loops are deformable to the identity in \( H \), so this twisted nontopologically Alice loop fails to carry topological monopole charge.
Recall that, in building singlevalued twisted Alice loops, we required $U(\varphi, \alpha)$ to be singlevalued in $\alpha$; we thus identified $h(\alpha)$ as a loop in $H$. Strictly, we do not need $h(\alpha)$ to be a loop; all we need is

$$h^{-1}(2\pi) U(\varphi) h(2\pi) = U(\varphi) . \quad (0.4)$$

We might still hope to build the fundamental $(1,0)$ monopole as a twisted Alice loop, exploiting this freedom in $h(\alpha)$. Were the twisted loop $U(\varphi, \alpha) = h^{-1}(\alpha/2) U(\varphi) h(\alpha/2) h^{-1}_o$ single-valued, it would carry fundamental $(-1,0)$ monopole charge, as it interpolates between $h^{-1}_o$ at $\varphi = 0$ and the nontrivial $(-1,0)$ antimonopole $h^{-1}(\alpha)$ at $\varphi = 2\pi$. However, this twisted loop candidate is not single-valued; it obeys instead $h^{-1}(2\pi) U(\varphi) h(2\pi) = U^{-1}(\varphi)$.

We thus cannot build a fundamental $(1,0)$ monopole as a twisted Alice loop in this model, where Alice behavior is nontopological and monopole charge is $Z_2 \times Z_2$.

This possibility to construct $U(\varphi, \alpha)$ single-valued in $\alpha$, without forcing $h(\alpha)$ to be a loop, always merits investigating. Indeed, in [10, 11], one of us constructed what is essentially the fundamental monopole in this model, by exploiting exactly such an accidental algebraic singlevaluedness. That construction (most clearly in section IIIA of [11], taking $F(r), \varphi$ as the spherical coordinates $\theta, \varphi$ at spatial infinity), is quite similar to the twisting constructions here. However, it describes a fundamentally point-like defect, and cannot be interpreted as a twisted loop.

**A TOPOLOGICALLY ALICE LOOP CARRYING ONLY DEPOSITED MONOPOLE CHARGE**

We consider a slightly modified canonical Alice string. Take $G$ to be $SU(3)$, with Higgs $\phi$ transforming according to $\phi \rightarrow g \phi g^T$ (giving fermions in this model a Majorana mass). When $\phi$ develops the vev $\langle \phi \rangle = \text{diag} (1, 1, -2)$, $SU(3)$ breaks to the residual symmetry $H = O(2)$, identical to that of the canonical Schwarz Alice string. Again we have $\pi_o(H) = Z_2$ and $\pi_1(H) = Z$, with topological strings and monopoles. We have the same Alice string as in the canonical case, making the $O(2)$ generator $T_z$ double-valued. This Alice behavior is again topological, as our topological criterion, that $\pi_o(H)$ acts nontrivially on $\pi_1(H)$, depends only on the unbroken symmetry group $H$.
Where we deviate from the canonical Alice string model is in the identification of twisted Alice loops as monopoles. Here, by the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $O(2)$ which can be unwound in $G$, here $SU(3)$. All nontrivial loops in $O(2)$ can be unwound in $SU(3)$; thus the fundamental monopole in this model has a loop in $O(2)$ of winding 1.

We now construct a monopole as a twisted Alice loop. We take $U(\varphi) = R_x(\varphi/2)$, as in the canonical Alice case, and $h(\alpha) = R_z(\alpha)$, a loop of winding 1. From Eq. (0.2), the twisted Wilson line

$$U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha) h^{-1}_o$$

generates a twisted Alice loop with single-valued condensate. $U(\varphi, \alpha)$ interpolates between $h_o^{-1}$ at $\varphi = 0$ and $h^{-2}(\alpha)$ at $\varphi = 2\pi$. This twisted Alice loop carries monopole charge of $-2$, which while nontrivial is not the fundamental antimonopole in this model. (Similarly, the inverse twisted Alice loop, with Wilson line $U^{-1}(\varphi, \alpha)$, carries monopole charge $+2$).

Again, we might still hope to build a fundamental monopole as a twisted Alice loop, by allowing $h(\alpha)$ above to be not a loop, but a curve obeying Eq. (0.4) This looser constraint still guarantees singlevaluedness in $\alpha$ of $U(\varphi, \alpha)$. Indeed, were the twisted loop $U(\varphi, \alpha) = h^{-1}(\alpha/2) U(\varphi) h(\alpha/2) h^{-1}_o$ single-valued in $\alpha$, with $h(\alpha) = R_z(\alpha)$ as above, it would carry fundamental antimonopole charge. This is because it interpolates between $h_o^{-1}$ at $\varphi = 0$ and the winding $-1$ loop $h^{-1}(\alpha)$ at $\varphi = 2\pi$. However, this twisted loop candidate is not single-valued in $\alpha$; it obeys instead $U(\varphi, 2\pi) = R^{-1}_x(\varphi) U(\varphi, 0)$. We thus cannot build a fundamental monopole as a twisted Alice loop in this model. Instead twisted Alice loops carry only the monopole charge which topological arguments ensure they must carry: because monopoles scatter into antimonopoles on transiting Alice loops, Alice loops must support deposited monopole charge, which arises in units of 2. Our twisting construction creates twisted Alice loops supporting exactly that deposited charge.

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