We consider spatiotemporal chaotic systems for which spatial correlation functions decay substantially over a length scale $\xi$ (the spatial correlation length) that is small compared to the system size $L$. Numerical simulations suggest that such systems generally will be extensive, with the fractal dimension $D$ growing in proportion to the system volume for sufficiently large systems ($L \gg \xi$). Intuitively, extensive chaos arises because of spatial disorder. Subsystems that are sufficiently separated in space should be uncorrelated and so contribute to the fractal dimension in proportion to their number. We report here the first numerical calculation that examines quantitatively how one important characterization of extensive chaos—the Lyapunov dimension density—depends on spatial disorder, as measured by the spatial correla-
tion length $\xi$. Surprisingly, we find that a representative extensively chaotic system does not act dynamically as many weakly interacting regions of size $\xi$.

More specifically, researchers have conjectured that the fractal dimension $D$ of a sufficiently-large homogeneous spatiotemporal chaotic system should obey the following simple scaling relation:

$$D = C(L/\xi)^d \quad \text{for } L \gg \xi.$$  

Here $C$ is a constant and $d$ is the spatial dimensionality of the system, e.g., $d = 2$ for a large-aspect-ratio convection experiment. Eq. (1) follows from an assumption of spatial disorder, that parts of a large homogeneous system are uncorrelated and hence dynamically independent when separated by distances larger than the length $\xi$. The volume $L^d$ of the system then acts as a gas of $N = L^d/\xi^d$ weakly-interacting regions of volume $\xi^d$ which contribute independently to the fractal dimension.

Eq. (1) predicts that spatiotemporal chaos is extensive since the fractal dimension $D$ is proportional to the system volume $L^d$. This has indeed been verified in numerical calculations of one-dimensional chaotic systems and is believed to hold generally. A more subtle prediction of Eq. (1) is that extensive chaos depends on the spatial disorder through a single lengthscale $\xi$ so that $D \propto 1/\xi^d$ for a fixed but sufficiently large system size $L$. It is useful to formulate this second prediction in terms of an intensive parameter so that the system size $L$ and details of the boundary conditions no longer enter. This can be done by introducing the intensive dimension density $\delta = D/L^d$, which is the fractal dimension per unit volume for an extensively chaotic system.
Eq. (1) then predicts that

\[ \delta \propto 1/\xi^d, \]

which we test in detail below. Researchers have shown for one case that this quantity is independent of the choice of boundary conditions for sufficiently large system sizes (see Fig. 10 of Ref. 9).

Eqs. (1) and (2) are important to verify and to understand for two reasons. First, they suggest a simple theoretical explanation for why chaos should be extensive, relating the complexity \( D \) of a dynamical system (roughly the minimum number of degrees of freedom needed to describe a system) to its spatial disorder. Second, Eq. (1) suggests a straightforward way to estimate the dynamical complexity of many interesting experimental systems by simply measuring the correlation length \( \xi \). This approach would have many benefits compared to time series methods since large spatiotemporal chaotic systems are high-dimensional, the length of a time series needed to estimate fractal dimensions above five is prohibitively large, and algorithms for estimating fractal dimensions from time series are notoriously unreliable. As an example, we note that scientists recently estimated a correlation length of \( \xi \approx 6 \text{ mm} \) from voltage measurements on the surface of a fibrillating pig heart. Since the radius of the heart was \( R \approx 25 \text{ mm} \), Eq. (1) would suggest that the fibrillating heart involved approximately \( D \geq 4\pi R^2/\pi \xi^2 \approx 70 \) independent regions, a result that is consistent with the failure of time-series algorithms to detect low-dimensional chaos.

To investigate the dependence of the dimension density \( \delta \) on the correlation length \( \xi \) we have
studied chaotic numerical solutions of the one-dimensional complex Ginzburg-Landau equation:

\[ \frac{\partial u(x,t)}{\partial t} = u + (1 + ic_1)\frac{\partial^2 u}{\partial x^2} - (1 - ic_3)|u|^2 u, \]  

in a periodic system of size \( L \). The field \( u(x,t) \) is complex-valued while the parameters \( c_1 \) and \( c_3 \) are real-valued. Eq. (3) is an experimentally relevant continuum dynamical model that holds universally near the onset of a Hopf bifurcation from a static homogeneous state to an oscillatory state. It is particularly valuable for testing Eq. (2) since there is an apparent phase transition in large systems—the L1 transition—as the parameter \( c_3 \) is varied for a fixed parameter \( c_1 \geq 1.8 \).

The correlation length \( \xi \) increases rapidly as one approaches the L1 transition from one side and so yields a valuable strong variation of \( \xi \) with parameters.

We integrated Eq. (3) using a pseudospectral method with time-splitting of the operators, carefully checking our results for convergence with respect to spatial and temporal resolutions and with respect to total integration times. Correlation lengths \( \xi \) were obtained by examining the asymptotic exponential decay \( A \exp(-x/\xi) \) of the magnitude of the complex two-point equal-time correlation function \( C(x) = \langle u^*(x + x', t)u(x', t) \rangle \), where the angle brackets indicate averaging over time \( t \) and space \( x' \). We calculated a particular fractal dimension, the Lyapunov dimension \( D_L \), via the Kaplan-Yorke formula which expresses \( D_L \) in terms of the spectrum of Lyapunov exponents. The exponents were obtained by an expensive but widely-used algorithm that involves integrating Eq. (3) together with many copies of the linearization of this equation around the solution \( u \). We used about 1200 CPU hours on CRAY YMP and on Thinking Machines CM5 supercomputers to obtain the results reported below. For most dynamical systems, the Lyapunov dimension is
believed to be the same as the information dimension $D_1$, which is just one of an infinity of fractal dimensions $D_q$ that characterize multifractal strange attractors.

Our results are presented in Figures 1-3 for the fixed parameter value $c_1 = 3.5$ and for various values of the parameter $c_3$ near the L1 transition. We have calculated correlation lengths $\xi$ for various system sizes up to $L = 262,144$ and Lyapunov dimensions $D_L$ for system sizes only up to $L = 1024$ (since the latter is far more expensive to calculate numerically). Figure 1(a) confirms the prediction of Eq. (1) concerning extensive chaos by showing that $D_L$ grows linearly with system size $L$ over the entire range $64 \leq L \leq 1024$; for $L = 1024$, we estimate a large fractal dimension of $D \approx 152$. The dimension density $\delta$ was estimated by a least-squares estimate of the slope of the linear portion of this curve. Similar extensive chaos curves were obtained for all $c_3$ values reported in this paper. Figure 1(b) shows how we calculated the correlation length $\xi$ from the asymptotic exponential decay of the spatial correlation function $C(x)$. Near the L1 transition, an unexpectedly large system size of $L/\xi > 1000$ was needed to find good exponential decay and to find a value of $\xi$ that was independent of the system size.

Fig. 2 summarizes how the Lyapunov dimension density and spatial correlation length vary with the parameter value $c_3$ for the particular value $c_1 = 3.5$. As $c_3$ decreases through the approximate position of the L1 transition ($c_3 \approx 0.74$), the Lyapunov dimension density (Fig. 2(a)) decreases continuously within our numerical accuracy. The correlation length $\xi$ has contrary behavior and increases by one order of magnitude close to the L1 transition (from 6 to 172). Careful numerical tests suggest that $\xi$ can not be estimated reliably below $c_3 \leq 0.79$, probably because it has become
too large. As we describe elsewhere in more detail, we find that the $\xi$ vs $c_3$ data are well described by an algebraic divergence of the form $\xi \propto (c_3 - c_3^*)^\alpha$ with exponent $\alpha = -2.4$ and with $c_3^* = 0.735$.

Our key result is given in Fig. 3, which presents the data of Fig. 2 in a way appropriate for testing Eq. (2). For spatial dimension $d = 1$, this equation predicts that the dimension density $\delta$ should be linear in the quantity $1/\xi$ while, in fact, the observed dependence is unambiguously nonlinear (the inset figure suggests that a $1/\sqrt{\xi}$ dependence may be more appropriate). If $\xi$ is truly diverging across the L1 transition while $\delta$ changes continuously, Eq. (2) must break down. This result is quite surprising since the chaos is extensive (Fig. 1(a)) and an intuitive reason for the extensive chaos is that remote parts of the system are uncorrelated and so contribute independently to the dynamics.

We conclude that Eqs. (1) and (2) are not correct. One explanation may be that the prefactor $C$ in Eq. (1) is not constant but depends on $\xi$ itself, i.e., details of the local dynamics over a region of size $\xi$ must be included in the analysis. Another reason may be that there is more than one lengthscale needed to characterize Ginzburg-Landau spatiotemporal chaos. A third possibility is that Eqs. (1) and (2) are, in fact, correct but involve a new, as-of-yet unidentified, lengthscale $\lambda$ in place of the correlation length $\xi$. Further analysis will be needed to distinguish between these possibilities and to understand how the dynamical complexity of an extensively chaotic system is related to its spatial disorder.

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Figures

FIG. 1. (a) Plot of the Lyapunov fractal dimension $D_L$ versus system size $L$ for Eq. (3) with periodic boundary conditions. We used the parameters values $c_1 = 3.5$ and $c_3 = 0.8$, a constant time step of $\Delta t = 0.05$, a spatial resolution of two Fourier modes per unit length, and a total integration time of $T = 50,000$ time units. The error in each fractal dimension is smaller than the size of the plotted points.

(b) Log-linear plot of the magnitude of the spatial correlation function $C(x) = \langle u^*(x + x', t)u(x', t) \rangle$ for spatiotemporal chaotic solutions of Eq. (3) for the parameter values $c_1 = 3.5$ and $c_3 = 0.8$. From left to right, the curves correspond to system sizes respectively of $L = 1024$, 16384, 32768, 131072, and 262144. A clear exponential decay of correlations is found but only for system sizes that are surprisingly large compared to the correlation length of $\xi = 126$. The space-time resolution and total integration time were the same as for Part (a).

FIG. 2. (a) Plot of the Lyapunov dimension density $\delta$ versus the parameter $c_3$ for fixed $c_1 = 3.5$. Dimension densities were obtained from the slopes of figures similar to Fig. 1(a). To within numerical accuracy, the curve is continuous across the L1 transition region.

(b) Plot of the spatial correlation length $\xi$ versus the parameter $c_3$ for fixed $c_1 = 3.5$. There is an apparent divergence which is the signature of long-range order arising from a phase transition. The $\xi$ values were obtained by least-squares fits to the linear portion of correlation plots similar to Fig. 1(b). Systems of increasing sizes up to $L = 262,144$ were studied until we had found
values of $\xi$ that no longer changed upon further increases in $L$. We were not able to calculate correlation lengths reliably for $c_3 < 0.79$ since the value of $\xi$ continued to change with increasing system size and with increasing integration time.

FIG. 3. Plot of the Lyapunov dimension density $\delta$ versus inverse correlation length $1/\xi$ for the parameter value $c_1 = 3.5$. The inset shows the same data plotted versus the inverse square root, $1/\sqrt{\xi}$. The dotted lines connecting adjacent points were added only to guide the eye in identifying possible linear trends. The nonlinear dependence on $1/\xi$ is inconsistent with the scaling law Eq. (1) for a gas of weakly coupled domains that are highly correlated over the range of the correlation length $\xi$. 
(a) 

(b)
