The Off-Shell Boundary State and Cross-Caps in the Genus Expansion of String Theory

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Abstract

In this paper we use the boundary state formalism for the bosonic string to calculate the emission amplitude for closed string states from particular D-branes. We show that the amplitudes are exactly those obtained from world-sheet sigma model calculations, and the construction of the boundary state automatically enforces the requirement for integrated vertex operators, even in the case of an off-shell boundary state. Using the boundary state and a similar expansion for the cross-cap, we produce higher order terms in the string loop expansion for the partition function of the quadratic backgrounds considered.
1 Introduction

The study of off shell string theory has been addressed many times in the literature within the context of background independent string field theory [1, 2, 3, 4, 5] which has been the subject of a considerable amount of interest in that it can provide useful information about the properties of unstable d-branes [6, 7, 8]. Despite this there are several subtleties that have been examined, and in particular a great deal of effort has been expended in determining an action for a tachyon field coupled to a bosonic string [6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18], and while great progress had been made the understanding of higher loop effects is incomplete at best.

A tractable problem within this genre is the study of the off-shell theory in the background of a quadratic tachyon potential, a problem that is similar in spirit and detail to the examination of string theory in the background of a constant electromagnetic field. [19] In this paper we combine these naturally compatible studies using the boundary state formalism [20, 21, 22, 23, 24]. It allows us to calculate the probability for a topological defect which supports these quadratic fields to emit any number of closed string states into its bulk space-time. The loss of conformal invariance introduced by the background tachyon field is naturally accommodated by a conformal transformation which induces a calculable change in the boundary state. This new boundary state can be shown to reproduce the sigma model expectation values for the insertion of a vertex operator at an arbitrary point on the string world-sheet.

Using the correspondence between the sigma model calculation and that in the operator formalism the question of higher genus surfaces with some number of boundaries interacting with the background fields is considered. The insertion of both loops and boundaries is included naturally in this method, and the results obtained are compared with known results.

Throughout this work the action under consideration is

\[ S(g, F, T_0, U) = \frac{1}{4\pi\alpha'} \int_{\Sigma} d\sigma d\phi \left( g_{\mu\nu} \partial^\alpha X^\mu \partial_\alpha X_\mu \right) \]

\[ + \int_{\partial\Sigma} d\phi \left( \frac{1}{2} F_{\mu\nu} \partial_\phi X^\mu + \frac{1}{2\pi} T_0 + \frac{1}{8\pi} U_{\mu\nu} X^\mu X^\nu \right), \quad (1) \]

where \( \alpha' \) is the inverse string tension, \( \Sigma \) is the string world-sheet, \( d\sigma d\phi \) is the integration measure of the string bulk, \( d\phi \) is the integration measure of the string world-sheet boundary, and \( \partial_\phi \) is the derivative tangential to that boundary. The field content in this are a constant \( U(1) \) gauge field \( F_{\mu\nu} \) and the tachyon profile \( T(X) = \frac{1}{2\pi} T_0 + \frac{1}{8\pi} U_{\mu\nu} X^\mu X^\nu \) is characterized by a constant, \( T_0 \) and a constant symmetric matrix \( U_{\mu\nu} \) which provides a simple generalization for the discussion given in [12].
2 The Boundary State

The virtue of the boundary state as a tool in the analysis of the action above is that it allows calculations that previously took careful integration to be reduced to algebraic manipulations. We wish to carefully construct the boundary state and to show that it reproduces with ease the particle emission amplitudes that would be obtained from the string sigma model. The starting point for this analysis is the action (1). By varying it, one obtains the equation

$$\left( \frac{1}{2\alpha'} g_{\mu\nu} \partial_\sigma + F_{\mu\nu} \partial_\phi + \frac{1}{4\pi} U_{\mu\nu} \right) X^\nu = 0$$ (2)

as the boundary condition for the string world-sheet. Recalling the conventions from the action, $\partial_\sigma$ is the derivative normal to the boundary and $\partial_\phi$ is the tangential to the boundary. We now create a state $|B\rangle$ that obeys the above condition as an operator equation. To do this we reparametrize the string world-sheet in terms of holomorphic and antiholomorphic variables $z = \sigma e^{i\phi}$ and $\bar{z} = \sigma e^{-i\phi}$ and use the standard mode expansion for $X$ as a function of $z$

$$X^\mu(z, \bar{z}) = x^\mu + p^\mu \ln |z^2| + \sum_{m \neq 0} \frac{1}{m} \left( \frac{\alpha^\mu_m}{z^m} + \frac{\bar{\alpha}^\mu_m}{\bar{z}^m} \right).$$ (3)

we find that in terms of the mode operators the boundary conditions read

$$\left( g + 2\pi\alpha' F + \frac{\alpha' U}{2} \right)_{\mu\nu} \alpha^\mu_n + \left( g - 2\pi\alpha' F - \frac{\alpha' U}{2} \right)_{\mu\nu} \bar{\alpha}^\mu_{-n} = 0.$$ (4)

The condition for the boundary state to obey (4) can then be restated in terms of (4) to be

$$\left[ \left( g + 2\pi\alpha' F + \frac{\alpha' U}{2} \right)_{\mu\nu} \alpha^\mu_n + \left( g - 2\pi\alpha' F - \frac{\alpha' U}{2} \right)_{\mu\nu} \bar{\alpha}^\mu_{-n} \right] |B\rangle = 0,$$ (5)

$$\left[ g_{\mu\nu} p^\mu - i \frac{\alpha'}{2} U_{\mu\nu} x^\nu \right] |B\rangle = 0.$$ (6)

To satisfy this it is clear that $|B\rangle$ must be a coherent state, and it is given by

$$|B\rangle = \mathcal{N} \prod_{n \geq 1} \exp \left( - \left( \frac{g - 2\pi\alpha' F - \frac{\alpha' U}{2}}{g + 2\pi\alpha' F + \frac{\alpha' U}{2}} \right)_{\mu\nu} \frac{\alpha^-_{-n} \bar{\alpha}^\nu_n}{n} \right) \exp \left( - \frac{\alpha'}{4} x^\mu U_{\mu\nu} x^\nu \right) |0\rangle = \mathcal{N} \prod_{n \geq 1} \exp \left( - \Lambda_{\mu\nu} \alpha^-_{-n} \bar{\alpha}^\nu_n \right) \exp \left( - \frac{\alpha'}{4} x^\mu U_{\mu\nu} x^\nu \right) |0\rangle$$ (7)
where $N$ is a normalization constant which must be determined, and we define

$$\Lambda_{\mu\nu}^n = \frac{1}{n} \frac{g - 2\pi \alpha' F - \frac{\alpha' L}{2}}{g + 2\pi \alpha' F + \frac{\alpha'^2 U}{2}}$$

for future convenience.

### 2.1 Conformal Transformation of the Boundary State

Clearly this boundary state is not conformally invariant due to the addition of the interaction with the tachyon field. The two cases where we expect conformal invariance are at the two fixed points of renormalization group flow, namely $U = 0$ and $U = \infty$ which correspond respectively to the case of Neumann or Dirichlet boundary conditions on the boundary of the string world sheet. Note that in the case of Dirichlet boundary conditions the interaction with the background electromagnetic field is eliminated, as would be expected from the sigma model point of view. Because of this it is interesting to examine how the boundary state transforms under the $PSL(2,\mathbb{R})$ symmetry that is broken by the presence of the $U$ term in the boundary state. In the two conformally invariant cases this leaves the action invariant. The action of $PSL(2,\mathbb{R})$ on the complex coordinates of the disk is to perform the mapping

$$z \rightarrow w(z) = \frac{az + b}{b^*z + a^*}$$

where $a$ and $b$ satisfy the relation

$$|a^2| - |b^2| = 1.$$

This transformation maps the interior of the unit disk to itself, the exterior to the exterior and the boundary to the boundary. Moreover, this transformation of the coordinates induces a mapping which intermixes the oscillator modes. To see this consider the definition of the oscillator modes

$$\alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint dz \frac{dz}{2\pi i} z^m \partial X^\mu(z)$$

where the contour is the boundary of the unit disk, and the mode expansion of $X$ is

$$\partial X^\mu(z) = -i \sqrt{\frac{\alpha'}{2}} \sum_m \frac{\alpha^\mu_m}{z^{m+1}}.$$

Now, using the fact that $X$ is a scalar, or equivalently the fact that $\partial X$ is a $(1,0)$ tensor, we see that

$$\alpha^\mu_m = \oint \frac{dz}{2\pi i} z^m \partial_w X^\mu(w) \frac{dw}{dz}.$$
Now, using the fact that a mode expansion for $X$ exists in terms of $w$ with coefficients $\alpha'_m$ in exactly the same way as (12), we see that

$$\alpha'^\mu_m = M^{(a,b)}_{mn} \alpha'_n$$

(14)

where

$$M^{(a,b)}_{mn} = \oint \frac{dz}{2\pi i} z^m (b^* z + a^*)^{n-1} \frac{1}{(az + b)^{n+1}}.$$  

(15)

The properties of the matrix $M$ are interesting and facilitate further study. The matrix has a block diagonal form so that creation and annihilation operators are not mixed by the conformal transformation, and with appropriate normalization of the oscillator modes it can be seen to be hermitian, or equivalently that it preserves the inner product on the space of operators. The exact form $M$ as a function of its indices can be easily obtained, but for the purposes of this discussion it is easier to simply note that with the rescaling $M_{mp} = \sqrt{\frac{\mu}{m}} M_{mp}$ for either $m, p > 0$ or $m, p < 0$ then $M_{mp}^{-1} = M_{mp}^\dagger$.

Using this information we obtain that the modification of the boundary state $|B_{a,b}\rangle = N \exp \left( \sum_{n=1,j,k=-\infty}^\infty \alpha'^-_n M^{(a,b)}_{n-k} A_{n\mu} M^{(a,b)*}_{-n-j} \tilde{\alpha}^{-j} \right) \exp \left( -\frac{\alpha'}{4} x^\mu U_{\mu\nu} x^\nu \right) |0\rangle$.  

(16)

In this equation and all following ones we drop the $'$ associated with the transformed oscillators for notational simplicity. Due to the intuition from the conformally invariant cases we conjecture that the proper definition of the boundary state to give the correct overlap with all closed string states is

$$|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) |B_{a,b}\rangle.$$  

(17)

This is just the boundary state (16) integrated over the Haar measure of $\text{PSL}(2,\mathbb{R})$.

### 2.2 Boundary State emission of one particle

Since we wish to show that the boundary state is an algebraized version of the action (1) we must calculate the emission probability for various particles from the boundary state above. This has been done in more detail in [25] but we recapitulate the results here for completeness.

The case of the tachyon is straightforward. We must evaluate the overlap of the Fock space ground state with the transformed boundary state (16). Here, and in subsequent
formulae we omit the momentum conserving $\delta$-functions, and the integration over the
transformation parameters for the boundary state. For a tachyon with momentum $p^{\mu}$
we find that the probability for emission from the boundary state is

$$
\langle 0, p^{\mu} | B_{a,b} \rangle = \mathcal{N} \exp \left( -p^{\mu} p^{\nu} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} M_{-n0}^{(a,b)} \Lambda_{\mu \nu}^{(a,b)*} M_{-n0}^{(a,b)*} \right)
$$

$$
= \mathcal{N} \exp \left( -p^{\mu} p^{\nu} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( g - 2\pi \alpha' F - \frac{\alpha' U}{2 n} \right) \mu_\nu \frac{|b|^{2n}}{|a|^{2n}} \right) \quad (18)
$$

In the above expression we have used the previously defined form for $\Lambda_{\mu \nu}^{n}$, the fact that
$M_{-n0}^{(a,b)} = \left( \frac{-b^*}{a} \right)^n$, and the relation $\alpha'' = \sqrt{\frac{\alpha'}{2}} p^{\mu}$.

Similarly, for an arbitrary massless state with polarization tensor $P_{\mu \nu}$ and momentum $p^{\mu}$

$$
| P_{\mu \nu} \rangle = P_{\mu \nu} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} | 0, p^{\mu} \rangle \quad (19)
$$

the overlap to be calculated is

$$
\langle P_{\mu \nu} | B_{a,b} \rangle = \mathcal{N} \exp \left( -p_{\mu} p_{\nu} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} M_{-n-1}^{(a,b)} \Lambda_{\mu \nu}^{(a,b)*} M_{-n-1}^{(a,b)*} \right)
$$

$$
+ p^{\alpha} p^{\beta} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} M_{-n-1}^{(a,b)} \Lambda_{\mu \alpha}^{(a,b)*} M_{-n-1}^{(a,b)*} \sum_{m=1}^{\infty} M_{-m0}^{(a,b)} \Lambda_{\beta \nu}^{m} M_{-m-1}^{(a,b)*} \right)
$$

$$
= \mathcal{N} \exp \left( -p^{\mu} p^{\nu} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( g - 2\pi \alpha' F - \frac{\alpha' U}{2 n} \right) \mu_\nu \frac{|b|^{2(n-1)}}{|a|^{2(n-1)}} \frac{1}{|a|^2} \right)
$$

$$
+ p^{\alpha} p^{\beta} \frac{\alpha' \alpha}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( g - 2\pi \alpha' F - \frac{\alpha' U}{2 n} \right) \mu_\alpha \frac{|b|^{2(n-1)}}{|a|^{2(n-1)}} \frac{-b^*}{|a|^2 a^*} \right)
$$

$$
\times \sum_{m=1}^{\infty} \left( g - 2\pi \alpha' F - \frac{\alpha' U}{2 m} \right) \beta_\nu \frac{|b|^{2(m-1)}}{|a|^{2(m-1)}} \frac{1}{|a|^2} \right) \quad (20)
$$

where again the explicit form of the matrices $M$ has been used in the last equality.

This kind of argument can be repeated indefinitely, but we present one more such
calculation which contains the germs of generality, which will prove useful to consider.
In particular the more general state $A$ with momentum $p^\mu$ is defined by

$$|A_{\mu\nu\delta}\rangle = A_{\mu\nu\delta} \alpha_{-a}^\mu \alpha_{-b}^\nu \alpha_{-c}^\delta |0, p^\mu\rangle,$$  \hspace{1cm} (21)

and its overlap with the boundary state is given by

$$\langle A_{\mu\nu\delta} | B \rangle = N \exp \left( -\frac{1}{2} \sum_{n=1}^{\infty} M_{-n0}^{(a,b)} \Lambda_{n}^{(a,b)*} \right) \times$$

$$A_{\mu\nu\delta} \sqrt{\alpha} \left[ \sum_{n} abM_{-n-a}^{(a,b)} \Lambda_{n}^{(a,b)*} \sum_{m} c M_{-m0}^{(a,b)*} \Lambda_{m}^{(a,b)*} \right.$$ 

$$+ p^\alpha \sum_{n} acM_{-n-a}^{(a,b)} \Lambda_{n}^{(a,b)*} \sum_{m} b M_{-m0}^{(a,b)*} \Lambda_{m}^{(a,b)*}$$ 

$$- p^\alpha p^\beta p^\gamma \sum_{n} aM_{-n-a}^{(a,b)} \Lambda_{n}^{(a,b)*} \sum_{m} b M_{-m0}^{(a,b)*} \Lambda_{m}^{(a,b)*}$$ 

$$\times \sum_{l} c M_{-l0}^{(a,b)*} \Lambda_{l}^{(a,b)*} \right].$$ \hspace{1cm} (22)

At this point the formulae become more abstruse and do not in general convey any more information than can already be discerned. It should be noted that the contractions of the various matrices look suspiciously like those of Green’s functions, which it will transpire that they are, but to see this requires a simple calculation. A special case of a more general formula proven in the next section shows that for $y = \frac{a z + b}{b z + a}$ subject to $|a|^2 - |b|^2 = 1$ we have that

$$\frac{1}{(k - 1)!} \delta^k z^d (y) |_{y=0} = k M_{-d-k}^{(a,b)*}.$$ \hspace{1cm} (23)

Note that since the transformation from $z$ to $y$ is one-to-one the above equation makes sense. This completes the analysis for the emission of one particle from the boundary state $|B_{a,b}\rangle$, however the question becomes more interesting for the emission of more than one particle.

### 2.3 Boundary State emission of many particles

As in the case of emission of one particle by the boundary state it is perhaps the most instructive to consider the case of the emission of two tachyons first, and then specialize to more complicated correlators. Ordering the operators appropriately for radial (as opposed to anti-radial) quantization and noting that the PSL(2,R) transformation is not sufficient to fix the location of both close string vertex operators we proceed to
calculate, using the previous definitions and mode expansion

\[
\langle B_{a,b} \rangle : \exp (i k^\mu X_\mu) : \omega | 0, p^\mu \rangle = N \langle 0 | \exp \left( - \frac{\alpha^\prime}{2} \sum_{l>0} \frac{1}{l} \left( \alpha^\mu - \bar{\alpha}^\mu \right) \right) \exp \left( \frac{i k^\mu x_{ij} + \sqrt{\frac{\alpha^\prime}{2}} k^\mu \bar{\alpha}^\mu \ln |\omega|^2}{|\omega|^2} \right) | 0, p^\mu \rangle
\]

\[
= N \exp \left( \frac{k^\mu x_{ij} + \sqrt{\frac{\alpha^\prime}{2}} k^\mu \bar{\alpha}^\mu \ln |\omega|^2}{|\omega|^2} \right) \exp \left( -p^\mu p^\nu \frac{\alpha^\prime}{2} \sum_{n=1}^\infty M_{n0}^{(a,b)} A^{n \mu \nu} M_{n0}^{(a,b)*} \right)
\]

\[
\exp \left( -p^\mu k^\nu \frac{\alpha^\prime}{2} \sum_{n=1, j=0}^\infty M_{n0}^{(a,b)} A^\nu A^{n \mu} M_{n0}^{(a,b)*} \right)
\]

\[
\exp \left( -k^\mu p^\nu \frac{\alpha^\prime}{2} \sum_{n=1, i=0}^\infty \bar{\omega}^i M_{n0}^{(a,b)} A^{n \mu \nu} M_{n0}^{(a,b)*} \right)
\]

\[
\exp \left( -p^\mu k^\nu \frac{\alpha^\prime}{2} \sum_{n=1, i, j=0}^\infty \bar{\omega}^i M_{n0}^{(a,b)} A^{n \mu \nu} M_{n0}^{(a,b)*} \right) . \quad (24)
\]

Upon closer inspection it is apparent that this very reminiscent of a pair of exponentiated greens functions. In fact, we will show in the next section that the expectation value of two tachyon vertex operators in the background of the boundary interaction exactly coincides with this.

The next natural quantity to calculate is the emission of a more general state in place of either, or both tachyons in the previous calculation. It is of course possible to demonstrate the overlap of an arbitrary string state explicitly, but the combinatorial nature of the result quickly renders the resulting expressions obscure. With this in mind we examine a state that contains the germ of generality and corresponds to the calculation done in the case of one particle emission.

\[
\langle B_{a,b} \rangle : \exp (i k^\mu X_\mu) : \omega | \omega^\alpha A_{\mu \delta} | \tilde{\alpha}^\mu A_{\mu \delta} | 0, p^\mu \rangle = \mathcal{A} T_2 \times A^{\mu \delta}
\]

\[
\left[ \sqrt{\frac{\alpha^\prime}{2}} \left( - \sum n M_{rj}^{(a,b)} \Lambda_{\gamma \gamma}^{r} M_{rj}^{(a,b)*} \bar{\omega}^j k^\gamma - k^\mu \frac{1}{\omega^\mu} - \sum n M_{rn}^{(a,b)} \Lambda^\gamma_{\nu r} M_{rn}^{(a,b)*} p^\gamma \right) \right.
\]

\[
\left. - \sum k^\gamma \omega^i M_{ri}^{(a,b)} \Lambda^i_{\gamma \gamma} M_{ri}^{(a,b)*} p^\gamma - k^\nu \frac{1}{\omega^\nu} - \sum p^\gamma M_{r0}^{(a,b)} \Lambda^\gamma_{\nu r} M_{r0}^{(a,b)*} p^\nu \right)
\]

\[
\left. - \sum k^\gamma \omega^j M_{rij}^{(a,b)} \Lambda^j_{\gamma \gamma} M_{rij}^{(a,b)*} q - k^\nu \frac{1}{\omega^\nu} - \sum p^\gamma M_{r0j}^{(a,b)} \Lambda^\gamma_{\nu r} M_{r0j}^{(a,b)*} q \right) \]
\[
+ \left( - \sum k^\gamma \omega^p M^{(ab)}_{r1} \Lambda^{r}_{\gamma \delta} M^{(ab)*}_{r2} q - k_\nu \frac{1}{\omega^q} - \sum p^\gamma M^{(ab)}_{r0} \Lambda^{r}_{\gamma \delta} M^{(ab)*}_{r2} q \right)
\]
\[
\left( - \sum n M^{(ab)}_{r1} \Lambda^{r}_{\mu \nu} M^{(ab)*}_{r2} p \right) \sqrt{\frac{\alpha'}{2}} + (p \leftrightarrow q, \nu \leftrightarrow \delta). \quad (25)
\]

In the above, \(A_{T2}\) is the result for the boundary state to emit two tachyons, which appears as a multiplicative factor and is calculated explicitly above [24].

Similarly, it is possible to calculate the analogous expression for the vertex which emits the complicated state at the point \(\omega\) on the disk, and using the standard commutation relationships as outlined previously we find
\[
\langle B_{a,b} : A_{\mu \nu \delta} \frac{1}{(n - 1)!} \partial^n X^\mu \frac{1}{(p - 1)!} \partial^p X^\nu \frac{1}{(q - 1)!} \partial^q X^\delta \exp (ik^\mu X^\mu) : |0, p^\nu\rangle = A_{T2} \times A_{\mu \nu \delta}
\]
\[
\left[ - \left( \sum \frac{1}{(n - 1)!} \frac{1}{(p - 1)!} \frac{m!}{(m - n)!} \omega^{m-n} M^{(ab)}_{r1} \Lambda^{r}_{\mu \nu} M^{(ab)*}_{r2} \frac{j!}{(j-p)!} \omega^{j-p} \right)
\right.
\]
\[
\left. + \left\{ \alpha' \left( - \sum \frac{1}{(n - 1)!} \frac{m!}{(m - n)!} \omega^{m-n} M^{(ab)}_{r1} \Lambda^{r}_{\mu \nu} M^{(ab)*}_{r2} \frac{j!}{(j-p)!} \omega^{j-p} \right)
\right.ight.
\]
\[
\left. - \sum \frac{1}{(n - 1)!} \frac{m!}{(m - n)!} \omega^{m-n} M^{(ab)}_{r1} \Lambda^{r}_{\mu \nu} M^{(ab)*}_{r2} \frac{j!}{(j-p)!} \omega^{j-p} \right)
\]
\[
\left. \left( - \sum p^\gamma M^{(ab)}_{r0} \Lambda^{r}_{\gamma \delta} M^{(ab)*}_{r2} \frac{1}{(p-1)!} \frac{j!}{(j-p)!} \omega^{j-p} + p_{\mu} (-1)^n \omega^{-n} \right)
\right]
\[
\left. \left( - \sum p^\gamma M^{(ab)}_{r0} \Lambda^{r}_{\gamma \delta} M^{(ab)*}_{r2} \frac{1}{(q-1)!} \frac{j!}{(j-q)!} \omega^{j-q} + p_{\mu} (-1)^q \omega^{-q} \right) \right) \right) \}
\]
\[
- \sum k^\gamma \omega^m M^{(ab)}_{r1} \Lambda^{r}_{\gamma \delta} M^{(ab)*}_{r2} \frac{1}{(q-1)!} \frac{j!}{(j-q)!} \omega^{j-q} + p_{\mu} (-1)^q \omega^{-q} \right) + (p \leftrightarrow q, \nu \leftrightarrow \delta). \quad (26)
\]

The above expression can be seen to be the same as that of the emission with the complicated vertex at the center, as the case of two tachyon emission would suggest.

### 3 Sigma Model

Having performed the calculations from the point of view of the raising and lowering operators it is now instructive to compare with what should be analogous results from sigma model calculations. We fix our convention that the functional integral is in all cases the average over the action given in (2),
\[
\langle \mathcal{O}(X) \rangle = \int \mathcal{D}X e^{-S(X)} \mathcal{O}(X). \quad (27)
\]
In addition, the greens function on the unit disk with Neumann boundary conditions is determined to be \[ G^{\mu\nu}(z, z') = -\alpha' g^{\mu\nu} \left( -\ln |z - z'| - \ln |1 - z\bar{z}'| \right), \] (28)
and it will be useful also to know the bulk to boundary propagator which is
\[ G^{\mu\nu}(\rho e^{i\phi}, e^{i\phi'}) = 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\rho^m}{m} \cos[m(\phi - \phi')]. \] (29)
The boundary to boundary propagator can be read off from (29) as the limit in which \( \rho \to 1 \). Throughout, we will use \( z = \rho e^{i\phi} \) as a parameterization of the points within the unit disk, so \( 0 \leq \rho \leq 1 \) and \( 0 \leq \phi < 2\pi \). Using the bulk to boundary propagator it is possible to integrate out the quadratic interactions on the boundary \[ G^{\mu\nu}(z, z') = -\alpha' g^{\mu\nu} \ln |z - z'| + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \frac{\left( g - 2\pi\alpha' F - \frac{\alpha' U}{2} \right)}{g + 2\pi\alpha' F + \frac{\alpha' U}{2}} \left[ \mu \nu \right] (\bar{z}z')^n 
+ \alpha' \sum_{n=1}^{\infty} \frac{\left( 2\pi\alpha' F + \frac{\alpha' U}{2} \right)}{g + 2\pi\alpha' F + \frac{\alpha' U}{2}} \{ \mu \nu \} (\bar{z}z')^n 
- \alpha' \sum_{n=1}^{\infty} \frac{\left( 2\pi\alpha' F + \frac{\alpha' U}{2} \right)}{g + 2\pi\alpha' F + \frac{\alpha' U}{2}} \{ \mu \nu \} (\bar{z}z')^n. \] (30)
Note that this expression is symmetric as it should be because the antisymmetry of lorentz indices in the final term is compensated by the antisymmetry of the coordinate term.

The first calculation that must be done to determine the normalization of the sigma model amplitudes is the partition function. In this approach the oscillator modes of \( X \) must be integrated out with the contributions from \( F \) and \( U \) treated as perturbations. Since both perturbations are quadratic, all the feynmann graphs that contribute to the free energy can be written and evaluated, and explicitly the free energy is given by
\[ \mathcal{F} = -\sum_{m=1}^{\infty} Tr \ln \left( g + 2\pi\alpha' F + \frac{\alpha' U}{2m} \right), \] (31)
see \[19, 27\] for further calculations done in this spirit. From (31) we immediately see that the partition function is given by
\[ Z = e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\det \left( g + 2\pi\alpha' F + \frac{\alpha' U}{2m} \right)} \int dx_0 e^{-\frac{1}{4\alpha'\mu_0^+\mu_0^-}}. \]
\[
\frac{1}{\text{det}(U/2)} e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\text{det}(g + 2\pi\alpha' F + \alpha' U/m)}.
\]

(32)

This expression is divergent, but using $\zeta$-function regularization \[11\] it can be reduced to

\[
Z = e^{-T_0} \sqrt{\det \left( \frac{g + 2\pi\alpha' F}{U/2} \right)} \det \Gamma \left( 1 + \frac{\alpha' U/2}{g + 2\pi\alpha' F} \right),
\]

(33)

where $\Gamma(g)$ is the $\Gamma$ function and the dependence of all transcendental functions on the matrices $U$ and $F$ is defined by their Taylor expansion.

3.1 Conformal transformation on the disk

We now wish to calculate the expectation value for vertex operators that correspond to different closed string states, however this is a process that must be done with some care. To calculate the emission of a closed string in the world-sheet picture one generally considers a disk emitting an asymptotic closed string state. This is really a closed string cylinder diagram. The standard method is to use conformal invariance to map the closed string state to a point on the disk, namely the origin, where a corresponding vertex operator is inserted. On the other hand it has been cogently argued that it is necessary to have an integrated vertex operator for closed string states to properly couple \[12\], in particular that the graviton must be produced by an integrated vertex operator to couple correctly to the energy momentum tensor. The distinction between a fixed vertex operator and an integrated vertex operator is moot in the conformally invariant case where the integration will only produce a trivial volume factor, however in the case we consider more care must be taken. We wish to consider arbitrary locations of the vertex operators on the string world sheet, and the natural measure to impose is that of the conformal transformations which map the origin to a point within the unit disk on the complex plane.

In other words we propose to allow the vertex operator corresponding to the closed string state to be moved from the origin by a conformal transformation that preserves the area of the unit disk, namely a PSL(2,R) transformation. The method to accomplish this is to go to a new coordinate system

\[
y = \frac{az + b}{b^*z + a^*}, \quad |a^2| - |b^2| = 1,
\]

(34)

and a vertex operator at the origin $y = 0$ would correspond to an insertion of a vertex operator at the point $z = -\frac{b}{a}$. It is worth noting that in the case of conformal invariance, that is when $U \to 0$ or $U \to \infty$ the greens function remains unchanged in form, the $y$ dependence coming from the replacement $z \to z(y)$. Even in the case of finite $U$ the only change to the greens function is the addition of a term that is harmonic within the
unit disk. The parameter of the integration over the position of the vertex operator would be to the measure on PSL(2,R), giving an infinite factor in the conformally invariant case \[4, 12, 28\]. From this argument we have a definite prescription for the calculation of vertex operator expectation values, which is to use the conformal transformation to modify the greens function, and calculate the expectation values of operators at the origin with this modified greens function.

3.2 Emission of one particle in sigma model

Now we will use this prescription to calculate the sigma model expectation values of some operators, and we will start with the simplest, that of the closed string tachyon. The vertex operator for the tachyon is \(e^{ip\mu X^\mu(y)}\), and it is inserted at the point \(y = 0\). The normal order prescription for all such operators is that any divergent pieces will be subtracted, but finite pieces will remain and by inspection we see that the appropriate subtraction from the greens function is

\[
: g^{\mu\nu}(z, z') : = g^{\mu\nu}(z, z') - g^{\mu\nu} \alpha' \ln |z - z'| \tag{35}
\]

Using (35) we can immediately see that

\[
\langle : e^{ip\mu X^\mu(y=0)} : \rangle = Z e^{-\frac{1}{2} p_\mu p_\nu g^{\mu\nu}(y(y))} \mid_{y=0} = Z e^{-\frac{1}{2} p_\mu p_\nu \sum_{n=1}^{\infty} \left( \frac{g - 2\pi \alpha' F + \frac{n}{2}}{g + 2\pi \alpha' U + \frac{n}{2}} \right) \mu\nu \frac{1}{n} \frac{|b_{2n}|}{a_{2n}}}. \tag{36}
\]

We recall that our procedure will necessitate an integration over the the parameters of the PSL(2,R) transformation, but comparison with (18) reveals that the normalization is fixed by

\[
\mathcal{N} = Z. \tag{37}
\]

Having obtained this result fixing the normalization it is natural to push the correspondence further as a check of its validity. We perform a similar analysis for the massless closed string excitations. In particular the graviton insertion at \(y = 0\) is given by

\[
\langle \mathcal{V}_h \rangle = \langle : -\frac{2}{\alpha'} h_{\mu\nu} \partial X^\mu \partial X^\nu e^{ip\mu X^\mu(y=0)} : \rangle \tag{38}
\]

where \(h\) is a symmetric traceless tensor and the normalization follows the conventions of \[29\]. This can be analyzed by the same techniques as for the tachyon, noting that there will be cross contractions between the exponential and the \(X\)-field prefactors.
Explicitly we obtain

\[
\langle \mathcal{V}_h \rangle = -\frac{2}{\alpha'} Z h_{\mu\nu} \left( \partial \partial' : G^{\mu\nu} (z(y), z'(y)) : \right. \\
\times \bar{\partial} : G^{\mu\nu} (z(y), z'(y)) : (i p_\alpha)(i p_\beta) \left. \right) e^{-\frac{4}{\alpha'} \sum p_\mu p_\nu \cdot G^{\mu\nu} (z(y), z'(y))} \mid_{y=0}
\]

\[
= Z h_{\mu\nu} \left( - \sum_{n=1}^{\infty} \left( \frac{g - 2 \pi \alpha' F - \frac{\alpha' U}{2n}}{g + 2 \pi \alpha' F + \frac{\alpha' U}{2n}} \right)^{\mu\nu} \left[ \frac{b^2(n-1)}{a^{2(n-1)} |a^2|^2} - b \right] \right)
\]
3.3 Sigma model emission of many particles

Having explored the emission of one particle and found many substantial similarities, we now look at the emission of two particles. We expect that this amplitude will depend upon the relative position of the two vertex operators, and since even in the case of conformal invariance there are not enough free parameters to fix two closed string vertex operators on the disk world sheet. We first calculate the expectation value of the emission of two tachyons, with momenta $p$ and $k$.

\[
\langle : e^{ik_n X^\mu} : \mid_\omega : e^{ip_n X^\nu} : \mid_0 \rangle = Z \exp \left( -\frac{k_\mu k_\nu}{2} G^{\mu\nu} (z(\omega), z(\omega)) \right) \\
\times \exp \left( \frac{p_\mu p_\nu}{2} G^{\mu\nu} (z(0), z(0)) \right) \exp \left( -\frac{k_\mu p_\nu}{2} G^{\mu\nu} (z(\omega), z(0)) \right) \\
= A_{T2\sigma}
\]

This is the necessary first step in determining a more arbitrary amplitude. To make contact with the more complicate amplitudes calculated in (23) and (24) we consider the expression

\[
\langle A_{\mu\nu\delta} : \frac{\partial^n}{(n-1)!} X^\mu : \frac{\bar{\partial}^p}{(p-1)!} X^\nu : \frac{\partial^q}{(q-1)!} X^\delta e^{ik_n X^\mu} : \mid_\omega : e^{ip_n X^\nu} : \mid_0 \rangle = A_{T2\sigma} A_{\mu\nu\delta} \left[ \left( \frac{ik_\alpha}{(n-1)!} \partial^n G^{\mu\alpha} (z(\omega), z'(\omega)) + \frac{ip_\alpha}{(n-1)!} \bar{\partial}^n G^{\mu\alpha} (z(\omega), z'(0)) \right) \right. \\
\times \left( \frac{ik_\beta}{(p-1)!} \bar{\partial}^p G^{\nu\beta} (z(\omega), z'(\omega)) + \frac{ip_\beta}{(p-1)!} \partial^p G^{\nu\beta} (z(\omega), z'(0)) \right) \right. \\
\times \left( \frac{ik_\gamma}{(q-1)!} \partial^q G^{\nu\gamma} (z(\omega), z'(\omega)) + \frac{ip_\gamma}{(q-1)!} \bar{\partial}^q G^{\nu\gamma} (z(\omega), z'(0)) \right) \\
\left. + \frac{\partial^n}{(n-1)!} \frac{\partial^p}{(p-1)!} G^{\mu\nu} (z(\omega), z'(\omega)) \right) \\
\times \left( \frac{ik_\gamma}{(q-1)!} \partial^q G^{\nu\gamma} (z(\omega), z'(\omega)) + \frac{ip_\gamma}{(q-1)!} \bar{\partial}^q G^{\nu\gamma} (z(\omega), z'(0)) \right) \\
\left. + \frac{\partial^n}{(n-1)!} \frac{\bar{\partial}^q}{(q-1)!} G^{\mu\nu} (z(\omega), z'(\omega)) \right) \\
\times \left( \frac{ik_\beta}{(p-1)!} \bar{\partial}^p G^{\nu\beta} (z(\omega), z'(\omega)) + \frac{ip_\beta}{(p-1)!} \partial^p G^{\nu\beta} (z(\omega), z'(0)) \right) \\
\left. + \frac{\bar{\partial}^n}{(n-1)!} \frac{\partial^q}{(q-1)!} G^{\mu\nu} (z(\omega), z'(\omega)) \right] 
\]

(43)

Also note that if we consider

\[
\langle : e^{ik_n X^\mu} : \mid_\omega A_{\mu\nu\delta} : \frac{\partial^n}{(n-1)!} X^\mu : \frac{\bar{\partial}^p}{(p-1)!} X^\nu : \frac{\partial^q}{(q-1)!} e^{ip_n X^\mu} : \mid_0 \rangle
\]

we see that it gives the above expression (43) with $\omega \leftrightarrow 0$. 

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To demonstrate the general equivalence of the boundary state approach with that of the sigma model the sums that appear in the general expressions of boundary state matrix elements must be shown to coincide with the expressions that appear above. To this end consider first the sum that appears in (24),

\[ \sum_{m=0}^{\infty} \omega^m M_{nm}^{(ab)} = \sum_{m=0}^{\infty} \int \frac{dz}{2\pi i} \omega^m z^n \frac{(b^* z + a^*)^{m-1}}{(az + b)^{m+1}} \]

\[ = \int \frac{dz}{2\pi i} \frac{1}{z^n} \frac{(b^* z + a^*)(a - \omega b^*)}{(az + b)^{m+1}} \left( z - \frac{a^* \omega - b}{-b^* \omega + a} \right)^{-1} \]

\[ = \left( \frac{a^* \omega - b}{-b^* \omega + a} \right)^c. \]

This derivation uses the normalization condition on \(a\) and \(b\), and can be seen to be equal to \(z^n(y)\) which is the inverse transform of (4).

The other sum that appears generally in this analysis is

\[ \sum_{m=0}^{\infty} \frac{m!}{(m-n)!} \omega^{m-n} M_{rm}^{(ab)} \]

as seen in (26). In the case \(n > m\) we have used the shorthand

\[ \frac{m!}{(m-n)!} = m(m-1)\ldots(m-n+1). \]

Now consider

\[ \sum_{m=0}^{\infty} \frac{m!}{(m-n)!} \omega^{m-n} M_{rm}^{(ab)} = \sum_{m=0}^{\infty} \int \frac{dz}{2\pi i} \frac{m!}{(m-n)!} \omega^{m-n} z^r \frac{(b^* z + a^*)^{m-1}}{(az + b)^{m+1}} \]

\[ = n! \int \frac{dz}{2\pi i} \frac{m!}{(m-n)!} \omega^{m-n} z^r \frac{(b^* z + a^*)^{m-1}}{(az + b)^{m+1}} \left( z - \frac{a^* \omega - b}{-b^* \omega + a} \right)^{-n-1} \]

\[ = \partial^n z^r (b^* z + a^*)^{n-1}(a - b^* \omega)^{-n+1} \bigg|_{z = \frac{a^* \omega - b}{-b^* \omega + a}} \]

\[ = \partial^n \left( \frac{a^* z - b}{-b^* z + a} \right)^r \bigg|_{z = \omega}. \]

The last equality in this can be shown by induction, the case for \(n = 1\) is trivial, and so we demonstrate the induction. First note that use of the Leibnitz rule gives

\[ \partial^k z^r (b^* z + a^*)^{k-1}(a - b^* \omega)^{-k+1} \bigg|_{z = \frac{a^* \omega - b}{-b^* \omega + a}} = \]

\[ \frac{(a^* \omega - b)^{n-k}}{(-b^* \omega + a)^{n+k}} \sum_{j=0}^{k} \binom{k}{j} n \ldots (n - (k - j - 1))(k - 1) \ldots (k - j)b^j(a^* \omega - b)^j. \]
Now consider for $k = k_0 + 1$ the expression can be manipulated

$$
\partial^k \left( \frac{a^* z - b}{-b^* z + a} \right)^n \Bigg|_{z=\omega} = \partial \left( \partial^{k_0} \left( \frac{a^* z - b}{-b^* z + a} \right)^n \right) \\
= \partial \left( \frac{(a^* z - b)^{n-k_0}}{(-b^* z + a)^{n+k_0}} \sum_{j=0}^{k_0} \binom{k_0}{j} n \ldots (n - (k_0 - j - 1)) \right) \\
(k_0 - 1) \ldots (k_0 - j) b^* j (a^* z - b)^j \\
= \frac{(a^* z - b)^{n-(k_0+1)}}{(-b^* z + a)^{n+(k_0+1)}} \sum_{j=0}^{k_0+1} \binom{k_0+1}{j} n \ldots (n - (k_0 - j)) \\
k_0 \ldots (k_0 + 1 - j) b^* j (a^* z - b)^j 
$$

as desired. This demonstrates the induction step, and the validity of (45). Note that similar results can be obtained for expressions with negative indices on $M_{nk}^{(a,b)}$ and negative powers of $\omega$. These are obtained from considering the boundary state on the right of the matrix elements. This have to be interpreted as a dual description of the boundary states presented. This is because radial quantization and the operator state correspondence imply that in this case the domain of interest is the complex plane with the unit disk excluded. This is equally a fundamental region of the plane, and the conformal transformation between the two is $\omega \rightarrow \frac{1}{\omega}$, a fact which is intimated at by the fact that $M_{-n-k}^{(a,b)} = M_{nk}^{*(a,b)}$.

Now we have demonstrated that the results obtained from the boundary state calculations exactly match those of the sigma model after the propagator including the boundary perturbations has been obtained, and the resulting expression has been transformed into a new coordinate system. This shows that the boundary state algebraizes all matrix elements that would otherwise be calculated in the sigma model. This observation will be important as we generalize these results to higher genus surfaces. We also remark that the result explicitly presented for the emission of two closed string states clearly generalizes to the emission of an arbitrary number of such particles. Mechanically this can be seen because the commutation of two such vertex operators to produce a normal-ordered expression produces the familiar logarithmic term, and the boundary state gives the $F$ and $U$ dependence within the inner product.
4 Higher Genus amplitudes

Now that the overlap of the boundary state with either single or multiple particle states we have the tools that are needed to proceed and determine higher order contributions in the string loop sense to the vacuum energy of the object described by the boundary state. We will proceed in the following manner, by utilizing a sewing construction to relate higher order amplitudes to products of lower order amplitudes. The procedure outlined as envisioned can produce an arbitrary number of interactions with the boundary state at the oriented tree level, and an arbitrary number of handles and interactions with the boundary state in the unoriented sector. As is well known the description of higher genus orientable surfaces is a more difficult subject and the construction will produce results that are implicit rather than explicit. The final result will be several terms in the Euler number expansion so that

\[ Z = Z_{\text{disk}} + Z_{\text{annulus}} + Z_{\text{MobiusStrip}} + \ldots \]  

(48)

with each term carrying the appropriate power of the open string coupling constant. The results in this section will be organized by Euler number, and where appropriate compared with other similar results in the literature.

4.1 $\chi = 1$

There are two surfaces with $\chi = 1$, the disk and $RP^2$. The non-orientable surface $RP^2$, see [30] for details in another context, has no interaction with the fields $F$ and $U$ and so is not of interest for this analysis. The disk by contrast has been analysed previously in this work and the contribution to the partition function for the boundary state is given by its overlap with the unit operator (equivalently the tachyon with zero momentum), as given in (32).

4.2 $\chi = 2$

There are several surfaces that have an Euler number of 2. The easiest to discuss in this is the torus, which is immaterial for the same reason that $RP^2$ was among the surfaces with $\chi = 1$, namely that it has no interactions with $F$ or $U$. Similarly the Klein bottle, the unoriented equivalent of the torus, will not contribute to the partition function. We are left with the annulus and with the Mobius strip, as the nontrivial contributions at this level. The annulus can be thought of as the tree level closed string exchange channel. The Mobius strip is the non-orientable analogue of the disk.

Considering first the annulus that was considered in detail in [25], we recapitulate some of the salient results. Suppressing for notational simplicity the integrations over
the parameters of the conformal transformations we have that

\[ Z_{\text{annulus}} = \langle B_{a,b} | \frac{1}{\Delta} | B_{a',b'} \rangle. \]  

(49)

Using the integral representation of the closed string propagator

\[ \frac{1}{\Delta} = \frac{1}{4\pi} \int \frac{d^2z}{|z|^2} z^{L_0-1} \bar{z}^{L_0-1} \]

and suppressing the integrals and, temporarily, the zero modes for notational clarity we have

\[
Z_{\text{annulus}} = Z_{\text{disk}}^2 \langle 0 \rangle \exp \left( -\alpha_{\mu}^a M_{n_i}^b \Lambda^n_{\mu \nu} M_{nj}^{(a,b)^*} \tilde{\alpha}^\nu \right) z \sum_{\alpha-n} \tilde{z} \sum_{\bar{\alpha}} \tilde{\bar{z}} \exp \left( -\alpha_{-\bar{k}}^a M_{-m-k} \Lambda^m_{-\bar{\gamma} \nu} M_{-m-l}^{(a,b)^*} \tilde{\bar{\alpha}}_{-\bar{l}} \right) \langle 0 \rangle 
\]

\[
= Z_{\text{disk}}^2 \langle 0 \rangle \exp \left( -\alpha_{\mu}^a M_{n_i}^b \Lambda^n_{\mu \nu} M_{nj}^{(a,b)^*} \tilde{\alpha}^\nu \right) \exp \left( -z^\nu \alpha_{-\bar{k}}^a M_{-m-k} \Lambda^m_{-\bar{\gamma} \nu} M_{-m-l}^{(a,b)^*} \tilde{\bar{\alpha}}_{-\bar{l}} \right) \langle 0 \rangle 
\]

\[
= \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} g^{\mu \nu} \delta_{rs} \left\{ r M_{\mu \nu} M^{(a,b)^*} M_{\nu \mu} M^{(a,b)^*} z^{k} \right\} \right) \times Z_{\text{disk}}^2 F(p). 
\]

(50)

Verifying the first equality requires that one use the Baker-Hausdorf formula for commutators of exponentials, and the second equality is an application of Wick’s theorem. The term in the last exponential is understood to have its powers defined by contraction of both the lorentz and oscillator indices, and the number \( F(p) \) is a gaussian factor dependant on the (otherwise implicit) momentum of each boundary state, which can be read off from the boundary conditions [3]. Explicitly the form of \( F(p) \) is given by

\[
F(p) = \exp \left\{ p^\mu p^\nu \left[ \left( \delta_{0j} g_{\mu \delta} - M_{n_0}^{(a,b)^*} \Lambda_n^\mu M^{(a,b)^*} \right) \right] \right\} 
\]

\[
\left( \frac{1}{\delta_{jk} g_{\nu \gamma} - j \tilde{z} j \Lambda^{(a,b)^*} \Lambda_{m^0} M^{(a,b)^*} z^l \Lambda^{\nu \gamma} M^{(a,b)^*} \Lambda_{m^0} M^{(a,b)^*}} \right)_{jk} 
\]

\[
\left( \delta_{0k} g_{\gamma \nu} - k \tilde{z} j \Lambda^{(a,b)^*} \Lambda_{n^0} M_{n^0}^{(a,b)^*} - g_{\mu \nu} \right). 
\]

(51)

In addition this is multiplied by terms coming from the zero mode part of the propagator. In the preceeding equations the oscillator index has been chosen as positive or zero to make the negative signs meaningful. In all cases, repeated indices indicate summation.
We note in passing that the cases of $U \to 0$ and $U \to \infty$ give a particularly simple form for the matrices $M \Lambda M^*$. We have

$$M^{(a,b)}_{km} \Lambda^k_{\mu\nu} M^{(a,b)*}_{kn} \bigg|_{U \to 0} = M^{(a,b)}_{km} \frac{1}{k} \left( \frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F} \right)_{\mu\nu} M^{(a,b)*}_{kn}$$

$$= \left( \frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F} \right)_{\mu\nu} \frac{1}{m} \delta_{mn}, \quad (52)$$

and similarly

$$M^{(a,b)}_{km} \Lambda^k_{\mu\nu} M^{(a,b)*}_{kn} \bigg|_{U \to \infty} = -g_{\mu\nu} \frac{1}{m} \delta_{mn}. \quad (53)$$

These results can be obtained by explicit contour integration using the definition of $M$. We can see that the $U = 0$ case gives the boundary state of a background gauge field and when $U = \infty$ a localized object appears. In fact this parameter $U$ interpolates between Neumann and Dirichlet boundary conditions.

It is worthwhile to check the result obtained in (50) in the known case where only the field $F$ is present. Then the boundary conditions enforce that $p = 0$, and with the above simplification we find

$$Z_{\text{annulus}}(F) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} g^{\mu\nu} \delta_{rs} \left[ r \delta_{rj} \left( \frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F} \right)_{\mu\nu} \right] \frac{1}{j} g^{\alpha\beta} \right. $$

$$\left. j^s \delta_{js} \left( \frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F} \right)_{\delta\nu} \right) \left. Z^2_{\text{disk}}(F(0)) \right)$$

$$\exp \left( -\sum_{r=0}^{\infty} Tr \ln \left( g - \frac{1 + |z|^{2r}}{1 - |z|^{2r}} 4\pi \alpha' F + 4\pi^2 \alpha'^2 F^2 \right) \right.$$

$$\left. - \sum_{r=0}^{\infty} Tr \ln \left( g(1 - |z|^{2r}) - \frac{1}{2} Tr \ln \left( g + 4\pi \alpha' F + 4\pi^2 \alpha'^2 F^2 \right) \right) \right)$$

$$\prod_{r=1}^{\infty} \left( 1 - |z|^{2r} \right)^{-D} \prod_{r=1}^{\infty} \det \left( g - \frac{1 + |z|^{2r}}{1 - |z|^{2r}} 4\pi \alpha' F + 4\pi^2 \alpha'^2 F^2 \right)^{-1}. \quad (54)$$

This result agrees upon the inclusion of the ghost contribution with that obtained in [19]. Note that the partition function for the disk is cancelled by the term constant in $r$ which is then summed using $\zeta$ function regularization, exactly as the Born-Infeld action was obtained in the first place.

In a similar method we can obtain the partition function for the Mobius strip in this background as well. We use the cross cap state elaborated on in [30].

$$|C\rangle = \exp \left( -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \alpha_{-n} \bar{\alpha}_{-n} \right) |0\rangle. \quad (55)$$
Using this in analogy with the development of (50) we find that the

\[ Z_{\text{mobius}} = \langle B_{a,b} | \frac{1}{\Delta} | C \rangle \]

\[ = Z_{\text{disk}} Z_{\mathbb{R}P^2} \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} g^{\mu\nu} \delta_{rs} \left\{ \tau M_{nr}^{(a,b)} \Lambda_{\mu\nu}^{n} M_{pj}^{(a,b)*} \frac{1}{s} \delta_{js} z^s \right\}^k \right)^{rs} . \]

(56)

As in the case of the annulus, we find that the contributions \( Z_{\text{disk}} \) cancel explicitly when we go to the \( U = 0 \) limit, where conformal invariance is restored. In that limit we find

\[ Z_{\text{mobius}}(F) = \prod_{m=1}^{\infty} \det \left( g + F \frac{1 + (-1)^m |z|^{2m}}{1 - (-1)^m |z|^{2m}} \right)^{-1} . \]

(57)

Finally it is amusing to check and make sure that an analogous calculation will go through and reproduce the known partition function for the Klein bottle. Instead of the two copies of the boundary state two cross caps are inserted, and the resulting expression

\[ Z_{K^2} = Z_{\mathbb{R}P^2}^2 \exp \left( - \sum_{r=1}^{\infty} g^{\mu\nu} \ln \left( g_{\mu\nu} (1 - |z|^{2r}) \right) \right) \]

(58)

which can be seen to reduce to Dedekind \( \eta \)-functions, in agreement with the known result [29].

4.3 \( \chi = 3 \)

To extend it beyond \( \chi = 2 \) the boundary state formalism requires careful contemplation. We propose the following method which allows the construction of states of arbitrary Euler number, and for the nonorientable sector in principle a complete description of the dynamics. The procedure proposed is as follows; using the sewing construction for higher genus amplitudes (described in [29] among others) and armed with the result proved earlier in this paper that the emission of any particle from the bosonic boundary state corresponds with the expectation of a vertex operator inserted at a definite position on the disk, we propose to add any number of interactions with the brane described by the boundary state and any number of cross caps.

To recapitulate, the idea of the sewing construction is to create a higher genus amplitude by joining two lower genus amplitudes by inserting a vertex operator on each of the lower genus amplitudes and summing over the vertex operator. Explicitly
the construction is
\[
\langle A_1 : \ldots : A_n, \ldots : A_1 : \ldots : V \rangle_M = \int_\omega \sum_V \omega^h \bar{\omega}^h \langle A_1 : \ldots : V \rangle_{M_1} \langle V : \ldots : A_n \rangle_{M_2} \tag{59}
\]
with \(M = M_1 \# M_2\) and \(\ldots\) represents arbitrary vertex insertions. This construction is tantamount to adding a closed string propagator between the two manifolds with vertex operators on them. Since we have shown that the emission of one particle from the disk with \(F\) and \(U\) on its boundary matches the overlap obtained from the boundary state
\[
\langle V | \mathcal{B}_{a,b} \rangle = \langle : V : | \mathcal{T}_{0, U, F} \rangle \tag{60}
\]
we can then use this to obtain the contribution of a boundary with the fields \(U\) and \(F\) at it. This sort of construction was considered in [31].

The novel feature presented here is the generalization of the boundary state and cross-cap operators through the state operator correspondence. The fact that sphere amplitude for three string scattering is conformally invariant is used, in combination with the fact that both \(|\mathcal{C}\rangle\) and \(|\mathcal{B}_{a,b}\rangle\) both have a well defined overlap with any closed string state allows us to take the expression
\[
\frac{1}{\Delta} |\mathcal{B}_{a,b}\rangle = \int \frac{dz \bar{z}}{|z|^4} \exp \left( -z^k \alpha^m_{-m-k} \Lambda^{m*}_{\gamma \delta} \Lambda_{-m-l}^l \bar{\alpha}^l \bar{z}^l \right) |0\rangle \tag{61}
\]
and its equivalent using \(|\mathcal{C}\rangle\) to (suppressing prefactors)
\[
\exp \left( -z^k \frac{\partial^k}{(k-1)!} X^{\gamma \delta} \Lambda^m_{-m-k} \Lambda^{m*}_{l} \frac{\partial^l}{(l-1)!} X^{\delta \bar{z}^l} \right) \tag{62}
\]
by use of the operator state correspondence. These states are inserted within expectation values to give higher genus contributions.

There are several different states with \(\chi = 3\). The most obvious are the four possible insertions of boundary states and cross caps, and the addition of a handle to either a boundary state or cross cap (thereby increasing from \(\chi = 1\) to \(\chi = 3\) because increasing the genus by 1 increases the Euler number by 2. Note that the state with three cross caps and the state with a cross cap and a handle are topologically equivalent.

To obtain the amplitude for three boundaries we calculate
\[
Z'_{\text{pants}} = \langle B_{a,b} \frac{1}{\Delta} : B_{a',b'} : \frac{1}{\Delta} B_{a'',b''} \rangle
\]
where \(: B_{a',b'} :\) is as given in (62). Noting that the coefficient of \(\alpha_m\) in \(\frac{\partial^n}{(n-1)!} X\) is
\[
\frac{\partial^n}{(n-1)!} X = \sum_{a=-\infty}^{\infty} D_{na} \alpha_a, \tag{63}
\]
\[
D_{na} = (-1)^{n-1} \frac{(a+1) \ldots (n+a-1)}{(n-1)!}, \tag{64}
\]
\[21\]
we proceed to calculate

\[ Z'_{\text{pants}} = Z_{\text{disk}}^3 F_0(p) \exp \left( \sum_k \frac{1}{k} \delta_{na} \left( nC_{nm}(1)mC_{am}(3) \right)^k \right) \]

\[ \exp \left( \sum_k \frac{1}{k} \delta_{na} \left( nC_{nm}(1)mD_{n'-a}C_{n'm'}(2)\bar{D}_{m'-m} \right)^k \right) \]

\[ \exp \left( \sum_k \frac{1}{k} \delta_{na} \left( nC_{nm}(3)mD_{n'a}C_{n'm'}(2)\bar{D}_{m'm} \right)^k \right) \]

\[ \exp \left( \sum_k \frac{1}{k} \delta_{na} \left( nC_{nm}(1)mD_{n'-j}C_{n'm'}(2)\bar{D}_{m'-m}C_{jk}(3)kD_{n'-a}C_{n'm'}(2)\bar{D}_{m'm} \right)^k \right) \]

\[ (65) \]

Where as in \[ F_0(p) \] is a complicated function which is gaussian in the momentum of the boundary state, the integrals are implicit, and the expression \( C_{nm}(i) \) is an abbreviation

\[ C_{nm}(i) = z_i^n M_{kn}^{(a,b)} \Lambda_{\mu \nu} M_{kn}^{(a,b)} z_i^m \] \[ (66) \]

with \( i \) an index reminding which integration from the closed string propagator \( z_i \) came from.

From this we see immediately that the contributions for the genus expansion become increasingly complicated as \( \chi \) increases. In the particularly simple case of a vanishing tachyon, this can be evaluated and one obtains a product of exponentials of hypergeometric functions. In particular for the case of the constant \( F \) field we obtain

\[ Z'_{\text{pants}}(F) = Z_{\text{disk}}^3 \exp \left( -\sum_n Tr \ln \left( 1 - |z_1 z_2|^2 F \left( g - \frac{2\pi \alpha'}{g + 2\pi \alpha'} \right)^2 \right) \right) \]

\[ \exp \left( -\sum_{na} Tr \ln \left( 1 - n|z_1|^{2n}|z_2|^2 F(-n + 1, -a + 1; 2; |z_2|^2) \left( g - \frac{2\pi \alpha'}{g + 2\pi \alpha'} \right)^2 \right) \right) \]

\[ \exp \left( -\sum_{na} Tr \ln \left( 1 - n|z_3|^{2n}|z_2|^2 F(n + 1, a + 1; 2; |z_2|^2) \left( g - \frac{2\pi \alpha'}{g + 2\pi \alpha'} \right)^2 \right) \right) \]

\[ \exp \left( -\sum_{nma} Tr \ln \left( 1 - n|z_4|^{2n}|z_2|^2 F(-n + 1, m + 2; |z_2|^2) \left( g - \frac{2\pi \alpha'}{g + 2\pi \alpha'} \right)^2 \right) \right) \]

\[ m|z_3|^{2m}|z_2|^2 F(m + 1, -a + 1; 2; |z_2|^2) \left( g - \frac{2\pi \alpha'}{g + 2\pi \alpha'} \right)^2 \] \[ (67) \]

In the above \( F(a, b; c; x) \) is the hypergeometric function defined by its series expansion

\[ F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a + n - 1)! (b + n - 1)! (c - 1)!}{n! (a - 1)! (b - 1)! (c + n - 1)!} x^n, \] \[ (68) \]
and the logarithm is interpreted as its series expansion, and both lorentz and oscillator
indices are summed over. Note that this expression has many of the properties that we
expect for the partition function on a twice punctured disk. In particular this depends
on three parameters (the \(z_i\) terms arising from the integration over the propagators
to the various boundary states) which can be identified as the Teichmuller parameters
for this surface. In the limit of any of these parameters going to zero the dominant
contribution is from the annulus amplitude. The analogous amplitude with any number
of cross-caps gives a similar expression with the following modifications, for each cross-
cap the argument in the hypergeometric expression acquire s a negative sign, and the
corresponding matrix of lorentz indices undergoes the substitution
\[
\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F} \rightarrow g.
\]

The other two diagrams that must be calculated are the corrections to the disk
and to \(RP^2\) which come from the addition of a handle. This addition is achieved
by taking the trace, weighted by a factor exponentiated to the level number (coming
from the propagator within the handle), which is an identical operation to taking the
expectation value of this operator on the torus. For this calculation it is necessary to
take the trace of an operator that generically has the normal ordered form
\[
\exp(-\alpha_n \mathcal{M}_{nm} \tilde{\alpha}_m) :
\]
where the indices on \(\mathcal{M}\) can be either positive or negative, with \(\mathcal{M}\) defined by
\[
\mathcal{M}_{mn} = D_{n'm'} C_{n'm'} \tilde{D}_{m'n}.
\]

After a considerable amount of algebra we find by summing over all states in the Fock
space that
\[
\text{Tr} \left( \omega^h \tilde{\omega}^\hbar : \exp(-\alpha_n \mathcal{M}_{nm} \tilde{\alpha}_m) : \right) = \prod_{n=1}^{\infty} \frac{1}{|1 - \omega^h_n|^2} \prod_{n=1}^{\infty} \frac{1}{1 - \left( \frac{|a|\omega^h_a}{1 - \omega^h_a} \mathcal{M}_{ab} \frac{|b|\omega^\hbar_b}{1 - \omega^\hbar_b} \mathcal{M}_{c-b} \right)^n}.
\]

This expression uses the convention that the sums within the denominator run over
positive and negative indices. This suppresses the contribution from the momentum
of the loop which is given by a gaussian, explicitly
\[
F(p) = \exp \left\{ pp \left[ \frac{1}{\delta_{0j} - \delta_{0j}} \left( \frac{1}{\delta_{kj} - \delta_{kj}} \right) \left( \frac{1}{\delta_{kl} - \delta_{kl}} \right) \right] \right\}.
\]

The specialization to the case of only interactions with a background \(F\) field is given
by substituting \(|z|^2 F(a + 1, b + 1; 2; |z|^2)\) for \(\mathcal{M}_{ab}\).
It is interesting at this point to compare the results for this procedure with those obtained by the standard method of constructing the Greens function on an arbitrary surface \([32]\), and then integrating out the boundary interaction as described previously \((30)\). The Greens function of a unit disk with Neumann boundary conditions with a puncture of radius \(\epsilon\) at \(z = 0\) and a puncture of radius \(\delta\) at \(z = re^{i\psi}\) is given by

\[
G'(z, z') = G(z, z') + (\ln \epsilon)^{-1} G(z, 0)G(z', 0) + (\ln \delta)^{-1} G(z, re^{i\psi})G(z', re^{i\psi}) - \text{Re} \left( 4\epsilon^2 \left( z - re^{i\psi} + \frac{\bar{z}}{1 - \bar{z}re^{i\psi}} \right) \left( z' - re^{-i\psi} + \frac{\bar{z}'}{1 - \bar{z}'re^{-i\psi}} \right) \right) - \text{Re} \left( 4\epsilon^2 \left( z^{-1} + \bar{z} \right) \left( z'^{-1} + \bar{z}' \right) \right) + O(\epsilon^2) + O(\epsilon^2\delta^2) + O(\delta^2). \tag{72}
\]

In the above the explicit form of the greens function for the disk \([25]\) has been substituted into the last two lines. Integrating out the background field \(F\) can be done by recasting this as a one dimensional \(3 \times 3\) matrix model. When this is done the interaction with a field on the boundary can be integrated out, much as was done for the \(2 \times 2\) case in \([19]\), and the resulting expression contains the lowest order terms (in the teichmuller parameter) of the hypergeometric functions obtained previously. Similarly there is a procedure for obtaining the Greens function for the disk with a handle added between balls of radius \(\epsilon\) centered at \(z = 0\) and \(z = re^{i\psi}\). This gives

\[
G'(z, z') = G(z, z') + (\ln \epsilon)^{-1} \left( G(z, 0) - G(z, re^{i\psi}) \right) \left( G(z', 0) - G(z', re^{i\psi}) \right) - \text{Re} \left[ 4\epsilon^2 \left( z^{-1} + \bar{z} \right) \left( \frac{1}{z' - re^{i\psi}} + \frac{\bar{z}'}{1 - \bar{z}'re^{i\psi}} \right) \right] + \left( z'^{-1} + \bar{z}' \right) \left( \frac{1}{z - re^{i\psi}} + \frac{\bar{z}}{1 - \bar{z}re^{i\psi}} \right) + O(\epsilon^2). \tag{73}
\]

As in the case of the disk with holes removed, this greens function can be then used to integrate out the quadratic purturbation, obtaining results that are consistant with those presented in \((70)\).

### 4.4 \(\chi = 4\)

As for \(\chi = 3\) there are a number of different surfaces of this genus that can be obtained with the insertion of handles, cross-caps, and boundaries. The method presented above provides a concrete proposal for the construction of these higher genus amplitudes for all \(\chi \geq 3\). The construction is particularly appropriate for what can be interpreted as tree level scattering amplitudes for an arbitrary number of closed strings emitted from the brane described by the boundary state.
5 Discussion and Conclusion

In this paper we have further explored the boundary state formalism [31] and discussed its extension to the off-shell case including interaction with a tachyon field of quadratic profile. The boundary state has been shown to reproduce the $\sigma$ model calculations for emission of any number of closed string states, as detailed in the correspondence

$$\langle V_1 : V_2 : \ldots | B_{a,b} \rangle = \langle : V_1 : V_2 : \ldots \rangle T_{a,b,U,F}.$$  \hfill (74)

This can be restated as the fact that the boundary state algebraizes the bosonic string propagator. It has been shown that the inner product of two of the boundary states also reproduces the $\sigma$ model calculations for a worldsheet of the appropriate genus. We also present a generalization of this to higher genus, the results of which become progressively more complicated. In the case of vanishing tachyon field we obtain the following expansion in the open string coupling constant $g_o$

$$Z_F = \sum x g_o^x Z_x$$

$$= g_o^{-1} \sqrt{\det (g + 2\pi \alpha' F)}$$

$$+ \prod_r \left(1 - |z^2|^r\right)^{-D} \prod_r \det \left(g - \frac{1 + |z^2|^r}{1 - |z^2|^r} 4\pi \alpha' F + 4\pi^2 \alpha'^2 F^2\right)^{-1}$$

$$+ \prod_r \left(1 - (-1)^r |z^2|^r\right)^{-D} \prod_r \det \left(g - \frac{1 + (-1)^r |z^2|^r}{1 - (-1)^r |z^2|^r} 2\pi \alpha' F\right)^{-1}$$

$$+ g_o \int \exp \left(- \sum_n Tr \ln \left(1 - |z_1 z_3|^{2n} \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)\right)$$

$$+ \sum_{na} Tr \ln \left(1 - n |z_1|^{2n} |z_2|^2 F(-n + 1, -a + 1; |z_2|^2) \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)$$

$$+ \sum_{na} Tr \ln \left(1 - n |z_3|^{2n} |z_2|^2 F(n + 1, a + 1; |z_2|^2) \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)$$

$$+ \sum_{nma} Tr \ln \left(1 - n |z_1|^{2n} |z_2|^2 F(-n + 1, m + 1; |z_2|^2) \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)$$

$$+ g_o \int \exp \left(- \sum_n Tr \ln \left(1 - |z_1 z_3|^{2n} \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)\right)$$

$$+ \sum_{na} Tr \ln \left(1 + n |z_1|^{2n} |z_2|^2 F(-n + 1, -a + 1; -|z_2|^2) \left(\frac{g - 2\pi \alpha' F}{g + 2\pi \alpha' F}\right)^2\right)$$
This is a generalization of the Born Infeld action taking into account higher loop stringy corrections, specifically including contributions from euler number $\chi = 3$ and including the contributions from non-orientable surfaces such as the Mobius strip. The construction presented in this work can be generalized without much effort to higher genus. It quickly becomes apparent that the simplifications obtained by the method of
encoding the Greens function in the boundary state are overwhelmed by the increase in the parameters associated with the various boundary states.

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