Wasserstein Distance-based Spectral Clustering with Application to Transaction Data

Yingqiu Zhu
School of Statistics, Renmin University of China
Danyang Huang*
School of Statistics, Renmin University of China
Yingying Li
The Hong Kong University of Science and Technology
Bo Zhang*
School of Statistics, Renmin University of China

March 8, 2022

Abstract

With the rapid development of online payment platforms, it is now possible to record massive transaction data. The economic behaviors are embedded in the transaction data for merchants using these platforms. Therefore, clustering on transaction data significantly contributes to analyzing merchants’ behavior patterns. This may help the platforms provide differentiated services or implement risk management strategies. However, traditional methods exploit transactions by generating low-dimensional features, leading to inevitable information loss. To fully use the transaction data, we utilize the empirical cumulative distribution of transaction amount to characterize merchants. Then, we adopt Wasserstein distance to measure the dissimilarity between any two merchants and propose the Wasserstein-distance-based spectral clustering (WSC) approach. Considering the sample imbalance in real datasets, we incorporate indices based on local outlier factors to improve the clustering performance. Furthermore, to ensure feasibility if the proposed method on large-scale datasets with limited computational resources, we propose a subsampling version of WSC. The associated theoretical properties are investigated to verify the efficiency of the proposed approach. The simulations and empirical study demonstrate that the proposed method outperforms feature-based methods in finding behavior patterns of merchants.

Keywords: Spectral clustering, Wasserstein distance, Empirical cumulative distribution function, Transaction data

*Danyang Huang and Bo Zhang are co-corresponding authors. They have contributed equally to this paper. Addresses: dyhuang@ruc.edu.cn (Danyang Huang) mabzhang@ruc.edu.cn (Bo Zhang). This work was supported by the National Natural Science Foundation of China (No. 12071477, 71873137) and building world-class universities (disciplines) of the Renmin University of China. The authors report there are no competing interests to declare.
1 Introduction

The rapid development of third-party online payment platforms such as eBay and Alipay, have resulted in an exponential increase in online transaction data. Massive transaction data are valuable to the customer relationship management (CRM) of enterprises, which aims at refining information related to customers to anticipate and respond to customer needs (Peppard, 2000). In fields of banking services, retailing, and e-commerce, a number of successful applications are based on CRM using transaction data (Park and Kim, 2003; Chan, 2008; Khajvand and Tarokh, 2011; Wu and Chou, 2011; Chiang, 2012; Zhang et al., 2014; Alborzi and Khanbabaei, 2016). For third-party online payment platforms, each registered merchant is a customer. These platforms pay emphasis on the CRM for merchants since it plays a critical role in both marketing management and risk controlling (Eisenmann and Barley, 2006; Lowry et al., 2006; Huo et al., 2011). However, efficient approaches to exploit transaction data are needed to understand merchant transaction behavior better.

An important technique in CRM is customer clustering, also called customer segmentation. Customer segmentation aims at dividing customers into distinct and homogeneous subgroups according to appropriate criteria (Dannenberg and Zupancic, 2009; Tsiptsis and Chorianopoulos, 2011). Clustering helps platforms learn more about merchants’ behavior and develop differentiated management strategies. Figure 1 shows an example of transaction data of merchants. Merchants’ business behaviors are embedded in their transactions. However, for merchants with different numbers of transactions, comparisons using complete transactions become difficult. Traditional clustering methods, such as the recency, frequency, and monetary value (RFM) model (Bult and Wansbeek, 1995), extract features from transaction data. These features describe how recent the customer’s last purchase is, purchasing frequency, and amount spent. RFM features have been widely applied in
customer segmentation (Hsu et al., 2012; Khobzi et al., 2014) and analysis of customer behavior (Khajvand and Tarokh, 2011; Dhandayudam and Krishnamurthi, 2013; Van Vlasselaer et al., 2015). However, feature-based methods, such as RFM, essentially reduce the raw transaction data to a low-dimensional vector, leading to inevitable information loss.

For example, as shown in Figure [1] if we adopt the monetary feature to characterize merchants, merchant A (monetary=100) and merchant B (monetary=100) can be viewed as similar merchants. However, an investigation of their transaction amount distributions shows obvious differences in their behavior patterns.

![Figure 1: Example of transaction data of merchants.](image)

An alternative approach to avoid huge information loss is to investigate the distribution among transaction data. To characterize the behavior pattern of each merchant, the empirical cumulative distribution function (ECDF) of the transaction data is considered a useful statistical tool in retaining sufficient information than low-dimensional features (Sakurai et al., 2008). Various dissimilarity measures are defined based on distribution functions, including Kullback-Leibler divergence (Kullback and Leibler, 1951), Rényi divergence (Rényi et al., 1961), Jensen-Shannon divergence (Lin, 1991) and Wasserstein distance (Vallender, 1974). Recently, Zhu et al. (2021) have employed the Kolmogorov–Smirnov statistic to measure the dissimilarity between ECDFs and proposed a clustering method for transaction data. However, it measures only the supreme difference between two ECDFs, which is a point-to-point distance, and does not capture the topology difference between distributions.
Among these measures, Wasserstein distance has been proved as a versatile tool (Del Barrio et al., 1999a; Piccoli and Rossi, 2014; Fournier and Guillin, 2015). Wasserstein distance measures the minimal efforts to reconfigure the probability mass of one distribution to recover another (Panaretos and Zemel, 2019). Panaretos and Zemel (2019) have argued the advantages of Wasserstein distance. First, it could well capture the distributions’ topological structure characteristics. Second, it has robust performance in dealing with data distributed irregularly over time. Third, it has fewer restrictions on distributions. For example, Kullback-Leibler divergence requires two distributions to have similar supports. In contrast, Wasserstein distance is applicable for continuous and discrete distributions having non-overlapping supports. Due to the appealing properties and theoretical results, Wasserstein distance has been widely applied in statistical inference (Rüschendorf, 1985; Del Barrio et al., 1999b; Piccoli and Rossi, 2016; Panaretos and Zemel, 2019).

Based on distance measurement for distribution functions, various clustering algorithms, such as K-means (Lloyd, 1982) and spectral clustering method (Hagen and Kahng, 2002) have been proposed. Spectral clustering methods, frequently used in community detection (Nascimento and De Carvalho, 2011), cluster objects based on eigen-decomposition of Laplacian matrices. Kannan et al. (2004) have developed criteria to verify the quality of spectral clustering and demonstrated that the spectral methods perform robustly in most cases. Von Luxburg et al. (2008) have established the statistical consistency of spectral clustering algorithms. In addition, Song et al. (2008), Chen and Cai (2011), and Law et al. (2017) have generalized spectral clustering methods for deep learning and parallel computation. Thus, spectral clustering methods could be applied in large-scale datasets. Inspired by their work, we expect to investigate spectral clustering methods for robust solutions in merchant clustering. In addition, Shah and Zaman (2010) have pointed out
that the sample imbalance in different clusters may negatively influence the performance of spectral clustering methods. Aksoylar et al. (2017) have argued that imbalanced clustering is similar to anomaly detection. Thus, we also consider incorporating anomaly detection methods, e.g., local outlier factor (LOF, Breunig et al. 2000), to boost the performance of spectral clustering against imbalanced data.

However, the computational complexity of spectral clustering is relatively high. Since it involves the singular value decomposition (SVD), the computational complexity is $O(n^3)$, where $n$ is the number of objects to be clustered (Halko et al., 2011). A typical solution for large-scale datasets is subsampling. Subsampling approaches have been applied as a promising tool to make inferences from large-scale datasets when computational resources are limited (Dhillon et al., 2013; Wang et al., 2018; Ai et al., 2021). Subsampling draws samples to approximate the estimates obtained from the whole dataset (Politis et al., 1999). Unlike applications of subsampling in supervised learning, the subsampling-based merchant clustering considers the distances among merchants selected in the subsample, including the distances between the selected and unselected samples. Thus, we could adopt subsampling to approximate the whole sample results when computational resources are limited.

This study proposes a novel spectral clustering method, called Wasserstein-distance-based spectral clustering (WSC), to analyze merchant transaction data. To the best of our knowledge, this is the first study to incorporate distribution distance in spectral clustering and provide a theoretical discussion about the spectral clustering of ECDFs. First, the corresponding empirical distribution functions are investigated to reflect the similarity between two merchants. Second, we build a modified spectral clustering framework based on the Wasserstein distance function. We theoretically analyze the statistical properties of WSC, including the convergence and clustering error rate. Since spectral clustering is
implemented on datasets involving two levels, i.e., transactions and merchants, we discuss the assumptions from those two aspects. Additionally, to deal with the sample imbalance issue, we adopt the LOF index in WSC implementation to boost clustering performance. Furthermore, to make inferences on large-scale datasets, we provide a subsampling version of WSC, improving the algorithm’s feasibility when computational resources are limited. The rest of the paper is organized as follows. In Section 2, we introduce WSC and its subsampling version in detail. Theoretical properties of the proposed method are discussed in Section 3. Next, we demonstrate the performance of WSC through simulations in Section 4. In Section 5, a real data example from a third-party payment platform is analyzed to illustrate its application. Finally, we conclude our work in Section 6.

2 Spectral Clustering based on Wasserstein distance

2.1 Definition of Wasserstein distance for transaction data

Given any two \( d \)-dimensional probability measures \( P_i \) and \( P_j \), the Wasserstein distance is defined as \( W_p(P_i, P_j) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \xi(dx, dy), \xi \in \mathbb{H}(P_i, P_j) \right\}^{1/p} \), where \( \mathbb{H}(P_i, P_j) \) denotes the set of all probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( P_i \) and \( P_j \); \( p \geq 1 \) is a positive integer and \( \xi(\cdot) \) denotes a possible coupling distribution of \( p_i \) and \( p_j \). The distance is well-defined if a finite \( p \)-th moment exists for the probability measures. For \( d > 1 \), there is no closed-form expression for the distance \( [\text{Panaretos and Zemel} 2019] \), while for \( d = 1 \), there exists an explicit expression for the distance. For any probability measure \( P_i \), let \( F_i(x) \) denote the corresponding cumulative distribution function (CDF) and \( F_i^{-1}(t) = \inf\{x : F_i(x) \geq t\} \) denote the quantile function. Then, the Wasserstein distance
with \( d = 1 \) can be expressed as

\[
W_p(P_i, P_j) = \left( \int_0^1 |F_i^{-1}(t) - F_j^{-1}(t)|^p dt \right)^{\frac{1}{p}}.
\]  

(1)

We define Wasserstein distance with \( d = 1 \) based on transaction data as follows. For each merchant \( i \) (\( 1 \leq i \leq n \)), let \( F_i(x) \) denote the underlying CDF of transaction amount. Then, the distance between any two merchants \( i \) and \( j \) (\( 1 \leq i \neq j \leq n \)) can be defined according to (1). For simplicity, \( p \) is set to 1 in this study.

Because the CDF of individual merchants are unknown, we adopt the ECDF to approximate the real distribution. For each merchant \( i \) (\( 1 \leq i \leq n \)), we assume \( v_i \) transactions, where \( v_i \) is a positive integer. For the \( q_i \)-th \( (1 \leq q_i \leq v_i) \) transaction, the transaction amount is recorded as \( m_{i,q_i} \in \mathbb{R}^1 \). Thus, the transaction amount vector of merchant \( i \) is \( M_i = (m_{i,1}, m_{i,2}, ..., m_{i,v_i}) \in \mathbb{R}^{v_i} \). The Wasserstein distance function can be accordingly defined on empirical measures. [Delattre et al. (2004) and Fournier and Guillin (2015)] have shown that Wasserstein distance between the empirical and corresponding true measures converges to 0 with increasing observed samples. For merchant \( i \), the ECDF is defined as

\[
\hat{F}_i(x) = v_i^{-1} \sum_{q_i=1}^{v_i} I(m_{i,q_i} \leq x),
\]

where \( I(\cdot) \) is an indicator function. Then, the Wasserstein distance function for transaction data could be defined based on the ECDFs.

**Definition 1. (Transaction-based Wasserstein distance function)** Let \( W(i, j) = \int_{\mathbb{R}} \left| \hat{F}_i(x) - \hat{F}_j(x) \right| dx \) denote the transaction-based Wasserstein distance between merchants \( i \) and \( j \) \( (1 \leq i \neq j \leq n) \). We illustrate this definition given finite observations of transactions. Let \( M^*_{i,j} = M_i \cup M_j \) denote the union of their transactions and \( v^*_{i,j} \) denote the number of distinct values of \( M^*_{i,j} \). Then \( M^*_{i,j} \) can be expressed as a sorted sequence \( \{m^*_{i,j,k} : 1 \leq k \leq v^*_{i,j}\} \) satisfying \( m^*_{i,j,k} \leq m^*_{i,j,k+1} \) for all \( 1 \leq k \leq v^*_{i,j} - 1 \). Then the distance
$W(i, j)$ could be calculated as

$$W(i, j) = \sum_{k=1}^{v_{i,j}^* - 1} \left| \hat{F}_i(m_{i,j,k}^*) - \hat{F}_j(m_{i,j,k}^*) \right| (m_{i,j,k+1}^* - m_{i,j,k}^*).$$

### 2.2 Spectral clustering based on Wasserstein distance

Merchant clustering aims to divide merchants into $K$ non-overlapping clusters, denoted as $K$ sets of merchant indices, $C_1, C_2, \ldots, C_K$. The number of clusters $K$ is pre-defined according to some prior knowledge. A typical spectral clustering (Hagen and Kahng, 2002) involves the following three steps, construction of a similarity matrix, calculation of the normalized Laplacian matrix, and clustering based on the eigenvectors of the Laplacian matrix. Specifically, we describe spectral clustering based on Wasserstein distance in detail in this section. The framework of the proposed method is illustrated in Figure 2.

**Figure 2:** WSC framework. First, we exploit Wasserstein distance to measure the dissimilarity between any two ECDFs of merchants to obtain a similarity matrix for all merchants. Second, a Laplacian matrix is derived from the similarity matrix. Last, we conduct K-means clustering on the $n$ rows of the Laplacian matrix for partitioning the $n$ merchants.

**Step 1. Construct a similarity matrix based on Wasserstein distance.** For $n$ merchants, let $S_n \in \mathbb{R}^{n \times n}$ ($1 \leq i \neq j \leq n$) denote the similarity matrix based on Wasserstein distance. The function $S(\cdot, \cdot)$ measures the similarity between two distributions. Then each element $S_{n,ij}$ is defined as $S_{n,ij} = S(i, j) = \exp \{ -W(i, j)/\sigma \}$, where $W(i, j)$ is the Wasser-
stein distance between merchants $i$ and $j$, and $\sigma$ is the scaling parameter for normalizing
the Wasserstein distance function. The diagonal elements of $S_n$ are set to 0.

In this step, an optional $K$-nearest neighbors (KNN) construction step can be applied
to reconstruct the similarity matrix. This is a widely applied method to improve the
performance of spectral clustering (Aksoylar et al., 2017). Let $k_0$ denote the threshold for
the neighbors of each merchant. Then, the reconstructed similarity matrix $S_n = [S_{n,ij}] \in \mathbb{R}^{n \times n}$ could be defined as

$$S_{n,ij} = \begin{cases} S(i,j), & i \in \mathcal{N}(j,k_0) \text{ or } j \in \mathcal{N}(i,k_0) \\ 0, & \text{otherwise} \end{cases}$$

where $\mathcal{N}(j,k_0)$ is the set of $k_0$ nearest neighbors of merchant $i$ measured by the Wasserstein
distance function. The set $\mathcal{N}(j,k_0)$ satisfies $|\mathcal{N}(j,k_0)| = k_0$, where $| \cdot |$ denotes the size of
the set; $W(i,j) \leq W(i',j)$, for any $i \in \mathcal{N}(j,k_0)$, and $i' \notin \mathcal{N}(j,k_0)$.

**Step 2. Obtain the normalized Laplacian matrix.** Based on the similarity matrix $S_n$, a normalized Laplacian matrix $L_n = D_n^{-1/2} S_n D_n^{-1/2}$ can be determined, where $D_n$ is a
diagonal matrix with $D_{n,ii} = \sum_{j=1}^{n} S_{ij}$, $1 \leq i \leq n$. $S_n$ is symmetric and all elements of $S_n$ are non-negative. Moreover, the Laplacian matrix $L_n$ is semi-definite.

**Step 3. Conduct clustering on $K$ eigenvectors of $L_n$.** First, we obtain $K$ largest
eigenvalues of $L_n$ for clustering. Let $\hat{\mathbf{v}}_n^1, \ldots, \hat{\mathbf{v}}_n^K$ denote the corresponding eigenvectors. We
can stack the eigenvectors to form a matrix $\hat{\mathbf{V}}_n = [\hat{\mathbf{v}}_n^1, \ldots, \hat{\mathbf{v}}_n^K] \in \mathbb{R}^{n \times K}$. Then, traditional
clustering methods, e.g., K-means, clusters the $n$ rows of $\hat{\mathbf{V}}_n$ into $K$ clusters. Since each row
of $\hat{\mathbf{V}}_n$ is related to an individual merchant, the partition of $n$ rows of $\hat{\mathbf{V}}_n$ indicates partitioning
of all merchants. Thus $K$ non-overlapping merchant index sets could be obtained. The
clustering algorithm is summarized in Appendix E.1.
The computational complexity of calculating the similarity matrix is at least \( O(n^2) \) because it requires calculating Wasserstein distances between each pair of merchants. However, by adopting a cover tree structure \( [\text{Beygelzimer et al.} \ 2006] \), the calculation time could be reduced to \( O(n \log(n)) \). See Appendix A.1 for a more detailed illustration.

**Remark 1. (Number of clusters)** To select an appropriate value for the number of clusters in a data-driven manner, we recommend two alternative approaches. First is the silhouette coefficient approach \( [\text{Rousseeuw} \ 1987] \). The second involves visualizing the eigenvalues arranged in descending order. If there is a sharp decrease between the \( K' \)-th and successive eigenvalues, \( K' \) could be an appropriate selection \( [\text{Filippone et al.} \ 2008] \).

### 2.3 Discussion on imbalanced data

The performance of spectral clustering may be unsatisfactory if the dataset is imbalanced, i.e., the size of at least one group is much smaller than the others \( [\text{Xuan et al.} \ 2013] \). The LOF method can overcome this problem. The key idea is to measure the probability of an object being an outlier by investigating its local density. According to LOF, we define \( k \)-distance and reachability distance to describe the local density for each object.

**Definition 2. (k-distance)** Given a threshold \( k_0 \), for each object \( i \) (\( 1 \leq i \leq n \)), the \( k \)-distance \( d_{k_0}(i) \) is the distance between this object and its \( k_0 \)-th nearest neighbor. It satisfies (a) for at least \( k_0 \) objects \( i' \) (\( 1 \leq i' \neq i \leq n \)), \( W(i', i) \leq d_{k_0}(i) \); (b) for at most \( k_0 - 1 \) objects \( i'' \) (\( 1 \leq i'' \neq i \leq n \)), \( W(i'', i) < d_{k_0}(i) \).

**Definition 3. (Reachability distance)** For any two objects \( i \) and \( j \), given the \( k \)-distances, the reachability distance for \( i \) is \( d^*(i, j) = \max\{d_{k_0}(j), W(i, j)\} \).

The reachability distance, based on the \( k \)-distance, could be viewed as a robust measurement to describe the local density for each object. For each object \( i \), given \( k_0 \), the local...
reachability density is \( l(i) = \left\{ \sum_{j \in \mathcal{N}(i, k_0)} d^*(i, j) \right\}^{-1} k_0 \), where \( \mathcal{N}(i, k_0) \) is the set of the \( k_0 \) nearest neighbors of object \( i \). Then, we can define the LOF index based on such notation.

**Definition 4. (LOF index)** For each object \( i \), LOF index is defined as \( \text{LOF}(i) = \left\{ k_0 l(i) \right\}^{-1} \sum_{j \in \mathcal{N}(i, k_0)} l(j) \).

Based on the above definition, one could verify that the range for LOF is \( [\epsilon^{-1}, \epsilon] \), where \( \epsilon = \max_{i \neq j} d^*(i, j) / \min_{i \neq j} d^*(i, j) \). LOF for a normal merchant is close to 1 while that for anomalous merchant can be very large. LOF has been shown to be robust for the identification of outliers (Lazarevic and Kumar, 2005; Li et al., 2015).

Next, we define the reconstructed similarity matrix \( S_n \) according to LOF as follows:

\[
S_{n,ij} = \begin{cases} 
S(i,j), & i \in \mathcal{N}(j, [\text{LOF}(j)]^{-1} k_0]) \text{ or } j \in \mathcal{N}(i, [\text{LOF}(i)]^{-1} k_0]) \\
0, & \text{otherwise}
\end{cases}
\]

where \([\cdot]\) represents the integer function. For “anomalous merchants”, merchants whose behavior patterns significantly differ from normal behavior have fewer investigated neighbors. Since there will be reduced connections between anomalous and normal ones, we can better distinguish anomalous from normal ones using the reconstructed similarity matrix.

### 2.4 Subsampling Wasserstein-distance-based spectral clustering

Because the spectral clustering requires Laplacian matrix decomposition, the computational complexity is \( O(n^3) \). Accordingly, it may be infeasible to implement WSC when the number of merchants \( n \) is extremely large. Therefore, for large datasets, we propose an approximate method based on subsampling, called *subsampling WSC* (SubWSC), as shown in Figure 3. This gives us the opportunity to apply WSC algorithm with limited computational resources in large datasets.
Figure 3: Comparison of SubWSC (lower panel) and WSC (upper panel) frameworks. SubWSC consists of the following steps. (1) Generate a sample by randomly selecting merchants. (2) Extract sampled columns with similarities between selected merchants (dark elements) and between selected and unselected merchants (grey elements). The solid lines in the network on the left side refer to the similarities involved in the sample. (3) Calculate Laplacian matrix $L_n$ using the sample. (4) Conduct SVD on $L_n^\top L_n$ and form a $n \times K$ matrix using its eigenvectors. (5) Conduct K-means clustering on the $n$ rows of the matrix to find clusters of all merchants.

In Figure 3, the key issue is to make inference for assigning all merchants based on those selected in the subsample. Specifically, let $n_s (n_s \geq K)$ denote the sample size. We can define a sampling matrix $C \in \mathbb{R}^{n \times n_s}$ to describe merchants selected into the sample. Each column of $C$ can be viewed as a one-hot code of a selected merchant. If merchant $i$ is selected as the $j$-th one of the sample, we have $C_{ij} = 1$ and $C_{i'j} = 0$ for $i' \neq i$. For simplicity, let $\phi(j)$ denote the corresponding order of the $j$-th sample, e.g., $\phi(j) = i$.

For convenience, we use mathcal letters to denote the notation used in the sample. For example, $L_n$ represents the normalized Laplacian matrix of the sample, calculated as $D_n^{-1/2} S_n C D_n^{-1/2} \in \mathbb{R}^{n \times n_s}$, where $D_{n_s}$ is a $n_s \times n_s$ diagonal matrix. For the $j$-th sample, where $\phi(j) = i$, we have $D_{n_s,jj} = D_{n,ii}$. Suppose that matrices $\hat{U}_n \in \mathbb{R}^{n \times n}$ and $\hat{V}_{n_s} \in \mathbb{R}^{n_s \times n_s}$ are eigenvectors of $L_n L_n^\top$ and $L_n^\top L_n$, respectively. We could find the $K$ largest eigenvalues and the corresponding eigenvectors, thus generating $\hat{U}_K \in \mathbb{R}^{n \times K}$ and $\hat{V}_{n_s} \in \mathbb{R}^{n_s \times K}$ from $\hat{U}_n$ and $\hat{V}_{n_s}$, respectively. Specifically, we focus on $\hat{U}_K = L_n \hat{V}_K \Sigma_K^{-1/2}$, where $\Sigma_K$ is a diagonal matrix consisting of $K$ largest eigenvalues of $L_n^\top L_n$. Each row of $\hat{U}_K$ refers to an individual merchant. Thus, the partition of $n$ rows of $\hat{U}_K$ can directly lead to the partitioning of all
Proposition 1. Given $S_n$, the computational complexity of SubWSC is $O(nn_s^2)$. 

The proof of Proposition 1 is given in Appendix A.2. Proposition 1 states that given a similarity matrix, SubWSC has a lower time cost. Thus, it could be applied to large datasets even with limited computational resources. For similarity matrix calculated based on the cover tree data structure, the overall computational complexity of clustering is $O(n \log n + nn_s^2)$. In real applications, the selection of sample size is a critical issue. The selection of SubWSC sample size is discussed in the next section.

3 Theoretical analysis

This section discusses the theoretical properties of the WSC method. Von Luxburg et al. (2008) have proven the consistency of general spectral clustering algorithms with a focus on independent data points. However, in this study, clustering is conducted for a multilevel data structure, where transactions belong to different merchants. Therefore, we adopt ECDF to extract information from the transaction level. For the theoretical investigation of WSC, we consider both the transaction and merchant level data. In this section, first, we introduce some basic notation and assumptions. Second, we prove the convergence of the WSC method for ECDFs of merchants. Third, we show that the clustering error rate of the proposed method converges to 0 as the number of merchants $n$ reaches infinity. Finally, we investigate the theoretical properties of SubWSC.
3.1 Notation and Basic Assumptions

For simplicity in technical proofs, all observed transactions are standardized to the range $[0, 1]$. This leads to $m_{i,v_i} \in [0, 1]$ for $1 \leq v_i \leq V_i$, $1 \leq i \leq n$. Thus, all Wasserstein distances can be standardized as $W(i,j) \in [0, 1]$. Furthermore, $\sigma$ is assumed to be 1 in theoretical discussion. Based on the similarity function, we define $d^*_{\min} = \min_{1 \leq i \leq n} \sum_{j \neq i} E[S(i,j)]$. Then, the following assumptions are introduced.

**Assumption 1 (Latent distribution):** There exist $K$ underlying distributions $\{F^*_1, \cdots, F^*_K\}$, such that, for each merchant $i$, $1 \leq i \leq n$, the CDF $F_i$ is identical to one of the $K$ distributions. Let $\gamma_i$ denote the relationship between merchant $i$ and the distributions. If the CDF of merchant $i$ is identical to $F^*_k$, then $\gamma_i = k$.

**Assumption 2 (Number of transactions):** Assume that $v_{\min} = \Omega(\log n)$, where $v_{\min} = \min\{v_i : i = 1, \cdots, n\}$. The notation $g(n) = \Omega(f(n))$ indicates the existence of positive constants $c_0$ and $n_0$ such that $g(n) \geq c_0 f(n)$ for all $n \geq n_0$ (Knuth 1976).

**Assumption 3 (Similarity matrix):** $d^*_{\min} = \Omega\left(n^{1/2} \log n\right)$.

According to Assumption 1, each merchant’s CDF can match exactly one of the $K$ underlying distributions, suggesting a particular behavior pattern. Thus, $n$ merchants can be partitioned according to their behavior patterns. Assumption 2 shows that the order of $v_{\min}$ should be no smaller than $\log n$. For example, if a dataset contains $n = 10,000$ merchants, Assumption 2 requires the minimal number of transactions per merchant to be no smaller than the order of $\log(n) = 9.21$ and is easily satisfied in real applications. Assumption 3 requires that the similarity matrix not be too sparse, which is reasonable and easily satisfied if there are no isolated clusters or outliers. In common finite mixture models (Shedden 2015), it is assumed that all observations follow $F^* = \sum_{k=1}^{K} w_k F^*_k$, which is a mixture of $K$ latent distributions, where $w_k$ is the weight for $F^*_k$ and $\sum_k w_k = 1$. Thus, it
can be verified that $d^*_\text{min} = \Omega(n)$ in finite mixture models. In this study, we assume that $d^*_\text{min} = \Omega\left(n^{1/2} \log n\right)$, which provides a more relaxed condition than $d^*_\text{min} = \Omega(n)$. Based on these assumptions, the theoretical analysis of the proposed method are investigated in the following subsection.

### 3.2 Theoretical analysis of WSC

As shown in Section 2.2, the spectral clustering depends on the eigenvectors of the Laplacian matrix. Thus, we first prove the convergence of the eigenvectors of the Laplacian matrix for WSC. Then, we discuss the clustering error rate of WSC to show the robustness of the proposed method.

First, we define some notations for establishing the WSC theory. We define an underlying matrix $L^*_n$ for $n$ merchants based on the similarity expectations within $S_n$. The detailed definition of $L^*_n$ can be found in Appendix A.3. With respect to $L^*_n$, we prove that its eigenvectors can directly indicate the correct assignments of merchants. Based on the underlying cluster labels $\{\gamma_i : 1 \leq i \leq n\}$, we define a membership matrix $Z_n = [Z_{n,ik}] \in \mathbb{R}^{n \times K}$, where $Z_{n,ik} = 1$ if $\gamma_i = k$, and 0 otherwise. Let $Z_{n,i \cdot}$ denote the $i$-th row of $Z_n$, $\lambda_1 \geq \cdots \geq \lambda_K$ denote the $K$ eigenvalues of $L^*_n$, and matrix $V_n = [V^1_n, \cdots, V^K_n] \in \mathbb{R}^{n \times K}$ denote the $K$ eigenvectors corresponding to $\lambda_1, \cdots, \lambda_K$. The $i$-th row of $V_n$ is denoted as $V_{n,i \cdot}$. Proposition 2 shows the linear relationship between $Z_n$ and $V_n$.

**Proposition 2. (Structure of eigenvectors)** There exists a matrix $\mu_0 \in \mathbb{R}^{K \times K}$ and diagonal matrix $\mu_1 \in \mathbb{R}^{n \times n}$ such that $V_n = \mu_1 Z_n \mu_0$. For any two merchants $i$ and $j$, $1 \leq i \neq j \leq n$, $Z_{n,i \cdot} = Z_{n,j \cdot}$ if and only if $V_{n,i \cdot} = V_{n,j \cdot}$.

The proof of Proposition 2 is provided in Appendix B.1. According to Proposition 2, if $V_{n,i \cdot} = V_{n,j \cdot}$, the corresponding merchants $i$ and $j$ belong to the same cluster. Further-
more, $V_n$ consists of only $K$ distinct rows, each of which is related to a single underlying distribution. K-means clustering on $V_n$ determines $K$ cluster centers corresponding to the $K$ underlying distributions to partition all merchants accurately. Then, we prove that $\hat{V}_n^1, \ldots, \hat{V}_n^K$, the eigenvectors derived from ECDFs of merchants, converge to those of the underlying Laplacian matrix $L_n^*$.

**Theorem 1. (Convergence of eigenvectors)** Based on Assumptions 1–3, there exist an orthogonal matrix $\Sigma$ and a constant $n_0$ such that for $n \geq n_0$,

$$||\hat{V}_n \Sigma - V_n||_F \leq \frac{c_0 \sqrt{K}}{\lambda_K \sqrt{\log n}}$$

holds with a probability of at least $\{1 - (2\sqrt{n})^{-1}\}^2$.

The proof of Theorem 1 is given in Appendix B.2. Theorem 1 provides an upper bound for the estimation error of the eigenvectors. From (2), the following conclusions are derived.

First, given $K$ and $\lambda_K$, the discrepancy between $\hat{V}_n \Sigma$ and $V_n$ measured by Frobenius norm is $O_p\left(\log^{-1/2} n\right)$. Thus, Theorem 1 indicates that the eigenvectors $\hat{V}_n$ converges to $V_n$ in probability as $n \to \infty$. If $V_n$ indicates the correct partition, the convergence of $\hat{V}_n$ suggests that the resulting cluster converges to the correct partitioning of all merchants. Second, the estimation error may be reduced if $\lambda_K$ is relatively large, which also suggests that $K$ is an appropriate number of clusters.

Based on the theoretical results mentioned above, we further discuss the clustering error in WSC. According to Proposition 2, the K-means clustering on $V_n$ leads to $K$ distinct cluster centers. For each merchant $i$, $1 \leq i \leq n$, the corresponding correct cluster center can be described using the vector $V_{n,i}$. For the K-means clustering on $\hat{V}_n$, let $c_i$ denote the center of the cluster assigned to merchant $i$. Intuitively, merchant $i$ is correctly clustered...
if $\|c_i - V_{n,i}\| < \|c_i - V_{n,j}\|$ holds for all $j$ with $\gamma_j \neq \gamma_i$. For the convenience of describing misclustering event, we provide the following sufficient condition for the correct assignment.

**Proposition 3. (Condition of correct assignment)** There exists a constant $c_0 > 0$ such that for each merchant $i$, $1 \leq i \leq n$, the clustering assignment is correct if $\|c_i - V_{n,i}\| < (2n)^{-1/2}\exp\left(-c_0/\sqrt{\log n}\right)$.

The proof of Proposition 3 is given in Appendix B.3. Based on Proposition 3, the clustering error rate is defined as $P_e = \frac{\#\{i : \|c_i - V_{n,i}\| \geq (2n)^{-1/2}\exp\left(-c_0/\sqrt{\log n}\right)\}}{n}$, where $\#\{\cdot\}$ denotes the number of elements within a set.

**Theorem 2. (Clustering error rate)** Under Assumptions 1–3, there exist constants $c_0$, $c_0'$, and $n_0$ such that for $n \geq n_0$,

$$P_e \leq \frac{c_0'K\exp\sqrt{\log n}}{\lambda_K^2 \log n}$$

holds with a probability of at least $\{1 - (2\sqrt{n})^{-1}\}^2$.

Thus, given $\lambda_K$ bounded above 0, the clustering error rate converges to 0 in probability as $n \to \infty$. This guarantees the theoretical property of the proposed method. The proof of Theorem 2 is given in Appendix B.4.

### 3.3 Theoretical analysis of SubWSC

This part discusses the convergence and clustering error rate of SubWSC in detail. We define an underlying matrix $L_n^*$ given a subsampling matrix $C$. The detailed definition of $L_n^*$ can be found in Appendix A.4. Corresponding to $\hat{U}_K$ and $\hat{V}_K$, matrices $U_K$ and $V_K$ can be defined based on $L_n^*$. A smaller sample size $n_s$ may lead to lower computational complexity, but also reduces the probability to cover merchants with $K$ different underlying
distributions. Thus, the selection of $n_s$ needs to be discussed. Moreover, to implement SubWSC, we have to find the $K$ non-zero eigenvalues of $(\mathcal{L}_n^*)^\top \mathcal{L}_n^*$ and the corresponding eigenvectors. The existence of the $K$ non-zero eigenvalues of $(\mathcal{L}_n^*)^\top \mathcal{L}_n^*$ requires that for each of the underlying distributions $F_1^*, \cdots, F_K^*$, there must be at least one relevant merchant selected in the subsample. For each $F_k^*$, $1 \leq k \leq K$, let $n_{s,k}$ denote the number of merchants relevant to this distribution within the sample. Then, an acceptable subsampling solution can be described by an event $e^* = \{n_{s,k} : n_{s,k} \geq 1, 1 \leq k \leq K, \sum_{k=1}^{K} n_{s,k} = n_s\}$.

With respect to $e^*$, we provide a discussion of $n_s$ in the following proposition. Let $\alpha = n^{-1} \min_{1 \leq k \leq K} \sum_{i=1}^{n} I(\gamma_i = k)$ describes the degree of imbalance among the whole dataset.

**Proposition 4.** For $\epsilon > 0$, the event $e^*$ happens with a probability of at least $1 - \epsilon$ if the sample size $n_s$ satisfies $n_s \geq \{\log(1 - \alpha)\}^{-1} \log(\epsilon/K)$.

The proof of Proposition 4 is given in Appendix C.1. Given a small value of $\epsilon$, a subsample with size $n_s$ satisfying Proposition 4 can cover $K$ different distributions with a high probability. Thus, the following conclusions are drawn. First, the required sample size is related to the number of clusters $K$. If merchants show different behavior patterns, large $n_s$ is recommended. Second, if the dataset is imbalanced, where $\alpha$ may be small, a relatively large $n_s$ is preferred. Last, for fixed $\alpha$ and $K$, we derive $n_s = \Omega(\log n)$. Thus, the minimal requirement of $n_s$ is of the order of $\log n$, which is feasible for large-scale datasets.

Based on Proposition 4, we analyze the theoretical properties of SubWSC. Similar to Section 3.2, we prove the convergence of $\hat{U}_K$ first. The detailed discussion is given in Appendix C.2. With respect to the subsampling version of the proposed method, we define the misclustering event and clustering error rate $P_e$ in Appendix C.3. The upper bound of $P_e$ is proved in the following theorem.

**Theorem 3.** (Clustering error rate of SubWSC) Assume that the sample size sat-
isfies $n_s \geq \{\log(1 - \alpha)\}^{-1} \log\{1/(\sqrt{nK})\}$. Let $\omega_1^2 \geq \cdots \geq \omega_K^2$ denote the $K$ eigenvalues of $\mathcal{L}_n^*(\mathcal{L}_n^*)^\top$. Then, under Assumptions 1–3, there exist constants $c_0$ and $n_0$ such that for $n \geq n_0$,

$$
P_e \leq \frac{c_0 K n_{\text{max}}}{\omega_K^2 n \log n}
$$

holds with a probability of at least $(1 - n^{-1/2})^3$.

The proof of this theorem is given in Appendix C.4. It suggests that the clustering performance of SubWSC is better when $\omega_K$ is larger. Given the eigenvalue $\omega_K$ is bounded away from 0, we derive $P_e = O_p(\log^{-1} n)$.

## 4 Numerical study

### 4.1 Simulated transaction data

In this section, we investigate the performance of the proposed method via simulation studies. To generate simulated transactions, we implement the following steps in each replication of the simulation.

**Step 1.** For cluster $k$, $1 \leq k \leq K$, the number of within-cluster merchants is set as $n_k$. The total number of merchants is $n = \sum_{k=1}^{K} n_k$.

**Step 2.** For each cluster $k$, $1 \leq k \leq K$, we specify a distribution $F_k^*$ as the ground truth. For any $k \neq k'$, $F_k^*$ is different from $F_{k'}^*$.

**Step 3.** For each merchant $i$, $1 \leq i \leq n$, define $v_i = \max\{v_{0i}, \log(n)\}$ as the number of transactions, where $v_{0i}$ is generated using a Poison distribution $\text{Poisson}(\beta)$.

**Step 4.** For each simulated merchant $i$, $1 \leq i \leq n$, let $\gamma_i$ denote the cluster it belongs to. Given $\gamma_i$, we generate $v_i$ samples $(m_{i,1}, \cdots, m_{i,v_i})$ from $F_{\gamma_i}^*$. Then, we use the absolute values $\{|m_{i,q}|\}_{q=1}^{v_i}$ as simulated transactions for merchant $i$. 


We consider two examples by specifying the distributions \( \{F_1^*, \ldots, F_K^*\} \). Figure 4 shows the empirical distributions of the generated clusters with continuous and discrete distributions.

**Example 1 (Continuous distributions):** We set \( K = 3 \). The ground truth \( F_1^*, F_2^* \) and \( F_3^* \) are \( N(2, 2^2) \), \( \text{Exp}(\frac{1}{2}) \) and \( \text{Gamma}(2, 1) \), respectively.

**Example 2 (Discrete distributions):** We again set \( K = 3 \). First, we set three basic distributions: \( F_1^* \) is set as \( N(4, 2^2) \); \( F_2^* \) is a mixture of \( \text{Exp}(\frac{1}{2}) \) and \( U[10, 12] \), with weights of 0.8 and 0.2, respectively; \( F_3^* \) is a mixture of \( \text{Exp}(\frac{1}{2}) \) and \( U[4, 6] \), with weights 0.3 and 0.7, respectively. Then for each merchant \( i, 1 \leq i \leq n \), the transaction amount \( m_{i,q} \) is replaced by the rounded value \( [m_{i,q}] \), \( 1 \leq q \leq v_i \), where \([·]\) denotes the integer function.

For each example, we compare the different methods under three settings: (a) \( n_1 = 30, n_2 = 50, n_3 = 75 \); (b) \( n_1 = 60, n_2 = 100, n_3 = 150 \); (c) \( n_1 = 120, n_2 = 200, n_3 = 300 \). The parameter related to the transaction amount \( \beta = 20, 50, 100 \). In the following parts, we denote the WSC method using the LOF index as “WSC+LOF”. We also consider an imbalanced case with \( n_1 = 20, n_2 = 100, n_3 = 500 \). In addition, to demonstrate the performance of SubWSC, we set the sample size \( n_s \) to range from 20 to 200 and are simulated under Setting (c) with \( \beta = 100 \) (Example 1) and \( \beta = 50 \) (Example 2).

Figure 4: Distributions of the generated clusters’ transaction amounts in Example 1 (upper row) and Example 2 (bottom row).
4.2 Comparison and evaluation

To demonstrate the clustering performance of the proposed method, we compare WSC with the following approaches:

**Standard K-means method** ([Lloyd, 1982](#)) with transaction-based features: The input features are *average amount of transactions* and *standard deviation of transactions*.

**Hierarchical clustering** ([HC, Defays 1977](#)) method based on the Wasserstein distance: It uses a $n \times n$ Wasserstein distance matrix with the complete-linkage agglomerative algorithm.

**Kolmogorov–Smirnov K-means Clustering** ([KSKC, Zhu et al. 2021](#)): This is a recently proposed method for clustering transaction data. A distance function based on the Kolmogorov–Smirnov statistic measures the dissimilarity between merchants represented using ECDF. An iterative heuristic algorithm then divides all merchants into $K$ clusters.

To quantitatively evaluate the clustering performance, we adopt the following indices: rank index (RI), cluster accuracy (CA), and normalized mutual information (NMI) ([Fahad et al. 2014](#)). $M = 100$ random replications are performed for effective evaluation. For the $m$th replication, $\text{RI}^{(m)}$ ($1 \leq m \leq M$) is recorded and $\overline{\text{RI}} = M^{-1} \sum_m \text{RI}^{(m)}$ can be determined. We calculate $\overline{\text{CA}}$ and $\overline{\text{NMI}}$ similarly. All simulations are conducted in Python using a MacBook Pro computer with a 3.1 GHz Intel Core i7 processor.

4.3 Simulation results

Numerical analysis was performed to verify the effectiveness of the proposed method using the simulated transaction data. Tables [1](#) and [2](#) show the clustering performances of different methods in Examples 1 and 2, respectively.

**Comparing clustering performances.** Based on Tables [1](#) and [2](#) we draw the following conclusions. First, WSC and WSC+LOF outperform the other methods in both the con-
Table 1: Performances of the four clustering methods for Example 1.

| Setting (a) | K-means | HC     | KSKC     | WSC     | WSC+LOF |
|-------------|---------|--------|----------|---------|---------|
| β=20        | .554 (.013) | .570 (.033) | .613 (.026) | .614 (.027) | .620 (.030) |
| RI          | .577 (.015) | .718 (.068) | .731 (.041) | .781 (.051) | .787 (.049) |
| β=100       | .606 (.017) | .879 (.051) | .860 (.038) | .917 (.043) | .928 (.041) |
| β=20        | .463 (.040) | .526 (.087) | .590 (.051) | .587 (.074) | .607 (.083) |
| CA          | .504 (.044) | .725 (.109) | .759 (.049) | .813 (.073) | .819 (.079) |
| β=100       | .544 (.043) | .902 (.055) | .887 (.035) | .942 (.038) | .948 (.036) |
| β=20        | .058 (.025) | .159 (.053) | .170 (.041) | .187 (.046) | .194 (.049) |
| NMI         | .128 (.033) | .397 (.100) | .397 (.072) | .491 (.080) | .498 (.084) |
| β=100       | .201 (.037) | .733 (.085) | .686 (.070) | .788 (.080) | .801 (.079) |

| Setting (b) | K-means | HC     | KSKC     | WSC     | WSC+LOF |
|-------------|---------|--------|----------|---------|---------|
| β=20        | .557 (.009) | .585 (.027) | .617 (.030) | .617 (.029) | .621 (.033) |
| RI          | .580 (.011) | .726 (.061) | .731 (.031) | .797 (.046) | .805 (.046) |
| β=100       | .607 (.012) | .883 (.047) | .887 (.031) | .931 (.022) | .936 (.021) |
| β=20        | .464 (.036) | .534 (.067) | .600 (.031) | .587 (.071) | .603 (.067) |
| CA          | .511 (.036) | .735 (.101) | .753 (.031) | .830 (.053) | .836 (.055) |
| β=100       | .542 (.033) | .911 (.047) | .903 (.025) | .948 (.018) | .952 (.018) |
| β=20        | .061 (.015) | .161 (.041) | .171 (.034) | .192 (.042) | .201 (.043) |
| NMI         | .126 (.023) | .410 (.083) | .376 (.048) | .514 (.068) | .518 (.073) |
| β=100       | .203 (.028) | .749 (.064) | .718 (.053) | .803 (.059) | .811 (.051) |

| Setting (c) | K-means | HC     | KSKC     | WSC     | WSC+LOF |
|-------------|---------|--------|----------|---------|---------|
| β=20        | .558 (.007) | .587 (.021) | .623 (.013) | .621 (.024) | .626 (.024) |
| RI          | .578 (.007) | .726 (.062) | .731 (.018) | .805 (.032) | .812 (.030) |
| β=100       | .607 (.009) | .892 (.042) | .891 (.017) | .933 (.016) | .947 (.051) |
| β=20        | .470 (.027) | .535 (.052) | .613 (.022) | .605 (.055) | .618 (.050) |
| CA          | .513 (.025) | .737 (.081) | .761 (.021) | .839 (.036) | .841 (.037) |
| β=100       | .546 (.026) | .917 (.032) | .912 (.011) | .950 (.013) | .957 (.014) |
| β=20        | .060 (.016) | .174 (.035) | .186 (.024) | .180 (.039) | .188 (.039) |
| NMI         | .127 (.015) | .421 (.070) | .408 (.033) | .516 (.046) | .522 (.048) |
| β=100       | .201 (.020) | .779 (.032) | .782 (.031) | .810 (.035) | .824 (.034) |
Table 2: Performances of the four clustering methods for Example 2.

| Setting (a) | K-means | HC | KSKC | WSC | WSC+LOF |
|------------|---------|----|------|-----|---------|
| β=20       | .544 (.013) | .730 (.017) | .741 (.036) | **.750 (.022)** | .761 (.023) |
| RI         | .546 (.014) | .783 (.029) | .817 (.034) | **.899 (.041)** | .929 (.044) |
| β=50       | .424 (.044) | .686 (.043) | .711 (.076) | **.747 (.064)** | .761 (.070) |
| CA         | .422 (.040) | .706 (.083) | .804 (.073) | **.865 (.062)** | .915 (.063) |
| β=20       | .062 (.029) | .507 (.048) | .522 (.073) | **.613 (.053)** | .635 (.047) |
| NMI        | .065 (.030) | .693 (.030) | .736 (.050) | **.848 (.073)** | .890 (.079) |

| Setting (b) | K-means | HC | KSKC | WSC | WSC+LOF |
|------------|---------|----|------|-----|---------|
| β=20       | .541 (.010) | .742 (.015) | .748 (.033) | **.762 (.035)** | .767 (.018) |
| RI         | .547 (.009) | .787 (.026) | .828 (.027) | **.928 (.041)** | .955 (.037) |
| β=50       | .425 (.034) | .696 (.041) | .713 (.070) | **.757 (.053)** | .779 (.051) |
| CA         | .427 (.028) | .713 (.080) | .830 (.046) | **.918 (.056)** | .956 (.041) |
| β=20       | .068 (.020) | .527 (.045) | .532 (.072) | **.623 (.075)** | .637 (.040) |
| NMI        | .070 (.021) | .703 (.025) | .745 (.041) | **.889 (.061)** | .927 (.056) |

| Setting (c) | K-means | HC | KSKC | WSC | WSC+LOF |
|------------|---------|----|------|-----|---------|
| β=20       | .544 (.008) | .764 (.010) | .759 (.029) | **.768 (.033)** | .771 (.011) |
| RI         | .547 (.007) | .786 (.024) | .829 (.021) | **.952 (.027)** | .967 (.021) |
| β=50       | .424 (.022) | .701 (.035) | .728 (.067) | **.755 (.041)** | .790 (.045) |
| CA         | .426 (.019) | .710 (.063) | .832 (.036) | **.946 (.027)** | .974 (.020) |
| β=20       | .067 (.013) | .655 (.025) | .656 (.062) | **.667 (.067)** | .675 (.024) |
| NMI        | .071 (.010) | .696 (.024) | .726 (.032) | **.921 (.049)** | .944 (.030) |

Continuous and discrete examples. Second, the increase in β leads to promising improvements in proposed methods. WSC can archive better clustering performances by collecting more transaction data per merchant. Furthermore, the clustering results in Figures 5 and 6 show that K-means fail in the two examples, while WCS+LOF provides the most similar results to the ground truth.

Performance on imbalanced datasets. The comparison results of the imbalanced setting are shown in Table 3. The performances of both WSC and WSC+LOF are competitive compared with other methods. However, WSC+LOF achieves relatively higher scores in terms of RI, CA and NMI than WSC. Thus, incorporating the LOF index can further improve the clustering performance of WSC in imbalanced datasets.

Performance of SubWSC. Figures 7 and 8 illustrate the clustering performances and time costs of SubWSC. With increasing sample size, the performance curves of SubWSC
gradually approach those of WSC+LOF. SubWSC also achieves comparable performances with small samples. In Example 1, when the sample size is 200 (32.3 % of the whole dataset), $\text{RI}$, $\text{CA}$, $\text{NMI}$ of SubWSC reach 90.2%, 93.0% and 80.7% of those without subsampling, respectively. Moreover, the time costs of SubWSC are significantly lower than those of WSC+LOF. Hence, the subsampling method requires much less computational resources than WSC+LOF. Accordingly, SubWSC provides a promising solution for clus-
Table 3: Performances of the four clustering methods under the imbalanced setting.

| Example 1 | K-means | HC   | KSKC | WSC   | WSC+LOF |
|-----------|---------|------|------|-------|---------|
| \(\beta=20\) | .471 (.010) | .503 (.037) | .491 (.014) | .508 (.020) | .514 (.022) |
| RI \(\beta=50\) | .479 (.011) | .615 (.061) | .566 (.019) | .635 (.026) | .650 (.025) |
| \(\beta=100\) | .484 (.012) | .694 (.066) | .640 (.014) | .747 (.043) | .781 (.037) |
| \(\beta=20\) | .477 (.022) | .507 (.073) | .502 (.028) | .510 (.052) | .516 (.052) |
| CA \(\beta=50\) | .491 (.027) | .636 (.072) | .597 (.028) | .653 (.049) | .681 (.053) |
| \(\beta=100\) | .500 (.027) | .709 (.077) | .667 (.036) | .797 (.060) | .832 (.057) |
| \(\beta=20\) | .025 (.009) | .078 (.039) | .089 (.023) | .105 (.026) | .113 (.029) |
| NMI \(\beta=50\) | .042 (.013) | .304 (.062) | .267 (.038) | .383 (.047) | .394 (.045) |
| \(\beta=100\) | .059 (.013) | .552 (.056) | .472 (.032) | .596 (.050) | .614 (.065) |

| Example 2 | K-means | HC   | KSKC | WSC   | WSC+LOF |
|-----------|---------|------|------|-------|---------|
| \(\beta=20\) | .472 (.031) | .654 (.064) | .636 (.043) | .683 (.019) | .694 (.027) |
| RI \(\beta=50\) | .465 (.014) | .671 (.024) | .659 (.057) | .696 (.020) | .704 (.019) |
| \(\beta=20\) | .465 (.047) | .662 (.075) | .643 (.043) | .691 (.045) | .699 (.051) |
| CA \(\beta=50\) | .464 (.034) | .673 (.057) | .663 (.065) | .706 (.052) | .714 (.055) |
| \(\beta=20\) | .042 (.022) | .493 (.065) | .471 (.056) | .583 (.041) | .595 (.039) |
| NMI \(\beta=50\) | .041 (.019) | .607 (.033) | .602 (.029) | .624 (.023) | .631 (.017) |

Clustering on massive datasets when computational resources are limited.

Figure 7: Clustering performance of SubWSC in Example 1 under Setting (c) with different sample size (simulation data size: 620 merchants).

Figure 8: Clustering performance of SubWSC in Example 2 under Setting (c) with different sample size (simulation data size: 620 merchants).
5 Empirical study

To demonstrate the advantages of the proposed method in real applications, we conduct an empirical study of merchant clustering using a real dataset. The dataset containing 134,772 transaction records of 1,069 merchants is provided by an anonymous third-party online payment platform. For better management of merchants, the payment platform demands a solution to filter out irregular merchants from normal merchants. Such partitioning will allow the platform to develop differentiated marketing strategies and risk control measures.

Irregular merchants can be classified into cash-out merchants, and speculative merchants. Cash-out merchants make false transactions to gain cash from credit cards they collect. Their risky behaviors may cause undesirable outcomes, e.g., credit card fraud. Speculative merchants do not run regular businesses on the platform but aim at earning the rewards provided by the platform to support normal merchants. The dataset consists of 155 cash-out merchants, 284 speculative merchants, and 630 normal merchants, labeled by the platform’s staff. The labels are only used in the evaluation of the clustering performance. The artificial recognition and labeling of merchants is time-consuming and has high labor costs, making it difficult to apply supervised learning methods, such as classification approaches. A data-driven clustering method would be more useful in detecting irregular merchants. Accordingly, the proposed WSC+LOF method is suitable for such applications.

Clustering result. According to the silhouette coefficients, $K$ is set to 3. The clustering results are given in Table 4. The result shows successful clustering of merchants, with the irregular merchants separated from the normal merchants. In addition, we compare the clustering performance of WSC+LOF with other methods. The K-means feature-based method fails to distinguish different types of merchants, suggesting limited performances

---

1 The platform shared the data on the condition of anonymity.
Table 4: Matching matrix corresponding to the result of WSC-LOF (upper panel) and performance of the three clustering methods (lower panel).

| Speculative merchants | Cluster 1 | Cluster 2 | Cluster 3 |
|------------------------|-----------|-----------|-----------|
| Cash-out merchants     | 269       | 0         | 15        |
| Normal merchants       | 8         | 120       | 27        |

| Performance Comparison |
|------------------------|
| RI         | CA         | NMI        |
| WSC+LOF    | 0.907      | 0.931      | 0.716     |
| KSKC       | 0.859      | 0.901      | 0.637     |
| K-means    | 0.465      | 0.617      | 0.099     |
| HC         | 0.747      | 0.804      | 0.504     |

in such applications. In contrast, the proposed method achieves the best performance in all the indices.

Visualization of the clusters. Figure 9 displays the clusters formed by the proposed method. The results show that WSC+LOF can serve as an effective tool to recognize heterogeneous behavior patterns. First, as shown in the distribution, cash-out merchants (Cluster 2) obtain large amounts of money. This is consistent with their motivations, namely gaining cash most efficiently. Second, compared to normal merchants (Cluster 3), speculative merchants (Cluster 1) prefer transactions in low amounts as they aim to earn rewards from the platform and minimize transaction recording costs.

Performance of SubWSC. Figure 10 displays the clustering performances and time costs of SubWSC for different sample sizes. For each sample size, the performance of SubWSC
is evaluated through 100 random replications. The horizons in Figure 10 denote the results of WSC+LOF, which involve the whole dataset. This result shows that SubWSC can achieve similar clustering performance to WSC+LOF with reduced time costs. For example, when the sample size is 300, SubWSC can achieve nearly 85% clustering performance of WSC+LOF in terms of CA at 17% time cost. Thus, SubWSC is a feasible solution for massive datasets with limited computational resources.

In addition, the proposed method can also be applied in other datasets besides transactions. In essence, it provides a data-driven solution to clustering objects through their empirical distributions. To illustrate its generality, we conducted a clustering analysis of soccer teams2 in the Premier League as an example, which can be found in Appendix D.

6 Concluding remarks

In this paper, we propose the WSC method as a useful tool for data mining based on empirical distributions. The merchants are characterized using ECDFs of the transaction amount, and the Wasserstein distance is incorporated to measure the dissimilarity between two ECDFs. We then compile a spectral clustering algorithm to divide merchants with different behavior patterns. We further propose SubWSC, a subsampling version with improved computational feasibility for large-scale datasets, especially when resources are

---

2European Soccer Database in Kaggle. https://www.kaggle.com/hugomathien/soccer
limited. The simulations and empirical studies demonstrate that WSC outperforms other clustering methods, especially feature-based methods.

In future work, the following topics will be discussed. First, the proposed method will be further extended via integrating more information, e.g., transaction time. Such integration may bring about more improvements in clustering performances. An alternative solution involves integrating various information using multivariate ECDFs with multidimensional Wasserstein distance. Second, we consider the generalization of WSC as another interesting future work. Besides transaction data, the proposed method can also be applied in other fields. A generalized framework based on WSC will be considered to use data distributions efficiently.

References

Ai, M., Wang, F., Yu, J., and Zhang, H. (2021), “Optimal subsampling for large-scale quantile regression,” *Journal of Complexity*, 62, 101512.

Aksoylar, C., Qian, J., and Saligrama, V. (2017), “Clustering and community detection with imbalanced clusters,” *IEEE Transactions on Signal and Information Processing over Networks*, 3, 61–76.

Alborzi, M. and Khanbabaei, M. (2016), “Using data mining and neural networks techniques to propose a new hybrid customer behaviour analysis and credit scoring model in banking services based on a developed RFM analysis method,” *International Journal of Business Information Systems*, 23, 1–22.

Beygelzimer, A., Kakade, S., and Langford, J. (2006), “Cover trees for nearest neighbor,” in *International Conference on Machine Learning*, p. 97–104.
Breunig, M. M., Kriegel, H.-P., Ng, R. T., and Sander, J. (2000), “LOF: Identifying Density-Based Local Outliers,” in ACM SIGMOD International Conference on Management of Data, p. 93–104.

Bult, J. R. and Wansbeek, T. (1995), “Optimal selection for direct mail,” Marketing Science, 14, 378–394.

Chan, C. C. H. (2008), “Intelligent value-based customer segmentation method for campaign management: A case study of automobile retailer,” Expert Systems with Applications, 34, 2754–2762.

Chen, X. and Cai, D. (2011), “Large scale spectral clustering with landmark-based representation,” in AAAI Conference on Artificial Intelligence, pp. 313–318.

Chiang, W.-Y. (2012), “To establish online shoppers’ markets and rules for dynamic CRM systems: an empirical case study in Taiwan,” Internet Research, 22, 613–625.

Dannenberg, H. and Zupancic, D. (2009), Customer segmentation, Gabler.

Defays, D. (1977), “An efficient algorithm for a complete link method,” The Computer Journal, 20, 364–366.

Del Barrio, E., Cuesta-Albertos, J. A., Matrán, C., and Rodríguez-Rodríguez, J. M. (1999a), “Tests of goodness of fit based on the L2-Wasserstein distance,” Annals of Statistics, 1230–1239.

Del Barrio, E., Giné, E., and Matrán, C. (1999b), “Central limit theorems for the Wasserstein distance between the empirical and the true distributions,” Annals of Probability, 27, 1009–1071.
Delattre, S., Graf, S., Luschgy, H., and Pages, G. (2004), “Quantization of probability distributions under norm-based distortion measures,” *Statistics & Decisions*, 22, 261–282.

Dhandayudam, P. and Krishnamurthi, I. (2013), “Customer behavior analysis using rough set approach,” *Journal of Theoretical and Applied Electronic Commerce Research*, 8, 21–33.

Dhillon, P., Lu, Y., Foster, D. P., and Ungar, L. (2013), “New subsampling algorithms for fast least squares regression,” in *Advances in Neural Information Processing Systems*, pp. 360–368.

Eisenmann, T. R. and Barley, L. (2006), “PayPal Merchant Services,” *Harvard Business School Case*, 806–188.

Fahad, A., Alshatri, N., Tari, Z., Alamri, A., Khalil, I., Zomaya, A. Y., Foufou, S., and Bouras, A. (2014), “A survey of clustering algorithms for big data: Taxonomy and empirical analysis,” *IEEE Transactions on Emerging Topics in Computing*, 2, 267–279.

Filippone, M., Camastra, F., Masulli, F., and Rovetta, S. (2008), “A survey of kernel and spectral methods for clustering,” *Pattern Recognition*, 41, 176–190.

Fournier, N. and Guillin, A. (2015), “On the rate of convergence in Wasserstein distance of the empirical measure,” *Probability Theory and Related Fields*, 162, 707–738.

Hagen, L. and Kahng, A. B. (2002), “New spectral methods for ratio cut partitioning and clustering,” *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 11, 1074–1085.
Halko, N., Martinsson, P.-G., and Tropp, J. A. (2011), “Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions,” *SIAM Review*, 53, 217–288.

Hsu, F.-M., Lu, L.-P., and Lin, C.-M. (2012), “Segmenting customers by transaction data with concept hierarchy,” *Expert Systems with Applications*, 39, 6221–6228.

Huo, H., Wei, Y., and Xin, S. (2011), “Risk analysis of the third-party payment business,” in *International Conference on Management Science & Industrial Engineering*, pp. 1143–1147.

Kannan, R., Vempala, S., and Vetta, A. (2004), “On clusterings: Good, bad and spectral,” *Journal of the ACM*, 51, 497–515.

Khajvand, M. and Tarokh, M. J. (2011), “Estimating customer future value of different customer segments based on adapted RFM model in retail banking context,” *Procedia Computer Science*, 3, 1327–1332.

Khobzi, H., Akhondzadeh-Noughabi, E., and Minaei-Bidgoli, B. (2014), “A new application of rfm clustering for guild segmentation to mine the pattern of using banks’ e-payment services,” *Journal of Global Marketing*, 27, 178–190.

Knuth, D. E. (1976), “Big omicron and big omega and big theta,” *ACM Sigact News*, 8, 18–24.

Kullback, S. and Leibler, R. A. (1951), “On information and sufficiency,” *Annals of Mathematical Statistics*, 22, 79–86.

Law, M. T., Urtasun, R., and Zemel, R. S. (2017), “Deep spectral clustering learning,” in *International Conference on Machine Learning*, pp. 1985–1994.
Lazarevic, A. and Kumar, V. (2005), “Feature bagging for outlier detection,” in ACM SIGKDD International Conference on Knowledge Discovery in Data Mining, pp. 157–166.

Li, L., Huang, L., Yang, W., Yao, X., and Liu, A. (2015), “Privacy-preserving LOF outlier detection,” Knowledge and Information Systems, 42, 579–597.

Lin, J. (1991), “Divergence measures based on the Shannon entropy,” IEEE Transactions on Information Theory, 37, 145–151.

Lloyd, S. (1982), “Least squares quantization in PCM,” IEEE Transactions on Information Theory, 28, 129–137.

Lowry, P. B., Wells, T. M., Moody, G. D., Humphreys, S., and Kettles, D. (2006), “Online payment gateways used to facilitate e-commerce transactions and improve risk management,” Communications of the Association for Information Systems, 17, 1–48.

Nascimento, M. C. and De Carvalho, A. C. (2011), “Spectral methods for graph clustering—a survey,” European Journal of Operational Research, 211, 221–231.

Panaretos, V. M. and Zemel, Y. (2019), “Statistical Aspects of Wasserstein Distances,” Annual Review of Statistics & Its Application, 6, 405–431.

Park, C.-H. and Kim, Y.-G. (2003), “A framework of dynamic CRM: Linking marketing with information strategy,” Business Process Management Journal, 9, 652–671.

Peppard, J. (2000), “Customer Relationship Management (CRM) in financial services,” European Management Journal, 18, 312–327.

Piccoli, B. and Rossi, F. (2014), “Generalized Wasserstein distance and its application to
transport equations with source,” *Archive for Rational Mechanics and Analysis*, 211, 335–358.

— (2016), “On properties of the generalized Wasserstein distance,” *Archive for Rational Mechanics and Analysis*, 222, 1339–1365.

Politis, D. N., Romano, J. P., and Wolf, M. (1999), *Subsampling*, Springer Science & Business Media.

Rényi, A. et al. (1961), “On measures of entropy and information,” in *Berkeley Symposium on Mathematical Statistics and Probability*, pp. 547–561.

Rousseeuw, P. J. (1987), “Silhouettes: A graphical aid to the interpretation and validation of cluster analysis,” *Journal of Computational and Applied Mathematics*, 20, 53–65.

Rüschendorf, L. (1985), “The Wasserstein distance and approximation theorems,” *Probability Theory and Related Fields*, 70, 117–129.

Sakurai, Y., Li, L., Chong, R., and Faloutsos, C. (2008), “Efficient distribution mining and classification,” in *SIAM International Conference on Data Mining*, pp. 632–643.

Shah, D. and Zaman, T. (2010), “Community Detection in Networks: The Leader-Follower Algorithm,” *arXiv: Machine Learning*.

Shedden, B. M. (2015), “Finite Mixture Modeling with Mixture Outcomes Using the EM Algorithm,” *Biometrics*, 55, 463–469.

Song, Y., Chen, W.-Y., Bai, H., Lin, C.-J., and Chang, E. Y. (2008), “Parallel spectral clustering,” in *Joint European conference on machine learning and knowledge discovery in databases*, pp. 374–389.
Tsiptsis, K. K. and Chorianopoulos, A. (2011), *Data mining techniques in CRM: Inside customer segmentation*, John Wiley & Sons.

Vallender, S. (1974), “Calculation of the Wasserstein distance between probability distributions on the line,” *Theory of Probability & Its Applications*, 18, 784–786.

Van Vlasselaer, V., Bravo, C., Caelen, O., Eliassi-Rad, T., Akoglu, L., Snoeck, M., and Baesens, B. (2015), “APATE: A novel approach for automated credit card transaction fraud detection using network-based extensions,” *Decision Support Systems*, 75, 38–48.

Von Luxburg, U., Belkin, M., and Bousquet, O. (2008), “Consistency of spectral clustering,” *Annals of Statistics*, 555–586.

Wang, H., Zhu, R., and Ma, P. (2018), “Optimal subsampling for large sample logistic regression,” *Journal of the American Statistical Association*, 113, 829–844.

Wu, R.-S. and Chou, P.-H. (2011), “Customer segmentation of multiple category data in e-commerce using a soft-clustering approach,” *Electronic Commerce Research and Applications*, 10, 331–341.

Xuan, L., Chen, Z., and Yang, F. (2013), “Exploring of clustering algorithm on class-imbalanced data,” in *International Conference on Computer Science & Education*, pp. 89–93.

Zhang, Y., Bradlow, E. T., and Small, D. S. (2014), “Predicting customer value using clumpiness: From RFM to RFMC,” *Marketing Science*, 34, 195–208.

Zhu, Y., Deng, Q., Huang, D., Jing, B.-Y., and Zhang, B. (2021), “Clustering based on Kolmogorov–Smirnov statistic with application to bank card transaction data,” *Journal of The Royal Statistical Society: Series C (applied Statistics)*, 70, 558–578.