HOMOLOGY AND $K$-THEORY OF DYNAMICAL SYSTEMS
III. BEYOND TOTALLY DISCONNECTED CASE

VALERIO PROIETTI AND MAKOTO YAMASHITA

Abstract. We study homological invariants of étale groupoids continuing on our previous work, but going beyond the ample case by incorporating resolutions in the space direction. We prove analogues of the Künneth formula and the Poincaré duality in this framework. For non-wandering Smale spaces, we show that Putnam’s homology is isomorphic to the groupoid homology with integer coefficients, and that the $K$-groups of $C^*$-algebras of stable and unstable equivalence groupoids have finite rank.

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Introduction

In this paper we study homological and $K$-theoretical invariants of dynamical systems, continuing our work [PY22, PY21]. Previously we have focused on totally disconnected systems, which are represented by ample groupoids, i.e., topological groupoids whose unit space is totally disconnected. Now we lift this restriction and work with topological dynamical systems of finite topological dimension.

A motivating class of examples is that of Smale spaces [Rue04], which capture hyperbolicity on compact metric spaces such as the dynamics of Anosov diffeomorphisms and more generally those on the basic sets of Axiom A diffeomorphisms. This has led to interesting intersection between the theory of dynamical systems and operator algebras.

More generally, topological groupoids have been an important framework to construct and understand the structure of operator algebras. Given a second countable locally compact groupoid $G$ with continuous Haar system, the convolution product on the space of compactly supported continuous functions gives a complex algebra with involution that can be completed to a $C^*$-algebra [Ren80]. This can be further generalized to incorporate continuous actions of $G$ on $C^*$-algebras, and can be analyzed by homological invariants such as the operator $K$-groups.

For Smale spaces, the $C^*$-algebras constructed from the stable and unstable equivalence relation of Smale systems capture interesting aspects of the homoclinic and heteroclinic structure of expansive dynamics [Put96, Tho10, Mat19].
Another interesting homological invariant for dynamical system is the homology of étale groupoids with finite cohomological dimension due to M. Crainic and I. Moerdijk, which is based on sheaves, derived formalism, and simplicial methods [CM00].

For ample groupoids, the connection between this homology and operator K-theory of the associated C*-algebra was recently popularized by H. Matui [Mat12, Mat16], and gained a lot of attention in the C*-algebra community. In particular, he conjectured that groupoid homology (suitably periodicized) and K-groups of the groupoid C*-algebra are isomorphic. While the original form of this conjecture had some counterexamples [Sca20], nonetheless it was still confirmed in many interesting cases by concrete computations [FKPS19,BDGW21,Dec21].

Our first contribution was to provide a more systematic picture behind this correspondence, and we constructed a homological spectral sequence whose E2-sheet is given by the homology groups of ample groupoids G, and abuts to the K-groups of crossed product of certain G-C*-algebras [PY22], based on the Meyer–Nest theory of triangulated categorical structure on equivariant KK-theory [MN10,Mey08]. When the groupoid has torsion-free stabilizers and satisfies (a stronger form of) the Baum–Connes conjecture as studied by J.-L. Tu [Tu99], the abutment is indeed the K-groups of the groupoid C*-algebra of G.

Although there is a wealth of interesting examples already in the framework of totally disconnected dynamical systems, it is clear that novel techniques for working with groupoid homology and K-theory in higher dimensions are needed, not only in view of Smale-type systems, but also to cover a larger range of examples relevant in operator algebras. We hope that, not just our results, but also the paradigm of combining methods from operator algebraic K-theory with those from algebraic topology and algebraic geometry as we sketch below, will lead to new developments in the study of higher dimensional dynamical systems.

We start with some general theorems on the Crainic–Moerdijk homology. Our first main results are the following analogues of the Künneth formula and Poincaré duality.

**Theorem A** (Theorem 2.1). Let G and H be étale groupoids, and S and T be G- and H-equivariant sheaves. Then there is a split short exact sequence

\[
0 \to \bigoplus_{a+b=k} H_a(G, S) \otimes H_b(H, T) \to H_k(G \times H, S \boxtimes T) \to \bigoplus_{a+b=k-1} \text{Tor}(H_a(G, S), H_b(H, T)) \to 0.
\]

**Theorem B** (Theorem 3.1). Let \( \tilde{G} \) be a locally compact groupoid such that \( \tilde{G}^x \) is homeomorphic to \( \mathbb{R}^n \), and \( \tilde{G}^x \) is at most a singleton for all \( x, y \) in \( \tilde{G}(0) \). Let us choose a generalized transversal \( T \) in \( \tilde{G}(0) \), so that \( G = \tilde{G}|_T \) is an étale groupoid. We have an isomorphism

\[
H_k(G, \mathbb{Z}) \cong H^{n-k}_c(\tilde{G}(0), \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \tilde{G}^x),
\]

where on the right-hand side cohomology is computed with coefficients in the orbit-wise orientation sheaf over the unit space of \( \tilde{G} \).

While the proof of Theorem A is a standard Eilenberg–Zilber type manipulation of multi-complexes, for Theorem B we make an essential use of the sheaf theoretic idea and derived formalism. This can be also interpreted as an analogue of Connes’s Thom isomorphism for K-groups [Con81].

Turning back to Smale spaces, I. F. Putnam developed a theory of homology based on an intricate resolution of Smale spaces by symbolic dynamics (shifts of finite type) [Put14], and proved analogues of the Lefschetz formula for his homology. In [PY21] we showed that the Crainic–Moerdijk homology with integer coefficients coincides with the Putnam homology for the non-wandering Smale spaces with totally disconnected stable sets. We can now generalize it to general non-wandering Smale spaces, as follows.

**Theorem C** (Theorem 5.5). Let \((X, \phi)\) be a non-wandering Smale space, and \( T \) be a transversal for the unstable equivalence relation groupoid \( R^u(X, \phi) \). Then Putnam’s homology \( H^*_u(X, \phi) \) is...
isomorphic to the groupoid homology of the étale groupoid \( R^u(X, \phi) \) with coefficients in the constant sheaf of integers:

\[
H^*_k(X, \phi) \cong H_k(R^u(X, \phi)|_T, \mathbb{Z}).
\]

The main ingredient in Putnam’s homology is a pair of Smale spaces \((Y, \psi)\) and \((Z, \zeta)\) that map onto \((X, \phi)\) such that: \((Y, \psi)\) has totally disconnected unstable sets, and the factor map induces a homeomorphism of stable sets (an \(s\)-bijective map); \((Z, \zeta)\) has totally disconnected unstable sets, and the factor map induces a homeomorphism of unstable sets (a \(u\)-bijective map). Then the iterated fiber products of these spaces give a simplicial-cosimplicial system of shifts of finite type, leading to this homology.

The cosimplicial part, that comes from \((Z, \zeta)\), gives a cosimplicial system of \(G\)-sheaves for the étale groupoid \( G = R^u(X, \phi)|_T \). We combine sheaf theoretic methods for the associated complex together with techniques behind our previous result to obtain the above theorem.

The above results lead to a number of concrete structural results on homology of Smale spaces. On one hand, combining Theorems A and C we obtain a completely general Kinneth formula for the homology theory of (non-wandering) Smale spaces, generalizing results in [PY21] and [DKW16]. On the other, combining Theorems B and C we solve a conjecture by Putnam [Put14 Question 8.3.2] (see also [APSG17]) in the case of Smale spaces whose unstable sets are homeomorphic to \( \mathbb{R}^n \) for some \( n \).

We next turn to the problem of comparison between homology and \( K\)-theory. Previously we proved that, for Smale spaces with totally disconnected stable sets (corresponding to the case where the étale groupoid \( R^s(X, \phi)|_T \) is ample), there is a homological spectral sequence whose \( E^2 \)-sheet is the Putnam homology and abuts to the \( K\)-groups of \( C^*R^u(X, \phi) \), the \( C^*\)-algebra of unstable equivalence relation groupoids [PY21]. Here we prove its dual analogue, for Smale spaces whose unstable sets are totally disconnected.

**Theorem D** (Corollary 5.10). Let \((X, \phi)\) be a non-wandering Smale space with totally disconnected unstable sets. There is a cohomological spectral sequence abutting to \( K_\bullet(C^*R^u(X, \phi)) \), with \( E_2 \)-sheet given by \( H^*_\bullet(X, \phi) \).

While the claim is quite analogous to our previous result, the proof turns out to be somewhat different: for the case of stable sets being totally disconnected, one starts with a factor map from a shift of finite type that maps onto \( X \) by an \( s\)-bijective map. This leads to an open subgroupoid of the étale groupoid \( R^u(X, \phi)|_T \), where the theory of projective resolutions in triangulated categories [MN10][Mey08] is applicable through restriction and induction functors between the equivariant KK-categories.

For the case of unstable sets being totally disconnected, one starts with a factor map from a shift of finite type by a \( u\)-bijective map. This leads to a totally disconnected space \( T_0 \) on which the étale groupoid \( G = R^u(X, \phi)|_T \) acts. Moreover, we obtain a simplicial system \( T_\bullet \) of totally disconnected \( G\)-spaces, with a compatible action of the symmetric groups, by taking the iterated pullback construction of the structure map \( T_0 \to T \). Morally speaking this represents a kind of injective resolution, which however does not behave as nicely as projective resolutions in the framework of KK-categories.

Here we instead adapt ideas behind G. Segal’s work [Seg68] on simplicial spaces and spectral sequences. Using a variation of geometric realization suitable in the setting of locally compact spaces, we obtain a spectral sequence analogous to the Atiyah–Hirzebruch spectral sequence but without CW-complexes around. The abutment to the desired \( K\)-groups, which corresponds to the Baum–Connes conjecture in the case of projective resolutions, comes from the sheaf theoretic comparison of cohomological invariants for the nerves of the associated transformation groupoid on the \( G\)-simplicial spaces \( T_\bullet \).

Finally, in the setting of general Smale spaces, while we cannot directly relate the homology to \( K\)-groups, the techniques underlying Theorem D still allow us to prove the following finiteness result for \( K\)-groups of stable and unstable equivalence relation groupoids for Smale spaces.
Theorem E (Theorem 5.11). Let $(X,\phi)$ be a non-wandering Smale space. Then the $K$-theory groups

$$K_\bullet(C^*R^s(X,\phi)), \quad K_\bullet(C^*R^u(X,\phi))$$

of the associated groupoid $C^*$-algebras are of finite rank.

This settles a conjecture in [KPW17], and has implications on the structure of the Ruelle algebras, i.e., the crossed products of $C^*R^s(X,\phi)$ and $C^*R^u(X,\phi)$ by the natural integer actions induced by $\phi$. It also shows that $C^*$-algebras of Smale spaces do not exhaust the class of (simple) classifiable, real rank zero $C^*$-algebras (which is a natural question in the context of the classification program of nuclear $C^*$-algebras, see [DS18, DGY20] for further details).

The paper is organized as follows. In Section 1 we recall the basic notions and fix our conventions used throughout the paper. We also briefly summarize our previous work to set the background for this work.

In Section 2 we prove a general Künneth formula (Theorem A). In Section 3 we first show a Poincaré duality-type result (Theorem B). We also present some concrete computations for notable examples of topological dynamical systems from nilpotent groups such as self-similar actions, and an analogue of solenoids in algebraic number fields.

Section 4 is the technical core of the paper, where we use methods from simplicial homotopy theory and manipulate sheaf resolutions to achieve approximation results in cohomology and $K$-theory. These results are useful in view of the (co)simplicial structure arising from factor maps of shifts of finite type.

Finally, in Section 5 we consider applications to Smale spaces. The main results here are the comparison theorem between Putnam’s homology and groupoid homology (Theorem C), and the finiteness result on the $K$-theory rank of Smale spaces (Theorem E).

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1. Preliminaries

We fix conventions in use throughout the paper. We only briefly recall definitions and mostly refer to and generally follow the treatment in [PY22, PY21].

1.1. Topological groupoids. We mainly work with second countable, locally compact, and Hausdorff groupoids. Given such a groupoid $G$, we denote its base space by $G^{(0)}$, with structure maps $s,r: G \to G^{(0)}$, and the $n$-th nerve space (for $n \geq 1$) given by

$$G^{(n)} = \{(g_1,\ldots,g_n) \in G^n : s(g_i) = r(g_{i+1})\}.$$

We say that $G$ is étale if $s$ and $r$ are local homeomorphisms, and ample if it is étale and its base space is totally disconnected.

When we work with operator algebras, we further assume that $G$ admits a continuous Haar system [Ren80]. A choice of continuous left Haar system is denoted by $(\lambda^x)_{x \in G^{(0)}}$, and the associated $C^*$-algebras are denoted by $C^*_c(G,\lambda)$ and $C^*_u(G,\lambda)$, for the universal and reduced model, respectively. Most concrete groupoids we consider will be amenable so that this distinction disappears. When the choice of $\lambda$ is already understood, we also write $C^*G$ and $C^*_cG$. When $G$ is étale, we take $\lambda$ to be the counting measure.

Following [CM00], we say that a (topological) groupoid homomorphism $f: H \to G$ is a Morita equivalence if

$$\{(x,g) \mid x \in H^{(0)}, g \in G, f(x) = r(g)\} \to G^{(0)}, \quad (x,g) \mapsto s(g)$$
is an étale surjection, and

$$\begin{array}{c}
H\xrightarrow{f} G \\
\big|_{(r,s)} \\
H^{(0)} \times H^{(0)} \xrightarrow{(f,f)} G^{(0)} \times G^{(0)}
\end{array}$$

is a pullback diagram. Two groupoids $G$, $H$ are Morita equivalent if there is another groupoid $K$ and Morita equivalence homomorphisms $K \to G$, $K \to H$. This is equivalent to the existence of a bibundle $H \curvearrowright X \curvearrowleft G$, see [PY22] and references therein.

In this paper, when $G$ is a groupoid which is not étale, we assume the existence of an abstract transversal $T \subset G^{(0)}$ such that $G|_T$ can be given an étale topology, and there is a Morita equivalence $G \sim G|_T$ (see [PS99] for more details on this). This assumption is satisfied in most examples arising from geometric or dynamical situations.

Given groupoid homomorphisms $f_i : H_i \to G$, we consider multiple groupoid pullbacks $H_1 \times_G \cdots \times_G H_n$ as we did in [PY22]. This is a groupoid whose base is given by

$$\{(x_1, g_1, x_2, \ldots, g_{n-1}, x_n) \mid x_i \in (H_i)^{(0)}, g_i \in G^{(f_i(x_i))}_{f_{i+1}(x_{i+1})}\},$$

and an arrow from $(x_1, g_1, \ldots, x_n)$ to $(x'_1, g'_1, \ldots, x'_n)$ is given by a tuple $(h_1, \ldots, h_n)$ for $h_i \in (H_i)_{x_i'}$ such that the diagram

$$\begin{array}{c}
f_1(x'_1) \leftarrow g'_1 \leftarrow f_2(x'_2) \leftarrow g'_2 \leftarrow \cdots \leftarrow g'_{n-1} \leftarrow f_n(x'_n) \\
f_1(h_1) \leftarrow f_2(h_2) \leftarrow \cdots \leftarrow f_n(h_n)
\end{array}$$

is commutative.

**Proposition 1.1.** Let $G$ be a topological groupoid, and $n$ be a nonnegative integer. Then the fibered product groupoid $G^{\times_G(n+1)}$, with respect to the copies of identity homomorphisms $G \to G$, is Morita equivalent to $G$.

**Proof.** We give explicit Morita equivalence homomorphisms between $G$ and $G^{\times_G(n+1)}$ in both ways (formally, we only need to know one). In one direction, $f : G^{\times_G(n+1)} \to G$ is given by

$$(G^{\times_G(n+1)})^{(0)} \to G^{(0)}, \quad (x_0, g_1, \ldots, x_n) \mapsto x_n$$

at the level of base, and

$$G^{\times_G(n+1)} \to G, \quad (g_0, \ldots, g_n) \mapsto g_n$$

at the level of arrows. In the other direction, $f' : G^{\times_G(n+1)} \to G$ is given by

$$G^{(0)} \to (G^{\times_G(n+1)})^{(0)}, \quad x \mapsto (x, \text{id}_x, \ldots, x)$$

at the level of base, and

$$G \to G^{\times_G(n+1)}, \quad (g, g, \ldots, g)$$

at the level of arrows. \qed

1.2. **Homology of étale groupoids.** We consider the homology of étale groupoids as defined by Crainic and Moerdijk [CM00]. A $G$-sheaf is a sheaf $F$ on $G^{(0)}$ endowed with a continuous action of $G$, modeled by a map of sheaves $s^* F \to r^* F$ on $G$. A $G$-sheaf $F$ is said to be (c-)soft when the underlying sheaf on $G^{(0)}$ has that property, that is, for any (compact) closed subset $S \subset G^{(0)}$ and any section $x \in \Gamma(S, F)$, there is an extension $\tilde{x} \in \Gamma(X, F)$.

Our standing assumption is that the cohomological dimension of any open sets of $G^{(0)}$ is bounded by some fixed integer $N$: to be precise, $H_k^c(U, F) = 0$ for any open subset $U \subset G^{(0)}$ and any $k > N$.

This is guaranteed when $G^{(0)}$ is a subspace of a metrizable space of (Lebesgue) topological dimension $N$, which covers all the concrete examples we consider. To see this, observe that the topological dimension of any compact subset $A \subset G^{(0)}$ is bounded by $N$ [Eng78, Section 3.1],
then the Čech cohomology $\check{H}^\bullet(A,F)$ will vanish in degree above $N$. Then, by paracompactness, $\check{H}^\bullet(A,F)$ agrees with the sheaf cohomology $H^\bullet(A,F) = R^\bullet\Gamma_A(F)$ for the right derived functor of the functor $\Gamma_A(F) = \Gamma(A,F)$, then we can combine [God73] Remarque II.4.14.1 and Théorème II.4.15.1 to get the claim.

Let $F$ be a $G$-sheaf. Then there is a bounded cohomological resolution $F^\bullet$ of $F$ by c-soft $G$-sheaves, and the homology with coefficient $F$, denoted $H_*^c(G,F)$, is defined as the homology of the total complex of the double complex $(C^i_\ell)_0 \leq i, j$ with terms

$$C^i_\ell = \Gamma_c(G^{(i)}, s^*F^\ell),$$

which has homological degree $i - j$. More generally, when $F_\bullet$ is a complex of $G$-sheaves bounded from below, take a resolution cohomological resolution $F^\bullet$ of $F_\bullet$ by a complex of c-soft $G$-sheaves bounded above in the cohomological degree. Then the hyperhomology with coefficient $F_\bullet$, denoted by $H_*^c(G,F_\bullet)$, is the homology of triple complex with terms

$$C^k_{i,j} = \Gamma_c(G^{(i)}, s^*F^k_j),$$

which has homological degree $i + j - k$.

When two étale groupoids $G$ and $H$ are Morita equivalent, there are natural correspondences between the $G$-sheaves and $H$-sheaves inducing an isomorphism of groupoid homology. In particular, if $f : H \to G$ is a Morita equivalence homomorphism, we have

$$H_*^c(G,F_\bullet) \cong H_*^c(H,f^*F_\bullet)$$

for any complex $F_\bullet$ of $G$-sheaves as above.

Recall that a groupoid is called elementary if it is the disjoint union of groupoids $K_i$, each of which is isomorphic to the product of a second countable locally compact space and a transitive principal groupoid on a finite set. An ample groupoid is an AF groupoid when it is an increasing union of elementary groupoids [Ren80] Definition III.1.1.1.

**Proposition 1.2** (cf. [Mat12] Theorem 4.11]). Let $G$ be an AF groupoid, and $F$ be a $G$-sheaf. Then $H_p^c(G,F) = 0$ for $p > 0$.

**Proof.** The proof is a close analogue of that for [Mat12] Theorem 4.11]. For each $i$, the groupoid $K_i$ is Morita equivalent to the quotient space $K_i \backslash K_i^{(0)}$ regarded as a trivial groupoid. The latter space is second countable and totally disconnected, hence has trivial groupoid homology in positive degrees for any coefficient sheaf (see [PY22], Proposition 1.8). Consider a class of $H_p^c(G,F)$, represented by a cycle $f \in \Gamma_c(G^{(p)}, s^*F)$. Since $f$ has compact support, there is an index $i$ with $f \in \Gamma_c(K_i^{(p)}, s^*F)$. Then we have $H_p^c(K_i,F) = 0$ by the Morita invariance of groupoid homology, thus $f$ must be a boundary. \qed

### 1.3. Derived functor formalism

We briefly recall the derived functor formalism of groupoid homology from [CM01a] Section 4]. Let $G$ and $G'$ be étale groupoids, and $\phi : G \to G'$ be a continuous groupoid homomorphism. Then, for each $x \in G^{(0)}$, the comma groupoid $x/\phi$ is defined as the groupoid whose objects are the pairs $(y,g')$, where $y \in G^{(0)}$ and $g' \in G'_x^{\phi(y)}$, and an arrow from $(y_1,g'_1)$ to $(y_2,g'_2)$ is given by $g \in G^{(0)}_{g_1}$ such that $\phi(g)g'_1 = g'_2$. This is an étale groupoid that comes with a homomorphism $\pi_x : x/\phi \to G$.

When $F$ is a $G$-sheaf, we consider a simplicial system of $G'$-sheaves, denoted by $B_*^c(\phi,F)$, which at the level of stalks is given by

$$B_n^c(\phi,F)_x = \Gamma_c((x/\phi)^{(n)}, s^*\pi_x^*F).$$

The left derived functor $L\phi_*F_\bullet$ for a homological complex of $G$-sheaves bounded from below $F_\bullet$ is represented by the total complex of the triple complex of $G'$-sheaves with terms $B_i^c(\phi,F^k_j)$ with homological degree $i + j - k$, where $F^k_j$ is a resolution of $F_j$ by c-soft $G$-sheaves. This is well defined up to quasi-isomorphism of $G'$-sheaves. The $n$-th derived functor, denoted by
\( L_n \phi^*_F \), is the \( G' \)-sheaf given as the \( n \)-th homology of \( \mathcal{L} \phi^*_F \). By construction the fiber of this sheaf is given by [CM00] Proposition 4.3

\[
(L_n \phi^*_F)_x = H_n(x/\phi, \pi^*_x F_\bullet).
\]

When \( G' \) is the trivial groupoid and \( \phi \) is the unique homomorphism \( G \to G' \), this recovers the definition of \( H_n(G, F_\bullet) \).

Besides the pullback functor (the inverse image functor) for sheaves, we will also make use of the direct image functor, simply defined as \( g_* F(U) = F(g^{-1}(U)) \) [God73]. In the setting of equivariant sheaves, \( g_* \) can also be defined, and it is still right adjoint to the pullback functor, see [CM00] Section 2.3 for details. It is worth noting that, if \( g \) is proper, then \( g_* \) coincides with the functor \( g^* \), sometimes called direct image functor with compact supports. In the usual setting of topological spaces, \( g_! \) is left adjoint to \( g^* \) whenever \( g \) is étale, however in the case of groupoids the corresponding statement requires additional hypotheses [CM00] Remark 5.2.

1.4. Triangulated category from groupoid equivariant KK-theory. Let us review the background for this paper by briefly summarizing the contents of [PY22, PY21]. We work in the equivariant Kasparov category \( KK^G \) for groupoids \( G \) as in the beginning of this Section. (For our examples \( G \) can be further assumed to be étale.)

Let \( H \) be an open subgroupoid of \( G \) satisfying \( H^{(0)} = G^{(0)} \). Then the induction functor \( \text{Ind}^G_H : KK^H \to KK^G \) is defined by

\[
\text{Ind}^G_H B = (C_0(G) \otimes C_0(G^{(0)})) \times H,
\]

where \( \otimes C_0(G^{(0)}) \) denotes a \( \mathbb{C}^* \)-algebraic balanced tensor product over \( C_0(G^{(0)}) \). This functor is left adjoint to the restriction functor \( \text{Res}^G_H : KK^G \to KK^H \).

Setting \( L = \text{Ind}^G_H \text{Res}^G_H \) and \( P_n = L^{n+1}A \) for \( n \geq 0 \), the adjunction guarantees that \( P_\bullet \) has the structure of an augmented simplicial object over \( A \). Then the theory of projective resolution in triangulated categories applies, and as a special case of the “ABC spectral sequence” [Mey08] for the functor \( A \mapsto K_\bullet(G \ltimes A) \), we obtain a spectral sequence whose \( E^2 \)-sheet displays the homology groups of the chain complex \( K_\bullet(G \ltimes P_\bullet) \), and the abutment is \( K_\bullet(G \ltimes P) \) for a naturally defined approximation \( P \) of \( A \).

Roughly speaking, the object \( P \) is constructed as follows: since the objects \( P_n \) satisfy a certain projectivity condition, it is possible to iterate the construction of exact triangles

\[
P_n \to N_n \to N_{n+1} \to P_n[1],
\]

where we use the convention \( A = N_0 \) and \((-)[1]\) denotes suspension. The \( N_k \)'s form an inductive system whose homotopy colimit we denote \( N \). Now the object \( P \) can be characterized as fitting a unique exact triangle

\[
P \to A \to N \to P[1].
\]

Finally, the triviality assumption for the stabilizers \( G \) implies that \( K_\bullet(G \ltimes P) \) can be identified with the left-hand side of the Baum–Connes conjecture with coefficients in \( A \), see [BP22]. Let us summarize the above as follows.

**Theorem 1.3** ([PY22] Corollary 3.6). Let \( G \) be an ample groupoid with torsion-free stabilizers satisfying the Baum–Connes conjecture with coefficients. Let \( H, A \), and \( P_n \) be as above. There exists a homological spectral sequence

\[
E^2_{pq} = H_p(K_\bullet(G \ltimes P_\bullet)) \Rightarrow K_{p+q}(G \ltimes A).
\]

**Remark 1.4.** In the case of \( H = G^{(0)} \), the \( E^1 \)-sheet of above spectral sequence becomes \( E^1_{pq} = K_q^G(C_0(G^{(p)}) \otimes C_0(G^{(0)})) \ A \). If moreover \( G \) is ample, the \( E^2 \)-sheet can be identified with \( H_p^G(K_q^G(A)) \), the groupoid homology of \( G \) with the coefficient \( G \)-sheaf corresponding to the \( C_c(G, \mathbb{Z}) \)-module \( K_q^G(A) \) [PY22 Section 3].
1.5. Smale spaces. A Smale space is a compact metric space \( X \) with a self-homeomorphism \( \phi \) satisfying a certain hyperbolicity condition \([\text{Rue04}]\). We denote the bracket map on \( X \) by \( [x, y] \) for \( d(x, y) < \epsilon_X \), so that \([x, y] \) is stably equivalent to \( x \) and unstably equivalent to \( y \), in the sense below. We assume that a Smale space is non-wandering.

The most essential feature of Smale spaces is given by the stable and unstable equivalence relations, defined as follows.

- given \( x, y \in X \), we say they are stably equivalent (denoted \( x \sim_s y \)) if
  \[
  \lim_{n \to \infty} d(\phi^n(x), \phi^n(y)) = 0;
  \]
- given \( x, y \in X \), we say they are unstably equivalent (denoted \( x \sim_u y \)) if
  \[
  \lim_{n \to \infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0.
  \]

The graph of the unstable equivalence relation has a structure of locally compact groupoid with a Haar system \([\text{Put96}]\), that we denote by \( R^u(X, \phi) \). Note that the stable equivalence relation groupoid satisfies \( R^s(X, \phi) = R^u(X, \phi^{-1}) \), and \( R^u(X, \phi) = R^u(X, \phi^k) \) holds for any \( k \geq 1 \). The orbits of \( R^s(X, \phi) \) are referred to as stable sets. Following the construction detailed in \([\text{PS99}]\), we obtain an étale groupoid by restricting \( R^s(X, \phi) \) to an appropriate subspace contained in a finite union of stable sets.

Given a finite directed graph \( G = (\mathcal{G}^0, \mathcal{G}^1, i, t : \mathcal{G}^1 \to \mathcal{G}^0) \), the associated shift of finite type is the Smale space with underlying metric space \( D \) double complex with terms \( C_n, \sigma \), which can be described using stable sets of \( (\Sigma, \sigma) \) \([\text{Put14}]\). The \( K_1 \)-group is trivial.

When \( (\Sigma, \sigma) \) is a shift of finite type, the groupoid \( R^u(\Sigma, \sigma) \) is AF. Moreover, the \( K_0 \)-group of \( C^* R^u(\Sigma, \sigma) \) is isomorphic to the stable dimension group \( D^s(\Sigma, \sigma) \) introduced by Krieger \([\text{Kri80}]\), which can be described using stable sets of \( (\Sigma, \sigma) \) \([\text{Put14}]\). Chapter 3). The \( K_1 \)-group is trivial.

1.6. Homology of Smale spaces. Let \((X, \phi)\) be a non-wandering Smale space. An \( s/u \)-bijective pair over \((X, \phi)\) is given by a Smale space \((Y, \psi)\) with totally disconnected unstable sets and an \( s \)-bijective map \( f : Y \to X \), and another Smale space \((Z, \zeta)\) with totally disconnected stable sets and a \( u \)-bijective map \( g : Z \to X \). Given such data, we set

\[
\Sigma_\mathcal{G} = \{e = (e_k)_{k \in \mathbb{Z}} \in (\mathcal{G}^1)^\mathbb{Z} \mid t(e_k) = i(e_{k+1})\}
\]

and homeomorphism given by \( \sigma(e)_k = e_{k+1} \).

A Smale space \((X, \phi)\) is isomorphic to a shift of finite type as above if and only if \( X \) is totally disconnected, and \( \phi \) is furthermore irreducible if and only if \( G \) is connected \([\text{Put14}]\) Theorem 2.2.8.

When \((\Sigma, \sigma)\) is a shift of finite type, the groupoid \( R^u(\Sigma, \sigma) \) is AF. Moreover, the \( K_0 \)-group of \( C^* R^u(\Sigma, \sigma) \) is isomorphic to the stable dimension group \( D^s(\Sigma, \sigma) \) introduced by Krieger \([\text{Kri80}]\), which can be described using stable sets of \( (\Sigma, \sigma) \) \([\text{Put14}]\). Chapter 3). The \( K_1 \)-group is trivial.

Let \((X, \phi)\) be a non-wandering Smale space. An \( s/u \)-bijective pair over \((X, \phi)\) is given by a Smale space \((Y, \psi)\) with totally disconnected unstable sets and an \( s \)-bijective map \( f : Y \to X \), and another Smale space \((Z, \zeta)\) with totally disconnected stable sets and a \( u \)-bijective map \( g : Z \to X \). Given such data, we set

\[
Y_n = Y \times_X \cdots \times_X Y, \quad \psi_n = (\psi \times \cdots \times \psi)|_{Y_n}.
\]

Then \((Y_n, \psi_n)\) is again a Smale space with totally disconnected unstable sets. We denote the restriction of \( f \times \cdots \times f \) to \( Y_n \) by \( f_n \). This map is again \( s \)-bijective \([\text{Put14}]\) Theorem 5.2.15].

Similarly, we define a Smale space with totally disconnected stable sets,

\[
Z_n = Z \times_X \cdots \times_X Z, \quad \zeta_n = (\zeta \times \cdots \times \zeta)|_{Z_n},
\]

together with the \( u \)-bijective map \( g_n : Z_n \to X \). Then we have that

\[
\Sigma_{L,M} = Y_L \times_X Z_M, \quad \sigma_{L,M} = (\psi_L \times \zeta_M)|_{\Sigma_{L,M}},
\]
is a Smale space with totally disconnected underlying space. Thus it is a shift of finite type.

By definition, the stable homology of \((X, \phi)\), denoted by \( H^s_*(X, \phi) \), is the homology of the double complex with terms \( D^s(\Sigma_{L,M}, \sigma_{L,M}) \) for \( L, M \geq 0 \) \([\text{Put14}]\). This construction is independent of the chosen \( s/u \)-bijective pair and it is functorial for \( s/u \)-bijective maps.

We will also work with a normalized double complex as originally used in \([\text{Put14}]\): the group \( D^s(\Sigma_{L,M}, \sigma_{L,M}) \) admits a natural action of \( S_{L+1} \times S_{M+1} \). We denote by \( D^s(\Sigma_{L,M}, \sigma_{L,M}), A \) the subgroup of elements \( x \) satisfying \( sx = (-1)^{|s|} x \) for \( s \in S_{M+1} \). When the map \( g : Z \to X \) is at most \( N \)-to-1, the group \( D^s(\Sigma_{L,M}, \sigma_{L,M}), A \) is trivial for \( M > N \).
Now, consider a Smale space \((X, \phi)\) whose stable sets are totally disconnected, and take an \(s\)-bijective map \(g: (\Sigma, \sigma) \to (X, \phi)\) from a shift of finite type. We then have étale groupoids \(G = R^s(\Sigma, \sigma)|_T\) and \(H = R^s(\Sigma, \sigma)|_{T_0}\) for appropriate transversals \(T \subset X\) and \(T_0 \subset \Sigma\) and such that \(g(T_0) = T\), such that \(H\) embeds as an open subgroupoid of \((G\, see [Put00] for the details). Then the construction of Section 1[-2] gives the following.

**Theorem 1.6 (PY21, Theorem 4.1).** Let \((X, \phi)\) be a non-wandering Smale space with totally disconnected stable sets. Then we have the homological spectral sequence

\[
E^2_{p,q} = E^3_{p,q} = H^s_p(X, \phi) \otimes K_q(\mathbb{C}) = K_{p+q}(C^*R^s(X, \phi)).
\]

In fact, Putnam’s homology agrees with the groupoid homology:

**Theorem 2.1.** Under the above setting, there is a split short exact sequence

\[
0 \to \bigoplus_{a+b=k} H_a(G, S) \otimes H_b(H, T) \to H_k(G \times H, S \boxtimes T) \to \bigoplus_{a+b=k-1} \text{Tor}(H_a(G, S), H_b(H, T)) \to 0.
\]

**Proof.** Let us take bicomplexes \(A = A_{a,i}\) and \(B = B_{b,j}\) computing \(H_\bullet(G, S)\) and \(H_\bullet(H, T)\), respectively. These are obtained from \(C\)-soft cohomological complex of sheaves \(\tilde{S}^\bullet\) and \(\tilde{T}^\bullet\), which quasi-isomorphic to \(S\) and \(T\) concentrated in degree 0. To obtain a homological complex we take the degree, so that \(A_{a,i} = \Gamma_c(G^{(a)}, \tilde{S}^{-i})\) for example.

Up to the identifications

\[
A_{k,i} \otimes B_{j,k} = \Gamma_c(G^{(k)}, \tilde{S}^{-i}) \otimes \Gamma_c(H^{(k)}, \tilde{T}^{-j}) \cong \Gamma_c((G \times H)^{(k)}, \tilde{S}^{-i} \boxtimes \tilde{T}^{-j}),
\]

the total complex of the triple complex \((A_{k,i} \otimes B_{j,k})\) computes \(H_k(G \times H, S \boxtimes T)\). The claim follows by a standard argument if we can show that this is quasi-isomorphic to the total complex of the quadruple complex \(A \otimes B = (A_{a,i} \otimes B_{b,j})_{a,b,i,j}\).

Now, observe that \(A \otimes B\) can be regarded as a bisimplicial object in the category of complexes, by totalizing in the \(i\)- and \(j\)-directions. By an Eilenberg–Zilber type theorem [GJ09, Theorem IV.2.4], for fixed \(q\), the total complex of the bisimplicial group \(C_q(a, b) = \bigoplus_{q=i+j} A_{a,i} \otimes B_{b,j}\) is chain homotopic to the Moore complex of the simplicial group \(C'_q(k) = \bigoplus_{q=i+j} A_{k,i} \otimes B_{k,j}\).

Now, take double complexes \(C_{k,q} = \bigoplus_{k=a+b} C_q(a, b)\) and \(C'_{k,q} = C'_q(k)\). Since the degree \(k\) is concentrated in \(k \geq 0\) while the degree \(q\) is in \(q \leq 0\), the spectral sequences \(E\) and \(E'\) associated with filtration by \(q\)-degree on \(\text{Tot} C\) and \(\text{Tot} C'\) are regular, in the sense that for any \(n\) there is \(s(n)\) such that we have \(E^r_{p,q} = 0\) for \(p + q = n, p < s(n)\). Then the spectral sequences converge to the total homologies, while we have the isomorphisms \(E^r_{p,q} = E'_{p,q}\) for \(r \geq 1\) by the above remark. We thus obtain the assertion. \(\square\)

**Remark 2.2.** Matui’s result [Mat16, Theorem 2.4] is a special case of the above result in the situation where the groupoids are totally disconnected and the coefficients are locally constant sheaves \(\mathbb{Z}\). (Note that his convention of homology \(H_n(G)\) differs from \(H_n(G, \mathbb{Z})\) unless \(G\) is totally disconnected.)
3. Poincaré duality

Suppose we have a locally compact groupoid $\tilde{G}$ such that $\tilde{G}^x$ is homeomorphic to $\mathbb{R}^n$, and $\tilde{G}^x_y$ is at most a singleton for all $x, y$ in $\tilde{G}^{(0)}$. Let us choose a generalized transversal $T$ in $\tilde{G}^{(0)}$, so that $G = \tilde{G}|_T$ is an étale groupoid. We also assume that groupoid homology for $G$ is definable.

Under our assumption, the structure map $s: \tilde{G} \to \tilde{G}^{(0)}$ is a model of the universal principal $\tilde{G}$-bundle $E\tilde{G} \to B\tilde{G}$. Then the Baum–Connes conjecture suggests a close relation between $H_*^{\text{groupoid}}(G, \mathbb{Z})$ and the compactly supported cohomology of the space $\tilde{G}^{(0)}$. Let us make this precise in the framework of groupoid homology.

3.1. **Poincaré duality-type theorem**. Now, let us consider a groupoid $\tilde{G}$ as in the beginning of this section, and take a transversal $T$ and the associated étale groupoid $G$ as before. We then consider the orbit-wise orientation sheaf $\mathcal{O}$ on $\tilde{G}^{(0)}$. Formally, its stalk at $x$ is given by

$$\mathcal{O}_x = (\wedge^n T_x \tilde{G}^x)/\mathbb{R}_+ \cong \mathbb{R}^\times/\mathbb{R}_+ \cong \{1, -1\}.$$ 

This sheaf has a natural action of $\mathbb{Z}/2\mathbb{Z}$. Note that $\mathcal{O}$ admits a global section (equivalently, it is trivializable) if and only if there is a global orientation on the orbits of $\tilde{G}$.

**Theorem 3.1.** Under the above setting, we have an isomorphism

$$H_k(G, \mathbb{Z}) \cong H^{n-k}_c(\tilde{G}^{(0)}, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O}).$$

**Proof.** The claim is equivalent to the isomorphism of groupoid homology groups

$$H_k(G, \mathbb{Z}) \cong H_{k-n}(\tilde{G}^{(0)}, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O}),$$

where we treat $\tilde{G}^{(0)}$ as a trivial groupoid on the right hand side.

Consider the $G$-space $E = \tilde{G}^T$. Then we have a morphism of groupoid

$$\phi: G \times E \to G$$

induced by the range map $\tilde{G}^T \to T$. We want to apply the constructions in [CM00] Section 4] to this setting. We have a Morita equivalence between $G \times E$ and $G^{(0)}$, induced by the source map $s: E \to \tilde{G}^{(0)}$. By [CM00] Corollary 4.6, we have

$$H_{p-n}(G \times E, s^* (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O})) \cong H^{n-p}_c(\tilde{G}^{(0)}, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O}).$$ (2)

Let us consider the $(G \times E)$-sheaf $F = s^* (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O})$, and its left derived functors $L_k \phi_* F$.

Fix $x \in T$. By our assumption on $\tilde{G}$, the object space of $x/\phi$ can be identified with the disjoint union of $\tilde{G}^y$ for $y \in G_x$. Given objects $g \in \tilde{G}^y$ and $g' \in \tilde{G}^x$ in $x/\phi$, there is an arrow from $g$ to $g'$ if and only if $g = g' g$ for the unique $g \in \tilde{G}^y$. In particular, $x/\phi$ is Morita equivalent to $G^x \cong \mathbb{R}^n$.

If we restrict the pullback sheaf $\pi^* x F$ to $\tilde{G}^x$, we get $s^* ((\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \mathcal{O})$, but this is isomorphic to $\mathbb{Z}$ by the global orientation on $\mathbb{R}^n$. We then have

$$H_k(x/\phi, \pi^* x F) \cong H^{n-k}_c(\mathbb{R}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (k = -n) \\ 0 & \text{(otherwise)} \end{cases}.$$ 

By (1), the $G$-sheaf $L_k \phi_* F$ on $G^{(0)} = T$ has the stalks isomorphic to $\mathbb{Z}$ when $k = -n$, and we have $L_k \phi_* F = 0$ otherwise. Of course, the same can be said about $L_n \phi_* \mathbb{Z}$. However, looking at the action of $\tilde{G}$, the extra factor $\mathcal{O}$ corrects the “sign” of the map $H_k(x/\phi, \pi^* x \mathcal{O}) \to H_k(x/\phi, \pi^* x \mathcal{O})$ induced by $g \in \tilde{G}^y$, and we have the isomorphism of $G$-sheaves between $L_{-n} \phi_* F$ and $\mathbb{Z}$.

Now, the Leray-type spectral sequence from [CM00] Theorem 4.4]

$$E^2_{pq} = H_p(G, L_q \phi_* F) \Rightarrow H_{p+q}(G \times E, F)$$

is degenerate at the $E^2$-sheet for degree reasons. Thus we get that $H_p(G, \mathbb{Z})$ is isomorphic to $H_{p-n}(G \times E, F)$. (A bit more conceptually, $L_\bullet \phi_* \mathbb{Z}$ is quasi-isomorphic to the degree shift of $\mathbb{Z}$, and groupoid hyperhomology and degree shift of coefficient commute.) Combined with (2), we obtain the assertion. □
Example 3.2. In the case of groupoids for substitution tilings, the above isomorphism appears in [PY22] Section 5.2.

Example 3.3. Let $\Gamma$ be a finitely generated group, and $\phi$ be an injective and surjective contracting virtual endomorphism of $\Gamma$. Then $\Gamma$ is virtually nilpotent, and admits a self-similar action $(\Gamma, X)$ where the alphabet set $X$ is a system of representatives of $\Gamma/\text{dom } \phi$. Its limit $\Gamma$-space $X_{\Gamma, X}$ can be identified with a nilpotent connected and simply connected Lie group $L$ on which $\Gamma$ acts by affine transformations in a proper and cocompact way [Nek05 Section 6.1]. In general, when $(\Gamma, X)$ is a contracting, recurrent, and regular self-similar action, we have the associated Smale space $S_{\Gamma, X}$ (the limit solenoid of $(\Gamma, X)$), and its unstable sets can be identified with $X_{\Gamma, X}$, see [Nek09]. Thus, under the above assumption on $(\Gamma, \phi)$, the unstable groupoid $G = R^u(S_{\Gamma, X})$ satisfies the assumption of Theorem 3.1.

Example 3.4. Let $M = L/\Gamma$ be an infra-nilmanifold, i.e., a quotient of a nilpotent, connected, and simply connected Lie group $L$ by a torsion-free group $\Gamma$ of affine automorphisms of $L$ such that $L = L \cap \Gamma$ is a finite index subgroup of $\Gamma$. Moreover, let $\psi$ be a hyperbolic affine automorphism of $M$ [Dek11]. Then $R^u(M, \psi)$ and $R^s(M, \psi)$ satisfy the assumption of Theorem 3.1. Indeed, an unstable set of $(M, \psi)$ can be identified with the subspace of the Lie algebra $L$ of $L$ spanned by eigenvectors of the “linear part” of $\psi$ for corresponding to its eigenvalues bigger than 1, while a stable set can be identified with the span of the other eigenvectors.

3.2. Expanding maps on compact manifolds. Let us describe a concrete example in the class of Example 3.3 that arises from a non-orientable surface. Consider the group $\Gamma$ of Example 3.3. Indeed, an unstable set of $M, \psi$ can be identified with the unstable groupoid $G = R^u(S_{\Gamma, X})$.

Turning to the groupoid homology, at degree $k = 0$ we have

$$H_0(G, \mathbb{Z}) \cong \mathbb{Z}[1/9] = \mathbb{Z}[1/3]$$

by [PY21] Theorem 4.1 and Proposition 6.3). More generally, as remarked in [PY21] Section 6.2, $H_k(G, \mathbb{Z})$ is the direct limit of group homology groups $H_k(\Gamma, \mathbb{Z})$ where the connecting map is induced by the Morita equivalence between $\Gamma \times \Omega_i$ and $g^i(\Gamma) \cong \Gamma$. Concretely, these maps are the transfer maps of group homology,

$$H_k(\Gamma, \mathbb{Z}) \to H_k(g^i(\Gamma), \mathbb{Z}),$$

see [Bro94] Section III.9.

At degree $k = 1$, let us write

$$H_1(\Gamma, \mathbb{Z}) = \Gamma^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad H_1(g^i(\Gamma), \mathbb{Z}) = g^i(\Gamma)^{ab} \cong 3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$
where the images of $a$ and $g(a)$ become the generator of the summand $\mathbb{Z}/2\mathbb{Z}$, and those of $b$ and $g(b)$ become generators of $\mathbb{Z}$ and $3\mathbb{Z}$ respectively. The transfer map is given by

$$H_1(\Gamma, \mathbb{Z}) \to H_1(g(\Gamma), \mathbb{Z}), \quad [a] \mapsto [g(a)], \quad [b] \mapsto 3[g(b)],$$

see for example [Bro94, Exercise III.9.2]. Thus, the inductive system $(H_1(g^i(\Gamma), \mathbb{Z}))_i$ can be identified with the constant system $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ whose connecting map is given by $(x, y) \mapsto (3x, y)$. In particular we obtain

$$H_1(G, \mathbb{Z}) \cong \mathbb{Z}[1/3] \oplus \mathbb{Z}/2\mathbb{Z}.$$  

For $k \geq 2$, we have $H_k(G, \mathbb{Z}) = 0$. One way to see this is to use the isomorphism

$$H_k(G, \mathbb{Z}) \cong H^{2-k}(Y, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

given by Theorem 3.1. This forces $H_k(G, \mathbb{Z}) = 0$ for $k > 2$ by degree reasons, and at $k = 2$ we have

$$H^0(Y, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \Gamma(Y, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = 0$$

because there is no global orientation on the orbits of $\tilde{G}$. (Alternatively, one can also use the duality between group homology and cohomology for $\Gamma$ to directly check $H_k(\Gamma, \mathbb{Z}) = 0$, see [Bro94, Section VIII.10].)

More generally, suppose that $M$ is an $n$-dimensional connected compact Riemannian manifold and $g: M \to M$ is an expanding map. Then $g$ admits a fixed point $x$, $\Gamma = \pi_1(M, x)$ is a torsion-free group of polynomial growth, and $\mathbb{R}^n$ is a universal cover for $M$, see [Nek05, Section 6.1].

With the virtual endomorphism $\phi$ represented by $g^{-1}$, we are in the setting of Example 3.3. Then the Smale space $S_{\Gamma, X}$ is given by $Y = \lim_{\leftarrow} g^{k} M$ and the associated self homeomorphism $\phi$ of $Y$. Again the groupoid $\tilde{G} = R^u(Y, \phi)$ is Morita equivalent to the étale groupoid $G = \Gamma \ltimes \Omega$ where $\Omega = \lim_{\leftarrow} \Gamma/g^i(\Gamma)$, and we have

$$H_k(G, \mathbb{Z}) \cong H^{n-k}(Y, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

Let us write this as a group cohomology with coefficient.

Since $G$ is a transformation groupoid, we also have

$$H_k(G, \mathbb{Z}) \cong H_k(\Gamma, C(\Omega, \mathbb{Z}))$$

with respect to the induced action of $\Gamma$ on $C(\Omega, \mathbb{Z})$. As $M$ is a model of the Eilenberg–MacLane space $K(\Gamma, 1)$, $\Gamma$ is a Poincaré duality group [Bro94, Section VIII.10], and we have

$$H_k(\Gamma, C(\Omega, \mathbb{Z})) \cong H^{n-k}(\Gamma, C(\Omega, \mathbb{Z}) \otimes D)$$

where $D$ is an infinite cyclic group endowed with the “sign” representation of $\Gamma$.

Again the Morita equivalence between $\Gamma \ltimes (\Gamma/g^i(\Gamma))$ and $g^i(\Gamma)$ leads to a presentation of these (co)homology groups as inductive limits of $H_k(g^i(\Gamma), \mathbb{Z}) \cong H_k(\Gamma, \mathbb{Z})$ with connecting maps given by the transfer maps. This corresponds to the isomorphism

$$K_* (\Gamma \ltimes C(\Omega)) \cong \lim_{\leftarrow} K_* (C^* g^i(\Gamma))$$

that follows from the Baum–Connes conjecture for coefficients for $\Gamma$, see also [Dec21] for the case of flat manifolds.

3.3. A generalization of solenoid. We have seen several examples in the previous section where our duality theorem has a fairly straightforward application. Here we consider a slightly more complicated system, obtained by generalizing the notion of solenoid. The argument in our proof of Poincaré duality will be useful in this setting too.

Let $p < q$ be two prime numbers, and put $N = pq$, $r = \frac{q}{p}$, $\Gamma = \mathbb{Z}[1/N]$. Then $\Gamma \ltimes \mathbb{R} \times \mathbb{Q}_p \times \mathbb{Q}_q$ (diagonally embedded) is a closed cocompact subgroup, hence the quotient

$$X = (\mathbb{R} \times \mathbb{Q}_p \times \mathbb{Q}_q)/\Gamma$$

is a compact metric space identified with the Pontryagin dual group of $\Gamma$.  

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Let $\phi$ be the self-homeomorphism of $X$ given by

$$\phi([x, y, z]) = [rx, ry, rz].$$

Then $(X, \phi)$ is a Smale space. The stable set are the orbits of the natural action of $Q$, and the unstable sets are those of the natural action of $R \times Q$.

For the fixed point $x_0 = [0, 0, 0]$ of $\phi$, the stable set $X^s(x_0)$ is the image of $Q$ in $X$, while the unstable set $X^u(x_0)$ is the image of $R \times Q$. From the description above, we see that the étale groupoid $R^u(X, \phi)|_{X^u(x_0)}$ is the transformation groupoid $\Gamma \ltimes Q$, with underlying arrow set given by

$$\{(0, 0, z_1), [0, 0, z_2] \mid z_1, z_2 \in Q, \exists x \in R, y \in Q : [0, 0, z_1] = [x, y, z_2]\}.$$

Similarly, $R^s(X, \phi)|_{X^s(x_0)}$ is the transformation groupoid $\Gamma \ltimes (R \times Q)$.

**Proposition 3.5.** Consider $C_c(Q, Z)$ as a $\Gamma$-module with respect to the induced action by translation on $Q$. The corresponding homology groups are

$$H_0(\Gamma, C_c(Q, Z)) \cong Z \left[\frac{1}{q}\right], \quad H_1(\Gamma, C_c(Q, Z)) \cong Z, \quad H_i(\Gamma, C_c(Q, Z)) \cong 0 \quad (i > 1).$$

**Proof.** For $k \in \mathbb{N}$, let $\Gamma_k \subset \Gamma$ denote the subgroup \{ $\{a/N^k \mid a \in Z\}$. On the one hand, because $\Gamma$ is the union of the increasing sequence of subgroups $\Gamma_k$, we have

$$H_i(\Gamma, C_c(Q, Z)) = \lim_{k \to \infty} H_i(\Gamma_k, C_c(Q, Z))$$

On the other, since $\Gamma_k$ is isomorphic to $Z$, we have

$$H_0(\Gamma_k, C_c(Q, Z)) \cong C_c(Q, Z) \Gamma_k, \quad H_1(\Gamma_k, C_c(Q, Z)) \cong C_c(Q, Z)^{\Gamma_k},$$

and $H_i(\Gamma_k, C_c(Q, Z)) = 0$ for $i > 1$. We thus need to understand the limits of invariants and coinvariants for $\Gamma_k$.

First, for the coinvariants we have

$$\lim_{k \to \infty} C_c(Q, Z) \Gamma_k = C_c(Q, Z) \Gamma.$$

Integration with respect to the normalized Haar measure of the additive group $Q$ defines an isomorphism between the group above and $Z[1/q]$.

As for the invariants, at $k = 1$, the $\Gamma_1$-invariant functions on $Q$ are linear combinations of the characteristic functions $\chi_{x_1, a_2, \ldots, a_n}$ on sets of the form

$$X_{1, a_2, \ldots, a_n} = \{a_0q^{-n} + \cdots + a_2q^{-2} + xq^{-1} \mid x \in Z_q\} \quad (n > 1, 0 \leq a_i < N).$$

Indeed, invariance under translation by $\frac{1}{p}$ is equivalent to invariance under $\frac{1}{p^2}$ and $\frac{1}{q}$. Invariance under $\frac{1}{q}$ gives sets of the above form by the density of $Z$ in $Z_q$, and by $\frac{1}{p} \in Z_q$ the invariance under $\frac{1}{p}$ is automatic. We also have similar descriptions for the $\Gamma_k$-invariant functions. Moreover, the connecting map is given by

$$C_c(Q, Z)^{\Gamma_k} \to C_c(Q, Z)^{\Gamma_{k+1}}, \quad f \mapsto \sum_{i=0}^{N-1} \tau_{-i} f,$$

where $\tau_t$ denotes the translation by $t \in \Gamma$. Thus the map characterized by

$$C_c(Q, Z)^{\Gamma_k} \to Z, \quad \chi_{x_{k, a_{k+1}, \ldots, a_n}} \mapsto 1$$

induces an isomorphism $\lim_{k \to \infty} C_c(Q, Z)^{\Gamma_k} \cong Z$. □

This gives a description of the groupoid homology $H_i(\Gamma, Z)$ for the étale groupoid $G = R^u(X, \phi)|_{X^u(x_0)}$. As for the groupoid $G' = \Gamma \ltimes (R \times Q) = R^u(X, \phi)|_{X^u(x_0)}$, we can use an argument similar to that of Poincaré duality to reduce its groupoid homology to $H_*(\Gamma, C_c(Q, Z))$.  

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Proposition 3.6. We have
\[ H_{-1}(G', \mathbb{Z}) \cong \mathbb{Z} \left[ \frac{1}{p} \right], \quad H_0(G', \mathbb{Z}) \cong \mathbb{Z}, \quad H_2(G', \mathbb{Z}) \cong 0 \quad (i > 0 \text{ or } i < -1) \]

Proof. In view of Proposition 3.5, it is enough to show \( H_i(G', \mathbb{Z}) \cong H_{i+1}(\Gamma, C_\ast(\mathbb{Q}_p, \mathbb{Z})) \). Write \( G = \Gamma \times \mathbb{Q}_p \), and consider the groupoid homomorphism \( \psi: G' \to G \), defined by forgetting the copy of \( \mathbb{R} \). For any \( x \in \mathbb{Q}_p = G(0) \), the comma groupoid \( x/\psi \) is Morita equivalent to the space \( \mathbb{R} \). Then the same argument as in the proof of Proposition 3.1 gives the claim. \( \Box \)

Remark 3.7. The above example belongs to the following class of Smale spaces from [Sch95, Section 7]. Let \( r \in \mathbb{Q} \subset \mathbb{R} \) be an algebraic number such that \( |gr| \neq 1 \) for any element \( g \) in the absolute Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \). Consider the field \( K = \mathbb{Q}(r) \) generated by \( r \), and let \( P_f^K \) and \( P^K \) be the set of finite and infinite places of the global field \( K \) respectively, and set
\[ P(r) = P^K_\infty \cup \{ v \in P^K_f \mid |r|_v \neq 1 \}. \]
Then the ring
\[ R_r = \{ a \in K \mid \forall v \in P^K_f \setminus P(r) : |a|_v \leq 1 \} \]
is a cocompact subring of the direct product of local fields \( \prod_{v \in P(r)} K_v \), and the quotient
\[ Y(r) = \left( \prod_{v \in P(r)} K_v \right) / R_r \]
can be identified with the Pontryagin dual of \( R_r \). The periodic points of the self-homeomorphism
\[ \phi: Y(r) \to Y(r), \quad [a_v]_{v \in P(r)} \mapsto [ra_v]_{v} \]
form a dense subset of \( Y(r) \) [Sch95 Section 5], and the \( K_v \)-direction is contracting (resp. expanding) for \( \phi \) if and only if \( |r|_v < 1 \) (resp. \( |r|_v > 1 \)). Hence \( (Y(r), \phi) \) is a non-wandering Smale space.

4. Symmetric simplicial spaces

In this section we leverage Segal’s work [Seg68] on simplicial spaces and spectral sequences to obtain cohomological approximation results in the setting of groupoids and dynamical systems. First we fix some definitions and conventions. Let \( A_\ast = (A_n)_{n=0}^\infty \) be a simplicial topological space, that is, a simplicial object in the category of topological spaces. Its thin geometric realization is defined to be
\[ |A_\ast| = \left( \prod_{n=0}^\infty \Delta^n \times A_n \right) / \sim, \]
where \( \Delta^n \) is the standard \( n \)-simplex and \( \sim \) is an equivalence relation coming from (co)degeneracy and (co)face maps.

Next further assume that \( A_\ast \) is a symmetric simplicial space, or equivalently, an \( S_\ast \)-space for the crossed simplicial group \( S_\ast \) [FL91], i.e., a simplicial space with compatible actions of \( S_{n+1} \) on \( A_n \). Alternatively, we can think of \( A_\ast \) as a contravariant functor on the category of (unordered) finite sets. Then we can form a “naive” geometric realization \( |A_\ast| \), as the quotient of \( |A_\ast| \) with respect to the orbit relations for the diagonal action of \( S_{n+1} \) on \( \Delta^n \times A_n \), for all \( n \). Moreover, we denote the image of \( \Delta^n \times A_n \) in \( |A_\ast| \) by \( |A_\ast|^{(n)} \).

4.1. Adaptation of Segal’s work. Let \( T \) be a locally compact space, and \( g: T_0 \to T \) be a proper continuous surjective map from another locally compact space. Then we obtain an \( S_n \)-space \( T_\ast \) by setting \( T_n \) to be the \( (n+1) \)-fold fiber product of \( T_0 \) over \( T \). Let us also denote by \( T_\ast^f \) the subset of points \( (x_0, \ldots, x_n) \) such that \( x_i \neq x_j \) for \( i \neq j \), i.e. the union of free \( S_{n+1} \)-orbits in \( T_n \). This is an open subset of \( T_n \).

Moreover, suppose that \( g \) is at most \( N \)-to-one, for some \( N \). Then \( |T_\ast| \) is a locally compact space endowed with a proper surjective map \( \tilde{g}: |T_\ast| \to T \). We also have \( |T_\ast| = |T_\ast|^{(N)}. \)
Proposition 4.1. Suppose that $g$ is as above. We have $H^n(T, Z) \cong H^n(\{ T_\ast \}, Z)$ induced by $\tilde{g}$.

Proof. For a topological space $X$, let $D^+(X)$ be the derived category of cochain complexes of sheaves of commutative groups which are bounded below up to quasi-isomorphisms. We thus have the right derived functor $R\tilde{g}_\ast : D^+(\{ T_\ast \}) \to D^+(T)$ of the direct image functor $\tilde{g}_\ast$. We also have the (term-wise application of) inverse image functor $\tilde{g}_\ast : D^+(T) \to D^+(\{ T_\ast \})$.

Since $\tilde{g}_\ast \{ T_\ast \} = \tilde{g}^* \mathbb{Z}_T$, we have a natural morphism $\mathbb{Z}_T \to R\tilde{g}_\ast \mathbb{Z}_T$ in $D^+(T)$ induced by the adjunction between $R\tilde{g}_\ast$ and $\tilde{g}_\ast$.

We first claim that $R^n\tilde{g}_\ast \mathbb{Z}_T = 0$ for $0 < n$. By the properness of $\tilde{g}$, for any $x \in T$ we have

$$(R^n\tilde{g}_\ast \mathbb{Z}_T)_x = H^n(\tilde{g}^{-1}(x), \mathbb{Z})$$

from the proper base change theorem, see [SD72, Corollaire 4.1.2]. Here $\tilde{g}^{-1}(x)$ is homeomorphic to $\Delta^m$ with $m = |\tilde{g}^{-1}(x)|$, hence the $n$-th cohomology group on the right hand side must vanish. This also shows $\tilde{g}_\ast \mathbb{Z}_T = \mathbb{Z}_T$.

Now, we have the Leray spectral sequence

$$E_2^{p,q} = H^p(T, R^q\tilde{g}_\ast \mathbb{Z}_T) \Rightarrow H^{p+q}(\{ T_\ast \}, \mathbb{Z}_T).$$

By the above claim we have $E_2^{p,q} = 0$ for $0 < q$, and $E_2^{0,p} = H^p(T, \mathbb{Z}_T)$. As the higher differentials are $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}$, we have $E_2^{0,0} = E_\infty^{0,0}$ and $E_2^{0,q} = 0$ for $q > 0$. This implies

$$H^n(\{ T_\ast \}, \mathbb{Z}_T) \cong E_\infty^{0,0} \cong H^n(T, \mathbb{Z}_T),$$

as claimed. \hfill \Box

Proposition 4.2. Suppose $g$ is as above. We then have $K^\bullet(\{ T_\ast \}) \cong K^\bullet(T)$ induced by $\tilde{g}$.

Proof. First, we may assume that $T$ is compact. Indeed, denoting the one-point compactification by $T^+$, we have a proper surjective map $T_0^+ \to T$. Then we have $\{ T_\ast \}^+ = \{ T_\ast \}$, hence it is enough to prove the claim for $\{ T_\ast \}$.

Consider the representable $K$-groups $RK^\bullet(X) = [X, KU_n]$, where $KU_n$ is the $K$-theory spectrum, i.e., $KU_2k = BU \times \mathbb{Z}$ and $KU_2k+1 = \Omega BU = U$ for the infinite unitary group $U = \lim U_k$. Then, this is a generalized cohomology as considered in [Seg68], while satisfying $RK^\bullet(X) \cong K^\bullet(X)$ for compact $X$. We thus need to check $RK^\bullet(\{ T_\ast \}) \cong RK^\bullet(T)$.

By [Seg68, Proposition 5.2], there is a spectral sequence

$$E_2^{pq} = H^p(X, F^q) \Rightarrow RK^{p+q}(X),$$

where $F^{2k} = \mathbb{Z}$ and $F^{2k+1} = 0$. (This is a version of the Atiyah–Hirzebruch spectral sequence that works without any CW-complex structure.) By Proposition 4.1, we have the isomorphism of $E_2$-sheet for $X = T$ and $X = \{ T_\ast \}$.

Following the scheme in [Seg68], we define a spectral sequence converging to $K_\bullet(G \ltimes C_0(\{ T_\ast \}))$. The filtration $\{ T_\ast \}^{(n)}$ on $\{ T_\ast \}$, combined with Proposition 4.2 induces a spectral sequence.

$$E_1^{pq} = K^{p+q}(\{ T_\ast \}^{(p)} \setminus \{ T_\ast \}^{(p-1)}) \Rightarrow K^{p+q}(T).$$

When $V$ is a vector space (over $\mathbb{Q}$) with a linear representation of $S_n$, let us denote its isotypic summand for the sign representation by $V_A$. This is the image of the projection

$$p_a = \frac{1}{n!} \sum_{s \in S_n} (-1)^{|s|} s$$

acting on $V$.

Lemma 4.3. Let $X$ be a locally compact Hausdorff space with an action of $S_n$, such that the stabilizer of any point $x \in X$ contains an odd permutation. Then $H^\bullet(X, \mathbb{Q})_A = 0$. 

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Proof. Let \( Y = X/S_n \), and consider the projection map \( \pi : X \to Y \). We then have \( H^\ast(X, \mathbb{Q}) \cong H^\ast(Y, \pi_*\mathbb{Q}) \). Indeed, \( \pi \) is a proper map with discrete fiber, hence the proper base change theorem applies and gives \( \mathcal{R}\pi_*\mathbb{Q} \cong \pi_*\mathbb{Q} \). Then the claim is a straightforward consequence of \( \mathcal{R}\pi_*\mathbb{Q} \cong \mathcal{R}\pi_*\mathbb{Q} \) for the canonical maps \( \pi^X \) and \( \pi^Y \) from \( X \) and \( Y \) to the singleton space.

Acting by the projection \( p_\ast \), we obtain

\[
H^\ast(X, \mathbb{Q})_A \cong H^\ast(Y, p_*\pi_*\mathbb{Q})
\]

Now, at the level of stalks we have

\[
(p_*\pi_*\mathbb{Q})_y = \Gamma(\pi^{-1}(y), \mathbb{Q})_A = 0
\]

for all \( y \in Y \), hence \( H^\ast(Y, p_*\pi_*\mathbb{Q}) = 0 \). \( \square \)

**Proposition 4.4.** We have

\[
(K^{p+q}(\lfloor T^p \rfloor \setminus \lfloor T^p \rfloor^{p-1})) \otimes \mathbb{Q} \cong (K^q(T^p) \otimes \mathbb{Q})_A
\]

with respect to the natural action of \( S_{p+1} \) on \( T^p \).

**Proof.** By construction, \( \lfloor T^p \rfloor^{p-1} \setminus \lfloor T^p \rfloor \) is the quotient of \( \Delta^p \times T^p_f \) by the diagonal action of \( S_{p+1} \). By \cite{EI2} Proposition 5.5, we have

\[
(K^{p+q}(\lfloor T^p \rfloor^{p-1}) \setminus \lfloor T^p \rfloor) \otimes \mathbb{Q} \cong (K^{p+q}(\Delta^p \times T^p_f)_{S_{p+1}}) \otimes \mathbb{Q}.
\]

On the right hand side, the action of \( S_{p+1} \) on \( K^{p+q}(\Delta^p \times T^p_f) \) can be identified with the twist of the natural one on \( K^q(T^p_f) \) by the sign representation. Moreover, rationalization and taking \( S_{p+1} \)-invariants commute, hence we have \( (K^q(T^p_f) \otimes \mathbb{Q})_A \).

It remains to show \( (K^q(T^p_f) \otimes \mathbb{Q})_A \cong (K^q(T^p) \otimes \mathbb{Q})_A \). By the 6-term exact sequence involving \( K^\ast(T^p_f) \otimes \mathbb{Q} \), \( K^\ast(T^p) \otimes \mathbb{Q} \), and \( K^\ast(T^p \setminus T^p_f) \otimes \mathbb{Q} \), which is \( S_{p+1} \)-equivariant, the claim is equivalent to \( (K^\ast(T^p \setminus T^p_f) \otimes \mathbb{Q})_A = 0 \).

Let \( X \) be the one point compactification of \( T^p \setminus T^p_f \). Then Lemma 4.3 implies that

\[
(K^i(X) \otimes \mathbb{Q})_A \cong \left( \bigoplus_k H^{i+2k}(X, \mathbb{Q}) \right)_A
\]

is trivial for \( i = 0, 1 \), hence we obtain the claim. \( \square \)

**Corollary 4.5.** We have a convergent cohomological spectral sequence

\[
E_1^{pq} = (K^q(T^p) \otimes \mathbb{Q})_A \Rightarrow K^{p+q}(T) \otimes \mathbb{Q},
\]

with the \( E_1 \)-differential \( E_1^{pq} \to E_1^{(p+1)q} \) induced by the simplicial structure of \( T^p \).

**4.2. Equivariant K-theory spectral sequence.** We now consider the case where \( G \) is an étale groupoid with torsion-free stabilizers satisfying the strong Baum–Connes \cite{BP22} conjecture, with base \( T = G(0) \). Suppose that \( T_0 \) is a \( G \)-space, mapping onto \( T \) by a surjective map which is at most \( N \)-to-one. Then \( \lfloor T \rfloor \) is also a locally compact \( G \)-space, and we have an equivariant \( \ast \)-homomorphism \( C_0(T) \to C_0(\lfloor T \rfloor) \).

**Proposition 4.6.** Under the above setting, the inclusion \( C^\ast T_0 \to G \rtimes C_0(\lfloor T \rfloor) \) induces an isomorphism \( K_\ast(G \rtimes C_0(\lfloor T \rfloor)) \cong K_\ast(C^\ast T_0) \).

**Proof.** By Theorem 1.3 and Remark 1.4 we have a spectral sequence

\[
E_1^{pq} = K^q(G^{(p)} \rtimes_T \lfloor T \rfloor) \cong K_q(G^{(p)} \rtimes C_0(\lfloor T \rfloor)) \Rightarrow K_{p+q}(G \rtimes C_0(\lfloor T \rfloor)),
\]

and a similar one for \( C^\ast T_0 \). Now, we observe that \( G^{(p)} \rtimes_T T_0 \) maps onto \( G^{(p)} \) by a finite-to-one map, and

\[
G^{(p)} \rtimes_T \lfloor T \rfloor \cong \lfloor G^{(p)} \rtimes_T T \rfloor.
\]

Combining this with Proposition 1.2 we have

\[
K^q(G^{(p)} \rtimes_T \lfloor T \rfloor) \cong K^q(G^{(p)}).
\]
Thus, the inclusion $C^*G \to G \times C_0([T_\bullet])$ induces an isomorphism of spectral sequences at the $E^1$-sheet.

This time we obtain the convergent spectral sequence

$$E^q_1 = K_{p+q}(G \times C_0([T_\bullet]^{|p|}) \land [T_\bullet]^{|p-1|})) \Rightarrow K_{p+q}(C^*G).$$ (3)

4.3. Soft resolution from simplicial totally disconnected space. Now, suppose that $G$ is an étale groupoid, and $T_0$ is a totally disconnected $G$-space with a proper structure map $g: T_0 \to T$ that is at most $N$-to-one. Our goal here is to show that, given these assumptions, groupoid homology for $G$ is definable. We denote the $(n+1)$-fold fiber product of $T_0$ over $T$ by $T_n$ and by $g_n$ the induced map $T_n \to T$.

**Proposition 4.7.** Let $F$ be a sheaf over $T_n$. Then $(g_n)_*F$ is a $(c)$-soft sheaf on $T$.

**Proof.** To check the $c$-softness of $(g_n)_*F$, we want to show the following property (see [God73, Théorème II.3.4.1]): for any point $x \in T$, there is a neighborhood $U$ of $x$ such that, for any closed set $S$ of $X$ contained in $U$ and $s \in \Gamma(S, (g_n)_*F)$, there is an extension $\tilde{s} \in \Gamma(U, (g_n)_*F)$ of $s$. We take $U$ to be a relatively compact open neighborhood of $x$.

By the second countability and total disconnectedness of $T_n$, $F$ is soft. Then, if $S$ is a closed set as above (hence compact), any section of $F$ on the compact set $g_n^{-1}(S)$ extends to a section on $g_n^{-1}(U)$. We thus need to show that the sections of $(g_n)_*F$ on $S$ can be identified with those of $F$ on $g_n^{-1}(S)$.

We have (see [God73, II.3.3.1])

$$\Gamma(S, (g_n)_*F) = \lim_{V \supset S} \Gamma(g_n^{-1}(V), F), \quad \Gamma(g_n^{-1}(S), F) = \lim_{V' \supset g_n^{-1}(S)} \Gamma(V', F),$$

where $V$ runs over open subsets of $U$ containing $S$, while $V'$ runs over those containing $g_n^{-1}(S)$. We thus have the equality of these spaces if the sets $g_n^{-1}(V)$ form a fundamental system of neighborhoods of $g_n^{-1}(S)$.

Let $V'$ be an open subset of $g_n^{-1}(U)$ containing $g_n^{-1}(S)$. Then $W = g_n^{-1}(U) \setminus V'$ is a compact subset of $T_n$ by the properness of $g_n$. Then $V = U \setminus g_n(W)$ is an open neighborhood of $S$ such that $g_n^{-1}(V) \subset V'$.

We also have the following correspondence for compactly supported sections.

**Proposition 4.8.** Let $F$ be a sheaf of commutative groups on $T_n$. Then $\Gamma_c(S, (g_n)_*F) = \Gamma_c(g_n^{-1}(S), F)$ for any open set $S \subset T$.

**Proof.** This follows from the properness of $g_n$. 

Now, let $F$ be a $G$-sheaf, and put $F^n = (g_n)_*(g_n^*F)$. This is a soft sheaf on $T$ by Proposition 4.7. By Proposition 4.8 we get

$$\Gamma_c((G \times T_n)^{(k)}, s^*g_n^*F) \cong \Gamma_c(G^{(k)}, s^*F^n).$$ (4)

Moreover, pullback along the structure maps of $(T_n)_n$ as a simplicial space induces a structure of cosimplicial object on $(F^n)_{n=0}^\infty$ in the category of sheaves over $T$. Namely, let $d^n_i: Z_n \to Z_{n-1}$ and $s^n_i: Z_n \to Z_{n+1}$ be the standard face and degeneracy maps. Then for any open set $U \subset T$,

$$\delta^n_i|_U : \Gamma(g_n^{-1}(U), g_n^*F) = \Gamma(U, F^{n-1}) \to \Gamma(g_n^{-1}(U), g_n^*F) = \Gamma(U, F^n), \quad \sigma \mapsto \sigma \circ d^n_i,$$

defines a morphism $\delta^n_i: F^{n-1} \to F^n$. Similarly we get $\sigma^n_i: F^{n+1} \to F^n$, and we obtain a cosimplicial structure on $(F^n)_{n=0}^\infty$.

**Proposition 4.9.** The cochain complex of $(F^n)_{n=0}^\infty$ is a resolution of $F$.

**Proof.** We need to check that, for each $t \in T$, the sequence of stalks

$$0 \to F_t \to F^0_t \to F^1_t \to \ldots$$
is exact. Put $Z_t = g_0^{-1}(t)$, which is a finite set by the properness of $g_0 = g$. Then $F^n_t = C(Z_t^{n+1}, F_t)$, and the above complex is given by the canonical simplicial structure on $(Z_t^{n+1})_{n=0}$. This is well-known to be exact.

To have a soft resolution of bounded length, let us introduce the “alternating” model. The natural action of $S_{n+1} \times T_n$ induces an action on $F^n$. We define a subsheaf $F^{n,a} \subset F^n$ as follows:

$$\Gamma(U, F^{n,a}) = \{ x \in \Gamma(U, F^n) \mid \text{supp } x \subset T^n_{n+1}, \forall s \in S_{n+1} : sx = (-1)^{|s|}x \}. $$

**Proposition 4.10.** The sheaf $F^{n,a}$ is again c-soft.

**Proof.** As in the proof of Proposition 4.7, let $U$ be a relatively compact open set of $T$, and $S$ be a compact subset of $U$. Given a section $x \in \Gamma(S, F^{n,a})$, we need to find an extension $\tilde{x} \in \Gamma(V, F^{n,a})$ of $x$ to some open neighborhood $V$ of $S$.

By Proposition 4.7 we can find an open neighborhood $V_0$ of $S$ and a section $\tilde{x}_0 \in \Gamma(V_0, F^n)$ restricting to $x$. Let $\tilde{x}_0 \in \Gamma(g^{-1}_n(V_0), g^{-1}_n F)$ be the section corresponding to $\tilde{x}_0$. It is enough to find a $S_{n+1}$-invariant open neighborhood $V'$ of $g^{-1}_n(S)$ such that the restriction $\tilde{x}_0|_{V'}$ satisfies $s(\tilde{x}_0|_{V'}) = (-1)^{|s|}\tilde{x}_0|_{V'}$ for $s \in S_{n+1}$. Indeed, the argument of the proof of Proposition 4.7 gives a neighborhood $V$ of $S$ satisfying $g^{-1}_n(V) \subset V'$, and the restriction of $\tilde{x}_0$ to $V$ is a section of $F^{n,a}$.

For each $u \in g^{-1}_n(S)$, find an open neighborhood $V_u$ such that $s^*\tilde{x}_0 = (-1)^{|s|}\tilde{x}_0$ holds on $V_u$ for any $s \in S_{n+1}$. Moreover, if $u$ is not in $T^n_{n+1}$, we ask $\tilde{x}_0$ to be trivial on $V_u$. This is possible since the stalks $s^*\tilde{x}_0(u)$ are the inductive limits of $s^*\tilde{x}_0$ around open neighborhoods of $u$. By compactness there are finitely many $u_1, \ldots, u_k$ such that $V' = \bigcup_i V_{u_i}$ covers $g^{-1}_n(S)$. □

A standard argument shows that $(F^{n,a})_n$ is a subcomplex of $(F^n)_n$. By assumption about the support and $T^n_{n+1}$, this is a bounded complex.

**Proposition 4.11.** The complex $(F^{n,a})_n$ is still a resolution of $F$.

**Proof.** Again we need to check the exactness of the augmented complex of stalks,

$$0 \to F_t \to F_t E^{0,a} \to F_t E^{1,a} \to \cdots \to F_t E^{N,a} \to 0.$$

The stalk of $F^{n,a}$ at $t$ is given by

$$F^{n,a}_t = \{ f \in C(Z_t^{n+1}, F_t) \mid \text{supp } f \subset Z_t^{n+1} \setminus D_{n+1}, \forall s \in S_{n+1} : sf = (-1)^{|s|}f \}$$

for $Z_t = g_0^{-1}(t)$, where $D_{n+1}$ is the degenerate part (the union of non-free orbits). A usual normalization argument for simplicial complexes (for example [352, Theorem VI.6.9]), together with the fact that $(Z_t^{n+1})_{n=0}$ is a contractible simplicial set, shows that the above augmented complex is indeed exact.

This allows us to define $H_\bullet(G, F)$, as follows.

**Proposition 4.12.** Let $F$ be a $G$-sheaf. Then the groupoid homology $H_\bullet(G, F)$ is well-defined as the homology of the double complex with components

$$\Gamma_c(G^{(k)}, s^* F^{n,a}), \quad n, k \geq 0$$

with total degree $k - n$.

5. **Application to Smale spaces**

Now we are ready to apply the above machinery to Smale spaces. Let us start with some preliminaries. We fix an $s/u$-bijective pair $(Y, \psi, f, Z, \zeta, g)$ over $(X, \phi)$ as before. Consider a subspace $T \subset X$ and suppose it is a transversal for the unstable equivalence relation. By Proposition 4.12, the étale groupoid $G = R^n(X, \phi)|_T$ indeed satisfies the assumption about $G$-sheaves to define homology.

**Proposition 5.1.** We can choose $T$ so that $T_n = g_n^{-1}(T)$ is a transversal for the unstable equivalence relation on $Z_n$ for all $n$. 

Proof. We choose a periodic point \( x_0 \in X \), and \( T \) to be a local stable set around \( x_0 \). Then \( T_n \) is a union of local stable sets around the points of \( g_n^{-1}(x_0) \) [Put15 Lemma 5.2.10]. Now, let \( z \in \mathbb{Z}_n \). Since \( T \) is a transversal, we can find \( z_1 \in T \) such that \( g_n(z) \sim_u x_1 \). As \( g_n \) is \( u \)-bijective, we can find \( z_1 \sim_u z \) such that \( g_n(z_1) = x_1 \). Our choice of \( T \) works if the system is mixing. If it is not, we use Smale’s decomposition theorem [Put10 Theorem 2.1 and the following discussion] and apply the argument above to each mixing factor.

In the following we will always use \( T \) satisfying the claim of Proposition 5.1. Put \( G = R^u(X, \phi)|_T \). Denote by \( \mathbb{Z} \) by the locally constant \( G \)-equivariant sheaf of integers over \( X \).

Proposition 5.2. The transversal \( T_n \subset Z_n \) has an action of \( G \) with anchor map \( g_n : T_n \to T \). We have \( G \ltimes T_n \cong R^u(Z_\mathbb{Z}_n, \zeta_n)|_{T_n} \).

Proof. Let \( z \in T_n \), \( x = g_n(z) \), and \( (x', x) \in G \). Since \( g_n \) is \( u \)-bijective, there is a unique point \( z' \in Z_n \) such that \( z' \sim_u z \) and \( g(z') = x' \). We define the action of \( (x', x) \) on \( z \) to be this \( z' \). Conversely, if \( z, z' \in T_n \) are unstably equivalent, \( z' \) is the image of the action of \( (g(z'), g(z)) \) on \( z \). This gives the identification of groupoids.

Recall that \( g_n \) is proper on stable sets [Put15 Theorem 5.2.5]. In particular, \( g_n \) is proper as a map \( T_n \to T \).

5.1. Comparison of homologies. Let us establish a generalization of Theorem 1.6

Lemma 5.3. For each \( M \), the natural map \( \Sigma_{0,M} \to Z_M \) is s-bijective. Moreover, we have

\[
\Sigma_{L,M} = \Sigma_{0,M} \times_{Z_M} \cdots \times_{Z_M} \Sigma_{0,M}
\]

Proof. Let \( z = (z_0, \ldots, z_M) \) and \( z' = (z'_0, \ldots, z'_M) \) be points of \( Z_M \) which are stably equivalent. This condition is equivalent to component-wise equivalences

\[
z_0 \sim_s z'_0, \ldots, \quad z_M \sim_s z'_M.
\]

Suppose that we are given a point in the fiber \( (\Sigma_{0,M})_z \), that is, a tuple of the form

\[
(y_0, z_0, \ldots, z_M) \quad (f(y_0) = g(z_0) = \cdots = g(z_M)).
\]

Write \( x = f(y_0) = g(z_0) \) and \( x' = g(z'_0) \). Then there is a unique point \( y'_0 \) in the stable equivalence class of \( y \) which maps to \( x' \). Then \( (y'_0, z'_0, \ldots, z'_M) \) is the unique point of \( \Sigma_{L,M} \) stably equivalent to \( (y_0, z_0, \ldots, z_M) \) and maps to \( z' \). The relation between \( \Sigma_{L,M} \) and the fibered product of \( \Sigma_{0,M} \) over \( Z_M \) is obvious.

Now, take a transversal \( T \subset X \) as in Proposition 5.1 and consider the étale groupoid \( G = R^u(X, \phi)|_T \). On the one hand, \( T_i = g_i^{-1}(T) \subset Z_i \) is a totally disconnected \( G \)-space. On the other, the factor map \( Y_j \to X \) defines an open subgroupoid \( H_j < G \) for each \( j \). Similarly, \( \Sigma_{j,i} \to Z_i \) defines an open subgroupoid \( H_{j,i} < G \ltimes T_i \).

Proposition 5.4. The groupoid \( H_j \) is Morita equivalent to \( H_0^{\times G(j+1)} \). Moreover, \( H_{j,i} \) is the transformation groupoid \( H_j \ltimes T_i \), which is also Morita equivalent to the multiple groupoid pullback of \( (j+1) \) copies of \( H_0 \) over \( G \ltimes T_i \).

Proof. The main ingredient of the proof is the following transversality [PY21 Proposition 2.9]: given \( y_0, \ldots, y_j \) such that \( f(y_i) \) are mutually unstably equivalent in \( X \), we can find \( y'_0, \ldots, y'_j \) such that \( y'_i \) is unstably equivalent to \( y_i \) in \( Y \), and additionally \( f(y'_i) = f(y'_i) \), for each \( i \). The first statement is proved in [PY21 Theorem 2.8]. The rest can be proved in similar ways.

Theorem 5.5. We have \( H_k(G, \mathbb{Z}) \cong H_k^u(X, \phi) \) for all \( k \in \mathbb{Z} \).
Proof. Consider the inclusion map \( f_j : H^{xG(j+1)} \to G^{xG(j+1)} \). We obtain a bicomplex of \( G^{xG(j+1)} \)-sheaves \( F_{j*} = \mathcal{L}(f_j)_! \mathbb{Z} \) [CM00, Section 4.2]. This is defined through a choice of c-soft resolution \( S^* \) of \( \mathbb{Z} \) as \( H^{xG(j+1)} \)-sheaves, then by taking the bicomplex \( B_*(f_{j*}, S^*) \), where \( B_n(f_{j*}, F) \) is defined as \( (\beta_n)\alpha_n F \) for

\[
G^0(n) \xrightarrow{\beta_n=\times\pi_2} K(n) \times_{G(0)} G^0(n_0) \xrightarrow{\alpha_n=r\pi_3} K(0)
\]

with \( K = H^{xG(j+1)} \), \( G' = G^{xG(j+1)} \), and \( \pi_i \) denoting the projections to \( K(n) \) and \( G' \).

By the Morita equivalence of Proposition [1] we can regard \( F_{j*} \) as a complex of \( G \)-sheaves. Suppose that \( S^* \) is given by a bounded c-soft resolution of \( \mathbb{Z} \) as a \( G^{xG(j+1)} \)-sheaf. Then the stalk of \( F_{j*}^i \) at \( x \in G(0) \) is given by

\[
(F_{j*}^i)_x = F_{j*}(S^*)_x = \Gamma_c((H^{xG(j+1)}(n) \times_{G(j)} (G^{xG(j+1)})_x), S_x^i),
\]

where we identified the base of \( G^{xG(j+1)} \) with \( G(j) \), and \( x \) with \( (x, id_x, \ldots, x) \) to make sense of \( (G^{xG(j+1)})_x \).

Recall that we have a bounded c-soft resolution \( F^{*,a} \) of the \( G \)-sheaf \( \mathbb{Z} \) from the simplicial totally disconnected \( G \)-space \( T_x \). We will use the corresponding \( G^{xG(j+1)} \)-sheaves \( S^* \) in the above scheme.

Moreover, we get a homological differential in the \( j \)-direction from the simplicial structure, hence we obtain a triple complex \( F_{**}^* \) of c-soft \( G \)-sheaves. We compute the groupoid hyperhomology \( \mathbb{H}_*(G, F_{**}^*) \) in two ways to obtain the claim of the theorem.

To relate this to the groupoid homology \( H_*(G, \mathbb{Z}) \), it is enough to see that the complex \( F_{**}^* \) is quasi-isomorphic to \( \mathbb{Z} \) (at degree 0), or equivalently, to the complex \( F^{*,a} \).

Given a \( G \)-sheaf \( S \) and any \( n \), we claim that the complex of \( G \)-sheaves \( F_{**}^n(S) \), defined analogously to \( F_{j*}^i \) above but with \( S \) instead of \( S^* \), form a homological resolution of \( S \). This is a stalk-wise computation, and we just need to check that the complex

\[
\cdots \to F_{1n}(S)_x \to F_{0n}(S)_x \to S_x \to 0
\]

is exact. From \( G \), we see that \( F_j(S)_x \) is the space of finitely supported maps from \( A_x^{j+1} \) to \( S_x \), where

\[
A_x = \{(h, g) \mid h \in H(n), g \in G_x, s(h) = r(g)\}.
\]

Then we can construct a contracting homotopy \( h : F_{j*}(S)_x \to F_{j+1,n}(S)_x \) for the above complex, by fixing \( a \in A_x \) and setting

\[
h(c)(a_0, \ldots, a_{j+1}) = \delta_{a,a_0} c(a_1, \ldots, a_{j+1}).
\]

Next, if we treat \( F_{**}^n(S) \) as a bicomplex, this is still a resolution of \( S \). Indeed the above already shows that this is quasi-isomorphic to a complex with terms \( S \) at each nonnegative degree. Taking into account the differential in the \( n \)-direction, we see that it is

\[
\cdots \to S \xrightarrow{id} S \xrightarrow{0} S \xrightarrow{0} 0.
\]

Using the above argument for \( S = F^{*,a} \), we get that \( F_{**}^n = (F_{j*}^i)_{i,j,n} \) is a resolution of \( F^{*,a} \), hence we get the isomorphism \( \mathbb{H}_*(G, F_{**}^*) \cong H_*(G, \mathbb{Z}) \).

It remains to show that our hyperhomology also computes \( H_*(X, \phi) \). By definition \( \mathbb{H}_*(G, F_{**}^*) \) is the homology of the quadruple complex with terms

\[
E_{jnk}^i = \Gamma_c(G^{(k)}, s^*F_{jn}^i)
\]

with total degree \( j + n + k - i \). The claim follows if we can show that the homology in the \((n,k)\)-direction gives \( D(\Sigma_{j,i})_A \) concentrated at degree 0 when we fix \( j \) and \( i \).

By the Morita equivalence between \( H_j \) and \( H^{xG(j+1)} \), this homology can be interpreted as \( \mathbb{H}_*(G, \mathcal{L}(f_j)) F^{*,a} \) if we regard \( F^{*,a} \) as an \( H_j \)-sheaf. Then by \( \mathbb{H}_*(G, \mathcal{L}(f_j), F^{*,a}) \cong H_*(H_j, F^{*,a}) \), our claim reduces to the computation of \( H_*(H_j, F^{*,a}) \).
Let $C_n$ denote the subgroup of $\Gamma_c(H_{j,i}^{(k)}, \mathbb{Z})$ consisting of the sections $x$ supported on the free orbits and such that $sz = (-1)^{|s|}x$ for $s \in S_{i+1}$. Then $H_*(H_j, F^{i,a})$ is the homology of $C_*$ by the c-softness of $F^{i,a}$. Hence we want to show

$$H_0(C_\ast) \cong D^s(\Sigma_{j,i}),A, \quad H_k(C_\ast) = 0 \quad (k > 0).$$

Let $H'_{j,i}$ be the quotient groupoid $H_{j,i}/S_{i+1}$. On the one hand, $H'_{j,i}$ is AF because $H_{j,i}$, which is Morita equivalent to $R^u(\Sigma_{j,i}, \sigma_{j,i})$, is AF. (Note that the quotient by $S_{i+1}$ does not create stabilizers.) On the other, $H_*(H_j, F^{i,a})$ can be regarded as the groupoid homology of $H'_{j,i}$ with the induced action of $F^{i,a}$. Then the higher homology groups vanish by Proposition 5.2.

Finally, the 0-th homology is the group of $H_j$-coinvariants of $C_0 = \Gamma_c(T, F^{i,a})$. By the concrete description of $D^s(\Sigma_{j,i}),A$ given by [Put14] Theorems 3.3.3 and 4.2.8, we have $H_0(C_\ast) \cong D^s(\Sigma_{j,i}),A$. More precisely, any element of the alternating subgroup of $D^s(\Sigma_{j,i})$ (with respect to the action of $S_{i+1}$) is represented by a linear combination of compact open subsets of a stable set with alternating coefficients. Such compact open sets must belong to the free part $T_i^f \subset T_i$.

Combining this with results from Sections 2 and 3, we obtain the following results.

**Theorem 5.6.** Let $(Y_1, \psi_1)$ and $(Y_2, \psi_2)$ be non-wandering Smale spaces. Then we have a split exact sequence

$$0 \to \bigoplus_{a+b=k} H^s_a(Y_1, \psi_1) \otimes H^b_k(Y_2, \psi_2) \to H^k(Y_1 \times Y_2, \psi_1 \times \psi_2) \to \bigoplus_{a+b=k-1} \text{Tor}(H^s_a(Y_1, \psi_1), H^b_k(Y_2, \psi_2)) \to 0.$$

**Theorem 5.7.** Let $(X, \phi)$ be a Smale space such that the unstable sets $X^u(x)$ are homeomorphic to $\mathbb{R}^n$, for some fixed $n$. Then we have an isomorphism

$$H^{n-k}_k(X, \phi) \cong H^{n-k}(X, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \langle \underline{\omega} \rangle),$$

where $\underline{\omega}$ is the unstable orientation sheaf.

This gives a partial answer to [Put14] Question 8.3.2, see also [APSG17] Remark at the end of Section 1.

**Example 5.8.** As a concrete example, let us consider the Smale spaces associated to hyperbolic toral automorphisms $(X, \phi) = (\mathbb{R}^2/\mathbb{Z}^2, A)$, where $A$ is a 2-by-2 matrix with integer entries and determinant equal to 1. This implies the $\mathbb{R}$-linear endomorphism associated to $A$ descends to a map of the 2-torus. Note that $A$ is called “hyperbolic” when its eigenvalues $\lambda_1, \lambda_2$ of $A$ satisfy $\lambda_1 > 1, \lambda_2 < 1$. The stable and unstable orbits of $A$ coincide with the lines spanned by the eigenvectors associated to $\lambda_1$ and $\lambda_2$.

Combining Theorems 5.6 and 5.7 we obtain

$$H^s_1(X, \phi) \cong \mathbb{Z}, \quad H^2_0(X, \phi) \cong \mathbb{Z}^2, \quad H^s_1(X, \phi) \cong \mathbb{Z}.$$

Note that this computation is much more involved if one follows the definition of Putnam homology (see [Put14] Section 7.4]). Even though the HK conjecture is only formulated for ample groupoids, it is worth pointing out that this homology calculation corresponds (after periodization) to the $K$-groups of the unstable $C^*$-algebra associated to $(X, \phi)$. Indeed, this algebra is the foliation $C^*$-algebra of the Kronecker flow along the $\lambda_1$-eigenline, hence it is Morita equivalent to the rotation algebra with angle the slope of the eigenline. The $K$-theory of this algebra is well-known to be $\mathbb{Z}^2$ in both even and odd degree.

**5.2 Smale spaces with totally disconnected unstable sets.** Now, let $(X, \phi)$ be a Smale space with totally disconnected unstable sets, and let us fix a $u$-bijective map $f: (\Sigma, \sigma) \to (X, \phi)$ from a shift of finite type. Let us take the constructions from Section 4.3 with $(\mathbb{Z}, \zeta) = (\Sigma, \sigma)$. In particular, we have an étale groupoid $G = R^u(X, \phi)|_T$ for a transversal $T$ as in Proposition 5.1 and the sheaves $(F^{n,a})_n$ giving a resolution of $\mathbb{Z}_T$.
Recall also that the Smale spaces \((\Sigma_n, \sigma_n)\) admit presentation by certain graphs \([\text{Put}14]\). One starts from a graph \(\mathcal{G}\) presenting \((\Sigma, \sigma)\) with vertex set \(\mathcal{G}^0\) and edge set \(\mathcal{G}^1\). Then one defines graphs \(\mathcal{G}_n\) representing \((\Sigma_n, \sigma_n)\), whose vertex sets are subsets of \((\mathcal{G}^0)^{n+1}\), and the edge sets are subsets of \((\mathcal{G}^1)^{n+1}\). There are natural actions of \(S_{n+1}\) on \((\mathcal{G}^i)^{n+1}\) inducing the one on \(\Sigma_n\).

**Proposition 5.9.** We have

\[
K_i(G \ltimes C_0(\bigcup \mathcal{T}_c^{(p)} \setminus \bigcup \mathcal{T}_c^{(p-1)})) \cong H_0(G, F^{p,a})
\]

when \(i \equiv p \mod 2\) and \(K_i(G \ltimes C_0(\bigcup \mathcal{T}_c^{(p)} \setminus \bigcup \mathcal{T}_c^{(p-1)})) = 0\) otherwise.

**Proof.** Put \(A = T_p/S_{p+1}\) and \(B = \bigcup \mathcal{T}_c^{(p)} \setminus \bigcup \mathcal{T}_c^{(p-1)}\). As \(A\) is a totally disconnected \(G\)-space, \(G \ltimes A\) is an ample groupoid. Moreover the \(G\)-\(C^*\)-algebra \(C_0(B)\) is naturally a \(G \ltimes \mathfrak{A}\)-\(C^*\)-algebra.

By Theorem [1.3] and Remark [1.4] we have a spectral sequence

\[
E_{pq}^2 = H_p(G \ltimes A, K_q(C_0(B))) \Rightarrow K_{p+q}(G \ltimes C_0(B)).
\]

Moreover, as in the proof of Theorem [5.5], \(G \ltimes A\) is an AF groupoid. By Proposition [1.2] we have \(E_{pq}^2 = 0\) for \(p > 0\) (for the above spectral sequence, hence it collapses at the \(E^2\)-sheet). It remains to identify \(H_0(G \ltimes A, K_q(C_0(B)))\) with \(H_0(G, F^{p,a})\) or 0 depending on the parity of \(q\).

By Morita invariance of groupoid homology, we may replace the transversal \(T^p \subset \Sigma_p\) by another \(S_{p+1}\)-invariant transversal. For each \(z \in \mathcal{G}^0 \subset (\mathcal{G}^0)^{p+1}\), let \(A_z\) be the space of one-sided infinite paths representing a local stable set in \(\Sigma_p\). As a new transversal, take one that can be identified with the union of the \(A_z\), and call it \(T'\). Then the free part \(T'^f\) is the subset of paths that pass through a vertex \(w \in \mathcal{G}^0\) which is in a free orbit of the \(S_{p+1}\) action.

Now, let \(B'\) be the quotient of \(\Delta^p \times T'^f\) which corresponds to \(B\) under the Morita equivalence. It is enough to check that the induced homomorphism

\[
j : K_q(C_0(B')) \to K_{q-p}(C_0(T'^f)),
\]

is injective, and identifies \(K_{q-p}(C_0(T'_p))\) with either \(\Gamma_c(T'_p, \mathbb{Z})\) or 0 depending on \((q-p) \mod 2\). By the above presentation of \(T'^f\), the \(S_{p+1}\)-\(C^*\)-algebra \(C_0(T'^f)\) is the union of \(S_{p+1}\)-invariant finite-dimensional subalgebras of the form \(C(K_n)\), where \(K_n\) is a free \(S_{p+1}\)-set. (Concretely, \(K_n\) is the set of paths of length \(n \in \mathcal{G}_p\) passing through a free orbit.) Let \(B_n\) be the quotient of \(\Delta^p \times K_n\). Then \(C_0(B_n)\) is again the union of \(C(B_n)\) which implies \(K_q(C_0(B)) = \lim_{n \to \infty} K_q(C(B_n))\). On \(K_q(C_0(B_n))\), the analogous claim about \(j\) is obvious. \(\square\)

We now have a dual analogue of the homological spectral sequence in Theorem [1.5].

**Corollary 5.10.** Let \((X, \phi)\) be as above. There is a cohomological spectral sequence abutting to \(K_*(C^* R^a(X, \phi))\), with \(E_2\)-sheet given by \(H_*(X, \phi)\). More precisely:

\[
E_{pq}^2 = H_{pq}(X, \phi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^* R^a(X, \phi)).
\]

**Proof.** The \(E_2\)-sheet of the spectral sequence \([3]\) is given by the cohomology of \(H_0(G, F^{p,a})\) for even \(q\), and trivial for odd \(q\).

We first claim that, up to inverting the degree, this cohomology is equal to \(H_*(G, \mathbb{Z})\). Indeed, as \((F^{p,a})_p\) is a resolution of \(\mathbb{Z}\), the groupoid homology is computed from the double complex with terms \(\Gamma_c(G^0, F^{p,a})\). When we fix \(p\), the resulting homological complex has homology groups

\[
H_q(G, F^{p,a}) \cong H_q(G \ltimes A, F^c)
\]

for \(A = T_p/S_{p+1}\) as before, and the \(G \ltimes A\)-sheaf \(F^c\) corresponding to \(F^{p,a}\). Again by Proposition [1.2] this homology is trivial for \(q > 0\). Then the homology of the total complex is the same as the cohomology of the cochain complex \(H_0(G, F^{**a})\).

Thus, we have

\[
E_{pq}^2 = H_{-p}(G, \mathbb{Z}) \quad (q \text{ even}),
\]

\[
E_{pq}^2 = 0 \quad (q \text{ odd}),
\]

which implies the claim. \(\square\)
5.3. Finiteness for general Smale spaces. In general, we cannot directly relate the $K$-groups of $C^* R^n(X, \phi)$ to the groupoid homology of $R^n(X, \phi)|_{T}$. However, the considerations from Section 4 still lead to the following finiteness result.

**Theorem 5.11.** Let $(X, \phi)$ be a Smale space. Then $K_\bullet(C^* R^n(X, \phi))$ is of finite rank.

**Proof.** Let us fix a $u$-bijective map $g: (Z, \zeta) \to (X, \phi)$ from another Smale space with totally disconnected stable sets, with $g$ at most $N$-to-one. Take a transversal $T \subset X$ as in Proposition 5.1 and consider the étale groupoid $G = R^n(X, \phi)|_T$ and the totally disconnected $G$-space $T_0 = g^{-1}(T)$. Let us take the constructions from Sections 4.3 and 4.2. Then the rationalization of the cohomological spectral sequence (3) becomes

$$E^1_{pq} = K_{p+q}(G \rtimes C_0([T_0\setminus \{\phi\}]\setminus T_0^{(p-1)})) \otimes \mathbb{Q} \Rightarrow K_{p+q}(C_\ast G) \otimes \mathbb{Q}. \quad (6)$$

Let us fix $p$. Again by Theorem 1.3, there is a homological spectral sequence

$$E^1_{pq} = K_{p+q}(G \rtimes C_0([T_0\setminus \{\phi\}]\setminus T_0^{(p-1)})) \Rightarrow K_{p+q}(G \rtimes C_0([T_0\setminus \{\phi\}]\setminus T_0^{(p-1)}))$$

compatible with the actions of $S_{p+1}$. By Proposition 4.4, rationalization at the $E^1$-sheet agrees with

$$(K_{p+q}(G \rtimes C_0(G^{\{\phi\}} \rtimes T_p)) \otimes \mathbb{Q})_A,$$

which converges to $(K_q(C^*(G \rtimes T_p)) \otimes \mathbb{Q})_A$. This implies that the spectral sequence (6) satisfies

$$E^1_{pq} \cong (K_q(C^*(G \rtimes T_p)) \otimes \mathbb{Q})_A.$$

The right hand side can be considered as a submodule of $K_q(C^*(R^n(Z_p, \zeta_p))) \otimes \mathbb{Q}$. On one hand, we know that $K_q(C^*(R^n(Z_p, \zeta_p)))$ is of finite rank by [PY21, Corollary 3.10]. On the other, we have $E^1_{pq} = 0$ for $p > N$, hence the assertion. \hfill \square

Recall that taking the inverse homeomorphism we get the stable equivalence relation:

$$R^s(X, \phi) = R^n(X, \phi^{-1}).$$

In particular, $K_\bullet(C^* R^n(X, \phi))$ is also of finite rank by the above theorem.

**Remark 5.12.** In [KPW17], Kaminker, Putnam, and Whittaker considered $K$-theoretic duality for the Ruelle algebras

$$\mathcal{R}_s(X, \phi) = C^*(R^s(X, \phi)) \rtimes_{\phi} \mathbb{Z}, \quad \mathcal{R}_u(X, \phi) = C^*(R^u(X, \phi)) \rtimes_{\phi} \mathbb{Z}.$$

They showed that these algebras are odd Spanier–Whitehead dual to each other, and that this duality implies $\mathcal{R}_s(X, \phi) \cong \mathcal{R}_u(X, \phi)$ under the assumption that $K_\bullet(C^* R^s(X, \phi))$ and $K_\bullet(C^* R^u(X, \phi))$ are of finite rank. They have also conjectured that this assumption is unnecessary, and our result above implies that this is indeed the case.

**Remark 5.13.** In view of the classification program of $C^*$-algebras and the results in [DST18, DGY20], it is natural to ask whether the range of $K$-theory on the class of (simple) classifiable, real rank zero $C^*$-algebras is exhausted by the (un)stable algebras of Smale spaces up to restricting to a transversal and passing to hereditary subalgebras. However, the above finiteness implies that this is not possible. Indeed, there are algebras as above whose $K$-groups are not of finite rank. For example, the simple AF algebra associated to the following graph

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or $A^{\otimes \infty}$ for a classifiable algebra $A$ whose $K$-groups have rank more than 1 will do.
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Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO, JAPAN
Email address: valerio@ms.u-tokyo.ac.jp

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, 0316 OSLO, NORWAY
Email address: makotoy@math.uio.no