Nonlocal Conformable-Fractional Differential Equations with a Measure of Noncompactness in Banach Spaces

1. Introduction

Many dynamical processes in physics, biology, economics, and other areas of applications can be governed by abstract ordinary differential evolution equations of the following form:

\[ \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)), \quad x(0) = x_0 + g(x), \quad t \in [0, T], \]

where \( d^\alpha(.)/dt^\alpha \) is the conformable fractional derivative of the order \( \alpha \in [0, 1] \). The linear part \( A \) is the infinitesimal generator of a uniformly continuous semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \). and \( g \) is a given function. The nonlocal condition attracts the attention of many authors in several works [27, 28]. The vector \( x_0 \) is an element of \( X \) and \( g: \mathcal{C} \to X \) is a given function, with \( \mathcal{C} \) is the space of continuous functions \( x(.) \) defined from \([0, T]\) into \( X \). Throughout this paper, we endow the space \( \mathcal{C} \) with the norm \( \| x \| = \sup_{t \in [0, T]} \| x(t) \| \). It is well known that the space \( (\mathcal{C}, \| . \|) \) is a Banach space. We also denote by \( \| . \| \) the norm in the space \( \mathcal{L}(X) \) of bounded operators defined form \( X \) into itself.
Our goal in this paper is to prove the existence of mild solutions for the Cauchy problem (2) by means of the Darbo–Sadovskii fixed point theorem without assuming the compactness of the family \( (T(t))_{t \geq 0} \) and the Lipschitz condition on the nonlocal part \( g \).

The content of this paper is organized as follows. In section 2, we recall some preliminary facts on the conformable fractional calculus and measure of noncompactness. Section 3 is devoted to prove the main result.

2. Preliminaries

Recalling some preliminary facts on the conformable fractional calculus.

**Definition 1** (see [10]). Let \( \alpha \in (0, 1] \). The conformable fractional derivative of order \( \alpha \) of a function \( x(\cdot) \) for \( t > 0 \) is defined as follows:

\[
\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\epsilon \to 0} \frac{x(t + \epsilon t^{1-\alpha}) - x(t)}{\epsilon},
\]

For \( t = 0 \), we adopt the following definition:

\[
\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{\epsilon \to 0^+} \frac{d^\alpha x(t)}{dt^\alpha}.
\]

The fractional integral \( I^\alpha(\cdot) \) associated with the conformable fractional derivative is defined by

\[
I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s)ds.
\]

**Theorem 1** (see [10]). If \( x(\cdot) \) is a continuous function in the domain of \( I^\alpha(\cdot) \), then we have

\[
\frac{d^\alpha I^\alpha(x)(t)}{dt^\alpha} = x(t).
\]

**Definition 2** (see [8]). The Laplace transform of a function \( x(\cdot) \) is defined by

\[
\mathcal{L}(x(t))(\lambda) = \int_0^{\infty} e^{-\lambda t} x(t)dt, \quad \lambda > 0.
\]

It is remarkable that the above transform is not compatible with the conformable fractional derivative. For this, the adapted transform is given by the following definition.

**Definition 3** (see [11]). The fractional Laplace transform of order \( \alpha \in (0, 1] \) of a function \( x(\cdot) \) is defined by

\[
\mathcal{L}_\alpha(x(t))(\lambda) = \int_0^{\infty} t^{\alpha-1} e^{-\lambda t^{\alpha}} x(t)dt, \quad \lambda > 0.
\]

The following proposition gives us the actions of the fractional integral and the fractional Laplace transform on the conformable fractional derivative, respectively.

**Proposition 1** (see [11]). If \( x(\cdot) \) is a differentiable function, then we have the following results:

\[
I^\alpha \left( \frac{d^\alpha x(\cdot)}{dt^\alpha} \right)(t) = x(t) - x(0),
\]

\[
\mathcal{L}_\alpha \left( \frac{d^\alpha x(t)}{dt^\alpha} \right)(\lambda) = \lambda \mathcal{L}_\alpha(x(t))(\lambda) - x(0).
\]

According to [15], we have the following remark.

**Remark 1.** For two functions \( x(\cdot) \) and \( y(\cdot) \), we have

\[
\mathcal{L}_\alpha \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) x(s)ds \right)(\lambda) = \mathcal{L}_\alpha(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda).
\]

Now, we recall some concepts on the Hausdorff measure of noncompactness.

**Definition 4** (see [29, 30]). For a bounded set \( B \) in a Banach space \( X \), the Hausdorff measure of noncompactness \( \sigma \) is defined as

\[
\sigma(B) = \inf \{ \varepsilon > 0 : B \text{ can be covered by a finite number of balls with radius } \varepsilon \}.
\]

The following lemma presents some basic properties of the Hausdorff measure of noncompactness.

**Lemma 1** (see [29, 30]). Let \( X \) be a Banach space and \( B, C \subset X \) be bounded. Then, the following properties hold.

1. \( B \) is precompact if and only if \( \sigma(B) = 0 \);
2. \( \sigma(B) = \sigma(\overline{B}) = \sigma(\text{conv}(B)) \), where \( \overline{B} \) and \( \text{conv}(B) \) mean the closure and convex hull of \( B \), respectively;
3. \( \sigma(B) \leq \sigma(C) \), where \( B \subset C \);
4. \( \sigma(B + C) \leq \sigma(B) + \sigma(C) \), where \( B + C = \{ x + y : x \in B, y \in C \} \);
5. \( \sigma(B \cup C) \leq \text{max}\{\sigma(B), \sigma(C)\} \);
6. \( \sigma(\lambda B) = |\lambda|\sigma(B) \) for any \( \lambda \in \mathbb{R} \), when \( X \) be a real Banach space;
7. If the operator \( Q : D(Q) \subset X \to Y \) is Lipschitz continuous with constant \( k \geq 0 \) then we have \( \rho(Q(B)) \leq k\sigma(B) \) for any bounded subset \( B \subset D(Q) \), where \( Y \) is another Banach space and \( \rho \) represents the Hausdorff measure of noncompactness in \( Y \).

**Definition 5** (see [30]). The operator \( Q : D(Q) \subset X \to X \) is said to be a \( \sigma \)-contraction if there exists a positive constant \( k < 1 \) such that \( \sigma(Q(B)) \leq k\sigma(B) \) for any bounded closed subset \( B \subset D(Q) \).

**Lemma 2** (see [29, 30] (Darbo–Sadovskii theorem)). Let \( B \subset X \) be a bounded, closed, and convex set. If \( Q : B \to B \) is a continuous and \( \sigma \)-contraction operator. Then, \( Q \) has at least one fixed point in \( B \).
Lemma 3 (see [31, 32]). Let $D \subset X$ be a bounded set, then there exists a countable set $D_0 \subset D$ such that $\sigma(D) \leq 2\sigma(D_0)$.

We denote by $\sigma_c$ the Hausdorff measure of non-compactness in the space $\mathcal{C}$ of continuous functions $x(.)$ defined from $[0, r]$ into $X$.

Lemma 4 (see [33]). Let $D_0 = \{x_n\} \subset \mathcal{C}$ be a countable set, then

1. $\sigma(D_0(t)) = \sigma(\{x_n(t)\})$ is Lebesgue integral on $[0, r]$,
2. $\sigma(\int_0^r D_0(s)ds) \leq 2 \int_0^r \sigma(D_0(s))ds$, where $\sigma(\int_0^r D_0(s)ds) = \sigma(\int_0^r x_n(s)ds)$.

Lemma 5 (see [29]). Let $D \subset \mathcal{C}$ be bounded and equi-continuous, then

1. $\sigma(D(t))$ is continuous on $[0, r]$,
2. $\sigma_c(D) = \max_{t \in [0, r]}(\sigma(D(t)))$.

3. Main Result

We first give the definition of mild solutions for the Cauchy problem (2). To do so, applying the fractional Laplace transform in equation (2), we obtain

$$\lambda \mathcal{L}_\alpha(x(t))(\lambda) = x_0 + g(x) + A\mathcal{L}_\alpha(x(t))(\lambda) + \mathcal{L}_\alpha(f(t, x(t)))(\lambda).$$

(12)

Then, one has

$$\mathcal{L}_\alpha(x(t))(\lambda) = (\lambda - A)^{-1} (x_0 + g(x)) + (\lambda - A)^{-1} \mathcal{L}_\alpha(f(t, x(t)))(\lambda).$$

(13)

Using the inverse fractional Laplace transform combined with Remark 1, we obtain

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right) \{x_0 + g(x)\} + \int_0^t s^\alpha T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$  

(14)

Motivate by the above calculus, we can introduce the following definition.

Definition 6. A function $x \in \mathcal{C}$ is called a mild solution of the Cauchy problem (2) if

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right) \{x_0 + g(x)\} + \int_0^t s^\alpha T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$  

(15)

To obtain the existence of mild solutions, we will need the following assumptions:

(H1) The function $f(t, .): X \rightarrow X$ is continuous, and for all $r > 0$ there exists a function $\mu_r \in L^\infty([0, r], \mathbb{R}^+)$ such that $\sup_{t \in [0, r]} ||f(t, x)|| \leq \mu_r(t)$, for all $t \in [0, r]$.

(H2) The function $f(., x): [0, r] \rightarrow X$ is continuous, for all $x \in X$.

(H3) The function $g: \mathcal{C} \rightarrow X$ is continuous and compact.

(H4) There exist positive constants $a$ and $b$ such that $||g(x)|| \leq a|x|_c + b$, for all $x \in \mathcal{C}$.

(H5) There exists a positive constant $L$ such that $\sigma(\int f(t, D_0)) \leq L\sigma(D_0)$, for any countable set $D_0 \subset X$ and $t \in [0, r]$.

Theorem 2. Assume that (H1) – (H5) hold, then the Cauchy problem (2) has at least one mild solution provided that

$$\sup_{t \in [0, r]} T\left(\frac{t^\alpha}{\alpha}\right) \max\left(a, \frac{4Lr^\alpha}{\alpha}\right) < 1.$$  

(16)

Proof. In order to use the Darbo–Sadovskii fixed point theorem, we put $B_{\delta} = \{x \in \mathcal{C}, ||x|| \leq \delta \}$ for $r > 0$ and define the operator $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ by

$$\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right) \{x_0 + g(x)\} + \int_0^t s^\alpha T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$  

(17)

The proof will be given in four steps.

Step 1. Prove that there exists a radius $\delta > 0$ such that $\Gamma: B_{\delta} \rightarrow B_{\delta}$.

Let $x \in \mathcal{C}$, we have

$$||\Gamma(x)(t)|| \leq \left|\left. T\left(\frac{t^\alpha}{\alpha}\right) \right| \{x_0 + g(x)\} \right| + \int_0^t s^\alpha \left|\left. T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) \right| \right|ds.$$  

(18)

Taking the supremum, we obtain

$$||\Gamma(x)||_c \leq \sup_{t \in [0, r]} T\left(\frac{t^\alpha}{\alpha}\right) \left|\left. \{x_0 + g(x)\} \right| \right| + \int_0^t s^\alpha \left|\left. \|f(s, x(s))\| \right| \right|ds.$$  

(19)

Using assumption (H4), we deduce that

$$||\Gamma(x)||_c \leq \sup_{t \in [0, r]} T\left(\frac{t^\alpha}{\alpha}\right) \left|\left. \{x_0\} + a|x|_c \right| \right| + b + \int_0^t s^\alpha \left|\left. \|f(s, x(s))\| \right| \right|ds.$$  

(20)

Hence, it suffices to consider $\delta$ as a solution of the following inequality:

$$\sup_{t \in [0, r]} T\left(\frac{t^\alpha}{\alpha}\right) \left|\left. \{x_0\} + ar + b + \frac{s^\alpha}{\alpha} \mu_r \right|_{L^\infty([0, r], \mathbb{R}^+)} \right| \leq \delta.$$  

(21)
\[
\delta \geq \sup_{t \in [0, r]} \left| T(t^{\alpha}/\alpha) \right| \left[ \| x_0 \| + b + \frac{t^\alpha}{\alpha} \mu^\delta_{L^p([0, r], \mathbb{R}^n)} \right].
\]

(22)

Step 2. Prove that \( \Gamma : B_\delta \rightarrow B_\delta \) is continuous.
Let \( (x_n) \subset B_\delta \) such that \( x_n \rightarrow x \) in \( B_\delta \). We have
\[
\Gamma(x_n)(t) - \Gamma(x)(t) = T \left( \frac{t^\alpha}{\alpha} \right) [g(x_n) - g(x)]
\]
\[
+ \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) \cdot \left[ f(s, x_n(s)) - f(s, x(s)) \right] ds.
\]

(23)

Then, by using a direct computation, we obtain
\[
\left\| \Gamma(x_n) - \Gamma(x) \right\|_{\infty} \leq \sup_{t \in [0, r]} \left[ T \left( \frac{t^\alpha}{\alpha} \right) \right] \left[ \| g(x_n) - g(x) \| + \int_0^t s^{\alpha-1} \left\| f(s, x_n(s)) - f(s, x(s)) \right\| ds \right].
\]

(24)

Using assumption \((H_1)\), we get \( \| s^{\alpha-1} [f(s, x_n(s)) - f(s, x(s))] \| \leq 2 \mu \delta(s) s^{\alpha-1} \) and \( f(s, x_n(s)) \rightarrow f(s, x(s)) \) as \( n \rightarrow +\infty \).

The Lebesgue dominated convergence theorem proves that \( \int_0^t s^{\alpha-1} \left\| f(s, x_n(s)) - f(s, x(s)) \right\| ds \rightarrow 0 \) as \( n \rightarrow +\infty \). According to continuity of the function \( g \), we deduce that \( \lim_{n \rightarrow +\infty} \| g(x_n) - g(x) \| = 0 \). Hence, \( \Gamma \) is continuous.

Step 3. Prove that \( \Gamma(B_\delta) \) is equicontinuous.
For \( x \in B_\delta \) and \( t_1, t_2 \in [0, r] \) such that \( t_1 < t_2 \), we have
\[
\Gamma(x)(t_2) - \Gamma(x)(t_1) = T \left( \frac{t_2^{\alpha} - t_1^{\alpha}}{\alpha} \right) - I \left[ T \left( \frac{t_1^{\alpha}}{\alpha} \right) \right] (x_0 + g(x))
\]
\[
+ \int_{t_1}^{t_2} s^{\alpha-1} T \left( \frac{t_2^\alpha - s^\alpha}{\alpha} \right) f(s, x(s)) ds
\]
\[
+ \int_{t_1}^{t_2} s^{\alpha-1} T \left( \frac{t_1^\alpha - s^\alpha}{\alpha} \right) f(s, x(s)) ds.
\]

(25)

By using assumptions \((H_1)\) and \((H_4)\), we obtain
\[
\left\| \Gamma(x)(t_2) - \Gamma(x)(t_1) \right\|
\leq \left( \| x_0 \| + a \delta + b + \frac{t_2^\alpha}{\alpha} \mu^\delta_{L^p([0, r], \mathbb{R}^n)} \right) \sup_{t \in [0, r]} \left[ T \left( \frac{t^\alpha}{\alpha} \right) \right]
\]
\[
\cdot \left[ T \left( \frac{t_2^\alpha - t_1^\alpha}{\alpha} \right) - I \right] + \sup_{t \in [0, r]} \left[ T \left( \frac{t_1^\alpha}{\alpha} \right) \right] \| \mu^\delta_{L^p([0, r], \mathbb{R}^n)} \left[ t_2^\alpha - t_1^\alpha \right] \right].
\]

(26)

The above inequality combined with the uniform continuity of the family \( (T(t))_{t \geq 0} \) proves that \( \Gamma(B_\delta) \) is equicontinuous on \([0, r]\).

Step 4. Prove that \( \Gamma : B_\delta \rightarrow B_\delta \) is a \( \sigma \)-contraction operator.
Let \( D \subset B_\delta \), then by Lemma 3 there exists a countable set \( D_0 \) such that \( D_0 = \{ x_0 \} \subset D \). Hence, \( \Gamma(D_0) \) becomes a countable subset of \( \Gamma(D) \). Thus, Lemma 3 proves that \( \sigma_e (\Gamma(D)) \leq 2 \sigma_e (\Gamma(D_0)) \). Since \( \Gamma(D_0) \) is bounded and equicontinuous, then by using Lemma 5, we obtain
\[
\sigma_e (\Gamma(D_0)) = \max_{t \in [0, r]} \left( \sigma (\Gamma(D_0)(t)) \right).
\]

(27)

Then, one has
\[
\sigma_e (\Gamma(D)) \leq 2 \sigma_e (\Gamma(D_0))
\]
\[
= 2 \max_{t \in [0, r]} \left( \sigma \left( T \left( \frac{t^\alpha}{\alpha} \right) \right) [x_0 + g(D_0)] \right)
\]
\[
+ \sigma \left( \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) f(s, D_0(s)) ds \right) \right).
\]

(28)

By using point (4) of Lemma 1, we deduce that
\[
\sigma_e (\Gamma(D)) \leq 2 \max_{t \in [0, r]} \sigma \left( T \left( \frac{t^\alpha}{\alpha} \right) \right) [x_0 + g(D_0)]
\]
\[
+ \sigma \left( \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) f(s, D_0(s)) ds \right) \right).
\]

(29)

Since \( g \) is compact, then \( T(t^\alpha/\alpha) [x_0 + g(D_0)] \) is relatively compact. Hence, using point (1) of Lemma 1 in the above inequality, we obtain
\[
\sigma_e (\Gamma(D)) \leq 2 \max_{t \in [0, r]} \left( \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) f(s, D_0(s)) ds \right).
\]

(30)

In view of Lemma 4, we get
\[
\sigma_e (\Gamma(D)) \leq 4 \max_{t \in [0, r]} \left( \int_0^t s^{\alpha-1} \sigma \left( T \left( \frac{t^\alpha - s^\alpha}{\alpha} \right) f(s, D_0(s)) ds \right) \right).
\]

(31)

Next, point (7) of Lemma 1 shows that
\[
\sigma_e (\Gamma(D)) \leq 4 \sup_{t \in [0, r]} \left| T \left( \frac{t^\alpha}{\alpha} \right) \right| \max_{t \in [0, r]} \left( \int_0^t s^{\alpha-1} \sigma (f(s, D_0(s))) ds \right).
\]

(32)

By using assumption \((H_3)\), we obtain
\[
\sigma_e (\Gamma(D)) \leq 4L \sup_{t \in [0, r]} \left( \left| T \left( \frac{t^\alpha}{\alpha} \right) \right| \right) \max_{t \in [0, r]} \left( \int_0^t s^{\alpha-1} \sigma (D_0(s)) ds \right).
\]

(33)
Hence, by using a direct computation combined with point (2) of Lemma 5, we obtain
\[
\sigma_c(\Gamma(D)) \leq 4L \sup_{t \in [0,T]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sigma_c(D) \int_0^T s^{\alpha-1} ds = \frac{4Lt^\alpha}{\alpha} \sup_{t \in [0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sigma_c(D).
\]
(34)

In consequence, we have
\[
\sigma_c(\Gamma(D)) \leq \frac{4Lt^\alpha}{\alpha} \sup_{t \in [0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sigma_c(D).
\]
(35)

Since \(4Lt^\alpha/\alpha \sup_{t \in [0,\tau]} |T(t^\alpha/\alpha)| < 1\), then \(\Gamma\) is a \(\sigma_c\)-contraction operator.

In conclusion, Lemma 2 shows that \(\Gamma\) has at least one fixed point, which is a mild solution of the Cauchy problem (2).

\(\square\)

Remark 2. We note that Theorem 2 improves Theorem 3 in [18] because in Theorem 2 we have not imposed the compactness of the family \((T(t))_{t>0}\) and the Lipschitz condition on the nonlocal part \(g\).

4. Conclusion

Without imposing the compactness condition on the semigroup family and the Lipschitz condition on the nonlocal condition, we have proved the existence of mild solutions for a class of conformable-fractional differential equations with nonlocal conditions in a Banach space. The main result is obtained by means of semigroup theory combined with the Darbo–Sadovskii fixed point theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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