Superconformal Primary Fields on a Graded Riemann Sphere

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Abstract

Primary superfields for a two dimensional Euclidean superconformal field theory are constructed as sections of a sheaf over a graded Riemann sphere. The transformation law is found to be the same as that of an $O(N)$ extended primary field. The construction is then applied to the $N = 3$ Neveu-Schwarz case. Various quantities in the $N = 3$ theory are calculated, such as elements of the super-Möbius group, and the two-point function. Applications of the construction to calculate three-point functions and fusion rules in a manifestly supersymmetric fashion are discussed.
1 Introduction

Two-dimensional conformal field theory has many applications in statistical mechanics and string theory. It also has a very rich algebraic nature, in a sense due to the symmetry algebra being infinite-dimensional. Exactly how this works in the bosonic case is extremely well understood.

The supersymmetrisation of two-dimensional conformal field theory is something that has been studied heavily from a string theory approach. The string is described by a two-dimensional conformal field theory, and the supersymmetrisation of the conformal field theory essentially admits fermions onto the string. This has mostly been studied from a Lagrangian point of view, where the Lagrangian exhibits the classical symmetries. Canonical quantisation can then be used to then obtain the quantum algebra.

In bosonic conformal field theory there is a way, using the high degree of symmetry, to obtain the quantum algebra by algebraic means, rather than from a Lagrangian. Extending this to a theory with one or two Grassmann variables has been covered extensively in the literature. Adding more supersymmetry to the theory has been studied, e.g. [29] [8] [9], but in nowhere near as much depth as the $N = 1, 2$ theories.

In two dimensional conformal field theory, it is found that one can always find a conformal transformation that maps the two-dimensional theory to a two dimensional theory that is flat. In the Euclidean case, the conformal transformations that map the plane to itself are precisely the holomorphic and anti-holomorphic transformations of the plane to itself. One can therefore build Euclidean two-dimensional conformal field
theory on a complex plane. To get a more ‘global’ picture of what is going on, the theory can then be conformally mapped to the Riemann sphere. Many of the properties of the conformal field theory can then be described by the properties of the Riemann sphere. The question then becomes how to build a theory on a ‘super Riemann sphere’, and how the properties of this object can be related to the properties of a superconformal field theory. In this note, the question is addressed, with particular attention payed to the $N = 3$ case.

Superconformal algebras in the classical case look like derivations on a polynomial ring in $\{z, z^{-1}, \theta_i\}$, where the $\theta_i$ are anticommuting ‘co-ordinates’, that preserve a differential form. This can be combined with the theory of extended graded manifolds, with the manifold in question being the Riemann sphere, to give a suitable setting for superconformal field theory (sections 2-4). One can then use this setting to construct and calculate various quantities in the field theory. The $N = 3$ Neveu-Schwarz case is studied in detail (sections 5-10).

## 2 The Graded Riemann Sphere

In this section, a Riemann sphere is considered, and how one can generalise it to the superconformal case. It should be mentioned that there are strictly speaking two approaches, which give rise to the same structure $\mathbb{C}[z, z^{-1}]$. Here, the algebraic structure $\mathbb{C}[z, z^{-1}]$, known as a Graded Manifold (of the extended type) will mostly be used. In some instances, it will become necessary to transfer to the analytic point of view, namely Supermanifolds $\mathbb{C}[z, z^{-1}]$.

First consider an ordinary Riemann sphere. Rather than consider it as a geometric object, one could consider it as a collection of open sets, $U_i$, and consider the functions $f : U_i \to \mathbb{C}$ that are holomorphic in each open set, denoted $f(U_i)$. Each $f(U_i)$ is a ring under addition in $\mathbb{C}$, and pointwise multiplication. Given an open subset $V \subseteq U$, one can construct a non-singular ring homomorphism $\rho_{UV} : f(U) \to f(V)$. These ring homomorphisms become the fundamental tools to work with. They give a way of comparing $f(U_i) \mid_{U_i \cap U_j}$ and $f(U_j) \mid_{U_i \cap U_j}$, and allow one to construct a sheaf of rings over the Riemann sphere, denoted $\mathcal{A}_0$. One can see that each $f(U_i)$ will be a subring of $\mathbb{C}[z, z^{-1}]$. One can then consider derivations on each $f(U_i)$, denoted $\text{Der}(U_i)$ which are $\mathbb{C}$-linear maps from $f(U_i)$ to $f(U_i)$ that obey the Leibniz rule. $\text{Der}(U_i)$ then forms a rank one module over $f(U_i)$. The map $\rho_{UV}$ induces a map $\rho_{UV, \ast} : \text{Der}(U) \to \text{Der}(V)$. One can use these $\rho_{UV, \ast}$ to construct a sheaf of abelian groups, namely the tangent sheaf, denoted $\mathcal{D}^1 \mathcal{A}_0$. It is, locally, a rank one $\mathcal{A}_0$-module. A section of it can be written locally as $g(z) \frac{\partial}{\partial z}$. On each $U_i$, one can also consider the $\mathcal{A}_0$ linear maps of $\text{Der}(U_i)$ into $\mathcal{A}_0$, denoted $\Omega^1(U_i)$. This is also a rank one $f(U_i)$-module, and the $\rho_{UV, \ast}$ induce a map $\rho_{UV}^\ast : \Omega^1(U) \to \Omega^1(V)$. Once again, the $\rho_{UV}^\ast$ can be used to construct a sheaf, denoted $\mathcal{D}^1 \mathcal{A}_0$. It is locally a rank one $\mathcal{A}_0$-module. Locally, a section can be written as $dz g(z)$. One can define the conformal condition as demanding that $\rho_{UV} : z' \mapsto z$, for $z$, $z'$ local co-ordinates in $U, V$ respectively, as having the property $\rho_{UV}^\ast dz' = dz \kappa(z)$ for some $\kappa \in \mathcal{A}_0$. Given this construction, a basis of infinitesimal transformations can be written down, namely $z' = z + az^{n+1}$ corresponding to a space of vector fields, which gives rise
to the Witt algebra. Phrasing the structure of a Riemann sphere in this way gives the most natural generalization to a graded Riemann sphere.

Similarly, an extended graded manifold can be defined, where each ring associated to each open set is no longer a subring of $\mathbb{C}[z, z^{-1}]$, but a larger ring containing Grassmann generators. The ring is no longer over $\mathbb{C}$, but over a complex, finitely generated unital Grassmann algebra, $B_L'$. Each ring associated to each open set is now a subring of $B_L'[z, z^{-1}, \theta_1, \ldots, \theta_N]$ (1)

It is worth noting that this is a slightly more general requirement than that of a graded manifold. In the graded manifold case, the ring is often still taken to be over $\mathbb{C}$, and so would look like $\mathbb{C}[z, z^{-1}, \theta_1, \ldots, \theta_N]$ (2) which, as a ring, is contained in (1). This can be seen by constructing a map $\pi : B_L' \mapsto \mathbb{C}$ by projecting onto the unital element in $B_L'$. The map defines an important quantity, namely the ‘body’ of an element of a Grassmann algebra. The approach of looking at an algebra over $B_L'$, rather than $\mathbb{C}$, as far as the author is aware, was first introduced in [28]. The ring (1) is the Neveu-Schwarz ring. This gives rise to a sheaf, denoted by $A_N$ for some positive integer $N$. The only condition on the ring associated to each open set, is that it be ‘holomorphic in $z$’. This is, in fact, quite a subtle analytic condition, a discussion of which will be postponed for a few paragraphs. Derivations are now replaced by the sheaf of graded derivations, $D_1A_N$ which is a left $A_N$ module. Accordingly, there is the sheaf of graded one-forms, $D_1A_N$, which are $A_N$ linear maps of $D_1A_N$ into $A_N$. $D_1A_N$ is a right $A_N$ module. For full details of this construction of derivations and one-forms, see [23].

There is now a question of a preserved one-form. The basis of differentials is now given by $(dz, d\theta_i)$. Rather than the one-form coming for free, it must now be defined. This one-form will define a generalised conformal structure [19]. Define the one form $\omega = dz - \sum_i d\theta_i \theta_i$. A transformation $(z, \theta_i) \mapsto (z', \theta'_i)$ is superconformal iff it is invertible and $\omega' = \omega \kappa (z, \theta_i)$ for $\kappa \in A_N$. All homomorphisms between open sets with intersection are demanded to be superconformal. An alternative basis for $D_1A_N$ is $(\omega, d\theta_i)$, which gives a corresponding dual basis of $D_1A_N$, namely $(\partial, D_i)$. Here, $\partial = \frac{\partial}{\partial z}$, and $D_i = \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial z}$. This is a convenient basis to work with. It can be readily shown from the superconformal condition, that

$$D_i z' - \sum_j \theta'_j D_i \theta'_j = 0$$

$$\kappa = \partial z' + \sum_j \theta'_j \partial \theta'_j$$

From here on summations are dropped, and summation convention should be assumed. In particular, considering infinitesimal transformations, once the $z'$ transformation is known, all the $\theta'_i$ transformations can almost be deduced. In the $N = 1$ case, there is a $\mathbb{Z}_2$ ambiguity. For higher $N$ this ambiguity becomes a continuous group, often interpreted as a gauge group in the physics literature [2]. $\kappa$ can also be given by a slightly different expression. The relation $[D_i, D_k] = 2\delta_{ik}\partial$, where the commutator is
graded, and \([3]\), are useful in showing that \(\det(D_i\theta'_j)^2 = (\partial z' + \theta'_i \partial \theta'_j)^N = \kappa^N\). This is an expression in a Grassmann ring over \(\mathbb{C}\), so it is not obvious that one can divide or take \(N\)-th roots. If the co-ordinate transformation on the intersection of two open neighbourhoods \((z, \theta) \mapsto (z', \theta'_i)\) involves a scale factor \(\kappa\), the inverse transformation will induce a scale factor \(\kappa'\), with \(\kappa \kappa' = \kappa' \kappa = 1\). Thus, \(\kappa\) has a unique \([32]\) inverse, and both \(\kappa\) and \(\kappa'\) have a component that is pure complex number, i.e. they have a non-zero ‘body’. Since an extended graded manifold framework is being used, this is not as trivial a fact as it might seem. \(N\)-th roots of \(\kappa\) can now be defined, as the binomial expansion around the ‘body’, \(e(\kappa)\). This expansion is finite due to the nilpotency of \(\kappa - e(\kappa)\). One finds that \(\omega' = \omega(\det(D_i\theta'_j))^{\frac{1}{2}} \zeta^2\), where \(\zeta\) is an \(N\)-th root of unity. Calculating the corresponding transformation on the basis of \(D^1A_N\), one finds that \(D'_i = (D'_i\theta_j)D_j\), and that \((D_i\theta'_j)^{-1} = (D'_i\theta_j)\).

There is a subtle point about the superconformal transformation. The map

\[
z' = z, \quad \theta'_i = -\theta_i
\]

is superconformal. If one restricts to transformations \(z' = z'(z)\) and uses the superconformal condition to deduce how the \(\theta_i\) can transform, the choice of possible minus manifests itself as a choice of spin structure. Considering the transformation \([5]\), one can ask what functions of \((1)\) are invariant under it. A basis can be chosen for this ring, namely \(\{z^n, z^n\theta_i, \ldots, z^n\theta_i\theta_j \cdots \theta_N\}\), for \(n \in \mathbb{Z}\). Without the minus sign, all these basis elements are transformed to themselves. With the minus sign, one finds that those elements with an odd number of \(\theta_i\) obtain a minus sign. Thus, only a subring is invariant. One can enlarge the invariant subring by introducing square roots and choosing the minus sign in the square root whenever one has an odd number of \(\theta_i\) in a basis element. Consider now a subring of

\[
B_{L'}[z^{\frac{1}{2}}, z^{-\frac{1}{2}}, \theta_1, \ldots, \theta_N]
\]

which has as a basis \(\{z^n, z^{n+\frac{1}{2}}\theta_i, z^n\theta_i\theta_j, z^{n+\frac{1}{2}}\theta_i\theta_j\theta_k \cdots \}\) \(n \in \mathbb{Z}\) and choose the negative square root under a superconformal transformation. Now all the basis elements are mapped to themselves under a superconformal transformation. This is the Ramond ring. It should be noted that with the analytic definition of functions that are ‘holomorphic in \(z'\) that will be used here (see below), the Ramond ring introduces a branch cut. It should also be noted that this construction can easily be extended to the case when some of the \(\theta_i\) have a minus sign in the transformation, and some do not.

Already the need for a more analytic idea of what is going on is apparent. This will be particularly important when the question of contour integration arises. Full details of this approach can be found in \([28, 26]\). Many the details are not mentioned here. Consider now a Grassmann algebra generated by \(L\) generators, \(B_L\). This algebra can be given a Banach algebra structure, with the resulting topology being Hausdorff. This splits into an even and odd part, \(B_L = B_{L0} \oplus B_{L1}\) as a vector space over \(\mathbb{C}\). Now the \(\theta\) ‘co-ordinates’ take values in \(B_{L1}\) and \(z\) in \(B_{L0}\). In particular, \(z\) can now have a nilpotent even part \(s(z)\) (the soul), but must have a proper complex number part \(\epsilon(z)\) (the body). Using the fact that a body exists, a ‘superdifferentiable’ \([20]\) function can be constructed. For \(L' = \{\text{the smallest integer not smaller than } \frac{1}{2}\}\), a continuation
can be specified, namely a continuation $Z_{L',L} : C^\infty(\epsilon(U), B_{L'}) \to G^\infty(U, B_L)$ for $U$ open in $B_L$. The algebra $B_{L'}$ can be associated to the algebra $B_L$ by the inclusion map $\iota_{L',L} : B_{L'} \to B_L$ which is the algebra homomorphism that maps the generators $\beta_i$ to $L'$ of the generators in $B_L$, and the unit in $B_{L'}$ to the unit in $B_L$. Note, $f(z)$ may be even or odd.

$$Z_{L',L}(f)(z) = \sum_{i=0}^{L} \frac{1}{i!} \iota_{L',L} \left( f^{(i)}(\epsilon(z)) \right) \times s(z)^i$$

(7)

Where $f^{(i)}$ denotes the i-th derivative. Now consider functions of the variables $(z, \theta_i)$ that can be written as

$$F(z, \theta_i) = Z_{L',L}(f_0)(z) + \sum_{i=1}^{N} Z_{L',L}(f_i)(z)\theta_i + \ldots$$

$$= \sum_{\mu=0}^{2^{L}-1} Z_{L',L}(f_\mu)(z)\theta_\mu$$

(8)

The statement that $F(z, \theta_i)$ is holomorphic in $z$ is the statement all the $f_\mu(w)$, are holomorphic in the complex variable $w$. These are examples of $GH^\infty$ functions. For a sheaf to be $GH^\infty$, its restriction maps, $\rho_{UV}$, must also be $GH^\infty$.

An interesting sheaf to look at is that formed by the one-form $\omega$. It shares similar properties to the one-form $dz$ on an ordinary Riemann sphere. Call the sections of this line bundle on the Riemann sphere $\mathcal{O}_0(1)$. If the change of co-ordinates on an intersection from one open set to another is $z \mapsto f(z)$, then the transition function for an element of $\mathcal{O}_0(1)$ is $f'(z)$. In conformal field theory, a primary field of weight $h$ can be thought of as a section of $\mathcal{O}_0(h) = \mathcal{O}_0(1)\otimes h$, where $h$ is a positive integer. In the super case, this gives something similar. It should be pointed out that $\omega$ will not give a line bundle. This would require the typical fibre to be a free $B_L$ module of rank one, rather than a line. As a result, it must be regarded as a sheaf. This gives rise to a sheaf of sections, $\mathcal{O}_N(1)$. On an intersection, the function $(\det D_i \theta'_j)^\frac{h}{2} \zeta^2$ is the sheaf’s homomorphism. This should be compared to the transition functions on a line bundle. ‘Uncharged’ primary superfields [12] can then be thought of as sections of $\mathcal{O}_N(h) = \mathcal{O}_N(1)^\otimes h$. It should be noted that now tensor products are taken over a graded ring, $\mathcal{A}_N$, so care must be taken with signs in the tensor product. For example, consider a tensor product between two graded left-$\mathcal{A}_N$ modules. Then

$$f_1 \otimes pf_2 = (-1)^{pf_1}pf_1 \otimes f_2$$

where $p \in \mathcal{A}_N$, and exponents of $(-1)$ give the parity of the associated element. Similar formulae should be used if one of the two modules, or both are right-$\mathcal{A}_N$ modules, with the obvious modifications in the exponent of $(-1)$.

There are other interesting sheaves also present. If one accepts the Berezin prescription for integration, then the integration ‘measure’ on the graded Riemann sphere is $\omega \otimes_i D_i$, which has a transformation rule $(\det D_i \theta'_j)^\frac{h}{2} \zeta^2$. As was first noticed in [29], there is also a $O(N)$ group present. This can be regarded as a sheaf in the following way.
Since the \( \{D_i\}, i = 1 \ldots N \), transform amongst themselves, one can consider the sheaf of supercovariant derivatives. Then the transformation law on intersections of open sets is \( D'_i = (D'_i \theta_j) D_j \). Since the superderivatives transform into one another, one can consider the sheaf of supercovariant derivatives. Call the sections of this sheaf \( \mathcal{C}_N \). The matrices \( D_i \theta'_j \) enjoy the following property

\[
(D_i \theta'_j)(D_k \theta'_j) = \delta_{ik} \left( \partial z' + \theta'_j \partial \theta'_j \right) = \delta_{ik} \kappa = \delta_{ik} (\det D_i \theta'_j)^\frac{2}{N} \zeta^2
\]  

(9)

The right hand side can be thought of as an \( N \times N \) identity matrix multiplying the scale factor of the superconformal transformation. Consider now a sheaf whose sections are \( \mathcal{O}(-\frac{1}{2}) \otimes \mathcal{C}_N \). Constructing \( \mathcal{O}(-\frac{1}{2}) \) requires taking a square root, and is very analogous to taking the square root of a line bundle. Hence, the choice of sign can be thought of as choosing a spin structure. The group homomorphisms on an intersection of open sets in this sheaf are given by

\[
M_{ij} = \frac{(D_i \theta'_j)}{\sqrt{\kappa}}
\]

(10)

and the sheaf itself is an \( \mathcal{A}_N \)-module of rank \( N \). Recall that on intersections, \( \kappa \) has an inverse \( \kappa' \), and so \( \kappa^{-\frac{1}{2}} \) is well defined on this intersection. There is still a question of a sign. Keeping in mind that \( \theta'_j = \pm \theta_j \) is a superconformal transformation, it is really the \( \theta_j \) that one would want to account for different spin structures, rather than \( \mathcal{O}(-\frac{1}{2}) \). Therefore, it seems reasonable to choose a plus sign for \( \kappa^{-\frac{1}{2}} \). Regarding the new homomorphism \( M_{ij} \) as a matrix acting on a free module of rank \( N \), it can be thought of as an element of \( \mathcal{O}(N) \), with the entries being even Grassmann elements. The new sheaf then gives rise to a fundamental representation of \( \mathcal{O}(N) \), and the \( M_{ij} \) the coefficients of a matrix with basis \( E^{ij} \). Call this sheaf \( \mathcal{G}_N \). This can give rise to other sheaves which are also \( \mathcal{O}(N) \) representations.

The manner in which this is done parallels what is often done for vector bundles, in particular frame bundles and spin bundles. The reason this treatment can be applied is that \( \mathcal{G}_N \) almost looks like a vector bundle, the only hindrance being that the ‘typical fibre’ would be a \( \mathcal{B}_L \) module rather than a vector space. Rather than an abelian group of rank \( N \) being associated to each open set, one can instead associate a group element of \( \mathcal{O}(N) \), just as is done with frame bundles with principal bundles, and retain the same \( M_{ij} \). This gives rise to a sheaf \( \mathcal{G}_N \). Considering, now a different representation \( \rho \) of \( \mathcal{O}(N) \) gives rise to a sheaf homomorphism (albeit of non-abelian groups)

\[
\sigma : (\mathbb{P}_1, \mathcal{G}_N) \rightarrow (\mathbb{P}_1, \mathcal{G}_N^\rho)
\]

(11)

where the group homomorphisms are now given by \( M_{ij} \rho(E^{ij}) \). Since the representation \( \rho \) has a vector space \( V \) associated to it, one can consider the sheaf which has the group homomorphisms given by \( M_{ij} \rho(E^{ij}) \), and stalk \( V \), and denote the sections of this sheaf by \( \mathcal{R}(G_N) \). An \( \mathcal{O}(N) \)-extended primary superfield, first introduced in [29], can then be defined as a section of \( \mathcal{O}_N(h) \otimes \mathcal{R}(G_N) \).
3 Contour Integration

Since one wishes to do conformal field theory in the setting presented above, a sensible question to ask is what closed contour integrals will look like, given the set of analytic functions $\{f_i\}$. All functions on a given open set look like

$$f(z) = \sum_{i=0}^{L} \frac{1}{i!} \epsilon(z) f_0(i) s(z)^i$$

where $f_0 \in C^\infty(U, B_{L'})$. This should be compared to the usual notion of a Taylor expansion, around $\epsilon(z)$. Note that if $L' = 0$, $B_{L'} = \mathbb{C}$, and $H^\infty$ functions are retrieved. In the following, the $\epsilon(z)$ will be suppressed (for clarity). It is a linear map, so one can see that the following workings are unaffected. The contour integral

$$\oint_C f(z) dz$$

needs to be considered. By the definition above, if $z$ is an ordinary complex number (i.e. has no soul), one finds

$$f(z) = \sum_{i=0}^{L} \frac{1}{i!} f_0(i) \epsilon(z) s(z)^i = f_0(z)$$

If the even co-ordinate, $w$, were to have soul as well as body, it would give an element of an even Grassmann algebra over a complex field, when evaluated at a point. Hence one can consider $w$ itself as being parameterised by a complex number $z$. Now consider a parameterisation $w = g(z) = b(z) + u(z)$, where $\epsilon(w) = b(z)$, $s(w) = u(z)$.

$$f(w) = \sum_{i=0}^{L} \frac{1}{i!} f_0(i) \epsilon(w) s(w)^i$$

$$= \sum_{i=0}^{p} \frac{1}{i!} (f_0(i) \circ b)(z) u(z)^i$$

$$= (f \circ g)(z)$$

where $p < L$ is the integer such that $u(z)^p \neq 0$, $u(z)^{p+1} = 0$. Using the definition of a contour integral given in [27], with $C_w = g(C_z)$,

$$\oint_{C_w} f(w) dw = \oint_{C_z} (f \circ g)(z) g'(z) dz$$

$$= \oint_{C_z} \left( \sum_{i=0}^{p} \frac{1}{i!} (f_0(i) \circ b)(z) u(z)^i \right) \left( \frac{d}{dz} b(z) + \frac{d}{dz} u(z) \right) dz$$

$$= \oint_{C_z} \left( \frac{d}{dz} b(z) \right) f_0(b(z)) + \sum_{i=1}^{p} \frac{1}{i!} \left( \frac{d}{dz} b(z) \right) (f_0(i) \circ b)(z) u(z)^i$$

$$= \sum_{i=0}^{p} \frac{1}{i!} (f_0(i) \circ b)(z) u(z)^i \left( \frac{d}{dz} u(z) \right) dz$$

(15)
All that has been done above is put all the definitions in and split up some summations. Note in the first summation, the chain rule can be used on the function \( b(z) \), and in the second summation, the chain rule can be used on \( u(z) \), giving

\[
\oint_{C_w} f(w)dw = \oint_{C_z} \left( \frac{d}{dz} b(z) \right)(f_0 \circ b)(z) + \sum_{i=1}^{p} \frac{1}{i!} \left( \frac{d}{dz} (f^{(i-1)} \circ b)(z) \right) u(z)^i \\
+ \sum_{i=0}^{p} \frac{1}{(i+1)!} (f^{(i)} \circ b)(z) \frac{d}{dz} \left( u(z)^{i+1} \right) \ dz
\]

\[
= \oint_{C_z} \left( \frac{d}{dz} b(z) \right)(f_0 \circ b)(z) + \sum_{i=1}^{p} \frac{1}{i!} \left( (f^{(i-1)} \circ b)(z) \cdot u(z)^i \right) \\
+ \frac{1}{(p+1)!} (f^{(p)} \circ b)(z) \frac{d}{dz} \left( u(z)^{p+1} \right) \ dz
\]  

The last term is in fact zero, due to the nilpotency of \( u(z) \). The term under the summation is a total derivative. As such, integrated around a closed contour, it vanishes identically. All that remains is

\[
\oint_{C_w} f(w)dw = \oint_{C_z} (f_0 \circ b)(z) b'(z) \ dz
\]

Thus, the contour integral can formally be treated as an integral in a normal complex number. It should be noted that all that has been used in this calculation is the chain rule and product rule over \( C^\infty \) functions.

4 The Preserved One-Form and Ramond Fields

Whilst generalising the bosonic setting in the previous sections, it was found that rather than a preserved one-form coming for free, it had to be specified. One could ask, what happens if another one-form is specified. In [21], other one-forms were considered. It was found that if one wanted a \( \mathbb{Z} \)-graded algebra, a one-form of the form \( dz - d\theta_i f(z)\theta_i \) had to have \( f(z) = z^n \). By making a change of variables, [21] then shows that one only need consider the cases \( n = 0, 1 \).

Consider, now, a different preserved one-form, namely \( \omega = dz - d\theta_i z\theta_i \). The dual derivations to \( (\omega, \theta_i) \) are \( (\partial, D_i) \), where \( D_i = \frac{\partial}{\partial \theta_i} + z\theta_i \frac{\partial}{\partial z} \). Now one finds that \( [D_i, D_j] = 2\delta_{ij}z\partial \). Requiring that under a transformation, \( \omega' = \omega \kappa(z, \theta_i) \) yields

\[
D_j z' - z'\theta_i' D_j \theta_i' = 0
\]

\[
\kappa = \partial z' + z'\theta_i' \partial \theta_i' = \left( \frac{\zeta}{z} \right)(\det D_i \theta_j') \hat{\Omega} \xi^2
\]

\[
D_i' = (D_i' \theta_j) D_j
\]

\[
(D_i \theta_j')(D_k \theta_j') = \delta_{ik}(\det D_i \theta_j') \hat{\Omega} \xi^2
\]

One can ask what an element of \( \mathcal{O}_N(h) \) may transform like, and what is the algebra of infinitesimal transformations associated to it. Consider the case \( N = 1 \). The field has a
transformation rule under \((z, \theta) \mapsto g(z, \theta) = (z', \theta')\)

\[
(U_g^{-1}\Phi U_g)(z', \theta') = \Phi'(z', \theta') = \Phi(z, \theta) \left(\frac{z'}{z}\right)^h (D\theta')^2
\]  

\(22\)

The superconformal condition, namely preserving the new one-form \(w\), imposes two types of transformation, a bosonic and a fermionic one. On the co-ordinates, \((z, \theta)\), the infinitesimal transformations are given by

\[
(z, \theta) \mapsto (z + az^{n+1}, \theta + \frac{a}{\theta}z^n) \quad \text{and} \\
(z, \theta) \mapsto (z + \epsilon \theta z^{r+1}, \theta - \epsilon z^r)
\]  

\(23\)

where \(r, n \in \mathbb{Z}\). These provide a basis for all infinitesimal transformations. Each one induces a transformation on the field, \((22)\), with \(a(n) = az^{n+1}, \epsilon(r) = \epsilon \theta z^{r+1}\)

\[
\delta_{a(n)} \Phi(z, \theta) = -a \left(z^{n+1} \partial_z + \frac{a}{\theta}z^n \partial_\theta + h(n+1)z^n\right) \Phi(z, \theta)
\]

\[
\delta_{\epsilon(r)} \Phi(z, \theta) = -\epsilon \left(\theta z^{r+1} \partial_z - z^r \partial_\theta + h(2r+1)\theta z^r\right) \Phi(z, \theta)
\]  

\(24\)

where \(n, r \in \mathbb{Z}\). These differential operators give rise to commutation relations

\[
[\delta_{a_1(m)}, \delta_{a_2(n)}] = (m-n)\delta_{a_2a_1(m+n)}
\]

\[
[\delta_{\epsilon_1(r)}, \delta_{\epsilon_2(s)}] = 2\delta_{\epsilon_2\epsilon_1(r+s)}
\]

\[
[\delta_{a(m)}, \delta_{\epsilon(r)}] = \left(\frac{m}{2} - r\right)\delta_{a(m+r)}
\]  

\(25\)

This gives a representation of the Ramond algebra. Note that no branch cut has been introduced. Another thing to note is that \(\delta_{a(0)}\) gives a \(l_0\) operator, which says that \(\theta\) scales like a field of weight zero, rather than a field of weight half. As a result, the expansion of \(\Phi(z, \theta)\) is now taken to be

\[
\Phi(z, \theta) = \phi_0(z) + \theta \phi_1(z) = \sum_{m \in \mathbb{Z}} \phi_{0m} z^{-m-h} + \theta \sum_{m \in \mathbb{Z}} \phi_{1m} z^{-m-h}
\]

Using this expansion, and writing the transformation of a field as

\[
(U_g \Phi U_g^{-1})(z, \theta) = \Phi(z, \theta) \left(\frac{z'}{z}\right)^h (D\theta')^2, \quad U_g = \exp(a_n L_n + \epsilon_r G_r)
\]  

\(26\)

one can find the action of the algebra on the modes of \(\Phi\) as

\[
[L_n, \phi_{0m}] = ((h-1)n - m) \phi_{0m+n} \quad [L_n, \phi_{1m}] = ((h-\frac{1}{2})n - m) \phi_{1m+n+n}
\]

\[
[G_r, \phi_{0m}] = \phi_{1m+r} \quad [G_r, \phi_{1m}] = ((2h-1)r - m) \phi_{0m+r}
\]

These are precisely the commutation relations one obtains from the \(N = 1\) Ramond OPEs from the usual method of introducing a branch cut. Rewriting the commutation relations in a more familiar way, and inserting the unique central extension, the algebra can be written down

\[
[L_m, L_n] = (m-n) L_{m+n} + \frac{C}{6} m(m^2-1) \delta_{m+n,0}
\]

\[
[L_m, G_r] = (m-r) G_{m+r} \quad [G_r, G_s] = 2L_{r+s} + \frac{3C}{4} (r^2 - \frac{1}{2}) \delta_{r+s,0}
\]  

\(27\)
Influenced by the form of the infinitesimal changes, OPEs can be postulated that give the above commutation relations, which read as

\[
\begin{align*}
L(w)\phi_0(z) & \sim \left( \frac{\partial}{(w-z)} + \frac{h}{(w-z)^2} \right) \phi_0(z) \\
L(w)\phi_1(z) & \sim \left( \frac{\partial}{(w-z)} + \frac{h + \frac{3}{2}}{(w-z)^2} - \frac{1}{2z(w-z)} \right) \phi_0(z) \\
G(w)\phi_0(z) & \sim \frac{1}{z(w-z)} \phi_1(z) \\
G(w)\phi_1(z) & \sim \left( \frac{\partial}{(w-z)} + \frac{2h}{(w-z)^2} - \frac{h}{z(w-z)} \right) \phi_0(z) \\
L(w)L(z) & \sim \frac{C}{(w-z)^4} + \frac{2L(z)}{(w-z)^2} + \frac{\partial L(z)}{(w-z)} \\
L(w)G(z) & \sim \frac{\partial G(z)}{(w-z)} + \frac{3G(z)}{2(w-z)^2} - \frac{G(z)}{2z(w-z)} \\
G(w)G(z) & \sim \frac{2zL(z)}{(w-z)} + \frac{2C}{3} \left( \frac{2z}{(w-z)^3} + \frac{1}{(w-z)^2} - \frac{1}{4z(w-z)} \right)
\end{align*}
\]

where \( L(z) = \sum_n L_n z^{-n-2} \), \( G(z) = \sum_r G_r z^{-r-1} \), \( z, r \in \mathbb{Z} \). This final set of OPEs demonstrate the drawbacks of the more abstract construction used in this paper of a Conformal Field Theory (namely via a section of a sheaf over some manifold), compared to the more usual approach of a free field realization. One can calculate the infinitesimal transformations of the field (the section obtained), and show the transformations close as a Lie algebra. One then has to ‘work backwards’ and try and construct OPEs and central charge terms that agree with the transformations and lie algebras calculated. It would be interesting to see if the Ramond field could be realized via a free field realization where usually one finds central terms and OPEs are explicitly calculable.

## 5 Classical \( N = 3 \) Algebra

Consider now that case of preserving the usual one-form, \( dz - d\theta_i \theta_i \), with three Grassmann variables. The superconformal condition is then \( \mathfrak{g}(Z) = (z, \theta_i) \), and writing a superconformal transformation as \( Z \mapsto g(Z) \), a representation of the group can be constructed via \( A_3 \), namely \( U_g f(Z) = (f \circ g^{-1})(Z) \). The infinitesimal transformations can be calculated, and a Lie superalgebra constructed. The infinitesimal transformations take the form of vector fields acting on functions.

The most general infinitesimal transformation on the \( z \) co-ordinate is

\[
z \mapsto z + a f(z, \theta_1, \theta_2, \theta_3) + c h(z, \theta_1, \theta_2, \theta_3)
\]

for \( f \) (\( h \)) some even (odd) function, and \( a \) (\( c \)) infinitesimal and of even (odd) parity. The functions \( f \) have analogues for transformations in the \( \theta_i \) co-ordinates. Breaking up \( f \) into superfield components gives eight different types of transformation.

\[
z \mapsto z + a(z) + \alpha_i(z) \theta_i + \frac{1}{2} a_{ij}(z) \theta_i \theta_j + \alpha_{123}(z) \theta_1 \theta_2 \theta_3
\]
The possible transformations are forced into only these eight types, and not some mix between them, by the superconformal condition.

An infinitesimal transformation most generally reads

\[ z' = z + \delta z \quad \theta'_j = \theta_j + \delta \theta_j \tag{31} \]

On substituting into (3), one finds that the superconformal condition reads

\[ D_i \delta z = \delta \theta_i + \sum_{j=1}^{3} \theta_j D_i \delta \theta_j \tag{32} \]

i.e. three equations, with three unknowns once \( \delta z \) has been specified. A basis for the infinitesimal \( z \) transformations is easily found, which is

\[ \delta z = \epsilon \theta_1 \theta_2 \theta_3 z^{n+\frac{1}{2}}, \delta z = a \theta_i \theta_j z^{n+1} \]

for \( i < j \), \( \delta z = \epsilon \theta_i z^{n+\frac{1}{2}} \), and \( \delta z = a z^{n+1} \). Given these eight types of transformation, precisely what the corresponding \( \delta \theta_i \) are, modulo possible \( \delta \theta_i \) if \( \delta z = 0 \), can be calculated explicitly. The case when \( \delta z = 0 \) is taken care of by the \( t_i \) generators below. Using this procedure, the infinitesimal generators of the \( N = 3 \) algebra can be calculated. The results are quite hefty, but the actual transformations give an intuitive idea of what each element of the algebra actually does. Summation convention is used in the following.

\[ z \mapsto z + a z^{n+1} \quad \theta_i \mapsto \theta_i + a \frac{1}{2}(n + 1) \theta_i z^n \]

\[ \Rightarrow l_m = -z^m \left( z \frac{\partial}{\partial z} + \frac{1}{2}(m + 1) \theta_i \frac{\partial}{\partial \theta_i} \right) \tag{33} \]

gives a vector field corresponding to an infinitesimal transformation when only \( a(z) \) is non-zero in (30). There are then the three single \( \theta \) terms, which can be found by considering the case when only one \( \alpha_i(z) \) is non-zero.

\[
\begin{align*}
z & \mapsto z - \epsilon \theta_1 z^{r+\frac{1}{2}} \\
\theta_1 & \mapsto \theta_1 + \epsilon z^{r+\frac{1}{2}} \\
\theta_2 & \mapsto \theta_2 - \epsilon (r + \frac{1}{2}) \theta_1 \theta_2 z^{r-\frac{1}{2}} \\
\theta_3 & \mapsto \theta_3 - \epsilon (r + \frac{1}{2}) \theta_1 \theta_3 z^{r-\frac{1}{2}}
\end{align*}
\]

gives rise to the vector field

\[ g^1_r = z^{r-\frac{1}{2}} (z \theta_1 \frac{\partial}{\partial z} - z \frac{\partial}{\partial \theta_1} + (r + \frac{1}{2}) \theta_1 \theta_2 \frac{\partial}{\partial \theta_2} + (r + \frac{1}{2}) \theta_1 \theta_3 \frac{\partial}{\partial \theta_3}) \]

Similarly

\[ g^i_r = z^{r-\frac{1}{2}} (z \theta_i \frac{\partial}{\partial z} - z \frac{\partial}{\partial \theta_i} + (r + \frac{1}{2}) \theta_i \theta_j \frac{\partial}{\partial \theta_j}) \]

It is worth noting that if one were not working on an extended graded manifold, but on a graded manifold (c.f. [11], [23]), then one would not be able to obtain the above vector field. The same statement holds for \( \psi_r \) below.
There are three double $\theta$ terms, e.g. $\theta_1\theta_2$ gives $t^3_n$

\[
\begin{align*}
  z &\mapsto z \\
  \theta_1 &\mapsto \theta_1 + a\theta_2 z^{n+1} \\
  \theta_2 &\mapsto \theta_2 - a\theta_1 z^{n+1} \\
  \theta_3 &\mapsto \theta_3 + a(n+1)\theta_1\theta_2\theta_3 z^n
\end{align*}
\]

A similar calculation applies to $t^1_n$ and $t^2_n$

\[
t^i_m = z^{m-1}(z\epsilon_{ijk}\partial_{\theta_j} - m\theta_1\theta_2\theta_3 \frac{\partial}{\partial \theta_i})
\]

These transformations leave the $z$ component unaltered, and as such have sometimes been interpreted in the physics literature [2] as a gauge group. The final term is similarly calculated, and is the three $\theta$ transformation

\[
\psi_r = -z^{r-\frac{1}{2}}(\theta_1\theta_2\theta_3 \frac{\partial}{\partial z} + \frac{1}{2}\epsilon_{ijk}\theta_i \frac{\partial}{\partial \theta_k})
\]

These vector fields, similarly calculated in [20], [7] then give rise to the commutation relations for the $N=3$ algebra without central extension.

\[
[t^i_m, t^j_n] = -\epsilon_{ijk}t^k_{m+n} \quad [t^i_m, \psi_s] = 0 \quad [t^i_m, g^j_r] = \delta_{ij}m\psi_{r+m} - \epsilon_{ijk}g^k_r
\]

\[
[l_m, \psi_s] = -(\frac{m}{2} + s)\psi_{m+s} \quad [l_m, t^i_n] = -nt^i_{m+n} \quad [g^i_r, \psi_s] = t^i_{r+s}
\]

\[
[l_m, g^i_r] = (\frac{m}{2} - r)g^i_{r+s} \quad [g^i_r, g^j_s] = 2\delta_{ij}s + \epsilon_{ijk}(r-s)t^k_{r+s}
\]

\[
[l_m, l_n] = (m-n)l_{m+n} \quad [\psi_m, \psi_n] = 0 \quad (34)
\]

Note in particular that the $t^i_n$ form an $su(2)$ loop algebra, which will be enhanced by a central extension in the quantum case to give an affine $su(2)$ algebra. One implication of this is that the representation theory will have to be very different to that of the $N=2$ case, where a $u(1)$ loop algebra appeared. The highest weight state must also be an $su(2)$ highest weight state. Since $U(1)$ is abelian, all its irreducible representations are one dimensional. The upshot of this is that the OPE can be easily adapted by including one more quantum number. Since $SU(2)$ is non-abelian, it will be seen that $su(2)$ generators will appear in the OPE.

6 Quantum $N=3$ Algebra

The quantum $N=3$ algebra was calculated from a Lagrangian approach, and canonically quantised in [2]. Whilst this section may look very technical, it should be stressed that essentially the same procedure is being used as in the well documented bosonic case, where the starting point is a section of a sheaf, namely $O_0(h)$. One plays the same
game, but now uses the section $O_N(h) \otimes \mathcal{R}(G_N)$. Since it is defined covariantly, one can then write down how it transforms. This then gives rise to infinitesimal transformations $\delta \Phi$, which close as a lie algebra. These relations can be written in terms of an operator $\mathbb{T}$ acting on $\Phi$, giving an OPE. From the $\delta \Phi$, an ansatz for the OPE of $\mathbb{T}$ with itself can be inferred. The action of the quantum algebras on primary fields is inherent in the $\mathbb{T}\Phi$ OPE. The commutation relations of the quantum algebra are then inherent in the $\mathbb{T}\mathbb{T}$ OPE as the modes. What must be checked from the first OPE, is that the primary superfield does indeed yield a highest weight vector.

Recall that for a primary field in the bosonic case, one performs a diffeomorphism from the Riemann sphere to itself that obeys the conformal condition, and looks at how the primary field transforms. More precisely, one considers a diffeomorphism $f$

$$f : \mathbb{P}_1 \rightarrow \mathbb{P}_1$$

$$z \mapsto f(z) = z'$$  \hspace{1cm} (35)

with the conformal condition

$$(f^*dz) = dz\kappa(z)$$  \hspace{1cm} (36)

One then calculates how $\phi \in O(h)$ transforms under a pull-back, where $\phi$ in local co-ordinates is $\phi(z)dz \otimes h$

$$(f^*\phi)(z) = \kappa^h(\phi \circ f)(z)dz \otimes h = \left(\frac{dz'}{dz}\right)^h \phi(z')dz \otimes h =: \phi'(z)dz \otimes h$$  \hspace{1cm} (37)

yielding the transformation law

$$(U_g \phi U_g^{-1})(z) = \left(\frac{dz'}{dz}\right)^h \phi(z') = \phi'(z)$$  \hspace{1cm} (38)

For the graded case, one has to consider an invertible sheaf morphism

$$f : (\mathbb{P}_1, \mathcal{A}_N) \rightarrow (\mathbb{P}_1, \mathcal{A}_N)$$

$$Z = (z, \theta_i) \mapsto Z' = (z', \theta'_i)$$ \hspace{1cm} (39)

such that $f$ (as well as $f^{-1}$) has a $GH^\infty$ action on the functions $\mathcal{A}_N$, and obeys the conformal condition

$$(f^*\omega) = \omega \kappa(Z)$$  \hspace{1cm} (40)

The transformation rule for the components of $\Phi \in O_N(h) \otimes \mathcal{R}(G_N)$ under a pull-back are then given by

$$\Phi'(Z) = \kappa^h(D_j \theta'_i) \sqrt{\kappa} g^{ij}(\Phi \circ f)(Z)$$ \hspace{1cm} (41)

The $g^{ij}$ are, up to a discrete subgroup, a representation of the lie group $O(N)$. The $g^{ij}$ explicitly realize the map \Box. This formula matches that found in \cite{29} for how a primary superfield transforms. One now writes down the transformation law as

$$(U_g \Phi U_g^{-1})(Z) = \kappa^h(D_j \theta'_i) \sqrt{\kappa} g^{ij} \Phi(Z')$$ \hspace{1cm} (42)
and parameterise the group action infinitesimally by

\[ U_g = \exp(a_n L_n + \alpha_i G_i^a + b_i T_i^a + \beta_r \psi_r) \]
\[ Z' = \exp(a_n l_n + \alpha_i g_i^a + b_i t_i^a + \beta_r \psi_r)Z \]

in a completely analogous way to \([26]\), to obtain the commutators of the superVirasoro operators on a primary field. For the \(N = 3\) case this yields \([15]\). One must now work backward to try and construct an OPE between a stress-energy tensor and primary superfield that yield these commutators.

For the \(N = 3\) case, the stress-energy tensor will be weight \(\frac{1}{2}\), and have superfield decomposition \([34]\)

\[ \mathbb{T}(Z) = \theta_1 \theta_2 \theta_3 L(z) + \frac{1}{2} \epsilon_{ijk} \theta_i G^k(z) + \theta_i T^i(z) + \psi(z) \]  

(43)

An OPE for \(N = 3\) is found in \([8]\), that, on contour integration, gives rise to the infinitesimal transformations of a primary superfield \([22]\). With \(Z_1 = (w, \chi_i)\), \(Z_2 = (z, \theta_i)\), this reads

\[ \mathbb{T}(Z_1) \Phi(Z_2) \sim \frac{h \theta_{12,1} \theta_{12,2} \theta_{12,3}}{Z_{12}^2} \Phi(Z_2) + \frac{\theta_{12,1} \theta_{12,2} \theta_{12,3}}{Z_{12}} \partial_w \Phi(Z_2) + \frac{\epsilon_{ijk} \theta_{12,i} \theta_{12,j} D_{2,k}}{4Z_{12}} \Phi(Z_2) + \frac{\theta_{12,i} J_i}{Z_{12}} \Phi(Z_2) \]  

(44)

where

\[ D_{2,i} = \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial z} \quad Z_{12} = (w - z - \chi_i \theta_i) \quad \theta_{12,i} = (\chi_i - \theta_i) \]

Where the \(J_i\) form an \(su(2)\) algebra \([35]\). The field, \(\Phi(Z)\), now also lives in an \(su(2)\) representation, say \(\mathcal{V}\). It is in fact an \(su(2)\) highest weight. \(\mathbb{T}\) can then be thought of as being an endomorphism of \(\mathcal{V}\), e.g. explicitly with \(su(2)\) indices \(\mathbb{T}(Z_1)^a_b \Phi(Z_2)^b\). This OPE is effectively a non-abelian version of the \(q\) term appearing in the \(N = 2\) case. In its place, another quantum number appears, which is the \(J_3\) eigenvalue. On the representation space, the action of \(T_0^\mathcal{V}\) on a highest weight state is identified with that of \(J_i\). The OPE can be split up into \(\theta\) components, according to \([13]\), and modes be taken of each of the operators, \(L(z), G^i(z), T^i(z), \psi(z)\), giving the formulae \([15]\), as required. Note in particular, how the classical algebra appears in the relations again. The extra terms are the \(h\) terms, which will give the \(L_0\) eigenvalue \(h\). The other extra terms, the \(J_i\), will give an action of \(su(2)\) on the primary field, and hence on the highest weight, which we know must be required from the classical analysis \([34]\), where a \(su(2)\) loop algebra appeared.

\[ [L_m, \Phi(Z)] = z^m \left( h(m + 1) + z \partial_z + \frac{1}{2}(m + 1) \theta_i \partial_{\theta_i} + \frac{1}{2z} m(m + 1) \epsilon_{ijk} \theta_i \theta_j \partial_{J_k} \right) \Phi(Z) \]
\[ [G_s^i, \Phi(Z)] = -z^{(s - \frac{1}{2})} \left( h(s + \frac{1}{2}) \theta_i + \frac{1}{2} \theta_i z \partial_z - \frac{1}{2z} z \partial_{\theta_i} + \frac{1}{2} (s + \frac{1}{2}) \theta_i \theta_j \partial_{\theta_j} + \right. \]
\[ (s + \frac{1}{2})(\epsilon_{ijk}\theta_j J_k) - \frac{1}{z}(s^2 - \frac{1}{4})\theta_1\theta_2\theta_3 J_i)\Phi(Z) \]

\[ [T^i_m, \Phi(Z)] = z^{(m-1)} \left( \frac{mh}{2} \epsilon_{ijk}\theta_j \theta_k - \frac{1}{2}(s-1)\theta_1\theta_2\theta_3 z \partial z + \frac{1}{2m}\theta_1\theta_2\theta_3 \partial_0, + z J_i - m(\theta_0\theta_0 J_0)\Phi(Z) \right) \]

\[ [\psi_s, \Phi(Z)] = z^{(s-\frac{1}{2})} \left( -\frac{h}{z}(s - \frac{1}{2})\theta_1\theta_2\theta_3 + \frac{1}{2}\theta_1\theta_2\theta_3 \partial z + \frac{1}{4}\epsilon_{ijk}(\theta_i\theta_j \partial_0 - \theta_i J_i)\Phi(Z) \right) \]

Note that

\[ [L_{-1}, \Phi(Z)] = \partial_2 \Phi(Z) \quad [G^i_{\frac{1}{2}}, \Phi(Z)] = \frac{1}{2}(\partial_0 - \theta_i \partial_z) \Phi(Z) \]

In particular, \( L_{-1} \) acts as a translation in \( z \) and \( G^i_{\frac{1}{2}} \) as a super-translation in the respective \( \theta_i \) direction. This allows vertex operators to be used, and an operator-state mapping employed \[22\, [36]. \] In the bosonic theory, vertex operators are characterised uniquely by their action on a vacuum \(|0\rangle\), which is annihilated by the raising operators, \( \{L_n : n \geq -1\} \). The vacuum cannot be invariant under the whole symmetry algebra without implying vanishing of the central extension. This generalises to \( N = 3 \), so that now \(|0\rangle\) is annihilated by \( \{L_n, G^i_r, T^j_m, \psi_s : n \geq -1, r \geq -\frac{1}{2}, m \geq 0, s \geq \frac{1}{2}\} \). To get the state associated to any vertex operator \( \Phi(Z) \), one looks at \( \lim_{Z \to 0} \Phi(Z)|0\rangle \). Given this, it can be seen from the relations [15] that the action of the raising operators on \(|\Phi\rangle\) is zero, e.g. for \( \{L_m : m > 0\} \)

\[ \lim_{Z \to 0}[L_m, \Phi(Z)]|0\rangle = 0 \]

The action of \( L_0 \) is given by

\[ [L_0, \Phi(Z)] = (h + z \partial_z + \frac{1}{2}(\theta_1 \partial \theta_1 + \theta_2 \partial \theta_2 + \theta_3 \partial \theta_3)) \Phi(Z) \]

\[ \Rightarrow \lim_{Z \to 0}[L_0, \Phi(Z)]|0\rangle = h|\Phi\rangle = L_0|\Phi\rangle \quad (47) \]

The action of \( T^i_0 \) is given by

\[ [T^i_0, \Phi(Z)] = (\frac{1}{2}\theta_3 \partial \theta_2 - \frac{1}{2}\theta_2 \partial \theta_3 + J_1)\Phi(Z) \]

\[ [T^j_0, \Phi(Z)] = (\frac{1}{2}\theta_1 \partial \theta_1 - \frac{1}{2}\theta_3 \partial \theta_3 + J_2)\Phi(Z) \]

\[ [T^k_0, \Phi(Z)] = (\frac{1}{2}\theta_2 \partial \theta_2 - \frac{1}{2}\theta_1 \partial \theta_1 + J_3)\Phi(Z) \]

\[ \Rightarrow \lim_{Z \to 0}[T^i_0, \Phi(Z)]|0\rangle = \lim_{Z \to 0}(J_i\Phi(Z)|0\rangle = J_i|\Phi\rangle = T^i_0|\Phi\rangle \quad (48) \]

where the vacuum is \( T^i_0 \) invariant.

Hence, on the highest weight state, the \( T^i_0 \) can be identified with the \( J_i \), so that \( T^3_0 \) gives rise to the \( q \) eigenvalue, and \( J^+ = J^+_0 + iJ^z_0 \) annihilates \(|\Phi\rangle\). As can be seen, \( \Phi(Z) \) is associated to a vector \(|\Phi\rangle\), which is a highest weight of the \( N = 3 \) field.

Rather than work explicitly with \[15\], one could simply consider what the infinitesimal transformations of the field are under an infinitesimal superconformal map

\[ (z, \theta_i) \mapsto (z + \delta z, \theta_i + \delta \theta_i) \]

(49)
This is useful to check closure as a lie algebra. It is useful to introduce the quantity $\nu(z) = \delta z + \theta_1 \delta \theta_1$. The transformation reads

$$\delta \Phi(Z) = h(\partial_z \nu(Z))\Phi(Z) + \nu(Z)\partial_z \Phi(Z) + \frac{1}{2} \sum_{j=1}^{3} (D_j \nu(Z))(D_j \Phi(Z)) +$$

$$((J_3 D_1 D_2 + J_1 D_2 D_3 + J_2 D_3 D_1)(\nu(Z)))\Phi(Z)$$

$$= \frac{2h}{3} (D_1 \delta \theta_1 + D_2 \delta \theta_2 + D_3 \delta \theta_3)\Phi(Z) + (\delta z)\partial_z \Phi(Z) +$$

$$\sum_{j=1}^{3} (\delta \theta_j) \partial_{\theta_j} \Phi(Z) + ((D_1 \delta \theta_2 - D_2 \delta \theta_1)J_3 + (D_2 \delta \theta_3 - D_3 \delta \theta_2)J_1 +$$

$$(D_3 \delta \theta_1 - D_1 \delta \theta_3)J_2)\Phi(Z)$$

(50)

The infinitesimal transformations form a Lie algebra, which can be calculated explicitly from (50).

$$[\delta_{\nu_1}, \delta_{\nu_2}]\Phi(Z) = \delta_{\nu_3}\Phi(Z)$$

$$\nu_3 = \nu_2(\partial_z \nu_1) - \nu_1(\partial_z \nu_2) + \frac{1}{2} \sum_{i=1}^{3} (D_i \nu_2)(D_i \nu_1)$$

(51)

It is worth noting that the algebra closes if and only if the $J_i$ satisfy the commutation relations $[J_i, J_j] = -\frac{1}{2} \epsilon_{ijk} J_k$.

This can then be used to construct an ansatz for an OPE of $T(Z_1)T(Z_2)$ (52), and then the modes calculated to give the commutators of the quantum theory (53).

$$T(Z_1)T(Z_2) = \frac{c}{Z_{12}} + \frac{\theta_{12,1} \theta_{12,2} \theta_{12,3}}{2Z_{12}^2} T(Z_2) +$$

$$\frac{\theta_{12,1} \theta_{12,2} \theta_{12,3}}{Z_{12}} \partial \nu T(Z_2) + \frac{\epsilon_{ijk} \theta_{12,i} \theta_{12,j} D_{2k}}{4Z_{12}} T(Z_2)$$

(52)

The first term gives rise to the central extension in the algebra, and arises in precisely the same way as the bosonic case. This OPE shows explicitly that $T(Z)$ is a weight $\frac{1}{2}$ field, although not primary. Since the central charge does not appear for the super M"{o}bius subalgebra, $T$ can be thought of as a quasiprimary superfield, in the trivial representation of $su(2)$. The modes of this can then be calculated to give the $N = 3$ algebra. Note that when the classical algebra expressions appear in (45), there are extra factors of $\frac{1}{2}$ appearing in (45). This corresponds to the extra factors of $\frac{1}{2}$ appearing in (53) when compared to the classical algebra.

$$[T^i_m, T^j_n] = -\frac{1}{2} \epsilon_{ijk} T^k_{m+n} + mc \delta_{ij} \delta_{m+n,0} \quad [T^i_m, \psi_s] = 0$$

$$[T^i_m, G^i_r] = \frac{1}{2} (\delta_{ij} m \psi_{r+m} - \epsilon_{ijk} G^k_{r+m}) \quad [L^i_m, \psi_s] = -(\frac{m}{2} + s) \psi_{m+s}$$

$$[L^i_m, T^j_n] = -n T^i_{m+n} \quad [G^i_r, \psi_s] = \frac{1}{2} T^i_{r+s} \quad [L^i_m, G^i_r] = (\frac{m}{2} - r) G^i_{r+m}$$

$$[G^i_r, G^j_s] = \frac{1}{2} \delta_{ij} L_{r+s} + \frac{1}{2} \epsilon_{ijk} (r - s) T^k_{r+s} - c(r^2 - \frac{1}{4}) \delta_{r+s,0} \delta_{ij}$$

$$[L^i_m, L^j_n] = (m - n) L^i_{m+n} - cm(m^2 - 1) \delta_{m+n,0} \quad [\psi_r, \psi_s] = c \delta_{r+s,0}$$

(53)
which agrees with [2].

7 The Neveu-Schwarz Algebra and its Verma Module

The $N = 3$ Neveu-Schwarz algebra is given by the above commutation relations where $m \in \mathbb{Z}$, $r \in \mathbb{Z} + \frac{1}{2}$. The basis can be changed so that the above relations are more useful for representation theory. Consider a change of variables

$$T^+_m = 2(iT^1_m - T^2_m) \quad T^-_m = 2(iT^1_m + T^2_m) \quad T^H_m = -2iT^3_m$$

$$G^+_r = 4(G^2_r - iG^1_r) \quad G^-_r = 4(G^2_r + iG^1_r) \quad G^H_r = 8iG^3_r \quad k = -4c \quad (54)$$

Then, the commutation relations become, for $x \in \{H, \pm\}$

$$[T^+_m, T^-_n] = 2T^H_{m+n} + 2km\delta_{m+n,0} \quad [T^H_m, T^+_n] = \pm T^+_{m+n} \quad [T^+_m, T^-_n] = 0$$

$$[T^H_m, T^H_n] = km\delta_{m+n,0} \quad [T^+_m, G^+_n] = 0 \quad [T^+_m, G^-_n] = -G^H_{r+m} \pm 8m\psi_{r+m}$$

$$[T^H_m, G^H_n] = -2G^+_{m+r} \quad [T^H_m, G^H_n] = 8m\psi_{m+r} \quad [T^H_m, G^-_n] = \pm G^H_{r+m}$$

$$[\psi_s, G^+_r] = \mp T^+_{r+s} \quad [\psi_s, G^-_r] = -2T^H_{r+s} \quad [G^+_r, G^+_s] = 0 \quad [T^+_m, \psi_s] = 0$$

$$[G^H_r, G^H_s] = -32L_{r+s} - 16k(r^2 - \frac{1}{4})\delta_{r+s,0} \quad [G^+_r, G^H_s] = 8(r-s)T^+_{r+s}$$

$$[G^-_r, G^-_s] = 16L_{r+s} + 8k(r^2 - \frac{1}{4})\delta_{r+s,0} + 8(r-s)T^H_{r+s}$$

$$[L_m, \psi_s] = -(\frac{m}{2} + s)\psi_{m+s} \quad [L_m, T^x_n] = -nT^x_{m+n} \quad [\psi_r, \psi_s] = -\frac{k}{4}\delta_{r+s,0}$$

$$[L_m, G^x_r] = (\frac{m}{2} - r)G^x_{r+m}$$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{k}{2}m(m^2 - 1)\delta_{m+n,0} \quad (55)$$

On the representations considered here, the algebra obeys hermiticity conditions

$$(\psi_r)^\dagger = -\psi_{-r} \quad (T^+_m)^\dagger = T^-_m \quad (T^-_m)^\dagger = T^+_m \quad (T^H_m)^\dagger = T^H_m$$

$$(G^H_r)^\dagger = -G^H_{-r} \quad (L_n)^\dagger = L_{-n} \quad (G^+_r)^\dagger = G^-_r \quad (G^-_r)^\dagger = G^+_r \quad (56)$$

The highest weight conditions on a vector $|\phi\rangle$ are then

$$T^x_m|\phi\rangle = 0 \quad G^x_r|\phi\rangle = 0 \quad L_m|\phi\rangle = 0 \quad \psi_r|\phi\rangle = 0 \quad T^+_0|\phi\rangle = 0 \quad (57)$$
for $m, r > 0$. The Cartan subalgebra is spanned by the elements $L_0, T_0^H$, such that $L_0 |\phi\rangle = h |\phi\rangle, T_0^H |\phi\rangle = q |\phi\rangle$. The algebra of raising operators, i.e. the algebra spanned by the elements giving the highest weight conditions, is generated by $T_0^+, G^{-\frac{1}{2}}, \psi_\frac{1}{2}$. Thus, a vector $|\chi\rangle$ with the properties $T_0^+ |\chi\rangle = 0, G^{-\frac{1}{2}} |\chi\rangle = 0$ and $\psi_\frac{1}{2} |\chi\rangle = 0$ will obey the highest weight conditions. Consider the Verma module $V(h, q)$ for a highest weight $|\varphi\rangle$, with $L_0 |\varphi\rangle = h |\varphi\rangle, T_0^H |\varphi\rangle = q |\varphi\rangle$. A vector $|\chi\rangle \neq |\varphi\rangle$ in the module defines a singular vector. The module itself admits a decomposition

$$V(h, q) = \bigoplus_{(m \geq 0)} \bigoplus_{(n \leq \frac{m}{2})} V_{m,n}$$

(58)

where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. This can be seen from the root structure [55], and the highest weight conditions. An example of a singular vector occurs when $(h, q, k) = (-\frac{1}{2}, -1, k)$. Under such conditions, a singular vector exists in $V_{\frac{1}{2}, 0}$.

$$|\chi\rangle = T_0^- G^{\frac{1}{2}} |\varphi\rangle$$

(59)

## 8 The Super Möbius Group

One might ask how exactly does the theory of the Möbius group generalise. In the bosonic case, the lie algebra of the group can be obtained by finding the globally defined vector fields on the Riemann sphere. The Riemann sphere can be considered as a pair of complex planes with transition function $w = \frac{1}{z}$ between them. In the graded Riemann sphere case, one can choose a homomorphism between rings of functions given by $(w, \chi_i) = (\frac{1}{z}, \frac{\theta_i \sqrt{-1}}{z})$. The south pole is $Z_s = (z, \theta_i) = (0, 0, 0, 0)$, and the north pole given by $Z_n = (\frac{1}{z}, \frac{\theta_i \sqrt{-1}}{z}) = (0, 0, 0, 0)$. One can then ask what are the globally defined graded vector fields. A basis of vector fields was calculated in section 5. It can be seen that many of them are divergent at the origin, or south pole. One must then check which vector fields are well behaved at both poles. As an example, consider the vector field [33]. This is clearly divergent for $m < -1$ at the south pole. To find out what $l_n$ looks like at the north pole, one uses the techniques of graded one-forms and vector fields [23] to find

$$l_m = w^{-m+1} \frac{\partial}{\partial w} - \frac{1}{2}(m - 1)w^{-m} \frac{\partial}{\partial \chi_j}$$

(60)

which is divergent for $m > 1$ at the north pole. Thus, one can conclude that $\{l_1, l_0, l_{-1}\}$ are globally defined. Similarly, one finds that the only other globally defined vector fields are $\{g^+_1, g^-_{-\frac{1}{2}}, t^0\}$. From the commutation relations, [34] it can be seen that the vector fields form a closed subalgebra, namely $osp(3, 2)$. One can then write down formal group elements by exponentiation.

$$\exp(\lambda l_1) : (z, \theta_i) \mapsto \frac{1}{1 - \lambda z}(z, \theta_i)$$

$$\exp(\lambda l_0) : (z, \theta_i) \mapsto (e^\lambda z, e^{\frac{\lambda}{2}} \theta_i)$$
\[
\exp(\lambda_{-1}) : (z, \theta_i) \mapsto (z + \lambda, \theta_i) \\
\exp(\epsilon g^i_{\frac{1}{2}}) : (z, \theta_i) \mapsto (z - \epsilon \theta_j, \theta_i, \theta_j + \epsilon) \quad \text{if } i \neq j \\
\exp(\epsilon g^i_{\frac{1}{2}}) : (z, \theta_i) \mapsto \frac{1}{1 + \epsilon \theta_j} (z, \theta_i, \theta_j + \epsilon z) \quad \text{if } i \neq j \\
\exp(\lambda_i t^i_0) : (z, \theta_i) \mapsto (z, M_{ij}(\lambda) \theta_j) \quad M_{ij}(\lambda) \in SO(3) \quad (61)
\]

In particular, the \(g^i_{\frac{1}{2}}\) give supersymmetry generators, the \(g^i_{\frac{1}{2}}\) give special superconformal transformations, and the \(t^i_0\) give an R-symmetry. Writing these transformations as \(Z \mapsto Z'\), the corresponding transformations on the field become

\[
e^{\lambda L_0} \Phi(Z) e^{-\lambda L_0} = e^{\lambda h} \Phi(Z') \\
e^{\epsilon G^i} \Phi(Z) e^{-\epsilon G^i} = e^{\frac{1}{2} \epsilon G^i} \Phi(Z') \\
e^{\rho G^i} \Phi(Z) e^{-\rho G^i} = \frac{1}{1 + \epsilon \theta_i} e^{-2 \rho \epsilon \theta_j \theta_j} \Phi(Z') \\
e^{\lambda T^i_0} \Phi(Z) e^{-\lambda T^i_0} = e^{\lambda L^i} \Phi(Z') \\
e^{\lambda L_1} \Phi(Z) e^{-\lambda L_1} = \frac{1}{1 - \lambda \beta} e^{(\frac{\lambda}{1 - \lambda \beta}) \epsilon_{ijk} \theta_i \theta_j \theta_k} \Phi\left(\frac{z}{(1 - \lambda \beta)^2}, \frac{\theta}{(1 - \lambda \beta)^2}\right) \quad (62)
\]

Using these formal group elements, any two points, \(V = (v, \beta_i)\) and \(U = (u, \alpha_i)\) say, can be mapped to the north and south poles respectively. In a conformal field theory formalism, usually the south pole is where the ‘in vacuum’ sits, and the north pole where the ‘out vacuum’ sits. The formal group element corresponding to this map is given by

\[
(z, \theta_i) \mapsto (z', \theta'_i) = \left(\frac{z - u, \theta_i - \alpha_i + (\alpha_i - \beta_i)(z - u)}{1 + (\alpha - \beta) \frac{v}{v - u}} - (1 + \frac{\alpha - \beta}{v - u})(\frac{z - u}{v - u})\right) \quad (63)
\]

where \(\beta = \sum_{i=1}^{3} \alpha_i \beta_i\).

To obtain this transformation, one can use \(g^i_{\frac{1}{2}}\) to send \(\alpha^i\) to 0, \(l_{-1}\) to move \(u\) to 0, \(g^i_{\frac{1}{2}}\) to send \(\beta_i\) to 0 when \(z = v\), and \(l_1\) to send \(v\) to \(\infty\). It is worth noting, that the only operators that have not been used are the \(t^i_0\) and \(l_0\). This degree of freedom is essentially a (complex) scale factor, and an \(SO(3)\) action on the \(\theta_i\). Thus, the even ‘co-ordinate’ of the third point can be sent anywhere one wishes, but on cannot quite do the same with the odd ‘co-ordinates’ of the third point. It should also be noted, that this construction should generalise to \(osp(N, 2)\), i.e. with an arbitrary number of odd co-ordinates.

The formal group element found \([63]\) implies that a correlation function of the form

\[
\langle 0 | \Phi_1(V) \Phi_2(Z) \Phi_3(U) | 0 \rangle \quad (64)
\]

can be superconformally mapped to a more typical presentation of the three-point function in conformal field theory:

\[
\langle 0 | \Phi_1(\infty) \Phi_2(Z') \Phi_3(0) | 0 \rangle = \langle \phi_1 | \Phi_2(Z') \phi_3 \rangle \quad (65)
\]
9 The Two-Point Function

In conformal field theory it is known that global conformal invariance is sufficient to solve for the two point function. This is indeed also the case for $N = 1, 2$. In this section, global $N = 3$ invariance is used to solve for the two-point function. This becomes quite a bit more complicated than smaller $N \leq 2$, calculationally, due to the presence of non-abelian R-symmetry, manifested by the presence of $su(2)$ generators in the theory. More precisely, the primary fields are $A_N \otimes \text{End} \mathcal{H} \otimes \mathcal{V}$ valued, where $\mathcal{V}$ is an $su(2)$ representation and $\mathcal{H}$ is the Hilbert space that $|0\rangle$ belongs to. The super-Virasoro operators are valued in $\text{End} \mathcal{H} \otimes \text{End} \mathcal{V}$.

The most convenient basis to work in is a 'charged' basis, where elements can be classified by their $su(2)$ charge, namely their $T_0$ eigenvalue. The basis is given by

$$\begin{align*}
\theta^+ &= 2(i\theta_1 - \theta_2) \\
\theta^- &= 2(i\theta_1 + \theta_2) \\
\theta^H &= i\theta_3
\end{align*}$$

The primary field $\Phi$ itself is the highest weight in an $su(2)$ representation, i.e. carries an $su(2)$ representation index, so that

$$\begin{align*}
J^+ \Phi(Z) &= (J^+)^A_B \Phi^B(Z) = 0 \\
J^H \Phi(Z) &= (J^H)^A_B \Phi^B(Z) = q \Phi^A(Z)
\end{align*}$$

In the following, $\Phi_i(Z)$ has conformal weight $h_i$ and spin $q_i$. The action of the twelve globally defined generators on $\Phi(Z)$ can then be given in Lie algebra form. The infinitesimal transformations are

$$\begin{align*}
[L_{-1}, \Phi] &= \partial_z \Phi \\
[G^\pm_{-\frac{1}{2}}, \Phi] &= \pm(\theta^\pm \partial_z + 8 \partial_{\theta^\pm})\Phi \\
[G^H_{-\frac{1}{2}}, \Phi] &= -4(\theta^H \partial_z + \partial_{\theta^H})\Phi \\
[T^H_0, \Phi] &= (\theta^- \partial_{\theta^-} - \theta^+ \partial_{\theta^+} + J^H)\Phi \\
[L_0, \Phi] &= (h + z\partial_z + \frac{1}{2}(\theta^+ \partial_{\theta^+} + \theta^- \partial_{\theta^-} + \theta^H \partial_{\theta^H}))\Phi \\
[T^\pm_0, \Phi] &= (\mp \frac{1}{2} \theta^\pm \partial_{\theta^H} + 4 \theta^H \partial_{\theta^\pm} + J^\pm)\Phi \\
[L_1, \Phi] &= (2hz + z(z\partial_z + \theta^+ \partial_{\theta^+} + \theta^- \partial_{\theta^-} + \theta^H \partial_{\theta^H}) + \frac{1}{8} \theta^+ \theta^- J^H \\
& \quad + \frac{1}{4} \theta^+ \theta^H J^- - \frac{1}{4} \theta^- \theta^H J^+)\Phi \\
[G^\pm_{\frac{1}{2}}, \Phi] &= (\pm 2h \theta^\pm + \theta^\pm z\partial_z + 8z\partial_{\theta^\pm} + \theta^\pm \theta^H \partial_{\theta^H} + \theta^+ \theta^- \partial_{\theta^\pm} \\
& \quad + 2 \theta^H \partial_{\theta^\pm} + \theta^\pm J^\pm)\Phi \\
[G^H_{\frac{1}{2}}, \Phi] &= (-8h \theta^H - 4 \theta^H z\partial_z - 4z\partial_{\theta^H} - 4 \theta^H \theta^- \partial_{\theta^-} - 4 \theta^H \theta^+ \partial_{\theta^+} \\
& \quad + \theta^- J^+ - \theta^+ J^-)\Phi
\end{align*}$$

(68)
Note that under the $T^H_0$ operator, $\theta^+$ and $\theta^-$ are ‘charged’, i.e. they possess a non-zero $T^H_0$ eigenvalue. The two point function, $\langle 0|\Phi_1(Z_1)\Phi_2(Z_2)|0 \rangle = \langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle$ is, as a function, a function of $Z_1 = (z, \theta_1)$ and $Z_2 = (w, \chi_i)$. Since the $\Phi_i$ are also highest weight vectors of $su(2)$ representations, $\mathcal{V}_i$, the two point function is an element of $\mathcal{V}_1 \otimes \mathcal{V}_2$. The $L_{-1}$ condition on the two-point function reads
\[ \mathcal{L}_{-1}\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle = (\partial_z + \partial_w)\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle = 0 \] (69)
implies that $\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle$ is a function of $(z - w)$ and $\theta_1, \chi_i$. Applying the $G^x_{-\frac{1}{2}}$ conditions yields similar equations to (69). These conditions show that $\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle$ is a function of
\[ s = (z - w + \frac{1}{8}(\theta^- \chi^+ + \theta^+ \chi^-) + \theta^H \chi^H) \] (70)
The $L_0$ condition gives a scaling condition, from which the most general form of the two point function can be seen to be
\[ \langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle = \frac{\alpha}{s^{h_1+h_2}} + \frac{\epsilon_+(\theta^+ - \chi^+)}{s^{h_1+h_2+\frac{1}{2}}} + (\text{two similar terms}) \]
\[ + \frac{b_+ H(\theta^+ - \chi^+)(\theta^H - \chi^H)}{s^{h_1+h_2+\frac{1}{2}}} + (\text{two similar terms}) \]
\[ + \frac{\eta(\theta^+ - \chi^+)(\theta^H - \chi^H)}{s^{h_1+h_2+\frac{1}{2}}} \] (71)
The $T^H_0$ condition includes $su(2)$ elements. It is worth writing this condition out explicitly, to illustrate the action of the elements. Putting in all the tensor products between $su(2)$ representations explicitly, the condition reads
\[ \left( (\mathbb{I} \otimes \mathbb{I}) (\theta^- \partial_{\theta^-} - \theta^+ \partial_{\theta^+} + \chi^- \partial_{\chi^-} - \chi^+ \partial_{\chi^+}) + J^H \otimes \mathbb{I} + \mathbb{I} \otimes J^H \right) \langle \Phi_1(Z_1) \otimes \Phi_2(Z_2) \rangle = 0 \] (72)
where
\[ \langle J^H \otimes \mathbb{I} + \mathbb{I} \otimes J^H \rangle \langle \Phi_1(Z_1) \otimes \Phi_2(Z_2) \rangle = \langle (J^H \Phi_1)(Z_1) \otimes \Phi_2(Z_2) \rangle + \langle \Phi_1(Z_1) \otimes (J^H \Phi_2)(Z_2) \rangle = (q_1 + q_2) \langle \Phi_1(Z_1) \otimes \Phi_2(Z_2) \rangle \] (73)
This condition gives three possible cases
\[ q_1 + q_2 = 0 \Rightarrow \text{only } (a, \epsilon_H, b_{+, -}, \eta) \text{ non-zero} \]
\[ q_1 + q_2 = 1 \Rightarrow \text{only } (\epsilon_+, b_{+H}) \text{ non-zero} \]
\[ q_1 + q_2 = -1 \Rightarrow \text{only } (\epsilon_-, b_{-H}) \text{ non-zero} \] (74)
Replacing $H$ with $+$ in (73), it can be seen that $J^+ \otimes \mathbb{I} + \mathbb{I} \otimes J^+$ annihilates $\langle \Phi_1(Z_1) \otimes \Phi_2(Z_2) \rangle$. The $T^+_{0}$ condition then gives - if $q_1 + q_2 = -1$, then $\epsilon_-, b_{-H} = 0$ - if $q_1 + q_2 = 0$, then $\epsilon_+, b_{+H} = 0$ - and gives no extra conditions if $q_1 + q_2 = 1$. Thus, the $q_1 + q_2 = -1$
case is irrelevant. $J^-$ is an operator that can cause calculational difficulties. The $T_0^-$ condition can be used to relate $\langle (J^-\Phi_1)\Phi_2 \rangle$ and $\langle \Phi_1(J^-\Phi_2) \rangle$.

Consider now the $L_1$ condition. This contains a term like

$$\theta^+\theta^H \langle (J^-\Phi_1)\Phi_2 \rangle + \chi^+\chi^H \langle \Phi_1(J^-\Phi_2) \rangle$$

The $T_0^-$ condition can be used to relate this to a term of the form

$$(\theta^+\theta^H - \chi^+\chi^H) \langle (J^-\Phi_1)\Phi_2 \rangle$$

Thus the condition implies that all those terms that cannot be factored by $(\theta^+\theta^H - \chi^+\chi^H)$ are zero. A similar condition arises for the $G^{x\frac{1}{2}}$ conditions. After much tedious algebra, one finds that

$$\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle = \left\{ \begin{array}{ll}
\frac{q_{h_1+h_2}}{\theta_{h_1}^+(\theta^+\theta^H)\theta_{h_2}^-(\theta^H\theta^+)} & \text{if } h_1 = h_2, \quad q_1 = q_2 = 0 \\
\frac{b_{h_1+h_2}}{\theta_{h_1}^+(\theta^+\theta^H)\theta_{h_2}^-(\theta^H\theta^+)} & \text{if } h_1 = h_2, \quad q_1 + q_2 = 1, \quad q_1, q_2 \neq 0 \\
0 & \text{otherwise}
\end{array} \right. \quad (75)$$

This has important applications to fusion. Considering the three-point function

$$\mathcal{F}_{123} = \langle \Phi_1(Z_1)\Phi_2(Z_2)\Phi_3(Z_3) \rangle$$

where $\Phi_1(Z_1)$ is a ‘probe field’ i.e. one can choose its $(h, q)$ parameters, call them $(h_1, q_1)$. $\Phi_2(Z_2)$ and $\Phi_3(Z_3)$ will have an OPE, which schematically looks like (i.e. omitting pole structure and other factors)

$$\Phi_2(Z_2)\Phi_3(Z_3) \sim \sum_n \Psi_n(Z_3) \quad (76)$$

that may be unknown, namely one may not know the $(h, q)$ of the $\Psi_n$. One can ask if the OPE between $\Phi_2(Z_2)$ and $\Phi_3(Z_3)$ can be deduced if one knows the values of $\mathcal{F}_{123}$, for all $h_1$ and $q_1$. From (75), one can see that for $\langle \Phi_1(Z_1), \Psi_n(Z_3) \rangle$ to be non-zero, a unique $(h_1, q_1)$ must be chosen. This choice determines the $(h, q)$ of $\Psi_n(Z_3)$. Thus one can make the statement that knowing when the three point function $\mathcal{F}_{123}$ vanishes is equivalent to knowing what $\Psi_n(Z_3)$ are in (76). These then give rise to the fusion rules.

One should note that global conformal invariance of the theory almost fixes three super co-ordinates, as can be seen from (63). In the mapping from (63) to (65) one can map $V$ and $U$ to the north and south poles. One can also map $z'$ from $Z' = (z', \theta'_i)$ in (63), to wherever desired, using $L_0$. There is not enough freedom to move the $\theta'_i$ wherever desired. Thus, one would expect that the three-point function could also be computed, up to an arbitrary function in $\theta'_i$. After expanding this function into components, this can be seen as being computable up to some arbitrary constants.

### 10 Other Applications

This section is strictly speaking a list of things that could be done, in the $N = 3$ theory. Since most of these things are very calculationally intensive, the author has not checked the details.
An interesting question is analysing the constraints that singular vectors give on three-point functions. If $|\chi\rangle$ is a singular vector in a module with highest weight $|\phi_1\rangle$, then what does the requirement that

$$\langle \phi_3 | \Phi(Z) | \chi \rangle$$

vanish imply about

$$\langle \phi_3 | \Phi(Z) | \phi_1 \rangle$$

Algebraically, this is in fact quite difficult, and the author has not managed to accomplish this. The main complication is that the composition between primary fields in the correlator are a tensor product between $su(2)$ representations $V$ and $V'$ and an $\text{End}H$ composition. This means that the only super-Virasoro operators that can be transferred across the tensor product are those that have $\text{End}V$ part proportional to the identity. The most obvious case where this applies is where all the fields have $q = 0$, i.e. they are all $su(2)$ singlets. Following [11], the lowering operators acting appearing in $|\chi\rangle$ can be rewritten in terms of operators that have commutator with a primary field in $(z, \theta_i)$ given by a polynomial in $(z, \theta_i)$, namely

$$L_m = -L_m + \frac{1}{z} L_{m+1} + \frac{1}{16z} (\theta^+ G_{m+\frac{1}{2}}^+ - \theta^- G_{m+\frac{1}{2}}^- + 2\theta^H G_{m+\frac{1}{2}}^H) +$$

$$\frac{m+1}{4z} (\theta^- \theta^H T_m^+ - \theta^+ \theta^H T_m^- - \theta^+ \theta^- T^H_m)$$

$$G_r^\pm = -G_r^\pm + \frac{1}{z} G_{r+1}^\pm - \frac{2\theta^H}{z} T_{r+\frac{1}{2}}^\pm - \frac{\theta^\pm}{z} T_r^\pm$$

$$G_r^H = -G_r^H + \frac{1}{z} G_{r+1}^H - \frac{\theta^-}{z} T_{r+\frac{1}{2}}^H + \frac{\theta^+}{z} T_{r+\frac{1}{2}}^-$$

$$T_m^\pm = -T_m^\pm + \frac{1}{z} T_{m+1}^\pm - \frac{\theta^\pm}{z} \theta_{m+\frac{1}{2}}$$

$$T_m^H = -T_m^H + \frac{1}{z} T_{m+1}^H + \frac{2\theta^H}{z} \theta_{m+\frac{1}{2}}$$

$$P_r = -\psi_r + 1 \nu \psi_{r+1}$$

One then finds

$$[L_m, \Phi(Z)] = h z^m, \quad [G_r^\pm, \Phi(Z)] = \pm 2h \theta^\pm z^{r-\frac{1}{2}}$$

$$[G_r^H, \Phi(Z)] = -8h \theta^H z^{r-\frac{1}{2}}, \quad [T_m^\pm, \Phi(Z)] = \pm h \theta^\pm \theta^H z^{m-1},$$

$$[T_m^H, \Phi(Z)] = -\frac{1}{4} h \theta^+ \theta^- z^{m-1}, \quad [P_r, \Phi(Z)] = \frac{1}{8} h \theta^+ \theta^- \theta^H z^{r-\frac{3}{2}}$$

Now one can commute the operators from $|\chi\rangle$ past $\Phi$ without introducing differential operators. The operators $L_m$ etc may now be re-expanded in terms of $L_m$ etc. Some of these operators will annihilate $\langle \phi_3 \rangle$. Those that do not, and are not diagonal, must be processed using the descent equations on $|\Phi(Z)\phi_1\rangle$ [6]. This then yields a set of polynomial equations giving conditions on the weights of the primary fields.
From a differential equation point of view, the question of singular vectors may not be such a difficult problem. As in [6], the lowering operators can be written as contour integrals, e.g.

\[
L_{-k}(z) = \frac{1}{2\pi i} \oint dw L(z)(w-z)^{1-k}
\]
\[
G_{-r}(z) = \frac{1}{2\pi i} \oint dw G(z)(w-z)^{1/2-r}
\]

etc. (79)

The condition of a singular vector, \(\mathcal{N}\phi_3\), where \(\mathcal{N}\) are some lowering operators, can be written as

\[
\langle \Phi_1(Z_1)\Phi_2(Z_2)(\mathcal{N}\Phi_3)(Z_3) \rangle = 0
\]

This then gives rise, via (79) and the OPE, to a differential equation on the three point function. One would expect the three point function to look like a product of powers of differences, as in the case of the two point function, e.g. \(s, (\theta^+ - \chi^+)\). As in the bosonic case, this should give rise to a polynomial in the \(h_i, q_i\) of the fields concerned. The difference now, is that the presence of \(J^-\) operators will give independent equations, e.g. \(F_{123} \in \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3\), hence \((\mathbb{I} \otimes \mathbb{I} \otimes J^-)\mathcal{F}\) is linearly independent of \((\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I})\mathcal{F}\). Other than this, the calculations should proceed precisely as in the bosonic case.

11 Conclusions

Starting from a graded Riemann sphere, a superconformal field theory was constructed. The construction roughly parallels that of the bosonic case, namely defining sections of a line bundle on a Riemann sphere, and rewriting the infinitesimal transformations of these sections as operator product expansions. Two ways were used to introduce a Ramond field, one by introducing a branch cut, the other by altering the preserved one-form. This suggests that looking at various sheaves on a graded Riemann sphere may be a potentially useful way of realizing fields in a superconformal field theory.

The super OPEs, together with an understanding of how the symmetries act on the graded Riemann sphere, were sufficient to compute the \(N = 3\) two-point function, up to multiplicative constants. In addition, it was illustrated how, in principle, the \(N = 3\) three point function and conditions given by singular vectors on the three point function could be calculated. It should be pointed out that the method of calculation was entirely in superfield formalism, and hence manifestly supersymmetric.

The only case that has really been studied here is the \(N = 3\) case, based on a Riemann Surface of genus zero. How this generalises to higher genus is an interesting question. An even more interesting question, is the \(N = 4\) case. Processing the \(N = 4\) theory through this machinery, does not produce the full OPE of the theory. There is a log term missing from the OPE corresponding to a \(U(1)\) charge. A question then arises, how to extend the framework of a graded Riemann sphere to incorporate this log term. Many parts of the \(N = 4\) theory will in fact look like the \(N = 3\) theory, since the \(N = 4\) currents arising from R-symmetry are a pair of commuting \(su(2)\) currents.
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[32] Assume otherwise, so that $\kappa \kappa' = 1 = \kappa' \kappa$, and $\kappa \kappa'' = 1 = \kappa'' \kappa$. Subtracting and factorising gives $(\kappa' - \kappa'')\kappa = 0 = \kappa(\kappa' - \kappa'')$. Using either inverse on these equations shows $\kappa' - \kappa'' = 0$.

[33] The commutators are graded, i.e. $[A, B] = AB - (-1)^{p(A)p(B)} BA$

[34] $\epsilon_{123} = 1$ and $\epsilon$ is antisymmetric in all its indices. Summation convention is used over repeated indices

[35] The commutation relations are $[J_i, J_j] = -\frac{1}{2} \epsilon_{ijk} J_k$. These are precisely the commutation relations of the sub-algebra formed by the $\frac{1}{2} t^i_0$
In the bosonic case, a vertex operator is characterised uniquely by its action on a vacuum, and can be defined by \( \phi(z)|0\rangle = \exp(zL_{-1})|\phi\rangle \). On a super-complex plane, this can be generalised to \( \Phi(Z)|0\rangle = \exp(zL_{-1})\exp(\theta_1G_{-\frac{1}{2}}^1)\ldots\exp(\theta_NG_{-\frac{1}{2}}^N)|\Phi\rangle \), giving the operator associated to a state.

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More precisely, these come from an ungraded involution on the algebra. On any representation of the algebra that admits a hermitian contragredient form, this involution then gives the adjoint with respect to that form.