Complexity of 3-manifolds

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Abstract

We give a summary of known results on Matveev’s complexity of compact 3-manifolds. The only relevant new result is the classification of all closed orientable irreducible 3-manifolds of complexity 10.

1. Introduction

In 3-dimensional topology, various quantities are defined, that measure how complicated a compact 3-manifold $M$ is. Among them, we find the Heegaard genus, the minimum number of tetrahedra in a triangulation, and Gromov’s norm (which equals the volume when $M$ is hyperbolic). Both Heegaard genus and Gromov norm are additive on connected sums, and behave well with respect to other common cut-and-paste operations, but it is hard to classify all manifolds with a given genus or norm. On the other hand, triangulations with $n$ tetrahedra are more suitable for computational purposes, since they are finite in number and can be easily listed using a computer, but the minimum number of tetrahedra is a quantity which does not behave well with any cut-and-paste operation on 3-manifolds. (Moreover, it is not clear what is meant by “triangulation”: do the tetrahedra need to be embedded? Are ideal vertices admitted when $M$ has boundary?)

In 1988, Matveev introduced \cite{Matveev88} for any compact 3-manifold $M$ a non-negative integer $c(M)$, which he called the complexity of $M$, defined as the minimum number of vertices of a simple spine of $M$. The function $c$ is finite-to-one on the most interesting sets of compact 3-manifolds, and it behaves well with respect to the most important cut-and-paste operations. Its main properties are listed below.

**additivity** \( c(M \# M^0) = c(M) + c(M^0) \);

**finiteness** for any $n$ there is a finite number of closed $\mathbb{P}^2$-irreducible $M$’s with $c(M) = n$, and a finite number of hyperbolic $N$’s with $c(N) = n$;

**monotonicity** $c(M_F) \leq c(M)$ for any incompressible $F$ cutting $M$ into $M_F$.

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We recall some definitions used throughout the paper. Let $M$ be a compact 3-manifold, possibly with boundary. We say that $M$ is hyperbolic if it admits (after removing all tori and Klein bottles from the boundary) a complete hyperbolic metric of finite volume (possibly with cusps and geodesic boundary). Such a metric is unique by Mostow’s theorem (see [35] for a proof). A surface in $M$ is essential if it is incompressible, $\partial$-incompressible, and not $\partial$-parallel. Thurston’s Hyperbolicity Theorem for Haken manifolds ensures that a compact $M$ with boundary is hyperbolic if and only if every component of $\partial M$ has $\chi \leq 0$, and $M$ does not contain essential surfaces with $\chi > 0$. The complexity satisfies also the following strict inequalities.

**filling** every closed hyperbolic $M$ is a Dehn filling of some hyperbolic $N$ with $c \langle N \rangle < c \langle M \rangle$;

**strict monotonicity** $c \langle M_F \rangle < c \langle M \rangle$ if $F$ is essential and $M$ is closed $\mathbb{P}^2$-irreducible or hyperbolic;

Some results in complexity zero already show that the finiteness property does not hold for all compact 3-manifolds.

**complexity zero** the closed $\mathbb{P}^2$-irreducible manifolds with $c = 0$ are $S^1 \# \mathbb{R} \mathbb{P}^3$ and $L(3,1)$. We also have $c(\mathbb{S}^2 \setminus S^1) = c(\mathbb{S}^2 \setminus S^1) = 0$. Interval bundles over surfaces and handlebodies also have $c = 0$.

The ball and the solid torus have therefore complexity zero. Moreover, the additivity property actually also holds for $\partial$-connected sums. These two facts together imply the following.

**stability** The complexity of $M$ does not change when adding 1-handles to $M$ or removing interior balls from it.

Note that both such operations that do not affect $c$ are “invertible” and hence topologically inessential. In what follows, a simplicial face-pairing $T$ of some tetrahedra is a triangulation of a closed 3-manifold $M$ when $M = \mathcal{T} \mathcal{T}$. Tetrahedra are therefore not necessarily embedded in $M$. A simplicial pairing $T$ is an ideal triangulation of a compact $M$ with boundary if $M$ is $\mathcal{T} \mathcal{T}$ minus open stars of all the vertices. The finiteness property above follows easily from the following.

**naturality** if $M$ is closed $\mathbb{P}^2$-irreducible and not $S^3 \# \mathbb{R} \mathbb{P}^3$ or $L(3,1)$, then $c \langle M \rangle$ is the minimum number of tetrahedra in a triangulation of $M$. If $N$ is hyperbolic with boundary, then $c \langle N \rangle$ is the minimum number of tetrahedra in an ideal triangulation of $N$. 

The beauty of Matveev’s complexity theory relies on the fact that simple spines are more flexible than triangulations: for instance spines can often be simplified by puncturing faces, and can always be cut along normal surfaces. In particular, we have the following result. An (ideal) triangulation $T$ of $M$ is minimal when $M$ cannot be (ideally) triangulated with fewer tetrahedra. A normal surface in $T$ is one intersecting the tetrahedra in normal triangles and squares, see [21].

**normal surfaces** let $T$ be a minimal (ideal) triangulation of a closed $\mathbb{P}^2$-irreducible (hyperbolic with boundary) manifold $M$ different from $S^3$, $\mathbb{R}P^3$, and $L(3;1)$. If $F$ is a normal surface in $T$ containing some squares, then $c(M_F) < c(M)$.

As an application of the previous properties, the following result was implicit in Matveev’s paper [30].

**Corollary 1.1.** Let $T$ be a minimal triangulation of a closed $\mathbb{P}^2$-irreducible 3-manifold $M$ different from $S^3, \mathbb{R}P^3, L(3;1)$. Then $T$ has one vertex only, and it contains no normal spheres, except the vertex-linking one.

Computers can easily handle spines and triangulations, and manifolds of low complexity have been classified by various authors. Closed orientable irreducible manifolds with $c \leq 6$ were classified by Matveev [29] in 1988. Those with $c = 7$ were then classified in 1997 by Ovchinnikov [37, 31], and those with $c = 8, 9$ in 2001 by Martelli and Petronio [25]. We present here the results we recently found for $c = 10$. The list of all manifolds with $c = 10$ has also been computed independently by Matveev [32], and the two tables (each consisting of 3078 manifolds) coincide. The closed $\mathbb{P}^2$-irreducible non-orientable manifolds with $c \leq 7$ have been listed independently in 2003 by Amendola and Martelli [4], and Burton [9].

Hyperbolic manifolds with cusps and without geodesic boundary were listed for all $c \leq 3$ in the orientable case by Matveev and Fomenko [34] in 1988, and for all $c \leq 7$ by Callahan, Hildebrand, and Weeks [10] in 1999. Orientable hyperbolic manifolds with geodesic boundary (and possibly some cusps) were listed for $c \leq 2$ by Fujii [18] in 1990, and for $c \leq 4$ by Frigerio, Martelli, and Petronio [15] in 2002.

All properties listed above were proved by Matveev in [30], and extended when necessary to the non-orientable case by Martelli and Petronio in [26], except the filling property, which is a new result proved below in Subsection 2.3. The only other new results contained in this paper are the complexity-10 closed census (also constructed independently by Matveev [32]), and the following counterexample (derived from that census) of a conjecture of Matveev and Fomenko [34] stated in Subsection 5.3.

**Proposition 1.2.** There are two closed hyperbolic fillings $M$ and $M^0$ of the same cusped hyperbolic $N$ with $c(M) < c(M^0)$ and $\text{Vol}(M) > \text{Vol}(M^0)$.

We mention the most important discovery of our census.
Proposition 1.3. There are 25 closed hyperbolic manifolds with $c = 10$ (while none with $c \leq 8$ and four with $c = 9$).

This paper is structured as follows: the complexity of a 3-manifold is defined in Section 2. We then collect in Section 3 and 4 the censuses of closed and hyperbolic 3-manifolds described above, together with the new results in complexity 10. Relations between complexity and volume of hyperbolic manifolds are studied in Section 5. Lower bounds for the complexity, together with some infinite families of hyperbolic manifolds with boundary for which the complexity is known, are described in Section 6. The algorithm and tools usually employed to produce a census are described in Section 7. Finally, we describe the decomposition of a manifold into bricks introduced by Martelli and Petronio in [25, 26], necessary for our closed census with $c = 10$, in Section 8. All sections may be read independently, except that Sections 7 and 8 need the definitions contained in Section 2.

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2. The complexity of a 3-manifold

We define here simple and special spines, and the complexity of a 3-manifold. We then show a nice relation between spines without vertices and Riemannian geometry, found by Alexander and Bishop [2]. Finally, we prove the filling property stated in the Introduction.

2.1. Definitions

We start with the following definition. A compact 2-dimensional polyhedron $P$ is simple if the link of every point in $P$ is contained in the graph $\ast$. Alternatively, $P$ is simple if it is locally contained in the polyhedron shown in Fig. 1-(3). A point, a compact graph, a compact surface are therefore simple. The polyhedron given by two orthogonal discs intersecting in their diameter is not simple. Three important possible kinds of neighborhoods of points are shown in Fig. 1. A point having the whole of $\ast$ as a link is called a vertex, and its regular neighborhood is shown in Fig. 1-(3). The set $V(P)$ of the vertices of $P$ consists of isolated points, so it is finite. Note that points, graphs, and surfaces do not contain vertices.

A compact polyhedron $P$ is a spine of a compact manifold $M$ with boundary if $M$ collapses onto $P$. When $M$ is closed, we say that $P$ is a spine if $M \cap P$ is an open ball. The complexity $c(M)$ of a compact 3-manifold $M$ is the minimal number of
vertices of a simple spine of $M$. As an example, a point is a spine of $S^3$, and therefore $c(S^3) = 0$. A simple polyhedron is special when every point has a neighborhood of one of the types (1)-(3) shown in Fig. 1 and the sets of such points induce a cellularization of $P$. That is, defining $S(P)$ as the set of points of type (2) or (3), the components of $P \cap S(P)$ should be open discs – the faces – and the components of $S(P) \cap V(P)$ should be open segments – the edges.

**Remark 2.1.** A special spine of a compact $M$ with boundary is dual to an ideal triangulation of $M$, and a special spine of a closed $M$ is dual to a 1-vertex triangulation of $M$, as suggested by Fig. 2. In particular, a special spine is a spine of a unique manifold. Therefore the naturality property of $c$ may be read as follows: every closed irreducible or hyperbolic manifold distinct from $S^3 \sharp \mathbb{R}P^3$, and $L(3;1)$ has a special spine with $c(M)$ vertices. Such a special spine is then called minimal.

### 2.2. Complexity zero

A handlebody $M$ collapses onto a graph, which has no vertices, hence $c(M) = 0$. An interval bundle $M$ over a surface has that surface as a spine, and hence $c(M) = 0$ again. Note that, by shrinking the fibers of the bundle, the manifold $M$ admits product metrics with arbitrarily small injectivity radius and uniformly bounded curvature. This is a particular case of a relation between spines and Riemannian geometry found by Alexander and Bishop [2]. A Riemannian 3-manifold $M$ is thin when its curvature-normalized injectivity radius is less than some constant $a_2 = 0.75$, see [2] for details. We have the following.
Proposition 2.2 (Alexander-Bishop [2]). A thin Riemannian 3-manifold has complexity zero.

2.3. The filling property

We prove here the filling property, stated in the Introduction. Recall from [30, 31] that by thickening a special spine \( P \) of \( M \) we get a handle decomposition \( \xi_P \) of the same \( M \). Normal surfaces in \( \xi_P \) correspond to normal surfaces in the (possibly ideal) triangulation dual to \( P \).

Theorem 2.3. Every closed hyperbolic manifold \( M \) is a Dehn filling of some hyperbolic \( N \) with \( c(N) < c(M) \).

Proof. Let \( P \) be a minimal special spine of \( M \), which exists by Remark 2.1. Take a face \( f \) of \( P \). By puncturing \( f \) and collapsing the resulting polyhedron as much as possible, we get a simple spine \( Q \) of some \( N \) obtained by drilling \( M \) along a curve. Since \( P \) is special, \( f \) is incident to at least one vertex. During the collapse, all vertices adjacent to \( f \) have disappeared, hence \( Q \) has less vertices than \( P \). This gives \( c(N) < c(M) \).

If \( N \) is hyperbolic we are done. Suppose it is not. Then it is reducible, Seifert, or toroidal. If \( N \) is reducible, the drilled solid torus is contained in a ball of \( M \) and we get \( N = M \# M^0 \) for some \( M^0 \), hence \( c(M) < c(N) < c(M) \) by the additivity property. Then \( N \) is irreducible. Moreover \( \partial N \) is incompressible in \( N \) (because \( M \) is not a lens space).

Then the 1-dimensional portion of \( Q \) can be removed, and we can suppose \( Q \) is a spine of \( N \) having only points of the type of Fig. 1.

Our \( N \) cannot be Seifert (because \( M \) is hyperbolic), hence its JSJ decomposition consists of some tori \( T_1; \ldots; T_k \). Each \( T_i \) is essential in \( N \) and compressible in \( M \). Each \( T_i \) can be isotoped in normal position with respect to \( \xi_Q \). Since \( Q \) is a spine, every normal surface in \( \xi_Q \) is normal also in \( \xi_P \). The only normal surface in \( \xi_P \) not containing squares is the vertex-linking sphere, therefore we have \( c(M_{T_i}) < c(M) \) for all \( i \) by the normal surfaces property. Each \( T_i \) is compressible in \( M \), hence either it bounds a solid torus or is contained in a ball. The latter case is excluded, otherwise \( M_T \) is the union of \( M \# M^0 \) and a solid torus, and \( c(M) < c(M_T) < c(M) \).

Therefore each \( T_i \) bounds a solid torus in \( M \). Each solid torus contains the drilled curve, hence they all intersect, and there is a solid torus \( H \) bounded by a \( T_i \) containing all the others. Therefore \( M_T = N^0 \setminus H \) where \( N^0 \) is a block of the JSJ decomposition, which cannot be Seifert, hence it is hyperbolic. We have \( c(N^0) = c(M_T) < c(M) \), and \( M \) is obtained by filling \( N^0 \), as required.

Remark 2.4. The proof Theorem 2.3 is also valid for \( M \) hyperbolike, i.e. reducible, atoroidal, and not Seifert.

3. Closed census

We describe here the closed orientable irreducible manifolds with \( c = 10 \), and the closed non-orientable \( \mathbb{P}^2 \)-irreducible ones with \( c = 7 \). Such manifolds are collected
| Complexity | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------|---|---|---|---|---|---|---|---|---|---|----|
| orientable |   |   |   |   |   |   |   |   |   |   |    |
| lens spaces| 3 | 2 | 3 | 6 | 10| 20| 36| 72| 136| 272| 528|
| other elliptic | . | . | 1 | 1 | 4 | 11| 25| 45| 78 | 142| 270|
| flat | . | . | . | . | 6 | . | . | . | . | . | . |
| Nil | . | . | . | . | 7 | 10| 14| 15| 15 | 15 | 15 |
| $\text{SL}_2\mathbb{R}$ | . | . | . | . | 39| 162| 513| 1416| 1416| 1416| 1416 |
| Sol | . | . | . | . | 5 | 9 | 23 | 39 | 39 | 39 | 39 |
| $\mathbb{H}^2\mathbb{R}$ | . | . | . | . | 2 | 2 | 8 | 8 | 8 | 8 | 8 |
| hyperbolic | . | . | . | . | . | 4 | 25 | 25 | 25 | 25 | 25 |
| not geometric | . | . | . | . | 4 | 35| 185| 777| 777| 777| 777 |
| total orientable | 3 | 2 | 4 | 7 | 14| 31| 74 | 175| 436| 1154| 3078 |
| non-orientable |   |   |   |   |   |   |   |   |   |   |    |
| flat | . | . | . | . | 4 | . | . | . | . | . | . |
| $\mathbb{H}^2\mathbb{R}$ | . | . | . | . | . | 2 | . | . | . | . | . |
| Sol | . | . | . | . | 1 | 1 | . | . | . | . | . |
| total non-orientable | . | . | . | . | 5 | 3 | . | . | . | . | . |

Table 1: The number of closed $\mathbb{P}^2$-irreducible manifolds of given complexity (up to 10 in the orientable case, and up to 7 in the non-orientable one) and geometry. Recall that there is no $\mathbb{P}^2$-irreducible manifold of type $S^2\mathbb{R}$, and no non-orientable one of type $S^3$, Nil, and $\text{SL}_2\mathbb{R}$.

in terms of their geometry, if any, in Table 1. The complete list of manifolds can be downloaded from [42].

### 3.1. The first 7 geometries

We recall [38] that there are eight important 3-dimensional geometries, six of them concerning Seifert manifolds. A Seifert fibration is described via its *normalized parameters* $F; (p_1,q_1)\vdots; (p_k,q_k)t$, where $F$ is a closed surface, $p_i>q_i>0$ for all $i$, and $t > k=2$ (obtained by reversing orientation if necessary). The Euler characteristic $\chi^{\text{orb}}$ of the base orbifold and the Euler number $e$ of the fibration are given respectively by

$$\chi^{\text{orb}} = \chi(F) \sum_{i=1}^k \frac{1}{p_i}$$

$$e = t + \sum_{i=1}^k \frac{q_i}{p_i}$$

and they determine the geometry of the Seifert manifold (which could have different fibrations) according to Table 1. The two non-Seifert geometries are the Sol and the hyperbolic ones [38].

The following result shows how to compute the complexity (when $c \leq 10$) of most manifolds belonging to the first 7 geometries. It is proved for $c \leq 9$ in [28], and completed for $c = 10$ here in Subsection 8.7. We define the norm $|p+q|$ of two coprime non-negative integers inductively by setting $\|0\|=0; 1\|=1; 1\|= 1; 0\|= 0$ and $p+q\|=

...
\[ p^2 + pq + q^2 = p^2 q + 1. \] A norm \( kA_k \) on matrices \( A \in GL_2(\mathbb{Z}) \) is also defined in [28].

**Theorem 3.1.** Let \( M \) be a geometric non-hyperbolic manifold with \( c(M) \leq 10 \):

(i) if \( M \) is a lens space \( L(p,q) \), then \( c(M) = p + q \);

(ii) if \( M \) is a torus bundle with monodromy \( A \) then \( c(M) = \min \{ k \in \mathbb{N} : kA_k + 5 < 6g \} \);

(iii) if \( M = S^2; (2;1); (3;1); (n;1); (m;1) \) with \( m > 5 \), we have \( c(M) = m \);

(iv) if \( M = S^2; (2;1); (3;1); (n;1); (m;1) \) is not of the type above, we have \( c(M) = p + q + 2 \);

(v) if \( M = F; (p_1,q_1); \cdots; (p_k,q_k) \) is not of the types above, then

\[
  c(M) = \max \{ 0, 1 + \chi(F) + 6 \chi(F) + \sum_{i=1}^{k} p_i q_i + 2 \}
\]

Note from Table 1 that a Seifert manifold with \( c < 6 \) has \( c_{\text{orb}} > 0 \) and one with \( c \leq 6 \) has \( c_{\text{orb}} < 0 \), whereas for higher \( c \) most Seifert manifolds have \( c_{\text{orb}} < 0 \).

**Remark 3.2.** Theorem 3.1, together with analogous formulas for some non-geometric graph manifolds, follows from the decomposition of closed manifolds into bricks, introduced in Section 8. The lists of all non-hyperbolic manifolds with \( c \leq 10 \) is then computed from such formulas by a computer program, available from [42]. A mistake in that program produced in [25] for \( c = 9 \) a list of 1156 manifolds instead of 1154 (two graph manifolds with distinct parameters were counted twice). Using Turaev-Viro invariants, Matveev has also recently checked that all the listed closed manifolds with \( c \leq 10 \) are distinct [32].

### 3.2. Hyperbolic manifolds

Table 3 shows all closed hyperbolic manifolds with \( c \leq 10 \). Each such manifold is a Dehn surgery on the chain link with 3 components shown in Fig. 3 with parameters shown in the table.

It is proved in [34] that every closed 3-manifold with \( c \leq 8 \) is a graph manifold, and that the first closed hyperbolic manifolds arise with \( c = 9 \). The hyperbolic manifolds

| \( e \) | \( c_{\text{orb}} > 0 \) | \( c_{\text{orb}} = 0 \) | \( c_{\text{orb}} < 0 \) |
|------|------------------|------------------|------------------|
| 0    | \( S^2 \)        | \( \mathbb{R} \) | \( H^2 \)        |
| \( e \neq 0 \) | \( S^3 \)        | Nil              | \( SL_2(\mathbb{R}) \) |

**Table 2:** The six Seifert geometries.
Table 3: The hyperbolic manifolds of complexity 9 and 10. Each such manifold is described as the surgery on the chain link with some parameters.
The symmetries of this link act transitively on the components, in such a way that to define the \((p=q=r=s=t=u)\)-surgery we do not need to associate a component to each parameter.

Figure 3: The chain link with 3 components.

with \(c = 9\) then turned out \([25]\) to be the 4 smallest ones known. The most interesting question about those with \(c = 10\) is then whether they are also among the smallest ones known, for instance comparing them with the closed census \([22]\) also used by SnapPea \([39]\). As explained in \([12]\), the manifolds in that census have all geodesics bigger than \(3\), and therefore some manifolds having \(c = 10\) are not present there (namely, those in Table \([3]\) corresponding to N. 16, 21, 24). We have therefore used SnapPea (in the python version) to compute a list of many surgeries on the chain link with 3 components (avoiding the non-hyperbolic ones, listed in \([27]\)), available from \([42]\), which contains many closed manifolds of volume smaller than 2 that are not present in SnapPea’s closed census. The entry “N.” in Table \([3]\) tells the position of the manifold in our table from \([42]\). The first 10 manifolds of the two lists nevertheless coincide and are also fully described in \([22]\), and they all have \(c \leq 10\), as Table \([3]\) shows.

3.3. Non-geometric manifolds

Every non-hyperbolic orientable manifold with \(c \leq 10\) is a graph manifold, i.e. its JSJ decomposition consists of Seifert or Sol blocks. A non-geometric orientable manifold whose decomposition contains a hyperbolic block with \(c \leq 11\) is constructed in \([3]\), and from our census now it follows that it cannot have \(c \leq 10\). Therefore we have proved the following.

Theorem 3.3. The first closed orientable irreducible manifold with non-trivial JSJ decomposition containing hyperbolic blocks has \(c = 11\).

All graph manifolds with \(c \leq 10\) are collected in Table \([4]\) according to their JSJ decomposition into fibering pieces, and to the type of fiberings of each piece.

3.4. The simplest manifolds

As the following discussion shows, in most geometries, the manifolds with lowest complexity are the “simplest” ones.

3.4.1. Elliptic

The elliptic manifolds of smallest complexity are \(S^3; \mathbb{R}P^3\); and \(L(3;1)\), having \(c = 0\). The first manifold which is not a lens space is \(S^2; L(3;1); \mathbb{C}^2(1)\); and has
| Complexity | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------|---|---|---|---|---|---|---|---|---|---|---|
| geometric  |   |   |   |   |   |   |   |   |   |   |   |
| lens spaces| 3 | 2 | 3 | 6 | 10| 20| 36| 72| 136| 272| 528|
| $S^2;3$    |   |   |   |   |   |   |   |   |   |   |   |
| $S^2;4$    |   |   |   |   |   |   |   |   |   |   |   |
| $S^2;5$    |   |   |   |   |   |   |   |   |   |   |   |
| $\mathbb{R}P^2;2$ |   |   |   |   |   |   |   |   |   |   |   |
| $\mathbb{R}P^2;3$ |   |   |   |   |   |   |   |   |   |   |   |
| $T$ or $K$ |   |   |   |   |   |   |   |   |   |   |   |
| $T;1$ or $K;1$ |   |   |   |   |   |   |   |   |   |   |   |
| $T$-fiberings over $S^1$ |   |   |   |   |   |   |   |   |   |   |   |
| $T$-fiberings over $I$ |   |   |   |   |   |   |   |   |   |   |   |
| non-geometric |   |   |   |   |   |   |   |   |   |   |   |
| $D;2 \rightarrow D;2$ |   |   |   |   |   |   |   |   |   |   |   |
| $A;1$      |   |   |   |   |   |   |   |   |   |   |   |
| $D;2 \rightarrow D;3$ |   |   |   |   |   |   |   |   |   |   |   |
| $S;1 \rightarrow D;2$ |   |   |   |   |   |   |   |   |   |   |   |
| $D;2 \rightarrow A;1 \rightarrow D;2$ |   |   |   |   |   |   |   |   |   |   |   |
| total      | 3 | 2 | 4 | 7 | 14| 31| 74| 175| 436| 1150| 3053|

Table 4: The type of graph manifolds of given complexity, up to 10. Here, $I;D;S;A;T;K$ denote respectively the closed interval, the disc, the Möbius strip, the annulus, the torus, and the Klein bottle. We denote by $X;n$ a block with base space the surface $X$ and $n$ exceptional fibers. We write $X$ for $X;0$. We have counted as $T$-fiberings only the Sol manifolds, not the manifolds also admitting a Seifert structure. There is a flat manifold with $c = 6$ counted twice, since it has two different fibrations, corresponding to the asterisks.
c = 2. It is the elliptic manifold with smallest non-cyclic fundamental group, having order 8 [31].

3.4.2. Flat

Every (orientable or not) flat manifold has $c = 6$. A typical way to obtain some flat 3-manifold $M$ is from a face-pairing of the cube: by taking a triangulation of the cube with 6 tetrahedra matching along the face-pairing, we get a minimal triangulation of $M$.

3.4.3. $H^2 \times \mathbb{R}$

The first manifolds of type $H^2 \times \mathbb{R}$ are non-orientable and have $c = 7$, and are also the manifolds of that geometry with smallest base orbifold [4], having volume $2\pi \chi^{orb} = \pi = 3$.

3.4.4. Sol

The first manifold of type Sol is also non-orientable and has $c = 6$, and it is the unique filling of the Gieseking manifold, the cusped hyperbolic manifold with smallest volume $1149$ and smallest complexity $1,014,911$ [10]. It is also the unique torus fibering whose monodromy $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is hyperbolic with $|\text{tr}A| < 2$ [4].

3.4.5. Hyperbolic

As we said above, the first orientable hyperbolic manifolds are the smallest ones known. It would be interesting to know the complexity of the first non-orientable closed hyperbolic manifold, whose volume is probably considerably bigger than in the orientable case, see [22].

4. Census of hyperbolic manifolds

We describe here the compact hyperbolic manifolds with boundary with $\chi = 0$ and $c \leq 7$, and the orientable ones with $\chi < 0$ and $c \leq 4$.

4.1. Manifolds with $\chi = 0$

Recall that we define a compact $M$ to be hyperbolic when it admits a complete metric of finite volume and geodesic boundary, after removing all boundary components with $\chi = 0$. Therefore, hyperbolic manifolds $M$ with $\chi(M) = 0$ have some cusps based on tori or Klein bottles, and those with $\chi(M) < 0$ have geodesic boundary and possibly some cusps. To avoid confusion, we define the topological boundary of $M$ to be the union of the geodesic boundary and the cusps.
Table 5: The number of cusped hyperbolic manifolds of given complexity, up to 7. The “topological boundary” indicates the tori $T$ and Klein bottles $K$ present as cusps.

Hyperbolic manifolds with $\chi(M) = 0$ and $c \leq 7$ were listed by Hodgson and Weeks in [10] and form the cusped census used by SnapPea. They are collected, according to their topological boundary, in Table 5. Hyperbolicity of each manifold was checked by solving Thurston’s equations, and all manifolds were distinguished computing their Epstein-Penner canonical decomposition [13]. In practice, volume, homology, and the length of the shortest geodesic are usually enough to distinguish two such manifolds.

4.2. Manifolds with $\chi < 0$

Equations analogous to Thurston’s were constructed by Frigerio and Petronio in [17] for an ideal triangulation $T$ of a manifold $M$ with $\chi(M) < 0$. A solution of such equations gives a realization of the hyperbolic structure of $M$ via partially truncated hyperbolic tetrahedra. One such tetrahedron is parametrized by its 6 interior dihedral angles $\alpha_1, \ldots, \alpha_6$. The sum of the 3 of them incident to a given vertex must be less or equal than $\pi$, and the vertex is truncated if the sum is less than $\pi$, or ideal if it is $\pi$. The compatibility equations ensure that identified edges all have the same length and that dihedral angles sum to $2\pi$ around each resulting edge. These equations, together with others checking the completeness of the cusps, realize the hyperbolic structure for $M$. Then Kojima’s canonical decomposition [24], analogous to Epstein-Penner’s, is a complete invariant which allows one to distinguish manifolds. In contrast with the case $\chi = 0$, there are plenty of manifolds having the same complexity that are not distinguished by volume, homology, Turaev-Viro invariants, and the canonical
Table 6: The number of orientable hyperbolic manifolds with non-empty geodesic boundary of given complexity, up to 4. The “topological boundary” indicates the genera of the boundary components, with zeroes correspond to cusps.

decomposition seems to be the only available tool, see Subsection 6.2. The results from [15] are summarized in Table 6.

Remark 4.1. The two censuses of hyperbolic manifolds described in this Section have a slightly more experimental nature than the closed census of Section 3 since solving hyperbolicity equations and calculating the canonical decomposition involve numerical calculations with truncated digits.

5. Complexity and volume of hyperbolic manifolds

We describe here some relations between the complexity and the volume of a hyperbolic 3-manifold.

5.1. Ideal tetrahedra and octahedra

As Theorem 5.1 below shows, there is a constant $K$ such that $\text{Vol}(M) < K \cdot c(M)$ for any hyperbolic $M$. Let $v_T = 1.0149 \ldots$ and $v_O = 3.6638 \ldots$ be the volumes respectively of the regular ideal hyperbolic tetrahedron and octahedron.

Theorem 5.1. Let $M$ be hyperbolic, with or without boundary. If $\chi(M) = 0$ we have $\text{Vol}(M) \leq v_T \cdot c(M)$. If $\chi(M) < 0$ we have $\text{Vol}(M) < v_O \cdot c(M)$.

Proof. First, note that by the naturality property of the complexity $c(M)$ is the minimum number of tetrahedra in an (ideal) triangulation. If $M$ is closed, take a minimal triangulation $T$ and straighten it. Tetrahedra may overlap or collapse to low-dimensional objects, having volume zero. Since geodesic tetrahedra have volume less than $v_T$, we get the inequality.

If $M$ is not closed, let $T$ be an ideal triangulation for $M$ with $c(M)$ tetrahedra. We can realize topologically $M$ with its boundary tori removed, by partially truncating each tetrahedron in $T$ (i.e. removing the vertex only in presence of a cusp, and an open star of it in presence of true boundary). Then we can straighten every truncated tetrahedron with respect to the hyperbolic structure in $M$. As above, tetrahedra may
overlap or collapse. In any case, the volume of each such will be at most $v_T$ if there is no boundary, and strictly less than $v_O$ in general, since any ideal tetrahedron has volume at most equal to $v_T$, and any partially truncated tetrahedron has volume strictly less than $v_O$ \[41\].

The constants $v_T$ and $v_O$ are the best possible ones, see Remark 6.9. A converse result of type $c(M) < K^0$ Vol($M$) is impossible, because for big $C$’s there are a finite number of hyperbolic manifolds with complexity less than $C$, and an infinite number of such with volume less than $C$.

5.2. First segments of $c$ and Vol

Complexity and volume give two partial orderings on the set $H$ of all hyperbolic 3-manifolds. By what was just said, they are globally qualitatively very different. Nevertheless, as noted in \[34\], they might have similar behaviours on some subsets of $H$. We propose the following conjecture.

**Conjecture 5.2.** Among hyperbolic manifolds with the same topological boundary, the ones with smallest complexity have volume smaller than the other ones.

The conjecture is stated more precisely as follows: let $M \Sigma$ be the set of hyperbolic manifolds having some fixed topological boundary $\Sigma$. Suppose $M^0 \subset M \Sigma$ is so that $c(M^0) > c(M)$ for all $M^0 \subset M \Sigma$. We conjecture that Vol($M^0$) > Vol($M$) for all $M^0 \subset M \Sigma$ having $c(M^0) > c(M)$. We now discuss our conjecture.

5.2.1. Closed case

The closed hyperbolic manifolds with smallest $c = 9$ are the four having smallest volume known, see Table 3. Therefore Conjecture 5.2 claims that these four are actually the ones having smallest volumes among all closed hyperbolic manifolds.

5.2.2. Connected topological boundary

In this case, Conjecture 5.2 is true, as the following shows.

**Theorem 5.3.** Among hyperbolic manifolds whose topological boundary is a connected surface, the ones with smallest volume are the ones with smallest complexity.

**Proof.** Among manifolds having one toric cusp, the figure-8 knot complement and its sibling are those with minimal volume $2v_T$ and minimal complexity 2. Among those with a Klein bottle cusp, the Gieseking manifold is the one with minimal volume $v_T$ and minimal complexity 1. Our assertion restricted to orientable 3-manifolds bounded by a connected surface of higher genus is proved in \[14\] combining the naturality property of the complexity with Miyamoto’s description \[56\] of all such manifolds with minimal volume. The same proof also works in the general case. \[\square\]
5.2.3. Experimental data

Conjecture 5.2 is true when restricted to the manifolds of Tables 3, 5, and 6 for all the boundary types involved (see [10], [39], and [15]). One sees from Table 3 that the manifolds of type $(K;K)$, $(T;T)$, $(K;K;T)$, $(T;T;T)$, $(K;K;K)$, and $(K;K;K;K)$ with smallest complexity have respectively $c = 2, 4, 4, 6, 6, 4$, and $6$. The manifolds with $c = 2$ are constructed with two regular ideal tetrahedra, and hence have volume $2\,v_T$. Those with $c = 4$ are constructed either with 4 regular ideal tetrahedra, hence having volume $4\,v_T = 4.05976\ldots$ (therefore Conjecture 5.2 claims that every other $M$ with the same topological boundary has volume bigger than $4\,v_T$). Those with $c = 6$ have volume $2\,v_D = 5.3334\ldots$, where $v_D = 2.6667\ldots$ is the volume of the “triangular ideal drum” used by Thurston [40] to construct the complement of the chain link of Fig. 3, which is the only orientable manifold among them.

Problem 5.4. Classify the hyperbolic (orientable) manifolds of smallest complexity among those having $\chi = 0$ and $k$ toric cusps, and compute their volume, for each $k$.

5.3. Matveev-Fomenko conjecture

As we mentioned above, the orderings given by $c$ and Vol are qualitatively different on the whole set $M$ of hyperbolic manifolds, but might be similar on some subsets of $M$. The following conjecture was proposed by Matveev and Fomenko in [34].

Conjecture 5.5 (Matveev-Fomenko [34]). Let $M$ be a hyperbolic manifold with one cusp. Among Dehn fillings $N$ and $N^0$ of $M$, if $c(N) < c(N^0)$ then $\text{Vol}(N) < \text{Vol}(N^0)$.

The complexity-10 closed census produces a counterexample to Conjecture 5.5.

Proposition 5.6. Let $N(p=q)$ be the $p=q$-surgery on the figure-8 knot. We have

\[
\begin{align*}
\text{Vol } N(p=2) &= 1,529,4773\ldots & c(N(p=2)) &= 11 \\
\text{Vol } N(7) &= 1,463,7766\ldots & c(N(7)) &> 11
\end{align*}
\]

Proof. We first note that $N(p=q) = N(p=q)$ is the $(p=q)$-surgery on the chain link. The manifold $N(7)$ does not belong to Table 8 (it is the manifold labeled as N.11 in our census of surgeries on the chain link of [42]), and hence has $c > 11$, whereas $N(p=2)$ is the manifold N.12 and has $c = 11$.

6. Lower bounds

Providing upper bounds for the complexity of a given manifold $M$ is relatively easy: from any combinatorial description of $M$ one recovers a spine of $M$ with $n$ vertices, and certainly $c(M) \leq n$. Finding lower bounds is a much more difficult task. The only
\(\partial\)-irreducible manifolds whose complexity is known are those listed in the censuses of Sections 3 and 4 and some infinite families of hyperbolic manifolds with boundary described below. In particular, for a closed irreducible \(M\), the value \(c(M)\) is only known when \(c(M) \leq 10\), i.e. for a finite number of manifolds.

6.1. The closed case

The only available lower bound for closed irreducible orientable manifolds is the following one, due to Matveev and Pervova. We denote by \(\|H_1(M)\|\) the order of the torsion subgroup of \(H_1(M)\), while \(b_1\) is the rank of the free part, i.e. the first Betti number of \(M\).

**Theorem 6.1 (Matveev-Pervova [33])**. Let \(M\) be a closed orientable irreducible manifold different from \(L(3;1)\). Then \(c(M) > 2\log \|H_1(M)\| + b_1 + 1\).

Recall that Theorem 3.1 holds only for \(c \leq 10\). Actually, the same formulas in the statement give an upper bound for \(c(M)\). Some such upper bounds for lens spaces, torus bundles, and simple Seifert manifolds were previously found by Matveev and Anisov, who proposed the following conjectures.

**Conjecture 6.2 (Matveev [31])**. We have

\[
c(L(p;q)) = p; q \quad 2 \quad c(S^2; (2;1); (2;1); (m;1)); \quad 1 = m
\]

**Conjecture 6.3 (Anisov [6])**. The complexity of a torus bundle \(M\) over \(S^1\) with monodromy \(A \in GL_2(\mathbb{Z})\) is \(c(M) = \min \{ 4k + 5; 6q \} \).

6.2. Families of hyperbolic manifolds with boundary of known complexity

The following corollaries of Theorem 5.1 were first noted by Anisov.

**Corollary 6.4 (Anisov [5])**. The complexity of a hyperbolic manifold decomposing into \(n\) ideal regular tetrahedra is \(n\).

**Corollary 6.5 (Anisov [5])**. The punctured torus bundle with monodromy \(A_n \in GL_2(\mathbb{Z})\) is \(c(M) = \min \{ 4k + 5; 6q \} \).

For each \(n > 2\), Frigerio, Martelli, and Petronio defined the family \(M_n\) of all orientable compact manifolds admitting an ideal triangulation with one edge and \(n\) tetrahedra.

**Theorem 6.6 (Frigerio-Martelli-Petronio [14])**. Let \(M \sqsubset M_n\). Then \(M\) is hyperbolic with a genus-\(n\) surface as geodesic boundary, and without cusps. It has complexity \(n\). Its homology, volume, Heegaard genus, and Turaev-Viro invariants also depend only on \(n\).
The manifolds in \( M_n \) are distinguished by their Kojima’s canonical decomposition (see Subsection 4.2), which is precisely the triangulation with one edge defining them. Therefore combinatorially different such triangulations give different manifolds.

**Theorem 6.7 (Frigerio-Martelli-Petronio [14, 16]).** Manifolds in \( M_n \) correspond bijectively to triangulations with one edge and \( n \) tetrahedra. The cardinality \( \#M_n \) grows as \( n^n \).

We say that a sequence \( a_n \) grows as \( n^n \) when there exist constants \( 0 < k < K \) such that \( n^k n < a_n < n^K n \) for all \( n \geq 0 \).

**Corollary 6.8 (Frigerio-Martelli-Petronio [16]).** The number of hyperbolic manifolds of complexity \( n \) grows as \( n^n \).

**Remark 6.9.** From the families introduced here we see that the inequalities of Theorem 5.1 cannot be strengthened. The torus bundles \( M \) above have \( \text{Vol}(M) = v_T c(M) \), and the manifolds in \( M_n \) have \( \text{Vol}(M) = v_n c(M) \), with \( v_n \) equal to the volume of a truncated truncated tetrahedral with all angles \( \pi = \frac{3n}{4} \), so that \( v_n \sim v_\infty \) for \( n \to \infty \).

The set \( M_n \) is also the set mentioned in Theorem 5.3 of all manifolds having both minimal complexity and minimal volume among those with a genus-\( n \) surface as boundary. We therefore get from Table 6 that \( \#M_n \) is 874;2340 for \( n = 2;3;4 \).

The class \( M_n \) is actually contained as \( M_n = M_{g,k} \) in a bigger family \( M_{g,k} \), defined in [16]. The set \( M_{g,k} \) consists of all orientable hyperbolic manifolds of complexity \( g + k \) with connected geodesic boundary of genus \( g \) and \( k \) cusps. Theorems 6.6 and 6.7 hold similarly for all such sets. For any fixed \( g \) and \( k \), \( M_{g,k} \) is the set of all manifolds with minimum complexity among those with that topological boundary. Therefore Conjecture 5.2 would imply the following.

**Conjecture 6.10 (Frigerio-Martelli-Petronio [16]).** The manifolds of smallest volume among those with a genus-\( g \) geodesic surface as boundary and \( k \) cusps are those in \( M_{g,k} \).

### 7. Minimal spines

We describe here some known results about minimal spines, which are crucial for computing the censuses of Sections 3 and 4.

#### 7.1. The algorithm

The algorithm used to classify all manifolds with increasing complexity \( n \) typically works as follows:

(i) list all special spines with \( n \) vertices (or triangulations with \( n \) tetrahedra);
(ii) remove from the list the many spines that are easily seen to be non-minimal, or not to thicken to an irreducible (or hyperbolic) manifold;

(iii) try to recognize the manifolds obtained from thickening the remaining spines;

(iv) eliminate from that list of manifolds the duplicates, and the manifolds that have already been found previously in some complexity-$n^0$ census for some $n^0 < n$.

Typically, step (1) produces a huge list of spines, 99.99::% of which are canceled via some quick criterion of non-minimality during step (2), and one is left with a much smaller list, so that steps (3) and (4) can be done by hand.

7.2. Cutting dead branches

Step (1) of the algorithm above needs a huge amount of computer time already for $c = 5$, due to the very big number of spines listed. Therefore one actually uses the non-minimality criteria (step (2)) while listing the special spines with $n$ vertices (step (1)), to cut many “dead branches”. Step (1) remains the most expensive one in terms of computer time, so it is worth describing it with some details.

A special spine or its dual (possibly ideal) triangulation $T$ (see Remark 2.1) with $n$ tetrahedra can be encoded roughly as follows. Take the face-pairing 4-valent graph $G$ of the tetrahedra in $T$. It has $n$ vertices and $2n$ edges. After fixing a simplex on each vertex, a label in $S_3$ on each (oriented) edge of $G$ encodes how the faces are glued. We therefore get $6^{2n}$ gluings (the same combinatorial $T$ is usually realized by many distinct gluings). Point (1) in the algorithm consists of two steps:

(1a) classify all 4-valent graphs $G$ with $n$ vertices;

(1b) for each graph $G$, fix a simplex on each vertex, and try the $6^{2n}$ possible labelings on edges.

Step (1b) is by far the most expensive one, because it contains many “dead branches”; most of them are cut as follows: a partial labeling of some $k$ of the $2n$ edges defines a partial gluing of the tetrahedra. If such partial gluing already fulfills some local non-minimality criterion, we can forget about every labeling containing this partial one.

Remark 7.1. A spine of an orientable manifold can be encoded more efficiently by fixing an immersion of the graph $G$ in $\mathbb{R}^2$, and assigning a colour in $\mathbb{Z}_2$ to each vertex and a colour in $\mathbb{Z}_3$ to each edge [7].

Local non-minimality criteria used to cut the branches are listed in Subsection 7.3. We discuss in Subsection 7.4 another powerful tool, which works in the closed case only: it turns out that most 4-valent graphs $G$ can be quickly checked a priori not to give rise to any minimal spine (of closed manifolds).
7.3. Local non-minimality criteria

We start with the following results.

**Proposition 7.2 (Matveev [30]).** Let \( P \) be a minimal special spine of a 3-manifold \( M \). Then \( P \) contains no embedded face with at most 3 edges.

**Proposition 7.3 (Matveev [30]).** Let \( P \) be a minimal special spine of a closed orientable 3-manifold \( M \). Let \( e \) be an edge of \( P \). A face \( f \) cannot be incident 3 times to \( e \), and it cannot run twice on \( e \) with opposite directions.

In the orientable setting, both Propositions 7.2 and 7.3 are special cases of the following. Recall that \( S(P) \) is the subset of a special spine \( P \) consisting of all points of type (2) and (3) shown in Fig. 1.

**Proposition 7.4 (Martelli-Petronio [25]).** Let \( P \) be a minimal spine of a closed orientable 3-manifold \( M \). Every simple closed curve \( \gamma \) bounding a disc in the ball \( M \setminus P \) and intersecting \( S(P) \) transversely in at most 3 points is contained in a small neighborhood of a point of \( P \).

Analogous results in the possibly non-orientable setting are proved by Burton [8].

7.4. Four-valent graphs

Quite surprisingly, some 4-valent graphs can be checked \textit{a priori} not to give any minimal special spine of closed 3-manifold.

**Remark 7.5.** The face-pairing graph of a (possibly ideal) triangulation is also the set \( S(P) \) in the dual special spine \( P \).

**Proposition 7.6 (Burton [8]).** The face-pairing graph \( G \) of a minimal triangulation with at least 3 tetrahedra does not contain any portions of the types shown in Fig. 4-(1,2,3), except if \( G \) itself is as in Fig. 4-(4).

A portion of \( G \) is of type shown in Fig. 4-(2,3,4) when it is as in that picture, with chains of arbitrary length. In the algorithm of Subsection 7.2, step (1b) can be therefore restricted to the \textit{useful} 4-valent graphs, i.e. the ones that do not contain the portions forbidden by Proposition 7.6. Table 7, taken from [8], shows that some 40\% of the graphs are useful.
8. Bricks

As shown in Sections 2 and 7, classifying all closed $\mathbb{P}^2$-irreducible manifolds with complexity $n$ reduces to listing all minimal special spines of such manifolds with $n$ vertices. Non-minimality criteria as those listed in Section 7 are then crucial to eliminate the many non-minimal spines (by cutting “dead branches”) and gain a lot of computer time. Actually, closed manifolds often have many minimal spines, and it is not necessary to list them all; a criterion that eliminates some, but not all, minimal spines of the same manifold is also suitable for us. This is the basic idea which underlies the decomposition of closed $\mathbb{P}^2$-irreducible manifolds into bricks, introduced by Martelli and Petronio in [25], and described in the orientable case in this Section. (For the nonorientable one, see [26].)

8.1. A quick introduction

The theory is roughly described as follows: every closed irreducible manifold $M$ decomposes along tori into pieces on which the complexity is additive. Each torus is marked with a $\theta$-graph in it, and the complexity of each piece is not the usual one, because it depends on that graphs. A manifold $M$ which does not decompose is a brick. Every closed irreducible manifold decomposes into bricks. The decomposition is not unique, but there can be only a finite number of such. In order to classify all manifolds with $c \leq 10$, one classifies all bricks with $c \leq 10$, and then assemble them in all possible (finite) ways to recover the manifolds.

For $c \leq 10$, bricks are atoroidal, hence either Seifert or hyperbolic. And the decomposition into bricks is typically a mixture of the JSJ, the graph-manifolds decomposition, and the thick-thin decomposition for hyperbolic manifolds. Very few closed manifolds do not decompose, i.e. are themselves bricks.

**Proposition 8.1.** There are 25 closed bricks with $c \leq 10$. They are: 24 Seifert manifolds of type $S^2; (2;1); (n;1); (n;1); 1$, and the hyperbolic manifold $N.34$ of Table 3.

Among closed bricks, we have Poincaré homology sphere $S^2; (2;1); (3;1); (5;1); 1$.

**Proposition 8.2.** There are 21 non-closed bricks with $c \leq 10$. 

---

| $n$ | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-----|----|----|----|----|----|----|----|----|----|
| useful | 2 | 4 | 12 | 39 | 138 | 638 | 3366 | 20751 | 143829 |
| all   | 4 | 10 | 28 | 97 | 359 | 1635 | 8296 | 48432 | 316520 |

*Table 7: Useful graphs among all 4-valent graphs with $n \leq 11$ vertices.*
There are 4978 closed irreducible manifolds with $c \leq 10$, see Table 1. Therefore $4953 = 4978 - 25$ such manifolds are obtained with the 21 bricks above.

Before giving precise definitions, we note that the layered triangulations [9, 23] of the solid torus $H$ are particular decompositions of $H$ into bricks. Our experimental results show the following.

**Proposition 8.3.** Every closed irreducible atoroidal manifold with $c \leq 10$ has a minimal triangulation containing a (possibly degenerate [9]) layered triangulation, except for some $(S^2; \#(1); \#(1); \#(1))$ and the hyperbolic N.34 of Table 3.

### 8.2. $\theta$-graphs in the torus

In this paper, a $\theta$-graph $\theta$ in the torus $T$ is a graph with two vertices and three edges inside $T$, having an open disc as a complement. That is, it is a trivalent spine of $T$. Dually, this is a one-vertex triangulation of $T$.

The set of all $\theta$-graphs in $T$ up to isotopy can be described as follows. After choosing a meridian and a longitude, every slope on $T$ (i.e. isotopy class of simple closed essential curves) is determined by a number $p=q \in \mathbb{Q}$. Those numbers are the ideal vertices of the Farey tessellation of the Poincaré disc sketched in Fig. 5-left. A $\theta$-graph contains three slopes, which are the vertices of an ideal triangle of the tesselation. This gives a correspondence between the $\theta$-graphs in $T$ and the triangles of the tesselation. Two $\theta$-graphs correspond to two adjacent triangles when they share two slopes, i.e. when they are related by a flip, shown in Fig. 5-right.

### 8.3. Manifolds with marked boundary

Let $M$ be a connected compact 3-manifold with (possibly empty) boundary consisting of tori. By associating to each torus component of $\partial M$ a $\theta$-graph, we get a manifold with marked boundary.
Let $M$ and $M^0$ be two marked manifolds, and $T \partial M; T^0 \partial M^0$ be two boundary tori. A homeomorphism $\psi : T \rightarrow T^0$ sending the marking of $T$ to the one of $T^0$ is an assembling of $M$ and $M^0$. The result is a new marked manifold $N = M \cup_{\psi} M^0$. We define analogously a self-assembling of $M$ along two tori $T; T^0 \partial M$, the only difference is that for some technical reason we allow the map to send one $\theta \rightarrow T$ either to $\theta^0 \rightarrow T$ itself or to one of the 3 other $\theta$-graphs obtained from $\theta^0$ via a flip.

8.4. Spines and complexity for marked manifolds

The notion of spine extends from the class of closed manifold to the class of manifolds with marked boundary. Recall from Subsection 2.1 that a compact polyhedron is simple when the link of each point is contained in $\partial$. A sub-polyhedron $P$ of a manifold with marked boundary $M$ is called a spine of $M$ if:

- $P \cap \partial M$ is simple,
- $M \cap (P \cap \partial M)$ is an open ball,
- $P \setminus \partial M$ is a graph contained in the marking of $\partial M$.

Note that $P$ is not in general a spine of $M$ in the usual sense. The complexity of a 3-manifold with marked boundary $M$ is of course defined as the minimal number of vertices of a simple spine of $M$. Three fundamental properties extend from the closed case to the case with marked boundary: complexity is still additive under connected sums, it is finite-to-one on orientable irreducible manifolds, and every orientable irreducible $M$ with $c(M) > 0$ has a minimal special spine. (Here, a spine $P$ of $M$ is special when $P \cap \partial M$ is: the spine $P$ is actually a special spine with boundary, with $\partial P = \partial M \setminus P$ consisting of all the $\theta$-graphs in $\partial M$.)

8.5. Bricks

An important easy fact is that if $M$ is obtained by assembling $M_1$ and $M_2$, and $P_i$ is a spine of $M_i$, then $P_1 \cap P_2$ is a spine of $M$. This implies the first part of the following result.

**Proposition 8.4 (Martelli-Petronio).** If $M$ is obtained by assembling $M_1$ and $M_2$, we have $c(M) \leq c(M_1) + c(M_2)$. If $M$ is obtained by self-assembling $N$, we have $c(M) \leq c(N) + 6$.

When $c(M) = c(M_1) + c(M_2)$ or $c(M) = c(N) + 6$, and the manifolds involved are irreducible, the (self)-assembling is called sharp. An orientable irreducible marked manifold $M$ is a brick when it is not the result of any sharp (self-)assembling.

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2To avoid confusion, the term skeleton was used in [25].

3This hypothesis is actually determinant only in one case, see [25].
Figure 6: If 4 edges disconnect $G$, then one of the two pieces is of one of these types.

**Theorem 8.5 (Martelli-Petronio [25]).** Every closed orientable irreducible $M$ is obtained from some bricks via a combination of sharp (self-)assemblings.

There are only a finite number of such combinations giving the same $M$.

**8.6. The algorithm that finds the bricks**

The algorithm described in Subsection 7.2 also works for classifying all bricks of increasing complexity, with some modifications, which we now sketch. As we said above, every brick with $c > 0$ has a minimal spine $P$ such that $P \setminus \partial M$ is special. The 4-valent graph $H = S(P \setminus \partial M)$ contains the $\theta$-graphs marking the boundary $\partial M$. By substituting (i.e. identifying) in $H$ each $\theta$-graph with a point, we get a simpler 4-valent graph $G$. We mark the edges of $G$ containing that new points with a symbol ?. It is then possible to encode the whole $P$ by assigning labels in $S_3$ on the remaining edges of $G$, as in Subsection 7.2. The spine $P$ is uniquely determined by such data.

Every edge of $G$ can have a label in $S_3 \cup \{?, g\}$, giving $7^{2n}$ possibilities to analyze during step (1b) of the algorithm (actually, they are $2^n (3 + 1)^{2n}$ by Remark 7.1). Although there are more possibilities to analyze than in the closed case ($7^{2n}$ against $6^{2n}$), the non-minimality criteria for bricks listed below are so powerful, that step (1b) is actually experimentally much quicker for bricks than for closed manifolds. This should be related with the experimental fact that there are much more manifolds than bricks.

**Proposition 8.6 (Martelli-Petronio [25]).** Let $P$ be a minimal special spine of a brick with $c > 3$. The 3 faces incident to an edge $e$ of $P$ are all distinct. A face can be incident to at most one $\theta$-graph in $\partial P$.

**Theorem 8.7 (Martelli-Petronio [25]).** Let $G$ be the 4-valent graph associated to a minimal special spine of a brick with $c > 3$. Then:

(i) no pair of edges disconnects $G$;

(ii) if $c \geq 10$ and a quadruple of edges disconnects $G$, one of the two resulting components must be of one of the forms shown in Fig. 6.

Point 2 of Theorem 8.7 is proved for $c \leq 9$ in [25] and conjectured there to be true for all $c$: its extension to the case $c = 10$ needed here is technical and we omit it. We can restrict step (1b) of the algorithm to the useful 4-valent graphs, i.e. the ones that are not forbidden by Theorem 8.7. Table 8 shows that only 2% of the graphs are useful for $c = 10; 11$. 


8.7. Bricks with $c \leq 10$.

We list here the bricks found. There are two kinds of bricks: the closed ones, and the ones with boundary. The closed ones correspond to the closed irreducible 3-manifolds that do not decompose.

**Theorem 8.8.** The closed bricks having $c \leq 10$ are:

- $S^2; (\alpha; 1); (\beta; 1); (\eta; 1)$; 1 with $m > 5m \leq 6$, having $c = m$;
- $S^2; (\alpha; 1); (\beta; 1); (\eta; 1)$; 1 not of the type above and with $5n; m; g \leq 6$, having $c = n + m$;
- the closed hyperbolic manifold $N.34$ from Table 3, with volume $175712$.

**Remark 8.9.** The manifolds $S^2; (\alpha; 1); (\beta; 1); (\eta; 1)$; 1 with $5n; m; g \leq 6$ or $4; 4; 4$ are not bricks. Actually, they are flat torus bundles, whereas every other such manifold is atoroidal.

In the following statement, we denote by $N (\alpha; \beta; \gamma)$ the following marked manifold: take the chain link of Fig. 3, perform an $\alpha$-surgery on one component, and if $\alpha = \theta (i)$, drill that component and mark the new torus with the $\theta$-graph containing the slopes $i; i + 1$. Do the same for $\beta$ and $\gamma$ (the choice of the components does not matter, see Fig. 3).

**Theorem 8.10.** The bricks with boundary having $c \leq 10$ are:

- $c = 0$: one marked $T [0; 1]$ and two marked solid tori;
- $c = 1$: one marked $T [0; 1]$;
- $c = 3$: one marked (pair of pants) $S^1$;
- $c = 8$: one marked $D; (\alpha; 1); (\beta; 1)$, and $N (1; 4; \theta (1))$;
- $c = 9$: four bricks of type $N (\alpha; \beta; \gamma)$, with $(\alpha; \beta; \gamma)$ being one of the following:
  - $(1; 4; \theta (1)); (1; \theta (2); \theta (2)); (\theta (1); \theta (1); \theta (2)); (\theta (2); \theta (2); \theta (2)); \theta (2); \theta (2); \theta (2)$.
The complement $M$ of this link is a hyperbolic manifold. On each cusp, there are two shortest loops of equal length, and hence two preferred $\theta$-graphs, the ones containing both loops. Up to symmetries of $M$, there are only 3 marked $M$’s with such preferred $\theta$-graphs, and these are the ones with $c = 10$.

Figure 7: The complement of a chain link with 4 components.

c = 10: eleven bricks of type $N(\alpha;\beta;\gamma)$, with $\langle \alpha;\beta;\gamma \rangle$ being one of the following:

- $(1;2;\theta(0))$ with $i \neq 3; 2; 1; \theta 3; (1;6;\theta(1))$;
- $(5;\theta(0);\theta(1)); (5;\theta(1);\theta(0)); (1;\theta(1);\theta(0));$
- $(1;\theta(0);\theta(1)); (2;\theta(0);\theta(1)); (\theta(1);\theta(0);\theta(1));$

and three marked complements of the same link, shown in Fig. 7.

Remark 8.11. Using the bricks with $c \leq 1$, one constructs every marked solid torus. This construction is the layered solid torus decomposition [9, 23]. An atoroidal manifold with $c \leq 10$ is either itself a brick, or it decomposes into one brick $B$ of Theorem 8.10 and some layered solid tori.

Remark 8.12. The generic graph manifold decomposes into some Seifert bricks with $c \leq 3$. As Theorem 3.1 suggests, the only exceptions with $c \leq 10$ are the closed bricks listed by Theorem 8.8 and some surgeries of the Seifert brick with $c = 8$.

Remark 8.13. Table 3 is deduced from Theorems 8.8 and 8.10 using SnapPea via a python script available from [42].

Remark 8.14. The proof of Theorem 3.1 from [28] extends to $c = 10$. One has to check that the new hyperbolic bricks with $c = 10$ do not contribute to the complexity of non-hyperbolic manifolds, at least for $c = 10$: we omit this discussion.

We end this Section with a conjecture, motivated by our experimental results, which implies that the decomposition into bricks is always finer than the JSJ.

Conjecture 8.15. Every brick is atoroidal.

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