ALGEBRAIC DE RHAM THEORY FOR RELATIVE COMPLETION OF \( SL_2(\mathbb{Z}) \)

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References

1. Introduction

Brown [2] defines multiple modular values to be periods of the coordinate ring \( \mathcal{O}(\mathcal{G}^{rel}) \) of the relative completion \( \mathcal{G}^{rel} \) of \( SL_2(\mathbb{Z}) \). In order for this period definition of multiple modular values to make sense, one needs an explicit \( \mathbb{Q} \)-de Rham theory for \( \mathcal{G}^{rel} \). In this paper, we provide such a theory, which enables us to explicitly construct iterated integrals of modular forms possibly of the second kind (see below) that may have singularities away from the cusp around which there is no monodromy. These newly constructed iterated integrals provide all multiple modular values, whereas previously only those multiple modular values that are iterated integrals of holomorphic modular forms have been studied by Brown [2] and Manin [16, 17].

The relative completion \( \mathcal{G}^{rel} \) of \( SL_2(\mathbb{Z}) \) with respect to the inclusion \( \rho : SL_2(\mathbb{Z}) \hookrightarrow SL_2(\mathbb{Q}) \) is an extension of \( SL_2 \) by a pronipotent group \( \mathcal{U}^{rel} \). The Lie algebra \( \mathfrak{u}^{rel} \) of \( \mathcal{U}^{rel} \) is freely and topologically generated by

\[
\prod_{n \geq 0} H^1(SL_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,
\]

where \( H \) is the standard representation of \( SL_2 \), and \( S^{2n}H \) its \( 2n \)-th symmetric power.

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1Refer to those correspond to classic modular forms in the holomorphic part, see below.
The first step is to construct an explicit \(\mathbb{Q}\)-de Rham structure on \(H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H)\). Recall that there is a mixed Hodge structure on \(H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H)\), which has weight and Hodge filtrations defined over \(\mathbb{Q}\): 
\[
W_{2n+1} H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H) = H^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2n} H);
\]
\[
W_{4n+2} H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H) = H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H),
\]
\[
F^{2n+1} H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H) \cong \left\{ \text{holomorphic part which consists of cohomology classes that correspond to classical modular forms} \right\}.
\]

Now we consider its \(\mathbb{Q}\)-de Rham structure \(H^1_{\text{dR}}(\mathcal{M}_{1,1}/\mathbb{Q}, S^{2n} \mathcal{H})\), where \(\mathcal{H}\) is the relative de Rham cohomology of the universal elliptic curve over \(\mathcal{M}_{1,1}\) with Gauss-Manin connection 
\[
\nabla_0 : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathcal{M}_{1,1}}(\log P)^3.
\]

The \(\mathbb{Q}\)-structure for the holomorphic part \(F^{2n+1} H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H)\) is classically well-known. In this holomorphic part, all \(\mathbb{Q}\)-de Rham classes correspond to classical modular forms of weight \(2n + 2\) with rational Fourier coefficients \([11, 21]\). We call these holomorphic modular forms. To obtain a complete \(\mathbb{Q}\)-de Rham basis of \(H^1_{\text{dR}}(\mathcal{M}_{1,1}/\mathbb{Q}, S^{2n} \mathcal{H})\), one needs to consider modular forms of the second kind. In Section 5.2, we find representatives of all \(\mathbb{Q}\)-de Rham classes in \(H^1_{\text{dR}}(\mathcal{M}_{1,1}/\mathbb{Q}, S^{2n} \mathcal{H})\). These classes correspond to modular forms of the second kind. Their representatives have at worst logarithmic singularities at the cusp and may have singularities with trivial residue at other points. This differs from the traditional approach using weakly modular forms, which allows arbitrary poles at the cusp (cf. Brown–Hain \([3]\)).

For each choice of a base point \(x\) of the moduli space \(\mathcal{M}_{1,1}\) of elliptic curves, we identify \(\text{SL}_2(\mathbb{Z})\) with the (orbifold) fundamental group \(\pi_1(\mathcal{M}_{1,1}, x)\). Denote the relative completion of \(\pi_1(\mathcal{M}_{1,1}, x)\) with respect to the inclusion \(\rho_x : \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Q})\) by \(\mathcal{G}_x\). It is isomorphic to \(G^{\text{rel}}\). Denote the Lie algebra of its unipotent radical by \(\mathfrak{u}_x\). One can construct canonical mixed Hodge structures, depending on the base point \(x\), on \(\mathcal{O}(\mathcal{G}_x)\) and on \(\mathfrak{u}_x\) that are compatible with their algebraic structures \([13, 14]\). It is achieved by finding a canonical flat connection on the Betti bundle \(\mathfrak{u}_B \to \mathcal{M}_{1,1}\) whose fiber over \(x\) is the Lie algebra \(\mathfrak{u}_x\). This connection is more general than the KZB connection in the elliptic curve case \([4, 11, 15]\).

To provide a \(\mathbb{Q}\)-de Rham theory for \(G^{\text{rel}}\), we construct in Section 3 a \(\mathbb{Q}\)-de Rham version of the canonical flat connection on \(\mathfrak{u}_{\text{dR}} \to \overline{\mathcal{M}}_{1,1}\) with a regular singularity at the cusp as follows. Starting from representatives of \(H^1_{\text{dR}}(\mathcal{M}_{1,1}/\mathbb{Q}, S^{2n} \mathcal{H})\) found earlier, we define a canonical flat connection, defined over \(\mathbb{Q}\), on a vector bundle \(\mathfrak{u}_{\text{dR}} \to \mathcal{M}_{1,1}\) whose fibers are the abelianizations of fibers of \(\mathfrak{u}_{\text{dR}}\). From this \(\mathbb{Q}\)-connection on \(\mathfrak{u}_{\text{dR}}\), we apply a Čech-de Rham version of Chen’s method of power series connections \([3]\) to obtain a sequence of canonical flat connections that converges to a canonical flat connection, defined over \(\mathbb{Q}\) with regular singularity at the cusp, on the vector bundle \(\mathfrak{u}_{\text{dR}}\). More specifically, we trivialize \(\mathfrak{u}_{\text{dR}}\) on the open cover of \(\overline{\mathcal{M}}_{1,1}\) consisting of \(\mathcal{M}_{1,1} - \{\rho\}\) and \(\mathcal{M}_{1,1} - \{\i\}\), and provide an inductive algorithm for constructing the connections on both opens, and for finding the gauge transformation on their intersection. By using this constructed de Rham bundle \(\mathfrak{u}_{\text{dR}}\) with connection, it is routine to construct a \(\mathbb{Q}\)-de Rham structure on \(\mathcal{O}(G^{\text{rel}})\) \([13, 7.6]\).

One of the main applications for this \(\mathbb{Q}\)-de Rham theory is the construction of all close\(^2\) iterated integrals of modular forms possibly of the second kind, which enables us to provide all multiple modular values. Previously Manin \([16, 17]\) and Brown \([2]\) only studied those multiple modular values.

\(^2\)Here \(P\) denotes the cusp in \(\overline{\mathcal{M}}_{1,1}\).

\(^3\)I.e. homotopy invariant.
that are (regularized) iterated integrals of holomorphic modular forms. In Section 7 we illustrate with some explicit examples of how to construct iterated integrals that provide the remaining multiple modular values in length two. This is achieved by carrying out the algorithm for constructing connections up to $u_{dR}/L^3(u_{dR})$.

In the final section, we provide an algebraic de Rham theorem for the relative completion of $SL_2(\mathbb{Z})$ by using tannakian formalism.

Part 1. Background

2. The Moduli Space $\overline{M}_{1,1}$ and Its Open Cover

2.1. The moduli stack $\overline{M}_{1,1}$ of stable elliptic curves. The moduli stack $\overline{M}_{1,1}/k$ of stable elliptic curves with one marked point is the stack quotient of $Y := A^2 - \{(0,0)\}$ by a $\mathbb{G}_m$-action

$$\lambda \cdot (u, v) = (\lambda^4 u, \lambda^6 v).$$

This $\mathbb{G}_m$-action is equivalent to a grading on the coordinate ring

$$\mathcal{O}(Y) := k[u, v] = \bigoplus_d \text{gr}_d \mathcal{O}(Y)$$

given by $\text{deg}(u) = 4$ and $\text{deg}(v) = 6$.

The discriminant function

$$\Delta := u^3 - 27v^2$$

has weight 12. The moduli stack $M_{1,1}$ of elliptic curves with one marked point is the stack quotient of

$$Y := A^2 - D$$

by the same $\mathbb{G}_m$-action, where $D$ is the discriminant locus defined by $\Delta = 0$.

Remark 2.1. One could think of $\overline{M}_{1,1}$ as a projective space, which would make our later discussions on $\overline{M}_{1,1}$ more natural. In the next section, we will define an “affine” open cover of $\overline{M}_{1,1}$ analogous to that of a projective space, and later use this open cover of $\overline{M}_{1,1}$ to compute Čech cohomology with twisted coefficients.

2.2. An open cover of $\overline{M}_{1,1}$. Define

$$Y_0 := \text{Spec } k[u, v, u^{-1}], \quad Y_1 := \text{Spec } k[u, v, v^{-1}].$$

Since $Y = A^2 - \{(0,0)\}$, we have $Y = Y_0 \cup Y_1$. The $\mathbb{G}_m$-action on $Y$: $\lambda \cdot (u, v) = (\lambda^4 u, \lambda^6 v)$, restricts to act on both $Y_0$ and $Y_1$. Note that for $i = 0, 1$, the coordinate rings

$$\mathcal{O}(Y_i) = \bigoplus_d \text{gr}_d \mathcal{O}(Y_i)$$

are graded with $u, v, u^{-1}, v^{-1}$ having weights $4, 6, -4, -6$, respectively. Define

$$U_0 := \mathbb{G}_m \backslash Y_0, \quad U_1 := \mathbb{G}_m \backslash Y_1$$

\footnote{The reason we choose this notation $\overline{Y}$, indicating the space being viewed as projective instead of affine throughout this paper, will be evident in Section 4, Remark 4.7.}
to be the stack quotients of the $\mathbb{G}_m$-action. Then

$$\mathcal{U} = \{U_0, U_1\}$$

forms an open cover of $\mathcal{M}_{1,1}$.

**Remark 2.2.** The cusp $P \in \mathcal{M}_{1,1}$, which corresponds to the isomorphism class of a nodal cubic, is in both $U_0$ and $U_1$.

Let $V_0$ be the affine subscheme of $\mathcal{Y}$ defined by $u = 1$. Its coordinate ring is $O(V_0) = O(\mathcal{Y})/I_u$, where $I_u$ is the graded ideal of $O(\mathcal{Y})$ generated by $u - 1$. Since $u$ has weight 4, the affine group scheme $\mu_4$ acts on $V_0 \simeq \mathbb{A}^1$. Similarly, define $V_1 := \text{Spec } O(\mathcal{Y})/I_v$, where $I_v$ is the graded ideal generated by $v - 1$. Since $v$ has weight 6, the affine group scheme $\mu_6$ acts on $V_1 \simeq \mathbb{A}^1$. Using the same argument of Lemma 3.2 in [3], we have

**Proposition 2.3.** The inclusions $V_0 \hookrightarrow Y_0$ and $V_1 \hookrightarrow Y_1$ induce isomorphisms of stacks over $\mathbb{Q}$

1. $\mu_4/\mathcal{Y} \simeq \mathbb{G}_m/\mathcal{Y} = U_0$,
2. $\mu_6/\mathcal{Y} \simeq \mathbb{G}_m/\mathcal{Y} = U_1$.

**Remark 2.4.** Let $Z$ be the affine subscheme of $\mathcal{Y}$ defined by $\Delta = 1$. Its projective closure is an elliptic curve that is isomorphic to the Fermat curve. Since $\Delta$ has weight 12, the affine group scheme $\mu_{12}$ acts on $Z$, Lemma 3.2 in [3] shows that $\mu_{12}/Z \simeq \mathbb{G}_m/Z = \mathcal{M}_{1,1}$. One could use $Z$ and its compactification to develop an algebraic de Rham theory for $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,1}$, see [3].

To develop a $\mathbb{Q}$-de Rham theory, we will use descriptions (2.1) of the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathcal{M}_{1,1}$ and work $\mathbb{G}_m$-equivariantly on $Y_0$, $Y_1$ and $\mathcal{Y}$. Other descriptions provided for these stacks involve roots of unity, which could be complicated to handle.

3. **Vector Bundles $S^{2n} \mathcal{H}$ on $\mathcal{M}_{1,1}$ and Their Canonical Extensions $S^{2n} \mathcal{H}$ on $\mathcal{M}_{1,1}$**

Since $O(\mathcal{Y})$ is a graded ring, one associates graded $O(\mathcal{Y})$-modules to coherent sheaves/vector bundles on $\mathcal{M}_{1,1}$.

3.1. **The Gauss-Manin connection on a rank two vector bundle $\mathcal{H}$ over $\mathcal{M}_{1,1}$**. Define a trivial rank two vector bundle $\mathcal{H}$ on $\mathcal{Y}$ by

$$\mathcal{H} := O_\mathcal{Y} S \oplus O_\mathcal{Y} T,$$

where the multiplicative group $\mathbb{G}_m$ acts on it by

$$\lambda \cdot S = \lambda S, \quad \lambda \cdot T = \lambda^{-1} T,$$

so $S$ and $T$ have weights $+1$ and $-1$ respectively. This vector bundle $\mathcal{H}$ and its restriction $\mathcal{H}$ to $Y$, descend to vector bundles $\mathcal{H}$ over $\mathcal{M}_{1,1}$ and $\mathcal{H}$ over $\mathcal{M}_{1,1}$. Note our abuse of notation that $\mathcal{H}$ and $\mathcal{H}$ might denote vector bundles over $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,1}$, or over $Y$ and $\mathcal{Y}$, depending on the context.

Define the connection on $\mathcal{H}$ and its symmetric powers $S^{2n} \mathcal{H} := \text{Sym}^{2n} \mathcal{H}$ by

$$S^{2n} \mathcal{H} := \bigoplus_{s+t=2n} O_\mathcal{Y} S^s T^t$$

by

$$\nabla_0 = d + \left( -\frac{1}{12} \frac{d\Delta}{\Delta} T + \frac{3\alpha}{2\Delta} S \right) \frac{\partial}{\partial T} + \left( \frac{u\alpha}{8\Delta} T + \frac{1}{12} \frac{d\Delta}{\Delta} S \right) \frac{\partial}{\partial S},$$

for

(3.1)
where \( \alpha = 2uv - 3wdu \) and \( \Delta = u^3 - 27v^2 \). It is \( \mathbb{G}_m \)-invariant, and has regular singularities along the discriminant locus \( D \). Note also that this connection \( \nabla_0 \), when pulled back to each \( \mathbb{G}_m \)-orbit, is the trivial connection \( d \). Therefore, it descends to a connection on \( \overline{H} \), and its symmetric powers

\[
S^{2n}\overline{H} := \text{Sym}^{2n}\overline{H}
\]

over \( \overline{\mathcal{M}}_{1,1} \). These bundles are the canonical extensions of \( H \) and \( S^{2n}H := \text{Sym}^{2n}H \) over \( \mathcal{M}_{1,1} \).

3.2. The local system \( \mathbb{H} \) over \( \mathcal{M}^\an_{1,1} \). Let \( \mathbb{H} := R^1\pi_*\mathbb{C} \), where \( \pi : \mathcal{E}^\an \to \mathcal{M}^\an_{1,1} \) is the universal elliptic curve. For any field \( k \subset \mathbb{C} \), algebraic de Rham theorem induces a natural isomorphism

\[
\mathcal{H} \cong \mathbb{H} \otimes_k \mathbb{C} \cong \mathbb{H}_B \otimes \mathbb{Q} \mathbb{C},
\]
of bundles with connections over \( Y^\an := Y(\mathbb{C}) \), where \( \mathbb{H}_B := R^1\pi_*\mathbb{Q} \) denotes the Betti realization of \( \mathbb{H} \), being endowed with the Gauss-Manin connection. For each \( n \), define \( 2n \)-th symmetric power \( S^{2n}\mathbb{H} := \text{Sym}^{2n}\mathbb{H} \) of \( \mathbb{H} \) over \( \mathcal{M}^\an_{1,1} \), then \( S^{2n}\mathbb{H} \) over \( \mathcal{M}_{1,1} \) is its de Rham realization.

For any point \( \tau \) in the upper half plane \( \mathfrak{h} \), define a lattice \( \Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau \), and an elliptic curve \( \mathcal{E}_\tau := \mathbb{C}/\Lambda_\tau \). Removing all the lattice points \( \Lambda_\tau := \{(\xi, \tau) \in \mathbb{C} \times \mathfrak{h} : \xi \in \Lambda_\tau \} \) from \( \mathbb{C} \times \mathfrak{h} \), there is a map

\[
\psi : \mathbb{C} \times \mathfrak{h} - \Lambda_\tau \to \mathcal{M}^\an_{1,1+\mathfrak{g}} \times \mathbb{C} \times \mathfrak{h} - \Lambda_\tau = \mathcal{M}^\an_{1,1+\mathfrak{g}},
\]

that induces an isomorphism

\[
\text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2 \otimes (\mathbb{C} \times \mathfrak{h} - \Lambda_\tau) \cong \mathbb{G}_m \otimes \mathcal{M}^\an_{1,1+\mathfrak{g}} = \mathcal{M}^\an_{1,2},
\]

where \( \varphi_\tau(\xi) \) and \( \varphi'_\tau(\xi) \) are the Weierstrass \( \varphi \)-function and its derivative, \( g_2(\tau) \) and \( g_3(\tau) \) are normalized Eisenstein series of weight 4 and 6.

On the left hand side, \( \mathbb{Z}^2 \) acts on \( \mathbb{C} \times \mathfrak{h} \) by:

\[
(m, n) \in \mathbb{Z}^2 : (\xi, \tau) \mapsto \left( \xi + (m, n) \left( \begin{array}{c} \tau \\ 1 \end{array} \right), \tau \right),
\]

and \( \text{SL}_2(\mathbb{Z}) \) acts compatibly by:

\[
\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : (\xi, \tau) \mapsto ((c\tau + d)^{-1}\xi, \gamma \tau).
\]

On the right hand side, the multiplicative group \( \mathbb{G}_m \) acts on \( \mathcal{M}^\an_{1,1+\mathfrak{g}} \) by

\[
\lambda \cdot (x, y, u, v) = (\lambda x, \lambda^3 y, \lambda^4 u, \lambda^6 v).
\]

One can pull back the local system \( \mathbb{H} \) through \( \pi : \mathcal{E}^\an \to \mathcal{M}^\an_{1,1} \) to \( \mathcal{E}^\an \), and then restrict to \( \mathcal{M}^\an_{1,2} \). One thus obtains a local system \( \mathbb{H} \) over \( \mathcal{M}^\an_{1,2} \) and its de Rham realization \( \mathcal{H} \) using the same process. The sections \( S \) and \( T \) of \( \mathcal{H} \) over \( \mathcal{M}_{1,2} \) correspond to de Rham classes represented by algebraic forms \( xdx/y \) and \( dx/y \) respectively (cf. [13 Prop. 2.1]). In particular, \( T \) corresponds to the abelian differential of an elliptic curve, which pulls back to \( 2\pi i \xi \) on \( \mathcal{E}_\tau \) under the map \( \psi \).

\(^5\)In [3 2.4], one forms \( \psi = \frac{1}{\pi} \frac{\partial}{\partial \tau} \) and \( \omega = \frac{3}{2\pi} \) are defined, and being used to write the same connection in a different form.

\(^6\)Here \( g_2(\tau) = 2G_4(\tau), g_3(\tau) = \frac{3}{4}G_6(\tau) \), where we use Zagier’s normalization for Eisenstein series and his notation \( G_4(\tau), G_6(\tau) \), see [19].

\(^7\)Compatible with the semi-direct product structure that is induced from the right multiplication of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{Z}^2 \).
Part 2. Relative Completion of $\text{SL}_2(\mathbb{Z})$

The relative completion $\mathcal{G}^{\text{rel}}$ of $\text{SL}_2(\mathbb{Z})$ with respect to the inclusion $\rho : \text{SL}_2(\mathbb{Z}) \hookrightarrow \text{SL}_2(\mathbb{Q})$, is an extension of $\text{SL}_2$ by a prounipotent group $\mathcal{U}^{\text{rel}}$. The Lie algebra $\mathfrak{u}^{\text{rel}}$ of $\mathcal{U}^{\text{rel}}$ is freely topologically generated by

$$\prod_{n \geq 0} H^1(\text{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,$$

where $H$ is the standard representation of $\text{SL}_2$, and $S^{2n}H$ its $2n$-th symmetric power.

We identify $H^1(\text{SL}_2(\mathbb{Z}), S^{2n}H)$ with $H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$. To develop an algebraic de Rham theory for $\mathcal{G}^{\text{rel}}$, it is necessary to find an algebraic de Rham structure $H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ on $H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$ first.

4. De Rham Cohomology with Twisted Coefficients

4.1. Algebraic de Rham theorem. Let $X$ be a smooth quasi-projective variety defined over $\mathbb{k}$. Without loss of generality one can assume $X = \overline{X} - P$, where $\overline{X}$ is smooth projective, and $P$ is a normal crossing divisor in $\overline{X}$. Given a vector bundle $(\mathcal{V}, \nabla)$ with flat connection over $X$, having regular singularities along $P$, denote by $\mathcal{V}$ the local system of horizontal sections of $\mathcal{V}^{\text{an}}$ over $X^{\text{an}}$. Define the twisted de Rham complex

$$\Omega^\bullet_X(\mathcal{V}) := \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{V},$$

and denote its hypercohomology by $H^\bullet_{\text{dR}}(X, \mathcal{V}) := \mathbb{H}^\bullet(X, \Omega^\bullet_X(\mathcal{V}))$. Deligne [7, Cor. 6.3] proved the following version of algebraic de Rham theorem for de Rham cohomology with twisted coefficients.

Theorem 4.1. There is an isomorphism

$$H^\bullet_{\text{dR}}(X, \mathcal{V}) \otimes_{\mathbb{k}} \mathbb{C} \cong H^\bullet(X^{\text{an}}, \mathcal{V}).$$

Remark 4.2. When $X$ is affine, the de Rham structure $H^\bullet_{\text{dR}}(X, \mathcal{V})$ can be computed as the cohomology $H^\bullet(\Gamma(X, \Omega^\bullet_X(\mathcal{V})))$ of global sections of the twisted de Rham complex with differential given by the connection $\nabla$.

Remark 4.3. One can replace $\Omega^\bullet_X$ by any complex that is quasi-isomorphic to $\Omega^\bullet_X$ or the direct image sheaf complex $\iota_* \Omega^\bullet_X$ (for the inclusion $i : X \hookrightarrow \overline{X}$), for example the logarithmic de Rham complex $\Omega^\bullet_X(\log P)$. Then the de Rham structure $H^\bullet_{\text{dR}}(X, \mathcal{V})$ can be computed as the hypercohomology

$$\mathbb{H}^\bullet(\overline{X}, \Omega^\bullet_X(\log P) \otimes_{\mathcal{O}_{\overline{X}}} \nabla)$$

of the twisted logarithmic de Rham complex

$$\Omega^\bullet_X(\log P) \otimes_{\mathcal{O}_{\overline{X}}} \nabla,$$

where $\nabla$ denotes the canonical extension of $\mathcal{V}$ to $\overline{X}$.

Example 4.4. Primary Example: $\mathbb{Q}$-de Rham structure $H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ on $H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$. Assume our field $\mathbb{k} = \mathbb{Q}$ and $n > 0$. Let $X = \mathcal{M}_{1,1}$, $\overline{X} = \overline{\mathcal{M}}_{1,1}$, and their coverings $Y = \mathcal{M}_{1,1} = \mathbb{A}^2 - D$, $\overline{Y} = \mathbb{A}^2 - \{(0,0)\}$. Let $P$ denotes the cusp in $\overline{\mathcal{M}}_{1,1}$, then

$$\overline{X} = G_m \| \overline{Y}, \quad \overline{X} - P = G_m \| (\overline{Y} - D) = G_m \| Y,$$

where the multiplicative group $G_m$ acts on $Y$ and $\overline{Y}$ as before in Section 2.
Let $\mathcal{V}$ be the bundle $S^{2n}\mathcal{H}$ over $\mathcal{M}_{1,1}$ defined in Section 3.1 and recall that we denoted by $\overline{\mathcal{V}} = S^{2n}\overline{\mathcal{H}}$ its canonical extension to $\overline{\mathcal{M}}_{1,1}$. The $\mathbb{Q}$-de Rham structure

$$H^1_{dR}(X, \mathcal{V}) = H^1_{dR}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$$

on $H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathcal{H})$, by Theorem 4.1 above and Remark 4.5 below, can be computed as the hypercohomology $\mathbb{H}^1(\mathcal{M}_{1,1}^{an}, \mathcal{F}^{*}_{2n})$ of the twisted logarithmic de Rham complex

$$\mathcal{F}^{*}_{2n} := (\Omega^{*}_{Y}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{G_m},$$

where the differential of the complex is induced by the Gauss-Manin connection $\nabla_0$ on $\mathcal{H}$ over $\mathcal{M}_{1,1}$ given explicitly by (3.1) in Section 3.1.

**Remark 4.5.** By Theorem 4.1 and Remark 4.3, $H^1_{dR}(Y, S^{2n}\mathcal{H})$ can be computed by the hypercohomology of the complex

$$\Omega^{*}_{Y}(\log D) \otimes S^{2n}\overline{\mathcal{H}}.$$

Since the differential $\nabla_0$ is $\mathbb{G}_m$-invariant, we obtain the subcomplex $\mathcal{F}^{*}_{2n} = (\Omega^{*}_{Y}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{G_m}$ of $\mathbb{G}_m$-invariant forms on $Y$, which descends to a complex $\mathcal{F}^{*}_{2n}$ on $\overline{\mathcal{M}}_{1,1}$. Since $\mathbb{G}_m$ is connected, this also computes $H^1_{dR}(Y, S^{2n}\mathcal{H})$. By computing the Leray spectral sequence for a $\mathbb{G}_m$-principle bundle $p: Y \to \mathcal{M}_{1,1}$, one gets natural isomorphisms

$$p^*: H^1_{dR}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \xrightarrow{\sim} H^1_{dR}(Y, S^{2n}\mathcal{H})$$

for $n > 0$, cf. Brown–Hain [3 §3].

### 4.2. Sections of sheaves $\mathcal{F}^{p}_{2n}$ in the twisted de Rham complex $\mathcal{F}^{*}_{2n}$.

Here we prepare ourselves for explicit computations later by writing down sections of sheaves $\mathcal{F}^{p}_{2n}$ over open sets $U_0$, $U_1$ and their intersection $U_{01}$.

First, we compute global sections of sheaves $\Omega^{p}_{Y}(\log D)$ in the logarithmic de Rham complex. By applying Deligne’s criterion ([8 §3.1]) for being a global section, we have

**Lemma 4.6.**

$$\Gamma(Y, \Omega^{p}_{Y}(\log D)) = \begin{cases} 
\mathcal{O}(Y) & p = 0, \\
\mathcal{O}(Y)^{\oplus} \mathcal{O}(Y) \frac{du}{\Delta} + \mathcal{O}(Y) \frac{dv}{\Delta} & p = 1, \\
\mathcal{O}(Y) \frac{d\phi}{\Delta} & p = 2, \\
0 & \text{otherwise.}
\end{cases}$$

**Proof.** In our case, a section $\phi$ is in the logarithmic complex if and only if $\Delta \cdot \phi$ and $\Delta \cdot d\phi$ are holomorphic on $Y$. The only interesting case is when $p = 1$. Assume that $\phi$ is a 1-form in the logarithmic complex, then $\phi := \Delta \cdot \phi$ is holomorphic, we can write it as

$$\phi = f du + g dv \in \mathcal{O}(Y) du \oplus \mathcal{O}(Y) dv,$$

where $f, g$ are polynomials in $u$ and $v$ as $\mathcal{O}(Y) = \mathbb{Q}[u,v]$. Since

$$d\phi = d \left( \frac{\phi}{\Delta} \right) = \frac{d\phi}{\Delta} + \phi \wedge \frac{d\Delta}{\Delta^2},$$

to make sure that $\Delta \cdot d\phi$ is holomorphic, the 2-form

$$\phi \wedge d\Delta = (f du + g dv) \wedge (3u^2du - 54vdv) = -(3u^2g + 54vf)du \wedge dv$$

has to be a multiple of $\Delta$, i.e. we need to have

$$(u^3 - 27v^2)(3u^2g + 54vf).$$
We call \((f, g)\) a solution if it satisfies the above condition. One can check that \((f, g) = (-3v, 2u)\) and \((f, g) = (3u^2, -54v)\) are solutions. Denote the highest degree of \(v\) in a polynomial \(f \in \mathbb{Q}[u, v]\) by \(\text{deg}_v(f)\), ignoring \(u\) or regarding \(u\) as a constant. To find other solutions, we apply a Euclidean algorithm to reduce the degrees \(\text{deg}_v(f)\) and \(\text{deg}_v(g)\) to 0. Applying Euclidean algorithm on \(f\) with \((-3v)\) from the \(f\)-component of the first solution \((-3v, 2u)\), we can write \(f = (-3v)q + f_1\), where \(q, f_1 \in \mathbb{Q}[u, v]\) with \(\text{deg}_v(f_1) < \text{deg}_v(-3v) = 1\). We define \(g_1 := g - 2uq\), then

\[
3u^2g_1 + 54v f_1 = 3u^2(g - 2uq) + 54v(f + 3vq) = (3u^2 g + 54v f) - 6q(u^3 - 27v^2),
\]

and we reduces the problem of finding \((f, g)\) to finding \((f_1, g_1)\), where \(\text{deg}_v(f_1) < \text{deg}_v(f)\) unless \(\text{deg}_v(g) = 0\). A similar process can reduce \(\text{deg}_v(g)\) by applying Euclidean algorithm on \(g\) with \((-54v)\) from the other solution \((3u^2, -54v)\). Note that we can continue this as long as \(\text{deg}_v(f)\) or \(\text{deg}_v(g)\) is positive, and \(\text{deg}_v(f) + \text{deg}_v(g)\) keeps decreasing after every reduction step. Eventually, we would have reduced to \((f, g)\) such that \(\text{deg}_v(f) = \text{deg}_v(g) = 0\) and \(\text{deg}_v(3u^2 g + 54v f)\) is at most 1. In this case, since on the left side \(\text{deg}_v(u^3 - 27v^2) = 2\), the right side polynomial \((3u^2 g + 54v f)\) has to be 0, which in turn implies \(f = g = 0\). This amounts to showing that \(\phi = fdu + gdv\) has to be an \(O(\mathcal{Y})\)-linear combination of

\[
-3vdu + 2udv = \alpha \quad \text{and} \quad 3u^2du - 54v dv = d\Delta,
\]

i.e.

\[
\varphi = \frac{\phi}{\Delta} \in O(\mathcal{Y})\frac{\alpha}{\Delta} \oplus O(\mathcal{Y})\frac{d\Delta}{\Delta}.
\]

Our next task is to apply Čech-de Rham theory to compute \(\mathbb{Q}\)-de Rham representatives of \(H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})\). In preparation, we compute the \(G_m\)-invariant sections on our open cover.

Since everything is graded by \(G_m\), it is easy to deduce from Lemma 4.6 by weight computations that global sections of \(\mathcal{F}_{2n}^p\) are

\[
\mathcal{F}_{2n}^p(\mathcal{M}_{1,1}) = \Gamma(\mathcal{Y}, (\Omega^p_{\mathcal{Y}}(\log D) \otimes S^{2n}\mathcal{H})^{G_m})
\]

\[
= \begin{cases} 
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(\mathcal{Y})) S^t \Delta^s & p = 0, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(\mathcal{Y})) S^t \Delta^s + \bigoplus_{s+t=2n} (\text{gr}_{t-s}O(\mathcal{Y})) \frac{d\Delta}{\Delta} S^t \Delta^s & p = 1, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(\mathcal{Y})) \frac{d\Delta}{\Delta} S^t \Delta^s & p = 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that for \(i = 0, 1\), we have \(O(\mathcal{Y}_i) = O(Y_i)\), and sections of \(\mathcal{F}_{2n}^p\) on \(U_i\) are

\[
\mathcal{F}_{2n}^p(U_i) = \Gamma(Y_i, (\Omega^p_{\mathcal{Y}}(\log D) \otimes S^{2n}\mathcal{H})^{G_m})
\]

\[
= \begin{cases} 
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(Y_i)) S^t \Delta^s & p = 0, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(Y_i)) S^t \Delta^s + \bigoplus_{s+t=2n} (\text{gr}_{t-s}O(Y_i)) \frac{d\Delta}{\Delta} S^t \Delta^s & p = 1, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s}O(Y_i)) \frac{d\Delta}{\Delta} S^t \Delta^s & p = 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(Y_{01} := Y_0 \cap Y_1 = \text{Spec} \mathbb{Q}[u, v, u^{-1}, v^{-1}]\), then the coordinate ring \(O(Y_{01}) = \mathbb{Q}[u, v, u^{-1}, v^{-1}]\) is graded with \(u, v, u^{-1}, v^{-1}\) having weights 4, 6, -4, -6, respectively. Denote degree \(n\) part of
\( \mathcal{O}(Y_{01}) \) by \( \text{gr}_s \mathcal{O}(Y_{01}) \). Since the \( \mathbb{G}_m \)-action on \( Y \) restricts to \( Y_{01} \), and the stack quotient of this action is \( U_{01} := \mathbb{G}_m \backslash Y_{01} \), we deduce similarly that

\[
\mathcal{F}_d^P(U_{01}) = \Gamma(Y_{01}, (\Omega^P_1(\log D) \otimes S^{2n}\mathcal{H}))^{\mathbb{G}_m}
\]

\[
= \left\{ \begin{array}{ll}
\bigoplus_{s+t=2n} (\text{gr}_{t-s} \mathcal{O}(Y_{01})) S^s T^t & p = 0, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s+2} \mathcal{O}(Y_{01})) S^s T^t & p = 1, \\
\bigoplus_{s+t=2n} (\text{gr}_{t-s+2} \mathcal{O}(Y_{01})) S^s T^t & p = 2, \\
0 & \text{otherwise.}
\end{array} \right.
\]


### 4.3. The Čech-de Rham complex \( \check{C}^*(\mathcal{U}, \mathcal{F}_d^P) \)

In this section, we construct a Čech-de Rham complex \( \check{C}^*(\mathcal{U}, \mathcal{F}_d^P) \) that computes the \( \mathbb{Q} \)-de Rham structure \( H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \) for the open cover \( \mathcal{U} = \{ U_0, U_1 \} \):

\[
\begin{aligned}
\mathcal{F}_d^2(U_0) \oplus \mathcal{F}_d^2(U_1) & \xrightarrow{\delta} \mathcal{F}_d^2(U_{01}) \\
(\nabla_0, \nabla_0) & \uparrow \\
\mathcal{F}_d^1(U_0) \oplus \mathcal{F}_d^2(U_1) & \xrightarrow{\delta} \mathcal{F}_d^2(U_{01}) \\
(\nabla_0, \nabla_0) & \uparrow \\
\mathcal{F}_d^0(U_0) \oplus \mathcal{F}_d^0(U_1) & \xrightarrow{\delta} \mathcal{F}_d^0(U_{01})
\end{aligned}
\]

where \( U_{01} \) is the intersection of \( U_0 \) and \( U_1 \); the horizontal differential \( \delta \) is the usual one for a Čech complex, and the vertical differential is \( \nabla_0 \) in the twisted de Rham complex \( \mathcal{F}_d^P \). Let

\[
D = \delta + (-1)^d \nabla_0
\]

be the (total) differential of the single complex \( sC^*(\mathcal{U}, \mathcal{F}_d^P) \) associated to the Čech-de Rham double complex \( \check{C}^*(\mathcal{U}, \mathcal{F}_d^P) \).

Recall from Example 4.4 that the \( \mathbb{Q} \)-structure \( H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \) is computed by the hypercohomology \( \mathbb{H}^1(\mathcal{M}_{1,1}, \mathcal{F}_d^P) \), where \( \mathcal{F}_d^P = (\Omega^P_1(\log D) \otimes S^{2n}\mathcal{H})^{\mathbb{G}_m} \). Since \( \mathcal{F}_d^P \) is coherent on \( \overline{\mathcal{M}}_{1,1} \) (see [18], or [9] for an algebraic proof), and the open cover \( \mathcal{U} = \{ U_0, U_1 \} \) is a “good cover”\(^8\) the Čech cohomology calculated for this open cover \( \mathcal{U} \) will give us the correct answer, and we have

\[
H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) = \mathbb{H}^1(\mathcal{M}_{1,1}, \mathcal{F}_d^P) = H^1(\mathcal{U}, sC^*(\mathcal{U}, \mathcal{F}_d^P)).
\]

\( \textbf{Remark 4.7.} \) One can compute the cohomology \( H^1(\mathcal{M}_{1,1}, \mathcal{F}_d^P) \) of global sections of \( \mathcal{F}_d^P \). One finds that its dimension equals that of \( \mathcal{M}_{2n+2} \), the \( \mathbb{Q} \) vector space spanned by \textit{holomorphic} modular forms of weight \( 2n+2 \) with rational Fourier coefficients. It does not equal the dimension of \( H^1(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{S}^{2n}\mathbb{H}) \). This suggests that one should not view \( Y \) with its \( \mathbb{G}_m \)-action as an affine variety and compute cohomology by global sections (cf. Remark 4.2). Instead one has to use hypercohomology of \( Y \) to compute \( H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \) if one wants representatives to have logarithmic singularities at the cusp.

### 5. Holomorphic Modular Forms and Modular Forms of the Second Kind

In this section, we apply the algebraic de Rham theory previously, and find explicitly all \( \mathbb{Q} \)-de Rham classes in the \( \mathbb{Q} \)-de Rham structure \( H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \) on \( H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}) \) for every \( n \). These \( \mathbb{Q} \)-de Rham classes are closely related to \textit{holomorphic} modular forms with rational Fourier coefficients.

\(^8\) In the sense of Bott and Tu [11, §8] that the (augmented) columns are exact in the Čech-de Rham complex.
By Eichler–Shimura, after tensoring $H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ with $\mathbb{C}$, one obtains a natural mixed Hodge structure on $H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$, which has weight and Hodge filtrations defined over $\mathbb{Q}$:

\begin{align}
W_{2n+1}^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H}) &= H^1_{\text{cusp}}(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H}); \\
W_{4n+2}^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H}) &= H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H}), \\
F_{2n+1}^2H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H}) &= M_{2n+2} \otimes \mathbb{Q}, \mathbb{C},
\end{align}

where $M_{2n}$ denotes the $\mathbb{Q}$-vector space spanned by holomorphic modular forms of weight $2n$ with rational Fourier coefficients.

The $\mathbb{Q}$-structure for the last part $F_{2n+1}^2H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$-- holomorphic modular forms -- is well known. Every $\mathbb{Q}$-de Rham cohomology class in $F_{2n+1}^2H^1(\mathcal{M}_{1,1}^{an}, S^{2n}\mathbb{H})$ can be represented by a global $\mathbb{G}_m$-invariant $\nabla_0$-closed 1-form on $\mathcal{M}_{1,1}$ with coefficients in $S^{2n}\mathcal{H}$. The explicit correspondence has been found in [11, §21], we record it here in our notation in the following section 5.1.

However, a global 1-form representative can not be found for the remaining $\mathbb{Q}$-de Rham classes if one insists that representatives have logarithmic singularities at the cusp (cf. Remark [17]). These remaining classes are the ones, under the Eichler–Shimura isomorphism, that involve anti-holomorphic cusps. In Section 5.2, we explain how to find and represent all $\mathbb{Q}$-de Rham classes including these remaining classes using Čech cocycles in the Čech-de Rham complex $\check{C}^*({\mathcal{U}}, F^*_{\mathbb{H}})$. Similar results were obtained by Brown–Hain [2] using weakly modular forms. Our representatives have the advantage of having logarithmic singularities at the cusp, which are better suited to computing regularized periods.

### 5.1. Holomorphic modular forms

Given a holomorphic modular form $f(\tau)$ of weight $2n + 2$ with rational Fourier coefficients, it corresponds to a polynomial $h(u, v) \in \mathbb{Q}[u, v]$ of weight $2n + 2$ (where $u$ has weight 4 and $v$ has weight 6). From Hain [11, §21], the cohomology class corresponding to $f(\tau)$ can be represented by a global 1-form

\[
h(u, v) \frac{\alpha}{\Delta} T^{2n} = h(u, v) \frac{2uv - 3v}{u^3 - 27v^2} T^{2n} \in \left( \Omega^1_Y(\log D) \otimes F_{2n}S^{2n}\mathcal{H} \right)^{G_m}.
\]

Analytically, the pullback of this form along the map $h \mapsto \mathcal{M}_{1,1}^{an} = Y^{an}$ defined by $\tau \mapsto (4\tau)^4g_2(\tau), (4\tau)^6g_3(\tau))$ is up to a rational multiple ($\frac{2}{\pi}$ to be precise),

\[
(2\pi)^2h(g_2(\tau), g_3(\tau)) T^{2n} \frac{dq}{q} = (2\pi)^{2n+1} f(\tau) T^{2n} d\tau, \quad \text{with} \quad q = e^{2\pi i \tau}.
\]

This 1-form is denoted by $\omega_f$ in Hain [10, §9.1], and by $f(\tau)$ in Brown [2, §2.1] to compute periods. We will adopt Hain’s notation $\omega_f$.

**Example 5.1.** The normalized cusp form $\Delta = q - 24q^2 + 252q^3 + \cdots$ of weight 12 is a rational polynomial $\Delta = u^3 - 27v^2$. So its corresponding class is represented by the 1-form

\[
\omega_\Delta = \frac{2uv - 3v}{u^3 - 27v^2} T^{10} = (2uv - 3v) T^{10}.
\]

### 5.2. Modular forms of the second kind

In this section, we will find all $\mathbb{Q}$-de Rham classes in $H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ as promised. We start by discussing what is a 1-cocycle in the single complex $\check{C}^*({\mathcal{U}}, F^*_{\mathbb{H}})$ associated to the Čech-de Rham double complex $\check{C}^*({\mathcal{U}}, F^*_{\mathbb{H}})$.

In the Čech-de Rham complex $\check{C}^*({\mathcal{U}}, F^*_{\mathbb{H}})$, every 1-cochain $\tilde{\omega}$ is of the form

\[
\tilde{\omega} = \begin{pmatrix}
(\omega^{(0)}, \omega^{(1)}) & 0 \\
0 & I
\end{pmatrix}
\]
where \( \omega^{(i)} \in \mathcal{F}_{2n}^{1}(U_{i}) \), and \( l \in \mathcal{F}_{2n}^{0}(U_{01}) \), so that \( (\omega^{(0)}, \omega^{(1)}) \in \mathcal{C}^{0}(\mathcal{U}, \mathcal{F}_{2n}^{1}) \) and \( l \in \mathcal{C}^{1}(\mathcal{U}, \mathcal{F}_{2n}^{0}) \). We often simply write it as \( \tilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l) \).

A 1-cochain \( \tilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l) \) in \( s\check{C}^{*}(\mathcal{U}, \mathcal{F}_{2n}^{*}) \) is a 1-cocycle whenever \( D\tilde{\omega} = 0 \); in other words, it is a cocycle if and only if \( \nabla_{0}\omega^{(0)} = -\nabla_{0}l = \omega^{(1)} - \omega^{(0)} = 0 \).

**Example 5.2.** Holomorphic modular forms as 1-cocycles. As was shown in the last section, a *holomorphic* modular form \( f \) with rational Fourier coefficients of weight \( 2n + 2 \) gives rise to a cohomology class \( [\omega_{f}] \in H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, S^{2n}\mathcal{H}) \), which can be represented by a *global* closed form \( \omega_{f} \). Denote by \( \omega_{f}^{(i)} \) the restriction of \( \omega_{f} \) to \( U_{i} \), \( i = 0, 1 \), and define

\[
\tilde{\omega}_{f} := (\omega_{f}^{(0)}, \omega_{f}^{(1)}; 0),
\]

then \( \tilde{\omega}_{f} \) is a 1-cocycle in our \( \check{C}^{*}(\mathcal{U}, \mathcal{F}_{2n}^{*}) \).

To find representatives for all \( \mathbb{Q} \)-de Rham classes in \( H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, S^{2n}\mathcal{H}) \), we use the second spectral sequence of the double complex \( \check{C}^{*}(\mathcal{U}, \mathcal{F}_{2n}^{*}) \), with the second page \( E_{2} \) given by \( H_{\nu_{0}} H_{3} \) and differentials \( d_{E} : E_{p}^{r,q} \to E_{p-r+1,q+r} \). Since the double complex \( \check{C}^{*}(\mathcal{U}, \mathcal{F}_{2n}^{*}) \) concentrates in the first two columns, this spectral sequence degenerates at \( E_{3} \) trivially.

We now use a spectral sequence zig-zag argument. Starting from \( l \) in the lower right corner, it extends to a 1-cocycle \( \tilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l) \) if it lives to \( E_{3} \):

\[
\begin{array}{ccc}
(0, 0) & \downarrow \delta & (\omega^{(0)}, \omega^{(1)}) \\
\nabla_{0}l \downarrow & \nabla_{0} \omega^{(1)} & \nabla_{0}\omega^{(0)} = 0
\end{array}
\]

i.e. \( \nabla_{0}\omega^{(0)} = \nabla_{0}\omega^{(1)} = 0 \) and \( \omega^{(1)} - \omega^{(0)} = \nabla_{0}l = 0 \). This is equivalent to the condition \( D\tilde{\omega} = 0 \).

Given a trivial class \( 0 \in H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, \mathcal{F}_{2n}^{*}) \), we can always choose \( l = 0 \) to represent it. Then by the conditions above, we would have a *global* closed form \( \omega \), which brings us back to the previous example. In fact, when \( n < 5 \), the corresponding group \( H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, S^{2n}\mathcal{H}) \) is trivial. The first interesting case occurs at \( n = 5 \), which is expected, since \( H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, S^{10}\mathcal{H}) \) corresponds to modular forms of weight \( 2n + 2 = 12 \), where a cusp form appears for the first time.

**Example 5.3.** First new \( \mathbb{Q} \)-de Rham class in \( H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, S^{2n}\mathcal{H}) \). When \( n = 5 \), one can find a nonzero class in \( H_{1}^{\text{dR}}(\mathcal{M}_{1, 1}, \mathcal{F}_{10}^{*}) \), cokernel of \( \delta \) on the 0-th row, represented by \( \frac{1}{uv} S^{10} \in \mathcal{F}_{10}^{0}(U_{01}) \). The reason is that \( \frac{1}{uv} = u^{-1}v^{-1} \), having negative powers for both \( u \) and \( v \), cannot be the difference of two elements in \( \mathbb{Q}[u, v][u^{-1}] \) and \( \mathbb{Q}[u, v][v^{-1}] \) respectively, where every monomial has a nonnegative power of at least one variable.

To extend the nontrivial class \( \frac{1}{uv} S^{10} \) through \( E_{2} \), one needs to find \( (\omega_{1, 1}^{(0)}, \omega_{1, 1}^{(1)}) \) such that

\[
\omega_{1, 1}^{(1)} - \omega_{1, 1}^{(0)} = \nabla_{0}l_{1, 1},
\]

where \( l_{1, 1} \) represents the class \( \frac{1}{uv} S^{10} \). One can choose \( l_{1, 1} \) to be \( \frac{1}{uv} S^{10} \) in this case since

\[
\nabla_{0} \left( \frac{1}{uv} S^{10} \right) = - \left( \frac{9\alpha}{u^{2}\Delta} + \frac{\alpha}{2v^{2}\Delta} \right) S^{10} - \frac{5\alpha}{4v^{3}} S^{9} T
\]
can be written as $\omega^{(1)}_{1,1} - \omega^{(0)}_{1,1}$ with

$$\omega^{(0)}_{1,1} = \frac{9\alpha}{u^2\Delta}S^{10} \in F^1_{10}(U_0),$$

and

$$\omega^{(1)}_{1,1} = -\frac{u\alpha}{2v^2\Delta}S^{10} - \frac{5\alpha}{4v\Delta}S^9T \in F^1_{10}(U_1).$$

One can easily compute and find that these two forms are $\nabla_0$-closed, so the cochain

$$\bar{\omega}_{1,1} = \begin{pmatrix} (\omega^{(0)}_{1,1}, \omega^{(1)}_{1,1}) & 0 \\ 0 & l_{1,1} \end{pmatrix}$$

lives to $E_3$ and represents a class in $H^1_{\text{dR}}(M_{1,1}, S^{10}\mathcal{H})$.

Using the same argument at the beginning of the above example, one easily gets

**Lemma 5.4.** The cohomology group $H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$ is spanned by classes $[\frac{1}{u^a v^b} S^t T^s]$ with positive integers $p$, $q$ such that $4p + 6q \leq 2n$, and nonnegative integers $s = n + 2p + 3q$, $t = n - 2p - 3q$.

**Remark 5.5.** Here $s$ and $t$ can be solved from $s + t = 2n$ ($S^t T^s \in S^{2n}\mathcal{H}$) and $s - t = 4p + 6q$ ($\frac{1}{u^a v^b} S^t T^s \in \mathcal{G}_{m_n}$-invariant, i.e. $\frac{1}{u^a v^b} \in \text{gr}_{t-s} \mathcal{O}(Y_{01})$).

Since we already found all the holomorphic modular forms in $H^1_{\text{dR}}(M_{1,1}, S^{2n}\mathcal{H})$, and based on Eichler-Shimura (5.1)-(5.3), the remaining classes span a vector space of dimension equal to the one spanned by (anti)holomorphic cusp forms. Therefore, by Lemma 5.6 below, if we were to find a class $[\bar{\omega}_{j,k}]$ in $H^1_{\text{dR}}(M_{1,1}, S^{2n}\mathcal{H})$ for each positive integer pair $(j, k)$ with $4j + 6k = 2n$, we will have found all the remaining classes.

**Lemma 5.6.** The number of positive integer pairs $(j, k)$ satisfying $4j + 6k = 2n$ is the same as the dimension of the space of cusp forms of weight $2n + 2$.

**Proof.** The dimension of cusp forms of weight $2n + 2$ is the number of normalized Hecke eigen cusp forms of the same weight. We can choose a basis to be $\Delta \cdot u^a v^b$ where $a$ and $b$ are nonnegative integers, $u$ and $v$ are normalized Hecke eigenform of weight 4 and 6 respectively as usual. It suffices to show a bijection between the set of integer pairs $(j, k)$ and the set of integer pairs $(a, b)$. Clearly the weight of modular forms provides us with the restriction $4a + 6b + 12 = 2n + 2$, or equivalently $4(a + 1) + 6(b + 1) = 2n$. Therefore, $j = a + 1$ and $k = b + 1$ gives us the bijection we need. □

The following result explains how to find representatives of all the remaining $\mathbb{Q}$-de Rham classes.

**Proposition 5.7.** Assume $n \geq 5$. For any pair of positive integers $j, k$ such that $4j + 6k = 2n$, there is a class $[\bar{\omega}_{j,k}]$ in $H^1_{\text{dR}}(M_{1,1}, S^{2n}\mathcal{H})$ represented by a Čech 1-cocycle

$$\bar{\omega}_{j,k} = \begin{pmatrix} (\omega^{(0)}_{j,k}, \omega^{(1)}_{j,k}) & 0 \\ 0 & l_{j,k} \end{pmatrix}$$

where

$$l_{j,k} = \sum_{s+t=2n} x_{s,t} S^t T^s \in F^0_{2n}(U_0),$$

with $x_{s,t} \in \text{gr}_{t-s} \mathcal{O}(Y_{01})$ and $x_{2n,0} = \frac{1}{u^a v^b}$. In other words, we can choose $l_{j,k}$ to be a $\mathbb{Q}$-linear combination of terms $\frac{1}{u^a v^b} S^t T^s$ starting with a term $\frac{1}{u^a v^b} \mathcal{G}_{m_n}$ so that $\nabla_0 l_{j,k}$ can be expressed as $\omega^{(1)}_{j,k} - \omega^{(0)}_{j,k}$, with both $\omega^{(0)}_{j,k}$ and $\omega^{(1)}_{j,k}$ being $\nabla_0$-closed.
Proof. Let us order the terms in \( l_{j,k} \) and \( \nabla_0 l_{j,k} \) by the power of \( S \), then \( S^{2n} \) has the highest order \( 2n \), while \( T^{2n} \) has the lowest order 0. By using (3.1), one can compute that

\[
\nabla_0 \left( \frac{1}{u^p v^q} S^T \right) = \frac{3t}{2u^p v^q} \frac{\alpha}{\Delta} S^{n+1} T^{t-1}
\]

(5.4)

\[
- \left( \frac{9p}{u^p+1} \frac{\alpha}{\Delta} + \frac{q}{2u^{p-2} v^{q+1}} \frac{\alpha}{\Delta} \right) S^T
\]

(5.5)

\[
- \frac{s}{8u^{p-1} v^q} \frac{\alpha}{\Delta} S^{n-1} T^{t+1}
\]

(5.6)

where the terms on the right hand side in (5.4), (5.5), (5.6) have orders \( s+1, s, s-1 \) respectively. Recall that our objective is to express \( \nabla_0 l_{j,k} \) as a difference \( \omega^{(1)}_{j,k} - \omega^{(0)}_{j,k} \). We call a term “bad” if its denominator has positive powers of both \( u \) and \( v \) (as we cannot write it as \( \omega^{(1)} - \omega^{(0)} \)). We will eliminate all bad terms appearing in \( \nabla_0 l_{j,k} \) by adding \( \nabla_0 \)-coboundaries.

To find \( l_{j,k} \), we start with \( \frac{1}{u^p v^q} S^{2n} \), then \( \nabla_0(\frac{1}{u^p v^q} S^{2n}) \) has terms of order \( 2n \) and \( 2n-1 \) (the \( (2n+1) \)-order term being 0). We can use order \( 2n-1 \) terms \( x_{2n-1,1} S^{2n-1} T \) to cancel the order \( 2n \) bad terms in \( \nabla_0(\frac{1}{u^p v^q} S^{2n}) \) since the order \( 2n \) term in \( \nabla_0(x_{2n-1,1} S^{2n-1} T) \) is just a rational multiple of \( x_{2n-1,1} S^{2n} \) by the formula in (5.4). Now \( \nabla_0(\frac{1}{u^p v^q} S^{2n} + x_{2n-1,1} S^{2n-1} T) \) has bad terms of order at most \( 2n-1 \).

We can repeat this process until the bad terms have order \( n+7 \), due to the fact that there always exist positive integer solutions for \( 4p+6q = 2r \) with \( 2r > 12 \), which correspond to bad terms of the form \( \frac{1}{u^p v^q} S^{n+r} T^{n-r} \). They are used to cancel rational multiples of \( \frac{1}{u^p v^q} S^{n+r+1} T^{n-r-1} \).

Now we need to vary the argument above to find the last term of order \( n+5 \), at which time the process terminates. We distinguish terms \( \frac{1}{u^p v^q} S^T \) and \( \frac{1}{u^p v^q} S^T \) by calling them “function” and “form” respectively. By our process above,

\[
\nabla_0 \left( \frac{1}{u^p v^q} S^{2n} + x_{2n-1,1} S^{2n-1} T + \cdots + x_{n+7,n-7} S^{n+7} T^{n-7} \right)
\]

(5.7)

has bad forms of order at most \( n+7 \) and at least \( n+6 \). In fact, the only \( \mathbb{G}_m \)-invariant forms of order \( n+7 \) is a linear combination of \( \frac{u^p}{v^q} S^{n+7} T^{n-7} \) and \( \frac{u^p}{v^q} S^{n+7} T^{n-7} \), neither of which is a bad form. We are thus left with only bad forms of order \( n+6 \) in (5.7). The only bad \( \mathbb{G}_m \)-invariant form of order \( n+6 \) up to a rational multiple, is \( \frac{1}{u^p v^q} S^{n+6} T^{n-6} \). To cancel bad forms of order \( n+6 \), we can choose a rational multiple of \( \frac{1}{u^p v^q} S^{n+5} T^{n-5} \) to be the last term of \( l_{j,k} \). Therefore, the only possible bad terms in \( \nabla_0(l_{j,k}) \) are the order \( n+5 \) and \( n+4 \) parts of \( \nabla_0(\frac{1}{u^p v^q} S^{n+5} T^{n-5}) \). Both these parts have no bad terms, which can be easily checked by formulas (5.5) and (5.6).

Eventually, we have a linear combination of bad terms

\[
l_{j,k} := \frac{1}{u^p v^q} S^{2n} + x_{2n-1,1} S^{2n-1} T + \cdots + x_{n+7,n-7} S^{n+7} T^{n-7} + x_{n+5,n-5} S^{n+5} T^{n-5}
\]

such that \( \nabla_0(l_{j,k}) \) has no bad terms.

After finding \( l_{j,k} \), it is routine to find \( \omega^{(0)}_{j,k} \) and \( \omega^{(1)}_{j,k} \) by putting all terms with powers of \( u \) in the denominator into \( \omega^{(0)}_{j,k} \) and putting all terms with powers of \( v \) in the denominator into \( \omega^{(1)}_{j,k} \). It remains to show that both \( \omega^{(0)}_{j,k} \) and \( \omega^{(1)}_{j,k} \) are \( \nabla_0 \)-closed. This follows from [3, Cor. 3.5]. Or one sees directly from the formulas (5.4)–(5.6) that all the terms in \( \omega^{(0)}_{j,k} \) and \( \omega^{(1)}_{j,k} \) have \( \alpha \) in the numerator and \( \Delta \) in the denominator, and easily checks that \( h \frac{\Delta}{S^T} \) is \( \nabla_0 \)-closed as long as it is a \( \mathbb{G}_m \)-invariant form. \( \square \)
Another Proof. Instead of the minimalist approach above where we only eliminate all bad terms that arise, we provide another efficient and canonical approach based on “Heads and Tails” in Brown–Hain [3, §4.1], eliminating all terms of positive order and leaving only the $T^{2n}$ term in $\nabla_0 l_{j,k}$.

Adapting from Lemma 4.1 in [3], we can easily show that there is a unique $\mathbb{Q}$-linear map $\phi : \gr_{-2n}O(Y_0) \to \mathcal{F}^{n}_2(U_0)$ such that if we write

$$\phi = \sum_{s+t=2n} \phi^{s,t} S^s T^t$$

where $\phi^{s,t} \in \text{Hom}(\gr_{-2n}O(Y_0), \gr_{t-s}O(Y_0))$, then we have

$$\phi^{2n,0}(l) = l \quad \text{and} \quad \nabla_0 \phi(l) = \nabla_0 \phi(l) \in \gr_{2n+2}O(Y_0) \frac{\alpha}{\Delta} T^{2n}.$$

In fact, we have

$$\nabla_0 \phi(l) = \frac{D^{2n+1}(l)}{(2n)!} \frac{3\alpha}{2\Delta} T^{2n},$$

where $D$ is the differential operator $g \partial / \partial g = (2\pi i)^{-1} \partial / \partial \tau$. It is obvious that every function in $\gr_{2n+2}O(Y_0)$ contains no bad terms. Therefore, we can set $l_{j,k}$ to be

$$l_{j,k} = \phi \left( \frac{1}{uv} \right) = \sum_{s+t=2n} \phi^{s,t} \left( \frac{1}{uv} \right) S^s T^t,$$

and choose $\omega^{(0)}_{j,k}, \omega^{(1)}_{j,k}$ accordingly so that $\omega^{(1)}_{j,k} - \omega^{(0)}_{j,k} = \nabla_0 l_{j,k}$.

Note that this proof provides a different 1-cocycle, but it represents the class $[\omega_{j,k}] \in H^1_{\text{dR}}(M_{1,1}, S^{2n} \mathcal{H})$ constructed in the previous proof.

Remark 5.8. For different pairs of integers $(j,k)$, the classes $[\omega_{j,k}] \in H^1_{\text{dR}}(M_{1,1}, S^{2n} \mathcal{H})$ are linearly independent because of Lemma 5.4 and the fact that $l_{j,k}$ starts with $\frac{1}{uv} S^{2n}$.

Example 5.9. The cocycle $\omega_{2,1}$ that represents $[\omega_{2,1}] \in H^1_{\text{dR}}(M_{1,1}, S^{14} \mathcal{H})$. We carry out the process in the above proposition when $(j,k) = (2,1)$ and $n = 7$. Starting with $\frac{1}{uv} S^{14}$ that represents $[\frac{1}{uv} S^{14}] \in H^1_{\text{dR}}(\mathcal{U}, J^0_{14})$, we have

$$\nabla_0 \left( \frac{1}{uv} S^{14} \right) = -\left( \frac{18}{uv} \frac{\alpha}{\Delta} + \frac{1}{2v^2} \frac{\alpha}{\Delta} \right) S^{14} - \frac{7}{4uv} \frac{\alpha}{\Delta} S^{13} T$$

with a bad term of order $13 = n + 6$, which can be eliminated by adding a multiple of the function $\frac{1}{uv} S^{12} T^2$ of order $n + 5 = 12$. Since

$$\nabla_0 \left( \frac{1}{uv} S^{12} T^2 \right) = \left( \frac{3}{uv} \frac{\alpha}{\Delta} - \left( \frac{9}{u^2} \frac{\alpha}{\Delta} + \frac{u}{2v^2} \frac{\alpha}{\Delta} \right) S^{12} T^2 - \frac{3}{2v} \frac{\alpha}{\Delta} S^{11} T^3,\right.$$n

we choose the correct multiple $\frac{7}{12}$ for $\frac{1}{uv} T^2 S^{12}$, and

$$\nabla_0 \left( \frac{1}{u^2 v^{14}} + \frac{7}{12} \frac{1}{uv} S^{12} T^2 \right) = -\left( \frac{18}{u^2} \frac{\alpha}{\Delta} + \frac{1}{2u} \frac{\alpha}{\Delta} \right) S^{14} - \left( \frac{21}{4u^2} \frac{\alpha}{\Delta} + \frac{7u}{24v^2} \frac{\alpha}{\Delta} \right) S^{12} T^2 - \frac{7}{8v} \frac{\alpha}{\Delta} S^{11} T^3$$

has no bad terms.
Define \( l_{2,1} = \frac{7}{24v^2} \Delta S^{14} + \frac{7}{8} \frac{1}{v^2} S^{12} T^2 \), \( \omega^{(0)}_{2,1} = \frac{18}{v^2} \Delta S^{14} + \frac{21}{4v^2} \Delta S^{12} T^2 \), and \( \omega^{(1)}_{2,1} = -\frac{1}{2v^2} \Delta S^{14} - \frac{7}{8} \Delta S^{11} T^3 \), then the cocycle 
\[
\tilde{\omega}_{2,1} := \begin{pmatrix} (\omega^{(0)}_{2,1}, \omega^{(1)}_{2,1}) & 0 \\ 0 & l_{2,1} \end{pmatrix}
\]
represents the class \([\tilde{\omega}_{2,1}]\) we are looking for.

6. A \( \mathbb{Q} \)-de Rham Structure on Relative Completion of \( \text{SL}_2(\mathbb{Z}) \)

Now we follow Hain [13, §7], and construct a \( \mathbb{Q} \)-de Rham structure on the relative completion \( \mathcal{G}_{rel} \) of \( \text{SL}_2(\mathbb{Z}) \).

We review the Betti version first. We view \( \text{SL}_2(\mathbb{Z}) \) as the (orbifold) fundamental group of \( \mathcal{M}_{1,1} \). For any base point \( x \in \mathcal{M}_{1,1} \), we have a Zariski dense monodromy representation \( \rho_x : \pi_1(\mathcal{M}_{1,1}, x) \to \text{SL}_2(\mathbb{Q}) \).

Denote the relative completion of \( \pi_1(\mathcal{M}_{1,1}, x) \) with respect to \( \rho_x \) by \( \mathcal{G}_x \). Denote the Lie algebra of its unipotent radical \( \mathcal{U}_x \) by \( \mathfrak{u}_x \). One can construct Betti \( \mathbb{Q} \)-structures on \( \mathcal{O}(\mathcal{G}_x) \) and on \( \mathfrak{u}_x \) that are compatible with their algebraic structures [13]. It is achieved by finding a (Betti) canonical flat connection 
\[
\nabla = \nabla_0 + \Omega
\]
on the bundle \( \mathfrak{u}_B \to \mathcal{M}_{1,1} \) whose fiber over \( x \) is the Lie algebra \( \mathfrak{u}_x \).

For the de Rham version, we will find in Section 6.4 a \( \check{\text{Č}} \)ech-de Rham 1-cochain \( \tilde{\Omega} \), interpreted as a connection form in Section 6.5. This provides us with connection forms \( \Omega^{(j)} \) on the opens \( U_j \), \( j = 0, 1 \) and a gauge transformation between the connections on the intersection \( U_{01} \). Note that all of these are defined over \( \mathbb{Q} \), and the connection forms have logarithmic poles at the cusp. From these data, we can first construct bundles on the two opens \( U_j \) with the corresponding connections
\[
\nabla = \nabla_0 + \Omega^{(j)}, \quad j = 0, 1,
\]
then glue them together on the intersection \( U_{01} \) via the gauge transformation. Therefore, we have constructed a bundle \( \mathfrak{u}_{\text{DR}} \to \mathcal{M}_{1,1} \) with a (de Rham) connection that is defined over \( \mathbb{Q} \) and has regular singularity at the cusp.

We start by describing the bundle \( \mathfrak{u} \) where this connection lives.

6.1. The bundle \( \mathfrak{u} \) of Lie algebras. Recall that the Lie algebra \( \mathfrak{u}^{rel} \) of \( \mathcal{U}^{rel} \) is freely topologically generated by 
\[
\prod_{n \geq 0} H^1(\text{SL}_2(\mathbb{Z}), S^{2n} H)^* \otimes S^{2n} H,
\]
where \( H \) is the standard representation of \( \text{SL}_2 \), and \( S^{2n} H \) its \( 2n \)-th symmetric power. Note that each fiber \( \mathfrak{u}_x \) of \( \mathfrak{u} \) is (abstractly) isomorphic to \( \mathfrak{u}^{rel} \). Define 
\[
\mathfrak{u}_1 := \prod_{n \geq 0} H^1(\mathcal{M}^{2n}_{1,1}, S^{2n} \mathbb{H})^* \otimes S^{2n} \mathbb{H}.
\]
This is a pro-vector bundle over \( \mathcal{M}_{1,1} \), whose fiber over \( x \) is the abelianization of the free Lie algebra \( \mathfrak{u}_x \). Let 
\[
\mathfrak{u}_n := \mathbb{L}_n(\mathfrak{u}_1)
\]
be the degree $n$ part of the free Lie algebra generated by $u_1$. For any $u \in L(u_1)$, we will often denote its degree $n$ part by $(u)_n$. Set
\[
  u := \lim_{n \to \infty} \bigoplus_{j=1}^{n} u_j,
\]
and
\[
  u^N := \lim_{n \geq N \to \infty} \bigoplus_{j=1}^{n} u_j,
\]
the parts of degree at least $N$ in the Lie algebra $u$.

Note that, by what we have done in Section 5, $u_1$ has its de Rham realization:
\[
  u_{1,\text{dR}} := \prod_{n \geq 0} H^1_{\text{dR}}(M_{1,1}, S^{2n}H)^* \otimes S^{2n}H,
\]
with connection $\nabla_0$. This gives rise to a $\mathbb{Q}$-structure $u_{\text{dR}}$ on the bundle $u$ with the same connection $\nabla_0$, but $(u_{\text{dR}}, \nabla_0) \otimes \mathbb{C}$ is not isomorphic to $(u_0, \nabla) \otimes \mathbb{C}$, where $\nabla = \nabla_0 + \Omega$ is the (Betti) canonical flat connection. In Section 6.4, we will see how to perturb the flat connection $\nabla_0$ on $u_{1,\text{dR}}$ to find a (de Rham) canonical flat connection $\tilde{\nabla} = \nabla_0 + \Omega$ on $u_{\text{dR}}$. From now on, we will work exclusively with de Rham realizations such as $u_{\text{dR}}$ and $u_{1,\text{dR}}$, but we will drop the subscripts $\text{dR}$ on them for notation simplicity.

6.2. The differential graded Lie algebra $K^\bullet(M_{1,1}; u)$. Denote by $K^\bullet(M_{1,1})$ the single complex associated to the Čech-de Rham double complex $\mathcal{C}^\bullet(u, \mathcal{F}_0^\bullet)$ where
\[
  \mathcal{F}_0^\bullet = (\Omega^\bullet \text{(log } D))^\mathbb{Z},
\]
and $D = \nabla_0 + \delta$ is the total differential in $\mathcal{C}^\bullet(u, \mathcal{F}_0^\bullet)$. Define a product on this complex based on the convention: if $\omega \in \mathcal{C}^p(u, \mathcal{F}_0^\bullet)$ and $\eta \in \mathcal{C}^r(u, \mathcal{F}_0^\bullet)$, then their product $\omega \wedge \eta \in \mathcal{C}^{p+r}(u, \mathcal{F}_0^{p+s})$ is defined by
\[
  (\omega \wedge \eta)(U_{a_0\cdots a_{p+r}}) = (-1)^p \omega(U_{a_0\cdots a_p}) \wedge \eta(U_{a_{p+1}\cdots a_{p+r}}),
\]
where on the right hand side forms are understood to be restricted to $U_{a_0\cdots a_{p+r}}$. Therefore, if $\tilde{\Omega}$ and $\tilde{\Omega}'$ are 1-cochains in $K^1(M_{1,1})$ with
\[
  \tilde{\Omega} = \begin{pmatrix}
    (\omega^{(0)}, \omega^{(1)}) & 0 \\
    0 & l
  \end{pmatrix}
\]
and
\[
  \tilde{\Omega}' = \begin{pmatrix}
    (\omega''^{(0)}, \omega''^{(1)}) & 0 \\
    0 & l'
  \end{pmatrix}
\]
their product is
\[
  \tilde{\Omega} \wedge \tilde{\Omega}' := \begin{pmatrix}
    (\omega^{(0)} \wedge \omega''^{(0)}, \omega^{(1)} \wedge \omega''^{(1)}) & 0 \\
    0 & l \cdot \omega''^{(1)} \wedge \omega^{(0)} + l' & 0
  \end{pmatrix}
\]
Note that this complex $K^\bullet(M_{1,1})$ does not compute $H^1_{\text{dR}}(M_{1,1/\mathbb{Q}})$ but its first cohomology has a nontrivial class $[\Delta]$. However, it computes $H^2_{\text{dR}}(M_{1,1/\mathbb{Q}})$, which is all we will need in Section 6.4.

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9The Čech-de Rham complex was defined in Section 4.3 with $n = 0$.
10cf. Brown–Hain §3 computations before Remark 3.1.
Let
\[ K^\bullet(M_{1,1}; u) := K^\bullet(M_{1,1}) \otimes u. \]

This is a differential graded Lie algebra with differential induced from the differential \( D \) of the complex \( K^\bullet(M_{1,1}) \). Define a product \([\cdot, \cdot]\) on \( K^\bullet(M_{1,1}; u) \) by:
\[ [\tilde{\Omega}_1 \otimes u, \tilde{\Omega}_2 \otimes v] := (\tilde{\Omega}_1 \wedge \tilde{\Omega}_2) \otimes [u, v], \]
where \( \tilde{\Omega}_1, \tilde{\Omega}_2 \in K^\bullet(M_{1,1}) \), \( u, v \in u \) with \([u, v]\) being the Lie product of \( u \) and \( v \).

6.3. The canonical 1-cocycle \( \tilde{\Omega}_1 \). We start with defining a canonical 1-cocycle
\[ \tilde{\Omega}_1 = (\tilde{\Omega}_1^{(0)}, \tilde{\Omega}_1^{(1)}; f_1) \in K^1(M_{1,1}; u_1) \]
or
\[ \tilde{\Omega}_1 = \begin{pmatrix} (\tilde{\Omega}_1^{(0)}, \tilde{\Omega}_1^{(1)}) & 0 \\ 0 & f_1 \end{pmatrix} \]
that represents the identity maps \( H^1_{dR}(M_{1,1}, S^{2n}\mathcal{H}) \to H^1_{dR}(M_{1,1}, S^{2n}\mathcal{H}) \) for every \( n > 0 \). It can be written as
\[ \tilde{\Omega}_1 := \prod_{n \geq 1} \tilde{\Omega}_{1,2n} \in K^1(M_{1,1}; u_1), \]
where we define \( \tilde{\Omega}_{1,2n} \) by using cocycles found in Section 5.

\[ \tilde{\Omega}_{1,2n} := \sum_f \tilde{\omega}_f X_f + \sum_{\{j,k:4j+6k=2n\}} \tilde{\omega}_{j,k} X_{j,k} \in K^1(M_{1,1}; S^{2n}\mathcal{H}) \otimes H^1_{dR}(M_{1,1}, S^{2n}\mathcal{H})^*, \]
with the first term on the right hand side is summing over Hecke eigenforms \( f \) of weight \( 2n + 2 \), and all \( X_f, X_{j,k} \in H^1_{dR}(M_{1,1}, S^{2n}\mathcal{H})^* \) form a dual basis for all \( Q \)-de Rham classes \( [\tilde{\omega}_f] \) and \( [\tilde{\omega}_{j,k}] \) in \( H^1_{dR}(M_{1,1}, S^{2n}\mathcal{H}) \).

For our purpose, we would prefer to write \( \tilde{\Omega}_{1,2n} \) in a different way: let
\[ \tilde{\omega}_f = \tilde{\Omega}_f \mathbb{T}^{2n} \quad \text{and} \quad \tilde{\omega}_{j,k} = \sum_m \tilde{\Omega}_{j,k}^m \mathbb{T}^{2n-m}, \]
with coefficients \( \tilde{\Omega}_f, \tilde{\Omega}_{j,k}^m \in K^1(M_{1,1}), 0 \leq m \leq 2n \). Set
\[ e_f := X_f \otimes \mathbb{T}^{2n} \quad \text{and} \quad e_{j,k}^m := X_{j,k} \otimes \mathbb{T}^{2n-m}, \]
then \( e_f, e_{j,k}^m \in u_1 \), and
\[ \tilde{\Omega}_{1,2n} = \sum_f \tilde{\Omega}_f e_f + \sum_{\{j,k:4j+6k=2n\}} \left( \sum_m \tilde{\Omega}_{j,k}^m e_{j,k}^m \right). \]

6.4. The connection 1-cochain \( \tilde{\Omega} \). One can follow the procedure in Hain [13] §7.3] to inductively define a connection 1-cochain \( \tilde{\Omega} \) from the canonical 1-cocycle \( \tilde{\Omega}_1 \). This procedure is modified from Chen’s method of power series connections [5]. Given a differential graded Lie algebra
\[ K^\bullet(X; u) = K^\bullet(X) \otimes u, \]
where $K^\bullet(X)$ is a complex whose second cohomology vanishes, i.e. $H^2(K^\bullet(X)) = 0$; and $u$ is a free Lie algebra generated by $u_1$, whose degree $n$ part is denoted by $u_n := \mathbb{L}_n(u_1)$, and parts of degree at least $N$ by $u^N := \lim_{n \to \infty} \bigoplus_{j=N}^{n} u_j$. There is a bracket $[,]$ on $K^\bullet(X;u)$ given by:

$$[\Omega_\beta \otimes e_\beta, \Omega_\gamma \otimes e_\gamma] := (\Omega_\beta \wedge \Omega_\gamma) \otimes [e_\beta, e_\gamma],$$

which is induced from the wedge product $\wedge$ on $K^\bullet(X)$ and the Lie bracket $[,]$ on $u$. The following result is used to inductively construct the (de Rham) connection form $\tilde{\Omega}$.

**Proposition 6.1.** Suppose we have a closed form

$$\Omega_1 \in K^1(X;u_1),$$

such that $D\Omega_1 = 0$. For any $n \geq 2$, we can find $\Xi_n \in K^1(X;u_n)$, and set $\Omega_n := \Omega_{n-1} + \Xi_n$, so that

$$D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n] \equiv 0 \mod u^{n+1}.$$  

**Proof.** Note that $D\Omega_1 = 0$, $[\Omega_1, \Omega_1] \in K^2(X;u_2)$ is closed. Since $H^2(K^\bullet(X)) = 0$, $[\Omega_1, \Omega_1]$ is thus exact. One can find $\Xi_2 \in K^1(X;u_2)$ such that $-D\Xi_2 = \frac{1}{2}[\Omega_1, \Omega_1]$ and $D\Omega_2 + \frac{1}{2}[\Omega_2, \Omega_2] \equiv 0 \mod u^4$.

Suppose for $n \geq 2$ we have already found $\Xi_2, \cdots, \Xi_n$, and $\Omega_n = \Omega_1 + \sum_{i=2}^n \Xi_i$, such that $D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n] \equiv 0 \mod u^{n+1}$. We claim that the degree $(n + 1)$ part

$$(D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n])[n+1] \in K^2(X;u_{n+1})$$

is closed. In fact,

$$D(D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n]) = \frac{1}{2}[D\Omega_n, \Omega_n] - \frac{1}{2}[\Omega_n, D\Omega_n]$$

$$= \frac{1}{2}(-\frac{1}{2}[\Omega_n, \Omega_n], \Omega_n) - \frac{1}{2}[\Omega_n, -\frac{1}{2}[\Omega_n, \Omega_n]]$$

$$= -\frac{1}{4}([\Omega_n, \Omega_n], \Omega_n] + \frac{1}{4}[\Omega_n, [\Omega_n, \Omega_n]] = 0 \mod u^{n+2}$$

where we have used Leibniz rule of $D$ on the first line, induction hypothesis on the second line, and both terms on the last line are 0 by Jacobi identity. Since $H^2(K^\bullet(X)) = 0$, $(D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n])[n+1] \in K^2(X;u_{n+1})$ is closed thus exact. We can find $\Xi_{n+1} \in K^1(X;u_{n+1})$ such that

$$-D\Xi_{n+1} = (D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n])[n+1] = \frac{1}{2}[\Omega_n, \Omega_n][n+1].$$

Define $\Omega_{n+1} := \Omega_n + \Xi_{n+1}$, then it is easy to check that

$$D\Omega_{n+1} + \frac{1}{2}[\Omega_{n+1}, \Omega_{n+1}] \equiv 0 \mod u^{n+2}$$

since $D\Omega_{n+1} + \frac{1}{2}[\Omega_{n+1}, \Omega_{n+1}] \equiv D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n] \equiv 0 \mod u^{n+1}$ and the degree $(n + 1)$ part on the left is $D\Xi_{n+1} + (\frac{1}{2}[\Omega_n, \Omega_n])[n+1] = 0$. \hfill \Box

**Example 6.2.** Since $K^\bullet(M_{1,1})$ computes the second cohomology $H^2_{\text{DR}}(M_{1,1}/\mathbb{Q})$ and it vanishes, the above proposition applies to our case $K^\bullet(M_{1,1};u)$. Since the canonical 1-cocycle

$$\tilde{\Omega}_1 \in K^1(M_{1,1};u_1)$$
is closed, one can recurrently define for \( n \geq 2 \),
\[
\tilde{\Omega}_n := \tilde{\Omega}_{n-1} + \tilde{\Xi}_n,
\]
with \( \tilde{\Xi}_n \in K^1(M_{1,1}; u_n) \) satisfying
\[
(D\tilde{\Xi}_n) = (D\tilde{\Omega}_{n-1} + \frac{1}{2}[\tilde{\Omega}_{n-1}, \tilde{\Omega}_{n-1}])_n = (\frac{1}{2}[\tilde{\Omega}_{n-1}, \tilde{\Omega}_{n-1}])_n,
\]
so that
\[
D\tilde{\Omega}_n + \frac{1}{2}[\tilde{\Omega}_n, \tilde{\Omega}_n] \in K^2(M_{1,1}; u^{n+1}).
\]
Define the connection 1-cochain
\[
\tilde{\Omega} := \lim_{\leftarrow} \tilde{\Omega}_n \in K^1(M_{1,1}; u).
\]
Then it is defined over \( \mathbb{Q} \) and satisfies that
\[
D\tilde{\Omega} + \frac{1}{2}[\tilde{\Omega}, \tilde{\Omega}] = 0.
\]
To fix notations in the following sections, let
\[
\tilde{\Omega}_n := \begin{pmatrix} (\Omega^{(0)}_n, \Omega^{(1)}_n) & 0 \\ 0 & F_n \end{pmatrix}
\]
and write it as
\[
\tilde{\Omega}_n = \sum_{i=1}^n \tilde{\Xi}_i,
\]
with \( \tilde{\Xi}_i(i \geq 2) \) defined as before, \( \tilde{\Xi}_1 := \tilde{\Omega}_1 \), and
\[
\tilde{\Xi}_i := \begin{pmatrix} (\Xi^{(0)}_i, \Xi^{(1)}_i) & 0 \\ 0 & f_i \end{pmatrix}
\]
so that \( \Omega^{(0)}_n = \sum_{i=1}^n \Xi^{(0)}_i, \Omega^{(1)}_n = \sum_{i=1}^n \Xi^{(1)}_i \) and \( F_n = \sum_{i=1}^n f_i \).

One can then formally write
\[
\tilde{\Omega} = \begin{pmatrix} (\Omega^{(0)}, \Omega^{(1)}) & 0 \\ 0 & F \end{pmatrix}
\]
as
\[
\tilde{\Omega} = \sum_{i=1}^\infty \tilde{\Xi}_i,
\]
where \( \Omega^{(0)} = \sum_{i=1}^\infty \Xi^{(0)}_i = \lim_{n} \Omega^{(0)}_n, \Omega^{(1)} = \sum_{i=1}^\infty \Xi^{(1)}_i = \lim_{n} \Omega^{(1)}_n \) and \( F = \sum_{i=1}^\infty f_i = \lim_{n} F_n \).
6.5. **Interpretation for \( \tilde{\Omega} \) as a connection form.** In this section, we interpret the connection 1-cochain

\[
\tilde{\Omega} = \begin{pmatrix}
(\Omega^{(0)}, \Omega^{(1)}) & 0 \\
0 & F
\end{pmatrix}
\]

as a connection form on the bundle \( u \). We will show that the condition

\[
D\tilde{\Omega} + \frac{1}{2}[\tilde{\Omega}, \tilde{\Omega}] = 0
\]

amounts to the facts that the 1-forms \( \Omega^{(i)} \) define flat connections

\[
\nabla = \nabla_0 + \Omega^{(i)}
\]

on the bundle \( u \) over the open subsets \( U_i \) of \( M_{1,1}, i = 0, 1 \), and that on the intersection \( U_{01} \) these two connections are algebraically gauge equivalent with gauge transformation

\[
g : U_{01} \rightarrow U_{\text{rel}} \hookrightarrow \text{Aut}(u_{\text{rel}})
\]

given by \( g := 1 + F \).

We set up the discussion as in Hain [13, §4.1]. Suppose \( \nabla_0 \) is a connection on an \( R \)-equivariant principle \( U \)-bundle \( X \times U \rightarrow X \). A 1-form \( \omega \) defines a connection \( \nabla \) on this bundle by

\[
\nabla s = \nabla_0 s + \omega s,
\]

where \( s \) is a section. This connection is flat if and only if

\[
\nabla_0 \omega + \omega \wedge \omega = 0.
\]

A gauge transformation \( g : X \rightarrow U \) changes the connection form \( \omega \) to a new one by

\[
\omega' = -\nabla_0 g \cdot g^{-1} + g \omega g^{-1}.
\]

In our case, \( X = M_{1,1}, R = \text{SL}_2(\mathbb{Z}) \), and the principle \( U \)-bundle is the bundle \( U \) over \( M_{1,1} \) whose fiber over \( x \) is the unipotent radical \( U_x \) of the relative completion \( G_x \). Since a prounipotent group is isomorphic to its Lie algebra as a group, one can think of this bundle \( U \) the same as its corresponding Lie algebra bundle \( u \). The connection on \( u \) is essentially given by

\[
\nabla = \nabla_0 + \tilde{\Omega},
\]

and provides us with a \( \mathbb{Q} \)-de Rham structure \( u_{\text{dR}} \) on \( u \) (or \( U_{\text{dR}} \) on \( U \)), which we now explain.

Since \( D\tilde{\Omega} + \frac{1}{2}[\tilde{\Omega}, \tilde{\Omega}] \in K^2(M_{1,1}; u) \) a priori has two parts

\[
\begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix}
\]

The top part being 0 means \( \nabla_0 \Omega^{(0)} + \Omega^{(0)} \wedge \Omega^{(0)} = 0 \) and \( \nabla_0 \Omega^{(1)} + \Omega^{(1)} \wedge \Omega^{(1)} = 0 \), which tells us that \( \Omega^{(i)} \) is flat on \( U_i \) for \( i = 0, 1 \). The lower part being 0 means

\[
(6.2) \quad \Omega^{(1)} - \Omega^{(0)} = \nabla_0 F + F \cdot \Omega^{(1)} - \Omega^{(0)} \cdot F = 0,
\]

where “\( \cdot \)” is a product on \( \mathcal{F}_0^\bullet(U_{01}) \otimes u \) induced by wedge product on \( \mathcal{F}_0^\bullet(U_{01}) \) and Lie bracket on \( u \). This equation \( (6.2) \) tells us that the 1-forms \( \Omega^{(0)} \) and \( \Omega^{(1)} \) are gauge equivalent on the intersection \( U_{01} \) of \( U_0 \) and \( U_1 \).
Proposition 6.3. The function
\[ g := 1 + F \in \mathcal{U}^{\mathbb{C}}(\mathcal{O}(U_{01})) \]
transforms \( \Omega^{(1)} \) to \( \Omega^{(0)} \). That is, on \( U_{01} \) we have
\[ \Omega^{(0)} = -\nabla_0 g \cdot g^{-1} + g \Omega^{(1)} g^{-1}. \]

Proof. Let \( g_n := 1 + F_n \), then it suffices to prove for every \( n \), we have
\[ \Omega^{(0)}_n = -\nabla_0 g_n \cdot g_n^{-1} + g_n \Omega^{(1)}_n g_n^{-1} \pmod{u^{n+1}}. \]

We prove this by induction. When \( n = 1 \), as \( \nabla_0(1) = d(1) = 0 \) and \( g_1^{-1} \equiv 1 \pmod{u^{1}} \), \(6.3\) becomes
\[ \Omega^{(0)}_1 = -\nabla_0 f_1 + \Omega^{(1)}_1, \]
which amounts to the fact that
\[ \bar{\Omega}_1 = \left[ \begin{array}{cc} (\Omega^{(0)}_1, \Omega^{(1)}_1) & 0 \\ 0 & f_1 \end{array} \right]. \]
is \( D \)-closed.

Assume that \(6.3\) holds for \((n - 1)\), we have
\[ \Omega^{(0)}_{n-1} = -\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1} \pmod{u^{n}}. \]
As for \( n \), we only need to prove that the degree \( n \) parts on both sides of \(6.3\) are the same. On the left hand side, it is \( \Xi^{(0)}_n \). One can easily show that \( g_n^{-1} \equiv g_{n-1}^{-1} \pmod{u^{n}} \), so we write the right hand side as
\[ -\nabla_0 g_n \cdot g_n^{-1} + g_n \Omega^{(1)}_n g_n^{-1} \]
\[ = -\nabla_0(g_{n-1} + f_n) \cdot (g_{n-1}^{-1} + u_n) + (g_{n-1} + f_n) \Omega^{(1)}_{n-1} + \Xi^{(1)}_n \]
with some \( u_n \in u^{n} \). Modulo terms in \( u^{n+1} \), the degree \( n \) part on the right side comes from
\[ -\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} - \nabla_0 f_n + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1} + \Xi^{(1)}_n \]
Or equivalently, the degree \( n \) part on the right hand side is
\[ \Xi^{(1)}_n - \nabla_0 f_n + (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1})n. \]

It remains to prove that the above is the same as \( \Xi^{(0)}_n \), which is equivalent to showing that
\[ \Xi^{(1)}_n - \Xi^{(0)}_n = \nabla_0 f_n = -(-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1})n. \]
Note that \( \Omega^{(0)}_n \) has terms only of degree less than \( n \), so one can add it to the right hand side without affecting the equality:
\[ (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1})n = (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1} - \Omega^{(0)}_{n-1})n. \]
By the induction hypothesis \(6.4\), the form in the parenthesis on the right has terms of degree \( n \) or higher. When multiplied by \( g_{n-1} \) on the right, its degree \( n \) part remains unchanged:
\[ (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \Omega^{(1)}_{n-1} g_{n-1}^{-1} - \Omega^{(0)}_{n-1})n = (-\nabla_0 g_{n-1} + g_{n-1} \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot g_{n-1}n) \]
Since \( g_{n-1} \), and thus \( \nabla_0 g_{n-1} \), has terms only of degree less than \( n \), we can remove them:
\[ (-\nabla_0 g_{n-1} + g_{n-1} \cdot \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot g_{n-1})n = (g_{n-1} \cdot \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot g_{n-1})n. \]
Plugging in $g_{n-1} = 1 + F_{n-1}$, and then removing the terms $\Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)}$ of degree less than $n$, we get:

$$(g_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} - g_{n-1})_n = (F_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} \cdot F_{n-1})_n.$$ 

The equation (6.5) now reduces to the equation

$$\Xi^{(1)}_n - \Xi^{(0)}_n - \nabla f_n + (F_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} \cdot F_{n-1})_n = 0.$$ 

This equation holds as it is the equation (6.1) in the recursive definition of $\tilde{\Xi}_n$ (cf. it also follows from the degree $n$ part of the equation (6.2)).

Recall that there is a Betti vector bundle $u_B \to \mathcal{M}_{1,1}$ with its (Betti) canonical flat connection $\nabla = \nabla_0 + \Omega$. Denote by $(u_B, \nabla)$ Deligne’s canonical extension of $(u_B, \nabla)$ to $\overline{\mathcal{M}}_{1,1}$.

**Theorem 6.4.** There is an algebraic de Rham vector bundle $u_{dR}$ over $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ endowed with connection

$$\nabla = \nabla_0 + \tilde{\Omega},$$

and an isomorphism

$$(u_{dR}, \nabla) \otimes \mathbb{Q} \subset (u_B, \nabla) \otimes \mathbb{Q}.$$ 

The algebraic de Rham vector bundle $u_{dR}$ and its connection $\tilde{\nabla}$ are both defined over $\mathbb{Q}$. Moreover, the connection $\tilde{\nabla}$ has a regular singularity at the cusp.

**Proof.** Recall that

$$\tilde{\Omega} = \begin{pmatrix} (\Omega^{(0)}, \Omega^{(1)}) & 0 \\ 0 & F \end{pmatrix}.$$ 

We construct trivial bundles

$$\begin{array}{ccc}
\mathcal{U}^{\text{rel}} \times U_0 & \text{and} & \mathcal{U}^{\text{rel}} \times U_1 \\
\downarrow & & \downarrow \\
U_0 & \text{and} & U_1
\end{array}$$

with connections $\nabla = \nabla_0 + \Omega^{(0)}$ and $\nabla = \nabla_0 + \Omega^{(1)}$, respectively. Define

$$g : U_{01} \to \mathcal{U}^{\text{rel}} \cong \text{Aut}(\mathcal{U}^{\text{rel}})$$

by $g = 1 + F$. After gluing these two bundles together on the intersection $U_{01}$ via the gauge transformation $g$, we obtain an algebraic vector bundle $u_{dR}$ over $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$. It is endowed with a connection $\tilde{\nabla} = \nabla_0 + \tilde{\Omega}$. This connection is defined over $\mathbb{Q}$, and has regular singularity at the cusp.

**Corollary 6.5.** The de Rham vector bundle $(u_{dR}, \tilde{\nabla})$ can be used to construct a $\mathbb{Q}$-de Rham structure on $\mathcal{O}(G^{\text{rel}})$ for each base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$.

**Proof.** We can define transport formula by using connection forms $\Omega^{(0)}$ and $\Omega^{(1)}$ on opens $U_0$ and $U_1$ respectively, then patching things together on their intersection via the gauge transformation $g$. One can then follow [13, §7.6] to find a $\mathbb{Q}$-de Rham structure on $\mathcal{O}(G^{\text{rel}})$ for each base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$.

**Remark 6.6.** In particular, one can choose the base point $x$ to be the unit tangent vector $\partial/\partial q$ at the cusp. This newly constructed $\mathbb{Q}$-de Rham structure will allow us to compute multiple modular values [2].
Part 3. Applications

7. Periods of Modular Forms

As mentioned in the introduction, we provide in this part newly constructed closed iterated integrals of modular forms. In particular, the modular forms that we consider include both holomorphic modular forms and modular forms of the second kind.

7.1. Twice iterated integrals of algebraic forms. Suppose we are given two algebraic 1-forms \( \omega \) and \( \eta \) on a Riemann surface \( M \). To construct a closed iterated integral of these two algebraic forms, one follows the recipe in Hain [12, §3]. Define

\[
\omega \cup \eta = \sum a_j p_j \quad a_j \in \mathbb{C}, \ p_j \in M
\]

if, locally, \( \omega = dF \) and

\[
(2\pi i)^{-1} \text{Res}_{p_j} F \eta = a_j.
\]

One can find another 1-form \( \xi \) such that

\[
(2\pi i)^{-1} \text{Res}_{p_j} \xi = -a_j.
\]

Then the twice iterated integral

\[
\int \omega \eta + \xi
\]

is a closed iterated integral on \( M \).

7.2. Twice iterated integrals of modular forms. Set \( \langle S, T \rangle = -\langle T, S \rangle = -1 \). This defines a skew-symmetric inner product \( \langle \cdot, \cdot \rangle \) on \( S^{2n} \mathcal{H} \), which induces a cup product on \( H_{dR}^1(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}) \) (cf. [3]). Now we are ready to construct our first example.

Example 7.1. First newly constructed iterated integral of modular forms. Recall from Example 5.1 that we have a global 1-form

\[
\omega_\Delta = \alpha T^{10} = (2udv - 3vdu) T^{10}
\]

that represents a class corresponding to the weight 12 cusp form \( \Delta = u^3 - 27v^2 \) in \( H_{dR}^1(\mathcal{M}_{1,1}, S^{10} \mathcal{H}) \). In Example 5.3 we found in \( H_{dR}^1(\mathcal{M}_{1,1}, S^{10} \mathcal{H}) \) another class represented by a Čech 1-cocycle

\[
\begin{pmatrix}
\omega_{1,1}^{(0)} & \omega_{1,1}^{(1)} \\
0 & l_{1,1}
\end{pmatrix}
\]

where \( \omega_{1,1}^{(0)} = \frac{9\alpha}{u^2} S^{10} \in \mathcal{F}_{10}^0(U_0) \), \( \omega_{1,1}^{(1)} = -\frac{9\alpha}{u^2} S^{10} - \frac{5\alpha}{u^2} S^{9} T \in \mathcal{F}_{10}^0(U_1) \), and \( l_{1,1} = \frac{1}{u} S^{10} \in \mathcal{F}_{10}^0(U_{01}) \). Using the interpretation in Section 6.5 we regard \( \omega_{1,1}^{(0)} \) and \( \omega_{1,1}^{(1)} \) on two opens \( U_0 \) and \( U_1 \) of \( \mathcal{M}_{1,1} \). We shall construct twice iterated integrals of \( \omega_\Delta \) and \( \omega_{1,1} \) on these two opens separately.

We first carry out the recipe in Section 7.1 locally on \( U_0 = \mathbb{G}_m \setminus Y_0 \). Contracting \( S^{10} \mathcal{H} \) by inner product and extracting 1-forms from \( \omega_\Delta \) and \( \omega_{1,1}^{(0)} \), we set

\[
\omega := 2udv - 3vdu, \quad \eta^{(0)} := \frac{9\alpha}{u^2 \Delta},
\]
which are algebraic forms on \( Y_0 = \mathbb{A}^2 - \{ u = 0 \} \). Note that \( \omega \cup \eta^{(0)} \) can only have residue at the \( \mathbb{G}_m \)-orbit \( u = 0 \), which corresponds to \([\rho] \in \mathcal{M}^a_{1,1}\). Now we work locally around \([\rho]\), i.e. over \( \mathbb{Q}[u][u^{-1}] \). Take a slice \( v = C \) close to \( u = 0 \), then we can write \( \omega \) locally as

\[
\omega = -3vdu = dF \quad \text{with} \quad F = -3vu,
\]

regarding \( v \) as a constant. Setting \( u = 0 \) in \( \Delta = u^3 - 27v^2 \), we have

\[
F\eta^{(0)} = (-3vu)\frac{9\alpha}{u^2\Delta} = \frac{27vu\alpha}{u^2\Delta} = -\frac{27vu(-3vdu)}{u^2(-27u^2)} = -3\frac{du}{u},
\]

and

\[
(2\pi i)^{-1} \text{Res}[\rho] F\eta^{(0)} = -3.
\]

Therefore,

\[
\int \omega\eta^{(0)} + 3\frac{du}{u}
\]

is a closed iterated integral on \( U_0 \).

Similarly on \( U_1 = \mathbb{G}_m \times Y_1 \), contracting \( S^{10}\mathcal{H} \) by inner product ignores the second term \(-\frac{5\alpha}{2v^2}\Delta S^9 T \) in \( \omega^{(1)}_1 \) since \((T^{10}, S^9 T) = 0 \). Extracting 1-forms from \( \omega_\Delta \) and the first term \(-\frac{u\alpha}{2v^2}\Delta S^{10} \) in \( \omega_\Delta^{(1)} \), we set

\[
\omega := 2udy - 3vdu, \quad \eta^{(1)} := -\frac{u\alpha}{2v^2\Delta},
\]

which are algebraic forms on \( Y_1 = \mathbb{A}^2 - \{ v = 0 \} \). Note that \( \omega \cup \eta^{(1)} \) can only have residue at the \( \mathbb{G}_m \)-orbit \( v = 0 \), which corresponds to \([i] \in \mathcal{M}^a_{1,1}\). Now we work locally around \([i]\), i.e. over \( \mathbb{Q}[v][v^{-1}] \). Take a slice \( u = C \) close to \( v = 0 \), then we can write \( \omega \) locally as

\[
\omega = 2udv = dF' \quad \text{with} \quad F' = 2uv,
\]

regarding \( u \) as a constant. Setting \( v = 0 \) in \( \Delta = u^3 - 27v^2 \), we have

\[
F'\eta^{(1)} = 2uv \cdot \left(-\frac{u\alpha}{2v^2\Delta}\right) = \frac{2uv\cdot u\alpha}{2v^2\Delta} = -\frac{2u^2v\cdot 2udv}{2v^2 \cdot u^3} = -2\frac{dv}{v},
\]

and

\[
(2\pi i)^{-1} \text{Res}[i] F'\eta^{(1)} = -2.
\]

Therefore,

\[
\int \omega\eta^{(1)} + 2\frac{dv}{v}
\]

is a closed iterated integral on \( U_1 \).

Remark 7.2. Note that \( \frac{dv}{v} = \frac{1}{3} \frac{d\alpha}{\Delta} + \frac{27u\alpha}{u^2\Delta} \) and \( dv = \frac{1}{2} \frac{d\alpha}{\Delta} + \frac{u^2\alpha}{2v^2\Delta} \). So if one were to transfer either the residue \(-3\) at \([\rho]\) from \( \omega \cup \eta^{(0)} \) or the residue \(-2\) at \([i]\) from \( \omega \cup \eta^{(1)} \) to the cusp \( P \), one would get the same residue \(-1\). This is expected since we are essentially computing a cup product of \( [\omega_\Delta] \) and \( [\omega_{1,1}] \), with both \( \eta^{(0)} \) and \( \eta^{(1)} \) representing the same class \( [\omega_{1,1}] \) on opens \( U_0 \) and \( U_1 \) of \( \mathcal{M}_{1,1} \).

It turns out that we have a nice description using our Čech complex.

Example 7.3. Čech description of the previous example. Recall from Example 5.2 that the class \([\omega_\Delta] \in H^1_{dR}(\mathcal{M}_{1,1}, S^{10}\mathcal{H}) \) can also be represented by a Čech 1-cocycle

\[
\tilde{\omega}_\Delta = (\omega^{(0)}_\Delta, \omega^{(1)}_\Delta; 0)
\]

where \( \omega^{(i)}_\Delta \) is the restriction of \( \omega_\Delta \) on \( U_i \).
Extracting 1-cochains of $K^\bullet(M_{1,1})$ the same way as in Section 6.3, we write

$$\tilde{\omega}_\Delta = \tilde{\Omega}_\Delta \otimes T^{10} \quad \text{and} \quad \tilde{\omega}_{1,1} = \tilde{\Omega}^{10}_{1,1} \otimes S^{10} + \tilde{\Omega}^9_{1,1} \otimes S^9T,$$

where

$$\tilde{\Omega}^{10}_{1,1} = \left( \begin{array}{ccc} (\alpha, \alpha) & 0 & 0 \\ 0 & 0 & \frac{1}{uv} \\ 0 & \frac{1}{uv} & 0 \end{array} \right), \quad \tilde{\Omega}^9_{1,1} = \left( \begin{array}{ccc} 0 & -\frac{5\alpha}{uv} & 0 \\ \frac{5\alpha}{uv} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

After taking the inner product, we can ignore the second term $\tilde{\Omega}^9_{1,1} \otimes S^9T$ in $\tilde{\Omega}_{1,1}$ since $\langle T^{10}, S^9T \rangle = 0$. It suffices to follow the procedure in Prop. 6.1 up to $n = 2$ when constructing twice iterated integrals. By the product formula defined in Section 6.2, we have

$$\tilde{\Omega}^\Delta \wedge \tilde{\Omega}^{10}_{1,1} = \left( \begin{array}{ccc} (0, 0) & 0 & 0 \\ 0 & 0 & -\frac{\alpha}{uv} \\ 0 & \frac{\alpha}{uv} & 0 \end{array} \right),$$

where $-\frac{\alpha}{uv} = -\frac{2udv - 3edv}{uv} = 3\frac{du}{u} - 2\frac{dv}{v}$. One can find another 1-cochain $\tilde{\Xi}$ of the form

$$\tilde{\Xi} := \left( \begin{array}{ccc} (\xi^{(0)}, \xi^{(1)}) & 0 \\ 0 & l \end{array} \right),$$

where $\xi^{(0)} = 3\frac{du}{u}$, $\xi^{(1)} = 2\frac{dv}{v}$ and $l = 0$, so that

$$D\tilde{\Xi} + \tilde{\Omega}_\Delta \wedge \tilde{\Omega}^{10}_{1,1} = 0.$$

These amount to the same information in the previous example to construct twice iterated integrals.

**Remark 7.4.** In general when constructing twice iterated integrals, after extracting 1-cochains of $K^\bullet(M_{1,1})$ from coefficients, one first computes their product

$$\tilde{\Omega}_\beta \wedge \tilde{\Omega}_\gamma = \left( \begin{array}{ccc} (\Phi^{(0)}, \Phi^{(1)}) & 0 & 0 \\ 0 & 0 & \xi \\ 0 & \xi & 0 \end{array} \right),$$

Then one wants to find a 1-cochain $\tilde{\Xi} := \left( \begin{array}{ccc} (\xi^{(0)}, \xi^{(1)}) & 0 \\ 0 & l \end{array} \right)$ so that $D\tilde{\Xi} + \tilde{\Omega}_\beta \wedge \tilde{\Omega}_\gamma = 0$. In our case, $\Phi^{(0)}$ and $\Phi^{(1)}$ are always 0, since all of our 1-forms involve $\alpha \frac{du}{u}$ and the wedge product of two such 1-forms is 0. For terms in $\xi$, one could put those of the form $\frac{du}{u}$ into $\xi^{(0)}$ and those of the form $\frac{dv}{v}$ into $\xi^{(1)}$. The rest of the terms in $\xi$ would be exact and can be written as $dl$ for some $l$. Then $\tilde{\Xi} = (\xi^{(0)}, \xi^{(1)}; l)$ would be sufficient to construct twice iterated integrals.

### 8. Tannaka Duality and the De Rham Tannaka Group $\mathcal{G}^{\text{dR}}$

Recall that $\mathcal{M}_{1,1} = \mathbb{G}_m \backslash Y$. To work with $\mathcal{M}_{1,1}$ is to work $\mathbb{G}_m$-equivariantly with $Y$. In this section we fix a base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$, and fix a lift $y \in \mathcal{Y}(\mathbb{Q})$ of $x$.

\[\text{\footnotesize{11}}\text{cf. In [3] Brown and Hain work with a complex }\Omega^\bullet(X, \mathcal{V}_n)\omega\text{ that each form in their complex involves }\omega, \text{ which is }\frac{du}{u}\text{ in our notation.}\]
(B) Define the category $\mathcal{C}_B$ to be the full subcategory of the category of $\mathbb{Q}$-local systems on $\mathcal{M}_{1,1}^{an}$ whose objects are local systems $\mathcal{V}$ over $\mathcal{M}_{1,1}^{an}$ admitting a filtration

$$0 = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_r = \mathcal{V}$$

such that each graded quotient $\mathcal{V}_m/\mathcal{V}_{m-1}$ is isomorphic to a finite direct sum of $S^{2n}\mathbb{H}$'s.  This is a $\mathbb{Q}$-linear tannakian category, with fiber functor $\omega_x$ given by taking the fiber at $x$.

Denote its Tannaka fundamental group by $\mathbb{G}_x^B = \text{Aut}_\mathbb{H}(\omega_x)$.  This is the relative completion of $\pi_1(\mathcal{M}_{1,1}^{an}, x)$.  In particular, $\mathbb{G}_x^B$ is isomorphic to $\mathcal{G}_{\text{rel}}$.  

(dR) Define the category $\mathcal{C}_{\text{dR}}$ to be the full subcategory of the category of regular connections on $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ with regular singularity at the cusp whose objects are vector bundles $\mathcal{V}$ over $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ that admit a filtration by connections

$$0 = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_r = \mathcal{V}$$

such that each graded quotient $\mathcal{V}_m/\mathcal{V}_{m-1}$ is isomorphic to a finite direct sum of $S^{2n}\mathbb{H}$'s with the Gauss-Manin connection $\nabla_0$.  This is a tannakian category, with fiber functor $\omega_x$ given by taking the fiber at $x$.  Denote its Tannaka fundamental group by $\mathbb{G}_x^{\text{dR}} = \text{Aut}_\mathbb{H}(\omega_x)$.

Remark 8.1.  It does not make sense to take the fiber at a stack/orbifold point $x \in \mathcal{M}_{1,1}$.  What we mean by that is to take the fiber at the lift $y \in Y$ of $x$.

Remark 8.2.  Note that the category of regular connections on $\overline{\mathcal{M}}_{1,1}$ with regular singularity at the cusp is not even an abelian category.  Our category $\mathcal{C}_{\text{dR}}$ is an abelian tensor category since every connection in $\mathcal{C}_{\text{dR}}$ has nilpotent residue at the cusp as the connection $\nabla_0$ on $\mathbb{H}$ does.

Remark 8.3.  The de Rham bundle $\mathfrak{u}_{\text{dR}}$ we constructed in section 6.5 is a pro-object of $\mathcal{C}_{\text{dR}}$.

Theorem 8.4.  There is a comparison isomorphism

$$\text{comp}_{\text{B, dR}} : \mathbb{G}_x^B \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathbb{G}_x^{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$$

for each base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$.

Proof.  A vector bundle $\mathcal{V} \otimes_{\mathcal{C}_{\text{dR}}^{\text{an}}} \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}$ over $\overline{\mathcal{M}}_{1,1}$ in $\mathcal{C}_{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$ restricts to a vector bundle over $\mathcal{M}_{1,1}$.  By Riemann-Hilbert correspondence, it corresponds to a local system $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{C}$ over $\mathcal{M}_{1,1}$ in $\mathcal{C}_B \otimes_{\mathbb{Q}} \mathbb{C}$.  This gives rise to a functor

$$\mathcal{C}_{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathcal{C}_B \otimes_{\mathbb{Q}} \mathbb{C},$$

which induces the map

$$\text{comp}_{\text{B, dR}} : \mathbb{G}_x^B \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathbb{G}_x^{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$$

for each base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$.

Since we can identify $\pi_1(\mathcal{M}_{1,1}^{an}, x)$ with $\text{SL}_2(\mathbb{Z})$ and embed it into $\text{SL}_2(\mathbb{Q})$, we have

$$1 \longrightarrow \mathcal{U}_x^B \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{G}_x^B \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \text{SL}_2 \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow 1$$

$$\downarrow \text{comp}_{\text{B, dR}} \quad \downarrow \text{comp}_{\text{B, dR}}$$

$$1 \longrightarrow \mathcal{U}_x^{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{G}_x^{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \text{SL}_2 \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow 1.$$

To prove that the vertical map in the middle is an isomorphism, it suffices to show that the vertical map on the left

$$\text{comp}_{\text{B, dR}} : \mathcal{U}_x^B \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathcal{U}_x^{\text{dR}} \otimes_{\mathbb{Q}} \mathbb{C}$$
between (pro)unipotent radicals is an isomorphism. This is equivalent to showing that the induced map
\[ \phi : u^B_x \otimes Q \to u^{dR}_x \otimes Q \]
between their Lie algebras is an isomorphism.

Surjectivity of \( \phi \) follows from Theorem 4.1 and the fact that \( u^B_x \) is a free Lie algebra. It remains to prove injectivity of \( \phi \).

In Section 6.3 we constructed a de Rham bundle \( u^{dR}_x \), which is an object in \( C^{dR} \). There is thus an action
\[ (8.1) \quad G^{dR}_x \to \text{Aut} \, u^{rel}_Q \]
of the de Rham Tannaka group on the fiber \( u^{rel}_Q \) over \( x \). By Theorem 6.4, after tensoring it with \( C \), it corresponds to an object \( u^B_x \otimes Q \) in \( C^B \otimes Q \), and thus an action
\[ (8.1) \quad G^B_x \times Q \to \text{Aut} \, u^{rel}_C \]
of the Betti Tannaka group on the complexified fiber \( u^{rel}_C \) over \( x \). This action factors through the action \( (8.1) \times Q \) and we have a diagram
\[
\begin{array}{ccc}
G^B_x \times Q & \stackrel{\text{comp}_{B, dR}}{\longrightarrow} & G^{dR}_x \times Q \\
\downarrow & & \downarrow \\
\text{Aut} u^{rel}_C.
\end{array}
\]
This induces a diagram
\[
\begin{array}{ccc}
U^B_x \times Q & \stackrel{\text{comp}_{B, dR}}{\longrightarrow} & U^{dR}_x \times Q \\
\downarrow & & \downarrow \\
\text{Aut} u^{rel}_C,
\end{array}
\]
and after taking Lie algebras
\[
\begin{array}{ccc}
u^B_x \otimes Q & \stackrel{\phi}{\longrightarrow} & u^{dR}_x \otimes Q \\
\downarrow & & \downarrow \\
\text{Der} u^{rel}_C.
\end{array}
\]
Since \( u^B_x \otimes Q \cong u^{rel}_C \) and the diagonal map above is the adjoint action which is injective, the horizontal map \( \phi \) is also injective.

\[ \Box \]

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