Linear pre-metric electrodynamics and deduction of the light cone

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Abstract. We formulate a general framework for describing the electromagnetic properties of spacetime. These properties are encoded in the ‘constitutive tensor of the vacuum’, a quantity analogous to that used in the description of material media. We give a generally covariant derivation of the Fresnel equation describing the local properties of the propagation of electromagnetic waves for the case of the most general possible linear constitutive tensor. We also study the particular case in which a light cone structure is induced and the circumstances under which such a structure emerges. In particular, we will study the relationship between the dual operators defined by the constitutive tensor under certain conditions and the existence of a conformal metric. Closure and symmetry of the constitutive tensor will be found as conditions which ensure the existence of a conformal metric. We will also see how the metric components can be explicitly deduced from the constitutive tensor if these two conditions are met. Finally, we will apply the same method to explore the consequences of relaxing the condition of symmetry and how this affects the emergence of the light cone.

Keywords: electrodynamics, pre-metric, light propagation

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1 Introduction

1.1 Motivation

In Einstein’s theory of gravity, General Relativity (GR), the fundamental variable describing the gravitational field is the spacetime metric. Einstein’s equations are partial differential equations which, together with suitable boundary conditions, determine the metric field from the energy-momentum distribution of matter.
In GR, the metric tensor fixes all properties of spacetime: its metric properties (allows to define lengths), its affine structure (allows to compare object at different points, by parallel transport) and also its electromagnetic properties (specifies how electromagnetic waves propagate and, in particular, defines the light cone at each event).

In searching for a better understanding of the structures which can be used to describe spacetime it is interesting and useful to distinguish each different role that the metric plays in GR. This approach has been followed before regarding the affine properties of spacetime. As a result, it is now clear that these properties are in general described by an affine connection, which is an object logically independent of the metric. It has been useful to consider the metric and the affine connection as independent fields describing different aspects of spacetime. In this way, it was possible to formulate more general, so-called metric-affine theories of gravity and spacetime which have the potential to describe physics beyond GR. In this more general framework, Einstein’s theory is recovered as a particular case when certain additional conditions (vanishing of torsion and non-metricity) are satisfied. When this happens the affine connection turns out to be a function of only the metric tensor, as in GR. On the other hand, the more general formalism allows us to search additionally for alternative points of view as, for example, the possibility to formulate a purely affine theory of gravity, in which the affine connection is considered to be the fundamental field and the metric would be a derived quantity.

One of the aims of the present work is to extend this line of thought also to the description of the electromagnetic properties of spacetime. For this, it is necessary to recognize as clearly as possible the role which the metric and affine structures play for the formulation of classical electrodynamics. It will turn out that the basic structure of electromagnetic theory, when properly formulated, is independent of metric and affine concepts. The electromagnetic properties of spacetime are encoded in the so-called constitutive tensor of spacetime, which is a quantity analogous to that used for the description of electrodynamics in material media. We consider the constitutive tensor of spacetime as independent of metric and affine structures, in the same way as the affine connection was considered to be independent of the metric, and develop the formalism for the description of the electromagnetic properties of spacetime. Not surprisingly, structures richer than in GR are found. In particular, we study some aspects of wave propagation in the case of an arbitrary constitutive tensor and derive the Fresnel equation describing the local properties of the propagation of light. In general, instead of a light cone structure related to quadratic forms, a general quartic structure is found. The particular case of Einstein’s theory corresponds to a particular choice of constitutive tensor, completely determined by the (conformal) metric.

We study the conditions which must be satisfied such that the electromagnetic properties of spacetime are compatible with a light cone structure. Finally, it is also possible (similarly to the case of purely affine theories) to consider the constitutive tensor as a more fundamental field describing spacetime and the conformal metric as a derived quantity.

That the conformal metric can indeed be considered as a consequence of the electromagnetic properties of spacetime should be intuitively clear. The causal properties of spacetime, described by the conformal metric in GR, are of an electromagnetic nature. Light rays are the physical realization of null vectors fields, null hypersurfaces are the regions through which electromagnetic waves can propagate. From an operational point of view it is clear that most of our knowledge of spacetime is extracted from the properties of electromagnetic fields and how they propagate. Today, we send electromagnetic waves towards (reflecting) objects to measure distances. We measure velocities by means of Doppler-shifted electromagnetic waves,
The properties of spacetime itself are then defined as those independent of the specific configuration of the fields and interpreted as describing the underlying ‘substrate’ on which we make measurements.

The idea of considering the spacetime metric as a secondary, derived field, is not new. Already in 1921 Eddington studied a ‘purely affine gravity’ model, in which the metric is defined in terms of a symmetric affine connection, see [64]. Thus, in Eddington’s theory, the causal properties of spacetime are derived from its affine properties. Other models have proposed to replace the metric by a trio of self-dual 2-forms [8, 9]. In [10] the metric is obtained from a solution of a theory formulated only in terms of a $SL(2, C)$ connection, a tetrad, and a scalar density. More recently, it has been shown by Barceló et al. [2, 3] that an effective metric can be derived (defined) for almost any lagrangian theory of scalar fields, provided one considers perturbations of the fields around some background configuration. If the theory depends of a single scalar field, then a metric can be uniquely defined (so that the equation for the field perturbation can be written as a Klein-Gordon equation with respect to that metric) [2]. If more scalar fields are involved, multiple metrics can be introduced in general, which can be described as refringence (birefringence, trirefringence, ...) in the sense that different fields of the theory would ‘see’ different effective metric structures [3]. In these models, the causal properties of space are again a manifestation of the dynamics of some more fundamental fields.

In a related context, there has been renewed interest in recent years for so-called ‘analogue models of gravity’, see [78]. These models are based on the results that some condensed matter systems, as, for instance, a moving fluid with acoustic perturbations, a moving dielectric with light, or a moving superfluid with quasiparticles, can be described in term of some ‘effective metric’. The metric is also here a derived, secondary object which turns out to be useful for a geometrical description (à la GR) of the system properties and depends on the more fundamental degrees of freedom of the system. The interest in these analogue models lies in the possibility to construct systems in which kinematical properties of physics in curved space could be simulated and tested in the laboratory. In [65], for instance, the possibility of constructing a ‘dielectric black hole analogs’ is discussed, i.e., dielectric materials in which the effective metric describes the analog of an event horizon, see also [40, 77]. Dynamics, on the other hand, is not likely to be simulated, since in general the dynamics of the effective metric can be completely different to that imposed by Einstein’s equations in GR.

My goal here is not to propose a concrete alternative spacetime theory, but the more modest one of studying a more general framework describing classical electrodynamics in which a metric tensor is not assumed as a basic fundamental field from the very beginning. This will raise the question on which fundamental structures classical electrodynamics can be based without using a metric. The framework will be called pre-metric (or sometimes metric-free) electrodynamics, and it shares many features and analogies with the theory of electrodynamics in a material medium. By studying such a general framework, the conformal properties of spacetime, i.e., the conformal spacetime metric with its nine independent components, can eventually emerge as an special case under certain particular circumstances. The study of the conditions under which a conformal metric structure is induced is also one of the subjects of this work. This framework could then be useful to develop a generalized theory of spacetime in which the properties of vacuum are treated in analogy to a material medium and in which GR with its causal structure could be recovered in suitable limiting cases.

The axiomatic approach to classical electrodynamics presented here has been developed from the original ideas of Kottler [36] and van Dantzig [76]. They seem to be the firsts who
recognized that the fundamental structure of Maxwell’s equations is independent of the metric and affine structure of spacetime. In what concerns the derivation of the spacetime metric from linear electrodynamics, in which we are interested here, Peres [56], already in 1962, wrote: ‘It is therefore suggested to consider the electromagnetic field as fundamental, and the metric field only as a subsidiary quantity’. This same idea was also developed by Toupin [71] and by Schönberg [62] who showed that a conformal metric structure is induced assuming that the ‘constitutive tensor’ defining the ‘spacetime relation’ between electromagnetic excitations and field strengths (see Section 2.6) satisfies the conditions of symmetry and closure (Section 4). Jadczyk [31] also showed that a spacetime metric can be introduced under the above mentioned conditions. However, only very recently and explicit derivation of the induced conformal metric has been given by Obukhov and Hehl [52].

We formulate a general framework for electrodynamics in an arbitrary linear electromagnetic medium\(^1\). These developments are useful for three different aspects at least. First, they make the fundamental structures of classical electrodynamics more transparent. Second, they may provide a basis for a deeper understanding and for generalized models of spacetime and its electromagnetic properties. This includes, for instance, the application of the formalism to study test theories, see [21] and references therein. Third, the formalism developed in Sections 2 and 3 can also be interpreted and used in optics as a general covariant theory of electrodynamics in inhomogeneous, anisotropic, and in general dissipative material media.

Finally, we study the particular case in which a light cone structure is induced and the circumstances under which such structure emerges. In particular, we will study the relationship between the dual operators defined by the constitutive tensor under certain conditions and the existence of a conformal metric. Closure and symmetry of the constitutive tensor will be found to be the conditions which ensure the existence of a conformal metric. We will also see how the metric components can be explicitly derived from the constitutive tensor if these two conditions are satisfied. We will also give an alternative, simpler, and more physical derivation of the metric based on direct use of our general results about the Fresnel equation describing the local properties of light propagation. Finally, we will apply the same method to explore the consequences of relaxing the condition of symmetry and how this affects the emergence of the light cone.

1.2 Maxwell-Lorentz equations in 3-vector notation

We start with electrodynamics within the framework of Special Relativity (SR), but in 3D formalism.

In the usual 3-dimensional vector notation, \( V := (V^x, V^y, V^z) \), the Maxwell-Lorentz equations in integral form read:

\[
\begin{align}
\int_{\partial V} D \cdot dS &= \int_V \tilde{\rho} dV, \\
\int_S H \cdot dr &= \int_S J \cdot dS + \frac{d}{dt} \left( \int_S D \cdot dS \right), \\
\int_{\partial V} B \cdot dS &= 0, \\
\int_S E \cdot dr &= -\frac{d}{dt} \left( \int_S B \cdot dS \right).
\end{align}
\]  

\(^1\) We use the word ‘medium’ in a general sense in order to refer to any ‘arena’ on which electromagnetic phenomena could take place. In particular, we include the vacuum as a particular electromagnetic medium. On the other hand, ‘material’ or ‘material medium’ will be used to refer to media with a known (atomic) substructure as, for instance, crystal, liquids, etc.
Here \( \mathbf{D} \) is the electric excitation (historically called ‘electric displacement’), \( \tilde{\rho} \) the electric charge density, \( \mathbf{H} \) the magnetic excitation (historically called ‘magnetic field’), \( \mathbf{J} \) the electric current density, \( \mathbf{B} \) the magnetic field strength and \( \mathbf{E} \) the electric field strength. The integrals above are defined over arbitrary volumes \( V \) with boundary \( \partial V \) and over arbitrary surfaces \( S \) with boundary \( \partial S \), respectively. The corresponding volume elements of the three-, two-, and one-dimensional regions are denoted by \( dV \), \( dS \), and \( dr \). Finally, the dot \( \cdot \) denotes the 3-dimensional scalar product of vectors, which is a metric-dependent operation.

Eq. (1a) summarizes the Gauss law. It implies that the field lines defined by \( \mathbf{D} \) can be open, the ends of which are located at points where charge is located. Eq. (2a) is usually interpreted as expressing the absence of magnetic monopoles in nature\(^2\). It implies that the magnetic lines defined by \( \mathbf{B} \) must be closed. Eq. (2b) summarizes Faraday’s induction law (‘a time-variation of an magnetic field induces an electric field’).

The field strengths \( \mathbf{E} \) and \( \mathbf{B} \) are operationally defined by means of the Lorentz force law. On test charges, the force density is given by

\[
\mathbf{F} = \tilde{\rho} \mathbf{E} + \mathbf{J} \times \mathbf{B}.
\]

Here \( \times \) is the 3-dimensional vector product, also a metric-dependent operation.

The Maxwell equations are completed by the relations

\[
\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B}.
\]

The constants \( \varepsilon_0 \) and \( \mu_0 \) are called permittivity and permeability of vacuum, respectively. The speed of light, i.e., the speed with which electromagnetic perturbations propagate, is given by \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \).

As it is well known, the Maxwell equations (1) and (2) can be written in differential form as

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \tilde{\rho}, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}.
\end{align*}
\]

The inhomogeneous Maxwell equations (5) are such that the conservation of electric charge is automatically satisfied, i.e.

\[
\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot \mathbf{J} = 0.
\]

Actually, Maxwell completed the electromagnetic equations known at his time by adding the ‘electric displacement’ term \( \frac{\partial \mathbf{D}}{\partial t} \) such that the resulting equations were consistent with charge conservation.

1.3 Electrodynamics in a material medium

It is well known that a macroscopic description of electromagnetic phenomena inside a material medium (treated as a continuum) can be achieved using macroscopic Maxwell equations which are of the same form as (5) and (6), but were now

\(^2\) For recent (unsuccessful) searches for magnetic monopoles, see [1, 22, 32] and references therein.
• \( \mathbf{D} \) and \( \mathbf{H} \) denote macroscopic field strengths related to the sources \( \rho \) and \( \mathbf{J} \), which are now external charge and current densities, respectively, and

• the relation between \((\mathbf{D}, \mathbf{H})\) and \((\mathbf{E}, \mathbf{B})\) is now not given by (4) but by a constitutive relations

\[
\mathbf{D} = \mathbf{D} [\mathbf{E}, \mathbf{B}], \quad \mathbf{H} = \mathbf{H} [\mathbf{E}, \mathbf{B}],
\]

which contain the information of the particular electromagnetic properties of the medium under consideration.

Among the many possible particular cases (nonlocal constitutive laws, ...) we recall here the case of a linear anisotropic medium for which the constitutive relations are usually written, in components (see for instance [38], page 313), as

\[
D_a = \varepsilon_{ab} E_b + \alpha_{ab} B_b, \quad H_a = \mu_{ab}^{-1} B_b + \beta_{ab} E_b.
\]

Here \( a, b, \ldots = 1, 2, 3 \), \( \varepsilon_{ab} \) is the permittivity tensor, and \( \mu_{ab} \) the permeability tensor of the medium. The tensors \( \alpha \) and \( \beta \) describe magneto-electrical properties. For examples of and further discussions on magneto-electric media, see [54].

Consider the case in which \( \alpha_{ab} \) and \( \beta_{ab} \) vanish. A non-magnetic medium corresponds to the subcase in which \( \mu_{ab} = \mu_0 \delta_{ab} \) so that the vacuum relation (4b) holds. Interesting properties of non-magnetic anisotropic media regarding propagation of plane electromagnetic waves include [38]:

• In general, for a triaxial crystal (i.e., when the three eigenvalues of \( \varepsilon_{ab} \) are different), a fourth order Fresnel equation determines the dispersion relation of plane waves.

• The wave vector \( \mathbf{k} \) and the ray vector \( \mathbf{s} \) are in general not parallel: \( \mathbf{k} \) is the vector normal to the wave front, \( \mathbf{s} \) is the direction of energy propagation.

• Birefringence in uniaxial crystals (two eigenvalues of \( \varepsilon_{ab} \) are equal): the Fresnel equation factorizes into two quadratic factors, one isotropic factor corresponding to ‘ordinary’ waves and an anisotropic one corresponding to ‘extraordinary’ waves.

Effects analogous to the above mentioned will be discussed in Section 3.

Other examples of local constitutive laws include:

• Double refraction induced by an electric field. This so-called Kerr effect can be induced if an isotropic material is placed in a constant electric field. The electric field breaks the isotropy of the medium producing a change in the dielectric constant, leading to effects similar to those observed in uniaxial crystals. This effect can be described by the nonlinear constitutive law corresponding to the following dielectric tensor:

\[
\varepsilon_{ab} = \varepsilon_0 \delta_{ab} + \alpha E_a E_b,
\]

with some constant \( \alpha \).

• Magneto-optical effects: The dielectric constant depends on the magnetic field strength \( \mathbf{H} \).
Table 1  Correspondence between vectors and exterior forms

| vector/scalar | $\tilde{\rho}$ | J | $\mathcal{H}$ | D | E | B |
|---------------|---------------|---|-------------|---|---|---|
| p-form        | $\rho$        | j | $\mathcal{D}$ | $\mathcal{E}$ | $\mathcal{B}$ | $\mathcal{H}$ |
| p             | 3             | 2 | 1           | 2 | 1 | 2 |

1.4 Maxwell-Lorentz equations in 3+1 exterior form notation

From the Maxwell-Lorentz equations in their integral form, see (1) and (2), one can see that the different fields appear associated with integrals over regions of different dimensionality. For the sources, we see that $\tilde{\rho}$ is integrated over 3-dimensional regions (volumes) while $J$ is integrated over surfaces, i.e., 2-dimensional regions. The excitations $\mathcal{H}$ and $\mathcal{D}$ are integrated over 1- and 2-dimensional domains, respectively. Finally, the field strengths $\mathcal{E}$ and $\mathcal{B}$ appears in the Maxwell-Lorentz equations under 1- and 2-dimensional integrals, respectively. From the theory of exterior forms, see [16], we know that a $p$-form is the natural object to be integrated over a $p$-dimensional domain. This means that the Maxwell-Lorentz equations can be reformulated in terms of exterior forms according to the identifications of Table 1. In terms of exterior forms, the Maxwell-Lorentz equations (1) and (2) are naturally expressed as

$$d\mathcal{D} = \rho, \quad d\mathcal{H} = \partial_t \mathcal{D} + j,$$

(11)

$$d\mathcal{B} = 0, \quad d\mathcal{E} + \partial_t \mathcal{B} = 0.$$  

(12)

Here $d = dx^a \wedge \partial_a$ denotes the 3-dimensional exterior derivative (we will use $d$ for the 4-dimensional exterior derivative). Note the internal consistency of these equations, since $d$ increases by one the rank of the exterior form on which it is applied. Conversely, taking the equations (11) and (12) as starting point, one can derive the corresponding field equations in terms of field components. We decompose each form as follows:

$$\rho = \frac{1}{3!} \rho_{abc} dx^a \wedge dx^b \wedge dx^c = \tilde{\rho} dx \wedge dy \wedge dz,$$

(13)

$$j = \frac{1}{2} j_{ab} dx^a \wedge dx^b = \frac{1}{2} \hat{\epsilon}_{abc} j^c dx^a \wedge dx^b,$$

(14)

$$\mathcal{H} = \mathcal{H}_a dx^a, \quad \mathcal{D} = \frac{1}{2} \mathcal{D}_{ab} dx^a \wedge dx^b = \frac{1}{2} \hat{\epsilon}_{abc} \mathcal{D}^c dx^a \wedge dx^b,$$

(15)

$$\mathcal{E} = E_a dx^a, \quad \mathcal{B} = \frac{1}{2} \mathcal{B}_{ab} dx^a \wedge dx^b = \frac{1}{2} \hat{\epsilon}_{abc} \mathcal{B}^c dx^a \wedge dx^b,$$

(16)

Here $\hat{\epsilon}_{abc} = \hat{\epsilon}_{(abc)}$, with $\hat{\epsilon}_{123} = 1$, is the 3-dimensional Levi-Civita symbol. Then, substituting (13)–(16) into the equations (11) and (12), one directly finds, see [63],

$$\partial_a \mathcal{D}^a = \rho, \quad \epsilon^{abc} \partial_b \mathcal{H}_c = \frac{\partial}{\partial t} \mathcal{D}^a + j^a,$$

(17)

$$\partial_a B^a = 0, \quad \epsilon^{abc} \partial_b E_c + \frac{\partial}{\partial t} B^a = 0.$$  

(18)
which generalize (5) and (6). Note that $D^a$ and $H^a$ are 3-dimensional vector densities. The formulation in terms of exterior forms has the advantage that the corresponding $p$-forms are independent of the 3-dimensional coordinate system used. In other words (11) and (12), and therefore also (17) and (18), are valid not only in cartesian coordinates but in any 3-dimensional coordinate system. Actually, if one considers the exterior forms in (13)–(16) as basic field variables, as we do here, then equations (11) and (12) are independent of any metric or affine structure of the 3-dimensional space.

1.5 Poincaré covariant Maxwell equations

As usually shown in textbooks on SR, the Maxwell equations (5) and (6) can be written covariantly under Poincaré transformations by defining the field strength $F_{ij} = -F_{ji}$ and the excitation $H_{ij} = -H_{ji}$ by

$$
F_{tx} := -E^x, \quad F_{ty} := -E^y, \quad F_{tz} := -E^z, \\
F_{xy} := B^z, \quad F_{yz} := B^x, \quad F_{zx} := B^y, \\
H_{tx} := H^x, \quad H_{ty} := H^y, \quad H_{tz} := H^z, \\
H_{xy} := D^z, \quad H_{yz} := D^x, \quad H_{zx} := D^y.
$$

Additionally, the electric current 4-vector density $\mathcal{J}^i$ is defined as

$$
\mathcal{J}^t := \rho, \quad \mathcal{J}^x := J^x, \quad \mathcal{J}^y := J^y, \quad \mathcal{J}^z := J^z.
$$

Then, we can write (5) and (6) as

$$
\frac{1}{2} \epsilon^{ijkl} \partial_j H_{kl} = \mathcal{J}^i, \quad \epsilon^{ijkl} \partial_j F_{kl} = 0,
$$

where $\epsilon^{ijkl}$ is the Levi-Civita symbol. The relation between $F_{ij}$ and $H_{ij}$, namely equation (4), is translated into

$$
H_{ij} = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \epsilon_{ijk} \eta^{km} \eta^{ln} F_{mn},
$$

where $\eta^{ij}$ are the components of the Minkowski metric in cartesian coordinates: $\eta_{ij} = \text{diag}(c^2, -1, -1, -1)$, $\eta^{ij} = \text{diag}(c^{-2}, -1, -1, -1)$. Inserting (25) into (24) we obtain (always in cartesian coordinates) the inhomogeneous Maxwell equations in a form which is often used in SR, namely

$$
\sqrt{\frac{\varepsilon_0}{\mu_0}} \partial_j F^{ij} = \mathcal{J}^i.
$$

1.6 Poincaré group invariance and ‘natural’ invariance

It is well known that the Maxwell equations (24) and (25) are form invariant under Poincaré transformations of the form

$$
x^i \rightarrow x'^i := \Lambda^i_j x^j + a^i,
$$

where $\Lambda^i_j$ are the elements of the Lorentz transformation matrix.
with \( \Lambda_i' \in SO(1, 3) \), i.e. satisfying

\[
\Lambda_i' \Lambda_j' \eta^{ij} = \eta^{i'j'}, \quad a^i \in \mathcal{R}.
\]

(28)

This form invariance of the Maxwell equations means that the physics of the electromagnetic fields remains the same on rotated frames and in frames moving with constant velocity with respect to each other (i.e. boosted frames), as well as at different points in spacetime.

However, a more careful analysis shows that the field equations (24) are actually form invariant under any coordinate transformation \( x^i \rightarrow x_i' = x_i'(x^i) \), provided \( F_{ij} \) and \( H_{ij} \) are considered as components of a second order (antisymmetric) tensor field, and \( \mathcal{J}^i \) as the components of a vector density field of weight +1. This feature of (24) is sometimes called ‘natural invariance’ of the Maxwell equations, see [59], chapter 3, for an extended discussion. The natural invariance shows, on the other hand, that the physical information about the equivalence of inertial frames under Lorentz transformations is contained exclusively in the spacetime relation between excitation and field strength (25). This is a property of the vacuum. Furthermore, we note that the Minkowski metric only appears in (25). Looking at (25) it is then clear that this invariance of the vacuum is a direct consequence of the form invariance of the Minkowski metric with respect to Poincaré transformations, see (28). In Section 2.7.1 we will give a more general definition of symmetries of a given electromagnetic medium.

### 1.7 Maxwell equations in curved spacetime

The traditional recipe to construct a theory including the interaction with the gravitational field is to ‘replace partial derivatives by covariant derivatives’, also called ‘minimal coupling’ to the gravitational field. This procedure ensures that the resulting equations are covariant under an arbitrary change of coordinates, see, for instance, [43] for more details and examples. The procedure is not free of ambiguities when one considers the electromagnetic potential as a fundamental variable, since then the Maxwell equations are of second order and a ‘normal ordering’ problem appears when applying the recipe above. This is a consequence of the fact that covariant derivatives do not commute in a curved spacetime.

A not widely recognized fact is, however, that in the case of electrodynamics the mentioned recipe is completely unnecessary since the Maxwell equations, when properly formulated, are ‘naturally covariant’, as we have seen in Section 1.6. For the (rather trivial) transition from a Minkowski space to a Riemannian space with metric \( g \) one just needs to replace the spacetime relation by

\[
H_{ij} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \hat{\epsilon}_{ijkl} \sqrt{|g|} g^{km} g^{ln} F_{mn},
\]

(29)

with \( g := \det (g_{ij}) \), so that \( F \) and \( H \) are tensors, as required by the natural invariance of (24). At every event one can find Riemann normal coordinates which will reduce the metric to its minkowskian form, i.e. \( g_{ij} \equiv \eta_{ij} \) and then (29) reduces to the form (4), in agreement with the equivalence principle.

Eq. (29) can be written in terms of the Hodge dual operator \( * \) of the metric \( g \), namely

\[
H = \sqrt{\frac{\varepsilon_0}{\mu_0}} \ast F, \quad (\ast F)_{ij} := \frac{1}{2} \hat{\epsilon}_{ijkl} \sqrt{|g|} g^{km} g^{ln} F_{mn}.
\]

(30)
The Maxwell equations in a Riemannian space are therefore, in terms of $F$ and $g$:

$$\partial_j \left( \sqrt{\frac{\varepsilon_0}{\mu_0}} \sqrt{\left| g \right|} g^{ik} g^{jl} F_{kl} \right) = \mathcal{J}^i, \quad \varepsilon^{ijkl} \partial_j F_{kl} = 0. \quad (31)$$

The fact that the Maxwell equations (24) are naturally invariant and that the metric structure of space does not enter in their formulation is not an accident. It is a consequence of the fundamental property that the basic structure of electrodynamics can be derived from counting procedures of charge and magnetic flux, with do not require a metric (nor an affine) structure of spacetime. This properties will be further clarified in the axiomatic approach of Section 2.

## 2 Electrodynamics on an arbitrary 4D-manifold

In this section, we would like to present an axiomatic construction of classical electrodynamics which intends to be as general as possible. Structures are only introduced when they are indispensable for the development and not earlier than necessary. This approach will provide us a very general framework which can then be applied to many different particular cases.

We model spacetime as a smooth 4-dimensional manifold $X$ which, at least in some neighborhood, admits a foliation into 3-dimensional submanifolds, parameterized by a monotonic ‘time’ parameter $\sigma$, see Fig. 1.

### 2.1 Charge conservation

Probably the most important and defining property of electromagnetic theory is the experimentally well tested fact of charge conservation. A basic property of electric charge is that it is an additive quantity which can be distributed in space, i.e., it is an extensive quantity. In nature, electric charge is know to be quantized, its fundamental quanta, the electric charges of

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3 For experiments testing charge conservation/violation see, for instance, [49, 67]
the quarks are $\pm e/3$ and $\pm 2e/3$, and those of the leptons $\pm e$, where $e$ denotes the elementary charge. In a classical field theory, we describe the distribution of an extensive quantity in terms of a current density, containing the information of how many charges are distributed in spacetime, and how they move.

Consider the total charge within some 3-dimensional region. This total charge should be independent of the orientation of any coordinate system used. Then, the electric charge distribution is described by the twisted electric current 3-form $J$, see [16] for definitions. For a 3-dimensional region $\Sigma_3$ of a hypersurface $h_\sigma$ one can interpret $\int_{\Sigma_3} J$ as the total charge contained in $\Sigma_3$. On the other hand, if the 3-dimensional region is of the type $\Omega_3 = \Sigma_2 \times [\sigma_1, \sigma_2]$ one can interpret $\int_{\Omega_3} J$ as the total charge crossing the 2-dimensional surface $\Sigma_2$ during the 'time' interval $[\sigma_1, \sigma_2]$. Consequently, the 3-form $J$ carries the dimension of charge, i.e. $[J] = q$.

Note that no concept of distance or parallel displacement, i.e., no metric or connection are necessary to define the concepts of charge and charge current 3-form specifically. Of course, quantities like 'charge per unit volume' and 'charge per unit area and unit time' are useful after one provides prescriptions for what 'unit volume', 'unit area' and 'unit time' are. However, the latter are concepts not needed to describe, e.g., how many electrons and therefore how much charge, are contained in a certain region. Clearly, the total charge is independent of the unit in which volume is measured.

The components $J_{\alpha\beta\gamma}$ of the electric current 3-form with respect to some coframe basis $\vartheta^\alpha$, $\alpha, \beta, \ldots = 0, 1, 2, 3$, are determined by

$$J = \frac{1}{3!} J_{\alpha\beta\gamma} \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma.$$  \hspace{1cm} (32)

If one associates a dimension $l$, in the sense of a segment to the coframe $\vartheta$, i.e. $[\vartheta] = l$, then the components $J_{\alpha\beta\gamma}$ carry the dimension $[J_{\alpha\beta\gamma}] = ql^{-3}$.

---

\[\text{i.e., a one dimensional extension on the manifold, not in the sense of a unit of length}\]
The conservation of electric charge is then expressed as the vanishing of the integral
\[ \oint_{\partial \Omega_4} J = 0, \quad \forall \Omega_4, \quad (33) \]
i.e., for any 3-dimensional boundary of a 4-dimensional region \( \Omega_4 \). In particular, for a region \( \Omega_4 = \Sigma_3 \times [\sigma_1, \sigma_2] \), the integral conservation law (33) requires the balance between the change of the charge in the region \( \Sigma_3 \) during the interval \([\sigma_1, \sigma_2]\) and the flux across its 2-dimensional boundary \( \partial \Sigma_3 \).

Since the region \( \Omega_4 \) in (33) is arbitrary, the Stokes theorem tells us that the current 3-form must be closed, i.e.,
\[ dJ = 0. \quad (34) \]

Now, according to the de Rham theorem, see for instance [16], the current 3-form is not only closed but also exact, i.e., it can be derived from some 2-form \( H \) by exterior derivation, if all its integrals vanish over 3-dimensional regions without boundaries. Under these assumptions, this means that there must be a 2-form \( H \) such that
\[ dH = J. \quad (35) \]
The twisted 2-form \( H \) is called the electromagnetic excitation and carries dimension of charge, \([H] = q\). As we will see, in a 3+1 decomposition its components can be identified with the metric-free generalization of the electric and magnetic excitations.

However, the conditions above are not enough to uniquely define the 2-form \( H \), since a ‘gauge’ transformation
\[ H \rightarrow H' := H + d\Psi \quad (36) \]
leaves (35) invariant, for an arbitrary twisted 1-form \( \Psi \).

Eq. (35) are thus the inhomogeneous Maxwell equations (4 equations). More than defining the values of \( H \), the above arguments show that the inhomogeneous Maxwell equations must be of the form (35), since this is the only kind of field equation which are compatible with electric charge conservation.

A single electromagnetic excitation will be picked out by the requirement that \( H = 0 \) for \( F = 0 \) for the spacetime/constitutive relation, see Sections 2.5 and 2.6.

### 2.2 Lorentz force

We assume now that the concept of force density is known from mechanics and use it to define the electromagnetic field strength 2-form \( F \), as usual, as force per unit charge. We define \( F \) by means of
\[ f_\alpha =: (e_\alpha, F) \wedge J, \quad (37) \]
where \( e_\alpha \) is a frame\(^5\) and \( f_\alpha \in \Lambda^4(X) \) are the corresponding components of the force density covector-valued 4-form in that frame\(^6\). From the definition (37) \( F \) is an untwisted 2-form, i.e., an

---

5 This is again a metric-independent quantity. A frame is just a basis of the tangent space

6 Remember, in classical mechanics force \( f_i = \frac{\partial L}{\partial \dot{x}_i} \) and momentum \( p_i = \frac{\partial L}{\partial x_i} \) are both covectors
intensive quantity. It describes how strongly the electromagnetic field acts on test currents. The dimension of $F$ is $[F] = \hbar / q$, with $\hbar$ denoting the dimension of an action. Equivalently $F$ has dimension of magnetic flux. The definition above is a very restrictive one, since it tells us that the force on test charges is determined by only the six independent components of the 2-form $F$, instead of the 16 independent quantities that a linear relation between force density and current density would admit in principle\(^7\). This assumption is part of our axiomatics, and it is suggested by the fact that the excitation is described by a field with 6 independent components and one therefore expects the field strength to have the same number of independent components. We know, of course, that this choice is reasonable since we know that in Maxwell’s theory, the Lorentz force is determined by the six independent components of the electric and magnetic field strengths.

It is true that in order to have a complete predictive theory, one still has to specify the relation between velocities and momenta of test currents. This is required in order to be able to predict the evolution of test currents in a given electromagnetic field. This relation includes the metric tensor in the known case of GR. However, we are interested here in the structure of the general electromagnetic theory, and not specifically in the mechanical ‘constitutive relation’.

### 2.3 Magnetic flux conservation

The next step in our axiomatic construction is to find conditions for the electromagnetic field strength to satisfy. The natural operation that can be done with a 2-form is to integrate it on a given 2-dimensional region. If we assume, in analogy to conservation of charge, that

$$\oint_{\partial \Omega_3} F = 0,$$

(38)

for an arbitrary 3-dimensional submanifold $\Omega_3$, then Stokes’ theorem provides us with a differential equation for $F$, namely that the field strength must be closed,

$$dF = 0,$$

(39)

and (at least in some neighborhood of each event) exact, i.e. $F = dA$. The untwisted 1-form $A$ is then the electromagnetic potential.

We take (38), or equivalently (39) as third axiom. It represents the homogeneous Maxwell equations (4 equations) and expresses the conservation of magnetic flux. In general, magnetic flux is not quantized, as it is the case of electric charge. In type II superconductors, however, as, e.g., in Niobium, quantized magnetic flux lines are possible.

This formulation does not admit magnetic charges (i.e. magnetic monopoles) in a natural way. Since the field strength $F$ is by its very definition an intensive quantity, a hypothetical magnetic charge density $\rho_m$, such that $dF = \rho_m$, would necessarily also be an intensive quantity (an untwisted 3-form), quite in contrast to the extensive nature of any charge-like quantity like, e.g., electric charge, energy-momentum, all twisted 3-forms.

Locally at least, see above, the homogeneous Maxwell equation (39) imply that the field strength $F$ can be derived from a untwisted 1-form $A$, the electromagnetic potential, such that

$$F = dA.$$

(40)

---

\(^7\) Consider for instance a relation of the form $f_\alpha = \Psi_\alpha \wedge J$. Then $\Psi_\alpha$ is a covector valued 1-form, and has therefore 16 independent components.
G. F. Rubilar, Linear pre-metric electrodynamics

Fig. 3 Conservation of magnetic flux in spacetime: For an arbitrary 3-dimensional integration domain $\Omega_3$, the integral $\int_{\partial \Omega_3} F$ vanishes.

Of course, $A$ is only determined up to a gauge transformation

$$A \rightarrow A' = A + d\Psi,$$

(41)

for any untwisted 0-form (i.e. scalar) $\Psi$.

The electromagnetic potential carries the same physical dimension as the electromagnetic field strength, namely $[A] = h/q$.

2.4 ‘Space-time’ decomposition

Consider a foliation characterized by the monotonic ‘time-like’ parameter $\sigma$. Consider also a vector field $n$ not lying on a $\sigma = \text{const.}$ surface, i.e. $n \parallel d\sigma \neq 0$, so that it can be used to ‘evolve’ the folia, see Fig. 1. In particular, one can rescale the vector field to make it fulfill the normalization condition $n \parallel d\sigma = 1$, which turns out to be useful.

We will call any $p$-form $\omega$ ‘3-dimensional with respect to $n$’ if $n \parallel \omega = 0$. The ‘time-like’ components of a ‘3-dimensional form with respect to $n$’ in a basis adapted to the folia $(e_0 \sim n)$ vanish. Then, given any $p$-form $\Psi$ one can define the $p$-form $\Psi_\parallel$ and the $(p-1)$-form $\Psi_\perp$, both of which are 3-dimensional with respect to $n$ in the above sense, as

$$\Psi_\parallel := n \parallel (d\sigma \wedge \Psi),$$

(42)

and

$$\Psi_\perp := n \parallel \Psi.$$  

(43)

For $p = 4$ we have $\Psi_\parallel = 0$. Similarly, for $p = 0$, $\Psi_\perp = 0$.

These two ‘3-dimensional’ quantities $\Psi_\parallel$ and $\Psi_\perp$ contain the complete information of the original form $\Psi$. The latter can be written as

$$\Psi = d\sigma \wedge \Psi_\perp + \Psi_\parallel.$$  

(44)

One may also call $\Psi_\parallel$ and $d\sigma \wedge \Psi_\perp$ the ‘transverse’ and ‘longitudinal’ parts of $\Psi$ with respect to $n$, respectively, see [27].

From their definition, $\Psi_\parallel$ and $\Psi_\perp$ satisfy the following properties
\[ (\Psi \land \Phi)_\perp = \Psi_\perp \land \Phi + (-1)^p \Psi_\perp \land \Phi_\perp, \] (45)

\[ (\Psi \land \Phi) = \Psi \land \Phi, \] (46)

\[ (\mathcal{L}_n \Psi)_\perp = \mathcal{L}_n \Psi, \] (47)

\[ \mathcal{L}_n \psi = \mathcal{L}_n \psi. \] (48)

Additionally, it is useful to define a 3-dimensional part of the exterior derivative, \(d\), such that for any \(p\)-form \(\Psi\)
\[ d\Psi := n \cdot (d\sigma \land \Psi). \] (49)

It has the following properties:

\[ d \] is 3-dimensional with respect to \(n\),
\[ d\Psi = n \cdot (d\Psi) \equiv 0, \] (50)

\[ \downarrow \] of a derivative,
\[ (d\Psi)_\perp = \mathcal{L}_n \Psi - d\Psi_\perp, \] (51)

\[ \downarrow \] of a derivative,
\[ d\Psi = d\Psi. \] (52)

By means of this general decomposition procedure, we decompose the Maxwell equations (35) and (39) in terms of ‘3-dimensional’ quantities. First decompose the fields \(J, H, F\) and \(A\) and introduce the notation
\[ j := - J_\perp, \quad \rho := \rho, \quad \mathcal{H} := H_\perp, \quad \mathcal{D} := H, \] (53)
\[ E := - F_\perp, \quad B := B, \quad \varphi := -A_\perp, \quad A := A. \] (54)

The 4-dimensional quantities can be reconstructed according to
\[ J = -j \land d\sigma + \rho, \quad H = -\mathcal{H}_\land d\sigma + \mathcal{D}, \] (55)
\[ F = E \land d\sigma + B, \quad A = -\varphi \land d\sigma + A. \] (56)
Now one can take the inhomogeneous Maxwell equation (35) and find
\[(dH - J)_\perp = (dH)_\perp - J = \mathcal{L}_n H - dH_\perp - J = \mathcal{L}_n \mathcal{D} - d\mathcal{H} + j,\]
(57)
and
\[dH - J = dH - H = dH - J = d\mathcal{D} - \rho.\]
(58)
Analogously, from the homogeneous Maxwell equations (39) we obtain
\[(dF)_\perp = \mathcal{L}_n F - dF_\perp = \mathcal{L}_n B + \mathcal{d}E,\]
(59)
and
\[dF = \mathcal{d}F = \mathcal{d}B.\]
(60)
Thus, we find the Maxwell equations in a 3+1 decomposed form to be
\[d\mathcal{D} = \rho, \quad d\mathcal{H} = \mathcal{L}_n \mathcal{D} + j, \quad d\mathcal{B} = 0, \quad d\mathcal{E} + \mathcal{L}_n \mathcal{B} = 0.\]
(61)
We can also decompose the law of charge conservation (34). It is important to note that no information is obtained from \(dJ\) since this quantity vanishes for any 3-form \(J\), as can be seen from the definition (42). We then compute
\[(dJ)_\perp = \mathcal{L}_n J - dJ_\perp = \mathcal{L}_n \rho + \mathcal{d}j,\]
(62)
so that charge conservation means
\[\mathcal{L}_n \rho + \mathcal{d}j = 0.\]
(63)
Finally, we decompose equation (40). We find
\[(dA)_\perp = \mathcal{L}_n A - dA_\perp = \mathcal{L}_n A + \mathcal{d}\varphi,\]
(64)
and
\[dA = \mathcal{d}A = \mathcal{d}A,\]
(65)
so that
\[E = -\mathcal{d}\varphi - \mathcal{L}_n A, \quad B = \mathcal{d}A.\]
(66)
The decomposed Maxwell equations (61) naturally generalize equations (11) and (12). The Lie derivative \(\mathcal{L}_n\) is the natural generalization of the time derivative since it measures the change of the (integral of the) corresponding field between different folia, according to the displacement induced by the vector field \(n\), see Fig. 4.
Under the displacement $x' \rightarrow x'' = x^i + a n^i(x)$ ($a$ being an infinitesimal constant parameter) the integral $\int_{\Omega_p} \Psi$ of a $p$-form $\Psi$ over the corresponding $p$-dimensional mapped region $\Omega'_p$ is given by $\int_{\Omega'_p} \Psi = \int_{\Omega_p} \Psi + a \int_{\Omega_p} \mathcal{L}_a \Psi$.

2.5 Measuring $H$

In this section, we discuss a general procedure for measuring the excitation $H$. This can be done by using an idealized object, namely an ideal conductor. This special material is assumed to have the following two properties:

1. In an ideal conductor, all charges are located on its surface. In other words, the ideal conductor is such that inside it no free charges can be found. If $\Omega_3$ is the 3-dimensional region describing the conductor, then the free charges are all located on $\partial \Omega_3$.

2. In the ‘rest frame’ of the conductor the electric excitation $\mathcal{D}$ vanishes. If at some event inside the conductor a volume element is spanned by the vectors $e_a, a, b, \ldots = 1, 2, 3$ and $e_0 = n$ is a vector pointing in the zeroth independent ‘time’ direction, then we assume $H_{ab} := e_b[e_a]H = 0$, or equivalently $H = d\sigma \wedge \mathcal{H}$, with $n|d\sigma = 1$ and $e_a|d\sigma = 0$, see (55).

With these assumptions one can measure the excitation by use of ‘Maxwellian double plates’. Consider two (uncharged) parallel plates made of an ideal conductor and locate them at the point $P$ where the excitation should be measured. The field strength (whatever value it may have) will induce, through the Lorentz force, surface charges in the conductor. Separate now the plates and measure the charge $Q$ induced in one of its surfaces. One can then integrate the inhomogeneous Maxwell equation over a volume with one side in one conducting plate and the other between the plates, see Fig. 5. Due to property one, in the limit of vanishing ‘thickness’ of $\Omega_3$, see Fig. 5, one finds

$$\int_{\Omega_3} dH = \int_{\partial \Omega_3} H = (e_2[e_1]H)|_P - (e_2[e_1]H)_{cond},$$

and $\int_{\Omega_3} J = Q$, so that

$$(H_{12})_P - (H_{12})_{cond} = Q.$$
by orienting the plates differently one can measure, e.g., $H_{13}$. Furthermore, by changing the state of motion of the conductor (i.e., different 4-velocities) one can measure components of $H$ which are, in the notation we are using, of the form $(e_1|e_0|H) = H_{01}$.

### 2.6 Constitutive relations

As it is clear from the last sections, the Maxwell equations in their form (35) and (39) are valid for any electromagnetic medium (i.e. including the vacuum). They describe the general features of electrodynamics which follow from charge and magnetic flux conservation. In this form, Maxwell equations are valid, for instance, in vacuum in Special Relativity, but also if gravitational effects are included in the context of General Relativity or alternative theories, as for instance in those formulated in a general metric-affine spacetime. They are also applicable to any material medium, if one interprets $J$ and $H$ as macroscopic quantities. In other words, (35) and (39) are general structures of electromagnetic theory. The additional structure that really defines the particular physical properties of the system under consideration, is the subject of this section.

We saw that the Maxwell equations (35) and (39) amount to only 8 independent equations for the 12 independent fields, $H$ and $F$. A further relation between the excitation and field

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**Fig. 5** Maxwellian plates. Here $e_1$ and $e_2$ span the surface element parallel to the plates

**Fig. 6** The spacetime/constitutive relation connecting excitation and field strength
strength, i.e., a connection between the field generated by the charges and the field acting on test currents is therefore necessary in order to make the theory complete. Such a relation is referred to as the constitutive relation when corresponding to some material medium, and as the spacetime relation when describing the vacuum properties. The formalism presented here can thus be used in both situations. In the first one, the current 3-form describes the so-called ‘external’ (‘free’) currents flowing through a material medium. In vacuum, $J$ represents the total current density. In any case, the relation between $H$ and $F$ describes the properties of the arena in which the electromagnetic phenomena of interest take place. Clearly, however, the spacetime relation is of more fundamental importance, since it describes a universal property of spacetime. Its knowledge allows, in principle, to deduce the constitutive relation of a material medium from the spacetime relation and the (charge) structure of the material, see the appendix B. Finally, it is also possible, when interested in material media, to abandon the deductive point of view and adopt a phenomenological description of the constitutive relation between the macroscopic field excitation related to the macroscopic (free) charges and the field strength without trying to deduce it from the underlying structure of the system. From this point of view, the only physical property which differentiates what we call vacuum and what we refer to as a material medium, is that the former does not (by definition) have any charge substructure. It is a truly physical question whether the charges described by the charge density $J$ are really the total charges within a certain region, or whether they are macroscopically averaged charges. It is important to realize that in any case, as soon as the considered charge is conserved, a corresponding electromagnetic excitation $H$ can be introduced by mean of the inhomogeneous Maxwell equations. As a result, $H$ will be just as ‘microscopic’ as the corresponding charge current $J$ is.

In general, spacetime/constitutive relations $H = H(F)$ can have any functional dependence, i.e., they can be non-linear and non-local. Non-local constitutive relations are common for material media because in general the macroscopic electromagnetic properties at some point of the material are influenced by the field and charge configuration at other points of it. Additionally, the finite propagation speed of the microscopical electromagnetic interaction between the different parts of the material leads to time-like non-locality. In many cases (typically, for slowly varying fields), however, this non-local effects are negligible and the medium can be treated as if the field excitation $H$ were depending only on the value of the field strength at the same point, i.e. with a local constitutive law. In what follows, we restrict ourselves only to this later case.

### 2.6.1 Linear constitutive relations

We now concentrate on the particular case in which the field strength and excitations are proportional. A great number of material media are known in which this is a good description for a variety of conditions, see for instance [38]. We also expect the spacetime relation (i.e. the vacuum relation) to be simple and, in particular, linear. Linearity is taken for granted in all traditional approaches to vacuum electrodynamics.

Given a local frame $\vartheta^\alpha$, with $\alpha = 0, 1, 2, 3$, we can decompose the exterior forms $H$ and $F$ as

\[
H = \frac{1}{2} H_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta, \quad F = \frac{1}{2} F_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta,
\]  

(69)
and write a general linear spacetime relation (constitutive law) as

\[ H_{\alpha\beta} = \frac{1}{2} \kappa^{\gamma\delta} F_{\gamma\delta} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} \chi_{\gamma^\delta} F_{\epsilon \theta}, \]

(70)

where \( \varepsilon_{\alpha\beta\gamma\delta} \) is the Levi-Civita symbol with \( \varepsilon_{0123} := 1 \) and \( \chi_{\alpha\beta\gamma\delta} \) is called the constitutive tensor density, which is an untwisted tensor density of weight +1, carrying dimension \( [\chi] = [H]/[F] = q^2/h \) or, equivalently, dimension of conductance or 1/(resistance).

From its definition, the constitutive tensor satisfies the following symmetry properties

\[ \chi_{\alpha\beta\gamma\delta} = -\chi_{\beta\alpha\gamma\delta} = -\chi_{\alpha\beta\delta\gamma}, \]

(71)

which means that it has 36 independent components. Due to these symmetry properties, the constitutive tensor can also be represented by a 6x6 matrix, say \( \chi^{IJ} \), where each index \( I, J, \ldots \) corresponds to a pair of antisymmetrized 4-dimensional indices, i.e., \( I \to [i_1i_2] \), etc. In this notation, e.g., \( \varepsilon^{IJ} \to \varepsilon^{[i_1i_2j_1j_2]} \). One can enumerate the 6-dimensional indices according to \( I, J, \ldots = 01, 02, 03, 23, 31, 12 \).

We want to identify some irreducible components of the constitutive tensor. Usually, irreducible pieces of a covariant object have different physical properties. Remember that so far no metric structure has been introduced. However, we always have the Levi-Civita tensor density \( \varepsilon_{\alpha\beta\gamma\delta} \) at our disposal, and thus we can decompose the constitutive tensor according to, see [61],

\[ \chi_{\alpha\beta\gamma\delta} = (1)\chi_{\alpha\beta\gamma\delta} + (2)\chi_{\alpha\beta\gamma\delta} + (3)\chi_{\alpha\beta\gamma\delta}, \]

(72)

where the different irreducible pieces \( (1)\chi, (2)\chi, \) and \( (3)\chi \) are determined by the symmetry properties

\[
\begin{align*}
(1)\chi_{\alpha\beta\gamma\delta} &= \chi_{\gamma^\delta\alpha\beta}, \\
(2)\chi_{\alpha\beta\gamma\delta} &= -\chi_{\gamma^\delta\alpha\beta}, \\
(3)\chi_{\alpha\beta\gamma\delta} &= \chi_{\alpha\beta\gamma\delta},
\end{align*}

(73)

\[
\begin{align*}
(1)\chi_{[\alpha\beta\gamma\delta]} &= 0, \\
(2)\chi_{[\alpha\beta\gamma\delta]} &= 0, \\
(3)\chi_{[\alpha\beta\gamma\delta]} &= 0,
\end{align*}

(74)

or, explicitly,

\[
\begin{align*}
(3)\chi_{\alpha\beta\gamma\delta} &= \chi_{[\alpha\beta\gamma\delta]}, \\
(2)\chi_{\alpha\beta\gamma\delta} &= \frac{1}{2} \left[ \chi_{\gamma^\delta\alpha\beta} - \chi_{\alpha\beta\gamma\delta} \right], \\
(1)\chi_{\alpha\beta\gamma\delta} &= \chi_{\alpha\beta\gamma\delta} - (2)\chi_{\alpha\beta\gamma\delta} - (3)\chi_{[\alpha\beta\gamma\delta]}.
\end{align*}

(75)

One can introduce an alternative, but equivalent, parametrization of the 15 independent components of the asymmetric piece \( (2)\chi \) in terms of a traceless second rank tensor \( S_{\alpha\beta} \) (thus, also with 15 independent components) [28] as

\[ (2)\chi_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} S_{\epsilon\beta} - \gamma_{\beta\delta\alpha\epsilon} S_{\epsilon\beta}, \]

(76)

\[ S_{\alpha\beta} = 0. \]

(79)

Now, contracting (79) with the Levi-Civita symbol, one finds that

\[ S_{\alpha\beta} = \frac{1}{4} \epsilon_{\alpha\gamma\delta\epsilon} (2)\chi_{\gamma^\delta\beta}. \]

(80)
which shows that the traceless tensor $S_{\alpha\beta}$ is uniquely determined by (79).

The third piece $\chi$ can always be written as

\[
(3)\chi_{\alpha\beta\gamma\delta} = \alpha(x) \epsilon_{\alpha\beta\gamma\delta},
\]

where \(\alpha(x)\) is uniquely determined to be

\[
\alpha := \frac{1}{4!} \hat{\epsilon}_{\alpha\beta\gamma\delta} \chi_{\alpha\beta\gamma\delta}.
\]

It is called an abelian axion field.

Denote the contribution of each piece of the constitutive tensor to the excitation as $^{(1)}H$, $^{(2)}H$, and $^{(3)}H$, such that

\[
^{(1)}H_{\alpha\beta} = \frac{1}{4} \hat{\epsilon}_{\alpha\beta\gamma\delta} (1)\chi_{\gamma\delta\epsilon\theta} F_{\epsilon\theta},
\]

and analogously for the other pieces. Then

\[
H = (1)H + (2)H + (3)H,
\]

and

\[
(3)H = \alpha F.
\]

Using (35) and (39), we find

\[
dH = d\left( (1)H + (2)H + (3)H \right)
\]

\[
= d\left( (1)H + (2)H \right) + d(\alpha \wedge F)
\]

\[
= d\left( (1)H + (2)H \right) + d\alpha \wedge F,
\]

showing that the axion piece contributes to the Maxwell equations only if $d\alpha \neq 0$.

Eq. (72) represents the irreducible decomposition of the corresponding $6 \times 6$ matrix with respect to the linear group into a symmetric traceless piece, an antisymmetric piece, and a trace piece. The Levi-Civita symbol serves as a kind of 'metric' in the 6-dimensional formulation ($\epsilon_{ijkl} \rightarrow \epsilon^{IJ}$) which is used to construct the traces of $\chi^{IJ}$. The constitutive tensor is therefore reduced as $\chi = (1)\chi + (2)\chi + (3)\chi$ in $36 = 20 + 15 + 1$ independent components, respectively.

No further decomposition of the constitutive tensor is possible at this point, since no additional geometric objects are available.

For later application, see Section 3.1, we abbreviate the action of the constitutive tensor by defining the operator $\# : \Lambda^2 \rightarrow \Lambda^2$ such that

\[
\# \Omega := \frac{1}{4} \hat{\epsilon}_{\alpha\beta\gamma\delta} \chi_{\gamma\delta\epsilon\theta} \Omega_{\epsilon\theta} \partial^\alpha \wedge \partial^\beta,
\]

for any 2-form $\Omega$ with frame components $\Omega_{\alpha\beta}$. Then we can rewrite our spacetime/constitutive relation (70) as

\[
H = \# F.
\]
Each irreducible piece is expected to describe different aspects of the medium. Additional information about the different properties of each piece will be obtained from the study of the electromagnetic energy-momentum current and of wave propagation. Notice for example that, if \((2)\chi \neq 0\) the Maxwell equations cannot follow as Euler-Lagrange equations from a Lagrangian of the usual form \(V := H \wedge F\), since \((2)\chi \neq 0\) drops out from \(V\) due to its symmetry properties. However, our intention is to try to develop our electromagnetic theory as generally as possible. Therefore we want to also include systems for which no Lagrange density can be found. Typical examples of such kind of physical systems are those including dissipative effects. One can therefore expect the ‘extra’ irreducible piece \((2)\chi\) to be related to some kind of intrinsic dissipative property of the medium. As we will see in Section 2.7, this is indeed the case. Furthermore, constitutive laws (for matter) with \((2)\chi \neq 0\) (non-vanishing “skewon fields”) have been discussed by Nieves and Pal [47, 48]. They yield \(T\)- and \(P\)-violating terms in the field equations.

Non-abelian axions were postulated for the first time by Peccei and Quinn [55], see also [82, 83]. Abelian axions coupling to electromagnetism were first considered by Ni [44] and correspond to a non-vanishing piece \((3)\chi\). There have been intensive experimental searches for axions, see [46, 66, 75] and references therein. To date, no evidence of such a field has been found. Constraints on the axion mass and coupling to the electromagnetic field have been obtained both from astrophysical observations as from laboratory experiments, see [46, 66, 75] for details. However, as we will see later, the axion-like term does not enter into important quantities as the energy-momentum current and the Fresnel equation. The axion remains a serious candidate for a particle search in experimental high energy physics and is a candidate for cold dark matter. The discussion about the possible existence of such field for some material medium has been rather controversial, see [72, 80, 81] and references therein. In this context the vanishing on the axion piece is referred to as the ‘Post constraint’ (PC), after the work of Post [59]. Sihvola and collaborators have correctly recognized that the axion piece is allowed by the basic structure of electrodynamics. However, if the medium is homogeneous, \(\alpha\) is constant\(^8\) and then the corresponding term drops out completely from the Maxwell field equations. Therefore, a possible axion field can only be detected if it is inhomogeneous (\(d\alpha \neq 0\)), or by its effects on the boundary separating two homogeneous media with different axion fields each. Some theoretical work on the reflection and transmission properties of this kind of medium can be found in [72] and references therein. No clear example of a material medium with a nontrivial axion-like term in its constitutive law has been reported in the literature.

Within a \(\chi\)-\(g\) formalism, i.e., when a (symmetric) constitutive tensor \(\chi\) is assumed for the description of the electromagnetic properties of spacetime in addition to a metric \(g\), astrophysical constraints on ‘nonmetric’ theories, i.e., in which the constitutive tensor differs from the one on a riemannian space, were considered in [45]. The particular case of deviations from the constitutive tensor of Minkowski vacuum is studied in [15, 20, 35].

2.6.2 Three dimensional decomposition of the constitutive tensor

In some particular applications in which a \(3 + 1\) decomposition is used, as for instance in nonrelativistic and/or noncovariant formulations of electrodynamics, see Section 2.6.3 for a particular example, it is convenient to express the constitutive tensor (36 components) in terms

\(^8\) by the very definition of an homogeneous medium
of four $3 \times 3$ matrices $A$, $B$, $C$, and $D$ (each with 9 independent components). We define\(^9\) them as follows:

\begin{align*}
A^{ba} &:= \chi^{0a0b}, \\
B^{ba} &:= \frac{1}{4} \epsilon_{acd} \chi^{cdef} \epsilon_{efb}, \\
C^b &:= \frac{1}{2} \epsilon_{acd} \chi^{cd0b}, \\
D^a &:= \frac{1}{2} \chi^{0a0d} \epsilon_{cdb}.
\end{align*}

(89)

Here $a, b, c, \ldots = 1, 2, 3$. The inverse relations are

\begin{align*}
\chi^{0a0b} &= A^{ba}, \\
\chi^{0abc} &= D^a \epsilon^{abc}, \\
\chi^{a0bc} &= \epsilon^{ab0d} C^d, \\
\chi^{abcd} &= \epsilon^{ab0d} \epsilon^{cdef} B^e F^f.
\end{align*}

(90)

Additionally, one can (3+1)-decompose the coordinate components of excitation and field strength as

\begin{align*}
D^a &:= (H_{23}, H_{31}, H_{12}), \\
\mathcal{H}_a &:= (H_{01}, H_{02}, H_{03}), \\
B^a &:= (F_{23}, F_{31}, F_{12}), \\
E_a &:= (F_{10}, F_{20}, F_{30}).
\end{align*}

(93-94)

With these definitions, the spacetime/constitutive relation can be written as

\begin{align*}
D^a &= -A^{ba} E_b + D^a_b B^b, \\
\mathcal{H}_a &= -C^b_a E_b + B^b_a B^b.
\end{align*}

(95)

From these relations, one sees that the components of $A$ correspond to the components of the usual dielectric tensor $\varepsilon$, whereas the components of $B$ correspond to those of the inverse permeability tensor $\mu$, see (9). Furthermore, using the 6-dimensional notation, we have

\begin{align*}
H_I &= (\mathcal{H}_a, D^a), \\
F_I &= (-E_a, B^a),
\end{align*}

(96)

and the spacetime relation is written as

\begin{equation}
H_I = \epsilon_{IJ} \chi^{JK} F_K,
\end{equation}

with

\begin{equation}
\chi^{IJ} = \begin{pmatrix} B_{ab} & D^a_a \\ C^a_b & A^{ab} \end{pmatrix}, \quad \epsilon_{IJ} = \begin{pmatrix} 0_3 & 1_3 \\ 1_3 & 0_3 \end{pmatrix}.
\end{equation}

(98)

Then the irreducible decomposition of $\chi$ can be found to be

\begin{align*}
(1) \chi^{IJ} &= \begin{pmatrix} B_{(ab)} & D^a_a \\ \frac{1}{2} (\mathcal{P}^b_a + \mathcal{P}^b_a) & A^{(ab)} \end{pmatrix}, \\
(2) \chi^{IJ} &= \begin{pmatrix} B_{[ab]} & D^a_a \\ \frac{1}{2} (\mathcal{D}^b_a - \mathcal{D}^b_a) & A^{[ab]} \end{pmatrix}, \\
(3) \chi^{IJ} &= \frac{1}{6} (C^c_c + D^c_c) \epsilon^{IJ}.
\end{align*}

(99-101)

\(^9\) The conventions in these definitions are taken such that they are consistent with those of [27]
We introduced the 3-dimensional traceless quantities
\( \mathcal{C}^a_b := C^a_b - \frac{1}{3} C^c_c \delta^a_b, \quad \mathcal{D}^a_b := D^a_b - \frac{1}{3} D^c_c \delta^a_b, \)
so that
\( \mathcal{C}^a_a = 0, \quad \mathcal{D}^a_a = 0. \)

In terms of \( S \), see (80), the second irreducible piece of the 3-dimensional matrices (89)-(90) are found to be given by
\[
(2) A^{ab} = \epsilon^{abc} S^0_c, \quad (2) B_{ba} = -\dot{\epsilon}_{abc} S^c_0, \\
(2) C^b_a = -S^b_a + \delta^b_a S^c_c, \quad (2) D^a_b = S^a_b - \delta^a_b S^c_c.
\]
Therefore
\[
(2) \chi^{IJ} = \left( \begin{array}{cc} (2) A_{ab} & (2) D^b_c \\ (2) C^b_a & (2) A^{ab} \end{array} \right) = \left( \begin{array}{cc} \dot{\epsilon}_{abc} S^c_0 + S^b_a - \delta^b_a S^c_c & -\epsilon^{abc} S^0_c \\ -S^a_b + \delta^a_b S^c_c & -\epsilon^{abc} S^0_c \end{array} \right),
\]
and
\[
(2) D^a = \epsilon^{abc} S^0_b E^c + (S^a_b - \delta^a_b S^c_c) B^b, \\
(2) \mathcal{F}_a = \dot{\epsilon}_{abc} S^0_b B^c + (S^a_b - \delta^a_b S^c_c) E_b.
\]

2.6.3 The asymmetric constitutive tensor of Nieves and Pal

Consider the particular case in which \( S^i_j \) is the traceless part of the product of a covector \( \omega \) and a vector \( v \), i.e.,
\[
S^i_j = \omega_i v^j - \frac{1}{4} (\omega_k v^k) \delta^i_j.
\]

Then we have
\[
(2) A^{ba} = \epsilon^{bac} \omega_c, \quad (2) B_{ba} = -\omega_b \dot{\epsilon}_{abc} v^c, \\
(2) C^b_a = -\left[ \omega_a v^b + \frac{1}{2} \delta^b_a (\omega_b v^0 - \omega_c v^c) \right], \\
(2) D^a_b = \omega_a v^b + \frac{1}{2} \delta^a_b (\omega_b v^0 - \omega_c v^c).
\]

As we mentioned, Nieves and Pal [47, 48] discussed P- and T-violating generalizations of the Maxwell-Lorentz equations for isotropic material media. The constitutive tensor they study has a nonvanishing skew-symmetric piece (2) \( \chi \). A 4-dimensional formulation of the constitutive tensor (Nieves and Pal used the 3+1 form of Maxwell’s equations in cartesian coordinates) can be given by considering, in addition to the Minkowski metric \( \eta \), a time-like vector \( v \), which explicitly breaks Lorentz invariance.

Consider, in addition to the usual ‘metric’ piece
\[
(1) \chi^{ijkl} = \sqrt{\frac{\varepsilon_0}{\mu_0}} (\eta^{ij} \eta^{kl} - \eta^{ik} \eta^{jl}) = 2 \sqrt{\frac{\varepsilon_0}{\mu_0}} \eta^{ijkl},
\]
an antisymmetric piece \( \chi \) defined by (79) and (109) for the particular case in which \( \omega \) and \( v \) are, in cartesian coordinates \( \omega_a = (\omega, 0, 0, 0) \) and \( v^a = (v, 0, 0, 0) \). This choice implies in particular that the material will look spatially isotropic in this frame. The corresponding 3-dimensional constitutive matrices \( A_{ab}, B_{ab}, C^a, \) and \( D^a_b \), according to their definitions (89) and (90), are then found to be

\[
A_{ab}^* = -\varepsilon_0 \delta_{ab}, \quad B_{ab}^* = \mu_0^{-1} \delta_{ab}, \quad C^a = -\frac{\omega v c^2}{2} \delta^a, \quad D^a_b = \frac{\omega v c^2}{2} \delta^a_b.
\]

Here we have also used \( \eta_{00}^* = c^2, \eta_{ab}^* = -\delta_{ab}, \eta_{00}^* = c^{-2} \) and \( \eta_{00}^* = -\delta_{00} \).

With these constitutive matrices, the 3+1 decomposition of the constitutive law takes the form

\[
D^a = \varepsilon_0 \delta_{ab} E_b + \frac{\omega v c^2}{2} B^a, \quad H_a = \frac{\omega v c^2}{2} E_a + \mu_0^{-1} \delta_{ab} B^b,
\]

or, in the vector notation used by Nieves and Pal [47, 48],

\[
D^* = \varepsilon_0 E + \gamma B, \quad H^* = \gamma E + \mu_0^{-1} B.
\]

Therefore, the constitutive relation of Nieves and Pal can be recovered provided \( \gamma = \omega v c^2 / 2 \).

Since we have introduced an additional timelike vector, it is natural that the material will violate \( T \) invariance, since \( v \) defines a preferred 'time direction'.

### 2.6.4 Nonlinear constitutive relations

We are mainly interested in linear electrodynamics. However, the formalism developed here can also be applied to nonlinear models, in the case that one concentrates on the properties of small perturbations of the electromagnetic configuration. This is, for instance, the case when one wishes to study propagation of electromagnetic waves in a nonlinear model. Examples of such models are the effective Heisenberg-Euler nonlinear theory [29, 30, 60] and Born-Infeld electrodynamics [5]. In this cases, our linear formalism can be applied by defining an effective constitutive tensor.

Consider an arbitrary local relation \( H = H(F) \). The corresponding Maxwell equation in absence of charges is written, in components, as

\[
\partial_j (\varepsilon^{ijkl} H_{kl}) = 0.
\]

Consider now the properties of a small perturbation \( \Delta F \) of the electromagnetic field around some background configuration \( \bar{F} \). We write the total electromagnetic field strength \( F = \bar{F} + \Delta F \). Then the field excitation can be written, to first order in the perturbation \( \Delta F \), as

\[
H_{ij}(F) = H_{ij}(\bar{F}) + \frac{1}{2} \frac{\partial H_{ij}}{\partial F_{kl}} \bigg|_{\bar{F}} \Delta F_{kl}.
\]

Inserting (119) into (118) and assuming that the background field \( \bar{F} \) is a solution of (118), i.e. \( \partial_j \left( \varepsilon^{ijkl} H_{kl}(\bar{F}) \right) = 0 \), we obtain an equation for the perturbation:

\[
\partial_j \left( \frac{1}{2} \varepsilon^{ijkl} \frac{\partial H_{kl}}{\partial F_{mn}} \bigg|_{\bar{F}} \Delta F_{mn} \right) = 0.
\]
We can write this equation in the same form as in the linear case, i.e. as

\[ \partial_j \left( \chi_{ijkl}^{\text{eff}} \Delta F_{kl} \right) = 0, \]  

(121)

with the 'effective constitutive tensor'

\[ \chi_{ijkl}^{\text{eff}} := \frac{1}{2} \epsilon_{ijmn} \frac{\partial H_{mn}}{\partial F_{kl}} |_{\bar{F}}. \]  

(122)

The tensor \( \chi_{ijkl}^{\text{eff}} \) will, in general, depend on the local constitutive law and on the background field \( \bar{F} \). This result shows that most of the results obtained for linear constitutive/spacetime relations can also be applied to every local electromagnetic theory provided one considers perturbations on some background solution. This is the case, for instance, of the propagation of waves in nonlinear media.

### 2.7 Symmetries and energy-momentum

#### 2.7.1 Symmetry of a linear medium

Here we define the concept of a symmetry of an electromagnetic medium. This definition applies to the linear case, when a constitutive tensor \( \chi \) is available.

To motivate our definition, consider the particular case in which the field configuration is such that the Lie-derivative of the electromagnetic field strength along some vector field \( \xi \) vanishes, i.e.

\[ \mathcal{L}_\xi F = 0. \]  

(123)

The geometric interpretation of the condition (123) is clear. It means that the electromagnetic field strength \( F \) is 'constant' along the direction \( \xi \). More precisely, the integral \( \int_{\Omega_2} F \) on some 2-dimensional surface \( \Omega_2 \) (which is the natural physical quantity associated to the 2-form \( F \)) is invariant under the displacement of the integration region \( \Omega_2 \) induced by \( \xi \). This property follows from the very definition of the Lie derivative, see, for instance, the discussion in [59] and Fig. 4 for \( \Psi = F \) and \( n = \xi \).

Now, in linear electrodynamics the excitation is determined by the field strength \( F \) and the constitutive tensor \( \chi \). It is clear that even if (the integral of) the field strength \( F \) is constant under the displacements defined by \( \xi \), i.e., if (123) holds, the excitation will not be constant unless the medium itself, i.e. the constitutive tensor, satisfies some condition. We will call this condition a symmetry condition for the medium. Clearly, the condition we are referring to is the vanishing of the Lie derivative of the constitutive tensor, since in this case (123) implies that \( H \) is also constant, i.e. \( \mathcal{L}_\xi H = 0 \). This lead us to the following definition:

**Definition:** A linear electromagnetic medium is said to have a symmetry under the displacement induced by a vector field \( \xi \) if the Lie derivative along \( \xi \) of its constitutive tensor vanishes, i.e. if

\[ \mathcal{L}_\xi \chi_{ijkl} = 0, \]  

(124)

for some vector field \( \xi \).

Notice that this condition implies the independent vanishing of the Lie derivative of each irreducible piece of the constitutive tensor, see the general decomposition formula (72). From
this definition and the property about the noncommutativity of the Lie derivative, it follows that if $\xi_1$ and $\xi_2$ are two vector fields describing symmetries of the medium, then the new vector $[\xi_1, \xi_2]$ is also a symmetry of the medium.

This immediately raises questions analogous to those one encounters in the study of isometries in GR, namely a) What is the maximum number $n_{\text{max}}$ of symmetries that a constitutive tensor allows? And then, assuming that $n_{\text{max}}$ is finite, b) What is the form of a ‘maximally symmetric’ constitutive tensor?, i.e., of a constitutive tensor allowing the maximum number of symmetries?

It can be speculated that the maximally symmetric constitutive tensor can correspond to $\chi = \chi_{\{\eta\}}$, i.e., to the (conformal) Minkowski vacuum, since we know that it is a highly symmetric case, see also the particular case below. This result would be interesting since it would provide a further way to ‘derive’ the vacuum spacetime relation by postulating that it has to be maximally symmetric. These questions will be investigated in the future.

2.7.2 Riemannian case

Here I show that in the special case for which $\chi = \chi_{\{g\}}$, i.e., for the ‘vacuum’ on a riemannian space\(^{10}\), the equation (124) defining the symmetries of the medium is equivalent to the conformal Killing equation for the metric $g$, i.e. to

$$\mathcal{L}_\xi \left(|g|^{1/4} g^{ij}\right) = 0,$$

or, explicitly

$$g^{m_j} \partial_m \xi^i + g^{m_m} \partial_m \xi^j + \xi^m \partial_m g^{ij} = \frac{1}{4} g^{ij} \left[2 \partial_m \xi^m + \xi^m g_{kl} \partial_m g^{kl}\right].$$

One proves this as follows. From (223) one finds

$$\mathcal{L}_\xi \chi^{ijkl}_{\{g\}} = 2 \left(|g|^{1/4} g^{k[i]}\right) \mathcal{L}_\xi \left(|g|^{1/4} g^{i]}l\right) + 2 \left(|g|^{1/4} g^{ij}\right) \mathcal{L}_\xi \left(|g|^{1/4} g^{ij}\right).$$

Contracting (127) with $\left(|g|^{-1/4} g_{ij}\right)$, one obtains

$$\left(|g|^{-1/4} g_{ij}\right) \mathcal{L}_\xi \chi^{ijkl}_{\{g\}} = g^{ik} g_{jl} \mathcal{L}_\xi \left(|g|^{1/4} g^{jl}\right) + 2 \mathcal{L}_\xi \left(|g|^{1/4} g^{ik}\right),$$

so that the condition (124) implies

$$g^{ik} g_{jl} \mathcal{L}_\xi \left(|g|^{1/4} g^{jl}\right) + 2 \mathcal{L}_\xi \left(|g|^{1/4} g^{ik}\right) = 0.$$

Contracting this equation with $g_{ik}$ one finds $g_{ij} \mathcal{L}_\xi \left(|g|^{1/4} g^{ij}\right) = 0$ which, when substituted back into (129), results in the conformal Killing equation for the metric $g$, namely (125). This means that the conformal Killing equation is a necessary condition for $\xi$ to be a symmetry of the medium. On the other hand, from (127) one sees that if $\xi$ is a conformal Killing vector then (125) is automatically satisfied. Therefore, (125) is also a sufficient condition.

In other words, in the riemannian case, all symmetries of the medium are conformal symmetries of the metric.

\(^{10}\) or a material medium in Minkowski space with an ‘effective optical metric’ $g$
2.7.3 Conservation of energy-momentum

In physics conservation laws play a central role. Among other properties, they allow the definition of conserved quantities in terms of which the description of the system and its evolution becomes simpler. In particular, energy and momentum are quantities associated to any field. In GR and similar theories they are the source of the gravitational field.

In field theory, the energy-momentum distribution and its flow are described by a covector valued 3-form which, in the electromagnetic case we will denote as $\Sigma^\alpha$. Usually, one also needs an additional vector field $\xi$ for constructing a frame-independent 3-form $Q := \Sigma^\alpha \xi^\alpha$ following the general scheme ‘conserved quantity’ $\sim$ ‘momentum’ $\times$ ‘vector field’. The vector field is usually related to some symmetry of the system. As an example of this general scheme recall the case of GR. There, a conserved quantity along the trajectory of a free test particle (i.e., a particle moving along a geodesic) can be defined provided the spacetime admits a symmetry. This symmetry is described by a Killing vector field $\xi_K$, i.e., an isometry of the spacetime metric, so that $L_{\xi_K} g_{ij} = 0$. The corresponding conserved quantity is then given by the projection (contraction) of the momentum of the test particle of mass $m$, $p_i := mg_{ij} dx^j / d\tau$, along the direction of the Killing vector i.e. $q := p_i \xi^i_K$.

Back to our original problem, the main property that $Q$ has to fulfill in order to be interpreted as energy-momentum is that it has to be related to the corresponding force law through a derivative of the form $\xi^\alpha f_\alpha \sim dQ$. If, for instance, $\xi$ happens to be a 4-velocity field, $\xi^\alpha f_\alpha$ can be interpreted as the rate of energy transfer to the particles via the Lorentz force. A second condition on $Q$ is that it must be conserved under some conditions, i.e. $dQ = 0$. One expects again the condition for energy-momentum conservation to be related to the symmetries of the system, at least when a Lagrangian is available, as we know from the Noether theorem. In linear electrodynamics this latter can happen when the irreducible piece $\chi^{(2)}$ of the constitutive tensor vanishes.

Therefore, to find an adequate energy-momentum $\Sigma_{\alpha}$ for the electromagnetic field, we try to express $\xi^\alpha f_\alpha$ as a total derivative of some 3-form $dQ$, with $Q := \Sigma_{\alpha} \xi^\alpha$.

First, we use the expression for the Lorentz force law (37), replace the current $J$ from the inhomogeneous Maxwell equation (35) and ’partially integrate’. One finds

$$
\xi^\alpha f_\alpha &= \xi^\alpha (e_\alpha | F) \wedge J \\
&= (\xi | F) \wedge J \\
&= (\xi | F) \wedge dH \\
&= -d[(\xi | F) \wedge H] + H \wedge d(\xi | F) \\
&= -d[(\xi | F) \wedge H] + H \wedge (L_\xi F).
$$

(130)

In the last step we have used the definition of the Lie derivative of $F$ and the homogeneous Maxwell equation, $dF = 0$.

The choice of $Q$ that equation (130) suggests, namely $Q = -(\xi | F) \wedge H$, is not what we are looking for since it would lead to conservation of $Q$ only if the field strength is constant in the direction $\xi$, i.e. if $L_\xi F = 0$, so that the last term in (130) vanish. But this is clearly a too restrictive condition.

11 Momentum is always a 1-form, i.e., a covector, see, for instance, Fig. 2 in [59]
To improve the situation we use now the fact that (130) can be rewritten as
\[ \xi^K F = d[Z - (\xi) F] + H \wedge (\mathcal{L}_\xi F) - dZ, \] (131)
where \( Z \) is an arbitrary 3-form.

To be sure that one is covering all the available possibilities, one should consider the most general 3-form \( Z \) that can be constructed using the available objects. In our case, they are only \( F, H, \) and \( \xi \). Under these conditions, \( Z \) can only be a linear combination of the terms \((\xi) F\) \& \( H \) and \((\xi) H\) \& \( F \). Therefore, we make the ansatz
\[ Z = a_1 (\xi) F \wedge H + a_2 (\xi) H \wedge F, \] (132)
with arbitrary constants \( a_1 \) and \( a_2 \). This implies
\[
\begin{align*}
dZ &= a_1 d(\xi) F \wedge H - a_1 (\xi) F \wedge dH + a_2 d(\xi) H \wedge F - a_2 (\xi) H \wedge dF \\
  &= a_1 (\mathcal{L}_\xi F) \wedge H - a_1 (\xi) F \wedge J + a_2 (\xi) H \wedge F \\
  &= a_1 (\mathcal{L}_\xi F) \wedge H - a_1 (\xi) f_\alpha + a_2 (\mathcal{L}_\xi H - (\xi) dH) \wedge F \\
  &= a_1 (\mathcal{L}_\xi F) \wedge H - a_1 (\xi) f_\alpha + a_2 (\mathcal{L}_\xi H) \wedge F - a_2 (\xi) J \wedge F \\
  &= a_1 (\mathcal{L}_\xi F) \wedge H + a_2 (\mathcal{L}_\xi H) \wedge F + (a_2 - a_1) (\xi) f_\alpha.
\end{align*}
\] (133)

Thus, we rewrite (130) as
\[
\xi^K f_\alpha = d[(a_1 - 1) (\xi) F \wedge H + a_2 (\xi) H \wedge F] + (1 - a_1) H \wedge (\mathcal{L}_\xi F) - a_2 (\mathcal{L}_\xi H) \wedge F - (a_2 - a_1) (\xi) f_\alpha.
\] (134)

This expression, which is an identity as soon as \( F \) and \( H \) satisfy the Maxwell equations and \( f_\alpha \) is given by (37), leads us to define
\[
\begin{align*}
Q &:= \frac{1}{1 - a_1 + a_2} [(a_1 - 1) (\xi) F \wedge H + a_2 (\xi) H \wedge F], \\
X &:= \frac{1}{1 - a_1 + a_2} [(1 - a_1) H \wedge (\mathcal{L}_\xi F) - a_2 (\mathcal{L}_\xi H) \wedge F],
\end{align*}
\] (135, 136)
so that
\[ \xi^K f_\alpha = dQ + X. \] (137)

The identity (137) summarizes all the possibilities to write \( \xi^K f_\alpha \) as a total derivative of the available fields plus some rest. The problem now is to find suitable values for \( a_1 \) and \( a_2 \) so that \( Q \) and \( X \) satisfy our requirements. In particular, we need \( X \) to vanish under some reasonable circumstances.

Up to here our results in this section are valid for any electromagnetic medium, since the spacetime/constitutive relation has not been used. To make up our minds about a reasonable choice of the constants it is useful to study the particular case of linear electrodynamics and compute \( X \) explicitly in terms of the field strength and the irreducible pieces of the constitutive tensor.

12 In principle, one could also use the potential \( A \) as a further object. However, it seems that no 3-form linear in \( \xi \), as required by the left hand side of (131), can be constructed using this 1-form together with \( F \) and \( H \).
Furthermore, we introduce the abbreviation $a := \frac{1}{4} (1 - a_1 + a_2)$ and compute

$$X = a [(1 - a_1) H_{ij} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi H_{ij}) F_{kl}] dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$
$$= a [(1 - a_1) H_{ij} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi H_{ij}) F_{kl}] \epsilon^{ijkl} \hat{e}$$
$$= 2a [(1 - a_1) H^{kl} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi H^{kl}) F_{kl}] \hat{e}$$
$$= a [(1 - a_1) \chi^{klij} F_{ij} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi \chi^{klij} F_{ij}) F_{kl}] \hat{e}$$
$$= a [(1 - a_1) \chi^{klij} F_{ij} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi \chi^{klij} F_{ij}) F_{kl}] \hat{e}$$
$$= a [(1 - a_1) \chi^{klij} - a_2 \chi^{ijkl}] F_{ij} (\mathcal{L}_\xi F_{kl}) - a_2 (\mathcal{L}_\xi \chi^{klij}) F_{ij} F_{kl}] \hat{e}$$
$$= a [(1 - a_1 - a_2) (1) \chi^{klij} + 3 \chi^{klij}) F_{ij} (\mathcal{L}_\xi F_{kl})$$
$$+ (1 - a_1 + a_2) (2) \chi^{klij} F_{ij} (\mathcal{L}_\xi F_{kl})$$
$$- a_2 \mathcal{L}_\xi (1) \chi^{klij} + 3 \chi^{klij}) F_{ij} F_{kl}] \hat{e}. \quad (138)$$

Therefore, we find

$$X = \frac{1}{4} F_{ij} \left[\frac{(1 - a_1 - a_2)}{(1 - a_1 + a_2)} (1) \chi^{klij} + 3 \chi^{klij}) (\mathcal{L}_\xi F_{kl}) + (2) \chi^{klij} (\mathcal{L}_\xi F_{kl})$$
$$- \frac{a_2}{(1 - a_1 + a_2)} \mathcal{L}_\xi (1) \chi^{klij} + 3 \chi^{klij}) F_{ij} F_{kl}] \hat{e}. \quad (139)$$

From this result, one recognizes first that no choice of $a_1$ and $a_2$ makes $X$ to vanish in general. Thus, no conservation law is possible, unless additional conditions are fulfilled. This is, however, what one expects. One also sees that the irreducible pieces enter differently in (139). In particular, notice that the factor $(1 - a_1 + a_2)$ in front of the irreducible piece $\chi^{klij}$ cancels completely. This means that the term $\chi^{klij} F_{ij} (\mathcal{L}_\xi F_{kl})$ will always be present in the decomposition of $X$, no matter which values of $a_1$ and $a_2$ one chooses. In other words, the irreducible piece $\chi^{klij}$ of the constitutive tensor will always prevent $Q$ from being a conserved quantity (in charge-free regions), unless of course a much more restrictive condition, like $\mathcal{L}_\xi F = 0$, is satisfied by the field configuration. This result is consistent with our interpretation of $\chi^{klij}$ as related to intrinsic dissipative properties of the medium.

The ‘best’ one can do is to choose

$$1 - a_1 - a_2 = 0, \quad (140)$$

so that the terms proportional to the symmetric irreducible pieces $\chi^{klij}$ and $\chi^{klij}$ in the first term on the right hand side of (139) vanish. In this case, after substituting $a_2 = 1 - a_1$ into (135) and (136), one finds that $Q$ and $X$ do not depend on $a_1$, so that one obtains a unique result, namely

$$Q = \frac{1}{2} [\xi H (\mathcal{L}_\xi F) - (\xi F) \wedge H], \quad (141)$$

$$X = \frac{1}{2} [H (\mathcal{L}_\xi F) - (\mathcal{L}_\xi H) \wedge F]. \quad (142)$$
Furthermore, for linear electromagnetic media we obtain, from (139),

\[X = -\frac{1}{8} F_{ij} \left[ 2 (2) \chi^{ijkl} (\mathcal{L}_\xi F_{kl}) + \mathcal{L}_\xi \left( (1) \chi^{ijkl} + (3) \chi^{ijkl} \right) F_{kl} \right] \hat{\epsilon}.
\] (143)

That the choice (141) and (142) is reasonable can be seen as follows. Consider the case in which \((2) \chi = 0\) so that a Lagrangian is available for the description of the system. In a charge-free region, (143) shows that \(Q\) is conserved provided \(\xi\) is a symmetry of the medium, i.e., \(\mathcal{L}_\xi \chi^{klij} = 0\). Furthermore, our choice agrees with the results valid in a Riemannian space.

Notice also that \(Q\) does not depend on the axion-like piece \((3) \chi\) entering the linear constitutive law. This can directly be seen using (141) and (85), since

\[
\left( \xi \right)^{(3)} H \wedge F - (\xi) F \wedge (\xi) H = (\xi) (\alpha F) \wedge F - (\xi) F \wedge (\alpha F)
\]

\[= \alpha (\xi) F \wedge F - \alpha (\xi) F \wedge F = 0.
\] (144)

However, the axion-like piece does appear in (143) and therefore can contribute to the non-conservation of \(Q\) provided \(\mathcal{L}_\xi \alpha \neq 0\).

2.7.4 Axiom 4: Energy-momentum tensor

The results in Section 2.7.3 show how to construct a 3-form which for a linear electromagnetic medium leads to conservation laws, under reasonable conditions. We take these results and generalize them to our fourth axiom, by means of which the kinematic energy-momentum 3-form \(k \Sigma_\alpha\) of any electromagnetic medium is defined by

\[k \Sigma_\alpha := \frac{1}{2} \left[ F \wedge (e_\alpha \wedge H) - H \wedge (e_\alpha \wedge F) \right].
\] (145)

Then the 3-form \(Q\) of Section 2.7.3 is given by \(Q = k \Sigma_\alpha \xi^\alpha\).

The adjective ‘kinematic’ is used here to emphasize that our definition of \(k \Sigma_\alpha\) was not based on dynamical properties of the system, but rather on kinematic arguments.

One can also now define the components \(k T_\alpha^\beta\) of the kinetic energy-momentum tensor density by

\[\vartheta^\beta \wedge k \Sigma_\alpha := k T_\alpha^\beta \hat{\epsilon},
\] (146)

where \(\hat{\epsilon} := \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3\). In particular, in a coordinate basis, with \(\mathcal{H}^{ij} := \epsilon^{ijkl} H_{kl}/2\),

\[k T_i^j = \frac{1}{4} \delta_i^j F_{kl} H^{kl} - F_{ik} \mathcal{H}^{jk}.
\] (147)

For a general electromagnetic medium, we summarize our results as

\[\xi^\alpha f_\alpha = dQ + X,
\] (148)

with

\[Q := k \Sigma_\alpha \xi^\alpha,
\] (149)

\[X = \frac{1}{2} \left[ H \wedge (\mathcal{L}_\xi F) - (\mathcal{L}_\xi H) \wedge F \right].
\] (150)
For a linear electromagnetic medium, we already showed that the axion-like piece \((3)\chi\) does not contribute to the kinematic energy-momentum tensor, i.e.,

\[
\kappa \Sigma_\alpha (\chi) = \kappa \Sigma_\alpha \left( (1)\chi + (2)\chi + (3)\chi \right) = \kappa \Sigma_\alpha (1)\chi + (2)\chi .
\] (151)

Additionally, we have shown, see (139) and (148), that the corresponding conservation law takes the form

\[
dQ = f_\alpha \xi^\alpha + \frac{1}{8} F_{ij} \left[ L_\xi \left( (1)\chi_{ijkl} + (3)\chi_{ijkl} \right) F_{kl} + 2(2)\chi_{ijkl} \left( L_\xi F_{kl} \right) \hat{\epsilon} \right],
\] (152)

for any vector field \(\xi\). In this equation the contributions of the three irreducible pieces \((1)\chi\), \((2)\chi\), and \((3)\chi\) are isolated from each other.

From these results, a consistent interpretation of (152) can be given as a ‘balance equation’ for the energy-momentum content of the electromagnetic field.

Recall first that even if \(\xi\) is a symmetry of the medium, in which case the second and third terms on the right hand side of (152) vanish, there is still a contribution proportional to the irreducible piece \((2)\chi\) of the constitutive tensor. This is interpreted as an intrinsic dissipative property of the medium generated by the irreducible piece \((2)\chi\).

In the case that \(\xi\) is a 4-velocity field the result above can be interpreted as an ‘energy balance’ equation for the change of the total electromagnetic energy. The integration of (152) over a 4-dimensional region of the form \(\Omega_4 = \Sigma \times [\sigma_0, \sigma]\) will produce\(^{13}\) at the left hand side the ‘change of the total energy of the electromagnetic field’ between the ‘times’ \(\sigma_0\) and \(\sigma\), i.e., \(\int_{\Sigma} Q - \int_{\Sigma_{\sigma_0}} Q\). The first term on the right hand side, \(f_\alpha \xi^\alpha\), will lead to the energy transferred from the electromagnetic field to the test particles via the Lorentz force, i.e., \(\int_{\Omega_4} f_\alpha \xi^\alpha\). The term proportional to \((2)\chi\) can be then interpreted as the rate at which the energy of the electromagnetic field is dissipated, \(\frac{1}{2} \int_{\Omega_4} F_{ij}^2 \chi_{ijkl} \left( L_\xi F_{kl} \right) \hat{\epsilon}\). The other terms will be proportional to the ‘time’ derivative of the constitutive tensor, \(\frac{1}{2} \int_{\Omega_4} F_{ij} L_\xi \left( (1)\chi_{ijkl} + (3)\chi_{ijkl} \right) F_{kl} \hat{\epsilon}\). When the medium is ‘time’ independent this term vanishes. If the time derivative does not vanish, it means that the properties of the medium are changing in ‘time’. Therefore, one could try to interpret this term as the energy per unit time needed to change the material properties\(^{14}\).

3 Wave propagation

We turn our attention to the wave propagation properties in our general framework of linear pre-metric electrodynamics.

Consider a region in spacetime without charges, i.e., \(J = 0\). Maxwell’s equations take then the form

\[
dF = 0 \quad dH = 0,
\] (153)

completed by the linear spacetime/constitutive relations (70). These equations will allow for solutions propagating in spacetime, the behavior of which is determined by the electromagnetic properties of the spacetime/medium.

\(^{13}\) We assume that the fields vanish at \(\partial \Sigma_\alpha\) (‘spatial infinity’)

\(^{14}\) Maybe with a change of some suitably defined ‘internal energy’ of the medium
Fig. 7 Characteristic surface for propagation of electromagnetic disturbances. The 1-form $q := d\Phi$ corresponds geometrically to two small parallel planes tangent to $S$ at each point $x$.

Since the constitutive tensor can have a very complicated spacetime dependence, many specific features of the propagation of waves over finite distances cannot be studied in general terms here. However, an important local property of the propagation of waves, namely the dispersion relation that the covectors tangent to a wave front must satisfy, can be derived in general. This is done by deriving the so-called Fresnel equation for the wave covectors.

### 3.1 Propagation of singularities

In the theory of partial differential equations, the propagation of waves is described by Hadamard discontinuities of solutions across a characteristic (wave front) hypersurface $S$ [17]. One can locally define $S$ by the equation $\Phi(x^i) = \text{const}$, for some function $\Phi(x)$. The Hadamard discontinuity of any function $F(x)$ across the hypersurface $S$ is defined as the difference $[F]_S(x) := F(x_+) - F(x_-)$, where $x_\pm := \lim_{\varepsilon \to 0} (x \pm \varepsilon)$ are points on the opposite sides of $S \ni x$. An electromagnetic wave is described as a solution of the vacuum Maxwell equations (153) for which the derivatives of $H$ and $F$ have discontinuities across the wave front hypersurface $S$.

Thus, in terms of field components, we have on the characteristic hypersurface $S$,

\begin{align}
[F_{ij}]_S &= 0, & [\partial_i F_{jk}]_S &= q_i f_{jk}, \\
[H_{ij}]_S &= 0, & [\partial_i H_{jk}]_S &= q_i h_{jk},
\end{align}

(154)

(155)

where $f_{ij}$ and $h_{ij}$ are the components of the 2-forms $f$ and $h$ describing the corresponding jumps of the derivatives of field strength and excitation across $S$, respectively, and the covector tangent to the wave front is given by

\[ q := d\Phi = q_i dx^i. \]

(156)

Notice that (154) and (155), taken together, are covariant conditions. In particular, although (154b) and (155b) may not look covariant at first sight, they are, provided (154a) and (155a)
are also satisfied. This can be seen by considering the transformation law of $\partial_i F_{jk}$ under coordinate transformations. Consider the coordinate transformation $x^i \to x'^i$. Then we have

$$\partial_i' F_{j'k'} = \frac{\partial}{\partial x'^{i}} \left( \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x^k}{\partial x'^{k'}} \right) F_{jk} + \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x^k}{\partial x'^{k'}} \partial_i F_{jk}. \tag{157}$$

Assuming that the coordinate transformation is smooth across $S$, i.e. $[\partial x^i/\partial x^i]_S = 0$, $[\partial^2 x'/\partial x^i\partial x^i]_S = 0$, we find that the jump of $\partial_i F_{jk}$ across $S$ is given by

$$[\partial_i' F_{j'k'}]_S = \frac{\partial}{\partial x'^{i}} \left( \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x^k}{\partial x'^{k'}} \right) [F_{jk}]_S + \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x^k}{\partial x'^{k'}} [\partial_i F_{jk}]_S, \tag{158}$$

which reduces to the transformation law of a tensor field, provided (154a) is satisfied, i.e.

$$[\partial_i' F_{j'k'}]_S = \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x^k}{\partial x'^{k'}} [\partial_i F_{jk}]_S. \tag{159}$$

We use (154), (155) and the Maxwell equations (153) to find the compatibility conditions that the field discontinuities most satisfy, and find

$$q \wedge h = 0, \quad q \wedge f = 0. \tag{160}$$

This ensures that the Maxwell equations (153) are satisfied. The Hadamard method used here is equivalent to the usual geometric optics limit made by expanding a solution for the electromagnetic potential of the form $A = ae^{i\Phi}$. The covector $q := d\Phi$ corresponds thus to the wave covector.

Assuming now that the constitutive tensor is regular on $S$ (in a geometric optics limit this means that we assume the scale of variations of the constitutive tensor to be much bigger than the scale of variation of the wave field) we obtain from (70) the corresponding relation between $h$ and $f$, namely

$$h = \# f. \tag{161}$$

Substituting (161) into (160a) and using (160b), we see that the equations (160) reduce to

$$q \wedge \# f = 0, \quad q \wedge f = 0. \tag{162}$$

Now, the general solution of (162b) is of the form

$$f = q \wedge a, \tag{163}$$

where the 1-form $a$ is defined up to ‘gauge transformations’

$$a \to a + \varphi q, \tag{164}$$

for any scalar function $\varphi$. This is an algebraic consequence of the gauge freedom in the definition of the electromagnetic potential. Then the compatibility condition (162a) reduces to

$$q \wedge \# (q \wedge a) = 0. \tag{165}$$
Furthermore, in order to allow for solutions, the fields \( f \) and \( h \) must satisfy some ‘integrability’ conditions. Since the form of the general solutions of (160) is \( f = q \wedge a \) and \( h = q \wedge b \), we find

\[
f \wedge f = 0, \quad h \wedge h = 0, \quad f \wedge h = 0,
\]

or, in components,

\[
\epsilon^{ijkl} f_{ij} f_{kl} = 0, \quad \epsilon^{ijkl} h_{ij} h_{kl} = 0, \quad \epsilon^{ijkl} f_{ij} h_{kl} = 0.
\]

### 3.2 Fresnel equation

As has been noticed before, see [51], not all the equations in (165) are independent. This fact makes the derivation of the Fresnel equation more involved.

Let us isolate the trivial parts of (165). We use a covector basis \( \vartheta^\alpha \) in order to write the covector \( a \) in terms of its frame components \( a^\alpha \) as \( a = a^\alpha \vartheta^\alpha \) and choose the covector \( \vartheta^0 \) as the covector \( q \), i.e. \( \vartheta^0 = q \). This can, of course, always be done at each point of spacetime, since we assume \( q \neq 0 \). With this choice, a gauge transformation (164) can then be completely accounted for by a transformation \( a_\hat{0} \rightarrow a_\hat{0} + \phi \). This means that the other frame components \( a_A, A, B, \ldots = \hat{1}, \hat{2}, \hat{3}, \) are gauge invariant. Since the term coming from the basis covector \( \vartheta^\hat{0} \) identically vanishes, i.e. \( q \wedge \# (q \wedge \vartheta^B) a_B \equiv 0 \), (165) can be written as

\[
q \wedge \# (q \wedge \vartheta^\hat{0}) a_B = 0.
\]

Furthermore, only 3 of the 4 components in the 3-form (168) are non-trivial. Multiplying (168) by \( \vartheta^\hat{0} \), one finds

\[
\vartheta^\hat{0} \wedge q \wedge \# (q \wedge \vartheta^\hat{0}) a_B \equiv 0.
\]

The remaining 3 components are

\[
\vartheta^A \wedge q \wedge \# (q \wedge \vartheta^B) a_B = 0.
\]

This equation can be written as

\[
\hat{\epsilon} W^{AB} a_B = 0,
\]

with a volume element \( \hat{\epsilon} \) that we choose to be \( \hat{\epsilon} := \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \) (it doesn’t matter very much which 4-form one uses, since the equation is homogeneous) and a \( 3 \times 3 \) matrix \( W^{AB} \). Using (87) and (170) we compute

\[
\vartheta^A \wedge q \wedge \# (q \wedge \vartheta^B) = \frac{1}{4} \vartheta^A \wedge \vartheta^\hat{0} \wedge \left( \epsilon_{\alpha \beta \gamma \delta} \gamma^\delta \vartheta^\alpha \delta^\beta_\delta \vartheta^\gamma_{\delta B} \right) \vartheta^\hat{0} \wedge \vartheta^\hat{1} \wedge \vartheta^\hat{2} \wedge \vartheta^\hat{3}
\]

\[
= \frac{1}{4} \epsilon^{\alpha \beta \gamma \delta} \delta^\beta_\delta \vartheta^\alpha \gamma^\delta \vartheta^\gamma_{\delta B} \vartheta^\hat{0} \wedge \vartheta^\hat{1} \wedge \vartheta^\hat{2} \wedge \vartheta^\hat{3}
\]

\[
= \delta^{A}_{[\gamma \delta]} \gamma^\delta \vartheta^\hat{0} \vartheta^\hat{B} \hat{\epsilon}
\]

\[
= \chi^{A \hat{0} \hat{B}} \hat{\epsilon}
\]

\[
= -\chi^{A \hat{0} \hat{B}} \hat{\epsilon}.
\]
Thus, since the negative sign is unimportant, we can define
\[ W^{AB} := \chi^{\delta\delta 0} B. \quad (173) \]

The corresponding Fresnel equation is then obtained from the vanishing of the determinant of \( W \), as the necessary and sufficient condition for existence of non-vanishing solutions \( a_B \). Now, since \( W \) is a \( 3 \times 3 \) matrix, its determinant is given by
\[
W := \det(W) \\
= \frac{1}{3!} \epsilon_{ABC} \epsilon_{DEF} W^{AD} W^{BE} W^{CF} \\
= \frac{1}{3!} \epsilon_{ABC} \epsilon_{DEF} \chi^{\delta\delta 0 D} \chi^{\delta\delta 0 E} \chi^{\delta\delta 0 F}. \quad (174)
\]

We apply now the following procedure to rewrite \( W \) as a fully 4-dimensional covariant expression. We ‘complete’ the 3-dimensional Levi-Civita symbols \( \epsilon_{ABC} \) to obtain the 4-dimensional one, by first using \( \epsilon_{ABC} \equiv \hat{\epsilon}_{0ABC} \) and then taking one of the \( \hat{0} \)-components of the constitutive tensors as fourth summation index. This leads us to consider the following identity:
\[
\hat{\epsilon}_{0ABC} \chi^{\delta\delta 0 D} \chi^{\delta\delta 0 E} \chi^{\delta\delta 0 F} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \chi^{\alpha\beta} \chi^{\gamma\delta} \chi^{\delta\delta}. \quad (175)
\]

This identity holds because on the right hand side of (175), due to the properties of the Levi-Civita symbol, one of the indices \( \alpha, \beta, \gamma \) or \( \delta \) must be zero, but on the other hand only \( \alpha \) and \( \beta \) will contribute, due to the (anti)symmetry properties of the constitutive tensor, see (71), which would otherwise make one of the two last \( \chi \)-factors vanish. Finally, the two remaining contributions are equal, canceling the factor 1/2 and proving the identity. This allows us to rewrite (174) as
\[
W = \frac{1}{3!} \frac{1}{2} \hat{\epsilon}_{\alpha\beta\gamma\delta} \hat{\epsilon}_{DEF} \chi^{\alpha\beta\gamma\delta} \chi^{\delta\delta}. \quad (176)
\]

We repeat the same procedure to complete the remaining 3-dimensional Levi-Civita symbol. Now, we can use the following identity:
\[
\hat{\epsilon}_{0DEF} \chi^{\alpha\beta\gamma\delta} \chi^{\gamma\delta\gamma\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta}. \quad (177)
\]

Using (177) in (176), we finally obtain
\[
W = \frac{1}{4!} \hat{\epsilon}_{\alpha\beta\gamma\delta} \hat{\epsilon}_{\gamma\delta\gamma\delta} \chi^{\alpha\beta\gamma\delta} \chi^{\gamma\delta\gamma\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta} \chi^{\delta\delta}. \quad (178)
\]

Since \( q^{\hat{0}} = q = q_i \, dx^i \), the above result can be written in coordinate components as
\[
W = \theta^2 \epsilon_{mnpq} \hat{\epsilon}_{rstu} \chi^{mnr} \chi^{jpsk} \chi^{lqtu} q_i q_j q_k q_l, \quad (179)
\]

with \( \theta := \det(e^a_i) \). We define the fourth order tensor density of weight +1, the ‘Fresnel tensor’ \( G^{ijkl} \), as
\[
G^{ijkl} := \frac{1}{4!} \epsilon_{mnpq} \hat{\epsilon}_{rstu} \chi^{mnr} \chi^{jpsk} \chi^{lqtu} q_i q_j q_k q_l, \quad G^{ijkl} = G^{(ijkl)}, \quad (180)
\]
which has 35 independent components. Then the Fresnel equation can finally be written as

$$G^{ijkl} q_i q_j q_k q_l = 0.$$  \hfill (181)

At each point, the Fresnel equation (181) defines in the space of wave vectors the wave (co-)vector surface, see for instance [38]. I would like to emphasize the generality of the above result. The Fresnel equation (181) is valid for any linear electromagnetic medium. This means for media that in general are inhomogeneous, anisotropic, and dissipative. As we have already discussed, this result can even be applied to study the properties of propagation of perturbation of any local medium, by using an effective constitutive tensor, see [53]. Furthermore, the above result is, as the whole formalism, generally covariant.

The general Fresnel equation (181) is in general a quartic equation in $q_i$ despite the fact that it was derived from a determinant of a $3 \times 3$ matrix quadratic in the wave vectors. This is because the remaining quartic term is multiplied by the trivial factor $q^2$, see (179). This corrects Denisov & Denisov [14] who claim that a particular case of the general linear constitutive law may yield a sixth order Fresnel equation.

### 3.2.1 Coordinate $3+1$ decomposition of the Fresnel equation

We rewrite now our general result (179) by performing a $3+1$ coordinate decomposition, and obtain

$$W = \theta^2 \left( q^a M + q^a q_b M^{ab} + q^a q_b q_c M^{abc} + q^a q_b q_c q_d M^{abcd} \right),$$  \hfill (182)

with

$$M := G^{0000}, \quad M^a := 4G^{000a}, \quad M^{ab} := 6G^{00ab}, \quad M^{abc} := 4G^{0abc}, \quad M^{abcd} := G^{abcd},$$  \hfill (183)

or explicitly in terms of the $3 \times 3$ matrices defined in (89) and (90):

$$M^a = \det A, \quad M^{ab} = -\hat{e}_{bcd} \left( A^{ab} A^{cc} C^d_e + A^{bc} A^{ee} D^d_e \right), \quad M^{abc} = \frac{1}{2} \hat{e}^{d(e(c)} \left[ B_{df} (A^{ab} D^e_d - D^c_e A^{bf}) + B_{df} (A^{ac} C^f_e) \right],$$

$$M^{ab} = \hat{e}^{d(e(c)} \left[ B_{df} (A^{ab} D^e_d - D^c_e A^{bf}) + B_{df} (A^{ac} C^f_e) \right], \quad M^{abcd} = \hat{e}^{e(f(g|h|d)} B_{hf} \left[ \frac{1}{2} A^{ab} B_{ge} - C^a_e D^b_g \right].$$  \hfill (187)

Each 3-dimensional tensor $M$ is totally symmetric, i.e. $M^{ab} = M^{ba}, M^{abc} = M^{bac}, M^{abcd} = M^{dabc}$. These results have been verified by using the Maple computer algebra system, together with its tensor package GrTensor15.

15 See http://grtensor.org
3.2.2 Properties of the Fresnel tensor density

First, one notices that the Fresnel equation is independent of the axion-like piece \( \chi \) of the constitutive tensor,

\[
G^{ijkl}(\chi) = G^{ijkl}(\chi^{(1)} + \chi^{(2)}),
\]

which, in particular, implies

\[
G^{ijkl}(\chi^{(3)}) = 0.
\]

Furthermore, due to the antisymmetry property of \( \chi \), one verifies that also

\[
G^{ijkl}(\chi^{(2)}) = 0.
\]

Actually, properties (191) and (192) generalize to

\[
G^{ijkl}(\chi^{(2)} + \chi^{(3)}) = 0,
\]

which can be verified, for instance, by using computer algebra. Notice that this identity is not trivial since \( G \) depends cubically on the constitutive tensor \( \chi \). The identity (193) shows that the symmetric piece \( \chi^{(1)} \) is indispensable for obtaining well-behaved wave properties: If \( \chi^{(1)} = 0 \), the Fresnel equation is trivially satisfied and thus no light cone structure could be induced.

Furthermore, since

\[
G^{ijkl}(\chi^{(1)} + \chi^{(2)}) \neq G^{ijkl}(\chi^{(1)}),
\]

the ‘skewon’ field does influence the Fresnel equation, and therefore, eventually, the light cone structure. An example of this general result can be found again in the asymmetric constitutive tensor studied by Nieves and Pal, see Section 2.6.3 and references [47, 48].

Actually, after some algebra, one finds

\[
G^{ijkl}(\chi^{(1)} + \chi^{(2)}) = G^{ijkl}(\chi^{(1)}) + \frac{1}{3} \chi^{(1)} \epsilon_{mnps} \epsilon_{rstu} (\chi^{(1)} mnr (\chi^{(2)} j|ps|k (\chi^{(2)} l)tu + \frac{1}{4} \epsilon_{mnps} \epsilon_{rstu} (\chi^{(2)} mnr (\chi^{(1)} j|ps|k (\chi^{(2)} l)tu),
\]

or, in a (more of less) obvious notation (see the definition (180)),

\[
G^{ijkl}(\chi, \chi, \chi) = G^{ijkl}(\chi^{(1)} \chi^{(1)} \chi^{(1)}) + 2 G^{ijkl}(\chi^{(1)} \chi^{(2)} \chi^{(2)}) + G^{ijkl}(\chi^{(2)} \chi^{(1)} \chi^{(2)}).
\]

The other terms vanish due to the symmetry properties of each irreducible piece.

Take now (195) and substitute the parametrization of \( \chi^{(2)} \) in terms of \( S_i^j \), see (79). After some lengthy but straightforward algebra, one finds that the two last contributions to the right hand side of (195) are actually equal, namely

\[
G^{ijkl}(\chi^{(1)}, \chi^{(2)}, \chi^{(2)}) = G^{ijkl}(\chi^{(2)}, \chi^{(1)}, \chi^{(2)}) = \frac{1}{3} \chi^{(1)} m(i|n|j S_m^k S_n^l).
\]

Therefore, the final result reads

\[
G^{ijkl}(\chi^{(1)} + \chi^{(2)}) = G^{ijkl}(\chi^{(1)}) + \frac{1}{3} \chi^{(1)} m(i|n|j S_m^k S_n^l),
\]
a very simple expression, indeed. This equation summarizes how the skewon piece ‘pertur-
bates’ the Fresnel equation that one would obtain only from the principal piece \((1)\chi\). The
skewon piece adds a second term which is linear in \((1)\chi\) and quadratic in \((2)\chi\).
For the particular case of Nieves and Pal, see (79) and (109), one finds
\[
G^{ijkl}q_iq_jq_kq_l = -V \left[ \frac{\varepsilon_0}{\mu_0} \left( \frac{\varepsilon_0}{\mu_0} (q \cdot q)^2 - (q \cdot q)(v \cdot v)(v \cdot q)^2 + (v \cdot q)^4 \right) \right].
\]
(199)
Compare this result with equation (5.7) in [48]. Use \(q_i = (\omega, -\vec{k}), v^i = (v, 0, 0, 0)\), \(g_{ij} = \eta_{ij} \equiv (c^2, -1, -1, -1)\), \(c^2 = 1/\varepsilon_0\mu_0\) and obtain
\[
G^{ijkl}q_iq_jq_kq_l = -\varepsilon_0^3 \left[ (\omega^2 - k^2c^2)^2 + \frac{c^4}{\varepsilon_0^2}k^2\omega^2v^4 \right].
\]
(200)
This results agrees\(^{16}\) with that in [48] (in their \(\varepsilon = \mu = 1\) case) for \(\zeta = cu^2/\varepsilon_0\).
Notice that in general, for \(\zeta \neq 0\), the Fresnel equation will have complex solutions. This
is again a manifestation of the dispersive properties described by the antisymmetric piece \((2)\chi\)
of the constitutive tensor.

### 3.3 Light rays

In a riemannian space, light rays are defined as the integral lines of the vector field defined by
\(V_{\text{riem}} := g^{ij}q_j\partial_i\), which is then a null vector. It satisfies \(V_{\text{riem}}|q = 0\), so that the vector \(V_{\text{riem}}\)
is orthogonal to the wave front \(S\). This definition makes direct use of a metric, it is a nontrivial
question whether it is possible to define light rays in our more general, pre-metric framework.

In analogy to the riemannian case, one can try to define light rays as the integral lines of a
vector field \(V\) which satisfies
\[
V|q = 0.
\]
(201)
Eq. (201) represents one condition for the 4 independent components of \(V\). Therefore, this
information is not enough to define light rays uniquely. We know, however, that in the riemannian
case the light rays defined by \(V_{\text{riem}}\) are also orthogonal to the polarization vector \(a\) (such
that \(f = q \land a\)), i.e. \(V_{\text{riem}}|a = 0\). The same is true for \(b\) with \(h = q \land b\), i.e. \(V_{\text{riem}}|b = 0\).
These known properties suggest to define the vector \(V\) for a given solution of the Maxwell
equations as those satisfying
\[
V|q = 0, \quad V|a = 0, \quad V|b = 0.
\]
(202)
The three conditions (202) determine then the vector \(V\) up to a scalar factor. This ambiguity
in the definition of \(V\) is however irrelevant, since the integral lines of \(V\) are independent of that
scalar factor\(^{17}\).

\(^{16}\) Nieves and Pal use a different system of units, compare, for instance, their vacuum equations (2.1) and (2.2)
with our Maxwell equations (1) and (2). Their equations can be obtained from our ones by substituting \(D = D_N\),
\(H = H_N/c, E = E_N/\varepsilon_0, B = B_N/(\varepsilon_0\mu), \rho = \rho_N, j = j_N, \) and \(t = t_N\), where the subindex \(N\) refers to the fields
in the notation of Nieves and Pal
\(^{17}\) If one computes the integral line by \(dx^i/dp = V^i\), then the freedom of choosing the scalar factor corresponds
to the freedom of choosing the parameter \(p\) along the curve. In other words, an arbitrary scalar factor can always be
absorbed by a reparametrization \(p' = p'(p)\)
One can construct immediately a solution of (202), since they imply that the coordinate components $V^i$ of $V$ have to be proportional to $\epsilon^{ijkl} q_j a_k b_l$. We therefore define

$$V^i := \epsilon^{ijkl} q_j a_k b_l.$$  \hfill (203)

This quantity is a vector density of weight $+1$. Again, the fact that (203) does not define a true vector field represents no problem for the definition of light rays as its integral lines. [Alternatively, one could try to ‘normalize’ $V$ in order to construct a true vector, for example by dividing $V$ by the density $|a|^4 := G^{ijkl} a_i a_j a_k a_l$, with $|a|$ being a kind of ‘norm’ of $a$. Obviously, this procedure only makes sense if $|a| \neq 0$.]

One can write (203) in a coordinate-free way as

$$V = \circ(q \wedge a \wedge b),$$  \hfill (204)

where $\circ$ is the (metric-free) operator mapping 3-form into vector densities of weight $+1$ according to $\circ \left(\alpha^\alpha \wedge \beta^\beta \wedge \gamma^\gamma\right) = \epsilon^{\alpha\beta\gamma} \epsilon_\delta$, so that $V = V^\alpha e_\alpha$.

Now, using the identities $q \wedge b = h = \# f = \# (q \wedge a)$ we can write (204) in terms of only $q$, $a$ and the operator $\#$ as

$$V = \circ \left[a \wedge \#(a \wedge q)\right],$$  \hfill (205)

or, in terms of the components of the constitutive tensor,

$$V^\alpha = \chi^{\alpha\beta\gamma\delta} a_\beta q_\gamma q_\delta.$$  \hfill (206)

As a consequence of our definitions, $V$ depends explicitly not only on the wave 1-form $q$ but also on the polarization 1-form $a$. This property agrees with the result that, for a general constitutive tensor, different polarization states will propagate in different directions. Birefringence is again a particular example of the above mentioned fact. One can also verify that the axion-like piece of the constitutive tensor drops out from $V$, so it depends only on $(1)\chi$ and $(2)\chi$.

To gain more physical insight on our definition of light rays, we evaluate the kinematic energy-momentum 3-form $k \Sigma_\alpha$ for our particular solutions. From its definition (145) and using $F \rightarrow f = q \wedge a$ and $H \rightarrow h = q \wedge b$, we find

$$k \Sigma_\alpha = \frac{1}{2} \left[q \wedge a \wedge (q_\alpha b - q b_\alpha) - q \wedge b \wedge (q_\alpha a - q a_\alpha)\right]$$

$$= \frac{1}{2} \left[q_\alpha q \wedge a \wedge b - q_\alpha q \wedge b \wedge a\right]$$

$$= q_\alpha q \wedge a \wedge b.$$  \hfill (207)

Using (204), we thus arrive at

$$\circ (k \Sigma_\alpha) = q_\alpha V.$$  \hfill (208)

In terms of the components $kT_\alpha^\beta$ of the kinetic energy-momentum density, see (146), one finds

$$kT_\alpha^\beta = q_\alpha V^\beta.$$  \hfill (209)

This result shows that $V$ determines the direction in spacetime in which the energy-momentum of the wave is transported. In other words, $V$ represents the 4-velocity of the energy transport.
3.4 Fresnel equation for $V$

Now, since $f = q \wedge a$ and $h = q \wedge b$, we see that the conditions (202) imply

$$V[h] = 0, \quad V[f] = 0. \quad (210)$$

Kiehn [34] uses (210) to define $V$ as an 'extremal vector field'. Furthermore, it can be shown that (210) are equivalent to (202). Take for instance (210b), then we have

$$V[f = V] (q \wedge a) = (V[q] a - q (V[a]) = 0. \quad (211)$$

Suppose now that the scalar $V[q]$ does not vanish, then (211) implies that $a$ is proportional to $q$. But this would imply that $f = q \wedge a \equiv 0$. Therefore, nontrivial solutions ($f \neq 0$) require (202a) to hold. Substituting this into (211) shows that also (202b) must be satisfied. Similarly, starting from (210a), the same argument shows that (202c) is required.

The equations (210) for the vector $V$ are similar to the Hadamard compatibility conditions (160) for the wave 1-form $q$. This suggests to apply a method similar to that used in Section 3.2 for deriving the Fresnel equation for $q$.

First, we solve (210b) for $f$. A general solution can be written as

$$f = V[c], \quad (212)$$

where $c$ is an arbitrary 3-form. We write the 4 independent components of $c$ as $c_{\alpha\beta\gamma} =: \epsilon_{\alpha\beta\gamma}\delta^\delta$. Using this in (210a) and (88) we find

$$V[\#(V[c])] = 0. \quad (213)$$

This equation is analogous to (165). As in the case of the wave 1-form $q$, not all the equations in (213) are independent, since interior product of $V$ with (213) vanishes identically. In analogy to the method used in Section 3.2, it is convenient to use a vector frame basis $e_\alpha$ and its dual $\vartheta^\alpha$, with the special 'gauge' $e_0 = V$. Then, after some algebra, one finds that (213) is equivalent to

$$M_{AB} c^B = 0, \quad (214)$$

with the $3 \times 3$ matrix $M_{AB}$ defined as

$$M_{AB} := \epsilon_{0ACD} \chi^{CDEF} \epsilon_{EFB0}. \quad (215)$$

Again, nontrivial solutions ($c^B \neq 0$) exist only if $\det(M_{AB}) = 0$, i.e. if and only if

$$\frac{1}{3!} c^{ABC} e^{DEF} M_{AD} M_{BE} M_{CF} = 0. \quad (216)$$

Using (215) and (216), and applying the same method as in Section 3.2 to complete two Levi-Civita symbols, we finally find also a quartic equation that now the components of $V$ must satisfy, namely

$$\mathcal{M}_{ijkl} V^i V^j V^k V^l = 0, \quad (217)$$
where
\[
\mathcal{M}_{ijkl} := \frac{1}{4!} \chi^{mnpq} \hat{e}_m \hat{e}_n \hat{e}_p \hat{e}_q \chi_{rstu} \chi_{uvhs}, \tag{218}
\]
is a totally symmetric tensor density of weight \(-1\).

Eq. (217) defines in the space of ray vectors the ray surface, see [38].

One can verify after some algebra that in the riemannian case the tensor density \(\mathcal{M}_{ijkl}\) reduces to
\[
\mathcal{M}_{ijkl} := 4|g|^{-1/2} g_{ij} g_{kl}, \tag{219}
\]
Additionally, one verifies from (206) that the vector density \(V\) reduces to
\[
V^i = 4|g|^{1/2} \left[ (g^{jk} a_k) (g^{il} a_l q_l) - (g^{il} q_l) (g^{jk} a_j a_k) \right]. \tag{220}
\]
Since in a riemannian space we can additionally choose a ‘Lorenz gauge’ for \(a\), namely \(g^{ij} a_i q_j = 0\), the first term vanishes and the above result shows that \(V^i \sim q^i := g^{ij} q_j\), as expected.

4 Light cone structure

In this section, we want to study the particular case of our general theory in which a light cone structure is induced. In terms of the Fresnel tensor \(G\), this means that the spacetime/medium is such that
\[
G^{ijkl} = G^{(ij} G^{kl)} = \frac{1}{3} \left( G^{ij} G^{kl} + G^{kj} G^{il} + G^{li} G^{jk} \right), \tag{221}
\]
for some symmetric second order tensor density \(G^{ij}\) of weight \(+1/2\). Then the Fresnel equation, which in general is quartic in the wave-covector, reduces to a second order equation, namely to a light cone condition:
\[
G^{ij} q_i q_j = 0. \tag{222}
\]

Obviously, we are interested in this subcase because in GR (and its special case of SR) the local properties of light propagation in vacuum are determined by the conformal structure of the underlying riemannian geometry. This implies that the wave covectors satisfy a condition (222), with \(G^{ij} \sim g^{ij}\). Here \(g^{ij}\) denotes the spacetime metric.

We are specifically interested in the following questions: a) What is the most general constitutive tensor leading to a light cone structure? b) Is it possible to obtain a light cone structure as a consequence of some metric-independent conditions? In other words, can a light cone structure be induced/deduced without postulating the existence of a metric from the very beginning? c) If this is the case, how can a conformal metric be explicitly constructed from the underlying constitutive tensor?
4.1 Looking for metric-independent conditions

It is clear that in order to induce a light cone structure additional information or assumptions must be added to the general pre-metric framework, in the context of which the constitutive tensor is arbitrary. We look for these additional conditions. First, we consider the particular case of vacuum in a riemannian space in order to see if it is possible to find some properties which can be then used as a guide in the pre-metric framework.

We know that the constitutive tensor corresponding to the vacuum in a riemannian space, i.e.

\[ \chi_{ijkl} = 2 \sqrt{\epsilon_0 \mu_0 |g|} g^{[i} g^{j]k} g^{l]}, \]

is such that the Fresnel equation reduces to the light cone condition. Note that \( \chi_{ijkl} \) is invariant under conformal transformations \( g_{ij} \rightarrow e^{\Psi(x)} g_{ij} \). As a consequence, only 9 of the 10 components of the metric can enter \( \chi \).

However, since the Fresnel equation is homogeneous, we see that

\[ \chi_{ijkl} = 2 f(x) \sqrt{|g|} g^{[i} g^{j]k} g^{l]}, \tag{224} \]

will also induce the same light cone structure, for any scalar function \( f(x) \). This field can be identified with a dilaton field. One uses the definition (180) of the Fresnel tensor for this particular case and, after some algebra, we obtain

\[ G_{ijkl} = \text{sgn}(g) f^3 \sqrt{|g|} g^{[i} g^{j]k} g^{l]}, \tag{225} \]

Let us summarize the properties of the constitutive tensor (224). Obviously \( \chi_{ijkl} = 0 \), because \( \chi_{ijkl} \) is symmetric under the exchange of the first and last pair of indices. Furthermore, \( \chi_{ijkl} = 0 \), i.e., no axion-like term is present. Additionally, one verifies that

\[ \frac{1}{8} \tilde{\epsilon}_{ijkl} \chi_{ijkl} \tilde{\epsilon}_{mnqp} \chi_{mnqp} = -f^2 \delta^i_j \delta^k_l. \tag{226} \]

In this expression the metric tensor does not explicitly appear, but the negative sign on the right hand side is due to the its lorentzian signature.

These two properties, can be rewritten in terms of the corresponding linear operator \( \kappa \) related to \( \chi_{ijkl} \), see (70). The symmetry property \( \chi_{ijkl} = 0 \) is equivalent to

\[ A \wedge (\kappa B) = (\kappa A) \wedge B, \tag{227} \]

for any 2-form \( A \) and \( B \), while the condition (226) can be written as

\[ \kappa^2 = -f^2 1. \tag{228} \]

From this example, which is realized in GR, we learn that the riemannian vacuum constitutive tensor satisfies some properties with can be written in a metric-independent way. As we will see in the next sections, an operator satisfying the two conditions above defines a duality operator.

Hence the question arises whether (227) and (228) can be taken as additional conditions in order to induce a light cone structure in the general pre-metric electrodynamic framework and whether these conditions are necessary and/or sufficient to induce a light cone structure. In fact, it turns out that there exists a relationship between dual operators and conformal metrics.
4.2 Reciprocity and closure

The two conditions (227) and (228) can be motivated already in the pre-metric framework. From the discussion of Section 2.7.3 it is clear that the condition of symmetry ensures that the medium/spacetime will not possess intrinsic dissipative properties. This can then be taken as an additional metric-independent condition, see, however, the discussion in Sections 2.6.3 and 3.2.2 about the constitutive tensor of Nieves and Pal, for which \( \lambda \neq 0 \). The condition of closure can be motivated as follows. An interesting feature of electrodynamics is its electric/magnetic reciprocity property. From the general expression of the kinetic energy-momentum current (145), one can see that the energy-momentum content of the electromagnetic field remains the same if one exchanges excitation \( H \) and field strength \( F \) in an appropriate way. Consider the transformation

\[
H \to \zeta F, \quad F \to -\frac{1}{\zeta} H. \tag{229}
\]

Here \( \zeta \) is a pseudoscalar carrying the same physical dimension as the constitutive tensor, i.e. \([\zeta] = [H/F] = q^2/\hbar\). Direct inspection of (145) shows that the energy-momentum current remains invariant under this transformation, i.e.

\[
k \Sigma_\alpha \to k \Sigma_\alpha. \tag{230}
\]

In terms of the 3-dimensional forms defined in Section 2.4, the transformation \( H \to \zeta F \) implies \( H \to -\zeta E \) and \( D \to \zeta B \) and \( F \to -\frac{1}{\zeta} H \) implies \( E \to \frac{1}{\zeta} H \) and \( B \to -\frac{1}{\zeta} D \). This clearly shows that (229) exchange electric and magnetic quantities.

So far, this invariance, which we call electric/magnetic reciprocity\(^{18}\), is a property only of the energy-momentum current. If one requires, as an additional postulate, the spacetime/constitutive relation also to be electric/magnetic reciprocal, a condition on the constitutive tensor can be found.

Take the spacetime/constitutive relation (70) and perform the transformation (229) while leaving the constitutive tensor unchanged. One finds

\[
\zeta F_{\alpha\beta} = \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} \chi^{\gamma\delta\epsilon\theta} (-1) \frac{1}{\zeta} H_{\epsilon\theta}. \tag{231}
\]

Use again (70) to replace the components of the excitation in terms of those of the field strength, and obtain

\[
-\zeta^2 F_{\alpha\beta} = \frac{1}{16} \epsilon_{\alpha\beta\gamma\delta} \chi^{\gamma\delta\epsilon\theta} \epsilon_{\epsilon\theta\mu\nu} \chi^{\mu\nu\lambda\rho} F_{\lambda\rho}. \tag{232}
\]

This expression leads to a condition for the constitutive tensor, namely to

\[
-\zeta^2 \delta[\alpha^{\beta} \delta^{\rho}] = \frac{1}{16} \epsilon_{\alpha\beta\gamma\delta} \chi^{\gamma\delta\epsilon\theta} \epsilon_{\epsilon\theta\mu\nu} \chi^{\mu\nu\lambda\rho}. \tag{233}
\]

Define now the dimensionless tensor

\[
\chi^{\alpha\beta\gamma\delta} := \frac{1}{\zeta} \chi^{\alpha\beta\gamma\delta}, \tag{234}
\]

\(^{18}\) In order to distinguish it from the conventional metric-dependent duality transformation \( F_{ij} \to \tilde{F}_{ij} := \frac{1}{2} \sqrt{-g} \epsilon_{ijkl} g^{km} g^{ln} F_{mn} \), under which the vacuum \( (J = 0) \) Maxwell equations in a riemannian space are invariant.
with
\[ \zeta^2 := -\frac{1}{96} (\hat{e}_{ijmn} \chi^{mnpq}) (\hat{e}_{pqrs} \chi^{rsti}). \] (235)

Then (233) can be rewritten as
\[ \frac{1}{8} \varepsilon_{\alpha\beta\gamma\delta} \chi^{\alpha} \gamma^\delta \varepsilon_{\theta\mu\nu} \chi^{\mu\nu\lambda\rho} = -2\delta^{[\lambda}_{\alpha} \delta^{\rho]}_{\beta}. \] (236)

It is convenient to define
\[ J_{\alpha\beta} := \frac{1}{2} \varepsilon_{\alpha\beta\eta\sigma} \chi^{\eta\sigma\gamma\delta}. \] (237)

Then the closure relation (236) reads
\[ \frac{1}{2} J_{\alpha\beta} \eta^\sigma J_{\eta^\sigma \lambda\rho} = -2\delta^{[\lambda}_{\alpha} \delta^{\rho]}_{\beta}. \] (238)

or, in the corresponding notation in terms of $6 \times 6$ matrices,
\[ J^2 = -1. \] (239)

We call this condition on the constitutive tensor closure relation. The negative sign in (229b) and therefore the one on the right hand side of (238) is a consequence of the negative relative sign of the two terms entering in the energy-momentum current, see (145). Below we will see that this negative sign will be responsible for the lorentzian signature of the induced metric.

Mathematically, this means that the operator $J$ represents an almost complex structure on the space of 2-forms.

4.3 Dual operators and metrics

A linear operator $J : \Lambda^2(X) \rightarrow \Lambda^2(X)$ acting on 2-forms is said to be a dual operator defining a complex structure if it satisfies the property of symmetry,
\[ A \wedge (JB) = (JA) \wedge B, \] (240)

for any 2-form $A$ and $B$, and if it is additionally closed such that
\[ J^2 = -1. \] (241)

Many people have studied this relationship, both as a useful tool in GR [6, 7] and as a method which could allow to consider the metric of spacetime as a secondary quantity constructed from some other fields, see for instance [8]. Clearly, the dimensionless part of the riemannian constitutive tensor $f^{-1} \chi_{ijkl}$ defines a dual operator which is just the Hodge dual of the metric $g$.

In the context of electrodynamics, it seems that Peres [56] was the first to try to reconstruct the conformally invariant part of the spacetime metric from the excitation $H$ and the field strength $F$. In his approach, however, the metric is assumed to exist such that the relation between $H$ and $F$ is just the same as the vacuum relation in a riemannian space. In our
notation, this condition is equivalent to the postulate that the operator $\kappa$ is a dual operator and that it also equals the Hodge dual of some metric that is to be determined. In this sense, the approach of Peres can be considered as a sort of ‘inverse problem’. Contrary to defining the operator $\kappa$ as the dual of the metric tensor, as is done in GR, Peres tried to determine a metric such that its Hodge dual operator coincides with $\kappa$. To that goal, it is assumed that the operator $\kappa$ satisfies the conditions of symmetry and closure, as a Hodge dual operator does. However, as we will discuss, Peres did not succeed in deriving the conformally metric since his result is not unique.

Toupin [71] and Schönberg [62] also studied how a conformal metric structure is induced and in particular how the conformally invariant part of the metric can be deduced from the spacetime/constitutive relation under the assumption of symmetry and closure. They seem to be the first who were able to show that a conformal metric structure is actually implied as a consequence of symmetry and closure. Brans [6, 7] also recognized that, within general relativity, it is possible to recover the metric from its Hodge dual operator. These structures were subsequently discussed by numerous people, by ’t Hooft [70], Harnett [18], and Obukhov & Tertychniy [50], amongst others, see also the references given there.

From these studies, it seems clear by now that there is a one-to-one correspondence between linear operators (of the kind of $\kappa$) satisfying symmetry and closure, and conformal metrics. If a metric is available, one can immediately construct the corresponding Hodge dual operator which, for a metric with lorentzian signature, will satisfy (240) and (241). On the other hand, the contrary is also valid. If an operator $J$ satisfying (240) and (241) is available, then a conformal metric structure is induced, such that the operator $J$ can be written as a Hodge dual operator. Furthermore, the metric components can be (re-)constructed by using the so-called Schönberg-Urbantke formula. Schönberg seems to be the first to derive this formula in the context of electrodynamics. This formula was also found by Urbantke [73, 74], but in a different context, namely in the framework of $SU(2)$ Yang-Mills theories. We will discuss some derivations of this important formula below.

Finally, by using the Schönberg-Urbantke formula in the context of linear pre-metric electrodynamics, Obukhov and Hehl [52] presented an explicit construction of the conformally invariant part of the metric tensor in terms of quantities parametrizing a constitutive tensor satisfying symmetry and closure.

After discussing these developments, an alternative procedure to deduce the conformal metric and its light cone structure will be presented. It relies on the general results about the Fresnel equation governing the local properties of electromagnetic waves. This alternative approach is therefore much more suitable to understand the physics underlying the emergence of the conformal structure of spacetime, since it directly involves the properties of the propagation of waves. Finally, this approach also allows to study how the dropping or modifying of the assumptions of symmetry and closure would affect the light cone structure. In particular, we will explore the consequences a possible asymmetric spacetime/constitutive relation (i.e., with skewon part) would have on the light cone, but maintaining the assumption of closure.

4.3.1 Schönberg-Urbantke formula

Here we discuss the successful derivation of of the conformal metric by Schönberg and the related work of Harnett.
In his work [62] Schönberg studied a different derivation of the metric from electromagnetism. Additionally, the analysis in [62] helps to better understand the group theoretical aspects of the emergence of a lorentzian metric structure and thus of the Lorentz group.

As we already saw before, the electromagnetic field strength $F$, with its six independent components, can be described as a vector in a 6-dimensional real vector space. We call this space $S_6$. One can map a basis of the space of 2-form, $dx^{i_1} \wedge dx^{i_2}$ to a basis $b^I$ of $S_6$ by means of the rule $[i_1 i_2] \rightarrow I = 01, 02, 03, 23, 31, 12$. Then one can write a vector $A \in S_6$ as $A = A_I b^I$.

An important role is played by an inner product $\epsilon$ on $S_6$ induced by the 4-dimensional Levi-Civita symbol $\epsilon^{ijkl}$. Given two vectors $A$ and $B$ of $S_6$, one can define their product as

$$\epsilon(A, B) := \epsilon^{IJ} A_I B_J.$$  

(242)

In 4-dimensional notation, this is equivalent to

$$\epsilon(A, B) \hat{\epsilon} := A \wedge B$$

(243)

or, in components,

$$\epsilon(A, B) = \frac{1}{4} \epsilon^{ijkl} A_{ij} B_{kl}.$$  

(244)

Thus, the Levi-Civita symbol acts as a metric in the 6-dimensional space $S_6$. Notice, however, that $\epsilon(A, B)$ is a 4-dimensional density of weight +1. Since the eigenvalues of $\epsilon^{IJ}$ are +1 and $-1$, each with multiplicity 3 (see (98b)), the signature of $\epsilon^{IJ}$ is $(+1, +1, +1, -1, -1, -1)$. This immediately shows that the 6-dimensional space $S_6$ naturally contains a $SO(3, 3)$ group structure. Transformation of $S_6$-vectors under the action of the $SO(3, 3)$ group leaves the product (242) invariant. It is important to emphasize that this group structure is always present, independent of any metric, affine, or whatever additional structure on the 4-dimensional manifold.

Using this product, one can express the assumption of symmetry of the constitutive tensor, i.e., $^{(2)} \chi = 0$ as

$$\epsilon(A, JB) = \epsilon(JA, B)$$

(245)

for all $A, B \in S_6$, because

$$\epsilon(A, JB) = \epsilon^{IJ} A_I \chi^{JK} B_K = \chi^{IK} A_I B_K$$

(246)

since $\chi^{IK} = \epsilon^{IJ} A_I J^J K$, and

$$\epsilon(JA, B) = \epsilon^{IJ} \chi^{JK} A_K B_J = \chi^{IK} A_K B_J = \chi^{KI} A_I B_K,$$

(247)

so that (245) requires symmetry of $\chi$, i.e., $\chi^{IJ} = \chi^{JI}$.

Furthermore, the additional condition of closure of the operator $J$, i.e.,

$$J^2 = -1,$$

(248)

is assumed to hold.
In the 6-dimensional real space $S_6$, the introduction of the linear operator $J$ satisfying (245) and (248) induces an almost complex structure. In order to be able to discuss this structure and in particular the eigenvectors of the operator $J$, consider the complex extension of $S_6$, i.e. the vector space of the complex 2-forms $S_6(C)$. The negative sign on the right hand side of (248) implies that the eigenvalues of $J$ are $\pm i$, and since $J$ is a real operator, each eigenvalue must have multiplicity 3. This means that the corresponding self-dual and anti-self-dual subspaces spanned by vectors with eigenvalues $+i$ and $-i$ are both 3-dimensional. We denote these subspaces as $S_3^+(C)$ and $S_3^-(C)$ respectively. The vectors of $S_3^+(C)$ will be denoted as $A^\pm$ and satisfy

$$J A^\pm = \pm i A^\pm. \quad (249)$$

Take any $A^\pm \in S_3^\pm(C)$ and write it as $A^\pm = A + i A'$ with $A$ and $A'$ real 6-dimensional vectors (i.e. $A, A' \in S_6$). Then from (249) one finds that

$$J (A + i A') = \pm i (A + i A'), \quad (250)$$

with implies

$$J A = \mp A', \quad J A' = \pm A. \quad (251)$$

Using (251) one can rewrite every element $S_3^\pm$ of $S_3^\pm(C)$, in terms of only the real $A \in S_6$ and the operator $J$, as

$$A^\pm = A \mp i J A. \quad (252)$$

The spaces $S_3^+(C)$ and $S_3^-(C)$ are orthogonal with respect to the product $\epsilon$. Take any $A^+ \in S_3^+(C)$ and $B^- \in S_3^-(C)$. Write them in the form (252), i.e. $A^+ = A - i J A$ and $B^- = B + i J B$, and compute their $\epsilon$-product. Using the symmetry and closure properties (245) and (248) one finds

$$\epsilon(A^+, B^-) = \epsilon(A - i J A, B + i J B) = \epsilon(A, B) + i \epsilon(A, J B) - i \epsilon(J A, B) + \epsilon(J A, J B) = \epsilon(A, B) + \epsilon(A, -B) = 0. \quad (253)$$

Consider a real operator $M$ acting on the real space $S_6$. If $A \in S_6$ then also $MA \in S_6$. However, the self-dual element $(MA)^+ of S_3^+$ will in general not be the result of the application of the linear operator $M$ on $A^+$, since

$$(MA)^+ = (MA) - i J (MA) \neq M(A - i J A) = M(A^+), \quad (254)$$

unless the operators $M$ and $J$ commute, i.e. $[M, J] = 0$. In other words, not every linear operator on $S_6$ is a linear operator in $S_3^+$. In particular, not every element of the group $O(3, 3)$ which leaves $\epsilon$ invariant defines a linear operator in the self-dual space $S_3^+$.

The corresponding subgroup of $O(3, 3)$ which commutes with $J$, i.e. the subgroup form by those operators $N$ such that

$$\epsilon(NA, NB) = \epsilon(A, B), \quad [N, J] = 0, \quad (255)$$
can be shown to be isomorphic to the Lorentz group. To prove this, consider the product $\epsilon$ restricted to the self-dual subspace $S^+_3$, which we denote as $\epsilon^+$, and is defined by

$$\epsilon^+ (A^+, B^+) := \epsilon (A^+, B^+).$$

(256)

Consider operators $N$ satisfying (255) which are therefore also linear operators of $S^+_3$. It is clear that they will also leave invariant the 3-dimensional product $\epsilon^+$, since in this case

$$\epsilon^+ ((NA)^+, (NB)^+) = \epsilon^+ (NA^+, NB^+) = \epsilon (NA^+, NB^+) = \epsilon (A^+, B^+) = \epsilon^+ (A^+, B^+).$$

(257)

This shows that the subgroup of $O(3,3)$ commuting with $J$ forms the invariance group of the 3-dimensional metric induced by $\epsilon^+$ in $S^+_3$. But since $S^+_3$ is a complex linear space, the corresponding invariance group of $\epsilon^+$ is clearly $SO(3, C)$, which is isomorphic to the Lorentz group $SO(3, 1)$. In other words, an operator $J$ satisfying symmetry (245) and closure (248) induces a $SO(3, C) \approx SO(3, 1)$ group structure, which manifest itself as the symmetry group of the natural metric structure on the self-dual space of $J$. Similarly, the same arguments can be repeated considering the anti-self-dual space $S^-_3$.

We consider now how to reconstruct the metric components of the corresponding induced Lorentzian metric. Consider a basis for each subspace. We denote the basis of $S^+_3$, $a, b, \ldots = 1, 2, 3$. Then the complex conjugate $S^\pm(a) := (S^a(a))^*$ is a basis of $S^\pm$, see (249).

Since $S_0(C) = S^+_3 \oplus S^-_3$, the six $S_0$-vectors $\{S^a_+(a), S^a_-(a)\}$ form a basis of $S_0(C)$. Then the orthogonality property (253) implies that the components of the metric $\epsilon$ of $S_0(C)$ in the basis $\{S^a_+(a), S^a_-(a)\}$ form a block-diagonal matrix, since $\epsilon \left( S^a_+(a), S^a_-(a) \right) = 0$.

One can use the exterior product of vectors in $S_0(C)$, which we denote by $\triangle$, and determine a volume element of $S^+_3$ as

$$W_+ := S^{(1)}_+ \triangle S^{(2)}_+ \triangle S^{(3)}_+ = \frac{1}{3!} \epsilon_{abc} S^a_+ \triangle S^b_+ \triangle S^c_+.$$  

(258)

Then $W_+$ is a 3-form on $S_0$, i.e. $W_+ \in \Lambda^3(S_0)$. Since $S^+_3$ is 3-dimensional, $W_+$ is, up to a scalar factor, independent of the choice of the basis $S^a_+$. Furthermore, any product of basis vectors $S^a_+$ of an order higher than 3 vanishes identically (e.g., $S^a_+ \triangle S^b_+ \triangle S^c_+ \triangle S^d_+ = 0$).

A volume element $W_-$ for $S^-_3$ can be similarly defined by using the basis $S^a_-$ of $S^-_3$, which then satisfies $W_- = (W_+)^*$. A 3-form $W \in \Lambda^3(S_0)$ has 20 independent components $W_{IJK}$, which correspond to a 4-dimensional tensor $W_{ijkl}$ of order six, with the symmetries $W_{ijklm} = -W_{jiklm} = -W_{ikljm} = -W_{ijkm} = -W_{ijkml} = -W_{ijlmk} = -W_{ijklm}$, etc. From this tensor, a symmetric second order 4-dimensional tensor density of weight +1 can be defined as

$$W_{ij} := \frac{1}{2} \epsilon^{klmn} W_{iklmnj},$$

(259)

which has thus 10 independent components. The remaining 10 components of $W$ can be mapped to a second order contravariant tensor density of weight +2, defined by

$$\overline{W}^{ij} := \frac{1}{4} \epsilon^{iklm} W_{klmnpq} \epsilon^{npqj}.$$  

(260)
Following (259) we define the tensor density $\mathcal{W}^+_{ij}$ associated to $W_+$ as
\[ \mathcal{W}^+_{ij} := \frac{1}{2} \varepsilon^{klmn} W^+_{iklmn} , \tag{261} \]
or, equivalently,
\[ \mathcal{W}^+_{ij} := \frac{1}{2 \cdot 3!} \varepsilon^{klmn} \hat{\varepsilon}_{abc} S^+_{ik} S^+_{lm} S^+_{nj} . \tag{262} \]

Furthermore, we saw that a change of the basis $S^+(a)$ of $S^+_3$ to a new one $S^+(a)'$ leads to a new volume $W^+_3$, which is necessarily proportional to $W^+_3$, i.e. $W^+_3 = a_3 W^+_3$ for some $a_3$. One can therefore define a new tensor density by
\[ \hat{\mathcal{W}}^+_{ij} := (\det W^+_{kl})^{-1/4} \mathcal{W}^+_{ij} , \tag{263} \]

Then $\hat{\mathcal{W}}^+_3$ is a tensor density of weight $-1/2$ which is independent of the choice of the basis $S^+(a)$ of $S^+_3$. Because of this important property, $\hat{\mathcal{W}}^+_{ij}$ must describe an intrinsic property of the self-dual space $S^+_3$ and therefore of the operator $J$. It is then tempting to identify the symmetric tensor density (263) with the conformally invariant part of the metric tensor $\sqrt{|g|}^{-1/4} g_{ij}$, which is also a tensor density of weight $-1/2$, i.e.,
\[ \hat{\mathcal{W}}^+_{ij} = a |g|^{-1/4} g_{ij} , \tag{264} \]
with some (in general complex) factor $a$.

That this identification is consistent with our expectations can be checked as follows. Consider the linear operator $L$ acting on vectors of $S_6(C)$, defined by
\[ (LA)_{ij} := \frac{1}{2} L^k_{ij} A_{kl} , \tag{265} \]
with
\[ L^k_{ij} := \hat{\mathcal{W}}^+_{im} \hat{\mathcal{W}}^+_{jn} \varepsilon^{mnkl} . \tag{266} \]
This operator is expected to be related to the original operator $J$, because the identification (264) would imply
\[ L^k_{ij} = a^2 |g|^{-1/2} g_{im} g_{jn} \varepsilon^{mnkl} , \tag{267} \]
which is proportional to the Hodge dual operator of the metric $g$, when applied to 2-forms.

Using (263) one re-writes the definition (266) of the operator $L$ as
\[ L^k_{ij} = (\det \mathcal{W}^+_{pq})^{-1/2} \mathcal{W}^+_{im} \mathcal{W}^+_{jn} \varepsilon^{mnkl} , \tag{268} \]
Furthermore, it can be proved that $L$ is a complex operator. From (268) and the definition (262) one can verify by direct but rather lengthy calculations that the elements of $S^+_3$ and $S^-_3$ are eigenvectors of $L$ with eigenvalues $+1$ and $-1$, respectively, i.e.
\[ LS^+ = S^+ , \quad LS^- = -S^- . \tag{269} \]
Since $S_{-3} = (S_{+3})^*$, the operator $L$ cannot be real. If it were, (269a) would imply $LS^{-} = S^{-}$, contradicting (269b). It is therefore clear that the real operator $J$ is just given by $J = iL$, because

$$(iL)S^{+} = iS^{+}, \quad (iL)S^{-} = -iS^{-},$$  \hspace{1cm} (270)

which coincides with (249). The property (270) is valid for any vector of $S_{3}^{\pm}$ and since $S_{6}(C) = S_{3}^{+} \oplus S_{3}^{-}$, it implies that the action of $iL$ and $J$ on any vector of $S_{6}(C)$ is exactly the same, hence they are the same operator.

Thus, we have proved that

$$J = iL,$$  \hspace{1cm} (271)

so that, from (268), we have

$$J_{ij}^{\ kl} = \left(-\det\ W_{pq}^{+}\right)^{-1/2} W_{im}^{+} W_{jn}^{+} \epsilon^{mnl}. $$  \hspace{1cm} (272)

With the identification (264) we write the operator $J$ as

$$J_{ij}^{\ kl} = \sqrt{-g} g_{im} g_{jn} \epsilon^{mnl}. $$  \hspace{1cm} (273)

From this expression, one sees that the metric $g_{ij}$ can be taken to be real, with lorentzian signature, since $J$ is a real operator. In other words, the results and (262) allow us to reconstruct the metric components as

$$g_{ij} \sim \epsilon^{klmn} \epsilon_{abc} S_{ik}^{+(a)} S_{lm}^{+(b)} S_{nj}^{+(c)}, $$  \hspace{1cm} (274)

with $g_{ij}$ real. The conformal factor remains of course undetermined.

### 4.3.2 Necessary and sufficient conditions for the constitutive tensor

The results discussed above can be summarized as follows. If the operator $J$, defined in (237) by the dimensionless part of the constitutive tensor, is symmetric and defines an almost complex structure, i.e., (240) and (241) are satisfied, then a lorentzian metric $g$ is determined, up to a conformal factor, such that

$$\chi^{ijkl} = 2 \sqrt{-g} g^{ik} g^{lj}, $$  \hspace{1cm} (275)

with $g := \det g_{ij} < 0$. In other words, the conditions (240) and (241) are sufficient to be able to write $\chi$ in the form (275)

But it is clear, see the discussion in Section 4.1, that (275) is sufficient to ensure that the corresponding operator $J$ satisfies (240) and (241). We thus conclude:

**The necessary and sufficient conditions to be able to write the dimensionless part $\chi$ of the constitutive tensor as**

$$\chi^{ijkl} = 2 \sqrt{-g} g^{ik} g^{lj}, $$  \hspace{1cm} (276)

**with a lorentzian metric $g$, are symmetry**

$$\chi^{ijkl} = \chi^{klji} $$  \hspace{1cm} (277)

**and closure**

$$\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} \chi^{\gamma\delta\theta} \epsilon_{\theta\mu\nu} \chi^{\mu\nu\lambda\rho} = -2\delta^{[\lambda}_{\alpha}\delta^{\rho]}_{\beta}. $$  \hspace{1cm} (278)
4.4 General solution of the closure relation

In order to be able to find an explicit expression for the metric in terms of quantities describing the components of the constitutive tensor, we have to solve the closure relation (233) or, equivalently, (238). We will solve this equations for the general case in which the constitutive tensor is asymmetric, so that we can later investigate the effect of relaxing the symmetry condition.

Let us now make the closure relation explicit. We turn back to the constitutive $6 \times 6$ matrix (239). We define dimensionless $3 \times 3$ matrices

\[
\mathcal{O} := \frac{A}{\zeta}, \quad \mathcal{C} := \frac{C}{\zeta}, \quad \mathcal{B} := \frac{B}{\zeta},
\]

In terms of these dimensionless matrices (we immediately drop the small circle for convenience), the closure relation reads

\[
\begin{align*}
\mathcal{A}^{ac} \mathcal{B}_{cb} + \mathcal{C}^{ac} \mathcal{C}_{cb} &= -\delta^a_b, \\
\mathcal{C}^{ac} \mathcal{A}_{cb} + \mathcal{A}^{ac} \mathcal{D}^b_c &= 0, \\
\mathcal{B}_{ac} \mathcal{C}^c_b + \mathcal{D}^c_a \mathcal{B}_{cb} &= 0, \\
\mathcal{B}_{ac} \mathcal{A}^{cb} + \mathcal{D}^c_a \mathcal{D}^b_c &= -\delta^b_a.
\end{align*}
\]

In $3 \times 3$ matrix notation, we then have

\[
\begin{align*}
\mathcal{A} \mathcal{B} + \mathcal{C}^2 &= -1_3, \\
\mathcal{C} \mathcal{A} + \mathcal{A} \mathcal{D} &= 0, \\
\mathcal{B} \mathcal{C} + \mathcal{D} \mathcal{B} &= 0, \\
\mathcal{B} \mathcal{A} + \mathcal{D}^2 &= -1_3.
\end{align*}
\]

Assume $\det \mathcal{B} \neq 0$. Then we can find the general non-degenerate solution as follows: Define the matrix $K_{ab}$ by

\[
K := \mathcal{B} \mathcal{C}, \quad \text{i.e.} \quad \mathcal{C} = \mathcal{B}^{-1} K,
\]

and substitute it into (285), then

\[
\mathcal{D} = -K \mathcal{B}^{-1}.
\]

Next, solve (283) with respect to $\mathcal{A}$:

\[
\mathcal{A} = -\mathcal{B}^{-1} - \mathcal{B}^{-1} K \mathcal{B}^{-1} \mathcal{B}^{-1}.
\]

Multiply (289) by $\mathcal{C}$ from the left and by $\mathcal{D}$ from the right, respectively, and find with (287) and (288),

\[
\begin{align*}
\mathcal{C} \mathcal{A} &= -\mathcal{B}^{-1} K \mathcal{B}^{-1} - \mathcal{B}^{-1} K \mathcal{B}^{-1} \mathcal{B}^{-1} \mathcal{B}^{-1} K \mathcal{B}^{-1}, \\
\mathcal{A} \mathcal{D} &= +\mathcal{B}^{-1} K \mathcal{B}^{-1} + \mathcal{B}^{-1} K \mathcal{B}^{-1} \mathcal{B}^{-1} K \mathcal{B}^{-1}.
\end{align*}
\]

Thus, (284) is automatically satisfied. Accordingly, only (286) has still to be checked. We compute its first and second term of its left side,

\[
\begin{align*}
\mathcal{B} \mathcal{A} &= -1_3 - K \mathcal{B}^{-1} K \mathcal{B}^{-1}, \\
\mathcal{D}^2 &= K \mathcal{B}^{-1} K \mathcal{B}^{-1}.
\end{align*}
\]
and find that it is fulfilled, indeed.

Summing up, we have derived the general solution of the closure relation (239) in terms of two arbitrary matrices $B$ and $K$ as

$$A = -B^{-1} - B^{-1} KB^{-1} K B^{-1},$$

$$C = B^{-1} K,$$

$$D = -K B^{-1},$$

(294) (295) (296)

or, in components,

$$A_{ab} = -B_{ab} - B_{ac} K_{cd} B_{de} K_{ef} B_{fb},$$

$$B_{ab} = B_{ba},$$

$$C_{a}^{b} = B_{ac} K_{cb},$$

$$D_{a}^{b} = -K_{ac} B_{cb}^{c}.$$  

(297) (298) (299) (300)

Here $B_{ab}$ are the components of the inverse $B^{-1}$, i.e. $B_{ab} B_{bc} = \delta_{a}^{c}$. The solution thus has $2 \times 9 = 18$ independent components. Alternatively, one can write the solution in a more compact notation as

$$A = -(1 + C^2) B^{-1},$$

$$D = -B B^{-1},$$

(301) (302)

which is parametrized by the arbitrary matrices $B$ and $C$ with altogether 18 independent components. In components, this means

$$A_{ab} = - (\delta_{a}^{d} + D_{a}^{c} D_{d}^{c}) B_{db},$$

$$B_{ab} = B_{ba},$$

$$C_{a}^{b} = -B_{ac} D_{a}^{c} B_{db},$$

$$D_{a}^{b} = D_{b}^{a}.$$  

(303) (304) (305) (306)

4.4.1 Explicit derivation of the metric components from the constitutive tensor

Here, the Schöenberg-Urbantke formula will be used to find an explicit expression for the conformal metric in terms of quantities parametrizing a constitutive tensor satisfying the assumptions of symmetry and closure. From the symmetry condition one finds that the constitutive tensor has the form (98), but with

$$A_{ab} = A_{ba}, \quad B_{ab} = B_{ba}, \quad D_{a}^{b} = C_{a}^{b},$$

(307)

so that $(2)_{X} = 0$, see (100). These symmetry conditions restrict the general solution of the closure relation found in the last section. From (299) and (300) we see that (307c) implies $-K_{bc} B^{ca} = B^{ca} K_{cb}$. Multiplying this equation by $B_{ad}$ and using (307c) we obtain $-K_{bd} = K_{db}$ which tells us that the matrix $K$ must be antisymmetric. Then (297) implies that $A$ is automatically symmetric, so no further conditions on $K$ arise from (307a). In summary, a symmetric constitutive tensor satisfying the closure relation is given by (297)–(300) with

$$B_{ab} = B_{ba}, \quad K_{ab} = -K_{ba},$$

(308)
As we saw in Section 4.3.1, to construct the metric using the Schönberg-Urbanke formula, we need a basis of the 3-dimensional space $S^+_3$ of self-dual 2-forms of the duality operator $J$.

To that end, we decompose the basis $b'$ of $S_6$ into two 3-dimensional column vectors, according to

$$b' = \left( \begin{array}{c} \beta^a \\ \gamma_b \end{array} \right), \quad a, b, \cdots = 1, 2, 3. \tag{309}$$

Then we can find their self-dual parts,

$$b'^+_1 = \frac{1}{2}(b' - iJb'), \tag{310}$$

and decompose them similarly as in (309), into

$$b'^+_1 = \left( \begin{array}{c} \beta^a \\ \gamma_b \end{array} \right). \tag{311}$$

Assuming, as we did before to find a solution of the closure relation, that the matrix $B$ is nonsingular, we can show that one can take $\gamma^+_1$ as a basis of the self-dual space $S^+_3$, since the remaining self-dual 2-forms $\beta^+$ can be written as linear combinations of the former. This is expected since $S_3$ is 3-dimensional. Explicitly, we have

$$\beta^+_a = (i\delta_a^b + B^{ac}K_{cb})B^{bd}\gamma^+_d. \tag{312}$$

This allows us to choose the 2-forms $\gamma^+_a$ or, equivalently, the triplet

$$S^+_a := -B^{ab}\gamma^+_b, \tag{313}$$

as basis of $S^+_3$.

The information of the constitutive tensor $\chi$ is now encoded into the triplet of 2-forms $S^+_a$.

The $(S^+_a)_{ij}$ are the components of the 2-form triplet $S^+_a = (S^+_a)_{ij} dx^i \wedge dx^j/2$. If we substitute the self-dual 2-forms $S^+_a$ into (274), we can display the metric explicitly in terms of the constitutive coefficients:

$$g_{ij} = \phi \left( \frac{\det B}{-k_b} - B_{ab} + (\det B)^{-1} k_a k_b \right). \tag{314}$$

Here $k^a := e^{abc}K_{bc}/2$, $k_a := B_{ab}k^b$ and $\phi$ is an arbitrary factor. The determinant of this metric is found to be $g = -\phi^4 (\det B)^2$ so that we verify that the metric in (314) has lorentzian signature.

4.4.2 Properties of the metric

The inverse of (314) can be found to be

$$g^{ij} = \frac{1}{\phi \det B} \left( \frac{1 - (\det B)^{-1}k_c k^c}{-k^a} \right) - k_b \frac{-(\det B)B^{ab}}{-(\det B)B^{ab}}. \tag{315}$$
With the help of (314) and (315), we can compute the Hodge duality operator $^*F$ attached to this metric. In terms of the components of the 2-form $F$, we have

$$^*F_{ij} := \frac{\sqrt{-g}}{2} \hat{e}_{ijkl} g^{km} g^{ln} F_{mn}. \quad (316)$$

This equation can be rewritten, in analogy to (70), by using the constitutive tensor $\chi\{g\}$, defined in (223), so that

$$^*F_{ij} = \frac{1}{4} \hat{e}_{ijkl} \chi^{klmn} F_{mn}, \quad (317)$$

In order to compare $\chi\{g\}$ with the original constitutive tensor $\chi$, we compute the corresponding 3-dimensional constitutive matrices of $\chi\{g\}$ according to (89) and (90).

Then straightforward calculations yield

$$A^{ab}_{\{g\}} = \sqrt{-g} \left( g^{00} g^{ab} - g^{0a} g^{0b} \right) A^{ab}, \quad (318)$$

$$\langle B_{\{g\}} \rangle_{ab} = \frac{1}{4} \sqrt{-g} \left( g^{ce} g^{df} - g^{de} g^{cf} \right) \hat{e}_{ace} \hat{e}_{dfb} = B_{ab}, \quad (319)$$

$$\langle C_{\{g\}} \rangle_{ab} = \frac{1}{2} \sqrt{-g} \left( g^{0c} g^{ad} - g^{0d} g^{ac} \right) \hat{e}_{bcd} = B^{ad} K_{db} = C_{a}^{b}. \quad (320)$$

Thus, $\chi\{g\} = \chi$, i.e., the metric extracted allows us to write the original duality operator $J$ as Hodge duality operator, $J = ^*$, when applied on 2-forms.

### 4.4.3 Alternative derivation

Here we provide an alternative derivation of the conformal metric, this time based on a direct computation of the Fresnel equation for the symmetric solution of the closure relation given by (297), (300) and (308). Using this equation and our general expressions (186)-(189) for the Fresnel equation in terms of the 3-dimensional constitutive matrices, we find after a straightforward calculation that

$$M = -\frac{1}{\det B} \left( 1 - k_{a}k_{a} \right)^{2}, \quad (321)$$

$$M^{a} = \frac{1}{\det B} 4k^{a} \left( 1 - \frac{k_{b}k_{b}}{\det B} \right), \quad (322)$$

$$M^{ab} = -\frac{1}{\det B} 4k^{a}k^{b} + 2B^{ab} \left( 1 - \frac{k_{c}k_{c}}{\det B} \right), \quad (323)$$

$$M^{abc} = -4B^{b(a}k^{c)}, \quad (324)$$

$$M^{abcd} = -\left( \det B \right) B^{(ab}B^{cd)}. \quad (325)$$

Substituting all this into the general Fresnel equation (182), we find

$$\mathcal{W} = -\frac{\theta^{2}}{\det B} \left[ q_{0}^{2} \left( 1 - \frac{k_{a}k_{a}}{\det B} \right) - 2q_{0}(q_{a}k^{a}) - (\det B) \left( q_{a}q_{b}B^{ab} \right) \right]^{2}. \quad (326)$$

Therefore we find that the Fresnel equation, $\mathcal{W} = 0$, can be written as

$$\left( g^{ij} q_{i}q_{j} \right)^{2} = 0, \quad (327)$$
where $g^{ij}$ is the (inverse) 4-dimensional conformal metric which arises from the duality operator and the closure relation. Direct comparison with (326) shows that

\begin{align*}
g^{00} &= \psi \left( 1 - \frac{k_a k^a}{\det B} \right), \\
g^{0a} &= -\psi k^a, \\
g^{ab} &= -\psi (\det B) B^{ab}.
\end{align*}

(328) (329) (330)

Here $\psi$ is the undetermined conformal factor.

Thus we indeed recover the null cone structure for the propagation of electromagnetic waves from our general analysis. The quartic surface degenerates to the null cone for the conformal metric $g$.

4.5 Relaxing the symmetry condition

Having verified and explicitly constructed the metric tensor from the constitutive tensor in the case in which it is symmetric and satisfies the closure relation, we want to study what consequences a nonvanishing skewon piece in the constitutive tensor could have on the emergence of a conformal structure. In particular, can the spacetime metric still be constructed? We have seen that the skewon piece does influence the light cone structure, and we also saw that an asymmetric constitutive tensor satisfying closure can accommodate 18 independent functions. Could this case, for instance, correspond to the emergence of two light cones, each with its 9 independent components?

Unfortunately, a computation of the Fresnel tensor and the Fresnel equation using the general asymmetric solution (297) to (300) of the closure relation does not help much to recognize a possible double light cone structure. The Fresnel equation is still a quartic equation, in general.

In what follows, we will study a particular case in order to try to get an idea of the qualitative properties that an additional skewon piece can induce. More details can be found in [61]. We will consider the case in which $K = 0$, which implies $D = 0$, $C = 0$ \footnote{i.e., a medium with no ‘magneto-electrical’ properties} and $A = -B^{-1}$.

Furthermore, we decompose the arbitrary matrix $B$ into its symmetric and antisymmetric parts,

\[ B_{ab} = b_{ab} + \epsilon_{abc} n^c, \quad \text{with} \quad b_{ab} := B_{(ab)}, \quad n^c := \epsilon^{cab} B_{[ab]}. \]

(331)

Note that $b_{ab}$ contributes only to $^{(1)}\chi$ and $n^c$ only to $^{(2)}\chi$. In this case, by substituting (331) into (182), we obtain:

\[ W = -\frac{q_0^2}{(\det b + n^2)} \left[ q_0^4 - 2q_0^2 (\bar{q}^2 - (qn)^2) + (\bar{q}^2 + (qn)^2)^2 \right]. \]

(332)

Here we used the abbreviations $n^2 := b_{ab} n^a n^b$, $\bar{q}^2 := \bar{b}_{ab} q_a q_b$, $qn := q_a n^a$, and $\bar{b}_{ab}$ is the (symmetric) matrix of the minors of $b_{ab}$.

In general, the right hand side of (332) is neither a square of a quadratic polynomial nor a product of two quadratic polynomials. In other words, neither a light cone nor a birefringence (double light cone) structure arises generically.
Formally, one can rewrite (332) as

$$W = -\frac{q_0^2}{(\det b + n^2)} \left[ q_0^2 + \left( qn + \sqrt{-q^2} \right)^2 \right] \left[ q_0^2 + \left( qn - \sqrt{-q^2} \right)^2 \right]$$

(333)

Thus, the question of the reducibility of the Fresnel equation translates into the algebraic problem of whether the square root $\sqrt{-q^2}$ is a real linear polynomial in $q_a$. There are three cases, depending on the rank of the $3 \times 3$ matrix $b_{ab}$, see [61].

(i) When $b_{ab}$ has rank 3, no factorization into light cones is possible (the roots $\alpha$ are complex), unless $n^a = 0$. This latter condition implies $(2) \chi = 0$, and the previous results are recovered.

(ii) When $b_{ab}$ has rank 2, i.e., $\det b = 0$, but at least one of the minors is nontrivial. Then, without loss of generality, one can assume:

$$b_{ab} = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(334)

In order to avoid complex solutions, we have to assume that the minor $b_{33} = -\mu^2 < 0$, so that $\sqrt{-q^2} = \mu q_3$. In this case we find

$$W = -\frac{q_0^2}{b_{11}(n^1)^2 + 2b_{12}n^1n^2 + b_{22}(n^2)^2} \times \left[ q_0^2 + \left( q_n^1 + q_2n^2 + q_3(n^3 + \mu) \right)^2 \right] \times \left[ q_0^2 + \left( q_n^1 + q_2n^2 + q_3(n^3 - \mu) \right)^2 \right].$$

(335)

i.e. two different light cones. We can read off, up to conformal factors, the components of the two corresponding 'metric' tensors defining the light cones:

$$g^{ij}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (n^1)^2 & n^1n^2 & n^1(n^3 + \mu) \\ 0 & n^1n^2 & (n^2)^2 & n^2n^3 \\ 0 & n^1(n^3 + \mu) & n^2n^3 & (n^3 + \mu)^2 \end{pmatrix},$$

(336)

$$g^{ij}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (n^1)^2 & n^1n^2 & n^1(n^3 - \mu) \\ 0 & n^1n^2 & (n^2)^2 & n^2n^3 \\ 0 & n^1(n^3 - \mu) & n^2n^3 & (n^3 - \mu)^2 \end{pmatrix}. $$

(337)

One verifies that $\det(g^{ij}_1) = \det(g^{ij}_2) = (n^1)^2(n^2)^2b_{33} = -(n^1)^2(n^2)^2\mu^2 < 0$, so that both metrics have the correct Lorentzian signature.

(iii) When the $3 \times 3$ matrix $b_{ab}$ has rank 1. In this case corresponds to the case 2 with $\mu = 0$. The Fresnel equation reduces to a single light cone, but the resulting metric is degenerated, since $\det(g^{ij}) = 0$.

Thus, we have demonstrated with a special case that some of the asymmetric solutions of the closure relation can yield birefringence.
We saw that the conditions of closure and symmetry of $\chi$ are sufficient for the existence of a well-defined light cone structure. If any of these conditions is violated, the light cone structure seems to be lost. The necessary conditions have still to be found.

5 Conclusions and prospects

In this work, we have developed a general framework for describing classical electrodynamics in a general 4-dimensional medium. We have paid special attention to identify the structures which are metric-independent, and which can therefore be applied to a great variety of particular cases, from generalized models describing the electromagnetic properties of spacetime to classical optics in material media.

One of the central results of this work is summarized in the Fresnel equation (181) and in particular in the Fresnel tensor (180). This equation determines the local properties of the propagation of waves, in particular it describes the geometry of the wave covectors. It is a remarkable result that this important equation could be derived for any linear medium. Furthermore, the result can also be used to describe the effective properties of electromagnetic perturbations in nonlinear media. Its structure, i.e., its dependence on the constitutive tensor of the corresponding medium is highly nontrivial. Cubic structures are not very common in physics. To the best of my knowledge, no previous derivation of the Fresnel equation has been given which is as general as that in Section 3.2, see also the early work of Tamm [69].

It is also important to emphasize that the Fresnel equation and the whole formalism is generally covariant. This means that no artificial or particular coordinate choices are necessary and that only quantities describing intrinsic properties, in our case the constitutive tensor, of the physical system under consideration enter in the formalism. Of course, inside a given medium, for example, one can choose specific coordinates which are useful for concrete calculations. This is usually the case if the medium possesses some symmetry, as defined in Section 2.7.1. Then an adapted coordinate system can be used which exploits this symmetry, so that calculations become simpler. This is, however, only a convenient choice, determined by the properties of the system (the constitutive tensor), but not a necessary a priori ingredient. In a material medium which is not isotropic, for instance, there is no a priori reason to use cartesian coordinates, the later are rather coordinates adapted to an isotropic medium, as for instance the minkowskian vacuum. Therefore, it is highly satisfactory to be able to describe the electromagnetic properties of a material medium in a generally covariant way.

We also studied the properties of the three different irreducible pieces which a general constitutive tensor can contain. We have proved that the first, the symmetric piece $^{(1)}\chi$ is essential for the medium to admit well behaved wave properties. This piece is therefore the most important object determining the geometry of wave covectors and the light cone structure under some particular conditions. The second piece $^{(2)}\chi$ was shown to describe dissipative properties of the medium. A nonvanishing $^{(2)}\chi$ piece can lead to dissipation of electromagnetic energy, even if the material’s properties are time independent. Furthermore, the skewon piece does influence the wave propagation and therefore the light cone structure. We have shown, studying some particular cases, that the skewon field can even give rise to a double light cone structure. The third possible irreducible piece $^{(3)}\chi$ of a constitutive tensor, the abelian axion, is more elusive. It does not contribute to the energy-momentum density of the electromagnetic field and does not affect the local properties of the electromagnetic waves and thus not the
light cone structure either. It is possible, however, that it could influence the properties of wave propagation over finite (long) distances. Examples of all the three irreducible pieces of a constitutive can be found in the literature. The skewon field has been discussed so far only for material media violating CP symmetry, but up to now not as a model for the classical electromagnetic properties of spacetime. If one follows the common assumption that a fundamental physical system should be describable in terms of an action principle, then the detection of a skewon piece could be interpreted as indicating an underlying substructure of the system under study. Of course, we do not claim here that the skewon field is actually nonvanishing in spacetime. We do believe, however, that it is a possible piece which can be used to quantify well defined, in general dissipative, properties of spacetime. Additionally, the skewon piece can be used to model possible effects violating local Lorentz invariance. Within the framework of GR, where a metric is present, one can show that the skewon piece necessarily violates local Lorentz invariance, see appendix C.

Furthermore, we have studied the conditions under which a light cone structure is induced. We have seen that, if the constitutive tensor is symmetric and satisfies a closure relation, a light cone is, in fact, induced. We also saw that the necessary and sufficient conditions for a constitutive tensor to be written as proportional to the Hodge dual operator of some metric, are symmetry and closure. This one-to-one relationship is valid when we formulate it in terms of the constitutive tensor. However, it is still an open question whether symmetry and closure are necessary conditions for defining a light cone, in the sense that the quartic Fresnel equation reduces to a quadratic equation for the wave covectors, as discussed at the beginning of Section 4. We proved that symmetry and closure are sufficient conditions for such a reduction. Strictly speaking, closure of the whole constitutive tensor is not a necessary condition since an additional axion piece does not disturb the Fresnel equation. However, it may be that closure of only the principal irreducible piece (1)χ and symmetry, i.e. (2)χ = 0, could be the necessary conditions. A proof of this conjecture is, however, still missing. Non-trivial examples supporting this conjecture can be found in [53].

Other interesting open issues which could be investigated are: We have seen that the constitutive tensor defines a certain generalization of the conformal properties of spacetime. Does it also defines a (generalized) affine structure? To answer this question, it would be interesting to study some additional properties of the propagation of waves, for instance, how the polarization vector propagates along a light ray. It should be possible to find an answer to this question within our general formalism and it is expected to involve some affine properties (under which parallel displacement is the polarization vector constant along the light ray?).

A  The situation so far

See Fig. 8 on page 777.

B  Electrodynamics in a material medium

In a material medium, treated as a continuum, one can describe the properties of electromagnetic fields by means of macroscopic Maxwell’s equations. A material medium can be defined in general terms as a region with a given total charge at macroscopic scales, but with a microscopic charge substructure. This substructure corresponds to the individual electrons, ions,
etc. which form the medium. The particular distribution and dynamics of the charges forming the material are in general unknown or difficult to model. Therefore, it is useful to describe the electrodynamics of materials only in term of the so called ‘external’ charges, which are those that can, in principle, be manipulated in experiments.

The procedure to define macroscopic Maxwell equations for the material medium starting from the microscopic ones consist then in the separation of the total charge in the sum of the two contributions originating ‘from the inside’ (bound or polarizational charge) and ‘from the outside’ (free or external charge):

\[ J = J^{\text{mat}} + J^{\text{ext}}. \]  

(338)

Bound charges and bound currents are inherent characteristics of matter determined by the medium itself. In contrast, external charges and external currents appear in general outside and inside matter and can be prepared for a specific purpose.

The external charge is assumed to be conserved, and consequently the bound charge is conserved, too:

\[ dJ^{\text{mat}} = 0, \quad dJ^{\text{ext}} = 0. \]  

(339)

As in the case of vacuum, see Refs. [24, 27, 52], this allows us to introduce the corresponding excitation \( H^{\text{mat}} \) as a ‘potential’ for the bound current:
\[ dH^{\text{mat}} = J^{\text{mat}}. \] (340)

In a (1+3) decomposition, the 6 components of \( H^{\text{mat}} \) are identified with the polarization \( P \) and magnetization \( M \).

Defining now the external excitation \( H^{\text{ext}} \) as

\[ H^{\text{ext}} := H - H^{\text{mat}}, \] (341)

we find the inhomogeneous Maxwell equation

\[ dH^{\text{ext}} = J^{\text{ext}}. \] (342)

Here, only external quantities are involved.

It remains to specify the constitutive law. In addition to the spacetime relation, \( H = H(F) \), the knowledge of the internal structure of a medium yields the macroscopic excitation \( H^{\text{mat}} \) (i.e., the polarization and magnetization) as a function of the electromagnetic field strength \( F \) (and possibly of some thermodynamical variables describing the material continuum). Then the constitutive law of the material is given by

\[ H^{\text{ext}} = H^{\text{ext}}(F) = H(F) - H^{\text{mat}}(F). \] (343)

C Local Lorentz invariance

Given a lorentzian metric, we can define the notion of local Lorentz invariance. Let \( T^{i_1 \ldots i_p} \) be the contravariant coordinate components of a tensor field and \( T^{\alpha_1 \ldots \alpha_p} := e_{i_1}^{\alpha_1} \ldots e_{i_p}^{\alpha_p} T^{i_1 \ldots i_p} \) its frame components with respect to an orthonormal frame \( e_{\alpha} = e_{i}^{\alpha} \partial_{i} \). A tensor is said to be local Lorentz invariant at a given point, if its frame components are invariant under a local Lorentz rotation of the orthonormal frame. Similar considerations extend to tensor densities.

There are only two geometrical objects which are numerically invariant under local Lorentz transformations: the Minkowski metric \( \eta_{\alpha \beta} \) and the Levi-Civita tensor density \( \hat{\epsilon}^{\alpha \beta \gamma \delta} \). Thus

\[ T^{\alpha \beta \gamma \delta} = \phi(x) \left( \eta^{\alpha \gamma} \eta^{\beta \delta} - \eta^{\beta \gamma} \eta^{\alpha \delta} \right) + \alpha(x) \epsilon^{\alpha \beta \gamma \delta} \] (344)

is the most general form of the frame components of a locally Lorentz invariant contravariant fourth rank tensor with the symmetries \( T^{ijkl} = -T^{jikl} = -T^{ijlk} \). Here \( \phi \) and \( \alpha \) are scalar and pseudo-scalar fields, respectively. Therefore, back in coordinate components, we find that

\[ \chi^{ijkl} = \phi(x) \sqrt{-g} \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right) + \alpha(x) \epsilon^{ijkl} \] (345)

is the most general form of a locally Lorentz invariant constitutive tensor allowed in a space with a lorentzian metric. Notice that this constitutive tensor is necessarily symmetric. In other words, any additional asymmetric piece \( \sim (2) \) \( \chi \) would violate local Lorentz invariance. This was found before in the particular examples of constitutive tensors studied by Nieves and Pal, see Section 2.6.3.
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References

[1] B. Abbott et al. (D0 Collaboration), A search for heavy pointlike Dirac monopoles, Phys. Rev. Lett. 81 (1998) 524–529.
[2] C. Barceló, S. Liberati, and M. Visser, Analogue gravity from field theory normal modes?, Class. Quantum Gravity 18 (2001) 3595–3610.
[3] C. Barceló, S. Liberati, and M. Visser, Refringence, field theory, and normal modes, Class. Quantum Gravity 19 (2002) 2961–2982. e-print Archive: gr-qc/0111059.
[4] F. Bopp, Prinzipien der Elektrodynamik, Z. Physik 169 (1962) 45–52.
[5] M. Born and L. Infeld, Foundations of the new field theory, Proc. Roy. Soc. (London) A 144 (1934) 425–451.
[6] C. H. Brans, Complex 2-form representation of the Einstein equations: The Petrov Type III solutions, J. Math. Phys. 12 (1971) 1616–1619.
[7] C. H. Brans, Complex structures and representations of the Einstein equations, J. Math. Phys. 15 (1974) 1559–1566.
[8] R. Capovilla, T. Jacobson, and J. Dell, General relativity without the metric, Phys. Rev. Lett. 63 (1989) 2325–2328.
[9] R. Capovilla, J. Dell, T. Jacobson, and L. Mason, Self-dual 2-forms and gravity, Class. Quantum Gravity 8 (1991) 41–57.
[10] R. Capovilla, J. Dell and T. Jacobson, A pure spin-connection formulation of gravity, Class. Quantum Gravity 8 (1991) 59–73.
[11] S. M. Carroll, G. B. Field, and R. Jackiw, Limits on a Lorentz- and parity-violating modification of electrodynamics, Phys. Rev. D 41 (1990) 1231–1240.
[12] R. Chanda, J. Nieves, P. Pal, Astrophysical constraints on axion and Majorana couplings, Phys. Rev. D 37 (1988) 2714–2721.
[13] L. Cooper and G. E. Stedman, Axion detection by ring lasers, Phys. Lett. B 357 (1995) 464–468. e-print Archive: hep-ph/9502256.
[14] V. I. Denisov and M. I. Denisov, Verification of Einstein’s principle of equivalence using a laser gyroscope in terrestrial conditions, Phys. Rev. D 60 (1999) 047301.
[15] J. Ellis et al., A search in Gamma-ray burst data for nonconstancy of the velocity of light, Ap. J. 535 (2000) 139–151.
[16] T. Frankel, The Geometry of Physics – An Introduction. Cambridge University Press, Cambridge (1999).
[17] J. Hadamard, Leçons sur la propagation des ondes et les équations de l’hydrodynamique (Hermann: Paris, 1903).
[18] G. Harnett, Metrics and dual operators, J. Math. Phys. 32 (1991) 84–91.
[19] G. Harnett, The bivector Clifford algebra and the geometry of Hodge dual operators, J. Phys. A 25 (1992) 5649–5662.
[20] M. P. Haugan and T. F. Kauffmann, New test of the Einstein equivalence principle and the isotropy of space, Phys. Rev. D 52 (1995) 3168–3175.
[21] M. Haugan and C. Lämmerzahl, On the experimental foundations of the Maxwell equations, Ann. Phys. (Leipzig) 11 (2000) 507–510.
[22] Y. D. He, Search for a Dirac magnetic monopole in high energy nucleus-nucleus collisions, Phys. Rev. Lett. 79 (1997) 3134–3137.
[23] F. W. Hehl, Yu. N. Obukhov, and G. F. Rubilar, Classical electrodynamics: A Tutorial on its Foundations. In: Quo vadis geodesia...? Festschrift for Erik W. Grafarend, F. Krumm and V. S. Schwarze (eds.) Univ. Stuttgart, ISSN 0933-2839 (1999) pp. 171–184. e-print Archive: physics/9907046.

[24] F. W. Hehl, Yu. N. Obukhov, and G. F. Rubilar, Spacetime metric from linear electrodynamics II, Ann. Phys. 11 (2000) Spec. Iss. SI 71-SI 78. e-print Archive: gr-qc/9911096.

[25] F. W. Hehl and Yu. N. Obukhov, How does the electromagnetic field couple to gravity, in particular to metric, nonmetricity, torsion, and curvature? In “Testing Relativistic Gravity in Space: Gyroscopes, Clocks, Interferometers ...”. Proceedings of the 220th Heraeus-Seminar, 22–27 August 1999 in Bad Honnef, C. Lämmerzahl et al. (eds.). Springer, Berlin (2000). e-print Archive: gr-qc/0001010.

[26] F. W. Hehl and Yu. N. Obukhov, On the energy-momentum current of the electromagnetic field in a pre-metric axiomatic approach. I. e-print Archive: gr-qc/0103020.

[27] F. W. Hehl and Yu. N. Obukhov, Foundations of classical electrodynamics (Birkhäuser: Boston, MA, 2002) To be published.

[28] F. W. Hehl, Yu. N. Obukhov, G. F. Rubilar, On a possible new type of a T odd skewon field linked to electromagnetism. In: Developments in Mathematical and Experimental Physics A. Macias, F. Uribe, and E. Diaz, eds. Volume A: Cosmology and Gravitation (Kluwer Academic/Plenum Publishers: New York, 2002) pp. 241–256. e-print Archive gr-qc/0203096.

[29] W. Heisenberg and H. Euler, Folgerungen aus der Dirac’schen Theorie des Positrons, Z. Phys. 98 (1936) 714–732.
[47] J. F. Nieves and P. B. Pal, P and CP-odd terms in the photon self-energy within a medium, Phys. Rev. D 39 (1989) 652–659.

[48] J. F. Nieves and P. B. Pal, The third electromagnetic constant of an isotropic medium, Am. J. Phys. 62 (1994) 207–216.

[49] E. B. Norman and J. N. Bahcall, Improved limit on charge conservation derived from GA–71 solar neutrino experiments, Phys. Rev. D 53 (1996) 4086–4088.

[50] Yu. N. Obukhov and S. I. Tertychniy, Vacuum Einstein equations in terms of curvature forms, Class. Quantum Gravity 13 (1996) 1623–1640.

[51] Yu. N. Obukhov, T. Fukui, and G. F. Rubilar, Wave propagation in linear electrodynamics, Phys. Rev. D 62 (2000) 044050. e-print Archive: gr-qc/0005018.

[52] Yu. N. Obukhov and F. W. Hehl, Space-time metric from linear electrodynamics, Phys. Lett. B 458 (1999) 466–470. e-print Archive gr-qc/9904067.

[53] Yu. N. Obukhov and G. F. Rubilar, Fresnel analysis of the wave propagation in nonlinear electrodynamics, Phys. Rev. D 66 (2002) 024042. e-print Archive: gr-qc/0204028.

[54] T. H. O'Dell, The electrodynamics of magneto-electric media, Selected of monographs on selected topics in solid state physics. Editor E. P. Wohlfarth. North-Holland, Amsterdam (1970).

[55] R. D. Peccei and H. R. Quinn, CP Conservation in the presence of pseudoparticles, Phys. Rev. Lett. 38 (1977) 1440–1443.

[56] A. Peres, Electromagnetism, geometry, and the equivalence principle, Ann. Phys. (USA) 19 (1962) 279–286.

[57] C. Piron and D. J. Moore, New aspects of field theory, Turk. J. Phys. 19 (1995) 202–216.

[58] J. Plebański, Electromagnetic waves in gravitational fields, Phys. Rev. 118 (1960) 1396–1408.

[59] E. J. Post, Formal Structure of Electromagnetics – General Covariance and Electromagnetics. North Holland, Amsterdam (1962) and Dover, Mineola, New York (1997).

[60] R. A. Puntigam, C. Lämmerzahl, and F. W. Hehl, Maxwell’s theory on a post-Riemannian spacetime and the equivalence principle, Class. Quantum Gravity 14 (1997) 1347–1356.

[61] G. F. Rubilar, Y. N. Obukhov, and F. W. Hehl, General covariant Fresnel equation and the emergence of the light cone structure in pre-metric electrodynamics, Int. J. Mod. Phys. D 11 (2002) No. 8 or 9, to appear. e-print Archive: gr-qc/0109012.

[62] M. Schönberg, Electromagnetism and gravitation, Rivista Brasileira de Fisica 1 (1971) 91–122.

[63] J. A. Schouten, Tensor Analysis for Physicists. 2nd ed., Dover, Mineola, New York (1989).

[64] E. Schrödinger, Space-Time structure. Reprinted with corrections, Cambridge University Press, Cambridge (1960).

[65] R. Schützhold, G. Plunien, and G. Soff, Dielectric black hole analogs, Phys. Rev. Lett. 88 (2002) 061101 (4 pages).

[66] G. E. Stedman, Ring-laser tests of fundamental physics and geophysics, Rep. Prog. Phys. 60 (1997) 615–688.

[67] R. I. Steinberg et al., Experimental test of charge conservation and the stability of the electron, Phys. Rev. 12 (1975) 2582–2586.

[68] J. L. Synge, Relativity: The General Theory, North Holland, Amsterdam (1960).

[69] I. E. Tamm, Relativistic crystaloptics in relation with the geometry of bi-quadratic form, J. of Russ. Phys.-Chem. Soc. 57 (1925) 209 –214 (in Russian). [Reprinted in: I. E. Tamm, Collected Papers (Nauka: Moscow, 1975) vol. 1, p. 33–61 (in Russian)].

[70] G. ’t Hooft, A chiral alternative to the vierbein field in general relativity, Nucl. Phys. B 357 (1991) 211–221.

[71] R. A. Toupin, Elasticity and electro-magnetics, in: Non-Linear Continuum Theories, C. I. M. E. Conference, Bressanone, Italy 1965. C. Truesdell and G. Grioli, coordinators. Pp.206–342.

[72] S. A. Tretyakov et al., Reply to comment on ’Reflection and transmission by a uniaxial bi-anisotropic slab under normal incidence of plane waves’, J. Phys. D 32 (1999) 2705–2706.

[73] H. Urbanke, A quasi-metric associated with SU(2) Yang-Mills field, Acta Phys. Austriaca Suppl. XIX (1978) 875–816.
[74] H. Urbantke, On integrability properties of $SU(2)$ Yang-Mills fields. I. Infinitesimal part, J. Math. Phys. 25 (1984) 2321–2324.
[75] K. van Bibber and D. Kinion, Review of Dark-Matter axion experiments, Nucl. Phys. B (Proc. Suppl.) 91 (2001) 376–383.
[76] D. van Dantzig, The fundamental equations of electromagnetism, independent of metrical geometry, Proc. Camb. Philos. Soc., 30 (1934) 421–427.
[77] M. Visser, Comment on ‘Relativistic effects of light in moving media with extremely low group velocity’, Phys. Rev. Lett., 85 (2000) 5252 (1 page).
[78] M. Visser, C. Barceló, and S. Liberati, Analogue models of and for gravity, Gen. Rel. Grav., 34 (2002) 1719–1734. e-print Archive: gr-qc/0111111.
[79] C. Wang, Mathematical Principles of Mechanics and Electromagnetism, Part B: Electromagnetism and Gravitation, Plenum Press, New York (1979).
[80] W.S. Weiglhofer and A. Lakhtakia, Further comments on the non-existence of linear non-reciprocal bi-isotropic media, J. Phys. A: Math. Gen. 31 (1998) 1113–1114.
[81] W.S. Weiglhofer and A. Lakhtakia, The Post constraint revisited, Arch. Elektron. Übertrag. 52 (1998) 276.
[82] S. Weinberg, A new light boson?, Phys. Rev. Lett. 40 (1978) 223–226.
[83] F. Wilczek, Problem of strong P and T invariance in the presence of instantons, Phys. Rev. Lett. 40 (1978) 279–282.