A GROSS-KOHNEN-ZAGIER THEOREM FOR NON-SPLIT CARTAN CURVES

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Abstract. Let $p$ be a prime number and let $E$ be rational an elliptic curve of conductor $p^2$ and odd analytic rank. We prove that the positions of its special points arising from non-split Cartan curves and imaginary quadratic fields where $p$ is inert are encoded in the Fourier coefficients of a Jacobi form of weight 6 and lattice index of rank 9, obtaining a result analogous to that of Gross, Kohnen and Zagier.

1. Introduction

Let $E/Q$ be an elliptic curve of conductor $N$ and odd analytic rank. Let $D$ be a negative fundamental discriminant prime to $2N$. Assume that $p$ splits in $Q(\sqrt{D})/Q$ for every $p | N$ (the Heegner condition). For each $r \in \mathbb{Z}/2N$ such that $r^2 \equiv D \pmod{4N}$, there exists a special cycle on the Jacobian of the modular curve $X_0^+(N)$, which under the modular parametrization maps to a point $Q^*_{D,r} \in E(Q) \otimes \mathbb{Q}$.

In their celebrated article [GKZ87], Gross, Kohnen and Zagier prove that the points $Q^*_{D,r}$ are aligned, and moreover, that their positions in the line are given by the Fourier coefficients of a Jacobi form $\phi_E$ of weight 2 and index $N$. Moreover, they show that the space generated by the points $Q^*_{D,r}$ is non-trivial if and only if $L'(E/Q,1) \neq 0$.

Later on, Zhang showed in [Zha01] how to obtain special points on general Shimura curves under more relaxed Heegner conditions, thus giving flexibility in the choice of the discriminants.

Consider the case $N = p^2$ where $p$ is an odd prime. Let $D$ be as above but now assume that $p$ is inert in $Q(\sqrt{D})/Q$. Then the construction of Zhang gives points $Q^+_{D,s} \in E(Q) \otimes \mathbb{Q}$ by considering special cycles on the Jacobian of the non-split Cartan curve $X^+_n(p)$. In this context, the main result of this article is the following analogue of [GKZ87, Theorem C].

Theorem (Theorem 10.4). Let $E/Q$ be an elliptic curve of conductor $p^2$ and odd analytic rank. There exists a positive definite even lattice $(\mathcal{L}_n, \beta)$ of rank 9 and a Jacobi form $\psi_E$ of weight 6 and lattice index $\mathcal{L}_n$ such that for every negative discriminant $D$ which is a non-square modulo $p$ we have

$$Q^+_{D,s} = c_{\psi_E} (\beta(s) - D/4p^2, s) Q,$$

for some $Q \in E(Q) \otimes \mathbb{Q}$ which is non-zero if and only if $L'(E/Q,1) \neq 0$.
Borcherds gave in [Bor99] an astonishing generalization of the result from Gross, Kohnen and Zagier. He considers Heegner divisors associated to certain even lattices, and proves that their generating series is modular. He then obtains (the “ideal statement” of) [GKZ87, Theorem C] by applying his modularity result to a specific lattice.

In order to use the result of Borcherds we need to find a concrete lattice whose Heegner divisors are the special cycles coming from \( X^+_{ns}(p) \), at least for discriminants prime with \( p \). This lattice is obtained by carefully studying the action of the non-split Cartan group on the set of quadratic forms giving rise to the special points. It turns out not to be stably isomorphic to a rank 1 lattice, which explains why our main result involves Jacobi forms of higher rank lattice index.

In the classical case the Jacobi form \( \phi_E \) is, by definition, a Hecke eigenform with the same eigenvalues as the modular form attached to \( E \). We are able to prove the analogous statement for \( \psi_E \) combining Theorem 10.4 with the formal distribution relations satisfied by the Heegner points on the non-split Cartan curves.

In addition, as an application of [Koh18], we relate the Fourier coefficients of \( \psi_E \) with certain coefficients of the Jacobi form \( \phi_E \). This gives a link with the classical theory, which is a particularity of the Atkin-Lehner sign of \( E \) at \( p \) being equal to 1.

We also focus on the explicit computation of the Jacobi form \( \psi_E \), as this was one of the main motivations for this article.

This article is organized as follows. Firstly, we review the non-split Cartan curves. In Section 4 we consider their special points and their relation with certain quadratic forms, which we study carefully in the next section. In Section 6 we define the relevant special cycles and consider the action of Hecke and Atkin-Lehner operators on them. Afterwards, we recall the results from [Zha01] which show that the non-split Cartan curve uniformizes the elliptic curve, and that give a formula for the height of the special cycles in this context. In Section 8 we review some facts about vector valued and Jacobi modular forms, and the connection between them. In the following section we explain how to build up the desired lattice in order to apply the results from Borcherds, relying on the results of Section 5. In Section 10 we prove the main results of this article, including Theorem 10.4. In the penultimate section we give the connection with classical Jacobi forms. We end this article with an explicit example, checking the validity of the conclusions we achieved.

2. Setting

Let \( p \) be an odd prime, and let \( \varepsilon \in \mathbb{Z}/4p^2 \) be a non-square modulo \( p \) such that \( \varepsilon \equiv 1 \pmod{4} \).

Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( p^2 \) and odd analytic rank. In particular, its local Atkin-Lehner sign \( w_p \) at \( p \) equals 1.

Let \( K \) be an imaginary quadratic field such that \( p \) is inert in \( K \) and let \( \mathcal{O} \subseteq \mathcal{O}_K \) be an order of discriminant \( D \) inside the ring of integers of \( K \). We will always assume (unless explicitly stated) that \( D \) is not divisible by \( p \).
3. Non-split Cartan curves

The non-split Cartan order is
\[ M_{ns} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : a \equiv d, b \equiv c \pmod{p} \right\}. \]

The non-split Cartan group modulo \( p \) is
\[ C_{ns} = \{ \overline{M} \in \text{GL}_2(\mathbb{F}_p) : M \in M_{ns} \}, \]
where \( \overline{M} \) denotes the reduction modulo \( p \) of \( M \). This is an abelian subgroup of \( \text{GL}_2(\mathbb{F}_p) \) isomorphic to \( \mathbb{F}_p^2 \). We also have the corresponding arithmetic non-split Cartan group \( \Gamma_{ns} = M_{ns} \cap \text{GL}_2^+(\mathbb{Q}) \).

We also consider
\[ M_{ns}^+ = M_{ns} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : a \equiv -d, b \equiv -c \pmod{p} \right\}, \]
and, as before,
\[ C_{ns}^+ = \{ \overline{M} \in \text{GL}_2(\mathbb{F}_p) : M \in M_{ns}^+ \}. \]

This group is the normalizer of \( C_{ns} \) inside \( \text{GL}_2(\mathbb{F}_p) \), and the quotient \( C_{ns}^+/C_{ns} \) is of order \( 2 \). This normalizer is a maximal subgroup of \( \text{GL}_2(\mathbb{F}_p) \). In addition, we set \( \Gamma_{ns}^+ = M_{ns}^+ \cap \text{GL}_2^+(\mathbb{Q}) \). Thus, the elements of \( \Gamma_{ns}^+ \) normalize \( \Gamma_{ns} \) and the quotient \( \Gamma_{ns}^+ / \Gamma_{ns} \) is also of order \( 2 \).

We consider the non-split Cartan (open) modular curve \( Y_{ns} = \Gamma_{ns} \backslash \mathcal{H} \) (resp. \( Y_{ns}^+ = \Gamma_{ns}^+ \backslash \mathcal{H} \)). Its compactification is given by \( X_{ns} = \Gamma_{ns} \backslash \mathcal{H}^* \) (resp. \( X_{ns}^+ = \Gamma_{ns}^+ \backslash \mathcal{H}^* \)), where \( \mathcal{H}^* \) is the union of the upper half plane \( \mathcal{H} \) and the cusps. Furthermore, we let \( J_{ns} \) (resp. \( J_{ns}^+ \)) denote the Jacobian of \( X_{ns} \) (resp. \( X_{ns}^+ \)).

The following result allows us to work with cycles of not necessarily degree 0 if we are willing to kill the torsion.

**Proposition 3.1.** The natural map \( J_{ns} \otimes \mathbb{Q} \to \text{Pic}(X_{ns}) \otimes \mathbb{Q} \) is an isomorphism.

**Proof.** Since the natural map \( J_{ns} \to \text{Pic}(X_{ns}) \) is injective, we only need to prove that after tensoring with \( \mathbb{Q} \) it is surjective. Let \( \xi \in \text{Pic}(X_{ns}) \otimes \mathbb{Q} \) be the Hodge class [Zha01, Section 6.2], which is a certain divisor of degree 1 that is a rational linear combination of cusps. By the Manin-Drinfeld theorem there exists \( \mathcal{M} \geq 1 \) such that \( \mathcal{M}\xi = 0 \). Given \( \mathcal{D} \in \text{Pic}(X_{ns}) \), let \( \mathcal{D}' = \mathcal{D} - \deg(\mathcal{D})\xi \in J_{ns} \otimes \mathbb{Q} \). Then \( \mathcal{D}' \otimes \mathcal{M} \) maps to \( \mathcal{D} \otimes \mathcal{M} - \deg(\mathcal{D})\xi \otimes \mathcal{M} = \mathcal{D} \otimes \mathcal{M} \). \( \square \)

4. Special points on non-split Cartan curves

An embedding \( \iota : K \to M_2(\mathbb{Q}) \) is an **optimal embedding** of \( \mathcal{O} \) into \( M_{ns} \) if
\[ \iota(K) \cap M_{ns} = \iota(\mathcal{O}). \]
Any such embedding gives rise to a unique point \( z \in \mathcal{H} \) fixed under the action of \( \iota(K^\times) \). The point \( z \in X_{ns}(\mathbb{C}) \) will be called a **special point** (in some situations also called CM or Heegner point) on the non-split Cartan curve. It is known that \( z \in X_{ns}(H_\mathcal{O}) \), where \( H_\mathcal{O} \) denotes the ring class field associated to the order \( \mathcal{O} \) (see for example [Koh18, Prop 3.1]).

Although this definition of special points has the merit of being very general, it is often convenient to have a concrete description of these points in terms of quadratic forms, as in [Gro84].
Fix $\omega \in K$ such that $\mathcal{O} = \mathbb{Z} + \omega \mathbb{Z}$. Let $\iota$ be an embedding of $\mathcal{O}$ into $M_{ns}$. Write

$$\iota(\omega) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (4.1)$$

Since $\iota(\omega) \in M_{ns}$ we know that $a, b, c, d \in \mathbb{Z}$ and that $a - d \equiv c - b \varepsilon \pmod{p}$. In addition, we have that $D = \text{tr}(\omega)^2 - 4 \text{nm}(\omega) = (a - d)^2 + 4bc$. If we set

$$A = c, \quad B = a - d, \quad C = -b,$$

and we let $v = [A, B, C]$ be the quadratic form given by $AX^2 + BXY + CY^2$, we know that $B \equiv A + C\varepsilon \equiv 0 \pmod{p}$ and that $v$ has discriminant $D$.

We then consider the sets of integral quadratic forms

$$Q_D = \{ [A, B, C] : B^2 - 4AC = D \},$$
$$Q_{ns} = \{ [A, B, C] : B \equiv A + C\varepsilon \equiv 0 \pmod{p} \},$$
$$Q_{ns,D} = Q_{ns} \cap Q_D.$$

**Proposition 4.2.** The map $\iota \mapsto v$ gives a bijection between the set of embeddings of $\mathcal{O}$ into $M_{ns}$ and $Q_{ns,D}$. Furthermore, $\iota$ is optimal if and only if $v$ is primitive.

**Proof.** To compute the reverse map, given $v = [A, B, C] \in Q_{ns,D}$ we let $\iota(\omega)$ be the matrix given by

$$\iota(\omega) = \begin{pmatrix} \frac{\text{tr}(\omega)-B}{2} & -C \\ A & \frac{\text{tr}(\omega)+B}{2} \end{pmatrix}. \quad (4.3)$$

Since both $\text{tr}(\omega)^2$ and $B^2$ are equal to $D$ modulo $2$, this matrix is integral, and it clearly belongs to $M_{ns}$. Furthermore, it has the same trace and discriminant as $\omega$, hence this defines an embedding $\iota$ of $\mathcal{O}$ into $M_{ns}$.

Let $t = \gcd(A, B, C)$. If the embedding fails to be optimal, there exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ with $\lambda_2 \notin \mathbb{Z}$ such that $\lambda_1 + \lambda_2 \iota(\omega) \in M_{ns}$. Using the notation from (4.1), we see that this implies that $\lambda_2(a - d), \lambda_2b, \lambda_2c \in \mathbb{Z}$. Then any prime dividing the denominator of $\lambda_2$ divides $t$, hence $\iota$ is not primitive.

Conversely, if $t \neq 1$ let $\lambda_1 = \frac{A\gamma}{2}, \lambda_2 = \frac{B\delta}{2}$. Using that $t$ is not divisible by $p$ (otherwise $p$ would divide $D$), an immediate computation gives that $\lambda_1 + \lambda_2 \iota(\omega) \in M_{ns} \setminus \iota(\mathcal{O})$, showing that the embedding is not optimal. \hfill $\square$

Given a quadratic form $v = [A, B, C]$ with negative discriminant, denote by $z_v$ the unique root in $\mathcal{H}$ of the polynomial $AX^2 + BX + C$.

**Proposition 4.4.** Under the bijection above, $z_\iota = z_v$.

**Proof.** Since $\iota(\omega)$ fixes $z_\iota$ using (4.3) it is easy to see that $z_\iota$ is a root of $AX^2 + BX + C$. \hfill $\square$

## 5. Quadratic Forms

Consider the action of $\text{SL}_2(\mathbb{Z})$ on the set of integral quadratic forms, which is given $[A, B, C] \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = [A', B', C']$, where

$$A' = A\alpha^2 + B\alpha\gamma + C\gamma^2,$$
$$B' = A(2\alpha\beta) + B(\alpha\delta + \beta\gamma) + C(2\gamma\delta),$$
$$C' = A\beta^2 + B\beta\delta + C\delta^2. \quad (5.1)$$
Lemma 5.2. Let \([A, B, C] \in \mathbb{Q}_{ns}\) and let \([A', B', C'] = [A, B, C] \cdot M\) where \(M \in \Gamma_{ns}\). Then, the following holds.

(1) \(A' \equiv A \pmod{p}\).
(2) \(B' \equiv B \pmod{2p}\).
(3) \(C' \equiv C \pmod{p}\).
(4) \(A - A' \equiv \varepsilon(C - C') \pmod{p^2}\).

In particular, \([A', B', C'] \in \mathbb{Q}_{ns}\).

Proof. Write \(M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\). In order to prove (1), using (5.1) we compute

\[A' = A\alpha^2 + B\alpha\gamma + C\gamma^2 \equiv A\alpha^2 - A/\varepsilon\gamma^2 \equiv A \pmod{p}\.

In the first congruence we have used that \(B \equiv A + C\varepsilon \equiv 0 \pmod{p}\) and in the second one we have used the fact that \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{ns}\). Items (2) and (3) follow similarly.

In order to prove (4), first note that the condition is equivalent to \(A' - C'\varepsilon \equiv A - C\varepsilon \pmod{p^2}\). We compute, again using (5.1),

\[A' - C'\varepsilon = A(\alpha^2 - \varepsilon\beta^2) + B(\alpha\gamma - \varepsilon\beta\delta) + C(\gamma^2 - \varepsilon\delta^2)\.

Since \(B \equiv A\gamma - \varepsilon\beta\delta \equiv 0 \pmod{p}\) we write

\[A' - C'\varepsilon \equiv A - A\gamma + A(\alpha^2 - \varepsilon\beta^2 - 1) + C(\gamma^2 - \varepsilon\delta^2 + \varepsilon) \pmod{p^2}.

As \((\alpha^2 - \varepsilon\beta^2 - 1)\) is divisible by \(p\) we can replace \(A\) with \(-C\varepsilon\) without changing the value of the expression modulo \(p^2\). Hence we get

\[A' - C'\varepsilon \equiv A - C\varepsilon + C(\gamma^2 - \varepsilon\delta^2 - \varepsilon - \epsilon\alpha^2 + \varepsilon^2\beta^2 + \varepsilon) \pmod{p^2}.

Completing squares we have

\[\gamma^2 - \varepsilon\delta^2 - \varepsilon - \epsilon\alpha^2 + \varepsilon^2\beta^2 + \varepsilon \equiv (\epsilon\beta - \gamma)^2 - \epsilon(\alpha - \delta)^2 - 2\alpha\delta + 2\beta\gamma + 2\varepsilon \pmod{p^2}\]

Because \(\alpha\delta - \beta\gamma = 1\) and \(\alpha \equiv \delta, \beta \varepsilon \equiv \gamma \pmod{p}\) we obtain that this last expression is 0 modulo \(p^2\) and thus

\[A' - C'\varepsilon \equiv A - C\varepsilon \pmod{p^2},\]

as we wanted. \(\square\)

In consequence it is natural to consider, for \(s \in \mathbb{Z}/2p^2\), the set

\[\mathbb{Q}_{ns,D,s} = \{[A, B, C] \in \mathbb{Q}_{ns,D} : A - C\varepsilon \equiv s \pmod{p^2}, B \equiv \varepsilon \pmod{2}\}.

By Lemma 5.2 the group \(\Gamma_{ns}\) acts on \(\mathbb{Q}_{ns,D,s}\).

Proposition 5.3. The set \(\mathbb{Q}_{ns,D,s}\) is non-empty if and only if \(s^2 \equiv \varepsilon D \pmod{4p^2}\). In that case the inclusion from \(\mathbb{Q}_{ns,D,s}\) to \(\mathbb{Q}_D\) gives a bijection

\[\mathbb{Q}_{ns,D,s}/\Gamma_{ns} \xrightarrow{\gamma} \mathbb{Q}_D/\text{SL}_2(\mathbb{Z}).\]

Proof. Let \([A, B, C] \in \mathbb{Q}_{ns,D,s}\). We have that

\[\varepsilon D = \varepsilon(B^2 - 4AC) \equiv \varepsilon(B^2 + (A - C\varepsilon)^2 - (A + C\varepsilon)^2) \equiv s^2 \pmod{4p^2},\]

which proves the “only if” part of the first claim.

We now prove that the map is injective. Let \([A, B, C],[A', B', C'] \in \mathbb{Q}_{ns,D,s}\) and suppose there exists a matrix \(M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) such that \([A, B, C] \cdot M = [A', B', C']\). We know that \(A + C\varepsilon \equiv A' + C'\varepsilon \equiv 0 \pmod{p}\) and \(A - C\varepsilon \equiv A' - C'\varepsilon \equiv 0 \pmod{p}\).
\( s \pmod{p} \), hence \( A \equiv A', C \equiv C' \pmod{p} \). Since \( B \) and \( B' \) are divisible by \( p \) and \( A, A', C, C' \) are not (because \( D \) is not divisible by \( p \)), by (5.1) we have

\[
A' \equiv A \pmod{p} \quad \Rightarrow \quad \alpha^2 - \gamma^2 / \varepsilon \equiv 1 \pmod{p},
\]

\[
C' \equiv C \pmod{p} \quad \Rightarrow \quad \delta^2 - \beta^2 \varepsilon \equiv 1 \pmod{p}.
\]

Combining these congruences and using that \( \det M = 1 \) we get

\[
\alpha - \delta)^2 - \varepsilon(\beta - \gamma / \varepsilon)^2 \equiv \alpha^2 - \gamma^2 / \varepsilon + \delta^2 - \beta^2 \varepsilon + 2(\beta \gamma - \alpha \delta) \equiv 2 - 2 \equiv 0 \pmod{p}.
\]

Since \( \varepsilon \) is a non-square modulo \( p \), we have \( \alpha \equiv \delta, \gamma \equiv \beta \varepsilon \pmod{p} \), and thus \( M \) belongs to \( \Gamma_{ns} \) as desired.

Now, assuming that \( s^2 \equiv \varepsilon D \pmod{4p^2} \), we prove the surjectivity, and in particular that \( Q_{ns,D,s} \) is non-empty. Let \([A, B, C] \in Q_D\). The matrices

\[
\begin{pmatrix}
B/2 & -C \\
A & -B/2
\end{pmatrix},
\begin{pmatrix}
0 & s/2 \\
s/2 & 0
\end{pmatrix}
\] \in GL_2(\mathbb{F}_p)

are conjugate, since they have the same characteristic polynomial, which has simple roots. Moreover, as \( \begin{pmatrix}
0 & s/2 \\
s/2 & 0
\end{pmatrix} \in C_{ns} \), which is abelian, and \( \det : C_{ns} \to \mathbb{F}_p^* \) is surjective, these matrices are actually conjugate by a matrix \( M \in SL_2(\mathbb{F}_p) \). Take a lift \( M \in SL_2(\mathbb{Z}) \), and let \([A', B', C'] = [A, B, C] \cdot M \). By construction, \( B' \equiv s \pmod{2} \) and \( A' - C' \varepsilon \equiv s \pmod{p} \). Now, both \( A' - C' \varepsilon \) and \( s \) are square roots of \( D \varepsilon \) modulo \( p^2 \) which are congruent (and non-zero) modulo \( p \), and thus they must be equivalent modulo \( p^2 \). This implies that \([A', B', C'] \in Q_{ns,D,s} \).

Recall that any element of \( \Gamma_{ns}^+ \setminus \Gamma_{ns} \) defines an involution \( W_p \) on \( X_{ns} \). In the same fashion as the \( \Gamma_0(N) \) case, this involution interchanges the square roots of \( \varepsilon D \pmod{p^2} \).

**Lemma 5.4.** Let \([A, B, C] \in Q_{ns} \) and let \([A', B', C'] = [A, B, C] \cdot M \), where \( M \in \Gamma_{ns}^+ \setminus \Gamma_{ns} \). Then, the following holds.

1. \( A' \equiv -A \pmod{p} \).
2. \( B' \equiv -B \pmod{2p} \).
3. \( C' \equiv -C \pmod{p} \).
4. \( A - A' \equiv -\varepsilon(C - C') \pmod{p^2} \).

In particular, \( Q_{ns,D,s} \cdot W_p = Q_{ns,D,-s} \).

**Proof.** The proof is exactly the same as in Lemma 5.2, but as we are taking \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{ns}^+ \setminus \Gamma_{ns} \) we have the congruences \( \alpha \equiv -\delta \pmod{p} \), \( \gamma \equiv -\varepsilon \beta \pmod{p} \). Finally, note that acting by \( M \) we change the value of \( s \) to \(-s \) modulo \( p^2 \). \( \Box \)

### 6. Special cycles

For the convenience of the reader, we recall the definitions

\[
Q_D = \{ [A, B, C] : B^2 - 4AC = D \},
\]

\[
Q_{ns} = \{ [A, B, C] : B \equiv A + C \varepsilon \equiv 0 \pmod{p} \},
\]

\[
Q_{ns,D} = Q_{ns} \cap Q_D,
\]

\[
Q_{ns,D,s} = \{ [A, B, C] \in Q_{ns,D} : A - C \varepsilon \equiv s \pmod{p^2}, B \equiv s \pmod{2} \}.
\]
We consider the sets of positive definite quadratic forms
\[
P_D = \{ [A, B, C] \in \mathbb{Q}_D : A > 0 \},
\]
\[
P_{ns,D,s} = \{ [A, B, C] \in \mathbb{Q}_{ns,D,s} : A > 0 \},
\]
and the special cycle
\[
P_{ns,D,s} = \sum_{v \in \mathcal{P}_{ns,D,s}/\Gamma_{ns}} z_v \in \text{Pic}(X_{ns}),
\]
which we denote by \( P_{D,s} \) for aesthetical purposes. It is understood to be zero if the set \( P_{ns,D,s} \) is empty.

**Remark 6.1.** Since \( p \) is inert in \( K \), there exists \( s \in \mathbb{Z}/2p^2 \) such that \( s^2 \equiv \varepsilon D \pmod{4p^2} \). In particular, by Proposition 4.2, the set \( P_{ns,D,s} \) is non-empty. Therefore under our setting the special cycles \( P_{D,s} \) are meaningful.

The formula for the action of the Hecke operators \( T_\ell \) on these special cycles is the same one that appears in [GKZ87, p.507] for the classical modular curve.

**Proposition 6.2.** Let \( \ell \neq p \) be a prime number. Then
\[
T_\ell P_{D,s} = P_{D,\ell^2,s\ell} + \left( \frac{D}{\ell} \right) P_{D,s} + \ell P_{D/\ell^2,s/\ell},
\]
(6.3)
where the last term is understood to be zero if \( \ell^2 \) does not divide \( D \).

**Proof.** This follows from [CV07, Corollary 6.6]. The computation in [CV07] is of local nature and works for general Shimura curves of level not divisible by \( \ell \), hence it applies in our setting. \( \square \)

In addition, we have the following formulas for the action of \( W_p \) and complex conjugation (denoted by a bar) on these special cycles.

**Proposition 6.4.** We have that
\[
P_{D,s} \cdot W_p = P_{D,-s} = \overline{P_{D,s}}.
\]

**Proof.** Since the action of \( W_p \) is given by a matrix of \( \text{SL}_2(\mathbb{Z}) \), which preserves positiveness, Lemma 5.4 implies that \( P_{ns,D,s} \cdot W_p = P_{ns,D,-s} \), proving the first equality. The second equality is [KP16, Proposition 3.3 (iii)]. \( \square \)

The following proposition tells us that for fundamental discriminants the cycles \( P_{D,s} \) are obtained as the traces of the special points on the non-split Cartan curve. In order to state it, we denote by \( H \) the Hilbert class field corresponding to the maximal order \( \mathcal{O}_K \) and, assuming that \( P_{ns,D,s} \) is non-empty, we let \( \iota \) be an optimal embedding corresponding via the bijection of Proposition 4.2 to an element in \( P_{ns,D,s} \). Moreover, we recall that under the identification of Proposition 3.1 we can view \( P_{D,s} \) as an element of \( J_{ns} \otimes \mathbb{Q} \).

**Proposition 6.5.** Suppose that \( D \) is fundamental. Then
\[
P_{D,s} = \sum_{\sigma \in \text{Gal}(H/K)} \sigma \cdot z_{\iota}.
\]
In particular, \( P_{D,s} \in J_{ns}(K) \otimes \mathbb{Q} \).
Proof. By Proposition 5.3 we have a bijection $\mathcal{P}_{ns,D,s}/\Gamma_{ns} \cong \mathcal{P}_D/\text{SL}_2(\mathbb{Z})$. Since $D$ is fundamental, the forms in $\mathcal{P}_D$ are necessarily primitive and then the set $\mathcal{P}_D/\text{SL}_2(\mathbb{Z})$ is in bijection with $\text{Gal}(H/K)$. Unraveling these identifications and using Proposition 4.4, the result follows. \[\square\]

7. Modular parametrization and Zhang’s formula

Recall that $E/\mathbb{Q}$ is an elliptic curve of conductor $p^2$ and such that $w_p = 1$. The following proposition shows that $E$ is uniformized by the non-split Cartan curve.

**Proposition 7.1.** There exists a non-zero Hecke-equivariant rational map $\pi^+: J_{ns}^+ \rightarrow E$.

**Proof.** The existence of such a map $\pi: J_{ns} \rightarrow E$ follows from [Zha01, Theorems 1.2.2, 1.3.1]. Since $w_p = 1$ this map factors through $J_{ns}^+$. \[\square\]

We define $P_{D,s}$ as the image of $P_{D,s}$ on $J_{ns}(K) \otimes \mathbb{Q}$, and the corresponding point $Q_{D,s}^+ = \pi^+(P_{D,s})$ on $E(K) \otimes \mathbb{Q}$.

**Remark 7.2.** The special cycles $P_{D,s}^+$ (and hence the points $Q_{D,s}^+$) do not depend on $s$, since $P_{D,-s} = P_{D,s}$ and whenever $\varepsilon D$ is a non-zero square in $\mathbb{Z}/2p^2$ it has exactly two square roots. Nevertheless, as this is a particularity of working with curves of level $p^2$, we consider the dependence on $s$ throughout the article.

**Proposition 7.3.** Let $\ell \neq p$ be a prime number. Then,

$$a_\ell(E) Q_{D,s}^+ = Q_{D,2s,\ell} + \left(\frac{D}{\ell}\right) Q_{D,s}^+ + \ell Q_{D,\ell^2,s/\ell^2}^+, \quad (7.4)$$

where the last term is understood to be zero if $\ell^2$ does not divide $D$.

**Proof.** This follows by applying $\pi^+$ to (6.3) and using the Hecke equivariance of this map. \[\square\]

**Proposition 7.5.** The points $Q_{D,s}^+$ belong to $E(\mathbb{Q}) \otimes \mathbb{Q}$.

**Proof.** If $D$ is fundamental, by Proposition 6.4 we have

$$2P_{D,s}^+ = P_{D,s} + P_{D,s} \cdot W_p = P_{D,s} + \overline{P_{D,s}}.$$

Therefore, using Proposition 6.5 we obtain that $P_{D,s}^+ \in J_{ns}(\mathbb{Q}) \otimes \mathbb{Q}$, and since $\pi^+$ is rational, we get that $Q_{D,s}^+ \in E(\mathbb{Q}) \otimes \mathbb{Q}$. Using (7.4) we see that the result holds for non-fundamental discriminants as well. \[\square\]

In this setting we have the following generalization of the Gross-Zagier formula by Zhang.

**Theorem 7.6.** [Zha01, Theorem 1.2.1] Assume that $D$ is fundamental and that $s^2 \equiv \varepsilon D \pmod{4p^2}$. Then

$$L'(E/K, 1) = c \sqrt{D} \left\langle Q_{D,s}^+, Q_{D,s}^+ \right\rangle,$$

where $\left\langle \cdot, \cdot \right\rangle$ denotes the Néron-Tate height pairing and $c$ is a non-zero constant which does not depend on $D$. 
8. Vector valued and Jacobi modular forms

Let \((L, \beta)\) be an even lattice and let \(L'\) denote its dual. The symmetric bilinear form \(\beta\) induces an integral quadratic form given by \(\beta(x) = \frac{1}{2} \beta(x, x)\). Its reduction modulo 1 induces a quadratic form on the finite group \(L' / L\) with values in \(\mathbb{Q}/\mathbb{Z}\), called the discriminant form. Let \(V_L\) be the group ring \(\mathbb{C}[L'/L]\) with standard basis \(\{e_s\}_{s \in L'/L}\). Let \(\rho_L\) be the representation of the metaplectic cover of \(SL_2(\mathbb{Z})\) associated to the discriminant form \((L'/L, \beta)\) (see [Bor99, Section 2]), and let \(\rho_L^*\) denote its dual.

Given \(k \in \frac{1}{2} \mathbb{Z}\), we say that a holomorphic function \(f : \mathcal{H} \to V_L\) is a vector valued modular form of weight \(k\) and type \(\rho_L^*\) if it is invariant under the \(k\)-slash operator induced by \(\rho_L^*\) (see [Boy15, Definition 3.49]) and is holomorphic at \(\infty\), i.e. if we consider the Fourier expansion

\[
f(\tau) = \sum_{s \in L'/L} \sum_{m \in \mathbb{Q}} c_{m,s} q^m e_s, \quad q = e^{2\pi i \tau},
\]

then \(c_{m,s} \neq 0\) implies that \(m \geq 0\). The space of such forms will be denoted by \(M_k(\rho_L^*)\), and is determined by the discriminant form and \(k\). The space of cuspforms, i.e. those forms for which \(c_{m,s} \neq 0\) implies that \(m > 0\), will be denoted by \(S_k(\rho_L^*)\).

Assume furthermore that \((L, \beta)\) is positive definite. Given \(k \in \mathbb{Z}\), we say that a holomorphic function \(\psi : \mathcal{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}\) is a Jacobi form of weight \(k\) and lattice index \(L\) if it is invariant under the \(k\)-slash operator induced by the action of the Jacobi group of \((L, \beta)\) (see [Boy15, Definition 3.29]), and is holomorphic at \(\infty\), i.e. if we consider the Fourier expansion

\[
\psi(\tau, z) = \sum_{s \in L'/L} \sum_{n \in \mathbb{Z}} c(n, s) q^n e^{\beta(s, z)}
\]

then \(c(n, s) \neq 0\) implies that \(n \geq \beta(s)\). The space of Jacobi forms of weight \(k\) and lattice index \(L\) will be denoted by \(J_{k,L}\). The space of Jacobi cuspforms, i.e. those forms for which \(c(n, s) \neq 0\) implies that \(n > \beta(s)\), will be denoted by \(S_{k,L}\).

By [Boy15, Theorem 3.5] we can identify Jacobi forms with vector valued modular forms. More precisely, if we denote by \(r\) the rank of the lattice \(L\), the following holds.

**Proposition 8.1.** Given \(\psi \in J_{k,L}\) and \(m \in \mathbb{Q}_{\geq 0}\) let \(c_{m,s} = c(\beta(s) + m, s)\) if \(m + \beta(s) \in \mathbb{Z}\) and \(c_{m,s} = 0\) otherwise. Then \(c_{m,s}\) depends only on the class of \(s\) modulo \(L\). Furthermore, the map

\[
\psi(\tau, z) = \sum_{s \in L'/L} \sum_{n \in \mathbb{Z}} c(n, s) q^n e^{\beta(s, z)} \mapsto \sum_{s \in L'/L} \sum_{m \in \mathbb{Q}} c_{m,s} q^m e_s
\]

gives an isomorphism \(J_{k+1/2,L} \simeq M_k(\rho_L^*)\), which preserves cuspforms.

In our applications we will need to work with vector valued modular forms associated to a non-positive definite lattice, which in principle do not correspond to Jacobi forms. In order to solve this, we resort to the concept of stably isomorphic lattices. Two (even) lattices \(L_1, L_2\) are said to be stably isomorphic if there exist (even) unimodular lattices \(U_1, U_2\) such that \(L_1 \oplus U_1 \simeq L_2 \oplus U_2\). In particular, the discriminant forms of \(L_1\) and \(L_2\) are the same and therefore

\[
M_k(\rho_{L_1}^*) \simeq M_k(\rho_{L_2}^*).
\]
Now [Nik79, Theorem 1.10.1] tell us that for any given lattice we can find a positive definite one that is stably isomorphic to it. Combining this with the aforementioned results, any vector valued modular form can be thought as a Jacobi form.

9. Finding the correct lattice

We let \( V = \{ x \in M_2(\mathbb{Q}) : \text{tr} x = 0 \} \), and we let \( \beta \) be the bilinear form in \( V \) given by \( \beta(x,y) = -\text{tr}(x \text{adj}(y))/4p^2 \). We consider the lattice

\[
L = \left\{ \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix} : A, B, C \in \mathbb{Z} \right\}.
\]

We let \( \text{SL}_2(\mathbb{Z}) \) act on \( V \) by conjugation. Then the map

\[
[A, B, C] \mapsto \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix}
\]

gives a \( \text{SL}_2(\mathbb{Z}) \)-equivariant bijection between the set of integral binary quadratic forms and \( L \). Under this bijection the set \( \mathcal{Q}_{ns} \) corresponds to the lattice \( L \cap M_{ns} = \{ (B^2 C - 2A - B) : A, B, C \in \mathbb{Z}, B \equiv 0 \pmod{p}, A \equiv -C \varepsilon \pmod{p} \} \).

The invariants of the action of \( \Gamma_{ns} \) on \( \mathcal{Q}_{ns} \) given by Lemma 5.2 suggest considering the lattice \( L_{ns} \) given by

\[
L_{ns} = \left\{ \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix} : A \equiv B \equiv C \equiv 0 \pmod{p}, \right. \]

\[
B \equiv 0 \pmod{2}, A \equiv \varepsilon C \pmod{p^2} \}.
\]

(9.1)

Notice that \( L_{ns} \) is an even lattice of signature \((2, 1)\). It has the following properties.

**Proposition 9.2.**

1. The dual lattice \( L_{ns}^\vee \) is equal to \( L \cap M_{ns} \).
2. The lattices \( L_{ns} \) and \( L_{ns}^\vee \) are \( \Gamma_{ns}^+ \)-invariant, and the action of \( \Gamma_{ns} \) on \( L_{ns}/L_{ns}^\vee \) is trivial.
3. The discriminant form \( (L_{ns}^\vee/L_{ns}, \beta) \) is isomorphic to \( (\mathbb{Z}/2p^2, s \mapsto s^2/4\varepsilon p^2) \).

**Proof.**

1. Take \( x_1 = \begin{pmatrix} B_1 & 2C_1 \\ -2A_1 & -B_1 \end{pmatrix}, x_2 = \begin{pmatrix} B_2 & 2C_2 \\ -2A_2 & -B_2 \end{pmatrix} \in V \). Then

\[
\beta(x_1, x_2) = \frac{2B_1 B_2 - 4A_1 C_2 - 4C_1 A_2}{4p^2}.
\]

For this to be integral for every \( x_1 \in L_{ns} \) we need that \( B_2 \equiv 0 \pmod{p} \) and \( A_1 C_2 + C_1 A_2 \equiv 0 \pmod{p^2} \). We know that \( A_1 = p\alpha \) and \( C_1 = p\gamma \) for some integers \( \alpha, \gamma \). Moreover, since \( A_1 \equiv C_1 \varepsilon \pmod{p^2} \) we get that \( \alpha \equiv \gamma \varepsilon \pmod{p} \) and that

\[
0 \equiv \alpha C_2 + \gamma A_2 \equiv \gamma (C_2 \varepsilon + A_2) \pmod{p}.
\]

Since we can choose some \( \gamma \) that is non-zero modulo \( p \) we obtain that \( C_2 \varepsilon = -A_2 \pmod{p} \). Thus we have proved that

\[
L_{ns}^\vee = \left\{ \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix} : A, B, C \in \mathbb{Z}, B \equiv 0 \pmod{p}, A \equiv -C \varepsilon \pmod{p} \right\}.
\]
which is equal to \( L \cap M_{\ns} \).

(2) This follows immediately by Lemmas 5.2 and 5.4.

(3) The map \( L_{\ns} / 2 \times \mathbb{Z}/p^2 \) given by

\[
\begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix} \mapsto (B \pmod{2}, A - C \varepsilon \pmod{p^2})
\]

has kernel \( L_{\ns} \), and is surjective. In fact, given \( s \in \mathbb{Z}/2p^2 \), take \( \left( \frac{B}{-2A - B} \right) \in L_{\ns} / 2 \times \mathbb{Z}/p^2 \) such that \( B \equiv s \pmod{2} \) and \( A - C \varepsilon \equiv s \pmod{p^2} \). As \( \varepsilon \) was chosen to be 1 modulo 4 and \( B \equiv A + C \varepsilon \equiv 0 \pmod{p} \) we obtain

\[
B^2 - 4AC \equiv (B^2 + (A - C \varepsilon)^2 - (A + C \varepsilon)^2) / \varepsilon \equiv s^2 / \varepsilon \pmod{4p^2}.
\]

This finishes the proof because \( \beta \left( \left( \frac{B}{-2A - B} \right) \right) = \frac{B^2 - 4AC}{4p^2} \).

\( \square \)

10. Main results

Under the identification given by Proposition 3.1, for \( d \in \mathbb{Q}_{>0} \) and \( s \in L_{\ns} / L_{\ns} \) we consider the cycle

\[
Y_{d,s} = \sum_{v \in s + L_{\ns} : \beta(v) = d} z_v \in J_{\ns} \otimes \mathbb{Q}.
\]

Under the identification given by part (3) of Proposition 9.2, we see that if \( Y_{d,s} \neq 0 \) then \( s^2 \equiv D \varepsilon \pmod{2p^2} \), where \( D / 4p^2 = d \) (this is valid for all negative discriminants, not necessarily prime to \( p \)). If \( d \geq 0 \) we set \( Y_{d,s} = 0 \).

We have the following modularity result, which is a consequence of the work Borcherds [Bor99].

**Proposition 10.1.** The generating series

\[
\sum_{s \in L_{\ns} / L_{\ns}} \sum_{m \in \mathbb{Q}} Y_{-m,s} q^m e_s
\]

is modular. More precisely, it lies in the space \((J_{\ns} \otimes \mathbb{Q}) \otimes S_3 / (\rho^*_L) \).

**Proof.** We follow [Bor99, Section 4] closely, using its notation.

Let \( G(L_{\ns}) \) be the Grassmanian of positive definite planes in \( L_{\ns} \otimes \mathbb{R} \), which is identified with \( \mathcal{H} \). Under this identification, given \( v \in L_{\ns}^* \) of negative norm we have that \( z_v \) is the image in \( \Gamma_{\ns} \backslash \mathcal{H} \) of the element of \( G(L_{\ns}) \) orthogonal to \( v \). For \( d < 0 \) we consider the Heegner divisors \( y_{d,s} \) given by

\[
y_{d,s} = \sum_{v \in s + L_{\ns} : \beta(v) = d} z_v \in \text{Pic} (X_{\ns}).
\]

Furthermore, we define a formal symbol \( y_{0,0} \), and if we write \( y_{d,s} \) with either \( d > 0 \) or \( d = 0 \) and \( s \neq 0 \) we understand that this is zero.

Let \( \Gamma(L_{\ns}) \) denote the group of automorphisms of the lattice \( L_{\ns} \) that act trivially on \( L_{\ns}^* / L_{\ns} \). The Heegner class group \( \text{HeegCl}(X_{\ns}) \) is the group generated by the divisors \( y_{d,s} \) quotiented by the subgroup of the so called principal Heegner divisors, which are the divisors of the form \( c_0,0 y_{0,0} + D \); here \( c_{0,0} \) is an integer and \( D \) is the divisor of a meromorphic automorphic form of weight \( c_{0,0} / 2 \) with respect to \( \Gamma(L_{\ns}) \) and some finite order character (the finite order condition is explained in the correction [Bor00]).
Given this setting, [Bor99, Theorem 4.5], combined with the fact that $S_{3/2}(\rho^*_L)$ has a basis of forms with rational coefficients (see [McG03, Theorem 5.6]), tell us that the generating series

$$\sum_{s \in L^\vee_{ns}/L_{ns}} \sum_{m \in \mathbb{Q}} y_{-m,s} q^m e_s$$

belongs to $(\text{HeegCl}(X_{ns}) \otimes \mathbb{Q}) \otimes S_{3/2}(\rho^*_L)$. In order to conclude, it suffices to prove that the natural map

$$\text{HeegCl}(X_{ns}) \otimes \mathbb{Q} \longrightarrow J_{ns} \otimes \mathbb{Q}$$

given by $y_{-m,s} \mapsto Y_{-m,s}$ is well defined.

As the cuspform $\Delta$ of weight 12 has no zeros and no poles on $\mathcal{H}$, the element $24y_{0,0}$ is a principal Heegner divisor and thus the element $y_{0,0} \in \text{HeegCl}(X_{ns}) \otimes \mathbb{Q}$ is trivial. Then it is enough to show that the divisor of a weight 0 meromorphic automorphic form for $\Gamma(L_{ns})$ is mapped to $0 \in J_{ns} \otimes \mathbb{Q}$ (we can omit the finite character condition since we have tensored with $\mathbb{Q}$).

Part (2) of Proposition 9.2 claims precisely that $\Gamma_{ns} \subseteq \Gamma(L_{ns})$, so a weight 0 meromorphic automorphic form for $\Gamma(L_{ns})$ is in particular automorphic for $\Gamma_{ns}$, and therefore its divisor is zero on $J_{ns} \otimes \mathbb{Q}$, as we wanted to prove. $\square$

In order to close the circle we need to show that the Fourier coefficients can be described in terms of special points. From now on we return to the original setting where $D$ is prime to $p$.

**Proposition 10.2.** Denote by $Y^+_{d,s}$ the projection of $Y_{d,s}$ onto $J^+_{ns} \otimes \mathbb{Q}$. Let $d = D/4p^2$ and $s \in L^\vee_{ns}/L_{ns}$. Then

$$Y^+_{d,s} = 2P^+_{D,s} \in J^+_{ns} \otimes \mathbb{Q}.$$

**Proof.** Unraveling the definitions, we see that there is a bijection

$$\mathbb{Q}_{ns,D,s} \longrightarrow \{v \in s + L_{ns} : \beta(v) = d\}$$

$$[A, B, C] \mapsto \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix}.$$

If $[A, B, C]$ is an indefinite form in $\mathbb{Q}_{ns,D,s}$, then $[-A, -B, -C] \in \mathcal{P}_{ns,D,-s}$, and they give rise to the same point in $\mathcal{H}$. Therefore, by Proposition 6.4,

$$Y_{d,s} = P_{D,s} + P_{D,-s} = P_{D,s} + P_{D,s} \cdot W_p \in J_{ns} \otimes \mathbb{Q}.$$

Then the result follows by projecting onto $J^+_{ns} \otimes \mathbb{Q}$. $\square$

As remarked by Borcherds [Bor99, Example 5.1], in the $\Gamma_0(N)$ case the corresponding lattice splits as the direct sum of a lattice generated by a vector of norm $2N$ and an even unimodular hyperbolic 2 dimensional even lattice. Therefore the space of vector valued modular forms with that lattice index corresponds to the space of Jacobi form of weight 2 and index $N$.

However, for $L_{ns}$ this is not the case anymore. In fact, using the criteria by Nikulin for the existence of even lattices with given signature and discriminant form (see [Nik79, Theorem 1.10.1]), part (3) of Proposition 9.2, and the fact that $\varepsilon$ is not a square modulo $p$ we see that $L_{ns}$ is not stably isomorphic to an even positive definite lattice of rank 1.
However, [Nik79, Corollary 1.10.2] implies that $L_{ns}$ is stably isomorphic to an even positive definite lattice $\mathcal{L}_{ns}$ of rank 9 (and this is the smallest possible dimension as the signature of the discriminant form is invariant modulo 8).

Piecing together the results of this section we obtain the following theorem.

**Theorem 10.3.** There exists $\psi \in (J_{ns}^+ \otimes \mathbb{Q}) \otimes S_{6, \mathcal{L}_{ns}}$ such that for every negative discriminant $D$ prime to $p$ we have

$$P_{D,s}^+ = c_\psi(\beta(s) - D/4p^2, s).$$

**Proof.** By Proposition 8.1 and (8.2) we have that

$$S_{6, \mathcal{L}_{ns}} \simeq S_{3/2}(\rho^*_{\mathcal{L}_{ns}}) \simeq S_{3/2}(\rho^*_{\mathcal{L}_{ns}}).$$

Then the result is a straightforward combination of Propositions 10.1 and 10.2. □

The next step is to project onto the elliptic curve $E$ in order to recover a statement about the special points on it. Given a special cycle $P_{D,s}^+$, recall that $Q^+_{D,s} = \pi^+(P_{D,s}^+) \in E(\mathbb{Q}) \otimes \mathbb{Q}$ (Proposition 7.5). The following is the main result of this article, which is analogous to [GKZ87, Theorem C].

**Theorem 10.4.** There exists a Jacobi form $\psi_E \in S_{6, \mathcal{L}_{ns}}$ such that for every negative discriminant $D$ prime to $p$ we have

$$Q^+_{D,s} = c_{\psi_E}(\beta(s) - D/4p^2, s) Q,$$

(10.5)

for some $Q \in E(\mathbb{Q}) \otimes \mathbb{Q}$ which is non-zero if and only if $L'(E/Q, 1) \neq 0$.

**Proof.** If $D$ is fundamental by Theorem 7.6 we have that $Q^+_{D,s} \neq 0$ if and only if $L'(E/K, 1) \neq 0$. In particular, if $L'(E/Q, 1) = 0$ all these cycles vanish. Furthermore, by Proposition 6.2 this is also true if $D$ is not fundamental. Then (10.5) holds letting $Q = 0$ and $\psi_E = 0$.

On the other hand, if $L'(E/Q, 1) \neq 0$, then $E(\mathbb{Q}) \otimes \mathbb{Q}$ has rank 1. Letting $Q$ be a generator, we identify $E(\mathbb{Q}) \otimes \mathbb{Q}$ with $Q$ and we define $\psi_E = \pi^+ \psi \in S_{6, \mathcal{L}_{ns}}$. The result follows immediately applying Theorem 10.3. □

Finally, we prove that the form $\psi_E$ lies in the expected Hecke eigenspace.

**Proposition 10.6.** The Jacobi form $\psi_E$ is an eigenform for $T_\ell$ for $\ell \neq p$ with eigenvalues $a_\ell(E)$.

**Proof.** Let $\chi = T_\ell \psi_E - a_\ell(E) \psi_E \in S_{6, \mathcal{L}_{ns}}$. Combining Proposition 7.3 with Theorem 10.4 and the explicit formulas for the action of the Hecke operators on Jacobi forms ([Ajo15, Theorem 2.6.1]), we easily check that the Fourier coefficients of $\chi$ associated to negative discriminants $D$ not divisible by $p$ vanish.

Using Proposition 8.1 we consider $\chi$ as a form in $S_{3/2}(\rho^*_{\mathcal{L}_{ns}})$. Let $H$ be the unique subgroup of $L^\vee_{ns}/L_{ns}$ of order $p$ which corresponds to the multiples of $2p$ inside $\mathbb{Z}/2p^2$. The group $H$ is isotropic, meaning that $\beta$ vanishes on $H$. In addition, $H^\perp$ corresponds to the subgroup of multiples of $p$ (which in turn correspond to the discriminants divisible by $p$). Then $\chi$ is supported in $H^\perp$, meaning that the components $\chi_s$ for $s \notin H^\perp$ vanish.

Let $M = H^\perp/H$. Then by [Bru14, Proposition 3.3] the form $\chi$ is an oldform, arising from the space of vector valued modular forms of weight $3/2$ and discriminant form $M$. Since $M$ has order 2 this space corresponds to the space of classical Jacobi form of weight 2 and index 1. Because the latter space is trivial we have that $\chi = 0$, as we wanted to prove. □
Remark 10.7. In [GKZ87] the authors obtain the Jacobi form $\psi_E$ using the correspondence between systems of eigenvalues of Jacobi forms and classical modular forms proved in [SZ88]. In particular, they do not need to prove (the analogous result of) Proposition 10.6; moreover, they use this correspondence to extend (the analogous result of) Theorem 10.4 to non-fundamental discriminants, knowing that it holds for fundamental ones. We want to stress that we are doing the exact opposite: we already know that Theorem 10.4 holds for all (not divisible by $p$) discriminants, and we leverage this to obtain that $\psi_E$ is a Hecke eigenform.

11. Relation with classical Jacobi forms

Denote by $\phi_E$ the classical Jacobi form of weight 2 and scalar index $p^2$ which corresponds to (the modular form corresponding to) $E$ via the Skoruppa-Zagier lift. The following proposition, valid since $w_p = 1$, relates the coefficients of $\psi_E$ with certain coefficients of $\phi_E$.

**Proposition 11.1.** There exists a non-zero constant $\kappa$ such that for every negative fundamental $D$ and $s \in \mathbb{Z}/2p^2$ with $s^2 \equiv \varepsilon D \pmod{4p^2}$,

$$Q_{Dp^2,r}^* = \kappa Q_{D,s}^+, \quad \text{(11.2)}$$

where $r \equiv p^2 s \pmod{2p^2}$. Furthermore,

$$c_{\phi_E} \left( (r^2 - Dp^2)/4p^2, r \right) = \kappa c_{\psi_E} \left( \beta(s) - D/4p^2, s \right). \quad \text{(11.3)}$$

**Proof.** The first part is a special case of [Koh18, Theorem 4.3]. More precisely, using the notation from [Koh18], if we take $M = f = 1$, then the special point $\gamma^{H^*}(P_p)$ (resp. $\gamma^{H}(P_1)$) is, by construction, our $Q_{Dp^2,r}^*$ with $r \equiv p^2 s \pmod{2p^2}$ (resp. our $Q_{D,s}^+$). Furthermore, the constant $\kappa$ such that $\hat{P}_1 = \kappa P_1$ does not depend on $D$. Then (11.2) follows, since $\gamma^{H^*}(P_p) = \hat{P}_1$.

As $p$ is inert in $K$, a quadratic form corresponding to $Q_{Dp^2,r}^*$ must be necessarily primitive. Hence (11.3) follows from the interpretation of [GKZ87, Theorem C] by Borcherds (see [Bor99, Example 5.1]) and Theorem 10.4. □

**Remark 11.4.** The results of this article can be generalized in a straightforward manner to include elliptic curves of conductor $N = p^2 M$ with $p \nmid M$ and odd analytic rank, giving the positions of the Heegner points induced by quadratic imaginary fields in which $p$ is inert but every prime dividing $M$ is split. For such curves we can certainly have $w_p = -1$. Although Theorem 10.4 continues to hold, the positions of the points in the line will not necessarily be given by the coefficients of a classical Jacobi form, as in Proposition 11.1.

12. An explicit example

We let $p = 17$ and $\varepsilon = 5$. Consider the elliptic curve

$$E : \quad y^2 + xy + y = x^3 - x^2 - 199x + 510.$$

It has conductor $289 = 17^2$ and rank 1. Our goal is to compute the Jacobi form $\psi_E$ alluded to in Theorem 10.4.

A generator of $E(\mathbb{Q})$, up to torsion, is given by the point $Q = [-12, 38]$. We compute for negative discriminants $D$ prime to 17 such that 17 is inert in $\mathbb{Q}(\sqrt{D})$
the special points $Q^+_D,s$ (which, according to Remark 7.2, do not depend on $s$) and we give the integer $m(D)$ such that, up to torsion, $Q^+_D,s = m(D)Q$.

These points can be computed using the non-split Cartan curve as explained in [Koh17], [KP16]. They are constructed by giving an explicit modular parametrization $\pi^+_+: X^+_n(17) \to E$, which amounts to finding an explicit cuspidal form for $\Gamma^+_n(17)$ with the same eigenvalues as $f_E$ for the Hecke operators, where $f_E$ is the cuspidal form of weight 2 and level 289 corresponding to $E$. In Table 1 below we record the computations for all valid discriminants of absolute value less than 200.

We now compute $\psi_E$, using SAGE ([The19]). We first compute the positive definite lattice $\mathcal{L}_n$ which is stably isomorphic to the lattice $L_n$ given by (9.1). This is done by using [Rau16, Algorithm 2.3], which adds to $L_n$ a copy of $E_8$ (the positive definite unimodular lattice of rank 8) and splits a two dimensional hyperbolic lattice $U$. Concretely, we have $L_n \oplus E_8 \simeq \mathcal{L}_n \oplus U$. As claimed above, $\mathcal{L}_n$ has rank 9. Its Gram matrix is given by

\[
\begin{pmatrix}
34 & -136 & -80 & 16 & -4 & -4 & 0 & 0 & 0 \\
-136 & 578 & 323 & -68 & 17 & 17 & 0 & 0 & 0 \\
-80 & 323 & 190 & -40 & 10 & 10 & 0 & 0 & 0 \\
16 & -68 & -40 & 12 & -3 & -3 & 0 & 0 & 0 \\
-4 & 17 & 10 & -3 & 2 & 0 & 0 & 0 & 0 \\
-4 & 17 & 10 & -3 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

Now we need to compute the space $S_{3/2}(\rho^\ast_{\mathcal{L}_n})$. We first, using (an enhanced version by Ehlen of) the algorithms of Williams derived from [Wil18], compute the space $S_{12+3}(\rho^\ast_{\mathcal{L}_n})$, since these algorithms do not work in weight $3/2$. By [Bru02], this space has dimension 296. Then we look for forms in that space such that the first Fourier coefficient vanishes for every $s \in \mathcal{L}_n^\vee/\mathcal{L}_n$ so that, when we divide by $\Delta$, the modular discriminant, we obtain the space $S_{3/2}(\rho^\ast_{\mathcal{L}_n})$. This space, according to [ES17], has dimension 7. Finally, using the identification given by Proposition 8.1, we search in that space for forms which have their first Fourier coefficients given by a fixed multiple of the values $m(D)$ computed in Table 1, and we see that this space is indeed one dimensional, spanned by the Jacobi form $\psi_E$.

We conclude our computations by verifying Proposition 11.1. Using the algorithms derived from [RSST16] we compute the Fourier coefficients of the Jacobi form $\phi_E$, which are given by applying the dualized Skoruppa-Zagier lift to the modular symbol corresponding to $E$. We verified that (11.3) holds for every $D$ of absolute value less than 200.

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References

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Table 1. Relative positions of special points

| D  | m(D) | D  | m(D) | D  | m(D) |
|----|------|----|------|----|------|
| -3 | 1    | -71| -3   | -139| 1    |
| -7 | -1   | -75| 3    | -143| -2   |
| -11| 3    | -79| -1   | -147| -5   |
| -12| 0    | -80| -2   | -148| 2    |
| -20| 2    | -88| 2    | -156| -8   |
| -23| 1    | -91| -2   | -159| 2    |
| -24| 2    | -92| -2   | -160| -2   |
| -27| 0    | -95| -4   | -163| 7    |
| -28| 2    | -96| -2   | -164| -4   |
| -31| 1    | -99| -3   | -167| 1    |
| -39| 4    | -107| -3 | -175| -3   |
| -40| 2    | -108| 0  | -176| -6   |
| -44| 0    | -112| 0  | -180| -2   |
| -48| -2   | -116| -2 | -184| 8    |
| -56| 0    | -124| -2 | -192| 2    |
| -63| -1   | -131| 3  | -199| -1   |

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