The existence of multi-vortices for a generalized self-dual Chern–Simons model

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Abstract
In this paper we establish the existence of multi-vortices for a generalized self-dual Chern–Simons model. Doubly periodic vortices, and topological and non-topological vortex solutions are constructed for this model. For the existence of doubly periodic vortex solutions, we establish an explicitly necessary and sufficient condition. It is difficult to find topological multi-vortex solutions due to the non-canonical structure of the equations. We overcome this difficulty by constructing a suitable sub-solution for the reduced equation. This technique maybe applied to the other problems with similar structures. For the existence of non-topological solutions we use a shooting argument.

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1. Introduction

In mathematical physics static solutions to gauge field equations with broken symmetry in two-space dimensions are often called vortices. Magnetic vortices play important roles in many areas of theoretical physics including superconductivity [1, 32, 40], electroweak theory [2–5] and cosmology [34, 44, 71]. The first and also the best-known rigorous mathematical construction of magnetic vortices was by Taubes [40, 69, 70], regarding the existence and uniqueness of static solutions of the Abelian Higgs model or the Ginzburg–Landau model [32]. Since then there has been much mathematical work on the existence and properties of such vortices. See, for example, [8, 10–12, 29, 47–49, 51, 54, 55, 59–61, 66, 72, 75]. It is also natural to consider the dyon-like vortices, often referred to as electrically charged magnetic vortices, carrying both magnetic and electric charges. Such dually charged vortices are very useful in several issues in theoretical physics such as high-temperature superconductivity [43, 50], Bose–Einstein condensates [37, 42], optics [13] and the quantum Hall effect [58].
It is now well known that there are no finite-energy dually charged vortices in two-space dimensions for the classical Yang–Mills–Higgs equations, Abelian or non-Abelian. This is known as the Julia–Zee theorem [41], the rigorous mathematical proof of which was carried out in [63]. To accommodate dually charged vortices, some effort has been made in [23, 24, 26, 27, 39, 45, 56, 57], where the Chern–Simons terms are introduced into the action Lagrangian. However, the full Chern–Simons–Higgs equations are very difficult to tackle. Only the radial case has been solved in [19]. Since the work of Hong et al [35] and Jackiw and Weinberg [38] was published, there have been some rigorous mathematical results on the existence of electrically charged vortices. In [35, 38], the Yang–Mills(or Maxwell) term is removed from the action Lagrangian density while the Chern–Simons term alone governs electromagnetism, which is physically sensible at large distance and low energies. When the Higgs potential takes a sextic form, the static equations of motion can be reduced to a Bogolmol’ny type system of first order equations [14], which enables one to make rigorous mathematical studies of such solutions. In such a setting, topological multi-vortices with quantized charges [62, 73], non-topological multi-vortices with fractional values of charges [17, 18, 20, 64] and periodic vortices [16, 22, 28, 46, 53, 67, 68] are all present.

Recently, a generalized Chern–Simons model was proposed in [9]. With a non-canonical kinetic term for the complex scalar field and a special choice of the Higgs potential, a generalized self-dual Chern–Simons equation can be obtained. Despite some numerical work in [9], up to now, there has been no rigorous mathematical analysis for this model.

The purpose of this paper is to carry out a rigorous mathematical analysis of this generalized self-dual Chern–Simons model. Specifically, we will establish the existence of doubly periodic multi-vortices, and topological and non-topological vortex solutions to the generalized self-dual Chern–Simons model. We reduce the problem to a semi-linear elliptic equation with Dirac source terms characterizing the locations of the vortices. To find the existence of vortices over a doubly periodic domain, we apply the method developed by Caffarelli and Yang [16] to construct suitable sub-solutions for the reduced equation. However, it is difficult to find the existence of topological solutions, due to the non-canonical structure of the reduced equations. To solve this difficulty, we find a technical sub-solution, which is useful for other problems with similar structures. For the existence of the non-topological solution, we apply a shooting argument, which was used in [20, 75].

The rest of our paper is organized as follows. In section 2, we formulate our problem and state our main results. In section 3, we prove the existence of periodic vortices. In sections 4 and 5, we give the proofs of topological and non-topological solutions, respectively.

2. Generalized Chern–Simons vortices

In this section we derive the generalized self-dual Chern–Simons equations, while in [9] only the radial case is considered. We adapt the notation in [38]. The (2+1)-dimensional Minkowski space metric tensor $g_{\mu\nu}$ is diag(1, −1, −1), which is used to raise and lower indices. The Lagrangian action density of the Chern–Simons-Higgs theory is given by the expression

$$ L = \frac{\kappa}{4} \varepsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + D_\mu \phi \overline{D^\mu \phi} - V(|\phi|), $$

(2.1)

where $D_\mu = \partial_\mu - i A_\mu$ is the gauge-covariant derivative, $A_\mu (\mu = 0, 1, 2)$ is a 3-vector field called the Abelian gauge field, $\phi$ is a complex scalar field called the Higgs field, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, is the induced electromagnetic field, $\alpha, \beta, \mu, \nu = 0, 1, 2, \kappa > 0$ is a constant referred to as the Chern–Simons coupling parameter, $\varepsilon^{012} = 1$, $V$ is the Higgs potential function and the summation convention over repeated indices is observed.
In [9], by changing the kinetic term of the Higgs field, the model (2.1) is modified into a non-canonical form,
\[ \mathcal{L}_G = \frac{\kappa}{4} e^{a\beta\gamma} A_\alpha F_{\beta\gamma} + w(|\phi|) D_\mu \phi D^\mu \phi - V(|\phi|). \] (2.2)
where \( w(|\phi|) \) is a function of the Higgs field. There is a natural geometry interpretation of the function \( w(|\phi|) \), that is, \( \phi \) take values in a curved manifold whose metric is given by the flat metric on \( \mathbb{R}^2 \) by the conformal factor \( w(|\phi|) \). Then the model (2.2) is a gauged nonlinear sigma model with a Chern–Simons term. This is another motivation for the modification. Pertaining to this aspect, vortices have been studied in [31] for the model for which \( w(\phi) \) corresponds to the round metric on \( \mathbb{C}P^1 \).

The Euler–Lagrange equations associated with the action density \( \mathcal{L}_G (2.2) \) are
\[ \kappa \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} = - j^\alpha = i w(|\phi|) (\phi \overline{D}_\alpha \phi - \overline{\phi} D_\alpha \phi), \] (2.3)
\[ w D_\mu D^\mu \phi + \partial_\mu w D^\mu \phi + \frac{\partial V}{\partial \phi} - |D_\mu \phi|^2 \frac{\partial w}{\partial \phi} = 0, \] (2.4)
where \( j^\alpha = (\rho, j) \) is the current density and \( F_{12} \) is the magnetic field.

We are interested in the static solutions of the equations (2.3) and (2.4) over a doubly periodic domain \( \Omega_1 \) such that the field configurations are subject to the 't Hooft boundary condition [36, 74, 75] under which periodicity is achieved through modulo gauge transformations and over the full plane \( \mathbb{R}^2 \). The \( \alpha = 0 \) component of equation (2.3) reads
\[ \kappa F_{12} = j^0 = \rho = - 2 A_0 |\phi|^2 w(\phi), \] (2.5)
the magnetic flux \( \Phi \) and the electric charge \( Q \) are related by the formula
\[ \Phi = \int F_{12} \, dx = \frac{1}{\kappa} \int \rho \, dx = \frac{Q}{\kappa}. \] (2.6)
Here and in the following, our integration is always conducted over the doubly periodic domain \( \Omega \) or the full plane \( \mathbb{R}^2 \). The energy density is then given by
\[ E = \frac{\kappa^2}{4} \frac{F_{12}^2}{|\phi|^2 w} + w |D_\rho \phi|^2 + V(|\phi|). \] (2.7)
As in [9], we choose special forms for \( w(|\phi|) \) and the Higgs potential function \( V(\phi) \) as follows
\[ w(|\phi|) = 3(1 - |\phi|^2)^2, \quad V(|\phi|) = \frac{3}{\kappa^2} |\phi|^2 (1 - |\phi|^2)^8. \]

Then the energy density (2.7) implies that a finite-energy solution of the solution to (2.3)–(2.4) over \( \mathbb{R}^2 \) satisfies the condition
\[ |\phi(x)| \to 1 \quad \text{as} \quad |x| \to +\infty \] (2.8)
or
\[ |\phi(x)| \to 0 \quad \text{as} \quad |x| \to +\infty, \] (2.9)
where the former is called topological and the latter is called non-topological, see [30, 66, 75].

The energy density (2.7) can be rewritten as
\[ E = \left[ \frac{\kappa}{2\sqrt{3}} \frac{F_{12}}{|\phi| (1 - |\phi|^2)^{3/2}} \mp \sqrt{3} \frac{1}{\kappa} |\phi| (1 - |\phi|^2)^{3/2} \right]^2 + 3(1 - |\phi|^2)^2 |D_1 \phi \pm i D_2 \phi|^2 \pm (1 - |\phi|^2) F_{12} \pm i 3(1 - |\phi|^2)^2 (D_1 \phi \overline{D_2 \phi} - D_2 \phi \overline{D_1 \phi}). \]
If \((\phi, A)\) is a finite-energy solution with winding number \(N\) of (2.3)–(2.4), following [40, 75], we can show that
\[
\int F_{12} \, dx = 2\pi N.
\]

Integrating over the doubly periodic domain \(\Omega\) or the full plane \(\mathbb{R}^2\), we have
\[
E(\phi, A) = \int \mathcal{E} \, dx
\]
\[
= \int \left\{ \frac{\kappa}{2\sqrt{3}} \frac{F_{12}}{|\psi|(1 - |\psi|^2)} \mp \frac{\sqrt{3}}{\kappa} |\psi|(1 - |\psi|^2)^2 \right\}^2 + 3(1 - |\psi|^2)^2 |D_1 \psi \pm iD_2 \psi|^2 \right\} \, dx
\]
\[
\pm 2\pi N \pm \int \text{Im} \left\{ \partial_j \varepsilon_{jk} \left( \left[ |\psi|^2 - (1 - |\psi|^2)^2 - 2 \right] \hat{\phi}(D_k \phi) \right) \right\} \, dx.
\]
(2.10)

Therefore, we can get the following lower bound of the energy
\[
E(\phi, A) \geq \pm 2\pi N.
\]

Then we see from (2.10) that such a lower bound is attained if and only if \((\phi, A)\) satisfies the following self-dual or anti-self-dual system
\[
D_1 \phi - iD_2 \phi = 0,
\]
(2.11)
\[
F_{12} + \frac{6}{\kappa^2} |\psi|^2 (1 - |\psi|^2)^2 = 0
\]
(2.12)
or
\[
D_1 \phi + iD_2 \phi = 0,
\]
(2.13)
\[
F_{12} - \frac{6}{\kappa^2} |\psi|^2 (1 - |\psi|^2)^2 = 0.
\]
(2.14)

It is easy to check that if \((\phi, A)\) is a solution of system (2.11)–(2.12), then \((\bar{\phi}, -A)\) is the solution of (2.13)–(2.14). In addition, in view of (2.5), any solution of (2.11)–(2.12) or (2.13)–(2.14) is also the solution of (2.3)–(2.4). Consequently, in the following we only consider (2.11)–(2.12).

To formulate our problem properly, as in [16, 40, 75], we can see that the zeros of \(\phi\) are isolated with integer multiplicities. These zeros are often referred to as vortices. Let the zeros of \(\phi\) be \(p_1, p_2, \ldots, p_m\) with multiplicities \(n_1, n_2, \ldots, n_m\), respectively. Then, \(\sum_{i=1}^{m} n_i = N\) gives the winding number of the solution and the total vortex number. We aim to look for \(N\)-vortex solutions of (2.11)–(2.12) such that, \(\phi\) has \(m\) zeros, say \(p_1, p_2, \ldots, p_m\) with multiplicities \(n_1, n_2, \ldots, n_m\), respectively, and \(\sum_{i=1}^{m} n_i = N\).

For the generalized Chern–Simons equations (2.11)–(2.12), we are interested in three situations. In the first situation the equations (2.11)–(2.12) will be studied over a doubly periodic domain \(\Omega\) such that the field configurations are subject to the 't Hooft boundary condition [36, 74, 75] under which periodicity is achieved through modulo gauge transformations. In the second and the third situations the equations are studied over the full plane \(\mathbb{R}^2\) under the topological condition (2.8) and non-topological condition (2.9), respectively.

The main results of this paper read as follows.
The existence of multi-vortices for a generalized self-dual Chern–Simons model 809

Theorem 2.1 (Existence of doubly periodic vortices). Let \( p_1, p_2, \ldots, p_m \in \Omega \), \( n_1, n_2, \ldots, n_m \) be some positive integers with \( \sum_{i=1}^{m} n_i = N \). There exists a critical value
\[
\kappa_c \in \left( 0, \sqrt{\frac{\Omega}{2 \pi N}} \right)
\]
of the coupling parameter such that the self-dual equations (2.11)–(2.12) admit a solution \((\phi, A)\) for which \( p_1, p_2, \ldots, p_m \) are zeros of \( \phi \) with multiplicities \( n_1, n_2, \ldots, n_m \), if and only if \( 0 < \kappa \leq \kappa_c \). When \( 0 < \kappa \leq \kappa_c \), the solution \((\phi, A)\) also satisfies the following properties.

The energy, magnetic flux and electric charge are given by
\[
E = 2\pi N, \quad \Phi = 2\pi N, \quad Q = 2\kappa \pi N. \tag{2.15}
\]
The solution \((\phi, A)\) is maximal in the sense that the magnitude of \( \phi \), \(|\phi|\), has the largest possible values among all the solutions to equations (2.11)–(2.12).

Let the prescribed data be denoted by \( S = \{ p_1, p_2, \ldots, p_m; n_1, n_2, \ldots, n_m \} \), where \( n_i \) may be zero for \( i = 1, \ldots, m \) and denotes the dependence of \( \kappa_c \) on \( S \) by \( \kappa_c(S) \). For \( S' = \{ p_1, p_2, \ldots, p_m; n'_1, n'_2, \ldots, n'_m \} \), we write \( S \leq S' \) if \( n_1 \leq n'_1, \ldots, n_m \leq n'_m \). Then \( \kappa_c \) is a decreasing function of \( S \) in the sense that
\[
\kappa_c(S) \geq \kappa_c(S'), \quad \text{if} \quad S \leq S'. \tag{2.16}
\]

Theorem 2.2 (Multiple existence of doubly periodic vortices). Let \( p_1, p_2, \ldots, p_m \in \Omega \), \( n_1, n_2, \ldots, n_m \) be some positive integers with \( \sum_{i=1}^{m} n_i = N \) and \( \kappa_c \) be given in theorem 2.1. If \( 0 < \kappa < \kappa_c \), then, in addition to the maximal solution \((\phi, A)\) given in theorem 2.1, the self-dual equations (2.11)–(2.12) have a second solution \((\phi, A)\) satisfying (2.15) and for which \( p_1, p_2, \ldots, p_m \) are the zeros of \( \phi \) with multiplicities \( n_1, n_2, \ldots, n_m \).

Theorem 2.3 (Topological solution). Let \( p_1, p_2, \ldots, p_m \in \mathbb{R}^2 \), \( n_1, n_2, \ldots, n_m \) be some positive integers with \( \sum_{i=1}^{m} n_i = N \). The self-dual equations (2.11)–(2.12) admit a topological solution \((\phi, A)\) such that the zeros of \( \phi \) are exactly \( p_1, p_2, \ldots, p_m \) with the corresponding multiplicities \( n_1, n_2, \ldots, n_m \). Moreover, the energy, magnetic flux and the charges are all quantized:
\[
E = 2\pi N, \quad \Phi = 2\pi N, \quad Q = 2\kappa \pi N. \tag{2.17}
\]
The solution is maximal in the sense that the Higgs field \( \phi \) has the largest possible magnitude among all the solutions with the same zero distribution and local vortex charges in the full plane.

Theorem 2.4 (Radially symmetric topological solution). For any point \( \tilde{x} \in \mathbb{R}^2 \) and a given integer \( N \geq 0 \), the self-dual equations (2.11)–(2.12) admit a unique topological solution \((\phi, A)\), which is radially symmetric about the point \( \tilde{x} \), such that \( \tilde{x} \) is the zero of \( \phi \) with multiplicities \( N \). Moreover, the energy, magnetic flux and charges are all quantized, given by (2.17).

Theorem 2.5 (Radially symmetric non-topological solution). For any point \( \tilde{x} \in \mathbb{R}^2 \) and a given integer \( N \geq 0 \), then for all \( \beta > 2N + 4 \), the self-dual equations (2.11)–(2.12) allow a non-topological solution \((\phi, A)\), which is radially symmetric about the point \( \tilde{x} \), such that \( \tilde{x} \) is the zero of \( \phi \) with the multiplicities \( N \) and realizing the prescribed decay properties,
\[
|\phi|^2 = O(r^{-\beta}), \quad |D_j \phi|^2 = O(r^{-2(\beta - j)}), \quad j = 1, 2, \quad F_{12} = O(r^{-\beta}) \tag{2.18}
\]
for large \( r = |x - \tilde{x}| > 0 \) and the corresponding values of energy, magnetic flux and electric charge are given by the formula
\[
E = 2\pi N + \pi \beta, \quad \Phi = 2\pi N + \pi \beta, \quad Q = \kappa (2\pi N + \pi \beta). \tag{2.19}
\]
Remark 2.1. Since the non-topological solution with arbitrary distributed vortices of the generalized Chern–Simons equations (2.11)–(2.12) is more involved, we will deal with it in a forthcoming paper.

3. Existence of doubly periodic vortices

In this section we aim to establish the existence of a vortex solution to the generalized Chern–Simons equations (2.11)–(2.12) over a doubly periodic domain $\Omega$. In other words, we present the proofs of theorem 2.1–2.2 in this section.

For convenience, we reduce the self-dual equation (2.11)–(2.12) to a scalar nonlinear elliptic equation with Dirac source terms. To this end, we complexify the variables $z = x^1 + ix^2$, $A = A_1 + iA_2$.

Let
$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$
Then from (2.11), we can get
$$F_{12} = -2\bar{\partial}\bar{\partial} \ln|\phi|^2 = -\frac{1}{2}\Delta \ln|\phi|^2.$$ (3.1)

Inserting (3.1) into (2.12) gives rise to the reduced equation
$$\Delta \ln|\phi|^2 = \lambda|\phi|^2((|\phi|^2 - 1)^5)$$ away from the zeros of $\phi$, where we write
$$\lambda \equiv \frac{12}{\kappa^2}$$ throughout this paper.

Counting all the multiplicities of the zeros of $\phi$, we write the prescribed zero set of $\phi$ as $Z(\phi) = \{p_1, \ldots, p_N\}$. Let $|\phi|^2 = e^u$. Then the generalized self-dual Chern–Simons equations (2.11)–(2.12) are transformed into the following scalar equation
$$\Delta u = \lambda e^u(e^u - 1)^5 + 4\pi N \sum_{j=1}^{\infty} \delta_{p_j}, \quad \text{in} \quad \Omega.$$ (3.3)

where $\delta_p$ is the Dirac distribution centred at $p \in \Omega$.

Conversely, if $u$ is a solution of (3.3), we can obtain a solution of (2.11)–(2.12) according to the transformation
$$\phi(z) = \exp\left(\frac{1}{2} u(z) + i \sum_{j=1}^{N} \text{arg}(z - p_j)\right),$$
$$A_1(z) = -2\text{Re}[i\bar{\partial}\ln \phi], \quad A_2(z) = -2\text{Im}[i\bar{\partial}\ln \phi].$$
Then it is sufficient to solve (3.3).

Let $u_0$ be a solution of the equation (see [7])
$$\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{N} \delta_{p_j}. \quad (3.4)$$

Setting $u = u_0 + v$, equation (3.3) can be reduced to the following equation
$$\Delta v = \lambda e^{u_0+v}(e^{u_0+v} - 1)^5 + \frac{4\pi N}{|\Omega|}.$$ (3.5)
It is easy to check that the function \( f(t) = e^t(e^t - 1)^5(t \in \mathbb{R}) \) has a unique minimal value \(-\frac{5}{6}e\). Then, if \( u \) is a solution of (3.5), we have

\[
\Delta u \geq -\frac{5}{6}e^\lambda + \frac{4\pi N}{|\Omega|}.
\] (3.6)

Integrating (3.6) over \( \Omega \), we have

\[
0 \geq -\frac{5}{6}e^\lambda |\Omega| + 4\pi N,
\] i.e.

\[
\lambda \geq \frac{6}{5}e^\lambda \frac{4\pi N}{|\Omega|},
\] (3.7)

which is a necessary condition for the existence of solutions to (3.3).

As in [16] or chapter 5 in [75] we can use a super- and sub-solution method to establish the existence results for (3.3).

To solve (3.5), we introduce the following iterative scheme

\[
\begin{cases}
(\Delta - K) v_n = \lambda e^{e^{u_0} + v_{n-1}}(e^{e^{u_0} + v_{n-1}} - 1)^5 - K v_{n-1} + \frac{4\pi N}{|\Omega|}, \\
n = 1, 2, \ldots, \\
v_0 = -u_0,
\end{cases}
\] (3.8)

where \( K > 0 \) is a constant to be determined.

**Lemma 3.1.** Let \( \{v_n\} \) be the sequence defined by (3.8) with \( K \geq 6\lambda \). Then

\[
v_0 > v_1 > v_2 > \cdots > v_n > \cdots \geq v_*
\] (3.9)

for any sub-solution \( v_* \) of (3.5). Therefore, if (3.5) has a sub-solution, the sequence \( \{v_n\} \) converges to a solution of (3.5) in the space \( C^k(\Omega) \) for any \( k \geq 0 \) and such a solution is the maximal solution of the equation.

**Proof.** We prove (3.9) by induction.

We prove the case \( n = 1 \) first. From (3.8) we have

\[
(\Delta - K) v_1 = Ku_0 + \frac{4\pi N}{|\Omega|},
\]

which implies \( v_1 \in C^\infty(\Omega) \cap C^\alpha(\Omega) \) for some \( 0 < \alpha < 1 \). Noting that \((\Delta - K)(v_1 - v_0) = 0 \) in \( \Omega \setminus \{p_1, p_2, \ldots, p_N\} \) and \( v_1 - v_0 < 0 \) on \( \partial \Omega_\varepsilon \), where \( \Omega_\varepsilon \) is the complement of \( \bigcup_{j=1}^N \{x \mid |x - p_j| < \varepsilon\} \) in \( \Omega \) for \( \varepsilon \) sufficiently small, using the maximum principle we have \( v_1 - v_0 < 0 \) in \( \Omega_\varepsilon \). Therefore, we have \( v_1 - v_0 < 0 \) in \( \Omega \).

Assume that \( v_0 > v_1 > \cdots > v_k \). From (3.8) and \( K \geq 6\lambda \) we obtain

\[
(\Delta - K)(v_{k+1} - v_k) = \lambda [e^{e^{u_0} + v_{k+1}}(e^{e^{u_0} + v_{k+1}} - 1)^5 - e^{e^{u_0} + v_k}(e^{e^{u_0} + v_k} - 1)^5] - K(v_k - v_{k-1})
\]

\[
= \lambda [e^{e^{u_0} + \xi}(e^{e^{u_0} + \xi} - 1)^5 - e^{e^{u_0} + \xi}(e^{e^{u_0} + \xi} - 1)] - K(v_k - v_{k-1})
\]

\[
\geq (6\lambda - K)(v_k - v_{k-1})
\]

\[
\geq 0,
\]

where \( v_k \leq \xi \leq v_{k-1} \). Using the maximum principle again, we have \( v_{k+1} < v_k \) in \( \Omega \).

Now we prove the lower bound in (3.9) in terms of the sub-solution \( v_* \) of (3.5), that is, \( v_* \in C^\infty(\Omega) \) and

\[
\Delta v_* \geq \lambda e^{e^{u_0} + \xi}(e^{e^{u_0} + \xi} - 1)^5 + \frac{4\pi N}{|\Omega|}.
\] (3.10)
Noting that \( v_0 = -u_0 \) and (3.10), we have
\[
\Delta (v_- - v_0) \geq \lambda e^{u_0 + v_-} - (e^{u_0 + v_-} - 1)^5 = \lambda e^{v_- - u_0} - (e^{v_- - u_0} - 1)^5 \quad \text{in} \quad \Omega \setminus \{p_1, \ldots, p_N\}.
\]
If \( \varepsilon > 0 \) is small, we see that \( v_- - v_0 < 0 \) on \( \partial \Omega \varepsilon \). Then, by the maximum principle, we obtain \( v_- - v_0 < 0 \) in \( \Omega \varepsilon \). Therefore, \( v_- - v_0 < 0 \) throughout \( \Omega \).

Now assume \( v_- < v_k \) for some \( k \geq 0 \). It follows from (3.9), (3.10) and the fact \( K > 6\lambda \) that
\[
(\Delta - K)(v_- - v_{k+1}) \geq \lambda [e^{u_0 + v_-} - (e^{u_0 + v_-} - 1)^5 - e^{u_0 + v_k} (e^{u_0 + v_k} - 1)^5] - K(v_- - v_k)
\]
\[
\geq (5\lambda - K)(v_- - v_k) \geq 0,
\]
where \( v_- \leq \xi \leq v_k \). Applying the maximum principle again, we get \( v_- < v_{k+1} \).

The convergence of the sequence \( \{v_n\} \) can be obtained by a standard bootstrap argument. Then lemma 3.1 follows.

In what follows we just need to construct a sub-solution of (3.5). Indeed, we have the following lemma.

**Lemma 3.2.** If \( \lambda > 0 \) is sufficiently large, equation (3.5) admits a sub-solution satisfying (3.10).

**Proof.** Choose \( \varepsilon > 0 \) sufficiently small such that the balls
\[
B(p_j, 2\varepsilon) = \{x \in \Omega | |x - p_j| < 2\varepsilon\}, \quad j = 1, 2, \ldots, N
\]
verify \( B(p_j, 2\varepsilon) \cap B(p_j, 2\varepsilon) = \emptyset \) if \( i \neq j \). Let \( f_\varepsilon \) be a smooth function defined on \( \Omega \) such that \( 0 \leq f_\varepsilon \leq 1 \) and
\[
f_\varepsilon = \begin{cases} 1, & x \in B(p_j, \varepsilon), \quad j = 1, 2, \ldots, N, \\ 0, & x \notin \bigcup_{j=1}^N B(p_j, 2\varepsilon), \\ \text{smooth connection,} & \text{elsewhere.}
\end{cases}
\]

Then,
\[
\bar{f}_\varepsilon = \frac{1}{|\Omega|} \int_\Omega f_\varepsilon \, dx \leq \frac{4\pi N \varepsilon^2}{|\Omega|}.
\] (3.11)
Define
\[
g_\varepsilon = \frac{8\pi N}{|\Omega|} (f_\varepsilon - \bar{f}_\varepsilon).
\]
It is easy to see that
\[
\int_\Omega g_\varepsilon \, dx = 0.
\]
Then we see that the equation
\[
\Delta w = g_\varepsilon
\] (3.12)
admits a unique solution up to an additive constant.
First, it follows from (3.11) that, for $x \in B(p_j, \epsilon)$,
\[
g_{\epsilon} \geq \frac{4\pi N}{|\Omega|} \left( 2 - \frac{8\pi N \epsilon^2}{|\Omega|} \right) > \frac{4\pi N}{|\Omega|} \tag{3.13}
\]
if $\epsilon$ is small enough. In the following we fix $\epsilon$ such that (3.13) is valid.

Next, we choose a solution of (3.12), say, $w_0$, to satisfy
\[
e^{u_0 + w_0} + w_0 \leq 1, \quad x \in \Omega.
\]
Hence, for any $\lambda > 0$,
\[
\Delta w_0 = g_{\epsilon} \geq \frac{4\pi N}{|\Omega|} \geq \lambda e^{u_0 + w_0}(e^{u_0 + w_0} - 1)^5 + \frac{4\pi N}{|\Omega|} \tag{3.14}
\]
for $x \in B(p_j, \epsilon), \ j = 1, 2, \ldots, N$.

Finally, set
\[
\mu_0 = \inf \left\{ e^{u_0 + w_0} \mid x \in \Omega \setminus \bigcup_{j=1}^N B(p_j, \epsilon) \right\},
\]
\[
\mu_1 = \sup \left\{ e^{u_0 + w_0} \mid x \in \Omega \setminus \bigcup_{j=1}^N B(p_j, \epsilon) \right\}.
\]
Then $0 < \mu_0 < \mu_1$ and $e^{u_0 + w_0}(e^{u_0 + w_0} - 1)^5 \leq \mu_0(\mu_1 - 1)^5 = -C_0 < 0$ for $x \in \Omega \setminus \bigcup_{j=1}^N B(p_j, \epsilon)$.

As a consequence, we can choose $\lambda > 0$ sufficiently large to fulfill (3.14) in the entire $\Omega$. Thus, $w_0$ is a sub-solution of (3.5). The proof of lemma 3.2 is complete.

Now we seek the critical value of the coupling parameter. We establish the following lemma.

**Lemma 3.3.** There is a critical value of $\lambda$, say, $\lambda_c$, satisfying
\[
\lambda_c \geq \frac{6^6 4\pi N}{5^5 |\Omega|}, \tag{3.15}
\]
such that, for $\lambda > \lambda_c$, equation (3.5) has a solution, while for $\lambda < \lambda_c$, equation (3.5) has no solution.

**Proof.** Assume that $v$ is a solution of (3.5). Then $u = u_0 + v$ satisfies (3.3) and is negative near the points $x = p_j, j = 1, \ldots, N$. Applying the maximum principle away from the points $x = p_j, j = 1, \ldots, N$, we see that $u < 0$ throughout $\Omega$.

Define
\[
\Lambda = \{ \lambda > 0 \mid \lambda \text{ is such that (3.5) has a solution} \}.
\]
Then we can prove that $\Lambda$ is an interval. To do so, we prove that, if $\lambda' \in \Lambda$, then $[\lambda', +\infty) \subset \Lambda$. Denote by $v'$ the solution of (3.5) at $\lambda = \lambda'$. Noting that $u_0 + v' < 0$, we see that $v'$ is a sub-solution of (3.5) for any $\lambda > \lambda'$. By lemma 3.1, we obtain $\lambda \in \Lambda$.

Let $\lambda_c = \inf \Lambda$. Then, by the necessary condition (3.7), we have $\lambda > \frac{6^6 4\pi N}{5^5 |\Omega|}$ for any $\lambda > \lambda_c$. Taking the limit $\lambda \to \lambda_c$, we obtain (3.15). Then lemma 3.3 follows.

Now we need to consider the critical case $\lambda = \lambda_c$. We use the method of [67] to deal with this case.
We first make a simple observation. We can show that the maximum solutions of (3.5) \( \{v_{\lambda} | \lambda > \lambda_c \} \) are a monotone family in the sense that \( v_{\lambda_1} > v_{\lambda_2} \) whenever \( \lambda_1 > \lambda_2 > \lambda_c \). Indeed, since \( u_0 + v_{\lambda} \) from (3.5) have
\[
\Delta v_{\lambda_2} = \lambda_2 e^{u_0 + v_{\lambda_2}} (e^{u_0 + v_{\lambda_2}} - 1)^5 + \frac{4\pi N}{|\Omega|} (\lambda_2 - \lambda_1) e^{u_0 + v_{\lambda_2}} (e^{u_0 + v_{\lambda_2}} - 1)^5 \\
\geq \lambda_1 e^{u_0 + v_{\lambda_2}} (e^{u_0 + v_{\lambda_2}} - 1)^5 + \frac{4\pi N}{|\Omega|} (\lambda_2 - \lambda_1) e^{u_0 + v_{\lambda_2}} (e^{u_0 + v_{\lambda_2}} - 1)^5
\]
for \( \lambda_1 > \lambda_2 > \lambda_c \). Therefore \( v_{\lambda_2} \) is a sub-solution of (3.5) with \( \lambda = \lambda_1 \). Hence \( v_{\lambda_1} > v_{\lambda_2} \) if \( \lambda_1 > \lambda_2 > \lambda_c \).

Set
\[
X = \left\{ v \in W^{1,2}(\Omega) \left| \int_{\Omega} v \, dx = 0 \right. \right\}.
\]
Then \( X \) is a closed subspace of \( W^{1,2}(\Omega) \) and
\[
W^{1,2}(\Omega) = \mathbb{R} \oplus X.
\]
In other words, for any \( v \in W^{1,2}(\Omega) \), there exits a unique number \( c \in \mathbb{R} \) and \( v' \in X \) such that \( v = c + v' \). In what follows, we will use the Trudinger–Moser inequality (see [7])
\[
\int_{\Omega} e^{v'} \, dx \leq C \exp \left( \frac{1}{16\pi} \int_{\Omega} |\nabla v'|^2 \, dx \right), \quad \forall v' \in X,
\]
where \( C \) is a positive constant depending only on \( \Omega \).

**Lemma 3.4.** Let \( v_{\lambda} \) be a solution of (3.5). Then \( v_{\lambda} = c_{\lambda} + v'_{\lambda} \), where \( c_{\lambda} \in \mathbb{R} \) and \( v'_{\lambda} \in X \). We have
\[
\|\nabla v'_{\lambda}\|_2 \leq C\lambda,
\]
where \( C \) is a positive constant depending only on the size of the torus \( \Omega \). Furthermore, \( \{c_{\lambda}\} \) satisfies the estimate
\[
|c_{\lambda}| \leq C(1 + \lambda + \lambda^2).
\]
In particular, \( v_{\lambda} \) satisfies
\[
\|v_{\lambda}\|_{W^{1,2}(\Omega)} \leq C(1 + \lambda + \lambda^2).
\]

**Proof.** Multiplying (3.5) by \( v'_{\lambda} \), integrating over \( \Omega \), and using the Schwarz and Poincaré inequalities, we have
\[
\|\nabla v'_{\lambda}\|_2^2 = -\int_{\Omega} \lambda e^{u_0 + v_{\lambda}} (e^{u_0 + v_{\lambda}} - 1)^5 v'_{\lambda} \, dx \\
\leq 2\lambda \int_{\Omega} |v'_{\lambda}| \, dx \leq 2\lambda |\Omega|^{1/2} \|v'_{\lambda}\|_2 \leq C\lambda \|\nabla v'_{\lambda}\|_2,
\]
which implies (3.17).

Noting the property \( u_0 + v_{\lambda} = u_0 + c_{\lambda} + v'_{\lambda} < 0 \), we have the upper bound,
\[
c_{\lambda} < -\frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx.
\]
From equation (3.5), we have
\[ \Delta v_\lambda = \lambda e^{u_\lambda + v_\lambda} (e^{u_\lambda + v_\lambda} - 1)^5 + \frac{4\pi N}{|\Omega|} \geq \lambda e^{u_\lambda + v_\lambda} (e^{u_\lambda + v_\lambda} - 1) + \frac{4\pi N}{|\Omega|}. \]

Integrating the above inequality over \( \Omega \) gives
\[ \lambda \int_\Omega e^{u_\lambda + v_\lambda} \, dx \geq \lambda \int_\Omega e^{2(u_\lambda + v_\lambda)} \, dx + 4\pi N > 4\pi N, \]
which is
\[ \lambda e^{c_\lambda} \int_\Omega e^{u_\lambda} e^{v_\lambda} \, dx \geq 4\pi N. \]

Then we have
\[ e^{c_\lambda} \geq C\lambda^{-1} \left( \int_\Omega e^{u_\lambda} e^{v_\lambda} \, dx \right)^{-1} \geq C\lambda^{-1} \left( \int_\Omega e^{v_\lambda} \, dx \right)^{-1} \geq C\lambda^{-1} \exp \left( -\frac{1}{16\pi} \| \nabla v_\lambda \|_2^2 \right), \]
where in the last inequality we have used the Trudinger–Moser inequality (3.16).

Now using (3.5) in the above inequality we can obtain a lower bound of \( c_\lambda \),
\[ c_\lambda \geq -C(1 + \lambda + \lambda^2). \]

Then (3.18) follows from (3.20) and (3.21). Combining (3.17) and (3.18), we obtain (3.19). Hence lemma 3.4 follows.

For the critical case we have the following result.

**Lemma 3.5.** The set of \( \lambda \) for which equation (3.5) has a solution is a closed interval. That is to say, at \( \lambda = \lambda_c \) (3.5) has a solution as well.

**Proof.** For \( \lambda_c < \lambda < \lambda_c + 1 \) (say), by lemma 3.4 the set \( \{v_\lambda\} \) is bounded in \( W^{1,2}(\Omega) \). Noting that \( \{v_\lambda\} \) is monotone with respect to \( \lambda \), we conclude that there exist \( v_* \in W^{1,2}(\Omega) \) such that
\[ v_\lambda \rightarrow v_* \text{ weakly in } W^{1,2}(\Omega) \text{ as } \lambda \rightarrow \lambda_c. \]

Therefore \( v_\lambda \rightarrow v_* \) strongly in \( L^p(\Omega) \) for any \( p \geq 1 \) as \( \lambda \rightarrow \lambda_c \). Using the Trudinger–Moser inequality (3.16) again we obtain \( e^{v_\lambda} \rightarrow e^{v_*} \) strongly in \( L^p(\Omega) \) for any \( p \geq 1 \) as \( \lambda \rightarrow \lambda_c \). Using this result in (3.5) and the \( L^2 \) estimates for the elliptic equations, we have \( v_* \in W^{2,2}(\Omega) \) and \( v_\lambda \rightarrow v_* \) strongly in \( W^{2,2}(\Omega) \) as \( \lambda \rightarrow \lambda_c \). Particularly, taking the limit \( \lambda \rightarrow \lambda_c \) in (3.5), we obtain that \( v_* \) is a solution of (3.5) for \( \lambda = \lambda_c \). Then the lemma follows.

Write
\[ P = \{ p_1, \ldots, p_m; n_1, n_2, \ldots, n_m \}, \quad P' = \{ p_1, \ldots, p'_m; n'_1, n'_2, \ldots, n'_m \}. \]

We denote the dependence of \( \lambda_c \) on \( P \) by \( \lambda_c(P) \). Consider the equation
\[ \Delta u = \lambda e^{u} (e^u - 1)^5 + 4\pi \sum_{j=1}^{m} n_j \delta_{p_j}, \quad (3.22) \]

**Lemma 3.6.** If \( P \leq P' \), we have \( \lambda(P) \leq \lambda(P') \).
Proof. It is sufficient to show that, if \( \lambda > \lambda_c(P) \), then \( \lambda \geq \lambda_c(P) \). Let \( u' \) be a solution of (3.22) with \( n_j = n'_j, \ j = 1, \ldots, m \) and \( u_0 \) satisfy

\[
\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{m} n_j \delta_{p_j},
\]

where \( N = n_1 + \cdots + n_m \). Setting \( u' = u_0 + v' \), we have

\[
\Delta v' = \lambda e^{u_0+v}(e^{u_0+v'} - 1)^5 + 4\pi N \frac{|\Omega|}{|\Omega|} + 4\pi \sum_{j=1}^{m} (n'_j - n_j) \delta_{p_j},
\]

which implies in particular that \( v' \) is a sub-solution of (3.5) in the sense of distribution and (3.9) holds pointwise. It is easy to check that the singularity of \( v' \) is at most of the type \( \ln |x - p_j| \). Hence, inequality (3.9) still results in the convergence of the sequence of \( \{v_n\} \) to a solution of (3.5) in any \( C^k \) norm. Indeed, by (3.9) we see that \( \{v_n\} \) converges almost everywhere and is bounded in \( L^2 \) norm. Applying the standard \( L^2 \) estimate, we see that the sequence converges in \( W^{1,2} \) to a strong solution of (3.5). Thus, a classical solution can be obtained.

Using a bootstrap argument again, we can obtain the convergence in \( C^k \) norm. This proves \( \lambda \geq \lambda_c(P) \). Therefore, \( \lambda(P) \leq \lambda(P') \).

From the above discussion we complete the proof of theorem 2.1.

Now we carry out the proof of theorem 2.2. It is easy to see that (3.5) is the Euler–Lagrange equation of the following functional

\[
I_\lambda(v) = \int_\Omega \left\{ \frac{1}{2} |\nabla v|^2 + \frac{\lambda}{6} (e^{u_0+v} - 1)^6 + 4\pi N \frac{|\Omega|}{|\Omega|} v \right\} \, dx.
\]  

(3.23)

Lemma 3.7. For every \( \lambda > \lambda_c \), problem (3.5) admits a solution \( v_\lambda \in W^{1,2}(\Omega) \) and it is a local minimum of the functional \( I_\lambda(v) \) defined by (3.23).

Proof. We apply the method in [67]. Since \( u_0 + v_* < 0 \), we see that \( v_* \) is a sub-solution of (3.5) for any \( \lambda > \lambda_c \). Define

\[
V = \{ v \in W^{1,2} \mid v \geq v_* \ \ a.e. \ in \ \Omega \}.
\]  

(3.24)

Then the functional \( I_\lambda \) is bounded form below on \( V \). We can study the following minimization problem

\[
\eta_0 = \inf \{ I_\lambda(v) \mid v \in V \}.
\]  

(3.25)

We will show that problem (3.25) admits a solution.

Let \( \{v_n\} \) be a minimizing sequence of (3.25). Then, by the decomposition formula,

\[
v_n = v'_n + c_n, \quad v'_n \in X, \quad c_n \in \mathbb{R}, \quad n = 1, 2, \ldots,
\]

we see that \( ||\nabla v_n||_2 \) is bounded since the definition of \( V \) gives a lower bound of \( \{c_n\} \). By the definition of \( I_\lambda(v) \) we have

\[
I_\lambda(v_n) \geq 4\pi c_n,
\]

which gives an upper bound of \( \{c_n\} \). Then \( \{v_n\} \) is a bounded sequence in \( W^{1,2}(\Omega) \). Without loss of generality, we may assume that \( \{v_n\} \) converges weakly to an element \( v \in W^{1,2}(\Omega) \) as \( n \to \infty \). Hence, \( v \) is a solution to problem (3.25). Using lemma 5.6.3 in [75] or the appendix of [67], we conclude that \( v \) is a solution of equation (3.5) and \( v \geq v_* \) in \( \Omega \). By the maximum principle we obtain the strict inequality \( v > v_* \) in \( \Omega \).
Next we prove that \( v \) is a local minimum of the functional (3.23) in \( W^{1,2}(\Omega) \). We use the approach of Brezis and Nirenberg [15] as in Tarantello [67] and Yang [75]. We argue by contradiction. Suppose otherwise \( v \) is not a local minimum of \( I_\lambda(v) \) in \( W^{1,2}(\Omega) \). Then, for any integer \( n \geq 1 \), we have

\[
\inf \left\{ I_\lambda(w) | w \in W^{1,2}(\Omega), \|w - v\|_{W^{1,2}(\Omega)} \leq \frac{1}{n} \right\} = \epsilon_n < I_\lambda(v). \tag{3.26}
\]

Similarly to the above, for any \( n \geq 1 \), we can conclude that the infimum of (3.26) is achieved at a point \( v_n \in W^{1,2}(\Omega) \). Then, by the principle of Lagrangian multipliers, we obtain that there exists a number \( \mu_n \leq 0 \) such that

\[-\Delta v_n + \lambda e^{\alpha v_n} (e^{\alpha v_n} - 1)^5 + \frac{4\pi N}{|\Omega|} = \mu_n (-\Delta v_n + v_n - v). \tag{3.27}\]

We rewrite the above equation the following form

\[
\Delta(v_n - v) = \frac{\lambda}{1 + |\mu_n|} \left[ e^{\alpha v_n} (e^{\alpha v_n} - 1)^5 - e^{\alpha v} (e^{\alpha v} - 1)^5 \right] + \frac{|\mu_n|}{1 + |\mu_n|} (v_n - v). \quad \text{Noting the fact that } \|v_n - v\|_{W^{1,2}(\Omega)} \to 0 \text{ as } n \to \infty \text{ and the Trudinger–Moser inequality (3.16), we see that the right-hand side of (3.27) converges to 0 as } n \to \infty. \text{ Then using the elliptic } L^2 \text{ estimate, we have } v_n \to v \text{ in } W^{2,2}(\Omega) \text{ as } n \to \infty. \text{ By embedding theorem we see that } v_n \to v \text{ in } C^0(\Omega) \text{ for any } 0 < \alpha < 1. \text{ Since } \Omega \text{ is compact and } v > v_n \text{ in } \Omega, \text{ we have } v_n > v_n \text{ for } n \text{ sufficiently large. This implies } v \in V \text{ for } n \text{ sufficiently large, which leads to } I_\lambda(v_n) \geq I_\lambda(v). \text{ Then we obtain a contradiction and the conclusion follows.}

In the following we show that the functional \( I_\lambda(v) \) satisfies the PS condition in \( W^{1,2}(\Omega) \).

**Lemma 3.8.** Any sequence \( \{v_n\} \subset W^{1,2}(\Omega) \) verifying

\[
I_\lambda(v_n) \to \alpha, \quad \|dI_\lambda(v_n)\|_d \to 0 \quad \text{as } n \to \infty \tag{3.28}
\]

admits a convergent subsequence, where we use \( \|\cdot\|_d \) to denote the norm of the dual space of \( W^{1,2}(\Omega) \).

**Proof.** By (3.28) we have

\[
\frac{1}{2} \|\nabla v_n\|^2 + \frac{\lambda}{6} \int_\Omega (e^{\alpha v_n} - 1)^6 \, dx + \frac{4\pi N}{|\Omega|} \int_\Omega v_n \, dx \to \alpha, \tag{3.29}
\]

\[
\left| \int_\Omega \nabla v_n \nabla \phi \, dx + \lambda \int_\Omega e^{\alpha v_n} (e^{\alpha v_n} - 1)^5 \phi \, dx + \frac{4\pi N}{|\Omega|} \int_\Omega \phi \, dx \right| \leq \varepsilon_n \|\phi\|_{W^{1,2}(\Omega)} \tag{3.30}
\]

as \( n \to \infty \), where \( \varepsilon_n \to 0 \) as \( n \to \infty \). Setting \( \phi = 1 \) in (3.30), we obtain

\[
\lambda \int_\Omega e^{\alpha v_n} (e^{\alpha v_n} - 1)^5 \, dx + 4\pi N \leq \varepsilon_n |\Omega|,
\]

which implies

\[
\frac{\varepsilon_n |\Omega|}{\lambda} \geq \frac{4\pi N}{\lambda} + \frac{\lambda}{6} \int_\Omega (e^{\alpha v_n} - 1)^6 \, dx
\]

\[
= \frac{4\pi N}{\lambda} + \frac{\lambda}{6} \int_\Omega (e^{\alpha v_n} - 1)^6 \, dx + \int_\Omega (e^{\alpha v_n} - 1)^5 \, dx
\]

\[
\geq \frac{4\pi N}{\lambda} - \frac{5}{6} |\Omega| + \frac{5}{6} \int_\Omega (e^{\alpha v_n} - 1)^6 \, dx.
\]
Then, it follows that
\[\int_{\Omega} (e^{16\rho + \kappa} - 1)^6 \, dx \leq \left(1 + \frac{6 \varepsilon_n}{\lambda}\right) |\Omega| - \frac{24 \pi N}{\lambda} \leq C. \quad (3.31)\]

Here and in the following we use \(C\) to denote a universal positive constant maybe different in different places. Hence, by (3.31) we have
\[\int_{\Omega} e^{6(\rho^2 + \kappa^2)} \, dx = \int_{\Omega} [(e^{16\rho + \kappa} - 1) + 1]^6 \, dx \leq 2^5 \left[\int_{\Omega} (e^{16\rho + \kappa} - 1)^6 \, dx + |\Omega|\right] \leq C. \quad (3.32)\]

Using Hölder inequality and (3.32), we have
\[\int_{\Omega} e^{2(\rho^2 + \kappa^2)} \, dx \leq \left(\int_{\Omega} e^{6(\rho^2 + \kappa^2)} \, dx\right)^{1/2} |\Omega|^{1/2} \leq C. \quad (3.33)\]

Likewise,
\[\int_{\Omega} e^{4(\rho^2 + \kappa^2)} \, dx \leq C. \quad (3.34)\]

Applying the decomposition formula \(v_n = v'_n + c_n\) in (3.29), we have
\[\frac{1}{2} \|
abla v'_n\|_2^2 + \frac{\lambda}{6} \int_{\Omega} (e^{2\rho^2 + \kappa^2} - 1)^6 \, dx + 4\pi N c_n \to \alpha, \quad (3.35)\]
as \(n \to \infty\). Then from (3.35) it follows that \(c_n\) is bounded from above. Since \(I_\alpha(v_n) \to \alpha\) as \(n \to \infty\), we may assume that for all \(n\),
\[\alpha - 1 < I_\alpha(v_n) < \alpha + 1, \quad (3.36)\]
which leads to
\[\alpha - 1 < \frac{1}{2} \|
abla v'_n\|_2^2 + \frac{\lambda}{6} \int_{\Omega} (e^{2\rho^2 + \kappa^2} - 1)^6 \, dx + 4\pi N c_n < \alpha + 1. \quad (3.37)\]

Therefore it follows from (3.32) and (3.36) that
\[\alpha - 1 + \frac{4\lambda \pi N}{\lambda} - \left(\frac{\lambda}{6} + \frac{\varepsilon_n}{\lambda}\right) |\Omega| < \frac{1}{2} \|
abla v'_n\|_2^2 + 4\pi N c_n < \alpha + 1. \quad (3.38)\]

Now we aim to get a lower bound for \(c_n\). Let \(\psi = v'_n\) in (3.30), we obtain
\[\|
abla v'_n\|_2^2 + \lambda \int_{\Omega} (e^{2\rho^2 + \kappa^2} - 1)^5 v'_n \, dx \leq \varepsilon_n \|
abla v'_n\|_2^2 \leq C \varepsilon_n \|
abla v'_n\|_2^2 \quad (3.39)\]
from which follows
\[\|
abla v'_n\|_2^2 + \lambda \int_{\Omega} e^{6(\rho^2 + \kappa^2)} v'_n \, dx \leq C \int_{\Omega} e^{6(\rho^2 + \kappa^2)} (e^{6\rho^2} - 1) v'_n \, dx \leq \lambda \int_{\Omega} e^{6(\rho^2 + \kappa^2)} v'_n \, dx + C \varepsilon_n \|
abla v'_n\|_2^2 \quad (3.40)\]

It is easy to see that (3.40) is equivalent to
\[\|
abla v'_n\|_2^2 + \lambda \int_{\Omega} e^{6(\rho^2 + \kappa^2)} (e^{6\rho^2} - 1) v'_n \, dx \leq \lambda \int_{\Omega} e^{6(\rho^2 + \kappa^2)} v'_n \, dx + C \varepsilon_n \|
abla v'_n\|_2^2 + C \int_{\Omega} e^{2(\rho^2 + \kappa^2)} + e^{2(\rho^2 + \kappa^2)} + e^{2(\rho^2 + \kappa^2)} + e^{2(\rho^2 + \kappa^2)} + 1) |v'_n| \, dx. \quad (3.41)\]

Now we deal with the right-hand side terms in (3.41). Using the Hölder and the Poincaré inequalities, we have
\[\int_{\Omega} e^{6(\rho^2 + \kappa^2)} v'_n \, dx \leq C \|
abla v'_n\|_2 \leq C \|
abla v'_n\|_2. \]
Applying the Hölder inequality, (3.32) and the Sobolev embedding theorem, we get
\[
\int_{\Omega} e^{\delta(\varphi + \phi)} |v_n'|^2 \, dx \leq \left( \int_{\Omega} e^{\delta(\varphi + \phi)} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v_n'|^6 \, dx \right)^{\frac{1}{3}} \leq C \|v_n'\|_6 \leq C \|\nabla v_n\|_2.
\]

All the other terms on the right-hand side of (3.39) can estimated in the same way and they all be bounded by \( C \|\nabla v_n\|_2 \). Then we have
\[
\|\nabla v_n\|_2 \leq C.
\]

Noting that
\[
\int_{\Omega} e^{\delta(\varphi + \phi)} (e^{\delta\varphi} - 1) v_n' \, dx \leq 0,
\]
we obtain from (3.40) that
\[
\|\nabla v_n\|_2 \leq C.
\]

Inserting (3.41) into (3.37), we see that \( c_n \) is bounded from below.

Then we can derive that \( \{v_n\} \) is uniformly bounded in \( W^{1,2}(\Omega) \). Without loss of generality, we may assume that there exists an element \( \nu \in W^{1,2}(\Omega) \) such that \( v_n \rightharpoonup v \) weakly in \( W^{1,2}(\Omega) \) and strongly in \( L^p(\Omega) \) for any \( p \geq 1 \).

Setting \( n \to \infty \) in (3.30), we have
\[
\int_{\Omega} \left\{ \nabla \nu \cdot \nabla \varphi + \lambda e^{\mu \nu}(e^{\mu \nu} - 1)^5 \varphi + \frac{4\pi N}{|\Omega|} \varphi \right\} \, dx = 0, \quad \forall \varphi \in W^{1,2}(\Omega).
\]

Then \( \nu \) is a critical point of the functional \( I_{\lambda} \).

Next we show that \( v_n \to v \) strongly in \( W^{1,2}(\Omega) \) as \( n \to \infty \).

Letting \( \varphi = v_n - v \) in (3.30) and (3.42) and subtracting the resulting expressions, we obtain
\[
\|\nabla (v_n - v)\|_2^2 + \lambda \int_{\Omega} \left[ e^{\delta(\varphi + \phi)} (e^{\delta\varphi} - 1)^5 - e^{\mu \nu}(e^{\mu \nu} - 1)^5 \right] (v_n - v) \, dx \\
\leq \varepsilon_n \|v_n - v\|_{W^{1,2}(\Omega)},
\]

which implies
\[
\|\nabla (v_n - v)\|_2^2 \\
\leq \lambda \int_{\Omega} \left[ e^{\delta(\varphi + \phi)} (e^{\delta\varphi} - 1)^5 - e^{\mu \nu}(e^{\mu \nu} - 1)^5 \right] |v_n - v| \, dx + \varepsilon_n \|v_n - v\|_{W^{1,2}(\Omega)} \\
\leq C \|v_n - v\|_2 + \varepsilon_n (\|v_n\|_{W^{1,2}(\Omega)} + \|v\|_{W^{1,2}(\Omega)}).
\]

Since the right-hand side of (3.43) tends to 0 as \( n \to \infty \), we have \( \nabla v_n \to \nabla v \) strongly in \( L^2(\Omega) \). Then we can obtain that \( v_n \to v \) strongly in \( W^{1,2}(\Omega) \) as \( n \to \infty \). Then the proof of lemma 3.8 is complete.

Next we establish the existence of secondary solutions of equation (3.5).

Let \( v_\lambda \) be the local minimum of \( I_{\lambda} \) obtained in lemma 3.7. Then there exists a positive constant \( \delta > 0 \) such that
\[
I_{\lambda}(v_\lambda) < I_{\lambda}(w), \quad \forall w \in W^{1,2}(\Omega), \quad \|w - v_\lambda\|_{W^{1,2}(\Omega)} \leq \delta.
\]

Here we assume that \( v_\lambda \) is a strict local minimum because otherwise we would already have additional solutions. Therefore we can assume that there admits a positive constant \( \delta_0 > 0 \) such that
\[
\inf \left\{ I_{\lambda}(w) \mid w \in W^{1,2}(\Omega), \quad \|w - v_\lambda\|_{W^{1,2}(\Omega)} = \delta_0 \right\} > I_{\lambda}(v_\lambda).
\]
We will show that the functional $I_\lambda$ possesses a ‘mountain pass’ structure. Indeed, since $u_0 + v_\lambda < 0$, we have
\[
I_\lambda(v_\lambda - c) - I_\lambda(v_\lambda) = \frac{\lambda}{6} \int_{\Omega} \left[ (e^{u_0 + v_\lambda} - c)^6 - (e^{u_0} - 1)^6 \right] \, dx - 4\pi N c
\]
\[
< \frac{\lambda}{6} |\Omega| - 4\pi N c \to -\infty \quad \text{as} \quad c \to +\infty.
\]
Then we can choose $c_0 > \delta_0$ sufficiently large such that
\[
I_\lambda(v_\lambda - c_0) < I_\lambda(v_\lambda) - 1 \quad \text{and} \quad |\Omega|^\frac{1}{2} c_0 > \delta_0.
\]
Denote by $P$ the set of all continuous paths in $W^{1,2}(\Omega)$ $\gamma(\cdot) : [0, 1] \to W^{1,2}(\Omega)$ connecting the points $v_\lambda$ and $v_\lambda - c_0$ with $\gamma(0) = v_\lambda$, $\gamma(1) = v_\lambda - c_0$. Define
\[
\alpha_0 = \inf_{\gamma \in P} \sup_{t \in [0,1]} \{ I_\lambda(\gamma(t)) \}.
\]
Then we have
\[
\alpha_0 > I_\lambda(v_\lambda). \tag{3.44}
\]
Therefore the functional $I_\lambda$ satisfies all the hypothesis of the mountain pass theorem of Ambrosetti–Rabinowitz [6]. Then we can conclude that $\alpha_0$ is a critical value of the functional $I_\lambda$ in $W^{1,2}(\Omega)$. Noting (3.44), we have an additional solution of equation (3.5). Thus we complete the proof of theorem 2.2.

4. Existence of topological solutions

In this section we establish the existence of topological solutions of the generalized self-dual Chern–Simons equations (2.11)–(2.12), i.e. we prove theorem 2.3. We will use a super- and sub-solution method to construct solutions. The key step is to find a suitable sub-solution to the reduced equation. This technique maybe applied to other problems with similar structures.

As in section 3, let $|\phi|^2 = e^u$, the prescribed zeros of $\phi$ be $p_1, \ldots, p_m$ with multiplicities $n_1, \ldots, n_m$, respectively, and $N = \sum_{s=1}^{m} n_s$. Then we arrive at the following governing equation
\[
\Delta u = \lambda e^u (e^u - 1)^5 + 4\pi \sum_{s=1}^{m} n_s \delta_{p_s} \quad x \in \mathbb{R}^2. \tag{4.1}
\]
Correspondingly, the topological condition (2.8) is changed into
\[
u(x) \to 0 \quad \text{as} \quad |x| \to +\infty. \tag{4.2}
\]
We define the background function
\[
u_0(x) = -\sum_{s=1}^{m} n_s \ln(1 + |x - p_s|^2). \tag{4.3}
\]
Then
\[
\Delta \nu_0 = 4\pi \sum_{s=1}^{m} n_s \delta_{p_s} - g, \tag{4.4}
\]
where $g = 4 \sum_{s=1}^{m} n_s (1 + |x - p_s|^2)^{-2}$. Let $v = u - \nu_0$, we have
\[
\Delta v = \lambda e^{\nu_0 + v} (e^{\nu_0 + v} - 1)^5 + g. \tag{4.5}
\]
The boundary condition (4.2) becomes
\[ v(x) \to 0 \quad \text{as} \quad |x| \to +\infty. \] (4.6)

It is easy to check that \( v^* = -u_0 \) is a super-solution to problem (4.5)–(4.6).

Next we construct a sub-solution to problem (4.5)–(4.6). The construction of sub-solution
is a crucial part of the proof. This technique maybe applied to the other problems with similar
structures.

**Lemma 4.1.** For any \( \lambda > 0 \), problem (4.5)–(4.6) admits a sub-solution.

**Proof.** It is shown in [62, 73] that for any \( \mu > 0 \) the equation
\[ \Delta u = \mu e^{u} (e^u - 1) + 4\pi \sum_{s=1}^{m} n_s \delta_{p_s}, \] (4.7)

has a topological solution \( u(\mu) \), namely, \( u(\mu) \) satisfies equation (4.7) and the boundary condition
\[ u(\mu) \to 0 \quad \text{as} \quad |x| \to +\infty. \]

Then for \( \mu = \lambda e^{-a} (e^{-a} - 1)^2, a > 0 \), problem (4.7) admits a solution \( u^* \) which satisfies
\[ \Delta (u^* - a) \geq \lambda e^{u^* - a} (e^{u^* - a} - 1)^2 + 4\pi \sum_{s=1}^{m} n_s \delta_{p_s}, \] (4.8)

and \( u^* \to 0 \) as \( |x| \to +\infty \) and \( u^* < 0 \) for all \( x \in \mathbb{R}^2 \). Since \( u_s < 0 \), for any \( a > 0 \), we have
\( e^{u_s - a} - 1 < e^{-a} - 1 < 0 \)

and
\( e^{u^* - a} - 1 > e^{u^* - a} - 1 \)

Then we obtain from (4.8) that
\[ \Delta (u^* - a - u_0) \geq \lambda e^{u^* - a} (e^{u^* - a} - 1)^2 + 4\pi \sum_{s=1}^{m} n_s \delta_{p_s}, \] (4.9)

Let \( v_s = u^*_s - a - u_0, \) from (4.9) we have
\[ \Delta v_s \geq \lambda e^{u^*_s - a} (e^{u^*_s - a} - 1)^2 + g \] (4.10)

and \( v_s \) satisfies \( v_s \to -a \) as \( |x| \to \infty \). Then we conclude that \( v_s \) is a sub-solution to problem (4.5)–(4.6). Then the lemma follows.

At this point we can establish a solution to problem (4.5)–(4.6) by the super-solution \( v^* \) and sub-solution \( v^* \).

Let \( B_r \) be a ball centred at the origin with radius \( r \) in \( \mathbb{R}^2 \), where \( r > |p_s|, s = 1, \ldots, m. \)

Consider the following boundary value problem:
\[ \Delta v = \lambda e^{u^*_s - a} (e^{u^*_s - a} - 1)^2 + g, \quad x \in B_r, \] (4.11)
\[ v = -u_0, \quad x \in \partial B_r. \] (4.12)

We first prove that problem (4.11)–(4.12) has a unique solution \( v \) satisfying \( v_s < v < v^* \).

It is easy to see that \( v^* = -u_0 \) and \( v_s = u^*_s - a - u_0 \) are a pair of ordered super-and
sub-solutions to problem (4.11)–(4.12).

We use the monotone iterative method. Let \( K > 0 \) be constant satisfying \( K \geq 6\lambda \). We first introduce an iteration sequence on \( B_r \).
\[ (\Delta - K) v_n = \lambda e^{u^*_s - a} (e^{u^*_s - a} - 1)^2 - K v_{n-1} + g \quad \text{in} \quad B_r, \] (4.13)
\[ v_n = v^* \quad \text{on} \quad \partial B_r, \quad n = 1, 2, \ldots, \] (4.14)
\[ v_0 = v^*. \] (4.15)
Lemma 4.2. Let \( \{v_n\} \) be the sequence defined by the iteration scheme (4.13). Then
\[
v^* > v_1 > v_2 > \cdots > v_n > \cdots > v_k.
\] (4.16)

Proof. We prove this lemma by induction.

For \( n = 1 \), \( v_1 \) satisfies
\[
(\Delta - K)v_1 = -Kv^* + g \quad \text{in} \quad B_r.
\] (4.17)

It is easy to see that the right-hand side of (4.14) belongs to \( L^p(B_r) \) for \( p > 2 \). Then by the standard theory, we have \( v_1 \in C^{1,\alpha}(\overline{B_r}) (0 < \alpha < 1) \). Near the set \( Q = \{p_1, \ldots, p_m\} \) we have \( v_1 < v^* \). In \( B_r \setminus Q \), we can get
\[
(\Delta(v_1 - v^*)) - K(v_1 - v^*) = 0.
\]

Then, by maximum principle we have \( v_1 < v^* \) in \( B_r \). Noting that \( v_* < v^* \), we have
\[
(\Delta - K)(v_* - v_1) = \lambda \varepsilon^{a(n+1)}(\varepsilon^{a(n+1)} - 1)^5 - K(v_* - v_1) = (5\lambda - K)(v_* - v_1) \geq 0.
\]

Here and what after we use \( \xi \) to denote an intermediate quantity from the mean value theorem. Hence by maximum principle again we have \( v_* < v_1 \) in \( B_r \).

Suppose that we have already obtained the inequality \( v_* < v_k, v_k < v_{k-1} \) for some \( k \geq 1 \). Then by (4.13) we have
\[
(\Delta - K)(v_{k+1} - v_k) = \lambda \varepsilon^{a(n+1)}(\varepsilon^{a(n+1)} - 1)^5 - \varepsilon^{a(n+1)}(\varepsilon^{a(n+1)} - 1)^5 - K(v_{k+1} - v_k) = (5\lambda - K)(v_{k+1} - v_k) \geq 0.
\]

Therefore we have \( v_{k+1} < v_2 \) in \( B_r \) by maximum principle. Similarly, we have
\[
(\Delta - K)(v_* - v_{k+1}) = \lambda \varepsilon^{a(n+1)}(\varepsilon^{a(n+1)} - 1)^5 - \varepsilon^{a(n+1)}(\varepsilon^{a(n+1)} - 1)^5 - K(v_* - v_{k+1}) = (5\lambda - K)(v_* - v_{k+1}) \geq 0.
\]

Hence we obtain \( v_* < v_{k+1} \) in \( B_r \). Then, we get (4.16). Hence lemma 4.2 follows.

Since \( v_* \) is a bounded function, we can get the existence of the pointwise limit
\[
v = \lim_{n \to \infty} v_n.
\] (4.18)

Let \( n \to \infty \) in (4.13) and by the elliptic estimate and embedding theorem we see that the limit (4.18) can be achieved in any strong sense and \( v \) is a smooth solution of (4.11)–(4.12). It is easy to see that the solution \( v \) is unique and \( v \) satisfies \( v_* < v < v^* \).

Now we denote by \( v^{(n)} \) the solution of (4.11)–(4.12) with \( r = n \) (\( n \) is large such that \( n > |p_s|, s = 1, \ldots, m \)). By the construction of \( v^{(n)} \), we have \( v^{(n+1)} \leq v^n \) in \( \partial B_{n+1} \). Then, \( v^{(n+1)} \) is a sub-solution of (4.11)–(4.12) with \( r = n \). Therefore, from lemma 4.2 we have \( v^{(n+1)} \leq v^{(n)} \) in \( B_n \) for any \( n \). Then for each fixed \( n_0 \geq 1 \), we have the monotone sequence \( v_{n_0} > v_{n_0+1} > \cdots > v_n > v_{n+1} > \cdots > v_0 \) in \( B_{n_0} \). Then we can see that the sequence \( \{v^{(n)}\} \) converges to a solution, say \( v \), of equation (4.5) over the full plane \( \mathbb{R}^2 \). By the elliptic \( L^p \) estimate, we have \( v \in W^{2,2}(\mathbb{R}^2) \). Then we find that \( v(x) \to 0 \) as \( |x| \to \infty \), which is
The existence of multi-vortices for a generalized self-dual Chern–Simons model 823

the topological boundary condition (4.6). Then we can find a topological solution \( u \) of (4.1) satisfying \( u < 0 \) in \( \mathbb{R}^2 \).

Now we show that \( v \) is maximal. Let \( \tilde{v} \) be another solution to (4.11)–(4.12). Then \( \tilde{v} \) satisfies

\[
\Delta (u_0 + \tilde{v}) = \lambda e^{u_0 + \tilde{v}} (e^{u_0 + \tilde{v}} - 1)^5 \quad \text{in} \quad \mathbb{R}^2 \setminus \{ p_1, \ldots, p_m \},
\]

\( u_0 + \tilde{v} = 0 \) at infinity, and \( u_0 + v < 0 \) in a small neighborhood of \( \{ p_1, \ldots, p_m \} \). Using maximum principle, we see that \( u_0 + \tilde{v} \leq 0 \). Then by lemma 4.2, we obtain \( \tilde{v} \leq v \), which is to say that \( v \) is maximal.

Let \( u \) be the solution of (4.1) obtained above. Define

\[
\theta(z) = \sum_{s=1}^{m} n_s \arg(z - z_s), \quad \phi(z) = \exp \left( \frac{1}{2} u(z) + i\theta(z) \right),
\]

\[
A_1(z) = -2\text{Re}[i\bar{\partial} \ln \phi], \quad A_2(z) = -2\text{Im}[i\bar{\partial} \ln \phi].
\]

Then \( (\phi, A) \) is a topological solution of system (2.11)–(2.12).

Hence the proof of theorem 2.3 is complete.

5. Existence of radially symmetric topological solutions and non-topological solutions

In this section we establish the existence of radially symmetric topological solutions and non-topological solutions for the generalized self-dual Chern–Simons equations (2.11)–(2.12), that is, we prove theorems 2.4–2.5. We use the method developed in [20, 75].

For convenience, we assume that the zeros of \( \phi \) concentrate at the origin with the multiplicities \( N \). Letting \( |\phi|^2 = e^u \), similar to section 3, we obtain the following governing equation

\[
\Delta u = \lambda e^u (e^u - 1)^5 + 4\pi N \delta(x), \quad x \in \mathbb{R}^2.
\]

Correspondingly, the topological condition (2.8) and non-topological condition (2.9) are changed to

\[
u(x) \to 0 \quad \text{as} \quad |x| \to +\infty
\]

and

\[
u(x) \to -\infty \quad \text{as} \quad |x| \to +\infty.
\]

To find the proof of theorem 2.4 and theorem 2.5, we first state the following theorem for (5.1).

**Theorem 5.1.** For \( N \geq 0 \), a radially symmetric solution of (5.1) is either trivial \( u \equiv 0 \), or negative \( u < 0 \). For every given \( N \) there exists a unique solution \( u = u(r) (r = |x|) \) which satisfies

\[
\lim_{r \to +\infty} u(r) = 0.
\]

All other solutions satisfy the behaviour

\[
\lim_{r \to +\infty} ru_r(r) = -\beta, \quad \beta > 2N + 4.
\]

Furthermore, for any \( \beta \in (2N + 4, +\infty) \), there exists at least one solution \( u \) of (5.1) realizing the behaviour (5.4).
From the first part of theorem 5.1, we can find the existence and uniqueness of the radially symmetric topological solution, then we can find the proof of theorem 2.4.

Let \( u \) be a solution obtained in the second part of theorem 5.1. Set
\[
\phi(z) = \exp\left(\frac{1}{2} u(z) + iN \arg z\right),
\]
(5.5)

\[
A_1(z) = -2\text{Re}[i\bar{\phi} \ln \phi], \quad A_2(z) = -2\text{Im}[i\bar{\phi} \ln \phi].
\]
(5.6)

Then we can construct the \( N \)-vortex radially non-topological solutions for the generalized Chern–Simons equations (2.11)–(2.12). When \( \beta \in (2N + 4, +\infty) \), let \( u \) be a solution obtained in theorem 5.1 such that (5.4) is satisfied. Hence by (2.11)–(2.12), and (5.5)–(5.6), we can obtain that the magnetic flux is
\[
\Phi = \int_{\mathbb{R}^2} F_{12} \, dx = \frac{\lambda}{2} \int_{\mathbb{R}^2} e^u(1 - e^u)^5 \, dx
\]

\[
= \pi \int_0^{+\infty} (u_{rr} + \frac{1}{r} u_r) \, dr = \pi \int_0^{r_{\infty}} (ru_r) \, dr = \pi (2N + \beta).
\]
(5.7)

Then the electric charge is
\[
Q = \kappa \Phi = \kappa \pi (2N + \beta).
\]

Noting (5.4)–(5.5) we can find
\[
|D_j \phi|^2 = \frac{1}{2} u^2 e^u = O(r^{-(2+\beta)}).
\]

Then it follows from (2.10) that the energy is
\[
E = \int_{\mathbb{R}^2} \mathcal{E} \, dx = \int_{\mathbb{R}^2} F_{12} \, dx = \pi (2N + \beta).
\]

Thus we complete the proof of theorem 2.5.

Now we only need to prove theorem 5.1.

Since we are interested in radially symmetric solutions of (5.1), setting \( r = |x| \), we obtain
\[
u_{rr}(r) + \frac{1}{r} u_r(r) + \lambda e^{u(r)}(1 - e^{u(r)})^5 = 0, \quad r > 0,
\]
(5.8)

\[
u(r) = 2N \ln r + O(1), \quad \text{for small } r > 0.
\]
(5.9)

Using new variables
\[
t = \ln r, \quad u(t) \equiv u(e^t),
\]
we transform (5.8)–(5.9) into
\[
u''(t) + \lambda e^{2t} e^{u(t)}(1 - e^{u(t)})^5 = 0, \quad -\infty < t < +\infty,
\]
(5.10)

\[
u(t) = 2Nt + O(1), \quad \text{as } \ t \to -\infty.
\]
(5.11)

To prove theorem 5.1, we first state the following theorem.

**Theorem 5.2.** There exists a unique solution to (5.10)–(5.11) such that \( u \leq 0 \), \( u' \geq 0 \), \( u'' < 0 \) in \( \mathbb{R} \) and
\[
\lim_{t \to +\infty} u(t) = 0.
\]
(5.12)

For any \( \beta \in (2N + 4, +\infty) \), problem (5.10)–(5.11) admits at least one solution such that \( u < 0 \), \( u'' < 0 \) in \( \mathbb{R} \) and
\[
\lim_{t \to +\infty} u'(t) = -\beta.
\]
(5.13)

Moreover, for any non-positive solution of (5.10)–(5.11) satisfying
\[
\liminf_{t \to +\infty} u(t) < 0,
\]
(5.14)

there exists some \( \beta \in (2N + 4, +\infty) \) such that \( u \) satisfies (5.13).
Suppose that \( u \) is a solution of (5.10)–(5.11) which becomes positive at some \( t = t_0 \). Then it follows from the maximum principle that \( u'(t_0) > 0 \). Therefore, \( u''(t) > 0 \) and \( u'(t) > 0 \) for all \( t > t_0 \). Then we have

\[
e^{u(t)} - 1 > e^{u(t_0)} - 1 > 0 \quad \text{for all } t > t_0.
\]

Using equation (5.10), there exist a positive constant \( \delta_0 \) depending on \( t_0 \) such that

\[
u''(t) > \delta_0 e^{u(t)}, \quad t > t_0.
\]

Then it is easy to see that \( u(t) \) blows up at the finite time \( t > t_0 \).

Hence by theorem 5.2, we can conclude the assertion of theorem 5.1.

In the following we only need to prove theorem 5.2.

Let

\[
g(u) = \begin{cases} 
e^u(1 - e^u)^5, & u \leq 0, \\
0, & u > 0.
\end{cases}
\]

To prove theorem 5.2, it is sufficient to prove the same result for the following problem

\[
u''(t) + \lambda e^{2t} g(u(t)) = 0, \quad -\infty < t < +\infty, \tag{5.15}
\]

\[u(t) = 2Nt + O(1), \quad \text{as } t \to \infty. \tag{5.16}
\]

First we establish the existence for the initial value problem (5.15).

**Lemma 5.1.** For any \( a \in \mathbb{R} \), there exists a unique solution \( u \) to problem (5.15) such that

\[
u(t) = 2Nt + a + o(1) \quad \text{as } t \to -\infty. \tag{5.17}
\]

Moreover, if \( u(t) \) is a solution of (5.15) in some interval, it can be extended to a global solution of (5.15) in \( \mathbb{R} \) which satisfies (5.17) for some \( a \in \mathbb{R} \).

**Proof.** It is easy to check that \( u(t) \) is a solution of (5.15) if and only if \( u(t) \) verifies

\[
u(t) = 2Nt + a - \lambda \int_{-\infty}^{t} (t - s) e^{2s} g(u(s)) \, ds. \tag{5.18}
\]

Let \( T < -\ln 2 \), we can get

\[
\int_{-\infty}^{T} (T - s) e^{2s} \, ds < \frac{1}{16}.
\]

Noting that \(|g(u)| + |g'(u)| < 7\), then by Picard iteration with \( u^0 = 2Nt + a \), we can establish the solution of (5.15) in the interval \((-\infty, T]\). Since \( g(u) \) is bounded, we can extend \( u \) to a solution of (5.15) in \( \mathbb{R} \).

Now we prove the uniqueness of the solution. Suppose that \( u^1, u^2 \) are two solutions of (5.15) in the interval \((-\infty, T]\). Let \( \tilde{u} = u^1 - u^2 \), we have

\[
|\tilde{u}(t)| = \left| -\lambda \int_{-\infty}^{t} (t - s) e^{2s} \left[ g(u^1(s)) - g(u^2(s)) \right] \, ds \right|
\leq \sup |g'(u)| \int_{-\infty}^{T} (T - s) e^{2s} \, ds \sup_{(-\infty, T]} |\tilde{u}|
\leq \frac{1}{2} \sup_{(-\infty, T]} |\tilde{u}|.
\]

Then we can get \( \sup_{(-\infty, T]} |\tilde{u}| = 0 \), which implies \( u^1 = u^2 \) in \((-\infty, T]\). By the unique continuation we have \( u^1 = u^2 \) in \( \mathbb{R} \).
Now we prove the second part of the lemma. Assume that \( u \) is a solution of \((5.15)\) in some interval. By \((5.15)-(5.16)\) we have
\[
\lim_{t \to -\infty} u'(t) = 2N
\]
and
\[
u'(t) = u'(0) + \lambda \int_{t}^{0} e^{2s} g(u(s)) \, ds.
\]
Then we have
\[
u'(t) = 2N - \lambda \int_{-\infty}^{t} e^{2s} g(u(s)) \, ds.
\]
Noting that
\[
\int_{0}^{t} \int_{-\infty}^{s} e^{2s_1} g(u(s_1)) \, ds_1 \, ds = \lambda \int_{-\infty}^{t} (t-s) e^{2s} g(u(s)) \, ds < +\infty
\]
we obtain
\[
u(t) = 2Nt + u(0) - \lambda \int_{0}^{t} \int_{-\infty}^{s_1} e^{2s_1} g(u(s_1)) \, ds_1 \, ds
\]
That is, \( u \) satisfies \((5.17)\) with
\[
\lambda = u(0) + \lambda \int_{-\infty}^{0} \int_{-\infty}^{s_1} e^{2s_1} g(u(s_1)) \, ds_1 \, ds.
\]
Then lemma 5.1 follows.

Now we investigate the behaviour of the solutions as \( t \to +\infty \). In the following we denote by \( u(t, a) \) the solution given by lemma 5.1. We use \( \cdot \) to denote the derivative with respect to \( t \) and subscript \( a \) to denote the derivative with respect to \( a \). We define the parameter sets:
\[
A^+ = \{ a \in \mathbb{R} \mid \exists t_0 \in \mathbb{R} \text{ such that } u(t_0) > 0 \},
\]
\[
A^0 = \{ a \in \mathbb{R} \mid u(t, a) \leq 0, \quad u'(t, a) \geq 0, \quad \forall t \in \mathbb{R} \},
\]
\[
A^- = \{ a \in \mathbb{R} \mid u(t, a) \leq 0, \quad \forall t \in \mathbb{R}, \quad \exists t_1 \in \mathbb{R} \text{ such that } u'(t_1) < 0 \}.
\]
It is easy to see that
\[
A^+ \cup A^0 \cup A^- = \mathbb{R}, \quad A^+ \cap A^0 = A^0 \cap A^- = A^+ \cap A^- = \emptyset
\]
Furthermore, we can obtain the following lemma.

**Lemma 5.2.**

1. If \( a \in A^+ \), then \( u' > 0 \) in the set \( \{ t \mid u(t, a) < 0, \quad \forall t \in (-\infty, t) \} \).
2. If \( a \in A^0 \), then \( u'' \leq 0 \) and \( u' \geq 0 \) in \( \mathbb{R} \) and \( \lim_{t \to +\infty} u(t, a) = 0 \).
3. If \( a \in A^- \), then \( u'' < 0, \quad u < 0 \) in \( \mathbb{R} \) and \( \lim_{t \to +\infty} u(t, a) = -\infty \).
4. \( A^+ \) is open and \( \left( \frac{\lambda}{e^{1/2}}, +\infty \right) \subset A^+ \).
5. \( A^- \) is open.
6. Let
\[
T > \frac{1}{2} \ln \frac{2(2N + 1)}{\lambda e^{-1}(1 - e^{-1})^2(e^2 - 1)}.
\]
then \( (-\infty, -\frac{\lambda}{e^{1/2}} - 2 - 2N) \subset A^- \).
7. \( A^0 \) is nonempty, bounded and closed.
Proof.

(1) Let \( a \in \mathcal{A}^+ \) and \( t_0 \) be first time such that \( u(t, a) \) hits the \( x \) axis from below. Then, \( u(t, a) < 0 \) for all \( t \in (-\infty, t_0) \). By equation (5.15) we have \( u'' = -\lambda e^{2s}g(u(t)) < 0 \) in \((-\infty, t_0) \). Hence \( u''(t, a) > 0 \) in \((-\infty, t_0) \).

(2) By the definition of \( \mathcal{A}^0 \), we see that the limit \( b = \lim_{t \rightarrow +\infty} u(t, a) \) exists and non-positive.

If \( b < 0 \), we have \( \lim_{t \rightarrow +\infty} u''(t, a) = -\lambda \lim_{t \rightarrow +\infty} e^{2s}e^b(1 - e^b)^5 = -\infty \), which leads to a contradiction.

(3) If \( u(t_0) = u'(t_0) = 0 \), then \( u(t, a) \equiv 0 \). Hence, if \( a \in \mathcal{A}^- \), then \( u(t, a) < 0 \) in \( \mathbb{R} \), and \( u''(t, a) < 0 \) by equation (5.15). Therefore, \( \lim_{t \rightarrow +\infty} u''(t, a) < 0 \), which implies \( \lim_{t \rightarrow +\infty} u(t, a) = -\infty \).

(4) Noting that \( u(t, a) \) is continuous in \( a \), if \( u(t_0, a_0) > 0 \), then we have \( u(t_0, a) > 0 \) when \( a \) is in a small neighborhood of \( a_0 \). Hence \( \mathcal{A}^+ \) is open. By (5.18) we have

\[
u(0, a) = a - \lambda \int_{-\infty}^{0} e^{2s}g(u(s)) ds \geq a - \frac{5^5 \lambda}{6^5 \frac{4}{3}}.\]

If \( a > \frac{5^5 \lambda}{6^5 \frac{4}{3}} \), then \( u(0, a) > 0 \), which says \( a \in \mathcal{A}^+ \).

(5) If \( a_0 \in \mathcal{A}^- \), then there exists \( \epsilon_0 \in \mathbb{R} \) such that \( u(t_0, a_0) < 0 \). Hence \( u(t_0, a) < 0 \) when \( a \) is close to \( a_0 \). By (3) we have \( u(t, a_0) < 0 \) for all \( t \leq t_0 \) and \( a \) close to \( a_0 \). By (5.18), we see that \( u \) cannot take a local negative minimum. Then \( u(t_0, a) < 0 \) and \( u'(t_0, a) < 0 \) implies \( u'(t, a) \leq 0 \) for all \( t > t_0 \). Hence, \( u(t, a) < 0 \) for all \( t > t_0 \) when \( a \) is close to \( a_0 \).

Then we see that \( \mathcal{A}^- \) is open.

(6) Let \( a < -\frac{5^5 \lambda}{6^5 \frac{4}{3}} - 2 - 2NT \). By (5.18) we have

\[
u(t, a) \leq 2Nt + a + \frac{5^5 \lambda}{6^5 \frac{4}{3}} < -2 \quad \text{for all} \quad t \in (-\infty, 0].\]

If \( a \notin \mathcal{A}^- \), since \( u(t, a) \) cannot assume a local minimum, there exist constants \( T_1 \) and \( T_2 \) such that \( T_2 < T_1 \), \( u(t, a) \leq -2 \) in \((-\infty, T_1] \), \( u(T_2, a) = -2, u'(T_2, a) \geq 0, u(t, a) \in [-2, -1] \) for all \( t \in [T_2, T_1] \), \( u(T_1, a) = -1, u'(T_1, a) \geq 0 \). Then we find that \( u''(t, a) = -\lambda e^{2s}g(u(t)) \leq 0 \) for all \( t \leq T_1 \), from which follows \( u'(t, a) \leq 2N \) for all \( t \in (-\infty, T_1] \). Then we have \( u(T_2, a) - u(0, a) \leq 2NT_2 \), which implies \( T_3 \geq \frac{u(T_2, a) - u(0, a)}{2N} \geq -2 - a - \frac{5^5 \lambda}{6^5 \frac{4}{3}} > T \).

Similarly, we have \( T_1 - T_2 \geq (1/2N) \). Hence, by the choice of \( T \), we have

\[
u'(T_1, a) = u'(T_2, a) - \lambda \int_{T_2}^{T_1} e^{2s}e^{u(s,a)}(1 - e^{u(s,a)})^5 ds \leq 2N - \lambda e^{-1}(1 - e^{-1})^5 \int_{T_2}^{T_1} e^{2s} ds \leq 2N - \frac{\lambda}{2}e^{-1}(1 - e^{-1})^2 e^{2T} (e^{\frac{T}{2}} - 1) < -1,\]

which leads to a contradiction. Therefore \( a \in \mathcal{A}^- \).

(7) By the assertions of (4)–(6), we can get (7).

Then the proof of of lemma 5.2 is complete.

Next we investigate the monotonicity of the solution with respect to \( a \).

Lemma 5.3. Let

\[ T_0(a) = \sup \{ T \in [-\infty, +\infty]| u(t, a) < 0, u'(t, a) > 0, \forall t \in (-\infty, T) \}. \]
Then,
\[ u_a(t, a) \geq \frac{1}{2N} u'(t, a) > 0, \quad \forall t \in (-\infty, T_0(a)). \]

**Proof.** It is easy to see that \( v(t, a) \equiv u_a(t, a) \) exists, is smooth and verifies
\[
\begin{align*}
v'(t, a) &= -\lambda e^{2t} g(u(t, a)) v(t, a), \quad -\infty < t < +\infty, \\
\lim_{t \to -\infty} v(t, a) &= 1, \quad \lim_{t \to -\infty} v'(t, a) = 0.
\end{align*}
\] (5.19) (5.20)

Let
\[ T_1(a) = \sup \{ \tau \in \mathbb{R} | v(t, a) > 0, \quad t \in (-\infty, \tau) \}. \]

Then by (5.20), we see that \( T_1(a) > -\infty \). Let \( w(t, a) = u'(t, a) \), from (5.15) we have
\[ \lim_{t \to -\infty} w(t, a) = 2N, \quad \lim_{t \to -\infty} w'(t, a) = 0. \]

Hence the function \( C(t, a) \equiv (w(t, a)/v(t, a)), \ t \in (-\infty, T_1(a)) \) satisfies \( \lim_{t \to -\infty} C(t, a) = 2N, \lim_{t \to -\infty} C'(t, a) = 0 \). Noting that \( w(t, a) \) satisfies
\[
\begin{align*}
w''(t, a) &= -\lambda e^{2t} g'(u(t, a)) w(t, a) - 2\lambda e^{2t} g(u(t, a)),
\end{align*}
\]

then we obtain
\[ C'(t, a) = \frac{2\lambda}{w(t, a)} \int_{-\infty}^{t} e^{2s} g(u(s, a)) v(s, a) ds \leq 0, \quad \forall t \in (-\infty, T_1(a)). \]

Hence \( C(t, a) \leq 2N, \forall t \in (-\infty, T_1(a)) \), which is \( u_a(t, a) \geq (1/2N)u'(t, a), \forall t \in (-\infty, T_1(a)). \) It is obvious that \( T_1(a) \geq T_0(a) \), thus lemma 5.3 follows.

Now we give a characterization of the sets \( A^+, A^-, \) and \( A^0 \).

**Lemma 5.4.** There exists a constant \( a_0 \) such that
\[ A^+ = (a_0, +\infty), \quad A^- = (-\infty, a_0) \quad \text{and} \quad A^0 = [a_0]. \]

**Proof.**

**Step 1.** We show that there exist two constants \( a_1, a_2 \) satisfying \( a_1 \leq a_2 \) such that
\[ A^+ = (a_2, +\infty), \quad A^- = (-\infty, a_1) \quad \text{and} \quad A^0 = [a_1, a_2]. \]

To prove \( A^+ = (a_2, +\infty), \) since \( A^+ \) is open, we just need to prove that if \( (b_1, b_2) \subset A^+, \) then \( b_2 \in A^+ \). For \( a \in (b_1, b_2) \), let \( z_0(a) \) be the first zero point of \( u \) on the \( t \) axis. Then we have \( u(z_0(a), a) = 0, u'(z_0(a), a) > 0 \) and by lemma 5.2 (1) \( u' > 0 \) in \( (-\infty, z_0(a)) \). Then by lemma 5.3 we obtain \( u_a \geq (1/2N)u' > 0 \) in \( (-\infty, z_0(a)] \). Using the implicit function theorem we see that \( z_0(a) \) is differentiable with respect \( a \) in the set \((b_1, b_2)\) and
\[
\frac{d}{da} z_0(a) = \frac{u_a(t, a)}{u'(t, a)} < 0.
\]

In view of (5.18) we obtain \( u(t, a) \leq 2Nt+a \) in \( (-\infty, z_0(a)], \) which leads to \( z_0(a) \geq -a/2N \) for \( b \in (b_1, b_2) \). Then we infer that \( z_0(b_2) \equiv \lim_{a \to b_2} z_0(a) \) exists and is finite. From continuity, we have \( u(z_0(b_2), b_2) = 0. \) If \( u'(z_0(b_2), b_2) \neq 0, \) then \( u(t, b_2) \equiv 0, \) which is impossible. Hence \( u'(z_0(b_2), b_2) \neq 0, \) which concludes that \( u(t, b_2) > 0 \) for \( t \) near \( z_0(b_2). \) Then \( b_2 \in A^+. \)

To prove \( A^- = (-\infty, a_1), \) it is sufficient to prove that if \( (b_1, b_2) \subset A^- \), then \( b_1 \in A^- \). For \( a \in A^- \), let \( z_1(a) \) be the first point such that \( u'(z_1(a), a) = 0 \) and \( m(a) = u(z_1(a), a) \) be the maximum of \( u(\cdot, a) \) in \( \mathbb{R}. \) Noting that for \( a \in A^-, u''(z_1(a), a) < 0, \) then again by the implicit function theorem we see that \( z_1(a) \) is a differentiable function on \( A^- \). Hence we have
\[
\frac{dm(a)}{da} = u'(z_1(a), a) \frac{z_1(a)}{m(a)} + u_a(z_1(a), a) = u_a(z_1(a), a) \geq 0, \quad \forall a \in (b_1, b_2).
\]
Then we obtain

\[ m(a) = \sup_{t \in \mathbb{R}} u(t, a) \leq m\left(\frac{1}{2}(b_1 + b_2)\right) < 0, \quad a \in (b_1, \frac{1}{2}(b_1 + b_2)). \]

Via continuity

\[ m(b_1) = \sup_{t \in \mathbb{R}} u(t, b_1) \leq m\left(\frac{1}{2}(b_1 + b_2)\right) < 0, \]

which implies \( b_1 \in A^- \cup A^0 \). From lemma 5.2 (2), we see that \( b_1 \notin A^0 \). Then \( b_1 \in A^- \).

Step 2. We show that \( a_1 = a_2 \). For \( a \in A^0 \), we have \( u'(t, a) > 0 \) in \( \mathbb{R} \) and by lemma 5.3 \( u_a(t, a) > 0 \) in \( \mathbb{R} \). Noting that \( \lim_{t \to \infty} u(t, a) = 0 \), for \( \forall \delta > 0 \), there exists a continuous function \( T_\delta(a) \) such that \( u(T_\delta(a)) = -\delta \) and \( u(t, a) > -\delta \) in \( (T_\delta(a), +\infty) \). Since \( g'(u) = e^u(1 - e^u)^4(1 - 6e^u) \leq 0 \) when \( u \in [-\ln 6, 0] \), we have

\[ u_a''(t, a) > -\lambda e^{2s}(1 - e^s)^4(1 - 6e^s)u_a(s, a) \geq 0, \quad t \in [T_{\ln 6}(a), +\infty), \quad a \in [a_1, a_2]. \]

Therefore, \( u_a \) is a non-negative convex function on \( [T_{\ln 6}(a), +\infty) \). Suppose, on the contrary, \( u_a(\infty, a) = 0 \) for some \( a \in [a_1, a_2] \). Then we see that \( C(t, a) = (u(t, a)/u_a(t, a)) \) satisfies

\[
C'(t, a) = -\frac{2\lambda}{u_a^2(t, a)} \int_{-\infty}^{t} e^{2s}e^{u(s, a)}(1 - e^{u(s, a)})^5 u_a(s, a) \, ds
\]

\[ = -\frac{2\lambda}{u_a^2(t, a)} \int_{-\infty}^{0} e^{2s}e^{u(s, a)}(1 - e^{u(s, a)})^5 u_a(s, a) \, ds \to -\infty \quad \text{as} \quad t \to +\infty, \]

which concludes that \( C(t, a) < 0 \) as \( t \) is sufficiently large. This contradicts the fact that \( C(t, a) > 0 \) for all \( t \in \mathbb{R} \). Therefore, for all \( t \in \mathbb{R} \) and by Fatou’s lemma that

\[ 0 = \lim_{t \to +\infty} (u(t, a_2) - u(t, a_1)) = \lim_{t \to +\infty} \int_{a_1}^{a_2} u_a(t, a) \, da \geq \int_{a_1}^{a_2} u_a(\infty, a) \, da, \]

which implies \( a_1 = a_2 \).

The proof of lemma 5.4 is complete.

**Lemma 5.5.** For \( a \in A^- \), the limit

\[ \beta(a) \equiv \lim_{t \to +\infty} u'(t, a) \]

exists and is positive and finite.

**Proof.** For \( a \in A^- \), \( u'(t, a) < 0 \) in \( \mathbb{R} \), then \( \beta(a) \) exists and \( \beta(a) \in (0, +\infty) \). We need to show that \( \beta(a) \) is finite. Suppose that \( \beta(a) > 3 \), then there exists a constant \( T > 1 \) such that \( u(t, a) \leq -3t \) \( \forall t \in (T, +\infty) \). When \( u < -2 \), \( g'(u) = e^u(1 - e^u)^4(1 - 6e^u) > 0 \). Then as \( t > T \), we have \( g(u(t, a)) = e^{u(t, a)}(1 - e^{u(t, a)})^5 \leq e^{-3t}(1 - e^{-3t})^5 \). Therefore, for all \( t > T \)

\[ u'(t, a) = u'(T, a) - \lambda \int_{T}^{t} e^{2s}e^{u(s, a)}(1 - e^{u(s, a)})^5 \, ds \]

\[ \geq u'(T, a) - \lambda \int_{T}^{t} e^{-s}(1 - e^{-3s})^5 \, ds \]

\[ \geq u'(T, a) - \lambda \int_{0}^{+\infty} e^{-s}(1 - e^{-3s})^5 \, ds. \]
Then, we have \( \beta(a) \leq u'(T, a) - \lambda \int_0^\infty e^{-s} (1 - e^{-3s})^5 ds < +\infty \), which concludes the lemma.

**Lemma 5.6.** For \( a \in A^- \), the functions \( e^{2t} e^{u(t,a)} (1 - e^{u(t,a)})^5 \) and \( e^{2t} [1 - (1 - e^{u(t,a)})^6] \) are both integrable on \( \mathbb{R} \). Moreover,

\[
\beta(a) + 2N = \lambda \int_\mathbb{R} e^{2t} e^{u(t,a)} (1 - e^{u(t,a)})^5 dt, \tag{5.21}
\]

\[
\frac{\beta^2}{2} - 2N^2 = \frac{\lambda}{3} \int_\mathbb{R} e^{2t} [1 - (1 - e^{u(t,a)})^6] dt. \tag{5.22}
\]

**Proof.** For \( a \in A^- \), \( u(t, a) < 0 \), then we have

\[
u'(t, a) = 2N - \lambda \int_{-\infty}^t e^{2s} e^{u(s,a)} (1 - e^{u(s,a)})^5 ds.
\]

In view of lemma 5.5, we can take the limit \( t \to +\infty \) in the above expression to get

\[-\beta(a) = 2N - \lambda \int_{-\infty}^{+\infty} e^{2s} e^{u(s,a)} (1 - e^{u(s,a)})^5 ds,
\]

which is (5.21).

Multiply (5.15) by \( u' \), and integrating the resulting equation over \([T, T]\), we have

\[
\frac{1}{2} u'^2(t, a) \bigg|_{t=-T}^{t=T} + \frac{\lambda}{6} \left[ e^{2t} [1 - (1 - e^{u(t,a)})^6] \right] \bigg|_{t=-T}^{t=T} = \frac{\lambda}{3} \int_{-T}^{T} e^{2s} [1 - (1 - e^{u(t,a)})^6] ds.
\]

(5.23)

Since \( u(t, a) < 0 \) for \( a \in A^- \), \( t \in \mathbb{R} \), it is easy to see that

\[
\lim_{t \to -\infty} \frac{\lambda}{6} \left[ e^{2t} [1 - (1 - e^{u(t,a)})^6] \right] = 0. \tag{5.24}
\]

Noting that \( u''(t, a) < 0 \) for \( a \in A^- \), \( t \in \mathbb{R} \), we have \( u'(t, a) > -\beta(a) \), which implies \( u'(t, a) > -\beta(a) \) for \( t \in \mathbb{R} \). For \( T \) sufficiently large such that \( u'(t, a) < 0 \) when \( t > T \), we can obtain

\[
\frac{1}{6} [1 - (1 - e^{u(s,a)})^6] = \int_{-\infty}^{u(s,a)} e^{u(s,a)} (1 - e^{u(s,a)})^5 du(s,a)
\]

\[
= \int_{-\infty}^{t} e^{u(s,a)} (1 - e^{u(s,a)})^5 u'(s,a) ds
\]

\[
\leq \beta(a) \int_{t}^{+\infty} e^{u(s,a)} (1 - e^{u(s,a)})^5 ds
\]

Then

\[
\frac{\lambda}{6} e^{2t} [1 - (1 - e^{u(t,a)})^6] \leq \beta(a) \int_{t}^{+\infty} e^{2s} e^{u(s,a)} (1 - e^{u(s,a)})^5 ds \to 0 \quad \text{as} \quad t \to +\infty.
\]

(5.25)

By (5.24) and (5.25), letting \( T \to +\infty \) in (5.23), we get (5.22). Then lemma 5.6 follows.

**Lemma 5.7.** \( \beta(a) \) is continuous in \( A^- \).
**Proof.** For any $\tilde{a} \in A^-$, we claim $\beta(\tilde{a}) > 2$. Suppose otherwise, noting that $u'' < 0$, we have $u'(t, \tilde{a}) \geq -\beta(\tilde{a}) > -2$ for $t \in \mathbb{R}$. Then we infer that there exists a positive constant $C$ such that $u(t, \tilde{a}) \geq -C - 2t$ for all $t > 0$. Let $T$ be a time so that $u(t, \tilde{a}) < -3$ for all $t > T$. Then we have

$$\lambda \int_T^{+\infty} e^{2\nu u(t, \tilde{a})} (1 - e^{u(t, \tilde{a})})^5 \, dt \geq \lambda \int_T^{+\infty} e^{2\nu (1 - e^{-C - 2t})^5} \, dt$$

$$= \lambda e^{-C} \int_T^{+\infty} (1 - e^{-C - 2t})^5 \, dt = +\infty,$$

which contradicts the finiteness of $\beta(\tilde{a})$. Therefore $\beta(\tilde{a}) > 2$.

Let $\delta = (\beta(\tilde{a}) - 2)/4$. Then there exists a positive constant $T_1$ such that $T_1 \geq 3/(2 + \delta)$, $u'(T_1, \tilde{a}) < -2(1 + \delta) - 2(1 + \delta)T_1$. Noting that $u(t, a)$ and $u'(t, a)$ are both continuous with respect to $a$, we have $u'(T_1, a) < -(2 + \delta)$ and $u(T_1, a) < -(2 + \delta)T_1$ when $a$ is close to $\tilde{a}$. Since $u'' < 0$, we have $u(t, a) \leq -(2 + \delta)t$ for $t \in [T_1, +\infty)$ when $a$ is close to $\tilde{a}$. Let

$$w(t) = \begin{cases} 
5\lambda \frac{e^{2\nu}}{6\nu} e^{2\nu}, & t \leq T_1, \\
\lambda e^{2\nu} (1 - e^{-(2+\delta)t})^5, & t > T_1.
\end{cases}$$

It is easy to see that $w(t) \in L^1(\mathbb{R})$. Noting that $\lambda e^{2\nu u(t, a)} (1 - e^{u(t, a)})^5 \leq w(t)$ for all $t \in \mathbb{R}$, then using the Lebesgue dominated control theorem and (5.21), we can find the continuity of $\beta(a)$. The proof of lemma 5.7 is complete.

Now we want to obtain the range of $\beta(a)$. We first investigate the behaviour of $\beta(a)$ as $a \to a_0$ and as $a \to -\infty$.

**Lemma 5.8.** There holds the limit $\lim_{a \to a_0} \beta(a) = +\infty$.

**Proof.** In view of (5.22), the continuity of $u(t, a)$, and the fact $u(t, a_0) \leq 0$, we can obtain

$$\liminf_{a \to a_0} \frac{1}{2} \beta^2(a) - 2N^2 = \liminf_{a \to a_0} \frac{\lambda}{3} \int_{\mathbb{R}} e^{2\nu} \left[1 - (1 - e^{u(t, a)})^6\right] \, dt$$

$$\geq \liminf_{T \to +\infty} \liminf_{a \to a_0} \frac{\lambda}{3} \int_0^T e^{2\nu} \left[1 - (1 - e^{u(t, a)})^6\right] \, dt$$

$$= \lim_{T \to +\infty} \frac{\lambda}{3} \int_0^T e^{2\nu} \left[1 - (1 - e^{u(t, a)})^6\right] \, dt$$

$$\geq \frac{\lambda}{3} \int_0^T e^{2\nu} \left[1 - (1 - e^{u(0, a_0)})^6\right] \, dt$$

$$\lim_{T \to +\infty} \int_0^T e^{2\nu} \, dt = +\infty.$$

Noting that $\beta(a) > 0$, we conclude the lemma.

**Lemma 5.9.** For $a \in A^-$, let $m(a) = \sup_{t \in \mathbb{R}} u(t, a)$. Then

$$\lim_{a \to -\infty} m(a) = -\infty.$$

**Proof.** Let $a \in A^-$, and $z_1(a)$ be the point such that $u'(z_1(a), a) = 0$. Then from (5.18) we have $u(t, a) \leq 2Nt + a$, which implies $m(a) = u(z_1(a), a) \leq 2Nz_1(a) + a$,
Lemma 5.10. There holds the limit
\[
\lim_{a \to -\infty} \beta(a) = 2N + 4. \tag{5.26}
\]
Moreover,
\[
\{\beta(a) | a \in A^-\} = (2N + 4, +\infty). \tag{5.27}
\]

Proof. Let \( a \ll -1 \) and \( T \gg 1 \) be two fixed constants. Then by (5.22), we have
\[
\frac{\beta^2}{2} - 2N^2 = \frac{\lambda}{3} \int_{-\infty}^{T} e^{u(a)} \left[ 1 - \left( 1 - e^{u(a)} \right)^5 \right] \, dt
\]
\[
= \frac{\lambda}{3} \int_{-\infty}^{T} e^{u(a)} \left[ 1 - e^{u(a)} \right] \, dt + \frac{\lambda}{3} \int_{T}^{\infty} \left[ 1 - e^{u(a)} \right] e^{u(a)} \left( 1 - e^{u(a)} \right)^5 \, dt
\]
\[
= \frac{\lambda}{3} \int_{-\infty}^{T} e^{u(a)} \left[ 1 - e^{u(a)} \right] \, dt + G(T^*, a) \frac{\lambda}{3} \int_{T}^{\infty} e^{u(a)} \left( 1 - e^{u(a)} \right)^5 \, dt
\]
where \( T^* \in (T, +\infty) \) and
\[
G(T^*, a) = \frac{1 - e^{u(T^*, a)}}{1 - e^{u(T^*, a)}}^5
\]
Then in view of (5.21), we obtain
\[
\frac{\beta^2}{2} - 2N^2 = 2G(T^*, a)(\beta(a) + 2N) + H(T, a), \tag{5.28}
\]
where
\[
H(T, a) = \frac{\lambda}{3} \int_{-\infty}^{T} e^{u(a)} \left[ 1 - e^{u(a)} \right] \, dt - G(T^*, a) e^{u(a)} \left( 1 - e^{u(a)} \right)^5 \left( 1 - e^{u(a)} \right)^5 \, dt.
\]
Using lemma 5.9, we infer that \( \lim_{a \to -\infty} H(T, a) = 0 \) and \( \lim_{a \to -\infty} G(T^*, a) = 1 \). It follows from (5.28) that
\[
\beta(a) = 2G(T^*, a) + \sqrt{\left( 2N + 2G(T^*, a) \right)^2 + 2H(T, a)}. \tag{5.29}
\]
Then, letting \( a \to -\infty \) in (5.29), we get (5.26).

Since \( \beta(a) \) is continuous in \( A^- \), we obtain

\[
\{ \beta(a) \mid a \in A^- \} \subset (2N + 4, +\infty).
\]  

Noting that as \( a \in A^- \), \( u(t, a) < 0 \), it is easy to check that

\[
\frac{1}{6} \left[ 1 - \left( 1 - e^{u(t,a)} \right)^6 \right] > 1.
\]

Then in view of (5.22), we have

\[
\beta^2(a) - 4N^2 = \frac{2\lambda}{3} \int_\mathbb{R} e^{2t} \left[ 1 - \left( 1 - e^{u(t,a)} \right)^6 \right] dt
\]

\[
= 4\lambda \int_\mathbb{R} e^{2t} \left[ 1 - \left( 1 - e^{u(t,a)} \right)^6 \right] e^{2t} e^{u(t,a)} \left( 1 - e^{u(t,a)} \right)^5 dt
\]

\[
> 4\lambda \int_\mathbb{R} e^{2t} e^{u(t,a)} \left( 1 - e^{u(t,a)} \right)^5 dt
\]

\[
= 4(\beta(a) + 2N),
\]

which implies \( \beta(a) > 2N + 4 \), that is

\[
\{ \beta(a) \mid a \in A^- \} \supset (2N + 4, +\infty).
\]  

Therefore (5.27) follows from (5.30) and (5.31). Then the proof of lemma 5.10 is complete.

Now combining lemmas 5.1–5.10, we can find theorem 5.2.

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The existence of multi-vortices for a generalized self-dual Chern–Simons model

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