Pseudo-Anosov mappings and toral automorphisms

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Abstract

For every irreducible automorphism $\phi \in \text{SL}_3(\mathbb{Z})$ of the 3-torus, for which the product of the expanding eigenvalues is positive, we construct a pseudo-Anosov mapping $f$ of an associated surface, semi-conjugate and almost-isomorphic to $\phi$, whose stretch factor is the product of the expanding eigenvalues of $\phi$. This shows that any norm-1 cubic Pisot number occurs as the stretch factor of a pseudo-Anosov mapping, proving a conjecture of Fried in degree 3. A similar construction works for the 4-torus on condition that $\phi$ has exactly two eigenvalues outside the unit circle (and whose product is positive). Furthermore, for any irreducible hyperbolic automorphism $\phi \in \text{SL}_n(\mathbb{Z})$ of the $n$-torus, $n \geq 4$, we construct a pseudo-Anosov mapping semiconjugate and almost-isomorphic to any sufficiently large power of $\phi$.

1 Introduction

Pseudo-Anosov diffeomorphisms of closed surfaces are of fundamental importance in Teichmüller theory, hyperbolic geometry and dynamics. They were introduced by Thurston [6] in his classification of surface homeomorphisms. However, it has remained challenging to give explicit constructions of families of pseudo-Anosov mappings.

In [3], Albert Fathi constructed the first examples of conjugations between pseudo-Anosov mappings of surfaces and toral automorphisms acting on invariant subsets of the torus. Shortly thereafter Pierre Arnoux [1] constructed an explicit map, based on the “Rauzy fractal”, from the surface of genus 3 onto the 3-torus, giving a semi-conjugation between a pseudo-Anosov mapping and a toral automorphism. Arnoux’s map is an almost isomorphism: a semiconjugacy which is surjective and almost-everywhere injective.

In this paper we generalize Arnoux’s construction:

Theorem 1.1. Let $n = 3$ or $4$ and $\phi \in \text{SL}_n(\mathbb{Z})$ be irreducible, with (in the case $n = 4$) two expanding eigenvalues, that is, two eigenvalues outside the unit circle. Assume in either case that the product of the expanding eigenvalues of $\phi$ is positive. There is a
closed orientable surface $\Sigma$, a pseudo-Anosov mapping $f : \Sigma \to \Sigma$, and a surjective map $\pi : \Sigma \to \mathbb{T}^n$, injective almost everywhere and conjugating $f$ to $\phi$, that is, so that the following diagram commutes:

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{f} & \Sigma \\
\downarrow{\pi} & & \downarrow{\phi} \\
\mathbb{T}^n & \xrightarrow{\pi} & \mathbb{T}^n.
\end{array}
$$

The stretch factor of $f$ is the absolute value of the product of the expanding eigenvalues of $\phi$.

The genus of $\Sigma$ is not controlled and may be larger than $n$.

**Theorem 1.2.** Let $n \geq 4$ and $\phi \in SL_n(\mathbb{Z})$ be irreducible and hyperbolic. For all sufficiently large $N$ there is a closed orientable surface $\Sigma$, a pseudo-Anosov mapping $f : \Sigma \to \Sigma$, and a surjective map $\pi : \Sigma \to \mathbb{T}^n$, injective almost everywhere, conjugating $f$ to $\phi^N$. The stretch factor of $f$ is the absolute value of the product of the expanding eigenvalues of $\phi^N$.

It seems likely that in Theorem 1.2, under the hypothesis that the product of the expanding eigenvalues is positive, we can take $N = 1$, but the proof of this would require finding a Markov partition for $\phi$ of an appropriate specialized type.

A biPerron number is an algebraic integer $\lambda > 1$ such that all Galois conjugates $z$ of $\lambda$ except $\lambda^{-1}$ are contained in the annulus $\lambda^{-1} < |z| < \lambda$. Fried [4] conjectured that if $\lambda$ is a biPerron number of norm $\pm 1$ (i.e., an algebraic unit), some power of $\lambda$ is the stretch factor of a pseudo-Anosov mapping. Theorem 1.1 confirms this conjecture for $\lambda$ of degree 3: a cubic biPerron unit $\lambda$ (equivalently, a cubic Pisot unit) is the leading eigenvalue of a matrix $\phi \in GL_3(\mathbb{Z})$, and $\phi$ (or $\phi^2$ if the norm of $\lambda$ is $-1$) satisfies the hypotheses of Theorem 1.1.

Theorem 1.2 by itself does not settle Fried’s conjecture: in the appendix we prove that there are biPerron units, no power of which is a Mahler measure (a Mahler measure is the absolute value of the product of the expanding eigenvalues of an integer matrix); in particular we show that $x_1 = 1 + \sqrt{2} + \sqrt{2 + \sqrt{2}}$, the leading root of $x^4 - 4x^3 - 2x^2 + 4x - 1$, is not a Mahler measure, and nor is any of its powers.

Our construction of invariant trees, which is an important step in the proofs, is very similar to that in the recent work of Coulbois and Minervino [2].

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1 A stronger conjecture, attributed by Curt McMullen to W. P. Thurston, is that we can take the power to be 1, that is, any biPerron unit is the stretch factor of a pseudo-Anosov.
2 Background

2.1 Markov partitions

Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ be the $n$-torus. Let $\phi \in \text{SL}_n(\mathbb{Z})$ be an orientation-preserving automorphism of $T^n$. We say $\phi$ is irreducible if it has no $\mathbb{Z}$-invariant proper subspace, equivalently, the characteristic polynomial is irreducible over $\mathbb{Z}$. We say $\phi$ is hyperbolic if it has no eigenvalues of modulus 1.

The unstable (or expanding) eigenspace $E^+$ is the direct sum of eigenspaces of $\phi$ for eigenvalues of modulus greater than 1. Likewise the stable (or contracting) eigenspace $E^-$ is the direct sum of eigenspaces of $\phi$ for eigenvalues of modulus less than 1.

A Markov partition $[5]$ for a hyperbolic automorphism $\phi$ of $T^n$ is a decomposition of $T^n$ into connected sets

$$T^n = \bigcup_{i=1}^k R_i,$$

referred to as rectangles$^2$ with the following properties

1. Each $R_i$ is the closure of its interior and has boundary of measure zero,
2. The interiors are disjoint: $\bar{R}_i \cap \bar{R}_j = \emptyset$ if $i \neq j$
3. Each set $R_i$ is a product: $R_i = U_i \times V_i$ where $U_i$ is parallel to $E^+$ and $V_i$ is parallel to $E^-$
4. $\phi(\bar{R}_i) \cap \bar{R}_j$ is either empty or is a union of subrectangles $\bar{U}_j \cap X_k$ (where $X_k$ is a translate of the image of $V_i$), that is, the image of $R_i$ maps completely over $R_j$, possibly multiple times, in the unstable direction.
5. $\phi^{-1}(\bar{R}_i) \cap \bar{R}_j$ is either empty or is a union of subrectangles $\bar{Y}_k \cap \bar{V}_j$ (where $Y_k$ is a translate of the image of $U_i$), that is, the preimage of $R_i$ maps completely over $R_j$, possibly multiple times, in the stable direction.

We define the $(k \times k)$ transition matrix $P \in M_k(\mathbb{Z})$ of the Markov partition by

$$P_{ij} = \ell$$

if $\phi(\bar{R}_i) \cap \bar{R}_j$ has $\ell$ such components.

Note that a Markov partition for $\phi$ is also a Markov partition for $\phi^{-1}$, and the transition matrix for $\phi^{-1}$ is given by the transposed matrix $P^t$.

The horizontal boundary of a rectangle $R = U \times V$ is $U \times (\partial V)$. The vertical boundary is $(\partial U) \times V$. The Markov properties 4. and 5. above can equivalently be stated as the fact the image of the vertical (resp. horizontal) boundary of a rectangle lies in the union of the vertical (resp. horizontal) boundaries.

2.2 Control points

The notion of control points is due to Thurston [7].

$^2$To distinguish this notion from the usual notion of rectangle, which we need below, we refer to a usual rectangle as a Euclidean rectangle.
Associated to a Markov partition is a strongly connected directed graph $G$, the transition graph: $G$ has a vertex (which for simplicity we also call $R_i$) for each rectangle $R_i$ and an edge from $R_i$ to $R_j$ for each component of $\phi(\hat{R}_i) \cap R_j$.

Choose for each vertex of $G$ a single outgoing edge; these outgoing edges form a subgraph $G_0$ with a unique cycle in each of its connected components. The subgraph $G_0$ defines a point in $u_i \in U_i$ for each $i$, called the control point, as follows. For each cycle $\gamma$ in $G_0$, and rectangle $R = U \times V$ on this cycle, the forward images of $U$ around the cycle give an expanding linear map of $U$, mapping over the original copy of $U$. This map has a unique fixed point in $U$. This is the control point in $U$. The preimages of this point define the control points on the entire component of $G_0$ containing $\gamma$.

The control points are forward invariant under $\phi$ by construction.

### 2.3 Pseudo-Anosov mappings

A singular measured foliation on a closed surface $\Sigma$ is a foliation, with a finite number of singular points (each of which is a “$k$-pronged” singularity for some $k \geq 3$), which is equipped with a transverse measure (a measure on any transversal which is invariant under sliding along leaves).

A diffeomorphism $\phi$ of a surface $\Sigma$ is said to be pseudo-Anosov if there is a pair of transverse singular measured foliations $\mathcal{F}^+, \mathcal{F}^-$, each preserved by $\phi$, and a real $\lambda > 1$ such that the corresponding measures are multiplied by $\lambda^{\pm 1}$ under $\phi$. The number $\lambda$ is the stretch factor of $\phi$; the dilatation of $\phi$ is $\lambda^2$, although some authors define the dilatation as $\lambda$; to avoid this ambiguity we refer to $\lambda$ as the stretch factor.

In more practical terms one can describe a pseudo-Anosov, up to conjugacy, as follows. There is a metric on $\Sigma$ in which $\Sigma$ is flat (Euclidean) with conical singularities having cone angles which are multiples of $\pi$ (cone angle $\pi k$ at a $k$-pronged singularity). The structure group consists of translations and rotations by $\pi$. In this metric $\Sigma$ is tiled by a finite number of Euclidean rectangles; restricted to a rectangle $\mathcal{F}^\pm$ are the foliations of the rectangle by horizontal and vertical lines, respectively; on each rectangle $\phi$ acts as a linear stretch map $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. The transverse measures are the vertical and horizontal Lebesgue measure, respectively.

Any pseudo-Anosov mapping $\phi$ has a Markov partition, which is a partition into Euclidean rectangles with the Markov property above: the forward image of the vertical sides of a rectangle lie in the union of vertical sides of the original rectangles, and the preimages of the horizontal sides lie in the union of the horizontal sides of the original rectangles. These rectangles necessarily have horizontal and vertical sides along the leaves of the unstable and stable foliations $\mathcal{F}^\pm$ respectively.

### 3 Construction

In this section we prove Theorem 1.1. We consider first the case $n = 3$. Let $\phi \in \text{SL}_3(\mathbb{Z})$ be irreducible. For convenience we assume (by replacing $\phi$ by $\phi^{-1}$ if necessary) that
E^+ has dimension 2 and E^- has dimension 1. By hypothesis \( \phi \) acts on \( E^+ \) and \( E^- \) preserving orientation. Let \( T^3 = \bigcup_{i=1}^{k} R_i \) be a Markov partition for \( \phi \), with rectangles \( R_i = U_i \times V_i \). Let \( P \) be its transition matrix. Let \( u_i \in U_i \) be a set of control points.

3.1 Tilings

The spaces \( E^\pm \) are linear spaces which are naturally linearly embedded in \( T^3 \) when we identify \( T^3 \) with \( \mathbb{R}^3/\mathbb{Z}^3 \). Likewise any translates of \( E^\pm \) are naturally linearly embedded in \( T^3 \).

Let \( U \) be a generic translate of \( E^+ \), generic in the sense that it and none of its forward images under \( \phi \) pass through the horizontal boundary \( U \times \partial V \) of any rectangle.

The intersections of \( U \) with the rectangles of the partition defines a tiling of the space \( U \) into tiles which are translates of the \( U_i \). This tiling has “finite local complexity”, that is, there are a finite number of different local patches of tiles (up to translation) of any given radius. We say two tiles have the same type if they are translates of \( U_i \) for the same \( i \). (Note that if a non-generic translate \( U \) passes through the horizontal boundary of a tile, it has two different tilings, which differ only on a bounded set.)

The tiling has a dual graph \( G_U \) whose vertices are the control points of the tiles, one per tile, and whose edges correspond to tiles intersecting nontrivially, that is, for which the intersection is nonempty and not a point. This graph \( G_U \) is connected, planar, and is defined for generic \( U \). By finite local complexity, \( G_U \) has bounded degree.

An example Markov partition and tiling is given in Figure 1 for an automorphism \( \phi = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \) with eigenvalues the roots of \( z^3 + 2z^2 + 2z - 1 \). The expanding eigenvalues are \( z_1, \bar{z}_1 \approx -1.17 \pm 1.2i \). In this example the Markov partition has three rectangles, and the corresponding \( U_i \)'s are similar to each other. Under \( \phi \) the tiles subdivide as shown. The areas of the tiles are the components of the left real eigenvector of \( M \), and are in proportion \( 1 : |z_1|^2 : |z_1|^4 \). The heights of the rectangles (i.e. the lengths of the \( V_i \)) are the components of the right real eigenvector of \( M \), and are the numerical frequencies of the three tile types in the tiling. It is not hard to reconstruct the Markov partition from this combinatorial data (see also Figure 4).

3.2 Spanning tree process

We construct an invariant spanning tree of \( G_U \), as follows.

The space \( U \) has image \( \phi(U) \) also parallel to \( E^+ \). For any tile \( T \) in \( U \), its image \( \phi(T) \subset \phi(U) \) is a finite connected set of tiles of \( \phi(U) \) (by the Markov property), called a “supertile”. Construct a spanning tree in the finite graph \( G_{\phi U} \cap \phi(T) \). We can construct this tree in a local manner, that is, if \( T \) and \( T' \) are tiles of the same type then we construct the same tree, up to translation, in the corresponding supertiles \( \phi(T) \) and \( \phi(T') \). In particular there is a single choice of spanning tree for every rectangle \( R_i \) of the partition.
Figure 1: A tiling and subdivision rule.
The union of these “level-1” spanning trees is a spanning forest of $G_U$ for any generic $U$, with one tree in each supertile.

Now for an edge in this forest from $u_T$ to $u_{T'}$ in $G_U$, choose a path from $\phi(u_T)$ to $\phi(u_{T'})$ in $G_{\phi U} \cap \phi(T \cup T')$ which is contained in the union of trees of $\phi(T)$ and $\phi(T')$, except for one new edge which crosses from $\phi(T)$ to $\phi(T')$. Again we make this choice in a local fashion: for an edge between a set of two tiles which is a translate of $T \cup T'$, use the same rule for its image. After we do this for all edge types currently present, there may be new edge types formed and we repeat the process for these edge types iteratively until no new edge types occur; there are only a finite number of possible adjacencies between tiles by finite local complexity, so we are done after a finite number $K$ of iterations. These choices define a spanning tree in $\phi^k(T)$ for any $k \geq 0$, and thus, in the limit as $k \to \infty$, a spanning tree in $G_U$ for almost every $U$. (A limit of spanning trees might have more than one infinite component, but this will happen only on a set of translates $U$ of measure zero, since almost surely the origin is contained in a growing sequence of supertiles whose union is all of $U$.)

Pulling the tree in $G_{\phi^k U}$ back under $\phi^{-k}$ gives a sequence of finer and finer trees in $U$. This sequence is convergent in the following sense. For any two control points $u, u'$ in $G_U$, there is a unique piecewise linear path in the $k$-th order refined tree from $u$ to $u'$, and these paths converge as $k \to \infty$: the path in the $k+1$-order tree from $u$ to $u'$ is obtained by replacing each edge of the $k$-th order path by a subpath in the $k+1$-order tree, and then cancelling any backtracks. Almost every point $p \in U$ has a unique sequence of order-$k$ subtiles containing it and nesting down to it. For almost every pair of points $p, p' \in U$ there is therefore a unique limiting path from $p$ to $p'$, which is the limit of the paths from the control point of a $k$th-order subtile containing $p$ to the control point of the $k$-th order subtile containing $p'$. These limiting paths don’t cross each other, since their approximations don’t cross each other. We say this collections of paths forms a space filling tree.

Note that the analog of the spanning tree in the one-dimensional contracting eigenspace $V$, built by the same construction, just consists in the space $V$ itself.

### 3.3 Eulerian tour

Our next goal is to define in a dynamically consistent way an Eulerian tour of the spanning tree constructed above.

Consider an edge $u_Tu_{T'}$ of the tree in $G_U$. On the path in the tree in $\phi(T \cup T')$ from $\phi(u_T)$ to $\phi(u_{T'})$ there is a unique edge crossing from the supertile $\phi(T)$ into the supertile $\phi(T')$. The image of this edge likewise is a path containing a unique edge crossing from $\phi^2(T)$ into $\phi^2(T')$, and so on. The intersection of the pullbacks to $T \cup T'$ of these forward images is a point on the common boundary of $T$ and $T'$. We call this the boundary control point between $T$ and $T'$. We augment the graph $G_U$ by adding a point in the middle of each edge corresponding to its boundary control point.

Note that two tiles $T, T'$ of the same type may have different neighbors in the tree in $G_U$ and thus a different set of boundary control points.
Figure 2: An invariant tree in the iterated subdivision of a tile. In this example, for convenience we added a set of additional vertices to the graph $G_{\mathcal{H}}$, at certain triple intersections of tiles; these points are also forward invariant. This significantly decreases the complexity of the tree. Each red tile now has one boundary control point; green tiles have one or two and purple tiles have one, two or three boundary control points.
A vertex $u_T$ of degree $\ell$ in $G_U$ has $\ell$ adjacent boundary control points. These are ordered cyclically using the planarity of $G_U$; we label them $b_1, \ldots, b_{\ell}$ in cyclic order. Now in the tree in $\phi(T)$, we can define an Eulerian tour, by “circumnavigating the tree keeping the tree on our left”. That is, when we arrive at a vertex $u_T$, take the next edge in cyclic order after the one we arrive on, and when we arrive at a boundary control point which has degree 1 in $\phi(T)$, we turn around and follow that edge in the opposite direction.

Similarly this procedure defines Eulerian tours in $\phi^k(T)$ for any $k$. If $T$ has boundary control points $b_1, \ldots, b_{\ell}$, the tour in $\phi^k(T)$ is a concatenation of $\ell$ “boundary-to-boundary” paths $\Gamma_{\phi^k(b_i), \phi^k(b_{i+1})}$ from $\phi^k(b_i)$ to $\phi^k(b_{i+1})$ for $i = 1, \ldots, \ell$ with cyclic indices. Moreover, $\Gamma_{\phi^k(b_i), \phi^k(b_{i+1})}$ is itself a concatenation of various boundary-to-boundary paths $\Gamma$ at level $k - 1$. The length of the $\Gamma$s at level $k$ is obtained by applying the appropriate subdivision matrix to the vector of lengths of the $\Gamma$s at level $k - 1$. This subdivision matrix $\tilde{M}$ has leading eigenvalue $\lambda$, the area growth (the product of the expanding eigenvalues of $\phi$), since the $\Gamma$ pass through all vertices of $G_U$. The relative lengths of the $\Gamma$s at level $k$ tend therefore as $k \to \infty$ to the components of the Perron eigenvector of $\tilde{M}$.

By pulling the paths $\Gamma_{\phi^k(b_i), \phi^k(b_{i+1})}$ back under $\phi^{-k}$ we get a tour of $T$ which is a concatenation of paths $\gamma_{b_i, b_{i+1}}^{(k)}$ in $T$ from $b_i$ to $b_{i+1}$ for each $i$ from 1 to $\ell$. We parameterize these paths by $\lambda^{-k}$ times the length of the corresponding $\Gamma$.

When $k \to \infty$ these paths $\gamma_{b_i, b_{i+1}}^{(k)}$ converge and the limiting paths $\gamma_{b_i, b_{i+1}}$ are then space filling and noncrossing, each filling out a subtile of $T$. Note that the subdivision matrix $\tilde{M}$ has as Perron eigenvector the vector of areas of these subtiles of the $T_i$. Thus in fact the lengths of the $\gamma_{b_i, b_{i+1}}$ are proportional to the corresponding subtile areas. Since this fact holds recursively for all subpaths, the paths $\gamma_{b_i, b_{i+1}}$ are in fact parameterized by area: in time $t$ they fill out an area $t$. They are thus injective almost everywhere. We call these paths the peano curves.

### 3.4 Euclidean rectangles

Note that the analog of the peano curve in a line $V$ parallel the stable eigenspace $E^-$ is simply a line segment $[c_1, c_2]$ from one boundary of a tile to its other boundary. It is parameterized by length.

For each tile $T$ of type $U$ and boundary control point $b_i$, the product of the domain of the peano curve $\gamma_{b_i, b_{i+1}}$ and the corresponding tile $[c_1, c_2]$ in $V$ defines a Euclidean rectangle $S_{(b_i, b_{i+1})}([c_1, c_2])$, which maps into (but not typically onto) the rectangle $R = U \times V$ in a space filling way. The images $R_{(b_i, b_{i+1})}(c_1, c_2) = \pi(S_{(b_i, b_{i+1})}([c_1, c_2]))$ form a refinement of the original Markov partition, which by construction is a new Markov partition for $\phi$.

The Euclidean rectangles $S_{(b_i, b_{i+1})}([c_1, c_2])$ are naturally glued along their boundaries, as follows. The point $b_i$ is a boundary point of $T$ and so comes from a boundary

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3 Here is the place where we use the fact that $\phi$ is orientation preserving: so the cyclic ordering is consistent at all scales.
Figure 3: The seven Peano curves in the example. Let $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ correspond to the paths colored red, yellow, green, cyan, dark blue, blue, light blue, respectively. The seven colored paths have forward images as follows: $a_0 \rightarrow a_1 a_2 a_3, a_1 \rightarrow a_4, a_2 \rightarrow a_5, a_3 \rightarrow a_6, a_4 \rightarrow a_5 a_6 a_5, a_5 \rightarrow a_6 a_0 a_1, a_6 \rightarrow a_2 a_3 a_4 a_1 a_2 a_3 a_4$ (compare these with the subdivision rule given in Figure 1).
segment \( \{b_i\} \times V \subset R \). Suppose \( T' \) is a tile adjacent to \( T \) and has \( b_i \) as a common boundary control point with \( T \). Then \( T' \) has a path \( \gamma_{b,b_i} \) ending at \( b_i \), and \( T' \) corresponds to a rectangle \( R' \) intersecting \( R \) somewhere along the segment \( \{b_i\} \times V \); this intersection is a closed interval in \( \{b_i\} \times V \). We glue the corresponding Euclidean rectangles \( S_{(b,b+1),(c_1,c_2)} \) and \( S_{(b,b),(c_1',c_2')} \) along the corresponding closed interval.

If a part of \( \{b_i\} \times V \) is not glued to any other tile that means that in the tree this boundary control point has degree 1; in this case we glue the corresponding subinterval of the boundary of \( S_{(b,b+1),(c_1,c_2)} \) to the same subinterval in its predecessor rectangle \( S_{(b-1,b),(c_1,c_2)} \). The entire vertical boundary of \( \{b_i\} \times V \) is thus glued to other rectangles in this way.

The horizontal boundaries of the Euclidean rectangles are glued as follows. When a (non-generic) plane \( U \) passes through a horizontal boundary of the partition, \( U \) will have two tilings, one coming from \( U^- \) (a translate of \( U \) just above) and one coming from \( U^+ \) (a translate just below). There are likewise two trees. However these two tilings and two trees differ only on a bounded set, by irrationality of \( E^+ \). Likewise in any forward image in \( \phi^k(U) \), the tilings and trees in \( \phi^k(U^-) \) and \( \phi^k(U^+) \) only differ in a bounded set. This implies that the pull-backs under \( \phi^{-k} \) of the trees differ in a smaller and smaller set; in the limit \( k \to \infty \) the trees are in fact identical, except possibly at a point, where they might connect differently; their corresponding peano curves will therefore differ at most at a single point. Call such a point a critical point.

We can now glue the upper horizontal boundary of each Euclidean rectangle to the lower boundary of another rectangle: for each point on the upper boundary, take the maximal interval around it avoiding the critical point, if any. This interval is glued to the lower boundary of another Euclidean rectangle.

The resulting quotient space is therefore a closed surface \( \Sigma \). The map \( \phi \) defines a pseudo-Anosov map \( f \) of this surface: it stretches the horizontal direction by \( \lambda \) and the vertical direction by \( \lambda^{-1} \). The surjective map \( \pi : \Sigma \to T^3 \) is by construction a semi-conjugation between \( f \) and \( \phi \). The fact that the peano curves are almost everywhere injective implies that \( \pi \) is an almost isomorphism.

See Figure 3 for the example. One can compute the genus of the surface in this example as follows. The rectangles of the Markov partition in Figure 4 have a common lower boundary, the plane \( E^+ \). The tops of the rectangles also meet on this common boundary, in a different order, see Figure 5. Comparing the superposition of Figure 3 and Figure 5 (see Figure 6), one can see that the tours are identical except at the two points where the three tiles meet (one such point is indicated by a black dot, the other is the start of the red tour in Figure 3). If we split the paths \( a_0, a_2, a_5 \) (the red, green and blue paths) at the black dot in Figure 5 so that \( a_0 \) is the concatenation of \( a_0' a_0'' \), and similarly \( a_2 = a_2' a_2'' \) and \( a_5 = a_5' a_5'' \), then starting from the yellow curve in Figure 5 the tour is \( a_1 a_1' a_2' a_0 a_3 a_4 a_0' a_0'' a_1 a_5 a_2 a_3 a_4 a_0' a_0'' a_1 a_2 a_3 a_4 a_0' a_0'' a_1 a_2 a_3 a_4 a_0' a_0'' a_1 a_2 a_3 a_4 a_0' a_0'' a_1 a_2 a_3 a_4 a_0' a_0''. Gluing a 20-gon with one of these tours along its upper boundary and the other along its lower boundary, both from left to right, one sees that the resulting surface has genus 3.
Figure 4: The images of the seven Euclidean rectangles in the example. These rectangles all have the same lower $z$ coordinate.

Figure 5: The seven Peano curves, as they would be arranged in a plane $U$ just below that in the previous figure.
3.5 Degree 4

The above procedure works in the same way when $\phi$ is a hyperbolic irreducible automorphism of $\mathbb{T}^4$, on condition that the stable and unstable eigenspaces have dimension 2, and $\phi$ preserves orientation on each of $E^\pm$. We define the tree and tour independently in both the stable and unstable eigenspaces (using $\phi^{-1}$ for the stable eigenspace), using the planarity of the graphs $G_U$ and $G_V$ to define the tours. The Euclidean rectangles are given by all possible products of the domains of $\gamma_{b_i,b_{i+1}}$ and $\gamma_{c_j,c_{j+1}}$, where these paths both come from the same rectangle $R = U \times V$ of the Markov partition. They are glued in the same way as above in the above discussion of the gluing of horizontal boundaries. The only small difference is that when a non-generic plane $U$ passes through a horizontal boundary of the partition, $U$ will have possibly (a finite number of) multiple tilings coming from nearby translates of $U$, not just two. However these all differ from each other in a bounded set as before.

4 Higher degree

When $E^+$ or $E^-$ has dimension $\geq 3$ there is a difficulty in the choice of an Eulerian tour of the tree. The tree can be constructed as in the above cases but the tree is no longer embedded in the plane and so there is not necessarily a canonical choice of cyclic ordering of the edges at a vertex. This has the potential to cause inconsistencies in defining the Eulerian tour: a choice of cyclic ordering at a vertex in a tile and its subtile might be incompatible, so that the pull-backs of the tours do not converge. For
a given tree there might in fact be no choice of compatible cyclic orientations. One can try to overcome these incompatibilities by either changing the choice of tree or choice of Markov partition, but we were not able to accomplish this in complete generality.

However we can modify the construction if we replace \( \phi \) by a sufficiently high power \( \phi^N \), as follows.

The Markov partition for \( \phi \) serves as a Markov partition for \( \phi^N \). Construct control points for \( \phi \) as before; these are also control points for \( \phi^N \). We now construct the boundary control points before constructing the tree: for every pair of tiles \( T, T' \) adjacent in \( G \cap U \), choose (in a locally defined manner as before) one edge connecting a vertex of \( G \cap \phi(U) \cap \phi(T) \) to an adjacent vertex of \( G \cap \phi(U) \cap \phi(T') \). This defines by iteration a boundary control point between any two adjacent tiles.

Now any tile type has a finite number of boundary control points \( b_1, \ldots, b_k \) depending only on \( \phi \) and not on \( N \). Choose an arbitrary cyclic ordering of these.

Define the tree as follows. We choose \( N \) large enough so that the \( \phi^N(b_i) \) are far apart from each other in the graph distance in \( \phi^N(T) \). In the image \( \phi^N(T) \cap G \), connect the \( \phi^N(b_i) \) using an embedded trivalent tree \( \tau \) as in Figure 7, so that the interior vertices \( v_i \) are also far apart and far from the boundary vertices, in graph distance. Such an embedded trivalent tree \( \tau \) can be found for \( N \) sufficiently large. At each trivalent interior vertex \( v \) of \( \tau \), the tour joining the \( \phi^N(b_i) \) induces a cyclic ordering on the three edges of \( \tau \) meeting at \( v \): we adjust the embedding around \( v \) (rewiring the branches if necessary in an annular neighborhood of \( v \)) so that the cyclic order of the \( \phi^N(b_i) \) is compatible with the cyclic order chosen at \( v \).

Now complete this trivalent tree to a spanning tree of \( \phi^N(T) \). The spanning tree is obtained by adding subtrees ("bushes") to \( \tau \), and each subtree has a tour well-defined from the order of its vertices.

The tour in \( \phi^N(T) \) is now defined from the cyclic orders of its vertices and is consistent by construction with that of its supertile \( T \).
The remainder of the construction proceeds as in the previous cases.

5 Questions

1. Can one control the genus of the surface $\Sigma$?
2. Our construction depends on several choices, even once the Markov partition is determined. How are the resulting surfaces and mappings related under these choices?
3. Can we take $N = 1$ in Theorem 1.2 by using an appropriate Markov partition?
4. Which biPerron units are Mahler measures? (See Theorem 6.1.)
5. Is there a pseudo-Anosov with stretch factor $1 + \sqrt{2} + \sqrt{2} + \sqrt{2}$? (See Theorem 6.1.)

6 Appendix

Theorem 6.1. The biPerron number $x_1 = 1 + \sqrt{2} + \sqrt{2} + \sqrt{2}$, the leading root of $x^4 - 4x^3 - 2x^2 + 4x - 1$, is not a Mahler measure, and nor is any power of $x_1$.

Proof. We prove that $x_1$ is not a Mahler measure; the proof for $x_1^n$ is identical. Let $x_1 = 1 + \sqrt{2} + \sqrt{2} + \sqrt{2} \approx 4.26$

$x_2 = 1 - \sqrt{2} - \sqrt{2} - \sqrt{2} \approx -1.17$

$x_3 = 1 + \sqrt{2} - \sqrt{2} + \sqrt{2} \approx 0.566$

$x_4 = 1 - \sqrt{2} + \sqrt{2} - \sqrt{2} \approx 0.351$

be the Galois conjugates of $x_1$. Let $F = \mathbb{Q}[\sqrt{2} + \sqrt{2}]$ be the splitting field of $x_1$. It has Galois group $G_F \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, \sigma_{10}, \sigma_{01}, \sigma_{11}\}$,

where $\sigma_{10}, \sigma_{01}$ maps $\sqrt{2}, \sqrt{2} + \sqrt{2}$ to $-\sqrt{2}, \sqrt{2} - \sqrt{2}$ and $\sqrt{2}, -\sqrt{2} + \sqrt{2}$ respectively, and $\sigma_{11} = \sigma_{10}\sigma_{01}$.

The $\sigma$ permute the roots as follows:

$\sigma_{10}(x_1, x_2, x_3, x_4) = (x_4, x_3, x_2, x_1)$

$\sigma_{01}(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2)$

$\sigma_{11}(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3)$.

Suppose $x_1$ is a Mahler measure: there is a monic polynomial $p(z) \in \mathbb{Z}[z]$ and $x_1 = \prod_{z \in I_1} z$ where $I_1$ is the set of roots of $p$ of modulus greater than 1. Let $K$ be the splitting field of $p$, and $G_K$ its Galois group over $\mathbb{Q}$.
Note that if an automorphism $\rho \in G_K$ fixes $x_1$, then it necessarily fixes $I_1$ as a set, by the Mahler measure property. Applying (lifts of) the $\sigma_i$ we find unique subsets $I_2, I_3, I_4$ of roots of $p$ with $x_j = \prod_{z \in I_j} z$. Note that $I_1$ and $I_2$ cannot be disjoint since $|x_2| > 1$. Write $I_1 = A \cup B$ and $I_2 = A \cup C$ where $A = I_1 \cap I_2$ is nonempty.

The group $G_F$ can be lifted to a subgroup of $G_K$. Choosing one such lift, let $\tilde{\sigma}$ denote the lift of $\sigma \in G_F$. Let $\mathcal{A}' = \tilde{\sigma}_{01}(\mathcal{A}), \mathcal{B}' = \tilde{\sigma}_{01}(\mathcal{B})$ and $\mathcal{C}' = \tilde{\sigma}_{01}(\mathcal{C})$. We have

\begin{align*}
I_3 &= \tilde{\sigma}_{01}(I_1) = \mathcal{A}' \cup \mathcal{B}' \\
I_4 &= \tilde{\sigma}_{01}(I_2) = \mathcal{A}' \cup \mathcal{C}'
\end{align*}

and

\begin{align*}
I_4 &= \tilde{\sigma}_{10}(I_1) = \mathcal{A}' \cup \mathcal{C}' \\
I_3 &= \tilde{\sigma}_{10}(I_2) = \mathcal{A}' \cup \mathcal{B}'.
\end{align*}

Thus $\tilde{\sigma}_{10}, \tilde{\sigma}_{01}$ exchange $A$ and $A'$ while $\tilde{\sigma}_{11}$ fixes both $A$ and $A'$.

Let $A = \prod_{z \in A} z$ and $A' = \prod_{z \in A'} z$. Then $AA'$ and $A + A'$ are invariant under all of $G_K$, and thus are rational integers. In particular $A, A' \in K$ are in the same quadratic extension of $\mathbb{Q}$, and so $A$ is a quadratic unit. Since $\sigma_{11}$ fixes $A$ and is the nontrivial automorphism of $\mathbb{Q}[\sqrt{2}]$, we know that $A$ is not in $\mathbb{Q}[\sqrt{2}] \setminus \mathbb{Q}$.

Suppose $A > 1$ is a quadratic unit not in $F$. There is a Galois conjugation in $K$ fixing $F$ and taking $A$ to one of $\pm 1/A$. If $A = 2 + \sqrt{3}$, for example, then $B = \prod_{z \in B} z = \frac{x_1}{2 + \sqrt{3}}$; a Galois conjugation would take this to $B'' = \frac{x_1}{2 - \sqrt{3}}$ which is larger than $x_1$, a contradiction, since $A \cup B$ are all the roots of modulus $> 1$.

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$\square$
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