Manipulation of coherent atom waves using accelerated two-dimensional optical lattices

Wei-Chih Ting, Dian-Jiun Han and Shin-Tza Wu

Department of Physics, National Chung Cheng University, Chiayi 621, Taiwan
E-mail: phystw@gmail.com

New Journal of Physics 12 (2010) 083059 (13pp)
Received 4 May 2010
Published 27 August 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/8/083059

Abstract. We study the dynamics of Bose–Einstein condensates in accelerated two-dimensional (2D) optical square lattices by numerically solving the Gross–Pitaevskii equation. We consider the regime with negligible mean-field interactions and examine in detail the pulses of atom clouds ejected from the condensate due to Landau–Zener tunnelling. The pulses exhibit patterned structures that can be understood from the momentum–space dynamics of the condensate. Apart from conceiving the realization of a pulsed 2D atom laser, we demonstrate that, by exploring the band structure of the lattice, the Landau–Zener tunnelling and Bragg reflection of the condensate inside the optical lattice can provide a means of manipulation of coherent atom waves.

Contents

1. Introduction 2
2. Two-dimensional (2D) atom laser 3
   2.1. Theoretical formulation 3
   2.2. Atom laser 4
   2.3. \( k \)-space dynamics 5
3. Manipulation of coherent atom waves 10
4. Conclusions and discussions 11
Acknowledgments 12
References 12

1 Author to whom any correspondence should be addressed.
1. Introduction

The past two decades have witnessed the fruitful interplay between condensed matter physics and ultracold atom physics. While cold atom systems have provided a new arena for condensed matter physics, at the same time, concepts and ideas from condensed matter physics have opened up new directions for cold atom physics. For instance, by loading ultracold atoms into optical potentials due to standing laser waves (i.e. optical lattices), it is possible to not only build up ‘quantum simulators’ for condensed matter systems (see, for example, [1]) but also control and manipulate cold atoms utilizing concepts from traditional solid-state systems (see, for instance, [2]). In this work, we wish to further explore the latter lines of investigation.

Electrons in solids are subject to periodic potentials due to the array of ion cores that constitutes the lattice. In ideal situations, the electron eigenfunctions take the form of a particular structure that is called the Bloch wave, and the corresponding eigenenergies cluster into energy bands that are separated by energy gaps (see, for example, [3]). As a consequence, electrons in solids can behave quite differently compared with those in free space. For instance, when a weak uniform static electric field is applied to the solid, instead of being uniformly accelerated, the electrons exhibit oscillatory motion (the Bloch oscillation; see, for example, [4]). This oscillatory motion can be understood as Bragg reflections of the Bloch states in momentum space (or $k$-space) at regions where energy gaps occur. If the bias field is large, the electrons can gain sufficient energy to lead to a finite probability of tunnelling across the energy gaps, resulting in transitions to higher energy bands (the Landau–Zener tunnelling; see, for example, [4]). In this work, we wish to consider the cold atom counterpart of the above construction. Namely, we shall consider a cloud of Bose–Einstein condensate subject to a periodic potential due to standing laser waves that form an optical lattice. We shall examine the dynamics of the condensate under the action of an external bias that corresponds to the uniform static electric field above. Experimentally, this bias can be furnished in a number of ways: one could make use of the direct gravity pull on the atoms, or introduce time-dependent frequency shifts between the lattice beams so that an effective acceleration on the atoms occurs (see, for example, [5, 6]). For spin-polarized condensates, it is also possible to generate the acceleration by means of magnetic field gradients [7].

The dynamics of ultracold atoms in 1D optical lattices subject to uniform accelerations have been investigated extensively [2]. In addition to the observation of Bloch oscillations [8], Wannier–Stark ladders [9] and Landau–Zener tunnelling [10] for condensates in accelerated 1D optical lattices, possible realizations for a 1D ‘atom laser’ have also been achieved [11] using such systems. By loading Bose condensed $^{87}$Rb atoms into a vertical 1D optical lattice, Anderson and Kasevich [11] demonstrated that for appropriate lattice strengths, pulses of atom clouds can tunnel out of the optical lattice due to the gravity pull. The generation of pulses of coherent atom waves thus suggests the system being a 1D pulsed atom laser [11, 12]. For 2D systems, there have also been studies of Bloch oscillations, Landau–Zener tunnelling (at lower energy bands; cf below) [13, 14] and Bloch–Zener oscillations [15] of ultracold atoms in accelerated 2D optical lattices. The richer dynamics of the condensates in 2D lattices has been proposed as a possible means of manipulation of the coherent atom waves [13]–[15]. Along these lines of investigation, we shall examine in this work the Landau–Zener tunnelling

2 For other realizations of atom lasers (involving and not involving accelerated optical lattices), see, for example, [7] and references therein.

3 There have also been related theoretical and experimental studies in photonic crystals [16]–[19].

New Journal of Physics 12 (2010) 083059 (http://www.njp.org/)
of atom waves within 2D optical lattices. More specifically, we will be interested in physical configurations similar to the 1D system considered by Anderson and Kasevich [11, 12]. To simplify the analysis, however, we will focus here on regimes where the nonlinear mean-field interaction is negligible (cf [11, 12]). The effects of the mean-field interaction in related problems are investigated in [20, 21].

In order to conceive a 2D atom laser, we shall consider a condensate loaded into a uniformly accelerated 2D optical square lattice in the regime where Landau–Zener tunnelling is the predominant effect. In particular, since the atoms need to tunnel out of the lattice to form the lasing pulses, the dynamics of the condensate at higher energy bands will be essential (cf [13, 14]). As we will see below, the more complicated energy landscape at higher energy bands turns out to enrich the structure for the output pulses of the atom laser. We shall study the dynamics of the condensate by numerically solving the Gross–Pitaevskii (GP) equation (see, for example, [6]). For appropriate accelerations, we shall find pulses of atom clouds tunnelling out of the condensate, which manifest a patterned structure. We will show that these are the consequences of Bragg reflection and Landau–Zener tunnelling at higher energy bands. For energy states in these higher energy bands, unlike their 1D counterparts [11], characteristics of Bloch waves are still manifest in the atom waves, even though they are quite close to free-particle states. By means of \( k \)-space analysis, we shall relate the propagation of the coherent atom waves in real space to the \( k \)-space dynamics of the condensate. As we will soon recognize, apart from a 2D atom laser with patterned pulses, our result can help us to envisage beam splitters for coherent atom waves.

In the following, we shall start in section 2.1 with an introduction to our theoretical formulation. Numerical results for our simulations will then be presented in section 2.2. In section 2.3, we will try to understand these results based on the \( k \)-space dynamics of the condensate. We will then demonstrate in section 3 how this analysis can help provide a useful tool for the manipulation of coherent atom waves. A brief conclusion is supplied at the end of the paper.

2. Two-dimensional (2D) atom laser

2.1. Theoretical formulation

Let us consider a Bose–Einstein condensate of atoms with atomic mass \( m \) in a 2D optical lattice [22]. We shall look into the regime where the low-temperature ground-state dynamics of the condensate can be described by the GP equation (see, for example, [6]),

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \Psi(\rho, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\rho, t) + U |\Psi(\rho, t)|^2 \right) \Psi(\rho, t),
\]

where \( \Psi \) is the condensate wave function, \( \rho = (x, y) \) is the 2D position vector, \( \nabla^2 \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) is the 2D Laplacian operator, \( V_{\text{ext}} \) is the total external potential acting on the condensate and \( U \) is the mean-field coupling constant. In order to examine the dynamics of the condensate in an accelerated optical lattice, here the external potential \( V_{\text{ext}} \) consists of two parts: one due to the optical lattice and the other from a uniform acceleration. The optical lattice potential that we shall consider is similar to that in [22],

\[
V_{\text{opt}}(x, y) = V_0 \left[ \cos^2(q_x x) + \cos^2(q_y y) \right],
\]

where \( V_0 \) is a real constant and \( q_x \) and \( q_y \) are wave numbers for the laser beams that build up the optical lattice. In experiments, the lattice beams usually have the same wavelength, so that
\( q_x = q_y = q \). We shall thus consider this case so that (2) corresponds to a 2D square lattice with lattice constant \( a = \pi / q \). If the lattice is uniformly accelerated with acceleration \( g \), there is the corresponding potential

\[ V_g(x, y) = -m g \cdot \rho. \]  

(3)

In order to maintain the phase coherence of the atom laser that we have in mind, it is crucial to avoid large potential strengths \([12]\). This is because for high lattice strengths, if one regards the condensate as a collection of individual atom clouds over the lattice sites, with increasing \( V_0 \), the wave function overlap between neighbouring sites would drop and, at the same time, the mean-field interaction \( U |\Psi|^2 \) would grow due to the more localized wave function and stronger coupling \( U \) \([23]\). These effects combine to reduce the phase coherence among the condensates on different lattice sites \([12, 24]\). This can be understood as a consequence of the numbersqueezing of the condensate on each lattice site due to the mean-field interaction when \( V_0 \) is large, which enhances the (uncorrelated) phase fluctuations among the lattice sites \([24]\). It is essential to recognize the role of the mean-field interaction in this decoherence mechanism: with \( U = 0 \), as long as the wave function overlap between adjacent sites remains finite, there would not be any decoherence even for large \( V_0 \). We will thus be interested in the regime where the mean-field interaction \( U |\Psi|^2 \) in (1) is negligible compared with the strength of the optical potential. In experiments, this would correspond to cases where well-resolved Bragg peaks are visible when the condensate is released from the optical lattice \([24, 25]\) (note, however, the caveat pointed out by Hadzibabic et al \([28]\)); namely, the system is supposed to be deep on the ‘superfluid’ side of the phase diagram. This would at the same time also justify our analysis of the condensate dynamics based on the GP equation, since for large potential strengths the system could make transitions into the Mott insulating phase \([25]\). Throughout this work, we will therefore consider the following GP equation,

\[
i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 \rho + V_0 \left[ \cos^2(qx) + \cos^2(qy) \right] - mg \cdot \rho \right) \Psi.
\]  

(4)

The time evolution of the condensate can then be analysed reliably using the Crank–Nicholson method \([29]\), which we shall now turn to.

2.2. Atom laser

Let us suppose we have initially in free space a stationary condensate that has a Gaussian wave function,

\[
\Psi(\rho, t_0) = \sqrt{\frac{N}{2\pi \sigma^2}} \exp\left( -\frac{\rho^2}{4\sigma^2} \right), \tag{5}
\]

where \( N \) is the total number of atoms, \( \rho = (x^2 + y^2)^{1/2} \) and \( \sigma \) is the width of the wave packet. In order to transfer the condensate into the lattice ground state, without turning on \( V_g \), the Gaussian wave packet is first subject to the optical lattice potential \( V_{\text{opt}} \) and set to evolve for a period of time. At time \( t = 0 \), we switch on \( V_g \) (see (3)) so that a static uniform acceleration \( g \) starts

\footnote{Since the mean-field interaction is proportional to the (s-wave) scattering length and the density of the condensate, besides tuning the lattice strength \( V_0 \) (such as in \([11, 12, 24, 25]\)), experimentally one could also attain this regime by using condensates with low densities and/or by reducing the scattering length via Feshbach resonances \([26, 27]\).}
acting on the condensate. It is then our main purpose to examine the subsequent dynamics of the condensate. As explained previously, to avoid complications from strong lattice potentials, we restrict ourselves here to potential strengths of the order of the recoil energy,

\[ E_R \equiv \frac{\hbar^2 q^2}{2m} = \frac{\hbar^2}{8ma^2}. \]  

(6)

We carried out numerical simulations for the above procedures based on the GP equation (4). The results for two different accelerations are presented in figure 1, where we have used \( \sigma = 2.5a \) and \( N = 2\pi \sigma^2 \) for the initial wave function (5) and \( V_0 = 1.0E_R \) for the optical potential (2). The acceleration is set to have the magnitude \( g = \frac{\pi E_R}{4ma} \), which has been chosen somewhat arbitrarily; one only has to make sure that it would lead to appreciable tunnelling probability for the condensate (see later). For convenience, we use the following time scale for the time evolution of the condensate,

\[ T_B = \frac{\frac{2\pi}{a}}{mg} = \frac{\hbar}{mga} = \frac{1}{g} \sqrt{8E_R \over m}, \]  

(7)

which is the time for a Bloch state to cross one complete Brillouin zone (namely, the Josephson time [11] or the Bloch period [14]) when \( g \) is along the lattice \((1, 0)\) direction\(^5\). We shall henceforth adopt a coordinate system with the \( x \)-axis along the lattice \((1, 0)\) direction and the \( y \)-axis along the \((0, 1)\) direction.

As shown in figure 1(a), for \( g \) along the \((1, 0)\) direction, pulses of atom clouds tunnel out of the condensate in a manner reminiscent of that in accelerated 1D optical lattices [11, 12]. The sequence of pulses generated are equally spaced in time by \( T_B \) and aligned entirely in the \((1, 0)\) direction. For \( g \) along the \((1, 1)\) direction (see figure 1(b)), however, the pulses have a more complicated structure. There are initially three pulses generated from the condensate in three different directions, which subsequently split into further sub-pulses. Clearly, this immediately brings up interesting possibilities for applications: besides a ‘patterned’ atom laser, one could also use the 2D optical lattice as a beam splitter for coherent atom waves, as we will soon demonstrate. In the following, we shall first examine the origin of this pattern and afterwards elaborate on its possible applications.

2.3. \( k \)-space dynamics

We shall now attempt to understand the above results from the viewpoint of the \( k \)-space dynamics of the condensate. This analysis will turn out to be very useful for real-space manipulations of atom waves using accelerated 2D optical lattices.

In the absence of lattice acceleration (i.e. \( g = 0 \)), the eigenstates of the GP equation (4) for the condensate are the Bloch states (see, for example, [3]). Each Bloch state is characterized by the quantum numbers \( k \) (the Bloch wave vector) and \( n \) (the band index). For the initial wave function (5), it corresponds in \( k \)-space to a Gaussian centering at \( k = (0, 0) \) with a spread \( \delta k_x = \delta k_y = (2\sigma)^{-1} \). Therefore, the subsequent time evolution of the condensate would yield wave packets that are superpositions of Bloch states with width \( \delta k \sim (2\sigma)^{-1} \). For the simulations in figure 1, we have \( \sigma = 2.5a \). Thus in \( k \)-space, instead of a single point, each atom cloud

\(^5\) Note that for \( g \) along arbitrary directions, \( T_B \) would in general not be identical to the period for the Bloch oscillation. In fact, it is possible to have open trajectories for Bloch oscillations [14]. In this case, the Bloch period does not exist.
Figure 1. Numerical results for accelerated condensates in a 2D square lattice with the acceleration $g$ along (a) the $(1,0)$ direction and (b) the $(1,1)$ direction. For both cases, we have $V_0 = 1.0E_R$ and $g = \frac{2E_R}{4ma}$, and the time scale $T_B$ is given by (7). Note that in (b), for convenience, the plots are made with respect to the coordinates $x' = (x + y)/\sqrt{2}$ and $y' = (y - x)/\sqrt{2}$, so that $g$ points along the $x'$-axis. In all plots, the coordinates are measured in units of the lattice constant $a$. 

New Journal of Physics 12 (2010) 083059 (http://www.njp.org/)
corresponds to a wave packet that spreads around a central value with area $\pi(\delta k)^2 \sim \frac{\pi}{25a^2}$, which is $\frac{1}{100} \simeq 0.32\%$ of the area of one single Brillouin zone. In the analysis given below, we shall ignore the spread in $k$ and focus on the center of each wave packet, as if the atom cloud were in a sharp Bloch state characterized by a single Bloch wave vector. In a more quantitative treatment, one can take into account the spreading by averaging over the distribution of the Bloch wave vectors \[8\].

In view of the above, the motion of the condensate under the action of a uniform static acceleration $g$ can then be dealt with using a semiclassical approach. Within each band (namely, for one specific $n$), the dynamics of the condensate follow the equations of motion (see, for example, \[3\]),

$$\frac{d\rho}{dt} = \frac{1}{\hbar} \nabla_k E_n, \quad (8)$$

$$\hbar \frac{dk}{dt} = mg, \quad (9)$$

where $\rho$ is, as before, the 2D position vector and $E_n$ is the eigenenergy for the Bloch state. As is clear from \[9\], under the action of a constant acceleration $g$, the Bloch state would proceed steadily in $k$-space along the direction determined by $g$. However, when the state arrived at zone boundaries where energy gaps exist, the state would have to either make transitions to higher energy bands (Landau–Zener tunnelling) or be reflected to other regions of the same zone (Bragg reflection), so that it could continue its course of evolution according to \[9\]. Note that these processes are not included in \[9\] since two or more energy bands can be involved. When only two states are involved, one finds that the tunnelling probability would be (see, for example, \[4\])

$$P = \exp \left( -\frac{\pi^2}{4} \frac{E_{\text{gap}}^2}{mgE_0} \right), \quad (10)$$

where $E_{\text{gap}}$ is the energy gap at the $k$-point in question and $E_0$ is the corresponding free-particle energy. If the acceleration $g$ is small, the tunnelling probability will be exponentially small. The state will then be Bragg reflected to another state within the original zone and subsequently continue to evolve in accordance with \[8\] and \[9\] until it meets another zone boundary. If the state has low probability for interband transitions, its motion will be limited to a single zone and the motion will be oscillatory in both real and $k$ spaces (which is the Bloch oscillation) \[14\]. However, for larger values of $g$, the tunnelling probability can become appreciable. The state can then tunnel into higher energy bands and give rise to pulses of atom clouds in real space moving with group velocities determined by \[8\]. As we will find out soon, the patterns of the atom pulses in figure 1 are the consequences of Bragg reflection and Landau–Zener tunnelling of the atom waves. Based on this observation, we will explain below how this picture can be used to carry out coherent manipulations of atom waves.

On the grounds of the foregoing analysis, we note that the state of the condensate migrates over the $k$-space under the pull of $g$ and hops around via Bragg reflections. It is thus very helpful to examine the $k$-space dynamics of the condensate using the extended-zone picture (see, for example, \[4\]). Figure 2 shows the extended-zone energy landscape over $k$-space obtained from direct diagonalization for the condensate in the 2D square lattice \[2\]. Energy gaps exist at regions where there are discontinuities in the energy contours, which appear as dark shading.
Figure 2. (a) The first 12 Brillouin zones for the 2D square lattice in the extended-zone scheme. The open circles represent reciprocal lattice points. (b) Contours for the corresponding energy landscape when $V_0 = 1.0E_R$. Here, $k_x$ and $k_y$ are both in units of $\pi/a$. For regions not shown here, one can infer the zone structure and energy landscape up to the 12th zone from the symmetry of the lattice.

lines in figure 2(b); the darker the shading, the larger the energy gap. Note that not every zone boundary has energy gaps. As in the extended-zone scheme, the zone index is identical to the band index, we shall henceforth use them interchangeably.

In our simulations in figure 1, we have chosen $g = \frac{\pi E_R}{4ma}$, which is sufficiently large to ensure that the tunnelling probability out of the first band is always appreciable for $V_0 = 1.0E_R$. With the initial wave function (5), the condensate starts at $t = 0$ from the state $\mathbf{k} = (0, 0)$ in the first zone. For figure 1(a), the acceleration points along the $(1, 0)$ direction. Therefore, according to (9), the condensate will proceed along the $k_x$-axis until it meets the zone boundary at $k_x = \frac{\pi}{a}$. The wave packet then becomes partly reflected back to the first zone and partly transmitted into the second zone, giving rise to the first tunnelling pulse shown in figure 1(a). The pulse that enters the second zone proceeds again according to (9) until it meets the zone boundary at $\mathbf{k} = (\frac{\pi}{a}, 0)$, where three Bragg planes intersect. From figure 2(b) it is clear that the energy gap at this point is tiny; thus the pulse can tunnel through entirely. As the wave packet moves along, its energy also grows. Eventually, the wave packet becomes essentially free, since it can overcome any energy gap that it encounters (see (10)). Note that this result is essentially identical to that for 1D optical lattices [11, 12].

For acceleration along the $(1, 1)$ direction (see figure 1(b)), the $k$-space dynamics for the condensate become more complicated. Again, the condensate starts at $t = 0$ from the state $\mathbf{k} = (0, 0)$ and evolves initially according to (9) along the $(1, 1)$ direction. However, at $t = T_B/\sqrt{2}$, the state reaches the point $\mathbf{k} = (\frac{\pi}{a}, \frac{\pi}{a})$, where bands 1–4 meet. Due to the large energy gap, the condensate can have a partial wave Bragg reflected back into the first band (to the point $\mathbf{k} = (\frac{-\pi}{a}, \frac{-\pi}{a})$) and a partial wave tunnelling into the second band (or the third band, since they are degenerate at this point; see figure 2(b)). For the partial wave entering into the second band, nevertheless, since the acceleration $g$ would drive the system to evolve along the $(1, 1)$ direction, the state has to either tunnel immediately into the fourth band or be Bragg reflected within band 2. The four atom clouds visible at $t = 2.0T_B$ in figure 1(b) represent each
of the above possibilities. The atom cloud at the top corner is the partial wave that is Bragg reflected back into the first band, executing an oscillatory motion in both real and \( k \)-spaces. The lower one is the partial wave that tunnels directly into the fourth band. The two side pulses are partial waves that are Bragg reflected to the points \( k = (\pm \frac{\pi}{a}, \frac{3\pi}{a}) \), which subsequently evolve under the acceleration \( g \) towards the states \((\frac{2\pi}{a}, 0)\) and \((0, \frac{2\pi}{a})\), respectively. For the three partial waves that tunnel out of the first band, the acceleration \( g \) continues to bring them forward along the \((1, 1)\) direction. The central pulse becomes essentially free, as there is no longer any appreciable energy gap along its \( k \)-space trajectory. For the two side pulses, however, as one can see from figure 2(b), there remain several gaps that they will cross.

Since the two side pulses are symmetric, it will suffice to focus our analysis on one of them. Let us consider the one on the right. After being Bragg reflected to the point \((-\frac{\pi}{a}, \frac{\pi}{a})\), the pulse evolves under the pull of \( g \) along the \((1, 1)\) direction. It will therefore come across an energy gap at \( k = (\frac{\pi}{3a}, \frac{2\pi}{3a}) \) at time \( t = \frac{2\pi}{3a} T_B \simeq 1.650 T_B \). For the result in figure 1(b), one can rule out any possibility of Bragg reflection at this point since the sub-pulses become visible only at \( t = 2.0 – 3.0 T_B \) (for a more detailed check, see later). The next energy gap that the pulse would meet occurs at \( t = \frac{3\pi}{\sqrt{2}} T_B \simeq 2.121 T_B \) at the point \( k = (\frac{\pi}{a}, \frac{3\pi}{a}) \). As we shall explain in greater detail below, at this point the pulse turns out to be partially Bragg reflected to the state \( k = (-\frac{\pi}{a}, \frac{3 \pi}{a}) \) and partially tunnels forward along its original trajectory. Therefore, the original pulse is split into two sub-pulses, resulting in the final pattern seen in figure 1(b). To substantiate the arguments in the preceding paragraph, one could consider an arbitrary Bloch state on the line segment between the \( k \) points \((-\frac{\pi}{a}, \frac{\pi}{a})\) and \((\frac{\pi}{a}, \frac{7\pi}{3a})\), and apply to the state the same acceleration \( g \) as above, but now in a controlled manner. For instance, starting from the point \( k = (-\frac{\pi}{2a}, \frac{3\pi}{2a}) \), we apply the acceleration \( g \) for the duration \( \Delta t = 1.0 T_B \) and then turn it off. This would send the pulse to the state \( k = (- 1 + 2\sqrt{2} \frac{\pi}{a}, \frac{3 + 2\sqrt{2}}{2} \frac{\pi}{a}) \simeq (0.9143\frac{\pi}{a}, 2.914\frac{\pi}{a}) \), which lies between the points \((\frac{\pi}{4a}, \frac{7\pi}{3a})\) and \((\frac{\pi}{a}, \frac{3\pi}{a})\), and then leave it to evolve freely under the lattice potential (2). We find that in this case the atom cloud remains a single one throughout its time evolution. If now the acceleration is kept on for a longer period of time, so that the pulse can go beyond the point \( k = (\frac{\pi}{a}, \frac{3\pi}{a}) \) slightly, one would observe that the two sub-pulses are generated. To confirm that Bragg reflection and Landau–Zener tunnelling do occur at \((\frac{\pi}{a}, \frac{3\pi}{a})\), one could repeat the above procedures but apply now a reduced acceleration along the \((1, 1)\) direction, say, \( g' = \frac{\pi}{20 ma} \). One would find that in this case only one pulse appears, even though an acceleration duration that could have sent the state beyond \( k = (\frac{\pi}{a}, \frac{3\pi}{a}) \) had been applied. What happens here is that the pulse is Bragg reflected from \((\frac{\pi}{a}, \frac{3\pi}{a})\) to the state \((-\frac{\pi}{a}, \frac{3\pi}{a})\) and it then tunnels into the next band under the action of \( g' \). To further endorse our argument, we note that the directions of the group velocities of the wave packets, according to (8), can be read off from the contours in figure 2(b). Hence, the sub-pulse that tunnels through from \( k = (-\frac{\pi}{3 a}, \frac{3 \pi}{3 a}) \) has a group velocity almost parallel to the \((0, 1)\) direction, while the one that tunnels forward from \( k = (\frac{\pi}{a}, \frac{3 \pi}{a}) \) is more free-particle-like (i.e. its real space trajectory bends downwards just like a projectile; see figure 1(b)). From the perspective of energetics, we note that along the \((1, 1)\) direction, tunnelling across the point \((-\frac{\pi}{a}, \frac{3 \pi}{a})\) would send the wave packet into the ninth band (degenerate with the tenth band), while across \((\frac{\pi}{a}, \frac{3 \pi}{a})\) it would bring it into the eleventh band (degenerate with the twelfth band). Taking into account the finite width of the wave packet, there are, in fact, other possible final states for the Bragg reflection [3]. However, they are suppressed here since the optical potential (2) has Fourier components only for \( \Delta k = (0, 0), (\pm 2\pi/a, 0) \) and \( (0, \pm 2\pi/a) \).
one finds that the former option is evidently energetically more favourable. It is also possible to confirm our argument based on Fourier analysis of the atom clouds [30].

Although a more quantitative analysis of the problem than that presented above is possible [17], our approach in the foregoing paragraph has its merits. Besides providing a check for our argument, it also alludes to one important point: in addition to considering condensates with zero (center-of-mass) initial velocity (as was done for the simulations in figure 1), one could also consider an initial wave function with a non-zero wave vector. Our analysis thus brings out a useful means of manipulation of coherent atom waves, as we shall now explain.

3. Manipulation of coherent atom waves

In order to prepare a condensate with non-zero Bloch wave vector \( \mathbf{k}_0 \), one can start from a stationary wave packet with the wave function (5). Without turning on the optical lattice potential, one applies first an acceleration \( g_0 \) (along the \( \mathbf{k}_0 \) direction) to the stationary condensate for the period of time \( \Delta t = \hbar k_0/mg_0 \) and then switches it off. The condensate will now have acquired the designated wave vector \( \mathbf{k}_0 \) and the wave function will be

\[
\Psi(\rho) = \sqrt{N} \exp \left( -\rho^2 / 4\sigma^2 \right) \exp(i \mathbf{k}_0 \cdot \rho).
\]

(11)

To project the state onto the lattice eigenstate with Bloch wave vector \( \mathbf{k}_0 \), one could switch on the optical lattice and allow the state (11) to evolve for a period of time. This would then transfer the condensate into the desired Bloch state. It should be noted that this procedure for state preparation would work better for weak lattice potentials (and \( \mathbf{k}_0 \) a distance away from any energy gaps, of course). For strong lattice potentials, the Gaussian state (11) may deviate enormously from the Bloch state \( \mathbf{k}_0 \). Thus the projection procedure may result in an atom cloud with extremely low density. To overcome this difficulty, one may first project the state onto a weak-potential eigenstate and afterwards increase the lattice strength steadily.

In figure 3, we demonstrate the manipulation of coherent atom waves utilizing the scheme proposed in the previous section. Here we start at \( t = 0 \) from the Gaussian wave packet (11) with \( \sigma = 2.5a, N = 2\pi \sigma^2 \) and \( \mathbf{k}_0 = (\pi / a, 5\pi / a) \). The state is left to evolve in the optical lattice (2) with \( V_0 = 1.0E_R \) for the period of time \( \Delta t = 1.0T_B \) and then switch it off. The condensate is then left to evolve in the lattice potential for another period of time \( \Delta t = 2.0T_B \). As one can see in figure 3, the condensate begins to split at \( t = 2.0T_B \) and two pulses of atom clouds become visible in the plot for \( t = 3.0T_B \). To further split the two pulses, at \( t = 4.0T_B \) we apply the acceleration \(-g\), which is opposite to that applied earlier. We turn off the acceleration after the duration \( \Delta t = 1.0T_B \) and allow the pulses to evolve in the lattice freely. The reverted acceleration brings the two pulses across the beam-splitting points \( \mathbf{k} = (\pm \pi / a, 2\pi / a) \). Therefore, as shown in figure 3, at \( t = 6.0T_B \) each of the two pulses splits again into two further sub-pulses. All these results are consequences of the processes expounded in the previous section. One can see that by exploring Bragg reflection and Landau–Zener tunnelling of the condensate in accelerated 2D optical lattices, we have furnished a beam splitter that may be useful for the manipulation of atom waves. By changing the lattice strength \( V_0 \) and/or the magnitude of the acceleration \( g \), one can tune the relative strengths between the reflected and transmitted pulses. Further manipulations of the optical atom cloud can be achieved by extensive exploration of the energy landscape for the optical
Figure 3. Numerical simulation of an accelerated condensate that starts from the initial wave function (11) with \( k_0 = \left( \frac{\pi}{2a}, \frac{5\pi}{2a} \right) \) in the same 2D optical lattice as in figure 1. An acceleration \( g = \frac{\pi E_R}{4 ma} \) is applied along the (1, 1) direction in a controlled manner, as described in the text. As in figure 1(b), here the time scale \( T_B \) is given by (7) and the plots are made against the rotated coordinates \( x' \) and \( y' \) in units of the lattice constant \( a \). Note that for different time frames we have shifted the coordinates to different ranges for convenience.

4. Conclusions and discussions

The interplay between cold atom physics and solid-state physics has stimulated exciting developments in the past few decades. In this work, we have studied the dynamics of a cloud
of ultracold atoms subject to an accelerated 2D optical square lattice by numerically solving the GP equation in the regime where mean-field interaction is negligible. We have shown that for sufficiently large accelerations the condensate can generate pulses of atom clouds that have structures richer than their 1D counterparts. Using a semiclassical picture, we have demonstrated that these structures can be understood from the $k$-space dynamics of the Bloch state. In particular, Bragg reflection and Landau–Zener tunnelling at higher energy bands are shown to be responsible for the pattern generated.

On the one hand, our result can help envisage 2D atom lasers that have patterned output pulses. These patterns are closely connected to the underlying lattice structures. On the other hand, we have shown that by appropriately controlling the magnitude, direction and duration of the lattice acceleration, one would be able to manipulate the atom wave by exploring the energy landscape of the optical lattice. We have demonstrated that a beam splitter for coherent atom waves can be furnished in this scheme.

An important issue that we have not addressed in this paper is the effect of the nonlinear mean-field interaction in (1). Besides decoherence of the condensate, as discussed in section 2.1 [12, 24], it has been shown that nonlinear effects can cause asymmetric Landau–Zener tunnelling and modulation instability for condensates in accelerated 2D optical lattices [20, 21]. We hope to investigate these problems in future publications. At the same time, the possible applications that ensue from this work in the fields of atom optics and quantum information sciences are also yet to be explored.

Acknowledgments

The authors thank Professors Sungkit Yip, Chung-Yu Mou and Chien-Hua Pao for valuable discussions. This research was supported by the NSC of Taiwan through grant numbers NSC 96-2112-M-194-011-MY3 and NSC 98-2112-M-194-001-MY3. It is also partly supported by the Center for Theoretical Sciences, Taiwan.

References

[1] Bloch I 2008 Science 319 1202
[2] Morsch O and Oberthaler M 2006 Rev. Mod. Phy. 78 179
[3] Ashcroft N W and Mermin N D 1976 Solid State Physics (Singapore: Thomson Learning)
[4] Ziman J M 1972 Principles of the Theory of Solids 2nd edn (Cambridge: Cambridge University Press)
[5] Raizen M, Salomon C and Niu Q 1997 Phys. Today 50(7) 30
[6] Pitaevskii L and Stringari S 2003 Bose–Einstein Condensation (Oxford: Oxford University Press)
[7] Couvert A, Jeppesen M, Kawalec T, Reinaudi G, Mathevet R and Guéry-Odelin D 2008 Europhys. Lett. 83 50001
[8] Ben Dahan M, Peik E, Reichel J, Castin Y and Salomon C 1996 Phys. Rev. Lett. 76 4508
[9] Wilkinson S R, Bharucha C F, Madison K W, Niu Q and Raizen M G 1996 Phys. Rev. Lett. 76 4512
[10] Bharucha C F, Madison K W, Morrow P R, Wilkinson S R, Sundaram B and Raizen M G 1997 Phys. Rev. A 55 R857
[11] Anderson B P and Kasevich M A 1998 Science 282 1686
[12] Anderson B P and Kasevich M A 1999 Proc. International School of Physics ‘Enrico Fermi’, Course CXL ed M Inguscio et al (Amsterdam: IOS Press) pp 439–52
[13] Kolovsky A R and Korsch H J 2003 Phys. Rev. A 67 063601
[14] Witthaut D, Keck F, Korsch H J and Mossmann S 2004 New J. Phys. 6 41
[15] Breid B M, Witthaut D and Korsch H J 2007 New J. Phys. 9 62
[16] Trompeter H, Krolikowski W, Neshev D N, Desyatnikov A S, Sukhorukov A A, Kivshar Y S, Pertsch T, Peschel U and Lederer F 2006 Phys. Rev. Lett. 96 053903
[17] Shchesnovich V S, Cavalcanti S B, Hickmann J M and Kivshar Y S 2006 Phys. Rev. E 74 056602
[18] Shchesnovich V S, Desyatnikov A S and Kivshar Y S 2008 Opt. Express 16 14076
[19] Dreisow F, Szameit A, Heinrich M, Pertsch T, Nolte S and Tünnermann A 2009 Phys. Rev. Lett. 102 076802
[20] Brazhnyi V A, Konotop V V and Kuzmiak V 2006 Phys. Rev. Lett. 96 150402
[21] Brazhnyi V A, Konotop V V, Kuzmiak V and Shchesnovich V S 2007 Phys. Rev. A 76 023608
[22] Greiner M, Bloch I, Mandel O, Hänsch T W and Esslinger T 2001 Phys. Rev. Lett. 87 160405
[23] Jaksch D, Bruder C, Cirac J I, Gardiner C W and Zoller P 1998 Phys. Rev. Lett. 81 3108
[24] Orzel C, Tuchman A K, Fenselau M L, Yasuda M and Kasevich M A 2001 Science 291 2386
[25] Greiner M, Mandel O, Esslinger T, Hänsch T W and Bloch I 2002 Nature 415 39
[26] Inouye S, Andrews M R, Stenger J, Miesner H-J, Stamper-Kurn D M and Ketterle W 1998 Nature 392 151
[27] Chin C, Grimm R, Julienne P and Tiesinga E 2010 Rev. Mod. Phys. 82 1225
[28] Hadzibabic Z, Stock S, Battelier B, Bretin V and Dalibard J 2004 Phys. Rev. Lett. 93 180403
[29] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 2002 Numerical Recipes in C++: The Art of Scientific Computing 2nd edn (Cambridge: Cambridge University Press)
[30] Ting W C, Han D J and Wu S T 2010 in preparation