On 3-Lie algebras with a derivation

Shuangjian Guo\textsuperscript{1}, Ripan Saha\textsuperscript{2}\textsuperscript{*}

1. School of Mathematics and Statistics, Guizhou University of Finance and Economics
   Guiyang 550025, P. R. of China
2. Department of Mathematics, Raiganj University
   Raiganj, 733134, West Bengal, India

ABSTRACT

In this paper, we study 3-Lie algebras with derivations. We call the pair consisting of a 3-Lie algebra and a distinguished derivation by the 3-LieDer pair. We define a cohomology theory for 3-LieDer pair with coefficients in a representation. We study central extensions of a 3-LieDer pair and show that central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation. We generalize Gerstenhaber’s formal deformation theory to 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivation.

Key words: 3-Lie algebra, derivation, representation, cohomology, central extension, deformation.

2020 MSC: 17A42, 17B10, 17B40, 17B56

1 Introduction

3-Lie algebras are special types of \( n \)-Lie algebras and have close relationships with many important fields in mathematics and mathematical physics\textsuperscript{[4, 5]}. The structure of 3-Lie algebras is closely linked to the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident \( M2 \)-branes and is applied to the study of the Bagger-Lambert theory. Moreover, the \( n \)-Jacobi identity can be regarded as a generalized Plucker relation in the physics literature. In particular, the metric 3-Lie algebras, or more generally, the 3-Lie algebras with invariant symmetric bilinear forms attract even more attention in physics. Recently, many more properties and structures of 3-Lie algebras have been developed, see \textsuperscript{[6, 7, 13, 18, 21, 25, 26]} and references cited therein.

\textsuperscript{*}Corresponding author: ripanjumaths@gmail.com
Derivations of types of algebra provide many important aspects of the algebraic structure. For example, Coll, Gertstenhaber, and Giaquinto \cite{9} described explicitly a deformation formula for algebras whose Lie algebra of derivations contains the unique non-abelian Lie algebra of dimension two. Amitsur \cite{1, 2} studied derivations of central simple algebras. Derivations are also used to construct homotopy Lie algebras \cite{24} and play an important role in the study of differential Galois theory \cite{19}. One may also look at some interesting roles played by derivations in control theory and gauge theory in quantum field theory \cite{3}. In \cite{12}, the authors studied algebras with derivations from an operadic point of view. Recently, Lie algebras with derivations (called LieDer pairs) are studied from a cohomological point of view \cite{23} and extensions, deformations of LieDer pairs are considered. The results of \cite{23} have been extended to associative algebras and Leibniz algebras with derivations in \cite{10} and \cite{11}.

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. The deformation theory was introduced by Gerstenhaber for rings and algebras \cite{14, 15}, and by Zhang for 3-Lie color algebras \cite{26}. They studied 1-parameter formal deformations and established the connection between the cohomology groups and infinitesimal deformations. Motivated by Tang’s \cite{23} terminology of LieDer pairs. Due to the importance of 3-Lie algebras, cohomology, and deformation theories, Our main objective of this paper is to study the cohomology and deformation theory of 3-Lie algebra with a derivation.

The paper is organized as follows. In Section 2, we define a cohomology theory for 3-LieDer pair with coefficients in a representation. In Section 3, we study central extensions of a 3-LieDer pair and show that isomorphic classes of central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation. In Section 4, we study formal one-parameter deformations of 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivations.

Throughout this paper, we work over the field $\mathbb{F}$ of characteristics 0.

\section{Cohomology of 3-LieDer pairs}

In this section, we define a cohomology theory for 3-LieDer pair with coefficients in a representation.

\textbf{Definition 2.1.} (\cite{10}) A 3-Lie algebra is a tuple $(L, [\cdot, \cdot, \cdot])$ consisting of a vector space $L$, a 3-ary skew-symmetric operation $[\cdot, \cdot, \cdot] : \wedge^3 L \to L$ satisfying the following Jacobi identity

$$
[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]],
$$

(2. 1)

for any $x, y, u, v, w \in L$. 

2
Definition 2.2. ([17]) A representation of a 3-Lie algebra \((L, [\cdot, \cdot, \cdot])\) on the vector space \(M\) is a linear map \(\rho : L \wedge L \to \mathfrak{gl}(M)\), such that for any \(x, y, z, u \in L\), the following equalities are satisfied

\[
\rho([x, y, z], u) = \rho(y, z)\rho(x, u) + \rho(z, x)\rho(y, u) + \rho(x, y)\rho(z, u),
\]
\[
\rho(x, y)\rho(z, u) = \rho(z, u)\rho(x, y) + \rho([x, y, z], u) + \rho(z, [x, y, u]).
\]

Then \((M, \rho)\) is called a representation of \(L\), or \(M\) is an \(L\)-module.

Definition 2.3. ([16]) Let \((L, [\cdot, \cdot, \cdot])\) be a 3-Lie algebra. A derivation on \(L\) is given by a linear map \(\phi : L \to L\) satisfying

\[
\phi_L([x, y, z]) = [\phi_L(x), y, z] + [x, \phi_L(y), z] + [x, y, \phi_L(z)], \quad \forall x, y, z \in L.
\]

We call the pair \((L, \phi_L)\) of a 3-Lie algebra and a derivation by a 3-LieDer pair.

Remark 2.4. Let \((L, [\cdot, \cdot, \cdot])\) be a 3-Lie algebra. For all \(x_1, x_2 \in L\), the map defined by

\[
ad_{x_1, x_2} x := [x_1, x_2, x], \quad \text{for all } x \in L,
\]
is called the adjoint map. From the Equation 2.1 it is clear that \(ad_{x_1, x_2}\) is a derivation. The linear map \(ad : L \wedge L \to \mathfrak{gl}(L)\) defines a representation of \((L, [\cdot, \cdot, \cdot])\) on itself. This representation is called the adjoint representation.

Definition 2.5. Let \((L, \phi_L)\) be a 3-LieDer pair. A representation of \((L, \phi_L)\) is given by \((M, \phi_M)\) in which \(M\) is a representation of \(L\) and \(\phi_M : M \to M\) is a linear map satisfying

\[
\phi_M(\rho(x, y)(m)) = \rho(\phi_L(x), y)(m) + \rho(x, \phi_L(y))(m) + \rho(x, y)(\phi_M(m)),
\]
for all \(x, y \in L\) and \(m \in M\).

Proposition 2.6. Let \((L, \phi_L)\) be a 3-LieDer pair and \((M, \phi_M)\) be a representation of it. Then \((L \oplus M, \phi_L \oplus \phi_M)\) is a 3-LieDer pair where the 3-Lie algebra bracket on \(L \oplus M\) is given by the semi-direct product

\[
[(x, m), (y, n), (z, p)] = ([x, y, z], \rho(y, z)(m) + \rho(z, x)(n) + \rho(x, y)(p)),
\]
for any \(x, y, z \in L\) and \(m, n, p \in M\).

**Proof.** It is known that \(L \oplus M\) equipped with the above product is a 3-Lie algebra.
Moreover, we have
\[
(\phi_L \oplus \phi_M)([(x, m), (y, n), (z, p)]) \\
= (\phi_L([x, y, z]), \phi_M(\rho(y, z)(m)) + \phi_M(\rho(z, x)(n)) + \phi_M(\rho(x, y)(p))) \\
= ([\phi_l(x), y, z], \rho(y, z)(\phi_M(m)) + \rho(\phi_L(x), z)(n) + \rho(\phi_L(x), y)(p)) \\
+ ([x, \phi_L(y), z], \rho(\phi_L(y), z)(m) + \rho(z, x)(\phi_M(n)) + \rho(x, \phi_L(y))(p)) \\
+ ([x, y, \phi_T(z)], \rho(y, \phi_L(z))(m) + \rho(z, \phi_L(x))(n) + \rho(x, y)(\phi_M(p))) \\
= [(\phi_L \oplus \phi_M)(x, m), (y, n), (z, p)] + [(x, m), (\phi_L \oplus \phi_M)(y, n), (z, p)] \\
+ [(x, m), (y, n), (\phi_L \oplus \phi_M)(z, p)].
\]

Hence the proof is finished. \(\square\)

Recall from [22] that let \(\rho\) be a representation of \((L, [\cdot, \cdot, \cdot])\) on \(M\). Denote by \(C^n(L, M)\) the set of all \(n\)-cochains and defined as
\[
C^n(L, M) = \text{Hom}((\wedge^2 L)^{\otimes n}, M), \quad n \geq 1.
\]

Let \(d^n : C^n(L, M) \to C^{n+1}(L, M)\) be defined by
\[
d^n f(X_1, \ldots, X_n, x_{n+1}) \\
= (-1)^{n+1} \rho(y_n, x_{n+1}) f(X_1, \ldots, X_{n-1}, x_n) \\
+ (-1)^{n+1} \rho(x_{n+1}, x_n) f(X_1, \ldots, X_{n-1}, y_n) \\
+ \sum_{j=1}^n (-1)^{j+1} \rho(x_j, y_j) f(X_1, \ldots, \hat{X}_j, \ldots, X_n, x_{n+1}) \\
+ \sum_{j=1}^n (-1)^j f(X_1, \ldots, \hat{X}_j, \ldots, X_n, [x_j, y_j, x_{n+1}]), \\
+ \sum_{1 \leq j < k \leq n} (-1)^j f(X_1, \ldots, \hat{X}_j, \ldots, X_{k-1}, [x_j, y_j, x_k] \wedge y_k) \\
+ x_k \wedge [x_j, y_j, x_k], X_{k+1}, \ldots, X_n, x_{n+1},
\]
for all \(X_i = x_i \wedge y_i \in \otimes^2 L, i = 1, 2, \ldots, n\) and \(x_{n+1} \in L\), it was proved that \(d^{n+1} \circ d^n = 0\). Therefore, \((C^*(L, M), d^*)\) is a cochain complex.

Observe that for trivial representation coboundary maps \(d^1\) and \(d^2\) are explicitly given as follows:
\[
d^1(f)(a, b, c) = [f(a), b, c] + [a, f(b), c] + [a, b, f(c)] - f([a, b, c]), \quad f \in C^1(L, M).
\]
\[
d^2(f)(a, b, c, d, e) = [a, b, f(c, d, e)] - f([a, b, c], d, e) + f(a, b, [c, d, e]) - f([a, b, c], d, e), \quad f \in C^2(L, M).
\]

In [20], the graded space \(C^*(L, L) = \bigoplus_{n \geq 0} C^{n+1}(L, L)\) of cochain groups carries a degree -1 graded Lie bracket given by \([f, g] = f \circ g - (-1)^{mn} g \circ f\), for \(f \in C^{m+1}(L, L), g \in C^m(L, L)\).
\[ C^{n+1}(L, L), \] where \( f \circ g \in C^{m+n+1}(L, L) \), and defined as follows:

\[
f \circ g(X_1, \ldots, X_{m+n}, x) = \sum_{k=1}^{m} (-1)^{(k-1)n} \sum_{\sigma \in S(k-1, n)} f(X_{\sigma(1)}, \ldots, X_{\sigma(k-1)}, g(X_{\sigma(k)}, \ldots, X_{\sigma(k+n-1)}), x_{k+n}) \\
\wedge y_{k+n}, X_{\sigma(k+n+1)}, \ldots, X_{\sigma(m+n)}, x) + \sum_{k=1}^{m} (-1)^{(k-1)n} \sum_{\sigma \in S(k-1, n)} (-1)^{\sigma} f(X_{\sigma(1)}, \ldots, X_{\sigma(k-1)}, x_{k+n}) \\
\wedge g(X_{\sigma(k)}, \ldots, X_{\sigma(k+n-1)}, y_{k+n}), X_{k+n+1}, \ldots, X_{m+n}, x) \sum_{\sigma \in S(m, n)} (-1)^{\sigma n} (-1)^{\sigma} f(X_{\sigma(1)}, \ldots, X_{\sigma(m)}), g(X_{\sigma(m+1)}, \ldots, X_{\sigma(m+n-1)}, X_{\sigma(m+n)}, x) \),
\]

for all \( X_i = x_i \wedge y_i \in \otimes^2 L, i = 1, 2, \ldots, m + n \) and \( x \in L \). Here \( S(k-1, n) \) denotes the set of all \((k-1, n)\)-shuffles. Moreover, \( \mu : \otimes^3 L \to L \) is a 3-Lie bracket if and only if \([\mu, \mu]\) = 0, i.e., \( \mu \) is a Maurer-Cartan element of the graded Lie algebra \((C^*(L, L), [\cdot, \cdot])\). where \( \mu \) is considered as an element in \( C^2(L, L) \). With this notation, the differential (with coefficients in \( L \)) is given by

\[
df = (-1)^n[\mu, f], \quad \text{for all } f \in C^n(L, L).
\]

In the next, we introduce cohomology for a 3-LieDer pair with coefficients in a representation.

Let \((L, \phi_L)\) be a 3-LieDer pair and \((M, \phi_M)\) be a representation of it. For any \( n \geq 2 \), we define cochain groups for 3-LieDer pair as follows:

\[
C^n_{3\text{-LieDer}}(L, M) := C^n(L, M) \oplus C^{n-1}(L, M).
\]

Define the space \( C^0_{3\text{-LieDer}}(L, M) \) of 0-cochains to be 0 and the space \( C^1_{3\text{-LieDer}}(L, M) \) of 1-cochains to be \( \text{Hom}(L, M) \). Note that \( \mu = [\cdot, \cdot, \cdot] \in C^2(L, L) \) and derivation \( \phi_L \in C^1(L, L) \). Thus, the pair \((\mu, \phi_L)\) \( \in C^2_{3\text{-LieDer}}(L, L) \). To define the coboundary map for 3-LieDer pair, we need following map \( \delta : C^n(L, M) \to C^n(L, M) \) by

\[
\delta f = \sum_{i=1}^{n} f \circ (Id_L \otimes \cdots \otimes \phi_L \otimes \cdots \otimes Id_L) - \phi_M \circ f.
\]

The following lemma shows maps \( \partial \) and \( \delta \) commute, and is useful to define the coboundary operator of the cohomology of 3-LieDer pair.

**Lemma 2.7.** The map \( \delta \) commute with \( d \), i.e., \( d \circ \delta = \delta \circ d \).

**Proof.** Note that in case of self representation, that is, when \((M, \phi_M) = (L, \phi_L)\), we have

\[
\delta(f) = -[\phi_L, f], \quad \text{for all } f \in C^n(L, L).
\]
Therefore, we have
\[(d \circ \delta)(f) = -d[\phi_L, f] = (-1)^n[\mu, [\phi_L, f]] = (-1)^n[[\mu, \phi_L], f] + (-1)^n[\phi_L, [\mu, f]] = (-1)^n[\phi_L, [\mu, f]] = (\delta \circ \delta)(f)\]

We are now in a position to define the cohomology of the 3-LieDer pair. We define a map \(\partial : C^n_{3\text{-LieDer}}(L, M) \rightarrow C^{n+1}_{3\text{-LieDer}}(L, M)\) by
\[
\partial f = (df, -\delta f), \quad \text{for all } f \in C^1_{3\text{-LieDer}}(L, M),
\]
\[
\partial (f_n, \overline{f}_n) = (df_n, df_n + (-1)^n\delta f_n), \quad \text{for all } (f_n, \overline{f}_n) \in C^n_{3\text{-LieDer}}(L, M).
\]

**Proposition 2.8.** The map \(\partial\) satisfies \(\partial \circ \partial = 0\).

**Proof.** For any \(f \in C^1_{3\text{-LieDer}}(L, M)\), we have
\[
(\partial \circ \partial)f = \partial(df, -\delta f) = ((d \circ d)f, -(d \circ \delta)f + (\delta \circ d)f) = 0.
\]
Similarly, for any \((f_n, \overline{f}_n) \in C^n_{3\text{-LieDer}}(L, M)\), we have
\[
(\partial \circ \partial)(f_n, \overline{f}_n) = \partial(df_n, df_n + (-1)^n f_n) = (d^2f_n, d^2\overline{f}_n + (-1)^n d\delta f_n + (-1)^n1\delta df_n) = 0.
\]
Hence the proof is finished. \(\square\)

Therefore, \((C^*_{3\text{-LieDer}}(L, M), \partial)\) forms a cochain complex. We denote the corresponding cohomology groups by \(H^*_\text{3-LieDer}(L, M)\).

### 3 Central extensions of 3-LieDer pairs

In this section, we study central extensions of a 3-LieDer pair. Similar to the classical cases, we show that isomorphic classes of central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation.

Let \((L, \phi_L)\) be a 3-LieDer pair and \((M, \phi_M)\) be an abelian 3-LieDer pair i.e, the 3-Lie algebra bracket of \(M\) is trivial.
Definition 3.1. A central extension of \((L, \phi_L)\) by \((M, \phi_M)\) is an exact sequence of 3-LieDer pairs

\[
0 \rightarrow (M, \phi_M) \xrightarrow{i} (\hat{L}, \phi_{\hat{L}}) \xrightarrow{p} (L, \phi_L) \rightarrow 0 \tag{3.1}
\]

such that \([i(m), \hat{x}, \hat{y}] = 0\), for all \(m \in M\) and \(\hat{x}, \hat{y} \in \hat{L}\).

In a central extension, using the map \(i\) we can identify \(M\) with the corresponding subalgebra of \(\hat{L}\) and with this \(\phi_M = \phi_{\hat{L}}|_M\).

Definition 3.2. Two central extensions \((\hat{L}, \phi_{\hat{L}})\) and \((\hat{L}', \phi_{\hat{L}}')\) are said to be isomorphic if there is an isomorphism \(\eta : (\hat{L}, \phi_{\hat{L}}) \rightarrow (\hat{L}', \phi_{\hat{L}}')\) of 3-LieDer pairs that makes the following diagram commutative

\[
\begin{array}{ccc}
0 & \rightarrow & (M, \phi_M) \\
\downarrow{Id_M} & & \downarrow{\eta} \\
0 & \rightarrow & (\hat{L}', \phi_{\hat{L}}')
\end{array}
\quad
\begin{array}{ccc}
(\hat{L}, \phi_{\hat{L}}) & \xrightarrow{p} & (L, \phi_L) \\
\downarrow{Id_L} & & \downarrow{q} \\
\hat{L}' & \rightarrow & (L, \phi_L) \\
\rightarrow & 0 & \rightarrow
\end{array}
\]

Let Eq. (3.1) be a central extension of \((L, \phi_L)\). A section of the map \(p\) is given by a linear map \(s : L \rightarrow \hat{L}\) such that \(p \circ s = Id_L\).

For any section \(s\), we define linear maps \(\psi : L \wedge L \wedge L \rightarrow M\) and \(\chi : L \rightarrow M\) by

\[
\psi(x, y, z) := [s(x), s(y), s(z)] - s([x, y, z]), \quad \chi(x) = \phi_{\hat{L}}(s(x)) - s(\phi_L(x)), \quad \text{for all } x, y, z \in L.
\]

Note that the vector space \(\hat{L}\) is isomorphic to the direct sum \(L \oplus M\) via the section \(s\). Therefore, we may transfer the structures of \(\hat{L}\) to \(L \oplus M\). The product and linear maps on \(L \oplus M\) are given by

\[
[(x, m), (y, n), (z, p)]_{\psi} = ([x, y, z], \psi(x, y, z)), \\
\phi_{L \oplus M}(x, m) = (\phi_L(x), \phi_M(m) + \chi(x)).
\]

Proposition 3.3. The vector space \(L \oplus M\) equipped with the above product and linear maps \(\phi_{L \oplus M}\) forms a 3-LieDer pair if and only if \((\psi, \chi)\) is a 2-cocycle in the cohomology of the 3-LieDer pair \((L, \phi_L)\) with coefficients in the trivial representation \(M\). Moreover, the cohomology class of \((\psi, \chi)\) does not depend on the choice of the section \(s\).

Proof. The tuple \((L \oplus M, \phi_{L \oplus M})\) is a 3-LieDer pair if and only if the following
Theorem 3.4. Let \((L, \phi_L)\) be a 3-LieDer pair and \((M, \phi_M)\) be an abelian 3-LieDer pair. Then the isomorphism classes of central extensions of \(L\) by \(M\) are classified by the second cohomology group \(H^2_{3\text{-LieDer}}(L, M)\).
Proof. Let \((\hat{L}, \phi_{\hat{L}})\) and \((\hat{L}', \phi_{\hat{L}'})\) be two isomorphic central extensions and the isomorphism is given by \(\eta: \hat{L} \to \hat{L}'\). Let \(s: L \to \hat{L}\) be a section of \(p\). Then
\[
p' \circ (\eta \circ s) = (p' \circ \eta) \circ s = p \circ s = Id_L.
\]
This shows that \(s' := \eta \circ s\) is a section of \(p'\). Since \(\eta\) is a morphism of 3-LieDer pairs, we have \(\eta|_M = Id_M\). Thus,
\[
\psi'(x, y, z) = [s'(x), s'(y), s'(z)] - s'([x, y, z])
= \eta([s(x), s(y), s(z)] - [x, y, z])
= \psi(x, y, z),
\]
and
\[
\chi'(x) = \phi_{\hat{L}'}(s'(x)) - s'(\phi_L(x))
= \phi_{\hat{L}'}(\eta \circ s(x)) - \eta \circ s(\phi_L(x))
= \phi_{\hat{L}}(s(x)) - s(\phi_L(x))
= \chi(x).
\]
Therefore, isomorphic central extensions give rise to the same 2-cocycle, hence, correspond to the same element in \(H^2_{3\text{-LieDer}}(L, M)\).

Conversely, let \((\psi, \chi)\) and \((\psi', \chi')\) be two cohomologous 2-cocycles. Therefore, there exists a map \(v: L \to M\) such that
\[
(\psi, \chi) - (\psi', \chi') = \partial v.
\]
The 3-LieDer pair structures on \(L \oplus M\) corresponding to the above 2-cocycles are isomorphic via the map \(\eta: L \oplus M \to L \oplus M\) given by \(\eta(x, m) = (x, m + v(x))\). This proves our theorem. \(\square\)

4 Extensions of a pair of derivations

It is well-known that derivations are infinitesimals of automorphisms, and a study \([\mathbb{S}]\) has been done on extensions of a pair of automorphisms of Lie-algebras. In this section, we study extensions of a pair of derivations and see how it is related to the cohomology of the 3-LieDer pair.

Let
\[
0 \longrightarrow M \xrightarrow{i} \hat{L} \xrightarrow{p} L \longrightarrow 0 \tag{4.1}
\]
be a fixed central extensions of 3-Lie algebras. Given a pair of derivations \((\phi_L, \phi_M) \in \text{Der}(L) \times \text{Der}(M)\), here we study extensions of them to a derivation \(\phi_L \in \text{Der}(\hat{L})\) which makes

\[
0 \xrightarrow{} (M, \phi_M) \xrightarrow{i} (\hat{L}, \phi_{\hat{L}}) \xrightarrow{p} (L, \phi_L) \xrightarrow{} 0
\]

(4.2)

into an exact sequence of 3-LieDer pairs. In such a case, the pair \((\phi_L, \phi_M) \in \text{Der}(L) \times \text{Der}(M)\) is said to be extensible.

Let \(s : L \to \hat{L}\) be a section of \(\text{Eq.}(4.1)\), we define a map \(\psi : L \otimes L \otimes L \to M\) by

\[
\psi(x, y, z) := [s(x), s(y), s(z)] - s([x, y, z]), \quad \chi(x) = \phi_L(s(x)) - s(\phi_L(x)), \quad \forall x, y, z \in L.
\]

Given a pair of derivations \((\phi_L, \phi_M) \in \text{Der}(L) \times \text{Der}(M)\), we define another map \(\text{Ob}_{(\phi_L, \phi_M)}^L : L \otimes L \otimes L \to M\) by

\[
\text{Ob}_{(\phi_L, \phi_M)}^L(x, y, z) := \phi_M(\psi(x, y, z)) - \psi(\phi_L(x), y, z) - \psi(x, \phi_L(y), z) - \psi(x, y, \phi_L(z)).
\]

**Proposition 4.1.** The map \(\text{Ob}_{(\phi_L, \phi_M)}^L : L \otimes L \otimes L \to M\) is a 2-cocycle in the cohomology of the 3-Lie algebra \(L\) with coefficients in the trivial representation \(\alpha\). Moreover, the cohomology class \(\{\text{Ob}_{(\phi_L, \phi_M)}^L\}\) \(\in H^2(L, M)\) does not depend on the choice of sections.

**Proof.** First observe that \(\psi\) is a 1-cocycle in the cohomology of the 3-Lie algebra \(L\) with coefficients in the trivial representation \(M\). Thus, we have

\[
(d\text{Ob}_{(\phi_L, \phi_M)}^M)(x, y, u, v, w) = -\text{Ob}_{(\phi_L, \phi_M)}^M(x, y, [u, v, w]) + \text{Ob}_{(\phi_L, \phi_M)}^M([x, y, u], v, w) + \text{Ob}_{(\phi_L, \phi_M)}^M(u, [x, y, v], w) + \text{Ob}_{(\phi_L, \phi_M)}^M(u, v, [x, y, w])
\]

\[
\begin{align*}
&= -\phi_M(\psi(x, y, [u, v, w])) + \psi(\phi_L(x), y, [u, v, w]) + \psi(x, \phi_L(y), [u, v, w]) + \psi(x, y, \phi_L([u, v, w])) + \phi_M(\psi([x, y, u], v, w)) - \psi(\phi_L([x, y, u]), v, w) - \psi([x, y, u], \phi_L(v), w) - \psi(x, \phi_L([u, y, v], w) + \phi_M(\psi(u, [x, y, v], w)) - \psi(x, y, \phi_L(u), v, [x, y, w]) - \psi(u, \phi_L(v), [x, y, w]) - \psi(u, v, \phi_L([x, y, w]))
\end{align*}
\]

\[
= \psi(\phi_L(x), y, [u, v, w]) + \psi(x, \phi_L(y), [u, v, w]) + \psi(x, y, \phi_L([u, v, w])) - \psi(\phi_L([x, y, u], v, w)) - \psi([x, y, u], \phi_L(v), w) - \psi([x, y, u], \phi_L([x, y, v], w)) - \psi(u, \phi_L([x, y, v], w)) - \psi(u, v, \phi_L([x, y, w])) = 0.
\]
Therefore, $Ob_{(\phi_L,\phi_M)}^L$ is a 2-cocycle. To prove the second part, let $s_1$ and $s_2$ be two sections of Eq.(4.1). Consider the map $u : L \rightarrow M$ given by $u(x) := s_1(x) - s_2(x)$. Then

$$\psi_1(x, y, z) = \psi_2(x, y, z) - u[x, y, z].$$

If $\text{Ob}_{(\phi_L,\phi_M)}^L$ and $\text{Ob}_{(\phi_L,\phi_M)}^M$ denote the one cocycles corresponding to the sections $s_1$ and $s_2$, then

$$\text{Ob}_{(\phi_L,\phi_M)}^L(x, y, z) = \phi_M(\psi_1(x, y, z)) - \psi_1(x, \phi_L(y), z) - \psi_1(x, y, \phi_L(z))$$

$$= \phi_M(\psi_2(x, y, z)) - \phi_M(u(x, y, z)) - \psi_2(x, \phi_L(y), z) + \psi_2(x, y, \phi_L(z)) + u(x, y, \phi_L(z))$$

$$= 2\text{Ob}_{(\phi_L,\phi_M)}^M(x, y, z) + d(\phi_M \circ u - u \circ \phi_L)(x, y, z).$$

This shows that the 2-cocycles $\text{Ob}_{(\phi_L,\phi_M)}^L$ and $\text{Ob}_{(\phi_L,\phi_M)}^M$ are cohomologous. Hence they correspond to the same cohomology class in $\in H^2(L, M)$.

The cohomology class $[\text{Ob}_{(\phi_L,\phi_M)}^L] \in H^2(L, M)$ is called the obstruction class to extend the pair of derivations $(\phi_L, \phi_M)$.

**Theorem 4.2.** Let Eq.(4.1) be a central extension of 3-Lie algebras. A pair of derivations $(\phi_L, \phi_M) \in \text{Der}(L) \times \text{Der}(M)$ is extensible if and only if the obstruction class $[\text{Ob}_{(\phi_L,\phi_M)}^L] \in H^2(L, M)$ is trivial.

**Proof.** Suppose there exists a derivations $\phi_L \in \text{Der}(\hat{L})$ such that Eq. (4.2) is an exact sequence of 3-LieDer pairs. For any $x \in L$, we observe that $p(\phi_L(s(x)) - s(\phi_L(x))) = 0$. Hence $\phi_L(s(x)) - s(\phi_L(x)) \in \ker(p) = im(i)$. We define $\lambda : L \rightarrow M$ by

$$\lambda(x) = \phi_L(s(x)) - s(\phi_L(x)).$$

For any $s(x) + a \in \hat{L}$, we have

$$\phi_L(s(x) + a) = s(\phi_L(x)) + \lambda(x) + \phi_L(a).$$

Since $\phi_L$ is a derivation, for any $s(x) + a, s(y) + b \in \hat{L}$, we have

$$\phi_M(\psi(x, y, z)) - \psi(\phi_L(x), y, z) - \psi(x, \phi_L(y), z) - \psi(x, y, \phi_L(z)) = -\lambda([x, y, z]),$$

or, equivalently, $\text{Ob}_{(\phi_L,\phi_M)}^L = \partial \lambda$ is a coboundary. Hence the obstruction class $[\text{Ob}_{(\phi_L,\phi_M)}^L] \in H^2(L, M)$ is trivial.

To prove the converse part, suppose $\text{Ob}_{(\phi_L,\phi_M)}^L = \partial \lambda$. We define a map $\phi_L : \hat{L} \rightarrow \hat{L}$ by

$$\phi_L(s(x) + a) = s(\phi_L(x)) + \lambda(x) + \phi_L(a).$$

Then $\phi_L$ is a derivation on $\hat{L}$ and Eq. (4.2) is an exact sequence of 3-LieDer pairs. Hence the pair $(\phi_L, \phi_M)$ is extensible. Thus, we obtain the following.
Theorem 4.3. If $H^2(L, M) = 0$, then any pair of derivations $(\phi_L, \phi_M) \in \text{Der}(L) \times \text{Der}(M)$ is extensible.

5 Formal deformations of 3-LieDer pairs

In this section, we study one-parameter formal deformations of 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivations.

Let $(L, \phi_L)$ be a 3-LieDer pair. We denote the 3-Lie bracket on $L$ by $\mu$, i.e., $\mu(x, y, z) = [x, y, z]$, for all $x, y, z \in L$. Consider the space $L[[t]]$ of formal power series in $t$ with coefficients from $L$. Then $L[[t]]$ is a $\mathbb{F}[[t]]$-module.

A formal one-parameter deformation of the 3-LieDer pair $(L, \phi_L)$ consists of formal power series

$$
\mu_t = \sum_{i=0}^{\infty} t^i \mu_i \in \text{Hom}(L \otimes^3 L)[[t]] \text{ with } \mu_0 = \mu,
$$

$$
\phi_t = \sum_{i=0}^{\infty} t^i \phi_i \in \text{Hom}(L, L)[[t]] \text{ with } \phi_0 = \phi_L,
$$

such that $L[[t]]$ together with the bracket $\mu_t$ forms a 3-Lie algebra over $\mathbb{F}[[t]]$ and $\phi_t$ is a derivation on $L[[t]]$.

Therefore, in a formal one-parameter deformation of 3-LieDer pair, the following relations hold:

$$
\mu_t(x, y, \mu_t(z, v, w)) = \mu_t(\mu_t(x, y, z), v, w) + \mu_t(z, \mu_t(x, y, v), w) + \mu_t(z, v, \mu_t(x, y, w)), \quad (5.1)
$$

$$
\phi_t(\mu_t(x, y, z)) = \mu_t(\phi_t(x), y, z) + \mu_t(x, \phi_t(y), z) + \mu_t(x, y, \phi_t(z)). \quad (5.2)
$$

Conditions Eqs.(5.1)-5.2 are equivalent to the following equations:

$$
\sum_{i+j=n} \mu_i(x, y, \mu_j(z, v, w)) \quad (5.3)
$$

$$
= \sum_{i+j=n} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)),
$$

and,

$$
\sum_{i+j=n} \phi_i(\mu_j(x, y, z)) \quad (5.4)
$$

$$
= \sum_{i+j=n} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)).
$$
For $n = 0$ we simply get $(L, \phi_L)$ is a 3-LieDer pair. For $n = 1$, we have

\[
\begin{align*}
\mu_1(x, y, [z, v, w]) &+ [x, y, \mu_1(z, v, w)] \\
\mu_1([x, y, z], v, w) &+ [\mu_1(x, y, z), v, w] + [z, \mu_1(x, y, v), w] \\
+ \mu_1(z, [x, y, v], w) &+ [z, v, \mu_1(x, y, w)] + \mu_1(z, [x, y, w]),
\end{align*}
\tag{5.5}
\]

and,

\[
\begin{align*}
\phi_1([x, y, z]) &+ \phi_L(\mu_1(x, y, z)) \\
\mu_1(\phi_L(x), y, z) &+ [\phi_1(x), y, z] + \mu_1(x, \phi_L(y), z) + [x, \phi_1(y), z] \\
+ \mu_1(x, y, \phi_L(z)) &+ [x, y, \phi_1(z)].
\end{align*}
\tag{5.6}
\]

The condition Eq.(5.5) is equivalent to $d(\mu_1) = 0$ whereas the condition Eq.(5.6) is equivalent to $d(\phi_1) + \delta(\mu_1) = 0$. Therefore, we have

\[\partial(\mu_1, \phi_1) = 0.\]

**Definition 5.1.** Let $(\mu_t, \phi_t)$ be a one-parameter formal deformation of 3-LieDer pair $(L, \phi_L)$. Suppose $(\mu_n, \phi_n)$ is the first non-zero term of $(\mu_t, \phi_t)$ after $(\mu_0, \phi_0)$, then such $(\mu_n, \phi_n)$ is called the infinitesimal of the deformation of $(L, \phi_L)$.

Hence, from the above observations, we have the following proposition.

**Proposition 5.2.** Let $(\mu_t, \phi_t)$ be a formal one-parameter deformation of a 3-LieDer pair $(L, \phi_L)$. Then the linear term $(\mu_1, \phi_1)$ is a 1-cocycle in the cohomology of the 3-LieDer pair $L$ with coefficients in itself.

**Proof.** We have showed that

\[\partial(\mu_1, \phi_1) = 0.\]

If $(\mu_1, \phi_1)$ be the first non-zero term, then we are done. If $(\mu_n, \phi_n)$ be the first non-zero term after $(\mu_0, \phi_0)$, then exactly the same way, one can show that

\[\partial(\mu_n, \phi_n) = 0.\]

□

Next, we define a notion of equivalence between formal deformations of 3-LieDer pairs.

**Definition 5.3.** Two deformations $(\mu_t, \phi_t)$ and $(\mu'_t, \phi'_t)$ of a 3-LieDer pair $(L, \phi_L)$ are said to be equivalent if there exists a formal isomorphism $\Phi_t = \sum_{i=0}^{\infty} t^i \phi_i : L[[t]] \to L[[t]]$ with $\Phi_0 = Id_L$ such that

\[
\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \quad \Phi_t \circ \phi_t = \phi'_t \circ \Phi_t.
\]
By comparing coefficients of $t^n$ from both the sides, we have
\[
\sum_{i+j=n} \phi_i \circ \mu_j = \sum_{p+q+r+l=n} \mu'_p \circ (\phi_q \otimes \phi_r \otimes \phi_l),
\]
\[
\sum_{i+j=n} \phi'_i \circ \phi_j = \sum_{p+q=n} \phi_p \circ \phi_q.
\]

Easy to see that the above identities hold for $n = 0$. For $n = 1$, we get
\[
\mu_1 + \phi_1 \circ \mu = \mu'_1 + \mu \circ (\phi_1 \otimes Id \otimes Id) + \mu \circ (Id \otimes Id \otimes \phi_1), \quad (5.7)
\]
\[
\phi_L \circ \Phi_1 + \phi'_1 = \phi_1 + \phi_1 \circ \phi_L. \quad (5.8)
\]

These two identities together imply that
\[
(\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial \phi_1.
\]

Thus, we have the following.

**Proposition 5.4.** The infinitesimals corresponding to equivalent deformations of the 3-LieDer pair $(L, \phi_L)$ are cohomologous.

**Definition 5.5.** A deformation $(\mu_t, \phi_t)$ of a 3-LieDer pair is said to be trivial if it is equivalent to the undeformed deformation $(\mu'_t = \mu, \phi'_t = \phi_L)$.

**Definition 5.6.** A 3-LieDer pair $(L, \phi_L)$ is called rigid, if every 1-parameter formal deformation $\mu_t$ is equivalent to the trivial deformation.

**Theorem 5.7.** Every formal deformation of the 3-LieDer pair $(L, \phi_L)$ is rigid if the second cohomology group of the 3-LieDer pair vanishes, that is, $H^2_{3-LieDer}(L, L) = 0$.

**Proof.** Let $(\mu_t, \phi_t)$ be a deformation of the 3-LieDer pair $(L, \phi_L)$. From the Proposition 5.2, the linear term $(\mu_1, \phi_1)$ is a 2-cocycle. Therefore, $(\mu_1, \phi_1) = \partial \Phi_1$ for some $\phi_1 \in C^2_{3-LieDer}(L, L) = \text{Hom}(L, L)$.

We set $\Phi_t = Id_L + t\Phi_1 : L[[t]] \to L[[t]]$ and define
\[
\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \quad \phi'_t = \Phi_t^{-1} \circ \phi_t \circ \Phi_t. \quad (5.9)
\]

By definition, $(\mu'_t, \phi'_t)$ is equivalent to $(\mu_t, \phi_t)$. Moreover, it follows from Eq.(5.7) that
\[
\mu'_t = \mu + t^2 \mu'_2 + \cdots \quad \text{and} \quad \phi'_t = \phi_L + t^2 \phi'_2 + \cdots.
\]

In other words, the linear terms are vanish. By repeating this argument, we get $(\mu_t, \phi_t)$ is equivalent to $(\mu, \phi_L)$.

Next, we consider finite order deformations of a 3-LieDer pair $(L, \phi_L)$, and show that how obstructions of extending a deformation of order $N$ to a deformation of order $(N+1)$ depends on the third cohomology class of the 3-LieDer pair $(L, \phi_L)$.
Definition 5.8. A deformation of order $N$ of a 3-LieDer pair $(L, \phi_L)$ consist of finite sums $\mu_t = \sum_{i=0}^{N} t^i \mu_i$ and $\phi_t = \sum_{i=0}^{N} t^i \phi_i$ such that $\mu_t$ defines 3-Lie bracket on $L[t]/(t^{N+1})$ and $\phi_t$ is a derivation on it.

Therefore, we have
\[
\sum_{i+j=n} \mu_i(x, y, \mu_j(z, v, w)) = \sum_{i+j=n} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)),
\]
and,
\[
\sum_{i+j=n} \phi_i(\mu_j(x, y, z)) = \sum_{i+j=n} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)),
\]
for $n = 0, 1, \ldots, N$. These identities are equivalent to
\[
[\mu, \mu_n] = -\frac{1}{2} \sum_{i+j=n, i,j>0} [\mu_i, \mu_j], \tag{5.10}
\]
\[-[\phi_L, \mu_n] + [\mu, \phi_n] = \sum_{i+j=n, i,j>0} [\phi_i, \mu_j]. \tag{5.11}
\]

Definition 5.9. A deformation $(\mu_t, \phi_t) = \sum_{i=0}^{N} t^i \mu_i, \phi_t = \sum_{i=0}^{N} t^i \phi_i$ of order $N$ is said to be extendable if there is an element $(\mu_{N+1}, \phi_{N+1}) \in C^2_{3\text{-LieDer}}(L, L)$ such that $(\mu'_t = \mu_t + t^{N+1} \mu_{N+1}, \phi'_t = \phi_t + t^{N+1} \phi_{N+1})$ is a deformation of order $N + 1$.

Thus, the following two equations need to be satisfied:
\[
\sum_{i+j=N+1} \mu_i(x, y, \mu_j(z, v, w)) = \sum_{i+j=N+1} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)), \tag{5.12}
\]
and,
\[
\sum_{i+j=N+1} \phi_i(\mu_j(x, y, z)) = \sum_{i+j=N+1} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)). \tag{5.13}
\]

The above two equations can be equivalently written as
\[
d(\mu_{N+1}) = -\frac{1}{2} \sum_{i+j=N+1, i,j>0} [\mu_i, \mu_j] = Ob^3, \tag{5.14}
\]
\[
d(\phi_{N+1}) + \delta(\mu_{N+1}) = -\sum_{i+j=N+1, i,j>0} [\phi_i, \mu_j] = Ob^2. \tag{5.15}
\]

Using the Equation \[5.14\] and \[5.15\] it is a routine but lengthy work to prove the following proposition. Thus, we choose to omit the proof.
Proposition 5.10. The pair \((Ob^3, Ob^2) \in C^3_{\text{3-LieDer}}(L, L)\) is a 3-cocycle in the cohomology of the 3-LieDer pair \((L, \phi_L)\) with coefficients in itself.

Definition 5.11. Let \((\mu_t, \phi_t)\) be a deformation of order \(N\) of a 3-LieDer pair \((L, \phi_L)\). The cohomology class \([\{Ob^3, Ob^2\}] \in H^3_{\text{3-LieDer}}(L, L)\) is called the obstruction class of \((\mu_t, \phi_t)\).

Theorem 5.12. A deformation \((\mu_t, \phi_t)\) of order \(N\) is extendable if and only if the obstruction class \([\{Ob^3, Ob^2\}] \in H^3_{\text{3-LieDer}}(L, L)\) is trivial.

Proof. Suppose that a deformation \((\mu_t, \phi_t)\) of order \(N\) of the 3-LieDer pair \((L, \phi_L)\) extends to a deformation of order \(N + 1\). Then we have

\[
\partial(\mu_{N+1}, \phi_{N+1}) = (Ob^3, Ob^2).
\]

Thus, the obstruction class \([\{Ob^3, Ob^2\}] \in H^3_{\text{3-LieDer}}(L, L)\) is trivial.

Conversely, if the obstruction class \([\{Ob^3, Ob^2\}] \in H^3_{\text{3-LieDer}}(L, L)\) is trivial, suppose that

\[
(Ob^3, Ob^2) = \partial(\mu_{N+1}, \phi_{N+1}),
\]

for some \((\mu_{N+1}, \phi_{N+1}) \in C^2_{\text{3-LieDer}}(L, L)\). Then it follows from the above observation that

\[
(\mu'_t = \mu_t + t^{N+1}\mu_{N+1}, \phi'_t = \phi_t + t^{N+1}\phi_{N+1})
\]

is a deformation of order \(N + 1\), which implies that \((\mu_t, \phi_t)\) is extendable. \(\Box\)

Theorem 5.13. If \(H^3_{\text{3-LieDer}}(L, L)\), then every finite order deformation of \((L, \phi_L)\) is extendable.

Corollary 5.14. If \(H^3_{\text{3-LieDer}}(L, L) = 0\), then every 2-cocycle in the cohomology of the 3-LieDer pair \((L, \phi_L)\) with coefficients in itself is the infinitesimal of a formal deformation of \((L, \phi_L)\).

ACKNOWLEDGEMENT

The paper is supported by the NSF of China (No. 12161013) and Guizhou Provincial Science and Technology Foundation (No. [2020]1Y005).

REFERENCES

[1] S.A. Amitsur, Derivations in simple rings, *Proc. Lond. Math. Soc.* (3) (1957)

[2] S.A. Amitsur, Extension of derivations to central simple algebras, *Commun. Algebra* (1982).

[3] V. Ayala, E. Kizil, I. de Azevedo Tribuzy, On an algorithm for finding derivations of Lie algebras, *Proyecciones* 31 (2012) 81-90.
[4] J. Bagger, N. Lambert, Gauge symmetry and supersymmetry of multiple \( M2 \)-branes, Phys. Rev. D 77 (6) (2008) 065008 (6 pages).

[5] J. Bagger, N. Lambert, Comments on multiple \( M2 \)-branes, J. High Energy Phys. (2) (2008) 105 (15 pages).

[6] C. Bai, L. Guo, Y. Sheng, Bialgebras, classical Yang-Baxter equation and Manin triples for 3-Lie algebras, Adv. Theor. Math. Phys. 23(2019) 27-74.

[7] R. Bai, W. Wu, Y. Li, Z. Li, Module extensions of 3-Lie algebras, Linear Multilinear A. 60 (4) (2012) 433-447.

[8] V.G. Bardakov, M. Singh, Extensions and automorphisms of Lie algebras, J. Algebra Appl. 16 (2017), 15 pp..

[9] V. Coll, M. Gerstenhaber, A. Giaquinto, An explicit deformation formula with non-commuting derivations, Ring theory 1989 (Ramat Gan and Jerusalem, 1988/1989) 396-403, Israel Math. Conf. Proc., 1, Weizmann, Jerusalem, 1989.

[10] A. Das, A. Mandal, Extensions, deformations and categorifications of AssDer pairs, preprint (2020), [arXiv.2002.11415](http://arxiv.org/abs/2002.11415).

[11] A. Das, Leibniz algebras with derivations, J. Homotopy Relat. Struct., 16 (2021) 245-274..

[12] M. Doubek, T. Lada, Homotopy derivations, J. Homotopy Relat. Struct. 11 (2016) 599-630.

[13] C. Du, C. Bai, L. Guo, 3-Lie bialgebras and 3-Lie classical Yang-Baxter equations in low dimensions, Linear Multilinear A. 66(8) (2018) 1633-1658.

[14] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. Math. 78, (1963) 267-288.

[15] M. Gerstenhaber, On the deformation of rings and algebras, Ann. Math. (2) 79 (1964) 59-103.

[16] V. Filippov, \( n \)-Lie algebras, Sib. Mat. Zh. 26 (1985) 126-140.

[17] S. Kasymov, On a theory of \( n \)-Lie algebras, Algebra Log. 26 (1987) 277-297.

[18] J. Liu, A. Makhlof, Y. Sheng, A new approach to representations of 3-Lie algebras and Abelian extensions, Algebr Represent Theor (20)(2017) 1415-1431.

[19] A. Magid, Lectures on differential Galois theory, University Lecture Series, 7. American Mathematical Society, Providence, RI, 1994.
[20] M. Rotkiewicz, Cohomology ring of \( n \)-Lie algebras, *Extr. Math.* 20 (2005), 219-232.

[21] Y. Sheng, R. Tang, Symplectic, product and complex structures on 3-Lie algebras, *J. Algebra* 508 (2018) 256-300.

[22] L. Takhtajan, Higher order analog of Chevalley-Eilenberg complex and deformation theory of \( n \)-algebras, *St. Petersburg Math. J.* 6 (1995) 429-438.

[23] R. Tang, Y. Frégier, Y. Sheng, Cohomologies of a Lie algebra with a derivation and applications, *J. Algebra*, 534 (2019) 65-99.

[24] T. Voronov, Higher derived brackets and homotopy algebras, *J. Pure Appl. Algebra* 202 (2005) 133-153.

[25] S. Xu, Cohomology, derivations and abelian extensions of 3-Lie algebras, *J. Algebra Appl.* 18(7) (2019) 1950130 (26 pages).

[26] T. Zhang, Cohomology and deformations of 3-Lie colour algebras, *Linear and Multilinear Algebra.* 63 (4) (2015) 651-671.