Does a Good Conservation of Jacobi’s Constant Imply a Good Orbital Numerical Integration?: I. Resonant Orbits.

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ABSTRACT

In this paper we show two examples of numerical orbital integrations (Planar Circular Restricted Three Body Problem) in which even though the conservation of Jacobi’s constant is near to 1 part in $10^8$, the integration proves to be wrong. That is, for some particular cases the Jacobi’s constant value is *insensitive* to very important qualitative (not quantitative) changes in the integrated orbit which are produced by numerical errors. We provide specific recommendations to test and ensure that a numerical code is working properly, regardless the numerical method employed.

**Key Words:** Orbits; Planetary Dynamics; Computer Techniques
I. INTRODUCTION

The planar circular restricted three-body problem (hereafter PCRTBP) consists of two bodies; a primary or central body (of mass $m_1$), a secondary or perturbing body (of mass $m_2$), both moving in circular orbits about their common center of mass, and a test particle moving under the gravitational effect of the other two masses without affecting their motion. In this scenario neither the total orbital energy nor the total angular momentum of the whole system are conserved since the test particle does not affect the motion of the other two bodies. In particular, the energy and angular momentum of the test particle’s orbit are not constants of the motion. However, the dynamical system still has an integral of the motion, the Jacobi’s constant, $C$ which, in a coordinate system in which $m_1$ and $m_2$ are always at rest, exclusively depends on the particle’s instantaneous position and velocity coordinates (see Eq. 2.5).

The planar circular restricted three-body problem is an excellent approach to a very wide variety of dynamical problems within the Solar System. Some examples are the orbits of comets when only the gravitational effects of the Sun and Jupiter are taken into account; non-collisional and collisional planetary rings affected by a planetary satellite; asteroid’s orbits subject to Jupiter’s gravitational perturbations, etc. Since there is no analytical solution to this problem, it is most common to solve the problem numerically. That is, to integrate (using some numerical method) the full Newtonian equations of motion for the system (see Eqs. 2.2 and 2.4) given the initial conditions for the three bodies and the secondary to primary mass ratio.

When integrating an orbit numerically in the PCRTBP scenario, in many instances there is no way to find out if the resulting orbit is correct or not, because we do not know the solution in advance. The only way to determine if the code is performing the integration correctly is by monitoring (at each instant of time) the value of the Jacobi’s constant which (of course) should remain constant throughout the integration. However, computers do not have infinite precision and therefore as the integration proceeds errors accumulate. Hence one can only expect that Jacobi’s constant is conserved to a reasonably large number of digits.

We want to show that an integration that gives a reasonably large number of unchanged digits in the value of Jacobi’s constant, does not necessarily imply a good performance of the code.
Since all numerical methods accumulate errors, these results are relevant to any numerical orbital integration in the PCRTBP scenario, regardless the numerical method used for the integration.

There is a large variety of numerical methods to integrate ordinary second order differential equations, a very popular one due to its high accuracy at low CPU cost is the Bulirsch-Stoer method. The code that performed all the orbital integrations shown in this paper is based on the implementation of this method that appears in the 2nd. edition of Numerical Recipes (Press et al. 1992).

II. THE FULL NEWTONIAN EQUATIONS OF MOTION OF THE SYSTEM

In what follows, the origin of the coordinate system is at the center of the primary point-mass, $m_1$. We use a Cartesian coordinate system to locate the position of the perturbing point-mass $m_2$ and the test particle. In this reference frame the primary is always at rest; the equation of motion for the perturber $m_2$ is:

$$\ddot{r}_2 = G(m_1 + m_2) \frac{r_2}{r_2^3} . \quad (2.1)$$

Where $r_2$ is the vector position of the secondary with respect to the primary and $r_2$ has time-dependent coordinates $(x_2, y_2)$. The double dot denotes the second time derivative in all of the equations, and $G$ is the gravitational constant. Writing the components of Eq. (2.1) one obtains:

$$\ddot{x}_2 = -G(m_1 + m_2) \frac{x_2}{(x_2^2 + y_2^2)^{3/2}} \quad (2.2a)$$

$$\ddot{y}_2 = -G(m_1 + m_2) \frac{y_2}{(x_2^2 + y_2^2)^{3/2}} \quad (2.2b)$$

The equation of motion for the particle is:

$$\ddot{r} = -Gm_1 \frac{r}{r^3} + Gm_2 \left( \frac{r_2 - r}{|r_2 - r|^3} - \frac{r_2}{r_2^3} \right) , \quad (2.3)$$

where $r$ with time dependent coordinates $(x, y)$, is the position vector of the test particle with respect to the primary. The last term on the right hand side of Eq. (2.3) is the indirect term, which accounts for the fact that the center of the primary is not an inertial
frame (e.g., Shu 1984; Murray and Dermott 1999) because it suffers accelerations due to its gravitational interaction with the secondary. Writing Eq. 2.3 by components one obtains:

\[ \ddot{x} = -Gm_1 \frac{x}{(x^2 + y^2)^{3/2}} + Gm_2 \left[ \frac{x_2 - x}{((x - x_2)^2 + (y - y_2)^2)^{3/2}} - \frac{x_2}{(x_2^2 + y_2^2)^{3/2}} \right] \]  
\[ \ddot{y} = -Gm_1 \frac{y}{(x^2 + y^2)^{3/2}} + Gm_2 \left[ \frac{y_2 - y}{((x - x_2)^2 + (y - y_2)^2)^{3/2}} - \frac{y_2}{(x_2^2 + y_2^2)^{3/2}} \right] \]  

Equations (2.4a) and (2.4b) form a coupled system of second order differential equations. For this dynamical system the only constant of the motion is Jacobi’s constant, \( C \). This constant is most commonly defined in a frame that rotates with the secondary’s mean orbital frequency, \( n_s \), and origin at the common center of mass of \( m_1 \) and \( m_2 \). In this frame it reads:

\[ C = n_s^2 (\ddot{x}^2 + \ddot{y}^2) + 2 \left( \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \right) - \ddot{v}_x^2 - \ddot{v}_y^2 , \]  

where \( \dddot{x} \) and \( \dddot{y} \) are the position coordinates of the particle in this rotating frame, \( \dddot{v}_x \) and \( \dddot{v}_y \) are the velocity components; \( r_1 \) and \( r_2 \) are the distances from the particle to \( m_1 \) and \( m_2 \) respectively, and \( \mu_1 = Gm_1 \) and \( \mu_2 = Gm_2 \).

Equations (2.2) and (2.4) are input into the numerical code in order to integrate in time, thus obtaining the positions and velocities of both, the perturber and the test particle at various time instants, \( t \), such that \( t = 0 \leq t \leq t_f \) where \( t_f \) denotes the time at which the integration is finished. Note that our code integrates the equations of motion with the origin at \( m_1 \) and includes the indirect terms. Hence the output are positions and velocities with respect to an origin at \( m_1 \) and the aforementioned Cartesian axes. At this point it is worth to mention that the output coordinates are then used to calculate the osculating elements for the particle’s orbit shown in all of the plots. Also, coordinates are transformed into the rotating frame (origin at the center of mass) in order to calculate Jacobi’s Constant. All this transformations, may of course introduce more errors, but this
only makes our point stronger because we have good conservation of Jacobi’s constant and still the integration proves to go eventually wrong.

III. THE ORBITS

To prove our point, we have chosen two different orbits whose semi-major axes are near the 2:1 and the 3:2 Inner Lindblad Resonances (ILR). In both cases the initial eccentricity of the test particle orbit is zero, \( e_0 = 0 \), and it starts at conjunction with the secondary, \( \lambda_{p0} = \lambda_{20} \), where the \( \lambda \)'s denote mean longitudes. For the two resonances considered, the initial semi-major axis of the test particle orbit is very close to the nominal location of each resonance. In general, the nominal location (semi-major axis) of a first order resonance is found through the commensurability of the mean orbital frequencies between the particle and the perturber (or secondary). This means that the ratio of their orbital frequencies (or periods) can be written as a ratio of two small integers, that is:

\[
    n_p = \frac{m}{m-1} n_2
\]

where \( m \) (integer) indicates the number of the resonance; \( n_p = \sqrt{Gm_1/a_p^3} \) is the test particle’s mean orbital frequency and \( n_2 = \sqrt{G(m_1 + m_2)/a_2^3} \) is the perturber’s mean orbital frequency. Solving for the semi-major axis of the particle one obtains:

\[
    a_n(m : m - 1) = Gm_1 \left[ \frac{1}{mn_2} \right]^{2/3},
\]

where subindex \( n \) stands for nominal, \( a \) denotes semi-major axis and the expression in parenthesis on the left hand side is the usual notation for a resonance. For the 2:1 ILR \( m = 2 \) and for the 3:2 ILR \( m = 3 \).

The actual initial semi-major axes for the particle’s orbit chosen for the present work, were found empirically by Espresate and Lissauer (2001) when trying to find the largest resonant forcing produced by the secondary on the test particle. For the 2:1 ILR case, this semi-major axis is given by:

\[
    a_p(2 : 1) = a_n(2 : 1)(1 + 1.5386878 \times 10^{-4})
\]
where $a_p$ was named, the perturbed center of the resonance in Espresate and Lissauer (2001) (note the misprint in Eq. 3 of Espresate and Lissauer 2001). This semi-major axis is slightly larger than the nominal and large perturbations occur. With this initial conditions the particle’s semi-major axis oscillates in such a way, that its average semi-major axis is the nominal resonant semi-major axis hence, on average, the commensurability of the orbital frequencies of the secondary and the particle is closer to 2. If the particle starts at the nominal semi-major axis its average semi-major axis is such that the frequencies commensurability is farther from (integer) 2 and hence the perturbations are smaller (see Espresate and Lissauer 2001).

For the 3:2 ILR case the perturbed center of the resonance was found at (Espresate 1997):

$$a_p(3:2) = a_n(3:2)(1 + 2.0565255 \times 10^{-4}).$$  \hfill (2.7)

Table I shows the initial conditions for the particle’s orbit in both cases; the semi-major axes are given in units of the distance between $m_1$ and $m_2$ and are given with a large number of digits that should all be taken into account.

[Table I]

As mentioned above our code uses the Bulirsch-Stoer method, which (like most numerical methods), requires an input parameter called $\varepsilon_p$ that represents the overall error tolerance and determines the accuracy of the integration.

Since the code is written in double precision, we start using a very small tolerance, $\varepsilon_p$, which is close to the machine double precision limit. For each of the subsequent integrations we increase the value of $\varepsilon_p$ thus requiring progressively less accuracy from the code. Table II shows the $\varepsilon_p$ parameter, and the length of each of the integrations, $t_f$, in units of the perturber’s orbital period. For a given ILR all of the integrations have exactly the same initial conditions for the particle’s orbit.

[Table II]

IV. RESULTS

Figure 1 shows the results of our best integration for the initial conditions in Table I (2:1 ILR, Run 1). That is, using the minimum error tolerance $\varepsilon_p = 5 \times 10^{-15}$. From top
to bottom the plots are: the semi-major axis in units of the separation between $m_1$ and $m_2$ as a function of time (in units of the perturber’s period); second panel is the eccentricity. Note that its maximum value coincides with the value given analytically by Franklin et al. (1984) for a 2:1 ILR and the mass ratio used in this work; third panel is the resonant argument, $\phi = 2\lambda_2 - \lambda_p - \omega$ where $\omega$ is the instantaneous periapse angle of the particle. As expected because the orbit is inside the resonance region, the resonant argument librates (as long as $e \neq 0$). Finally the lowest panel shows the fractional conservation of Jacobi’s constant which by the end of the integration has a fractional change a little over $\sim 10^{-10}$. Hence the constant is conserved up to 9 digits approximately. It is interesting to note that there is a time interval in which the semi-major axis, the eccentricity, and the resonant argument stay almost constant as it happens for the 2:1 exact resonance according to Winter and Murray (1997). Nevertheless, the particle falls out of exact resonance and moves back to the libration region and later repeats the cycle.

[FIG.1]

Figure 2 shows the same quantities as Fig. 1 but for Run 2, an integration using a tolerance 10 times larger than in Run 1 ($\epsilon = 5 \times 10^{-14}$). Still the integration looks good although Jacobi’s conservation is almost 4 times worst than in Run 1.

[FIG.2]

Figure 3 shows the same plots as Fig. 2 but for Run 3. Again we have increased the tolerance by a factor of 10 but now there is a very notable change in the behavior of the integration. The eccentricity never makes it back to zero as in the previous integrations. It remains oscillating between the maximum and half the maximum and seems to continue like that. Jacobi’s conservation however is of 7 digits or near to 1 part in $10^8$ which is still acceptable in most published works. It can be seen that when the qualitative change occurs, Jacobi’s constant does not show any signal (whatsoever) that something has started to go wrong. Since the change in the orbit is not quantitatively large, this is not a surprise, but certainly the integration is qualitatively wrong. In particular, the resonant argument does not seem to be librating as clearly as in the previous two runs specially after the first cycle.

[FIG.3]

Figure 4 shows the results for $\epsilon = 5 \times 10^{-12}$ in which a slow increase in the lowest points of the oscillations in eccentricity starts to become visible and also no libration of
the resonant argument is visible. Note that the number of maxima in the eccentricity oscillations is 6. The conservation of Jacobi’s constant could still be considered acceptable up to 6 digits approximately.

[FIG. 4]

Figures 5 and 6 are the results from Runs 5 and 6 and it really becomes a disaster. Specially (of course) for Run 6 in which the tolerance is simply too large and the lower limit of the eccentricity oscillation is frankly increasing. Note also that in Fig. 5 the number of maxima for the eccentricity is 9 for the same time interval while in Run 4, there are only 4 maxima. The conservation of Jacobi’s constant is of only 5 digits approximately. For Run 6 the number of maxima in the eccentricity oscillation is 14 and the amplitude keeps decreasing. In Run 6 one has a really bad conservation of Jacobi’s constant of up to 4 digits only.

[FIGS. 5 and 6]

Figures 7, 8 and 9 from Run 7, 8 and 9 respectively, show exactly the same kind of deterioration than that at the 2:1 ILR as one increases the $\varepsilon$ value. The same qualitative changes which again are not reflected in the conservation of Jacobi’s constant. Also the overall conservation seems to be a little better for the same values of $\varepsilon$ than in the runs for the 2:1 ILR case.

[FIGS. 7, 8 and 9]

The question that arises now is if one can trust ‘blindly’ an integration with $\varepsilon = 5 \times 10^{-15}$. The answer of course is NO. Even with this small tolerance, errors keep accumulating (however slowly) and eventually the code will start doing things wrong. That is the purpose of Runs 10 and 11 in Table II. The results of Run 10 are shown in Fig. 10. It can be seen that eventually the integration suffers a qualitative change of the same kind, but much later in time. When this happens (close to $t = 375,000$ orbits), Jacobi’s constant is being conserved up to 8 digits (!) which is usually viewed as a good numerical performance. The change is not significant, quantitatively speaking and that is why Jacobi’s constant does not show any significant variation. Nevertheless the results are wrong because that is not the correct orbital behavior.

[FIGS. 10 and 11]

Finally, Figure 11 shows a long integration for the 3:2 ILR which shows a very similar effect after $1.7 \times 10^5$ orbits, which is earlier than for the 2:1 resonance using the same $\varepsilon$.
value.

CONCLUSIONS

In this paper we have shown orbital numerical integrations for two resonant orbits in the (PCRTBP scenario) in which, although Jacobi’s constant conservation may be as good as 1 part in $10^8$ the results are wrong. A qualitative change in the orbit eventually shows up. When the qualitative change occurs, the eccentricity and the semi-major axis keep oscillating near the correct values producing no significant quantitative changes in the value of Jacobi’s constant. It is the accumulation of errors during the integration what causes this qualitative change in the integrated orbit which turns out to be incorrect. That is, in both cases at the largest accuracy the eccentricity always makes it back to its initial value, and so does the semi-major axis (Fig. 1 and 2). As one requires less accuracy from the numerical code in order to speed the performance, this qualitative change shows up earlier in the integration; the less the accuracy, the sooner it shows up.

In other words, based on the Jacobi’s constant conservation and requiring 7 or 8 digits to remain unchanged, do not guaranty that the orbit is being integrated correctly. Qualitative changes may happen and will not show on the Jacobi’s constant value.

Regardless the numerical method used, all computers accumulate errors and therefore the behavior for these two orbits should be similar. May be with a different method (or different computer) the qualitative change will happen later in time for a given $\varepsilon$ value, or may be earlier in time. It depends upon the combination of the type of computer itself and the numerical method employed. However, most surely it will eventually occur.

In view of this results, we warmly recommend to all people that make orbital numerical integrations in the PCRTBP scenario, particularly to those who use the implementation of the Bulirsch-Stoer method in Press et al. 1992, the following: i) first determine to total time, $t_f$ of the integration you plan to perform; ii) use the $\varepsilon$ value you will use for your integration and iii) with those quantities, perform an integration using the initial conditions of any of the two orbits presented here. If during your whole integration interval the eccentricity behaves as shown in Fig. 1, then you maybe more confident that your method and your computer are most probably working well during the time interval of your interest. If not, try to reduce the $\varepsilon$ value or the time length of your integration.

We recommend these two orbits because first of all we already have their initial con-
ditions, and also because they are good “cheaters”, they can give you a good conservation of Jacobi’s constant but still be wrong.

Therefore, if your code integrates properly during $t_f$ any of these two orbits (using an $eps$ of your choice), you may be a bit more confident that during your time interval your code is doing things correctly. Specially if you are not working near resonances, where errors are consistently amplified. Needless to say that if you have enough time, you can check your results, specially qualitative behavior of the orbit for different values of $eps$ and check how reliable is the number of digits you are conserving in Jacobi’s constant during the whole integration length. In other words, try to find out first if the orbit you want to integrate is not one good “cheater” like these two.

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TABLES

TABLE I

| $a_p/a_2$       | $e_0$ | $\lambda_{p_0}$ | $\lambda_{2_0}$ | $m_1/m_2$ | ILR |
|-----------------|-------|------------------|------------------|-----------|-----|
| 0.63005724618926 | 0.0   | 0.0              | 0.0              | $1 \times 10^{-6}$ | 2:1 |
| 0.76329951620447 | 0.0   | 0.0              | 0.0              | $1 \times 10^{-6}$ | 3:2 |

TABLE II

| RUN No. | $eps$         | $t_f$ | ILR |
|---------|---------------|-------|-----|
| 1       | $5 \times 10^{-15}$ | 40,000 | 2:1 |
| 2       | $5 \times 10^{-14}$ | 40,000 | 2:1 |
| 3       | $5 \times 10^{-13}$ | 40,000 | 2:1 |
| 4       | $5 \times 10^{-12}$ | 40,000 | 2:1 |
| 5       | $5 \times 10^{-11}$ | 40,000 | 2:1 |
| 6       | $5 \times 10^{-10}$ | 40,000 | 2:1 |
| 7       | $5 \times 10^{-15}$ | 40,000 | 3:2 |
| 8       | $5 \times 10^{-13}$ | 40,000 | 3:2 |
| 9       | $5 \times 10^{-10}$ | 40,000 | 3:2 |
| 10      | $5 \times 10^{-15}$ | $1 \times 10^6$ | 2:1 |
| 11      | $5 \times 10^{-15}$ | $4 \times 10^5$ | 3:2 |
FIGURE CAPTIONS

Figure 1. Shown are the results from Run 1 (Table II) and the initial conditions in Table I for the 2:1 case. From top to bottom the plots are: the semi-major axis in units of the separation between \( m_1 \) and \( m_2 \) as a function of time (in units of the perturber’s period). The eccentricity \( e(t) \) as a function of time; the resonant argument (in degrees) \( \phi = 2\lambda_2 - \lambda_p - \omega \), as a function of time. The vertical axis has limits \([-180, 180]\). The bottom panel shows the fractional conservation of Jacobi’s constant which by the end of the integration has a fractional change a little over \( \sim 10^{-10} \). Hence the constant is conserved up to 9 digits approximately.

Figure 2. Same plots as in Fig. 1 but from results of Run 2 (Table II). Note in the bottom panel that Jacobi’s constant conservation is a little worse when the value of \( \text{eps} \) is increased.

Figure 3. Same plots as in Fig. 1 but from results of Run 3 (Table II). Note there is a qualitative change in the orbit due to the accumulation of numerical errors which is not reflected in any way on the panel at the bottom, where the fractional change of Jacobi’s constant is shown. At the end of the integration the Jacobi’s constant conserves to approximately 7 digits. Note also that now the eccentricity reaches its maximum 5 times whereas in the previous two figures it happens only twice.

Figure 4. Results from Run 4 (Table II) displayed in the same way as in previous figures. The number of maxima reached by the eccentricity is now almost 7 and it actually does not make it to 0.02, the maximum now (not noticeable in this resolution) is slightly lower than 0.02.

Figure 5. Results from Run 5 (Table II). Now the eccentricity shows 9 maxima and just like in the previous two figures, the resonant argument (third panel from the top) is circulating except during the first cycle. The conservation of Jacobi’s constant is of approximately 5 digits.

Figure 6. Note the appearance of 14 maxima in the eccentricity (second panel from the
top) and the minima are increasing. However the whole amplitude of the oscillation in semi-major axis and eccentricity are smaller than in Figs. 4 and 5. This integration has a very questionable conservation of Jacobi’s constant of approximately 4 digits.

Figure 7. Same as in Fig. 1 but for the 3:2 resonance case, Run 7 in Table I. Panels show the same quantities as in the previous figures. Note that the semi-major axis and the eccentricity have a very similar qualitative behavior as for the 2:1 case (see Fig. 1). However the amplitude and the period of the periodic oscillations are smaller. Jacobi’s constant is conserved to almost one more digit than in the integration for the 2:1 case, using the same value of $\epsilon$ (see Fig. 1).

Figure 8. Results from Run 8 with an error tolerance a 100 times larger than that in Fig. 7. As in the 2:1 case, the integration is qualitatively different, the eccentricity (second panel from the top) never makes it back to zero as in Fig. 7 but still Jacobi’s constant is conserved to approximately 8 digits (!) and shows no sign when the integration starts to go wrong. Also note the increase in the number of maxima reached by the eccentricity.

Figure 9. Results from Run 9 with an error tolerance 1000 times larger than in the previous case. The number of maxima in the eccentricity has increased noticeably and shows the same behavior as that observed in Fig. 6 for the 2:1 case. Still the value of Jacobi’s constant is slightly better conserved than in Run 6.

Figure 10. A long term integration (million orbits) for the 2:1 case for the smallest error tolerance. It can be seen that approximately at $t \sim 3.75 \times 10^5$, the code starts to go wrong in the same way as it did for Run 3 at $t \sim 1 \times 10^4$. At this point, Jacobi’s constant is being conserved up to 8 digits, which is more accurate than what is reported in the literature some times.

Figure 11. A long term integration ($4 \times 10^5$ orbits), for the 3:2 ILR case with the smallest error tolerance. The code starts to misbehave at $t \sim 1.7 \times 10^5$ orbits, which is almost a factor of 2 sooner than for the 2:1 case (see Fig. 10). Even though Jacobi’s constant is better conserved than in Run 10. The graph at the bottom shows rapid oscillations (which are not observed in the same graph in Fig. 10) that may indicate a numerical instability of some sort.
\[ a_0 = a_p(2:1) \]
\[ \varepsilon_{ps} = 5 \times 10^{-15} \]

Fig. 1
\[ a_0 = a_p (2:1) \]
\[ \epsilon p s = 5 \times 10^{-14} \]

**Fig. 2**
\( a_0 = a_p (2:1) \)

\( \epsilon_p s = 5 \times 10^{-13} \)

Fig. 3
Fig. 4

\[ a_0 = a_p (2:1) \]
\[ \epsilon_{ps} = 5 \times 10^{-12} \]
$a_0 = a_p (2:1)$

$\epsilon p s = 5 \times 10^{-11}$
\[ a_0 = a_p (2:1) \]
\[ \varepsilon_{PS} = 5 \times 10^{-10} \]
$a_0 = a_p(3:2)$

$\varepsilon_p s = 5 \times 10^{-15}$

Fig. 7
$a_0 = a_p (3:2)$

$\epsilon_p = 5 \times 10^{-13}$

Fig. 8
$a_0 = a_p (3:2)$

$\varepsilon_{ps} = 5 \times 10^{-10}$

Fig. 9
\[ a_0 = a_p (2:1) \]

\[ \epsilon ps = 5 \times 10^{-15} \]
$a_0 = a_p(3:2)$

$\epsilon_{ps} = 5 \times 10^{-15}$

**Fig. 11**