ABSTRACT. To any continuous eigenvalue of a Cantor minimal system $(X, T)$, we associate an element of the dimension group $K^0(X, T)$ associated to $(X, T)$. We introduce and study the concept of irrational miscibility of a dimension group. The main property of these dimension groups is the absence of irrational values in the additive group of continuous spectrum of their realizations by Cantor minimal systems. The strong orbit equivalence (respectively orbit equivalence) class of a Cantor minimal system associated to an irrationally miscible dimension group $(G, u)$ (resp. with trivial infinitesimal subgroup) with trivial rational subgroup, have no non-trivial continuous eigenvalues.

1. Introduction

The study of orbit equivalence in measurable dynamics was initiated by H. Dye [8], who proved in particular that any two ergodic probability measure preserving transformations are orbit equivalent. Therefore the spectrum of an ergodic probability measure preserving systems is independent of its orbit equivalence class.

In topological dynamics, the study of orbit equivalence was initiated by M. Boyle [1] and was developed for Cantor minimal systems by T. Giordano, I. Putnam and C. Skau in [13] where the notion of strong orbit equivalence was also introduced; the two notions of orbit equivalence and strong orbit equivalence were characterized using two dimension groups $K^0(X, T)$ and $K^0_m(X, T)$ naturally associated to a Cantor minimal system $(X, T)$. Since then, the study of topological orbit equivalence has been very active (see for example [7], [11], [12], [23], [24]).

The topological version of Dye’s theorem proved in [13], states that a uniquely ergodic Cantor minimal system is orbit equivalent either to an odometer or to a Denjoy system [13]. Odometers are rotations on a Cantor set and therefore automorphic systems (that is, minimal isometries on compact topological groups). Denjoy systems (in special

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cases they are known as Sturmian systems), are almost one to one extensions of irrational rotations on the unit circle.

For a non-uniquely ergodic Cantor minimal system, \((X, T)\), the situation is more complicated as not every such system is orbit equivalent to an almost automorphic system. Related to this phenomenon is the nature of the spectra

\[
\text{Sp}(T) = \{ \lambda \in S^1 : \exists f \in C(X, \mathbb{C}) ; f \circ T = \lambda f \}
\]

(in the literature often termed the continuous spectrum, a standard object in operator theory meaning something completely different).

Recall (see [13], section 2.3) that for a Cantor minimal system \((X, T)\), its associated pointed dimension group, \((K_0(X, T), K_0^+(X, T), [1_X])\), is a complete invariant for strong orbit equivalence.

Information about the relationship between (strong) orbit equivalence class of Cantor minimal systems and their spectra can be derived from the value group of their invariant measures on clopen sets, and also from the rational subgroup, \(\mathbb{Q}(K_0^0(X, T), [1_X])\), of their dimension group. In [23], Ormes proved that for any \(\lambda = e^{2\pi i \theta}\) in the spectrum of a Cantor minimal system \((X, T)\) there exists \(f \in C(X, \mathbb{Z})\) such that \(f - \theta 1_X\) is a real coboundary. From Theorem 2.2 the function \(f\) can be chosen to be the indicator function of a clopen set. Consequently, for any \(0 \leq \theta < 1\), such that \(e^{2\pi i \theta} \in \text{Sp}(T)\), there exists a clopen set whose measure is \(\theta\) with respect to all invariant probability measures on \(X\). Equivalently,

\[
\text{Sp}(T) \subseteq \bigcap_{\tau \in S(K_0^{0}(X, T), [1_X])} \tau(K_0^0(X, T)),
\]

where \(S(K_0^{0}(X, T), [1_X])\) denotes the set of normalized traces (states) on \(K_0^0(X, T)\). This result has been recently stated in [4, Theorem 1.] as well. (The reverse inclusion fails in general, as there are many uniquely ergodic weakly mixing systems.)

Therefore, if a Cantor minimal system has an eigenvalue \(\exp(2\pi i \theta)\) for some irrational \(\theta\), there will be an irrational number in the set of values of its traces. But not all the irrational numbers which appear in the intersection are eigenvalues for \((X, T)\). If the intersection contains only rational numbers (the pointed dimension group is then called irreationally mixing), there will be no irrational eigenvalues (the converse fails), and when this occurs, it is relatively easy to decide when all systems orbit or strongly orbit equivalent to it are weakly mixing. This applies to various classes of simple dimension groups, for example, if \(G\) has \(n > 1\) pure traces, and either rank \((G/\text{Inf}(G)) = n\) or
if rank $G = n + 1$ and $G$ is finitely generated (as an abelian group), then all realizations of $G$ by Cantor minimal systems cannot be even Kakutani equivalent to a Cantor minimal system with irrational eigenvalues. There are large classes of such dimension groups already in the literature.

The rational subgroup of a dimension group corresponds to the rational part of the spectrum. Indeed, it was proved in [24] that $\exp(2\pi i 1/p) \in \text{Sp}(T)$ if and only if $1/p \in \mathbb{Q}(K^0(X, T), [1_X])$. When the system is weakly mixing, the rational subgroup is isomorphic to $\mathbb{Z}$ (a rank one cyclic subgroup). We describe in Proposition [4.1] (resp. [4.3]), strong orbit equivalent (resp. orbit equivalent) classes of Cantor minimal systems which are weakly mixing.

Let $(B, V)$ be a Bratteli diagram whose associated dimension group has a trivial rational subgroup. In Theorem [4.4] we show that there exists a telescoping $\tilde{B}$ of $B$ and a proper ordering on $\tilde{B}$ whose associated Bratteli-Vershik system is weakly mixing.

1.1. **Topological dynamical systems.** A *topological dynamical system* is a pair $(X, T)$ wherein $X$ is a compact metric space and $T$ is a self-homeomorphism. A topological dynamical system is said to be *topologically transitive* if there exists a point in $X$ with dense orbit; it is *minimal* when every point has dense orbit. Two dynamical systems $(X, T)$ and $(Y, S)$ are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$. If $h$ is merely a continuous surjection then $(Y, S)$ is a *factor of $(X, T)$* and the function $h$ is called a *factor map*, and $(X, T)$ an *extension of $(Y, S)$*. An *almost one to one extension* of a minimal system is an extension $h : X \to Y$ for which the set of points with unique pre-image is residual in $Y$; this is equivalent to there being at least one point with unique pre-image.

By a *Cantor system*, we mean a topological dynamical system on a totally disconnected metric space without isolated points. As an example one may consider a subshift system [15]; let $A = \{1, \ldots, \ell\}$ and $\Omega = A^\mathbb{Z}$ of all bi-infinite sequences on the finite alphabet $A$ with the product topology. Let $(\Omega, T)$ be the topological dynamical system defined by the left shift map. If $X$ is any $T$-invariant closed subset of $\Omega$, $(X, T)$ is called a *subshift*. *Substitutions* are Cantor minimal subshifts.

*Kronecker systems* are minimal systems on a compact metric group; examples include irrational rotations on the unit circle and *odometers* on a shift space. The odometer associated to the sequence $(a_1, a_2, \ldots)$, $a_i \geq 2$ is the pair $(X, T)$ with $X = \prod_{n=1}^{\infty} \{0, 1, \ldots, a_n-1\}$ endowed with the product topology and $T$ is defined by addition of $(1, 0, 0, \ldots)$ with
An almost one to one extension of a Kronecker system is called an almost automorphic system.

A complex number $\lambda = e^{2\pi i \theta}$ is a continuous eigenvalue for $(X, T)$ if there exists a continuous function $f : X \to S^1$ such that $f \circ T = \lambda f$; then $f$ is said to be the corresponding eigenfunction for the system. An eigenfunction is a factor map from $X$ into the unit circle; this yields a factor map onto the unit circle iff $\theta$ is an irrational number. The set of all eigenvalues of $(X, T)$ is called its spectrum, denoted $\text{Sp}(T)$. This is sometimes called the continuous spectrum, but this term has been preempted by operator theory. Every topological dynamical system has $\lambda = 1$ in its spectrum, and if $(X, T)$ is a minimal system and $\text{Sp}(T) = \{1\}$, then $(X, T)$ is weakly mixing.

Definition 1.1. Two Cantor minimal systems, $(X, T)$ and $(Y, S)$, are orbit equivalent if there exists a homeomorphism $F : X \to Y$ such that $F(\mathcal{O}_T(x)) = \mathcal{O}_S(F(x))$ for all $x \in X$.

When two systems are orbit equivalent, it follows from the definitions that there exists an integer-valued function $n : X \to \mathbb{Z}$ such that for each point $x$ in $X$, $F(T(x)) = S^{n(x)}(F(x))$, and similarly, there exists a map $m : Y \to \mathbb{Z}$ such that $F(T^m(x)) = S(F(x))$. The maps $m$ and $n$ are called orbit cocyles. Since the systems are minimal, the two cocycles are uniquely defined.

Definition 1.2. Two Cantor minimal systems are strongly orbit equivalent if there exists an orbit equivalence $F$ such that the two orbit cocycles $m$ and $n$ arising from $F$ have at most one point of discontinuity.

Definition 1.3. Let $(X, T)$ be a Cantor minimal system and $U$ be a clopen subset of $X$. The first return map $T_U : U \to U$ is defined by $T_U(x) = T^{r_U(x)}(x)$ where

$$r_U(x) = \inf\{n \in \mathbb{Z}^+ : T^n(x) \in U\}.$$ 

The new Cantor minimal system, $(U, T_U)$, is called the induced system of $(X, T)$ with respect to $U$.

1.2. Ordered Bratteli diagrams. A Bratteli diagram is an infinite directed graph which consists of a vertex set, $V$, and an edge set, $E$, such that $V$, $E$ are unions of countably many non-empty finite sets,

$$V = V_0 \cup V_1 \cup \cdots ; \quad V_0 = \{v\}, \quad E = E_1 \cup E_2 \cup \cdots .$$

There are two maps, the range and the source, $r, s : E \to V$, with $r(E_n) \subset V_n$ and $s(E_n) \subset V_{n-1}$. A vertex $u \in V_n$ is connected to $v \in V_{n+1}$ if there is an edge $e \in E$ such that $r(e) = u$ and $s(e) = v$. 

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So for any \( n \in \mathbb{N} \), there is a \( |V_n| \times |V_{n-1}| \) incidence matrix \( A_n \). Each \( a_{ij} \) thus counts the number of edges from \( v_j \in V_n \) to \( v_i \in V_{n+1} \). The diagram obtained by considering the vertices of \( V_n \) as the nodes arranged horizontally in the \( n \)th level of a diagram. Then the \( i \)th node in the \( n \)th level will be connected to the \( j \)th node in the \( n + 1 \)st level by \( a_{ij} \) edges.

Let \( \{n_k\}_{k=0}^\infty \) be an increasing sequence of natural numbers with \( n_0 = 0 \). We may telescope the diagram along this sequence, obtaining a new one, by taking \( V' \) and \( E' \) to be the sets of vertices and edges such that \( V_k' = V_{n_k} \), and choosing the incidence matrix between two consecutive levels \( k' \) and \( (k+1)' \) to be \( A_k' = A_{n_k} A_{n_k+1} \cdots A_{n_{k+1}} \). A Bratteli diagram is \textit{simple} if it admits a telescoping such that all the resulting incidence matrices have (strictly) positive entries. The matrices \( A_n \) yield a direct limit partially ordered abelian group, known as the dimension group (associated to the Bratteli diagram); see below.

\textbf{Definition 1.4.} An \textit{ordered Bratteli diagram}, \( B(V, E, \geq) \), is a Bratteli diagram \( (V, E) \) with a partial ordering \( \geq \) on its edges such that two edges \( e \) and \( e' \) are comparable iff \( r(e) = r(e') \); In other words, each set \( r^{-1}(v) \), \( v \in V \setminus V_0 \), is linearly ordered. The edge with the biggest (least) number in the ordering is called the \textit{max edge} (\textit{min edge}).

There is an induced ordering on any telescoped diagram. For any two positive integers \( l \) and \( k \) with \( k < l \), the set \( E_{k+1} \circ E_{k+2} \circ \cdots \circ E_{l} \) running from \( V_k \) to \( V_l \) can be ordered as follows: \( (e_{k+1}, e_{k+2}, \ldots, e_l) > (f_{k+1}, f_{k+2}, \ldots, f_l) \) iff there exists some \( i \) with \( k + 1 \leq i \leq l \) such that for all \( i < j \leq l \), \( e_j = f_j \) and \( e_i > f_i \).

Let \( (V, E, \geq) \) be an ordered Bratteli diagram and \( X_B \) denote the associated infinite path space,

\[
X_B = \{(e_1, e_2, \cdots) : e_i \in E_i, \quad r(e_i) = s(e_{i+1}) ; \quad i = 1, 2, \ldots \}.
\]

Two paths are \textit{cofinal} if almost all their edges agree. The usual compact topology on the path space \( X_B \) has the set of cylinder sets as a basis, where cylinder sets are of the form,

\[
U(e_1, e_2, \ldots, e_k) = \{(f_1, f_2, \ldots) \in X_B : \quad f_i = e_i, \quad 1 \leq i \leq k \}.
\]

Equipped by this topology, \( X_B \) is a compact Hausdorff space with a countable basis consisting of clopen sets; it is called \textit{Bratteli compactum}. When \( (V, E) \) is a simple Bratteli diagram, then \( X_B \) has no isolated points, so is a Cantor set. Let \( X_B^{\text{max}} \) denote the set of all those elements \( (e_1, e_2, \ldots) \) in \( X_B \) such that each \( e_n \) is a maximal edge, and define \( X_B^{\text{min}} \) analogously. An ordered Bratteli diagram is called \textit{properly ordered} if \( (V, E) \) is a simple Bratteli diagram and \( X_B^{\text{max}} \) and \( X_B^{\text{min}} \)
each contain only one element; when this occurs, the maximal and minimal paths are denoted \( x_{\text{max}} \) and \( x_{\text{min}} \) respectively. For any Bratteli diagram, there exists an ordering which makes it properly ordered [21].

Let \((V, E, \leq)\) be a simple properly ordered Bratteli diagram. The \textit{Vershik (or adic) map} is the (minimal) homeomorphism \( \varphi_B : X_B \to X_B \) wherein \( \varphi_B(x_{\text{max}}) = x_{\text{min}} \), and for any other point \((e_1, e_2, \ldots) \neq x_{\text{max}}, \) the map sends the path to its successor [21]; in particular, let \( k \) be the smallest number that \( e_k \) is not a max edge, let \( f_k \) be the immediate successor of \( e_k \), and then \( \varphi_B((e_1, e_2, \ldots)) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots) \), where \((f_1, \ldots, f_{k-1})\) is the minimal edge in \( E_1 \circ E_2 \circ \cdots \circ E_{k-1} \) which has the same source as \( f_k \).

Using Kakutani-Rokhlin partitions for Cantor minimal systems, Herman, Putnam and Skau proved the following result.

\textbf{Theorem 1.5.} [21] Let \((X, T)\) be a Cantor minimal system. Then \( T \) is topologically conjugate to a Vershik map on a Bratteli compactum \( X_B \) of a properly ordered Bratteli diagram \((V, E, \leq)\). Furthermore, given \( x_0 \in X \) we may choose the conjugating map \( f : X \to X_B \) so that \( f(x_0) \) is the unique infinite maximal path in \((V, E, \leq)\).

1.3. Dimension groups.

\textbf{Definition 1.6.} A \textit{unital (or pointed) partially ordered group} is a triple \((G, G_+, u)\) where

- \( G \) is an abelian group,
- \( G_+ \) is a subset of \( G \) such that \( G_+ \cap (-G_+) = \{0\}, \ G_+ + G_+ \subseteq G_+, \ G_+ - G_+ = G, \)
- \( u \) is an element of \( G_+ \) such that for any \( g \in G \) there exists \( n \in \mathbb{Z}^+ \) such that \( (nu - g) \in G_+. \)

A partially ordered group is a \textit{dimension group} if

- \((G, G^+)\) is \textit{unperforated:} if \( g \in G \) and \( ng \in G^+ \) for some \( n \in \mathbb{N}, \) then \( g \in G^+. \)
- \((G, G^+)\) satisfies the \textit{Riesz interpolation property:} if \( g_1, g_2, h_1, h_2 \in G \) and \( g_i \leq h_j, \ i, j = 1, 2, \) then there exists a \( g \in G \) such that \( g_i \leq g \leq h_j \) for all \( i, j. \)

A subgroup \( J \) of the dimension group \( G \) is called an \textit{order ideal} if \( J = J^+ - J^+, \) where \( J^+ = J \cap G^+ \) with the property that \( 0 \leq a \leq b \in J \) implies \( a \in J. \) A dimension group is called \textit{simple} if it has no non-trivial order ideal. In a simple dimension group, any \( u \in G^+ \setminus \{0\} \) is an order unit [21].
A homomorphism \( p : G \to \mathbb{R} \) is a state (or normalized trace) if \( p(G^+) \geq 0 \) and \( p(u) = 1 \). The collection of all (normalized) traces on \( G \) is a compact convex subset of \( \mathbb{R}^G \) denoted by \( S(G, u) \).

Recall that an element \( g \) of a partially ordered group \( G \) (with an order unit \( u \)) is infinitesimal if \( -\epsilon u \leq g \leq \epsilon u \), for all \( 0 < \epsilon \in \mathbb{Q} \). The set of all infinitesimals of \( G \) constitutes the infinitesimal subgroup of \( G \), denoted \( \text{Inf}(G) \).

The rational subgroup of a simple dimension group \((G, G_+)\) with order unit \( u \) is
\[
\mathbb{Q}(G, u) := \{ g \in G : \exists p \in \mathbb{Z}, n \in \mathbb{N} \text{ such that } pg = nu \}.
\]

Let \((G, u)\) be a simple noncyclic dimension group. Recall [9] that the double dual map \( g \mapsto \hat{g} \), where \( \hat{g}(\tau) = \tau(g) \), is an order preserving affine representation, \( G \to \text{Aff}(S(G, u)) \).

For \((G, u)\), a countable unperforated partially ordered abelian group with order unit, let \( J(G, u) \) denote the subgroup of \( G \) given by
\[
J(G, u) = \{ g \in G : \hat{g} \text{ is constant} \}
\]
and \( I(G, u) \) be the subgroup of \( \mathbb{R} \) defined as
\[
I(G, u) = \bigcap_{\tau \in S(G, u)} \tau(G).
\]
For any \( g \in J(G, u) \) let \( \Phi(g) \) be the real number given by \( \hat{g}(\tau) \), for any (all) \( \tau \in S(G, u) \).

**Proposition 1.7.** Let \((G, u)\) be a countable unperforated partially ordered abelian group with order unit. Then
\[
\{ \tau \in S(G, u) : \ker \tau = \text{Inf } G \}
\]
is a dense \( G_\delta \) in \( S(G, u) \).

**Proof.** For each \( g \in G \setminus \text{Inf } G \), define \( U_g := \{ \sigma \in S(G, u) : \sigma(g) \neq 0 \} = (\hat{g}^{-1}(0))^c \). This is clearly open in \( S(G, u) \), and is also dense: take any element \( \tau \in S(G, u) \) with \( \tau(g) = 0 \), since \( g \not\in \text{Inf } G \), there exists \( \sigma \in S(G, u) \) such that \( \sigma(g) \neq 0 \); then \( \phi_n := (1/n)\sigma + (1 - 1/n)\tau \to \tau \), and \( \phi_n(g) \neq 0 \), so \( \phi_n \in U_g \).

As \( S(G, u) \) is compact, \( \bigcap_{g \in G \setminus \text{Inf } G} U_g \) is a dense \( G_\delta \), and this is precisely the set of traces \( \tau \) with \( \ker \tau = \text{Inf } G \). \( \square \)
Corollary 1.8. Let \((G, u)\) be a countable unperforated partially ordered abelian group with order unit \(u\). Then
\[
\bigcap_{\tau \in S(G,u)} \tau(G) = \{ \lambda \in \mathbb{R} \mid \exists g \in G \text{ such that } \hat{g} = \lambda 1 \}.
\]

Proof. The right side is clearly contained in the left. Suppose \(\alpha\) belongs to the left side but not the right. By Proposition 1.7, there exists a trace \(\tau \in S(G,u)\) such that \(\ker \tau = \text{Inf} G\). There exists \(g \in G\) such that \(\tau(g) = \alpha\). Since \(\alpha\) does not belong to the right side, \(\hat{g}\) is not constant, so there exists \(\sigma \in S(G,u)\) such that \(\sigma(g) \neq \alpha\).

Let \(F\) be the subfield of \(\mathbb{R}\) generated by \(\tau(G) \cup \sigma(G)\); this is countable, so there exists \(\beta\) in the open interval \((0,1)\) such that \(\{1, \beta\}\) is linearly independent over \(F\). Set \(\phi = \beta \tau + (1 - \beta) \sigma\). As \(\alpha\) belongs to the left side, there exists \(h \in G\) such that \(\phi(h) = \alpha\). This yields \(\alpha = \beta \tau(h) + (1 - \beta) \sigma(h)\), whence \((\tau(h) - \sigma(h))\beta + (\sigma(h) - \alpha) = 0\). As \(\sigma(h), \tau(h)\), and \(\alpha = \tau(g)\) belong to \(F\), we deduce \(\tau(h) = \sigma(h) = \alpha\).

Thus \(\tau(g) = \tau(h)\), so \(g - h \in \text{Inf} G\). Hence \(\sigma(g) = \sigma(h)\), and the latter equals \(\alpha\), contradicting \(\sigma(g) \neq \alpha\).

□

Corollary 1.9. If \((G, u)\) is a simple dimension group, then we have the following short exact sequence:
\[
0 \longrightarrow \text{Inf} (G) \longrightarrow J(G, u) \overset{\Phi}{\longrightarrow} I(G, u) \longrightarrow 0.
\]

Proof. The proof is a straightforward consequence of Corollary 1.8 and the definitions of \(J(G, u), I(G, u),\) and \(\Phi\).

Let \((X, T)\) be a Cantor minimal system and \(C(X, \mathbb{Z})\) be the abelian group of all continuous integer-valued functions on \(X\). Denote by \(\partial_T C(X, \mathbb{Z})\) the subgroup of all elements, \(g\), in \(C(X, \mathbb{Z})\) which can be represented in the form \(g = f - f \circ T\) for some \(f \in C(X, \mathbb{Z})\); an alternative notation is \(g = (I - T)f\). Each element of \(\partial_T C(X, \mathbb{Z})\) is called an integer coboundary. Define \(K^0(X, T) = C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})\) and set \(K^0_+(X, T)\) to be the semigroup of equivalence classes of nonnegative functions, That is,
\[
K^0_+(X, T) = \{ [f] : f \geq 0 \}.
\]

Denote by \([1_X]\), the equivalence class of the constant function 1 on \(X\). Then
\[
(K^0(X, T), K^0_+(X, T), [1_X])
\]
is a simple non-cyclic dimension group. Moreover, any simple non-cyclic dimension group, \(G\), can be realized as \(K^0(X, T)\) for some Cantor minimal system \((X, T)\) [21, Theorem 6.2]. There is a bijective correspondence between \(S(K^0(X, T), [1_X])\) and \(M_T(X)\), the space of all invariant measures on \((X, T)\). For a Cantor minimal system \((X, T)\), we have that \(\{f : \int f \, d\mu = 0 \ \forall \ \mu \in M_T(X)\}\) is a subgroup of \(C(X, \mathbb{Z})\) containing \(\partial_T C(X, \mathbb{Z})\). Then

\[
\text{Inf}(K^0(X, T)) = \{f : \int f \, d\mu = 0 \ \forall \ \mu \in M_T(X)\} / \partial_T C(X, \mathbb{Z})
\]

and

\[
K^0_m(X, T) = C(X, \mathbb{Z}) / \{f : \int f \, d\mu = 0 \ \forall \ \mu \in M_T(X)\}.
\]

Indeed, \(K^0_m(X, T) = K^0(X, T) / \text{Inf}(K^0(X, T))\), and both \(K^0(X, T)\) and \(K^0_m(X, T)\) are simple dimension groups (with the obvious orderings), the latter with order unit \([1_X]\).

**Theorem 1.10.** [13] Two Cantor minimal systems, \((X, T)\) and \((Y, S)\), are orbit equivalent iff

\[(K^0_m(X, T), K^0_m(X, T)^+, [1_X]) \simeq (K^0(Y, S), K^0_m(Y, S)^+, [1_Y]).\]

**Theorem 1.11.** [13] Two Cantor minimal systems, \((X, T)\) and \((Y, S)\), are strongly orbit equivalent iff

\[(K^0(X, T), K^0(X, T)^+, [1_X]) \simeq (K^0(Y, S), K^0(Y, S)^+, [1_Y]).\]

## 2. Spectra and real coboundaries

Let \((X, T)\) be a Cantor minimal system. If \(E(X, T)\) denotes as in [22], the subgroup of all real numbers \(\theta\) that \(\exp(2\pi i \theta) \in \text{Sp}(T)\), we construct in this section an embedding \(\Theta\) of \(E(X, T)\) into \(K^0(X, T)\) (Corollary 2.6) and study properties of the image \(\Theta(E(X, T))\).

Recall that a continuous function \(f : X \to \mathbb{R}\) is a real \(T\)-coboundary if \(f = g - g \circ T\) for some \(g \in C(X, \mathbb{R})\). Real co-boundaries were characterized in [18] by Gottschalk & Hedlund and studied in particular in [23] and [25].

The following lemma is well known, but as it plays an important role in Theorem 2.2 we include a proof.

**Lemma 2.1.** Let \(X\) be a totally disconnected compact space, and suppose that \(f : X \to \mathbb{S}^1\) is continuous. For every \(\epsilon > 0\), there exists a continuous function \(F_\epsilon : X \to [-\epsilon, 1]\) such that \(f = \exp(2\pi i F_\epsilon)\).
Proof. With suitable choice of analytic branches of the logarithm, for each \( x \in X \), there exists a neighbourhood \( U_x \) and a continuous function \( E_x : U_x \to [-\delta, 1 + \eta] \), \( \delta, \eta \geq 0 \), such that \( f(y) = \exp(2\pi i E_x(y)) \), for \( y \in U_x \). Then compactness of \( X \) yields a finite open covering by \( \{U_{x(j)}\}_j \), and total disconnectedness yields a finite disjoint clopen covering \( \{V_j\}_{j=1}^n \) with \( V_j \subseteq U_{x(j)} \). To finish the proof, set \( F_\varepsilon = \sum_{j=1}^n \chi_{V_j} E_{x(j)} |U_{x(j)}| \) such that \( F_\varepsilon \). □

Let \((X, T)\) be a Cantor minimal system; as in [4], denote by \( E(X, T) \) the subgroup of real numbers consisting of all \( \theta \) such that \( \exp(2\pi i \theta) \in \text{Sp}(T) \). This is the additive group of (continuous) eigenvalues.

**Theorem 2.2.** Let \((X, T)\) be a transitive topological dynamical system, and suppose that \( X \) is totally disconnected. Then \( \theta \in E(X, T) \), \( 0 < \theta < 1 \), if and only if there exists a clopen set \( U = U_0 \) such that

\[
1_{U_0} - \theta \cdot 1 \text{ is a real coboundary.}
\]

Moreover, for every \( \mu \in \mathcal{M}(X, T) \), \( \mu(U_0) = \theta \).

**Proof.** Let \( f \) be a continuous eigenfunction with respect to \( \exp(2\pi i \theta) \). The function \( |f| \) is continuous and constant on orbits, at least one of which is dense. By replacing \( f \) by \( f/|f| \) if necessary, we may assume that \( |f| = 1 \). Choose \( 0 < \epsilon < \min\{\theta, 1 - \theta\} \) and write \( f = \exp(2\pi i F_\varepsilon) \) by the previous lemma. Then for all \( x \in X \), there exists an integer \( k(x) \) such that \( F_\varepsilon(x) - F_\varepsilon(Tx)) = -\theta + k(x) \).

We have \(-\epsilon \leq F_\varepsilon(T(x)) \leq 1 \) and \( \epsilon + \theta < 1 \). Thus

\[-1 < -1 + (\theta - \epsilon) \leq k(x) \leq \epsilon + F_\varepsilon(x) + \theta < 2.\]

But \( k(x) \) is an integer-valued function, so \( k(x) \in \{0, 1\} \); in addition, \( k(x) = F_\varepsilon(x) - F_\varepsilon \circ T(x) + \theta \) is continuous. Hence \( k = 1_U \) for a clopen subset \( U \) of \( X \).

For any invariant measure \( \mu \)

\[
\mu(U) = \int_X 1_U \, d\mu = \int_X k \, d\mu = \int_X (F_\varepsilon - F_\varepsilon \circ T + \theta) \, d\mu = \theta.
\]

The converse is straightforward. If \( 1_U - \theta \cdot 1 \) is a real coboundary, that is an element of \((1 - T)C(X, \mathbb{R})\), then \( \lambda := \exp 2\pi i \theta \) is an eigenvalue of \( T \). Explicitly, if \( 1_U - \theta \cdot 1 = (1 - T)F \) where \( F \) is real-valued and continuous, then we see

\[
f := \exp 2\pi i F = \exp 2\pi i (F \circ T + 1_U - \theta \cdot 1)
= f \circ T \cdot 1 \cdot \lambda^{-1}.
\]

Hence \( f \circ T = \lambda f \).

In the case that \( \theta = 1/q \), we take \( F = \sum_{j=0}^{q-1} 1_{T^jU}f/q \). □
Remark 2.3. Let \((G, u) = (K^0(X, T), [1_X])\). With the notation of sections 2, 3, the conclusion of Theorem 2.2 can be rephrased as follows: If \(\theta \in E(X, T) \cap (0, 1)\), then there exists \(g \in G^+\) such that \(\hat{g} = \theta \cdot 1\).

Let us say that a dynamical system has sufficiently many measures if every nonempty open set has nonzero measure for at least one invariant probability measure. Every minimal system obviously has this property.

Corollary 2.4. Let \((X, T)\) be a topologically transitive dynamical system where \(X\) is totally disconnected space, and there are sufficiently many \(T\)-invariant measures. If \(p\) and \(q\) are relatively prime positive integers such that \(\theta = \frac{p}{q} \in E(X, T) \cap (0, 1)\), then there exists a clopen set \(U_{\theta}\) of \(X\) such that \(q[1_{U_{\theta}}] = p[1_X]\) in \(K^0(X, T)\).

Proof. Suppose that \(\theta = \frac{p}{q}\) belongs to \(E(X, T) \cap (0, 1)\). For \(0 < \epsilon < 1\), set \(F := F_{\epsilon}\). First assume that \(p = 1\). Let \(U = U_{\theta}\) be given by Theorem 2.2. We claim that \(U, TU, \ldots, T^{q-1}U\) are pairwise disjoint.

To prove the claim, we have by the preceding, \(F - F \circ T = 1_U - \theta\) and \(F \circ T - F \circ T^2 = 1_{TU} - \theta\). Also since \(f \circ T^2 = \lambda^2 f\), it follows that \(F - F \circ T^2 = 1_V - 2\theta\) for some clopen \(V\). However, \(F - F \circ T^2 = (F - F \circ T) + (F \circ T - F \circ T^2) = 1_U + 1_{TU} - 2\theta\), or \(1_U + 1_{TU} = 1_V\); this forces \(U \cap TU = \emptyset\). The same reasoning may be applied to any pair of iterates of \(T^iU\), \(0 \leq i \leq q - 1\), and the claim follows.

By the claim and the fact that for any invariant measure \(\mu\), \(\mu(T^iU) = 1/q\), we have \(\mu(\bigcup_{i=0}^{q-1}T^iU) = 1\). Since sufficiently many measures exist, \(X = \bigcup_{i=0}^{q-1}T^iU\). As \(1_{TU}\) are all equivalent in \(K^0(X, T)\), it follows that \(q[1_U] = [X]\).

If \(p \neq 1\), there exists a positive integer \(s\) such that \(ps \equiv 1 \mod q\). Then we can apply the previous to \(T^s\) (we no longer need \(T^s\) to be topologically transitive, since we have already constructed a suitable logarithm of \(f\)).

Lemma 2.5. Let \((X, T)\) be a topologically transitive Cantor system and \(\theta \in E(X, T), 0 < \theta < 1\). If \(f, g\) are two continuous integer-valued functions on \(X\) such that \(f - \theta 1\) and \(g - \theta 1\) are real valued coboundaries, then \(f - g\) is an integer valued coboundary.

Proof. Let \(\lambda = \exp 2\pi i \theta\). We have \(F = f - \theta \cdot 1 + F \circ T\) and \(G = g - \theta \cdot 1 + G \circ T\), \(F, G \in C(X, \mathbb{R})\). As \(f\) is integer-valued, this yields \(\exp (2\pi i F) = \lambda^{-1} \exp (2\pi i F \circ T) = \lambda^{-1} \exp (2\pi i F) \circ T\).
Hence \( \exp(2\pi iF) \) and (with similar arguments) \( \exp(2\pi iG) \) are two eigenfunctions for \( T \) with the same eigenvalue \( \lambda \). As \( T \) is transitive, they are proportional; hence there exists \( t \in [0, 1) \) such that \( \exp(2\pi iF) = \exp(2\pi it) \cdot \exp(2\pi iG) \). Therefore, \( h := F - G - t1 \in C(X, \mathbb{Z}) \) and

\[
\partial_T h = \partial_T F - \partial_T G = \partial_T (F - G) = f - g
\]

which proves the lemma.

For \( x \in \mathbb{R} \), let us denote by \([x]\) (respectively \(\{x\}\)) the largest (respectively, smallest) integer less (respectively larger) than \( x \) and recall that \( \{x\} \) is the fractional part of \( x \).

**Corollary 2.6.** Let \( (X, T) \) be a Cantor minimal system. For each \( \theta \in E(X, T) \), let \( \Theta(\theta) \) be defined by

\[
\Theta(\theta) = \begin{cases} 
[\theta][1_X] + [1_{U(\theta)}] & \text{if } \theta \geq 0, \\
[\theta][1_X] + [1_{U(\theta)}] & \text{if } \theta < 0.
\end{cases}
\]

Then \( \Theta : E(X, T) \to K^0(X, T) \) is an injective homomorphism.

**Proof.** By Theorem 2.2 and Lemma 2.5, \( \Theta \) is well-defined. Now we prove that \( \Theta : E(X, T)^+ \to K^0(X, T)^+ \) is a semigroup homomorphism. Recall first that if \( \theta_1 \) and \( \theta_2 \) are positive, then \( \theta_1 + \theta_2 = [\theta_1 + \theta_2] + \{\theta_1 + \theta_2\} \). So if \( 0 < \{\theta_1\} + \{\theta_2\} < 1 \), then \( \{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\} \) and \( [\theta_1 + \theta_2] = [\theta_1] + [\theta_2] \). And if \( 1 \leq \{\theta_1\} + \{\theta_2\} < 2 \) then \( [\theta_1 + \theta_2] = [\theta_1] + [\theta_2] + 1 \) and \( \{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\} - 1 \).

Therefore, to check that \( \Theta : E(X, T)^+ \to K^0(X, T)^+ \) is a semigroup homomorphism, it suffices to consider the following cases.

1. If \( 0 \leq \theta_1 + \theta_2 < 1 \). By Theorem 2.2, there exist clopen sets \( U_{\theta_1}, U_{\theta_2}, U_{\theta_1 + \theta_2} \) and continuous real valued functions \( F_1, F_2 \) and \( F \) such that \( 1_{U_{\theta_1 + \theta_2}} = F - F \circ T + (\theta_1 + \theta_2)1_X \) and \( 1_{U_{\theta_1}} = F_1 - F_1 \circ T, i = 1, 2 \). So

\[
1_{U_{\theta_1 + \theta_2}} - (1_{U_{\theta_1}} + 1_{U_{\theta_2}}) = (F - F_1 - F_2) - (F - F_1 - F_2) \circ T
\]

and therefore, \( 1_{U_{\theta_1 + \theta_2}} - (1_{U_{\theta_1}} + 1_{U_{\theta_2}}) \in (1 - T)C(X, \mathbb{R}) \). So by Lemma 2.5, \([1_{U_{\theta_1 + \theta_2}}] = [1_{U_{\theta_1}}] + [1_{U_{\theta_2}}]\) in \( K^0(X, T) \).
(2) If $1 \leq \theta_1 + \theta_2 < 2$. Set $\theta = \theta_1 + \theta_2 - 1$. By Theorem 2.2, there exists clopen sets $U_{\theta}, U_{\theta_j}$, real valued functions $F_j, j = 1, 2$ and $F$ such that for $j = 1, 2$, $1_{U_{\theta_j}} - \theta_j = F_j - F_j \circ T$ and $1_{U_{\theta}} - \theta_1 = F - F \circ T$. Then $f_j = \exp(2\pi i F_j)$, $j = 1, 2$ and $f = \exp(2\pi i F)$ are eigenfunctions with respect to $\theta_j$ and $\theta$. As $f_1 f_2 f^{-1}$ is eigenfunction with respect to the eigenvalue 1, there exists $t \in \mathbb{R}$ such that $\exp(2\pi i t) f_1 f_2 f^{-1} = 1$ and therefore, $F_1 + F_2 - F - t \in C(X, \mathbb{Z})$. Then

$$1_{U_{\theta_1}} + 1_{U_{\theta_2}} - 1_{U_{\theta}} - 1_X = (1 - T)(F_1 + F_2 - F - t) = (1 - T)(F_1 + F_2 - F).$$

Hence,

$$[1_{U_{\theta_1}}] + [1_{U_{\theta_2}}] = [1_{U_{\theta}}] + [1_X] = [1_X].$$

Therefore, if $0 \leq \{\theta_1\} + \{\theta_2\} < 1$, then $\{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\}$ and $\{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\}$ and by case 1) we have

$$\Theta(\theta_1 + \theta_2) = \Theta(\theta_1) + \Theta(\theta_2).$$

If $1 \leq \{\theta_1\} + \{\theta_2\} < 2$, then $\{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\} + 1$ and $\{\theta_1 + \theta_2\} = \{\theta_1\} + \{\theta_2\} - 1$. Then by case 2), we will get (2.1).

Moreover, $\Theta(0) = [0]$. So we have a semigroup homomorphism which can be extended to a group homomorphism $\Theta : E(X, T) \to K^0(X, T)$. Theorem 2.2 implies that the kernel of $\Theta$ is trivial.

If $(X, T)$ is a Cantor minimal system and $(G, u) = (K^0(X, T), [1_X])$, let us write (see Section 2.3) to simplify notation,

$$J(X, T) = J(G, u), \quad I(X, T) = I(G, u).$$

Note that the group $I(X, T)$ is denoted by $I(G, G^+, u)$ in [4].

Then by Section 2.3, Theorem 2.2 and Corollary 2.6, $E(X, T)$ is a subgroup of $\text{Im}(\Phi) = I(X, T)$ and $\Theta(E(X, T)) \subset J(X, T)$. Denoting $\Phi^{-1}(E(X, T))$ by $H(X, T)$, we have the following corollary.

**Corollary 2.7.** Suppose $(X, T)$ is a Cantor minimal system. Then the short exact sequence of abelian groups,

$$\text{Inf}(K^0(X, T)) \longrightarrow H(X, T) \longrightarrow E(X, T)$$

splits (positively).
The following proposition, already stated in [13] and proved in [24], is a consequence of Corollary 2.4 and Lemma 2.5.

**Proposition 2.8.** Let \((X, T)\) be a Cantor minimal system and let \(\Theta\) be the injective homomorphism defined in Corollary 2.6. Then
\[
\Theta(E(X, T) \cap \mathbb{Q}) = \mathbb{Q}(K^0(X, T), [1_X]).
\]

**Proof.** By Corollary 2.4 if \(\theta = p/q \in E(X, T)\), then there exists a clopen set \(U\) such that \(q[1_U] = p[1_X]\). So from the definition of \(\mathbb{Q}(K^0(X, T), [1_X])\), the result follows from Corollary 2.6. \(\square\)

**Remark 2.9.** For a Cantor minimal system \((X, T)\), the rational spectrum, \(E(X, T) \cap \mathbb{Q}\) is therefore invariant under strong orbit equivalence. This was already observed in [13], [24].

**Theorem 2.10.** Let \((X, T)\) be a Cantor minimal system and \(\Theta\) be as in Corollary 2.6. Then \(K^0(X, T)/\Theta(E(X, T))\) is torsion-free.

**Proof.** Let \(g \in K^0(X, T)\). Let us show that if \(ng \in \Theta(E(X, T))\) for some \(n \in \mathbb{N}\) then \(g\) itself belongs to \(\Theta(E(X, T))\). Since \(\Theta(E(X, T))\) is totally ordered we can assume that \(ng\) is positive (otherwise, replace it by \(-ng\)) and as \(K^0(X, T)\) is unperforated, \(g \in K^0(X, T)^+ \setminus \{0\}\). By the assumption, there exists \(\theta \in E(X, T) \cap \mathbb{R}^+\) such that \(ng = \Theta(\theta)\).

If \(\theta \in \mathbb{Q}\), then by Proposition 2.8, \(ng \in \mathbb{Q}(K^0(X, T), [1_X])\), hence \(g \in \mathbb{Q}(K^0(X, T), [1_X])\). By Proposition 2.8 again, \(g \in \Theta(E(X, T))\).

We can therefore assume \(\theta \notin \mathbb{Q}\) and that there exists a proper clopen set \(U\) of \(X\) such that \(ng = [\theta][1_X] + [1_U]\) for some clopen subset \(U\) such that \(1_U - \{\theta\}1_X\) is a real coboundary.

Suppose first that \(n > [\theta]\). Then
\[
ng = [\theta][1_X] + [1_U] \leq (n-1)[1_X] + [1_U] \leq n[1_X].
\]
So, as \(K^0(X, T)\) is unperforated, \(0 < g \leq [1_X]\). Moreover, \(ng \neq n[1_X]\) which yields that \(g \neq [1_X]\) and by [16] Lemma 2.5, there exists a proper clopen set \(V\) of \(X\) such that \(g = [1_V]\).

Thus \(n1_V - [\theta]1_X - 1_U = G - G \circ T\), for some \(G \in C(X, \mathbb{Z})\). As \(\{\theta\}1_X - 1_U = F - F \circ T\) for some \(F \in C(X, \mathbb{R})\), we have
\[
1_V - \frac{\theta}{n}1_X = 1_V - \frac{[\theta] + \{\theta\}}{n}1_X = \frac{F + G}{n} - \frac{F + G}{n} \circ T.
\]
which by Theorem 2.2 means that $\theta/n \in E(X, T)$ and
$$\Theta(\theta/n) = [1_V] = g.$$

If $n \leq \lfloor \theta \rfloor$, write $\lfloor \theta \rfloor = kn + s$ with $0 \leq s < n$ and $k \geq 1$. As $ng = (kn + s)[1_X] + [1_U]$, we have
$$n(g - k[1_X]) = s[1_X] + [1_U].$$
In particular, $g - k[1_X] \geq 0$ and since $s < n$, the previous case implies that $g - k[1_X] \in \Theta(E(X, T))$. \hfill \Box

Let $(X, T)$ be a Cantor minimal system with $\text{Inf}(K^0(X, T)) = \{0\}$. Maintaining our notation, $J(X, T) \cong I(X, T)$; by Corollary 1.9, we can identify $I(X, T)/E(X, T)$ with $J(X, T)/\Theta(E(X, T))$.

**Corollary 2.11** ([4], Theorem 1). Let $(X, T)$ be a Cantor minimal system with $\text{Inf}(K^0(X, T)) = \{0\}$.

Then the quotient group $I(X, T)/E(X, T)$ is torsion-free.

**Proof.** Let $g \in K^0(X, T)$. If for some $n \in \mathbb{N}$, there exists $\theta \in E(X, T)$ with $ng = \Theta(\theta)$, then for every $\tau \in S(K^0(X, T), [1_X])$,
$$n\tau(g) = \tau(ng) = \tau(\Theta(\theta)) = \theta.$$
Hence $g \in J(X, T)$; so the torsion elements of $K^0(X, T)/\Theta(E(X, T))$ and of $J(X, T)/\Theta(E(X, T))$ are the same. By Theorem 2.10 and the above identification, the corollary is proved. \hfill \Box

**Remark 2.12.** For the identification between $J(X, T)/\Theta(E(X, T))$ and $I(X, T)/E(X, T)$ to be true, $\text{Inf}(K^0(X, T))$ has to be trivial. In [4], an example of a Cantor minimal system for which neither $\text{Inf}(K^0(X, T))$ nor the torsion subgroup of $I(X, T)/E(X, T)$ are trivial, is described.

### 3. Irrational Miscibility

Let $G$ be a simple dimension group. With notation and results of Section 2, recall that we have for $u \in G^+ \setminus \{0\}$, a short exact sequence
$$0 \rightarrow \text{Inf}(G) \longrightarrow J(G, u) \longrightarrow I(G, u) \longrightarrow 0$$
where $J(G, u) = \{g \in G : \hat{g} \text{ is constant} \}$ and $\Phi(g) = \hat{g}$.

**Definition 3.1.** Let $(G, u)$ be a non-cyclic simple dimension group. We say that $(G, u)$ is *irrationally miscible* if $\Phi(J(G, u)) \subseteq \mathbb{Q}$. Moreover, $G$ is *globally irrationally miscible* if for all choices of order units $u$, $(G, u)$ is irrationally miscible.

The following is an immediate consequence.
Proposition 3.2. Let \((G, u)\) be a non-cyclic simple dimension group with order unit \(u\). Then \((G, u)\) is irrationally miscible iff \(\bigcap_{\tau \in S(G,u)} \tau(G) \subseteq \mathbb{Q}\).

Remark 3.3. In general, we cannot take the intersection over the set of pure traces, \(\bigcap_{\partial \in \partial S(G,u)} \tau(G)\), as shown by the following drastic example.

Let \(K\) be the set of real algebraic numbers; this is a countable subfield of the reals. Equipped with the sums of squares ordering, this is well-known (and easily proved) to be a simple dimension group and a partially ordered ring with 1 as order unit, and its pure traces with respect to the order unit 1 are given by \(r \mapsto \gamma(r)\) where \(\gamma : K \to K\) is a Galois automorphism (we view the second copy of \(K\) as a subgroup of the reals).

In particular, for every pure trace \(\gamma\) on \((K, 1)\), we have \(\gamma(K) = K\) (viewing the latter as a subgroup of \(\mathbb{R}\)), and thus \(\bigcap_{\gamma \in \partial S(G,u)} \gamma(K) = K\).

However, \(\bigcap_{\tau \in S(K,1)} \tau(K) = \mathbb{Q}\).

It suffices to show \(\{ \lambda \in \mathbb{R} \mid \hat{g} = \lambda I\} = \mathbb{Q}\). Select \(g \in K\); if \(\hat{g} = \lambda I\), then for every Galois automorphism, we have that \(\gamma(g)\) is the same constant, \(\lambda\); hence \(g\) belongs to the fixed point subgroup of the Galois group, hence is a rational number.

If we replace the order unit 1 by any other order unit, the same conclusions apply, as is easy to verify, since \(K\) is a field. Hence \(K\) with the sums of squares ordering is globally irrationally miscible.

Of course, we can obtain a simpler example with quadratic squares ordering. There are two pure traces, each has range \(K\) itself, but the intersection over all the traces of their images is just \(\mathbb{Q}\), via the same argument. The infinite-dimensional example is more interesting.

Before giving sufficient conditions for a non-cyclic simple dimension group to be globally irrationally miscible, let us describe the implication of this property for dynamical systems.

Let \((X, T)\) be a Cantor minimal system and \((G, u) = (K^0(X, T), [1_X])\). By results in section 3, we have

\[ \Theta(E(X, T)) \subseteq J(X, T) \quad \text{and} \quad \Phi \circ \Theta(\theta) = \theta, \quad \forall \theta \in E(X, T). \]

The condition \(\bigcap_{\tau \in S(G,u)} \tau(G) \subseteq \mathbb{Q}\) is not affected when we factor out the infinitesimals from \(G\). Therefore, if \((K^0(X, T), [1_X])\) is irrationally
miscible, then the additive group of continuous eigenvalues of any Cantor minimal systems orbit equivalent to \((X, T)\) is contained in \(\mathbb{Q}\).

**Proposition 3.4.** If \(G\) is a simple dimension group with a rational-valued trace, then \(G\) is globally irrationally miscible.

**Proof.** Follows immediately from Proposition 3.2 above, since there exists a trace \(\tau\) such that \(\tau(G) \subset \mathbb{Q}\), and renormalizations (changing the order unit) do not affect this property. \(\square\)

Recall that a matrix has *equal column sums* if all its column sums are the same.

**Corollary 3.5.** Let \(\{A_n\}\) be a sequence of nonnegative integer matrices, with equal column sum such that the direct limit \(G\) of \(A_n : \mathbb{Z}^{f(n)} \rightarrow \mathbb{Z}^{f(n+1)}\) is simple (and non cyclic). Then \(G\) is globally irrationally miscible.

**Proof.** Let \(c_n\) be the column sum of \(A_n\). For \(a = [a(n), n] \in G\) and \(c(n) = \prod_{1 \leq n} c_1\) then \(\tau(a) = \frac{c(n)}{\sum_n a(n)}\) defines a trace on \(G\), and \(\tau(G) \subset \mathbb{Q}\). \(\square\)

**Example 3.1.** The condition of Corollary 3.5 applies when all the matrices \(\{A_n\}\) are circulant. For example, let \((B, V)\) be the Bratelli diagram with \(|V_n| = 2\) for all \(n \geq 2\) and for any \(n \geq 1\), \(A_n\) be the symmetric matrix, \(\begin{bmatrix} k_n & \ell_n \\ \ell_n & k_n \end{bmatrix}\). By Corollary 3.5, the associated dimension group, \(G\) is irrationally miscible. Moreover, \(\text{Inf} (G)\) is trivial. Therefore any Cantor minimal system \((X, T)\) with \(K^0(X, T)\) order isomorphic to \(G\) has only rational continuous eigenvalues.

Recall that for an abelian group \(H\), rank \(H\) denotes the dimension of the rational vector space \(H \otimes \mathbb{Q}\).

**Proposition 3.6.** Let \(G\) be a simple dimension group with \(n\) pure traces, \(n > 1\).

If either \(\text{rank} G/\text{Inf} (G) = n\) or both \(\text{rank} G/\text{Inf} (G) = n + 1\) and \(G/\text{Inf} (G)\) is finitely generated, then \(G\) is globally irrationally miscible.

**Proof.** Let \(u\) be an order unit of \(G\) and \(\{\sigma_i : 1 \leq i \leq n\}\) be the set of pure (normalized) traces on \(G\). Let \(\tilde{G}\) denote both the (simple dimension) group \(G/\text{Inf} (G)\) and the dense image of \(G\) in \(\text{Aff}(S(G, u)) \simeq \mathbb{R}^n\).
Let \( \hat{\Phi} \) denote the restriction to \( \hat{G} \) of the map \( \Phi \) from \( \text{Aff}(S(G, u)) \) to \( \mathbb{R}^{n-1} \), given by
\[
\Phi(f) = (\sigma_2(f) - \sigma_1(f), \ldots, \sigma_n(f) - \sigma_1(f)), \quad f \in \text{Aff}(S(G, u)).
\]

If \((G, u)\) is not irrationally miscible, then there exists an irrational number \( \lambda \) and \( g \in G \) such that \( \hat{g} = \lambda \hat{u} \). Then the \( \ker \hat{\Phi} \) contains both \( \hat{g} \) and \( \hat{u} \), so has rank at least 2 and thus \( \text{Im}(\hat{\Phi}) \) has rank at most \( \text{rank}(\hat{G}) - 2 \).

As \( \Phi \) is continuous and surjective, \( \hat{\Phi}(\hat{G}) \) is dense in \( \mathbb{R}^{n-1} \) and therefore must be of rank at least \( n - 1 \). Hence, we obtain a contradiction if \( \text{rank}(\hat{G}) = n \).

If \( \text{rank}(\hat{G}) = n + 1 \) and \( \hat{G} \) is finitely generated, then we note that a free dense subgroup of \( \mathbb{R}^{n-1} \) must have rank at least \( n \), again reaching a contradiction. \( \square \)

If \( G \) is a simple dimension group with an order unit \( u \), we denote (as in the proof of the above proposition) by \( \hat{G} \) both the (simple dimension) group \( G/\text{Inf}(G) \) and the dense image of \( G \) in \( \text{Aff}(S(G, u)) \approx \mathbb{R}^n \). Set \( J = \{ \hat{g} \in \hat{G} : \hat{g} \text{ is constant} \} \).

**Proposition 3.7.** Let \( G \) be a simple dimension group with order unit \( u \) such that \( \text{rank}(G/\text{Inf}(G)) < \infty \). Let \( l \) be the number of pure traces of \((G, u)\). Then \( l \leq \text{rank}(\hat{G}) \) and \( \text{rank}(\hat{G}/J) \geq l - 1 \). If \( G \) itself has finite rank, then
\[
\text{rank } G \geq \text{rank } J + l - 1;
\]
and if \( G/\text{Inf}(G) \) is finitely generated, then \( \text{rank } G/J \geq l \).

**Proof.** Since \( \text{Inf}(G) \subset J \), \( G/J \approx (G/\text{Inf}(G))/(J/\text{Inf}(G)) \), whence the former is of finite rank. Let \( \{\sigma_1, \ldots, \sigma_l\} \) be the set of pure traces of \( G \) and let \( \Phi : \text{Aff}(S(G, u)) \to \mathbb{R}^{l-1} \) be given (as in the proof of Proposition 3.6) by\[
\Phi(f) = (\sigma_2 - \sigma_1(f), \ldots, \sigma_l - \sigma_1(f)), \quad f \in \text{Aff}(S(G, u)).
\]

Then \( \Phi(G) \) is a dense subgroup of \( \mathbb{R}^{l-1} \), and as \( J \subset \ker(\Phi) \), \( \Phi(G/J) \) is isomorphic to a dense subgroup of \( \mathbb{R}^{l-1} \). Hence, \( \text{rank}(G/J) \geq l - 1 \). If \( G/\text{Inf}(G) \) is finitely generated, then \( \text{rank}(G/J) \geq l \) (as a dense subgroup of \( \mathbb{R}^{l-1} \) must have rank at least \( l \)). \( \square \)
If \((X, T)\) is a Cantor minimal system and \((G, u) = (K^0(X, T), [1_X])\), then by results in section 3, \(H(X, T) = \Theta(E(X, T)) \subset J(X, T)\) and therefore,
\[
\text{rank } (E(X, T)) \leq \text{rank } (J(X, T)).
\]
If \((X, T)\) has \(n\) ergodic invariant measures, then the proposition above yields
\[
\text{(3.1) } \text{rank } (K^0(X, T)) \geq \text{rank } (E(X, T)) + n - 1
\]
or
\[
\text{rank } (K^0(X, T)) \geq \text{rank } (E(X, T)) + n
\]
if \(K^0(X, T)\) is finitely generated.

4. **Weak mixing Bratteli-Vershik systems**

A topological dynamical system \((X, T)\) is **weakly mixing** if the product system \((X \times X, T \times T)\) is topologically transitive, or equivalently, if for any two nonempty open sets \(U\) and \(V\) of \(X\), the set \(N(U, V) := \{n \in \mathbb{Z} : T^n U \cap V \neq \emptyset\}\) is **thick** \([10]\). Recall that a subset of \(\mathbb{Z}\) is **thick** if
\[
\forall k \in \mathbb{N}, \exists n; n, n + 1, \ldots, n + k \in N(U, V).
\]
By \([27]\), Proposition 3], \((X, T)\) is weakly mixing if and only if for any pair of nonempty open sets \(U\) and \(V\) there exist \(n \in \mathbb{N}\) such that \(n, n + 1 \in N(U, V)\).

Recall that for a minimal system, \((X, T)\), to be weakly mixing, it is necessary and sufficient that its additive continuous spectrum \(E(X, T)\) is equal to \(\mathbb{Z}\) (see \([15]\) for example).

We will denote by \(\mathcal{WM}\), the collection of weakly mixing Cantor minimal systems.

For a simple dimension group \((G, u)\), let \(\text{SOE}(G, u)\) denote the class of Cantor minimal systems \((X, T)\) such that \((K^0(X, T), [1_X])\) is order isomorphic to \((G, u)\). Recall (see Theorem \([1.11]\)) that two Cantor minimal systems belong to \(\text{SOE}(G, u)\) if and only if they are strongly orbit equivalent.

Similarly, for a simple dimension group \((G, u)\) whose infinitesimals subgroup is trivial, let \(\text{OE}(G, u)\) be the class of all Cantor minimal systems \((X, T)\) such that \((K^0(X, T)/\text{Inf } (K^0(X, T)), [1_X])\) is order isomorphic to \((G, u)\). Two Cantor minimal systems belong to \(\text{OE}(G, u)\) if and only if they are orbit equivalent (see Theorem \([1.10]\)).
Proposition 4.1. Let \((G, u)\) be a simple dimension group with order unit \(u\). If \(\mathbb{Q}(G, u) = \mathbb{Z}\) and \((G, u)\) is irrationally miscible, then \(\text{SOE}(G, u) \subset W.M\).

Remark. In other words, if \((X, T)\) is a minimal Cantor system such that \((K^0(X, T), [1_X])\) is irrationally miscible and has trivial rational spectrum, then \((X, T)\) is weakly mixing.

Proof. Let \((X, T)\) be a Cantor minimal system with \((K^0(X, T), [1_X]) = (G, u)\). As \((G, u)\) is irrationally miscible and by Theorem 2.2, \(E(X, T) \subseteq \mathbb{Q}\). So by Proposition 2.8, \(\Theta(E(X, T)) = \mathbb{Q}(G, u) = \mathbb{Z}\). □

The following theorem is a direct corollary of [24, Theorem 6.1].

Theorem 4.2. Let \((G, u)\) be a simple dimension group whose rational subgroup \(\mathbb{Q}(G, u)\) is equal to \(\mathbb{Z}\). Then there exists a topologically weakly mixing Cantor minimal system \((X, T)\) belonging to \(\text{SOE}(G, u)\).

Proof. Let \((Y, \nu, T)\) be a measurable weakly mixing system on a Lebesgue space. By [24, Theorem 6.1], there exists a Cantor minimal system \((X, S)\) with \((K^0(X, S), [1_X]) \simeq (G, u)\) and \(\mu \in \mathcal{M}_S(X)\) such that \((X, S, \mu)\) is isomorphic to \((Y, \nu, T)\). In particular, \(E(X, S) = \mathbb{Z}\). □

Proposition 4.3. Let \((G, u)\) be a simple non-cyclic dimension group with trivial infinitesimal subgroup. If \((G, u)\) is irrationally miscible and \(\mathbb{Q}(G, u) \subset \mathbb{Z}\) then

\[\text{OE}(G, u) \subset W.M.\]

Proof. Recall that by Proposition 3.2 if \((H, v)\) is a simple non-cyclic dimension group, then \((H, v)\) is irrationally miscible if and only if \((H/\text{Inf}(H), [v])\) is also irrationally miscible. Moreover, if \(x \in \mathbb{Q}(H, v)\) then the image \([x]\) in \(H/\text{Inf}(H)\) belongs to \(\mathbb{Q}(H/\text{Inf}(H), [v])\).

Let \((X, T)\) be a Cantor minimal system belonging to \(\text{OE}(G, u)\). Then \((K^0(X, T), [1_X])\) is irrationally miscible and

\[\Theta(E(X, T)) \subset \mathbb{Q}(K^0(X, T), [1_X]).\]

As \(\mathbb{Q}(K^0(X, T), [1_X]) = \mathbb{Z}\), it follows that \((X, T)\) is weakly mixing. □

In the rest of this section, we are going to prove that:

Theorem 4.4. Let \(B = (V, E)\) be a simple Bratteli diagram such that \(\mathbb{Q}(K^0(V, E), [v_0]) = \mathbb{Z}\). Then there exists a telescoping \(\tilde{B} = (\tilde{V}, \tilde{E})\) of \(B\) and a proper ordering on \(\tilde{B}\) whose associated Bratteli-Vershik system is weakly mixing.
To prove this theorem, we will use Corollary 4.6 and Lemma 4.7. The first one is a result of [24] which we prove for sake of completeness. It gives a characterization of Bratteli diagrams whose associated dimension group has a trivial rational subgroup. Lemma 4.7 gives a technical characterization for a Bratteli-Vershik transformation to be weakly mixing.

**Lemma 4.5.** Let \((G, u)\) be a simple dimension group with order unit \(u\). Then its rational subgroup \(\mathbb{Q}(G, u)\) is trivial if and only if the equation \(nx = u\) is not solvable for any \(x \in G, x \neq u, \) and \(n \in \mathbb{Z}\).

**Proof.** If \(\mathbb{Q}(G, u) = \mathbb{Z}\), then (1) is clearly satisfied. By Proposition 3.4, we only have to prove that (1) implies \(\mathbb{Q}(G, u) = \mathbb{Z}\).

If not, let \(x \in G\) such that \(px = qu\) with \((p, q) = 1\). As there exists \(a, b \in \mathbb{Z}\) with \(ap + bq = 1\) we have \(bpx = bqu = (1 - ap)u\). Hence, \(p(bx + au) = u\), which contradicts (1).

**Lemma 4.5** forces a combinatorial property for any Bratteli diagram \(B = (V, E)\) with \(K^0(V, B) = G\). That is Corollary 4.6.

Let \(B = (V, E)\) be a Bratteli diagram with \(V = \bigsqcup_{k \geq 0} V_k, V_0 = \{v^0\}, V_k = \{v^k_1, v^k_2, \ldots, v^k_{n(k)}\}\).

For \(k \geq 1\) and \(1 \leq j \leq n(k)\), let \(h^k_j\) denote the number of (finite) paths from \(v^0\) to \(v^k_j\). We assume that \(h_k = (h^k_1, \ldots, h^k_{n(k)}) \in \mathbb{N}^{n(k)}\).

**Corollary 4.6.** Let \(B = (V, E)\) be as above and \(K^0(V, E)\) its associated (simple) dimension group. Then \(\mathbb{Q}(K^0(V, E), [v^0]) = \mathbb{Z}\) if and only if for all \(k \geq 1\), \(\gcd(h^k_1, \ldots, h^k_{n(k)}) = 1\).

**Lemma 4.7.** Let \((B, \leq)\) be a properly ordered Bratteli diagram such that the minimal path \(e_{\min} = (e_1, e_2, \ldots)\) satisfies

\[ r(e_k) = v^k_1, \quad \forall k \geq 1. \]

Then the associated Bratteli-Vershik system \((X_B, T_B)\) is weakly mixing if and only if the following condition \(C(4.7)\) is satisfied:

\[ \forall k \geq 1, \exists n; \{n, n + 1\} \subset N(C(e^k), C(e^k)) \quad C(4.7) \]

where \(C(e^k)\) is the cylinder set defined by the finite path \(e^k = (e_1, \ldots, e_k)\).
Proof. By [27, Proposition 3] condition $C(4.7)$ is necessary. We prove its sufficiency in two steps.

**Step 1:** Condition $C(4.7)$ implies that
\[
\forall k \geq 1, \forall a, b, \exists n; \{n, n + 1\} \subset N(C(a), C(b)), \quad C_1(4.7)
\]
where $a, b$ are two finite paths from $v^0$ to $v^k_1$.

To see this, let us first notice that if $\{n, n + 1\} \subset N(C(e_k^0), C(e_k^0))$ and $0 \leq i \leq h^k_1$, then
\[
\{n + i, n + i + 1\} \subset N(C(e_k^0), C(T_i^i e_k^0)).
\]
As for $a, b \in E(v^0, v^k_1)$, with $a < b$, there exists $0 \leq i \leq h^k_1$ and $1 \leq j \leq h^k_1 - i$ such that $a = T_i^i e_k^0$ and $b = T_j^j + i e_k^0$, condition $C_1(4.7)$ follows.

**Step 2:** Condition $C_1(4.7)$ implies that $(X_B, T_B)$ is weakly mixing.

As the cylinders form a basis for the topology on $X_B$, it is enough to show that for any two cylinder sets $U$ and $V$, $N(U, V)$ contains two consecutive integers. Then for $k$ large enough there exist two finite paths $a, b \in E(v^0, v^k_1)$ such that $C(a) \subset U$ and $C(b) \subset V$. As $N(C(a), C(b)) \subset N(U, V)$, step 2 is proved.

Let us recall that if $\{h_1, h_2, \ldots, h_n\}$ is a set of $n$ positive integers having greatest common divisor one, then for every integer $m$ bigger than $(\min(h_i) - 1)(\max(h_i) - 1)$, there exist $x_1, x_2, \ldots, x_n \in \mathbb{Z}^+$ such that $m = \sum_{j=1}^n x_j h_j$. This result was proved in 1935 by I. Schur, but not published until 1942 by A. Brauer [2]. This will be used in the proof of Proposition 4.4.

**Proof of Theorem 4.4.**

Let $B = (V, E)$ with $V_0 = \{v^0\}$ and $V_k = \{v^k_1, \ldots, v^k_n(k)\}$. By Lemma 4.6 and Schur’s result, there exist for each $k \geq 1$, some $\ell_k \in \mathbb{N}$ and $x^k_1, \ldots, x^k_n(k) \in \mathbb{N} \cup \{0\}$ such that
\[
(4.1) \quad \ell_k h^k_1 + 1 = \sum_{j=1}^{n(k)} x^k_j h^k_j, \quad x^k_j \geq 0.
\]
By induction, we can construct a sequence
\[
1 = k_1 < k_2 < k_3 < \ldots
\]
such that
\[ \#e(v^k_{m}, v^{k+1}_m) > \ell_{km} + x^k_m, \]
\[ \#e(v^k_{j}, v^{k+1}_j) > x^k_j, \quad 2 \leq j \leq n(k). \]

where for \( p < q \), \( \#e(v^p_j, v^q_i) \) is the number of finite paths from \( v^p_j \) to \( v^q_i \).

So by telescoping the diagram along the sequence \((k_m)_{m \geq 1}\), we can assume that for every \( k \geq 1 \), (4.1) holds and
\[ \#e(v^k_1, v^{k+1}_1) > \ell_k + x^k_1, \]
\[ \#e(v^k_j, v^{k+1}_j) > x^k_j, \quad 2 \leq j \leq n(k). \]

To complete the proof of the theorem, we define an ordering \( \leq \) on \( B = (V, E) \) satisfying condition \( C(4.7) \) and such that \((B, \leq)\) is properly ordered. For \( k \geq 1 \) and \( 2 \leq j \leq n(k+1) \), we order the edges of \( r^{-1}(v^k_{j+1}) \) from left to right. To describe the linear order on \( r^{-1}(v^k_1) \), let us introduce the following notation.

For \( 1 \leq j \leq n(k) \), let \( \max^k_j = \#e(v^k_1, v^{k+1}_j) \) and
\[ e(v^k_i, v^{k+1}_i) = \{e(i, j) : 1 \leq j \leq \max^k_i\}. \]

Then the order on \( r^{-1}(v^k_{j+1}) \) is determined by ordering the pairs \( \{(i, j) : 1 \leq j \leq \max^k_i, 1 \leq i \leq n(k)\} \) as follows:
\[
(1, 1) < (1, 2) < \cdots < (1, \ell_k + x^k_1) < \\
(2, 1) < (2, 2) < \cdots < (2, x^k_2) < \\
\vdots \\
(n(k), 1) < (n(k), 2) < \cdots < (n(k), x^k_{n(k)}) < \\
(1, \ell_k + x^k_1 + 1) < \cdots < (1, \max^k_1) < \\
(2, x^k_2 + 1) < \cdots < (2, \max^k_2) < \\
\vdots \\
(n(k), x^k_{n(k)} + 1) < \cdots < (n(k), \max^k_{n(k)}).
\]

It is easy to check that with this ordering \((B, \leq)\) is properly ordered with a single min path (resp. max path) going through \( v^k_1 \) (resp. \( v^k_{n(k)} \)) for all \( k \geq 1 \).

To finish the proof, let us verify that condition \( C(4.7) \) is satisfied. Let \( e = (e_1, e_2, \ldots) \) be the minimal path of \( X_B \). Then by the above definition of the ordering on \( B \), \( e_n \in e(v^{n-1}_1, v^n_1) \) is equal to \( e_n(1, 1) \). So
belongs to the cylinder set \( C(e^k) \), where \( e^k = (e_1, e_2, \ldots, e_k) \). Since \( \#e(v^0, v_1^k) = h_1^k \),

\[
T_B^{\ell_k h_1^k}(e) = (e_1, e_2, \ldots, e_k, e_{k+1}(1, \ell_k + 1), \ldots) \in C(e^k)
\]

and therefore,

\[
\ell_k h_1^k \in \mathcal{N}(C(e^k), C(e^k)).
\]

Let \( a = (e_1, \ldots, e_k, e_{k+1}(1, \ell_k + 1), \ldots) \in C(e^k) \). By (4.1),

\[
T_B^{\ell_k h_1^k + 1}(a) = (e_1, \ldots, e_k, e_{k+1}(1, \ell_k + x_1^k + 1), \ldots)
\]

also belongs to \( C(e^k) \) and therefore, condition \( C(4.7) \) is verified. □

We adopt the following notation for Proposition 4.8.

- \( \Sigma_k = \{ \text{paths from } v_0 \text{ to } v_k \in V_k, i = 1, \ldots, n(k) \} \).
- \( \Sigma_Z^k = \text{shift space with the finite alphabet } \Sigma_k \).
- \( \pi_k : X_B \to \Sigma_k \) truncation map, restricts each infinite path of \( X_B \) to its first \( k \) coordinates.
- \( Y_k = \{ (\pi_k(T_B^n x))_{n \in \mathbb{Z}} : x \in X_B \} \subseteq \Sigma_Z^k \).
- \( S_k \) is the shift map on \( Y_k \).

It is well-known that \( \pi_k \) is a factor map and \( (X_B, T_B) \) is an extension of \( (Y_k, S_k) \).

**Proposition 4.8.** Let \( (B, \leq) = (V, E, \leq) \) be a properly ordered Bratteli diagram and \( (X_B, T_B) \) the associated Bratteli-Vershik system. With the above notation, \( (X_B, T_B) \) is weakly mixing if and only if \( (Y_k, S_k) \) is weakly mixing for all \( k \geq 1 \).

**Proof.** If \( (X_B, T_B) \in \mathcal{WM} \) then \( (Y_k, S_k) \in \mathcal{WM} \) as \( \pi_k \) is a factor map. Conversely, it is enough to observe that for any two cylinder sets \( U = [e_1, \ldots, e_k] \) and \( V = [f_1, \ldots, f_k] \) of \( X_B, \mathcal{N}(U, V) \) is thick. Let us denote by \( \bar{U} \) (resp. \( \bar{V} \)) the cylinder of \( Y_k \) given by \( \bar{U} = \{ y \in Y_k : y_0 = (e_1, \ldots, e_k) \} \) (resp. \( \bar{V} = \{ y \in Y_k : y_0 = (f_1, \ldots, f_k) \} \)). If \( (Y_k, S_k) \in \mathcal{WM} \) then \( \mathcal{N}(\bar{U}, \bar{V}) \) is thick, and therefore \( \mathcal{N}(U, V) \) has the same property. □

One class of examples of simple dimension groups satisfying both the conditions of Proposition 4.1 occurs when we combine the gcd= 1 condition with Corollary 3.5.

For example, let \( G \) be an extension, \( \mathbb{Z} \to G \to \mathbb{Z}[1/3] \) with the strict ordering induced by the map to \( \mathbb{Z}[1/3] \); an easy example is the stationary direct limit implemented by the matrix \( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \); this yields a
non-split extension. By choosing an element \( u \in G^+ \), not an integer multiple of any other elements of \( G \), \( \mathbb{Q}(G, u) = \mathbb{Z} \). Moreover, the unique state \( \tau \) of \( G, u \) satisfies \( \tau(G) \subset \mathbb{Q} \). By Corollary 3.5 and Proposition 4.1, \( \text{SOE}(G, u) \subset \text{WM} \).

On the other hand, \( G/\text{Inf} \) is isomorphic to \( \mathbb{Z}[1/3] \), so \( (X, T) \) is orbit equivalent to the 3-odometer, which is not weakly mixing. Recall that by [13, Corollary 2], any uniquely ergodic Cantor minimal system is orbit equivalent to a minimal Cantor system which is a Denjoy’s or an odometer, hence is not weakly mixing.

However, when the system is not uniquely ergodic, somewhat more interesting phenomena can occur and we may have a system which is not even orbit equivalent to a non-weakly mixing system. For instance, if we consider \( G \) to be the direct limit implemented by the matrices \( \begin{pmatrix} k_n & l_n \\ l_n & k_n \end{pmatrix} \), \( n \geq 1 \), then \( \text{Inf}(G) \) is trivial and by Proposition 3.4, there exists a rational valued trace on it. It is not hard to choose a sub-diagram which satisfies the gcd= 1 condition as well. Then by Corollary 4.3, any Cantor minimal system orbit equivalent to the Vershik system associated to this sub-diagram, is weakly mixing.

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