Stochastic interpretation of Kadanoff-Baym equations

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We show that the nonperturbative quantum transport equations, the so called ‘Kadanoff-Baym equations’, within the non-equilibrium real time Green’s function description can be understood as the ensemble average over stochastic equations of Langevin type. For this we couple a free scalar boson quantum field to an environmental heat bath with some given temperature $T$. The inherent presence of noise and dissipation related by the fluctuation-dissipation theorem guarantees that the modes or particles become thermally populated on average in the long-time limit. This interpretation leads to a more intuitive physical picture of the process of thermalization and of the interpretation of the Kadanoff-Baym equations. One also immediately understands that the non-equilibrium real time Green’s function description, the so called ‘Kadanoff-Baym equations’, within perturbative equations of motion (by various approximations) is resolved by a clear physical necessity of damping within the one-particle propagator. The occurrence of such ill-defined terms arising solely in a strictly perturbative expansion in out of equilibrium quantum field theory has a natural interpretation in analogy to Fermi’s golden rule.

I. MOTIVATION

Non-equilibrium many body theory had been traditionally a major topic of research for describing various transport processes in condensed matter physics and nuclear physics. Over the last years a lot of interest for non-equilibrium quantum field theory has now emerged in particle physics. A very powerful diagrammatic tool is the non-equilibrium Green’s function technique by means of non-equilibrium Green’s functions for describing a quantum system also beyond thermal equilibrium [2]. The resulting causal and non-perturbative equations of motion (by various approximations), the so called Kadanoff-Baym equations [1], have to be considered as an ensemble average over the initial density matrix characterizing the preparation of the initial state of the system. Typically, if the system is close to thermal equilibrium, the (initial or resulting) density matrix allows for thermal fluctuations which should be inherent to the transport process under consideration.

If fluctuations are physically present in the course of the evolution, then, according to the famous fluctuation-dissipation theorem [4], also dissipation must be present. Or, in turn, if the (quantum) system behaves dissipatively, as a consequence, there must exist fluctuations. Again, the Kadanoff-Baym equations have to be understood as an ensemble average over all the possible fluctuations. This inherent stochastic aspect of the Kadanoff-Baym equations is what we want to point out and thus provide, as we believe, some new physical insight into its merely complex structure [2]. Some of our conclusions are already known and can be found to some extent in various different studies [3, 5]. A new aspect, however, is the intimate connection of the Kadanoff-Baym equations to Langevin like processes.

As a reminder of a Langevin process let us first briefly review the description of classical Brownian motion. Consider a classical system described by the variables $Q$ and $P$ interacting with a (heat) bath of oscillators of various frequencies [6] according to the hamiltonian

$$H = H_S(Q, P) + \sum_\nu \left( \frac{p_\nu^2}{2m_\nu} + \frac{m_\nu \omega_\nu^2}{2} q_\nu^2 \right) - Q \cdot \sum_\nu \Gamma_\nu \dot{q}_\nu.$$  

From the resulting equations of motion the bath degrees of freedom can formally be integrated as

$$q_\nu(t) = q_\nu(0) \cos \omega_\nu t + \frac{p_\nu(0)}{m_\nu \omega_\nu} \sin \omega_\nu t + \int_0^t dt' \sin \omega_\nu (t-t') \Gamma_\nu Q(t'),$$  

so that the system’s variables obey the effective Langevin-like equation of motion

$$\ddot{Q} + \frac{\partial H_S^{(m)}}{\partial Q} + 2 \int_0^t dt' \Gamma(t-t') \dot{Q}(t') = \xi(t).$$  

The bare hamiltonian $H_S$ is slightly modified as

$$H_S^{(m)}(Q, P) = H_S(Q, P) + \left( \sum_\nu \frac{(-\Gamma_\nu^2)}{2m_\nu \omega_\nu^2} \right) Q^2.$$  

On the other hand a Stokes-type frictional term (in general nonlocal in time) with a dissipation kernel

$$\Gamma(t-t') = \frac{1}{2} \sum_\nu \frac{\Gamma_\nu^2}{m_\nu \omega_\nu^2} \cos \omega_\nu (t-t')$$  

enters into the left hand side of (1.3) supplemented by a source term $\xi(t)$ on the right hand side being given as...
\[ \xi(t) = \sum_{\nu} \Gamma_{\nu} \left( \left[ q_{\nu}(0) - \frac{\Gamma_{\nu}}{m_{\nu}\omega_{\nu}} Q(0) \right] \cos \omega_{\nu} t + \frac{p_{\nu}(0)}{m_{\nu}\omega_{\nu}} \sin \omega_{\nu} t \right) . \]

In \( \xi \) especially the initial conditions of the bath oscillators enter. In general, they are not known exactly, but will follow some statist distribution. Hence, \( \xi(t) \) has to be interpreted as a stochastic or ‘noisy’ source driving the fluctuations of the systems degrees of freedom. Indeed, if the initial conditions are randomly choosen by a thermal distribution \( P[q(0), p(0)] \approx e^{-H_{B}/k_{B}T} \), \( \xi(t) \) is completely specified by a Gaussian distribution (‘central limit theorem’) with zero mean and the correlation kernel \( I \)

\[ I(t - t') = \langle \xi(t)\xi(t') \rangle = k_{B}T \sum_{\nu} \frac{\Gamma_{\nu}^{2}}{m_{\nu}\omega_{\nu}^{2}} \cos \omega_{\nu}(t - t') \approx 2k_{B}T \delta(t - t') . \]

\( \langle \ldots \rangle \) denotes the average over all possible realizations of the stochastic variable \( \xi(t) \). The dissipation kernel \( \Gamma \) and the random force \( \xi(t) \) have the same microscopic origin. In fact, as just derived, there exists a simple relation between them, called the fluctuation-dissipation theorem. The evolution of the systems degrees of freedom are thus supplemented by an interplay between a dissipative term and a fluctuating source. On the average, they should then equilibrate thermally. As a special application consider a heavy ‘Brownian’ particle with mass \( M \) placed in a thermal environment obeying an effective Langevin equation as just stated, i.e.,

\[ M \ddot{x} + 2 \int_{-\infty}^{t} dt' \Gamma(t - t') \dot{x}(t') = \xi(t) . \]

(Here we assume that the friction kernel \( \Gamma(t) = \Gamma(-t) \) is ‘well behaved’ meaning that its Fourier transform \( \Gamma(\omega) \geq 0 \) and has only some finite extension in time.) To convince oneself that such a description in the long time limit is in accordance with the equipartition condition \( \langle p^{2} \rangle / (2M) = T/2 \), one has to solve \([1.7]\) by means of the retarded and advanced Green’s function:

\[ \tilde{G}_{\text{ret}}(\omega) = \frac{1}{-i \omega + \left( \frac{1}{\pi} \Gamma(\omega) + \frac{1}{\pi M} P \int d\omega' \frac{\Gamma(\omega')}{\omega' - \omega} \right) + \epsilon} = \tilde{G}_{\text{av}}^{*}(\omega) . \]

One now finds in the long-time limit on the average the desired property

\[ \lim_{t \to \infty} \langle p^{2}(t) \rangle = \int d\omega \tilde{G}_{\text{ret}}(\omega) \tilde{I}(\omega) \tilde{G}_{\text{av}}(\omega) = T M . \]

Hence, irrespective of the detailed form of the friction kernel \( \Gamma(t) \), the equipartition condition is automatically fulfilled as long as the noise kernel \( I(t) \) fulfills the fluctuation-dissipation relation \([1.6]\).

For a further physical motivation let us now return to quantum field theory and already point out some similarities. One of the major present topics in quantum field theory at finite temperature or near thermal equilibrium concerns the evolution and behavior of the long wavelength modes. These modes often lie entirely in the non-perturbative regime. Therefore solutions of the classical field equations in Minkowski space have been widely used in recent years to describe long-distance properties of quantum fields that require a non-perturbative analysis. A justification of the classical treatment of the long-distance dynamics of bosonic quantum fields at high temperature is based on the observation that the average thermal amplitude of low-momentum modes is large. For a weakly coupled quantum field the occupation number of a mode with wave vector \( \vec{p} \) and frequency \( \omega_{\vec{p}} \) is given by the Bose distribution

\[ n(\omega_{\vec{p}}) = \left( e^{\omega_{\vec{p}}/T} - 1 \right)^{-1} . \]

For temperatures \( T \) much higher than the (dynamical) mass scale \( m^{*} \) of the quantum field, the occupation number for the low-momentum modes \( \omega_{\vec{p}} \ll T \) becomes large and approaches the classical equipartition limit

\[ n(\omega_{\vec{p}}) \to 0 \quad T / m^{*} \gg 1 . \]

The classical field equations (which can be understood as a coherent state approximation to the full quantum field theory) should provide a good approximation for the dynamics of such highly occupied modes. At a closer look, however, the cogency of this heuristic argument suffers considerably. The thermodynamics of a classical field is only defined if an ultraviolet cut-off \( k_{c} \) is imposed on the momentum \( \vec{p} \) such as a finite lattice spacing \( a \). Many, if not most, thermodynamical properties of the classical field depend strongly on the value of the cut-off parameter \( k_{c} \) and diverge in the continuum limit \( (k_{c} \to \infty) \). In a correct semi-classical treatment of the soft modes the hard modes thus cannot be neglected, but it should incorporate their influence in a consistent way. In a recent paper of one of us \([3]\) it was shown how to construct an effective semi-classical action for describing not only the classical behavior of the long wavelength modes below some given cutoff \( k_{c} \), but taking into account also perturbatively the interaction among the soft and hard modes. By integrating out the ‘influence’ of the hard modes on the two-loop level for standard \( \phi^{4} \)-theory the emerging semi-classical equations of motion for the soft fields can be derived from an effective action and become stochastic equations of motion of Langevin type \([3]\):

\[ \frac{\partial^{2} \phi}{\partial t^{2}} + \left( k^{2} + \tilde{m}^{2} + \sum_{i=a,b,c} \mu_{i,k_{c}} \right) \phi + \left( \frac{\tilde{g}^{2}}{6} + \mu^{(d)}_{a,k_{c}} \right) \phi^{3} \]
+ \mu_{3,k_c} \otimes \phi^5 + \sum_{i=c,d,e} \eta_{k_i}^{(i)} \phi \approx \sum_{N=1}^{3} \xi_N \otimes \phi^{N-1}. \quad (1.12)

(Here \( \otimes \) denotes a convolution in momentum space. The coefficients \( \mu_i \) as well as the damping coefficients \( \eta_i \) depend on \( k \) and \( t \), as well as on the momentum cut-off \( k_c \).) These resemble in its structure the analogous expression to eq. (1.7). The hard modes act as an environmental heat bath. They also guarantee that the soft modes become, on average, thermally populated with the same temperature as the heat bath. Equivalently to (1.7) we will see that a similar relation exists for the average fluctuation of the amplitude squared. For the semi-classical regime where \( |\vec{p}| \ll T \) it will yield

\[
\langle \langle \phi(\vec{p}, t \to \infty)^2 \rangle \rangle \approx \frac{V}{E_{p}} T \approx \frac{V}{E_{p}} n(\vec{p}, t), \quad (1.13)
\]

where \( V \) is the volume of the system. Such kind of Langevin description for the non-perturbative evolution of (super-)soft modes (on a scale of \( |\vec{p}| \sim g^2 T \)) in non-Abelian gauge theories has recently been put forward [3].

In analogy to the Langevin description stated above we want to formulate in the following the effect of the heat bath on the evolution of the system degrees of freedom by means of the CTPGF technique. It is the intention of our study to provide in a self-contained way new insight into the dissipative and stochastic character of the CTPGF approach in thermal and non-equilibrium field theory. For this we discuss a free scalar field theory interacting with a heat bath. In the next section we will review the CTPGF technique and evaluate the average characteristic properties of the CTP propagator resulting from the (non-equilibrium) equations of motion, the Kadanoff-Baym equations. Within a rather simple rearrangement of the interaction kernels stemming from the heat bath, it becomes transparent to identify the dynamically generated mass shift, the dissipation and the fluctuation terms, all of them contributing to the dynamical evolution of the system. We also show that the fluctuation-dissipation theorem emerges naturally and is thus stated in microscopic terms. We then introduce in the third section the concept of a stochastic generating functional with external noise. In this way the analogy of thermalization in quantum field theory to a Langevin-like process becomes apparent. We thus can point out the influence of the noise correlation function on the propagator which contains the occupation number. As motivated above, one also immediately realizes that the long wavelength modes with momenta \( |\vec{k}| \ll T \) and energies \( |\omega| \ll T \) behave as classical propagating modes for a weakly interacting theory. We also briefly mention an attempt how to to derive an effective Boltzmann-Langevin equation for the phase-space occupation number of the bosonic particles from first principles. Section IV is devoted to the issue of so called pinch singularities emerging in a strictly perturbative evaluation in out of equilibrium quantum field theory. We will give first a very clear physical picture for their occurrence. We then show how the ‘problem’ of pinch singularities is cured naturally within the (non-perturbative) processes of thermalization and damping and providing a bridge to standard kinetic theory. We will end our findings with a brief summary and conclusions in section V.

II. KADANOFF-BAYM EQUATIONS

In this section we study the thermalization of a simple quantum field theory. For this purpose we couple a system of free scalar fields to an environmental heat bath of temperature \( T \). We start with the closed time path action for this scalar field \( \phi \) \[3\]:

\[
S = \int d^4x \frac{1}{2} \left[ \phi^+ (-\Box - m^2) \phi^+ - \phi^- (-\Box - m^2) \phi^-
- \phi^+ \Sigma^{++} \phi^+ - \phi^- \Sigma^{+-} \phi^- - \phi^- \Sigma^{--} \phi^+ - \phi^- \Sigma^{--} \phi^- \right]. \quad (2.1)
\]

The time integration starts at some fixed time \( t_0 \) at which the system starts to evolve from some initial density matrix. The interaction among the system and the heat bath is stated by an interaction kernel involving a self energy (or ‘mass’) operator \( \Sigma \) resulting effectively from integrating out the heat bath degrees of freedom. Clearly, this self energy operator is the only quantity which might drive the system towards equilibrium. We assume that \( \Sigma \) is solely determined by the properties of the heat bath. Ideally such a scenario holds for the linear response regime.

In \[23\] the self energy contribution from the heat bath is parametrized in the Keldysh notation by the four self energy parts

\[
\Sigma^{++}(x_1, x_2) = \Theta(t_1 - t_2) \Sigma^{>}(x_1, x_2) + \Theta(t_2 - t_1) \Sigma^{<}(x_1, x_2)
\]

\[
\Sigma^{--}(x_1, x_2) = \Theta(t_2 - t_1) \Sigma^{>}(x_1, x_2) + \Theta(t_1 - t_2) \Sigma^{<}(x_1, x_2)
\]

\[
\Sigma^{+-}(x_1, x_2) = -\Sigma^{<}(x_1, x_2)
\]

\[
\Sigma^{-+}(x_1, x_2) = -\Sigma^{>}(x_1, x_2)
\]

where \( t_1 \) and \( t_2 \) are the time components of the vectors \( x_1 \) and \( x_2 \), respectively. With these definitions at hand one sees that in principle only two self energy parts are independent, e.g. \( \Sigma^{<} \) and \( \Sigma^{>}. \) One can also easily show that the difference \( \Sigma^{>}(x_1, x_2) - \Sigma^{<}(x_1, x_2) \) is real while the sum \( \Sigma^{>}(x_1, x_2) + \Sigma^{<}(x_1, x_2) \) is purely imaginary.

If, as assumed, the heat bath stays at thermal equilibrium at some fixed temperature \( T \) and the self energy is solely determined by the property of this heat bath, then the important relation

\[
\Sigma^{>}(k) = e^{k_0/T} \Sigma^{<}(k), \quad (2.6)
\]
The expectation value is defined via path integrals as advanced self energies will become crucial below when we will show that the modes of our system in the long-time limit will indeed equilibrate at the temperature of the heat bath.

In order to explicitly explore the causal structure, it is useful to introduce the more physical retarded and advanced self energies

\[ \Sigma^{\text{ret}}(x_1, x_2) := \Theta(t_1 - t_2) \left[ \Sigma^>(x_1, x_2) - \Sigma^<(x_1, x_2) \right], \]
\[ \Sigma^{\text{av}}(x_1, x_2) := \Theta(t_2 - t_1) \left[ \Sigma^<(x_1, x_2) - \Sigma^>(x_1, x_2) \right]. \]

Given the fields \( \phi^+ \) and \( \phi^- \) on the two branches of the CTP contour, four two-point functions can be defined:

\[ D^{ab}(x_1, x_2) := -i(\phi^a(x_1) \phi^b(x_2)), \quad a, b = +, - . \] (2.7)

The expectation value is defined via path integrals as

\[ \langle \mathcal{O} \rangle := \frac{1}{N} \int \mathcal{D}[\phi^+, \phi^-] \mathcal{O} e^{iS[\phi^+, \phi^-]} \rho[\phi^+, \phi^-] \] (2.8)

where \( \rho \) is the density matrix of the system for the initial time \( t_0 \).

When evaluating the two-point functions by using (2.8) one finds

\[ D^{++}(x_1, x_2) = \Theta(t_1 - t_2) \Theta(t_2 - t_1) D^<(x_1, x_2), \]
\[ D^{--}(x_1, x_2) = \Theta(t_1 - t_2) \Theta(t_2 - t_1) D^<(x_1, x_2), \]
\[ D^{+-}(x_1, x_2) = -i(\phi^- (x_2) \phi^+ (x_1)) =: D^<(x_1, x_2), \]
\[ D^{-+}(x_1, x_2) = -i(\phi^+ (x_1) \phi^- (x_2)) =: D^>(x_1, x_2). \] (2.9)

In addition, one also now introduces retarded and advanced propagators and for an off-diagonal element, say

\[ D^< := D^{+-} . \] (2.13)

The equation of motion for the retarded propagator then reads

\[ (-\Box - m^2 - \Sigma^{\text{ret}}) D^{\text{ret}} = \delta \] (2.14)

A similar one is given for the advanced propagator. Please note already the analogy to the solving of the Langevin equation carried out in the introduction by introducing the retarded and advanced propagator. Additional information now comes from the equation of motion of the propagator \( D^< \), which reads

\[ (-\Box - m^2 - \Sigma^{\text{av}} - \Sigma^{\text{ret}}) D^{\text{av}} = 0 , \] (2.15)

This is just the famous Kadanoff-Baym equation \[2.14\] and \[2.13\] determine the complete and causal (non-equilibrium) evolution for the two-point functions. To get now more physical insight into the (effective) action given in \[2.1\] and in the resulting Kadanoff-Baym equations we now introduce the following real valued quantities:

\[ s(x_1, x_2) := \frac{1}{2} \text{sgn}(t_1 - t_2) \left( \Sigma^>(x_1, x_2) - \Sigma^<(x_1, x_2) \right) \]
\[ = s(x_2, x_1), \] (2.16)
\[ a(x_1, x_2) := \frac{1}{2} \left( \Sigma^>(x_1, x_2) - \Sigma^<(x_1, x_2) \right) \]
\[ = -a(x_2, x_1), \] (2.17)
\[ I(x_1, x_2) := -\frac{1}{2i} \left( \Sigma^>(x_1, x_2) + \Sigma^<(x_1, x_2) \right) \]
\[ = I(x_2, x_1). \] (2.18)

Our notion for \( s \) and \( a \) serves as a reminder for the respective symmetry properties. It basically represents the standard decomposition of the real and imaginary part of the Fourier transform of the retarded self energy operator \( \Sigma^{\text{ret}} \). \( s \) yields a (dynamical) mass shift for the \( \phi \) modes caused by the interaction with the modes of the heat bath, while \( a \) is responsible for the damping, i.e. dissipation of the \( \phi \) fields. The important thing to point out will be that \( I \) characterizes the fluctuations.

We first note that on account of (2.16)-(2.18) together with (2.23) the CTP action (2.1) can be written as

\[ S = \frac{1}{2} \left[ (\phi^+ \phi^+ - \phi^- \phi^-) - (\phi^+ \phi^- + \phi^- \phi^+) \right] + i \left[ (\phi^+ \phi^- - \phi^- \phi^+) \right] . \] (2.19)

This expression is identical to the influence functional given by Feynman and Vernon \[2.13\]. To the exponential factor in the path integral (2.8) the \((s+a)\) term contributes a phase while the \( I \) term causes an exponential damping and thus signals nonunitary evolution. Following the ideas of Feynman and Vernon one can therefore indeed identify \( I \) as a ‘noise’ correlator (see next section). The equivalence of the CTP formalism and the influence functional approach has indeed been pointed out already in \[2.3\] on a strictly formal level. Our discussion here is inspired on more physical intuition.

The two relevant equations of motion are stated as

\[ (-\Box - m^2 - s - a) D^{\text{ret}} = \delta , \] (2.20)
\[ (-\Box - m^2 - s + a) D^< + (a + iI) D^{\text{av}} = 0 . \] (2.21)

We see that the last equation (the ‘Kadanoff-Baym’ equation) is the only one where \( I \) occurs. Hence the retarded (and advanced) propagators are determined by \( s \) and \( a \) while the number density is additionally influenced by the fluctuations \( I \).
For the interpretation of $s$ and $a$ we consider the long-time behavior of these equations. In this case we can assume that the system becomes translational invariant in time and space and the boundary terms are no longer important. For $\bar{D}^{\text{ret}}$ one immediately finds
\[
\bar{D}^{\text{ret}}(k) = \frac{1}{k^2 - m^2 - \bar{s}(k) - \bar{a}(k)} = (\bar{D}^{\text{av}}(k))^* .
\] (2.22)
From this the spectral function is given by
\[
\bar{A}(k) := \frac{i}{2} [\bar{D}^{\text{ret}}(k) - \bar{D}^{\text{av}}(k)] = \frac{i \bar{a}(k)}{|k^2 - m^2 - \bar{s}(k)|^2 + |\bar{a}(k)|^2}.
\] (2.23)
Note that $\bar{a}(k)$ is purely imaginary since $a(x_1, x_2)$ is real and antisymmetric. For the same reason $\bar{s}$ is real. Inspecting the spectral function it becomes obvious that $\bar{s} \equiv \Re \bar{\Sigma}^{\text{ret}}$ contributes an (energy dependent) mass shift while $\bar{a} = i \Im \bar{\Sigma}^{\text{ret}}$ causes the damping of propagating modes $\bar{k}$. $\bar{a}$ is related to the commonly used damping rate $\bar{\Gamma}$ via
\[
\bar{\Gamma}(k) = \frac{i \bar{a}(k)}{k_0}.
\] (2.24)
For $\bar{D}^\prec$ one finds in the long-time limit the relation
\[
\bar{D}^\prec(k) = \bar{D}^{\text{ret}}(k) \bar{\Sigma}^\prec \bar{D}^{\text{av}}(k)
\]
\[
= \bar{D}^{\text{ret}}(k) \left[ -\bar{a}(k) - i \bar{I}(k) \right] \bar{D}^{\text{av}}(k)
\]
\[
= \frac{-\bar{a}(k) - i \bar{I}(k)}{i \bar{a}(k)} \bar{A}(k) = -2i n(k) \bar{A}(k).
\] (2.25)
We note that the first equation stated in (2.25) is sometimes denoted in the literature as a generalized fluctuation-dissipation theorem. We will now outline its intimate connection to a standard Langevin process in the following.

To proceed we need a relation between $\bar{a}$ and $\bar{I}$, i.e. a relation between the damping and the noise term. In our case it is a simple consequence of the KMS condition $\bar{\Sigma}^\prec$ using the definitions (2.17), (2.18):
\[
n(k) = \frac{\bar{\Sigma}^\prec(k)}{\bar{\Sigma}^\succ(k) - \bar{\Sigma}^\prec(k)} = \frac{1}{e^{\hbar a/kT} - 1} \equiv n_B(k_0),
\] (2.26)
which indeed shows that the phase space occupation number of the soft modes in the long-time limit becomes a Bose distribution with the temperature of the heat bath. (2.26) is independent of the explicit and detailed form of the self energy but is solely determined by the KMS condition. If one now assumes that the coupling between the bath and the system becomes very weak, i.e. $\bar{a}, \bar{\Gamma}, \bar{I} \to 0$, the expression (2.26)
\[
n(k) \left( \to \frac{0^+}{0^-} \right) \to \frac{1}{e^{\hbar a/kT} - 1} \quad (2.27)
\]
is still preserved as long as the KMS condition is fulfilled.

It is now very illuminating to explicitly write down the relation between $\bar{a}(k)$ and $\bar{I}(k)$
\[
\bar{I}(k) = \frac{\bar{\Sigma}^\succ(k) + \bar{\Sigma}^\prec(k)}{\bar{\Sigma}^\succ(k) - \bar{\Sigma}^\prec(k)} i \bar{a}(k) = \coth \left( \frac{k_0}{2T} \right) i \bar{a}(k).
\] (2.28)
In the high temperature (classical) limit one gets
\[
\bar{I}(k) = \frac{T}{k_0} 2i \bar{a}(k),
\] (2.29)
or, employing (2.24),
\[
\bar{I}(k) = 2T \bar{\Gamma}(k).
\] (2.30)
Recalling our discussion of Brownian motion in the introduction this compares favorably well with (1.6)! Indeed as we shall see in the next section we can define a quantity which obeys a Langevin equation very similar to the one for the Brownian particle. The physical meaning of $I$ as a ‘noise’ correlator will become obvious. The relation (2.28) thus already represents the generalized fluctuation-dissipation relation (6) from a microscopic point of view by the various definitions of $\bar{I}, \bar{a}$ and $\bar{\Gamma}$ through the parts $\bar{\Sigma}^\prec$ and $\bar{\Sigma}^\succ$ of the self energy.

We close this section by remarking that in all applications typically the major goal of the Kadanoff-Baym equations had been to derive a standard kinetic transport equation for the (semi-classical) phase-space distribution $f(\vec{x}, \vec{k}, t)$ which should be valid for weak coupling and a nearly homogeneous system. If the system is in a general off-equilibrium state the two-point functions depend not only on the so called microscopic variable $u := x_1 - x_2$ but additionally on the macroscopic center-of-mass variable $X := (x_1 + x_2)/2$. Performing a standard Wigner transformation for $\bar{D}^\prec$ and making a gradient expansion up to linear order in the center-of-mass variable $X$ one finds the well-known relativistic kinetic transport equation in the quasi-particle approximation for the semi-classical (on-shell) one-particle phase-space distribution $f_0$:
\[
k_0 \partial_k^\gamma f(\vec{x}, \vec{k}, t) = \frac{1}{2} \left( i \bar{\Sigma}^\prec(\vec{x}, k) [f(\vec{x}, \vec{k}, t) + 1] - i \bar{\Sigma}^\succ(\vec{x}, k) f(\vec{x}, \vec{k}, t) \right) \bigg|_{k_0 = \omega_k^0}
\] (3.1)
resembling in its form the Boltzmann equation. Here all energies have to be evaluated onshell. In this form it is obvious that $i \bar{\Sigma}^\prec/2\omega_k^0$ can be interpreted as the production rate, while $i \bar{\Sigma}^\succ/2\omega_k^0$ stands for the absorption rate for modes with the respective energy.

### III. STOCHASTIC INTERPRETATION AND SOME CONSEQUENCES

To see the connection between the formalism presented in the previous section (1) and stochastic equations we decompose the influence action $S$ as given in (2.19) in
its real and imaginary part and rewrite the generating functional \( I \)

\[
Z[j^+, j^-] := \int D[\phi^+ \phi^-] \rho[\phi^+, \phi^-] \times e^{\text{Re} S[\phi^+, \phi^-] + i \int \phi^+ j^+ \phi^- j^- - \frac{1}{2} (\phi^+ - \phi^-) I (\phi^+ - \phi^-)}
\]

\[
= \frac{1}{N} \int D[\xi] e^{-\frac{\xi}{2} \int \xi^{-1} \xi} \int D[\phi^+, \phi^-] \rho[\phi^+, \phi^-] \times e^{\text{Re} S[\phi^+, \phi^-] + i \int \phi^+ j^+ \phi^- j^- + i \xi (\phi^+ - \phi^-)}
\]

\[
= \frac{1}{N} \int D[\xi] e^{-\frac{\xi}{2} \int \xi^{-1} \xi} Z'[j^+ + \xi, j^- - \xi]
\]

\[
\equiv \langle Z'[j^+ + \xi, j^- - \xi] \rangle \tag{3.1}
\]

with

\[
\tilde{N} := \int D\xi e^{-\frac{\xi}{2} \int \xi^{-1} \xi} \tag{3.2}
\]

The generating functional \( Z[j^+, j^-] \) can thus be interpreted as a new stochastic generating functional \( Z'[j^+ + \xi, j^- - \xi] \) averaged over a random (noise) field \( \xi \) which is Gaussian distributed with the width function \( I \), i.e.

\[
\langle \mathcal{O} \rangle := \frac{1}{N} \int D\xi \mathcal{O} e^{-\frac{\xi}{2} \int \xi^{-1} \xi} \tag{3.3}
\]

From the last definition we find that the (ensemble) average over the noise field vanishes

\[
\langle \xi \rangle = 0 \tag{3.4}
\]

while the noise correlator is given by

\[
\langle \xi \rangle = I \tag{3.5}
\]

The action entering the definition of \( Z' \) is no longer \( S \), but only the real part of the influence action \( (2.19) \).

From this new stochastic functional \( Z' \) a Langevin equation for a classical \( \phi \) field can be derived \( [6] \). Noting that the fields \( \langle \phi^+ \rangle \xi \) on the upper branch and \( \langle \phi^- \rangle \xi \) on the lower branch are equal (and denoted as \( \phi \xi \) in the following), its equation of motion derived from \( Z' \) takes the form

\[
(-\Box - m^2 - s) \phi \xi - a \phi \xi = -\xi \tag{3.6}
\]

This, indeed, represents a standard Langevin equation.

The spatial Fourier transform of the Langevin equation \( (3.6) \) then takes the form

\[
\hat{\phi}(\vec{k}, t) + (m^2 + \vec{k}^2 - 2\Gamma(\vec{k}, \Delta t = 0)) \phi(\vec{k}, t)
\]

\[
+ 2 \int dt' \Gamma(\vec{k}, t - t') \phi(\vec{k}, t') = \xi(\vec{k}, t) \tag{3.7}
\]

The analogy between this Langevin equation \( (3.7) \) and the one for a single classical oscillator is obvious. The important difference, however, is the fact that the corresponding relations \( (2.23) \) and \( (3.6) \) between the respective noise kernel \( I \) and friction kernel \( \Gamma \) only agree in the high temperature limit.

One can now ask to what extend the classical equations of motion \( (3.7) \) together with \( (3.5) \) are an approximation for the full quantum problem given by the equation of motion \( (2.21) \) for \( D^< \). Inverting \( (3.6) \) one finds for the correlation function in the long-time limit

\[
- i \langle \langle \phi^+ \rangle \xi \langle \phi^- \rangle \xi \rangle = - i D^< \langle \xi \xi \rangle D^\text{av} = - i D^< I D^\text{av} \tag{3.8}
\]

Note that \( (3.8) \) is indeed the relation \( (1.13) \) advocated in the introduction to hold in the (semi)-classical regime. One has to compare with the full quantum correlation function \( D^< \) of \( (2.23) \). One thus has that \( (-a - iI) \) is approximated by \( -iI \). Of course this is justified, if \( |a| \ll I \) holds. Using the microscopic quantum version \( (2.28) \) of the fluctuation-dissipation theorem this is equivalent to \( \coth \left( \frac{\gamma}{2T} \right) \gg 1 \). Thus in the high temperature limit or – turning the argument around – for low frequency modes, i.e. for \( k_0 \ll T \), the classical solution yields a good approximation to the full quantum case. To be more precise: In simulations one has to solve the classical Langevin equation \( (3.9) \) and calculate \( n \)-point functions by averaging over the random sources. This has been raised in \( [13] \). We would also like to refer to the works in \( [17] \) for some practical examples regarding the mathematical evaluation of expectation values in the transition from quantum to classical field theories using dimensional reduction techniques.

One can also write down the equations of motion for the quantum two-point functions with external noise \( \xi \) by introducing the ‘noisy’ propagators

\[
D^\text{ab}(x_1, x_2) := -i \langle \phi^a(x_1) \phi^b(x_2) \rangle \xi \tag{3.9}
\]

One recovers that \( D^\text{ret} \) and \( D^\text{av} \) obey the same equations of motion as \( D^\text{ret} \) and \( D^\text{av} \), respectively, and are thus the same. Only the equation of motion for \( D^< \) and hence for the occupation number is modified compared to the one for \( D^< \). Averaging this equation over the noise fields according to \( (3.3) \) one indeed rederives the Kadanoff-Baym equation \( (2.21) \). This demonstrates that the Kadanoff-Baym equation can be interpreted as an ensemble average over fluctuating fields which are subject to noise, the latter being correlated by the sum of self energies \( \Sigma^< \) and \( \Sigma^> \), i.e. from a transport theoretical point of view the sum of production and annihilation rate. We want to note once more that the ‘noisy’ or fluctuating part denoted by \( \xi \) inherent to the structure of the Kadanoff-Baym equation \( (2.15) \) guarantees that the modes or particles become correctly (thermally) populated, as can be realized by inspecting \( (2.26) \) or \( (2.27) \).

What is changed, if we replace our toy model of a free system coupled to an external heat bath by a self-coupled and thus nonlinear closed system? In an interacting field theory of a closed system the Kadanoff-Baym equations
formally have exactly the same structure as in our toy model. The important difference, however, is that the self energy operator is now described fully (within the appropriate approximative scheme) by the system variables, i.e. it is expressed as a convolution of various two-point functions. Hence, an underlying simple stochastic process, as in our case an external stochastic Gaussian process, cannot really be extracted. However, we emphasize again that the emerging structure of the Kadanoff-Baym equations is identical. The decomposition of the self energy operator into its three physical parts (mass shift $\delta$, damping $\alpha$, and fluctuation kernel $I$) can immediately be taken over. Hence these three parts keep their clear physical meaning also for a nonlinear closed system.

We close our discussion by noting that one can also pursue to derive a standard kinetic transport equation for the (semi-classical) phase-space distribution $f(\vec{x}, \vec{k}, t)$ including fluctuations\footnote{This derived kinetic transport process (3.10) has the physical meaning also for a nonlinear closed system.} . In analogy to (2.31) one gets

\[
\begin{align*}
&k_{\mu} \partial_{\kappa}^\mu k_{\kappa}(\vec{x}, \vec{k}, t) = \frac{1}{2} \left[i \Sigma^< (X, k) [f_{\kappa}(\vec{x}, \vec{k}, t) + 1]ight. \\
&\left. - i \Sigma^> (X, k) f_{\kappa}(\vec{x}, \vec{k}, t) \right]_{k_0 = \omega_0^0} + \omega_0^0 \bar{\mathcal{F}}^\text{phen}(\vec{x}, \vec{k}, t). \tag{3.10}
\end{align*}
\]

This derived kinetic transport process\footnote{This derived kinetic transport process (3.10) has the structure of the phenomenologically inspired Boltzmann-Langevin equation \cite{10}: In order to describe fluctuations around the average, Bixon and Zwanzig \cite{16} postulated a stochastic (classical) Boltzmann equation in analogy to the Langevin equation for a Brownian particle. By adding a fluctuating collision term to the linearized Boltzmann equation (of the form of equation (2.31)) they obtained the following scheme

\[
\frac{d}{dt} f_{\xi}(t) = -\bar{\Gamma}(f_{\xi}(t) - f_{\text{eq}}) + \bar{\mathcal{F}}^\text{phen}(t) \tag{3.11}
\]

for a system near equilibrium. The correlation function of the fluctuating collision term was guessed on the basis of the fluctuation-dissipation theorem. Also it was shown that the stochastic Boltzmann equation provides a basis for describing hydrodynamic fluctuations. On the other hand, the equations of Bixon and Zwanzig were not derived but obtained on the basis of intuitive arguments. Instead, our approach carried out in \cite{5} has to be considered as a clear derivation from first principles. Indeed it shows (nearly) a one to one correspondence to the phenomenologically introduced scheme. However, also some severe interpretational difficulties in the interpretation of the fluctuating phase-space density remain. We refer the interested reader to our discussion in \cite{5}.

**IV. NONPERTURBATIVE RESOLUTION OF PINCH SINGULARITIES**

We have seen in the previous sections that modes or quasi-particles become thermally populated by a non-perturbative interplay between noise and dissipative terms entering the Kadanoff-Baym equations. It is non-perturbative as the equations of motion explicitly resum the self energy contributions by a Schwinger-Dyson equation defined on the real time path contour. In addition we have observed in eq. (2.27) that seemingly ill-defined expressions of the form $\theta'/\theta'$ are well-behaved in the sense of a weak coupling limit.

Rather similar ill-defined expressions result from so called pinch singularities within the context of strictly perturbative expansions in real time non-equilibrium field theory first raised in ref. \cite{7}. It is the purpose of this section to demonstrate explicitly how pinch singularities are absent within a necessary non-perturbative context \cite{18}.

\[
\Sigma_p = \quad \rightarrow \quad \rightarrow \quad \xrightarrow{2}
\]

**FIG. 1.** Lowest order self energy term in $\phi^4$-theory which contributes to the pinch problem (sunset diagram). The imaginary part of the sunset diagram can be identified with a scattering amplitude.

Employing the diagrammatic CTP rules potential ‘pinch singularities’ can arise in strictly perturbative expressions. Consider in the following a weakly interacting scalar $\phi^4$-theory. The initial state in the far past (assuming a homogeneous and stationary system) is prepared by specifying the momentum occupation number $\tilde{n}(\vec{p})$ of the (initially non interacting) onshell particles. Note that this occupation number depends only on the three momentum $\vec{p}$. The occupation number $\tilde{n}(\vec{p})$ enters the (free) propagator

\[
D_0^> (p) = -2\pi i \delta(p^2 - m^2) \\
\times |\Theta(p_0)\tilde{n}(\vec{p}) - \Theta(-p_0)(1 + \tilde{n}(\vec{p}))|	ag{4.1}
\]

The free retarded and advanced propagator are given by

\[
D_0^{\text{ret/av}} (p) = \frac{1}{p^2 - m^2 \pm i\epsilon \text{sgn}(p_0)}. \tag{4.2}
\]

A specified self energy insertion $\Sigma_p$ in a strictly perturbative expansion is given by a convolution of these free propagators. As a characteristic example the ‘sunset’ graph arising in scalar $\phi^4$-theory is illustrated in fig. 2. We choose this particular graph as an example since the self energies $\Sigma_p^{</\rangle}(\vec{p}, p_0 = E_p)$ do not vanish onshell. The strictly perturbative corrections to the propagators within the context of the Schwinger-Keldysh formalism take the form

\[
D_p^{\text{ret/av}} = D_0^{\text{ret/av}} + D_0^{\text{ret/av}} \Sigma_p^{\text{ret/av}} D_0^{\text{ret/av}} + \ldots \\
=: D_0^{\text{ret/av}} + \Delta D_p^{\text{ret/av}}. \tag{4.3}
\]

\[
D_p^{<} = D_0^{<} + D_0^{\text{ret/av}} \Sigma_p^{<} D_0^{<} + D_0^{\text{ret/av}} \Sigma_p^{<} D_0^{\text{ret/av}} + \ldots =: D_0^{<} + \Delta D_p^{<}. \tag{4.4}
\]
As $D_{\text{ret}}^\text{eff}$ contains a pole at $p_0 = \pm E_p - i\epsilon$ and $D_{\text{av}}^\nu$ a pole at $p_0 = \pm E_p + i\epsilon$ the product of both in the expression for $\Delta D^\nu$ is ill-defined, if $\Sigma_p^\nu (\vec{p}, p_0 = E_p = \sqrt{m^2 + p^2})$ does not vanish onshell. Transforming such an expression back into a time representation, the contour has to pass between this pair of two infinitely close poles. In fact, all three contributions to $\Delta D^\nu$ are ill-defined. On the other hand the perturbative corrections $\Delta D^\nu_{\text{perturbed}}$ to the free retarded/advanced propagator are free of any pinch singularities as the emerging poles are all located at the same side of the contour. The three perturbative corrections can be further rearranged to extract the part carrying the pinch singularity

$$\Delta D_{\text{pinch}}^\nu(p) = D_{\text{ret}}^\nu(p) \left[ \Theta(p_0) ((1 + \hat{n}(\vec{p}))\Sigma_p^\nu (p) - \hat{n}(\vec{p})\Sigma_0^\nu (p)) + \Theta(-p_0) ((1 + \hat{n}(\vec{p}))\Sigma_p^\nu (p) - \hat{n}(\vec{p})\Sigma_0^\nu (p)) \right] D_{\text{av}}^\nu (p).$$

(4.5)

The expression in the square brackets is familiar from standard kinetic theory (compare with $2.31$). Apart from a trivial factor one can interpret

$$\Gamma_{\text{eff}}(\vec{p}) := \frac{1}{2E_p} \left[ (1 + \hat{n}(\vec{p}))i\Sigma_p^\nu (p) - \hat{n}(\vec{p})i\Sigma_0^\nu (p) \right] \bigg|_{p_0 = E_p}$$

(4.6)

as the net effective rate for the change of the occupation number per time. For an equilibrium situation the occupation number is given by the Bose distribution and the self energy insertions fulfill the KMS condition (2.3). Hence, for the equilibrium case the whole bracket exactly vanishes and no pinch singularities emerge within a strictly perturbative expansion. It was observed and proven already by Landsman and van Weert that such ill-defined terms do cancel each other, to each order in perturbation theory, if the system stays at thermal equilibrium (11). This is not the case for a general non-equilibrium configuration. This seemingly ‘severe’ problem arising for systems out of equilibrium was first raised by Altherr and Seibert (17).

In a subsequent paper Altherr (19) tried to ‘cure’ this problem by hand by introducing a finite width for the ‘unperturbed’ free CTP propagator $D_0$ so that the expressions are at least defined in a mathematical sense. Such a procedure, of course, already represents some mixing of non-perturbative effects within the ‘free’ propagator. As we have seen from our discussions in the previous sections a damping term (resulting in a non vanishing width) and the associated noise guarantee that the propagator becomes thermally populated at long times. Hence, for any system which moves towards thermal equilibrium and thus behaves dissipatively, the full propagator must have some finite width (due to collisions or more generally due to damping). Within his modified perturbative approach, Altherr (11) also showed that seemingly higher order diagrams do contribute to a lower order in the coupling constant, as some of the higher order diagrams involving pinch terms will receive factors of the form $1/\Gamma_n$, $n \geq 1$ reducing substantially the overall power in the coupling constant. In his particular case Altherr investigated the dynamically generated effective mass (the ‘tadpole’ contribution) within standard $\phi^4$-theory. (For the hard modes the onshell damping $\Gamma$ is of the order of $o(q^4T)$.) Therefore he concluded that power counting arguments might in fact be much less trivial for systems out of equilibrium. We will come back to his observation below.

Comparing (4.3) and (4.4) with the Boltzmann equation (2.31) one is already tempted to speculate that the occurring pinch singularity must be connected to the change in time in occupation number. Hence, we will first now elaborate on the physical reason for the occurrence of pinch singularities. Within standard scattering theory one would think about the probability for a particle of some initial momentum state to be scattered into another momentum state. Therefore we ask, how the occupation number $\hat{n}$ has changed after a long time. The occupation number for the out-states can be readily extracted from $D^\nu$ by means of the formula (18)

$$n(\vec{p}, t \rightarrow \infty)^{\text{(out)}} = \langle a_{\vec{p}}^{\dagger} \rho^{\text{(out)}} (\vec{p}) \rangle = \left( \frac{E_p}{2E_p} + \frac{1}{2} \int \frac{d^4y}{(2\pi)^4} e^{ip\cdot y} \Theta(t - t') \right) \int d^3x \rho^{\text{(out)}} (\vec{x}, t') \bigg|_{t'=t}.$$  

(4.7)

We now assume that the interaction is switched on at a time $t = -T/2$ and switched off at $t = T/2$, i.e. we replace $\Sigma_p^{\nu < (out)} (x_1, x_2)$ by

$$\Theta(\frac{t}{2} - t_1) \Theta(\frac{t}{2} - t_2) \Sigma_p^{\nu < (out)} (x_1, x_2) \Theta(t_1 + \frac{t}{2}) \Theta(t_2 + \frac{t}{2})$$

and assume that the duration time $T$ is large but finite. This procedure regulates the pinch singularities to a finite value. Invoking a couple of manipulations (18) one ends with the rather ‘obvious’ result

$$\Delta n(\vec{p})^{\text{(out)}}_\text{pinch} \approx \Gamma_{\text{eff}}(\vec{p}) \int \frac{dp_0}{2\pi} \frac{4}{(p_0 - E_p)^2} \sin^2 \left( \frac{T}{2} (p_0 - E_p) \right) \approx \Gamma_{\text{eff}}(\vec{p}) \cdot T.$$  

(4.8)

We thus have demonstrated the bridge between the occurrence of pinch singularities within the context of the CTP formalism and Fermi’s golden rule in elementary quantum scattering theory. The effective rate $\Gamma_{\text{eff}}$ is therefore analogous to the transition probability per unit time. Indeed one can easily understand in physical terms that one has to expect such a singularity in perturbation theory: Staying strictly within the first order contribution the particles remain populated with the initially prepared non-equilibrium occupation number (since this quantity enters the free propagator (1.1)) and scatter for an infinitely long time. Therefore, the resulting shift $\Delta n(\vec{p})^{\text{(out)}}$ should scale with $\Gamma_{\text{eff}}(\vec{p}) \cdot T$ with $\Gamma_{\text{eff}}(\vec{p})$ held fixed. However, looking at the Boltzmann equation (2.31), the occupation number and the collision rate
do not stay constant during the dynamical evolution of the system, but will be changed on a timescale of roughly $1/\Gamma$. The quasi-particles are not really asymptotic states.

Subsequently, pinch singularities are formally cured by a resummation procedure. The propagators will become dressed and supplemented by a finite (collisional or more generally damping) width. This represents already a non-perturbative effect which only can be achieved by a resummation of Dyson-Schwinger type. As a first attempt (proposed by Baier et al. [23]), one might resum the full series of (4.3,4.4) using the self energy $\Sigma_p$. With the definitions $\Gamma_p(\vec{p}, p_0) := \frac{1}{2\pi} \{ [\Sigma^S_p(\vec{p}, p_0) - \Sigma^S_p(\vec{p}, p_0)]\}$ and $\text{Re} \Sigma_p := \text{Re} \Sigma^{\text{ret}} = \text{Re} \Sigma^{\text{av}}$ one finds [3]

$$D^{<} = D^{\text{ret}} \left( (D^{\text{ret}}_0)^{-1} D^{<}_0 (D^{\text{av}}_0)^{-1} \right) D^{\text{av}} + D^{\text{ret}} \Sigma_p^{<} D^{\text{av}} = (-2i) \frac{\Gamma^{\text{ret}}_p}{(p^2 - m^2 - \text{Re} \Sigma^{\text{ret}}_p)^2 + p^2 \Gamma^{\text{ret}}_p^2} \Sigma^{<}_p - \Sigma^{<}_p = n_\Sigma(\vec{p}, p_0). \quad (4.9)$$

(The first term on the r.h.s. of the first equation represents a boundary term [3] of the (initial) propagator $D^{<}_0$. If there would be no interaction, i.e. $\Sigma_p \equiv 0$, then this boundary term just gives the free and undisturbed propagator. On the other hand, if $\Sigma_p$ is nonvanishing onshell, as assumed, this boundary term vanishes within the carried out resummation [3]. Only the second term then contributes and yields the result stated in the second line of (4.9).

Hence the resummation of the series [1,3] of ill-defined terms results in a well-defined expression. The quantity $n_\Sigma(\vec{p}, p_0)$ has to be interpreted as the ‘occupation number’ demanded by the self energy parts [3]. If the equilibrium KMS conditions (2.3) apply for the self energy part, then $n_\Sigma(\vec{p}, p_0) \rightarrow n_B(p_0)$ becomes just the Bose distribution function. For a general non-equilibrium situation, however, this factor deviates from the Bose distribution. Comparing (1.1) with (1.9), the astonishing thing to note is that in the initial non-equilibrium distribution $n$ has dropped out completely!

Calculating $\Sigma_p$ on a purely perturbative level the initial occupation number $\tilde{n}$ enters via the free propagator (1.1). This however cannot be the whole truth in a dynamically evolving system. It is important to make sure that such a system is prepared at some finite initial time $t_0$. (If $t_0$ would be taken as $t_0 \rightarrow -\infty$ the system would already have reached equilibrium long time ago. Time reversal symmetry is explicitly broken, so that the propagators in principle have to depend on both time arguments explicitly before the system has reached a final equilibrium configuration. Therefore the simple use of Fourier transforms, as carried out in [1,11,18,21], and which in fact has led to the pinch singularities in (4.3), is dubious [3].) The initial out of equilibrium distribution $\tilde{n}(t_0)$ cannot stay constant during the evolution of the system as it has to evolve towards the Bose distribution. As long as the system is not in equilibrium, the propagator thus cannot be stationary. In addition, the self energy parts $\Sigma^<$ and $\Sigma^>$ do also evolve with time. Hence they should depend on the evolving distribution function and not persistently on the initial one, $\tilde{n}$, which enters $\Sigma_p$ in (1.9). Thus, the resummation of (1.9) does not cover all relevant contributions. The self energy operators must be evaluated using the fully dressed and temporally evolving one-particle propagators.

\[ \begin{array}{c}
\end{array} \]

**FIG. 2.** Dyson-Schwinger equation with fully dressed propagators (skeleton expansion).

The solution to these demands is, of course, the description of the system by means of appropriate (quantum) transport equations like the Kadanoff-Baym equations. Graphically this is illustrated in fig. 2. In addition to the sunset diagram we have also included the Hartree diagram there which in a perturbative scheme is the one which arises first. The difference to the resummation of (1.9) is the fact that the propagators entering into the self-energy operators are now also the fully dressed ones.

Unfortunately, the full quantum transport equations are generally hard to solve and thus are not so much of practical use. Yet one need not be that pessimistic. For weak coupling, i.e. if the damping width is sufficiently small compared to the quasi-particle energy one can take the Markov approximation to obtain standard kinetic equations. For the situation of fig. 2, one gets

$$\begin{align*}
(E_p \partial_t - \vec{p} \partial_{\vec{x}} - \partial_{\vec{p}} f(\vec{x}, t; \vec{p})) f(\vec{x}, t; \vec{p}) &= (4.10) \\
\frac{1}{2} [\Sigma^<(\vec{x}, t; \vec{p}, E_p)] f(\vec{x}, \vec{p}, t) + 1 - i\Sigma^>(\vec{x}, t; \vec{p}, E_p) f(\vec{x}, \vec{p}, t] \\
\text{Here } f \text{ denotes the semi-classical non-equilibrium phase-space distribution of quasi-particles. } m(\vec{x}, t) \text{ denotes the sum of the bare and the dynamical (space time dependent) mass generated by the Hartree term. Within this Markovian approximation the fully dressed and resummed propagators are given by [3]}
\end{align*}$$

\[ D^{\text{ret}/\text{av}}(\vec{x}, t; p) \approx \frac{1}{p^2 - m^2(\vec{x}, t) - \text{Re} \Sigma(\vec{x}, t; p) + ip_0 \Gamma(\vec{x}, t; p)} , \quad (4.11) \]

\[ D^<(\vec{x}, t; p) \approx (-2i) f(\vec{x}, t; \vec{p}) \times \frac{p_0 \Gamma(\vec{x}, t; p)}{(p^2 - m^2(\vec{x}, t) - \text{Re} \Sigma(\vec{x}, t; p))^2 + p^2 \Gamma^2(\vec{x}, t; p)} . \quad (4.12) \]

We emphasize that in (4.12) the instantaneous non-equilibrium phase space distribution function $f(t)$ enters and not the initial one, $\tilde{n}$. The dynamically generated mass as well as the collisional self energy contribution $\Sigma$ can thus be evaluated with these propagators. Higher order terms leading to the pinch singularities are explicitly resummed and lead to finite and transparent results.
One can now understand the observations made by Altherr [19]. He has found, starting from some non-equilibrium distribution \( \tilde{n} \), that higher order diagrams contribute to the same order in the coupling constant as the lowest order one. Indeed, in his investigation, the particular higher order diagrams where nothing but the perturbative contributions of the series in (4.4) for the dressed or resummed one-particle propagator \( D^\omega \). The only difference is that he has employed a ‘free’ propagator modified by some finite width in order that each of the terms in the series (4.4) becomes well defined. The reason for the higher order diagrams to contribute to the same order is that the initial out-of-equilibrium distribution \( \tilde{n} \) cannot stay constant during the evolution of the system as it has to evolve towards the Bose distribution. If \( \tilde{n} - n_B \) is of order \( o(1) \), it is obvious that there must exist contributions which perturbatively attribute to the temporal change of the distribution function and contribute to the same order \( o(1) \). In fact, in our prescription (4.12) \( \tilde{n} \) has simply been substituted by the actual phase space distribution \( f \). Then calculating e.g. the tadpole diagram, as discussed in the particular case of [19], one has to stay within lowest order in the skeleton expansion, but with the fully dressed propagator.

V. SUMMARY AND CONCLUSIONS

Our study provides new intuitive insight into non-equilibrium quantum field theory and the process of thermalization. In our discussions we have elucidated on the stochastic aspects inherent to the (non-) equilibrium quantum transport equations, the so called Kadanoff-Baym equations. For this we have started with a simple model, where we have coupled a free scalar boson quantum field to an external heat bath with some given temperature \( T \). We have isolated a term denoted by \( I \) which solely characterizes the (thermal and quantum) fluctuations inherent to the underlying transport process. By introducing a stochastic generating functional the emerging stochastic equations of motion can then be seen as generalized (quantum) Langevin processes. Long wavelength modes with momenta and energies \( |\vec{k}|, \omega \ll T \) then behave as classical propagating modes for a weakly interacting theory. The important observation is that there the average occupation number of the soft modes becomes large and approaches the classical equipartition limit. We hope that our detailed analysis of the real time description of the soft modes within the Schwinger-Keldysh formalism can attribute to this subject. The understanding of the behavior of the soft modes is crucial e.g. for the issue of baryon number violation in the hot electroweak theory [1].

We also have demonstrated how so called pinch singularities are regulated within the non-perturbative context of the thermalization process. These singularities do (and have to) appear in the perturbative evaluation of higher order diagrams within the CTP description of non-equilibrium quantum field theory. Their occurrence signals the occurrence of (onshell) damping or dissipation. This necessitates in the description of the evolution of the system by means of non-perturbative transport equations. In the weak coupling regime this corresponds to standard kinetic theory. In this case we have given a prescription of how the dressed propagators can be approximated in a very transparent form.

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