Heat distribution of relativistic Brownian motion

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Abstract. Understanding the statistical behavior of the heat in stochastic systems gives us insight into the thermodynamics of such systems. Using the recently proposed relativistic stochastic thermodynamics, we investigate the statistics of the heat of a Relativistic Ornstein–Uhlenbeck particle, comparing with the classical case. The results are exact through numerical integration of the Fokker–Planck of the joint distribution, and are validated by numerical simulations.

1 Introduction

Understanding the behavior of the heat exchanged by a system has always had an important role in physics. Since the beginnings of thermodynamics, understanding how a system loses or gains energy from the surroundings was important to develop early thermal machines [1]. With today’s technological advances, we are able to thermodynamically interact on an increasingly smaller scale, ranging from micrometer to nanometers. Typically, these systems are far from equilibrium, where the thermodynamic functionals, such as heat, entropy, or work, are treated as fluctuating quantities. The field of the investigations of the thermodynamics of such fluctuating systems is well known in the literature as the stochastic thermodynamics [2–5].

Heat is a fundamental quantity in stochastic thermodynamics, i.e., the energy naturally exchanged between the system and the surroundings, in a disordered way. As a random variable, characterization of the statistics of heat for diffusive systems was carried out in many different models [6–22]. These works bring physical insights into the thermodynamics of classical diffusive systems, however, as far as we known, they only deals with non-relativistic systems.

Relativistic diffusive systems are expected to be found in nature and can appear in quite different phenomena such as cosmic jets [23,24], quark-muon plasma produced by heavy ion collisions [25,26], and graphene in a semiclassical regime [27,28]. From a more mathematical point of view, these relativistic diffusive systems can be modeled by the relativistic Brownian motion [29–31] which are a particular case of Brownian motion with nonlinear friction term [32]. One well-studied case is the relativistic Ornstein–Uhlenbeck [33], where its relaxation properties were studied in [34,35].

Recently, Pal and Deffner proposed a stochastic thermodynamic framework for the relativistic Brownian motion [23]. Their model is based on the relativistic Ornstein–Uhlenbeck case, where they define heat, work, and entropy for the relativistic Brownian particle. This relativistic version of the classical stochastic thermodynamics quantities [5] was shown to satisfy the conservation of energy and the fluctuation theorem version of the second law of thermodynamics. Since heat is a fundamental quantity, we thus want to understand its behavior for this new relativistic case. A similar approach was carried out for the work in the quantum relativistic regime [36]. It is meaningful to check if the relativistic stochastic thermodynamics leads to a consistent statistical behavior of the heat. Moreover, it is also interesting to compare with the classical case. How different is the behavior of the heat between the classical and relativistic system? In addition, it is important to notice that the same definition of heat was also proposed by Koide and Kodama in [25].

In the present paper, by means of the stochastic thermodynamics framework [23], we investigate the heat distribution for the relativistic Ornstein–Uhlenbeck model. We obtain the heat distribution in two distinct limits, the exact relativity limit, and the ultra-relativistic limit. The results are exact through numerical integration. We use the variational formula for the Fokker–Planck [37,38], and we integrate it numerically. We also use the path integral formalism [39–42] to deal with the ultra-relativistic case. The results are compared with numerical simulations of the stochastic process and are found in agreement. Moreover, we investigate the asymptotic case \( t \to \infty \), the relaxation process and we compare directly the relativistic cases with the classical counterpart.
The paper is organized as follows: in Sect. 2 we define the model and its thermodynamics. In Sect. 3 we studied the heat fluctuations of the ultra-relativistic case by path integrals. In Sect. 4 we obtain through numerical integration of the Fokker–Planck equation of the joint distribution the exact relativistic limit. In Sect. 5 we discuss the equilibrium case for the ultra-relativistic and exact case, and in Sect. 6 we studied the relaxation process for the two cases, starting with a delta distribution, we see the evolution of the heat distribution through the equilibrium. In Sect. 7 we compare the heat fluctuations of the ultra-relativistic case and exact case, with the classical underdamped Brownian motion. We conclude in Sect. 8 with a discussion of the results.

2 Relativistic Ornstein–Uhlenbeck

As a first model to study the heat fluctuations of a relativistic particle, we study the free case, often called relativistic Ornstein–Uhlenbeck [31]. Despite being the most simple situation for the relativistic Brownian particle, we will see that the nonlinear dependency on the momentum can lead us to non-trivial results.

A relativistic particle is a particle with an absolute velocity not exceeding the speed of light $|V(t)| < c$. Its stochastic behavior for the momentum is defined in the inertial frame of the environment and the corresponding stochastic equation is (unless explicitly stated, we assume $c = 1$)

$$dp = -\Gamma(p)dt + \sqrt{2\gamma M T}dW_t,$$

where

$$\Gamma(p) = \gamma M \frac{p}{\sqrt{p^2 + M^2}} = \gamma MV(t),$$

is the nonlinear drift term of the relativistic Ornstein–Uhlenbeck model [33], and $V(t) = p/\sqrt{p^2 + M^2} < 1$ is the relativistic velocity of the particle. Here we are using the same notation in [23]. Writing in terms of the absolute velocity, the interpretation of the drift term is simple: it is just the drift $-\gamma V$, but now with the velocity constrained by $V < c$.

For $M \gg p$ we find the classical case, while for $p \gg M$ we have the ultra-relativistic limit, where the velocity of the particle is close to the speed of light. The constants above are: the drift $\gamma$, the temperature of the surrounding $T$, and the rest mass of the particle $M$. The integrated noise $dW_t$ is a delta-correlated Wiener process [41], and models the fast degrees of freedom of the environment. According to [23, 25] the heat functional is given by

$$Q[p(t)] = \int_0^t \frac{p}{\sqrt{p^2 + M^2}} \left( -\Gamma(p) + \sqrt{2\gamma M T}dW_t \right) d\tau$$

$$= \sqrt{p_0^2 + M^2} - \sqrt{p_f^2 + M^2},$$

where $Q < 0$ means that the particle is losing energy to the environment, while $Q > 0$ says that the particle is absorbing energy from the environment. The balance of energy is the same as for the classical case. The drift term is responsible for the negatives values, while the noise term yields positive values. What changes, comparing with the non-relativistic case, is the nontrivial dependence on $p$ in the drift term. Notice that the first law of thermodynamics in the absence of work, i.e., $Q[p] = \Delta E$, is valid since $E = \sqrt{p^2 + M^2}$ is the relativistic energy.

We assume that the particle is initially in thermal equilibrium with the environment, with the initial distribution given by the Juttner distribution [43]

$$\rho(p_0, 0) = \rho_0 \exp \left( -\beta \sqrt{p^2 + M^2} \right),$$

which is the correct equilibrium thermal distribution of the relativistic Ornstein–Uhlenbeck model, as verified by a microscopic collision simulation in [43].

2.1 Heat functional

The heat, being a functional of the trajectory, has the conditional probability

$$P(Q|Q = Q[p]) = \delta(Q - Q[p]).$$

It emphasizes that the random values of $Q$ are given by the trajectory-dependent formula $Q[p]$. In the studied case, the heat only depends on the initial and final points of the trajectory. Therefore, the probability distribution for the heat will be given by

$$P(Q, t) = \int dp_0 dp\rho(p_0) \int_{p(0) = p_0}^{p(t) = p_f} Dp e^{-A[p]\delta(Q - Q[p(t)])}\rho(p_0, 0)\rho(p_f, 0),$$

$$= \int dp_0 P(Q, p_0, t),$$

where the path integral is over all the possible continuous non-differentiable trajectories [42] and $A[p(t)]$ is the stochastic action (see Appendix).

3 Ultra-relativistic regime

The ultra-relativistic regime occurs when the Brownian particle has a speed close to $c$, or equivalently $p \gg M$. In this regime the energy is given by a linear dependency on the momentum. As a consequence, the equation for the particle’s momentum will be simplified. Interestingly, the ultra-relativistic Brownian motion, can describe the behavior of the charge carriers on a graphene plate [27, 28] where, instead of the constant $c$, we have the Fermi velocity $v_F$. The analysis presented here can be cast as a simplified version of such a system as well. Here we will follow the notation by [27], since we do not have mass, we now make $c \neq 1$. 
In the ultra-relativistic regime, \( E = c|p| \) and the Langevin equation becomes

\[
dp = -\gamma c \frac{p}{|p|} dt + \sqrt{2\gamma T} dW_t, \tag{6}
\]

where the drift term is now similar to a dry friction force [28, 44]. Notice that \( c \) is no longer equals one here. (Because we do not have mass, it is important to use another parameter in the equations.) Following the definitions given in Sect. 2, the heat exchanged between the particle and the environment will be

\[
Q[p] = \Delta E = c(|p| - |p_0|), \tag{7}
\]

which is the first law of stochastic thermodynamics [23, 25]. The heat distribution is then given by

\[
P(Q) = \int dp \int dp_0 \rho(p_0) P[p, t|p_0] \delta(Q - Q[p]), \tag{8}
\]

where we remove the Dirac delta of the path integral in Eq. (5), making the path integral become the conditional probability. The conditional probability is written in as a path integral [40, 41]

\[
P[p, t|p_0] = \int_{p(0)=p_0}^{p(t)=p_t} Dp \ e^{-\mathcal{A}(p(t))}, \tag{9}
\]

which is a sum over all trajectories that are continuous and non-differentiable [42]. The result of this path integral is

\[
P[p_t, t|p_0] = \frac{1}{\int dp P[p, t|p_0]} \left( e^{-\gamma c p_t} \frac{1}{\sqrt{4\pi D t}} \ e^{-\gamma c t} \left( \frac{|p_t| - |p_0|}{\sqrt{2\gamma c}} \right) \right.
\]

\[
+ \frac{\gamma c}{4D} e^{-\gamma c |p_t|} \text{erfc} \left( \frac{|p_t|}{\sqrt{2\gamma c}} \right)
\]

\[
\times \left[ \frac{1}{\sqrt{4D}} \left( \frac{|p_t| + |p_0|}{\sqrt{2\gamma c}} - \gamma c \sqrt{t} \right) \right], \tag{10}
\]

where erfc is the complementary error function. We derive the above result in Appendix A. This conditional probability can be checked by numerical simulations of the Langevin Eq. (6). In the heat distribution formula, we can rewrite the Dirac delta to find a more convenient formula for the heat distribution

\[
P(Q) = \int \frac{d\lambda}{2\pi} e^{i\lambda Q} Z(\lambda), \tag{11}
\]

where

\[
Z(\lambda) = \int dp \int dp_0 \rho(p_0) e^{-i\lambda c (|p_t| - |p_0|)} P[p, t|p_0], \tag{12}
\]

is the characteristic function of the heat, where we use the Juttner initial equilibrium distribution \( \rho(p_0) = \rho_0 \exp(-\beta c |p_0|) \). Due to the complicated dependence on \( p_0 \), \( Z(\lambda) \) cannot be solved analytically. We find \( Z(\lambda) \) by numerically integrating over \( p_t \) and \( p_0 \). The result is plotted in Fig. 1. Notice that \( Z(0) = 1 \), meaning that the distribution is properly normalized. Given \( Z(\lambda) \) we use Eq. (11) to numerically integrate \( \lambda \) and find the heat distribution. The distribution is plotted in Fig. 2a.

The heat distribution in Fig. 2a is symmetrical, meaning that the particle has no tendency to absorb or lose energy from the bath. This behavior is encountered in its classical version, where \( \Gamma(p) = -\gamma p \). It happens because we start in thermal equilibrium using the Jutt-
tner distribution. Moreover, compared with the classical case, the distribution is smoother around $Q = 0$, meaning that, for a given trajectory, the chance of the particle absorbing or losing energy is larger than the classical case [6].

4 Exact solution

To solve the model exactly, a path integral technique can only give us an approximate result. Thus, we opt to use the Fokker–Planck formalism, which can be solved by numerical integration. We also compare the Fokker–Planck result with numerical simulations of the Langevin equation. The stochastic equation for the momentum is

$$\frac{dp}{dt} = -\gamma M \frac{p}{\sqrt{p^2 + M^2}} dt + \sqrt{2\gamma MT} dW_t. \quad (13)$$

The joint heat distribution $P(Q,p)$ can be solved exactly via the Fokker–Planck equation. One such approach was used in the derivation of the heat and work distributions of a Brownian particle in a double well potential [20,21]. One considers the stochastic equation for the heat

$$dQ = \frac{p}{\sqrt{p^2 + M^2}} dp$$

$$= -\gamma M \frac{p^2}{p^2 + M^2} dt + \frac{p}{\sqrt{p^2 + M^2}} \sqrt{2\gamma MT} dW_t,$$

where we replaced $dp$ by the Langevin equation Eq. (13). With the above equation, we can construct the Fokker–Planck for the joint distribution $P(Q,p)$. The Fokker–Planck for $P(Q,p)$ and its initial condition are

$$\frac{\partial}{\partial t} P(Q,p) = -\nabla \cdot (F P(Q,p) - B \cdot \nabla P(Q,p)),$$

$$P(Q,p_0, t = 0) = \delta(Q) \rho(p_0), \quad (15)$$

where $\nabla = (\partial_p, \partial_Q)$, and

$$F = \left( \begin{array}{c} -\frac{\gamma M p}{\sqrt{p^2 + M^2}} \\ -\frac{\gamma M p}{p^2 + M^2} \end{array} \right), \quad B = \left( \begin{array}{c} 2\gamma MT \frac{\gamma MT p}{\sqrt{p^2 + M^2}} \\ 2\gamma MT p \frac{\gamma MT p}{p^2 + M^2} \end{array} \right). \quad (16)$$

Equation (15) can be solved numerically giving $P(Q,p)$ which can be integrated numerically over $p$ to give the heat distribution $P(Q)$. We solve Eq. (15) numerically through the Finite Elements method [45], implemented by FEniCS [37], which uses the variational formula of the PDE, also known as the weak form [45]. In our case, the variational formula is

$$\int \partial_t P(Q,p) \nu d\Sigma$$

$$= \int (\mathbf{F} \cdot \nabla P(Q,p)) d\Sigma - \int B \nabla P(Q,p) \cdot \nabla \nu d\Sigma,$$

where $\nu$ is a test function, and we are integrating in the domain $d\Sigma = dpdQ$. In this variational formula, the boundary condition in the weak form is satisfied by

$$\int (\mathbf{F} \cdot \mathbf{B} P(Q,p)) \cdot \hat{n} \nu d\Sigma = 0 \quad (18)$$

which ensures the no flux boundary conditions over the integrated region. We implement the time evolution by an Euler scheme. The result $P(Q,p)$ is plotted in Fig. 3. With $P(Q,p)$ we integrate it numerically over $p$, obtaining the heat distribution $P(Q)$ plotted in Fig. 4a. The relative error between the Fokker–Planck solution and the numerical simulation is in Fig. 4c, showing the agreement between the two approaches.

The behavior of the heat distribution is clear. The particle in average will not absorb or gain energy from the environment and it is equally probable to lose or to gain energy since the distribution is symmetrical. This shows that the system is naturally close to equilibrium, which is expected since we assume an initially thermalized distribution. Moreover, comparing with the ultra-relativistic case, we see a peak around $Q = 0$ meaning that the tendency of zero average heat is stronger than for the ultra-relativistic case.

5 Equilibrium case

In the previous section, we start the dynamics in equilibrium, but allows the particle to be in any final state.
Fig. 4 a Heat distribution in time $t = 1$. Note the peak behavior in $Q = 0$ a feature common in many stochastic thermodynamics systems. The inset figure is the heat distribution in the small interval $[-1,1]$. The red solid line is the Fokker–Planck result, while the open circles are the simulation of the Langevin equation.  

**b** Relative error between simulation and numerical solution. All constants are set to one using the conditional probability $P[p_t,t|p_0]$. Now, we want to study the limit $t \to \infty$ where

$$\lim_{t \to \infty} P[p_t,t|p_0] = \frac{\exp(-\beta E)}{Z(\beta)},$$

i.e. we are assuming that the particle is in equilibrium after a long process ($t \to \infty$). In the above equation, $E$ is the energy and $Z$ is the partition function for this equilibrium distribution.

By considering this case, we are able to find analytical results for the heat distribution in the ultra-relativistic.

### 5.1 Ultra-relativistic

For the ultra-relativistic case, the conditional probability can be obtained analytically, as is showed in Appendix A, and we can check directly that $\lim_{t \to \infty} P[p_t,t|p_0] = \rho_0(pt)$

$$P(Q) = \int \frac{d\lambda}{2\pi} e^{i\lambda Q} \frac{Z(\beta - i\lambda)Z(\beta + i\lambda)}{Z(\beta)^2}.$$

This explicitly shows that this heat distribution has an exponential tail. Moreover, this is the same result obtained by [9] for an classical underdamped Brownian particle with an harmonic force. Curiously, the ultra-relativistic Brownian motion has the same heat behavior of a classical particle in a harmonic potential for $t \to \infty$.

### 5.2 Exact

Due the dispersion relation $E = \sqrt{M^2 + p^2}$, the equilibrium distribution does not have a closed formula for its partition function,

$$Z = \int \exp\left(-\beta \sqrt{p^2 + M^2}\right) dp.$$  

Nevertheless, we can integrate numerically and compare with the ultra-relativistic case. Using Eq. 21, to find the heat equilibrium heat distribution, we can compare both equilibrium heat distributions in Fig. 5.

By comparing the two heat distributions, one can see that the distributions are very similar with the finite time case in the previous section. This occurs because in both cases the initial distribution is in equilibrium, and the dynamics for the finite time does not give enough fluctuations to able us to see any qualitative difference.

Still, by analyzing the equilibrium case, we find the heat distribution analytically, and surprisingly it has the same form of an classical underdamped Brownian particle. This coincidence is counter-intuitive, since the ultra-relativistic case is more distant to the classical case if we compare with the exact limit.
6 Relaxation process

Starting from the equilibrium distribution, it is hard to see the relaxation process of the distribution. To study the relaxation process, instead of the equilibrium distribution, we start with a delta distribution in the momentum $\rho_0(p_0) = \delta(p_0)$, which is the distribution where we have all the knowledge about the momentum of the relativistic particle. Moreover, it can represent the limit $t \to 0$ for the distribution $\exp\left(-p^2/(2Dt)\right)(2\pi Dt)^{-1/2}$.

6.1 Ultra-relativistic

For the ultra-relativistic case, with the initial delta distribution $\rho_0(p_0) = \delta(p_0)$, the characteristic function can be obtained analytically and simplifies to

$$Z(\lambda) = \int dp P[p, t|0]e^{-i\lambda|p|}$$

$$= \frac{\gamma}{2(\gamma + \gamma T\lambda)} \text{N} \left( \text{erfc} \left( -\frac{c\gamma}{2\sqrt{T}} \right) - \text{erfc} \left( \frac{c(2\sqrt{2}\gamma + 2i\sqrt{2}\gamma T\lambda - \gamma \sqrt{T})}{2\sqrt{\gamma T}} \right) \right. \left. \times e^{-c^2(\gamma + \gamma T\lambda)(\sqrt{T^2 - 2\gamma T\lambda})} \right)$$

$$+ e^{-2\lambda t(-\gamma T\lambda + i\gamma)} \text{erfc} \left( \frac{c(t + 2i\gamma T\lambda)}{2\sqrt{\gamma T}} \right)$$

$$N = \int dp P[p, t|0]$$

$$= 1 \left( -e^{-c^2(\gamma + \gamma T t)} \text{erfc} \left( \frac{c\gamma(2\sqrt{2} - \sqrt{t})}{2\sqrt{\gamma T}} \right) \right. \left. + \left( \frac{1}{\sqrt{\pi \gamma T t}} - 1 \right) \text{erfc} \left( \frac{c\gamma}{2\sqrt{\gamma T}} + 2 \right) \right).$$

This characteristic function can be integrated numerically, giving us the heat distribution, which is plotted for different times in Fig. 6.

With the equilibrium distribution as the initial, there is only a trivial fluctuation theorem since the probability is symmetric, giving $P(Q) = P(-Q)$. And by analyzing Fig. 6 one can see that this is also the case for a delta distribution. Note that, as time pass, and the heat distribution tends to equilibrium, the fluctuations around the mean increase. This can be explained by the fact that the initial distribution has no fluctuations in the position of the particle in opposition with the equilibrium distribution. To better illustrate this point, we also recalculate the distribution with a different initial momentum value, $p_0 > 0$. The result is plotted in Fig. 7 and we can see that we again have $P(Q) = P(-Q)$, but now, the evolution of the distribution is slightly different due to $p_0 \neq 0$. There is now more fluctuation in the earlier times (observe the fatter tails for very short times), which corresponds to the evolution of the momentum towards its mean.
Fig. 8 Relaxation process of the heat distribution in the exact limit. All constants are set to one. The evolution of the heat distribution is very similar with the ultra-relativistic case, where the fluctuations only broadens as time pass. Inset: the heat distribution in the interval $Q \in [-5, 20]$. Note that for $t = 100$ the fluctuations in the tail are greater than in $t = 10$.

6.2 Exact

For the exact case, the procedure is analog with the initial distribution, we solve using the FENICs with initial probability distribution

$$P(Q, p_0, t = 0) = \delta(p_0) \delta(Q), \tag{26}$$

which is the only difference of the previous approach. The result can be seen in Fig. 8.

The relaxation process is very similar to the ultra-relativistic case: the fluctuations increase as time passes, as one can see in the inset of Fig. 8, and the peak of the heat distribution decreases. Moreover, the distribution is symmetric around $Q = 0$, yielding a trivial Fluctuation Theorem $P(Q) = P(-Q)$. Notice that the distribution in Fig. 8 does not have a sharp peak in $Q = 0$, in contrast with the full equilibrium case (see Fig. 5). We point out that this absence of the peak comes from numerical error: without enough data points, the distribution is smoothed.

7 Relativistic vs classical

The heat distribution in the underdamped case was calculated explicitly in [6], and has the same functional formula of an overdamped Brownian motion with a harmonic force, which is calculated in [18]. For the same values, we can plot the two cases to search for differences. We want to answer the question: Is there an intrinsic relativistic effect in the heat distribution?

$$P(Q)_{\text{classical}} = \frac{\sqrt{\coth(\gamma t) + 1}}{\sqrt{2} \pi T} K_0 \left( \frac{|Q| \sqrt{\coth(\gamma t) + 1}}{\sqrt{2} T} \right). \tag{27}$$

Plotting the ultra-relativistic case and the exact case together with the classical in Fig. 9 we can visualize some differences.

Physically, we find that in the Relativistic Stochastic Thermodynamics the heat shares similar statistical behavior compared to the classical (non-relativistic) case [6] (see also [46,47]). The relativistic heat distribution for the free particle has a sharp distribution around $Q = 0$, meaning that the particle neither absorbs nor gains energy on average from the environment. This result is also consistent with the classical case. Moreover, another similarity is the symmetry of the distribution, which is $P(Q) = P(-Q)$ showing that there is no preferred direction for the heat. One can expect this result since no external force or internal potential is interacting with the particle. Therefore, the proposed relativistic stochastic thermodynamics leads to well-behaved physical properties for the heat.

8 Conclusion

In the present paper, we have study the energy exchanged between a Relativistic diffusive particle and a thermal environment. We solve for the heat distribution in two different regimes, the ultra-relativistic and the exact relativistic limit. We first calculate the heat distribution in the ultra-relativistic limit through the use of path integrals. We find exact results through numerical integration of the characteristic function. For the exact limit, such an approach is not available, thus we use
FEniCS [37] to numerically solve the Fokker–Planck of the joint distribution \( P(Q,p) \), and then integrate over \( p \) to find \( P(Q) \). Both results are in essence exact and agree with numerical simulations. We then calculate the equilibrium case, where \( P[p_t, t|p_0] \rightarrow \rho_0(p_t) \) becomes the canonical distribution, analyzing the heat distribution for the asymptotic time. Moreover, we studied the relaxation process, choosing a delta distribution as initial condition, this allows us to find an analytic expression for the characteristic function of the heat in the ultra-relativistic limit and an exact numerical distribution for the exact relativistic case, and also allow us to see the evolution of the heat distribution toward equilibrium. By the end, we compare the ultra-relativistic and exact relativistic cases with the heat in the classical Brownian motion, finding differences between the distributions. Moreover, all heat distributions obtained in the paper obey a trivial fluctuation theorem \( P(Q) = P(-Q) \).

The methods presented here can be generalized to higher dimensions, which is a more realistic scenario. A promising case is in the graphene chip [27], where the charge carriers could obey the 2D Ultra-relativistic Langevin equation. Thus, one can study the protocols in the ultra-relativistic case, perform work on the system, and even study thermal machines protocols. Moreover, as we showed, the fluctuations of heat are different, and can lead to different results in the efficiency of such machines. Hence, the present study can also serve as a starting point for the study of such machines. By calculating the heat and work distribution, one can investigate the efficiency of such a case. This will be a theme of future work.

Moreover, a more realistic scenario can also take into account interactions and multiplicative noise [48]. Both complications can also be treated with the methods exposed herein.

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**Author contributions**

PVP (graduate student under supervision by WAMM) contributed with most of the ideas, calculations and writings (75%). WAMM contributed with ideas, discussions, and text corrections and adjustment (25%).

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### Appendix A: Path integral for ultra-relativistic case

In Eq. 10 the conditional probability can be derived by means of the path integral technique. Which states that

\[
P[p_t, t|p_0] = \int_{p(0)=p_0}^{p(t)=p_t} Dp \ e^{-A[p(t)]}
\]  

(A1)

where \( A[p(t)] \) is the stochastic action [40–42], in the Stratonovich prescription is given by

\[
A[p(t)] = \frac{1}{4D} \int_0^t \left( \dot{p} + \gamma c \frac{p}{|p|} \right)^2 d\tau - \frac{\gamma c}{2} \int_0^t \frac{\partial}{\partial p} |p| d\tau,
\]  

(A2)

where \( D = \gamma T \). By noticing that \( p/|p| = \text{sign}(p) = 2H(p)-1 \) and \( H'(p) = \delta(p) \), we can rewrite the action as

\[
A[p(t)] = \frac{1}{4D} \int_0^t \left( \dot{p}^2 - \alpha \delta(p) \right) d\tau + \frac{\gamma c}{2D} \left( \gamma c t + |p_t| - |p_0| \right)
\]  

(A3)

where, \( \alpha = 4D\gamma c \). The conditional probability can be rewritten as

\[
P[p_t, t|p_0] = e^{-\frac{1}{2D} \int_0^t (\dot{p}^2 - \alpha \delta(p)) d\tau + \frac{\gamma c}{2D} \left( \gamma c t + |p_t| - |p_0| \right)} K[p_t, t|p_0],
\]  

(A4)

where \( K[p_t, t|p_0] \) will be the path integral

\[
K[p_t, t|p_0] = \int_{p(0)=p_0}^{p(t)=p_t} Dp \exp \left( -\frac{1}{4D} \int_0^t (\dot{p}^2 - \alpha \delta(p)) d\tau \right)
\]  

(A5)

which has the same structure of a quantum mechanical propagator of a particle with a delta potential [49–52]. Thus, following [52] we review the derivation of this path integral.

To solve Eq. (A5) we expand the potential, obtaining

\[
K[p_t, t|p_0] = K_0[p_t, t|p_0] + K_1[p_t, t|p_0]
\]  

(A6)

where

\[
K_0[p_t, t|p_0] = \int_{p(0)=p_0}^{p(t)=p_t} Dp e^{\int_0^t \frac{\alpha}{2} \dot{p}^2 d\tau} = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(p_t-p_0)^2}{4D t}} 
\]  

(A7)

\[
K_1[p_t, t|p_0] = \sum_{n=1}^{\infty} \left( \frac{-1}{2D} \right)^n \frac{(-\alpha)^n}{n!} \int Dp \ e^{-\int_0^t \dot{p}^2 d\tau} \times \left( \int_0^t \delta(p) d\tau \right)^n.
\]  

(A8)

Therefore, we only have to solve \( K_1 \). To do this, note that

\[
\left( \int_0^t \delta(p) d\tau \right)^n = \int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n \prod_{k=1}^n \delta(p(t_k))
\]  

(A9)

\[
= n! \int_0^t dt_1 \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \prod_{k=1}^n \delta(p(t_k))
\]  

(A10)
where in the second line, we just reordered the time. The path integral in $K$ is a Wiener path integral that describes a Markovian stochastic process, thus we have the property

$$K_0[p_t, t|p_0, 0] = K_0[p, t|p_0, 0]K_0[p_0, t|p_{t-1}, t_0] \ldots K_0[p_{t-1}, t_0|p_t, 0],$$

(A11)

therefore, in $K_1$ we have the expression

$$\int dp_n \int dp_{n-1} \ldots \int dp_1 K_0[p_t, t|p_n, t_n] \delta(p_n)$$

$$\prod_{k=2}^n K_0[p_k, t_k|p_{k-1}t_{k-1}] K_0[p_t, t|p_0, 0] \delta(p_k) \delta(p_1)$$

(A12)

$$= K_0[p, t|0, t_n] \prod_{k=2}^n K_0[0, t_k|0, t_{k-1}] K_0[0, t_1|p_0, 0],$$

(A13)

Thus, we have

$$K_1 = \sum_{n=1}^\infty \left( \frac{\alpha}{4D} \right)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots$$

$$\times \int_0^{t_2} dt_1 K_0[p_t, t|0, t_n]$$

$$\times \prod_{k=2}^n K_0[0, t_k|0, t_{k-1}] K_0[0, t_1|p_0, 0].$$

(A14)

Note that we have convolutions between the propagators. Then, by making the Laplace’s transform

$$\tilde{K}_1 = \int_0^\infty e^{-st} K_1[p_t, t|p_0] dt,$$

(A15)

we can get rid of the time integrals, giving

$$\tilde{K}_1 = \sum_{n=1}^\infty \left( \frac{\alpha}{4D} \right)^n \tilde{K}(p_t, s) \tilde{K}(0, s)^{n-1} \tilde{K}(p_0, s)$$

(A16)

where

$$\tilde{K}(p, s) = \int_0^\infty e^{-st} e^{-\frac{p^2}{4\piDt}} dt = \exp \left( -|p| \sqrt{\frac{s}{D}} \right) \frac{\sqrt{1}}{\sqrt{s}},$$

(A17)

then

$$\tilde{K}_1 = \sum_{n=1}^\infty \left( \frac{\gamma c}{\sqrt{4D}} \right)^n \frac{1}{\sqrt{4D}} \exp \left( -(|p_t| + |p_0|) \sqrt{\frac{s}{D}} \right)$$

$$\times \left[ \frac{1}{\sqrt{s}} \right]^{n+1}.$$ (A18)

The above sum is solved exactly, since $\sum_{n=1}^\infty (a)^n = a/(a - 1)$, and then we can use the inverse Laplace transform to find the desired $K_1$.

$$K_1[p_t, t|p_0] = \frac{c\gamma}{\sqrt{4D}} L_1^{-1} \left[ \frac{\exp \left( -(|p_t| + |p_0|) \sqrt{s} \right)}{\sqrt{s} \sqrt{4D} - \gamma c} \right],$$

(A19)

where $L_1^{-1}$ is the inverse Laplace transform. Note that

$$L_1^{-1} \left[ \frac{e^{-a\sqrt{s}}}{\sqrt{s} \sqrt{4D} + b} \right] = e^{a} e^{\frac{b^2}{4}} \text{erfc} \left( \sqrt{\frac{b}{2}} + \frac{a}{2\sqrt{2}} \right).$$ (A20)

Thus, using the above formula, $K_1$ becomes

$$K_1[p_t, t|p_0] = \frac{c\gamma}{\sqrt{4D}} e^{\frac{c^2\gamma^2}{2D}} \exp \left( -\frac{\gamma c}{2D} (|p_t| + |p_0|) \right) \text{erfc}$$

$$\times \left[ \frac{1}{\sqrt{4D}} \left( \frac{|p_t| + |p_0|}{\sqrt{2}} - \gamma \sqrt{c} \sqrt{t} \right) \right].$$ (A21)

Therefore, we have

$$K[p_t, t|p_0] = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(p_t - p_0)^2}{4\piDt}} + \frac{c\gamma}{4D} e^{\frac{c^2\gamma^2}{2D}} t e^{-\frac{c^2\gamma^2}{2D} (|p_t| + |p_0|)} \text{erfc}$$

$$\times \left[ \frac{1}{\sqrt{4D}} \left( \frac{|p_t| + |p_0|}{\sqrt{2}} - \gamma \sqrt{c} \sqrt{t} \right) \right].$$ (A22)

and finally the conditional probability

$$P[p_t, t|p_0] = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(p_t - p_0)^2}{4\piDt}} e^{-\frac{c\gamma}{\sqrt{4D}} \left( -|p_0| + |p_t| + \frac{c^2\gamma^2}{2D} \right)}$$

$$+ \frac{c\gamma}{4D} e^{-\frac{c\gamma}{\sqrt{4D}}} \text{erfc} \left[ \frac{1}{\sqrt{4D}} \left( \frac{|p_t| + |p_0|}{\sqrt{2}} - \gamma \sqrt{c} \sqrt{t} \right) \right].$$

(A23)

Note that this distribution recovers the equilibrium one for $t \to \infty$, that is

$$\lim_{t \to \infty} P[p_t, t|p_0] = \frac{c\gamma}{2D} e^{-\frac{c\gamma}{\sqrt{4D}}} \frac{1}{2} e^{-\frac{c\gamma}{2D} |p_t|},$$

(A24)

as it should be.

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