Effectively Open Real Functions

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Abstract. A function $f$ is continuous iff the pre-image $f^{-1}[V]$ of any open set $V$ is open again. Dual to this topological property, $f$ is called open iff the image $f[U]$ of any open set $U$ is open again. Several classical Open Mapping Theorems in Analysis provide a variety of sufficient conditions for openness.

By the Main Theorem of Recursive Analysis, computable real functions are necessarily continuous. In fact they admit a well-known characterization in terms of the mapping $V \mapsto f^{-1}[V]$ being effective: Given a list of open rational balls exhausting $V$, a Turing Machine can generate a corresponding list for $f^{-1}[V]$. Analogously, effective openness requires the mapping $U \mapsto f[U]$ on open real subsets to be effective.

By effectivizing classical Open Mapping Theorems as well as from application of Tarski’s Quantifier Elimination, the present work reveals several rich classes of functions to be effectively open.

1 Introduction

Computability theory over the reals started by investigating single numbers. When real functions were later considered it turned out that continuity was a necessary condition for computability. A function $f : X \to Y$ between topological spaces is continuous iff, for any open set $V \subseteq Y$, its pre-image $f^{-1}[V] \subseteq X$ is open again. In the case of open $X \subseteq \mathbb{R}^n$ and $Y = \mathbb{R}^m$ this means that, for any countable union of $n$-dimensional open rational Euclidean balls

$$V = \bigcup_{j} B(y_j, r_j), \quad y_j \in \mathbb{Q}^m, \ r_j \in \mathbb{Q}_{>0}, \quad B(y, r) := \{u \in \mathbb{R}^m : |y - u| < r\},$$

$U := f^{-1}[V] \subseteq \mathbb{R}^n$ is also a countable union of $n$-dimensional open rational Euclidean balls $B(x_k, s_k)$. Moreover $f$ is computable in the sense of iff the mapping $V \mapsto f^{-1}[V]$ on hyperspaces of open subsets is effective in that, given a list of (centers $x_k$ and radii $r_k$) of open rational Euclidean balls $B(x_k, r_k) \subseteq \mathbb{R}^n$ exhausting $V$, one can compute a corresponding list of open rational Euclidean balls $B(y_{\ell}, s_{\ell}) \subseteq \mathbb{R}^n$ exhausting $f^{-1}[V]$; cf. Lemma 6.1.7 in [24].

So to speak ‘dual’ to continuity is openness: The function $f$ is open if, rather than its pre-image, its image $f[U] \subseteq Y$ is open for any open set $U \subseteq X$. While for example any constant $f$ lacks the latter property, conditions sufficient for its presence are given by a variety of well-known Open Mapping Theorems for instance in Functional Analysis, Complex Calculus, Real Analysis, or Algebraic Topology.

The classical duality of continuity and openness raises the question whether and to what extent it carries over to the computable setting. For the first two aforementioned theorems, effectivized versions (in the sense of Recursive Analysis) have been established respectively in [24] and [13]; see Theorem 2 below. It is indeed natural to consider, similarly to continuity and computability, also effective openness in the following sense:

**Definition 1.** Let $X \subseteq \mathbb{R}^n$ be r.e. open. Call an open function $f : X \to \mathbb{R}^m$ effectively open if, from any two lists $(x_j)_{j \in \mathbb{N}} \subseteq \mathbb{Q}^n$ and $(r_j)_{j \in \mathbb{N}} \subseteq \mathbb{Q}_{>0}$ such that $U = \bigcup_j B(x_j, r_j) \subseteq X$, a Turing Machine can compute two similar lists $(y_j)_{j \in \mathbb{N}} \subseteq \mathbb{Q}^m$ and $(s_j)_{j \in \mathbb{N}} \subseteq \mathbb{Q}_{>0}$ such that $f[U] = \bigcup_{j} B(y_j, s_j)$.

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1 that is, the union of certain open rational balls $B(z_j, t_j)$ whose centers $z_j$ and radii $t_j$ form computable rational sequences; cf. [24]
In the convenient language of Type-2 Theory of Effectivity [22], this amounts to the mapping $U \mapsto f[U]$ on open Euclidean subsets being $(\theta^n \rightarrow \theta^m)$–computable. Here, $\theta^n$ denotes a canonical representation for the hyperspace $\mathcal{O}^d$ of open subsets of $\mathbb{R}^d$; cf. Definition 5.1.15 in [22].

Apart from its natural duality to continuity and computability, openness and effective openness arise in the foundation of CAD/CAE [9] in connection with regular sets — i.e., roughly speaking, full-dimensional but not necessarily convex [16] ones — as essential prerequisites for computations thereon; cf. Proposition 1.1d-f) and Section 3.1 in [24].

The present work proves several rich and important classes of functions to be effectively open and thus applicable to such problems. Our claims proceed in analogy to those of classical Open Mapping Theorems. An example due to P. HERTLING illustrates the idea:

**Theorem 2.**

a) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be complex differentiable and non-constant. Then $f$ is open.

b) Let $f$ furthermore be computable. Then it is effectively open.

c) Claim b) holds even uniformly in $f$, that is, the mapping $(f, U) \mapsto f[U]$ with domain

$$\{(f, U) \mid f : \mathbb{C} \rightarrow \mathbb{C} \text{ complex differentiable non-constant, } U \subseteq \mathbb{C} \text{ open}\}$$

is $([\theta^2 \rightarrow \theta^2] \times [\theta^2 \rightarrow \theta^2])$–computable.

**Proof.** a) is well-known in Complex Analysis; see, e.g., [20, pp.231–233]. For b) and c), cf. Corollary 4.4 and Theorem 4.3 in [13], respectively. □

Here, $\theta^2$ denotes the Cauchy representation for the set $\mathbb{C}$ of complex numbers, identified with $\mathbb{R}^2$; and $[\theta^n \rightarrow \theta^m]$ is a natural representation for continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$; see Definitions 4.1.17 and 6.1.1 in [22].

In the spirit of the above result, we present in Section 2 several classical Open Mapping Theorems from Real Analysis and Algebraic Topology; and in Sections 3 and 4 according effectivizations. More precisely, proof-mining reveals several classes of computable open functions on Euclidean space to be effectively open. We focus on claims similar to Theorem 2b), that is, for fixed $f$ but uniformly in $U$. Section 5 takes a different approach in devising ‘from scratch’ proofs that computable open semi-algebraic functions are effectively open; here, arguments are based on Algebra and Tarski’s Quantifier Elimination. Section 6 finally investigates the general relation between computability and effective openness. We conclude in Section 7 with a strengthening of [24, Theorem 3.9] based on the results from Section 4.

### 2 Classical Open Mapping Theorems

We start with a characterization of open functions resembling that of continuous ones:

**Lemma 3.** Let $X \subseteq \mathbb{R}^n$ be open and denote $\overline{B}(x, s) := \{v \in \mathbb{R}^n : |v - x| \leq s\}$.

a i) A function $f : X \rightarrow \mathbb{R}^m$ is continuous iff the mapping

$$\text{Moc}_f : X \times \mathbb{N} \rightarrow \mathbb{R}, \quad (x, k) \mapsto \sup \{s \geq 0 : f[\overline{B}(x, s) \cap X] \subseteq B(f(x), 2^{-k})\}$$

is strictly positive;

a ii) equivalently: to any $(x, k) \in X \times \mathbb{N}$, there exists an $\ell \in \mathbb{N}$ such that

$$f[\overline{B}(x, 2^{-\ell}) \cap X] \subseteq B(f(x), 2^{-k}).$$

(2)
b) If \( f \) is continuous, \( x \in X \) and \( k \in \mathbb{N} \). Then for some \( s := \text{Moc}_f(x,k) \geq s \) is positive.

a) Let \( f \) be strictly positive and \( V \subseteq \mathbb{R}^m \) open. To show that \( f^{-1}[V] \) is open again, let \( x \in f^{-1}[V] \) arbitrary. As \( y := f(x) \in V \) and \( V \) is open, \( B(y,2^{-k}) \subseteq V \) for some \( k \in \mathbb{N} \). Then for \( s := \text{Moc}_f(x,k)/2 \), the open set \( U := B(x,s) \cap X \) satisfies

\[
\begin{align*}
x \in U \subseteq f^{-1}[f[U]] & \subseteq f^{-1}[B(y,2^{-k})] \subseteq f^{-1}[V]
\end{align*}
\]

That is, an entire open ball around \( x \) lying within \( f^{-1}[V] \).

Conversely let \( f \) be continuous, \( x \in X \) and \( k \in \mathbb{N} \). Therefore the pre-image \( U := f^{-1}[V] \) of \( V := B(f(x),2^{-k}) \) is open and contains \( x \). In particular \( B(x,s) \subseteq U \) for some \( s > 0 \) and \( \text{Moc}_f(x,k) \geq s \) is positive.

b) If \( f \) is open, then its image \( f[U] \) of the open set \( U := B(x,2^{-k}) \cap X \neq \emptyset \) is open again and thus contains, around the point \( f(x) \in f[U] \), some non-empty ball \( B(f(x),s) \) entirely; hence \( \text{Moc}_f(x,k) \geq s > 0 \).

Conversely let \( U \) denote an open subset of \( X \). To any \( y \in f[U] \), consider \( x \in U \) with \( y = f(x) \) and \( k \in \mathbb{N} \) such that \( B(x,2^{-k}) \subseteq U \). Then \( s := \text{Moc}_f(x,k)/2 \) satisfies

\[
B(y,s) \subseteq f[B(x,2^{-k}) \cap X] \subseteq f[U].
\]

Therefore \( f[U] \) is open.

b) Follows as in a ii). In particular it holds that \( f[U] = \bigcup_{x \in U} B(f(x),2^{\text{Moc}_f(x,k_x)}) \) for open \( U \subseteq X \) whenever \( k_x \in \mathbb{N} \) satisfies \( B(x,2^{-k_x}) \subseteq U \).
c) Any continuously differentiable (\(C^1\)) \(f : X \to \mathbb{R}^m\) is open, provided its Jacobian \(f'(x) = ((\partial_j f_i))_i\) has rank \(m\) for all \(x \in X\).

\(\text{d) Whenever continuous } f : X \to \mathbb{R}^n \text{ satisfies local injectivity (i.e., to each } x \in X \text{ there exists } \varepsilon > 0 \text{ such that the restriction } f\bigg|_{B(x,\varepsilon)} \text{ is injective), then it is open.}\)

Claim \(\text{d) generalizes Domain Invariance from Algebraic Topology where often injectivity is presumed globally.}\)

\textbf{Proof} (Fact \(\text{6}).\)

a) Exploit the one-dimensional range and apply the Intermediate Value Theorem: For an open ball \(U := B(x, 2^{-k}) \subseteq X\), \(f[U]\) is connected and thus a real interval. As furthermore \(f\) has by prerequisite no local extrema, any \(y \in f[U]\) is accompanied by \(y_-, y_+ \in f[U]\) such that \(y_- < y < y_+\). This implies \((y_-, y_+) \subseteq f[U]\) and reveals that \(f[U]\) is open.

b) follows from c), as the Jacobian of \(f(x) = A \cdot x + b\) is \(A \in \mathbb{R}^{m \times n}\) (independent of \(x\)) and \(\text{rank}(A) = m\) is equivalent to \(f\) being surjective.

d) See for example \(\text{[22 THEOREM 4.3]}\) where (for \(r = \varepsilon\)) the proof proceeds by showing that the topological degree \(d(\Omega, f, y)\) of \(f\) with respect to domain \(\Omega := B(x, r)\) is non-zero for all \(y\) in some \(s\)-ball around \(f(x)\). This guarantees that \(f|_{\Omega}\) attains any such value \(y \in B(f(x), s)\), that is, \(f[\Omega] \) contains \(B(f(x), s)\). For \(\varepsilon < 2^{-k}\), this implies \(\text{Moc}_f(x, k) \geq s/2 > 0\) and by Lemma \(\text{[6b]}\) yields openness of \(f\). \(\square\)

Regarding c), \(f\) has no chance of being locally injective whenever \(n > m\) so that d) is not applicable in that case. Instead, exploiting differentiability, recall the Inverse Function Theorem from Real Analysis:

\textbf{Fact 7} Let \(U \subseteq \mathbb{R}^n\) be open, \(f : U \to \mathbb{R}^m\) continuously differentiable, and \(x_0 \in U\) such that \(\text{rank } f'(x_0) = m\). Then there exists a continuously differentiable local right inverse to \(f\) at \(x_0\), that is, \(\delta > 0\) and a \(C^1\) function

\[ g : B(f(x_0), \delta) \subseteq \mathbb{R}^m \to U \quad \text{such that} \]
\[ g(f(x_0)) = x_0, \quad f \big(g(y)\big) = y \quad \forall y \in B(f(x_0), \delta) \]  

(5)

If \(n = m\), then \(g\) is unique and locally left inverse to \(f\), i.e., \(g(f(x)) = x\) on \(B(x_0, \varepsilon)\) for some \(\varepsilon > 0\).

In particular, \(f[U]\) covers the open ball \(B(f(x_0), \delta) \subseteq \mathbb{R}^m\). By taking \(U := B(x_0, 2^{-k}) \subseteq X\), we obtain \(\text{Moc}_f(x, k) \geq \delta/2 > 0\) and Fact \(\text{[6d]}\) finally follows with Lemma \(\text{[6b]}\).

\section{Effective Continuity, Effective Openness}

The present section is about an effectivization of Lemma \(\text{[6]}\). While positivity of \(\text{Moc}_f/\text{Moo}_f\) is trivially equivalent to the existence of an according \(\text{moc}/\text{moo}\), respectively, similar equivalences are by no means obvious with respect to computability. In fact for this purpose, both \(\text{moc}\) and \(\text{moo}\) have to be allowed to become \textit{multi-valued} in the sense \(\text{[22 DEFINITION 3.1.3.4]}\) that the integer \(\ell\) returned by a Type-2 Machine computing \(\text{moc}(x, k)\) or \(\text{moo}(x, k)\) may depend, rather than on the value of the argument \(x\) itself, also on the particular choice of rational approximations for \(x\). Such effects are well-known in Recursive Analysis, see for instance \(\text{[22 EXAMPLE 4.1.10 or THEOREM 6.3.7]}\).

Also recall, e.g. from \(\text{[22]}\), that \(\varphi_\prec\) is a representation for \(\mathbb{R}\) connected to lower computability in that it encodes rational approximations to the real number under consideration \textit{from below}. Furthermore, \(\nu\) denotes the standard notation of \(\mathbb{N}\).
Theorem 8. Let \( X \subseteq \mathbb{R}^n \) be r.e. open. Parallel to (the numbering in) Lemma \( \mathbf{3} \) we have:

a) Fix some effective (i.e., \((\nu \rightarrow \varnothing^n)\)-computable) enumeration \( (x_j) \) of a dense sub-set \( \mathbf{x} \) of \( X \). A function \( f : X \rightarrow \mathbb{R}^m \) is computable and if the real sequence \( \{f(x_j)\} \) is computable and \( \{f(x_k)\} \) is effectively open iff the real sequence \( \{f(x_j)\} \) is computable if the real sequence \( \{f(x_k)\} \) is computable and \( \{f(x_l)\} \) is effectively open.

b) A computable \( f : X \rightarrow \mathbb{R}^m \) is effectively open iff \( f \) is continuous and \( (\varnothing^n \times \nu \rightarrow \varnothing_n)\)-computable.

Claim b) is closely related to Theorem 6 in [11]. Extending Definition 4, multi-valued functions \( \text{moc/moc} \) in the sense of Claims a ii) and b ii) will in the sequel also be called moduli of continuity/openness, respectively. Before turning to the proof, we provide in Lemma 10 some tools on multi-valued computability which turn out to be useful.

By the Main Theorem of Computable Analysis, any computable real function \( f \) on a compact domain is continuous and thus bounded. However, the present work also considers multi-valued functions like moduli of continuity; and such functions can in general be unbounded even on compact domains.

Example 9. For a rational sequence \((x_j)\) with \(|x_j - x| < 2^{-j}\) for all \( j = 0, 1, \ldots \), let

\[
F((x_j) ) := \left\lfloor \frac{1}{x_j + 1} \right\rfloor
\]

Then, \( F \) is a computable realization of a multi-valued, unbounded function \( f : [0, 1] \rightarrow \mathbb{N} \).

Item b) below basically says that such unpleasant cases can always be avoided by passing to another computable multi-valued function. To this end, we call \( \tilde{f} : X \rightarrow Y \) a sub-function of \( f : X \rightarrow Y \) if \( \tilde{f}(x) \subseteq f(x) \) for all \( x \in X \) and remark that, according to Definition 3.1.3.4], if \( \tilde{f} \) is computable then so are all its super-functions \( f \).

Lemma 10. Let \( X \subseteq \mathbb{R}^n \) be r.e. open and let \( \varnothing_d \) denote the signed digit representation.

a) The multi-valued partial mapping \( G : \subseteq \Omega^n \times \mathbb{R}^n \rightarrow \mathbb{N} \), \( \text{Graph}(G) := \{(U, x, k) : \mathbb{B}(\mathbf{x}, 2^{-k}) \subseteq U \} \) is \((\varnothing^n \times \nu \rightarrow \nu)\)-computable.

b) To every \((\varnothing^n \rightarrow \varnothing^n)\)-computable multi-valued \( f : X \rightarrow \mathbb{R}^m \), there exists a multi-valued \((\varnothing^n \rightarrow \varnothing^n)\)-computable sub-function \( \tilde{f} \) with which the image \( f[K] := \bigcup_{x \in K} f(x) \subseteq \mathbb{R}^m \) of any compact subset \( K \) of \( X \) is bounded.

c) For \( m = 1 \) and the function \( \tilde{f} \) from b), an upper bound \( N \in \mathbb{N} \) on \( f[K] \subseteq \mathbb{R}^m \) can be found effectively; formally: The multi-valued mapping \( \mathbf{x} \rightarrow N \) with \( \tilde{f}(K) \subseteq [-N, +N] \) is \((\varnothing^n \rightarrow \nu)\)-computable.

Claims b) and c) also hold uniformly in \( p \) for parametrized computable functions \( p \mapsto f(p, \cdot) : X \rightarrow \mathbb{R}^m \).

Proof. a) By [24, Lemma 4.1b], the property \( \overline{\mathbf{B}(\mathbf{x}, 2^{-k})} \subseteq U \) is semi-decidable with respect to \( k \in \mathbb{N} \) for \( \mathbf{x} \) given by a \( \varnothing^n \)-name and \( U \in \Omega^n \) by a \( \theta^n \)-name. So dove-tailed search can effectively find an appropriate \( k \) whenever \( \mathbf{x} \in U \).

b) Let \( F : \subseteq \Sigma^* \rightarrow \Sigma^* \) denote some computable (single-valued) realization of \( f \). Exploiting \( \varnothing \equiv \varnothing_d \) according to [22, Theorem 7.2.5.1], we pre-compose \( F \) with a computable function \( G \) converting \( \varnothing_d \)-names to \( \varnothing^n \)-names, \( F \circ G \) therefore realizes a \((\varnothing_d \rightarrow \varnothing^n)\)-computable sub-function \( \tilde{f} \) of \( f \), defined by \( \tilde{f}(\mathbf{x}) = \{ \varnothing^n(\bar{x}) : \varnothing_d(\bar{x}) = x \} \). By [22, Exercise 7.2.9], the collection \( K \subseteq (\Sigma^*)^n \) of all \( \varnothing_d \)-names \( \bar{x} \) of all \( x \in K \) is in particular compact. Being Cantor-continuous, \( F \circ G \) maps \( K \) to a compact set \( (F \circ G)[K] \) whose image under \( \varnothing^n \), namely the set \( f[K] \), is again compact by admissibility of \( \varnothing^n \).

\( ^2 \) like, for instance, \( X \cap \mathbb{Q} \)
c) Rather than carefully adapting the proof of for example [22, THEOREMS 7.1.5], we slightly modify the Type-2 Machine $M$ computing $F \circ G$ in b) to operate as follows: Upon input of an $g^{\text{id}}_{\text{ad}}$-name for $x \in X$ and while calculating rational approximations $y_j$ to $y = f(x)$ with $|y_j - y| < 2^{-j}$, idly loop $\lceil |y_0| + 1 \rceil$ times before actually outputting the first symbol of that $g$-name for $y$ and then proceeding like $M$.

This new machine $M$ will thus satisfy $\text{dom}(M) = \text{dom}(\bar{M})$ and $\text{Time}_{\bar{M}}(\bar{\sigma})(1) \geq \bar{f}(x)$ for any $g^{\text{id}}_{\text{ad}}$-name $\bar{\sigma}$ of $x \in X$. In particular, $\text{Time}^{\bar{K}}_{\bar{M}}(1) \in \mathbb{N}$ is an upper bound on $\bar{f}[K]$ where $\bar{K} \subseteq \Sigma^\omega$ denotes the collection of all $g^{\text{id}}_{\text{ad}}$-names $\bar{f}$ for all $x \in K$.

According to [22, EXERCISE 7.2.9], $K \to \bar{K}$ is $(\kappa^2 \to \kappa^2)$-computable; and [22, EXERCISE 7.1.4a] implies that, from a $\kappa^2$-name of $\bar{K}$, one can effectively obtain an upper bound $N$ on $\text{Time}^{\bar{K}}_{\bar{M}}(1)$. $\square$

### 3.1 Proof of Theorem 8

This section proves the several claims made in Theorem 8.

**Claim.** Let $X \subseteq \mathbb{R}^n$ be r.e. open, $(x_j)$, a computable sequence dense in $X$, and $f : X \to \mathbb{R}^m$ computable. Then the sequence $(f(x_j))$ is computable, and $f$ admits a computable multi-valued modulus of continuity.

**Proof.** The first sub-claim is immediate. For the second one, let $x \in X$ and $k \in \mathbb{N}$ be given. From these, $\theta^k$-compute $U := f^{-1}[B(f(x), 2^{-k})] \cap X$ by virtue of [22, THEOREM 6.2.4.1 and COROLLARY 5.1.18.1]. Then invoke Lemma [10a) to obtain some $\ell \in G(U, x)$. This satisfies Equation (2) because $f[V] \subseteq U$ is equivalent to $V \subseteq f^{-1}[U]$. $\square$

**Claim.** Let $X \subseteq \mathbb{R}^n$ be r.e. open, $(z_j)$, a computable sequence dense in $X$, $f : X \to \mathbb{R}^m$ such that $(f(z_j))$ is computable, and $\text{moc} : X \times \mathbb{N} \Rightarrow \mathbb{N}$ a computable multi-valued modulus of continuity. Then, $f$ is computable.

**Proof.** First note that $f$ is continuous by Lemma [5b). We show that it furthermore admits effective evaluation: Given a sequence $x_\ell \in X$ of rational vectors with $|x - x_\ell| < 2^{-\ell}$ for some $x \in X$, one can computably obtain a sequence $y_\ell$ such that $|f(x) - y_\ell| < 2^{-k}$.

Indeed, calculate by prerequisite $\ell \in \text{moc}(x, k)$; then search (dove-tailing) for some $j$ with $z_j \in X$ and $|z_j - x_{\ell+1}| < 2^{\ell-\ell_1}$; finally let $y_\ell := f(z_j)$. It follows $|z_j - x| < 2^{-\ell}$ and thus, by Equation (2), $|f(x) - y_\ell| < 2^{-k}$. $\square$

**Claim.** Let $X \subseteq \mathbb{R}^n$ be r.e. open, $f : X \to \mathbb{R}^m$ computable and effectively open. Then, $\text{Moo}_f : X \times \mathbb{N} \to \mathbb{R}$ is $(\theta^0 \times \nu \to g_\prec)$-computable.

**Proof.** The mapping $(x, k) \mapsto B(f(x), 2^{-k}) \cap X =: U$ is $(\theta^0 \times \nu \to \theta^0)$-computable since $X$ is r.e. open [22, COROLLARY 5.1.18.1]. By assumption on effective openness of $f$, one can therefore obtain a $\theta^0$-name for the open set $V := f[U] \ni y := f(x)$. Then searching all rational $s \geq 0$ satisfying $\overline{B}(f(x), s) \subseteq V$ is possible due to [22, LEMMA 4.1b)] and yields lower approximations to (i.e., a $g_\prec$-name for) the value $\text{Moo}_f(x, k)$. $\square$

**Claim.** Let $\text{Moo}_f : X \times \mathbb{N} \to \mathbb{R}$ be strictly positive and $(\theta^0 \times \nu \to g_\prec)$-computable; then there is a computable multi-valued moo.

**Proof.** From a $g_\prec$-name of $s := \text{Moo}_f(x, k) > 0$, obtain some $\ell \in \mathbb{N}$ with $2^{-\ell} < s$; compare [22, EXAMPLE 4.1.10]. $\square$

For the converse claims in Theorem 8b), the prerequisite of a computable $f$ can actually be relaxed to continuity with computable values on a computable dense subset. This resembles conditions (9a) and (9b) in [11] and is, without (9c) therein, more general than requiring computability of $f$. 

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Claim 11 Let \( X \subseteq \mathbb{R}^n \) be r.e. open, \((x_i)_i\) a dense computable sequence in \( X \), \( f : X \to \mathbb{R}^m \) continuous, the sequence \((f(x_i))_i\) computable, and moo a computable multi-valued modulus of openness. Then \( f \) is effectively open.

Proof. From Lemma 10b) we already know that \( f \) is open. The goal is thus to \( \theta_\omega \)-compute \( f[U] \), given a \( \theta_\omega \)-name of some open \( U \subseteq X \). The proof of Lemma 5 ii) has revealed that

\[
f[U] \supseteq \bigcup_{x \in U} B\left(f(x), 2^{-\ell_x}\right) \supseteq \bigcup_{j: x_j \in U} B\left(f(x_j), 2^{-\ell_{x_j}}\right) =: V \tag{6}
\]

for arbitrary \( \ell_x \) in moo \((x, G(U, x))\) with \( G \) from Lemma 10a). \( V \) is indeed contained in \( f[U] \) as the union to the right ranges only over certain \( x \in U \) compared to all in the left one. Being only a countable union, \( V \) can be \( \theta_\omega \)-computed according to Example 5.1.19.1 in [22]. More precisely, the \( \theta_\omega \)-name of \( U \) permits enumeration of all \( j \) such that \( x_j \in U \) by virtue of Lemma 10a); the multi-valued mapping \( h : (U, x) \mapsto \text{moo}(x, G(U, x)) \) is \((\theta_\omega \times \varnothing) \rightarrow \nu)\)-computable; and the multi-valued mapping \( j \mapsto B(f(x_j), 2^{-h(U, x_j)}) \) is \((\nu \Rightarrow \theta_\omega)\)-computable since \( j \mapsto x_j, j \mapsto f(x_j) \) both are by assumption.

To complete the proof of Claim 11 we shall show that in fact the reverse inclusion \( "f[U] \subseteq V" \) holds as well for any \( \ell_{x_j} \in h(U, x_j) \) with \( h \) denoting the computable subfunction according to Lemma 10b). So take arbitrary \( y \in f[U] \), \( y = f(x) \) with \( x \in U \). If \( x = x_j \) for some \( j \), then \( y \in V \) by definition anyway. If \( x \) does not occur within the sequence \((x_j)_i\), consider some compact ball \( B := \overline{B(x, r)} \) sufficiently small to be contained in \( U \). By the parametrized version of Lemma 10(i), there exists an upper bound \( L \in \mathbb{N} \) for \( h(U, \cdot) \) on \( B \). Exploiting continuity, \(|f(x) - f(z)| < 2^{-L} \) for all \( z \) sufficiently close to \( x \). In particular for an appropriate \( z = x_j \) and any \( \ell \in h(U, x_j) \leq L \) by choice of \( L \), it holds that \( f(x) \in B(f(x_j), 2^{-L}) \subseteq B(f(x_j), 2^{-\ell}) \). The latter term occurs in the right hand side union of (6); we have thus proven an arbitrary \( y = f(x) \in f[U] \) to lie in \( V \). \( \square \)

4 Effectivized Open Mapping Theorems

Here come the already announced effectivizations of the classical claims from Fact 6.

Theorem 12. Let \( X \subseteq \mathbb{R}^n \) be r.e. open.

a) Every computable open \( f : X \to \mathbb{R} \) (i.e., with one-dimensional range) is effectively open. More generally whenever a computable open \( f : X \to \mathbb{R}^m \) maps open balls \( B \subseteq X \) to convex sets \( f[B] \subseteq \mathbb{R}^m \), then it is effectively open.

b) Any surjective computable affinely linear mapping is effectively open.

c) If computable \( f : X \to \mathbb{R}^m \) is \( C^1 \), \( f \) is computable and has rank \( m \) everywhere, then \( f \) is effectively open.

d) Let both \( f : X \to \mathbb{R}^n \) and \( h : X \to \mathbb{N} \) be computable such that, for any \( x \in X \) and \( \ell \in h(x) \), the restriction \( f|_{B(x, 2^{-\ell}) \cap X} \) is injective. Then \( f \) is effectively open.

e) Suppose \( X \) is bounded, \( f : \overline{X} \to \mathbb{R}^n \) computable and locally injective; then \( f|_X \) is effectively open. The same holds if \( f : X \to \mathbb{R}^n \) is computable and globally injective.

Proof. Claims a) and d) will be proven in Subsections 4.1 and 4.2 respectively. Claim b) follows from c) just like in the classical case; and c) in turn, once again similarly to the classical case, is a consequence of the effectivized Inverse Function Theorem 4 below.

Claim d) implies the second part of e) as, \( h(x) := 0 \) will do. For the first part of e) observe that, \( f \) being locally injective on a compact domain, finitely many out of the balls \( B(x, \varepsilon(x)) \) suffice to cover \( X \). Therefore there exists one \( \varepsilon_0 > 0 \) common to all \( x \in X \) such that \( f|_{B(x, \varepsilon_0) \cap X} \) is injective; w.l.o.g. \( \varepsilon_0 = 2^{-k_0} \) for some \( k_0 \in \mathbb{N} \). As \( k_0 \) is just an integral constant, \( h(x) := k_0 \) defines a computable function; now apply Item d). \( \square \)

\footnote{Here we do not need to find this bound effectively.}
The following is a computable counterpart to Fact 4.

**Theorem 13 (Effectivized Inverse Function Theorem).** Let $U \subseteq \mathbb{R}^n$ be r.e. open, $f : U \to \mathbb{R}^m$ computable with computable derivative, and $x_0 \in U$ computable such that rank $f'(x_0) = m$. Then there exists a computable local right inverse to $f$, that is, a computable function $g$ with computable derivative satisfying $\delta$ and $\text{dom}(g) = B(f(x_0), \delta) \subseteq \mathbb{R}^m$ for some rational $\delta > 0$.

Moreover, such a $\delta = 2^{-k} > 0$ is uniformly computable from $x_0$: formally:

For r.e. open $X \subseteq \mathbb{R}^n$ and computable, continuously differentiable $f : X \to \mathbb{R}^m$ with computable derivative $f' : X \to \mathbb{R}^m \times \mathbb{N}$, the multi-valued mapping $I : \mathbb{D}^n \times X \to \mathbb{N} \times \mathbb{N}$ with

$$\text{Graph}(I) := \{(U, x_0, k) \mid x_0 \in U \subseteq X, \text{ rank } (f'(x_0)) = m, \exists g : B(f(x_0), 2^{-k}) \to U \text{ satisfying } \delta)\}$

is $(\theta^n \times g^n \Rightarrow \nu)$-computable.

Similarly to the classical case, $f[U]$ in particular covers the open ball $B(f(x_0), \delta) \subseteq \mathbb{R}^m$. Setting $\text{moo}(x, k) := I(B(x, 2^{-k}), x)$ therefore proves Theorem 12c) by virtue of Theorem 8b).

We emphasize that Theorem 13 can be generalized to hold even uniformly in $(f, f')$. Furthermore the multi-valued computation is extendable to yield not only $\delta$ but also $g$ and $g'$. As the domain of these partial functions varies, an according formalization however requires an appropriate representation such as $\delta_1$ from Exercise 6.1.11 in 89 and is beyond our present interest.

Let us also point out that, although the proofs to Theorem 14 (in Subsection 4.3) as well as the one to Theorem 14c+d proceed by presenting according algorithms, they are not necessarily constructive in the intuitionistic sense since the correctness of these algorithms relies on Brouwer’s Fixed-Point Theorem.

4.1 Proof of Theorem 12c)

Claim. Let $X \subseteq \mathbb{R}^n$ be r.e. open. If computable open $f : X \to \mathbb{R}^m$ maps open Euclidean balls to convex sets, then it is effectively open.

Proof. Recall that a $\theta^n$-name for $U \subseteq X$ is (equivalent to) a list of all closed rational Euclidean balls $\overline{B}_i = B(z_i, r_i)$ contained in $U$ 89 DEFINITION 5.1.15.1 and EXERCISE 5.1.7.

Since it is easy to obtain a $\psi^n$-name for each such $\overline{B}_i$, one can $\psi^n$-compute $\overline{f}([B_i])$ by virtue of Theorem 6.2.4.3 in 89. In fact $f([B_i]) = \overline{f}([B_i])$ since $f$ is continuous and $\overline{B}$ compact; cf. 89 LEMMA 4.4d). The prerequisite asserts $\overline{f}([B_i])$ to be convex, and its closure is thus convex and even regular by Proposition 1.1f) in 89. By virtue of 89 Theorem 4.12a), the $\psi^n$-name for $\overline{f}([B_i])$ can hence be converted into a matching $\theta^n$-name, that is, a $\theta^n$-name for $\overline{f}([B_i])$, a subset of $f[U]$ as $\overline{B}_i \subseteq U$ and $\overline{f}([B_i]) = f([B_i])$.

Doing so for all $\overline{B}_i$ listed in the $\theta^n$-name of $U$ and taking their countable union according to 89 EXERCISE 5.1.19], we thus obtain a $\theta^n$-name for some open subset $V$ of $f[U]$. To see that $V$ in fact coincides with $f[U]$, consider some $y \in f[U], y = f(x)$ with $x \in U$. Then some entire ball $B(x, s)$ is contained inside of $U$. By density, there exist $z \in \mathbb{Q}^n$ and $0 < r \in \mathbb{Q}$ such that $x \in B(z, r) \subseteq \overline{B}(z, r) \subseteq B(x, s)$. This $\overline{B}_i$ will thus occur in the input and, in the union constituting $V$, lead to a term $\overline{f}([B_i]) \supseteq f([B_i]) \supseteq f(x) = y$, cf. 89 LEMMA 4.2i)]. As $y \in f[U]$ was arbitrary, this proves $V \supseteq f[U]$. □
4.2 Proof of Theorem 12d)

Regarding Theorem 12d), the claim follows uniformly in \( f \) from the below

**Lemma 14.** Fix r.e. open \( X \subseteq \mathbb{R}^n \). The multi-valued mapping \( H : C(X, \mathbb{R}^n) \times X \times \mathbb{N} \Rightarrow \mathbb{N} \)

with \( \text{Graph}(H) := \{(f, x, k, \ell) \mid f : X \rightarrow \mathbb{R}^n \text{ injective on } B(x, 2^{-k}) \subseteq X, \}

\( B(f(x), 2^{-\ell}) \subseteq f(B(x, 2^{-k}))\} \)

is \(((\theta^o \rightarrow \theta^o) \times \theta^o \times \nu \Rightarrow \nu)\)-computable.

**Proof.** The mapping’s value is indeed an open set because of Fact 6.2.4.2 and 5.1.13.2 in [22]. Recall its proof based on Theorem 4.3 in [22] together with Theorem 3.1(d4+d5) therein. The latter reveal that, for each \( \Omega := B(x, 2^{-k-1}) \) — observe \( \Omega \subseteq X - f[\overline{\Omega}] \) covers \( B(f(x), r) \) where \( r > 0 \) denotes the distance of \( f(x) \) to the set \( K := f[\partial \Omega] \). The sphere boundary \( \partial \Omega \) being obviously \( \kappa^o \)-computable from \( (x, k) \), \( K \)'s distance function is uniformly computable by virtue of Theorem 6.2.4.4 in [22]. In particular, one can effectively evaluate this function at \( f(x) \) and thus obtain the aforementioned \( r \). From this it is easy to get some \( \ell \in \mathbb{N} \) with \( 2^{-\ell} < r \). \( \square \)

4.3 Proof of Effectivized Inverse Function Theorem

An important part in the proof of Theorem 13 relies on the following result on computability of unique zeros of real functions. It generalizes Corollary 6.3.5 in [22] from one to higher dimensions.

**Lemma 15.** Consider the class of continuous real functions \( f \) in \( n \) variables on the closed unit ball \( B(0, 1) \subseteq \mathbb{R}^n \) attaining the value zero in exactly one point. Hereon, the \( B \)-valued function \( Z_u \), defined by

\[ Z_u(f) = x \quad :\Leftrightarrow \quad x \text{ is the (unique) zero of } f, \]

is \((\theta^o \rightarrow \theta) \rightarrow \theta^o\)-computable.

**Proof.** By Theorems 6.2.4.2 and 5.1.13.2 in [22] one can, given a \( \theta^o \rightarrow \theta \)-name of \( f \), \( \psi^\theta \)-compute the set \( f^{-1} \{\emptyset\} \subseteq B \). This computation actually yields a \( \kappa^\theta \)-name of this set which, by prerequisite, consists of exactly one point. Now apply Exercise 5.2.3 in [22]. \( \square \)

Recall the second claim from Theorem 13 which shall be proven first:

**Claim 16** Let \( X \subseteq \mathbb{R}^n \) be r.e. open, \( f : X \rightarrow \mathbb{R}^m \) computable with computable derivative \( f' : X \rightarrow \mathbb{R}^m \times \mathbb{R}^n \). Then the multi-valued mapping \( \Gamma : \Theta^o \times X \Rightarrow \mathbb{N} \) with

\[ \text{Graph}(\Gamma) := \{(U, x_0, k, \ell) \mid x_0 \in U \subseteq X, \text{rank } f'(x_0) = m, \}

\( \exists g : B(f(x_0), 2^{-\ell}) \rightarrow U \text{ satisfying } \beta \}\}

is \((\theta^o \times \theta^o \Rightarrow \nu)\)-computable.

**Proof.** Given a \( \theta^\circ \)-name of \( U \subseteq X \) and \( x_0 \in U \), determine according to Lemma 11(b) some \( k_0 \in \mathbb{N} \) such that \( B(x_0, 2^{-k_0}) \subseteq U \). Exploit differentiability of \( f \) to write

\[ f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + r(x) \quad (7) \]

with computable and computably differentiable \( r \) satisfying \( r(x)/|x| \rightarrow 0 \) as \( x \rightarrow x_0 \).
Since the computable matrix-valued function $x \mapsto f'(x)$ was required to have rank $m$ in $x_0$, certain $m$ of its columns are linearly independent. In fact, one can effectively find a regular $m \times m$ submatrix $A = A(x_0)$ of $f'(x_0)$: by dovetailing w.r.t. all (finitely many) possible candidates and looking for one with non-zero determinant.

For ease of notation, suppose that $f'(x_0)$ is of the form $(A|B)$ with $B \in \mathbb{R}^{(n-m) \times m}$. Continuity of the function $x \mapsto \det A(x)$ with non-zero value at $x_0$ yields that $A(x)$ is regular on a whole ball around $x_0$; $x \mapsto \det A(x)$ even being computable, a corresponding radius $2^{-k_1} \leq 2^{-k_0}$ can in fact be found effectively.

By (computable) translation, it suffices to prove the claim for the computable function

$$f(x_0 + x) \in \mathbb{R}^m.$$

Indeed, any local right inverse $\tilde{g} : B(f(x_0), \hat{\delta}) \subseteq \mathbb{R}^m \to \tilde{U} := B(0, 2^{-k_0}) \subseteq \mathbb{R}^m$ for this restriction can straight-forwardly (and computably) be extended to one for $f$ by letting $g(y) := (\tilde{g}(y), 0) + x_0 \in \mathbb{R}^n$.

$A = f'(0)$ being regular, $c := \min_{|x|=1} |A \cdot x|$ is non-zero and, according to Corollary 6.2.5 in [22], can be effectively calculated from the given data.

Effective continuity of $r'(\cdot)$ together with $r'(0) = (0)_{i,j}$ implies that one can computably find an integer $k_2 \geq k_1$ satisfying $|r'(z)| \leq c/2$ for all $|z| \leq 2^{-k_2}$. Here, $\|B\| := \sqrt{\sum_j \sum_{i,j} |b_{ij}|^2}$ denotes the square sum norm on matrices which is known to be submultiplicative: $|B \cdot x| \leq \|B\| \cdot |x|$. Consequently, by taking the norm on both sides of the Mean Value Theorem

$$r(y+h) - r(y) = \left( \int_0^1 r'(y + th) \, dt \right) \cdot h,$$

it follows with $h := x - y$ that for all $x, y \in B(0, 2^{-k_2})$ we have:

$$|r(x) - r(y)| \leq \left( \int_0^1 \| r'(y + th) \| \, dt \right) \cdot |h| \leq \frac{c}{2} \cdot |x - y|. \quad (8)$$

This asserts injectivity of $\tilde{f} : \overline{B}(0, 2^{-k_2})$. Indeed, $\tilde{f}(x) = \tilde{f}(y)$ implies with Equation (7) that $A \cdot x + r(x) = A \cdot y + r(y)$ and thus

$$c \cdot |x - y| \leq |A \cdot (y - x)| = |r(x) - r(y)| \leq \frac{c}{2} \cdot |x - y| :$$

a contradiction for $x \neq y$.

We may thus apply Lemma 14 to obtain some $\ell \in \mathbb{N}$ such that any $y \in B(\tilde{f}(0), 2^{-\ell})$ is the image of one and exactly one $x \in B(0, 2^{-k_2}) \subseteq \tilde{U}$. Finally setting $\tilde{g}(y) := x$ shows that $\tilde{f}$ does have a local right inverse.

The first part of Theorem 13 claims the right inverse we have just constructed to be computable and differentiable with computable derivative:

**Claim 17** Let $U \subseteq \mathbb{R}^n$ be r.e. open, $f : U \to \mathbb{R}^m$ computable with computable derivative, and $x_0 \in U$ computable such that rank $f'(x_0) = m$. Then there exists a computable $C^1$ function $g$ with computable derivative on some open ball $B(f(x_0), \delta) \subseteq \mathbb{R}^m$ satisfying (5).

**Proof.** Recall from the proof of Claim 10 the reduction from the case $n \geq m$ to the case $n = m$ leading to a function $\tilde{f}$ instead of $f$ which turned out to be injective on some
2−k′-ball around x0. Let Ũ := B(x0, 2−k′−1) and apply it to f the classical Inverse Mapping Theorem, in particular the last line of Fact. It asserts f to have on some (possibly smaller) open ball B(x0, ˜ε) ⊆ B(x0, 2−k′−1) around x0 a unique and continuously differentiable local inverse g. For any y from its domain B(f(x0), δ), the value g(y) is according to Equation 5 the unique x ∈ B(x0, ˜ε) with f(x) = y. Since f is injective on B(x0, 2−k′) ⊇ Ũ ⊇ B(x0, ˜ε), g(y) is the unique zero of x → f(x) − y on Ũ. Computing y → g(y) can thus be performed by finding this zero by virtue of Lemma to actually apply it, straight-forward scaling and translation effectively reduces Ũ = B(x0, 2−k′−1) to to B(0, 1).

Differentiability of g is asserted already classically. Moreover, the Chain Rule of Differentiation yields the formula g′(y) = f′(g(y))−1 which (Cramer’s Rule and computability of determinants) reveals that g′ is computable as well.

□

5 Computable Open Semi-Algebraic Functions are Effectively Open

Open Mapping Theorems give conditions for continuous functions to be open. However being only sufficient, they miss many continuous open functions.

Example 18. Let f : R3 → R2, (x, y, z) ↦→ (x3 + z2, y3 + z2). Then f is open although no item from Fact is applicable: a) fails due to the 2D range, b) fails due to nonlinearity, c) fails because f′(0) = 0, and d) fails as f lacks injectivity everywhere.

Section 4 of the present work provided effectivizations of those classical results where the prerequisites were strengthened from continuity to computability in order to assert, in addition to openness, effective openness. They therefore cannot be applied to cases such as Example where the classical theorems fail already. The main result of this section is of a different kind in that it requires openness in order to conclude effective openness. It is concerned with semi-algebraic functions in the sense of, e.g., [1] Section 2.4.2.

Definition 19. Let F ⊆ R denote a field. A set S ⊆ Rn is basic semi-algebraic over F if

S = \{ x ∈ Rn : \prod_{i=1}^{k} p_i(x) ≥ 0 ∧ \ldots ∧ \prod_{i=1}^{\ell} q_i(x) ≥ 0 ∧ q_1(x) > 0 ∧ \ldots ∧ q_l(x) > 0 \}

for certain k, \ell ∈ N, p_1, \ldots, q_\ell ∈ F[X_1, \ldots, X_n], that is, if S is the set of solutions to some finite system of polynomial inequalities both strict and non-strict with coefficients from F. S is semi-algebraic over F if it is a finite boolean combination (intersection and union) of basic semi-algebraic sets over F. A partial function f : F ⊆ Rn → Rm is semi-algebraic over F if Graph(f) = \{(x, y) : x ∈ dom(f) ∧ y = f(x)\} ⊆ Rn+m is semi-algebraic over F. In the case F = R, the indication “over F” may be omitted.

The class of semi-algebraic functions is very rich:

Example 20. a) Any rational function f ∈ R(X_1, \ldots, X_n) is semi-algebraic.

b) The roots of a univariate polynomial p = \sum_{i=0}^{n-1} p_i \cdot x^i ∈ R[X], considered as a partial function of its coefficients (p_0, \ldots, p_{n-1}), are semi-algebraic.

c) For semi-algebraic f and g, both composition g ◦ f and juxtaposition (f, g) are again semi-algebraic. Projection Rn+m → Rn, (x, y) ↦→ x is also semi-algebraic.

Proof. a) Let f = p/q with co-prime p, q ∈ R[X_1, \ldots, X_n]. Observe that

(x, y) ∈ Graph(f) ⇔ q(x) ≠ 0 ∧ p(x) = y \cdot q(x)

which is a boolean combination of polynomial inequalities. For b) and c) as well as for further examples of semi-algebraic functions, refer to [1] Section 2.4.2].

The main result of the present section thus covers many more in addition to Example.

Theorem 21. Let f : Rn → Rm be computable, open, and semi-algebraic over R, with open dom(f) := X. Then X is r.e. and f is effectively open.
5.1 Applications of Quantifier Elimination to Recursive Analysis

Quantifier elimination is an important tool in the algebraic framework of computability and complexity [21]. Its reliance on (in)equality as a decidable primitive seemingly renders it useless for the framework of Recursive Analysis. It does however have interesting consequences to non-uniform computability as revealed in this section.

The following Lemma will be applied to $E := \mathbb{R}$ and $F := \mathbb{R}_c$, the set of computable real numbers, a real closed field [22 COROLLARY 6.3.10], but might be of independent interest and is therefore formulated a bit more generally.

**Lemma 22.**

a) Let $F$ denote a real closed field with field extension $E$ and $f \in F[X_1, \ldots, X_n]$. If $g \in E[X_1, \ldots, X_n]$ divides $f$ considered as polynomial over $E$, then $\lambda g \in F[X_1, \ldots, X_n]$ for some non-zero $\lambda \in E$.

b) Let $F$ denote a field with extension $E$. If $f, g \in F[X_1, \ldots, X_n]$ and $h \in E[X_1, \ldots, X_n]$ is a gcd of $f$ and $g$ considered as polynomials over $E$, then $\lambda h \in F[X_1, \ldots, X_n]$ for some non-zero $\lambda \in E$.

c) Let $\emptyset \neq X \subseteq \mathbb{R}^n$ be r.e. open. Suppose $p, q \in \mathbb{R}[X_1, \ldots, X_n]$ are coprime with $q(x) \neq 0$ for all $x \in X$ and such that $p/q : X \to \mathbb{R}$ is computable. Then $\lambda p, \lambda q \in \mathbb{R}_c[X_1, \ldots, X_n]$ for some non-zero $\lambda \in E$; that is, the coefficients of the rational function $p/q$ may w.l.o.g. be presumed computable.

**Proof.**

b) In the uni-variate case $n = 1$, this follows from the Euclidean Algorithm since its calculation of the gcd uses only arithmetic operations $+,-,\times,\div$ and thus remains within the coefficient field of the input polynomials $f$ and $g$. In the multi-variate case, the gcd is still well-defined (up to multiples $\lambda \in E$) based on unique factorization in $E[X_1, \ldots, X_n]$ [6 EXERCISE 4.8.2.9]. Moreover, it can be calculated via Gröbner Bases [6 PROPOSITIONS 4.8.3.13+14], again using only arithmetic operations and thus remaining within the field $F$.

a) The equation $f = g \cdot h$ of $n$-variable polynomials over $E$ translates to a finite bilinear system of (in-)equalities for the $O(d^n)$ coefficients of $g$ and those of $h$ with coefficients from $F$, $d := \deg (g)$. The absolute terms for instance must satisfy $g \cdot \lambda = \deg (\lambda g)$ for some non-zero $\lambda \in E$.

b) By prerequisite, $y_j := p(x_j)/q(x_j) \in \mathbb{R}_c$ for each $x_j \in \prod X_j$, $j = 1, \ldots, d^n$. Expanding the equations $p(x_j) - y_j \cdot q(x_j) = 0$ in the multinomial standard basis yields a homogeneous system of linear equations with respect to the coefficients of both $p$ and $q$ to be solved for.

On the other hand the system itself is composed from (products of components of) computable reals $x_j$ and $y_j$. It follows from [26 COROLLARY 15] that this system admits a computable non-zero solution $\hat{p}, \hat{q} \in \mathbb{R}_c[X_1, \ldots, X_n]$. In particular, $\hat{p}/\hat{q}$ is defined and coincides with $p/q$ almost everywhere on $X$.

For $h := \gcd(\hat{p}, \hat{q})$, $\hat{p} := p/h$ and $\hat{q} := q/h$ are coprime and, based on Items a) and b), still belong to $\mathbb{R}_c[X_1, \ldots, X_n]$. Moreover it holds that $\hat{p} \cdot q = \hat{q} \cdot p$ by uniqueness of multivariate polynomials on the grid $X_1 \times \ldots \times X_n$ (e.g. Schwartz-Zippel Lemma). As $q$ divides $\hat{p} \cdot q = \hat{q} \cdot p$, coprimality with $p$ requires it to divide $\hat{q}$. Similarly $\hat{q}$ divides $q$. Thus $\hat{q} = \lambda q$ for some non-zero $\lambda \in E$ and consequently $\hat{p} = \lambda p$.

It is well-known in Recursive Analysis that equality of reals lacks even semi-decidability. Surprisingly it becomes decidable for rational arguments to real polynomial equations:

**Proposition 23.**

a) Let $p \in \mathbb{R}[X_1, \ldots, X_n]$ denote an $n$-variable polynomial.

Then \( \{ x \in \mathbb{Q}^n : p(x) = 0 \} \) is decidable in the classical (i.e., Type-1) sense.
b) Let $\Psi(X_1, \ldots, X_n)$ denote a finite Boolean combination of polynomial equalities and inequalities in variables $X_1, \ldots, X_n$ with computable real coefficients. Then \( \{ x \in \mathbb{Q}^n : \Psi(x) \} \) is (classically) semi-decidable.

c) Let $X \subseteq \mathbb{R}^n$ be open and semi-algebraic over $\mathbb{R}_c$. Then $X$ is r.e., that is, $\theta_2^\mathcal{C}$-computable.

Proof. b) Without loss of generality, $\Psi$ consists — apart from equalities — of strict inequalities only; otherwise replace any "$p(x) \leq 0$" with "$p(x) < 0 \lor p(x) = 0$".

Since $p \in \mathbb{R}[X_1, \ldots, X_n]$ is computable by assumption, strict inequalities are obviously semi-decidable; and equalities are even decidable by virtue of

a) Let

$$p(x) = \sum_{k_1=0}^{d_1} \cdots \sum_{k_n=0}^{d_n} a_{(k_1, \ldots, k_M)} \cdot x_1^{k_1} \cdots x_n^{k_n}$$

with $a_{(k_1, \ldots, k_M)} \in \mathbb{R}_c$. Choose\(^4\) among these $a_k$ a basis \( \{ b_0 = 1, b_1, \ldots, b_m \} \) for the finite-dimensional $\mathbb{Q}$-vector space $V := \{ q_0 + q_k a_k + \ldots + q_{(d_1, \ldots, d_n)} a_{(d_1, \ldots, d_n)} : q_k \in \mathbb{Q} \}$. Consequently, each coefficient of $p$ is of the form $a_k = \sum_{i=0}^{m} A_i k_i b_i$ with fixed $A_i k_i \in \mathbb{Q}$. Now for given $x \in \mathbb{Q}^n$,

$$0 = p(x) = \sum_{i=0}^{m} b_i \left( \sum_{k_1=0}^{d_1} \cdots \sum_{k_n=0}^{d_n} A_i k_i \cdot x_1^{k_1} \cdots x_n^{k_n} \right)$$

holds if and only if $R_i(x) = 0$ for all $i = 0, \ldots, m$ because the $b_i$ are linearly independent over $\mathbb{Q}$. The equalities $R_i(x) = 0$ in turn are of course decidable by means of exact rational arithmetic.

c) Let $x \in \mathbb{Q}^n$ and $0 < r \in \mathbb{Q}$. Then "$B(x, r) \subseteq X$" is equivalent to

$$\forall y \in \mathbb{R}^n : \left( \sum_{i=1}^{n} (y_i - x_i)^2 < r^2 \Rightarrow y \in X \right)$$

a first-order formula $\Phi(x, r)$ in the language of ordered fields with coefficients by assumption from the real closed field $\mathbb{R}_c$. By Tarski’s Quantifier Elimination\(^5\), there exists an equivalent quantifier-free formula $\Psi(x, r)$ over $\mathbb{R}_c$ \footnote{Tarski’s Quantifier Elimination allows for the elimination of quantifiers in first-order logic, reducing the formula to a quantifier-free one.} but for rational $(x, r)$, $\Psi(x, r)$ is semi-decidable according to b) and $X = \bigcup \{ B(x, r) : x \in \mathbb{Q}^n, 0 < r \in \mathbb{Q}, B(x, r) \subseteq X \}$ is therefore $\theta_2^\mathcal{C}$-computable.

5.2 Proof of Theorem\(^27\) and Consequences

Proof (Theorem\(^27\)). The domain of $f$ is semi-algebraic over $\mathbb{R}_c$ according to \footnote{Tarski’s Quantifier Elimination for the real numbers allows for the elimination of quantifiers in formulas, reducing them to quantifier-free forms.} and r.e. due to Proposition\(^23\)). Similarly to the proof there, we observe:

$$\overline{B}(f(x), s) \subseteq f \left[ B(x, 2^{-k}) \cap X \right] \iff \exists v \in \mathbb{R}^n \forall y \in \mathbb{R}^n \exists u \in \mathbb{R}^m : (x, v) \in \text{Graph}(f) \land (u, y) \in \text{Graph}(f) \land \left( \sum_{j=1}^{m} (y_j - v_j)^2 \leq s^2 \Rightarrow \sum_{i=1}^{n} (x_i - u_i)^2 < 2^{-2k} \right)$$

and observe the strong non-uniformity inherent in this step; for example a still open problem of number theory asks whether $e \cdot \pi$ or $e + \pi$ is rational \footnote{This problem remains unsolved in number theory.} (p.153).
Since the latter is a first-order formula $\Phi(x, s)$, by assumption with coefficients from $\mathbb{R}_c$, there exists by [1] Theorem 2.74 an equivalent quantifier-free formula $\Psi(x, s)$ again over $\mathbb{R}_c$. This in turn is semi-decidable for rational $(x, s)$ by virtue of Proposition [23]) so that $\text{Moo}_f$ can be approximated from below on $Q^n$. Now apply the Lemma below. □

Example 5 illustrated that in Lemma 3(b) as well a Theorem 8(b), it does not suffice to consider $\text{Moo}_f$ only on a dense subset of $X$. On the other hand if $f$ is already asserted as open, then computability of $\text{Moo}_f$ on rationals already does guarantee effective openness:

**Lemma 24.** Let $X \subseteq \mathbb{R}^n$ be r.e. open; furthermore let $f : X \to \mathbb{R}^m$ be computable and open. If $\text{Moo}_f : (X \cap Q^n) \times \mathbb{N} \to \mathbb{R}$ is $(\nu^n \times \nu \to \theta_c)$–computable, then $f$ is effectively open.

**Proof.** The goal is to $\theta_c$—compute $f[U]$, given a $\theta_c$—name of some open $U \subseteq X$. To this end observe that, similarly to the proof of Claim 11

$$f[U] \equiv \bigcup_{x \in U} B\left(f(x), \text{Moo}_f(x, G(U, x))/2\right) \supseteq \bigcup_{x' \in U \cap Q^n} B\left(f(x'), \text{Moo}_f(x', G(U, x'))/2\right) =: V$$

with $G$ from Lemma 10(a). And, again, the countable union $V$ can be $\theta_c$–computed because $Q^n \times R \ni (x, r) \mapsto B(f(x), r)$ is $(\nu^n \times \nu \to \theta_c)$–computable and the multi-valued mapping $h : Q^n \ni z \mapsto \text{Moo}_f(z, G(U, z))/2$ is $(\nu^n \to \theta_c)$–computable by assumption. It remains to show that, again, the reverse inclusion “$f[U] \subseteq V$” holds as well for a suitable computable subfunction. More precisely w.l.o.g. replace $G$ from Lemma 10(b) by $G$ according to Lemma 10(b) such that $G(U, \cdot)$ is bounded on compact subsets of $\mathbb{R}^n$. Now consider some $x \in U \setminus Q^n$. We show that then $f(x) \in V$.

Let some compact ball $\overline{B} := \overline{B}(x, r)$ be contained in $U$ and take an upper bound $L \in \mathbb{N}$ for $G(U, \cdot)$ on $\overline{B}$. By assumption, $\delta := \text{Moo}_f(x, L + 1)$ is strictly positive. The computable $f$ is continuous so that, for some $0 < r' \leq \min\{r, 2 \cdot L^{-1}\}, f(x') \in B(f(x), \delta/2)$ whenever $x' \in \overline{B} := \overline{B}(x, r'), Q^n$ being dense in $U$, there exists some rational $x' \in \overline{B}$. Now observe that

1) $\overline{B}(x, 2 \cdot (L + 1)) \subseteq \overline{B}(x', 2 \cdot L)$ by choice of $x'$, thus $f[\overline{B}(x, 2 \cdot (L + 1))] \subseteq f[\overline{B}(x', 2 \cdot L)]$;
2) from continuity of $f$ it follows $\overline{B}(f(x'), s - \frac{s}{2}) \subseteq \overline{B}(f(x), s)$ for any $s \geq \frac{\delta}{2}$;
3) combining i) and ii) yields $\text{Moo}_f(x', L) + \frac{\delta}{2} \geq \text{Moo}_f(x, L + 1) = \delta$ because, by

Equation 3, $\text{Moo}_f(x', L)$ is the supremum of feasible radii $s$.
4) Any $x' \in G(U, x')$ has $x' \leq L$ by choice of $L$; therefore
5) $\text{Moo}_f(x', L) \geq \text{Moo}_f(x', L)$ as $\text{Moo}_f(x', \cdot)$ is monotonic according in Equation 3.

We conclude that $\delta' := \text{Moo}_f(x', \ell') \geq \frac{\delta}{2}$ and $f(x) \in B(f(x'), \delta') \subseteq V$. □

**Corollary 25.** If the rational functions $f_i \in \mathbb{R}(X_1, \ldots, X_m)$ are computable for $i = 1, \ldots, n$ and the function $(f_1, \ldots, f_m) : \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is open, then it is effectively open.

**Proof.** According to Example 20(b), $f_i$ as well as its domain is semi-algebraic; in fact semi-algebraic over $\mathbb{R}_c$ by virtue of Lemma 22(b). Now apply Theorem 21.

In Theorem 21 $f$ was explicitly required to be semi-algebraic over $\mathbb{R}_c$; yet it seems reasonable, similarly to Lemma 22(b), to

**Conjecture 26.** Let $F \subseteq \mathbb{R}$ be a real closed subfield. Furthermore let $f : \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous and semi-algebraic (over $\mathbb{R}$) with $\text{dom}(f)$ semi-algebraic over $F$ and such that $f(x) \in F$ whenever $x \in F^n \cap \text{dom}(f)$. Then $f$ is semi-algebraic already over $F$. **
6 Effective Openness and Computability

The preceding sections presented sufficient conditions for a computable function \( f \) to be effectively open. The present one aims more generally at the logical relation between openness, continuity, effective openness, and computability of real functions.

The two classical properties for instance are well-known mutually independent: Continuity does not imply openness; nor does openness require continuity. (Counter-)Examples c) and d) below reveal that the same still holds under effectivized prerequisites.

**Example 27.**

a) There exists a function \( h : \mathbb{R} \to \mathbb{R} \) such that \( h[(a, b)] = (0, 1) \) for any \( a < b \).

b) There exists an open but not effectively open real function.

c) There exists a computable but not open real function; e.g. \( f : \mathbb{R} \to \mathbb{R}, x \mapsto 0 \).

d) There exists an effectively open but uncomputable real function.

e) There exist open functions \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) such that \( f_1 \neq f_2 \) but \( f_1[U] = f_2[U] \) for any open \( U \subseteq \mathbb{R} \).

**Proof.** a) Cf., e.g., item no.100 in the GUIDE preceding [10].

b) Let \( u \) be right-uncomputable and \( v > u \) be left-uncomputable. Let \( g : \mathbb{R} \to \mathbb{R}, g(x) = u + (v - u)x \). Then, with \( h \) from a), \( g \circ h : \mathbb{R} \to \mathbb{R} \) has image \((u, v)\) for any non-empty open \( U \) and is thus open; but under \( \theta^{-}\)-computable \( U := (0, 1) \), this image lacking \( \theta^{-}\)-computability [22 EXAMPLE 5.1.17.2a)\] reveals that \( g \circ h \) is not effectively open.

d) The function \( h \) from a) is open but maps the compact interval \([0, 1] \subseteq \mathbb{R} \) to the non-compact interval \((0, 1)\)

\[
(0, 1) = h\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) \subseteq h[0, 1] \not\subseteq h[\left(-1, \frac{1}{2}\right)] = (0, 1)
\]

and thus cannot be continuous nor computable. For \( \theta^{-}\)-computing \( U = h[U] \), it suffices to output a \( \theta^{-}\)-name of \((0, 1)\) [22 EXAMPLE 5.1.17.2c)] independent of the input \( U \neq \emptyset \). The test “\( U \neq \emptyset \)” is obviously semi-decidable, formally: \( \mathcal{D} \setminus \{\emptyset\} \) is \( \theta^{-}\)-r.e. [22 DEFINITION 3.1.3.2].

e) Let \( f_1 := h \) from a) and \( f_2 := g \circ h \) with \( g : \mathbb{R} \to \mathbb{R}, g(y) = y^3 \), open as composition of two open functions. As \( h[\left(0, \frac{1}{2}\right)] = (0, 1) \), there is some \( x \in (0, 1) \) such that \( h(x) = y := \frac{1}{2} \). Then \( f_2(x) = \frac{1}{2} \) reveals that \( f_1 \neq f_2 \).

Attempts to strengthen Examples c) and d) immediately raise the following

**Question 28.**

a) Is there a computable, open but not effectively open real function?

b) Is there a continuous, effectively open but uncomputable real function?

Regarding Theorem [12a]), a putative example for Question [28a]) must have domain and range both of dimension at least two, that is, a graph living in \( \mathbb{R}^d \) for some \( d \geq 4 \). Moreover its graph cannot be semi-algebraic because of Theorem [21] Concerning candidates to [28b]), the following result allows to restrict research to functions with one-dimensional range on domains of dimension at least two.

**Theorem 29.**

a) On r.e. open \( X \subseteq \mathbb{R} \), any continuous and effectively open \( f : X \to \mathbb{R} \) is also computable.

b) Let \( X \subseteq \mathbb{R}^n \) be r.e. open, \( f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m \) continuous and effectively open but not computable.

Then some \( f_i : X \to \mathbb{R} \), too, is continuous and effectively open, but not computable.

**Proof.** b) Recall that the projections \( \text{pr}_i : \mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto x_i \) are computable (hence continuous) and open; even effectively open: Theorem [12b) or b). By closure under composition, the component functions \( f_i = \text{pr}_i \circ f : X \to \mathbb{R} \) are therefore continuous and effectively open themselves. Regarding that a vector-valued \( f \) is computable iff its components are [24 Lemma 4.1.19.5], it follows that at least some \( f_i \) cannot be computable.
a) To evaluate $f$ at a given $x \in X$, we are given two monotonic sequences $(u_j)_j$ and $(v_j)_j$ of rational numbers converging to $x$ from below and above, respectively. As $x \in X$ is open, the entire interval $[u_j, v_j]$ belongs to $X$ for some $J \in \mathbb{N}$; and, since $(u_j)$ and $(v_j)$ are respectively increasing and decreasing to $x$, also $x \in [u_j, v_j] \subseteq X$ for all $j \geq J$. In fact, such $J$ can be found effectively because the property $"[u_j, v_j] \subseteq X"$ is semi-decidable by virtue of [24, Lemma 4.1b)].

Now, for each $j \geq J$, $\theta_{c,j}$-compute the open intervals $U_j := (u_j, u_{j+1})$ and $V_j := (v_{j+1}, v_j)$ as well as (by prerequisite) their images $f[U_j]$ and $f[V_j]$ and choose rational numbers $a_j \in f[U_j]$ and $b_j \in f[V_j]$. According to Lemma 30 below, both sequences $(a_j)$ and $(b_j)$ converge to $f(x)$ monotonically from different sides; and comparing $a_j$ to $b_j$ immediately reveals which one constitutes the lower and which one the upper approximations.

The following lemma can be regarded as a one-dimensional converse to Fact [64] because it implies that, for arbitrary open $X \subseteq \mathbb{R}$, a continuous open function $f : X \to \mathbb{R}$ is injective on any connected component of $f$.

**Lemma 30.** Let $X \subseteq \mathbb{R}$ be open and connected, $f : X \to \mathbb{R}$ continuous and open. Then $f$ is either strictly increasing or strictly decreasing.

**Proof.** Let $a, b, c \in X$, $a < b < c$. W.l.o.g. presuming $f(a) < f(c)$ — otherwise consider the negative inverse of $f$ — we show $f(a) < f(b) < f(c)$. Now suppose for instance that $f(b) > f(c)$. As $f$ is continuous on $[a, c]$, it attains its maximum therein at some $x \in [a, c]$ with a value $f(x) \geq f(b) > \max\{f(a), f(c)\}$; in particular, $x \in (a, c)$. Therefore, the interval $f([a, c])$ is closed on its upper end contradicting that $f$ is open. By considering the minimum of $f$ on $[a, c]$, the case $f(b) < f(a)$ similarly raises a contradiction. \(\Box\)

## 7 Conclusion

The present work investigated conditions for an open function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be effectively open in the sense that the image mapping $U \mapsto f[U]$ is ($\theta^n_m \to \theta^n_m$)–computable. This property is so to speak dual to function computability because the latter holds for $f : \mathbb{R}^n \to \mathbb{R}^m$ iff the pre-image mapping $V \mapsto f^{-1}[V]$ is ($\theta^n_m \to \theta^n_m$)–computable.

**Remark 31.** This characterization of computable real functions gave in **Definition** 6.1.6 or [22] rise to a natural representation — equivalent to many other ones [22, Lemmas 6.1.7 and 6.1.10] — for the space $C(X, \mathbb{R}^m)$ of all (not necessarily computable) continuous functions $f : X \to \mathbb{R}^m$, namely by $\theta^n_m$-encoding, for each open rational ball $B \subseteq \mathbb{R}^m$, the open set $f^{-1}[B]$.

Analogy might suggest to represent the family of all (not necessarily computable) open functions $f : X \to \mathbb{R}^m$ by $\theta^n_m$-encoding, for each open rational ball $B \subseteq \mathbb{R}^n$, the open set $f[B]$. However Example [27b] reveals that such a representation would not be well-defined.

It was already mentioned that effectively open functions have applications in computations on regular sets such as in solid modeling. For instance when encoding bounded regular $R \subseteq \mathbb{R}^d$ as a list of open rational balls with union dense in $R$ (representation $\mathcal{T}_\mathbb{Z}$), this will render not only union and intersection computable but also pre-image and image $R \mapsto g[R]$ under computable effectively open functions $g$ [24, Theorem 3.9]. According to Theorem [12b], that requirement on $g$ is satisfied by any computably differentiable function with regular derivative everywhere. However some $g$ might be computably differentiable and open with $g'(x)$ occasionally singular. The following result based on Sard’s Theorem asserts that, even then, $R \mapsto g[R]$ is $\theta^n_{\mathbb{Z}}$–computable:

**Theorem 32.** Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be computable, open, and $C^1$ with computable derivative $g'$. Then its image mapping on bounded regular sets $R \mapsto g[R]$ is ($\theta^n_\mathbb{Z} \to \theta^n_\mathbb{Z}$)–computable.
Proof. Let \( U_0 := \{ x \in \mathbb{R}^n : \text{rank} (g'(x)) = m \} \) denote the set of regular points of \( g \).

Consider the function \( G := \text{rank} \circ g' : \mathbb{R}^n \to \mathbb{N}, x \mapsto \text{rank} (g'(x)) \). Because of its discrete range, the pre-image \( G^{-1}([m-\frac{1}{2}, \infty)) \) obviously coincides with \( U_0 \). Moreover, being the composition of the lower semi-computable rank-function — see [25, PROPOSITION 6] or [26, THEOREM 7(i)] — with computable \( g' \), \( G \) is in particular lower semi-continuous and \( U_0 \subset \mathbb{R}^n \) therefore open [20, DEF. 2.8]; in fact r.e. open, see Lemma 33 below.

By Theorem 12c), at least the the restriction \( g|_{U_0} \) is thus effectively open. So given as \( \vartheta_n \)-name for \( R \subset \mathbb{R}^n \) a \( \vartheta_n \)-name for open \( U \subset \mathbb{R}^n \) with \( \overline{U} = R \), \( \vartheta_n \)-compute \( U \cap U_0 \) according to [22, COROLLARY 5.1.18.1]; then exploit effective openness of \( g|_{U_0} \) to \( \vartheta_n \)-compute \( V := g[U \cap U_0] \).

We claim that this yields a valid \( \vartheta_n \)-name for the regular set \( g[R] \), i.e., it holds that \( \overline{V} = g[R] \). To this end, observe that \( \overline{U_0} = \mathbb{R}^n \); for if \( A_0 := \mathbb{R}^n \setminus U_0 \) had non-empty interior, then the set \( V_0 := \{ g|_{A_0} \} \) of critical values would be open (since \( g \) is open by prerequisite) and non-empty rather than having measure zero as Sard’s Theorem asserts [18]. \( U_0 \) thus being dense, [24, LEMMA 4.3(c) and LEMMA 4.4(d)] imply \( U \cap U_0 = R \) and \( \overline{V} = g[R] \).

**Lemma 33.** Let \( h : \mathbb{R}^d \to \mathbb{R} \) be lower semi-computable.

Then the mapping \( \mathbb{R} \ni \alpha \mapsto h^{-1}([\alpha, \infty)) \subset \mathbb{R}^d \) is \((\vartheta> \to \vartheta_<^d)\)-computable.

**Proof.** Recall that lower semi-computability of \( h \) means that evaluation of \( h \) at some \( x \in \mathbb{R}^d \), given open rational balls \( B_j \ni x \) of radius \( r_j \to 0 \), yields rational numbers \( \beta_j \) tending from below to \( h(x) \).

So feed into this \( h \)-oracle all open rational balls \( B_j \subset \mathbb{R}^d \) and, whenever the answer \( \beta_j \) is strictly greater than \( \alpha \) (semi-decidable, given a \( \vartheta_> \)-name for \( \alpha \),) report this \( B_j \). The resulting sequence obviously covers exactly \( h^{-1}([\alpha, \infty)) \) and consequently is a \( \vartheta_<^d \)-name for this set.

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