AN EFFICIENT D-N ALTERNATING ALGORITHM FOR SOLVING AN INVERSE PROBLEM FOR HELMHOLTZ EQUATION

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ABSTRACT. Data completion known as Cauchy problem is one most investigated inverse problems. In this work we consider a Cauchy problem associated with Helmholtz equation. Our concerned is the convergence of the well-known alternating iterative method [25]. Our main result is to restore the convergence for the classical iterative algorithm (KMF) when the wave numbers are considerable. This is achieved by, some simple modification for the Neumann condition on the under-specified boundary and replacement by relaxed Neumann ones. Moreover, for the small wave number $k$, when the convergence of KMF algorithm’s [25] is ensured, our algorithm can be used as an acceleration of convergence.

In this case, we present theoretical results of the convergence of this relaxed algorithm. Meanwhile it, we can deduce the convergence intervals related to the relaxation parameters in different situations. In contrast to the existing results, the proposed algorithm is simple to implement converges for all choice of wave number.

We approach our algorithm using finite element method to obtain an accurate numerical results, to affirm theoretical results and to prove it’s effectiveness.

1. Introduction. The Helmholtz equation is act as a time-independent form of the wave equation. Therefore, it can be arises in wide range for the many branches in the science and engineering which depend on stationary oscillating processes. Especially, in physical phenomena such as aeroacoustics, vibration phenomena, wave
propagation, electromagnetic waves, seismic inversion, as well as the underwater acoustic description.

This explains the extensive works about the approximation equation that applied in numerous fields [10, 14, 8, 16, 24, 26, 38]. Generally, these physical problems can be written in the partial differential equation form as follows:

$$\Delta u + k^2 u = f,$$

with some boundary conditions and where $k$ denoting the wave number.

For solving these kinds of the problems, we must entirely know each of the geometry of the concern domain, the substances properties, the exterior source acting in the solution domain and boundary conditions. Subsequently, all these are indicating to establish the problems of the type direct and well-posed.

Direct boundary value problems for Helmholtz equations with Dirichlet, Neumann or Robin conditions have been widely studied in the last century. Regrettably, there are not many physical and engineering topics applicable to this class. In fact, only in some internal points of the assumed domain or in some areas of the boundary, the known boundary data appear, leading to the Cauchy problem, which is a classical example of the inverse problem and is the severely ill-posed problem in the sense of Hadamard [15]. This means that one of these conditions is not met by the solution (existence, uniqueness, stability). Numerical approaches to solve inverse problems has been extensively investigated [9, 1, 27, 36, 34, 35, 31, 32, 37].

In order to solve direct and inverse problems governed by the Helmholtz equation, numerical methods have been proposed. The authors noted in [17, 18] that the key explanation behind the consistency of the numerical solution for the direct problem of Helmholtz depends substantially on the physical parameter $k$. Attention was given in [5, 20, 29] that the iterative approach suggested by [25] fails to find the desired approximate solution when the wave number values become high.

In this work, we propose a new efficient relaxed algorithm. It is based on the relaxation of a Neumann condition on the under-specified boundary part. Then it follows an idea basically suggested in [21] to solve the Cauchy problem for Laplace equation. This approach keeps not only the differential equation but introduces no modulation of the domain.

In the same way as in the KMF algorithm [25], two well-posed auxiliary problems for the equation governing the original problem are solved alternately without altering the domain of study. An appropriate choice of the relaxation parameter of one of the boundary conditions at each iteration guarantees the regularizing character of our algorithm. We apply the idea of the Cauchy problem associated with the Helmholtz equation, and we give the interval in which the relaxation parameter must be chosen to ensure convergence, based on the value of the wave number used.

The rest of the paper is organized as follows: the mathematical formulation is presented in Section 2, the description of the KMF algorithm is made in Section 3 while in Section 4, our algorithm is described. We present the convergence outcomes in section 5 and we illustrate the key convergence theorem that relies on the wave number and relaxation factor values. We prove that the suggested algorithm restore the convergence for the alternating iterative algorithm. We also announce a convergence acceleration results. Finally, Section 6 is dedicated to numerical results discussions.

2. **Mathematical formulation.** Consider a bounded domain $\Omega \subset \mathbb{R}^d$ where $d \in \{1, 2, 3\}$. Let $\Gamma$, the Lipschitz boundary of $\Omega$, be such that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. The
Cauchy problem for the Helmholtz equation that we consider is as follows:

\[ \Delta u + k^2 u = 0, \quad \text{in } \Omega, \quad (2.1) \]

where the real positive constant \( k \) denotes the wave number. This equation is completed by boundary conditions described by:

\[ u = f_1, \quad \text{on } \Gamma_1, \quad (2.2) \]
\[ \partial_\nu u = f_2, \quad \text{on } \Gamma_1, \quad (2.3) \]
\[ u = g_1, \quad \text{on } \Gamma_2. \quad (2.4) \]

The derivative of \( u \) in the outward normal direction is denoted by \( \partial_\nu \) while \( f_1, f_2 \) and \( g_1 \) are given functions.

3. **Description of algorithm.** The KMF algorithm for solving Cauchy problems is an alternating iterative procedure, based on the principle of the alternate resolution of two well-posed problems built from the original one. Koslov et al. [25] were at the origin of this algorithm. Then it was implemented and improved by relaxation scheme in [21, 22, 23]. After that, different studies have been done using this algorithm for solving ill-posed problems governed by partial differential equations [2, 4, 6, 7, 9, 11, 12, 13]. The KMF alternating procedure for the Helmholtz problem Eqs. (2.1) to (2.4) is based on solving alternately the following two well-posed problems:

\[ \Delta u + k^2 u = 0, \quad \text{in } \Omega, \quad (3.1) \]
\[ u = v, \quad \text{on } \Gamma_0, \quad (3.2) \]
\[ \partial_\nu u = f_2, \quad \text{on } \Gamma_1, \quad (3.3) \]
\[ u = g_1, \quad \text{on } \Gamma_2. \quad (3.4) \]

and

\[ \Delta u + k^2 u = 0, \quad \text{in } \Omega, \quad (3.5) \]
\[ \partial_\nu u = \eta, \quad \text{on } \Gamma_0, \quad (3.6) \]
\[ u = f_1, \quad \text{on } \Gamma_1, \quad (3.7) \]
\[ u = g_1, \quad \text{on } \Gamma_2. \quad (3.8) \]

where \( v \) and \( \eta \) are two functions to be modified at each iteration.

We describe this algorithm in its easy to understand form, first presented in [21]

**Algorithm 1:**

1. For \( m = 0 \), choose the initial estimate \( \eta = \eta^0 \), then
2. Solve problem Eqs. (3.5) to (3.8) to find \( u^{(2m)} \).
3. Take \( v = u^{(2m)} \) and solve problem Eqs. (3.1) to (3.4) to find \( u^{(2m+1)} \).
4. If \( \frac{||u^{(2m+2)} - u^{(2m)}||_{L^2(\Gamma_0)}}{||u^{(2m+2)}||_{L^2(\Gamma_0)}} < \varepsilon \) then stop.
5. Else, \( m \leftarrow m + 1 \),
6. take \( \eta = \eta^{(m)} = \partial_\nu u^{(2m-1)} \big|_{\Gamma_0} \) and go to step 2.

Several studies to solve the Cauchy problem for the Helmholtz equation [20, 29] have been made based on the application of Algorithm 1. In particular, it was used by Marin et al. [29] to solve Cauchy problem for modified Helmholtz equation (the case when \( k \) is pure imaginary). They also reported that Algorithm 1 does not
converge where \( k \) is real as is the case in the problem Eq. (2.1). This remark was confirmed by Johansson-Kozlov in [19] where they claimed that lack of coercivity leads to non-convergence of the alternating method. In the sequel, we will propose the relaxation alternating algorithm for solving the problem Eqs. (2.1) to (2.4) and we show that this algorithm converge for all values of the real \( k \).

4. Description of the modified algorithm. In this section, we consider the relaxation algorithm initially introduced by Jourhmane-Nachaoui [21, 22] to solve Cauchy problem in a bioelectric field. Their approach was aimed at showing an improvement of the mathematical algorithm based on the use of a relaxation factor and then proving the convergence of the FEM numerical implementation of the algorithm. Then, the authors show that the relaxed algorithm is convergent in the case where the Cauchy problem is governed by the Poisson equation [23]. The number of iterations needed to achieve convergence to the solution of the considered Cauchy problem is considerably reduced by this procedure.

Recently, this relaxation was accurately investigated to solve inverse Cauchy problem arising in many applications [11, 6, 30]. In particular this relaxation was used for solving Cauchy problem for modified Helmholtz [3, 20, 28]. In the following, we will describe the relaxation algorithm to solve Cauchy Helmholtz problem. The aim of this relaxed algorithm is to enable the convergence in the case where Algorithm 1 diverges (large values of the wave number \( k \)). In the remainder cases, the goal is to accelerate the convergence compared to Algorithm 1.

We modify Algorithm 1 to solve the problem Eq. (2.1) and Eqs. (2.2) to (2.4) by relaxing the Neumann condition Eq. (3.6) using a relaxation parameter \( \theta \in [0, 2] \). Our modified algorithm, presenting the same computational schemes as Algorithm 1, is summarized in the following:

**Algorithm 2:**

1. For \( m = 0 \), choose the initial estimate \( \eta = \eta^0 \), then
2. Solve problem Eqs. (3.5) to (3.8) to find \( u^{(2m)} \).
3. Take \( v = u^{(2m)} \) and solve problem Eqs. (3.1) to (3.4) to find \( u^{(2m+1)} \).
4. If \( \frac{||u^{(2m+2)} - u^{(2m)}||_{L^2(\Gamma_0)}}{||u^{(2m+1)}||_{L^2(\Gamma_0)}} \leq \varepsilon \) stop.
5. Else, \( m \leftarrow m + 1 \),
6. take
   \[
   \eta^{(m)} = \theta \partial_\nu u^{(2m-1)} |_{\Gamma_0} + (1 - \theta) \eta^{(m-1)} |_{\Gamma_0}, \tag{4.1}
   \]
   and go to step 2.

**Remark 1.** Note that
- Algorithm 2 and Algorithm 1 have the same structure, the only difference is in the way we calculate \( \eta^{(m)} \).
- The value \( \theta = 1 \) in Eq. (4.1) corresponds to the alternating iterative Algorithm 1. Thus Algorithm 2 is a generalisation of Algorithm 1.
- for \( \theta \in [0, 1] \) (resp. \( \theta \in [1, 2] \)) in Eq. (4.1) correspond to under- (resp. over)-relaxation of the alternating iterative Algorithm 1. This is why we can expect an acceleration of convergence for \( \theta \) belonging to a sub-interval of \([0, 2]\).

5. Convergence results. In this section, we will discuss the convergence of the relaxation Algorithm 2 for all values of \( k \). Let us denote by \( u^{(n)} = u - u^{(n)} \) where
u is solution of the problem defined by Eqs. (2.1) to (2.4). Then \( w^{(n)} \) is obtained as a solution of the following problems
\[
\Delta w^{(2m)} + k^2 w^{(2m)} = 0, \quad \text{in } \Omega, \tag{5.1}
\]
\[
\partial_y w^{(2m)} = \theta \partial_y w^{(2m-1)} + (1 - \theta) \partial_y w^{(2(m-1))}, \quad \text{on } \Gamma_0, \tag{5.2}
\]
\[
w^{(2m)} = 0, \quad \text{on } \Gamma_1, \tag{5.3}
\]
\[
w^{(2m)} = 0, \quad \text{on } \Gamma_2, \tag{5.4}
\]
and \( w^{(2m+1)} \) is solution of the following problem
\[
\Delta w^{(2m+1)} + k^2 w^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{5.5}
\]
\[
w^{(2m+1)} = w^{(2m)}, \quad \text{on } \Gamma_0, \tag{5.6}
\]
\[
\partial_y w^{(2m+1)} = 0, \quad \text{on } \Gamma_1, \tag{5.7}
\]
\[
w^{(2m+1)} = 0, \quad \text{on } \Gamma_2. \tag{5.8}
\]
It is clear that \( u^{(n)} \) converges to \( u \) is equivalent that \( w^{(n)} \) converges to 0. Thus why in the following we will find conditions under which the convergence of \( w^{(n)} \) to 0 will be ensured.

For simplicity and without loss of generality, we suppose that the Cauchy problem is posed in a rectangular domain, i.e. \( \Omega = [0,a] \times [0,b] \), with boundaries \( \Gamma_0 = [0,a] \times \{b\} \), \( \Gamma_1 = [0,a] \times \{0\} \), \( \Gamma_2 = \{0\} \times [0,b] \cup \{a\} \times [0,b] \).

Consider the case \( m = 0 \), choose \( \eta^{(0)} \) and let us derive a solution \( w^{(0)} \) of the problem defined by Eqs. (5.1) to (5.4) using the method of separation of variables. The basic idea is to seek a solution expressed in the separable form

\[
w^{(0)}(x,y) = X(x)Y(y),
\]
where \( X \) and \( Y \) are functions of \( x \) and \( y \) respectively. Then,
\[
w^{(0)}(x,y) = [A \sin(\beta x) + B \cos(\beta x)] [C \sinh(\alpha y) + D \cosh(\alpha y)],
\]
with
\[
\alpha^2 - \beta^2 + k^2 = 0. \tag{5.9}
\]
Taking into account the boundary conditions Eqs. (5.3) to (5.4) we have
\[
w^{(0)}(0,y) = 0 \quad \Rightarrow \quad B = 0,
\]
\[
w^{(0)}(a,y) = 0 \quad \Rightarrow \quad \sin(\beta a) = 0 \quad \text{and thus their exists } n \text{ such that } \beta_n = \beta = \frac{n\pi}{a},
\]
then,
\[
\forall n \geq 1, \quad w^{(0)}(x,y) = A_n \sin(\beta_n x) [C_n \sinh(\alpha_n y) + D_n \cosh(\alpha_n y)],
\]
is solution of the Helmholtz equation. Then,
\[
w^{(0)}(x,0) = 0 \quad \Rightarrow \quad D_n = 0.
\]
Since the problem Eq. (5.1) and( Eq. (5.3)) to (5.4) is linear and homogeneous, then by superposition principal, the solution is given by
\[
w^{(0)}(x,y) = \sum_{n=1}^{\infty} A_n \sin(\beta_n x) \sinh(\alpha_n y). \tag{5.10}
\]
Then using \( \partial_y w^{(0)}(x, b) = \eta^{(0)}(x) \), differentiating \( w^{(0)}(x, y) \) in the last equation with respect to \( y \), multiply by \( \sin(\beta_n x) \) the two members of this boundary condition, we obtain, after integration on \([0, a] \) and using the orthogonality character of \( \{\sin(n\pi x)\}_{n=1}^{\infty} \):

\[
A_n^{(0)} = \frac{2}{a\alpha_n \cosh(\alpha_n b)} \int_0^a \eta^{(0)}(x) \sin \left( \frac{n\pi x}{a} \right) dx, \quad n \geq 1.
\]

From Eq. (5.9), taking \( \lambda_n = (\alpha_n)^2 = (\beta_n)^2 - k^2 = \left( \frac{n\pi}{a} \right)^2 - k^2, \forall n \geq 1 \), then if \( 0 < k < \frac{\pi}{a} \) we have \( \lambda_n > 0, \forall n > 1 \) and if \( k \geq \frac{\pi}{a} \) then \( \exists n_0 > 0 \), such that \( \lambda_n \geq 0, \forall n \geq n_0 \), and \( \lambda_n < 0 \), for \( n < n_0 \). Thus Eq. (5.10) can be written as,

\[
w^{(0)}(x, y) = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} A_n^{(0)} \sin(\beta_n x) \sinh(\sqrt{\lambda_n} y), & 0 < k < \frac{\pi}{a}, \\
\sum_{n=1}^{\infty} A_n^{(0)} \sin(\beta_n x) \sinh(\sqrt{\lambda_n} y), & k \geq \frac{\pi}{a},
\end{array} \right.
\]

(5.11)

where \( A_n^{(0)} \) is rewritten as

\[
A_n^{(0)} = \left\{ \begin{array}{ll}
\frac{C_n^{(0)}}{\sqrt{\lambda_n} \cosh(\sqrt{\lambda_n} b)}, & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
\frac{C_n^{(0)}}{\sqrt{-\lambda_n} \cosh(\sqrt{-\lambda_n} b)}, & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0,
\end{array} \right.
\]

(5.12)

with

\[
C_n^{(0)} = \frac{2}{a} \int_0^a \eta^{(0)}(x) \sin \left( \frac{n\pi x}{a} \right) dx.
\]

Note that \( n_0 \) is the is the smallest integer such that, \( \frac{nn\pi}{a} \geq k \).

Having construct the approximation \( w^0(x, y) \), we apply the separation of variables principle to Eq. (5.5) and using conditions (5.7) and (5.8) with help of superposition principal to find the expression of \( w^{(1)}(x, y) \)

\[
w^{(1)}(x, y) = \sum_{n=1}^{\infty} A_n^{(1)} \sin(\beta_n x) \cosh(\alpha_n y).
\]

(5.13)

Using condition (5.6) we find that

\[
A_n^{(1)} = A_n^{(0)} \tanh(\alpha_n b), \quad \forall n \geq 1.
\]

Which can be rewritten as

\[
A_n^{(1)} = \left\{ \begin{array}{ll}
A_n^{(0)} \tanh(\sqrt{\lambda_n} b), & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A_n^{(0)} \tanh(\sqrt{-\lambda_n} b), & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{array} \right.
\]

(5.14)

Next, taking \( \eta^{(1)} = \theta \partial_y w^{(1)} \big|_{\Gamma_0} + (1 - \theta) \cdot \eta^{(0)} \) on \( \Gamma_0 \) in Eq. (5.2), we find \( w^{(2)} \) using the same previous technique:

\[
w^{(2)}(x, y) = \sum_{n=1}^{\infty} A_n^{(2)} \sin(\beta_n x) \sinh(\alpha_n y).
\]

(5.15)

Condition Eq. (5.2) imply that

\[
A_n^{(2)} = A_n^{(1)} \theta \tanh(\alpha_n b) + A_n^{(0)} (1 - \theta), \quad \forall n \geq 1.
\]
Replacing $A^{(1)}_n$ by its formula in (5.14) we obtain

$$
A^{(2)}_n = \begin{cases} 
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{\lambda} b) \right)^2 + (1 - \theta) \right), & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{-\lambda} b) \right)^2 + (1 - \theta) \right), & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{cases}
$$

(5.16)

Having construct $w^{(2)}$ and using the same procedure, we express $w^{(3)}$ as

$$
w^{(3)}(x, y) = \sum_{n=1}^{\infty} A^{(3)}_n \sin(\beta_n x) \cosh(\alpha_n y).
$$

(5.17)

Then, by using the condition Eq. (5.6) we have

$$A^{(3)}_n = A^{(2)}_n \tanh(\alpha_n b), \quad \forall n \geq 1.
$$

Using (5.16) we get

$$
A^{(3)}_n = \begin{cases} 
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{\lambda} b) \right)^2 + (1 - \theta) \right) \left( \tanh(\sqrt{\lambda} b) \right), & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{-\lambda} b) \right)^2 + (1 - \theta) \right) \left( \tanh(\sqrt{-\lambda} b) \right), & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{cases}
$$

(5.18)

From the computation of $A^{(3)}_n$, $A^{(3)}_n$ and $A^{(3)}_n$ we formulate the following hypothesis for $m \geq 0$:

$$
w^{(2m)}(x, y) = \begin{cases} 
\sum_{n=1}^{\infty} A^{(2m)}_n \sin(\beta_n x) \sinh(\sqrt{\lambda} y), & 0 < k < \frac{\pi}{a}, \\
\sum_{n=1}^{\infty} A^{(2m)}_n \sin(\beta_n x) \sinh(\sqrt{-\lambda} y) \\
+ \sum_{n=n_0+1}^{\infty} A^{(2m)}_n \sin(\beta_n x) \sinh(\sqrt{\lambda} y), & k \geq \frac{\pi}{a},
\end{cases}
$$

(5.19)

where

$$
A^{(2m)}_n = \begin{cases} 
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{\lambda} b) \right)^2 + (1 - \theta) \right)^m, & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A^{(0)}_n \left( \theta \left( \tanh(\sqrt{-\lambda} b) \right)^2 + (1 - \theta) \right)^m, & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0,
\end{cases}
$$

(5.20)

and

$$
w^{(2m+1)} = \begin{cases} 
\sum_{n=1}^{\infty} A^{(2m+1)}_n \sin(\beta_n x) \cosh(\sqrt{\lambda} y), & 0 < k < \frac{\pi}{a}, \\
\sum_{n=1}^{\infty} A^{(2m+1)}_n \sin(\beta_n x) \cosh(\sqrt{-\lambda} y) \\
+ \sum_{n=n_0+1}^{\infty} A^{(2m+1)}_n \sin(\beta_n x) \cosh(\sqrt{\lambda} y), & k \geq \frac{\pi}{a},
\end{cases}
$$

(5.21)
where

\[
A_n^{(2m+1)} = \begin{cases} 
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{\lambda_n} b) \right)^2 + (1 - \theta) \right)^m (\tanh(\sqrt{\lambda_n} b)) 
& \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{-\lambda_n} b) \right)^2 + (1 - \theta) \right)^m (\tanh(\sqrt{-\lambda_n} b)) 
& \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{cases}
\]

We can state the following lemma.

**Lemma 5.1.** Consider the two well-posed problems for Helmholtz equation Eqs. (5.1) to (5.4) and Eqs. (5.5) to (5.8) then for \( m \geq 0 \) \( w^{(m)} \) is given form formulas Eq. (5.19) and Eq. (5.21).

**Proof.** It is clear that this is satisfied for \( m = 1 \). Suppose that Eq. (5.19) and Eq. (5.21) are satisfied until \( m = m_0 \). Now for \( m_0 + 1 \) we have

\[
w^{(2(m_0+1))}(x, y) = \sum_{n=1}^{\infty} A_n^{(2(m_0+1))} \sin(\beta_n x) \sinh(\alpha_n y).
\]

Applying condition Eq. (5.2) to \( w^{(2(m_0+1))}(x, y) \) we obtain

\[
A_n^{(2(m_0+1))} = A_n^{(2m_0+1)} \theta \tanh(\alpha_n b) + A_n^{(2m_0)}(1 - \theta)
\]

Denotes by \( M_0 = (2(m_0 + 1)) \), then by induction hypothesis we have

\[
A_n^{(M_0)} = \begin{cases} 
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{\lambda_n} b) \right)^2 + (1 - \theta) \right)^{m_0} (\tanh(\sqrt{\lambda_n} b)) 
& \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
+ A_n^{(0)} \left( \theta \left( \tanh(\sqrt{-\lambda_n} b) \right)^2 + (1 - \theta) \right)^{m_0} (1 - \theta),
\end{cases}
\]

and

\[
A_n^{(M_0)} = \begin{cases} 
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{\lambda_n} b) \right)^2 + (1 - \theta) \right)^{m_0} (\tanh(\sqrt{-\lambda_n} b)) 
& \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0,
\end{cases}
\]

which leads to

\[
A_n^{(M_0)} = \begin{cases} 
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{\lambda_n} b) \right)^2 + (1 - \theta) \right)^{m_0+1} , & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
A_n^{(0)} \left( \theta \left( \tanh(\sqrt{-\lambda_n} b) \right)^2 + (1 - \theta) \right)^{m_0+1} , & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{cases}
\]

We have also

\[
w^{2(m_0+1)+1} = \sum_{n=1}^{\infty} A_n^{(2(m_0+1)+1)} \sin(\beta_n x) \cosh(\alpha_n y).
\]

Using condition Eq. (5.6) we obtain

\[
A_n^{(2(m_0+1)+1)} = A_n^{(2(m_0+1))} \tanh(\alpha_n b), \forall n \geq 1.
\]
Then using formulas Eq. (5.24) we obtain
\[
A^{2(m_0+1)+1}_n = \begin{cases} 
\left[ A^{(0)}_n \left( \theta \left( \tanh(\sqrt{n}b) \right)^2 + (1 - \theta) \right)^{m_0+1} \right] \tanh(\sqrt{n}b), \\
\text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0), \\
\left[ A^{(0)}_n \left( \theta \left( \tanh(\sqrt{-n}b) \right)^2 + (1 - \theta) \right)^{m_0+1} \right] \tanh(\sqrt{-n}b), \\
\text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0.
\end{cases}
\]

This completes the proof of the lemma. \[\square\]

**Theorem 5.2.** Consider the Cauchy problem for Helmholtz equation Eqs. (2.1) to (2.4). The following statements hold

(i) Algorithm 1 converge for \(0 < k \leq \frac{\pi}{a} \sqrt{1 + \left( \frac{a^4}{b^2} \right)^2}\) and diverges for \(k > \frac{\pi}{a} \sqrt{1 + \left( \frac{a^4}{b^2} \right)^2}\).

(ii) For \(k < \frac{\pi}{a}\), for all \(\theta \in (0, 2)\), the sequence \((w^{(m)})_{m \geq 0}\) defined by Eqs. (5.19) to (5.21) converges to 0 independently of the initial value \(\eta^{(0)}\).

(iii) For \(\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \sqrt{1 + \left( \frac{a^4}{b^2} \right)^2}\), for all \(\theta \in (0, \theta^*_2)\) with \(\theta^*_2 = \frac{2}{1 + \left( \tan(\sqrt{-\lambda_1}b) \right)^2}\), the sequence \((w^{(m)})_{m \geq 0}\) defined by Eqs. (5.19) to (5.21) converges to 0 independently of the initial value \(\eta^{(0)}\).

(iv) For \(k > \frac{\pi}{a} \sqrt{1 + \left( \frac{a^4}{b^2} \right)^2}\), for all \(\theta \in (0, \theta^*_3)\), with \(\theta^*_3 = \frac{2}{1 + \left( \tan(\sqrt{-\lambda_1}b) \right)^2}\), the sequence \((w^{(m)})_{m \geq 0}\) defined by Eqs. (5.19) to (5.21) converges to 0 independently of the initial value \(\eta^{(0)}\).

**Proof.** Note that, since \(\forall m \geq 0\), we have
\[
A^{(2m+1)}_n = A^{(2m)}_n \tanh(\alpha_n b), \forall n \geq 1.
\]
to show that \((w^{(m)})_{m \geq 0}\) converges it suffice to show that the subsequence \((w^{(2m)})_{m \geq 0}\) converges. Thus we will show, under which conditions \(\lim_{m \to \infty} \|w^{(2m)}\|_{L^2(\Omega)} = 0\).

We have
\[
\|w^{(2m)}\|^2_{L^2(\Omega)} = \int_0^b \int_0^a |w^{(2m)}(x,y)|^2 \, dx \, dy.
\]

Using the Parseval’s identity, we get
\[
\|w^{(2m)}\|^2_{L^2(\Omega)} = \int_0^b \frac{a}{2} \sum_{n=1}^\infty |A^{(2m)}_n| \sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n y})^2 \, dy,
\]
\[
= \frac{a}{2} \sum_{n=1}^\infty \left[ A^{(2m)}_n \right]^2 \int_0^b \sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n y})^2 \, dy.
\]
where \( \text{sgn} \) denotes the sign function of real number. Since
\[
\int_0^b \sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}y)^2 \, dy
\]
\[
= \int_0^b \frac{e^{2\sqrt{\text{sgn}(\lambda_n)\lambda_n}y} + e^{-2\sqrt{\text{sgn}(\lambda_n)\lambda_n}y} - 2}{4} \, dy,
\]
\[
= \frac{1}{2\sqrt{\text{sgn}(\lambda_n)\lambda_n}} \left( \sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) \cosh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) - \sqrt{\text{sgn}(\lambda_n)\lambda_n}b \right).
\]
This last equation and replacing \( A_n^{(2m)} \) by its formula implies,
\[
\|w^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2} \sum_{n=1}^{\infty} \left[ A_n^{(0)} \right]^2 \left[ \frac{\sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) \cosh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) - b}{2\sqrt{\text{sgn}(\lambda_n)\lambda_n}} \right]^2 
\times \left[ \theta \left( \tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)^2 + 1 - \theta \right) \right]^{2m}.
\]
(5.25)

In order to show the convergence in \( L^2(\Omega) \) of the sequence \( (w^{(m)})_{m \geq 0} \) to 0, it suffices to show that
\[
| \theta \left( \tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)^2 + 1 - \theta \right) | < 1, \quad \text{for all } n \geq 1.
\]
(5.26)

Note that we have
\[
| \theta \left( \tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)^2 + 1 - \theta \right) | = \begin{cases} 
| \theta \left( \tanh(\sqrt{\lambda_n}b) \right)^2 + 1 - \theta, & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0) , \\
| -\theta \left( \tanh(\sqrt{-\lambda_n}b) \right)^2 + 1 - \theta, & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0. 
\end{cases}
\]

for (i), we know that \( \text{Algorithm 1} \) corresponds to taking \( \theta = 1 \) in \( \text{Algorithm 2} \). This correspond to take \( \theta = 1 \) in (5.25) and in this case we have convergence if the following condition is satisfied:
\[
| \tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) | < 1, \quad \forall n \geq 1.
\]
(5.27)

Now, using the fact that \( \tanh(ix) = i\tan(x) \), we have
\[
\tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b) = \begin{cases} 
\tanh(\sqrt{\lambda_n}b), & \text{if } k < \frac{\pi}{a} \text{ or } (k \geq \frac{\pi}{a} \text{ and } n \geq n_0) , \\
i\tan(\sqrt{-\lambda_n}b), & \text{if } k \geq \frac{\pi}{a} \text{ and } n < n_0. 
\end{cases}
\]

As it can be seen, the convergence of the sequence \( (w^{(2m)})_{m \geq 0} \) is conditioned by the values of \( k \). Firstly, if \( k < \frac{\pi}{a} \), the inequality (5.27) is equivalent to
\[
| \tanh(\sqrt{\lambda_n}b) | < 1, \quad \forall n \geq 1.
\]
This condition is satisfied for all \( n \), therefore \( \text{Algorithm 1} \) converges for \( 0 \leq k < \frac{\pi}{a} \).

Secondly, if \( k \geq \frac{\pi}{a} \), we need to consider two cases: \( 1 \leq n < n_0 \) and \( n \geq n_0 \). Note that if \( n \geq n_0 \) condition (5.27) is equivalence to
\[
| \tanh(\sqrt{\lambda_n}b) | < 1,
\]
which is satisfied \( \forall n \geq n_0 \). Thus the convergence for the case \( k \geq \frac{\pi}{a} \) is conditioned by the case \( 1 \leq n < n_0 \). In this case, (5.27) is equivalent to
\[
| i\tan(\sqrt{-\lambda_n}b) | < 1,
\]
which is equivalent to

\[-1 < \tan(\sqrt{-\lambda_n b}) < 1,\]

giving

\[-\frac{\pi}{4} < k^2 - \left(\frac{n\pi}{a}\right)^2 b < \frac{\pi}{4}.\]

Thereby, we obtain

\[\frac{\pi}{a} \leq k < \frac{\pi}{a} \sqrt{n^2 + \left(\frac{a}{4b}\right)^2}.\]

Since

\[\frac{\pi}{a} \sqrt{n^2 + \left(\frac{a}{4b}\right)^2} = \tan(\sqrt{\lambda_n b}) \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2},\] for \(1 \leq n < n_0,\)

\(k\) must satisfy

\[\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2}.\]

Thus (5.27) will be satisfied if \(\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2}\), therefore the first part of assertion (i) is valid.

Now if \(k > \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2}, \) |\(\tanh(\sqrt{\lambda_n b})\)\| \(\geq 1, \) \(\forall n \geq 1,\) this implies that the sequence \((w^{(2m)})\) diverges.

For the convergence of the relaxed Algorithm 2, consider first the case (ii) : \(k < \frac{\pi}{a} \).

To fulfill the condition (5.26), the relaxation variable \(\theta\) must satisfy the following inequality,

\[-2 < \theta \left(\tanh(\sqrt{\lambda_n b})^2 - 1\right) < 0, \] for all \(n \geq 1,\]

which is equivalent to

\[0 < \theta < \frac{2}{1 - (\tanh(\sqrt{\lambda_n b}))^2}, \] for all \(n \geq 1.\]

This leads to

\[0 < \theta < \min \left(\frac{2}{1 - (\tanh(\sqrt{\lambda_n b}))^2}, 2\right), \] for all \(n \geq 1.\]

Since

\[\frac{2}{1 - (\tanh(\sqrt{\lambda_n b}))^2} > 2, \] for all \(n \geq 1,\]

then

\[0 < \theta < 2.\]

Thus, the assertion (ii) is valid.
(iii) : \( \frac{\pi}{\alpha} \leq k \leq \frac{\pi}{\beta} \sqrt{1 + (\frac{a}{\beta})^2} \).

In this case, to ensure the convergence condition (5.26), the relaxation variable \( \theta \) must satisfy the following inequality,

\[
-2 < -\theta \left( \tanh(\sqrt{\text{sgn}(\lambda_n)b}^2 + 1) \right) < 0, \quad \forall n \geq 1
\]  

(5.28)

Now, Consider the two cases \( 1 \leq n < n_0 \) and \( n \geq n_0 \)

If \( 1 \leq n < n_0 \), condition (5.28) becomes

\[
-2 < -\theta \left( \tanh(-\lambda_n b)^2 + 1 \right) < 0,
\]

and this implies

\[
0 < \theta < \frac{2}{1 + (\tanh(-\lambda_n b))^2}.
\]

Since

\[
\frac{2}{1 + (\tanh(-\lambda_n b))^2} > \frac{2}{1 + (\tanh(-\lambda_1 b))^2},
\]

we obtain

\[
0 < \theta < \frac{2}{1 + (\tanh(-\lambda_1 b))^2}.
\]  

(5.29)

If \( n \geq n_0 \), (5.28) becomes

\[
-2 < -\theta \left( \tanh(\sqrt{-\lambda_n b})^2 - 1 \right) < 0,
\]

which gives

\[
0 < \theta < \frac{2}{1 - (\tanh(\sqrt{-\lambda_n b}))^2}.
\]

Since

\[
\frac{2}{1 - (\tanh(\sqrt{-\lambda_n b}))^2} > \frac{2}{1 - (\tanh(\sqrt{-\lambda_1 b}))^2},
\]

we obtain

\[
0 < \theta < \frac{2}{1 - (\tanh(\sqrt{-\lambda_1 b}))^2}.
\]

This gives

\[
0 < \theta < \min\left( \frac{2}{1 - (\tanh(\sqrt{-\lambda_n b}))^2}, 2 \right).
\]

Since,

\[
\frac{2}{1 - (\tanh(\sqrt{-\lambda_n b}))^2} > 2,
\]

thus, we have

\[
0 < \theta < 2.
\]

This last inequality combined with (5.29), imply that Algorithm 2 converges for all \( \theta \in (0, \theta_2^*) \), where \( \theta_2^* = \frac{2}{1 + (\tanh(-\lambda_1 b))^2} \), then the assertion (iii) is valid.
(iv) \( k > \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{b}\right)^2} \).

In this case, we use the same steps as in (iii), we obtain \( \theta_3^* = \theta_2^* \). Thus assertion (iv) is valid. We obtain converge for \( \theta \in (0, \theta_3^*) \) and it remains to show that \( \theta_3^* < 1 \), which we show by a contradiction argument.

Suppose that \( \theta_3^* \geq 1 \), this is equivalent to

\[
\frac{2}{1 + \left(\tan(\sqrt{-\lambda_1 b})\right)^2} \geq 1,
\]

this lead to

\[
\left(\tan(\sqrt{-\lambda_1 b})\right)^2 \leq 1.
\]

Since

\[-\lambda_1 = k^2 - \left(\frac{\pi}{a}\right)^2,
\]

We obtain

\[
k \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2},
\]

which is in contradiction with assumption that \( \frac{\pi}{a} \leq k \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2} \). Therefore, \( \theta_3^* < 1 \).

In the following theorem we show that in the case where Algorithm 1 converges, we can find an interval in which the choice of \( \theta \) makes Algorithm 2 faster than Algorithm 1, giving a convergence acceleration character to our Algorithm 2.

**Theorem 5.3.** Let the sequence \( (w^{(m)}_\theta)_{m \geq 0} \) be given by Eqs. (5.19) to (5.21), then the following statements hold

(i) For \( k < \frac{\pi}{a} \), the convergence of the sequence \( (w^{(m)}_\theta)_{m \geq 0} \) is faster than the one of sequence \( (w^{(m)}_1)_{m \geq 0} \) \( \forall \theta \in (1, \theta_1^*) \), where \( \theta_1^* = \min \left( \frac{1 + \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}{1 + \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}, 2 \right) \).

(ii) For \( \frac{\pi}{a} \leq k \leq \frac{\pi}{a} \sqrt{1 + \left(\frac{a}{4b}\right)^2} \), the convergence of sequence \( (w^{(m)}_\theta)_{m \geq 0} \) is faster than the convergence of \( (w^{(m)}_1)_{m \geq 0} \) \( \forall \theta \in (\theta_2^*, 1) \cup (1, \min(\theta_1^*, \theta_2^*)) \), where \( \theta_2^* = \min \left( \frac{1 + \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}{1 - \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}, 2 \right) \) and \( \theta_1^* = \frac{1 - \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}{1 + \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2} \).

**Proof.**

(i) \( k < \frac{\pi}{a} \), let us recall that in this case convergence is ensured \( \forall \theta \in (0, 2) \).

In order to show this result, the following inequality must be satisfied:

\[
| \theta \left(\frac{\tan(\sqrt{\text{sgn}(\lambda_n)\lambda_n})}{\sqrt{\text{sgn}(\lambda_n)\lambda_n}}\right)^2 + 1 - \theta | < | \left(\frac{\tan(\sqrt{\text{sgn}(\lambda_n)\lambda_n})}{\sqrt{\text{sgn}(\lambda_n)\lambda_n}}\right)^2 |,
\]

which is equivalent to

\[
- \left(\frac{\tan(\sqrt{\lambda_n b})}{\sqrt{\lambda_n b}}\right)^2 < \theta \left(\frac{\tan(\sqrt{\lambda_n b})}{\sqrt{\lambda_n b}}\right)^2 + 1 - \theta < \left(\frac{\tan(\sqrt{\lambda_n b})}{\sqrt{\lambda_n b}}\right)^2,
\]

and this lead to

\[
1 < \theta < \frac{1 + \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}{1 - \left(\frac{\tan(\sqrt{-\lambda_1 b})}{\sqrt{-\lambda_1 b}}\right)^2}, \quad \text{for all } n \geq 1.
\]
Since 

\[
\frac{1 + (\tanh(\sqrt{\lambda_n b}))^2}{1 - (\tanh(\sqrt{\lambda_n b}))^2} > \frac{1 + (\tanh(\sqrt{\lambda_1 b}))^2}{1 - (\tanh(\sqrt{\lambda_1 b}))^2}, \quad \text{for all } n > 1,
\]

we obtain

\[
1 < \theta < \frac{1 + (\tanh(\sqrt{\lambda_1 b}))^2}{1 - (\tanh(\sqrt{\lambda_1 b}))^2},
\]

and since, we have

\[
0 < \theta < 2,
\]

we get

\[
1 < \theta < \min \left( \frac{1 + (\tanh(\sqrt{\lambda_n b}))^2}{1 - (\tanh(\sqrt{\lambda_n b}))^2}, \frac{1 + (\tanh(\sqrt{\lambda_1 b}))^2}{1 - (\tanh(\sqrt{\lambda_1 b}))^2} \right). \tag{5.32}
\]

Therefore, using (5.32) and taking \( \theta_1 = \min \left( \frac{1 + (\tanh(\sqrt{\lambda_n b}))^2}{1 - (\tanh(\sqrt{\lambda_n b}))^2}, \frac{1 + (\tanh(\sqrt{\lambda_1 b}))^2}{1 - (\tanh(\sqrt{\lambda_1 b}))^2} \right) \), the assertion (i) is valid.

(ii) : \( \frac{\pi}{n} \leq k \leq \frac{\pi}{n} \sqrt{1 + \left( \frac{\pi}{n} \right)^2} \), we recall that in this case we have convergence \( \forall \theta \in (0, \theta_2^a) \).

Consider the case where \( 1 \leq n < n_0 \), then from (5.30) we get

\[
-\left( \tan(\sqrt{-\lambda_n b}) \right)^2 < -\theta \left( \tan(\sqrt{-\lambda_n b}) \right)^2 + 1 - \theta < \left( \tan(\sqrt{-\lambda_n b}) \right)^2, \tag{5.33}
\]

which is equivalent to

\[
\frac{1 - (\tan(\sqrt{-\lambda_n b}))^2}{1 + (\tan(\sqrt{-\lambda_n b}))^2} < \theta < 1. \tag{5.34}
\]

Since

\[
\frac{1 - (\tan(\sqrt{-\lambda_n b}))^2}{1 + (\tan(\sqrt{-\lambda_n b}))^2} > \frac{1 - (\tan(\sqrt{-\lambda_1 b}))^2}{1 + (\tan(\sqrt{-\lambda_1 b}))^2},
\]

we have

\[
\theta_3^o < \theta < 1, \tag{5.35}
\]

with \( \theta_3^o = \frac{1 - (\tan(\sqrt{-\lambda_1 b}))^2}{1 + (\tan(\sqrt{-\lambda_1 b}))^2} \).

If \( n \geq n_0 \), inequality (5.30) has the same form as in (5.31). So using the same argument as in the proof of assertion (i), we have \( (1, \min(\theta_2^a, \theta_2^o)) \). Finally, combining results for the two cases we obtain \( \theta \in (\theta_3^o, 1) \cup (1, \min(\theta_2^a, \theta_2^o)) \), which shows assertion (ii) and completes the proof of theorem.

6. **Numerical results and discussion.** The goal of this section is to confirm the theoretical results in particular that the Algorithm 1 does not converge for \( k \geq k^* \), where \( k^* \) is the limit value given in Theorem 5.2. While Algorithm 2 converge without restriction on \( k \). Moreover, we study the influence of the relaxation parameter \( \theta \) on the convergence. The well-posed boundary value problems involved in each iteration are solved using the finite element method.
Figure 1. Results obtained by Algorithm 1 and Algorithm 2, at $y = b$ for $k = \sqrt{15}$.

6.1. Example. For the first numerical computations, we take on the boundary $\Gamma_1$

$$f_1(x) = \left(3 \sin(\pi x) + \frac{\sin(3\pi x)}{19} + 9 \exp(-30(x - b)^2)\right)x^2(a - x)^2. \quad (6.1)$$

The goal is to reconstruct the following boundary data on $\Gamma_0$:

$$ue(x) = 2 \left(8 \sin(\pi x) + \frac{\sin(3\pi x)}{17} + 20 \exp(-50(x - b)^2)\right)x^2(a - x)^2. \quad (6.2)$$

First, the direct problem Eq. (2.1) with boundary data Eq. (2.2), Eq. (2.4) and Eq. (6.2) is solved to obtain $f_2$, the normal derivative on $\Gamma_1$. Then we consider the Cauchy problem Eqs. (2.1) to (2.4) to reconstruct $ue$ on $\Gamma_0$. We take $a = 1$, $b = 0.2$ and thus $k^* = \frac{\pi}{b} \sqrt{1 + \left(\frac{a}{b}\right)^2} \simeq 5.029$.

Figure 2. Algorithm 2: Variation of iterations number at the convergence for $k = \sqrt{15}$.

For the wave number $k$, we take $k = \sqrt{15} < k^*$ and $k = \sqrt{25.5}$, $k = \sqrt{35}$ and $k = \sqrt{52}$. The initial estimate taken for both Algorithm 1 and Algorithm 2 is
\( \eta^0 = 0 \). As a stopping criterion, we take the following stopping criteria is used
\[
\frac{\| u^{(2n)} - u^{(2n-1)} \|_{L^2(\Gamma_0)}}{\| u^{(2)} \|_{L^2(\Gamma_0)}} < \varepsilon.
\] (6.3)

The first results, concerning the case \((k = \sqrt{15})\), all algorithms converge to a good approximation of the exact solution \(u^e(x)\) (see Fig. 1) for the same tolerance \(\varepsilon = 1.0 \times 10^{-5}\). For Algorithm 2 the number of iterations varies according to \(\theta \in ]0, 2]\) (see Fig. 2). Moreover, we obtain an optimal \(\theta^{opt} = 1.6\), which allows the convergence with only 637 iterations when Algorithm 1 required more than 1000 iterations (see Fig. 3). As we inspected Algorithm 2 accelerated significantly, by more the \(1/3\), of the convergence. This confirms the benefit of this relaxed algorithm.

![Figure 3. Comparison of relative errors in Algorithm 2, for \(\theta = 1\), \(\theta = 1.6\) in the case \(k = \sqrt{15}\).](image)

![Figure 4. Stopping criteria and relative error (6.3) in Algorithm 1 for \(k = \sqrt{25.5}\).](image)

Next, we present numerical results for \((k \geq k^*)\). Starting by seeing that for \(k = \sqrt{25.5}\), Algorithm 1 does not converge as we can see in Fig. 4. In contrast, as
can be seen in Fig. 5, Algorithm 2 converges to a good approximation. A numerical study, similar to the one that allowed us to determine the optimal \( \theta \) for \( k = \sqrt{15} \), gives for the case \( k = \sqrt{25.5} \) an optimal parameter \( \theta^{op} = 0.98 \). Moreover, this convergence is realized with an optimal number of 1000 iterations (see Fig. 6).

![Figure 5. Exact and reconstructed solutions obtained by Algorithm 2, at \( y = b \) for \( k = \sqrt{25.5} \).](image)

![Figure 6. Comparison of relative errors in Algorithm 2, for \( \theta = 0.1 \) and \( \theta = 0.98 \) in the case \( k = \sqrt{25.5} \).](image)

Now, we test Algorithm 2 with \( k = \sqrt{35} \) and \( k = \sqrt{52} \). As it can be seen in Fig. 7, the reconstructed solution on \( \Gamma_0 \) is closed to the solution \( u_e \) in both cases Algorithm 2 converges to a good approximation in both cases \( k = \sqrt{35} \) and \( k = \sqrt{52} \).

We then proceeded to verify the theoretical results given in Theorem 5.2. In Table 1 we give, for each wave number \( k \), the appropriate interval where the parameter \( \theta \) must belong so that Algorithm 2 converges when Algorithm 1 diverges. Moreover, we observe that the interval of convergence narrows as the wave number increases. This confirm the theoretical results.
We also observe that the optimal parameter $\theta_{op}$ making the convergence faster is permanently located near the upper end of the convergence interval. This facilitates the choice of the relaxation parameter to have an acceleration of convergence in the case where Algorithm 1 converges or to have a rapid convergence in the case where it diverges, it is therefore sufficient to take a parameter $\theta$ close to the upper end of the convergence interval.

| Wave numbers | Algorithm 1 | Algorithm 2 | Relaxed intervals |
|--------------|-------------|-------------|-------------------|
| $k = \sqrt{15}$ | converges | converges | (0, 1.6168) |
| $k = \sqrt{25.5}$ | diverges | converges | (0, 0.9893) |
| $k = \sqrt{35}$ | diverges | converges | (0, 0.5791) |
| $k = \sqrt{52}$ | diverges | converges | (0, 0.1450) |

Table 1. Convergence intervals for different values of $k$. 

Figure 7. Exact and reconstructed solutions from Algorithm 2, at $y = b$ (a) $k = \sqrt{35}, \theta = 0.5$ (b) $k = \sqrt{52}, \theta = 0.14$. 

(a) 

(b)
We also observe that the convergence intervals given in Theorem 5.2 are optimal, this means that as soon as \( \theta > \theta^*_i, i = 2, 3 \), Algorithm 2 diverges. For example in Fig. 8, we see clearly that Algorithm 2 diverges for \( \theta = 1.62 \) which is greater but very close to \( \theta^*_2 = 0, 1.6168 \).

**Figure 8.** Diverges of the Algorithm 2 for \( \theta = 1.62 \) in the case \( k = \sqrt{15} \).

### 6.2. Numerical stability

To simulate the influence of errors in the measured data, we examine here the stability of our Algorithm 2, considering that the Cauchy data are noisy. This is done by adding to the exact boundary conditions a Gaussian noise with mean zero with a noise level \( \delta \) in the following form:

\[
\begin{align*}
    f_1^\delta &= f_1 + (\delta f_1), \\
    f_2^\delta &= f_2 + (\delta f_2),
\end{align*}
\]

We denote by \( u^\delta \) the solution obtained by solving the inverse Cauchy problem with noisy data \( f_1^\delta \) and \( f_2^\delta \). We compare the exact solution \( u_e \) the approximate solution without noise and the approximate one \( u^\delta \) with different noise level \( \delta \in \{2.5 \times 10^{-2}, 5 \times 10^{-2}, 10^{-1}, 2 \times 10^{-1}\} \).

**Figure 9.** Exact and reconstructed noisy solutions obtained by Algorithm 2, at \( y = b \) for \( k = \sqrt{15} \) and \( \theta = 1.6 \).
Figure 10. Exact and reconstructed noisy solutions obtained by Algorithm 2, at \( y = b \) for \( k = \sqrt{52} \) and \( \theta = 0.14 \).

As we can see in Fig. 9 and Fig. 10, the obtained solutions \( u^\delta \) are not so far from the exact solution. Indeed, the error verify the following inequality

\[
\|ue - u^\delta\|_{L^2(\Gamma_0)} \leq \delta
\]

which shows that Algorithm 2 is stable since the computed solutions are obtained with an error which is of the same order as the noise.

7. Conclusions. We have explored the Cauchy problem associated with the Helmholtz equation in this paper. Based on the relaxation of the usual boundary condition in the KMF algorithm, we established an efficient algorithm. The breakthrough lies in the fact that this evolved algorithm converges without adding any artificial interior boundaries or regularization parameters for all wave number values in the Helmholtz equation.

Furthermore, our relaxed parameters are completely exhibited as function of the given data. Moreover, we concluded that our suggested algorithm has enormous impact on reducing the rate of convergence in case \( k \) is small. While, it restore the convergence to the original KMF algorithm when \( k \) is large. The main advantage of our modification algorithm is that it provides accurate convergence intervals through the theoretical results.

The theoretical study is supported by many numerical experiments which confirm each other’s. In addition, we demonstrated the computational stability of the algorithm that was established. This reliability is shown by numerical examples where the variance in the solutions relating to noisy data has been shown to be of the same order as the noise in the data.

For a future work, we will improve a rate of convergence for our modification algorithm using adaptive mesh techniques especially for the case when wave numbers \( k \) are big.

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