Abstract—The robust state estimation problem is on how to design robust filters for estimating an unknown state of uncertain systems. This paper considers this problem for multi-agent systems with multiplicative noise and degraded measurements over corrupted channels. Employing a covariance intersection fusion method, we propose a distributed robust Kalman filter with stochastic gains, which enables a sequence of upper bounds of conditional mean square error given channel noise to be calculated online. Considering the limitation of step-wise optimization, for better performance, we propose a switching fusion scheme based on a sliding window method, which provides an online design of covariance intersection weights by solving a semi-definite programming problem. Compared to the filter fusing latest estimates, the one based on the switching fusion method has a smaller upper bound of the conditional mean square error. We present a robust collective observability condition, which degenerates to the traditional collective observability condition for time-varying stochastic systems if there is no measurement degradation or multiplicative noise. Under this condition and strong connectivity, we prove that the mean square errors of two filters are both uniformly upper bounded by a constant matrix over a finite transient time, which depends on the system observability and the network size. Different to existing results, some requirements including stability for the systems and observability of the sub-systems are not needed for our results. Finally, a numerical simulation is provided to validate the theoretical results.

Index Terms—Multi-agent system, distributed filtering, robust Kalman filter, multiplicative noise, degraded measurement, corrupted channel

1. INTRODUCTION

Recently, networked state estimation problems over multi-agent systems are drawing more and more attention due to their broad range of applications [1–3]. Two main frameworks, namely the centralized and the distributed, have been considered. Compared to the centralized framework, the distributed one shows more robustness in network structure, better performance in energy saving and stronger ability for parallel processing. Thus, a growing number of researchers are focusing on the study of distributed state estimation problems [4–8]. Because of the complexity of network environments and the uncertainties of physical systems, networked robust filters are needed [9–11].

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For stochastic multi-agent systems, it is difficult to handle uncertain system dynamics in the filter design [9, 11–17]. Multiplicative noise exists in many situations like signal transmission and sampling, amplitude modulation, etc [18]. For systems with multiplicative noise, the uncertainties will dramatically increase when the nominal system is unstable. Thus, to design effective filters for state estimation is even more challenging in this case. In the literature, [12, 14] studied the centralized estimation problems for systems with multiplicative noise and parameter uncertainties, robust centralized filters were provided and analyzed. [13, 16] studied the distributed fusion problem for uncertain systems with correlated noise and bounded uncertainties, and provided robust filters. Yet, they paid little attention to estimation performance. Besides, $H_{\infty}$-based distributed robust estimation algorithms were studied in [17], where the conditions to guarantee estimation performance seem difficult to be verified in practice. Although many networked filters have been proposed, robust estimation for unstable system dynamics suffering uncertainties still needs further investigation.

In distributed estimation, generally, each individual agent has limited observability, which in addition is seriously deteriorated if measurements are degraded. The degradation usually comes from sensor or communication limitations [19–21]. One such example is communication fading. A detailed study on Kalman filter with measurement degradations was given in [22]. For distributed filters based on degraded measurements, [19] studied the filtering problem for sensor networks with stochastic gain degradation. A minimum-variance recursive filter was proposed for sensors connected over a complete graph, which seems difficult to be met in large-scale networks. In [20], a distributed filter was proposed for a state-saturated system with degraded measurements and quantization effects. Under the assumption that the solutions of two difference equations exist, the upper boundedness of the estimation covariance was guaranteed. Nevertheless, the conditions seem not easy to be verified as they need to hold at each time instant. Therefore, it is necessary to develop robust distributed filters, and find easy-to-check conditions on degraded measurements to guarantee estimation stability.

In the existing literature [19–21, 23–28], communication channels of agents are supposed to be noise or disturbance free, which however is difficult to be met in applications like sensor networks [29]. The uncertainty induced by channel corruption makes it more difficult to design and analyze distributed filters. The distributed filter parameters, such as filtering gain and fusion weight, have a big influence in many aspects, such as precision of estimates, and boundedness of
mean square estimation error. [30] investigated the design of distributed filters with constant filtering gains and fusion weights, and analyzed the conditions to ensure mean square boundedness of estimation error. [23] proposed a novel distributed filter with constant weights by combining a diffusion step with the Kalman filter, and analyzed the performance of the filter by assuming that each sub-system is observable, which is a restrictive condition for large-scale networks. Compared to constant distributed filtering parameters, time-varying ones can sometimes give better performance [31–33]. Some global knowledge was required in the existing literature [19, 20, 26], where each agent should know the statistics of non-neighbors. The above literature all assumed perfect communications of agents, i.e., the channels should be noise free. Although [25] studied the case that state estimates suffer channel noise, the parameter matrices were required to be perfectly transmitted. Therefore, designing time-varying filter parameters in a distributed manner over corrupted channels needs further investigation.

The main contributions of this paper are summarized in the following.

- By employing the covariance intersection (CI) method and the knowledge of local imprecise statistics, we design a robust distributed Kalman filter for a class of multi-agent systems with uncertain dynamics and degraded measurements over corrupted communication channels. We prove conditional consistency of the filter, which means that a sequence of upper bounds of conditional mean square estimation error given channel noise can be calculated by each agent online. Moreover, we show that the upper bounds are minimized by the designed filtering gains which are stochastic but adapted to the sigma algebras generated by channel noise.

- We propose a robust collective observability condition for multi-agent systems with multiplicative noise and degraded measurements. We show that the robust collective observability condition degenerates to the traditional collective observability condition of time-varying systems if there is no measurement degradation or multiplicative noise. Under mild conditions including robust collective observability and strong connectivity, we prove that the mean square error is uniformly upper bounded by a constant matrix after a finite transient time, which depends on the network size and system observability.

- Based on past estimates from neighboring agents, we provide a sliding-window fusion method with adaptive CI weights by solving a semi-definite programming problem online. Considering practical limitations for step-wise optimization, a switching fusion strategy is provided. Compared to the filter fusing latest estimates, the one based on the sliding-window method inherits conditional consistency and has a smaller upper bound of the conditional mean square error given the same channel noise. The distributed robust Kalman filter with sliding-window fusion is proven to have mean square boundedness of the estimation error.

The results of this paper make significant contributions compared to the existing literature. In particular, first, compared to [25] where the transmitted state estimates suffer channel noise, this paper studies a more general case of channel corruption, which can add noise to both transmitted state estimates and parameter matrices. A new robust CI method is then provided to obtain a conditionally consistent estimate. Moreover, this paper removes two requirements in many existing studies that the nominal systems have to be stable [19, 20, 26] and that each sub-system is observable [21, 23, 24]. In addition, different from existing results [19, 20, 25–28] using the latest estimates from neighbors, the switching fusion scheme of this paper considers the limitation of step-wise optimization and will utilize the past information more efficiently.

The remainder of this paper is organized as follows: Section 2 is on the problem formulation. Section 3 considers the filter design. Section 4 studies the mean square boundedness of the estimation error for the proposed filter. Section 5 provides a sliding-window fusion method. Section 6 is on numerical simulation. The conclusion is given in Section 7.

### Notations

Superscript $T$ represents transpose. The notation $A \succeq B$ ($A > B$), where $A$ and $B$ are both symmetric matrices, means that $A - B$ is a positive semidefinite (positive definite) matrix. $I_n$ is an $n$-dimensional vector with all elements of one. $I_n$ stands for the identity matrix with $n$ rows and $n$ columns. $\mathbb{R}^n$ is the set of $n$-dimensional real vectors. $\mathbb{N}$ stands for the set of natural numbers. $E\{x\}$ denotes the mathematical expectation of the stochastic vector $x$, and $\text{Cov}\{x\} = E\{(x - E\{x\})(x - E\{x\})^T\}$. $\text{blockdiag}\{\cdot\}$ and $\text{diag}\{\cdot\}$ represent the diagonalizations of block elements and scalar elements, respectively. $\text{Tr}(P)$ stands for the trace of matrix $P$, $\rho(A)$ denotes the spectral radius of $A$ and $\|A\|_2 = \sqrt{\rho(A^T A)}$. $\lambda_{\text{max}}(A)$ is the maximal eigenvalue of matrix $A$, $\sup \sigma(\cdot)$ stands for the supremum operator. $\sigma(\cdot)$ is the minimal $\sigma$-algebra operator generated by a collection of subsets.

### 2. Problem Formulation

#### A. Motivating example

In a spatially distributed physical system, a state vector usually consists of elements over a large geographical area. The evolution of the state is related to spatial and temporal system dynamics. Agents located at different positions can collaborate based on their intermittent measurements on partial elements of the state. The state elements, the evolution of state vector and the measurements are often polluted by noise. Thus it is necessary to develop robust and reliable distributed filtering algorithms for state estimation.

A motivating example of a random dynamic field driven by noise $w_k(p)$ and monitored by a multi-agent system is shown in Fig. 1, where $p \in \mathbb{R}^2$ [34]. The variable $x_k^i(p)$ stands for the temperature in station $i$ at $k$th time instant. In the figure, colors of the field represent values of $x_k^i(p)$. In Fig. 1(c), $\gamma_{k,i}$ is the measurement degradation factor, $y_{k,i}$ is the measurement vector, and $\hat{x}_{k,i}$ is the state estimate of the overall state $x_k = [x_{k,1}, \ldots, x_{k,n}]^T$. The problem considered in this paper is how to design a distributed robust filter based on
the corrupted measurements \(\{y_{k,i}, k \in \mathbb{N}, i = 1, \ldots, 4\}\) and the collaboration of agents, such that the overall temperature field \(x_k\) can be effectively estimated by each agent.

B. Preliminaries

Consider a stochastic multi-agent system with \(N > 0\) agents:

\[
\begin{aligned}
x_{k+1} &= (A_k + F_k \varepsilon_k)x_k + w_k, \\
y_{k,i} &= \gamma_{k,i} C_{k,i} x_k + v_{k,i},
\end{aligned}
\]

where \(x_k \in \mathbb{R}^n\) denotes the state vector, \(w_k \in \mathbb{R}^n\) the independent process noise with zero mean, \(\varepsilon_k \in \mathbb{R}\) the independent multiplicative noise with zero mean, \(y_{k,i} \in \mathbb{R}^{m_i}\) the measurement vector, \(v_{k,i} \in \mathbb{R}^{m_i}\) the independent measurement noise with zero mean and \(\gamma_{k,i} \in \mathbb{R}\) the independent random fading factor lying in the interval \([0,1]\) with \(E[\gamma_{k,i}] = \tau_{k,i}\), where \(\tau_{k,i}\) is a known scalar with \(0 < \tau_{k,i} \leq 1, i = 1, 2, \ldots, N\). The initial state \(x_0\) is generated from an unknown distribution, \(A_k, F_k\) and \(C_{k,i}\) are known matrices with appropriate dimensions. Also, \(F_k, k \in \mathbb{N}\), are non-singular.

We model the agent communications as a directed graph \(G = (\mathcal{V}, \mathcal{E}, \mathcal{A})\), which consists of the nodes \(\mathcal{V} = \{1, 2, \ldots, N\}\), the links \(\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}\), and the weighted adjacency matrix \(\mathcal{A} = [a_{i,j}]\), where \(a_{i,j} > 0, a_{i,j} \geq 0, \sum_{j \in \mathcal{V}} a_{i,j} = 1\). If \(a_{i,j} > 0, j \neq i\), there is a link \((i, j)\) in \(\mathcal{E}\), through which node \(i\) can directly receive messages from node \(j\). In this case, node \(j\) is called a neighbor of node \(i\). The neighbor set of node \(i\), including itself, is denoted \(\mathcal{N}_i\). The graph \(G\) is called strongly connected if for any two nodes \(i_1, i_2\), there exists a directed path from \(i_1\) to \(i_2\):

\((i_{t-1}, i_t), \ldots, (i_2, i_3), (i_1, i_2)\).

In a distributed scheme, if the communication channels are perfect, then one agent will receive the accurate messages transmitted from its neighbors. However, due to the complexity of environment and the physical restriction of channels, the received messages may be corrupted by some disturbances or noise. Let \(\{\hat{x}_{k,j}, \hat{P}_{k,j}\}\) be the pair that agent \(j\) sent out at time \(k\), where \(\hat{x}_{k,j} \in \mathbb{R}^n\) and \(\hat{P}_{k,j} \in \mathbb{R}^{n \times n}\). Due to the existence of channel noise, we assume that the pair \(\{\hat{x}_{k,i,j}, \hat{P}_{k,i,j}\}\) received by agent \(i\) from agent \(j\) satisfies the following condition:

\[
\begin{aligned}
\hat{x}_{k,i,j} &= \bar{x}_{k,j} + \varepsilon_{k,i,j}, j \in \mathcal{N}_i, \\
\hat{P}_{k,i,j} &= \tilde{P}_{k,j} + \tilde{D}_{k,i,j}, j \in \mathcal{N}_i,
\end{aligned}
\]

where \(\varepsilon_{k,i,j} \in \mathbb{R}^n\) is the noise added into the state estimate \(\hat{x}_{k,j}\) and \(\tilde{D}_{k,i,j} \in \mathbb{R}^{n \times n}\) is the noise added into the parameter matrix \(\tilde{P}_{k,i,j}\).

Assume \((\Omega, \mathcal{F}, P)\) denote the basic probability space. \(\mathcal{F}_k\) stands for a filtration of \(\sigma\)-algebra \(\mathcal{F}\). A discrete-time sequence \(\{\xi_k, \mathcal{F}_k\}\) is said to be adapted if \(\xi_k\) is measurable to \(\mathcal{F}_k\). The detailed knowledge on probability can refer to [35]. We require the following assumption on noise and initial estimates.

**Assumption 2.1.** The following conditions hold

1) The initial value \(x_0\), its estimates \(\hat{x}_{0,i}\), and the noise \(\varepsilon_i, \gamma_{k,i}, v_{k,i}\), \(\varepsilon_{k,i,j}, \tilde{D}_{k,i,j}\) are independent both in time and space, \(i, j \in \mathcal{V}, k = 0, 1, \ldots\).

2) There exist known matrices \(Q_k, R_{k,i}, P_0\) and scalars \(\mu_k, \varphi_{k,i}, i = 1, 2, \ldots, N\), such that

\[
\begin{aligned}
E[w_k w_k^T] &\leq Q_k, \inf_{k \in \mathbb{N}} Q_k \geq Q > 0, \sup_{k \in \mathbb{N}} Q_k \leq Q < \infty, \\
E[\varepsilon_k^2] &\leq \mu_k, E[x_0 x_0^T] \leq P_0, \text{Cov}[\varepsilon_{k,i}] \leq \varphi_{k,i}, \\
E[v_{k,i} v_{k,i}^T] &\leq R_{k,i}, \sup_{k \in \mathbb{N}} \tau_{k,i}^2 C_{k,i} R_{k,i}^{-1} C_{k,i} \leq \infty, \\
E[(\hat{x}_{0,i} - x_0) (\hat{x}_{0,i} - x_0)^T] &\leq P_0, \forall i \in \mathcal{V}.
\end{aligned}
\]

3) The channel noise \(\varepsilon_{k,i,j}\) and \(\tilde{D}_{k,i,j}\) are bounded, subject to \(\sup_{i} |\varepsilon_{k,i,j}| \leq T_{i,j}, -D_{i,j} \leq \tilde{D}_{k,i,j} \leq D_{i,j}\), where \(T_{i,j}\) and \(D_{i,j}\) are positive semi-definite matrices. \(\tilde{D}_{k,i,j}\) is a real-valued symmetric matrix.
The exact covariance information of the stochastic uncertainties is not required. Bounds and statistics are known only to individual agents, thus the conditions in 2) of Assumption 2.1 are milder than [19, 20, 26], where each agent has full knowledge of the statistics over the global system. For a symmetric matrix \( \tilde{P}_{k,i,j} \), through some encoding protocol, agent \( j \) can transmit the upper triangular part to agent \( i \), which can recover the whole matrix with the corresponding decoding protocol. Hence, the requirement on symmetry of noise matrix \( D_{k,i,j} \) is reasonable. Since agent \( i \) can obtain its own estimate \( \{ \hat{x}_{k,i}, \tilde{P}_{k,i} \} \) accurately, we denote \( \tilde{e}_{k,i,i} = 0 \) and \( D_{k,i,i} = 0 \).

Let \( \tilde{x}_k \) be the estimate of system state \( x_k \) and \( \tilde{e}_k = \tilde{x}_k - x_k \) be the state estimation error. We need the following definitions in this paper.

**Definition 2.1.** [36] (Consistency) The pair \( \{ \hat{x}_k, P_k \} \) is consistent if there is a deterministic sequence \( \{ P_k \} \) such that \( E[\tilde{e}_k \tilde{e}_k^T] \leq P_k \).

**Definition 2.2.** (Conditional Consistency) The pair \( \{ \hat{x}_k, P_k \} \) is conditionally consistent if there is an adapted sequence \( \{ P_k, K_k \} \), such that \( E[\tilde{e}_k \tilde{e}_k^T | K_k] \leq P_k \), where \( K_k \) is a \( \sigma \)-algebra.

Due to the unknown correlation between agent estimates, estimation error covariances of individual agents can not be obtained in a distributed manner [21, 32, 33, 37]. The consistency in Definition 2.1 has two benefits. First, the estimation precision of each agent can be evaluated online. Second, a CI structure for each agent can be considered. However, due to the influence of channel noise, the consistency method in [27, 28, 32, 33] can not be utilized. Thus, we provide the conditional consistency in Definition 2.2, which generalizes the concept of consistency, and it permits \( P_k \) to be stochastic. Note that the pair \( \{ \hat{x}_k, E[P_k] \} \) is consistent, if \( \{ \hat{x}_k, P_k \} \) is conditionally consistent.

**C. Problem**

In this paper, we consider a three-stage distributed filtering structure for each agent \( i, i \in V \), consisting of time prediction, measurement update and local fusion:

\[
\begin{align*}
\hat{x}_{k,i} &= A_{k-1}\hat{x}_{k-1,i}, \\
\tilde{x}_{k,i} &= \hat{x}_{k,i} + K_{k,i}(y_{k,i} - \tau_{k,i}C_{k,i}\tilde{x}_{k,i}), \\
\tilde{x}_{k,i,j} &= \sum_{j \in N_i} W_{k,i,j}\tilde{x}_{k,i,j},
\end{align*}
\]

where \( \tilde{x}_{k,i} \), \( \tilde{x}_{k,i,j} \) and \( \hat{x}_{k,i} \) are the state prediction, state update and state estimate of agent \( i \) at time \( k \), respectively. \( \tilde{x}_{k,i,j} \) is the noisy estimate received by agent \( j \) from agent \( i \). \( \tilde{e}_{k,i,j} \) is the channel noise of link \((i, j)\) of graph \( G \). Besides, \( K_{k,i,j} \) is the filtering gain parameter matrix, \( W_{k,i,j} \) is the local fusion parameter matrix. Both \( K_{k,i,j} \) and \( W_{k,i,j} \) remain to be designed.

**Remark 2.1.** Different from many existing results [19, 30, 33, 38], measurements and measurement matrices are not transmitted in our setting. The advantages of this protocol lie in many aspects including security and energy saving.

In this paper, we consider three essential subproblems:

(a) How to design the parameters \( K_{k,i} \) and \( W_{k,i,j} \) in (3) based on the local knowledge and the corrupted messages from neighboring agents such that the filter is conditionally consistent given channel noise?

(b) Which conditions on system structure and noise statistics can enable the mean square estimation error of the distributed filter to be bounded?

(c) How to design fusion scheme based on past estimates from neighbors for better performance?

### 3. Filter Design

In this section, we provide a distributed design of the parameters \( K_{k,i} \) and \( W_{k,i,j} \) for the filter (3), based on available noise statistical information and system structural information. Denote \( \tilde{e}_{k,i} = \tilde{x}_{k,i} - x_k, \tilde{e}_{k,i} = \tilde{x}_{k,i} - x_k, \tilde{e}_{k,i,j} = \tilde{x}_{k,i,j} - x_k, \tilde{e}_{k,i} = \tilde{x}_{k,i} - x_k \). Lemmas 3.1 – 3.2 in the following are needed for further analysis.

**Lemma 3.1.** The estimation errors \( \tilde{e}_{k,i}, \tilde{e}_{k,i}, \tilde{e}_{k,i,j} \) and \( \tilde{e}_{k,i} \) satisfy the following iterations:

\[
\begin{align*}
\tilde{e}_{k,i} &= A_{k-1}\tilde{e}_{k-1,i} - \tau_{k,i}C_{k,i}\tilde{e}_{k,i}, \\
\tilde{e}_{k,i} &= \left( I_n - \tau_{k,i}K_{k,i}C_{k,i} \right)\tilde{e}_{k,i} \\
&\quad + K_{k,i}(y_{k,i} - \tau_{k,i}C_{k,i}\tilde{x}_{k,i}), \\
\tilde{e}_{k,i,j} &= \tilde{e}_{k,i,j} + \tilde{e}_{k,i,j}, j \in N_i.
\end{align*}
\]

**Proof.** See Appendix A. \(\square\)

**Lemma 3.2.** Under Assumption 2.1, it holds that \( E[\tilde{x}_{k,i}^T \tilde{x}_{k,i}] \leq \Pi_{k}, \forall k \in N \), where \( \Pi_{k} \) is recursively calculated through \( \Pi_{k+1} = A_k\Pi_kA_k^T + \mu_k F_k\Pi_k F_k^T + Q_k \), with \( \Pi_0 = P_0 \).

**Proof.** See Appendix B. \(\square\)

By employing the CI method, the following lemma provides a time-varying choice of the matrices \( W_{k,i,j} \). Based on the choice, the conditional consistency given channel noise is guaranteed for (3).

**Lemma 3.3.** Consider the multi-agent system (1) satisfying Assumption 2.1. For the filter (3), if \( K_{k,i} \) is adapted to \( W_{k,i,j} = \sigma(D_{k,i,j}, 1 \leq t \leq k, i, j, \in N) \), and

\[
W_{k,i,j} = a_{i,j}P_{k,i} \tilde{P}_{k,i,j} + D_{k,i,j} + Y_{k,i,j}^{-1},
\]

then the pairs \( \{ \tilde{x}_{k,i}, \tilde{P}_{k,i} \}, \{ \tilde{x}_{k,i}, \tilde{P}_{k,i} \}, \{ \tilde{x}_{k,i}, \tilde{P}_{k,i} \} \) are all conditionally consistent given \( W_{k,i,j} \), where \( P_{k,i}, \tilde{P}_{k,i}, \tilde{P}_{k,i,j} \) and \( P_{k,i,j} \) satisfy

\[
\begin{align*}
\tilde{P}_{k,i} &= A_{k-1}\tilde{P}_{k-1,i}A_{k-1}^T + \mu_k F_{k-1}^\Gamma F_{k-1} + Q_{k-1}, \\
\tilde{P}_{k,i} &= (I - \tau_{k,i}K_{k,i}C_{k,i})\tilde{P}_{k,i} (I - \tau_{k,i}K_{k,i}C_{k,i})^T \\
&\quad + K_{k,i} \left( R_{k,i} + \varphi_{k,i}C_{k,i}I_k C_{k,i}^T \right) K_{k,i}, \\
\tilde{P}_{k,i,j} &= \tilde{P}_{k,i} + D_{k,i,j}, j \in N_i, \\
P_{k,i} &= \left( \sum_{j \in N_i} a_{i,j} \left( \tilde{P}_{k,j} + D_{k,j} + Y_{k,j}^{-1} \right) \right)^{-1}.
\end{align*}
\]

**Proof.** See Appendix C. \(\square\)
Here \( \{a_{i,j}\} \), i.e., the elements of adjacency matrix \( A \), play the role of weights of covariance intersection fusion. A further investigation on the fusion weights is studied in Section 5. Different from the existing literature [19, 20, 25–27], the parameter matrices \( P_{k,i}, P_{k,i}, R_{k,i}, P_{k,i}, \) are random due to channel noise \( \mathcal{N}_{k,i,j} \). Meanwhile, Lemma 3.3 shows that \( P_{k,i} \) is adapted to the \( \sigma \)-algebra \( \mathcal{W}_k \), we then can design a filtering gain parameter \( K_{k,i} \) adapted to the \( \sigma \)-algebra \( \mathcal{W}_k \) such that the bound of the conditional mean square error, i.e., \( P_{k,i} \), is minimized. The design of \( K_{k,i} \) is provided in the following lemma.

**Lemma 3.4.** The solution \( K_{k,i}^* = \arg \min_{K_{k,i}} \text{Tr}\{P_{k,i}\} \) is given by

\[
K_{k,i}^* = \frac{\tau_{k,i} P_{k,i} C_{k,i}^T \Xi_{k,i}}{\tau_{k,i} P_{k,i} C_{k,i}^T P_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i}},
\]

where \( \Xi_{k,i} = \sigma_{k,i}^2 C_{k,i} P_{k,i} C_{k,i}^T + R_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i} \).

Furthermore, \( K_{k,i}^* \) is adapted to \( \mathcal{W}_k \).

**Proof.** See Appendix D.

Considering the multi-agent system (1) with filter (3) and parameters as given in Lemmas 3.3 and 3.4, we obtain the distributed robust Kalman filter (DRKF) given in Algorithm 1. Different from [21, 37], the implementation of this filter simply depends on the local information \( \{C_{k,i}, R_{k,i}, \varphi_{k,i}, \tau_{k,i}\} \) and the information \( \{\hat{x}_{k,i,j}, \hat{P}_{k,i,j}, j \in \mathcal{N}_i\} \) from neighbors. Thus, it obeys a fully distributed design and implementation.

**Algorithm 1** Distributed robust Kalman filter (DRKF):

**Prediction:** Each agent carries out a state prediction:

\[
\hat{x}_{k,i} = A_{k,i} \hat{x}_{k,i-1},
\]

\[
\hat{P}_{k,i} = A_{k,i} P_{k,i-1} A_{k,i}^T + \mu_{k,i} F_{k,i-1} \Pi_k F_{k,i-1}^T + Q_{k,i} - \Pi_k A_{k,i},
\]

**Update:** Each agent uses its own measurements to update its estimate:

\[
\hat{x}_{k,i} = \tilde{K}_{k,i}(y_{k,i} - \tau_{k,i} C_{k,i} \hat{x}_{k,i}),
\]

\[
K_{k,i} = \frac{\tau_{k,i} P_{k,i} C_{k,i}^T (\tau_{k,i} C_{k,i} P_{k,i} C_{k,i}^T + R_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i})^{-1}}{\tau_{k,i} P_{k,i} C_{k,i}^T P_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i}},
\]

\[
\hat{P}_{k,i} = (I - \tilde{K}_{k,i} C_{k,i}) \hat{P}_{k,i}.
\]

**Fusion:** Each agent receives and fuses the corrupted pair \( \{\hat{x}_{k,i,j}, \hat{P}_{k,i,j}\} \) given in (2):

\[
\hat{x}_{k,i} = \sum_{j \in \mathcal{N}_i} a_{i,j}(\hat{P}_{k,i,j} + D_{i,j} + \Sigma_{i,j})^{-1} \hat{x}_{k,i,j},
\]

\[
\hat{P}_{k,i} = (\sum_{j \in \mathcal{N}_i} a_{i,j}(\hat{P}_{k,i,j} + D_{i,j} + \Sigma_{i,j})^{-1})^{-1}.
\]

**4. Bounded Mean Square Error**

In this section, we find mild conditions to guarantee boundedness of mean square errors. Denote \( \Phi_{j,k} := \Phi_{j,s} \Phi_{s,k}, \Phi_{k+1,k} = A_{k,i} \Phi_{k,k} = I_n \). For Algorithm 1, Assumptions 4.1 and 4.2 are needed.

**Assumption 4.1.** (Robust collective observability) There exists an integer \( N > 0 \) and a constant \( \alpha > 0 \) such that for \( k \in \mathbb{N} \),

\[
\sum_{i=1}^{k+N} \sum_{j=k}^{j=k} \Phi_{j,k}^T C_{j,i} R_{j,i}^{-1} C_{j,i} \Phi_{j,k} \geq \alpha I_n,
\]

where \( C_{j,i} = \tau_{j,i} C_{j,i} j, j \in \mathcal{N}, i \in \mathcal{V}, \)

\[
R_{j,i} = R_{j,i} + \varphi_{j,i} C_{j,i} C_{j,i}^T,
\]

\[
\varphi_{j,i} = \|P_0\|_2 \sum_{i=0}^{j-1} \bar{a}_i + \sum_{s=1}^{j-1} \sum_{l=s}^{\infty} \bar{q}_{s-1} \bar{a}_l + \bar{q}_j,
\]

\[
\bar{q}_j = \|Q_j\|_2.
\]

Assumption 4.1 provides a condition based on the system structure and noise statistics. It can be regarded as a distributed version of the observability condition with multiplicative noise in [39]. The condition does not require that each sub-system should be observable [21, 23, 24]. Moreover, if \( \varphi_{k,i} \equiv 0 \), \( \forall k \in \mathcal{N}, i \in \mathcal{V}, \) Assumption 4.1 corresponds to the collective observability condition for time-varying stochastic system [27].

In the next, a requirement on the multiplicative noise is studied. Denote the time sequence set of non-zero multiplicative noise by

\[
\mathbb{K}_T = \{k_i = \min_{\mu_k > 0} \{k | k \geq k_{t-1}, k, t \in \mathbb{N}\} \}.
\]

**Assumption 4.2.** There exist positive scalars \( \lambda_1, \lambda_2, M, \) and \( \varrho \in (0, 1) \), such that

\[
\lambda_1 I_n \leq A_{k,i}^T A_{k,i} \leq \lambda_2 I_n, k \in \mathbb{N} \tag{7}
\]

\[
\prod_{t=s}^{l} \rho_{k,t} \leq M \varrho^{l-s}, 0 \leq s \leq l < \infty \tag{8}
\]

\[
\sup_{t \in \mathbb{N}} \|\mu_{k+1,t} F_{k+1,t} Q_{k+1,t} F_{k+1,t}^T\|_2 < \infty, \tag{9}
\]

where \( k_i \in \mathbb{K}_T \) and

\[
\rho_{k,t} = \frac{\mu_{k+1,t}}{\mu_{k,t}} \|F_{k+1,t} \Phi_{k+1,t} \Phi_{k+1,t} F_{k+1,t}^T\|_2^2 + \mu_{k+1,t} \|F_{k+1,t} \Phi_{k+1,t} \|_2^2.
\]

\[
Q_{k+1,t} k_i = \sum_{k=k}^{k+1} \Phi_{k+1,t} \Phi_{k+1,t} Q_{k+1,t} \Phi_{k+1,t}^T k_i.
\]

Compared to [19, 20, 26], (7) is a milder condition, which permits our nominal system to be unstable. For the case \( \{k \} \mu_k > 0, k \in \mathbb{N} \) being a finite set or in the absence of multiplicative noise, (8) and (9) are naturally satisfied for sufficiently small \( \mu_k \) replacing the points \( \mu_k = 0 \). Assumptions 4.1 and 4.2 concern the worst cases for the system with uncertain noise statistics. If the statistics are accurately known, they can be replaced to milder conditions.

For further analysis, we need Lemmas 4.1 – 4.2.

**Lemma 4.1.** Suppose Assumption 4.1 hold, then

\[
\sum_{k=1}^{k+N} \sum_{j=k}^{j=k} \Phi_{j,k}^T C_{j,i} R_{j,i}^{-1} C_{j,i} \Phi_{j,k} \geq \alpha I_n, \tag{10}
\]

where \( R_{k,i} := R_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T \).

**Proof.** See Appendix E.

**Lemma 4.2.** Suppose Assumption 4.2 hold, then

\[
\sup_{k \in \mathbb{N}} \{\mu_k F_{k} \Pi_k C_{k}^T\} < \infty.
\]
Proof. See Appendix F.

Let $e_{k,i} := \tilde{x}_{k,i} - x_i$ be the estimation error of agent $i$ by Algorithm 1. Theorem 4.1 states boundedness of mean square error of Algorithm 1 for the multi-agent system (1).

**Theorem 4.1.** Consider the multi-agent system (1) satisfying Assumption 2.1, 4.1 – 4.2. For Algorithm 1, if the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is strongly connected, then there exists a positive scalar $\bar{\eta}$ such that

$$\sup_{k \geq N + \bar{\eta}} \lambda_{\max}(E\{e_{k,i}e_{k,i}^T\}) \leq \frac{\bar{\eta}}{\alpha}, \forall i \in \mathcal{V},$$

where $\alpha$ is given in Assumption 4.1.

**Proof.** Denote

$$S_{k,i} := P_{k,i}^{-1},$$
$$\tilde{Q}_k := \mu_k F_k \Pi_k F_k^T + Q_k,$$
$$G_{k,i} := \sum_{j \in \mathcal{V}} a_{ij} \tilde{C}_j^T R_{k,j}^{-1} \tilde{C}_{k,j},$$
$$\tilde{R}_{k,i} := R_{k,i} + \varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T.$$  

By Assumption 2.1,

$$\tilde{P}_{k,i,j} + D_{k,i,j} + T_{k,i,j} = \tilde{P}_{k,j} + D_{k,j,i} + D_{k,j} + T_{k,j} \geq \tilde{P}_{k,j} + T_{k,j} \geq \tilde{P}_{k,j}.$$  

As $\inf_{k \in \mathcal{K}} Q_k \geq \underline{Q}$ and $\sup_{k \in \mathcal{K}} \left[\frac{1}{2} r_{k,i}^T C_{k,i}^T R_{k,i}^{-1} C_{k,i}\right] < \infty$ in Assumption 2.1, there exists a scalar $\vartheta_0 > 0$, such that $\tilde{P}_{k,i,j} + D_{k,i,j} + T_{k,i,j} \leq (1 + \vartheta_0) P_{k,i,j}$.

According to Algorithm 1 and Lemma 4.2,

$$S_{k,i} = \sum_{j \in \mathcal{V}} a_{ij} \tilde{P}_{k,i,j} + D_{k,i,j} + T_{k,i,j}^{-1} \leq (1 + \vartheta_0) P_{k,i}^{-1},$$

where $0 < \eta < 1$. This inequality is obtained by Lemma 1 in [32] considering Assumption 4.2 and $\frac{1}{1+\vartheta_0} < 1$. Let $a_{ij,k}$ be the $(i,j)$th element of $A^k$. By recursively applying (11) $k$ times with $k \geq N + \bar{\eta}$, we have

$$S_{k,i} \geq \eta^k \Phi_{k,0}^{-T} \sum_{j \in \mathcal{V}} a_{ij,k} S_{0,j} \Phi_{k,0}^{-1} + \frac{S_{k,i}}{1 + \vartheta_0},$$

where

$$S_{k,i} = \sum_{s=1}^{k} \eta^{s-1} \Phi_{k,k-s+1}^{-T} \sum_{j \in \mathcal{V}} a_{ij,s} \tilde{S}_{k,s+1,j} \Phi_{k,k-s+1}^{-1},$$

with $\tilde{S}_{k,j} = \tilde{C}_j^T R_{k,j}^{-1} \tilde{C}_{k,j}$. Due to that first term of the right-hand side of (12), it follows that

$$S_{k,i} \geq \frac{\tilde{S}_{k,i}}{1 + \vartheta_0}, \forall k \geq N + \bar{\eta}.$$  

Since $\mathcal{G}$ is strongly connected, it is similar to [27] that $a_{ij,s} > 0$ for $s \geq N - \bar{\eta}$. Supposing $\bar{L} = N + \bar{\eta}$, we obtain

$$S_{k,i} \geq \sum_{s=1}^{\bar{L}} \eta^{s-1} \Phi_{k,k-s+1}^{-T} \sum_{j \in \mathcal{V}} a_{ij,s} \tilde{S}_{k,s+1,j} \Phi_{k,k-s+1}^{-1} \geq a_{\min} \eta^{\bar{L}-1} \sum_{s=1}^{\bar{L}} \Phi_{k,k-s+1}^{-T} \sum_{j \in \mathcal{V}} \tilde{S}_{k,s+1,j} \Phi_{k,k-s+1}^{-1} \geq a_{\min} \eta^{\bar{L}-1} \sum_{j=1}^{\bar{\eta}} \sum_{s=1}^{\bar{L}+1} \Phi_{k,k-s+1}^{-T} \tilde{S}_{k,s+1,j} \Phi_{k,k-s+1}^{-1},$$

where $a_{\min} = \min_{i,j \in \mathcal{V}} a_{ij,s} > 0, s \in \{N_1, \ldots, \bar{L} \}$.

According to Assumption 4.2, there exists a scalar $\bar{\beta} > 0$, such that $\Phi_{k,k-L+1} = \beta I_n$, $\forall k \geq 0$. For the equality of (14), thanks to Lemma 4.1 and $\bar{L} = N + \bar{\eta}$, it holds that

$$\sum_{j=1}^{\bar{\eta}} \sum_{s=1}^{\bar{L}+1} \Phi_{k,k-s+1}^{-T} \tilde{S}_{k,s+1,j} \Phi_{k,k-s+1}^{-1} = \Phi_{k,L}^{-1} \sum_{j=1}^{\bar{\eta}} \sum_{s=1}^{\bar{L}+1} \Phi_{k,k-L+1}^{-T} \tilde{S}_{k,s+1,j} \Phi_{k,k-L+1}^{-1} \geq a_{\min} \eta^{\bar{L}-1} \alpha \beta I_n, \forall k \geq N + \bar{\eta}.$$  

Summing up (14) and (15) yields

$$\tilde{S}_{k,i} \geq a_{\min} \eta^{\bar{L}-1} \alpha \beta I_n, \forall k \geq N + \bar{\eta}.$$  

Let $S_k(\alpha) = a_{\min} \eta^{\bar{L}-1} \alpha \beta I_n$. In light of (13), it holds that $P_{k,i}^{-1} = S_{k,i} \geq S_k(\alpha), \forall k \geq N + \bar{\eta}$. Hence, $\sup_{k \geq \bar{L}} P_{k,i} \leq S_k(\alpha)$. Since the filter is conditionally consistent, $\sup_{k \geq \bar{L}} E\{(\tilde{x}_{k,i} - x_i)(\tilde{x}_{k,i} - x_i)^T | W_k\} \leq S_k^{-1}(\alpha)$. Taking mathematical expectation on its both sides and denoting $\bar{\eta} = \eta^{\frac{1}{\alpha^2}} > 0$, the conclusion of this theorem holds.

**Theorem 4.1** thus states that mean square boundedness of Algorithm 1 is guaranteed under mild conditions, including robust collective observability and strong connectivity. Theorem 4.1 states also that the mean square error of Algorithm 1 can be upper bounded after a finite time $N + \bar{\eta}$, where $N$ is a parameter in the assumption of robust collective observability and $N$ is the network size. Besides, it reveals that a larger $\alpha$ can lead to a smaller bound of mean square error. Thus, increasing observability (w.r.t. $C_{k,i}$) and reducing noise interference (w.r.t. $R_{k,i}$) can both contribute to improving estimation performance.

5. DISTRIBUTED FUSION: A SLIDING WINDOW METHOD

Since the estimates $\{\tilde{x}_{k,i,j}, \tilde{P}_{k,i,j} \in \mathcal{N}_i\}$ obtained by agent $i$ have been corrupted by channel noise through (2), designing a distributed filter simply based on latest estimates may lead to performance degradation if these estimates have been seriously deteriorated. In this case, instead of using the scheme in Section 3 and many existing literature [19–21, 23–27], we consider to fuse the past received estimates from neighbors, which may lead to a better estimate than that of simply fusing current estimates. We set a sliding window with
length $L \geq 1$ to cover which period that the past estimates can be employed. For $l = 0, \ldots, L$, we denote

$$
\hat{x}_{k-l,j} := \hat{x}_{k-l,i,j},
\hat{P}_{k-l,j} := \hat{P}_{k-l,i,j} + D_{i,j} + Y_{i,j},
$$

(17)

By Lemma 3.3, $\{\hat{x}_{k,j}, \hat{P}_{k-j}\}$ is conditionally consistent given $\mathcal{V}_k = \sigma(\hat{D}_{t;i,j}, 1 \leq t \leq k, i, j, \in \mathcal{V})$. If the length of sliding window is $L$, then from times $k - L + 1$ to $k$, agent $i$ will have the messages $\{\hat{x}_{t;i,j}, \hat{P}_{t;i,j}\}_{t=k-L+1}^{t=k-1}$ from agent $j$. We denote

$$
\begin{align*}
&\langle \hat{x}_{k,j}^{(1)}, \hat{P}_{k,j}^{(1)} \rangle := (f_0(\hat{x}_{k,j}), g_0(\hat{P}_{k,j})) := (\hat{x}_{k,j}, \hat{P}_{k,j}) \\
&\langle \hat{x}_{k,j}, \hat{P}_{k,j} \rangle := (f_l(\hat{x}_{k-1,j}), g_1(\hat{P}_{k-1,j})) \\
&\vdots \\
&\langle \hat{x}_{k,j}^{(L)}, \hat{P}_{k,j}^{(L)} \rangle := (f_{L-1}(\hat{x}_{k-L+1,j}), g_{L}(\hat{P}_{k-L+1,j}),
\end{align*}
$$

(18)

where for $l = 1, \ldots, L - 1,

$$
\begin{align*}
&f_l(\hat{x}_{k-1,j}) = f_l(f_{l-1}(\hat{x}_{k-1,j})) \\
&g_l(\hat{P}_{k-1,j}) = g_l(g_{l-1}(\hat{P}_{k-1,j})) \\
&f_1(\hat{x}_{k-1,j}) = A_{k-1} \hat{x}_{k-1,j} \\
&g_1(\hat{P}_{k-1,j}) = A_{k-1} \hat{P}_{k-1,j} A_{k-1}^T + Q_{k-1} + \mu_k F_k^{-1} \Pi_k F_k^{-T}.
\end{align*}
$$

(19)

Then we obtain the predicted pairs in Table I by employing $\{\hat{x}_{t;i,j}, \hat{P}_{t;i,j}\}_{t=k-L+1}^{t=k-1}$.

Thus, at time $k$, based on the local knowledge and the information received from neighbors, agent $i$ can fuse the messages $\{\hat{x}_{t;i,j}, \hat{P}_{t;i,j}, j \in \mathcal{V}_i\}_{t=k-L+1}^{t=k-1}$ to obtain a better estimate of $x_k$. By Lemma 3.1, $\{\hat{x}_{t;i,j}, \hat{P}_{t;i,j}, j \in \mathcal{V}_i\}_{t=k-L+1}^{t=k-1}$ are all conditionally consistent given $\mathcal{W}_k = \sigma(\hat{D}_{t;i,j}, 1 \leq t \leq k, i, j, \in \mathcal{V})$.

### Table I
**Predicted messages based on past received pairs**

| Messages in the window $(k - L, k]$ | Messages predicted up to time $k$ |
|------------------------------------|---------------------------------|
| $\langle \hat{x}_{k,j}, \hat{P}_{k,j} \rangle$ | $\langle \hat{x}_{k,j}^{(1)}, \hat{P}_{k,j}^{(1)} \rangle$ |
| $\langle \hat{x}_{k-1,j}, \hat{P}_{k-1,j} \rangle$ | $\langle \hat{x}_{k-1,j}^{(2)}, \hat{P}_{k-1,j}^{(2)} \rangle$ |
| $\vdots$ | $\vdots$ |
| $\langle \hat{x}_{k-L+1,j}, \hat{P}_{k-L+1,j} \rangle$ | $\langle \hat{x}_{k-L+1,j}^{(L)}, \hat{P}_{k-L+1,j}^{(L)} \rangle$ |

Let

$$
\hat{x}_{k,i} = P_{k,i} \sum_{s=1}^{L} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1} \hat{x}_{k,j}^{(s)},
$$

(20)

$$
P_{k,i} = \sum_{s=1}^{L} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1},
$$

(21)

where $a_{s,i,j,k}^{(s)}$ is the $(i,j)$th element of $\bar{\bar{A}}_k \in \mathbb{R}^{N \times NL}$ which is the CI fusion weight matrix for $\{\hat{x}_{k,j}^{(s)}, \hat{P}_{k,j}^{(s)}, j \in \mathcal{V}_i\}_{t=k-L+1}^{t=k-1}$. In the following, the design of $\bar{\bar{A}}_k$ is studied. By the proof of Lemma 3.3 and (18), $\{\hat{x}_{k,j}^{(s)}, \hat{P}_{k,j}^{(s)}, j \in \mathcal{V}_i\}_{t=k-L+1}^{t=k-1}$ are conditionally consistent given $\mathcal{W}_k = \sigma(\hat{D}_{t;i,j}, 1 \leq t \leq k, i, j, \in \mathcal{V})$. Thus, we aim to design the CI fusion weights to improve the estimation performance in the sense of obtaining a smaller $P_{k,i}$. Based the method in our previous work [27], the design of $\bar{\bar{A}}_k$ is given by solving the following optimization problem

**Problem 1**

$$
\{a_{s,i,j,k}^{(s)} | j \in \mathcal{N}_i, s = 1, \ldots, L\} = \arg \min a_{s,i,j,k}^{(s)} \Delta P_{k,i}^{-1},
$$

(22)

subject to $\Delta P_{k,i} > 0, 0 \leq a_{s,i,j,k}^{(s)} \leq 1, \sum_{s=1}^{L} \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} = 1, a_{s,i,j,k}^{(s)} = 1$, where

$$
\Delta P_{k,i} = \sum_{s=1}^{L} \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1} - \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} P_{k,i}^{-1}.
$$

(23)

The idea is to design $a_{s,i,j,k}^{(s)}$ so as to obtain a smaller fused matrix $P_{k,i}$. According to [27], Problem 1 is convex and equivalent to a semidefinite programming problem, which can be effectively solved by many existing algorithms if the problem is feasible. Otherwise, we can choose $\bar{\bar{A}}_k = (A \in \mathbb{R}^{N \times (N-1)L})$ to maintain the running of our algorithm. However, due to practical limitations such as battery capability and energy consumption for sensor networks, it is not desirable to solve the online optimization problem (22) at each time. Suppose each agent have the ability to solve (22) at time instants $\{k_s\}_{s=1}^{\infty}$, subject to

$$
\text{mod}(k_s, \Delta_k, i) = 0
$$

where $\text{mod} (a, b)$ is the remainder operator of $\frac{a}{b}$ and $\Delta_k, i \in \mathbb{Z^+}$ is the interval length where agent $i$ can not carry out the optimization problem (22). In other words, at time instants, $\{k_s\}_{s=1}^{\infty}$, each agent employs (20) to obtain a fused estimate. For other instants, it utilizes the fusion methods in Algorithm 1 based on the latest estimates from neighbors. Then, we provide the distributed robust Kalman filter with sliding-window fusion in Algorithm 2.

**Algorithm 2** Distributed robust Kalman filter with sliding-window fusion (DRKF-SWF):

**Prediction:** Each agent carries out a state predictor

$$
\hat{x}_{k,i} = A_{k-1} \hat{x}_{k-1,i},
\hat{P}_{k,i} = A_{k-1} \hat{P}_{k-1,i} A_{k-1}^T + \mu_k F_k^{-1} \Pi_k F_k^{-T} + Q_{k-1},
$$

(21)

**Update:** Each agent uses its own measurements to update the estimation

$$
\hat{x}_{k,i} = \hat{x}_{k,i} + K_{k,i}(y_{k,i} - \tau_{k,i} C_k x_{k,i}),
\hat{P}_{k,i} = \tau_{k,i} \hat{P}_{k,i} + \tau_{k,i} C_k C_k^T + R_{k,i} + \varphi_{k,i} C_k \Pi_k C_k^T.
$$

(22)

**Local Fusion:** Each agent fuses the corrupted pair $\{\hat{x}_{k,i}, \hat{P}_{k,i}\}$ given in (2):

if $\text{mod}(k, \Delta_k, i) = 0$

$$
\hat{x}_{k,i} = P_{k,i} \sum_{s=1}^{L} \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1} \hat{x}_{k,j}^{(s)},
\hat{P}_{k,i} = (\sum_{s=1}^{L} \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1})^{-1},
$$

(23)

where $\hat{x}_{k,j}^{(s)}, \hat{P}_{k,j}^{(s)}$ are given in (18), and $\{a_{s,i,j,k}^{(s)}\}_{s=1}^{L}$ are given by solving Problem 1;

else

$$
\hat{x}_{k,i} = P_{k,i} \sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1} \hat{x}_{k,j}^{(s)},
\hat{P}_{k,i} = (\sum_{j \in \mathcal{N}_i} a_{s,i,j,k}^{(s)} (\hat{P}_{k,j}^{(s)})^{-1})^{-1}.
$$

(24)
The following lemma shows that Algorithm 2 is also conditionally consistent.

**Lemma 5.1.** Consider the multi-agent system (1) satisfying Assumption 2.1. For Algorithm 2, the pairs \( \{\tilde{x}_{k,i}, \tilde{P}_{k,i}\}, \{\tilde{x}_{k,i}, \tilde{P}_{k,i}\} \) are conditionally consistent given \( \mathcal{W}_k \).

**Proof.** It is similar as the proof of Lemma 3.3 by considering the CI fusion in (20) and the fact that \( K_{k,i} \) is adapted to \( \mathcal{W}_k = \sigma(\mathcal{D}_{t,i,j}, 1 \leq t \leq k, i, j, \in \mathcal{V}) \).

The relationship between Algorithm 1 and Algorithm 2, we provide the following lemma.

**Lemma 5.2.** Under the same \( \sigma \)-algebra \( \mathcal{W}_k = \sigma(\mathcal{D}_{t,i,j}, 1 \leq t \leq k, i, j, \in \mathcal{V}) \), for Algorithms 1 - 2, it holds that

\[
P_{k,i}^A \preceq P_{k,i}^B
\]

where \( P_{k,i}^A \) and \( P_{k,i}^B \) are the \( P_{k,i} \) matrix of Algorithm 2 and Algorithm 1, respectively. Furthermore, let \( T \) be the time length of interest, then Algorithm 2 degenerates to Algorithm 1 if \( \Delta_{k,i} > T \).

**Proof.** The proof can be conducted by an inductive way. For convenience, we simply show the case at the time instants \( \{k\} \) subject to \( \text{mod}(k, \Delta_{k,i}) = 0 \). If Problem 1 is feasible, the constraint of Problem 1 \( \Delta P_{k,i} > 0 \) ensures that Algorithm 2 has a smaller \( R_{k,i} \). Otherwise, the setting \( A_k = (A \times 0^{N \times (N-1)}) \) guarantees (24). If \( \Delta_{k,i} > T \), there is no time instant \( k \) such that \( \text{mod}(k, \Delta_{k,i}) = 0 \). Thus, the fusion scheme of Algorithm 2 is the same as Algorithm 1.

Lemma 5.2 shows that compared to Algorithm 1 fusing latest estimates, Algorithm 2 employing the sliding-window method has a smaller upper bound of the mean square error at each time instant by paying more computational resources for solving an online optimization problem. A larger sliding window parameter \( L \) can lead to a smaller objective function of Problem 1, yet as the increase of optimization variables and prediction operations in (19), the computation will increase as well. The time length \( \Delta_{k,i} \) also influences the estimation performance, since a larger \( \Delta_{k,i} \) requires that agent \( i \) stops obtaining adaptive fusion weights for a longer time. In practical applications, the parameters \( L \) and \( \Delta_{k,i} \) can be chosen based on the computational and communication ability of agents. The mean square boundedness of estimation error for Algorithm 2 is investigated in the following.

**Theorem 5.1.** Consider the multi-agent system (1) satisfying Assumptions 2.1, 4.1 - 4.2. For Algorithm 2, if the directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) is strongly connected, then there exists a positive scalar \( \tilde{\eta} \) such that

\[
\sup_{k \leq N+\tilde{\eta}} \lambda_{\max}(E(\tilde{e}_{k,i}^T \tilde{e}_{k,i})) \leq \frac{\tilde{\eta}}{\alpha}, \forall i \in \mathcal{V},
\]

where \( \alpha \) is given in Assumption 4.1.

**Proof.** By Lemma 5.2, the conclusion is proved by referring to the proof of Theorem 4.1.

In this section, we study an example to validate the effectiveness of the proposed algorithm and the theoretical results developed in this paper. For the temperature field in Fig. 1, we suppose that the initial state \( x_0 \) and measurement noise of agents are generated by independent standard normal distributions. The fading factors \( \gamma_{k,i} \) follow independent uniform distributions, \( i = 1, 2, 3, 4 \). The time sequence \( \{t_k\} \) lies in the interval \([0, 10]\) with uniform sampling step 0.1. The statistics of system noise and fading factors are known. The matrices and scalars corresponding to (1) are assumed to be

\[
A_k = \begin{pmatrix}
1.05 \times (1 - 0.01 \times t_k) & -0.1 \\
0.1 & 0.98
\end{pmatrix}
\]

\[
F_k = I_4, Q_k = 0.1 \times I_2, P_0 = I_2, \mu_k = 0.1 \times (t_k + 2)^{-1}
\]

\[
R_{k,1} = 0.07, R_{k,2} = 0.08, R_{k,3} = R_{k,4} = 0.09
\]

\[
\tau_{k,1} = 0.85, \varphi_{k,1} = 0.8 \times 10^{-3}, C_{k,1} = (0, 1)
\]

\[
\tau_{k,2} = 0.15, \varphi_{k,2} = 0.8 \times 10^{-3}, C_{k,2} = (0, 1)
\]

\[
\tau_{k,3} = 0.20, \varphi_{k,3} = 0.8 \times 10^{-3}, C_{k,3} = (0, 1)
\]

\[
\tau_{k,4} = 0.85, \varphi_{k,4} = 0.8 \times 10^{-3}, C_{k,4} = (1, 0)
\]

The initial estimate setting is \( \hat{x}_{i,0} = I_2 \) and \( P_{i,0} = 100 \times I_2 \), \( \forall i \in \mathcal{V} \). The weighted adjacency matrix is

\[
A = [a_{i,j}] = \begin{pmatrix}
0.3 & 0.7 & 0 & 0 \\
0 & 0.4 & 0.6 & 0 \\
0 & 0.3 & 0.7 & 0 \\
0.4 & 0.4 & 0 & 0.3
\end{pmatrix}
\]

The channel noise is assumed to be mutually independent and uniformly distributed over \([-1, 1]\). Then we choose \( Y_{i,j} = \mathcal{D}_{i,j} = I_2, i, j \in \mathcal{V} \). We conduct Monte Carlo experiments, in which \( N_t = 100 \) Monte Carlo trials are performed. We
Fig. 3. Comparison of tracking performance for the proposed filter DRKF together with filters from the literature.

Fig. 4. Consistent estimates of DRKF-SWF with $L = 2$ and $\Delta_{k,i} = 5$.

Fig. 5. Comparison of DRKF and DRKF-SWF.

denote

$$MSE_k = \frac{1}{|V|} \sum_{i \in V} \sum_{j=1}^{N_i} (\hat{x}_{k,i,j} - x_k)^T (\hat{x}_{k,i,j} - x_k)$$

$$\text{Tr}(P_k) = \frac{1}{|V|} \sum_{i \in V} \sum_{j=1}^{N_i} \text{Tr}(P_{k,i,j}),$$

(25)

where $\hat{x}_{k,i,j}$ and $P_{k,i,j}$ are the state estimate and parameter matrix of the $j$th trail of agent $i$ at the $k$th time instant. For better illustration, instead of providing conditional consistency, we focus on consistency, i.e., we show that $\text{Tr}(P_k)$ is an upper bound of $MSE_k$ defined in (25).

The mean square boundedness and consistency of DRKF are given in Fig. 2, which shows that the mean square error keeps bounded and the consistency of estimates holds for each agent. We compare the proposed DRKF with some other algorithms including centralized Kalman filter (CKF), centralized robust Kalman filter (CRKF), collaborative scalar-gain estimator (CSGF) [30] and distributed state estimation with consensus on the posteriors (DSEA-CP) [32]. Note that CRKF can be obtained from DRKF by removing the fusion stage and utilizing all observations of the agents. The MSEs of the algorithms are shown in Fig. 3, which indicates that DRKF achieves better estimation accuracy than CSGF, DSEA-CP and DRKF. Fig. 4 indicates that the DRKF-SWF provides bounded mean square estimation errors and consistent estimates for each agent. By setting $\Delta_{k,i} = \Delta_k$, $i \in V$, Fig. 5 shows that the proposed DRKF-SWF with sliding-window length $L = 2$ provides smaller upper bounds than DRKF by decreasing the interval length $\Delta_k$.

7. Conclusion

This paper studied the distributed robust state estimation problem for a class of discrete-time stochastic multi-agent systems with multiplicative noise and degraded measurements over corrupted communication channels. Employing local imprecise statistics, we first proposed a three-staged distributed robust Kalman filter. Then, under some mild conditions, we proved that the mean square errors are uniformly bounded by a constant matrix after a finite transient time. The finite time is related with the collective observability and the network size. A switching fusion scheme based on a sliding-window fusion method was proposed to obtain a smaller upper bound of the mean square error by considering the computational ability of agents.

Appendix A

Proof of Lemma 3.1

According to the system dynamics (1) and filter (3), in the prediction stage, we have $\epsilon_{k,i} = \bar{x}_{k,i} - x_k = A_{k-1} \epsilon_{k-1,i} - w_{k-1} - \epsilon_{k-1,F_{k-1}}x_{k-1}$. In the filtering update stage, one can obtain $\hat{x}_{k,i} = \hat{x}_k + K_{k,i} (y_i - C_{k,i} \epsilon_{k,i} + u_k - \tau_{k,i} C_{k,i} \bar{x}_{k,i}) = (I_n - \tau_{k,i} K_{k,i} C_{k,i}) \hat{x}_k + K_{k,i} u_k + (\gamma_{k,i} - \tau_{k,i}) K_{k,i} C_{k,i} \bar{x}_k$. The transmission error satisfies $\epsilon_{k,i,j} = \bar{x}_{k,j} + \epsilon_{k,i,j} - x_k = \epsilon_{k,j} + \epsilon_{k,i,j}$. In the local fusion stage, $\epsilon_{k,i} = \sum_{j \in N_i} W_{k,i,j} \epsilon_{k,i,j}$.
APPENDIX B
PROOF OF LEMMA 3.2
Proof. We use an inductive method to prove this lemma. At the initial time instant, it follows from Assumption 2.1 that $E\{x_0x_0^T\} \leq P_0 = \Pi_0$. Suppose at time $k$ that $E\{x_kx_k^T\} \leq \Pi_k, \forall k \geq 0$. According to (1), $x_k$ is adapted to $F_{k-1}$. By Assumption 2.1, we have $E\{e_kx_k\} = 0$, and $E\{u_kx_k\} = 0$. For $E\{x_{k+1}^2x_k^T\}$, it holds that $E\{x_kx_kx_k^T\} = E\{x_k^2\}E\{x_kx_k^T\}$, then

\[
E\{x_{k+1}x_k^T\} = E\{(A_k + F_k)x_kT\}(A_k + F_kx_k)^T + E\{w_kx_k^T\} + E\{\{A_k + F_kx_k\}x_k^T\}(A_k + F_kx_k)^T
\]
\[
\leq A_kE\{x_kx_k^T\}A_k^T + E\{x_k^2\}F_kE\{x_k^T\}F_k^T + E\{w_kw_k^T\}
\]
\[
\leq A_k\Pi_k A_k^T + \mu_k F_k\Pi_k F_k^T + Q_k = \Pi_{k+1}.
\]
Hence, we obtain $E\{x_{k+1}x_k^T\} \leq \Pi_{k+1}$.

APPENDIX C
PROOF OF LEMMA 3.3
Proof. At the initial time instant, under Assumption 2.1, we have $E\{|\bar{x}_{k-1} - x_0\|^2|W_0\} = E\{|\bar{x}_{k-1} - x_0\|^2|W_0\} \leq P_0$. To finish the proof, we use an inductive method. Suppose at the $(k-1)$th time instant, $E\{|\bar{x}_{k-1} - x_{k-1}\|^2|W_{k-1}\} \leq P_{k-1}$, for which the proof is similar to Appendix B. Therefore, we have

\[
E\{|\bar{x}_{k-1} - x_{k-1}\|^2|W_{k-1}\} \leq P_{k-1},
\]

In the measurement update stage, according to Lemma 3.1, we have $E\{\bar{x}_{k-1} - x_{k-1}\} = 0$, and $E\{\bar{x}_{k-1}^T \bar{x}_{k-1}^T|W_0\} = 0$. Since $v_{k,i}$ and $\gamma_{k,i}$ are mutually independent and $K_{k,i}$ is adapted to $W_k$, we have

\[
E\{\{\bar{x}_{k-1} - x_{k-1}\}^T (\bar{x}_{k-1} - x_{k-1})\}|W_k\}
\]
\[
\leq A_{k-1}E\{e_{k-1}e_{k-1}^T|W_{k-1}\}A_{k-1}^T + \mu_{k-1} F_{k-1}E\{x_{k-1}x_{k-1}^T\}F_{k-1}^T
\]
\[
\leq A_{k-1}E\{e_{k-1}e_{k-1}^T|W_{k-1}\}A_{k-1}^T + \mu_{k-1} F_{k-1}\Pi_{k-1}F_{k-1}^T + Q_{k-1} = \tilde{P}_{k-1}.
\]

As the communication channels are imperfect and the messages received by each agent are polluted by the channel noise in (2). According to Assumption 2.1 and Lemma 3.1, we have

\[
E\{\{\tilde{x}_{k-1} - x_{k-1}\}(\tilde{x}_{k-1} - x_{k-1})\}|W_k\}
\]
\[
\leq E\{\{\tilde{x}_{k-1} - x_{k-1}\}(\tilde{x}_{k-1} - x_{k-1})\}|W_k\} + E\{e_{k-1}e_{k-1}^T|W_k\}
\]
\[
\leq \tilde{P}_{k-1} + \text{sup}\{e_{k-1},\tilde{e}_{k-1}\}.
\]

APPENDIX D
PROOF OF LEMMA 3.4
Proof. According to Lemma 3.3, we have

\[
\tilde{P}_{k,i} = (I_n - \tau_{k,i} K_{k,i} C_{k,i})\tilde{P}_{k,i}(I_n - \tau_{k,i} K_{k,i} C_{k,i})^T
\]
\[
+ K_{k,i} (\varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}) K_{k,i}^T
\]
\[
= P_{k,i} - \tau_{k,i} K_{k,i} C_{k,i} \tilde{P}_{k,i} - \tau_{k,i} \tilde{P}_{k,i} C_{k,i} K_{k,i}^T
\]
\[
+ \tau_{k,i}^2 K_{k,i} C_{k,i} \tilde{P}_{k,i} C_{k,i}^T K_{k,i}^T
\]
\[
+ K_{k,i} (\varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}) K_{k,i}^T
\]
\[
= P_{k,i} - \tau_{k,i} K_{k,i} C_{k,i} \tilde{P}_{k,i} - \tau_{k,i} \tilde{P}_{k,i} C_{k,i} K_{k,i}^T
\]
\[
+ \tau_{k,i}^2 K_{k,i} C_{k,i} \tilde{P}_{k,i} C_{k,i}^T K_{k,i}^T
\]
\[
+ K_{k,i} (\varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}) K_{k,i}^T
\]

where $K_{k,i}^* = \tau_{k,i} K_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}$ and $\varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}$ is minimized when $K_{k,i}^* = \varphi_{k,i} C_{k,i} \Pi_k C_{k,i}^T + R_{k,i}$. Thus, (27) shows that $\text{Tr(}\tilde{P}_{k,i})(k)$ is minimized. As a result, $\tilde{P}_{k,i} = (I - \tau_{k,i} K_{k,i} C_{k,i})\Pi_k C_{k,i}^T$. Since $K_{k,i}^*$ is a measurable function of $\tilde{P}_{k,i}$, which is adapted to $W_k$. Thus, $K_{k,i}^*$ is adapted to $W_k$.

APPENDIX E
PROOF OF LEMMA 4.1
According to Assumption 2.2, we have $\Pi_{k+1} = A_k \Pi_k A_k^T + \mu_k F_k \Pi_k F_k^T + Q_k$. Taking 2-norm operator on the both sides yields $\|\Pi_{k+1}\|_2 \leq \|\Pi_k\|_2 (\|A_k\|_2 + \mu_k \|F_k\|_2) + \|Q_k\|_2$. Denote $A_{k,i} = \|A_k\|_2$ and $F_k = \|F_k\|_2$. Let $\varphi_k C_{k,i} \Pi_k C_{k,i}^T = R_k C_{k,i}^T$. If (6) is satisfied, (10) will be satisfied.

APPENDIX F
PROOF OF LEMMA 4.2
Proof. According to Lemma 3.2 and the notations in Assumption 2.2, we have $\Pi_{k+1} = \Phi_{k+1, k} \Pi_k \Phi_{k+1, k}^T + Q_{k+1} + \mu_k \Phi_{k+1, k} F_k \Pi_k F_k^T \Phi_{k+1, k}^T$. Multiplying the left-hand side of $\Pi_{k+1}$ by $\mu_k F_k \Pi_k$ and the right-hand side of $\Pi_{k+1}$ by $F_k^T \Phi_{k+1, k}^T$ yields

\[
\mu_k F_k \Pi_k \Phi_{k+1, k}^T + Q_{k+1}
\]
\[
= \mu_k F_k \Pi_k \Phi_{k+1, k}^T + Q_{k+1}
\]
\[
+ \mu_k F_k \Pi_k \Phi_{k+1, k}^T F_k^T \Phi_{k+1, k}^T
\]
\[
+ \mu_k F_k \Pi_k \Phi_{k+1, k}^T F_k^T \Phi_{k+1, k}^T
\]
\[
= \mu_k F_k \Pi_k \Phi_{k+1, k}^T + Q_{k+1}
\]
\[
+ \mu_k F_k \Pi_k \Phi_{k+1, k}^T F_k^T \Phi_{k+1, k}^T
\]
\[
= \mu_k F_k \Pi_k \Phi_{k+1, k}^T + Q_{k+1}
\]
\[
+ \mu_k F_k \Pi_k \Phi_{k+1, k}^T F_k^T \Phi_{k+1, k}^T
\]
According to (40), the conditions (8) and (9), we have

\[ \Delta_{k+1} = \sum_{t=1}^{k+1} e_t^2 \]

where \( Q_{k+1-k} = \sum_{t=k}^{k+1} e_t^2 \). Denote

\[ \Delta_{k+1} = \frac{1}{\mu_{k+1}} F_{k+1} T_{k+1-k} F_{k+1} T_{k+1-k} \]

Taking 2-norm operator on both sides of (28) yields

\[ \left\| \Delta_{k+1} \right\|_2 \leq \left\| \frac{1}{\mu_{k+1}} F_{k+1} T_{k+1-k} F_{k+1} T_{k+1-k} \right\|_2 \]

\[ \leq \mu_{k+1} \left\| \Delta_{k+1} \right\|_2 + \left\| \frac{1}{\mu_k} Q_{k+1-k} F_{k+1} T_{k+1-k} \right\|_2 \]

\[ \leq \mu_{k+1} \left\| \Delta_{k+1} \right\|_2 + \left\| \mu_{k+1} F_{k+1} T_{k+1-k} \right\|_2 \]

According to (40), the conditions (8) and (9), we have \( \sup_{k_t \in N} \left\| \Delta_{k_t} \right\|_2 < \infty \), i.e., \( \Delta_{k_t} \) is uniformly upper bounded. □

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