Evaporating Quantum Lukewarm Black Holes Final State From Back-Reaction Corrections of Quantum Scalar Fields

H. Ghaffarnejad, H. Neyad, and M. A. Mojahedi

Department of Physics, Semnan University, P. O. Box 35195-363, Iran

Received ; accepted }

1 hghafarnejad@yahoo.com.
2 hsniad@yahoo.com.
3 amirmojahed2@gmail.com.
We obtain renormalized stress tensor of a mass-less, charge-less dynamical quantum scalar field, minimally coupled with a spherically symmetric static Lukewarm black hole. In two dimensional analog the minimal coupling reduces to the conformal coupling and the stress tensor is found to be determined by the nonlocal contribution of the anomalous trace and some additional parameters in close relation to the work presented by Christensen and Fulling. Lukewarm black holes are a special class of Reissner-Nordström-de Sitter space times where its electric charge is equal to its mass. Having the obtained renormalized stress tensor we attempt to obtain a time-independent solution of the well known metric back reaction equation. Mathematical derivations predict that the final state of an evaporating quantum Lukewarm black hole reduces to a remnant stable mini black hole with moved locations of the horizons. Namely the perturbed black hole (cosmological) horizon is compressed (extended) to scales which is smaller (larger) than the corresponding classical radius of the event horizons. Hence there is not obtained an deviation on the cosmic sensor-ship hypothesis.

*Subject headings:* Hawking Radiation; Lukewarm Black hole; Back reaction equation; Reissner Nordström de Sitter; Noncommutative quantum gravity; stability
1. Introduction

Semiclassical approach of quantum gravity theory is known as quantum matter field theory propagated on a curved space-time, in which a classically treated curved space-time is perturbed by a suitable quantum matter field (Birrell and Davies 1982). A fundamental problem in this version of the quantum gravity theory, is calculation of renormalized expectation value of quantum matter stress tensor operator $< \hat{T}_{\mu\nu} >_{ren}$. Renormalization theory give us a suitable theoretical prediction, in which expectation value of a singular quantum field stress tensor operator reduces to a nonsingular quantity contained an anomalous trace. This nonsingular stress tensor treats as source in RHS of the Einstein's gravity equation such as follows.

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi \{ T^{class}_{\mu\nu} + < \hat{T}_{\mu\nu} >_{ren} \}$$  \hspace{1cm} (1)

where $G_{\mu\nu}$ is Einstein tensor with the perturbed metric $g_{\mu\nu} = \hat{g}_{\mu\nu} + \Delta g_{\mu\nu}$ and the background metric $\hat{g}_{\mu\nu}$, $\Lambda$ is positive cosmological constant and $T^{class}_{\mu\nu}$ is classical baryonic matter or non-baryonic dark matter field stress tensor. Non-minimally coupled scalar dark matter fields with a negative value of equation of state parameter may to be come originally from effects of conformal frames. The latter case of the matter is a good candidate to explain positivity accelerated expansion of the universe and to remove the naked singularity of the universe in quantum cosmological approach. See (Nozari and Sadatian 2009) and references therein. The above equation which is written in units $G = h = c = 1$ is called the metric back-reaction equation. There are presented several methods for the renormalization prescription, namely dimensional regularization, point splitting, adiabatic and Hadamared renormalization prescriptions (Birrell and Davies 1982). The latter method has distinctions with respect to the other methods of the renormalization prescriptions. Hadamared renormalization prescription is described in terms of Hadamared states and it predicts few conditions on unknown quantum vacuum state of an arbitrary interacting quantum
field (Brown 1984; Bernard and Folacci 1986; Ghafarnejad and Salehi 1997). Hence it provides the most direct and logical approach to the renormalization problem for practical calculations. Furthermore it is well defined for both massive and massless fields.

Renormalization theory is still establish the covariant conservation of stress tensor operator expectation value of quantum field contained with a non vanishing trace anomaly, namely $\nabla^\nu < \hat{T}_{\mu \nu} >_{\text{ren}} = 0$. This anomaly is obtained in terms of geometrical objects such as $R_{\mu \nu} R^{\mu \nu}$, $R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta}$, $\Box R$, and $R^2$ for conformal coupling massless quantum field propagated on four dimensional curved space time (Christensen 1976; Adler, Liberman and Ng 1977; Wald 1978; Birrell and Davies 1982; Brown 1984; Bernard and Folacci 1986; Ghafarnejad and Salehi 1997; Parker and Toms 2009). In two dimension the conformal coupling reduces to minimal coupling and so the quantity of trace anomaly is obtained in terms of the Ricci scalar $R = R^\beta_\beta$ which for a massless scalar matter field become:

$$< \hat{T}^\mu_\mu >_{\text{ren}} = \frac{R}{24\pi}.$$  

(2)

The main problem in the equation (1) is to fined $< \hat{T}_{\mu \nu} >_{\text{ren}}$ coupled with an arbitrary non-static and non-spherically symmetric dynamical metric. But there are many degrees of freedom and inherent complexity on four dimensional solutions of equation (1). There are obtained in detail only for class of four dimensional spherically symmetric space times which are treated as two dimensional curved space times, because the spherically symmetric condition on four dimensional space times eliminates the extra degrees of freedom of Equation (1) (Christensen and Fulling 1977). Two dimensional analog of the renormalization theory and solutions of the back-reaction equation is used to determine final state of spherically symmetric dilatonic and also non-dilatonic evaporating black holes metric by several authors. For instance Strominger et al were obtained a nonsingular metric for final state of an evaporating two dimensional dilatonic massive black hole (Alwis 1992; Banks et al 1992; Callan et al 1992; Russo et al 1992; Piran and Strominger 1993). It is shown in (Lowe and O’Loughlin 1993) that an evaporating two dimensional dilatonic
Reissner-Nordström black hole reduces to a remnant, stable nonsingular space time. Evaporating dilatonic Schwarzschild de Sitter black holes final state whose size is comparable to that of the cosmological horizon is in thermal equilibrium (Bousso and Hawking 1998). It is obtained that final state of a non-dilatonic Schwarzschild-de Sitter evaporating black hole reduces to a remnant stable object with a nonsingular metric (Ghafarnejad 2006; Ghaffarnejad 2007). It is shown by Balbinot et al. that the Hawking evaporation (Hawking 1974; Hawking 1975) of the two dimensional non-dilatonic Schwarzschild black hole is stopped (Balbinot and Brown 1984; Balbinot 1984; Balbinot 1985; Balbinot 1986; Balbinot and Barletta 1989). Back reaction corrections of conformally invariant quantum scalar field in the Hartle Hawking vacuum state (Hartle and Hawking 1976) was used to determine quantum perturbed metric of a non-dilatonic Reissner Nordström black hole by Wang et al. (Wang and Huang 2001). They followed the York approach where a small quantity $\epsilon$ is introduced to solve the metric back-reaction equation (1) by applying the perturbation method (York 1985).

Furthermore noncommutative quantum field theory in curved space times and so generalized uncertainty principle derived from string theory (Amati et al. 1987, 1988, 1989, 1990; Capozziello et al. 2000; Snyder 1947; Seiberg and Witten 1999; Douglas and Nekrasov 2001), is other quantum gravity approach in which the space-time points might be noncommutative (Aschieri et al. 2005; Calmet and Kobakhidze 2005, 2006; Chamseddine 2001). The latter quantum gravity model is also predicts remnant stable mini-quantum black hole where the Hawking radiation process finishes when black hole approaches to its Planck scale with a nonzero temperature (Nicolini P. et al. 2006; Nozari and Mehdipour 2005, 2008).

According to the perturbation method presented by the York, we solve in this paper, two dimensional analog of the metric back-reaction equation (1) and determine final state of an evaporating Lukewarm black hole. This kind of a black hole is a special class of Reissner-
Nordström de sitter spherically symmetric static black hole where mass parameter is equal to the charge parameter. According to the work presented by Christensen and Fulling  
\cite{Christensen_1977} we obtain the renormalized stress tensor components of black hole Hawking radiation in terms of a nonlocal contribution of the trace anomaly. The plan of this paper is as follows.

In section 2, we define Lukewarm classical black hole metric and obtain locations of its event horizons. In section 3, we derive thermal radiation stress tensor operator expectation value of a massless, charge-less quantum matter scalar field propagating on the black hole metric. Having the obtained Hawking radiation quantum stress tensor, we solve back-reaction metric equation (1) in the section 4 and obtain locations of the quantum perturbed horizons. Section 5 denotes to the concluding remarks.

2. Lukewarm Black Hole Metric

Reissner Nordström de Sitter space times with Lorentzian line element is given by

\[ ds^2 = -\Omega(r)dt^2 + \frac{dr^2}{\Omega(r)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \]  

where

\[ \Omega(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} \]  

and \( M, Q \) are the mass and charge of the black hole respectively. \( \Lambda \) is the positive cosmological constant. Lukewarm black holes are a particular class of Reissner Nordström-de Sitter, with \( Q = M \). For \( 4M < \sqrt{3/\Lambda} \) we have three distinct horizons, namely black hole event horizon at \( r = r_h \), inner Cauchy horizon at \( r = r_{ca} \), and cosmological horizon at \( r = r_c \), where

\[ r_{ca} = \frac{1}{2} \sqrt{3/\Lambda} \left( -1 + \sqrt{1 + 4M \sqrt{\Lambda/3}} \right) \]  
\[ r_h = \frac{1}{2} \sqrt{3/\Lambda} \left( 1 - \sqrt{1 - 4M \sqrt{\Lambda/3}} \right) \]
and
\[ r_c = \frac{1}{2} \sqrt{3/\Lambda} \left( 1 + \sqrt{1 - 4M\sqrt{\Lambda}/3} \right). \]  
\(7\)

While the event horizon is formed by the gravitational potential of the black hole, the cosmological horizon is formed as a result of the expansion of the universe due to the cosmological constant (Gibbons and Hawking 1977; Breen and Ottewill 2011). An observer located between the two horizons is causally isolated from the region within the event horizon, as well as from the region outside the cosmological horizon. The above line element is exterior metric of a spherically symmetric static body with mass \(M\) and charge \(Q\). It is solution of the equation (1) under the condition \(\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} = 0\) where \(T_{\mu\nu}^{\text{class}}\) is stress tensor of classical electromagnetic field of a point charge \(Q\) and it is given in \((t, r, \theta, \varphi)\) coordinates such as follows:

\[ T^{(\text{class}) \mu \nu} = \frac{1}{8\pi} \left( \frac{Q}{r^2} \right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]  
\(8\)

In advanced time Eddington-Finkelestein coordinates \((v, r, \theta, \varphi)\) where

\[ dv = dt + \frac{dr}{\Omega(r)} \]  
\(9\)

one can obtained classical electromagnetic field stress tensor (8) such as follows.

\[ T^{\text{class}}_{vv}(v, r) = \frac{\Omega^{-1}(x) - \Omega(x)}{128M^2x^4} \]  
\(10\)

with

\[ \Omega(x) = 1 - \frac{1}{x} + \frac{q^2}{4x^2} - \frac{\varepsilon x^2}{4}, \]  
\(11\)

\[ T^{\text{class}}_{vr} = T^{\text{class}}_{rr} = \frac{\Omega^{-1}(x)}{128M^2x^4} \]  
\(12\)

and

\[ T^{\text{class}}_{\theta\theta} = -\frac{1}{32\pi x^2}, \quad T^{\text{class}}_{\varphi\varphi} = \sin^2 \theta T^{\text{class}}_{\theta\theta} \]  
\(13\)
where we defined
\[ x = \frac{r}{2M}, \quad q = \frac{Q}{M}, \quad \varepsilon = \frac{16M^2\Lambda}{3} > 0. \]  

(14)

Locations of the classical event horizons defined by (5), (6) and (7) become respectively
\[ x_{ca} = \frac{1 - \sqrt{1 + \varepsilon}}{\sqrt{\varepsilon}}, \quad x_b = \frac{1 - \sqrt{1 - \varepsilon}}{\sqrt{\varepsilon}}, \quad x_c = \frac{1 + \sqrt{1 - \varepsilon}}{\sqrt{\varepsilon}} \]  

(15)

where
\[ x_b x_c = \frac{1}{\sqrt{\varepsilon}} \]  

(16)

and in case \( 0 < \varepsilon < 1 \) we have
\[ x_{ca} \approx \frac{\varepsilon}{8} - \frac{1}{2}, \quad x_b \approx \frac{1}{2} + \frac{\sqrt{\varepsilon}}{8}, \quad x_c \approx \frac{2}{\sqrt{\varepsilon}} - \frac{1}{2} - \frac{\sqrt{\varepsilon}}{8}. \]  

(17)

Applying (11) with \( q = 1 \), we obtain locations of the horizons and quasi-flat regions of the black hole space time, from the equations \( \Omega(x) = 0 \) and \( \frac{d\Omega(x)}{dx} = 0 \) respectively. These conditions reduce to the following relations.
\[ \varepsilon_e(x) = \frac{4}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}. \]  

(18)

and
\[ \varepsilon_q(x) = \frac{2}{x^2} - \frac{1}{x^3}. \]  

(19)

Diagrams of the functions defined by (18) and (19) are given by dash-lines and solid line in figure 1, respectively. These diagrams are valid for \( 0 < \varepsilon < 1 \). In case \( \varepsilon \geq 1 \) locations of the black hole and the cosmological horizons reach to each others and so cases to instability of the black hole.

In the next section we derive the Hawking thermal radiation of a quantum Lukewarm black hole minimally coupled with a linear two dimensional, massless, charge-less, quantum scalar field. We will consider that the interacting quantum scalar field to be charge-less and so has not electromagnetic action with the classical electric field stress tensor \( T_{\mu\nu}^{\text{class}} \) defined by (8). So we can suppose that the electric charge of the black hole is not perturbed by the
quantum scalar field. Also we will assume that the quantum scalar field is propagated in s (spherically) mode on the spherically symmetric background metric (3) and so its $g_{tt}$ and $g_{rr}$ components are perturbed by the renormalized expectation value of quantum field stress tensor operator $<\hat{T}_{\mu\nu}[\hat{\phi}]>_{ren}$. Applying the latter assumption one can use two dimensional analog of the quantum field back-reaction corrections on the metric such as follows.

3. Black Hole Hawking Radiation

According to the work presented by Christensen and Fulling (Christensen and Fulling 1977) we will fined here general solution of the covariant conservation equation defined by

$$\nabla_{\nu}S^{\nu}_{\mu} = 0, \quad S^{\nu}_{\mu} = <\hat{T}^{\nu}_{\mu}>_{ren}$$

(20)

under the anomaly condition (2). Assuming $\theta, \phi = constant$, two dimensional analog of the metric (3) described in the advanced time Eddington-Finkelestein coordinates (9), become

$$ds^2 = -\Omega(r)dv^2 + 2dvdr.$$  

(21)

Applying (21) the corresponding Ricci scalar become $R = \Omega''(r)$ where the over prime $'$ denotes to differentiation with respect to radial coordinate $r$ and hence the anomaly condition (2) become

$$S^{v}_{v}(r) + S^{r}_{r}(r) - \Omega''(r)/24\pi = 0.$$  

(22)

Nonzero components of second kind Christoffel symbols are obtained as

$$\Gamma^{v}_{vv} = \frac{\Omega'(r)}{2} = -\Gamma^{r}_{vr} = \Gamma^{r}_{rv}, \quad \Gamma^{r}_{vv} = \frac{\Omega(r)\Omega'(r)}{2}.$$  

(23)

Applying (23), the covariant conservation equation defined by (20) leads to the following differential equations.

$$S^{\nu'}_{v} + \Omega'(S^{r}_{r} - S^{v}_{v})/2 - \Omega\Omega'S^{v}_{r}/2 = 0$$  

(24)
and

\[ S''_r + \Omega' S'_v/2 = 0. \quad (25) \]

Using

\[ S_v = S_v, \quad S'_v = S_r, \quad S'_r = S_v + \Omega S_{rr}, \quad S'_r = S_v + \Omega S_{rv} \quad (26) \]

with \( S_{vr} = S_{rv} \) the equations (22), (24) and (25) become respectively

\[ \Omega S_{rr} + 2 S_{vr} = \frac{\Omega''}{24\pi}, \quad (27) \]

\[ S_{vv} + \Omega S_{rv} = C_1 \quad (28) \]

and

\[ S'_{vr} + \frac{3}{2} \Omega' S_{rr} + \Omega S'_{rr} = 0 \quad (29) \]

where \( C_1 \) is integral constant. Applying (27) and (29) we obtain

\[ S_{rr}(r) = \frac{1}{\Omega^2(r)} \left\{ C_2 - \frac{1}{24\pi} \int^r \Omega(\tilde{r})\Omega''(\tilde{r}) d\tilde{r} \right\} \quad (30) \]

where \( C_2 \) is also integral constant. Using (27) and (30) one can show

\[ S_{vr}(r) = S_{rv}(r) = - \frac{C_2}{2\Omega(r)} + \frac{1}{48\pi} \left\{ \Omega''(r) + \frac{1}{\Omega(r)} \int^r \Omega(\tilde{r})\Omega''(\tilde{r}) d\tilde{r} \right\}. \quad (31) \]

Applying (28) and (31) we obtain

\[ S_{vv}(r) = C_1 + \frac{C_2}{2} - \frac{1}{48\pi} \left\{ \Omega(r)\Omega''(r) + \int^r \Omega(\tilde{r})\Omega''(\tilde{r}) d\tilde{r} \right\}. \quad (32) \]

Using (4) and (14) with \( q = 1 \), \( 0 < \varepsilon < 1 \), the stress tensor components defined by (30), (31) and (32) can be rewritten as

\[ S_{\mu\nu}(v, r) = \frac{1}{96\pi M^2} \times \left( \begin{array}{c} 48\pi M^2(2C_1 + C_2) - 2B(x) - 12A(x) \\ \frac{2B(x)+12A(x)-48\pi M^2C_2}{\Omega(x)} \\ \frac{2B(x)+12A(x)-48\pi M^2C_2}{\Omega(x)} - 12A(x) \end{array} \right) \quad (33) \]
where we defined
\[
\Omega(x) = 1 - \frac{1}{x} + \frac{1}{4x^2} - \frac{\varepsilon x^2}{4},
\]
\[
A(x) = \frac{1}{6} \int^x \Omega(\bar{x})\Omega'''(\bar{x})d\bar{x} = \frac{1}{24} - \frac{1}{4x^4} + \frac{1}{2x^4} - \frac{1}{3x^3} - \frac{\varepsilon}{8x^2} + \frac{\varepsilon}{4x}
\]
and
\[
\Omega(x)\Omega''(x) = B(x) = \frac{3}{8x^6} - \frac{2}{x^5} + \frac{7}{2x^4} - \frac{2}{x^3} - \frac{\varepsilon}{2x^2} + \frac{\varepsilon}{x} - \frac{\varepsilon}{2} + \frac{\varepsilon^2 x^2}{8}.
\]

Now we should determine the integral constants \(C_1\) and \(C_2\). For the determination of these constants we require the regularity of \(S_{\nu}^\mu\) at the black hole horizon in a coordinate system which is regular there. The stress tensor \(S_{\nu}^\mu\), as measured in a local Kruskal coordinate system at black hole horizon, will be finite if \(S_{vv}\) and \(S_{tt} + S_{rr}\), are finite as \(x \to x_b\) and

\[
\lim_{x \to x_b} (x - x_b)^{-2}|S_{uu}| < \infty,
\]

where \((u, v)\) are null coordinates (Christensen and Fulling 1977). We find easily
\[
S_{uu} = \frac{1}{4}(S_{tt} + \Omega^2 S_{rr} - 2\Omega S_{tr})
\]
where
\[
S_{tt} = S_{vv} + \Omega^2 S_{rr} - 2\Omega S_{rv}
\]
and
\[
S_{tr} = S_{rt} = S_{vr} - \Omega S_{rr}
\]
are obtained by applying (9) and definition \(S_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu S_{\alpha\beta}\). Applying (30), (38), (39) and (40) we obtain
\[
S_{uu}(x) = 156\pi M^2 C_2 + 24\pi M^2 C_1 - 27A(x) - 5B(x)/2.
\]

For a fixed \(\varepsilon\) as \(0 < \varepsilon < 1\), diagram of the figure 1 determines locations of the unperturbed black hole and cosmological horizons \(x_b, x_c\) where \(x_b < x_c\). Having this obtained black hole horizon radius \(x_b\), and (41), the initial condition \(S_{uu}(x_b) = 0\) reduces to
\[
2C_1 + 13C_2 = \frac{54A(x_b) + 5B(x_b)}{24\pi M^2}.
\]
For fields describing a gas of massless bosons (without spin, charge, or internal degrees of freedom) moved in quasi flat regions of a two dimensional curved space, the density and the flux are actually equal, so that $S^t_t(x_q) + S^r_r(x_q) = 0$, (Christensen and Fulling 1977) which in terms of the $(v, r)$ coordinates become

$$2\Omega(x_q)S_{rv}(x_q) - S_{vv}(x_q) = 0$$

(43)

where $x_q$ obtained from $\Omega'(x_q) = 0$, (see figure 1 ) defines quasi-flat regions of two dimensional version of the space time (3). Applying (33) the initial condition (43) become

$$2C_1 + 3C_2 = \frac{18A(x_c) + 3B(x_c)}{24\pi M^2}. \quad (44)$$

Using (42) and (44) one can obtain

$$C_1 = \frac{18[13A(x_c) - 9A(x_b)] + 13B(x_c) - 15B(x_b)}{480\pi M^2} \quad (45)$$

and

$$C_2 = \frac{18[3A(x_b) - A(x_c)] + 5B(x_b) - B(x_c)}{240\pi M^2}. \quad (46)$$

We are now in a position to show that the stress tensor (33) defined in the quasi flat region $x = x_q$ can be decomposed in terms of thermal equilibrium $S^{(e)}_{\mu\nu}$ and radiating $S^{(r)}_{\mu\nu}$ stress energy tensors of massless and charge-less bosonic gas respectively as

$$S^{(e)}_{\nu\mu}(t, r) = \frac{\pi}{12} T^2 \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

(47)

and

$$S^{(r)}_{\nu\mu}(t, r) = \frac{\pi}{12} T^2 b \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

(48)

where

$$\frac{T_b}{T_s} = 4\sqrt{B(x_q) + 12A(x_q) - 16.2A(x_b) + 6.4A(x_c) - 3.9B(x_c) + 4.5B(x_b)}, \quad (49)$$
and

\[
\frac{T_c}{T_S} = 2 \sqrt{54A(x_b) - 18A(x_c) + 5B(x_b) - B(x_c) - 2B(x_q) - 24A(x_q)}
\] (50)

are defined as the black hole radiation and the cosmological thermal equilibrium temperatures respectively. \( T_S = \frac{1}{8\pi M} \) is the well known Schwarzschild black hole temperature. Now we seek to obtain time-independent solutions of the back reaction equation (1) by applying (10), (11), (12) and (33) in case \( q = 1 \).

4. Back Reaction Equation

Applying the advanced-time Eddington-Finkelestein coordinates \((v, r, \theta, \varphi)\), defined by (9), the quantum perturbed metric (3) is taken to have the form

\[
ds_f^2 = -e^{2\psi(r)} F(r) dv^2 + 2e^{\psi(r)} dv dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\] (51)

with

\[
F(r) = 1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2} - \frac{1}{3} \lambda(r) r^2
\] (52)

in which \( \psi, m \) are assumed to be depended alone to the radial coordinate \( r \), because the perturbed metric should still be static and spherically symmetric. The index \( f \) denotes to the word final state of quantum perturbed evaporating Lukewarm Black hole. The perturbed metric (51) leads to the static metric (3) under the following boundary conditions:

\[
\psi(x_b; \varepsilon = 0) = 0, \quad m(x_b; \varepsilon = 0) = M, \quad \lambda(x_b; \varepsilon = 0) = \Lambda
\] (53)

where \( x_b = \frac{1}{2} \) is obtained from (15) under the condition \( \varepsilon = 0 \). Applying (51) and definitions

\[
\frac{m(r)}{M} = \rho(x), \quad \lambda(x) = \frac{3\varepsilon \sigma(x)}{16M^2}, \quad q = 1 = \frac{Q}{M}, \quad x = \frac{r}{2M}
\] (54)

the \((v, r)\) components of the Einstein’s tensor become
\[ G_{vv}(x) = -e^{2\psi(x)} \frac{1}{x^2} \left( 1 - \frac{\rho(x)}{x} + \frac{1}{4x^2} - \frac{\varepsilon \sigma(x)x^2}{4} \right) \]
\times \left( \rho'(x) + \frac{1}{4x^2} + \frac{3\varepsilon \sigma(x)x^2}{4} + \frac{\varepsilon \sigma'(x)x^3}{4} \right), \tag{55} \]
\[ G_{vr}(x) = G_{rv}(x) = e^{\psi(x)} \frac{\rho'(x)}{x^2} + \frac{\varepsilon \sigma'(x)x}{4} + \frac{1}{4x^4} + \frac{3\varepsilon \sigma(x)}{4} \] \tag{56}
\[ G_{rr}(x) = -2e^{2\psi(x)} \frac{1}{x^2} \]
\[ \frac{\rho'(x)}{x^2} + \frac{\varepsilon \sigma'(x)x}{4} + \frac{1}{16x^4} \]
\[ + \pi \frac{[1 - \Omega^2(x)]}{16x^2} + \Omega(x)[4\pi M^2(2C_1 + C_2) - A(x) - B(x)/6] = 0 \] \tag{58}
\[ \Omega(x)e^{\psi(x)} \left( \frac{1}{16x^4} + \frac{\rho'(x)}{4x^2} + \frac{\varepsilon \sigma'(x)x}{16} \right) + 4\pi M^2C_2 - \frac{\pi}{16x^2} - A(x) - \frac{B(x)}{6} = 0, \] \tag{59}
and
\[ \psi'(x) = \frac{16x^4[A(x) - 8\pi M^2C_2] - \pi \Omega(x)}{8x^3 \Omega^2(x)} \] \tag{60}
where \(\Omega(x)\) is given by (34). Solution of the equation (60) can be obtained directly by integrating. It is useful that, we obtain behavior of the solution \(\psi(x)\) at neighborhood of its singular points, namely \(x = 0\) and \(x_{b,c}\) where \(\Omega(x_{b,c}) = 0\). However we obtain
\[ \psi(x < x_b) \simeq C_{\psi} + 0.24 \ln \left( 4 - \frac{1}{x} \right) + \frac{0.3125}{4x - 1}, \] \tag{61}
\[
\psi(x \to x_b) \simeq C_\psi - \frac{2x_b^3[A(x_b) - 8\pi M^2 C_2]}{(x - x_b)} \tag{62}
\]

and

\[
\psi(x \to x_c) \simeq C_\psi + \frac{x_c^3[A(x_c) - 8\pi M^2 C_2]}{2(x_c - x)} \tag{63}
\]

where

\[
\Omega(x < x_b) \simeq \frac{1}{4x^2} - \frac{1}{x}, \quad 0 < \varepsilon < 1, \tag{64}
\]

\[
\Omega(x \to x_b) \simeq 1 - \frac{x_b}{x}, \quad \Omega(x \to x_c) \simeq 1 - \frac{x^2}{x_c^2} \simeq 2(1 - \frac{x}{x_c}) \tag{65}
\]

and \(C_\psi\) is integral constant which is determined by the initial conditions (53) such as follows.

Applying \(\psi(x_b) = 0\) where \(x_b = \frac{1}{2}\) with \(\varepsilon = 0\) the solution (61) leads to

\[
C_\psi \simeq 2.07 \times 10^{-3}, \quad e^{C_\psi} \approx 1. \tag{66}
\]

Inserting (59) the equation (58) become

\[
\frac{\rho(x)}{x} + \frac{\varepsilon x^2 \sigma(x)}{4} = \frac{H(x)}{G(x)} \tag{67}
\]

where

\[
H(x) = (1 + 4x^2)[\pi/4x^2 + 4[A(x) + B(x)/6] - 16\pi M^2 C_2]
+ \{\pi + 64\pi M^2 x^2(2C_1 - C_2)\Omega(x) - [\pi + 16x^2(A(x) + B(x)/6)]\Omega^2(x)\}e^{-\psi(x)} \tag{68}
\]

and

\[
G(x) = \pi - 64\pi M^2 C_2 x^2 + 16x^2[A(x) + B(x)/6]. \tag{69}
\]

One can rewrite the equation (59) as

\[
\frac{\rho'(x)}{x^2} + \frac{\varepsilon x \sigma'(x)}{4} = Z(x) \tag{70}
\]
where we defined
\[ Z(x) = \frac{\pi + 16x^4[A(x) + B(x)/6] - 64\pi M^2 C_2 x^4 - \Omega(x)e^\psi(x)}{4x^4 \Omega(x)e^\psi(x)}. \] (71)

Applying (67), (70) and identity
\[ \frac{2\rho(x)}{x^3} - \frac{\varepsilon\sigma(x)}{4} = \frac{\rho'}{x^2} + \frac{\varepsilon x\sigma'}{4} - \left[ \frac{1}{x} \left( \frac{\rho(x)}{x} + \frac{\varepsilon\sigma(x)x^2}{4} \right) \right]' \] (72)
we obtain
\[ \frac{2\rho(x)}{x^3} - \frac{\varepsilon\sigma(x)}{4} = Z(x) - \left[ \frac{H(x)}{xG(x)} \right]'. \] (73)

Using (67) and (73) we obtain exactly
\[ \rho(x) = \frac{x^3}{3} \left[ Z(x) + \frac{1}{x} \left( \frac{H(x)}{G(x)} \right)' \right] \] (74)
and
\[ \sigma(x) = \frac{4}{3\varepsilon} \left[ -Z(x) + \frac{1}{x} \left( \frac{H(x)}{G(x)} \right)' + \frac{H(x)}{x^2G(x)} \right]. \] (75)

Applying (35), (36), (61), (62), (63), (64), (65), and (66) one obtain
\[ H(x < x_b) \simeq 0.42 \frac{x^6}{x^b} \left( 1 - \frac{1}{x^{1.76}} \right), \quad G(x < x_b) \simeq 9.3 \frac{x^3}{x^3} \left( \frac{0.18}{x} - 1 \right), \] (76)
\[ H(x \to x_b) \simeq \pi \exp \left\{ \frac{2x^3[A(x_b) - 8\pi M^2 C_2]}{(x - x_b)} \right\} \] (77)
\[ H(x \to x_c) \simeq (1 + 4x^2)[\pi/4x^2_c + 4A(x_c) + 2B(x_c)/3 - 16\pi M^2 C_2] \times \pi \exp \left\{ -\frac{x^2[A(x_c) - 8\pi M^2 C_2]}{2(x_c - x)} \right\} \] (78)
\[ G(x \to x_{b,c}) = \pi - 64\pi M^2 C_2 x_{b,c}^2 + 16x_{b,c}^2[A(x_{b,c}) + B(x_{b,c})/6], \] (79)
\[ Z(x < x_b) \simeq 2.27 \frac{x^3}{x^3} \left( 1 - \frac{0.11}{x} \right), \] (80)
Applying (77), (79) and (81) the equations defined by (73) and (75) become respectively

\[
Z(x \to x_b) \simeq \left[ \frac{\pi}{4x_b^3} + 4x_b[A(x_b) + B(x_b)/6] - 16\pi M^2 C_2 x_b \right] \\
\times (x - x_b)^{-1} \exp \left\{ \frac{2x_b^3 [A(x_b) - 8\pi M^2 C_2]}{(x - x_b)} \right\} 
\]

(81)

and

\[
Z(x \to x_c) \simeq \left[ \frac{\pi}{8x_c^3} + 2x_c[A(x_c) + B(x_c)/6] - 8\pi M^2 C_2 x_c \right] \\
\times (x_c - x)^{-1} \exp \left\{ - \frac{x_c^3 [A(x_c) - 8\pi M^2 C_2]}{2(x_c - x)} \right\} 
\]

(82)

Using (76) and (80), the equations defined by (74) and (75) become respectively

\[
\rho(x < x_b) \simeq 0.76 \left(1 - \frac{0.11}{x}\right) - \frac{2.51 \times 10^{-3}}{x^{5.52}(0.18 - x)^2} 
\]

(83)

and

\[
\sigma(x < x_b) \simeq -\frac{4}{3\varepsilon} \left\{ \frac{2.27}{x^3} \left(1 - \frac{0.11}{x}\right) + \frac{0.043}{x^{5.76}(0.18 - x)} + \frac{7.53 \times 10^{-5}}{x^{8.52}(0.18 - x)^2} \right\}. 
\]

(84)

Applying (77), (79) and (81) the equations defined by (73) and (75) become respectively

\[
\rho(x \to x_b) \simeq \left\{ \frac{\pi}{12} + \frac{4x_b^3 [A(x_b) + B(x_b)/6]}{3} - \frac{16\pi M^2 C_2 x_b^4}{3} \right\} \\
\times \left\{ \frac{2x_b^3 [A(x_b) - 8\pi M^2 C_2]}{(x - x_b)^2} \right\} 
\]

(85)

\[
\sigma(x \to x_b) \simeq -\frac{4}{3\varepsilon} \left\{ \frac{\pi}{3x_b^3} + 4x_b[A(x_b) + B(x_b)/6] - 16\pi M^2 x_b C_2 \right\} \\
+ \frac{2\pi x_b^2 [A(x_b) - 8\pi M^2 C_2]}{G(x_b)(x - x_b)^2} \exp \left\{ \frac{2x_b^3 [A(x_b) - 8\pi M^2 C_2]}{(x - x_b)} \right\}, 
\]

(86)

\[
\rho(x \to x_c) \simeq \left\{ \frac{\pi}{24} + \frac{8x_c^4 [A(x_c) + B(x_c)/6]}{3} - 8\pi M^2 C_2 x_c^4 / 3 \right\} \\
- \frac{\pi x_c^3 (1 + 4x_c^2)[\pi/4 + 4A(x_c) + 2B(x_c)/3 - 16\pi M^2 C_2][A(x_c) - 8\pi M^2 C_2]}{2G(x_c)(x_c - x)^2} \\
\times \exp \left\{ -\frac{x_c^3 [A(x_c) - 8\pi M^2 C_2]}{2(x_c - x)} \right\} 
\]

(87)

and
\[ \sigma(x \to x_c) \simeq - \frac{4}{3\varepsilon} \left( \frac{\pi/8x_c^3 + 2x_c[A(x_c) + B(x_c)/6] - 8\pi M^2 C_2 x_c}{x_c - x} \right. \\
\left. + \frac{\pi x_c^2(1 + 4x_c^2)[\pi/4x_c^2 + 4A(x_c) + 2B(x_c)/3 - 16\pi M^2 C_2][A(x_c) - 8\pi M^2 C_2]}{2G(x_c)(x_c - x)^2} \right) \times \exp \left\{ \frac{-x_c^3[A(x_c) - 8\pi M^2 C_2]}{2(x_c - x)} \right\} \]  

(88)

Having the above obtained solutions we are now in a position to write the quantum perturbed Lukewarme black hole metric (51) defined in \((t, r, \theta, \varphi)\) coordinates as

\[ ds_f^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \]  

(89)

in which

\[ dt = e^{\psi(r)}dv - \frac{dr}{F(r)} \]  

(90)

and \(\psi(r)\) with \(r = 2Mx\), is given by (61), (62) and (63). \(F(r)\) defined by (52) and (54) as

\[ F(x) = 1 + \frac{1}{4x^2} - \frac{\rho(x)}{x} - \frac{\varepsilon \sigma(x)x^2}{4} \]  

(91)

is given exactly by applying (83), (84), (85), (86), (87) and (88). It will be useful that we choose a numerical value for \(x_{b,c}\) from the figure 1 such as follows.

Experimental limits on the cosmological constant is obtained as \([\text{Kenyon 1991}]\)

\[ |\Lambda| \leq 10^{-54} \text{cm}^{-2} \]  

(92)

and order of magnitude of Schwarzschild radiuses for a galaxy and the Sun is given by \((2M \sim 10^{16} \text{cm})\) and \((2M \sim 3 \times 10^5 \text{cm})\) respectively. So whose corresponding coupling parameter \(\varepsilon = \frac{16M^2 \Lambda}{3}\) will be obtain as \(\varepsilon_{\text{galaxy}} \simeq 1.33 \times 10^{-22}\) and \(\varepsilon_{\text{sun}} \simeq 1.2 \times 10^{-43}\) respectively which are very small digits. As a numerical result we use here \(\varepsilon = 10^{-22}\) and obtain

\[ (x_b, x_c) \simeq (0.5, 10^{11}) \]  

(93)

\[ A(x_b) = A(x_c) \simeq 0 \quad B(x_b) \simeq 48, \quad B(x_c) \simeq -3.75 \times 10^{-23} \]  

(94)
and

\[ \pi M^2 C_2 \approx 1, \quad G(x_b) \approx -15, \quad G(x_c) \approx 12.57. \]  \hfill (95)

Using (83), (84), (85), (86), (87), (88) and the above numerical values the equation (91) leads to

\[ F(x < 0.5) \approx \frac{0.62}{x^{6.52}(0.18 - x)^2}; \]  \hfill (96)

\[ F(x \to 0.5) \approx \frac{42.53(x - 0.45)}{(x - 0.5)^2} \exp \left\{ \frac{2}{0.5 - x} \right\}; \]  \hfill (97)

\[ F(x \to 10^{11}) \approx \frac{2.13 \times 10^{56}}{(1 - \frac{x}{10^{11}})^2} \exp \left\{ \frac{4 \times 10^{12}}{1 - \frac{x}{10^{11}}} \right\}. \]  \hfill (98)

The solution (96) dose not vanished in regions \(0 < x < 0.5\). The solution (97) vanishes at \(x \approx 0.45\). This is location of the perturbed black hole event horizon where \(x_b = 0.5\) is classical unperturbed radius of the Lukewarm black hole event horizon. It is seen easily that the solution (98) converges to a zero value (the perturbed cosmological event horizon) at limits \(x >> 10^{11}\). These solutions predict that the interacting quantum field back reaction corrections on the perturbed Lukewarm static black hole metric cause to shift the location of event horizons. In other word the cosmic sensor-ship hypothesis is still saved in the presence of the quantum field perturbations on a curved background metric. As a future work the authors will be attempt to seek a time dependent version of perturbation solutions of the problem. Particularly stability prediction of an evaporating Lukewarm black hole encourages us to seek unperturbed solutions of the back reaction equation of the problem by using the Wheeler-DeWitt canonical quantum gravity approach. Result of this work together with results of several works pointed in the introduction predict remnant stable mini quantum black holes where the cosmic sensor-ship hypothesis is still valid.
5. Concluding Remarks

Two dimensional analog of the Hawking thermal radiation stress tensor of the quantum perturbed spherically symmetric static Lukewarm back hole is derived, by applying the Christensen and Fulling method. Then the obtained stress tensor, is used to solve a time-independent version of the well known metric back-reaction equation defined in a perturbed Lukewarm metric. According to the York's hypothesis (York 1985), we assume here that the massless and charge-less quantum scalar fields propagated on the background metric are in s (spherically) modes and so $(t, r)$ components of the metric are perturbed only. This leads still to save its spherically symmetric property and to assume that the mass and cosmological parameter of the Lukewarm black hole to be chosen as slowly varying radial dependence functions. However, mathematical derivations predict a shrunk black hole horizon with an extended cosmological horizon with respect to the corresponding classical horizons location. Particularly these quantum field perturbations do not cause violations of the cosmic sensor-ship hypothesis.
REFERENCES

Adler S., Liberman J. and Ng Y. J., Ann. Phys. (N.Y.) 106, 279 (1977)

de Alwis S. P., Phys. Rev. D46, 5429 (1992)

Amati D, Ciafaloni M. and Veneziano G., Phys. Lett. B216, 41 (1989).

Amati D, Ciafaloni M. and Veneziano G., Phys. Lett. B197, 81 (1987).

Amati D, Ciafaloni M. and Veneziano G., In. J. Mod. Phys. A3, 1615 (1988).

Amati D, Ciafaloni M. and Veneziano G., Nucl. Phys. B347, 530 (1990).

Aschieri P. et al, Class. Quant. Grav.22, 3511 (2005)[hep-th/0504183].

Balbinot R., Phys.Rev.D33,1611 (1986)

Balbinot R. and Barletta A., Class. Quant. Grav.6, 203; Grav.6, 195 (1989)

Balbinot R., Phys. Lett.B136, 337 (1984)

Balbinot R., Nuovo Cimento B 86 , 31 (1985)

Balbinot R. and Brown M. R., Phys. Lett. A 100, 80 (1984)

Banks T., Dabholkar A. , Douglas M. R. and O'Loughlin M.; Phys. Rev. D45, 3607 (1992)

Birrell and Davies, *Quantum Fields in Curved space* ( Cambridge University press, Cambridge, England, 1982)

Bousso R. and Hawking S.W., Phys. Rev. D57, 2436, (1998)

Brown M. R., J. Math. Phys, 25, 136 (1984)

Breen C. and Ottewill A. C., gr-qc/1112.3048v1 (2011)
Bernard D. and Folacci A., Phys. Rev. D34, 2286 (1986)

Callan. C. G. Jr, Giddings S. B., Harvey J. A. and Strominger A., Phys. Rev. D45, R1005 (1992).

Calmet. X and Kobakhidze A., Phys. Rev. D72, 045010 (2005), hep-th/0506157.

Calmet. X and Kobakhidze A., Phys. Rev. D74, 047702 (2006), hep-th/0605275.

Capozziello S., Lambiase G. and Scarpetta G., Int. J. Theor. Phys. 39, 15 (2000).

Chamseddine A. H., Phys. Lett. B504, 33 (2001), hep-th/0009153.

Christensen S. M., Phys. Rev. D14, 2490 (1976)

Christensen S. M. and Fulling S. A., Phys. Rev. D15, 2088 (1977)

Douglas N. and Nekrasov N. A., Rev. Mod. Phys. 73, 977 (2001).

Ghafarnejad H. and Salehi H., Phys. Rev.D56, 4633 (1997); 57, 5311 (E) (1998)

Ghafarnejad H., Phys. Rev. D47, 104012 (2006)

Ghaffarnejad H., Phys. Rev. D75, 084009 (2007)

Gibbons G. W. and Hawking S. W., phys. Rev.D15, 2738 (1977)

Hawking S. W., Nature, 248, 30 (1974)

Hawking S. W., Commun. Math. Phys. 43, 199 (1975)

Hartle J. B. and Hawking S. W., Phys. Rev. D13, 2188 (1976)

Kenyon I. R., General Relativity (Oxford University press, New York, 1991).

Lowe D. A. and M. O’Loughlin, Phys. Rev. D48, 3735 (1993)
Nicolini P., Smailagic A. and Spallucci E., Phys. Lett. B 632, 547 (2006), \[gr-qc/0510112\].

Nozari K. and Mehdipour S. H., Mod. Phys. Lett.A 20, 2937 (2005), \[gr-qc/0809.3144\].

Nozari K. and Mehdipour S. H., Class. Quant. Grav. 25, 175015 (2008), \[gr-qc/0801.4074\].

Nozari K. and Sadatian S. D., Mod. Phys. Lett.A24, 3143 (2009), \[gr-qc/090.0241\].

Parker L. and Toms D., *Quantum Field Theory in Curved Spacetime* (Cambridge University, 2009)

Piran T. and Strominger A., Phys. Rev. D48, 4729 (1993)

Russo J.G., Susskind L. and Thorlacius L., Phys. Rev. D46, 3444 (1992); Phys. Lett. B292, 13 (1992)

Snyder H. S., Phys. Rev.71 38 (1947).

Seiberg N. and Witten E. , JHEP 9909, 73, 977 (2001).

Wald R. M., Phys. Rev. D17, 1477 (1978).

Wang B. and Huang C. G., Phys. Rev.D63, 124014 (2001)

York, J. W. Jr., Phys. Rev.D31, 775 (1985)

This manuscript was prepared with the AAS \LaTeX\X macros v5.2.
Fig. 1.— Dash-lines describe the black hole and cosmological horizon radiiues obtained from the equation $\Omega(x) = 0$ with $q = 1$, namely the equation (18). Solid line defines quasi flat regions of the space time (3) which is obtained from the equation $\Omega'(x) = 0$ with $q = 1$, namely the equation (19). Values with $0 < \varepsilon < 10^{-22}$ is not shown here, because the diagram has large variations.