IMPROVED SOBOLEV INEQUALITIES INVOLVING WEIGHTED MORREY NORMS AND THE EXISTENCE OF NONTRIVIAL SOLUTIONS TO DOUBLY CRITICAL ELLIPTIC SYSTEMS INVOLVING FRACTIONAL LAPLACIAN AND HARDY TERMS

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ABSTRACT. In this paper, we prove two new improved Sobolev inequalities involving weighted Morrey norms in \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) and \( D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n) \). For instance, the corresponding inequality in \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) states that: there exists \( C = C(n, s, \alpha, \eta_1, \eta_2) > 0 \) such that for each \( (u, v) \in \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \), \( p \in [2, 2^*_s(\alpha)) \) and \( \theta \in (\theta, \frac{s}{2^*_s(\alpha)}) \), it holds that

\[
\left( \int_{\mathbb{R}^n} \frac{|u|^q |v|^q}{|y|^\alpha} \, dy \right)^{\frac{1}{q}} \leq C \left( \|u\|_{\dot{H}^s(\mathbb{R}^n)} \|v\|_{\dot{H}^s(\mathbb{R}^n)} \right)^{\frac{1}{2}} \left( (uv) \right)^{\frac{s}{2^*_s(\alpha) - 2}} \left( \|v\|_{\dot{H}^s(\mathbb{R}^n)} \right)^{\frac{s}{2^*_s(\alpha) - 2}} \left( \|u\|_{\dot{H}^s(\mathbb{R}^n)} \right)^{\frac{s}{2^*_s(\alpha) - 2}},
\]

where \( s \in (0, 1) \), \( 0 < \alpha < 2s < n \), \( \eta_1 + \eta_2 = 2^*_s(\alpha) := \frac{2(n-\alpha)}{n-2s} \), \( 1 < \eta_1 \leq \eta_2 < \eta_1 + \frac{s}{2} \), \( \theta = \max \left\{ \frac{2}{2^*_s(\alpha)}, \frac{2 \eta_2}{2^*_s(\alpha) - 2}, \frac{2(\frac{n-\alpha}{2} - \frac{s}{2})}{2^*_s(\alpha) - 2}, \frac{\eta_2}{2^*_s(\alpha) - 2} \right\} \), \( t = 1 - \frac{\eta_2 - \eta_1}{\alpha} \) and \( r = \frac{2\eta_2}{2^*_s(\alpha)} \). This inequality, together with its counterpart in \( D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n) \) extend similar Sobolev inequality in \( \dot{H}^s(\mathbb{R}^n) \) as well as in \( D^{1,p}(\mathbb{R}^n) \) obtained by G. Palatucci and A. Pisante [Calc. Var., 50 (2014)] to the product spaces \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) and \( D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n) \), respectively.

With the help of the inequality (1), we succeed in obtaining some new existence results for doubly critical elliptic systems involving fractional Laplacian and Hardy terms.

1. Introduction and main results. In this paper, we establish two new improved Sobolev inequalities involving weighted Morrey norms in the product spaces \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) and \( D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n) \), respectively and use the inequalities to prove the existence of nontrivial solutions to the following doubly critical elliptic systems involving fractional Laplacian (or p-Laplacian) and Hardy terms:

\[
\begin{align*}
(-\Delta)^s u - \gamma_1 \frac{u}{|x|^{2s}} &= \left[ I_\mu * F_\alpha (\cdot, u) \right] (x) f_\alpha (x, u) + \frac{\eta_1}{2^*_s(\beta)} \frac{|u|^{\gamma_1 - 2} u |v|^{\gamma_2}}{|x|^\beta} \\
(-\Delta)^s v - \gamma_2 \frac{v}{|x|^{2s}} &= \left[ I_\mu * F_\alpha (\cdot, v) \right] (x) f_\alpha (x, v) + \frac{\eta_2}{2^*_s(\beta)} \frac{|v|^{\gamma_2 - 2} v |u|^{\gamma_1}}{|x|^\beta} 
\end{align*}
\]

(1)

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where $s \in (0, 1)$, $0 \leq \alpha, \beta < 2s < n$, $\mu \in (0, n)$, $\gamma_1, \gamma_2 < \gamma_H$, $\eta_1, \eta_2 > 1$, $\eta_1 + \eta_2 = 2^*_\mu(\beta) = \frac{2(n-\beta)}{n-2s}$, $I_{n,\lambda}(x) = \frac{|x|^{-n}}{|x|^{\mu(\beta)}}$, $F_{\alpha}(x, u) = \frac{|u(x)|^{2^*_\mu(\alpha)}}{|x|^{\mu(\alpha)}}$, $F_{\beta}(x, u) = \frac{|u(x)|^{2^*_\mu(\beta)-2}u(x)}{|x|^{\mu(\beta)}}$, $(-\Delta)^s$ is the fractional Laplacian operator, $2^*_\mu(\alpha) = (1 - \frac{\mu}{n}) \cdot 2^*_\mu(\alpha)$ and $\delta_\mu(\alpha) = (1 - \frac{\mu}{n})\alpha$.

Here $\gamma_H$ is the best constant in the fractional Hardy inequality (see (5) below), $2^*_\mu(\beta)$ is the critical fractional Hardy-Sobolev exponent and $(2^*_\mu(\alpha), \delta_\mu(\alpha))$ is a pair of critical exponents in the sense of fractional Hardy-Sobolev inequality and Hardy-Littlewood-Sobolev inequality

We first recall some terminology. For $s \in (0, 1)$, $(-\Delta)^s$ is defined on the Schwartz class (i.e. space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^n$) by

$$(\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi), \forall \xi \in \mathbb{R}^n,$$

where $\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi x} u(x) dx$ is the Fourier transform of $u$. For $s \in (0, 1)$, the Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$||u||_{\dot{H}^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u(x)|^2 dx \right)^{\frac{1}{2}}.$$

(See [31] for the basics on the fractional Laplacian). The space $D^{1,p}(\mathbb{R}^n)$ is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm $||u||_{D^{1,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}$.

Let $\Omega$ be an open set in $\mathbb{R}^n$, $p \in [1, +\infty)$ and $\gamma \in [0, n]$, the Morrey space $L^{p,\gamma}(\Omega)$ is defined as

$$L^{p,\gamma}(\Omega) = \left\{ u : ||u||_{L^{p,\gamma}(\Omega)} = \sup_{R > 0, x \in \Omega} \left( R^{\gamma-n} \int_{B_R(x) \cap \Omega} |u(y)|^p dy \right)^{\frac{1}{p}} < +\infty \right\},$$

where $B_R(x) = \left\{ y \in \mathbb{R}^n : |y-x| < R \right\}$ denotes the open ball centered at $x$ with radius $R$ in $\mathbb{R}^n$. We see that $L^{p,\gamma}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for any $p \geq 1$ and $L^{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. See [11, 23, 29, 27, 32, 34] for more informations on Morrey spaces.

In [32], G. Palatucci et al. discovered the embeddings

$$\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{s,n}(\mathbb{R}^n) \hookrightarrow L^{p,\frac{n}{2}(n-2s)}(\mathbb{R}^n)$$

and established the improved Sobolev inequality

$$||u||_{L^{s,n}(\mathbb{R}^n)} \leq C ||u||_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta} ||u||_{L^{p,\frac{n}{2}(n-2s)}(\mathbb{R}^n)}^\theta, \forall u \in \dot{H}^s(\mathbb{R}^n),$$

where $s \in (0, 1)$, $n > 2s$, $1 \leq p < 2^*_s := \frac{2n}{n-2s}$, $\max\left\{ \frac{n}{2s}, 1 - \frac{1}{p} \right\} < \theta < 1$ and $C = C(n, s) > 0$. Using the inequality (3), they gave a direct proof of the existence of optimizers and the compactness up to translation of optimizing sequences for the usual Sobolev embedding

$$||u||_{L^{s,n}(\mathbb{R}^n)} \leq \Lambda_n, \forall u \in \dot{H}^s(\mathbb{R}^n),$$

where $\Lambda_n = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u(x)|^2 dx}{\int_{\mathbb{R}^n} |u(x)|^{2^*_s} dx}$. Moreover, they gave a more transparent proof of the profile decomposition of bounded sequences in $\dot{H}^s(\mathbb{R}^n)$.

For $s \in (0, 1)$, $n > 2s$ and $\gamma_H := 4^s \frac{p^2}{s(1-2s)}$ the fractional Hardy inequality is as follows:

$$\gamma_H \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}} dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u(x)|^2 dx, \forall u \in \dot{H}^s(\mathbb{R}^n),$$

(5)
where $\Gamma$ denotes the Gamma function and $\gamma_H$ is the best constant for (5) (See (2.1) in [14]). Let $\gamma < \gamma_H$, then (5) shows that
\[
||u||_{\gamma} := \left( \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} u|^2 \, dx - \gamma \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2\gamma}} \, dx \right)^{\frac{1}{2}}
\] (6)
defines an equivalent norm of $\dot{H}^s(\mathbb{R}^n)$. In [12], S. Dipierro et al. studied the minimization problem involving the Hardy potential
\[
\Lambda_{n,s}(\gamma) = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \left[ |(-\Delta)^{\gamma/2} u|^2 - \gamma \frac{u^2}{|x|^{2\gamma}} \right] \, dx}{\left( \int_{\mathbb{R}^n} |u(x)|^{2s} \, dx \right)^{\frac{1}{2^s}}},
\] (7)
for $s \in (0, 1)$, $n > 2s$, $2^*_s = \frac{2n}{n-2s}$ and $\gamma < \gamma_H$. By using inequality (3) and symmetric decreasing rearrangement technique, they obtained a minimizer of $\Lambda_{n,s}(\gamma)$.

The minimization problems
\[
\Lambda_{n,s}(\gamma, \alpha) = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \left[ |(-\Delta)^{\gamma/2} u|^2 - \gamma \frac{u^2}{|x|^{2\gamma}} \right] \, dx}{\left( \int_{\mathbb{R}^n} \frac{|u(x)|^{2s(\alpha)}}{|x|^{\alpha}} \, dx \right)^{\frac{1}{2s(\alpha)}}},
\] (8)
and
\[
\Lambda_{n,s,\mu}(\gamma, \alpha) = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \left[ |(-\Delta)^{\gamma/2} u|^2 - \gamma \frac{u^2}{|x|^{2\gamma}} \right] \, dx}{\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2s(\mu)}|u(y)|^{2s(\alpha)}}{|x-y|^{\mu}} \, dx \, dy \right)^{\frac{1}{2s(\mu)}}},
\] (9)
where $s \in (0, 1)$, $0 \leq \alpha < 2s < n$, $\mu \in (0, n)$, $2^*_s(\alpha) = \frac{2(n-\alpha)}{n-2s}$, $\gamma^*_\mu(\alpha) = (1 - \frac{\mu}{2n}) \cdot 2^*_s(\alpha)$, $\delta_\mu(\alpha) = (1 - \frac{\mu}{2n}) \alpha$ and $\gamma < \gamma_H$, are related to partial differential equations involving fractional Laplacian and critical singularities. Clearly, (8) is more general than (7).

N. Ghoussoub et al. in [15] proved that (8) is achieved provided either $\alpha = 0$ and $0 \leq \gamma < \gamma_H$ or $\alpha > 0$ and $\gamma < \gamma_H$ respectively by using the $s$-harmonic extension for fractional Laplacian developed by L. Caffarelli et al. in [6]. Alternatively, J. Yang in [41] directly solved (8) with $\alpha > 0$ and $\gamma = 0$ by establishing an improved Sobolev inequality involving Morrey norms. The method of [12] is applicable to (9) with $\alpha = 0$. However, (9) with $\alpha > 0$ can not be solved by the method of [15] and [12, 41] as the term $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2s(\mu)}|u(y)|^{2s(\alpha)}}{|x-y|^{\mu}} \, dx \, dy$ is nonlocal and the usual Morrey space $L^{p,\gamma}(\mathbb{R}^n)$ is not good enough to deal with this problem.

To solve (9) with $\alpha > 0$, a special weighted Morrey space is needed. For $p \in [1, +\infty)$, $\gamma, \lambda > 0$ and $\gamma + \lambda \in (0, n)$, we say a Lebesgue measurable function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the weighted Morrey space $L^{p,\gamma+\lambda}(\mathbb{R}^n, |y|^{-\lambda})$ if
\[
||u||_{L^{p,\gamma+\lambda}(\mathbb{R}^n, |y|^{-\lambda})} := \sup_{R > 0} \left\{ R^{\gamma+\lambda-n} \int_{B_R(x)} \frac{|u(y)|^p}{|y|^{p\lambda}} \, dy \right\}^{\frac{1}{p}} < +\infty.
\]
In [24], we discovered the embeddings
\[
\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*_s(\alpha)}(\mathbb{R}^n, |y|^{-\alpha}) \hookrightarrow L^{p,\frac{n}{2s(n-2s)}+\frac{s}{2}r}(\mathbb{R}^n, |y|^{-\frac{s}{2}r})
\] (10)
and established the improved Sobolev inequality
\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2^*_s(\alpha)}}{|y|^{\alpha}} \, dy \right)^{\frac{1}{2^*_s(\alpha)}} \leq C ||u||_{\dot{H}^s(\mathbb{R}^n)} ||u||_{L^{p,\frac{n}{2s(n-2s)}+\frac{s}{2}r}(\mathbb{R}^n, |y|^{-\frac{s}{2}r})}^{1-\frac{s}{2}} \quad \forall u \in \dot{H}^s(\mathbb{R}^n)
\] (11)
where $s \in (0,1)$, $0 < \alpha < 2s < n$, $p \in [1, 2s^*(\alpha))$, $r = \frac{2\alpha}{2s^*(\alpha)}$, $\max\{\frac{2}{2s^*(\alpha)}, 1 - \frac{\alpha}{s^*(\alpha) - \frac{n}{2}}\} < \theta < 1$ and $C = C(n, s, \alpha) > 0$. Using (10) and (11), we solved (8) and (9) in a direct way (See Proposition 4.1 of [24]). Furthermore, we get the existence of a nontrivial solution to some doubly critical equation involving fractional Laplacian and a Hardy term by using (10) and (11) (See [24]). Refined Sobolev inequalities with Coulomb norm are studied in e.g. [3, 28, 37].

To study the existence of nontrivial solutions to the elliptic systems (1), improved Sobolev inequalities in product spaces are needed.

In this paper, we establish two improved Sobolev inequalities in the spaces $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ and $D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n)$ involving weighted Morrey norms, respectively.

Our main results in this aspect are the following Theorems 1.1-1.4.

**Theorem 1.1.** Let $s \in (0,1)$, $0 < \alpha < 2s < n$ and $\eta_1 + \eta_2 = 2s^*(\alpha)$ satisfy $1 < \eta_1 \leq \eta_2 < \eta_1 + \frac{\alpha}{2}$. Then there exists a constant $C = C(n, s, \alpha, \eta_1, \eta_2) > 0$ such that for each $(u, v) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, $p \in [2, 2s^*(\alpha))$ and $\theta \in (\tilde{\theta}, \frac{2\eta_1}{2s^*(\alpha)})$, it holds that

$$
\left( \int_{\mathbb{R}^n} \frac{|u|^{\eta_1}|v|^{\eta_2}}{|y|^\alpha} \, dy \right)^{\frac{1}{\eta_2}} \leq C ||u||^{\frac{\theta}{H^s(\mathbb{R}^n)}} ||v||^{\frac{2s^*(\alpha) - \eta_2}{2s^*(\alpha) - \frac{n}{2}}} ||(uv)||^{\frac{n}{L^2}} \tilde{\theta}^{\frac{n}{L^2}} (n - 2s^*(\alpha))(\mathbb{R}^n, |y|^{-\frac{n}{2}}),
$$

where $\tilde{\theta} = \max\left\{\frac{2}{2s^*(\alpha)}, 1 - \frac{\alpha}{s^*(\alpha) - \frac{n}{2}}\right\}$.

**Theorem 1.2.** Let $s \in (0,1)$, $0 < \alpha < 2s < n$ and $\eta_1 + \eta_2 = 2s^*(\alpha)$ satisfy $1 < \eta_1 \leq \eta_2 < \eta_1 + \frac{\alpha}{2}$. If $\{(u_k, v_k)\}$ is a bounded sequence in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with

$$
\frac{1}{\inf \lambda_k \int_{\mathbb{R}^n} \frac{|u_k(y)|^{\eta_1}|v_k(y)|^{\eta_2}}{|y|^\alpha} \, dy} \geq C > 0
$$

for some constant $C$, then there exists, up to a subsequence, a family of positive numbers $\{\lambda_k\}$ such that

$$
(\tilde{u}_k(x), \tilde{v}_k(x)) : = (\lambda_k^{-\frac{2s}{\alpha}} u_k(\lambda_k x), \lambda_k^{-\frac{2s}{\alpha}} v_k(\lambda_k x)) \rightharpoonup (\tilde{u}, \tilde{v}) \text{ in } H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)
$$

for some $\tilde{u}, \tilde{v} \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$.

**Theorem 1.3.** Let $2 \leq p < n$, $0 < \alpha < p$ and $\eta_1 + \eta_2 = p^*(\alpha) := \frac{p(n-\alpha)}{n-p}$ satisfy $1 < \eta_1 \leq \eta_2 < \eta_1 + \alpha$. Then there exists a constant $C = C(n, p, \alpha, \eta_1, \eta_2) > 0$ such that for each $(u, v) \in D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n)$, $m \in [p, p^*(\alpha))$ and $\theta \in (\tilde{\theta}, \frac{2\eta_1}{p^*(\alpha)})$, it holds that

$$
\left( \int_{\mathbb{R}^n} \frac{|u|^{\eta_1}|v|^{\eta_2}}{|y|^\alpha} \, dy \right)^{\frac{1}{\eta_2}} \leq C ||u||^{\frac{\theta}{D^{1,p}(\mathbb{R}^n)}} ||v||^{\frac{2s^*(\alpha) - \eta_2}{D^{1,p}(\mathbb{R}^n)}} ||uv||^{\frac{n}{L^2}} \tilde{\theta}^{\frac{n}{L^2}} (n - p^*(\alpha))(\mathbb{R}^n, |y|^{-\frac{n}{2}}),
$$

where $\tilde{\theta} = \max\left\{\frac{p}{p^*(\alpha)}, \frac{2\eta_1}{p^*(\alpha) - \frac{n}{2}}\right\}$, $t = 1 - \frac{\eta_2 - \eta_1}{\alpha}$ and $r = \frac{p\alpha}{p^*(\alpha)}$. 


Theorem 1.4. Let $2 \leq p < n$, $0 < \alpha < p$ and $\eta_1 + \eta_2 = p^*(\alpha) := \frac{p(n-\alpha)}{n-p}$ satisfy $1 < \eta_1 \leq \eta_2 < \eta_1 + \alpha$. If $\{(u_k, v_k)\}$ is a bounded sequence in $D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n)$ with
\[
\inf_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \frac{|u_k(y)|^\eta_1 |v_k(y)|^\eta_2}{|y|^{\alpha}} dy \geq C > 0
\]
for some constant $C$, then there exists, up to a subsequence, a family of positive numbers $\{\lambda_k\}$ such that
\[
(\tilde{u}_k(x), \tilde{v}_k(x)) := (\lambda_k^{\frac{\alpha}{n-p}} u_k(\lambda_k x), \lambda_k^{\frac{\alpha}{n-p}} v_k(\lambda_k x)) \rightharpoonup (\bar{u}, \bar{v}) \text{ in } D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n)
\]
for some $(\bar{u}, \bar{v}) \in D^{1,p}(\mathbb{R}^n) \times D^{1,p}(\mathbb{R}^n)$ with $\bar{u} \neq 0$ and $\bar{v} \neq 0$.

Remark 1. The improved Sobolev inequalities (12) and (13) are new. (10) and Young’s inequality show under the same notations as in Theorem 1.1 that
\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^\theta |v(y)|^\theta}{|y|^{\alpha}} dy \right)^{\frac{1}{\theta}} \leq C \left( ||u||_{H^{s}(\mathbb{R}^n)} + ||v||_{H^{s}(\mathbb{R}^n)} \right),
\]
which can be proved by Hölder’s inequality and the fractional Hardy-Sobolev inequality (See (22) of this paper). Take $u = v$ and $\eta_1 = \eta_2 = \frac{2s}{n}$ in (12), we get
\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2s} |v(y)|^{2s}}{|y|^{\alpha}} dy \right)^{\frac{1}{2s}} \leq C \left( ||u||_{H^{s}(\mathbb{R}^n)} + ||v||_{H^{s}(\mathbb{R}^n)} \right),
\]
SO (12) covers (11) and (11) refines the fractional Hardy-Sobolev inequality
\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2s} |v(y)|^{2s}}{|y|^{\alpha}} dy \right)^{\frac{1}{2s}} \leq C ||u||_{H^{s}(\mathbb{R}^n)}.
\]
We will see later that (12) yields a direct proof of the existence of minimizer and the compactness up to dilation of minimizing sequences for the minimization problem
\[
S_{n,s}(\eta_1, \eta_2, \alpha) = \inf_{(u,v) \in X, uv \neq 0} \left( \int_{\mathbb{R}^n} \frac{|u(y)|^\eta_1 |v(y)|^\eta_2}{|y|^{\alpha}} dy \right)^{\frac{1}{\eta_1 + \eta_2}},
\]
where $X = H^{s}(\mathbb{R}^n) \times H^{s}(\mathbb{R}^n)$.

Applying Theorem 1.2, we can prove the existence of nontrivial weak solutions to the elliptic systems (1). Also, the counterpart Theorems 1.3-1.4 of Theorems 1.1-1.2 are useful in dealing with p-Laplacian systems.

The fractional and other non-local operators arise in many practical problems (See [5, 31] and the references therein). The Hardy type potentials $\frac{1}{|x|^\alpha}$ arise, e.g. in nonrelativistic quantum mechanics, molecular physics, quantum cosmology, linearization of combustion models and so on; the system (1) with power type nonlinearities and $s = 1$ is related to Bose-Einstein condensate (See [9]). The problem of multiple critical exponents has been extensively studied (See [8, 9, 13].
Theorem 1.5. In [13], R. Filippucci et al. studied the doubly critical equation of Emden-Fowler type:

$$-\Delta_p u - \frac{\kappa u^{p-1}}{|x|^\rho} = u^{p^* - 1} + \frac{u^{p^*(\alpha) - 1}}{|x|^{\alpha}}$$

in $$\mathbb{R}^n, u \geq 0, \ u \in D^{1,p}(\mathbb{R}^n) \quad (14)$$

where $$n \geq 2, p \in (1, n), \alpha \in (0, p), \ p^* = \frac{np}{n-p}, \ p^*(\alpha) = \frac{p(n-\alpha)}{n-p}, \ 0 \leq \kappa < \bar{\kappa} = \left(\frac{n-p}{p}\right)^p$$

and $$\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$$ is the p-Laplacian of $$u$$. Based on truncation skills, they obtained a nontrivial weak solution to (14) by using the mountain pass lemma and concentration analysis of the corresponding ($$PS$$) sequence. The works of [15, 24, 42] and [2] were devoted to the fractional Laplacian and fractional p-Laplacian equations involving different critical nonlinearities, respectively. One can refer to [33, 40] for fractional Laplacian systems involving different critical nonlinearities. For the cases of the standard Laplacian, biharmonic and p-biharmonic operators, one can refer to [7, 10, 9, 16, 17, 18, 22, 26, 38].

Let $$s \in (0, 1), \ n > 2s$$ and $$\gamma_i < \gamma_H$$ for $$i = 1, 2$$, we define two equivalent norms $$|| \cdot ||_{\gamma_i}$$ ($$i = 1, 2$$) on $$\dot{H}^s(\mathbb{R}^n)$$ (See (6)) and denote the related inner products of $$u, \phi \in \dot{H}^s(\mathbb{R}^n)$$ by

$$\langle u, \phi \rangle_{\gamma_i} = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi dx - \gamma_i \int_{\mathbb{R}^n} \frac{|u\phi|}{|x|^{2s}} dx, \ i = 1, 2.$$

For $$X := \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n)$$ and $$(u, v), (\phi, \psi) \in X$$, we define

$$\langle (u, v), (\phi, \psi) \rangle_X = \langle u, \phi \rangle_{\gamma_1} + \langle v, \psi \rangle_{\gamma_2}, \ ||(u, v)||^2 = ||u||_{\gamma_1}^2 + ||v||_{\gamma_2}^2.$$

Denote

$$B_{\alpha}(u, u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2\alpha_\mu}(\alpha)}{|x|^\alpha} \frac{|u(y)|^{2\alpha_\mu}(\alpha)}{|y|^\alpha} dx dy, \ \forall u \in \dot{H}^s(\mathbb{R}^n) \quad (16)$$

where $$s \in (0, 1), \ 0 \leq \alpha < 2s < n, \ \mu \in (0, n), \ 2\alpha_\mu(\alpha) = (1 - \frac{\alpha}{2s})2_\mu(\alpha)$$ and $$\delta_\mu(\alpha) = (1 - \frac{\alpha}{2s})\alpha$$. By (25) below, $$B_{\alpha}(u, u)$$ is well-defined for any $$u \in \dot{H}^s(\mathbb{R}^n)$$. Let

$$B_{\alpha}(u \pm v) = B_{\alpha}(u, u) + B_{\alpha}(v, v), \ \forall (u, v) \in X. \quad (17)$$

Then the energy functional associated to system (1) is defined as:

$$I(u, v) = \frac{1}{2} ||(u, v)||^2 - \frac{1}{2^{2_\mu}(\beta)} \int_{\mathbb{R}^n} \frac{|u|^{n_1} v^{n_2}}{|x|^{\beta}} dx - \frac{1}{2^{2_\mu}(\alpha)} B_{\alpha}(u \pm v).$$

We say $$(u, v) \in X$$ is a weak solution to (1) if

$$\langle I'(u, v), (\phi, \psi) \rangle_X := \sum_{i=1}^{2} \langle u, \phi \rangle_{\gamma_i} - \frac{1}{2^{2_\mu}(\beta)} \int_{\mathbb{R}^n} \eta_1 |u|^{n_1-2u} \phi |v|^{n_2} + \eta_2 |v|^{n_2-2v} \psi |u|^{n_1}$$

$$- \int_{\mathbb{R}^n} \left[ I_\mu * F_\alpha(\cdot, u) \right] (x) f_\alpha(x, u) \phi(x) + \left[ I_\mu * F_\alpha(\cdot, v) \right] (x) f_\alpha(x, v) \psi(x) dx = 0$$

for any $$(\phi, \psi) \in X$$, where $$\langle \cdot, \cdot \rangle_X$$ was defined in (15). Clearly, a critical point of $$I$$ in $$X$$ is a weak solution to (1). We say a weak solution $$(u, v) \in X$$ to (1) is a semi-trivial solution if $$u \equiv 0$$ or $$v \equiv 0$$ or $$u \equiv v$$. A weak solution $$(u, v) \in X$$ to (1) is called a nontrivial solution if $$u \not\equiv 0, v \not\equiv 0$$ and $$u \not\equiv v$$. Our existence result to (1) is as follows.

Theorem 1.5. Let $$(0, 1)$$ and $$\eta_1 + \eta_2 = 2_\mu(\beta)$$ satisfy $$1 < \eta_1 < \eta_2 < \eta_1 + \frac{\alpha}{2}$$, then system (1) possesses at least a nontrivial weak solution provided either (I) $$0 < \alpha, \beta < 2s < n, \ mu \in (0, n)$$ and $$\gamma_1 \leq \gamma_2 < \gamma_H$$ or (II) $$\alpha = 0 < \beta < 2s < n,$$
\( \mu \in (0, n) \) and \( 0 \leq \gamma_1 \leq \gamma_2 < \gamma_H \); if \( \eta_1 = \eta_2 = \frac{2s(\beta)}{2} \) and \( \gamma_1 \neq \gamma_2 \), then the previous conclusion remains true if (I) or (II) holds.

**Remark 2.** Theorem 1.5 indicates that we can relax the lower bound of \( \gamma_1, \gamma_2 \) in system (1) provided \( \alpha, \beta > 0 \) while the conditions \( \gamma_1, \gamma_2 \geq 0 \) are needed if \( \alpha = 0 \). Inequality (11) is good enough to deal with the doubly critical equation studied in [24], but not sufficient to system (1) which contains the coupled terms. One of the main difficulties to prove the existence of nontrivial solutions to (1) is to prove that critical point \( (u, v) \in X \) of the energy functional \( I(u, v) \) of (1) is nontrivial. To deal with this difficulty, we do need the inequality (12) (See Theorem 1.1).

The proof of Theorem 1.1 is based on a careful estimate of the Riesz potentials on weighted \( L_p \) spaces established in [35] and the Calderon-Zygmund type techniques combining with a precise control on \( A_{p,q} \)-constant associated to the weights in terms of the Morrey norm appeared in the inequality. This idea comes from e.g. [32] in which inequality (3) was proved. However, the improved inequality (12) involves the product space \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \), thus for \( (u, v) \in \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \), we need estimate the Riesz potentials corresponding to \( u \) and \( v \) in a same weighted \( L_p \) space while keeping the product \( uv \) in some weighted Morrey space. Moreover, it is difficult to control the \( A_{p,q} \)-constant as the parameters affect each other. Theorem 1.3 is proved in the same way but an extra pointwise potential estimate is needed since the Riesz potential representation of usual Sobolev functions in \( \dot{H}^s(\mathbb{R}^n) \) is no longer applicable to functions in \( D^{1,p}(\mathbb{R}^n) \). Theorem 1.2 and Theorem 1.4 are direct applications of Theorem 1.1 and Theorem 1.3, respectively.

There are several difficulties to prove Theorem 1.5. Firstly, the truncation techniques used in [13, 15] are not applicable because of the convolution terms in (1). Secondly, (1) is invariant under the transformation \((u(x), v(x)) \mapsto (\lambda^{\frac{n-2}{2}} u(\lambda x), \lambda^{\frac{n-2}{2}} v(\lambda x))\) with \( \lambda > 0 \), hence the compactness of any \((PS)_c\) sequence with \( c > 0 \) does not hold. In fact, if the compactness of \((PS)_c\) sequence holds for some \( c > 0 \) and \( \{(u_k, v_k)\} \subset X \) is a \((PS)_c\) sequence, then \((u_k, v_k) \to (u, v)\) strongly in \( X \) up to a subsequence. However, \((\tilde{u}_k, \tilde{v}_k) = (k^{\frac{n-2}{2}} u_k(kx), k^{\frac{n-2}{2}} v_k(kx))\) is also a \((PS)_c\) sequence and \((\tilde{u}_k, \tilde{v}_k) \to (0, 0)\) strongly in \( X \), which contradicts to \( c > 0 \). Thirdly, we have trouble in ruling out the “vanishing” of the corresponding \((PS)\) sequence due to an asymptotic competition between the two critical nonlinearities. The three above difficulties appeared also for equations studied in [24]. In [24], these difficulties were overcame by using the embeddings (10) and the inequality (11). So we adapt the same method as [24] did. (1) has a special difficulty that it contains coupled terms \( \frac{|u|^{\gamma_1-2} u^{|\gamma_2|}}{|x|^2} \) and \( \frac{|v|^{\gamma_2-2} |u|^{|\gamma_1|}}{|x|^2} \), which can not be dealt with by (11) but (12).

Now, we give the outline of the proof for Theorem 1.5. We use the Mountain pass lemma to find critical points of \( I(u, v) \) on \( X = \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \). Since (1) contains doubly critical exponents, we require that the Mountain pass level \( c < c^* \) for some suitable threshold value \( c^* \) as usual. This is crucial in ruling out the “vanishing” of the corresponding \((PS)_c\) sequence. To this end, we introduce two minimization problems

\[
S_{n,s,\mu}(\alpha) = \inf_{(u,v)\in X\setminus\{(0,0)\}} \frac{||(u,v)||^2}{B_{\alpha}(u+v)} \frac{1}{2^s(\alpha)}
\]
Then Theorem 1.2 can be used to prove that the Mountain pass level $c < c^*$ for

$$c^* = \min \left\{ \frac{2^\#(\alpha)}{2 \cdot 2^\#(\alpha)} S_{n,s,\mu}(\alpha) \frac{2^\#(\alpha)-1}{2^{(n-\beta)}}, \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta) \right\}. $$

Then as usual, the Mountain pass lemma gives a $(PS)_c$ sequence $\{(u_k, v_k)\}$ for $I$ at level $c > 0$. That is,

$$I(u_k, v_k) \rightarrow c < c^*, \quad I'(u_k, v_k) \rightarrow 0 \text{ strongly in } X'. $$

Clearly, $\{(u_k, v_k)\}$ is bounded so we may assume that $(u_k, v_k) \rightarrow (u, v)$ in $X$ for some $(u, v) \in X$. But it may occur that $u \equiv 0$ or $v \equiv 0$. Denote $d_1 = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{|u_k(x)|^2}{|x|^\beta} \, dx$ and $d_2 = \lim_{k \rightarrow \infty} B_n(u_k \oplus v_k)$. By (18), (19) and (20), we have

$$d_1 \frac{2^\#(\alpha)}{2 \cdot 2^\#(\alpha)} A_1 \leq d_2, \quad d_2 \frac{2^\#(\alpha)}{2 \cdot 2^\#(\alpha)} A_2 \leq d_1 $$

where $A_1 = \Lambda_{n,s}(\eta_1, \eta_2, \beta) - \frac{2(n-\beta)}{2^{(n-\beta)}} - \frac{2^\#(\alpha)-1}{2^{(n-\beta)}} \Lambda_{n,s,\mu}(\alpha) - \frac{2^\#(\alpha)}{2^{(n-\beta)}} \Lambda_{n,s,\mu}(\alpha)$. Since $c < c^*$, we have $A_1 > 0, A_2 > 0$. Thus (21) implies that $d_1 \geq \varepsilon_0 > 0$ and $d_2 \geq \varepsilon_0 > 0$ as $c > 0$. Thus we find a $K > 0$ large such that $k \geq K$ implies

$$\int_{\mathbb{R}^n} \frac{|u_k(y)|^2 |v_k(y)|^{\beta/2}}{|y|^\beta} \, dy > \varepsilon_0/2.$$

Then Theorem 1.2 gives a sequence of positive numbers $\{\lambda_k > 0\}$ such that

$$(\tilde{u}_k(x), \tilde{v}_k(x)) := (\frac{n-2s}{2} u_k(\lambda_k x), \frac{n-2s}{2} v_k(\lambda_k x)) \rightarrow (\tilde{u}, \tilde{v}) \text{ in } X = H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$$

for some $(\tilde{u}, \tilde{v}) \in X$ with $\tilde{u} \neq 0, \tilde{v} \neq 0$. Moreover, we can check that $\tilde{u} \neq \tilde{v}$ and

$$\lim_{k \rightarrow +\infty} I(\tilde{u}_k, \tilde{v}_k) = c, \quad I'(\tilde{u}_k, \tilde{v}_k) = \lim_{k \rightarrow +\infty} I'(\tilde{u}_k, \tilde{v}_k) = 0 \text{ strongly in } X'.$$

Thus it is enough to get minimizers of (18) and (19), respectively. To this end, we need some kinds of compactness. Problem (18) can be solved by using inequality (11) (See Lemma 4.1 below). However, (11) is not good enough to deal with (19). Thanks to Theorem 1.2, we can prove the existence of minimizers for $\Lambda_{n,s}(\eta_1, \eta_2, \alpha)$ in $X$ in a direct way. Moreover, Theorem 1.2 is useful to rule out the “vanishing” of the corresponding $(PS)$ sequence.

**Notation.** For instance, the dual space of the Banach space $X$ is denoted by $X'$. $(PS)$ sequence denotes “Palais-Smale” sequence in short. $B_R(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < R \}$. We use $\rightarrow$ and $\rightharpoonup$ to denote the strong and weak convergence in the related spaces, respectively. $N = \{1, 2, \cdots\}$ is the set of natural numbers. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. By saying a function is “measurable”, we always mean that it is “Lebesgue measurable”. “$\Lambda$” denotes the Fourier transform and “$\mathbb{F}$” denotes the inverse Fourier transform. Generic fixed and numerical constants will be denoted by $C$ (with subscript in some case) and they will be allowed to vary within a single line or formula.
2. Preliminaries. In this section, we give some preliminary results.

**Lemma 2.1.** (Fractional Hardy-Sobolev inequalities: Lemma 2.1 of [15])

Let \( s \in (0, 1) \) and \( 0 \leq \alpha \leq 2s < n \). Then there exist positive constants \( c \) and \( C \) such that

\[
\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2^*_\alpha}} \leq c \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx, \quad \forall u \in \dot{H}^s(\mathbb{R}^n). \tag{22}
\]

Moreover, if \( \gamma < \gamma_H = 4^n \frac{\Gamma(n \alpha - \alpha)}{\Gamma(n - \alpha)^2} \), then

\[
C \left( \int_{\mathbb{R}^n} \frac{|u|^{2^*_\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2^*_\alpha}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx - \gamma \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2\alpha}} \, dx, \quad \forall u \in \dot{H}^s(\mathbb{R}^n). \tag{23}
\]

**Proposition 1.** (Hardy-Littlewood-Sobolev inequality, Theorem 4.3 in [25]) Let \( t, r > 1 \) and \( \mu \in (0, n) \) with \( \frac{1}{t} + \frac{\mu}{n} + \frac{1}{r} = 2 \), \( f \in L^t(\mathbb{R}^n) \) and \( h \in L^r(\mathbb{R}^n) \). There exists a sharp constant \( C(t, n, \mu, r) \), independent of \( f, h \) such that

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) h(y) \frac{dx \, dy}{|x-y|^\mu} \right| \leq C(t, n, \mu, r) \|f\|_{L^t(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}. \tag{24}
\]

If \( t = r = \frac{2n}{2n - \mu} > 1 \), then \( C(t, n, \mu, r) = \pi^{\frac{n}{\mu}} \frac{\Gamma\left(\frac{n}{2} - \frac{\mu}{2}\right)}{\Gamma\left(\frac{n - \mu}{2}\right)} \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n - \frac{\mu}{2}\right)} \right\}^{-1 + \frac{n}{\mu}} \).

Let \( s \in (0, 1) \), \( 0 \leq \alpha < 2s < n, \mu \in (0, n) \). \( \forall u \in \dot{H}^s(\mathbb{R}^n) \), take \( t = r = \frac{2n}{2n - \mu} > 1 \) and \( f(\cdot) = h(\cdot) = \frac{|u(\cdot)|^{2^*_{\mu}(\alpha)}}{|\cdot|^{\mu/\alpha}(\cdot)} \) in (24). Then Lemma 2.1 implies that \( f, h \in L^{\frac{2n}{2n - \mu}}(\mathbb{R}^n) \) and for the \( B_u(\cdot, \cdot) \) introduced in (16), we have

\[
B_u(u, u) \leq C(n, \mu) \left( \int_{\mathbb{R}^n} \frac{|u|^{2^*_{\mu}(\alpha)}}{|x|^{\alpha}} \, dx \right)^{\frac{2n - \mu}{n}} \leq C\|u\|_{\dot{H}^s(\mathbb{R}^n)}^{2\frac{2^*_{\mu}(\alpha)}{n}}, \quad \forall u \in \dot{H}^s(\mathbb{R}^n). \tag{25}
\]

**Lemma 2.2.** (Lemma 2.3 of [24]) Let \( s \in (0, 1) \) and \( 0 < r < s < \frac{n}{2} \). If \( \{u_k\} \) is a bounded sequence in \( \dot{H}^s(\mathbb{R}^n) \) and \( u_k \rightharpoonup u \) in \( \dot{H}^s(\mathbb{R}^n) \), then

\[
\frac{|u_k|}{|x|^r} \rightarrow \frac{|u|}{|x|^r} \quad \text{in } L^2_{loc}(\mathbb{R}^n), \quad \frac{u_k}{|x|^r} \rightarrow \frac{u}{|x|^r} \quad \text{in } L^2_{loc}(\mathbb{R}^n).
\]

**Lemma 2.3.** Let \( s \in (0, 1) \), \( 0 < \alpha < 2s < n, \eta_1, \eta_2 > 1, \eta_1 + \eta_2 = 2^*_\alpha \). If \( \{(u_k, v_k)\}_{k \in \mathbb{N}} \) is a bounded sequence in \( X = \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) and \( (u_k, v_k) \rightharpoonup (u, v) \) in \( X \), then we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \frac{|u_k|^{\eta_1} |v_k|^{\eta_2}}{|x|^{\alpha}} \, dx = \int_{\mathbb{R}^n} \frac{|u|^{\eta_1} |v|^{\eta_2}}{|x|^{\alpha}} \, dx + \lim_{k \to \infty} \int_{\mathbb{R}^n} \frac{|u_k - u|^{\eta_1} |v_k - v|^{\eta_2}}{|x|^{\alpha}} \, dx.
\]

**Proof.** The proof is similar to that of Theorem 2 in [4] when using \( j : \mathbb{C} \to \mathbb{C} \) defined by \( j(s + it) := \frac{|t|^{\eta_1} |t|^{\eta_2}}{|x|^{\alpha}} \), for \( s, t \in \mathbb{R} \), in that theorem. Here \( i \) is the imaginary unit. One can also refer to Lemma 2.4 in [19] or Lemma 2.3 in [9].

3. Proof of Theorems 1.1-1.4. In this section, we give the detailed proof of Theorems 1.1-1.4.

The following norm inequality for Riesz potentials on weighted Lebesgue space is the crucial ingredient in proving Theorem 1.1 and Theorem 1.3.

**Lemma 3.1.** (Theorem 1 in [35], or Theorem D in [30]) Suppose that \( n \geq 1 \), \( 0 < \tilde{s} < n \), \( 1 < \tilde{p} \leq \tilde{q} < +\infty \), \( \tilde{p}' = \frac{\tilde{p}}{\tilde{p} - \tilde{s}} \) and that \( V \) and \( W \) are nonnegative measurable functions on \( \mathbb{R}^n \). If, for some \( \sigma > 1 \),
\[
|Q|^{\frac{\tilde{s}}{\tilde{p}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q V^\sigma dy \right)^{\frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q W^{(1 - \tilde{p}')\sigma} dy \right)^{\frac{1}{\tilde{p}'}} \leq C_\sigma \tag{26}
\]
for all cubes \( Q \subset \mathbb{R}^n \), then for any function \( f \in L^\tilde{p}(\mathbb{R}^n, W(y)) \), we have
\[
\left( \int_{\mathbb{R}^n} |\ell_s f(y)|^{\tilde{q}} V(y) dy \right)^{\frac{1}{\tilde{q}}} \leq CC_\sigma \left( \int_{\mathbb{R}^n} |f(y)|^{\tilde{p}} W(y) dy \right)^{\frac{1}{\tilde{p}}}, \tag{27}
\]
where \( C = C(\tilde{p}, \tilde{q}, n) \) and \( \ell_s f(y) = \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{\tilde{s}}} dz \) is the Riesz potential of order \( \tilde{s} \).

Let \( p \in [1, +\infty) \), \( \gamma, \lambda > 0 \) and \( \gamma + \lambda \in (0, n) \), then the following fundamental properties about \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \) hold via Hölder’s inequality:

1. \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \hookrightarrow L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \) for \( \rho = \frac{n}{\gamma + \lambda} > 1 \);

2. For any \( p \in (1, +\infty) \), we have \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \hookrightarrow L^{1,\tilde{p} + \frac{\tilde{s}}{\tilde{p}}}(\mathbb{R}^n, |y|^{-\lambda}) \);

3. Let \( p \geq 2 \) and \( u, v \in L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \), then \( uv \in L^{p/2,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \) since
\[
||uv||_{L^{p/2,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda})} \leq ||u||_{L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda})} ||v||_{L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda})} < +\infty.
\]

**Proof of Theorem 1.1.** For \( u \in \dot{H}^{s}(\mathbb{R}^n) \), we have \( \hat{g}(\xi) := |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^n) \) and \( ||u||_{\dot{H}^{s}(\mathbb{R}^n)} = ||\hat{g}||_{L^2(\mathbb{R}^n)} \) by Plancherel’s theorem. Thus, \( u(x) = \langle \frac{1}{|x|^\sigma} \rangle^v \ast g(x) = \ell_s g(x) \), where \( \ell_s g(x) = \int_{\mathbb{R}^n} \frac{g(z)}{|x-z|^{\gamma + s}} dz \). Similarly, let \( v = \dot{H}^{s}(\mathbb{R}^n) \) and \( \hat{f}(\xi) := |\xi|^s \hat{v}(\xi) \in L^2(\mathbb{R}^n) \), we have \( v(x) = \langle \frac{1}{|x|^\sigma} \rangle^v \ast f(x) = \ell_s f(x) \).

**Case (i):** \( \eta_1 \leq \eta_2 \).

Firstly, let \( \eta_1 + \eta_2 = 2^*_s(\alpha) \) with \( 1 < \eta_1 < \eta_2 < \eta_1 + \frac{2}{s} \) and define \( t := 1 - \frac{(\eta_2 - \eta_1)s}{\alpha} \in (0, 1) \). Take \( \hat{s} = s, \tilde{p} = 2, \sigma = \frac{2}{\alpha} > 1 \),
\[
\max \left\{ 2, 2\eta_1 - \frac{t\alpha}{s}, 2\eta_1 - \frac{2t}{2^*_s(\alpha) - \frac{2}{s} \cdot 2^*_s(\alpha) - \frac{2}{s}}, 2\eta_1 - t \cdot 2^*_s(\alpha) \right\} < \tilde{q} < 2\eta_1,
\]
\[
W(y) \equiv 1 \text{ and } V(y) = \frac{|u||t \cdot 2^*_s(\alpha) - \frac{2}{s} \cdot 2^*_s(\alpha) - \frac{2}{s}}{|y|^\alpha} \text{ in Lemma 3.1, then condition (26) becomes}
\]
\[
|Q|^{\frac{\tilde{s}}{\tilde{p}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q V^\sigma dy \right)^{\frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q W^{(1 - \tilde{p}')\sigma} dy \right)^{\frac{1}{\tilde{p}'}} \leq C_\sigma \tag{28}
\]
For any cubes \( Q \subset \mathbb{R}^n \) centered at \( x \), there exists \( R > 0 \) such that \( B_{\eta_1}(x) \subset Q \subset B_R(x) \). We deduce \( \frac{s}{n} + 1 > \frac{\alpha}{s} > 0 \) by \( \tilde{q} < 2\eta_1 < 2^*_s(\alpha) \), it results that
\[
|Q|^{\frac{\tilde{s}}{\tilde{p}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q V^\sigma dy \right)^{\frac{1}{\tilde{p}'}} \leq |B_R(x)|^{\frac{\tilde{s}}{\tilde{p}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|B_R(x)|} \int_{B_R(x)} V^\sigma dy \right)^{\frac{1}{\tilde{p}'}} \leq CR^{\frac{\tilde{s}}{\tilde{p}} + \frac{1}{\tilde{p}'}} \left( R^{-n} \int_{B_R(x)} V^\sigma dy \right)^{\frac{1}{\tilde{p}'}}. \tag{29}
\]
Secondly, we verify condition (28). From (29), it is sufficient to verify
\[
R^{-n} \int_{B_R(x)} V^\sigma \, dy = R^{-n} \int_{B_R(x)} \frac{|u|^{(n-\frac{q}{2})\sigma} |v|^{(2q-\frac{2}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy \\
= R^{-n} \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{\sigma \alpha}} \cdot \frac{|v|^{(2q-\frac{2}{2})\sigma}}{|y|^{(1-t)\sigma \alpha}} \, dy \\
\leq R^{-n} \left[ \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy \right]^{t} \left[ \int_{B_R(x)} \frac{|v|^{(2q-\frac{2}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy \right]^{1-t} \\
= R^{-n} \left[ \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy \right]^{t} \left[ \int_{B_R(x)} |v|^{2} \, dy \right]^{1-t} \\
\leq R^{-n} \left[ \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy \right]^{t} \left[ C ||v||_{H^s(\mathbb{R}^n)}^{2} \right]^{1-t} \quad \text{(30)}
\]
From $2\eta_1 - t \cdot 2^*_s(\alpha) < \tilde{q} < 2\eta_1$, we have $0 < (\eta_1 - \frac{\tilde{q}}{2}) \frac{\sigma}{T} < 1$. Moreover, we have
\[
\frac{1}{1-(n-\frac{q}{2})\frac{\sigma}{T}} < n \quad \text{by} \quad \tilde{q} > 2\eta_1 - \frac{2\nu(2\alpha-\frac{q}{2})}{2^*_s(\alpha)} \cdot 2^*_s(\alpha). \quad \text{Recall that} \quad \rho \in (0, 1), \quad \text{it results that}
\]
\[
\int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{\sigma \alpha}} \, dy = \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{(1-t)\sigma \alpha}} \, dy \\
\leq \left[ \int_{B_R(x)} \frac{1}{|y|^{1-(n-\frac{q}{2})\frac{\sigma}{T}}} \, dy \right]^{1-(n-\frac{q}{2})\frac{\sigma}{T}} \left[ \int_{B_R(x)} \frac{|uv|^{(n-\frac{q}{2})\sigma}}{|y|^{(1-t)\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\frac{\sigma}{T}} \\
= CR^{\sigma \alpha + n - (n-\frac{q}{2})\frac{\sigma}{T}} \left[ \int_{B_R(x)} \frac{|uv|}{|y|^{\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\frac{\sigma}{T}}, \quad \text{(31)}
\]
where $r := \frac{\alpha(1-\rho)}{(n-\frac{q}{2})} = \frac{2\alpha}{2^*_s(\alpha)}$. Put (31) into (30), we get
\[
R^{-n} \int_{B_R(x)} V^\sigma \, dy \leq CR^{-\rho \sigma \alpha + n(t-1) - n(n-\frac{q}{2})\sigma} \left[ \int_{B_R(x)} \frac{|uv|}{|y|^{\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\sigma} ||v||_{H^s(\mathbb{R}^n)}^{2(1-t)}.
\]
Therefore,
\[
R^{\frac{n}{q} - \frac{2}{2}} \left( R^{-n} \int_{B_R(x)} V^\sigma \, dy \right)^{\frac{1}{q}} \leq R^{\frac{n}{q} - \frac{2}{2}} \left\{ CR^{-\rho \sigma \alpha + n(t-1) - n(n-\frac{q}{2})\sigma} \left[ \int_{B_R(x)} \frac{|uv|}{|y|^{\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\sigma} ||v||_{H^s(\mathbb{R}^n)}^{2(1-t)} \right\}^{\frac{1}{q}} \\
= C \left\{ R^{\rho \sigma (s+\frac{q}{2}) - \rho \sigma \alpha + n(t-1) - n(n-\frac{q}{2})\sigma} \left[ \int_{B_R(x)} \frac{|uv|}{|y|^{\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\sigma} ||v||_{H^s(\mathbb{R}^n)}^{2(1-t)} \right\}^{\frac{1}{q}} \\
= C \left\{ R^{\rho \sigma (s+\frac{q}{2}) - \rho \sigma \alpha + n(t-1) - n(n-\frac{q}{2})\sigma} \left[ \int_{B_R(x)} \frac{|uv|}{|y|^{\sigma \alpha}} \, dy \right]^{(n-\frac{q}{2})\sigma} \right\} \frac{|w|}{||w||_{H^s(\mathbb{R}^n)}}^{2(1-t) \sigma \alpha + 2n}
Then, for any \( s \in (0, 1) \), it follows that

\[
\eta_1 = \eta_2 = \frac{2s(\alpha)}{2s(\alpha) - 2s}.
\]

Take \( \tilde{s} = s, \tilde{p} = 2, \max\{2, \frac{2s(\alpha)}{2s(\alpha) - 2s}, \frac{2s(\alpha) - 2s}{2s(\alpha)}, \frac{2s(\alpha) - 4s}{2s(\alpha)}\} \) and \( \tilde{q} = \frac{2s(\alpha)}{2s(\alpha) - 2s} \), then \( (u,v) \) is a constant.

**Proof of Theorem 1.2.** For any bounded sequence \( \{u_k,v_k\} \subset \dot{H}^s(\mathbb{R}^n) \times \dot{H}^\theta(\mathbb{R}^n) \) with

\[
\inf_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \frac{|u_k(y)|^{2s}|v_k(y)|^{2s}}{|y|^{2s}} \, dy \geq C > 0,
\]

then for any \( \theta = \frac{q}{2s(\alpha)} \) satisfying \( \tilde{\theta} < \theta < \frac{2s(\alpha)}{2s(\alpha) - 2s} \), we have

\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2s}|v(y)|^{2s}}{|y|^{2s}} \, dy \right)^{\frac{1}{2s(\alpha)}} \leq C\|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{2}{2s(\alpha)}} \|v\|_{\dot{H}^\theta(\mathbb{R}^n)}^{\frac{2}{2s(\alpha)}} \|(uv)\|_{L^{\frac{n}{n-2s+r}}(\mathbb{R}^n,|y|^{-r})},
\]

where \( s \in (0, 1) \), \( 0 < \alpha < 2s < n \), \( r = \frac{2s(\alpha)}{2s(\alpha) - 2s} \), \( C = C(n, s, \alpha, \eta_1, \eta_2) > 0 \) is a constant, \( \tilde{\theta} = \max\left\{ \frac{2s(\alpha)}{2s(\alpha) - 2s}, \frac{n(2s(\alpha) - 2s)}{2s(\alpha)} - \frac{2s(\alpha) - 2s}{2s(\alpha)}, \frac{2s(\alpha) - 4s}{2s(\alpha)} \right\} \) and \( t = 1 - \frac{(2s - n)\alpha}{\alpha} \). We can check that \( 0 < \frac{2s(\alpha) - 2s}{2s(\alpha) - 2s} < \frac{2s - n}{s} \frac{1}{2s(\alpha) - 2s} < 1 \) and hence \( \tilde{\theta} = \max\left\{ \frac{2s - n}{s} \frac{1}{2s(\alpha) - 2s}, \frac{2s(\alpha) - 4s}{2s(\alpha)} \right\} \).
take \( p = 2 \) in (10) and inequality (12), then there exists \( \tilde{C} > 0 \) such that
\[
0 < \tilde{C} \leq \frac{|(u_k v_k)|}{L^{1,n-2s+r}(\mathbb{R}^n, |y|^{-r})} \leq \frac{|u_k||L^{2,n-2s+r}(\mathbb{R}^n, |y|^{-r})|v_k||L^{2,n-2s+r}(\mathbb{R}^n, |y|^{-r})} \leq \tilde{C}^{-1},
\]
where \( r = \frac{2a}{2s+1} \). For any \( k \geq 1 \), we may find \( \lambda_k > 0 \) and \( x_k \in \mathbb{R}^n \) such that
\[
\lambda_k^{-2s+r} \int_{B_{\lambda_k}(x_k)} \left| \frac{(u_k v_k(y))}{|y|^{r}} \right| dy > \left| \frac{|(u_k v_k)|}{L^{1,n-2s+r}(\mathbb{R}^n, |y|^{-r})} \right| \frac{\tilde{C}}{2k} \geq C_1 > 0.
\]
Let \( \tilde{u}_k(x) = \frac{n-2s}{\lambda_k^2} \) \( u_k(\lambda_k x) \), \( \tilde{v}_k(x) = \frac{n-2s}{\lambda_k^2} v_k(\lambda_k x) \) and \( \tilde{x}_k = \frac{x_k}{\lambda_k} \), then
\[
\int_{B_{1}(\tilde{x}_k)} \left| \frac{|\tilde{u}_k \tilde{v}_k(x)|}{|x|^{r}} \right| dx \geq C_1 > 0. \tag{32}
\]
We claim that \( \{\tilde{x}_k\} \) is bounded. Otherwise, \( |\tilde{x}_k| \to +\infty \), then for any \( x \in B_1(\tilde{x}_k) \), \( |x| \geq |\tilde{x}_k| - 1 \) for \( k \) large. By Hölder’s inequality, we have
\[
\int_{B_{1}(\tilde{x}_k)} \left| \frac{|\tilde{u}_k \tilde{v}_k(x)|}{|x|^{r}} \right| dx \leq \frac{1}{(|\tilde{x}_k| - 1)^r} \int_{B_{1}(\tilde{x}_k)} \left| \tilde{u}_k \tilde{v}_k(x) \right| dx
\]
\[
\leq \left( \frac{\int_{B_{1}(\tilde{x}_k)} \left| \tilde{u}_k(x) \right|^{2s} dx \right)^{\frac{1}{2s}} \left( \frac{\int_{B_{1}(\tilde{x}_k)} \left| \tilde{v}_k(x) \right|^{2s} dx \right)^{\frac{1}{2s}} \right)^{\frac{1}{2}} \leq \frac{C}{(|\tilde{x}_k| - 1)^r} \frac{|\tilde{u}_k||H^{s}(\mathbb{R}^n)|}{|\tilde{v}_k||H^{s}(\mathbb{R}^n)|} \leq \frac{C}{(|\tilde{x}_k| - 1)^r} \to 0, \quad \text{as} \quad k \to +\infty
\]
which contradicts to (32). In the last inequality, we use the fact \( ||u_k||_{H^s} = ||u||_{H^s} \) and \( ||v_k||_{H^s} = ||v||_{H^s} \). Hence, \( \{\tilde{x}_k\} \) is bounded, from (32) we find \( R > 0 \) such that
\[
\int_{B_{R}(0)} \left| \frac{|\tilde{u}_k \tilde{v}_k(x)|}{|x|^{r}} \right| dx \geq C_1 > 0. \tag{33}
\]
Since \( ||\tilde{u}_k||_{H^s(\mathbb{R}^n)}, ||\tilde{v}_k||_{H^s(\mathbb{R}^n)} \leq C \), there exists \( (\tilde{u}, \tilde{v}) \in \dot{H}^{s}(\mathbb{R}^n) \times \dot{H}^{s}(\mathbb{R}^n) \) such that
\[
(\tilde{u}_k, \tilde{v}_k) \to (\tilde{u}, \tilde{v}) \text{ in } \dot{H}^{s}(\mathbb{R}^n) \times \dot{H}^{s}(\mathbb{R}^n), \quad (\tilde{u}_k, \tilde{v}_k) \to (\tilde{u}, \tilde{v}) \text{ a.e. on } \mathbb{R}^n \times \mathbb{R}^n
\]
up to a subsequence. According to \( s = \frac{na}{2\sigma(\alpha)} < s \) and Lemma 2.2, we have
\[
\frac{|\tilde{u}_k|}{|x|^{\sigma}} \to \frac{|\tilde{u}|}{|x|^{\sigma}} \text{ in } L^2_{loc}(\mathbb{R}^n), \quad \frac{|\tilde{v}_k|}{|x|^{\sigma}} \to \frac{|\tilde{v}|}{|x|^{\sigma}} \text{ in } L^2_{loc}(\mathbb{R}^n).
\]
Therefore, (33) leads to
\[
\int_{B_{R}(0)} \frac{|\tilde{u}(\tilde{x})(x)|}{|x|^{r}} dx \geq C_1 > 0 \quad \text{and so} \quad \tilde{u} \neq 0, \tilde{v} \neq 0.
\]

**Proof of Theorem 1.3.** For \( n \geq 3 \) and \( u \in C^\infty_0(\mathbb{R}^n) \), we have
\[
u(x) = \Delta^{-1} \Delta u = C_1 \int_{\mathbb{R}^n} \frac{\Delta u(y)}{|x-y|^{n-2}} dy = C_2 \int_{\mathbb{R}^n} \frac{(x-y)\nabla u(y)}{|x-y|^{n}} dy.
\]
Thus, \( |u(x)| \leq |C_2| \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n}} dy \leq C \epsilon_t(|\nabla u|)(x) \), where \( C_1, C_2, C \) are positive constants (depending on \( n \)). These inequalities hold for \( n = 2 \) via the logarithmic kernel (see [32]). By density of \( C^\infty_0(\mathbb{R}^n) \) in \( D^{1,p}(\mathbb{R}^n) \), it is also true for any \( u \in D^{1,p}(\mathbb{R}^n)(n \geq 2) \).

**Case (i):** \( \eta_1 \neq \eta_2 \).
Let \( \eta_1 + \eta_2 = p^*(\alpha) \) with \( 1 < \eta_1 < \eta_2 < \eta_1 + \alpha \) and denote \( t := 1 - \frac{m-n}{\sigma} \in (0, 1) \). Take \( s = 1, \tilde{p} = p, \sigma = \frac{\alpha}{\alpha - 1} > 1, \max \left\{ p, 2\eta_1 - t\alpha, 2\eta_1 - \frac{t(1-s)}{\sigma(\alpha - 1)} \right\} > p^*(\alpha) \)

However, (11) is not applicable to problem (19) as the appearance of the coupled 
\[ \rho \]
the proof of Theorem 1.1 with Lemma 4.2.

Theorem 1.1 with 
\[ \eta \]
We may verify as Lemma 2.3 that 
\[ \sum \]
Let 
\[ \int \]
\[ \eta \]
\[ \eta \]
\[ \eta \]
Problem (18) can be solved by using the embeddings (10) and inequality (11).

Let \( \tilde{u} \) be a minimizing sequence of \( \Lambda \) while \( \sigma = \frac{p}{\alpha} > 1 \) in Lemma 3.1. The rest is similar to that of Theorem 1.1 with \( t := \frac{q}{p - \alpha} \) while \( r := \frac{p(1 - t) \alpha}{p - \alpha} = \frac{p \alpha}{p - \alpha} \).

Proof of Theorem 1.4. Imitate the proof of Theorem 1.2, we shall replace (12) by (13). We omit the details.

4. Solving the minimization problems (18) and (19). In this section, we solve the minimization problems (18) and (19).

Problem (18) can be solved by using the embeddings (10) and inequality (11). However, (11) is not applicable to problem (19) as the appearance of the coupled term \( \int_{\mathbb{R}^n} \frac{|u|^{q_1} |v|^{q_2}}{|y|^{\alpha}} \, dx \) in the definition of \( \Lambda_{n, s}(\eta_1, \eta_2, \alpha) \). We will use Theorem 1.2 to solve (19).

Lemma 4.1. Let \( s \in (0, 1) \), \( X = \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) and \( S_{n, s, \mu}(\alpha) \) be defined in (18).

1. If \( 0 < \alpha < 2s < n \), \( \mu \in (0, n) \) and \( \gamma_1, \gamma_2 < \gamma_H \), then \( S_{n, s, \mu}(\alpha) \) is attained in \( X \setminus \{(0, 0)\} \); (2) If \( n > 2s \), \( \mu \in (0, n) \) and \( 0 \leq \gamma_1, \gamma_2 < \gamma_H \), then \( S_{n, s, \mu}(0) \) is attained in \( X \setminus \{(0, 0)\} \).

Proof. (1) The proof is similar to that of Proposition 4.1-(1) in [24]; (2) If \( \alpha = 0 \) and \( 0 \leq \gamma_1, \gamma_2 < \gamma_H \), we shall combine the symmetric decreasing rearrangement technique with inequality (3). One can also refer to [12, 13, 24].

Lemma 4.2. Let \( s \in (0, 1) \), \( 0 < \alpha < 2s < n \), \( \gamma_1, \gamma_2 < \gamma_H \) and \( \eta_1 + \eta_2 = 2s \alpha \) satisfy \( 1 < \eta_1 \leq \eta_2 < \eta_1 + \frac{\alpha}{s} \). Then \( \Lambda_{n, s}(\eta_1, \eta_2, \alpha) \) defined in (19) is attained in \( \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \).

Proof. Let \( \{(u_k, v_k)\} \subset X = \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) be a minimizing sequence of \( \Lambda_{n, s}(\eta_1, \eta_2, \alpha) \), i.e.

\[ \int_{\mathbb{R}^n} \frac{|u_k(y)|^{\eta_1} |v_k(y)|^{\eta_2}}{|y|^\alpha} \, dy = 1, \quad ||(u_k, v_k)||^2 \to \Lambda_{n, s}(\eta_1, \eta_2, \alpha). \]

By Theorem 1.2, there exists a family of numbers \( \{\lambda_k > 0\} \) such that
\[ (\tilde{u}_k(x), \tilde{v}_k(x)) := \left( \lambda_k^{\frac{n - 2s}{2}} u_k(\lambda_k x), \lambda_k^{\frac{n - 2s}{2}} v_k(\lambda_k x) \right) \to (\tilde{u}, \tilde{v}) \text{ in } \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \]
for some \( (\tilde{u}, \tilde{v}) \in \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) with \( \tilde{u} \neq 0, \tilde{v} \neq 0 \). Since \( \Lambda_{n, s}(\eta_1, \eta_2, \alpha) \) is invariant under the dilations given by \( \lambda_k \), we have

\[ \int_{\mathbb{R}^n} \frac{|\tilde{u}_k(x)|^{\eta_1} |\tilde{v}_k(x)|^{\eta_2}}{|x|^\alpha} \, dx = 1, \quad ||(\tilde{u}_k, \tilde{v}_k)||^2 \to \Lambda_{n, s}(\eta_1, \eta_2, \alpha). \]

We may verify as Lemma 2.3 that

\[ 1 = \int_{\mathbb{R}^n} \frac{|\tilde{u}_k|^{\eta_1} |\tilde{v}_k|^{\eta_2}}{|x|^\alpha} \, dx = \int_{\mathbb{R}^n} \frac{|\tilde{u}_k - \tilde{u}|^{\eta_1} |\tilde{v}_k - \tilde{v}|^{\eta_2}}{|x|^\alpha} \, dx + \int_{\mathbb{R}^n} \frac{|\tilde{u}|^{\eta_1} |\tilde{v}|^{\eta_2}}{|x|^\alpha} \, dx + o(1). \]
By the weak convergence \((\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v})\) in \(X\) and \(\tilde{u} \neq 0\), \(\tilde{v} \neq 0\), we have
\[
\Lambda_{n,s}(\eta_1, \eta_2, \alpha) = \lim_{k \to \infty} \frac{\|\tilde{u}_k \eta_1^{|s|} \tilde{v}_k \eta_2^{|s|} dx}{\|\tilde{u}_k - \tilde{u} \eta_1^{|s|} \tilde{v}_k - \tilde{v} \eta_2^{|s|} dx} \geq \Lambda_{n,s}(\eta_1, \eta_2, \alpha)
\]

\[
\geq \frac{1}{2} \left[ \int_{\mathbb{R}^n} \left| \tilde{u}_k \eta_1^{|s|} \tilde{v}_k \eta_2^{|s|} dx \right|^2 \right]^{1/2} - \frac{1}{2} \left[ \int_{\mathbb{R}^n} \left| \tilde{u}_k - \tilde{u} \eta_1^{|s|} \tilde{v}_k - \tilde{v} \eta_2^{|s|} dx \right|^2 \right]^{1/2}.
\]

Here we use the fact that \((a + b)^{\frac{2}{2+\alpha}} \leq a^{\frac{2}{2+\alpha}} + b^{\frac{2}{2+\alpha}}, \forall a \geq 0, b \geq 0\) and \(\frac{2}{2+\alpha} < 1\). So we have
\[
\int_{\mathbb{R}^n} \left| \tilde{u}_k \eta_1^{|s|} \tilde{v}_k \eta_2^{|s|} dx \right|^2 = 1, \lim_{k \to \infty} \int_{\mathbb{R}^n} \left| \tilde{u}_k - \tilde{u} \eta_1^{|s|} \tilde{v}_k - \tilde{v} \eta_2^{|s|} dx \right|^2 = 0.
\]

It results to \(\|\tilde{u}, \tilde{v}\| \leq \Lambda_{n,s}(\eta_1, \eta_2, \alpha) \leq \|\tilde{u}, \tilde{v}\|^2\). Consequently,
\[
\Lambda_{n,s}(\eta_1, \eta_2, \alpha) = \|\tilde{u}, \tilde{v}\|, \lim_{k \to \infty} \left|\|\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}\|\right|^2 = 0.
\]

Formula (A.11) in [36] implies that \(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} \tilde{u}^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} \tilde{u}^2 dx\), hence, \(\|\tilde{u}, \tilde{v}\|\) is also a minimizer of \(\Lambda_{n,s}(\eta_1, \eta_2, \alpha)\), we can assume \(\tilde{u} \geq 0\) and \(\tilde{v} \geq 0\).

**Remark 3.** To prove Lemma 4.2, (11) does not work because of the coupled term \(\int_{\mathbb{R}^n} \left| \frac{\kappa_n}{|x|^{n-2s}} u \right|^2 dx\) in the definition of \(\Lambda_{n,s}(\eta_1, \eta_2, \alpha)\). Indeed, let \(\{u_k, v_k\} \subset X\) be a minimizing sequence of \(\Lambda_{n,s}(\eta_1, \eta_2, \alpha)\), i.e. \(\int_{\mathbb{R}^n} \left| \frac{\kappa_n}{|x|^{n-2s}} u_k \right|^2 dx = 1\) and \(\|u_k, v_k\|^2 \to \Lambda_{n,s}(\eta_1, \eta_2, \alpha)\), we have \(1 \leq \left( \int_{\mathbb{R}^n} \frac{|\kappa_n|^2}{|x|^{2s-\alpha}} dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^n} \frac{|\kappa_n|^2}{|x|^{2s-\alpha}} dx \right)^{\frac{1}{2}} \). Similar to [24], (11) implies that there exist \(\nu_k, \mu_k\) such that \(\kappa_k(x) = \nu_k^{\frac{n-2s}{2}} u_k(\nu_k x) \to \kappa \neq 0\), \(\kappa_k(x) = \mu_k^{\frac{n-2s}{2}} v_k(\mu_k x) \to \kappa \neq 0\), which gives \(\|u_k, v_k\|^2 \to \Lambda_{n,s}(\eta_1, \eta_2, \alpha)\), however, we cannot deduce \(\int_{\mathbb{R}^n} \left| \frac{\kappa_n}{|x|^{n-2s}} u_k \right|^2 dx = 1\).

5. **Proof of Theorem 1.5.** We shall now use the minimizers of \(S_{n,s,\mu}(\alpha)\) and \(\Lambda_{n,s}(\eta_1, \eta_2, \alpha)\) obtained in Lemmas 4.1-4.2, to prove the existence of a nontrivial weak solution for system (1). For any \((u, v) \in X = H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)\), the energy functional associated to (1) is:
\[
I(u, v) = \frac{1}{2} \left|\|u, v\|\right|^2 - \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{|\kappa_n|}{|x|^{2s}} u \right|^2 dx - \frac{1}{2} \cdot 2^\alpha f(u, v) \tag{34}
\]

where \(\|u, v\|\) and \(B_{u}(u, v)\) were defined in (15) and (17). Fractional Sobolev and Hardy-Sobolev inequalities yield that \(I \in C^1(X, \mathbb{R})\) such that for any \((\phi, \psi) \in X\)
\[
\langle I'(u, v), (\phi, \psi) \rangle_X = \sum_{i=1}^{2} \int_{\mathbb{R}^n} u \phi_i - \frac{1}{2^\alpha} \int_{\mathbb{R}^n} \left| \frac{\kappa_n}{|x|^{2s}} u \phi \right|^2 + \left| \frac{\kappa_n}{|x|^{2s}} v \psi \right|^2 \right|_{dx} \phi \psi dx,
\]

where \(\langle , \rangle_X\) was defined in (15).
Lemma 5.1. (Mountain pass lemma, [1]) Let $(E, \| \cdot \|)$ be a Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the following conditions:

(1) $I(0) = 0$,

(2) There exist $\rho, r > 0$ such that $I(z) \geq \rho$ for all $z \in E$ with $\|z\| = r$,

(3) There exist $t_0 \in E$ such that $\lim_{t \to +\infty} \sup I(tv_0) < 0$.

Let $t_0 > 0$ be such that $\|tv_0\| > r$ and $I(t_0v_0) < 0$, and define

$$c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t)),$$

where $\Gamma = \left\{ g \in C^0([0,1], E) : g(0) = 0, g(1) = t_0v_0 \right\}$. Then, $c \geq \rho > 0$ and there exists a (PS) sequence $\{z_k\} \subset E$ for $I$ at level $c$, i.e. $\lim_{k \to \infty} I(z_k) = c$ and $\lim I'(z_k) = 0$ strongly in $E'$.

We now use Lemma 5.1 to prove the following Proposition.

Proposition 2. Let $s \in (0,1)$, $0 < \alpha, \beta < 2s < n$, $\mu \in (0, n)$, $\gamma_1, \gamma_2 < \gamma_H$, $\eta_1, \eta_2 > 1$ and $\eta_1 + \eta_2 = 2^*(\beta)$. Consider the functional $I$ defined in (34) on the Banach space $X = H^s(\mathbb{R}^n) \times H^{s}(\mathbb{R}^n)$. Then there exists a (PS) sequence $\{(u_k, v_k)\} \subset X$ for $I$ at some $c \in (0, c^*)$, i.e.

$$\lim_{k \to +\infty} I(u_k, v_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k, v_k) = 0 \quad \text{strongly in } X',$$

where $c^* := \min \left\{ \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{s,n,\mu}(\alpha)^{-\frac{\mu}{2}} \frac{2^\mu(\alpha)-1}{2} \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta)^{-\frac{n-\beta}{2}} \right\}$.

Proof. We now verify the conditions of Lemma 5.1. For any $(u, v) \in X$,

$$I(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{2s(\beta)} \int_{\mathbb{R}^n} \frac{|u|^2 |v|^{2s}}{|x|^\beta} dx - \frac{1}{2 \cdot 2^\mu(\alpha)} B_\alpha(u \oplus v),$$

$$\geq \frac{1}{2} \|(u, v)\|^2 - C_1 \|(u, v)\|^{2^*(\beta)} - C_2 \|(u, v)\|^2 \cdot 2^\mu(\alpha).$$

Since $s \in (0,1)$, $0 < \alpha, \beta < 2s < n$ and $\mu \in (0, n)$, we have that $2^*(\beta) > 2$ and $2 \cdot 2^\mu(\alpha) > 2^*(\alpha) > 2$. Therefore, there exists $r > 0$ small enough such that

$$\inf_{\|u\| = r} I(u, v) > 0 = I(0, 0),$$

so (1) and (2) of Lemma 5.1 are satisfied. From

$$I(t(u, v)) = \frac{t^2}{2} \|(u, v)\|^2 - \frac{t^{2^*(\beta)}}{2s(\beta)} \int_{\mathbb{R}^n} \frac{|u|^2 |v|^{2s}}{|x|^\beta} dx - \frac{t^{2 \cdot 2^\mu(\alpha)}}{2 \cdot 2^\mu(\alpha)} B_\alpha(u \oplus v),$$

we derive that $\lim_{t \to +\infty} I(t(u, v)) = -\infty$ for any $(u, v)$ in $X$. Consequently, for any fixed $V_0 = (v_0, \bar{v}_0) \in X$, there exists $t_0 > 0$ such that $\|tV_0V_0\| = \|tV_0(v_0, \bar{v}_0)\| > r$ and $I(tV_0V_0) < 0$. So (3) of Lemma 5.1 is satisfied.

Using Lemmas 4.1-4.2, we obtain a minimizer $U_\alpha = (u_\alpha, \bar{u}_\alpha) \in X$ for $S_{s,n,\mu}(\alpha)$ and $W_\beta = (w_\beta, \bar{w}_\beta) \in X$ for $\Lambda_{n,s}(\eta_1, \eta_2, \beta)$ respectively. So there exist

$$V_0 := \begin{cases} U_\alpha, & \text{if } \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{s,n,\mu}(\alpha)^{-\frac{\mu}{2}} \frac{2^\mu(\alpha)-1}{2} \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta)^{-\frac{n-\beta}{2}}; \\
W_\beta, & \text{if } \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{s,n,\mu}(\alpha)^{-\frac{\mu}{2}} \frac{2^\mu(\alpha)-1}{2} \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta)^{-\frac{n-\beta}{2}} \end{cases}$$
and \( t_0 > 0 \) such that \(|t_0 V_0| > r\) and \( I(t_0 V_0) < 0\). We can define
\[
c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t))
\]
where \( \Gamma := \{ g(t) = (g_1(t), g_2(t)) \in C^0([0,1], X) : g(0) = (0,0), g(1) = t_0 V_0 \} \).

Clearly, we have \( c > 0 \). If \( V_0 = U_\alpha \), we can derive that \( 0 < c < \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{n,\mu}(\alpha) \frac{2^\mu(\alpha)}{2^\mu(\alpha)-1} \). In fact, for \( U_\alpha = (u_\alpha, \tilde{u}_\alpha) \in X \) and every \( t \geq 0 \), we have
\[
I(tU_\alpha) \leq f_1(t) := \frac{t^2}{2} ||(u_\alpha, \tilde{u}_\alpha)||^2 - \frac{t^2 \cdot 2^\mu(\alpha)}{2 \cdot 2^\mu(\alpha)} B_\alpha(u_\alpha + \tilde{u}_\alpha).
\]

We also compute that \( f_1(t) \) attains its maximum at \( \hat{t} = \left( \frac{||(u_\alpha, \tilde{u}_\alpha)||^2}{B_\alpha(u_\alpha + \tilde{u}_\alpha)} \right)^{\frac{1}{2^\mu(\alpha)-1}} \) and
\[
\sup_{t \geq 0} f_1(t) = \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{n,\mu}(\alpha) \frac{2^\mu(\alpha)}{2^\mu(\alpha)-1}.
\]

The equality does not hold in (36), otherwise, we would have that \( \hat{I} = \sup_{t \geq 0} I(tU_\alpha) = \sup_{t \geq 0} f_1(t) \). Let \( t_1 > 0 \) where \( \sup_{t \geq 0} I(tU_\alpha) \) is attained. We have
\[
f_1(t_1) \geq \frac{t_1^2 \cdot 2^\mu(\beta)}{2^\mu(\beta)} \int_{\mathbb{R}^n} \frac{|u_\alpha|^{\eta_1}|\tilde{u}_\alpha|^{\eta_2}}{|x|^\beta} dx = f_1(\hat{t}),
\]
which means that \( f_1(t_1) > f_1(\hat{t}) \) since \( t_1 > 0 \). This contradicts the fact that \( \hat{t} \) is the unique maximum point of \( f_1(t) \). Thus, we have
\[
\sup_{t \geq 0} I(tU_\alpha) < \sup_{t \geq 0} f_1(t) = \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{n,\mu}(\alpha) \frac{2^\mu(\alpha)}{2^\mu(\alpha)-1}.
\]

If \( V_0 = W_\beta \), similarly, we can verify \( \sup_{t \geq 0} I(tW_\beta) < \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta) \frac{n-\beta}{2(n-\beta)} \). So we have \( 0 < c < c^* := \min \left\{ \frac{2^\mu(\alpha)-1}{2 \cdot 2^\mu(\alpha)} S_{n,\mu}(\alpha) \frac{2^\mu(\alpha)}{2^\mu(\alpha)-1}, \frac{2s-\beta}{2(n-\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta) \frac{n-\beta}{2(n-\beta)} \right\} \).

Since (1)-(3) of Lemma 5.1 are satisfied, there exists a \( (PS) \) sequence \( \{u_k, v_k\} \subset X \) for \( I \) at some \( c \in (0, c^*) \), that is, \( \lim_{k \to \infty} I(u_k, v_k) = c \) and \( \lim \right. \right. \left. \right. I'(u_k, v_k) = 0 \) in \( X' \).

**Proof of Theorem 1.5. Case (A):** \( \eta_1 + \eta_2 = 2s(\beta) \) satisfy \( 1 < \eta_1 < \eta_2 < \eta_1 + \frac{\beta}{s} \).

\textbf{(A-I)} The case \( s \in (0, 1), 0 < \alpha, \beta < 2s < n, \mu \in (0, n) \) and \( \gamma_1 \leq \gamma_2 < \gamma_H \).

Let \( \{ (u_k, v_k) \}_{k \in \mathbb{N}} \) be a \( (PS) \) sequence as in Proposition 2, i.e. \( I(u_k, v_k) \to c \) and \( I'(u_k, v_k) \to 0 \) strongly in \( X' \) as \( k \to +\infty \). Then, we have
\[
I(u_k, v_k) = \frac{1}{2} ||(u_k, v_k)||^2 - \frac{1}{2s(\beta)} \int_{\mathbb{R}^n} \frac{|u_k|^{\eta_1}|v_k|^{\eta_2}}{|x|^\beta} dx - \frac{1}{2 \cdot 2^\mu(\alpha)} B_\alpha(u_k + v_k)
\]
\[
= c + o(1)
\]
and
\[
(I'(u_k, v_k), (u_k, v_k))_X = \|(u_k, v_k)\|^2 - \int_{\mathbb{R}^n} \frac{|u_k|^{\eta_1} |v_k|^{\eta_2}}{|x|^\beta} dx - B_\alpha(u_k \oplus v_k)
\]
\[= o(1)\|(u_k, v_k)\|, \tag{38}
\]
where \(B_\alpha(u_k \oplus v_k)\) was defined in (17).

From (37) and (38), if \(2 \cdot 2^{\#}(\alpha) \geq 2^*_s(\beta) > 2\), we have
\[
c + o(1)\|(u_k, v_k)\| = I(u_k, v_k) - \frac{1}{2^*_s(\beta)} (I'(u_k, v_k), (u_k, v_k))_X \\
\geq \left( \frac{1}{2} - \frac{1}{2^*_s(\beta)} \right)\|(u_k, v_k)\|^2.
\]
If \(2^*_s(\beta) > 2 \cdot 2^{\#}(\alpha) > 2\), we have
\[
c + o(1)\|(u_k, v_k)\| = I(u_k, v_k) - \frac{1}{2 \cdot 2^{\#}(\alpha)} (I'(u_k, v_k), (u_k, v_k))_X \\
\geq \left( \frac{1}{2} - \frac{1}{2 \cdot 2^{\#}(\alpha)} \right)\|(u_k, v_k)\|^2.
\]
Thus, \(\{(u_k, v_k)\}\) is bounded in \(X\), then from (38) there exists a subsequence, still denoted by \(\{(u_k, v_k)\}\), such that
\[
\|(u_k, v_k)\|^2 \to b, \int_{\mathbb{R}^n} \frac{|u_k|^{\eta_1} |v_k|^{\eta_2}}{|x|^\beta} dx \to d_1, \quad B_\alpha(u_k \oplus v_k) \to d_2, \quad b = d_1 + d_2.
\]
By the definition of \(\Lambda_{n,s}(\eta_1, \eta_2, \beta)\) and \(S_{n,s,\mu}(\alpha)\), we get
\[
d_1^{2^{\#}(\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta) \leq b, \quad d_2^{2^{\#}(\alpha)} S_{n,s,\mu}(\alpha) \leq b.
\]
Therefore \(d_1^{2^{\#}(\beta)} \Lambda_{n,s}(\eta_1, \eta_2, \beta) \leq d_1 + d_2\) and \(d_2^{2^{\#}(\alpha)} S_{n,s,\mu}(\alpha) \leq d_1 + d_2\). It results that
\[
d_1^{2^{\#}(\beta)} \left( \Lambda_{n,s}(\eta_1, \eta_2, \beta) - d_1^{2^{\#}(\beta) - 2^{\#}(\beta)} \right) \leq d_2, \quad d_2^{2^{\#}(\alpha)} \left( S_{n,s,\mu}(\alpha) - d_2^{2^{\#}(\alpha) - 2^{\#}(\alpha)} \right) \leq d_1. \tag{39}
\]
We claim that \(\Lambda_{n,s}(\eta_1, \eta_2, \beta) - d_1^{2^{\#}(\beta) - 2^{\#}(\beta)} > 0\) and \(S_{n,s,\mu}(\alpha) - d_2^{2^{\#}(\alpha) - 2^{\#}(\alpha)} > 0\). Indeed, by the fact that \(c + o(1)\|(u_k, v_k)\| = I(u_k, v_k) - \frac{1}{2} (I'(u_k, v_k), (u_k, v_k))_X\), we have
\[
\left( \frac{1}{2} - \frac{1}{2^*_s(\beta)} \right) \int_{\mathbb{R}^n} \frac{|u_k|^{\eta_1} |v_k|^{\eta_2}}{|x|^\beta} dx + \left( \frac{1}{2} - \frac{1}{2 \cdot 2^{\#}(\alpha)} \right) B_\alpha(u_k \oplus v_k) = c + o(1)\|(u_k, v_k)\|.
\]
Let \(k \to +\infty\), we have
\[
\left( \frac{1}{2} - \frac{1}{2^*_s(\beta)} \right) d_1 + \left( \frac{1}{2} - \frac{1}{2 \cdot 2^{\#}(\alpha)} \right) d_2 = c, \tag{40}
\]
It results to \(d_1 \leq \frac{2(\alpha - \beta)}{2^{\#}(\alpha)} c\) and \(d_2 \leq \frac{2 \cdot 2^{\#}(\alpha)}{2^{\#}(\alpha) - 1} c\). Recall that \(0 < c < c^*\), we have
\[
\Lambda_{n,s}(\eta_1, \eta_2, \beta) - d_1^{2^{\#}(\beta) - 2^{\#}(\beta)} \geq A_1 > 0, \quad S_{n,s,\mu}(\alpha) - d_2^{2^{\#}(\alpha) - 2^{\#}(\alpha)} \geq A_2 > 0,
\]
where \( A_1 = \Lambda_{n,s}(\eta_1, \eta_2, \beta) - \left[ \frac{2(n-\beta)}{2s-\beta} \right] \frac{2n^2}{2s^2-\beta^2} \) and \( A_2 = S_{n,s,\mu}(\alpha) - \left[ \frac{2^\mu}{2^\alpha - \mu} \right] \frac{2^\alpha}{2^\alpha - \mu} \). Thus (39) gives

\[
d_1^{\frac{2(\beta)}{2s-\beta}} A_1 \leq d_2, \quad d_2^{\frac{1}{\frac{2^\mu}{2^\alpha - \mu}}} A_2 \leq d_1.
\]

If \( d_1 = 0 \) and \( d_2 = 0 \), then (40) implies that \( c = 0 \), a contradiction to \( c > 0 \). Therefore \( d_1 > 0 \) and \( d_2 > 0 \), we can choose \( \varepsilon_0 > 0 \) such that \( d_1 \geq \varepsilon_0 > 0 \) and \( d_2 \geq \varepsilon_0 > 0 \), so there exists a \( K > 0 \) large such that \( k \geq K \) and

\[
\int_{\mathbb{R}^n} \frac{|u_k(y)|^{\eta_1}|v_k(y)|^{\eta_2}}{|y|^\beta} dy > \varepsilon_0/2, \quad B_\alpha(u_k \mp v_k) > \varepsilon_0/2.
\]

Let \( k \geq K \), by Theorem 1.2, there exists a family of numbers \( \{\lambda_k > 0\} \) such that

\[
(\tilde{u}_k(x), \tilde{v}_k(x)) := (\lambda_k^{\frac{n-2}{2}} u_k(\lambda_k x), \lambda_k^{\frac{n-2}{2}} v_k(\lambda_k x)) \rightarrow (\tilde{u}, \tilde{v}) \quad \text{in} \quad \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n)
\]

for some \((\tilde{u}, \tilde{v}) \in \dot{H}^s(\mathbb{R}^n) \times \dot{H}^s(\mathbb{R}^n) \) with \( \tilde{u} \neq 0, \tilde{v} \neq 0 \).

In addition, the fact \(|((\tilde{u}_k, \tilde{v}_k))|^2 = ||u_k, v_k||^2 \leq C \) implies that \(||\tilde{u}_k|^{\eta_1-2}\tilde{u}|^{\eta_2}||^2 \) and \(||\tilde{v}_k|^{\eta_2-2}\tilde{v}|^{\eta_1}||^2 \) are bounded in \( L^{2^*(\beta)}(\mathbb{R}^n, |x|^{-\beta}) \), therefore,

\[
|\tilde{u}_k|^{\eta_1-2}\tilde{u}|^{\eta_2} \rightarrow |\tilde{u}|^{\eta_1-2}\tilde{u}|^{\eta_2} \quad \text{in} \quad L^{2^*(\beta)}(\mathbb{R}^n, |x|^{-\beta}),
\]

\[
|\tilde{v}_k|^{\eta_2-2}\tilde{v}|^{\eta_1} \rightarrow |\tilde{v}|^{\eta_2-2}\tilde{v}|^{\eta_1} \quad \text{in} \quad L^{2^*(\beta)}(\mathbb{R}^n, |x|^{-\beta}). \quad (41)
\]

For any \((\phi, \psi) \in X\), by Lemma 2.9 of [24], we have

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^n} [I_\mu \ast F_\alpha(\cdot, \tilde{u}_k)](x) f_{s}(x, \tilde{u}_{\phi}(x)) \phi(x) dx = \int_{\mathbb{R}^n} [I_\mu \ast F_\alpha(\cdot, \tilde{u})](x) f_{s}(x, \tilde{u}) \phi(x) dx,
\]

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^n} [I_\mu \ast F_\alpha(\cdot, \tilde{v}_k)](x) f_{s}(x, \tilde{v}_{\psi}(x)) \psi(x) dx = \int_{\mathbb{R}^n} [I_\mu \ast F_\alpha(\cdot, \tilde{v})](x) f_{s}(x, \tilde{v}) \psi(x) dx. \quad (42)
\]

Finally, we need to check that \( \{(\tilde{u}_k, \tilde{v}_k)\} \) is also a \((PS)\) sequence for \( I \) at energy level \( c \). Since the norms in \( H^s(\mathbb{R}^n) \) and \( L^{2^*(\beta)}(\mathbb{R}^n, |x|^{-\alpha}) \) are invariant under the special dilations \( \tilde{u}_k(x) = \lambda_k^{\frac{-n-2}{2}} u_k(\lambda_k x) \) and \( \tilde{v}_k(x) = \lambda_k^{\frac{-n-2}{2}} v_k(\lambda_k x) \), we have \( \lim_{k \to +\infty} I(\tilde{u}_k, \tilde{v}_k) = \lim_{k \to +\infty} I(u_k, v_k) = c \). Moreover, \( \forall (\phi, \psi) \in X \), we have

\[
(\phi_k(x), \psi_k(x)) = (\lambda_k^{\frac{2s-n}{2}} \phi(\frac{x}{\lambda_k}), \lambda_k^{\frac{2s-n}{2}} \psi(\frac{x}{\lambda_k})) \in X.
\]

From \( I'(u_k, v_k) \to 0 \) in \( X' \), we can derive that

\[
\lim_{k \to +\infty} \langle I'(u_k, v_k), (\phi, \psi) \rangle_X = \lim_{k \to +\infty} \langle I'(u_k, v_k), (\phi_k, \psi_k) \rangle_X = 0.
\]

Thus, (41) and (42) lead to \( \langle I'(\tilde{u}, \tilde{v}), (\phi, \psi) \rangle_X = \lim_{k \to +\infty} \langle I'(\tilde{u}_k, \tilde{v}_k), (\phi, \psi) \rangle_X = 0 \). If \( \tilde{u} \equiv \tilde{v} \), we deduce from \( \eta_1 < \eta_2 \) and \( \gamma_1 \leq \gamma_2 \) that \( 0 \leq (\gamma_2 - \gamma_1) \int_{\mathbb{R}^n} |\tilde{v}^2| dx = \frac{\eta_2}{2^*(\beta)} \int_{\mathbb{R}^n} |\tilde{v}^{2^*(\beta)}| dx < 0 \), this contradiction implies \( \tilde{u} \neq \tilde{v} \). Hence \((\tilde{u}, \tilde{v})\) is a nontrivial weak solution to (1).

(A-II) The case \( s \in (0, 1), \alpha = 0 < \beta < 2s < \eta \), \( \mu \in (0, n) \) and \( 0 \leq \gamma_1 \leq \gamma_2 < \gamma_H \). Since \( \beta > 0 \), Theorem 1.2 is still effective, so we can get a nontrivial weak solution to (1) as (A-I). Notice that \( S_{n,s,\mu}(0) \) is attained provided \( 0 \leq \gamma_1, \gamma_2 < \gamma_H \), see Lemma 4.1.

Case (B): \( \eta_1 = \eta_2 = \frac{2^*(\beta)}{2} \) and \( \gamma_1 \neq \gamma_2 \). The proof is similar to that of Case (A), but we shall restrict \( \gamma_1 \neq \gamma_2 \) since \( \eta_1 = \eta_2 \).
Remark 4. By using Theorem 1.4, we can proceed as in the proof of Theorem 1.5 to derive the existence of a nontrivial solution \((u, v) \in D^{1, p}(\mathbb{R}^n) \times D^{1, p}(\mathbb{R}^n)\) to the following doubly critical system involving p-Laplacian in \(\mathbb{R}^n\) with Hardy terms:

\[
\begin{aligned}
\mathcal{L}_{\kappa_1} u &:= \left( \int_{\mathbb{R}^n} \frac{|u(y)|^{p^*_\mu(\alpha)}|y|^\delta_\mu}{|x|^\delta_\mu(\alpha)} dy \right) \frac{|u(x)|^{p^*_\alpha(\beta)}u(x)}{|x|^\delta_\alpha(\beta)} + \frac{\eta_1}{p^*(\beta)} \frac{|u|^{\eta_1 - 2}u|v|^{\eta_2}}{|x|^\beta}, \\
\mathcal{L}_{\kappa_2} v &:= \left( \int_{\mathbb{R}^n} \frac{|v(y)|^{p^*_\mu(\alpha)}|y|^\delta_\mu}{|x|^\delta_\mu(\alpha)} dy \right) \frac{|v(x)|^{p^*_\alpha(\beta)}v(x)}{|x|^\delta_\alpha(\beta)} + \frac{\eta_2}{p^*(\beta)} \frac{|v|^{\eta_2 - 2}v|u|^{\eta_1}}{|x|^\beta}
\end{aligned}
\]  
\tag{43}

where \(\mathcal{L}_{\kappa_1} u := -\Delta_p u - \kappa_1 \frac{|u|^{p^*_\mu(\alpha)}}{|x|^\delta_\mu(\alpha)}\), \(n \geq 2\), \(p \in (1, n)\), \(\kappa_2 < \kappa := [(n - p)/p]^\mu (i = 1, 2)\), \(\mu \in (0, n), 0 \leq \alpha, \beta < p, \eta_1, \eta_2 > 1, \eta_1 + \eta_2 = p^*(\beta) = \frac{p(n - \beta)}{n - p}, \delta_\mu(\alpha) = (1 - \frac{\mu}{2n})\alpha\) and \(\delta_\mu(\alpha) = (1 - \frac{\mu}{2n})\alpha\). In fact, we have

Theorem 5.2. Let \(\eta_1 + \eta_2 = p^*(\beta)\) satisfy 1 < \(\eta_1 < \eta_2 < \eta_1 + \beta\), then system (43) possesses at least a nontrivial weak solution provided either (I) \(p \in [2, n)\), 0 < \(\alpha, \beta < p\), 0 < \(\mu < n\) and \(\kappa_1 \leq \kappa < \kappa\) or (II) \(p \in [2, n)\), \(\alpha = 0 < \beta < p\), 0 < \(\mu < n\) and 0 < \(\kappa_1 \leq \kappa < \kappa\): If \(\eta_1 = \eta_2 = \frac{p^*(\beta)}{2}\) and \(\kappa_1 \neq \kappa_2\), then the previous conclusion remains true if (I) or (II) holds.

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