Circuit Complexity of Visual Search

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Abstract
We investigate computational hardness of the feature search and conjunction search through the lens of circuit complexity. Let \( x_1, \ldots, x_n \) (resp., \( y_1, \ldots, y_n \)) be Boolean variables each of which takes the value one if and only if a neuron at place \( i \) detects a common feature (resp., another common feature). We then simply formulate the feature search and the conjunction search as Boolean functions

\[
F_{TR}^n(x_1, \ldots, x_n) = \bigvee_{i=1}^n x_i
\]

and

\[
CONJ^n(x_1, \ldots, x_n, y_1, \ldots, y_n) = \bigvee_{i=1}^n x_i \land y_i,
\]

respectively. We employ a threshold circuit or a discretized circuit (such as a sigmoid circuit or a ReLU circuit where activation functions and weights are discretized) as our theoretical models of neural networks, and consider the following four computational resources of circuits: [i] the number of neurons (size), [ii] the number of levels (depth), [iii] the maximum number of concurrently active neurons that are allowed to output non-zero values during computation (energy), and [iv] synaptic weight resolution (weight).

We first prove that any threshold circuit \( C \) of size \( s \), depth \( d \), energy \( e \) and weight \( w \) satisfies

\[
\log(rk(M_C)) \leq ed \log(s + \log w + \log n),
\]

where \( rk(M_C) \) is the rank of the communication matrix \( M_C \) of a 2n-variable Boolean function that \( C \) computes. Thus, such a threshold circuit \( C \) is able to compute only a Boolean function of which communication matrix has rank bounded by a product of logarithmic factors of \( s, w \) and linear factors of \( d, e \). Since \( CONJ_n \) has rank \( 2^n \), it holds that

\[
n \leq ed \log(s + \log w + \log n),
\]

which implies an exponential lower bound on the size of even sublinear-depth threshold circuit if energy and weight are sufficiently small. Since \( F_{TR} \) is computable by a single threshold gate independently of \( n \), our result suggests that capacity limitation for the feature and conjunction search may be inherently different, and the depth and the number of active neurons that are allowed to output non-zero values during computation (energy), and [iv] synaptic weight resolution (weight).

We next consider a discretized circuit, and prove that a similar inequality also holds for a discretized circuit \( C: rk(M_C) = O(ed \log(s + \log w + \log n)^3) \). We obtain this bound by showing that a discretized circuit can be simulated by a threshold circuit with moderate increase of size, depth, energy and weight. Thus, if we consider the number of non-zero output values as a measure for sparse activity of a neural network, our results suggest that larger depth helps neural networks to acquire sparse activity, which may shed light on the reason for the success of deep learning.

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1 Introduction

Background. DiCarlo and Cox argued that constructing good internal representations is crucial to perform object recognition for neural networks in the brain [8]. Here, an internal representation is given by a feedforward neural network, and described by a vector in a very high dimensional space, where each axis is one neuron’s activity (e.g., firing rate over an \( \sim 200 \) ms interval) and the dimensionality equals to the number (e.g., \( \sim 1 \) million) of
neurons performing the task. They call representations good if a given pair of two images that are hard to distinguish at the input space is mapped to the high dimensional space in which the resulting two internal representations are now easy to separate by simple classifiers such as a linear classifier. This perspective is important not only for the brain computation but machine learning based on artificial neural networks such as deep learning, because an artificial neural network successfully obtained through a deep learning process would provide such good internal representations.

In this paper, we consider visual search tasks, perceptual tasks related to visual information processing, and investigate how computationally hard we construct good internal representations for the feature search and conjunction search. Visual search asks a subject to look for a target within a background of distractors, and to decide whether a target exists or not as quickly and accurately as possible. Feature search is a type of visual search where a salient primary feature, such as a color or an orientation, easily distinguishes a target from the distractors. Finding a red object among blue objects, or finding a letter X among letters Os are examples of the feature search. Conjunction search is a similar task, but a target and distractors share a feature. Finding a red letter X among red letter Os and blue letter Xs is an example of the conjunction search.

In the early studies of visual search, Treisman and Gelade [46] reported that time taken for decisions, the so-called reaction time, for the feature search is constant to (that is, independent of) the number of distractors, while the counterpart for conjunction search increases in the number of distractors, which suggests that neural mechanisms behind the feature search and conjunction search are different. They proposed a theory claiming that the information processing for visual search consists of two stages, that is, preattentive (parallel) stage and attentive (serial) stage. While a limited set of basic features can be processed in parallel, other searches require serial search, and produce the set-size effect. In the last decades, further studies revealed that their theory cannot completely explain the issue, and new theories have tried to account for a variety of experimental results designed since then (see, for example, [9, 15, 40, 52, 53]). The research is still actively ongoing, and we have, to the best of our knowledge, no general theory that is able to account for numerous experimental results. In the paper [34], Nakayama and Martini applied the concept of the DiCarlo and Cox formulation to the domain of visual search, and proposed a direction of future research for visual search that comprehensively handle a variety of pattern recognition problems including the object recognition and visual search tasks. They suggested that any pattern recognition task has necessary detailness (the number of characteristics) and scope (the number of objects) to describe, and there is a tradeoff between them: one can have either lots of dimensions or lots of objects but not both due to limitations of computational resources for neural networks.

We explore a computational aspect of the feature search and conjunction search through the lens of circuit complexity. Circuit complexity is a branch of computational complexity theory in theoretical computer science that aims at clarifying complexity of a Boolean function in terms of computational resources for logic circuits. Let $C$ be a model of a logic circuit, and $f$ be a Boolean function for discussion. In a typical lower bound argument, we introduce a complexity measure for Boolean functions, and show that any Boolean function computable by a circuit chosen from $C$ has low complexity that is bounded by its available resources. If $f$ has inherent complexity that requires circuits to possess high complexity, we can show that no circuit in $C$ is able to compute $f$ if the computational resources are short.

We believe that lower bound arguments provide insights into neural computation. Besides the famous Marr's three-level approach towards understanding the brain computation
(computational level, algorithmic level, and implementation level) [31], Valiant [51] added an additional requirement that it has to incorporate some understanding of the quantitative constraints that are faced by cortex. We expect that a lower bound argument could exhibit such a quantitative constraint through a complexity measure. Maass et al. [29] pointed out a difficulty to uncover a neural algorithm employed by the brain, because its hardware could be extremely adopted to the task, and consequently the algorithm vanishes: even if we know to the last detail its precise structure, connectivity, and vast array of numerical parameters (an artificial neural network given by deep learning is the case), it is still hard to extract an algorithm implemented in the network. A lower bound argument does not provide a description of an explicit neural algorithm, but could give a hint for computational natures behind neural computation, because a lower bound argument necessarily deals with every algorithm which a theoretical model of a neural network is able to implement.

We below make simplifications in the visual search tasks and neural networks, and then show that a tradeoff exists among several computational resources.

**Our Theoretical Model.** In the papers [23, 24, 25], Maass and Legenstein defined Boolean functions related to pattern recognition of which input space consists of neurons detecting common features, and investigated them from the viewpoint of circuit complexity. We follow their formulation, and simplify the feature search and conjunction search as follows.

We assume that an input space consists of a group of neurons \( x = (x_1, \ldots, x_n) \), where \( n \) is the number of neurons, each of which detects a primary common feature at a specific place \( i, 1 \leq i \leq n \), and \( x_i = 1 \) (i.e., \( x_i \) is active) if and only if the feature is detected at the place \( i \). We then formulate the feature search as \( \text{FTR}_n(x) = \bigvee_{i=1}^n x_i \), which is identical to the OR function of \( n \) Boolean variables. For conjunction search, we introduce, besides \( x_1, \ldots, x_n \), another group of neurons \( y = (y_1, \ldots, y_n) \) each of which detects another primary common feature at the place \( i \). We then formulate the conjunction search as \( \text{CONJ}_n(x) = \bigvee_{i=1}^n x_i \wedge y_i \) which is evaluated to one if and only if there is a place \( i \) where both features are detected.

We first consider a threshold circuit as our circuit model. A threshold circuit is a feedforward logic circuit where a basic computational element is a simple integrate-and-fire neuron equipped with the threshold function. A threshold circuit is traditionally considered to be a simple theoretical model of a neural network [32, 42], and have received considerable attention in circuit complexity for decades [33, 38].

We consider the following four computational resources well-studied in circuit complexity: [i] the number of neurons (size), [ii] the number of levels (depth), [iii] the maximum number of active neurons allowed to support an internal representation (energy), and [iv] synaptic weight resolution (weight). It is plausible that neural networks have restricted amount of these resources. Clearly, the visual system in the brain has a limited number of neurons, and the number of neurons performing a particular visual information processing could be further restricted. The depth is related to the reaction time for performing tasks, and hence is favorable to be small. The energy was introduced in [48] for threshold circuits based on the biological fact that the energy cost of a single spike is known to be high in average, and hence a relatively small number of neurons out of a large population in the nervous system can be simultaneously active [10, 20, 37]. Experimental results show that the maximum synaptic weight is reported to be less than 10 mv for some areas of the brain [3]. Although a single synapse weight might be an arbitrary real number between 0 mv and 10 mv, it is unlikely that a neuron can distinguish between synaptic weights, say, 0.5 mv and 0.5 ± 2⁻²⁰ mv against neural noise. Thus, we may assume that a neuron has bounded degree of resolution.

**Discussion on Known Results.** Threshold circuits have received considerable attention in circuit complexity. It is known that even polynomial-size and constant-depth threshold
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circuits can compute a variety of Boolean functions including basic arithmetic operations such as addition, multiplication, division, etc. \cite{35,34,11,15}. Also, a number of lower bound arguments have developed for threshold circuits under some restrictions on computational resources including size, depth, energy and weight \cite{1,2,5,12,13,17,20,30,35,41,47,49,50}. Currently, it is a long-standing open problem in circuit complexity to show an exponential lower bound on the size of depth-2 threshold circuits without any other restrictions for an explicit function \cite{5}. Also, the complement of CONJ\(_n\), the disjointness function, is well-studied in the literature \cite{4}, and lower bounds on the size of threshold circuits computing the disjointness function are known \cite{28,56}.

We can observe that such known results already imply a difference between FTR\(_n\) and CONJ\(_n\). Since the input space of FTR\(_n\) is linearly separable, even a threshold circuit of a single gate can compute FTR\(_n\), for any \(n\). Thus, FTR\(_n\) is computable independently of the number of inputs. Contrasting with FTR\(_n\), the size \(s\) of threshold circuit computing CONJ\(_n\) is not independent of the number of input variables: it holds that

\[
n / \log(n + 1) \leq s
\]  

(1)

even if the depth, energy and weight are unlimited \cite{28}. Thus, CONJ\(_n\) requires threshold circuits to have almost linear number of gates in \(n\), which suggests that computational capacity for the feature and conjunction search are different.

**Our Results for Circuit Complexity.** Our main result strengthens the observation above, and shows that the difference is much sharper when we take all the four resources into account.

Let \(C\) be a threshold circuit with Boolean input variables \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\). We employ the so-called communication matrix which is a well-known tool used in a wide variety of research areas including circuit complexity (see, for example, \cite{19,21}). The communication matrix \(M_C\) of \(C\) is a \(2^n \times 2^n\) matrix where each row (resp., each column) is indexed by an assignment \(a \in \{0,1\}^n\) to \(x\) (resp., \(b \in \{0,1\}^n\) to \(y\)), and the value \(M_C[a,b]\) is defined to be the output of \(C\) given \(a\) and \(b\). We use the rank of a communication matrix as our complexity measure, and denote by \(\text{rk}(M_C)\) the rank of \(M_C\) over \(F_2\).

The following theorem is our main result:

**Theorem 1.** Let \(s, d, e\) and \(w\) be integers satisfying \(2 \leq s, d, 10 \leq e, 1 \leq w\). If a threshold circuit \(C\) computes a Boolean function of \(2n\) variables, and has size \(s\), depth \(d\), energy \(e\) and weight \(w\), then it holds that

\[
\log(\text{rk}(M_C)) \leq ed(\log s + \log w + \log n).
\]  

(2)

The theorem implies that \(C\) is able to compute a Boolean function whose communication matrix has rank bounded by a product of logarithmic factors of \(s, w\) and linear factors of \(d, e\). Since CONJ\(_n\) has full rank \(2^n\) \cite{18}, the theorem implies that

\[
n \leq ed(\log s + \log w + \log n)
\]  

(3)

holds if a threshold circuit \(C\) computes CONJ\(_n\). Arranging Eq. (3), we can obtain a lower bound \(2^{n/(ed)}/(w^m) \leq s\) which is exponential in \(n\) if a product of \(e\) and \(d\) is sub-linear and \(w\) is sub-exponential. For example, we can obtain an exponential lower bound \(s = 2^{O(n^{1/3})}\) even for threshold circuits of depth \(n^{1/3}\), energy \(n^{1/3}\) and weight \(2^{o(n^{1/3})}\).

While several exponential lower bounds are known for threshold circuits, all the arguments are designated for constant-depth threshold circuits, and these lower bounds cannot be
meaningful for threshold circuits of non-constant depth. To the best of our knowledge, our lower bound is the first nontrivial exponential lower bound for threshold circuits of non-constant depth. The known result most related to ours is the one in [49] showing $s = 2^{\Omega(n/e^d)}$ for size $s$, depth $d$ and energy $e$ of any unrestricted-weight threshold circuit computing a high bounded-error communication complexity function $f$. Although Our lower bound depends on the weight, it can be exponential even if $d$ is sub-linear, and works for much weaker condition: the rank $\Omega(n)$ suffices.

We also show that the lower bound in Eq. (3) is tight up to a constant factor. More formally, we show that $\text{CONJ}_n$ is computable by a threshold circuit of size $s$, depth $d$, energy $e$ and weight $w$ that satisfy $ed(\log s + \log w + \log n) \leq 4n + o(n)$ if $ed = o(n/\log n)$. Thus, the theorem neatly captures a computational aspect of threshold circuits computing $\text{CONJ}_n$, and suggests that there exists a tradeoff relating $\log s + \log w$, $d$, and $e$.

**Simulating Discretized Circuits.** In addition, we consider a discretized circuit such as a sigmoid circuit or a ReLU circuit where activation functions and weights are discretized, and define a similar notion for the energy and weight of a discretized circuit. We then observe that the lower bound for threshold circuits can be applied to such discretized circuits by showing that any discretized circuit can be simulated by a threshold circuit with moderate increase of size, depth energy and weight. Combining Eq. (2) with this simulation, we obtain an inequality

$$rk(M_C) = O(ed(\log s + \log w + \log n)^3)$$

for a discretized circuit $C$ of size $s$, depth $d$, energy $e$ and a weight $w$, where the rank is bounded by a product of polylogarithmic factors of $s, w$ and linear factors of $d, e$. Maass et al. [27] showed that a sigmoid circuit can be simulated by a threshold circuit, but their simulation is optimized to be depth-efficient, and do not care about energy. Thus, it does not fit in our purpose.

**Implication for Visual Search.** Equation (3) shows that computational capacity for $\text{CONJ}_n$ is bounded by a product of logarithmic factors of $s, w$ and linear factors of $d, e$. In particular, the contribution of even a hundred millions of gates to the product is only $< 27$, while in contrast the energy linearly affect the product. Thus the energy plays more important role in its computational capacity than the size. However, the number of neurons that can be substantially active concurrently is limited to possibly fewer than 1% in the brain [26]. Thus, although we simplified both tasks and neural networks, these observations suggest that neural networks may face more severe capacity limitation for performing the conjunction search than Eq. (1) implies.

**Implication for Machine Learning.** The size, depth, energy and weight are important parameters also for artificial neural networks. The size and depth are major topics on success of deep learning. The energy is related to important techniques for deep learning method such as regularization, sparse coding, or sparse autoencoder [14] [22] [35]. The weight resolution is closely related to chip resources in neuromorphic hardware systems [39], and quantization schemes received attention [7] [13].

Theorem 1 implies that a threshold circuit is able to compute a Boolean function of bounded rank. Thus, we can consider Eq. (3) as a bound on its concept class, a set of Boolean functions computable by threshold circuits of size $s$, depth $d$, energy $e$ and weight $w$. According to the bound, $c$ times larger depth is comparable to $2^c$ times larger size. Thus, large depth could enormously help neural networks to increase its expressive power. Also, the bound suggests that increasing depth could also help a neural network to acquire sparse activity when we have hardware constraints on both the number of neurons and the weight
resolutions. These observations may shed some light on the reason for the success of deep learning.

Organization. The rest of the paper is organized as follows. In Section 2, we define terms needed for analysis. In Section 3, we present our main lower bound result. In Section 4, we show the tightness of the bound by constructing a threshold circuit computing \( \text{CONJ}_n \). In Section 5, we show that a discretized circuit can be simulated by a threshold circuit. In Section 6, we conclude with some remarks.

2 Preliminaries

For an integer \( n \), we denote by \([n]\) a set \( \{1, 2, \ldots, n\} \). The base of the logarithm is two unless stated otherwise. In Section 2.1, we give definitions of \( \text{FTR}_n \) and \( \text{CONJ}_n \). In Section 2.2, we define terms on threshold circuits and discretized circuits. In Section 2.3, we define communication matrix, and show some related propositions.

2.1 Modeling Visual Search as Boolean Functions

Let \( x = (x_1, x_2, \ldots, x_n) \) be Boolean input variables indicating if \( n \) neurons detects a primary feature, and \( y = (y_1, y_2, \ldots, y_n) \) be Boolean input variables indicating if \( n \) neurons detects another primary feature, where the subscript \( i \in [n] \) specifies a place. We formulate the feature search and the conjunction search as

\[
\text{FTR}_n(x) = \bigvee_{i=1}^n x_i \quad \text{and} \quad \text{CONJ}_n(x) = \bigvee_{i=1}^n x_i \land y_i,
\]

respectively.

It is easy to observe that the input space of \( \text{FTR}_n \) is linearly separable by a hyperplane \( x_1 + x_2 + \cdots + x_n = 1/2 \). In the rest of the paper, we thus investigate computational hardness of \( \text{CONJ}_n \).

2.2 Circuit Model

2.2.1 Threshold Circuits

Let \( k \) be a positive integer. A threshold gate \( g \) with \( k \) input variables \( \xi_1, \xi_2, \ldots, \xi_k \) has weights \( w_1, w_2, \ldots, w_k \), and a threshold \( t \). We define the output \( g(\xi_1, \xi_2, \ldots, \xi_k) \) of \( g \) as

\[
g(\xi_1, \xi_2, \ldots, \xi_k) = \text{sign} \left( \sum_{i=1}^{k} w_i \xi_i - t \right) = \begin{cases} 1 & \text{if } t \leq \sum_{i=1}^{k} w_i \xi_i; \\ 0 & \text{otherwise} \end{cases}
\]

To evaluate the weight resolution, we assume single synaptic weight to be discrete, and that \( w_1, w_2, \ldots, w_n \) are integers. The weight \( w_g \) of \( g \) is defined as the maximum of the absolute values of \( w_1, w_2, \ldots, w_k \). In other words, we assume that \( w_1, w_2, \ldots, w_k \) are \( O(\log w_g) \)-bit coded discrete values. Throughout the paper, we allow a gate to have both positive and negative weights, although biological neurons are either excitatory (all the weights are positive) or inhibitory (all the weights are negative). As mentioned in [28], this relaxation has basically no impact on circuit complexity investigations, unless one cares about constant blowup in computational resources. This is because a single gate with positive and negative weights can be simulated by a pair of excitatory and inhibitory gates.

A threshold circuit \( C \) is a combinatorial circuit consisting of threshold gates, and is expressed by a directed acyclic graph. The nodes of in-degree 0 correspond to input variables,
and the other nodes correspond to gates. Let $G$ be a set of the gates in $C$. For each gate $g \in G$, the level of $g$, denoted by $\text{lev}(g)$, is defined as the length of a longest path from an input variable to $g$ on the underlying graph of $C$. For each $l \in [d]$, we define $G_l$ as a set of gates in the $l$th level:

$$G_l = \{g \in G \mid \text{lev}(g) = l\}.$$  

Given an input assignment $(a, b) \in \{0, 1\}^{2n}$ to $(x, y)$, the outputs of the gates in $C$ are inductively determined from the bottom level.

In this paper, we consider a threshold circuit $C$ for a Boolean function $f : \{0, 1\}^{2n} \rightarrow \{0, 1\}$. Thus, $C$ has $2n$ Boolean input variables $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, and a unique output gate, denoted by $g^{\text{clf}}$, which is a linear classifier separating internal representations given by the gates in the lower levels (possibly together with input variables).

Consider a gate $g$ in $C$. Let $w_1^g, w_2^g, \ldots, w_n^g$ (resp., $w_1^{\text{clf}}, w_2^{\text{clf}}, \ldots, w_n^{\text{clf}}$) be the weights for $x_1, x_2, \ldots, x_n$ (resp., $y_1, y_2, \ldots, y_n$), and $t_g$ be threshold of $g$. For each gate $h$ directed to $g$, let $w_{n,g}$ be a weight of $g$ for the output of $h$. Then the output $g(x, y)$ of $g$ is defined as

$$g(x, y) = \text{sign}(p_y(x, y) - t_g)$$

where $p_y(x, y)$ denotes a potentials of $g$ invoked by the input variables and gates:

$$p(x, y) = \sum_{i=1}^n w_i^g x_i + \sum_{i=1}^n w_i^y y_i + \sum_{l=1}^{\text{lev}(g)-1} \sum_{h \in G_l} w_{h,g} h(x, y).$$

We sometimes write $p^g_y(x)$ (resp., $p^y_g(y)$) for the potential invoked by $x$ (resp., $y$):

$$p^g_y(x) = \sum_{i=1}^n w_i^g x_i \quad \text{and} \quad p^y_g(y) = \sum_{i=1}^n w_i^y y_i.$$

Although the inputs to $g$ are not only $x$ and $y$ but the outputs of gates in the lower levels, we write $g(x, y)$ for the output of $g$, because $x$ and $y$ inductively decide the output of $g$. We say that $C$ computes a Boolean function $f : \{0, 1\}^{2n} \rightarrow \{0, 1\}$ if $g^{\text{clf}}(a, b) = f(a, b)$ for every $(a, b) \in \{0, 1\}^{2n}$.

Let $C$ be a threshold circuit. We define size $s$ of $C$ as the number of the gates in $C$, and depth $d$ of $C$ as the level of $g^{\text{clf}}$. We define the energy $e$ of $C$ as

$$e = \max_{(a, b) \in \{0, 1\}^{2n}} \sum_{g \in G} g(a, b).$$

We define weight $w$ of $C$ as the maximum of the weights of the gates in $C$: $w = \max_{g \in G} w_g$.

### 2.2.2 Discretized Circuits

Let $\varphi$ be an activation function. Let $\delta$ be a discretizer that maps a real number to a number representable by a bitwidth $b$. We define a discretized activation function $\delta \circ \varphi$ as a composition of $\varphi$ and $\delta$, that is, $\delta \circ \varphi(x) = \delta(\varphi(x))$ for any number $x$. We say that $\delta \circ \varphi$ has silent range for an interval $I$ if $\delta \circ \varphi(x) = 0$ if $x \in I$, and $\delta \circ \varphi(x) \neq 0$, otherwise. For example, if we use the ReLU function as the activation function $\varphi$, then $\delta \circ \varphi$ has silent range for $I = (-\infty, 0)$ for any discretizer $\delta$. If we use the sigmoid function as the activation function $\varphi$ and linear partition as discretizer $\delta$, then $\delta \circ \varphi$ has silent range for $I = (-\infty, t_{\text{max}}]$ where $t_{\text{max}} = \ln(1/(2^b - 1))$ where $\ln$ is the natural logarithm.
Let $\delta \circ \varphi$ be a discretized activation function with silent range. A $(\delta \circ \varphi)$-gate $g$ with $k$ input variables $\xi_1, \xi_2, \ldots, \xi_k$ has weights $w_1, w_2, \ldots, w_k$ and a threshold $t$, where each of the weights and threshold are discretized by $\delta$. The output $g(\xi_1, \xi_2, \ldots, \xi_k)$ of $g$ is then defined as

$$g(\xi_1, \xi_2, \ldots, \xi_k) = \delta \circ \varphi \left( \sum_{i=1}^{k} w_i \xi_i - t \right).$$

A $(\delta \circ \varphi)$-circuit is a combinatorial circuit consisting of $(\delta \circ \varphi)$-gates except that the top gate $g^{2^{|F|}}$ is a threshold gate, that is, a linear classifier. We define size and depth of a $(\delta \circ \varphi)$-circuit as the maximum number of gates outputting non-zero values in the circuit:

$$e = \max_{(a, b) \in \{0, 1\}^n} \sum_{y \in C} \left\| g(a, b) \neq 0 \right\|$$

where $[P]$ for a statement $P$ denote a notation of the function which outputs one if $P$ is true, and zero otherwise. We define weight $w$ of $C$ as $w = 2^{2b}$, where $2b$ is the bitwidth possibly needed to represent a potential value invoked by a single input of a gate in $C$.

### 2.3 Communication Matrix and its Rank

Let $Z \subseteq \{0, 1\}^n$. For a Boolean function $f : Z \times Z \rightarrow \{0, 1\}$, we define a communication matrix $M_f$ over $Z$ as a $2^{|Z|} \times 2^{|Z|}$ matrix where each row and column are indexed by $a \in Z$ and $b \in Z$, respectively, and each entry is defined as $M_f(a, b) = f(a, b)$. We denote by $rk(M_f)$ the rank of $M_f$ over $\mathbb{F}_2$. If a circuit $C$ computes $f$, we may write $M_C$ instead of $M_f$.

If a Boolean function $f$ does not have an obvious separation of the input variables to $x$ and $y$, we may assume a separation so that $rk(M_f)$ is maximized.

Let $k$ and $n$ be natural numbers such that $k \leq n$. Let

$$Z_k = \{ a \in \{0, 1\}^n \mid \text{The number of ones in } a \text{ is at most } k \}.$$ 

A $k$-disjointness function DISJ$_{n,k}$ over $Z_k$ is defines as follows:

$$\text{DISJ}_{n,k}(x, y) = \bigwedge_{i=1}^{n} x_i \lor \neg y_i,$$

where the input assignments are chosen from $Z_k$. The book [18] contains a simple proof showing DISJ$_{n,k}$ has full rank.

- **Theorem 2** (Theorem 13.10 [18]). $rk(M_{\text{DISJ}_{n,k}}) = \sum_{i=0}^{k} \binom{n}{i}$. In particular, $rk(M_{\text{DISJ}_{n,n}}) = 2^n$.

CONJ$_n$ is the complement of DISJ$_{n,n}$. We can obtain the same bound for CONJ$_n$, as follows:

- **Corollary 3**. $rk(M_{\text{CONJ}_{n,n}}) = 2^n$.

We also use well-known facts on the rank. Let $A$ and $B$ be two matrices of same dimensions. We denote by $A + B$ the summation of $A$ and $B$, and by $A \circ B$ the Hadamard product of $A$ and $B$.

- **Fact 1**. For two matrices $A$ and $B$ of same dimensions, we have
  (i) $rk(A + B) \leq rk(A) + rk(B)$;
  (ii) $rk(A \circ B) \leq rk(A) \cdot rk(B)$.
3 Lower Bound for Threshold Circuits

In this section, we give the inequality relating the rank of the communication matrix to the size, depth, energy and weight.

\textbf{Theorem 4 (Theorem 1 restated).} Let \( s, d, e \) and \( w \) be integers satisfying \( 2 \leq s, d, 10 \leq e, 1 \leq w \). Suppose a threshold circuit \( C \) computes a Boolean function of \( 2n \) variables, and has size \( s \), depth \( d \), energy \( e \), and weight \( w \). Then it holds that

\[ \log(rk(M_C)) \leq ed(\log s + \log w + \log n). \]

We prove the theorem by showing that \( M_C \) is a sum of matrices each of which corresponds to an internal representation that arises in \( C \). Since \( C \) has bounded energy, the number of internal representations is also bounded. We then show by the inclusion-exclusion principle that each matrix corresponding an internal representation has bounded rank. Thus, Fact 1 implies the theorem.

\textbf{Proof.} Let \( C \) be a threshold circuit that computes a Boolean function of \( 2n \) variables, and has size \( s \), depth \( d \), energy \( e \) and weight \( w \). Let \( G \) be a set of the gates in \( C \). For \( l \in [d] \), let \( G_l \) be a set of the gates in \( l \)-th level of \( C \). Without loss of generality, we assume that \( G_d = \{ g_{\text{clf}} \} \). We evaluate the rank of \( M_C \), and prove that

\[ rk(M_C) \leq \left( \frac{e_{apr} \cdot s}{e - 1} \right)^{e-1} \cdot \left( \frac{e_{apr} \cdot s}{e - 1} \cdot (2nw + 1)^{e-1} \right)^{d-1} \cdot (2nw + 1) \]

\[ \leq \left( \frac{e_{apr} \cdot s}{e - 1} \cdot (2nw + 1)^{d-1} \right)^{ed} \]

where \( e_{apr} \) is the Napier’s constant. The last inequality holds if \( 10 \leq e \). Taking the logarithm of the inequality, we obtain the theorem.

Let \( P_l \) be a subset of \( G_l \) for each \( l \in [d] \). Given an input \( (a, b) \in \{0, 1\}^{2n} \), we say that an internal representation \( P = (P_1, P_2, \ldots, P_d) \) arises for \( (a, b) \) if

\[ g(a, b) = \begin{cases} 1 & \text{if } g \in P_l; \\ 0 & \text{if } g \notin P_l \end{cases} \]

for every \( l \in [d] \). We denote by \( P^*(a, b) \) the internal representation that arises for \( (a, b) \in \{0, 1\}^{2n} \). We then define \( P_1 \) as a set of internal representations that arise for \( (a, b) \) such that \( g_{\text{clf}}(a, b) = 1 \):

\[ P_1 = \{ P^*(a, b) \mid g_{\text{clf}}(a, b) = 1 \}. \]

Note that, for any \( P = (P_1, P_2, \ldots, P_d) \in P_1(C) \), \( |P_1| + |P_2| + \cdots + |P_{d-1}| \leq e - 1 \) and \( |P_d| = 1 \). Thus we have

\[ |P_1| \leq \sum_{k=0}^{e-1} \binom{s}{k} \leq \left( \frac{e_{apr} \cdot s}{e - 1} \right)^{e-1}. \quad (4) \]

For each \( P \in P_1 \), let \( M_P \) be a \( 2^n \times 2^n \) matrix such that, for every \( (a, b) \in \{0, 1\}^{2n} \),

\[ M_P(a, b) = \begin{cases} 1 & \text{if } P = P^*(a, b); \\ 0 & \text{if } P \neq P^*(a, b). \end{cases} \]
By the definitions of $P_1$ and $M_P$, we have

$$M_C = \sum_{P \in P_1} M_P,$$

and hence Fact 1(i) implies that

$$\text{rk}(M_C) \leq \sum_{P \in P_1} \text{rk}(M_P).$$

Thus Eq. (4) implies that

$$\text{rk}(M_P) \leq \left( \left( \frac{c_{\text{npr}} \cdot s}{e-1} \right)^{e-1} \cdot \max_{P \in P_1} \text{rk}(M_P) \right)^{d-1} \cdot (2nw+1)^{e-1}.$$
Since $g$ receives a value between $-w$ and $w$ from a single input, we have $|R| \leq 2nw + 1$. For $r = (r_1, r_2, \ldots, r_z) \in R_1 \times R_2 \times \cdots \times R_z$, we define $R(r) = X(r) \times Y(r)$ as a combinatorial rectangle where

$$X(r) = \{x \mid \forall k \in [z], p_{r_k}(x) = r_k\}$$

and

$$Y(r) = \{y \mid \forall k \in [z], t_{r_k} \leq r_k + p_{r_k}(y)\}.$$ 

Clearly, all the rectangles are disjoint, and hence $M[T]$ can be expressed as a sum of rank-1 matrices given by $R(r)$'s taken over all the $r$'s. Thus Fact 1(i) implies that its rank is at most $|R_1| \times |R_2| \times \cdots \times |R_z| \leq (2nw + 1)^z$. \hfill \blacktriangle

Following the inclusion-exclusion principle, we define a $2^n \times 2^n$ matrix

$$H[T_i] = \sum_{T_i \subseteq T \subseteq T_i'} (-1)^{|T| - |T_i|} M[T].$$

We below show that $M_P$ is a Hadamard product of $H[T_1], H[T_2], \ldots, H[T_d]$:

\begin{itemize}
    \item [\blacktriangledown] Claim 6.
    \end{itemize}

$$M_P = H[T_1] \circ H[T_2] \circ \cdots \circ H[T_d].$$

**Proof.** Consider an arbitrary fixed assignment $(a, b) \in \{0, 1\}^{2^n}$. We show that $M_P(a, b) = 0$ if

$$H[T_1](a, b) \circ H[T_2](a, b) \circ \cdots \circ H[T_d](a, b) = 0,$$

and $M_P(a, b) = 1$, otherwise. We here write $P^* = (P_1^*, P_2^*, \ldots, P_d^*)$ to denote $P^*(a, b)$ for a simpler notation.

Suppose $M_P(a, b) = 0$. In this case, we have $P \neq P^*$, and hence there exists a level $l \in [d]$ such that $P_l \neq P_l^*$ while $P_{l'} = P_{l'}^*$ for every $l' \in [l - 1]$. Clearly, it suffices to show that $H[T_i](a, b) = 0$. For such $l$, it holds that for every $g \in G_l$, $\tau[g, P]$ is identical to $\tau[g, P^*]$. We consider two cases: $P_l \setminus P_l^* \neq \emptyset$ and $P_l \subseteq P_l^*$.

Consider the case where $P_l \setminus P_l^* \neq \emptyset$, then there exists $g \in P_l \setminus P_l^*$. Since $g \notin P_l^*$, we have

$$\tau[g, P^*](a, b) = \tau[g, P](a, b) = 0,$$

and hence $M[T](a, b) = 0$ for every $T$ such that $T_i \subseteq T$. Thus $H[T_i](a, b) = 0$. Consider the other case where $P_l \subseteq P_l^*$. Let $T_i^* = \{\tau[g, P^*] \mid g \in P_l^*\}$. Clearly, $M[T](a, b) = 1$ if $T$ satisfies $T_i \subseteq T \subseteq T_i^*$, and $M[T](a, b) = 0$ if $T$ satisfies $T_i^* \subseteq T$. Thus, we have

$$H[T_i](a, b) = \sum_{T_i \subseteq T \subseteq T_i^*} (-1)^{|T| - |T_i|} M[T](a, b) = \sum_{T_i \subseteq T \subseteq T_i^*} (-1)^{|T| - |T_i|}.$$

Consequently, the binomial theorem implies that

$$H[T_i](a, b) = \sum_{k=0}^{T_i^* - |T_i|} \binom{|T_i^* - |T_i|}{k} (-1)^k = (1 - 1)^{|T_i^* - |T_i|} = 0,$$

as desired.
Suppose $M_P(a, b) = 1$. In this case, we have $P = P^*$. Thus, for every $l \in [d]$, we have $M[T_l](a, b) = 1$, while $M[T](a, b) = 0$ for every $T$ satisfying $T_l \subset T$. Therefore,

$$\sum_{T_l \subset T \subset T'} (-1)^{|T| - |T_l|} M[T](x, y) = M[T_l](x, y) = 1,$$

which implies $H[T_l](a, b) = 1$ for every $l \in [d]$. Therefore

$$H[T_1](a, b) \circ H[T_2](a, b) \circ \cdots \circ H[T_d](a, b) = 1.$$  

We finally evaluate $rk(M_P)$. Claim 6 and Fact 1(ii) imply that

$$rk(M_P) \leq \prod_{l=1}^{d} rk(H[T_l]).$$  \hspace{1cm} (5)

Fact 1(i) and Claim 5 imply that

$$rk(H[T_l]) \leq \sum_{T_l \subset T \subset T'} rk(M[T]) \leq \left(\frac{c_{np} \cdot s}{e - 1}\right)^{e-1} \cdot (2nw + 1)^{e-1}$$  \hspace{1cm} (6)

for every $l \in [d - 1]$, and

$$rk(H[T_d]) \leq 2nw + 1.$$  \hspace{1cm} (7)

Eqs. (5)-(7) imply that

$$rk(M_P) \leq \left(\frac{c_{np} \cdot s}{e - 1}\right)^{e-1} \cdot (2nw + 1)^{d-1} \cdot (2nw + 1)$$

as desired.

Combining Theorems 3 and 4, we obtain the following corollary:

**Corollary 7.** Let $s, d, e$ and $w$ be integers satisfying $2 \leq s, d$, $10 \leq e$, $1 \leq w$. Suppose a threshold circuit $C$ of size $s$, depth $d$, energy $e$, and weight $w$ computes $\text{CONJ}_n$. Then it holds that

$$n \leq ed(\log s + \log w + \log n).$$

Equivalently, we have $2^{n/(ed)}/(nw) \leq s$.

## 4 Tightness of the Inequality

In this section, we show that the lower bound given in Theorem 3 is almost tight if the depth and energy are small. The following theorem gives an explicit construction of a threshold circuit computing $\text{CONJ}_n$.

**Theorem 8.** For any integers $e$ and $d$ such that $2 \leq e$ and $2 \leq d$, $\text{CONJ}_n$ is computable by a threshold circuit of size

$$s \leq (e - 1)(d - 1) \cdot 2^{\frac{n}{e-1} \cdot \frac{1}{d-1}},$$

depth $d$, energy $e$ and weight

$$w \leq \frac{n}{(e - 1)(d - 1)}.$$
Substituting $s, d, e$ and $w$ of a threshold circuit given in Theorem 8 to the right hand side of Eq. (3), we have

$$ed(\log s + \log w + \log n) \leq ed \left( \frac{n}{(e-1)(d-1)} + \log(e-1)(d-1) + \log \left( \frac{n}{(e-1)(d-1)} \right) + \log n \right) \leq 4n + O(ed \log n).$$

Thus it almost matches the left hand side of Eq. (2) if $ed = o(n/ \log n)$.

In the construction, We separate $[n]$ into several blocks, and give a set of threshold gates for each block so that at most one of the gates outputs one for any input assignment. We then give a layered structure where the gates for each block.

**Proof.** We prove the theorem by constructing the desired circuit $C$. For simplicity, we assume that $n$ is divisible by $(e-1)(d-1)$. Let $m = n/((e-1)(d-1))$. We then separate $[n]$ into $(e-1)(d-1)$ blocks $B_{k,l}$, $k \in [e-1]$ and $l \in [d-1]$, each of which contains $m$ integers. We denote by $\text{CONJ}_{B_{k,l}}$ the function over the variables in $B_{k,l}$. Clearly,

$$\text{CONJ}_n(x, y) = \bigvee_{k \in [e-1]} \left( \bigvee_{l \in [d-1]} \text{CONJ}_{B_{k,l}}(x, y) \right).$$

Our construction is of two steps: (i) For each $B_{k,l}$, we construct $2^m$ gates, exactly one of which outputs one if there exists $i \in B_{k,l}$ such that $x_i \land y_i = 1$, and none of which outputs one, otherwise; and (ii) we complete the construction by arranging the gates given in (i). After we complete the construction of $C$, we show that $C$ computes $\text{CONJ}_n$ and evaluate its size, depth, energy and weight.

(i) **Construction of the gates for $B_{k,l}$**

Consider a fixed $B_{k,l}$. Recall that we denote by $[P]$ for a statement $P$ a notation of the function which outputs one if $P$ is true, and zero otherwise. It is easy to observe that

$$\text{CONJ}_{B_{k,l}}(x, y) = \bigvee_{B \subseteq B_{k,l}} F_{B_{k,l}}^B(x, y)$$

where

$$F_{B_{k,l}}^B(x, y) = \bigwedge_{i \in B} [x_i = 1] \land \bigwedge_{i \notin B} [x_i = 0] \land \bigvee_{i \in B} y_i.$$

For any assignment $a \in \{0, 1\}^n$, we define $B^*(a)$ as

$$B^*(a) = \{ i \in B_{k,l} \mid a_i = 1 \}.$$

Then, for every $(a, b) \in \{0, 1\}^{2n}$,

$$F_{B_{k,l}}^B(a, b) = \begin{cases} 1 & \text{if } B = B^*(a) \text{ and } \exists i \in B, b_i = 1; \\ 0 & \text{otherwise.} \end{cases}$$

We below show that $F_{B_{k,l}}^B$ is computable by a threshold gate.

**Claim 9.** For any $B \subseteq B_{k,l}$, $F_{B_{k,l}}^B$ is computable by a threshold gate with weights $w_i^x, w_i^y$ and the threshold $t$ defined as follows: For every $i \in [n]$,

$$w_i^x = \begin{cases} 0 & \text{if } i \notin B_{k,l}; \\ |B| & \text{if } i \in B \subseteq B_{k,l}; \\ -|B| & \text{if } i \in B_{k,l} \setminus B, \end{cases}$$

$$w_i^y = \begin{cases} 0 & \text{if } i \notin B_{k,l}; \\ |B| & \text{if } i \in B \subseteq B_{k,l}; \\ -|B| & \text{if } i \in B_{k,l} \setminus B, \end{cases}$$

$$t = \sum_{B \subseteq B_{k,l}} \sum_{i \in B} w_i^x + \sum_{B \subseteq B_{k,l}} \sum_{i \notin B} w_i^y.$$
and

\[ w^y_i = \begin{cases} 
0 & \text{if } i \not\in B_{k,l}; \\
1 & \text{if } i \in B \subseteq B_{k,l}; \\
0 & \text{if } i \in B_{k,l} \setminus B,
\end{cases} \]

and \( t = |B|^2 + 1 \).

**Proof.** Suppose \( F_{k,l}^B(x,y) = 1 \), that is \( B = B^*(x) \) and there exists \( i^* \in B \) such that \( y_{i^*} = 1 \). Then we have

\[ \sum_{i \in B_{k,l}} w_x^x x_i + \sum_{i \in B} w_y^y y_i \geq |B| \cdot |B| + 1 = t, \]

and hence \( g_{k,l}^B \) outputs one.

Suppose \( F_{k,l}^B(x,y) = 0 \). Consider the case where \( B \neq B^*(x) \). If there exists \( i^* \in B^*(x) \setminus B \), then

\[ \sum_{i \in B_{k,l}} w_x^x x_i \leq \sum_{i \in B} w_x^x x_i + w_x^{x^*} \leq |B| \cdot |B| - |B|. \]

Thus

\[ \sum_{i \in B_{k,l}} w_x^x x_i + \sum_{i \in B} w_y^y y_i \leq (|B| \cdot |B| - |B|) + |B| \leq |B|^2 < t, \]

and hence \( g_{k,l}^B \) outputs zero. If \( B^*(x) \subseteq B \), then

\[ \sum_{i \in B_{k,l}} w_x^x x_i \leq (|B| - 1) \cdot |B|. \]

Thus

\[ \sum_{i \in B_{k,l}} w_x^x x_i + \sum_{i \in B} w_y^y y_i \leq ((|B| - 1) \cdot |B|) + |B| \leq |B|^2 < t, \]

and hence \( g_{k,l}^B \) outputs zero.

Consider the other case where \( y_i = 0 \) for every \( i \in B_{k,l} \). Then

\[ \sum_{i \in B} w_y^y y_i = 0. \]

Thus

\[ \sum_{i \in B_{k,l}} w_x^x x_i + \sum_{i \in B} w_y^y y_i \leq |B| \cdot |B| < t, \]

and hence \( g_{k,l}^B \) outputs zero. ▶

(ii) **Arrangement of the gates**

We construct \( C \) from the bottom level. Firstly, we put in the first level a threshold gate computing \( F_{k,1}^B \) for every \( k \in [e - 1] \) and \( B \subseteq B_{1,1} \). For each \( l, 2 \leq l \leq d - 1 \), we then add in the \( l \)th level a threshold gate computing \( F_{k,l}^B \) for every \( k \in [e - 1] \) and \( B \subseteq B_{l,l} \), and connect
the outputs of all the gates in the lower level to every \( g_{k,l}^B \) with weight \(-|B|\). Lastly we add \( g^{\text{sh}} \) that computes a conjunction of all the gates \( g_{k,l}^B \):

\[
g^{\text{sh}}(x,y) = \text{sign} \left( \sum_{k \in [e-1]} \sum_{l \in [d-1]} \sum_{B \subseteq B_{k,l}} g_{k,l}^B(x,y) - 1 \right).
\]

We have completed the construction of \( C \).

(iii) Evaluation of \( C \)

We here show that \( C \) compute \( \text{CONJ}_n \). By construction, the following claim is easy to verify.

\[\blacktriangleright \text{Claim 10.} \]

\[ g_{k,l}^B(x,y) = \begin{cases} F_{k,l}^B(x,y) & \text{if every gate in the levels } 1,\ldots,l-1 \text{ outputs zero;} \\ 0 & \text{otherwise.} \end{cases} \]

\[\text{Proof.} \] Let \( w_1^x, \ldots, w_n^x \) and \( w_1^y, \ldots, w_n^y \) be the weights of \( g_{k,l}^B \) for \( x \) and \( y \), respectively, and \( t = |B|^2 + 1 \) be the threshold.

If every gate in the levels \( 1,\ldots,l-1 \) outputs zero, then Claim 9 implies that \( g_{k,l}^B \) is identical to a threshold function computing \( F_{k,l}^B \), and hence \( g_{k,l}^B(x,y) = F_{k,l}^B(x,y) \). Otherwise, there is a gate outputting one in the lower levels. Since \( g_{k,l}^B \) receives an output from the gate in the lower level, the potential of \( g_{k,l}^B \) is at most

\[
\sum_{i \in B_{k,l}} w_i^x x_i + \sum_{i \in B} w_i^y y_i - |B| \leq |B| \cdot |B| + |B| - |B| < t.
\]

Thus, \( g_{k,l}^B \) outputs zero.

\[\blacktriangleleft \]

Suppose \( \text{CONJ}(x,y) = 0 \). In this case, we have \( x_i \land y_i = 0 \) for every \( i \in [n] \). Thus, for every \( k \in [e-1], l \in [d-1] \) and \( B \subseteq B_{k,l} \), \( F_{k,l}^B(x,y) = 0 \). Therefore, Claim 10 implies that no gate in \( C \) outputs one.

Suppose \( \text{CONJ}(x,y) = 1 \). In this case, there exists \( i^* \in [n] \) such that \( x_{i^*} \land y_{i^*} = 1 \). Without loss of generality, we assume that \( i^* \in B_{k^*,l^*} \) for some \( k^* \in [e-1] \) and \( l^* \in [d-1] \) such that \( x_{i^*} \land y_{i^*} = 0 \) for every \( i \in B_{k,l}, 1 \leq k \leq e-1 \) and \( 1 \leq l \leq l^* - 1 \). The claim implies that \( g_{k^*,l^*}^B(x,y) = F_{k^*,l^*}^B(x,y) \) for every \( B \subseteq B_{k^*,l^*} \). Since \( x_{i^*} \land y_{i^*} = 1 \), we have \( F_{k^*,l^*}^B(x,y) = 1 \), where \( B^* = B^*(x) \). Therefore, \( g_{k^*,l^*}^B(x,y) = 1 \), which implies that \( g^{\text{sh}}(x,y) = 1 \).

We lastly evaluate the size, depth, energy and weight of \( C \). Since we have \((e-1)(d-1)\) blocks, and we make \( 2^m - 1 \) gates for each block, we have

\[
s \leq (e-1)(d-1)(2^m-1) + 1 \leq (e-1)(d-1)2^{\log_2\frac{n}{(e-1)(d-1)}} - 1
\]

where the additional one corresponds to the output gate. Since the gates \( g_{k,l}^B \) are placed in \( l \)-th level for \( l \in [d-1] \), the level of \( g^{\text{sh}} \) is clearly \( d \), and hence \( C \) has depth \( d \). Claim 10 implies that if there is a gate outputting one in level \( l \) then no gate in higher levels outputs one. In addition, at most one of \( g_{k^*,l^*}^B \) for any \( B \subseteq B_{k,l} \) outputs one. Therefore, at most \( e-1 \) gates in the \( l \)-th level outputs one, followed by \( g^{\text{sh}} \) outputting one. Thus \( C \) has energy \( e \). The weight of any gate is at most

\[
|B_{k,l}| \leq \frac{n}{(e-1)(d-1)}.
\]
5 Simulating Discretized Circuits

In this section, we show that any discretized circuit can be simulated by a threshold circuit with moderate increase of size, depth, energy, and weight. Thus a similar inequility holds also for discretized circuits. Recall that we define energy and weight of discretized circuits differently from the ones of threshold circuits (See Section 2.2.2).

Theorem 11. Let $\delta$ be a discretizer and $\varphi$ be an activation function such that $\delta \circ \varphi$ has silent range. If a $(\delta \circ \varphi)$-circuit $C$ of size $s$, depth $d$, energy $e$ and weight $w$ computes a Boolean function $f$, then it hold that

$$\log(rk(M_C)) = O(ed(log s + log w + log n)^3).$$

In the inequality, the linear factors of $e, d$ and polylogarithmic factors of $s, w$ are comparable.

Our simulation is based on a binary search over potentials of a discretized gate. We use a conversion technique from a linear decision tree to a threshold circuit presented in [18].

Proof. Let $\delta$ be a discretizer and $\varphi$ be an activation function such that $\delta \circ \varphi$ has silent range for $I$. In the proof, we only consider an open interval $I = (t_{\min}, t_{\max})$, because the proof for the other cases are similar. Let $C$ be a $(\delta \circ \varphi)$-circuit of size $s$, depth $d$, energy $e$ and weight $w$. We obtain the desired threshold circuit $C'$ by showing a procedure by which any $(\delta \circ \varphi)$-gate $g$ in $C$ can be safely replaced by a set of threshold gates.

Let $g$ be an arbitrary $(\delta \circ \varphi)$-gate in $C$ that computes $g(x, y) = \delta \circ \varphi(p(x, y))$ We first consider $[t_{\max}, \infty)$. Let $P_g$ be a set of potential values for which $g$ outputs a non-zero value:

$$P_g = \{p(a, b) \mid (a, b) \in \{0, 1\}^{2n}, t_{\max} \leq p(a, b)\}.$$

Since the activation function and weights are discretized, we have $|P_g| = O((s + n)w)$.

We operate the binary search over $P_g$, and make a threshold gate that outputs one for every input $(a, b)$ such that $p(a, b)$ takes a particular value in $P$. For any $Q \subseteq P_g$, we define $\text{mid}(Q)$ as a median of integers in $Q$, and $Q^+$ (resp., $Q^-$) be the upper (resp., lower) half of $Q$:

$$Q^+ = \{p \in Q \mid p \leq \text{mid}(Q)\} \quad \text{and} \quad Q^- = \{p \in Q \mid \text{mid}(Q) < p\}.$$

If $Q$ contains even number of values, we take the greater one of two median values.

Let $s$ be a binary string. We inductively construct a threshold gate $g_s$ on the length of $s$. For two strings $t$ and $s$, we write $t \prec s$ if $t$ is a proper prefix of $s$. We denote by $t0$ (resp., $t1$) is a string $t$ followed by 0 (resp., by 1).

For the base of our construction, we consider the empty string $\epsilon$. Let $P_\epsilon = P_g$. We make a threshold gate $g_\epsilon$ computing

$$g_\epsilon(x, y) = \text{sign}(p(x, y) - t_{\epsilon}[x])$$

where $t_{\epsilon}[x] = \text{mid}(P_\epsilon)$.

Suppose we have constructed gates $g_t$ for every $t$ satisfying $|t| \leq k - 1$. Consider a string $s$ of length $k$. By the induction hypothesis, we have gates $g_t$ for every $t, t \prec s$. Let $s'$ be a string obtained by dropping the last symbol of $s$. Then we define

$$P_s = \begin{cases} P_{s'}^+ & \text{if the last symbol of } s \text{ is } 1; \\ P_{s'}^- & \text{if the last symbol of } s \text{ is } 0. \end{cases}$$
Let \( W = 3(s + n)w \). We make a threshold gate \( g_\mathbf{s} \) as follows:

\[
g_\mathbf{s}(\mathbf{x}, \mathbf{y}) = \text{sign} \left( p(\mathbf{x}, \mathbf{y}) + \sum_{t \prec s} w_{t, s} \cdot g_t(\mathbf{x}, \mathbf{y}) - t_0[s] \right)
\]

where \( w_{t, s} \) is the weight for the output of \( g_t \), and defined as

\[
w_{t, s} = \begin{cases} 
W & \text{if } t1 \text{ is a prefix of } s; \\
-W & \text{if } t0 \text{ is a prefix of } s; \\
0 & \text{otherwise}, 
\end{cases}
\]

and \( t_0[s] = \text{mid}(P_s) - W \cdot N_1(s) \), where \( N_1(s) \) is the number of ones in \( s \).

We repeatedly apply the above procedure until \( |P_s| = 1 \). Since we apply the binary search over \( P \), we have \( O(|P|) = O((s + n)w) \) gates, and the length of \( |s| \) is \( O(\log(s + n) + \log w) \) for any gate \( g_s \).

Consider the strings \( s \) for which we have constructed \( g_s \). Let

\[
S_g = \{ s \mid 2 \leq |P_s| \} \quad \text{and} \quad L_g = \{ s \mid |P_s| = 1 \}.
\]

For each \((a, b) \in \{0, 1\}^{2n}\), we denote by \( s^*(a, b) \in L_g \) the unique string satisfying \( P_{s^*(a, b)} = \{ p(a, b) \} \). The following claims shows that \( g_s \) are useful to simulate \( g \).

\( \triangleright \) Claim 12. Let \((a, b) \in \{0, 1\}^{2n}\) be an input assignment satisfying \( t_{\max} \leq p(a, b) \).

(i) For \( s \in S_g \), \( g_s \) outputs one if and only if \( s1 \) is a prefix of \( s^*(a, b) \).

(ii) For \( s \in L_g \), \( g_s \) outputs one if and only if \( s = s^*(a, b) \).

**Proof.** Consider an arbitrary input assignment \((a, b) \in \{0, 1\}^{2n} \) satisfying \( t_{\max} \leq p(a, b) \). For notational simplicity, we write \( s^* \) for \( s^*(a, b) \).

Proof of (i). We verify the claim by induction on the length of \( s \). For the base case, consider \( \epsilon \). It suffices to show that \( g_\epsilon(a, b) = 1 \) if the first symbol of \( s^* \) is 1, and \( g_\epsilon(a, b) = 0 \) otherwise. If the first symbol is 1, we have \( p(a, b) \in P^+_\epsilon \), which implies that \( t_{\max} \leq p(a, b) \). Thus, \( g_\epsilon(a, b) = 1 \). Similarly, if the first symbol is 0, we have \( p(a, b) \in P^-_{\epsilon} \), which implies that \( p(a, b) < t_{\max} \). Thus, \( g_\epsilon(a, b) = 0 \).

We assume for the induction hypothesis that \( g_t \) outputs one if and only if \( t1 \) is a prefix of \( s^*(a, b) \) for every \( t \) of length at most \( k - 1 \) for a positive integer \( k \). We below consider a string \( s \) of length \( k \).

We first verify that \( g_s \) outputs zero if \( s \) itself is not a prefix of \( s^*(a, b) \). If \( s \) is not a prefix of \( s^* \), there exists a prefix \( t' \) of \( s \) such that \( t'0 \) (resp., \( t'1 \)) is a prefix of \( s \) while \( t'1 \) (resp., \( t'0 \)) is not a prefix of \( s^* \).

Consider the case where \( t'0 \) is a prefix of \( s^* \) (that is, \( t'1 \) is a prefix of \( s \)). In this case, the induction hypothesis implies that \( g_{t'0}(a, b) = 0 \). In addition, since \( t'1 \) is a prefix of \( s \), we have \( w_{t', s} = W \). Thus, the potential of \( g_s \) for \((a, b) \) is at most

\[
p(a, b) + \sum_{t \prec s} w_{t, s} \cdot g_t(a, b) - t_0[s] \leq p(a, b) + W \cdot (N_1(s) - 1) - \text{mid}(P_s) - W \cdot N_1(s)
\]

\[
\leq p(a, b) - \text{mid}(P_s) - W
\]

which is less than zero, because \( p(a, b) \leq (s + n)w \).

Consider the case where \( t'1 \) is a prefix of \( s^* \) (that is, \( t'0 \) is a prefix of \( s \)). In this case, the induction hypothesis implies that \( g_{t'1}(a, b) = 1 \). In addition, since \( t'0 \) is a prefix of \( s \),
implies that depth of $g$ and threshold of $P$ above procedure: It suffices to consider a gate obtained by multiplying $p\not= p$.

**Proof.**

Since $s = P$.

**Claim 13.** For any $(a, b) \in \{0, 1\}^{2n}$ satisfying $p(a, b) < t_{\text{max}}$, every gate $g_s$ output zero.

**Proof.**

Since $p(a, b) < t_{\text{max}} \leq \text{mid}(P_g)$, $g_s(a, b) = 0$, which implies by a similar argument to Claim 12, for all the gates $g_s$ such that the $s$ contains a symbol 1. If $s$ is of only 0s, we have $p(a, b) < t_{\text{max}} \leq \text{mid}(Q)$ for any $Q \subseteq P_g$, which implies that $g_s(a, b) = 0$. ▲

Claims 12 and 13 imply that we can safely replace $g$ by $g_s, s \in S_g \cup L_g$, by connecting the output of each gate $g_s, s \in L_g$, with weight $w_{g_s, g'}(s \otimes \varphi(p))$ where $p$ is the unique value in $P_s$ to every gate $g'$ which the gate $g$ is originally connected to in $C$. Since $|S_g \cup L_g| = O((s+n)w)$, and the length of $s$ is $O(\log(s+n) + \log w)$, the size and depth respectively increase by these factors.

We can make another set of threshold gates for $(-\infty, t_{\text{min}}]$ in a similar manner to the above procedure: It suffices to consider a gate obtained by multiplying $-1$ by the weights and threshold of $g$ together with an interval $[-t_{\text{min}}, \infty)$.

Using the above procedure, we replace every gate $g \in G \setminus \{g^2\}$ in $C'$ by a set of threshold gates, and complete the construction of $C'$. Clearly, the size of $C'$ is $O(s \cdot (s+n)w)$. Since $g_s$ receives the outputs of $g_t$ for every $t < s$ and the length of $s$ is $O(\log(s+n) + \log w)$, the depth of $C'$ is $O(d \cdot (\log(s+n) + \log w))$. Claims 12 and 13 imply that the energy of $C'$ is $O(e \cdot (\log(s + n) + \log w))$. Clearly, the weight of $C'$ is $W = O((s+n)w)$. Thus Theorem 4 implies that

\[ rk(M_C) = O(e(\log(s+n) + \log w) \cdot d(\log(s+n) + \log w) \cdot (\log s + \log w + \log n)) \]

\[ = O(ed(\log s + \log n + \log w)^3), \]
as desired.

6 Conclusions

In this paper, we simplified visual search tasks and neural networks, and investigated circuit complexity of the tasks for threshold circuits and discretized circuits. We first consider threshold circuits, and obtained a tradeoff relating size, depth, energy and weight, where linear factors of the depth and energy and logarithmic or logarithmic factors of size and weight are comparable. The tradeoff implies exponential lower bounds on the size of threshold circuits of even sub-linear depth if the energy is sub-linear and weight is sub-exponential. We then consider discretized circuits, and show that a discretized circuit can be simulated by a threshold circuit. The simulation implies a similar inequality relating size, depth, energy and weight where linear factors of the depth and energy and polylogarithmic factors of size and weight are comparable.

Since we simplified and ignored many aspects of the tasks and neural computation, our results are not enough to explain the issue on the set-size effect. For example, we assume that a neural network is completely feedforward, their computation proceeds in discrete time steps, and no neural noise exists. We also formulated the visual search tasks as very simple Boolean functions. In particular, although we introduced Boolean variables for inputs of the conjunction search, we ignore spatial information of them in the formulation. It would be interesting if we can obtain more sophisticated tradeoff by considering more real situations for visual search tasks. It would be also interesting if a similar tradeoff exists for other neural information processing. Our proof is based on a simple rank argument, and so, contrasting with the current situation of circuit complexity, specifying computational power of threshold circuits of small size, depth, energy and weight is possibly within a reach of conventional analytical tools.

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