The effects of wind on the generation of surface wave in a shallow water

Rifky Fauzi1,∗, L. Hari Wiryanto1

1Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10
Bandung, Indonesia
E-mail: *rifkyfauzi9@gmail.com

Abstract. In this paper, the generation of surface waves by wind is modeled using an extended
shallow water model. Jeffreys’ sheltering mechanism is incorporated to model the wind action
on the surface. This modification allows the model to have a wave grow or decay mechanism
under the action of wind. The wave generation or damping is investigated by calculating critical
wind velocity to excite oscillatory waves on the surface. The result shows that critical wind is
proportional to a numerical constant, namely the sheltering coefficient, which related to the
capability of wave overcome the wind gust. Thus this model has the convenience that the
critical wind speed may comparable to physical data for a suitable choice of this constant.

1. Introduction
Air and water can be considered as two-layer of fluids with the lighter fluid float on top of the
heavier ones. By assuming that the fluids are irrotational, incompressible, and inviscid, the flow
of both fluids can be expressed by a term of potential functions satisfying the Laplace equation
with corresponding boundaries. In [1], it was shown that by expressing the water and air in
terms of potential function with certain boundary conditions, the motion of the interface is
obtained. The motion of this interface is dependent on the thickness of both fluids. The air
thick can be assumed to be infinite, meaning that up above the air, there is not a boundary.
Meanwhile, the lower fluid can be separated by two cases, which is for deepwater i.e., the depth
is infinite and shallow water where the depth is finite.

By this assumption, a motion of monochromatic wave on the interface can be obtained whose
speed depends on physical parameters such as characteristic velocities, gravity, densities, and
fluid thicknesses. This wave motion contains instability in case the air stream related to water
flow is greater than a certain number. This occurrence is well-known as Kelvin-Helmholtz
instability [2], i.e., the instability developed by fluid flow at a different speed. In this paper, it
will later be called wave instability under the action of an air stream. By dividing two cases
related to water depth, a critical wind velocity to excite simple waves has been obtained for
deep and shallow water cases. Comparison to data obtained from [3] for a deep case and [4]
for shallow water show that this approach has a disagreement with the critical wind to excite a
wave on the water surface.

As an alternative, a modified model based on the shallow water equation (SWE), which
includes wind source term, is proposed to study this instability. This modified model is obtained
by revisiting the pressure term of SWE then embedding Jeffreys’s sheltering theory to obtain
wind forcing terms. Linear stability is then conducted to calculate critical wind velocity for a surface wave to be excited.

2. Irrotational motion of two layer fluids

Suppose two fluids be incompressible, irrotational and inviscid with \( \rho_1 \) and \( \rho_2 \) are the density for lower and upper fluid respectively. The fluids is separated by an interface \( \eta(x, t) \) at \( y = 0 \) and bounded below at \( y = -h_1 \) and above \( y = h_2 \). The velocity potential \( \Phi^1(x, y, t) \) and \( \Phi^2(x, y, t) \) for lower and upper fluid respectively satisfy Laplace equation as follows

\[
\Phi^{(i)}_{xx} + \Phi^{(i)}_{yy} = 0 \tag{1}
\]

with boundary condition

\[
\eta_t + \Phi^{(i)}_x \eta_x = \Phi^{(i)}_y \quad \text{at} \ y = 0 \tag{2}
\]

and

\[
\frac{P_i}{\rho_i} = \Phi^{(i)}_t + \frac{1}{2}(\Phi^{(i)}_x^2 + \Phi^{(i)}_y^2) + g\eta + \text{constant} \quad \text{at} \ y = \eta(x, t) \tag{3}
\]

meanwhile

\[
\Phi^{(1)}_y = 0 \quad \text{at} \ y = -h_1 \quad \text{and} \quad \Phi^{(2)}_y = 0 \quad \text{at} \ y = h_2 \tag{4}
\]

Equation 1 together with boundary condition 2, 3 and 4 describe flow for each fluid.

In order to simplify the investigation, several adjustments are made. The potential function \( \Phi^{(i)} \) is a perturbation of uniform flow \( U_i \). The potentials are also needed to finite at the lateral boundary. Hence, the Laplace equation and its boundary can be rewritten as

\[
\Phi^{(i)}(x, y, t) = U_i x + \phi^{(i)}(x, y, t) \tag{5}
\]

Meanwhile, the wave on the interface is described as

\[
\eta(x, t) = a e^{-i(kx - \omega t)} \tag{6}
\]

where \( k \) is wave number and \( \omega \) is wave speed. Substitute equation 5 in to equation 1 and boundary conditions 2, 3 and 4 then neglect the higher order terms one obtain

\[
\phi^{(i)}_{xx} + \phi^{(i)}_{yy} = 0 \tag{7}
\]

with

\[
\eta_t + U_i \eta_x = \phi^{(i)}_y \quad \text{at} \ y = 0, \tag{8}
\]

\[
\frac{P_i}{\rho_i} = \phi^{(i)}_t + U_i \phi^{(i)}_x + g\eta + \text{constant} \quad \text{at} \ y = \eta(x, t) \tag{9}
\]

and

\[
\phi^{(1)}_y = 0 \quad \text{at} \ y = -h_1 \quad \text{and} \quad \phi^{(2)}_y = 0 \quad \text{at} \ y = h_2 \tag{10}
\]

Assuming that the velocity potentials are separable functions \( \phi^{(i)}(x, y, t) = A_i(y)B_i(x, t) \) then following precedures in [5] gives

\[
\Phi^{(i)}(x, y, t) = U_i x + i(\omega - U_i) \frac{(-1)^{i+1} A_i \eta(x, t)}{k} \tag{11}
\]

where

\[
A_i = \coth(kh_i) \tag{12}
\]

and \( k \) is wave number. Equation 11 is solution for equation 7 along with boundaries 8 and 10.
To complete the procedure, substituting equation 11 and equation 6 in to equation 9 then equate the equations provide a necessary condition for the problem as follows

$$\omega = \frac{(\rho_2 U_2 A_2 + \rho_1 U_1 A_1)k}{\rho_2 A_2 + \rho_1 A_1} \pm \sqrt{\frac{g k (\rho_1 - \rho_2)}{\rho_2 A_2 + \rho_1 A_1}} - \frac{\rho_1 \rho_2 (U_1 - U_2)^2 A_1 A_2 k^2}{(\rho_2 A_2 + \rho_1 A_1)}$$  \hspace{1cm} (13)

Clearly, $\omega$ can be imaginary. Meaning that surface wave in equation 6 with wave number $k$ satisfies equation 9 with potential function in the form of equation 11 has phase velocity $\omega$ provided by equation 13. If the right-hand side of equation 13 is a real number, the wave propagates with speed $\omega^\pm$.

It can be seen that the term containing root in equation 13 can be imaginary provided that

$$(U_1 - U_2)^2 > \frac{g (\rho_1 - \rho_2)}{k \rho_1 \rho_2} \tanh(kh_1) \tanh(kh_2) (\rho_2 \coth(kh_2) + \rho_1 \coth(kh_1))$$  \hspace{1cm} (14)

If equation 14 is satisfied, wave speed in equation 13 would be imaginary which is $\omega = \omega_r + i \omega_i$. As a consequence, surface wave in equation 6 can be rearranged as

$$\eta(x, t) = a e^{i \omega t} e^{-i k x - \omega_r t}$$  \hspace{1cm} (15)

The first two-term in equation 15 represents wave amplitude which depends on time, while the rest represent periodic waves. Flow satisfying this condition generates the so-called Kelvin-Helmholtz (KH) Instability [3]. This critical wind velocity is a necessary condition for the monochromatic wave to be amplified by air stream over the surface of the water.

As for upper fluid (air) is unbounded, the dispersion relation in equation 13 now reads

$$\omega = \frac{(\rho_2 U_2 + \rho_1 U_1 \coth(kh_1))k}{\rho_2 + \rho_1 \coth(kh_1)} \pm \sqrt{\frac{g k (\rho_1 - \rho_2)}{\rho_2 + \rho_1 \coth(kh_1)}} - \frac{\rho_1 \rho_2 (U_1 - U_2)^2 \coth(kh_1) k^2}{(\rho_2 + \rho_1 \coth(kh_1))^2}$$  \hspace{1cm} (16)

For case there is no flow in both fluids i.e. $U_1 = U_2 = 0$, the obtained dispersion relation similar to [1]. Further, for an additional $\rho_2 = 0$, the obtained dispersion relation equivalent to linear wave theory. Hence from equation 16, critical wind speed for a wave to be excited is

$$(U_1 - U_2)^2 > U_{crit} = \frac{g (\rho_1 - \rho_2)}{\rho_1 \rho_2} \tanh(kh_1) (\rho_2 + \rho_1 \coth(kh_1))$$  \hspace{1cm} (17)

for surface waves in $h_1$ depth.

For deep water example, the critical wind to excite the disturbance on the free surface is $U_1 - U_2 = 650 \text{ m/s}$ or $23.4 \text{ km/h}$. Such wind velocity according to [6] and [3] is not comparable to actual condition which is only about 1.1 m/s wind velocity to generate waves. Meanwhile for shallow water case where $kh_1 \ll 1$ [7]. For $\rho_1 = 1000 \text{ kg/m}^3$, $\rho_2 = 1.29 \text{ kg/m}^3$, $g = 9.81 \text{ m/s}^2$ one can have critical wind plot for different $kh$ as seen in Figure 2. The critical wind velocity shown in the figure has a extremely different comparing to actual data from [4] which is only about $4 \text{ m/s}$.

This discrepancy may occur due to the assumption that the upper fluid (air) is incompressible and irrotational whilst air is actually compressible and, more importantly, irregular. In this paper, the air motion or wind is modeled by studying its effect on the surface. The effect of wind above the surface, according to [6], will generate pressure variation on the surface. The following section will incorporate the pressure fluctuation due to wind in a shallow-water model.
3. Wind forced shallow water model

In this section, Jeffreys sheltering theory is used to modify the well-known shallow water equation. This depth-integrated based model is relatively simpler than the former model to solve. A numerical method based on staggered grids [8], or predictor-corrector scheme [9] can be employed to solve the shallow water model. Not only this model has interesting phenomena such as wave breaking, but it also covers numerous conveniences to include bottom friction effect or various cross-section area.

The shallow water equation with additional eddy viscosity in the flat bottom as in [8] reads

$$\frac{\partial \eta}{\partial t} + \frac{\partial (hv)}{\partial x} = 0,$$

$$\frac{\partial (hv)}{\partial t} + \frac{\partial (hv^2)}{\partial x} + \frac{1}{\rho} \frac{\partial F_P}{\partial x} = -C_f v |v| + \nu \frac{\partial}{\partial x} \left( h \frac{\partial v}{\partial x} \right)$$

(18)

where $h(x,t) = h_0 + \eta(x,t)$ denoting water depth, $v$ denoting horizontal water velocity and constant $\nu$ is eddy viscosity coefficient. While the force due to hydrostatic pressure $F_P$ distribution is

$$F_P(x, z, t) = \int_0^{h(x,t)} \left( \rho g (h(x,t) - z) + P_a(x,t) \right) \, dz$$

(19)

Assuming that there is no external pressure, $P_a = 0$ on the surface should give a standard shallow water equation.

The Jeffreys’ sheltering theory states that the pressure on the surface is proportional to the slope of the surface facing the wind. Suppose that wind blows with constant velocity $U^* > 0$. Thus the surface pressure experiencing wind is given by

$$P_a(x, t) = \sigma P U^2 \frac{\partial \eta}{\partial x}$$

(20)
with \( \rho_a \) denoting air density, \( U' \) wind velocity average at 10-meter height, and \( s \) is a numerical coefficient smaller than unity. This mechanism describes the interaction between air and water by assuming that the wave facing directly to the wind experience the most significant effect.

This theory provides an explanation of wind action on the water surface, rather than modeling the motion of the air since it is actually irregular, according to [10]. It was mentioned in [11] that the knowledge of shear effect due to wind action, as in [12], is still not widely explored. Meanwhile, the sheltering theory has been used several times in [13] to study freak wave and in [14] to study three-wave modulations both in deep water. Therefore, Jeffreys’ sheltering theory is the potential to be applied in shallow water.

In order to include the wind pressure to shallow water equation, equation 20 need to be substituted into equation 19 then integrate the equation. Then substituting the result to equation 18 gives

\[
\frac{\partial \eta}{\partial t} + \frac{\partial (hv)}{\partial x} = 0, \quad \frac{\partial (hv)}{\partial t} + \frac{\partial (hv^2)}{\partial x} + gh \frac{\partial h}{\partial x} = -C_f v|v| + \nu \frac{\partial}{\partial x} \left( h \frac{\partial v}{\partial x} - \frac{s \rho_a U'^2}{\rho} h \frac{\partial \eta}{\partial x} \right)
\] (21)

Equation 21 is a modified shallow water equation which includes wind and viscosity effect. According to [2] the viscosity has a damping effect on the waves. In such a way, viscosity has the ability to reduce the amplitude. Based on the two last terms in the shallow water equation, it can be identified qualitatively that the wind term has the ability to counterbalance the effect of viscosity. A quantitative investigation should be made to confirm this argument.

To study the amplification mechanism due to wind, the surface wave is assumed to be small of order \( \epsilon \) which is small comparing to unity. Thus the wave height can be rewritten as equation

\[
h(x,t) = h_0 + \epsilon \eta(x,t) \quad \text{and} \quad v(x,t) = \epsilon u.
\]

Substituting to shallow water equation 21 gives

\[
\frac{\partial \eta}{\partial t} + h_0 \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} - \frac{s \rho_a U'^2}{\rho} \frac{\partial^2 \eta}{\partial x^2}
\] (22)

Equation 22 is obtained by neglecting the smaller order terms of order \( \epsilon^2 \). Now, suppose that the surface displacement and water velocity is given as oscillatory profile as follows

\[
\eta(x,t) = \hat{\eta} e^{ikx - \omega t}, \quad u(x,t) = \hat{u} e^{ikx - \omega t}
\] (23)

Substituting equation 23 into equation 22 gives the following system of equation

\[
\begin{bmatrix}
\omega & i(hk) + s \rho_a U'^2 k^2 / \rho \\
i(gk) + s \rho_a U'^2 k^2 / \rho & \omega + \nu k^2
\end{bmatrix}
\begin{bmatrix}
\hat{\eta} \\
\hat{u}
\end{bmatrix} = 0
\] (24)

The necessary condition for the system to have a nontrivial solution is

\[
\left( \omega + \frac{\nu k^2}{2} \right)^2 + gh_0 k^2 + i(h_0 \frac{s \rho_a U'^2}{\rho} - k^3) = \frac{\nu^2 k^4}{4}
\] (25)

Suppose that

\[
\sigma^2 = gh_0 k^2 + i(h_0 \frac{s \rho_a U'^2}{\rho} - k^3), \quad \theta = \frac{\nu k^2}{\sigma}, \quad x\sigma = \omega + \frac{\nu k^2}{2}
\] (26)

Substituting into equation 25 and assume that the viscosity is small i.e. \( \nu \ll 1 \) thus one obtain \( x = \pm i \). So that the wave speed is given by

\[
\omega = -\frac{\nu k^2}{2} \pm i\sigma
\] (27)
where
\[
\sigma_r = \frac{1}{2} \left( a + (a^2 + b^2)^{1/2} \right)^{1/2}, \quad \sigma_i = \frac{1}{2} \left( -a + (a^2 + b^2)^{1/2} \right)^{1/2}
\] (28)
meanwhile
\[
a = gh_0 k^2, \quad b = h_0 s \rho a U'^2 \rho k^3
\] (29)
The growth (decay) of the wave depends on the following critical value
\[
\omega_r = \frac{\nu k^2}{2} \pm \sigma_i
\] (30)
while the wave propagating with \( \omega_i \). As for \( \omega_r > 0 \) one have
\[
U'^2 c > \frac{\nu \rho}{s h_0 \rho} e^2
\] (31)
where \( e^2 = gh_0 + \nu^2 \). Following the preceding section, the critical wind for wave to be excited is
\[
U > \left( \frac{27 g \nu \rho}{4 \rho} \right)^{1/3} \left( \frac{1}{s} \right)^{1/3}
\] (32)
which has been simplified due to small \( \nu \). Using the constants in the former analysis with additional \( \nu = 1.004 \times 10^{-6} \), stability curve for different value of \( s \) is shown in Figure 3.

**Figure 2.** The figure shows the critical wind velocities (m/s) curve for the various value of \( s \) for the shallow water model. Wind velocity lies in the area above the curve induces instability on the surface of the water.
4. Conclusion
Critical wind velocity to excite pure wave on the water surface for two different approaches has been obtained. The first approach is by employing the irrotational assumption for both fluids. Thereby critical wind velocity to induce wave on the interface is obtained, which so-called Kelvin-Helmholtz instability. It shows that only extreme windy event is able to induce wave on the water surface. Differently, Jeffreys’ sheltering mechanism is embedded into the shallow water equation hence obtain critical wind, which depends on the eddy viscosity and sheltering coefficient. This approach has convenience for choosing the coefficient to obtain comparable critical wind with actual data. This result, however, might be used reversely to estimate the sheltering coefficient associated with the actual wind, which induces waves on shallow water area.

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