EMBEDDING 3-MANIFOLDS WITH BOUNDARY INTO CLOSED 3-MANIFOLDS

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ABSTRACT. We prove that there is an algorithm which determines whether or not a given 2-polyhedron can be embedded into some integral homology 3-sphere.

This is a corollary of the following main result. Let $M$ be a compact connected orientable 3-manifold with boundary. Denote $G = \mathbb{Z}$, $G = \mathbb{Z}/p\mathbb{Z}$ or $G = \mathbb{Q}$. If $H_1(M; G) \cong G^k$ and $\partial M$ is a surface of genus $g$, then the minimal group $H_1(Q; G)$ for closed 3-manifolds $Q$ containing $M$ is isomorphic to $G^{k-g}$.

Another corollary is that for a graph $L$ the minimal number $\text{rk} H_1(Q; \mathbb{Z})$ for closed orientable 3-manifolds $Q$ containing $L \times S^1$ is twice the orientable genus of the graph.

1. Introduction and main results

Let $M$ be a compact orientable 3-manifold with boundary. In Theorem 1.3, we find the minimal rank of $H_1(Q; \mathbb{F})$ for all closed 3-manifolds $Q$ containing $M$ (i.e. such that $M$ embeds into $Q$) in terms of homology of $M$. Here $\mathbb{F}$ is one of the fields $\mathbb{Q}$ or $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$. Theorem 1.4 is an integral version of Theorem 1.3. The following are two corollaries.

Corollary 1.1. Suppose $G = \mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}$. There exists an algorithm that for any given (finite) 2-polyhedron $P$ tells if $P$ is embeddable into some $G$-homology 3-sphere (the sphere is not fixed in advance).

According to [5], the existence of an algorithm recognizing embeddability of 2-polyhedra in $\mathbb{R}^3$ is unknown, cf. [2].

Corollary 1.2. Let $L$ be a connected graph of genus $g(L)$. Suppose $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$. The minimal number $\dim H_1(Q; \mathbb{F})$ for closed orientable 3-manifolds $Q$ containing $L \times S^1$ equals to $2g(L)$.

Here the genus of graph $g(L)$ is the minimal $g$ such that $L$ embeds into a surface of genus $g$. To prove these corollaries, we use the classification of 3-thickenings of 2-polyhedra [9, 4, 11]. According to [5], the existence of an algorithm recognizing embeddability of 2-polyhedra in $\mathbb{R}^3$ is unknown, cf. [2].

Corollary 1.2. Let $L$ be a connected graph of genus $g(L)$. Suppose $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$. The minimal number $\dim H_1(Q; \mathbb{F})$ for closed orientable 3-manifolds $Q$ containing $L \times S^1$ equals to $2g(L)$.

Here the genus of graph $g(L)$ is the minimal $g$ such that $L$ embeds into a surface of genus $g$. To prove these corollaries, we use the classification of 3-thickenings of 2-polyhedra [9, 4, 11]. In particular, from the cited papers we derive Lemma 1.8 stating that all orientable 3-thickenings of a given 2-polyhedron are algorithmically constructible.

Theorem 1.3. Let $M$ be a compact connected 3-manifold with orientable boundary. Denote $g := \text{rk} H_1(\partial M; \mathbb{Z})/2$. Take a field $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$. Suppose $M$ is orientable or $\mathbb{F} = \mathbb{Z}_2$.

(a) If $M$ is embedded into a closed 3-manifold $Q$, then $\dim H_1(Q; \mathbb{F}) \geq \dim H_1(M; \mathbb{F}) - g$.

(b) There is a closed 3-manifold $Q$ containing $M$ such that $\dim H_1(Q; \mathbb{F}) = \dim H_1(M; \mathbb{F}) - g$ and $Q$ is orientable if $M$ is orientable.

Part (a) is simple: it follows from the Mayer-Vietoris sequence, see the proof at the end of the introduction. Proof of part (b) (i.e., the construction of ‘minimal’ $Q$) is given in §2. It is based on symplectic linear algebra and Poincaré’s theorem on the image of the mapping class group of a surface $P$ in $\text{Aut}(H_1(P; \mathbb{Z}))$. 

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Theorem 1.4. Let $M$ be a compact connected orientable 3-manifold with boundary. Denote $g := \text{rk} H_1(\partial M; \mathbb{Z})/2$.

(a) If $M$ is embedded into a closed 3-manifold $Q$, then $H_1(Q; \mathbb{Z})$ has a sub-quotient isomorphic to $C(M) := \mathbb{Z}^{\text{rk} H_1(M; \mathbb{Z}) - g} \oplus \text{Tors} H_1(M; \mathbb{Z})$.

(b) Suppose $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^m$. Then there is a closed orientable 3-manifold $Q$ containing $M$ such that $H_1(Q; \mathbb{Z}) \cong C(M) = \mathbb{Z}^{m-g}$.

(c) There is a compact connected orientable 3-manifold $M$ with boundary which is not embeddable into any closed 3-manifold $Q$ such that $H_1(Q; \mathbb{Z}) \cong C(M)$.

Here $\text{rk} X$ and $\text{Tors} X$ are, respectively, the rank and the torsion subgroup of an abelian group $X$. Again, part (a) is essentially known and part (b) is new; it is proved after Theorem 1.3(b) in §2. We present an example for part (c) in §3.

Remark. Suppose a closed orientable 3-manifold $Q$ contains $M$ and $H_1(Q; \mathbb{Z}) \cong C(M)$. Then for each field $F = \mathbb{Z}_p$ and $F = \mathbb{Q}$ we get $\dim H_1(Q; F) = \dim H_1(M; F) - \text{rk} H_1(\partial M; \mathbb{Z})/2$, while the proof of Theorem 1.3(b) generally provides different ‘minimal’ manifolds for different fields.

Corollary 1.5. Let $M$ be a compact orientable 3-manifold with boundary and suppose $G = \mathbb{Z}$, $G = \mathbb{Z}_p$ or $G = \mathbb{Q}$. Then $M$ embeds into some $G$-homology 3-sphere if and only if $H_1(M; G) \oplus H_1(M; G) \cong H_1(\partial M; G)$.

Corollary 1.5 is straightforward. Corollaries 1.1 and 1.2 are proved below in this section. The construction of the ‘minimal’ $Q$ in Corollary 1.2 is simpler than the general construction in Theorem 1.3. However, the lower estimation here is harder and is reduced to the lower estimation in Theorem 1.3 by the following lemma. This lemma is proved in §4.

Lemma 1.6. Let $L$ be a connected graph. Suppose that the product $L \times S^1$ is embeddable into a 3-manifold $Q$. Suppose that either $Q$ is orientable or $L$ is not homeomorphic to $S^1$ or $I$. Then the regular neighborhood of $L \times S^1$ in $Q$ is homeomorphic to the product $K \times S^1$ for a certain 2-manifold $K$ containing $L$. If $Q$ is orientable, then $K$ is also orientable.

For instance, let $K_5$ be the complete graph on 5 vertices. Corollary 1.2 implies that $K_5 \times S^1$ is embeddable into a certain closed orientable 3-manifold $Q$ such that $\dim H_1(Q; \mathbb{F}) = 2$ and is not embeddable into any closed orientable 3-manifold with the first homology group of dimension 0 or 1. This result was obtained by A. Kaibkhanov (unpublished). The non-embeddability of $K_5 \times S^1$ into $S^3$ was stated by M. Galecki and T. Tucker (as far as the author knows, unpublished) and proved by M. Skopenkov in [12].

The structure of the paper is as follows. Now we prove Corollary 1.2, Theorems 1.3(a) and 1.4(a). In this section we also prove Corollary 1.1 for which we will need Lemma 1.8 below. In §2 we prove Theorems 1.3(b) and 1.4(b). In §3 we provide an example which proves Theorem 1.4(c). In §4 we prove Lemmas 1.6 and 1.8. The proof of both lemmas uses the classification of 3-dimensional thickenings of 2-polyhedra [9].

Example 1.7. For $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$ denote $r(M; \mathbb{F}) := \dim H_1(M; \mathbb{F}) - \dim H_1(\partial M; \mathbb{F})/2$.

(a) Let $\Xi$ be a surface of genus $g$ with $h$ holes. Then $r(\Xi \times S^1; \mathbb{F}) = 2g$.

(b) Let $\Xi$ be a connected sum of $k \mathbb{R}P^2$'s with $h$ holes. Then $r(\Xi \times S^1; \mathbb{Z}_2) = k$.

Proof of Corollary 1.2 modulo Theorem 1.3 and Lemma 1.6. Since $S^1 \times S^1 \subset S^3$ and $I \times S^1 \subset S^3$, it suffices to consider the case when $L$ is not homeomorphic to $S^1$ or $I$. Corollary 1.2 now follows from Lemma 1.6, Example 1.7(a) and Theorem 1.3. 

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1 The non-embeddability of $K_5 \times S^1$ into $S^3$ could be proved in a simpler way using the van Kampen theorem if we assumed that $S^3 \setminus U(K_5 \times S^1)$ is homeomorphic to a disjoint union of solid tori. (Here $U(K_5 \times S^1)$ denotes the regular neighborhood of $K_5 \times S^1$ in $S^3$.) However, this assumption is not trivial to prove and becomes wrong if we replace $K_5$ by some other graph $G$ such that $G \times S^1$ embeds into $S^3$. For example, let $G$ be a point. Take a knotted embedding $S^1 \subset S^3$. Then $S^3 \setminus U(S^1)$ is not homeomorphic to a solid torus.
Proof of Theorems 1.3(a) and 1.4(a). Suppose that \( M \subset Q \), where \( M \) is a 3-manifold with boundary and \( Q \) is a closed 3-manifold. In this paragraph, the homology coefficients are \( \mathbb{Z}, \mathbb{Z}_p \) or \( \mathbb{Q} \). Let \( i: H_1(\partial M) \to H_1(M) \), \( I: H_1(M) \to H_1(Q) \) be the inclusion-induced homomorphisms. From the exact sequence of pair \((Q, M)\) we obtain that \( H_1(Q) \) has a subgroup isomorphic to \( H_1(M)/\ker I \). From the Mayer-Vietoris sequence for \( Q = M \cup_{\partial M} (Q - M) \) we obtain \( \ker I \subset \im i \). So \( H_1(M)/\im i \) is a quotient of \( H_1(Q)/\ker I \).

Let us prove Theorem 1.3(a); here the coefficients are \( \mathbb{F} = \mathbb{Z}_p \) or \( \mathbb{F} = \mathbb{Q} \). By the known ‘half lives – half dies’ lemma, \( \dim \im i = \dim H_1(\partial M; \mathbb{F})/2 = g \), see [1] p.158, [3] Lemma 3.5. Thus \( \dim H_1(Q; \mathbb{F}) \geq \dim H_1(M; \mathbb{F}) - g \).

To prove Theorem 1.4(a), it is left to check that \( C(M) \cong K := H_1(M; \mathbb{Z})/\im i \). Indeed, we obtain that \( \rk K = g \) by the universal coefficients formula and the argument from the previous paragraph for \( \mathbb{Q} \)-coefficients, and \( \text{Tors } K = Tors H_1(M, \partial M; \mathbb{Z}) = Tors H_1(M; \mathbb{Z}) \) by the exact sequence of pair \((M, \partial M)\) and Poincaré duality. ■

Let \( P \) be a (finite) polyhedron. If a 3-manifold \( M \) is a regular neighborhood of \( P \subset M \), then the pair \((M, P)\) is called a 3-thickening of \( P \) [10]. If we say that two thickenings are homeomorphic, we mean that they are homeomorphic in the category of thickenings, i.e. the homeomorphism in question is relative to the polyhedron embedded into each thickening.

The following lemma is known to specialists, but the author has not found any proof in literature. This lemma is proved by combining [9] and [11] (also see [4]); we prove it in §4.

**Lemma 1.8.** Each polyhedron \( P \) has (up to homeomorphism) a finite number of orientable 3-thickenings. There exists an algorithm that for a given polyhedron \( P \) constructs all its orientable 3-thickenings (i.e., constructs their triangulations), or tells that the polyhedron has none.

**Proof of Corollary 1.3 modulo Corollary 1.2 and Lemma 1.8.** Clearly, \( P \) is embeddable into an orientable 3-manifold \( Q \) if and only if there exists an orientable 3-thickening of \( P \) which is embeddable into \( Q \). So the algorithm for Corollary 1.3 is as follows. First, the algorithm constructs all orientable 3-thickenings of \( P \) with the help of Lemma 1.8. If there are no such thickenings, then \( P \) is not embeddable into any orientable 3-manifold, and the algorithm gives the negative answer. Otherwise, the algorithm checks the condition of Corollary 1.2 for each orientable 3-thickening of \( P \) and gives the positive answer if the condition was fulfilled for at least one 3-thickening. ■

**Remark.** Our methods do not lead to an algorithm for embeddability of 2-polyhedra into \( \mathbb{R}^3 \) because we do not deal with the fundamental group, which is presumably much harder to do.

2. **Proof of Theorems 1.3(b), 1.4(b) (construction of a manifold \( Q \))**

In this section give a proof of Theorem 1.3(b) and then slightly modify it to prove Theorem 1.4(b).

**Proofs of Theorem 1.3(b).** Denote \( \mathbb{F} := \mathbb{Z}_p \) or \( \mathbb{F} := \mathbb{Q} \). In the current proof, if coefficients in a homology group are omitted, they are assumed to be in \( \mathbb{F} \).

Let \( X \subset \mathbb{R}^3 \) be the standardly embedded disjoint union of handlebodies such that \( \partial X \cong \partial M \) and let \( i: H_1(\partial M) \to H_1(M), \; i': H_1(\partial X) \to H_1(X) \) be the inclusion-induced homomorphisms. We construct the required manifold \( Q \) as a union of \( X \) and \( M \) along certain diffeomorphism \( f: \partial X \to \partial M \). Consider the Mayer-Vietoris sequence

\[
H_1(\partial M) \stackrel{i \oplus i' f^{-1} \circ f}{\longrightarrow} H_1(M) \oplus H_1(x) \to H_1(Q) \to \widetilde{H}_0(\partial M) = 0.
\]

It follows that

\[
H_1(Q) \cong \frac{H_1(M) \oplus H_1(X)}{(i \oplus i' f^{-1} \circ f)H_1(\partial M)}.
\]

Suppose the map \( i \oplus i' f^{-1} \) is a monomorphism. Then \( \dim H_1(Q) = \dim H_1(M) - g \) as required. So our goal now is to construct a map \( f: \partial X \to \partial M \) such that \( i \oplus i' f^{-1} \) is a monomorphism.

Let us introduce new notation. For \( G = \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{Z}_p \), a bilinear form \( \omega: G^{2g} \otimes G^{2g} \to G \) is called symplectic if it is non-degenerate, skew-symmetric and, when \( G = \mathbb{Z} \), unimodular. A submodule
Lemma 2.1. Let \( \omega \) be a symplectic form on \( \mathbb{Z}^{2g} \).

(a) Denote by \( \phi : \mathbb{Z}^{2g} \to \mathbb{Z}_p^{2g} \) the homomorphism which applies mod \( p \) reduction to each component and by \( \omega_p \) the symplectic form on \( \mathbb{Z}_p^{2g} \) which is mod \( p \) reduction of \( \omega \). For each \( \mathbb{Z}_p \)-Lagrangian \( A \subset \mathbb{Z}_p^{2g} \) there exists a \( \mathbb{Z} \)-Lagrangian \( B \subset \mathbb{Z}^{2g} \) such that \( \phi B = A \).

(b) Denote by \( \phi : \mathbb{Z}^{2g} \to \mathbb{Q}^{2g} \) the inclusion and by \( \omega_0 \) the symplectic form on \( \mathbb{Q}^{2g} \) defined by the restriction \( \omega|_{\mathbb{Z}^{2g}} \equiv \omega \). For each \( \mathbb{Q} \)-Lagrangian \( A \subset \mathbb{Q}^{2g} \) there exists a \( \mathbb{Z} \)-Lagrangian \( B \subset \mathbb{Z}^{2g} \) such that \( \text{Lin} \phi B = A \).

Proof of Lemma 2.1. Part (b) is obvious. Let us prove part (a). Recall that \( \phi \) is the reduction mod \( p \). Take a set of generators \( \{e_i, f_i\}_{i=1}^g \) for \( \mathbb{Z}^{2g} \) such that \( \omega(e_i, f_i) = \delta_{ij} \). Then \( \text{Lin} \{\phi e_i\}_{i=1}^g \) is a \( \mathbb{Z}_p \)-Lagrangian, hence there exists \( h_{e_i} \in \text{Sp}(2g, \mathbb{Z}_p) \) taking \( \{\phi e_i\}_{i=1}^g \) to \( A \) because \( \text{Sp}(2g, \mathbb{Z}_p) \) acts transitively on \( \mathbb{Z}_p \)-Lagrangians. Since mod \( p \) reduction maps \( \text{Sp}(2g, \mathbb{Z}) \) epimorphically onto \( \text{Sp}(2g, \mathbb{Z}_p) \) [7, Theorem VII.21], we can find \( h \in \text{Sp}(2g, \mathbb{Z}) \) such that \( \phi h = h_{e_i} \). Then \( B := \{he_i\}_{i=1}^g \) is the required \( \mathbb{Z} \)-Lagrangian.

Continuation of proof of Theorem [13](b). Denote by \( \cap : H_1(\partial M; \mathbb{Z}) \times H_1(\partial M; \mathbb{Z}) \to \mathbb{Z} \) the intersection form and by \( \cap : H_1(\partial M) \times H_1(\partial M) \to \mathbb{F} \) the induced form (as in Lemma 2.1); \( \cap|_F \) coincides with the \( \mathbb{F} \)-coefficients intersection form on \( H_1(\partial M) \). It is well known that \( \dim \ker i = g \) and \( \cap|_{\ker i} \equiv 0 \) (the last assertion is analogous to [1] p.158). In other words, \( \ker i \) is Lagrangian with respect to \( \cap \). Clearly, there exists another Lagrangian \( A \subset H_1(\partial M) \) such that \( \ker i \cap A = \{0\} \). Let \( \phi \) be the homomorphism from Lemma 2.1. By Lemma 2.1(a) or Lemma 2.1(b) (depending on what coefficient field \( \mathbb{F} \) we are working with) we obtain a Lagrangian submodule \( B \subset H_1(\partial M; \mathbb{Z}) \) such that \( \text{Lin} \phi B = A \) (if \( \mathbb{F} = \mathbb{Z}_p \), this is equivalent to \( \phi B = A \)).

Recall the Poincaré theorem [8] that for a handle sphere \( S \) every automorphism of \( H_1(S; \mathbb{Z}) \) preserving the intersection form is induced by some self-diffeomorphism of \( S \).

Denote \( i_Z : H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \) the inclusion-induced homomorphism; then \( \ker i_Z \) is generated by the meridians and is a \( \mathbb{Z} \)-Lagrangian in \( H_1(\partial X) \). Thus there exists a diffeomorphism \( f : \partial X \to \partial M \) such that \( f_* \ker i_Z = B \). (Indeed, suppose that \( \partial X \cong \partial M \) is connected. Pick any diffeomorphism \( h_1 : \partial X \to \partial M \). Then \( K := h_1^- \ker i_Z \subset H_1(\partial M, \mathbb{Z}) \) is a \( \mathbb{Z} \)-Lagrangian. By the Poincaré theorem and because \( \text{Sp}(2g, \mathbb{Z}) \) acts transitively on \( \mathbb{Z} \)-Lagrangians there exists a self-diffeomorphism \( h_2 \) of \( \partial M \) such that \( h_2^* K = B \). Now take \( f := h_2 h_1 \). If \( \partial M \) is not connected, apply this construction componentwise.)

Because \( X \) is a disjoint union of handlebodies, \( \ker i' = \text{Lin} \phi \ker i_Z \) (if \( \mathbb{F} = \mathbb{Z}_p \) and not \( \mathbb{Q} \), then \( \ker i' = \phi \ker i_Z \)). So

\[
\ker i' f_*^{-1} = f_* \text{Lin} \phi \ker i_Z = \text{Lin} \phi f_* \ker i_Z = \text{Lin} \phi B.
\]

Recall that \( \ker i \cap \text{Lin} \phi B = \{0\} \). Therefore \( i \oplus i' f_*^{-1} \) is monomorphic. □

Proof of Theorem [7](b). We use notation similar to the previous proof and work with \( \mathbb{Z} \)-coefficients here. Recall that \( \ker i \) is a \( \mathbb{Z} \)-Lagrangian, i.e. \( \cap|_{\ker i} \equiv 0 \) and \( H_1(\partial M)/\ker i \cong \mathbb{Z}^g \) [1, p.158]; thus we can find a set of generators \( \{x_1, \ldots, x_g\} \subset H_1(\partial M) \) such that \( \{x_1, \ldots, x_g\} \) generate \( \ker i \) and \( \{x_g, \ldots, x_{2g}\} \) also generate a Lagrangian. Then there exists a diffeomorphism \( f : \partial X \to \partial M \) such that \( \ker i' f_*^{-1} \) is generated by \( \{x_{g+1}, \ldots, x_{2g}\} \). This is done analogously to the proof of Theorem [13](b) using the Poincaré theorem [8]. By construction we obtain

\[
H_1(Q) \cong \frac{H_1(M) \oplus H_1(X)}{(i \oplus i' f_*^{-1}) H_1(\partial M)} \cong \frac{H_1(M)}{i H_1(\partial M)} \oplus \frac{H_1(X)}{(i' f_*^{-1}) H_1(\partial M)} \cong \frac{H_1(M)}{i H_1(\partial M)} \cong C(M).
\]

The second group in the direct sum is obviously zero for \( X \) a disjoint union of handlebodies. The last isomorphism is shown in the proof of Theorems [13](a), [14](a). □
In this section we omit $\mathbb{Z}$-coefficients. Theorem 1.4(c) is implied by the following two lemmas.

**Lemma 3.1.** There exists a connected orientable 3-manifold $M$ such that

1. $\partial M$ is a torus and $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$.
2. Let $l$ and $m$ generate $H_1(M)$ and $2m = 0$. For some generators $a$, $b$ of $H_1(\partial M) \cong \mathbb{Z} \oplus \mathbb{Z}$ the inclusion-induced homomorphism $i : H_1(\partial M) \to H_1(M)$ is given by $i(a) = 2l$, $i(b) = m$.

**Proof.** Let $D := D^2 \times S^1$ be a solid torus and $D'$ its copy. Cut out from $D$ another solid torus which lies inside $D$ and runs twice along the parallel of $D$ (see Figure 1). Glue the result to $D'$ along $\partial D = \partial D'$. It is easily seen that the orientable 3-manifold $M$ obtained satisfies (1), (2). The generators of $H_1(M)$ as in (2) are shown on Figure 1.

**Lemma 3.2.** Consider a manifold $M$ from Lemma 3.1. Then $C(M) = \mathbb{Z}_2$ (the group $C(M)$ is introduced in Theorem 1.4) but $M$ is not embeddable into any closed 3-manifold $Q$ such that $H_1(Q) \cong \mathbb{Z}_2$.

**Proof.** Obviously, $C(M) = \mathbb{Z}_2$. Suppose to the contrary that there is an embedding $M \subset Q$. Denote by $X$ the closure of $Q \setminus M$ and by $i' : H_1(\partial X) = H_1(\partial M) \to H_1(X)$ the inclusion-induced homomorphism. It follows from the Mayer-Vietoris sequence that

$$H_1(Q) \cong \frac{H_1(M) \oplus H_1(X)}{(i \oplus i')H_1(\partial M)},$$

thus, $H_1(Q)$ contains the subgroup $R := \frac{H_1(M)}{i(\ker i')}$. First, suppose $Q$ is orientable. Then the rank of $\ker i'$ is equal to 1, so $\ker i'$ is generated by $pa + qb$ for some $p, q \in \mathbb{Z}$. Notice that $i(pa + qb) = 2pl + qm$. We obtain that $R$ is generated by $l$ and $m$ with the following two relations: $2m = 0$, $2pl + qm = 0$. Clearly, $R \neq 0$ and $R \neq \mathbb{Z}_2$ since the determinant of the matrix $\left( \begin{smallmatrix} 2 & q \\ 0 & p \end{smallmatrix} \right)$ is divisible by 4 but never equals $\pm 2$ or $\pm 1$, as it should be when $R \cong \mathbb{Z}_2$ or $R = 0$.

The case of non-orientable $Q$ is analogous. We have now to consider the cases $\rk \ker i' = 0$ and $\rk \ker i' = 2$. In the first case, $R = \mathbb{Z} \oplus \mathbb{Z}_2$. In the second case, the matrix of relations for $R$: $(\begin{smallmatrix} 2p & 2r \\ q & s \end{smallmatrix})$ is such that all of its 2$\times$2-minors are divisible by 4. This again implies that $R \neq 0$ and $R \neq \mathbb{Z}_2$.

**Remark.** The manifold $M$ constructed in Lemma 3.1 is embeddable into a 3-manifold $Q$ with $H_1(Q) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and into $S^1 \times S^2$ with $H_1(S^1 \times S^2) \cong \mathbb{Z}$ (both manifolds are obtained by gluing a solid torus to $M$ appropriately). These two manifolds verify Theorem 1.4(b) for this particular manifold $M$: the first manifold $Q$ when $\mathbb{F} \neq \mathbb{Z}_2$, and $S^1 \times S^2$ when $\mathbb{F} = \mathbb{Z}_2$.

4. Proofs of Lemmas 1.6, 1.8

We will use results from \cite{9}; let us state them here briefly and prove Lemma 1.6 after that. The proof of Lemma 1.8 uses the same results and is given at the end of this section.

A classification of 3-thickenings of 2-polyhedra \cite{9}.

Let $P$ be a 2-polyhedron. By $P'$ we denote the 1-subpolyhedron which is the set of points in $P$ having no neighborhood homeomorphic to 2-disk. By $P''$ we will denote a (finite) set of points of $P'$ having no neighborhood homeomorphic to a book with $n$ sheets for some $n \geq 1$. Take a point in any component of $P'$ containing no point of $P''$. Denote by $F$ the union of $P''$ and these points.
Suppose that \( \cup_{A \in F} \text{lk } A \) is embeddable into \( S^2 \). (Here \( \text{lk} \) denotes link of a point.) Take a collection of embeddings \( \{ g_A : \text{lk } A \to S^2 \}_{A \in F} \). Take the closure \( d \subset P' \) of a connected component of \( P' \setminus \partial P'' \) and denote its ends by \( A, B \in F \) (possibly, \( A = B \)). Now \( d \) meets \( \text{lk } A \cup \text{lk } B \) at two points (distinct, even when \( A = B \)). If for each such \( d \) the maps \( g_A \) and \( g_B \) give the same or the opposite orders of rotation of the pages of the book at \( d \) then the collection \( \{ g_A \} \) is called faithful. Two collections of embeddings \( \{ f_A : \text{lk } A \to S^2 \}, \{ g_A : \text{lk } A \to S^2 \} \) are called isopositioned, if there is a family of homeomorphisms \( \{ h_A : S^2 \to S^2 \}_{A \in F} \) such that \( h_A \circ f_A = g_A \) for each \( A \in F \). This relation preserves faithfulness. Denote by \( E(P) \) the set of faithful collections up to isposition.

Suppose that \( M \) is a 3-thickening of \( P \). Take any point \( A \in F \) and consider its regular neighborhood \( R_M(A) \). Since \( \partial R_M(A) \) is a sphere, we have a collection of embeddings \( \{ \text{lk } A \to \partial R_M(A) \}_{A \in F} \). Since for each closure \( d \subset P' \) of a connected component of \( P' \setminus \partial P'' \) the regular neighborhood of \( d \) is embedded into \( M \), this collection of embeddings is faithful. The class \( e(M) \in E(P) \) of this collection is called the \( e \)-invariant of \( M \).

\[ \text{Theorem 4.1.} \] Thickenings \( M_1, M_2 \) of \( P \) are homeomorphic relative to \( P \) if and only if \( w_1(M_1)|_P = w_1(M_2)|_P \) and \( e(M_1) = e(M_2) \).

\[ \text{Proof of Lemma 4.1.} \] Without loss of generality we may assume that \( Q \) is a regular neighborhood of \( L \times S^1 \). Due to Theorem 4.1, it is sufficient to construct a 2-manifold \( K \) containing \( L \) such that

(a) \( K \times S^1 \) is a regular neighborhood of \( L \times S^1 \), \( e(K \times S^1) = e(Q) \) and

(b) \( w_1(K \times S^1)|_{L \times S^1} = w_1(Q)|_{L \times S^1} \).

First, let us construct a 2-manifold \( K \) satisfying (a). Take a triangulation of the graph \( L \); we will work with this triangulation only and denote it by the same letter \( L \). For each vertex \( v \) in \( L \) consider an arbitrarily oriented 2-disk \( D^2_v \). Consider the edges \( e_1, \ldots, e_n \) containing \( v \). The embedding \( L \times S^1 \subset Q \) defines a cyclic ordering of \( e_1, \ldots, e_n \). Take a disjoint union of \( n \) arcs in \( \partial D^2_v \) (each arc corresponding to an edge \( e_i \)) such that the cyclic ordering of the arcs is the same as that of the edges.

For each edge \( e \) connecting vertices \( u \) and \( v \) connect \( D^2_u \) and \( D^2_v \) with a strip \( D^1 \times D^1 \), gluing it along the two arcs that correspond to \( e \). The strip can be glued in two ways: we can either twist it or not (with respect to the orientations on \( D^2_u \) and \( D^2_v \)). After gluing a strip for each edge of \( L \), we get a union of disks and strips that is a 2-manifold; denote it by \( K \). The manifold \( K \) depends on choosing the twists. However, any such \( K \) satisfies (a), no matter what the twists are.

By choosing the twists, let us obtain the property (b).

If \( Q \) is orientable, glue all the strips without twists. Then \( K \) is orientable, and \( w_1(K \times S^1)|_{L \times S^1} = w_1(Q)|_{L \times S^1} = 0 \).

Now let us choose the twists in the other case: \( L \) is not homeomorphic to \( S^1 \) or \( I \) (and \( Q \) is not necessarily orientable). Denote the set of all edges of \( L \) by \( E \). Take a point \( 0 \in S^1 \). Take a set of cycles \( c_1, \ldots, c_s, c \in Z(L; \mathbb{Z}_2) \) such that \( [c_1], \ldots, [c_s], [c] \in H_1(L; \mathbb{Z}_2) \) is a basis. Represent \( w_1(Q)|_{L \times \{0\}} \) as a cochain \( \{ a_e \in \{0, 1\} \}_{e \in E} \) so that for all \( k, 1 \leq k \leq s \), \( \sum_{e \in t_k} a_e \text{ mod } 2 = \langle w_1(Q)|_{L \times \{0\}}, c_k \rangle \).

For each edge \( e \in E \), twist the corresponding strip if \( a_e = 1 \), and do not twist the corresponding strip if \( a_e = 0 \). We now obtain \( w_1(K \times S^1)|_{L \times \{0\}} = w_1(Q)|_{L \times \{0\}} \) by construction. We claim that the constructed \( K \) satisfies (b).

Indeed, take a vertex \( v \) of degree at least 3. This can be done, because \( L \) is not homeomorphic to \( S^1 \) or \( I \). The homology classes of

\[ c_i \times \{O\}, \quad 1 \leq i \leq s, \quad \text{and} \quad \{v\} \times S^1 \]

form a basis of \( H_1(K \times S^1; \mathbb{Z}_2) \). But

\[ \langle w_1(Q), \{v\} \times S^1 \rangle = 0 = \langle w_1(K \times S^1), \{v\} \times S^1 \rangle \]

because the regular neighborhood of \( \{v\} \times S^1 \) in \( Q \) is orientable (the orientation is defined by the orientation on \( S^1 \) and the cyclic ordering of the link of \( v \) because \( \text{deg } v \geq 3 \)). Thus we obtain \( w_1(K \times S^1)|_{L \times S^1} = w_1(Q)|_{L \times S^1} \), and the proof is finished. \( \blacksquare \)
Proof of Lemma 1.8. Let $P$ be a 2-polyhedron. We use the notation from the beginning of this section. Take a faithful collection $\{g_A\}_{A \in F}$ of embeddings. If the phrase from the definition of faithfulness: ‘the maps $g_A$ and $g_B$ give the same or the opposite orders of rotation of the pages of the book at $d$’ is true even in the form ‘the maps $g_A$ and $g_B$ always give the opposite orders of rotation of the pages at $d$’, then the collection $\{g_A\}$ is called orientably faithful. Two collections $\{f_A\}, \{g_A\}$ are called orientably isopositioned, if there is a family of orientation-preserving homeomorphisms $\{h_A : S^2 \to S^2\}_{A \in F}$ such that $h_A \circ f_A = g_A$ for each $A \in F$. This relation preserves the property of being orientably faithful. Denote by $SE(P)$ the set of orientably faithful collections up to orientable isoposition.

An orientable 3-thickening $M$ of $P$ induces an $se$-invariant $se(M) \in SE(P)$. It is an oriented version of the $e$-invariant and is defined analogously. The following is essentially proved in [11] and [4]: every class $c \in SE(P)$ is an $se$-invariant of some orientable 3-thickening of $P$. These papers give an algorithm for construction of such thickening. Moreover, if two orientable 3-thickenings $M_1, M_2$ of $P$ have the same $se$-invariants $se(M_1) = se(M_2) \in SE(P)$, they are homeomorphic (this follows from Theorem 3, since the Stiefel-Whitney classes are zeros in the orientable case).

The set $SE(P)$ is obviously finite. Hence the number of orientable 3-thickenings of $P$ is finite. The algorithm for construction of all orientable 3-thickenings of $P$ is as follows. For each class $c \in SE(P)$ build a corresponding orientable 3-thickening using the construction from [11], [4]. Theorem 4.1 guarantees that we will obtain all orientable 3-thickenings as result. ■

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