Shannon entropy as a measure of directional emission in microcavity

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We propose a noble notion of the directional emission in microcavity lasers. First, Shannon entropy of the far-field profiles in the polar coordinate can quantify the degree of unidirectionality of the emission, while previous notions about the unidirectionality can not efficiently measure in the robust range against a variation of the deformation parameter. Second, a divergence angle of the directional emission is defined phenomenologically in terms of full width at half maximum, and it is barely applicable to a complicated peak structure. However, Shannon entropy of semi-marginal probability of the far-field profiles in the cartesian coordinate can present equivalent results, and moreover it is applicable to even the cases with a complicated peak structure of the emission.

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Microcavity lasers have recently garnered considerable attention owing to their applicability as optimal candidate models for studying wave chaos [1, 2] and non-Hermitian quantum systems [1, 3, 4], as well as their optoelectronic applications and photonics [5]. In particular, the key to such applications is that microcavity lasers have high quality (Q) factors and directional emission simultaneously [6, 7]. High-Q modes guarantee a low threshold lasing; in addition, they can be adopted for bio-molecule detections [8, 9] and nano particles [10, 11]. However they are limited by isotopic emissions and low output power. Hence, directional emission is also required to support a high output power, and also facilitate easy coupling to a waveguide for optoelectronic circuits. Therefore, to date, several microcavity lasers have been studies to achieve these properties simultaneously [12–17].

In these studies, the discussions about a high Q-factor have been addressed quantitatively and systemically, because the Q-factor can be well defined by $Q = \frac{f_r}{\Delta f}$ where $f_r$ is a resonance frequency and $\Delta f$ is a resonance width, or equivalently, $Q = \frac{f_r}{f_i}$ in the context of a complex eigenfrequency $f_c = f_r +jf_i$. However, regarding directional emission, we consider that its definition is relatively subtle and it remains not well established up to date. In this Letter, thus, we introduce new measures of the unidirectional emission by exploiting the notion of Shannon entropy, and this suggestion holds that an entropic measure of the unidirectional emission is more accurate and efficient than former alternatives. It is possible to validate these results by demonstrating that, former alternatives fail to detect the degree of the directional emission, our new methods can.

To further engage this discussion, we consider a prevalent and basic limaçon-shaped microcavity laser as a candidate for the directional emission with a high Q-factor [6, 7, 18]. In this reason, it has been extensively studied to date. The geometrical boundary of the limaçon-shaped microcavity is defined as follows:

$$R(\theta) = R_0(1 + \chi \cos \theta),$$

where $\theta$ is the angle in the polar coordinate, $\chi$ is the deformation parameter, and $R_0(=1)$ is the radius of circles at $\chi = 0$. Only TM modes of limaçon-shaped microcavity laser with an effective index of refraction $n = 3.3$ are treated in ray simulation. Some of the representative far-field profiles (FFPs) in the limaçon-shaped cavity is shown in Fig. 1. The figures (a), (b), and (c) in Fig. 1 are plotted in the cartesian coordinate within the range of $|x| \leq 5$ and $|y| \leq 5$ at each deformation $\chi = 0.43$, $\chi = 0.454$, and $\chi = 0.478$, respectively. The subfigures (d), (e), and (f) in Fig. 1 are FFPs plotted in the polar coordinate, corresponding to FFPs in the cartesian coordinate, respectively. We call the angles in the range of $|\theta| \leq \frac{\pi}{2}$ as ‘emission window’, which was first introduced by Refs. [19, 20] for the definition of unidirectionality.

It was reported that the deformation parameter $\chi = 0.43$ is the optimal value for unidirectional emission [6]. Moreover, they have concluded that the results are robust against any variation of the deformation parameter in the range of $0.41 \leq \chi \leq 0.49$ [6]. Consistent with their results, the overall profiles of all figures in Fig. 1 appear similar each other. However, a closer examination might suggest that FFPs at $\chi = 0.454$ exhibit a lager unidirectionality than the others.

To validate this suggestion, we address three types of measure for the unidirectionality. The blue square ($U_W$) and red circles ($U_C$) in Fig. 2(a) are measures of the unidirectionality associated with the emission windows. More precisely, the unidirectionality $U_C$ marked by blue squares is defined as follow [21, 22]:

$$U_C = \frac{\int_0^{\frac{\pi}{2}} I(\theta) \cos \theta d\theta}{\int_0^{\frac{\pi}{2}} I(\theta) d\theta}.$$

Here, $I(\theta)$ denotes the intensity of angular distribution of FFPs [23]. The $\cos \theta$ as a window function determines
FIG. 1: Far-field profiles in limaçon-shaped cavity. The figures (a), (b), and (c) are the far-field profiles in the cartesian coordinate within \( x \in [-5, 5] \) and \( y \in [-5, 5] \) at each deformation \( \chi = 0.43, \chi = 0.454, \) and \( \chi = 0.478 \), respectively. The figures (d), (e), and (f) are also far-field profiles plotted in the polar coordinate corresponding to the far-field profiles in the cartesian coordinate. The angles in the range of \( |\theta| \leq \frac{\pi}{4} \) is the emission window.

The extent to which the emission directionality deviates from unidirectionality. Actually, the positive and negative \( UC \) represent tendencies toward a forward and backward emission, respectively, and \( UC = 0 \) corresponds to the bidirectional or isotropic emission of the microcavity laser. In our case, the values of \( UC \) almost increase linearly from \( UC \approx 0.3 \) to \( UC \approx 0.35 \) in the range of \( 0.0426 \leq \chi \leq 0.0478 \). This result implies that the unidirectionality increases as the deformation parameter \( \chi \) increases, unlike our observation with naked eyes. Note that we here discretize \( 2\pi \) into 3600 pieces for numerical calculation, i.e., \( d\theta \sim \Delta \theta = 0.1 \).

The other measure of the unidirectionality \( UW \) marked by red circles is also defined as follow [10, 20]:

\[
U_W = \frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} I(\theta)d\theta}{\int_{-\pi}^{\pi} I(\theta)d\theta}
\]

(3)

where the integral range of numerator runs from \(-\frac{\pi}{4}\) to \(\frac{\pi}{4}\). The angles in this range of \(|\theta| \leq \frac{\pi}{4}\) is so-called the emission window as mentioned before.

Hence, the meaning of this definition is clearly ratio between accumulated intensities of FFPs within emission window and the total intensity of FFPs. The values of \( U_W \) also increase almost linearly from \( U_W \approx 0.56 \) to \( U_W \approx 0.59 \) in the range of \( 0.0426 \leq \chi \leq 0.0478 \). Consequently, these values of \( U_W \) are also inconsistent with our observation. We can conjecture that this discrepancy is attributed to the fact that \( U_W \) increases proportionally to the integral value of the emission window, regardless of the detailed structure of the emission window. Furthermore, it should be noticed that \( UC \) exhibits a similar trend with \( U_W \).

Next, we introduce entropic unidirectionality. Accordingly, we first have to obtain a Shannon entropy associated with FFPs. Shannon entropy is a relevant measure of the average amount of information for a random variable with a given probability distribution function [24]. It was firstly developed and utilized in communication theory [24]. However, recently it has been also exploited in various areas such as bio-systems, economics, atomic physics, and microcavity laser [25].

The discrete Shannon entropy for the intensity of FFPs is formally defined as:

\[
S_N = -\frac{1}{\log K} \sum_{i=1}^{K} \rho_i \log \rho_i,
\]

(4)

where \( \rho_i \) represents the probability distribution obtained under the normalization condition \( \sum_{i=1}^{K} I(\theta_i) = 1 \). That is, the random variable \( X \) is an angular coordinate \( \Theta \) with the probability distribution \( \{\rho_i\} = \{P(\Theta = \theta_i)\} \). In addition, the \( \frac{1}{\log K} \) is a normalization factor such that...
FIG. 3: FFPs in the oval-shaped microcavity laser. The subfigures (a), (b), and (c) are FFPs in the cartesian coordinate restricted in the range of \( x \in [-5, 5] \) and \( y \in [-5, 5] \) at each deformation \( \varepsilon = 0.043 \), \( \varepsilon = 0.05 \), and \( \varepsilon = 0.058 \), respectively. The subfigures (d), (e), and (f) are also FFPs plotted in the polar coordinate corresponding to FFPs in the cartesian coordinate.

The value of Shannon entropy is restricted in the range of \( 0 \leq S_N \leq 1 \), with \( K = 3600 \) as mentioned above. The orange circles in Fig. 2 (b) represent normalized Shannon entropy \( (S_N) \) calculated by this definition. The value of \( S_N \) has a local maximum (0.898) at \( \chi = 0.43 \) and local minimum (0.888) at \( \chi = 0.454 \), respectively.

Our recent works have confirmed that Shannon entropy can be beneficial in measuring the delocalization of the given probability distributions [29]. Hence, the minimum (or maximum) value of the Shannon entropy indicate the maximum localization (or delocalization) of the intensity of FFPs in the polar coordinate under the normalization condition \( \sum_{i=1}^{K} I(\theta_i) = 1 \). Moreover, the maximal entropy, \( \log K \), corresponds to the isotropic distribution that is not a bidirectional emission in the polar coordinate.

As a result, by satisfying the normalized Shannon entropy, we can define a new measure of the unidirectionality given by

\[ U_S = 1 - S_N, \quad (5) \]

i.e., it is related to a redundancy of Shannon entropy of the intensity of FFPs.

The green squares represent the value of the \( U_S \). These plots reveal that the value of \( U_S \) at \( \chi = 0.43 \) has a local minimum rather than a local maximum. On the other hand, the values of \( U_S \) have a local maximum with value 0.898 at \( \chi = 0.454 \). This result validates the conjecture that by exploiting the Shannon entropy, our measure of unidirectionality is coincident with our observations in Fig. 1.

For more generality of our argument and to understand the discrepancy between Fig. 2(a) and Fig. 2(b), let us consider an oval-shaped microcavity laser with an effective index of refraction \( n = 3.3 \) for TE mode [30]. The geometrical boundary condition of an oval shaped-cavity, which is the deformed from an ellipse, is defined as follow:

\[ \frac{x^2}{a^2} + (1 + \varepsilon x) \frac{y^2}{b^2} = 1. \quad (6) \]

For convenience, we substitute the deformation parameter \( \chi \) to \( \varepsilon \). It was reported that optimized condition for a directional emission is \( a = 1.0 \), \( b = 1.03 \), and \( \varepsilon = 0.05 \) where \( a \) and \( b \) are major and minor axis of an ellipse, and \( \varepsilon \) is the deformation parameter, respectively [30]. According to their results, we conduct ray simulation in the range of \( 0.043 \leq \varepsilon \leq 0.058 \) at the fixed value of \( a = 1.0 \) and \( b = 1.03 \). We plot some of the representative FFPs, i.e., Fig. 3(a), (b), and (c) in the cartesian coordinate restricted in \( x \in [-5, 5] \) and \( y \in [-5, 5] \) at each deformation \( \varepsilon = 0.043 \), \( \varepsilon = 0.05 \), and \( \varepsilon = 0.058 \). The corresponding FFPs in the polar coordinate are displayed in Fig. 3(d), (e), and (f).

FIG. 4: Three types of measure of the unidirectionality in the oval-shaped microcavity. The blue squares and red circles in (a) are measures of the unidirectionality associated with the emission window. Both of them have local maximal values at \( \varepsilon = 0.05 \). The green squares (or orange circles) in (b) are entropic unidirectionality (or normalized Shannon entropy). The values of green squares also have local maximum at \( \varepsilon = 0.05 \).

On the contrast to Fig. 1, we can easily notice that
the overall FFPs depending on the deformation parameter vary manifestly, and it can be naturally expected that the local maximum of the unidirectionality is obtained at \( \varepsilon = 0.05 \) by observing Fig. 5(b) and (c), and comparing other subfigures in Fig. 4. In this case, the local maximum values of \( U_C, U_W \), and \( U_S \) are attained at \( \varepsilon = 0.05 \) simultaneously, i.e., entropic measure of the unidirectionality \( U_S \) agrees well with the former measure of the unidirectionality \( (U_C, U_W) \) related to the emission window. Consequently, this fact implies that \( U_C \) and \( U_W \) only can detect the unidirectionality when the overall FFPs vary significantly with a manifest variation of the emission window. However, our noble measure for the unidirectionality by employing the Shannon entropy can efficiently capture the unidirectionality in any case. Notice that the relative difference between the local maximum and the local minimum of \( U_S \) in Fig. 4 is much larger than that of Fig. 2. This fact coincides with our initial intuition.

![Image](58x413 to 180x501)

**FIG. 5:** (a) The blue squares are markers of the divergence angle \( D_A \) and the red circles are those of Shannon entropy of semi-marginal probabilities \( (S_y) \) in the oval-shaped microcavity laser at each \( \varepsilon = 0.043 \), \( \varepsilon = 0.05 \), and \( \varepsilon = 0.058 \). These two plots show similar trends and this results support our previous assumption. Three insets are semimarginal probabilities related to \( S_y \). (b) The black squares are markers of Shannon entropy of semi-marginal probabilities \( (S_y) \) in the limaçon-shaped microcavity laser at \( \chi = 0.43 \), \( \chi = 0.454 \), and \( \chi = 0.478 \). Three insets are also semi-marginal probabilities related to \( S_y \).

Another aspect for definition of the directional emission along with the unidirectionalism is a so-called ‘divergence angle’. This concept presents an analogy for the intensity of FFPs, \( I(x, y) \), in the cartesian coordinate as a joint probability distribution function under the normalization condition \( \int \int I(x, y)dx\,dy = 1 \), i.e., the random variables \( X \) and \( Y \) are components of the cartesian coordinate with the joint probability distribution function \( \rho(x, y) = P(X = x, Y = y) \). The integral \( \int \rho(x, y)dx \) is performed over the interval \( x \in [0, 5] \) to solely handle the forward emission. This is why we call it as semi-marginal probability distribution. Note that we have discretized \((x, y)\)-coordinate in the range \( x \in [-5, 5] \) and \( y \in [-5, 5] \) into \( 1000 \times 1000 \) grid for the numerical calculation. Then, the discrete Shannon entropy from the \( \rho(y_j) \) is defined by

\[
S_y = \frac{1}{\log K} \sum_{y_j=1}^{K} \rho(y_j) \log \rho(y_j),
\]

where the semi-marginal distribution \( \rho(y_j) = \sum_{i=1}^{K} \rho(x_i, y_j) \) and \( K = 1000 \).

The blue squares in Fig. 5(a) represent the divergence angle \( (D_A) \) and the red spheres in Fig. 5(a) represent the Shannon entropy \( (S_y) \) of the semi-marginal probabilities in the oval-shaped microcavity laser at each \( \varepsilon = 0.043 \), \( \varepsilon = 0.05 \), and \( \varepsilon = 0.058 \). Note that these two plots show similar trends and this results support our previous assumption. Three insets in Fig. 5 represent semi-marginal probability densities \( (\rho_y) \). All peaks of \( \rho_y \) are located at \( \varepsilon \approx \pm 1 \), and careful examination reveals that the third one is more spread out than the others.

In the case of complex peak structures, as illustrated in Fig. 4(c), (e), and (f), we can barely define \( D_A \) (i.e., FWHM). However, we can quantitatively and systematically measure the spread of the emission peak by exploiting \( S_y \), and the obtained results are presented in Fig. 4(b). The black squares in Fig. 4(b) indicate Shannon entropy \( (S_y) \) of semi-marginal probability \( (\rho_y) \) in the limaçon shaped-cavity at \( \chi = 0.43 \), \( \chi = 0.454 \), and \( \chi = 0.478 \). The minimum value of \( S_y \) in Fig. 4(a) is substantially larger than that of \( S_y \) in Fig. 4(b), which can be confirmed by comparison between Fig. 4(c) and Fig. 4(e).

Consequently, we conclude that the oval-shaped microcavity laser exhibits larger unidirectionality and smaller \( S_y \) (i.e., \( D_A \)) than the limaçon-shaped microcavity laser, thereby resulting in a better directional emission. In contrast, the limaçon-shaped microcavity laser has more robust range against any variation of the deformation parameter than the oval-shaped microcavity laser. We have present noble measures for a directional emission in microcavity lasers by exploiting the Shannon en-

| TABLE I: Comparison between previous and our results on the directional emission |
|-----------------|-----------------|-----------------|
|                | **Previous results** | **Our results** |
| Unidirectionality | Restricted       | **Always possible** |
| Divergence angle  | Restricted       | **Always possible** |
entropy. There are primarily two aspects of the directional emission—the unidirectionality and the divergence angle. Shannon entropy obtained from the normalized intensity of angular distributions of FFPs can measure the unidirectionality even when former notions can not effectively detect in the robust with respect to variations of the deformation parameter. Shannon entropy obtained from the semi-marginal probability densities of FFPs in the cartesian coordinates can provide equivalent results to the divergence angle; moreover, it can be applied to even in the case of complicated peak structures. Table I summarizes our results.

Furthermore, our noble measures can be applicable to any microcavity lasers. We hope that our results can help to design and to modulate a microcavity lasers for a better directional emission.

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