The zero forcing number of the complement of a graph

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Abstract

Motivated in part by an observation that the zero forcing number for the complement of a tree on \( n \) vertices is either \( n - 3 \) or \( n - 1 \) in one exceptional case, we consider the zero forcing number for the complement of more general graphs under some conditions, particularly those that do not contain complete bipartite induced subgraphs. We also move well beyond trees and completely study the possible zero forcing numbers for the complements of unicyclic graphs, and examine the zero forcing number for the complements of some specific families of graphs containing more than one cycle.

Keywords Zero forcing number, complements, bipartite graphs, trees, unicyclic graphs
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1 Introduction

Zero forcing is an iterative graph coloring process where an initial set of blue vertices eventually color all vertices of a graph using a color change rule. The zero forcing number of a graph was first defined in 2008 by the AIM Minimum Rank Special Graphs Work Group, as a method for bounding the maximum nullity of a graph [1]. In [1], the authors determine the zero forcing number of the complement of a tree by proving that the zero forcing number is equal to the maximum nullity for the complement of a tree. However, this proof does
not directly involve properties of zero forcing. Since 2008, zero forcing has been a topic of much study, and has grown in to a graph parameter of independent interest (for example, see [2, 3, 4, 7, 8, 10, 11]). In this paper, we determine the zero forcing number of the complements of graphs through direct zero forcing arguments, including a new proof of the zero forcing number of the complement of a tree which does not involve maximum nullity. This result follows from a more general result, which gives a lower bound for the zero forcing number of graphs in terms of the smallest complete bipartite graph which they do not contain as a subgraph.

Let $G$ be a graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. The set of vertices adjacent to a vertex $v \in V(G)$ is called the neighborhood of $v$, denoted $N_G(v)$. The complement of a graph $\overline{G}$ is the graph with vertex set $V(G)$ and edge set $E(\overline{G}) = \{v_iv_j : v_i, v_j \in V(G) \text{ and } v_i v_j \not\in E(G)\}$. An independent set is a set of vertices such that no two vertices in the set are adjacent. The complete bipartite graph $K_{r,s}$ is the graph on $r+s$ vertices such that $V(G) = A \cup B$, where $A$ and $B$ are independent sets of size $r$ and $s$, respectively, and $E(G) = \{v_i v_j : v_i \in A \text{ and } v_j \in B\}$. Suppose $H$ is an induced subgraph of a graph $G$. Then it follows easily from the definitions of graph complements and induced subgraphs that $\overline{\overline{G}}$ is an induced subgraph of $\overline{G}$.

In the zero forcing process, denote an initial set of blue vertices $B \subset V(G)$ and let the remaining vertices of the graph be white. The color change rule states that a blue vertex $b \in B$ can force a white vertex $w$ to become blue if $w$ is the only white neighbor of $b$. A zero forcing set is any initial set of blue vertices in a graph such that repeated application of the color change rule results in every vertex of the graph being blue. The zero forcing number of a graph $G$, denoted $Z(G)$ is the minimum size of a zero forcing set. A minimum zero forcing set of a graph $G$ is a zero forcing set $B$ such that $|B| = Z(G)$.

In each application of the color change rule, we assume all possible forces occur simultaneously. Note that for each white vertex which is forced, there may be multiple blue vertices which could have forced it. However, we can assume in this case that one of the blue vertices was chosen to force the white vertex. A forcing chain is a sequence of $k$ vertices in the graph $(v_1, v_2, \ldots, v_k)$ such that during the zero forcing process, $v_i$ forces $v_{i+1}$ for all $1 \leq i \leq k-1$ and $1 \leq k \leq n$. Each zero forcing set and each choice of forces during the applications of the color change rule results in a set of forcing chains. Note that the number of chains is equal to the size of the zero forcing set and the chains are all disconnected. The reversal of a forcing chain $(v_1, v_2, \ldots, v_k)$ is the sequence of vertices $(v_k, v_{k-1}, \ldots, v_1)$. It was shown in [9] that the set of forcing chain reversals are themselves a set of zero forcing chains for the same graph, with the last vertices in each zero forcing chain forming a new zero forcing set of the same size.

The zero forcing number of the complement of a graph has previously been studied in [6]. In this paper, the authors show that the zero forcing number of a graph on $n$ vertices is at most $n - 3$ as long as the complement of the graph is connected. They also bound the sum of the zero forcing numbers of a graph and it complement in terms of $n$ and the minimum degree.

In this paper, we begin in Section 2 by providing a lower bound on the zero forcing number of the complement of a graph which does not contain $K_{r,s}$ as a subgraph. In Section 3, we apply this result to bipartite graphs, bounding the zero forcing number of the complement...
of a complete bipartite graph from above and below, and showing that each number in this interval can be achieved. We also provide a new proof of the characterization of the zero forcing number of the complement of a tree. In Section 4 the zero forcing number of the complement of unicyclic graphs on $n$ vertices is determined to be $n - 2$, $n - 3$ or $n - 4$ and we provide a complete characterization of when each of these values is achieved. In Section [4,1] we compute the zero forcing number of the complement of specific graphs which are unicyclic. Finally, in Section 5 we turn our attention to the complement of graphs with more than one cycle and compute the zero forcing number of the complement for several specific families.

2 Background and Key Result

As mentioned in the introduction, one of the numerous contributions from the work in [1] was a determination of the zero forcing number of the complement of a tree. The method of proof given in [1] did not rely on the rules of zero forcing, but rather by first establishing the maximum nullity for the complement of a tree then noting equality between the zero forcing number and the maximum nullity must also hold in this extreme case.

Part of the purpose of this work is to go back to the complement of trees and prove directly that if $T$ is a tree on $n$ vertices, then either $Z(T) = n - 3$ or $Z(T) = n - 1$, if $T$ is a star on $n$ vertices. As our analysis unfolded in this study, it became clear ([12]) that our techniques could be extended to a much broader class of graphs well beyond the case of trees. For instance trees may be considered as the class of bipartite graphs that do not contain $K_{2,2}$ as an induced subgraph. This leads to the following subsection which contains a key result that will be used throughout our work.

2.1 Graphs which do not contain a $K_{r,s}$ subgraph

In this subsection we focus on the zero forcing number of the complement of graphs that do not contain a specific complete bipartite graph as an induced subgraph.

**Theorem 2.1.** Let $G$ be a graph on $n$ vertices which does not contain $K_{r,s}$ as a subgraph for some $r + s \leq n$. Then $Z(G) \geq n - r - s + 1$.

*Proof.* Suppose that $Z(G) \leq n - r - s$ and let $B$ be a zero forcing set of size $n - r - s$ for $G$. Let $W = V - B$, the set of initially white vertices and let $W = \{w_1, \ldots, w_{r+s}\}$. There must exist a vertex $v_1 \in B$ whose only white neighbor in $G$ lies in $W$, or else no forcing occurs. Let $v_1$ force $w_1$. Then in $G$, $v_1$ is adjacent to $\{w_2, \ldots, w_{r+s}\}$. In order for forcing to continue, there exists $v_2 \in B$ whose only remaining white neighbor in $G$ lies in $W - w_1$. Note it is possible that $v_2 = w_1$, but $v_2$ cannot be $v_1$ or $\{w_2, \ldots, w_{r+s}\}$. Let the vertex which $v_2$ forces be $w_2$. Then $v_2$ must be adjacent in $G$ to $\{w_3, \ldots, w_{r+s}\}$. Continuing in this manner, we can find vertices $v_1, \ldots, v_{r+s}$ such that $v_i$ forces $w_i$ in $G$ for $1 \leq i \leq r + s$. Further, for $1 \leq i \leq r + s - 1$, $v_i$ is adjacent to $\{w_{i+1}, \ldots, w_{r+s}\}$ in $G$. Note in each case it is possible that $v_i = w_j$ for some $1 \leq j \leq i - 1$, but $v_i$ cannot be $v_k$ for $2 \leq k \leq i - 1$ or $w_j$ for $i \leq j \leq r + s$. Therefore the vertices $\{v_1, \ldots, v_r\}$ and $\{w_{r+1}, \ldots, w_{r+s}\}$ contain $K_{r,s}$ as a subgraph. \qed
As mentioned previously, Theorem 2.1 represents an important result that will be referenced a number of times in this work. In particular, we may apply Theorem 2.1 to immediately deduce that if \( G \) is bipartite and does not contain \( K_{2,2} \) as an induced subgraph then \( Z(G) \) is either \( n - 3 \) or \( n - 1 \) (see Proposition 3.3 in the next section devoted to bipartite graphs).

3 Bipartite graphs

We now turn our attention to the study of bipartite graphs. Unless otherwise stated, we will assume throughout this section that \( G \) is a connected bipartite graph with disjoint independent sets \( X,Y \subset V(G) \) such that \( V(G) = X \cup Y \) and \( |X| = r, |Y| = s \) where \( r \leq s \in \mathbb{Z}_+ \). Note, this gives \( |V(G)| = n = r + s \). We first state a known result about general graphs that will be useful in this section.

**Lemma 3.1.** [1] For a graph \( G \) on at least 3 vertices, \( Z(G) \geq |G| - 2 \) if and only if \( G \) does not contain \( P_2 \cup P_2 \cup P_2, P_3 \cup P_2, P_4, \kappa, \) or a dart as an induced subgraph.

The following observation bounds the zero forcing number of the complement of any bipartite graph in terms of the size of the larger component on \( G \) and the number of vertices of \( G \).

**Observation 3.2.** Suppose the independent sets \( X \) and \( Y \) in the graph \( G \) satisfy \( 1 < |X| = r \leq |Y| = s \). Then \( s - 1 \leq Z(G) \leq r + s - 1 \).

Focusing our attention to trees, we re-prove the following known result about the zero forcing number of the complement of a tree, using Theorem 2.1.

**Proposition 3.3.** Let \( T \) be a tree on \( n \geq 4 \) vertices which is not the star graph. Then \( Z(G) = n - 3 \). For the star graph \( K_1,n-1 \), with \( n \geq 3 \), \( Z(K_1,n-1) = n - 1 \).
Proof. First observe, that for the star graph \( K_{1,n-1} \), \( \overline{K_{1,n-1}} \) is \( K_{n-1} \cup K_1 \). Since for any \( r \geq 2 \) \( Z(K_r) = r - 1 \), it is clear \( Z(\overline{K_{1,n-1}}) = n - 2 + 1 = n - 1 \). Let \( T \) be a tree that is not a star. Since trees do not contain \( K_{2,2} \), \( Z(T) \geq n - 3 \) by Theorem 2.1. To show equality, we describe a zero forcing set \( B \) for \( \overline{T} \) of size \( n - 3 \). Since \( T \) is not a star is must have diameter at least 3. Let \( v_0 \) be a leaf in \( T \). Then there exists an induced path on four distinct vertices in \( T \) on the vertices \( \{v_0, v_1, v_2, v_3 \} \). Define \( B = V(T) \setminus \{v_0, v_1, v_2 \} \). Since \( v_2 \) is in \( B \) and is adjacent to \( v_0 \) in \( T \) and not adjacent to either \( v_1 \) nor \( v_3 \), it follows that \( v_2 \) can force \( v_0 \). Then \( v_0 \) can now force \( v_3 \), as \( v_0 \) is not adjacent to \( v_1 \) in \( \overline{T} \). Finally, \( v_1 \) can be forced. Thus \( B \) is a zero forcing set of size \( n - 3 \). \( \square \)

In fact, the above result holds for more than trees. We now provide a generalization to all \( K_{2,2} \)-free graphs which are not the star graph.

**Proposition 3.4.** If \( G \) is a connected \( K_{2,2} \)-free bipartite graph with \( |G| = n \geq 4 \) vertices and \( G \neq K_{1,n-1} \), then \( Z(\overline{G}) = n - 3 \).

**Proof.** Observe that if \( \overline{G} \) is disconnected, then \( \overline{G} \) is the disjoint union of two complete graphs which implies \( G \) is the complete bipartite graph \( K_{r,s} \) which contradicts our assumptions of \( G \) being \( K_{2,2} \)-free and \( G \neq K_{1,n-1} \). Now let \( G \) be a connected bipartite graph on \( n \geq 4 \) vertices that is \( K_{2,2} \)-free and assume \( G \neq K_{1,r} \). By Theorem 2.1, we know \( Z(\overline{G}) \geq n - 3 \). Assume \( Z(\overline{G}) = n - 2 \). From Lemma 3.1, \( \overline{G} \) does not contain \( P_3 \) as an induced subgraph and therefore \( \text{diam}(\overline{G}) = 2 \). This implies \( G \) is a connected cograph \([5]\). However, it is well-known that any connected cograph has a disconnected complement (since any cograph is either the union or join of two smaller cographs). \( \square \)

We now consider bipartite graphs in general. To start, we compute the zero forcing number of the complement of the complete bipartite graph. Recall that \( \overline{K_{r,s}} = K_r \cup K_s \).

**Proposition 3.5.** For \( n \geq 4 \), \( Z(\overline{K_{r,s}}) = r + s - 2 \).

As noted in Observation 3.2, for any connected bipartite graph \( G \), we have \( s - 1 \leq Z(\overline{G}) \leq n - 1 \), where \( s \) is the size of the larger independent set of \( G \). By Proposition 3.3, we have \( Z(\overline{K_{1,n-1}}) = n - 1 \) for the star graph and \( Z(T) = n - 3 \) for a tree graph. Therefore, we sought to determine if every positive integer between the bounds \( s - 1 \) and \( n - 2 \) is achievable by some connected bipartite graph. The following results address that question with Theorem 3.9 providing an affirmative answer. First, we prove some helpful lemmas.

**Lemma 3.6.** Let \( H = \overline{G} \). Then, there exists a minimum zero forcing set \( F \) of \( H \) where \( |F \cap Y| \geq s - 1 \).

**Proof.** If no vertex in \( X \) can force a vertex in \( Y \) for all minimum zero forcing sets, then \( Y \subseteq F \) or a vertex in \( Y \) must be forced by a vertex in \( Y \). In the latter case \( |F \cap Y| = s - 1 \).

Suppose there is a minimum zero forcing set \( F \) such that there is a vertex in \( X \) that forces a vertex in \( Y \). Then all but at most one of the vertices that get forced last in their forcing chain are in \( Y \). Hence the reversal with respect to \( F \) corresponds to a zero forcing set that contains at least \( s - 1 \) vertices from \( Y \). \( \square \)
Proof. By Lemma 3.6, there exists a minimum zero forcing set $F$ for $H'$ such that $|F \cap (Y \cup \{v\})| \geq (s + 1) - 1 = s$. If $v$ doesn’t force, then $F \setminus \{v\}$ is a zero forcing set for $H$ with a similar forcing sequence as the forcing sequence corresponding to $F$. If $v$ forces the vertex $a$ in $S$, then $(F \setminus \{v\}) \cup \{a\}$ is a zero forcing set for $H$. In either case $Z(H') \geq Z(H)$.

Lemma 3.8. Let $H = \overline{G}$, and let $H'$ be the graph constructed by $H$ by adding a vertex $v$ such that $N_{H'}(v) = S \cap \{b\}$ where $b$ is a vertex in a minimum zero forcing set of $H'$ which contains at least $s - 1$ vertices in $Y$. Then $Z(H') \geq Z(H)$.

Proof. By Lemma 3.6, there exists a minimum zero forcing set $F$ for $H'$ such that $|F \cap (Y \cup \{v\})| \geq (s + 1) - 1 = s$. If $v$ doesn’t force then $F \setminus \{v\}$ is a zero forcing set for $H$ with a similar for sequence as the forcing sequence corresponding to $F$. If $v$ forces the vertex $a$ in $Y$, then $(F \setminus \{v\}) \cup \{a\}$ is a zero forcing set for $H$. In either case $Z(H') \geq Z(H)$.

We now apply Lemmas 3.6, 3.7, and 3.8 to prove the following theorem.

Theorem 3.9. For $n \geq 4$ with $n = r + s$ and $r \geq 2$, there exists a graph $H'$ which is the complement of a connected bipartite graph on $n$ vertices with disjoint independent sets such that $|X| = r$, $|Y| = s$ where $n = r + s$, $r \leq s$, and $Z'(H') = i$ for all $i \in \{s - 1, s, \ldots, r + s - 2\}$.

Proof. For completeness, observe that a graph containing two cliques with one edge between the two cliques has zero forcing number equal to $s - 1$.

Consequently, we establish, via induction, that there exist such a graph $H'$ with $Z(H') = i$ for $i \in \{s, s + 1, \ldots, r + s - 2\}$. Let $n = 4$. If $r = 2$ and $s = 2$ then $Z(P_4) = 1$, and $Z(C_4) = 2$.

Now suppose the statement is true for all such $k$ with $k > 4$. Let $k + 1 = r_{k+1} + s_{k+1}$. Let $1 < r_{k+1} \leq s_{k+1}$. By the hypothesis there exists $H'_k$ such that $Z(H'_k) = i$ for each $i \in \{s_{k+1} - 1, \ldots, r_{k+1} + s_{k+1} - 3\}$. Assume that the corresponding cliques associated with $H'_k$ are induced on the vertices $X_k$ and $Y_k$. Let $F_k$ be a minimum zero forcing set for $H'_k$ which contains $s - 1 = s_{k+1} - 2$ vertices from $Y_k$. Such a set $F_k$ exists by Lemma 3.6.

Case 1: Suppose $Y_k \subseteq F_k$. Let $H'_{k+1}$ be the graph constructed from $H'_k$ by adding the vertex $v$ such that $N_{H'_{k+1}}(v) = Y_k$. Then $Z(H'_{k+1}) \geq Z(H'_k)$ by Lemma 3.7. Moreover $(F_k \setminus \{x\}) \cup \{v\}$ is a zero forcing set for $H'_{k+1}$ where $x \in Y_k$.

Case 2: Suppose $Y_k$ is not a subset of any minimum zero forcing set of $H'_k$. Let $H'_{k+1}$ be the graph constructed from $H'_k$ by adding the vertex $v$ such that $N_{H'_{k+1}}(v) = Y_k \cup \{b\}$ where $b \in F_k \cap X_k$. Such a $b$ exists since $Y_k$ is not a subset of $F_k$ and $Z(H'_k) \geq s_k$. By Lemma 3.8 $Z(H'_{k+1}) \geq Z(H'_k)$. Also $|F_k \cap Y_{k+1}| = s_{k+1} - 2$. Moreover $(F_k \setminus \{b\}) \cup \{v\}$ is a zero forcing set for $H'_{k+1}$ where $v$ forces $b$ after the first force in $Y_{k+1}$. The vertex that forces $x$ is not adjacent to $b$ as that would contradict there does not exist a minimum zero forcing set for $H'_k$ which contains $Y_k$.

Finally, by Proposition 3.5 $Z(K_{r_{k+1}, s_{k+1}}) = r_{k+1} + s_{k+1} - 2$. This completes the proof. 


4 Unicyclic Graphs

In Section 3, Proposition 3.3 gives us the zero forcing number for the complement of any acyclic graph. The next logical step is to examine graphs that contain precisely one cycle. Such a graph is called a unicyclic graph.

**Definition 4.1.** A unicyclic graph is a connected graph containing exactly one cycle such that every vertex on the single cycle is adjacent to at least one vertex in a single tree.

![Figure 2: The general form of a unicyclic graph](image)

Let $G$ be a unicyclic graph with vertices $v_1, v_2, \ldots, v_n$ on the single cycle, and forests $B_1, B_2, \ldots, B_n$, where each forest contains $m_i$ vertices excluding $v_i$. Note that in $G$, any vertex of $B_k$ with $k \neq i$ is nonadjacent to $v_i$ otherwise we create a cycle. A simple lower bound for the zero forcing number of the complement of this graph is therefore achieved by bounding the minimum degree of a vertex in this graph. If we let $m_{\max}$ be the maximum size of a forest in $G$ and consider the degree of $v_{\max}$, a vertex on the cycle in $G$ that is adjacent to all vertices in the maximum sized forest $B_{\max}$, we observe the following lower bound.

**Observation 4.2.** Let $G$ be a unicyclic graph as labeled in Figure 2. Then,

$$n + \sum_{j=1}^{n} m_j - m_{\max} - 3 \leq Z(G).$$

Not only can this bound can be improved upon, but we can also find the exact zero forcing number of the complement of any unicyclic graph. This is the main topic for what follows in this section.

**Remark 4.3.** If $G$ is a tree, we know $Z(G) = n - 3$ or in the special case when $G$ is a star, $Z(G) = n - 1$. We can view a unicyclic graph $G$ as being a tree with an additional edge, and therefore, $\overline{G}$ being the complement of a tree minus a single edge. So, combining this observation with the inequality for any graph $G$:

$$-1 \leq Z(G) - Z(G - e) \leq 1$$

we have

$$n - 4 \leq Z(\overline{G}) \leq n - 2$$

for any unicyclic graph $G$. It is also helpful to note that the only graph on $n$ vertices whose zero forcing number is $n - 1$ is the complete graph on $n$ vertices.
We now characterize the unicyclic graphs $G$ with $Z(\overline{G}) = n - 4, n - 3, n - 2$.

**Proposition 4.4.** Suppose $G$ is a connected unicyclic graph, $|G| \geq 5$ with a cycle of length 4 or more. Then $\overline{G}$ contains $P_4$ as an induced subgraph.

Proof. Label the vertices of the cycle of $G$ as $v_1, \ldots, v_k$ with $k \geq 4$ as in Figure 2. If $k \geq 5$, then the subgraph induced by the vertices $\{v_1, v_2, v_3, v_4\}$ is $P_4$. On the other hand if $k = 4$, then since $|G| \geq 5$, there exists a vertex $u$ in $G$ not on the cycle but adjacent to a vertex on the cycle. Suppose, without loss of generality that $u$ is adjacent to $v_4$. Then the subgraph induced by the vertices $\{v_2, v_3, v_4, u\}$ is $P_4$. Thus in either case $G$ contains $P_4$ as an induced subgraph, hence so does $\overline{G}$. □

**Corollary 4.5.** If $G$ is a unicyclic graph of size $n \geq 5$ such that $G$ contains a cycle of length at least four, then $Z(\overline{G}) \leq n - 3$.

**Proposition 4.6.** Let $G$ be a unicyclic graph on $n \geq 4$ vertices. Then, $Z(\overline{G}) = n - 2$ if and only if $G = K_{1,n-1} + e$ or in the special case when $n = 4$, $G$ could also be isomorphic to $C_4$.

Proof. Let $G$ be a unicyclic graph. Assume $n \geq 5$ and that $G = K_{1,n-1} + e$. Then $\overline{G}$ has the structure of $(K_{n-1} - e) \cup \{v_0\}$ where $v_0$ is the center (or vertex of maximum degree) of $G$. So,

$$Z(\overline{G}) = Z(K_{n-1} - e) + Z(\{v\}) = (n - 3) + 1 = n - 2.$$ 

Now suppose $Z(\overline{G}) = n - 2$. If $G$ contains a cycle of length 4 or more and $n \geq 5$, then by Corollary 4.5 we get $Z(\overline{G}) \leq n - 3$. Hence $G$ must contain a cycle of length 3. Label the vertices on the 3-cycle of $G$ as $a, b, c$. Assume, that at least two vertices on this 3-cycle have degree at least 3, say $\text{deg}(a) \geq 3$ and $\text{deg}(b) \geq 3$. Let $u$ (and $v$) be a neighbour of $a$ (of $b$) not on the cycle. Now consider the subgraph of $G$ induced by the vertices $u, a, b, v$, which induces a $P_4$. Hence $\overline{G}$ contains $P_4$ as an induced subgraph. Thus $Z(\overline{G}) \leq n - 3$, which is a contradiction. Hence at most one vertex on the 3-cycle has degree at least three. Since $n \geq 4$, assume vertex $a$ has degree at least 3. It is straightforward to deduce that if $G$ is not isomorphic to $K_{1,n-1} + e$, then $G$ will contain an induced $P_4$. Finally, if $n = 4$, then the complement of $G$ is either a cycle of 4 vertices or is a path on 3 vertices with an additional isolated vertex. In either case the zero forcing number is two as needed. This completes the proof. □

**Remark 4.7.** From our previous proposition, all other unicyclic graphs have $Z(\overline{G}) = n - 3$ or $n - 4$. In fact, those unicyclic graphs with $Z(\overline{G}) = n - 4$ are contained among those with $C_4$ as an induced subgraph, as the next proposition provides.

**Proposition 4.8.** Let $G$ be a unicyclic graph and $Z(\overline{G}) = n - 4$. Then, $C_4$ is an induced subgraph of $G$.

Proof. Let $G$ be unicyclic and $Z(\overline{G}) = n - 4$. If $G$ contains a cycle $C_j$ for $j > 4$, then $G$ is $K_{2,2}$-free, and so $Z(\overline{G}) \geq n - 3$, by Theorem 2.4, contradicting our assumption of the zero forcing number. So, $G$ must contain the cycle $C_4$ as an induced subgraph. □
From the proof of the previous proposition we observe that if $G$ is a unicyclic graph that contains $C_\ell$ with $\ell \geq 5$ as an induced subgraph, then $Z(\overline{G}) = n - 3$. Before we state the next result, we note that if $n = 3, 4$, then it is straightforward to verify the possible list of zero forcing numbers of the complements of such unicyclic graphs.

**Theorem 4.9.** Suppose $G$ is a unicyclic graph on $n \geq 5$ vertices. Assume $G$ contains $C_4$ as an induced subgraph. Then the following statements hold:

1. If at most one vertex on the cycle $C_4$ has degree 2, then $Z(\overline{G}) = n - 4$;
2. If exactly two vertices on the cycle $C_4$ have degree 2, and if
   (a) these vertices are adjacent, then $Z(\overline{G}) = n - 4$, or
   (b) these vertices are not adjacent, then $Z(\overline{G}) = n - 3$;
3. If exactly three vertices on the cycle $C_4$ have degree 2, then $Z(\overline{G}) = n - 3$.

**Proof.** Suppose the vertices on the cycle are labeled consecutively as $\{a, b, c, d\}$.

Consider $G$ as in (1). First assume that no vertex on the cycle $C_4$ has degree two and let $\{\alpha, \beta, \gamma, \delta\}$ be neighbours of $\{a, b, c, d\}$, respectively not on the cycle (see Figure 3). Let

\[Z = V(\overline{G}) \setminus \{\alpha, b, c, d\}.\]

We claim that $Z$ is a zero forcing set for $\overline{G}$. By definition of $Z$ if follows that vertex $a$ in $\overline{G}$ has only one white neighbour, namely $c$. So $a$ can force vertex $c$. From here we note that vertex $c$ now has only one white neighbour, namely $\alpha$. So $c$ can force $\alpha$. At this stage we observe that $\beta$ has only one white neighbour, namely $d$, since $\beta$ is adjacent to $b$ in $G$. Thus $\beta$ can force $d$, and finally the last white vertex ($b$) can be forced.

Hence $Z(\overline{G}) \leq n - 4$, but from the remarks at the beginning of this section we also know $Z(\overline{G}) \geq n - 4$, from which we conclude that $Z(\overline{G}) = n - 4$. If only one vertex has degree two
on the cycle $C_4$, then a very similar argument can be applied as used above to verify that $Z(\overline{G}) = n - 4$. In this case, assume that vertex $c$ has degree 2, and that vertices $\{\alpha, \delta\}$ are neighbours of vertices $\{a, d\}$, respectively not on the cycle. Now let $Z = V(\overline{G}) \setminus \{\alpha, \delta, b, d\}$. As above it follows that $Z$ is a minimum zero forcing set for $\overline{G}$.

Suppose $G$ satisfies (2a). Assume the two adjacent vertices on the cycle with degree two are $c$ and $d$. Using a similar approach as in the previous case, suppose $\alpha$ and $\beta$ are neighbours of $a$ and $b$, respectively not on the cycle. Let $Z = V(\overline{G}) \setminus \{\alpha, \beta, b, d\}$. We claim that $Z$ is a zero forcing set for $\overline{G}$. Since $a$ is not adjacent to any of $\alpha$, $b$, or $d$ in $\overline{G}$ it follows that $a$ can force $\beta$. From here we have the following possible sequence of forces: $c$ can force $\alpha$; $\beta$ can force $\alpha$, and finally $b$ can forces $d$. Hence $Z$ is zero forcing set for $\overline{G}$, from which it follows that $Z(\overline{G}) = n - 4$ as $n - 4$ is a lower bound as observed above.

Regarding (2b), assume that the two non-adjacent vertices on the cycle with degree two are $b$ and $d$, and let $\alpha$ and $\gamma$ be neighbours of $a$ and $c$, respectively not on the cycle. Denote the forests adjacent to $a$ and $c$ by $F_1$ and $F_3$, respectively (see Figure 4). To prove that the zero forcing number of $\overline{G}$ is $n - 3$, we will establish that no subset of vertices with size $n - 4$ can be a zero forcing set for $\overline{G}$. Let $B \subset V(\overline{G})$ with $|B| = n - 4$ and consider the size of the set $(V(F_1) \cup V(F_3)) \cap B$. If $B = V(F_1) \cup V(F_3)$, then it is clear that no forces are possible. Now assume $|(V(F_1) \cup V(F_3)) \setminus B| = 1$. Thus exactly one vertex on the cycle of $G$ is blue, which leads to two possible cases: either $a$ is blue or $b$ is blue. In either case neither $a$ nor $b$ can perform an initial force. If there is a vertex $x$ that is blue and in $V(F_1) \cup V(F_3)$, then $x$ has at least two white neighbours among the vertices on the cycle in either case. Hence $B$ is not a zero forcing set for $\overline{G}$. Next assume that $|(V(F_1) \cup V(F_3)) \setminus B| = 2$. In this case there are exactly two vertices on the cycle that are blue, which leads to three possible situations: (i) $a$ and $c$ are blue; (ii) $b$ and $d$ are blue; or (iii) $a$ and $b$ are blue.

First consider the case in which $a$ and $c$ are blue. Then it follows that no vertex in $V(F_1) \cup V(F_3)$ can apply a force (as both $b$ and $d$ are white). So assume that $a$ can force a vertex in $V(F_1) \cup V(F_3)$, call it $x$. This can only happen if $x$ is in $F_1$, is a neighbour of $a$, and both $x$ and $\alpha$ are white, and they are the only white vertices in $V(F_1) \cup V(F_3)$. Then $c$ can force $\alpha$. However, no other forces are possible as both $b$ and $d$ are white. So now

![Figure 4: Unicyclic graph containing $C_4$ satisfying item (2b)](image)

Figure 4: Unicyclic graph containing $C_4$ satisfying item (2b)
assume that \( b \) and \( d \) are blue. Then since \( \{b, d\} \) are joined to the vertices in \( V(F_1) \cup V(F_3) \) in \( \overline{G} \), neither \( b \) nor \( d \) can perform a force at this initial stage. Assume there is a vertex \( x \), in \( F_1 \) that performs the first force. Then it follows that the remaining two white vertices \( (u \) and \( v) \) must also be in \( F_3 \) (since the vertices of \( F_1 \) are joined to the vertices in \( F_3 \)), that \( x = \alpha \) and \( x \) must force \( c \). Since \( c \) has at least two white neighbours (including \( \alpha \)), \( c \) cannot perform a force. So there must be another vertex \( y \) that can perform a force at this stage. Since no blue vertex in \( F_3 \) can force, it follows that \( y \) is in \( F_1 \) and \( y \) is not \( \alpha \). In this case the only possible force is for \( y \) to force vertex \( a \). In this case we must have that both \( \alpha \) and \( y \) adjacent \( u \) and \( v \) in \( G \). Hence \( G \) contains two 4-cycles, which is a contradiction. The last case \( (iii) \) follows in a similar manner.

Now assume \(|(V(F_1) \cup V(F_3)) \setminus B| = 3 \). In this case there are two cases to consider since there is exactly one white vertex on the cycle. Suppose either \( a \) is white or \( b \) is white. In either case, we can apply arguments similar to the above case to deduce that \( B \) was a zero forcing set, then \( G \) would contain more than one 4-cycle, which is not possible. Finally, suppose \(|(V(F_1) \cup V(F_3)) \setminus B| = 4 \). Then all white vertices are among the vertices of \( V(F_1) \cup V(F_2) \). Therefore either none of the vertices \( \{a, b, c, d\} \) can perform a force or, say all of the vertices in \( F_1 \) are adjacent to \( a \). In the latter case, \( a \) could force a vertex in \( F_3 \), only if there is just one white vertex in \( F_3 \). After this no further forces are possible. Additionally, at least one of \( F_1 \) or \( F_3 \) must contain at most one white vertex, since the vertices of \( F_1 \) are joined to the vertices of \( F_3 \) in \( \overline{G} \). Suppose \( F_1 \) contains at most one white vertex, say \( u \), and \( x, y, z \) are three white vertices in \( F_3 \). Then the first force is accomplished by a vertex \( w \) in \( F_3 \), and in this case \( w \) forces \( u \). Thus \( w \) must be adjacent to all of \( x, y, z \) in \( G \). The next possible force is by, say \( w' \) and \( w' \) can force \( x \). Hence \( w' \) must be adjacent to both \( y \) and \( z \) in \( G \). Thus the vertices \( w, y, z, w' \) form a 4-cycle in \( G \), which is not possible. Consequently, we have argued that there is not a zero forcing set for \( \overline{G} \) of size \( n - 4 \), so we may conclude that \( Z(\overline{G}) = n - 3 \).

Finally, suppose \( (3) \) holds. Then we can argue is a manner similar to \( (2b) \) whereby we consider all possible subsets \( B \) of size \( n - 4 \) and verify that none of them can be a minimum zero forcing set by breaking the argument depending on the overlap between \( B \) and the vertices in the forest adjacent to exactly one vertex on the cycle.

This completes the proof. \( \square \)

**Corollary 4.10.** All unicyclic graphs \( G \) with \( G \neq K_{1,n-1} + e \) and containing \( C_\ell, \, \ell \geq 5 \) as an induced subgraph, or satisfy \( (2b) \) or \( (3) \) in Theorem 4.9 have \( Z(\overline{G}) = n - 3 \).

### 4.1 Families of unicyclic graphs

Following the results in the previous subsection and in the interest of providing illustrative examples of families of unicyclic graphs, we present the following results. We remark that Proposition 4.11 was proven previously in [1] with minimum rank techniques. We provide a different proof of the result using the techniques developed in this work.

**Proposition 4.11.** If \( n \geq 5 \), \( Z(\overline{C_n}) = n - 3 \) while \( Z(\overline{C_3}) = 3 \) and \( Z(\overline{C_4}) = 2 \).

**Proof.** The complement \( \overline{C_4} \) is the empty graph \( E_4 \) and clearly \( Z(E_3) = 3 \). The complement \( \overline{C_4} \) is two disjoint copies of \( K_2 \). Thus, \( Z(\overline{C_4}) = 2 \). For \( n \geq 5 \), since \( C_n \) is 2–regular, \( \overline{C_n} \) is
\[n - 3\)-regular. The lower bound follows from the fact that 
\[n - 3 = \delta(C_n) \leq Z(C_n).\] 
Since \(n \geq 5\), \(C_n\) and hence \(\overline{C_n}\) contain \(P_4\) as an induced subgraph. Using Lemma 3.1 it follows 
that \(Z(C_n) \leq n - 3\).

\[\square\]

**Definition 4.12.** A \((m,n)\)-sunlet graph is a cycle on \(n\) vertices with \(m\) pendant vertices 
adjacent to \(m\) of the \(n\) vertices on the cycle (see Figure 5).

![Figure 5: The \((m,n)\)-sunlet graph with vertices \(v_{i-1}, v_i, v_{i+1}\), and \(x_i\) labeled](image)

**Proposition 4.13.** Let \(G\) be a \((m,n)\)-sunlet graph. \(Z(G) = n + m - 3\).

**Proof.** Note that in \(\overline{G}\) each vertex on the cycle in \(G\) is adjacent to \(n - 3\) vertices that are 
also on the cycle in \(G\) and is either adjacent to \(m\) or \(m - 1\) vertices which are leafs in \(G\). Also, each vertex that is a pendant in \(G\) is adjacent to \(n + m - 2\) vertices in \(\overline{G}\). Thus, the 
minimum degree in \(\overline{G}\) is \(n + m - 4\) and we have \(Z(\overline{G}) \geq n + m - 4\). Those vertices in \(\overline{G}\) that 
have degree \(n + m - 4\) are those vertices which are adjacent in \(G\) to one pendant vertex \(x_i\) as 
in Figure 5. Let \(v_i\) be such a vertex. Consider a set of initially blue vertices \(B'\) containing \(v_i\), 
all but one of it’s neighbors in \(\overline{G}\), and \(x_i\). Then, \(v_{i-1}\) will be forced by \(v_{i+2}\), and any vertex 
adjacent to \(v_{i+1}\) in \(\overline{G}\) will force \(v_{i+1}\). Therefore, \(B'\) is a zero forcing set of size \(n + m - 3\) and 
we have \(n + m - 4 \leq Z(\overline{G}) \leq n + m - 3\).

We now show a zero forcing set for \(\overline{G}\) cannot have size \(n + m - 4\). Consider a zero forcing 
set of \(\overline{G}\) of size \(n + m - 4\). Then, there must exists a blue vertex in this set that has exactly 
one white neighbor, otherwise forcing would not begin. To have a zero forcing set of this 
size, we must have a vertex \(v_i\) of minimum degree with all but one of its neighbors colored 
blue. So, consider a set of initial blue vertices \(B\) containing \(v_i\) in figure 5 and all but one 
of it’s neighbors in \(\overline{G}\). The remaining white neighbor of \(v_i\) is forced in the first round. This 
gives us \(n + m - 4 + 1 = n + m - 3\) blue vertices and 3 white vertices which, by observation, 
must be \(x_i, v_{i-1},\) and \(v_{i+1}\). However, \(v_{i+2}\) is adjacent to \(v_{i-1}\) and \(x_i\); \(v_{i-2}\) is adjacent to \(x_i\) 
and \(v_{i+1}\); \(v_1, \ldots, v_{i-3}, \ldots, v_{i+3}, \ldots, v_n\) are adjacent to \(x_i\) and both \(v_{i-1}\) and \(v_{i+1}\); and finally 
\(x_j\) where \(j \neq i\) is adjacent to at least one of \(\{v_{i-1}, v_{i+1}\}\) and \(x_i\).

Therefore, we can not form a zero forcing set with the minimum degree number of vertices 
and thus \(Z(\overline{G}) = n + m - 3\). 

\[\square\]

**Definition 4.14.** The \(n\)-sunlet graph is the graph on \(2n\) vertices obtained by attaching \(n\) 
pendant edges to a cycle graph \(C_n\), i.e. the coronas \(C_n \circ K_1\).
Notice, the $n$–sunlet graph can be viewed as a $(n,n)$–sunlet graph, and thus, we obtain the following corollary.

**Corollary 4.15.** If $n \geq 3$, $Z(C_n \odot K_1) = 2n - 3$.

## 5 Graphs with More than One Cycle

The next natural progression of our study of the zero forcing number of graph complements is examining graphs with more than one cycle. First, in Section 5.1, we present results related to cactus graphs, a structural extension of the unicyclic graphs we explored in Section 4. Then, in Section 5.2, we give the zero forcing number for the complements of some well known families of graphs, including the wheel graph $W_n$, the friendship graph $F_n$, the Dutch Windmill Graph $D_n^{(m)}$, and other families which contain more than one cycle.

### 5.1 Cactus Graphs

Cactus graphs are a special case of graphs with more than one cycle. They can be thought of as a generalization of unicyclic graphs. While unicyclic graph contain a single cycle incident to a number of forests, cactus graphs contain multiple cycles which share at most one vertex in common, and each cycle is incident to a number of forests.

**Definition 5.1.** A cactus (sometimes called a cactus tree) is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, a cactus is a connected graph in which every edge belongs to at most one simple cycle, or (for nontrivial cactus) in which every block (maximal subgraph without a cut-vertex) is an edge or a cycle.

![A Cactus Graph](image)

**Figure 6: A Cactus Graph**

Similarly to in Observation 4.2, one can easily derive a lower bound on the zero forcing number of the complement of any cactus graph. First, observe that any vertex in a cactus graph $G$ is incident to $r$ cycles and incident to a forest on which it has $m$ neighbors (note that $r$ or $m$ could be 0). In the complement, the degree of such a vertex is $n - 1 - 2r - m$. Let $C(G)$ denote the number of cycles in a graph and let $m_{\text{max}}$ denote the number of vertices in the largest forest in the graph. The maximum possible $r$ value is $C(G)$ and the maximum possible $m$ value is $m_{\text{max}}$. Therefore, since the zero forcing number of a graph is bounded below by the minimum degree, we observe the following result.
Observation 5.2. Let $C(G)$ denote the number of cycles in a graph. Let $m_{max}$ denote the size of the largest tree in a graph. Let $G$ denote a cactus graph on $n$ vertices. Then,

$$n - 2C(G) - m_{max} - 2 \leq Z(G).$$

Using similar ideas to that in Lemma 3.1 and Observation 5.2, we can obtain a result giving us the zero forcing number of the complement of a cactus graph, provided the graph meets certain subgraph constraints.

Proposition 5.3. Suppose $G$ is a cactus graph containing no $C_4$ and at least one vertex of degree 2 that is incident to a cycle. Then, either $G = C_3$ or $Z(G) = n - 3$.

Proof. Let $z$ be a vertex of degree 2 in $G$ that is incident to a cycle. Observe that $z$ has degree $n - 3$ in $\overline{G}$. Consider a zero forcing set $B$ which contains $z$ and all but one of its neighbor in $\overline{G}$. The neighbor of $z$ which is not in $B$ will forced in the first round. Let the two remaining vertices be $x$ and $y$ and observe that they are the neighbors of $z$ on a cycle in $G$.

Consider the neighbors of $x$ and $y$ in $G$ which are not $x, y, z$. If neither vertex has another neighbor in $G$, then $G$ is $C_3$ and $Z(C_3) = 3$. So suppose at least one of $x$ or $y$ has such a neighbor $v$. Since $G$ does not contain a $C_4$ subgraph, $v$ is not adjacent to both $x$ and $y$ in $G$. Also, $v$ is not adjacent to $z$, since $z$ has degree 2. Therefore, in $\overline{G}$, $v$ is either in $B$ or is forced in the first round. In the next round, $v$ can force either $x$ or $y$, whichever vertex its is not adjacent to in $G$. Finally, the remaining vertex can be forced by any of its neighbors.

Now, suppose there exists a zero forcing set $B$ for $\overline{G}$ such that $|B| = n - 4$. Since there are four initially white vertices, in order for a force to occur, three of these vertices must be adjacent to a vertex $v$ in $G$ while the remaining vertex is nonadjacent. After this force, there are 3 remaining white vertices in $\overline{G}$: the neighbors of $v$ in $G$. In order for a force to occur, there must be a vertex $w$ in $G$ so that $w$ is adjacent to two of the neighbors of $v$ and nonadjacent to the other neighbor of $v$ in $G$. If such a vertex exists then $v$, two of its neighbors, and $z$ form a $C_4$. This is impossible. Thus, $Z(\overline{G}) \geq n - 3.$

Finally, we present a result on a family of cactus graphs which we call $(n, m_1, m_2, \ldots, m_n)$-cycle-bracelet graphs.

Definition 5.4. A $(n, m_1, m_2, \ldots, m_n)$–cycle-bracelet graph is a graph consisting of $n$ cycles of sizes $m_1, m_2, \ldots, m_n$ where $C_{m_i}$ shares exactly one vertex with $C_{m_{i-1}}$ and exactly one vertex with $C_{m_{i+1}}$ for all $1 \leq i \leq n$ (see Figure 7).

Proposition 5.5. Let $G$ be a $(n, m_1, m_2, \ldots, m_n)$-cycle-bracelet graph. Then

$$Z(\overline{G}) = \sum_{i=1}^{n} m_i - n - 3.$$
Figure 7: A \((n, m_1, m_2, \ldots, m_n)\)-cycle-bracelet graph

\textit{Proof.}: Note that every vertex in a \((n, m_1, m_2, \ldots, m_n)\)-cycle-bracelet graph \(G\) has degree 2 or 4. So, in \(\overline{G}\), every vertex has degree \(\sum_{i=1}^{n} m_i - n - 3\) or \(\sum_{i=1}^{n} m_i - n - 5\). Consider a vertex \(w\) of degree 2 and its two neighbors \(x\) and \(y\) in \(G\). In the cycle containing \(w, x,\) and \(y\), let \(w_1\) be the other neighbor of \(x\) and \(w_2\) be the other neighbor of \(y\) in \(G\). If a zero forcing set \(B\) of \(\overline{G}\) contains \(w\) and all but one of its neighbors in \(\overline{G}\), then the remaining neighbor is forced by \(w\). This leaves only two white vertices, \(x\) and \(y\). Note in \(G\), \(w_1\) is adjacent to \(x\) but not adjacent to \(y\) and \(w_2\) is adjacent to \(y\) but not adjacent to \(x\). Therefore, \(w_1\) can force \(y\) and \(w_2\) can force \(x\). Thus, \(Z(\overline{G}) \leq \sum_{i=1}^{n} m_i - n - 3\).

Let \(v\) be a vertex with degree \(\sum_{i=1}^{n} m_i - n - 5\) in \(\overline{G}\) and let \(x_1, x_2, x_3, x_4\) be the neighbors of \(v\) in \(G\). Assume there exists a zero forcing set \(B\) of \(\overline{G}\) which contains \(v\), all but one of the neighbors of \(v\) in \(\overline{G}\), and \(x_4\). Note that \(|B| = \sum_{i=1}^{n} m_i - n - 4\). In the first round, the remaining neighbor of \(v\) is forced by \(v\). This leaves 3 white vertices, \(x_1, x_2, x_3\). The vertex \(v\) can not force any of these vertices in \(\overline{G}\). Any other vertex in the graph is adjacent to at least two of \(x_1, x_2, x_3\), so the entire graph can not be forced. Therefore, any zero forcing set of \(\overline{G}\) must contain at least \(\sum_{i=1}^{n} m_i - n - 3\) vertices. \(\square\)

5.2 Miscellaneous Graphs

We now determine the zero forcing number of the complement of the Wheel graph, Seashell graph, Friendship graph, and various Harp graphs. First, we provide the definitions for these graph families.

The \textit{wheel graph}, denoted \(W_n\), of order \(n \geq 4\), sometimes simply called an \(n\)-wheel, is a graph that contains a cycle of order \(n - 1\) and for which every graph vertex in the cycle is connected to one other graph vertex known as the hub. The edges of a wheel which include the hub are called spokes. \(W_n\) can also be defined as the graph join \(K_1 \vee C_{n-1}\). The \textit{Seashell graph} on \(n\) vertices, denoted \(SE_n\), is a cycle on \(n\) vertices \(C_n\) with one vertex on the cycle that is adjacent to all other vertices in the graph. The \textit{Friendship graph}, denoted \(F_n\), can
be constructed by joining \( n \) copies of the cycle graph \( C_3 \) with a common vertex. The Dutch Windmill graph, denoted \( D_n^{(m)} \), consists of \( m \) copies of \( C_n \) with a vertex in common. The One Stringed Harp Graph is the cycle graph \( C_n \) on \( n \) vertices with an extra edge connecting two vertices on the cycle that were previously nonadjacent. The \( \left\lfloor \frac{n-2}{2} \right\rfloor - \text{Harp graph} \) on \( n \) vertices is the cycle \( C_n \) on \( n \) vertices with \( \left\lfloor \frac{n-2}{2} \right\rfloor \) extra edges positioned such that \( v_{i-k} \) is adjacent to \( v_{i+k+2} \) for \( 0 \leq k \leq \left\lfloor \frac{n-2}{2} \right\rfloor - 1 \).

Figure 8: The Wheel, Seashell, and Dutch Windmill graphs

Proposition 5.6. For \( n \geq 4 \), \( Z(W_n) = n - 2 \).

Proof. The complement \( \overline{W_n} \) consists of a single isolated vertex and \( \overline{C_n} \). By Proposition 4.11, the zero forcing number of \( \overline{C_n} \) is \( n - 3 \). In order to color the isolated vertex we must add another vertex to the zero forcing set. \( \square \)

Proposition 5.7. Let \( n \geq 4 \). \( Z(SE_n) = n - 3 \).

Proof. In \( \overline{SE_n} \), there is one isolated vertex, \( v_i \), the vertices \( v_{i-1} \) and \( v_{i+1} \) have degree \( n - 3 \) and all other vertices degree \( n - 4 \). In order for any forces to occur, any zero forcing set \( B \) must include \( v_i \), as well as at least one vertex \( v_j \) \( (i \neq j) \), and all but one of it’s neighbors. We choose a vertex of minimum degree. Therefore, \( |B| \geq n - 4 + 1 = n - 3 \). Now, we show such a set of vertices does indeed form a zero forcing set. Let \( B \) include \( v_{i-2} \), which has degree \( n - 4 \), and all but one of its neighbors, as well as \( v_i \). This leaves 3 white vertices in \( \overline{C} \). In the first round, the remaining white neighbor of \( v_{i-2} \) can be forced. Then the blue vertex \( v_{i-4} \) forces \( v_{i-1} \) and finally, any vertex can force \( v_{i-3} \). Therefore, \( B \) is a zero forcing set. \( \square \)

Proposition 5.8. Let \( n \geq 4 \) and \( m \geq 2 \). \( Z(D_n^{(m)}) = nm - m - 2 \).

Proof. There is one vertex of degree \( nm - 3m \) in \( \overline{D_n^{(m)}} \) and all other vertices have degree \( nm - m - 2 \). Consider a vertex of degree \( nm - m - 2 \), labeled \( v \) in the figure below in \( \overline{D_n^{(m)}} \). Let this vertex and all but one of its neighbor be contained in an initially blue set \( B \). The remaining neighbor of \( v \) can be forced by \( v \) in the first round. After this, \( nm - m - 1 \) vertices are blue and only 2 are white. With the vertices labeled as in Figure 9, vertex 3 forces vertex 1 and finally, any vertex which is adjacent to vertex 2 can force it. Therefore, \( B \) is a zero forcing set of \( \overline{D_n^{(m)}} \).
We now show that $Z(D_{m}^{(m)}) > nm - m - 3$, by showing that if $D_{m}^{(m)}$ contains 4 white vertices, no forcing can occur. Since any blue vertex of degree $nm - m - 2$ is nonadjacent to at most 2 white vertices, they are adjacent to at least two white vertices. Therefore, none of these vertices can perform a force. So, the only vertex that could potentially force is vertex $c$, noted in the figures below, if it is adjacent to only one of the white vertices. The white vertices are therefore arranged in exactly one of the following cases according to Figure 10:

**Case 1:** Vertex $c$ forces vertex $A$. Then, any vertex on the graph is adjacent to either $B$ and $E$, $B$ and $D$, $D$ and $E$ or $B$, $D$, and $E$ and so we can not force the entire graph.

**Case 2:** Vertex $c$ forces vertex $A$. Then, any vertex on the graph is adjacent to either $B$ and $E$, $B$ and $D$, $D$ and $E$ or $B$, $D$, and $E$ and so we can not force the entire graph.

**Case 3:** Vertex $c$ forces vertex $D$. Then, any vertex on the graph is adjacent to either $A$ and $B$, $A$ and $E$, $B$ and $E$, or $A$, $B$, and $E$ and so we can not force the entire graph.

**Case 4:** Vertex $c$ forces vertex $A$. Then, any vertex on the graph is adjacent to either $B$ and $E$, $B$ and $D$, $D$ and $E$ or $B$, $D$, and $E$ and so we can not force the entire graph.

**Case 5:** Vertex $c$ forces vertex $A$. Then, any vertex on the graph is adjacent to either $B$ and $E$, $B$ and $D$, $D$ and $E$ or $B$, $D$, and $E$ and so we can not force the entire graph.

**Proposition 5.9.** For $n \geq 2$, $Z(F_{n}) = 2n - 1$.

*Proof.* In $F_{n}$, there is a single isolated vertex $v_{i}$ and $2n$ vertices of degree $2n - 2$. In order for any forces to occur, any zero forcing set $B$ must include $v_{i}$, as well as at least one vertex $v_{j}$ ($i \neq j$) and and all but one of it’s neighbors. Therefore, $|B| \geq 2n - 2 + 1 = 2n - 1$. Such a set $B$ is a zero forcing set since $v_{j}$ can force its one white neighbor and then any vertex adjacent to the last white vertex can force it. 

\[\square\]
Figure 11: The Friendship graphs $F_2$, $F_3$ and $F_4$, the $\lfloor \frac{n-2}{2} \rfloor$-Harp graph, and the One Stringed Harp graphs

**Proposition 5.10.** Let $G$ be the One Stringed Harp Graph with the edge $v_iv_j$. Then,

$$Z(G) = \begin{cases} n - 4 & \text{if } v_j = v_i \pm 3 \\ n - 3 & \text{otherwise} \end{cases}$$

**Proof.** Label the vertices of $G$ as $v_1, \ldots, v_n$. In order to force anything in $G$, a vertex and all except one of its neighbors must be colored blue. Let $B$ be a set of initially blue vertices containing a vertex $v_i$ of lowest degree in $G$ and all but one of it’s neighbors. In the first round, the white neighbor of $v_i$ is forced. Since the minimum degree in $G$ is $n - 4$, this leaves only 3 white vertices: the three vertices that are adjacent to $v_i$ in $G$, $v_{i-1}$, $v_{i+1}$, and $v_j$. In order to force another vertex, there must be a blue vertex $v_k$, which is a neighbor of $v_i$ in $G$ and is adjacent to only one white vertex. This means $v_k$ is adjacent in $G$ to two of $v_{i-1}$, $v_{i+1}$ and $v_j$. It can not be the case that $v_k$ is adjacent to $v_{i-1}$ and $v_{i+1}$, because that would mean $v_i = v_k$. So either $v_k$ is adjacent to $v_j$ and $v_{i-1}$ or to $v_j$ and $v_{i+1}$ in $G$. Thus, $v_k = v_{i-2}$ and $v_j = v_{i-3}$ or $v_k = v_{i+2}$ and $v_j = v_{i+3}$. So, $n - 4 \leq Z(G)$ if $v_j = v_{i\pm 3}$ and $n - 3 \leq Z(G)$ otherwise.

Now, suppose $v_j = v_{i\pm 3}$. In the second round, $v_{i\pm 2}$ will force $v_{i\mp 1}$. Next, $v_{j\mp 1}$ will force $v_{i\pm 1}$. Finally, any vertex adjacent to the remaining white vertex can force it. Otherwise, if $v_j \neq v_{i\pm 3}$, then a zero forcing set $B$ must also contain $v_j$. In this case, $v_{i-1}$ is forced by $v_{i+2}$ while $v_{i+1}$ is forced by $v_{i-2}$. Therefore, in both cases, $B$ is a zero forcing set.

**Proposition 5.11.** Let $G$ be the $\lfloor \frac{n-2}{2} \rfloor$-Harp graph on $n$ vertices. $Z(G) = n - 4$.

**Proof.** On the graph $G$, there are 2 vertices of degree $n - 3$ if $n$ is even and 3 vertices of degree $n - 3$ if $n$ is odd. The remaining vertices have degree $n - 4$. So, we obtain $n - 4 = \delta(G) \leq Z(G)$.

Let the $\lfloor \frac{n-2}{2} \rfloor$ extra edges in $G$ be such that $v_{i-k}$ is adjacent to $v_{i+k+2}$ for $0 \leq k \leq \lfloor \frac{n-2}{2} \rfloor - 1$. Consider the set of initially blue vertices $B$ containing the vertex $v_i$ and all except one of it’s $n - 4$ neighbors in $G$, and note $|B| = n - 4$. The remaining neighbor of $v_i$ is forced in the
first round. This leaves 3 white vertices, the neighbors of $v_i$ in $G$, $v_{i-1}, v_{i+1},$ and $v_{i+2}$. Since $v_{i+3}$ is nonadjacent to $v_{i-1}$ and $v_{i+2}$ in $\overline{G}$, it forces $v_{i+1}$. The vertex $v_{i-2}$ forces $v_{i+2}$ and any vertex which is adjacent to $v_{i-1}$ can force it. Thus, $B$ is a zero forcing set.

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