A Search Algorithm for Simplicial Complexes

Subhrajit Bhattacharya*

August, 2016

Abstract

We present the ‘Basic S*’ algorithm for computing shortest path through a metric simplicial complex. In particular, given a metric graph, $G$, which is constructed as a discrete representation of an underlying configuration space (a larger “continuous” space/manifold typically of dimension greater than one), we consider the Rips complex, $R(G)$, associated with it. Such a complex, and hence shortest paths in it, represent the underlying metric space more closely than what the graph does. While discrete graph representations of continuous spaces is convenient for motion planning in configuration spaces of robotic systems, the metric induced in them by the ambient configuration space is significantly different from the metric of the configuration space itself. We remedy this problem using the simplicial complex representation. Our algorithm requires only an abstract graph, $G = (V,E)$, and a cost/length function, $d : E \to \mathbb{R}_+$, as inputs, and no global information such as an embedding or a global coordinate chart is required. The complexity of the Basic S* algorithm is comparable to that of Dijkstra’s search, but, as the results presented in this paper demonstrate, the shortest paths obtained using the proposed algorithm represent/approximate the geodesic paths in the original metric space significantly more closely.

1 Introduction

Computing shortest path in a configuration space is fundamental to motion planning problems in robotics. While continuous methods for path planning does exist [Zef96, RK91, CRC03, HLK07], they suffer from drawbacks, especially in presence of obstacles/holes in the configuration space, such as difficulty in imposing arbitrary optimality criteria (potential/vector field methods [RK91, HLK07]), large search space (variational methods [MKK12, LS10, DK99]), termination at local optimum [RK91, KV88, KK92, CRC03] due to non-convex search spaces, and in general lack of rigorous guarantees when the configuration space has an arbitrary topology (non-contractible spaces) or non-trivial geometry (non-convex, general metric spaces).

A robust and popular alternative to the continuous approaches is the discrete approach of graph search-based planning. The basic idea behind the approach is to sample points from the configuration space, and construct a graph by connecting “neighboring” vertices with edges (representing actions taking the system from one sampled configuration to another). Any trajectory in the original configuration space is approximated by a path in the graph [BM07]. One can thus employ any search algorithm like Dijkstra’s [Dij59], A* [HNR68], D* [Ste95], ARA* [HZ07] or R* [LS08] to search for the optimal path in the graph from a start vertex to a goal vertex. Such a discrete approach for motion planning in graphs is indifferent to the underlying topology/geometry of the configuration space (hence suitable for use in arbitrary configuration spaces), comes with guarantees on algorithmic completeness, termination and optimality in the graph (or bounds on sub-optimality), and are extremely fast. Such conveniences are precisely the reason that graph search-based approaches have been extremely popular in solving motion planning problems on real robotic systems such as motion planning for autonomous vehicles [FBLD08, UAB* 08, MBB* 08, HKB* 09], planning for robotic
arms such as PR2 in cluttered environments [CCL10], multi-robotic systems [SPL15, BLK10], and motion planning for systems involving cables [BKH+15, KBK14].

However the major drawback of using such discrete graph-based approaches in motion planning is that the computed paths remain constrained to the graph, which constitutes a small (1-dimensional) subset of the original configuration space. This means that paths that are optimal in the graph need not be optimal in the original configuration space. This issue is typically not remedied by reducing size of the discretization (see Figure 1). In recent years there have been significant effort in trying to remedy this issue in specific classes of configuration spaces or graphs. All such approaches fall under the general category of what is known as “any-angle path planning” algorithms [UK15].

The method proposed in this paper, in the same spirit, may be considered as an any-angle path planning algorithm. Instead of planning paths in a graph, we propose an algorithm for finding shortest paths through simplicial complexes. In particular, given a graph, we consider the Rips complex of the graph, and compute shortest path in that complex (Figure 2). The unique features of our proposed method are as follows:

- While the input to our algorithm is a metric graph (i.e., a graph with specified edge costs/lengths), the underlying structure on which we compute an optimal path is a metric simplicial complex (for a given graph we consider its Rips complex). More generally, our algorithm can be used to compute shortest paths in metric simplicial complexes (not necessarily a Rips complex of a metric graph).
- The input graph can be an arbitrary, abstract metric graph. In particular, we do not require the underlying metric space (whose discrete representation is the graph) to be a subset of flat/Euclidean space (unlike what is required by Theta* [NDKF07, NKT10], ANYA [HG13] and Simplicial Dijkstra [YL11]). Informally speaking, our method can deal with graphs with “non-uniform traversal costs” – both non-homogeneous and anisotropic.
- Our method does not require the graph to be embedded in some continuous space or an Euclidean space. In particular, we do not need coordinates for the vertices as input to the algorithm. The only input required to our algorithm is the abstract graph, $G = (V, E)$, and a cost/length function, $d : E \rightarrow \mathbb{R}_+$. Embeddings are constructed locally for simplices as required, and no other data, besides $G$ and $d$, are required. This is in contrast to [YL11, FS07].
- Our algorithm is designed for simplicial complexes of arbitrary dimensions and does not require any specific kind of discretization, as long as the simplicial complex covers the entire original configuration space (in particular, any arbitrary triangulation of a 2-dimensional configuration space is sufficient). This, once again, is in contrast to [FS07].
- We consider an accurate geometric model in computing the distances, based on local embedding of simplices in a model Euclidean space. This is in contrast to [YL11, FS07]. The accurate model allows us to guarantee that the cost/length of shortest paths computed using the proposed algorithm approaches the true geodesic distance on Riemannian manifolds as the discretization size is made finer.
- Our algorithm is local, requiring the abstract graph, $G = (V, E)$, and a cost/length function, $d : E \rightarrow \mathbb{R}_+$, only. No global information such as line-of-sight or global embedding is required.

Our focus is the development of an algorithm that can compute optimal paths in arbitrary metric spaces represented by an abstract metric graphs such that the computed path is not restricted to the graph and represents the true geodesic path in the underlying metric space as closely as possible. Our algorithm is local, requiring only the abstract graph, $G = (V, E)$, and a cost/length function, $d : E \rightarrow \mathbb{R}_+$. We use the Dijkstra’s search as the backbone for our algorithm, and develop techniques to incorporate simplicial data into it. More efficient versions of the algorithm (incorporating features of heuristic, randomized, incremental and any-time search algorithms) are within the scope of future work. We believe that in context of robot motion planning, the proposed algorithm is a first, formal use of a finite element method (FEM) [DLT12], where the role of simplicial complexes is well-appreciated.

1.1 Outline of Paper

In the next sub-section we introduce some preliminary notations and definitions. Following that we introduce the main Basic S* algorithm, the sub-procedures involved in it, and a path reconstruction algorithm. Some
and finally we present simulation results. For better readability, many of the detailed proofs and derivations have been moved to the appendix.

1.2 Preliminaries

![Diagram of a graph and its Rips complex](image)

**Figure 2:** A graph (left), its Rips complex (right), and shortest paths in those. In this example, \( V = \{a, b, c, d, e, f, g, h, i, j, k, l, m\} \) is the vertex set, \( C_0 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}, \{k\}, \{l\}, \{m\}\} \) is the set of 0-simplices, \( C_1 = \{\{a, b\}, \{b, c\}, \{c, a\}, \{c, d\}, \{c, b\}, \{b, d\}, \{b, e\}, \{e, c\}, \{b, f\}, \{e, f\}, \{f, h\}, \{e, h\}, \{h, l\}, \{d, g\}, \{g, j\}, \{j, k\}, \{j, l\}, \{k, l\}, \{l, i\}, \{l, m\}, \{k, m\}\} \) is the edge set (the set of 1-simplices), \( C_2 = \{\{a, b, c\}, \{b, c, d\}, \{b, c, e\}, \{e, c, d\}, \{b, e, d\}, \{e, b, f\}, \{f, h, e\}, \{i, j, l\}, \{j, i, k\}, \{k, l, m\}, \{k, g, j\}\} \) is the set of 2-simplices, and \( C_3 = \{\{b, c, d, e\}\} \) is the set of 3-simplices.

**Definition 1** (Simplicial Complex [Hat01] – Combinatorial Definition). A simplicial complex, \( C \), constructed over a set \( V \) (the vertex set) is a collection of sets \( C_n \), \( n = 1, 2, \ldots \), such that

i. An element in \( C_n \), \( n \geq 0 \) is a subset of \( V \) and has cardinality \( n + 1 \) (i.e., For all \( \sigma \in C_n \), \( \sigma \subseteq V \), \( |\sigma| = n + 1 \)). \( \sigma \) is called a “\( n \)-simplex”.

ii. If \( \sigma \in C_n \), \( n \geq 1 \), then \( \sigma - v \in C_{n-1} \), \( \forall v \in \sigma \). Such a \((n-1)\)-simplex, \( \sigma - v \), is called a “face” of the simplex \( \sigma \).

The simplical complex is the collection \( C = \{C_0, C_1, C_2, \ldots \} \). We also define \( C_* = C_0 \cup C_1 \cup C_2 \cup \cdots \subseteq P(V) \).

In general, a \( n \)-simplex is a set containing \( n + 1 \) elements, \( \sigma = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n\} \), where \( v_i \in V \), \( i = 0, 1, \ldots, n \).

In algebraic topology, one imparts group or vector space structures on these sets via operation completions, and defines linear maps between those (the boundary maps). However, for the purpose of this paper we do not require such algebraic constructions.

**Definition 2** (Rips Complex of a Graph, \( \mathcal{R}(G) \)). If \( G \) is an undirected graph with \( V \) its vertex set and \( E \) its edge set, we define the Rips Complex of the graph, \( \mathcal{R}(G) \), to be the simplicial complex with an \( n \)-simplex consisting of every \((n+1)\)-tuple of vertices that are all connected to each other (a clique).

In notations, \( \mathcal{R}(G) = \{C_0, C_1, C_2, C_3, \ldots\} \) is such that \( C_0 = \{\{v\} \mid v \in V\} \) and for \( \sigma \in C_n, n > 1 \) and \( a, b \in \sigma \), we have \{a, b\} \in E \). Also, define \( C_* = C_0 \cup C_1 \cup C_2 \cup \cdots \).
The set \( C_0 \) is the set of 0-simplices consisting of singleton sets, each containing a single vertex from \( V \), while the set \( C_1 = E \). From now on, whenever we refer to a simplicial complex, unless otherwise specified, we will refer to the Rips complex \( \mathcal{R}(G) = \{C_0, C_1, C_2, \cdots \} \), for a given graph \( G = (V, E) \). Figure 2 illustrates the Rips complex of a graph with an explicit example of the sets \( C_0, C_1, C_2, \cdots \).

## 2 Basic S* Algorithm

The idea behind the S* search algorithm is very similar to the standard search algorithms such as Dijkstra’s or A*. However, instead of restricting paths to the graph \( G \), it allows paths to pass through simplices of \( \mathcal{R}(G) \). Specifically, for a particular vertex, \( u \in V \), when updating its “came-from” vertex and its minimum distance from the start, we don’t just replace the earlier values with the new lower value. Instead, we construct the maximal simplices (Definition 3) containing the vertex \( w \), the expanding vertex, and other already expanded vertices, and find the shortest path through those simplex. This is illustrated in the example of Figure 3.

The complete pseudocode for the Basic S* search algorithm (with a single start vertex and no specified goal vertex) is given in Algorithm 1.

### Algorithm 1: Basic S*

\[
(d, cfp) = \text{Basic\textunderscore S}^* \ (G, d, s)
\]

**Inputs:**
- a. Graph \( G \) with vertex set \( V \) and (undirected) edge set \( E = C_1 \)
- b. A length/cost function on the edge set, \( d : E \to \mathbb{R}_+ \)
- c. Start vertex, \( s \in V \)

**Outputs:**
- a. The distances from \( s \) to every vertex in the graph, \( \bar{d} : V \to \mathbb{R}_+ \)
- b. Came-from-point map, \( cfp : V \to C_0 \times \mathbb{R}^k \)

1. Initiate \( d \): Set \( \bar{d}(v) := \infty \) for all \( v \in V \) // distances (implicit declaration for any un-visited vertex).
2. Set \( \bar{d}(s) \leftarrow 0 \) // start vertex.
3. Initiate \( cfp \): Set \( cfp(v) := \emptyset \) for all \( v \in V \) // came-from point (implicit declaration for any un-visited vertex).
4. Set \( Q := V \) // Set of un-expanded vertices.

while \( Q \neq \emptyset \) AND stopping criterion not met)

6. Set \( q := \arg\min_{v \in Q} \bar{d}(q) \) // Maintained by a heap data-structure.
7. if \( (\bar{d}(q) = \infty) \) // cannot reach any other vertex.
8. break
9. Set \( Q \leftarrow Q \setminus \{q\} \) // Remove \( q \) from \( Q \).

for each \((u \in \mathcal{N}_G(q))\) // For each neighbor of \( q \) (both expanded and un-expanded).

11. Set \( S := \{y \mid y \in \mathcal{N}_G(q), y \in \mathcal{N}_G(u), y \notin Q\} \) // expanded common neighbors of \( q \) and \( u \).
12. Set \( \mathcal{MS} := \text{MaximalSimplices}_{\mathcal{R}(G)}(S; \{u, q\}) \) // maximal simplices attached to \( \{u, q\} \).

for each \((\sigma \in \mathcal{MS})\)

14. Set \((d', (\sigma', \bar{w}')) := \text{DistanceThroughSimplex}_{(d, \bar{w})}(\sigma, u) \) // distance to \( u \) through simplex \( \sigma \).

if \((d' < \bar{d}(u))\) // Found better \( \bar{d} \)-score for this neighbor.
16. Set \( \bar{d}(u) \leftarrow d' \)
17. Set \( cfp(u) \leftarrow (\sigma', \bar{w}') \)
18. if \((u \notin Q)\) // \( u \) is already expanded.
19. Set \( Q \leftarrow Q \cup \{u\} \). // “un-expand” \( u \).

20. return \( \bar{d}, cfp \).

Key features of the algorithm:

1. Similar to Dijkstra’s or A* search, we maintain a list of “un-expanded” vertices (open list), \( Q \), in a heap data structure with the \( \bar{d} \)-values (which, in search algorithm literature, has be traditionally called \( g \)-score) of the vertices being the heap keys. At each of the while loop iterations (starting on Line 5), the un-expanded vertex with lowest \( \bar{d} \)-value is popped (“expanded”) and the \( \bar{d} \)-value of its neighbors are checked for improvement (loop starting on Line 10).

2. Unlike Dijkstra’s algorithm, however, we do not restrict this update only to unexpanded neighbors of
Figure 3: Comparison between Dijkstra’s search algorithm for graph, $G$, and the Basic S* search algorithm for $R(G)$. (a)-(c) shows steps in a typical graph search algorithm. In Basic S*, the steps (a)-(b) are the similar, but step (c) is replaced by the step (d).
q, and instead check for update for all neighbors, u, in \( \mathcal{N}_G(q) \). This is because there may be obtuse simplices, that get generated at a later stage, through which the distance to an already expanded vertex may be shorter. This is described in more details in Figure 4.

![Figure 4](image)

Figure 4: Illustration of a scenario where improvement of the \( \bar{d} \)-value of a vertex, \( q_1 \), is possible even after it has been expanded. This prompts us to check for potential update of all the neighbors of an expanding vertex rather than only the un-expanded neighbors (Line 10 of Algorithm 1).

3. When vertex \( q \) is being expanded, in order to compute a candidate for updating the \( \bar{d} \)-value of a neighboring vertex \( u \), we generate all maximal simplices (in \( \mathcal{R}(G) \)) consisting of vertices \( q, u \) and only other expanded vertices (Line 12, and procedure ‘MaximalSimplices’ is described in more details in Section 2.1).

4. We compute a potential \( \bar{d} \)-value, \( d' \), for updating \( u \) for paths through each of these maximal simplices attached to \( \{q, w\} \), and choose the lower out of that as the candidate to test against for update. This is computed by the ‘DistanceThroughSimplex’ procedure described in Section 2.2. This procedure also returns the pair of data, \( (\sigma', \vec{w}') \), which represents a “came-from point” inside simplex \( \sigma' \). The cfp is however mostly irrelevant in context of this algorithm.

5. This is in contrast to the corresponding step in a graph search algorithm, where the potential value to test against is simply the sum of the \( \bar{d} \)-value at \( q \) and the length/cost of the edge \( \{q, u\} \).

2.1 Computing Attached Maximal Simplices

**Definition 3** (Maximal Simplices and Maximal Simplices Attached to a Simplex).

a. *Maximal Simplex Constructed Out of a Set of Vertices:* A maximal simplex constructed out of a set of vertices, \( S \subseteq C_0 \), is a subset \( \chi \subseteq S \), such that \( \chi \) is a simplex in \( \mathcal{R}(G) \), and \( \chi \) is not a face of any higher-dimensional simplex constituting of the vertices from \( S \). We refer to the set of maximal simplices created out of \( S \) as \( \mathcal{M}(S) \subseteq C_* \). Formally, \( \mathcal{M}(S) = \{ \chi \mid \chi \subseteq S, \chi \in C_*, \chi \cup a \notin C_*, \forall a \in S - \chi \} \).

b. *Neighbors of a Simplex:* A vertex \( a \in C_0 \) is called a neighbor of a simplex \( \sigma \in C_* \), if \( \{a, b\} \in C_1 \) for all \( b \in \sigma \) (i.e., \( a \) is connected to every vertex in \( \sigma \) by a 1-simplex). Thus, if \( \sigma \in C_n \), then \( \{a\} \cup \sigma \in C_{n+1} \). The set of all neighbors of \( \sigma \) in \( G \) is \( \mathcal{N}_G(\sigma) = \bigcap_{v \in \sigma} \mathcal{N}_G(v) \) (where \( \mathcal{N}_G(v) \) is the set of neighbors of \( v \) in \( G \), excluding \( v \) itself).
c. Attached Maximal Simplex Constructed Out of a Set of Neighbors: If \( S \subseteq \mathcal{N}_G(\sigma) \) is a set of neighbors of a simplex \( \sigma \in C_* \), then the set of maximal simplices constructed out of \( S \) and attached to \( \sigma \) is the set \( \mathcal{M}(S; \sigma) := \mathcal{M}(S \cup \sigma) \). See Figure 5.

The following property is obvious.

**Lemma 1.** If \( \sigma \in C_* \) is a simplex, and \( S \) a set of neighbors of \( \sigma \), (i.e., every vertex in \( S \) is connected to every vertex in \( \sigma \)), then, \( \mathcal{M}(\sigma \cup S) = \{ \sigma \cup \chi \mid \chi \in \mathcal{M}(S) \} \).

![Figure 5](image.png)

(a) In this sub-complex, the set of maximal simplices constructed out of the set \( U = \{a, b, c, d, e, f\} \) is \( \mathcal{M}(U) = \{\{a, b, e, c\}, \{a, e, d\}, \{a, c, f\}\} \). This is also the set of maximal simplices attached to \( \sigma = \{a, c\} \) and constructed out of \( S = \{b, e, d, f\} \).

(b) The set \( S = \{b, e, d, f\} \) is a set neighbors of the 1-simplex \( \sigma = \{a, c\} \). \( \mathcal{M}(S) = \{\{d, e\}, \{b, e\}, \{f\}\} \). It is easy to observe that \( \mathcal{M}(U) = \{\sigma \cup \chi \mid \chi \in \mathcal{M}(S)\} \).

Furthermore, since \( d \) and \( b \) are not connected in the complex, \( \mathcal{M}(S) \) can be partitioned into subset of simplices containing \( d \) but not \( b \), containing \( b \) but not \( d \), and containing neither.

The direct way of computing \( \mathcal{M}(S; \sigma) \) will be to check if \( \sigma \cup \alpha \) is a simplex in \( C_* \) for every \( \alpha \in \mathcal{P}(S) \) (the power set of \( S \)). However the complexity of this algorithm would be \( O(2^{|S|}) \). The development of a more efficient algorithm for procedure \texttt{MaximalSimplices} relies on the following observation:

**Lemma 2.** Suppose \( \sigma \in C_* \) and \( S \) is a set of neighbors of \( \sigma \). Identify two vertices, \( a, b \in S \) such that \( \{a, b\} \notin C_1 \) – i.e., \( a \) and \( b \) not connected (if such a pair does not exist, then \( \sigma \cup S \) is the only maximal simplex). Then the set \( \mathcal{M}(S; \sigma) \) of maximal simplices constructed out of \( S \) and attached to \( \sigma \) can be partitioned into three parts:

(i) Maximal simplices containing \( a \), but not containing \( b \): \( \mathcal{M}(S \cap \mathcal{N}_G(a); \sigma \cup \{a\}) \)
(Note: \( S \cap \mathcal{N}_G(a) \) does not contain \( b \), since \( a \) and \( b \) are not connected).

(ii) Maximal simplices containing \( b \), but not containing \( a \): \( \mathcal{M}(S \cap \mathcal{N}_G(b); \sigma \cup \{b\}) \).

(iii) Maximal simplices containing neither \( a \) nor \( b \): \( \mathcal{M}(S \setminus \mathcal{N}_G(a, b); \sigma) \).
Algorithm 2: Construct Attached Maximal Simplices

\[
\mathcal{MS} = \text{MaximalSimplices}_{\mathcal{R}(G)}(S; \sigma)
\]

Inputs: a. Rips complex, \( \mathcal{R}(G) = \{C_0, C_1, C_2, \ldots \} \), of graph \( G = (V, E) \).
b. Simplex \( \sigma \subset C_n \).
c. A set of neighbors, \( S \subset \mathcal{N}_G(\sigma) \).

Outputs: a. The set of maximal simplices constructed out of \( S \) and attached to \( \sigma \).

1. if \( |S| > 1 \)
2. for each \( (m \in S, n \in S, m \not= n) \)
3. if \( \{m, n\} \notin C_i \)
4. \( a := m \), \( b := n \) \( // a, b \in S \) such that they are not connected.
5. break
6. if \( a, b \) are undefined
7. \( \mathcal{MS} := \{\sigma \cup S\} \)
8. else
9. \( \mathcal{MS}_{(i)} := \text{MaximalSimplices}_{\mathcal{R}(G)}(S \cap \mathcal{N}_G(a); \sigma \cup \{a\}) \)
10. \( \mathcal{MS}_{(ii)} := \text{MaximalSimplices}_{\mathcal{R}(G)}(S \cap \mathcal{N}_G(b); \sigma \cup \{b\}) \)
11. \( \mathcal{MS}_{(iii)} := \text{MaximalSimplices}_{\mathcal{R}(G)}(S - \{a, b\}; \sigma) \)
12. \( \mathcal{MS} := \mathcal{MS}_{(i)} \cup \mathcal{MS}_{(ii)} \cup \mathcal{MS}_{(iii)} \)
13. return \( \mathcal{MS} \).

Factoring in the computational overhead for the search of the pair \( \{a, b\} \) and the computation in computing set intersections (which is an \( O(k \log k) \) operation for sets of size \( k \) maintained using a heap), the complexity of this algorithm is \( O(|S|^3 + |S|^2 \log |S|) \sim O(|S|^3) \).

### 2.2 Distance of a Vertex Through a Simplex

**Definition 4** (A Pointed Simplex). A pointed simplex is a simplex, \( \sigma \), with a preferred vertex, \( u \in \sigma \), called the apex of the simplex.

Without loss of generality, we refer to the vertices of a \((n-1)\)-simplex, \( \sigma \), as \( v_0, v_1, v_2, \ldots, v_{n-1} \), with \( v_0 \) being the apex whenever \( \sigma \) is pointed, and arbitrarily chosen ordering for \( v_1, v_2, \ldots, v_{n-1} \).

**Definition 5** (A Metric Simplex). A metric \((n-1)\)-simplex is an \((n-1)\)-simplex, \( \sigma \), with a metric defined on the set \( \sigma, d: \sigma \times \sigma \to \mathbb{R}_{\geq 0} \), satisfying all the axioms of a metric.

With \( \sigma = \{v_0, v_1, v_2, \ldots, v_{n-1}\} \), for brevity we will write \( d(v_i, v_j) = d_{i,j} = d_{j,i} \) for all \( v_i, v_j \in \sigma \). Thus, a metric \((n-1)\)-simplex is defined by the pair \((\sigma, d)\).

**Definition 6** (An Euclidean Realizable Metric Simplex). A metric \((n-1)\)-simplex, \((\sigma, d)\), is called Euclidean realizable if its constituent vertices can be isometrically embedded in an Euclidean space (i.e., the Euclidean distance between the embedded vertices are equal to the distances between the vertices in the metric simplex).

**Proposition 1** (Canonical Euclidean Realization of a Metric Simplex). There is an unique embedding of a Euclidean realizable metric \((n-1)\)-simplex, \((\sigma = \{v_0, v_1, \ldots, v_{n-1}\}, d)\), given by \( e: \sigma \to \mathbb{R}^{n-1} \) such that

1. The embedded point for the \( j \)-th vertex has non-zero value for the first \( j \) coordinates, with the \( j \)-th coordinate being non-negative, and zero for the rest. That is, \( v_j := e(v_j) = [v_{j,0}, v_{j,1}, \ldots, v_{j,j-1}, 0, \ldots, 0] \), \( v_{j,j-1} \geq 0 \), \( j = 0, 1, 2, \ldots, n-1 \).
2. \( \|v_i - v_j\| = d(v_i, v_j) = d_{i,j} \).

Explicitly, the embedding can be written using the following recursive formula:

\[
v_{j,k} = \begin{cases} 
    d_{j,0}^2 - d_{j,k+1}^2 + v_{k+1,k}^2 + \sum_{p=0}^{k-1} (v_{k+1,p}^2 - 2v_{j,p}v_{k+1,p}) / 2v_{k+1,k}, & \text{for } k < j - 1, \\
    \sqrt{d_{j,0}^2 - \sum_{p=0}^{k-2} v_{j,p}^2}, & \text{for } k = j - 1, \\
    0, & \text{for } k \geq j.
\end{cases}
\]
where, $\sum_{p=0}^{\beta} h(p) = 0$ whenever $\beta < \alpha$. Using (11), the computation of $v_{1,0}$, $v_{2,0}, v_{2,1}, v_{3,0}, v_{3,1}, v_{3,2}, \cdots, v_{j,0}, v_{j,1}, \cdots, v_{j,j-1}, v_{j+1,0}, \cdots, \cdots$ can be made in an incremental manner, with the computation of a term in this sequence requiring only the previous terms.

The proof of this proposition is constructive, and the construction appears in Appendix A.1. An illustration of this embedding of a simple 2-simplex is shown in Figures 3(d) and 6.

**Definition 7** (Canonical Euclidean Realization of Metric Simplex). The map $e$ described in Proposition 1 is referred to as the canonical Euclidean realization of the metric simplex $(\sigma, d)$, and will be referred to as $E_t(\sigma): \sigma \rightarrow \mathbb{R}^{n-1}, v_i \mapsto v_i$.

**Corollary 1** (Embedding Dimension for an Euclidean Realizable Metric Simplex). A metric $(n-1)$-simplex, $(\sigma, d)$, is Euclidean realizable iff its constituent vertices can be isometrically embedded in $\mathbb{R}^{n-1}$.

The proof of the above corollary follows using rigidity argument and dimension analysis.

### 2.2.1 Spherical Extrapolation for Computing Unrestricted $d$-distance of Apex:

**Proposition 2.** Given the canonical Euclidean realization, $E_t(\sigma) = e: v_j \mapsto v_j = [v_{j,0}, v_{j,1}, \cdots, v_{j,j-1}, 0, \cdots, 0]$, of a pointed Euclidean realizable metric simplex, $(\sigma, d)$ (with apex $v_0$), and given a map $\mathcal{d}: \{v_1, v_2, \cdots, v_{n-1}\} \rightarrow \mathbb{R}^+$, one can compute a point $o = [o_0, o_1, \cdots, o_{n-2}] \in \mathbb{R}^{n-1}$, using the following formula:

$$
o_0 = \frac{-V + \sqrt{V^2 - 4UW}}{2U}, \quad o_k = A_k o_0 + B_k, \quad k = 1, 2, \cdots, n-2
$$

where,

$$
A_k = \frac{v_{1,0} - v_{2,0}}{v_{2,1}} - \sum_{p=1}^{k-1} v_{k+1,p} A_p, \quad B_k = \frac{1}{2v_{2,1}} \left( v_{2,0}^2 + v_{2,1}^2 - v_{1,0}^2 - \bar{d}_1^2 \right)
$$

$$
U = 1 + \sum_{p=1}^{n-2} A_p^2, \quad V = 2 \left( -v_{1,0} + \sum_{p=1}^{n-2} A_p B_p \right), \quad W = v_{1,0}^2 - \bar{d}_1^2 + \sum_{p=1}^{n-2} B_p^2
$$

$\bar{d}_j := d(v_j)$

A real solution to (2) exists iff a point $o$ exists in the same Euclidean space as the embedded metric simplex satisfying $||o - v_j|| = \bar{d}_j$. In that case $v_0$ and $o$ are points lying on or on the opposite sides of the hyperplane containing $v_1, v_2, \cdots, v_{n-1}$.

The proof, once again, is constructive, and appears in Appendix A.2.

Given an Euclidean realizable metric simplex, $(\sigma, d)$, with apex $v_0$, we can construct its canonical Euclidean realization, $e := E_t(\sigma)$, using equations (11). Also, given the map $\mathcal{d}$, we can compute the coordinate of $o$

**Definition 8** (Unrestricted $d$-distance of Apex). Given the canonical Euclidean realization, $E_t(\sigma) = e: v_j \mapsto v_j$, of a pointed Euclidean realizable metric $(n-1)$-simplex, $(\sigma, d)$, with apex $v_0$, we compute the point $o \in \mathbb{R}^{n-1}$ satisfying the given distances $||o - e(v_j)|| = \bar{d}_j, \quad j = 1, 2, \cdots, n-1$ using equations (2). We thus define the unrestricted $d$-distance of $v_0$ to be $\bar{D}^{ph}_{(d,\mathcal{d})}(\sigma, v_0) = ||o - e(v_0)||$.

$\bar{D}^{ph}_{(d,\mathcal{d})}(\sigma, v_0)$ is the length of the line segment connecting $v_0$ and $o$ in the Euclidean realization (Figure 6). This line, $\overset{\leftrightarrow}{v_0 o}$, intersects the hyperplane, $H_0$, containing $v_1, v_2, \cdots, v_{n-1}$ at a general point that can be written as $\sum_{i=1}^{n-1} w_i v_i$, where $\sum_{i=1}^{n-1} w_i = 1$. The following is a simple geometric consequence, and a derivation appears in Appendix A.3.

**Proposition 3.** The point at which the line connecting $v_0$ and $o$ intersects the hyperplane $H_0$ is given by $i_0 = \sum_{i=1}^{n-1} w_i v_i$, with

$$
w_k = \frac{w^t_k}{\sum_{i=1}^{n-1} w_i}, \quad k = 1, 2, \cdots, n-1.
$$
Figure 6: Spherical Extrapolation: Euclidean realization of a metric simplex \( \sigma \) of unrestricted \( \overline{d} \)-distance for (b) is computed by computing the unrestricted \( \overline{d} \)-distance through the face opposite to \( v_1 \). This is computed by a completely separate Euclidean realization of the metric simplex \( (v_0, v_2, v_3, d) \).

where, \( w'_j \) can be computed recursively using the formula \( w'_j = \frac{d_{j-1} - \sum_{i=j+1}^{n-1} w'_{i,j-1}}{w'_{j,j-1}} \). Note that the terms in the sequence \( w'_{n-1}, w'_{n-2}, \ldots, w'_1 \) can be computed in an incremental manner.

If all the \( w_j \), \( j = 1, 2, \ldots, n-1 \) are non-negative, then the line intersects the hyperplane inside (on the boundary of) the Euclidean realization of the face of the simplex containing \( v_1, v_2, \ldots, v_{n-1} \). Otherwise it intersects outside.

Definition 9 (Intersection Point in Spherical Extrapolation). For the weights computed using equation (28), we introduce the map \( \overline{W}_{(d, \overline{d})}^{ph}(\sigma, v_0) : \sigma - \{v_0\} \rightarrow \mathbb{R} \), \( v_i \mapsto w_i, i = 1, 2, \ldots, n-1 \).

The above method of computing unrestricted \( \overline{d} \)-distance of apex and the weights, \( w_j \), relies on the construction of the point \( o \), and identifying the \( \overline{d} \)-distances as the distances from that point. That’s precisely the reason that we refer to this method of computation “spherical”. In the following sub-section we introduce an alternative to this computation.

2.2.2 Linear Extrapolation for Computing Unrestricted \( \overline{d} \)-distance of Apex: Instead of computing a point, \( o \in \mathbb{R}^{n-1} \), from which the distances of the points \( v_j \) are \( \overline{d}_j \), one can compute a \( (n-2) \)-dimensional hyperplane, \( I \), from which the distances of the points \( v_j \) are \( \overline{d}_j \). This gives us an alternative way of computing the \( \overline{d} \)-distance of the apex, \( v_0 \) (Figure 7). The following proposition summarizes the computation, the proof of which appears in Appendix A.4.

Proposition 4. Given the canonical Euclidean realization, \( \mathcal{E}_d(\sigma) = e : v_j \mapsto v_j = [v_{j,0}, v_{j,1}, \ldots, v_{j,j-1}, 0, \ldots, 0] \), of a pointed Euclidean realizable metric simplex, \( (\sigma, d) \) (with apex \( v_0 \)), and given a map \( \overline{d} : \{v_1, v_2, \ldots, v_{n-1}\} \rightarrow \mathbb{R}_+ \), one can compute a hyperplane, \( I \), described by the equation \( u \cdot x + \mu = 0 \), where, \( x \in \mathbb{R}^{n-1} \) is a point on the hyperplane, \( u = [u_0, u_1, \cdots, u_{n-2}] \in \mathbb{R}^{n-1} \) is an unit vector orthogonal to the plane, and \( \mu \) is a constant using the following formulae:

\[
\mu = -\frac{Q + \sqrt{Q^2 - 4PR}}{2P},
\]

\[
u_k = M_k \mu + N_k, \quad k = 0, 1, 2, \ldots, n-2
\]
we introduce the map \( \overline{d}_{(d,\overline{d})}(\sigma, v_0) \geq 0 \).

**Definition 10** (Unrestricted \( \overline{d} \)-distance of Apex). Given the canonical Euclidean realization, \( \mathcal{E}_d(\sigma) = e : v_j \mapsto v_j \), of a pointed Euclidean realizable metric \((n - 1)\)-simplex, \((\sigma, d)\), with apex \( v_0 \), we compute the plane, \( I \), described by the equation \( u \cdot x + \mu = 0 \) (with \( u \) an unit vector) satisfying \( u \cdot v_j + \mu = \overline{d}_j \), \( j = 1, 2, \cdots, n - 1 \), using equations (4). We thus define the unrestricted \( \overline{d} \)-distance of \( v_0 \) to be \( \overline{d}_{(d,\overline{d})}(\sigma, v_0) = u \cdot v_0 + \mu \).

**Proposition 5.** The point at which the perpendicular dropped from \( v_0 \) to the plane, \( I \), intersects the hyperplane \( H_0 \) is given by \( \mathbf{h}_0 = \sum_{i=1}^{n-1} w_i \mathbf{v}_i \), with

\[
\sum_{i=1}^{n-1} w_i \mathbf{v}_i = w_k \mathbf{v}_k, \quad k = 1, 2, \cdots, n-1.
\]

where, \( w_j \) can be computed recursively using the formula \( w_j = \frac{u_j - \sum_{i=1}^{n-1} w_i/v_i}{v_j} \). Note that the terms in the sequence \( w_0, w_1, \cdots, w_0 \) can be computed in an incremental manner.

If all the \( w_j \), \( j = 1, 2, \cdots, n-1 \) are non-negative, then the line intersects the hyperplane inside (or on the boundary of) the Euclidean realization of the face of the simplex containing \( v_1, v_2, \cdots, v_{n-1} \). Otherwise it intersects outside.

**Definition 11** (Intersection Point in Linear Extrapolation). For the weights computed using equation (5), we introduce the map \( \overline{d}_{(d,\overline{d})}(\sigma, v_0) : \sigma - \{v_0\} \to \mathbb{R}, \mathbf{v}_i \mapsto w_i, \quad i = 1, 2, \cdots, n-1 \).

A feature of the linear method is that the \( \overline{d} \)-distances being distances from a plane, there is no notion of triangle inequality or other properties of a metric that can be defined on \( \overline{d} \). This may or may not
be desirable under different situations. Given a fine-enough discretization, both the spherical and linear approaches should approximate the underlying metric relatively well. In our implementation we use only the spherical extrapolation.

2.3 Algorithm for Computing $d$-distance Through Simplex

As discussed, if some of the weights given by $\overrightarrow{W}(d,\mathcal{D})(\sigma,v_0)$ are negative, then the line of the shortest length from the apex to $o$ (in spherical extrapolation) or $l$ (in linear extrapolation) will not pass through the face simplex constituting of $v_1,v_2,\ldots,v_{n-1}$. In that case we need to compute the length of the shortest path that intersects $H_0$ inside the simplex constituting of $v_1,v_2,\ldots,v_{n-1}$. Such a path, clearly, will pass through one of the faces opposite to $v_1,v_2,\ldots$ or $v_{n-1}$ and is the $d$-distance of $v_0$ restricted to the face (see Figure 6(b) or 7(b)). Thus, our algorithm for computing $d$-distance through the simplex is one that computes and compares the $d$-distance of $v_0$ restricted to the faces until a set of non-negative weights are found. The pseudocode is given below.

**Algorithm 3: Compute Distance Through Simplex**

\[
(d',(\sigma',\overrightarrow{w}')) = DistanceThroughSimplex(d,\mathcal{D})(\sigma,u)
\]

**Inputs:**
- a. The metric simplex, $\sigma$, with distance between vertices $d : \sigma \times \sigma \rightarrow \mathbb{R}_+$.
- b. Apex, $u \in \sigma$. (Thus, letting $\sigma = \{v_0:=u,v_1,v_2,\ldots,v_{n-1}\}$.)
- c. $d$-values, $\mathcal{D} : \sigma - \{u\} \rightarrow \mathbb{R}_+$.

**Outputs:**
- a. $d$-distance to $u$ through simplex $\sigma$.
- b. Came-from point, $(\sigma',\overrightarrow{w}')$, on a subsimplex $\sigma' \subseteq \sigma - \{u\}$.

1. If $(\sigma,u)$ exists in lookup table with the specified $d$-values for $\sigma - \{u\}$, return its distance-through-simplex value and came-from point.
2. \[\overrightarrow{w} := \overrightarrow{W}_{(d,\mathcal{D})}(\sigma,u)\] // weights associated with $v_i,i=1,2,\ldots,n-1$
3. if $\overrightarrow{w}(v_i) \geq 0, \forall i = 1,2,\ldots,n-1$ // all weights are non-negative
4. \[d' := \overrightarrow{D}_{(d,\mathcal{D})}(\sigma,u),\]
5. $\sigma' := \{v \in \sigma - \{u\} | \overrightarrow{w}(u) > 0\}$ // simplex constituting of vertices with non-zero weights
6. $\overrightarrow{w}'(v) := \overrightarrow{w}(v), \forall v \in \sigma'$ // weights describing came-from point in simplex $\sigma'$
7. else
8. \[d' := \infty\]
9. foreach $(i \in \{1,2,\ldots,n-1\})$
10. if $\overrightarrow{w}(v_i) < 0$ // Need to check $d$-value through face.
11. \[(d'',(\sigma'',\overrightarrow{w}'')) = DistanceThroughSimplex(d,\mathcal{D})(\sigma - v_i, u)\] // distance through face opposite to $v_i$.
12. Insert an entry for $(\sigma - v_i, u)$ with the specified $d$-values for $\sigma - \{v_i,u\}$ into lookup table,
13. with $d''$ its distance-through-simplex value and $(\sigma'',\overrightarrow{w}'')$ its came-from point.
14. if $d'' < d'$$d' := d''$
15. \[(\sigma',\overrightarrow{w}') := (\sigma'',\overrightarrow{w}'')\] // came-from point.
16. return $(d',(\sigma',\overrightarrow{w}'))$

where, the “*” in $\overrightarrow{W}$ and $\overrightarrow{D}$ refers to either “sph” or “lin” depending on the chosen extrapolation method.

Note that the procedure also return the pair $(\sigma',\overrightarrow{w})$, where $\sigma'$ is a simplex (consisting of vertices with non-zero weights) and $\overrightarrow{w} : \sigma' \rightarrow (0,1]$ is a map that associates weights to the vertices in the simplex. These two pieces of information, $(\sigma',\overrightarrow{w})$, together describes a point inside the simplex $\sigma'$ as the convex combination $\sum_{v \in \sigma'} \overrightarrow{w}(v) \cdot v$.

During the recursive calls to $DistanceThroughSimplex$, it is possible that the procedure is called on the same simplex multiple times (by different faces). In order to avoid re-computation of the $d$-value through the same simplex multiple times, we maintain a lookup table of the simplices (as a hash table maintained globally across all calls to $DistanceThroughSimplex$), and return the $DistanceThroughSimplex$ value if it exists. The entries in the table (the hash key) are distinguished by the simplex vertices (unordered), the simplex’s apex as well as the $d$-value of the other vertices.
3 Path Reconstruction

As described, abstract point inside a simplicial complex can be described by two pieces of information: i. a $m$-simplex, $\rho = \{v_0, v_1, \ldots, v_m\} \in C_*$, inside (or on the boundary of) which the point lies, and ii. a set of positive weights associated with each vertex such that they add up to 1 (we represent the weights by the map $\overline{w}: \sigma \rightarrow \mathbb{R}_{\geq 0}$, with $\sum_{i\in \rho} \overline{w}(v_i) = \sum_{i=0}^{m} \overline{w}(v_i) = 1$). The point itself would then be the abstract convex combination $\sum_{i\in \rho} \overline{w}(v_i) = \sum_{i=0}^{m} \overline{w}(v_i) v_i$. The pair $p = (\sigma, \overline{w})$ described a point, and a (piece-wise linear) path in a simplicial complex can thus be described as a sequence of points $p_i = (\sigma_i, \overline{w}_i), i = 0, 1, 2, \ldots$

The primary output of the Basic $S^*$ algorithm (Algorithm 1) is a $d$-value for every vertex in $G$. In order to find the shortest path connecting $s \in V$ with some arbitrary $g \in V$ (which has been expanded), like any search algorithm, we need to reconstruct a path. The basic reconstruction algorithm is as follows:

**Algorithm 4: Reconstruct Path**

\[
\{p_i\}_{i=0,1,2,\ldots} = \text{ReconstructPath}((R(G), d, \overline{w}))(g)
\]

**Inputs:**
- a. The Rips complex of graph, $G = (V, E)$.
- b. Length/cost of the edges, $d: V \rightarrow \mathbb{R}_{+}$.
- c. The $d$-values of the vertices (obtained as output of $S^*$ algorithm).
- d. The vertex, $g$, to find the shortest path from.

**Outputs:**
- a. A sequences of points, $p_i = (\sigma_i, \overline{w}_i)$ for $i = 0, 1, 2, \ldots$.

1. $\overline{w}_0(g) := 1$ // weight map, $\overline{w}: \{g\} \rightarrow [0, 1]$, $g \rightarrow 1$
2. $p_0 := (\{g\}, \overline{w}_0)$ // 0-simplex consisting only of $g$, with a weight of 1 associated with it.
3. $\mu := \emptyset$
4. $i := 0$
5. **while** ($p_i \neq (\{s\}, \overline{w}: \{s\} \rightarrow (0, 1], s \rightarrow 1)$) 
6. \(p_{i+1}, \mu') := \text{Compute Came From Point}(R(G), d, \overline{w})(p_i, \mu)
7. $\mu \leftarrow \mu'$
8. $i \leftarrow i + 1$
9. **return** \(\{p_i\}_{i=0,1,2,\ldots}\)

The procedure $\text{Compute Came From Point}$ takes in a point, $p_i = (\sigma_i, \overline{w}_i)$, on a simplex $\sigma_i$, and returns another point, $p_{i+1} = (\sigma_{i+1}, \overline{w}_{i+1})$ on a different simplex that it came from (Figure 8). As a result, the relationship between the two simplices, $\sigma_i$ and $\sigma_{i+1}$ is that they both are sub-simplices of a same maximal simplex. Thus, given $p_i$, in order to compute $p_{i+1}$, we construct all the maximal simplices attached to $\sigma_i$ (Line 4 of Algorithm 5), interpret the point $p_i$ as a vertex in each of the maximal simplices (Lines 8, 9), and

![Figure 8: Came-from point.](image_url)
construct pointed metric simplices using faces of each of them and $p_i$ as apex. The complete pseudocode is as follows:

**Algorithm 5: Compute Came-From Point in Path Reconstruction**

\[ ((\sigma', \overline{w}'), \mu') = \text{ComputeCameFromPoint}(R(G), s, d, \overline{w}) ((\sigma, \overline{w}), \mu) \]

**Inputs:**
- a. The Rips complex of graph, $G = (V, E)$.
- b. Length/cost of the edges, $d : V \rightarrow \mathbb{R}_+$.
- c. The $\overline{d}$-values of the vertices (obtained as output of S* algorithm).
- d. A point, $p = (\sigma, \overline{w})$, in the complex.
- e. Maximal simplex, $\mu$, to be ignored.

**Outputs:**
- a. A point, $p' = (\sigma', \overline{w}')$, in the complex.
- b. The maximal simplex, $\mu'$, through which the line $\overline{p_i}p_{i+1}$ passes.

```plaintext
1. $d^{ex} := d$ // extended distance map.
2. $S := N(\sigma)$ // common neighbors of all the vertices in $\sigma$.
3. $d' := \infty$ // distance to $p$.
4. $\mathcal{MS} = \text{MaximalSimplices}_{R(G)}(S; \sigma)$ // maximal simplices attached to $\sigma$
5. foreach ($\rho \in \mathcal{MS} - \{\mu\}$) // all attached maximal simplices, except $\mu$.
6. $\overline{w}^{ex} := \overline{w}$ ; $\overline{w}^{ex}(\nu) := 0$, $\forall \nu \in \rho - \sigma$ // extend domain of $\overline{w}$ to the simplex $\rho$.
7. $e := \overline{E}_d(\rho)$ // canonical Euclidean embedding of $\rho$.
8. $p := \sum_{v \in \rho} \overline{w}^{ex}(v) e(v)$ // coordinates of $p = (\sigma, \overline{w})$ in canonical embedding of $\rho$.
9. $d^{ex}(v, p) = d^{ex}(p, v) := \|e(v) - p\|$, $\forall v \in \rho$ // extend domain of distance/cost function to include $\{v, p\}$.
10. foreach ($u \in \rho$).
11. $\gamma := \rho - \{u\}$ // face of $\rho$ opposite to $u$.
12. if $(\gamma \notin \mu)$ // ignore any subsimplex of $\mu$.
13. $(d'', (\sigma'', \overline{w}'')) := \text{DistanceThroughSimplex}(d^{ex}, \overline{w}^{ex})(\gamma \cup \{p\}, p)$ // $d'$-distance of $p$ through simplex $\gamma \cup \{p\}$ with $p$ as apex.
14. if $(d'' < d')$
15. $d' \leftarrow d''$
16. $(\sigma', \overline{w}') \leftarrow (\sigma'', \overline{w}'')$
17. $\mu' \leftarrow \rho$
18. return $((\sigma', \overline{w}'), \mu')$
```

### 4 Analysis

#### 4.1 Correctness

We construct maximal simplices attached to every edge, $\{u, q\}$, in Line 12 of the Basic S* algorithm (Algorithm 1) and use it for computing and testing for update of the $d'$-value of a vertex $u$ that is a neighbor to the expanding vertex, $q$. Since the edge itself is a sub-1-simplex of each of those maximal simplices, the distance, $d'$, through the maximal simplex (Line 14) cannot be greater than the distance obtained by simply adding $d(\{u, q\})$ to $d(q)$ (consequence of triangle inequality). As a result we have the following proposition.

**Proposition 6** (\(d\)-value Bounded Above by Graph Search Value). The Basic S* algorithm (Algorithm 1) cannot return a higher value for the distance to any vertex ($d$-values) than the Dijkstra’s search algorithm.

Furthermore, a simplicial complex with constant metric inside each simplex gives a piece-wise linear approximation of any Riemannian manifold [Jos08]. The canonical Euclidean embedding of the simplices indeed induces such constant metrics on the simplices (any constant Riemannian metric can be converted to Euclidean metric through appropriate scalings [Pet03]). Thus, the proposed method is appropriate for computing shortest paths in discrete representations of Riemannian manifolds. As a direct consequence, we have the following proposition. A more formal proof is under the scope of future work.

**Proposition 7** (\(d\)-value Approaches Path Metric on Riemannian Manifold). Suppose $M$ is a Riemannian manifold (possibly with boundaries), and $R(G)$ is a simplicial approximation of the manifold (with the cost
of the edges in G set to their lengths on the manifold). Then the cost/distance between two points on the manifold computed using the Basic S* Algorithm converges to the actual distance between the same points on the manifold as the size of the simplices (lengths/costs of edges in G) approach zero.

4.2 Complexity

The Basic S* algorithm (Algorithm 1) has an overall structure very similar to Dijkstra’s algorithm. If the graph, G, has |V| counts of vertices, and if all of those are expanded, then the main while block of the algorithm (starting at Line 5) will loop for |V| times. Inside each loop, the following processes happen:

i. The vertex with lowest \( \bar{d} \)-value is extracted from set Q (Line 6). The size of Q (open list) is of the order of a constant power of the size of V, and since Q is maintained in a heap data structure, the complexity of this step is \( O(\log |Q|) \sim O(\log |V|^4) \sim O(\log |V|) \).

ii. We loop through each neighbor of each vertex to check for updates (Line 10). Assuming average degree of each vertex is \( D \), this loops for \( O(D) \) times. Inside each of these sub-loops, the following computations happen:

   a. The algorithm for computing the set of maximal simplices attached to each edge (Line 12), as discussed in Section 2.1, is of complexity \( O(|S|^4) \sim O(D^3) \) (where \( S \) is the set of neighbors of an edge).

   b. The duality between maximal simplices and the vertices tells us that the average number of maximal simplices attached to each edge is also \( D \). Thus, the innermost loop (starting at Line 13 that includes the “DistanceThroughSimplex” call) loops for \( O(D) \) times.

Thus, the overall complexity of the algorithm is \( O(|V| (\log(|V|) + D^2)) \sim O(|V| (\log(|V|) + D^4)) \).

4.2.1 Complexity as Size of Configuration Space Increases Keeping Dimension Constant: If the average degree of the vertices is finite and constant (for a simplicial discretization of a finite-dimensional configuration space that indeed is the case), and if \( |V| \to \infty \), then the complexity is simply \( O(|V| \log |V|) \).

4.2.2 Complexity as Dimension of Configuration Space Increases Keeping Diameter Constant: If \( N \) is the dimension of the configuration space, and the diameter of the configuration space is held constant, then as \( N \to \infty \) we have \(|V| \to O(e^N), D \to O(e^N)\). Thus, the complexity of the algorithm as \( N \to \infty \) is \( O(e^N (\log(e^N) + e^{4N})) \sim O(e^N) \).

5 Results

Simple Demonstration: As a very simple demonstration, we present a comparison between the progress of search in a graph constructed out of an uniform triangulation (using equilateral triangles) of an Euclidean plane with a single obstacle (Figure 9).

Shortest Path on a 2-sphere: We next present the result of computing the shortest path (geodesic) on the surface of an unit sphere. We use the usual spherical coordinates, \( (\phi, \theta) \), where \( \phi \in [0, \pi] \) is the latitudinal angle measured from the positive Z axis, and \( \theta \in [0, 2\pi] \) is the longitudinal angle measured from the positive X axis (Figure 10(a)).

The matrix representation of the Riemannian metric tensor [Jos08] in this coordinate system is \( g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 \phi \end{bmatrix} \) (the round metric). Thus, an infinitesimal segment at \( (\phi, \theta) \) of extent \( \Delta \phi \) along the \( \phi \) direction and \( \Delta \theta \) along the \( \theta \) direction will be of length/cost \( \Delta l = \sqrt{\Delta \phi^2 + \sin^2 \phi \Delta \theta^2} \). In particular, using a discrete graph representation (Figure 10(b)) of the coordinate chart, if two neighboring vertices, \( v_1 \) and \( v_2 \), have spherical coordinates \( (\phi_1, \theta_1) \) and \( (\phi_2, \theta_2) \) respectively, we compute the cost/length of the edge connecting those vertices as \( d(v_1, v_2) = \sqrt{(\phi_2 - \phi_1)^2 + \sin^2 \left( \frac{\phi_1 + \phi_2}{2} \right) (\theta_2 - \theta_1)^2} \).
We construct the graph, $G$, out of a vertex set that has vertices placed on an uniform square lattice, with the separation of the neighboring vertices in the $\phi$ or $\theta$ direction being equal to $\pi/f$, where $f$ is the “fineness” of the discretization (larger the value of $f$, finer is the discretization). Figure 10(b) shows such a discretization of the chart with $f = 8$. Comparison of paths computed using Dijkstra’s search on this graph and the Basic S* search are shown in Figure 10. As we increase the fineness value, $f$, it is to be noted that the graphs for lower fineness are not subsets of the graphs of higher fineness. Thus, interestingly, the cost of the paths computed using Dijkstra’s search increase with fineness (Figure 10(d)), but the cost of the paths computed using Basic S* decreases and approaches the geodesic path (great circle) on the sphere.

6 Conclusion

We presented the Basic S* algorithm for computing optimal path through simplical complexes, in particular, Rips complexes of graphs constructed as discrete representation of an arbitrarily configuration space. The currently proposed algorithm is the basic version of it, with a structure similar to Dijkstra’s. However, incorporating heuristic functions, any-time/incremental computations and randomized searches is possible within this framework. The presented results illustrate the effectiveness of the algorithm in computing paths that more closely represent geodesic paths in some 2-dimensional configuration spaces. In future, results in 3-dimensional configuration spaces and comparisons with other “any-angle” search algorithms will be provided. Also, more formal proofs of the algorithmic correctness propositions will be given.
Figure 9: Comparison between Dijkstra’s search and Basic S* search using a graph constructed out of an uniform triangulation of an Euclidean plane with a single obstacle. Expanded vertices colored in red and green, while vertices in open list are in blue. 

Top row: Progress of Dijkstra’s search in (a)-(c). Second row: Progress of Basic S* search in (e)-(g). Color of the vertices, (i), indicates the relative error between the distance to a vertex as computed by the search algorithm and the true length of shortest path (geodesic distance) to the point using the Euclidean metric on the plane (green: low error, red: high error). Note how the Basic S* search algorithm computes a distance that more closely represents the underlying path metric induced by the Euclidean metric.

(i) Vertex color legend for the relative error between the distance computed by search algorithm (figures (a)-(c): Dijkstra’s, figures (e)-(g): Basic S*) and the true length of shortest path computed using Euclidean metric.

(j) Close-up view of the path computed in (h) using Basic S* search. Blue dots show the points \( \{p_i\}_{i=0,1,...} \) returned by the \textit{ReconstructPath} procedure. Notice how they lie on the boundaries of the maximal 2-simplices.
(a) The spherical coordinates.

(b) The latitude-longitude, $(\phi, \theta)$, coordinate chart. A triangulation graph (light gray) on the coordinate chart, with fineness, $f = 8$, is shown. The background color indicates the determinant of the metric tensor, which corresponds to cost of edges in a particular region (higher: red, lower: green).

(c), (b): Optimal paths computed on the surface of a unit sphere using Dijkstra's/A* search (red) and Basic S* search (blue). For reference, the figure also shows the geodesic path (green) and the path corresponding to a straight line segment on the $(\phi, \theta)$ coordinate chart (magenta). (b) shows the paths on the $(\phi, \theta)$ coordinate chart computed for a discretization of fineness $f = 8$. (c) shows the paths on the surface of a sphere for various values of fineness, $f = 8, 16, 24, \ldots, 80$.

(d) The total cost/length of paths computed using Dijkstra’s (red) and Basic S* (blue) using different fineness values for the discretization (data points at $f = 8, 16, 24, \ldots, 80$). The cost of two reference paths (geodesic on sphere: green, straight line segment on coordinate chart: magenta) are also shown.

Figure 10: Computation of optimal path connecting two points on a 2-sphere.
Appendix

A Euclidean Realization of Metric Simplices

Given a metric \((n-1)-simplex, (\sigma, d)\), one can construct an isometric embedding of the simplex in the Euclidean \((n-1)\)-space, \(e: \sigma \to \mathbb{R}^{n-1}\) such that \(\|e(v_i) - e(v_j)\| = d_{ij}\) and \(v_i := e(v_i)\) has all its last \(n-1-i\) coordinates set to zero. The explicit construction can be described as follows:

Suppose,

\[
\begin{align*}
  v_0 &= [0, 0, \ldots, 0] \\
  v_1 &= [v_{1,0}, 0, \ldots, 0] \\
  v_2 &= [v_{2,0}, v_{2,1}, \ldots, 0] \\
  \vdots \\
  v_j &= [v_{j,0}, v_{j,1}, \ldots, v_{j,j-1}, 0, \ldots, 0] \\
  \vdots \\
  v_{n-1} &= [v_{n-1,0}, v_{n-1,1}, \ldots, v_{n-1,n-1}, v_{n-1,n-2}] 
\end{align*}
\]

From the above, one gets for \(0 \leq l < j \leq n-1\),

\[
\begin{align*}
  d^2_{j,l} &= \|v_j - v_l\|^2 = \begin{cases} 
    \sum_{p=0}^{j-l-1} v_{j,p}^2, & \text{when } l = 0 \\
    \sum_{p=0}^{j-l} (v_{j,p} - v_{l,p})^2 + \sum_{p=l}^{j-1} v_{j,p}^2, & \text{when } 0 < l < j \leq n-1 
  \end{cases} 
\end{align*}
\]

A.1 Recursive Formula for Computing \(v_{j,k}\)

Using (6) we compute \(v_{j,k}\) for \(0 \leq k < j\), \(0 < j\) as follows:

For \(k = 0\), \(j = 1\) clearly,

\[
v_{1,0} = \pm d_{1,0}
\]

For \(k = 0\), \(1 < j \leq n-1\),

\[
\begin{align*}
  d^2_{j,1} - d^2_{j,0} &= (v_{j,0} - v_{1,0})^2 + \sum_{p=1}^{j-1} v_{j,p}^2 - \sum_{p=0}^{j-1} v_{j,p}^2 \\
  &= v_{1,0}^2 - 2v_{j,0}v_{1,0} \\
  \Rightarrow v_{j,0} &= (d^2_{j,0} - d^2_{j,1} + v_{1,0}^2) / 2v_{1,0}
\end{align*}
\]

For \(0 < k < j - 1\), \(1 < j \leq n-1\),

\[
\begin{align*}
  d^2_{j,k+1} - d^2_{j,0} &= \sum_{p=0}^{k} (v_{j,p} - v_{k+1,p})^2 + \sum_{p=k+1}^{j-1} v_{j,p}^2 - \sum_{p=0}^{j-1} v_{j,p}^2 \\
  &= \sum_{p=0}^{k} (v_{k+1,p}^2 - 2v_{j,p}v_{k+1,p}) + (v_{k+1,k}^2 - 2v_{j,k}v_{k+1,k}) \\
  \Rightarrow v_{j,k} &= (d^2_{j,0} - d^2_{j,k+1} + v_{k+1,k}^2 + \sum_{p=0}^{k-1} (v_{k+1,p}^2 - 2v_{j,p}v_{k+1,p})) / 2v_{k+1,k}
\end{align*}
\]

For \(k = j - 1\), \(1 < j \leq n-1\),

\[
\begin{align*}
  d^2_{j,0} &= \sum_{p=0}^{j-1} v_{j,p}^2 \\
  \Rightarrow v_{j,j-1} &= \pm \sqrt{d^2_{j,0} - \sum_{p=0}^{j-2} v_{j,p}^2}
\end{align*}
\]

The important property of the formulae in equations (7), (8), (9) and (10) is that the computation of a term in the sequence \(v_{1,0}, v_{2,0}, v_{2,1}, v_{3,0}, v_{3,1}, v_{3,2}, \ldots, v_{j,0}, v_{j,1}, \ldots, v_{j,j-1}, v_{j+1,0}, \ldots, \) requires only the values of the terms appearing before it. Thus, one can compute these values incrementally starting with \(v_{1,0}\). Furthermore, inserting a new vertex (say the \((n+1)^{th}\) vertex, \(v_n\)) to a simplex requires us to compute only the new coordinates \(v_{n,0}, v_{n,1}, \ldots, v_{n,n-1}\) for the Euclidean realization of the new extended simplex.
With the understanding that \( \sum_{p=0}^{\beta} h(p) = 0 \) whenever \( \beta < \alpha \), equations (7)–(10) can be written more compactly to give the coordinates of the embedded vertices as,

\[
V_{j,k} = \begin{cases} 
\left( d_{j,0}^2 - d_{j,k+1}^2 + v_{k+1,k}^2 + \sum_{p=0}^{k-1} (v_{k+1,p}^2 - 2\nu_{p}v_{k+1,p}) \right) / 2\nu_{k+1,k}, & \text{for } k < j - 1, \\
\pm \sqrt{d_{j,0}^2 - \sum_{p=0}^{j-2} v_{j,p}^2}, & \text{for } k = j - 1, \\
0, & \text{for } k \geq j. 
\end{cases}
\] (11)

We choose the positive solution for coordinates \( V_{j,j-1} \). The computation of \( V_{1,0}, V_{2,0}, V_{2,1}, V_{3,0}, V_{3,1}, V_{3,2}, \ldots, V_{j,0}, V_{j,1}, \ldots, V_{j,j-1}, V_{j+1,0}, \ldots, \) can be made in an incremental manner, with the computation of a term in this sequence requiring only the previous terms.

**Lemma 3.** Equation (11) does not have a real solution iff the metric simplex, \((\sigma,d)\), is not Euclidean realizable.

**Lemma 4.** If \( V_{j,j-1} = 0 \) for some \( j \), and all the coordinates are real, then \((\sigma,d)\) has a degenerate Euclidean realization.

### A.2 Coordinate Computation for a Point with given Distances to \( \{V_j\}_{j=1,2,...,n-1} \)

Given a metric \((n-1)\)-simplex, \((\sigma,d)\), and the canonical Euclidean realization \( \mathcal{E}_\sigma(\sigma) = e : v_i \mapsto v_i \), we consider an additional point, \( o \), and its Euclidean realization in the same Euclidean space,

\[
o = [o_0, o_1, \ldots, o_{n-2}] \in \mathbb{R}^{n-1} \] (12)

Along with given its distances to all vertices in \( \sigma \), except \( v_0 \): \( d : \sigma - v_0 \rightarrow \mathbb{R}_+ \). For convenience, we write \( d(v_i) = d_i, i = 1, 2, \ldots, n - 1 \). Thus, for \( 0 < l \leq n - 1 \)

\[
d_l^2 = \sum_{p=0}^{l-1} (o_p - v_{l,p})^2 + \sum_{p=l}^{n-2} o_p^2 \] (13)

Thus, for \( l = 2, 3, \ldots, n - 1 \),

\[
d_l^2 - d_1^2 = \sum_{p=0}^{l-1} (o_p - v_{1,p})^2 + \sum_{p=1}^{n-2} o_p^2 - (o_0 - v_{1,0})^2 - \sum_{p=1}^{n-2} o_p^2 = -2o_0v_{1,0} + v_{1,0}^2 + 2o_0v_{1,0} - v_{1,0}^2 + \sum_{p=0}^{l-1} (-2o_pv_{1,p} + v_{1,p}^2) = 2(v_{1,0} - o_0) + 2(v_{1,0} - o_0) + \left( \sum_{p=0}^{l-1} v_{1,p}^2 - v_{1,0}^2 \right) \] (14)

\[
\Rightarrow \quad o_{l-1} = \frac{v_{1,0} - o_0}{v_{1,0} - o_0} o_0 - \sum_{p=1}^{l-2} w_{l-1,p} o_p + \frac{1}{w_{l-1}} \left( \sum_{p=0}^{l-1} v_{1,p}^2 - v_{1,0}^2 + d_1^2 - d_l^2 \right) \]

\[
\Rightarrow \quad o_k = \alpha_k o_0 + \sum_{p=1}^{k-1} \alpha_k p o_p + \beta_k \quad (\text{with } k = l - 1) \]

where, \( \alpha_k, o_k = \frac{v_{1,0} - v_{k+1,0}}{v_{k+1,0}}, p = 1, 2, \ldots, k-1 \) and \( \beta_k = \frac{1}{w_{k+1}} \left( \sum_{p=0}^{k} v_{k+1,p}^2 - v_{1,0}^2 + d_1^2 - d_k^2 \right) \).

Letting

\[
o_l = A_l o_0 + B_l \] (15)

we have,

\[
o_k = \alpha_k o_0 + \sum_{p=1}^{k-1} \alpha_k p o_p + \beta_k = \alpha_k o_0 + \sum_{p=1}^{k-1} \alpha_k o_p(A_p o_0 + B_p) + \beta_k = \left( \alpha_k o_0 + \sum_{p=1}^{k-1} \alpha_k o_p A_p \right) o_0 + \left( \beta_k + \sum_{p=1}^{k-1} \alpha_k p B_p \right) \] (16)

Thus, \( A_k \) and \( B_k \) can be determined iteratively as follows:

\[
A_k = \alpha_k o_0, \quad B_k = \beta_k + \sum_{p=1}^{k-1} \alpha_k p B_p \] (17)
We can now determine \( o_0 \) from the expression for \( \overline{d}_1^2 \):

\[
\overline{d}_1^2 = (o_0 - v_{1,0})^2 + \sum_{p=1}^{n-2} o_p^2 = (o_0^2 - 2v_{1,0}o_0 + v_{1,0}^2) + \sum_{p=1}^{n-2}(A_p^2o_0^2 + 2A_pB_po_0 + B_p^2) \\
= \left(1 + \sum_{p=1}^{n-2} A_p^2\right) o_0^2 + 2(-v_{1,0} + \sum_{p=1}^{n-2} A_pB_p) o_0 + \left(v_{1,0}^2 - \overline{d}_1^2 + \sum_{p=1}^{n-2} B_p^2\right) = 0
\]  \hspace{1cm} (18)

We need to choose the solution of \( o \) such that \( v_0 \) and \( o \) lies on the opposite sides of the hyperplane containing the points \( v_1, v_2, \ldots, v_{n-1} \). Since we have chosen \( v_{1,0} \) to be positive, and since \( v_0 \) is the origin, choosing the higher value of \( o_0 \) will satisfy this condition. Thus,

\[
o_0 = -V + \sqrt{V^2 - 4UW} \\
= \frac{-V + \sqrt{V^2 - 4UW}}{2U}
\]  \hspace{1cm} (19)

where, \( U = \left(1 + \sum_{p=1}^{n-2} A_p^2\right) \), \( V = 2(-v_{1,0} + \sum_{p=1}^{n-2} A_pB_p) \), \( W = \left(v_{1,0}^2 - \overline{d}_1^2 + \sum_{p=1}^{n-2} B_p^2\right) \).

**Lemma 5.** (19) does not have a real solution iff \( o \) is not embeddable in the same Euclidean space as an Euclidean realization of \((\sigma, d)\) satisfying the distance relations \( \overline{d} \).

### A.3 Intersection Point of Plane and Line Segment

A general point on the (affine) hyperplane, \( H_0 \), containing points \( v_1, v_2, \ldots, v_{n-1} \) can be written as \( \sum_{i=1}^{n-1} w_i v_i \), where \( \sum_{i=1}^{n-1} w_i = 1 \). A point on the line, \( L \), joining \( v_0 \) and \( o \) can be written as \( w_0v_0 + w_o o \) (since \( v_0 = [0, 0, \ldots, 0] \)), where \( w_0 + w_o = 1 \). Thus, the point of intersection of \( H_0 \) and \( L \) is

\[
i_0 = \sum_{i=1}^{n-1} w_i v_i = w_o o \\
\Rightarrow \sum_{i=1}^{n-1} w_i v_{i,j-1} = w_o o_{j-1}, \text{ for } j = 1, 2, \ldots, n - 1 \\
\Rightarrow \sum_{i=j}^{n-1} w_i v_{i,j-1} = w_o o_{j-1} \text{ (since } v_{i,j-1} = 0 \text{ for } i \leq j - 1) \\
\Rightarrow w_j = \frac{w_o o_{j-1} - \sum_{i=j+1}^{n-1} w_i v_{i,j-1}}{v_{j,j-1}} \\
\Rightarrow w_{j}' = \frac{o_{j-1} - \sum_{i=j+1}^{n-1} w_i' v_{i,j-1}}{v_{j,j-1}}, \text{ where, } w_{j}' = \frac{w_j}{w_o}
\]  \hspace{1cm} (20)

The above gives a recursive formula that lets us compute the terms in the sequence \( w_{n-1}' , w_{n-1}' , \ldots , w_1' \) in an incremental manner, with computation of each term requiring the knowledge of the previous terms in the sequence only.

Since \( \sum_{i=1}^{n-1} w_i = 1 \) and \( w_{j}' = \frac{w_j}{w_o} \), we have the following

\[
w_j = \frac{w_{j}'}{\sum_{i=1}^{n-1} w_i'}
\]  \hspace{1cm} (21)

**A.3.1 Sign of the Wights:** In general, let \( H_{k_1,k_2,\ldots,k_p} \) be the \((n-1-p)\)-dimensional hyperplane containing all the points in the set \( \{v_i\}_{i=0,1,\ldots,n-1} \setminus \{v_{k_j}\}_{j=1,2,\ldots,p} \) (i.e., the hyperplane of the subsimplex not containing \( v_{k_1}, v_{k_2}, \ldots, v_{k_p} \)).

**Lemma 6.**

\[
w_k \begin{cases} < 0 & \text{iff } i_0 \text{ and } v_k \text{ lie on the opposite sides of } H_{0,k} \text{ in } H_0 \\ = 0 & \text{iff } i_0 \text{ lies on } H_{0,k} \\ > 0 & \text{iff } i_0 \text{ and } v_k \text{ lie on the same sides of } H_{0,k} \text{ in } H_0 \end{cases}
\]  \hspace{1cm} (22)
A.4 Computation of Plane with Given Distances from \( \{v_j\}_{j=1,2,...,n-1} \)

Given a metric \((n - 1)\)-simplex, \((\sigma, d)\), and the canonical Euclidean realization \(E_d(\sigma) = e : v_i \mapsto v_i\), we consider a hyperplane, \(I\), described by the equation, \(u \cdot x + \mu = 0\), where, \(x \in \mathbb{R}^{n-1}\) is a point on the hyperplane, \(u = [u_0, u_1, \cdots, u_{n-2}] \in \mathbb{R}^{n-1}\) is an unit vector orthogonal to the plane, and \(\mu\) is a constant.

Distance of the point \(v_j\) from the plane is \(d_j\). Thus, for \(j = 1, 2, \cdots, n - 1\), we have,

\[
\begin{align*}
    \mathbf{u} \cdot \mathbf{v}_j + \mu &= d_j \\
    \Rightarrow \quad \sum_{p=0}^{j-1} u_p v_{j,p} + \mu &= d_j \\
    \Rightarrow \quad u_{j-1} &= \frac{d_j - \mu - \sum_{p=0}^{j-2} u_p v_{j,p}}{v_{j,j-1}} \\
    \Rightarrow \quad u_k &= \frac{d_{k+1} - \mu - \sum_{p=0}^{k-1} u_p v_{k+1,p}}{v_{k+1,k}}
\end{align*}
\]

Letting \(u_l = M_l \mu + N_l\) we have

\[
    u_k = \frac{d_{k+1} - \mu - \sum_{p=0}^{k-1} (M_p \mu + N_p) v_{k+1,p}}{v_{k+1,k}}
\]

Thus, we have the following recursive equation for \(M_k\) and \(N_k\),

\[
    M_k = -\left(1 + \frac{1}{v_{k+1,k}} \sum_{p=0}^{k-1} M_p v_{k+1,p}\right), \quad N_k = \left(d_{k+1} - \frac{1}{v_{k+1,k}} \sum_{p=0}^{k-1} N_p v_{k+1,p}\right)
\]

for \(k = 0, 1, 2, \cdots, n - 2\). With the understanding that \(\sum_{p=\alpha}^{\beta} h(p) = 0\) whenever \(\beta < \alpha\), we have \(M_0 = -1, N_0 = d_1\).

Since \(u\) is an unit vector, \(\sum_{j=0}^{n-1} u_j^2 = 1\). Thus, \(\left(\sum_{j=0}^{n-1} M_j^2\right) \mu^2 + \left(2 \sum_{j=0}^{n-1} M_j N_j\right) \mu + \left(\sum_{j=0}^{n-1} N_j^2 - 1\right) = 0\).

Thus,

\[
    \mu = \frac{-Q + \sqrt{Q^2 - 4PR}}{2P}
\]

where, \(P = \sum_{j=0}^{n-1} M_j^2\), \(Q = 2 \sum_{j=0}^{n-1} M_j N_j\) and \(R = \sum_{j=0}^{n-1} N_j^2 - 1\). We choose the positive sign before the square root since we want the plane satisfying (23) that is farthest from the origin.

Equations (26), (27) and (24) computes the required plane, \(I\), which is at a distance \(d_j\) from \(v_j\), \(j = 1, 2, \cdots, n - 1\).

Similar to what described in Section A.3, the point at which the perpendicular (in the direction of \(u\)) dropped from \(v_0\) onto the hyperplane \(I\) intersects the hyperplane \(H_0\) is given by \(i_0 = \sum_{i=1}^{n-1} w_i v_i\), with

\[
    w_k = \frac{w'_k}{\sum_{i=1}^{n-1} w'_i}, \quad k = 1, 2, \cdots, n - 1.
\]

where, \(w'_j\) can be computed recursively using the formula \(w'_j = \frac{u_{j-1} - \sum_{i=1}^{n-1} w'_i v_{i,j-1}}{v_{j,j-1}}\).
References

[BKH+15] Subhrajit Bhattacharya, Soonkyum Kim, Hordur Heidarsson, Gaurav Sukhatme, and Vijay Kumar. A topological approach to using cables to separate and manipulate sets of objects. *International Journal of Robotics Research*, 34(6):799–815, April 2015. DOI: 10.1177/0278364914562236.

[BLK10] Subhrajit Bhattacharya, Maxim Likhachev, and Vijay Kumar. Multi-agent path planning with multiple tasks and distance constraints. In *Proceedings of IEEE International Conference on Robotics and Automation (ICRA)*, Anchorage, Alaska, 3-8 May 2010.

[BM07] J.A. Bondy and U.S.R. Murty. *Graph theory*. Graduate texts in mathematics. Springer, 2007.

[CCL10] Benjamin Cohen, Sachin Chitta, and Maxim Likhachev. Search-based planning for manipulation with motion primitives. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, 2010.

[CRC03] David C Conner, Alfred Rizzi, and Howie Choset. Composition of local potential functions for global robot control and navigation. In *Proc. Int’l Conf. on Intelligent Robots and Systems (IROS)*, pages 3546–3551, 2003.

[Dij59] Edsger W. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1:269–271, 1959.

[DK99] Jaydev P. Desai and Vijay Kumar. Motion planning for cooperating mobile manipulators. *Journal of Robotic Systems*, 16:557–579, 1999.

[DLT12] G. Dhatt, E. Lefrançois, and G. Touzot. *Finite Element Method*. ISTE. Wiley, 2012.

[FBLD08] David Ferguson, Christopher R. Baker, Maxim Likhachev, and John M Dolan. A reasoning framework for autonomous urban driving. In *Proceedings of the IEEE Intelligent Vehicles Symposium (IV 2008)*, pages 775–780, Eindhoven, Netherlands, June 2008.

[FS07] Dave Ferguson and Anthony Stentz. Field d*: An interpolation-based path planner and replanner. In *Robotics research*, pages 239–253. Springer, 2007.

[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[HG13] D. Harabor and A. Grastien. An optimal any-angle pathfinding algorithm. In *Proceedings of the International Conference on Automated Planning and Scheduling*, pages 308–311, 2013.

[HKB+09] Dennis W. Hong, Shawn Kimmel, Rett Boehling, Nina Camoriano, Wes Cardwell, Greg Janmanan, Alex Purcell, Dan Ross, and Eric Russel. Development of a Semi-Autonomous Vehicle Operable by the Visually Impaired. pages 455–467. 2009.

[HLK07] M. A. Hsieh, S. Loizou, and V. Kumar. Stabilization of multiple robots on stable orbits via local sensing. In *IEEE International Conference on Robotics and Automation*, Rome, Italy, May 2007.

[HNR68] P. E. Hart, N. J. Nilsson, and B. Raphael. A formal basis for the heuristic determination of minimum cost paths. *IEEE Transactions on Systems, Science, and Cybernetics*, SSC-4(2):100–107, 1968.

[HZ07] Eric A. Hansen and Rong Zhou. Anytime heuristic search. *Journal of Artificial Intelligence Research (JAIR)*, 28:267–297, 2007.

[Jos08] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer, 2008.

[KBK14] Soonkyum Kim, Subhrajit Bhattacharya, and Vijay Kumar. Path planning for a tethered mobile robot. In *Proceedings of IEEE International Conference on Robotics and Automation (ICRA)*, Hong Kong, China, May 31 - June 7 2014.
[KK92] J. O. Kim and P. K. Khosla. Real-time obstacle avoidance using harmonic potential functions. *Robotics and Automation, IEEE Transactions on*, 8(3):338–349, Jun 1992.

[KV88] Pradeep Khosla and Richard Volpe. Superquadric artificial potentials for obstacle avoidance and approach. Philadelphia, Apr 1988.

[LS08] Maxim Likhachev and Anthony Stentz. R* search. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, 2008.

[LS10] Andrea L’Afflitto and Cornel Sultan. Calculus of variations for guaranteed optimal path planning of aircraft formations. pages 1972–1977, May 2010.

[MBB+08] Michael Montemerlo, Jan Becker, Suhrid Bhat, Hendrik Dahlkamp, Dmitri Dolgov, Scott Ettinger, Dirk Haehnel, Tim Hilden, Gabe Hoffmann, Burkhard Huhnke, Doug Johnston, Stefan Klupp, Dirk Langer, Anthony Levandowski, Jesse Levinson, Julien Marcil, David Orenstein, Johannes Paefgen, Isaac Penny, Anna Petrovskaya, Mike Pfueger, Ganymed Stanek, David Stavens, Antone Vogt, and Sebastian Thrun. Junior: The stanford entry in the urban challenge. *J. Field Robot.*, 25(9):569–597, September 2008.

[MKK12] Daniel Mellinger, Alex Kushleyev, and Vijay Kumar. Mixed-integer quadratic program trajectory generation for heterogeneous quadrotor teams. In *Proceedings of the 2012 IEEE International Conference on Robotics and Automation (ICRA)*, pages 477–483, 2012.

[NDKF07] Alex Nash, Kenny Daniel, Sven Koenig, and Ariel Felner. Theta*: Any-angle path planning on grids. In *AAAI*, pages 1177–1183. AAAI Press, 2007.

[NKT10] Alex Nash, Sven Koenig, and Craig A. Tovey. Lazy theta*: Any-angle path planning and path length analysis in 3d. In Maria Fox and David Poole, editors, *AAAI*. AAAI Press, 2010.

[Pet03] Anton Petrunin. Polyhedral approximations of riemannian manifolds. *Turkish Journal of Mathematics*, 27(1):173–188, 2003.

[RK91] E. Rimon and D.E. Koditschek. The construction of analytic diffeomorphisms for exact robot navigation on star worlds. *Trans. of the American Mathematical Society*, 327(1), Sept. 1991.

[SPL15] Siddharth Swaminathan, Mike Phillips, and Maxim Likhachev. Planning for multi-agent teams with leader switching. In *ICRA*, pages 5403–5410. IEEE, 2015.

[Ste95] A. Stentz. The focussed D* algorithm for real-time replanning. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*, pages 1652–1659, 1995.

[UAB+08] Christopher Urmson, Joshua Anhalt, Hong Bae, J. Andrew (Drew) Bagnell, Christopher R. Baker, Robert E Bittner, Thomas Brown, M. N. Clark, Michael Darmos, Daniel Demitrish, John M Dolan, David Duggins, David Ferguson, Tugrul Galatali, Christopher M Geyer, Michele Gittleman, Sam Harbaugh, Martial Hebert, Thomas Howard, Sascha Kolski, Maxim Likhachev, Bakhtiar Litkouhi, Alonzo Kelly, Matthew McNaughton, Nick Miller, Jim Nickolaou, Kevin Peterson, Brian Plinick, Ragunathan Rajkumar, Paul Rybski, Varsha Sadekar, Bryan Salesky, Young-Woo Seo, Sanjiv Singh, Jarrod M Snider, Joshua C Struble, Anthony (Tony) Stentz, Michael Taylor, William (Red) L. Whittaker, Ziv Wolkowicki, Wende Zhang, and Jason Ziglar. Autonomous driving in urban environments: Boss and the urban challenge. *Journal of Field Robotics Special Issue on the 2007 DARPA Urban Challenge, Part I*, 25(1):425–466, June 2008.

[UK15] Tansel Uras and Sven Koenig. An empirical comparison of any-angle path-planning algorithms. In *Proceedings of the Eighth Annual Symposium on Combinatorial Search (SOCS)*, pages 206–211, Dead Sea, Israel, June 2015.

[YL11] D. S. Yershov and S. M. LaValle. Simplicial dijkstra and a* algorithms for optimal feedback planning. In *2011 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 3862–3867, Sept 2011.
[Zef96] Milos Zeefran. *Continuous methods for motion planning*. PhD thesis, University of Pennsylvania, Philadelphia, PA, USA, 1996. Supervisor-Vijay Kumar.