Measurements of observables replaced by “evaluations” in Quantum Theory

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Abstract. In quantum physics there are circumstances where the direct measurement of particular observables encounters difficulties; in some of these cases, however, its value can be evaluated, i.e. it can be inferred by measuring another observable characterized by perfect correlation with the observable of interest. Though an evaluation is often interpreted as a measurement of the evaluated observable, we prove that the two concepts cannot be identified in quantum physics, because the identification yields contradictions. Then, we establish the conceptual status of evaluations in Quantum Theory and the role can be ascribed to them.

1. Introduction

In quantum physics there are circumstances where some difficulties encountered in measuring observables are outflanked by exploiting the correlations existing among observables.

As an example, we can consider a typical Stern–Gerlach experiment for a spin-1/2 particle, where the gradient of the magnetic field is oriented along $z$. The $z$ component $S_z$ of the spin is an observable which pertains to an internal degree of freedom of the particle, so that it is difficult to concretely design a direct measurement of $S_z$. However, the observable $T_{up}$ which localizes the particle in the upper exit of the magnet is perfectly correlated, according to the laws of Quantum Mechanics, with the values of $S_z$; hence, the value $+1/2$ of $S_z$ is inferred for the out-coming particles localized in the upper exit by a measurement of $T_{up}$.

Such a method is used also to outflank the obstacles raised by the fundamental principles of Quantum Mechanics. For instance, in a typical double-slit experiment the observable $W$ that indicates the slit the particle passes through is represented by a localization operator $\hat{W}$ which does not commute with the operator $\hat{Q}_F$ representing the impact position on the final screen, because $W$ and $Q_F$ are positions at different times. Therefore, the measurement of $W$ is in principle forbidden for a particle whose final position $Q_F$ is measured. However, under suitable conditions [1], an observable $T_W$ exists such that $[\hat{T}_W, \hat{Q}_F] = 0$, and whose outcomes are perfectly correlated with the outcomes of $W$ in every simultaneous measurement of $T_W$ and $W$; so, by measuring $T_W$ and $Q_F$ together, which slit information is inferred from the outcome of $T_W$, via the perfect correlation between $T_W$ and $W$ without measuring $W$, while $Q_F$ is directly measured.

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A priori, to assign an observable $E$ the value obtained as actual outcome of a correlated observable $T$ should be distinguished from a direct measurement of $E$; so, we call it evaluation of $E$ by $T$. The different statuses of measurement and evaluation in Quantum Theory are established in section 2.1 and 2.2.

In this work we address the problem of establishing to what extent evaluations can be interpreted as real measurements. In fact, in the experiment of Stern&Gerlach, for instance, the localization of the particle by $T_{up}$ is interpreted as a valid measurement of the spin $S_z$. But, from a theoretical point of view the problem exists. Indeed in section 2.3 we show that to identify the evaluation of an observable with its measurement leads to contradictions in Quantum Mechanics.

Now, since evaluations are diffusely practiced in quantum physics, the task of establishing how they are related to measurements cannot be overlooked in Quantum Theory.

Then, in section 3 we shows that evaluations behave as perfect simulations of the measurement of the evaluated observable; more precisely, we show that the physical consequences of the occurrence of every measurement’s outcome of an observable are physically indistinguishable from the consequences of the occurrence of the same outcome for its evaluation.

However, in section 4 we point out that the interpretation of evaluations as simulations does not apply if the evaluation of an observable $E$ by $T$ is performed together with the measurement of another observable $F$ which does not commute with the evaluated observable $E$. In this more general case we prove that a unique joint probability $p_D(E&F)$ exists which rules over a value assignment for $E$ consistent with the simultaneous occurrence of outcomes of actually performed measurements of $F$; furthermore, we prove that just to assign $E$ the values evaluated by $T$ realizes such a unique probability. As a consequence, though $E$ does not commute with $F$, the evaluated observable $E$ can be assigned the value evaluated by $T$ without violating logical consistency and such an assignment consistently extends the actual measurements’ outcomes.

The impossibility of identifying evaluations with measurement proved in section 2 is then explained on the basis of our results.

2. “Evaluations” in Quantum Theory
Here, in section 2.1, we formally establish the concept of evaluation within Quantum Theory. To do this, in section 2.1 we have to make explicit some implications of the standard interpretation of Quantum Theory. Section 2.3 shows that evaluations cannot be identified with measurements, because the identifications provokes contradictions in Quantum Mechanics.

2.1. Basic formalism
Let $H$ be the Hilbert space of the quantum theory of the investigated physical system. Given any observable $A$, let $A$ be the corresponding self-adjoint operator, the expected value of $A$ is $Tr(\rho A)$, where the density operator $\rho$ is the quantum state of the system [2].

Given a quantum state $\rho$, by support of $\rho$ we mean any concrete subset $S(\rho)$ of specimens of the physical system [3], whose quantum state is $\rho$. Given a support $S(\rho)$, by $A(S(\rho))$ we denote the concrete subset of all specimens in $S(\rho)$ which actually undergo a measurement of $A$. In the following, we shall write simply $A$ instead of $A(S(\rho))$ to avoid a cumbersome notation, whenever no confusion is likely.

By elementary observable we mean any observable $E$ having only 0 or 1 as possible outcomes, and hence represented by a projection operator $\hat{E}$: the expected value $Tr(\rho \hat{E})$ of an elementary observable $E$ coincides with the probability that outcome 1 occurs in a measurement of $E$. By $E$ we denote the set of all elementary observables, and by $\hat{E}(H)$ the set of all projection operators of $H$.

Fixed any support $S(\rho)$, in correspondence with every elementary observable $E$ we define the following extensions of $E$ in $S(\rho)$.
- the set $E$ of the specimens in $S(\rho)$ which actually undergo a measurement of $E$;
- the set $E_1 \subseteq E$ (resp., $E_0 \subseteq E$) for which the outcome 1 (resp., 0) of $E$ has been obtained.

In agreement with Quantum Mechanics, we assume that the following statements hold [3].

(2.1.i) If $E$ is an elementary observable, then for every $\rho$ a support $S(\rho)$ exists such that $E \neq \emptyset$.

(2.1.ii) For every support $S(\rho)$, $E_1 \cap E_0 = \emptyset$ and $E_1 \cup E_0 = E$, for every $\rho$.

(2.1.iii) If $\text{Tr}(\rho E) \neq 0$ then a support $S(\rho)$ exists such that $E_1 \neq \emptyset$, and if $\text{Tr}(\rho E) \neq 1$, then a support $S(\rho)$ exists such that $E_0 \neq \emptyset$.

In Quantum Theory [2], if $\hat{B} = f(\hat{A})$ holds for two self-adjoint operators $\hat{A}$ and $\hat{B}$, then a measurement of the observable $B$, henceforth denoted by $f(A)$, can be performed by measuring $A$ and then transforming the outcome $a$ by the function $f$ into the outcome $b = f(a)$ of $B$. As a consequence, the following statements hold in Quantum Theory.

If $\hat{B} = f(\hat{A})$ then $x \in A$ implies $x \in B$. 

(2.2)

If $[\hat{A}, \hat{B}] = 0$, then a third self-adjoint operator $\hat{C}$ and two functions $f$ and $g$ exist so that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$ [2]. Therefore, $A$ and $B$ can be measured together if the corresponding operators commute with each other. Thus, the following implications hold.

(2.1.iv) $\{E_j\} \subseteq \mathcal{E}$ and $[\hat{E}_j, \hat{E}_k] = 0 \ \forall j, k$ imply $\forall \rho \exists S(\rho)$ such that $\cap_j E_j \neq \emptyset$.

(2.1.v) If $[\hat{A}, \hat{B}] = 0$ and $\hat{D} = f(\hat{A}, \hat{B})$ then $x \in A \cap B$ implies $x \in D$, $\forall S(\rho)$.

(2.1.vi) If $\hat{F}, \hat{G} \in \mathcal{E}$ and $\hat{F} \perp \hat{G} = 0$, i.e. if $\hat{F} \perp \hat{G}$, then $\hat{F}_1 \cap \hat{G}_1 = 0$, $\forall S(\rho)$, $\forall \rho$.

i.e. in every simultaneous measurement of $F$ and $G$ the outcome 1 for $F$ and 1 for $G$ are mutually exclusive; in this case the projection $\hat{E} = \hat{F} + \hat{G}$ belongs to $\hat{E}(\mathcal{H})$; conversely, if $\hat{F} \perp \hat{G}$ then $\hat{E} = \hat{F} + \hat{G}$ represents an elementary observable $E$ whose measurement’s outcome can be the sum of the simultaneous outcomes of $F$ and $G$.

### 2.2. Evaluations of Elementary Observables in Quantum Theory

In general, given two elementary observables $E$ and $T$ we say that $E$ can be evaluated by $T$ if, according to Quantum Mechanics, the following perfect correlation holds:

“the outcome of $T$ is 1 if and only if the outcome of $E$ is 1 in every simultaneous measurement”.

By evaluation of $E$ by $T$ we mean to assign $E$ the value obtained as outcome of an actual measurement of $T$. Then we can give the following formal definition.

**Definition 2.1.** Given $E, T \in \mathcal{E}$, the elementary observable $E$ can be evaluated by $T$ when the system is assigned the state $\rho$, written $E \prec \rho \succ T$, if

(i) a support $S(\rho)$ exists such that $T \cap E \neq \emptyset$ (simultaneous measurability),

(ii) if $x \in E \cap T$ then $[x \in T_1$ if and only if $x \in E_1]$ and $[x \in T_0$ if and only if $x \in E_0]$, $\forall S(\rho)$.

If an evaluation of $E$ by $T$ were identifiable with a measurement of $E$ in the state $\rho$, then $T_1 \subseteq E_1$ and $T_0 \subseteq E_0$ should hold for any $S(\rho)$. But the relation $E \prec \rho \succ T$ is symmetric; thus the identification would be fully expressed by the following statement.

$$E \prec \rho \succ T \text{ if and only if } T_1 = E_1 \text{ and } T_0 = E_0, \forall S(\rho).$$ (3.1)

Another relation $E \leftrightarrow T$ can be defined as follows

**Definition 2.2.** The relation $T \leftrightarrow E$ holds if $[\hat{T}, \hat{E}] = 0$ and $\hat{T} \rho = \hat{E} \rho$ hold.
Relation $\leftrightarrow$ is stronger than $\prec \rho \succ$; indeed the following proposition holds.

**Proposition 2.1.** Given $E,T \in \mathcal{E}$, if $E \leftrightarrow T$ holds then $E \prec \rho \succ T$ holds too.

**Proof.** Condition (i) in def. 2.1 follows from (2.1.iv). Then condition (ii) holds if the probability of the pairs $(1,0)$ and $(0,1)$ in a simultaneous measurement of $T$ and $F$ is zero, i.e., if $Tr(\rho T[1 - \hat{T}]) = 0$ and $Tr(\rho [1 - \hat{T}]\hat{T}) = 0$, i.e. if $\hat{T}\rho = T\hat{E}\rho$ and $\hat{E}\rho = \hat{E}\hat{T}\rho$.

2.3. Evaluations are not measurements

Now we shall single out seven elementary observables $E^\alpha, E^\beta, F, G^\alpha, G^\beta, L^\alpha, L^\beta$ of a particular quantum system, chosen so that they are all measurable together if the identification (2.3) of evaluations with measurements holds. Then we show that their simultaneous outcomes $\eta^\alpha, \eta^\beta, \phi, \gamma^\alpha, \gamma^\beta, \lambda^\alpha, \lambda^\beta$ must satisfy the following constraints, where $f$ is the function $f(\xi) = 2\xi - 1$.

$$
\begin{align*}
\text{i) } & f(\eta^\alpha)f(\phi) = -f(\gamma^\alpha)f(\lambda^\alpha), \\
\text{ii) } & f(\eta^\beta)f(\phi) = -f(\gamma^\beta)f(\lambda^\beta), \\
\text{iii) } & f(\eta^\alpha)f(\gamma^\alpha) = -f(\lambda^\alpha)f(\gamma^\alpha), \\
\text{iv) } & f(\eta^\beta)f(\gamma^\beta) = -f(\lambda^\beta)f(\gamma^\beta),
\end{align*}
$$

(2.4)

Since each factor $f(\xi)$ in (2.4) must be $-1$ or $+1$, these constraints are contradictory, because, by elementary algebra, they imply $f(\gamma^\alpha)f(\gamma^\beta) = -f(\gamma^\alpha)f(\gamma^\beta)$. Thus, in Quantum Mechanics evaluations cannot be identified with measurements.

To realize such a program, we consider a quantum system described in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$, where each $\mathcal{H}_k$ is $\mathbb{C}^2$. The following projection operators represent seven elementary observables\(^2\).

\[
\begin{align*}
\hat{E}^\alpha & = \frac{1}{2} \left[ \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right] \otimes 1_2 \otimes 1_3 \otimes 1_4; & \hat{E}^\beta & = \frac{1}{2} \left[ \begin{array}{cc}
i & 1 \\
1 & i
\end{array} \right] \otimes 1_2 \otimes 1_3 \otimes 1_4; \\
\hat{F} & = 1_1 \otimes \frac{1}{2} \left[ \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right] \otimes 1_3 \otimes 1_4; & \hat{G}^\alpha & = 1_1 \otimes 1_2 \otimes \frac{1}{2} \left[ \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right] \otimes 1_4; & \hat{G}^\beta & = 1_1 \otimes 1_2 \otimes \frac{1}{2} \left[ \begin{array}{cc}
i & 1 \\
1 & i
\end{array} \right] \otimes 1_4; \\
\hat{L}^\alpha & = 1_1 \otimes 1_2 \otimes 1_3 \otimes \frac{1}{2} \left[ \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right] \otimes 1_4; & \hat{L}^\beta & = 1_1 \otimes 1_2 \otimes 1_3 \otimes \frac{1}{2} \left[ \begin{array}{cc}
i & 1 \\
1 & i
\end{array} \right] \otimes 1_4.
\end{align*}
\]

Let the physical system be assigned the pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$, where

\[
\psi_0 = \frac{1}{\sqrt{2}} \left( \left[ \begin{array}{c} 1 \\ 1 \\
1 \\
1
\end{array} \right] \otimes \left[ \begin{array}{c} 0 \\ 1 \\
0 \\
1
\end{array} \right] \otimes \left[ \begin{array}{c} 0 \\ 1 \\
0 \\
1
\end{array} \right] \otimes \left[ \begin{array}{c} 0 \\ 1 \\
0 \\
1
\end{array} \right] \right).
\]

The four projection operators $\hat{E}^\alpha, \hat{F}, \hat{G}^\beta, \hat{L}^\alpha$ commute with each other; hence, by (2.1.iv), all the corresponding elementary observables can be measured together, i.e. a support $\mathcal{S}(\rho_0)$ and a specimen $x_0 \in \mathcal{S}(\rho_0)$ exist such that

\[
x_0 \in E^\alpha \cap F \cap G^\beta \cap L^\alpha.
\]

Let $\eta^\alpha, \phi, \gamma^\beta, \lambda^\alpha$ be their respective outcomes relative to $x_0$. Now, the projection operator

\[
\hat{M} = \frac{1 - f(\hat{E}^\alpha)f(\hat{F})f(\hat{L}^\alpha)}{2}
\]

\(^2\) Our argument makes use of the mathematical setting adopted by Greenberger, Horne, Shimony and Zeilinger (GHSZ) [4] to prove that a given set of premises are in contradiction with Quantum Theory. In [3] we proved that if the premises of GHSZ theorem are modified, then the contradiction does not necessarily arise. The hypotheses of the present argument, i.e. identification (2.3), are different from the premises of GHSZ. Thus the occurrence of a contradiction needs an explicit proof.
is a function of $\hat{E}^\alpha, \hat{F}, \hat{L}^\alpha$; therefore $x_0 \in \mathbf{M}$ because of (2.1.v) and $\mu = \frac{1}{2}[1 - f(\eta^\alpha)f(\phi)f(\lambda^\alpha)]$ must be the outcome of the elementary observable $M$ so measured on $x_0$.

But $[M, \hat{G}^\alpha] = 0$ trivially holds; moreover, a direct calculation shows that the equation $\hat{M}\rho_0 = \hat{G}^\alpha\rho_0$ is satisfied; then $M \cup \hat{G}^\alpha$ holds and Prop.2.1 implies $M \prec \rho_0 \succ G^\alpha$. If the identification (2.3) holds, then $x_0$ belongs to $G^\alpha$ and $\gamma^\alpha = \mu = \frac{1}{2}[1 - f(\eta^\alpha)f(\phi)f(\lambda^\alpha)]$ is to be identified as the outcome of the measurement of $G^\alpha$ on $x_0$. Then, from $f(\gamma^\alpha) \equiv 2\gamma^\alpha - 1$ we find (2.4.i). Then, $x_0 \in \mathbf{E}^\alpha \cap \mathbf{F} \cap G^\alpha \cap G^\beta \cap \mathbf{L}^\alpha$ and the constraint (2.4.i) must hold.

Now we derive (2.4.ii). By defining $N = \frac{1 - f(\hat{E}^\beta)f(\hat{G}^\alpha)}{2}$, we can verify that $N \rho_0 \rightarrow E^\beta$. Then, following the argument which led us to (2.4.i), we obtain that $x_0 \in \mathbf{E}^\alpha \cap \mathbf{E}^\beta \cap \mathbf{F} \cap G^\alpha \cap G^\beta \cap \mathbf{L}^\alpha$ and (2.4.ii) holds.

Similarly, by defining $\hat{R} = \frac{1 - f(\hat{E}^\beta)f(\hat{G}^\alpha)}{2}$, it turns out that $R \rho_0 \rightarrow L^\beta$ holds. Then, we can imply that $x_0 \in \mathbf{E}^\alpha \cap \mathbf{E}^\beta \cap \mathbf{F} \cap G^\alpha \cap G^\beta \cap L^\alpha \cap L^\beta$ and (2.4.iii) hold.

But we can also define $\hat{S} = \frac{1 + f(\hat{E}^\alpha)f(\hat{G}^\beta)}{2}$. The elementary observable $S$ turns out to satisfy the relation $S \rho_0 \rightarrow L^\beta$, which implies (2.4.iv).

Then all the constraints (2.4) must hold for the simultaneous measurement of $E^\alpha, E^\beta, F, G^\alpha, G^\beta, L^\alpha, L^\beta$ on the specimen $x_0$. Thus, identification (2.3) cannot hold in Quantum Mechanics.

3. Evaluations as simulations of measurements

The impossibility of identifying evaluations with measurements, proved in section 2, opens the question of establishing the conceptual status of the evaluations in Quantum Theory; since evaluations are diffusely practiced in quantum physics, it is important give an answer to such a question. In this section we show that an evaluation of $E$ by $T$ works as a perfect simulation of a measurement of $E$.

A way to understand which is the conceptual status of an evaluation of $E$ by $T$ is to compare the physical implications of the occurrences of the outcomes of $E$ with the physical implications of the corresponding outcomes of $T$. Such a comparison amounts to compare the conditional probabilities $P(F \mid E) = \text{Tr}(\rho \hat{F} \hat{E})/\text{Tr}(\rho \hat{E})$, $P(F \mid E') = \text{Tr}(\rho \hat{F} \hat{E'})/\text{Tr}(\rho \hat{E})$ established by Quantum Theory with the conditional probabilities $P(F \mid T) = \text{Tr}(\rho \hat{F} \hat{T})/\text{Tr}(\rho \hat{T})$, $P(F \mid T') = \text{Tr}(\rho \hat{F} \hat{T'})/\text{Tr}(\rho \hat{T'})$, where $E' \equiv 1 - E$ and $T' \equiv 1 - T$.

These conditional probabilities are defined whenever $[\hat{F}, \hat{T}] = [\hat{F}, \hat{E}] = 0$; therefore the domain of the comparison is $\mathcal{F}(E,T) = \{F \in \mathcal{E} \mid [F, \hat{T}] = [F, \hat{E}] = 0\}$. The following statement follows from prop. 2.1.

If $T \not\rho_0 \rightarrow E$ then $P(F \mid T) = \frac{\text{Tr}(\rho \hat{F} \hat{T})}{\text{Tr}(\rho \hat{T})} = \frac{\text{Tr}(\rho \hat{F} \hat{E})}{\text{Tr}(\rho \hat{E})} = P(F \mid E)$, $\forall F \in \mathcal{F}(E,T)$. (3.1)

Now, from Def. 2.2 it follows that $T \not\rho_0 \rightarrow E$ holds iff $T' \not\rho_0 \rightarrow E'$; so (3.1) extends to

If $T \not\rho_0 \rightarrow E$ then $P(F \mid T') = \frac{\text{Tr}(\rho \hat{F} \hat{T'})}{\text{Tr}(\rho \hat{T'})} = \frac{\text{Tr}(\rho \hat{F} \hat{E})}{\text{Tr}(\rho \hat{E})} = P(F \mid E')$, $\forall F \in \mathcal{F}(E,T)$. (3.2)

Therefore, in the case that $T \not\rho_0 \rightarrow E$ holds, the effects of an evaluation of $E$ by $T$ are indistinguishable from the effects of the occurrence of the same outcome in a direct measurement of $E$. In other words, the measurement of $E$ is perfectly simulated by a measurement of an evaluating observable $T$.

Remark 3.1. In fact, once fixed the evaluated observables $E$, our results are obtained for evaluating observables $T$ such that $T \not\rho_0 \rightarrow E$; these observables form a subset of all evaluating
observables evaluating $E$ which have to satisfy the weaker condition $T \prec \rho \succ E$. However, the converse of Prop. 2.1, i.e. the implication $T \prec \rho \succ E$ implies $T \leftarrow \rho \rightarrow E$, immediately follows from the principle of Quantum Mechanics which establishes that $T$ and $E$ are measurable together if and only if $[\hat{T}, \hat{E}] = 0$. Therefore, if such a principle is assumed, our results holds for all evaluating observables.

4. To evaluate $E$ while incompatible observables are measured

Let $T$, $E$ and $F$ be elementary observables such that $T \leftarrow \rho \rightarrow E$ and $[\hat{T}, \hat{E}] = 0$, so that $T$ can be measured together with $F$; in the case in which $F \notin \mathcal{F}(E, T)$, i.e. if $[\hat{F}, \hat{E}] \neq 0$, a measurement of $T$ simultaneous to a measurement of $F$ cannot be interpreted as a simulation of a measurement of $E$, according to section 3.1, because there is no conditional probability $P(F \mid E)$ to be compared with $P(F \mid T)$.

The double slit experiment described in the introduction is an emblematic example of this circumstance. In that case it is possible to evaluate the which slit observable $W$ because $P(F(\Delta) \mid W)$ does not exist since $[\hat{W}, \hat{F}(\Delta)] \neq 0$.

In the present section we address the interpretative lack which occurs in a general situation where

$$T \leftarrow \rho \rightarrow E, \quad F \in \mathcal{F}(T) \equiv \{ F \in \mathcal{E} \mid [\hat{T}, \hat{F}] = 0 \}, \quad \text{but} \quad [\hat{E}, \hat{F}] \neq 0,$$

and $E$ is evaluated by $T$ simultaneously to a measurement of $F$.

In section 4.1 we establish results which allow us to provide the problem with an answer we formulate in section 4.2. Remark 3.1 applies also in this more general case.

4.1. Consistency of assignment by evaluations

Let us start with the following result of Cassinelli and Zanghì [5].

**Lemma 4.1.** Let $\hat{A}$ be a Von Neumann algebra$^3$, and let $\Pi(\hat{A})$ be the set of all projection operators in $\hat{A}$. Given a projection operator $\hat{E} \in \Pi(\hat{A})$, for every density operator $\rho$ the function

$$p_\rho(E \&) : \Pi(\hat{A}) \rightarrow [0, 1], \quad p_\rho(E \& F) = Tr(\rho \hat{E} \hat{F} \hat{E})$$

is the unique functional which satisfies the following conditions.

(i) If $F \in \Pi(\hat{A})$ and $[\hat{F}, \hat{E}] = 0$ then $p_\rho(E \& F) = Tr(\rho \hat{E} \hat{F})$; (ii) if $\{ \hat{F}_j \}_{j \in J} \subseteq \Pi(\hat{A})$ is any countable family of projection operators such that $\sum_{j \in J} \hat{F}_j = \hat{F} \in \Pi(\hat{A})$, then $p_\rho(E \& F) = \sum_{j \in J} p_\rho(E \& \hat{F}_j)$.

Since the set $\hat{F}(\hat{T}) = \{ \hat{F} \in \mathcal{E}(\mathcal{H}) \mid [\hat{F}, \hat{T}] = 0 \}$ is just the set of all projection operators of the Von Neumann algebra $\hat{A}(\hat{T}) = \{ \hat{T} \}'$, and hence $\hat{F}(\hat{T})$ generates $\hat{A}(\hat{T})$, the following theorem can be proved by means of Lemma 4.1.

**Theorem 4.1.** Let $T$ be an elementary observable. Given $E \in \mathcal{F}(T)$, for every quantum state $\rho$ the mappings

$$p_\rho(E \&) : \mathcal{F}(T) \rightarrow [0, 1], \quad p_\rho(E \& F) = Tr(\rho \hat{E} \hat{F} \hat{E}),$$

are the unique functionals which satisfy the following conditions.

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$^3$ A Von Neumann algebra $[6]$ is a subset $\hat{A}$ of bounded linear operators of the Hilbert space $\mathcal{H}$ such that $\hat{A} = (\hat{A})' \equiv \hat{A}'$, where $\hat{A}'$ denotes the commutant of $\hat{A}$, i.e. the set of all bounded linear operators $\hat{B}$ of $\mathcal{H}$ such that $[\hat{B}, \hat{A}] = 0$ for all $\hat{A} \in \hat{A}$. The theory of Von Neumann algebras $[6]$ shows that if $\Pi(\hat{A})$ is the set of all projection operators in the Von Neumann algebra $\hat{A}$, then $\hat{A} = \Pi(\hat{A})''$. 

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(C.1) If \( F \in \mathcal{F}(T) \) and \([\hat{F}, \hat{E}] = 0\) then \( p_\rho(E\&F) = Tr(\rho\hat{E}\hat{F})\) and \( p_\rho(E'\&F) = Tr(\rho\hat{E}'\hat{F}')\);

(C.2) if \( \{F_j\}_{j \in J} \subseteq \mathcal{F}(T) \) is any countable family such that \( \sum_{j \in J} F_j = \hat{F} \in \mathcal{F}(T) \), then \( p_\rho(E\&F) = \sum_{j \in J} p_\rho(E\&F_j) \) and \( p_\rho(E'\&F) = \sum_{j \in J} p_\rho(E'\&F_j) \).

Theorem 4.1 implies that if \( E \) can be assigned values which extend its measured values to the case that \( E \) is not measured, in a way which is logically consistent with the outcomes of whatever actually measured observable \( F \in \mathcal{F}(T) \), then it is necessary that such an assignment be ruled over by the probability

\[
p_\rho(E\&F) = Tr(\rho\hat{E}\hat{F}) \quad \text{(resp., } p_\rho(E'\&F) = Tr(\rho\hat{E}'\hat{F}')\text{)}
\]

of assigning \( E \) value 1 (resp., 0) and obtaining value 1 for \( F \).

However, the agreement with probability \( p_\rho \) is not sufficient. Logical consistency requires that the condition must hold too.

(C.3) \( p_\rho(E'\&F) + p_\rho(E\&F) = Tr(\rho\hat{F}) \) for all \( F \in \mathcal{F}(T) \)

In general, (C.3) does not hold. Indeed, we have

\[ Tr(\rho\hat{F}) = Tr(\rho[\hat{E} + \hat{E}'][\hat{E} + \hat{E}']) = p_\rho(E\&f) + p_\rho(E'\&f) + 2ReTr((\hat{E}\hat{F}\hat{E}')\).
\]

Therefore, if \([\hat{F}, \hat{E}] \neq 0\) the presence of a non-vanishing interference term \(2ReTr((\hat{E}\hat{F}\hat{E}')\) cannot be excluded, and (C.3) is violated.

The following theorem proves that to assign \( E \) just the outcome of the evaluation of \( E \) by \( T \) guarantees all conditions (C.1), (C.2), (C.3).

**Theorem 4.2.** Given two elementary observables \( E, T \in \mathcal{E} \), if \( E \leftrightarrow \rho \to T \), so that \( T \) can evaluate \( E \), then for all \( F \in \mathcal{F}(T) \) the equalities \( p_\rho(E\&F) = Tr(\rho\hat{T}\hat{F}) \) and \( p_\rho(E'\&F) = Tr(\rho\hat{T}'\hat{F}) \) hold. Furthermore, also (C.3) holds.

**Proof.** If \( E \leftrightarrow \rho \to T \), then \( \hat{E}\rho = \hat{T}\rho \) and \( \rho\hat{E} = \rho\hat{T} \) hold; therefore \( p_\rho(E\&F) = Tr(\rho\hat{E}\hat{F}) = Tr(\hat{E}\rho\hat{F}) = Tr(\hat{T}\rho\hat{F}) = Tr(\rho\hat{T}\hat{F}) \) holds because \([\hat{F}, \hat{T}] = 0\). Analogously, since \( E \leftrightarrow \rho \to T \) implies \( E' \leftrightarrow \rho \to T' \), \( p_\rho(E'\&F) = Tr(\rho\hat{T}'\hat{F}) \) is proved. Then \( p_\rho(E\&F) + p_\rho(E'\&F) = Tr(\rho\hat{T}\hat{F}) + Tr(\rho\hat{T}'\hat{F}) = Tr(\rho\hat{F}) \); thus also (C.3) is proved.

4.2. Conclusions

In virtue of theorem 4.2 we can state the following conclusions, which are valid whenever \( E \leftrightarrow \rho \to T \).

(Ev.1) It is logically consistent to assign \( E \) the value obtained by an evaluation by \( T \) together with simultaneous measurement of whatever family \( \{F_j\} \) of observables from \( \mathcal{F}(T) \), also if \([\hat{F}, \hat{E}] \neq 0\).

(Ev.2) The probability which rules over such an assignment and the occurrences of the outcomes extends the probability established by Quantum Theory for actually measured outcomes.

(Ev.3) The consistency is guaranteed within the domain \( \mathcal{F}(T) \subseteq \mathcal{E} \), not elsewhere.

According to these conclusions, to assign \( E \) the value evaluated by \( T \) cannot provoke contradiction if the evaluation is performed together with whatever measurement of an observable \( F \), also if \([\hat{F}, \hat{E}] \neq 0\). Thus, our conclusions (Ev) contribute to fulfill the interpretative lack about evaluations which arises in the circumstances where conditions (4.1) hold.

As further implications of our results, we can deduce that it is possible to exploit evaluations for consistently assigning non-commuting observables simultaneous values. Indeed, let us consider two elementary observables \( E_1, E_2 \) such that \( E_1 \leftrightarrow \rho \to T_1 \) and \( E_2 \leftrightarrow \rho \to T_2 \), so that they can
be evaluated by \( T_1 \) and \( T_2 \), respectively, in the state \( \rho \). If \([\hat{T}_1, \hat{T}_2] = 0\) and \( E_1, E_2 \in \mathcal{F}(T_1) \cap \mathcal{F}(T_2)\), the evaluations of both \( E_1 \) and \( E_2 \) can be obtained simultaneously by measuring together \( T_1 \) and \( T_2 \). Therefore \( E_1 \) and \( E_2 \) can be consistently assigned simultaneous values, also if \([\hat{E}_1, \hat{E}_2] \neq 0\).

Meaningful examples are developed in [1],[7],[8]. These results do no conflict with the impossibility of identifying evaluations with measurement proved in section 2.3; on the contrary, they explain it. Therein the observables actually measured are \( E^\alpha, F, G^\beta, L^\alpha \). Hence, also the elementary observables \( M \) and \( N \) represented by the projection operators \( \hat{M} = \frac{1 - f(\hat{E}^\alpha)f(\hat{F})f(\hat{L}^\alpha)}{2} \) and \( \hat{N} = \frac{1 - f(\hat{F})f(\hat{G}^\beta)f(\hat{L}^\alpha)}{2} \) are actually measured according to (2.1.v). Since \( \hat{N} = \frac{1 - f(\hat{F})f(\hat{G}^\beta)f(\hat{L}^\alpha)}{2} \), \( \mathcal{F}(E^\beta) \cap \mathcal{F}(N) \) hold, \( G^\alpha \) and \( E^\beta \) can be consistently assigned values. However, \( L^\beta \) cannot be consistently assigned the value evaluated by \( R \), though \( R \leftarrowarrow E^\beta \), because \( R \) is not actually measured. Indeed, since \( \hat{R} = \frac{1 - f(\hat{E}^\beta)f(\hat{F})f(\hat{G}^\beta)}{2} \), the measurement of \( R \) entails the measurement of \( E^\beta \) and \( G^\alpha \); but \([\hat{E}^\beta, \hat{E}^\alpha] \neq 0\), \([\hat{G}^\alpha, \hat{G}^\beta] \neq 0\) and \( E^\alpha \), \( G^\beta \) are actually measured. Thus, according to our results, a consistent value assignment also to \( E^\beta \) is not guaranteed, and indeed an inconsistency arises in attempting it.

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