A COMPACT $G_2$-CALIBRATED MANIFOLD WITH FIRST BETTI NUMBER $b_1 = 1$

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Abstract. We construct a compact formal 7-manifold with a closed $G_2$-structure and with first Betti number $b_1 = 1$, which does not admit any torsion-free $G_2$-structure, that is, it does not admit any $G_2$-structure such that the holonomy group of the associated metric is a subgroup of $G_2$. We also construct associative calibrated (hence volume-minimizing) 3-tori with respect to this closed $G_2$-structure and, for each of those 3-tori, we show a 2-dimensional family of non-trivial associative deformations.

1. Introduction

A 7-manifold $M$ is said to admit a $G_2$-structure if there is a reduction of the structure group of its frame bundle from the linear group $GL(7, \mathbb{R})$ to the exceptional Lie group $G_2$. A $G_2$-structure is equivalent to the existence of a certain type of a non-degenerate 3-form $\varphi$ (the $G_2$ form) on the manifold. Indeed, by [16] a manifold $M$ with a $G_2$-structure comes equipped with a Riemannian metric $g$, a cross product $P$, a 3-form $\varphi$, and orientation, which satisfy the relation

$$\varphi(X, Y, Z) = g(P(X, Y), Z),$$

for every vector fields $X, Y, Z$ on $M$.

If the 3-form $\varphi$ is covariantly constant with respect to the Levi-Civita connection of the metric $g$ or, equivalently, the intrinsic torsion of the $G_2$-structure vanishes [36], then the holonomy group of $g$ is contained in $G_2$, and the 3-form $\varphi$ is closed and cocallosed [16]. In this case, the $G_2$-structure is said to be torsion-free. The first complete examples of metrics with holonomy $G_2$ were obtained by Bryant and Salamon in [6], while compact examples of Riemannian manifolds with holonomy $G_2$ were constructed first by Joyce [27], and then by Kovalev [30], Kovalev and Lee [31], and Corti, Haskins, Nordström, Pacini [10]. More recently, a new construction of compact manifolds with holonomy $G_2$ has been given in [29] by gluing families of Eguchi-Hanson spaces.

A $G_2$-structure is called calibrated (or closed) if the 3-form $\varphi$ is closed [23], and a $G_2$-structure is said to be cocalibrated (or coclosed) if the 3-form $\varphi$ is coclosed. These two classes of $G_2$-structures are very different in nature, the closed condition of the $G_2$ form being much more restrictive; for example, Crowley and Nördstrom in [11] prove that coclosed $G_2$-structures always exist on closed spin manifolds and satisfy the parametric $h$-principle.

Compact $G_2$-calibrated manifolds have interesting curvature properties. It is well known that a $G_2$ holonomy manifold is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated $G_2$ manifold, both the Einstein condition [8] and scalar-flatness

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are equivalent to the holonomy being contained in $G_2$. In fact, Bryant in [5] shows that the scalar curvature is always non-positive.

All the known examples in the literature of compact 7-manifolds admitting a closed $G_2$ form, which is not coclosed, have first Betti number strictly bigger than one. The first example of a compact $G_2$-calibrated manifold that does not have holonomy $G_2$ was obtained in [14]. This example is a nilmanifold, that is a compact quotient of a simply connected nilpotent Lie group by a lattice, endowed with an invariant calibrated $G_2$-structure. In [9] Conti and the first author classified the nilpotent 7-dimensional Lie algebras that admit a calibrated $G_2$-structure. All those examples are non-formal. Other examples were given in [15]. They are formal compact solvable manifolds with first Betti number $b_1 = 3$.

In this paper, we show a compact formal 7-manifold with a closed $G_2$-structure and with first Betti number $b_1 = 1$ not admitting any torsion-free $G_2$-structure. To our knowledge, this manifold is the first example of compact $G_2$-calibrated manifold that satisfies all these properties.

To construct such a manifold, we start with a compact 7-manifold $M$ equipped with a closed $G_2$ form $\varphi$ and with first Betti number $b_1(M) = 3$. Then we quotient $M$ by a finite group preserving $\varphi$ to obtain an orbifold $\widetilde{M}$ with an orbifold closed $G_2$ form $\tilde{\varphi}$ and with first Betti number $b_1(\widetilde{M}) = 1$ (Proposition 14). We resolve the singularities of the 7-orbifold $\widetilde{M}$ to produce a smooth 7-manifold $\tilde{M}$ with a closed $G_2$ form $\tilde{\varphi}$, with first Betti number $b_1(\tilde{M}) = 1$ and such that $(\tilde{M}, \tilde{\varphi})$ is isomorphic to $(\widetilde{M}, \tilde{\varphi})$ outside the singular locus of $\widetilde{M}$ (Theorem 17). The idea of this construction stems from our study of the original Joyce’s techniques on “$G_2$-orbifold resolutions” [27, 28] that allowed him the construction of compact Riemannian manifolds with holonomy $G_2$. (There “$G_2$-orbifold” means an orbifold with an orbifold closed and co-closed $G_2$ form.)

Next, we prove that $\tilde{M}$ has the aforementioned properties. More precisely, using the concept of 3-formal minimal model, introduced in [17] as an extension of formality [12] (see Section 3 for details) we prove that the 7-manifold $\tilde{M}$ is formal (Proposition 20). On the other hand, we show that $\tilde{M}$ has fundamental group $\pi_1(\tilde{M}) = \mathbb{Z}$ (Proposition 19), this resulting from the careful choice of the action of the finite group acting on $M$. Finally, using this last result and that $b_1(\tilde{M}) = 1$, we prove that if $\tilde{M}$ carries a $G_2$ form such that the holonomy group of the associated metric is a subgroup of $G_2$, then $\tilde{M}$ has a finite covering which is a product of a 6-dimensional simply connected Calabi-Yau manifold and a circle, and so there exist a closed 2-form $\omega$ and a closed 1-form $\eta$ on $\tilde{M}$ such that $\omega^3 \wedge \eta \neq 0$ at every point of $\tilde{M}$. But we see that this is not possible by the cohomology of $\tilde{M}$ determined in Proposition 18. This shows that $\tilde{M}$ does not admit any torsion-free $G_2$-structure (Theorem 21).

Now, let us recall that for each 7-manifold $N$ with a $G_2$-structure $\phi$, one may define a special class of 3-dimensional orientable submanifolds of $N$ called associative 3-folds (see section 7 for details). Their tangent spaces are subalgebras of the cross-product algebras induced by $\phi$ on the tangent spaces of $N$; in fact, these latter subalgebras are isomorphic to $\mathbb{R}^3$ with the standard vector product. If the $G_2$-structure $\phi$ is closed, then $\phi$ is a
calibration and every associative 3-fold is a minimal submanifold of \(N\) (moreover, locally volume-minimizing in its homology class \([28, \text{Proposition 3.7.2}]\)).

For the compact 7-manifold \(M\) with the closed \(G_2\) form \(\varphi\) mentioned above, we consider a non-trivial involution of \(M\) preserving \(\varphi\), and we construct an example of a 2-dimensional family of associative volume-minimizing 3-tori in \(\tilde{M}\) (Proposition \[24\]).

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2. Orbifolds

In this section we collect some basic facts and definitions concerning \(G_2\) forms on smooth manifolds and on orbifolds (see \([1, 5, 16, 23, 25, 26, 28, 36]\) for details).

Let us consider the space \(O\) of the Cayley numbers (or octonions) which is a non-associative algebra over \(\mathbb{R}\) of dimension 8. We can identify \(\mathbb{R}^7\) with the subspace of \(O\) consisting of pure imaginary Cayley numbers. Then, the product on \(O\) defines on \(\mathbb{R}^7\) the 3-form \(\varphi_0\) given by

\[
\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245},
\]

where \(\{e^1, \ldots, e^7\}\) is the standard basis of \((\mathbb{R}^7)^*\). Here, \(e^{127}\) stands for \(e^1 \wedge e^2 \wedge e^7\), and so on. The stabilizer of \(\varphi_0\) under the standard action of \(\text{GL}(7, \mathbb{R})\) on \(\Lambda^3(\mathbb{R}^7)^*\) is the Lie group \(G_2\), which is one of the exceptional Lie groups, and it is a compact, connected, simply connected, simple Lie subgroup of \(\text{SO}(7)\) of dimension 14.

Note that \(G_2\) acts irreducibly on \(\mathbb{R}^7\) and preserves the standard metric and orientation for which \(\{e^1, \ldots, e^7\}\) is an oriented and orthonormal basis. The \(\text{GL}(7, \mathbb{R})\)-orbit of \(\varphi_0\) is open in \(\Lambda^3(\mathbb{R}^7)^*\), so \(\varphi_0\) is a stable 3-form on \(\mathbb{R}^7\) \([24]\).

Definition 1. Let \(V\) be a real vector space of dimension 7. A 3-form \(\varphi \in \Lambda^3(V^*)\) on \(V\) is a \(G_2\) form (or \(G_2\)-structure) on \(V\) if there is a linear isomorphism \(u: (V, \varphi) \rightarrow (\mathbb{R}^7, \varphi_0)\) such that \(u^* \varphi_0 = \varphi\), where \(\varphi_0\) is given by \([1]\).

A \(G_2\)-structure on a 7-dimensional smooth manifold \(M\) is a reduction of the structure group of its frame bundle from \(\text{GL}(7, \mathbb{R})\) to the exceptional Lie group \(G_2\). Gray in \([20]\) proved that a smooth 7-manifold \(M\) carries \(G_2\)-structures if and only if \(M\) is orientable and spin.

The presence of a \(G_2\)-structure is equivalent to the existence of a differential 3-form \(\varphi\) (the \(G_2\) form) on \(M\), which can be defined as follows. Denote by \(T_p(M)\) the tangent space to \(M\) at \(p \in M\), and by \(\Omega^*(M)\) the algebra of the differential forms on \(M\).

Definition 2. Let \(M\) be a smooth manifold of dimension 7. A \(G_2\) form on \(M\) is a differential 3-form \(\varphi \in \Omega^3(M)\) such that, for each point \(p \in M\), \(\varphi_p\) is a \(G_2\) form on \(T_p(M)\) (in the sense of Definition \([1]\)) that is, for each \(p \in M\), there is a linear isomorphism \(u_p: (T_p(M), \varphi_p) \rightarrow (\mathbb{R}^7, \varphi_0)\) satisfying \(u_p^* \varphi_0 = \varphi_p\), where \(\varphi_0\) is given by \([1]\).
Therefore, if \( \varphi \) is a \( G_2 \) form on \( M \), then \( \varphi \) can be locally written as \([1]\) with respect to some (local) basis \( \{e^1, \ldots, e^7\} \) of the (local) 1-forms on \( M \).

Note that there is a 1-1 correspondence between \( G_2 \)-structures and \( G_2 \) forms on \( M \). In fact, if \( \varphi \in \Omega^2(M) \) is a \( G_2 \) form on \( M \), the subbundle of the frame bundle whose fibre at \( p \in M \) consists of the isomorphisms \( u_p:\ (T_p(M), \varphi_p) \rightarrow (\mathbb{R}^7, \varphi_0) \), such that \( u_p^*\varphi_0 = \varphi_p \), defines a principal subbundle with fibre \( G_2 \), that is a \( G_2 \)-structure on \( M \).

Since \( G_2 \subset \text{SO}(7) \), a \( G_2 \) form on \( M \) determines a Riemannian metric and an orientation on \( M \). Let \( \varphi \) be a \( G_2 \) form on \( M \). Denote by \( g_{\varphi} \) the Riemannian metric induced by \( \varphi \), and by \( \nabla_{\varphi} \) the Levi-Civita connection of \( g_{\varphi} \). Let \( \star_{\varphi} \) be the Hodge star operator determined by \( g_{\varphi} \) and the orientation induced by \( \varphi \).

**Definition 3.** We say that a manifold \( M \) has a closed \( G_2 \)-structure if there is a \( G_2 \) form \( \varphi \) on \( M \) such that \( \varphi \) is closed, that is \( d\varphi = 0 \). A manifold \( M \) has a coclosed \( G_2 \)-structure if there is a \( G_2 \) form \( \varphi \) on \( M \) such that \( \varphi \) is coclosed, i.e. \( d(\star_{\varphi}\varphi) = 0 \). A \( G_2 \) form \( \varphi \) on \( M \) is torsion-free if \( \nabla_{\varphi}\varphi = 0 \).

**Orbifold \( G_2 \) forms.**

**Definition 4.** A (smooth) \( n \)-dimensional orbifold is a Hausdorff, paracompact topological space \( X \) endowed with an atlas \( \{(U_p, \tilde{U}_p, \Gamma_p, f_p)\} \) of orbifold charts, that is \( U_p \subset X \) is a neighbourhood of \( p \in X \), \( \tilde{U}_p \subset \mathbb{R}^n \) an open set, \( \Gamma_p \subset \text{GL}(n, \mathbb{R}) \) a finite group acting on \( \tilde{U}_p \), and \( f_p: \tilde{U}_p \rightarrow U_p \) is a \( \Gamma_p \)-invariant map with \( f_p(0) = p \), inducing a homeomorphism \( \tilde{U}_p/\Gamma_p \cong U_p \). Moreover, the charts are compatible in the following sense:

If \( q \in U_q \cap U_p \), then there exist a connected neighbourhood \( V \subset U_q \cap U_p \) and a diffeomorphism \( F: f_q^{-1}(V)_0 \rightarrow f_p^{-1}(V) \), where \( f_p^{-1}(V)_0 \) is the connected component of \( f_p^{-1}(V) \) containing \( q \), such that \( F(\sigma(x)) = \rho(\sigma)(F(x)) \), for any \( x \), and \( \sigma \in \text{Stab}_{\Gamma_p}(q) \), where \( \rho: \text{Stab}_{\Gamma_p}(q) \rightarrow \Gamma_q \) is a group isomorphism.

For each \( p \in X \), let \( n_p = \#\Gamma_p \) be the order of the orbifold point (if \( n_p = 1 \) the point is smooth, also called non-orbifold point). The singular locus of the orbifold is the set \( S = \{p \in X \mid n_p > 1\} \). Therefore \( M - S \) is a smooth \( n \)-dimensional manifold. The singular locus \( S \) is stratified: if we write \( S_k = \{p \mid n_p = k\} \), and consider its closure \( \overline{S_k} \), then \( \overline{S_k} \) inherits the structure of an orbifold. In particular \( S_k \) is a smooth manifold, and the closure consists of some points of \( S_l \), \( l \geq 2 \).

We say that the orbifold is locally oriented if \( \Gamma_p \subset \text{GL}(n, \mathbb{R}) \) for any \( p \in X \). As \( \Gamma_p \) is finite, we can choose a metric on \( \tilde{U}_p \) such that \( \Gamma_p \subset \text{SO}(n) \). An element \( \sigma \in \Gamma_p \) admits a basis in which it is written as

\[
\sigma = \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix} \right),
\]

for \( \theta_1, \ldots, \theta_r \in (0, 2\pi) \). In particular, the set of points fixed by \( \sigma \) is of codimension \( 2r \). Therefore the set of singular points \( S \cap U_p \) is of codimension \( \geq 2 \), and hence \( X - S \) is connected (if \( X \) is connected). Also we say that the orbifold \( X \) is oriented if it is locally oriented and \( X - S \) is oriented.

A natural example of orbifold appears when we take a smooth manifold \( M \) and a finite group \( \Gamma \) acting on \( M \) smoothly and effectively. Then \( \widetilde{M} = M/\Gamma \) is an orbifold. If \( M \) is
oriented and the action of \( \Gamma \) preserves the orientation, then \( \hat{M} \) is an oriented orbifold. Note that for every \( \hat{p} \in \hat{M} \), the group \( \Gamma_{\hat{p}} \) is the stabilizer of \( p \in M \), with \( \hat{p} = \hat{\pi}(p) \) under the natural projection \( \hat{\pi}: M \to \hat{M} \).

Let \( X \) be an orbifold of dimension \( n \). An orbifold \( k \)-form \( \alpha \) on \( X \) consists of a collection of differential \( k \)-forms \( \alpha_p \) (\( p \in X \)) on each open \( \tilde{U}_p \) which are \( \Gamma_p \)-equivariant and that match under the compatibility maps between different charts.

The space of orbifold \( k \)-forms on \( X \) is denoted by \( \Omega^k_{\text{orb}}(X) \). The wedge product of orbifold forms and the exterior differential \( d \) on \( X \) are well defined. Thus, we have

\[
d: \Omega^k_{\text{orb}}(X) \longrightarrow \Omega^{k+1}_{\text{orb}}(X).
\]

The cohomology of \( (\Omega^k_{\text{orb}}(X), d) \) is the cohomology of the topological space \( X \) with real coefficients, \( H^*(X) \) (see [7, Proposition 2.13]).

**Remark 1.** Suppose that \( X = M/\Gamma \) is an orbifold, where \( M \) is a smooth manifold and \( \Gamma \) is a finite group acting smoothly and effectively on \( M \). Then, the definition of orbifold forms implies that any \( \Gamma \)-invariant differential \( k \)-form \( \alpha \) on \( M \) defines an orbifold \( k \)-form \( \hat{\alpha} \) on \( X \), and vice-versa. Moreover, it is straightforward to check that the exterior derivative on \( M \) preserves \( \Gamma \)-invariance. Thus, if \( (\Omega^k(M))^{\Gamma} \) denotes the space of the \( \Gamma \)-invariant differential \( k \)-forms on \( M \), then \( H^k(M)^{\Gamma} \subset H^k(M) \) is the subspace of the de Rham cohomology classes of degree \( k \) on \( M \) such that each of these classes has a representative that is a \( \Gamma \)-invariant differential \( k \)-form, then we have

\[
\Omega^k_{\text{orb}}(X) = (\Omega^k(M))^{\Gamma}, \quad H^k(X) = H^k(M)^{\Gamma}.
\]

**Definition 5.** Let \( X \) be a 7-dimensional orbifold. We call \( \varphi \in \Omega^3_{\text{orb}}(X) \) an orbifold \( G_2 \) form on \( X \) if, for each \( p \in X \), \( \varphi_p \) is a \( G_2 \) form (in the sense of Definition 2) on the open \( \tilde{U}_p \subset \mathbb{R}^7 \) of the orbifold chart \((U_p, \tilde{U}_p, \Gamma_p, \varphi_p)\). If in addition \( \varphi \) is also closed (\( d\varphi = 0 \)) we call \( \varphi \) an orbifold closed \( G_2 \) form.

An orbifold \( G_2 \)-structure can also be defined as a reduction of the orbifold frame bundle from \( \text{GL}(7, \mathbb{R}) \) to \( G_2 \), as in the case of smooth manifolds.

If \( M \) is a smooth 7-manifold with a closed \( G_2 \) form \( \varphi \), and \( \Gamma \) is a finite group acting effectively on \( M \) and preserving \( \varphi \), then \( \varphi \) induces an orbifold closed \( G_2 \) form on the 7-orbifold \( \hat{M} = M/\Gamma \).

**Definition 6.** Let \( X \) be a 7-dimensional orbifold with an orbifold closed \( G_2 \) form \( \varphi \). A closed \( G_2 \) resolution of \((X, \varphi)\) consists of a smooth manifold \( \tilde{X} \) with a closed \( G_2 \) form \( \tilde{\varphi} \) and a map \( \pi: \tilde{X} \to X \) such that:

- \( \pi \) is a diffeomorphism \( \tilde{X} - E \to X - S \), where \( S \subset X \) is the singular locus and \( E = \pi^{-1}(S) \) is the exceptional locus.
- \( \tilde{\varphi} \) and \( \pi^*\varphi \) agree in the complement of a small neighbourhood of \( E \).

### 3. Formality of manifolds and orbifolds

In this section we review some definitions and results about formal manifolds and formal orbifolds (see [3, 12, 13, 17] for more details).
We work with differential graded commutative algebras, or DGAs, over the field $\mathbb{R}$ of real numbers. The degree of an element $a$ of a DGA is denoted by $|a|$. A DGA $(A, d)$ is said to be minimal if:

1. $A$ is free as an algebra, that is $A$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
2. there is a collection of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well ordered set $I$, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$, and each $da_\tau$ is expressed in terms of the previous $a_\mu$, $\mu < \tau$. This implies that $da_\tau$ does not have a linear part.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. In our context, the main example of DGA is the de Rham complex $(\Omega^\ast(M), d)$ of a smooth manifold $M$, where $d$ is the exterior differential.

The cohomology of a differential graded commutative algebra $(A, d)$ is denoted by $H^\ast(A)$. This space is naturally a DGA with the product inherited from that on $A$ while the differential on $H^\ast(A)$ is identically zero. A DGA $(A, d)$ is connected if $H^0(A) = \mathbb{R}$, and it is 1-connected if, in addition, $H^1(A) = 0$.

We say that $(\bigwedge V, d)$ is a minimal model of a differential graded commutative algebra $(A, d)$ if $(\bigwedge V, d)$ is minimal and there exists a morphism of differential graded algebras $\rho : (\bigwedge V, d) \rightarrow (A, d)$ inducing an isomorphism $\rho^* : H^\ast(\bigwedge V) \rightarrow H^\ast(A)$ on cohomology. In [22], Halperin proved that any connected differential graded algebra $(A, d)$ has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved by Deligne, Griffiths, Morgan and Sullivan [12, 21, 39].

A minimal model of a connected smooth manifold $M$ is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^\ast(M), d)$ of differential forms on $M$. If $M$ is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to the space $V^i$ of generators in degree $i$, for any $i$. The latter also happens when $i > 1$ and $M$ is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$ (see [12]).

We say that a DGA $(A, d)$ is a model of a manifold $M$ if $(A, d)$ and $M$ have the same minimal model. In this case, if $(\bigwedge V, d)$ is the minimal model of $M$, we have

$$(A, d) \overset{\nu}{\leftarrow} (\bigwedge V, d) \overset{\rho}{\rightarrow} (\Omega^\ast(M), d),$$

where $\rho$ and $\nu$ are quasi-isomorphisms.

A minimal algebra $(\bigwedge V, d)$ is formal if there exists a morphism of differential algebras $\psi : (\bigwedge V, d) \rightarrow (H^\ast(\bigwedge V), 0)$ inducing the identity map on cohomology. A DGA $(A, d)$ is formal if its minimal model is formal. A smooth manifold $M$ is formal if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, symmetric spaces, flag manifolds, and compact Kähler manifolds.

The formality property of a minimal algebra is characterized as follows.
Theorem 7 ([12]). A minimal algebra \((\bigwedge V, d)\) is formal if and only if the space \(V\) can be decomposed into a direct sum \(V = C \oplus N\) with \(d(C) = 0\), \(d\) is injective on \(N\) and such that every closed element in the ideal \(I\) generated by \(N\) in \(\bigwedge V\) is exact.

This characterization of formality can be weakened using the concept of \(s\)-formality introduced in [17].

Definition 8. A minimal algebra \((\bigwedge V, d)\) is \(s\)-formal \((s > 0)\) if for each \(i \leq s\) the space \(V^i\) of generators of degree \(i\) decomposes as a direct sum \(V^i = C^i \oplus N^i\), where the spaces \(C^i\) and \(N^i\) satisfy the following conditions:

1. \(d(C^i) = 0\),
2. the differential map \(d : N^i \rightarrow \bigwedge V\) is injective, and
3. any closed element in the ideal \(I_s = I(\bigoplus_{i \leq s} N^i)\), generated by the space \(\bigoplus_{i \leq s} N^i\) in the free algebra \(\bigwedge(\bigoplus_{i \leq s} V^i)\), is exact in \(\bigwedge V\).

A smooth manifold \(M\) is \(s\)-formal if its minimal model is \(s\)-formal. Clearly, if \(M\) is formal then \(M\) is \(s\)-formal for every \(s > 0\). The main result of [17] shows that sometimes the weaker condition of \(s\)-formality implies formality.

Theorem 9 ([17]). Let \(M\) be a connected and orientable compact differentiable manifold of dimension \(2n\) or \((2n - 1)\). Then \(M\) is formal if and only if it is \((n - 1)\)-formal.

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 9 implies that any simply connected compact manifold of dimension at most six is formal. (This result was proved earlier in [34].)

Note that Crowley and Nordström in [11] have introduced the Bianchi-Massey tensor on a manifold \(M\), and they prove that if \(M\) is a closed \((n - 1)\)-connected \((4n - 1)\)-manifold, with \(n \geq 2\), then \(M\) is formal if and only if the Bianchi-Massey tensor vanishes.

For later use, we recall here the following characterization of the \(s\)-formality of a manifold.

Lemma 10 ([13]). Let \(M\) be a manifold with minimal model \((\bigwedge V, d)\). Then \(M\) is \(s\)-formal if and only if there is a map of differential algebras

\[\vartheta : (\bigwedge V^{\leq s}, d) \rightarrow (H^*(M), d = 0),\]

such that the map \(\vartheta^* : H^*(\bigwedge V^{\leq s}, d) \rightarrow H^*(M)\) induced on cohomology is equal to the map \(\varphi^* : H^*(\bigwedge V^d, d) \rightarrow H^*(\bigwedge V, d) = H^*(M)\) induced by the inclusion \(\varphi : (\bigwedge V^{\leq s}, d) \rightarrow (\bigwedge V, d)\).

In particular, \(\vartheta^* : H^i(\bigwedge V^{\leq s}) \rightarrow H^i(M)\) is an isomorphism for \(i \leq s\), and a monomorphism for \(i = s + 1\).

Definition 11. Let \(X\) be an orbifold. A minimal model for \(X\) is a minimal model \((\bigwedge V, d)\) for the DGA \((\Omega^*_{orb}(X), d)\). The orbifold \(X\) is formal if its minimal model is formal.

For a simply connected orbifold \(X\), the dual of the real homotopy vector space \(\pi_i(X) \otimes \mathbb{R}\) is isomorphic to the space \(V^i\) of generators in degree \(i\), for any \(i\), where \(\pi_i(X)\) is the homotopy group of order \(i\) of the underlying topological space in \(X\). In fact, the proof
where \( A \) of the form

\[
6 \times \mathbb{Z} \times 2
\]

connected compact orbifold of dimension at most \( n \) or \( 2n \). Let \( \Gamma \) be the discrete subgroup of \( G \) consisting of real matrices whose entries \( (x_1, x_2, \ldots, x_7) \in 2\mathbb{Z} \times \mathbb{Z}^6 \), that is \( x_i \) are integer numbers and \( x_1 \) is even. It is easy to see that \( \Gamma \) is a subgroup of \( G \). So the quotient space of right cosets

\[
M = \Gamma \backslash G
\]

(3)

Moreover, the proof of Theorem 9 given in [12] only uses that the cohomology \( H^\ast(M) \) is a Poincaré duality algebra. By [37], we know that the singular cohomology of an orbifold also satisfies a Poincaré duality. Thus, Theorem 9 also holds for compact connected orientable orbifolds, that is we have

**Proposition 12.** Let \( X \) be a connected and orientable compact orbifold of dimension \( 2n \) or \( (2n-1) \). Then \( X \) is formal if and only if it is \((n-1)\)-formal. In particular, any simply connected compact orbifold of dimension at most 6 is formal.

4. A 7-orbifold with an orbifold closed \( G_2 \) form

Let \( G \) be the connected nilpotent Lie group of dimension 7 consisting of real matrices of the form

\[
a = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
\]

where \( A_1 \) and \( A_2 \) are the matrices

\[
A_1 = \begin{pmatrix}
1 & -x_2 & x_1 & x_4 & -x_1x_2 & x_6 \\
0 & 1 & 0 & -x_1 & x_5 & 1/2 x_1^2 \\
0 & 0 & 1 & 0 & -x_2 & -x_4 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
A_2 = \begin{pmatrix}
1 & -x_3 & x_1 & x_5 & -x_1x_3 & x_7 \\
0 & 1 & 0 & -x_1 & x_4 & 1/2 x_1^2 \\
0 & 0 & 1 & 0 & -x_3 & -x_5 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where \( x_i \in \mathbb{R} \), for any \( i \in \{1, \ldots, 7\} \). Then, a global system of coordinate functions \( \{x_1, \ldots, x_7\} \) for \( G \) is given by \( x_i(a) = x_i \), with \( i \in \{1, \ldots, 7\} \). Note that if a matrix \( A \in G \) has coordinates \( a_i \), then the change of coordinates of \( a \in G \) by the left translation \( L_A \) are given by

\[
L_A^i(x_i) = x_i \circ L_A = x_i + a_i, \quad i = 1, 2, 3,
L_A^4(x_4) = x_4 + a_2x_1 + a_4,
L_A^5(x_5) = x_5 + a_3x_1 + a_5,
L_A^6(x_6) = x_6 - 1/2 a_2x_1^2 - a_1x_4 - a_1a_2x_1 + a_6,
L_A^7(x_7) = x_7 - 1/2 a_3x_1^2 - a_1x_5 - a_1a_3x_1 + a_7.
\]

A standard calculation shows that a basis for the left invariant 1–forms on \( G \) consists of

\[
\{dx_1, dx_2, dx_3, dx_4 - x_2dx_1, dx_5 - x_3dx_1, dx_6 + x_1dx_4, dx_7 + x_1dx_5\}.
\]

Let \( \Gamma \) be the discrete subgroup of \( G \) consisting of matrices whose entries \( (x_1, x_2, \ldots, x_7) \in 2\mathbb{Z} \times \mathbb{Z}^6 \), that is \( x_i \) are integer numbers and \( x_1 \) is even. It is easy to see that \( \Gamma \) is a subgroup of \( G \). So the quotient space of right cosets

\[
M = \Gamma \backslash G
\]
is a compact 7-manifold. Hence the forms \( dx_1, dx_2, dx_3, dx_4 - x_2 dx_1, dx_5 - x_3 dx_1, dx_6 + x_1 dx_4, dx_7 + x_1 dx_5 \) descend to 1-forms \( e^1, e^2, e^3, e^4, e^5, e^6, e^7 \) on \( M \) such that
\[
de^i = 0, \quad i = 1, 2, 3, \quad de^4 = e^{12}, \quad de^5 = e^{13}, \quad de^6 = e^{14}, \quad de^7 = e^{15}, \quad (4)
\]
and such that at each point of \( M, \{e^1, e^2, e^3, e^4, e^5, e^6, e^7\} \) is a basis for the 1-forms on \( M \). Here, \( e^{12} \) stands for \( e^1 \wedge e^2 \), and so on.

**Lemma 13.** The nilmanifold \( M \) defined by \( (2) \) is diffeomorphic to the mapping torus \( M_\nu \) of the diffeomorphism of the 6-torus \( \nu: T^6 = \mathbb{R}^6/\mathbb{Z}^6 \rightarrow T^6 = \mathbb{R}^6/\mathbb{Z}^6 \), induced by the linear automorphism of \( \mathbb{R}^6 \) associated to the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 \\
2 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 2 & -2 & 1
\end{pmatrix}.
\]

**Proof.** Consider the projection
\[
p: M \rightarrow S^1 = \mathbb{R}/2\mathbb{Z}
\]
\[
[(x_1, \ldots, x_7)] \mapsto (x_1 + 2\mathbb{Z}). \quad (5)
\]
The fiber over \( x_1 + 2\mathbb{Z} \in S^1 \) is the set of equivalence classes of \( \mathbb{R}^6 \) by the equivalence relation
\[
(x_2, \ldots, x_7) \sim (x_2 + a_2, x_3 + a_3, x_4 + a_2 x_1 + a_4, x_5 + a_3 x_1 + a_5, x_6 - \frac{1}{2} a_2 x_1^2 + a_6, x_7 - \frac{1}{2} a_3 x_1^2 + a_7).
\]
The quotient \( \mathbb{R}^6/\sim \) is then the 6-torus \( \mathbb{R}^6/\Lambda(x_1) \) with lattice \( \Lambda(x_1) \subset \mathbb{R}^6 \) given by the span over \( \mathbb{Z} \) of the columns of the matrix
\[
B(x_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 0 & 0 \\
0 & x_1 & 0 & 1 & 0 & 0 \\
-\frac{1}{2} x_1^2 & 0 & 0 & 1 & 0 \\
0 & -\frac{1}{2} x_1^2 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
The fiber \( p^{-1}(x_1 + 2\mathbb{Z}) = \mathbb{R}^6/\Lambda(x_1) \) can be identified with the standard torus \( T^6 = \mathbb{R}^6/\mathbb{Z}^6 \), by the diffeomorphism
\[
f_{x_1}: \mathbb{R}^6/\Lambda(x_1) \rightarrow \mathbb{R}^6/\mathbb{Z}^6
\]
\[
[(x_2, \ldots, x_7)] \mapsto [B(x_1)^{-1}(x_2, \ldots, x_7)].
\]
Therefore, \( p^{-1}([0, 2]/2\mathbb{Z}) \cong ([0, 2] \times T^6)/\nu \), for an appropriate diffeomorphism \( \nu: \{0\} \times T^6 \cong \{2\} \times T^6 \), that we describe next.

The manifold \( M \) is obtained by gluing the boundaries of \( p^{-1}([0, 2]/2\mathbb{Z}) \) via the map
\[
h: p^{-1}(0 + 2\mathbb{Z}) \rightarrow p^{-1}(2 + 2\mathbb{Z}),
\]
\[
[(x_2, x_3, x_4, x_5, x_6, x_7)] \in \mathbb{R}^6/\Lambda(x_1) \mapsto [(x_2, x_3, x_4, x_5, x_6 - 2x_4, x_7 - 2x_5)] \in \mathbb{R}^6/\Lambda(x_1),
\]
that corresponds to the matrix
\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

Thus $M$ is the manifold obtained from $[0,2] \times T^6$ by identifying the ends $\{0\} \times T^6 \cong \{2\} \times T^6$ by the diffeomorphism $\nu$ of $T^6$ induced by the linear automorphism of $\mathbb{R}^6$

\[
(x_2, \ldots, x_7) \rightarrow E (x_2, \ldots, x_7)^T,
\]

where
\[
E = B(2)^{-1} C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

Swapping the coordinates $(x_2, \ldots, x_7)$ to the order $(x_2, x_4, x_5, x_3, x_6, x_7)$, we get the matrix in the statement. \hfill \Box

Now we consider the action of the finite group $\mathbb{Z}_2$ on $G$ given by
\[
\rho: G \rightarrow G \\
(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7),
\]

where $\rho$ is the generator of $\mathbb{Z}_2$. This action satisfies the condition $\rho(a \cdot b) = \rho(a) \cdot \rho(b)$, for $a, b \in G$, where $\cdot$ denotes the natural group structure of $G$. This follows since $\rho$ is the conjugation by the matrix
\[
J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_1 = \text{diag}(1, -1, -1, 1, 1, -1), \quad J_2 = \text{diag}(1, 1, -1, -1, -1, 1),
\]

i.e. $\rho(a) = ja^{-1} j$. Moreover, $\rho(\Gamma) = \Gamma$. Thus, $\rho$ induces an action on the quotient $M = \Gamma \backslash G$. Denote by
\[
\rho: M \rightarrow M
\]
the $\mathbb{Z}_2$-action. Then, the induced action on the 1-forms $e^i$ is given by
\[
\rho^* e^i = -e^i, \quad i = 1, 2, 5, 6, \quad \rho^* e^j = e^j, \quad j = 3, 4, 7.
\]

**Proposition 14.** The quotient space $\widetilde{M} = M/\mathbb{Z}_2$ is a compact 7-orbifold with first Betti number $b_1(\widetilde{M}) = 1$, and with an orbifold closed $G_2$ form.

**Proof.** Since the $\mathbb{Z}_2$-action on $M$ is smooth and effective, the quotient space $\widetilde{M} = M/\mathbb{Z}_2$ is a 7-orbifold, which is compact because $M$ is compact. Moreover, using Nomizu’s theorem \[35\], from (4) we have that the first de Rham cohomology group of $M$ is $H^1(M) = \langle [e_1], [e_2], [e_3] \rangle$. Then, as a consequence of (2) and from (7), the first cohomology group of $\widetilde{M}$ is
\[
H^1(\widetilde{M}) = H^1(M)^{\mathbb{Z}_2} = \langle [e_1], [e_2], [e_3] \rangle^{\mathbb{Z}_2} = \langle [e_3] \rangle.
So the first Betti number of \( \hat{M} \) is \( b_1 = 1 \).

We define the 3-form \( \varphi \) on \( M \) given by
\[
\varphi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.
\] (8)

Clearly, \( \varphi \) is a closed \( G_2 \) form on \( M \) which is \( \mathbb{Z}_2 \)-invariant. Indeed, on the right-hand side of (8) all the terms, except the last 3 terms, are closed. But \( d(e^{257} + e^{347} + e^{356}) = 0 \), and so \( \varphi \) is closed. Moreover, each term on the right-hand side of (8) is \( \mathbb{Z}_2 \)-invariant. Thus \( \varphi \) induces an orbifold closed \( G_2 \) form \( \hat{\varphi} \) on \( \hat{M} \).

Denote by
\[
\hat{\pi} : M \to \hat{M}
\]
the natural projection. The singular locus \( S \) of \( \hat{M} \) is the image by \( \hat{\pi} \) of the set \( S' \) of points in \( M \) that are fixed by the \( \mathbb{Z}_2 \)-action defined by (8). So \( S \) consists of all the 3-dimensional spaces \( S_a = \hat{\pi}(S'_a) = S'_a/\mathbb{Z}_2 \), where
\[
S'_a = \{ (a_1, a_2, x_3, x_4, a_5, a_6, x_7) \mid (x_3, x_4, x_7) \in T^3 \} \subset M,
\]
and
\[
a = (a_1, a_2, a_5, a_6) \in A = \{0, 1\} \times (\{0, 1/2\})^3.
\]

Therefore, there are \( 2^4 = 16 \) components of the singular locus of the orbifold.

The set \( S'_a \) is included in the fiber \( p^{-1}(0+2\mathbb{Z}) \) or \( p^{-1}(1+2\mathbb{Z}) \) of the projection \( p \) defined by (3). Thus, \( S'_a \) is a Lie subgroup of \( T^6 \), hence it is abelian and so isomorphic to a 3-torus \( T^3 \). Consequently, \( S \) is a disjoint union of 16 copies de \( T^3 \).

5. Local model around the singular locus

To desingularize the orbifold \( \hat{M} = M/\mathbb{Z}_2 \) considered in Proposition 14, we study here each of the 16 connected components \( S_a \) (defined before) of the singular locus \( S \) of \( \hat{M} \).

For \( a = (a_1, a_2, a_5, a_6) \in A = \{0, 1\} \times (\{0, 1/2\})^3 \), consider the element \( a \in G \) given by \( a = (a_1, a_2, 0, 0, a_5, a_6, 0) \in G \). Then, the left translation \( L_a : G \to G \) is such that \( L_a(\Gamma) = \Gamma \), and so it induces a diffeomorphism \( L_a : M \to M \) that preserves the \( G_2 \) form \( \varphi \) on \( M \) defined by (8), and it satisfies
\[
L_a(\rho(b)) = a \rho(b) = \rho(a) \rho(b) = \rho(ab) = \rho(L_a(b)),
\]
for every \( b \in M \). So \( L_a : M \to M \) defines an orbifold diffeomorphism \( L_a : \hat{M} \to \hat{M} \) sending \( S_0 \) to \( S_a \), where \( 0 = (0, 0, 0, 0) \in A \). Therefore, doing the desingularization around \( S_0 \), we can translate it to the other \( S_a \) via the orbifold diffeomorphism \( L_a \).

From now on, we focus on \( S_0 = \{ (0, 0, x_3, x_4, 0, 0, x_7) \} \subset \hat{M} \). We consider the corresponding set
\[
S' = S'_0 = \{ (0, 0, x_3, x_4, 0, 0, x_7) \} \subset M,
\]
which is a fixed locus of the \( \mathbb{Z}_2 \)-action (given by (8)) and isomorphic to a 3-torus \( T^3 \).

The following proposition allows us to show an appropriate local model around \( S_0 \) that we will use in the next section to desingularize \( S_0 \).
Proposition 15. There exist neighbourhoods $U'$ and $U''$ of $S'$ in the manifold $M$ with $U'' \subset U'$, and there are closed $G_2$ forms $\phi$ and $\psi$ on $M$ and $U'$, respectively which are invariant by the $\mathbb{Z}_2$-action given by (\ref{eqn:z2_action}), and such that $\phi$ is equal to the $G_2$ form $\varphi$, defined by (\ref{eqn:varphi}), outside $U' \cong T^3 \times B^4_\epsilon$ and $\phi = \psi$ is the standard $G_2$ form on $U'' \cong T^3 \times B^4_\epsilon$.

Proof. We define a small neighbourhood $U'$ of $S'$ in $M$ as follows. A point in $U'$ is given by $(x_1, \ldots, x_7)$, with $(x_1, x_2, x_3, x_6)$ small and such that, under the equivalence relation given by the action of $\Gamma$ on the points of $U'$,

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \sim (x_1, x_2, x_3 + a_3, x_4 + a_4, x_5 + a_3 x_1, x_6, x_7 + a_7 - \frac{1}{2} a_3 x_1^2).$$

It is natural to introduce on $U'$ the coordinates $(x'_1, \ldots, x'_7)$ defined by

$$\begin{align*}
x'_5 &= x_5 - x_1 x_3, \\
x'_7 &= x_7 + \frac{1}{2} x_3 x_1^2, \\
x'_j &= x_j, \quad j \neq 5, 7.
\end{align*}$$

That is,

$$U' \cong T^3 \times B^4_\epsilon,$$

where $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$ has coordinates $(x'_3, x'_4, x'_7)$, and $B^4_\epsilon \subset \mathbb{R}^4$ has coordinates $(x'_1, x'_2, x'_5, x'_6)$.

Note that for $\epsilon < \frac{1}{2}$, the neighbourhoods $L_a(U') \cap L_b(U') = \emptyset$, for any $a, b \in \mathbb{A}$ distinct.

If we restrict the 1-forms $e^1, \ldots, e^7$ to $S'$, by setting $x'_1 = x'_2 = x'_5 = x'_6 = 0$, we get

$$\begin{align*}
e^5|_{S'} &= dx_5 - x_3 dx_1 = dx'_5, \\
e^7|_{S'} &= dx_7 = dx'_7,
\end{align*}$$

since $dx'_7 = dx_7 + \frac{1}{2} x_1^2 dx_3 + x_1 x_3 dx_1$ and $dx'_5 = dx_5 - x_1 dx_3 - x_3 dx_1$.

Thus, $e^j|_{S'} = dx'_j$, $1 \leq j \leq 7$, and the restriction $\varphi|_{S'}$ to $S' \subset U'$ of the closed $G_2$ form $\varphi$ on $M$ given by (\ref{eqn:varphi}), that is

$$\varphi = e^{123} + e^{145} + e^{167} - e^{246} + e^{257} + e^{347} + e^{356}$$

$$= e^{347} + e^3(e^{12} + e^{56}) - e^4(e^{15} - e^{26}) + e^7(e^{16} + e^{25})$$

coincides with the restriction $\psi|_{S'}$ to $S'$ of the standard $G_2$ form $\psi$ on $U' \cong T^3 \times B^4_\epsilon$ given by

$$\psi = dx'_3 \wedge (dx'_4 + dx'_5) - dx'_4 \wedge (dx'_5 - dx'_6) + dx'_7 \wedge (dx'_1 - dx'_2),$$

that is, we have $\psi|_{S'} = \varphi|_{S'}$. Here, $dx'_12$ stands for $dx'_1 \wedge dx'_2$, and so on. Moreover, using (\ref{eqn:z2_action}) and (\ref{eqn:varphi}), one can check that the $G_2$ form $\psi$ on $U' \cong T^3 \times B^4_\epsilon$ is invariant by the $\mathbb{Z}_2$-action.

Now let us modify the $G_2$-structure $\varphi$ on $M$ inside $U' \cong T^3 \times B^4_\epsilon$ so that it is equal to the 3-form $\psi$ given by (\ref{eqn:psi}) on a smaller neighbourhood $U''$ of $S'$. The 3-form $\psi - \varphi$ is closed on $U'$, and it satisfies the condition $(\psi - \varphi)|_{T^3 \times \{0\}} = 0$, hence it defines the zero de
Rham cohomology class on $U'$. So $\psi - \varphi = d\alpha$, for some 2-form $\alpha$ on $U'$. Moreover, as $|\psi - \varphi| \leq Cr$, where $r$ is the radial coordinate of $B^4_\epsilon \subset \mathbb{R}^4$, we can take $|\alpha| \leq Cr^2$. Indeed, following the standard procedure of [19, p. 542], we can use the homotopy operator to determine $\alpha$. Write the 3-form $\varphi - \psi$ as

$$\psi - \varphi = \beta_0 \wedge dr + \beta_1,$$

for some closed forms $\beta_0$ and $\beta_1$. The 2-form $\alpha = \int_0^r \beta_0 \, dr$ is smooth and satisfies $d\alpha = \psi - \varphi$.

On $B^4_\epsilon$ consider a bump function $\rho(r)$ such that $\rho(r) = 1$ for $r \leq \epsilon/2$, and $\rho(r) = 0$ for $\epsilon \geq r \geq 3\epsilon/4$. Define the 3-form $\phi$ on $M$ by

$$\phi = \varphi + d(\rho \alpha).$$

Then $\phi = \varphi$ outside $U'$ and $\phi = \psi$ in

$$U'' \cong T^3 \times B^4_{\epsilon/2}.$$

Moreover, $|d\rho| \leq C/\epsilon$ for a uniform constant, so $|d(\rho \alpha)| \leq C\epsilon$. For $\epsilon > 0$ small enough, $\phi$ is non-degenerate, hence it defines a closed $G_2$ form on $M$. Now, using (12), one can check that the $G_2$ form $\phi$ is still $\mathbb{Z}_2$-invariant \(\Box\)

As a consequence of Proposition 15 we have the following corollary.

**Corollary 16.** There exist neighbourhoods $U$ and $V$ of $S_0$ in the orbifold $\hat{M} = M/\mathbb{Z}_2$ with $V \subset U$, and there are orbifold closed $G_2$ forms $\hat{\psi}$ and $\hat{\psi}$ on $\hat{M} = M/\mathbb{Z}_2$ and $U$, respectively such that $\hat{\psi} = \hat{\psi}$ outside $U$, and $\hat{\psi} = \hat{\psi}$ in the neighbourhood $V$ of $S_0$. Moreover, the singular locus $S$ of $\hat{M}$ is covered by the disjoint union $\bigsqcup_{a \in A} L_a(U)$.

**Proof.** We define the neighbourhoods $U$ and $V$ of $S_0$ by

$$U = U'/\mathbb{Z}_2 \cong T^3 \times (B^4_\epsilon/\mathbb{Z}_2), \quad V = U''/\mathbb{Z}_2 \cong T^3 \times (B^4_{\epsilon/2}/\mathbb{Z}_2),$$

where $U'$ and $U''$ are given by (10) and (13), respectively. Consider the closed $G_2$ forms $\psi$ and $\phi$ defined by (11) and (12), respectively. By Proposition 15, both these forms are $\mathbb{Z}_2$-invariant, and hence they descend to orbifold closed $G_2$ forms $\hat{\psi}$ and $\hat{\psi}$ on $U$ and $\hat{M}$, respectively and they satisfy the required conditions.

As we have noticed in the proof of Proposition 15 we have $L_a(U') \cap L_b(U') = \emptyset$, for any $a,b \in A$ distinct. So, $S \subset \bigsqcup_{a \in A} L_a(U)$. \(\Box\)

**Remark 2.** Note that the $G_2$ form $\psi$ given by (11) can be defined as the restriction to $U'$ of the $G_2$ form $\Psi$ on $T^3 \times \mathbb{C}^2$ defined by (17) (see below). Firstly, we see that in the coordinates $(x_1', \ldots, x_7')$, defined by (9), the action of $\mathbb{Z}_2$ on $U'$ is given by

$$(x_1', x_2', x_5', x_6') \mapsto (-x_1', -x_2', -x_5', -x_6'),$$

and fixing $(x_3', x_4', x_7')$. Introduce now the complex coordinates

$$z_1 = x_1' + ix_2',$$
$$z_2 = x_5' + ix_6',$$

so that $U' \cong T^3 \times B^4_\epsilon$, where $B^4_\epsilon \subset \mathbb{C}^2$, and the action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ is given by

$$\rho: \mathbb{C}^2 \to \mathbb{C}^2,$$

$$\rho(z_1, z_2) \mapsto (-z_1, -z_2).$$

(15)
The natural SU(2)-structure on \( \mathbb{C}^2 \) is given by the Kähler form \( \omega \) and the \((2,0)\)-form \( \Omega \) defined, respectively by

\[
\omega = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = dx'_{12} + dx'_{56},
\]

\[
\Omega = dz_1 \wedge dz_2 = (dx'_{15} - dx'_{26}) + i(dx'_{25} + dx'_{16}).
\]  

The action of \( Z_2 \) on \( \mathbb{C}^2 \) given by \( (15) \) preserves both these forms. The standard closed \( G_2 \)-structure on \( T^3 \times \mathbb{C}^2 \) is given by

\[
\Psi = dx'_{345} + dx'_{3} \wedge \omega - dx'_{4} \wedge \Re\Omega + dx'_{7} \wedge \Im\Omega.
\]  

So the restriction \( \Psi|_V \), to \( U' \) coincides with the \( 3 \)-form \( \psi \) defined by \( (11) \). Then, Corollary \( (16) \) implies that

\[
\hat{\Psi}|_V = \hat{\phi} = \hat{\psi}
\]  

in the neighbourhood \( V \) of \( S_0 \), where \( \hat{\Psi} \) is the orbifold closed \( G_2 \) form induced by \( \Psi \) on \( T^3 \times (\mathbb{C}^2/Z_2) \), and \( \hat{\Psi}|_V \) is the restriction of \( \hat{\Psi} \) to \( V \).

6. RESOLVING THE SINGULAR LOCUS

In this section we desingularize the singular locus \( S \) of \( \hat{M} \) to get a smooth compact 7-manifold \( \hat{M} \) diffeomorphic to \( \hat{M} \) outside \( S \), and such that \( \hat{M} \) has the required properties, i.e. with first Betti number \( b_1(\hat{M}) = 1 \), with no torsion-free \( G_2 \)-structures and with a closed \( G_2 \) form \( \hat{\varphi} \) such that \( \hat{\varphi} = \hat{\varphi} \) outside a neighbourhood of \( S \), where \( \hat{\varphi} \) is the orbifold closed \( G_2 \) form on \( \hat{M} \) given in the proof of Proposition \( (14) \).

**Theorem 17.** There exists a smooth compact manifold \( \hat{M} \) with a closed \( G_2 \) form \( \hat{\varphi} \) and with first Betti number \( b_1(\hat{M}) = 1 \), such that \( (\hat{M}, \hat{\varphi}) \) is a closed \( G_2 \) resolution of \( (\hat{M}, \hat{\varphi}) \) (in the sense of Definition \( (5) \)).

**Proof.** We know that doing the desingularization around the component \( S_0 \) of \( S \), we can translate it to the other components \( S_a \) of the singular locus \( S \) via the diffeomorphism \( L_a : \hat{M} \to \hat{M} \) defined in the section \( (5) \).

Let \( V \) be the neighbourhood of \( S_0 \) given by \( (14) \) with the orbifold closed \( G_2 \) form \( \hat{\Psi}|_V \), induced on \( V \) by the \( G_2 \) form \( \Psi \) defined by \( (17) \). In order to desingularize \( S_0 \), we shall replace the factor \( B_{1/2}/\mathbb{Z}_2 \) of \( V \) by a smooth 4-manifold that agrees with \( B_{1/2}^3/\mathbb{Z}_2 \) in a neighbourhood of its boundary.

Firstly we consider the complex orbifold \( X = \mathbb{C}^2/\mathbb{Z}_2 \). By Remark \( (2) \) we know that the action of \( Z_2 \) on \( \mathbb{C}^2 \) defined by \( (15) \) preserves the natural integrable SU(2)-structure \((\omega, \Omega)\) on \( \mathbb{C}^2 \) given by \( (16) \). (Thus \( \mathbb{Z}_2 \) is a finite subgroup of SU(2).) We resolve the singularity of \( X = \mathbb{C}^2/\mathbb{Z}_2 \) to get a smooth manifold \( \tilde{X} \) with a (non-integrable) SU(2)-structure. This goes as follows: take the blow-up \( \mathbb{C}^2 \) of \( \mathbb{C}^2 \) at the origin. This is given by

\[
\mathbb{C}^2 = \left\{ (z_1, z_2, [w_1, w_2]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid w_1 z_2 = w_2 z_1 \right\}.
\]

Now we quotient \( \tilde{C}^2 \) by \( \mathbb{Z}_2 \) in order to get a smooth manifold

\[
\tilde{X} = \mathbb{C}^2/\mathbb{Z}_2.
\]

and a map \( \pi : \tilde{X} \to X \) such that \( \pi \) is a diffeomorphism \( \tilde{X} - \pi^{-1}(0) \to X - \{0\} \).
The $(2,0)$-form $\Omega = dz_1 \wedge dz_2$ on $X$ extends to a no-where vanishing $(2,0)$-form on $\tilde{X}$, that we call $\Omega$ again. This can be easily checked as follows, using the two affine charts. For the first one, we take $w_1 = 1$, $w_2 = w$, $z_1 = z$, $z_2 = wz$, so the chart of $\tilde{C}^2$ is parametrized by $(z,w) \in \mathbb{C}^2$. The quotient by $\mathbb{Z}_2$ is given by $(z,w) \mapsto (-z,w)$, so $\tilde{X}$ is parametrized by $(u,w) \in \mathbb{C}^2$, with $u = z^2$. The form $\Omega$ in these coordinates $(u,w)$ has the following expression

$$\Omega = dz_1 \wedge dz_2 = dz \wedge (wz) = \frac{1}{2} du \wedge dw.$$  

Thus $\Omega$ is non-zero and is defined on the whole chart. The computations for other chart are similar.

On $\tilde{C}^2$ there exists a Kähler form $\tilde{\omega}$ defined as 

$$\tilde{\omega} = \omega + \delta \omega_{\mathbb{C}P^1},$$  

where $\omega$ is the Kähler form on $\mathbb{C}^2$ given by $\omega_{\mathbb{C}P^1}$, $\delta > 0$ is a small real number, and $\omega_{\mathbb{C}P^1}$ is the natural Fubini-Study form of $\mathbb{C}P^1$. We take the $(1,1)$-form $\omega_{\mathbb{C}P^1}$ on $\tilde{B}_{\epsilon/2}$ in $\tilde{C}^2$, where $\tilde{B}_{\epsilon/2}$ is the blow-up of $B_{\epsilon/2} \subset \mathbb{C}^2$ at the origin, and we restrict $\omega_{\mathbb{C}P^1}$ to

$$A = \tilde{B}_{\epsilon/2} - B_{\epsilon/4} \cong (\epsilon/4, \epsilon/2) \times S^3,$$

hence it must be exact on $A$. By the $\partial \bar{\partial}$-lemma, we have $\omega_{\mathbb{C}P^1} = \partial \bar{\partial} h$, for some function $h$ on $A$. Now we take a bump function $\rho(r)$ which is identically zero for $r \geq \epsilon/2 - \eta$, and identically one for $r \leq \epsilon/4 + \eta$, for a small $\eta > 0$, and set 

$$\tilde{\omega}' = \omega + \delta \partial \bar{\partial} (\rho h).$$

For small $\delta > 0$, $\tilde{\omega}'$ is still a Kähler form on $A$. Moreover, for $r \geq \epsilon/2 - \eta$ we have that $\tilde{\omega}' = \omega$ and, for $r \leq \epsilon/4 + \eta$, we have that $\tilde{\omega}'$ coincides with the standard Kähler form $\tilde{\omega}$ of $\tilde{C}^2$. All the construction is $\mathbb{Z}_2$-invariant, so $\tilde{\omega}'$ descends to the resolution $\tilde{X}$.

We define the $G_2$ form $\tilde{\Psi}$ on $T^3 \times (\tilde{B}_{\epsilon/2}^4/\mathbb{Z}_2)$ by 

$$\tilde{\Psi} = dx_3' \wedge dx_3' + \tilde{\omega}' - dx_4' \wedge \Re \Omega + dx_4' \wedge \Im \Omega.$$  

Thus, for $\epsilon/2 - \eta \leq r < \epsilon/2$, we have $\tilde{\Psi} = \tilde{\Psi}$ on $T^3 \times (\tilde{B}_{\epsilon/2}^4/\mathbb{Z}_2)$, and hence $\tilde{\Psi} = \tilde{\phi}$ on $T^3 \times (\tilde{B}_{\epsilon/2}^4/\mathbb{Z}_2)$ by $[18]$. Now we glue $T^3 \times (\tilde{B}_{\epsilon/2}^4/\mathbb{Z}_2)$ endowed with this $G_2$ form $\tilde{\Psi}$ to $\tilde{M} - (T^3 \times (B_{\epsilon/2}^4/\eta/\mathbb{Z}_2))$ with the $G_2$ form $\tilde{\phi}$ given in Corollary $[16]$ These two glue nicely to give a $G_2$ form $\tilde{\varphi}$ on the resulting smooth manifold $\tilde{M}$.

The map $\pi: \tilde{X} \to X$ defines a map that we denote by the same symbol $\pi: \tilde{M} \to \hat{M}$, which satisfies the conditions of Definition $[16]$. Thus, $(\hat{M}, \hat{\varphi})$ is a closed $G_2$-resolution of $(\tilde{M}, \tilde{\varphi})$.

Finally, that $b_1(\hat{M}) = 1$ follows from the following proposition. 

\begin{proposition}
There is an isomorphism 

$$H^*\left(\hat{M}\right) \cong H^*\left(\tilde{M}\right) \oplus \left(\bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i]\right),$$

\end{proposition}
where \([E_i] \in H^2(\widetilde{M})\) is the class of the exceptional divisor \(E_i \subset \widetilde{X} = \mathbb{C}^2/\mathbb{Z}_2\) with \(1 \leq i \leq 16\).

Proof. Let \(V \subset \widetilde{M}\) be a neighbourhood of the exceptional divisors, that is \(V = \bigsqcup_i V_i\), where \(V_i \cong T^3 \times (B^4/\mathbb{Z}_2) \sim T^3\). Let \(U\) be the complement \(\widetilde{M} - \bigsqcup_i E_i\). Then \(U \cap V = \bigsqcup_i (U \cap V_i)\), where \(U \cap V_i \sim T^3 \times (S^3/\mathbb{Z}_2)\), where \(~\sim~\) means homotopy equivalent.

Let \(\tilde{V} \subset \widetilde{M}\) be the preimage of \(V\) under \(\pi : \widetilde{M} \to \widetilde{M}\). Then \(\tilde{V} = \bigsqcup_i \tilde{V}_i\) with \(\tilde{V}_i \cong T^3 \times (\hat{B}^4/\mathbb{Z}_2) \sim T^3 \times E_i\) and \(E_i \cong \mathbb{C}P^1 \cong S^2\).

The map \(\pi^* : H^k(\widetilde{M}) \to H^k(\widetilde{M})\) is injective. In fact, let \(\alpha \in H^k(\widetilde{M})\) be a non-zero element. As the cohomology of \(\widetilde{M}\) is a Poincaré duality algebra, there is some \(\beta \in H^7-k(\widetilde{M})\) such that \(\alpha \cdot \beta = [\widetilde{M}]\). Applying \(\pi^*\), and noting that \(\pi : \widetilde{M} \to \widetilde{M}\) is a degree 1 map, we have that \(\pi^* \alpha \cdot \pi^* \beta = [\widetilde{M}]\). Then \(\pi^* \alpha \neq 0\).

We write the Mayer-Vietoris sequences associated to \(\widetilde{M} = U \cup V\) and \(\widetilde{M} = U \cup \tilde{V}\) as

\[
\begin{align*}
\rightarrow H^{k-1}(U \cap V) \xrightarrow{\delta^{k-1}_1} H^k(\widetilde{M}) \xrightarrow{\delta^k} H^k(U) \oplus H^k(V) \xrightarrow{\delta^k} H^k(U \cap V) \xrightarrow{\delta^k} H^{k+1}(U \cap V) \rightarrow \\
\rightarrow H^{k-1}(U \cup \tilde{V}) \xrightarrow{\delta^{k-1}_2} H^k(\tilde{M}) \xrightarrow{\delta^k} H^k(U \cup \tilde{V}) \xrightarrow{\delta^k} H^{k+1}(U \cup \tilde{V}) \rightarrow \\
\xrightarrow{Q} \xrightarrow{\bigoplus_{i=1}^{16} H^{k-2}(T^3) \otimes [E_i]} \end{align*}
\]

where \(Q\) is the cokernel of \(\pi^*\). It is clear that \(\text{im} \delta^{k-1}_1 = \text{im} \delta^{k-1}_2\). This happens for all \(k\). So \(\ker \delta^k_1 = \ker \delta^k_2\). Applying the snake lemma to the diagram above (looking at it vertically), we have an exact sequence

\[
0 \to \text{im} \delta^{k-1}_1 \to \text{im} \delta^{k-1}_2 \to \ker f \to \ker \delta^k_1 \to \ker \delta^k_2 \to \text{im} f \to 0.
\]

This concludes that \(f\) is an isomorphism. Therefore there is an exact sequence

\[
0 \to H^*(\widetilde{M}) \to H^*(\widetilde{M}) \xrightarrow{\bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i]} \to 0,
\]

where \([E_i] \in H^2(\widetilde{M})\) is the class of the exceptional divisor \(E_i \subset \mathbb{C}^2/\mathbb{Z}_2\) (1 \(\leq i \leq 16\)).

Now let us construct a splitting of the above exact sequence. For this, we take the Thom form \(\eta_i\) of each of the exceptional divisors \(E_i \subset \mathbb{C}^2/\mathbb{Z}_2\). Let \(E\) be one of these exceptional divisors. The Thom form of \(E\) is a compactly supported 2-form \(\eta\) on a neighbourhood of \(E\) such that \([\eta] = [E]\). Moreover \(\eta^2\) represents a 4-form such that \(\int_F \eta^2 = |F| \cdot |E|^2 = -2\) for each fiber \(F = \{p\} \times \mathbb{C}^2/\mathbb{Z}_2\) of \(\hat{V} = T^3 \times \mathbb{C}^2/\mathbb{Z}_2\). If \(\lambda\) is the bump 4-form on the origin of \(\mathbb{C}^2\), pulled-back to \(\mathbb{C}^2/\mathbb{Z}_2\), then \([\eta^2] = -2[\lambda]\). Pulling-back to \(\hat{V} = T^3 \times \mathbb{C}^2/\mathbb{Z}_2\), we have that \([\eta^2] = -2[T^3]\). With this we construct the compactly supported cohomology of \(\hat{V}_i = T^3 \times \mathbb{C}^2/\mathbb{Z}_2\) as the forms \(\Lambda(e^3, e^4, e^7) \wedge [\eta_i]\). This gives the splitting. \(\square\)
The algebra structure of $H^*(\widehat{M})$ can be described explicitly as follows. Under the isomorphism given in Proposition 18, i.e.

$$H^*(\widehat{M}) \cong H^*(\widehat{M}) \oplus \left( \bigoplus_{i=1}^{16} H^*(T^3) \otimes [E_i] \right),$$

the elements of $H^*(\widehat{M})$ multiply following its algebra structure. Moreover, an element $\alpha \in H^*(\widehat{M})$ and $\beta \otimes [E_j]$ multiply as $\alpha \cdot (\beta \otimes [E_j]) = (i_j^* \alpha \wedge \beta) \otimes [E_j]$, where $i_j : S_j \subset \widehat{M}$ is the inclusion of the $j$-th component $S_j$ of the singular locus. Finally,

$$(E_j) \cdot [E_j] = -2[\lambda] = -2e^{1256},$$

since it is the Poincaré dual of the $T^3$ given by coordinates $(x_3, x_4, x_7)$. So $(\beta \otimes [E_j]) \cdot (\gamma \otimes [E_j]) = -2\beta \wedge \gamma \wedge [\lambda] \in H^*(\widehat{M})$. In sum,

$$(\alpha_1, \sum_j \beta_{1j} \otimes [E_j]) \cdot (\alpha_2, \sum_j \beta_{2j} \otimes [E_j]) =$$

$$= \left( \alpha_1 \wedge \alpha_2 - 2 \sum_j \beta_{1j} \wedge \beta_{2j} \wedge \lambda, \sum_j (\beta_{1j} \wedge \alpha_2 + \alpha_1 \wedge \beta_{2j}) \otimes [E_j] \right).$$

To complete the proof of Theorem 17, we compute the Betti numbers of $\widehat{M}$. By Nomizu’s theorem 

$$H^2(M) = \langle [e^{16}], [e^{17}], [e^{23}], [e^{24}], [e^{25} + e^{34}], [e^{35}], [e^{27} - e^{45} - e^{36}] \rangle,$$

$$H^3(M) = \langle [e^{136}], [e^{146}], [e^{147}], [e^{157}], [e^{167}], [e^{234}], [e^{235}], [e^{236} + e^{245}],$$

$$[e^{237} + e^{345}], [e^{246}], [e^{357}], [e^{247} + e^{256} + e^{346}], [e^{257} + e^{347} + e^{356}] \rangle,$$

and thus

$$H^2(\widehat{M}) = H^2(M) \mathbb{Z}_2 = \langle [e^{16}], [e^{25} + e^{34}] \rangle,$$

$$H^3(\widehat{M}) = H^3(M) \mathbb{Z}_2 = \langle [e^{136}], [e^{146}], [e^{157}], [e^{167}], [e^{235}],$$

$$[e^{236} + e^{245}], [e^{246}], [e^{257} + e^{347} + e^{356}] \rangle.$$

Then, Proposition 14 and Proposition 18 imply that the Betti numbers of $\widehat{M}$ are as follows:

$$b_1(\widehat{M}) = b_1(\widehat{M}) = 1,$$

$$b_2(\widehat{M}) = b_2(\widehat{M}) = 16 = 18,$$

$$b_3(\widehat{M}) = b_3(\widehat{M}) + 16 b_1(T^3) = 56.$$

**Proposition 19.** The compact manifold $\widehat{M}$ has fundamental group $\pi_1(\widehat{M}) = \mathbb{Z}$.

*Proof.* Let $\widehat{\pi} : M \to \widehat{M}$ be the quotient map. Fix $p_0 \in M$ be the point with coordinates $(0, \ldots, 0)$, and let $q_0 = \widehat{\pi}(p_0)$ be the image of $p_0$ under the projection $\widehat{\pi}$. Let $\gamma_1, \ldots, \gamma_7$ be the loops on $M$, where $\gamma_i$ is the image under $\widehat{\pi}$ of the path from $p_0$ to $e_i = (0, \ldots, (i), \ldots, 0)$. These are generators of the fundamental group $\pi_1(M, p_0)$ subject to the relations

$$[\gamma_1, \gamma_2] = \gamma_4, [\gamma_1, \gamma_3] = \gamma_5, [\gamma_1, \gamma_4] = \gamma_6, [\gamma_1, \gamma_5] = \gamma_7,$$

(19)
and the fact the the other commutators are zero, i.e. \( \gamma_2, \gamma_4 \) commute, etc.

It is easy to see that any loop \( \bar{\alpha} \) on \( \hat{M} \) lifts to \( M \) (non-uniquely). The (closed) portions of \( \bar{\alpha} \) that lie in the orbifold locus lift uniquely. The (open) part of \( \bar{\alpha} \) that lies off the orbifold locus lift to two possible paths (since over there \( \hat{\pi} \) is a double covering). Take any of those lifts. The result is a continuous path \( \alpha \) on \( M \) such that \( \bar{\alpha} = \hat{\pi} \circ \alpha \). This concludes that \( \pi_1(\hat{M}, q_0) \) is generated by the images \( \gamma_i = \hat{\pi} \circ \bar{\gamma}_i, 1 \leq i \leq 7 \).

Now recall that \( \mathbb{Z}_2 \) acts by \( \tilde{e} \). Under it, the image of \( \gamma_1 \) is the same as the path from \((0,0,\ldots,0)\) to \((\frac{1}{2},0,\ldots,0)\) followed by the same path in the reversed direction. This is contractible, hence \( \bar{\gamma}_1 = 0 \). The same happens with \( \gamma_2 \), so \( \bar{\gamma}_2 = 0 \). Using the relations \( V \), we conclude that \( \pi_1(\hat{M}, q_0) = \langle \gamma_3 \rangle \). Therefore \( \pi_1(\hat{M}) \cong \mathbb{Z} \), since \( b_1(M) = 1 \).

Now we prove that the resolution process does not alter the fundamental group. Let us treat the case of one of the orbifold locus \( S_0 \cong T^3 \subset \hat{M} \). Let \( \pi : \hat{M} \to M \) be the resolution map. Take \( U \) a neighbourhood of \( S_0 \), and \( V = \hat{M} - S_0 \). Consider \( \hat{U} = \pi^{-1}(U) \) and \( \hat{V} = \pi^{-1}(V) \). Then by Seifert Van-Kampen, \( \pi_1(\hat{M}) \) is the amalgamated sum of \( \pi_1(U) \) and \( \pi_1(V) \) over \( \pi_1(U \cap V) \). And \( \pi_1(\hat{M}) \) is the amalgamated sum of \( \pi_1(\hat{U}) \) and \( \pi_1(\hat{V}) \) over \( \pi_1(\hat{U} \cap \hat{V}) \). Note that \( \hat{V} \cong V, \hat{U} \cap \hat{V} \cong U \cap V, \) and \( U \sim T^3, \hat{U} \sim T^3 \times \mathbb{CP}^1 \), so that \( \pi_1(\hat{U}) \cong \pi_1(U) \). Therefore \( \pi_1(\hat{M}) \cong \pi_1(\hat{M}) \cong \mathbb{Z} \).

Next, we complete the properties of \( \hat{M} \) proving that it is formal, and that it does not admit any torsion-free \( G_2 \)-structure.

**Proposition 20.** The compact manifold \( \hat{M} \) is formal.

**Proof.** We are going first to check that the orbifold \( \hat{M} \) is formal. Note that the cohomology group \( H^3(\hat{M}) \) of \( \hat{M} \) decomposes as

\[
H^3(\hat{M}) = A \oplus B,
\]

where \( A = \langle [e^{136}], [e^{235}] \rangle \) and \( B = \langle [e^{146}], [e^{157}], [e^{167}], [e^{246}], [e^{236} + e^{245}], [e^{237} + e^{347} + e^{356}] \rangle \). Then, the multiplication by \( [e^3] \) vanishes on \( A \), and it defines an isomorphism \( [e^3] : H^2(\hat{M}) \to A \). Moreover, the multiplication by \( [e^3] \) is injective on \( B \to H^4(\hat{M}) \). For this just check that the map \( H^3(\hat{M}) \times H^3(\hat{M}) \to \mathbb{R}, (\alpha, \beta) \mapsto \int \alpha \wedge \beta \wedge e^3 \) has matrix (on the given basis of \( H^3(\hat{M}) \)) of the form

\[
\begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & * \\
\vdots & & \vdots & \\
1 & * & \ldots & *
\end{pmatrix}.
\]

On the other hand, with respect to the basis \( \alpha_1 = [e^{16}], \alpha_2 = [e^{25} + e^{41}] \) of \( H^2(\hat{M}) \), we have \( \alpha_1^2 = 0 \) and \( \alpha_2^2 = 2[e^{2345}] = -2[e^3] \wedge [e^{236} + e^{245}] \), but \( \alpha_1 \wedge \alpha_2 = 2[e^{256}] \neq 0 \).

Since \( M \) is a compact nilmanifold, the minimal model of \( M \) is the minimal DGA \( (\bigwedge V, d) \), where \( V = \langle e^1, \ldots, e^7 \rangle \) and the differential \( d \) is defined by \( \Box \). Let \( F = \mathbb{Z}_2 \) be the finite group acting on \( M \), and on the minimal model. So \( ((\bigwedge V)^F, d) \) is a model (not minimal) of \( \hat{M} = M/F \). Let \( \rho : (\bigwedge W, d) \to ((\bigwedge V)^F, d) \) be a minimal model of \( \hat{M} \).
According with notation of Definition 8 we write $W^i = C^i + N^i$, $i \leq 3$. Then,

$$W^1 = C^1 \equiv \langle a_1 \rangle,$$

$$W^2 = C^2 \equiv \langle b_1, b_2 \rangle,$$

$$W^3 = C^3 \oplus N^3, \text{ where } C^3 = \langle c_1, c_2, c_3, c_4, c_5, c_6 \rangle, \text{ } N^3 = \langle \eta_1, \eta_2 \rangle,$$

the differential $d$ is given by $d(C^i) = 0$, $d\eta_1 = b_1^2$, $d\eta_2 = b_2^2 + 2a_1c_5$, and the morphism $\rho : (\bigwedge W, d) \to (\bigwedge V, d)$ of differential algebras is defined by

$$\rho(a_1) = e^3, \quad \rho(b_1) = e^{16}, \quad \rho(b_2) = e^{25} + e^{34},$$

$$\rho(c_1) = e^{146}, \quad \rho(c_2) = e^{157}, \quad \rho(c_3) = e^{167},$$

$$\rho(c_4) = e^{246}, \quad \rho(c_5) = e^{236} + e^{245}, \quad \rho(c_6) = e^{257} + e^{347} + e^{356},$$

$$\rho(\eta_i) = 0, \quad i = 1, 2.$$

Now we can prove that $\tilde{M}$ is 3-formal, and so it is formal by Proposition 12. For this we have to look at the closed elements of $\mathcal{I}(N^3) \subset \bigwedge W \leq 3$, and check that the image through $\rho$ is exact. We only look at elements of degree at most 7. Those are

$$a_1(\lambda_1\eta_1 + \lambda_2\eta_2), \quad (\mu_1b_1 + \mu_2b_2)(\lambda_1\eta_1 + \lambda_2\eta_2), \quad (\lambda_1\eta_1 + \lambda_2\eta_2)(\sum_{i=1}^{6} \nu_ic_i),$$

and the product of these by closed elements, where $\lambda_i$, $\mu_i$, and $\mu_j$ are real numbers with $i = 1, 2$ and $1 \leq j \leq 6$. Note that if $\alpha$ is one of those elements, then $\rho(\alpha) = 0$.

To check the formality of $\tilde{M}$, now we have to work out the 3-minimal model of it, with the algebra structure of $H^*(\tilde{M})$ given above. Therefore, the minimal model of $\tilde{M}$ must be a differential graded $(\bigwedge Z, \tilde{d})$, being $Z$ the graded vector space $Z = \bigoplus_i Z^i$ with

$$Z^1 = W^1,$$

$$Z^2 = W^2 \oplus S, \quad S = \langle B_i | 1 \leq i \leq 16 \rangle,$$

$$Z^3 = W^3 \oplus T, \quad T = \langle C_i^4, C_i^7 | 1 \leq i \leq 16 \rangle,$$

and the differential $\tilde{d}$ is given by $\tilde{d}(W^i) = d(W^i)$, $\tilde{d}(B_i) = 0$, $\tilde{d}(C_i^4) = 0 = \tilde{d}(C_i^7)$. Now, we define the map of differential algebras $\tilde{\theta} : (\bigwedge Z \leq 3, d) \to (H^*(\tilde{M}), d = 0)$ by

$$\tilde{\theta}(a_1) = [e^3],$$

$$\tilde{\theta}(b_1) = [e^{16}],$$

$$\tilde{\theta}(c_1) = [e^{146}],$$

$$\tilde{\theta}(c_4) = [e^{246}],$$

$$\tilde{\theta}(C_i^4) = [e^4] \otimes [E_i],$$

$$\tilde{\theta}(C_i^7) = [e^7] \otimes [E_i],$$

$$\tilde{\theta}(\eta_i) = 0,$$

where $1 \leq i \leq 16$ and $j = 1, 2$. One can check that the map $\tilde{\theta}$ is such that the map $\tilde{\theta}^* : H^*(\bigwedge V \leq s, d) \to H^*(\tilde{M})$ induced on cohomology is equal to the map $\tau^* : H^*(\bigwedge V \leq s, d) \to H^*(\bigwedge V, \tilde{d}) = H^*(\tilde{M})$ induced by the inclusion $\iota : (\bigwedge V \leq s, d) \to (\bigwedge V, \tilde{d})$. So $\tilde{M}$ is 3-formal by Lemma 10 and, by Theorem 9, $\tilde{M}$ is formal. □

**Theorem 21.** The compact manifold $\tilde{M}$ does not admit any torsion-free $G_2$-structure.
Proof. We prove the theorem by contradiction. Suppose that \( \tilde{M} \) admits a torsion-free \( G_2 \)-structure with associated metric \( g \). Then, the holonomy group of \( g \) is a subgroup of \( G_2 \). By [28, Theorem 10.2.1] the only connected Lie subgroups of \( G_2 \) that can arise as holonomy of the Riemannian metric \( g \) are \( G_2 \), \( SU(3) \), \( SU(2) \) and \( \{1\} \). Since \( b_1(\tilde{M}) = 1 \) and \( \pi_1(\tilde{M}) = \mathbb{Z} \), the holonomy group of \( g \) must be \( SU(3) \).

Therefore, \( \tilde{M} \) has a finite covering \( N \times S^1 \) with \( N \) being a 6-dimensional simply connected Calabi-Yau manifold. Indeed, by Proposition 1.1.1 of [27] we know that \( (\tilde{M}, g) \) must admit as Riemannian finite cover a product \( N \times S^1 \), for some compact, simply connected 6-manifold \( N \). Since the holonomy group of the induced metric on the finite cover is the product of the holonomy group of \( N \) and the trivial group, the induced metric on \( N \) is Ricci-flat and its holonomy group is \( SU(3) \). That is \( N \) is a Calabi-Yau manifold.

Thus, on \( N \times S^1 \) and, consequently on \( \tilde{M} \) there exist a closed 2-form \( \omega \) and a closed 1-form \( \eta \) such that \( \omega^3 \wedge \eta \neq 0 \) at every point. But this is not possible by the cohomology of \( \tilde{M} \). First, we see that it is not possible in \( H^*(\tilde{M}) \) since we must have \( \eta = e^3 \), and we know that \( H^2(\tilde{M}) = \langle [e^{16}], [e^{25} + e^{34}] \rangle \). Then we use Proposition [28] \( \square \)

7. Associative 3-folds in \( \tilde{M} \)

The closed \( G_2 \) form \( \tilde{\varphi} \) constructed on \( \tilde{M} \) defines an associative calibration on \( \tilde{M} \). This means that, for any \( p \in \tilde{M} \), we have that every oriented 3-dimensional subspace \( V \) of the tangent space \( T_p\tilde{M} \) satisfies \( \tilde{\varphi}(p)|_V = \lambda \operatorname{vol}_V \), for some \( \lambda \leq 1 \), where the volume form \( \operatorname{vol}_V \) is induced from the restriction to \( V \) of the inner product \( g_{\tilde{\varphi}} \) at \( p \) (see [23] and [28, §3.7]). The 3-dimensional orientable submanifolds \( Y \subset \tilde{M} \) calibrated by the \( G_2 \) form \( \tilde{\varphi} \), i.e. those submanifolds \( Y \subset \tilde{M} \) that satisfy \( \tilde{\varphi}(p)|_{T_pY} = \operatorname{vol}_Y(p) \), for each \( p \in Y \) and for some unique orientation of \( Y \), are often called associative 3-folds. Every compact calibrated submanifold \( Y \) is volume-minimizing in its homology class, in particular \( Y \) is minimal [28, Proposition 3.7.2].

We shall produce examples of associative 3-folds in \( \tilde{M} \) from the fixed locus of a \( G_2 \)-involution of the compact manifold \( M = \Gamma \backslash G \) defined in [3], applying the following.

**Proposition 22** ([28, Proposition 10.8.1]). Let \( N \) be a 7-manifold with a closed \( G_2 \) form \( \phi \), and let \( \sigma : N \to N \) be an involution of \( N \) satisfying \( \sigma^* \phi = \phi \) and such that \( \sigma \) is not the identity map. Then the fixed point set \( P = \{ p \in N \mid \sigma(p) = p \} \) is an embedded associative 3-fold. Furthermore, if \( N \) is compact then so is \( P \).

**Remark 3.** Note that Proposition 10.8.1 in [28] is stated for the \( G_2 \)-structures that are closed and coclosed, but the coclosed condition is not used in the proof.

Recall from section 4 the 7-dimensional Lie group \( G \), and consider on \( G \) the involution given by

\[
\sigma : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, x_3, x_4, -x_5, \frac{1}{2} - x_6, x_7).
\]

The involution \( \sigma \) is equivariant with respect to the left multiplications by elements of the subgroup \( \Gamma \subset G \). Indeed, for each \( a \in G \) and \( A \in \Gamma \) we may write, noting the properties
of the $\mathbb{Z}_2$-action $\rho$ on $G$ defined by (3),
\[ L_A(\sigma(a)) = L_A(L_\rho(\rho(a))) = L_\rho(L_A(\rho(a))) = L_\rho(\rho(A') \cdot \rho(a)) = \rho(L_A'(a)) = \sigma(L_A'(a)), \]
where $A' = \rho(A)$ and $L_\rho$ denotes the left translation by an element with coordinates $(x_i) = (0, 0, 0, 0, \frac{1}{2}, 0)$ in $G$. Therefore, $\sigma$ descends to the quotient manifold $M = \Gamma \backslash G$. The induced map on $M$, still denoted by $\sigma$, commutes with $\rho$ and so $\sigma$ descends to the orbifold $\tilde{M} = M/\mathbb{Z}_2$. From now on, we denote by $\tilde{\sigma}$ the involution of $\tilde{M}$ induced by $\sigma$.

The fixed locus $\tilde{P}$ of $\tilde{\sigma}$ is the image by the natural projection $\tilde{\pi}: M \to \tilde{M}$ of the set $P$ of points in $M$ that are fixed by the involution $\sigma: M \to M$. Thus, $\tilde{P}$ consists of all the 3-dimensional spaces $\tilde{P}_b = \tilde{\pi}(P_b) = P_b/\mathbb{Z}_2$, where
\[ P_b = \{(b_1, b_2, x_3, x_4, b_5, b_6, x_7) \mid (x_3, x_4, x_7) \in T^3\} \subset M, \]
and $b = (b_1, b_2, b_5, b_6) \in \mathbb{B} = \{0, 1\} \times \{0, 1/2\} \times \{0, 1/2\} \times \{1/4, 3/4\}$. Hence, $P$ is a disjoint union of 16 copies of a 3-torus $T^3$. Now one can check that the fixed locus $\tilde{P}$ of $\tilde{\sigma}$ consists of 8 disjoint copies of $T^3$ since in the orbifold $\tilde{M}$ the points of coordinates $(b_1, b_2, x_3, x_4, b_5, 1/4, x_7)$ and $(b_1, b_2, x_3, x_4, b_5, 3/4, x_7)$ are the same. Observe that the fixed loci $P$ of $\sigma$ and $S'$ of $\rho$ do not intersect, and hence the fixed locus $\tilde{P}$ of $\tilde{\sigma}$ and the singular locus $S$ of the orbifold $\tilde{M}$ also do not intersect.

**Proposition 23.** Each of the eight disjoint copies of 3-tori in $\tilde{M}$, which are the fixed locus $\tilde{P}$ of $\tilde{\sigma}$, define eight embedded, associative (calibrated by $\tilde{\varphi}$), minimal 3-tori in $\tilde{M}$.

**Proof.** Since the $G_2$ form $\varphi$ on $M$ defined in (3) is preserved by the involution $\sigma$ of $M$, each of the 16 torus $P_b$ in $M$ fixed by $\sigma$ is an associative 3-fold in $(M, \varphi)$ by Proposition 22. Now we know that the $\mathbb{Z}_2$-action $\rho$ on $M$ preserves the $G_2$ form $\varphi$ on $M$, and induces the $G_2$ form $\tilde{\varphi}$ on $\tilde{M}$ (see section 4), so that the pull-back of $\tilde{\pi}$ sends $\tilde{\varphi}$ to $\varphi$. Thus, the 2-to-1 projection map $\tilde{\pi}: M \to \tilde{M}$ outside the set $S'$ of points in $M$ fixed by $\rho$ is a local isomorphism of the closed $G_2$-structures and hence also a local isometry of the induced metrics. Consequently, $\tilde{\pi}$ preserves the associative calibrated property of submanifolds, and so each of the eight copies of $T^3$ is an associative (and minimal) 3-fold in $\tilde{M}$. Furthermore, as we mentioned above, these 3-tori do not meet the singular locus $S$ of $\tilde{M}$.

To complete the proof, let us recall that the $G_2$-structure $\tilde{\varphi}$ on $\tilde{M}$ agrees, away from a neighbourhood $U$ of $S$, with the $G_2$-structure $\tilde{\varphi}$ induced on $\tilde{M}$ from $M$. It follows that the above 3-tori lift diffeomorphically to the resolution $\tilde{M}$ and define 8 embedded, associative (calibrated by $\tilde{\varphi}$), minimal 3-tori in $\tilde{M}$. □

McLean [33] studied the deformation problem for several types of calibrated submanifolds. For compact associative 3-folds, the problem may be expressed as a non-linear elliptic PDE, with index zero, if the $G_2$ form is closed and coclosed. This result was generalized by Akbulut and Salur to arbitrary $G_2$ forms [2, Theorem 6]. It follows that any compact associative 3-fold in $\tilde{M}$ is either rigid or, otherwise, has infinitesimal associative
deformations which in general need not arise from the actual deformations (as the linear part of the deformation problem may have a nontrivial cokernel).

We next show that the 3-tori in the present example do have associative deformations.

**Proposition 24.** Each of the eight associative 3-tori in $\widetilde{M}$ arising from the fixed locus of $\sigma$ has a smooth 2-dimensional family of non-trivial associative deformations.

**Proof.** As in the previous sections, in light of the symmetry by left translations, it suffices to consider just one component $Y_0$ of the fixed locus of $\sigma$. A tubular neighbourhood of $Y_0$ in $\widetilde{M}$ is isometric to a tubular neighbourhood of the image of $Y_0$ in the smooth locus $M \setminus S$. As the projection $M \to \widetilde{M}$ is a local isometry away from the preimage of $S$ we may work on $M$ with the $G_2$-structure $\varphi$ and consider a component of the preimage of $Y_0$ which by abuse of notation we continue to denote by $Y_0 \subset M$. We may choose $Y_0$ to be defined by $x_1 = x_2 = x_5 = 0$, $x_6 = \frac{1}{3}$, then the associative 3-torus $Y_0$ is contained in the fiber $p^{-1}(0 + 2\mathbb{Z})$ of the projection $p : M \to \mathbb{R}/2\mathbb{Z}$ (see (5)).

Every fiber $p^{-1}(x_1)$ has a natural structure of a complex 3-torus $\mathbb{C}^3/\Lambda(x_1)$, where the complex coordinates on $\mathbb{C}^3$ are given by $x_2 + ix_3$, $x_4 + ix_5$, $x_6 + ix_7$. Moreover, these latter 3-tori are biholomorphic to the standard 3-torus $p^{-1}(0) = \mathbb{C}^3/\mathbb{Z}^6$ because the linear isomorphisms $B(x_1)$ and $C$ are contained in the image of $\text{SL}(3, \mathbb{R})$ under the chain of natural embeddings of groups $\text{SL}(3, \mathbb{R}) \subset \text{SL}(3, \mathbb{C}) \subset \text{SL}(6, \mathbb{R})$. The complex 3-form $(e^2 + ie^3) \wedge (e^4 + ie^5) \wedge (e^6 + ie^7)|_{p^{-1}(x_1)}$ induces on each complex torus $p^{-1}(x_1)$ a holomorphic trivialization of the canonical bundle of $(3, 0)$-forms. The closed 2-form $e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7|_{p^{-1}(x_1)}$ induces on $p^{-1}(x_1)$ a Ricci-flat Kähler metric which depends non-trivially on $x_1$ and when $x_1 = 0$ coincides with the ‘usual’ Kähler metric on $\mathbb{C}^3/\mathbb{Z}^6$. Thus each fibre $p^{-1}(x_1)$ has a torsion-free $\text{SU}(3)$ (Calabi–Yau) structure compatible with the torsion-free $G_2$-structure on $M$.

It is easy to check that $Y_0$ is a special Lagrangian 3-torus in the Calabi–Yau threefold $Z_0 = p^{-1}(0)$. Furthermore, the special Lagrangian tori

$$Y(a, b) = \{(0, y_1, y_2, a, \frac{1}{3} + b, y_3) \in \mathbb{R}^3 \mid (y_1, y_2, y_3) \in \mathbb{Z}^6\}$$

in $p^{-1}(0)$ are associative in $(M, \varphi)$ as $\phi|_{Y(a, b)} = dx_3 \wedge dx_4 \wedge dx_7|_{Y(a, b)} = e^3 \wedge e^4 \wedge e^7|_{Y(a, b)}$. For small $a, b$ the $Y(a, b)$ induce well-defined non-trivial associative deformations of $Y_0$ in $(\widetilde{M}, \tilde{\varphi})$. \hfill $\Box$

**References**

[1] A. Adem, J. Leida and Y. Ruan, *Orbifolds and string theory*, Cambridge Univ. Press, 2007.

[2] S. Akbulut and S. Salur, Deformations in G2 manifolds, *Adv. Math.* 217 (2008), 2130–2140.

[3] G. Bazzoni, I. Biswas, M. Fernández, V. Muñoz, A. Tralle, Homotopic properties of Kähler orbifolds, to appear in *Special Metrics and Group Actions in Geometry*, Springer INDAM.

[4] R. L. Bryant, Metrics with exceptional holonomy, *Ann. of Math.* 126 (1987), 525–576.

[5] R. L. Bryant, Some remarks on G2-structures, *Proceedings of Gökova Geometry-Topology Conference 2005*, Gökova Geometry/Topology Conference (GGT), Gökova, 2006, pp. 75–109.

[6] R. L. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.* 58 (1989), 829–850.

[7] G. Cavalcanti, M. Fernández, V. Muñoz, Symplectic resolutions, Lefschetz property and formality, *Advances Math.* 218 (2008), 576–599.

[8] R. Cleyton, S. Ivanov, On the geometry of closed G2-structures, *Comm. Math. Phys.* 270 (2007), 53–67.
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[9] D. Conti, M. Fernández, Nilmanifolds with a calibrated $G_2$-structure, *Differ. Geom. Appl.* 29 (2011), 493–506.

[10] A. Corti, M. Haskins, J. Nordström, T. Pacini, $G_2$-manifolds and associative submanifolds via semi-Fano 3-folds, *Duke Math. J.* 164 (10) (2015), 1971–2092.

[11] D. Crowley, J. Nordström, New invariants of $G_2$-structures, *Geom. Topol.* 19 (2015), 2949–2992.

[12] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* 29 (1975), 245–274.

[13] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory*, Springer, 2002.

[14] M. Fernández, An example of a compact calibrated manifold associated with the exceptional Lie group $G_2$, *J. Differ. Geom.* 26 (1987), 367–370.

[15] M. Fernández, A family of compact solvable $G_2$-calibrated manifolds, *Tohoku Math. J.* 39 (2007), 193–218.

[16] M. Fernández, A. Gray, Riemannian manifolds with structure group $G_2$, *Annali di Mat. Pura Appl.* 32 (1982), 19–45.

[17] M. Fernández, V. Muñoz, Formality of Donaldson submanifolds, *Math. Z.* 250 (2005), 149–175.

[18] M. Fernández, V. Muñoz, Erratum: Formality of Donaldson submanifolds, *Math. Z.* 257 (2007), 465–466.

[19] R. Gompf, A new construction of symplectic manifolds, *Annals of Math.* (2) 142 (1995), 537–696.

[20] A. Gray, Vector cross products on manifolds, *Trans. Amer. Math. Soc.* 141 (1969), 465–504.

[21] P. Griffiths and J. Morgan, *Rational homotopy theory and differential forms*, Progress in Math. 16, Birkhäuser, 1981.

[22] S. Halperin, *Lectures on minimal models*, Mém. Soc. Math. France 230, 1983.

[23] R. Harvey and H. B. Lawson, Calibrated geometries, *Acta Math.* 148 (3) (1982) 47–157.

[24] N. Hitchin, The geometry of three-forms in six and seven dimensions, *J. Differ. Geom.* 55 (2000), 547–576.

[25] N. Hitchin, Stable forms and special metrics, in *Global Differential Geometry: The Mathematical Legacy of Alfred Gray* volume 288 of Contemp. Math., Amer. Math. Soc., 2001, pp. 70–89.

[26] N. Hitchin, Special holonomy and beyond, in *Strings and Geometry*, volume 3 of Clay Math. Proc., Amer. Math. Soc., 2004, pp. 159–175.

[27] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy $G_2$. I, II, *J. Differ. Geom.* 43 (1996), 291–328, 329–375.

[28] D. D. Joyce, *Compact manifolds with special holonomy*, OUP, Oxford, 2008.

[29] D. D. Joyce and S. Karigiannis, A new construction of compact torsion-free $G_2$-manifolds by gluing families of Eguchi-Hanson spaces, arXiv:1707.09325 [math.DG].

[30] A. Kovalev, Twisted connected sums and special Riemannian holonomy, *J. Reine Angew. Math.* 565 (2003), 125–160.

[31] A. G. Kovalev and N.-H. Lee, $K3$ surfaces with non-symplectic involution and compact irreducible $G_2$-manifolds, *Math. Proc. Camb. Phil. Soc.* 151 (2011), 193–218.

[32] J. McKay, *Graphs, singularities, and finite groups*, Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 183–186, Proc. Sympos. Pure Math., 37, Amer. Math. Soc., Providence, R.I., 1980.

[33] R. C. McLean, Deformations of calibrated submanifolds, *Comm. Anal. Geom.* 6 (1998), 707–747.

[34] J. Neisendorfer and T. Miller, Formal and coformal spaces, *Illinois. J. Math.* 22 (1978), 565–580.

[35] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* 59 (1954), 531–538.

[36] S. Salamon, *Riemannian geometry and holonomy groups*, Longman Scientific and Technical, Harlow Essex, U.K., 1989.

[37] I. Satake, On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. USA* 42 (1956), 359–363.

[38] P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980.

[39] D. Sullivan, Infinitesimal computations in topology, *Inst. Hautes Études Sci. Publ. Math.* 47 (1978), 269–331.
