ON FREE ENERGIES OF THE POTTS MODEL ON THE CAYLEY TREE

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Abstract. For the Potts model on the Cayley tree, some explicit formulae of the free energies and entropies (according to vector-valued boundary conditions (BCs)) are obtained. They include translation-invariant, periodic, Dobrushin-like BCs, as well as those corresponding to weakly periodic Gibbs measures.

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1. Introduction and definitions

On Cayley trees, not only Gibbs measures but also the free energy (and the entropy) depend on the BC. A study of this dependence is given in [1]. It is shown there that for all previously known BCs the free energies exist. Later, in [2] a construction of new Gibbs measures (called alternating Gibbs measures) is presented and their corresponding free energies are given. Moreover, it was proved that free energy of some alternating Gibbs measures may not exist.

The purpose of this paper is to study free energies of the Potts model on the Cayley tree.

The Cayley tree $\Gamma^k$ (See [2]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$, where $V$ is the set of vertices of $\Gamma^k$, $L$ is the set of edges of $\Gamma^k$ and $i$ is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest neighboring vertices, and we write $l = \langle x, y \rangle$.

The distance $d(x, y), x, y \in V$ on the Cayley tree is defined by $d(x, y) = \min\{d|\exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle\}$.

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \{x \in V \mid d(x, x^0) \leq n\}$, $L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}$.

It is known that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups $\{e, a_i\}, i = 1, \ldots, k + 1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$. 
Denote by $S(x)$ the set of direct successors of $x \in G_k$. Let $S_1(x)$ be the set of all nearest neighboring vertices of $x \in G_k$, i.e. $S_1(x) = \{ y \in G_k : \langle x, y \rangle \}$ and $x_\downarrow$ denotes the unique element of the set $S_1(x) \setminus S(x)$.

We consider models where spin takes values from the set $\Phi = \{1, 2, \ldots, q\}$, $q \geq 2$. A configuration $\sigma$ is defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

The Hamiltonian of the Potts model has the form

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)},$$  \hspace{1cm} (1.1)$$

where $J \in R$, $\delta_{uv}$ is the Kronecker symbol.

We identify the set $\Phi$ by the set $\{\sigma_1, \ldots, \sigma_q\}$, where $\sigma_i \in \mathbb{R}^{q-1}$ such that

$$\sigma_i\sigma_j = \begin{cases} -\frac{1}{q-1}, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Then we have

$$\delta_{\sigma(x)\sigma(y)} = \frac{q-1}{q} \left( \sigma(x)\sigma(y) + \frac{1}{q-1} \right).$$  \hspace{1cm} (1.2)$$

Using this formula the Hamiltonian of the Potts model can be reduced to the Hamiltonian of the Ising model with $q$ spin values:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \sigma(x)\sigma(y).$$  \hspace{1cm} (1.3)$$

We fix a basis $\{e_1, \ldots, e_{q-1}\}$ on $\mathbb{R}^{q-1}$, such that $e_i = \sigma_i$, $i = 1, 2, \ldots, q-1$. It is clear that

$$\sum_{i=1}^{q} \sigma_i = 0.$$  \hspace{1cm} (1.4)$$

We note that if $h = (h_1, \ldots, h_{q-1})$, then

$$h\sigma_i = \begin{cases} -\frac{q}{q-1}h_i - \frac{1}{q-1} \sum_{j=1}^{q-1} h_j, & \text{if } i = 1, \ldots, q-1, \\ -\frac{1}{q-1} \sum_{j=1}^{q-1} h_j, & \text{if } i = q. \end{cases}$$

Define a finite-dimensional distribution of a probability measure $\mu$ in the volume $V_n$ as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \right\},$$  \hspace{1cm} (1.5)$$

where $\beta = 1/T$, $T > 0$ is the temperature, $h_x \in \mathbb{R}^{q-1}$,
and $Z_{n}^{-1}$ is the normalizing factor, i.e.

$$Z_{n} = Z_{n}(\beta, h) = \sum_{\sigma \in \Omega_{n}} \exp \left( -\beta H_{n}(\sigma_{n}) + \sum_{x \in W_{n}} h_{x} \sigma(x) \right).$$

The collection of vectors $h = \{ h_{x} \in \mathbb{R}^{q-1}, x \in V \}$ stands for (generalized) BC.

The following limit (if it exist) is called free energy corresponding to BC $h$:

$$E(\beta, h) = -\lim_{n \to \infty} \frac{1}{\beta |V_{n}|} \ln Z_{n}(\beta, h).$$

We say that probability distributions (1.5) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$ we have

$$\sum_{\sigma^{(n)} \in \Phi^{W_{n}}} \mu_{n}(\sigma_{n-1} \vee \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}),$$

where $\sigma_{n-1} \vee \sigma^{(n)}$ is the concatenation of the configurations.

In this case, there exists a unique $\mu$ on $\Phi^{V}$ such, that for all $n$ and $\sigma_{n} \in \Phi^{V_{n}}$ we have

$$\mu(\{ U_{n} = \sigma_{n} \}) = \mu_{n}(\sigma_{n}).$$

Such measure is called a limiting Gibbs measure corresponding to Hamiltonian (1.3) and to the vector-valued function $h_{x}, x \in V$.

The next statement describes the condition on $h_{x}$ ensuring that $\mu_{n}(\sigma_{n})$ are compatible.

**Theorem 1.** Measures (1.5) satisfy (1.6) if only if for all $x \in V \setminus \{ x^{0} \}$ the following equation holds:

$$h_{x} = \sum_{y \in S(x)} F(h, \theta),$$

where $F : h = (h_{1}, \ldots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta) = (F_{1}, \ldots, F_{q-1}) \in \mathbb{R}^{q-1}$ is defined as

$$F_{i} = \ln \left( \frac{(\theta - 1)e^{h_{i}} + \sum_{j=1}^{q-1} e^{h_{j}} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_{j}}} \right), \quad \theta = \exp(J\beta).$$

**Proof.** In [8, p.106] this theorem is proved for Hamiltonian (1.1) (i.e. without change (1.2)). Here we shall prove (1.7) for Hamiltonian (1.3), i.e. with change (1.2). We show this because, a formula obtained in this proof will be helpful to find a general form of the free energy.

Substituting (1.5) in (1.6), in view of (1.2) we get

$$\frac{Z_{n-1}}{Z_{n}} \sum_{\sigma^{(n)} \in \Phi^{W_{n}}} \exp \left( J\beta \sum_{x \in W_{n-1}} \left( \sum_{y \in S(x)} \sigma(x) \sigma(y) \right) + \sum_{x \in W_{n-1}} \left( \sum_{y \in S(x)} h_{y} \sigma(y) \right) \right) = \exp \left( \sum_{x \in W_{n-1}} h_{x} \sigma(x) \right), \sigma(x) \in \Phi.$$
Consequently,
\[
\frac{Z_{n-1}}{Z_n} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \sum \exp(J \beta \sigma(x) \sigma(y) + h_y \sigma(y)) = \prod_{x \in W_{n-1}} \exp(h_x \sigma(x)), \sigma(x) \in \Phi. 
\]  
(1.8)

Fix \(x \in W_{n-1}\) and rewrite (1.8) for \(\sigma(x) = \sigma_i, i = 1, \ldots, q - 1\) and \(\sigma(x) = \sigma_q\). Then dividing each of them by the last one we get

\[
\prod_{y \in S(x)} \sum_{\sigma(y)} \exp((J \beta \sigma_i + h_y)\sigma(y)) = \exp(h_x \sigma_i), \sigma(y) \in \Phi, i = 1, 2, \ldots, q - 1. 
\]  
(1.9)

Notice that \(h_x \sigma_i - h_x \sigma_q = \frac{q}{q-1}h_{x,i}\), where \(h_x = (h_{x,1}, \ldots, h_{x,q-1})\). Now we change the variables as \(\frac{1}{q-1}h_{x,i} \rightarrow h_{x,i}\) then we get the equation (1.7). For a proof of the inverse statement see [8, p.107]. □

Let \(G_k/G_k^* = \{H_1, \ldots, H_r\}\) be the quotient group, where \(G_k^*\) is a normal subgroup of index \(r \geq 1\).

**Definition 1.** A set of vectors \(h = \{h_x, x \in G_k\}\) is said to be \(G_k^*\)-periodic, if \(h_{yx} = h_x\) for any \(x \in G_k\) and \(y \in G_k^*\).

**Definition 2.** A set of vectors \(h = \{h_x, x \in G_k\}\) is said to be \(G_k^*\)-weakly periodic, if \(h_x = h_{ij}\) for \(x \in H_i\) and \(x \downarrow \in H_j\) for any \(x \in G_k\).

**Definition 3.** A measure \(\mu\) is said to be \(G_k^*\)-periodic (weakly periodic), if it corresponds to the \(G_k^*\)-periodic (weakly periodic) set of vectors \(h\). The \(G_k\)-periodic measure is said to be translation-invariant.

In this paper we compute free energies which correspond to translation-invariant, periodic, weakly periodic and some non-periodic BCs (Gibbs measures).

## 2. Formula of free energy

The following theorem gives a general form of the free energy.

**Theorem 2.** For BCs satisfying (1.6), the free energy is given by the formula

\[
E(\beta, h) = -\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x), 
\]

(2.1)

where

\[
a(x) = \frac{1}{q^\beta} \sum_{i=1}^q \ln \left( \sum_{u=1}^q \exp \left( (J \beta \sigma_i + h_x)\sigma_u \right) \right). 
\]

(2.2)

**Proof.** In view of (1.9) we have

\[
\prod_{y \in S(x)} \sum_{u=1}^q \exp(J \beta \sigma_i \sigma_u + \sigma_u h_y) = b(x) \exp(\sigma_i h_x), \; i = 1, \ldots, q.
\]
Multiplying all such equalities and using (1.4) we obtain

\[ b^q(x) = \prod_{i=1}^{q} \prod_{y \in S(x)} \sum_{u=1}^{q} \exp(J \beta \sigma_i \sigma_u + \sigma_u h_y) = \prod_{y \in S(x)} \prod_{i=1}^{q} \sum_{u=1}^{q} \exp(J \beta \sigma_i + h_y) \sigma_u, \]

hence

\[ b(x) = \prod_{y \in S(x)} \left( \sum_{u=1}^{q} \exp(J \beta \sigma_i + h_y) \sigma_u \right)^{1/q}. \]

Let \( A_n = \prod_{x \in W_n} b(x). \) It is clear that \( Z_n = A_{n-1} Z_{n-1}. \) Consequently

\[ Z_n = \prod_{x \in W_{n-1}} b(x) Z_{n-1} = \prod_{x \in W_{n-1}} b(x) \prod_{x \in W_{n-2}} b(x) Z_{n-2} = \ldots = \prod_{x \in V_{n-1}} b(x), \]

and

\[ \ln Z_n = \sum_{x \in V_{n-1}} \ln b(x). \]

Hence we obtain that

\[ a(x) = \frac{1}{q \beta} \sum_{i=1}^{q} \ln \left( \sum_{u=1}^{q} \exp\{J \beta \sigma_i + h_x \sigma_u\} \right). \]

□

3. Free energies corresponding to some BCs

3.1. Translation-invariant case. Consider translation-invariant set of vectors \( h_x, \) i.e. \( h_x = h = (h_1, h_2, \ldots, h_{q-1}) \in \mathbb{R}^{q-1}, \forall x \in G_k. \) Then from (1.7) we obtain

\[ h_i = k \ln \left( \frac{(\theta - 1) e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right), \quad i = 1, \ldots, q - 1. \]  

(3.1)

Denoting \( z_i = \exp(h_i), i = 1, \ldots, q - 1, \) we get from (3.1)

\[ z_i = \left( \frac{(\theta - 1) z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k, \quad i = 1, 2, \ldots, q - 1. \]

(3.2)

For \( k = 2 \) we denote

\[ x_1(m) = \frac{\theta - 1 - \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m}, \quad x_2(m) = \frac{\theta - 1 + \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m}, \]

where

\[ \theta \geq \theta_m = 1 + 2 \sqrt{m(q - m)}, \quad m = 1, \ldots, q - 1. \]

It is easy to see that

\[ \theta_m = \theta_{q-m} \quad \text{and} \quad \theta_1 < \theta_2 < \cdots < \theta_{\left\lfloor \frac{q}{2} \right\rfloor - 1} < \theta_{\left\lfloor \frac{q}{2} \right\rfloor} \leq q + 1. \]

Let \( k = 2, J > 0, \) then the following statements are known (see [6]).
1. If $\theta < \theta_1$, then the system of equations (3.1) has a unique solution $h_0 = (0, 0, \ldots, 0)$;
2. If $\theta_m < \theta < \theta_{m+1}$ for some $m = 1, \ldots, \lfloor \frac{q}{2} \rfloor - 1$, then the system of equations (3.1) has solutions
   \[ h_0 = (0, 0, \ldots, 0), \ h_{1i}(s), \ h_{2i}(s), \ i = 1, \ldots, \left(\frac{q-1}{s}\right), \]
   \[ h_{1i}(q-s), \ h'_{2i}(q-s), \ i = 1, \ldots, \left(\frac{q-1}{q-s}\right), \ s = 1, 2, \ldots, m, \]
   where $h_{ji}(s)$ (resp. $h'_{ji}(q-s)$) $j = 1, 2$ is a vector with $s$ (resp. $q-s$) coordinates equal to $2 \ln x_j(s)$ (resp. $2 \ln x_j(q-s)$) and remaining $q-s-1$ (resp. $s-1$) coordinates equal to 0. The number of such solutions is equal to
   \[ 1 + 2 \sum_{s=1}^{m} \left(\frac{q}{s}\right); \]
3. If $\theta_{\lfloor \frac{q}{2} \rfloor} < \theta \neq q + 1$, then there are $2^{q-1} - 1$ solutions to (3.1);
4. If $\theta = q + 1$ then the number of solutions is as follows
   \[ \begin{cases} 
   2^{q-1}, & \text{if } q \text{ is odd} \\
   2^{q-1} - \left(\frac{q-1}{2}\right), & \text{if } q \text{ is even}; 
   \end{cases} \]
5. If $\theta = \theta_m$, $m = 1, \ldots, \lfloor \frac{q}{2} \rfloor$, $(\theta_{\lfloor \frac{q}{2} \rfloor} \neq q + 1)$ then $h_{1i}(m) = h_{2i}(m)$. The number of solutions is equal to
   \[ 1 + \left(\frac{q}{m}\right) + 2 \sum_{s=1}^{m-1} \left(\frac{q}{s}\right). \]

Thus any solution of (3.1) has the form
\[ h = (h_x, h_x, \ldots, h_x, 0, 0, \ldots, 0), \ m \geq 0 \]
up to a permutation of coordinates.

In this subsection we shall calculate free energies and entropy $S(\beta, h) = -\frac{dE(\beta, h)}{dT}$ for the set of translation-invariant vectors $h_x = h$, with $h$ given by (3.4).

Case $m = 0$. In this case $h = h_0 = (0, 0, \ldots, 0) \in \mathbb{R}^{q-1}$. From (2.1) we have
\[ E_{TI}(\beta, h_0) = -a(x) = -\frac{1}{q\beta} \sum_{i=1}^{q} \ln \left(\sum_{u=1}^{q} \exp(J\beta \sigma_i \sigma_u)\right) = \]
\[ -\frac{1}{q\beta} \sum_{i=1}^{q} \ln \left(\exp(J\beta) + (q-1) \exp\left(\frac{J\beta}{1-q}\right)\right) = \]
\[ -\frac{1}{q\beta} q \ln \left(\exp(J\beta) + (q-1) \exp\left(\frac{J\beta}{1-q}\right)\right) = -J - \frac{1}{\beta} \ln \left(1 + (q-1) \exp\left(\frac{J\beta}{1-q}\right)\right). \]
Remark 1. We note that the entropy for \(2 \ln x_1(m)\) and \(2 \ln x_2(m)\) can be calculated by the formula (3.4) replacing \(h_\ast\) by \(2 \ln x_1(m)\) and \(2 \ln x_2(m)\) respectively.

Remark 2. Notice that under any permutations of the coordinates of the vector \(h_x\) the free energy and entropy do not change.

In Fig.1 graphs of the free energies (3.5) are shown.
3.2. A non-translation-invariant BCs. In this subsection we consider non-translation-invariant BCs (Gibbs measures) constructed by N.Ganikhadj aev in [3]. Here one considers the half tree. Namely the root $x^0$ has $k$ nearest neighbors. Consider an infinite path $\pi = \{x^0 = x_0 < x_1 < \ldots \}$ (the notation $x < y$ meaning that pathes from the root to $y$ go through $x$). Take two different solutions $h^1_\ast$ and $h^2_\ast$ of (3.1) (having the form (3.4)). Associate to this path a collection $h^\pi_\pi$ of vectors given by the condition

$$h^\pi_x = \begin{cases} h^1_x, & \text{if } x \prec x_n, \ x \in W_n; \\
^2_x, & \text{if } x_n \preceq x, \ x \in W_n, \end{cases}$$

(3.7)

$n = 1, 2, \ldots$ where $x \prec x_n$ (resp. $x_n \prec x$) means that $x$ is on the left (resp. right) from the path $\pi$.

For a given infinite path $\pi$ we put

$$W^{\pi}_n = \{x \in W_n : x \prec x_n\},$$

where $n = 1, 2, \ldots$.

**Lemma 1.** For any $\pi$ there exists the limit

$$\lim_{n \to \infty} \frac{|W^{\pi}_n|}{|W_n|} = a^\pi.$$  

(3.8)

**Proof.** Put

$$\Delta^\pi_1 = W^{\pi}_1; \quad \Delta^\pi_n = W^{\pi}_n \cap S(x_{n-1}),$$

where $n \geq 2$. 

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**Fig. 1.** Case $q = 3$. The free energy $F(\beta, h_0)$ the bold solid line (line 1); the free energy $F(\beta, (2 \ln x_1(1), 0))$ solid line (line 2); the free energy $F(\beta, (2 \ln x_2(1), 0))$ the dotted line (line 3); the free energy $F(\beta, (2 \ln x_1(2), 2 \ln x_1(2)))$ the dashed line (line 4); the free energy $F(\beta, (2 \ln x_2(2), 2 \ln x_2(2)))$ the dotted-dashed line (line 5), where $x_i(m), i = 1, 2, m = 1, 2$ defined in (3.3).
It is easy to see that
\[ |W_n^\pi| = \sum_{i=1}^{n} |\Delta_i^\pi| \cdot |W_{n-1}| \]
and \(0 \leq |\Delta_i^\pi| \leq k-1\), for any \(i, \pi\).

Let \(a_n^\pi = \frac{|W_n^\pi|}{|W_n|}\). Then using \(|W_n| = k^{n-1}(k+1)\) we get
\[ a_{n+1}^\pi - a_n^\pi = \frac{|W_{n+1}^\pi|}{|W_{n+1}|} - \frac{|W_{n}^\pi|}{|W_n|} = \frac{\Delta_{n+1}^\pi}{k^n(k+1)} \geq 0, \]
i.e. the sequence \(a_n^\pi\) is a monotone non-decreasing. It is easy to see that \(a_n^\pi < 1\).

Consequently \(\lim_{n \to \infty} |W_n^\pi| = a^\pi\) exists and is finite. □

Denote \(V_n^l = \bigcup_{i=1}^{n} W_n^\pi\). By the Stolz-Cesáro theorem (see e.g. [5]) we have
\[ \lim_{n \to \infty} \frac{|V_n^l|}{|V_n|} = \lim_{n \to \infty} \frac{|W_n^\pi|}{|W_n|} = a^\pi. \tag{3.9} \]

Now we calculate the free energy for the set of vectors \(h_x^\pi\). In view of (3.9) we get
\[
E_G(\beta, m, h_x^\pi) = -\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x) = -\frac{1}{q\beta} \lim_{n \to \infty} \frac{|V_n^l|}{|V_n|} \sum_{i=1}^{q} \ln \left( \sum_{u=1}^{q} \exp\{ (J\beta \sigma_i + h_1^x)\sigma_u \} \right) - \\
-\frac{1}{q\beta} \lim_{n \to \infty} \left( 1 - \frac{|V_n^l|}{|V_n|} \right) \sum_{i=1}^{q} \ln \left( \sum_{u=1}^{q} \exp\{ (J\beta \sigma_i + h_2^x)\sigma_u \} \right) = \]
\[ a^\pi E_{TI}(\beta, m, h_1^x) + (1 - a^\pi) E_{TI}(\beta, m, h_2^x), \tag{3.10} \]
where \(E_{TI}(\beta, m, h)\) is defined in (3.5).

Using (3.6) and (3.10) we get the following formula for the corresponding entropy:
\[ S_G(\beta, m, h_x^\pi) = a^\pi S_{TI}(\beta, m, h_1^x) + (1 - a^\pi) S_{TI}(\beta, m, h_2^x). \]

3.3. Periodic BCs. In this subsection we consider periodic BCs and will be calculating free energies for them.

We consider the case \(q = 3\). It is known (see [9]) that there are only \(G_k^{(2)}\)-periodic Gibbs measures, where \(G_k^{(2)}\) is the set of all words of even lengths. The corresponding set of vectors \(h = \{ h_x \in R^{q-1} : x \in G_k \}\) has the form
\[ h_x = \begin{cases} h_1^1, & \text{if } x \in G_k^{(2)}, \\
h_1^2, & \text{if } x \in G_k \setminus G_k^{(2)}, \end{cases} \]
where \(h_1^1 = (h_1^1, h_2^1), h_1^2 = (h_1^2, h_2^2)\).
In view of (1.7) we have

\[
\begin{align*}
\begin{cases}
    h_1^1 &= k \ln \left( \frac{\theta \exp(h_1^2) + \exp(h_1^2) + 1}{\exp(h_1^2) + \exp(h_1^2) + \theta} \right) \\
    h_1^2 &= k \ln \left( \frac{\exp(h_1^2) + \theta \exp(h_1^2) + 1}{\exp(h_1^2) + \exp(h_1^2) + \theta} \right) \\
    h_2^1 &= k \ln \left( \frac{\theta \exp(h_1^2) + \exp(h_1^2) + 1}{\exp(h_1^2) + \exp(h_1^2) + \theta} \right) \\
    h_2^2 &= k \ln \left( \frac{\exp(h_1^2) + \theta \exp(h_1^2) + 1}{\exp(h_1^2) + \exp(h_1^2) + \theta} \right).
\end{cases}
\end{align*}
\]  

(3.11)

For \( k = 3 \), \( J < 0 \) it is known (see [4]) that if \( 0 < \theta < \frac{1}{4} \) then the system (3.11) has at least two solutions of the form: \( h = (h_1, h_1, h_2, h_2) \). Now for such set of periodic vectors \( h_x \), we shall describe free energies. From [22] we have

\[
a(x) = \begin{cases}
    d(h^1), & \text{if } x \in G_k^{(2)} \\
    d(h^2), & \text{if } x \in G_k \setminus G_k^{(2)},
\end{cases}
\]

where

\[
d(h) = \frac{1}{q\beta} \sum_{i=1}^{q} \left( \sum_{u=1}^{q} \exp \{ (J \beta \sigma_i + h) \sigma_u \} \right).
\]

(3.12)

Denote

\[
V_{even,n} = \{ x \in V_n : x \in G_k^{(2)} \}, \quad V_{odd,n} = \{ x \in V_n : x \in G_k \setminus G_k^{(2)} \}.
\]

It is easy to check that for \( n = 2p \),

\[
|V_{even,2p}| = \frac{k^{2p+1} - 1}{k - 1}, \quad |V_{odd,2p}| = \frac{k^{2p} - 1}{k - 1},
\]

for \( n = 2p + 1 \),

\[
|V_{even,2p+1}| = \frac{k^{2p} - 1}{k - 1}, \quad |V_{odd,2p+1}| = \frac{k^{2p+2} - 1}{k - 1},
\]

also we have

\[
|V_n| = \frac{(k + 1)^{kn} - 2}{(k - 1)}.
\]

Using these formulas we calculate the free energy:

\[
E_{per}(\beta, h_x) = -\lim_{n \to \infty} \frac{|V_{even,n}| d(h^1) + |V_{odd,n}| d(h^2)}{|V_n|} =
\]

\[
-\lim_{n \to \infty} \begin{cases}
    \frac{|V_{even,2p}| d(h^1) + |V_{odd,2p}| d(h^2)}{|V_{2p}|} & \text{if } n = 2p \\
    \frac{|V_{even,2p+1}| d(h^1) + |V_{odd,2p+1}| d(h^2)}{|V_{2p+1}|} & \text{if } n = 2p + 1,
\end{cases}
\]

\[
- \frac{1}{k+1} \begin{cases}
    k d(h^1) + d(h^2) & \text{if } n = 2p, p \to \infty \\
    d(h^1) + kd(h^2) & \text{if } n = 2p + 1.
\end{cases}
\]
From this equality it follows that if $d(h^1) \neq d(h^2)$, then for the periodic BCs a free energy does not exist. For $k = q = 3$ and fixed $\theta = \frac{1}{5}$, using a computer analysis one can see that $d(h^1) \neq d(h^2)$.

**Remark 3.** It is known (see [1]) that for periodic Gibbs measures of the Ising model on Cayley trees the free energies exist. But for the Potts model we proved that free energy of periodic Gibbs measures may not exist.

### 3.4. Weakly periodic BCs.

Construct a weakly periodic BC. For $A \subset \{1, 2, ..., k+1\}$ we consider $H_A = \{x \in G_k : \sum_{j \in A} w_j(x) = \text{even}\}$, where $w_j(x)$ is the number of $a_j$ in a word $x$, $G_k/H_A = \{H_A, G_k \setminus H_A\}$ is a quotient group. For simplicity, we set $H_0 = H_A$, $H_1 = G_k \setminus H_A$. The $H_A$ - weakly periodic sets of vectors $h = \{h_x \in R^{n-1} : x \in G_k\}$ have the following form

$$h_x = \begin{cases} 
    h_1, & \text{if } x_1 \in H_0, x \in H_0 \\
    h_2, & \text{if } x_1 \in H_0, x \in H_1 \\
    h_3, & \text{if } x_1 \in H_1, x \in H_0 \\
    h_4, & \text{if } x_1 \in H_1, x \in H_1.
\end{cases} \quad (3.13)$$

Here $h_i = (h_{i1}, h_{i2}, ..., h_{iq-1})$, $i = 1, 2, 3, 4$. By (1.7), we have

$$\begin{align*}
    h_1 &= (k - |A|)F(h_1, \theta) + |A|F(h_2, \theta), \\
    h_2 &= (|A| - 1)F(h_3, \theta) + (k + 1 - |A|)F(h_4, \theta), \\
    h_3 &= (|A| - 1)F(h_2, \theta) + (k + 1 - |A|)F(h_1, \theta), \\
    h_4 &= (k - |A|)F(h_4, \theta) + |A|F(h_3, \theta).
\end{align*} \quad (3.14)$$

In [7] it was shown that for $|A| = k$, $k \geq 6$ the system of equations (3.14) has at least two (not translation-invariant) solutions, which generate sets of vectors $h_x$ of the form of (3.13), where all coordinates of vectors $h_i, i = 1, 2, 3, 4$ are equal and $h_i \neq h_j$ for $i \neq j$. Now for such weakly periodic sets of vectors $h_x$, we calculate the corresponding free energy.

We introduce the following

$$\begin{align*}
    A_n &= |\{(x, y) \in L_n : x \in H_0, y = x_1 \in H_0\}|, \\
    B_n &= |\{(x, y) \in L_n : x \in H_0, y = x_1 \in H_1\}|, \\
    C_n &= |\{(x, y) \in L_n : x \in H_1, y = x_1 \in H_0\}|, \\
    D_n &= |\{(x, y) \in L_n : x \in H_1, y = x_1 \in H_1\}|,
\end{align*}$$

where $L_n$ is a set of edges in $V_n$.

It is known from [1] that

$$\begin{align*}
    \lim_{n \to \infty} \frac{A_n}{|V_n|} &= \lim_{n \to \infty} \frac{D_n}{|V_n|} = \frac{1}{2(k+1)}, \\
    \lim_{n \to \infty} \frac{B_n}{|V_n|} &= \lim_{n \to \infty} \frac{C_n}{|V_n|} = \frac{k}{2(k+1)}.
\end{align*}$$
Using these formulas we calculate the free energy:

\[ E_{WP}(\beta, q, h_x) = - \lim_{n \to \infty} \frac{1}{|V_n|} \left( A_n d(h_1) + B_n d(h_2) + C_n d(h_3) + D_n d(h_4) \right) = \]

\[ \frac{1}{2(k+1)} \left( E_{TI}(\beta, q - 1, h_1) + k E_{TI}(\beta, q - 1, h_2) + k E_{TI}(\beta, q - 1, h_3) + E_{TI}(\beta, q - 1, h_4) \right), \]

where \( E_{TI}(\beta, m, h_x) \) is defined in (3.5). Now the entropy is

\[ S_{WP}(\beta, q, h_x) = \frac{1}{2(k+1)} \left( S_{TI}(\beta, q - 1, h_1) + k S_{TI}(\beta, q - 1, h_2) + k S_{TI}(\beta, q - 1, h_3) + S_{TI}(\beta, q - 1, h_4) \right), \]

where \( S_{TI}(\beta, m, h_x) \) is defined in (3.6).

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