Approximate Embedding of Large Polygons into $\mathbb{Z}^2$

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Abstract. Let $\mathbb{Z}^2$ denote the standard lattice in the plane $\mathbb{R}^2$. We prove that given a finite subset $S \subset \mathbb{R}^2$ and $\varepsilon > 0$, then for all sufficiently large dilations $t > 0$ there exists a rotation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ around the origin such that $\text{dist}(\rho(tz), \mathbb{Z}^2) < \varepsilon$, for all $z \in S$. The result, in a larger generality, has been proved in 2006 by Tamar Ziegler (improving earlier results by Furstenberg, Katznelson, Weiss). The proof presented in the paper is short and self-contained; it is not based on the Furstenberg correspondence principle.

1. Conjecture and result

For $z \in \mathbb{R}^k$, denote by $||z|| = \text{dist}(z, \mathbb{Z}^k)$ the Euclidean distance of $z$ from the standard lattice set $\mathbb{Z}^k \subset \mathbb{R}^k$.

For a finite metric space $X = (X,d)$ and $k \geq 1$, denote by $\Sigma_k(X)$ the collection of isometric embeddings $f : X \to \mathbb{R}^k$. Set

$$\tau_k(X) = \inf_{f \in \Sigma_k(X)} \left( \max_{x \in X} ||f(x)|| \right).$$

A metric space $X$ is called $k$-flat if it embeds isometrically into $\mathbb{R}^k$ (i.e., if $\Sigma(X) \neq \emptyset$). By definition, $\tau_k(X) = \infty$ if $X$ is not $k$-flat. Clearly $\tau_k(X) < \sqrt{k}$ if $X$ is $k$-flat.

Given a metric space $X = (X,d)$ and $t > 0$, we write $tX$ for the metric space $(X,td)$.

The following theorem is restatement of a special case of T. Ziegler’s result, [7, Theorem 1.3], which superseded an earlier result [3] by Furstenberg, Katznelson and Weiss.

Theorem 1.1. For $k \geq 2$ and any finite $k$-flat metric space $X$

$$\lim_{t \to +\infty} \tau_k(tX) = 0.$$  \hspace{1cm} (1.2)

We present a new proof of Theorem 1.1 which works only for even $k$. The proof is short and self-contained, it is not based on the Furstenberg correspondence principle (the way the approaches in the papers mentioned above are). Our approach may lead to explicit estimates on the speed convergence in (1.2) to 0. (This direction is not pursued in this paper).

Our proof of Theorem 1.1 is based on the following two theorems.

Theorem 1.2. The set $C$ of natural numbers $k$ for which the claim of Theorem 1.1 holds is closed under addition.

Theorem 1.3. $2 \in C$, i.e. for any finite 2-flat metric space $X$

$$\lim_{t \to +\infty} \tau_2(tX) = 0.$$
Theorem 1.3 is an immediate consequence of Theorem 3.1 in the next section. The proof of Theorem 1.2 is presented in Section 5.

I am indebted to Professor Mihalis Kolountzakis for directing me to Ziegler’s paper [7] on the next day the first version of the current note appeared on arXiv.

2. TRIANGLES AND EMBEDDINGS

By a triangle we mean a metric space of cardinality 3. Since every triangle is 2-flat, we have the following corollary.

**Corollary 2.1.** For every triangle \(X\), \(\lim_{t \to +\infty} \tau_2(tX) = 0\).

By the separation of a finite metric space \(X = (X, d)\) (notation: \(\text{sep}(X)\)) we mean the minimal positive distance in it: \(\text{sep}(X) = \min_{a,b \in X, a \neq b} d(a, b)\).

The following proposition shows that the value \(\tau_2(X)\) does not need to be small for triangles \(X\) with large separation \(\text{sep}(X)\) (cf. Corollary 2.1).

**Proposition 2.1.** For all triangles \(X_t = \{(0, 0), (0, t), (t + \frac{1}{2}, 0)\} \subset \mathbb{R}^2, \) with \(t > 0\), we have \(\tau_2(X_t) \geq \frac{1}{8}\) and \(\text{sep}(X_t) = t\).

In the proof of Proposition 2.1, we use the fact that the formula \(||x - y|| = \text{dist}(x - y, Z^2)\) \((x, y \in \mathbb{R}^2)\) defines a pseudometric on \(\mathbb{R}^2\).

**Proof of Proposition 2.1.** Let \(f \in \Sigma_2(X_t)\) and set \(A = (0, 0), B = (0, t), C = (t + \frac{1}{2}, 0), D = (t, 0); A, B, C, D \in \mathbb{R}^2\).

We have to show that \(m := \max ||f(A)||, ||f(B)||, ||f(C)|| \geq \frac{1}{8}\).

Since \(f\) is an isometry, \(|C - D|_2 = \frac{1}{2}\) implies \(|f(C) - f(D)|_2 = \frac{1}{2}\), and hence \(\frac{1}{2} = ||f(C) - f(D)|| \leq ||f(C) - f(A)|| + ||f(D) - f(A)||\).

Note that \(||f(D) - f(A)|| = ||f(B) - f(A)||\) because the corresponding vectors in \(\mathbb{R}^2\) are perpendicular and have the same length. It follows that \(\frac{1}{2} \leq ||f(C) - f(A)|| + ||f(B) - f(A)|| \leq 2 ||f(A)|| + ||f(B)|| + ||f(C)|| \leq 4m\),

whence \(m \geq \frac{1}{8}\), completing the proof of Proposition 2.1. \(\Box\)

**Remark 2.1.** One can show that \(\lim_{t \to +\infty} \tau_k(X_t) = 0\), for \(k \geq 3\). In general, if \(X_{s,t} = \{(0, 0), (0, t), (s, 0)\} \subset \mathbb{R}^2\), then \(\lim_{s \to +\infty} \lim_{t \to +\infty} \tau_k(X_{s,t}) = 0\), for \(k \geq 3\).

This does not extend to \(k = 2\) in view of Proposition 2.1.
3. Central result

Denote by $Z^2$ the standard lattice in the plane $\mathbb{R}^2$. Denote by $\mathcal{R}$ the set of rotations $\rho: \mathbb{R}^2 \to \mathbb{R}^2$ around the origin. The central result of the paper is the following theorem.

**Theorem 3.1.** Let $S \subset \mathbb{R}^2$ be a finite set. Then, for every $\varepsilon > 0$, there exists $T > 0$ such that for every dilation $t > T$ there exists a rotation $\rho \in \mathcal{R}$, $\rho = \rho(\varepsilon, t)$, such that

$$||\rho(tz)|| = \text{dist}(\rho(tz), Z^2) < \varepsilon, \text{ for all } z \in S.$$ 

The above theorem is translated into Theorem 3.2 below using the language of complex numbers. The proof is provided for Theorem 3.2 (rather than for Theorem 3.1) because it utilizes the field structure of $\mathbb{C}$, and the alternative proof is somewhat longer.

One identifies $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, the lattice $Z^2$ becomes the set $\mathbb{Z}[i]$ of Gaussian integers, and one sets the notation:

$$||z|| = \min_{w \in \mathbb{Z}[i]} |z - w|, \text{ for } z \in \mathbb{C},$$

and

$$||V|| = \max_{1 \leq k \leq r} ||z_k||, \text{ where } V = (z_k)_{k=1}^r \in \mathbb{C}^r.$$ 

Denote by $T = \{ z \in \mathbb{C} | |z| = 1 \}$ the unit circle in $\mathbb{C}$.

**Theorem 3.2.** Let $V = (z_k)_{k=1}^r \in \mathbb{C}^r$ be a vector, $r \geq 1$. Then, for every $\varepsilon > 0$, there exists $T > 0$ such that for every $t > T$ there is (a rotation) $\theta \in T$ such that

$$||\theta tV|| < \varepsilon.$$ 

Note that $||\theta tV|| < \varepsilon \iff ||\theta tz_k|| < \varepsilon$, for all $1 \leq k \leq r$.

The central idea of the proof (of Theorem 3.2) is conveyed in Section 5 where a special case (when $V$ is typical in the sense of Definition 4.2 of the next section) is treated.

The proof of Theorem 3.2 is completed in Section 7.

In the final Section 7 we state a conjecture extending Theorem 1.1 (the lattice $Z^2$ is replaced by an arbitrary syndetic subset of $\mathbb{R}^2$).

4. Notation and definitions

In what follows we do not distinguish between vectors $V = (z_k)_{k=1}^r \in \mathbb{C}^r$ and $r$-tuples of its entries with appropriate $r$.

**Definition 4.1.** Let $\mathbb{F} \subset \mathbb{C}$ be a subfield of complex numbers. An $r$-tuple (or a vector) $V = (z_k)_{k=1}^r \in \mathbb{C}^r$ is called $\mathbb{F}$-generic if its $r$ entries are linearly independent over $\mathbb{F}$.

In particular, the entries of any $\mathbb{F}$-generic vector must be distinct.

**Notation.** For any vector $V = (z_k)_{k=1}^r = (z_1, z_2, \ldots, z_r) \in \mathbb{C}^r$, set

$$\text{Re}(V) = (\text{Re}(z_k))_{k=1}^r \in \mathbb{R}^r, \quad \text{Im}(V) = (\text{Im}(z_k))_{k=1}^r \in \mathbb{R}^r,$$

(where $\text{Re}(\cdot)$, $\text{Im}(\cdot)$ stand for real and imaginary parts, respectively), and let

$$\text{Re Im}(V) = (\text{Re}(z_1), \ldots, \text{Re}(z_r), \text{Im}(z_1), \ldots, \text{Im}(z_r)) \in \mathbb{R}^{2r}$$

denote the result of concatenation of $\text{Re}(V)$ and $\text{Im}(V)$.
Definition 4.2. A vector \( V = (z_k)_{k=1}^r = (z_1, z_2, \ldots, z_r) \in \mathbb{C}^r \) is called \textit{typical} if the vector \( \text{Re Im}(V) \in \mathbb{R}^{2r} \) is \( \mathbb{Q} \)-generic in the sense of Definition 4.1.

We will use the following lemma.

Lemma 4.1. Let \( V = (z_k)_{k=1}^r \in \mathbb{C}^r \) be a typical \( r \)-tuple, and let \( \varepsilon > 0 \) be given. Then there is a positive number \( L = L(V, \varepsilon) > 0 \) such that for any other \( r \)-tuple \( W = (w_k)_{k=1}^r \in \mathbb{C}^r \) there exists a number \( s \in [0, L] \), such that \( \| W + sV \| < \varepsilon \) (equivalently, \( \| w_k + sz_k \| < \varepsilon \), for all \( 1 \leq k \leq r \)).

Proof. Set \( V' = \text{Re Im}(V) = (v_k)_{k=1}^{2r} \in \mathbb{R}^{2r} \) and \( W' = \text{Re Im}(W) = (u_k)_{k=1}^{2r} \in \mathbb{R}^{2r} \).

In order to prove the lemma, it is enough to establish the existence of a positive number \( L = L(V', \varepsilon) \) (not depending on \( W' \)) such that

\[
\min_{0 \leq s \leq L} \| W' + sV' \| < \varepsilon
\]

(equivalently, \( \text{dist}(W' + sV', \mathbb{Z}^{2r}) < \varepsilon \)).

This existence follows from the minimality of the linear flow

\[ T^tX = X + tV' \mod 1, \quad X \in \mathbb{R}^{2r}/\mathbb{Z}^{2r}, \]

on the torus \( \mathbb{R}^{2r}/\mathbb{Z}^{2r} \) in the direction \( V' \). (Since \( V \) is typical, all the entries \( v_k \) of the vector \( V' \) are linearly independent, and the minimality holds, see e.g. [5, Proposition 1.5.1] from the dynamical point of view, or [6, §9, Example 9.3 and Exercise 9.27], from the number theory perspective).

Indeed, the minimality of the linear flow implies that all orbits become \( \varepsilon \)-dense in the same finite time \( L \). \( \square \)

5. Proof of Theorem 3.2. Typical case

In this section we prove Theorem 3.2 under the added assumption that the \( r \)-tuple

\( V = (z_k)_{k=1}^r \in \mathbb{C}^r \)

is typical (Definition 4.2). Let \( \varepsilon > 0 \) be given. Set \( W = -itV = (-it z_k)_{k=1}^r \). By Lemma 4.1, there exists \( L > 0 \) such that for every \( t > 0 \) there exists \( s(t) \in [0, L] \) such that

\[
\| (-it + s(t))V \| < \varepsilon/2, \quad t \in \mathbb{R}.
\]

Since multiplication by \( i \) is an isometry in \( \mathbb{C} \), we obtain

\[
\| (t + is(t))V \| < \varepsilon/2, \quad t \in \mathbb{R}.
\]

For \( t > 0 \), set

\[
\theta_t = e^{is(t)/t} \in T = \{ z \in \mathbb{C} \mid |z| = 1 \}.
\]

Then

\[
\theta_t t = e^{is(t)/t} \cdot \left( 1 + \frac{is(t)}{t} + O(t^{-2}) \right) \cdot t = t + is(t) + o(1) \quad (\text{as } t \to \infty),
\]

whence, for large \( t > 0 \), we obtain

\[
\| \theta_t tV - (t + is(t))V \| < \varepsilon/2.
\]
The inequalities (5.1) and (5.2) imply that \(|\theta t V| < \varepsilon\) for large \(t\), completing the proof of Theorem 3.2 (under the added assumption that \(V\) is typical).

6. The case of \(V\) being \(\mathbb{Q}[i]\)-generic

In this section we prove Theorem 3.2 under the added assumption that the \(r\)-tuple
\[ V = (z_k)_{k=1}^r \in \mathbb{C}^r \]
is \(\mathbb{Q}[i]\)-generic (Definition 4.1) where \(\mathbb{Q}[i] = \{ z \in \mathbb{C} \mid \text{Re}(z), \text{Im}(z) \in \mathbb{Q} \} \).

Note that this condition on \(V\) (\(\mathbb{Q}[i]\)-genericity) is weaker than the one imposed in the preceding section (for \(V\) to be typical).

In view of the result in the preceding section, it is enough to show that \(\theta V\) is typical for some \(\theta \in \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \).

Proposition 6.1. Let \( V = (z_k)_{k=1}^r \in \mathbb{C}^r \) be a \(\mathbb{Q}[i]\)-generic vector. Then for all but at most a countable set of \(\phi \in [0, 2\pi)\) the vector \(e^{i\phi} V\) is typical.

The proof (in the end of the section) is a combination of Lemmas 6.1 and 6.2 below.

Lemma 6.1. Let \( V = (z_k)_{k=1}^r \in \mathbb{C}^r \) be a \(\mathbb{Q}[i]\)-generic vector. Then for all but at most a countable set of \(\phi \in [0, 2\pi)\) the vector
\[ W_\phi = (e^{i\phi} z_1, e^{i\phi} z_2, \ldots, e^{i\phi} z_r, e^{-i\phi} \bar{z}_1, e^{-i\phi} \bar{z}_2, \ldots, e^{-i\phi} \bar{z}_r) \in \mathbb{C}^{2r} \]
is also \(\mathbb{Q}[i]\)-generic.

Proof. It is enough to show that for every non-zero vector
\[ 0 \neq U = (a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r) \in (\mathbb{Q}[i])^{2r} \]
the scalar product
\[ U \cdot W_\phi = \sum_{k=1}^r a_k e^{i\phi} z_k + \sum_{k=1}^r b_k e^{-i\phi} \bar{z}_k = e^{i\phi} \sum_{k=1}^r a_k z_k + e^{-i\phi} \sum_{k=1}^r b_k \bar{z}_k \]
may vanish only for a finitely many \(\phi \in [0, 2\pi)\).

Indeed, the alternative is that \(U \cdot W_\phi = 0\) for all \(\phi\) (since \(U \cdot W_\phi\) is analytic in \(\phi\)), and hence both sums
\[ S_1 = \sum_{k=1}^r a_k z_k; \quad S_2 = \sum_{k=1}^r b_k \bar{z}_k \]
vanish (since the functions \(e^{i\phi}\) and \(e^{i\phi}\) are linearly independent over \(\mathbb{C}\)). The \(\mathbb{Q}[i]\)-genericity of \(V\) and the fact that \(S_1 = 0\) imply that all \(a_k = 0\). On the other hand, \(S_2 = 0\) implies \(\bar{S}_2 = \sum_{k=1}^r b_k \bar{z}_k = 0\), and hence all \(b_k = 0\). We conclude that \(U = 0\), a contradiction. \(\square\)

Lemma 6.2. Let \( V = (z_k)_{k=1}^r \in \mathbb{C}^r \) be a vector and assume that the vector
\[ W = (z_1, \ldots, z_r, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_r) \in \mathbb{C}^{2r} \]
is \(\mathbb{Q}[i]\)-generic. Then \(V\) is typical.
Proof. It is easy to see that the $2r$ entries of the vector $W$ and the $2r$ entries of the vector
\[ U = \text{Re} \text{Im}(V) = (\text{Re}(z_1), \ldots, \text{Re}(z_r), \text{Im}(z_1), \ldots, \text{Im}(z_r)) \in \mathbb{R}^{2r} \]
generate the same vector space over $\mathbb{Q}[i]$. Denote by $d$ its dimension.

We have $d = 2r$ because $W$ is $\mathbb{Q}[i]$-generic. It follows that $U$ is also $\mathbb{Q}[i]$-generic (otherwise we would have $d < 2r$). Since $U$ is a real vector, it is in fact $\mathbb{Q}$-generic, i.e. $V$ is typical. This complete the proof of Lemma 6.2. \hfill \Box

Proof of Proposition 6.1. The proof is obtained by combining statements of Lemmas 6.1 and 6.2. \hfill \Box

7. General case

It remains to consider the case when the vector $V$ is not $\mathbb{Q}[i]$-generic. (Recall that the case of $\mathbb{Q}[i]$-generic $V$ is considered in the previous section). Without loss of generality, we may assume (after a rearrangement of the entries of $V$ if needed) that we have a representation
\[ V = (z_1, z_2, \ldots, z_m, w_1, w_2, \ldots, w_n) \in \mathbb{C}^r, \quad r = m + n, \quad m, n \geq 1, \]
where the first $m$ entries $z_1, z_2, \ldots, z_m$ are linearly independent over $\mathbb{Q}[i]$ and each of the consequent $n$ entries $w_k$ is a linear combination of the first $m$ ones:
\[ w_j = \sum_{k=1}^{m} f_{j,k} z_k; \quad 1 \leq j \leq n; \quad f_{j,k} \in \mathbb{Q}[i]. \]

Select an integer $M > \sum_{k=1}^{m} \sum_{j=1}^{n} |f_{j,k}|$ such that
\[ Mf_{j,k} \in \mathbb{Z}[i], \quad \text{for all} \ 1 \leq j \leq n, 1 \leq k \leq m. \]

Let $\varepsilon > 0$ be given. Since the vector $V_0 = \frac{1}{M}(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ is $\mathbb{Q}[i]$-generic, it follows from the result stated in the beginning of Section 7 that for all sufficiently large $t > 0$ there exists a point $\theta_t \in \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ such that for all large $t > 0$
\[ \| \frac{1}{M} \theta_t V_0 \| < \frac{\varepsilon}{2mM^2}, \]
or, equivalently,
\[ \| \frac{\theta_t z_k}{M} \| < \frac{\varepsilon}{2mM^2}, \quad \text{for all} \ 1 \leq k \leq m. \] (7.1)

It follows that
\[ \| \theta_t z_k \| < \frac{\varepsilon}{2mM} < \varepsilon, \quad \text{for all} \ 1 \leq k \leq m, \]
and
\[ \| \theta_t w_j \| = \| \sum_{k=1}^{m} \theta_t f_{j,k} z_k \| \leq \sum_{k=1}^{m} \| \theta_t (Mf_{j,k}) \frac{z_k}{M} \| \leq \sum_{k=1}^{m} (Mf_{j,k}) \| \frac{\theta_t z_k}{M} \| \leq \sum_{k=1}^{m} M^2 \frac{\varepsilon}{2mM^2} = \frac{\varepsilon}{2} < \varepsilon, \quad \text{for all} \ 1 \leq j \leq n. \]

We conclude $\| \theta_t V \| < \varepsilon$ for large $t$, completing the proof of Theorem 3.2.
8. Proof of Theorem 1.2

Given that $m, n \in \mathcal{C}$, we have to show that $p = m + n \in \mathcal{C}$, i.e. that, given a $p$-flat metric space $X = (X, d)$ with $p = m + n$, then $\lim_{t \to +\infty} \tau_p(tX) = 0$.

Without loss of generality, we assume that $X \subset \mathbb{R}^p = \mathbb{R}^m \times \mathbb{R}^n$. Let $\varepsilon > 0$ be given. We have to show that there exists $\psi \in \Sigma_p(tX)$ such that $\|\psi(x)\| < \varepsilon$, for all $x \in X$.

Denote by $\pi_1, \pi_2$ corresponding projections $\pi_1 : \mathbb{R}^p \to \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^p \to \mathbb{R}^n$. Let $X_1 = \pi_1(X) \in \mathbb{R}^m$, $X_2 = \pi_2(X) \in \mathbb{R}^n$.

Since $m, n \in \mathcal{C}$, for large $t > 0$ there are isometric embeddings $\phi_{1,t} : tX_1 \to \mathbb{R}^m$, $\phi_{2,t} : tX_2 \to \mathbb{R}^n$, such that

$$\|\phi_{1,t}(u)\| < \varepsilon/2, \quad \text{for all } u \in X_1,$$

and

$$\|\phi_{2,t}(v)\| < \varepsilon/2, \quad \text{for all } v \in X_2.$$

Set the maps $\psi_1 : X \to \mathbb{R}^m$, $\psi_2 : X \to \mathbb{R}^n$, as the compositions

$$\psi_1 : = \phi_{1,t} \circ M_t \circ \pi_1, \quad \psi_2 : = \phi_{2,t} \circ M_t \circ \pi_2$$

where $M_t$ is a multiplication by $t$ operator.

Then the map $\psi = \psi_1 \times \psi_2$ is an isometric embedding, $\psi : tX \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^p$, such that

$$\|\psi(x)\| \leq \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2} < \varepsilon,$$

for all $x \in X$. (Recall that by definition $tX = (X, td)$, for metric spaces $X = (X, d)$).

This completes the proof of Theorem 1.2.

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