Dynamical systems and numerical analysis: the study of measures generated by uncountable I.F.S.

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Abstract Measures generated by Iterated Function Systems composed of uncountably many one-dimensional affine maps are studied. We present numerical techniques as well as rigorous results that establish whether these measures are absolutely or singular continuous.

Keywords Iterated Function Systems · Singular measures · Fourier transform · Invariant measures

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1 Introduction

In this paper I want to describe an example of the fruitful interplay between the theory of dynamical systems and numerical analysis: I want to show how theoretical questions such as singularity (or continuity) with respect to Lebesgue of a dynamical measure can be attacked from a numerical point of view, and vice versa how particular features in the numerical analysis of measures can be properly explained by concepts in the theory of dynamical systems.
Let us consider the iteration of maps $\phi_\lambda : X \to X$ from a compact metric space $X$ to itself, labelled by the variable $\lambda$, which belongs to a measure space $\Lambda$, on which the probability measure $\sigma$ is given. This process gives rise to what is called an Iterated Function System, or I.F.S., [1, 10] with invariant measure $\mu$, which can be defined as follows. Consider the transfer operator $T$ on the space $C(X)$ of continuous functions on $X$, via

$$(Th)(x) = \int d\sigma(\lambda)(h \circ \phi_\lambda)(x).$$

(1)

Then, let $T^*$ be the adjoint operator in the space of regular Borel measures on $X$. An invariant measure of the I.F.S. is the fixed point of $T^*$,

$$T^*(\mu) = \mu.$$  

(2)

Suitable hypotheses can be formulated in order for $\mu$ to be unique, given the choice of the set of maps $\phi_\lambda$ and of the distribution $\sigma$ [18]. We are interested in the nature of $\mu$: is it of pure type? If so, is it pure point, singular, or absolutely continuous? What characteristics of $\phi_\lambda$ and $\sigma$ have importance in this regard? This problem belongs to a classical topic of research in dynamical systems, that looks for a.c.i.m., that is, absolutely continuous invariant measures. Indeed, it is my opinion that singular continuous measures are equally, if not more interesting, in many respects.

The plan of this paper is to study this problem in a class of maps of the real line. We will derive algorithms to reduce the quest for an invariant measure to a fixed point problem. If in addition there exists an attractive fixed point in a suitable space of densities, absolute continuity of the measure will follow. We will also describe the Fourier transform of the measure $\mu$, its Mellin transform and its Sobolev dimension, that will also lead to numerical and theoretical methods to determine absolute continuity or singularity of the invariant measure.

2 Infinite affine Iterated Function Systems

Iterated Function Systems have been originally introduced and studied in [1, 10], although in some form their history goes much further back in time, see [21]. They have become a versatile mathematical tool with applications to image compression [2, 12], quantum dynamics [15] and much more [5]. In its simplest form, an I.F.S. is a finite collection of contractive maps of a space $X$ into itself: when iterated randomly, these maps produce a stochastic process in $X$ with invariant measure $\mu$. Interesting results and applications are found for affine one dimensional maps of the kind

$$\phi_{\delta,\beta}(x) = \delta(x - \beta) + \beta.$$  

(3)

Each of these maps has the fixed point $\beta$ and contractions ratio $0 < \delta < 1$. When dealing with finite collections of affine one-dimensional maps, the prob-