RG flow in 2d field theory coupled to gravity

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Abstract: The renormalization group flow in two-dimensional field theories is modified if they are coupled to gravity. Beta function coefficients are changed, the \( c \)-theorem is no longer strictly valid, and flows from fixed points with central charge \( c > 25 \) to fixed points with \( c < 25 \) are forbidden. This is discussed in general and at two examples, the Kosterlitz-Thouless phase transition and the Wess-Zumino-Witten model. A possible application to string cosmology is pointed out.

1. Introduction

Consider a renormalizable two-dimensional field theory with coupling constants \( \lambda^i \) on a surface with fixed background metric \( g_{\alpha\beta} \). The coupling constants will typically “run” under a change of scale by the factor \( e^\tau \), where \( \tau \) is “renormalization group time”. The question addressed in this talk is: how is this flow \( \lambda^i(\tau) \) modified if the theory is coupled to gravity, i.e., if \( g_{\alpha\beta} \) is taken to be a dynamical variable?

Why is this an interesting question? First of all, some physical systems like the 3d Ising model or QCD are conjectured to have descriptions in terms of random surfaces. In investigating these systems, one is often interested in properties like fixed points, critical coefficients, or phase diagrams. In other words, one is interested in properties of the RG flow of the corresponding 2d field theory on a surface with fluctuating geometry. Second, as I will explain, RG trajectories “in the presence of gravity” can be regarded as time-dependent classical solutions of string theory. Conversely, interpreting string solutions as RG trajectories may shed new light on the former.

How can we define scale-dependent coupling constants in theories with gravity, where the scale itself is a dynamical variable and is integrated over? To address this question, we will first discuss the Kosterlitz-Thouless transition in the sine-Gordon model coupled to gravity. We will introduce a method to determine the flow that also agrees with matrix model and light-cone gauge results. As an application, the flow from a free theory to a WZW model with central charge \( c \) will then be discussed, with emphasis on the case \( c \geq 25 \). Finally, the modification of the flow in the general bosonic 2d field theory due to gravity and a connection with string cosmology will be discussed. Some speculative thoughts will conclude this talk.

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2. The sine–Gordon model coupled to gravity

Consider a free scalar field $x$ with a sine–Gordon interaction $\cos px$ with sine–Gordon momentum $p = \sqrt{2} + \epsilon, \epsilon \ll 1$. At $p = \sqrt{2}$ the interaction becomes marginal and the Kosterlitz–Thouless transition takes place. The action is proportional to

$$\int \sqrt{g} \left\{ (\partial x)^2 + m \cos(\sqrt{2} + \epsilon) x \right\}.$$ 

Coupled to gravity, the theory is described in the approach of David, Distler and Kawai \[1\] by the action (up to some coefficients)

$$S = \int \sqrt{\hat{g}} \left\{ (\partial x)^2 + \partial \phi^2 + 2\sqrt{2} \hat{R} \phi + m \cos(\sqrt{2} + \epsilon) x \ e^{\epsilon \phi} - m^2 \phi \ \partial x^2 + ... \right\}.$$ 

Here, $\phi$ is the Liouville mode, related to the conformal factor $e^{\alpha \phi}$. Its kinetic term is induced by the conformal anomaly. We have ignored the cosmological constant, which plays no role in the following (see appendix of \[2\]). $\hat{g}$ is an arbitrarily chosen background metric that nothing must depend on; in particular, the combined $(x, \phi)$ theory must be scale invariant. This is guaranteed to first order in $m, \epsilon$ by the “gravitational dressing” $e^{\epsilon \phi}$, to second order by the $O(m^2)$ term \[3\], and so on. But while the scale $\sqrt{g}$ is fictitious, there is a physical scale: $\sqrt{ge^{\alpha \phi}}$, with $\alpha = -\sqrt{2}$. Therefore, a shift of $\phi$ by a constant $\lambda$,

$$\phi \rightarrow \phi + \lambda,$$ 

is a scale transformation. So let us make make $m$ and $\epsilon$ $\lambda$–dependent such that the shift (5) is absorbed. $m(\lambda)$ and $\epsilon(\lambda)$ are what we call “running coupling constants.” They are easy to find in this example:

$$m(\lambda) = m_0 e^{-\lambda}, \quad \epsilon(\lambda) = \epsilon_0 - \frac{1}{\sqrt{2}} \lambda m^2.$$ 

Here, $m_0$ and $\epsilon_0$ are small initial parameters. In deriving $\epsilon(\lambda)$, a $\lambda m^2 \partial x^2$ term has been absorbed in a redefinition of $x$ and in a shift of $\epsilon$. Defining ‘dot’ as $\frac{d}{d\lambda}$, we get the lowest order “beta functions”

$$\dot{\epsilon} \sim -m^2, \quad \dot{m} \sim -\epsilon m.$$ 

Fig. 1: Kosterlitz–Thouless transition in the sine–Gordon model coupled to gravity
The resulting flow diagram is shown in fig. 1. It is qualitatively the same as the Kosterlitz–Thouless diagramm of the flat–space sine–Gordon model, with a diagonal phase boundary at $\epsilon \propto m$. Upon working out the coefficients, one finds that the overall “velocity” of the flow is cut in half by gravity. More generally, quadratic beta functions as in (8) are multiplied by the factor $-2/(Q\alpha)$ (for $c = 1$, $Q = 2\sqrt{2}$ and $\alpha = -\sqrt{2}$).

This agrees with the light–cone gauge result [3]. The KT transition in the presence of gravity has also been observed in the matrix models [4]. This (and agreements for other models) confirms our method of finding the flow in theories coupled to gravity: First, we add the new field $\phi$ to the theory without gravity; then we make the combined theory scale invariant; finally, we interpret $\tau = \frac{\alpha}{2}\phi$ as “RG time” (see also [3]).

3. The nonlinear sigma model with Wess–Zumino term

Let us now apply this method to models with $c > 1$, and in particular with $c > 25$, coupled to gravity. Of course, bosonic theories with $c > 1$ cannot be consistently coupled to gravity, due to the tachyon that appears in the spectrum of the target space theory. We should really be discussing supersymmetric theories coupled to supergravity with the tachyon projected out. But to simplify things, let us stick to the bosonic case and just ignore the tachyon.

Consider as an example the nonlinear sigma model with a WZ term. The fields lie in an $N$–dimensional target space, the group space $G$, with a fixed metric $\hat{G}_{ij}$ and curvature

$$\hat{R}^{(N)}_{ij} = \frac{1}{4}f_{imn}f_{jmn} = \frac{1}{4}c_G \hat{G}_{ij},$$

where $f_{ijk}$ are the structure constants and $c_G$ is the value of the quadratic Casimir operator in the adjoint representation. The WZ term can be represented by the antisymmetric tensor field $\hat{B}_{ij}$ with field strength

$$\hat{H}_{ijk} = \nabla_i[B_{jk}] = k f_{ijk}.$$

The RG flow in the model without gravity leads from a free theory ($\lambda \sim \infty$) in the UV to the WZW model in the IR. Let us now “turn on” gravity. We thus add the Liouville mode $\phi$ to the theory, with kinetic term $\partial \phi^2 + \hat{R}^{(2)}(\Phi, \phi)$. Here, $\Phi$ is the “dilaton”. The next step, making the combined theory scale invariant, corresponds to solving the equations of motion of 3+1 dimensional string theory (= conformal invariance conditions), with metric and $B$ fields given by $G_{ij}, B_{ij}$ and $G_{0i} = B_{0i} = 0, G_{00} = \pm 1$ where roman indices run from 1 to 3. Following [3, 6], let us make the ansatz

$$G_{ij}(\vec{x}, \phi) = e^{2\lambda(\phi)}\hat{G}_{ij}(\vec{x}), \quad (1)$$
$$\Phi(\vec{x}, \phi) = \Phi(\phi), \quad (2)$$
$$H_{ijk} = \hat{H}_{ijk} = k f_{ijk}. \quad (3)$$

This ansatz is consistent as a consequence of the group symmetry; one can also show that there are no solutions with $\phi$-dependent $k$. Let us assume that the central charge $c$ of the WZW model is greater than 25 such that $G_{00}$ can be set to $-1$ (this is achieved by either choosing a large enough group or adding a number of free fields to the model). Then the
equations for $\lambda(\phi)$ and $Q \equiv -\dot{\Phi}(\phi)$ (dots represent derivatives with respect to $\phi$) come out to be to order $\alpha'$:

$$\ddot{\lambda} + Q \dot{\lambda} = -\frac{1}{6N} \frac{\partial c(\lambda)}{\partial \lambda}, \quad (4)$$

$$Q^2 = N \dot{\lambda}^2 + \frac{1}{3} [c(\lambda) - 25], \quad (5)$$

$$\dot{Q} = -N \dot{\lambda}^2. \quad (6)$$

Here we have used the $c$–function

$$c(\lambda) = N - 3(R^{(N)} - \frac{1}{12} H^2) = N + \frac{3}{4} N c_G (-e^{-2\lambda} + \frac{1}{3} k^2 e^{-6\lambda}). \quad (7)$$

For comparison, the standard RG flow towards the IR region is determined by

$$\dot{\lambda} = \beta(\lambda) = -\frac{1}{6N} \frac{\partial c(\lambda)}{\partial \lambda}. \quad (8)$$

c(\lambda) approaches $N$ from below for $\lambda \to \infty$ ($\lambda = \infty$, $c = N$ is a trivial fixed point) and has a minimum (corresponding to the WZW model [8]) at

$$e^{2\lambda} = |k|, \quad \bar{c}_{WZW} = N - \frac{c_G N}{2 |k|} + O(\frac{1}{k^2}) = \frac{N |k|}{|k| + \frac{1}{2} c_G}.$$ 

We are of course restricted to large enough $\lambda$ so that higher order corrections in $\alpha' (\sim 1/k)$ can be neglected.

Let us briefly discuss the flow eqs. (4)–(6) in the presence of gravity (fig.2; see also [6, 7, 9]). For $Q > 0$ they describe the damped motion of a particle in the potential $c(\lambda)$. The flow again interpolates between the free theory $\lambda \to \infty$ and the WZW model. In the limit $c_{WZW} \to \infty$ of weak coupling to gravity, $Q \to \infty$ and after rescaling time the standard flow equation without gravity is recovered. If $c_{WZW} > 25$, $\lambda$ settles down – after possible oscillations – at the WZW fixed point. As $c_{WZW}$ approaches 25 from above, the friction coefficient $Q$ decreases to zero near the fixed point ($\dot{\lambda} \to 0$). If $N > 25$ but $c_{WZW} < 25$, there is no stable IR fixed point. Indeed, eq.(6) implies that $Q$ decreases until a new fixed point is reached. But the WZW fixed point with $c < 25$ cannot be reached since it would correspond to imaginary $Q$, and there is no other fixed point with $c \geq 25$. So $Q$ keeps decreasing, becomes negative, and the flow diverges due to anti–damping.
4. General models and string theory

Let us now extend these results to the general 2\textit{d} sigma model with \( N \) fields \( x^i \) and Lagrangian

\[ G_{ij}(\vec{x})\partial_\alpha x^i \partial_\beta x^j + B_{ij}(\vec{x})\partial_\alpha x^i \partial_\beta x^j \epsilon^{\alpha\beta} + \hat{R}\Phi(\vec{x}) + \ldots, \]

coupled to gravity. From the preceding, an RG trajectory in the presence of gravity can then be identified with a solution of the \( N + 1 \) dimensional bosonic string low–energy effective equations as follows. A string solution is given by the target space fields \((\mu, \nu = 0, 1, \ldots, N+1)\)

\[ G_{\mu\nu}(\vec{x}, t), \ B_{\mu\nu}(\vec{x}, t), \ \phi(\vec{x}, t) \]

At least locally, diffeomorphism symmetry and the gauge symmetry associated with the antisymmetric tensor field can be used to set

\[ B_{0i} = 0, \ G_{00} = \pm 1, \ G_{0i} = 0. \quad (9) \]

Then the string solution becomes a trajectory \( \vec{\lambda}(t) \) (with time being the parameter along the trajectory), where

\[ \vec{\lambda} = \{G_{ij}(\vec{x}), B_{ij}(\vec{x}), \ldots\} \]

parametrize the space of coupling constants of 2\textit{d} bosonic sigma models with \( N \)-dimensional euclidean target space. As shown in [9], in the vicinity of CFT’s with central charge \( c \), \( \vec{\lambda}(t) \) obeys the equation of motion

\[ \ddot{\vec{\lambda}} + Q \dot{\vec{\lambda}} = \begin{cases} \overline{-\beta} & \text{for } c > 25 \\ +\beta & \text{for } c < 25, \end{cases} \quad \text{with} \quad Q^2 = \frac{1}{3}|c - 25|, \quad (10) \]

where \( \overline{\beta} \) are the exact \( \beta \)-functions of the sigma model with \( N \)-dimensional target space. \( \overline{\beta} \) is essentially the gradient of the \( c \)-function which thus plays the role of a potential. This is the generalization of the flow equations of the last section (with RG time \( \tau = \frac{\alpha}{2} t \); if one wants \( \alpha \) to be real one is restricted to \( c \leq 1 \) or \( c \geq 25 \)). \( Q \), the time derivative of the spatially constant mode of the dilaton, again plays the role of a friction coefficient. It becomes small in the vicinity of “critical” \( (c = 25) \) string vacua. There are two disconnected sectors of string solutions, corresponding to euclidean \( (c < 25) \) or minkowskian \( (c > 25) \) target space. Solutions cannot interpolate between euclidean and minkowskian signature [9]. In particular, they cannot interpolate between CFT’s with \( c > 25 \) and CFT’s with \( c < 25 \), as has already been seen at the example of the last section.

5. Summary and outlook

Let us summarize the effects of fluctuating geometry on the flow that we have discussed. First, beta function coefficients are modified – e.g., the “velocity” of the flow in the KT transition is cut in half. Second, the flow equations become second order in derivatives. One consequence of this is that flows may oscillate around IR stable fixed points (minima of the \( c \)-function), rather than running straight into the fixed point. As the central charge \( c \) of the fixed point approaches 25 from above, the oscillations become less and less damped. If \( c \) drops below 25, they become anti–damped and the fixed point is not reached.
These results have been derived using the string equations of motion. Conversely, one may regard cosmological string solutions as RG trajectories in the presence of gravity. Starting from an UV theory with \( c \gg 25 \), which corresponds to the very early universe, the world would have “flown” to some IR fixed point with \( c = 25 \) which corresponds to a string vacuum. This proposal has to overcome some obstacles already at the level of genus zero. First, it seems to conflict with the suggestion that in 2\( d \) gravity we should throw out half of the string solutions, corresponding to the “wrong Liouville dressing” [10]. It must also be explained, e.g., why the world is not stuck in a vacuum with \( c > 25 \).

Our treatment of cosmological string solutions suggests a method to assign probabilities to different string vacua. One could think of the evolution of the universe as the trajectory of a Roulette ball rolling on a board (theory space) with holes (minima of the potential \( c \sim \) string vacua) of different sizes. Not knowing the initial conditions of the universe (initial parameters of the ball), one would of course bet on the largest hole. Whether such a “largest hole” can be identified remains to be seen.

The above discussion has been restricted to genus zero. An interesting question is how the RG flow is modified by topology fluctuations on the world-sheet. One modification, the Fischler–Susskind effect [11], is already well-known. It is also natural to expect that the flow with topology fluctuations is described by the quantum mechanical, rather than classical motion of a particle in the potential \( c \), since surfaces of higher topology correspond to loop diagrams in string theory. It would be very interesting to look for signatures of this in the matrix model results, like “tunneling” between various IR stable fixed points.

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