Existence and upper semicontinuity of random attractors for the 2D stochastic convective Brinkman–Forchheimer equations in bounded domains

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ABSTRACT

In this work, we discuss the large time behaviour of the solutions of two-dimensional stochastic convective Brinkman–Forchheimer (SCBF) equations on bounded domains. Under the functional setting $V \hookrightarrow H \hookrightarrow V'$, where $H$ and $V$ are appropriate separable Hilbert spaces and the fact that the embedding $V \hookrightarrow H$ is compact, we establish the existence of random attractors in $H$ for the stochastic flow generated by 2D SCBF equations perturbed by additive noise. We prove the upper semicontinuity of the random attractors for 2D SCBF equations in $H$, when the coefficient of random term approaches zero. Moreover, we obtain the existence of random attractors in a more regular space $V$, using the pullback flattening property. The existence of random attractors ensures the existence of invariant compact random set and hence we show the existence of an invariant measure for 2D SCBF equations. Finally, we also comment on the uniqueness of invariant measures.

1. Introduction

The study of asymptotic behaviour of dynamical systems is one of the most important areas of mathematical physics. A comprehensive investigation on the attractors for the deterministic infinite-dimensional dynamical systems has been carried out in Refs. [16,52,56], etc. However, their corresponding stochastic versions also have great importance, therefore the analysis of the infinite-dimensional random dynamical system (RDS) is also a predominant branch of stochastic partial differential equations (SPDEs). A detailed study as well as an elaborate literature is available in Ref. [2] on the generation of RDSs for stochastic ordinary differential equations and SPDEs. The existence of random attractors for a large class of SPDEs like stochastic reaction–diffusion equations, the stochastic $p$-Laplace equation and stochastic porous media equations, etc. driven by general additive noise is established in Ref. [26]. As far as the stochastic Navier–Stokes equations (SNSEs) are concerned, the notion of random attractors was introduced in Refs. [8,19,20], etc. and the authors established the existence of random attractors for 2D SNSE on bounded domains. Since the generation of RDS and random attractors for Navier–Stokes equations (NSEs) is a very
vast area of research, we restrict ourselves to those works which are relevant to the results of this paper. The existence of random attractors for several physically relevant stochastic models is proved in the works [3,4,9,10,17,37,44,45,58], etc. and the references therein.

In the functional setting \( \mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V} \), where \( \mathbb{H} \) and \( \mathbb{V} \) are appropriate separable Hilbert spaces (see Section 2 for details on the function spaces), since we do not have enough tools to find an absorbing set in a more regular space than \( \mathbb{V} \) (for the forcing \( f \in \mathbb{H} \)), we are not able to prove the existence of random attractors in \( \mathbb{V} \) using compactness arguments. To resolve this problem, the authors in Ref. [39] introduced a method to find the existence of random attractors using the pullback flattening property and this method became successful to prove the existence of random attractors for 2D SNSE as well as stochastic reaction–diffusion equations in \( \mathbb{H}^{1} \)-norm. The authors in the works [23,40,59,60], etc. obtained the existence of random attractors in \( \mathbb{H}^{1} \)-norm for different stochastic models appearing in fluid mechanics by verifying the pullback flattening property. The author in Ref. [58] proved the existence of random attractors for stochastic 3D damped NSE on bounded domains with additive noise by verifying the pullback flattening property. It appears to us that the results obtained in the work [58] may not hold true on bounded domains due to the technical difficulties discussed in the works [34,48], etc. (commutativity of the Helmholtz–Hodge projection with \(-\Delta\) and the non-zero boundary condition for the projected nonlinear damping term, see Remark 4.8).

Furthermore, in the study of random attractors, one more property of the random attractors was introduced in Ref. [15], which is the upper semicontinuity of random attractors. Roughly speaking, if \( \mathcal{A} \) is a global attractor for the deterministic system and \( \mathcal{A}_\varepsilon \) is a random attractor for the corresponding stochastic system perturbed by a small noise, we say that these attractors have the property of upper semicontinuity if \( \lim_{\varepsilon \to 0} d(\mathcal{A}_\varepsilon, \mathcal{A}) = 0 \), where \( d \) is the Hausdorff semidistance given by \( d(\mathcal{A}, \mathcal{B}) = \sup_{y \in \mathcal{A}} \inf_{z \in \mathcal{B}} \rho(y, z) \), for any \( \mathcal{A}, \mathcal{B} \subset X \), on a Polish space \((X, \rho)\). After introducing the concept of upper semicontinuity, the authors in Ref. [15] proved the upper semicontinuity of random attractors for 2D SNSE and stochastic reaction–diffusion equations. The existence and upper semicontinuity of a pullback attractor for stochastic retarded 2D NSE on a bounded domain are obtained in Ref. [33]. The author in Ref. [57] established the upper semicontinuity of random attractors for non-compact RDSs and applied this result to a stochastic reaction–diffusion equation on the whole space. The upper semicontinuity of random attractors for stochastic \( p \)-Laplacian equations on unbounded domains is obtained in Ref. [43].

The main aim of this article is to study the asymptotic behaviour of solutions of the stochastic version of the following system perturbed by additive noise. Let \( \mathcal{O} \subset \mathbb{R}^2 \) be a bounded domain with \( C^2 \)-boundary \( \partial \mathcal{O} \) and consider the following convective Brinkman–Forchheimer (CBF) equations in \( \mathcal{O} \) with homogeneous Dirichlet boundary conditions:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p &= f \text{ in } \mathcal{O} \times (0, \infty), \\
\nabla \cdot u &= 0 \text{ in } \mathcal{O} \times (0, \infty), \\
\mathbf{u} &= 0 \text{ on } \partial \mathcal{O} \times (0, \infty), \\
u(0) &= u_0 \text{ in } \mathcal{O}, \\
\int_{\mathcal{O}} p(x, t) dx &= 0 \text{ in } (0, \infty). 
\end{align*}
\]
The convective Brinkman-Forchheimer equations (1) describe the motion of incompressible fluid flows in a saturated porous medium. Here, $u(x, t) \in \mathbb{R}^2$ represents the velocity field at time $t$ and position $x$, $p(x, t) \in \mathbb{R}$ denotes the pressure field and $f(x, t) \in \mathbb{R}^2$ is an external forcing. The final condition in (1) is imposed for the uniqueness of the pressure $p$. The positive constant $\mu$ represents the Brinkman coefficient (effective viscosity), and the positive constants $\alpha$ and $\beta$ represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. The absorption exponent $r \in [1, \infty)$ and $r = 3$ is known as the critical exponent. In this work, we consider $r \in [1, 3]$ only. For $r > 3$, we have to restrict ourselves to periodic domains (see Remark 4.8). For $\alpha = \beta = 0$, we obtain the classical 2D NSE. Thus, one can consider the system (1) as damped NSEs with the linear and nonlinear damping terms $\alpha u$ and $\beta |u|^{r-1} u$, with $r > 1$, respectively.

The global solvability of the system (1) in two- and three-dimensional bounded domains is available in Refs. [1,34,47], etc. The existence of global attractors for 2D deterministic CBF equations in $\mathbb{H}$ and $V$ on unbounded Poincaré domains is proved in Refs. [46,49], respectively. For a sample literature on the attractors for two- and three-dimensional CBF equations and damped NSEs, the interested readers are referred to see Refs. [29,31,32,34,49], etc. On the stochastic counterpart, the existence of a unique pathwise strong solution to two- and three-dimensional stochastic convective Brinkman–Forchheimer (SCBF) equations driven by multiplicative noise is obtained in Ref. [48]. The authors in Ref. [37] proved the existence of random attractors in $\mathbb{H}$ for 2D and 3D SCBF equations perturbed by additive rough noise on unbounded Poincaré domains. In Ref. [35], authors considered the system (1) on 3D periodic domains and demonstrated the existence and upper semicontinuity of random attractors for $r \geq 3$ (for $r = 3$ with $2\beta \mu \geq 1$). The existence of RDSs and random attractors in $\mathbb{H}$ for a large class of locally monotone SPDEs perturbed by additive Lévy noise is obtained in Ref. [27]. In this work, we establish the existence of a random attractor in $\mathbb{H}$ for 2D SCBF equations for $r \in [1, 3]$ on bounded domains. Furthermore, we consider the stability of global attractor and prove that the random attractors for the 2D SCBF system with small additive noise will converge to the global attractor of the unperturbed 2D CBF system, when the parameter of the perturbation tends to zero. For $r \in [1, 3]$, we also obtain the random attractors in $V$ using the method introduced in Ref. [39] (by proving the pullback flattening property of the solution).

The rest of the paper is organized as follows. In the next section, we define the function spaces that are needed for the global solvability of the system (1). We introduce the linear and nonlinear operators along with their properties in the same section. Furthermore, we provide the definitions and results on RDSs and random attractors. The 2D SCBF equations are also considered in the same section and we discuss the global solvability results. The metric as well as RDSs for our model is constructed in Section 3. The existence of random attractors in $\mathbb{H}$ for 2D SCBF equations is proved in Section 4 by establishing absorbing balls in $\mathbb{H}$ and $V$, and then using an abstract result provided in Ref. [20, Theorem 3.11] (Theorems 4.5, 4.6 and 4.7). In Section 5, we establish the upper semicontinuity of the random attractors in $\mathbb{H}$ (Theorem 5.1). We also remark that such results hold true even on Poincaré domains (Remark 5.2). The existence of random attractors in $V$ for 2D SCBF equations using pullback flattening property is proved in Section 6 (Theorems 6.1 and 6.2). Note that the existence of random attractors ensures the existence of invariant compact
random set and hence we show the existence of an invariant measure for 2D SCBF equations in Section 7 (Theorem 7.3) by invoking Ref. [20, Corollary 4.4]. A comment on the uniqueness of invariant measures is also given (Remark 7.4).

2. Mathematical formulation and preliminaries

In this section, we present the necessary function spaces needed to obtain the existence and uniqueness of solutions as well as asymptotic behaviour of solutions of the system (1). In our analysis, the parameter $\alpha$ does not play a major role, therefore we set $\alpha$ to be zero in (1) for the rest of our work.

2.1. Function spaces

We denote $C_0^\infty (\mathcal{O}; \mathbb{R}^2)$ for the space of all infinitely differentiable functions ($\mathbb{R}^2$-valued) with compact support in $\mathcal{O} \subset \mathbb{R}^2$. Let us define

$$V := \{u \in C_0^\infty (\mathcal{O}; \mathbb{R}^2) : \nabla \cdot u = 0\},$$

$$H := \text{the closure of } V \text{ in the Lebesgue space } L^2(\mathcal{O}; \mathbb{R}^2),$$

$$\mathcal{V} := \text{the closure of } V \text{ in the Sobolev space } H^1_0(\mathcal{O}) = H^1(\mathcal{O}; \mathbb{R}^2),$$

$$\tilde{L}^p := \text{the closure of } V \text{ in the Lebesgue space } L^p(\mathcal{O}) = L^p(\mathcal{O}; \mathbb{R}^2)$$

for $p \in (2, \infty)$. Then, for the $C^2$-boundary $\partial \mathcal{O}$, we characterize the spaces $H$, $V$, and $\tilde{L}^p$ as $H = \{u \in L^2(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0\}$, with the norm $\|u\|_H^2 := \int_\mathcal{O} |u(x)|^2 \, dx$, where $n$ is the outward normal to $\partial \mathcal{O}$, and $u \cdot n|_{\partial \mathcal{O}}$ should be understood in the sense of trace in $H^{-1/2}(\partial \mathcal{O})$ (cf. Ref. [54, Theorem 1.2, Chapter 1]), $V = \{u \in H^1_0(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0\}$, with the norm $\|u\|_V^2 := \int_\mathcal{O} |\nabla u(x)|^2 \, dx$, and $\tilde{L}^p = \{u \in L^p(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0\}$, with the norm $\|u\|_{\tilde{L}^p}^p := \int_\mathcal{O} |u(x)|^p \, dx$, respectively. Let $(\cdot, \cdot)$ denote the inner product in the Hilbert space $H$ and $(\cdot, \cdot)$ represent the induced duality between the spaces $V$ and its dual $V'$ as well as $\tilde{L}^p$ and its dual $\tilde{L}^p'$, where $(1/p) + (1/p') = 1$. Note that $H$ can be identified with its dual $H'$. Moreover, we have the Gelfand triple $V \hookrightarrow H \cong H' \hookrightarrow V'$ with dense and continuous embedding, and the embedding $V \hookrightarrow H$ is compact.

2.2. Linear operator

Let $P_p : L^p(\mathcal{O}) \to \tilde{L}^p$ denote the Helmholtz–Hodge projection [25]. For $p = 2$, $P := P_2$ becomes an orthogonal projection and for $2 < p < \infty$, $P_p$ is a bounded linear operator. Since $\mathcal{O}$ has $C^2$-boundary, $P$ maps $H^1(\mathcal{O})$ into itself (see Ref. [54, Remark 1.6]). Let us define

$$\begin{cases}
Au := -P \Delta u, & u \in D(A), \\
D(A) := \mathcal{V} \cap H^2(\mathcal{O}).
\end{cases}$$

It should be noted that the operator $A$ is a non-negative self-adjoint operator in $H$ with $\mathcal{V} = D(A^{1/2})$ and

$$\langle Au, u \rangle = \|u\|^2_\mathcal{V} \text{ for all } u \in \mathcal{V} \text{ so that } \|Au\|_\mathcal{V} \leq \|u\|_\mathcal{V}.$$  (2)
For the bounded domain $\mathcal{O}$, the operator $A$ is invertible and its inverse $A^{-1}$ is bounded, self-adjoint and compact in $\mathbb{H}$. Thus, using spectral theorem, the spectrum of $A$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues. Moreover, there exists an orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathbb{H}$ consisting of eigenfunctions of $A$ such that $Ae_k = \lambda_k e_k$, for all $k \in \mathbb{N}$. We know that any $u \in \mathbb{H}$ can be expressed as $u = \sum_{k=1}^\infty (u, e_k)e_k$ and hence $Au = \sum_{k=1}^\infty \lambda_k (u, e_k)e_k$, for all $u \in D(A)$. Thus, we deduce that

$$\|\nabla u\|_{\mathbb{H}}^2 = \langle Au, u \rangle = \sum_{k=1}^\infty \lambda_k |(u, e_k)|^2 \geq \lambda_1 \sum_{k=1}^\infty |(u, e_k)|^2 = \lambda_1 \|u\|_{\mathbb{H}}^2 \quad (3)$$

for all $u \in \mathbb{V}$ and

$$\|A u\|_{\mathbb{H}}^2 = \langle Au, Au \rangle = \sum_{k=1}^\infty \lambda_k^2 |(u, e_k)|^2 \geq \lambda_1 \sum_{k=1}^\infty \lambda_k |(u, e_k)|^2 = \lambda_1 \|\nabla u\|_{\mathbb{H}}^2 \quad (4)$$

for all $u \in D(A)$.

### 2.3. Bilinear operator

Let us define the trilinear form $b(\cdot, \cdot, \cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by

$$b(u, v, w) = \int_\mathcal{O} (u(x) \cdot \nabla) v(x) \cdot w(x) \, dx = \sum_{i,j=1}^2 \int_\mathcal{O} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w(x) \, dx.$$  

If $u, v$ are such that the linear map $b(u, v, \cdot)$ is continuous on $\mathbb{V}$, the corresponding element of $\mathbb{V}'$ is denoted by $B(u, v)$. We also denote $B(u) = B(u, u) = \mathcal{P}[(u \cdot \nabla)u]$. An integration by parts yields

$$\begin{cases}
 b(u, v, v) = 0 & \text{for all } u, v \in \mathbb{V}, \\
 b(u, v, w) = -b(u, w, v) & \text{for all } u, v, w \in \mathbb{V}.
\end{cases} \quad (5)$$

The following well-known inequality is due to Ladyzhenskaya [42, Lemma 1, Chapter I]:

$$\|v\|_{L^4(\mathcal{O})} \leq 2^{1/4} \|v\|_{L^2(\mathcal{O})}^{1/2} \|\nabla v\|_{L^2(\mathcal{O})}^{1/2}, \quad v \in \mathbb{H}^1(\mathcal{O}). \quad (6)$$

Furthermore, an application of the Gagliardo–Nirenberg inequality [51, Theorem 2.2] yields the following generalization of (6):

$$\|v\|_{L^p(\mathcal{O})} \leq C \|v\|_{L^2(\mathcal{O})}^{2/p} \|\nabla v\|_{L^2(\mathcal{O})}^{1-(2/p)}, \quad v \in \mathbb{H}^1(\mathcal{O}), \quad (7)$$

for all $p \in [2, \infty)$. Thus, it is immediate that $\mathbb{V} \subset \tilde{L}^{r+1}$, for all $r \in [1, \infty)$. Also, for $p \in [2, 4]$, using Poincaré’s inequality, we conclude that

$$\|v\|_{L^p(\mathcal{O})} \leq C \|v\|_{L^2(\mathcal{O})}^{1-(2/p)} \|\nabla v\|_{L^2(\mathcal{O})}^{2/p}, \quad v \in \mathbb{H}^1(\mathcal{O}). \quad (8)$$

Using Ladyzhenskaya’s inequality, it is immediate that $B$ maps $\tilde{L}^4$ (and so $\mathbb{V}$) into $\mathbb{V}'$ and

$$|\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \|u\|_{\tilde{L}^4}^2 \|\nabla v\|_{\mathbb{H}} \leq \sqrt{2} \|u\|_{\mathbb{H}} \|\nabla u\|_{\mathbb{H}} \|v\|_{\mathbb{V}}$$
for all $v \in V$, so that
\[
\|B(u)\|_V \leq \sqrt{2} \|u\|_H \|\nabla u\|_H \leq \frac{\sqrt{2}}{\lambda_1^{1/4}} \|u\|^2_V \text{ for all } u \in V,
\] (9)
using the Poincaré inequality. Also, we need the following estimate on the trilinear form $b$ in the sequel (see Ref. [55, Chapter 2, Section 2.3]):
\[
|b(u, v, w)| \leq C \|u\|^{1/2}_H \|u\|^{1/2}_V \|v\|^{1/2} \|v\|^{1/2} \|w\| \text{ for all } u, v \in V, w \in H.
\] (10)

### 2.4. Nonlinear operator
Let us now consider the operator $C(u) := \mathcal{P}(|u|^{r-1}u)$. It is immediate that $\langle C(u), u \rangle = \|u\|^{r+1}_{L^{r+1}}$ and the map $C(\cdot) : L^{r+1} \rightarrow L^{(r+1)/r}$. Note that
\[
C'(u)v = \begin{cases} \mathcal{P}(v) & \text{for } r = 1, \\ \mathcal{P}(|u|^{r-1}v) + (r-1)\mathcal{P} \left( \frac{u}{|u|^{3-r}}(u \cdot v) \right) & \text{if } u \neq 0, \\ 0 & \text{if } u = 0 \\ \mathcal{P}(|u|^{r-1}v) + (r-1)\mathcal{P}(|u|^{r-3}(u \cdot v)) & \text{for } r \geq 3 \end{cases}
\]
for all $u, v \in L^{r+1}$, where $C'(\cdot)$ denotes the Gateaux derivative of $C(\cdot)$. Also, for any $r \in [1, \infty)$ and $u_1, u_2 \in V$, we have (see Ref. [48, subsection 2.4])
\[
\langle C(u_1) - C(u_2), u_1 - u_2 \rangle \geq 0.
\] (11)

### 2.5. Notations and preliminaries
In this subsection, we introduce the basic notions and preliminaries on RDSs. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ be a given filtered probability space.

**Definition 2.1:** Suppose that $X$ is a Polish space, that is, a metrizable complete separable topological space, $\mathcal{B}$ is its Borel $\sigma$-field and $\mathcal{G} := (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a metric DS. A map $\varphi : \mathbb{R}^+ \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ is called a *measurable RDS* (on $X$ over $\mathcal{G}$), if and only if

(i) $\varphi$ is $(\mathcal{B}(\mathbb{R}^+)) \otimes \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}$-measurable;
(ii) $\varphi$ is a $\theta$-cocycle, that is,
\[
\varphi(t+s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x));
\]
(iii) $\varphi(t, \omega) : X \rightarrow X$ is continuous.

The map $\varphi$ is said to be *continuous* if and only if for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$, $\varphi(t, \omega, \cdot) : X \rightarrow X$ is continuous.

Now we recall the notion of an absorbing random set from the works [8,18]. Let $\mathcal{D}$ be the class of closed and bounded random sets on $X$. 
Definition 2.2: A random set $A(\omega)$ is said to absorb another random set $B(\omega)$ if and only if for all $\omega \in \Omega$, there exists a time $t_B(\omega) \geq 0$ such that
$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset A(\omega) \text{ for all } t \geq t_B(\omega).$$
The smallest time $t_B(\omega) \geq 0$ for which the above inclusion holds is called the *absorption time* of $B(\omega)$ by $A(\omega)$.

A random set $A(\omega)$ is called $D$-absorbing if and only if $A(\omega)$ absorbs every $D(\omega) \in D$.

Definition 2.3: A random set $A(\omega)$ is a random $D$-attractor if and only if

(i) $A$ is a compact random set,
(ii) $A$ is $\varphi$-invariant, that is, $P$-a.s.,
$$\varphi(t, \omega)A(\omega) = A(\theta_t\omega),$$
(iii) $A$ is $D$-attracting, in the sense that, for all $D(\omega) \in D$ it holds that
$$\lim_{{t \to \infty}} d(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) = 0,$$
where $d$ is the Hausdorff semidistance.

Definition 2.4 ([2, Remark 1.1.8]): Given an RDS $\varphi$. Then the mapping
$$(\omega, x) \mapsto (\theta_t(\omega), \varphi(t, \omega)x) =: \Theta_t(\omega, x), \quad t \in \mathbb{R}^+,$$
is a measurable DS on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ which is called the skew product of metric DS $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ and the cocycle $\varphi(t, \omega)$ on $X$. Conversely, every such measurable skew product DS $\Theta$ defines a cocycle $\varphi$ on its $x$ component, thus a measurable RDS.

Definition 2.5 ([12]): Let $\varphi$ be a given RDS over a metric DS $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. A probability measure $\eta$ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is called an invariant measure for $\varphi$ if and only if

(i) $\Theta_t$ preserves $\eta$ (that is, $\Theta_t(\eta) = \eta$) for each $t \in \mathbb{R}^+$;
(ii) the first marginal of $\eta$ is $P$, that is, $\pi_\Omega(\eta) = P$, where $\pi_\Omega : \Omega \times X \ni (\omega, x) \mapsto \omega \in \Omega$.

Definition 2.6 ([39]): A RDS $\theta$-cocycle $\varphi$ on a Banach space $X$ is said to be *pullback flattening* if for every random $\mathcal{D}$-bounded set $B = \{B(\omega), \omega \in \Omega\}$ in $X$, for $\delta > 0$ and $\omega \in \Omega$, there exists a $T_0(B, \delta, \omega) > 0$ and a finite-dimensional subspace $X_\delta$ of $X$ such that

(i) $\bigcup_{{t \geq T_0}} P_\delta \varphi(t, \theta_{-t}, B(\theta_{-t}\omega))$ is bounded and
(ii) $\| (I - P_\delta)(\bigcup_{{t \geq T_0}} \varphi(t, \theta_{-t}, B(\theta_{-t}\omega))) \|_X < \delta$,

where $P_\delta : X \to X_\delta$ is a bounded projection and (ii) is understood in the sense that $\| (I - P_\delta)\varphi(t, \theta_{-t}\omega, x_0) \|_X < \delta$, for all $x_0 \in B(\theta_{-t}\omega)$ and $t \geq T_0$.

Theorem 2.7 ([39]): Suppose that an RDS $\theta$-cocycle $\varphi$ is pullback flattening and has a random bounded absorbing set. Then it has a unique random attractor.
2.6. SCBF equations

In this subsection, we provide the abstract formulation of CBF equations (1) and discuss its stochastic counterpart. We take the external forcing $f$ appearing in (1) independent of time.

2.6.1. Abstract formulation

On taking orthogonal projection $\mathcal{P}$ onto the first equation in (1), we obtain

$$\begin{cases} \frac{du}{dt} + \mu A u + B(u) + \beta C(u) = f, & t \geq 0, \\ u(0) = u_0, \end{cases} \tag{12}$$

where $u_0 \in \mathbb{H}$ and $f \in \mathbb{V}$. A random perturbation of the abstract deterministic 2D CBF equations (12) is given by

$$\begin{cases} \frac{du_{\varepsilon}}{dt} + [\mu A u_{\varepsilon} + B(u_{\varepsilon}) + \beta C(u_{\varepsilon})]dt = f dt + \varepsilon dW(t), & t \geq 0, \\ u_{\varepsilon}(0) = u_0 \end{cases} \tag{13}$$

for $r \geq 1$ and $\varepsilon \in (0, 1]$, where we assume that $u_0 \in \mathbb{H}$, $f \in \mathbb{H}$ and $W(t)$, $t \in \mathbb{R}$, is a two-sided $\mathbb{H}$-valued Wiener process with its reproducing kernel Hilbert space (RKHS) $\mathbb{K}$ (a cylindrical Wiener process on $\mathbb{K}$, cf. Ref. [13]) defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$. Remember that RKHS of a centred Gaussian measure $\nu$ on a separable Banach space $X$ is a unique Hilbert space $(\mathbb{K}, \| \cdot \|_\mathbb{K})$ such that $\mathbb{K} \hookrightarrow X$ continuously and for each $\Psi \in X^*$, the random variable $\Psi$ on the probability space $(X, \nu)$ is Gaussian with mean 0 and variance $\|\Psi\|_\mathbb{K}^2$ [21].

In this paper, we assume that RKHS $\mathbb{K}$ satisfies the following assumption:

**Assumption 2.8:** $\mathbb{K} \subset \mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$ is a Hilbert space such that for some $\delta \in (0, 1/2)$,

$$A^{-\delta} : \mathbb{K} \rightarrow \mathbb{V} \cap \mathbb{H}^2(\mathcal{O}) \text{ is Hilbert – Schmidt.} \tag{14}$$

**Remark 2.9:** Since $D(A) = \mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$, Assumption 2.8 can be reformulated in the following way also (see Ref. [11]). $\mathbb{K}$ is a Hilbert space such that $\mathbb{K} \subset D(A)$ and for some $\delta \in (0, 1/2)$, the map

$$A^{-\delta - 1} : \mathbb{K} \rightarrow \mathbb{H} \text{ is Hilbert-Schmidt.} \tag{15}$$

Since $\mathcal{O}$ is a bounded domain, then $A^{-s} : \mathbb{H} \rightarrow \mathbb{H}$ is Hilbert-Schmidt if and only if $\sum_{j=1}^{\infty} \lambda_j^{-2s} < \infty$, where $A e_j = \lambda_j e_j$, $j \in \mathbb{N}$ and $e_j$ is an orthogonal basis of $\mathbb{H}$. On bounded domains, we know that $\lambda_j \sim j$ (cf. Ref. [24, p. 54]) and hence $A^{-s}$ is Hilbert-Schmidt if and only if $s > 1/2$. In other words, with $\mathbb{K} = D(A^{s+1})$, the embedding $\mathbb{K} \hookrightarrow \mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$ is Hilbert-Schmidt if and only if $s > 1/2$. Thus, Assumption 2.8 is satisfied for any $\delta > 0$. In fact, the condition (14) holds if and only if the operator $A^{-(s+1+\delta)} : \mathbb{H} \rightarrow \mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$ is Hilbert-Schmidt. The requirement of $\delta < 1/2$ in Assumption 2.8 is necessary because we need (see Subsection 3.2) the corresponding Ornstein–Uhlenbeck process to take values in $\mathbb{V} \cap \mathbb{H}^2(\mathcal{O})$. 
3. RDS generated by 2D SCBF equations

In this section, we construct the metric dynamical system and the RDS for the model (13).

3.1. Wiener process

Let us denote $X = V \cap H^2(\mathcal{O})$ and let $E$ denote the completion of $A^{-\delta}X$ with respect to the image norm $\|x\|_E = \|A^{-\delta}x\|_X$, for $x \in X$, where $\| \cdot \|_X = \| \cdot \|_V + \| \cdot \|_{H^2}$. Note that $E$ is a separable Banach space (see Ref. [5]).

For $\xi \in (0, 1/2)$, we set

$$\|\omega\|_{C^{\xi}_0(R, E)} = \sup_{t \neq s \in \mathbb{R}} \frac{\|\omega(t) - \omega(s)\|_E}{|t - s|^\xi (1 + |t| + |s|)^{1/2}}.$$

We also define

$$C^{\xi}_1(R, E) := \{ \omega \in C(R, E) : \omega(0) = 0, \|\omega\|_{C^{\xi}_1(R, E)} < \infty \},$$

$$\Omega(\xi, E) := \{ \omega \in C^\infty_0(R, E) : \omega(0) = 0 \} C^{\xi}_1(R, E).$$

The space $\Omega(\xi, E)$ is a separable Banach space. Let us denote by $\mathcal{F}$, the Borel $\sigma$-algebra on $\Omega(\xi, E)$. For $\xi \in (0, 1/2)$, there exists a Borel probability measure $\mathbb{P}$ on $\Omega(\xi, E)$ (see Ref. [6]) such that the canonical process $w_t, t \in \mathbb{R}$, defined by

$$w_t(\omega) := \omega(t), \quad \omega \in \Omega(\xi, E), \quad (16)$$

is an $E$-valued two-sided Wiener process.

For $t \in \mathbb{R}$, let $\mathcal{F}_t := \sigma\{w_s : s \leq t\}$. Then there exists a bounded linear map $W_t : K \rightarrow L^2(\Omega(\xi, E), \mathcal{F}_t, \mathbb{P})$ (cf. Ref. [12, Subsection 6.1]). Moreover, the family $\{W_t\}_{t \in \mathbb{R}}$ is a cylindrical Wiener process on a filtered probability space $(\Omega(\xi, E), \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ (see also Ref. [13]).

On the space $\Omega(\xi, E)$, we consider a flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega(\xi, E), \quad t \in \mathbb{R}.$$

This flow leaves the space invariant. It is obvious that for each $t \in \mathbb{R}$, $\theta_t$ preserves $\mathbb{P}$.

3.2. Ornstein–Uhlenbeck process

In this subsection, we define an Ornstein–Uhlenbeck process under Assumption 2.8 (for more details, see Ref. [37, Section 3]). For $\delta$ as in Assumption 2.8, $\mu, \beta > 0$, $\alpha \geq 0$, $\xi \in (\delta, 1/2)$ and $\omega \in C^{\xi}_1(R, E)$, we define

$$z_\omega(\omega)(t) := \int_{-\infty}^t (\mu A + \alpha I)^{1+\delta} e^{-(t-\tau)(\mu A + \alpha I)} [(\mu A + \alpha I)^{-\delta} \omega(t) - (\mu A + \alpha I)^{-\delta} \omega(\tau)] d\tau$$

(17)
for any \( t \geq 0 \). Note that (17) is well defined since \( \omega \in C_{1/2}^\xi (\mathbb{R}, E) \). Hence, if \( \omega \in C^\infty (\mathbb{R}, E) \) with \( \omega (0) = 0 \), then \( z_{\alpha} (t) \) is the solution of the following equation:

\[
\frac{dz_{\alpha} (t)}{dt} + (\mu A + \alpha I) z_{\alpha} = \frac{d\omega (t)}{dt}, \quad t \in \mathbb{R}.
\]  

(18)

Analogous to our definition (16) of the Wiener process \( w(t) \), \( t \in \mathbb{R} \), we can view the formula (17) as a definition of a process \( z_{\alpha} (t) \), \( t \in \mathbb{R} \), on the probability space \((\Omega, \xi, \mathcal{F}, \mathbb{P})\). Equation (18) suggests that this process is an Ornstein–Uhlenbeck process. In fact we have the following result:

**Proposition 3.1 ([12, Proposition 6.10]):** The process \( z_{\alpha} (t) \), \( t \in \mathbb{R} \), is stationary Ornstein–Uhlenbeck process on \((\Omega, \xi, \mathcal{F}, \mathbb{P})\). It is a solution of the equation

\[
\frac{dz_{\alpha} (t)}{dt} + (\mu A + \alpha I) z_{\alpha} = dW(t), \quad t \in \mathbb{R},
\]  

(19)

that is, for all \( t \in \mathbb{R} \),

\[
z_{\alpha} (t) = \int_{-\infty}^{t} e^{-(t-s)(\mu A + \alpha I)} dW(s),
\]  

(20)

\( \mathbb{P} \)-a.s., where the integral is an Itô integral on the M-type 2 Banach space \( X \) in the sense of Ref. [7]. In particular, for some \( C \) depending on \( X \),

\[
\mathbb{E} \left[ \| z_{\alpha} (t) \|^2_X \right] = \mathbb{E} \left[ \int_{-\infty}^{t} e^{-(t-s)(\mu A + \alpha I)} \| dW(s) \|^2_X \right] \leq C \int_{-\infty}^{t} e^{-(t-s)(\mu A + \alpha I)} \| e^{-sA} \|^2_{\gamma (K, X)} ds
\]

\[
= C \int_{0}^{\infty} e^{-2\alpha s} \| e^{-\mu s A} \|^2_{\gamma (K, X)} ds.
\]  

(21)

Moreover, \( \mathbb{E} [\| z_{\alpha} (t) \|^2_X] \) tends to 0 as \( \alpha \to \infty \).

**Remark 3.2:** By Ref. [37, Proposition 3.1], we write the following result for the Ornstein–Uhlenbeck process given in Proposition 3.1:

\[
z_{\alpha} (\theta t \omega) (s) = z_{\alpha} (\omega) (t + s), \quad t, s \in \mathbb{R},
\]

and

\[
z_{\alpha} \in L^q (a, b; X),
\]  

(22)

where \( q \in [1, \infty] \).

Since by Proposition 3.1, the process \( z_{\alpha} (t) \), \( t \in \mathbb{R} \), is \( X \)-valued stationary and ergodic. Hence, by the strong law of large numbers, we have (see Ref. [22] for a similar argument)

\[
\lim_{t \to \infty} \frac{1}{t} \int_{-t}^{0} \| z_{\alpha} (s) \|^2_X ds = \mathbb{E} \left[ \| z_{\alpha} (0) \|^2_X \right], \quad \mathbb{P} \text{-a.s., on } C_{1/2}^\xi (\mathbb{R}, X).
\]  

(23)

It follows from Proposition 3.1 that we can find \( \alpha_0 \) such that for all \( \alpha \geq \alpha_0 \),

\[
\mathbb{E} \left[ \| z_{\alpha} (0) \|^2_X \right] \leq \frac{\mu^2 \lambda_1}{16},
\]  

(24)

where \( \lambda_1 \) is the constant appearing in the Poincaré inequality (3).
By $\Omega_\alpha(\xi, E)$, we denote the set of those $\omega \in \Omega(\xi, E)$ for which the equality (23) holds true. Therefore, we fix $\xi \in (\delta, 1/2)$ and set

$$\Omega := \hat{\Omega}(\xi, E) = \bigcap_{n=0}^{\infty} \Omega_n(\xi, E).$$

For reasons that will become clear later, we take as a model of a metric DS the quadruple $(\Omega, \hat{\mathcal{F}}, \hat{\mathcal{P}}, \hat{\theta})$, where $\hat{\mathcal{F}}, \hat{\mathcal{P}}, \hat{\theta}$ are, respectively, the natural restrictions of $\mathcal{F}, \mathcal{P}$ and $\theta$ to $\Omega$.

**Proposition 3.3:** The quadruple $(\Omega, \hat{\mathcal{F}}, \hat{\mathcal{P}}, \hat{\theta})$ is a metric DS. For each $\omega \in \Omega$, the limit in (23) exists.

Let us now formulate an immediate and important consequence of the above result.

**Corollary 3.4:** For each $\omega \in \Omega$, there exists $t_0 = t_0(\omega) \geq 0$ such that

$$\frac{8}{\mu} \int_{-t}^{0} \|z_\alpha(s)\|^2_X ds \leq \frac{\mu \lambda_1 t}{2}, \quad t \geq t_0.$$  
(25)

Also, since the embedding $X \hookrightarrow V$ is a contraction, we have

$$\frac{8}{\mu} \int_{-t}^{0} \|z_\alpha(s)\|^2_V ds \leq \frac{\mu \lambda_1 t}{2}, \quad t \geq t_0.$$  

**3.3. RDS**

Under Assumption 2.8 and $\delta \in (0, 1/2)$, we take a fixed $\mu, \beta > 0$ and some parameter $\alpha \geq 0$. We also fix $\xi \in (\delta, 1/2)$.

Denote by $\nu_\varepsilon(t) = u_\varepsilon(t) - \varepsilon z_\alpha(t)$. For convenience, we write $\nu_\varepsilon(t) = \nu_\varepsilon(t)$ and $z_\alpha(\omega)(t) = z(t)$. Then $\nu_\varepsilon(t)$ satisfies the following abstract RDS:

$$\begin{align*}
\frac{d\nu_\varepsilon}{dt} &= -\mu A\nu_\varepsilon - B(\nu_\varepsilon + \varepsilon z) - \beta C(\nu_\varepsilon + \varepsilon z) + \varepsilon \alpha z + f, \\
\nu_\varepsilon|_{t=0} &= u_0 - \varepsilon z(0).
\end{align*}$$  
(26)

Because $z_\alpha \in C_{1/2}(\mathbb{R}, X)$, $z_\alpha(0)$ is a well-defined element of $V$. In what follows, we provide the definition of weak as well as strong solutions (in the deterministic sense) for the system (26).

**Definition 3.5:** Assume that $r \in [1, 3]$, $u_0 \in H$ and $f \in V'$. Let $T > 0$ be any fixed time, a function $\nu_\varepsilon(\cdot, 0; \omega, u_0 - \varepsilon z(0))$ is called a weak solution of the problem (26) on time interval $[0, T]$, if

$$\nu_\varepsilon \in C([0, T]; H) \cap L^2(0, T; V),$$

$$\frac{d\nu_\varepsilon}{dt} \in L^2([0, T]; V')$$

for $r \in [1, 3)$ and

$$\frac{d\nu_\varepsilon}{dt} \in L^2([0, T]; \tilde{\mathcal{L}}^{4/3})$$

for $r = 3$. 


and it satisfies

(i) for any $\psi \in \mathbb{V}$,

$$
\left\langle \frac{dv_e}{dt}, \psi \right\rangle = -\langle \mu A v_e + B(v_e + \varepsilon z) + \beta C(v_e + \varepsilon z), \psi \rangle + \langle f, \psi \rangle + \langle \varepsilon \alpha z, \psi \rangle
$$

(27)

for all $t \in [0, T]$.

(ii) $v_e(\cdot, 0; \omega, u_0 - \varepsilon z(0))$ satisfies the following initial data:

$$
v_e|_{t=0} = u_0 - \varepsilon z_0(0).
$$

**Definition 3.6:** Assume that $r \in [1, 3], u_0 \in \mathbb{V}$ and $f \in \mathbb{H}$. Let $T > 0$ be any fixed time, a function $v_e(\cdot, 0; z, u_0 - \varepsilon z(0))$ is called a strong solution of the problem (26) on time interval $[0, T]$, if

$$
v_e \in C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)), \quad \frac{dv_e}{dt} \in L^2(0, T; \mathbb{H})
$$

and it satisfies (26) as an equality in $\mathbb{H}$ for a.e. $t \in [0, T]$.

The system (26) is a pathwise deterministic system. By a standard Galerkin method (see Refs. [47,48]), one can obtain that if $z \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A))$, then for all $t > 0$, and for every $u_0 \in \mathbb{H}, f \in \mathbb{V}$ and $\omega \in \Omega$, (26) has a unique solution in the sense of Definition 3.5. In addition, for $u_0 \in \mathbb{V}$ and $f \in \mathbb{H}$, (26) has a unique solution in the sense of Definition 3.6. For the next two results, we take $f$ is dependent on $t$.

**Theorem 3.7:** Assume that, for some $T > 0$ fixed, $u_0^n \rightarrow u_0$ in $\mathbb{H}$,

$$
z_n \rightarrow z \text{ in } L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A)), \quad f_n \rightarrow f \text{ in } L^2(0, T; \mathbb{V}).
$$

Let us denote by $v_e(t, 0; \omega, u_0 - \varepsilon z(0))$, the solution of the problem (26) and by $v_e(t, 0; \omega, u_0^n - \varepsilon z_n(0))$, the solution of the problem (26) with $z, f, u_0$ being replaced by $z_n, f_n, u_0^n$, respectively. Then,

$$
v_e(\cdot, 0; \omega, u_0^n - \varepsilon z_n(0)) \rightarrow v_e(\cdot, 0; z, u_0 - \varepsilon z(0)) \text{ in } C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}).
$$

In particular, $v_e(T, 0; \omega, u_0^n - \varepsilon z_n(0)) \rightarrow v_e(T, 0; \omega, u_0 - \varepsilon z(0))$ in $\mathbb{H}$.

**Proof:** See the proof of Ref. [37, Theorem 3.9].

**Theorem 3.8:** Assume that, for some $T > 0$ fixed, $u_0^n \rightarrow u_0$ in $\mathbb{V}$,

$$
z_n \rightarrow z \text{ in } L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A)), \quad f_n \rightarrow f \text{ in } L^2(0, T; \mathbb{H}).
$$

Let us denote by $v_e(t, 0; z, u_0 - \varepsilon z(0))$, the solution of the problem (26) and by $v_e(t, 0; \omega, u_0^n - \varepsilon z_n(0))$, the solution of the problem (26) with $z, f, u_0$ being replaced by $z_n, f_n, u_0^n$, respectively. Then

$$
v_e(\cdot, 0; \omega, u_0^n - \varepsilon z_n(0)) \rightarrow v_e(\cdot, 0; \omega, u_0 - \varepsilon z(0)) \text{ in } C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)).
$$

In particular, $v_e(T, 0; \omega, u_0^n - \varepsilon z_n(0)) \rightarrow v_e(T, 0; \omega, u_0 - \varepsilon z(0))$ in $\mathbb{V}$. 


Proof: In order to simplify the proof, we introduce the following notations:

\[ v_n(t) = v_\varepsilon(t, 0; \omega, u_0^n - \varepsilon z_n(0)), \quad v(t) = v_\varepsilon(t, 0; \omega, u_0 - \varepsilon z(0)), \quad y_n(t) = v_n(t) - v(t), \]

and

\[ \hat{z}_n(t) = z_n(t) - z(t), \quad \hat{f}_n(t) = f_n(t) - f(t), \quad t \in [0, T]. \]

It is easy to see that \( y_n(\cdot) \) solves the following initial value problem:

\[
\begin{align*}
\frac{dy_n(t)}{dt} &= -\mu Ay_n(t) - B(v_n(t) + \varepsilon z_n(t)) + B(v(t) + \varepsilon z(t)) - \beta C(v_n(t) + \varepsilon z_n(t)) \\
&\quad + \beta C(v(t) + \varepsilon z(t)) + \varepsilon \alpha \hat{z}_n(t) + \hat{f}(t), \\
y_n(0) &= u_0^n - u_0.
\end{align*}
\]

(28)

Taking the inner product with \( Ay_n(\cdot) \) to the first equation in (28), and then using (2) and (5), we get

\[
\frac{1}{2} \frac{d}{dt} \| y_n(t) \|_V^2
\]

\[
= -\mu \| Ay_n(t) \|_H^2 - b(y_n(t), v_n(t) + \varepsilon z_n(t), Ay_n(t)) - \varepsilon b(\hat{z}_n(t), v_n(t) \\
+ \varepsilon z_n(t), Ay_n(t)) - b(v(t) + \varepsilon z(t), y_n(t), Ay_n(t)) - \varepsilon b(v(t) + \varepsilon z(t), \hat{z}_n(t), Ay_n(t)) \\
- \beta (C(v_n(t) + \varepsilon z_n(t)) - C(v(t) + \varepsilon z(t)), Ay_n(t)) \\
+ \varepsilon \alpha (\hat{z}_n(t), Ay_n(t)) + \hat{f}(t), Ay_n(t))
\]

(29)

for a.e. \( t \in [0, T] \). Now, using (3), (10), \( 0 < \varepsilon \leq 1 \) and Young’s inequality, we have

\[
|b(y_n, v_n + \varepsilon z_n, Ay_n)| \leq C \| y_n \|_V \| v_n + \varepsilon z_n \|_V^{1/2} \| Ay_n + \varepsilon Az_n \|_H^{1/2} \| Ay_n \|_H \\
\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| y_n \|_V^2 \| v_n + \varepsilon z_n \|_V \| Ay_n + \varepsilon Az_n \|_H, \\
\]

(30)

\[
|\varepsilon b(\hat{z}_n, v_n + \varepsilon z_n, Ay_n)| \leq |b(\hat{z}_n, v_n + \varepsilon z_n, Ay_n)| \\
\leq C \| \hat{z}_n \|_V \| v_n + \varepsilon z_n \|_V^{1/2} \| Ay_n + \varepsilon Az_n \|_H^{1/2} \| Ay_n \|_H \\
\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| \hat{z}_n \|_V^2 \| v_n + \varepsilon z_n \|_V \| Ay_n + \varepsilon Az_n \|_H, \\
\]

(31)

\[
|b(v + \varepsilon z, y_n, Ay_n)| \leq C \| v + \varepsilon z \|_V \| y_n \|_V^{1/2} \| Ay_n \|_H^{3/2} \\
\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| v + \varepsilon z \|_V^2 \| y_n \|_V^2, \\
\]

(32)

\[
|\varepsilon b(v + \varepsilon z, \hat{z}_n, Ay_n)| \leq |b(v + \varepsilon z, \hat{z}_n, Ay_n)| \leq C \| v + \varepsilon z \|_V \| \hat{z}_n \|_V^{1/2} \| A\hat{z}_n \|_H^{1/2} \| Ay_n \|_H \\
\leq \frac{\mu}{14} \| Ay_n \|_H^2 + C \| v + \varepsilon z \|_V^2 \| \hat{z}_n \|_V \| A\hat{z}_n \|_H. \\
\]

(33)

Now by using Taylor’s formula, Hölder’s inequality, Sobolev’s embedding and Young’s inequality, we find

\[
\beta |(C(v_n + \varepsilon z_n) - C(v + \varepsilon z), Ay_n)|
\]
\[
\begin{aligned}
\text{Using } 0 < \varepsilon \leq 1, \text{ Hölder’s and Young’s inequalities, we estimate}

\varepsilon \alpha |\hat{z}_n, Ay_n| & \leq \alpha |Ay_n||\hat{z}_n| \leq \frac{\mu}{14} |Ay_n| + C||\hat{z}_n|, \\
(\hat{f}_n, Ay_n) & \leq |Ay_n| \hat{f}_n \leq \frac{\mu}{14} |Ay_n| + C|\hat{f}_n|.
\end{aligned}
\]
Then by an application of Gronwall’s inequality, we obtain
\[
\|y_n(t)\|_V^2 \leq \left( \|y_n(0)\|_V^2 + C \int_0^T \beta_n(s) ds \right) e^{C \int_0^T \alpha_n(s) ds}
\]
for all \( t \in [0, T] \). On the other hand, we find
\[
\int_0^T \beta_n(s) ds \\
\leq T^{1/2} \|v_n + \varepsilon z_n\|_{L^2(0,T;V)} \|v_n + \varepsilon z_n\|_{L^2(0,T;D(A))} \|\hat{z}_n\|_{L^2(0,T;V)}^2 \\
+ T \left[ \|v + \varepsilon z\|_{L^2(0,T;V)} + \|v_n + \varepsilon z_n\|_{L^2(0,T;V)} \right] \|\hat{z}_n\|_{L^2(0,T;V)}^2 + \|\hat{f}_n\|_{L^2(0,T;H)} \\
+ T^{1/2} \|v + \varepsilon z\|_{L^2(0,T;V)}^2 \|\hat{z}_n\|_{L^2(0,T;V)}^2 \|\hat{z}_n\|_{L^2(0,T;D(A))}.
\]
Hence \( \int_0^T \beta_n(s) ds \to 0 \) as \( n \to \infty \). Moreover, we have
\[
\int_0^T \alpha_n(s) ds = T^{1/2} \|v_n + \varepsilon z_n\|_{L^2(0,T;V)} \|v_n + \varepsilon z_n\|_{L^2(0,T;D(A))} + T \|v_n + \varepsilon z_n\|_{L^2(0,T;V)}^2 \\
+ T \|v + \varepsilon z\|_{L^2(0,T;V)}^2 + \|v + \varepsilon z\|_{L^2(0,T;V)}^4 < \infty.
\]
Since, \( \|y_n(0)\|_V = \|u_0^n - u_0\|_V \to 0 \) and \( \int_0^T \beta_n(s) ds \to 0 \) as \( n \to \infty \), and for all \( n \in \mathbb{N}, \int_0^T \alpha_n(s) ds < \infty \), then (38) asserts that \( \|y_n(t)\|_V \to 0 \) as \( n \to \infty \) uniformly for \( t \in [0, T] \). Since \( v_n(\cdot) \) and \( v(\cdot) \) are continuous in \( V \), we also have
\[
v_\varepsilon(\cdot, 0; \omega, u_0^n - \varepsilon z_0(0)) \to v_\varepsilon(\cdot, 0; \omega, u_0 - \varepsilon z(0)) \text{ in } C([0, T]; V).
\]
From (37), we also infer
\[
\mu \int_0^T \|Ay_n(t)\|_{D(\Omega)}^2 \leq \|y_n(0)\|_V^2 + C \sup_{s \in [0, T]} \|y_n(s)\|_V^2 \int_0^T \alpha_n(s) ds + C \int_0^T \beta_n(s) ds \to 0,
\]
as \( n \to \infty \) and therefore
\[
v_\varepsilon(\cdot, 0; \omega, u_0^n - \varepsilon z_n(0)) \to v_\varepsilon(\cdot, 0; \omega, u_0 - \varepsilon z(0)) \text{ in } L^2(0, T; D(A)),
\]
which completes the proof.

**Definition 3.9:** We define a map \( \varphi^\varepsilon : \mathbb{R}^+ \times \Omega \times V \to V \) by
\[
(t, \omega, u_0) \mapsto v_\varepsilon^\alpha(t) + \varepsilon z_\alpha(t) \in V,
\]
where \( v_\varepsilon^\alpha(t) = v_\varepsilon(t, 0; \omega, u_0 - \varepsilon z_\alpha(0)) \) is a solution to the problem (26) with the initial condition \( u_0 - \varepsilon z_\alpha(0) \).

**Proposition 3.10** ([37, Proposition 3.11]): If \( \alpha_1, \alpha_2 \geq 0 \), then \( \varphi^\varepsilon_{\alpha_1} = \varphi^\varepsilon_{\alpha_2} \).

Proposition 3.10 says that the map \( \varphi^\varepsilon \) does not depend on \( \alpha \) and hence from now on, we will denote it by \( \varphi_\varepsilon \).
Theorem 3.11: \((\varphi_{\epsilon}, \theta)\) is an RDS.

**Proof:** All the properties with the exception of the cocycle one of an RDS follow from Theorem 3.8. Hence we only need to show that for any \(u_0 \in \mathbb{V}\),

\[
\varphi_{\epsilon}(t + s, \omega)u_0 = \varphi_{\epsilon}(t, \theta_s \omega)\varphi_{\epsilon}(s, \omega)u_0, \quad t, s \in \mathbb{R}^+.
\] (40)

Remaining proof of this theorem is similar to that of Ref. [12, Theorem 6.15] and hence we omit it here. ■

Let us now define, for \(u_{\epsilon}(s) \in \mathbb{V}\), \(\omega \in \Omega\), and \(t \geq s\),

\[
u_{\epsilon}(t; s, u_{\epsilon}(s)) := \varphi_{\epsilon}(t - s; \theta_s \omega)u_{\epsilon}(s) = v_{\epsilon}(t, s; \omega, u_{\epsilon}(s) - \epsilon z(s)) + z(t),
\] (41)

then for each \(s \in \mathbb{R}\) and each \(u_{\epsilon}(s) \in \mathbb{H}\) or \(u_{\epsilon}(s) \in \mathbb{V}\), the process \(u_{\epsilon}(t), \ t \geq s\), is a solution to the problem (13).

**4. Random attractors for 2D SCBF equations in \(\mathbb{H}\)**

The existence of random attractors in \(\mathbb{H}\) for 2D SCBF equations is established in this section. We consider the RDS \(\varphi_{\epsilon}\) over the metric DS \((\Omega, \hat{\mathcal{F}}, \hat{P}, \hat{\theta})\).

**Lemma 4.1:** For each \(\omega \in \Omega\),

\[
\lim_{t \to -\infty} \sup \|z(\omega)(t)\|_H^2 e^{\mu \lambda_1 t + (8/\mu) \int_0^t \|z(\xi)\|_V^2 d\xi} = 0.
\]

**Proof:** Let us fix \(\omega \in \Omega\). By Corollary 3.4, we can find \(t_0 \leq 0\) such that for \(t \leq t_0\),

\[
\frac{8}{\mu} \int_t^0 \|z(s)\|_V^2 ds \leq -\frac{\mu \lambda_1 t}{2}, \quad t \leq t_0.
\] (42)

Since \(A\) is the generator of an analytic semigroup on \(\mathbb{V}\), one can apply [9, Proposition 2.11] and find \(\rho_1 = \rho_1(\omega) \geq 0\) such that

\[
\frac{\|z(t)\|_V}{|t|} \leq \rho_1 \text{ for } t \leq t_0.
\] (43)

Therefore, we have, for every \(\omega \in \Omega\),

\[
\lim_{t \to -\infty} \sup \|z(\omega)(t)\|_H^2 e^{\mu \lambda_1 t + (8/\mu) \int_0^t \|z(\xi)\|_V^2 d\xi}
\]

\[
\leq \lim_{t \to -\infty} \sup_{\lambda_1} \frac{1}{\lambda_1} \|z(\omega)(t)\|_V^2 e^{\mu \lambda_1 t + (8/\mu) \int_0^t \|z(\xi)\|_V^2 d\xi}
\]

\[
\leq \frac{\rho_1^2}{\lambda_1} \lim_{t \to -\infty} \sup |t|^2 e^{\mu \lambda_1 t/2} = 0,
\]

which completes the proof. ■
Lemma 4.2: For each $\omega \in \Omega$,

$$\int_{-\infty}^{0} \left\{ 1 + \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 \right\} e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\zeta)\|_{V}^2 d\zeta} dt < \infty.$$ 

Proof: Since for $t_0 \leq 0$,

$$\int_{-\infty}^{0} \left\{ 1 + \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 \right\} e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\zeta)\|_{V}^2 d\zeta} dt < \infty,$$

therefore, it is sufficient to prove that the integral

$$\int_{-\infty}^{t_0} \left\{ 1 + \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 \right\} e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\zeta)\|_{V}^2 d\zeta} dt < \infty.$$ 

Due to (42), we obtain

$$\int_{-\infty}^{t_0} e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\zeta)\|_{V}^2 d\zeta} dt \leq \int_{-\infty}^{t_0} e^{\mu \lambda_1 t/2} dt < \infty.$$ 

Using (42) and (43), we deduce that

$$\int_{-\infty}^{t_0} \left\{ \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 \right\} e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\zeta)\|_{V}^2 d\zeta} dt$$

$$\leq \int_{-\infty}^{t_0} \left\{ \rho_1^2 |t|^2 + \rho_1^{r+1} |t|^{r+1} + \rho_1^4 |t|^4 \right\} e^{\mu \lambda_1 t/2} dt < \infty,$$

which completes the proof. \hfill \blacksquare

Definition 4.3: A function $\kappa : \Omega \to (0, \infty)$ belongs to class $\mathcal{K}$ if and only if

$$\lim_{t \to \infty} \sup_{t \in [0, \infty)} [\kappa(\theta_{-t}\omega)]^2 e^{-\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\omega(s))\|_{V}^2 ds} = 0,$$ 

(44)

where $\lambda_1$ is the first eigenvalue of the Stokes operator $A$.

We denote by $\mathcal{D}\mathcal{R}$, the class of all closed and bounded random sets $D$ on $\mathbb{H}$ such that the radius function $\Omega \ni \omega \mapsto \kappa(D(\omega)) := \sup\{\|x\|_{\mathbb{H}} : x \in D(\omega)\}$ belongs to the class $\mathcal{R}$.

By Corollary 3.4, we infer that the constant functions belong to $\mathcal{R}$. The class $\mathcal{R}$ is closed with respect to sum, multiplication by a constant and if $\kappa \in \mathcal{R}, 0 \leq \bar{\kappa} \leq \kappa$, then $\bar{\kappa} \in \mathcal{R}$.

Proposition 4.4: Define the functions $\kappa_i : \Omega \to (0, \infty), \ i = 1, 2, 3, 4, 5, 6$, by the following formulae, for $\omega \in \Omega$,

$$[\kappa_1(\omega)]^2 := \|z(\omega)(0)\|_{\mathbb{H}}^2, \quad [\kappa_2(\omega)]^2 := \sup_{s \leq 0} \|z(\omega)(s)\|_{\mathbb{H}}^2 e^{\mu \lambda_1 s + (8/\mu) \int_{s}^{0} \|z(\omega(\zeta))\|_{V}^2 d\zeta},$$

$$[\kappa_3(\omega)]^2 := \int_{-\infty}^{0} \|z(\omega)(t)\|_{V}^2 e^{\mu \lambda_1 t + (8/\mu) \int_{t}^{0} \|z(\omega(\zeta))\|_{V}^2 d\zeta} dt,$$
\[
[\kappa_4(\omega)]^2 := \int_{-\infty}^{0} \|z(\omega)(t)\|_{V}^{r+1} e^{\mu \xi_1 t + (8/\mu) \int_{t}^{0} \|\omega(\zeta)\|_V^2 d\zeta} dt,
\]
\[
[\kappa_5(\omega)]^2 := \int_{-\infty}^{0} \|z(\omega)(t)\|_{V}^{4} e^{\mu \xi_1 t + (8/\mu) \int_{t}^{0} \|\omega(\zeta)\|_V^2 d\zeta} dt,
\]
\[
[\kappa_6(\omega)]^2 := \int_{-\infty}^{0} e^{\mu \xi_1 t + (8/\mu) \int_{t}^{0} \|\omega(\zeta)\|_V^2 d\zeta} dt.
\]

Then all these functions belong to the class \(K\).

**Proof:** Recall by Remark 3.2 that \(z(\theta_{-t} \omega)(s) = z(\omega)(s - t)\). Thus, we find

\[
\limsup_{t \to \infty} [\kappa_1(\theta_{-t} \omega)]^2 e^{-\mu \xi_1 t + (8/\mu) \int_{0}^{t} \|\omega(\zeta)\|_V^2 ds} = \limsup_{t \to \infty} \|z(\theta_{-t} \omega)(0)\|^2 e^{-\mu \xi_1 t + (8/\mu) \int_{0}^{t} \|\omega(\zeta)\|_V^2 ds}
\]
\[
= \limsup_{t \to \infty} \|z(\omega)(-t)\|^2 e^{-\mu \xi_1 t + (8/\mu) \int_{-t}^{0} \|\omega(\zeta)\|_V^2 ds}.
\]

Using Lemma 4.1, we have, \(\kappa_1 \in K\). It can be easily seen that

\[
[\kappa_2(\theta_{-t} \omega)]^2 = \sup_{s \leq 0} \|z(\theta_{-t} \omega)(s)\|^2 e^{\mu \xi_1 s + (8/\mu) \int_{s}^{0} \|\omega(\xi_{1} t + (8/\mu) \int_{0}^{\xi t} \|\omega(\zeta)\|_V^2 d\zeta} dt
\]
\[
= \sup_{s \leq 0} \|z(\omega)(s - t)\|^2 e^{\mu \xi_1 (s - t) + (8/\mu) \int_{s - t}^{0} \|\omega(\zeta)\|_V^2 d\zeta} dt
\]
\[
= \sup_{s \leq 0} \|z(\omega)(s - t)\|^2 e^{\mu \xi_1 (s - t) + (8/\mu) \int_{s - t}^{0} \|\omega(\zeta)\|_V^2 d\zeta} dt e^{-\mu \xi_1 t}
\]
\[
= \sup_{\sigma \leq -t} \|z(\omega)(\sigma)\|^2 e^{\mu \xi_1 \sigma + (8/\mu) \int_{0}^{\xi t} \|\omega(\zeta)\|_V^2 d\zeta} dt e^{-\mu \xi_1 t}
\]

and

\[
\limsup_{t \to \infty} [\kappa_2(\theta_{-t} \omega)]^2 e^{-\mu \xi_1 t + (8/\mu) \int_{t}^{0} \|\omega(\zeta)\|_V^2 ds} = \limsup_{t \to \infty} \sup_{\sigma \leq -t} \|z(\omega)(\sigma)\|^2 e^{\mu \xi_1 \sigma + (8/\mu) \int_{0}^{\xi t} \|\omega(\zeta)\|_V^2 d\zeta} d\zeta
\]
\[
= \limsup_{\sigma \to -\infty} \|z(\omega)(\sigma)\|^2 e^{\mu \xi_1 \sigma + (8/\mu) \int_{0}^{\xi t} \|\omega(\zeta)\|_V^2 d\zeta} d\zeta
\]
\[
= 0,
\]

which implies \(\kappa_2 \in K\) (in view of Lemma 4.1). From the previous part of the proof, we infer that

\[
\left\{ [\kappa_3(\theta_{-t} \omega)]^2 + [\kappa_4(\theta_{-t} \omega)]^2 + [\kappa_5(\theta_{-t} \omega)]^2 + [\kappa_6(\theta_{-t} \omega)]^2 \right\} e^{-\mu \xi_1 t + (8/\mu) \int_{0}^{\xi t} \|\omega(\zeta)\|_V^2 ds}
\]
\[
= \int_{-\infty}^{-t} \left\{ \|z(\omega)(t)\|^2 + \|z(\omega)(s)\|^2 e^{\mu \xi_1 t + (8/\mu) \int_{t}^{0} \|\omega(\zeta)\|_V^2 ds} + \|z(\omega)(t)\|^4 + 1 \right\}
\]
Since from Lemma 4.2, we have

\[
\int_{-\infty}^{0} \left\{ \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 + 1 \right\} e^{\mu \lambda_{1} t + (8 / \mu) \int_{s}^{0} \|z(\xi)\|_{V}^2 d\xi} dt < \infty.
\]

By the Lebesgue monotone theorem, we conclude that as \( t \to \infty \)

\[
\int_{-\infty}^{-t} \left\{ \|z(t)\|_{V}^2 + \|z(s)\|_{V}^{r+1} + \|z(t)\|_{V}^4 + 1 \right\} e^{\mu \lambda_{1} t + (8 / \mu) \int_{s}^{0} \|z(\xi)\|_{V}^2 d\xi} dt \to 0.
\]

This implies that \( \kappa_{3}, \kappa_{4}, \kappa_{5}, \kappa_{6} \in \mathcal{R} \), which completes the proof.

\[\Box\]

**Theorem 4.5:** Assume that \( r \in [1, 3], f \in \mathbb{H} \) and Assumption 2.8 holds. Then there exists a family \( \mathcal{B}_0 = \{B_0(\omega) : \omega \in \Omega \} \) of \( \mathcal{D}, \mathcal{R} \)-random absorbing sets in \( \mathbb{H} \) corresponding to the RDS \( \varphi_{e} \).

**Proof:** Let \( D \) be a random set from the class \( \mathcal{D}, \mathcal{R} \). Let \( \kappa_{D}(\omega) \) be the radius of \( D(\omega) \), that is, \( \kappa_{D}(\omega) := \sup\{\|x\|_{H} : x \in D(\omega)\}, \omega \in \Omega. \)

Let \( \omega \in \Omega \) be fixed. For given \( s \leq 0 \) and \( u_0 \in \mathbb{H} \), let \( v_{e}(\cdot) := v_{e}(\cdot, s; \omega, u_0 - \varepsilon z(s)) \) be the unique weak solution of (26) on time interval \([s, \infty)\) with the initial condition \( v_{e}(s) = u_{0} - \varepsilon z(s) \). Multiplying the first equation of (26) by \( v_{e}(\cdot) \) and integrating the resulting equation over \( \mathcal{O} \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v_{e}(t)\|_{H}^2 = -\mu \|v_{e}(t)\|_{V}^2 - b(v_{e}(t) + \varepsilon z(t), v_{e}(t) + \varepsilon z(t), v_{e}(t))
\]

\[
- \beta(C(v_{e}(t) + \varepsilon z(t), v_{e}(t)) + \varepsilon \alpha(z(t), v_{e}(t)) + (f, v_{e}(t))
\]

\[
= -\mu \|v_{e}(t)\|_{V}^2 - \varepsilon b(v_{e}(t), z(t), v_{e}(t)) - \varepsilon z(t) - \beta \|v_{e}(t) + \varepsilon z(t)\|_{L^{r+1}}^{r+1}
\]

\[
+ \varepsilon \alpha(z(t), v_{e}(t)) + (f, v_{e}(t)). \quad (45)
\]

Using \( 0 < \varepsilon \leq 1 \), Hölder’s inequality, Young’s inequality, Sobolev’s embedding, (3) and (6), we have

\[
|\varepsilon b(v_{e}, z, v_{e})| \leq \|v_{e}\|_{L^4}^{2} \|z\|_{V} \leq \sqrt{2} \|v_{e}\|_{L^2} \|v_{e}\|_{V} \|z\|_{V} \leq \frac{\mu}{8} \|v_{e}\|_{V}^{2} + \frac{4}{\mu} \|z\|_{V}^{2} \|v_{e}\|_{H}^{2},
\]

\[
\times e^{\mu \lambda_{1} t + (8 / \mu) \int_{s}^{0} \|z(\xi)\|_{V}^2 d\xi} dt.
\]
Thus from (45), we deduce

\[ \frac{1}{2} \frac{d}{dt} \| v_e(t) \|_{H}^{2} + \frac{\mu}{2} \| v_e(t) \|_{V}^{2} + \frac{\beta}{2} \| v_e(t) \|_{V}^{2} + \| \varepsilon z(t) \|_{E}^{r+1} \leq \frac{4}{\mu} \| v_e(t) \|_{H}^{2} + \frac{4\varepsilon^{2}}{\mu \lambda_{1}} \| z(t) \|_{V}^{2} + C \varepsilon^{r+1} \| z(t) \|_{V}^{r+1} + \frac{4\alpha^{2}}{\mu \lambda_{1}} \| f \|_{H}^{2}, \]

and

\[ \frac{d}{dt} \| v_e(t) \|_{H}^{2} + \mu \lambda_{1} \| v_e(t) \|_{H}^{2} \leq \frac{8}{\mu} \| v_e(t) \|_{H}^{2} + \frac{8\varepsilon^{4}}{\mu \lambda_{1}} \| z(t) \|_{V}^{4} + \frac{8\alpha^{2}}{\mu \lambda_{1}} \| f \|_{H}^{2} \]

for a.e. \( t \in [0, T] \). We infer from the classical Gronwall inequality that

\[ \| v_e(0) \|_{H}^{2} \leq \| v_e(s) \|_{H}^{2} + \int_{s}^{0} \left\{ \frac{8\varepsilon^{4}}{\mu \lambda_{1}} \| z(\xi) \|_{V}^{4} + C \varepsilon^{r+1} \| z(\xi) \|_{V}^{r+1} \right\} e^{\mu \lambda_{1} t - (8/\mu) \int_{t}^{0} \| z(\xi) \|_{V}^{2} d\xi} dt \]

\[ \leq 2 \| u_0 \|_{H}^{2} e^{\mu \lambda_{1} + (8/\mu) \int_{0}^{0} \| z(\xi) \|_{V}^{2} d\xi} + 2\varepsilon^{2} \| z(s) \|_{H}^{2} e^{\mu \lambda_{1} + (8/\mu) \int_{s}^{0} \| z(\xi) \|_{V}^{2} d\xi} + \int_{s}^{0} \left\{ \frac{8\varepsilon^{4}}{\mu \lambda_{1}} \| z(t) \|_{V}^{4} + C \varepsilon^{r+1} \| z(t) \|_{V}^{r+1} \right\} e^{\mu \lambda_{1} t - (8/\mu) \int_{t}^{0} \| z(\xi) \|_{V}^{2} d\xi} dt. \]
Let us set for $\omega \in \Omega$,  
\[
[k_{11}(\omega)]^2 = 2 + 2\varepsilon^2 \sup_{s \leq 0} \left\{ \|z(s)\|_{2}^{2} e^{\mu \lambda_{1} t + (8/\mu) \int_{s}^{0} \|z(\xi)\|_{2}^{2} d\xi} \right\} + \int_{-\infty}^{0} \left\{ 8\varepsilon^{4} \frac{\mu \lambda_{1}}{\mu^{4}} \|z(t)\|_{V}^{4} \right\} dt + C\varepsilon^{r+1} \|z(t)\|_{V}^{r+1} + \frac{8\lambda_{2}^{2}\varepsilon^{2}}{\mu \lambda_{1}} \|z(t)\|_{V}^{2} + \frac{8}{\mu \lambda_{1}} \|f\|_{H_{1}}^{2} e^{\mu \lambda_{1} t + (8/\mu) \int_{s}^{0} \|z(\xi)\|_{2}^{2} d\xi} dt, \tag{48}
\]

\[
\kappa_{12}(\omega) = \varepsilon \|z(\omega)(0)\|_{H_{1}}. \tag{49}
\]

By Lemma 4.2 and Proposition 4.4, we infer that both $k_{11}$ and $k_{12}$ belong to the class $\mathcal{R}$ and also that $k_{13} := k_{11} + k_{12}$ belongs to the class $\mathcal{R}$ as well. Therefore, the random set $B_{0}$ defined by
\[
B_{0}(\omega) = \{ u_{\varepsilon} \in H : \|u_{\varepsilon}\|_{H_{1}} \leq k_{13}(\omega) \}
\]

belongs to the family $\mathcal{D}_{\mathcal{R}}$.

We will show now that $B_{0}$ absorbs $D$. Let $\omega \in \Omega$ be fixed. Since $k_{13}(\omega) \in \mathcal{R}$, there exists $t_{D}(\omega) \geq 0$ such that
\[
[k_{D}(\theta_{-t}\omega)]^{2} e^{-\mu \lambda_{1} t + (8/\mu) \int_{s}^{0} \|z(\omega)(s)\|_{2}^{2} ds} \leq 1 \text{ for } t \geq t_{D}(\omega).
\]

Thus, if $u_{0} \in D(\theta_{-t}\omega)$ and $s \leq -t_{D}(\omega)$, then for any $\varepsilon \in (0, 1]$ by (47), we get
\[
\|v_{\varepsilon}(0, s; \omega, u_{0} - \varepsilon z(s))\|_{H_{1}} \leq k_{11}(\omega). \tag{50}
\]

Moreover, we have
\[
\|u_{\varepsilon}(0, s; \omega, u_{0})\|_{H_{1}} \leq \|v_{\varepsilon}(0, s; \omega, u_{0} - \varepsilon z(s))\|_{H_{1}} + \varepsilon \|z(\omega)(0)\|_{H_{1}} \leq k_{13}(\omega). \tag{51}
\]

This implies that $u_{\varepsilon}(0, s; \omega, u_{0}) \in B_{0}(\omega)$, for all $s \leq -t_{D}(\omega)$. This proves that $B_{0}$ absorbs $D$.

Furthermore, integrating (46) over $(-1, 0)$, we find for any $\omega \in \Omega$, there exists a $\kappa_{14}(\omega) \geq 0$ such that
\[
\int_{-1}^{0} \left[ \|v_{\varepsilon}(t)\|_{V}^{2} + \|v_{\varepsilon}(t) + \varepsilon z(t)\|_{H_{1}}^{2} \right] dt \leq \kappa_{14}(\omega) \tag{52}
\]

for all $s \leq -t_{D}(\omega)$.

**Theorem 4.6:** Assume that $r \in [1, 3], f \in H_{2}$ and Assumption 2.8 holds. Then there exists a family $\mathcal{B}_{1}(\omega) : \omega \in \Omega$ of $\mathcal{D}_{\mathcal{R}}$-random absorbing sets in $V$ corresponding to the RDS $\varphi_{\epsilon}$.

**Proof:** Let $\omega \in \Omega$ be fixed. For given $s \leq 0$ and $u_{0} \in H$, let $v_{\varepsilon}(\cdot) := v_{\varepsilon}(\cdot; s; \omega, u_{0} - \varepsilon z(s))$ be the unique solution of (26) on time interval $[s, \infty)$ with the initial condition $v_{\varepsilon}(s) = u_{0} - \varepsilon z(s)$. Multiplying the first equation of (26) by $Av_{\varepsilon}(\cdot)$ and then integrating the resulting equation over $\mathcal{O}$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v_{\varepsilon}(t)\|_{V}^{2} + \mu \|Av_{\varepsilon}(t)\|_{H_{1}}^{2} = -b(v_{\varepsilon}(t) + \varepsilon z(t), v_{\varepsilon}(t) + \varepsilon z(t), Av_{\varepsilon}(t))
\]
Hence for 

\[ \frac{d}{dt} \| v_{\varepsilon}(t) \|^2_{L^2} + \mu \| A v_{\varepsilon}(t) \|^2_{L^2} \]

Using \( 0 < \varepsilon \leq 1 \), Hölder’s and Young’s inequalities, we estimate

\[
|b(v_{\varepsilon} + \varepsilon z, v_{\varepsilon} + \varepsilon z, A v_{\varepsilon})| \leq C \| v_{\varepsilon} + \varepsilon z \|_{L^2}^{1/2} \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| A v_{\varepsilon} + \varepsilon A z \|_{L^2}^{1/2} \| A v_{\varepsilon} \|_{L^\infty}
\]

\[
\leq C \| v_{\varepsilon} + \varepsilon z \|_{L^2}^{1/2} \| v_{\varepsilon} + \varepsilon z \|_{L^\infty}
\]

\[
\times (\| A v_{\varepsilon} \|_{L^2}^{1/2} + \varepsilon^{1/2} \| A z \|_{L^2}^{1/2}) \| A v_{\varepsilon} \|_{L^\infty}
\]

\[
\leq \frac{\mu}{8} \| A v_{\varepsilon} \|_{L^2}^2 + C \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| v_{\varepsilon} + \varepsilon z \|_{L^2}^4
\]

\[
+ \varepsilon C \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| A z \|_{L^\infty} \| A v_{\varepsilon} \|_{L^\infty}^2
\]

\[
\leq \frac{\mu}{8} \| A v_{\varepsilon} \|_{L^2}^2 + C \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| A z \|_{L^\infty} \| v_{\varepsilon} \|_{L^2}^4
\]

\[
+ \varepsilon C \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| A z \|_{L^\infty} \| v_{\varepsilon} \|_{L^2}^2
\]

\[
+ \varepsilon^4 C \| v_{\varepsilon} + \varepsilon z \|_{L^2}^2 \| z \|_{L^4}^4 + \varepsilon^3 C \| v_{\varepsilon} + \varepsilon z \|_{L^\infty} \| A z \|_{L^\infty} \| z \|_{L^2}^2,
\]

(54)

\[
\varepsilon \alpha \| (z, A v_{\varepsilon}) \| \leq \varepsilon \alpha \| z \|_{L^\infty} \| A v_{\varepsilon} \|_{L^\infty} \leq \frac{\mu}{8} \| A v_{\varepsilon} \|^2_{L^2} + \varepsilon^2 C \| z \|^2_{L^2}
\]

(55)

\[
|f, A v_{\varepsilon})| \leq \| f \|_{L^\infty} \| A v_{\varepsilon} \|_{L^\infty} \leq \frac{\mu}{8} \| A v_{\varepsilon} \|^2_{L^2} + C \| f \|^2_{L^2},
\]

(56)

\[
|\langle C(v_{\varepsilon} + \varepsilon z), A v_{\varepsilon} \rangle| \leq \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \| A v_{\varepsilon} \|_{L^\infty} \leq \frac{\mu}{8} \| A v_{\varepsilon} \|^2_{L^2} + C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2},
\]

(57)

For \( r \in [1, 2] \), we estimate the final term from (57) using (7) and (3) as

\[
C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \leq C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
= C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
\leq C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
\leq C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} \|^2_{L^2} + \varepsilon^2 C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \| z \|^2_{L^2}.
\]

(58)

For \( r \in (2, 3] \), we estimate the final term from (57) using (7) and (3) as

\[
C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \leq C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
= C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
\leq C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} + \varepsilon z \|^2_{L^2}
\]

\[
\leq C \| v_{\varepsilon} + \varepsilon z \|^2 \| v_{\varepsilon} \|^2_{L^2} + \varepsilon^4 C \| v_{\varepsilon} + \varepsilon z \|^2_{L^2} \| z \|^4_{L^2}.
\]

(59)

Hence for \( r \in [1, 2] \), we infer from the inequalities (54)–(58) that for any \( \omega \in \Omega \),

\[
\frac{d}{dt} \| v_{\varepsilon}(t) \|^2_{L^2} + \mu \| A v_{\varepsilon}(t) \|^2_{L^2}
\]
\[ \leq \left[ C\|v_\varepsilon(t) + \varepsilon z(t)\|_{\Omega_1}^2 \|v_\varepsilon(t)\|_V^2 + \varepsilon C\|v_\varepsilon(t)\|_V \right. \\
\left. + \varepsilon \|z(t)\|_{\Omega_1} \|Az(t)\|_{\Omega_1} + \|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-2}^{2r-2} \right] \\
\times \|v_\varepsilon(t)\|_V^2 + \varepsilon^4 C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 + \varepsilon^3 C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 + \varepsilon^2 C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 + \varepsilon C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 + \varepsilon C\|\varepsilon z(t)\|_V^2 + C\|f\|_{\Omega_1}^2. \] 

Thus, it is immediate that

\[ \frac{d}{dt}\|v_\varepsilon(t)\|_V^2 \leq S_1(t)\|v_\varepsilon(t)\|_V^2 + S_2(t), \] 

where

\[ S_1 = C\|v_\varepsilon + \varepsilon z\|_{\Omega_1}^2 \|v_\varepsilon\|_V^2 + \varepsilon C\|v_\varepsilon\|_{\Omega_1} \|Az\|_{\Omega_1} + C\|v_\varepsilon\|_{2r-2}, \] 

\[ S_2 = \varepsilon^4 C\|v_\varepsilon + \varepsilon z\|_{\Omega_1}^2 \|z\|_V^4 + \varepsilon^3 C\|v_\varepsilon + \varepsilon z\|_{\Omega_1} \|Az\|_{\Omega_1} \|z\|_V^2 \] 

\[ + \varepsilon^2 C\|v_\varepsilon + \varepsilon z\|_{2r-4}^2 \|z\|_V^2 + \varepsilon^2 C\|\varepsilon z\|_V^2 + C\|f\|_{\Omega_1}^2. \]

For \( r \in (2, 3) \), we infer from the inequalities (54)–(57) and (59) that for any \( \omega \in \Omega \),

\[ \frac{d}{dt}\|v_\varepsilon(t)\|_V^2 + \mu\|Av_\varepsilon(t)\|_{\Omega_1}^2 \leq \left[ C\|v_\varepsilon(t) + \varepsilon z(t)\|_{\Omega_1}^2 \|v_\varepsilon(t)\|_V^2 + \varepsilon C\|v_\varepsilon(t) + \varepsilon z(t)\|_{\Omega_1} \|Az(t)\|_{\Omega_1} \right. \\
\left. + \|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 \|v_\varepsilon(t)\|_V^2 + \varepsilon^4 C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 \right] \] 

\[ + \|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 \|v_\varepsilon(t)\|_V^2 + \varepsilon^3 C\|v_\varepsilon(t) + \varepsilon z(t)\|_{2r-4}^2 \|z(t)\|_V^2 + \varepsilon^2 C\|z(t)\|_V^2 + C\|f\|_{\Omega_1}^2. \] 

Thus, it is immediate that

\[ \frac{d}{dt}\|v_\varepsilon(t)\|_V^2 \leq \tilde{S}_1(t)\|v_\varepsilon(t)\|_V^2 + \tilde{S}_2(t), \] 

where

\[ \tilde{S}_1 = C\|v_\varepsilon + \varepsilon z\|_{\Omega_1}^2 \|v_\varepsilon\|_V^2 + \varepsilon C\|v_\varepsilon\|_{\Omega_1} \|Az\|_{\Omega_1} + C\|v_\varepsilon\|_{2r-4}^2 \|v_\varepsilon\|_V^2, \] 

\[ \tilde{S}_2 = \varepsilon^4 C\|v_\varepsilon + \varepsilon z\|_{\Omega_1}^2 \|z\|_V^4 + \varepsilon^3 C\|v_\varepsilon + \varepsilon z\|_{\Omega_1} \|Az\|_{\Omega_1} \|z\|_V^2 \] 

\[ + \varepsilon^2 C\|v_\varepsilon + \varepsilon z\|_{2r-4}^2 \|z\|_V^2 + \varepsilon^2 C\|\varepsilon z\|_V^2 + C\|f\|_{\Omega_1}^2. \]
From Theorem 4.5, we get for any \( \varepsilon \in (0, 1) \) and for all \( t \in [-1, 0] \),
\[
\|v_\varepsilon(t, s, \omega, u_0 - \varepsilon z(s))\|_{\mathbb{H}} \leq \kappa_{11}(\omega)
\]
for any \( s \leq -(t_D(\omega) + 1) \). Therefore, for \( s \leq -(t_D(\omega) + 1) \), using (22), (52) and (68), we obtain
\[
\int_{-1}^{0} \| v_\varepsilon(t) \|_V^2 dt < \infty, \quad \int_{-1}^{0} s_1(t) dt < \infty, \quad \int_{-1}^{0} s_2(t) dt < \infty,
\]
\[
\int_{-1}^{0} \tilde{s}_1(t) dt < \infty, \quad \int_{-1}^{0} \tilde{s}_2(t) dt < \infty.
\]
Hence, by the uniform Gronwall lemma [56, Lemma 1.1], we infer that for any \( \varepsilon \in (0, 1) \), \( r \in [1, 3] \) and \( \omega \in \Omega \), there exists \( \kappa_{15}(\omega) \geq 0 \) such that
\[
\|v_\varepsilon(0, s, \omega, u_0 - \varepsilon z(s))\|_{\mathbb{V}} \leq \kappa_{15}(\omega)
\]
for any \( s \leq -(t_D(\omega) + 1) \). Moreover, integrating (60) and (64) over \((-1, 0)\), we find for any \( \varepsilon \in (0, 1) \), \( r \in [1, 3] \) and \( \omega \in \Omega \), there exists \( \kappa_{16}(\omega) \geq 0 \) such that
\[
\mu \int_{-1}^{0} \|a v_\varepsilon(t)\|_{\mathbb{H}}^2 dt \leq \kappa_{16}(\omega)
\]
for any \( s \leq -(t_D(\omega) + 1) \).
\[\blacksquare\]

Thanks to the compactness of \( \mathbb{V} \) in \( \mathbb{H} \), from Theorems 4.5 and 4.6 and the abstract theory of random attractors [20, Theorem 3.11], we immediately conclude the following result:

**Theorem 4.7:** Suppose that \( r \in [1, 3] \), \( f \in \mathbb{H} \) and Assumption 2.8 holds. Then the cocycle \( \varphi_\varepsilon \) corresponding to the 2D SCBF equations with additive noise (13) has a random attractor \( \hat{A}_\varepsilon = \{A_\varepsilon(\omega) : \omega \in \Omega\} \) in \( \mathbb{H} \).

**Remark 4.8:** For \( r > 3 \), one can prove the existence of random attractors for the 2D SCBF equations (13) on periodic domain in \( \mathbb{H} \) (cf. Ref. [36]) using the equality
\[
\int_{\mathcal{O}} (-\Delta v(x)) \cdot |v(x)|^{r-1} v(x) dx
= \int_{\mathcal{O}} |\nabla v(x)|^2 |v(x)|^{r-1} dx + 4 \left[ \frac{r-1}{(r+1)^2} \right] \int_{\mathcal{O}} |\nabla |v(x)||^{(r+1)/2} dx
= \int_{\mathcal{O}} |\nabla v(x)|^2 |v(x)|^{r-1} dx + \frac{r-1}{4} \int_{\mathcal{O}} |v(x)|^{r-3} |\nabla v(x)|^2 dx.
\]
But this equality may not be useful in the context of bounded domains (cf. Refs. [34,48], etc.), since \( \mathcal{P}(|u|^{r-1} u) \) need not be zero on the boundary, and \( \mathcal{P} \) and \( -\Delta \) are not necessarily commuting (for a counterexample, see Example 2.19, Ref. [53]). In order to prove the existence of random absorbing set for 2D SCBF equations in \( \mathbb{V} \), we do not use the above equality for \( r \in [1, 3] \). Due to this technical difficulty, it appears to us that the results obtained in the works [30,58] (random exponential attractors and random attractors for 3D damped NSE) hold true on periodic domains only.
5. Upper semicontinuity of \( \mathcal{DR} \)-random attractors in \( H \)

In this section, we prove the upper semicontinuity of random attractors in \( H \). The existence of random attractors for the stochastic system (13) in \( H \) is proved in Theorem 4.7 and the existence of global attractors for the deterministic system (12) in \( H \) is established in Ref. [49, Theorem 3.7]. Upper semicontinuity results for SNSEs are obtained in Ref. [15] and stochastic the Cahn–Hilliard–Navier–Stokes system is established in Ref. [44]. Now, using the similar techniques in the work [15], we state and prove the following theorem on the upper semicontinuity of the random attractors:

**Theorem 5.1:** Suppose that \( r \in [1,3], f \in H \) and Assumption 2.8 is satisfied. Also, assume that the deterministic system (12) has a global attractor \( \hat{A} \) and its small random perturbed dynamical system (13) possesses a \( \mathcal{DR} \)-random attractor \( \hat{A}_\varepsilon = \{ A_\varepsilon(\omega) : \omega \in \Omega \} \), for any \( \varepsilon \in (0,1] \). If the following conditions hold:

(K1) For each \( t_0 \geq 0 \) and \( \omega \in \Omega \)

\[
\lim_{\varepsilon \to 0^+} d(\varphi_\varepsilon(t_0, \theta-t_0 \omega)u_0, S(t)u_0) = 0
\]

uniformly on bounded sets of \( H \), where \( \varphi_\varepsilon \) is a RDS and \( S(t) \) is a semigroup generated by (13) and (12), respectively, with the same initial condition \( u_0 \).

(K2) There exists a compact set \( K \subset H \) such that

\[
\lim_{\varepsilon \to 0^+} d(A_\varepsilon(\omega), K) = 0
\]

for \( \omega \in \Omega \).

Then \( \hat{A}_\varepsilon \) and \( \hat{A} \) have the property of upper semicontinuity, that is,

\[
\lim_{\varepsilon \to 0^+} d(\hat{A}_\varepsilon(\omega), \hat{A}) = 0
\]

for \( \omega \in \Omega \).

Furthermore, if for \( \varepsilon_0 \in (0,1] \), we have that for \( \omega \in \Omega \) and all \( t_0 > 0 \)

\[
\varphi_\varepsilon(t_0, \theta-t_0 \omega)u_0 \to \varphi_{\varepsilon_0}(t_0, \theta-t_0 \omega)u_0 \quad \text{as} \quad \varepsilon \to \varepsilon_0
\]

uniformly on bounded sets of \( H \), then the convergence (73) is upper semicontinuous in \( \varepsilon \), that is,

\[
\lim_{\varepsilon \to \varepsilon_0} d(\hat{A}_\varepsilon(\omega), \hat{A}_{\varepsilon_0}(\omega)) = 0
\]

for \( \omega \in \Omega \).

**Proof:** To prove the property of upper semicontinuity for our system, we only need to verify the conditions (K1) and (K2).
Step I. Verification of (K₂): Let us introduce
\[ v_\varepsilon(t) = u_\varepsilon(t) - \varepsilon z(t), \]
where \( u_\varepsilon(t) \) and \( z(t) \) are the unique solutions of (13) and (19), respectively. Also from (22), we have \( z \in L^\infty_{loc}([t_0, \infty); V) \cap L^2_{loc}([t_0, \infty); D(A)). \) Clearly, \( v_\varepsilon \) satisfies
\[ \frac{dv_\varepsilon}{dt} = -\mu Av_\varepsilon - B(v_\varepsilon + \varepsilon z) - \beta C(v_\varepsilon + \varepsilon z) + \varepsilon \alpha z + f. \] (75)
From Theorems 4.5 and 4.6, we observe that there exists \( \hat{k}_\varepsilon(\omega) \in \hat{\mathcal{R}} \)-class such that
\[ ||v_\varepsilon(0)||_V \leq \hat{k}_\varepsilon(\omega). \]
If we call \( K_\varepsilon(\omega) \), the ball in \( V \) of radius \( \hat{k}_\varepsilon(\omega) + \varepsilon ||z(0)||_V \), we have a compact (since \( V \hookrightarrow H \) is compact) \( D(\hat{\mathcal{R}}) \)-absorbing set in \( H \) for \( \varphi_\varepsilon \). Furthermore, there exists a \( \hat{k}_d \) independent of \( \omega \in \Omega \) such that
\[ \lim_{\varepsilon \to 0^+} \hat{k}_\varepsilon(\omega) \leq \hat{k}_d, \]
which verifies Ref. [15, Lemma 1] and hence (K₂) follows.

Step II. Verification of (K₁): In order to verify the assertion (K₁), it is enough to prove that the solution \( \varphi_\varepsilon(t, \omega) u_0 \) of system (13) converges to the solution \( S(t)u_0 \) of the unperturbed system (12) in \( H \) as \( \varepsilon \to 0^+ \) uniformly on bounded sets of initial conditions, that is, for each \( \omega \in \Omega \), any \( t_0 \geq 0 \) and any bounded subset \( G \subset H \), we have
\[ \lim_{\varepsilon \to 0^+} \sup_{u_0 \in G} ||\varphi_\varepsilon(t_0, \theta_{-t_0}\omega) u_0 - S(t_0)u_0||_H = 0. \] (76)
For any \( u_0 \in G \), let \( u_\varepsilon(t) = \varphi_\varepsilon(t + t_0, \theta_{-t_0}) u_0 \) and \( u(t) = S(t + t_0)u_0 \), respectively, be the unique weak solutions of the systems (13) and (12) with initial condition \( u_0 \) at \( t = -t_0 \). Also, let for \( T \geq 0 \),
\[ y_\varepsilon(t) = u_\varepsilon(t) - u(t), \quad t \in [-t_0, T]. \]
Clearly, \( y_\varepsilon(\cdot) \) satisfies
\[ \begin{cases} dy_\varepsilon(t) + \{\mu Ay_\varepsilon(t) + B(y_\varepsilon(t) + u(t)) - B(u(t)) \\
+ \beta C(y_\varepsilon(t) + u(t)) - \beta C(u(t))\}dt = \varepsilon dW(t), \end{cases} \]
\[ y_\varepsilon(-t_0) = 0, \]
in \( V' \) for all \( t \in [-t_0, T]. \) Let us introduce \( \eta_\varepsilon(\cdot) = y_\varepsilon(\cdot) - \varepsilon z(\cdot) \), where \( z(\cdot) \) is the solution of (19), then \( \eta_\varepsilon(\cdot) \) satisfies the following equation in \( V' \):
\[ \begin{cases} \frac{d\eta_\varepsilon}{dt} = -\mu A\eta_\varepsilon - B(\eta_\varepsilon + \varepsilon z + u) + B(u) - \beta C(\eta_\varepsilon + \varepsilon z + u) + \beta C(u) + \varepsilon \alpha z, \\
\eta_\varepsilon(-t_0) = -\varepsilon z(-t_0), \end{cases} \] (77)
or
\[ \begin{cases} \frac{d\eta_\varepsilon}{dt} = -\mu A\eta_\varepsilon - B(\eta_\varepsilon, \eta_\varepsilon) - \varepsilon B(\eta_\varepsilon, z) - B(\eta_\varepsilon, u) - \varepsilon B(z, \eta_\varepsilon) - \varepsilon^2 B(z, z) - \varepsilon (z, u) \\
- B(u, \eta_\varepsilon) - \varepsilon B(u, z) - \beta C(\eta_\varepsilon + \varepsilon z + u) + \beta C(u) + \varepsilon \alpha z, \\
\eta_\varepsilon(-t_0) = -\varepsilon z(-t_0). \end{cases} \] (78)
Taking the inner product of the first equation of (78) with \( \eta_e(\cdot) \) in \( \mathbb{H} \) and making use of (5), we get

\[
\frac{1}{2} \frac{d}{dt} \| \eta_e(t) \|_{\mathbb{H}}^2 = -\mu \| \eta_e(t) \|_{\mathbb{H}}^2 - \epsilon b(\eta_e(t), z(t), \eta_e(t)) - b(\eta_e(t), u(t), \eta_e(t)) \\
+ \epsilon^2 b(z(t), \eta_e(t), z(t)) + \epsilon b(z(t), \eta_e(t), u(t)) + \epsilon b(u(t), \eta_e(t), z(t)) \\
- \beta [C(\eta_e(t) + \epsilon z(t) + u(t)) - C(u(t)), \eta_e(t) + \epsilon z(t) + u(t) - u(t)] \\
+ \epsilon \beta [C(\eta_e(t) + \epsilon z(t) + u(t)) - C(u(t)), z(t)] + \epsilon \alpha (z(t), \eta_e(t))
\]  

(79)

for a.e. \( t \in [-t_0, T] \). Making use of Hölder’s inequality, Sobolev’s embedding and Young’s inequality, we get

\[
|\epsilon b(\eta_e, z, \eta_e)| \leq \epsilon \| \eta_e \|_{\overline{\mathbb{H}}^4}^2 \| z \| \| v \| \leq \epsilon \sqrt{2} \| \eta_e \|_{\overline{\mathbb{H}}} \| \eta_e \| \| v \| \| z \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + \epsilon^2 C \| z \|_{\overline{\mathbb{V}}} \| \eta_e \|_{\overline{\mathbb{H}}},
\]

(80)

\[
|b(\eta_e, u, \eta_e)| \leq \| \eta_e \|_{\overline{\mathbb{H}}}^2 \| u \| \| v \| \leq \sqrt{2} \| \eta_e \|_{\overline{\mathbb{H}}} \| \eta_e \| \| u \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + C \| u \|_{\overline{\mathbb{V}}}^2 \| \eta_e \|_{\overline{\mathbb{H}}},
\]

(81)

\[
|\epsilon b(z, \eta_e, z)| \leq \epsilon \| z \|_{\overline{\mathbb{H}}}^2 \| \eta_e \| \| v \| \leq \epsilon C \| z \|_{\overline{\mathbb{V}}} \| \eta_e \| \| v \| \| z \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + \epsilon^2 C \| z \|_{\overline{\mathbb{V}}},
\]

(82)

\[
|\epsilon b(z, \eta_e, u)| \leq \epsilon \| z \|_{\overline{\mathbb{H}}} \| \eta_e \| \| u \| \| v \| \leq \epsilon C \| z \|_{\overline{\mathbb{V}}} \| u \| \| v \| \| u \| \| v \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + \epsilon^2 C \| z \|_{\overline{\mathbb{V}}},
\]

(83)

\[
|\epsilon b(u, \eta_e, z)| \leq \epsilon \| u \|_{\overline{\mathbb{H}}} \| \eta_e \| \| z \| \| v \| \leq \epsilon C \| u \|_{\overline{\mathbb{V}}} \| z \| \| v \| \| \eta_e \| \| v \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + \epsilon^2 C \| u \|_{\overline{\mathbb{V}}}^2 \| z \|_{\overline{\mathbb{V}}}.
\]

(84)

By (11), we have

\[
- \beta [C(\eta_e + \epsilon z + u) - C(u), \eta_e + \epsilon z + u - u] \leq 0.
\]

(85)

Now, we consider

\[
|\epsilon \beta [C(\eta_e + \epsilon z + u) - C(u), z]| \leq |\epsilon \beta [C(u_e), z]| + |\epsilon \beta [C(u), z]| \\
\leq \epsilon \beta \left\{ \| u_e \|_{\overline{\mathbb{H}}^{r+1}} + \| u \|_{\overline{\mathbb{H}}^{r+1}} \right\} \| z \|_{\overline{\mathbb{H}}^{r+1}}
\]

(86)

and

\[
\epsilon \alpha (z(t), \eta_e(t)) \leq \epsilon \alpha \| z \|_{\overline{\mathbb{H}}} \| \eta_e \|_{\overline{\mathbb{H}}} \leq \epsilon \alpha C \| z \|_{\overline{\mathbb{V}}} \| \eta_e \| \| v \| \leq \frac{\mu}{12} \| \eta_e \|_{\overline{\mathbb{H}}}^2 + \epsilon^2 \alpha^2 C \| z \|_{\overline{\mathbb{V}}}. 
\]

(87)

Combining (80)–(87) and using in (79), we deduce that

\[
\frac{d}{dt} \| \eta_e(t) \|_{\overline{\mathbb{H}}}^2 \\
\leq C \{ \epsilon^2 \| z(t) \|_{\overline{\mathbb{V}}} + \| u(t) \|_{\overline{\mathbb{V}}}^2 \} \| \eta_e(t) \|_{\overline{\mathbb{H}}} + \epsilon^2 C \{ \| z(t) \|_{\overline{\mathbb{V}}} + 2 \| u(t) \|_{\overline{\mathbb{V}}}^2 + \alpha^2 \} \| z(t) \|_{\overline{\mathbb{V}}} + \epsilon \beta C \left\{ \| u_e(t) \|_{\overline{\mathbb{H}}^{r+1}} + \| u(t) \|_{\overline{\mathbb{H}}^{r+1}} \right\} \| z(t) \|_{\overline{\mathbb{H}}^{r+1}}
\]
It can be easily seen that

\[
\int_{-t_0}^t \alpha_\varepsilon(s) \, ds \leq \varepsilon^2 (t + t_0) \| \mathbf{z} \|_{L^\infty([-t_0, t]; V)}^2 + \| \mathbf{u} \|_{L^2([-t_0, t]; V)}, \]

\[
\int_{-t_0}^t \beta_\varepsilon(s) \, ds \leq \varepsilon \left\{ (t + t_0) \| \mathbf{z} \|_{L^\infty([-t_0, t]; V)}^2 + 2 \| \mathbf{u} \|_{L^2([-t_0, t]; V)} + (t + t_0) \alpha_\varepsilon^2 \right\} \| \mathbf{z} \|_{L^\infty([-t_0, t]; V)}^2
\]

\[
+ \beta \| \mathbf{u}_e \|_{L^\infty([-t_0, t]; L^2)} \| \mathbf{u}_e \|_{L^2([-t_0, t]; V)} + \beta \| \mathbf{u} \|_{L^\infty([-t_0, t]; \mathbb{H})} \| \mathbf{u} \|_{L^2([-t_0, t]; V)}
\]

\[
+ (t + t_0) \| \mathbf{z} \|_{L^\infty([-t_0, t]; V)}^2.
\]

Since \( \mathbf{u}_e, \mathbf{u} \in L^\infty_{loc}((-t_0, \infty); \mathbb{H}) \cap L^2_{loc}((-t_0, \infty); V) \), and \( \mathbf{z} \in L^\infty_{loc}([-t_0, \infty); V) \), therefore \( \int_{-t_0}^t \beta_\varepsilon(s) \, ds \) and \( \int_{-t_0}^t \alpha_\varepsilon(s) \, ds \) both are finite. Hence, by (89), we immediately have

\[
\lim_{\varepsilon \to 0^+} \| \eta_\varepsilon(t) \|_{\mathbb{H}}^2 = 0,
\]

which completes the proof of (76) by taking \( t = 0 \). Hence (K2) is verified.

Since both conditions (K1) and (K2) hold for our model, the property of upper semicontinuity (72) holds true in \( \mathbb{H} \).

**Step III. Proof of (73):** In order to prove (73), it is enough to prove that for any bounded subset \( G \subset \mathbb{H} \), we have

\[
\lim_{\varepsilon \to 0^+} \sup_{\mathbf{u}_0 \in G} \| \varphi_\varepsilon(t_0, \theta_{-t_0} \omega) \mathbf{u}_0 - \varphi_{\varepsilon_0}(t_0, \theta_{-t_0} \omega) \mathbf{u}_0 \|_{\mathbb{H}} = 0.
\]

For any \( \mathbf{u}_0 \in G \), let us take \( \mathbf{u}_e(t) = \varphi_\varepsilon(t + t_0, \theta_{-t_0}) \mathbf{u}_0 \) and \( \mathbf{u}_{e_0}(t) = \varphi_{\varepsilon_0}(t + t_0, \theta_{-t_0}) \mathbf{u}_0 \). Let \( \mathbf{u}_e(\cdot) \) be the unique weak solution of the system (13) and \( \mathbf{u}_{e_0}(\cdot) \) be the unique weak solution
of the system (13) when ε is replaced by ε₀, with initial condition u₀ at t = −t₀. Also, let
\[ w(t) = u_ε(t) - u_{ε₀}(t), \quad t \in [0, T]. \]

Clearly, w(·) satisfies
\[
\begin{cases}
\frac{dw(t)}{dt} + [μAw(t) + B(w(t) + u_{ε₀}(t)) - B(u_{ε₀}(t)) + βC(w(t) + u_{ε₀}(t)) - βC(u_{ε₀}(t))]dt = ε^*dW(t), \\
w(−t₀) = 0,
\end{cases}
\]
in \( V' \) for a.e. \( t \in [−t₀, T] \), where \( ε^* = ε - ε₀ \). Let us introduce \( q(·) = w(·) - ε^*z(·) \), where \( z(·) \) is the unique solution of (19). Then \( q(·) \) satisfies the following equation in \( V' \):
\[
\begin{cases}
\frac{dq}{dt} = -μAQ - B(q, q) - ε^*B(q, z) - B(q, u_{ε₀}) - ε^*B(z, z) - ε^*(z, u_{ε₀}) - B(u_{ε₀}, q) - ε^*B(u_{ε₀}, z) - βC(q + ε^*z + u_{ε₀}) + βC(u_{ε₀}) + ε^*αz, \\
q(−t₀) = -ε^*z(−t₀).
\end{cases}
\]

The above system is similar to (78) and a calculation similar to (88) yields
\[
\|q(t)\|_{V'}^2 \leq \|q(−t₀)\|_{V'}^2 + C \int_{−t₀}^{t} p_ε(s)\|q(s)\|_{V'}^2ds + ε^*C \int_{−t₀}^{t} q_ε(s)ds \quad \text{for } t \in [−t₀, T],
\]
where
\[
p_ε = (ε^*)^2\|z\|_{V'}^2 + \|u_{ε₀}\|_{V'}^2, \\
q_ε = ε^*\left(\|z\|_{V'}^2 + 2\|u_{ε₀}\|_{V'}^2 + α^2\right)\|z\|_{V'}^2 + β\|u_{ε₀}\|_{H^α}^{-1}\|u_{ε₀}\|_{V'}^2 + β\|u_{ε₀}\|_{H^α}^{-1}\|u_{ε₀}\|_{V'}^2 + \|z\|_{V'}^2.
\]

Then applying the Gronwall inequality, we deduce that
\[
\|q(t)\|_{V'}^2 \leq (ε^*)^2C\|z(−t₀)\|_{V'}^2 + ε^*C \int_{−t₀}^{t} q_ε(s)ds + e^{∫_{−t₀}^{t} p_ε(s)ds}.
\]
Since \( u_{ε}, \ u_{ε₀} \in L^∞_{loc}([-t₀, ∞); \mathbb{H}) \cap L^2_{loc}([-t₀, ∞); \mathbb{V}) \), and \( z \in L^∞_{loc}([-t₀, ∞); \mathbb{V}) \), therefore \( ∫_{−t₀}^{t} p_ε(s)ds \) and \( ∫_{−t₀}^{t} q_ε(s)ds \) both are finite. Hence, by (94), we immediately have
\[
\lim_{ε → ε₀} \|q(t)\|_{H^α}^2 = 0,
\]
which completes the proof of (90) by taking \( t = 0 \). Since (73) holds true, (74) follows immediately.

**Remark 5.2:** The upper semicontinuity of random attractors for non-compact RDSs with an application to a stochastic reaction–diffusion equation on the whole space is discussed in Ref. [57]. Furthermore, the existence and upper semicontinuity of random attractor for 2D and 3D SCBF equations driven by linear multiplicative noise on the whole space are established in Ref. [38]. One can obtain the upper semicontinuity property of the random attractors for the 2D SCBF equations (13) on Poincaré domains also (see Ref. [37]), by using Ref. [57, Theorem 3.1].
6. Random attractors for 2D SCBF equations in $\mathbb{V}$

In this section, we prove the existence of random attractors in the more regular space $\mathbb{V}$. To show the existence of random attractors in $\mathbb{V}$, we prove that our RDS satisfies pullback flattening property in $\mathbb{V}$. Here, we denote by $\mathcal{D}\mathcal{R}$, the class of all closed and bounded random sets $D(\omega)$ on $\mathbb{V}$ such that the radius function $\Omega \ni \omega \mapsto \kappa(D(\omega)) := \sup\{\|x\|_\mathbb{V} : x \in D(\omega)\}$ belongs to the class $\mathcal{R}$.

We have discussed in Subsection 2.2 that there exists an orthonormal basis $\{e_k\}_{k=1}^\infty$ of $\mathbb{H}$ consisting of eigenfunctions of $A$ corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^\infty$ such that

$$Ae_k = \lambda_k e_k.$$  

Let us denote by $\mathbb{H}_m = \text{span}\{e_1, e_2, \ldots, e_m\}$ and let $P_m : \mathbb{H} \to \mathbb{H}_m$ be the $\mathbb{H}$ orthogonal projection onto $\mathbb{H}_m$.

Since the existence of random absorbing sets in $D(A^s), s > 1/2$, is not available for the 2D SCBF equations (13) (for $f \in \mathbb{H}$), the compactness arguments cannot be used to obtain the existence of random attractors in $\mathbb{V}$. Therefore, in order to establish the existence of random attractors in $\mathbb{V}$, we prove that the cocycle $\varphi_\varepsilon$ corresponding to the 2D SCBF equations (13) satisfies the pullback flattening property in $\mathbb{V}$.

**Theorem 6.1:** Suppose that $r \in [1, 3], f \in \mathbb{H}$ and Assumption 2.8 holds. Then for any $\varepsilon \in (0, 1)$, the cocycle $\varphi_\varepsilon$ corresponding to the 2D SCBF equations with additive noise (13) satisfies the pullback flattening property in $\mathbb{V}$.

**Proof:** For given $s \leq 0$, let $\hat{\mathcal{B}} = \{\hat{B}(\omega) : \omega \in \Omega\} \in \mathcal{D}\mathcal{R}$, $\delta > 0$, and

$$v_\varepsilon(t) = v_\varepsilon(t, s, \omega, u_0 - z(s)) = \varphi_\varepsilon(t - s; \theta_s \omega)u_0 - z(t) = : v_{\varepsilon,1}(t) + v_{\varepsilon,2}(t),$$

where $u_0 \in \hat{\mathcal{B}}(\theta_s \omega)$, $v_{\varepsilon,1}(t) = P_m v_\varepsilon(t)$ and $v_{\varepsilon,2}(t) = v_\varepsilon(t) - P_m v_\varepsilon(t) = Q_m v_\varepsilon(t).$ Remember that for $\psi \in D(A)$, we have

$$\begin{align*}
P_m \psi &= \sum_{j=1}^m (\psi, e_j) e_j, \\
A P_m \psi &= \sum_{j=1}^m \lambda_j (\psi, e_j) e_j, \\
Q_m \psi &= \sum_{j=m+1}^\infty (\psi, e_j) e_j, \\
A Q_m \psi &= \sum_{j=m+1}^\infty \lambda_j (\psi, e_j) e_j,
\end{align*}$$

$$\|AQ_m\psi\|^2_\mathbb{H} = \sum_{j=m+1}^\infty \lambda_j^2 |(\psi, e_j)|^2 \geq \lambda_{m+1} \sum_{j=m+1}^\infty \lambda_j |(\psi, e_j)|^2 = \lambda_{m+1} \|Q_m\psi\|^2_\mathbb{V}$$

and

$$\|AP_m\psi\|^2_\mathbb{H} = \sum_{j=1}^m \lambda_j^2 |(\psi, e_j)|^2 \leq \lambda_m \sum_{j=1}^m \lambda_j |(\psi, e_j)|^2 = \lambda_m \|P_m\psi\|^2_\mathbb{V}.$$  

That is, one can deduce

$$\|AQ_m\psi\|_\mathbb{H} \geq \sqrt{\lambda_{m+1}} \|Q_m\psi\|_\mathbb{V} \text{ and } \|AP_m\psi\|_\mathbb{H} \leq \sqrt{\lambda_m} \|P_m\psi\|_\mathbb{V}. \tag{95}$$
Taking the inner product of the first equation of (26) with $A_{v_{e,2}}(\cdot)$, we get

$$
\frac{1}{2} \frac{d}{dt} \| v_{e,2}(t) \|_V^2 = -\mu \| A_{v_{e,2}}(t) \|_{\mathbb{H}}^2 - b(v_e(t) + \varepsilon z(t), v_e(t) + \varepsilon z(t), A_{v_{e,2}}(t))
$$

$$
- \beta(C(v_e(t) + \varepsilon z(t)), A_{v_{e,2}}(t)) + \varepsilon \alpha(z(t), A_{v_{e,2}}(t)) + (f, A_{v_{e,2}}(t)).
$$

(96)

Using $0 < \varepsilon \leq 1, (10)$, Hölder’s and Young’s inequalities, we obtain

$$
\left| b(v_e + \varepsilon z, v_e + \varepsilon z, A_{v_{e,2}}) \right| \leq C \| v_e + \varepsilon z \|_V^{1/2} \| v_e + \varepsilon z \|_V \| A_{v_e} + \varepsilon A z \|_{\mathbb{H}}^{1/2} \| A_{v_{e,2}} \|_{\mathbb{H}}
$$

$$
\leq \frac{\mu}{8} \| A_{v_{e,2}} \|_{\mathbb{H}}^2
$$

$$
+ C[\| v_e + \varepsilon z \|_V^2 \| v_e + \varepsilon z \|_V + \| A_{v_{e,2}} \|_{\mathbb{H}}^2 + \| A z \|_{\mathbb{H}}^2].
$$

(97)

$$
\left| (C(v_e + \varepsilon z), A_{v_{e,2}}) \right| \leq \| v_e + \varepsilon z \|_V \| A_{v_{e,2}} \|_{\mathbb{H}} \leq \frac{\mu}{8} \| A_{v_{e,2}} \|_{\mathbb{H}}^2 + C \| v_e + \varepsilon z \|_V^{2 r - 2},
$$

(98)

$$
\varepsilon \alpha(z, A_{v_{e,2}}) \leq \alpha \| z \|_{\mathbb{H}} \| A_{v_{e,2}} \|_{\mathbb{H}} \leq \frac{\mu}{8} \| A_{v_{e,2}} \|_{\mathbb{H}}^2 + C \| z \|_V^2,
$$

(99)

$$
(f, A_{v_{e,2}}) \leq \| f \|_V \| A_{v_{e,2}} \|_{\mathbb{H}} \leq \frac{\mu}{8} \| A_{v_{e,2}} \|_{\mathbb{H}}^2 + C \| f \|_V^2.
$$

(100)

Using the inequalities (97)–(100) in (96) and then using (95), we deduce that for any $\varepsilon \in (0, 1]$ and $\omega \in \Omega$,

$$
\frac{d}{dt} \| v_{e,2}(t) \|_V^2 + \mu \lambda_{m+1} \| v_{e,2}(t) \|_V^2
$$

$$
\leq C \| v_e(t) + \varepsilon z(t) \|_V^4 \| v_e(t) + \varepsilon z(t) \|_V^4 + C \| A v_e(t) \|_{\mathbb{H}}^2
$$

$$
+ C \| A z(t) \|_{\mathbb{H}}^2 + C \| v_e(t) + \varepsilon z(t) \|_V^2 \| v_e(t) + \varepsilon z(t) \|_V^{2 r - 2}
$$

$$
+ C \| z(t) \|_V^2 + C \| f \|_V^2.
$$

Thus, it is immediate that

$$
\frac{d}{dt} \left[ e^{\mu \lambda_{m+1} t} \| v_{e,2}(t) \|_V^2 \right] \leq \left[ C \| v_e(t) + \varepsilon z(t) \|_V^4 \| v_e(t) + \varepsilon z(t) \|_V^4 + C \| A v_e(t) \|_{\mathbb{H}}^2
$$

$$
+ C \| A z(t) \|_{\mathbb{H}}^2 + C \| v_e(t) + \varepsilon z(t) \|_V^2 \| v_e(t) + \varepsilon z(t) \|_V^{2 r - 2}
$$

$$
+ C \| z(t) \|_V^2 + C \| f \|_V^2 \right] e^{\mu \lambda_{m+1} t}.
$$

(101)

From Theorems 4.5 and 4.6, we get for any $\varepsilon \in (0, 1]$ and $t \in [-1, 0]$, there exists $\kappa_{17}(\omega) \geq 0$ and $\kappa_{18}(\omega) \geq 0$ such that

$$
\| v_e(t, s; \omega, u_0 - z(s)) \|_{\mathbb{H}} \leq \kappa_{17}(\omega) \quad \text{and} \quad \| v_e(t, s; \omega, u_0 - z(s)) \|_V \leq \kappa_{18}(\omega)
$$

(102)

for any $s \leq -(t_D(\omega) + 2)$. Hence by an application of the uniform Gronwall Lemma [56, Lemma 1.1], using (70) and (102) in (101), we deduce that for any $\varepsilon \in (0, 1]$ and for any
\[ \omega \in \Omega, \text{ there exists } \kappa_{19}(\omega) \geq 0 \text{ such that} \]
\[ \|v_{s,2}(0, s; \omega, u_0 - z(s))\|_V \leq \kappa_{19}(\omega)e^{-\mu \lambda_{m+1}} \] \hspace{1cm} (103)
for any \( s \leq -(t_D(\omega) + 2) \). Therefore, for sufficiently large \( m \), we get
\[ \|Q_m v_{s}(0, s; \omega, u_0 - z(s))\|_V \leq \kappa_{19}(\omega)e^{-\mu \lambda_{m+1}} \]
for any \( \delta > 0 \), \( \omega \in \Omega \) and any \( s \leq -(t_D(\omega) + 2) \).

From Theorems 4.5, 4.6, 6.1 and 2.7, we immediately conclude the following result:

**Theorem 6.2:** Suppose that \( r \in [1, 3], f \in \mathbb{H} \) and Assumption 2.8 holds. Then for any \( \varepsilon \in (0, 1] \), the cocycle \( \varphi_\varepsilon \) corresponding to the 2D SCBF equations with additive noise (13) has a \( \mathcal{D} \mathcal{R} \)-random attractor \( G_\varepsilon = \{ G_\varepsilon(\omega) : \omega \in \Omega \} \) in \( V \).

**Remark 6.3:** For \( r > 3 \), the proof of Theorem 6.1 will remain same. Since the proof of Theorem 6.1 depends on the proof of Theorems 4.5 and 4.6, we have to restrict ourselves to periodic domain for proving the existence of random attractors (for the 2D SCBF equations (13), when \( r > 3 \)) in \( V \).

### 7. Invariant measures

In this section, we discuss the existence of an invariant measure for the 2D SCBF equations (13), which is a direct consequence of Ref. [20, Corollary 4.4] along with Theorems 4.7 and 6.2. Let \( \varphi_\varepsilon \) be the RDS corresponding to the 2D SCBF equations (13), which is defined by (41). Let us define the transition operator \( P_t \) by
\[ P_t f(x) = \int \hspace{1cm} (104) \]
for all \( f \in \mathcal{B}_b(\mathbb{H}) \), where \( \mathcal{B}_b(\mathbb{H}) \) is the space of all bounded and Borel measurable functions on \( \mathbb{H} \). A proof similar to Ref. [12, Proposition 3.8] yields the following result:

**Lemma 7.1:** The family \( \{ P_t \}_{t \geq 0} \) is Feller, that is, \( P_t f \in C_b(\mathbb{H}) \) if \( f \in C_b(\mathbb{H}) \), where \( C_b(\mathbb{H}) \) is the space of all bounded and continuous functions on \( \mathbb{H} \). Furthermore, for any \( f \in C_b(\mathbb{H}) \), \( P_t f(x) \to f(x) \) as \( t \downarrow 0 \).

**Definition 7.2:** A Borel probability measure \( \nu \) on \( \mathbb{H} \) is called an invariant measure for a Markov semigroup \( \{ P_t \}_{t \geq 0} \) of Feller operators on \( C_b(\mathbb{H}) \) if and only if
\[ P_t^* \nu = \nu, \hspace{1cm} t \geq 0, \]
where \( (P_t^* \nu)(\Gamma) = \int_{\mathbb{H}} P_t(x, \Gamma) \nu(dx) \) for \( \Gamma \in \mathcal{B}(\mathbb{H}) \) and the \( P_t(x, \cdot) \) is the transition probability, \( P_t(x, \Gamma) = P_t(\chi_\Gamma)(x), \hspace{1cm} x \in \mathbb{H} \).

Using similar arguments as in the proof of Ref. [20, Theorem 5.6], one can prove that \( \varphi_\varepsilon \) is a Markov RDS, that is, \( P_{t+s} = P_t P_s \), for all \( t, s \geq 0 \). Since, we know by Ref. [20, Corollary
4.4] that if a Markov RDS on a Polish space has an invariant compact random set, then there exists a Feller invariant probability measure \( \nu \) for \( \varphi_\varepsilon \).

In Theorems 4.7 and 6.2, we have proved the existence of random attractors in \( \mathbb{H} \) and in \( \mathbb{V} \), respectively. By the definition of random attractors, it is immediate that there exists an invariant compact random set in \( \mathbb{H} \) as well as in \( \mathbb{V} \). A Feller invariant probability measure for a Markov RDS \( \varphi \) on \( \mathbb{H} \) is, by definition, an invariant probability measure for the semigroup \( \{P_t\}_{t \geq 0} \) defined by (104). Hence, we have the following result on the existence of invariant measures for the 2D SCBF equations (13).

**Theorem 7.3:** There exists an invariant measure for the 2D SCBF equations (13) in \( \mathbb{H} \).

**Remark 7.4:** 1. In Theorem 6.2, we have also proved that there exists a random attractor in \( \mathbb{V} \) and hence there exists an invariant compact random set in \( \mathbb{V} \). Invoking Ref. [20, Corollary 4.4], we obtain the existence of an invariant measure for the 2D SCBF equations (13) in \( \mathbb{V} \) as well.

2. In this work, \( \{W(t)\}_{t \in \mathbb{R}} \) is an \( \mathbb{H} \)-valued Wiener process with RKHS \( K \) satisfying Assumption 2.8. In particular, \( K \subset \mathbb{H} \) and the natural embedding \( i : K \hookrightarrow \mathbb{H} \) is a Hilbert-Schmidt operator. For a fixed orthonormal basis \( \{w_k\}_{k \in \mathbb{N}} \) of \( K \) and a sequence \( \{\beta_k\}_{k \in \mathbb{N}} \) of independent Brownian motions defined on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}) \) such that \( W(t) \) can be written in the following form:

\[
W(t) = \sum_{k=1}^{\infty} \beta_k(t)w_k, \quad t \geq 0.
\]

Moreover, there exists a covariance operator \( J \in \mathcal{L}(\mathbb{H}) \) associated with \( W(t) \) defined by

\[
\langle h_1, h_2 \rangle = \mathbb{E}\{\langle W(1), h_1 \rangle_{\mathbb{H}} \langle W(1), h_2 \rangle_{\mathbb{H}}\}, \quad h_1, h_2 \in \mathbb{H}.
\]

It is well known from Ref. [21] that \( J \) is a non-negative self-adjoint and trace class operator in \( \mathbb{H} \) which implies that \( \{W(t)\}_{t \in \mathbb{R}} \) is a non-degenerate white noise. Furthermore, \( J = ii^* \) and \( K = R(J^{1/2}) \), where \( R(J^{1/2}) \) is the range of the operator \( J^{1/2} \) (see Ref. [14]). Note that

\[
\sum_{k=1}^{\infty} \|iw_k\|_{\mathbb{H}}^2 = tr[J] < \infty.
\]

The existence of a unique ergodic and strongly mixing invariant measure for SCBF equations subjected to non-degenerate additive noise (Assumption 2.8) is proved in Ref. [48] using the exponential stability of solutions. By establishing the irreducibility and strong Feller property of the Markov semigroup associated with the solutions of 2D SCBF equations perturbed by white noise, the uniqueness of invariant measures and ergodicity is established in Ref. [41]. The uniqueness of invariant measure for SCBF equations driven by additive degenerate noise (\( \sigma W(\cdot) \), where \( W(\cdot) \) is a cylindrical Wiener process taking values in \( \mathbb{H} \) with covariance operator \( I \) and \( \sigma \) satisfying highly degenerate but essentially elliptic condition, cf. Ref. [50, Assumption 2.6] via the asymptotic coupling method [28] is established in [50].

**Remark 7.5:** In order to prove the upper semicontinuity of random attractors when \( \varepsilon \to 0 \), we considered the small noise assumption (that is, \( \varepsilon \in (0, 1) \)). It is worth mentioning
here that small noise assumption is not required to prove the existence of unique random attractor (in both $H$ and $V$) as well as the existence of unique invariant measure (in both $H$ and $V$) for the 2D SCBF equations (13).

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**References**

[1] S.N. Antontsev and H.B. de Oliveira, *The Navier–Stokes problem modified by an absorption term*, Appl. Anal. 89(12) (2010), pp. 1805–1825.

[2] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[3] P. Bates, H. Lisei, and K. Lu, *Attractors for stochastic lattice dynamical systems*, Stoch. Dyn. 6(1) (2006), pp. 1–21.

[4] P. Bates, K. Lu, and B. Wang, *Random attractors for stochastic reaction–diffusion equations on unbounded domains*, J. Differ. Equ. 246 (2009), pp. 845–869.

[5] Z. Brzeźniak, *Stochastic partial differential equations in M-type 2 banach spaces*, Potential Anal. 4 (1995), pp. 1–45.

[6] Z. Brzeźniak, *On Sobolev and Besov spaces regularity of Brownian paths*, Stoch. Stoch. Rep. 56(1–2) (1996), pp. 1–15.

[7] Z. Brzeźniak, *Stochastic convolution in banach spaces*, Stoch. Stoch. Rep. 61 (1997), pp. 245–295.

[8] Z. Brzeźniak, M. Capiński, and F. Flandoli, *Pathwise global attractors for stationary random dynamical systems*, Probab. Theory Relat. Fields 95 (1993), pp. 87–102.

[9] Z. Brzeźniak, T. Caraballo, J.A. Langa, Y. Li, G. Lukaszewicz, and J. Real, *Random attractors for stochastic 2D Navier–Stokes equations in some unbounded domains*, J. Differ. Equ. 255 (2013), pp. 3897–3919.

[10] Z. Brzeźniak, B. Goldys, and Q.T. Le Gia, *Random attractors for the stochastic Navier–Stokes equations on the 2D unit sphere*, J. Math. Fluid Mech. 20 (2018), pp. 227–253.

[11] Z. Brzeźniak and Y.H. Li, *Asymptotic behaviour of solutions to the 2D stochastic Navier–Stokes equations in unbounded domains-new developments*, in Albeverio S, Ma Z-M, Rockner M. editors. *Recent Developments in Stochastic Analysis and Related Topics*, World Sci. Publ., Hackensack, NJ, 2004, pp. 78–111.

[12] Z. Brzeźniak and Y. Li, *Asymptotic compactness and absorbing sets for 2D stochastic Navier–Stokes equations in some unbounded domains*, Trans. Am. Math. Soc. 358(12) (2006), pp. 5587–5629.

[13] Z. Brzeźniak and S. Peszat, *Stochastic two dimensional Euler equations*, Ann. Probab. 29(4) (2001), pp. 1796–1832.
[14] Z. Brzeźniak and J. van Neerven, *Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem*, Studia Math. 143 (2000), pp. 43–74.

[15] T. Caraballo, J.A. Langa, and J.C. Robinson, *Upper semicontinuity of attractors for small random perturbations of dynamical systems*, Commun. Partial Differ. Equ. 23(9–10) (1998), pp. 1557–1581.

[16] I. Chueshov, *Dynamics of Quasi-Stable Dissipative Systems*, Springer-Verlag, Cham, 2015.

[17] H. Crauel, *Global random attractors are uniquely determined by attracting deterministic compact sets*, Ann. Mat. Pura Appl. 176 (1999), pp. 57–72.

[18] H. Crauel, *Random Probability Measures on Polish Spaces*, Stochastics Monographs, Vol. 11, Taylor & Francis, London, 2002.

[19] H. Crauel, A. Debussche, and F. Flandoli, *Random attractors*, J. Dyn. Differ. Equ. 9(2) (1995), pp. 307–341.

[20] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Relat. Fields 100 (1994), pp. 365–393.

[21] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and Its Applications, Vol. 44, Cambridge University Press, Cambridge, 1992.

[22] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Mathematical Society Lecture Note Series, Vol. 229, Cambridge University Press, Cambridge, 1996.

[23] X. Feng and B. You, *Random attractors for the two-dimensional stochastic g-Navier–Stokes equations*, Stochastics 92(4) (2020), pp. 613–626. doi: 10.1080/17442508.2019.1642340

[24] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

[25] D. Fujiwara and H. Morimoto, *An Lr-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), pp. 685–700.

[26] B. Gess, W. Liu, and M. Röckner, *Random attractors for a class of stochastic partial differential equations driven by general additive noise*, J. Differ. Equ. 251(4–5) (2011), pp. 1225–1253.

[27] B. Gess, W. Liu, and A. Schenke, *Random attractors for locally monotone stochastic partial differential equations*, J. Differ. Equ. 269 (2020), pp. 3414–3455.

[28] M. Hairer, J.C. Mattingly, and M. Scheutzow, *Asymptotic coupling and a general form of Harris’s theorem with applications to stochastic delay equations*, Probab. Theory Relat. Fields. 149 (2011), pp. 223–259.

[29] K.W. Hajduk and J.C. Robinson, *Energy equality for the 3D critical convective Brinkman–Forchheimer equations*, J. Differ. Equ. 263 (2017), pp. 7141–7161.

[30] Z. Han and S. Zhou, *Random exponential attractor for the 3D non-autonomous stochastic damped Navier–Stokes equation*, J. Dyn. Differ. Equ. (2021), pp. 1–17.

[31] A. Ilyin, K. Patni, and S. Zelik, *Upper bounds for the attractor dimension of damped Navier–Stokes equations in $\mathbb{R}^2$*, Discrete Contin. Dyn. Syst. 36(4) (2016), pp. 2085–2102.

[32] A.A. Ilyin and E.S. Titi, *Sharp estimates for the number of degrees of freedom for the damped-driven 2-D Navier–Stokes equations*, J. Nonlinear Sci. 16(3) (2006), pp. 233–253.

[33] X. Jia and X. Ding, *Random attractors for stochastic retarded 2D-Navier–Stokes equations with additive noise*, J. Funct. Spaces 2018 (2018), p. 3105239.

[34] V.K. Kalantarov and S. Zelik, *Smooth attractors for the Brinkman–Forchheimer equations with fast growing nonlinearities*, Commun. Pure Appl. Anal. 11 (2012), pp. 2037–2054.

[35] K. Kinra and M.T. Mohan, *Large time behavior of the deterministic and stochastic 3D convective Brinkman–Forchheimer equations in periodic domains*, J. Dyn. Differ. Equ. 1 (2021), pp. 1–42.

[36] K. Kinra and M.T. Mohan, *Convergence of random attractors towards deterministic singleton attractor for 2D and 3D convective Brinkman–Forchheimer equations*, Evol. Equ. Control Theory 11(5) (2022), pp. 1701–1744.

[37] K. Kinra and M.T. Mohan, *Random attractors for 2D and 3D stochastic convective Brinkman–Forchheimer equations in some unbounded domains*, 2020, arXiv:2010.08753, https://arxiv.org/pdf/2010.08753.pdf

[38] K. Kinra and M.T. Mohan, *Existence and upper semicontinuity of random pullback attractors for 2D and 3D non-autonomous stochastic convective Brinkman–Forchheimer equations on whole space*, Differ. Integr. Equ. (2022), https://arxiv.org/pdf/2105.13770.pdf
[39] P.E. Kloeden and J.A. Langa, *Flattening squeezing and the existence of random attractors*, Proc. Roy. Soc. 463 (2007), pp. 163–181.

[40] P.E. Kloeden, J.A. Langa, and J. Real, *Pullback V-attractors of the 3-dimensional globally modified Navier–Stokes equations*, Commun. Pure Appl. Anal. 6(4) (2007), p. 937.

[41] A. Kumar and M.T. Mohan, *Large deviation principle for occupation measures of the two dimensional stochastic convective Brinkman–Forchheimer equations*, Stoch. Anal. Appl. (2021), pp. 1–43. doi:10.1080/07362994.2021.2005626

[42] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[43] J. Li, Y. Li, and H. Cui, *Existence and upper semicontinuity of random attractors for stochastic p-Laplacian equations on unbounded domains*, Electron. J. Differ. Equ. 87 (2014), p. 27.

[44] F. Li and B. You, *Random attractor for the stochastic Cahn–Hilliard–Navier–Stokes system with small additive noise*, Stoch. Anal. Appl. 36(3) (2018), pp. 546–559.

[45] H. Liu and H. Gao, *Ergodicity and dynamics for the stochastic 3D Navier–Stokes equations with damping*, Commun. Math. Sci. 16(1) (2018), pp. 97–122.

[46] M.T. Mohan, *The $H^1$-compact global attractor for the two dimensional convective Brinkman–Forchheimer equations in unbounded domains*, J. Dyn. Control Syst. 28 (2021), pp. 791–816. doi:10.1007/s10883-021-09545-2

[47] M.T. Mohan, *On the convective Brinkman–Forchheimer equations*, Submitted for publication.

[48] M.T. Mohan, *Stochastic convective Brinkman–Forchheimer equations*, Submitted for publication, https://arxiv.org/abs/2007.09376

[49] M.T. Mohan, *Asymptotic analysis of the 2D convective Brinkman–Forchheimer equations in unbounded domains: global attractors and upper semicontinuity*, Submitted for publication, https://arxiv.org/abs/2010.12814

[50] M.T. Mohan, *Asymptotic log-Harnack inequality for the stochastic convective Brinkman–Forchheimer equations with degenerate noise*, Submitted for publication, https://arxiv.org/pdf/2008.00955.pdf

[51] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa 3(13) (1959), pp. 115–162.

[52] J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, Cambridge, 2001.

[53] J.C. Robinson, J.L. Rodrigo, and W. Sadowski, *The Three-Dimensional Navier–Stokes Equations, Classical Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.

[54] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1977.

[55] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd ed., CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1995.

[56] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Applied Mathematical Sciences, Vol. 68, Springer, New York, 1998.

[57] B. Wang, *Upper semicontinuity of random attractors for non-compact random dynamical systems*, Electron. J. Differ. Equ. 139 (2009), p. 18.

[58] B. You, *The existence of a random attractor for the three dimensional damped Navier–Stokes equations with additive noise*, Stoch. Anal. Appl. 35(4) (2017), pp. 691–700.

[59] W. Zhao, *$H^1$-random attractors and random equilibria for stochastic reaction-diffusion equations with multiplicative noises*, Commun. Nonlinear Sci. Numer. Simul. 18(10) (2013), pp. 2707–2721.

[60] W. Zhao, *$H^1$-random attractors for stochastic reaction–diffusion equations with additive noise*, Nonlinear Anal. 84 (2013), pp. 61–72.