THE SCATTERING MATRIX WITH RESPECT TO AN HERMITIAN MATRIX OF A GRAPH

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Abstract

Recently, Gnutzmann and Smilansky [5] presented a formula for the bond scattering matrix of a graph with respect to a Hermitian matrix. We present another proof for this Gnutzmann and Smilansky’s formula by a technique used in the zeta function of a graph. Furthermore, we generalize Gnutzmann and Smilansky’s formula to a regular covering of a graph. Finally, we define an $L$-function of a graph, and present a determinant expression. As a corollary, we express the generalization of Gnutzmann and Smilansky’s formula to a regular covering of a graph by using its $L$-functions.

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1 Introduction

Ihara zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [9]. Originally, Ihara presented p-adic Selberg zeta functions of discrete groups, and showed that its reciprocal is an explicit polynomial. Serre [13] pointed out that the Ihara zeta function is the zeta function of the quotient $T/\Gamma$ (a finite regular graph) of the one-dimensional Bruhat-Tits building $T$ (an infinite regular tree) associated with $GL(2, k_p)$.

A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada [10][17]. Hashimoto [8] treated multi-variable zeta functions of bipartite graphs. Bass [1] generalized Ihara’s result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial. Various proofs of Bass’ Theorem were given by Stark and Terras [15], Foata and Zeilberger [3], Kotani and Sunada [10]. Sato [12] defined the second weighted zeta function of a graph by using not an infinite product but a determinant.

In this paper, we another proof for the Gnutzmann and Smilansky’s formula on the bond scattering matrix of a graph by using not an infinite product but a determinant. The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph (see [2, 4, 5, 11]). Smilansky [14] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs, and expressed some determinant on the bond scattering matrix of a graph $G$ by using the characteristic polynomial of its Laplacian. Recently, Gnutzmann and Smilansky [5] presented a formula for the bond scattering matrix of a graph with respect to a Hermitian matrix.

In this paper, we another proof for the Gnutzmann and Smilansky’s formula on the bond scattering matrix of a graph with respect to a Hermitian matrix. by a technique used in the zeta function of a graph, and treat some related topics. In Section 2, we review the Ihara zeta function and the bond scattering matrix of a graph $G$. In Section 3, we present another proof for the Gnutzmann and Smilansky’s formula by a technique used in the zeta function of a graph. In Section 4, we express a new zeta function of $G$ on the bond scattering matrix of $G$ with respect to a Hermitian matrix by using the Euler product. In Section 5, we generalize the Gnutzmann and Smilansky’s formula to a regular covering of $G$. In Section 6, we define an $L$-function of $G$, and present its determinant expression. As a corollary, we express the generalization of the Gnutzmann and Smilansky’s formula to a regular covering of $G$ by using its $L$-functions.

2 The zeta functions and the bond scattering matrix of a graph

Graphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $b = (u, v) \in D(G)$, set $u = o(b)$ and $v = t(b)$. Furthermore, let $b^{-1} = (v, u)$ be the inverse of $b = (u, v)$.

A path $P$ of length $n$ in $G$ is a sequence $P = (b_1, \ldots, b_n)$ of $n$ arcs such that $b_i \in D(G)$, $t(b_i) = o(b_{i+1})$ for $1 \leq i \leq n - 1$, where indices are treated mod $n$. Set $|P| = n$, $o(P) = o(b_1)$ and $t(P) = t(b_n)$. Also, $P$ is called an $(o(P), t(P))$-path. We say that a path $P = (b_1, \ldots, b_n)$ has a backtracking or back-scatter if $b_{i+1} = t_{i-1}^-$ for some $i(1 \leq i \leq n - 1)$. A $(v, w)$-path is called a $v$-cycle (or $v$-closed path) if $v = w$. The inverse cycle of a cycle $C = (b_1, \ldots, b_n)$ is the cycle $C^{-1} = (b_n, \ldots, b_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are called equivalent if there exists $k$ such that $f_j = e_{j+k}$ for all $j$. The inverse cycle of $C$ is in general not equivalent to $C$. Let $[C]$ be the equivalence class which contains a cycle $C$. Let $B^r$ be the cycle obtained by going $r$ times around a cycle $B$. Such a cycle is called a power of $B$. A cycle $C$ is reduced if both $C$ and $C^2$ have
no backtracking. Furthermore, a cycle $C$ is prime if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph $G$ corresponds to a unique conjugacy class of the fundamental group $\pi_1(G)$ of $G$ at a vertex $u$ of $G$. Furthermore, an equivalence class of prime cycles of a graph $G$ is called a primitive periodic orbit of $G$ (see [14]).

The Ihara zeta function of a graph $G$ is a function of a complex variable $t$ with $|t|$ sufficiently small, defined by

$$Z(G, t) = Z_G(t) = \prod_{[p]} (1 - t^{|[p]|})^{-1},$$

where $[p]$ runs over all equivalence classes of prime, reduced cycles of $G$ (see [9]).

**Theorem 1 (Ihara; Bass)** Let $G$ be a connected graph. Then the reciprocal of the Ihara zeta function of $G$ is given by

$$Z(G, t)^{-1} = (1 - t^2)^{-1} \det(I - tA(G) + t^2(D - I)),$$

where $r$ and $A(G)$ are the Betti number and the adjacency matrix of $G$, respectively, and $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = v_i = \deg u_i$ where $V(G) = \{u_1, \ldots, u_n\}$.

Let $G$ be a connected graph and $V(G) = \{u_1, \ldots, u_n\}$. Then we consider an $n \times n$ matrix $W = (w_{ij})_{1 \leq i, j \leq n}$ with $ij$ entry the complex variable $w_{ij}$ if $(u_i, u_j) \in E(G)$, and $w_{ij} = 0$ otherwise. The matrix $W = W(G)$ is called the weighted matrix of $G$. Furthermore, let $w(u_1, u_2) = w_{12}, \ldots, w(u_n, u_1) = w_{nn} \in V(G)$ and $w(b) = w_{ij}, b = (u_i, u_j) \in E(G)$. For each path $P = (e_1, \ldots, e_r)$ of $G$, the norm $w(P)$ of $P$ is defined as follows: $w(P) = w(e_1)w(e_2)\cdots w(e_r)$.

Let $G$ be a connected graph with $n$ vertices and $m$ unoriented edges, and $W = W(G)$ a weighted matrix of $G$. Two $2m \times 2m$ matrices $B = B(G) = (B_{e,f})_{e,f \in R(G)}$ and $J_0 = J_0(G) = (J_{e,f})_{e,f \in R(G)}$ are defined as follows:

$$B_{e, f} = \begin{cases} \{w(f)\} & \text{if } t(e) = o(f), \\ \{0\} & \text{otherwise}, \end{cases} \quad J_{e, f} = \begin{cases} 1 & \text{if } f = \hat{e}, \\ 0 & \text{otherwise}. \end{cases}$$

Then the second weighted zeta function of $G$ is defined by

$$Z_1(G, w, t) = \det(I_n - t(B - J_0))^{-1}. $$

If $w(e) = 1$ for any $e \in E(G)$, then the zeta function of $G$ is the Ihara zeta function of $G$.

**Theorem 2 (Sato)** Let $G$ be a connected graph, and let $W = W(G)$ be a weighted matrix of $G$. Then the reciprocal of the second weighted zeta function of $G$ is given by

$$Z_1(G, w, t)^{-1} = (1 - t^2)^{m-n} \det(I_n - tW(G) + t^2(D - I_G)),$$

where $n = |V(G)|$, $m = |E(G)|$ and $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(b) = u_i} w(e), V(G) = \{u_1, \ldots, u_n\}$.

Next, we state the bond scattering matrix of a graph. Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{u_1, \ldots, u_n\}$ and $D(G) = \{b_1, \ldots, b_m, b_{m+1}, \ldots, b_{2m}\}$ such that $b_{m+j} = b_j^{-1} (1 \leq j \leq m)$. The Laplacian (matrix) $L = L(G)$ of $G$ is defined by

$$L = L(G) = -A(G) + D.$$

Let $\lambda$ be an eigenvalue of $L$ and $\psi = (\psi_1, \ldots, \psi_n)$ the eigenvector corresponding to $\lambda$. For each arc $b = (u_j, u_i)$, one associates a bond wave function

$$\psi_b(x) = a_b e^{i\pi x / 4} + a_{b-1} e^{-i\pi x / 4}, \quad x = \pm 1$$

under the condition

$$\psi_b(1) = \psi_j, \psi_b(-1) = \psi_i.$$

We consider the following three conditions:
1. **uniqueness**: The value of the eigenvector at the vertex $u_j$, $\psi_j$, computed in the terms of the bond wave functions is the same for all the arcs emanating from $u_j$.

2. $\psi$ is an eigenvector of $L$;

3. **consistency**: The linear relation between the incoming and the outgoing coefficients (1) must be satisfied simultaneously at all vertices.

By the uniqueness, we have

$$a_{b_1} e^{i\pi/4} + a_{b_1}^{-1} e^{-i\pi/4} = a_{b_2} e^{i\pi/4} + a_{b_2}^{-1} e^{-i\pi/4} = \cdots = a_{b_j} e^{i\pi/4} + a_{b_j}^{-1} e^{-i\pi/4},$$

where $b_1, b_2, \ldots, b_d_j$ are arcs emanating from $u_j$, and $d_j = \text{deg } u_j$, $i = \sqrt{-1}$.

By the condition 2, we have

$$- \sum_{k=1}^{d_j} (a_{b_k} e^{-i\pi/4} + a_{b_k}^{-1} e^{i\pi/4}) = (\lambda - v_j) \sum_{k=1}^{d_j} (a_{b_k} e^{i\pi/4} + a_{b_k}^{-1} e^{-i\pi/4}).$$

Thus, for each arc $b$ with $o(b) = u_j$,

$$a_b = \sum_{t(c)=u_j} \sigma_{b,c}^{(u_j)}(\lambda) a_c, \quad (1)$$

where

$$\sigma_{b,c}^{(u_j)}(\lambda) = i(\delta_{b^{-1},c} - \frac{2}{d_j - 1 + (1 - \lambda/d_j)}),$$

and $\delta_{b^{-1},c}$ is the Kronecker delta. The bond scattering matrix $U(\lambda) = (U_{ef})_{e,f \in D(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} \sigma_{e,f}^{(t(f))} & \text{if } t(f) = o(e), \\ 0 & \text{otherwise} \end{cases}$$

By the consistency, we have

$$U(\lambda) a = a,$$

where $a = t(a_{b_1}, a_{b_2}, \ldots, a_{b_{2m}})$. This holds if and only if

$$\det(I_{2m} - U(\lambda)) = 0.$$

**Theorem 3 (Smilansky)** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then the characteristic polynomial of the bond scattering matrix of $G$ is given by

$$\det(I_{2m} - U(\lambda)) = \frac{2^n i^n \det(\lambda I_n + A(G) - D)}{\prod_{j=1}^{d} (d_j - id_j + \lambda i)} = \prod_{[p]} (1 - a_p(\lambda)),$$

where $[p]$ runs over all primitive periodic orbits of $G$, and

$$a_p(\lambda) = \sigma_{b_1,b_n}^{(t(b_n))} \sigma_{b_{n-1},b_{n-1}}^{(t(b_{n-1}))} \cdots \sigma_{b_2,b_1}^{(t(b_1))}, \quad p = (b_1, b_2, \ldots, b_n)$$

Mizuno and Sato [11] presented another proof for this Smilansky’s formula by using the determinant expression of the second weighted zeta function of a graph.
3 The scattering matrix of a graph with respect to a Hermitian matrix

Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{1, \ldots, n\}$ and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+j} = e_j^{-1}$ ($1 \leq j \leq m$). Furthermore, let an Hermitian matrix $H = H(G) = (H_{uv})_{u, v \in V(G)}$ be given as follows:

$$H_{uv} = \begin{cases} h_f e^{2i\gamma_f} & \text{if } f = (u, v) \in D(G), \\ 0 & \text{otherwise}, \end{cases}$$

where, for each $f \in D(G)$,

$$h_f = h_{f^{-1}} \geq 0 \text{ and } \gamma_f = -\gamma_{f^{-1}} \in [-\pi/2, \pi/2].$$

If $H_{uv} = H_f$ is real and negative, then we choose $\gamma_f = \pi/2$ if $u \geq v$ and $\gamma_f = -\pi/2$ if $u < v$. Set

$$h(u, v) = h_{uv} = h_f \text{ and } \gamma(u, v) = \gamma_{uv} = \gamma_f \text{ for } f = (u, v) \in D(G).$$

Now, let $\lambda$ be an eigenvalue of $H$ and $\psi = (\psi_1, \ldots, \psi_n)$ the eigenvector corresponding to $\lambda$. For each arc $b = (u, v)$, one associates a bond wave function

$$\psi_b(x) = \frac{e^{i\gamma_b}}{\sqrt{h_b}}(a_{b-1}e^{ix/4} + a_be^{-ix/4}), \quad x = \pm 1$$

under the condition

$$\psi_b(1) = \psi_u, \psi_b(-1) = \psi_v.$$

We consider the following three conditions:

1. uniqueness: The value of the eigenvector at the vertex $u$, $\psi_u$, computed in the terms of the bond wave functions is the same for all the arcs emanating from $u$.

2. $\psi$ is an eigenvector of $H$;

3. consistency: The linear relation between the incoming and the outgoing coefficients (1) must be satisfied simultaneously at all vertices.

By the uniqueness 1, we have

$$\frac{e^{i\gamma_{b_1}}}{\sqrt{h_{b_1}}}(a_{b_1-1}e^{i\pi/4} + a_{b_1}e^{-i\pi/4}) = \frac{e^{i\gamma_{b_2}}}{\sqrt{h_{b_2}}}(a_{b_2-1}e^{i\pi/4} + a_{b_2}e^{-i\pi/4}) = \ldots$$

$$= \frac{e^{i\gamma_{b_d}}}{\sqrt{h_{b_d}}}(a_{b_d-1}e^{i\pi/4} + a_{b_d}e^{-i\pi/4}) = \psi_u,$$

where $b_1, b_2, \ldots, b_d$ are arcs emanating from $u$, and $d = \deg u$, $i = \sqrt{-1}$.

By the condition 2, we have

$$(H_{uu} - \lambda)\psi_u + \sum_{v \in E_u} H_{uv}\psi_v = 0,$$

and so,

$$(H_{uu} - \lambda)e^{i\gamma_{b_1}}/\sqrt{h_{b_1}}(a_{b_1-1}e^{i\pi/4} + a_{b_1}e^{-i\pi/4}) = -\frac{1}{d} \sum_{k=1}^d H_{b_j} e^{i\gamma_{b_k}}/\sqrt{h_{b_k}}(a_{b_k-1}e^{i\pi/4} + a_{b_k}e^{-i\pi/4}),$$
where \( \mathcal{E}_u = \{ f \in D(G) \mid o(f) = u \} \). Thus, for each arc \( b \) with \( o(b) = u \),

\[
a_b^{-1} = ia_b - 2 \sum_{k=1}^{d} \frac{\sqrt{h_b} \sqrt{h_b_k}}{H_{uu} - \lambda - i\Gamma_u} e^{i(\gamma_{bk} + \gamma_{b^{-1}})} a_{bk},
\]

where

\[
\Gamma_u = \sum_{k=1}^{d} h_{bk}.
\]

Let \( e = b^{-1}, f = b_k \) and

\[
\sigma_{e,f}^{(u)}(\lambda) = i\delta_{e^{-1}f} - 2 \frac{\sqrt{h_e} \sqrt{h_f}}{H_{uu} - \lambda - i\Gamma_u} e^{i(\gamma_f + \gamma_e)},
\]

where \( \delta_{e^{-1}f} \) is the Kronecker delta. Then we have

\[
a_e = \sum_{o(f) = u} \sigma_{e,f}^{(u)}(\lambda) a_f
\]

for each arc \( e \) such that \( t(e) = u \). The bond scattering matrix \( U(\lambda) = (U_{ef})_{e,f \in D(G)} \) of \( G \) is defined by

\[
U_{ef} = \begin{cases} 
\sigma_{e,f}^{(t(e))} & \text{if } t(e) = o(f), \\
0 & \text{otherwise}
\end{cases}
\]

By the consistency \( 3 \), we have

\[
U(\lambda) a = a,
\]

where \( a = \{a_1, a_{b_2}, \ldots, a_{b_m}\} \). This holds if and only if

\[
det(I_{2m} - U(\lambda)) = 0.
\]

We present another proof of Theorem 4 by using the technique on the Ihara zeta function, which is different from a proof in [5].

**Theorem 4 (Gnutzmann and Smilansky)** Let \( G \) be a connected graph with \( n \) vertices \( 1, \ldots, n \) and \( m \) edges. Then, for the bond scattering matrix of \( G \),

\[
det(I_{2m} - U(\lambda)) = \frac{(-1)^{n^2m} \det(\Lambda_n - H)}{\prod_{j=1}^{n} (H_{jj} - \lambda - i\Gamma_j)}.
\]

**Proof.** The argument is an analogue of Watanabe and Fukumizu’s method [18].

Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges, \( V(G) = \{1, \ldots, n\} \) and \( D(G) = \{b_1, \ldots, b_m, b_1^{-1}, \ldots, b_m^{-1}\} \). Set \( d_j = \deg j \) and

\[
x_j = \frac{2}{H_{jj} - \lambda - i\Gamma_j}
\]

for each \( j = 1, \ldots, n \). Furthermore, for \( e \in D(G) \), let

\[
w(e) = \sqrt{h_e} e^{i\gamma_e}.
\]

Then we have

\[
\sigma_{e,f}^{(t(e))}(\lambda) = i\delta_{e^{-1}f} - x_{t(e)} w(e) w(f).
\]

Now, we consider a \( 2m \times 2m \) matrix \( B = (B_{ef})_{e,f \in D(G)} \) given by

\[
B_{ef} = \begin{cases} 
x_{o(f)} w(e) w(f) & \text{if } t(e) = o(f), \\
0 & \text{otherwise}
\end{cases}
\]
Let $K = (K_{i,j})_{1 \leq i \leq 2m:1 \leq j \leq n}$ be the $2m \times n$ matrix defined as follows:

$$K_{i,j} := \begin{cases} x_j w(b_i) & \text{if } o(b_i) = j, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, we define two $2m \times n$ matrices $L = (L_{i,j})_{1 \leq i \leq 2m:1 \leq j \leq n}$ and $M = (M_{i,j})_{1 \leq i \leq 2m:1 \leq j \leq n}$ as follows:

$$L_{i,j} := \begin{cases} w(b_i) & \text{if } t(b_i) = j, \\ 0 & \text{otherwise}, \end{cases}$$

$$M_{i,j} := \begin{cases} w(b_i) & \text{if } o(b_i) = j, \\ 0 & \text{otherwise}. \end{cases}$$

Note that

$$K = M \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} = MX. \quad (3)$$

Furthermore, we have

$$L^T K = B \quad (4)$$

and

$$M^T L = H. \quad (5)$$

Note that

$$H_{uv} = w(u,v)^2 \text{ if } (u,v) \in D(G).$$

But, since

$$U_{ef} = \begin{cases} -x_{t(e)} w(e) w(f) & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\ i - x_{t(e)} w(e) w(f) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$U(\lambda) = iJ_0 - B.$$

Furthermore, if $A$ and $B$ are an $r \times s$ and an $s \times r$ matrix, respectively, then we have

$$\det(I_r - AB) = \det(I_s - BA).$$

Thus,

$$\det(I_{2m} - uU(\lambda)) = \det(I_{2m} - u(iJ_0 - B))$$

$$= \det(I_{2m} - iuJ_0 + uL^T K)$$

$$= \det(I_{2m} + uL^T K(I_{2m} - iuJ_0)^{-1}) \det(I_{2m} - iuJ_0)$$

$$= \det(I_n + uL^T K(I_{2m} - iuJ_0)^{-1}L) \det(I_{2m} - iuJ_0).$$

Arrange arcs of $D(G)$ as follows: $b_1, b_1^{-1}, \ldots, b_m, b_m^{-1}$. Then we have

$$\det(I_{2m} - iuJ_0) = \det\begin{pmatrix} 1 & -iu & \cdots & 0 \\ -iu & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} = (1 + u^2)^m.$$
Furthermore,

$$(I_{2m} - iuJ_0)^{-1} = \begin{bmatrix} 1 & -iu & \ldots & 0 \\ -iu & 1 & \ddots & \\ \vdots & \ddots & \ddots & \\ 0 & \ddots & \ddots & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{1 + u^2} \begin{bmatrix} 1 & iu & \ldots & 0 \\ iu & 1 & \ddots & \\ \vdots & \ddots & \ddots & \\ 0 & \ddots & \ddots & 1 \end{bmatrix}$$

$$= \frac{1}{1 + u^2} (I_{2m} + iuJ_0).$$

Therefore, it follows that

$$\det(I_{2m} - uU(\lambda)) = (1 + u^2)^m - n \det((1 + u^2)I_n + u^2 KL + iu^2 iKJ_0L).$$

But, we have

$$^t KL = X^t ML = XH.$$ 

Furthermore, we have

$$^t KJ_0L = X^t MJ_0L.$$ 

Then, for $u, v \in V(G)$, we have

$$(^t MJ_0L)_{uv}$$

$$= \delta_{uv} \sum_{c(e) = u} (^t M)_{ue}(J_0)_{ee^{-1}}(L)_{ee^{-1}v}$$

$$= \delta_{uv} \sum_{c(e) = u} w(e) \cdot 1 \cdot w(e^{-1})$$

$$= \delta_{uv} \sum_{c(e) = u} h_e e^{i\gamma_e} \sqrt{h_e} e^{-i\gamma_e}$$

$$= \delta_{uv} \sum_{c(e) = u} h_e = \delta_{uv} \Gamma_u.$$ 

Now, let

$$D_L = \begin{bmatrix} \Gamma_1 & 0 \\ \cdot & \cdot \\ 0 & \Gamma_n \end{bmatrix}.$$ 

Then

$$^t KJ_0L = XD_L.$$ 

Thus,

$$\det(I_{2m} - uU(\lambda)) = (1 + u^2)^m - n \det((1 + u^2)I_n + uXH + iu^2 XD_L).$$
Substituting \( u = 1 \), we obtain
\[
\det(I_{2m} - U(\lambda)) = 2^{m-n} \det(2I_n + XH + iXD) \\
= 2^{m-n} \det\left[ \begin{array}{ccc} 2 & i\Gamma_u & \cdots \\ \cdots & H_{uu} - \lambda - i\Gamma_u & \cdots \\ \cdots & \cdots & H_{uv} \end{array} \right] \\
= \prod_{u=1}^{2m} (-\lambda I_n + H) \\
= \prod_{u=1}^{(-1)^n2m} \det(\lambda I_n - H).
\]
\]

\[\Box\]

4 The Euler product with respect to the scattering matrix

We present the Euler product for the determinant formula of the scattering matrix \( U(\lambda) \) of a graph.

**Theorem 5** Let \( G \) be a connected graph with \( m \) edges, and \( H = H(G) = (H_{uv})_{u,v \in V(G)} \) an Hermitian matrix defined in Section 2. Then the characteristic polynomial of the bond scattering matrix of \( G \) induced from \( H \) is given by
\[
\det(I_{2m} - uU(\lambda)) = \prod_{|C|}(1 - w_C u^{|C|}),
\]
let \( c \) runs over all equivalence classes of prime cycles in \( G \), and
\[
w_C = \sigma^{(t(e_1))}_1 \sigma^{(t(e_2))}_2 \cdots \sigma^{(t(e_n))}_n, \quad C = (b_1, b_2, \ldots, b_n)
\]

**Proof.** Let \( D(G) = \{b_1, \cdots, b_{2m}\} \) such that \( b_{m+j} = b_j^{-1}(1 \leq j \leq m) \). Set \( U = U(\lambda) \). Since
\[
\log \det(I - uF) = \text{Tr} \log(I - uF),
\]
for a square matrix \( F \), we have
\[
\log \det(I - uU) = \text{Tr} \log(I - uU) = -\sum_{k=1}^{\infty} \frac{\text{Tr}(U^k)}{k} u^k.
\]
Here,
\[
\text{Tr}(U^k) = \sum_C w_C,
\]
where \( C \) runs over all cycles of length \( k \) in \( G \), and
\[
w_C = \sigma^{(t(e_1))}_1 \sigma^{(t(e_2))}_2 \cdots \sigma^{(t(e_k))}_k, \quad C = (b_1, b_2, \ldots, b_k)
\]
Thus,
\[
\frac{d}{du} \log \det(I_{2m} - uU) = \sum_{k=1}^{\infty} \frac{\text{Tr}(U^k)}{k} u^k = \sum_C w_C u^{|C|},
\]
where \( C \) runs over all cycles in \( G \).
Now, let \( C \) be any cycle in \( G \). Then there exists exactly one prime cycle \( D \) such that

\[ C = D^l. \]

Thus, we have

\[ u \frac{d}{du} \log \det(I_{2m} - uU) = - \sum_D \sum_{k=1}^{\infty} w_D^k u^{k|D|}, \]

and so,

\[ \frac{d}{du} \log \det(I_{2m} - uU) = - \sum_D \sum_{k=1}^{\infty} w_D^k u^{k|D| - 1}, \]

where \( D \) runs over all prime cycles in \( G \). Therefore, it follows that

\[
\log \det(I_{2m} - uU) = - \sum_D \sum_{k=1}^{\infty} \frac{w_D^k}{k|D|} u^{k|D|} \\
= - \sum_D \sum_{k=1}^{\infty} \frac{|D|}{k|D|} w_D^k u^{k|D|} \\
= - \sum_D \sum_{k=1}^{\infty} \frac{1}{k} w_D^k u^{k|D|} \\
= \sum_D \log(1 - w_D u^{|D|}).
\]

Hence,

\[
\det(I_{2m} - uU(\lambda)) = \prod_{|C|} (1 - w_C u^{|C|}),
\]

\( \square \)

5 Scattering matrix of a regular covering of a graph

Let \( G \) be a connected graph, and let \( N(v) = \{ w \in V(G) \mid (v, w) \in D(G) \} \) denote the neighbourhood of a vertex \( v \) in \( G \). A graph \( H \) is a covering of \( G \) with projection \( \pi : H \rightarrow G \) if there is a surjection \( \pi : V(H) \rightarrow V(G) \) such that \( \pi|_{N(v')}: N(v') \rightarrow N(v) \) is a bijection for all vertices \( v \in V(G) \) and \( v' \in \pi^{-1}(v) \). When a finite group \( \Pi \) acts on a graph \( G \), the quotient graph \( G/\Pi \) is a graph whose vertices are the II-orbits on \( V(G) \), with two vertices adjacent in \( G/\Pi \) if and only if some two of their representatives are adjacent in \( G \). A covering \( \pi : H \rightarrow G \) is regular if there is a subgroup \( B \) of the automorphism group \( \text{Aut} H \) of \( H \) acting freely on \( H \) such that the quotient graph \( H/B \) is isomorphic to \( G \).

Let \( G \) be a graph and \( \Gamma \) a finite group. Then a mapping \( \alpha : D(G) \rightarrow \Gamma \) is an ordinary voltage assignment if \( \alpha(v, u) = \alpha(u, v)^{-1} \) for each \( (u, v) \in D(G) \). The pair \( (G, \alpha) \) is an ordinary voltage graph. The derived graph \( G^\alpha \) of the ordinary voltage graph \( (G, \alpha) \) is defined as follows: \( V(G^\alpha) = V(G) \times \Gamma \) and \( ((u, h), (v, k)) \in D(G^\alpha) \) if and only if \( (u, v) \in D(G) \) and \( k = h \alpha(u, v) \). The natural projection \( \pi : G^\alpha \rightarrow G \) is defined by \( \pi(u, h) = u \). The graph \( G^\alpha \) is a derived graph covering of \( G \) with voltages in \( \Gamma \) or a \( \Gamma \)-covering of \( G \). Note that \( |\mathcal{E}_{(u, h)}| = |\mathcal{E}_u| \) for each \( (u, h) \in V(G^\alpha) \). The natural projection \( \pi \) commutes with the right multiplication action of the \( \alpha(e), e \in D(G) \) and the left action of \( \Gamma \) on the fibers: \( g(u, h) = (u, gh), g \in \Gamma \), which is free and transitive. Thus, the \( \Gamma \)-covering \( G^\alpha \) is a \( |\Gamma| \)-fold regular covering of \( G \) with covering transformation group \( \Gamma \). Furthermore, every regular covering of a graph \( G \) is a \( \Gamma \)-covering of \( G \) for some group \( \Gamma \) (see [6]).

Let \( G \) be a connected graph, \( \Gamma \) be a finite group and \( \alpha : D(G) \rightarrow \Gamma \) be an ordinary voltage assignment. In the \( \Gamma \)-covering \( G^\alpha \), set \( v_g = (v, g) \) and \( e_g = (e, g) \), where \( v \in V(G), e \in D(G), g \in \Gamma \). For \( e = (u, v) \in D(G) \), the arc \( e \) emanates from \( u \) and terminates at \( v_{\alpha(e)} \). Note that \( e_g^{-1} = (e^{-1})_{\alpha(e)} \).
Let $G$ be a connected graph, $\Gamma$ be a finite group and $\alpha : D(G) \rightarrow \Gamma$ be an ordinary voltage assignment. Furthermore, let $H = H(G) = (H_{uv})_{u, v \in V(G)}$ be an Hermitian matrix such that

$$H_{uv} = \begin{cases} h_{f}e^{2i\gamma_{f}} & \text{if } f = (u, v) \in D(G), \\ 0 & \text{otherwise}, \end{cases}$$

where, for each $f \in D(G)$,

$$h_{f} = h_{f-1} \geq 0 \text{ and } \gamma_{f} = -\gamma_{f-1} \in [\pi/2, \pi/2].$$

We give the function $\tilde{h} : G(\alpha) \rightarrow \mathbb{R}$ and $\tilde{\gamma} : G(\alpha) \rightarrow [\pi/2, \pi/2]$ induced from $h$ and $\gamma$, respectively, as follows:

$$\tilde{h}(u_{g}, v_{k}) = h_{uv} \text{ and } \tilde{\gamma}(u_{g}, v_{k}) = \gamma_{uv} \text{ if } (u, v) \in D(G) \text{ and } k = g\alpha(u, v).$$

Furthermore, we consider the Hermitian matrix $\hat{H} = H(G(\alpha)) = (H_{uv})_{u, v \in V(G(\alpha))}$ of $G^\alpha$ induced from $H$. At first, let

$$H_{u_{g}u_{g}} = H_{uu} \text{ for each } g \in \Gamma.$$ 

For $(u_{g}, v_{k}) \in D(G^\alpha)$, we have

$$H_{u_{g}v_{k}} = \tilde{h}(u_{g}, v_{k})e^{2i\tilde{\gamma}(u_{g}, v_{k})} = h_{uv}e^{2i\gamma_{uv}}.$$ 

Thus,

$$H_{u_{g}v_{k}} = \begin{cases} h_{uv}e^{2i\gamma_{uv}} & \text{if } (u, v) \in D(G) \text{ and } k = g\alpha(u, v), \\ H_{uu} & \text{if } u = v \text{ and } k = g, \\ 0 & \text{otherwise} \end{cases}.$$

Next, we consider the bond wave function of the regular covering $G^\alpha$ of $G$. Let $V(G) = \{v_{1}, \ldots, v_{n}\}$, $\overline{D}(G) = \{e_{1}, \ldots, e_{m}, e_{i}^{-1}, \ldots, e_{m}^{-1}\}$ and $\overline{\Gamma} = \{g_{1} = 1, g_{2}, \ldots, g_{p}\}$. Let $\lambda$ be a eigenvalue of $\tilde{H} = H(G^\alpha)$, and let $\tilde{\phi} = (\tilde{\phi}_{v_{1}g_{1}}, \ldots, \tilde{\phi}_{v_{1}g_{p}}, \ldots, \tilde{\phi}_{v_{n}g_{1}}, \ldots, \tilde{\phi}_{v_{n}g_{p}})$ be the eigenvector corresponding to $\lambda$, where $\tilde{\phi}_{v_{i}g_{j}}$ corresponds to the vertex $(v_{i}, g_{j})$ ($1 \leq i \leq n; 1 \leq j \leq p$) of $G^\alpha$. Furthermore let $b_{g} = (v_{g}, z_{g\alpha(b_{g})})$ be any arc of $G^\alpha$, where $b = (v, z) \in D(G)$, $g \in \Gamma$. Then the bond wave function of $G^\alpha$ is

$$\phi_{b_{g}}(x) = e^{i\gamma_{b_{g}}}\sqrt{h_{b_{g}}}(a_{b_{g}}^{+}e^{i\pi/4} + a_{b_{g}}e^{-i\pi/4}), \quad x = \pm 1, \quad i = \sqrt{-1}$$

under the condition

$$\phi_{b_{g}}(1) = \phi_{v_{g}} \text{ and } \phi_{b_{g}}(-1) = \phi_{z_{g\alpha(b_{g})}}.$$ 

By (1), we have

$$a_{b_{g}}^{+} = \begin{cases} i\delta_{b_{g}^{-1}e_{g}} & -2\sum_{o(e_{g}) = v_{g}}H_{v_{g}v_{g}}^{-1/2}\sum_{o(e_{g}) = z_{g\alpha(b_{g})}}H_{z_{g\alpha(b_{g})}z_{g\alpha(b_{g})}}^{-1/2}e^{i(\gamma_{b_{g}} + \gamma_{v_{g}})}a_{e_{g}} \\ \sum_{o(e_{g}) = v_{g}}\sigma_{b_{g}e_{g}}^{(v_{g})}a_{e_{g}} \end{cases}$$

for each arc $b_{g}$ with $o(b_{g}) = v_{g}$, where

$$\sigma_{b_{g}e_{g}}^{(v_{g})} = i\delta_{b_{g}^{-1}e_{g}} - 2\frac{H_{v_{g}v_{g}}}{H_{v_{g}v_{g}}^{-1/2}\sum_{o(e_{g}) = v_{g}}H_{v_{g}v_{g}}^{-1/2}\sum_{o(e_{g}) = z_{g\alpha(b_{g})}}H_{z_{g\alpha(b_{g})}z_{g\alpha(b_{g})}}^{-1/2}e^{i(\gamma_{b_{g}} + \gamma_{v_{g}})}a_{e_{g}}$$

and

$$\hat{h}_{e_{g}} = \hat{h}(e_{g}), \quad \hat{\gamma}_{e_{g}} = \hat{\gamma}(e_{g}).$$
By the definitions of $\hat{h}$, $\hat{\gamma}$ and $\hat{\mathbf{H}}$, we have
\[
\sigma_{b\gamma}^{(e)} = i\hat{\delta}_{b\gamma} - 2\frac{\sqrt{\hat{h}_e\hat{h}_b}}{\hat{H}_{vv} - \lambda - i\Gamma_v} e^{i(\gamma_b + \gamma_e)} = \sigma_{be}^{(v)} = \sigma_{be}^{(t(b))}.
\]
Note that $\mathcal{E}_{v,g} = \mathcal{E}_v$. Thus,
\[
a_{b\gamma}^{-1} = \sum_{\alpha(e_g) = \gamma} \sigma_{be}^{(t(b))} a_{e\gamma}.
\]
Therefore, the bond scattering matrix $\hat{\mathbf{U}}(\lambda) = (U(e_g, f_h))_{e_g, f_h \in D(G^\alpha)}$ of $G^\alpha$ is given by
\[
U(e_g, f_h) = \begin{cases} \sigma_{ef}^{(t(e))} & \text{if } t(f_h) = \alpha(e_g), \\ 0 & \text{otherwise}. \end{cases}
\]
But, we have
\[
x_{v,g} = \frac{2}{\hat{H}_{vv} - \lambda - i\Gamma_v} = x_v
\]
for $v_g \in V(G^\alpha)$. Furthermore, let $\hat{w} : D(G^\alpha) \rightarrow \mathbb{C}$ be given as follows:
\[
\hat{w}(e_g) = \sqrt{\hat{h}_e} e^{i\gamma_{e_g}} \text{ for each } e_g \in D(G^\alpha).
\]
Then we have
\[
\hat{w}(e_g) = \sqrt{\hat{h}_e} e^{i\gamma_{e_g}} = w(e), \quad e_g \in D(G^\alpha).
\]
For $g \in \Gamma$, let the matrix $\mathbf{H}_g = (H_{uv}^{(g)})$ be defined by
\[
H_{uv}^{(g)} = \begin{cases} h_{uv} e^{2i\gamma_{uv}} & \text{if } \alpha(u, v) = g \text{ and } (u, v) \in D(G), \\ 0 & \text{otherwise}. \end{cases}
\]
Furthermore, let $\mathbf{U}_g = (U^{(g)}(e, f))$ be given by
\[
U^{(g)}(e, f) = \begin{cases} \sigma_{ef}^{(t(e))} & \text{if } t(e) = \alpha(f) \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise}, \end{cases}
\]
Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \ldots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element $a_{ij}$ replaced by the matrix $a_{ij} \mathbf{B}$.

**Theorem 6** Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ unoriented edges, $\Gamma$ be a finite group and $\alpha : D(G) \rightarrow \Gamma$ be an ordinary voltage assignment. Set $|\Gamma| = p$. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of $\Gamma$, and $f_i$ be the degree of $\rho_i$ for each $i$, where $f_1 = 1$.

If the $\Gamma$-covering $G^\alpha$ of $G$ is connected, then, for the bond scattering matrix of $G^\alpha$,
\[
\det(\mathbf{I}_{2mp} - \hat{\mathbf{U}}(\lambda)) = \det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) \prod_{i=2}^k \det(\mathbf{I}_{2m f_i} - \sum_{h} \rho_i(h) \otimes \mathbf{U}_h)^{f_i},
\]
\[
= \frac{2^{mp}(-1)^p \det(\lambda \mathbf{I}_n - \mathbf{H})}{\prod_{u \in V(G)} (\hat{H}_{uu} - \lambda - i\Gamma_u)} \prod_{i=2}^k \det(\lambda \mathbf{I}_{n f_i} - \sum_{h \in \Gamma} \rho_i(h) \otimes \mathbf{H}_h - \mathbf{I}_{f_i} \otimes \text{diag}(\mathbf{H}))^{f_i},
\]
where
\[
\text{diag}(\mathbf{H}) = \begin{bmatrix} H_{v_1v_1} & 0 \\ 0 & \ddots \\ 0 & H_{v_nv_n} \end{bmatrix}.
\]
Proof. Let $|\Gamma|=p$. By Theorem 4, for the bond scattering matrix of $\Gamma^\alpha$, we have

$$\det(I_{2mp}-\hat{U}(\lambda)) = \frac{2^{mp}(-1)^{np}\det(\lambda I_{np}-H(G^\alpha))}{\prod_{u\in V(G)}(H_{uu}-\lambda-i\Gamma_u)^p}.$$ 

Let $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+j} = e_j^{-1}(1 \leq j \leq m)$ and $\Gamma = \{1 = g_1, g_2, \ldots, g_p\}$. Arrange arcs of $G^\alpha$ in $p$ blocks: $(e_1, 1), (e_2, 1), \ldots, (e_{2m}, g_2), \ldots; (e_1, g_p), \ldots, (e_{2m}, g_p)$. We consider the matrix $U(\lambda)$ under this order. For $h \in \Gamma$, the matrix $P_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $U(g_i, f_{g_i}) \neq 0$ if and only if $t(e, g_j) = o(f, g_i)$. Furthermore, $t(e, g_j) = o(f, g_i)$ if and only if $(o(f), g_j) = (t(e), g_i) = (t(e), g_i \alpha(e))$. Thus, $t(e) = o(f)$ and $\alpha(e) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus, we have

$$U(\lambda) = \sum_{h \in \Gamma} P_h \otimes U_h.$$ 

Furthermore, we have

$$\text{diag}(H(G^\alpha)) = I_p \otimes \text{diag}(H).$$

Let $\rho$ be the right regular representation of $\Gamma$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be all inequivalent irreducible representations of $\Gamma$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_1 = 1$. Then we have $\rho(g) = P_g$ for $g \in \Gamma$. Furthermore, there exists a nonsingular matrix $P$ such that $P^{-1} \rho(g) P = (1) \oplus f_2 \circ \rho_2(g) \oplus \cdots \oplus f_k \circ \rho_k(g)$ for each $g \in \Gamma$ (see [12]). Thus, we have

$$P^{-1} P_g P = (1) \oplus f_2 \circ \rho_2(g) \oplus \cdots \oplus f_k \circ \rho_k(g).$$

Putting $F = (P^{-1} \otimes I_{2m}) \hat{U}(\lambda)(P \otimes I_{2m})$, we have

$$F = \sum_{g \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(g) \oplus \cdots \oplus f_k \circ \rho_k(g)\} \otimes U_g.$$ 

Note that $U(\lambda) = \sum_{g \in \Gamma} U_g$ and $1 + f_2^2 + \cdots + f_k^2 = p$. Therefore it follows that

$$\det(I_{2mp}-\hat{U}(\lambda)) = \det(I_{2m}-U(\lambda)) \prod_{i=2}^k \det(I_{2mf_i} - \sum_g \rho_i(g) \otimes U_g)^{f_i}.$$ 

Next, let $V(G) = \{v_1, \ldots, v_n\}$. Arrange vertices of $G^\alpha$ in $p$ blocks: $(v_1, 1), (v_1, 2), \ldots, (v_1, g_p), (v_2, 1), \ldots, (v_2, g_p), \ldots; (v_n, 1), \ldots, (v_n, g_p)$. We consider the matrix $H(G^\alpha)$ under this order.

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $((u, g_i), (v, g_j)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $g_j = g_i \alpha(u, v)$. If $g_j = g_i \alpha(u, v)$, then $\alpha(u, v) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$H(G^\alpha) = \sum_{h \in \Gamma} P_h \otimes H_h + I_p \otimes \text{diag}(H).$$ 

Putting $E = (P^{-1} \otimes I_n) H(G^\alpha)(P \otimes I_n)$, we have

$$E = \sum_{h \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)\} \otimes H_h + I_p \otimes \text{diag}(H).$$
Let \( \rho \) be a finite group and \( \alpha \) be a connected graph with \( n \) vertices and \( m \) unoriented edges, \( \Gamma \) be a finite group and \( \alpha : D(G) \rightarrow \Gamma \) be an ordinary voltage assignment. Furthermore, let \( H = H(G) = (H_{uv})_{u,v \in V(G)} \) be an Hermitian matrix such that

\[
H_{uv} = \begin{cases} 
  h_f e^{2i\gamma_f} & \text{if } f = (u,v) \in D(G), \\
  0 & \text{otherwise},
\end{cases}
\]

where, for each \( f \in D(G) \),

\[
h_f = h_{f - 1} \geq 0 \text{ and } \gamma_f = -\gamma_{f - 1} \in [-\pi/2, \pi/2].
\]

Let \( \rho \) be a unitary representation of \( \Gamma \) and \( d \) its degree. The \( L \)-function of \( G \) associated with \( \rho \) and \( \alpha \) is defined by

\[
Z_H(G, \lambda, \rho, \alpha) = \det(I_{2md} - \sum_{h \in \Gamma} \rho(h) \bigotimes U_h)^{-1}.
\]

If \( \rho = 1 \) is the identity representation of \( \Gamma \), then

\[
Z_H(G, \lambda, 1, \alpha) = \det(I_{2m} - U)^{-1}.
\]

A determinant expression for the \( L \)-function of \( G \) associated with \( \rho \) and \( \alpha \) is given as follows. For \( 1 \leq i, j \leq n \), the \((i, j)\)-block \( F_{i,j} \) of a \( dn \times dn \) matrix \( F \) is the submatrix of \( F \) consisting of \( d(i - 1) + 1, \ldots, di \) rows and \( d(j - 1) + 1, \ldots, dj \) columns.

**Theorem 7** Let \( G \) be a connected graph with \( n \) vertices and \( m \) unoriented edges, \( \Gamma \) be a finite group and \( \alpha : D(G) \rightarrow \Gamma \) be an ordinary voltage assignment. If \( \rho \) is a unitary representation of \( \Gamma \) and \( d \) is the degree of \( \rho \), then the reciprocal of the \( L \)-function of \( G \) associated with \( \rho \) and \( \alpha \) is

\[
Z_H(G, \lambda, \rho, \alpha)^{-1} = \frac{2^{md}(-1)^{nd}}{\prod_{u \in V(G)}(H_{uu} - \lambda - i\Gamma_u)d} \det(\lambda I_{np} - \sum_{g \in \Gamma} \rho(g) \bigotimes H_g - I_d \bigotimes \text{diag}(H)).
\]

**Proof.** The argument is an analogue of Watanabe and Fukumizu’s method \cite{18}. 

\[\boxed{}\]
Let $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ such that $e_{m+i} = e_i^{-1}(1 \leq i \leq m)$. Note that the $(e, f)$-block $\left(\sum_{g \in \Gamma} U_g \otimes \rho(g)\right)_{ef}$ of $\sum_{g \in \Gamma} U_g \otimes \rho(g)$ is given by

$$\left(\sum_{g \in \Gamma} U_g \otimes \rho(g)\right)_{ef} = \begin{cases} \rho(\alpha(e))\rho(\sigma)^{(e)(f)}_{ef} & \text{if } t(e) = o(f), \\ 0_d & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, two $2m \times 2m$ matrices $B_g = (B_{ef}^{(g)})_{e, f \in D(G)}$ and $J_g = (J_{ef}^{(g)})_{e, f \in D(G)}$ are defined as follows:

$$B_{ef}^{(g)} = \begin{cases} x_{o(f)}w(e)w(f) & \text{if } t(e) = o(f) \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise.} \end{cases}$$

$$J_{ef}^{(g)} = \begin{cases} 1 & \text{if } f = e^{-1} \text{ and } \alpha(e) = g, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$U_g = iJ_g - B_g \text{ for } g \in \Gamma.$$}

Let $K = (K_{ij})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ be the $2md \times nd$ matrix defined as follows:

$$K_{ij} := \begin{cases} v_{o(e_i)}w(e_i)I_d & \text{if } o(e_i) = v_j, \\ 0_d & \text{otherwise.} \end{cases}$$

Furthermore, we define two $2md \times nd$ matrices $L = (L_{ij})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ and $M = (M_{ij})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ as follows:

$$L_{ij} := \begin{cases} w(e_i)\rho(\alpha(e_i)) & \text{if } t(e_i) = v_j, \\ 0_d & \text{otherwise,} \end{cases}$$

$$M_{ij} := \begin{cases} w(e_i)I_d & \text{if } o(e_i) = v_j, \\ 0_d & \text{otherwise.} \end{cases}$$

Then we have

$$K = M(X \otimes I_d) = MX_d,$$

where

$$X_d = X \otimes I_d.$$}

Furthermore, we have

$$L^tK = \sum_{h \in \Gamma} B_h \otimes \rho(h) = B_p \quad (6)$$

and

$$L^tML = \sum_{g \in \Gamma} H_g \otimes \rho(g), \quad (7)$$

where

$$B_p = \sum_{g \in \Gamma} B_g \otimes \rho(g).$$

Thus,

$$\det(I_{2md} - u \sum_{g \in \Gamma} \rho(g) \otimes U_g) = \det(I_{2md} - u \sum_{g \in \Gamma} U_g \otimes \rho(g))$$

$$= \det(I_{2md} - u \sum_{g \in \Gamma} (iJ_g - B_g) \otimes \rho(g))$$

$$= \det(I_{2md} - iu \sum_{g \in \Gamma} J_g \otimes \rho(g) + u \sum_{g \in \Gamma} B_g \otimes \rho(g)).$$

Now, let

$$J_p = \sum_{g \in \Gamma} J_g \otimes \rho(g).$$

Note that

$$J_p^2 = I_{2md}.$$
Then we have
\[
\det(I_{2md} - u \sum_{g \in \Gamma} \rho(g) \otimes U_g)
\]
= \det(I_{2md} - iu J_\rho + u B_\rho)
= \det(I_{2md} + u B_\rho (I_{2md} - iu J_\rho)^{-1}) \det(I_{2md} - iu J_\rho)
= \det(I_{2md} + u L^t K (I_{2md} - iu J_\rho)^{-1}) \det(I_{2md} - iu J_\rho)
= \det(I_{nd} + u L^t K (I_{2md} - iu J_\rho)^{-1} L) \det(I_{2md} - iu J_\rho).
\]

But, we have
\[
\det(I_{2md} - iu J_\rho) = \det \left( \begin{array}{ccc}
I_d & -iu \rho(\alpha(e_1^{-1})) & 0 \\
-iu \rho(\alpha(e_1^{-1})) & I_d & \\
0 & & \ddots
\end{array} \right) = (1 + u^2)^{md}.
\]

Furthermore, we have
\[
(I_{2md} - iu J_\rho)^{-1}
\]
= \[
\left( \begin{array}{ccc}
I_d & -iu \rho(\alpha(e_1^{-1})) & 0 \\
-iu \rho(\alpha(e_1^{-1})) & I_d & \\
0 & & \ddots
\end{array} \right)^{-1}
\]
= \[
\frac{1}{1+u^2} \left( \begin{array}{ccc}
I_d & iu \rho(\alpha(e_1^{-1})) & 0 \\
iu \rho(\alpha(e_1^{-1})) & I_d & \\
0 & & \ddots
\end{array} \right)
\]
= \frac{1}{1+u^2} (I_{2md} + iu J_\rho).
\]

Thus, we have
\[
\det(I_{2md} - u \sum_{g \in \Gamma} \rho(g) \otimes U_g)
\]
= \(1 + u^2)^{md} \det(I_{nd} + u/(1 + u^2) L^t K (I_{2md} + iu J_\rho) L)
= \(1 + u^2)^{md-nd} \det((1 + u^2)I_{nd} + u L^t K + iu^2 K J_\rho L).
\]

Now, we have
\[
L^t K L = X_d L^t M L = X_d \sum_{g \in \Gamma} H_g \otimes \rho(g).
\]

Furthermore,
\[
L^t K J_\rho L = X_d L^t M J_\rho L.
\]
Then we have

\[(^tMJ \rho L)_{uv} = \delta_{uv} \sum_{\alpha(e) = u} (^tM)_{ue} (J \rho)_{ee^{-1}} (L)_{e^{-1}v} \]

and

\[= \delta_{uv} \sum_{\alpha(e) = u} w(e) J_d \rho(\alpha(e)) w(e^{-1}) \rho(\alpha(e^{-1})) \]

Furthermore, since

\[\lambda \sum_{\alpha(e) = u} h_e I_d = \delta_{uv} \Gamma_u I_d. \]

Thus,

\[^tKJ \rho L = X(D \Gamma \otimes I_d), \]

where

\[D \Gamma = \begin{bmatrix} \Gamma_{v_1} & 0 \\ \vdots & \ddots \\ 0 & \Gamma_{v_n} \end{bmatrix}. \]

Therefore, it follows that

\[ \det(I_{2md} - u \sum_{g \in V} \rho(g) \otimes U_g) \]

\[= (1 + u^2)^{m-n} \det((1 + u^2)I_{nd} + u X_d \sum_{g \in V} H_g \otimes \rho(g) + i u^2 X_d(D \Gamma \otimes I_d)). \]

Substituting \(u = 1\), we obtain

\[ \det(I_{2md} - \sum_{g \in V} \rho(g) \otimes U_g) \]

\[= 2^{m-n} \det(2I_{nd} + X_d \sum_{g \in V} H_g \otimes \rho(g) + i X_d(D \Gamma \otimes I_d)) \]

\[= 2^{m-n} \det(X_d) \det(2X_d^{-1} + \sum_{g \in V} H_g \otimes \rho(g) + i D \Gamma \otimes I_d). \]

Then we have

\[ \det(X_d) = \det(X \otimes I_d) = (\det(X))^d = \frac{2^{nd}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i \Gamma_u)^d}. \]

Furthermore, since

\[X_d^{-1} = X^{-1} \otimes I_d, \]

we have

\[(2X_d^{-1} + i D \Gamma \otimes I_d)_{uu} = (2 \frac{H_{uu} - \lambda - i \Gamma_u}{2} + i \Gamma_u) \otimes I_d \]

\[= (H_{uu} - \lambda) \otimes I_d. \]

That is,

\[2X_d^{-1} + i D \Gamma \otimes I_d = -\lambda I_{nd} + \text{diag}(H) \otimes I_d. \]

Therefore, it follows that

\[ \det(I_{2md} - \sum_{g \in V} \rho(g) \otimes U_g) \]

\[= \frac{2^{nd}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i \Gamma_u)^d} \det(-\lambda I_{nd} + \sum_{g \in V} H_g \otimes \rho(g) + \text{diag}(H) \otimes I_d) \]

\[= \frac{(-1)^{nd} 2^{nd}}{\prod_{u \in V(G)} (H_{uu} - \lambda - i \Gamma_u)^d} \det(\lambda I_{nd} - \sum_{g \in V} \rho(g) \otimes H_g - I_d \otimes \text{diag}(H)). \]
By Theorems 6 and 7 the following result holds.

**Corollary 1** Let $G$ be a connected graph with $m$ edges, $\Gamma$ be a finite group and $\alpha : D(G) \rightarrow \Gamma$ be an ordinary voltage assignment. Then

$$\det(I_{2mp} - \tilde{U}(\lambda)) = \prod_\rho Z_H(G, \lambda, \rho, \alpha)^{-\deg \rho},$$

where $\rho$ runs over all inequivalent irreducible representations of $\Gamma$ and $p = |\Gamma|$.

### 7 Example

We give an example. Let $G = K_3$ be the complete graph with three vertices 1, 2, and 3, and six arcs $e_1, e_2, e_3, e^{-1}_1, e^{-1}_2, e^{-1}_3$, where $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_1)$. Furthermore, let $H = \begin{bmatrix} a & b e^{2i\alpha} & b e^{2i\alpha} \\ b e^{-2i\alpha} & a & b e^{2i\alpha} \\ b e^{-2i\alpha} & b e^{2i\alpha} & a \end{bmatrix}$, where $a > 0$, $b > 0$ and $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have

$$x_1 = x_2 = x_3 = \frac{2}{a - \lambda - 2ib}.$$  

Set $x = \frac{2}{a - \lambda - 2ib}$. Considering $U(\lambda)$ under the order $e_1, e_2, e_3, e^{-1}_1, e^{-1}_2, e^{-1}_3$, we have

$$U(\lambda) = \begin{bmatrix} -xe^{2i\alpha} & -xe^{2i\alpha} & -xe^{2i\alpha} \\ -xe^{2i\alpha} & -xe^{2i\alpha} & -xe^{2i\alpha} \\ -xe^{2i\alpha} & -xe^{2i\alpha} & -xe^{2i\alpha} \\ i - xb & -xb & -xb \\ -xb & i - xb & -xb \\ -xb & -xb & i - xb \end{bmatrix}.$$ 

By Theorem 4, we have

$$\det(I_3 - U(\lambda)) = \frac{2^3(-1)^3}{(a - \lambda - 2ib)^3} \det(\lambda I_3 - H)$$

$$= \frac{1}{(a - \lambda - 2ib)^3} \begin{bmatrix} \lambda - a & -be^{2i\alpha} & -be^{2i\alpha} \\ -be^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\ -be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{bmatrix}$$

$$= \frac{1}{(a - \lambda - 2ib)^3} \left\{ (\lambda - a)^3 - 3b^2(\lambda - a) - b^3(e^{2i\alpha} + e^{-2i\alpha}) \right\}$$

$$= \frac{1}{(a - \lambda - 2ib)^3} \left\{ (\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3 \cos 2\alpha \right\}.$$ 

Next, let $\Gamma = Z_3 = \{1, \tau, \tau^2\} (\tau^3 = 1)$ be the cyclic group of order 3, and let $\lambda : D(K_3) \rightarrow Z_3$ be the ordinary voltage assignment such that $\alpha(e_1) = \tau$, $\alpha(e^{-1}_2) = \tau^2$ and $\alpha(e_2) = \alpha(e_3) = \alpha(e^{-1}_3) = 1$. Then the $Z_3$-coverng $K_3^\alpha$ of $K_3$ is the cycle graph of length 9.

The characters of $Z_3$ are given as follows: $\chi_i(\tau^j) = (\xi^i)^j$, $0 \leq i, j \leq 2$, where $\xi = \frac{-1 + \sqrt{3}i}{2}$. Then we have

$$H_1 = \begin{bmatrix} 0 & 0 & b e^{2i\alpha} \\ b e^{-2i\alpha} & b e^{2i\alpha} & 0 \end{bmatrix}, \quad H_\tau = \begin{bmatrix} 0 & b e^{2i\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_{\tau^2} = \begin{bmatrix} 0 & b e^{-2i\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Now, by Theorem 7,

$$\zeta_H(K_3, \lambda, \chi_1, \alpha)^{-1} = \frac{2^3(-1)^3}{(a - \lambda - 2ib)^3} \det(\lambda I_3 - \sum_{j=0}^2 \chi_1(\tau^j)H_{\tau^j} - \text{diag}(H))$$

$$= \frac{-8}{(a - \lambda - 2ib)^3} \left[ \begin{array}{ccc} \lambda - a & -b\xi e^{2i\alpha} & -be^{2i\alpha} \\
-b\xi^2 e^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\
-be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{array} \right]$$

$$= \frac{-8}{(a - \lambda - 2ib)^3} \{(\lambda - a)^3 - 3b^2(\lambda - a) - b^3(\xi e^{2i\alpha} + \xi e^{-2i\alpha})\}$$

Similarly, we have

$$\zeta_H(K_3, \lambda, \chi_2, \alpha)^{-1} = \frac{2^3(-1)^3}{(a - \lambda - 2ib)^3} \det(\lambda I_3 - \sum_{j=0}^2 \chi_2(\tau^j)H_{\tau^j} - \text{diag}(H))$$

$$= \frac{-8}{(a - \lambda - 2ib)^3} \left[ \begin{array}{ccc} \lambda - a & -b\xi e^{2i\alpha} & -be^{2i\alpha} \\
-b\xi^2 e^{-2i\alpha} & \lambda - a & -be^{2i\alpha} \\
-be^{-2i\alpha} & -be^{-2i\alpha} & \lambda - a \end{array} \right]$$

$$= \frac{-8}{(a - \lambda - 2ib)^3} \{(\lambda - a)^3 - 3b^2(\lambda - a) - b^3(\xi e^{2i\alpha} + \xi e^{-2i\alpha})\}$$

By Corollary 1, it follows that

$$\det(I_{18} - \tilde{U}(\lambda)) = \det(I_6 - U(\lambda)) \zeta_H(K_3, \lambda, \chi_1, \alpha)^{-1} \zeta_S(K_H, \lambda, \chi_2, \alpha)^{-1}$$

$$= \frac{-512}{(a - \lambda - 2ib)^3} \{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3 \cos 2\alpha\}$$

$$\times \{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3 \cos (\alpha + \pi/3)\} \{(\lambda - a)^3 - 3b^2(\lambda - a) - 2b^3 \cos 2(\alpha + 2\pi/3)\}.$$

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