Abstract
In this paper we study the existence and uniqueness of Nash equilibria (solution to competition-wise problems, with several controls trying to reach possibly different goals) associated to linear partial differential equations and show that, in some cases, they are also the solution of suitable single-objective optimization problems (i.e. cooperative-wise problems, where all the controls cooperate to reach a common goal). We use cost functions associated with a particular linear parabolic partial differential equation and distributed controls, but the results are also valid for more general linear differential equations (including elliptic and hyperbolic cases) and controls (e.g. boundary controls, initial value controls,...).

Keywords Nash equilibria · Cooperative controls · Noncooperative controls · Noncooperative game · Linear partial differential equations · Optimal control · Adjoint system · Multiobjective optimization · Single-objective optimization

1 Introduction
Nash equilibria are solutions of a noncooperative multiobjective optimization strategy first proposed by Nash (see [1]). Since it originated in game theory and economics, the notion of player is often used. For an optimization problem with $N$ objectives or functionals $J_i$ to minimize, a Nash strategy consists in having $N$ players or controls $v_i$, each optimizing his own criterion. However, each player has to optimize his criterion given that all the other criteria are fixed by the rest of the players. When no player can further improve his criterion, it means that the system has reached a Nash equilibrium state.
There are many problems involving Nash equilibria governed by partial differential equations. As explained in [2], they arise in very many fields of Environment and Engineering. The authors of the mentioned paper give, as an example, a case where the solution to the state equation corresponds to the concentration of chemicals in a lake and each control corresponds to a local agent or local plant (each having its own interest or functional to minimize, sometimes over different domains).

Of course, there are other strategies for multiobjective optimization, such as the Pareto cooperative strategy [3], the Stackelberg hierarchical strategy [4] or the Stackelberg–Nash strategy [2]. For the case of the Stackelberg–Nash strategy the problems under consideration may also include requirements of approximate or exact controllability (see [2, 5, 6]).

To the best of our knowledge, [7] and [8] are the first articles dealing with the theoretical and numerical study of Nash equilibria for differential games associated to partial differential equations. Based on [7] and [9], we deal here with a general linear case with \( N \) cost functions and controllers and show how, in some cases, the Nash equilibria (solution to differential games associated to multiobjective optimization problems with several noncooperative controllers), are also the solution of single-objective optimization problems (where all the controllers cooperate to reach a common goal). We use cost functions associated with linear parabolic partial differential equations and distributed controls, but other kinds of linear differential equations (e.g. elliptic, hyperbolic,..) and controls (e.g. boundary controls, initial value controls,...) can be also used, using the same technique (see, for instance, [10, 11]). A preliminary preprint with some of the results studied here can be seen in [12].

The fact that a noncooperative game (i.e a competition-wise problem) can be seen as a (cooperative) single-objective optimization problem (i.e. a noncompetition-wise problem) is very interesting, not only because of the curious noncooperative-cooperative equivalence, but also because of the huge amount of software to compute solutions and literature written about the latter kind of problems, that could be used in the framework of, apparently, a different type of problems.

The solutions of classical optimal control problems governed by parabolic partial differential equation are solutions of optimality systems typically given by a forward parabolic system (the state system) coupled with a backward parabolic system (the adjoint system). In the case of Nash equilibria with \( N \) players acting on similar parabolic problems, there are also associated optimality systems which are now typically given by a forward parabolic system (the state system) coupled with \( N \) backward parabolic systems (the adjoint system). Optimality systems composed of more than one forward and/or backward coupled parabolic differential equations arise also in other contexts as, for instance, insensitizing control of parabolic problems (see [13–15]).

This paper deals with the case of deterministic differential equations. There are not many works in the literature regarding Nash equilibria associated to the multiobjective control of problems governed by stochastic differential equations (with noise generating disturbances in the cost functionals). An example is [16], where the authors study Nash equilibria in some stochastic differential games.

In Sect. 2 we formulate the problem and give an optimality system providing a necessary and sufficient condition for the Nash equilibria. The existence and uniqueness of Nash equilibria is studied in Sect. 3. In Sect. 4 we show the equivalence, in
some cases, between the noncooperative multiobjective differential games defining
the Nash equilibria and suitable (cooperative) single-objective optimization problems.
Finally, in Sect. 5 we give a summary of the major results of the paper.

2 Formulation of the Problem

Let us consider $T > 0, \Omega \subset \mathbb{R}^d$ a bounded and smooth open set with $d \in \{1, 2, 3\}$, and
two subsets $\Gamma_1, \Gamma_2 \subset \partial \Omega$ such that $\partial \Omega = \Gamma_1 \cup \Gamma_2$. We define $Q = \Omega \times (0, T), \Sigma_1 = \Gamma_1 \times (0, T), \Sigma_2 = \Gamma_2 \times (0, T)$ and the control Hilbert spaces $U_i = L^2(\omega_i \times (0, T))$ and $U = U_1 \times \cdots \times U_N$, where $N \in \mathbb{N}$ and $\omega_i \subset \Omega$, for $i \in \{1, \ldots, N\}$. Finally, we consider the functionals $J_i : U \to \mathbb{R}$, with $i \in \{1, \ldots, N\}$, given by

$$J_i(v_1, \ldots, v_N) = \frac{\alpha_i}{2} \int_{\omega_i \times (0, T)} |v_i|^2 dx dt + \frac{1}{2} \int_Q \rho_i(x) |y - y_i,d|^2 dx dt + \frac{1}{2} \int_{\Omega} \eta_i(x) |y(T) - y_i,T|^2 dx,$$

for every $v = (v_1, \ldots, v_N) \in U$, where $\alpha_i > 0, \rho_i, \eta_i \in L^\infty(\Omega)$ such that $\rho_i, \eta_i \geq 0$, the function $y = y(v)$ is defined as the solution of

$$\begin{cases}
\frac{\partial y}{\partial t} - \Delta y = f + \sum_{i=1}^{N} v_i \chi_{\omega_i} & \text{in } Q, \\
y(0) = y_0 & \text{in } \Omega, \\
y = g_1 & \text{on } \Sigma_1, \\
\frac{\partial y}{\partial n} = g_2 & \text{on } \Sigma_2,
\end{cases}$$

with $f, g_1, g_2, y_0, y_i,d$ and $y_i,T$ being smooth enough functions and $\chi_{\omega} : \Omega \to \mathbb{R}$ the characteristic function (with values 1 in $\omega$ and 0 in $\Omega \setminus \omega$) for any $\omega \subset \Omega$.

This generalizes the typical examples in the literature of 2 controls (instead of $N$), $\rho_i = k_i \chi_{\omega_d,i}$ and $\eta_i = l_i \chi_{\omega_T,i}$, where $k_i, l_i > 0, \omega_d,i, \omega_T,i \subset \Omega$. A special case is when $\omega_{T1} \cap \omega_{T2} \neq \emptyset$ and/or $\omega_{d1} \cap \omega_{d2} \neq \emptyset$. This case is a competition-wise problem, with each control (or player) trying to reach (possibly) different goals over a common domain. In some sense this is the case where the behavior of the solution $y$ associated to a Nash equilibrium is most difficult to forecast.

Remark 1 Most of the results to follow are also valid for more general linear operators such as, for instance,

$$\mathcal{L} \varphi = \frac{\partial \varphi}{\partial t} - \nabla \cdot (A(x)\nabla \varphi) + V \cdot \nabla \varphi + c(x) \varphi.$$

The technique is also valid for different type of controls such as, for instance, boundary or initial controls.

□
Now, given \( i \in \{1, \ldots, N\} \), for every \((w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_{i-1} \times \mathcal{U}_{i+1} \times \cdots \times \mathcal{U}_N\) we consider the optimal control problem

\[
(CP_i) \quad \begin{cases} 
\text{Find } u_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N) \in \mathcal{U}_i, \text{ such that } \\
J_i(w_1, \ldots, w_{i-1}, u_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N), w_{i+1}, \ldots, w_N) \\
\leq J_i(w_1, \ldots, w_{i-1}, v_i, w_{i+1}, \ldots, w_N), \forall v_i \in \mathcal{U}_i.
\end{cases}
\]

The (unique) solution \( u_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N) \) of problem \((CP_i)\) is characterized by

\[
\frac{\partial J_i}{\partial v_i}(w_1, \ldots, w_{i-1}, u_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N), w_{i+1}, \ldots, w_N) = 0.
\]

Therefore, a Nash equilibrium is a \( N \)-tuple \((u_1, \ldots, u_N) \in \mathcal{U}\) such that \( u_i = u_i(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N) \) for all \( i \in \{1, \ldots, N\} \), i.e. \((u_1, \ldots, u_N)\) is a solution of the coupled (optimality) system:

\[
\frac{\partial J_i}{\partial v_i}(u_1, \ldots, u_N) = 0, \quad \forall i \in \{1, \ldots, N\}.
\]

In the linear case studied here, this system of equations is a necessary and sufficient condition for \( u \) to be a Nash equilibrium. In general this system is only a necessary condition, although in some nonlinear cases (see, e.g. [17]), the functionals are convex and system (2) is also a sufficient condition.

Following [7] it is easy to prove that, if \( i \in \{1, \ldots, N\} \),

\[
\frac{\partial J_i}{\partial v_i}(v) = \alpha_i v_i + p_i(v)\chi_{\omega_i} \in \mathcal{U}_i,
\]

where for any \( v = (v_1, \ldots, v_N) \in \mathcal{U} \) the function \( p_i = p_i(v) \) is the solution of the adjoint system

\[
\begin{cases}
-\frac{\partial p_i}{\partial t} - \Delta p_i = \rho_i(y - y_i,d) & \text{in } Q, \\
p_i(T) = \eta_i(y(T) - y_i,T) & \text{in } \Omega, \\
p_i = 0 & \text{on } \Sigma_1, \\
\frac{\partial p_i}{\partial n} = 0 & \text{on } \Sigma_2
\end{cases}
\]

and \( y = y(v) \) is the solution of (1).
Therefore, system (2) is equivalent to the (optimality) system

\[
\begin{align*}
\left\{ \begin{array}{l}
u_i = - \frac{1}{\alpha_i} p_i \chi_{\omega_i}, \quad i \in \{1, \ldots, N\} \\
\frac{\partial y}{\partial t} - \Delta y = f + \sum_{i=1}^{N} u_i \chi_{\omega_i} \quad \text{in } Q, \\
y(0) = y_0 \quad \text{in } \Omega, \\
y = g_1 \quad \text{on } \Sigma_1. \\
\frac{\partial y}{\partial n} = g_2 \quad \text{on } \Sigma_2;
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
- \frac{\partial p_i}{\partial t} - \Delta p_i = \rho_i(y - y_i, d) \quad \text{in } Q, \\
p_i(x, T) = \eta_i(y(T) - y_i, T) \quad \text{in } \Omega, \\
p_i = 0 \quad \text{on } \Sigma_1, \\
\frac{\partial p_i}{\partial n} = 0 \quad \text{on } \Sigma_2.
\end{array} \right.
\]

3 Existence and Uniqueness of Solution of Nash Equilibria

Let us consider \( \alpha = (\alpha_1, \ldots, \alpha_N) \). It is obvious that

\[
v = (v_1, \ldots, v_N) \longrightarrow \left( \frac{\partial J_1}{\partial v_1}(v_1, \ldots, v_N), \ldots, \frac{\partial J_N}{\partial v_N}(v_1, \ldots, v_N) \right) \in \mathcal{U}
\]

is an affine mapping of \( \mathcal{U} \). Therefore, there exist a linear continuous mapping \( A_\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{U}) \) and a vector \( b \in \mathcal{U} \) such that

\[
\left( \frac{\partial J_1}{\partial v_1}(v_1, \ldots, v_N), \ldots, \frac{\partial J_N}{\partial v_N}(v_1, \ldots, v_N) \right) = A_\alpha v - b.
\]

Let us identify mapping \( A_\alpha \): For every \( v = (v_1, \ldots, v_N) \in \mathcal{U} \), the linear part of the affine mapping in relation (3) is defined by

\[
A_\alpha v = (\alpha_1 v_1 + \tilde{p}_1 \chi_{\omega_1}, \ldots, \alpha_N v_N + \tilde{p}_N \chi_{\omega_N}),
\]

where \( \tilde{p}_i = \tilde{p}_i(v), i \in \{1, \ldots, N\} \), is the solution of

\[
\begin{align*}
\left\{ \begin{array}{l}
- \frac{\partial \tilde{p}_i}{\partial t} - \Delta \tilde{p}_i = \rho_i \tilde{y} \quad \text{in } Q, \\
\tilde{p}_i(x, T) = \eta_i \tilde{y}(T) \quad \text{in } \Omega, \\
\tilde{p}_i = 0 \quad \text{on } \Sigma_1, \\
\frac{\partial \tilde{p}_i}{\partial n} = 0 \quad \text{on } \Sigma_2.
\end{array} \right.
\]

and \( \tilde{y} = \tilde{y}(v) \) is the solution of

\[
\begin{cases}
\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = \sum_{i=1}^{N} v_i \chi_{\omega_i} & \text{in } Q, \\
\tilde{y}(0) = 0 & \text{in } \Omega, \\
\tilde{y} = 0 & \text{on } \Sigma_1, \\
\frac{\partial \tilde{y}}{\partial n} = 0 & \text{on } \Sigma_2.
\end{cases}
\]

**Proposition 1**  
Mapping \( A_{\alpha} : \mathcal{U} \to \mathcal{U} \) is linear and continuous. Furthermore, if \( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} \) is sufficiently large, it is also \( \mathcal{U} \)-elliptic, i.e., there exists \( C > 0 \) such that

\[
(A_{\alpha} v, v) \geq C ||v||^2,
\]

where \((\cdot, \cdot)\) and \( || \cdot || \) represent the canonical scalar product and norm of the Hilbert space \( \mathcal{U} \), respectively.

**Proof**  
It is obvious that \( A_{\alpha} \) is a linear mapping and it is easy to show that it is continuous (see [18]).

Let us consider \( v = (v_1, \ldots, v_N) \in \mathcal{U} \) and \( w = (w_1, \ldots, w_N) \in \mathcal{U} \). We have then

\[
(A_{\alpha} v, w) = \left( (\alpha_1 v_1 + \tilde{p}_1 \chi_{\omega_1}, \ldots, \alpha_N v_N + \tilde{p}_N \chi_{\omega_N}), (w_1, \ldots, w_N) \right)
\]

\[
= \sum_{i=1}^{N} \int_{\omega_i \times (0, T)} (\alpha_i v_i + \tilde{p}_i(v)) w_i \, dx \, dt.
\]

Let us focus on the term \( \int_{\omega_i \times (0, T)} \tilde{p}_i(v)w_i \, dx \, dt \), following the approach in [7] and [19]. We have

\[
\int_{\omega_i \times (0, T)} \tilde{p}_i(v)w_i \, dx \, dt
\]

\[
= \int_Q \tilde{p}_i(v) \left( \frac{\partial}{\partial t} \tilde{y}(0, \ldots, w_i, \ldots, 0) - \Delta \tilde{y}(0, \ldots, w_i, \ldots, 0) \right) \, dx \, dt
\]

\[
= \int_Q \left( - \frac{\partial}{\partial t} \tilde{p}_i(v) - \Delta \tilde{p}_i(v) \right) \tilde{y}(0, \ldots, w_i, \ldots, 0) \, dx \, dt
\]

\[
+ \int_{\Omega} \eta_i \tilde{y}(T; v) \tilde{y}(T; 0, \ldots, w_i, \ldots, 0) \, dx
\]

\[
= \int_Q \rho_i \tilde{y}(v) \tilde{y}(0, \ldots, w_i, \ldots, 0) \, dx \, dt + \int_{\Omega} \eta_i \tilde{y}(T; v) \tilde{y}(T; 0, \ldots, w_i, \ldots, 0) \, dx.
\]
Then,

\[
(\mathcal{A}_\alpha v, w) = \sum_{i=1}^{N} \left( \alpha_i \int_{Q} \rho_i \tilde{y}(v) \tilde{y}(0, \ldots, w_i, \ldots, 0) dxdt + \int_{Q} \eta_i \tilde{y}(T; v) \tilde{y}(T; 0, \ldots, w_i, \ldots, 0) dxdt \right).
\]

Since the mapping \( v \mapsto \tilde{y}(v) \) is linear and continuous from \( \mathcal{U} \) to \( C([0, T]; L^2(\Omega)) \) (see, e.g., [18]), it is easy to prove there exists a constant \( c > 0 \) such that

\[
||\tilde{y}(v)||_{L^2(Q)} + ||\tilde{y}(T; v)||_{L^2(\Omega)} \leq c||v||.
\]

Therefore,

\[
(\mathcal{A}_\alpha (v, v) \geq \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} ||v||^2 - \sum_{i=1}^{N} c^2 (||\rho_i||_{L^\infty(\Omega)} + ||\eta_i||_{L^\infty(\Omega)}) ||v|| ||v||_{L^4} \geq C||v||^2,
\]

with

\[
C = \left( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} - \sum_{i=1}^{N} c^2 (||\rho_i||_{L^\infty(\Omega)} + ||\eta_i||_{L^\infty(\Omega)}) \right).
\]

Notice that \( C > 0 \) if \( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} > \sum_{i=1}^{N} c^2 (||\rho_i||_{L^\infty(\Omega)} + ||\eta_i||_{L^\infty(\Omega)}) \), which proves that \( \mathcal{A}_\alpha \) is \( \mathcal{U} \)-elliptic in that case and completes the proof. \( \square \)

**Remark 2** Let us consider \( \alpha = (\alpha_1, \ldots, \alpha_N) \). Operator \( \mathcal{A}_\alpha : \mathcal{U} \to \mathcal{U} \) can be rewritten as

\[
\mathcal{D}_\alpha v + \mathcal{B} v,
\]

where \( \mathcal{D}_\alpha : \mathcal{U} \to \mathcal{U} \) is the homeomorphism defined by

\[
\mathcal{D}_\alpha v = (\alpha_1 v_1, \ldots, \alpha_N v_N)
\]

and

\[
\mathcal{B} v = (\tilde{p}_1 \chi_{\omega_1}, \ldots, \tilde{p}_N \chi_{\omega_N}),
\]

with

\[
(\mathcal{B} v, w) = \sum_{i=1}^{N} \left( \int_{Q} \rho_i \tilde{y}(v) \tilde{y}(0, \ldots, w_i, \ldots, 0) dxdt + \int_{\Omega} \eta_i \tilde{y}(T; v) \tilde{y}(T; 0, \ldots, w_i, \ldots, 0) dx \right).
\]
If $N = 1$ (classical single-objective optimization problem), then operator $B$ is non-negative. But for $N \geq 2$ (which is the case of the multiobjective problem of interest in this work) this property does not hold in general. For instance, if $N = 2$ (just for the sake of simplicity) and $\eta_1 = \eta_2 = 0$, then

$$(Bv, v) = \int_Q \tilde{y}(v_1, v_2) \left( \rho_1 \tilde{y}(v_1, 0) + \rho_2 \tilde{y}(0, v_2) \right) dx dt.$$ 

Let us consider $v_1$ positive and $v_2$ negative. Then it is well-known that $\tilde{y}(v_1, 0) > 0$ and $\tilde{y}(0, v_2) < 0$ in $Q$. Let us suppose $\tilde{y}(v_1, v_2) = \tilde{y}(v_1, 0) + \tilde{y}(0, v_2) > 0$ (if $\tilde{y}(v_1, v_2) = 0$ we can change $v_1$ or $v_2$; and the case $\tilde{y}(v_1, v_2) < 0$ can be done similarly).

Now, if $\rho_1 = 1$ and

$$\rho_2 > \frac{\int_Q \tilde{y}(v_1, v_2) \tilde{y}(v_1, 0) dx dt}{-\int_Q \tilde{y}(v_1, v_2) \tilde{y}(0, v_2) dx dt},$$

then $(Bv, v) < 0$. □

Let us identify $b$: The constant part of the affine mapping (3) is given by the function $b \in U$ defined by $b = -(\tilde{p}_1 \chi_{\omega_1}, \ldots, \tilde{p}_N \chi_{\omega_N})$, where $\tilde{p}_i, i \in \{1, \ldots, N\}$, is the solution of

$$\begin{cases}
-\frac{\partial \tilde{p}_i}{\partial t} - \Delta \tilde{p}_i = \rho_i(\tilde{y} - y_i, d) & \text{in } Q, \\
\tilde{p}_i(T) = \eta_i(\tilde{y} - y_i, T) & \text{in } \Omega, \\
\tilde{p}_i = 0 & \text{on } \Sigma_1, \\
\frac{\partial \tilde{p}_i}{\partial n} = 0 & \text{on } \Sigma_2,
\end{cases}$$

and $\tilde{y}$ is the solution of

$$\begin{cases}
\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = f & \text{in } Q, \\
\tilde{y}(0) = y_0 & \text{in } \Omega, \\
\tilde{y} = g_1 & \text{on } \Sigma_1, \\
\frac{\partial \tilde{y}}{\partial n} = g_2 & \text{on } \Sigma_2.
\end{cases} \tag{4}$$

Notice that, for any $v \in \mathcal{U}$, $y(v) = \tilde{y}(v) + \bar{y}$ and $p_i(v) = \tilde{p}_i(v) + \bar{p}_i$.

**Theorem 1** If $\min_{i \in \{1, \ldots, N\}} \{\alpha_i\}$ is sufficiently large, there exist a unique Nash equilibrium of the problem defined in Sect. 2.

**Proof** As showed above, the Nash equilibria are characterized by the solutions of (2), which are also characterized by the solutions $u \in \mathcal{U}$ of

$$a_\alpha(u, v) = L(v), \ \forall \ v \in \mathcal{U},$$

$\square$ Springer
where \( a_\alpha(\cdot, \cdot) : \mathcal{U} \times \mathcal{U} \to \mathbb{R} \) is defined by

\[
a_\alpha(v, w) = (A_\alpha v, w) \quad \forall \ v, w \in \mathcal{U},
\]

and \( L : \mathcal{U} \to \mathbb{R} \) by

\[
L(v) = (b, v), \quad \forall \ v \in \mathcal{U}.
\]

Proposition 1 proves that mapping \( a_\alpha(\cdot, \cdot) \) is bilinear, continuous and, if \( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} \) is sufficiently large, it is also \( \mathcal{U} \)-elliptic. Furthermore, mapping \( L \) is (obviously) linear and continuous. Thus, by the (well-known) Lax-Milgram Theorem, system (2) has a unique solution or, equivalently, there exists a unique Nash equilibrium of the problem defined in Sect. 2, if \( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} \) is sufficiently large.

The discretization of the problem considered above and the development of suitable algorithms to get a numerical solution approximating the Nash equilibria can follow the approaches in [7] and [19]. Due to the properties proven in Proposition 1, the conjugate gradient algorithm is well-suited for the numerical resolution. All the details about how to do this can be seen in [7].

We have seen that, if \( \alpha = (\alpha_1, \ldots, \alpha_N) \in (0, \infty)^N \) and \( \min_{i \in \{1, \ldots, N\}} \{\alpha_i\} \) is sufficiently large, equation (2), or equivalently \( A_\alpha u = b \), has a unique solution \( u \in \mathcal{U} \). The following mathematical question arises: Is there existence and uniqueness of solution for some \( \alpha \in \mathbb{R}^N \) not satisfying those requirements? The answer is yes, as we can see in the following theorem.

**Theorem 2** Let us take \( \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) \in \Phi = \{ (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N : \alpha_i \neq 0 \ \forall \ i \in \{1, \ldots, N\} \} \) and \( \alpha = \gamma \tilde{\alpha} \), with \( \gamma > 0 \). Then, the equation \( A_\alpha u = b \) has a unique solution for every \( \gamma > 0 \) except, at most, a sequence of values of \( \gamma \) converging to 0.

**Proof** Following Remarks 2 and 4, if \( \alpha = \gamma \tilde{\alpha} \), equation (2), or equivalently \( A_\alpha u = b \), can be rewritten as

\[
\begin{align*}
u + (D_\alpha)^{-1} B u &= (D_\alpha)^{-1} b, 
\end{align*}
\]

where \( (D_\alpha)^{-1} : \mathcal{U} \to \mathcal{U} \) is the homeomorphism defined by

\[
(D_\alpha)^{-1} v = (\alpha_1^{-1} v_1, \ldots, \alpha_N^{-1} v_N)
\]

and the operator \( (D_\alpha)^{-1} B : \mathcal{U} \to \mathcal{U} \) is compact (this property can be easily proven by applying the Aubin–Lions Compactness Lemma; see, for instance, [20]).

Then, according to the Fredholm’s alternative (see Theorem 6.6 of [21]), equation (5) has a unique solution for any \( b \in \mathcal{U} \) (i.e. \(-1 \) belongs to the resolvent set of \( (D_\alpha)^{-1} B \) or, equivalently, it does not belong to its spectrum) if, and only if, \( v = 0 \) is the unique solution to the homogeneous equation

\[
v + (D_\alpha)^{-1} B v = 0 \ (\in \mathcal{U}),
\]
i.e. $-1$ is not an eigenvalue of $(D_\alpha)^{-1} B$. Now, if $\sigma$ is the spectrum of $(D_\alpha)^{-1} B$, then $\sigma \setminus \{0\}$ is, at most, a sequence converging to 0 (see Theorem 6.8 of [21]), which proves the theorem.

**Remark 3** The condition "for any $b \in U$" used in the proof of Theorem 2 is stronger than necessary, since $b$ cannot be any arbitrary element of $U$ (due to the smoothness property of the parabolic equations). This implies that the equation $A_\alpha u = b$ could have a unique solution for more elements $\alpha \in U$ (not only for those proven in the proof of the mentioned theorem).

### 4 Equivalent Single-Objective Control Problems

In this section we will show that, in some cases, the solution of noncooperative differential games defining Nash equilibria, are the solution of suitable optimization problems, where all the controls cooperate to minimize a suitable single-objective cost function.

Let us consider the subfamily of problems defined in Sect. 2, for which there exists $\rho \in L^\infty(Q)$ and $\eta \in L^\infty(\Omega)$ such that $\rho_i = \rho$ and $\eta_i = \eta$ in $Q$, for all $i \in \{1, \ldots, N\}$. Therefore, in this case the functional $J_i$, with $i \in \{1, \ldots, N\}$, is given by

\[
J_i(v_1, \ldots, v_N) = \frac{\alpha_i}{2} \int_{\omega_i \times (0,T)} |v_i|^2 \mathrm{d}x \mathrm{d}t + \frac{1}{2} \int_Q \rho |y - y_i,d|^2 \mathrm{d}x \mathrm{d}t + \frac{1}{2} \int_{\Omega} \eta |y(T) - y_i,T|^2 \mathrm{d}x,
\]

with $y = y(v)$ being the solution of (1).

As in the general case studied in Sect. 2, a Nash equilibrium is a $N$-tuple $(u_1, \ldots, u_N) \in U = U_1 \times \cdots \times U_N$ solution of (2), where

\[
\frac{\partial J_i}{\partial v_i}(v) = \alpha_i v_i + p_i(v) \chi_{\omega_i} \in U_i, \ \forall i \in \{1, \ldots, N\}
\]

for any $v = (v_1, \ldots, v_N) \in U$ and $p_i = p_i(v)$ is now the solution of

\[
\begin{align*}
-\frac{\partial p_i}{\partial t} - \Delta p_i &= \rho (y - y_i,d) \quad \text{in } Q, \\
p_i(T) &= \eta (y(x, T) - y_i,T) \quad \text{in } \Omega, \\
p_i &= 0 \quad \text{on } \Sigma_1, \\
\frac{\partial p_i}{\partial n} &= 0 \quad \text{on } \Sigma_2.
\end{align*}
\]
Therefore, system (2) is equivalent in this case to

\[
\begin{align*}
\frac{\partial y}{\partial t} - \Delta y &= f + \sum_{i=1}^{N} u_i \chi_{\omega_i} \quad \text{in } Q, \\
y(0) &= y_0 \quad \text{in } \Omega, \\
y &= g_1 \quad \text{on } \Sigma_1, \\
\frac{\partial y}{\partial n} &= g_2 \quad \text{on } \Sigma_2; \\
\frac{\partial p_i}{\partial t} - \Delta p_i &= \rho (y - y_{i,d}) \quad \text{in } Q, \\
p_i(T) &= \eta(y(T) - y_{i,T}) \quad \text{in } \Omega, \quad i \in \{1, \ldots, N\} \\
p_i &= 0 \quad \text{on } \Sigma_1, \\
\frac{\partial p_i}{\partial n} &= 0 \quad \text{on } \Sigma_2.
\end{align*}
\]

Again

\[
\left( \frac{\partial J_1}{\partial v_1}(v_1, \ldots, v_N), \ldots, \frac{\partial J_N}{\partial v_N}(v_1, \ldots, v_N) \right) = A_\alpha v - b.
\]

and now

\[
A_\alpha v = (\alpha_1 v_1 + \tilde{p} \chi_{\omega_1}, \ldots, \alpha_N v_N + \tilde{p} \chi_{\omega_N}),
\]

where \( \tilde{p} = \tilde{p}(v) \) is the solution of

\[
\begin{align*}
-\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} &= \rho \tilde{y} \quad \text{in } Q, \\
\tilde{p}(x, T) &= \eta \tilde{y}(T) \quad \text{in } \Omega, \\
\tilde{p} &= 0 \quad \text{on } \Sigma_1, \\
\frac{\partial \tilde{p}}{\partial n} &= 0 \quad \text{on } \Sigma_2.
\end{align*}
\]

**Proposition 2** For the family of problems studied in Sect. 4 mapping \( A_\alpha : \mathcal{U} \rightarrow \mathcal{U} \) is linear, continuous self-adjoint and \( \mathcal{U} \)-elliptic.

**Proof** Following the proof of Proposition 1, \( A_\alpha \) is a linear and continuous mapping. Furthermore, given \( v = (v_1, \ldots, v_N) \in \mathcal{U} \) and \( w = (w_1, \ldots, w_N) \in \mathcal{U}, \)

\[
(A_\alpha v, w) = \sum_{i=1}^{N} \left( \alpha_i \int_{\omega_i \times (0,T)} v_i w_i \, dx \, dt + \int_{Q} \rho_i \tilde{y}(v)\tilde{y}(0, \ldots, w_i, \ldots, 0) \, dx \, dt \right.
\]

\[
\left. + \int_{\Omega} \eta \tilde{y}(T; v)\tilde{y}(T; 0, \ldots, w_i, \ldots, 0) \, dx \right).
\]
\[ \sum_{i=1}^{N} \alpha_i \int_{\omega_i \times (0,T)} v_i w_i \, dx \, dt + \int_{\Omega} \rho \tilde{y}(v) \tilde{y}(w) \, dx \, dt + \int_{\Omega} \eta \tilde{y}(T; v) \tilde{y}(T; w) \, dx = (v, A_\alpha w) \]

and

\[ (A_\alpha v, v) \geq \min_{i \in \{1, \ldots, N\}} \{ \alpha_i \} ||v||^2, \]

which proves that \( A_\alpha \) is self-adjoint and \( \mathcal{U} \)-elliptic.

\[ \square \]

**Remark 4** Notice that, as in Remark 2, \( A_\alpha v \) can be rewritten as

\[ D_\alpha v + Bv, \]

and now, with the assumptions given in this section, operator \( B \) is self-adjoint and non-negative, since

\[ (Bv, w) = \sum_{i=1}^{N} \int_{\Omega} \rho \tilde{y}(v) \tilde{y}(w) \, dx \, dt + \int_{\Omega} \eta \tilde{y}(T; v) \tilde{y}(T; w) \, dx. \]

The constant part of the affine mapping (3) is given by the function \( b \in \mathcal{U} \) defined by \( b = - (p_1 \chi_{\omega_1}, \ldots, p_N \chi_{\omega_N}) \), where \( p_i, i \in \{1, \ldots, N\} \), is now the solution of

\[
\begin{align*}
- \frac{\partial p_i}{\partial t} - \Delta p_i &= \rho (\bar{y} - y_{i,d}) \quad \text{in } Q, \\
p_i(T) &= \eta (\bar{y} - y_{i,T}) \quad \text{in } \Omega, \\
p_i &= 0 \quad \text{on } \Sigma_1, \\
\frac{\partial p_i}{\partial n} &= 0 \quad \text{on } \Sigma_2,
\end{align*}
\]

and \( \bar{y} \) is the solution of (4).

**Theorem 3** There exist a unique Nash equilibrium of the problem defined in Sect. 4.

**Proof** The proof follows the one of Theorem 1, taking into account that in this case \( A_\alpha \) is unconditionally \( \mathcal{U} \)-elliptic. \[ \square \]

The discretization of the problem considered above and the development of suitable algorithms to get a numerical solution approximating the Nash equilibria are given in [7], where numerical examples are also showed.

**Theorem 4** The (unique) Nash equilibrium \( u = (u_1, \ldots, u_N) \in \mathcal{U} \) of the problem defined in Sect. 4 is the (unique) solution of the following optimal control problems:
• Find $u \in \mathcal{U}$ such that $J(u) = \min_{v \in \mathcal{U}} J(v)$, where

$$J(v) = \sum_{i=1}^{N} \frac{\alpha_i}{2} \int_{\omega_i \times (0,T)} |v_i|^2 \, dx \, dt$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \left( \int_{Q} \rho |y(0, \ldots, v_i, \ldots, 0) - y_{i,d}|^2 \, dx \, dt + \int_{\Omega} \eta |y(T; 0, \ldots, v_i, \ldots, 0) - y_{i,T}|^2 \, dx \right)$$

$$+ 2 \sum_{i,j=1, i < j}^{N} \left( \int_{Q} \rho y(0, \ldots, v_i, \ldots, 0) y(0, \ldots, v_j, \ldots, 0) \, dx \, dt \right)$$

$$\int_{\omega_T} \eta y(T; 0, \ldots, v_i, \ldots, 0) y(T; 0, \ldots, v_j, \ldots, 0) \, dx \right).$$

• Given $j, p \in \{1, \ldots, N\}$, find $u \in \mathcal{U}$ such that $J_{j,p}(u) = \min_{v \in \mathcal{U}} J_{j,p}(v)$, where

$$J_{j,p}(v) = \sum_{i=1}^{N} \frac{\alpha_i}{2} \int_{\omega_i \times (0,T)} |v_i|^2 \, dx \, dt$$

$$+ \frac{1}{2} \int_{Q} \rho |y(v) - y_{j,d}|^2 \, dx \, dt + \frac{1}{2} \int_{\Omega} \eta |y(T; v) - y_{p,T}|^2 \, dx$$

$$+ 2 \sum_{i=1(i \neq j)}^{N} \int_{Q} \rho (y_{j,d} - y_{i,d}) \tilde{y}(0, \ldots, v_j, \ldots, 0) \, dx \, dt$$

$$+ 2 \sum_{i=1(i \neq p)}^{N} \int_{\Omega} \eta (y_{p,T} - y_{i,T}) \tilde{y}(T; 0, \ldots, v_j, \ldots, 0) \, dx.$$

**Proof** We have seen previously that there exists a unique Nash equilibrium, which is the solution $u \in \mathcal{U}$ of

$$(A_{\alpha} u, v) = (b, v) \quad \forall \, v \in \mathcal{U}.$$ 

Then, because of the properties of $A_{\alpha}$ given in Theorem 2, we have (see, e.g., [22, Theorem 2.44]) that

$$\tilde{J}(u) = \min_{v \in \mathcal{U}} \tilde{J}(v),$$

with
\[ \tilde{J}(v) = \langle A_\alpha v, v \rangle - 2(b, v) = \sum_{i=1}^{N} \alpha_i \int_{\omega_i \times (0, T)} |v_i|^2 \, dx \, dt + \int_{Q} \rho |\tilde{y}(v)|^2 \, dx \, dt + \int_{\Omega} \eta |\tilde{y}(T; v)|^2 \, dx + 2 \sum_{i=1}^{N} \int_{Q} \rho (\tilde{y} - y_i, d) \tilde{y}(0, \ldots, v_i, \ldots, 0) \, dx \, dt + 2 \sum_{i=1}^{N} \int_{\Omega} \eta (\tilde{y}(T) - y_i, T) \tilde{y}(T; 0, \ldots, v_i, \ldots, 0) \, dx. \]

Then, using that \( \tilde{y}(v) = \sum_{i=1}^{N} \tilde{y}(0, \ldots, v_i, \ldots, 0) \), we have that
\[
\tilde{J}(v) = \sum_{i=1}^{N} \left( \alpha_i \int_{\omega_i \times (0, T)} |v_i|^2 \, dx \, dt + \int_{Q} \rho \left[ |\tilde{y}(0, \ldots, v_i, \ldots, 0)|^2 + 2(\tilde{y} - y_i, d) \tilde{y}(0, \ldots, v_i, \ldots, 0) \right] \, dx \, dt \right) + 2 \sum_{i, j=1}^{N} \left( \int_{Q} \rho \tilde{y}(0, \ldots, v_i, \ldots, 0) \tilde{y}(0, \ldots, v_j, \ldots, 0) \, dx \, dt \right) + \int_{\Omega} \eta (\tilde{y}(T) - y_i, T) \tilde{y}(T; 0, \ldots, v_i, \ldots, 0) \, dx \right) \].

Hence, using that \( y = \tilde{y} + \bar{y} \) we have that
\[ \tilde{J}(v) = J(v) - C, \]
where
\[ C = \sum_{i=1}^{N} \left( \rho \int_{Q} (\tilde{y} - y_i, d)^2 \, dx \, dt + \int_{\Omega} \eta (\tilde{y} - y_i, T)^2 \, dx \right) \]
is a constant (independent of \( v \)), which completes the proof of the first part of the theorem.

In order to prove the second part of the theorem, given \( j \in \{1, \ldots, N\} \), let us focus on the following terms of \( \tilde{J}(v) \):
\[
\int_{Q} \rho |\tilde{y}(v)|^2 \, dx \, dt + 2 \sum_{i=1}^{N} \int_{Q} \rho (\tilde{y} - y_i, d) \tilde{y}(0, \ldots, v_i, \ldots, 0) \, dx \, dt = \int_{Q} \rho \left[ |\tilde{y}(v)|^2 + 2(\tilde{y} - y_j, d)(\tilde{y}(v) - \tilde{y}(v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_N)) \right] \, dx \, dt
\]
\[ +2 \sum_{i=1(i\neq j)}^{N} \int_Q \rho(\bar{y} - y_{i,d}) \tilde{y}(0, \ldots, v_i, \ldots, 0) dx \, dt \]

\[ = \int_Q \rho \left[ |\tilde{y}(v)|^2 + 2(\bar{y} - y_{j,d}) \tilde{y}(v) \right] dx \, dt \]

\[ +2 \sum_{i=1(i\neq j)}^{N} \int_Q \rho(y_{j,d} - y_{i,d}) \tilde{y}(0, \ldots, v_i, \ldots, 0) dx \, dt. \]

Something similar can be done with other terms of \( \tilde{J}(v) \), so that

\[ \tilde{J}(v) = J(v) - C_{j,p}, \]

where

\[ C_{j,p} = \int_Q \rho(\bar{y} - y_{j,d})^2 dx \, dt + \int_{\Omega} \eta(\bar{y} - y_{p,T})^2 dx, \]

which completes the proof. \( \square \)

5 Conclusions

This paper studies Nash equilibria of noncooperative differential games with several players (controllers), each one trying to minimize his own cost function defined in terms of a general class of linear partial differential equations. We give results of existence and uniqueness of Nash equilibria and show how, in some cases, the corresponding Nash equilibria (solution to competition-wise problems, with each control trying to reach possibly different goals), are also the solution of suitable single-objective optimization problems (i.e. cooperative-wise problems, where all the controls cooperate to reach a common goal). A natural question arises: Are there Nash equilibria associated to nonlinear problems than can be also characterized as the solutions of single-objective problems? This is an open problem for interested researchers. One should take into account that in the linear case studied here, the optimality system (2) is a necessary and sufficient condition for the Nash equilibria, but in general it is only a necessary condition. Most of the works dealing with nonlinear problems find solutions of the corresponding optimality system (which are not proved to be Nash equilibria). However, in [17] the existence of Nash equilibria is proved for a nonlinear problem, by showing that the functionals are convex and the optimality system is also a sufficient condition.

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