OPTIMAL CONSTANTS AND EXTREMISERS FOR SOME SMOOTHING ESTIMATES

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Abstract. We establish new results concerning the existence of extremisers for a broad class of smoothing estimates of the form

$$\|\psi(\|\nabla\|) \exp(it\phi(\|\nabla\|)) f \|_{L^2(w)} \leq C \|f\|_{L^2},$$

where the weight $w$ is radial and depends only on the spatial variable; such a smoothing estimate is of course equivalent to the $L^2$-boundedness of a certain oscillatory integral operator $S$ depending on $(w, \psi, \phi)$.

Furthermore, when $w$ is homogeneous, and for certain $(\psi, \phi)$, we provide an explicit spectral decomposition of $S^*S$ and consequently recover an explicit formula for the optimal constant $C$ and a characterisation of extremisers. In certain well-studied cases when $w$ is inhomogeneous, we obtain new expressions for the optimal constant.

1. Introduction

For real-valued functions $\Phi(\xi)$ and $\nabla = \nabla_x$, it is easy to see that the solutions $u(t, x) = \exp(it\Phi(\nabla)) f(x)$ to the Cauchy problem of linear dispersive equations

$$\begin{cases}
(i\partial_t + \Phi(\nabla)) u(t, x) = 0, \\
u(0, x) = f(x) \in L^2(\mathbb{R}^d)
\end{cases}$$

preserve the $L^2$-norm of the initial data $f$, that is, we have $\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ for any fixed time $t \in \mathbb{R}$. But if we integrate the solution in $t$, we get an extra gain of regularity in $x$. For example, we have the estimates

$$\|\Psi(x, \nabla) \exp(-it\Delta) f\|_{L^2_x(\mathbb{R} \times \mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}$$

for the Schrödinger equation (the case $\Phi(\xi) = |\xi|^2$), where

- $[A]$ $\Psi(x, \nabla) = (1 + |x|^2)^{-1/2}(1 - \Delta)^{1/4}$ ($d \geq 3$),
- $[B]$ $\Psi(x, \nabla) = |x|^{a-1}|\nabla|^a$ ($a \in (1 - \frac{d}{2}, \frac{1}{2})$, $d \geq 2$),
- $[C]$ $\Psi(x, \nabla) = (1 + |x|^2)^{-s/2}|\nabla|^{1/2}$ ($s > \frac{1}{2}$, $d \geq 2$).

The estimate of type $[A]$ is due to Kato and Yajima [25] (see also [7]). Type $[B]$ is due to Kato and Yajima [25] for $a \in [0, \frac{1}{2})$ for $d \geq 3$, $a \in (0, \frac{1}{2})$ for $d = 2$ and Sugimoto [44] for $a \in (1 - \frac{d}{2}, \frac{1}{2})$ for all $d \geq 2$ (see also [7]). Type $[C]$ is due to Kenig, Ponce and Vega [26] (see also [7] and [12]).

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These estimates are often called smoothing estimates, and their local version was first proved by Sjölin \[42\], Constantin and Saut \[16\], and Vega \[46\]. There is a vast literature on this subject, including Ben-Artzi and Devinatz \[5,6\], Hoshiro \[23,24\], Kenig, Ponce and Vega \[26,27,28,29,30,31\], Linares and Ponce \[32\], Sugimoto \[45\], Walther \[47\], Ruzhansky and Sugimoto \[38\].

Rather less is known about the optimal constant for smoothing estimates. In Simon \[40\] and Watanabe \[49\], explicit optimal constants were given for type [B] smoothing estimates. In significantly greater generality (under radial assumptions on $\Phi$ and $\Psi$, and further mild conditions), Walther \[48\] established an expression for the optimal constant involving a double supremum; see Theorem 1.1 below. Our purpose in this paper is to provide a number of results which build on these works, concerning both the optimal constant and extremising initial data. Our results complement the recent body of work concerning optimal Strichartz estimates; see, for example, Christ and Shao \[15\], Fanelli, Vega and Visciglia \[17,18\], Foschi \[19\], Ramos \[37\], Bennett et al. \[8\], Bez and Rogers \[9\].

To each spatial dimension $d \geq 2$, radial weight $w : [0, \infty) \to [0, \infty)$, smoothing function $\psi : [0, \infty) \to [0, \infty)$, dispersion relation $\phi : [0, \infty) \to \mathbb{R}$, and $f \in L^2(\mathbb{R}^d) \setminus \{0\}$, let $C_d(w, \psi, \phi; f)$ be the quantity given by

$$C_d(w, \psi, \phi; f) = \frac{\|w(|x|)^{1/2}\psi(|\nabla|) \exp(it\phi(|\nabla|))f\|_{L^2_t, x(\mathbb{R} \times \mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}}.$$

Of course,

$$C_d(w, \psi, \phi) = \sup_{f \in L^2(\mathbb{R}^d) \setminus \{0\}} C_d(w, \psi, \phi; f)$$

is the optimal constant $C \in (0, \infty]$ for which the smoothing estimate

$$\|w(|x|)^{1/2}\psi(|\nabla|) \exp(it\phi(|\nabla|))f\|_{L^2_t, x(\mathbb{R} \times \mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$$

holds for all $f \in L^2(\mathbb{R}^d)$.

Throughout the paper, we assume $(w, \psi, \phi)$ satisfies the basic regularity condition that, for each $k \in \mathbb{N}_0$, the function $\alpha_k : [0, \infty) \to [0, \infty)$ is continuous, where

$$\alpha_k(\varrho) = \frac{\varrho \psi(\varrho)}{|\phi'(\varrho)|} \int_0^\infty J_{\nu(k)}(r\varrho)^2rw(r) \, dr.$$

Here, $J_\nu$ is the Bessel function of the first kind of order $\nu$, and

$$\nu(k) = \frac{d}{2} + k - 1$$

for each $k \in \mathbb{N}_0$. Implicitly, of course, this means that we are assuming that $\phi$ is differentiable. We shall also assume throughout the paper that $\phi$ is injective. Note that each $\alpha_k$ is continuous if $w$ is integrable, $\psi$ is continuous and $\phi$ is continuously differentiable; but as will become clear we do not restrict ourselves to integrable weights.

**Theorem 1.1.** \[48\] We have $C_d(w, \psi, \phi) = (2\pi \sup_{k \in \mathbb{N}_0} \sup_{\varrho \in [0, \infty)} \alpha_k(\varrho))^{1/2}$. 
We may define an equivalence relation \( \approx \) on the set of \((w, \psi, \phi)\) described above by

\[
(w, \psi, \phi) \approx (\tilde{w}, \tilde{\psi}, \tilde{\phi}) \quad \text{if and only if} \quad w = \tilde{w}, \quad \psi^2 / |\phi'| = \tilde{\psi}^2 / |\tilde{\phi}'|.
\]

Clearly, by Theorem 1.1, we have that

\[
(C_d(w, \psi, \phi) = C_d(\tilde{w}, \tilde{\psi}, \tilde{\phi}) \quad \text{whenever} \quad (w, \psi, \phi) \approx (\tilde{w}, \tilde{\psi}, \tilde{\phi}),
\]

so that the optimal constant is unchanged within each equivalence class. We can also explain this fact by the comparison principle discussed in Ruzhansky and Sugimoto [38], where non-radial functions \((w, \psi, \phi)\) are treated as well. All explicit values of \(C_d(w, \psi, \phi)\) in the sequel are given for the case \(\phi(r) = r^2\) corresponding to the Schrödinger equation. This is for simplicity and we emphasise that further optimal constants are immediately available via (1.3).

Theorem 1.1 leaves open several natural questions which we shall address in this paper. Firstly, we shall consider the existence and nature of extremisers for (1.2); that is, \(f \in L^2(\mathbb{R}^d) \setminus \{0\}\) for which

\[
C_d(w, \psi, \phi; f) = C_d(w, \psi, \phi).
\]

In order to state our first main result in this direction, Theorem 1.2 below, let us introduce the notation

\[
(1.4) \quad \alpha = \sup_{k \in \mathbb{N}_0} \sup_{\rho \in (0, \infty)} \alpha_k(\rho).
\]

**Theorem 1.2.** An extremiser for (1.2) exists if and only if there exists \(k_0 \in \mathbb{N}_0\) and a set \(S \subset (0, \infty)\) of positive Lebesgue measure such that \(\alpha_{k_0}(\rho) = \alpha\) for all \(\rho \in S\).

We can, for example, deduce from Theorem 1.2 the non-existence of extremisers for a broad class of smoothing estimates for weights \(w\) which are integrable. For this we will establish the following.

**Theorem 1.3.** Suppose \(w \in L^1(0, \infty)\) and

\[
\rho \mapsto \frac{\rho \psi(\rho)^2}{|\phi'(\rho)|}
\]

is real analytic on \((0, \infty)\). Then \(\alpha_k\) is real analytic on \((0, \infty)\) for each \(k \in \mathbb{N}_0\).

As a sample application, by combining Theorems 1.2 and 1.3 we shall show the following.

**Corollary 1.4.** Suppose \(w \in L^1(0, \infty)\) and

\[
\rho \mapsto \frac{\rho \psi(\rho)^2}{|\phi'(\rho)|}
\]

is real analytic on \((0, \infty)\). If \(\alpha_k\) is non-constant for each \(k \in \mathbb{N}_0\), then there are no extremisers to (1.2). In particular, if \(w \neq 0\) and

\[
(1.5) \quad \frac{\psi(\rho)^2}{|\phi'(\rho)|}
\]

is asymptotically constant as \(\rho\) tends to zero and asymptotically nonzero constant as \(\rho\) tends to infinity, then there are no extremisers to (1.2).
The hypotheses of Corollary 1.4 are satisfied in many classical smoothing estimates. For example, Simon showed in [10] that for the Schrödinger equation with 
\((w(r), \psi(r), \phi(r)) = (1 + r^2)^{-1}, r^{1/2}, r^2)\), we have

\[(1.6) \quad C_d(w, \psi, \phi) = (\pi/2)^{1/2}\]

for each \(d \geq 3\), that is, the optimal constant for smoothing estimate of type [C] with \(s = 1\). Corollary 1.4 tells us immediately that (1.6) has no extremisers.

In [10], Simon further established that for \((w(r), \psi(r), \phi(r)) = (r^{-2}, 1, r^2)\), we have

\[(1.7) \quad C_d(w, \psi, \phi) = (\pi/(d - 2))^{1/2}\]

for each \(d \geq 3\), that is, the optimal constant for smoothing estimate of type [B] with \(a = 0\). Of course, here the weight is not integrable and we shall see that any nonzero radial initial data will be an extremiser. In fact, we provide a comprehensive analysis of the case where the weight is radial and homogeneous. In order to describe our results, it is convenient to let the linear operator \(S\) be given by

\[Sf(x, t) = w(|x|)^{1/2} \int_{\mathbb{R}^d} \exp(i x \cdot \xi + t \phi(|\xi|)) \psi(|\xi|) f(\xi) \, d\xi\]

for appropriate (say Schwartz) functions \(f : \mathbb{R}^d \to \mathbb{C}\). Note that

\[Sf(x, t) = (2\pi)^d w(|x|)^{1/2} \psi(|x|) \exp(it\phi(|x|)) f(x),\]

where, \(\hat{f}\), the Fourier transform of \(f\), is given by

\[\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot \xi) \, dx.\]

Therefore,

\[\|S\| = (2\pi)^d/2 C_d(w, \psi, \phi),\]

where \(\|S\|\) denotes the \(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})\) operator norm of \(S\). Our main result concerning \(S\) is the following.

**Theorem 1.5.** Let \((w(r), \psi(r), \phi(r)) = (r^{-2(1-a)}, r^a, r^2)\), where \(a \in (1 - \frac{d}{2}, \frac{1}{2})\). For each \(k \in \mathbb{N}_0\) we have

\[S^*Sf(\eta) = 2^{d-1} \pi^{d+1/2}(-1)^k \frac{\Gamma(\frac{1}{2} - a) \Gamma(\frac{d}{2} + a - 1) \Gamma(2 - a - \frac{d}{2})}{\Gamma(1 - a) \Gamma(\frac{d}{2} - a + k) \Gamma(2 - a - \frac{d}{2} - k)} f(\eta),\]

where

\[(1.8) \quad f(\eta) = P(\eta)|f_0(|\eta||\eta|^{-d/2-k+1/2}\]

and \(P\) is any solid spherical harmonic of degree \(k\), and \(f_0\) is any element of \(L^2(0, \infty)\). Consequently, the operator norm of \(S^*S\) is the largest eigenvalue

\[2^{d-1} \pi^{d+1/2} \frac{\Gamma(\frac{1}{2} - a) \Gamma(\frac{d}{2} + a - 1)}{\Gamma(1 - a) \Gamma(\frac{d}{2} - a)},\]

and this is attained if and only if \(S^*S\) is evaluated on any radial function.

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1 we have, of course, chosen to suppress the dependence of \(S\) on \(w, \psi\) and \(\phi\).
Underpinning Theorem 1.5 is the compactness of the operator $L^2(S^{d-1}) \rightarrow L^2(S^{d-1})$ which is the analogue of $S^* S$ restricted to $S^{d-1}$. In particular, with $(w, \psi, \phi)$ as in Theorem 1.5 let $T$ be the operator given by

$$Tf(\eta) = |\eta|^a \int_{S^d} \frac{|\xi|^a}{|\xi - \eta|^{d+2a-2}} \delta(|\xi|^2 - |\eta|^2) f(\xi) \, d\xi$$

and note that

$$T = \frac{1}{2\pi \gamma(d+2a-2)} S^* S,$$

where

$$\gamma(\lambda) = \frac{\pi^{d/2} \lambda \Gamma(\frac{1}{2} \lambda)}{\Gamma(\frac{1}{2} (d-\lambda))}.$$ 

The identity (1.10) follows from the expression

$$\frac{1}{\lambda^d} \chi(\xi) = \frac{\gamma(\lambda)}{\lambda^d}$$

for the Fourier transform of a Riesz potential, valid for $\lambda \in (0, d)$. Switching to polar coordinates, for $\eta \neq 0$, it follows that

$$Tf(\eta) = \frac{1}{2} \int_{S^{d-1}} \frac{1}{|\theta - \eta'|^{d+2a-2}} f(|\eta|\theta) \, d\sigma(\theta),$$

where $\eta' = |\eta|^{-1}\eta$.

We now define $T_S$ to be the analogue of the operator $T$ restricted to functions on the unit sphere, given by

$$T_S f(\omega) = \frac{1}{2} \int_{S^{d-1}} \frac{1}{|\theta - \omega|^{d+2a-2}} f(\theta) \, d\sigma(\theta)$$

for each $f \in L^2(S^{d-1})$.

**Theorem 1.6.** If $(w(r), \psi(r), \phi(r)) = (r^{-2(1-a)}, r^a, r^2)$, where $a \in (1 - \frac{d}{2}, \frac{d}{2})$, then the operator $T_S : L^2(S^{d-1}) \rightarrow L^2(S^{d-1})$ is compact. In fact, if $k \in \mathbb{N}_0$ and $P$ is a solid spherical harmonic of degree $k$, then $T_S P = \lambda_k P$, where

$$\lambda_k = \frac{\pi^{d/2} \Gamma(\frac{1}{2} - a)}{2^{2a} \Gamma(-a + \frac{d}{2})} (-1)^k \Gamma(\frac{1}{2} - a) \Gamma(2 - a - \frac{d}{2} - k) \Gamma(-a + \frac{d}{2} + k).$$

The sequence of eigenvalues $(\lambda_k)_{k \geq 0}$ is a decreasing sequence converging to zero and hence the operator norm of $T_S$ is equal to

$$\frac{\pi^{d/2} \Gamma(\frac{1}{2} - a)}{2^{2a} \Gamma(-a + \frac{d}{2})}.$$

**Remark.** Theorems 1.5 and 1.6 have been stated with each component of $(w, \psi, \phi)$ as a homogeneous function. It is crucial to the proofs that $w$ is homogeneous; however, Theorems 1.5 and 1.6 may be extended to $\psi$ and $\phi$ satisfying

$$\psi(r)^2 = \lambda |\phi'(r)| r^{1-\mu},$$

where $w(r) = r^{-\mu}$, for some $\mu \in (1, d)$, and $\lambda$ is some non-negative constant. In this case, the eigenvalues appearing in Theorems 1.5 and 1.6 should be multiplied by $2\lambda$. These facts will be clear from the arguments in Section 4 and we omit the details.
From Theorem 1.5 and the duplication formula
\[ 2^{2x-1} \Gamma(x) \Gamma(x + \frac{1}{2}) = \pi^{1/2} \Gamma(2x) \quad (x > 0) \]
(see [21, p.240]), it follows that for \((w, \psi(r), \phi(r)) = (r^{-2(1-a)}, r^a, r^2)\), we have
\begin{equation}
C_d(w, \psi, \phi) = \left( \pi 2^{2a-1} \frac{\Gamma(1-2a) \Gamma(\frac{d}{2}+a-1)}{\Gamma(1-a) \Gamma(\frac{d}{2}-a)} \right)^{1/2}
\end{equation}
for each \(d \geq 2\) and each \(a \in (1 - \frac{d}{2}, \frac{d}{2})\), that is, the optimal constant for smoothing estimate of type \([B]\) with general \(a\), and
\[ C_d(w, \psi, \phi) = C_d(w, \psi, \phi; f) \]
precisely when \(f \in L^2(\mathbb{R}^d)\) is radial. The case \(a = 0\) and \(d \geq 3\) is the optimal constant in (1.7) due to Simon.

Our argument leading to Theorem 1.5 essentially proceeds by multiplying out the \(L^2(\mathbb{R}^{d+1})\) norm of \(Sf\), an idea which has been fruitful on several occasions in understanding Lebesgue space norms of oscillatory integral operators when the exponent is an even integer. In this particular case of \((w, \psi, \phi)\), this approach is different to (and more straightforward) than the approach of Walther in proving Theorem 1.1. We note, however, that in earlier work, Watanabe [49] (see also [11]) used the multiplying out approach to show that radial input functions are extremisers in the homogeneous case \((w(r), \psi(r), \phi(r)) = (r^{-2(1-a)}, r^a, r^2)\), and gave an expression of the optimal constant. One should view Theorems 1.5 and 1.6 as extensions of this result in [49]. We mention a different extension in very recent work of Ozawa and Rogers [34] where the sharp Hardy–Littlewood–Sobolev inequality on the sphere, due to Lieb, is used to establish certain angular refinements with optimal constants and characterisations of extremisers.

Our final contribution in this paper is to explicitly compute the quantity \(\alpha\) in (1.4) (and hence the optimal constant in the associated smoothing estimate) in certain cases where the weight is inhomogeneous. The finiteness of \(C_d(w, \psi, \phi)\) when
\begin{equation}
(w(r), \psi(r), \phi(r)) = ((1 + r^2)^{-1}, (1 + r^2)^{1/4}, r^2)
\end{equation}
and \(d \geq 3\), that is, the smoothing estimate of type \([A]\), motivated the considerations of optimal constants for smoothing estimates by Simon in [40], which led to (1.6) and (1.7). However, the value of \(C_d(w, \psi, \phi)\) for \((w, \psi, \phi)\) in (1.14) was left open in [40]. We compute the value of \(\alpha\), and hence \(C_d(w, \psi, \phi)\), in this case, and the closely related case where
\begin{equation}
(w(r), \psi(r), \phi(r)) = ((1 + r^2)^{-1}, (1 + r)^{1/2}, r^2),
\end{equation}
in spatial dimensions \(d = 3\) and \(d = 5\).

**Theorem 1.7.** If \((w(r), \psi(r), \phi(r)) = ((1 + r^2)^{-1}, (1 + r^2)^{1/4}, r^2)\) then
\begin{equation}
C_3(w, \psi, \phi) = \pi^{1/2} \quad \text{and} \quad C_5(w, \psi, \phi) = (\pi/2)^{1/2}.
\end{equation}
If \((w(r), \psi(r), \phi(r)) = ((1 + r^2)^{-1}, (1 + r)^{1/2}, r^2)\) then
\begin{equation}
C_3(w, \psi, \phi) = \pi^{1/2} \quad \text{and} \quad C_5(w, \psi, \phi) = (2\pi a_0(\varrho_0))^{1/2},
\end{equation}
where \(\varrho_0\) is the unique positive solution of
\[(3 + 2\varrho + 2\varrho^2 + \varrho^3) \sinh \varrho = \varrho(3 + 2\varrho + \varrho^2) \cosh \varrho.\]
Key to our proof of Theorem 1.7 is the monotonicity of certain quantities involving modified Bessel functions of the first kind, $I_{\nu}(g)$ and $K_{\nu}(g)$. We will use monotonicity properties in both the argument $g$ and the index $\nu$.

We remark that the optimal constants for smoothing estimates of type [A] with $d = 3, 5$, type [B], and type [C] with $s = 1$ have been thus explicitly determined, but those for other cases are still left open.

In all of the above cases where we have found the optimal constant (including the case of homogeneous weights in (1.12)), it is true that

\begin{equation}
C_d(w, \psi, \phi) = \left(2\pi \sup_{g \in [0, \infty)} \alpha_0(g) \right)^{1/2};
\end{equation}

that is, the supremum in $k \in \mathbb{N}_0$ in (1.14) is attained at $k = 0$. We shall see that the supremum in $g$ may be attained in several ways; see the remarks at the end of Section 5.

It is conceivable that one could find a geometric characterisation of the $(w, \psi, \phi)$ under which (1.16) is true. This is suggested by earlier work of several authors in the case of weighted $L^2$ estimates for solutions of the Helmholtz equation, or weighted $L^2$ estimates for the Fourier extension operator associated to the unit sphere, where boundedness is known to be equivalent to the $L^\infty$-boundedness of an $X$-ray transform applied to the weight $w$; see, for example, [1], [2], [10], [33]. This viewpoint led to the simple example of $(w, \psi, \phi)$ at the end of Section 5 where (1.16) fails. In this example, the weight is supported away from the origin, unlike the weights of the form $w(r) = r^{-\lambda}$, $w(r) = (1 + r^2)^{-\lambda/2}$ or $w(r) = (1 + r)^{-\lambda}$ considered above for which (1.16) holds.

Organisation. In the subsequent section, we introduce some notation and facts concerning spherical harmonics and Bessel functions of the first kind. In Section 3 we prove Theorems 1.2 and 1.3. Section 4 is concerned with the case of homogeneous weights where Theorems 1.5 and 1.6 are proved and several further remarks are given. Finally, in Section 5 we prove Theorem 1.7.

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2. Preliminaries and notation

The notation $A \lesssim_{p_1, \ldots, p_m} B$ means that $A \leq CB$, where the constant $C$ depends on at most the parameters $p_1, \ldots, p_m$. Also, $A \sim_{p_1, \ldots, p_m} B$ means $A \lesssim_{p_1, \ldots, p_m} B$ and $B \lesssim_{p_1, \ldots, p_m} A$.

We use $d\sigma$ throughout as the induced Lebesgue measure on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$.

2.1. Spherical harmonic decomposition of $L^2(\mathbb{R}^d)$. Let $k \in \mathbb{N}_0$. Write $\mathcal{H}_k$ for the space of solid spherical harmonics; that is, the space of polynomials on $\mathbb{R}^d$ with
complex coefficients which are homogeneous of degree $k$ and harmonic. Also, we let $H_k$ denote the space of all linear combinations of functions of the form

$$
\xi \mapsto P(\xi)f_0(|\xi|)|\xi|^{-d/2-k+1/2},
$$

where $P \in A_k$ and $f_0 \in L^2(0, \infty)$. It will be convenient to fix an orthonormal basis 

$$
\{P^{(k,1)}, \ldots, P^{(k,a_k)}\}
$$

of $A_k$, so that each $f \in H_k$ may be written

$$
f(\xi) = \sum_{m=1}^{a_k} P^{(k,m)}(\xi)f_0^{(m)}(|\xi|)|\xi|^{-d/2-k+1/2},
$$

where $f_0^{(m)} \in L^2(0, \infty), 1 \leq m \leq a_k$.

We shall use the complete orthogonal direct sum decomposition

$$
L^2(\mathbb{R}^d) = \bigoplus_{k=0}^{\infty} H_k
$$

in the sense that the $H_k$ are closed mutually orthogonal subspaces of $L^2(\mathbb{R}^d)$, and each $f \in L^2(\mathbb{R}^d)$ may be expressed as $\sum_{k=1}^{\infty} f_k$ where $f_k \in H_k$ for each $k \in \mathbb{N}_0$. We refer the reader to [43] for further details.

2.2. Properties of the Bessel function $J_\nu$. For $\Re \nu > -\frac{1}{2}$ and $z \in \mathbb{C}$ such that $\arg(z) \in (\pi, \pi)$, the Bessel function $J_\nu$ is given by the expression

$$
J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} e^{izt}(1 - t^2)^{\nu - \frac{1}{2}} \, dt.
$$

Mostly we are concerned with $J_\nu(r)$ when $r \in [0, \infty)$. For $\nu \in \frac{1}{2}\mathbb{N}$ it is well-known that explicit formulae in terms of elementary functions for $J_\nu$ are available; for example,

$$
J_{1/2}(r) = \left(\frac{2}{\pi r}\right)^{1/2} \sin(r), \quad \text{and} \quad J_{3/2}(r) = \left(\frac{2}{\pi r}\right)^{1/2} \left(\frac{\sin(r)}{r} - \cos(r)\right),
$$

which we need on several occasions.

We conclude this section with two asymptotic results concerning $J_\nu$.

**Theorem 2.1.** Suppose $\nu > -\frac{1}{2}$. Then

$$
|J_\nu(r) - (\frac{2}{\pi r})^{1/2} \cos(r - \frac{\pi}{2} \nu - \frac{\pi}{4})| \lesssim \nu \, r^{-3/2}
$$

for all $r \geq 1$.

For a proof of Theorem 2.1 see [43].

**Theorem 2.2.** Suppose $w \in L^1(0, \infty)$ and $\nu > -\frac{1}{2}$. Then

$$
\varrho \int_{0}^{\infty} J_\nu(\varrho r)^2 rw(r) \, dr
$$

tends to zero as $\varrho$ tends to zero, and tends to $\frac{1}{\pi} \|w\|_{L^1(0,\infty)}$ as $\varrho$ tends to infinity.
Proof. We use the Bessel function asymptotics in Theorem 2.1. In particular, it follows that for all $r > 0$ we have
\[ r^{1/2} J_{\nu(k)}(r) = (\frac{2}{\pi})^{1/2} \cos(r - \ell) + E(r), \]
where
\[ |E(r)| \lesssim_d, k (1 + r)^{-1} \]
and $\ell = \frac{\pi}{2} \nu(k) + \frac{\pi}{4}$. Therefore
\[ r J_{\nu(k)}(r)^2 = \frac{\pi}{2} \cos^2(r - \ell) + \tilde{E}(r) \]
for all $r > 0$. Here $\tilde{E}$ also satisfies an estimate of the form (2.4) and therefore
\[ \int_0^\infty \tilde{E}(rg) w(r) \, dr \to 0 \quad \text{as } g \to \infty \]
by the dominated convergence theorem and since $w \in L^1(0, \infty)$. For the main term we have
\[ 4 \int_0^\infty \cos^2(rg - \ell) w(r) \, dr \]
\[ = e^{-2it} \int_0^\infty e^{2irew(r)} \, dr + e^{2it} \int_0^\infty e^{-2irew(r)} \, dr + 2 \int_0^\infty w(r) \, dr. \]

The first two terms on the right-hand side tend to zero as $g$ tends to infinity by the Riemann–Lebesgue lemma, again using $w \in L^1(0, \infty)$. Hence
\[ \int_0^\infty \cos^2(rg - \ell) w(r) \, dr \to \frac{1}{2} \|w\|_{L^1(0, \infty)} \quad \text{as } g \to \infty \]
and it follows that
\[ g \int_0^\infty J_{\nu(k)}^2(rg) r w(r) \, dr \to \frac{1}{2} \|w\|_{L^1(0, \infty)} \]
as $g \to \infty$ as claimed.

Also, note that $rg J_{\nu(k)}(rg)^2 \lesssim_d, k 1$ uniformly in $g > 0$ by (2.5), and it follows immediately from the dominated convergence theorem and the boundedness of the Bessel function that
\[ g \int_0^\infty J_{\nu(k)}^2(rg) r w(r) \, dr \to 0 \]
as $g \to 0$.

3. Extremisers: Proofs of Theorems 1.2 and 1.3

Theorem 1.2 will follow from a re-visit of Walther’s proof of Theorem 1.1 discussed in [48], which we briefly recall now. The first step is an application of Plancherel in time for each fixed $x \in \mathbb{R}^d$. To see this explicitly, first note that
\[ Sf(x, t) = \int_\mathbb{R} \exp(it\phi) \tilde{f}(x)(\phi) \, d\phi, \]
where
\[ \tilde{f}(x)(\phi) = \frac{w(|x|)^{1/2} \psi(\phi^{-1}(\phi)) \phi^{-1}(\phi)^{d-1}}{\phi'(\phi^{-1}(\phi))} \int_{S^{d-1}} \exp(i\phi^{-1}(\phi)x \cdot \theta) f(\phi^{-1}(\phi)\theta) \, d\sigma(\theta) \]
for \( q \in \phi((0, \infty)) \), and \( \hat{f}[x](q) = 0 \) otherwise. Therefore,
\[
\|Sf\|_{L^2_t,(\mathbb{R} \times \mathbb{R}^d)}^2 = 2\pi \|\hat{f}\|_{L^2_x,\mathbb{R}^d}^2.
\]
Orthogonality considerations (see \[47\] Sect. 4.2.3) lead to
\[
\|Sf\|_{L^2_t,(\mathbb{R} \times \mathbb{R}^d)}^2 = 2\pi \sum_{k=0}^{\infty} a_k \|f^{(k,m)}\|_{L^2_{\alpha,\mathbb{R}^d}}^2,
\]
where \( f = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} f^{(k,m)} \) and
\[
(f^{(k,m)}(\xi) = P^{(k,m)}(\xi)f_0^{(k,m)}(\xi)|\xi|^{-d/2-k+1/2}
\]
for some \( f_0^{(k,m)} \in L^2(0, \infty) \).

If \( k \in \mathbb{N}_0 \) and \( P \in \mathfrak{A}_k \) then we have
\[
\hat{P}a(x) = \frac{(2\pi)^{d/2}}{\pi^k} P(x)J_{\nu(k)}(|x||x|^{-\nu(k)})
\]
for each \( x \in \mathbb{R}^d \). From this and certain changes of variables we obtain
\[
\|\hat{f}^{(k,m)}\|_{L^2_{\alpha,\mathbb{R}^d}}^2 = (2\pi)^d \int_0^{\infty} \alpha_k(\varrho)\|f_0^{(k,m)}(\varrho)\|^2 \varrho d\varrho
\]
(see \[48\] Sect. 6.1) and consequently,
\[
\|Sf\|_{L^2_t,(\mathbb{R} \times \mathbb{R}^d)}^2 = (2\pi)^{d+1} \sum_{k=0}^{\infty} a_k \sum_{\varrho>0}^{} \sup_{\varrho>0} \alpha_k(\varrho) \int_0^{\infty} |f_0^{(k,m)}(\varrho)|^2 \varrho d\varrho
\]
\[
\leq 2(\pi)^{d+1} \sum_{k=0}^{\infty} \sum_{\varrho>0} \alpha_k(\varrho) \int_0^{\infty} |f_0^{(k,m)}(\varrho)|^2 \varrho d\varrho
\]
\[
\leq 2(\pi)^{d+1} \sum_{k=0}^{\infty} \sum_{\varrho>0} \alpha_k(\varrho) \int_0^{\infty} |f_0^{(k,m)}(\varrho)|^2 \varrho d\varrho = (2\pi)^{d+1} \|f\|_{L^2(\mathbb{R}^d)}^2.
\]

Let us see that the constant \((2\pi)^{d+1} \alpha\) in the above estimate is optimal, given that each \( \alpha_k \) is continuous. To begin, let \( \varepsilon > 0 \). Then there exist \( k_0 \in \mathbb{N}_0 \) and \( \varrho_0 > 0 \) such that \( \alpha - 2\varepsilon < \alpha_{k_0}(\varrho_0) \leq \alpha \) and by continuity there exists \( \delta > 0 \) such that \( \alpha - \varepsilon < \alpha_{k_0}(\varrho) \leq \alpha \) for each \( \varrho \in [\varrho_0 - \delta, \varrho_0 + \delta] \). Now let \( f \in \mathfrak{A}_{k_0} \) be given by
\[
f(\xi) = P(\xi)f_0(\xi)|\xi|^{-d/2-k_0+1/2},
\]
where \( P \) is any element of \( \mathfrak{A}_{k_0} \) normalised so that \( \|P\|_{L^2(\mathbb{R}^d)} = 1 \) and \( f_0 \) is any nonzero element of \( L^2(0, \infty) \) which is supported on \([\varrho_0 - \delta, \varrho_0 + \delta] \). Using equality \((3.3)\) we get
\[
\|Sf\|_{L^2_t,(\mathbb{R} \times \mathbb{R}^d)}^2 = (2\pi)^{d+1} \int_0^{\infty} \alpha_{k_0}(\varrho)\|f_0(\varrho)\|^2 \varrho d\varrho
\]
\[
\geq (2\pi)^{d+1}(\alpha - \varepsilon)\|f_0\|_{L^2(0, \infty)}^2 = (2\pi)^{d+1}(\alpha - \varepsilon)\|f\|_{L^2(\mathbb{R}^d)}^2,
\]
and consequently the constant \((2\pi)^{d+1} \alpha\) cannot be bettered.
Proof of Theorem 1.3. Suppose $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ satisfies
\begin{equation}
C_d(w, \psi, \phi) = C_d(w, \psi, \phi; f)
\end{equation}
so that the inequalities in (3.5) and (3.8) are both equalities. As above, we write $f = \sum_{k=0}^{\infty} f_k$, where $f_k = \sum_{m=1}^{a_k} f^{(k,m)}$ and $f^{(k,m)}$ is given by (3.4). Let $F^{(k)}(\varrho) = \sum_{m=1}^{a_k} |f_0^{(k,m)}(\varrho)|^2$ so that
\[
\int_0^\infty F^{(k)}(\varrho) \, d\varrho = \|f_k\|_{L^2(\mathbb{R}^d)}^2.
\]
Also, let
\[
\mathcal{K} = \{k \in \mathbb{N}_0 : \sup_{\varrho > 0} \alpha_k(\varrho) = \alpha\}.
\]
From equality in (3.5), it follows that $f_k$ must be zero for $k \notin \mathcal{K}$. So $f = \sum_{k \in \mathcal{K}} f_k$ and since $f \neq 0$ there exists $k_0 \in \mathcal{K}$ such that $f_{k_0} \neq 0$. From equality in (3.4), we see that for all $k \in \mathcal{K}$ we must have
\[
\alpha_k(\varrho) = \alpha \quad \text{for all } \varrho \in \text{supp} F^{(k)}.
\]
Now $F^{(k_0)} \in L^1(0, \infty) \setminus \{0\}$ and hence the desired conclusion holds by taking $S = \text{supp} F^{(k_0)}$.

For the converse, suppose we are given a set $S$ of positive Lebesgue measure and $k_0 \in \mathbb{N}_0$ such that $\alpha_{k_0}(\varrho) = \alpha$ for each $\varrho \in S$. Let $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ be given by
\[
f(\xi) = P(\xi) f_0(\xi) |\xi|^{-d/2-k_0+1/2},
\]
where $P$ is any element of $\mathfrak{A}_{k_0}$ normalised so that $\|P\|_{L^2(\mathbb{R}^d)} = 1$ and $f_0$ is any nonzero function in $L^2(0, \infty)$ which is supported on $S$. Then it is clear from (3.3) that we have equality in both (3.4) and (3.5) and hence (3.6) holds for such $f$. \hfill \Box

Proof of Theorem 1.3. It is clearly enough to prove that $\tilde{\alpha} : \mathcal{S} \to \mathbb{C}$ is complex analytic on the strip $\mathcal{S}$, where
\[
\tilde{\alpha}(z) = \int_0^\infty J_\nu(rz)^2 rw(r) \, dr
\]
and
\[
\mathcal{S} = \{z \in \mathbb{C} : \text{Re}(z) > 0 \text{ and } \text{Im}(z) \in (-1, 1)\}.
\]
Here, $J_\nu$ denotes the usual analytic extension of the Bessel function to the half-plane $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$, given by (2.2), and $\nu \geq 0$ is fixed. To this end, for each $N \in \mathbb{N}$, let $\tilde{\alpha}_N : \mathcal{S} \to \mathbb{C}$ be given by
\[
\tilde{\alpha}_N(z) = \int_0^N J_\nu(rz)^2 rw(r) \, dr,
\]
for $z \in \mathcal{S}$. We claim that each $\tilde{\alpha}_N$ is complex analytic on $\mathcal{S}$ and $\tilde{\alpha}_N$ converges uniformly to $\tilde{\alpha}$ on every compact subset of $\mathcal{S}$. From the claim, it follows that $\tilde{\alpha}$ is complex analytic on $\mathcal{S}$ as required.

To see that our claim is true, let $\mathfrak{D} \subset \mathcal{S}$ be compact and note that $|\text{Re}(z)| \geq \varepsilon$, for all $z \in \mathfrak{D}$, where $\varepsilon$ is some strictly positive constant depending on $\mathfrak{D}$. From Theorem 2.1 (see also Watson [50], page 199) it follows that
\[
|J_\nu(z)| \lesssim_{\nu} (1 + |\text{Re}(z)|)^{-1/2}
\]
for each \( z \in \mathfrak{D} \), and therefore,
\[
|\tilde{\alpha}(z) - \tilde{\alpha}_N(z)| \lesssim_{\nu} \int_N^\infty \frac{rw(r)}{1 + r|\text{Re}(z)|} \, dr \lesssim_{\nu} \int_N^\infty w(r) \, dr.
\]
Hence, \( \sup_{z \in \mathfrak{D}} |\tilde{\alpha}(z) - \tilde{\alpha}_N(z)| \to 0 \) uniformly as \( N \to \infty \) as required.

Finally, a straightforward argument using the complex analyticity and boundedness properties of \( J_\nu \) on \( \mathfrak{S} \), shows that each \( \tilde{\alpha}_N \) is complex analytic on \( \mathfrak{S} \). This completes the proof of our claim, and hence Theorem 1.3.

\[\square\]

Proof of Corollary 1.4. Fix \( k \in \mathbb{N}_0 \). Since \( \alpha_k \) is analytic, the pre-image set \( \alpha_k^{-1}(\alpha) \) is either equal to \((0, \infty)\) or it has Lebesgue measure zero. This follows because the zero set of an analytic function on \((0, \infty)\) is either \((0, \infty)\) or contains only isolated points. In the latter case, the zero set is countable and hence has Lebesgue measure zero. Hence, by Theorem 1.2, if each \( \alpha_k \) is non-constant then no extremisers exist.

From Theorem 2.2 and our hypotheses on the ratio in (1.5), we know that \( \alpha_k(\hat{\nu}) \to 0 \) as \( \hat{\nu} \to 0 \) and \( \alpha_k(\hat{\nu}) \) tends to a strictly positive number as \( \hat{\nu} \to \infty \). This means each \( \alpha_k \) is not constant and therefore no extremisers exist. \( \square \)

4. Homogeneous weights: Proof of Theorems 1.5 and 1.6

Let \((w(r), \psi(r), \phi(r)) = (r^{-2(1-a)}, r^a, r^2)\) where \( a \in (1 - \frac{d}{2}, \frac{1}{2}) \) and \( d \geq 2 \). We first prove Theorem 1.6 concerning \( T_\mathfrak{S} \), which we recall is given by
\[
T_\mathfrak{S} f(\omega) = \frac{1}{2} \int_{S^{d-1}} \frac{1}{\theta - \omega |^{d+2a-2}} f(\theta) \, d\sigma(\theta).
\]
We remark that if \( f \) is constant then the rotation invariance of \( d\sigma \) clearly implies that \( f \) is an eigenvector of \( T_\mathfrak{S} \) with an explicitly computable eigenvalue. In order to extend this to the full strength of Theorem 1.6 we use the Funk–Hecke theorem.

**Theorem 4.1 (Funk–Hecke).** Let \( k \in \mathbb{N}_0 \) and let \( P \) be a spherical harmonic of degree \( k \). Then, for each unit vector \( \omega \),
\[
\int_{S^{d-1}} F(\omega \cdot \theta) P(\theta) \, d\sigma(\theta) = P(\omega) \frac{[3d-2]}{C_{d,k}(1)} \int_{-1}^1 F(t) C_{d,k}(t) (1 - t^2)^{-\frac{d-3}{2}} \, dt
\]
holds whenever the complex-valued function \( F \) is integrable on \([-1, 1]\) with respect to the weighted Lebesgue measure \((1 - t^2)^{-\frac{d-3}{2}} \, dt\).

Here, \( C_{d,k} \) is the Gegenbauer (or ultraspherical) polynomial of degree \( k \) associated with \( \frac{d-2}{2} \), defined via the generating function
\[
(1 - 2st + t^2)^{-\frac{d-2}{2}} = \sum_{k=0}^\infty C_{d,k}(s) t^k
\]
for \( |s| \leq 1 \) and \( |t| < 1 \) (see, for example, [43]). For a proof of the Funk–Hecke theorem, see [39].
Proof of Theorem 1.6. Let $P$ be a spherical harmonic of degree $k$ and note that

$$T_S P(\omega) = \frac{1}{2} \int_{S^{d-1}} \frac{1}{|\theta - \omega|^{d+2a-2}} P(\theta) d\sigma(\theta)$$

$$= 2^{-\frac{d+2a}{2}} \int_{S^{d-1}} \frac{1}{(1 - \theta \cdot \omega)^{\frac{d+2a-2}{2}}} P(\theta) d\sigma(\theta),$$

and thus, by the Funk–Hecke Theorem,

$$(4.1) \quad T_S P(\omega) = P(\omega) 2^{-\frac{d+2a}{2}} \left| \frac{|S^{d-2}|}{C_{d,k}(1)} \right| \int_{-1}^{1} (1 - t)^{-\frac{d+2a-2}{2}} C_{d,k}(t)(1 - t^2)^{\frac{d-3}{2}} dt.$$ 

We have

$$C_{d,k}(1) = \frac{\Gamma(d - 2 + k)}{k! \Gamma(d - 2)},$$

which can be found in [41], and therefore, using the formula in terms of the Gamma function for the integral in (4.1) from [20] (page 795), we obtain

$$\left| \frac{|S^{d-2}|}{C_{d,k}(1)} \right| \int_{-1}^{1} (1 - t)^{-\frac{d+2a-2}{2}} C_{d,k}(t)(1 - t^2)^{\frac{d-3}{2}} dt$$

$$= (-1)^k 2^{-\frac{d+2a}{2}} \pi^{\frac{d-1}{2}} \frac{\Gamma\left(\frac{1}{2} - a\right) \Gamma\left(2 - a - \frac{d}{2}\right)}{\Gamma(2 - a - \frac{d}{2} - k) \Gamma(-a + \frac{d}{2} + k)}$$

which is equal to $\lambda_k$. Hence, $T_S P = \lambda_k P$, as claimed.

Since

$$\frac{\lambda_k}{\lambda_{k+1}} = \frac{-a + \frac{d}{2} + k}{a - 1 + \frac{d}{2} + k}$$

is strictly larger than one for $a \in (1 - \frac{d}{2}, \frac{1}{2})$, it follows that $(\lambda_k)_{k\geq 0}$ is a decreasing sequence. To complete the proof of Theorem 1.6 it remains to show that $\lambda_k \to 0$ as $k \to \infty$. For this, in the case $a + \frac{d}{2} \notin \mathbb{Z}$ we have

$$\Gamma(2 - a - \frac{d}{2} - k) \Gamma(-a + \frac{d}{2} + k) = \Gamma(1 - s) \Gamma(t + s),$$

where

$$s = -1 + a + \frac{d}{2} + k \quad \text{and} \quad t = 1 - 2a.$$

By the Euler reflection formula (using that $s \notin \mathbb{Z}$),

$$\Gamma(1 - s) \Gamma(s) = \frac{\pi}{\sin(\pi s)},$$

and therefore

$$(4.2) \quad \left| \frac{1}{\Gamma(1 - s) \Gamma(t + s)} \right| \leq \frac{\Gamma(s)}{\Gamma(t + s)}.$$ 

Using Stirling's formula

$$\lim_{x \to \infty} \frac{\Gamma(x + 1)}{\sqrt{2\pi x} x/e^x} = 1$$

it follows that $\frac{\Gamma(s)}{\Gamma(t + s)} \to 0$ as $s \to \infty$, provided that $t > 0$. We have $t > 0$ since $a < \frac{1}{2}$ and it follows that $\lambda_k \to 0$ as $k \to \infty$ in this case.
In the remaining case \( a + \frac{d}{2} \in \mathbb{Z} \), note that \( d \geq 4 \) since \( a \in \left( 1 - \frac{d}{2}, \frac{1}{2} \right) \). If we let \( m \) be the integer given by \[ m = a + \frac{d}{2} - 2 \]
then \( m \in (-1, \frac{d-2}{2}) \). Repeatedly using the identity
\[ \Gamma(x+1) = x\Gamma(x) \]
it follows that
\[ \left| \frac{\Gamma(2 - a - \frac{d}{2})}{\Gamma(2 - a - \frac{d}{2} - k)} \right| = (m+k)(m+k-1) \cdots (m+1) \]
and therefore
\[ \left| \frac{\Gamma(2 - a - \frac{d}{2})}{\Gamma(2 - a - \frac{d}{2} - k)\Gamma(-a + \frac{d}{2} + k)} \right| = \frac{(m+k)(m+k-1) \cdots (m+1)}{(-m+3+d+k)!} \leq \frac{1}{-m+3+d+k}. \]
It follows that \( \lambda_k \to 0 \) as \( k \to \infty \) in this case too. \( \Box \)

**Proof of Theorem 1.5.** Suppose
\[ f(\eta) = P(\eta)f_0(|\eta|)|\eta|^{-d/2-k+1/2}, \]
where \( P \) is any spherical harmonic of degree \( k \), and \( f_0 \) is any element of \( L^2(0, \infty) \).

By Theorem 1.6
\[ Tf(\eta) = T_5P(\eta')f_0(|\eta|)|\eta|^{-d/2-k+1/2} = \lambda_k P(\eta')f_0(|\eta|)|\eta|^{-d/2+1/2} = \lambda_k f(\eta), \]
where \( \eta' = |\eta|^{-1}\eta \). Theorem 1.5 now follows from (1.10). \( \Box \)

We conclude this section with several remarks on the homogeneous weight case.

**Remarks.** (1) When \( a = 0 \) one may proceed slightly differently. In this case one can check that
\[ \lambda_k = \frac{(d-2)\pi^{d/2}}{(d+2k-2)\Gamma\left(\frac{d}{2}\right)} \]
so it suffices to show that
\[ \int_{S^{d-1}} \frac{1}{|\theta - \eta'|^{d-2}} P(\theta) \, d\sigma(\theta) = 2\lambda_k P(\eta') \]
for each nonzero \( \eta \). By a limiting argument, since \( \theta \mapsto (1 - \theta \cdot \omega)^{2-d} \in L^1(S^{d-1}) \), it suffices to prove that
\[ \lim_{t \to 1} \int_{S^{d-1}} \frac{1}{|\theta - t\eta'|^{d-2}} P(\theta) \, d\sigma(\theta) = 2\lambda_k P(\eta'). \]

Expanding the kernel as a power series we have
\[ \frac{1}{|\theta - t\omega|^{d-2}} = \frac{1}{(1 - 2(\theta \cdot \omega)t + t^2)^{(d-2)/2}} = \sum_{\ell=0}^{\infty} C_{d,\ell}(\theta \cdot \omega)t^\ell \]
for $|t| < 1$. Crucially, we have that the operator $P_{d,\ell}$ given by
\[
P_{d,\ell}F(\omega) = \frac{1}{2}(d - 2) + \ell \int_{S^{d-1}} C_{d,\ell}(\omega \cdot \theta) F(\theta) \, d\sigma(\theta)
\]
for $F \in L^2(S^{d-1})$ is the orthogonal projection from $L^2(S^{d-1})$ to the subspace of functions on $S^{d-1}$ which arise as the restriction of harmonic polynomials of $d$ variables and homogeneous of degree $\ell$. A proof of this fact may be found in [41] (see Corollary 4.2). So
\[
\int_{S^{d-1}} \frac{1}{|\theta - t\eta'|^{d-2}} P(\theta) \, d\sigma(\theta) = \sum_{\ell=0}^{\infty} \frac{1}{2}(d - 2) + \ell P_{d,\ell}(\eta') t^\ell = \frac{1}{2}(d - 2)|S^{d-1}| P(\eta') t^k
\]
and (4.3) now follows.

(2) One may show that $T_2$ is compact without identifying an explicit spectral decomposition using a more direct argument. In particular, it suffices to show the strong operator convergence
\[
\lim_{t \to 0} \|T_2 - T_2^t\| = 0,
\]
where
\[
T_2^t f(\omega) = \frac{1}{2} \int_{S^{d-1}} \frac{1 - \chi(0,\omega)(|\theta - \omega|)}{|\theta - \omega|^{d+2a-2}} f(\theta) \, d\sigma(\theta),
\]
because each $T_2^t$ is compact (the kernel $(\theta,\omega) \mapsto |\theta - \omega|^{-(d+2a-2)}(1 - \chi(0,\omega)(|\theta - \omega|) \in L^2(S^{d-1} \times S^{d-1})$ and compactness follows from the standard argument for Hilbert–Schmidt kernels on bounded domains).

To see (4.4), for each $f \in L^2(S^{d-1})$, Cauchy–Schwarz implies
\[
|(T_2 - T_2^t)f(\omega)|^2 \leq \int_{S^{d-1}} \frac{\chi(0,\omega)(|\theta - \omega|)}{|\theta - \omega|^{d+2a-2}} \, d\sigma(\theta) \int_{S^{d-1}} \frac{|f(\theta)|^2}{|\theta - \omega|^{d+2a-2}} \, d\sigma(\theta)
\]
\[
= \frac{1}{2} \int_{S^{d-1}} \frac{\chi(0,\omega)(|\theta - e_1|)}{|\theta - e_1|^{d+2a-2}} \, d\sigma(\theta) \int_{S^{d-1}} \frac{|f(\theta)|^2}{|\theta - e_1|^{d+2a-2}} \, d\sigma(\theta)
\]
so that
\[
\|T_2 - T_2^t\|_{L^2(S^{d-1})} \lesssim a_{d,\ell} \|f\|_{L^2(S^{d-1})} \int_{S^{d-1}} \frac{\chi(0,\omega)(|\theta - e_1|)}{|\theta - e_1|^{d+2a-2}} \, d\sigma(\theta).
\]
Here we have used the restriction $a \in (1 - \frac{d}{2}, \frac{1}{2})$ to obtain the finiteness of the integral
\[
\int_{S^{d-1}} \frac{1}{|e_1 - \omega|^{d+2a-2}} \, d\sigma(\omega).
\]
Now
\[
\int_{S^{d-1}} \frac{\chi(0,\omega)(|\theta - e_1|)}{|\theta - e_1|^{d+2a-2}} \, d\sigma(\theta) \sim a_{d,\ell} \int_{1 - \frac{1}{2}e^2 < t < 1} \frac{1}{(1 - t)^{1/2a}} \, dt \sim a_{d,\ell} \in 1 - 2a
\]
and since $a \in (1 - \frac{d}{2}, \frac{1}{2})$ we get (4.4).
For each $k \in \mathbb{N}_0$ let $\beta_k$ be given by

$$
\beta_k(\varrho) = \varrho \int_0^\infty J_\nu(kr^2) \frac{r}{1 + r^2} \, dr
$$

for $\varrho \in [0, \infty)$. The following lemma concerning the shape of each $\beta_k$ is key to our proof of Theorem 1.7. The modified Bessel functions of the first kind, $I_\nu$ and $K_\nu$, are given by

$$
I_\nu(r) = i^{-\nu} J_\nu(i r) \quad \text{and} \quad K_\nu(\varrho) = \frac{\pi}{2 \sin(\nu \pi)} (I_{-\nu}(\varrho) - I_\nu(\varrho)).
$$

We shall need the following special cases

$$
I_{1/2}(r) = (\frac{r}{2\pi})^{1/2} \sinh(r), \quad K_{1/2}(r) = (\frac{2}{\sqrt{\pi}})^{1/2} e^{-r}
$$

and

$$
I_{3/2}(r) = (\frac{2r}{\sqrt{\pi}})^{1/2} (\cosh(r) - r^{-1} \sinh(r)), \quad K_{3/2}(r) = (\frac{2r}{\sqrt{\pi}})^{1/2} (1 + r^{-1}) e^{-r}.
$$

**Lemma 5.1.** For each $k \in \mathbb{N}_0$ and $\varrho \in [0, \infty)$ we have

$$
\beta_k(\varrho) = \varrho I_{\nu(k)}(\varrho) K_{\nu(k)}(\varrho).
$$

Furthermore, $\beta_k$ is nonnegative, strictly concave, tends to zero as $\varrho$ tends to zero, and tends to $\frac{1}{2}$ as $\varrho$ tends to infinity.

**Proof:** The identity (5.3) can be found in [20] (page 671, formula 6.535), and the claimed limits for $\beta_k$ follow immediately from Theorem 2.2. The strict increasingness and concavity of $\beta_k$ follows from work of Hartman [21] for $\nu(k) > \frac{1}{4}$. This covers all $k \in \mathbb{N}_0$ and $d \geq 3$ except for $(k, d) = (0, 3)$, however a direct calculation using (5.1) reveals that

$$
\varrho I_{\nu(0)}(\varrho) K_{\nu(0)}(\varrho) = \frac{1}{4} (1 - e^{-2\varrho})
$$

in this case and the desired conclusion holds in this case too. For $\nu(k) > \frac{1}{4}$, the point is that $\varrho \mapsto \varrho^{1/2} I_{\nu(k)}(\varrho)$ and $\varrho \mapsto \varrho^{1/2} K_{\nu(k)}(\varrho)$ are linearly independent solutions of

$$
x''(\varrho) - (1 + (\nu(k)^2 - \frac{1}{4}) \varrho^{-2}) x(\varrho) = 0,
$$

which is valid for each $0 < \lambda < 2\nu + 1$. One can find (4.3) in Watson [50] (page 403, formula (2)), or prove it directly from (3.2). In fact,

$$
\alpha_k = 2^{2(\nu - 1)} \frac{\Gamma(1 - 2\nu) \Gamma(\nu(k) + a)}{\Gamma(1 - a) \Gamma(\nu(k) + 1 - a)}
$$

and it is straightforward to check that this is decreasing in $k$. We also remark that (4.3) has appeared in related work [13] and [14], where the emphasis is not on obtaining optimal constants.
a special case of the Whittaker differential equation. See Theorem 4.1 of [21] for precisely the result that \( \phi \rightarrow qI_{\nu(k)}(q)K_{\nu(k)}(q) \) is strictly increasing and strictly concave on \((0, \infty)\). We also note that earlier work of Hartman and Watson [22] gives the strict increasingness for all \( \nu(k) \geq \frac{1}{2} \).

Remark. If \( (w(r), \psi(r), \phi(r)) = ((1+r^2)^{-1}, r^{1/2}, r^2) \) then \( \alpha_k(q) = \frac{1}{2} \beta_k(q) \). It follows from Lemma 5.1 that \( \alpha = \frac{1}{3} \), and this shows how Theorem 1.1 recovers the optimal constant in [1.6] (due to Simon [40]).

Proof of Theorem 1.7. First, we consider the case \( (w(r), \psi(r), \phi(r)) = ((1+r^2)^{-1}, (1+r)^{1/2}, r^2) \). By Lemma 5.1 it follows that

\[
\alpha_k(q) = \frac{1}{2}(1 + q)I_{\nu(k)}(q)K_{\nu(k)}(q).
\]

Of course, by Lemma 5.1 we know that \( \phi \rightarrow qI_{\nu(k)}(q)K_{\nu(k)}(q) \) is strictly increasing on \((0, \infty)\). However, \( \phi \rightarrow I_{\nu(k)}(q)K_{\nu(k)}(q) \) is strictly decreasing on \((0, \infty)\). This fact was proved by Phillips and Malin [50] when \( \nu(k) \in \mathbb{N} \) and recently Penfold, Vanden-Broeck and Grandison [53] for all \( \nu(k) \geq 0 \) (see also work of Baricz [3] who extended this to \( \nu(k) \geq \frac{1}{2} \) with a short proof). However, we may immediately reduce considerations to the case \( k = 0 \) because the function \( \nu \rightarrow I_{\nu}(q)K_{\nu}(q) \) is strictly decreasing on \([0, \infty)\) for each fixed \( \nu > 0 \) (see, for example, [4]).

When \( d = 3 \), from (5.1) we have

\[
\alpha_0(q) = \frac{1 + q}{4q^2}(1 - e^{-2q})
\]

and it is straightforward to check this is strictly decreasing for \( q \in (0, \infty) \). Hence \( \alpha = \alpha_0(0) = \frac{1}{2} \) in this case, or equivalently, \( C_3(w, \psi, \phi) = \pi^{1/2} \) as claimed.

When \( d = 5 \), using (5.2) we obtain

\[
\alpha_0(q) = \frac{1}{2}q^{-3}(1 + q^2) e^{-q}(q \cosh q - \sinh q).
\]

We claim that \( \alpha_0 \) has a unique global maximum on \((0, \infty)\). To see this, note that

\[
\alpha_0'(q) = \frac{1}{2}q^{-4}(1 + q) e^{-q}((3 + 2q + 2q^2 + q^3) \sinh q - q(3 + 2q + q^2) \cosh q)
\]

and so it suffices to show that

\[
\Upsilon(q) = (3 + 2q + 2q^2 + q^3) \sinh q - q(3 + 2q + q^2) \cosh q
\]

has a unique positive root. Now

\[
\Upsilon'(q) = (q - 2)(q(1 + q) \cosh q - (1 + q + q^2) \sinh q)
\]

and it is straightforward to check that \( \Upsilon'(q) > 0 \) for \( q \in (0, 2) \) and \( \Upsilon'(q) \leq 0 \) for \( q \in [2, \infty) \). Since \( \Upsilon(0) = 0 \) and

\[
\Upsilon(q) \leq (3 - q) \cosh q < 0
\]

for \( q > 3 \) it follows that \( \Upsilon \) has a unique positive root. It follows that \( \alpha = \alpha_0(q_0) \), where \( q_0 \) is the unique positive solution of \( \Upsilon(q_0) = 0 \), and hence \( C_5(w, \psi, \phi) = (2\pi \alpha_0(q_0))^{1/2} \) as claimed.

Now suppose \( (w(r), \psi(r), \phi(r)) = ((1+r^2)^{-1}, (1+r^2)^{1/4}, r^2) \). Again, from monotonicity in the index, we may reduce considerations to computing \( \alpha = \sup_{q \in [0, \infty)} \alpha_0(q) \).
When $d = 3$, we may simply observe that $\psi(r) \leq (1 + r)^{1/2}$ and the above considerations immediately give that $\alpha = \alpha_0(0) = \frac{1}{2}$, or equivalently $C_3(w, \psi, \phi) = \pi^{1/2}$.

When $d = 5$, we have

$$\alpha_0(\rho) = -\frac{1}{4} \rho^{-3}(1 + \rho^2)^{-1/2}(3 + 6 \rho + 6 \rho^2 + 4 \rho^3 + 2 \rho^4 - 3 \rho^2 - \rho^2(e^{2\rho} - 7))$$

and using the Maclaurin series for $e^{2\rho}$ it follows that $\alpha_0'(\rho) > 0$ for all $\rho \in (0, \infty)$. Therefore, using Theorem 2.2

$$\alpha = \lim_{\rho \to \infty} \alpha_0(\rho) = \frac{1}{4}$$

and hence $C_5(w, \psi, \phi) = (\pi/2)^{1/2}$. $\square$

It is now clear that (1.10) holds for every $(w, \psi, \phi)$ considered to this point. In the homogeneous case considered in Section 4, $\sup_{\rho \in [0, \infty)} \alpha_0(\rho)$ is attained everywhere since $\alpha_0$ is constant (in fact, each $\alpha_k$ is constant in this case). For the inhomogeneous cases considered in Theorem 1.7, the supremum is attained at a unique point (if we allow $\rho = \infty$). We remark that other types of “intermediate” behaviour are possible, including cases where $\alpha_0$ is locally constant. For an explicit (albeit somewhat artificial) example, consider $(w(r), \psi(r), \phi(r)) = (r^{-2}(\mu - \cos(\rho)), 1, r^2)$, where $\mu > 1$ is some fixed constant, and for simplicity let $d = 3$. In this case we have

$$\alpha_k(\rho) = \frac{\mu}{2(2k + 1)} - \frac{1}{2} \int_0^{\infty} J_{k+1/2}(r^2) \frac{\cos(r^2/\rho)}{r} \, dr,$$

where we have made use of (4.5). If we let $\Lambda$ be the tent function given by $\Lambda(r) = (2 - |r|)\chi_{[-2,2]}(r)$, then $\Lambda = \chi_{[-1,1]} * \chi_{[-1,1]}$. Since the Fourier transform of $r \mapsto \frac{1}{r} \sin(r)$ is $\pi \chi_{[-1,1]}$, using the formula (2.3), an explicit computation leads to

$$\alpha_0(\rho) = \frac{\mu}{2} - \frac{1}{4} \Lambda(1/\rho).$$

Thus, $\alpha_0(\rho)$ takes the constant value $\frac{\mu}{2}$ for $\rho \in [0, \frac{1}{2}]$, and, for $\rho \in [\frac{1}{2}, \infty)$ coincides with the decreasing function $\frac{1}{2}(\mu - 1) + \frac{1}{2} \rho^{-1}$. For $k \geq 1$ we have

$$\alpha_k(\rho) \leq \frac{\mu + 1}{2(2k + 1)} < \frac{\mu}{3},$$

where the first inequality follows by trivially estimating the trigonometric part of the weight and (4.5), and the second is true since $\mu > 1$. Hence

$$\alpha = \sup_{\rho \in [0, \infty)} \alpha_0(\rho) = \frac{\mu}{2}$$

which is attained for any $\rho \in [0, \frac{1}{2}]$.

We conclude with the particular case with $d = 3$ and

$$(w(r), \psi(r), \phi(r)) = (\frac{1}{2}N\chi_{I(N)}(r), r^{1/2}, r^2),$$

where $I(N) = (1 - \frac{1}{N}, 1 + \frac{1}{N})$ and $N$ is some fixed positive number which will be taken sufficiently large. As we will see, this is an example where (1.10) is not true. Note that $\alpha \leq N$ in this case, which follows from (1.6). Firstly, we have

$$2\pi \alpha_0(\rho) = 1 - \frac{1}{4} N \rho^{-1}(\sin(2\rho(1 + N^{-1})) - \sin(2\rho(1 - N^{-1})))$$
and therefore
\[ \sup_{\rho \in [0, \infty)} \alpha_0(\rho) \leq \frac{1}{\pi} \]
for each \( N \). We now claim that there exists \( \rho_0 > 0 \) such that, for \( N \) sufficiently large,
\[ \alpha_1(\rho_0) > \frac{1}{\pi}, \tag{5.4} \]
from which it is clear that \((1.16)\) is not true in this case. To see this claim, first note that \( \Xi(\rho_0) = \rho_0^{-1} \sin(\rho_0) - \cos(\rho_0) > 1 \) for some \( \rho_0 \in (0, \pi) \), since \( \Xi(\pi) = 1 \), \( \Xi'(\pi) < 0 \) and by smoothness considerations. Also,
\[ \alpha_1(\rho_0) = \frac{1}{4} N \rho_0 \int_{1-N^{-1}}^{1+N^{-1}} J_{3/2}(\rho_0 r)^2 r \, dr \to \frac{1}{2} \rho_0 J_{3/2}(\rho_0)^2 = \frac{1}{\pi} \Xi(\rho_0)^2 \]
as \( N \) tends to infinity, from which \((5.4)\) follows.

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