On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator

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Abstract: In this paper, we aim to study the oscillatory behavior of a class of even-order advanced differential equations with a non-canonical operator. In addition, we present results on the asymptotic behavior of this type of equations and provide an example that illustrates our main results.

Keywords: oscillation; even-order; advanced differential equations; asymptotic behavior

1. Introduction

In recent decades, many authors have studied problems of a number of different classes of advanced differential equations including the asymptotic and oscillatory behavior of their solutions, see \([1–8]\) and the references cited therein. For some more recent oscillation results, see \([9–20]\). The interest in studying advanced differential equations is also caused by the fact that they appear in models of several areas in science. In \([21–23]\), singular systems of differential equations are used to study the dynamics and stability properties of electrical power systems. Some additional mathematical background on this can be found in \([24]\). Systems of differential equations with delays are used to study additional properties of electrical power systems in \([25,26]\). Non-linear advanced differential equations can be used to describe complex dynamical networks, see \([27–29]\), and bring new insight to their stability. Furthermore, this type of equations can be also used in the modeling of dynamical networks of interacting free-bodies, see \([30]\). Finally, properties of advanced differential equations are used in the study of singular differential equations of fractional order, see \([31,32]\). Several other examples in Physics can be found in \([33]\). In this paper, we consider an even-order non-linear advanced differential equation with a non-canonical operator of the following type:

\[
L_y + q(y(\eta(v))) = 0, \quad L_y := \left( a(v) \left( y^{(x-1)}(v) \right)^{\frac{1}{x}} \right),
\]

where \(v \geq v_0, x\) is even and \(\beta\) is a quotient of odd positive integers. The operator \(L_y\) is said to be in canonical form if \(\int_{v_0}^{\infty} a^{1/\beta}(s) \, ds = \infty\); otherwise, it is called noncanonical. Throughout this work, we suppose that:

\(C1\): \(a \in C^1([v_0, \infty), \mathbb{R})\), \(a(v) > 0, a'(v) \geq 0, \)
\(C2\): \(q, \eta \in C([v_0, \infty), \mathbb{R}), q(v) \geq 0, \eta(v) \geq v, \lim_{v \to \infty} \eta(v) = \infty, \)
\(C3\): \(g \in C(\mathbb{R}, \mathbb{R})\) such that \(g(x)/x^{\beta} \geq k > 0\), for \(x \neq 0\) and under the condition

\[
\zeta(v) = \int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \, ds < \infty.
\]
Definition 1. The function \( y \in C^{k-1}[s, \infty) \), \( s \geq s_0 \), is called a solution of (1), if \( \left( y^{(k-1)}(v) \right)^{\beta} \in C^1[v, \infty), \) for \( a \in C^1([v_0, \infty), \mathbb{R}) \), \( a(v) > 0 \) and \( y(v) \) satisfies (1) on \([v, \infty)\).

Definition 2. Let

\[ D = \{ (v, s) \in \mathbb{R}^2 : v \geq s \geq v_0 \} \text{ and } D_0 = \{ (v, s) \in \mathbb{R}^2 : v > s \geq v_0 \}. \]

A kernel function \( H_i \in C(D, \mathbb{R}) \) is said to belong to the function class \( \mathfrak{H} \), written by \( H \in \mathfrak{H} \), if, for \( i = 1, 2 \),

(i) \( H_i(v, s) > 0 \), on \( D_0 \) and \( H_i(v, s) = 0 \) for \( v > v_0 \) with \( (v, s) \notin D_0 \);

(ii) \( H_i(v, s) \) has a continuous and nonpositive partial derivative \( \partial H_i/\partial s \) on \( D_0 \) and there exist functions \( \tau, \vartheta \in C^1([v_0, \infty), (0, \infty)) \) and \( h_i \in C(D_0, \mathbb{R}) \) such that

\[
\frac{\partial}{\partial s} H_1(v, s) + \frac{\tau'(s)}{\tau(s)} H_1(v, s) = h_1(v, s) H_1^\beta/(\beta+1)(v, s)
\]

and

\[
\frac{\partial}{\partial s} H_2(v, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(v, s) = h_2(v, s) \sqrt{H_2(v, s)}.
\]

Next we will discuss the results in [34–36]. Actually, our purpose in this article is to complement and improve these results. Agarwal et al. in [34,35] studied the even-order nonlinear advanced differential equations

\[
\left( \left( y^{(k-1)}(v) \right)^{\beta} \right)' + q(v) y^{\beta}(\eta(v)) = 0.
\]

By means of the Riccati transformation technique, the authors established some oscillation criteria of (5). Grace and Lalli [36] investigated the second-order neutral Emden–Fowler delay dynamic equations

\[
y^{(k)}(v) + q(v) y(\eta(v)) = 0,
\]

and established some new oscillation for (5) under the condition

\[
\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty.
\]

To prove this, we apply the previous results to the equation

\[
y^{(k)}(v) + \frac{\partial}{\partial v} y(\lambda v) = 0, \ v \geq 1.
\]

if we set \( \kappa = 4 \) and \( \lambda = 2 \), then by applying conditions in [34–36] on Equation (8), we find the results in [35] improves those in [36]. Moreover, the those in [34] improves results in [35,36]. Thus, the motivation in our paper is to complement and improve results in [34–36]. We will use the following methods:

- Integral averaging technique.
- Riccati transformations technique.
- Method of comparison with second-order differential equations.

We will also use the following lemmas from (1):

Lemma 1 ([3]). If \( y^{(i)}(v) > 0, i = 0, 1, ..., \kappa, \) and \( y^{(\kappa+1)}(v) < 0, \) then

\[
\frac{y(v)}{v^{k}/k!} \geq \frac{y'(v)}{v^{k-1}/(k-1)!}.
\]
Lemma 2 ([19]). Suppose that $y \in C^k([v_0, \infty), (0, \infty))$, $y^{(k)}$ is of a fixed sign on $[v_0, \infty)$, $y^{(k)}$ not identically zero and there exists a $v_1 \geq v_0$ such that

$$y^{(k-1)}(v) y^{(k)}(v) \leq 0,$$

for all $v \geq v_1$. If we have $\lim_{v \to \infty} y(v) \neq 0$, then there exists $v_0 \geq v_1$ such that

$$y(v) \geq \frac{\theta}{(k-1)!} v^{k-1} |y^{(k-1)}(v)|,$$

for every $\theta \in (0, 1)$ and $v \geq v_0$.

Lemma 3 ([2]). Let $\beta$ be a ratio of two odd numbers, $V > 0$ and $U$ are constants. Then

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1} V^{\beta-1}}{x^\beta}, \quad V > 0.$$

Lemma 4. Suppose that $y$ is an eventually positive solution of (1). Then, there exist three possible cases:

(S1) $y(v) > 0$, $y'(v) > 0$, $y''(v) > 0$, $y^{(r)}(v) > 0$, $y^{(k)}(v) < 0$,

(S2) $y(v) > 0$, $y''(v) > 0$, $y^{(r+1)}(v) < 0$ for all odd integer $r \in \{1, 3, \ldots, k-3\}$, $y^{(k-1)}(v) > 0$, $y^{(k)}(v) < 0$,

(S3) $y(v) > 0$, $y^{(r-2)}(v) > 0$, $y^{(r-1)}(v) < 0$, $L_y \leq 0$,

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

2. Oscillation Criteria

Theorem 1. Assume that (2) holds. If the differential equations

$$\left(\frac{(k-2)a_1^\beta(v)}{(\theta v^{k-2})^\beta} (y'(v))^\beta\right)' + kq(v) y^\beta(v) = 0, \quad \forall \theta \in (0, 1),$$

(9)

$$y''(v) + y(v) \frac{1}{(k-4)!} \int_v^\infty (\zeta - v)^{k-4} \left(\frac{1}{a(\zeta)} \int_\zeta^\infty q(s) \, ds\right)^{1/\beta} \, d\zeta = 0,$$

(10)

and

$$\left(a(v) (y'(v))^\beta\right)' + y^\beta(v) kq(v) \left(\frac{\xi(v)}{\xi(v)}\right)^\beta \left(\frac{\theta_1}{(k-2)}\right)^{\beta} \eta^{k-2}(v) = 0, \quad \theta_1 \in (0, 1)$$

(11)

are oscillatory for every constant $\theta, \theta_1 \in (0, 1)$, then every solution of (1) is either oscillatory or satisfies

$$\lim_{v \to \infty} y(v) = 0.$$

Proof. Assume to the contrary that $y$ is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have three possible cases (S1), (S2) and (S3). Let case (S1) hold. Using Lemma 2, we find

$$y'(v) \geq \frac{\theta}{(k-2)!} v^{k-2} y^{(k-1)}(v),$$

(12)
for every \( \theta \in (0, 1) \) and for all large \( v \). We set

\[
\varphi (v) := \tau (v) \left( \frac{a (v) \left( y^{(k-1)} (v) \right) }{y^\beta (v)} \right),
\]

and observe that \( \varphi (v) > 0 \) for \( v \geq v_1 \), where \( \tau \in C^1 ([v_0, \infty), (0, \infty)) \) and

\[
\varphi' (v) = \tau' (v) \frac{a (v) \left( y^{(k-1)} (v) \right) }{y^\beta (v)} + \tau (v) \frac{\left( a \left( y^{(k-1)} \right) \right) ' (v)}{y^\beta (v)}
- \beta \tau (v) \frac{y^{\beta-1} (v) y' (v) a (v) \left( y^{(k-1)} (v) \right) ^\beta}{y^{2\beta} (v)}.
\]

Using (12) and (13), we obtain

\[
\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) + \tau (v) \frac{\left( a \left( y^{(k-1)} \right) \right) ' (v)}{y^\beta (v)}
- \beta \tau (v) \frac{(\kappa - 2)! \, \theta v^{\kappa-2} a (v) \left( y^{(k-1)} (v) \right) ^{\beta+1}}{\kappa^{\beta+1}}
\leq \frac{\tau' (v)}{\tau (v)} \varphi (v) + \tau (v) \frac{\left( a \left( y^{(k-1)} \right) \right) ' (v)}{y^\beta (v)}
- \beta \theta v^{\kappa-2} \frac{(\kappa - 2)! \, \theta v \left( y^a (v) \right) }{\kappa^{\beta+1}} \varphi (v) \frac{\kappa^{\beta+1}}{v}.
\]

From (1) and (14), we obtain

\[
\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) - k \tau (v) \frac{q (v) y^\beta (\eta (v))}{y^\beta (v)} - \frac{\beta \theta v^{\kappa-2}}{(\kappa - 2)! \, \theta v \left( y^a (v) \right) } \varphi (v) \frac{\kappa^{\beta+1}}{v}.
\]

Note that \( y' (v) > 0 \) and \( \eta (v) \geq v \), thus, we find

\[
\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) - k \tau (v) q (v) - \frac{\beta \theta v^{\kappa-2}}{(\kappa - 2)! \, \theta v \left( y^a (v) \right) } \varphi (v) \frac{\kappa^{\beta+1}}{v}.
\]

If we set \( \tau (v) = k = 1 \) in (15), then we find

\[
\varphi' (v) + \frac{\beta \theta v^{\kappa-2}}{(\kappa - 2)! \, \theta v \left( y^a (v) \right) } \varphi (v) \frac{\kappa^{\beta+1}}{v} + q (v) \leq 0.
\]

From [37], we can see that Equation (9) is non-oscillatory, which is a contradiction.

Let case \( S_2 \) hold. If we set

\[
\psi (v) := \vartheta (v) \frac{y' (v)}{y (v)},
\]

we see that \( \psi (v) > 0 \) for \( v \geq v_1 \), where \( \vartheta \in C^1 ([v_0, \infty), (0, \infty)) \). By differentiating \( \psi (v) \), we find

\[
\psi' (v) = \frac{\vartheta' (v)}{\vartheta (v)} \psi (v) + \vartheta (v) \frac{y'' (v)}{y (v)} - \frac{1}{\vartheta (v)} \psi (v)^2.
\]
Now, by integrating (1) from \( v \) to \( m \) and using \( y' (v) > 0 \), we get
\[
a (m) \left( y^{(\kappa-1)} (m) \right)^{\hat{\beta}} - a (v) \left( y^{(\kappa-1)} (v) \right)^{\hat{\beta}} = - \int_{v}^{m} q (s) \xi (y (\eta (s))) \, ds.
\]
By virtue of \( y' (v) > 0 \) and \( \eta (v) \geq v \), we get
\[
a (m) \left( y^{(\kappa-1)} (m) \right)^{\hat{\beta}} - a (v) \left( y^{(\kappa-1)} (v) \right)^{\hat{\beta}} \leq -ky^{\hat{\beta}} (v) \int_{v}^{u} q (s) \, ds.
\]
Letting \( m \to \infty \), we see that
\[
a (v) \left( y^{(\kappa-1)} (v) \right)^{\hat{\beta}} \geq ky^{\hat{\beta}} (v) \int_{v}^{\infty} q (s) \, ds
\]
and so
\[
y^{(\kappa-1)} (v) \geq y (v) \left( \frac{k}{a (v)} \int_{v}^{\infty} q (s) \, ds \right)^{1 / \hat{\beta}}.
\]
Integrating again from \( v \) to \( \infty \), \( \kappa - 4 \) times, we get
\[
y'' (v) + \frac{y (v)}{(\kappa - 4)!} \int_{v}^{\infty} (\zeta - v)^{\kappa-4} \left( \frac{k}{a (\zeta)} \int_{\zeta}^{\infty} q (s) \, ds \right)^{1 / \hat{\beta}} \, d\zeta \leq 0. \hspace{1cm} \text{(17)}
\]
From (16) and (17), we obtain
\[
\psi' (v) \leq \frac{\psi' (v)}{\psi (v)} \psi (v) - \frac{\psi (v)}{(\kappa - 4)!} \omega (s) - \frac{1}{\psi (v)} \psi (v)^2, \hspace{1cm} \text{(18)}
\]
where
\[
\omega (s) = \int_{s}^{\infty} (\zeta - v)^{\kappa-4} \left( \frac{k}{a (\zeta)} \int_{\zeta}^{\infty} q (s) \, ds \right)^{1 / \hat{\beta}} \, d\zeta.
\]
If we now set \( \psi (v) = k = 1 \) in (18), then we obtain
\[
\psi' (v) + \psi^2 (v) + \frac{1}{(\kappa - 4)!} \omega (s) \zeta \leq 0.
\]
From [37], we see Equation (10) is non-oscillatory, which is a contradiction.
Let case (S₃) hold. By recalling that \( a (v) \left( y^{(\kappa-1)} (v) \right)^{\hat{\beta}} \) is non-increasing, we obtain
\[
a^{1 / \hat{\beta}} (s) y^{(\kappa-1)} (s) \leq a^{1 / \hat{\beta}} (v) y^{(\kappa-1)} (v), s \geq v \geq v_1.
\]
Dividing the latter inequality by \( a^{1 / \hat{\beta}} (s) \) and integrating the resulting inequality from \( v \) to \( u \), we get
\[
y^{(\kappa-2)} (u) \leq y^{(\kappa-2)} (v) + a^{1 / \hat{\beta}} (v) y^{(\kappa-1)} (v) \int_{v}^{u} a^{-1 / \hat{\beta}} (s) \, ds.
\]
Letting \( u \to \infty \), we obtain
\[
0 \leq y^{(\kappa-2)} (v) + a^{1 / \hat{\beta}} (v) y^{(\kappa-1)} (v) \zeta (v).
\]
Thus,
\[
\frac{-a^{1 / \hat{\beta}} (v) y^{(\kappa-1)} (v) \zeta (v)}{y^{(\kappa-2)} (v)} \leq 1. \hspace{1cm} \text{(19)}
\]
Furthermore, we get
\[
\left( \frac{y^{(k-2)}(v)}{\zeta(v)} \right)' \geq 0,
\]  
(20)
due to (19). Now define
\[
\phi(v) = \frac{a(v) \left( y^{(k-1)}(v) \right)^{\beta}}{(y^{(k-2)}(v))^\beta},
\]  
(21)
we see that \( \phi(v) < 0 \) for \( v \geq v_1 \), and
\[
\phi'(v) = \frac{\left( a(v) \left( y^{(k-1)}(v) \right)^{\beta} \right)'}{(y^{(k-2)}(v))^\beta} - \frac{\beta a(v) \left( y^{(k-1)}(v) \right)^{\beta+1}}{(y^{(k-2)}(v))^{\beta+1}}.
\]
It follows from (1) and (19) that
\[
\phi'(v) = \frac{-kq(v) y^{\beta} (\eta(v))}{(y^{(k-2)}(v))^\beta} - \frac{\beta \phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.
\]
From Lemma 2, we find
\[
y(v) \geq \frac{\theta_1}{(k-2)!} v^{k-2} y^{(k-2)}(v).
\]  
(22)
Thus, we have
\[
\phi'(v) = \frac{-kq(v) y^{\beta} (\eta(v))}{(y^{(k-2)}(v))^\beta} - \frac{\beta \phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.
\]
From (22), we obtain
\[
\phi'(v) \leq -kq(v) \left( \frac{\theta_1 v^{k-2}(v)}{(k-2)!} \right)^\beta \left( \frac{\eta(v)}{\zeta(v)} \right)^\beta - \frac{\beta \phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.
\]  
(23)
From [37], we can see that Equation (11) is non-oscillatory, which is a contradiction. Theorem 1 is proved. 

Remark 1. It is well known (see [15]) that if
\[
\int_{v_0}^{\infty} \frac{1}{a(v)} \, dv < \infty, \text{ and } \liminf_{v \to \infty} \left( \int_{v_0}^{v} \frac{1}{a(s)} \, ds \right) \int_{v}^{\infty} \left( \int_{v_0}^{v} \frac{1}{a(s)} \, ds \right)^2 q(s) \, ds > \frac{1}{4},
\]
then Equations (9)–(11) with \( \beta = 1 \) are oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with \( \beta = 1 \).

Theorem 2. Let \( \beta = k = 1 \) and assume that (2) holds. If for \( \theta, \theta_1 \in (0,1) \)
\[
\liminf_{v \to \infty} \left( \int_{v_0}^{v} \frac{\theta s^{x-2}}{(k-2)! a(s)} \, ds \right) \left( \int_{v_0}^{v} \frac{\theta s^{x-2}}{(k-2)! a(s)} \, ds \right)^2 q(s) \, ds > \frac{1}{4},
\]  
(24)
with
\[
\int_{v_0}^{\infty} \frac{\theta v^{x-2}}{(k-2)! a(v)} \, dv < \infty,
\]
and if
\[ \liminf_{t \to \infty} \frac{1}{t (k-1)!} \int_t^\infty (\zeta - v)^{k-4} \left( \frac{1}{\sigma (\zeta)} \int_0^\infty q (s) \, ds \right)^{1/\beta} \, d\zeta dv > \frac{1}{4}, \]  
(25)

\[ \liminf_{t \to \infty} \left( \int_t^\infty \frac{1}{\sigma (s)} \, ds \right)^{-1} \int_t^\infty \left( \int_t^\infty \frac{1}{\sigma (s)} \, ds \right)^2 \frac{\theta (s)}{\zeta (s) (k-2)!} \, \frac{\zeta (s) (k-2)!}{\zeta (s) (k-2)!} \, ds > \frac{1}{4}, \]  
(26)

then every solution of (1) is either oscillatory or satisfies \( \lim_{v \to \infty} y (v) = 0 \).

In the next theorem, we employ the integral averaging technique to establish a Philos-type oscillation criteria for (1):

**Theorem 3.** Let (2) holds. If there exist positive functions \( \tau, \theta \in C^1 ([v_0, \infty), \mathbb{R}) \) such that

\[ \limsup_{v \to \infty} \frac{1}{H_1 (v, v_1)} \int_{v_1}^v (H_1 (v, s) k \tau (s) q (s) - \pi (s)) \, ds = \infty, \]  
(27)

\[ \limsup_{v \to \infty} \frac{1}{H_2 (v, v_1)} \int_{v_1}^v \left( H_2 (v, s) \frac{\theta (s)}{(k-4)!} - \frac{\theta (s)}{h^2 (v, s)} \right) \, ds = \infty, \]  
(28)

and,

\[ \limsup_{v \to \infty} \frac{1}{H_3 (v, v_1)} \int_{v_1}^v \left( H_3 (v, s) kq (s) \left( \frac{\theta (s)}{(k-2)!} \right)^{\beta} \xi (s) - \pi (s) \right) \, ds = \infty, \]

where

\[ \pi (s) = \frac{k^\beta+1 (v, s) H_1^\beta (v, s) ((k-2)!)^\beta}{(\beta+1)^\beta} \frac{\tau (s) a (s)}{(\theta s^{k-2})^\beta} \]

and

\[ \tilde{\pi} (s) = \frac{\beta^\beta+1 H_3 (v, s) a^1(v)}{(\beta+1)^{\beta+1}} \frac{1}{a^{1/\beta} (s) \zeta (s)}. \]

Then every solution of (1) is either oscillatory or satisfies \( \lim_{v \to \infty} y (v) = 0 \).

**Proof.** Assume to the contrary that \( y \) is a positive solution of (1). Then, we can suppose that \( y (v) = y (\sigma (v)) \) is positive for all \( v \geq v_1 \) sufficiently large. From Lemma 4, we have three possible cases (\( S_1 \)), (\( S_2 \)) and (\( S_3 \)). Assume that (\( S_1 \)) holds. From Theorem 1, we get that (13) holds. Multiplying (13) by \( H_1 (v, s) \) and integrating the resulting inequality from \( v_1 \) to \( v \) we find that

\[ \int_{v_1}^v H_1 (v, s) k \tau (s) q (s) \, ds \leq \varphi (v_1) H_1 (v, v_1) + \int_{v_1}^v \left( \frac{\partial}{\partial s} H_1 (v, s) + \frac{\tau (s) H_1 (v, s)}{\tau (s)} \right) \varphi (s) \, ds \]

\[ - \int_{v_1}^v \frac{\beta \theta s^{k-2}}{(k-2)! (\tau (s) a (s))^{\beta+1}} H_1 (v, s) \varphi (s) \, ds. \]

From (3), we get

\[ \int_{v_1}^v H_1 (v, s) k \tau (s) q (s) \, ds \leq \varphi (v_1) H_1 (v, v_1) + \int_{v_1}^v h_1 (v, s) H_1^{\beta/ (\beta+1)} (v, s) \varphi (s) \, ds \]

\[ - \int_{v_1}^v \frac{\beta \theta s^{k-2}}{(k-2)! (\tau (s) a (s))^{\beta}} H_1 (v, s) \varphi (s) \, ds. \]  
(29)

Using Lemma 3 with \( V = \beta \theta s^{k-2} / ((k-2)! (\tau (s) a (s))^{1/2}) \), \( U = h_1 (v, s) H_1^{\beta/ (\beta+1)} (v, s) \)
And \( y = \varphi (s) \), we get
\[
 h_1 (v, s) H_1^{\beta / (\beta + 1)} (v, s) \varphi (s) - \frac{\beta \theta s^{\kappa - 2}}{(\kappa - 2)! \left( \tau (s) a(s) \right)^{\beta}} H_1 (v, s) \varphi^{\beta + 1} (s)
\]
\[
\leq h_1^{\beta + 1} (v, s) H_1^\beta (v, s) \left( (\kappa - 2)! \left( \tau(s) a(s) \right)^{\beta} \right) \frac{\phi_{s - 2}^{\beta}}{\theta s^{\kappa - 2}},
\]
which, with (29) gives
\[
\frac{1}{H_1 (v, v_1)} \int_{v_1}^v (H_1 (v, s) k \tau (s) q (s) - \pi (s)) ds \leq \varphi (v_1),
\]
which contradicts (27). Assume that (S₂) holds. From Theorem 1, we get that (18) holds. Multiplying (18) by \( H_2 (v, s) \) and integrating the resulting inequality from \( v_1 \) to \( v \), we obtain
\[
\int_{v_1}^v H_2 (v, s) \frac{\theta (s)}{(\kappa - 4)!} \omega (s) ds \leq \psi (v_1) H_2 (v, v_1)
\]
\[
+ \int_{v_1}^v \left( \frac{\partial}{\partial s} H_2 (v, s) + \frac{\theta' (s)}{\theta (s)} H_2 (v, s) \right) \psi (s) ds
\]
\[
- \int_{v_1}^v \frac{1}{\theta (s)} H_2 (v, s) \varphi^2 (s) ds.
\]
Thus, from (4), we obtain
\[
\int_{v_1}^v H_2 (v, s) \frac{\theta (s)}{(\kappa - 4)!} \omega (s) ds \leq \psi (v_1) H_2 (v, v_1) + \int_{v_1}^v h_2 (v, s) \sqrt{H_2 (v, s) \psi (s)} ds
\]
\[
- \int_{v_1}^v \frac{1}{\theta (s)} H_2 (v, s) \varphi^2 (s) ds
\]
\[
\leq \psi (v_1) H_2 (v, v_1) + \int_{v_1}^v \frac{\theta (s) h_2^2 (v, s)}{4} ds
\]
and so
\[
\frac{1}{H_2 (v, v_1)} \int_{v_1}^v \left( H_2 (v, s) \frac{\theta (s)}{(\kappa - 4)!} \omega (s) - \frac{\theta (s) h_2^2 (v, s)}{4} \right) ds \leq \psi (v_1),
\]
which contradicts (28). Assume that (S₃) holds. Using (19) and (21), we see that
\[
- \phi (v) \zeta^{\beta} (v) \leq 1
\]
due to (30). Multiplying this inequality by \( \zeta^{\beta} (v) \) and integrating the resulting inequality from \( v_1 \) to \( v \), we get
\[
\zeta^{\beta} (v) \phi (v) - \zeta^{\beta} (v_1) \phi (v_1) + \beta \int_{v_1}^v a^{-1/\beta} (s) \zeta^{\beta - 1} (s) \phi (s) ds
\]
\[
\leq - \int_{v_1}^v k q (s) \left( \frac{\theta_1 \eta s^{\kappa - 2} (s)}{(\kappa - 2)!} \right)^{\beta} \zeta^{\beta} (\eta (s)) ds - \beta \int_{v_1}^v \frac{\phi^{\beta/\beta + 1} (s)}{a^{1/\beta} (s)} \zeta^{\beta} (s) ds.
\]
Multiplying (31) by $H_3(v,s)$, we find that
\[
\int_{v_1}^{v} H_3(v,s) k q(s) \left( \frac{\theta s^{k-2} (s)}{(k-2)!} \right)^\beta \zeta^\beta (\eta(s)) \, ds \leq \zeta^\beta (v_1) \phi(v_1) H_3(v,v_1) - \zeta^\beta (v) \phi(v) H_3(v,v_1)
\]
\[
+ \int_{v_1}^{v} \beta a^{-1/\beta} (s) \zeta^{\beta-1} (s) \phi(s) H_3(v,s) \, ds
\]
\[
- \int_{v_1}^{v} \frac{\beta \phi^{\beta+1} (s)}{a^{1/\beta} (s)} \zeta^\beta (s) H_3(v,s) \, ds.
\]

Using Lemma 3 with $V = \zeta^\beta (s) H_3(v,s) / a^{1/\beta} (s)$, $U = a^{-1/\beta} (s) \zeta^{\beta-1} (s) H_3(v,s)$ and $y = \phi(s)$, we get
\[
\beta a^{-1/\beta} (s) \zeta^{\beta-1} (s) \phi(s) H_3(v,s) - \frac{\beta \phi^{\beta+1} (s)}{a^{1/\beta} (s)} \zeta^\beta (s) H_3(v,s)
\]
and easily, we find that
\[
\frac{1}{H_3(v,v_1)} \int_{v_1}^{v} \left( H_3(v,s) k q(s) \left( \frac{\theta s^{k-2} (s)}{(k-2)!} \right)^\beta \zeta^\beta (\eta(s)) - \eta (s) \right) \, ds \leq \zeta^\beta (v_1) \phi(v_1) + 1,
\]
which contradicts (27). This completes the proof. $\square$

**Example 1.** We consider the equation
\[
\left( u^5 y''' (v) \right)' + v q_0 y (3v) = 0, \quad v \geq 1,
\]
where $q_0 > 0$ is a constant. Note that $\beta = 1$, $\kappa = 4$, $a(v) = v^5$, $q(v) = v q_0$ and $\eta(v) = 3v$. If we set $k = 1$, then condition (24) becomes
\[
\liminf_{v \to \infty} \left( \int_{v_0}^{v} \frac{\theta s^{k-2} (s)}{(k-2)! a(s)} \, ds \right)^{-1} \int_{v_0}^{v} \left( \int_{v_0}^{v} \frac{\theta s^{k-2} (s)}{(k-2)! a(s)} \, ds \right)^2 q(s) \, ds
\]
\[
= \liminf_{v \to \infty} \left( 4v^2 \right) \int_{v}^{\infty} \frac{q_0}{16s^3} \, ds = \liminf_{v \to \infty} \left( 4v^2 \right) \left( \frac{q_0}{32v^2} \right)
\]
\[
= \frac{q_0}{8} > \frac{1}{4},
\]
while condition (25) becomes
\[
\liminf_{v \to \infty} v \int_{v_0}^{v} \frac{1}{(k-4)!} \int_{v}^{v} (s-v)^{k-4} \left( \frac{1}{a(\xi)} \int_{\xi}^{v} q(s) \, ds \right)^{1/\beta} d\xi dv = \liminf_{v \to \infty} v \left( \frac{q_0}{4v} \right)
\]
\[
= \frac{q_0}{4} > \frac{1}{4},
\]
and hence condition (26) is satisfied. Therefore, from Theorem 2, all solutions of Equation (32) are oscillatory if $q_0 > 2$.

**Remark 2.** One can easily see that the results obtained in [18,19] cannot be applied to conditions in Theorem 2, so our results are new.
Remark 3. We can generalize our results by studying the equation in the form

\[
\left( a(v) \left( y^{(k-1)}(v) \right)^{\beta} \right)' + \sum_{i=1}^{j} q_i(v) y^{\beta}(\eta_i(v)) = 0, \text{ where } v \geq v_0, \ j \geq 1.
\]

For this we leave the results to researchers interested.

3. Conclusions

In this article we provided three new Theorems on the oscillatory and asymptotic behavior of a class of even-order advanced differential equations with a non-canonical operator in the form of (1).

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following cases:

\[ (S_1) \quad y(v) > 0, \ y'(v) > 0, \ y^{(k-2)}(v) > 0, \ y^{(k-1)}(v) \leq 0, \ \left( a(v) \left( y^{(k-1)}(v) \right)^{\beta} \right)' \leq 0, \]

\[ (S_2) \quad y(v) > 0, y^{(r)}(v) < 0, y^{(r+1)}(v) > 0, \forall r \in \{1, 3, \ldots, k-3\}, \]

and \( y^{(k-1)}(v) < 0, \ \left( a(v) \left( y^{(k-1)}(v) \right)^{\beta} \right)' \leq 0. \)

For all this there is some research in progress.

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