SCALE INVARIANT FOURIER RESTRICTION TO A HYPERBOLIC SURFACE

BETSY STOVALL

Abstract. This result sharpens the bilinear to linear deduction of Lee and Vargas for extension estimates on the hyperbolic paraboloid in $\mathbb{R}^3$ to the sharp line, leading to the first scale-invariant restriction estimates, beyond the Stein–Tomas range, for a hypersurface on which the principal curvatures have different signs.

1. Introduction

We consider the Fourier restriction/extension problem for the hyperbolic paraboloid

$$S := \{(\tau, \xi) \in \mathbb{R}^{1+2} : \tau = \xi_1 \xi_2\}.$$  

We denote by $\mathcal{E}$ the extension operator,

$$\mathcal{E}f(t, x) := \int_{\mathbb{R}^2} e^{i(t,x) \cdot (\xi_1 \xi_2 \xi)} f(\xi) \, d\xi.$$  \hspace{1cm} (1.1)

For consistency of exponents, we will consider the problem of establishing $L^r \to L^{2s}$ extension estimates for $\mathcal{E}$, and we are primarily interested in the case when $r = s'$.

In [4, 7], Lee and Vargas independently established an essentially optimal $L^2$-based bilinear adjoint restriction estimate for $S$. This result states that if $f$ and $g$ are supported in $1 \times 1$ rectangles that are separated from one another by a distance $1$ in the horizontal direction and $1$ in the vertical direction, then

$$\|\mathcal{E}f \mathcal{E}g\|_s \lesssim \|f\|_r \|g\|_r.$$  \hspace{1cm} (1.2)

This two-parameter separation of the tiles is both necessary and troublesome. On the one hand, necessity can be seen by considering the case when each of $f_\pm$ is supported on a $\frac{1}{4}$-neighborhood of $(\pm 1, 0)$. On the other hand, the separation leads to difficulty in deducing linear restriction estimates from the bilinear ones. Indeed, the natural analogue of the Whitney decomposition approach of [3] leads to a sum in two scales, length and width, rather than a single distance scale, leading to a loss of the scaling line in the distinct approaches of Lee [4] and Vargas [7].

The purpose of this note is to overcome this obstacle and recover the sharp line.

Theorem 1.1. With $\mathcal{E}$ as in (1.1), assume that the estimate

$$\|\mathcal{E}f \mathcal{E}g\|_s \lesssim \|f\|_r \|g\|_r$$  \hspace{1cm} (1.3)

holds for some $\frac{3}{2} < s < 2$ and $r < s'$, whenever $f$ and $g$ are supported on $1 \times 1$ rectangles that are separated from one another by a distance $1$ in both the horizontal and vertical directions. Then $\mathcal{E}$ is of restricted strong type $(s', 2s)$, and consequently of strong type $(\tilde{s}', 2\tilde{s})$ for all $\tilde{s} > s$.  

To put the hypothesis on $s$ in context, we recall that for $s \leq \frac{3}{2}$, linear extension estimates are known to be impossible, and for $s \geq 2$, they are already known,\cite{6}.

As is well-known, a (local, linear) $L^{r_0} \to L^{2s_0}$ extension estimate for some $r_0 > s_0'$ allows us, by interpolation with the $L^2$-based bilinear extension estimate (1.2), to establish the $L^r$-based bilinear extension estimate (1.3) for some $r > s_0$ and $r < s'$. Replacing $s_0$ with $s$ is a loss (the extent of which depends on $1 - r_0^{-1} - s_0^{-1}$), but $r < s'$ is a gain in the sense that the corresponding linear extension estimate $E : L^r \to L^{2s}$ is false.

In \cite{4,7}, Lee and Vargas independently used the bilinear extension estimate (1.2) to prove that

$$\|E f\|_{2s} \lesssim \|f\|_{L^r}, \quad (1.4)$$

for all $s > \frac{5}{2}$, $r > s'$, and $f$ supported in the unit ball. In \cite{1}, Cho–Lee used Guth’s polynomial partitioning argument from \cite{2} to prove (1.4) for $f$ supported in the unit ball and $2s = r > 3.25$; this was subsequently improved by Kim \cite{3} to the range $2s > 3.25$ and $r > s'$. Using these results and the discussion in the preceding paragraph, Theorem 1.1 immediately yields the following slight improvement on Kim’s result.

**Corollary 1.2.** For $2s > 3.25$, the extension operator $E$ is bounded from $L^{s'}$ to $L^{2s}$.

To the author’s knowledge, this is the first scalable restriction estimate for a negatively curved hypersurface, beyond the Stein–Tomas range ($s = 2$).

**Terminology.** A constant will be said to be admissible if it depends only on $s, r$. The inequality $A \lesssim B$ means that $A \leq CB$ for some implicit constant $C$, and implicit constants will be allowed to change from line to line. A dyadic interval is an interval of the form $[m2^{-n}, (m + 1)2^{-n}]$, for some $m, n \in \mathbb{Z}$, and $I_n$ denotes the set of all dyadic intervals of length $2^{-n}$. A tile is a product of two dyadic intervals, and $D_{J,K}$ denotes the set of all $2^{-J} \times 2^{-K}$ tiles.

**Outline of proof.** To prove our restricted strong type estimate, it suffices to bound the extension of a characteristic function. Our starting point is the bilinear to linear deduction of Vargas \cite{7}, which shows that, under the hypotheses of Theorem 1.1, the extension of the characteristic function of a set $\Omega$ with roughly constant fiber length obeys the scalable restriction estimate $\|E \chi_{\Omega}\|_{2s} \lesssim |\Omega|^\frac{1}{2}$. In \cite{7}, off-scaling estimates are obtained by subdividing a set $\Omega$ in the unit cube into subsets having constant fiber length. Off-scaling contributions from those subsets with very short fibers are small (because the sets themselves are small), and adding these amounts to summing a geometric series.

We wish to remain on the sharp line, so must be more careful. Our first step, taken in Section 2, is to understand when Vargas’s constant fiber estimate can be improved. To this end, we prove a dichotomy result: If $\Omega$ has constant fiber length, then either $\Omega$ is highly structured (more precisely, $\Omega$ is nearly a tile), or we have a better bound on the extension of $\chi_{\Omega}$. Roughly speaking, this reduces matters to controlling the extension of a union of tiles $\tau_k$ having heights $2^{-k}$, which is the task of Section 3. We can estimate

$$\|E \chi_{\bigcup \tau_k}\|_{2s} \lesssim \left( \sum \|E \chi_{\tau_k}\|_{2s}^2 \right)^{\frac{1}{2}} + \text{off-diagonal terms},$$
where the off-diagonal terms involve products $E \chi^\tau_k E \chi^\tau_{k'}$, with $|k-k'|$ large. Boundedness of the main term follows from Vargas's estimate and convexity ($2s > s'$). It remains to bound the off-diagonal terms, for which it suffices to prove a bilinear estimate with decay:

$$\|E^{\tau_k} E^{\tau_{k'}}\|_s \lesssim 2^{-c_0 |k-k'|} \max\{|\tau_k|, |\tau_{k'}|\} \frac{1}{s'},$$

and we prove this by combining the bilinear extension estimate for separated tiles with a further decomposition.

Of course, we have lied. In Section 2, our dichotomy is not that a constant fiber length set $\Omega$ is either a tile or has zero extension, and so we still have remainder terms that must be summed. To address this, we argue more quantitatively than has been suggested above: Any constant fiber length set can be approximated by a union of tiles, where the number of tiles and tightness of the approximation depends on how sharp is our estimate $\| E\chi_\Omega \|_{2s} \lesssim |\Omega|^\frac{1}{s'}$; then we must bound extensions of sets $\bigcup_k \bigcup_{\tau \in T_k} \tau$, where $T_k \subseteq D_{j(k),k}$ may be large (but fortunately, not too large).

Acknowledgements. This work has been supported in part by a grant from the National Science Foundation (DMS-1600458), and was carried out in part while the author was in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, during the Spring of 2017, a visit that was supported in part by a National Science Foundation grant to MSRI (DMS-1440140). The author would like to thank Sanghyuk Lee, Andreas Seeger, and Ana Vargas, from whom she learned of this problem and some of its history.

2. An inverse problem related to Vargas’s linear estimate

To prove Theorem 1.1, it suffices to prove that $\| E\chi_\Omega \|_{2s} \lesssim |\Omega|^\frac{1}{s'}$, for all measurable sets $\Omega$. By scaling, it suffices to consider $\Omega$ contained in the unit cube, $\tau_0$. In [7], Vargas proved the following.

Theorem 2.1 (Vargas, [7]). For each $K \geq 0$, let

$$\Omega(K) := \{ \xi \in \Omega : H^1(\pi_{1, \xi}) \sim 2^{-K} \}.$$

Then under the hypotheses of Theorem [3], for any measurable set $\Omega' \subseteq \Omega(K)$,

$$\| E\chi_{\Omega'} \|_{2s} \lesssim |\Omega(K)|^\frac{1}{s'}.$$  \hspace{1cm} (2.1)

This version differs slightly from the one stated in [7], but it follows from the same proof. In proving the next proposition, we will review Vargas’s argument, so the reader may verify the above-stated version below.

Our first step is to solve an inverse problem: Characterize those sets $\Omega = \Omega(K)$ for which the inequality in (2.1) can be reversed. We record here a useful rescaling of the bilinear estimate (1.3), namely, for $f, g$ supported on tiles in $D_{j,k}$ that are separated by a distance $2^{-2j}$ in the vertical direction and $2^{-j}$ in the horizontal direction,

$$\| EfEg \|_s \lesssim 2^{-(j+k)(2 - \frac{2}{s} - \frac{2}{s'})} \| f \|_r \| g \|_r.$$ \hspace{1cm} (2.2)

Proposition 2.2. Assume that the hypotheses of Theorem [7] hold. Let $\Omega \subseteq \tau_0$ be a measurable set, and assume that $\Omega = \Omega(K)$ for some integer $K \geq 0$. Choose a nonnegative integer $J$ such that $|\pi_{1,\Omega}| \sim 2^{-J}$, and let $\varepsilon \lesssim 1$ denote the smallest dyadic number such that

$$\| E\chi_{\Omega'} \|_{2s} \leq \varepsilon |\Omega|^\frac{1}{s'},$$
for every measurable $Ω′ ⊆ Ω$. Then $Ω = \bigcup_{0 < δ ≤ ζ} Ω_δ$, with the union taken over dyadic $δ$. For each $δ$, $Ω_δ$ is contained in a union of $O(δ^{-C})$ tiles $τ ∈ T_δ ⊆ D_{J,K}$, and for each subset $Ω′ ⊆ Ω_δ$, $|Eχ_{Ω′}|_{L^2} ≤ \frac{δ|Ω|^{\frac{1}{2}}}{δ^{1}}$.

**Proof of Proposition 2.2.** Our decomposition will be done in three stages. Our first decomposition will be of $Ω$ into sets $Ω_1^n$, with $π_1 (Ω_1^n)$ nearly an interval, $I ∈ I_1$. Our second decomposition will be of $Ω_1^n$ into sets $Ω_2^n, \rho ≤ η$, each of which is nearly a product of $I$ with a set of measure $2^{-K}$. Our third decomposition will be of $Ω_2^n, \rho$ into sets $Ω_3^n, \rho, δ ≤ ρ$, each of which is nearly a product of $I$ with an interval in $I_1$.

The product of two dyadic intervals is a tile, so we take $Ω_3 := \bigcup_{\rho ≥ δ} \bigcup_{η ≥ ρ} Ω_3^n, \rho, δ$; the $(\log δ)^{-2}$ factor that arises from taking this union is harmless.

Let $S := π_1 (Ω)$. We know that $|S| \sim 2^{-J}$ and that $S ⊆ [-1, 1]$. Let $ξ_1 ∈ S$, and for each $0 < η < \varepsilon$, let $I_η (ξ_1)$ be the maximal dyadic interval $I ⊇ ξ_1$ satisfying $|I ∩ Ω| ≥ η^C |I|$, if such an interval exists. We record that $|I_η (ξ_1)| ≤ η^{-C} 2^{-J}$, and if $ξ_1$ is a Lebesgue point of $S$, then $|I_η (ξ_1)| > 0$. Let

$$T_η := \{ξ_1 \in S : |I_η (ξ_1)| ≥ η^{C} 2^{-J}\},$$

and let $S_η := T_η \setminus T_{2η}$, for dyadic $0 < η < \varepsilon$. Then a.e. (indeed, every Lebesgue) point of $S$ is contained in a unique $S_η$. We set $Ω_1^n := Ω ∩ π_1^{-1} (S_η)$.

**Lemma 2.3.** For each $0 < η ≤ \varepsilon$, $S_η$ is contained in a union of $O(η^{-2C})$ dyadic intervals $I ∈ I_1$, and for each $η < \varepsilon$ and each subset $Ω′ ⊆ Ω_1^n$, $|Eχ_{Ω′}|_{L^2} ≤ η^2 |Ω|^{\frac{1}{2}}$.

**Proof of Lemma 2.3.** All of the conclusions except for the bound on the extension $Eχ_{Ω′}$ with $Ω′ ⊆ Ω_1^n$ are immediate. To establish this bound, we optimize Vargas’s proof of Theorem 2.1. The argument is largely the same as that in [7], so we will be brief. Performing a Whitney decomposition in each variable $ξ_1, ξ_2$ separately and applying the almost orthogonality lemma from [5] (for which it is important that $s ≤ 2$),

$$|Eχ_{Ω′}|_{L^2}^2 \leq \sum_{k,j} \left( \sum_{τ, τ′ ∈ D_{j,k}} |Eχ_{Ω′ ∩ τ} Eχ_{Ω′ ∩ τ′}|_{L^2}^2 \right)^{\frac{1}{2}},$$

where we say that $τ ∼ τ′$ if $τ$ and $τ′$ are $2^{-J}$ separated in the horizontal direction and $2^{-k}$ separated in the vertical direction.

From our hypothesis that we have bilinear extension estimates for some $r < s′ < 2s$,

$$|Eχ_{Ω′}|_{L^2}^2 \leq \sum_{k,j} 2^{-(j+k)(2^{-r} - \frac{1}{2})} \sum_{τ} |τ|^{\frac{1}{4}} |Ω′ ∩ τ|^{\frac{1}{4}} \lesssim \sum_{k,j} 2^{-(j+k)(2^{-r} - \frac{1}{2})} \max_{τ ∈ D_{j,k}} |τ|^{\frac{1}{4}} |Ω′|^{\frac{1}{4}}.$$

To bound this double sum, Vargas used the inequality

$$|Ω′ ∩ τ| \lesssim \min \{2^{-j}, 2^{-J}\} \min \{2^{-k}, 2^{-K}\}.$$

The definition of $Ω_1^n$ will allow us to improve on this bound.

Take $C$ exactly as in the definition of $T_η$. For $I_j ∈ I_1$, we have the trivial bound

$$|I_η ∩ S_η| ≤ \min \{|I_j|, |S_η|\} ≤ \min \{2^{-j}, 2^{-J}\},$$

but when $|j - J| < C \log η^{-1}$, we get a dramatic improvement. Indeed, if $η^{C} 2^{-J} ≤ 2^{-j} ≤ η^{-C} 2^{-J}$, then $|S_η ∩ I_j| < η^{C} |I_j|$, and if $η^{2C} 2^{-J} < 2^{-j} < η^{C} 2^{-J}$, then
\(|S_\eta \cap I_j| \lesssim \eta^{2C} 2^{-J}\); in each case, were the stated bound to fail, we could find a point \(\xi_1 \in S_\eta\) that belonged to some \(S_{\eta'}\) with \(\eta' > \eta\), a contradiction. As
\[
|\Omega' \cap (I_j \times I_k)| \leq |S_\eta \cap I_j| \min\{2^{-k}, 2^{-K}\},
\]
the above improvement and the inequalities \(r < s' < 2s\) lead to
\[
\|E_{\chi_{\Omega'}}\|_{2s} \lesssim \eta^{C'} 2^{-(J+K)(\frac{1}{2} - \frac{1}{s})} |\Omega'|^\frac{1}{s} \lesssim \eta^{C'} |\Omega|^\frac{1}{s},
\]
for \(C' > 0\) some admissible constant dictated by \(C, r, s\); we can reverse engineer \(C\) so that \(C' = 2\).

Now for our second decomposition. Although \(\pi_1(\Omega_\eta^1)\) (roughly) projects down to a small number of intervals, an individual horizontal slice \(\pi_2^{-1}(\xi_2) \cap \Omega_\eta^1\) might be much smaller. Our next step is to decompose into sets where the size of a nonempty slice is roughly comparable to the size of the projection of the whole. (Sets with this property are nearly products.)

Fix \(0 < \eta \leq \varepsilon\). For dyadic \(0 < \rho \leq \eta\), we define
\[
V_\rho = \{\xi_2 \in \pi_2(\Omega_\eta^1) : H^1(\pi_2^{-1}(\xi_2) \cap \Omega_\eta^1) \geq \rho C 2^{-J}\},
\]
and set \(U_\eta := V_\eta, U_\rho := V_\rho \backslash V_2\rho\), for \(\rho < \eta\). We define \(\Omega_{\eta,\rho}^2 := \pi_2^{-1}(U_\rho) \cap \Omega_\eta^1\).

**Lemma 2.4.** For each \(0 < \rho < \eta \leq \varepsilon\), and each subset \(\Omega' \subseteq \Omega_{\eta,\rho}^2\), \(\|E_{\chi_{\Omega'}}\|_{2s} \lesssim \rho^2 |\Omega|^\frac{1}{s}\).

**Proof of Lemma 2.4.** The proof is similar to Lemma 2.3, the only difference being that the bound on the intersection of a tile \(\tau \in \mathcal{D}_{j,k}\) with \(\Omega' \subseteq \Omega_{\eta,\rho}^2\) is
\[
|\tau \cap \Omega'| \lesssim \min\{\rho C^2 2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\}.
\]
Alternately, one may deduce this directly from Vargas’ Theorem 2.1 by interchanging the indices, and then using the size estimate \(|\Omega_{\eta,\rho}^2| \lesssim \rho C 2^{-(J+K)} \sim \rho^C |\Omega|\), for \(\rho < \eta\).

Now our third decomposition. A single \(\Omega_{\eta,\rho}^2\) is “nearly” a product, but \(\pi_2(\Omega_{\eta,\rho}^2)\) might be far from an interval. This can be fixed in a similar manner to the decomposition of \(\Omega\) into the \(\Omega_\eta^1\). Indeed, we perform exactly the same decomposition as before, only interchanging the roles of the indices.

We complete the proof by taking unions as described at the outset. The factors of \(\eta^2\) and \(\rho^2\) in Lemmas 2.3 and 2.4 (and the factor of \(\delta^2\) in the analogue for \(\Omega_{\eta,\rho,\delta}^3\)) mean that the resulting factor of \((\log \delta^{-1})^2\) is indeed harmless. \(\square\)

### 3. Extensions of characteristic functions of near tiles

We will complete the proof of Theorem 1.1 by summing the extensions of the sets that arise in Proposition 2.2. Let \(\mathcal{K}(\varepsilon)\) denote the collection of all \(K \in \mathbb{Z}_{\geq 0}\) for which \(\varepsilon\) is the smallest dyadic number for which \(\|E_{\chi_{\Omega'}}\|_{2s} \leq \varepsilon |\Omega(K)|^\frac{1}{s}\) holds for all \(\Omega' \subseteq \Omega(K)\).

**Lemma 3.1.** For \(\varepsilon > 0, 0 < \delta \leq \varepsilon\), under the hypotheses of Theorem 1.1
\[
\| \sum_{K \in \mathcal{K}(\varepsilon)} E_{\chi_{\Omega_{\delta}(K)}} \|_{2s}^2 \lesssim (\log \delta^{-1})^2 \sum_{K \in \mathcal{K}(\varepsilon)} \|E_{\chi_{\Omega_{\delta}(K)}}\|_{2s}^2 + \delta |\Omega|^\frac{1}{s}.
\]
Proof of Lemma 3.1. It suffices to prove
\[
\| \sum_{K \in \mathcal{K}} \mathcal{E} \chi_{\Omega(K)} \|_{L^2}^2 \lesssim \sum_{K \in \mathcal{K}} \| \mathcal{E} \chi_{\Omega(K)} \|_{L^2}^2 + \delta^2 |\Omega| \frac{\delta}{T},
\]
with \( \mathcal{K} \subseteq \mathcal{K}(\epsilon) \) chosen so that \( \mathcal{K} \) and \( J(K) \) are both \( A \log \delta^{-1} \)-separated, with \( A \) a sufficiently large admissible constant. (\( A \) will be much larger than the constant \( C \) in Proposition 2.2.) Since \( s < 2 \), the triangle inequality gives
\[
\| \sum_{K \in \mathcal{K}} \mathcal{E} \chi_{\Omega(K)} \|_{L^2}^2 = \int \| \sum_{K \in \mathcal{K}} \prod_{i=1}^{4} \mathcal{E} \chi_{\Omega(K_i)} \|_2^2 \lesssim \sum_{K \in \mathcal{K}} \| \mathcal{E} \chi_{\Omega(K)} \|_{L^2}^2 + \sum_{i=1}^{4} \| \prod_{i=1}^{4} \mathcal{E} \chi_{\Omega(K_i)} \|_2^2,
\]
where \( \sum' \) indicates a sum taken on quadruples \( \mathcal{K} = (K_1, K_2, K_3, K_4) \in \mathcal{K}^4 \), with at least two entries distinct. We take a moment from the proof of Lemma 3.1 to prove the following.

Lemma 3.2. If \( K, K' \in \mathcal{K} \), and \( J := J(K), J' := J(K') \), then
\[
\| \mathcal{E} \chi_{\Omega(K)} \mathcal{E} \chi_{\Omega(K')} \|_{L^2} \lesssim 2^{-c_0|K-K'|} \max(|\Omega(K)|, |\Omega(K')|) \frac{\delta}{T},
\]
for some admissible constant \( c_0 > 0 \).

Proof of Lemma 3.2. If \( K = K' \), the inequality is a trivial consequence of Cauchy–Schwarz and (2.1). If \( J = J' \), again apply Cauchy–Schwarz and (2.1), since
\[
|\Omega(K)| \frac{\delta}{T} \sim 2^{-c_0|K-K'|} \max(|\Omega(K)|, |\Omega(K')|) \frac{\delta}{T}.
\]
Thus it remains to consider the cases when \( J \) and \( J' \), and likewise, \( K \) and \( K' \), differ. By symmetry, it suffices to consider the cases \( J < J', K < K' \); and \( J > J', K < K' \).

If \( J < J' \) and \( K < K' \), then \( |\Omega(K)| \sim 2^{-(J+K)} \geq 2^{-|K-K'|}|\Omega(K')| \), so (3.1) follows from Theorem 2.1 and Cauchy–Schwarz.

Thus we may assume that \( K < K' \) and \( J > J' \). By Proposition 2.2 and the separation condition on \( \mathcal{K} \), it suffices to prove that
\[
\| \mathcal{E} \chi_{\Omega(K)} \mathcal{E} \chi_{\Omega(K')} \|_{L^2} \lesssim 2^{-c_0|K-K'|}|\Omega(K)| \frac{\delta}{T} \max(|\Omega(K)|, |\Omega(K')|) \frac{\delta}{T},
\]
for tiles \( \tau \in \mathcal{T}_j(K), \tau' \in \mathcal{T}_j(K') \).

Note that our conditions on \( J, J', K, K' \) mean that \( \tau \) is taller than \( \tau' \), and \( \tau' \) is wider than \( \tau \). By translating, we may assume that the \( y \)-axis forms the center line of \( \tau \) and that the \( x \)-axis forms the center line of \( \tau' \). Recalling that our tiles are contained in \( 2\tau_0 \), we decompose:
\[
\tau = \bigcup_{k=0}^{K'} \tau_k, \quad \tau_k = \tau \cap \{ \xi : |\xi_2| \sim 2^{-k} \}, \quad \tau' = \bigcup_{j=0}^{J'} \tau'_j, \quad \tau'_j = \tau' \cap \{ \xi : |\xi_1| \sim 2^{-j} \}.
\]

By the (2-parameter) Littlewood–Paley square function estimate (the two-parameter version can be proved using Khintchine’s inequality), the fact that \( s < 2 \), and the
triangle inequality,
\[ \|\mathcal{E}\chi_{\tau}(K)\mathcal{E}\chi_{\tau'}(K')\|_{s}^{2} \lesssim \sum_{k=0}^{K'} \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\mathcal{E}\chi_{\tau'_{j}\cap\Omega(K')}\|_{2s}^{2}. \tag{3.3} \]

We begin with the sum over those terms with \( k = K' \). By Cauchy–Schwarz and \([2.1]\),
\[ \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\mathcal{E}\chi_{\tau'_{j}\cap\Omega(K')}\|_{s} \lesssim \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\|_{2s} \|\mathcal{E}\chi_{\tau'_{j}\cap\Omega(K')}\|_{2s} \lesssim \sum_{j=0}^{J} |\tau_{K'}|^{\frac{1}{2}^{r'}} |\tau'_{j}|^{\frac{1}{2}^{r}}. \]

Because of the way the \( \tau'_{j} \) were defined, we have at most two nonempty \( \tau'_{j} \) with \( j \leq J' \). This, combined with the bound \(|\tau'_{j}| \leq \min\{2^{-(j-J')}, 1\}|\tau'_{j}| \) gives \( \sum_{j} |\tau'_{j}|^{\frac{1}{2}^{r}} \lesssim |\tau'|^{\frac{1}{2}^{r}} \) (despite the fact that \( s < s' \)). Since \(|\tau_{K'}| \sim 2^{-(K'-K)|\tau'|}, |\tau| \sim |\Omega(K)|, \) and \(|\tau'| \sim |\Omega(K')|\),
\[ \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\mathcal{E}\chi_{\tau'_{j}\cap\Omega(K')}\|_{s}^{2} \lesssim 2^{-(K'-K)|\tau'|} |\Omega(K)|^{\frac{1}{2}^{r'}} |\Omega(K')|^{\frac{1}{2}^{r}}. \]

In the case \( j = J \), a similar argument implies that
\[ \sum_{k=0}^{K'} \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\mathcal{E}\chi_{\tau'_{J}\cap\Omega(K')}\|_{s}^{2} \lesssim 2^{-(J-J')|\tau'|} |\Omega(K)|^{\frac{1}{2}^{r'}} |\Omega(K')|^{\frac{1}{2}^{r}} \sim 2^{-(K'-K)|\tau'|} |\Omega(K)|^{\frac{1}{2}^{r'}}. \]

In the cases \( k < K' \) and \( j < J \), we have a gain, due to our bilinear extension estimate. If \( k < K' \) and \( j < J, \tau_{k} \) is a (subset of four) tile(s) in \( \mathcal{D}_{j, \max(k, K'), K} \), \( \tau_{j} \) is a (subset of four) tile(s) in \( \mathcal{D}_{\max(j, J'), K'} \), and these tiles are separated by a distance \( 2^{-k} \) in the vertical direction \( 2^{-j} \) in the horizontal direction. These tiles are contained in separated tiles in \( \mathcal{D}_{j, k} \), so by the hypotheses of our theorem, for any \( r < r_{0} \),
\[ \|\mathcal{E}\chi_{\tau_{k}\cap\Omega(K)}\mathcal{E}\chi_{\tau'_{j}\cap\Omega(K')}\|_{s} \lesssim 2^{-(j+k)(\frac{1}{2}^{r'} - \frac{1}{2}^{r})} |\tau_{k}\cap\Omega(K)|^{\frac{1}{2}^{r}} |\tau'_{j}\cap\Omega(K')|^{\frac{1}{2}^{r'}}. \]

From our observation above that we have at most two values of \( j \) (resp. \( k \)) in our sum with \( j \leq J' \) (resp. \( k \leq K' \)), our assumption that \( r < s' \) gives
\[ \sum_{j=0}^{J} \sum_{k=0}^{K'} 2^{-(j+k)(\frac{1}{2}^{r'} - \frac{1}{2}^{r})} |\tau_{k}\cap\Omega(K)|^{\frac{1}{2}^{r}} |\tau'_{j}\cap\Omega(K')|^{\frac{1}{2}^{r'}} \leq \sum_{j=0}^{J} \sum_{k=0}^{K'} 2^{-(j+k)(\frac{1}{2}^{r'} - \frac{1}{2}^{r})} |\tau_{k}|^{\frac{1}{2}^{r}} |\tau'_{j}|^{\frac{1}{2}^{r'}} \lesssim 2^{-(J'+K)(\frac{1}{2}^{r'} - \frac{1}{2}^{r})} |\Omega(K)|^{\frac{1}{2}^{r}} |\Omega(K')|^{\frac{1}{2}^{r}} \sim \delta^{-C} 2^{-(J'+K)(\frac{1}{2}^{r'} - \frac{1}{2}^{r})} |\Omega(K)|^{\frac{1}{2}^{r}} |\Omega(K')|^{\frac{1}{2}^{r}}, \]
which, by \((3.3)\), is stronger than \((3.2)\). \qed

We return to the proof of Lemma \((3.1)\)

Let \( K_{1}, K_{2}, K_{3}, K_{4} \in K \), not all equal. Rearranging indices if needed, we may assume that \( N_{1} := K_{1} + J(K_{1}) \) is minimal among all \( N_{i} \) and that \(|K_{1} - K_{4}| \geq... \)
\( \frac{1}{2} |K_i - K_j| \) for all \( i, j \). Thus \( |\Omega(K_1)| \) is maximal. By Hölder’s inequality and Lemma 3.2,

\[
|\prod_{i=1}^{4} E_{\Omega_i(K_i)}|_2 \lesssim 2^{-c_0} |K_1 - K_4| |\Omega(K_1)|^{\frac{1}{2s}}.
\]

Therefore

\[
\sum' \| \prod_{i=1}^{4} E_{\Omega_i(K_i)} \|_2 \lesssim \sum_{K_1 \in K} \sum_{K_1 \neq K_4 \in K} |K_4 - K_1|^{2s} 2^{-c_0} |K_1 - K_4| |\Omega(K_1)|^{\frac{1}{2s}}.
\]

Since \( 2s > s' \) and \( K \) is \( A \log \delta^{-1} \)-separated for some very large \( A \), our error term is bounded by \( \delta^C |\Omega|^{\frac{1}{2s}} \). \( \square \)

**Proof of Theorem 1.1.** We decompose \( \Omega \) by fiber length, and decompose the fiber lengths according to the exactness of Vargas’s estimate:

\[
\| E_{\Omega}(K) \|_{2s} \leq \sum_{0 < \varepsilon < 1} \sum_{0 < \delta \leq \varepsilon} \sum_{K \in K(\varepsilon)} \| E_{\Omega_i(K)} \|_{2s} \lesssim \sum_{0 < \varepsilon < 1} \sum_{0 < \delta \leq \varepsilon} \left[ (\log \delta^{-1}) \sum_{K \in K(\varepsilon)} \| E_{\Omega_i(K)} \|_{2s}^{\frac{1}{2s}} + \delta |\Omega|^{\frac{1}{2s}} \right] \lesssim \sum_{0 < \varepsilon < 1} \sum_{0 < \delta \leq \varepsilon} \left[ (\log \delta^{-1}) 2^s \sum_{K \in K(\varepsilon)} |\Omega(K)|^{\frac{1}{2s}} + \varepsilon |\Omega|^{\frac{1}{2s}} \right] \lesssim \sum_{0 < \varepsilon < 1} \sum_{0 < \delta \leq \varepsilon} \log \delta^{-1} |\Omega|^{\frac{1}{2s}} \lesssim |\Omega|^{\frac{1}{2s}}.
\]

where, for the second to last inequality we are using the fact that \( 2s > s' \) and the triangle inequality for \( \ell^{\frac{1}{2s}} \) to sum the volumes of the \( \Omega(K) \). \( \square \)

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