Fuzzy epidemic model in a population having critical density dependent growth

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Abstract. The epidemic growth model is an important tool used in predicting the future of a population and the spread of disease in the population. An epidemic model is usually formed in a differential equation or a system consisting of several differential equations. The biological complexity in the underlying population affects the complexity of the epidemic model. One example of biological complexity is the Allee effect which reflects the critical density dependent of the population growth. In this paper we discuss a Logistic epidemic by considering this Allee effect on the population. Dynamic analysis is performed by determining fixed point and its stability analysis in crisp condition. We found the Basic Reproduction Ratio (BRR) for the model. The properties of the solution of the model are explored by the use of its numerical solution. Since we also consider the fuzziness of parameters and variables in the model, the numerical solution is generated using a modified Runge-Kutta method. This is done to explore the effect of inaccuracy and uncertainty which often occur in epidemiological problems.

1. Introduction
Epidemiological models have been developed and explored by many researchers in their respective fields. Covering both epidemiology in humans, animals and plants. One example of study that addresses epidemiology in humans is the characterization of the endocrine disruptive effects on human health: the role of epidemiological cohorts [1]. Research examples for cases of epidemiology in animals includes epidemiological studies of pet cross section related to human leptospirosis cases in Nicaragua [2] and the works of [3] [4] [5] for the case of Rabies transmission. Furthermore, an example of research in the case of plant epidemiology is the discussion of mathematical models of the application of protective and curative fungicides and stability analysis [6].

Mathematical epidemiological models are expected to provide an understanding of the dynamics of epidemics and can be used as a basis for reducing the possibility of spreading outbreaks and stopping infections. Many factors are considered to have contribution in the dynamics of disease transmission. The typical of population growth can have an impact on the dynamics of the spread of epidemic diseases. Birth and death rates not only affect population changes, but also affect disease epidemics, and areas with high population density will increase disease incidence.

In real cases the epidemiological model always experiences inaccuracies or uncertainties related to the nature of the state variables involved, both parameters and initial conditions. Therefore, to model epidemic problems and as a prediction procedure for epidemiology of infectious diseases, a fuzzy theory approach is used. Fuzzy Theory is a mathematical framework used to represent uncertainty, obscurity, inaccuracy, lack of information, and partial truth [7].
Research on population growth and epidemiological models has been carried out. However, not many mathematical models discuss the Allee effect on the spread of the epidemic. This study will discuss the epidemic model in a population, where population growth is critically influenced by a certain relationship, such as Allee effect. In this case, it is the relationship between population density and population growth rate. The main cause of this effect is the difficulty of finding partners between individual species with low population densities [8].

Analytical and numerical solutions of the model will be obtained for a crisp condition. The epidemic model will be analyzed for its fixed point and stability, and the Basic Reproduction Ratio (BRR) of the model. This is done to find out whether there will be an epidemic or not in the course of the disease transmission. The solution for uncertain condition will be discussed using the fuzzy approach and theories therein.

2. Methods
In this section, we present some preliminary theory and tools that will be used in the preceding analysis of the epidemic model. A model of epidemic having Allee effect is introduced and the presence of inaccuracy and uncertainty is deficted using the fuzzy approach.

2.1. Epidemiological mathematical models
The epidemic model of logistic population growth with the Allee effect can be expressed in the form of a system of equations:

From equation (1) it can be seen that a large number of vulnerable individuals (S) are affected by intrinsic growth rate (r), carrier capacity (K) and Allee threshold (A), and will decrease due to the presence of a number of individuals who are susceptible to infection through contact with infected individual (βSI). The magnitude of the number of people infected (I) is affected by the presence of a number of vulnerable individuals who are infected through contact with infected individuals (βSI), which decreases due to the growth rate of infected individuals (γI) and natural mortality rates (μI). Whereas the magnitude of the number of recovered individuals (R) is affected by the level of infected individuals (γI) which is reduced due to the natural death rate (μR).

2.1.1 Fixed point, stability, and instability. Intuitively, a dynamic system is said to be in equilibrium if the exchange rate does not change over time. That means the population is in equilibrium if it stays the same. Mathematically can be stated as follows: suppose x(t) shows the population at time t; if x(t) is a constant function, or equivalently dx/dt = 0 for all t, then the population is in equilibrium. The equilibrium point or fixed point can also be defined as follows [9]:

the value x̄ is called a fixed point if f(x̄) = 0. If x̄ is a fixed point, the constant function x(t) = x̄ is called a fixed solution or steady state solution of differential equations (2).

2.1.2 Basic reproduction ratio (R0). The basic reproduction ratio is an important threshold or quantity in epidemiology which states the average number of individuals infected secondary to infection from individuals with primary infection occurring in vulnerable populations. This threshold is very useful to indicate whether the epidemic occurs or not. If R0 < 1, the infection caused by an infected individual
produces less than one new infected individual on average during infection and infection cannot grow (non-epidemic). On the other hand, if $R > 1$, the infection caused by an infected individual produces more than one new infected individual on average and the disease spreads within the population (epidemic) [10].

2.2. Population growth model with Allee effect

The Allee effect was first introduced by Warder Clyde Allee in 1930. He observed the fact that population growth in small populations can occur positively, but there may also be a negative growth or extinctions in the population. This observation is related to the initial density of the population. He further concluded that there is a critical threshold value where if the initial density of the population is below this critical value, the population will disappear and if it is above this critical value, the population has a positive growth phase. In other words, the population with this kind of biological complexity has a population threshold size, where a small population could extinct deterministically. This critical value is often referred to as Allee threshold [11].

2.3. Fuzzy theory

The Fuzzy Set Theory was first published by Lotfi A. Zadeh in 1965. Fuzzy set theory provides a mathematical framework that represents uncertainty, obscurity, inaccuracy, lack of information and partial truth. Basically, the fuzzy set theory is an extension of classical or crisp set theory. In a crisp set theories, the existence of an element in a set of $A$ will only have two possible membership, namely to become a member of $A$ or not to become a member. A value that shows how much the membership level of an element $x$ in a set $A$ is known as membership value, notated with $\mu_A(x)$. In strict sets, two possible membership levels can be stated with

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$  

The fuzzy set states that an object can be a member of several sets with different membership degrees ($\mu$) [7]. The fuzzy set $\tilde{A}$ in the speech universe $U$ can be defined as a set of ordered pairs, namely $\tilde{A} = \{(x, \mu_A(x)) | x \in U\}$ is the membership degree of $x$ in set $\tilde{A}$ which has intervals between 0 and 1.

The set of all elements included in the fuzzy set $\tilde{A}$ with at least $\alpha$ degrees is called $\alpha$-level or $\alpha$-cut, and denoted by $[\tilde{A}]^\alpha$, that is,

$$[\tilde{A}]^\alpha = \{ x \in U | \mu_A(x) \geq \alpha \}.$$  

So, the set $[\tilde{A}]^\alpha$ consists of $U$ elements whose membership degree is greater than $\alpha$. The biggest level is $\alpha = 1$, and determines the set of $U$ that is fully owned by $\tilde{A}$. The set of $\alpha$-cut can provide a different way to consider the fuzzy set. Each fuzzy set can be represented by the aggregation of the $\alpha$-cut set. In this sense, all fuzzy sets can be broken down into $\alpha$-cut clusters [13].

The following is the characteristic function of the simplest form fuzzy number, i.e. the triangular fuzzy number which has a triangular shape.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ; x < a \\ \frac{x-a}{b-a} & ; a \leq x \leq b \\ \frac{x-c}{b-c} & ; b \leq x \leq c \\ 0 & ; x > c \end{cases}.  \tag{3}$$  

The function of a triangular fuzzy number characteristic has a triangular graph representation with $[a, b]$ as the base of the triangular and point $(b, 1)$ as a single node. Therefore, real numbers $a$, $b$ and $c$
define a triangular fuzzy number which will be denoted by \((a;b;c)\). Graphs of the characteristic functions of triangular curves are as follows:

\[
\begin{array}{c}
\begin{array}{c}
\mu(x) \\
\hline
a & b & c
\end{array}
\end{array}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Representation of characteristic functions of triangular curves.}
\end{figure}

\(\alpha\)-cut of a triangular fuzzy number has the following form:

\[
\left[ a_{\alpha}^{c}, a_{\alpha}^{a} \right] = \left[ (b-a)\alpha + a, (b-c)\alpha + c \right].
\] (4)

3. Results and discussion

A fixed point is obtained when the growth rate of each compartment reaches zero or when there is a "zero growth rate", which is a fixed condition in which the number of individuals in a particular population does not grow or decrease near zero, the equation system (1) becomes

\[
\begin{align*}
\frac{dS}{dt} &= rS \left( 1 - \frac{S}{K} \right) - \beta SI = 0 \\
\frac{dI}{dt} &= \beta SI - \gamma I - \mu I = 0 \\
\frac{dR}{dt} &= \gamma I - \mu R = 0.
\end{align*}
\] (5)

The non-endemic fixed point is a situation where no individual is infected with a disease in the population, denoted by \(E\). The endemic fixed point is a condition where there are infected individuals and infectious diseases in the population, denoted by \(E\). By completing the equation system (5) using Maple 2016 software, the following fixed points are obtained:

\[
E_1 = \left( S_1, I_1, R_1 \right) = \left( S_1, 0, 0 \right) = \left( K, 0, 0 \right); E_2 = \left( S_2, I_2, R_2 \right) = \left( S_2, 0, 0 \right) = \left( A, 0, 0 \right); \text{and}
\]

\[
E^* = \left( S^*, I^*, R^* \right), \text{ where } S^* = \frac{\gamma + \mu}{\beta}, I^* = -\frac{1}{AK\beta^2} \left( r(A\beta - \gamma - \mu)(K\beta - \gamma - \mu) \right), \text{ and}
\]

\[
R^* = -\frac{1}{AK\beta^2\mu} \left( \gamma r(-A\beta + \gamma + \mu)(-K\beta + \gamma + \mu) \right).
\]

To find the value of the Basic Reproduction Ratio \((R)\), the Next Generation Approach Method is used. The \(R\) value of the equation system (1) are

\[
R_0 = MD^{-1} = \frac{\beta K}{\gamma + \mu} \quad \text{and} \quad R_0 = MD^{-1} = \frac{\beta A}{\gamma + \mu}
\]

with \(\beta K\) and \(\beta A\) are the average number of healthy people infected by infective individuals per unit of time, and \(\frac{1}{\gamma + \mu}\) can be interpreted as the average length of infection.
One way to determine the stability of a fixed point is to look at the eigenvalue of the fixed point. The Jacobian matrix of the system equation (1) is used as the first step in finding the eigenvalue of each fixed point.

- **Analysis stability of non-endemic fixed point** $E'_1$

The Jacobian matrix $E'_1$ has the characteristic equation of $t$ in the form

$$P_{E'_1}(\lambda) = \left(\lambda + r \left(\frac{K}{A} - 1\right) \right) \left(-\beta K + \gamma + \lambda + \mu\right)(\lambda + \mu) = 0.$$  

From the characteristics equation above, three real roots are obtained, namely

$$\lambda_1 = \frac{r(A-K)}{A}, \lambda_2 = K\beta - \mu - \gamma, \lambda_3 = -\mu.$$  

The root of the characteristic equation $P_{E'_1}(\lambda) = 0$ is the eigenvalue of the fixed point $E'_1$. The type of stability for the model at a fixed point $(K,0,0)$ is *saddle point (unstable)* because $\lambda_3 < 0 < \lambda_2$ and $\lambda_1$. The condition needed for stability at a fixed point $(K,0,0)$ is $K\beta < \mu + \gamma$ or $R_0 < 1$.

- **Analysis stability of non-endemic fixed point** $E'_2$

The Jacobian matrix $E'_2$ has the characteristic equation of $t$ in the form

$$P_{E'_2}(\lambda) = \left(\lambda - r \left(1 - \frac{A}{K}\right) \right) \left(-A\beta + \gamma + \lambda + \mu\right)(\lambda + \mu) = 0.$$  

From the characteristics equation above, three real roots are obtained, namely

$$\lambda_1 = -\frac{r(A-K)}{K}, \lambda_2 = A\beta - \mu - \gamma, \lambda_3 = -\mu.$$  

The root of the characteristic equation $P_{E'_2}(\lambda) = 0$ is the eigenvalue of the fixed point $E'_2$. The type of stability for the model at a fixed point $(A,0,0)$ is *saddle point (unstable)* because $\lambda_3 < 0 < \lambda_2$. The condition needed for stability at a fixed point $(A,0,0)$ is $A\beta < \mu + \gamma$ or $R_0 < 1$.

- **Analysis stability of endemic fixed point** $E^e$

The characteristic equation of the Jacobian matrix $E^e$ is

$$P_{E^e}(\lambda) = -\frac{1}{K\beta} \left(\frac{\left((K\beta r)^2 - (\mu + \gamma - \lambda)(\mu + \gamma - 2\lambda)(\gamma + \mu)(\gamma + \mu r - A\beta^2 K \lambda^2)(\lambda + \mu)\right)}{\gamma + \mu \gamma + \mu^2 - \lambda^2} \right) = 0.$$  

From the characteristics equation above, obtained

$$a_0 = 1; a_1 = \frac{r \left(R_{0h} + R_{0v} - 2\right)}{R_{0h} R_{0v}} > 0 \iff R_{0h} + R_{0v} - 2 > 0 \iff R_{0h} + R_{0v} > 2;$$

$$a_2 = \frac{(R_{0h} + R_{0v} - 1)(\gamma + \mu)}{R_{0h} R_{0v}} > 0 \iff R_{0h} + R_{0v} - 1 > 0 \iff R_{0h} + R_{0v} > 1.$$
Numerical simulation is done by determining parameters and variables hypothetically. Following are the parameter values and the variables used in the model are shown sequentially by Table 1 and Table 2.

| Table 1. Parameter values. |
|-----------------------------|
| Symbol | Value |
| 1 | $\beta$ | 0.000375 |
| 2 | $\gamma$ | 0.125 |
| 3 | $\mu$ | 0.25 |
| 4 | $r$ | 0.0407 |
| 5 | $K$ | 1000 |
| 6 | $A$ | 100 |

| Table 2. Variables. |
|---------------------|
| Symbol | Value |
| 1 | $S(0)$ | 800 |
| 2 | $I(0)$ | 450 |
| 3 | $R(0)$ | 300 |

The graph of the solution of the system equation (1) is shown in Figure 2.

![Graph of the dynamics of the logistics epidemic model with Allee effect.](image)

The solution of the epidemic model in the growth of the logistics population with the Allee effect using a fuzzy approach is then searched using the $\alpha$-cut method. The following are parameter values and the variables used in the model are shown sequentially by Table 3 and Table 4.
Table 3. Parameter Values.

| Symbol | Value    |
|--------|----------|
| 1      | $\beta$  | 0.0000375 |
| 2      | $\gamma$ | 0.125     |
| 3      | $\mu$    | 0.25      |
| 4      | $r$      | 0.0407    |
| 5      | $K$      | 1500      |
| 6      | $A$      | 300       |

Table 4. Variables.

| Symbol | Value   |
|--------|---------|
| $S(0)$ | $S(0)_1$ | 1598 |
|        | $S(0)_2$ | 1498 |
|        | $S(0)_3$ | 1398 |
| $I(0)$ | $I(0)_1$ | 1050 |
|        | $I(0)_2$ | 1000 |
|        | $I(0)_3$ | 950  |
| $R(0)$ | $R(0)_1$ | 850  |
|        | $R(0)_2$ | 800  |
|        | $R(0)_3$ | 750  |

By substituting parameter values and variables into the model, a graph of the dynamics of the epidemic model on the growth of the logistic population with the Allee effect is shown in Figure 3.

Figure 3. Graph the dynamics of the logistic epidemic model with the Allee effect using the $\alpha$-cut method.

Figure 3 shows that numerical simulation results using the $\alpha$-cut method is quantitatively different to the crisp condition in Figure 2, but qualitatively gives similar long-term prediction result.
4. Conclusion
We conclude that the solution of the model in this paper is critically affected by the initial value. An initial value greater than the critical threshold will cause the solution to rise towards carrying capacity $K$ asymptotically. That means that there is an increase in the population and is finally stable at the point $K$. However, a smaller initial may cause the solution to go down to zero asymptotically. That means that the population is finally extinct. We showed that mathematical models with crisp and fuzzy theory approach provide solutions that are quantitatively different, but qualitatively give the same long-term results. Giving initial value is very influential on the graph of the solution to be obtained from the model. In giving a fuzzy initial value greater than Allee threshold, the more iterations, the graph of the solution becomes crisp in the end. The same thing applies to giving an initial value smaller than Allee threshold, in which the population is finally extinct. In giving a fuzzy initial value greater than Allee threshold, the more iterations, the graph of the solution becomes crisp.

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