Correlation of Excursion Sets for Non-Gaussian CMB Temperature Distributions

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ABSTRACT

We present a method, based on the correlation function of excursion sets above a given threshold, to test the Gaussianity of the CMB temperature fluctuations in the sky. In particular, this method can be applied to discriminate between standard inflationary scenarios and those producing non-Gaussianity such as topological defects. We have obtained the normalized correlation of excursion sets, including different levels of noise, for 2-point probability density functions constructed from the Gaussian, $\chi^2_n$ and Laplace 1-point probability density functions in two different ways. Considering subdegree angular scales, we find that this method can distinguish between different distributions even if the corresponding marginal probability density functions and/or the radiation power spectra are the same.

Key words: cosmology: cosmic microwave background - anisotropies - Gaussianity

1 INTRODUCTION

The temperature field $T(\theta, \phi)$, associated with the cosmic microwave background (CMB), is usually assumed to be a homogeneous and isotropic Gaussian random field, i.e. the n-point probability density function is a multivariate Gaussian characterized by the 2-point correlation function $C(\theta)$ or equivalently the radiation power spectrum $C_l$. The form of the temperature field at recombination is related to the initial density field coming from quantum fluctuations during an inflationary era in the very early stages of the history of the Universe. However, other sources of density fluctuations (such as topological defects) may emerge at these early times and generate a non-Gaussian random field. In any case future high precision maps should be searched for any traces of non-Gaussianity since both foregrounds and systematic errors can leave non-Gaussian imprints as well.

Whereas very accurate temperature maps can be obtained for the standard inflationary model (at the level of $\lesssim 1\%$ in the radiation power spectrum), this is not the case for scenarios based on topological defects. In particular, the best known of these models is the one generated by cosmic strings (Turok 1996), which results in temperature maps approximately Gaussian on scales that span from relative small (a few arcminutes according to Gott III et al. 1990) to large angular scales. Monopoles and textures generate maps which are also Gaussian at large angular scales but not at small scales (Shellard 1996). The texture and monopole maps show an asymmetric distribution with a positive tail decreasing more slowly than in the Gaussian case (Turok 1996).

Taking into account the present uncertainty in characterizing the temperature distributions for some non-standard scenarios, in this paper we will explore several non-Gaussian models. We consider a $\chi^2_n$ model for the temperature distribution function that satisfies the previous two properties, asymmetry and an exponential tail, and a Laplace model that represents a symmetric distribution with exponential tails. Moreover, we will also study a toy model which has the property of having a Gaussian marginal distribution but a non-Gaussian 2-point probability density function, a property that appears in the cosmic string maps (Gott III et al. 1990). As it is shown in figures 12a,b of that paper, the temperature 1-point probability density function of cosmic strings is Gaussian on scales above a few arcminutes whereas the one of the temperature gradient shows a clear non-Gaussian behaviour. On the other hand, the patchy behaviour in a CMB map associated to such network can be represented by the mentioned toy model (see §2.2).
In the analysis of the 2D temperature maps, we will consider the correlation of the excursion sets (or regions) above a certain threshold. For the 3D matter density field, such correlations were introduced by Kaiser (1984) assuming a Gaussian 2-point probability density function to study the correlation function of rich clusters. He found an amplification with respect to the correlation function associated with the underlying matter fluctuations. This was applied (Martínez-González & Sanz 1988) to test models of biased structure formation in several scenarios. Coles (1989) modeled the matter density field with a 1D Gaussian, log-normal and a $\chi^2_1$ distributions (the 2 non-Gaussian distributions were used in an attempt to allow for non-linear evolution) to study the correlation function of both peaks and excursion sets (in this last case, the Politzer & Wise 1984 approximation was assumed). The excursion sets are easier to identify in the maps than the peaks and they carry very similar information at high thresholds. In fact, the number of maxima and excursion sets above a threshold coincides asymptotically.

For the 2D temperature field, non-Gaussian statistics have been applied by Coles & Barrow (1987) to obtain the mean size and frequency of occurrence of the excursion sets above a given level for different random fields. Such properties depend only on the 1-point probability density function. On the other hand, Pomplio et al. (1995) used a multifractal analysis of the temperature scans to explore the possibility of distinguishing between Gaussian fluctuations and non-Gaussian ones produced by a network of cosmic strings. Finally, Kogut et al. (1995) used the peak correlation function in the COBE-DMR 2-year sky maps to test for a class of non-Gaussian models. In the last paper, the temperature maps are generated using a spherical harmonic decomposition whose multipole components $a_m$ are drawn from Gaussian, log-normal and $\chi^2_n$ distributions. They find that the 2-point correlation of peaks is a better discriminator between Gaussian and non-Gaussian temperature fields than the genus and the 3-point correlation function. Kogut et al. (1996) also carried out a similar analysis with the DMR 4-year sky maps.

In a previous paper, we studied some geometrical properties of the CMB maxima above a given level (mean number, curvature and ellipticity) assuming Gaussian temperature fluctuations (Barreiro et al. 1997), to discriminate among standard CDM models. In this paper, we will use the correlation of excursion sets to test the Gaussianity of the CMB temperature fluctuations in the sky, including the presence of noise. An advantage of using this quantity is that it is easy to identify the maxima (i.e. pixels above the fixed threshold) in temperature maps. These correlations are amplified by a purely statistical effect with respect to those associated with the temperature field. They carry information on the 2-point probability density function, allowing us to distinguish between two different distributions even if the underlying 1-point probability density function is the same. The latter point will become clear with the use of some reference toy models.

We discuss, in §2, the different 2-point probability density functions generated in two different ways. §3 is dedicated to calculating the correlation of excursion sets for different models in the case of an ideal experiment whereas in §4 we include noise. Finally, in §5 and §6, we give the main results and conclusions of the paper, respectively.

2 TWO POINT DISTRIBUTION FUNCTIONS
We are interested in several Gaussian and non-Gaussian distributions that could represent the temperature field produced in different cosmological scenarios. Inflationary models will generate Gaussian random fields whereas other models such as those derived from topological defects will in general give rise to non-Gaussian ones. We shall introduce the chi-squared probability density function as a simple distribution that asymptotically contains the Gaussian one. It has already been used in different cosmological contexts, in relation to the large scale structure (Coles 1989), to study topological properties of the CMB (Coles & Barrow 1987) and as a test of Gaussianity in the COBE-DMR four-year sky maps (Kogut et al. 1995).

The aim of this paper is to calculate the 2-point correlation of the excursion sets above a given threshold. In order to do this, we need to know the corresponding 2-point probability density functions (hereinafter, 2-pdf’s). The Gaussian, $\chi^2_n$ with $n$ degrees of freedom and Laplace 2-pdf’s are obtained from the 1-pdf’s (see figure 1) in two different ways: one will follow the ”standard” procedure whereas the other represents a simple ”toy” ansatz (Berry 1973, Jones 1996).

2.1 Standard procedure
a) Gaussian
The 2-pdf for a Gaussian field with zero mean, variance $\sigma^2$ and correlation $\tau$ is given by:

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\tau^2}} e^{-x_1^2+2\tau x_1x_2-2x_2^2/(2\sigma^2(1-\tau^2))}.$$  (1)

This distribution is symmetric and extends from $-\infty$ to $\infty$.

b) Chi-squared $\chi^2_n$
It is possible to generalize the univariate $\chi^2_n$ distribution with $n$ degrees of freedom to the bivariate case as follows: let us define the random vector $y \equiv (y_1, y_2) = \sigma_x^2 (x_1^2, x_2^2)$ where $\sigma_x$ is a constant and $x \equiv (x_1, x_2)$ is a random variable described by a bivariate Gaussian satisfying $x_1^2 + x_2^2 = 1$, $\langle x_1x_2 \rangle = \tau$. Then, it is possible to calculate the 2-pdf for

Figure 1. The Gaussian (solid), $\chi^2_1$ (short-dashed), $\chi^2_{30}$ (dotted), $\chi^2_{60}$ (long-dashed) and Laplace (dotted-short dashed) 1-pdf’s with zero mean and unit variance are shown.
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The generalization has been used by the first author to describe the statistical properties of echoes diffracted from rough surfaces. If we have a process with correlation \( \tau \) and described by a 1-pdf \( f(x) \), with mean \( \mu \) and variance \( \sigma^2 \), then, the distribution:

\[
f(x_1, x_2) = f(x_1)\delta(x_1 - x_2)\tau + f(x_1)f(x_2)(1 - \tau)
\]

is a 2-pdf with correlation \( \tau \) and marginal distributions given by the original 1-pdf. Here \( \delta(x - x_0) \) is the Dirac delta distribution centered on \( x_0 \). This distribution extends in the same range as the \( f(x) \) being considered and satisfies the interesting properties: \( f(x_1, x_2) = f(x_1)\delta(x_1 - x_2) \) for \( \tau \to 1 \) and \( f(x_1, x_2) = f(x_1)f(x_2) \) for \( \tau \to 0 \), i.e., the same limits that can be obtained for the bivariate Gaussian. This generalization is a simple example which shows that knowledge of the 1-pdf and the correlation allows many possibilities for the 2-pdf.

We will construct this 2-pdf for the Gaussian and non-Gaussian fields considered previously, and we will compare the correlation of regions obtained in this way with those obtained from the standard distributions given in \( \S 2.1 \). A CMB map associated to this toy model represents a surface consisting of flat regions separated by vertical walls whose rms height is \( 2\sigma \), \( \sigma \) being the dispersion of the underlying 1-pdf (Berry, 1973). In particular, for the Gaussian 1-pdf this could mimic the patchy behaviour given by the Kaiser-Stebbins effect produced by a network of long straight strings (Kaiser 

\& Stebbins 1984, Pompilio et al. 1995). For a simulation of the Kaiser-Stebbins effect see Magueijo \& Lewin (1997).

\section{3 CORRELATION OF EXCURSION SETS}

Kaiser (1984) has calculated the 2-point correlation function of excursion sets above a threshold \( \nu \sigma \) for a 3D Gaussian field of variance \( \sigma^2 \) and mean value zero. We will follow the same procedure for the distributions considered in this paper.

The probability that a randomly chosen point lies above a certain level \( \nu \sigma \) is:

\[
P_1 = \int_{\nu\sigma}^{\infty} f(x) dx .
\]

If we choose another point at distance \( r \) from the first, the probability that the field at both points takes a value exceeding that threshold is:

\[
P_2 = \int_{\nu\sigma}^{\infty} \int_{\nu\sigma}^{\infty} f(x_1, x_2, \tau) dx_1 dx_2 ,
\]

where \( \tau \) is the correlation of the field.

Therefore, the 2-point correlation function for the excursion sets, \( \xi_{>\nu} \), is:

\[
1 + \xi_{>\nu}(\tau) = \frac{P_2}{P_1} .
\]

\( \xi_{>\nu}(\tau) \) gives the fractional excess probability that a point \( x_2 \) lies above \( \nu \sigma \), given that \( x_1 \) also exceeds that threshold and \( |x_1 - x_2| = r \).

Next we shall introduce the normalized correlation \( C_{>\nu}(\tau) \) associated to the characteristic function of an excursion set \( h(x) \) defined by:

\[
C_{>\nu}(\tau) = \frac{\int_{\nu\sigma}^{\infty} \int_{\nu\sigma}^{\infty} f(x_1, x_2, \tau) dx_1 dx_2}{\int_{\nu\sigma}^{\infty} f(x_1, x_2) dx_1 dx_2} .
\]
\[ h(x) = \begin{cases} 1 & \text{if } f(x) > \nu \\ 0 & \text{otherwise.} \end{cases} \] (12)

Then the correlation \( C_{>\nu}(r) \) is the standard correlation coefficient of \( h(x) \): \( C_{>\nu}(r) \equiv \langle h(x_1)h(x_2) \rangle - \langle h(x_1) \rangle \langle h(x_2) \rangle / \sigma^2_h(x_1) \sigma_h(x_2) \) with \( \sigma^2_h(x) \equiv \langle h^2(x) \rangle - \langle h(x) \rangle^2 \). It is straightforward to obtain the relation:

\[ C_{>\nu}(r) = \frac{P_2}{1-P_1} = \xi_{>\nu}(r). \] (13)

We note that \( P_2(\tau=1) = P_1 \) for the standard Gaussian, standard chi-squared and toy model distributions. We have also proved that this result is valid for high thresholds in the standard Laplace case, and it applies for the \( \nu \)'s considered in the present paper.

\( \xi_{>\nu}(r) \) at zero lag (\( \tau = 1 \)) only contains information about the 1-pdf, however \( C_{>\nu}(0) = 1 \) for any distribution. Since the information from the 2-pdf is only encoded in the shape of the correlation of excursion sets and not in its normalization, it is better to use \( C_{>\nu}(r) \) to test the discriminating power of the 2-pdf.

The standard and toy model procedures will be considered separately.

### 3.1 Standard procedure

a) Gaussian

For a Gaussian field \( P_1 \) is given by:

\[ P_1 = \frac{1}{2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right), \] (14)

where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \) is the complementary error function.

One of the integrals which appears in \( P_2 \) can also be evaluated, resulting:

\[ 1 + \xi_{>\nu} = \frac{\sqrt{2}}{\pi} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right) \int_\nu^{\infty} dy e^{-\frac{\nu^2}{2(1+y^2)}} \] (15)

in agreement with Kaiser (1984). This last expression must be evaluated numerically when estimating \( C_{>\nu} \).

b) Non-Gaussian

As explained in \( \S 2.1 \) the chi-squared distribution has mean value \( \mu \), different from zero, and it must be centered if we want this distribution to describe the temperature field. To take this into account, the integrals appearing in \( P_1 \) and \( P_2 \) have been evaluated between \( \mu + \nu \sigma \) and \( \infty \). This procedure is completely equivalent to centering the \( \chi^2_n \) distribution and taking the limits as shown in equations (9) and (10). Then, \( P_1 \) is given by:

\[ P_1 = 1 - \frac{\gamma \left( \frac{2}{\sigma^2}, \sqrt{2} \right) \left( \nu + \sqrt{2} \right)}{\Gamma \left( \frac{2}{\sigma^2} \right)}, \] (16)

where \( \gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt \) is the incomplete gamma function.

For the Laplace case, we have:

\[ P_1 = 1 - \frac{1}{2} e^{-\sqrt{2} \nu}, \quad \nu < 0, \]  

\[ P_1 = \frac{1}{2} e^{-\sqrt{2} \nu}, \quad \nu > 0. \]
4 CORRELATION OF EXCURSION SETS INCLUDING NOISE

In a real experiment, the presence of instrumental noise modifies the correlations calculated in §3. In addition, the result will depend on the amplitude of the signal and then we need to choose specific cosmological models. Whereas for the standard inflationary scenario with Gaussian temperature fluctuations, the radiation power spectrum can be determined very accurately once the matter content is specified, this is not the case for other alternative scenarios such as topological defects. We will assume the same radiation assumption. In particular, we consider in the present section are known, are at least as large as the ones found under this assumption. We expect that the differences in a more realistic case, where the appropriate power spectra are known, are at least as large as the ones found under this assumption. In particular, we consider in the present section a flat CDM model (baryon content $\Omega_b = 0.05$, Hubble constant $h=0.5$, cosmological constant $\Lambda = 0$) with adiabatic fluctuations and a Harrison-Zel’dovich primordial spectrum, kindly provided by N.Sugiyama. The $C_l$’s have been normalized to the COBE 2-year maps (Cayón et al. 1996: this normalization does not appreciably change with the 4-year data) being the signal dispersion $\sigma_s = 3.5 \times 10^{-5}$ for an antenna of FWHM=10’ and a smoothing of the same width. We also include the effect of noise that is assumed to be Gaussian white noise. Following standard observational procedures, we have filtered signal plus noise with a Gaussian with the same width as the antenna.

The angular correlation function for the cosmological signal with a Gaussian beam profile is given by:

$$C(\alpha, \sigma_f) = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)C_l e^{-2\ell(\ell+1)\sigma_s^2/\ell} P_b(\cos \alpha),$$  

and its normalized correlation $\tau_s$:

$$\tau_s = \frac{C(\alpha, \sigma_f)}{\sigma_s^2},$$  

$$\sigma_s^2 = C(0, \sigma_f),$$  

where $\sigma$ is the Gaussian dispersion of the antenna ($\sigma_f = 0.425 \times \text{FWHM}$).

The correlation $\tau_N$ and the dispersion $\sigma_N$ for the filtered noise are very well approximated by ($\sigma_f \lesssim 0.1 \text{ rad}$):

$$\tau_N \simeq e^{-\sigma_s^2/4\sigma_f^2},$$  

$$\sigma_N^2 \simeq \frac{A_N}{4\pi\sigma_f}.$$  

We choose different levels of noise, fixing the noise amplitude $A_N = (0, 1.9, 17) \times 10^{-15}$ in order to obtain $\sigma_N = (0, 1.3) \times 10^{-5}$ after smoothing with a 10’ FWHM Gaussian window. These levels of noise cover the range of sensitivities expected for future experiments (e.g. MAP, Planck Surveyor Mission).

The presence of noise produces distortions in the distributions considered. The new distribution is given by a sum of a Gaussian process (coming from the noise) with dispersion $\sigma_N$ plus the process that characterizes the signal with dispersion $\sigma_s$. The new 1-pdf and 2-pdf are given by:

$$f(z) = \int f_N(z-y)f_s(y)dy,$$  

$$f(z_1, z_2) = \int \int f_N(z_1 - y_1, z_2 - y_2)f_s(y_1, y_2)dy_1dy_2,$$  

where $f_N$ and $f_s$ refer to the distributions of the noise and the signal respectively. The integrals are evaluated over the range covered by the given signal distribution. The variance of the new process is given by $\sigma^2 = \sigma_N^2 + \sigma_s^2$ and its correlation, for angles greater than the coherence angle $\theta^c_N = \sqrt{\sigma_N^2}$, can be approximated by the expression $\tau_N \simeq \tau_s/(1+SNR^{-2})$ where $SNR = \sigma_s/\sigma_N$ is the signal–to–noise ratio. We note that we are interested in temperature thresholds $\nu_0$ in units of $\sigma_s$, which is related to the threshold $\nu_1$, in units of $\sigma_s$, by $\nu_1 = \nu_0/\sqrt{1+SNR^{-2}}$.

We will again consider both procedures separately.

4.1 Standard procedure

a) Gaussian

For a signal described by a Gaussian distribution, the new process remains Gaussian with dispersion $\sigma_N$ and correlation $\tau_N$ defined as before. Therefore, the functional form of the correlation of excursion sets is the same as the one without noise given by equations (14) and (15), where the threshold and correlation are now $\nu_0$ and $\tau_0$, respectively.

b) Non-Gaussian

$P_1$ and $P_2$ are again given by equations (9) and (10). For $P_1$ we have:

$$P_1 = \frac{1}{2} \int f_s(y) \text{erfc} [g(y)] dy,$$  

where $g(y) = (\nu_0 \sqrt{1+SNR} + (\nu_1 + y) SNR)/\sqrt{2}$ with $\mu = \sqrt{2}\sigma_s$ and $\sigma = 0$ for the $\chi^2_N$ and Laplace cases, respectively. For the Laplace distribution $P_1$ can be evaluated analytically.

On the other hand, if we assume $\tau_N \ll 1$, some of the integrals which appear in $P_2$ can be evaluated, resulting in :

$$P_2 = \frac{1}{4} \int f_s(y_1, y_2) \text{erfc} [g(y_1)] \text{erfc} [g(y_2)] dy_1dy_2.$$  

This approximation is valid for $\alpha \gtrsim \theta^c_N$. To obtain a better estimation of the normalized correlation of regions for small angles when noise is present, we have evaluated $P_2$ as a power series considering terms up to second order in $\tau_N$ and then interpolated using its known value at $\tau = 1$ (again $P_2(\tau = 1) = P_1$).

4.2 Toy model procedure

In this case $P_1$ and $P_2$ are also given by equations (26) and (27) where $f_s(y)$ is the 1-pdf corresponding to the signal (Gaussian, $\chi^2_N$ or Laplace) and $f_s(y_1, y_2)$ is the 2-pdf constructed using the toy model procedure. In particular, assuming $\tau_N \ll 1$ we obtain for $P_2$:

$$P_2 = \frac{\tau_s}{4} \int f_s(y)(\text{erfc} [g(y)])^2 dy + (1 - \tau_s) P_1^2.$$  

In figure 2, $C_{>\nu}$ versus $\tau_s$ are plotted for different levels of noise and thresholds $\nu_0$ (for $\Omega = 1$ and FWHM=10’), for the Gaussian, $\chi^2_N$ with $n=1,30,60$ and Laplace distributions following the standard procedure. The main effect of the noise is to produce a rapid fall of the normalized correlation.
of regions at small scales. However, it is still possible to see clear differences between Gaussian and non-Gaussian distributions, up to levels of SNR≈1. As before, increasing the threshold $\nu_s$ produces an amplification of the differences.

5 RESULTS

We have applied the calculations of the previous sections to open and flat CDM models ($\Omega = 1, 0.3, 0.1$, with $\sigma_N(10') = 3.5 \times 10^{-5}$) and considered three different levels of noise ($\sigma_N(10') = (0, 1, 3) \times 10^{-5}$).

Besides the instrumental noise, our calculations are affected by statistical errors with whole sky coverage. Since we are considering very small angular scales, we expect that the cosmic variance will introduce a small uncertainty in our results.

As we have already pointed out, increasing the threshold $\nu_s$ enhances the difference between the considered correlations. However, this also produces a rapid fall in the number of excursion sets above $\nu_s$ and, thus, higher statistical errors. On the other hand, increasing the width of the antenna improves the SNR but drastically decreases the number of spots above $\nu_s$. This shows that we must find a compromise for choosing the optimal parameters for our analysis. In particular, we have taken an antenna FWHM=10' and a threshold $\nu_s = 2.5$ to present our results.

The normalized correlation of excursion sets above a threshold gives us different information from that of the 2-point correlation of the temperature fluctuations, since we obtain additional information about the 2-pdf. For this reason, our method is of special interest for discriminating between different distributions of the temperature fluctuations for a given power spectrum. In fact, as we have already pointed out, even if we have two fields described by the same 1-pdf and correlation, we can show that it is possible to discriminate between them through the correlation of regions.

In the upper part of figure 3, we plot pairs of standard and toy model distributions derived from the same 1-pdf for $\Omega = 1$, FWHM=10', $\nu_s = 2.5$ and no noise. The drawn errors have been estimated for all the distributions from high resolution Gaussian simulations of the sky. We expect them to be a good approximation to the real errors for the $\chi^2_30$ and $\chi^2_{60}$ distributions. In the rest of the cases, even if the errors were underestimated, we don’t expect our results to be very much affected since the difference between distributions is, generally, very clear. In addition, performing a simple Poissonian calculation of errors suggests that they are larger in the Gaussian case. Using a $\chi^2$ test, we obtain a null result for the hypothesis that each pair of curves is derived from the same population, with a very high confidence level ($\gtrsim 99\%$). Considering open models, $\Omega = 0.3, 0.1$, or introducing noise up to $\sigma_N(10') = 3 \times 10^{-5}$ does not significantly affect these results.

On the other hand, we can compare the Gaussian and non-Gaussian standard distributions. For $\Omega = 1$, FWHM=10', $\nu_s = 2.5$ and no noise (see lower plots of figure 3), we can distinguish between the Gaussian and non-Gaussian ($\chi^2_30$, $\chi^2_{60}$ and Laplace) standard distributions with a confidence level $\gtrsim 99\%$. Increasing the level of noise up to $\sigma_N(10') = 3 \times 10^{-5}$ or changing the power spectrum does not appreciably modify these results.

We can also try to discriminate between flat and open models for a given standard distribution through $C_{>\nu}$. For $\nu_s = 2.5$, FWHM=10' and no noise, we can differentiate between the flat and open ($\Omega = 0.1$ or $\Omega = 0.3$) models with a confidence level $\sim 99\%$. In Figure 4, we have represented the Gaussian, $\chi^2_{30}$ and Laplace distributions for the three considered values of $\Omega$, $\nu_s = 2.5$, FWHM=10' and no noise.

Considering a higher threshold $\nu_s$ amplifies the difference among the considered distributions. However it also produces a rapid fall in the number of excursion sets above $\nu$ and thus higher statistical errors, what leads to smaller confidence levels in the separation of the models. On the other hand, the smaller the threshold the closer $C_{>\nu}$ for the different distributions, since in the limit $\nu \rightarrow -\infty$ all the pixels are above $\nu$ and the correlation of regions is zero for all the distributions.
6 CONCLUSIONS

We have presented a method for testing the Gaussianity of the CMB temperature fluctuations in the sky based on the normalized correlation function of excursion sets above a given threshold. It can be used to distinguish between standard inflationary scenarios of generation of density fluctuations and those based on topological defects as well as to search for any trace of non-Gaussianity due to systematic errors or foregrounds. Such a technique was first introduced in cosmology by Kaiser (1984) to study the correlation of galaxy clusters in the context of biased scenarios of galaxy formation.

As an application, we have constructed 2-pdf’s from the Gaussian, $\chi^2$ and Laplace 1-pdf’s in two different ways. From these, we have obtained the normalized correlation of excursion sets including different levels of noise. This correlation contains additional information to that of the simple radiation power spectrum.

Our main conclusion is that, using the correlation of excursion sets above high thresholds (e.g. $\nu = 2, 3$) on sub-degree scales, it is possible to discriminate between different distributions even if the 1-pdf and correlation are the same. In particular, we are able to clearly distinguish between the Gaussian and $\chi^2_0$ cases even in the presence of certain levels of noise within the range of sensitivities expected for future experiments. Increasing the threshold amplifies the differences among the considered distributions but also the statistical errors, leading to smaller confidence levels.

Finally, the correlation of regions can also be used to compare different $\Omega$ models though it is less efficient than directly using the radiation power spectrum.

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APPENDIX A: THE 2-PDF FOR THE $\chi^2_0$ DISTRIBUTION

We can find the 2-pdf for $\chi^2_0$ in the following way. First, we construct the 2-pdf for $\chi^2_0$, where X is a bivariate Gaussian process with zero mean. If $f_X(x_1, x_2)$ denotes its 2-pdf then:

$$f_{\chi^2_0}(y_1, y_2) = \int J \ f_X(x_1 = x_1(y_1), x_2 = x_2(y_2))\ , \quad (A1)$$

being $|J|$ the Jacobian of the transformation $(x_1, x_2) \rightarrow (y_1, y_2)$, with $y_1 = x_1^2, y_2 = x_2^2$:

$$f_{\chi^2_0}(y_1, y_2) = \frac{1}{2\pi\sigma_x^2\sqrt{1 - \tau_x^2}} \frac{1}{\sqrt{y_1 y_2}} \ e^{-\frac{y_1 + y_2}{2\sigma_x^2(1 - \tau_x^2)}} \ cos\left(\frac{\tau_x\sqrt{y_1 y_2}}{\sigma_x^2(1 - \tau_x^2)}\right), \quad (A2)$$

where $\sigma_x^2$ and $\tau_x$ are the variance and correlation of the Gaussian process, respectively.

Taking into account that the characteristic function of a given distribution is defined as the Fourier transform:

$$\phi(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy_1 dy_2 f(y_1, y_2) e^{i(t_1 y_1 + t_2 y_2)} \ , \quad (A3)$$

we get for $\chi^2_0$:

$$\phi_{\chi^2_0}(t_1, t_2) = \frac{1}{\sqrt{1 - 2i\sigma_x^2(t_1 + t_2) - 4\sigma_x^2(1 - \tau_x^2)t_1 t_2}} \ . \quad (A4)$$

On the other hand, the characteristic function of a sum of n independent processes is the product of their characteristic functions, so we have for $\chi^0_n$:

$$\phi_{\chi^0_n}(t_1, t_2) = \left[\phi_{\chi^0_1}(t_1, t_2)\right]^n \ . \quad (A5)$$

This distribution has variance $\sigma^2 = 2n\sigma^2_x$, correlation $\tau \equiv \tau_x^n$ and mean $\mu \equiv n\sigma_x^2 \equiv \sqrt{n\sigma^2}$. Then, we can write $\phi_{\chi^0_n}$ in terms of $\sigma$ and $\tau$:

$$\phi_{\chi^0_n}(t_1, t_2) = \frac{1}{1 - i\sqrt{\bar{\sigma}(t_1 + t_2) - \frac{2}{\bar{\sigma}}\sigma^2(1 - \tau)t_1 t_2}} \ . \quad (A6)$$

Inverting eq.(A3) we obtain the 2-pdf for a chi-squared process with n degrees of freedom given by equations (2-4) of §2.1.
