Coulombic contribution to the flux of angular momentum in general relativity

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The radiated flux of angular momentum in electromagnetism depends on both radiative and Coulombic aspects of the field. We show that the same is true for angular momentum radiated by gravitational waves in general relativity. We rely on the Landau-Lifschitz formulation of general relativity, and generalize the standard expression for the angular-momentum flux, which is restricted to periodic sources at rest in the reference frame in which the flux is evaluated. Our generalized expression is valid for any source and in any reference frame, and it reduces to the standard expression in its domain of applicability. Our generalized expression reveals that the Coulombic piece of the gravitational field impacts the angular momentum flux through the “momentum aspect”, the time-independent coefficient of the $1/r$ term in the time-space components of the metric tensor near future null infinity. The momentum aspect is nonzero when the source of gravitational waves is moving with respect to the asymptotic rest frame. This new contribution to the flux of angular momentum is illustrated for a binary system with moving center-of-mass; it appears at 2.5 post-Newtonian order beyond the leading-order contribution to the flux. When the motion corresponds to the recoil of the source due to the radiation of linear momentum, the correction appears at 6 post-Newtonian order. We conclude that the new contribution is not of immediate concern for gravitational wave observations.

I. INTRODUCTION

Direct gravitational wave observations are now a reality [1–6]. Critical in analyzing the enormous amount of data gathered by gravitational wave observatories such as LIGO and Virgo is a solid theoretical understanding of gravitational waveforms. Since numerical simulations are time-consuming (and therefore costly) and do not necessarily aid one’s understanding of the physical processes at play, analytic models are helpful. Most analytic models of gravitational waveforms use an iterative procedure: starting with a Newtonian description of the source dynamics, corrections of higher and higher order in powers of $(v/c)^2$ are incorporated into the conservative dynamics, together with the system’s radiative losses; here $v$ is a typical velocity scale, and $c$ is the speed of light. This process approximates the source’s motion better and better, as well as the resulting gravitational wave emission. Among the key ingredients in this procedure are balance laws that determine the rate at which energy, linear momentum, and angular momentum change as a result of gravitational radiation. These balance laws are usually provided by the Landau-Lifschitz formulation of the Einstein field equations. This formulation is of course not unique. Nonetheless, its strength lies with the fact that it provides balance laws with a precise and physically sensible definition for the quantities that appear on both sides of the equation.

Our interest in this paper is with the balance law for angular momentum,

$$\frac{d}{dt} J^{ab} = -\mathcal{J}^{ab}, \quad (1.1)$$

where $J^{ab}$ is the total angular momentum tensor for the spacetime, and $\mathcal{J}^{ab}$ is the flux of angular momentum at infinity. This equation describes the rate at which gravitational waves remove angular momentum from the system. Ashtekar and Bonga and shown recently that in the case of electrodynamics, the radiated flux of angular momentum depends not only on the fields’s radiative modes, but also on its Coulombic aspects [7, 8]. Specifically, the flux of angular momentum contains a term that is quadratic in the radiative modes, and one that is the product of radiative modes and Coulombic aspects. These Coulombic aspects manifest themselves when the total charge of the source of electromagnetic radiation is nonzero. For instance, the angular momentum radiated by a charged spinning sphere with variable angular velocity is entirely due to the interaction between the Coulombic and radiative aspects of the electromagnetic field [9].
Ashtekar and Bonga suggested that the same might be true for general relativity. In this paper, we examine the Landau-Lifschitz balance law for angular momentum and show that this is indeed the case: the flux of angular momentum in general relativity depends on both the radiative and Coulombic aspects of the gravitational field. Based on an analogy with Maxwell’s theory, one might anticipate that the Coulombic aspects manifest themselves through the mass of the source of gravitational radiation. This turns out not to be true: the Coulombic aspects appear through what we call the “momentum aspect”, the time-independent coefficient of the $1/r$ term in the time-space components of the metric tensor near future null infinity. This implies that the Coulombic aspects are relevant when the source of gravitational waves is moving relative to the reference frame in which the flux of angular momentum is measured. The expressions for the fluxes of energy and linear momentum are not altered by the Coulombic aspects; only radiative modes contribute to those fluxes.

To derive this result we generalize the standard expression for the flux of angular momentum, which is restricted to periodic sources of gravitational waves that are at rest relative to the reference frame in which the flux is evaluated [10, 11]. Our generalized expression is valid in any frame, and there is no restriction to periodic sources.

An issue that we had to face in our calculations is that the angular-momentum balance equation is in general ill-defined in harmonic coordinates. This is because the total angular momentum $J^{ab}$ is in general infinite for asymptotically flat spacetimes, with a time-dependent infinite contribution that makes the balance equation meaningless. (The infinite contribution can be identified by calculating the angular momentum for a finite region of space, and plucking out the terms that diverge when taking a limit to an infinite region.) When, however, the harmonic gauge is specialized to the transverse-traceless gauge (tt gauge) by imposing additional coordinate conditions, the balance equation becomes well-defined, because the infinite contribution to $J^{ab}$ becomes time-independent and therefore drops out of the equation. We shall therefore adopt the tt gauge in the main body of the paper.

Returning to the Coulombic contribution to the flux of angular momentum, we find two new terms. The first can be written as the time derivative of an angular integral involving the momentum aspect and the field’s radiative modes. This term is matched by an identical term appearing on the left-hand side of Eq. (1.1). This term is therefore physically uninteresting, as it leaves no imprint on the source dynamics. The second term is the angular integral of a cross-product between the momentum aspect and the density of radiated linear momentum. This term does influence the source dynamics whenever the source has a nonzero center-of-mass momentum. In the absence of center-of-mass motion, we recover the standard expression for the flux [10, 11], up to time-derivative terms that vanish on the average for a periodic source of gravitational waves. An explicit calculation of the flux for a gravitationally bound binary system with nonzero center-of-mass momentum shows that this contribution persists for circular orbits. The Coulombic contribution, however, is cubic in the gravitational potentials. As a consequence, its leading post-Newtonian (PN) order is rather high: It makes its first appearance at 5PN order, that is, 2.5PN order beyond the leading-order flux. When the center-of-mass velocity corresponds to the recoil due to the emission of linear momentum, the Coulombic contribution to the flux occurs at 8.5PN order, or 6PN order beyond the leading-order term. Even for black hole merger remnants with high recoil velocities (as high as a few percent of the speed of light [12–14]), the Coulombic contribution to the angular-momentum flux is numerically sub-dominant. Thus, while conceptually important, the Coulombic correction to the flux is is not of immediate concern for gravitational wave observations.

In situations in which the Coulombic contributions may be found to be dynamically important, they could be eliminated by performing a coordinate transformation to a frame in which the source is at rest. In many circumstances this would be the simplest thing to do. But in other circumstances, for example in the case of a recoiling source, it may be simpler to let the source recede and calculate the flux of angular momentum with the Coulombic term included.

This paper is organized as follows. In Sec. II we lay out our notation and conventions and introduce the harmonic and tt gauges for the gravitational potentials. Section III is the heart of the paper, in which we present our generalized expression for the flux of angular momentum. In Sec. IV we construct the total angular momentum that appears on the left hand side of Eq. (1.1). We show that one of the Coulombic contributions to the flux, the one equal to a time derivative, is matched precisely by a corresponding term in $dJ^{ab}/dt$. The physical relevance of the remaining Coulombic contribution is illustrated in Sec. V, in which we calculate the flux of angular momentum for a binary system in an eccentric orbit with nonzero center-of-mass momentum. We summarize our results and offer more discussion in Sec. VI. The calculations in the main body of the paper were carried out with the help of GRTensorIII [15] working under Maple, and are presented with almost no details of derivation. In the Appendices we present alternative versions of these calculations, following the methods of [7, 8], with all details provided.

II. PRELIMINARIES

In this section we spell out the conventions used in the main body of this paper, and introduce the harmonic and tt gauges for the gravitational potentials.
A. Wave-zone expansion

We work with the quasi-Lorentzian coordinates \( x^\alpha = (x^0, x^a) = (ct, x, y, z) \), and relate the Cartesian coordinates \( x^a = (x, y, z) \) in the usual way to spherical coordinates \((R, \theta, \varphi)\). We use \( R \) to denote the Euclidean distance \( \sqrt{x^2 + y^2 + z^2} \) instead of the more usual \( r \), which is reserved for orbital radius in later portions of the paper. The gravitational potentials are defined by \( \bar{h}^{\alpha \beta} := \eta^{\alpha \beta} - \sqrt{-g} g^{\alpha \beta} \), and in the far-away wave zone they are expanded in powers of \( 1/R \) as

\[
\bar{h}^{00} = \frac{4G}{c^2} \left[ \frac{1}{R} f^{00}(\tau, \theta, \varphi) + \frac{1}{R^2} f^{00}_1(\tau, \theta, \varphi) + \cdots \right],
\]

(2.1a)

\[
\bar{h}^{0a} = \frac{4G}{c^3} \left[ \frac{1}{R} f^{0a}(\tau, \theta, \varphi) + \frac{1}{R^2} f^{0a}_1(\tau, \theta, \varphi) + \cdots \right],
\]

(2.1b)

\[
\bar{h}^{ab} = \frac{4G}{c^4} \left[ \frac{1}{R} f^{ab}(\tau, \theta, \varphi) + \frac{1}{R^2} f^{ab}_1(\tau, \theta, \varphi) + \cdots \right],
\]

(2.1c)

where \( \tau := t - R/c \) is retarded time. The potentials are assumed to satisfy the harmonic-gauge condition \( \partial_\beta \bar{h}^{\alpha \beta} = 0 \), and the expansions imply that the potentials satisfy the wave equation \( \Box \bar{h}^{\alpha \beta} = -(16\pi G/c^4) \tau^{\alpha \beta} \), with an effective energy-momentum tensor \( \tau^{\alpha \beta} \) that falls off at least as fast as \( 1/R^2 \).

To first order in the gravitational potentials, the metric and its inverse are given by

\[
g^{\alpha \beta} = \eta^{\alpha \beta} + \bar{h}^{\alpha \beta} - \frac{1}{2} \bar{h} \eta^{\alpha \beta},
\]

(2.2a)

\[
g_{\alpha \beta} = \eta_{\alpha \beta} - \bar{h}^{\alpha \beta} + \frac{1}{2} \bar{h} \eta_{\alpha \beta},
\]

(2.2b)

where \( \bar{h} := \eta_{\alpha \beta} \bar{h}^{\alpha \beta} \).

We introduce the three-dimensional basis vectors

\[
N^a = [\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta], \quad \theta^a = [\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta], \quad \varphi^a = [-\sin \varphi, \cos \varphi, 0].
\]

(2.3)

The metric on a unit 2-sphere is given by

\[
ds^2 = \Omega_{AB} d\theta^A d\theta^B = d\theta^2 + \sin^2 \theta \, d\varphi^2,
\]

(2.4)

and the matrix inverse of \( \Omega_{AB} \) is denoted \( \Omega^{AB} \). The Jacobian matrix between the Cartesian and spherical coordinates is given by

\[
\frac{\partial x^a}{\partial R} = N^a, \quad \frac{\partial x^a}{\partial \theta^A} = RN^a_A,
\]

(2.5)

and

\[
\frac{\partial R}{\partial x^a} = N_a, \quad \frac{\partial \theta^A}{\partial x^a} = \frac{1}{R} N^A_a,
\]

(2.6)

where \( N^a_A := \partial_A N^a \) and \( N^A_a := \Omega^{AB} \delta_{ab} N^b_B \).

B. Harmonic and tt gauges

At order \( 1/R \) the harmonic-gauge condition implies

\[
f^{00} = \frac{1}{c} f^{00}_a N_a, \quad f^{0a} = \frac{1}{c} f^{0a}_b N_b,
\]

(2.7)

in which an overdot indicates differentiation with respect to \( \tau \). These equations integrate to

\[
f^{00} = m(\theta, \varphi) + \frac{1}{c} f^{00}_a N_a, \quad f^{0a} = p^a(\theta, \varphi) + \frac{1}{c} f^{0a}_b N_b,
\]

(2.8)
where we refer to \( m(\theta, \varphi) \) as the mass aspect and \( p^a(\theta, \varphi) \) as the momentum aspect; these functions play the role of integration constants.\(^1\) These equations reveal that in addition to the mass and momentum aspects, \( f^{00} \) and \( f^{0a} \) are both determined by \( f^{ab} \). At the next order, \( \mathcal{O}(1/R^2) \), the harmonic-gauge condition implies

\[
\bar{f}^{00}_1 = \frac{1}{c} \hat{f}^{00}_1 N_a + f^{0a} N_a - N^A \partial_A f^{0a}, \quad \bar{f}^{0a}_1 = \frac{1}{c} \hat{f}^{ab}_1 N_b + f^{ab} N_b - N^A \partial_A f^{ab}.
\] (2.9)

As we show below, the harmonic gauge can always be refined to a transverse-traceless gauge (tt gauge)\(^2\), defined by

\[
f^{ab}_{tt} N_b = 0 = \delta_{ab} f^{ab}_{tt}.
\] (2.10)

A priori, this tt gauge is not defined beyond order 1. Nonetheless, the gauge freedom at order 1/\( R^2 \) is severely restricted by Eq. (2.10). In the tt gauge, the potentials can be expressed in terms of the two polarizations \( f_+ (\tau, \theta, \varphi) \) and \( f_\times (\tau, \theta, \varphi) \), defined by

\[
f_+ := \frac{1}{2} (\theta_a \theta_b - \varphi_a \varphi_b) f^{ab}, \quad f_\times := \frac{1}{2} (\theta_a \varphi_b + \varphi_a \theta_b) f^{ab}.
\] (2.11)

In terms of the polarizations, the gravitational potentials in the tt gauge are given by

\[
f^{00}_{tt} = m(\theta, \varphi), \quad f^{0a}_{tt} = p^a(\theta, \varphi), \quad f^{ab}_{tt} = f_+ (\theta^a \theta^b - \varphi^a \varphi^b) + f_\times (\theta^a \varphi^b + \varphi^a \theta^b).
\] (2.12)

We now establish the statement that the harmonic gauge can always be refined to the tt gauge. A gauge transformation generated by the vector \( \zeta^a \) produces a change

\[
\bar{h}^{\alpha\beta}_{\text{new}} = \bar{h}^{\alpha\beta}_{\text{old}} - \partial^\alpha \zeta^\beta - \partial^\beta \zeta^\alpha + (\partial_\mu \zeta^\mu) n^{\alpha\beta}
\] (2.13)

in the potentials. The transformation preserves the harmonic-gauge condition when \( \zeta^a \) satisfies the wave equation \( \Box \zeta^a = 0 \). To order 1/\( R \) the solution that preserves the fall-off behavior of \( \bar{h}^{\alpha\beta} \) can be expressed as

\[
\zeta^0 = \frac{4G}{c^3 R} \alpha(\tau, \theta, \varphi), \quad \zeta^a = \frac{4G}{c^3 R} \beta^a(\tau, \theta, \varphi),
\] (2.14)

and it is easy to show that the transformation produces

\[
f^{ab}_{\text{new}} = f^{ab}_{\text{old}} + \beta^a N^b + N^a \beta^b + (\dot{\alpha} - \dot{\beta} c N_c) \delta^{ab}.
\] (2.15)

The other relations issued from the transformation are redundant. The conditions of Eq. (2.10) can always be imposed on \( f^{ab}_{\text{new}} \) with the choices

\[
\dot{\alpha} = -\frac{1}{4} (\delta_{ab} + N_a N_b) f^{ab}_{\text{old}}, \quad \dot{\beta}^a = -\dot{\alpha} N^a - N_b f^{ab}_{\text{old}}.
\] (2.16)

The constants of integration for \( \alpha \) and \( \beta^a \) have no impact on the potentials at order 1/\( R \), so that the tt gauge is completely fixed at order 1/\( R \). They do change the potentials at order 1/\( R^2 \), so that \( f^{ab}_{\text{tt}} \) contains residual gauge freedom in the tt gauge. In particular, they change the time-independent part of \( f^{ab}_{\text{tt}} \). (For more details regarding the residual gauge freedom in the tt gauge, see App. A.1.)

### III. Flux of Angular Momentum

This section contains the main results of this paper. We specialize to the tt gauge from the start and only describe the main steps involved in carrying out the calculations. A more detailed derivation of these results is presented in App. A, where we also present the flux of angular momentum for a general harmonic gauge.

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1. The names “mass aspect” and “momentum aspect” for these quantities are motivated by a calculation of the Landau-Lifschitz mass \( M \) and linear momentum \( P^a \). In the tt gauge, we find that \( M = (4\pi)^{-1} \int m(\theta, \varphi) d\Omega \) and \( P^a = (4\pi)^{-1} \int p^a(\theta, \varphi) d\Omega \), where \( d\Omega := \sin \theta d\theta d\phi d\alpha \).

2. Note that the nomenclature for this gauge is somewhat ambiguous: the potentials are transverse in the sense that the potentials are orthogonal to \( N^a \), but not in the sense that \( \partial_\mu h^{ab} = 0 \). The first condition is local in space, whereas the second condition is global. In the limit \( R \to \infty \) with \( \tau \) held fixed, the local and global conditions give the same expressions for the angular components of \( \bar{h}^{ab} \), up to a time-independent function that does not enter any physical observable. For a more detailed discussion on this issue, see [7, 8]. To remind ourselves that the gauge specified in Eq. (2.10) is transverse in the local sense (but not in the global sense), we name it tt gauge instead of TT gauge (which would refer to the global condition).
A. Result

As reviewed, for example, in Sec. 6.1.3 of [16], the flux of angular momentum is given by

$$\mathcal{T}^{ab} = \oint [x^a (-g) t_{00}^a \hat{h} - x^b (-g) t_{00}^b] \, dS_c,$$  \hspace{1cm} (3.1)

where the integral is over a 2-sphere \((\tau, R) = \text{constant}\) in the limit \(R \to \infty\), and \((-g) t_{00}^a\) are the spatial components of the Landau-Lifshitz energy-momentum pseudotensor. The relevant terms of the pseudotensor, after expanding in powers of \(\hat{h}^{\alpha\beta}\) to cubic order and specializing to the harmonic gauge, are given by

$$(-g) t_{00}^a = \frac{\epsilon^4}{16\pi G} \left[ \frac{1}{2} \eta^{\mu\nu} \eta_{\rho\sigma} \partial_\rho \hat{h}^{\alpha\beta} \partial_\sigma \hat{h}^{\kappa\nu} - 2 \eta^{\mu\nu} \eta_{\rho\sigma} \partial_\rho \hat{h}^{\alpha\beta} \partial_\sigma \hat{h}^{\kappa\nu} + \eta_{\rho\sigma} \eta^{\alpha\beta} \partial_\rho \hat{h}^{\kappa\nu} \partial_\sigma \hat{h}^{\mu\nu} = - \frac{1}{2} \eta^{\mu\nu} \hat{h} \eta^{\alpha\beta} \partial_\nu \hat{h} \partial_\alpha \hat{h} \right].$$  \hspace{1cm} (3.2)

Note that terms cubic in \(\hat{h}^{\alpha\beta}\) that do not contribute to the flux are omitted from this expression. It is not obvious that the integral of Eq. (3.1) is finite in the limit \(R \to \infty\), given that most terms are quadratic in \(\hat{h}^{\alpha\beta}\) and that \(\hat{h}^{\mu\nu} \sim R^{-1}\). However, the antisymmetry of the integrand and the harmonic gauge conditions will ensure that \(\mathcal{T}^{ab}\) is indeed finite.

Introducing the notation \(\partial_\alpha h_{bc} = -\frac{1}{2} \eta_{\alpha\rho} h_{bc} + \hat{\theta}_\alpha h_{bc}\), where the operator \(\hat{\theta}_\alpha\) bypasses the dependence on retarded time \(\tau\) and operates only on the remaining spatial dependence, we insert the potentials of Eq. (2.1) and their derivatives into the integral of Eq. (3.1). We exploit the harmonic-gauge conditions of Eqs. (2.8) and (2.9) to eliminate \(f^{00}, f^{0a}, \hat{f}^{00}, \) and \(\hat{f}^{0a}\) from the expression. We observe that after involvement of the gauge conditions, the flux integrand no longer involves \(f_1^{\alpha\beta}\) and can be expressed entirely in terms of \(m, p^a,\) and \(f^{ab}\).

This last observation implies that the flux integrand is not gauge-invariant: When \(f^{ab}\) is decomposed into trace, longitudinal, longitudinal-transverse, and tt pieces (as defined in Sec. 11.1.3 of [16]) we find that the flux integrand implicates all pieces, and not just the tt pieces of the potentials (as is the case for the fluxes of energy and linear momentum). This is clear from the expressions in App. A, which present the flux in a general harmonic gauge.

Specializing to the tt gauge, we find that the angular-momentum flux can be expressed as

$$\mathcal{T}^{ab} = \oint t^{ab} \, d\Omega,$$  \hspace{1cm} (3.3)

where \(d\Omega := \sin \theta \, d\theta \, d\phi\) is the volume element on the unit two-sphere, and the flux integrand is decomposed into radiative, cubic, and time-derivative terms:

$$t^{ab} = t^{ab}_{\text{rad}} + t^{ab}_{\text{cubic}} + t^{ab}_{\text{deriv}}.$$

In terms of the momentum aspect \(p^a\) and the radiative modes encoded in \(f^{ab}_{tt}\), the explicit expressions are

$$t^{ab}_{\text{rad}} = \frac{G}{\pi c^3} \left[ - (N^a \dot{f}^{bp} - N^b \dot{f}^{ap}) \dot{\varphi} q_{fp} + (N^a \dot{p} q_{fp} - N^b \dot{p} q_{fp}) f_{pq} - \frac{1}{2} (N^a \dot{p} f_{pq} - N^b \dot{p} f_{pq}) \dot{f}_{pq} \right],$$  \hspace{1cm} (3.5a)

$$t^{ab}_{\text{cubic}} = \frac{2G^2}{\pi c} \dot{f}_{pq} \dot{f}_{pq} (N^a p^b - p^a N^b),$$  \hspace{1cm} (3.5b)

$$t^{ab}_{\text{deriv}} = \frac{G}{\pi c} (N^a \dot{f}^{bc} - N^b \dot{f}^{ac}) p^c.$$  \hspace{1cm} (3.5c)

In these relations it is understood that \(f^{ab}\) is presented in the tt gauge.

Let us comment on the main properties of this result. First, given that \(f^{ab}\) is a function of retarded time and two angular variables, \(\hat{\theta}_a\) acts as an angular derivative in Eq. (3.5a). Second, note that in the tt gauge, the angular-momentum flux depends on the momentum aspect \(p(\varphi, \phi)\) and is independent of the mass aspect \(m(\varphi, \phi)\). When
the momentum aspect vanishes, the cubic and time-derivative terms vanish as well. This is not true in a general harmonic gauge, because generically the flux depends both on the mass and momentum aspects (see App. A). Third, the purely radiative terms are quadratic in \( f^{ab} \) and are of the form \([\text{time derivative of the gravitational potentials}] \times [\text{angular derivative of the gravitational potentials}]\). The expression for these radiative terms is different from the “standard” expression found in [11, 16]. The standard expression applies only in the asymptotic rest-frame of a periodic source of gravitational waves, whereas our result is not restricted to periodic sources nor to the asymptotic rest-frame. Our result relies only on the \(1/R\) fall-off of the gravitational potentials and the tt gauge conditions. When we restrict it to periodic sources, we find that the difference between the expressions disappears when time-averaged over a period. We will show this in detail in Sec. III B. Fourth, the cubic terms are a cross product between \( p^a \), the momentum aspect, and \( \dot{f}^{pq} N^a \), the density of radiated linear momentum. The appearance of terms cubic in the potentials does not occur in electrodynamics and is due to the non-linearity of general relativity. Fifth, the time-derivative term involves the momentum aspect and as we shall see in Sec. IV, it is matched by an identical term on the left hand side of the balance equation (1.1); this term does not affect the source dynamics. The physical relevance of the cubic terms on the source dynamics will be discussed in Sec. V, where we consider the example of a binary system with a nonzero center-of-mass momentum.

B. Comparison with the standard expression

The “standard expression” for the angular-momentum flux integrand, as given in [11] or Eq. (12.47) of [16], is

\[
t_{\text{stan}}^{ab} = \frac{G}{\pi c^5} \left[ f^{ap} \dot{f}^b_p - f^{bp} \dot{f}^a_p - \frac{1}{2} \left( x^c \partial_p f_{pq} - x^b \partial_p f_{cq} \right) \dot{f}^{pq} \right],
\]

and it differs from Eq. (3.5). This standard expression applies only to sources that are at rest relative to the reference frame in which the flux is evaluated, so that in this context, \( p^a(\theta, \varphi) = 0 \) and \( t^{ab} = t^{ab}_{\text{rad}} \). Furthermore, in [11] the standard expression is obtained after averaging the flux integrand over a period of oscillation of the gravitational potentials; it is meant to apply on the average to a periodic source of gravitational waves.

The difference between Eq. (3.5) and the standard expression is

\[
\Delta t^{ab} := t^{ab}_{\text{rad}} - t^{ab}_{\text{stan}},
\]

and it is easy to show that this is the sum of a divergence and a time derivative:

\[
\Delta t^{ab} = \frac{G}{\pi c^5} \left( -\partial_p v^{abp} + w^{ab} \right),
\]

where

\[
v^{abp} := \left( x^a \dot{f}^b_{cp} - x^b \dot{f}^a_{cp} \right) f^{cp}, \quad w^{ab} := \left( x^a \partial_p f^b_q - x^b \partial_p f^a_q \right) f^{pq}.
\]

By virtue of the tt gauge, \( v^{abp} \) is tangent to the 2-sphere with respect to its last index, which implies that \( \partial_p v^{abp} \) is actually a divergence on the 2-sphere; its integral over the angles (\( \theta, \varphi \)) therefore vanishes. The term \( w^{ab} \) necessarily vanishes when averaged over a period of oscillation, and we conclude that Eq. (3.5) and the standard expression are equivalent (after averaging) when applied to periodic sources at rest relative to the reference frame in which the flux of angular momentum is evaluated.

In contrast with the standard expression, Eq. (3.5) applies to sources of gravitational waves that are moving, and it is not restricted to periodic sources of gravitational waves. Its domain of applicability is therefore wider than the standard expression.

C. Complex representation of the radiative terms

As we will show in detail in App. A, the radiative contribution to the flux integrand, given by Eqs. (3.5a), can be expressed neatly in terms of a complex combination of the two polarizations,

\[
f := f_+ + i f_x.
\]

We find that

\[
t^{xy}_{\text{rad}} = \frac{G}{\pi c^5} \left[ -i \sin \theta \rho + cc \right],
\]
\[ t^\varphi_{\text{rad}} = \frac{G}{\pi c^5} \left[ (-\sin \varphi + i \cos \theta \cos \varphi) \rho + cc \right], \]  
\[ t^{\varphi \varphi}_{\text{rad}} = \frac{G}{\pi c^5} \left[ (\cos \varphi + i \cos \theta \sin \varphi) \rho + cc \right], \]  
where cc denotes “complex conjugate” and

\[ \rho := \frac{1}{2} \partial f \partial f - \frac{3}{2} \partial \bar{f} \partial f. \]  

Here, \( \partial \) is the “eth” differential operator, which acts as

\[ \partial \eta = \frac{1}{2} \left( \partial_0 + \frac{i}{\sin \theta} \partial_\varphi - s \cos \theta \sin \theta \right) \eta \]  
on an object of spin-weight \( s \). The field \( f \) has spin-weight \( +2 \), and its complex conjugate \( \bar{f} \) has spin-weight \(-2 \). These expressions are significantly more compact than those involving the polarizations \( f_+ \) and \( f_\times \),

\[ t^\varphi_{\text{rad}} = \frac{G}{\pi c^5} \left[ (2 \sin \theta \partial_0 f_\times + 2 \cos \theta \cos \varphi \partial_\varphi f_+ - \partial_\varphi f_+) \hat{f}_+ - (2 \sin \theta \partial_0 f_+ + 2 \cos \theta \cos \varphi \partial_\varphi f_+ + \partial_\varphi f_+ \hat{f}_+ \right], \]  
\[ t^{\varphi \varphi}_{\text{rad}} = \frac{G}{\pi c^5} \left[ \sin \varphi \partial_0 f_+ - \frac{\sin \varphi}{\sin \theta} \partial_\varphi f_+ + \frac{4 \cos \theta \sin \varphi}{\sin \theta} f_+ - 2 \cos \theta \cos \varphi \partial_\varphi f_+ + \frac{2 \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{2 \cos^2 \theta \cos \varphi}{\sin \theta} f_+ \right] \hat{f}_+ \]  
\[ + \left( 2 \cos \theta \cos \varphi \partial_\varphi f_+ - \frac{2 \sin \varphi}{\sin \theta} \partial_\varphi f_+ + \frac{2 \cos^2 \theta \cos \varphi}{\sin \theta} f_+ + \sin \varphi \partial_0 f_+ + \cos \theta \cos \varphi \partial_\varphi f_+ - \frac{4 \cos \theta \sin \varphi}{\sin \theta} f_+ \right) \hat{f}_x \]  
\[ + \left( 2 \cos \theta \sin \varphi \partial_0 f_+ + \frac{2 \cos \varphi}{\sin \theta} \partial_\varphi f_+ + \frac{2 \cos^2 \theta \sin \varphi}{\sin \theta} f_+ - \cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ \right) \hat{f}_x \]  

\[ t^{\varphi \varphi}_{\text{rad}} = \frac{G}{\pi c^5} \left[ \sin \varphi \partial_0 f_+ + \frac{\sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \sin \varphi}{\sin \theta} f_+ - 2 \cos \theta \cos \varphi \partial_\varphi f_+ - \frac{2 \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{2 \cos^2 \theta \sin \varphi}{\sin \theta} f_+ \right] \hat{f}_+ \]  
\[ + \left( 2 \cos \theta \sin \varphi \partial_\varphi f_+ + \frac{2 \cos \varphi}{\sin \theta} \partial_\varphi f_+ + \frac{2 \cos^2 \theta \sin \varphi}{\sin \theta} f_+ - \cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ \right) \hat{f}_x \]  

\[ t^{\varphi \varphi}_{\text{rad}} = \frac{G}{\pi c^5} \left[ (-\cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ + 2 \cos \theta \sin \varphi \partial_0 f_+ - \frac{2 \cos \varphi}{\sin \theta} \partial_0 f_+ - \frac{2 \cos^2 \theta \sin \varphi}{\sin \theta} f_+ \right] \hat{f}_+ \]  
\[ + \left( 2 \cos \theta \cos \varphi \partial_0 f_+ + \frac{2 \cos \varphi}{\sin \theta} \partial_0 f_+ + \frac{2 \cos^2 \theta \cos \varphi}{\sin \theta} f_+ - \cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \cos \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ \right) \hat{f}_x \]  

\[ t^{\varphi \varphi}_{\text{rad}} = \frac{G}{\pi c^5} \left[ (-\cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \sin \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ + 2 \cos \theta \sin \varphi \partial_0 f_+ - \frac{2 \cos \varphi}{\sin \theta} \partial_0 f_+ - \frac{2 \cos^2 \theta \sin \varphi}{\sin \theta} f_+ \right] \hat{f}_+ \]  
\[ + \left( 2 \cos \theta \cos \varphi \partial_0 f_+ + \frac{2 \cos \varphi}{\sin \theta} \partial_0 f_+ + \frac{2 \cos^2 \theta \cos \varphi}{\sin \theta} f_+ - \cos \varphi \partial_\varphi f_+ + \frac{\cos \theta \cos \varphi}{\sin \theta} \partial_\varphi f_+ - \frac{4 \cos \theta \cos \varphi}{\sin \theta} f_+ \right) \hat{f}_x \]  

### IV. ANGULAR MOMENTUM

In this section we examine the angular-momentum tensor \( J^{ab} \), which satisfies the balance equation \( \dot{J}^{ab} = -\gamma^{ab} \) as a matter of identity. As reviewed in Sec. 6.1.3 of [16], the angular-momentum tensor is defined as an integral over a 2-sphere \((\tau, R) = \text{constant}\)

\[ J^{ab} = \frac{c^3}{8\pi G} \int \left( R N^a \delta_m H^{b|mk} - H^{|ab|k} \right) R^2 N_k \, d\Omega + \frac{c^2}{8\pi G} \frac{\partial}{\partial r} \int R^3 N^a H^{b|00k} N_k \, d\Omega, \]  

where \( H^{\mu\nu\lambda} = 2g^{\mu\nu} g^{\lambda\lambda} \) with \( g^{\mu\nu} \) the inverse densitized metric. We will write this as

\[ J^{ab} = \oint j^{ab} \, d\Omega, \]  

where the integrand \( j^{ab} \) depends on \( \tau, \theta, \) and \( \varphi \).

The calculation of \( J^{ab} \) requires the potentials of Eqs. (2.1) expanded through order \( R^{-3} \). After imposition of the harmonic gauge conditions, the integrand becomes independent of \( f_2^\beta \), and can be expressed entirely in terms of \( f_1^\alpha \) and \( f_1^{\alpha\beta} \).

A surprising outcome of this calculation is that \( j^{ab} \) is seen to diverge as \( R \) when \( R \to \infty \). In a general harmonic gauge, the diverging contribution is complicated, and there appears to be no way of guaranteeing that its integral is a time-independent constant. When the gauge is refined to the \(tt\) gauge of Sec. II B, however, the expression simplifies to

\[ j^{ab}_{\text{div}} = -\frac{R}{4\pi c} \left( x^a \partial_\tau f^{bc} - x^b \partial_\tau f^{ac} \right) + \frac{R}{2\pi} \left( N^a p^b - p^a N^b \right). \]
The first set of terms within brackets is equal to \( \partial_\tau (x^a f^{bc} - x^b f^{ac}) \), and because \( f^{ac} \) is tangent to the 2-sphere by virtue of the tt gauge, it gives no contribution to the integral. Hence, we find that

\[
J_{\text{div}}^{ab} = \frac{1}{2\pi} \oint (x^a p^b - p^a x^b) \, d\Omega. \tag{4.4}
\]

In spite of the fact that it diverges when \( R \to \infty \), this contribution to the total angular momentum comes with a compelling interpretation as the integral of \( x \times \dot{p} \). It is an infinite constant that is never implicated in the angular-momentum balance equation. The constancy of \( J_{\text{div}}^{ab} \) in the tt gauge provides a strong argument in favor of presenting \( \tau^{ab} \) in this gauge.

The remaining contributions to \( J_{\text{lin}}^{ab} \) are finite in the limit \( R \to \infty \). The expression involves \( f_1^{ab} \) in addition to the mass aspect \( m \), momentum aspect \( p^a \), and the radiative terms encoded in \( f_{\text{tt}}^{ab} \). Simplifying the result as for the diverging piece, we find that the terms that do not vanish after integrating are

\[
J_{\text{lin}}^{ab} = \frac{G}{\pi c^3} (N^a f_1^{bc} - N^b f_1^{ac}) p^c - \frac{G}{\pi c^3} (N \cdot p) \left( x^a \partial_\tau f^{bc} - x^b \partial_\tau f^{ac} \right)
- \frac{3}{4\pi} \left( N^a f_1^{bc} - N^b f_1^{ac} \right) N_c
- \frac{G}{16\pi c^4} \left( m - \frac{1}{c} N \cdot p \right) \left( N^a f_1^{bc} - N^b f_1^{ac} \right) N_c, \tag{4.5}
\]

where it is understood that \( f^{ab} \) is presented in the tt gauge. The first line involves leading-order potentials only, while the second line implicates next-to-leading order potentials. It is interesting to observe that the first set of terms, on the other hand, makes a genuine contribution to \( J_{\text{div}}^{ab} \). Furthermore, note that \( J_{\text{lin}}^{ab} \) contains only terms that are linear and quadratic in the gravitational potentials. This is in contrast to the flux of angular momentum, which also contains cubic terms. From this simple observation it is immediately clear that the cubic terms in the flux do not have a matching counterpart in \( J_{\text{lin}}^{ab} \), and that they therefore affect the source dynamics.

\[\text{V. EXAMPLE: BINARY SYSTEM WITH MOVING CENTER-OF-MASS}\]

\[\text{A. Post-Newtonian equations}\]

To the first few post-Newtonian (pN) orders, the solution to \( \Box \tilde{h}^{\alpha\beta} = -(16\pi G/c^4)\tau^{\alpha\beta} \) in the wave zone can be expressed as a multipole expansion (see, for instance, Eq. (7.25) in [16]). In the far-away wave zone the expression reduces to Eq. (2.1) with

\[
f^{00} = M + \frac{1}{c} \dot{M} N_j + \frac{1}{2c^2} \dot{M}^{jk} N_j N_k + O(c^{-3}), \tag{5.1a}
\]

\[
f^{0a} = P^a + \frac{1}{c} \dot{P}^{aj} N_j + \frac{1}{2c^2} \dot{P}^{ajk} N_j N_k + O(c^{-3}), \tag{5.1b}
\]

\[
f^{ab} = Q^{ab} + \frac{1}{c} \dot{Q}^{abj} N_j + \frac{1}{2c^2} \dot{Q}^{abjk} N_j N_k + O(c^{-3}), \tag{5.1c}
\]

where the multipole moments

\[
M^L := c^{-2} \int \tau^{00} x^L d^3 x, \quad P^{aL} := c^{-1} \int \tau^{0a} x^L d^3 x, \quad Q^{abl} := \int \tau^{ab} x^L d^3 x, \tag{5.2}
\]

are evaluated at the retarded time \( \tau \). The multi-index \( L := c_1 c_2 \cdots c_\ell \) contains a number \( \ell \) of individual indices, and \( x^L := x^{c_1} x^{c_2} \cdots x^{c_\ell} \). The conservation statements \( \partial_\tau \tau^{\alpha\beta} = 0 \) imply a number of identities among the multipole moments, including the facts that the total mass \( M \), total momentum \( P^a \), and total angular-momentum \( J^{ab} = c^{-1} \int (x^a \tau^{0b} - x^b \tau^{0a}) d^3 x \) are constants (independent of \( \tau \)). We also have

\[
P^a = \dot{M}^a, \quad P^{ab} = \frac{1}{2} (M^{ab} - J^{ab}) \tag{5.3}
\]

and

\[
Q^{ab} = \frac{1}{2} \dot{M}^{ab}, \quad Q^{abc} = \frac{1}{2} (\dot{P}^{abc} + \dot{F}^{bac} - \dot{P}^{cab}). \tag{5.4}
\]
These identities are not exact, but they hold to the first few PN orders required in the following discussion. Note that the total mass \( M \) includes all forms of energy in the system, including rest masses, kinetic energies, and gravitational potential energy. Similar statements can be made about the total momentum \( P^a \) and the total angular momentum \( J^{ab} \).

We consider a system of \( N \) point masses, labelled by \( A = 1, \ldots, N \). The mass of body \( A \) is denoted \( m_A \), its position is \( \mathbf{r}_A \), and its velocity is \( \mathbf{v}_A \). The system’s mass density is

\[
c^{-2} r^{00} = \sum_{A=1}^{N} m_A \delta(\mathbf{x} - \mathbf{r}_A) + O(c^{-2}),
\]

and its current density is

\[
c^{-1} r^{0a} = \sum_{A=1}^{N} m_A v_A^a \delta(\mathbf{x} - \mathbf{r}_A) + O(c^{-2}).
\]

### B. Momentum aspect

We assume that the system’s center-of-mass (CM) moves relative to the origin of the spatial coordinates, so that the system’s total momentum \( \mathbf{P} \) does not vanish. We let \( \mathbf{w} := \mathbf{P}/M \) be the CM’s velocity, and express the position of body \( A \) as \( \mathbf{r}_A = \mathbf{w} t + \mathbf{\bar{r}}_A \), where \( \mathbf{\bar{r}}_A \) is its position relative to the CM. Similarly, the velocity of body \( A \) is written as \( \mathbf{v}_A = \mathbf{w} + \mathbf{\bar{v}}_A \), where \( \mathbf{\bar{v}}_A \) is the velocity relative to the CM.

With this decomposition of the motion, it is easy to collect all the terms in \( f^{0a} \) that make up the momentum aspect \( \mathbf{p}^a(\theta, \varphi) \): we look for terms in the multipole moments \( P^{aL} \) that give rise to a time-independent contribution after the required number of differentiations. For \( P^a \) we evidently get \( P^a = M w^a \), for \( P^{a\beta} \) we have \( P^{a\beta} = M w^a w^\beta t + \) relative-motion terms \( + O(c^{-2}) \), and for \( P^{a\beta\gamma} \) we get \( P^{a\beta\gamma} = M w^a w^\beta w^\gamma t^2 + \) relative-motion terms \( + O(c^{-2}) \). From all this we obtain

\[
\mathbf{p}^a = M w^a \left[ 1 + \frac{1}{c}(\mathbf{w} \cdot \mathbf{N}) + \frac{1}{c^2}(\mathbf{w} \cdot \mathbf{N})^2 + O(c^{-3}) \right].
\]

We observe that for a single particle of mass \( m \) moving with a constant velocity \( \mathbf{w} \), the momentum aspect would build up to \( \mathbf{p}^a = M w^a/(1 - \mathbf{w} \cdot \mathbf{N}/c) \) with \( M = m(1 - w^2)^{-1/2} \); this is the familiar expression that can be obtained on the basis of the Liénard-Wiechert potentials.

### C. Leading post-Newtonian contributions to the flux

The leading PN contribution to the radiative piece of the flux is obtained by inserting the potentials of Eq. (5.1), projected to the \( \eta \) gauge, within Eq. (3.5a), and then integrating over the 2-sphere. We find

\[
\mathcal{F}^{ab}_{\text{rad}} = \frac{2G}{5c^4} \left( \frac{M (\langle \alpha \rangle_p \langle \beta \rangle_p)}{M_{\langle \alpha \beta \rangle}} - \frac{\langle \alpha \rangle_p \langle \beta \rangle_p}{\langle \alpha \rangle_p \langle \beta \rangle_p} \right),
\]

where \( M^{(ab)} := M^{ab} - \frac{1}{3} M^p \delta^{ab} \) is the tracefree piece of \( M^{ab} \). This expression is identical to Eq. (12.68c) in [16], which was obtained on the basis of the “standard expression” for the radiative flux. This agreement follows because the difference between the two expressions, given by \( (G/\pi c^5) \int \hat{\mathbf{w}}^{ab} d\Omega \) with \( \hat{w}^{ab} \) defined in Eq. (3.9), produces a contribution of order \( c^{-7} \) to the flux.

Performing the same manipulations for the cubic terms of Eq. (3.5b), we find that the leading post-Newtonian contribution is given by

\[
\mathcal{F}^{ab}_{\text{cubic}} = \frac{4}{105} \frac{M G^2}{c^{10}} (g^a w^b - w^a g^b),
\]

where

\[
g^a = 4 \tilde{M}_c^a (2w^c \tilde{P}_d^c - 3w^d \tilde{M}_d^c) + 4 \tilde{M}_{ac}^d (4 \tilde{P}_{dc}^d - 5 \tilde{P}_{cd}^d)
+ 2 \tilde{M}_c^c (9 \tilde{P}^{ad}_{cd} - 10 \tilde{P}^{da}_{cd}) - 2 \tilde{M}_{cd} (17 \tilde{P}^{ac}_{cd} - 22 \tilde{P}^{da}_{cd}).
\]

(5.10)
It should be noticed that this comes with a factor of $c^{-10}$; it therefore constitutes a 2.5PN correction to the leading contribution to the flux, given by the radiative term, which itself occurs at 2.5PN order.

Because the time-derivative terms of Eq. (3.5c) are matched by identical terms in $J^{ab}$, they have no dynamical consequences. For the sake of completeness, however, we give their leading expression for a post-Newtonian source:

$$
\mathcal{T}^{ab}_{\text{deriv}} = -\frac{4}{15} \frac{GM}{c^5} \left[ \langle \dot{M}_p^a w^b - \dot{M}_p^b w^a \rangle w^p + 4 (\ddot{F}^{abp}_p - \ddot{F}^{bap}_p) w^p w^p - 2 (\dddot{F}^{abp}_p w^b - \dddot{F}^{bap}_p w^a) + 2 (\dddot{F}^{abp}_p w^b - \dddot{F}^{bap}_p w^a) \right].
$$

(D. Binary system)

In the remainder of this section we specialize the foregoing expressions to a binary system. We let $r := r_1 - r_2$ and $v := v_1 - v_2$ be the relative positions and velocities. We express the position of each body as $r_1 = \mathbf{r} + \mathbf{a}$, $r_2 = \mathbf{r} + \mathbf{b}$, and the velocities as $v_1 = \mathbf{v} + \mathbf{a}$, $v_2 = \mathbf{v} + \mathbf{b}$. The mass parameters $M := m_1 + m_2 + O(c^{-2})$, $\eta := m_1 m_2 / M^2$, and $\Delta := (m_1 - m_2) / M$. The relevant multipole moments are then

$$
M^{ab} = \sum_{A=1}^{2} m_A r_A^b r_A^b = M w^a w^b t^2 + \eta M r^a r^b,
$$

and

$$
P^{abc} = \sum_{A=1}^{2} m_A v_A^a r_A^b v_A^c = M w^a w^b w^c t^2 + \eta M v^a (w^b r^c + w^c r^b) t + \eta M w^a v^b r^c - \eta \Delta M v^a r^b r^c.
$$

We take the binary system to be on an eccentric orbit in the $x$-$y$ plane of the coordinate system. The relative motion is described in terms of the basis vectors $\mathbf{n} := [\cos \Phi, \sin \Phi, 0]$ and $\mathbf{\lambda} := [-\sin \Phi, \cos \Phi, 0]$, where $\Phi(t)$ is the orbital phase. We have that $r = r \mathbf{n}$ and $v = \dot{r} \mathbf{n} + r \dot{\Phi} \mathbf{\lambda}$, with

$$
r = \frac{p}{1 + e \cos \Phi}, \quad \dot{r} = \sqrt{\frac{GM}{p}} e \sin \Phi, \quad \dot{\Phi} = \sqrt{\frac{GM}{p^3}} (1 + e \cos \Phi)^2.
$$

The motion is governed by two orbital parameters, the semilatus rectum $p$ and the eccentricity $e$. The orbit’s major axis is aligned with the $x$ direction.

We insert these expressions within the multipole moments, evaluate the time derivatives, and substitute the results into Eq. (5.8) and (5.9). The resulting expressions are lengthy and we simplify the results by performing an average over an orbital period, according to

$$
\langle f \rangle = (1 - e^2)^{3/2} \frac{1}{2\pi} \int_0^{2\pi} (1 + e \cos \Phi)^{-2} f(\Phi) \, d\Phi,
$$

where $f$ is any function of the orbital phase. For the radiative terms we obtain the standard results

$$
\langle \mathcal{T}^{xy}_{\text{rad}} \rangle = \frac{4}{5} \frac{G^{7/2} \eta^2 M^{9/2}}{c^{5/2}} (1 - e^2)^{3/2} (8 + 7e^2),
$$

\langle \mathcal{T}^{yz}_{\text{rad}} \rangle = 0,

\langle \mathcal{T}^{zx}_{\text{rad}} \rangle = 0.

We point out that before averaging, the radiative contribution to the angular-momentum flux carries a dependence upon the CM velocity $\mathbf{w}$. This dependence, however, goes away after the orbital average.

For the cubic contribution to the flux, we observe that a number of terms are proportional to $t$ before averaging; these originate from terms proportional to $\dot{t}$ in $P^{abc}$. The orbital average eliminates these when we take into account the fact that $t$ changes very little in the course of an orbit — the natural time scale corresponds to the radiation-reaction time — so that it can be formally treated as a constant when performing the average. We find that the averaged quantities are linear and quadratic in $\mathbf{w}$:

$$
\langle \mathcal{T}^{xy}_{\text{cubic}} \rangle = \frac{G^5 \eta^2 M^6}{c^{10}} (1 - e^2)^{3/2} \left[ \frac{32}{105} e^2 (13 + 2e^2) w^x w^y + \frac{2}{15} \Delta \sqrt{\frac{GM}{p}} e (312 + 456e^2 + 37e^4) w^x \right],
$$

(5.17a)
\[ \langle \mathcal{T}_{\text{cubic}}^{yz} \rangle = \frac{G^5 \eta^2 M^6}{c^{10} \eta^2} \left(1 - \epsilon^2\right)^{3/2} \left[ -\frac{8}{105} (192 + 606 \epsilon^2 + 77e^4) w^y w^z - \frac{2}{15} \sqrt{\frac{GM}{p}} e^{(312 + 456 \epsilon^2 + 37e^4)w^z} \right], \] (5.17b)
\[ \langle \mathcal{T}_{\text{cubic}}^{xx} \rangle = \frac{G^5 \eta^2 M^6}{c^{10} \eta^2} \left(1 - \epsilon^2\right)^{3/2} \left[ \frac{8}{105} (192 + 554 \epsilon^2 + 69e^4) w^x w^z \right]. \] (5.17c)

For a circular binary the expressions reduce to
\[ \langle \mathcal{T}_{\text{cubic}}^{xy} \rangle = 0, \] (5.18a)
\[ \langle \mathcal{T}_{\text{cubic}}^{yz} \rangle = -\frac{512 G^5 \eta^2 M^6}{35 c^{10} \eta^2} w^y w^z, \] (5.18b)
\[ \langle \mathcal{T}_{\text{cubic}}^{xx} \rangle = \frac{512 G^5 \eta^2 M^6}{35 c^{10} \eta^2} w^x w^z. \] (5.18c)

The vanishing of the \( xy \)-component of the cubic term after averaging implies that there is no tension with the expectation that \( \mathcal{P} = \Phi \mathcal{T}^{xy} \) for a circular orbit, where \( \mathcal{P} \) is the flux of energy.

Finally, because \( \mathcal{T}_{\text{deriv}}^{ab} \) is an overall time derivative, its average over an orbital cycle is guaranteed to vanish:
\[ \langle \mathcal{T}_{\text{deriv}}^{ab} \rangle = 0. \] (5.19)

**VI. CONCLUSION**

Balance laws play a critical role in creating accurate analytic models of gravitational waveforms. They relate the rate of change of the source’s energy, linear momentum, and angular momentum to the radiative losses measured by fluxes at infinity. In this paper we generalized the standard expression for the flux of angular momentum from periodic sources of gravitational waves with a fixed center-of-mass [10, 11] to arbitrary sources with nonzero center-of-mass momentum. In the absence of center-of-mass motion, our generalized flux deviates from the standard expression by a time-derivative term that appears at 1PN order beyond the leading-order flux; for periodic sources of gravitational waves, this additional contribution to the flux vanishes on the average.

When the source has a nonzero center-of-mass momentum, we find that the flux of angular momentum is not merely determined by the radiative modes of the gravitational field. Instead, it receives an additional contribution from an interplay between Coulombic and radiative modes. This result bears a resemblance to the situation in electrodynamics; in this case also the flux of angular momentum includes an interplay between the radiative modes of the field and the electromagnetic charge aspect [8]. A critical difference between electromagnetism and general relativity is the manner in which the Coulombic aspects manifest themselves. In electrodynamics, the interplay involves the charge aspect, a direct analogue of the mass aspect \( m(\theta, \phi) \). In general relativity, on the other hand, the interplay does not implicate the mass aspect, but it involves the momentum aspect \( p(\theta, \phi) \). While the momentum aspect can be eliminated by a choice of reference frame, the mass aspect — and the charge aspect in electrodynamics — cannot be eliminated. This is an essential difference in the ways that the Coulombic aspects of the field manifest themselves in the flux of angular momentum.

The last observation indicates that the additional contribution to the flux of angular momentum in general relativity, which is cubic in the gravitational potentials and involves the momentum aspect \( p \), is purely kinematical in nature. It reflects the center-of-mass motion of the source of gravitational waves in the reference frame in which the flux is evaluated. As stated, this motion can always be eliminated by a coordinate transformation, and implementing this transformation will also eliminate the additional contribution to the flux. In this way the standard expression for the flux is recovered, up to the time-derivative term that vanishes on the average for a periodic source of gravitational waves.

The kinematical origin of our additional contribution to the flux of angular momentum suggests another way to obtain its expression. One begins with a source of gravitational waves at rest, and introduce a coordinate transformation that describes a boost with center-of-mass velocity \( \mathbf{w} \). Under this boost the 3-surfaces of constant \( t \) become tilted, the 2-surfaces of constant \( t \) and \( R \) become deformed spheres, and the gravitational potentials transform in accordance with their tensorial properties. Accounting for all these changes in a calculation of the angular-momentum flux would return our generalized expression. We hasten to admit that we didn’t carry out this alternative calculation, for the

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3 The simplest way to achieve this is to perform a Lorentz transformation of the quasi-Lorentzian coordinates. But the coordinate freedom implies that this prescription is not unique. There are many transformations that correspond, asymptotically, to a boost with velocity \( \mathbf{w} \).
simple reason that it would be much harder to proceed in this way. The methods adopted in this paper are simpler, because they allow us to focus our attention entirely on an expansion of the gravitational potentials near future null infinity, and to encapsulate all the details of the coordinate transformation within the momentum aspect $p(\theta, \phi)$. But we are confident that the alternative calculation would reproduce our results, and reveal more fully the kinematic origin of our additional terms.

The calculations presented in this paper are grounded in a very essential way on the Landau-Lifshitz formulation of general relativity, which provides a foundation for the conservation law of Eq. (1.1). This approach is not unique, and one might consider an alternative approach based on a Bondi-Sachs expansion of the metric [17–19], and improve the rigor by exploiting the Penrose compactification of future null infinity [20]. It is likely that the gauge issues encountered here — in particular, the fact that the conservation law is in general ill-defined in harmonic coordinates — would have a parallel in the alternative approach based on Bondi-Sachs coordinates (although it is not clear how this would manifest itself in the geometric approach pioneered by Penrose). One would also have to investigate the consequences of the BMS group [21–23] and probably choose “good cuts” of future null infinity [24]. We leave such an exploration to future work.

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Appendix A: Flux of angular momentum

These appendices provide details that were omitted in the main text. First, we lay down the notation and re-introduce the harmonic and $tt$ gauges. Next, we derive an expression for the flux of angular momentum in a general harmonic gauge, and then specialize it to the $tt$ gauge. In Appendix B we calculate the angular-momentum tensor in a general harmonic gauge. We work with geometrized units in which $G = c = 1$, and restore these factors in appropriate places.

1. Notation and gauge conditions

The derivations in these appendices rely on the following decomposition of the linearized metric perturbation, $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h$ (see also [7]):

$$h_{\alpha\beta} = 2\phi \nabla_\alpha t \nabla_\beta t + 2A_1 \nabla_\alpha t \nabla_\beta R + 2A_2 \nabla_\alpha t m_\beta + 2\bar{A}_2 \nabla_\alpha \bar{m}_\beta$$

$$+ B_{11} \nabla_\alpha R \nabla_\beta R + 2B_{12} \nabla_\alpha R m_\beta + 2\bar{B}_{12} \nabla_\alpha \bar{m}_\beta + B_{22} m_\alpha m_\beta + \bar{B}_{22} \bar{m}_\alpha \bar{m}_\beta + 2C_{22} m_\alpha \bar{m}_\beta,$$

where $m_\alpha$ is a (complex) null vector on the two-sphere,

$$m_\alpha = \frac{1}{\sqrt{2}} R (\nabla_\alpha \theta + i \sin \theta \nabla_\alpha \varphi),$$

normalized such that $m^\alpha \bar{m}_\alpha = 1$. The metric perturbation has zero spin-weight; consequently the components $\phi, A_1, B_{11}$ and $C_{22}$ have spin-weight 0, $A_2$ and $B_{12}$ have spin-weight $-1$, and $B_{22}$ has spin-weight $-2$. We focus on the behavior of the perturbation in the far-away wave zone, in which the Cartesian components of $h_{\alpha\beta}$ can be expanded in powers of $R^{-1}$. It follows that each component defined by Eq. (A1) admits a similar expansion. For example,

$$\phi(t, R, \theta, \varphi) = \phi_0(t, \theta, \varphi) + \frac{\phi_1(t, \theta, \varphi)}{R^2} + \cdots$$

$$A_1(t, R, \theta, \varphi) = \frac{A_1^0(t, \theta, \varphi)}{R} + \frac{A_1^1(t, \theta, \varphi)}{R^2} + \cdots,$$

and so on. As previously, $\tau$ denotes retarded time: $\tau = t - R/c$. Note that $B_{22}^0$ (and $B_{22}^{\bar{0}}$) are related to the polarizations $f_+$ and $f_\times$ defined by (2.11):

$$B_{22}^0 = \frac{4G}{c^4} (f_+ - if_\times) = \frac{4G}{c^4} \bar{f} \iff f_+ = \frac{c^4}{4G} \text{Re} B_{22}^0 \text{ and } f_\times = \frac{c^4}{4G} \text{Im} B_{22}^0.$$


where we have reinstated $G$ and $c$ for clarity.

Under a general gauge transformation generated by the co-vector $\zeta_\alpha$, the metric perturbation changes according to $h_{\alpha\beta} \rightarrow h_{\alpha\beta} + 2\nabla_\alpha \zeta_\beta$. Decomposing the vector as $\zeta_\alpha = -\Lambda \nabla_\alpha t + \zeta_1 \nabla_\alpha R + \zeta_2 m_\alpha + \zeta_3 \bar{m}_\alpha$, we find that the components of the gauge-transformed metric are given by

\[
\begin{align*}
\phi &\rightarrow \tilde{\phi} = \phi - \Lambda \\
A_1 &\rightarrow \tilde{A}_1 = A_1 + \Lambda - \partial_R \Lambda + \dot{\zeta}_1 \\
A_2 &\rightarrow \tilde{A}_2 = A_2 - \frac{\sqrt{2}}{R} \partial_\alpha \Lambda + \dot{\zeta}_2 \\
B_{11} &\rightarrow \tilde{B}_{11} = B_{11} + 2\partial_R \zeta_1 - 2\zeta_1 \\
B_{12} &\rightarrow \tilde{B}_{12} = B_{12} + \frac{\sqrt{2}}{R} \partial_\alpha \zeta_1 - \dot{\zeta}_2 + \partial_R \zeta_2 - \frac{\zeta_2}{R} \\
B_{22} &\rightarrow \tilde{B}_{22} = B_{22} + \frac{2\sqrt{2}}{R} \partial_\alpha \zeta_2 \\
C_{22} &\rightarrow \tilde{C}_{22} = C_{22} + \frac{2\sqrt{2}}{R} \Re \partial_\alpha \zeta_2 + 2\frac{\zeta_1}{R} .
\end{align*}
\]

When the components of $\zeta_\alpha$ are of the form

\[
\begin{align*}
\Lambda(t, r, \theta, \varphi) &= \frac{\Lambda^0(t, \theta, \varphi)}{R} + \frac{\Lambda^1(t, \theta, \varphi)}{R^2} + \mathcal{O}(R^{-3}) \\
\zeta_1(t, r, \theta, \varphi) &= \frac{\zeta_1^0(t, \theta, \varphi)}{R} + \frac{\zeta_1^1(t, \theta, \varphi)}{R^2} + \mathcal{O}(R^{-3}),
\end{align*}
\]

with $\zeta_2$ (and $\bar{\zeta}_2$) admitting a similar expansion, the gauge transformation preserves the fall-off conditions of the gravitational potentials. Note that the leading-order terms cannot depend on $r$ as this would spoil the desired fall off in the far-away wave zone. When the gravitational potentials are required to satisfy the harmonic gauge conditions both before and after the transformation, we must insist that $\Lambda = \zeta_1 = \bar{\zeta}_2 = 0$. In addition, the components of the metric perturbation must satisfy

\[
0 = \partial^\alpha h_{\alpha\beta} = -\frac{1}{R} \left[ \left( \phi^0 + A_1^0 + \frac{1}{2} B_{11}^0 + C_{22}^0 \right) \nabla_\alpha t + \left( 2\phi^0 + 2A_1^0 + B_{11}^0 \right) \nabla_\alpha R + \left( A_2^0 + B_{12}^0 \right) m_\alpha + \ldots \right] \\
+ \frac{1}{R^2} \left[ \left( A_1^0 + 2\sqrt{2} \Re A_2^0 - \phi^0 - A_1^0 + \frac{1}{2} B_{11}^0 - C_{22}^0 \right) \nabla_\alpha t + \left( -\phi^0 + A_1^0 + \frac{3}{2} B_{11}^0 - C_{22}^0 \right) \nabla_\alpha R + \left( 2\sqrt{2} \Re A_2^0 - A_1^0 - B_{12}^0 \right) m_\alpha + \ldots \right] \\
+ \frac{1}{R^3} \left[ \left( 2\sqrt{2} \Re A_2^0 - \phi^0 - A_2^0 + \frac{1}{2} B_{11}^0 - C_{22}^0 \right) \nabla_\alpha t + \left( -2\phi^0 + B_{11}^0 + 2\sqrt{2} \Re A_2^0 - B_{12}^0 \right) \nabla_\alpha R + \left( \sqrt{2} \Re A_2^0 - 2B_{11}^0 + B_{12}^0 \right) m_\alpha + \ldots \right] \\
+ \mathcal{O}\left( \frac{1}{R^4} \right).
\]

The $1/R$-part of the harmonic gauge condition is equivalent to the leading-order linearized Einstein's equation — see Eqs. (3.17), (3.22) and (3.23) in [7]. This is not surprising given the assumed fall-off conditions, which are motivated by the fact that the linearized Einstein equation takes the form of a wave equation.

There is a residual gauge freedom in the harmonic gauge. We will use this freedom to specialize to the tt gauge, which is defined by Eq. (2.10),

\[
\tilde{h}^{ab} N_b = \mathcal{O}(R^{-2}) , \quad \tilde{h}^{ab} \delta_{ab} = \mathcal{O}(R^{-2}) .
\]

In terms of the components of the metric perturbation, we impose

\[
\phi^0 + \frac{1}{2} B_{11}^0 - C_{22}^0 = 0 , \quad B_{12}^0 = 0 , \quad \phi^0 - \frac{1}{2} B_{11}^0 = 0
\]

(A8)
to achieve the tt gauge, so that
\begin{equation}
B_{11}^0 \equiv 2\phi^0, \quad C_{22}^0 \equiv 2\phi^0, \quad B_{12}^0 \equiv 0; \quad (A10)
\end{equation}
all other components are \( \tau \)-independent (which follows directly by combining the above conditions with the harmonic gauge conditions). In terms of the mass and momentum aspects introduced in Sec. II B, we have that
\begin{equation}
\phi^0 \equiv \frac{1}{4} B_{11}^0 \equiv \frac{1}{4} C_{22}^0 \equiv \frac{G}{2} m(\theta, \varphi) + \frac{1}{c} N_0 p^0(\theta, \varphi), \quad A_1^0 \equiv -\frac{2G}{c} N_0 p^0(\theta, \varphi), \quad A_2^0 = -\frac{4G}{c} m_a p^0(\theta, \varphi), \quad (A11)
\end{equation}
where we again make the factors of \( G \) and \( c \) explicit. The tt gauge is implemented by setting the leading-order parts of \( \zeta_a \) to
\begin{equation}
\dot{\Lambda}^0 = \phi^0 - \frac{1}{2} C_{22}^0, \quad \dot{\zeta}_1 = -\phi^0 + \frac{1}{2} B_{11}^0, \quad \dot{\zeta}_2 = B_{12}^0. \quad (A12)
\end{equation}
This assignment does not spoil the harmonic gauge condition, which requires that \( \zeta_a \) satisfies \( \Box \zeta_a = 0 \). This can be verified explicitly using
\begin{equation}
\Box \zeta_a = \frac{1}{R^3} \left( \left[ -4\theta \Delta \Lambda^0 - 2\dot{\Lambda}_1 \right] \nabla_a \tau + \left[ 4\theta \Delta (\zeta_1^0 - \Lambda^0) + 2(\zeta_1^0 - \dot{\Lambda}_1) - 2\zeta_1^0 - 4\sqrt{2} \text{Re} \dot{\zeta}_2^0 \right] \nabla_a R \\
+ \left[ 4\theta \Delta \zeta_2^0 + 2\sqrt{2} \text{Re} \zeta_1^0 + 2\zeta_1^1 \right] m_a + \left[ 4\theta \Delta \zeta_2^0 + 2\sqrt{2} \text{Re} \zeta_1^0 + 2\zeta_1^1 \right] \bar{m}_a \right) + O \left( \frac{1}{R^4} \right). \quad (A13)
\end{equation}
The constants of integration in the components of \( \zeta_a^0 \) have no impact on the leading-order metric perturbations, as can be seen from the transformations in Eqs. (A5a)–(A5g). They do, however, influence the next-to-leading order perturbations, which are not completely gauge fixed (but still severely restricted) in the tt gauge. In particular, under a gauge transformation that preserves the tt gauge, the next-to-leading order perturbations transform as
\begin{align}
\phi^1 &\rightarrow \tilde{\phi}^1 = \phi^1 - \dot{\Lambda}_1 = \phi^1 + 2\theta \Delta \Lambda^0, \quad (A14a) \\
A_1^1 &\rightarrow \tilde{A}_1^1 = A_1^1 + \Lambda^0 + \dot{\Lambda}_1 + \zeta_1^1 = A_1^1 + \Lambda^0 + 2\theta \Delta \Lambda^0 + \zeta_1^1 - 2\zeta_1^0 + 2\sqrt{2} \text{Re} \dot{\zeta}_2^0, \quad (A14b) \\
A_2^1 &\rightarrow \tilde{A}_2^1 = A_2^1 - \sqrt{2} \theta \Lambda^0 + \zeta_1^1 = A_2^1 - \sqrt{2} \theta \Lambda^0 - 2\zeta_1^0 - 2\sqrt{2} \text{Im} \dot{\zeta}_2^0, \quad (A14c) \\
B_{11}^1 &\rightarrow \tilde{B}_{11}^1 = B_{11}^1 - 4\zeta_1^0 + 4\theta \Delta \zeta_1^0 - 4\sqrt{2} \text{Re} \dot{\zeta}_2^0, \quad (A14d) \\
B_{12}^1 &\rightarrow \tilde{B}_{12}^1 = B_{12}^1 + \sqrt{2} \theta \zeta_1^0 - 2\zeta_2^0 - \zeta_2^1 = B_{12}^1 + 2\sqrt{2} \theta \zeta_1^0 - 2\zeta_2^0 + 2\sqrt{2} \text{Re} \dot{\zeta}_2^0, \quad (A14e) \\
B_{22}^1 &\rightarrow \tilde{B}_{22}^1 = B_{22}^1 + 2\sqrt{2} \theta \zeta_2^0 \quad (A14f) \\
C_{22}^1 &\rightarrow \tilde{C}_{22}^1 = C_{22}^1 + 2\sqrt{2} \text{Re} \dot{\zeta}_2^0 + 2\zeta_2^0, \quad (A14g)
\end{align}
where we used Eq. (A13) to write the gauge transformation in terms of the leading-order pieces of \( \zeta_a \). Note that \( \zeta_1^0 = 0 \) in the tt gauge. Consequently, the time derivative of the next-to-leading order perturbations are gauge fixed in the tt gauge. The only gauge freedom left is to change their \( \tau \)-independent parts.

2. Flux of angular momentum

The flux of angular momentum, given by Eq. (3.1), refers to a Cartesian chart. We write it as
\begin{equation}
\mathcal{T}^{ab} = 2 \lim_{R \to \infty} \int \mathcal{N}^{[a[t]} \mathcal{N}^b_c \hat{d} \Omega, \quad (A15)
\end{equation}
where \( t^{ab} := R^3 (-g_{ab})^{tt} \). Because the calculations performed in this Appendix do not refer to such a chart, we find it useful to write the flux in the \( A \)-direction \( (A = \{x, y, z\}) \) as
\begin{equation}
\mathcal{T}^{(A)} := F^{(A)}_{ab} \mathcal{T}^{ab} = 2 \lim_{R \to \infty} \int \mathcal{F}^{(A)}_{ab} \mathcal{N}^a \mathcal{N}^{bc} \hat{d} \Omega, \quad (A16)
\end{equation}
where
\begin{equation}
F^{(A)}_{ba} := \begin{cases} 2 \tilde{g}^{[b} z^{a]} & \text{for } A = x \\ 2 \tilde{g}^{[b} \tilde{w}^{a]} & \text{for } A = y \\ 2 \tilde{g}^{[b} \tilde{g}^{a]} & \text{for } A = z, \end{cases} \quad (A17)
\end{equation}
with $\hat{x}^a$, $\hat{y}^b$, and $\hat{z}^c$ denoting the unit vectors in the $x, y, z$ directions, respectively. We next write the flux as

$$T_{(A)} = \lim_{R \to \infty} \oint \left( g_{(A)} \hat{t}^{ab} m_a N_b + cc \right) d\Omega,$$  \hspace{1cm} (A18)

by expressing the rotational Killing vector fields $K_{(A)}^a = F^b_{(A)} N_b$ as

$$K_{(A)}^a = R \left( g_{(A)} m^a + \tilde{g}_{(A)} \tilde{m}^a \right)$$  \hspace{1cm} (A19)

with $g_{(A)}(\theta, \varphi)$ satisfying $\tilde{\partial} g_{(A)} = 0$ and taking the explicit form

$$g_{(A)} = \begin{cases} -\frac{1}{\sqrt{2}} \sin \varphi + \frac{i}{\sqrt{2}} \cos \theta \cos \varphi & \text{for } A = x \\ \frac{1}{\sqrt{2}} \cos \varphi + \frac{i}{\sqrt{2}} \cos \theta \sin \varphi & \text{for } A = y \\ -\frac{i}{\sqrt{2}} \sin \theta & \text{for } A = z \end{cases}.$$  \hspace{1cm} (A20)

Let us first focus on the terms in the flux that are quadratic in the gravitational potentials, which we denote by a subscript $(2)$. Substituting Eq. (A1) into the expression for the Landau-Lifshitz pseudotensor in Eq. (3.2), and discarding terms that decay faster than $R^{-4}$, we find after a long but straightforward calculation that

$$t^{ab}_{(2)} m_a N_b = \frac{R^3}{16\pi} \left[ \hat{A}_2 \left( \hat{A}_1 + \hat{B}_11 + \hat{C}_22 \right) + \hat{B}_12 \left( -2\phi - \hat{A}_1 + \hat{C}_22 \right) - \hat{B}_22 \left( \hat{A}_2 + \hat{B}_2 \right) + \left( \phi + \frac{1}{2} \hat{B}_11 - \hat{C}_22 \right) \left( \frac{1}{\sqrt{2}R} \tilde{\partial} (\phi + \frac{1}{2} \hat{B}_11 - \hat{C}_22) + \sqrt{\frac{2}{R}} \tilde{\partial} \hat{A}_2 - \sqrt{\frac{2}{R}} \tilde{\partial} \hat{B}_2 \right) + \frac{\sqrt{2}}{R} \left( \phi - \frac{1}{2} \hat{B}_11 - \hat{C}_22 \right) \tilde{\partial} (\phi - \frac{1}{2} \hat{B}_11 - \hat{C}_22) - \frac{1}{\sqrt{2}R} \left( \phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22 \right) \tilde{\partial} (\phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22) \right) + (\phi + \frac{1}{2} \hat{B}_11) \left( \partial_R \hat{A}_2 + \partial_R \hat{B}_2 + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{B}_2 \right) - \frac{1}{\sqrt{2}R} \left( \phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22 \right) \tilde{\partial} (\phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22) + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 - \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 \right) + \hat{B}_2 \left( \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{B}_2 - \frac{1}{\sqrt{2}R} \tilde{\partial} \hat{B}_2 \right) + \frac{1}{\sqrt{2}R} \left( \phi - \frac{1}{2} \hat{B}_11 + \hat{C}_22 \right) \tilde{\partial} (\phi - \frac{1}{2} \hat{B}_11 + \hat{C}_22) + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{B}_2 \right) + \hat{B}_2 \left( \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{B}_2 - \frac{1}{\sqrt{2}R} \tilde{\partial} \hat{B}_2 \right) + \frac{1}{\sqrt{2}R} \left( \phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22 \right) \tilde{\partial} (\phi + \frac{1}{2} \hat{B}_11 + \hat{C}_22) + \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 - \frac{\sqrt{2}}{R} \tilde{\partial} \hat{A}_2 \right) \right].$$  \hspace{1cm} (A21)

At first glance it appears that in the first line there are terms that scale as $R$, but use of the harmonic gauge conditions at leading order in $R^{-1}$ reveals that these in fact cancel:

$$\hat{A}_2 \left( \hat{A}_1 + \hat{B}_11 + \hat{C}_22 \right) + \hat{B}_12 \left( -2\phi - \hat{A}_1 + \hat{C}_22 \right) - \hat{B}_22 \left( \hat{A}_2 + \hat{B}_2 \right) = \left( \hat{A}_1 + \hat{B}_11 + \hat{C}_22 \right) \left( \hat{A}_2 + \hat{B}_2 \right) - \hat{B}_22 \left( \hat{A}_2 + \hat{B}_2 \right) = 0.$$  \hspace{1cm} (A22)

Hence, the flux of angular momentum is finite in the limit $R \to \infty$, keeping $\tau$ fixed. This conclusion is a nontrivial check of the validity (of some of the terms) in Eq. (A21).

Next, we focus on the terms cubic in the gravitational potentials, which are given simply by

$$t^{ab}_{(3)} m_a N_b = -\frac{1}{16\pi} (\hat{A}_2 + \hat{B}_2) \left( \hat{B}_2 + \hat{B}_2 \right) + O(R^{-1}).$$  \hspace{1cm} (A22)

Combining the quadratic and cubic contributions, expanding all terms in Eq. (A21) in powers of $R^{-1}$, and making use of the leading-order harmonic gauge conditions repeatedly, we find that

$$t^{ab} m_a N_b = \left( \hat{A}_1 \left( \frac{1}{\sqrt{2}} \tilde{\partial} (\phi - \frac{3}{2} \hat{B}_11 - 3\hat{C}_22) - \hat{A}_1 + 2\hat{B}_11 + \sqrt{\frac{2}{R}} \tilde{\partial} \hat{B}_11 \right) + \hat{B}_11 \left( \frac{1}{\sqrt{2}} \tilde{\partial} (\phi + \frac{1}{2} \hat{B}_11 - \hat{C}_22) + 2\hat{A}_2 - \hat{B}_2 - \frac{1}{\sqrt{2}R} \tilde{\partial} \hat{B}_2 + \hat{A}_2 + \hat{B}_2 \right) \right).$$
\[ + \hat{B}_{12}^0 \left( \sqrt{2} \partial (\hat{A}_0^0 + \hat{B}_{12}^0) - \hat{B}_{22}^0 \right) \]
\[ + \hat{B}_{12}^0 \left( -\phi^0 + \frac{1}{2} \hat{B}_{11}^0 - 2C_{02}^0 + \sqrt{2} \partial (\hat{A}_0^0 + \hat{B}_{12}^0) - 2\phi^1 - 2\hat{A}_1^0 + \hat{B}_{11}^0 \right) \]
\[ + \frac{\sqrt{2}}{2} \hat{B}_{22}^0 \hat{\theta}_{B_0}^0 + \hat{B}_{22}^0 \left( \sqrt{2} \partial (\phi^0 - \frac{1}{2} \hat{B}_{11}^0) - A_2^0 - \frac{1}{\sqrt{2}} \partial B_{22}^0 - (\hat{A}_1^0 + \hat{B}_{12}^0) \right) \]
\[ - (A_0^0 + B_{12}^0) \left( \hat{B}_{22}^0 \hat{\eta}_{B_0}^0 + O(R^{-1}) \right) . \]

This expression features many higher-order terms, and all of them can be traced back to the first line in Eq. (A21). Consequently, all such terms are differentiated with respect to retarded time. This allows us to use the \( R^{-1} \) part of the harmonic gauge condition in Eq. (A7) to express them in terms of leading-order pieces of the metric perturbation. Thus, \( 2\dot{\phi}_1^0 + 2A_1^0 + \hat{B}_{11}^0 = -\phi^0 + A_0^0 + \frac{1}{2} \hat{B}_{11}^0 - C_{02}^0 + 2\sqrt{2} \Re \partial (\hat{A}_0^0 + \hat{B}_{12}^0) \), and \( A_1^0 + \hat{B}_{12}^0 = \sqrt{2} \partial (\phi^0 - \frac{1}{2} \hat{B}_{11}^0) + \sqrt{2} \partial B_{22}^0 + 2\hat{B}_{12}^0 \).

Simplifying the resulting expression once more using the leading-order harmonic gauge conditions, we arrive at the final result
\[
\frac{t^{ab}m_a N_b}{16\pi} = \frac{1}{16\pi} \int \Re \left[ g_{(A)} \left( \frac{1}{\sqrt{2}} \hat{B}_{22}^0 \hat{\eta}_{B_0}^0 - \frac{3}{\sqrt{2}} \hat{B}_{22}^0 \hat{\eta}_{B_0}^0 \right) - \frac{1}{2} \hat{A}_1^0 \hat{B}_{22}^0 - A_2^0 \hat{B}_{22}^0 \right] d\Omega , \tag{A23}
\]

where we have reinstated \( G \) and \( c \). The first two terms involve the radiative modes only, the third term is cubic in the metric perturbation, and the last term is a total time derivative since \( A_0^0 = 0 \) in the tt gauge. Recall that \( A_0^0 \approx \tilde{m}_a p^a \), so that the last two terms are proportional to the momentum aspect \( p^a \). Using Eqs. (A4) and (A11), we have explicitly verified that the radiative, cubic, and time-derivative terms match the results reported in Sec. III. From this expression, it is clear that when the radiative modes are time-independent, that is, when \( \hat{B}_{22}^0 = 0 \), the flux of angular momentum vanishes. This is as expected, since in the linearized theory the Bondi news tensor is directly proportional to \( \hat{B}_{22}^0 \), and the flux of angular momentum must be zero when the news tensor vanishes.

Appendix B: Total angular momentum

With the decomposition of the metric given in App. A1, we provide here a detailed derivation of the total angular momentum \( J^{ab} \), whose time derivative appears in the balance equation. Our starting point is the definition for angular momentum in Eq. (4.1).

The following results will be useful:
\[
H^{[0]a b} N_k = \tilde{h}^{[0]a b} - \tilde{h}^{[0]a b} N_k \tag{B1}
\]
\[
N^{[a} H^{b]000} N_k = -N^{[a} \tilde{h}^{b]k} N_k - N^{[a} \tilde{h}^{b]k} N_k \tilde{h}^{00} + N^{[a} \tilde{h}^{b]k} \tilde{h}^{0k} N_k \tag{B2}
\]
\[
N^{[a} \partial_m H^{b]m0} N_k = -N^{[a} N^{b]} \partial_m \tilde{h}^{b]0} + N^{[a} \partial_m \left( \tilde{h}^{b]0} \tilde{h}^{m0} - \tilde{h}^{b]0} \tilde{h}^{m0} \right) N_k . \tag{B3}
\]

We split the calculation into two pieces: a part that is linear in the gravitational potentials, and a part that is quadratic. Collecting the terms linear in the gravitational potentials, we find that
\[
J_{\text{linear}}^{ab} = \frac{1}{8\pi} \int R \left[ - (\dot{A}_2 + \dot{B}_{12}) + \phi R \dot{A}_2 - \frac{1}{R} \dot{A}_2 \right] N^{[a} \tilde{m}^{b]} + cc \right] \tag{B4}
\]
\[
\frac{1}{8\pi} \oint R^3 \left[ -\frac{1}{R} (\dot{\hat{A}}_2 + \dot{\hat{B}}_{12}^0) - \frac{1}{R^2} (\ddot{A}_1 + \ddot{B}_{12} + 2\ddot{A}_2) - \frac{1}{R^3} (4\dddot{A}_1 + 6\dddot{B}_{12} + 3\dddot{A}_2 + \ldots) \right] N^{[\alpha \beta]} + cc \right] .
\]

This term appears to diverge quadratically in \( R \) in the limit \( R \to \infty \). The first term vanishes, however, by virtue of the harmonic gauge conditions. And the second term can be rewritten as

\[
\dot{\hat{A}}_2 + \dot{\hat{B}}_{12} + 2\ddot{A}_2 = \sqrt{2} \partial_\theta (\phi^0 - \frac{1}{2} \dddot{B}_{11}^0) + \sqrt{2} \partial_\phi B_{22}^0 + 2(\dddot{A}_2 + \dddot{B}_{12}^0). \tag{B5}
\]

With this we get

\[
J^{(A)}_{\text{linear,div}} = \frac{1}{8\pi} \oint R \Re \left[ g^{(A)}(\theta, \varphi) \left( \sqrt{2} \partial_\theta (\phi^0 - \frac{1}{2} \dddot{B}_{11}^0) + \sqrt{2} \partial_\phi B_{22}^0 + 2(\dddot{A}_2 + \dddot{B}_{12}^0) \right) \right] \tag{B6}
\]

\[
= \frac{1}{8\pi} \oint R \Re \left[ \sqrt{2} \partial_\theta (\phi^0 - \frac{1}{2} \dddot{B}_{11}^0) + \sqrt{2} \partial_\phi B_{22}^0 - 2g(\dddot{A}_2 + \dddot{B}_{12}^0) \right] \tag{B7}
\]

\[
= \frac{1}{8\pi} \oint R \Re \left[ -2g(\dddot{A}_2 + \dddot{B}_{12}^0) \right] \tag{B8}
\]

when we calculate the angular momentum in a particular \( A \)-direction. We integrated by parts to go from the first to the second line, and to go to the third we used that \( \partial_\theta g = 0, \partial_\phi g \) is entirely imaginary (so that \( \Re (\partial_\phi g) = 0 \)), and \( \phi^0 \) and \( B_{11}^0 \) are real. The divergent piece of \( J^{(A)}_{\text{linear}} \) is nonzero, even in the tt gauge. However, as was shown in Sec. IV, in the tt gauge we can interpret this term as \( x \times p \), so that when considering sources with \( p = 0 \), \( J^{(A)}_{\text{linear}} \) is finite. While the total angular momentum is formally divergent even in the tt gauge, the time derivative of \( J^{(A)}_{\text{linear}} \), which appears on the left-hand side of the angular-momentum balance equation, is finite. This is true in any harmonic gauge, given that \( A_2^0 + B_{12}^0 = 0 \) by the harmonic gauge condition. However, we shall see that this is not true for the contribution to \( J^{ab} \) that is quadratic in the gravitational potentials. In this case, imposition of the tt gauge is necessary to make sense of the total angular momentum.

The finite part of \( J^{ab}_{\text{linear}} \) in the limit \( R \to \infty \) is

\[
J^{(A)}_{\text{linear,fin}} = -\frac{1}{8\pi} \oint \Re \left[ g(\dddot{A}_2 + \dddot{B}_{12} + 3\dddot{A}_2) \right] d\Omega, \tag{B9}
\]

and using the harmonic gauge condition \( \dot{\hat{A}}_2 + \dot{\hat{B}}_{12} = \sqrt{2} \partial_\theta (\phi^1 - \frac{1}{2} B_{11}) + \dot{\hat{B}}_{12} + \sqrt{2} \partial_\phi B_{22}, \) we find that with similar manipulations as above,

\[
J^{(A)}_{\text{linear,fin}} = -\frac{1}{8\pi} \oint \Re \left[ g(B_{12}^0 + 3\dddot{B}_{12}) \right] d\Omega . \tag{B10}
\]

This expression is valid in a generic harmonic gauge, but it is \textit{gauge-dependent}. In the tt gauge, there is residual gauge freedom in \( A_2^1 \) and \( B_{12}^1 \), so that the integrand of \( J^{ab}_{\text{linear}} \) is not completely gauge fixed. Nonetheless, the residual gauge freedom in the integrand vanishes after the integration over the two-sphere has been carried out, so that \( J^{ab}_{\text{linear}} \) is completely gauge fixed in the tt gauge. This follows from substituting Eqs.(A14c) and (A14e) into the above expression, integrating by parts, and using \( \partial_\theta g = 0 \) and \( \Re (\partial_\phi g) = 0 \).

Moving on to the part quadratic in the gravitational potentials, we find

\[
J^{ab}_{\text{quad}} = \frac{1}{8\pi} \oint \left[ R^3 \left[ \partial_\tau (A_2 - B_{12})(\phi + A_1 + \frac{1}{2} B_{11}) - C_{22}(A_2 + B_{12}) \right] \right.
\]

\[
+ g (A_1 B_{12} - A_2 (\phi + \frac{1}{2} B_{11} - C_{22}) \right]
\]

\[
\left. + \frac{1}{R} \left( (A_2 (\phi - \frac{1}{2} B_{11}) - \dddot{A}_2 B_{22} + \partial_\phi (A_2 B_{12} - A_2 B_{12})) \right) \right] N^{[\alpha \beta]} + cc \right) d\Omega.
\tag{B11}
\]

The second and third lines are manifestly finite in the limit \( R \to \infty \), given that they decay as \( R^3 \cdot R^{-3} \), whereas the first line is not:

\[
J^{(A)}_{\text{quad,div}} = \frac{1}{4\pi} \oint R \Re \left[ g(A_2 (\phi + A_1 + \frac{1}{2} B_{11}) \right] d\Omega . \tag{B12}
\]

In contrast to the diverging linear piece, this term is time dependent so that in general it persists in the balance equation. This illustrates the important fact that the balance equation is not well-defined in a generic harmonic
gauge in the limit $R \to \infty$. In the tt gauge, however, $J^{(A)}_{\text{quad, div}} = 0$ because $\tilde{A}^0_2 = 0$. Consequently, the balance equation is well-defined in the tt gauge. We shall therefore, from now on, specialize to the tt gauge.

We find after using the harmonic gauge conditions several times that the quadratic term can be written as

$$J^{(A)}_{\text{quad, lin}} = \frac{1}{8\pi} \int \text{Re} \left[ g(A) \left( \tilde{A}_2^1(4\phi^0 + 2A^0_1) - \sqrt{2}\tilde{B}_2^0(4\phi^0 + A^0_1) - A^0_2\tilde{B}_2^0 \right) \right] d\Omega \quad (B13)$$

in the tt gauge. Despite the appearance of higher order terms, this expression is completely gauge fixed since the residual gauge freedom in $\tilde{A}_2^1$ is only a time-independent function. The total angular momentum in the tt gauge is therefore given by

$$J^{(A)} = J_{\text{lin}}^{(A)} + J^{(A)}_{\text{quad}} \equiv \frac{c^3}{8\pi G} \int \text{Re} \left[ g(A) \left( -2R\tilde{A}_2^0 - 3\tilde{A}_2^1 - \tilde{B}_2^0 - \frac{1}{2}\tilde{B}_2^1(4\phi^0 + 2A^0_1) + \sqrt{2}\tilde{B}_2^0 A^0_1 - A^0_2\tilde{B}_2^0 \right) \right] d\Omega,$$

and there is no residual gauge freedom left. Its time derivative is

$$\dot{J}^{(A)} \equiv \frac{c^3}{8\pi G} \int \text{Re} \left[ g(A) \left( 2\dot{B}_2^1 - \frac{1}{2}\tilde{B}_2^1(4\phi^0 + 2A^0_1) + \sqrt{2}\tilde{B}_2^0 A^0_1 - \frac{1}{2}A^0_2\tilde{B}_2^0 \right) \right] d\Omega,$$  \quad (B14)

where we used the harmonic gauge conditions once more and performed similar manipulations as in the rest of this appendix to simplify the result. The first two terms involve parts of the gravitational potentials that decay as $R^{-2}$, as well as the mass and momentum aspects. The third term is a product of the radiative mode and the radial component of the momentum aspect. The last term matches exactly the total time-derivative term that appears in the flux of angular momentum (see eq. (A24)). As pointed out before, this shows clearly that this term has no dynamical consequences.

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