Gravitational scattering on a global monopole reexamined

Waldemar Puszkarz*

Department of Physics and Astronomy,
University of South Carolina,
Baltimore, SC 29208

(5 November 1995)

Abstract

We critically reexamine the gravitational scattering of scalar particles on a global monopole studied recently. The original investigation of Mazur and Papavassiliou is extended by considering different couplings of the scalar field to the space-time curvature and by varying the dimension of the space-time where the Klein-Gordon (KG) field lives. A universal behavior of the leading term in the scattering amplitude as a function of this dimension is revealed.

*Electronic address: puszkarz@math.sc.edu

Submitted to Physical Review D
1 Introduction

Recently, the gravitational scattering on a global monopole has been studied [1, 2, 3]. In the most recent paper [3], the scattering of fermions has been investigated and the correct scattering amplitude for it derived. The other papers [1, 2] deal with scalar particles in a monopole background. Their analysis of the scattering of these particles on the global monopole leads to equations describing the total scattering cross-section for this process. However, a brief examination of those equations suffices to notice that they differ, having in common a serious pathology: in the limit corresponding to the Minkowski space-time they result in infinite total cross-sections for the process under consideration. Thus, one can rightly suspect that this very unphysical result is due to some errors in the derivation of the discussed formulas rather than an intrinsic pathology of the problem. To show that it is indeed the case is in part the goal of the present paper. Before we embark on this, let us mention key errors in [1, 2]. In the approach of Mazur and Papavassiliou [1], the optical theorem is used to obtain the expressions for the total scattering cross-section. Since, as shown below, this theorem is not valid for the case under study, its application renders formulas describing the total cross-section (equations (26) and (28) in [1]) incorrect. In the paper by Lousto, a simple error in the calculation of an integral is made (formulas (27), (28), and (25) in [2]), which nevertheless brings about dramatic consequences. The present report is organized as follows. Since its major purpose is to reexamine and elucidate points that lead to wrong results in the above referred papers, we perform this task in the very next section where we also find out how the inclusion of interaction between the scalar field and the curvature of the global monopole space-time affects the amplitude for the process under consideration. We extend the original investigations [1, 2] in Section 3. Here, we demonstrate that the leading term in the amplitude can be obtained as a special case of the scattering on the global monopole in an arbitrary number of dimensions \( D = 1 + d \) of the underlying space-time with \( d > 1 \). In the conclusions we summarize the main results of the work presented. Following this, two appendices gather explicit derivations of some relevant formulas, which we did not find suitable to include in the main body of the paper.

2 Critique and Correction of Previous Results

For the sake of simplicity, in what follows, we will study massless scalar particles in the global monopole background. We will adopt a system of units in which \( c = G = \hbar = 1 \).

Let us start from the Lagrangian

\[
L = \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Psi + \xi R \Psi^2,
\]

where \( \nabla_\alpha \) is a covariant derivative with respect to the metric \( g_{\alpha\beta} \) and \( R \) stands for the scalar curvature of the space-time where the scalar field \( \Psi \) propagates. The parameter \( \xi \) is a coupling constant.

The equation of motion derived from (1) reads

\[
\left( \nabla^2 - \xi R \right) \Psi = 0, \tag{2}
\]

where

\[
\nabla^2 = \frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} \partial_\beta g^{\alpha\beta} \right)
\]
and \( g = \text{det}(g_{\alpha\beta}) \).

We will first consider the minimal coupling case, i.e., with \( \xi = 0 \). Since the metric of the global monopole
\[
ds^2 = -dt^2 + dr^2 + b^2 r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{3}
\]
is static, the stationary solutions to (2) can be assumed as \( \Psi(\vec{r}, t) = \Phi(\vec{r})e^{-iEt} \). (See [4] for the derivation of the metric and [1] for a list of references.) Upon defining \( k^2 = E^2 \), the KG equation reduces to
\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{L^2}{b^2 r^2} + k^2 \right] \Phi(\vec{r}) = 0, \tag{4}
\]
where \( L^2 \) is the Laplacian on a two-dimensional sphere and \( k^2 > 0 \) as we are interested in the scattered states only.

Due to the spherical symmetry of the problem under study, as a complete set of eigenfunctions on \( S^2 \) one can employ the Legendre polynomials \( P_l(\cos \theta) \). Let us recall that they satisfy
\[
L^2 P_l(\cos \theta) = l(l + 1) P_l(\cos \theta). \tag{5}
\]

Looking for a solution to (2) as a series \( \sum_{l=0}^{\infty} a_l R_l(r) P_l(\cos \theta) \), brings us, upon the separation of variables, to the following equation for the radial wave function
\[
R'' + \frac{2}{r} R' + \left( k^2 - \frac{l(l+1)}{b^2 r^2} \right) R = 0, \tag{6}
\]
the primes denoting the differentiation with respect to \( r \). By demanding that \( R \) is regular at the origin, one can single out the unique solution \( R_l(r) = r^{-(l+1)/2} J_{l+1/2}(kr) \), where \( \nu(l) = b^{-1} \sqrt{(l+1/2)^2 - (1-b^2)/4} \) (see [1] for more details).

The general scattering solution is a superposition of the incoming wave function \( \Phi_{\text{in}} = e^{ikz} = e^{ikr \cos \theta} \) and the scattered one \( \Phi_{\text{scat}} = \Phi_{\text{in}} + \Phi_{\text{scat}} \),
\[
\Phi = \Phi_{\text{in}} + \Phi_{\text{scat}}, \tag{7}
\]
where \( \Phi_{\text{scat}} = e^{ikr f(\theta)} \) as \( r \to \infty \). We are primarily interested in the function \( f(\theta) \), the scattering amplitude that contains information on the differential cross-section of the scattering process. Using the partial wave analysis, one arrives at the following formula for this amplitude [see Eq.(B13)],
\[
f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)(e^{2i\delta_l} - 1) P_l(\cos \theta), \tag{8}
\]
where a phase shift \( \delta_l \) is a function of \( b \)
\[
\delta_l = \delta_l(b) = \frac{\pi}{2} \left( l + 1/2 - \nu(l) \right) = \frac{\pi}{2} \left( l + 1/2 - b^{-1} \sqrt{(l+1/2)^2 - (1-b^2)/4} \right). \tag{9}
\]

It is here that our analysis departs significantly from the one in [1]. First of all, the sum
\[
\Sigma_0 = \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta), \tag{10}
\]
as shown in Appendix A, produces a Dirac-delta term which was overlooked in [1]. Besides, as we will shortly see, even if we neglect this term, the remaining part of the amplitude leads to a differential scattering cross-section

\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \] (11)

that being strongly divergent for some values of \( \theta \), results in an infinite total scattering cross-section \( \sigma \) for our process. The optical theorem,

\[ \sigma = \frac{4\pi}{k} \text{Im} f(0), \] (12)

usually invoked to relate \( f(\theta) \) to \( \sigma \), even if naively valid in the sense that both sides of it are infinite, does not provide any meaningful information on the scattering process under consideration. Moreover, the optical theorem is formally (naively) valid only due to the presence of the delta term \( \Sigma_0 \) that was omitted in [1].

We will now proceed to compute

\[ \Sigma_r = \sum_{l=0}^{\infty} (2l + 1)e^{2i\delta_l} P_l(\cos \theta) \] (13)

by employing the method of [1]. Since \( e^{2i\delta_l} = e^{i\pi\alpha}(1 + i\pi a^2/2bz + O(z^{-3})) \), where \( \alpha = 1 - b^{-1} \), \( a^2 = (1 - b^2)/4 \), and \( z = l + 1/2 \), if we use \( a^2 \) as an expansion parameter, \( \Sigma_r \) can be well approximated by the first two terms in this expansion, i.e., \( \Sigma_r = \Sigma_1 + \Sigma_2 \), where

\[ \Sigma_1 = 2 \sum_{l=0}^{\infty} z(l) e^{i\pi\alpha(l)} P_l(\cos \theta) = \frac{2}{i\pi} \frac{d}{d\alpha} \sum_{l=0}^{\infty} e^{i\pi\alpha(l)} P_l(\cos \theta) = \frac{-2i \sin \pi \alpha}{[2(\cos \pi \alpha - \cos \theta)]^{3/2}}. \] (14)

and

\[ \Sigma_2 = \frac{i\pi a^2}{b} \sum_{l=0}^{\infty} e^{i\pi\alpha(l)} P_l(\cos \theta) = \frac{i\pi a^2 b^{-1}}{[2(\cos \pi \alpha - \cos \theta)]^{1/2}}. \] (15)

To show that \( \sum_{l=0}^{\infty} e^{i\pi\alpha(l)} P_l(\cos \theta) = [2(\cos \pi \alpha - \cos \theta)]^{-1/2} \), one makes use of the generating function for the Legendre polynomials \( F(h, \theta) = \sum_{l=0}^{\infty} h^l P_l(\cos \theta) = (1 - 2h \cos \theta + h^2)^{-1/2} \). Therefore, in our approximation the scattering amplitude for \( \theta < \pi \alpha \) and \( \theta > \pi \alpha \) is

\[ f_-(\theta) = -\frac{i}{k} \left[ \delta(1 - \cos \theta) + \frac{1}{2\sqrt{2(\cos \theta - \cos \pi \alpha)}} \left( \frac{\pi a^2}{b} + \frac{\sin \pi \alpha}{\cos \theta - \cos \pi \alpha} \right) \right], \quad (16a) \]

and

\[ f_+(\theta) = \frac{1}{2k\sqrt{2(\cos \pi \alpha - \cos \theta)}} \left[ \frac{\pi a^2}{b} - \frac{\sin \pi \alpha}{\cos \pi \alpha - \cos \theta} \right], \quad (16b) \]

correspondingly. As seen from the last equations, the differential cross-section is singular for \( \theta = 0, \pi \alpha \). It seems also to be vanishing for \( \theta \) such that \( |\cos \pi \alpha - \cos \theta| = b \sin \pi \alpha/\pi a^2 \). However, this effect is obviously an artifact of the approximation used and if the omitted terms are present, the scattering cross-section is not likely to vanish, at least not for the angles satisfying the above condition.
For $\theta \neq 0$, the differential cross-section simplifies to

$$d\sigma = \frac{\sin^2 \pi \alpha}{64k^2} \left| \frac{\omega}{\sin^3 \frac{\pi}{2} \sin^3 \left( \frac{\pi}{2} + \pi \alpha \right)} \right|^2 \left[ 1 - \frac{2\pi a^2 \sin \frac{\pi}{2} \sin \left( \frac{\pi}{2} + \pi \alpha \right)}{b \sin \pi \alpha} \right]^2,$$

(17)

where $\omega = \theta - \pi \alpha$. Clearly, the main contribution to the cross-section comes from the vicinity of the ring $\theta = \pi \alpha$, which corresponds to the limit $\omega \to 0$ in Eq. (17) and should physically be understood as the condition $\omega \ll \pi \alpha$. It is in this limit that the singularity of the scattering cross-section for a non-forward scattering is displayed in its full transparency. In a good approximation the differential cross-section is given here by

$$d\sigma \approx \frac{1}{64k^2 \sin \pi \alpha \sin^3 \omega / 2}.$$

(18a)

As seen from this formula, the singularity so strong is bound to yield an infinite total cross-section even if the delta term is discarded. In the regime $\alpha \ll 1$, which seems to be the most physically plausible, the leading approximation to $d\sigma / d\Omega$ for $\omega$ not so close to 0 behaves as

$$d\sigma \approx \frac{\cos^2 \alpha}{64k^2 \sin^6 \omega / 2} \left( 1 + \sin^2 \omega / 2 \right)^2.$$

(18b)

It is through Eqs. (17) and (18) that the physical signature of the monopole on the scattering of scalar particles is most prominently exhibited.

It is a good point to address some results of [2] pertinent to our study. Contrary to Eq. (28) in there, the total scattering cross-section is not finite as shown above. The roots of this error are in the omission of the absolute value brackets in Eq. (25) of the discussed paper and the subsequent application of the Cauchy principal value method to calculate an integral whose integrand does not surrender to the trick: it does not change sign at the singular point with the consequence of infinities adding up instead of cancelling out. The question whether the total scattering cross-section is to be regularized will not be considered in this report. Although in the addressed paper this issue seems to be brought up by the error, one should not neglect it. We limit ourselves to two comments only. On the one hand, it is not a new problem: a similar singularity occurs in the Rutherford formula for the Coulombic scattering as it does in the celebrated Aharonov-Bohm scattering [5]. In both cases the total cross-section is infinite. On the other hand, it is clear that if one wants to make any reasonable use of the total cross-section in situations like the discussed ones, a regularization cannot be avoided.

Let us now consider the case $\xi \neq 0$. Since for our metric the scalar curvature is $R = -2(1-1/b^2)r^{-2}$ (see [1]), Eq. (4) is now replaced by

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{L^2}{b^2 r^2} - \frac{2\xi (1-1/b^2)}{r^2} + k^2 \right] \Phi(\tilde{r}) = 0$$

(19)

and the radial part of it can be brought to

$$R'' + \frac{2}{r} R' + \left[ \frac{1}{r^2} \left( l(l+1)/b^2 + 2\xi (1-1/b^2) \right) + k^2 \right] R = 0$$

(20)

whose solution is $R_{\nu(l, \xi)} = r^{-1/2} J_{\nu(l, \xi)}(kr)$, where $\nu(l, \xi) = b^{-1} \sqrt{(l+1/2)^2 - (1-b^2)(1+8\xi)/4}$. We see that the only way $\xi$ affects our previous solution is by the change of the parameter $a^2$, namely $a^2 \to a'^2 = a^2 (1+8\xi)$. For $\xi = -1/8$, $a'^2 = 0$, causing formulas (16) and (17) to simplify considerably. For the massless, conformally coupled scalar field $\xi = 1/6$ and $a'^2 = 7(1-b^2)/12$ is slightly more than twice as large as for the massless, non-coupled field. (Similar observations in congruence with ours are made in [2].)
3 Higher Dimensional Scattering

We will now establish a universal behavior of the leading term in the differential scattering cross-section for the scattering on the global monopole in an arbitrary number of dimensions.

To this end, let us consider the metric of the global monopole in \( D = 1 + d \) dimensions \((d \geq 2)\)
\[
ds^2 = -dt^2 + dr^2 + b^2 r^2 d\Omega_{d-1}^2,
\]
(21)
where \( d\Omega_{d-1}^2 \) is the metric on a \((d - 1)\)-dimensional sphere and \( b^2 \) represents the monopole defect which in \((1+2)\)- and \((1+3)\)-dimensional spacetimes is the angular and the solid angle defect, respectively. For \( d > 3 \) one has to do with defects that result in deficits on spheres of higher dimensions. In what follows, we will assume \( d > 2 \).

For simplicity, let us limit ourselves to the minimal coupling case so that the KG equation for \( \Psi = e^{iEt} \Phi(\vec{r}) \) reduces to
\[
\left[ \frac{1}{r^d} \frac{\partial}{\partial r} \left( r^d \frac{\partial}{\partial r} \right) - \frac{L_{d-1}^2}{b^2 r^2} + k^2 \right] \Phi(\vec{r}) = 0
\]
(22)
[see the derivation of Eq.(4)], where \( L_{d-1}^2 \) is the Laplacian on the \((d - 1)\)-dimensional sphere. The eigenfunctions \( Y \) of \( L_{d-1}^2 \) satisfy the equation
\[
L_{d-1}^2 Y = l(l + d - 2) Y
\]
(23)
and in the case of spherical symmetry are known as the Gegenbauer polynomials \( G_s^l(\cos \gamma) \), where \( s = \frac{d-2}{2} \) and \( \gamma \) is an angle between two arbitrarily chosen directions from the center of symmetry. In what follows, we will treat \( \gamma \) as the angle between an incoming and a scattered wave.

We will seek the solution for \( \Phi(\vec{r}) \) as a series
\[
\Phi(\vec{r}) = \sum_{l=0}^{\infty} a_l R_s^l(r) G_s^l(\cos \gamma)
\]
(24)
the radial part of which satisfies
\[
\frac{1}{r^d} \frac{d}{dr} \left( r^d \frac{d}{dr} \right) R_s^l + \left( k^2 - \frac{l(l + d - 2)}{b^2 r^2} \right) R_s^l = 0.
\]
(25)
By substituting \( R_s^l = r^{-(d-2)/2} R_l \) one obtains from (25)
\[
R_l'' + \frac{1}{r} R_l' + \left( k^2 - \frac{\nu^2(l, s)}{r^2} \right) R_l = 0,
\]
(26)
where \( \nu^2(l, s) = \frac{2l + d - 1}{b^2} \). The solution regular at \( r = 0 \) is \( R_s^l(r) = r^{-s} J_{\nu(l, s)}(kr) \).

The amplitude of the scattering cross-section, as found in Appendix B, is
\[
f(\gamma) = \frac{C(d)}{k^{(d-1)/2}} \sum_{l=0}^{\infty} (l + s)(e^{2i\delta_l(s)} - 1) G_s^l(\cos \gamma) = \frac{C(d)}{k^{(d-1)/2}} \left( \Sigma_0^s + \Sigma_r^s \right),
\]
(27)
where \( C(d) \) is constant for a fixed \( d \) and \( \Sigma_0^s = \sum_{l=0}^{\infty} (l + s) G_s^l(\cos \gamma) \) is zero (unless \( \gamma = 0 \) when it is infinite as a Dirac delta as shown in Appendix A) and will be omitted from further considerations.
The other term, \( \Sigma_s^r = \sum_{l=0}^{\infty} (l + s)e^{2i\delta_l(s)}G_l^s(\cos \gamma) \), will be studied in a greater detail for it is the one that contains the leading term of \( f(\gamma) \). Now, 

\[
\delta_l(s) = \frac{\pi}{2} (l + s - \nu(l, s)) = \frac{\pi}{2} \left[ z - \frac{z}{b} \sqrt{1 - a^2/z^2} \right] = \frac{\pi}{2} (\alpha z + a^2/2bz + O(z^{-3})),
\]

where \( \alpha = 1 - b^2 \), \( a^2 = s^2(1 - b^2) \), and \( z = l + s \). We are now ready to find the leading term \( lt \) in \( \Sigma_s^r \),

\[
\Sigma_s^r = \sum_{l=0}^{\infty} z(l) e^{i\pi \alpha z(l)} (1 + i\pi a^2/2bz + O(z^{-2})) G_l^s(\cos \gamma),
\]

that is,

\[
lt = \sum_{l=0}^{\infty} z(l) e^{i\pi \alpha z(l)} G_l^s(\cos \gamma) = \frac{1}{i\pi} \frac{d}{d\alpha} h^s(\gamma, \alpha),
\]

where

\[
h^s(\gamma, \alpha) = \sum_{l=0}^{\infty} e^{i\pi \alpha z(l)} G_l^s(\cos \gamma) = \left[ 2(\cos \pi \alpha - \cos \gamma) \right]^{-s}.
\]

The last formula can be worked out by performing straightforward manipulations on the generating function for the Gegenbauer polynomials

\[
G(x, t) = \sum_{l=0}^{\infty} t^l G_l^s(x) = (1 - 2xt + t^2)^{-s}.
\]

Finally,

\[
lt = \frac{-is \sin \pi \alpha}{2^s(\cos \pi \alpha - \cos \gamma)^{s+1}},
\]

and the leading term of \( f(\gamma) \) in its complete form reads

\[
flt = \frac{-i(d - 2)\Gamma(\frac{d + 2}{2}) e^{-i\frac{\pi}{2}(d-1)} \sin \pi \alpha}{2^s \sqrt{\pi k^{d-1}} (\cos \pi \alpha - \cos \gamma)^{\frac{d-1}{2}}}.
\]

The last formula is valid for \( d > 2 \) and \( \gamma > \pi \alpha \). It is easy to analytically extend it to \( \gamma < \pi \alpha \), but since this changes only the exponent, the final result [Eq. (36)] remains unaffected. For \( d = 2 \) as shown in [6]

\[
f(\gamma) \propto \frac{\sin \pi \alpha}{k^{\frac{d-1}{2}} (\cos \pi \alpha - \cos \gamma)^{\frac{d}{2}}},
\]

As opposed to \( d > 2 \), the scattering amplitude on the monopole in \((1 + 2)\)-dimensional space-time is known in its complete exact form. Therefore we have shown that in the first approximation

\[
f(\gamma) \propto \frac{\sin \pi \alpha}{k^{\frac{d-1}{2}} (\cos \pi \alpha - \cos \gamma)^{\frac{d}{2}}},
\]

thus exhibiting a universal behavior in the sense that it can be described by a single unique formula which as a function of \( d \) applies to all dimensions \( d > 1 \).
4 Conclusions

We have shown that the optical theorem as employed in [1] for the gravitational scattering on the global monopole is not valid, thus producing the wrong total cross-section for the process under study. The situation here is similar to the Coulomb scattering where the total cross-section is divergent due to the long-range nature of this interaction and is characteristic to the scattering on the global monopole in any number of dimensions in which the global monopoles are conceivable. For the (1 + 2)-dimensional monopole it was first noticed in [3]. It is the cone-like structure of these space-times that provides the long-range interaction thereby making even high angular momenta contribute in a non-negligible manner to the total scattering cross-section. Although the appearance of the delta term in the scattering amplitude is the main obstruction in a meaningful application of the optical theorem, the theorem would not be valid even if this term were left out. Furthermore, we have provided the resolution to the problem of the infinite total cross-section arising in the Minkowski space-time limit reported in [2]. This effect is completely spurious as caused by erroneous calculations. We have also demonstrated that the coupling of the scalar field to a non-zero curvature of the monopole space-time can affect the scattering amplitude leading in some instances to its simplification. A nice feature of the amplitude is that its leading term is given by a single universal formula valid for all space-time dimensions that allow for the existence of global monopoles. In all of them, the total scattering cross-section is an ill-defined quantity and, unless a reasonable regularization is proposed, one should not invoke it to describe the scattering process discussed throughout this paper.

Acknowledgments

I would like to thank Professor Pawel O. Mazur for introducing me to the physics of monopoles and his encouragement to write up the results of this work. I am particularly indebted to him for the critical reading of the manuscript, which helped clarify a few points and correct some formulas. I am also beholden to Carlos O. Lousto for pointing me to his paper. This work was partially supported by the NSF grant No.13020 F167.

Appendix A

We will show here that $\Sigma^s_0$ is equal to some Dirac-delta term. Using the generating function for the Gegenbauer polynomials

$$G(h, \gamma) = \sum_{l=0}^{\infty} h^l G^s_l (\cos \gamma) = (1 - 2h \cos \gamma + h^2)^{-s}, \quad (A1)$$

it is straightforward to see that

$$\Sigma(h, x) = sG + h \frac{\partial G}{\partial h} = \sum_{l=0}^{\infty} (l + s) h^l G^s_l (\cos \gamma) = \frac{s(1 - h^2)}{(1 - 2hx + h^2)^{s+1}}, \quad (A2)$$

where $x = \cos \gamma$, and $s = \frac{d-2}{2}$. Now,

$$\Sigma^s_0 = \sum_{l=0}^{\infty} (l + s) G^s_l (\cos \gamma) = \lim_{h \to 1} \Sigma(h, x)$$
which equals $\infty$ if $x = 1$ or 0 otherwise. This clearly demonstrates that $\Sigma_0^\delta$ is equal to a Dirac delta term. For the $(1 + 3)$-dimensional space-time $\Sigma_0^\delta = \sqrt{2}\Sigma_0$, so $\Sigma_0 = \sum_{l=0}^\infty (2l + 1)P_l(\cos \theta)$ must be proportional to some Dirac delta as well. To establish the coefficient of proportionality, we will employ the orthogonality relation for the Legendre polynomials
\begin{equation}
\int_{-1}^{1} P_l(x)P_\nu(x)\, dx = \frac{2}{2l + 1}\delta_\nu.
\end{equation}
By multiplying $\Sigma_0$ by $P_\nu(x)$ and integrating over $x$, one obtains 2 due to (A3). Therefore one is lead to conclude that $\Sigma_0 = 2\delta(1 - \cos \theta)$.

**Appendix B**

Below we present the phase-shift method, also known as the partial wave analysis, applied to a spherically symmetric scattering in $D = 1 + d$ ($d > 2$) space-time dimensions.

In $D$ dimensions, the stationary Schrödinger equation describing a single freely propagating particle of mass $m$ and energy $E$ reads
\begin{equation}
(\nabla_D^2 + k^2)\Psi(\vec{r}) = 0,
\end{equation}
where $k^2 = 2mE$ and $\nabla_D^2 = \frac{1}{r^d}\frac{\partial}{\partial r} \left(r^d \frac{\partial}{\partial r}\right) - \frac{L_d^2}{r^2}$, $L_{d-1}$ being the Laplacian on a $(d - 1)$-dimensional sphere. By separating variables $\Psi(\vec{r}) = R(r)Y(\gamma)$, one obtains
\begin{equation}
L_{d-1}Y_i^d(\gamma) = l(l + d - 2)Y_i^d, \tag{B2}
\end{equation}
\begin{equation}
R''(r) + \frac{d}{r}R'(r) + \left(k^2 - \frac{l(l + d - 2)}{r^2}\right)R(r) = 0, \tag{B3}
\end{equation}
where $\gamma$ is an angle between two arbitrary directions from the center of symmetry which in our case is the center of scattering. In what follows, we will think of $\gamma$ as the angle between an incoming and a scattered wave. The solutions to (B2) in terms of functions of $\cos \gamma$ are called the Gegenbauer polynomials. We will use them from now on employing the notation $Y_i^d(\gamma) = G_i^s(\cos \gamma)$, where $s = \frac{d-2}{2}$.

Let us now work out the solutions to (B3). We will denote them by $R_i^s$ by analogy to $G_i^s$. Upon substituting $R_i^s = r^{-s}l_i$, Eq. (B3) reduces to
\begin{equation}
R''_i + \frac{1}{r}R'_i + \left(k^2 - \frac{(l + s)^2}{r^2}\right)R_i = 0, \tag{B4}
\end{equation}
that is to the well known Bessel equation. Being interested in the solutions regular at $r = 0$, we choose $R_i^s(r) = r^{-s}J_{l+s}(kr)$. Once we know the solutions to (B3), the most complete solution to (B1) can be found as a series $\Psi(\vec{r}) = \sum_{l=0}^\infty a_lR_i^s(r)G_i^s(\cos \gamma)$. Knowing that the asymptotic behavior of $J_{\pm \nu}(y)$ for $y \to \infty$ is
\begin{equation}
J_{\pm \nu}(y) \sim \sqrt{\frac{2}{\pi y}} \cos \left(y \pm \frac{\pi \nu}{2} - \frac{\pi}{4}\right), \tag{B5}
\end{equation}
one obtains that for $r \to \infty$
\begin{equation}
\Psi(\vec{r}) \sim \sum_{l=0}^\infty b_l \cos \left(kr - \frac{\pi}{2}l - \frac{\pi}{4}(d - 1)\right)G_i^s(\cos \gamma) \frac{1}{r^{(d-1)/2}}. \tag{B6}
\end{equation}
In the presence of a potential $V(r)$ modifying Eq. (B1) through its impact on $k^2$, one should expect phase shifts in the asymptotic form of $\Psi(\vec{r})$

$$\Psi^V(\vec{r}) = \sum_{l=0}^{\infty} c_l \cos \left( kr - \frac{\pi}{2} l - \frac{\pi}{4} (d-1) + \delta_l(s) \right) G^*_l(\cos \gamma) \frac{1}{r^{(d-1)/2}}. \quad (B7)$$

On the other hand, one can compose $\Psi^V(\vec{r})$ of the incoming wave $e^{ikr}$ and the scattered one

$$\Psi^V(\vec{r}) = e^{ikr} + \frac{f(\gamma)}{r^{d-1}}, \quad (B8)$$

where $f(\gamma)$ is the scattering amplitude and the incoming wave propagates along the $z$-axis. It is this axis with respect to which the angle $\gamma$ is measured. In an arbitrary number of dimensions, $e^{ikz}$ can be expanded as

$$e^{ikz} = 2^s \Gamma(s) \sum_{l=0}^{\infty} (s+l)! \frac{J_{l+s}(kr)}{(kr)^s} G^*_l(\cos \gamma) \quad (B9).$$

Using (B9), (B5), and (B8), the asymptotics of $\Psi^V(\vec{r})$ can be expressed in terms of $f(\gamma)$ as

$$\Psi^V(\vec{r}) = \frac{1}{r^{d-1}^{\frac{1}{2}}} \left[ f(\gamma) e^{ikr} + \frac{2^{d-3} \Gamma(s)}{\sqrt{\pi k^{d-1} / 2}} \sum_{l=0}^{\infty} (2l + d - 2)! \cos(kr - \frac{\pi}{2} l - \frac{\pi}{4} (d-1)) G^*_l(\cos \gamma) \right]. \quad (B10)$$

Since this is to be equal to (B7), one obtains two equations for the coefficients of $e^{ikr}$ and $e^{-ikr}$. It is from these equations that we arrive at

$$c_l = 2^{d-3} \frac{l! e^{i\delta_l(s)} (2l + d - 2) \Gamma(s)}{\sqrt{\pi k^{d-1} / 2}} \quad (B11)$$

to finally get

$$f(\gamma) = \frac{2^{d-3} \Gamma(\frac{d-2}{2}) e^{-i\frac{\pi}{4} (d-1)} (2l + s)!}{\sqrt{\pi k^{d-1}/2}} \sum_{l=0}^{\infty} (l + s) \left( e^{i\delta_l(s)} - 1 \right) G^*_l(\cos \gamma). \quad (B12)$$

One can easily apply (B12) to the case $d = 3$. Indeed, then $G^*_l = P_l$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which leads to

$$f(\gamma) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) \left( e^{i\delta_l} - 1 \right) P_l(\cos \gamma). \quad (B13)$$

References

[1] P. O. Mazur and J. Papavassiliou, Phys. Rev. D44, 1317 (1991).

[2] C. O. Lousto, Class. Quantum Grav. 9, 2417 (1992).

[3] H. Ren, Phys. Lett. B324, 149 (1994).

[4] M. Barriola and A. Vilenkin, Phys. Rev. Lett. 63, 341 (1989).

[5] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

[6] S. Deser and R. Jackiw, Comm. Math. Phys. 118, 495 (1988).