ASYMPTOTIC BEHAVIOUR OF RESONANCE EIGENVALUES OF THE SCHÖDINGER OPERATOR WITH A MATRIX POTENTIAL

SEDEF KARAKILIÇ, SETENAY AKDUMAN, AND DIDEM COŞKAN

ABSTRACT. We will discuss the asymptotic behaviour of the eigenvalues of a Schrödinger operator with a matrix potential defined by the Neumann boundary condition in $L^2_m(F)$, where $F$ is a $d$-dimensional rectangle and the potential is an $m \times m$ matrix with $m \geq 2$, $d \geq 2$, when the eigenvalues belong to the resonance domain, roughly speaking they lie near the planes of diffraction.

1. Introduction

In this paper, we consider the Schrödinger operator with a matrix potential $V(x)$ defined by the differential expression

$$L\phi = -\Delta \phi + V\phi$$

(1)

and the Neumann boundary condition

$$\frac{\partial \phi}{\partial n}|_{\partial F} = 0,$$

(2)

in $L^2_m(F)$ where $F$ is the $d$ dimensional rectangle $F = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_d]$, $\partial F$ is the boundary of $F$, $m \geq 2$, $d \geq 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary $\partial F$, $\Delta$ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, $x = (x_1, x_2, \ldots, x_d) \in R^d$, $V$ is a real valued symmetric matrix $V(x) = (v_{ij}(x))$, $i, j = 1, 2, \ldots, m$, $v_{ij}(x) \in L^2(F)$, that is, $V(x) = V(x)$.

We denote the operator defined by (1)-(2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of $L(V)$ by $\Lambda_N$ and $\Psi_N$, respectively.

The eigenvalues of the operator $L(0)$ which is defined by the differential expression (1) when $V(x) = 0$ and the boundary condition (2) are $|\gamma|^2$, and the
corresponding eigenspaces are \( E_\gamma = \text{span}\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\} \), where

\[
\begin{align*}
\gamma &= (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma + 0}{2}, \\
\frac{\Gamma + 0}{2} &= \left\{ \left( \frac{n_1 \pi}{a_1}, \frac{n_2 \pi}{a_2}, \ldots, \frac{n_d \pi}{a_d} \right) : n_k \in \mathbb{Z}^+ \cup \{0\}, k = 1, 2, \ldots, d \right\}, \\
\Phi_{\gamma,j}(x) &= (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0), j = 1, 2, \ldots, m,
\end{align*}
\]

and the non-zero component of \( \Phi_{\gamma,j}(x) \) is \( u_\gamma(x) = \cos \frac{n_1 \pi}{a_1} x_1 \cos \frac{n_2 \pi}{a_2} x_2 \cdots \cos \frac{n_d \pi}{a_d} x_d \), which stands in the \( j \)th component. In particular, \( u_0(x) = 1 \) when \( \gamma = (0, 0, \ldots, 0) \).

It can be easily calculated that the norm of \( u_\gamma(x) \), \( \gamma \in \frac{\Gamma + 0}{2} \), in \( L_2(F) \) is \( \sqrt{\frac{\mu(F)}{|A_\gamma|}} \), where \( \mu(F) \) is the measure of the \( d \)-dimensional parallelepiped \( F \), \( |A_\gamma| \) is the number of vectors in \( A_\gamma = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : |\alpha_k| = |\gamma^k|, k = 1, 2, \ldots, d \} \). \( \frac{\Gamma}{2} = \left\{ \left( \frac{n_1 \pi}{a_1}, \frac{n_2 \pi}{a_2}, \ldots, \frac{n_d \pi}{a_d} \right) : n_k \in \mathbb{Z}, k = 1, 2, \ldots, d \right\} \).

From now on, \( \langle , , \rangle \) and \( (, ,) \) will denote the inner products in \( L_2^d(F) \) and \( L_2(F) \), respectively.

Since \( \{u_\gamma(x)\}_{\gamma \in \frac{\Gamma + 0}{2}} \) is a complete system in \( L_2(F) \), for any \( q(x) \) in \( L_2(F) \) we have

\[
q(x) = \sum_{\gamma \in \frac{\Gamma + 0}{2}} \frac{|A_\gamma|}{\mu(F)} (q, u_\gamma) u_\gamma(x). \tag{3}
\]

In our study, it is convenient to use the equivalent decomposition (see [4])

\[
q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{4}
\]

where \( q_\gamma = \frac{1}{\mu(F)} (q(x), u_\gamma(x)) \) for the sake of simplicity. That is, the decomposition (3) and (4) are equivalent for any \( d \geq 2 \). Thus, according to (4), each matrix element \( v_{ij}(x) \in L_2(F) \) of the matrix \( V(x) \) can be written in its Fourier series expansion

\[
v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x), \tag{5}
\]

\[
v_{ij\gamma} = \frac{v_{ij}(u_\gamma, u_\gamma)}{\mu(F)}, \quad (v_{ij}, u_\gamma) = \frac{1}{\mu(F)} \int_F v_{ij}(x) u_\gamma(x) dx \quad \text{and} \quad v_{ij0} = \frac{1}{\mu(F)} \int_F v_{ij}(x) dx \quad i, j = 1, 2, \ldots, m.
\]

We assume that \( l > \frac{(d+20)(d-1)}{2} + d + 3 \) and the Fourier coefficients \( v_{ij\gamma} \) of \( v_{ij}(x) \) satisfy

\[
\sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}|^2 (1 + |\gamma|^2) < \infty, \tag{6}
\]

for each \( i, j = 1, 2, \ldots, m \). Let \( \rho \) be a large parameter, \( \rho \gg 1 \) and \( \alpha \) be a positive number with \( 0 < \alpha < \frac{1}{\pi^2d^2} \) then for \( \Gamma(\rho^\alpha) = \{ \gamma \in \frac{\Gamma}{2} : 0 \leq |\gamma| < \rho^\alpha \} \) and \( p = l - d \).
the condition (6) implies that
\[ v_{ij}(x) = \sum_{\gamma \in \Gamma(p^\alpha)} v_{ij\gamma}(x) + O(p^{-\alpha}). \tag{7} \]
Here \( O(p^{-\alpha}) \) is a function in \( L_2(F) \) with norm of order \( p^{-\alpha} \).

Furthermore, by (6), we have
\[ M_{ij} = \sum_{\gamma \in \frac{1}{p} \Gamma} |v_{ij\gamma}| < \infty, \tag{8} \]
for all \( i, j = 1, 2, \ldots, m \).

Notice that, if a function \( q(x) \) is sufficiently smooth \( q(x) \in W^2_1(F) \) and the support of \( \nabla q(x) = \left( \frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d} \right) \) is contained in the interior of the domain \( F \), then \( q(x) \) satisfies condition (6) (See [7]). There is also another class of functions \( q(x) \), such that \( q(x) \in W^2_1(F) \),
\[ q(x) = \sum_{\gamma \in \Gamma} q_{\gamma} u_{\gamma}(x), \]
which is periodic with respect to a lattice
\[ \Omega = \{(m_1a_1, m_2a_2, \ldots, m_da_d) : m_k \in \mathbb{Z}, k = 1, 2, \ldots, d \} \]
and thus it also satisfies condition (6).

As in [17]-[22], we divide \( R^d \) into two domains: Resonance and Non-resonance domains. In order to define these domains, let us introduce the following sets:

Let \( 0 < \alpha < \frac{1}{d+20} \), \( \alpha_k = 3^k \alpha, k = 1, 2, \ldots, d - 1 \) and
\[ V_b(\rho^{\alpha_1}) = \left\{ x \in R^d : |x|^2 - |x + b|^2 < \rho^{\alpha_1} \right\} \]
\[ E_1(\rho^{\alpha_1}, p) = \bigcup_{b \in \Gamma(p^{\alpha_1})} V_b(\rho^{\alpha_1}) \]
\[ U(\rho^{\alpha_1}, p) = R^d \setminus E_1(\rho^{\alpha_1}, p) \]
\[ E_k(\rho^{\alpha_k}, p) = \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p^{\alpha_k})} \left( \bigcap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k}) \right) \]
where \( b \neq 0, \gamma_i \neq 0, i = 1, 2, \ldots, k \) and the intersection \( \bigcap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k}) \) in \( E_k \) is taken over \( \gamma_1, \gamma_2, \ldots, \gamma_k \) which are linearly independent vectors and the length of \( \gamma_i \) is not greater than the length of the other vector in \( \Gamma \sqcap \gamma_i R \). The set \( U(\rho^{\alpha_1}, p) \) is said to be a non-resonance domain, and the eigenvalue \( |\gamma|^2 \) is called a non-resonance eigenvalue if \( \gamma \in U(\rho^{\alpha_1}, p) \). The domains \( V_b(\rho^{\alpha_1}) \), for \( b \in \Gamma(p^{\alpha_k}) \) are called resonance domains and the eigenvalue \( |\gamma|^2 \) is a resonance eigenvalue if \( \gamma \in V_b(\rho^{\alpha_k}) \).
As noted in [20]-[21], the domain $V_b(\rho^{\alpha_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is, 

\[
\frac{\mu((V_b(\rho^{\alpha_1}) \setminus E_2) \cap B(q))}{\mu(V_b(\rho^{\alpha_1}) \cap B(q))} \to 1, \text{ as } \rho \to \infty,
\]

where $B(\rho) = \{ x \in \mathbb{R}^d : |x| = \rho \}$, if 

\[
2\alpha_2 - \alpha_1 + (d + 3)\alpha < 1, \quad \alpha_2 > 2\alpha_1, \quad (9)
\]

hold. Since $0 < \alpha < \frac{1}{d + 20}$, the conditions in (9) hold.

In most cases, it is important to know the asymptotic behavior of the eigenvalues of the Schrödinger operator $L(V)$. In this paper, [3] and [8], we construct the asymptotic formulas in the high energy region for eigenvalues of the operator $L(V)$.

In [3], we obtain the asymptotic formulas of arbitrary order for the eigenvalue of $L(V)$ corresponding to the non-resonance eigenvalues $|\gamma|^2$ of $L(0)$ in arbitrary dimension $d \geq 2$.

In [8], we constructed the high energy asymptotics of arbitrary order for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when $\gamma$ belongs to the special single resonance domains $V_b(\rho^{\alpha_1}) \setminus E_2$, where $\delta$ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right), \ldots, e_d = \left(0, \ldots, \frac{\pi}{a_d}\right), d \geq 2$.

In this paper, we study the case for which $|\gamma|^2$ is a resonance eigenvalue. More precisely, in Theorem (1) and (2) of Section (2), we assume that $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}((\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d - 1$ and $\gamma \notin V_{\gamma_k}(\rho^{\alpha_k})$ for $k = 1, 2, \ldots, d$ and prove that the corresponding eigenvalue of $L(V)$ is close to the sum of the eigenvalue of the matrix $V_0$ and the eigenvalue of the matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ (See (14)).

In Section (3), this time we assume that $\gamma \in V_b(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{1}{\pi} \setminus \{e_1, e_2, \ldots, e_d\}$, that is, $\gamma$ is in a single resonance domain and we prove the main result Theorem (7) which gives a connection between the eigenvalues of $L(V)$ corresponding to a single resonance domain and the eigenvalues of the Sturm-Liouville operators.

Note that, the case $\delta = e_i, \quad i = 1, 2, \ldots, d$, was considered in [8], by a different but simpler method and better formulas were obtained.

2. Asymptotic Formulas for the Eigenvalues in the Resonance Domain

We assume that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1, 2, \ldots, d$, and $|\gamma|^2$ is a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d - 1$, such that $|\gamma| \sim \rho$ where $|\gamma| \sim \rho$ means that $|\gamma|$ and $\rho$ are asymptotically equal, that is, there exist $c_1, c_2$ satisfying the inequality $c_1\rho \leq |\gamma| \leq c_2\rho, \quad c_i, \quad i = 1, 2, 3, \ldots$
are positive real constants which do not depend on \(\rho\). To obtain the asymptotic formulas for the eigenvalues of \(L(V)\) corresponding to \(|\gamma|^2\) we use the binding formula (see (9) in [3])

\[
(\Lambda_N - |\gamma|^2)\langle \Psi_N, \Phi_{\gamma,j} \rangle = \langle \Psi_N, V\Phi_{\gamma,j} \rangle. 
\]  \hspace{1cm} (10)

Now, we decompose \(V(x)\Phi_{\gamma,j}(x)\) with respect to the basis \(\{\Phi_{\gamma,i}(x)\}_{\gamma \in \mathbb{Z}, i = 1, 2, \ldots, m}\). By definition of \(\Phi_{\gamma,j}(x)\), it is obvious that

\[
V(x)\Phi_{\gamma,j}(x) = (v_{1j}(x)u_{\gamma}(x), \ldots, v_{mj}(x)u_{\gamma}(x)). 
\]  \hspace{1cm} (11)

Substituting the decomposition (7) of \(v_{ij}(x)\) in (11), we get

\[
V(x)\Phi_{\gamma,j}(x) = \left( \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{1j,\gamma}u_{\gamma}(x)u_{\gamma}(x), \ldots, \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{mj,\gamma}u_{\gamma}(x)u_{\gamma}(x) \right) + O(\rho^{-\alpha}).
\]

Since \(\gamma\) does not belong to the domains \(V_{ek}(\rho^\alpha)\), for each \(k = 1, 2, \ldots, d\), we may use the following equation

\[
\sum_{\gamma \in \Gamma(\rho^\alpha)} v_{ij,\gamma}u_{\gamma}(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{ij,\gamma}u_{\gamma,\gamma}(x)
\]

which is proved in [9] (see equation (18) in [9]), and obtain

\[
V(x)\Phi_{\gamma,j}(x) = \left( \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{1j,\gamma}u_{\gamma}(x)u_{\gamma}(x), \ldots, \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{mj,\gamma}u_{\gamma}(x)u_{\gamma}(x) \right) + O(\rho^{-\alpha}).
\]  \hspace{1cm} (12)

Substituting (12) into (10), we obtain

\[
<\Psi_N, \Phi_{\gamma,j}> = \frac{<\Psi_N, V\Phi_{\gamma,j}>}{(\Lambda_N - |\gamma|^2)}
\]

\[
= \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^\alpha)} v_{ij,\gamma} <\Psi_N, \Phi_{\gamma,\gamma,i}> (\Lambda_N - |\gamma|^2) + O(\rho^{-\alpha})
\]  \hspace{1cm} (13)

for every vector \(\gamma \in \mathbb{Z}\), satisfying the condition

\[
|\Lambda_N - |\gamma|^2| > \frac{1}{2} \rho^{\alpha_1}.
\]

Letting \(p_1 = \left\lfloor \frac{p_0}{\rho} \right\rfloor\), that is, \(p_1\) is the integer part of \(\frac{p_0}{\rho}\), we define the following sets

\[
B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{b : b = \sum_{i=1}^{k} n_i \gamma_i, n_i \in \mathbb{Z}, |b| < \frac{1}{2} \rho^{\frac{1}{2} n_k + 1}\},
\]

\[
B_k(\gamma) = \gamma + B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{\gamma + b : b \in B_k(\gamma_1, \gamma_2, \ldots, \gamma_k)\},
\]

\[
B_k(\gamma, p_1) = B_k(\gamma) + \Gamma(p_1 \rho^\alpha).
\]
Let $h_\tau$, $\tau = 1, 2, \ldots, b_k$ denote the vectors of $B_k(\gamma, p_1)$, $b_k$ the number of the vectors in $B_k(\gamma, p_1)$. By its definition, it can easily be obtained that $b_k = O(\rho^{\frac{3\alpha}{2}})$, since $\alpha_k = 3^k \alpha$, $2 \leq k \leq d$. We define the $mb_k \times mb_k$ matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ by

$$C = \begin{bmatrix} |h_1|^2I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & |h_2|^2I - V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & \vdots & \ddots & \vdots \\ V_{h_{b_k} - h_1} & V_{h_{b_k} - h_2} & \cdots & |h_{b_k}|^2I - V_0 \end{bmatrix},$$

(14)

where $V_{h_\tau-h_\xi}$, $\tau, \xi = 1, 2, \ldots, b_k$ are the $m \times m$ matrices defined by

$$V_{h_\tau-h_\xi} = \begin{bmatrix} v_{11h_\tau-h_\xi} & v_{12h_\tau-h_\xi} & \cdots & v_{1mh_\tau-h_\xi} \\ v_{21h_\tau-h_\xi} & v_{22h_\tau-h_\xi} & \cdots & v_{2mh_\tau-h_\xi} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1h_\tau-h_\xi} & v_{m2h_\tau-h_\xi} & \cdots & v_{mmh_\tau-h_\xi} \end{bmatrix}.$$  

(15)

Writing equation (13) for all $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1, 2, \ldots, b_k$ and $j = 1, 2, \ldots, m$, we get

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{m,j} > = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^s)} v_{ijg_\tau} < \Psi_N, \Phi_{h_\tau, -g_\tau} > + O(\rho^{-\alpha_k}).$$  

(16)

Similar system of equations for quasi-periodic boundary condition was investigated in [19], [21] and [22]. More recently, in [22], Lemma 2.2.1. states that for $\gamma \in (\bigcap_{i=1}^{k} V_\gamma(\rho^s)) \setminus E_{k+1}$, $h_\tau \in B_k(\gamma, p_1)$ and $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma(\rho^s)$, if $h_\tau - \gamma \notin B_k(\gamma, p_1)$ then

$$||\gamma|^2 - |h_\tau - \gamma - \gamma_1 - \cdots - \gamma_s|^2| > \frac{1}{5} \rho^{\alpha_k+1},$$

(17)

for $s = 0, 1, 2, \ldots, p_1 - 1$.

Thus, if an eigenvalue $\Lambda_N$ of $L(V)$ satisfies

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2} \rho^{\alpha_k},$$

(18)

then by (17) and (18), we have

$$|\Lambda_N - |h_\tau - \gamma - \gamma_1 - \cdots - \gamma_s|^2| > \frac{1}{6} \rho^{\alpha_k+1}.$$  

(19)

Now, we prove that if (18) holds then

$$O(\rho^{-\alpha_k}) = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma(\rho^s)} v_{ijg_\tau} < \Psi_N, \Phi_{h_\tau, -g_\tau} >$$

(20)
for any \( j = 1, 2, \ldots, m \). Here we remark that \( \gamma \neq 0 \). If it were the case, then we would have from \( h_\tau - \gamma t \notin B_k(\gamma, p_1) \) that \( h_\tau \notin B_k(\gamma, p_1) \) which is a contradiction. So, to prove \( (20) \), we argue as Theorem 2.2.2 (a) of \cite{22}. Since \( \Lambda_N \) satisfies the inequality \( (18) \), by \( (19) \) (for \( s = 0 \)) we have \( | \Lambda_N - | h_\tau - \gamma t |^2 | > \frac{1}{6} \rho^{\alpha k+1} \). Using this, in the equation \( (13) \) instead of \( \gamma \) we write \( h_\tau - \gamma t \) to get

\[
< \Psi_N, \Phi_{h_\tau - \gamma t, j} > = \sum_{i=1}^{m} \sum_{i \in \Gamma(\rho^\alpha)} v_{i j_1} < \Psi_N, \Phi_{h_\tau - \gamma t - \gamma j_1, j_1} > \left( \frac{1}{\Lambda_N - | h_\tau - \gamma t |^2} \right) + O(\rho^{-\rho_\alpha}). \tag{21}
\]

Substituting this equation \((21)\) into the right hand side of \((20)\), we obtain

\[
\sum_{\gamma \in \Gamma(\rho^\alpha)} \frac{v_{ij \gamma t}}{h_\tau - \gamma \notin B_k(\gamma, p_1)} \frac{v_{ij \gamma t}}{\Lambda_N - | h_\tau - \gamma t |^2} \sum_{i_1 \in \Gamma(\rho^\alpha)} \sum_{i_1 \in \Gamma(\rho^\alpha)} v_{i_1 j_1} < \Psi_N, \Phi_{h_\tau - \gamma t - \gamma_1, j_1} > + O(\rho^{-\rho_\alpha}).
\]

In this manner, iterating \( p_1 \) times, we get

\[
\sum_{\gamma \in \Gamma(\rho^\alpha)} \frac{v_{ij \gamma t}}{h_\tau - \gamma \notin B_k(\gamma, p_1)} \frac{v_{ij \gamma t}}{\Lambda_N - | h_\tau - \gamma t |^2} \sum_{i_1, i_2, \ldots, i_{p_1} = 1}^{p_1} \gamma_i \gamma_2 \cdots \gamma_{p_1} \in \Gamma(\rho^\alpha) \Rightarrow v_{i j_1 \gamma t_1} = \Psi_N, \Phi_{h_\tau - \gamma t - \gamma_1 j_1} = \frac{1}{\Lambda_N - | h_\tau - \gamma t - \gamma_1 |^2} \left( \frac{1}{\Lambda_N - | h_\tau - \gamma t - \gamma_1 |^2} \right) \cdots \left( \frac{1}{\Lambda_N - | h_\tau - \gamma t - \gamma_1 |^2} \right) + O(\rho^{-\rho_\alpha}).
\]

Taking norm of both sides of the last equality, using \( (19) \), the relation \((8)\) and the fact that \( p_1 \alpha k+1 \geq p_2 \alpha_2 > p_\alpha \), we obtain

\[
| \sum_{\gamma \in \Gamma(\rho^\alpha)} \frac{v_{ij \gamma t}}{h_\tau - \gamma \notin B_k(\gamma, p_1)} | = O(\rho^{-\rho_\alpha}),
\]

which implies \((20)\). Therefore, the equation \((16)\) becomes

\[
(\Lambda_N - | h_\tau |^2) < \Psi_N, \Phi_{h_\tau, j} > = \sum_{i=1}^{m} \sum_{i \in \Gamma(\rho^\alpha)} v_{ij \gamma t} < \Psi_N, \Phi_{h_\tau - \gamma t, j} > + O(\rho^{-\rho_\alpha}). \tag{22}
\]

Since \( h_\tau - \gamma t \in B_k(\gamma, p_1) \), using the notation \( h_\xi = h_\tau - \gamma t \), the decomposition \( (22) \) can be written as

\[
(\Lambda_N - | h_\tau |^2) < \Psi_N, \Phi_{h_\tau, j} > = \sum_{i=1}^{m} \sum_{i \in \Gamma(\rho^\alpha)} v_{ij \gamma t - h_\xi} < \Psi_N, \Phi_{h_\xi, j} > + O(\rho^{-\rho_\alpha}). \tag{23}
\]
Isolating the terms where $h_{\tau} - h_{\xi} = 0$ in (23), we get
\[
(\Lambda_N - |h_{\tau}|^2) < \Psi_N, \Phi_{h_{\tau},j} > = \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{h_{\tau},i} >
+ \sum_{i=1}^{m} \sum_{h_{\tau} - h_{\xi} \in \Gamma(\rho^{\alpha})} v_{ijh_{\tau} - h_{\xi}} < \Psi_N, \Phi_{h_{\xi},i} >
+ O(\rho^{-p\alpha}).
\]

Writing the equation (24) for all $j = 1, 2, \ldots, m$ and for any $\tau = 1, 2, \ldots, b_k$, we get the system of equations
\[
[(\Lambda_N - |h_{\tau}|^2)I - V_0]A(N, h_{\tau}) = \sum_{\xi=1}^{b_k} V_{h_{\tau} - h_{\xi}} A(N, h_{\xi}) + O(\rho^{-p\alpha}),
\]
where $I$ is an $m \times m$ identity matrix, $V_{h_{\tau} - h_{\xi}}$ is given by (15),
\[
O(\rho^{-p\alpha}) = (O(\rho^{-p\alpha}), \ldots, O(\rho^{-p\alpha}))
\]
is an $m \times 1$ vector and $A(N, h_{\xi})$ is the $m \times 1$ vector
\[
A(N, h_{\xi}) = < \Psi_N, \Phi_{h_{\xi},1} >, < \Psi_N, \Phi_{h_{\xi},2} >, \ldots, < \Psi_N, \Phi_{h_{\xi},m} >
\]
for any $\xi = 1, 2, \ldots, b_k$. Letting $\lambda_{N,\tau} = \Lambda_N - |h_{\tau}|^2$, we have
\[
\begin{bmatrix}
\lambda_{N,1}I - V_0 & -V_{h_{1} - h_{2}} & \cdots & -V_{h_{b_k} - h_{1}} \\
-V_{h_{2} - h_{1}} & \lambda_{N,2}I - V_0 & \cdots & -V_{h_{b_k} - h_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
-V_{h_{b_k} - h_{1}} & -V_{h_{b_k} - h_{2}} & \cdots & \lambda_{N,b_k}I - V_0
\end{bmatrix}
\begin{bmatrix}
A(N, h_1) \\
A(N, h_2) \\
\vdots \\
A(N, h_{b_k})
\end{bmatrix}
= \begin{bmatrix}
O(\rho^{-p\alpha}) \\
O(\rho^{-p\alpha}) \\
\vdots \\
O(\rho^{-p\alpha})
\end{bmatrix}.
\]

We may write the system (27) as
\[
[\Lambda_N I - C]A(N, h_1, h_2, \ldots, h_{b_k}) = O(\rho^{-p\alpha}),
\]
where $I$ is an $mb_k \times mb_k$ identity matrix, $C$ is given by (14). $A(N, h_1, h_2, \ldots, h_{b_k})$ is the $mb_k \times 1$ vector
\[
A(N, h_1, h_2, \ldots, h_{b_k}) = (A(N, h_1), A(N, h_2), \ldots, A(N, h_{b_k}))
\]
and the right side of the system (28) is the $mb_k \times 1$ vector whose norm is
\[
|O(\rho^{-p\alpha})| = O(\sqrt{b_k}\rho^{-p\alpha}).
\]

**Theorem 1.** Let $| \gamma |^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d - 1$ where $| \gamma | \sim \rho$, and $\Lambda_N$ an eigenvalue
of the operator $L(V)$ for which (18) holds and its corresponding eigenfunction $\Psi_N$ satisfies

$$|\Phi_{\gamma,i}, \Psi_N>|>|c_4\rho^{-c_\alpha}.$$  \hfill (31)

Then there exists an eigenvalue $\eta_s(\gamma)$, $1 \leq s \leq mb_k$ of the matrix $C$ such that

$\Lambda_N = \eta_s(\gamma) + O(\rho^{-(p-c-\frac{d}{2})\alpha}).$

**Proof.** Since (18) is satisfied, (28) holds. Then multiplying both sides of the equation (28) by $[\Lambda_N - C]^{-1}$, then taking norm of both sides and by (30), we get

$$|A(N, h_1, h_2, \ldots, h_{b_k})| \leq ||[\Lambda_N - C]^{-1}||O(\sqrt{b_k}\rho^{-p\alpha}).$$  \hfill (32)

Using the fact that $\gamma$ is one of $h_1, h_2, \ldots, h_r$ (See definition of $B_k(\gamma, p_1)$) and hence by (31) and (32), we obtain

$$c_5\rho^{-c_\alpha} < |A(N, h_1, h_2, \ldots, h_{b_k})| \leq ||[\Lambda_N - C]^{-1}||\sqrt{b_k}c_6\rho^{-p\alpha}.$$  \hfill (33)

Since $[\Lambda_N - C]^{-1}$ is symmetric matrix with the eigenvalues $\frac{1}{\Lambda_N - \eta_s(\gamma)}$, $s = 1, \ldots, mb_k$, we have

$$\max_{s=1,\ldots,mb_k} |\Lambda_N - \eta_s(\gamma)|^{-1} = ||[\Lambda_N - C]^{-1}|| > c_7c_8^{-1}b_k^{-\frac{1}{2}}\rho^{-c_\alpha+p\alpha},$$

where $b_k = O(\rho^{\frac{d}{2}3^{2\alpha}})$, thus

$$\min_{s=1,2,\ldots,mb_k} |\Lambda_N - \eta_s(\gamma, \lambda_i)| \leq c_9\rho^{-(p-c-\frac{d}{2})\alpha},$$

and

$$\Lambda_N = \eta_s(\gamma, \lambda_i) + O(\rho^{-(p-c-\frac{d}{2})\alpha}).$$

**Theorem 2.** Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in \bigcap_{i=1}^{k} V_i, (\rho^{\alpha_i}) \setminus E_{k+1}$, $k = 1, 2, \ldots, d-1$ where $|\gamma| \sim \rho$, $\eta_s(\gamma)$ an eigenvalue of the matrix $C$ such that $|\eta_s(\gamma) - |\gamma|| < \frac{3}{8}\rho^{\alpha_i}$. Then there is an eigenvalue $\Lambda_N$ of the operator $L(V)$ satisfying

$$\Lambda_N = \eta_s(\gamma) + O(\rho^{-p\alpha+\frac{d}{2}3^{2\alpha}+\frac{d+1}{4}}).$$  \hfill (34)

**Proof.** By the general perturbation theory, there is an eigenvalue $\Lambda_N$ of the operator $L(V)$ such that $|\Lambda_N - |\gamma||^2 < \frac{1}{2}\rho^{2\alpha_1}$ holds. Thus one can use the system (28) and we prove the theorem for this eigenvalue $\Lambda_N$.

Let $\eta_s$, $s = 1, 2, \ldots, mb_k$ be an eigenvalue of the matrix $C$ and $\theta_s = (\theta_1^s, \theta_2^s, \ldots, \theta_{mb_k}^s)$ the corresponding normalized eigenvector, where $\theta_s^\tau = (\theta_s^{\tau_1}, \theta_s^{\tau_2}, \ldots, \theta_s^{\tau_{mb_k}})$, $\tau = 1, 2, \ldots, b_k$. Multiplying the equation (28) by $\theta_s$, since $C$ is symmetric (see (14) and (15)), we get

$$|\Lambda_N - \eta_s||A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| = |O(\rho^{-p\alpha}) \cdot \theta_s|.$$

$$\min_{s=1,2,\ldots,mb_k} |\Lambda_N - \eta_s||A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| \leq c_9\rho^{-(p-c-\frac{d}{2})\alpha},$$

and

$$\Lambda_N = \eta_s(\gamma, \lambda_i) + O(\rho^{-(p-c-\frac{d}{2})\alpha}).$$
By using $b_k = O(\rho^{d/4})$, (30) and the Cauchy Schwartz Inequality for the right hand side of (34), we have

$$|A_N - \eta_s||A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| = O(\rho^{-\alpha + \frac{d}{4}}).$$

(35)

So we need to prove that

$$|A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s| > c_{10} \rho^{-\frac{d-1}{4}},$$

(36)

from which the theorem follows.

For this purpose, we first consider the decomposition of the matrix $C$ as

$$C = A + B;$$

where

$$A = \begin{bmatrix} |h_1|^2 I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & |h_{b_k}|^2 I \end{bmatrix}, \quad B = \begin{bmatrix} V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & \ddots & \ddots & \vdots \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & V_0 \end{bmatrix}.$$ (37)

The eigenvalues and the corresponding eigenspaces of the matrix $A$ are $|h_{\tau}|^2$ and $E_\tau = \text{span}\{e_j : (\tau - 1)m + 1 \leq j \leq \tau m\}$, respectively, where

$$\{e_j = (0, \ldots, 0, 1, 0, \ldots, 0)\}_{j=1}^{m_{b_k}}$$

is the standard basis of $R^{m_{b_k}}$. Now, we use the following notation

$$\theta_s(h_{\tau,j}) = \theta_s \cdot e_j = \theta_{s,j}^\tau, \quad \text{if} \quad (\tau - 1)m + 1 \leq j \leq \tau m,$$

(38)

for $\tau = 1, 2, \cdots, b_k$.

Multiplying $(A + B)\theta_s = \eta_s \theta_s$ by $e_j$, since $A$ and $B$ are symmetric, we get

$$(\eta_s - |h_{\tau}|^2)\theta_s(h_{\tau,j}) = \theta_s \cdot B e_j$$

(39)

and $(\tau - 1)m + 1 \leq j \leq \tau m$, and $\tau = 1, 2, \cdots, b_k$.

On the other hand, if we consider the sum of the elements in the $i$-th row of the matrix $B$, by (8)

$$\sum_{\tau=1}^{b_k} \sum_{j=1}^{m} v_{ij}h_{\tau,h_{\tau}} < \sum_{j=1}^{m} M_{ij},$$

(40)

for all $i = 1, 2, \ldots, m$. Since $B$ is a symmetric matrix and by (40), the sum of elements in each row of $B$ is less then $M = \max_{i=1,2,\ldots,m} \{\sum_{j=1}^{m} M_{ij}\}$, the eigenvalues of $B$ are also less then $M$ from which we have $\|B\| \leq M$.

Thus, by (26), (36), (38), we have

$$|A(N, h_1, \ldots, h_{b_k}) \cdot \theta_s| = |\langle \psi_N, \sum_{\tau=1}^{b_k} \sum_{j=1}^{m} \theta_s(h_{\tau,j})\phi_{h_{\tau,j}} \rangle|,$$

(41)
which, together with Parseval’s relation, imply

\[ 1 = \left\| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \Phi_{h_{\tau,i}} \right\|^2 \]

\[
= \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
+ \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2.
\] (42)

Now we estimate the first summation in the expression (42):

\[
\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau=1}^{b_k} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
= \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
+ \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s - |h_{\tau}|^2| < \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 < 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
+ 2 \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s - |h_{\tau}|^2| < \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2. \] (43)

Using Bessel’s inequality, Parseval’s relation, orthogonality of the functions \( \Phi_{h_{\tau,i}}(x) \), \( \tau = 1, 2, \ldots, b_k \), \( i = 1, 2, \ldots, m \), the binding formula (39) and \( \| B \| \leq M \), we have

\[
\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \left| \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} > \right|^2 \\
\leq \left\| \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \Phi_{h_{\tau,i}} \right\|^2 \\
= \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} |\theta_s(h_{\tau,i})|^2 \| \Phi_{h_{\tau,i}} \|^2 \\
= \sum_{\tau:|\eta_s - |h_{\tau}|^2| \geq \frac{1}{2} \rho^{2\alpha_1}} \sum_{i=1}^{m} \frac{|\theta_s(h_{\tau,i}) B e_i|^2}{|\eta_s - |h_{\tau}|^2|^2} = O(\rho^{-2\alpha_1}). \] (44)
The assumption $|\eta_s - |\gamma|^2| < \frac{3}{8} \rho^{\alpha_1}$ of the theorem and $|\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}$ imply that $||\gamma|^2 - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}$. So by the well-known formula

$$\frac{1}{\lambda_N - |h_r|^2} = \frac{1}{\lambda_N - |\gamma|^2} \left\{ \sum_{n=0}^{k} \frac{|h_r|^2 - |\gamma|^2}{\lambda_N - |\gamma|^2} n + O(\rho^{-(k+1)\alpha_1}) \right\},$$

for $|\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}$, and $||\gamma|^2 - |h_r|^2| < \frac{1}{8} \rho^{2\alpha_1}$, using (39), we have

$$\sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_{\tau,i}} \rangle^2$$

$$= \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_{\tau,i}} \rangle}{|\lambda_N - |h_r|^2|}$$

$$\leq \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} (k + 1) \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_{\tau,i}} \rangle}{|\lambda_N - |\gamma|^2|}$$

$$+ \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} (k + 1) \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_{\tau,i}} \rangle}{|\lambda_N - |\gamma|^2|}$$

$$\vdots$$

$$+ \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} (k + 1) \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_{\tau,i}} \rangle}{|\lambda_N - |\gamma|^2|}$$

To calculate the order of each term in (44), we use Bessel's inequality and the orthogonality of $\Phi_{h_{\tau,i}}$. So we have

$$2 \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} (k + 1)$$

$$\times \sum_{\tau : |\eta_s - |h_r|^2| < \frac{1}{8} \rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_{\tau,i}} \rangle}{|\lambda_N - |\gamma|^2|}^2$$

$$= 2 \sum_{N : |\lambda_N - |\gamma|^2| \geq \frac{1}{8} \rho^{2\alpha_1}} (k + 1)$$

$$\frac{(k + 1)}{|\lambda_N - |\gamma|^2|^{2(r+1)}}$$
\[
\times \left| \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_\tau,i} > (|h_\tau|^2 - |\gamma|^2)^r \right|^2
\leq c_{11}(\rho^{2\alpha_1})^{-2(r+1)}(k + 1)
\times \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} \left| \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) (|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_\tau,i} > \right|^2
\leq c_{12}(\rho^{2\alpha_1})^{-2(r+1)}(k + 1)
\times \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) (|h_\tau|^2 - |\gamma|^2)^r V \Phi_{h_\tau,i} >^2
\leq c_{13}(\rho^{2\alpha_1})^{-2(r+1)}(k + 1)
\times \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) (|h_\tau|^2 - |\gamma|^2)^r \| V \Phi_{h_\tau,i} >^2
= c_{14}(\rho^{2\alpha_1})^{-2(r+1)}(k + 1)
\times \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) (|h_\tau|^2 - |\gamma|^2)^r \| V \Phi_{h_\tau,i} >^2
\leq c_{15}(\rho^{2\alpha_1})^{-2(r+1)}(\frac{1}{2} \rho^{\alpha_1})^{2r}(k + 1)
\times \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \| V \Phi_{h_\tau,i} >^2 = O(\rho^{-2(r+1)\alpha_1}),
\]

for \( r = 0, 1, 2, \ldots, k \). Now let \( K \) be the number of \( h_\tau \) satisfying \(|\eta_s - |h_\tau|^2| < \frac{1}{8} \rho^{\alpha_1} \), then the order of the last summation in (46) is:

\[
\sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} (k + 1)
\times \left| \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \sum_{i=1}^m \theta_s(h_{\tau,i}) < \Psi_N, V \Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1}) \right|^2
\leq K \sum_{N:|A_N - |\gamma|^2| \geq \frac{1}{2} \rho^{\alpha_1}} (k + 1)
\times \left| \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} O(\rho^{-(k+1)\alpha_1})^2 \cdot |\theta_s(h_{\tau,i})|^2 \cdot | < \Psi_N, V \Phi_{h_\tau,i} > \right|^2
\leq c_{16} \cdot K \cdot \rho^{-2(k+1)\alpha_1} \cdot \sum_{\tau:|\eta_\tau - |h_\tau|^2| < \frac{1}{2} \rho^{\alpha_1}} \| V(x) \Phi_{h_\tau,i} >^2
\leq c_{17} \cdot K^2 \cdot M^2 \cdot \rho^{-2(k+1)\alpha_1} = K^2 \cdot 0(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}).
since \( K = O(\rho^{2\alpha}) \) and we can always choose \( k \) in \( O(\rho^{-2(k+1)\alpha_1}) \) such that
\[
K^2 \cdot O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),
\]
which together with the estimations (44), (45) and (46) imply
\[
O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N - |\gamma||^2 \geq \frac{1}{2} \rho^{2\alpha_1}} b_k \sum_{k=1}^m \theta_s(h_{\tau,i}) \langle \Psi_N, \Phi_{h_{\tau,i}} \rangle \leq 2, \]
Therefore, from the decomposition (42) we have
\[
1 - O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N - |\gamma||^2 < \frac{1}{2} \rho^{2\alpha_1}} b_k \sum_{k=1}^m \theta_s(h_{\tau,i}) \langle \Psi_N, \Phi_{h_{\tau,i}} \rangle \leq 2.
\]
Since the number of indexes \( N \) satisfying \( |\Lambda_N - |\gamma||^2 < \frac{1}{2} \rho^{2\alpha_1} \) is less than \( \rho^{d-1} \), we have
\[
1 - O(\rho^{-2\alpha_1}) \leq \rho^{d-1} \max_{N:|\Lambda_N - |\gamma||^2 < \frac{1}{2} \rho^{2\alpha_1}} \left\{ |\sum_{k=1}^m \theta_s(h_{\tau,i}) \langle \Psi_N, \Phi_{h_{\tau,i}} \rangle \leq 2 \right\}
\]
which implies together with the relation (41) that
\[
|A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s|^2 \geq \frac{1 - O(\rho^{-2\alpha_1})}{\rho^{d-1}}.
\]
It follows from the equation (35) and the estimation (48) that
\[
\Lambda_N = \eta_s + \frac{O(\rho^{-2\alpha_1}) - \frac{1}{2} \rho^{2\alpha_1}}{O(\rho^{-2\alpha_1})},
\]
that is, (36) holds.

3. Asymptotic Formulas for the Eigenvalues in a Single Resonance Domain

Now, we investigate in detail the eigenvalues of \( L(V) \) in a single resonance domain. In order the inequalities
\[
0 < \alpha < \frac{1}{d+20}, \quad 2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \quad (49)
\]
and
\[
\alpha_2 > 2\alpha_1, \quad (50)
\]
to be satisfied, we can choose \( \alpha, \alpha_1 \) and \( \alpha_2 \) as follows
\[
\alpha = \frac{1}{d+p}, \quad \alpha_1 = \frac{p_1}{d+p}, \quad \alpha_2 = \frac{2p_2 + 1}{d+p},
\]
where \( p_2 = \left[ \frac{p_1^2}{3} \right] - 1 \). Let \( \gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2, \ \delta \in \frac{p_1}{2} \setminus \{ e_i \} \), where \( \delta \) is minimal in its direction. Consider the following sets:

\[
B_1(\delta) = \{ b : b = n\delta, n \in \mathbb{Z}, |b| < \frac{1}{2} \rho^{\frac{1}{2} \alpha_2} \}, \\
B_1(\gamma) = \gamma + B_1(\delta) = \{ \gamma + b : b \in B_1(\delta) \}, \\
B_1(\gamma, p_1) = B_1(\gamma) + \Gamma(p_1 \rho^\alpha).
\]

As before, denote by \( h_\tau, \tau = 1, 2, ..., b_1 \) the vectors of \( B_1(\gamma, p_1) \), where \( b_1 \) is the number of vectors in \( B_1(\gamma, p_1) \). Then the matrix \( C(\gamma, \delta) = (c_{ij}), i, j = 1, 2, ..., mb_1 \) is defined by

\[
C(\gamma, \delta) = \begin{bmatrix}
|h|_1^2I - V_0 & V_{h_2 - h_1} & \cdots & V_{h_2 - h_1} \\
& \vdots & \ddots & \vdots \\
& & \ddots & |h|_{b_1}^2I - V_0 \\
V_{h_2 - h_1} & V_{h_2 - h_1} & \cdots & |h|_{b_1}^2I - V_0
\end{bmatrix},
\]

(51)

where \( V_{h_2 - h_1}, \tau, \xi = 1, 2, ..., b_1 \) are the \( m \times m \) matrices defined by (15).

Also we define the matrix \( D(\gamma, \delta) = (d_{ij}) \) for \( i, j = 1, 2, ..., ma_1 \), where \( h_1, h_2, ..., h_{a_1} \) are the vectors of \( B_1(\gamma, p_1) \setminus \{ \gamma + n\delta : n \in \mathbb{Z} \} \), and \( a_1 \) is the number of vectors in \( B_1(\gamma, p_1) \setminus \{ \gamma + n\delta : n \in \mathbb{Z} \} \). Clearly \( a_1 = O(\rho^{\frac{1}{2} \alpha_2}) \).

**Lemma 3.** a) If \( \eta_{j_s} \) is an eigenvalue of the matrix \( C(\gamma, \delta) \) such that \(|\eta_{j_s} - |h_s|^2| < M \) for \( s = 1, 2, ..., a_1, 1 + (s - 1)m \leq j_s \leq ms \), then

\[
|\eta_{j_s} - |h_r|^2| > \frac{1}{4} \rho^{\alpha_2}, \ \forall \tau = a_1 + 1, a_1 + 2, ..., b_1.
\]

b) If \( \eta_{j_s} \) is an eigenvalue of the matrix \( C(\gamma, \delta) \) such that \(|\eta_{j_s} - |h_s|^2| < M \) for \( s = a_1 + 1, a_1 + 2, ..., b_1 \) and \( 1 + (s - 1)m \leq j_s \leq ms \), then

\[
|\eta_{j_s} - |h_r|^2| > \frac{1}{4} \rho^{\alpha_2}, \ \forall \tau = 1, 2, ..., a_1.
\]

**Proof.** First we prove

\[
||h_s|^2 - |h_r|^2| \geq \frac{1}{3} \rho^{\alpha_2}, \ \forall s \leq a_1, \ \forall \tau > a_1.
\]

(52)

By definition, if \( s \leq a_1 \) then \( h_\tau = \gamma + n\delta \), where \(|n\delta| < \frac{1}{2} \rho^{\frac{1}{2} \alpha_2} + p_1 \rho^\alpha \). If \( \tau > a_1 \) then \( h_\tau = \gamma + s'\delta + a \), where \(|s'\delta| < \frac{1}{7} \rho^{\frac{1}{2} \alpha_2}, a \in \Gamma(p_1 \rho^\alpha) \setminus \delta R \). Therefore

\[
|h_\tau|^2 - |h_s|^2 = 2\gamma \cdot a + 2s' \delta \cdot a + 2s' \gamma \cdot \delta + |s'\delta|^2 + |a|^2 - 2n\gamma \cdot \delta - |n\delta|^2.
\]

Since \( \gamma \notin V_\delta(\rho^{\alpha_1}) \), \(|a| < p_1 \rho^\alpha \), we have

\[
2\gamma \cdot a > \rho^{\alpha_2} - \epsilon_0 \rho^{2\alpha_2}.
\]

The relation \( \gamma \in V_\delta(\rho^{\alpha_1}) \) and the inequalities for \( s' \) and \( n \) imply that

\[
2s' \gamma \cdot \delta + 2s' \gamma \cdot a + |a|^2 - 2n\gamma \cdot \delta = O(\rho^{\frac{1}{2} \alpha_2 + \alpha_1}),
\]
Thus (52) follows from these relations, since $\frac{1}{2}\alpha_2 + \alpha_1 < \alpha_2$ and $\frac{1}{2}\alpha_2 + \alpha < \alpha_2$.

The eigenvalues of $D(\gamma, \delta)$ and $C(\gamma, \delta)$ lay in $M$-neighborhood of the numbers $|h_k|^2$ for $k = 1, 2, ..., a_1$ and for $k = 1, 2, ..., b_1$, respectively. The inequality (52) shows that one can enumerate the eigenvalues $\eta_j$ ($j = 1, 2, ..., mb_1$) of $C$ in the following way:

$$\eta_j \equiv \eta_{j_s}, \quad j_s \leq ma_1, \quad 1 + (s - 1)m \leq j_s \leq sm$$

when for $s \leq a_1$, $\eta_j$ lay in $M$-neighborhood of $|h_s|^2$ and

$$\eta_j \equiv \eta_{j_t}, \quad j_t \geq ma_1, \quad 1 + (\tau - 1)m \leq j_t \leq \tau m$$

when for $\tau > a_1$, $\eta_j$ lay in $M$-neighborhood $|h_{\tau}|^2$. Then by (52), we get

$$|\eta_{j_s} - |h_{\tau}|^2| > \frac{1}{4} \rho^\alpha,$$  \hfill (53)

for $s \leq a_1, \tau > a_1$ and $s > a_1, \tau \leq a_1$. \hfill \Box

Now, using the notation $h_s = \gamma - (\frac{s}{2})\delta$ if $s$ is even, $h_s = \gamma + (\frac{s-1}{2})\delta$ if $s$ is odd, for $s = 1, 2, ..., a_1$, (without loss of generality assume that $a_1$ is even) and using the orthogonal decomposition of $\gamma \in \mathbb{R}^2$, $\gamma = \beta + (1 + \nu(\beta))\delta$, where $\beta \in H_\delta \equiv \{x \in \mathbb{R}^d : x \cdot \delta = 0\}$, $l \in \mathbb{Z}$, $\nu \in [0, 1)$ we can write the matrix $D(\gamma, \delta)$ as

$$D(\gamma, \delta) = |\beta|^2 I + E(\gamma, \delta),$$  \hfill (54)

where $I$ is a maximal identity matrix and $E(\gamma, \delta)$ is

$$E(\gamma, \delta) = \begin{bmatrix}
(v_{+\delta})^2 I + v_0 & v_{+\delta} & v_{-\delta} & \cdots & v_{-\delta} \\
(v_{-\delta})^2 I + v_0 & v_{-\delta} & v_{+\delta} & \cdots & v_{+\delta} \\
v_{+\delta} & v_{+\delta} & (v_{+\delta})^2 I + v_0 & \cdots & \vdots \\
v_{-\delta} & v_{-\delta} & \cdots & (v_{-\delta})^2 I + v_0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
v_{-\delta} & \vdots & \cdots & \vdots & (v_{-\delta})^2 I + v_0 & \cdots & (v_{+\delta})^2 I + v_0
\end{bmatrix}.$$  

Denote $n_k = \frac{k}{2}$ if $k$ is even, $n_k = \frac{k-1}{2}$ if $k$ is odd. The system

$$\{e^{i(n_k+\nu)t} : k = 1, 2, ...\}$$

is a basis in $L^2_\nu[0, 2\pi]$. Let $T(\gamma, \delta) \equiv T(P(t), \beta)$ be the operator in $\ell_2$ corresponding to the Sturm-Liouville operator $T$, generated by

$$-|\delta|^2 Y''(t) + P(t)Y(t) = \mu Y(t),$$  \hfill (55)

$$Y(t + 2\pi) = e^{i2\pi \nu(\beta)}Y(t),$$

where $P(t) = (p_{ij}(t)), p_{ij}(t) = \sum_{k=1}^{\infty} v_{iju_k}\delta e^{in_k t}, v_{iju_k}\delta = (v_{ij}(x), \frac{1}{|A_{nu_k}|} \sum_{A_{nu_k} \in A_{nu_k}} e^{i(\alpha, x)}),$ $t = x \cdot \delta$. It means that $T(\gamma, \delta)$ is the infinite matrix $(Te^{i(l+nu+\nu)t}, e^{i(l+nu+\nu)t})$, $k, m = 1, 2, ...$.  


To find the relation between the eigenvalues of \( L(V) \) in a single resonance domain and the eigenvalues of the Sturm-Liouville operators defined by (55), we need the following theorems.

**Theorem 4.** Let \( \gamma \in V_\delta((\rho^{a_1}) \setminus E_2 \) and \( |\gamma| \sim \rho \). Then, for any eigenvalue \( \eta_{j_s} (\gamma) \) of the matrix \( C(\gamma, \delta) \) satisfying

\[
|\eta_{j_s} - |h_s|^2| < M, \quad \gamma \in \gamma \in V_\delta((\rho^{a_1}) \setminus E_2)
\]

there exists an eigenvalue \( \bar{\eta}_{k(j_s)} \) of the matrix \( D(\gamma, \delta) \) such that

\[
\eta_{j_s} = \bar{\eta}_{k(j_s)} + O(\rho^{-\frac{1}{2}a_2}).
\]

**Proof.** Let \( \eta_{j_s} \) be an eigenvalue of the matrix \( C(\gamma, \delta) \) satisfying (56) and \( \theta_{j_s} = (\theta^1_{j_s}, \theta^2_{j_s}, ..., \theta^m_{j_s}) \) be the corresponding normalized eigenvector, \( |\theta_{j_s}| = 1 \). Now, we consider the decomposition \( C = A + B \) and the matrices \( A, B \) which are defined in (37). Writing the binding formula (39) for \( \eta_{j_s} \) and using (38), we get

\[
(\eta_{j_s} - |h_s|^2)\theta_{j_s}(h_{\tau,i}) = \theta_{j_s} \cdot Be_i,
\]

\( \tau = 1, 2, ..., b_1, \quad 1 + (\tau - 1)m \leq i \leq \tau m. \)

For simplicity, we use the following notation in the sequel:

\[
e_{\xi,k} = e_k \quad \text{if} \quad 1 + (\xi - 1)m \leq k \leq \xi m, \quad \xi = 1, \ldots, b_1,
\]

\[
Be_i \cdot e_{k_1} = Be_{\tau,i} \cdot e_{\xi,k_1} = b(\tau, i, \xi, k_1).
\]

Thus, substituting the orthogonal decomposition

\[
Be_i = Be_{\tau,i} = \sum_{\xi = 1, 2, ..., b_1} b(\tau, i, \xi, k_1) e_{\xi,k_1}
\]

into the formula (57), we get

\[
(\eta_{j_s} - |h_s|^2)\theta_{j_s}(h_{\tau,i}) = \theta_{j_s} \cdot \sum_{\xi = 1, 2, ..., b_1} b(\tau, i, \xi, k_1) e_{\xi,k_1}
\]

\[
= \sum_{\xi = 1, 2, ..., b_1} b(\tau, i, \xi, k_1) \theta_{j_s} \cdot e_{\xi,k_1}
\]

\[
= \sum_{\xi = 1, 2, ..., b_1} b(\tau, i, \xi, k_1) \theta_{j_s} \theta_{j_s}(h_{\xi,k_1}).
\]

It is clear that

\[
b(\tau, i, \xi, k_1) = \begin{cases} 
0 & \text{if} \quad \xi = \tau, \\
\frac{1}{v_{k_1}i\xi - h_\tau} & \text{if} \quad \xi \neq \tau,
\end{cases}
\]

which implies

\[
\sum_{\xi = 1, 2, ..., b_1} b(\tau, i, \xi, k_1) = \sum_{\xi = 1, 2, ..., b_1} \frac{1}{v_{k_1}i\xi - h_\tau}.
\]
Thus one has

\[
(\eta_{js} - |h_\tau|^2)\theta_{js}(h_\tau, i) = \sum_{\xi=1,\ldots,b_1} \sum_{\nu \in k_i h_\xi - h_\tau} \theta_{js}(h_\xi, k_1)
\]

\[
= \sum_{\xi=1,\ldots,a_1} \sum_{\nu \in k_i h_\xi - h_\tau} \theta_{js}(h_\xi, k_1)
\]

\[
+ \sum_{\xi=a_1+1,\ldots,b_1} \sum_{\nu \in k_i h_\xi - h_\tau} \theta_{js}(h_\xi, k_1).
\]

(58)

Now, writing the equation (58) for all \(h_\tau, \tau = 1, 2, \ldots, a_1\), we get the system of linear algebraic equations:

\[
(\eta_{js} - |h_1|^2)\theta_{js}(h_1, i) - \sum_{\xi=1,\ldots,a_1} \sum_{\nu \in k_i h_\xi - h_1} \theta_{js}(h_\xi, k_1)
\]

\[
= \sum_{\xi=a_1+1,\ldots,b_1} \sum_{\nu \in k_i h_\xi - h_1} \theta_{js}(h_\xi, k_1)
\]

\[
(\eta_{js} - |h_2|^2)\theta_{js}(h_2, i) - \sum_{\xi=1,\ldots,a_1} \sum_{\nu \in k_i h_\xi - h_2} \theta_{js}(h_\xi, k_1)
\]

\[
= \sum_{\xi=a_1+1,\ldots,b_1} \sum_{\nu \in k_i h_\xi - h_2} \theta_{js}(h_\xi, k_1)
\]

\[
\vdots
\]

\[
(\eta_{js} - |h_{a_1}|^2)\theta_{js}(h_{a_1}, i) - \sum_{\xi=1,\ldots,a_1} \sum_{\nu \in k_i h_\xi - h_{a_1}} \theta_{js}(h_\xi, k_1)
\]

\[
= \sum_{\xi=a_1+1,\ldots,b_1} \sum_{\nu \in k_i h_\xi - h_{a_1}} \theta_{js}(h_\xi, k_1)
\]

(59)

Using the binding formula (57), the relation (53), and \(\|B\| \leq M\), for any \(\tau = 1, 2, \ldots, a_1\), we find

\[
| \sum_{\xi=a_1+1,\ldots,b_1} v_{k_1 h_\xi - h_\tau} \theta_{js}(h_\xi, k_1) | = | \sum_{\xi=a_1+1,\ldots,b_1} v_{k_1 h_\xi - h_\tau} \theta_{js} \cdot B \xi \cdot k_1 |
\]

\[
\leq \sum_{\xi=a_1+1,\ldots,b_1} | v_{k_1 h_\xi - h_\tau} | | \theta_{js} | \|B\| | e_{\xi,k_1} |
\]

\[
\leq 4\rho^{-\alpha_2} M \sum_{\xi=a_1+1,\ldots,b_1} | v_{k_1 h_\xi - h_\tau} |
\]

\[
\leq 4\rho^{-\alpha_2} M^2
\]

(60)
and
\[
\sum_{\tau=a_1+1,\ldots,b_1 \atop i=1,2,\ldots,m} |\theta_{j_s}(h_\tau,i)|^2 = \sum_{\tau=a_1+1,\ldots,b_1 \atop i=1,2,\ldots,m} \frac{|\theta_{j_s} \cdot B e_\tau,i|^2}{(\eta_{j_s} - |h_\tau|^2)^2}
\]
\[
= \sum_{\tau=a_1+1,\ldots,b_1 \atop i=1,2,\ldots,m} |B \theta_{j_s} \cdot e_\tau,i|^2 \frac{1}{(\eta_{j_s} - |h_\tau|^2)^2}
\]
\[
\leq 16M^2 \rho^{-2\alpha_2} = O(\rho^{-2\alpha_2}). \tag{61}
\]

By \[\ref{60}\] and \[\ref{54}, \ref{59}\] becomes
\[
[\theta^1_{j_s}, \theta^2_{j_s}, \ldots, \theta^{a_1}_{j_s}]^t = (D(\gamma, \delta) - \eta_{j_s} I)^{-1} [O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})]^t. \tag{62}
\]

By the Parseval’s identity and \[\ref{61}\], we get
\[
\sum_{\tau=1,2,\ldots,a_1 \atop i=1,2,\ldots,m} |\theta_{j_s}(h_\tau,i)|^2 = \sum_{\tau=1,2,\ldots,b_1 \atop i=1,2,\ldots,m} |\theta_{j_s}(h_\tau,i)|^2 - \sum_{\tau=a_1+1,\ldots,b_1 \atop i=1,2,\ldots,m} |\theta_{j_s}(h_\tau,i)|^2
\]
\[
\geq 1 - O(\rho^{-2\alpha_2}).
\]

Now, taking norm of both sides in \[\ref{62}\] and using the above inequality we have
\[
\sqrt{1 - O(\rho^{-2\alpha_2})} < (\sum_{\tau=1,2,\ldots,a_1 \atop i=1,2,\ldots,m} |\theta_{j_s}(h_\tau,i)|^2)^{\frac{1}{2}} \leq \|(D(\gamma, \delta) - \eta_{j_s} I)^{-1} O(\sqrt{a_1} \rho^{-\alpha_2}).
\]

Thus
\[
max|\eta_{j_s} - \tilde{\eta}_{k(j_s)}|^{-1} > \frac{\sqrt{1 - O(\rho^{-2\alpha_2})}}{\sqrt{a_1} \rho^{-\alpha_2}},
\]
or
\[
min|\eta_{j_s} - \tilde{\eta}_{k(j_s)}| = O(\sqrt{a_1} \rho^{-\alpha_2}) = O(\rho^{-\frac{3}{4} \alpha_2}),
\]
where the maximum (minimum) is taken over all \(\tilde{\eta}_{k(j_s)} \), \(s = 1,2,\ldots,a_1\). So the result follows.

**Theorem 5.** For any eigenvalue \(\tilde{\eta}_\tau\) of the matrix \(D(\gamma, \delta)\), there exists an eigenvalue \(\eta_{j_s(\tau)}\) of the matrix \(C(\gamma, \delta)\) such that
\[
\eta_{j_s(\tau)} = \tilde{\eta}_\tau + O(\rho^{-\frac{3}{4} \alpha_2}).
\]
Proof. Define the matrix \( D' = D'(\gamma, \delta) \) by

\[
D' = \begin{pmatrix}
|h_{11}|^2 - V_0 & v_{h_{12}} - h_{21} & \cdots & v_{h_{1a_1}} - h_{a_11} & 0 & 0 & \cdots & 0 \\
v_{h_{21}} - h_{12} & |h_{22}|^2 I - V_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
v_{h_{a_11}} - h_{1a_1} & v_{h_{a_12}} - h_{a_12} & \cdots & |h_{a_11}|^2 I - V_0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & |h_{a_12}|^2 I - V_0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & |h_{a_11}|^2 I - V_0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & |h_{b_11}|^2 I \\
\end{pmatrix}
\] (63)

So that the spectrum of the matrix \( D' \) is

\[
\text{spec}(D') = \text{spec}(D(\gamma, \delta)) \bigcup \{ |h_{a_1+1}|^2, |h_{a_1+2}|^2, \ldots, |h_{b_1}|^2 \}
\]

\[
= \{ \bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_{ma_1}, |h_{a_1+1}|^2, |h_{a_1+2}|^2, \ldots, |h_{b_1}|^2 \}
\]

Let us denote by \( \Upsilon_\tau = (\Upsilon_1^\tau, \Upsilon_2^\tau, \ldots, \Upsilon_{a_1}^\tau, 0, \ldots, 0)_{mb_1 \times 1}, \) \( \Upsilon_i^\tau = (\Upsilon_{i1}^\tau, \Upsilon_{i2}^\tau, \ldots, \Upsilon_{im}^\tau)_{m \times 1} \) the normalized eigenvector corresponding to the \( \tau \)-th eigenvalue of the matrix \( D' \), for \( \tau = 1, 2, \ldots, ma_1 \) and by \( \{ e_{k,i} \}_{i=1,2,\ldots,m} \) the eigenvector corresponding to the \( k \)-th eigenvalue \( |h_k|^2 \) of \( D' \), for \( k = a_1 + 1, a_1 + 2, \ldots, b_1 \).

Now, using (62) from the previous theorem, we have

\[
(D' - \eta_{j_\tau})[\theta_{j_\tau}^1, \theta_{j_\tau}^2, \ldots, \theta_{j_\tau}^{b_1}]^t = (D(\gamma, \delta) - \eta_{j_\tau})[\theta_{j_\tau}^1, \theta_{j_\tau}^2, \ldots, \theta_{j_\tau}^{a_1}]^t, (|h_{a_1+1}|^2 - \eta_{j_\tau})\theta_{j_\tau}^{a_1+1}, \ldots, (|h_{b_1}|^2 - \eta_{j_\tau})\theta_{j_\tau}^{b_1}]
\]

Taking inner product of both sides of the last equality by \( \Upsilon_\tau \) for \( \tau = 1, 2, \ldots, ma_1 \), using that \( D' \) is symmetric and \( D' \Upsilon_\tau = \bar{\eta}_\tau \Upsilon_\tau \), we have

\[
(\eta_{j_\tau(\tau)} - \bar{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_\tau}^k \cdot \Upsilon^k_\tau = \sum_{k=1}^{a_1} O(\rho^{-\alpha_2}) \Upsilon^k_\tau,
\] (64)

For the right hand side of the equation (64) using the Cauchy-Schwarz inequality, we get

\[
| \sum_{k=1}^{a_1} O(\rho^{-\alpha_2}) \Upsilon^k_\tau | \leq \sqrt{\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})^2} \sqrt{\sum_{k=1}^{a_1} |\Upsilon^k_\tau|^2} \leq \sqrt{a_1 (\rho^{-\alpha_2})^2} = O(\sqrt{a_1 \rho^{-\alpha_2}}),
\]

where \( a_1 = O(\rho^{\frac{3}{4} \alpha_2}) \). Thus, the equation (64) can be written as

\[
(\eta_{j_\tau(\tau)} - \bar{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_\tau}^k \cdot \Upsilon^k_\tau = O(\rho^{-\frac{3}{4} \alpha_2}).
\] (65)
In order to get the result, we need to show that for any \( \tau = 1, 2, \ldots, ma_1 \) there exists \( \theta_{j_\tau} \) such that

\[
\left| \sum_{k=1}^{a_1^k} \theta_{j_\tau}^{k} \cdot \mathcal{Y}_\tau^{k} \right| = \left| \theta_{j_\tau} \cdot \mathcal{Y}_\tau \right| > \sqrt{\frac{1 - O(\rho^{-2\alpha_2})}{ma_1}} > c_1 \alpha_1^2 \alpha_2. \tag{66}
\]

For this, we consider the orthogonal decomposition \( \mathcal{Y}_\tau = \sum_{s=1}^{mb_1} (\mathcal{Y}_\tau \cdot \theta_{j_s}) \theta_{j_s} \) and the Parseval’s identity

\[
1 = \sum_{s=1}^{mb_1} |\mathcal{Y}_\tau \cdot \theta_{j_s}|^2 = \sum_{s=1}^{ma_1} |\mathcal{Y}_\tau \cdot \theta_{j_s}|^2 + \sum_{s=ma_1+1}^{mb_1} |\mathcal{Y}_\tau \cdot \theta_{j_s}|^2.
\]

First, let us show that

\[
\sum_{s=ma_1+1}^{mb_1} |\mathcal{Y}_\tau \cdot \theta_{j_s}|^2 = O(\rho^{-2\alpha_2}). \tag{67}
\]

Using the decomposition \( \mathcal{Y}_\tau = \sum_{k=1,2,\ldots,a_1}^{k} \mathcal{Y}_\tau \cdot e_{k,i} \cdot e_{k,i} \), the binding formula \((57)\) for \( C(\gamma, \delta) \) and \( A \), the relation \((53)\), and the Bessel’s inequality we obtain the estimation

\[
\sum_{s=ma_1+1}^{mb_1} |\mathcal{Y}_\tau \cdot \theta_{j_s}|^2 \\
= \sum_{s=ma_1+1}^{mb_1} \left| \left( \sum_{k=1,2,\ldots,a_1}^{k} \mathcal{Y}_\tau^{k} e_{k,i} \right) \cdot \theta_{j_s} \right|^2 \\
= \sum_{s=ma_1+1}^{mb_1} \left| \sum_{k=1,2,\ldots,a_1}^{k} \mathcal{Y}_\tau^{k} (e_{k,i} \cdot \theta_{j_s}) \right|^2 \\
\leq 16 \sum_{s=ma_1+1}^{mb_1} \rho^{-2\alpha_2} \left( \sum_{k=1,2,\ldots,a_1}^{k} |\mathcal{Y}_\tau^{k}| \left| \theta_{j_s} \cdot B_{k,i} \right| \right)^2 \\
\leq \sum_{s=ma_1+1}^{mb_1} 16 |a_1| m \rho^{-2\alpha_2} \left( \sum_{k=1,2,\ldots,a_1}^{k} |\mathcal{Y}_\tau^{k}|^2 \left| \theta_{j_s} \cdot B_{k,i} \right|^2 \right) \\
\leq 16 \rho^{-2\alpha_2} |a_1| m \sum_{k=1,2,\ldots,a_1}^{k} |\mathcal{Y}_\tau^{k}|^2 \sum_{s=ma_1+1}^{mb_1} \left| \theta_{j_s} B_{k,i} \right|^2
ASYMPTOTIC BEHAVIOUR OF RESONANCE EIGENVALUES 507

\[ 16 \rho^{-2\alpha} |a_1 |m \sum_{k=1, \ldots, \alpha_1 \atop i=1, \ldots, m} |Y^k_i|^2 |B e_{k,i}|^2 \leq 16 \rho^{-2\alpha} |a_1 |MM^2 \sum_{k=1,2, \ldots, \alpha_1 \atop i=1,2, \ldots, m} |Y^k_i|^2 \]

\[ \leq 16 |a_1 |M \rho^{-2\alpha} M^2 = O(\rho^{-\frac{3}{2} \alpha}). \]

Therefore one has

\[ \sum_{s=1}^{\max} |Y_\tau \cdot \theta_j|^2 = 1 - O(\rho^{-\frac{3}{2} \alpha}) \]

from which it follows that there exists an eigenvector \( \theta_j(\tau) \) such that \( (66) \) holds. Dividing both sides of \( (65) \) by \( (66) \) we get the result

\[ \eta_j(\tau) = \eta_\tau + O(\rho^{-\frac{3}{2} \alpha}). \]

\[ \square \]

**Theorem 6.** For every eigenvalue \( \zeta_s \) of the Sturm-Liouville operator \( T(\gamma, \delta) \), there exists an eigenvalue \( \tilde{\zeta}_s \) of the matrix \( E(\gamma, \delta) \) such that

\[ \zeta_s = \tilde{\zeta}_s + O(\rho^{-\frac{3}{2} \alpha}). \]

**Proof.** Decompose the infinite matrix \( T(\gamma, \delta) \) as \( T(\gamma, \delta) = \tilde{A} + \tilde{B} \) where the matrix \( \tilde{A} \) is defined by

\[
\tilde{A} = \begin{bmatrix}
(l + v)^2|\delta|^2 & I + V_0 \\
(l - 1 + v)^2|\delta|^2 & I + V_0 \\
& \ddots \\
& & (l - \frac{a_1}{2} + v)^2|\delta|^2 & I + V_0
\end{bmatrix}
\]

and \( \tilde{B} = T(\gamma, \delta) - \tilde{A} \). Let \( \zeta_s \) be an eigenvalue of \( T(\gamma, \delta) \), and \( \Theta_s = (\Theta_s^1, \Theta_s^2, \Theta_s^3, \ldots) \), \( \Theta_s^r = (\Theta_s^{1}, \ldots, \Theta_s^{m}) \) be the corresponding normalized eigenvector, that is, \( T \Theta_s = \zeta_s \Theta_s \). \( \text{span} \{ e_i : (\tau - 1)m + 1 \leq i \leq \tau m \} \) is the eigenspace of the matrix \( \tilde{A} \) which corresponds to the eigenvalue \( |(\tau' + v)\delta|^2 \), where \( \tau' = l - \frac{\tau}{2} \) if \( \tau \) is even, \( \tau' = l + \frac{\tau - 1}{2} \) if \( \tau \) is odd, for \( \tau = 1, 2, \ldots \) and \( \{ e_i \} \) is the standard basis for \( l_2 \).

One can easily verify that

\[ \left( \zeta_s - |(\tau' + v)\delta|^2 \right) \Theta_s^\tau = \Theta_s \cdot \tilde{B} e_{\tau,i}, \]

where \( e_{\tau,i} \equiv e_i \), if \( (m - 1)\tau + 1 \leq i \leq m\tau \).

Using the orthogonal decomposition \( \tilde{B} e_{\tau,i} = \sum_{j=1}^{\infty} \sum_{k=1}^{m} (\tilde{B} e_{\tau,i} \cdot e_{k,j}) e_{k,j} \) \( (69) \) reduces to

\[ \left( \zeta_s - |(\tau' + v)\delta|^2 - |v_{ii0}|^2 \right) \Theta_s = \sum_{j=1}^{\infty} \sum_{k=1}^{m} (\tilde{B} e_{\tau,i} \cdot e_{k,j}) \Theta_s^{kj} \]
and since $\tilde{B}c_{r,i} \cdot e_{k,j} = v_{ij(n_k-n_r)\delta}$ for $k \neq r$,

$$(\zeta_s - (\tau' + v)\delta)^2 \Theta_s^{r,i} = \sum_{j=1}^{m} a_j \sum_{k=1}^{a_1} v_{ij(n_k-n_r)\delta} \Theta_s^{k,j} = \sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ij(n_k-n_r)\delta} \Theta_s^{k,j}. \quad (70)$$

Now take any eigenvalue $\zeta_s$ of $T(\gamma, \delta)$, satisfying $|\zeta_s - |(i' + v)\delta|^2| < sup|P(t)|$ for $s = 1, 2, ..., \frac{m_1}{2}$, where $i' = l - \frac{s}{2}$ if $s$ is even, $i' = l + \frac{s-1}{2}$ if $s$ is odd. The relations $\gamma \in V_{\delta}(\rho^\alpha)$ (\(\delta \neq e_i\)) and $\gamma = \beta + (l + v)\delta$, $\beta \cdot \delta = 0$ imply

$$|2\gamma \cdot \delta + |\delta|^2| = |(l + v)|\delta|^2 + |\delta|^2 < \rho^\alpha, \quad |l| < c_1 \rho^{\alpha_1}.$$ 

Therefore, using the definition of $i'$ and $\tau'$, we have

$$|(i' + v)\delta| < \frac{|a_1\delta|}{4} + c_2 \rho^{\alpha_1}$$

for $s = 1, 2, ..., \frac{a_1}{2}$ and

$$|(\tau' + v)\delta| > \frac{|a_1\delta|}{2} - c_2 \rho^{\alpha_1}$$

for $\tau > a_1$. Since $|a_1| > c_2 \rho^{\alpha_2}$ and $a_2 > 2a_1$, we have

$$\left|\left|(i' + v)\delta \right|^2 - \left|(\tau' + v)\delta \right|^2\right| > c_2 \rho^{\alpha_2}$$

(71)

for $s \leq \frac{a_1}{4}, \tau > a_1$, which implies

$$|\zeta_s - |(\tau' + v)|\delta|^2| = ||\zeta_s - |(i' + v)|\delta|^2| - |(\tau' + v)| \delta|^2| - |(i' + v)|\delta|^2| > c_2 \rho^{\alpha_2},$$

for $s = 1, 2, ..., \frac{a_1}{2}, \tau > a_1$.

Since $\tilde{B}$ corresponds to the operator $P : Y \rightarrow P(t)Y$ in $L^2_\delta[0, 2\pi]$, which has norm $sup|P(t)| \leq M$. Using this, equation 69 and 72, we have for the right hand side of 70 that

$$\left|\sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ij(n_k-n_r)\delta} \Theta_s^{k,j}\right| \leq \left|\sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ij(n_k-n_r)\delta}\right| \frac{\left|\Theta_s \cdot \tilde{B}e_{k,j}\right|}{\left|\zeta_s - |(k' + v)\delta|^2\right|}$$

$$\leq \sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ij(n_k-n_r)\delta} \left|\Theta_s\right| \left|\tilde{B}ight| \left|\Theta_s\right| = \left|M \rho^{-\alpha_2}\sum_{j=1}^{m} \sum_{k=a_1+1}^{\infty} v_{ij(n_k-n_r)\delta}\right|$$

$$\leq c_2 \rho^{-\alpha_2},$$

(73)

Therefore writing the equation 70 for all $\tau = 1, 2, ..., a_1$, and using 73 we get the following system

$$(E(\gamma, \delta) - \zeta_s I) [\Theta_s^1, \Theta_s^2, ..., \Theta_s^{a_1}] = [O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})],$$

(74)
where $I$ is an $ma_1 \times ma_1$ identity matrix. Using $\Theta_s = \sum_{\tau=1}^{\infty} \Theta^\tau e_{\tau,i}$, the formula (69) and the inequality (72), we have

$$\sum_{\tau=a_1+1}^{\infty} |\Theta^\tau_s|^2 = \sum_{\tau=a_1+1}^{\infty} \left| \frac{\Theta_s \cdot B e_{\tau,i}}{\xi_s - (\tau + v)\delta} \right|^2 = O(\rho^{-2\alpha_2})$$

and thus

$$\sum_{\tau=1}^{a_1} |\Theta^\tau_s|^2 = 1 - O(\rho^{-2\alpha_2}). \quad (75)$$

Multiplying both sides of (74) by $(E(\gamma, \delta) - \xi_s I)^{-1}$,

$$[\Theta^1_s, \Theta^2_s, \ldots, \Theta^{a_1}_s] = (E(\gamma, \delta) - \xi_s I)^{-1}[O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})],$$

then taking norm of both sides and using (75), we get

$$\sqrt{1 - O(\rho^{-2\alpha_2})} = \|(E(\gamma, \delta) - \xi_s I)^{-1}\|O(\sqrt{a_1} \rho^{-\alpha_2})$$

or

$$\min_{\tau} |\xi_s - \xi_s^\tau| = \frac{O(\sqrt{a_1} \rho^{-\alpha_2}) \cdot \sqrt{m}}{\sqrt{1 - O(\rho^{-2\alpha_2})}} = O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the minimum is taken over all eigenvalues $\xi_s^\tau$ of the matrix $E(\gamma, \delta)$. Thus, the result follows.

**Theorem 7.** (Main result) For every $\beta \in H_\delta$, $|\beta| \sim \rho$ and for every eigenvalue $\xi_s(\nu(\beta))$ of the Sturm-Liouville operator $T(\gamma, \delta)$, there is an eigenvalue $\Lambda_N$ of the operator $L(V)$ satisfying

$$\Lambda_N = |\beta|^2 + \xi_s + O(\rho^{-\frac{1}{2}\alpha_2}).$$

**Proof.** From Theorem 6 and the definition of $E(\gamma, \delta)$, there exists an eigenvalue $\bar{\eta}_{\tau(s)}$ of the matrix $D(\gamma, \delta)$, where $\gamma$ has a decomposition $\gamma = \beta + (\tau + \nu(\beta) \delta)$, satisfying $\bar{\eta}_{\tau(s)} = |\beta|^2 + \xi_s + O(\rho^{-\frac{3}{4}\alpha_2})$. Therefore, the result follows from Theorem 5 and Theorem 2. \qed

**References**

[1] Atalgin, Ş., Karakılıç, S. and Veliev, Ö. A., Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator, *Turk J Math.*, 26 (2002) 215–227.

[2] Berezin, F. A. and Shubin, M. A., The Schrödinger Equation, Kluwer Academic Publishers, Dordrecht, 1991.

[3] Coşkun, D. and Karakılıç, S., High energy asymptotics for eigenvalues of the Schrödinger operator with a matrix potential, *Mathematical Communications*, 16(2) (2011).

[4] Feldman, J., Knoerrer, H. and Trubowitz, E., The Perturbatively Stable Spectrum of the Periodic Schrödinger Operator, *Invent. Math.*, 100 (1990) 259–300.

[5] Feldman, J., Knoerrer, H. and E. Trubowitz, The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator, *Comment. Math. Helvetica*, 66(1991) 557–579.

[6] Friedlander, L., On the Spectrum for the Periodic Problem for the Schrödinger Operator, *Communications in Partial Differential Equations*, 15(1990) 1631–1647.
[7] Hald, O. H. and McLaughlin, J.R., Inverse Nodal Problems: Finding the Potential from Nodal Lines, *Memoirs of AMS*, 572, 119 (1996) 0075–9266.

[8] Karakılıç, S. and Akduman, S., Eigenvalue Asymptotics for the Schrödinger Operator with a Matrix Potential in a Single Resonance Domain, *Filomat*, 29(1) (2015) 21–38.

[9] Karakılıç, S., Atılıgan, Ş. and Veliev, O. A., Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator with Dirichlet and Neumann Boundary Conditions, *Reports on Mathematical Physics* (ROMP), 55(2) (2005) 221–239.

[10] Karakılıç, S., Veliev, O. A. and Atılıgan, Ş., Asymptotic Formulas for the Resonance Eigenvalues of the Schrödinger Operator, *Turkish Journal of Mathematics*, 29(4) (2005) 323–347.

[11] Karpeshina, Y., Perturbation Theory for the Schrödinger Operator with a non-smooth Periodic Potential, *Math. USSR-Sb.*, 71 (1992) 701–123.

[12] Karpeshina, Y., Perturbation series for the Schrödinger Operator with a Periodic Potential near Planes of Diffraction, *Communication in Analysis and Geometry*, 4(3) (1996) 339–413.

[13] Karpeshina, Y., On the Spectral Properties of Periodic Polyharmonic Matrix Operators, *Advanced in Mathematical Physics*, 1(2) (2002) 221–239.

[14] Karpeshina, Y., Perturbation Theory for the Periodic Multidimensional Schrödinger Operator and the Bethe-Sommerfeld Conjecture, *International Journal of Contemporary Mathematical Sciences*, 2(5) (2007) 19–87.

[15] Veliev, O. A., Multidimensional periodic Schrödinger operator: Perturbation theory and applications., *Springer*, Vol. 263 2015.

Current address: Sedef Karakılıç: Dokuz Eylül University, İzmir Turkey.
E-mail address: sedef.erim@deu.edu.tr
ORCID Address: http://orcid.org/0000-0002-0407-0271

Current address: Setenay Akduman: İzmir Demokrasi University, İzmir Turkey.
E-mail address: setenay.akduman@idu.edu.tr
ORCID Address: http://orcid.org/0000-0003-2492-3734

Current address: Didem Coşkan: Dokuz Eylül University, İzmir Turkey.
E-mail address: coskan.didem@gmail.com
ORCID Address: http://orcid.org/0000-0003-2358-198X