The $n$-fold reduced bar construction

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Abstract

This paper is about a correspondence between monoidal structures in categories and $n$-fold loop spaces. We develop a new syntactical technique whose role is to substitute the coherence results, which were the main ingredients in the proofs that the Segal-Thomason bar construction provides an appropriate simplicial space. The results we present here enable more common categories to enter this delooping machine. For example, such is the category of finite sets with two monoidal structures brought by the disjoint union and Cartesian product.

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1 Introduction

A correspondence between monoidal structures in categories and loop spaces is initially established by Stasheff in [21]. Since then, a connection of various algebraic structures on a category with 1-fold, 2-fold, $n$-fold, and infinite loop spaces is a subject of many papers (see [13], [20], [15], [23], [11], [9], [3], and references therein). The categories in question are usually equipped with one or several monoidal structures, and natural transformations providing symmetry, braiding, or some other kind of interchange between these structures. There are two main approaches to the subject. One is operadic and the other is through the Segal-Thomason bar construction, which we simply call reduced bar construction, as in [23]. The latter, to which we will keep throughout the paper, is an approach to the Quillen plus construction and it is the initial step connecting various monoidal categories with loop spaces.

The $n$-fold reduced bar construction based on an $n$-fold monoidal category $\mathcal{M}$ is an iteration of a construction of a simplicial object based on a monoid in a category whose monoidal structure is given by finite products. The goal is
to obtain a lax functor $\overline{WM}$ from $(\Delta^o)^n$, the $n$th power of the opposite of the simplicial category, to the category $Cat$, of categories and functors, such that

$$\overline{WM}(k_1, \ldots, k_n) = M^{k_1 \cdots k_n}.$$  

Moreover, some particular arrows of $(\Delta^o)^n$, which are built out of face maps corresponding to projections, have to be mapped by $\overline{WM}$ to identities.

Then, by applying Street’s rectification to $\overline{WM}$ (see [22]) one obtains a functor $V$, with the same source and target as $\overline{WM}$, such that

$$B(V(k_1, \ldots, k_n)) \simeq (BM)^{k_1 \cdots k_n},$$

where $B$ is the classifying space functor. The composition $N \circ V$, where $N$ is the nerve functor, is a multisimplicial set. Up to group completion (see [20] and [17]), this is sufficient for the realization of this multisimplicial set to be an $n$-fold delooping of $BM$. A thorough survey of the results concerning these matters is given in [19].

A definition of $n$-fold monoidal category is usually inductive. One starts with the 2-category $Cat$ whose monoidal structure is given by 2-products. The 0-cells of a 2-category $Mon(Cat)$ are pseudomonoids (or monoids) in $Cat$, i.e., monoidal (or strict monoidal) categories. Then one makes a choice what to consider to be the 1-cells of $Mon(Cat)$, i.e., how strictly they should preserve the monoidal structure. The monoidal structure of $Mon(Cat)$ is again given by 2-products. A pseudomonoid (or a monoid) in $Mon(Cat)$ is a (strict) two-fold monoidal category and if we iterate this procedure with the same degree of strictness, we obtain one possible notion of $n$-fold monoidal category.

Joyal and Street, [11], dealt with such a concept having in its basis a certain 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. They showed that such a degree of strictness leads to a sequence of categorial structures starting with monoidal categories, then they have braided monoidal categories as the two-fold monoidal categories and symmetric monoidal categories as the $n$-fold monoidal categories for $n \geq 3$.

Balteanu, Fiedorowicz, Schwänzl, and Vogt, [3], considered a variant of $Mon(Cat)$ consisting of strict monoidal categories, monoidal functors (in which the interchange between multiplicative structures need not be invertible), and monoidal transformations. This was an important advance leading to a definition of $n$-fold monoidal categories without stabilization at $n = 3$. However, they did not laxify the appropriate interchanges for units, which were treated in their work as strict as possible.

The notion of iterated monoidal structures they deal with is not so natural, and this is not because they missed the right definition of $n$-fold monoidal structure, the one given in [2]. The reason they stopped at that notion was probably the impossibility to prove an appropriate coherence result for the more general case. One monoidal unit prevented Kelly and Mac Lane, [12], to prove the full coherence for symmetric monoidal closed categories. The situation brought by diversifying the monoidal units in the case of $n$-fold monoidal structures is far more complicated.

The idea of [8] and [18] was to laxify the interchanges for units as much as coherence admits. Trimble and the second author showed that a coherence result for pseudocommutative pseudomonoids, for which some structural constraints are invertible, in a 2-category of symmetric monoidal categories, lax symmetric
monoidal functors, and monoidal transformations is sufficient for the reduced bar construction.

In this paper we consider the variant of $\text{Mon}(\text{Cat})$ in which the interchange between multiplicative structures and interchange between units need not be invertible, i.e., a 2-monoidal category of monoidal categories, lax monoidal functors, and monoidal natural transformations. This is the basis used by Aguiar and Mahajan, [2], for the definition of the notion of $n$-fold monoidal category. The possibility of defining $n$-fold monoidal structures with respect to such a basis is much less explored, perhaps because of difficulties in proving the corresponding coherence results. Such a coherence result usually guarantees commutativity of all the diagrams in $n$-fold monoidal categories relevant for the reduced bar construction.

This is the first proof of such a result that is not of the form: prove the coherence and do not worry about the lax conditions. We develop a new syntactical technique whose role is to substitute the coherence results. The correctness of the reduced bar construction, i.e., that $\overline{W}M$ is a lax functor, is guaranteed by commutativity of certain diagrams. Our main idea is to check this directly.

We consider the two steps that seem to be necessary in the proof of correctness of the reduced bar construction. These steps are roughly sketched below and precisely given in Sections 4, 6 and 8. It turns out that the definition of $n$-fold monoidal category given in [2] provides these two steps. We start with checking the correctness of the reduced bar construction based on a two-fold monoidal category, i.e., 2-monoidal category of [2], or duoidal category of [4] and [5]. The first step in this case is trivial, and the second step, which may be simply modified and used for the $n$-fold case, is more involved.

Then we check the correctness of the reduced bar construction based on a three-fold monoidal category, i.e., 3-monoidal category of [2]. We go through two steps that are in spirit the same as in the two-fold case. Neither of these steps is now trivial but, as it is mentioned above, the second is just a modification of the corresponding step in the two-fold case. The combinatorial structure of $n$-fold monoidal categories, defined by iterating this procedure, as it is already shown in [2], stabilizes at $n = 3$. Hence, an $n$-fold monoidal category, for $n \geq 3$, may be envisaged as a sequence of $n$ monoidal structures in a category, such that its every subsequence of length three corresponds to a three-fold monoidal category. The correctness of the reduced bar construction based on an $n$-fold monoidal category is obtained as simple modifications of the results mentioned above.

Our techniques are very much syntactical. We rely on a syntactical nature of the simplicial category presented by its generating arrows and equations. These equations are easily turned into rewrite rules, which are useful for some normalization techniques. Also, we try to point out the combinatorial core of the subject. This is the reason why our definition of the reduced bar construction $\overline{W}M$, although it covers the one of [3], is given in different terms. From a composition of functors involved in the definition of $\overline{W}M$ we abstract a shuffle of $n$ sequences, whose members are generators of the simplicial category. Then we consider some available transpositions turning this shuffle into a shuffle obtained by concatenating these $n$ sequences in a desired order. The first step in the proof of correctness of the reduced bar construction shows that the equations of $n$-fold monoidal categories suffice to consider any two applications of available
transpositions from one shuffle to the other to be equal. This is a consequence of some naturality assumptions in the two-fold case. In the $n$-fold case, for $n \geq 3$, we need some additional equations brought by the assumptions on 1-cells of $\text{Mon}_2(\text{Cat})$. Roughly speaking, these equations guarantee that the following two applications of transpositions in our shuffles, which correspond to the Yang-Baxter equation, are equal.

\[ \begin{array}{c} \text{Sequence 1} \\ \text{Sequence 2} \end{array} \]

The sequences that constitute a shuffle may be transformed according to the equations of the simplicial category. Let $\Phi'$ be the result of such a transformation of a sequence $\Phi$. The second step in the proof of correctness of the reduced bar construction shows that the equations of the $n$-fold monoidal categories suffice to consider the permutation of $\Phi$ or of $\Phi'$ with a member of another sequence to be equal. All these equations are already present in the two-fold case.

Hence, the equations of $n$-fold monoidal categories guarantee the correctness of the reduced bar construction. On the other hand, these equations are also necessary if one proves the correctness through these two steps. Our work may be characterized as the process of defining the $n$-fold monoidal categories just from the correctness of the reduced bar construction based on a multiple monoidal structure. We believe there are no further possibilities to laxify the notion of an $n$-fold monoidal category preserving the reduced bar construction.

With respect to the reduced bar construction, our result generalizes all the results mentioned above. It does not involve coherence results whose proofs in the case of $n$-fold monoidal categories are lengthy and complicated. The two steps of our proofs mentioned above are pretty straightforward. This paper, except for some basic categorial definitions, is self-contained.

To conclude, we mention that the interchanges between the monoidal structures required for the $n$-fold monoidal categories are usually brought about by braiding and symmetry. It is pointed out in [8] and [18] that a bicartesian structure (a category with all finite coproducts and products) brings the desired interchanges but the corresponding coherence result required some unusual properties of such a category—a coproduct of terminal objects should be terminal. Our results show that this coherence is not necessary anymore and that every bicartesian category, for every $n$, may be conceived as an $n$-fold monoidal category in $n + 1$ different ways. Although a bicartesian category is already $\infty$-monoidal, since it is symmetric monoidal (in two ways), this fact is interesting—there is an $\omega \times \omega$-indexed family of reduced bar constructions based on such a category. We discuss these matters in more details at the end of section 7. Also, this gives a positive answer to the second question of [18, Section 8].

2 The two-fold monoidal categories

The notion of two-fold monoidal category that we use in this paper is defined in [10, Section 4]. It appears in [2, Section 6.1] under the name "2-monoidal"
category and in both [4 Section 2.2] and [5 Section 3] under the name duoidal category. The notion appears as the second iterate of the inductive definition mentioned in the introduction. It slightly generalizes the notion of bimonoidal intermuting category introduced in [8 Section 12]. The difference between these two notions is that, in bimonoidal intermuting categories, the arrows $\beta$ and $\tau$ from below are required to be isomorphisms. The motivation behind this invertibility requirement is a coherence result in the style of Kelly and Mac Lane (see [12]), which is proved in [8].

Let $\text{Mon}(\text{Cat})$ be the 2-category whose 0-cells are the monoidal categories, 1-cells are the monoidal functors, and 2-cells are the monoidal transformations (see [14 XI.2]). The monoidal structure of $\text{Mon}(\text{Cat})$ is given by 2-products (see [6 7.4]).

**Definition.** A two-fold monoidal category is a pseudomonoid in $\text{Mon}(\text{Cat})$.

The unfolded form of this definition is given in Section 9 (Appendix). In this paper we are interested in strict monoidal structures and we now give a more symmetric definition of two-fold strict monoidal categories. A two-fold strict monoidal category is a category $\mathcal{M}$ equipped with two strict monoidal structures $\langle \mathcal{M}, \otimes_1, I_1 \rangle$ and $\langle \mathcal{M}, \otimes_2, I_2 \rangle$ together with the arrows $\kappa: I_1 \to I_2$, $\beta: I_1 \to I_1 \otimes_2 I_1$, $\tau: I_2 \otimes_1 I_2 \to I_2$, and a natural transformation $\iota$ given by the family of arrows

$$\iota_{A, B, C, D}: (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D),$$

such that the following twelve equations hold:

1. $\iota \circ (1 \otimes_1 \iota) = \iota \circ (\iota \otimes_1 1)$,
2. $\iota \circ (1 \otimes_1 \beta) = 1$,
3. $\iota \circ (\beta \otimes_1 1) = 1$,
4. $\tau \circ (1 \otimes_1 \tau) = \tau \circ (\tau \otimes_1 1)$,
5. $\tau \circ (1 \otimes_1 \kappa) = 1$,
6. $\tau \circ (\kappa \otimes_1 1) = 1$,
7. $(1 \otimes_2 \iota) \circ \iota = (\iota \otimes_2 1) \circ \iota$,
8. $(1 \otimes_2 \tau) \circ \iota = 1$,
9. $(\tau \otimes_2 1) \circ \iota = 1$,
10. $(1 \otimes_2 \beta) \circ \iota = (\beta \otimes_2 1) \circ \beta$,
11. $(1 \otimes_2 \kappa) \circ \iota = 1$,
12. $(\kappa \otimes_2 1) \circ \beta = 1$.

### 3 The reduced bar construction

Here we will only give a definition of the reduced bar construction based on a strict monoidal category. We refer to [13 Section 6] for the complete analysis of this construction.

Let $\Delta$ (denoted by $\Delta^+$ in [14]) be the topologist’s simplicial category defined as in [14 VII.5] for whose arrows we take over the notation used in that book. In order to use geometric dimension, the objects of $\Delta$, which are the nonempty ordinals $\{1, 2, 3, \ldots\}$ are rewritten as $\{0, 1, 2, \ldots\}$. Hence, for $n \geq 1$ and $0 \leq i \leq n$, the source of $\delta^i_n$ is $n-1$ and the target is $n$, while for $n \geq 1$ and $0 \leq i \leq n-1$, the source of $\sigma^p_n$ is $n$ and the target is $n-1$. When we speak of $\Delta^p$, then we denote its arrows $(\delta^i_n)^p: n \to n-1$ by $d^p_i$ and $(\sigma^i_n)^p: n-1 \to n$ by $s^p_i$.

The arrows of $\Delta^p$ satisfy the following basic equations:

$\delta^p_i \circ d^p_j = d^p_{l-1} \circ d^p_j$, when $l-1 \geq j$,

$s^p_{j+1} \circ s^p_l = s^p_{l+1} \circ s^p_{j}$, when $l+1 > j$,
whose monoidal structure is given by finite products. Let $\mathcal{M}$ be a strict monoidal category, hence a monoid in $\text{Cat}$. Thus, suffices to put any composite of $d$'s into the above form (cf. the proof of [14, VII.5, Lemma 3]).

\[ d^n_i \circ s^n_i = \begin{cases} 
  s_{i-1}^{n-1} \circ d_{j-1}^{n-1}, & \text{when } j \leq l - 1, \\
  1, & \text{when } l \in \{j, j - 1\}, \\
  s_{i-1}^{n-1} \circ d_{j+1}^{n-1}, & \text{when } j \geq l + 2.
\end{cases} \]

These particular identities whose left-hand sides are treated as redexes and the right-hand sides as the corresponding contracta serve to define the normal form (see below). The definition of the natural transformation $\omega$ (the ultimate ingredient in our construction) is completely based on this normal form. We use some syntactical techniques in this paper—it is therefore important how we represent the arrows by terms. However, we will never write brackets to denote the association of the binary operation of composition, and appropriate identity arrows could be considered present in a term or deleted from it, if necessary. The following proposition is analogous to [14, VII.5, Proposition 2].

**Proposition 3.1.** The category $\Delta^{op}$ is generated by the arrows $d^n_i : n \to n - 1$ for $n \geq 1$, $0 \leq i \leq n$, and $s^n_i : n - 1 \to n$ for $n \geq 1$, $0 \leq i \leq n - 1$, subject to the basic equations of $\Delta^{op}$.

**Proof.** As in the lemma preceding [14, VII.5, Proposition 2], one can prove that every arrow of $\Delta^{op}$ has a unique representation of the form $1$ or $s_{l_1} \circ \ldots \circ s_{l_k} \circ d_{j_1} \circ \ldots \circ d_{j_m}$ (with the superscripts omitted) for $k + m \geq 1$, $l_1 > \ldots > l_k$, $j_1 \geq \ldots \geq j_m$. The basic equations of $\Delta^{op}$ (read from the left to the right as reduction rules) suffice to put any composite of $d$'s and $s$'s into the above form (cf. the proof of [14, Coherence in S4 Section 3]).

We call the arrows $1_n$, $d^n_i$, and $s^n_i$ basic arrows of $\Delta^{op}$. Also, we call the above representation of an arrow $f$ of $\Delta^{op}$ the normal form of $f$ and denote it by $f^{\text{n.f.}}$. This normal form does not completely correspond to the normal form given in the mentioned lemma of [14 VII.5]—by varying the directions of the reduction rules corresponding to the first two basic equations of $\Delta^{op}$ one may obtain other possible normal forms.

**Remark 3.2.** If $f_1, \ldots, f_k$ are basic, non-identity arrows of $\Delta^{op}$ such that $f_k \circ \ldots \circ f_1$ is defined and not a normal form, then there is $1 \leq i \leq k - 1$ such that $f_{i+1} \circ f_i$ is the left hand side of one of the basic equations of $\Delta^{op}$.

By [14 XL.3, Theorem 1], we may regard $\text{Cat}$ as a strict monoidal category whose monoidal structure is given by finite products. Let $\mathcal{M}$ be a strict monoidal category, hence a monoid in $\text{Cat}$. The reduced bar construction (see [23]) based on $\mathcal{M}$ is functor $\overline{\mathcal{W}}\mathcal{M} : \Delta^{op} \to \text{Cat}$ defined as follows.

\[ \overline{\mathcal{W}}\mathcal{M}(n) = \mathcal{M}^n, \]

\[ \overline{\mathcal{W}}\mathcal{M}(d^n_i)(A_1, A_2, \ldots, A_n) = (A_2, \ldots, A_n), \]

\[ \overline{\mathcal{W}}\mathcal{M}(d^n_i)(A_1, \ldots, A_{n-1}, A_n) = (A_1, \ldots, A_{n-1}), \]

and for $1 \leq i \leq n - 1$ and $0 \leq j \leq n - 1$,

\[ \overline{\mathcal{W}}\mathcal{M}(d^n_i)(A_1, \ldots, A_i, A_{i+1}, \ldots, A_n) = (A_1, \ldots, A_i \otimes A_{i+1}, \ldots, A_n), \]

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\[ \overline{\mathcal{W}}(s^n_0)(A_1, \ldots, A_j, A_{j+1}, \ldots, A_{n-1}) = (A_1, \ldots, A_j, I, A_{j+1}, \ldots, A_{n-1}), \]

where \( \otimes \) is the tensor and \( I \) is the unit of the strict monoidal category \( \mathcal{M} \).

We denote by \( \overline{\mathcal{W}}\mathcal{M}^n : \Delta^op \to \text{Cat} \) the reduced bar construction based on the \( n \)-th power of the strict monoidal category \( \mathcal{M} \) (which is again a strict monoidal category with the structure defined component-wise). When \( \mathcal{M} \) is a two-fold strict monoidal category (or an \( n \)-fold, in general), then we denote by \( \overline{\mathcal{W}}\mathcal{M}_1 : \Delta^op \to \text{Cat} \) the reduced bar construction based on the \( 1 \)-th monoidal structure of \( \mathcal{M} \).

### 4 The two-fold reduced bar construction

For \( k \geq 1 \), let \( f_k \ldots f_1 \) be a sequence of basic arrows of \( \Delta^op \) such that the composition \( f_k \circ \ldots \circ f_1 \) is defined. We say that \( \Phi = (f_k, 1) \ldots (f_1, 1) \) is a sequence of colour 1 and we abbreviate \( f_k \circ \ldots \circ f_1 \) by \( \circ \Phi \). We define a sequence of colour 2 (or of any other colour) in the same manner. We assume that, if necessary, identities could always be added to, or deleted from sequences of any colour. However, for measuring the length of such a sequence, only non-identity members are taken into account.

Let \( \Phi \) be a sequence of colour 1 and let \( \Gamma \) be a sequence of colour 2. Suppose that \( \circ \Phi : n_1 \to n_2 \) and \( \circ \Gamma : m_1 \to m_2 \). Let \( \Theta \) be a shuffle of these two sequences. For example, let \( \Phi \) be \( (d^1_2, 1)(d^1_3, 1) \), let \( \Gamma \) be \( (d^2_3, 2)(s^3_0, 2)(d^3_1, 2) \), and let \( \Theta \) be the following shuffle

\[ (d^2_3, 2)(d^1_3, 1)(d^1_2, 1)(s^3_0, 2)(d^3_1, 2). \]

We have \( \circ \Phi : 3 \to 1 \) and \( \circ \Gamma : 3 \to 2 \).

For every member \( (f, 1) \) of \( \Theta \), we define its inner power to be the target of its right-closest \((g, 2)\) in \( \Theta \). We may assume that such \((g, 2)\) exists since we can always add an identity of colour 2 to the right of \((f, 1)\) in \( \Theta \). For every member \( (g, 2) \) of \( \Theta \), we define its outer power to be the target of its right-closest \((f, 1)\) in \( \Theta \). For \( \Theta \) defined as above, for example, we have that the inner power of \((d^1_2, 1)\) is 3 and the outer power of \((d^2_3, 2)\) is 1.

Let \( \mathcal{M} \) be a two-fold strict monoidal category. We define a functor

\[ \overline{\mathcal{W}}\mathcal{M}_2 : \mathcal{M}^{n_2 \cdot m_2} \to \mathcal{M}^{n_1 \cdot m_1} \]

in the following way: replace in \( \Theta \) every \((f, 1)\) whose inner power is \( i \) by \( \overline{\mathcal{W}}\mathcal{M}_1^i(f) \), and every \((g, 2)\) whose outer power is \( o \) by \( (\overline{\mathcal{W}}\mathcal{M}_2(g))^o \), and insert \( o \)'s. For \( \Theta \) as above, we have that \( \overline{\mathcal{W}}\mathcal{M}_2 \) is

\[ \overline{\mathcal{W}}\mathcal{M}_2(d^2_1) \circ \overline{\mathcal{W}}\mathcal{M}_1^i(d^1_2) \circ \overline{\mathcal{W}}\mathcal{M}_1^j(d^1_1) \circ (\overline{\mathcal{W}}\mathcal{M}_2(s^3_0))^3 \circ (\overline{\mathcal{W}}\mathcal{M}_2(d^3_1))^3, \]

which gives that \( \overline{\mathcal{W}}\mathcal{M}_2(A, B, C, D, E, F, G, H, J) \) is the ordered pair

\[ (I_2 \otimes I_1 \otimes I_2, ((A \otimes_2 B) \otimes_1 (D \otimes_2 E) \otimes_1 (G \otimes_2 H)) \otimes_2 (C \otimes_1 F \otimes_1 J)). \]

For basic arrows \( f : n \to n' \) and \( g : m \to m' \) of \( \Delta^op \) we define a natural transformation

\[ \chi(f, g) : \overline{\mathcal{W}}\mathcal{M}^{m'}_1(f) \circ (\overline{\mathcal{W}}\mathcal{M}_2(g))^n \to (\overline{\mathcal{W}}\mathcal{M}_2(g))^n \circ \overline{\mathcal{W}}\mathcal{M}^{m'}_1(f) \]

to be the identity natural transformation except in the following cases:
The length of every normalizing path starting with \( \Theta \) is

\[ \text{Proof.}\]

Every normalizing path starting with \( \text{Proposition 4.1.} \) is a normalizing path of length 4 starting with \( \Theta \) as in the example given above.

\[ \text{where} \quad k \text{ and} \quad \text{is a natural transformation from} \]

\[ \text{Here} \quad 1 \text{ denotes the n-tuple of 1's and} \quad 1 \text{ is a tuple of 1's whose length can be}
\]
easily calculated in all the cases, but we will not write the exact length to avoid overlong expressions.

\[ \text{For} \quad j \geq 0, \text{ let} \quad \Theta_0, \ldots, \Theta_j \text{ be shuffles of} \quad \Phi \quad \text{and} \quad \Gamma \text{ such that} \quad \Theta_0 = \Theta \text{ and} \quad \Theta_j = \Gamma \Phi \quad (a \text{ shuffle obtained by concatenating} \quad \Gamma \quad \text{and} \quad \Phi), \text{ and if} \quad j > 0, \text{ then for}
\]
every \( 0 \leq i \leq j - 1 \) we have that \( \Theta_i = \Pi(f,1)(g,2)\Lambda \) and \( \Theta_{i+1} = \Pi(g,2)(f,1)\Lambda \).

We call \( \Theta_0, \ldots, \Theta_j \) a normalizing path starting with \( \Theta \). Its length is \( j \). For example,

\[ \Theta_0 = (d_2,2)(d_1,1)(d_3,2)(d_2,1)(d_1,2), \quad \Theta_1 = (d_2,2)(d_1,1)(s_0,2)(d_2,1)(d_1,2), \]

\[ \Theta_2 = (d_2,2)(d_2,1)(s_0,2)(d_2,1)(d_2,1), \quad \Theta_3 = (d_2,2)(s_0,2)(d_1,1)(d_1,2)(d_1,1), \]

\[ \Theta_4 = (d_2,2)(s_0,2)(d_1,2)(d_2,1)(d_1,1) \]

is a normalizing path of length 4 starting with \( \Theta \) as in the example given above.

**Proposition 4.1.** Every normalizing path starting with \( \Theta \) has the same length.

**Proof.** The length of every normalizing path starting with \( \Theta \) is

\[ \sum_{(f,1) \in \Theta} k(f,1), \]

where \( k(f,1) \) is the number of \( (g,2) \)'s to the right of \( (f,1) \) in \( \Theta \).

If \( \Theta_i = \Pi(f,1)(g,2)\Lambda \) and \( \Theta_{i+1} = \Pi(g,2)(f,1)\Lambda \), then

\[ \varphi_i = \overline{\mathcal{W}}M_{\Pi} \chi(f,g) \overline{\mathcal{W}}M_{\Lambda} \]

is a natural transformation from \( \overline{\mathcal{W}}M_{\Theta_i} \) to \( \overline{\mathcal{W}}M_{\Theta_{i+1}} \). In order to define \( \overline{\mathcal{W}}M_{\Pi} \) and \( \overline{\mathcal{W}}M_{\Lambda} \), we may assume that \( \Pi \) and \( \Lambda \) are two-coloured since we can always add identities, if necessary. Let

\[ \varphi(\Theta_0, \ldots, \Theta_j) = \begin{cases} \varphi_{j-1} \circ \cdots \circ \varphi_0, & \text{when} \quad j \geq 1, \\ 1, & \text{when} \quad j = 0. \end{cases} \]

Suppose \( \Theta'_0, \ldots, \Theta'_j \) is another normalizing path starting with \( \Theta \). Then \( \varphi(\Theta_0, \ldots, \Theta_j) \) is again a natural transformation from \( \overline{\mathcal{W}}M_{\Theta_0} \) to \( \overline{\mathcal{W}}M_{\Gamma \Phi} \). We can show that these natural transformations are in fact the same.

**Theorem 4.2.** \( \varphi(\Theta_0, \ldots, \Theta_j) = \varphi(\Theta'_0, \ldots, \Theta'_j) \).

**Proof.** We prove this fact by induction on \( j \geq 0 \). If \( j = 0 \), then \( \varphi(\Theta_0) = \varphi(\Theta'_0) = 1 \). If \( j > 0 \) and \( \Theta_1 = \Theta'_1 \), then we apply the induction hypothesis to
the sequences of shuffles $\Theta_1, \ldots, \Theta_j$ and $\Theta'_1, \ldots, \Theta'_j$. Suppose now $\Theta_1 \neq \Theta'_1$ and $\varphi_0 = \overline{\mathcal{M}}_{\Pi_1} \chi(f, g) \overline{\mathcal{M}}_\Lambda, \varphi'_0 = \overline{\mathcal{M}}_{\Pi'} \chi(f', g') \overline{\mathcal{M}}_{\Lambda'}$. Then either

$$\Theta = \Pi_1(f', 1)(g', 2)\Pi_2(f, 1)(g, 2)\Lambda \quad \text{or} \quad \Theta = \Pi(f, 1)(g, 2)\Lambda_1(f', 1)(g', 2)\Lambda_2.$$ 

In the first case, by naturality we have

$$\varphi_0 = \overline{\mathcal{M}}_{\Pi_1} \chi(f', g') \overline{\mathcal{M}}_{\Pi_2} \chi(f, g) \overline{\mathcal{M}}_\Lambda \circ \overline{\mathcal{M}}_{\Pi_1} \chi(f', g') \overline{\mathcal{M}}_{\Pi_2} \chi(f, g) \overline{\mathcal{M}}_\Lambda$$

and by applying the induction hypothesis twice we obtain the following commutative diagram, in which $\Xi$ is $\Pi_1(g', 2)f_1(1)\Pi_2(g, 2)f_1(1)\Lambda$.

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{\Theta_0} & \overset{\varphi_0}{\longrightarrow} & \overline{\mathcal{M}}_{\Theta'_0} \\
\overline{\mathcal{M}}_{\Theta_1} & \overset{\varphi_1}{\longrightarrow} & \overline{\mathcal{M}}_{\Theta'_1} \\
\vdots & \overset{\text{ind. hyp.}}{\longrightarrow} & \vdots \\
\overline{\mathcal{M}}_{\Gamma_{\Phi}} & \overset{\varphi_{j-1}}{\longrightarrow} & \overline{\mathcal{M}}_{\Gamma_{\Phi'}}
\end{array}$$

We proceed analogously in the second case.

By Theorem 4.2, the following definition is correct.

**Definition.** Let $\varphi_{\Theta} : \overline{\mathcal{M}}_{\Theta} \to \overline{\mathcal{M}}_{\Gamma_{\Phi}}$ be $\varphi(\Theta_0, \ldots, \Theta_j)$, for an arbitrary normalizing path $\Theta_0, \ldots, \Theta_j$ starting with $\Theta$.

We are ready to define a function from the objects of $(\Delta^{op})^2$ to the objects of $\text{Cat}$ and from the arrows of $(\Delta^{op})^2$ to the arrows of $\text{Cat}$, relying on a two-fold strict monoidal category $\mathcal{M}$. Our goal is to show that these two functions make a lax functor from $(\Delta^{op})^2$ to $\text{Cat}$.

**Definition.** A two-fold reduced bar construction $\overline{\mathcal{M}}$ is defined on objects of $(\Delta^{op})^2$ as:

$$\overline{\mathcal{M}}(n, m) = M^{n \cdot m},$$

and on arrows of $(\Delta^{op})^2$ as:

$$\overline{\mathcal{M}}(f, g) = \overline{\mathcal{M}}_{\Phi \Gamma},$$

where $\Phi$ is a sequence of colour 1 such that $\circ \Phi = f$ and $\Gamma$ is a sequence of colour 2 such that $\circ \Gamma = g$.

This definition is correct since both $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$ are functors, which implies that for every other $\Phi'$ such that $\circ \Phi' = f$ and $\Gamma'$ such that $\circ \Gamma' = g$, we have the same value for $\overline{\mathcal{M}}(f, g)$. Also, in general, $\overline{\mathcal{M}}$ is not a functor from $(\Delta^{op})^2$ to $\text{Cat}$ since it does not preserve composition (it preserves identities). To
make a lax functor (see [22]) out of \( \mathcal{W} \mathcal{M} \), for every composable pair of arrows \((f_1, g_1), (f_2, g_2)\) of \((\Delta^o)^2\), we have to define a natural transformation

\[
\omega_{(f_2, g_2), (f_1, g_1)} : \mathcal{W} \mathcal{M}(f_2, g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) \rightarrow \mathcal{W} \mathcal{M}(f_2 \circ f_1, g_2 \circ g_1),
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{W} \mathcal{M}(f_3, g_3) \circ \mathcal{W} \mathcal{M}(f_2, g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) & \xrightarrow{\omega_{(f_3, g_3), (f_2, g_2)}} & \mathcal{W} \mathcal{M}(f_3 \circ f_2, g_3 \circ g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) \\
\mathcal{W} \mathcal{M}(f_3 \circ f_2, g_3 \circ g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) & \xrightarrow{\omega_{(f_3, g_3), (f_2, g_2)}} & \mathcal{W} \mathcal{M}(f_3 \circ f_2 \circ f_1, g_3 \circ g_2 \circ g_1) \\
\end{array}
\]

(\text{Diag 4.1})

**Definition.** Let \( \Phi_1 \) and \( \Phi_2 \) be sequences of colour 1 and \( \Gamma_1 \) and \( \Gamma_2 \) be sequences of colour 2 such that \( \circ \Phi_1 = f_1^{nf} \), \( \circ \Phi_2 = f_2^{nf} \), \( \circ \Gamma_1 = g_1^{nf} \) and \( \circ \Gamma_2 = g_2^{nf} \). We define

\[
\omega_{(f_2, g_2), (f_1, g_1)} \text{ as } \varphi_{\Gamma_2, \Phi_2, \Gamma_1, \Phi_1}.
\]

**Note.** Observe that \( \mathcal{W} \mathcal{M}(f_2, g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) = \mathcal{W} \mathcal{M}_{\Gamma_2, \Phi_2, \Gamma_1, \Phi_1} \), and \( \mathcal{W} \mathcal{M}(f_2 \circ f_1, g_2 \circ g_1) = \mathcal{W} \mathcal{M}_{\Gamma_2, \Phi_2, \Gamma_1, \Phi_1} \).

Let \( \Phi_1, \Phi_2, \Gamma_1 \) and \( \Gamma_2 \) be as above, and let \( \Phi_3 \) and \( \Gamma_3 \) be sequences of colour 1 and 2 respectively such that \( \circ \Phi_3 = f_3^{nf} \) and \( \circ \Gamma_3 = g_3^{nf} \). By the definitions of \( \omega \) and \( \mathcal{W} \mathcal{M} \), we have that

\[
\begin{align*}
\omega_{(f_3, g_3), (f_2, g_2), (f_1, g_1)} & = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \mathcal{W} \mathcal{M}_{\Gamma_1, \Phi_1} \quad \text{and} \\
\mathcal{W} \mathcal{M}(f_3, g_3) \circ \mathcal{W} \mathcal{M}(f_2, g_2) \circ \mathcal{W} \mathcal{M}(f_1, g_1) & = \mathcal{W} \mathcal{M}_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2, \Gamma_1, \Phi_1}.
\end{align*}
\]

Also, \( \omega_{(f_3, g_3), (f_2, g_2), (f_1, g_1)} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \), where \( \circ \Phi' = (f_3 \circ f_2)^{nf} \) and \( \circ \Gamma' = (g_3 \circ g_2)^{nf} \), and

\[
\omega_{(f_3, g_3), (f_2, g_2), (f_1, g_1)} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \varphi_{\Gamma_2, \Phi_2, \Gamma_1, \Phi_1}.
\]

By Theorem 4.2 we have that

\[
\varphi_{\Gamma_3, \Gamma_2, \Phi_2, \Gamma_1, \Phi_1} \circ \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \mathcal{W} \mathcal{M}_{\Gamma_1, \Phi_1} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \mathcal{W} \mathcal{M}_{\Gamma_1, \Phi_1} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \mathcal{W} \mathcal{M}_{\Gamma_1, \Phi_1}.
\]

Hence, to prove that Diag 4.1 commutes, it suffices to show that

\[
(i) \quad \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} \quad \text{and} \quad (ii) \quad \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2} = \varphi_{\Gamma_3, \Phi_3, \Gamma_2, \Phi_2}.
\]

**Lemma 4.3.** If \( \Phi \) and \( \Phi' \) are sequences of colour 1 such that \( \circ \Phi = \circ \Phi' \) is a basic equation of \( \Delta^o \), and \( g \) is a basic arrow of \( \Delta^o \), then \( \varphi_{\Phi(g, 2)} = \varphi_{\Phi'(g, 2)} \).
Now we compute.

We have two normalizing paths. The first one is starting with $\Phi(a)$ and we give a detailed proof for three cases, first of which is trivial and in the remaining two we need some of the equations (1)–(6). The rest is done analogously.

1.1. Suppose $\circ \Phi = \circ \Phi'$ is $d_{m-1}^j \circ d_{i-1}^m = d_{i-1}^m \circ d_{j}^m$ for $j \leq l - 2$.

1.1.1. Suppose $g$ is $s_{i}^m$.

We have two normalizing paths. The first one is starting with $\Phi(g, 2)$ and it is

$$(d_{m-1}^j, 1)(d_{j}^m, 1)(s_{i}^m, 2), \ (d_{m-1}^j, 1)(s_{i}^m, 2)(d_{j}^m, 1), \ (s_{i}^m, 2)(d_{m-1}^j, 1)(d_{j}^m, 1).$$

Now we compute $\varphi_{\Phi(g, 2)}$, and we note that $\varphi_{\Phi(g, 2)}$ is formally $\varphi_{\Phi(g, 2), (1, m-1, 2)}$ (we repeatedly use such an abbreviation throughout the paper):

$$\varphi_{\Phi(g, 2)} = (\chi(d_{m-1}^j, s_{i}^m) \varphi_{\Phi(g, 2), (1, m-1, 2)} \chi(d_{m-1}^j, s_{i}^m))$$

$$= (\chi(d_{m-1}^j, s_{i}^m) \varphi_{\Phi(g, 2), (1, m-1, 2)} \chi(d_{m-1}^j, s_{i}^m))$$

$$= \left((1^{(j-1)m}, 1^i, \tau, 1) \varphi_{\Phi(g, 2), (1, m-1, 2)} \left((1^{(j-1)m}, 1^i, \tau, 1) \right)\right)$$

$$= (1^{(j-1)m}, 1^i, \tau, 1) \circ (1^{(l-2)m}, 1^i, \tau, 1). \ (\text{since } l - 1 > j)$$

On the other hand, the second normalizing path starting with $\Phi'(g, 2)$ is

$$(d_{m-1}^j, 1)(d_{j}^m, 1)(s_{i}^m, 2), \ (d_{m-1}^j, 1)(s_{i}^m, 2)(d_{j}^m, 1), \ (s_{i}^m, 2)(d_{m-1}^j, 1)(d_{j}^m, 1),$$

and therefore

$$\varphi_{\Phi'(g, 2)} = (\chi(d_{m-1}^j, s_{i}^m) \varphi_{\Phi(g, 2), (1, m-1, 2)} \chi(d_{m-1}^j, s_{i}^m))$$

$$= (\chi(d_{m-1}^j, s_{i}^m) \varphi_{\Phi(g, 2), (1, m-1, 2)} \chi(d_{m-1}^j, s_{i}^m))$$

$$= \left((1^{(l-2)m}, 1^i, \tau, 1) \varphi_{\Phi(g, 2), (1, m-1, 2)} \left((1^{(j-1)m}, 1^i, \tau, 1) \right)\right)$$

$$= (1^{(l-2)m}, 1^i, \tau, 1) \circ (1^{(j-1)m}, 1^i, \tau, 1).$$

Since $j - 1 \neq l - 2$, we see that these two tuples of arrows are the same, i.e., we have:

$$\varphi_{\Phi(g, 2)} = (1^{(j-1)m+i}, \tau, 1^{(l-j-1)m-1}, \tau, 1) = \varphi_{\Phi'(g, 2)}.$$

1.1.2. Suppose $g$ is $d_{j}^m$.

$$\varphi_{\Phi(g, 2)} = (1^{(j-1)(m-1)+i-1}, \tau, 1^{(l-j-1)(m-1)-1}, \tau, 1) = \varphi_{\Phi'(g, 2)}.$$
We now compute \( \varphi \)
\[ \varphi = (\chi(d_j^{-1}, s_i^m) \overline{W_M(d_{j+1}^{-1}, 1)}) \circ (\overline{W_M(d_{j-1}^{-1}, 1)} \chi(d_j^l, s_i^m)) \]
\[ = \left(1^{(j-1)m}, 1^i, \tau, \vec{I}\right) \overline{W_M}(d_{j+1}^{-1}) \circ (\overline{W_M}(d_{j-1}^{-1}) (1^{jm}, 1^i, \tau, \vec{I})) \]
\[ = \phi \circ (1^{(j-1)m}, 1^i, \tau, \vec{I}) \]
\[ = \phi \circ (1^{(j-1)m} \circ (1^i, \tau, \vec{I})) \]
\[ = \phi \circ (1^{(j-1)m} \circ (1^i, \tau, \vec{I})) \]

On the other hand, the normalizing path starting with \( \Phi'_{(g, 2)} \) is
\[ (d_j^{-1}, 1)(d_j^m, 1)(s_i^m, 2), \ (d_j^{-1}, 1)(s_i^m, 2)(d_j^m, 1), \ (s_i^m, 2)(d_j^{-1}, 1)(d_j^m, 1). \]

We now compute \( \varphi_{\Phi'(g, 2)} \):
\[ \varphi_{\Phi'(g, 2)} = (\chi(d_j^{-1}, s_i^m) \overline{W_M}(d_{j+1}^m)) \circ (\overline{W_M}(d_{j-1}^m, 1) \chi(d_j^l, s_i^m)) \]
\[ = \left(1^{(j-1)m}, 1^i, \tau, \vec{I}\right) \overline{W_M}(d_{j+1}^m) \circ (\overline{W_M}(d_{j-1}^m) (1^{jm}, 1^i, \tau, \vec{I})) \]
\[ = \phi \circ (1^{(j-1)m}, 1^i, \tau, \vec{I}) \]
\[ = \phi \circ (1^{(j-1)m} \circ (1^i, \tau, \vec{I})) \]
\[ = \phi \circ (1^{(j-1)m} \circ (1^i, \tau, \vec{I})) \]

Since, by (4), we have that \( \tau \circ (1 \otimes 1 \tau) = \tau \circ (1 \otimes 1 1) \), we conclude that \( \varphi_{\Phi'(g, 2)} = \phi \).

1.2.2. Suppose \( g \) is \( d_j^m \).
\[ \varphi_{\Phi'(g, 2)} = (1^{(j-1)(m-1)+i-1}, \tau \circ (1 \otimes 1 i), \vec{I}) \]
\[ = (1^{(j-1)(m-1)+i-1}, \tau \circ (1 \otimes 1 1), \vec{I}) = \phi_{\Phi'(g, 2)}, \text{ by (1)}. \]

2. Suppose \( \circ \Phi = \circ \Phi' \) is \( s_j^{n+1} \circ s_i^m = s_i^{n+1} \circ s_j^n \) for \( j \leq l \).

2.1. Suppose \( g \) is \( s_i^m \).
\[ \phi_{\Phi'(g, 2)} = (1^{jm+i}, \kappa, 1^{(l-j+1)m-1}, \kappa, \vec{I}) = \phi_{\Phi'(g, 2)}. \]

2.2. Suppose \( g \) is \( d_i^m \).
\[ \phi_{\Phi'(g, 2)} = (1^{(m-1)+i-1}, \beta, 1^{(l-j+1)(m-1)-1}, \beta, \vec{I}) = \phi_{\Phi'(g, 2)}. \]

3. Suppose \( \circ \Phi = \circ \Phi' \) is \( d_j^m \circ s_i^m = s_i^{n-1} \circ d_j^{n-1} \) for \( j \leq l - 1 \).

3.1. Suppose \( g \) is \( s_i^{n-1} \).
\[ \phi_{\Phi'(g, 2)} = (1^{(j-1)m+i}, \tau, 1^{(l-j)m-1}, \kappa, \vec{I}) = \phi_{\Phi'(g, 2)}. \]

3.1.1. Suppose \( g \) is \( s_i^{n} \).
\[ \phi_{\Phi'(g, 2)} = (1^{(j-1)m+i}, \tau, 1^{(l-j)m-1}, \kappa, \vec{I}) = \phi_{\Phi'(g, 2)}. \]

3.1.2. Suppose \( g \) is \( d_j^m \).
\[ \phi_{\Phi'(g, 2)} = (1^{(j-1)(m-1)+i-1}, \beta, 1^{(l-j)(m-1)-1}, \beta, \vec{I}) = \phi_{\Phi'(g, 2)}. \]
3.2. Suppose \( o\Phi = o\Phi' \) is \( d^n_j \circ s^n_j = 1 \).

3.2.1. Suppose \( g \) is \( s^m_i \).

\[
\varphi_{\Phi(g,2)} = (1^{(j-1)m+1}, \tau \circ (1 \otimes_1 \kappa), \bar{1})
\]
\[
= (1^{(j-1)m+1}, 1, \bar{1}) = \varphi_{\Phi'(g,2)}, \quad \text{by (5).}
\]

3.2.2. Suppose \( g \) is \( d^m_i \).

We now have this normalizing path starting with \( \Phi(g,2) \):

\[
(d^n_j, 1)(s^n_j, 1)(d^m_i, 2), \quad (d^n_j, 1)(d^m_i, 2)(s^n_l, 1), \quad (d^m_i, 2)(d^n_j, 1)(s^n_j, 1).
\]

Since \( \varphi_{\Phi'(g,2)} = \varphi_{1(g,2)} = \bar{1} \), we ought to compute \( \varphi_{\Phi(g,2)} \):

\[
\varphi_{\Phi(g,2)} = \left( \chi(d^n_j, d^m_i) \mathbf{WM}_{(s^n_j, 1)} \right) \circ \left( \mathbf{WM}_{(d^n_j, 1)} \chi(s^n_j, d^m_i) \right)
\]
\[
= \left( \chi(d^n_j, d^m_i) \mathbf{WM}_{(s^n_j)} \right) \circ \left( \mathbf{WM}^{-1}_{(d^n_j, 1)}(d^m_i) \chi(s^n_j, d^m_i) \right)
\]
\[
= \left( 1^{(j-1)(m-1)}, 1^{i-1}, \tau, \bar{1}, \mathbf{WM}_{(s^n_j)} \right) \circ \left( \mathbf{WM}^{-1}_{(d^n_j, 1)}(1^{j(m-1)}, 1^{i-1}, \beta, \bar{1}) \right)
\]
\[
= \left( 1^{(j-1)(m-1)+i-1}, \tau \circ (1 \otimes_1 \beta), \bar{1} \right) = (\varphi_{\Phi'(g,2)} = (1^{(j-1)(m-1)+i-1}, 1, \bar{1}) = \varphi_{\Phi'(g,2)}).
\]

3.3. Suppose \( o\Phi = o\Phi' \) is \( d^n_j \circ s^n_j = 1 \).

3.3.1. Suppose \( g \) is \( s^m_i \).

\[
\varphi_{\Phi(g,2)} = (1^{(j-1)m+1}, \tau \circ (\kappa \otimes_1 1), \bar{1})
\]
\[
= (1^{(j-1)m+1}, 1, \bar{1}) = \varphi_{\Phi'(g,2)}, \quad \text{by (6).}
\]

3.3.2. Suppose \( g \) is \( d^m_i \).

\[
\varphi_{\Phi(g,2)} = (1^{(j-1)(m-1)+i-1}, \tau \circ (\beta \otimes_1 1), \bar{1})
\]
\[
= (1^{(j-1)(m-1)+i-1}, 1, \bar{1}) = \varphi_{\Phi'(g,2)}, \quad \text{by (3).}
\]

3.4. Suppose \( o\Phi = o\Phi' \) is \( d^n_j \circ s^n_j = s^{n-1}_l \circ d^{n-1}_{j-1} \) for \( j \geq l + 2 \).

3.4.1. Suppose \( g \) is \( s^m_i \).

\[
\varphi_{\Phi(g,2)} = (1^{lm+i}, \kappa, 1^{(j-l-1)m-1}, \tau, \bar{1}) = \varphi_{\Phi'(g,2)}.
\]

3.4.2. Suppose \( g \) is \( d^m_i \).

\[
\varphi_{\Phi(g,2)} = (1^{(l-m)+i-1}, \beta, 1^{(j-l-1)(m-1)-1}, \tau, \bar{1}) = \varphi_{\Phi'(g,2)}.
\]

Lemma 4.4. If \( \Psi \) and \( \Psi' \) are sequences of colour 1 such that \( o\Psi \) is \( o\Psi' \), and \( g \) is a basic arrow of \( \Delta o\Psi \), then \( \varphi_{\Psi(g,2)} = \varphi_{\Psi'(g,2)} \).

Proof. Let \( \mu(\Psi) \) be a “distance” from \( o\Psi \) to \( o\Psi \). For example, \( \mu(\Psi) \) can be defined as the ordered pair

\[
(n, m),
\]

where \( n \) is the number of subsequences of \( \Psi \) that are of the form \( (d, 1)(s, 1) \), i.e., \( s \) precedes \( d \) looking from the right to the left (not necessary immediately) in \( \Psi \),
and $m$ is the number of subsequences of $\Psi$ of the form $(s_i,1)(s_j, 1)$ with $i \leq j$, or $(d_i,1)(d_j,1)$ with $i < j$. Suppose that our set of “distances” is lexicographically ordered.

We proceed by induction on $\mu(\Psi)$. If $\mu(\Psi) = (0,0)$, then $\Psi = \Psi'$ and we are done. If $\mu(\Psi) > (0,0)$, then, by Remark 3.2, $\Psi$ must be of the form $\Psi_2 \Phi \Psi_1$, where $\Phi = \Phi'$ is a basic equation of $\Delta^op$. Then we have

$$\varphi_{\Psi_2 \Phi \Psi_1, (g,2)} = \varphi_{\Psi_2 \Phi \Psi_1, (g,2)} \circ W \varphi_{\Phi, (g,2)} W \varphi_{\Psi_1, (g,2)},$$

(by Theorem 4.2)

$$= \varphi_{\Psi_2 \Psi_1, (g,2)} \circ W \varphi_{\Psi_1, (g,2)} W \varphi_{\Psi_2 \Phi, (g,2)},$$

(by Lemma 4.3 and functoriality of $W \varphi_{\Phi_1}$)

$$= \varphi_{\Psi_2 \Phi \Psi_1, (g,2)},$$

(by Theorem 4.2)

$$= \varphi_{\Psi_2 \Psi_1, (g,2)},$$

(by the ind. hyp. since $\mu(\Psi_2 \Phi \Psi_1) < \mu(\Psi_2 \Phi \Psi_1')) \not<

We can prove now (i) by induction on the length of $\Gamma_1$ where in the induction step we use Lemma 4.4. We can prove (ii) in a dual manner using the equations (7)-(12) for the proof of a lemma dual to Lemma 4.3. So, we have:

**Theorem 4.5.** The two-fold reduced bar construction $W \varphi_{\Phi_1}$ together with the natural transformation $\omega$, makes a lax functor from $(\Delta^op)^2$ to $\text{Cat}$.

5 The three-fold monoidal categories

The notion of three-fold monoidal category that we use in this paper is defined in [2] Section 7.1 under the name 3-monoidal category. In order to define this notion we first define what the arrows between the two-fold monoidal categories are.

**Definition.** A two-fold monoidal functor between two-fold monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a 5-tuple $\langle F, \sigma^1, \delta^1, \delta^2, \sigma^2 \rangle$, where for $i \in \{1,2\}$,

$$\sigma^i_{A,B} : FA \otimes_i F B \to F(A \otimes_i B)$$

$$\delta^i : I^p_i \to FI^p_i$$

are arrows of $\mathcal{D}$ natural in $A$ and $B$, such that $\langle F, \sigma^1, \delta^1 \rangle$ and $\langle F, \sigma^2, \delta^2 \rangle$ are monoidal functors between, respectively, the first and the second monoidal structures of $\mathcal{C}$ and $\mathcal{D}$. Moreover, the structure brought by the arrows $\kappa, \beta, \tau$ and $\iota$ is preserved, which means that the following four diagrams commute (with the superscripts $\mathcal{C}$ and $\mathcal{D}$ omitted):

$$
\begin{array}{ccc}
I_1 & \xrightarrow{\kappa} & I_2 \\
\delta_1 \downarrow & & \delta_2 \\
F I_1 & \xrightarrow{F \kappa} & F I_2 \\
\end{array}
$$

$$
\begin{array}{ccc}
I_1 & \xrightarrow{\beta} & I_1 \otimes_2 I_1 \\
\delta_1 \downarrow & & \delta_1 \otimes_2 \delta_1 \\
F I_1 \otimes_2 F I_1 & \xrightarrow{F \beta} & F(I_1 \otimes_2 I_1) \\
\end{array}
$$

$$
\begin{array}{ccc}
I_2 \otimes_1 I_2 & \xrightarrow{\tau} & I_2 \\
\delta_2 \otimes_1 \delta_2 \downarrow & & \delta^2 \\
F I_2 \otimes_1 F I_2 & \xrightarrow{F \tau} & F(I_2 \otimes_1 I_2) \\
\end{array}
$$

$$
\begin{array}{ccc}
I_1 \otimes_2 I_1 & \xrightarrow{\sigma^1} & I_2 \\
\delta^1 \otimes_2 \delta^1 \downarrow & & \delta^2 \\
F I_1 \otimes_2 F I_1 & \xrightarrow{F \sigma^1} & F(I_1 \otimes_2 I_1) \\
\end{array}
$$
are two-fold monoidal and, moreover, the following equations hold:

\[(FA \otimes_2 FB) \otimes_1 (FC \otimes_2 FD) \xrightarrow{\sigma^2 \otimes_1 \sigma^2} (FA \otimes_1 FC) \otimes_2 (FB \otimes_1 FD)\]

\[\xrightarrow{\ i\ } (FA \otimes_1 FC) \otimes_2 (FB \otimes_1 FD) \xrightarrow{\ \sigma^1 \otimes_2 \sigma^1\ } F(A \otimes_1 C) \otimes_2 F(B \otimes_1 D)\]

\[\xrightarrow{\ \sigma^1\ } F(A \otimes_1 C) \otimes_2 F(B \otimes_1 D) \xrightarrow{\ \sigma^2\ } F((A \otimes_2 B) \otimes_1 (C \otimes_2 D)) \xrightarrow{\ \underline{\ F}\ } F((A \otimes_1 C) \otimes_2 (B \otimes_1 D))\]

Let \(Mon_2(Cat)\) be the 2-category whose 0-cells are the two-fold monoidal categories, 1-cells are the two-fold monoidal functors, and 2-cells are the two-fold monoidal transformations, i.e., monoidal transformations with respect to both structures. The monoidal structure of \(Mon_2(Cat)\) is yet again given by 2-products.

**Definition.** A *three-fold monoidal* category is a pseudomonoid in \(Mon_2(Cat)\).

Hence, a three-fold monoidal category consists of the following:

1. a two-fold monoidal category \(\mathcal{M}\),
2. two-fold monoidal functors \(\otimes_3: \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) and \(I_3: 1 \to \mathcal{M}\),
3. two-fold monoidal transformations \(\alpha_3, \beta_3, \) and \(\lambda_3\) such that the structure \((\mathcal{M}, \otimes_3, I_3, \alpha_3, \beta_3, \lambda_3)\) satisfies the pseudomonoid conditions.

In an unfolded form, this means that a three-fold monoidal category is a category \(\mathcal{M}\) equipped with three monoidal structures \((\mathcal{M}, \otimes_1, I_1)\), \((\mathcal{M}, \otimes_2, I_2)\), and \((\mathcal{M}, \otimes_3, I_3)\) such that

**[1-2]** \((\mathcal{M}, \otimes_1, I_1), (\mathcal{M}, \otimes_2, I_2), \kappa: I_1 \to I_2, \beta: I_1 \to I_1 \otimes_1 I_1, \tau: I_2 \otimes_1 I_2 \to I_2, \) and \(\iota: (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)\),

**[2-3]** \((\mathcal{M}, \otimes_2, I_2), (\mathcal{M}, \otimes_3, I_3), \kappa': I_2 \to I_3, \beta': I_2 \to I_2 \otimes_2 I_2, \tau': I_3 \otimes_2 I_3 \to I_3, \) and \(\iota': (A \otimes_3 B) \otimes_2 (C \otimes_3 D) \to (A \otimes_2 C) \otimes_3 (B \otimes_2 D)\),

**[1-3]** \((\mathcal{M}, \otimes_1, I_1), (\mathcal{M}, \otimes_3, I_3), \kappa'': I_1 \to I_3, \beta'': I_1 \to I_1 \otimes_3 I_1, \tau'': I_3 \otimes_1 I_3 \to I_3, \) and \(\iota'': (A \otimes_1 B) \otimes_1 (C \otimes_3 D) \to (A \otimes_1 C) \otimes_3 (B \otimes_1 D)\),

are two-fold monoidal and, moreover, the following equations hold:

\[(13) \quad \kappa' \circ \kappa = \kappa'',\]
\[(14) \quad \beta' \circ \kappa = (\kappa \otimes_3 \kappa) \circ \beta'',\]
\[(15) \quad \tau' \circ (\kappa'' \otimes_2 \kappa'') \circ \beta = \kappa'',\]
\[(16) \quad \iota' \circ (\beta'' \otimes_2 \beta'') \circ \beta = (\beta \otimes_3 \beta) \circ \beta'',\]
\[(17) \quad \kappa' \circ \tau = \tau'' \circ (\kappa' \otimes_1 \kappa'),\]
\[(18) \quad \beta' \circ \tau = (\tau \otimes_3 \tau) \circ \iota'' \circ (\beta' \otimes_1 \beta'),\]
\[(19) \quad \tau' \circ (\tau'' \otimes_2 \tau'') \circ \iota = \tau'' \circ (\tau' \otimes_1 \tau'),\]
\[(20) \quad \iota' \circ (\tau'' \otimes_2 \iota'') \circ \iota = (\iota \otimes_3 \iota) \circ \iota'' \circ (\iota' \otimes_1 \iota').\]

*Note.* The last eight equations represent the four commutative diagrams given above, with \(F\) replaced by the two-fold monoidal functors \(I_3\) and \(\otimes_3\).

As in the case of two-fold monoidal categories, here we are interested only in *three-fold strict monoidal* categories, i.e., when the structures \((\mathcal{M}, \otimes_1, I_1), (\mathcal{M}, \otimes_2, I_2),\) and \((\mathcal{M}, \otimes_3, I_3)\) are strict monoidal.
6 The three-fold reduced bar construction

Let $\Phi$, $\Gamma$, and $H$ be sequences of colour 1, 2, and 3, respectively, such that $\sigma \Phi : n_s \to n_t$, $\sigma \Gamma : m_s \to m_t$, and $\sigma H : p_s \to p_t$. Let $\Theta$ be a shuffle of these three sequences. For example, let $\Phi$ be $(d_2^2, 1)(d_1^3, 1)$, let $\Gamma$ be $(d_1^2, 2)$, let $H$ be $(s_1^3, 3)(s_2^3, 3)$, and let $\Theta$ be the following shuffle

$$(d_2^2, 1)(s_1^3, 3)(d_1^3, 1)(d_1^2, 2)(s_2^3, 3).$$

For every member $(f, 1)$ of $\Theta$, we define its inner power to be the product of the targets of its right-closest $(g, 2)$ and right-closest $(h, 3)$ in $\Theta$. We may assume again that such $(g, 2)$ and $(h, 3)$ exist since we can always add an identity of colour 2 and an identity of colour 3 to the right of $(f, 1)$ in $\Theta$. For every member $(g, 2)$ of $\Theta$, we define its inner power to be the target of its right-closest $(h, 3)$ in $\Theta$, and we define its outer power to be the target of its right-closest $(f, 1)$ in $\Theta$. For every member $(h, 3)$ of $\Theta$, we define its outer power to be the product of the targets of its right-closest $(f, 1)$ and right-closest $(g, 2)$ in $\Theta$. For $\Theta$ as above, for example, we have that the outer power of $(s_2^3, 3)$ is 6.

Let $M$ be a three-fold strict monoidal category. We define a functor

$$\overline{WM}_\Theta : M_{n_s, m_s, p_s} \to M_{n_t, m_t, p_t},$$

in the following way: replace in $\Theta$ each $(f, 1)$ whose inner power is $i$ by $\overline{WM}_1^i(f)$, every $(g, 2)$ whose inner power is $i$ and outer power is $o$ by $(\overline{WM}_2^o(g))^n$ and every $(h, 3)$ whose outer power is $o$ by $(\overline{WM}_3(h))^n$, and insert $o$'s. For $\Theta$ as above, we have that $\overline{WM}_\Theta$ is

$$\overline{WM}_1^2(d_2^2) \circ (\overline{WM}_3(s_2^3))^2 \circ \overline{WM}_1^3(d_1^3) \circ (\overline{WM}_2^2(d_1^2))^3 \circ (\overline{WM}_3(s_1^3))^6,$$

which gives that $\overline{WM}_\Theta(A, B, C, D, E, F)$ is the 3-tuple

$$((A \otimes_2 B) \otimes_1 (C \otimes_2 D), I_3, (I_3 \otimes_2 I_3) \otimes_1 (I_3 \otimes_2 I_3)).$$

For basic arrows $f : n \to n'$, $g : m \to m'$ of $\Delta^{op}$ and $w \geq 0$ we define a natural transformation

$$\chi_{w}^{1,2}(f, g) : \overline{WM}_1^{m_w} \circ (\overline{WM}_2^n(g))^n \Rightarrow (\overline{WM}_2^n(g))^n \circ \overline{WM}_1^{m_w}(f)$$

to be the identity natural transformation except in the following cases:

| $f$ | $g$ | $\chi_{w}^{1,2}(f, g)$ |
|-----|-----|------------------|
| $s_{n+1}^n$ | $s_{n+1}^n$ | $(1)^{(n+1)w}, 1^{w}, 1^{w}$ |
| $d_{ij}^n, 1 \leq j \leq n - 1$ | $s_{n+1}^n$ | $(1)^{(j+1)w}, 1^{w}, 1^{w}$ |
| $s_{n+1}^n$ | $d_{ij}^m, 1 \leq i \leq m - 1$ | $(1)^{(m+1)w}, 1^{w}, 1^{w}$ |
| $d_{ij}^n, 1 \leq j \leq n - 1$ | $d_{ij}^m, 1 \leq i \leq m - 1$ | $(1)^{(i+1)w}, 1^{w}, 1^{w}$ |

Table 2: $\chi_{w}^{1,2}$ in nontrivial cases.
In order to simplify some calculations and improve the presentation of the paper, we introduce the following formal operation of multiplication (always from the right) of tuples representing the natural transformations by 0-1 matrices having in each column exactly one entry equal to 1 and all the other entries equal to 0, which is derived from the standard multiplication of matrices. For example,

\[(1, \kappa, 1) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = (1, \kappa^2, 1^2, \kappa^2, 1^2)\]

Note that the tuples of the third column of Table 2 are obtained as a result of multiplication of the tuples in the third column of Table 1 (where \(m\) is replaced by \(p\) by \(1\)).

Multiplication of the tuples in the third column of Table 1 by the matrix

\[I_u \otimes I_{m'} \otimes (1, \ldots, 1),\]

where \(I_k\) is the \(k \times k\) identity matrix and \(\otimes\) is the Kronecker product of matrices.

For basic arrows \(g: m \to m', h: p \to p'\) of \(\Delta^{op}\) and \(u \geq 0\) we define a natural transformation

\[\chi^{2,3}(g, h): (\overline{\mathcal{M}}^k_3(g))^u \circ (\overline{\mathcal{M}}_3(h))^{um} \to (\overline{\mathcal{M}}_3(h))^{um'} \circ (\overline{\mathcal{M}}^p_2(g))^u\]

to be the identity natural transformation except in the following cases:

| \(g\)  | \(h\)  | \(\chi^{2,3}(g, h)\) |
|-------|-------|---------------------|
| \(s^{m+1}_1\) | \(s^{p+1}_k\) | \((1^i(p+1), 1^k, \kappa', 1^u)\) |
| \(d^{m}_i, 1 \leq i \leq m - 1\) | \(s^{p+1}_k\) | \((1^{(i-1)(p+1)}, 1^k, \tau', 1^u)\) |
| \(s^{m+1}_1\) | \(d^{p}_k, 1 \leq k \leq p - 1\) | \((1^{(i-1)(p-1)}, 1^{k-1}, \beta', 1^u)\) |
| \(d^{m}_i, 1 \leq i \leq m - 1\) | \(d^{p}_k, 1 \leq k \leq p - 1\) | \((1^{(i-1)(p-1)}, 1^{k-1}, \tau', 1^u)\) |

Table 3: \(\chi^{2,3}u\) in nontrivial cases.

Note that the tuples of the third column of this table are obtained as a result of multiplication of the tuples in the third column of Table 1 (where \(m\) is replaced by \(p\), \(n\) is replaced by \(m\), \(i\) is replaced by \(k\), \(j\) is replaced by \(i\), and \(\kappa\), \(\beta\), \(\tau\), and \(i\) are replaced by \(\kappa', \beta', \tau', \) and \(i'\)) by the matrix

\[(1, \ldots, 1) \otimes I_{m'} \otimes I_{p'}\]

Finally, for basic arrows \(f: n \to n', h: p \to p'\) of \(\Delta^{op}\) and \(v \geq 0\) we define a natural transformation

\[\chi^{1,3}(f, h): \overline{\mathcal{M}}^{op}_1(f) \circ (\overline{\mathcal{M}}_3(h))^{nv} \to (\overline{\mathcal{M}}_3(h))^{n'v} \circ \overline{\mathcal{M}}^{op}_1(f)\]

to be the identity natural transformation except in the following cases:
Proof. Consider the following table in which $d_y^n$ is such that $0 < y < x$.

| $f$ | $g$ | $h$ | component | EQ |
|-----|-----|-----|-----------|----|
| $s_j^{n+1}$ | $s_k^{p+1}$ | $s_k^{p+1}$ | $j(m+1)(p+1)+i(p+1)+k+1$ | (13) |
| $d_j^n$ | $s_k^{p+1}$ | $d_k^n$ | $j(m+1)(p-1)+i(p-1)+k$ | (14) |
| $s_j^{n+1}$ | $d_j^n$ | $d_k^n$ | $j(m-1)(p+1)+(i-1)(p+1)+k+1$ | (15) |
| $d_j^n$ | $s_k^{p+1}$ | $d_k^n$ | $j(m-1)(p-1)+(i-1)(p-1)+k+1$ | (16) |
| $s_j^{n+1}$ | $d_j^n$ | $d_k^n$ | $(j-1)(m+1)(p+1)+i(p+1)+k+1$ | (17) |
| $d_j^n$ | $s_k^{p+1}$ | $d_k^n$ | $(j-1)(m+1)(p-1)+i(p-1)+k$ | (18) |
| $d_j^n$ | $d_j^n$ | $d_k^n$ | $(j-1)(m-1)(p+1)+(i-1)(p+1)+k+1$ | (19) |
| $d_j^n$ | $d_j^n$ | $d_k^n$ | $(j-1)(m-1)(p-1)+(i-1)(p-1)+k$ | (20) |

Table 4: $\chi_n^{1,3}$ in nontrivial cases.

As in the previous cases, the tuples of the third column of this table are obtained as a result of multiplication of the tuples in the third column of Table 4 (with some necessary replacements) by a certain matrix, in this case that matrix is $I_n \otimes (1, \ldots, 1) \otimes I_{p'}$.

Lemma 6.1. For basic arrows $f: n \to n'$, $g: m \to m'$, and $h: p \to p'$ of $\Delta^p$ the following diagram commutes:

\[
\begin{array}{ccc}
\chi_{n'}^{1,2}(f, g)(\mathcal{W}\mathcal{M}_3(h)) & \mathcal{W}\mathcal{M}_{(f,1), (g,2), (h,3)} & \chi_{n'}^{1,2}(g, h) \\
\mathcal{W}\mathcal{M}_{(g,2), (f,1), (h,3)} & \mathcal{W}\mathcal{M}_{(f,1), (h,3), (g,2)} & \chi_{n'}^{1,2}(g, h) \\
\mathcal{W}\mathcal{M}_{(h,3), (f,1), (g,2)} & \mathcal{W}\mathcal{M}_{(f,1), (g,2), (h,3)} & \chi_{n'}^{1,2}(g, h) \\
\end{array}
\]

Proof. Consider the following table in which $d_y^n$ is such that $0 < y < x$.
This gives a list of all nontrivial cases for \( f, g, \) and \( h \). In this table we point out the component of the two \( n' \cdot m' \cdot p' \)-tuples of arrows, representing the left-hand side and the right-hand side of the above diagram, where we use one of the equations (13)-(20). In all the other components, the left-hand side is equal to the right-hand side by simple categorial arguments.

As an illustration of these arguments, here we give a proof for one of the cases from the table, namely when \( f = d^n_i, g = s^{m+1}_j, \) and \( h = d^p_k \). At the left hand side of the diagram we have the following

\[
\chi^{1,2}_n(d^n_j, s^{m+1}_i)(\mathcal{WM}_3(d^n_k))^{nm} = (1^{(j-1)(m+1)(p-1)}(1^{(p-1)}), \tau^{p-1}, \bar{1}, \bar{1}), \quad (L1)
\]

\[
(\mathcal{WM}_2^{n-1}(s^{m+1}_i))^{n-1} \chi^{1,3}_m(d^n_j, d^p_k) =
(\mathcal{WM}_2^{n-1}(s^{m+1}_i))^{n-1} (1^{(j-1)m(p-1)}, (1^{k-1}, \beta', \bar{1}, \bar{1})), \quad (L2)
\]

\[
\chi^{2,3}_m(s^{m+1}_i, d^p_k)\mathcal{WM}_1^{np}(d^n_j) = ((1^{(p-1)}, 1^{k-1}, \beta', \bar{1}))^{n+1}, \quad (L3)
\]

while at the right hand side we have:

\[
\mathcal{WM}_1^{(m+1)(p-1)}(d^n_j, \chi^{2,3}_m(s^{m+1}_i, d^p_k)) = \mathcal{WM}_1^{(m+1)(p-1)}(d^n_j)(1^{(p-1)}, 1^{k-1}, \beta', \bar{1})^{n+1} =
((1^{(p-1)}, 1^{k-1}, \beta', \bar{1})^{j-1}, 1^{(p-1)+k-1}, \beta' \otimes 1_{\beta'}, \bar{1}, 1^{(p-1)}, 1^{k-1}, \beta', \bar{1})^{n-j-1}), \quad (D1)
\]

\[
\chi^{1,3}_m(d^n_j, d^p_k)(\mathcal{WM}_2(s^{m+1}_i))^{n} = (1^{(j-1)(m+1)(p-1)}, (1^{k-1}, \beta', \bar{1})^{m+1}, \bar{1}), \quad (D2)
\]

\[
(\mathcal{WM}_3(d^n_k))^{(n-1)(m-1)} \chi^{1,2}_n(d^n_j, s^{m+1}_i) = \mathcal{WM}_3(d^n_k)(n-1)(m-1) \circ (1^{(j-1)(m+1)p}, 1^{i}), \quad (D3)
\]

Now we take a look at all entries that are not equal to 1 (non-identities). For example, in \( L1 \) the non-identities are at positions

\[(j-1)(m+1)(p+1)+i(p-1)+l, \quad 1 \leq l \leq p-1,\]

and those entries are equal to \( \tau \). By comparing the non-identities for \( L1, L2, L3, L1, L2, \) and \( L3 \), we get that the only difference is at position \((j-1)(m+1)(p+1)+i(p-1)+k\), where we have that \( i' \circ 1 \circ \tau \) must be equal to \( (\tau \otimes 3) \circ i' \circ (\beta' \otimes 1 \beta'), \) which is exactly our equation (18).

Let \( \Theta_0, \ldots, \Theta_j \) for \( j \geq 0 \) be shuffles of \( \Phi, \Gamma, \) and \( H \) such that \( \Theta_0 = \Theta \) and \( \Theta_j = H \Phi \), and if \( j > 0 \), then for every \( 0 \leq i \leq j-1 \) we have that
\( \Theta_i = \Pi(f, 1)(g, 2) \Lambda \) and \( \Theta_{i+1} = \Pi(g, 2)(f, 1) \Lambda \), or \( \Theta_i = \Pi(g, 2)(h, 3) \) and \( \Theta_{i+1} = \Pi(h, 3)(g, 2) \Lambda \), or \( \Theta_i = \Pi(f, 1)(h, 3) \Lambda \) and \( \Theta_{i+1} = \Pi(h, 3)(f, 1) \Lambda \). We call \( \Theta_0, \ldots, \Theta_j \) a normalizing path starting with \( \Theta \). Its length is \( j \) and Lemma 4.1 still holds.

If \( \Theta_j = \Pi(f, 1)(g, 2) \Lambda \) and \( \Theta_{i+1} = \Pi(g, 2)(f, 1) \Lambda \), then for \( w \) being the target of the leftmost member of \( \Lambda \) of colour 3 we have that

\[
\varphi_i = \overline{\mathcal{M}}_{\Pi} \chi_{w}^{1,2}(f, g) \overline{\mathcal{M}}_{\Lambda}
\]

is a natural transformation from \( \overline{\mathcal{M}}_{\Theta_i} \) to \( \overline{\mathcal{M}}_{\Theta_{i+1}} \). We define \( \varphi_i \) analogously in the other two possibilities for the pair \( \Theta_i, \Theta_{i+1} \) relying on \( \chi_{u}^{2,3}(g, h) \) or \( \chi_{v}^{1,3}(f, h) \), for \( u \) being the target of the leftmost member of \( \Lambda \) of colour 2 and \( v \) being the target of the leftmost member of \( \Lambda \) of colour 1.

We define \( \varphi(\Theta_0, \ldots, \Theta_k) \) as in the two-fold case and for \( \Theta_0', \ldots, \Theta_j' \) being another normalizing path starting with \( \Theta \), we can show the following.

**Theorem 6.2.** \( \varphi(\Theta_0, \ldots, \Theta_j) = \varphi(\Theta_0', \ldots, \Theta_j') \).

**Proof.** We proceed by induction on \( j \geq 0 \). If \( j = 0 \), then \( \varphi(\Theta_0) = \varphi(\Theta_0') = 1 \).

If \( j > 0 \), then we are either in the situation as in the proof of Theorem 4.2 and we proceed analogously, or for some basic arrows \( f : n \to n', g : m \to m' \), and \( h : p \to p' \) of \( \Delta^o \) we have that

\[
\varphi_0 = \overline{\mathcal{M}}_{\Pi} \chi_{p'}^{1,2}(f, g) \overline{\mathcal{M}}_{(h, 3)\Lambda} \quad \text{and} \quad \varphi_0' = \overline{\mathcal{M}}_{\Pi(f, 1)} \chi_{n}^{2,3}(g, h) \overline{\mathcal{M}}_{\Lambda}.
\]

In the latter case, we use Lemma 6.1 and the induction hypothesis twice to obtain the following commutative diagram.

By Theorem 6.2, the following definition is correct.

**Definition.** Let \( \varphi_{\Theta} : \overline{\mathcal{M}}_{\Theta} \to \overline{\mathcal{M}}_{\Pi \Gamma \Phi} \) be \( \varphi(\Theta_0, \ldots, \Theta_j) \), for an arbitrary normalizing path \( \Theta_0, \ldots, \Theta_j \) starting with \( \Theta \).

**Definition.** A three-fold reduced bar construction \( \overline{\mathcal{M}} \) is defined on objects of \( (\Delta^o)^3 \) as:

\[
\overline{\mathcal{M}}(u, m, p) = \mathcal{M}^{n,m,p},
\]

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and on arrows of $(\Delta^p)^3$ as:
\[ \overline{WM}(f, g, h) = \overline{WM}_{H\Phi\varphi}, \]
where $\Phi$ is a sequence of colour 1 such that $\circ \Phi = f$, $\Gamma$ is a sequence of colour 2 such that $\circ \Gamma = g$, and $H$ is a sequence of colour 3 such that $\circ H = h$.

**Definition.** Let $(f_1, g_1, h_1)$, $(f_2, g_2, h_2)$ be a composable pair of arrows of $(\Delta^p)^3$. Let $\Phi_1$ and $\Phi_2$ be sequences of colour 1, let $\Gamma_1$ and $\Gamma_2$ be sequences of colour 2, and let $H_1$ and $H_2$ be sequences of colour 3 such that $\circ \Phi_1$ is $f_1^{nf}$, $\circ \Phi_2$ is $f_2^{nf}$, $\circ \Gamma_1$ is $g_1^{nf}$, $\circ \Gamma_2$ is $g_2^{nf}$, $\circ H_1$ is $h_1^{nf}$, and $\circ H_2$ is $h_2^{nf}$. We define a natural transformation
\[ \omega((f_1, g_1, h_1), (f_2, g_2, h_2)) : \overline{WM}(f_2, g_2, h_2) \circ \overline{WM}(f_1, g_1, h_1) \to \overline{WM}(f_2 \circ f_1, g_2 \circ g_1, h_2 \circ h_1), \]
\[ \omega((f_2, g_2, h_2), (f_1, g_1, h_1)) = df \varphi H_2 f_2 H_1 \Gamma_1 \varphi_1. \]

Let $\Phi_1$, $\Phi_2$, $\Gamma_1$, $\Gamma_2$, $H_1$, and $H_2$ be as in the definition above, and let $(f_3, g_3, h_3)$ be an arrow of $(\Delta^p)^3$ such that $(f_2, g_2, h_2)$ and $(f_3, g_3, h_3)$ are composable. Let $\Phi_3$, $\Gamma_3$, and $H_3$ be sequences of colour 1, 2, and 3, respectively, such that $\circ \Phi_3$ is $f_3^{nf}$, $\circ \Gamma_3$ is $g_3^{nf}$, and $\circ H_3$ is $h_3^{nf}$. By the definitions of $\omega$ and $\overline{WM}$, we have that
\[ \omega((f_3, g_3, h_3), (f_2, g_2, h_2)) \overline{WM}(f_1, g_1, h_1) = \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1, \]
\[ \overline{WM}(f_3, g_3, h_3) \omega((f_2, g_2, h_2), (f_1, g_1, h_1)) = \overline{WM} H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1. \]
Also,
\[ \omega((f_2, g_2, h_2), (f_1, g_1, h_1)) = \varphi \Gamma_2 \Gamma_1 \varphi_1 \Gamma_1 \varphi_1, \]
where $\circ \Phi'$ is $(f_2 \circ f_1)^{nf}$, $\circ \Gamma'$ is $(g_2 \circ g_1)^{nf}$, $\circ H'$ is $(h_2 \circ h_1)^{nf}$, and
\[ \omega((f_2, g_2, h_2), (f_1, g_1, h_1)) = \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1, \]
where $\circ \Phi''$ is $(f_2 \circ f_1)^{nf}$, $\circ \Gamma''$ is $(g_2 \circ g_1)^{nf}$ and $\circ H''$ is $(h_2 \circ h_1)^{nf}$.

By Theorem 6.2 we have that
\[ \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 \circ \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 = \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1, \]
\[ = \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 \circ \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1. \]
Hence, to prove that the three-fold reduced bar construction $\overline{WM}$, together with the natural transformation $\omega$, makes a lax functor from $(\Delta^p)^3$ to $\text{Cat}$ (see Diag 4.1 in Section 4), it suffices to show that
\[ (i) \quad \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 = \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 \quad \text{and} \]
\[ (ii) \quad \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1 = \varphi H_3 \Gamma_3 \varphi H_2 \Gamma_2 \varphi H_1 \Gamma_1 \varphi_1. \]
To prove $(i)$ and $(ii)$ we use the same arguments as in the two-fold case. Let $x$, $y$, and $z$ be three different elements of the set $\{1, 2, 3\}$ such that $x < y$. Note that the position of $(1, z)$ in the two shuffles of the lemma below is irrelevant; $(1, z)$ serves just to keep $\varphi$ correctly defined and to introduce the parameter $q$.

**Lemma 6.3.** If $\Phi$ and $\Phi'$ are sequences of colour $x$ such that $\circ \Phi = \circ \Phi'$ is a basic equation of $\Delta^p$, and $g$ is a basic arrow of $\Delta^p$, then for every $q \geq 0$ we have that $\varphi g((1, q), (1, z)) = \varphi g((q, y),(1, z))$. 

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Proof. Suppose the target of $\sigma \Phi$ is $n'$ and the target of $g$ is $m'$. If $x = 1$, $y = 2$, and $z = 3$, then we proceed as in Lemma 4.3 with all the cases modified so that the tuples representing the natural transformations are multiplied by the matrix $I_{n'} \otimes I_{m'} \otimes (1, \ldots, 1)$. For example, Case 1.1.1 now reads

$$\varphi_{(g,2)}(1_{q,3}) = (1^{(j-1)mq}, 1^q, \tau^q, 1^{(l-1)m-1}, \tau^q, I) = \varphi_{(g,2)}(1_{q,3}).$$

If $x = 2$, $y = 3$, and $z = 1$, we again proceed as in Lemma 4.3 with all the cases modified so that $\kappa$, $\beta$, $\tau$, and $\iota$ are replaced by $\kappa''$, $\beta''$, $\tau''$, and $\iota''$, and the tuples representing the natural transformations are multiplied by the matrix $(1, \ldots, 1) \otimes I_{n'} \otimes I_{m'}$. For example, Case 1.1.1 now reads

$$\varphi_{(g,3)}(1_{q,1}) = ((1^{(j-1)mq}, 1^q, \tau^q, 1^{(l-1)m-1}, \tau^q, I)^{\otimes q}) = \varphi_{(g,3)}(1_{q,1}).$$

If $x = 1$, $y = 3$, and $z = 2$, we modify all the cases of Lemma 4.3 so that $\kappa$, $\beta$, $\tau$, and $\iota$ are replaced by $\kappa''$, $\beta''$, $\tau''$, and $\iota''$, and the tuples representing the natural transformations are multiplied by the matrix $I_{n'} \otimes (1, \ldots, 1) \otimes I_{m'}$. For example, Case 1.1.1 now reads

$$\varphi_{(g,3)}(1_{q,2}) = (1^{(j-1)mq}, 1^q, \tau^q, 1^{(m-1)}, \tau^q, I) = \varphi_{(g,3)}(1_{q,2}).$$

By relying on Lemma 6.3, we can prove a lemma analogous to Lemma 4.4 and this suffices for the proof of (i) by induction on the sum of lengths of $H_1$ and $\Gamma_1$. We can prove (ii) in a dual manner. Hence, we have:

**Theorem 6.5.** The three-fold reduced bar construction $\mathcal{WM}$, together with the natural transformation $\omega$, makes a lax functor from $(\Delta^{op})^3$ to $\text{Cat}$.

### 7 The $n$-fold monoidal categories

The notion of $n$-fold monoidal category that we use in this paper is defined in [2] Section 7.6, under the name $n$-monoidal category. Before we define the notion of $(n + 1)$-fold monoidal category, for $n \geq 3$, we first define what the arrows between the $n$-fold monoidal categories are. For this inductive definition we assume that an $n$-fold monoidal category, for $n \geq 3$, is a category $\mathcal{M}$ equipped with $n$ monoidal structures $\langle \mathcal{M}, \otimes_1, I_1 \rangle, \ldots, \langle \mathcal{M}, \otimes_n, I_n \rangle$ such that for every $1 \leq k < l < m \leq n$, the category $\mathcal{M}$ with the structures $\langle \mathcal{M}, \otimes_k, I_k \rangle$, $\langle \mathcal{M}, \otimes_l, I_l \rangle$ and $\langle \mathcal{M}, \otimes_m, I_m \rangle$ is three-fold monoidal. Hence, for every $1 \leq k < l \leq n$, the category $\mathcal{M}$ with the structures $\langle \mathcal{M}, \otimes_k, I_k \rangle$, and $\langle \mathcal{M}, \otimes_l, I_l \rangle$ is two-fold monoidal. We denote by $\kappa_{k,l}: I_k \to I_l$, $\beta_{k,l}: I_k \to I_k \otimes_l I_l$, $\tau_{k,l}: I_l \otimes_k I_k \to I_l$ and $\iota_{k,l}: (A \otimes_l B) \otimes_k (C \otimes_l D) \to (A \otimes_k C) \otimes_l (B \otimes_k D)$ the required arrows of $\mathcal{M}$.

**Definition.** An $n$-fold monoidal functor between two $n$-fold monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a $(2n + 1)$-tuple $(F, \sigma^1, \delta^1, \ldots, \sigma^n, \delta^n)$, where for $k \in \{1, \ldots, n\}$,

$$\sigma^k_{A,B}: F A \otimes^D_k F B \to F(A \otimes^C_k B) \quad \text{and} \quad \delta^k: I^C_k \to F I^D_k$$
are arrows of $\mathcal{D}$ natural in $A$ and $B$, such that $(F, \sigma^k, \delta^k)$ is a monoidal functor between the $k$th monoidal structures of $\mathcal{C}$ and $\mathcal{D}$. Moreover, for every $1 \leq k < l \leq n$, the following four diagrams commute (with the superscripts $\mathcal{C}$ and $\mathcal{D}$ omitted):

Let $\text{Mon}_n(\text{Cat})$ be the 2-category whose 0-cells are the $n$-fold monoidal categories, 1-cells are the $n$-fold monoidal functors, and 2-cells are the $n$-fold monoidal transformations, i.e., monoidal transformations with respect to all $n$ structures. The monoidal structure of $\text{Mon}_n(\text{Cat})$ is again given by 2-products.

**Definition.** An $(n+1)$-fold monoidal category is a pseudomonoid in $\text{Mon}_n(\text{Cat})$.

By this inductive definition, it is clear that an $n$-fold monoidal category satisfies the assumptions given above, which we may take as an unfolded form of this definition. As in the case of two-fold and three-fold monoidal categories, we are only interested in $n$-fold strict monoidal categories, i.e., when the structures $\langle M, \otimes_1, I_1 \rangle, \ldots, \langle M, \otimes_n, I_n \rangle$ are strict monoidal.

It is easy to verify that every $n$-fold monoidal category in the sense of [3] is an $n$-fold strict monoidal category in our sense. Also, every braided monoidal category is a two-fold monoidal category, in our sense, with both monoidal structures being the same, with all the $i$ arrows obtained by braiding, and with $\kappa$ being the identity arrow. Similarly, for every $n$, a symmetric monoidal category is $n$-fold monoidal with all $n$ monoidal structures being the same.

On the other hand, it is not true that every $n$-fold strict monoidal category in our sense is an $n$-fold monoidal in the sense of [3]. It is not only the case that the difference would appear in arrows that involve the units. The arrows of the form

\[ A \otimes_i B \to A \otimes_j B \quad \text{and} \quad A \otimes_i B \to B \otimes_j A, \]

for $i < j$, show that the axiomatization of $n$-fold monoidal categories given in [3] leads to a non-conservative extension of its fragment without the units.
These arrows are not presumed by our definition. Hence, the categories would be different in their unit-free fragments too.

One important example where these interchanges between the monoidal structures appear not as a consequence of braiding or symmetry is the case of bicartesian categories, i.e., categories with all finite coproducts and products. If we denote the nullary and binary coproducts of a bicartesian category by 0 and +, and nullary and binary products by 1 and $\times$, then the unique arrows

$$\kappa: 0 \to 1, \quad \beta: 0 \to 0 \times 0, \quad \tau: 1 + 1 \to 1$$

of this category together with the arrows

$$\iota_{A,B,C,D}: (A \times B) + (C \times D) \to (A + C) \times (B + D),$$

which are canonical in the coproduct-product structures, guarantee that such a category may be conceived as a two-fold monoidal with the first monoidal structure given by + and 0, and the second given by $\times$ and 1. Furthermore, such a category may be conceived as an $n$-fold monoidal category in $n + 1$ different ways by taking first $0 \leq k \leq n$ monoidal structures to be the one given by + and 0, and the remaining $n - k$ monoidal structures to be the one given by $\times$ and 1.

As a consequence of this fact and the results of the next section, we have that there is an $\omega \times \omega$-indexed family of reduced bar constructions based on a bicartesian category (strictified in both monoidal structures). This is related to Adams’ remark on $E_\infty$ ring spaces given in [1, §2.7] where the bicartesian category $\text{FinSet}$ of finite sets and functions, with disjoint union as + and Cartesian product as $\times$, is mentioned. By Segal’s opinion “most fundamental $\Gamma$-space” arises from this category under disjoint union (see [20, §2]).

According to our results, it is possible to combine the disjoint union and Cartesian product in the category $\text{FinSet}$ to obtain other simplicial spaces. Since we have the initial (and a terminal) object in this category, its classifying space is contractible and all the other realizations of simplicial sets in question are path-connected. Hence, the induced $H$-space structures are grouplike, and there is no need for group completion when one starts to deloop $\text{FinSet}$ with respect to the disjoint union and then continue to deloop it with respect to Cartesian product.

8 The $n$-fold reduced bar construction

In Sections 4 and 6, we have defined the $n$-fold reduced bar construction for $n = 2$ and $n = 3$. We define, in the same manner, the $n$-fold reduced bar construction for $n = r > 3$. Let $\Phi^1, \ldots, \Phi^r$ be sequences of colours $1, \ldots, r$, respectively, such that for every $1 \leq k \leq r$, $\Phi^k: n_k \to n_k^t$. Let $\Theta$ be a shuffle of these $r$ sequences. Note that here, and in all the cases below, $k \in \{1, \ldots, r\}$ is just an upper index and not a power of $n$.

For every member $(f, k)$ of $\Theta$, we define its inner power and its outer power to be

$$\prod_{1 \leq j < k} t_j \quad \text{and} \quad \prod_{k < j \leq r} t_j,$$

respectively, where $t_l$ is the target of its right-closest member of $\Theta$ of colour $l$. We assume that the empty product is 1. This definition is in accordance with
the corresponding definitions for two and three-fold cases; the difference is that the powers always equal to 1 are not mentioned there.

Let \( \mathcal{M} \) be an \( r \)-fold strict monoidal category. We define a functor

\[
\overline{W}_\mathcal{M}_\Theta : \mathcal{M}^{n_1 \cdots n_r} \to \mathcal{M}^{n'_1 \cdots n'_r}
\]

in the following way: replace in \( \Theta \) every \((f,k)\) whose inner power is \( i \) and outer power is \( o \) by \( (\overline{W}_\mathcal{M}_k(f))^o \), and insert \( \circ \)'s.

For basic arrows \( f : n \to n' \) and \( g : m \to m' \) of \( \Delta^{op} \), \( 0 \leq k < l \leq r \), and \( u, v, w \geq 0 \), we define a natural transformation

\[
\chi_{\mathcal{M}^{n_m \omega}}^{k,l}(f,g) : (\overline{W}_\mathcal{M}_k^{m_n \omega}(f))^u \circ (\overline{W}_\mathcal{M}_l^{m_n \omega}(g))^v \to (\overline{W}_\mathcal{M}_k^{m_n \omega}(g))^{uv} \circ (\overline{W}_\mathcal{M}_l^{m_n \omega}(f))^u
\]

to be the identity natural transformation except in the following cases:

\[
\begin{array}{cccc}
\text{Type} & f & g & \chi_{\mathcal{M}^{n_m \omega}}^{k,l}(f,g) \\
\hline
s_j^{n+1} & s_j^{m+1} & ((1^{(m+1)uw}, 1_{k,l}^v, \mathbf{1})^u) \\
\hline
s_j^n, 1 \leq j \leq n - 1 & s_j^{m+1} & ((1^{(j-1)(m+1)uw}, 1_{k,l}^v, \mathbf{1})^u) \\
\hline
d_i^n, 1 \leq i \leq m - 1 & d_i^{m+1} & ((1^{(j-1)(m-1)uw}, 1_{k,l}^v, \mathbf{1})^u) \\
\hline
d_i^n, 1 \leq j \leq n - 1 & d_i^{m+1} & ((1^{(j-1)(m-1)uw}, 1_{k,l}^v, \mathbf{1})^u) \\
\end{array}
\]

Note that the tuples of the third column of the table above are obtained as a result of multiplication of the tuples in the third column of Table II (where \( \kappa, \beta, \tau, \) and \( \iota \) are replaced by \( \kappa_{k,l}, \beta_{k,l}, \tau_{k,l}, \text{ and } \iota_{k,l} \)) by the matrix

\[
\begin{pmatrix}
1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1 \\
\end{pmatrix}
\]

For the following lemma, which is analogous to Lemma 6.1, we assume that \( f : n \to n' \), \( g : m \to m' \), and \( h : p \to p' \) are basic arrows of \( \Delta^{op} \), that \( 1 \leq a < b < c \leq r \), that \( \Lambda \) is a shuffle of sequences of colours \( 1, \ldots, r \) with only identity arrows in it, and that

\[
u = \prod_{1 \leq i \leq a} t_i, \quad \nu_1 = \prod_{a < i \leq b} t_i, \quad \nu_2 = \prod_{b < i \leq c} t_i, \quad \nu = \prod_{b < i \leq r} t_i,
\]

where \( t_i \) is the target of the leftmost member of \( \Lambda \) of colour \( k \).

For example, if \( r = 7 \), \( a = 2 \), \( b = 4 \), \( c = 5 \), and

\[
\Lambda = (1_2, 1)(1_3, 2)(1_4, 3)(1_5, 4)(1_6, 5)(1_7, 6)(1_4, 7),
\]
then \( u = 2 \), \( v_1 = 3 \), \( v_2 = 1 \), and \( w = 20 \).

**Lemma 8.1.** _The following diagram commutes:_

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Definition. Let $\Theta$, the following theorem is proved in the same manner as Theorem 8.2, relying on Lemma 8.1 instead of Lemma 6.1.

Proof. The tuples representing the natural transformations of the left-hand side and the right-hand side of this diagram are obtained by multiplying the corresponding tuples of the diagram in Lemma 6.1 (where $\kappa$, $\kappa'$, and $\kappa''$ are replaced by $\kappa_{a,b}$, $\kappa_{b,c}$, and $\kappa_{a,c}$, etc.) by the matrix

$$
\begin{pmatrix}
(1,\ldots,1) \otimes I_{u'} \otimes (1,\ldots,1) \otimes I_{v'}(1,\ldots,1) \otimes I_{w'} \otimes (1,\ldots,1).
\end{pmatrix}
$$

Hence, Lemma 6.1 directly implies this lemma.

Let $\Theta_0,\ldots,\Theta_j$, for $j \geq 0$, be shuffles of $\Phi^1,\ldots,\Phi^r$ such that $\Theta_0 = \Theta$ and $\Theta_j = \Phi^r \cdots \Phi^1$, and if $j > 0$, then for every $0 \leq i \leq j - 1$ we have that for some $1 \leq k < l \leq r$, $\Theta_i = \Pi(f,k)(g,l)\Lambda$ and $\Theta_{i+1} = \Pi(g,l)(f,k)\Lambda$. We call $\Theta_0,\ldots,\Theta_j$ a normalizing path starting with $\Theta$. Its length is $j$ and Lemma 4.1 still holds.

For $u$, $v$, and $w$ being respectively

$$
\prod_{1 \leq z < k} t_z, \prod_{k < z < l} t_z, \prod_{l < z \leq r} t_z,
$$

where $t_z$ is the target of the leftmost member of $\Lambda$ of colour $z$, we have that

$$
\varphi_\Lambda = \WM_{\Pi} \chi^{k,l}_{u,v,w}(f,g) \WM_{\Lambda},
$$

is a natural transformation from $\WM_{\Theta_j}$ to $\WM_{\Theta_{j+1}}$. We define $\varphi(\Theta_0,\ldots,\Theta_j)$ as in the two-fold case and for $\Theta'_0,\ldots,\Theta'_j$ being another normalizing path starting with $\Theta$, the following theorem is proved in the same manner as Theorem 6.2, relying on Lemma 8.1 instead of Lemma 6.1.

**Theorem 8.2.** $\varphi(\Theta_0,\ldots,\Theta_j) = \varphi(\Theta'_0,\ldots,\Theta'_j)$.

By Theorem 8.2, the following definition is correct.

**Definition.** Let $\varphi_\Theta : \WM_{\Theta} \to \WM_{\Theta',\Theta''}$ be $\varphi(\Theta_0,\ldots,\Theta_j)$, for an arbitrary normalizing path $\Theta_0,\ldots,\Theta_j$ starting with $\Theta$.

Note again that in the following definition, $1, 2, \ldots, r$ are the upper indices and not the powers of $n$:

**Definition.** An $r$-fold reduced bar construction $\WM$ is defined on objects of $(\Delta^p)^r$ as:

$$
\WM(n^1,\ldots,n^r) = \mathcal{M}^{n^1 \cdots n^r},
$$

26
and on arrows of \((\Delta^op)'\) as:

\[
\overline{\mathcal{M}}(f^1, \ldots, f') = \mathcal{M}_{\Phi^0 \cdots \Phi^1},
\]

where \(\Phi^i\) is a sequence of colour \(i\) such that \(\circ \Phi^i = f^i\).

**Definition.** Let \((f_1^1, \ldots, f_1^r), (f_2^1, \ldots, f_2^r)\) be a composable pair of arrows of \((\Delta^op)'\) and let for every \(i \in \{1, \ldots, r\}\) and every \(j \in \{1, 2\}\), the sequence \(\Phi^i_j\) of colour \(i\) be such that \(\circ \Phi^i_j = (f^i_j)^{nf}\). We define a natural transformation

\[
\omega(f_1^1, \ldots, f_1^r), (f_2^1, \ldots, f_2^r) : \overline{\mathcal{M}}(f_2^1, \ldots, f_2^r) \circ \overline{\mathcal{M}}(f_1^1, \ldots, f_1^r) \to \overline{\mathcal{M}}(f_2^1 \circ f_1^1, \ldots, f_2^r \circ f_1^1),
\]

\[
\omega(f_2^1, \ldots, f_2^r), (f_1^1, \ldots, f_1^r) = d' \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2.
\]

Let \(\Phi^1_1, \ldots, \Phi^1_r\) and \(\Phi^2_1, \ldots, \Phi^2_r\) be as above, and let \((f_1^1, \ldots, f_1^r)\) be an arrow of \((\Delta^op)'\) such that \((f_2^1, \ldots, f_2^r)\) and \((f_1^1, \ldots, f_1^r)\) are composable. Let \(\Phi^1_1, \ldots, \Phi^1_r\) be sequences of colour \(1, \ldots, r\), respectively, such that \(\circ \Phi^1_i = (f^1_i)^{nf}\). By the definitions of \(\omega\) and \(\overline{\mathcal{M}}\), we have that

\[
\omega(f_1^1, \ldots, f_1^r), (f_2^1, \ldots, f_2^r) \overline{\mathcal{M}}(f_1^1, \ldots, f_1^r) = \Phi^1_1 \cdots \Phi^1_r \cdots \Phi^1_r \overline{\mathcal{M}}(f_1^1, \ldots, f_1^r)
\]

and

\[
\overline{\mathcal{M}}(f_1^1, \ldots, f_1^r) \omega(f_2^1, \ldots, f_2^r), (f_1^1, \ldots, f_1^r) = \overline{\mathcal{M}} \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2 \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2.
\]

Also,

\[
\omega(f_2^1 \circ f_1^1, \ldots, f_2^r \circ f_1^1), (f_1^1, \ldots, f_1^r) = \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i
\]

where \(\circ \Phi^i_i = (f^i_i)^{nf}\), and

\[
\omega(f_2^1, \ldots, f_2^r), (f_1^1 \circ f_1^1, \ldots, f_1^1 \circ f_1^1) = \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i \Phi^1_2 \cdots \Phi^1_i \cdots \Phi^1_i,
\]

where \(\circ \Phi^i_i = (f^i_i)^{nf}\).

By Theorem 8.2 we have that

\[
\overline{\mathcal{M}} \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i = \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i \cdots \Phi^1_i \circ \overline{\mathcal{M}} \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i
\]

Hence, to prove that the \(r\)-fold reduced bar construction \(\overline{\mathcal{M}}\), together with the natural transformation \(\omega\), makes a lax functor from \((\Delta^op)'\) to \(\mathcal{C}at\) (see Diag 4.1 in Section 4), it suffices to show that

\[
(i) \quad \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i = \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i \quad \text{and}
\]

\[
(ii) \quad \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i = \Phi^1_2 \cdots \Phi^1_2 \cdots \Phi^1_i \cdots \Phi^1_i
\]

To prove \((i)\) and \((ii)\) we use the same arguments as in the two-fold case. Let \(\Lambda\) be a shuffle of sequences of colours \(1, \ldots, r\) with only identity arrows in it. Let \(1 \leq k < l \leq r\) and let \(u, v, w\) be respectively

\[
\prod_{1 \leq z < k} t_z, \quad \prod_{k < z < l} t_z, \quad \prod_{l < z \leq r} t_z,
\]

where \(t_z\) is the target of the leftmost member of \(\Lambda\) of colour \(z\).

**Lemma 8.3.** If \(\Phi\) and \(\Phi'\) are sequences of colour \(k\) such that \(\circ \Phi = \circ \Phi'\) is a basic equation of \(\Delta^op\), and \(g\) is a basic arrow of \(\Delta^op\), then we have that \(\Phi_{(g, l)}\Lambda = \Phi'_{(g, l)}\Lambda\).
Proof. Suppose the target of \( \Phi \) is \( n' \) and the target of \( g \) is \( m' \). We proceed as in Lemma 4.3 with all the cases modified so that \( \kappa, \beta, \tau, \) and \( \iota \) are replaced by \( \kappa_{k,l}, \beta_{k,l}, \tau_{k,l}, \) and \( \iota_{k,l}, \) and the tuples representing the natural transformations are multiplied by the matrix

\[
(1, \ldots, 1) \otimes I_{n'} \otimes (1, \ldots, 1) \otimes I_{m'} \otimes (1, \ldots, 1).
\]

So, for example, Case 1.1.1 now reads

\[
\varphi_{(g,l)\lambda} = \varphi_{\Phi(g,l)\lambda} - ( (1, b(m-w), v, (1, c(w-i(m-w)), (1, c(w-i(m-w)), (1, c(w-i(m-w)))), (1, c(w-i(m-w))))).
\]

By relying on Lemma 8.3, we can prove a lemma analogous to Lemma 4.4 and this suffices for the proof of (i) by induction on the sum of lengths of \( \Phi_1', \ldots, \Phi_r' \). We can prove (ii) in a dual manner. So, we have:

**Theorem 8.5.** The \( r \)-fold reduced bar construction \( \overline{W}M \), together with the natural transformation \( \omega \), makes a lax functor from \((\Delta^\text{op})^r\) to \( \text{Cat} \).

We see, by analyzing this result, that the conditions imposed by the definition of \( n \)-fold monoidal categories are not only sufficient, but they are also necessary for the proof of correctness of the \( n \)-fold reduced bar construction. If one proves this through the steps established by our Theorem 8.2 and Lemmata analogous to Lemma 8.3, then all the combinatorial structure of \( n \)-fold monoidal categories is used.

For \( m \geq 1 \), consider the arrows \( i_1, \ldots, i_m : m \to 1 \) of \( \Delta^\text{op} \) given by the following diagrams.

For functors \( F_i : A \to B_i \), \( 1 \leq i \leq m \), let \( \{F_1, \ldots, F_m\} : A \to B_1 \times \ldots \times B_m \) be the functor obtained by the Cartesian structure of \( \text{Cat} \). It is easy to verify that for every \( l \in \{0, \ldots, r-1\} \) and every \( k \geq 0 \), the functor \( W : \Delta^\text{op} \to \text{Cat} \) defined as \( \overline{W}M \) is such that

\[
(W(i_1), \ldots, W(i_m)) : W(m) \to (W(1))^m
\]

is the identity. This means that \( \overline{W}M \) is Segal’s lax functor according to the definition given in [19, Section 4]. Let \( V \) be a rectification of \( \overline{W}M \) obtained by [22, Theorem 2]. Following the results of [19], up to group completion, this is sufficient for the realization of \( B \circ V \) to be an \( n \)-fold delooping of the classifying space \( BM \) of \( \mathcal{M} \).

**9 Appendix**

By the definition given in Section 2, a two-fold monoidal category consists of the following:
1. a monoidal category \(\mathcal{M} \otimes I_1, \alpha_1, \rho_1, \lambda_1\) (here \(\alpha_1, \rho_1,\) and \(\lambda_1,\) respectively, denote associativity, right and left identity natural isomorphisms),

2. monoidal functors \(\otimes_2: \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) and \(I_2: 1 \to \mathcal{M},\)

3. monoidal transformations \(\alpha_2, \rho_2,\) and \(\lambda_2\) such that \((\mathcal{M}, \otimes_2, I_2, \alpha_2, \rho_2, \lambda_2)\) satisfies the pseudomonoid conditions (i.e., the equations of a monoidal category).

That \(\otimes_2\) is a monoidal functor means that there is a natural transformation \(\iota\) given by the family of arrows

\[
i_{A,B,C,D}: (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D),
\]

and an arrow \(\beta: I_1 \to I_1 \otimes_2 I_1\) such that the following three diagrams commute:

\[
\begin{align*}
\xymatrix{
(A \otimes_1 (B \otimes_1 C)) \otimes_2 (D \otimes_1 (E \otimes_1 F)) & ((A \otimes_1 B) \otimes_1 C) \otimes_2 ((D \otimes_1 E) \otimes_1 F) \\
1 \otimes_1 \iota & \iota \\
(A \otimes_2 D) \otimes_1 ((B \otimes_1 C) \otimes_2 (E \otimes_1 F)) & ((A \otimes_1 B) \otimes_2 (D \otimes_1 E)) \otimes_1 (C \otimes_2 F) \\
\iota \otimes_1 \iota & 1 \\
(A \otimes_2 D) \otimes_1 ((B \otimes_2 E) \otimes_1 (C \otimes_2 F)) & ((A \otimes_2 D) \otimes_1 (B \otimes_2 E)) \otimes_1 (C \otimes_2 F) \\
\alpha_1 & \alpha_1
}
\end{align*}
\]

\[
\begin{align*}
\xymatrix{
(A \otimes_2 B) \otimes_1 (I_1 \otimes_2 I_1) & (A \otimes_1 I_1) \otimes_2 (B \otimes_1 I_1) \\
1 \otimes_1 \beta & 1 \\
(A \otimes_2 B) \otimes_1 I_1 & A \otimes_2 B \\
\rho_1 & \rho_1
}
\end{align*}
\]

\[
\begin{align*}
\xymatrix{
(I_1 \otimes_2 I_1) \otimes_1 (A \otimes_2 B) & (I_1 \otimes_1 A) \otimes_2 (I_1 \otimes_1 B) \\
\beta \otimes_1 1 & \lambda_1 \otimes_2 \lambda_1 \\
I_1 \otimes_1 (A \otimes_2 B) & A \otimes_2 B \\
\lambda_1 & \lambda_1
}
\end{align*}
\]

That \(I_2\) is a monoidal functor means that there are arrows \(\tau: I_2 \otimes_1 I_2 \to I_2\) and \(\kappa: I_1 \to I_2\) such that the following diagrams commute:

\[
\begin{align*}
\xymatrix{
I_2 \otimes_1 (I_2 \otimes_1 I_2) & (I_2 \otimes_1 I_2) \otimes_1 I_2 \\
1 \otimes_1 \tau & \iota \otimes_1 1 \\
I_2 \otimes_1 I_2 & I_2 \otimes_1 I_2 \\
\tau & \tau
}
\end{align*}
\]

\[
\begin{align*}
\xymatrix{
I_2 \otimes_1 I_1 & I_2 \otimes_1 I_2 \\
I_2 \otimes_1 I_2 & I_2 \otimes_1 I_2 \\
\kappa \otimes_1 1 & \kappa \otimes_1 1
}
\end{align*}
\]

\[
\begin{align*}
\xymatrix{
I_2 \otimes_1 I_1 & I_2 \otimes_1 I_2 \\
I_2 \otimes_1 I_2 & I_2 \otimes_1 I_2 \\
\rho_1 & \rho_1
}
\end{align*}
\]

\[
\begin{align*}
\xymatrix{
I_2 \otimes_1 I_2 & I_2 \otimes_1 I_2 \\
I_2 \otimes_1 I_2 & I_2 \otimes_1 I_2 \\
\lambda_1 & \lambda_1
}
\end{align*}
\]
That $\alpha_2$ is a monoidal transformation means that the following diagrams commute:

$$(A \otimes_2 (B \otimes_2 C)) \otimes_1 (D \otimes_2 (E \otimes_2 F)) \xrightarrow{\alpha_2 \otimes_1 \alpha_2} ((A \otimes_2 B) \otimes_2 C) \otimes_1 ((D \otimes_2 E) \otimes_2 F)$$

That $\rho_2$ is a monoidal transformation means that the following diagrams commute:

$$(A \otimes_2 (B \otimes_2 I_2)) \otimes_1 (B \otimes_2 (I_2 \otimes_2 I_2)) \xrightarrow{\rho_2 \otimes_1 \rho_2} ((A \otimes_1 B) \otimes_2 (I_2 \otimes_2 I_2)$$

Finally, that $\lambda_2$ is a monoidal transformation means that the following diagrams commute:

$$(I_2 \otimes_2 A) \otimes_1 (I_2 \otimes_2 B) \xrightarrow{\lambda_2 \otimes_1 \lambda_2} ((I_2 \otimes_1 I_2) \otimes_2 (A \otimes_1 B)$$
References

[1] J.F. Adams, *Infinite loop spaces*, Annals of Mathematics Studies vol. 90, Princeton University Press, Princeton, 1978

[2] M. Aguiar and S. Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010

[3] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt, *Iterated monoidal categories*, *Advances in Mathematics*, vol. 176 (2003), pp. 277-349

[4] M. Batanin and M. Markl, *Centers and homotopy centers in enriched monoidal categories*, *Advances in Mathematics*, vol. 230 (2012), pp. 1811-1858

[5] T. Booker and R. Street, *Tannaka duality and convolution for duoidal categories*, *Theory and Applications of Categories*, vol. 28 (2013), pp. 166-205

[6] F. Borceux, *Handbook of Categorical Algebra I*, Cambridge University Press, Cambridge, 1994

[7] K. Došen and Z. Petrić, *Coherence for modalities*, *Journal of Pure and Applied Algebra*, vol. 215 (2011), pp. 1606-1628 [arXiv:0809.2494]

[8] ———, *Intermutation*, *Applied Categorical Structures*, vol. 20 (2012), pp. 43-95 [arXiv:math/0701325]

[9] G. Dunn, *E_\infty*-monoidal categories and their group completions, *Journal of Pure and Applied Algebra*, vol. 95 (1994), pp. 27-39

[10] R. Garner *Understanding the small object argument*, *Applied Categorical Structures*, vol. 17 (2009), pp. 247-285

[11] A. Joyal and R. Street, *Braided tensor categories*, *Advances in Mathematics*, vol. 102 (1993), pp. 20-78

[12] G.M. Kelly and S. Mac Lane, *Coherence in closed categories*, *Journal of Pure and Applied Algebra*, vol. 1 (1971), pp. 97-140, 219

[13] S. Mac Lane, *Natural associativity and commutativity*, *Rice University Studies, Papers in Mathematics*, vol. 49 (1963), pp. 28-46

[14] ———, *Categories for the Working Mathematician*, Springer, Berlin, 1971 (expanded second edition, 1998)

[15] J.P. May, *E_\infty*-spaces, group completions, and permutative categories, *New Developments in Topology* (G. Segal, editor), London Mathematical Society Lecture Note Series, vol. 11, Cambridge University Press, Cambridge, 1974, pp. 61-93
[16] J.E. McClure and J.H. Smith, *Operads and cosimplicial objects: An introduction*, *Axiomatic, Enriched and Motivic Homotopy Theory* (J.P.C. Greenlees, editor), Kluwer Academic Publishers, Dordrecht, 2002, pp. 133-171

[17] D. McDuff and G. Segal, *Homology fibrations and the “group-completion” theorem*, *Inventiones mathematicae*, vol. 31 (1976), pp. 279-284

[18] Z. Petrić and T. Trimble, *Symmetric bimonoidal intermuting categories and $\omega \times \omega$ reduced bar constructions*, *Applied Categorical Structures*, vol. 22 (2014), pp. 467-499 (arXiv:0906.2954)

[19] Z. Petrić, *Segal’s multisimplicial spaces*, preprint (2014) (arXiv:1407.3914)

[20] G. Segal, *Categories and cohomology theories*, *Topology*, vol. 13 (1974), pp. 293-312

[21] J.D. Stasheff, *Homotopy associativity of H-spaces, I, II*, *Transactions of the American Mathematical Society*, vol. 108 (1963), pp. 275-292, 293-312

[22] R. Street, *Two constructions on lax functors*, *Cahiers de topologie et géométrie différentielle*, vol. 13 (1972), pp. 217-264

[23] R.W. Thomason, *Homotopy colimits in the category of small categories*, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 85, 91 (1979), pp. 91-109