The landscape of bounds for binary search trees

Parinya Chalermsook\(^1\), Mayank Goswami\(^1\), László Kozma\(^2\), Kurt Mehlhorn\(^1\), and Thatchaphol Saranurak\(^3\)

\(^1\) Max-Planck Institute for Informatics, Germany. \{parinya,mgayank,mehlhorn\}@mpi-inf.mpg.de
\(^2\) Saarland University, Germany. kozma@cs.uni-saarland.de
\(^3\) KTH Royal Institute of Technology, Sweden. thasar@kth.se

Abstract

Binary search trees (BSTs) with rotations can adapt to various kinds of structure in search sequences, achieving amortized access times substantially better than the \(\Theta(\log n)\) worst-case guarantee. Classical examples of structural properties include static optimality, sequential access, working set, key-independent optimality, and dynamic finger, all of which are now known to be achieved by the two famous online BST algorithms (Splay and Greedy). Beyond the insight on how “efficient sequences” might look like, structural properties are important as stepping stones towards proving or disproving dynamic optimality, the elusive 1983 conjecture of Sleator and Tarjan that postulates the existence of an asymptotically optimal online BST. A BST can be optimal only if it satisfies all “sound” properties (those achieved by the offline optimum).

In this paper, we introduce novel properties that explain the efficiency of sequences not captured by any of the previously known properties, and which provide new barriers to dynamic optimality. We also establish connections between various properties, old and new. For instance, we show the following.

- A tight bound of \(O(n \log d)\) on the cost of Greedy for \(d\)-decomposable sequences, improving our earlier \(n^{2O(d^2)}\) bound (FOCS 2015). The result builds on the recent lazy finger result of Iacono and Langerman (SODA 2016). On the other hand, we show that lazy finger alone cannot explain the efficiency of pattern avoiding sequences even in some of the simplest cases.
- A hierarchy of bounds using multiple lazy fingers, addressing a recent question of Iacono and Langerman.
- The optimality of the Move-to-root heuristic in the key-independent setting introduced by Iacono (Algorithmica 2005).
- A new tool that allows combining any finite number of sound structural properties. As an application, we show an upper bound on the cost of a class of sequences that all known properties fail to capture.
- The equivalence between two families of BST properties. The observation on which this connection is based was known before – we make it explicit, and apply it to classical BST properties. This leads to a clearer picture of the relations between BST properties and to a new proof of several known properties of Splay and Greedy that is arguably more intuitive than the current textbook proofs.

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

1 Introduction

In the dynamic BST model a sequence of keys are accessed in a binary search tree, and after each access, the tree can be reconfigured via a sequence of rotations and pointer moves starting from the root. (There exist several alternative but essentially equivalent cost models, see e.g. \([1][34]\).) Two classical online algorithms in this model are the Splay tree of Sleator and Tarjan \([31]\) and Greedy, an algorithm discovered independently by Lucas \([21]\) and Munro \([26]\) and turned into an online algorithm by Demaine al. \([1]\).
Our understanding of the BST model goes far beyond the usual paradigm of worst-case complexity. For broad classes of access sequences the $\Theta(\log n)$ worst-case bound is too pessimistic, and both Splay and Greedy are able to achieve better amortized access times. Understanding the kinds of structure in sequences that facilitate efficient access has been the main focus of BST research in the past decades. The description of useful structure is typically given in the form of formulaic bounds.

Given an access sequence $X = (x_1, \ldots, x_m) \in [n]^m$, a formulaic BST bound (or simply BST bound) is a function $f(X)$ computable in polynomial time, intended to capture the access cost of BST algorithms on sequence $X$. We say that a BST bound $f(X)$ is sound if $\text{OPT}(X) \leq O(f(X) + |X|)$ where $\text{OPT}(X)$ is the optimal cost achievable by an offline algorithm. The bound $f(X)$ is achieved by algorithm $A$ if the cost of accessing sequence $X$ by $A$ (denoted $A(X)$) is at most $O(f(X) + |X|)$. BST bounds play two crucial roles:

(i) They shed light on the structures that make sequences efficiently accessible by BST algorithms. For instance, the dynamic finger bound $\sum_i \log |x_i - x_{i+1} + 1|$ intuitively captures the “encoding length” of the distances between consecutive accesses (algorithms can take advantage of the proximity of keys).

(ii) A sound BST bound is a concrete intermediate step towards the dynamic optimality conjecture \cite{31}, which postulates that a simple online algorithm can asymptotically match the optimum on every access sequence, i.e. that it can be $O(1)$-competitive. This has been conjectured for both Splay and Greedy, but the conjecture remains unsettled after decades of research. An $O(1)$-competitive algorithm needs to achieve all sound BST bounds. Proposing concrete bounds and verifying whether candidate algorithms such as Splay or Greedy achieve them has been so far the main source of progress towards dynamic optimality.

Several such bounds appear in the literature: besides the classical dynamic finger \cite{9,31}, working set \cite{19,31}, unified bound \cite{14,31}, etc. recently studied bounds include lazy finger and weighted dynamic finger \cite{4,20}, and bounds pertaining to pattern-avoidance \cite{7}. In some cases the interrelation between these bounds is unclear (i.e. whether one subsumes the other).

**Our contributions.** In this paper we systematically organize the known bounds into a coherent picture, and study the pairwise relations between bounds. We introduce new sound BST bounds (in fact, a hierarchy of them). Some of these bounds serve as bridges between existing bounds, whereas others explain the easiness of certain sequences, hitherto not captured by any known bound. (We only focus on bounds defined on access sequences, we ignore therefore the deque \cite{32}, and split \cite{22} conjectures, that concern other operations.)

We highlight in this section the contributions that we find most interesting, with an informal discussion of their implications. We refer to §2 for a more precise definition of the bounds considered in this paper. Our current knowledge of sound BST bounds and their relations, i.e. the “landscape” of BST bounds is presented in Figure 2 In the following, let $X \in [n]^m$ be an arbitrary access sequence.

**Lazy finger results.** Our first set of contributions is a study of lazy finger bounds, their generalizations, and their connections with other BST bounds. The lazy finger bound \cite{4}, denoted $LF(X)$, captures the “proximity” of successive accesses in a reference tree. Bose et al. \cite{4} proved that lazy finger generalizes the classical dynamic finger bound. It was recently shown \cite{20} that $\text{GREEDY}(X) \leq O(LF(X))$. We prove a new connection between lazy finger and a recently studied \cite{7} decomposability parameter $d(X)$.

**Theorem 1.** For permutation sequence $X \in [n]^n$ and decomposability parameter $d = d(X)$, we have $LF(X) \leq O(n \log d)$.

As a corollary, we obtain the tight bound $\text{GREEDY}(X) \leq O(n \log d)$ which improves the earlier bound of $O(n^2 O(d^2))$ and resolves an open question from \cite{7}. We remark that $d(X)$ is a
natural parameter whose special case \( d(X) = 2 \) includes the well-known traversal sequences.

Next, inspired by \[13\], we define the \( k \)-lazy finger parameter \( LF^k(X) \).

**Theorem 2.** Let \( X \in [n]^m \) be a sequence and \( k \in \mathbb{N} \). Then \( OPT(X) \leq O(\log k) \cdot LF^k(X) \).

This improves the \( O(k) \cdot LF^k(X) \) bound, which is implicit in \[13\]. Moreover, our bound is tight in the sense that there exists \( X \) for which \( OPT(X) \geq \Omega(\log k) \cdot LF^k(X) \). Remark that \( LF^1(X) \geq LF^2(X) \geq \ldots \geq LF^n(X) \), giving a hierarchy of sound BST bounds. For \( k \geq 2 \), this bound is not known to be achieved by any online algorithm.

The bounds in the \( k \)-lazy finger hierarchy are not implied by each other. In fact, we show a strongest possible separation between \( LF^k \) and \( LF^{k-1} \). That is, for any \( k \), there is a sequence \( X \) for which \( LF^{k-1}(X)/LF^k(X) \geq \Omega((\log (n/k))) \). This result yields a large number of intermediate steps towards dynamic optimality: A candidate algorithm not only needs to achieve the finger bounds for constantly many fingers, but also has to match the asymptotic ratio of \( O(\log k) \).

We show an application of multiple lazy fingers by giving a new upper bound on \( OPT \): Let \( m(X) \) denote the monotone complexity parameter of \( X \). In \[7\], we showed that \( greedy(X) \leq |X| \cdot 2^{O(k^2)} \) for \( k = m(X) \). Here we show that \( OPT(X) \leq O(\log k) \cdot LF^k(X) \leq O(k \log k) \cdot |X| \), raising the open question of whether there is any online algorithm matching this bound.

**Interleave results.** We introduce a simulation technique that allows combining any finite number of sound BST bounds.

Let \( X^{(1)}, \ldots, X^{(\ell)} \) be a collection of sequences where \( X^{(i)} \in [n_i]^m \); (each \( X^{(i)} \) is a sequence of length \( m_i \) on key space of size \( n_i \)). We consider a natural way to compose these sequences by using a sequence \( \tilde{X} \in [\ell]^m \) as a template, where \( m = \sum_i m_i \) and \( n = \sum_i n_i \). The “composed sequence” \( S = (S_1, \ldots, S_m) \in [n]^m \) is defined as \( S_i = X^{(X_{t_i})} + N_i \), where \( N_i = \sum_{i=1}^{X_{t_i}-1} n_i \), and \( \sigma(t) = |\{t' \leq t : \tilde{X}_{t'} = X_{t_i}\} | \). Denote the composed sequence by \( X = \tilde{X}\{X_1, \ldots, X_{\ell}\} \). Intuitively, the sequences are interleaved spatially, and the order in which they produce the next element of the composed sequence is governed by the template sequence. See Figure 1 for illustration.

**Theorem 3.** Let \( X = X_1, \ldots, X_{\ell} \), and let \( f, f_1, \ldots, f_\ell \) be sound BST bounds. Then \( f(X) = f(\tilde{X}) + \sum_{i=1}^{\ell} f_i(X_i) \) is also a sound BST bound.

This result allows us to analyze the optimum of natural classes of sequences whose easiness was not implied by any of the known bounds.

**Key-independent setting.** We revisit the key-independent setting, in which Iacono showed that optimality is equivalent to working set \[19\]. We show the following.

**Theorem 4.** In the key-independent case Move-to-root is optimal.

This may seem surprising, since Move-to-root is a rather simple heuristic, not guaranteed to achieve even sublinear amortized access time. The result is nevertheless consistent with intuition, since in other key-independent models (e.g. list update problem) heuristics similar to Move-to-root have already been known to be asymptotically optimal \[30\]. Moreover, in such key-independent problems, “useful structure” has typically been described via bounds resembling the working set bound (see e.g. \[1,2,27\]), which in the key-independent BST case is indeed the “full story” of optimality.

**Relations between classical bounds.** We make explicit the equivalence between two popular notions in BST bounds, namely, information-theoretic proximity with weighted elements (such as in weighted dynamic finger) and proximity of keys in a reference tree (such

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**Figure 1** The sequence \( X = (1,5,3,2,4) \) (left) resulting from composing the sequences \( X^{(1)} = (1,3,2) \) (circles) and \( X^{(2)} = (2,1) \) (squares) with the template \( X = (1,2,1,1,2) \) (right).
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as in static optimality and lazy finger). This equivalence has implicitly appeared several times in the literature, but it has not been applied to some of the classical BST bounds.

By making the connection explicit, the landscape of known bounds becomes clearer. In particular, the following become obvious: (i) static optimality is just an access lemma with fixed weight function, and (ii) static finger is an unweighted version of static optimality. Using these observations, we prove some of the known properties of Splay and Greedy in a way that is arguably simpler and more intuitive than existing textbook proofs.

Open problems. Some of the sound BST bounds presented in the paper are not known to be achieved by online algorithms. In particular, does any online algorithm achieve the $k$-lazy finger (times $O(\log k)$) bound when $k \geq 2$? Does any online algorithm achieve the interleave bound? Are there broad classes of linear cost sequences not captured by any of the known bounds? Does any online algorithm achieve the bound of $O(k \log k)|X|$, where $k = m(X)$? These questions serve as concrete intermediate steps for proving or disproving dynamic optimality.

Our result for the decomposability parameter of a sequence is tight. The bounds for general pattern avoidance and monotone pattern parameter (defined in §2) are not known to be tight.

There exist other ways of composing sequences (different from the operation used in our interleave bound). Do these operations similarly lead to composite BST bounds? In particular, if $X$ is the merge of $X_1$ and $X_2$ (i.e. $X$ can be partitioned into two subsequences $X_1$ and $X_2$), does a linear cost of both $X_1$ and $X_2$ imply the linear cost of $X$?

2 Dictionary of BST bounds

In this section we list the BST bounds considered in the paper, marking those that are new with *. Let $S = (s_1, \ldots, s_m) \in [n]^m$ be an access sequence. A weight function $w : [n] \rightarrow \mathbb{R}^+$ maps elements to positive reals. For convenience, denote $w[i_1 : i_2] = \sum_{i = \min(i_1, i_2)}^{\max(i_1, i_2)} w(i)$, let $W = w[1 : n]$, and $w(X) = \sum_{i \in X} w(i)$ for any set $X \subset [n]$. The following bounds can be defined for any access sequence $S \in [n]^m$.

Basic bounds.

Balance: The balance bound is $B(S) = m \log n$. It describes the fact that accesses take amortized $O(\log n)$ time. We can generalize it with weights as follows.

Weighted Balance*: For any weight function $w$, let $WB_w(S) = \sum_{j = 1}^m \log \frac{W[j]}{w(s_j)}$. The weighted balance bound is $WB(S) = \min_w WB_w(S)$. Note that for the uniform weight function $w_i = 1$, $WB_w(S) = B(S)$. The reader might observe the similarity of the weighted balance bound with the access lemma, a statement that bounds the amortized cost of a single access. The access lemma has been used to prove properties of Splay and other algorithms [6,7,31]. We observe that matching the bound $WB$ is a weaker condition than satisfying the access lemma, since here the weights are fixed throughout the sequence of accesses, whereas the access lemma makes no such assumption.

Entropy: Let $m_i$ be the number of times element $i$ is accessed. The entropy bound [31] is $H(S) = \sum_{i=1}^n m_i \log \frac{m_i}{m}$. The weighted entropy bound [31] is $H_w(S) = \sum_{i=1}^n m_i \log \frac{w[i]}{m}$. The weighted entropy bound* is $H_w(S) = \sum_{i=1}^n m_i \log \frac{w[i]}{m_i}$.

Locality in keyspace.

Static Finger: this bound depends on the distances from a fixed key. For an arbitrary element $f \in [n]$, let $SF_f(S) = \sum_{j = 1}^m |f - s_j| + 1$. The static finger bound [31] is $SF(S) = \min_f SF_f(S)$. We define a weighted version as follows.

For any weight function $w$ and any element $f$, let $WSF_{w,f}(S) = \sum_{j = 1}^m \log \frac{w[f,s_j]}{w[f,s_j]}$. The weighted static finger* bound is $WSF(S) = \min_w WSF_w(S)$. 


Figure 2 BST bounds and relations between them. Each box represents a BST bound. Arrows indicate asymptotic domination: the source of the arrow is “smaller” than the target. Bounds grouped into larger boxes are asymptotically equivalent. Shaded boxes indicate bounds that are either new or strengthened in this paper (with the exception of the $k$-avoiding bound for which we prove in this paper a stronger lower bound only). The following remarks are in order.

1. We define pattern-avoiding bounds in a parameterized way, such that they are well-defined for all access sequences. Let $k$ be an arbitrary positive integer. If $X$ is $k$-avoiding, then the value of the $k$-avoiding bound for $X$ is $2^{o(n)}O^{(k)}$, and otherwise the value of the bound is defined to be $+\infty$. Similarly, the value of the $k$-decomposable bound is $O(n \log k)$ if $X$ is $k$-decomposable, and $+\infty$ otherwise. We define two $k$-monotone bounds, a strong, and a weak bound. Both bounds are set to $+\infty$ if $X$ is not $k$-monotone. In other cases, the value of the strong, respectively weak, $k$-monotone bound is $O(nk \log k)$, respectively $n^{2O(k^2)}$. The traversal and sequential bounds are defined similarly: the value of the bound is $O(n)$ for a permutation sequence $X \in \{n\}^n$, if $X$ is a preorder traversal, respectively monotone increasing sequence, and $+\infty$ otherwise.

2. Boxes with a parameter $k$ indicate families of bounds: there is a different bound defined for each value of $k$. Bounds in the same box for different values of $k$ are not always comparable, i.e., the parameterized families of bounds are not necessarily increasing or decreasing with $k$.

3. In case of the arrows from ($k$-lazy finger $\times O(\log k)$) to ($k$-monotone), and from the stronger ($k$-monotone) to the weaker ($k$-monotone), it is meant that the former bound with a given fixed value of $k$ dominates the latter bound with the same value $k$.

4. The arrows from (lazy finger) to ($k$-decomposable), from (OPT) to ($k$-lazy finger $\times O(\log k)$), and from (OPT) to ($k$-avoiding) indicate that the former dominates the latter for all values of $k$.

5. The arrows from ($k$-monotone) to (sequential), from ($k$-decomposable) to (traversal), and from ($k$-lazy finger $\times O(\log k)$) to (lazy finger) indicate domination for any constant $k \geq 2$. 
Dynamic Finger: these bounds depend on the distances between consecutive accesses. The dynamic finger bound \[D(S) = \sum_{j=2}^{m} \log(|s_j - s_{j-1}| + 1).\]

For any weight function \(w\), let
\[WDF_w(S) = \sum_{j=2}^{m} \log \frac{w(s_j)}{\min\{w(s_{j-1}), w(s_j)\}}.\]

The weighted dynamic finger bound \[WDF_w(S) = WDF(S) = \min_w WDF_w(S).\]

Locality in time.

Working set: For any \(j \leq m\), let the last touch time of the element \(s_j\) be \(\rho_S(j) = \max\{k < j \mid s_k = s_j\}\). If \(j\) is the first time that \(s_j\) is accessed, then we set \(\rho_S(j) = 0\).

The working set at time \(j\) is defined as \(w_S(j) = \{s_i \mid \rho_S(j) < i \leq j\}\). In words, it is the set of distinct elements accessed since the last touch time of the current element.

The working set bound \[WS(S) = \sum_{j=1}^{m} \log(|w_S(j)|)\]

Locality in a reference tree.

Static Optimality: For any fixed BST \(T\) on \([n]\), let
\[SO_T(S) = \sum_{j=1}^{m} d_T(s_j),\]
where \(d_T(s_j)\) is the depth of \(s_j\) in \(T\) i.e. the distance from the root to \(s_j\). The static optimality bound \([31]\) is
\[SO(S) = \min_T SO_T(S).\]

Fixed Finger*: For any fixed BST \(T\) and any element \(f\), let
\[FF_{T,f}(S) = \sum_{j=1}^{m} d_T(f, s_j)\]
where \(d_T(f, s_j)\) is the distance from \(f\) to \(s_j\) in \(T\). The fixed finger* bound is
\[FF(S) = \min_{f,T} FF_{T,f}(S).\]

Lazy Finger: The previous two bounds capture the proximity of an access to the root and to a fixed key \(f\) respectively. The lazy finger bound \([4]\) captures the proximity of consecutive accesses in a reference tree. For any fixed BST \(T\) on \([n]\), let
\[LF_{T}(S) = \sum_{j=2}^{m} d_T(s_{j-1}, s_j)\]
where \(d_T(s_{j-1}, s_j)\) is the distance from \(s_{j-1}\) to \(s_j\) in \(T\). The lazy finger bound is
\[LF(S) = \min_T LF_T(S).\]

\(k\)-Lazy Finger*: We generalize the lazy finger bound to allow multiple fingers. Our definition is inspired by \([4,13]\). Let \(k \in \mathbb{N}\) and \(T\) be a binary search tree on \([n]\). A finger strategy consists of a sequence \(\vec{f} \in [k]^m\) where \(f_t \in [k]\) specifies the finger that will serve the request \(s_t\), and an initial vector \(\vec{\ell} \in [n]^k\) where \(\ell_t \in [n]\) specifies the initial location of finger \(i\). The cost of strategy \((\vec{f}, \vec{\ell})\) is
\[LF_{T,f,\ell}(S) = \sum_{t=1}^{m} (1 + d_T(s_t, s_{\sigma(t,t)}))\]
where \(\sigma(i,t)\) is the location of finger \(i\) before time \(t\) and \(\sigma(i,1) = \ell_t\). Let \(LF_{T,f,\ell}(S) = \min_{f,\ell} LF_{T,f,\ell}(S)\).

In other words, for a fixed BST \(T\) on key set \([n]\), \(LF_T^x(S)\) is the optimal \(k\)-server solution that serves access sequence \(S\) in tree \(T\). We define \(LF^x(S) = \min_T LF_T^k(S)\). It is clear form the definition that \(LF^1(S) \geq LF^2(S) \geq \ldots \geq LF^n(S) = m\).

Unified Bound: The unified bound \([14,31]\) computes for every access the minimum among the static finger, static optimality, and working set bounds. It is defined as
\[UB = \min_{T,f} \sum_{j=1}^{m} \log(\min\{|f - s_j| + 1, d_T(s_j), |w_S(j)|\}).\]

The bound should not be confused with the unified conjecture \([5]\), which subsumes the working set and dynamic finger bounds but is not currently known to be achieved by OPT.

The described bounds are summarized in Table 1 in \(\S 15\). We defer the definition of key-independent bounds to \(\S 15.3\).

Pattern avoidance.

Pattern avoidance bounds are, in some sense, different from the other BST bounds; they capture a more “global” structure, whereas other bounds all measure a broadly understood “locality of reference”.

The pattern avoidance parameter \(p(X)\) is the smallest integer such that \(X\) avoids some permutation pattern \(\sigma\) of length \(p(X)\). If \(k \geq p(X)\), we say that \(X\) is \(k\)-avoiding. The following are special cases of this parameter.

The monotone pattern parameter \(m(X)\) is the smallest integer such that \(X\) avoids one of the patterns \((1, \ldots, m(X))\) or \((m(X), \ldots, 1)\). If \(k \geq m(X)\), we say that \(X\) is \(k\)-monotone.
The monotone pattern parameter of sequential access is $m(X) = 2$.

The decomposability parameter is defined for permutation access sequences ($X \in [n]^n$). Parameter $d(X)$ is the smallest integer such that $X$ avoids all simple permutations of length $d(X) + 1$ and $d(X) + 2$.

For a traversal sequence $X$ (i.e. the preorder sequence of some BST) we have $d(X) = 2$.

We refer to [7] for more details on pattern-avoiding bounds. The fact that $OPT$ is linear for traversal and for sequential access is well-known.

**Theorem 5** ([7]). Let $X$ be a permutation input sequence in $[n]^n$, let $p = p(X)$, let $d = d(X)$, and $m = m(X)$. The following relations hold:

- $\text{GREEDY}(X) \leq n2^{O(d^2)}$ and $\text{OPT}(X) \leq O(m \log d)$
- $\text{GREEDY}(X) \leq n2^{O(m^2)}$

**Known relations between bounds.** We refer to Figure 2 for illustration. By definition, the weighted bounds are stronger than their unweighted counterparts because of the uniform weight function mapping all elements to 1, therefore $WB(S) \leq B(S)$, $WSF(S) \leq SF(S)$, $WDF(S) \leq DF(S)$ for any sequence $S$. In [7] it is shown that pattern-avoidance bounds are incomparable with the dynamic finger and working set bounds.

**Theorem 6.** The following relations hold for any sequence $S$:

- $SO(S) = \Theta(H(S))$.
- $LF(S) = \Theta(WDF(S))$.
- $UB(S) = \Theta(WS(S))$.

**Known competitiveness of algorithms.** We only focus on the complexity of the Splay and Greedy algorithms. Other algorithms considered in the literature include Tango trees [12] and Multi-splay [33]. We summarize the bounds known to be achieved by Splay and Greedy in Theorem 22 deferred to §A. These facts are also illustrated in Figure 2.

3 New equivalences between classical bounds

In this section we revisit some of the classical BST bounds and establish new equivalences. In §3.1 we prove that the weighted balance, weighted static finger, static optimality, and fixed finger bounds are equivalent. As an application, a new proof is presented that Splay and Greedy satisfy these properties in §3.2. In §3.3 we show that Move-to-root is the optimal algorithm, when key values are randomly permuted.

3.1 Static optimality and equivalent bounds

The conceptual message of this section is that the weighted version of information-theoretic proximity is equivalent to proximity in a reference tree. We start by stating the technical tools that allow the conversion between the two settings.

**Weight ⇒ Tree:** We refer to the randomized construction of a BST from an arbitrary weight function due to Seidel and Aragon [29].

**Lemma 7** ([29]). Given a weight function $w$, there is a randomized construction of a BST $T_w$ with the following properties:

- the expected depth of element $i$ is $E[d_{T_w}(i)] = \Theta(\log \frac{w(i)}{w(i)'})$, and
- the expected distance from element $i$ to $j$ is $E[d_{T_w}(i,j)] = \Theta(\log \frac{w[i,j]}{\min(w[i],w(j)'})$. 

...
When only the first property is needed, we can use a deterministic construction, see §C.

**Tree ⇒ Weight:** Given a tree \( T \), the following assignment of weights is folklore.

**Lemma 8.** Let \( T \) be a BST and define \( w(i) = 4^{-d_R(i)} \) for all \( i \in [n] \). Then for any key \( i \in [n] \), \( \sum_{j \in T_i} w(j) = \Theta(w(i)) \).

**Proof.** From the BST property, there are at most \( 2^d \) nodes that are at depth \( d \) in the subtree \( T_i \). Therefore, \( w(i) \leq \sum_{j \in T_i} w(j) \leq \sum_{d'=0}^{\infty} 2^d 4^{-d_R(i)-d'} \leq 2w(i) \).

We show that for any sequence \( S \), the following bounds are equivalent: weighted balance, static optimality, weighted static finger, and fixed finger. We defer the proofs to §D. We remark that all proofs in this section are inspired by the equivalence between the weighted dynamic finger and lazy finger bounds in [4].

**Theorem 9.** For all sequences \( S \), we have \( WB(S) = SO(S) = WSF(S) = FF(S) \) (up to constant factors).

**Discussion.** We can interpret the theorem as follows: (i) Any algorithm satisfying the access lemma [31] obviously satisfies static optimality, because \( SO = WB \), and \( WB \) is equivalent to the access lemma with the restriction that the weight function is fixed throughout the sequence. (ii) In static BSTs, fixing the finger at the root is the best choice up to a constant factor, because \( SO = FF \). (iii) \( SO \) is now obviously stronger than \( SF \) because \( SO = WSF \), and \( WSF \) is the weighted version of \( SF \).

### 3.2 New proofs of static optimality

In this section we give a simple direct proof that Splay and Greedy achieve static optimality. By Theorem 9, this implies that the other bounds are also achieved. These facts are well-known [17,31], but we find the new proofs to provide additional insight. We present the proof for Splay and defer the proof for Greedy to §E.

We use a potential function with a clear combinatorial interpretation.

**Min-depth potential function.** Fix a BST \( R \), called a reference tree. Let \( T \) be the current tree maintained by our BST algorithm (either Splay or Greedy). Let \( T(i) \) denote the set of elements in the subtree rooted at \( i \). For each element \( i \), the potential of \( i \) with respect to \( R \) is \( \varphi_R(i) = -2 \cdot \min_{j \in T(i)} d_R(j) \). The min-depth potential of \( T \) with respect to \( R \) is \( \varphi_R(T) = \sum_{i=1}^n \varphi_R(i) \). We will drop the subscript \( R \) for convenience. We present an easy but crucial fact.

**Fact 1.** For any interval \([a, b]\) and any BST \( R \), there is a unique element \( c \in [a, b] \) with smallest depth in \( R \).

**Proof.** Suppose there are at least two elements \( c \) and \( c' \) with smallest depth. Then the lowest common ancestor \( LCA(c, c') \) would have smaller depth, which is a contradiction.

**Theorem 10.** The amortized cost of splay for accessing element \( i \) is \( O(d_R(i)) \).

**Proof.** Let \( \varphi^b(i) \) and \( \varphi^a(i) \) be the potential of \( i \) before and after splaying \( i \). We have \( \varphi^a(i) = 0 \) because \( i \) is the root, and \( \varphi^a(i) \geq -2d_R(i) \).

For each zigzag or zigzag step (see [31] for the description of the Splay algorithm), let \( x, y, z \) be the elements in the step where \( d_T(x) > d_T(y) > d_T(z) \). Let \( \varphi(i) \) and \( \varphi'(i) \) be the potential before and after the step, and let \( T \) and \( T' \) be the tree before and after the step. It suffices to prove that the cost is at most \( 3(\varphi'(x) - \varphi(x)) \). This is because by telescoping, the total cost for splaying \( i \) will be \( O(\varphi^a(i) - \varphi^b(i)) = O(d_R(i)) \), and the amortized cost in the final zig step is trivially at most \( 1 + 2(\varphi'(x) - \varphi(x)) \). We analyze the two cases.
Zigzag: we have that $T(x)$ and $T'(z)$ are such that $T(x) \cap T'(z) = \emptyset$ and $T(x), T'(z) \subset T'(x)$. By Fact 1, either $\varphi(x) + 2 \leq \varphi'(x)$ or $\varphi'(z) + 2 \leq \varphi(x)$, so we have $2 \leq 2\varphi'(x) - \varphi(x) - \varphi'(z)$. Therefore, the amortized cost is

\[
2 + \varphi'(x) + \varphi'(y) + \varphi'(z) - \varphi(x) - \varphi(y) - \varphi(z) = 2 + \varphi'(y) + \varphi'(z) - \varphi(x) - \varphi(y) \\
\leq 2 + \varphi'(x) + \varphi'(z) - 2\varphi(x) \\
\leq (2\varphi'(x) - \varphi(x) - \varphi'(z)) + \varphi'(x) + \varphi'(z) - 2\varphi(x) \\
= 3(\varphi'(x) - \varphi(x)).
\]

Zigzag: we have that $T'(y)$ and $T'(z)$ are such that $T'(y) \cap T'(z) = \emptyset$ and $T'(y), T'(z) \subset T'(x)$. By Fact 1, we have $2 \leq 2\varphi'(x) - \varphi'(y) - \varphi'(z)$. Therefore, the amortized cost is

\[
2 + \varphi'(x) + \varphi'(y) + \varphi'(z) - \varphi(x) - \varphi(y) - \varphi(z) = 2 + \varphi'(y) + \varphi'(z) - \varphi(x) - \varphi(y) \\
\leq (2\varphi'(x) - \varphi'(y) - \varphi'(z)) + \varphi'(y) + \varphi'(z) - \varphi(x) - \varphi(y) \\
\leq 2(\varphi'(x) - \varphi(x)).
\]

It is instructive to observe the similarity between the min-depth potential and the *sum-of-logs potential* [31], which is essentially the “soft-min” version of min-depth, if the weights are set to $4^{-\delta n(i)}$. Such a weight assignment is used in proving static optimality e.g. in [15].

### 3.3 Move-to-root is optimal when elements are randomly permuted

Let $\pi : [n] \to [n]$ be a permutation. For any sequence $S \in [n]^m$, we denote by $\pi(S) = (\pi(s_1), \ldots, \pi(s_m))$ the permuted sequence of $S$ by $\pi$. The *key-independent optimality bound* is defined as $\text{KI-OPT}(S) = E_\pi[\text{OPT}(\pi(S))]$, where the expectation is over the uniform random distribution of permutations of size $n$.

**Theorem 11** ([19]). $\text{KI-OPT}(S) = \Theta(\text{WS}(S) + n \log n)$ for any sequence $S$.

This shows that the expected cost over a random order of the elements, of the optimal algorithm is equivalent to the working set bound up to a constant factor if the length of the sequence is $m \geq n \log n$. In this section we show that in the key-independent setting even the very simple heuristic that just rotates the accessed element to the root, is optimal. This algorithm is called Move-to-root [3], and we denote its total cost for accessing a sequence $S$ from initial tree $T$ by $\text{MTR}_T(S)$.

**Definition 12.** For any sequence $S$ and any initial tree $T$, the *key-independent move-to-root bound* is $\text{KI-MTR}_T(S) = E_\pi[\text{MTR}_T(\pi(S))]$ where $\pi$ is a random permutation.

The following theorem shows that key-independent move-to-root (starting from a balanced tree), key-independent optimality, and working set bounds are all equivalent when the length of the sequence is $m \geq n \log n$. The proof is deferred to §F.

**Theorem 13.** Let $T$ be a BST of logarithmic depth. Then, $\text{KI-MTR}_T(S) = \Theta(\text{KI-OPT}(S) + n \log n) = \Theta(\text{WS}(S) + n \log n)$ for any sequence $S$.

We remark that Move-to-root is known to have another property related to randomized inputs: If accesses are drawn independently from some distribution, then Move-to-root achieves static optimality for that distribution [3].

---

1 The term $n \log n$ is missing in [19].
4 A new landscape via $k$-lazy fingers

In this section we study the lazy finger bound and its generalization to multiple fingers. In §G.1 we argue that $\text{OPT}(X) \leq O(\log k) \cdot \text{LF}^k(X)$ for all sequences $X$, refining the result of [13] which had an overhead factor of $O(k)$ instead of $O(\log k)$. This bound is essentially tight: We show in §G that there is a sequence $X$ for which $\text{OPT}(X) = \Theta(\log k) \cdot \text{LF}^k(X)$.

4.1 Applications of lazy fingers

Application 1: Lazy fingers and decomposability.

First, we give necessary definitions. Let $\sigma = (\sigma(1), \ldots, \sigma(n))$ be a permutation. For $a, b : 1 \leq a < b \leq n$, we say that $[a, b]$ is a block of $\sigma$ if $\{\sigma(a), \ldots, \sigma(b)\} = \{c, \ldots, d\}$ for some integer $c, d \in [n]$. A block partition of $\sigma$ is a partition of $[n]$ into $k$ blocks $[a_i, b_i]$ such that $(\bigcup_j [a_i, b_i]) \cap \mathbb{N} = [n]$. For such a partition, for each $i = 1, \ldots, k$, consider a permutation $\sigma_i \in S_{b_i - a_i + 1}$ obtained as an order-isomorphic permutation when restricting $\sigma$ on $[a_i, b_i]$. For each $i$, let $q_i \in [a_i, b_i]$ be a representative element of $i$. The permutation $\tilde{\sigma} \in [k]^k$ that is order-isomorphic to $\{\sigma(q_1), \ldots, \sigma(q_k)\}$ is called a skeleton of the block partition. We may view $\sigma$ as a deflation $\tilde{\sigma}(\sigma_1, \ldots, \sigma_k)$.

Now we provide a recursive definition of $d$-decomposable permutations. We refer to [7] for more details. A permutation $\sigma$ is $d$-decomposable if $\sigma = (1)$, or $\sigma = \tilde{\sigma}(\sigma_1, \ldots, \sigma_d)$ for some $d' \leq d$ and each permutation $\sigma_i$ is $d$-decomposable.

Lemma 14. Let $S$ be a $k$-decomposable permutation of length $n$. Then $\text{LF}(S) \leq 4(|S| - 1) [\log k]$.

Proof. It is sufficient to define a reference tree $T$ for which $\text{LF}(S)$ achieves such bound. We remark that the tree will have auxiliary elements. We construct $T$ recursively. If $S$ has length one, $T$ has a single node and this node is labeled by the key in $S$. Clearly, $\text{LF}(S) = 0$.

Otherwise, let $S = S[S_1, \ldots, S_j]$ with $j \in [k]$ the outermost partition of $S$. Denote by $T_i$ the tree for $S_i$ that has been inductively constructed. Let $T_0$ be a BST of depth at most $[\log j]$ and with $j$ leaves. Identify the $i$-th leaf with the root of $T_i$ and assign keys to the internal nodes of $T_0$ such that the resulting tree is a valid BST. Let $r_i$ be the root of $T_i$, $0 \leq i \leq j$ and let $r = r_0$ be the root of $T$. Then

\[
\text{d}_T(r, s_1) \leq [\log k] + d_{T_1}(r_1, s_1)
\]

\[
\text{d}_T(r, s_n) \leq [\log k] + d_{T_j}(r_j, s_n)
\]

\[
\text{d}_T(s_{i-1}, s_i) \leq \begin{cases} 
\text{d}_{S_{\ell}}(s_{i-1}, s_i) & \text{if } s_{i-1}, s_i \in S_{\ell} \\
2 [\log k] + d_{T_{\ell}}(r_{\ell}, s_{i-1}) + d_{T_{\ell+1}}(r_{\ell+1}, s_i) & \text{if } s_{i-1} \in S_{\ell} \text{ and } s_i \in S_{\ell+1},
\end{cases}
\]

and hence

\[
\text{LF}_T(S) = \text{d}_T(r, s_0) + \sum_{i \geq 2} \text{d}_T(s_{i-1}, s_i) + \text{d}_T(s_n, r)
\]

\[
\leq 2j [\log k] + \sum_{1 \leq \ell \leq j} \text{LF}_{T_{\ell}}(S_{\ell}) \leq 2j [\log k] + \sum_{1 \leq \ell \leq j} 4(|S_{\ell}| - 1) [\log k]
\]

\[
\leq (2j - 4j + 4 \sum_{1 \leq \ell \leq j} |S_{\ell}|) [\log k] \leq 4(|S| - 1),
\]

where the last inequality uses $j \geq 2$.

Combining this with the Iacono-Langerman result [20], we conclude that for any $k$-decomposable sequence $S$, we have $\text{GREEDY}(S) \leq O(|S| \log k)$. This strengthens our earlier result [7] that $\text{GREEDY}(S) \leq n^2 2^{O(k^2)}$. 


Application 2: Improved relation between $\text{OPT}(S)$ and $m(S)$.

Let $k = m(S)$, i.e. the smallest integer such that $S$ is $k$-monotone. In [7] we show that $\text{OPT}(S) \leq \text{GREEDY}(S) \leq |S|2^{O(k^2)}$. Here we show the substantially stronger bound $\text{OPT}(S) \leq O(|S|k \log k)$, raising the obvious open question of whether any online BST can match this bound.

Lemma 15. Let $S$ be a $k$-monotone sequence of length $n$. Then $\text{LF}^k(S) = O(nk)$.

Proof. $S$ avoids $(k+1, \ldots, 1)$ or $(1, \ldots, k+1)$. Assume that the first case holds (the argument for the other case is symmetric). Then, $S$ can be partitioned into $k$ subsequences $S_1, \ldots, S_k$, each of them increasing, furthermore, such a partition can be computed online. We argue that $\text{LF}^k(S) = O(nk)$ for any BST $T$.

Let $T$ be any binary search tree containing $[n]$ as elements. Consider $k$ lazy fingers $f_1, \ldots, f_k$ in $T$. We define the strategy for the fingers as follows: When $s_j$ is accessed, if $s_j \in S_i$, then move finger $f_i$ to serve this request. Observe that $f_i$ only needs to do in-order traversal in $T$ (since the subsequence $S_i$ is increasing), which takes at most $O(n)$ steps. Thus, $\text{LF}^k(S) = O(nk)$.

By Theorem 2, we can simulate $k$-finger with an overhead factor of $O(\log k)$, concluding with the following theorem.

Theorem 16. Let $S$ be a $k$-monotone sequence. Then $\text{OPT}(S) = O(|S| \cdot k \log k)$.

5 Combining easy sequences

Recall that for any sequence $S \in [n]^m$, $\text{OPT}(S)$ denotes the optimal cost for executing $S$ on a BST which contains $[n]$ as elements. Let $P = ([a_1, b_1], \ldots, [a_k, b_k])$ be a partitioning of $[n]$ into $k$ intervals. That is, $a_1 = 1, b_k = n$ and $b_i = a_{i+1} - 1$. Given $P$, we can define $S_1, \ldots, S_k$ and $\tilde{S}$ as follows. For each $1 \leq i \leq k$, $S_i$ is obtained from $S$ by restriction to $[a_i, b_i]$. That is, for each $s_j \in [a_i, b_i]$ starting from $j = 1$ to $m$, we append $s_j - a_i + 1$ to the sequence $S_i$. Let $m_i$ be the length of $S_i$ and hence $\sum_{i=1}^{k} m_i = m$. Next, we define $\tilde{S} = (\tilde{s}_1, \ldots, \tilde{s}_m) \in [k]^m$. For each $j \leq m$, if $s_j \in [a_i, b_i]$, then $\tilde{s}_j = i$.

The main theorem of this section is the following. See Appendix H for the proof.

Theorem 17 (Time-interleaving Bound). For any sequence $S$ and a partition $P$ of $[n]$ into $k$ intervals, $\text{OPT}(S) \leq \sum_{i=1}^{k} \text{OPT}(S_i) + 3\text{OPT}(\tilde{S})$.

This theorem bounds the optimal cost of any sequence $S$ that can be obtained by “interleaving” $k$ sequences $S_1, \ldots, S_k$ according to $\tilde{S}$. To illustrate the power of this result, we give a bound for the optimal cost of the “tilted grid” sequence. Let $\ell = \sqrt{n}$ and consider the point set $P = \{(i\ell + (j-1), j\ell + i-1) : i, j \in [\ell]\}$. It is easy to check that there are no two points aligning on $x$ or $y$ coordinates, therefore $P$ corresponds to a permutation $S \in [n]^m$. In [7] it is shown that none of the known bounds imply $\text{OPT}(S) = O(n)$. We observe that the tilted grid sequence can be seen as a special case of a broad family of “perturbed grid”-type sequences, amenable to similar analysis.

Corollary 18. Let $S$ be the tilted grid sequence. Then $\text{OPT}(S) = O(n)$.

Proof. For $1 \leq i \leq \sqrt{n}$, let $[a_i, b_i] = [1 + (i-1)\sqrt{n}, i\sqrt{n}]$. Then $S_i$ is a sequential access of length $\sqrt{n}$, and $\tilde{S}$ is a sequential access of length $\sqrt{n}$ repeated $\sqrt{n}$ times. So $\text{OPT}(S_i) = O(\sqrt{n})$ and $\text{OPT}(\tilde{S}) = O(n)$. By Theorem 17 $\text{OPT}(S) \leq \sum_{i=1}^{\sqrt{n}} \text{OPT}(S_i) + 3\text{OPT}(\tilde{S}) = O(n)$.
6 Separations between BST bounds

In this section we show examples that separate the bounds to the largest extent possible. For lack of space, we defer the proofs to §II.

We first discuss the gap between \( \text{OPT} \) and other BST bounds. When we say that a BST bound \( f \) is tight, we mean that there are infinitely many sequences \( X \) for which \( \text{OPT}(X) \geq \text{cf}(X) \) for some constant \( c \) (not depending on \( X \)). It is in this sense that many classical bounds (working set, static optimality, dynamic finger) are tight. We emphasize that many bounds, such as \( k \)-lazy finger, \( k \)-monotone, and \( k \)-avoiding, are in fact families of bounds (parameterized by \( k \in \mathbb{N} \)), e.g. the \( k \)-monotone bounds are given by \( \{f_k\} \) where \( f_k(X) = |X| \cdot k \log k \) if \( X \) avoids \((k, \ldots, 1) \) or \((1, \ldots, k) \). Therefore, the concept of tightness for these bounds is somewhat different. Our results are summarized in the following theorem.

\[ \text{Theorem 19.} \] For each \( k \) (possibly a function that depends on \( n \)), there are infinitely many sequences \( S_1, S_2, S_3 \), for which the following holds:

- \( LF^k(S_1) \cdot \log k \leq \text{OPT}(S_1) \),
- \( m(S_2) = k \), and \( \text{OPT}(S_2) = \Omega(|S_2| \cdot \log k) \),
- \( p(S_3) = k \), and \( \text{OPT}(S_3) = \Omega(|S_3| \cdot \sqrt{k}) \).

The results are derived using information-theoretic arguments. For \( k \)-monotone bounds, this technique cannot prove \( \text{OPT}(S) \geq |S| \cdot g(k) \) for a super-logarithmic function \( g(k) \). For \( k \)-avoiding bounds, the information-theoretic limit is \( |S| \cdot k \).

Next, we show a strong separation in the hierarchy of lazy finger bounds. The results are summarized in the following theorem.

\[ \text{Theorem 20.} \] For any \( k \) and infinitely many \( n \), there is a sequence \( S_k \) of length \( n \), such that:

- \( LF^{k-1}(S_k) = \Omega(n \log(n/k)) \),
- \( LF^k(S_k) = O(n) \) (independent of \( k \))
- \( S_k \) avoids \((k + 1, k, \ldots, 1) \)
- \( \text{OPT}(S_k) = O(n) \) (independent of \( k \))

This theorem also implies a weak separation in the class of lazy finger bounds and monotone pattern bounds: For constant \( k \), a sequence \( S_k \) is linear when applying the monotone BST bound, but the lazy finger bound with \( k - O(1) \) fingers would not give better than \( \Omega(n \log n) \).

Finally, we show that lazy fingers are not strong enough to subsume the classical bounds.

\[ \text{Theorem 21.} \] For any \( k \) and infinitely many \( n \), there are sequences \( S_1 \) and \( S_2 \) of length \( n \), such that:

- \( \text{WS}(S_1) = o(LF^k(S_1)) \), and
- \( LF^k(S_2) = o(\text{WS}(S_2)) \).
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A Bounds known to be achieved by Splay and Greedy

Theorem 22. The following relations hold for any sequence $S$:

- [31] Splay achieves the $WB$, $SO$, $SF$, $WS$ bounds.
- [32] Splay achieves the sequential bound.
- [10] Splay achieves the $DF$ bound.
- [17] Greedy achieves the $WB$, $SO$, $SF$, $WS$, and sequential bounds.
- [20] Greedy achieves the $WDF$ bound.
- [7] Greedy achieves the traversal, $k$-avoiding, and weak $k$-monotone bounds.
### Table of BST bounds

| Bounds                  | Acronym | Formula                                                                 |
|-------------------------|---------|-------------------------------------------------------------------------|
| Balance                 | $B$     | $m \log n$                                                              |
| Weighted Balance        | $WB$    | $\min_w \sum_{j=1}^m \log \frac{W(s_j)}{w(s_j)}$                      |
| Entropy                 | $H$     | $\sum_{j=1}^m n_i \log \frac{n_i}{m}$                                 |
| Static Finger           | $SF$    | $\min_f \sum_{j=1}^m \log(|f - s_j| + 1)$                             |
| Weighted Static Finger* | $WSF$   | $\min_{w,f} \sum_{j=1}^m \log \frac{w(f)}{\min[w(f),w(s_j)]}$        |
| Unified Bound           | $UB$    | $\min_{T,f} \sum_{j=1}^m \log(\min(|f - s_j| + 1, d_T(s_j), |w_S(j)|))$  |
| Dynamic Finger          | $DF$    | $\sum_{j=2}^m \log(|s_j - s_{j-1}| + 1)$                              |
| Weighted Dynamic Finger | $WDF$   | $\min_w \sum_{j=2}^m \log \frac{w(s_{j-1},s_j)}{\min[w(s_{j-1}),w(s_j)]}$ |
| Working Set             | $WS$    | $\sum_{j=2}^m \log(|w_S(j)|)$                                         |
| Static Optimality       | $SO$    | $\min_T \sum_{j=1}^m d_T(s_j)$                                        |
| Fixed Finger*           | $FF$    | $\min_{T,f} \sum_{j=1}^m d_T(f, s_j)$                                 |
| Lazy Finger             | $LF$    | $\min_T \sum_{j=2}^m d_T(s_{j-1}, s_j)$                               |

*Table 1 Summary of BST bound definitions*
**Theorem 23.** Given a weight function $W$, there is a deterministic construction of a BST $T_w$ such that the depth of every key $i \in [n]$ is $d_{T_w}(i) = O(\log \frac{w}{w(i)})$.

**Proof.** Let $w_1, \ldots, w_n$ be a sequence of weights. We show how to construct a tree in which the depth of element $\ell$ is $O(\log \frac{w[1 : \ell]}{\min(w_1, w_{\ell})})$.

For $i \geq 1$, let $j_i$ be minimal such that $w[1 : j_i] \geq 2^i w_1$. Then $w[1 : j_i - 1] < 2^i w_1$ and $w[j_i - 1 + 1 : j_i] \leq 2^{i-1} w_1 + w_{\ell}$.

Let $T_i$ be the following tree. The right child of the root is the element $j_i$. The left subtree is a tree in which element $\ell$ has depth $O(2^{i-1} w_1 / w_{\ell})$.

The entire tree has $w_1$ in the root and then a long right spine. The trees $T_i$ hang off the spine to the left. In this way the depth of the root of $T_i$ is $O(i)$.

Consider now an element $\ell$ in $T_i$. Assume first that $\ell \neq j_i$. The depth is

$$O \left( i + \log \frac{2^{i-1} w_1}{w_{\ell}} \right) = O \left( i + \log \frac{2^{i-1} w_1}{\min(w_1, w_{\ell})} \right) = O \left( \log \frac{2^{i-1} w_1}{\min(w_1, w_{\ell})} \right) = O \left( \frac{w[1 : \ell]}{\min(w_1, w_{\ell})} \right).$$

For $\ell = j_i$, the depth is

$$O (i) = O \left( \log \frac{2^i w_1}{w_1} \right) = O \left( \log \frac{w[1 : j_i]}{\min(w_1, w_{\ell})} \right).$$
D Missing proofs from §3.1

Theorem 24. \( \text{WB}(S) = O(\text{SO}(S)) \) for any sequence \( S \).

Proof. \( \text{WB}(S) = O(\text{SO}(S)) \): (note\(^2\)) Fix any BST \( T \). It suffices to show the existence of a weight function \( w \) such that \( \text{WB}_w(S) = O(\text{SO}_T(S)) \). Define the weight function \( w(i) = 4^{-d_T(i)} \). Observe first that \( W \leq 2 \) from Lemma \( \Box \) applied at the root \( r \in [n] \). This implies that for each key \( i \in [n] \), we have \( \log \frac{w(i)}{w(a)} = O(d_T(i)) \).

\( \text{SO}(S) = O(\text{WB}(S)) \): Fix any weight function \( w \). It suffices to show the existence of a BST \( T \) such that \( \text{SO}_T(S) = O(\text{WB}_w(S)) \). By choosing \( T_w \) according to the distribution in Lemma \( \Box \), we have \( \mathbb{E}[\text{SO}_T(S)] = \mathbb{E}[\sum_{j=1}^m d_T(s_j)] = \sum_{j=1}^m \mathbb{E}[d_T(s_j)] = \Theta(\sum_{j=1}^m \log \frac{w(s_j)}{w(a)}) = \Theta(\text{WB}_w(S)) \). Since the expectation satisfies the bound, there exists a tree \( T \) satisfying the bound. \( \blacksquare \)

Theorem 25. \( \text{WSF}(S) = O(\text{FF}(S)) \) for any sequence \( S \).

Proof. Fix any BST \( T \) and any element \( f \). It suffices to show the existence of a weight function \( w \) such that \( \text{WSF}_{w,f}(S) = O(\text{FF}_{T,f}(S)) \). In particular, for all \( i \), we show that \( \log \frac{w[i]}{\min\{w(f), w(i)\}} = \Theta(d_T(f, i)) \) when we set \( w(i) = 4^{-d_T(i)} \). Let \( a = \text{LCA}(f, i) \) and hence \( d_T(f, i) = d_T(f, a) + d_T(a, i) \).

From Lemma \( \Box \) applied at node \( a \), we have \( w[j : i] \leq \sum_{j \in T_a} w(j) \leq 2w(a) \). Therefore,

\[
\log \frac{w[f : i]}{\min\{w(f), w(i)\}} \leq \log \frac{2w(a)}{\min\{w(f), w(i)\}} \\
\leq \log \frac{2 \cdot 4^{-d_T(a)}}{\min\{4^{-d_T(f)}, 4^{-d_T(i)}\}} \\
\leq O(1) + 2 \cdot \max\{d_T(f) - d_T(a), d_T(i) - d_T(a)\} \\
= O(1) + 2 \cdot \max\{d_T(f, a), d_T(i, a)\} \\
\leq O(1) + 2 \cdot d_T(f, i).
\]

\( \blacksquare \)

Theorem 26. \( \text{FF}(S) = O(\text{WSF}(S)) \) for any sequence \( S \).

Proof. Fix any weight function \( w \) and any element \( f \). It suffices to show a BST \( T \) such that \( \text{FF}_{T,f}(S) = O(\text{WSF}_{w,f}(S)) \). By invoking Lemma \( \Box \) with weight \( w \), we have a distribution of BST \( T_w \) that satisfies \( \mathbb{E}[\text{FF}_{T_w}(S)] = \mathbb{E}[\sum_{j=1}^m d_{T_w}(f, s_j)] = \sum_{j=1}^m \mathbb{E}[d_{T_w}(f, s_j)] = O(\sum_{j=1}^m \log \frac{w[f : s_j]}{\min\{w(f), w(s_j)\}}) = \Theta(\text{WSF}_{w,f}(S)) \). Since the expectation satisfies the bound, there must exist a tree \( T \) satisfying the bound. \( \blacksquare \)

Theorem 27. \( \text{WSF}(S) = O(\text{FF}(S)) \) for any sequence \( S \).

So far, we have shown the equivalences between two pairs \( \text{WB} \leftrightarrow \text{SO} \) and \( \text{FF} \leftrightarrow \text{WSF} \). The following two (very simple) connections complete the proof.

Theorem 28. \( \text{FF}(S) = O(\text{SO}(S)) \) for any sequence \( S \).

Proof. Fix any tree \( T \). We set \( f \) to be the root of \( T \), then \( \text{FF}_{T,f}(S) = \text{SO}_T(S) \). \( \blacksquare \)

\(^{2}\) In \cite{31} it is shown that \( \text{WB}(S) = O(\text{H}(S)) \) and since \( \text{H}(S) = \Theta(\text{SO}(S)) \), this implies that \( \text{WB}(S) = O(\text{SO}(S)) \). We include a proof for completeness.
Theorem 29. \(WB(S) = O(FF(S))\) for any sequence \(S\).

Proof. Fix any weight function \(w\) associated with the weighted static finger bound. It suffices to show a weight function \(w\) such that \(WB_w(S) = O(FF_{T,f}(S))\). In particular, for all \(i\), we show that \(\log W_w(i) = \Theta(d_T(f,i))\) when we set \(w(i) = 4^{-d_T(f,i)}\). This follows because the number of elements with distance \(d\) from \(i\) is at most \(3^d\), and so \(W = \sum_{i=1}^{n} w(i) \leq \sum_{d<\infty} 3^d \cdot 4^{-d} = O(1)\). □
The landscape of bounds for binary search trees

\section{The static optimality of Greedy}

In the geometric view where Greedy is defined \cite{11}, the role of the subtree $T(i)$ rooted at $i$ is played by the \textit{neighborhood} $T(i)$ of element $i$. The definition of neighborhood can be found in \cite{6,17}.

\textbf{Theorem 30.} The amortized cost of Greedy for accessing element $i$ is $O(d_R(i))$.

\textit{Proof.} Suppose that Greedy, for accessing $i$, touches $x_{-l}, x_{-l+1}, \ldots, x_0 = i, x_1, \ldots, x_r$. We only bound the amortized cost for touching $x_0, x_1, \ldots, x_r$. The analysis for the other elements is symmetric. Let $\varphi(x)$ and $\varphi'(x)$ be the potential of an element $x$ before, respectively after accessing $i$. Define $\varphi(x_i) = \varphi'(x_i) = 0$ for $i > r$. Note that $\varphi(i), \varphi'(i) \leq 0$ for all $i$. Also, for any $j \in T(i)$, $\varphi(j) \leq \varphi(i)$. The crucial observation is that, for any $i \leq r - 3$,

$$2 + \varphi'(x_i) + \varphi'(x_{i+2}) \leq 2\varphi(x_{i+3}).$$

This follows from Fact 1 and the facts that $T'(x_i) \cap T'(x_{i+2}) = \emptyset$ and $T'(x_i), T'(x_{i+2}) \subset T'(x_{i+3})$. Therefore, the amortized cost is

$$\sum_{0 \leq i \leq r} 1 + \varphi'(x_i) - \varphi(x_i) = \sum_{0 \leq i \leq r} (4 + \varphi'(x_i) + \varphi'(x_{i+1}) + \varphi'(x_{i+2}) + \varphi'(x_{i+3}) - \sum_{0 \leq i \leq r} \varphi(x_i))$$

$$\leq \sum_{0 \leq i \leq r} (2 + \varphi'(x_i) + \varphi'(x_{i+2})) + (2 + \varphi'(x_{i+1}) + \varphi'(x_{i+3})) - \sum_{0 \leq i \leq r} \varphi(x_i)$$

$$\leq \sum_{0 \leq i \leq r} \varphi(x_{i+3}) + \varphi(x_{i+4}) - \sum_{0 \leq i \leq r} \varphi(x_i)$$

$$\leq -\varphi(x_0) - \varphi(x_1) - \varphi(x_2)$$

$$\leq -3\varphi(x_0) = O(d_R(x_0)).$$
Lemma 31 (§3.3). Let \( i < k \). Then \( i \) is an ancestor of \( k \) after serving \( S \) if either \( i \) was accessed and there was no access to a key in \([i+1, \ldots, k]\) after the last access to \( i \) or \( i \) and \( k \) were not accessed and \( i \) is an ancestor of \( k \) in the initial tree.

This implies that we obtain the same final tree if we delete all but the last access to each element from the access sequence.

We use the following property of the Move-to-root algorithm.

Fact 2. Let \( S = (s_1, \ldots, s_m) \) be an access sequence. Before accessing \( s_j \), for any \( j' < j \), the keys \( s_{j'}, s_{j'+1}, \ldots, s_j \) form a connected component containing the root of \( T' \).

Now we are ready to prove the main theorem of the section.

Theorem 32. For any sequence \( S \) and any initial tree \( T \), \( KI-MTR_T(S) = O(WS(S) + f(n)) \) where \( f(n) = \sum_{i=1}^{n} d_T(i) \) depends only on \( n \) (and not on \( m \)).

Proof. We first analyze the cost of non-first accesses \( s_{\pi(j)} \). Consider the accesses \( s_{\rho_1(j)}, s_{\rho_1(j)+1}, \ldots, s_{j-1} \) where \( s_{\rho_1(j)} = s_j \), and \( s_{\ell}, s_{\ell'} \) for all \( \ell, \ell' \in \{\rho_1(j), \ldots, j-1\} \): These elements are stored in a connected subtree \( T' \) containing the root and \( T' \) is also formed if we deleted all but the last access to any element from this sequence. More formally, let \( J \subseteq \{\rho_1(j), \ldots, j-1\} \) be the set indices \( i \) for which \( s_i \) is the last appearance of its key in \( w_S(j) \). So we have \( |J| = |w_S(j)| \).

The expected access cost of \( s_j \) is exactly the expected number of its ancestors in \( T' \):

\[
\sum_{i \in J} Pr_{\pi}[ \pi(s_i) \text{ is ancestor of } \pi(s_{\rho_1(j)}) ]
\]

If \( \pi(s_i) > \pi(s_j) \), then the key \( \pi(s_i) \) is an ancestor of \( \pi(s_j) \) only if \( \pi(s_{\ell}) = \min \{\pi(s_{\ell'})\}_{\ell, \ell' \geq \ell} \) (due to Lemma 31). Otherwise, if \( \pi(s_{\ell}) < \pi(s_j) \), it is an ancestor only if \( \pi(s_{\ell}) = \max \{\pi(s_{\ell'})\}_{\ell, \ell' \geq \ell} \). In any case, this probability is exactly \( \frac{1}{|J \cap [\ell, j]|} \). Therefore, the expected access cost of \( s_j \) is at most \( \sum_{i \in J} \frac{1}{|J \cap [\ell, j]|} = O(\log |J|) = O(\log |w_S(j)|) \).

Consider a first access \( s_j \). By Fact 2 the elements \( s_1, s_2, \ldots, s_{j-1} \) form a connected subtree \( T' \) containing the root. An argument similar to the one above shows that the expected depth of \( T' \) is \( O(\log |w_S(j)|) \). So the expected length of the search path of \( s_j \) is at most \( O(\log |w_S(j)|) + d_T(s_j) \).

Thus, the total cost is \( \sum_{j=1}^{m} O(\log |w_S(j)|) + O(\sum_{i=1}^{n} d_T(i)) = O(WS(S) + f(n)) \).

It follows that key-independent Move-to-root (starting from a balanced tree), key-independent optimality, and working set bounds are all equivalent when the length of the sequence is \( m \geq n \log n \).

Corollary 33. Let \( T \) be a BST of logarithmic depth. Then, \( KI-MTR_T(S) = \Theta(KI-OPT(S) + n \log n) = \Theta(WS(S) + n \log n) \) for any sequence \( S \).

Proof. We trivially have \( KI-OPT(S) \leq KI-MTR_T(S) \). By Theorems 11 and 32 we have \( KI-MTR_T(S) = O(WS(S) + n \log n) = O(KI-OPT(S) + n \log n) \). This sandwiches the quantities.
G Proofs from Section 4

G.1 Simulating $k$-Lazy fingers

\textbf{Theorem 34.} For any sequence $S$, $\text{OPT}(S) \leq O(\log k) \cdot LF^k(S)$.

To prove this theorem, we refine the result in [13] which shows how to simulate a $k$-finger BST using a standard BST. In a $k$-finger BST there are $k$ pointers, and each of these can move to its parent or to one of its children. Moreover, the position of a finger is maintained while performing rotations.

When we execute an access sequence using $k$-finger BST, each finger is initially at the root. We specify an initial tree. For each access, one of the $k$ fingers must move to the accessed element. After each access the fingers remain in their position, instead of moving to the root, as in the standard BST model.

The cost for accessing a sequence is the total number of finger moves and rotations.

Given an online $k$-finger algorithm $A_{kBST}$, we define an online BST algorithm $A_{sim}(A_{kBST})$ that simulates $A_{kBST}$. In [13] it is shown that this can be achieved with a factor $O(k)$ increase in cost. We strengthen the result by showing the following result.

\textbf{Theorem 35.} There is a BST algorithm for simulating a $k$-finger BST with overhead factor $O(\log k)$.

From Theorem 35 we immediately get Theorem 34 as a corollary. This is because the model which defines the $k$-lazy finger bound $LF^k(\cdot)$ is exactly $k$-finger BST except that the tree is static. $LF^k(S)$ is the cost of $k$-finger BST on the optimal static tree. If we can simulate any $k$-finger BST algorithm with overhead $O(\log k)$, then we can indeed simulate any $k$-finger BST algorithm on any static tree, including the optimal tree. Therefore, $\text{OPT}(S) \leq LF^k(S) \cdot O(\log k)$ for any sequence $S$.

The rest of this section is devoted to the proof of Theorem 35. We use the approach from [13]. We are simulating a $k$-finger BST $T$ using a standard BST $T'$. The ingredients of the proof are: (1) To make sure that each element with a finger on it in $T$ has depth at most $O(\log k)$ in $T'$. (In [13], each finger may have depth up to $O(k)$ in $T'$.) (2) To implement a deque data structure within $T'$ so that each finger in $T$ can move to any of its neighbors with cost $O(\log k)$ amortized. (In [13], this cost is $O(1)$ amortized.) Therefore, to move a finger $f$ to its neighbor $x$ in $T$, we can simply access $f$ from the root of $T'$ in $O(\log k)$ steps, and then move $f$ to $x$ in $T'$ in $O(1)$ amortized steps. Hence, the overhead factor is $O(\log k)$.

G.1.1 DequeBST

We describe how to implement a deque in the BST model, called \textit{dequeBST}. This is be used later in the simulation. The following construction appears to be folklore, and it is the same as the one used in [13]. As we have not found an explicit description in the literature, we include it here for completeness.

\textbf{Lemma 36.} The minimum and maximum element from a dequeBST can be deleted in $O(1)$ amortized operations.

\textit{Proof.} The simulation is inspired by the well-known simulation of a queue by two stacks with constant amortized time per operation ([24, Exercise 3.19]). We split the deque at some position (determined by history) and put the two parts into structures that allow us to access the first and the last element of the deque. It is obvious how to simulate the deque
operations as long as the sequences are non-empty. When one of the sequences becomes empty, we split the other sequence at the middle and continue with the two parts. A simple potential function argument shows that the amortized cost of all deque operations is constant.

Let $\ell_1$ and $\ell_2$ be the length of the two sequences, and define the potential $\Phi = |\ell_1 - \ell_2|$. As long as neither of the two sequences are empty, for every insert and delete operation both the cost and the change in potential are $O(1)$. If one sequence becomes empty, we split the remaining sequence into two equal parts. The decrease in potential is equal to the length of the sequence before the splitting (the potential is zero after the split). The cost of splitting is thus covered by the decrease of potential.

The simulation by a BST is easy. We realize both sequences by chains attached to the root. The right chain contains the elements in the second stack with the top element as the right child of the root, the next to top element as the left child of the top element, and so on.

G.1.2 Extended Hand

To describe the simulation precisely, we borrow terminology from [13]. Let $T$ be a BST with a set $F$ of $k$ fingers $f_1, \ldots, f_k$. For convenience we assume the root of $T$ to be one of the fingers. Let $S(T, F)$ be the Steiner tree with terminals $F$. A knuckle is a connected component of $T$ after removing $S(T, F)$. Let $P(T, F)$ be the union of fingers and the degree-3 nodes in $S(T, F)$. We call $P(T, F)$ the set of pseudofingers. A tendon $\tau_{x,y}$ is the path connecting two pseudofingers $x, y \in P(T, F)$ (excluding $x$ and $y$) such that there is no other $z \in P(T, F)$ inside. We assume that $x$ is an ancestor of $y$.

The next definitions are new. For each tendon $\tau_{x,y}$, there are two half tendons, $\tau_{x,y}^<, \tau_{x,y}^>$ containing all elements in $\tau_{x,y}$ which are less than $y$ and greater than $y$ respectively. Let $H(T, F) = \{\tau_{x,y}^<, \tau_{x,y}^> \mid \tau_{x,y} \text{ is a tendon}\}$ be the set of all half tendons.

For each $\tau \in H(T, F)$, we can treat $\tau$ as an interval $[\min(\tau), \max(\tau)]$ where $\min(\tau), \max(\tau)$ are the minimum and maximum elements in $\tau$ respectively. For each $f \in P(T, F)$, we can treat $f$ as an trivial interval $[f, f]$.

Let $E(T, F) = P(T, F) \cup H(T, F)$ be the set of intervals defined by all pseudofingers $P(T, F)$ and half tendons $H(T, F)$. We call $E(T, F)$ an extended hand. Note that when we treat $P(T, F) \cup H(T, F)$ as a set of elements, such a set is exactly $S(T, F)$. So $E(T, F)$ can be viewed as a partition of $S(T, F)$ into pseudofingers and half-tendons.

We first state two facts about the extended hand.

- **Lemma 37.** Given any $T$ and $F$ where $|F| = k$, there are $O(k)$ intervals in $E(T, F)$.

  *Proof. Note that $|P(T, F)| \leq 2k$ because there are $k$ fingers and there can be at most $k$ nodes with degree 3 in $S(T, F)$. Consider the graph where pseudofingers are nodes and tendons are edges. That graph is a tree. So $|H(T, F)| = O(k)$ as well.**

- **Lemma 38.** Given any $T$ and $F$, all the intervals in $E(T, F)$ are disjoint.

  *Proof. Suppose that there are two intervals $\tau, x \in E(T, F)$ that intersect each other. One of them, say $\tau$, must be a half tendon. Because the intervals of pseudofingers are of length zero and they are distinct, they cannot intersect. We write $\tau = \{t_1, \ldots, t_k\}$ where $t_1 < \cdots < t_k$. Assume w.l.o.g. that $t_i$ is an ancestor of $t_{i+1}$ for all $i < k$, and so $t_k$ is an ancestor of a pseudofingers $f$ where $t_k < f$.**

\[^3\text{In [13], they define a hand which involves only the pseudofingers.} \]
Suppose that \( x \) is a pseudofinger and \( t_j < x < t_{j+1} \) for some \( j \). Since \( t_j \) is the first left ancestor of \( t_{j+1}, \) \( x \) cannot be an ancestor of \( t_{j+1} \) in \( T \). So \( x \) is in the left subtree of \( t_{j+1} \). But then \( t_{j+1} \) is a common ancestor of two pseudofingers \( x \) and \( f \), and \( t_{j+1} \) must be a pseudofinger which is a contradiction.

Suppose next that \( x = \{x_1, \ldots, x_t\} \) is a half tendon where \( x_1 < \cdots < x_t \). We claim that either \([x_1, x_2] \subset [t_j, t_{j+1}]\) for some \( j \) or \([t_1, t_k] \subset [x_{j'}, x_{j'+1}]\) for some \( j' \). Suppose not. Then there exist two indices \( j \) and \( j' \) where \( t_j < x_{j'} < t_{j+1} < x_{j'+1} \). Again, \( x_{j'} \) cannot be an ancestor of \( t_{j+1} \) in \( T \), so \( x_{j'} \) is in the left subtree of \( t_{j+1} \). We know either \( x_{j'} \) is the first left ancestor of \( x_{j'+1} \) or \( x_{j'+1} \) is the first right ancestor of \( x_{j'} \). If \( x_{j'} \) is an ancestor of \( x_{j'+1} \), then \( x_{j'+1} < t_{j+1} \) which is a contradiction. If \( x_{j'+1} \) is the first right ancestor of \( x_{j'} \), then \( t_{j+1} \) is not the first right ancestor of \( x_{j'} \) and hence \( x_{j'+1} < t_{j+1} \) which is a contradiction again. Now suppose w.l.o.g. \([x_1, x_t] \subset [t_j, t_{j+1}]\). Then there must be another pseudofinger \( f' \) in the left subtree of \( t_{j+1} \), hence \( \tau \) cannot be a half tendon, which is a contradiction.

### G.1.3 The structure of the simulating BST

In this section, we describe the structure of the BST \( T' \) that we maintain given a \( k \)-finger BST \( T \) and the set of fingers \( F \).

For each half tendon \( \tau \in H(T, F) \), let \( T'_{\tau} \) be the tree with \( \min(\tau) \) as a root which has \( \max(\tau) \) as a right child. \( \max(\tau) \)'s left child is a subtree containing the remaining elements \( \tau \setminus \{\min(\tau), \max(\tau)\} \). We implement a \textit{dequeBST} on this subtree as defined in §G.1.1. By Lemma 38, intervals in \( E(T, F) \) are disjoint and hence they are totally ordered. Since \( E(T, F) \) is an ordered set, we can define \( T''_{\tau} \) to be a balanced BST such that its elements correspond to elements in \( E(T, F) \). Let \( T_E' \) be the BST obtained from \( T''_{\tau} \) by replacing each node \( a \) in \( T''_{\tau} \) that corresponds to a half tendon \( \tau \in H(T, F) \) by \( T''_{\tau} \). That is, suppose that the parent, left child, and right child are \( a_u, a_l, \) and \( a_r \) respectively. Then the parent in \( T_E' \) of the root of \( T''_{\tau} \) which is \( \min(\tau) \) is \( a_u \). The left child in \( T_E' \) of \( \min(\tau) \) is \( a_l \) and the right child in \( T_E' \) of \( \max(\tau) \) is \( a_r \).

The BST \( T' \) has \( T_E' \) as its top part and each knuckle of \( T \) hangs from \( T_E' \) in a determined way.

**Lemma 39.** Each element corresponding to pseudofinger \( f \in P(T, F) \) has depth \( O(\log k) \) in \( T_E' \), and hence in \( T' \).

**Proof.** By Lemma 37, \( |E(T, F)| = O(k) \). So the depth of \( T_{\tau} \) is \( O(\log k) \). For each node \( a \) corresponding to a pseudofinger \( f \in P(T, F) \), observe that the depth of \( a \) in \( T_E' \) is at most twice the depth of \( a \) in \( T''_{\tau} \) by the construction of \( T''_{\tau} \).

### G.1.4 The cost for simulating the \( k \)-finger BST

We finally prove Theorem 35. That is, we prove that whenever one of the fingers in a \( k \)-finger BST \( T \) moves to its neighbor or rotates, we can update the maintained BST \( T' \) to have the structure as described in the last section with cost \( O(\log k) \).

We state two observations which follow from the structure of our maintained BST \( T' \) described in §G.1.3. The first observation follows immediately from Lemma 36.

**Fact 3.** For any half tendon \( \tau \in H(T, F) \), we can insert or delete the minimum or maximum element in \( T''_{\tau} \) with cost \( O(1) \) amortized.

Next, it is convenient to define a set \( A \), called \textit{active set}, as a set of pseudofingers, the roots of knuckles whose parents are pseudofingers, and the minimum or maximum of half tendons.
Fact 4. When a finger \( f \) in a \( k \)-finger BST \( T \) moves to its neighbor or rotates with its parent, the extended hand \( E(T, F) = P(T, F) \cup H(T, F) \) is changed as follows.

1. There are at most \( O(1) \) half tendons \( \tau \in H(T, F) \) whose elements are changed. Moreover, for each changed half tendon \( \tau \), either the minimum or maximum is inserted or deleted. The inserted or deleted element \( a \) was or will be in the active set \( A \).
2. There are at most \( O(1) \) elements added or removed from \( P(T, F) \). Moreover, the added or removed elements were or will be in the active set \( A \).

Lemma 40. Let \( a \in A \) be an element in the active set. We can move \( a \) to the root with cost \( O(\log k) \) amortized. Symmetrically, the cost for updating the root \( r \) to become some element in the active set is \( O(\log k) \) amortized.

Proof. There are two cases. If \( a \) is a pseudofinger or a root of a knuckle whose parent is pseudofinger, we know that the depth of \( a \) was \( O(\log k) \) by Lemma 39. So we can move \( a \) to root with cost \( O(\log k) \). Next, if \( a \) is the minimum or maximum of a half tendon \( \tau \), we know that the depth of the root of the subtree \( T_\tau \) is \( O(\log k) \). Moreover, by Lemma 40 we can delete \( a \) from \( T_\tau \) (make \( a \) a parent of \( T_\tau \)) with cost \( O(1) \) amortized. Then we move \( a \) to root with cost \( O(\log k) \) worst-case. The total cost is then \( O(\log k) \) amortized. The proof for the second statement is symmetric.

Lemma 41. When a finger \( f \) in a \( k \)-finger BST \( T \) moves to its neighbor or rotates with its parent, the BST \( T' \) can be updated accordingly with cost \( O(\log k) \) amortized.

Proof. According to 4, we separate our cost analysis into two parts.

For the fist part, let \( a \in A \) be the element to be inserted into a half tendon \( \tau \). By Lemma 40 we move \( a \) to root with cost \( O(\log k) \) and then insert \( a \) as a minimum or maximum element in \( T_\tau \) with cost \( O(\log k) \). Deleting \( a \) from some half tendon with cost \( O(\log k) \) is symmetric.

For the second part, let \( a \in A \) be the element to be inserted into a half tendon \( \tau \). By Lemma 40 again, we move \( a \) to root and move back to the appropriate position in \( T'_E \) with cost \( O(\log k) \). We also need rebalance \( T'_E \) but this also takes cost \( O(\log k) \).

Proof. [Proof of Theorem 39] We describe the simulation algorithm \( A_{sim} \) with overhead \( O(\log k) \). Let \( A_{kBST} \) be an algorithm for the \( k \)-finger BST \( T \). Whenever there is an update in the \( k \)-finger BST \( T \) (i.e. a finger moves to its neighbor or rotates), \( A_{sim}(A_{kBST}) \) updates the BST \( T' \) according to Lemma 41 with cost \( O(\log k) \) amortized. \( T' \) is maintained so that its structure is as described in §G.1.3. By Lemma 39 we can access any finger \( f \) of \( T \) from the root of \( T' \) with cost \( O(\log k) \). Therefore, the cost of \( A_{sim}(A_{kBST}) \) is at most \( O(\log k) \) times the cost of \( A_{kBST} \).

G.2 Lazy finger bounds with auxiliary elements

Recall that \( LF(S) \) is defined as the minimum over all BSTs \( T \) over \([n]\) of \( LF_T(S) \). It is convenient to define a slightly stronger lazy finger bound that also allows auxiliary elements. Define \( \overline{LF}(S) \) as the minimum over all binary search trees \( T \) that contains the keys \([n]\) (but the size of \( T \) can be much larger than \( n \)). We define \( \overline{LF}^k(S) \) as the \( k \)-lazy finger bound when the tree is allowed to have auxiliary elements. We argue that the two definitions are equivalent.

Theorem 42. For any integer \( k \), \( LF^k(S) = \Theta(\overline{LF}^k(S)) \) for all \( S \).
Proof. It is clear that $\overline{LF}^k(S) \leq LF^k(S)$. We only need to show the converse.

Let $T$ be the binary search tree (with auxiliary elements) such that $LF^k_T(S) = \overline{LF}^k(S)$. Denote by $\hat{f}$ the optimal finger strategy on $T$. Let $[n] \cup X$ be the elements of $T$ where $X$ is the set of auxiliary elements in $T$. For each $a \in [n] \cup X$, let $d_T(a)$ be the depth of key $a$ in $T$, and let $w(i) = 4^{-d_T(i)}$. For any two elements $i$ and $j$ and set $Y \subseteq [n] \cup X$, let $w_Y[i : j]$ be the sum of the weight $\sum_{k \in Y \cap [i,j]} w(i)$. For any $i, j \in [n] \cup X$ such that $i \leq j$, we have

$$\lg \frac{w_{[n] \cup X}[i : j]}{\min(w(i), w(j))} = O(d_T(i,j)),$$

where $d_T(i,j)$ is the distance from $i$ to $j$ in $T$. So, this same bound also holds when considering only keys in $[n]$. That is, for $i, j \in [n]$, we have

$$\lg \frac{w_{[n]}[i : j]}{\min(w(i), w(j))} = O(d_T(i,j)).$$

Given the weight of $\{w(a)\}_{a \in [n]}$, the BST $T'$ (without auxiliary elements) is constructed by invoking Lemma [1]. We bound the term $LF^k_T(S)$ (using strategy $\hat{f}$) by

$$O(\sum_{i} d_T(s_{s_{\sigma(f,t)}, s_t}) = O(\sum_{i} \lg \frac{w_{[n]}[s_t : s_{s_{\sigma(f,t), s_t}}]}{\min(w(s_t), w(s_{s_{\sigma(f,t), s_t}}))}) = O(LF^k_T(S))$$

where $S = (s_1, \ldots, s_m)$. Therefore, $LF^k(S) \leq LF^k_T(S) = O(LF^k_T(S)) = O(\overline{LF}^k(S))$. $\blacksquare$
H.1 Auxiliary Elements

For any $X \subseteq \mathbb{Q}$ finite subset of rational numbers, we let $\text{OPT}_X(S)$ be the optimal cost for executing $S$ on a BST which contains $[n] \cup X$ as elements. We call $X$ auxiliary elements. Let $\text{OPT} \_\text{aux}(S) = \min_X \text{OPT}_X(S)$. The following theorem shows that auxiliary elements never improve the optimal cost.

\begin{theorem} \label{thm:auxiliary-elements}
\text{OPT}(S) = \text{OPT}_X(S) for any sequence $S$ and auxiliary elements $X$.
\end{theorem}

\begin{proof}
It is clear that $\text{OPT}_X(S) \leq \text{OPT}(S)$, because we can start with the initial tree such that no element in $X$ is above any element in $[n]$, and then use an optimal algorithm on a BST with $[n]$ as elements to arrange the tree without touching any element in $X$.

Next, to show that $\text{OPT}(S) \leq \text{OPT}_X(S)$, we will prove that for any $Y \subseteq \mathbb{Q}$ and another number $y \in \mathbb{Q} \setminus Y$, $\text{OPT}_Y(S) \leq \text{OPT}_{Y \cup \{y\}}(S)$. It suffices to prove that, given an optimal BST algorithm $A$ for executing $S$ on a BST $T_A$ with $[n] \cup Y \cup \{y\}$ as elements, we can obtain another BST algorithm $B$ on a BST $T_B$ with $[n] \cup Y$ as elements whose cost for execute $S$ is at most the cost of $A$ which is $\text{OPT}_{Y \cup \{y\}}(S)$.

Let $y' \in [n] \cup Y$ be either the predecessor or successor of $y$. At each time, the algorithm $B$ arranges the elements such that the depth of $y'$ is $d_{T_B}(y') = \min\{d_{T_A}(y'), d_{T_A}(y)\}$ and the relative depth of other elements $[n] \cup Y \setminus \{y'\}$ in $T_B$ are equivalent as the corresponding elements in $T_A$. Observe that, for any element $i \in [n] \cup Y$, the search path of $i$ in $T_B$ is a subset of the search path of $i$ in $T_A$. So the cost of $B$ at any access is at most of the cost of $A$. This conclude the proof.
\end{proof}

H.2 Proof of Interleaving Bound

Before we can prove Theorem 17, we need one lemma.

\begin{lemma} \label{lem:interleaving-bound}
For any sequence $S \in [n]^m$, there is a BST algorithm $A$ on a BST $T_A$ with $[n] \cup X$ as elements where $X$ is some set of auxiliary elements and all elements in $[n]$ are maintained as leaves in $T_A$. Moreover, the cost of $A$ is at most $3\text{OPT}(S)$.
\end{lemma}

\begin{proof}
Let $B$ be an optimal BST algorithm on a BST $T_B$ with $[n]$ as elements for executing $S$. To construct $T_A$, for each element $i$, we replace $i$ in $T_B$ with three elements $i_L, i_R, i$ where $i - 1 < i_L < i < i_R < i + 1$ and $i_L$ is always a parent of $i_R$ which is always a parent of $i$. Therefore, $d_{T_B}(i) = 3d_{T_B}(3)$ and $i$ is leaf, for each $i \in [n]$. After each access, if $B$ rearranges $T_B$, we rearrange $T_A$ accordingly, which costs at most 3 times as much.
\end{proof}

\begin{proof}[Proof of Theorem 17]
Let $P = ([a_1, b_1], \ldots, [a_k, b_k])$ be the partition of $[n]$ from the theorem. We show a BST algorithm $A$ on a BST $T_A$ with some auxiliary elements such that the cost for accessing $S$ is at most $\sum_{i=1}^k \text{OPT}(S_i) + 3\text{OPT}(\tilde{S})$.

To describe $T_A$, we construct a BST $\tilde{T_A}$ that has $[k]$ as leaves using Lemma 44. Then, for each leaf $i \in [k]$, we replace $i$ with the root of a subtree $T^{(i)}_{\tilde{T_A}}$ containing elements in $[a_i, b_i]$. That is, the auxiliary elements in $\tilde{T_A}$ are always above elements in $T^{(i)}_{\tilde{T_A}}$ for all $i \in [k]$. To describe the algorithm $A$, if an element $x \in [a_i, b_i]$ is accessed, we first access the root of $T^{(i)}_{\tilde{T_A}}$ using an algorithm from Lemma lem:integer as leaves, and then we access $x$ inside $T^{(i)}_{\tilde{T_A}}$ using the optimal algorithm for executing $S_i$. By Lemma lem:integer as leaves, the total cost spent in $\tilde{T_A}$ is $3\text{OPT}(\tilde{S})$, and the total cost spent in $T^{(i)}_{\tilde{T_A}}$ is $\text{OPT}(S_i)$ for each $i$.
\end{proof}
Omitted Proofs from Section 6

I.1 Proof of Theorem 19

All bounds are derived via information-theoretic arguments. To avoid the need to do probabilistic analysis, we use instead the language of Kolmogorov complexity, which is applicable to a specific input, rather than a distribution on inputs. First, we state the following proposition.

Lemma 45 (\[7\]). For any sequence \(S\), let \(K(S)\) denote Kolmogorov complexity of \(S\). We have \(K(S) = O(\text{OPT}(S))\).

We also use the following standard fact in the theory of Kolmogorov complexity.

Lemma 46. Let \(S\) be a subset of sequences in \([n]^m\). There exists a sequence \(S' \in S\) with \(K(S') = \Omega(\log |S|)\).

We are now ready to derive all bounds.

Theorem 47 (\(k\)-lazy fingers). For each \(k\) (possibly a function that depends on \(n\)), there are infinitely many sequences \(X\), for which \(LF^k(S) \cdot \log k \leq \text{OPT}(S)\).

Proof. Let \(J \subseteq [n] : |J| = k\). We choose a sequence \(S \in J^m\) that has Kolmogorov complexity at least \(K(S) = \Omega(m \log k)\), so we have \(\text{OPT}(S) = \Omega(m \log k)\). Now we argue that the lazy finger cost is low. Choose the reference tree as an arbitrary balanced tree \(T\). Notice that the lazy finger bound is \(LF^k_J(S) = O(m)\): Initial fingers can be chosen to be the location of keys in \(J\). Afterwards, each key in \(J\) is served by its private finger.

Theorem 48 (Monotone). For each \(k\), there are infinitely many sequences \(S\) for which \(m(S) = k\), and \(K(S) \geq |S| \cdot \log k\).

Proof. This follows from the result of Regev \[28\] (who proved much more general results) which implies that for sufficiently large \(n\), the number of permutations \(S \in [n]^n\) that avoid \((1, \ldots, k)\) is at least \(k^{\Omega(n)}\). Therefore, there exists a permutation \(S\) with \(K(S) = \Omega(n \log k)\).

Pattern avoidance.

We argue that there is a permutation sequence \(S\) of size \(n\) that avoids a pattern of size \(k\) (an indication of easiness) but nevertheless has high Kolmogorov complexity \(K(S) = \Omega(n \sqrt{k})\).

Let \(\pi\) be a permutation of size \(k\). Let \(S_{\pi}(n)\) be the number of permutations of size \(n\) avoiding \(\pi\), and let \(L(\pi) = \lim_{n \to \infty} S_{\pi}(n)^{1/n}\). Let \(ex(n, \pi)\) be the maximum number of non-zeros of an \(n \times n\) matrix that avoids \(\pi\), and let \(c(\pi) = ex(n, \pi)/n\). Cibulka \[8\] shows that \(c(\pi) = O(L(\pi)^{1.5})\) and \(L(\pi) = c(\pi)^2\); for a simpler proof of the second result see \[16\].

Theorem 49 (\[8,16\]). \(c(\pi) = L(\pi)^{\Theta(1)}\).

In \[16\], Fox also shows the surprising result that there exists a permutation \(\pi\) of size \(k\) such that \(c(\pi) = 2^{\Omega(k^{1/4})}\). It is also shown in \[16\] that for any \(\pi\) of size \(k\), \(c(\pi) = 2^{O(k)}\), improving the celebrated result of \(2^{O(k \log k)}\) by Marcus and Tardos \[23\]. Geneson and Tian \[18\] improve the lower bound as follows.

Theorem 50 (\[18\]). There exists \(\pi\) of size \(k\) where \(c(\pi) = 2^{\Omega(k^{1/2})}\).
Combining Theorem 49 and Theorem 50, we have that there exists \( \pi \) such that \( L(\pi) = 2^\Omega(k^{1/3}) \). By the definition of \( L(\pi) \), we conclude that there is a permutation \( \pi \) of size \( k \) and infinitely many integers \( n \) such that \( S_\pi(n) = \Omega(2^{n\sqrt{k}}) \). Let \( P \) be such a set of permutations. We know that there must be a permutation \( S \in P \) such that the Kolmogorov complexity of \( S \) is \( K(S) = \Omega(n\sqrt{k}) \).

### 1.2 Proof of Theorem 20

Let \( n \) be an integer multiple of \( k \) and \( \ell = n/k \). Consider the tilted \( k \)-by-\( \ell \) grid \( S_k \). The access sequence is \( 1, \ell + 1, \ldots, \ell \cdot (k - 1) + 1, 2, \ell + 2, \ldots, (k - 1)\ell + 2, \ldots, (k - 1)\ell + \ell \). It is clear that this sequence avoids \( (k + 1, k, \ldots, 1) \), so the third part of the theorem follows easily.

We now show that \( LF_k(S_k) = O(n) \). We write \( S_k = (s_1, \ldots, s_n) \). The idea is to partition the keys into blocks, and use each finger to serve only the keys inside blocks. In particular, for each \( i = 1, \ldots, k \), denote by \( B_i \subseteq [n] \) the set of keys in \([\ell(i - 1) + 1, \ell \cdot i] \). We create a reference tree \( T \) and argue that \( LF_k(T, S_k) = O(n) \). Let \( T_0 \) be a BST of height \( O(\log k) \) and with \( k \) leaves. Each leaf of \( T_0 \) corresponds to the keys \( \{\ell \cdot (i - 1) + j + 1\}_{j=1}^{k} \). The non-leaves of \( T_0 \) are assigned arbitrary fractional keys that are consistent with the BST properties. For each \( i \), path \( P_i \) is defined as a BST with key \( \ell \cdot (i - 1) + 1 \) at the root, where for each \( j = 0, \ldots, (\ell - 1) \), the key \( \ell(i - 1) + j \) has \( \ell(i - 1) + (j + 1) \) as its only (right) child. The final tree \( T \) is obtained by hanging each path \( P_i \) as a left subtree of a leaf \( \ell \cdot (i - 1) + \frac{1}{2} \). The \( k \)-server strategy is simple: The \( i \)th finger only takes care of the elements in block \( B_i \). The cost for the first access in block \( B_i \) is \( O(\log k) \), and afterwards, the cost is only \( O(1) \) per access. So the total access cost is \( O(\frac{k}{\ell} \log k + n) = O(n) \).

To see that \( OPT(S_k) = O(n) \), let \( S_k^{(i)} \) be obtained from \( S_k \) by restriction to \( B_i \) for each \( i = 1, \ldots, k \). Let \( \tilde{S}_k = (\tilde{s}_1, \ldots, \tilde{s}_n) \in [k]^n \) where \( \tilde{s}_j = i \) iff \( s_j \in B_i \). By Theorem 17 \( OPT(S_k) \leq \sum_{i=1}^{k} OPT(S_k^{(i)}) + O(OPT(\tilde{S}_k)) \). But, for each \( i \), \( OPT(S_k^{(i)}) = O(\ell) \) because \( S_k^{(i)} \) is just a sequential access of size \( \ell \). \( OPT(\tilde{S}_k) = O(k \cdot \ell) = O(n) \) because \( \tilde{S}_k \) a sequential access of size \( k \) repeated \( \ell \) times. Therefore, \( OPT(S_k) = O(n) \).

The rest of this section is devoted to proving the following:

**Theorem 51.** \( LF^{k-1}(S_k) = \Omega(\frac{n}{k} \log(k/n)) \)

A finger configuration \( \tilde{f} = (f(1), \ldots, f(k - 1)) \in [n]^{k-1} \) specifies to which keys the fingers are currently pointing. Let \( T \) be a reference tree. Any finger strategy can be described by a sequence \( \tilde{f}_1, \ldots, \tilde{f}_t \), where \( \tilde{f}_t \) is a configuration after element \( s_t \) is accessed. Just like in a general \( k \)-server problem, we may assume w.l.o.g. the following:

**Fact 5.** For each time \( t \), the configurations \( \tilde{f}_t \) and \( \tilde{f}_{t+1} \) differ at exactly one position. In other words, we only move the finger that is used to access \( s_{t+1} \).

We see the input sequence \( S_k \) as having \( \ell \) phases: The first phase contains the subsequence \( 1, \ell + 1, \ldots, \ell(k - 1) + 1 \), and so on. Each phase is a subsequence of length \( k \).

**Lemma 52.** For each phase \( p \in \{1, \ldots, \ell\} \), there is time \( t \in [(p - 1)\ell + 1, p \cdot \ell] \) such that \( s_t \) is accessed by finger \( j \) such that \( f_{t-1}(j) \) and \( f_t(j) \) are in different blocks, and \( f_{t-1}(j) < f_t(j) \). That is, this finger moves to the block \( B_{b'} \), \( b = t \mod k \), from some block \( B_{b''} \), where \( b' < b \), in order to serve \( s_t \).

**Proof.** Suppose not. By 7, this implies that each finger is used to access each element only once in this phase (because accesses in this phase are done in blocks \( B_1, \ldots, B_k \) in this order). This is impossible because we only have \( k - 1 \) fingers. ■
The landscape of bounds for binary search trees

For each phase $p \in [ℓ]$, let $t_p$ denote the time for which such a finger moves across the blocks from left to right; if they move more than once, we choose $t_p$ arbitrarily. Let $J = \{t_p\}_{p=1}^{ℓ}$. For each finger $j \in [k-1]$, each block $i \in [k]$ and block $i' \in [k]$ : $i < i'$, let $J(j,i,i')$ be the set containing the time $t$ for which finger $f(j)$ is moved from block $B_i$ to block $B_{i'}$ to access $s_i$. Let $c(j,i,i') = |J(j,i,i')|$. Notice that $\sum_{j,i,i'} c(j,i,i') = \frac{n}{k} = ℓ$, due to the lemma. Let $P(j,i,i')$ denote the phases $p$ for which $t_p \in J(j,i,i')$.

Lemma 53. $\sum_{j,i,i' \in J(j,i,i')} c(j,i,i') \geq n/2k$ if $n = \Omega(k^4)$.

Proof. There are only at most $k^3$ triples $(j,i,i')$, so the terms for which $c(j,i,i') < 16$ contribute to the sum at most $16k^3$. This means that the sum of the remaining is at least $n/k - 16k^3 \geq n/2k$ if $n$ satisfies $n = \Omega(k^4)$.

From now on, we consider the sets $J'$ and $J'(j,i,i')$ that only concern those $c(j,i,i')$ with $c(j,i,i') \geq 16$ instead.

Lemma 54. There is a constant $\eta > 0$ such that the total access cost during the phases $P(j,i,i')$ is at least $\eta c(j,i,i') \log c(j,i,i')$.

Once we have this lemma, everything is done. Since the function $g(x) = x \log x$ is convex, we apply Jensen’s inequality to obtain:

$$\frac{1}{|J'|} \sum_{j,i,i'} \eta c(j,i,i') \log c(j,i,i') \geq \eta \left(\frac{n}{2k|J'|}\right) \log(n/2k|J'|).$$

Note that the left side is the term $\mathbb{E}[g(x)]$, while the right side is $g(\mathbb{E}(x))$. Therefore, the total access cost is at least $\frac{\eta n}{2k} \log(n/2k)$, due to the fact that $|J'| \leq k^3$. We now prove the lemma.

Proof. [Proof of Lemma 54] We recall that, in the phases $P(j,i,i')$, the finger-$j$ moves from block $B_i$ to $B_{i'}$ to serve the request at corresponding time. For simplicity of notation, we use $\tilde{J}$ and $C$ to denote $J(j,i,i')$ and $c(j,i,i')$ respectively. Also, we use $\tilde{J}$ to denote the finger-$j$. For each $t \in \tilde{J}$, let $a_t \in B_i$ be the key for which the finger $f$ moves from $a_t$ to $s_t$ when accessing $s_t \in B_{i'}$. Let $J = \{t_1, \ldots, t_C\}$ such that $a_{t_1} < a_{t_2} < \ldots < a_{t_C}$. Let $R$ be the lowest common ancestor in $T$ of keys in $[a_{t_C/2}+1, a_{t_C}]$.

Lemma 55. For each $r \in [1, \ldots, \lfloor C/2 \rfloor]$, the access cost of $s_{t_r}$ and $s_{t_{C-r}}$ is together at least $\min\{d_T(R,s_{t_r}),d_T(R,s_{t_{C-r}})\}$.

Proof. Let $u_r$ be the lowest common ancestor between $a_{t_r}$ and $s_{t_r}$. If $s_{t_r}$ is in the subtree rooted at $R$, then the cost must also be at least $d_T(u_r, s_{t_r}) \geq d_T(R, s_{t_r})$ as $u_r$ must be an ancestor of $R$ (because $a_{t_r} < a_{t_C/2} < a_{t_C} < s_{t_r}$). Otherwise, we know that $s_{t_r}$ is outside of the subtree rooted at $R$, and so is $s_{t_{C-r}}$. On the other hand, $a_{t_{C-r}}$ is in such subtree, so moving the finger from $a_{t_{C-r}}$ to $s_{t_{C-r}}$ must touch $R$, therefore costing at least $d_T(R, s_{t_{C-r}})$.

Lemma 55 implies that, for each $r = 1, \ldots, \lfloor C/2 \rfloor$, we pay the distance between some element $v_r \in \{s_{t_r}, s_{t_{C-r}}\}$ to $R$. The total such costs would be $\sum_r d_T(R, v_r)$. Applying the fact that (i) $v_r$’s are different and (ii) there are at most $3^d$ vertices at distance $d$ from a vertex $R$, we conclude that this sum is at least $\sum_r d_T(R, v_r) \geq \Omega(C \log C)$. 

1.3 Working set and $k$-lazy finger bounds are incomparable

We show the following.

 ► Theorem 56.

 1. There exists a sequence $S$ such that $WS(S) = o(LF^k(S))$, and
 2. There exists a sequence $S'$ such that $LF^k(S') = o(WS(S'))$.

The sequence $S'$ above is straightforward: For $k = 1$, just consider the sequential access $1, \ldots, n$ repeated $m/n$ times. For $m$ large enough, the working set bound is $\Omega(m \log n)$. However, if we start with the finger on the root of the tree which is just a path, then the lazy finger bound is $O(m)$. The $k$-lazy finger bound is always less than lazy finger bound, so this sequence works for the second part of the theorem.

The existence of the sequence $S$ is slightly more involved (the special case for $k = 1$ was proved in [4]), and is guaranteed by the following theorem, the proof of which comprises the remainder of this section.

 ► Theorem 57. For all $k = O(n^{1/2-\epsilon})$, there exists a sequence $S$ of length $m$ such that $WS(S) = O(m \log k)$ whereas $LF^k(S) = \Omega(m \log(n/k))$.

We construct a random sequence $S$ and show that while $WS(S) = O(m \log k)$ with probability one, the probability that there exists a tree $T$ such that $LF^k_T(S) \leq cm \log_2(n/k)$ is less than $1/2$ for some constant $c < 1$. This implies the existence of a sequence $S$ such that for all trees $T$, $LF_T^k(S) = \Omega(m \log(n/k))$.

The sequence is as follows. We have $Y$ phases. In each phase we select $2k$ elements $R_i = \{r_j^i\}_{j=1}^{2k}$ uniformly at random from $[n]$. We order them arbitrarily in a sequence $S_i$, and access $[S_i]_{X/2k}$ (access $S_i X/2k$ times). The final sequence $S$ is a concatenation of the sequences $[S_i]_{X/2k}$ for $1 \leq i \leq Y$. Each phase has $X$ accesses, for a total of $m = XY$ accesses overall. We will choose $X$ and $Y$ appropriately later.

**Working set bound.** One easily observes that $WS(S) = O(Y (2k \log n + (X - 2k) \log(2k)))$, because after the first $2k$ accesses in a phase, the working set is always of size $2k$. We choose $X$ such that the second term dominates the first, say $X \geq 5k \log n \log 2k$. We then have that the working set bound is $O(XY \log k) = O(m \log k)$, with probability one.

**$k$-lazy finger bound.** Fix a BST $T$. We classify the selection of the set $R_i$ as being $d$-good for $T$ if there exists a pair $r_j^i, r_t^i \in R_i$ such that their distance in $T$ is less than $d$. The following lemma bounds the probability of a random selection being $d$-good for $T$.

 ► Lemma 58. Let $T$ be any BST. The probability that $R_i$ is $d$-good for $T$ is at most $8k^2 3^d / n$.

*Proof.* We may assume $8k^2 3^d / n < 1$ as the claim is void otherwise. We compute the probability that a selection $R_i$ is not $d$-good first. This happens if and only if the balls of radius $d$ around every element $r_j^i$ are disjoint. The volume of such a ball is at most $3^d$, so we can bound this probability as
\[ P[R_i \text{ is not } d\text{-good for } T] = \Omega^{2k-1} \left( 1 - \frac{3^d}{n} \right) \]
\[ \geq \left( 1 - \frac{2k3^d}{n} \right)^{2k} \]
\[ \Rightarrow P[R_i \text{ is } d\text{-good for } T] \leq 1 - \left( 1 - \frac{2k3^d}{n} \right)^{2k} \]
\[ = 1 - \exp \left( 2k \ln \left( 1 - \frac{2k3^d}{n} \right) \right) \]
\[ \leq 1 - \exp \left( -8k^23^d/n \right) \]
\[ \leq 8k^23^d/n, \]
where the last two inequalities follow from \( \ln(1-x) > -2x \) for \( x \leq 1/2 \) (note that \( 8k^23^d/n < 1 \) implies \( 2k3^d/n \leq 1/2 \)) and \( e^x > 1 + x \), respectively.

Observe that if \( R_i \) is not \( d\text{-good} \), then the \( k\)-lazy finger bound of the access sequence \([S_i]\) is \( \Omega(d(X-k)) = \Omega(dX) \). This is because in every occurrence of \( S_i \), there will be some \( k \) elements outside the \( 2k \) total that will be outside the \( d \)-radius balls centered at the current \( k \) fingers.

We call the entire sequence \( S \) \( d\text{-good} \) for \( T \) if at least half of the sets \( R_i \) are \( d\text{-good} \) for \( T \). Thus if \( S \) is not \( d\text{-good} \), then \( LF_k^d(S) = \Omega(XYd) \).

\textbf{Lemma 59.} \( P[S \text{ is } d\text{-good for } T] \leq \left( \frac{32k^23^d}{n} \right)^{Y/2} \).

\textit{Proof.} By the previous lemma and by definition of goodness of \( S \), we have that

\[ P[S \text{ is } d\text{-good for } T] \leq \left( \frac{Y}{Y/2} \right) \left( \frac{8k^23^d}{n} \right)^{Y/2} \]
\[ \leq 4^{Y/2} \left( \frac{8k^23^d}{n} \right)^{Y/2} \]
\[ = \left( \frac{32k^23^d}{n} \right)^{Y/2}. \]

The theorem now follows easily. Taking a union bound over all BSTs on \([n]\), we have

\[ P[S \text{ is } d\text{-good for some BST } T] \leq 4^n \left( \frac{32k^23^d}{n} \right)^{Y/2}. \]

Now set \( Y = 2n \). We have that

\[ P[\exists \text{ a BST } T : LF_k^d(S) \leq md/4] \leq 4^n \left( \frac{32k^23^d}{n} \right)^n. \]

Putting \( d = \log_3 \frac{n}{2k^23^d} \) gives that for some constant \( c < 1 \),

\[ P[\exists \text{ a BST } T : LF_k^d(S) \leq c(m \log(n/k))] \leq 4^n \left( \frac{32k^23^d}{n} \right)^n = 1/2 \]

which implies that with probability at least 1/2 one of the sequences in our random construction will have \( k\)-lazy finger bound that is \( \Omega(m \log(n/k)) \). The working set bound is always \( O(m \log k) \). This establishes the theorem.