Length scales and eddy viscosities in turbulent free shear flows

Gioacchino Cafiero
Turbulence, Mixing and Flow control group,
Dept of Aeronautics, Imperial College London (UK)

Martin Obligado
Univ. Grenoble Alpes, CNRS, Grenoble INP, LEGI, 38000 Grenoble, France

John Christos Vassilicos
Turbulence, Mixing and Flow control group,
Department of Aeronautics, Imperial College London (UK)

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Abstract

In the present paper, we address the important point of the proportionality between the longitudinal integral lengthscale ($L$) and the characteristic mean flow width ($\delta$) using experimental data of an axisymmetric wake and a turbulent planar jet. This is a fundamental hypothesis when deriving the self-similar scaling laws in free shear flows, irrespective of turbulence dissipation scaling. We show that $L/\delta$ is indeed constant, at least in a range of streamwise distances between 15 and 50 times the characteristic inlet dimension ($L_{ref}$, nozzle width or wake generator size). Furthermore, we revisit turbulence closure models such as the Prandtl mixing length and the constant eddy viscosity in the light of the non-equilibrium dissipation scalings. We show that the mixing length model, with $l_m \sim \delta$, does not comply with the scalings stemming from the non-equilibrium version of the theory; we instead obtain $l_m \sim \delta \sqrt{Re_G/Re_0\delta}$, where $Re_G$ and $Re_0\delta$ are a global and local Reynolds number, respectively. Similarly, the eddy viscosity model holds in the case of the non-equilibrium version of the theory provided that the eddy viscosity is constant everywhere, not only across sections orthogonal to the streamwise direction as in the equilibrium case. We conclude comparing the results of the different models with each other and with experimental data and with an improved model (following Townsend) that accounts for the intermittency of the flow and corrects for the eddy viscosity variation across the flow boundaries.
INTRODUCTION

Free shear flows are of significant importance in many natural and industrial applications. They are also of great interest for fundamental research, as it is one of the few cases in turbulence where mean quantities can be predicted under a small, physically based, set of hypotheses. The theory, derived by Townsend \cite{2} and later by George \cite{3}, requires the self-preservation of some turbulence quantities in the mean momentum and streamwise kinetic energy equations. In order to close the equations, an \textit{ad hoc} assumption is required to model the dissipation term in the kinetic energy equation. The closure usually chosen is the one consistent with the Richardson-Kolmogorov cascade. One assumes that the centreline turbulence dissipation rate $\varepsilon$ can be described as $\varepsilon = C_{\varepsilon}K^{3/2}/L$, where $K$ is the turbulent kinetic energy, $L$ is an integral length-scale of turbulence (usually taken to be the longitudinal one) and $C_{\varepsilon}$ is a dimensionless coefficient which may depend on boundary conditions but it is independent of Reynolds number at high enough Reynolds number values. Finally, the integral lengthscale $L$ is assumed to be proportional to a mean flow profile width such as the wake/jet width $\delta$.

Focusing on the free shear flows that will be investigated in the present work, namely the axisymmetric wake and the planar jet, this theoretical approach leads to the following streamwise evolutions of the centreline velocity (jet) or velocity deficit (wake) $u_0$ and the jet or wake width $\delta$:

\begin{align}
  u_0 &\sim (x - x_0)^a, \\
  \delta &\sim (x - x_0)^b,
\end{align}

with $a = -1/2$ and $b = 1$ for the planar jet and $a = -2/3$ and $b = 1/3$ for the axisymmetric wake \cite{2, 3, 4} ($x$ is the streamwise coordinate and $x_0$ is a virtual origin).

Previous to Townsend \cite{2} and George \cite{3}, researchers were already able to predict these exact same streamwise dependencies of $\delta$ and $u_0$ under different assumptions. A closure of the mean momentum equation can be given by assuming that the relevant component of the Reynolds shear stress tensor, $<u'v'>$, is related to the streamwise mean flow velocity $\bar{u}$ by

\begin{equation}
  <u'v'> = -\nu_T \frac{\partial \bar{u}}{\partial y}
\end{equation}

where $y$ is the spreading direction of the flow and $\nu_T$ is the eddy viscosity. The modelling of $\nu_T$ has been the focus of intense research during the first half of the 20th century (\cite{5}, \cite{6}).
and has been studied for many free shear flows. The most basic and common hypotheses used are Prandtl’s mixing length hypothesis,
\[ \nu_T = l_m^2 \frac{\partial \pi}{\partial y}, \]
where \( l_m \) is the mixing length, and a constant eddy viscosity hypothesis
\[ \nu_T = L_{eddy} U_{eddy} \]
where \( L_{eddy} \) and \( U_{eddy} \) are characteristic scales of length and velocity, respectively, which may depend on \( x \) but are constant along \( y \).

Equations (1) and (2) are retrieved with \( l_m \sim \delta \) and \( L_{eddy} \sim \delta \). In this sense, the Richardson-Kolmogorov cascade (which is one of the pillars of the Townsend-George approach given that it adopts \( \varepsilon \sim K^{3/2}/\delta \)) is consistent with both the constant eddy viscosity and the Prandtl’s mixing length hypotheses.

Recent works have unveiled the presence of turbulence dissipation scalings in free shear flows which are at odds with the Richardson-Kolmogorov scaling for \( \varepsilon \) (9, 10, 11, 12). Direct numerical simulations (DNS) and experiments for axisymmetric wakes and experimental data for the plane jet suggest the presence of free shear flows that follow nonequilibrium scalings for dissipation, at least for a large portion of the flow. The dissipation parameter \( C_\varepsilon \) is no longer independent of Reynolds number (though it does remain independent of the fluid’s kinematic viscosity \( \nu \)), but scales as \( \text{Re}_G / \text{Re}_\delta \) where \( \text{Re}_G = U_{ref} L_{ref} / \nu \), \( (U_{ref} \text{ being the free stream or inlet velocity and } L_{ref} \text{ a characteristic inlet lengthscale such as the wake generator’s size or the nozzle width) is the global Reynolds number and } \text{Re}_\delta = \sqrt{K_0 \delta / \nu} \) \( (K_0 \text{ being the turbulent kinetic energy on the flow centreline}) \) is a local Reynolds number. As evidenced by Dairay et al. (10) and Cafiero and Vassilicos (9), the application of the non-equilibrium dissipation scaling, makes it possible to use a smaller number of assumptions than Townsend (2) and George (3) and leads to new exponents for eq. (1) and (2). In the planar jet case, \( a = -1/3 \) and \( b = 2/3 \) while in the axisymmetric wake case, \( a = -1 \) and \( b = 1/2 \). It is important to explicitly notice that both in the classical and the non equilibrium dissipation scaling the assumption \( L \sim \delta \) is needed. In table I we summarise the scalings stemming from the Richardson-Kolmogorov equilibrium dissipation and the non-equilibrium dissipation versions of the theory.

One easily checks that the Prandtl mixing length hypothesis cannot lead to (1) and (2) with non-equilibrium exponents if \( l_m \sim \delta \). The non-equilibrium scalings can however be
TABLE I: Summary of the mean flow \( (u_0) \), characteristic flow width \( (\delta) \), mixing length \( (l_m) \) and eddy viscosity \( (\nu_T) \) scalings obtained according to the equilibrium and the non-equilibrium versions of the Townsend-George theory for the axisymmetric wake and turbulent planar jet cases.

\[
\begin{align*}
\text{Dissipation Scaling} & \quad \text{Equilibrium} & \quad (Re_G/Re_0)^{m} K_0^{3/2}/\delta \\
\text{Power laws exponents: Axisymmetric Wake} & \quad a = -2/3, b = 1/3 & \quad a = -1, b = 1/2 \\
\text{Power laws exponents: Planar Jet} & \quad a = -1/2, b = 1 & \quad a = -1/3, b = 2/3 \\
\text{Mixing length } l_m & \quad \sim \delta & \quad \sim \delta \sqrt{Re_G/Re_0} \\
\text{Eddy viscosity } \nu_T & \quad \sim u_0\delta & \quad \sim U_{ref} L_{ref}
\end{align*}
\]

retrieved if \( l_m \sim \delta \sqrt{Re_G/Re_0} \) where \( Re_0 = u_0\delta/\nu \) (see table I). As for the constant eddy viscosity, it can still be used to obtain non-equilibrium scaling exponents \( a \) and \( b \) but only if \( \nu_T \) is constant throughout the flow so that \( \nu_T \sim U_{ref} L_{ref} \), not only across sections of the flow orthogonal to the streamwise coordinate as in \( \nu_T \sim u_0\delta \).

In this work we first and foremost address one key aspect of the Townsend-George approach using experimental data for a turbulent axisymmetric wake at \( Re_G = 40000 \) and a planar jet at \( Re_G = 20000 \): the important question of the proportionality of \( L \) and \( \delta \) in free shear flows. Secondly, we also ask whether the equilibrium and non-equilibrium mixing length and eddy viscosity models imply different mean flow profiles and how the different models perform in predicting these profiles.

**EXPERIMENTAL SETUP**

The experiments are carried out in two different facilities at Imperial College London. A schematic representation of both the planar jet and the axisymmetric wake flows is provided in figure I(I) and I(II), respectively. The planar jet flow is generated using a centrifugal blower which collects air from the environment and then forces it into a plenum chamber. In order to reduce the inflow turbulence intensity level and remove any bias due to the feeding circuit, the air passes through two sets of flow straighteners before entering a convergent
duct (having area ratio equal to about 8). At the end of the duct there is a letterbox slit with aspect ratio $w/L_{\text{ref}} = 31$ (with $L_{\text{ref}} = 15$ mm). In order to produce a top hat entrance velocity profile, the two longest sides of the slit are filleted with a radius $r = 2L_{\text{ref}}$, following the careful recommendation by [13]. The jet exhausts in still ambient air and is confined in the spanwise direction by two perspex walls of size $100L_{\text{ref}} \times 100L_{\text{ref}}$ placed in $x - y$ planes ($L_{\text{ref}}$ is the slot width in the $y$ direction). The rotational speed of the blower is controlled using an in-house PID controller to produce an inlet Reynolds number $Re_G = 20000$. Single (SW) and Cross (XW) wire measurements are taken along the jet centreline in the range $x/L_{\text{ref}} = 14 - 50$ with $2L_{\text{ref}}$ spacing. Both SW and XW are driven by a Dantec Streamline constant temperature anemometer (CTA). Data are sampled at a frequency of 50 KHz, with measurements lasting 60 s and 120 s respectively in the SW and XW cases.

The wake flows are generated in the low-turbulence wind tunnel at Imperial College London. The measurement test section is 3 ft x 3 ft ($\approx 91$ cm x 91 cm) and length 4.25 m. The plates employed for these experiments have a reference length $L_{\text{ref}} = \sqrt{A_{\text{plate}}} = 64$ mm, with thickness 1.25 mm, $A_{\text{plate}}$ being the frontal area of the plate. The plate is suspended in the centre of the wind tunnel normal to the laminar free stream using four 1 mm diameter piano wires. The free-stream velocity was kept fixed at $U_{\text{ref}} = 10$ m/s using a PID controller. For that value, the velocity fluctuations around the mean are below 0.1% when the plate is not in place. The velocity signal is measured using a one component hot-wire (herein referred to as SW) driven by a Dantec Streamline constant temperature anemometer (CTA). Data are sampled at a frequency of 20 KHz. Each measurement lasts for 60 s, which was
deemed to be sufficient to converge the integral scales. Finally, a X-wire probe was used to estimate the kinetic energy only for centreline measurements. The centreline kinetic energy is calculated by assuming axial symmetry, i.e. $K_0 = 0.5 \left( <u_x'^2> + 2 <u_r'^2> \right)$ where $u_x'$ and $u_r'$ are streamwise and radial fluctuating velocities. More details about the experimental set-up can be found in [10, 11].

The longitudinal integral lengthscale $L$ is calculated converting the anemometer time signal into space using the frozen turbulence hypothesis and using the autocorrelation of the fluctuating streamwise velocity. Comparison of the results with those obtained using the expression proposed by [14], i.e. $L = \pi E_u(k = 0)/u'^2$, shows minimum discrepancies. The estimate of the turbulent dissipation rate $\varepsilon$ is obtained from its isotropic surrogate, i.e. $\varepsilon_{ISO} = 15 \nu (\partial u'/\partial x)^2$, by integrating the one dimensional spectrum of the velocity signal $F^{(1)}_{11}$ as follows

$$\overline{(\partial u'/\partial x)^2} = \int_0^\infty k^2 F^{(1)}_{11} dk. \tag{6}$$

In both the experiments, we took care of reducing the noise at the high wavenumber end of the spectrum. As suggested by [11], we fit the portion of the spectrum at frequencies higher than Kolmogorov’s frequency with an exponential law. It must be however remarked that the contribution of this portion of the spectrum to the integral in equation (6) is always less than 6%.

RESULTS

Townsend [2], George [3], Dairay et al [10] and Cafiero & Vassilicos [9] assumed that the dissipation lengthscale $C_\varepsilon K^{3/2}/\varepsilon$ could be interchangeably taken to be proportional to the integral lengthscale $L$ or the characteristic flow width $\delta$ without loss of validity of their results, at least in terms of scaling. It is then pertinent to investigate this assumption by looking at data measured for two different wake generating bodies, a square and a fractal (see [10, 11] for more details), as well as for the planar jet. Data are taken along the flows’ centreline. The inlet Reynolds numbers are $Re_G = 40000$ and 20000 for the wakes and jet, respectively. Figure 2a supports the assumption of proportionality between $L$ and $\delta$, at least in the range of streamwise distances $15 \leq x/L_{ref} \leq 50$, which contains the region where the non-equilibrium dissipation scaling holds as reported in [9, 11]. A constant value of the ratio $L/\delta$ is attained both in the wake and the jet cases, but the value of the constant seems to
FIG. 2: a) Streamwise profiles of $L/\delta$ for the planar jet (red circles), the axisymmetric fractal wake (blue squares) and the axisymmetric square wake (black triangles). The ratio is calculated along the centreline of the flow. Inlet Reynolds numbers $Re_G = 40000$ (wakes) and 20000 (jet). b) Non-equilibrium dissipation constant $C_{NE,\delta} = (U_{ref}L_{ref})\varepsilon\delta^2/K_0$ for the the planar jet (red circles-continuous) and the axisymmetric fractal wake (blue squares-continuous). Non-equilibrium dissipation constant $C_{NE,L} = (U_{ref}L_{ref})\varepsilon L^2/K_0$ for the planar jet (red circles-dotted) and the axisymmetric fractal wake (blue squares-dotted).}

Lateral Profiles

In this section we investigate the consequences of the application of a different turbulence dissipation scaling on the lateral mean flow profiles. The scalings (1) and (2) and stemming from the non-equilibrium version of the theory, can also be obtained with the mixing length,

$$l_m \sim \delta \sqrt{Re_G/Re_0\delta},$$

(7)
but not with \( l_m \sim \delta \) (see table I).

Similarly, for the constant eddy viscosity model, the non-equilibrium version of the theory returns the right exponent \( a \) and \( b \) provided that \( \nu_T \) is not only constant across a section orthogonal to the mean flow as in the equilibrium case, but throughout. Introducing into equation (3) the scalings of \( \langle u'v' \rangle \) stemming from the non-equilibrium version of the theory (see [10] and [9]), we obtain

\[
-\nu_T \frac{\partial \pi}{\partial y} = u_0^2 \frac{d\delta}{dx},
\]

(8)

for the planar jet case [9] and

\[
-\nu_T \frac{\partial \pi}{\partial y} = U_{ref} u_0 \frac{d\delta}{dx},
\]

(9)

for the axisymmetric wake case [10], [11]. Introducing the power laws (1) and (2) with the non-equilibrium values of \( a \) and \( b \) reported in table I,

\[
\nu_T \sim U_{ref} L_{ref} \sim \text{const},
\]

(10)

both for the planar jet and for the axisymmetric wake case, as opposed to

\[
\nu_T \sim u_0 \delta,
\]

(11)

obtained from the equilibrium version of the Townsend-George theory (which actually requires one more assumption to conclude, see [10] and [9]).

It is then important to determine whether the differences in mixing length and in the eddy viscosity reflect in different mean flow profiles for different turbulent dissipation scalings. Furthermore, it is also important to determine whether the mean flow profiles obtained with the Prandtl mixing length or with the constant eddy viscosity models are consistent with the experimental data, and more or less so depending on turbulence dissipation scaling.

Mixing-length based models [6, 8] have largely proven to be inadequate for correctly predicting lateral mean flow profiles. Townsend [7], comparing with the experimental results obtained in the turbulent planar wake of a square cylinder, finds that a constant eddy viscosity \( \nu_T \sim u_0 \delta \) best represents his results. We also aim at comparing the mixing length based model with a constant value of the eddy viscosity, but by taking into account the non-equilibrium modification of these two models. Furthermore, following Townsend’s approach [7], we also correct the eddy viscosity to account for the intermittency of the flow. In the
FIG. 3: a) Mean flow velocity profiles for the axisymmetric wake in the range of streamwise distances $15 \leq x/L_{ref} \leq 50$ (symbols). Black, red and blue lines are representative of equations $12$, $13$ and $16$, respectively. b) Intermittency factor in the range of streamwise distances $15 \leq x/L_{ref} \leq 50$ rescaled with respect to its local maximum. The continuous line is representative of equation $15$. Data are plotted against the similarity coordinate $\eta = r/\delta$. The inlet Reynolds number is $Re_G = 40000$.

following we compare our experimental data with mean flow profiles predicted by Prandtl’s mixing length and constant eddy viscosity models for the axisymmetric wake and the planar jet.

The detailed derivations of the profiles stemming from these two models modified to take into account the non-equilibrium cascade are reported in the appendix.

**Axisymmetric Wake**

For an axisymmetric wake, when using Prandtl’s mixing length (eq. $4$) to model the Reynolds shear stress it is possible to show that the mean flow profile can be described as $6$ (see Appendix)

$$\frac{U}{U_{max}} = \sqrt{x} f(\eta) = \left(1 - \left(\frac{\eta}{\eta_0}\right)^{3/2}\right)^2, \quad (12)$$

where $\eta = y/\sqrt{x}$ and $\eta_0$ ($y = y_0$) is the point where $f \to 0$ (already a non physical result) on the boundary of the wake.

When, instead, a constant value of the eddy viscosity is used (eq. $5$), the following form
of the mean flow can be obtained \[15\]:

\[
\frac{U}{U_{\text{max}}} = e^{-k\eta^2},
\]

(13)

with \( \eta = y/\delta \); \( k \) depends on the turbulence dissipation scaling (see Appendix). It is worth remarking that the profiles obtained in equations (12)-(13) are valid for both turbulence dissipation scalings (equilibrium and non-equilibrium).

Figure 3a shows the mean flow velocity profiles rescaled with the maximum at each streamwise location plotted against the similarity variable \( r/\delta \). A comparison of the two proposed models (eq. 12-13) suggests that Prandtl’s mixing length model significantly overestimates the velocity profile for values of \( r/\delta > 1 \). A constant value of the eddy viscosity seems to follow more closely the physics of the problem. As also showed by Townsend \[7\], a significant improvement of the eddy viscosity model can be obtained by accounting for the intermittency factor \( \gamma \) calculated as the inverse of the kurtosis of the time derivative of the streamwise fluctuating velocity, i.e.,

\[
\gamma = \frac{1}{\text{Kurt}[du'/dt]}.
\]

(14)

This is due to the fact that the eddy viscosity cannot be non-zero beyond the turbulent/non-turbulent interface, where there are no vortical fluctuations at all. Following Townsend, we use a functional form for the intermittency as

\[
\frac{\gamma}{\gamma_{\text{max}}} = \frac{1}{(1 + (\eta/\alpha_1)^2 + (\eta/\alpha_2)^4 + (\eta/\alpha_3)^6)}
\]

(15)

with \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) parameters to be determined, and we redefine the eddy viscosity as \( \nu_{T}^I = \gamma \nu_{T} \). Equation (15) requires that the intermittency factor is self preserving; a condition satisfied in our experiments as can be observed from Figure 3b where we plot the intermittency profiles obtained in the range of streamwise distances \( 10 \leq x/L_{\text{ref}} \leq 50 \) normalised with respect to the local maximum at each streamwise location. The continuous line is representative of the fit of equation (15) and shows a remarkable agreement with the experimental data.

Hence, we modify the eddy viscosity accounting for the intermittency of the flow, i.e. \( \nu_{T}^I = \nu_{T} \gamma \), with \( \nu_{T} \) a constant value and we find a solution to the self-similar equation of the form

\[
\frac{U}{U_{\text{max}}} = e^{-k\eta^2\left(\frac{1}{2} + \frac{1}{4}(\eta/\alpha_1)^2 + \frac{1}{8}(\eta/\alpha_2)^4 + \frac{1}{16}(\eta/\alpha_3)^6\right)}.
\]

(16)
FIG. 4: a) Mean flow velocity profiles for the turbulent planar jet in the range of streamwise distances $15 \leq x/L_{ref} \leq 50$ (symbols). Black and blue lines are representative of the solution obtained with Prandtl (1925), Eddy viscosity (EV) assumption or introducing the intermittency function, respectively. b) Intermittency function rescaled with respect to the local maximum at each streamwise position $x/L_{ref}$ in the range $15 \leq x/L_{ref} \leq 50$. The black line is representative of the fit $\frac{\gamma}{\gamma_{max}} = f(\eta)^{m}$. Data are plotted against the similarity coordinate $\eta = y/\delta$. The inlet Reynolds number is $Re_G = 20000$.

We seek for the coefficients $\alpha_1$, $\alpha_2$, $\alpha_3$ and $k$ which optimise equations 15 and 16 simultaneously. The continuous blue line in figure 3a shows that the introduction of the intermittency factor significantly improves the results, particularly in proximity of the wake boundaries. Equation (16) and variations of it have been extensively used to fit turbulent wakes [16] from bluff plates and wind turbines [17] as *ad hoc* modifications to a Gaussian profile.

**Planar Jet**

We follow a similar procedure in the planar jet case. The application of Prandtl’s mixing length model leads to the following form of the jet momentum equation

$$kF''u^2 + FF' = 0$$

(17)

where $F'(\eta) = f(\eta)$, $f(\eta) = \tau(x,y)/u_0(x)$, and $k$ a constant value which depends on the turbulence dissipation scaling (this dependence is reported in the Appendix). Nevertheless,
as for the axisymmetric wake, there is no substantial difference in the functional form of the
mean flow profile for the equilibrium and non-equilibrium cases. As reported by Abramovich
[18], Tollmien [19] was the first to find a numerical solution for equation (17). We also solve
the equation numerically, comparing the results with our experimental measurements and
with the results obtained with the eddy viscosity assumption with/without the intermittency
correction.

The adoption of the turbulent viscosity model, leads to the following momentum equation
in similarity variables

\[ FF'' + F'^2 = 2kF''' \]  

(18)

(relations for \( k \) are reported in the Appendix). This equation can be solved to obtain the
velocity profile \( f(\eta) \)

\[ f(\eta) = \text{sech}^2(\eta\sqrt{k}). \]  

(19)

In Figure 4a we report the experimental data obtained from the planar jet experiment in the
range of streamwise distances \( 15 \leq x/L_{ref} \leq 50 \), along with the mean flow profiles predicted
by the Prandtl mixing length model (black) and the eddy viscosity model (red). Despite
very little differences throughout the whole range of lateral locations, it can be argued that
the Prandtl model slightly underestimates the mean flow profile at small values of \( \eta \).

Even though the eddy viscosity model shows good agreement with the experimental data,
we decided to try and improve it by accounting for the intermittency of the flow, hence
introducing \( \nu'_I = \nu_T \gamma \). As already discussed in the axisymmetric wake case, this requires
that the intermittency function is self-similar. In Figure 4b we compare the data obtained in
the range \( 15 \leq x/L_{ref} \leq 50 \) rescaled with respect to the local maximum at each streamwise
location \( (\gamma_{\text{max}}) \). It can be concluded that \( \gamma \) is indeed self-similar. The momentum equation
for the turbulent planar jet hence modifies as

\[ \left( F'F \right)' = 2k\gamma F'' \]  

(20)

where \( F'(\eta) = f(\eta) \). The solution will depend on the choice of \( \gamma \); we decide to introduce a
function \( \gamma(\eta) = f(\eta)^m \), with \( m \neq 1 \) \( (m = 1 \) results in a sinusoidal function for \( f(\eta) \) \) and
we integrate numerically equation (20). The choice of a different intermittency function is
mainly driven by the fact that equation (15) does not lead to a closed solution of equation
(20), and we considered a better choice by relating the intermittency function directly to
the mean flow. This can also be instrumental in future investigations aimed at relating
the intermittency function $\gamma$ to the turbulence cascade. As evidenced in Figure 4, the
solution obtained with the introduction of the intermittency function (blue line) gives a
slight improvement to the fit of the experimental data.

CONCLUSIONS

Using experimental data from a turbulent planar jet and two axisymmetric turbulent
wakes, we find evidence for the proportionality of the integral lengthscale $L$ and the char-
acteristic flow width $\delta$ in the range of streamwise distances $15 \leq x/L_{ref} \leq 50$. This is a
fundamental hypothesis when deriving the self-similar scaling laws in turbulent free shear
flows, and it is now established in the region where the turbulent dissipation scaling is of
the now known non-equilibrium type.

We then revisit the turbulence closure models such as Prandtl’s mixing length [1] and
constant eddy viscosity in the light of the non-equilibrium dissipation scalings. In this
framework, the cornerstone of Prandtl’s (1925) model, i.e. $l_m \sim \delta$, is not valid. We show
that $l_m \sim \delta \sqrt{Re_G/Re_0}$ instead. The scalings (1) and (2) stemming from the non-equilibrium
cascade agree with the eddy viscosity model $\nu_T \sim U_{ref}L_{ref}$ rather than $\nu_T \sim u_0 \delta$, hence
implying a constant value of the eddy viscosity everywhere.

However, we demonstrate that these differences do not lead to different mean flow profiles.
A systematic comparison of the mean flow profiles predicted by Prandtl (1925) and the eddy
viscosity models with the experimental data for the axisymmetric wake and the turbulent
planar jet reveals the inadequacy of Prandtl’s mixing length hypothesis to correctly predict
the mean flow behaviour even where the turbulence dissipation has a non-equilibrium cascade
scaling. Furthermore, following Townsend [7] we show that the prediction can be further
improved by accounting for the intermittency of the flow, particularly in the axisymmetric
wake case. In agreement with Townsend [7], we find that rescaling the eddy viscosity $\nu_T$ with
the intermittency function $\gamma$ provides a better representation of the mean flow behaviour as
it accounts for the eddy viscosity drop across the turbulent flow’s intermittent boundaries.
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APPENDIX A: GOVERNING EQUATIONS

In this Appendix we report about the equations that lead to the definition of the lateral profiles for the axisymmetric wake and the planar jet flows, following [10] and [9] respectively. We start by describing the axisymmetric wake, then particularise the radial profiles according to the different closure models for the Reynolds stresses.

Axisymmetric wake

In the thin shear layer approximation, the momentum balance reads

\[
U_{ref} \frac{\partial}{\partial x} (U_{ref} - U) = -\frac{1}{r} \frac{\partial}{\partial r} (r \langle u' v' \rangle).
\] (21)

Assuming that \( U_s = U_{ref} - U = u_0 f(\eta) \), and substituting in equation (21),

\[
U_{ref} \frac{\partial}{\partial x} (f(\eta)) - U_{ref} U_s f'(\eta) \eta \frac{1}{\delta} \frac{\partial}{\partial x} \delta = -\frac{1}{r} \frac{\partial}{\partial r} (r \langle u' v' \rangle).
\] (22)

Momentum flux constancy \( U_s \delta^2 = U_{ref} \theta^2 \) can be differentiated to get

\[
\frac{\partial}{\partial x} U_s = -2 \frac{U_s \delta}{\delta} \frac{\partial}{\partial x} \delta,
\] (23)

so we rewrite the momentum equation as

\[
\frac{2U_{ref}U_s}{\delta} \frac{\partial}{\partial x} (\delta f(\eta)) + U_{ref} U_s f'(\eta) \eta \frac{1}{\delta} \frac{\partial}{\partial x} \delta = \frac{1}{r} \frac{\partial}{\partial r} (r \langle u' v' \rangle).
\] (24)

The solution of equation (24) depends on the Reynolds stress modelling. As reported by Goldstein [6], the classical streamwise dependent eddy viscosity based on Prandtl’s mixing length (Prandtl (1925)),

\[
\nu_T = l_m^2 \frac{\partial}{\partial r} (U_{ref} - U),
\] (25)

leads to the following equation,

\[
f' = \sqrt{\left(\frac{\eta f}{k}\right)},
\] (26)
with \( k = \frac{U_{ref}^2}{\delta^2 \nu_T} \). This equation has the solution,

\[
\frac{U}{U_{ref}} = f(\eta) = \left(1 - \left(\frac{\eta}{\eta_0}\right)^{3/2}\right)^2,
\]

where \( \eta_0 = (3k)^{1/3} \) is the point at the boundary of the wake (and therefore where \( f \to 0 \)). Now, depending on the properties of the turbulent cascade, two different closures can be obtained:

i) Richardson Kolmogorov cascade:

In this case, we have \( l_m = C\delta \), with \( C \) a constant. Furthermore, the streamwise scalings are \( u_0 = A U_{ref} (\frac{x-x_0}{\theta})^{-2/3} \) and \( \delta = B \theta (\frac{x-x_0}{\theta})^{1/3} \). Adding the integral form of momentum conservation \( (u_0 \delta^2 = U_{ref} \theta^2) \), we get that \( k = 3C^2 \) and \( \eta_0 = 9C^{2/3} \).

ii) Non-equilibrium cascade:

In this case, we have \( l_m = C\delta \sqrt{Re_G/Re_0} = C\sqrt{U_{ref} L_{ref} \frac{\delta}{\nu_T}} \), and again \( C \) is a constant. In this case, the streamwise scalings are \( u_0 = A U_{ref} (\frac{x-x_0}{L_{ref}})^{-1} \left(\theta/\theta_{ref}\right)^2 \) and \( \delta = B \sqrt{L_{ref}(x-x_0)} \). Therefore, the constant becomes \( k = 2 \left(\frac{C}{B}\right)^2 \) and \( \eta_0 = 6 \left(\frac{C}{B}\right)^{2/3} \).

Conversely, the adoption of a turbulent eddy viscosity model, \( \nu_T = \text{constant} \) delivers a substantially different lateral velocity profile

\[
\frac{U}{U_{ref}} = e^{-\frac{k\eta^2}{2}},
\]

with \( \eta = y/\delta \) and \( k = \frac{U_{ref}^4 \delta^4}{\nu_T} \). Once more, two different closures can be obtained:

i) Richardson Kolmogorov cascade:

We have \( \nu_T = C u_0 \delta \), with \( C \) constant, and we find that \( k = \frac{1}{3} \frac{B^3}{C} \).

ii) Non-equilibrium cascade:

In this case, we have \( \nu_T = C U_{ref} L_{ref} \). The constant becomes \( k = \frac{1}{2} \frac{B^2}{C} \).

On the other hand, Townsend [7], studying the planar wake, suggested that the quality of the fit could be further improved by accounting for the intermittency of the flow \( \gamma \). He proposed the use of \( \gamma \) in a modified eddy viscosity \( \nu_T^I = \gamma \nu_T \) where

\[
\frac{\gamma}{\gamma_{max}} = \frac{1}{(1 + (\eta/\alpha_1)^2 + (\eta/\alpha_2)^4 + (\eta/\alpha_3)^6)}.
\]

leading to the following correction of equation (28)

\[
\frac{U}{U_{max}} = e^{-k\eta^2 \left(\frac{1}{2} + \frac{1}{4} (\eta/\alpha_1)^2 + \frac{1}{8} (\eta/\alpha_2)^4 + \frac{1}{32} (\eta/\alpha_3)^6\right)},
\]

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where $k$ remains unchanged from the previous case, not corrected by the intermittency.

**Planar Jet**

In the thin shear layer approximation, the streamwise momentum equation for the planar jet flow is

$$U \frac{\partial}{\partial x}(U) + V \frac{\partial}{\partial y}(U) = - \frac{\partial}{\partial y}(\langle u'v' \rangle), \tag{31}$$

and similarly to the wake flow, we can assume that the mean flow is self similar $U = u_0 f(\eta)$. This condition, along with continuity,

$$\frac{\partial}{\partial x}(U) + \frac{\partial}{\partial y}(V) = 0, \tag{32}$$

implies that the lateral velocity is self-similar as well. Casting equations (31) and (32) together and using the self similarity of the mean flow we obtain

$$\frac{\partial}{\partial x}(\delta) \left( f^2 + \frac{f'}{\eta} \int_0^\eta f(\eta)d\eta \right) = - \frac{\partial}{\partial y}(\langle u'v' \rangle). \tag{33}$$

Introducing $F = f'$ and rearranging the equation, we get

$$\frac{\partial \delta}{\partial x} \left( F'F \right) = - \frac{\partial}{\partial y} u'v'. \tag{34}$$

Equation (34) is then particularized depending on the closure for the Reynolds shear stresses. Prandtl’s mixing length model leads to the following equation,

$$kF'^2 + FF' = 0, \tag{35}$$

with $k = 2 \frac{r^2}{d^2}$. This equation has no known analytical solution, hence we solve it numerically. We can again relate the constant $k$ to model constants depending on the type of turbulence cascade:

i) Richardson Kolmogorov cascade:

The mixing length is $l_m = C\delta$, with $C$ constant. The streamwise scalings are $u_0 = AU_j(z-x_0)\frac{1}{2}$ and $\delta = Bh(z-x_0)$, and therefore we get that $k = 2C^2/B$.

ii) Non-equilibrium cascade:

In this case, we have $l_m = C\delta \sqrt{Re_G/Re_0\delta}$, and again $C$ is a constant. In this case, the streamwise scalings are $u_0 = AU_{ref} \left( \frac{z-x_0}{h} \right)^{1/3}$ and $\delta = BL_{ref} \left( \frac{z-x_0}{h} \right)^{2/3}$. Adding the integral form of momentum conservation ($u_0^2\delta = U_j^2 h$), the constant becomes $k = 3\sqrt{\frac{C}{B\delta}}$.  

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Assuming now a constant turbulent eddy viscosity model, the momentum equation particularizes as follows

\[
\left( F'^2 + F'' F \right) = k F''',
\]

with \( k = \frac{\delta u_0 \delta}{2 \nu_T} \). Finally,

\[
\left( F' F \right)' = k \gamma F''.
\]

This equation can be solved to obtain the velocity profile \( f(\eta) = F'(\eta) \),

\[
f(\eta) = \text{sech}^2(\eta \sqrt{k}).
\]

Again, depending on the properties of the turbulent cascade, both \( \nu_T \) and \( k \) will adopt different values:

i) Richardson Kolmogorov cascade:

The eddy viscosity takes the form \( \nu_T = C u_0 \delta \) with \( C \) a constant. Therefore, we get \( k = 2CA^2 \).

ii) Non-equilibrium cascade:

We have \( \nu_T = C U_{\text{ref}} L_{\text{ref}} \), and the constant becomes \( k = 3CA^3 \).

Similarly to the axisymmetric wake case, we also study the case of modified eddy viscosity \( \nu_T' = \gamma \nu_T \). Equation (36) then becomes

\[
\left( F' F \right)' = k \gamma F'',
\]

whose solution depends on the choice of \( \gamma \). We propose a function \( \gamma = (f(\eta))^m \) (with \( m \neq 1 \)), relating the intermittency to the mean flow profile. As there is no known closed solution, we numerically solve equation (39).

\[ \text{gioacchino.caﬁero@polito.it, Dipartimento di Ingegneria Meccanica e Aerospaziale, Politecnico di Torino, Italy} \]
\[ \text{martin.obligado@univ-grenoble-alpes.fr} \]
\[ \text{j.c.vassilicos@imperial.ac.uk} \]

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