Universality Classes, Statistical Exclusion Principle and Properties of Interacting Fermions

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Abstract

We point to the possibility of existence of the statistical-spin-liquid state as the state which differs from either Fermi or Luttinger liquid states. In the statistical spin liquid the double occupancies are excluded from the physical space. Each of the above three cases (Fermi, Luttinger and spin liquids) represents an universality class for the interacting many-particle fermion systems. The properties of the spin liquid such as the chemical potential, the entropy, and the magnetization curve, as well as the quasiparticle structure are briefly discussed.

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In the theory of correlated systems the principal problem is to transform the microscopic model of interacting particles into an effective approach with interaction among quasiparticles and/or collective excitations. Haldane [1] in his seminal paper provided an unified framework of the effective theory of interacting fermions in a normal (metallic) state. The Fermi surface, whose existence is a basic postulate of the approach, is considered as a d-1 dimensional collection of points, where the momentum distribution function $n_k$ has singularities. Obviously, this might be either a step discontinuity (as in the Fermi liquid case), or any weaker singularity (as in the case of Luttinger liquid). The Fermi surface in both cases must obey the Luttinger theorem which is taken as an additional postulate. Starting from these two postulates one can describe the low-energy long-wave-length excitations of the system solely in terms of the fluctuating Fermi surface.

Explicitly, one introduces the local Fermi wave vector $k_{FQ\sigma}(x)$ related to the undistorted Fermi wave vector $k_{FQ\sigma}^0(x)$ by

$$k_{FQ\sigma}(x) = k_{FQ\sigma}^0(x) + \delta k_{FQ\sigma}(x), \quad (1)$$

where vector $Q$ with d-1 components, where d is the space dimension, describes a coarse-grained point of the Fermi surface, $\sigma = \pm 1$ is the spin quantum number, and $x$ is a position variable. It turns out [1] that only normal to the surface fluctuations $\delta k_{FQ\sigma}(x)$ give contributions to the physical quantities such as the local fluctuation in the total particle density, which is defined as $\delta n_{\sigma}(x) = \sum_Q \delta n_{Q\sigma}(x) = \sum_Q \delta k_{FQ\sigma}(x) \Delta \nu_Q$, where $\Delta \nu_Q$ characterizes the Fermi surface discontinuity in the ground state occupation number $n_k$ distribution. Moreover, the normal fluctuations of the Fermi surface obey the commutation relations of a Kac-Moody algebra [1]. Therefore, one can write down the most general effective Hamiltonian for interacting fermions in the following bosonized form

$$\Delta H^{eff} = \frac{1}{2} \sum_q \sum_{Q_1Q_2} \sum_{\sigma\sigma'} \Gamma^{\sigma\sigma'}_{Q_1Q_2}(q) \delta n_{Q_1\sigma}(q) \delta n_{Q_2\sigma'}(-q). \quad (2)$$

The operator $\delta n_{Q\sigma}(q) = \sum_x e^{iqx} \delta n_{Q\sigma}(x)$ is the bosonlike Tomonaga-Luttinger density wave operator and $\Gamma^{\sigma\sigma'}_{Q_1Q_2}$ is the positive defined matrix of elements given by [1].
The quantities $v_{FQ_\sigma}$ are the Fermi velocities on separate points of the Fermi surface and the scaling variable $\Lambda$ is the distance of the two closest points on it; it is the cut-off parameter in the theory. An universal behavior of the system depends on how the interaction part of the effective Hamiltonian (2) behaves in the scaling limit $\Lambda \to 0$. The purpose of this paper is to point out that in this limit a systematic classification of the interacting fermion liquids into three physically distinct classes takes place, as well as to specify briefly each of them.

To address this fundamental question we suppose that the interaction part of (2) scales as

$$\Lambda^{d-1} \frac{f(k_{FQ_1\sigma}, k_{FQ_2\sigma'}; q)}{(v_{FQ_1\sigma} v_{FQ_2\sigma'})^{1/2}} \sim \Lambda^{d-1} \Lambda^\alpha,$$

where $\alpha$ is the exponent characterizing the type of the singularity. When $\alpha < d - 1$, then the Landau Fermi liquid is stable [1]. For $\alpha = d - 1$ the interaction part of the Hamiltonian (2) is finite and comparable to the kinetic energy when $\Lambda \to 0$. Therefore this part is called the relevant variable. Since this interaction couples two different points of the Fermi surface, the bosonized Hamiltonian must be diagonalized via the Bogolyubov transformation in order to determine the excitation spectrum. In effect, this leads to the Luttinger liquid type behavior, i.e. the Luttinger liquid is the stable fixed point for the interacting electrons in $d$ dimensions [2]. However, when $\alpha > d - 1$ the interacting part becomes infinite in the scaling limit $\Lambda \to 0$ and therefore, the diagonalization procedure is impossible to perform.

Since the conventional methods are useless in the case when $\alpha > d - 1$, we propose to extend the Pauli exclusion principle in order to project out from the Hilbert space the states which lead to the infinite energy contribution. More precisely, we take the forward scattering amplitude for particles with opposite spins as divergent with the exponent $\alpha > d - 1$. Note, that for particles with spin one-half this is the only nontrivial possibility due to the existence of the Pauli exclusion principle, but for particles with higher spin or additional internal symmetries (such as orbital degeneracy), additional processes may appear in $\Gamma$. Instead of
dealing with the infinite amplitude from the beginning we take this amplitude to be a finite number ($U_s$) and put at the end of calculations $U_s \to \infty$; this limit corresponds to the case with a singular scattering forward amplitude and, consequently, to the statistical exclusion discussed earlier. Thus the change of the total system energy is expressed as follows

$$\delta E = \sum_{k\sigma} \epsilon_{k\sigma} < \delta n_{k\sigma}> + U_s \sum_k < \delta n_{k\uparrow}\delta n_{k\downarrow}>$$

$$+ \{marginally - relevant - terms\},$$

where $\epsilon_{k\sigma} = \epsilon_k - \sigma h$ is the dispersion relation for particles with spin $\sigma$ moving in an applied magnetic field $h$. The brackets $<>$ mean taking both quantum and thermal averages. Within the model defined by the first term in (5), the wave vector $k$ is a good quantum number. Hence, Eq.(5) defines a new effective model of interacting fermions which is exactly soluble in the scaling limit. In order to find the exact partition function and the momentum distribution functions we have to minimize the thermodynamic potential with respect to both $< \delta n_{k\sigma}> \equiv \delta n_{k\sigma}$ and with $< \delta n_{k\uparrow}\delta n_{k\downarrow}> \equiv \delta n_{kd}$, where the entropy is defined by

$$\delta S = k_B \sum_k \left[ \sum_{\sigma} (\delta n_{k\sigma} - \delta n_{kd}) \ln(\delta n_{k\sigma} - \delta n_{kd}) + \delta n_{kd} \ln \delta n_{kd} +$$

$$+ (1 - \delta n_{k\uparrow} - \delta n_{k\downarrow} + \delta n_{kd}) \ln(1 - \delta n_{k\uparrow} - \delta n_{k\downarrow} + \delta n_{kd}) \right].$$

The resulting distribution functions are

$$\delta n_{k\sigma} - \delta n_{kd} = \frac{e^\beta U_s e^{\beta (\epsilon_k - \mu)} \cosh(\beta h)}{1 + e^\beta U_s e^{\beta (\epsilon_k - \mu)}[e^{\beta (\epsilon_k - \mu)} + 2 \cosh(\beta h)]]}$$

$$[1 + \sigma \tanh(\beta h)],$$

and

$$\delta n_{kd} = \frac{1}{1 + e^\beta U_s e^{\beta (\epsilon_k - \mu)}[e^{\beta (\epsilon_k - \mu)} + 2 \cosh(\beta h)]]}.$$
This model has interesting properties as a function of $U_s$. Namely, at $T = 0$ and in the absence of the magnetic field the distribution function $\delta n_{k\sigma}$ has two steps for $U < W$, where $W$ is the bandwidth of the single particles energies $\epsilon_k$. One of the steps is located at the Fermi level $\mu$; below it all $k$ states are singly occupied down to the second step located at energy $(\mu - U_s)$. For energies below the second step, the doubly occupied $k$ configurations are admissible.

To see the effect of this double-step distribution function on the physical properties we calculate the chemical potential $\mu$ as a function of temperature $T$, with the value of $U_s$ as a parameter. For simplicity we take the constant density of states of the width $W$ and within the energy interval $-W/2 \leq \epsilon \leq W/2$; the gravity center of the band is chosen at zero energy. In Fig.1 we present our numerical results for $\mu$. The case with $U_s = 0$ corresponds to the Fermi liquid fixed point. In that case, the chemical potential decreases to $-\infty$ with growing temperature. A quite different behavior is observed in the $U_s \rightarrow \infty$ limit. Then, the chemical potential increases indefinitely. However, for any intermediate value of $U_s$ the chemical potential is represented by a retrograde curve, which starts decreasing at sufficiently high temperatures. Thus, even though for any finite $U_s$ the Fermi surface singularity is now $\Delta\nu_{Q\sigma} = 1/2$, only the $U_s \rightarrow \infty$ limit leads to a non-Fermi liquid behavior at high temperatures. This happens because for finite $U_s$ a crossover to the Fermi liquid behavior takes place whenever $k_B T \geq U_s$. Also, as displayed in Fig.2, the high-temperature entropy for the half filled band is equal to $2k_B \ln 2$ per particle for $U_s < \infty$. As was shown before [3], only in the case where the double occupancies are suppressed entirely, the entropy is $k_B \ln 2$ per particle. The $T \rightarrow \infty$ entropy value $2k_B \ln 2$ per carrier characterizes the Fermi liquid universality class from the statistical point of view, whereas the value $k_B \ln 2$ is characteristic of a different class of universality. The singular interaction ($\alpha > d - 1$) causes the breakdown of both the Fermi and the Luttinger liquid pictures where $|\Delta\nu_{Q\sigma}| = 1$. Instead, a new liquid is stable. We call this liquid the statistical spin liquid because the singular dynamical interaction is transmuted into the statistical interaction between the particles with opposite spins and with the same $k$ number [3,4]. This statistical interaction removes half of the...
available single-particle states from the physical space and is characterized by the index of the Fermi surface singularity $|\Delta \nu_{Q\sigma}| = 1/2$. This liquid is an example of many-particle system with the fractional exclusion statistics, which takes into account the spin degrees of freedom.

In the low temperature regime one can perform the Sommerfeld expansion for thermodynamic quantities. For example, the temperature dependence of the chemical potential in the paramagnetic case has the form

$$\mu = \epsilon_F - k_B T \ln 2 - \frac{\pi^2}{6} \left( \frac{\rho'}{\rho} \right) (k_B T)^2 + O(T^4),$$

(9)

where $\epsilon_F$ is the Fermi energy and $\rho$ ($\rho'$) is the value of the density of states (its derivative) taken at the Fermi energy. In the statistical spin liquid case a linear in $T$ term appears. This unconventional type of $\mu$ dependence might be observed in the thermopower measurements.

We have also calculated the internal energy per particle, which is

$$\frac{E}{N} = \frac{E_0}{N} + \frac{\pi^2}{6} (k_B T)^2 \rho + O(T^4),$$

(10)

where $E_0$ is the ground state energy. Thus, the specific heat is $C_V = \gamma_{sl} T$, where $\gamma_{sl} = (1/3) \pi^2 k_B^2 \rho$. It is a linear function in $T$, as in the Fermi liquid case, but the $\gamma$ coefficient is one half of that for the Fermi liquid. This is due to the exclusion of the half of the total number of states.

It is also interesting to inquire how the system magnetization changes for an arbitrary value of $U_s$. Using the formula (7) we find that the magnetization of the system is expressed by

$$m = \frac{1}{N} \sum_{k} \left[ \delta n_{k\uparrow} - \delta n_{k\downarrow} \right] =$$

$$\frac{1}{N} \sum_{k} \frac{e^{\beta U_s} e^{\beta (\epsilon_k - \mu)} \cosh(\beta h)}{1 + e^{\beta U_s} e^{\beta (\epsilon_k - \mu)}[e^{\beta (\epsilon_k - \mu) + 2 \cosh(\beta h)]} \tanh(\beta h).$$

(11)

In Fig.3 we present our numerical solution for the magnetization per particle as a function of the applied magnetic field for selected values of $U_s > 0$. The magnetization curve saturates faster for larger $U_s$ values. Also, in the statistical spin liquid case, only one-half of the
magnetic field is needed to saturate the moment, as compared to that for the Fermi liquid. This is easy to understand because in the present case each \( k \) state is singly occupied. In the \( U_s \to \infty \) limit the total magnetization curve coincides with that for \( nN \) localized moments [3,4].

The single particle Green function for the Fermi liquid has simple poles which correspond to the energy of quasiparticles. A marginal interaction renormalizes the spectral weight, as well as the energy spectrum of quasiparticles, but the analytical structure of the Green function remains unaltered. The Luttinger liquid Green function has the branch cuts instead of simple poles [7]. Therefore, the quasiparticle spectrum is not present and only the collective excitations emerge. In the statistical spin liquid case the propagator has again simple poles. However, the spectral weight function is renormalized even if the marginal interactions are not included. Namely, one can easily find that the Green function for the spin liquid is of the form

\[
G_{\sigma}^{R(A)}(k, \omega) = \frac{1}{2\pi} \frac{1 - n_{k\sigma}}{\omega - \epsilon_k \pm i\delta}.
\]

To derive this propagator one utilizes the commutation algebra of creation and annihilation operators \( (b_{k\sigma}^+ \text{ and } b_{k\sigma}, \text{ respectively}) \) in which the singly occupancy principle is incorporated explicitly; this yields

\[
\{b_{k\sigma}, b_{k'\sigma'}^+\} = \delta_{k,k'} \left[ (1 - n_{k\sigma})\delta_{\sigma,\sigma'} + b_{k\sigma}^+ b_{k\sigma} (1 - \delta_{\sigma,\sigma'}) \right].
\]

Since these operators do not commute to a number the statistical factor appears in the numerator of the Green function. Surprisingly enough, the interaction of the power \( \alpha \) stronger then \( d - 1 \) restores the quasiparticle structure again. As in the Fermi liquid, we can include the marginal interaction as a perturbation on the statistical spin liquid state.

One may view the singularity of the scattering amplitude from a different prospective. The double-occupancy projection may be regarded as a bone fide instability of the Fermi surface (cf.Fig.1). This is clear to see in the case with one particle per one available \( k \) state (i.e. the case with one particle per atom). Then, the ground state of the spin liquid is that of
a Mott insulator, since all available states are singly occupied and the configurational entropy 
\(k_B \ln 2\) per particle) is that of a Mott insulator in the spin disordered phase. Therefore, 
the Fermi liquid - spin liquid boundary may be considered as a Mott-Hubbard boundary 
for this particular band filling \(n = 1\). The scaling properties near this boundary as a 
function of \(n\) have been studied recently \([8]\) from the spin liquid side; the corresponding 
situation from the Fermi liquid side has been studied for the Hubbard model system both 
at \(T = 0\) long time ago \([9]\) and for \(T > 0\) later \([10]\). The Mott localization appears as 
an instability of the Fermi surface against normal fluctuations and is characterized by the 
condition \(k_{FQ\sigma}^0 \approx \delta k_{FQ\sigma}(x)\). The nature of the collective excitations near this instability, as 
well as the role of the Luttinger liquid stability in the regime in between the Fermi- and the 
spin-liquid states, all treated as separate condensed phases, needs to be explored.

In summary, we noted that the effective interaction with strong enough singularity can 
lead to an instability of the Luttinger liquid state with respect to the statistical spin liquid 
state in the same manner as the Fermi liquid becomes unstable with respect to the Lut-
ttinger liquid. The three states: the Fermi, the Luttinger, and the statistical spin liquids 
are the only three universality classes arising from the power-law scaling behavior of the 
interaction between the quasiparticles. We investigated also the statistical properties and 
the quasiparticle spectrum of such spin liquid and compared them with those for the Fermi 
and the Luttinger liquids. We did not address, how such a singular interaction might arise.
Hatsugai and Kohmoto \([11]\) pointed to the long-range nature of the interaction as a possible 
source of such a singular behavior. The presence of the van-Hove singularities on the Fermi 
surface enhances the scattering amplitude via the vanishing Fermi velocities in Eq.(2), and 
may stabilize either the Luttinger or the statistical spin liquids.

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FIGURES

Fig. 1 Temperature dependence of the chemical potential for different values of $U_s$. The limit $U_s \to \infty$ represents the statistical spin liquid.

Fig. 2 Temperature dependence of the entropy. The $T = 0$ value for $U_s/W > 1$ is $k_B \ln 2$ per particle.

Fig. 3 Magnetization curves for different $U_s$. The $U_s = \infty$ curve corresponds to that for localized moments.