Realizing the Quantum Hall System in String Theory

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Abstract

In a recent paper Bernevig, Brodie, Susskind and Toumbas constructed a brane realization of the Quantum Hall fluid. Since then it has been realized that the Quantum Hall system is very closely related to non–commutative Chern Simons theory and this suggests alternative brane constructions which we believe are more reliable and clear. In this paper a brane construction is given for the non–commutative Chern Simons Matrix formulation of the Quantum Hall system as described by in recent papers by Susskind, Polychronakos and by Hellerman and Van Raamsdonk. The system is a generalized version of Berkooz’s “Rigid Light Cone Membrane” which occurs as an excition of the DLCQ description of the M5–brane in a background 3–form field. The original construction of Berkooz corresponds to the fully filled $\nu = 1$ state of the QH system. To change the filling fraction to $\nu = 1/(k + 1)$ a system of $k$ background D8-branes is required. Quasi–hole excitations can be generated by passing a D6-brane though the Rigid Membrane.
1 Chern Simons Matrix Theory and the Quantum Hall System

According to \[1, 2, 3, 4, 5\], the quantum hall system at filling fraction \(\nu = 1/(k + 1)\) can be described by Abelian non–commutative Chern Simons Theory at level \(k\).

\[
S = \frac{k}{4\pi} \int d^3 y \epsilon^{\mu\nu\lambda} \left[ A_\mu \star \partial_\nu A_\lambda + \frac{2}{3} A_\mu \star A_\nu \star A_\lambda \right]
\]  

(1.1)

where the star–product is the usual Moyal product with non–commutativity parameter \(\theta\). The parameters of the Quantum Hall system are the magnetic field \(B\) and the filling fraction \(\nu\) given by

\[
\nu = 1/(k + 1) \\
B = (k + 1)/\theta
\]

(1.2)

Note that the connection between level and filling fraction is slightly different than given in \[1\], ie \(\nu = 1/k\). The shift of \(k\) by one is a quantum effect found by Polychronakos.

Alternatively, the theory may be described by a matrix model involving classical Hermitian matrix variables \(X^i, A_0\). The index \(i\) runs over the spatial directions \(i = 1, 2\) and \(A_0\) is a matrix valued connection which implements gauge invariance under unitary transformations in the matrix space. The Lagrangian for the matrix theory is

\[
L = BTr \left\{ \epsilon_{ij} (\dot{X}^i + [A_0, X^i]) X^j + 2\theta A_0 \right\}
\]

(1.3)

The equation of motion for \(A_0\) (Gauss law constraint) is

\[
[X^1, X^2] = i\theta
\]

(1.4)

This equation can only be solved if the matrices are infinite dimensional. This corresponds to an infinite number of electrons on an infinite plane. There are many reasons to want to regulate the system by taking the number of electrons to be finite, forming a finite droplet of Quantum Hall fluid with a boundary. Polychronakos has given an elegant modification of the system which accomplishes this \[2\]. Following Polychronakos we introduce a set of bosonic degrees of freedom \(\psi_n\) where the index \(n\) runs over \((n = 1, 2, ..., N)\). The

\footnote{This Lagrangian appears not to make sense for \(k = 0\) but we remind the reader that it is a formal expression in which the definition of the level is regulator-dependent. In what follows we will use only the explicitly matrix-regulated version.}
matrices $X^i, A_0$ are now $N \times N$ Hermitian matrices. An additional term in the action is introduced

$$L_\psi = \psi^\dagger (i\dot{\psi} - A_0 \psi)$$

(1.5)

The Gauss law constraint becomes

$$[X^1, X^2] = i\theta \left( I - \frac{1}{k+1} \psi \psi^\dagger \right)$$

(1.6)

where $I$ is the unit matrix and $\psi \psi^\dagger$ represents the matrix with components $\psi_m \psi_n^\dagger$. This equation no longer requires infinite dimensional matrices. The constraint is best understood in the following way. Take the trace to get

$$\sum_m \psi_m \psi_m^\dagger = N(k+1)$$

(1.7)

This expression is intended to be read as quantum-ordered and tells us that there must be exactly $Nk$ quanta of the $\psi$ field present. These $Nk$ quanta reside at the boundary of the droplet and provide the needed boundary degrees of freedom that are implicit in a Chern Simons theory. The traceless part of the equation is the $SU(N)$ generator and tells us that the state has to be invariant under the operations

$$X \rightarrow u^\dagger Xu$$

(1.8)

This tells us that in forming states from the Fock space of the oscillators $\psi, X$ we must contract all indices to form $SU(N)$ singlets. The states of this system have been analyzed \[3\] and shown to be in one to one correspondence with the states of the Laughlin theory.

## 2 The Level Shift

In the original paper on the QH system and non–commutative Chern Simons Theory \[1\] the connection between level and filling factor was given as

$$\nu = \frac{1}{k}.$$  

(2.1)

Subsequently Polychronakos discovered a quantum correction modifies this to

$$\nu = \frac{1}{k+1}.$$  

(2.2)

In this section we will give a simple derivation of this shift based on the wave functions given in \[3\]. The argument is due to Jeong-Hyuck Park and Dongsu Bak. \[6\].
The first step is to find an operator in the matrix theory which represents the area of the Quantum Hall droplet. For a uniform droplet of electrons it is easily seen that the right expression is

\[ \text{Area} = \frac{2\pi}{N} \sum_n (X_n)^2. \]  \hspace{1cm} (2.3)

The matrix analogue of this is

\[ \text{Area} = \frac{2\pi}{N} \text{Tr}(X)^2 \]  \hspace{1cm} (2.4)

Following [3] we define the $N \times N$ matrix of harmonic oscillator operators

\[ A_{mn} \equiv \sqrt{\frac{B}{2}} (X_1 + iX_2)_{mn} \]  \hspace{1cm} (2.5)

The ground state found of the droplet is given by the state [3]

\[ |k\rangle = \{e^{\sum_i (\psi_i^\dagger \psi_i + A_i^\dagger A_i)} |0\rangle \)  \hspace{1cm} (2.6)

Now observe that $\text{Tr}(X)^2$ is given by

\[ \text{Tr}(X)^2 = \frac{2}{B} (\text{Tr}A^\dagger A + \frac{1}{2} N^2) \]  \hspace{1cm} (2.7)

The expression $\text{Tr}A^\dagger A$ merely counts the total number of $A^\dagger$ that appear in (2.6). The term $12N^2$ is the zero-point fluctuation of the $N^2$ oscillators. This zero-point oscillation is the cause of the level shift.

It is easily seen that for large $N$ the entire expression becomes

\[ \text{Tr}(X)^2 = \frac{1}{B} (k + 1)N^2 \]  \hspace{1cm} (2.8)

and from (2.4)

\[ \text{Area} = \frac{2\pi}{B} (k + 1)N \]  \hspace{1cm} (2.9)

which corresponds to a filling fraction $\nu = 1/(k + 1)$.

The shift in the connection between level and filling factor would seem to undo the relation between filling factor and statistics found in [1]. However there is a compensating shift of statistics in matrix models that was also overlooked in [1]. The gauge invariant variables in an $SU(N)$ invariant matrix model are the eigenvalues of the matrices. The measure on the space of the eigenvalues involves a so called Vandermonde determinant which may be absorbed into the wave invariant functions. The result is to interchange Fermi and Bose statistics [3]

\[ ^2 \text{This was explained to us by A. Polychronakos} \]
In particular note that filling factor 1 is described by the simplest possible theory with \( k = 0! \)

## 3 Rigid Open Membranes

The existence of a 5+1 dimensional quantum field theory called the (0,2) theory is essential for the consistency of string theory. The theory may be thought of as the low energy description of an M5-brane in 11 dimensional M-theory. The only concrete construction of the (0,2) theory was given in [9] and consists of a DLCQ description obtained by considering Matrix Theory [11] in the background of a longitudinal 5-brane. In this description the elementary momentum carriers are D0-branes.

The theory has also been studied in the background of a 3-form field strength \( H_{+ij} \) by Berkooz [10] who finds that the momentum carriers blow up into “rigid open membranes” with boundaries on the 5-brane. The process is similar to that by which strings in a background \( B_{\mu\nu} \) field expand and form a dipole of size equal to their momentum [7]. The effect is also a version of the Myers effect [8]. In this paper we will see that Berkooz’s Rigid Open Membranes are ideal for modeling the Quantum Hall system [12].

The system studied in [10] consists of a single M5-brane wrapped on the compact light like direction \( x^− \). The other directions of the world volume are \( x^+, X^1, X^2, X^3, X^4 \). The system may also be thought of as a D4-brane in 2a string theory. The background \( H \) field has components

\[
H_{+12} = H_{+34} = H = 0
\]  

We will consider a DLCQ excitation carrying \( N \) units of \( P_− \). That is

\[
P_− = N/R
\]  

In the appropriate decoupling limit the corresponding matrix theory can be described in terms of \( N \times N \) matrices \( X^i \) (i=1,2,3,4) and two complex \( N \)-vectors \( Q, \tilde{Q} \). The \( X^i \) transform as adjoints of \( U(N) \) and the \( Q^i \) as fundamentals. The \( X \) may be thought of as describing the strings connecting the D0-branes and the \( Q \) as describing strings connecting the D0 and D4 branes. Following Berkooz we define

\[
X = X^1 + iX^2 \\
\tilde{X} = X^3 + iX^4
\]  

4
The decoupling limit studied by Berkooz involves letting the 11 dimensional Planck mass $M_p$ and the field $H$ tend to infinity. In this limit the Hamiltonian vanishes and the only equations of motion which survives are the vanishing of the D and F terms:

$$[X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] + QQ^\dagger - (\tilde{Q})^\dagger(\tilde{Q}) = \theta$$

$$[X, \tilde{X}] + Q\tilde{Q} = 0$$

(3.4)

where $\theta$ is given by

$$\theta = \frac{H}{RM_p^6}.$$  

(3.5)

If we specialize Berkooz’s equations to the case of an open membrane oriented in the $X^1, X^2$ plane then

$$\tilde{X} = \tilde{Q} = 0$$

(3.6)

and the equations take the form

$$[X^1, X^2] + iQQ^\dagger = i\theta.$$  

(3.7)

The important thing to notice is that this equation is the same as eq(1.6) with the replacement

$$Q \rightarrow \sqrt{\frac{\theta}{k}}\psi.$$  

(3.8)

Since there is no Lagrangian this system is identical to the $k = 0$ version of Polychronakos’ system. Thus Berkooz’s Rigid open membrane is a quantum Hall bubble with filling fraction 1. The term “rigid” is being used by Berkooz in the same way as “incompressible” is used in the Quantum Hall context.

4 Filling Fraction 1/n

In the limit $N \rightarrow \infty$ the D0-branes form an infinite 2-brane with a distant boundary at infinity. Equivalently they form an infinite Quantum Hall Droplet at filling fraction $\nu = 1$. To change the filling fraction we need to introduce something that will induce a Chern Simons term at level $k$ on the membrane. Fortunately Brodie has told us how to do that [14]. Consider adding a stack of D8-branes (in the type 2a description). The D8-branes lie in the directions $X^1, X^2, \ldots, X^8$ and are displaced from the D4-brane along the $X^9$ axis. On one side of the D8’s the vacuum is the conventional flat vacuum of type 2a string theory. On the other side there is a nonvanishing 10-form field sourced by the D8’s. In this region the vacuum is described by massive 2a gravity.
Let us begin with the D4 and its attached rigid open membrane on the trivial vacuum side. Now transport $k$ of the D8’s past the D4 system $\mathbb{3}$. Two important effects occur, both of which are encoded in Chern Simons terms induced on branes in the massive 2a theory $\mathbb{15}$, $\mathbb{14}$. The induced term on a $p$–brane has the formal structure

$$L_p = kA \wedge F^2. \quad (4.1)$$

In particular for a D0-brane the term is

$$L_0 = kA_0 \quad (4.2)$$

which is just a chemical potential for string ends. It indicates that the system formed from $N$ D0-branes must have $kN$ fundamental strings ending on it. This is also known as the Hanany Witten effect $\mathbb{16}$. The other end of the string can be on the 8-branes but it does not have to be. In fact if the 8-brane is moved well past the D4 system the stable configuration will involve strings which end on the D4-brane. In other words we will find that the number of $Q$ quanta will be $kN$ in exact agreement with eq.(1.7).

Furthermore the rigid membrane will also have a Chern Simons 2-brane term induced which according to (4.1) will have the form

$$L_2 = kA \wedge F. \quad (4.3)$$

Actually this is only correct if there is no background $H$ field. In the presence of the $H$ field ( B field in the 2a language) the Chern Simons term must become non–commutative $\mathbb{13}$. Evidently then, the effects induced on the open rigid membrane are exactly what are needed to turn the system into that studied in $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$. In other words the system becomes the Quantum Hall System at filling $\nu = 1/(k + 1)$.

5 Six-Branes and Quasiholes

The D6-brane plays an interesting role in the brane/QH correspondence. Recall that in Laughlin’s theory the Quantum Hall fluid will support quasiholes of fractional charge $\nu$. In $\mathbb{1}$ these quasihole states were constructed by modifying the Gauss law constraint (1.4) to allow an explicit source

$$[X^1, X^2] = i\theta(1 + \nu P) \quad (5.1)$$

$^3$The system of 8-branes and 4-branes is BPS and therefore stable. Adding the D0-branes leads to a non-BPS configuration but will not destabilize the configuration since it is a localized perturbation
where $P$ is a projection operator of rank one in the matrix space. If the projection operator projects onto a localized coherent state in the oscillator representation of the matrix space then the quasihole is localized by the coordinates of the center of the coherent state. From eq.(1.6) we see that we can accomplish the same thing in the regularized theory of [2] by exciting a single $\psi$ quantum. In the brane representation this corresponds to adding an additional string connecting the rigid open membrane to the D4-brane.

Another way to view the quasihole in the standard Quantum Hall framework is to begin with a magnetic monopole on one side of the plane containing the electrons. If we adiabatically pass the monopole through the plane, say at the origin, the effect is to push each electron to an orbit of one higher unit of angular momentum. This leaves a hole at the origin which has a fractional charge $\nu$.

The obvious candidate to replace the monopole in the 10 dimensional type 2a string theory is the D6-brane oriented in the $X^3, \ldots, X^8$ direction. Suppose we pass the D6-brane through the rigid open membrane piercing it at some location. If the correspondence holds true it should create a quasihole at that point. In other words it should leave behind a string end on the membrane. This is again an example of the Hanany–Witten effect which requires just such a string to form when a D6-brane is passed through a D2-brane.

After this work was completed we became aware of a similar brane construction by Oren Bergman, John Brodie, and Yuji Okawa. The setup that these authors use is similar but not identical to the one reported here and the conclusions generally agree [17].

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