ON 2-FOLD COVERS OF GRAPHS

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Abstract

A regular covering projection \(\varphi: \tilde{X} \to X\) of connected graphs is \(G\)-admissible if \(G\) lifts along \(\varphi\). Denote by \(\tilde{G}\) the lifted group, and let \(\text{CT}(\varphi)\) be the group of covering transformations. The projection is called \(G\)-split whenever the extension \(\text{CT}(\varphi) \to \tilde{G} \to G\) splits. In this paper, split 2-covers are considered, with a particular emphasis given to cubic symmetric graphs. Supposing that \(G\) is transitive on \(X\), a \(G\)-split cover is said to be \(G\)-split-transitive if all complements \(\bar{G} \cong G\) of \(\text{CT}(\varphi)\) within \(\tilde{G}\) are transitive on \(\tilde{X}\); it is said to be \(G\)-split-sectional whenever for each complement \(\bar{G}\) there exists a \(\bar{G}\)-invariant section of \(\varphi\); and it is called \(G\)-split-mixed otherwise.

It is shown, when \(G\) is an arc-transitive group, split-sectional and split-mixed 2-covers lead to canonical double covers. Split-transitive covers, however, are considerably more difficult to analyze. For cubic symmetric graphs split 2-cover are necessarily canonical double covers (that is, no \(G\)-split-transitive 2-covers exist) when \(G\) is 1-regular or 4-regular. In all other cases, that is, if \(G\) is \(s\)-regular, \(s = 2, 3\) or 5, a necessary and sufficient condition for the existence of a transitive complement \(\bar{G}\) is given, and moreover, an infinite family of split-transitive 2-covers based on the alternating groups of the form \(A_{12k+10}\) is constructed.

Finally, chains of consecutive 2-covers, along which an arc-transitive group \(G\) has successive lifts, are also considered. It is proved that in such a chain, at most two projections can be split. Further, it is shown that, in the context of cubic symmetric graphs, if exactly two of them are split, then one is split-transitive and the other one is either split-sectional or split-mixed.

Keywords: graph, cubic graph, symmetric graph, \(s\)-regular group, regular covering projection.

1 Introductory remarks

Let \(\varphi: \tilde{X} \to X\) be a regular covering projection of connected (simple) graphs. Comparing symmetry properties of a given base graph \(X\) and a covering graph \(\tilde{X}\) has become quite an active area of research in recent years. The motivation stems from problems related to construction and classification of certain classes of graphs and maps on surfaces, counting the number of graphs in certain families, producing lists of graphs with a given degree of symmetry, inductive approach to studying the structure of graphs via inspection of smaller graphs arising as quotients relative to a semiregular group of automorphisms.

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etc. References are too numerous to be listed here, but see, for instance, \[1, 7, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 31, 33, 34, 36, 37, 39, 42, 44\].

Questions related to symmetry properties of \(X\) and \(X\) are intimately linked with the problem of lifting and projecting automorphisms \([3, 11, 28, 30]\). Let \(g \in \text{Aut} \ X\) and \(\tilde{g} \in \text{Aut} \ X\) be automorphisms with the property that \(\varphi g = \tilde{g} \varphi\) (note that functions are composed on the right); such an automorphism \(\tilde{g}\) is known as a lift of \(g\) while \(g\) is the projection of \(\tilde{g}\) along \(\varphi\). If all elements of a subgroup \(G \leq \text{Aut} \ X\) have a lift, then the collection of all such lifts forms the lifted group \(\tilde{G} \leq \text{Aut} \ X\); the projection \(\varphi\) is then called \(G\)-admissible. In particular, the group of covering transformations \(\text{CT}(\varphi) \leq \text{Aut} \ X\) is the lift of the trivial group, and \(\tilde{G}\) is isomorphic to an extension of \(\text{CT}(\varphi)\) by \(G\). We remark that any abstract group extension can be viewed as a lifting problem; this adds further motivation to the topic.

A \(G\)-admissible regular covering projection \(\varphi: \tilde{X} \to X\) is called split relative to \(G\) (or \(G\)-split for short) whenever the extension \(\text{CT}(\varphi) \to \tilde{G} \to G\) is split. This case deserves special attention: since there exists a complement \(\tilde{G} \cong G\) of \(\text{CT}(\varphi)\) within \(\tilde{G}\) we can compare actions of two isomorphic groups, \(G\) on \(X\) and \(\tilde{G}\) on \(\tilde{X}\), where \(\tilde{G}\) projects isomorphically onto \(G\) along \(\varphi\). A restrictive situation such as this makes it possible to derive a lot more information about graphs and their symmetries.

However, the analysis can still be quite complicated. This is due to the fact that several complements \(\tilde{G}\) of \(\text{CT}(\varphi)\) might exist within \(\tilde{G}\), which moreover differ in their actions on \(\tilde{X}\). Two particular cases of split covers seem to stand out. The first one occurs when there exists a complement \(\tilde{G}\) which is sectional. By this we mean that there is a section of \(\tilde{X}\) – a set of vertices containing exactly one point from each fibre – which is invariant under the action of \(\tilde{G}\). The second particular case occurs when there exists a complement \(\tilde{G}\) which is transitive on \(\tilde{X}\) (\(G\) itself must then be transitive on \(X\)). Two particularly interesting extremal cases of split covers can now be defined along these lines. A projection \(\varphi\) is called split-sectional if all complements of \(\text{CT}(\varphi)\) within \(\tilde{G}\) are sectional, and is split-transitive if all complements are transitive. A projection is called split-mixed whenever both sectional and transitive complements exist. Of course, it can also happen that each complement of \(\text{CT}(\varphi)\) within \(\tilde{G}\) is neither sectional nor transitive. This fourth possibility is perhaps less interesting, and definitely the most difficult of all to analyze.

In this paper we restrict ourselves to 2-covers, that is, to regular covering projections \(\varphi: \tilde{X} \to X\) where \(\text{CT}(\varphi)\) is isomorphic to the cyclic group \(\mathbb{Z}_2\), with a particular emphasis given to such covering projections in the context of cubic symmetric graphs. Now, if the projection is \(G\)-split, the lifted group \(\tilde{G}\) is necessarily isomorphic to the direct product \(G \times \mathbb{Z}_2\). In other words, complements \(\tilde{G}\) of \(\text{CT}(\varphi)\) are normal in \(\tilde{G}\). In addition, let us assume that \(G\) is transitive on \(X\). Then any complement is either sectional or transitive in its action on \(\tilde{X}\). Thus, apart form the possibility that \(\varphi: \tilde{X} \to X\) is non-split relative to \(G\), there are three possible types of \(G\)-split 2-covers: split-sectional, split-transitive, and split-mixed. Obviously, these various kinds of lifts do depend on all four items involved: the graph \(X\), the cover \(\tilde{X}\), the actual projection \(\varphi: \tilde{X} \to X\), and the group \(G\). For example, the cube \(Q_3\) is obtained as a sectional split 2-cover of \(K_4\) relative to the alternating group \(A_4\); on the other hand, \(Q_3\) can be viewed as a mixed split 2-cover relative to the symmetric group \(S_4\). For details see Section 2 where a thorough analysis of all these possibilities is given in the context of some cubic symmetric graphs of small order.

The rest of this article is organized as follows. In Section 3 split lifts with a sectional a complement are considered (for general graphs). In Section 4 transitive complements are considered in the context of cubic symmetric graphs. In Section 5 we consider consecutive
2-lifts and show that within split covers only two such lifts are possible for cubic symmetric graphs. We end the paper with Section 6 where we propose a short list of problems for future research.

2 Examples

We now give examples of all four types of 2-covers, using the complete graph $K_4$ on 4 vertices and the Petersen graph $O_3$ as base graphs. Throughout this article most of our examples will be given in the context of cubic symmetric graphs. We will therefore use standard notation for these graphs from the extended Foster Census, see [5, 6], together with commonly used names for some of them where appropriate. By $F_n A$, $F_n B$, etc. we refer to the corresponding graphs of order $n$ in the Foster census of all cubic symmetric graphs [5, 6], where the symbol $F_n A$ is conveniently shortened to $F_n$ whenever a unique such graph exists. For example, $K_4$ and $O_3$ are the graphs $F_4$ and $F_{10}$, respectively.

Before starting with examples, some additional terminology is in order. A group of automorphisms $G$ of a graph $X$ is arc-transitive (also symmetric) if $G$ acts transitively on the arcs in $X$, and $s$-regular whenever it acts regularly on the $s$-arcs in $X$; here, an $s$-arc in $X$ is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of $X$ with the property that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$ (see [7]). The graph $X$ is said to be $s$-regular whenever its full automorphism group $Aut X$ acts regularly on $s$-arcs in $X$. For instance, $K_4$ has only two arc-transitive groups of automorphisms, the alternating group $A_4$ (which is 1-regular) and the symmetric group $S_4$ (which is 2-regular). Similarly, the Petersen graph $O_3$ has only two arc-transitive groups of automorphisms, $A_5$ and $S_5$, which are 2- and 3-regular, respectively.

Given a graph $X$, the canonical double cover $CDC(X) \to X$ is a 2-cover which can be reconstructed by the constant $Z_2$-voltage assignment $\zeta(x) = 1$, $x \in A(X)$. (See [21] for an extensive treatment of voltage graphs.) Note that $CDC(X)$ is connected if and only if $X$ is connected and not bipartite. Note that along the canonical double cover any subgroup $G$ of the full automorphism group $Aut X$ lifts to a group isomorphic to $G \times Z_2$, which has a sectional complement $G \cong G$ to $CT(\varphi)$, see Section 3.

Example 2.1 The complete graph $K_4$ is not bipartite and its canonical double cover $\varphi: CDC(K_4) \to K_4$ is therefore connected. In fact, $CDC(K_4)$ is isomorphic to the cube $Q_3$ of order 8 (F8 in Foster notation); its full automorphism group is $S_4 \times Z_2$.

Clearly, $\varphi$ is a split 2-cover relative to the 1-regular group $A_4$ and relative to the 2-regular group $S_4$ of $K_4$; both have sectional complements. Moreover, no complement of $\varphi$ relative to $A_4$ is transitive since 8 is not a divisor of $|A_4|$. Hence $\varphi$ is a split-sectional 2-cover relative to $A_4$.

Let $S_4 \times Z_2 = S_4 \times \langle a \rangle$ and let $T = A_4 \times \langle ca \rangle$, where $S_4$ is a sectional complement of $\varphi$ relative to the 2-regular automorphism group $S_4$ of $K_4$ and $c$ is an involution in $S_4 \setminus A_4$. Then, $T \cong S_4$ is a transitive complement relative to $S_4$. This implies that $\varphi$ is a split-mixed 2-cover relative to $S_4$.

Example 2.2 The Petersen graph $F_{10}$ is not bipartite and hence it has a connected canonical double cover $\varphi_1: CDC(F_{10}) \to F_{10}$. There are precisely two connected cubic symmetric graphs of order 20 (see [5, 8, 9]), the dodecahedron $F_{20} A$ and the Desargues graph $F_{20} B$. The dodecahedron is 2-regular with the full automorphism group $A_5 \times Z_2$.
The Deasargues graph, which is the canonical double cover \( CDC(F_{10}) \) is 3-regular with the full automorphism group \( S_5 \times Z_2 \). Note that the only arc-transitive subgroups of \( \text{Aut} F_{10} \) are the 2-regular group \( A_5 \) and the 3-regular group \( S_5 \). Clearly, \( A_5 \times Z_2 \) contains only one subgroup isomorphic to \( A_5 \). Thus, \( \varphi_1 \) is a split-sectional 2-cover relative to \( A_5 \).

On the other hand, a similar argument to the one given in Example 2.1 shows that \( \varphi_1 \) is a split-mixed 2-cover relative to \( S_5 \).

The 2-regular full automorphism group \( A_5 \times Z_2 \) of the dodecahedron \( F_{20A} \) contains a normal subgroup \( Z_2 \). The quotient graph of \( F_{20A} \) corresponding to the orbits of \( Z_2 \) must be the Petersen graph \( F_{10} \) because there is only one connected cubic symmetric graph of order 10. Thus, \( F_{20A} \) is a split 2-cover of \( F_{10} \) relative to the 2-regular group \( A_5 \) of \( F_{10} \).

Denote this 2-cover by \( \varphi_2: F_{20A} \to F_{10} \). The full automorphism group \( S_5 \) of \( F_{10} \) cannot lift along \( \varphi_2 \) because \( F_{20A} \) is 2-regular. Since \( F_{20A} \) is not bipartite, \( A_5 \) is transitive on \( F_{20A} \), and the uniqueness of \( A_5 \) in \( A_5 \times Z_2 \) implies that \( \varphi_2 \) is a transitive 2-cover relative to \( A_5 \).

This example needs some further comments which we here give without proof. For details see Section 5. As \( F_{20A} \) is not bipartite it has the canonical double cover \( \varphi_3: F_{40} \to F_{20A} \), where \( F_{40} \) is the unique connected cubic symmetric graph of order 40. Note that \( F_{40} \) is 3-regular (see [5]) and has \( A_5 \times Z_2 \times Z_2 \) as a 2-regular group of automorphisms. Then, \( \varphi_3 \) is a split-sectional 2-cover relative to the 1-regular automorphism group \( A_5 \) of \( F_{20A} \) and a split-mixed 2-cover of \( F_{20A} \) relative to the 2-regular automorphism group \( A_5 \times Z_2 \) of \( F_{20A} \). Consider now the subgroup \( A_5 \times Z_2 \times Z_2 \) of \( \text{Aut} F_{40} \). One may choose a normal subgroup \( Z_2 \) in \( A_5 \times Z_2 \times Z_2 \) fixing the bipartition sets of \( F_{40} \) setwise. The quotient graph with respect to the action of this \( Z_2 \) is bipartite, and so it must be the Deasargues graph \( F_{20B} \). Denote this projection, which is clearly not a CDC, by \( \varphi_4: F_{40} \to F_{20B} \). Then, \( \varphi_4 \) is a transitive 2-cover of \( F_{20B} \) relative to the 2-regular automorphism group \( A_5 \times Z_2 \) of \( F_{20B} \). We obtain the following figure: Thus, there are two different ‘chains’

![Diagram](image)

Figure 1: Consecutive split 2-covers of the Petersen graph relative to \( A_5 \).

of consecutive 2-covers from the Petersen graph \( F_{10} \) to the connected cubic symmetric graph \( F_{40} \) relative to \( A_5 \). This idea will be used to prove that a ‘chain’ of consecutive split 2-covers of a connected cubic graph relative to an arc-transitive group of automorphisms cannot have ‘length’ more than 2 (see Theorem 5.3 in Section 5).

**Example 2.3** In view of Examples 2.1 and 2.2 there exist transitive, sectional and mixed split 2-covers. In fact, infinitely many examples exist for each of these three cases. By [16, Theorem 4.4], there is an infinite family of 2-regular cubic graphs, denoted there by \( EO_{p^3} \), where \( p = \pm 1(\text{mod } 5) \) is a prime. Each graph \( EO_{p^3} \) is not bipartite and has \( Z_p^3 \times A_5 \) as the full automorphism group. Since the cover is bipartite, any vertex-transitive subgroup of automorphisms has a subgroup of index 2 fixing the bipartition sets setwise, and since the group \( Z_p^3 \times A_5 \) has no subgroup of index 2, the canonical double cover of \( EO_{p^3} \) is a
split-sectional 2-cover relative to $\mathbb{Z}_p^6 \times A_5$.

Again by [16, Theorem 4.4], there is an infinite family of 3-regular cubic graphs arising as covers of the Petersen graph, denoted there by $EO_{p^6}$, where $p > 5$ is a prime. The graphs $EO_{p^6}$ are not bipartite and have $\mathbb{Z}_p^6 \times S_5$ as the full automorphism group. Since $\mathbb{Z}_p^6 \times S_5$ has a transitive subgroup of index 2, the canonical double cover of $EO_{p^6}$ is a split-mixed 2-cover relative to $\mathbb{Z}_p^6 \times S_5$ by Proposition [30,2]. Constructing infinitely many split-transitive 2-covers is a bit more complicated, see Theorem [16,2] in Section [3].

Example 2.4 To end this section we would like to give an example of non-split 2-cover, although this paper concentrates on split 2-covers.

The Möbius-Kantor graph $F_{16}$ is a 2-cover of the 3-cube $F_8$. By [18, Theorem], its full automorphism group $Aut F_{16} \cong \mathbb{Z}_2 \times S_4$ lifts. We claim that this cover is non-split. Suppose that $H = (\mathbb{Z}_2 \times S_4) \times \mathbb{Z}_2$ is a subgroup of $Aut F_{16}$. Let $T$ be the unique non-trivial normal 2-subgroup of $S_4$. Then $T$ has order 4, and since $T$ is characteristic in $S_4$, it is normal in $H$. It follows that the quotient graph of $F_{16}$ corresponding to the orbits of $T$ is the complete graph $K_4$, and $H/T \cong \mathbb{Z}_2 \times S_3 \times \mathbb{Z}_2$ can be viewed as a subgroup of $Aut K_4$ – because $T$ is the kernel of the action of $H$ on the set of orbits of $T$, which is impossible since $Aut K_4 \cong S_4$. This implies that the 2-cover is non-split relative to the automorphism group $Aut F_{16}$.

3 Sectional complements in split 2-covers

Let $\varphi: \mathcal{X} \to X$ be a 2-cover of connected graphs, and let $G \leq Aut X$ be a vertex-transitive group that lifts along $\varphi$ to $\tilde{G}$. The case when $\varphi$ is $G$-split with a sectional complement can be conveniently described on the base graph by an appropriate choice of voltages. In fact, the following holds (for a general version see [30]).

Proposition 3.1 Let $\varphi: \mathcal{X} \to X$ be a 2-cover of connected graphs, and let $G \leq Aut X$ be vertex-transitive.

1. Then $G$ lifts along $\varphi$ if and only if, for any voltage assignment by which the projection $\varphi$ is reconstructed, the set of all 0-voltage closed walks is invariant under the action of $G$.

2. Moreover, $\varphi$ is $G$-split with a sectional complement if and only if $\varphi$ can be reconstructed by a voltage assignment $\zeta: A(X) \to \mathbb{Z}_2$ which satisfies the following condition: for each automorphism $g \in G$ and each walk $W$ in $X$ we have $\zeta(W^g) = \zeta(W)$.

Proof. As for the first part, obviously the condition that the set of 0-voltage walks is invariant under the action of $G$ is a necessary one. Similarly, it can also be seen that it is sufficient, see [30].

We now prove the second part. Suppose that $\varphi: \mathcal{X} \to X$ is reconstructed by the voltage assignment in $\mathbb{Z}_2$ satisfying $\zeta(W^g) = \zeta(W)$ for all walks $W$ in $X$ and all $g \in G$. Then the set of closed walks with trivial voltage is clearly invariant under the action of $G$, and hence $G$ lifts. For each vertex $v$ of $X$, let $fib(v) = \{v_0, v_1\}$ denote the fibre over $v$. Choose a base vertex $b \in V(X)$. For each $g \in G$, let $\tilde{g}$ be the lift of $g$ which maps the vertex $b_0 \in fib(b)$ to the vertex $\tilde{b}_0 \in fib(\tilde{b})$. Then $\tilde{g}$ preserves the set of vertices labelled by 0, that is, the 0-section. Indeed, let $u \in V(X)$ be an arbitrary vertex, and let $W$ be the walk from $b$ to $u$ with $\zeta(W) = 0$. (Note that such a walk always exists.) Let $\tilde{W}$ be its lift with $b_0$ as the initial vertex. The terminal vertex of $\tilde{W}$ is $u_0 \in fib(u)$ because $\zeta(W) = 0$. The walk $W^g$
from \( b^9 \) to \( u^9 \) also has trivial voltage. Its lift \( \tilde{W}^9 \) starting at \( b_0^9 \) terminates at \( u_0^9 \). Since \( \tilde{g} \) is the lift of \( g \), it maps the walk \( \tilde{W} \) to the walk \( \tilde{W}^9 \). Hence \( \tilde{g} \) maps \( u_0 \) to \( u_0^9 \). It follows that the 0-section is invariant under the action of \( \{ \tilde{g} \mid g \in G \} \). Moreover \( \{ \tilde{g} \mid g \in G \} \) must be a group, indeed a subgroup of index 2 in \( \tilde{G} \), as required.

To show the converse, let \( \tilde{G} \) be a complement of \( \text{CT}(\varphi) \) preserving a section of \( \varphi: \tilde{X} \to X \). This covering projection can be reconstructed by a voltage assignment in \( \mathbb{Z}_2 \) in such a way that this particular section is labelled by 0. Consider now an arbitrary walk \( W \) in \( X \), and let \( W_0 \) be its lift in \( \tilde{X} \) starting at the corresponding vertex from the 0-section. Since \( \tilde{G} \) preserves the 0-section and the 1-section, it is obvious that \( W_0^9 \) has the initial vertex in the 0-section and its terminal vertex belongs to the same section as the terminal vertex of \( \tilde{W} \). Since the lift of \( W^9 \) starting in the 0-section must be \( W_0^9 \), it follows that \( \zeta(W^9) = \zeta(W) \), as required.

A typical example illustrating Proposition 3.1 is the canonical double cover \( \text{CDC}(X) \to X \). Recall that any group \( G \leq \text{Aut}X \) lifts to \( \text{CDC}(X) \) as \( G \times \mathbb{Z}_2 \), and there exists a sectional complement. Further examples of split 2-covers with a sectional complement which are not canonical double covers are given at the end of this section. Note that a canonical double cover need not be split-sectional, for transitive complements might exist as well; hence it can be split-mixed, but clearly not split-transitive. So let us consider the question of when does an arbitrary \( G \)-split 2-cover with a sectional complement have another complement which is transitive on the vertex set of the covering graph. The following holds.

**Proposition 3.2** Let \( \varphi: \tilde{X} \to X \) be a \( G \)-split 2-cover with a sectional complement. Then \( \text{CT}(\varphi) \) has a transitive complement within the lifted group \( \tilde{G} \) if and only if \( G \) has a vertex-transitive subgroup of index 2.

**Proof.** Let \( \tilde{G} = \text{CT}(\varphi) \times G \), where \( G \) has two orbits on the vertex set of \( \tilde{X} \). Suppose that \( \tilde{G} = \text{CT}(\varphi) \times K \), where \( K \) acts transitively on the vertex set of \( \tilde{X} \). Since \( \tilde{G} \) and \( K \) are of index 2 in \( \tilde{G} \), the intersection \( K \cap \tilde{G} \) is of index 2 in \( G \) and in \( K \). Consequently, the projection \( H \leq G \) of \( K \cap \tilde{G} \) is a subgroup of index 2 in \( G \). Moreover, \( H \) must be vertex-transitive. Indeed, since the group \( K \cap \tilde{G} \) is of index 2 in \( K \) it has at most two orbits on the vertex set of \( \tilde{X} \). In fact, as \( K \cap \tilde{G} \) is contained in \( \tilde{G} \), it cannot be transitive. So \( K \cap \tilde{G} \) must have two orbits and is therefore transitive on the set of all fibers. Consequently, the projection \( H \) is transitive on \( X \).

Conversely, let \( H \leq G \) be a vertex-transitive subgroup of index 2. Then there is an element \( g \in G \) such that \( G = \langle H, g \rangle \). Denote by \( c \in \text{CT}(\varphi) \) the nonidentity element. Then \( \tilde{G} = \langle c \rangle \times \tilde{G} \), and the lift of \( H \) can be written as \( \tilde{H} = \langle c \rangle \times \tilde{H} \) such that \( \tilde{H} \leq \tilde{G} \). Consider now the group \( K = \langle \tilde{H}, c\tilde{g} \rangle \), where \( \tilde{g} \in \tilde{G} \). Obviously, \( K \) is isomorphic to \( G \) and acts transitively on the vertex set of \( \tilde{X} \).

See Example 2.2 with \( F40 \to F20A \) and \( G = A_5 \times \mathbb{Z}_2 \) for an illustration of the above proposition. As already mentioned, a \( G \)-split 2-cover with a sectional complement need not be a canonical double cover, see Construction 3.3 below. However, if \( G \) is also edge-transitive (in addition to being vertex-transitive), then a \( G \)-split 2-cover with a sectional complement is necessarily a canonical double cover.

**Proposition 3.3** Let \( \varphi: \tilde{X} \to X \) be a \( G \)-admissible 2-cover of connected graphs. If \( G \leq \text{Aut}X \) is vertex and edge-transitive then \( \varphi \) is \( G \)-split with a sectional complement if and
only if \( \varphi \) is the canonical double cover. (Equivalently, if \( \varphi \) is split, then it is split-transitive relative to \( G \) if and only if \( \varphi \) is not the canonical double cover).

**Proof.** If \( \varphi \) is a canonical double cover, then clearly \( \varphi \) is \( G \)-split with a sectional complement. For the converse we may assume, by Proposition 3.1, that \( \varphi \) is reconstructed by a voltage assignment \( \zeta : A(X) \to \mathbb{Z}_2 \) such that \( \zeta(W^g) = \zeta(W) \) for all walks \( W \) and all \( g \in G \). As \( G \) is assumed edge-transitive, all edges (and hence arcs) have equal voltage. Since \( \tilde{X} \) is assumed connected we have \( \zeta(x) = 1 \) for all arcs, that is, \( \varphi \) is the canonical double cover. \( \blacksquare \)

As long as we choose to consider, say, arc-transitive groups (later on we shall in fact restrict our considerations to arc-transitive cubic graphs), the question of whether a given \( G \)-admissible 2-cover is split with a sectional complement is solved (in some sense). All we need is to check whether a given 2-cover is indeed canonical. This is algorithmically easy, as the following proposition shows.

**Proposition 3.4** Let \( X \) be a connected graph and \( \varphi : \tilde{X} \to X \) a 2-fold covering projection arising from a voltage assignment \( \zeta : A(X) \to \mathbb{Z}_2 \). Then \( \varphi \) is the canonical double cover if and only if each odd length cycle in \( X \) has voltage 1 and each even cycle in \( X \) has voltage 0. Moreover, it suffices to test this condition on the set of base cycles only.

**Proof.** It is easy to see that two voltage assignments \( \zeta_1, \zeta_2 : A(X) \to \mathbb{Z}_2 \) are equivalent if and only if \( \zeta_1(W) = \zeta_2(W) \) for each closed walk \( W \), and the proof is immediate. \( \blacksquare \)

We now turn to the question of existence of split 2-covers of a vertex-transitive graph \( X \) with sectional complement which are not canonical double covers. In view of Proposition 3.1, this can only happen if the vertex-transitive group \( G \) in question is not edge-transitive. We give below two examples of split 2-covers (of cubic vertex-transitive graphs) which are not canonical double covers.

**Construction 3.5** Let \( n \) be odd and let \( X \) be the circulant \( X \cong \text{Cay}(\mathbb{Z}_{2n}, \{1, 2n - 1, n\}) \). Define the voltage assignment \( \zeta : A(X) \to \mathbb{Z}_2 \) in such a way that all edges \([i, i + 1] \) \( i \in \mathbb{Z}_{2n} \) receive voltage 0 and edges \([i, i + n] \) receive voltage 1. Moreover, by Proposition 3.3, the respective 2-cover \( \varphi : \tilde{X} \to X \) is not the canonical double cover since the cycle \((0, 1, 2 \ldots n, 0) \) of \( X \) has even length but voltage 1 (see Figure 2). Let \( G = \text{Aut } X \cong D_{4n} \), let \( H \cong \mathbb{Z}_{2n} \) be the cyclic subgroup of index 2 in \( G \), and let \( K \cong D_{2n} \) be the dihedral subgroup of index 2 in \( G \). Obviously, the group \( G \) lifts by Proposition 3.1 and the corresponding 2-cover is split. Moreover, by Proposition 3.3 the projection \( \varphi \) is split-sectional relative to \( H \) and relative to \( K \), and split-mixed relative to \( G \).

**Construction 3.6** Let \( n \) be odd, and let \( X = C_{2n} \square K_2 \) be the cartesian product of \( C_{2n} \) with \( K_2 \), viewed as the Cayley graph \( \text{Cay}(G, \{b, ab, c\}) \), where the group \( G = \langle a, b, c \mid a^n = b^2 = c^2 = 1, a^b = a^{-1}, a^c = a, b^c = b \rangle \) is isomorphic to \( D_{2n} \times \mathbb{Z}_2 \). Define the voltage assignment \( \zeta : A(X) \to \mathbb{Z}_2 \) ‘consistent’ with the regular action of \( G \) in such a way that all \( b \)-edges and all \( c \)-edges receive voltage 1 and all \( ab \)-edges receive voltage 0. The corresponding 2-cover is not the canonical double cover because \( X \) contains an even length cycle with voltage 1. In fact, any of the two \( 2n \)-cycles obtained from \( b \)-edges and \( ab \)-edges is of this kind (see Figure 3). Also, this 2-cover has a sectional complement relative to \( G \). It is split-sectional.
4 Split 2-covers with a transitive complement

We start this section by constructing a graph from a finite group $G$ relative to a subgroup $H$ of $G$ and a union $D$ of some double cosets of $H$ in $G$ such that $D^{-1} = D$, the so called coset graph \[35,32\].

The coset graph $Cos(G, H, D)$ of $G$ with respect to $H$ and $D$ is defined to have vertex set $[G : H]$, the set of right cosets of $H$ in $G$, and edge set $\{(Hg, Hdg) \mid d \in D\}$. The graph has valency $|D|/|H|$ and is connected if and only if $D$ generates the group $G$. For $g \in G$, denote by $R(g)_H$ the right multiplication of $g$ on $[G : H]$, that is, $R(g)_H : Hx \mapsto Hxg$, $x \in G$. Clearly, $R(g)_H$ is an automorphism of $Cos(G, H, D)$. Let $R(G)_H = \{R(g)_H \mid g \in G\}$. Then $R(G)_H \leq \text{Aut}(Cos(G, H, D))$. Note that it may happen that $R(g)_H$ is the identity automorphism of $Cos(G, H, D)$ for some $g \neq 1$ in $G$. Actually, $R(G)_H \cong G$ if and only if $H_G = 1$, where $H_G$ is the largest normal subgroup of $G$ in $H$. Since $R(G)_H \leq \text{Aut}(Cos(G, H, D))$, we have that $Cos(G, H, D)$ is vertex-transitive and $R(G)_H$ acts arc-transitively on the coset graph if and only if $D$ is a single double coset. (Note that the concept of a coset graph is equivalent to the concept of an orbital graph \[40,35\].)

Conversely, let $X$ be a graph and let $A$ be a vertex-transitive subgroup of $\text{Aut}(X)$. By \[35\], the graph $X$ is isomorphic to a coset graph $Cos(A, H, D)$, where $H = A_u$ is the stabilizer of $u \in V(X)$ and $D$ consists of all elements of $A$ which map $u$ to its neighbors. It is easy to show that $H_A = 1$ and that $D$ is a union of some double cosets of $H$ in $A$ satisfying $D = D^{-1}$. Assume that $A$ acts arc-transitively on $V(X)$ and that $g \in A$ interchanges $u$ and one of its neighbors. Then, $g^2 \in H$ and $D = HgH$. The valency of $X$ is $|D|/|H| = |H : H \cap H^g|$.

Let $X$ be a connected cubic symmetric graph with $G$ as an $s$-regular subgroup of $\text{Aut}(X)$. By \[311\] Proposition 2-Proposition 5], the stabilizer $G_v$ of $v \in V(X)$ in $G$ is isomorphic to $\mathbb{Z}_3$, $S_3$, $S_3 \times \mathbb{Z}_2$, $S_4$, or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.

Now we consider split 2-covers of a connected cubic graph relative to an $s$-regular group of automorphisms with a transitive complement. Let $X$ have a connected split 2-cover $Y$ such that an $s$-regular group of automorphisms $G$ lifts to $\mathbb{Z}_2 \times G$ with $G$ acting on $Y$ transitively. Clearly, $|V(Y)| = 2|V(X)|$. If $s = 1$ then $|G| = 3|V(X)|$ and since $G$ is transitive on $Y$, $|Y|$ is a divisor of $|G|$, that is, $2|V(X)| \mid 3|V(Y)|$, which is impossible. If $s = 4$ then $\mathbb{Z}_2 \times G$ is 4-regular and $G$ is 3-regular, which is also impossible because a 3-regular group of automorphisms cannot be contained a 4-regular group of automorphisms (see \[311\]). This implies the following theorem.
Theorem 4.1 Let $X$ be a connected cubic graph with $G$ as an $s$-regular automorphism group, where $s = 1, 4$. Then each $G$-split 2-cover must be the canonical double cover.

Theorem 4.2 Let $X$ be a connected cubic graph with \{u, v\} as an edge and with $G$ as an $s$-regular group of automorphisms, where $s = 2, 3, 5$, and let $H = G_u$ be the stabilizer of $u \in V(X)$ in $G$. Then, $X$ has a connected $G$-split 2-cover with a transitive complement if and only if there is an element $b \in G$ interchanging $u$ and $v$ and $H$ has a subgroup $L$ of index 2 such that

$$b^2 \in L, \quad \langle b, L \rangle = G, \quad \text{and} \quad |L : L^b \cap L| = 3. \quad (1)$$

Moreover, the covering graph is isomorphic to the coset graph $\text{Cos}(G, L, LbL)$.

PROOF. Suppose first that $X$ has a connected split 2-cover $Y$ such that $G$ lifts to $\tilde{G} \times \mathbb{Z}_2$ with $\tilde{G}$ acting on $Y$ transitively. In the sequel we may identify $\tilde{G}$ with $G$ for convenience. Then, $G$ is $(s - 1)$-regular on $V(Y)$ because it is $s$-regular on $V(X)$. Since $s = 2, 3, 5$, the group $G$ is arc-transitive on $Y$. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the fibres of $u$ and $v$ in the 2-cover $Y$ of $X$. By identifying each vertex in $V(X)$ with its fibre in $Y$, one has $H = G_u = G_{\{u_1, u_2\}}$, where $G_{\{u_1, u_2\}}$ is the subgroup of $G$ fixing the fibre $\{u_1, u_2\}$ setwise. Since $u$ and $v$ are adjacent, $u_1$ is adjacent to a vertex in $\{v_1, v_2\}$, say $v_1$. The arc-transitivity of $G$ on $Y$ implies that there is an element $b \in G$ interchanging $u_1$ and $v_1$. It follows that $b$ interchanges $u$ and $v$. Let $L = G_{u_1}$. Since $Y$ is a connected cubic graph, one has $G = \langle b, L \rangle$, $b^2 \in L$ and $|L : L^b \cap L| = 3$. This shows that (1) is a necessary condition.

For the converse, suppose that there is an element $b \in G$ interchanging $u$ and $v$ and that $H$ has a subgroup $L$ of index 2 such that (1) is satisfied.

Since $G$ is arc-transitive on a given connected cubic graph $X$, one has $H_G = 1$, $|H : H^b \cap H| = 3$, and $X \cong \text{Cos}(G, H, HbH)$. Without loss of generality we may identify $X$ with this coset graph, $X = \text{Cos}(G, H, HbH)$. Set $\bar{X} = \text{Cos}(G, L, LbL)$. Clearly, $\bar{X}$ is a connected cubic graph with $G$ as an $(s - 1)$-regular automorphism subgroup since $H_G = 1$ implies that $L_G = 1$. Thus, $L$ is isomorphic to $\mathbb{Z}_3, S_3, S_4$ for $s = 2, 3, 5$, respectively. Since $H$ is isomorphic to $S_3, S_3 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2$ for $s = 2, 3, 5$, respectively, there is an involution $r \in H$ such that $H = L \rtimes \langle r \rangle$, where $r$ normalizes $L$. Note that for $s = 3, 5$ the involution $r$ centralizes $L$ (and hence $H = L \rtimes \langle r \rangle$), but for $s = 2$ it does not. Set

$$K = G \times \langle c \rangle, \quad D = \langle L, rc \rangle, \quad (2)$$

where $\langle c \rangle \cong \mathbb{Z}_2$. Then, $D \leq K$, $\langle D, bD \rangle = \langle D, b \rangle = K$ and $(D, bD)^{-1} = D$, $b \in L$. Since $rc$ normalizes $L$ (but does not centralize) for $s = 2$, and centralizes $L$ for $s = 3, 5$, one has $D \cong S_3, S_3 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2$ for $s = 2, 3, 5$, respectively. We claim that

$$\bar{X} := \text{Cos}(K, D, DbD)$$

is a connected cubic graph with $K$ as an $s$-regular group of automorphisms. To prove this we need to show that $D_K = 1$ and that $|D : D^b \cap D| = 3$.

Suppose on the contrary that $D_K \neq 1$. Then, $D_K \not< D$, $D_K \cap G \not< G$, and $D_K \cap L \not< L$. Let $s = 2$. Then $D \cong S_3$, and since $D_K \not< D$, one has that $D_K = L$ is the unique Sylow 3-subgroup of $D$. Hence $D_K \leq H$, which contradict the fact that $H_G = 1$. Let $s = 3$. Assume that $N \leq D_K$ is a minimal normal subgroup of $K$. Then $N$ is an elementary
abelian 2- or 3-group. Since $D = L \times \langle rc \rangle \cong S_3 \times \mathbb{Z}_2$, the group $N$ is either the Sylow 3-subgroup of $L$, or else $N = \langle rc \rangle$. In the former case one has $N \leq H_G$, a contradiction. As for the latter case, we have $rc \in Z(K)$ because $|N| = 2$ and hence $(rc) \leq Z(G)$. It follows that $(r) \leq H_G$, a contradiction. Let $s = 5$. Then $D = L \times \langle rc \rangle \cong S_4 \times \mathbb{Z}_2$. Clearly, $L \cong S_4$ has only one non-trivial normal subgroup of order 4, say $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $M \leq D_K$ be a minimal normal subgroup of $K$. Then $M$ is a 2- or 3-group. Since $M \triangleleft D$, the group $M$ cannot be a 3-group, and one may easily obtain that $M = T$, $M = T \times \langle rc \rangle$, or $M = \langle rc \rangle$.

For the first two cases, $T = M \cap G \triangleleft G$, and for the last case $\langle rc \rangle \triangleleft G$, contrary to $H_G = 1$. Thus, $D_K = 1$.

To prove $|D : D^b \cap D| = 3$, it suffices to show that $D^b \cap D$ is a Sylow 2-subgroup of $D$. Since $|H : H^b \cap H| = |L : L^b \cap L| = 3$, the groups $H^b \cap H$ and $L^b \cap L$ are Sylow 2-subgroups of $H$ and $L$, respectively. Since $b^2 \in L \leq H$, one has $(H^b \cap H)^b = H^b \cap H$ and $(L^b \cap L)^b = L^b \cap L$, that is, $b$ normalizes $H^b \cap H$ and $L^b \cap L$. Let $s = 2$. Then $L \cong \mathbb{Z}_3$, $H = L \times \langle r \rangle$, and $D = L \times \langle rc \rangle \cong S_3$. This implies that $L^b \cap L = 1$ and $H^b \cap H \cong \mathbb{Z}_2$. Assume that $H^b \cap H = \langle h \rangle$. Since $b$ normalizes $H^b \cap H$ and since $c \in Z(K)$, one has $h^b = h$ and $(ch)^b = ch$. Note that $h$ is a product of $r$ and an element in $L$; this holds since $H \cong S_3$. Thus, $ch \in D$, and so $D^b \cap D$ contains the Sylow 2-subgroup $(ch)$ of $D$. Suppose $D^b \cap D = D$. Then $D^b = D$, and hence $b$ normalizes the unique Sylow 3-subgroup $L$ of $D$, that is, $L^b = L$. Thus, $L \leq H^b \cap H$, which is impossible because $H^b \cap H$ is a 2-group. It follows that $D^b \cap D$ is a Sylow 2-subgroup of $D$, as required. So let us consider the remaining two cases where $s = 3$ or 5. Then $L \cong S_3$, $H = L \times \langle r \rangle$ and $D = L \times \langle rc \rangle$ for $s = 3$, and $L \cong S_4$, $H = L \times \langle r \rangle$ and $D = L \times \langle rc \rangle$ for $s = 5$. In both cases, let $T$ be a Sylow 2-subgroup of $L$. Then, $H$ has a unique Sylow 2-subgroup containing $T$, that is, $T \times \langle r \rangle$. Since $L^b \cap L \leq H^b \cap H$ and the groups $H^b \cap H$ and $L^b \cap L$ are Sylow 2-subgroups of $H$ and $L$, respectively, one has $H^b \cap H = (L^b \cap L) \times \langle r \rangle$. Thus, $(L^b \cap L) \times \langle r \rangle = H^b \cap H = (H^b \cap H)^b = (L^b \cap L) \times \langle r^b \rangle$. It follows that there exists an $\ell \in L^b \cap L$ such that $r^b = \ell r$. So, $(L^b \cap L) \times \langle rc \rangle = (L^b \cap L) \times \langle r \rangle \langle c \rangle = (L^b \cap L) \times \{ r, c \} = (L^b \cap L) \times \langle rc \rangle$, implying that $D^b \cap D$ contains the Sylow 2-subgroup $(L^b \cap L) \times \langle rc \rangle$ of $D$. Note that a Sylow 3-subgroup of $D$ is also a Sylow 3-subgroup of $H$. This implies that $D^b \cap D = D$ because $H^b \cap H$ is a Sylow 2-subgroup of $H$. It follows that $D^b \cap D$ is a Sylow 2-subgroup of $D$, as required.

Thus, the claim is true, that is, $\overline{X} = \text{Cos}(K, D, DbD)$ is a connected cubic graph with $K$ as an $s$-regular group of automorphisms. Recall that $K = G \times \langle c \rangle$, $H = L \times \langle r \rangle$, and $D = L \times \langle rc \rangle$. For $g \in G$ we have $Dgc = Drcgc = Drg$. Thus, $G$ is transitive on $\overline{X}$. Let $X$ be the quotient graph of $\overline{X}$ corresponding to the orbits of $\langle c \rangle$. Then $X$ is a 2-cover of $\overline{X}$, with $G$ projecting to a group isomorphic to $G$. Recall that $X = \text{Cos}(G, H, HbH)$. For a vertex $Dk \in V(\overline{X})$, denote by $\overline{Dk}$ the vertex of $\overline{X}$ corresponding to $Dk$, that is, $\overline{Dk} = \{ Dk, Dkc \}$, the orbit of $\langle c \rangle$ containing $Dk$. Define a map from $X$ to $\overline{X}$ by $\alpha: Hg \mapsto Dg$ for $g \in G$. Assume $Hg_1 = Hg_2$. Since $H = L \cup Lr$, one has $Lg_1 \cup Lrg_1 = Lg_2 \cup Lrg_2$. It follows that either $Lg_1 = Lg_2$ and $Lrg_1 = Lrg_2$ or $Lg_1 = Lrg_2$ and $Lrg_1 = Lg_2$. This implies that $\overline{Dg_1} = \{ Dg_1, Dg_1c \} = \{ Lg_1 \cup Lrg_1c, Lrg_1 \cup Lg_1c \} = \{ Lg_2 \cup Lrg_2c, Lrg_2 \cup Lg_2c \} = \overline{Dg_2}$ because $D = L \cup Lrc$. Thus, $\alpha$ is well defined. By transitivity of $G$ on $\overline{X}$, the mapping $\alpha$ is surjective and so bijective because $|V(\overline{X})| = |V(X)|$. Take an edge $(Hg, Htg)$ in $\overline{X}$, where $g \in G$ and $t \in HbH$. Since $H = \langle L, r \rangle$ and $D = \langle L, rc \rangle$, one has $t \in DbD$ or $tc \in DbD$. If $t \in DbD$ then $\{ Dg, Dt \}$ is an edge of $\overline{X}$, and if $tc \in DbD$ then $\{ Dg, Dtgc \}$ is also an edge. In both cases, $(Dg, Dt)$ is an edge of $X$. Thus, $\alpha$ is an isomorphism from $X$ to $\overline{X}$. This means that $\overline{X}$ is a 2-cover of $X$, with $G$ projecting to a group isomorphic to $G$.

As with the isomorphism $\alpha$, one can easily show that the map from $\text{Cos}(G, L, LbL)$ to
\[ \mathcal{X} = \text{Cos}(K, D, DbD), \] defined by \( \beta: Lg \mapsto Dg \), is an isomorphism. This completes the proof. \[ \square \]

We say that an ordered pair groups \((H, K)\) is isomorphic to an ordered pair groups \((H', K')\) if \( H \cong H' \) and \( K \cong K' \); this is denoted by \((H, K) \cong (H', K')\) for short. By the constructions of the automorphisms \( \alpha \) and \( \beta \) in Theorem 4.2 we have

**Corollary 4.3** Let \( L \leq H \leq G \) with \( H_G = 1 \). Suppose that \((H, L) \cong (S_3, \mathbb{Z}_3), (S_3 \times \mathbb{Z}_2, S_3)\) or \((S_4 \times \mathbb{Z}_2, S_4)\). Then, \( G \) has an element \( b \) satisfying \( b^2 \in L \), \( \langle b, L \rangle = G \), and \( |H : H^b \cap H| = |L : L^b \cap L| = 3 \) if and only if the projection \( \varphi : Lg \mapsto Hg \) from the coset graph \( \text{Cos}(G, L, LbL) \) to the coset graph \( \text{Cos}(G, H, HbH) \) is a split 2-cover with \( G \) as a transitive complement.

In the remainder of this section we construct an infinite family of split-transitive 2-covers of cubic symmetric graphs relative to 2-regular groups of automorphisms. We will need two classical results regarding transitive permutation groups, by Jordan and Marggraf [43]. Let \( G \) be a transitive permutation group on a set \( \Omega \). A nonempty subset \( \Delta \) of \( \Omega \) is called a block for \( G \) if, for each \( g \in G \), either \( \Delta^g = \Delta \) or \( \Delta^g \cap \Delta = \phi \). A group \( G \) is said to be primitive if \( G \) has no block \( B \) such that \( 1 < |B| < |\Omega| \).

**Proposition 4.4** [43] Thm. 13.8 A primitive group of degree \( n \), which contains a cycle of degree \( m \) with \( 1 < m < n \), is \( (n - m + 1) \)-fold transitive.

**Proposition 4.5** [43] Thm. 13.9 Let \( p \) be a prime and \( G \) a primitive group of degree \( n = p + k \) with \( k \geq 3 \). If \( G \) contains an element of degree \( p \) and order \( p \), then \( G \) is either alternating or symmetric.

The construction of split-transitive 2-covers is based on the alternating group \( A_{2k+10} \), where \( k \) is a nonnegative integer. Let us first define the following permutations in \( A_{12k+10} \):

\[
\begin{align*}
a_k &= \prod_{i=1}^{4k+3} (3i - 2, 3i, 3i - 1), \\
r_k &= \left( \prod_{i=1}^{2k+1} (6i - 2, 6i - 1, 6i + 1, 6i + 2) \right) (1, 2), \\
b_k &= \left( \prod_{i=1}^{2k+1} (6i - 3, 6i, 6i - 2, 6i + 1, 6i - 1, 6i + 2) \right) (12k + 9, 12k + 10).
\end{align*}
\]

Clearly, \( L_k = \langle a_k \rangle \cong \mathbb{Z}_3 \), \( H_k = \langle a_k, r_k \rangle \cong S_3 \), and \( H_k^{b_k} \cap H_k = \langle r_k \rangle \cong \mathbb{Z}_2 \).

**Theorem 4.6** The projection \( \varphi : L_kg \mapsto H_kg, g \in A_{12k+10} \), is a split-transitive 2-cover from the coset graph \( \text{Cos}(A_{12k+10}, L_k, L_kb_Lk) \) to the coset graph \( \text{Cos}(A_{12k+10}, H_k, H_kb_Hk) \) relative to \( A_{12k+10} \).

**Proof.** If \( A_{12k+10} = \langle a_k, b_k \rangle \) then the theorem is true by Corollary 4.3 because of cubic symmetry of \( A_{12k+10} \) implies that there is a unique complement in the 2-cover \( \varphi \). Let \( G_k = \langle a_k, b_k \rangle \). To finish the proof, it suffices to show that \( G_k = A_{12k+10} \).

Let \( i_1, i_2, \ldots, i_\ell, j_1, j_2, \ldots, j_m \) be distinct numbers. Let \( x = (i_1 i_2 \ldots i_\ell) \) and \( y = (j_1 j_2 \ldots j_m) \) be cycle permutations with the first entry \( i_1 \) and \( j_1 \) distinguished, respectively. By \( (i_1 i_2 \ldots i_\ell y) \) or \( (x y) \) we denote the cyclic permutation \( (i_1 i_2 \ldots i_\ell j_1 j_2 \ldots j_m) \).
It is easy to see that $G_k$ is transitive on $\Omega = \{1, 2, \cdots, 12k+10\}$. Now, use an induction on $k$ to claim that

$$b_k b_k^{a_k} = (e_k f_k) \quad (3)$$

where $e_k$ is a cycle of length $4k + 4$ on $\{3\ell + 1 \mid 0 \leq \ell \leq 4k + 3\}$ with the first entry $12k + 7$ and the last entry $12k + 10$ and $f_k$ is a cycle of length $4k + 3$ on $\{3\ell \mid 1 \leq \ell \leq 4k + 3\}$ with the first entry $12k + 6$ and the last entry $12k + 9$.

If $k = 0$ then $b_0 b_0^{a_0} = (7 1 4 10 6 3 9) = (e_0 f_0)$, where $e_0 = (7 1 4 10)$ is a cycle on $\{1, 4, 7, 10\}$ and $f_0 = (6 3 9)$ is a cycle on $\{3, 6, 9\}$. The claim is true. By induction hypothesis, assume that $[3]$ is true for $k = m \geq 1$, that is, $b_m b_m^{a_m} = (e_m f_m)$, where $e_m$ is a cycle of length $4m + 4$ on $\{3\ell + 1 \mid 0 \leq \ell \leq 4m + 3\}$ with the first entry $12m + 7$ and the last entry $12m + 10$ and $f_m$ is a cycle of length $4m + 3$ on $\{3\ell \mid 1 \leq \ell \leq 4m + 3\}$ with the first entry $12m + 6$ and the last entry $12m + 9$.

Let $k = m + 1$. Set

$$a' = (12m + 10 12m + 11 12m + 12),$$

$$a'' = (12m + 13 12m + 14 12m + 15)(12m + 16 12m + 17 12m + 18),$$

$$b' = (12m + 9 12m + 13 12m + 10 12m + 12),$$

$$b'' = (12m + 11 12m + 14)(12m + 15 12m + 18),$$

$$b'' = (12m + 16 12m + 19)(12m + 17 12m + 20)(12m + 21 12m + 22).$$

Then, $a_{m+1} = a_m a' a''$ and $b_{m+1} = b_m b' b''$. Furthermore, $a_m a'' = a'' a_m$, $a_m b'' = b'' a_m$, $b_m b'' = b'' b_m$, and $b_m a'' = a'' b_m$. Thus, $b_m b_m' a_m = b_{m+1} a_{m+1} = b_m b' b'' a' a'' = b_m a_m c_1$, where $c_1 = (b')^{a_m} b'' a''$. Note that $(b')^{a_m} = (12m + 7 12m + 13 12m + 10 12m + 12)$ and by computation, one has

$$c_1 = (12m + 7 12m + 14 12m + 12)(12m + 17 12m + 21 12m + 22 12m + 19)$$

$$= (12m + 11 12m + 15 12m + 16 12m + 20 12m + 18 12m + 13).$$

Similarly, $a_{m+1} b_m a_m = a_m b_m a_m (a')^{a_m} a'' = a_m b_m a_m c_2$ where

$$c_2 = (12m + 7 12m + 15 12m + 11)(12m + 18 12m + 20 12m + 22 12m + 19)$$

$$= (12m + 12 12m + 14 12m + 16 12m + 21 12m + 17 12m + 13);$$

$$a_{m+1}^2 b_m a_m = a_m^2 b_m a_m (a')^{a_m} a'' = a_m^2 b_m a_m c_3$$

where

$$c_3 = (12m + 11 12m + 14)(12m + 12 12m + 15)(12m + 13 12m + 16)$$

$$= (12m + 17 12m + 20)(12m + 18 12m + 21)(12m + 19 12m + 22);$$

$$b_{m+1} a_{m+1}^2 b_m a_m = b_m a_m (b')^{a_m} a_m b'' = b_m a_m b_m a_m c_4$$

where

$$c_4 = (12m + 6 12m + 16 12m + 22 12m + 18 12m + 12)$$

$$= (12m + 7 12m + 15 12m + 21 12m + 19 12m + 13).$$

Thus, $b_{m+1} b_{m+1}^{a_{m+1}} = b_m b_m^{a_m} c_4 = (e_m f_m) c_4$. Recall that $e_m$ is a cycle of length $4m + 4$ on $\{3\ell + 1 \mid 0 \leq \ell \leq 4m + 3\}$ with the first entry $12m + 7$ and the last entry $12m + 10$ and $f_m$ is a cycle of length $4m + 3$ on $\{3\ell \mid 1 \leq \ell \leq 4m + 3\}$ with the first entry $12m + 6$ and the last entry $12m + 9$. It follows that $b_{m+1} b_{m+1}^{a_{m+1}} = (12m + 19 12m + 13 e_m 12m + 16 12m + 22 12m + 18 12m + 12 f_m 12m + 15 12m + 21)$. Set $e_{m+1} = (12m + 19 12m + 13 e_m 12m + 16 12m + 22)$ and $f_{m+1} = (12m + 18 12m + 12 f_m 12m + 15 12m + 21)$. Then, $b_{m+1} b_{m+1}^{a_{m+1}} = (e_{m+1} f_{m+1})$. 

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where $e_{m+1}$ is a cycle of length $4(m+1)+4 = 4m+8$ on $\{3\ell+1 \mid 0 \leq \ell \leq 4(m+1)+3\}$ with the first entry $12(m+1)+7 = 12m+19$ and the last entry $12(m+1)+10 = 12m+22$ and $f_{m+1}$ is a cycle of length $4(m+1)+3 = 4m+7$ on $\{3\ell \mid 1 \leq \ell \leq 4(m+1)+3\}$ with the first entry $12(m+1)+6 = 12m+18$ and the last entry $12(m+1)+9 = 12m+21$. Thus, (3) is true for any $k \geq 0$.

Recall that $\Omega =\{1,2,\ldots, 12k+10\}$. Let $\Omega_1 = \{3\ell+1 \mid 0 \leq \ell \leq 4k+3\}$, $\Omega_2 = \{3\ell +2 \mid 0 \leq \ell \leq 4k+2\}$ and $\Omega_3 = \{3\ell \mid 1 \leq \ell \leq 4k+3\}$. By (3), $b_k b_k^{\alpha_k} = (e_k f_k)$ is a cyclic permutation on $\Omega_1 \cup \Omega_3$ of length $8k+7$ with $e_k$ and $f_k$ cyclic permutations on $\Omega_1$ and $\Omega_3$ respectively, and it fixes the set $\Omega_2$ pointwise. Thus, $(b_k b_k^{\alpha_k})^{\alpha_k} = (e_k' f_k')$ is a cyclic permutation of length $8k+7$ on $\Omega_1 \cup \Omega_2$ with $e_k'$ and $f_k'$ cyclic permutations on $\Omega_2$ and $\Omega_1$ respectively. Remember that $G_k$ is transitive on $\Omega$. We now prove that $G_k$ is primitive. Let $\Delta$ be a block of $G_k$ with $|\Delta| > 1$. Assume $2 \in \Delta$. Suppose $\Delta \subseteq \Omega_2$. Since $|\Delta| > 1$, let $2 \notin 3\ell+2 \in \Omega_2$. By considering the permutation $(b_k b_k^{\alpha_k})^{\alpha_k}$, $\Delta$ contains at least one element in $\Omega_1$, a contradiction. Thus, there exist $x \in \Omega_1 \cup \Omega_3$ and $x \in \Delta$. Since $b_k b_k^{\alpha_k} = (e_k f_k)$ fixes 2 and is cyclic on $\Omega_1 \cup \Omega_3$, one has $\Omega_1 \cup \Omega_3 \subseteq \Delta$, implying that $|\Delta| \geq 8k+8$. Since $|\Delta|$ is a divisor of $12k+10$, one has $|\Delta| = 12k+10$. Thus, $G_k$ is primitive on $\Omega$. By Proposition 4.4, $G_{10} = A_{10}$ and for $k \geq 1$, by Proposition 4.4, $G_k$ is $(4k+4)$-transitive because $G_k$ contains a cycle of length $8k+7$. Since $4k+3 > 6$, $G_k = A_{12k+10}$.

## 5 Chains of consecutive 2-covers

We start with some terminology. Let

$$X_n \xrightarrow{\varphi_n} X_{n-1} \cdots X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0$$

be a chain of consecutive regular covering projections. Suppose further that there is a chain of groups $G = G_0, G_1, G_2, \ldots, G_n$ such that $G_j \leq \text{Aut} \ X_j$ is the lift of $G_{j-1} \leq \text{Aut} \ X_{j-1}$ along $\varphi_j$ for each $j \in \{1, 2, \ldots, n\}$. We then say that the above chain of covers is $G$-admissible. A chain with this property is denoted by $C(X,G)$. In particular, if all extensions $\text{CT}(\varphi_j) \rightarrow G_j \rightarrow G_{j-1}$ are split, then the chain is said to be $G$-split-admissible. Further, a $G$-split-admissible chain $C(X,G)$ is said to be split-sectional and split-transitive provided all extensions $\text{CT}(\varphi_j) \rightarrow G_j \rightarrow G_{j-1}$ are, respectively, split-sectional and split-transitive.

Let $G = G_0$ be an arc-transitive group of automorphisms of a symmetric graph $X = X_0$. The length of the pair $(X,G)$ is the largest integer $n$ such that there exists a $G$-admissible chain of $n$ consecutive 2-covers $X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0$. In particular, the split-length of the pair $(X,G)$ is the largest integer $n$ such that there exists a $G$-split-admissible chain of $n$ consecutive 2-covers $X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0$. Analogously, the sectional-length and the transitive-length of the pair $(X,G)$ is the largest integer $n$ such that there exists, respectively, a sectional $G$-split-admissible chain and a transitive $G$-split-admissible chain of $n$ consecutive 2-covers $X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0$.

Note that for a $G$-split-admissible chain $C(X,G)$ we have that $G_j \cong G \times \mathbb{Z}_2^n$ for every $1 \leq j \leq n$. Moreover, since $G_n$ is arc-transitive on $X_n$ we have that $G_n = \langle H_n, b \rangle$, where $H_n$ is the stabilizer of a vertex in $G_n$ and $b$ maps this vertex to one of its neighbors. Since $G_n \cong G \times \mathbb{Z}_2^n$ has at least $n$ generators, one has $n \leq |H_n|$. Recall that $|H_n|$ is equal to the order of stabilizers in $G$. Hence,
Proposition 5.1  The split-length of a symmetric graph relative to an arc-transitive group of automorphisms must be finite.

Lemma 5.2  Let $G$ be an arc-transitive group of automorphisms of a symmetric graph $X$, and let $\mathcal{C}(X, G)$ be a $G$-admissible chain

$$X_2 \overset{\psi_2}{\rightarrow} X_1 \overset{\psi_1}{\rightarrow} X_0$$

of two consecutive 2-covers where $\psi_2$ is split-transitive or split-sectional, respectively. Then there exists a $G$-admissible chain $\mathcal{C}'(X, G)$

$$X_2 \overset{\psi_1'}{\rightarrow} X_1' \overset{\psi_1'}{\rightarrow} X_0$$

of two consecutive 2-covers such that $\psi_1'$ is split-transitive or split-sectional, respectively.

Proof. Denote the respective lifted groups in the above sequence by $G = G_0, G_1, G_2$, and let $CT(\psi_1) = \langle c_1 \rangle$ and $CT(\psi_2) = \langle c_2 \rangle$. Since $\psi_2$ is split, we have $G_2 = G_1 \times \langle c_2 \rangle$, where $G_1 \cong G_1$. Since $G_1$ contains $\langle c_1 \rangle \cong \mathbb{Z}_2$ as a normal subgroup, $G_2$ contains $\langle c_1 \rangle \times \langle c_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\langle c_1 \rangle \leq G_1$ projects onto $\langle c_1 \rangle$ along $\psi_2$. Of course, $\langle c_1 \rangle$ is normal in $G_2$ and hence acts semiregularly on $X_2$. Moreover, observe that the orbits of $\langle c_1 \rangle \times \langle c_2 \rangle$ contain no edges. Therefore, quotienting by the action of $\langle c_1 \rangle$ gives rise to a covering projection $\bar{\psi}_2: X_2 \rightarrow X_1'$, where the quotient graph $X_1'$ is again simple. (Note: it might happen that $X_1' = X_1$.) Let $\langle c_2 \rangle$ be the projection of $\langle c_2 \rangle$ along $\psi_2$. Then the group $G_2$ projects onto $G_1 = G_1/\langle c_1 \rangle \cong G_0 \times \langle c_2 \rangle$. Quotienting by $\langle c_2 \rangle$ gives rise to a covering projection $\bar{\psi}_1: X_1' \rightarrow X_0$.

If $\psi_2$ is split-transitive, then $\psi_1'$ is split with a transitive complement since $G_1/\langle c_1 \rangle$ is transitive on $X_1'$ which projects isomorphically onto $G_0$. In fact, $\psi_1'$ is also split-transitive. Suppose on the contrary that there exists a sectional complement $H$ to $\langle c_2 \rangle$ within $G_1$. Clearly, its lift $\bar{H}$ along $\bar{\psi}_2$ is not transitive on $X_2$, and $G_2 = \bar{H} \times \langle c_2 \rangle$. Now $H$ projects onto $G_1$ along $\psi_2$. Hence $\bar{\psi}_2$ is split-mixed, a contradiction.

If $\psi_2$ is split-sectional, then $\psi_1'$ is split since $G_1/\langle c_1 \rangle$ is a complement to $\langle c_2 \rangle$ which projects onto $G_0$ along $\psi_1'$. We now show that $G_1/\langle c_1 \rangle$ is indeed sectional. First note that by Proposition 3.3 the projection $\psi_2$ is the canonical double cover. Hence $X_2$ is bipartite. Now the group $G_1$ preserves the bipartition sets of $X_2$, which implies that $G_1/\langle c_1 \rangle$ also preserves the bipartition sets of the graph $X_1'$. Thus, $\psi_1'$ is split with a sectional complement $G_1/\langle c_1 \rangle$. In fact, $\psi_1'$ is also split-sectional. Suppose on the contrary that there exists a transitive complement $H$ to $\langle c_2 \rangle$ within $G_1$. Clearly, $G_2 = \bar{H} \times \langle c_2 \rangle$ where $\bar{H}$ is the lift of $H$ along $\bar{\psi}_2$. Thus, $\bar{H}$ is a sectional complement to $\langle c_2 \rangle$ in $G_2$ because $\psi_2$ is assumed split-sectional. Consequently, $\bar{H}$ fixes the bipartition sets of $X_2$ setwise. Hence $\bar{H} = G_1$ because $G_2$ has a unique intransitive index 2 subgroup. But then $H = G_1/\langle c_1 \rangle$, which is impossible as $H$ is transitive on $X_1'$ and $G_1/\langle c_1 \rangle$ is not. This completes the proof.

Remark. As for the case when in Lemma 5.2 the projection $\psi_2$ is split-mixed, from the proof it is easy to see that there exist two chains

$$X_2 \overset{\psi_2}{\rightarrow} X_1' \overset{\psi_1'}{\rightarrow} X_0$$
$$X_2' \overset{\psi_2'}{\rightarrow} X_1'' \overset{\psi_1''}{\rightarrow} X_0$$
such that \( \varphi'_1 \) is split-transitive or split-mixed and \( \varphi''_1 \) is split-sectional or split-mixed. However, it may happen that neither \( \varphi'_1 \) nor \( \varphi''_1 \) is split-mixed. See Example 2.2.

In what follows we shall restrict ourselves to cubic symmetric graphs. Theorem 5.3 below gives us some partial information about the structure of consecutive 2-covers of such graphs. As a consequence we show in Corollary 5.4 that the split-length of a cubic symmetric graph relative to an arc-transitive group of automorphisms is at most 2.

**Theorem 5.3** Let \( G \) be a arc-transitive group of automorphisms of a symmetric cubic graph \( X \), and let \( \mathcal{C}(X, G) \) be a \( G \)-admissible chain

\[
X_n \xrightarrow{\varphi_n} X_{n-1} \cdots X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0
\]

of consecutive 2-covers. Then at most two of 2-covers in the chain are split. In particular, if precisely two of them are split, then one is split-transitive and the other is split-sectional or split-mixed, and this can happen only if \( G \) is \( s \)-regular for \( s = 2, 3, 5 \).

**Proof.** Clearly, at most one of the 2-covers in the chain \( \mathcal{C}(X, G) \) can be split with a sectional complement (that is, split-sectional or split-mixed). Namely, suppose that \( \varphi_i : X_i \to X_{i-1} \) is the first such 2-cover. Then by Proposition 3.3 the projection \( \varphi_i \) is the canonical double cover; hence all \( X_j, j \geq i \), are bipartite. Because graphs are assumed connected, all \( \varphi_j, j \geq i \), are split-transitive or non-split.

Applying Lemma 5.2 we may without loss of generality assume that all split-transitive covers come first followed by all others. We now show that the initial subchain containing only split-transitive covers is of length at most 1. Suppose on the contrary that in the subchain

\[
X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0
\]

both \( \varphi_1 \) and \( \varphi_2 \) are split-transitive.

Suppose that \( G \) is \( s \)-regular. First note that \( s \neq 1, 4 \). Namely, Theorem 4.1 implies that each split 2-cover of \( X_0 \) relative to \( G \) is the canonical double cover. Hence \( X_1 \) is bipartite. But then \( X_2 \) is disconnected, a contradiction. Also, note that in this case at most one split 2-cover exists in \( \mathcal{C}(X, G) \).

We may therefore restrict ourselves to the case \( s \in \{2, 3, 5\} \). In what follows we shall be using the following fact: a vertex-transitive index 2 subgroup in an \( s \)-regular group, where \( s \geq 2 \), must be \( (s-1) \)-regular.

Denote now the corresponding lift of \( G \) along \( \varphi_1 \) by \( G_1 = \tilde{G} \times \langle c_1 \rangle \), where \( \text{CT}(\varphi_1) = \langle c_1 \rangle \), and the lift of \( G_1 \) along \( \varphi_2 \) by \( G_2 = \tilde{G} \times \langle c_2 \rangle = \tilde{G} \times \langle \tilde{c}_1 \rangle \times \langle c_2 \rangle \), where \( \text{CT}(\varphi_2) = \langle c_2 \rangle \), and \( \tilde{c}_1 \) and \( \tilde{G} \) project to \( c_1 \) and \( G \) along \( \varphi_2 \), respectively. Then \( G_1 \) and \( G_2 \) are \( s \)-regular, and the transitive complements \( \tilde{G} \) within \( G_1 \) and \( \tilde{G}_1 \) within \( G_2 \) are \( (s-1) \)-regular. Similarly, the transitive complement \( \tilde{G}_1 = \tilde{G} \times \langle \tilde{c}_1 \rangle \) to \( \langle c_2 \rangle \) within \( G_2 \), and the lift \( \tilde{G} \times \langle c_2 \rangle \) of \( \tilde{G} \) along \( \varphi_2 \), are \( (s-1) \)-regular. Suppose that \( \tilde{G} \) is not transitive. Then the group \( \tilde{G} \) lifts along \( \varphi_2 \) to \( \tilde{G} \times \langle c_2 \rangle \) with \( \tilde{G} \) as a sectional complement. Hence \( \varphi_2 \) is the canonical double cover, by Proposition 5.3. But then, again by Proposition 5.3, the projection \( \varphi_2 \) cannot be split-transitive relative to \( G_1 \), a contradiction. We conclude that \( \tilde{G} \) is transitive on \( X_2 \). This is an immediate contradiction if \( s = 2 \) because \( \tilde{G} \times \langle c_2 \rangle \) is then 1-regular and cannot contain a transitive subgroup of index 2. If \( s = 3 \), then \( \tilde{G} \) is \( (s-2) \)-regular. But that is again a contradiction: if \( s = 5 \), we get a contradiction because a 5-regular group \( G_2 \) cannot contain a 3-regular subgroup, by [11]; if \( s = 3 \) we get a contradiction because...
by \([8]\), a 3-regular group \(G_2\) cannot contain three 2-regular subgroups (namely, \(\hat{G} \times \langle e_2 \rangle\), \(\hat{G} \times \langle \tilde{e}_1 \rangle\), and \(\hat{G} \times \langle \tilde{e}_1, c_2 \rangle\)).

This shows that there is at most one split-transitive 2-cover in the chain, and the proof is complete. ■

Let \(X\) be a cubic symmetric graph, and let \(G\) be an arc-transitive subgroup of \(\text{Aut} \ X\). We say that \(G\) is of type \((2^1)\) if it is 2-regular and if it contains an involution flipping an edge. Similarly, we say that \(G\) is of type \((2^2)\) if it is 2-regular and if it contains no involution flipping an edge. If the full automorphism group \(\text{Aut} \ X\) is of type \((2^1)\) (respectively, of type \((2^2)\)) and \(\text{Aut} \ X\) contains no other arc-transitive subgroups, then \(X\) is called of type \((2^1)\) (respectively, of type \((2^2)\)). Next, we say that \(X\) is of type \((2^1, 3)\) if \(\text{Aut} \ X\) is 3-regular and contains a 2-regular subgroup of type \((2^1)\), but no other arc-transitive subgroups. And finally, we say that \(X\) is of types \((3)\) or \((5)\), respectively, if it is 3-regular or 5-regular, respectively, and if, furthermore, \(\text{Aut} \ X\) contains no other arc-transitive subgroups. With this terminology, we have the following corollary.

**Corollary 5.4** Let \(X\) be a connected cubic symmetric graph and let \(G\) be an \(s\)-regular subgroup of \(\text{Aut} \ X\). If \(s \in \{1, 4\}\) then the split-length of \((X, G)\) is at most 1. If \(s \in \{2, 3, 5\}\), then the split-length is at most 2, and moreover, if it is 2 then \(X\) is one of the following types: \((5)\), \((3, 2^1)\), or \((2^1, 3)\) (with \(G\) a 2-regular subgroup of \(\text{Aut} \ X\)).

**Proof.** Using Theorem 4.1 we have that \(s \in \{2, 3, 5\}\) if the split-length of \((X, G)\) is 2. Furthermore, \(G\) has no 1-regular and no 4-regular subgroups. (If otherwise, Theorem 4.1 implies that both covers in the chain must be canonical double covers, a contradiction.)

Suppose that \(G\) is of type \((2^2)\). If there exists a split-transitive 2-cover \(\psi: \tilde{X} \rightarrow X\) relative to \(G\), then the complement \(\tilde{G} \cong G\) of \(CT(\psi)\) within \(\tilde{G} \cong G \times \mathbb{Z}_2\) is 1-regular on \(\tilde{X}\). But this is a contradiction since \(\tilde{G}\) is also of type \((2^2)\), and by [11, Theorem 3] a group of type \((2^2)\) does not contain a 1-regular subgroup.

Using the list in [8] of all possible types of cubic symmetric graphs, this leaves us with \((5)\), \((3, 2^1)\), or \((2^1, 3)\) as the only possible types of \(X\). As for the case when \(X\) is of type \((2^1, 3)\), suppose that \(G\) is 3-regular. If there exists a split-transitive 2-cover \(\psi: \tilde{X} \rightarrow X\) relative to \(G\), then the complement \(\tilde{G} \cong G\) of \(CT(\psi)\) within \(\tilde{G} \cong G \times \mathbb{Z}_2\) is 2-regular on \(\tilde{X}\), and [11, Theorem 4] implies that \(G \cong H \times \mathbb{Z}_2\) where \(H\) is a 2-regular subgroup of \(\text{Aut} \ X\). But then \(X\) is a split-transitive 2-cover of some smaller cubic symmetric graph, a contradiction since by Theorem 5.3 there is at most one split-transitive 2-cover in the corresponding chain. This completes the proof of Corollary 5.4. ■

### 6 Concluding remarks

Let \(X\) be a connected symmetric graph and let \(G\) be an \(s\)-arc-transitive subgroup of \(\text{Aut} \ X\). By Proposition 5.1 the split-length of \((X, G)\) is finite. Furthermore, by Corollary 5.4 if \(X\) is cubic, then the split-length of \((X, G)\) is at most 2. This suggests the following natural problem.

**Problem 6.1** Let \(X\) be a symmetric graph with valency greater than 3, and let \(G\) be an arc-transitive subgroup of \(\text{Aut} \ X\). Find the split-length of \((X, G)\).

**Problem 6.2** Let \(X\) be a connected symmetric graph with valency greater than 2, and let \(G\) be an arc-transitive group of automorphisms of \(X\). Can the length of a chain of
consecutive non-split 2-covers relative to \((X, G)\) be infinite? In particular, what can we say about the case when \(X\) is cubic?

This brings us to (non-split) lengths of the pair \((X, G)\). Let \(C_{2n}\) be the cycle of length \(2n\) with \(n \geq 2\). Then, \(\text{Aut} \ C_{2n}\) is arc-transitive and isomorphic to the dihedral group \(D_{4n}\). Clearly, \(\text{Aut} \ C_{2n}\) has a normal subgroup of order 2 (the center of \(\text{Aut} \ C_{2n}\)), and so \(C_{2n}\) is a 2-cover of \(C_n\) with \(\text{Aut} \ C_n\) lifting to \(\text{Aut} \ C_{2n}\). If \(n\) is even, this 2-cover is non-split relative to \(\text{Aut} \ C_n\), for otherwise the center of \(\text{Aut} \ C_{2n}\) would contain at least 4 elements which is not the case. Thus, for any even \(n \geq 2\) the length of consecutive non-split 2-covers relative to \((C_n, \text{Aut} \ C_n)\) is infinite.

As for cubic graphs, by [5], there is a unique cubic symmetric graph \(F_{40}\) of order 40 and a unique such graph \(F_{80}\) of order 80. Using the computer software package MAGMA [4], one can show that \(F_{80}\) is a non-split 2-cover of \(F_{40}\) relative to \(\text{Aut} \ F_{40}\), and that \(F_{40}\) is a non-split 2-cover of the Desargues graph \(F_{20B}\) relative to \(\text{Aut} \ F_{20B}\). Since \(F_{20B}\) is a split 2-cover of the Petersen graph \(F_{10}\), this brings to length of a chain of consecutive 2-covers to 3. We do not know, however, if such a chain of length 4 exists in the case of cubic symmetric graphs. We would like to propose the following problem.

**Problem 6.3** Let \(X\) be a connected cubic symmetric graph, and let \(G\) be an arc-transitive subgroup of \(\text{Aut} \ X\). Is there an upper bound on the length of the chain of consecutive 2-covers relative to \((X, G)\)?

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