ON THE CHVÁTAL-JANSON CONJECTURE

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Abstract. In a recent paper, Svante Janson has considered a conjecture suggested by Vašek Chvátal dealing with the probability that a binomial random variable with parameters $n$ and $m/n$ - where $m$ is an integer - exceeds its expectation $m$. Albeit Janson has provided a proof of this conjecture for large $n$, we show that the result actually holds for each $n \geq 2$.

1. Introduction

By assuming that $B(n, m/n)$ denotes a binomial random variable with parameters $n$ and $m/n$, Janson (2021) introduces the following conjecture suggested by Vašek Chvátal in a personal communication.

Conjecture 1 (Chvátal). For any fixed $n \geq 2$, as $m$ ranges over $\{0, \ldots, n\}$, the probability $q_m := P(B(n, m/n) \leq m)$ is the smallest when $m = \left\lfloor \frac{2n}{3} \right\rfloor$ where $\lfloor \cdot \rfloor$ represents the nearest integer function.

It is worth noting that the conjecture may have interesting applications, since the probability that a binomial random variable exceeds its expected value has generally an important role in the machine learning literature (see e.g. Doerr, 2018, Greenberg and Moliri, 2014, Vapnik, 1998). Such a probability has even a connection with an equation given by Ramanujan, as emphasized by Jogdeo and Samuels (1968). For further results on the topic, see Pelekis and Ramon (2016), Slud (1977).

Janson (2021) has proven that, for large $n$, Conjecture 1 actually holds and the probabilities $q_m$ have a unique minimum. More precisely, Janson (2021) provides the following theorem.

Theorem 1. There exists a $n_0$ such that for each $n_0 \geq n$: i) $q_m$ is minimum for $m = \left\lfloor \frac{2n}{3} \right\rfloor$ and ii) $q_m \geq q_{m+1}$ if and only if $m + \frac{1}{3} < \frac{2n}{3}$.

However, Janson (2021) remarks that, even if it is possible in principle computing an explicit value for $n_0$ in the proof of Theorem 1 and numerically checking the statement for $n < n_0$, such an issue is not practically feasible. Actually, Janson (2021, Remark 1.5) wishes for a general proof of Theorem 1.

In the present contribution, we give a plain proof of Theorem 1 for each $n \geq 2$. The proof is achieved by means of different methods with respect to those adopted by Janson (2021), which are actually based on the version for integer-valued random variables of the asymptotic Edgeworth expansion for probabilities in the central limit theorem - as proposed by Esseen (1945). Indeed, our proof shares similarities.

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with the approach introduced by Rigollet and Tong (2011, Appendix B) for assessing that \( q_m \geq q_{m+1} \) for \( 0 \leq m < n/2 \) and \( n \geq 2 \).

2. NOTATIONS AND PRELIMINARIES

Let \((U_i)_{1 \leq i \leq n}\) be \( n \) independent copies of a Uniform random variable on \([0, 1]\). If \((U_{(i)})_{1 \leq i \leq n}\) represent the order statistics corresponding to \((U_i)_{1 \leq i \leq n}\), it obviously holds

\[
q_m = P(\sum_{i=1}^{n} I_{U_i \leq m/n} \leq m) = P(U_{(m+1)} > m/n).
\]  

On the basis of (1), for each \( m \leq n - 1 \) it follows that

\[
q_m = (m+1) \binom{n}{m+1} \int_{m/n}^{1} x^m(1-x)^{n-m-1} \, dx,
\]

since the probability density function \( f_{m+1} \) of \( U_{(m+1)} \) is given by

\[
f_{m+1}(x) = (m+1) \binom{n}{m+1} x^m(1-x)^{n-m-1} I_{[0,1]}(x)
\]

(see e.g. Feller, 1971, Section I.7).

**Lemma 1.** Let us assume that \( n \geq 2 \) and \( m \leq n - 2 \). Then

\[
q_m \geq q_{m+1} \iff \int_{m/n}^{(m+1)/n} x^{m+1}(1-x)^{n-m-2} \, dx \geq b_m,
\]

where \( b_m = \frac{(m/n)^{m+1}(1-m/n)^{n-m-1}}{n-m-1} \). In addition, \( q_m \geq q_{m+1} \) is equivalent to

\[
\int_{0}^{1} (1 + t/m)^{m+1}(1 - t/(n - m))^{n-m-2} \, dt \geq \frac{n-m}{n-m-1},
\]

or to

\[
\int_{0}^{1} (1 - v/(m+1))^{m}(1 + v/(n-m-1))^{n-m-1} \, dv \geq 1.
\]

**Proof.** On the basis of (2) and by using the definition of the binomial coefficient, it follows that \( q_m \geq q_{m+1} \) is equivalent to

\[
\frac{m+1}{n-m-1} \int_{m/n}^{1} x^m(1-x)^{n-m-1} \, dx \geq \int_{(m+1)/n}^{1} x^{m+1}(1-x)^{n-m-2} \, dx.
\]

Integrating by part the left-hand side of the previous inequality, the expression reduces to

\[-b_m + \int_{m/n}^{1} x^{m+1}(1-x)^{n-m-2} \, dx \geq \int_{(m+1)/n}^{1} x^{m+1}(1-x)^{n-m-2} \, dx\]

and the main result follows. As to (3), from the main result and by means of the substitution \( x = m/n + t/n \), it reads

\[
\int_{0}^{1} (m/n + t/n)^{m+1}(1 - m/n - t/n)^{n-m-2} \, dt \geq nb_m,
\]
which provides (3) by suitably dividing both sides by \((m/n)^{m+1}(1-m/n)^{n-m-2}\).

As to (4), by multiplying both sides of (3) by \((n-m-1)/(n-m)\) and integrating by parts the corresponding left-hand side, it reads

\[
\int_0^1 (1 + \frac{t}{m})^m (1 - \frac{t}{n-m})^{n-m-1} dt \geq (1 - \frac{1}{n-m})^{n-m-1}(1 + \frac{1}{m})^m.
\]

By dividing both sides of the previous inequality by the quantity in the right-hand side, the expression reduces to

\[
\int_0^1 \frac{(t + m)}{1+m} (\frac{n-m-t}{n-m-1})^{n-m-1} dt \geq 1,
\]

which readily provides (4) by considering the transformation \(t = 1 - v\)．

\[\square\]

**Lemma 2.** For a given \(n \geq 3\), let \(m\) be an integer in \([1, n - 2]\). For each \(v \in ]0, 1]\), the function \(g_v\) defined on \([1, n - 2]\) and such that

\[g_v(m) = (1 - \frac{v}{m + 1})^m (1 + \frac{v}{n-m-1})^{n-m-1}\]

is decreasing. Moreover, the function

\[h : m \mapsto \int_0^1 (1 - \frac{v}{m + 1})^m (1 + \frac{v}{n-m-1})^{n-m-1} dv\]

is decreasing on \([1, n - 2]\).

**Proof.** For a given \(v \in ]0, 1]\) and by denoting with \(x\) a real number in \([1, n - 2]\), it suffices to prove that \(g'_v(x) < 0\). Since

\[g'_v(x) = g_v(x) \left[ \log(1 - \frac{v}{x+1}) + \frac{vx}{(x+1)(1-x)} - \log(1 + \frac{v}{n-x-1}) + \frac{v}{1 + \frac{v}{n-x-1}} \right],\]

it holds

\[g'_v(x) < 0 \iff \frac{vx}{(x+1)(1-x)} + \frac{v}{1 + \frac{v}{n-x-1}} < \log(1 + \frac{v}{1 - \frac{v}{x+1}}).\]

In addition, since

\[\log\left(1 + \frac{v}{1 - \frac{v}{n-x-1}}\right) = \log\left[\left(1 + \frac{v}{n-x-1}\right)\left(1 + \frac{v}{1 - \frac{v}{x+1}}\right)\right]\]

\[= \log\left(1 + \frac{v}{n-x-1}\right) + \log\left(1 + \frac{vx}{x+1}\right),\]

in order to prove that \(g'_v(x) < 0\) it suffices to show that

\[\frac{v}{1 + \frac{v}{n-x-1}} < \log\left(1 + \frac{v}{n-x-1}\right)\]

\[(5)\]

and

\[\log\left(1 + \frac{vx}{x+1}\right) > \frac{vx}{1 - \frac{v}{x+1}}\]

\[(6)\]

By assuming that \(c = \frac{v}{n-x-1}\), inequality (5) follows from \(\log(1+c) > c/(1+c)\), which holds for each \(c > 0\), while inequality (6) is equivalent to

\[\log\left(1 + \frac{v}{x+1-v}\right) - \frac{v}{x+1-v} + \frac{v}{(x+1)(x+1-v)} > 0\]
which, by assuming that \( c = v/(x + 1 - v) \), reduces to

\[
\log(1 + c) - c + \frac{c^2}{v(c + 1)} > 0.
\]

Inequality (7) holds since

\[
\log(1 + c) - c + \frac{c^2}{v(c + 1)} \geq \log(1 + c) - c = \log(1 + c) - \frac{c}{c + 1}
\]

and Lemma is proven.

\[\square\]

3. Proof of the Chvátal-Janson conjecture

In this section we provide a proof of Theorem 1 for \( n \geq 2 \). On the basis of Lemma 1 and Lemma 2, for a given \( n = 3s + r \), where \( s \geq 1 \) and \( r \in \{0, 1, 2\} \), it suffices to prove that for \( r = 0 \) it holds

\[
\int_0^1 (1 - \frac{v}{2s})^{2s-1}(1 + \frac{v}{s})^s \, dv \geq 1 > \int_0^1 (1 - \frac{v}{2s+1})^{2s}(1 + \frac{v}{s-1})^{s-1} \, dv,
\]

while for \( r = 1, 2 \) it holds

\[
\int_0^1 (1 - \frac{v}{2s+1})^{2s+1}(1 + \frac{v}{s + r - 1})^{s+r-1} \, dv \geq 1
\]

and

\[
1 > \int_0^1 (1 - \frac{v}{2s+2})^{2s+1}(1 + \frac{v}{s + r - 2})^{s+r-2} \, dv.
\]

By considering the inequalities (8) (i.e. when \( r = 0 \)) and by applying the Bernoulli inequality \((1 + c)^s \geq 1 + sc\) which holds for each \( c > -1 \), it follows that the first inequality in (8) is true for each \( s \geq 1 \) since

\[
\int_0^1 (1 - \frac{v}{2s})^{2s-1}(1 + \frac{v}{s})^s \, dv = \int_0^1 [(1 - \frac{v}{2s})^2(1 + \frac{v}{s})]^s(1 - \frac{v}{2s})^{-1} \, dv
\]

\[
= \int_0^1 [1 - \frac{3v^2}{4s^2} + \frac{v^3}{4s^3}]^s(1 - \frac{v}{2s})^{-1} \, dv
\]

\[
\geq \int_0^1 (1 - \frac{3v^2}{4s^2} + \frac{v^3}{4s^2})(1 - \frac{v}{2s})^{-1} \, dv
\]

\[
\geq \int_0^1 (1 - \frac{3v^2}{4s^2} + \frac{v^3}{4s^2})(1 + \frac{v}{2s} + \frac{v^2}{4s^2}) \, dv
\]

\[
= 1 + \frac{5}{96s^2} - \frac{1}{80s^3} + \frac{1}{96s^4} > 1.
\]

The first inequality in the previous expression follows from \((1 - c)^{-1} > 1 + c + c^2\), which holds for each \( c \in [0, 1] \). In turn for \( r = 0 \), as to the second inequality in (8) and by assuming that \( I_s = \int_0^1 (1 - \frac{v}{2s+1})^{2s}(1 + \frac{v}{s-1})^{s-1} \, dv \), it reads
\[ I_s = \int_0^1 \left[ (1 - \frac{v}{2s+1})^2 (1 + \frac{v}{s-1}) \right]^s (1 + \frac{v}{s-1})^{-1} \, dv \]

\[ = \int_0^1 \frac{2s+1}{2s+1-v} \exp \left( (2s+1) \log(1 - \frac{v}{2s+1}) + (s-1) \log(1 + \frac{v}{s-1}) \right) \, dv \]

\[ = \int_0^1 \frac{2s+1}{2s+1-v} \exp \left( \sum_{k=2}^{\infty} \frac{v^k}{k} \left( (s-1)^{k-1} - \frac{1}{(2s+1)^{k-1}} \right) \right) \, dv \]

\[ < \int_0^1 \frac{2s+1}{2s+1-v} \exp \left( 3 \sum_{k=2}^{\infty} \frac{v^k}{k} \left( (s-1)^{k-1} - \frac{1}{(2s+1)^{k-1}} \right) \right) \, dv, \]

since \( \sum_{k=4}^{\infty} \frac{v^k}{k} \left( (s-1)^{k-1} - \frac{1}{(2s+1)^{k-1}} \right) \) < 0. By adopting the notation

\[ \gamma(s, v) = \sum_{k=2}^{\infty} \frac{v^k}{k} \left( (s-1)^{k-1} - \frac{1}{(2s+1)^{k-1}} \right), \]

it should be remarked that

\[ \gamma(s, v) = -\frac{3v^2 s}{2(s-1)(2s+1)} + \frac{v^3 s(s+2)}{(s-1)^2(2s+1)^2} < 0 \]

for each \( s \geq 2 \) and \( v \in [0, 1] \). Since \( \exp(c) < 1 + c + \frac{c^2}{2} \) for \( c < 0 \), it follows

\[ I_s < \int_0^1 \frac{2s+1}{2s+1-v} \exp \left( \gamma(s, v) \right) \, dv \]

\[ < \int_0^1 \frac{2s+1}{2s+1-v} \left( 1 + \gamma(s, v) + \frac{\gamma(s, v)^2}{2} \right) \, dv. \]

Moreover, since

\[ \frac{2s+1}{2s+1-v} = \frac{1}{1 - \frac{v}{2s+1}} = 1 + \frac{v}{2s+1} + \frac{v^2}{(2s+1)^2} \cdot \frac{1}{1 - \frac{v}{2s+1}} \]

\[ \leq 1 + \frac{v}{2s+1} + \frac{5v^2}{4(2s+1)^2}, \]

it also follows

\[ I_s < \int_0^1 \left( 1 + \frac{v}{2s+1} + \frac{5v^2}{4(2s+1)^2} \right) \left( 1 + \gamma(s, v) + \frac{\gamma(s, v)^2}{2} \right) \, dv. \]

By computing the integral and by means of tedious algebraic manipulations, it holds

\[ I_s < 1 + \frac{1 + 3s(-11s^3 + (s+4)^2)}{(s-1)^4(2s+1)^6} + \frac{s^6(29 + 7s - 15s^2/2)}{(s-1)^4(2s+1)^6}. \]

Since the numerators of the two fractions in the previous expressions are negative for \( s \geq 3 \), it holds that \( I_s < 1 \) for \( s \geq 3 \). Finally, by direct computation it also follows that \( I_2 < 1 \) and hence Theorem 1 holds true for \( r = 0 \). As to (9), i.e. when \( r = 1, 2 \), let us assume that

\[ J_r = \int_0^1 \left( 1 - \frac{v}{2s+1} \right)^{2s} \left( 1 + \frac{v}{s-r-1} \right)^{s+r-1} \, dv. \]
By remarking that for each \( s \) it holds \((1 + \frac{v}{s+1})^s \leq (1 + \frac{v}{s})^s\), it follows \( J_1 \leq J_2 \).

Hence, in order to prove (9) it suffices to show \( J_1 \geq 1 \). It holds

\[
J_1 = \int_0^1 \left(1 - \frac{v}{2s+1}\right)^{2s}(1 + \frac{v}{s})^s \, dv
\]

\[
= \int_0^1 \left[(1 - \frac{v}{2s+1})^2(1 + \frac{v}{s})^s\right] \, dv
\]

\[
= \int_0^1 \left[(1 - \frac{2v}{2s+1} + \frac{v^2}{(2s+1)^2})(1 + \frac{v}{s})^s\right] \, dv
\]

\[
= \int_0^1 \left[(1 + \frac{v(1 - 2v)}{s(2s+1)} + \frac{v^2}{(2s+1)^2}(1 + \frac{v}{s})^s\right] \, dv.
\]

By applying Bernoulli inequality, it reads

\[
J_1 \geq \int_0^1 \left(1 + \frac{v(1 - 2v)}{2s+2} + \frac{sv^2}{(2s+1)^2}(1 + \frac{v}{s})\right) \, dv
\]

\[
= 1 - \frac{1}{6(2s+1)} + \frac{s}{3(2s+1)^2} + \frac{1}{4(2s+1)^2} = 1 + \frac{1}{12(2s+1)^2},
\]

which obviously implies (9). Finally, we prove (10). By adopting the notation

\[
H_r = \int_0^1 (1 - \frac{v}{2s+2})^{2s+1}(1 + \frac{v}{s + r - 2})^{s+r-2} \, dv,
\]

since for each \( v \) it holds \((1 + \frac{v}{s+1})^{s-1} \leq (1 + \frac{v}{s})^s\), it also follows that \( H_1 \leq H_2 \) and hence it suffices to show the case \( H_2 < 1 \). To this aim, similarly to the the proof of the inequality \( I_s < 1 \), it reads

\[
H_2 = \int_0^1 (1 - \frac{v}{2s+2})^{2s+1}(1 + \frac{v}{s})^s \, dv
\]

\[
= \int_0^1 \frac{2s+2}{2s+2-v} \exp \left((2s+2)\log(1 - \frac{v}{2s+2}) + s\log(1 + \frac{v}{s})\right) \, dv
\]

\[
= \int_0^1 \frac{2s+2}{2s+2-v} \exp \left(\sum_{k=2}^{\infty} \frac{v^k}{k} \left(\frac{(-1)^{k-1}}{s^{k-1}} - \frac{1}{(2s+2)^{k-1}}\right)\right) \, dv
\]

\[
< \int_0^1 \frac{2s+2}{2s+2-v} \exp \left(\sum_{k=2}^{3} \frac{v^k}{k} \left(\frac{(-1)^{k-1}}{s^{k-1}} - \frac{1}{(2s+2)^{k-1}}\right)\right) \, dv,
\]

since \( \sum_{k=4}^{\infty} \frac{v^k}{k} \left(\frac{(-1)^{k-1}}{s^{k-1}} - \frac{1}{(2s+2)^{k-1}}\right) < 0 \). By assuming that

\[
\lambda(s, v) = \sum_{k=2}^{3} \frac{v^k}{k} \left(\frac{(-1)^{k-1}}{s^{k-1}} - \frac{1}{(2s+2)^{k-1}}\right),
\]
it should be noticed that $\lambda(s, v) < 0$ for each $s \geq 2$ and $v \in ]0, 1[$. By considering the inequality $\exp(c) < 1 + c + \frac{c^2}{2}$ for $c < 0$, it follows

$$H_2 < \int_0^1 \frac{2s + 2}{2s + 2 - v} \exp(\lambda(s, v)) \frac{dv}{v}$$

$$< \int_0^1 \frac{2s + 2}{2s + 2 - v} (1 + \lambda(s, v) + \frac{\lambda(s, v)^2}{2}) \frac{dv}{v}$$

$$< \int_0^1 \left(1 + \frac{v}{2s + 2} + \frac{6v^2}{5(2s + 2)^2}\right) \left(1 + \lambda(s, v) + \frac{\lambda(s, v)^2}{2}\right) \frac{dv}{v},$$

since it holds $\frac{2s + 2}{2s + 2 - v} < 1 + \frac{v}{2s + 2} + \frac{6v^2}{5(2s + 2)^2}$ for each $s \geq 2$ and $v \in ]0, 1[$. By evaluating the previous integral and by suitable algebraic manipulations, the following inequality holds for $s \geq 2$

$$H_2 < 1 + \frac{25s^5 - 8(s^4 + s^3) + 8s^2 + 22s + 18}{64s(s + 1)^6}.$$ 

The right-hand side of the previous inequality is obviously less than 1. Moreover, a direct computation provides $H_2 < 1$ for $s = 1$. Therefore, Theorem 1 is proven.

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