On the transition from parabolicity to hyperbolicity for a nonlinear equation under
Neumann boundary conditions

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An integro differential equation which is able to describe the evolution of a large class of dissipative models, is considered. By means of an equivalence, the focus shifts to the perturbed sine-Gordon equation that in superconductivity finds interesting applications in multiple engineering areas. The Neumann boundary problem is considered, and the behaviour of a viscous term, defined by a higher-order derivative with small diffusion coefficient $\varepsilon$, is investigated. The Green function, expressed by means of Fourier series, is considered, and an estimate is achieved. Furthermore, some classes of solutions of the hyperbolic equation are determined, proving that there exists at least one solution with bounded derivatives. Results obtained prove that diffusion effects are bounded and tend to zero when $\varepsilon$ tends to zero.

1 Introduction

Evolution problems of several physical models such as motions of viscoelastic bodies or fluids, [1,2], heat conduction, [3,4], sound propagation [5], and biological systems such as neural communications [6–9], can be modelled by means of the following integro differential equation:

\begin{equation}
\mathcal{L}u \equiv u_t - \varepsilon u_{xx} + au + \delta \int_0^t e^{-\beta(t-\tau)} u(x, \tau) \, d\tau = F(x, t, u)
\end{equation}

which is part of a more general class of mathematical physical models typical of materials with memory [10–13](and references therein).

Furthermore, equation (1.1) can also describe specific phenomena in superconductivity (see, f.i. [14–17]). Indeed, when in (1.1) one assumes
\[ a = \alpha - \frac{1}{\varepsilon} \quad \delta = -\frac{a}{\varepsilon} \quad \beta = \frac{1}{\varepsilon} \]
and \( F \) is such that
\[ F(x, t, u) = -\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} \left[ \text{sen} u(x, \tau) - \gamma \right] d\tau, \]
integral equation (1.1) turns into the so called perturbed sine-Gordon equation
\[ u_{xx} - u_{tt} = \sin u + \gamma + \alpha u_t - \varepsilon u_{xxt}. \]

Equation (1.4) models the flux dynamics in the Josephson junction where two superconductors are separated by a thin insulating layer. Term \( \alpha u_t \) denotes the dissipative normal electron current flow across the junction, while the third term such as \( \varepsilon u_{xxt} \) represents the flowing of quasi-particles parallel to the junction [18]. The value range for \( \alpha \) and \( \varepsilon \) depends on the material of the real junction and in the vast majority of cases it results \( 0 < \alpha, \varepsilon < 1 \) [19, 20]. If, however, the resistance of the junction is so low as to completely shorten the capacitance, the case \( \alpha \) large with respect to 1 arises [21].

Josephson junctions find application in many fields of practical interest. For example, it is possible to diagnose heart and/or blood circulation problems using magnetocardiograms employing superconducting quantum interference devices (SQUIDs). Another area where SQUIDs are used is magnetoencephalography (MEG)- where they allow to make inferences and analyze magnetic fields generated by electric currents inside brains. In geophysics, they are used as gradiometers or as gravitational wave detectors ([21] and reference therein) and play an important role in the study of the potential virtues of superconducting digital electronics [22]. SQUIDs are also used in nondestructive testing as a convenient alternative to ultra sound or x-ray methods ([19, 23–25] and reference therein) and can be used as fast, switchable meta-atoms [26], too. Besides, superconductor technology and its applications are in continuous development. As an example, superconductivity powers the Tevatron collider at Fermilab to produce the top quark, and CERNs Large Hadron Collider (LHC) to discover the Higgs boson and advanced superconducting magnets are already being developed for future collider projects [27].

Of course, previously mentioned equations are to be complemented by initial-boundary problems (many examples can be found, f.i. in [7, 28, 29]) and a particular attention will be given to the Neumann problem since it is meaningful in several scientific fields. In mathematical biology, for instance, those boundary conditions occur when a two-species reaction diffusion system is subjected to flux boundary conditions [9]. The same conditions are present in case of pacemakers [30] and they are applied to study distributed FitzHugh-Nagumo systems as well [31]. Moreover, in superconductivity, in the case of a Josephson junction,
they can refer to the phase gradient value and they are proportional to the magnetic field \[32,33\].

1.1 Mathematical considerations, state of the art and aim of the paper

The connection between integro differential equation (1.1) and equation (1.4) allows us to extend the analysis performed on one model to the other. In particular, our attention will be devoted to the parabolic third order linear operator

(1.5) \[ \mathcal{L}u \equiv \partial_{xx}(\varepsilon u_t + u) - \partial_t(u_t + \alpha u) \]

which characterizes equation (1.4). Since in many cases the evolution is described by deep interactions between wave propagation and diffusion, operator $\mathcal{L}$ can also be considered as a linear hyperbolic operator perturbed by viscous terms that are described by higher-order derivatives with small diffusion coefficients $\varepsilon$. Indeed, when $\varepsilon \equiv 0$, the operator (1.5) turns into a hyperbolic one:

(1.6) \[ \mathcal{L}_0 U \equiv U_{xx} - \partial_t(U_t + \alpha U) \]

and the influence of the dissipative terms, represented by $\varepsilon \partial_{xxt}$ on the wave behavior has to be estimated.

From this small parameter depends the transmission times of the signals as well, at least when the main process is the diffusion one. In order to obtain some knowledge about the dynamics, the order of diffusion effects and the time-intervals must be estimated. This analysis leads to an evaluation of the non linear parabolic - hyperbolic boundary layers.

Similar problems, for Dirichlet conditions, have already been studied. In particular, in \[34\], for $\alpha = 0$, an asymptotic approximation was established by means of the two characteristic times: slow time $\tau = \varepsilon t$ and fast time $\theta = t/\varepsilon$. Moreover, for equations of type (1.4), in \[35\] an analytical analysis proved that the surface damping has little influence on the behaviour of the oscillator, confirming numerical results showed in \[36\]. Other numerical investigations on the influence of surface losses can be found in \[37\], too.

As for the Neumann problem, fast and slow diffusion effects have been investigated in \[38\] and furthermore, an estimate allows us to evaluate the behaviour when time $t$ tends to infinity.

Moreover, in \[29,39–41\] a class of equations characterized by an operator like (1.5) is considered and theorems of existence and uniqueness of solutions related to Dirichlet, Neumann and pseudoperiodic boundary conditions are proved.

Finally, we would like to point out that the linear operator $\mathcal{L}$ has already been examined by means of a convolutions of Bessel functions and Fourier series,
and in this case, the related Green Function \( G = \sum_{n=1}^{\infty} G_n \) has been estimated by means of constants also depending on parameter \( \varepsilon \) \([39, 42]\) or considering \( G = \sum_{n=1}^{\infty} \frac{G_n}{n^2} \) \([38]\).

Our purpose is to analyse the error \( d(x, t, \varepsilon) \) represented by the difference between the solution \( u \) related to the Neumann boundary problem \( \mathcal{P}_\varepsilon \) for equation (1.4) and the wave solution \( U \) deduced from \( \mathcal{P}_\varepsilon \) when \( \varepsilon \to 0 \):

\[
(1.7) \quad d(x, t, \varepsilon) = u(x, t, \varepsilon) - U(x, t).
\]

For this purpose, it appears necessary to determine an estimate for the Green function proving that it is bounded by terms independent from \( \varepsilon \) and vanishing when \( t \to \infty \).

Moreover, some classes of solutions of the hyperbolic equation

\[
(1.8) \quad U_{xx} - \partial_t(U_t + \alpha U) = \sin U + \gamma
\]

will be determined showing that there exists at least one solution with bounded derivatives.

Finally, by applying a Gronwall inequality, it will be possible to highlight the infinite time-interval where the diffusion effects vanish for \( \varepsilon \to 0 \).

The paper is organized as follows:

In section 2, the mathematical problem is stated and thanks to the Green Function, determined through Fourier series, the solution related to the remainder term \( d \) is shown.

In section 3, by means of lemmas on hyperbolic and circular terms, in theorem 3.1 the following estimate \( H = \sum_{n=1}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq B e^{-m t} \) is proved where \( B \) and \( m \) are two positive constants independent from \( \varepsilon \).

In section 4, many classes of solutions of (1.8) are determined, and an example of solution with bounded derivatives is shown.

Finally, in section 5, an estimate for the remainder term is achieved by proving that the diffusion effects are of the order of \( \varepsilon^{1-k/\alpha} \log \varepsilon^{-k} \) with \( 0 < k < \alpha \).

## 2 Statement of the problem and solution of the nonlinear problem

Letting \( T \) be an arbitrary positive constant and letting

\[
\Omega = \{(x, t) : 0 \leq x \leq \ell, \quad 0 \leq t \leq T\},
\]

we point our attention to the following Neumann problem:

\[
(2.9) \quad \begin{align*}
\partial_{xx}(\varepsilon u_t + u) - \partial_t(u_t + \alpha u) &= \sin u + \gamma, & (x, t) \in \Omega; \\
u(x, 0) &= h_0(x), & u_t(x, 0) &= h_1(x), \quad x \in [0, \ell], \\
u_x(0, t) &= \varphi_0(t), & u_x(\ell, t) &= \varphi_1(t), & 0 \leq t \leq T.
\end{align*}
\]
When \( \varepsilon \equiv 0 \), problem (2.9) turns into the Neumann problem related to operator \( L_0 \) defined in (1.6) with the same initial boundary conditions:

\[
\begin{aligned}
U_{xx} - \partial_t(U_t + \alpha U) &= \sin U + \gamma & (x, t) \in \Omega, \\
U(x, 0) &= h_0(x), & U_t(x, 0) = h_1(x), & x \in [0, \ell], \\
U_x(0, t) &= \varphi_0(t), & U_x(\ell, t) = \varphi_1(t), & 0 \leq t \leq T.
\end{aligned}
\]  

(2.10)

The influence of the dissipative term on the wave behaviour of \( U \) can be estimated when the difference \( d \) defined in (1.7) is evaluated.

So, let us consider the following problem related to the remainder term \( d \):

\[
\begin{aligned}
\partial_{xx} (\varepsilon \partial_t + 1) d - \partial_t (\partial_t + \alpha) d &= F(x, t, d), & (x, t) \in \Omega, \\
d(x, 0) &= 0, & d_t(x, 0) = 0, & x \in [0, \ell], \\
d_x(0, t) &= 0, & d_x(\ell, t) = 0, & 0 \leq t \leq T.
\end{aligned}
\]  

(2.11)

with

\[
F(x, t, d) = \sin(d + U) - \sin U - \varepsilon U_{xxt}.
\]  

(2.12)

If one assumes:

\[
\begin{aligned}
\gamma_n &= \frac{n\pi}{\ell}, & h_n &= \frac{1}{2}(\varepsilon \gamma_n^2) + \frac{1}{2}(\alpha \gamma_n^2), \\
\omega_n &= \sqrt{h_n^2 - \gamma_n^2}, \\
H_n(t) &= e^{-h_n t} \sinh(\omega_n t) \omega_n
\end{aligned}
\]  

(2.13)

by means of standard techniques, the Green function \( G = G(x, t) \) of problem (2.11) is given by

\[
G(x, t, \xi) = \frac{1}{\ell} \frac{1 - e^{-\alpha t}}{\alpha} + \frac{2}{\ell} \sum_{n=1}^{\infty} H_n(t) \cos \gamma_n \xi \cos \gamma_n x.
\]  

(2.14)

Moreover, by means of standard methods related to integral equations and thanks to the fixed point theorem, owing to basic properties of the Green function \( G \) already proved in [42], it is possible to prove that problem (2.11) admits a unique regular solution in \( \Omega \) and it results (see, f.i [41, 43, 44]):

\[
d(x, t) = -\frac{1}{\ell} \int_0^t d\tau \int_0^\ell \left[ \frac{1 - e^{-\alpha (t-\tau)}}{\alpha} \right] F(\xi, \tau, d(\xi, \tau)) d\xi
\]  

\[
- \frac{2}{\ell} \int_0^t d\tau \int_0^\ell H(x, \xi, t-\tau) F(\xi, \tau, d(\xi, \tau)) d\xi.
\]  

(2.15)
where

\begin{equation}
H = \sum_{n=1}^{\infty} H_n(t) \cos \gamma_n \xi \cos \gamma_n \chi.
\end{equation}

3 Estimates related to the Green Function

In order to obtain an estimate of function \(d(x, t, \varepsilon)\), it appears necessary to evaluate the Green Function.

So that, let us assume \(0 < \alpha < 1\) and \(0 < \varepsilon < 1\) and let us denote by \(N_1\) the minimum natural number larger than \(\frac{\ell}{\pi} \sqrt{1 - \alpha \varepsilon}\), and by \(N_2\) the maximum natural number smaller than \(\frac{\ell}{\pi} \sqrt{1 + \alpha \varepsilon}\).

It is necessary to distinguish two cases: when \(N_1 < 1\) or \(N_1 > 1\).

If \(N_1 < 1\), function \(H_n(t)\) in (2.13) contains trigonometric functions for \(N_1 \leq n \leq N_2\) and hyperbolic terms for \(n \geq N_2 + 1\).

When \(N_1 > 1\), interval \([1, N_1 - 1]\) with hyperbolic terms must be considered, too.

Firstly, in the hypothesis of \(N_1 > 1\), we analyze hyperbolic terms both in \([1, N_1 - 1]\) and in \([N_2 + 1, \infty]\) proving the following lemma:

**Lemma 1.** Whatever \(0 < \alpha < 1\) and \(0 < \varepsilon < 1\) may be, there exist positive constants \(B_i, (i = 1, 2)\) independent from \(\varepsilon\), such that:

\begin{equation}
\sum_{n=N_2+1}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq B_1 e^{-\frac{t}{4}},
\end{equation}

\begin{equation}
\sum_{n=1}^{N_1-1} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq B_2 e^{\frac{-t}{(\pi \varepsilon \sqrt{c})(1 + \sqrt{1 - a \varepsilon c})}} \quad (N_1 > 1).
\end{equation}

**Proof.** Let us start with considering the infinite interval \((N_2 + 1, \infty)\). If \(c\) denotes an arbitrary positive constant less than 1, let \(N_c\) be the integer part of \(\ell/(\pi \varepsilon \sqrt{c})(1 + \sqrt{1 - a \varepsilon c})\).

Interval \((N_2 + 1, \infty)\) will be divided into intervals \((N_2 + 1, N_c - 1)\); \((N_c, \infty)\) and it will result:

\begin{equation}
\sum_{n=N_2+1}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} = \sum_{n=N_2+1}^{N_c-1} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} + \sum_{n=N_c}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} = H' + H''
\end{equation}

In order to evaluate \(H'\) we state that

\begin{equation}
e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} = e^{-(h_n - \omega_n) t} \int_0^t e^{-2\omega_n \tau} \, d\tau \leq t \, e^{-(h_n - \omega_n) t}
\end{equation}
and

\[(3.21) \quad h_n - \omega_n = h_n - h_n \sqrt{1 - \frac{\gamma_n^2}{h_n^2}} \geq h_n - h_n \left(1 - \frac{\gamma_n^2}{2h_n^2}\right).\]

So that, since \(n \geq N_2 + 1 > \ell/(\pi \varepsilon)\) and \(a \varepsilon < 1\), it results:

\[(3.22) \quad e^{-(h_n-\omega_n)t} \leq e^{-t \frac{\gamma_n^2}{h_n^2}} \leq e^{-t \frac{\gamma_n^2}{2h_n^2}}.\]

Moreover, taking into account that

\[(3.23) \quad e^{-x} \leq \frac{\nu^x}{(ex)^\nu} \quad \forall \ x > 0; \ \forall \nu > 0,
\]

one has:

\[(3.24) \quad H' \leq \sum_{n=N_2+1}^{N_2-1} te^{-\frac{\gamma_n^2}{h_n^2}} \leq B' e^{-\frac{\gamma_n^2}{2h_n^2}}\]

where \(B' = \frac{4\varepsilon}{c} (N_c - N_2 - 1) \leq \frac{8\ell}{\pi \sqrt{c}}\).

As for \(H''\), when \(n \geq N_c\), one has \(\gamma_n \geq \frac{1}{\varepsilon \sqrt{c}} (1 + \sqrt{1 - a \varepsilon}) \Rightarrow \varepsilon \sqrt{c} \gamma_n^2 + a \sqrt{c} - 2\gamma_n \geq 0\), so that \(\frac{2\gamma_n}{\alpha + \varepsilon \gamma_n} \leq \sqrt{c} \Rightarrow \frac{\alpha}{\gamma_n} \leq \sqrt{c}\). As a consequence it results:

\[(3.25) \quad \omega_n = h_n \sqrt{1 - \frac{\gamma_n^2}{h_n^2}} \geq h_n \sqrt{1 - c} \geq e^{\varepsilon \pi^2 n^2 \sqrt{1 - c}} \]

\(B'' = \frac{8\ell^2}{c} \left(\frac{\pi \sqrt{c}/\ell}{\pi \sqrt{1 - c}}\right)^{(1-\eta)} \frac{1}{\varepsilon \nu} \leq \frac{4\ell^2 (\pi \sqrt{c}/\ell)^{(1-\eta)}}{\pi^2 \sqrt{1 - c}} \varepsilon^{-\eta} \zeta(1 + \eta) e^{-\frac{\gamma_n^2}{2h_n^2}}\)

and considering (3.22) as well, if \(B'' = \frac{8\ell^2 (\pi \sqrt{c}/\ell)^{(1-\eta)}}{e \pi^2 \sqrt{1 - c}} \zeta(1 + \eta)\), since \(t \geq 1\), it results:

\[(3.26) \quad H'' \leq B'' e^{1-\eta} e^{-\frac{\gamma_n^2}{2h_n^2}}.\]

Finally, according to (3.20), (3.21) and (3.23), one has:

\[(3.27) \quad \sum_{n=1}^{N_1-1} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq \sum_{n=1}^{N_1-1} te^{-t \frac{\gamma_n^2}{h_n^2}} \leq B_2 e^{-\frac{\gamma_n^2}{2h_n^2}}.\]
where, $B_2 = (N_1 - 1) \frac{2\pi^2}{\epsilon^2} \leq \frac{4\pi a}{\ell^2}$. Therefore, since (3.24), (3.26), and (3.27) hold, denoting by $B_1 = \max\{B', B''\}$, the theorem is proved.

Now, letting

$$\omega_0 = \sqrt{\gamma_n^2 - \alpha^2}; \quad \tilde{\omega}_n = \sqrt{\gamma_n^2 - h_n^2}; \quad h_1 = \frac{1}{2} (\alpha + \frac{\pi^2}{\ell^2})$$

(3.28)

in order to evaluate circular terms, we will firstly prove the following inequality involving difference $R(n, t)$. Then, attention will be devoted to all the terms of series in $H_n(t)$.

**Lemma 2.** Whatever $0 < \alpha < 1$ and $0 < \varepsilon < 1$ may be, there exists a positive constant $B_3$ independent from $\varepsilon$, such that, letting $0 < \eta < 1$, $0 < h < 1/2$, $1 < k < 3/2$, it results:

$$\sum_{n=1}^{N_2} \left[ e^{-h_n t} \frac{\sin(\tilde{\omega}_n t)}{\omega_n} - e^{-h_1 t} \frac{\sin(\omega_0 t)}{\omega_0} \right] \leq B_3 \varepsilon \rho e^{-\frac{4\pi^2}{\ell^2}}$$

(3.29)

where $\rho = \min\{\eta, 1 - \eta, 1/2 - h, 3/2 - k\}$.

**Proof.** Indicating by $\hat{R}(n, s)$ the Laplace transform of function $R(n, t)$, one deduces that

$$\hat{R}(n, s) = \frac{(s + h_1)^2 + \omega_0^2 - (s + h_n)^2 - \tilde{\omega}_n^2}{[(s + h_1)^2 + \omega_0^2] [(s + h_n)^2 + \tilde{\omega}_n^2]}$$

So that, denoting by $\tilde{g} = \frac{\pi^2}{\ell^2} (2\alpha + \frac{\pi^2}{\ell^2})$, it results

$$R(n, t) = \int_0^t e^{-h_1 (t-\tau)} \sin(\omega_0 (t-\tau)) e^{-h_n \tau} \sin(\tilde{\omega}_n \tau) d\tau + \int_0^t e^{-h_1 (t-\tau)} \sin(\omega_0 (t-\tau)) e^{-h_n \tau} \tilde{g} \left[ \frac{\sin(\tilde{\omega}_n \tau) - \cos(\tilde{\omega}_n \tau)}{\omega_n} \right] d\tau$$

(3.30)

Indicating by

$$g_0 = \sqrt{\gamma_n^2 - \alpha^2} \quad g = \frac{\pi^2}{4\ell^2} (2\alpha + \frac{\pi}{\ell})$$

(3.31)

one has:

$$\omega_0 \geq n g_0, \quad \tilde{g} \leq g.$$

(3.32)
Cases \( n = 1, n = N_2, \) and \( 2 \leq n \leq N_2 - 1 \) will be considered and the value of function \( R(n, t) \) in correspondence of \( n \equiv \bar{n} \) will be denoted by \( R(\bar{n}, t) \).

When \( n = 1 \), from (3.30), if \( g_1 = 32g/[(\alpha \varepsilon)^2 g_0] \), one has:

\[
R(1, t) \leq e^{-\frac{\pi^2}{\ell^2}} \varepsilon g_1
\]

Moreover, since

\[
h_n - h_1 = \frac{\pi^2(n^2 - 1)\varepsilon}{2\ell^2},
\]

for \( n = N_2 \), denoting by \( \eta \) a constant such that \( 0 < \eta < 1 \), let

\[
g_{N_2} = g / (2g_0 \varepsilon N_2)
\]

\[
A' = 4 \varepsilon N_2 \varepsilon \varepsilon \pi^2 \eta^2 (N_2^2 - 1) \leq 4,
\]

by means of (3.23), one obtains:

\[
R(N_2, t) \leq e^{-\frac{\pi^2}{\ell^2}} \varepsilon^2 t^2 g_{N_2} + A' e^{-\frac{\pi^2}{\ell^2}} t^{1-\eta} \varepsilon^2.
\]

So that, there exists a positive constant \( A'' \), independent from \( \varepsilon \) such that:

\[
R(N_2, t) \leq A'' \varepsilon^2 e^{-\frac{\pi^2}{\ell^2}}
\]

Now, let us consider \( 2 \leq n \leq N_2 - 1 \). Since \( \tilde{\omega}_n \geq \gamma_n \sqrt{1 - h_n/\gamma_n} = n\Phi_n \), indicating by

\[
s = \sqrt{\pi} \sqrt{\frac{2\sqrt{1-a\varepsilon - \varepsilon^2 \pi \ell}}{1-\pi/\ell + \sqrt{1-a\varepsilon}}} = \frac{(\Phi_n)_{n=N_2-1}}{\sqrt{\pi}},
\]

\[
p = \sqrt{\frac{4[\pi(\beta - \ell) - \pi^2 a]}{2\sqrt{\pi \ell}}} = (\Phi_n)_{n=2}
\]

and by \( q = \min \{ s, p \} \), it is possible to choose a positive constant \( g_2 \) depending on parameter \( a \), but independent from \( \varepsilon \), such that \( g_2 \leq q \). In such a way, for all \( 2 \leq n \leq N_2 - 1 \) one has:

\[
\tilde{\omega}_n \geq n \sqrt{\varepsilon} g_2.
\]
So that, it results:

\[
\begin{align*}
&\left(3.40\right) \\
&\sum_{n=2}^{N_2-1} R(n, t) \leq \sum_{n=2}^{N_2-1} \frac{g}{n^2 g_0 g_2} \sqrt{\varepsilon t} + \\
&\sum_{n=2}^{N_2-1} \left( \frac{\pi^2 \varepsilon}{E_0} h_n + \frac{\pi^2 \varepsilon n}{E_0} \right) \int_0^t |\sin(\omega_0(t - \tau))| e^{-\varepsilon(h_n - h_1)\tau} d\tau.
\end{align*}
\]

Considering \(0 < \eta < 1\), and indicating by \(h\), and \(k\) two constants such that \(0 < h < 1/2\) \(< k < 3/2\),

let

\[
C_1 = \frac{\pi^2 \alpha}{\ell^2 g_0^2 g_2} \left( \frac{2h\ell^2}{e \pi^2} \right)^h, \quad C_2 = \pi^4 \varepsilon^2 \left( \frac{2k\ell^2}{e \pi^2} \right)^k, \quad C_3 = \frac{2\pi^2}{\ell^2 g_0^2} \left( \frac{2\eta\ell^2}{e \pi^2} \right)^\eta.
\]

The analysis lead us to consider asymptotic behavior when variable \(t\) tends to infinity. Therefore it does not effect generality if we consider \(t \geq 1\). In this hypothesis and since \(\int_0^t |\sin(\omega_0(t - \tau))| d\tau \leq 2/\omega_0\), one obtains:

\[
\begin{align*}
&\left(3.41\right) \\
&\sum_{n=2}^{N_2-1} R(n, t) \leq \sum_{n=2}^{N_2-1} \frac{g}{n^2 g_0 g_2} \sqrt{\varepsilon t} + \\
&\sum_{n=2}^{N_2-1} \left[ C_1 \frac{\varepsilon^{1/2 - h}}{n(n^2 - 1)^h} + C_2 \frac{n \varepsilon^{3/2 - k}}{(n^2 - 1)^k} + C_3 \frac{\varepsilon^{1 - \eta}}{(n^2 - 1)^\eta} \right].
\end{align*}
\]

So, it is possible to choose, in this case as well, a positive constant \(C_4\) independent from \(\varepsilon\), such that:

\[
\begin{align*}
&\left(3.42\right) \\
&\sum_{n=2}^{N_2-1} R(n, t) \leq C_4 \left[ \sqrt{\varepsilon t} + \varepsilon^{1/2 - h} + \varepsilon^{3/2 - k} + \varepsilon^{1 - \eta} \right] e^{-\frac{a_2}{\varepsilon}}.
\end{align*}
\]

For this and since \(3.33\) and \(3.37\), \(3.29\) holds.

**Remark** Naturally, by tracing back this demonstration, it is possible to prove that for \(N_1 > 1\) the Lemma 2 holds even if interval \([1, N_2]\) has to be replaced with \([N_1, N_2]\).

So that, for any value of \(N_1\), the following theorem can be proved:
\textbf{Teorema 3.1.} Whatever \(0 < a < 1\) and \(0 < \varepsilon < 1\) may be, there exists a positive constant \(B\), independent from \(\varepsilon\), such that:

\[(3.43) \quad \sum_{n=1}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq B e^{-m t}\]

being \(m = \max\{a/4, \alpha \ell^2/(2\pi^2)\}\).

\textit{Proof.} Hyperbolic terms have been evaluated by means of lemma (1). Besides, if \(N_1 < 1\), it remains to be proven that there exists a positive constant \(B_4\) independent from \(\varepsilon\) such that:

\[(3.44) \quad \sum_{n=1}^{N_2} H_n = \sum_{n=1}^{N_2} e^{-h_n t} \frac{\sin(\tilde{\omega}_n t)}{\tilde{\omega}_n} \leq B_4 e^{-\frac{\alpha}{4}}\]

But (3.44) follows on application of lemma 2 and since

\[(3.45) \quad \sum_{n=1}^{N_2} H_n(t) \leq R(n, t) + e^{-\frac{\alpha}{4} t} \sum_{n=1}^{\infty} H_n^0(t)\]

where \(H_n^0(t) = \frac{\sin(\omega_0 t)}{\omega_0}\).

Similar estimates hold also for \(N_1 > 1\), namely when circular terms appear only for \(n \in [N_1, N_2]\).

So that, since \(e^{-\frac{\alpha}{4}} \leq e^{-\frac{\alpha}{4} t}\) and \(e^{-2(\varepsilon \pi^2/\ell^2)} \leq e^{-\varepsilon \pi^2/\ell^2}\), from (3.17), (3.18) and (3.44), theorem is proved.

\[\square\]

4 \ Explicit solutions of the hyperbolic equation

Let us consider the semilinear second order equation:

\[(4.46) \quad U_{xx} - U_{tt} - \alpha U_t = \sin U + \gamma\]

When \(\alpha = \gamma = 0\), (4.46) represents the sine-Gordon equation and there exists a wide literature about its classes of soliton solutions [45–47]. Besides, if \(\alpha \neq 0\) or \(\gamma \neq 0\), other solutions are shown in [48, 49].

Now, let \(f\) be an arbitrary function, and let us consider the following function \(\Pi(f)\):

\[(4.47) \quad \Pi(f) = 2 \arctan e^f\]

so that
\[ \sin \Pi(f) = \frac{1}{\cosh(f)}, \quad \cos \Pi(f) = -\tanh(f). \]

By means of function (4.47) it is possible to find a class of solutions of equation (4.46).

Indeed, it is possible to verify that the following function:

\[ U = 2 \Pi[f(\xi)] \quad \text{with} \quad \xi = \frac{x - t}{\alpha} \]

is a solution of (4.46) provided that one has:

\[ -\alpha U_t = \sin U + \gamma. \]

Moreover, since (4.48) and being \( \Pi = \frac{1}{\cosh f} \), it results:

\[ -\alpha U_t = 2 \frac{f'}{\cosh f}; \quad \sin U = -2 \frac{\tanh f}{\cosh f}. \]

So, from (4.50), one deduces that function \( f \) must satisfy the following equation:

\[ \frac{df}{-\tanh f + \gamma/2 \cosh f} = d\xi. \]

According to value of constant \( \gamma \), it is possible to find explicit classes of solution (see f.i. [35]), and we will consider as an example \( \gamma = 0 \) and \( \gamma = 1 \).

So, indicating by \( f_0, \xi_0 \) the respectively initial data of function \( f \) and variable \( \xi \), when \( \gamma = 0 \), for \( r_0 = e^{\xi_0} \sinh f_0 \), since (4.51) it results:

\[ \sinh f = r_0 e^{-\xi}, \]

and one has:

\[ U = 4 \arctan(y + \sqrt{y^2 + 1}) \quad \text{with} \quad y = r_0 e^{-\xi}. \]

Besides, if \( \gamma = 1 \), since \( \int \frac{df}{1/2 \cosh f - \tanh f} = \frac{-2}{\sinh f - 1} + \text{const} \), denoting by \( r_0 = \xi_0 + \frac{2}{\sinh f_0 - 1} \), it results:

\[ U = 4 \arctan(y + \sqrt{y^2 + 1}) \quad \text{with} \quad y = \frac{2 + r_0 - \xi}{r_0 - \xi}. \]
Our aim is to find a solution of problem (2.10) whose derivatives are bounded. In order to do so, let us consider the following initial boundary problem:

\[
\begin{aligned}
U_{xx} - \partial_t(U_t + \alpha U) &= \sin U \quad (x, t) \in \Omega, \\
U(x, 0) &= 2 \arctan(e^{x/\alpha}), \quad U_t(x, 0) = -\frac{2}{\alpha} \frac{e^{x/\alpha}}{1 + e^{2x/\alpha}}, \quad x \in [0, \ell], \\
U_x(0, t) &= \frac{2}{\alpha} \frac{e^{-t/\alpha}}{1 + e^{-2x/\alpha}}, \quad U_x(\ell, t) = \frac{2}{\alpha} \frac{e^{(\ell-t)/\alpha}}{1 + e^{2(\ell-t)/\alpha}}, \quad 0 \leq t \leq T,
\end{aligned}
\]

which admits the solution

\[
(4.56) \quad U = 2 \arctan e^{(x-t)/\alpha}.
\]

It results:

\[
(4.57) \quad U_{xxt} = -\frac{2}{\alpha^3} \frac{e^{2\xi} - 6 + e^{-2\xi}}{(e^{\xi} + e^{-\xi})^3}
\]

which is bounded for all \((x, t) \in \Omega\).

5 Estimates for the remainder term

Let \(U\) be a solution of the reduced problem (2.10) and let us assume

\[
(5.58) \quad S(t) = \sup_{0 \leq x \leq \ell} |d(x, t)|.
\]

It is possible to state:

**Teorema 5.2.** If there exists a positive constant \(b\) such that

\[
(5.59) \quad |U_{xxt}(x, t)| \leq b,
\]

then, indicating by \(T_\varepsilon = \log(1/\varepsilon^k)\), with \(0 < k < \alpha\), for all \(1 \leq t < T_\varepsilon\) there exists a positive constant \(\Gamma\), independent from \(\varepsilon\), such that:

\[
(5.60) \quad 0 \leq S(t) \leq \Gamma \varepsilon^{1-k/\alpha} \log(\frac{1}{\varepsilon^k}).
\]

**Proof.** Let us consider function \(F\) defined in (2.12):

\[
(5.61) \quad F = \sin (d + U) - \sin U - \varepsilon U_{xxt}.
\]
According to (5.59), function $F(x,t,u)$ satisfies the following inequality:

$$|F(x,t)| \leq |d(x,t)| + \varepsilon b.$$  \hspace{1cm} (5.62)

So that, by means of (2.15)-(2.16) and (3.43) and since $t \leq \log(\frac{1}{\varepsilon k})$, one obtains:

$$d(x,t) \leq C \varepsilon \log(\frac{1}{\varepsilon k}) + \int_0^t \left[ \frac{1}{\alpha} + Be^{-m(t-\tau)} \right] |d(\tau,\xi)| \, d\tau.$$  \hspace{1cm} (5.63)

being $C = (B + \frac{1}{\alpha})b$.

So, considering that:

$$\int_0^t \left[ \frac{1}{\alpha} + Be^{-m(t-\tau)} \right] d\tau \leq \frac{1}{\alpha} \log(\frac{1}{\varepsilon k}) + \frac{B}{m},$$

by means of a linear generalization of Gronwall’s inequality (see f.i. (50)), it results:

$$d(x,t) \leq C \varepsilon \log(\frac{1}{\varepsilon k}) \left[ \frac{B}{m} + \frac{1}{\alpha} \right] = \Gamma \varepsilon^{1-k/\alpha} \log(\frac{1}{\varepsilon k})$$  \hspace{1cm} (5.64)

from which theorem is proved.

\[ \square \]

**Remark** Estimate (5.60) specifies the infinite time-intervals where the effects of diffusion vanish when $\varepsilon \rightarrow 0$. Indeed, the evolution of the superconductive model is characterized by diffusion effects which are of the order of $\varepsilon^{1-k/\alpha} \log(\frac{1}{\varepsilon k})$ with $0 < k < \alpha$.

**Remark** Formula (4.57) shows that the class of functions satisfying hypotheses of Theorem 5.2 is not empty.

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