Dynamics of the Shapovalov mid-size firm model

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1. Introduction

Consider the model proposed by V.I. Shapovalov in 2004 \cite{22}, describing the behavior of a mid-size firm:

\begin{align*}
\dot{x} &= -\sigma x + \delta y, \\
\dot{y} &= \mu x + \mu y - \beta xz, \\
\dot{z} &= -\gamma z + \alpha xy.
\end{align*}

(1)

Here $\alpha$, $\beta$, $\sigma$, $\delta$, $\mu$, $\gamma$ are some positive parameters, the variables $x$, $y$, $z$ denote the growth of three main factors of production: the loan amount $x$, fixed capital $y$, and the number of employees $z$ (as an increase in human capital). Later similar systems were studied, e.g., in \cite{26}.

As part of the study of (1) system, \cite{22, 4} was set the task of nonlinear analysis of this system and its limit dynamics in order to predict the stability of Shapovalov model (1) and determine the conditions when the system has some predictable dynamics (Shapovalov problem of a mid-size firm dynamics forecasting). The non-triviality of this problem lies in the fact that the system has an unstable equilibrium and chaotic dynamics.

2. System transformation

System (1) can be reduced to the Lorenz-like system

\begin{align*}
\dot{x} &= -cx + cy, \\
\dot{y} &= rx + y - xz, \quad \text{where } c = \frac{\sigma}{\mu}, r = \frac{\delta}{\sigma}, b = \frac{\gamma}{\mu}, \\
\dot{z} &= -bz + xy,
\end{align*}

(2)

using the following coordinate transformation

\begin{equation}
(x, y, z) \rightarrow \left( \frac{\mu}{\sqrt{\alpha\beta}} x, \frac{\mu\sigma}{\delta\sqrt{\alpha\beta}} y, \frac{\mu\sigma}{\delta\beta} z \right), \quad t \rightarrow \frac{t}{\mu}.
\end{equation}

(3)

System (2) differs from the classical Lorentz system \cite{19} in the sign of the coefficient at $y$ in the second equation, which is 1 here, while in the Lorentz system this coefficient is -1.

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Accordingly, the inverse transformation

\[(x, y, z) \rightarrow \left( \frac{\sqrt{\alpha\beta}}{\mu} x, r\frac{\mu}{\sqrt{\alpha\beta}} y, z \right), t \rightarrow \mu t \]  \hspace{1cm} (4)

reduces system (2) to system (1) with coefficients \( \sigma = c\mu, \delta = rcmu, \gamma = b\mu \). Remark, that transformation (3) and (4) does not change the direction of time (see, corresponding discussion in [14]).

3. Sustainability analysis

Further, we analyze the system (2) and apply the inverse transformation (4) to obtain conditions on the parameters of Shapovalov system (1). To solve the Shapovalov problem, using the standard stability analysis of dynamical systems, we calculate the equilibria of system (1). System (1) always has three equilibria

\[O^{(1)}_1 = (0, 0, 0), O^{(1)}_{2,3} = \left( \pm \sqrt{\frac{\gamma\mu(\sigma + \delta)}{\alpha\beta\sigma}}, \pm \sqrt{\frac{\gamma\mu\sigma(\sigma + \delta)}{\alpha\beta\delta}}, \mu(\sigma + \delta) \right), \]  \hspace{1cm} (5)

For the Jacobian matrix of system (1)

\[ J = \begin{pmatrix} -\sigma & \delta & 0 \\ \mu - \beta z & \mu & -\beta x \\ \alpha y & \alpha x & -\gamma \end{pmatrix}, \]  \hspace{1cm} (6)

the characteristic polynomial \( \det(J - Is) \) has form

\[ \chi(s, x, y, z) = s^3 + p_{1}^{(1)}(x, y, z)s^2 + p_{2}^{(1)}(x, y, z)s + p_{3}^{(1)}(x, y, z), \]  \hspace{1cm} (7)

where

\[ p_{1}^{(1)}(x, y, z) = \sigma + \gamma - \mu, \]
\[ p_{2}^{(1)}(x, y, z) = \sigma(\gamma - \mu) - \mu(\gamma + \delta) + \alpha^2x^2 + \alpha\delta z, \]  \hspace{1cm} (8)
\[ p_{3}^{(1)}(x, y, z) = -\gamma\mu(\sigma + \delta) + \alpha^2\sigma x^2 + \alpha^2\delta xy + \alpha\delta\gamma z. \]

**Lemma 1.** The equilibrium state \( O^{(1)}_1 = (0, 0, 0) \) of system (1) is unstable for all parameter values.

**Lemma 2.** If one of the relations

\[
\begin{bmatrix}
 r > c(3 - (c + b)) \\
 b - (c + 1)
\end{bmatrix}, \quad b > c + 1,
\]
\[
\begin{bmatrix}
 r < c(3 - (c + b)) \\
 b - (c + 1)
\end{bmatrix}, \quad 3 - c < b < c + 1
\]  \hspace{1cm} (9)

holds for system (2) then the equilibria \( O^{(1)}_{2,3} \) of system (1) are stable.

If both relations (9) are not satisfied, then the equilibria \( O^{(1)}_{2,3} \) of system (1) are unstable.
4. Analytical localization of attractor

Using the ideas presented in [11, 13, 12], we can prove the following.

**Lemma 3.** If $\gamma < 2\sigma$, then for any solution of system (1) we have the following estimate

$$\liminf_{t \to +\infty} \left[ z(t) - \frac{\alpha}{2\delta} x^2(t) \right] \geq 0.$$  \hspace{1cm} (10)

*Proof.* For system (2) and the Lyapunov function

$$V(x, z) = z - \frac{x^2}{2c},$$

we have

$$\dot{V}(x(t), z(t)) = -bV(x(t), z(t)) + \left(1 - \frac{b}{2c}\right)x^2(t).$$

If $b < 2c$ then

$$V(x(t), z(t)) \geq \exp(-bt)V(x(0), z(0)),$$

that yields estimate (10). Thus, the global attractor is located in the positive invariant set representing a parabolic cylinder (Fig. 1)

$$\Omega_1 = \left\{ (x, y, z) \in \mathbb{R}^3 \mid z \geq \frac{x^2}{2c} \right\}.$$  \hspace{1cm} (11)

By (4) we get the statement of Lemma for system (1). \hfill \Box

![Figure 1](image-url)  

Figure 1: Localization of the chaotic attractor of system (2) with parameters $r = 7.3171$, $b = 2.0909$, $c = 3.7273$ by the parabolic cylinder $\Omega_1$.

**Theorem 1.** If $\gamma > 2\mu$ and $\gamma < 2\sigma$, then all solutions of system (1) eventually fall in a bounded closed set.
Proof. For system (2) we consider Lyapunov function

$$V(x, y, z) = \frac{1}{2} \left[ A x^2 - 2 B x y + y^2 + (z - (r + (A + B)c - B))^2 \right], \quad (12)$$

where $A$ and $B$ are an arbitrary positive parameters.

If $A > B^2$ then

$$V(x, y, z) = \frac{1}{2} \left[ A(x - \frac{B}{A} y)^2 + (1 - \frac{B^2}{A})y^2 + (z - (r + (A + B)c - B))^2 \right] \to \infty$$
as $|(x, y, z)| \to \infty$. For an arbitrary solution $u(t) = (x(t), y(t), z(t))$ of system (2) by Lemma 3 we have

$$\dot{V}(x, y, z) = -(Ac + Br)x^2 - (Bc - 1)y^2 - (b - 2Bc)z^2 + (r + (A + B)c - B)bz - 2c B z \left( z - \frac{x^2}{2c} \right)$$

$$\leq -(Ac + Br)x^2 - (Bc - 1)y^2 - (b - 2Bc)z^2 + (r + (A + B)c - B)bz.$$

In order to have $Bc - 1 > 0$, $b - 2Bc > 0$, we choose $B \in \left( \frac{1}{c}, \frac{b}{2c} \right)$ under the assumptions $b > 2$, $b < 2c$. Suppose that $\varepsilon \in (0, b - 2Bc)$ and $\lambda = \min \{ Ac + Br, Bc - 1, (b - 2Bc) - \varepsilon \} > 0$. Then

$$\dot{V}(x, y, z) = -(Ac + Br)x^2 - (Bc - 1)y^2 - (b - 2Bc - \varepsilon)z^2 - \varepsilon z^2 + (r + (A + B)c - B)bz$$

$$= -(Ac + Br)x^2 - (Bc - 1)y^2 - (b - 2Bc - \varepsilon)z^2 - \left( \frac{\varepsilon}{2} - \frac{r + (A + B)c - B}{2\sqrt{\varepsilon}} \right) b^2$$

$$+ \frac{(r + (A + B)c - B)^2 b^2}{4\varepsilon} \leq -\lambda (x^2 + y^2 + z^2) + \frac{(r + (A + B)c - B)^2 b^2}{4\varepsilon}.$$

Suppose that $x^2 + y^2 + z^2 \geq R^2$. Then a positive $\varkappa$ exists such that

$$\dot{V}(x, y, z) \leq -\lambda R^2 + \frac{(r + (A + B)c - B)^2 b^2}{4\varepsilon} < -\varkappa \quad \text{for} \quad R^2 > \frac{1}{\lambda} \frac{(r + (A + B)c - B)^2 b^2}{4\varepsilon}.$$

We choose $\eta > 0$ such that

$$\{(x, y, z) \mid V(x, y, z) \leq \eta \} \supset \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2 \}.$$

Thus, the relation $x^2 + y^2 + z^2 \leq R^2$ implies that

$$A(x - \frac{B}{A} y)^2 + (1 - \frac{B^2}{A})y^2 + (z - (r + (A + B)c - B))^2 =$$

$$A(x - \frac{B}{A} y)^2 + (1 - \frac{B^2}{A})y^2 + z^2 - 2(r + (A + B)c - B) z + (r + (A + B)c - B)^2 \leq 2\eta.$$

According to the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$A(x - \frac{B}{A} y)^2 \leq 2A \left( x^2 + \frac{B^2}{A^2} y^2 \right) \leq 2A \left( 1 + \frac{B^2}{A^2} \right) R^2$$

and, since

$$-2(r + (A + B)c - B) z \leq 2(r + (A + B)c - B) |z| \leq 2(r + (A + B)c - B) R,$$

it is sufficient to choose

$$\eta \geq \frac{1}{2} \left[ (2A + 2 + \frac{B^2}{A}) R^2 + 2(r + (A + B)c - B) R + (r + (A + B)c - B)^2 \right].$$
Hence, if \( A > B^2 \) and if \( R, \eta \) are chosen as shown above, then system (2) has the following compact ellipsoidal absorbing set:

\[
B_0 = \left\{ (x, y, z) \middle| V(x, y, z) = \frac{1}{2} \left[ Ax^2 - 2Bxy + y^2 + (z - (r + (A + B)c - B))^2 \right] \leq \eta \right\}.
\]

By (4) we get the statement of Theorem for system (1). Thus, system (1) generates a dynamical system, the solutions of system (1) exist for \( t \in [0, +\infty) \) and system (1) possesses a global attractor, which contains all equilibria (see e.g. [15, 2]).

The obtained absorbing set can be further refined (see, e.g., [26, 27]).

The problem of forecasting for established (limiting) behaviour of the dynamical system can be solved via localization of attractors of this system [15]. While trivial attractors (stable equilibrium points) can be easily found analytically or numerically, the search of periodic and chaotic attractors can turn out to be a challenging problem. For numerical localization of an attractor, one needs to choose an initial point in the basin of attraction and observe how the trajectory, starting from this initial point, after a transient process visualizes the attractor. Self-excited attractors, even coexisting in the case of multistability [21], can be revealed numerically by the integration of trajectories, started in small neighbourhoods of unstable equilibria, while hidden attractors have the basins of attraction, which are not connected with equilibria and are “hidden somewhere” in the phase space [13, 7, 15, 5]. Remark that in numerical computation of trajectory over a finite-time interval, it is difficult to distinguish a sustained chaos from a transient chaos (a transient chaotic set in the phase space, which can persist for a long time) [3]. The time interval for reliable computation with 16 significant digits and error \( 10^{-4} \) is estimated as \([0; 36]\), and reliable computation for a longer time interval, e.g. \([0; 10000]\) in [17], is a challenging task that requires significant increase of the precision of the floating-point representation and the use of supercomputers [4]. Analytical aspects of this problem are concerned with the so-called shadowing theory (see e.g. [20]) which for some classes of systems can guarantee the existence of the “true” trajectory in the vicinity of its approximation. Analytical procedures for determining of the global stability areas are able to mitigate the influence of computer errors and thus make a reliable forecast of system dynamics.

5. Global stability

From the mathematical point of view, the problem of search of conditions for the ultimate stationary behavior of Shapovalov model (1) corresponds to the tend of all its trajectories toward a stationary set. The nonlinearity of this system and the presence of one unstable equilibria \( O_1(0, 0, 0) \) for all values of its parameters makes this problem nontrivial. Furthermore, for certain values of the parameters of this system, it was shown in [22] that this problem has a negative solution, and chaotic behavior is observed in system (1). In our work, using special analytical methods, we demonstrate the effective obtaining of analytical conditions for the global stability of system (1) when all its trajectories tend to equilibria.

**Theorem 2.** If for system (1) relations

\[
\begin{align*}
(\gamma - \sigma) \left( \frac{\gamma}{\mu} + 1 \right) &< \delta < (\gamma + \sigma) \left( \frac{\gamma}{\mu} - 1 \right), \\
2\mu &< \gamma < 2\sigma,
\end{align*}
\]

hold, then any solution tends to some equilibrium state for \( t \to +\infty \).
Proof. Consider system (2) obtained of (1) by changing variables (3). For system (2) the eigenvalues of Jacobian matrix have cumbersome expressions. So we follow the ideas from [24, 9, 10] and use transformation $S$

$$S = \begin{pmatrix}
-\frac{1}{b+1} & 0 & 0 \\
-\frac{b}{b+1} & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix} \tag{14}
$$

where $a = \frac{c}{\sqrt{(1+b)(c-b)+rc}}$, and condition

$$rc > (b+1)(b-c) \tag{15}$$

is satisfied. Then the symmetrized Jacobian matrix of this system $\frac{1}{2}(SJS^{-1} + (SJS^{-1})^*)$ has the following eigenvalues

$$\lambda_2 = -b, \lambda_{1,3} = -\frac{c-1}{2} \pm \frac{1}{2} \left( (2b+1-c)^2 + a^2 \left( \frac{b+1}{c} x + y \right)^2 + \left( az - \frac{2b}{a} \right)^2 \right)^{\frac{1}{2}}. \tag{16}$$

The inequalities

$$2(\lambda_1 - \lambda_2) \geq 2b+1-c+|2b+1-c| \geq 0 \tag{17}$$

imply $\lambda_1 \geq \lambda_2$. From (16) we get the ratio

$$2(\lambda_1 + \lambda_2) = -(c-1+2b) + \left( (2b+1-c)^2 + a^2 \left( \frac{b+1}{c} x + y \right)^2 + \left( az - \frac{2c}{a} \right)^2 \right)^{\frac{1}{2}}. \tag{18}$$

Using the famous inequality $\sqrt{k+l} \leq \sqrt{k} + \frac{l}{2\sqrt{k}}, \forall k > 0, l \geq 0$, we get an estimate

$$2(\lambda_1 + \lambda_2) \leq -(c-1+2b) + [(c+1)^2 + 4cr]\frac{1}{2} +$$

$$+ \frac{2}{[(c+1)^2 + 4cr]^{\frac{3}{2}}} \left[ -cz + \frac{a^2 z^2}{4} + \frac{a^2}{4} \left( \frac{b+1}{c} x + y \right)^2 \right]. \tag{19}$$

We introduce the function $V(x, y, z) = \frac{\theta(x,y,z)}{[(c+1)^2 + 4cr]^\frac{3}{2}}$, where

$$\theta(x, y, z) = h_1 x^2 + (-bh_2 + h_3) y^2 + h_3 z^2 + \frac{h_2}{4b} x^4 - h_2 x^2 y - h_2 h_4 x y - \frac{c}{b} z, \tag{20}$$

$h_j (j = 1, 4)$ are some positive real parameters. Then

$$2(\lambda_1 + \lambda_2) + 2\dot{V} \leq -(c-1+2b) - s(c-1) + (1-s) [(c+1)^2 + 4cr]\frac{1}{2} +$$

$$+ \frac{2(1-s)}{[(c+1)^2 + 4cr]^{\frac{3}{2}}} \left[ W(x, y, z) + \dot{\theta} \right], \tag{21}$$

where $W(x, y, z) = -cz + \frac{a^2 z^2}{4} + \frac{a^2}{4} \left( \frac{b+1}{c} x + y \right)^2$, and the function $\theta(x, y, z)$ is such that relation

$$W(x, y, z) + \dot{\theta} \leq 0, \quad \forall x, y, z \geq \frac{x^2}{2c} \tag{22}$$

holds. By (19), we get the following estimate

$$2(\lambda_1 + \lambda_2 + \dot{V}) \leq -(c-1+2b) + [(c+1)^2 + 4cr]\frac{1}{2}. \tag{23}$$
The relation $\lambda_1 + \lambda_2 + \dot{V} < 0$ is satisfied if for the parameters of system (2) the following condition

$$-(c - 1 + 2b) + \left[(c + 1)^2 + 4cr\right]^\frac{1}{2} < 0$$

holds. This is the case if relations

$$
\begin{cases}
  cr < (b + c)(b - 1), \\
  c + 2b > 1, \\
  b > 1
\end{cases}
$$

are true. From here, taking into account (15) and the condition of Theorem 1, we obtain domain of parameters

$$
\begin{cases}
  (b + 1) \left(\frac{b}{c} - 1\right) < r < \left(\frac{b}{c} + 1\right)(b - 1), \\
  2 < b < 2c
\end{cases}
$$

in which system (2) is globally stable.

By (4) we get the statement of Theorem for system (1).

Note that by the Fishing principle [11, 15, 16] we obtain condition $3c > 2b - 1$ which is necessary and sufficient for the existence of homoclinic orbit (i.e. for this condition there exist a certain $r$ such that system (2) has a homoclinic orbit). This conditions is valid under the second conditions from (26) and the one can use a binary search [16].

![Figure 2: Parameters of system (2) corresponding absorbing set $B$ and stability region (26).](image)

The figure 2 shows the domain of values of parameters (26), for which system (2) has an absorbing set $B$. At each point in this region, you can choose the value of the parameter $r$ such that if the first relation in (26) is satisfied, then system (2) and, accordingly, system (1), is stable, otherwise, it is unstable. The points $N_i = N_i(c, b, r)$, $(i = 1, 5)$ correspond to the values of the parameters of system (2), at which the various behaviors are demonstrated. So the point $N_1(1.9524, 0.4762, 0.2439)$ corresponds to Shapovalov’s parameters [22]

$$
\alpha = 5, \beta = 8, \gamma = 1, \delta = 1, \mu = 2.1, \sigma = 4.1,
$$
and the point $N_2(1.9524, 1.1117, 0.2439)$ corresponds to parameters

$$\alpha = 4, \beta = 8, \gamma = 2.3345, \delta = 1, \mu = 2.1, \sigma = 4.1$$

(28)

from [4]. For the values of parameters (27) and (28) it is not possible to construct an absorbing set and, accordingly, in these cases, system [1] may not be dissipative. The points $N_3, N_4, N_5$ lie in the region of dissipativity. The point $N_3(2.1000, 3.1500, 4.0000) \in D_1$ corresponds to the stable regime, $D_1$ is the domain in which the inequalities $(b + 1) \left(\frac{b}{c} - 1\right) < r < (\frac{b}{c} + 1) (b - 1)$ are necessary to ensure the stability of system (1). The point $N_4(3.7273, 2.0909, 7.3171) \in D_2$ corresponds to the unstable regime, in the domain $D_2$ the inequalities $0 < r < (\frac{b}{c} + 1) (b - 1)$ imply the stability of system [2]. Note that for the classical parameters of the Lorenz system $\sigma = 10, b = 8/3, r = 24$ (the point $N_5 \in D_2$) system (1) is dissipative and its equilibrium states are unstable.

Thus, we showed that inside absorbing set $\mathcal{B}$ all the trajectories of system [2] not only enter in $\mathcal{B}$, but also tend to stationary set defined by the relations (26). A similar conclusion follows for system [1] by inverse transformation (4) to system [2].

6. Chaotic dynamics

Along with the statement of the problem of studying the stability of Shapovalov model [1] showed in [22] that this system exhibits chaotic behavior for the values of parameters (27), which, taking into account the transformation (3), correspond the following values of the parameters of system (1) $r = 0.2439, b = 0.4762, c = 1.9524$. Using numerical experiments, we have analyzed the chaotic dynamics of system (1) and visualize a self-excited attractor for Shapovalov’s values of parameters (27). On this attractor, along with the corresponding solution of system (1), we have obtained some estimates, which allows us to calculate such characteristics as the dimension of the attractor and entropy.

For a dynamical system $\varphi^t(u_0) = u(t, u_0)$ generated by system (1) we consider the concept of the finite-time Lyapunov dimension [6, 8], which is convenient for carrying out numerical experiments with finite time:

$$\dim_{\text{FTL}}(t, u) = j(t, u) + \frac{\text{LE}_i(t, u) + \cdots + \text{LE}_i(t, u)(t, u)}{|\text{LE}_i(t, u)|},$$

where $j(t, u) = \max\{m : \sum_{i=1}^{m} \text{LE}_i(t, u) \geq 0\}$, and $\{\text{LE}_i(t, u)\}_{i=1}^3$ is an ordered set of finite-time Lyapunov exponents (FTLEs). Then the finite-time Lyapunov dimension $\dim_{\text{FTL}}$ of dynamical system generated by (1) on compact invariant set $\mathcal{A}$ is defined as: $\dim_{\text{FTL}}(t, \mathcal{A}) = \sup_{u \in \mathcal{A}} \dim_{\text{FTL}}(t, u)$. According to Douady–Oesterl´e theorem, for any fixed $t > 0$ the FTLD is an upper estimate of the Hausdorff dimension: $\dim_{\text{H}} \mathcal{A} \leq \dim_{\text{FTL}}(t, \mathcal{A})$. The best estimation is called the Lyapunov dimension [6]:

$$\dim_{\text{LY}} \mathcal{A} = \inf_{t \geq 0} \sup_{u \in \mathcal{A}} \dim_{\text{FTL}}(t, u).$$

In Fig. [3] is shown the grid of points $\mathcal{C}_{\text{grid}}$ filling the attractor: the grid of points fills cuboid $\mathcal{C} = [-1, 1] \times [-2.5, 2.5] \times [0.1, 2.5]$ rotated by 45 degrees around the $z$-axis, with the distance between points equals to 0.5 (see Fig. [3]). The considered time interval is $[0, T = 500], k = 1000, \tau = 0.5$, and the integration method is MATLAB ode45 with predefined parameters. The infimum on the time interval is computed at the points $\{t_k\}_1^N$ with time step $\tau = t_{i+1} - t_i = 0.5$. Note that if for a certain time $t = t_k$ the computed trajectory is out of the cuboid, the corresponding value of finite-time local Lyapunov dimension is not taken into account in the computation of maximum of the finite-time local Lyapunov dimension (e.g. if there are trajectories with initial conditions in cuboid, which tend to infinity).

For the considered set of parameters we use MATLAB realization of the adaptive algorithm of finite-time Lyapunov dimension and Lyapunov exponents computation [8] and obtain maximum
of the finite-time local Lyapunov dimensions at the points of grid \( \max_{u \in C_{\text{grid}}} \dim_{L}(t, u) \), at the time points \( t = t_k = 0.5 \kappa \) (\( k = 1, \ldots, 1000 \)). For parameters \( r = 0.2439, b = 0.4762, c = 1.9524 \) we get \( \max_{u \in C_{\text{grid}}} \dim_{L}(100, u) = 2.0699 \) \( \max_{u \in C_{\text{grid}}} \dim_{L}(500, u) = 2.0676 \).

Figure 3: Localization of the chaotic attractor of system (2) with parameters \( r = 0.2439, b = 0.4762, c = 1.9524 \) by the cuboid \( C \) and the corresponding grid of points \( C_{\text{grid}} \).

This estimation is consistent with hypothesis on Lyapunov dimension of a self-excited attractor and refined the result obtained in [23].

Conclusion

Complexity of analysis of dynamics of financial and economic systems is due to possible presence of multistability, when for different initial data the system trajectories could converge to distinct attractors. The coexistence of local attractors complicates forecasting the behavior of a dynamical system and estimating of its quantitative characteristics, for instance, the Lyapunov dimension of the attractor.

Recent results obtained in the field of nonlinear methods of dynamical systems theory allow one to successfully analyze global and local dynamics using analytical procedures. The above studies have extensive applications including the analysis of global dynamics in a number of economic models [18]. One of the efficient methods for analysis of these models is the method of discovering global attractors by construction of their absorbing sets, with subsequent implementation of effective analytical and numerical procedures of investigation for bounded sets of initial conditions.

In this paper, we performed a global analysis of stability of the Shapovalov model. We derived the absorbing set, which made it possible to localize the global attractor, and the domains of parameters for which stability and instability are observed. To characterize the chaotic dynamics, we numerically estimate the Lyapunov dimension of attractor.

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