Energy-momentum and angular momentum of Gödel Universes

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Abstract

We discuss the Einstein energy-momentum complex and the Bergmann-Thomson angular momentum complex in general relativity and calculate them for space-time homogeneous Gödel universes. The calculations are performed for a dust acausal Gödel model and for a scalar-field causal Gödel model. It is shown that the Einstein pseudotensor is traceless, not symmetric, the gravitational energy ”density” is negative and that the gravitational Poynting vector vanishes. Significantly, the total (gravitational and matter) energy ”density” for the acausal model is zero while for the causal model it is negative. The Bergmann-Thomson angular momentum complex does not vanish for both Gödel models.

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I. INTRODUCTION

The problem of the energy-momentum of gravitational field has a very long tradition in general relativity. The point is that the gravitational field can be made locally vanish and so one is always able to find the frame in which the energy-momentum of gravitational field is zero while in the other frame it is not. In other words, the physical objects which can describe this situation cannot be tensors, i.e., the objects which vanish in all the frames provided they vanish in at least one of them. The proposed quantities which actually fulfill the conservation law of matter appended with gravitational field are called energy-momentum complexes while their gravitational parts are called gravitational field pseudotensors. An energy-momentum complex is then the sum of the obvious energy-momentum tensor of matter and an appropriate pseudotensor. Unfortunately, the choice of the gravitational field pseudotensor is not unique and because of that quite a few definitions of these pseudotensors have been proposed. Historically, one of the earliest definitions was given by Einstein followed by Landau-Lifshitz [1], Møller [2], Papapetrou [3], Bergmann-Thomson [4], Weinberg [5] and Bak-Cangemi-Jackiw [6], for example. Among them only those of Landau-Lifshitz, Weinberg and Bak-Cangemi-Jackiw are symmetric, but only in holonomic frames. In particular, the Einstein pseudotensor is not symmetric. The problem of the energy-momentum of the gravitational field can also be extended to the standard field theory problem of the angular momentum. The appropriate expressions have been proposed of which the Bergmann-Thomson angular momentum complex [4] being the most widely used.

Because of the freedom of a choice of pseudotensors and the fact that they usually give different results for the same type of spacetime some authors [7–9] have proposed an alternative approach to the problem in which they defined the quantities which describe the generalized energy-momentum content of the gravitational field and which are tensors. These quantities are called gravitational superenergy tensors and gravitational supermomentum tensors.

It seems interesting to make a comparative analysis of the results which can be ob-
tained in the energy pseudotensor approach with that of the superenergy tensor approach for various models of spacetime. The question arises whether the appropriate physical conclusions obtained on the level of the energy-momentum are preserved on the level of the supermomentum and vice versa.

In this context the canonical superenergy tensor and the canonical angular supermomentum tensor for space-time homogeneous universes of Gödel type have been calculated and discussed by these authors recently [10–12]. The task of this paper is to calculate the appropriate pseudotensors (complexes) and make the comparison of the physical results.

It is not random that we have chosen Gödel universes as the example models to compare the results. Firstly, Gödel universes rotate and so they should have non-zero angular momentum. Secondly, they possess closed timelike curves (CTCs) which is a big peculiarity and may have interesting consequences onto the results. In particular, the CTCs should be avoided according to the Hawking’s chronology protection conjecture [13] and this somehow may be related to the energy-momentum and the angular momentum in the same way as it was the case for the superenergy and the supermomentum in Ref. [10].

On the other hand, following Hawking’s chronology protection conjecture it has been shown that it is possible to avoid CTCs in many gravitational theories. This is the case in minimally coupled to gravity scalar field theories [14], in quadratic gravity theories [15], in five-dimensional gravity theories [16], or in string/M-theory inspired gravitational theories [17–19]. In Ref. [19], for example, it has been shown that CTCs can be avoided for brane models with the negative total effective energy density. One should also emphasize that Gödel universes attracted attention of many authors recently, just in the context of conventional gravity theory [20,21].

In this paper we study field theoretical quantities such as the energy-momentum and the angular momentum for Gödel universes. In order to fulfill the task we apply the Einstein energy-momentum complex of gravitation and matter and the Bergmann-Thomson angular momentum complex. These quantities seem to be the best of all which have been proposed so far, including the so-called “quasi-local quantities”. We perform our analysis in orthonor-
mal frames (anholonomic frames) which requires adopting the original expressions to these frames. The obvious way to express Einstein complex and Bergmann-Thomson complex in anholonomic frames is to use the formalism of the tensor-valued (or pseudotensor-valued) exterior differential forms which we do in Section II. In Section III we apply the obtained quantities for Gödel spacetimes. In Section IV we conclude and make some comparison of energetic quantities with superenergetic quantities for Gödel spacetimes which have been obtained earlier [10].

II. ENERGY-MOMENTUM AND ANGULAR MOMENTUM IN GENERAL RELATIVITY

As it was already mentioned in the Introduction the gravitational field does not possess the proper definition of an energy-momentum tensor and an angular momentum tensor and one usually defines some energy-momentum pseudotensors. The thorough investigations of the energy-momentum problem in general relativity suggest [2,22] that the most satisfactory of all the possible gravitational energy pseudotensors already listed in the Introduction is the canonical gravitational energy-momentum pseudotensor of Einstein \( \varepsilon t^i_k \) (see e.g. [1]). In consequence, the best of all the proposed gravitational angular momentum pseudotensors is considered to be the Bergmann-Thomson pseudotensor [4] since it is constructed of the Einstein pseudotensor. We follow this point of view and will discuss a particular application of these pseudotensors to Gödel universes.

Independently, in general relativity one can also introduce the canonical gravitational superenergy tensor and the canonical gravitational angular supermomentum tensor. This was done in a series of one of the authors’ papers [8,9]. It appeared that the idea of the superenergy and the angular supermomentum tensors was universal: to any physical field which possesses an energy-momentum tensor or a pseudotensor which is constructed out of the Levi-Civita connection one can always attribute a corresponding superenergy tensor and a corresponding angular supermomentum tensor.
The canonical superenergy and angular supermomentum tensors prove very useful for the local analysis of the gravitational and matter fields. They also admit suitable global integral superenergetic quantities for gravity and matter [9].

In this paper we confine ourselves to the analysis of the energetic quantities for Gödel spacetimes. In fact, we calculate the energy-momentum “densities” and the angular momentum “densities” for these spacetimes. In order to calculate these quantities we use the expressions for Einstein energy-momentum pseudotensor and Bergmann-Thomson angular momentum complex in an anholonomic form. The appropriate formulas which are valid in an arbitrary frame \((\theta^i)\) can be obtained by the application of the tensor-valued differential forms [23,24].

In the language of the differential forms the Einstein equations read as

\[
\frac{1}{2} \Omega^j_k \wedge \eta_{ij}^k = -\chi T_i, \quad (2.1)
\]

where

\[
\Omega^k_i = \frac{1}{2} R^k_{lmn} e^m \wedge e^n \quad (2.2)
\]

is the curvature 2-form of the Riemannian (or Levi-Civita) connection 1-form \(\omega^i_k = \Gamma^i_{kl} \theta^l\), \(\chi = 8\pi \ (G = c = 1)\), and

\[
\eta_{ij}^k = g^{kl} \eta_{ijl} = g^{kl} e^r \eta_{ijr} = g^{kl} e^r \sqrt{|g|} \epsilon_{ijr} \quad (2.3)
\]

is a pseudotensorial 1-form with \(\epsilon_{ijr}\) being the totally antisymmetric Levi-Civita pseudotensor. In the following we will use an anholonomic Lorentzian frame \((e^i)\) defined by

\[
g = \eta_{ik} e^i \otimes e^k, \quad (2.4)
\]

where \(g\) is an arbitrary spacetime metric and \(\eta_{ik}\) is Minkowski metric. In (2.1) \(T_i := T^k_i \eta_k\) is the energy-momentum 3-form of matter with \(T^k_i\) being the symmetric energy-momentum tensor of matter, and

\[
\eta_i = \frac{1}{3} e^i \wedge \eta_{ij} = \frac{1}{6} e^j \wedge e^k \wedge \eta_{ijk} \quad (2.5)
\]
is a pseudotensorial 3-form.

Decomposing (2.1) in the basis of the 3-forms $\eta$, one can easily get the Einstein equations in an ordinary tensorial form

$$G_{ik} = \chi T_{ik},$$

(2.6)

where

$$G_{ik} := R_{ik} - \frac{1}{2} g_{ik} R$$

(2.7)

are the components of the Einstein tensor. It is known that the Einstein equations (2.1) can also be transformed to the superpotential form

$$d \left( \frac{1}{2\chi} \eta_{ij}^k \wedge \omega^j_k \right) = T_i + \frac{1}{2\chi} \left( \eta_{pj}^k \wedge \omega^j_k \wedge \omega^p_i + \eta_{ij}^p \wedge \omega^k_p \wedge \omega^j_k \right).$$

(2.8)

The equations (2.8) are independent of coordinates (or frames) and define the canonical 3-form of the gravitational energy-momentum

$$E_t := \frac{1}{2\chi} \left( \eta_{pj}^k \wedge \omega^j_k \wedge \omega^p_i + \eta_{ij}^p \wedge \omega^k_p \wedge \omega^j_k \right),$$

(2.9)

and the 2-form

$$F_U := \frac{1}{2\chi} \eta_{ij}^k \wedge \omega^j_k$$

(2.10)

gives the so-called Freud superpotentials.

The sum

$$E_t + T_i := E K_i$$

(2.11)

composes the 3-form $E K_i$ which we call the canonical Einstein energy-momentum complex of gravitation and matter. From (2.8) and (2.11) we have

$$E K_i = d F U_i.$$ 

(2.12)

A troublesome fact is that the 3-form $E t_i$ and, in consequence, the 3-forms $d F U_i$ and $E K_i$ are non-tensorial. This means gravitational energy-momentum is not localizable. In fact, only
the global energy-momentum can be properly defined in the asymptotically flat spacetimes (at null and spatial infinity). From (2.12) one immediately gets the local, differential energy-momentum conservation laws for gravity and matter (also called weak conservation laws – they hold in any reference frame) in the form

\[ dE_Ki = 0. \]  
(2.13)

The integration of (2.13) over a compact 4-dimensional domain \( \Omega \) leads to Synge’s integral conservation laws \[27\]

\[ \int_{\partial V} (Et_i + T_i) = 0, \]  
(2.14)

where \( \partial V \) denotes a 3-dim outward oriented boundary of the 4-dim domain \( V \).

It is interesting to note that the integrals on the left-hand side of (2.14) have no geometrical meaning, but they are zero in any reference frame, i.e., they behave like scalars. Moreover, for a closed system [2], after the appropriate choice of the domain \( V \) one obtains from (2.14) the ordinary conservation laws for energy-momentum of matter and gravitation.

In the basis of the 3-forms \( \eta_i \) one can decompose the canonical 3-form of the gravitational energy-momentum as follows

\[ Et_i = E t_i^q \eta_q, \]  
(2.15)

and its components form the energy-momentum pseudotensor of Einstein. In a Lorentzian frame \((e^i)\) we have

\[ Et_i^q = \frac{1}{2\chi} \left( g^{kl}_{ij} \eta^{qrs} \eta_{jklr} \gamma^j_{is} + g^{pl}_{ij} \eta^{qrs} \eta_{jplr} \gamma^k_{qs} \gamma^j_{ls} \right), \]  
(2.16)

where \( \gamma \)'s denote Ricci rotation coefficients, i.e., Levi-Civita connection in this frame. Let us also mention that in a Lorentzian frame \((e^i)\) one has \( g = -1, \eta^{0123} = 1, \eta_{0123} = -1, g^{ik} = \eta^{ik}, g_{ik} = \eta_{ik} \).

In section III we will use the formula (2.16) to calculate the energy-momentum “densities” for Gödel spacetimes in an appropriate Lorentzian frame.
Now we turn into the problem of angular momentum in general relativity which is more complicated than the problem of energy-momentum (see e.g. [25]). The main new obstacle is that the coordinates \((x^i)\) do not form the components of any global radius vector \(\vec{r}\) so even an ordinary field theoretical matter angular momentum

\[
mM^{i\alpha a} = \sqrt{|g|} \left( x^i T^{\alpha a} - x^k T^{ia} \right)
\]  

(2.17)
does not form a tensor density. In general relativity one can define the radius vector only locally. For example, the normal coordinates \((y^i)\) form the components of the local radius vector \(\vec{r}\) with respect to their origin.

In the following we will define the components \((r^i)\) of the local radius vector \(\vec{r}\) with respect to the Lorentzian frame \((e^i)\) by

\[
Dr^i = e^i,
\]  

(2.18)
where \(D\) is the exterior covariant derivative.

In the normal coordinates at the point \(P\), \(\text{NC}(P)\), this gives the equality between the normal coordinates and the local radius vector

\[
r^i = y^i.
\]  

(2.19)

Apart from this first obstacle there is another. In general, it is difficult to define invariantly the angular momentum in an asymptotically flat spacetimes and also the resulting global angular momentum integrals in radiative spacetimes do not converge (see e.g. [25]). However, these problems can be avoided in the case of closed systems provided one applies e.g., the definition of the angular momentum given by Bergmann and Thomson. This is what we now call the Bergmann-Thomson angular momentum complex. Because of the fact that it is closely related to the Einstein energy-momentum complex we call it canonical, too. Using the Bergmann-Thomson angular-momentum complex, one can also reflect the temporal changes of the global angular momentum in asymptotically flat spacetimes [26].
Bearing in mind all the arguments for the Bergmann-Thomson angular momentum complex we will apply this complex to calculate angular momentum densities for Gödel spacetimes in a Lorentzian frame \( (e^i) \). In order to get a suitable formula in a Lorentzian frame we start with equations (2.11) and (2.12) with raised index \( i \) to get

\[
\begin{align*}
  r^i \left( E^k t^k + T^k \right) - r^k \left( E^i t^i + T^i \right) &= r^i d_F U^k - r^k d_F U^i, \\
  \text{or} \\
  r^i \left( E^k t^k + T^k \right) - r^k \left( E^i t^i + T^i \right) + dr^i \wedge_F U^k - dr^k \wedge_F U^i &= d \left( r^i U^k - r^k U^i \right). 
\end{align*}
\]

The equations (2.21) hold in any reference frame (both holonomic and anholonomic) and give the local, differential conservation laws for the angular momentum of gravitation and matter

\[
\begin{align*}
  d \left[ r^i \left( E^k t^k + T^k \right) - r^k \left( E^i t^i + T^i \right) + dr^i \wedge_F U^k - dr^k \wedge_F U^i \right] &= 0, \\
  \text{and the integral Synge’s conservation laws} \ [27] \\
  \int_{\partial V} \left[ r^i \left( E^k t^k + T^k \right) - r^k \left( E^i t^i + T^i \right) + dr^i \wedge_F U^k - dr^k \wedge_F U^i \right] &= 0. 
\end{align*}
\]

The 3-form (2.21) gives the “densities” of the total canonical angular momentum for gravitation and matter. Decomposing it in the basis \( \eta \) one can obtain the antisymmetric in the first two indices components \( _{BT} M^{ika} = -_{BT} M^{kia} \) of the canonical Bergmann-Thomson angular momentum complex of gravitation and matter in Lorentzian frames as follows

\[
\begin{align*}
  d \left( r^i d_F U^k - r^k d_F U^i \right) :=_{BT} M^{ikt} \eta_t, 
\end{align*}
\]

where

\[
\begin{align*}
  _{BT} M^{ika} &= \frac{1}{2X} \left[ \eta_m m n g^{r s} \gamma^m_{r t} \left( \eta^{a i m} g^{k l} - \eta^{a k m} g^{i l} \right) + \eta^{a b s} \eta_{l m r s} \gamma^m_{n t} g^{n r} r^p \left( g^{i l} \gamma^k_{t p} - g^{k l} \gamma^i_{t p} \right) \\
  &\quad + \left( r^i g^{k l} - r^k g^{i l} \right) \eta^{a p m} \left( \eta_{j m n} g^{r m} \gamma^t_{l s} \gamma^i_{t p} - \eta_{j m n} g^{r m} \gamma^t_{l s} \gamma^j_{t p} - \frac{1}{2} \eta_{j m n} g^{r m} R^j_{l t n p} \right) \right],
\end{align*}
\]

and as \( g^{k l} \) one should take Minkowski metric \( \eta^{k l} \). In order to get gravitational part of this complex only, one should subtract the material part (2.17). In Section III we will use the formula (2.25) in order to calculate the canonical angular momentum densities for Gödel spacetimes.
III. ENERGY-MOMENTUM AND ANGULAR MOMENTUM COMPLEXES OF

GÖDEL UNIVERSES

Following Ref. [10] we will perform the calculations of the energy-momentum complex and the angular momentum complex for generalized Gödel spacetimes in a Lorentzian frame \((e^i)\) defined by

\[
\begin{align*}
    e^0 &= dt' + H(x)dy \\
    e^1 &= dx \\
    e^2 &= D(x)dy \\
    e^3 &= dz,
\end{align*}
\]

where

\[
H(x) = e^{mx}, \quad D(x) = \frac{e^{mx}}{\sqrt{2}},
\]

and \(m = \text{const.}\) The appropriate line element in the coordinates \((t', x, y, z)\) reads as

\[
ds^2 = -[dt' + H(x)dy]^2 - D(x)^2dy^2 + dx^2 + dz^2.
\]

The only non-vanishing Ricci rotation coefficients [10] in the Lorentzian frame (3.1) are

\[
\begin{align*}
    \gamma^0_{12} = \gamma^1_{20} &= \gamma^1_{02} = \frac{m}{\sqrt{2}}, \\
    \gamma^0_{21} = \gamma^2_{10} &= \gamma^2_{01} = -\frac{m}{\sqrt{2}}, \\
    \gamma^1_{22} &= -\gamma^2_{12} = -m.
\end{align*}
\]

According to (3.15) and (3.10) one has to put \(m = \sqrt{2}\Omega\) for an acausal Gödel model, and \(m = 2\Omega\) for a causal model [14,15].

In order to learn about causality one has to make a change of coordinates from \((t', x, y, z)\) into \((t, r, \psi, z)\) as follows

\[
x = \frac{1}{m} \ln \left[ \cosh (mr) + \cos \psi \right]
\]
\[ y = -\frac{\sqrt{2}}{m} \frac{\sin \psi \sinh (mr)}{\cosh (mr) + \cos \psi} \]  
(3.6)

\[ t' = t + \frac{\sqrt{2}}{m} \left[ 2 \arctg \left( e^{-mr \tan \frac{\psi}{2}} \right) - \psi \right] \]  
(3.7)

\[ z = z \]  
(3.8)

which brings the metric (3.3) into the form

\[ ds^2 = -dt^2 - 2H(r)dt \, d\psi + G(r) \, d\psi^2 + dr^2 + dz^2, \]  
(3.9)

where

\[ G(r) = D^2(r) - H^2(r) = \left[ \frac{1}{m} \sinh (mr) \right]^2 - \left[ \frac{4\Omega}{m^2} \sinh^2 \left( \frac{mr}{2} \right) \right]^2 \]
\[ = \frac{4}{m^2} \sinh^2 \left( \frac{mr}{2} \right) \left[ 1 + \left( 1 - \frac{4\Omega^2}{m^2} \right) \sinh^2 \left( \frac{mr}{2} \right) \right], \]  
(3.10)

with \( m \) and \( \Omega \) constants. In fact, \( m \) is a parameter which may distinguish between causal and acausal Gödel spacetimes. For a perfect-fluid source it is restricted by [14]

\[ 0 \leq m^2 \leq 2\Omega^2, \]  
(3.11)

while for a scalar field as the source of gravity it has the values [14]

\[ 2\Omega^2 \leq m^2 \leq 4\Omega^2. \]  
(3.12)

Taking

\[ m^2 = 2\Omega^2, \]  
(3.13)

one gets the original Gödel spacetime [11] in which we have an acausal region and \( G(r) \) in Eq. (3.10) becomes negative. This region appears for a radial coordinate

\[ r > r_c, \quad \text{where} \quad \sinh^2 \left( \frac{mr_c}{2} \right) = 1. \]  
(3.14)

However, in the case of the scalar field source one can take

\[ 4\Omega^2 = m^2, \]  
(3.15)
which gives

\[ G(r) = D^2(r) - H^2(r) > 0 \]  \hspace{1cm} (3.16)

and the term in front of \(d\psi^2\) in the metric (3.9) remains positive. The conditions (3.15) and (3.16) remove CTCs to a point which is formally at \(r_c = \infty\). We call the model given by the condition (3.15) the causal Gödel spacetime. In fact, there is a larger class of such causal Gödel models [28,29] for which there are no CTCs for any value of the radial coordinate \(r > 0\).

The only nonvanishing components of the Riemann tensor in a Lorentzian frame (3.1) permitted by the space-time homogeneity of the Gödel universe are [14,15]

\[ R_{0101} = R_{0202} = \frac{1}{4} \left( \frac{H'}{D} \right)^2 = \Omega^2, \hspace{1cm} R_{1212} = \frac{3}{4} \left( \frac{H'}{D} \right)^2 - \frac{D''}{D} = 3\Omega^2 - m^2, \]  \hspace{1cm} (3.17)

where \(m = \sqrt{2}\Omega\) for the acausal model, and \(m = 2\Omega\) for the causal model, \((\ldots)' = \partial/\partial x\).

Using (3.4) one can easily calculate the Einstein energy-momentum pseudotensor (2.16). Its non-vanishing components are

\[ e^{00} = \frac{m^2}{16\pi}, \hspace{1cm} e^{11} = -\frac{m^2}{16\pi}, \hspace{1cm} e^{22} = -\frac{m^2}{16\pi}, \hspace{1cm} e^{33} = \frac{m^2}{16\pi}, \hspace{1cm} e^{02} = -\frac{m^2\sqrt{2}}{16\pi}, \]  \hspace{1cm} (3.18)

which according to (3.13) and (3.15) give

\[ e^{00} = \frac{\Omega^2}{8\pi}, \hspace{1cm} e^{11} = -\frac{\Omega^2}{8\pi}, \hspace{1cm} e^{22} = -\frac{\Omega^2}{8\pi}, \hspace{1cm} e^{33} = \frac{\Omega^2}{8\pi}, \hspace{1cm} e^{02} = -\frac{\sqrt{2}\Omega^2}{8\pi}, \]  \hspace{1cm} (3.19)

for an original acausal Gödel spacetime [11], and

\[ e^{00} = \frac{\Omega^2}{4\pi}, \hspace{1cm} e^{11} = -\frac{\Omega^2}{4\pi}, \hspace{1cm} e^{22} = -\frac{\Omega^2}{4\pi}, \hspace{1cm} e^{33} = \frac{\Omega^2}{4\pi}, \hspace{1cm} e^{02} = -\frac{\sqrt{2}\Omega^2}{4\pi}, \]  \hspace{1cm} (3.20)

for a causal Gödel spacetime [14]. In Ref. [21] the calculation of the Landau-Lifshitz and Möller pseudotensors were performed in holonomic coordinates for the acausal model and they give different results from ours. However, in the orthonormal frames the Landau-Lifshitz and Einstein pseudotensors coincide and the results should be the same (see e.g. [23]).
From (3.19) and (3.20) one can conclude that in both cases the Einstein pseudotensor is traceless, but (as expected) not symmetric, and that the gravitational energy “density”

\[ \epsilon_g := E t_i^k v^i v_k \]  

is in both cases negative \((v^i = \delta^i_0, v_k = g_{k0} \text{ for Gödel universes})\), i.e.,

\[ \epsilon_g = -\frac{\Omega^2}{8\pi} < 0 \]  

for the acausal model, and

\[ \epsilon_g = -\frac{\Omega^2}{4\pi} < 0 \]  

for the causal model. This can have an interesting connection to the results of the calculation for brane universes [19] where it was shown that CTCs avoidance (and so the causality) is possible provided the total effective energy density is \textit{negative} for these models. Also, in both models all the components of the gravitational Poynting 4-vector

\[ \gamma P^i = \left( \delta^i_k + v^i v_k \right) E t_i^k v^j \]  

identically vanish in the Lorentzian coreper (3.1). This means that we have no gravitational energy flux which seems to be related to the fact that the magnetic part of the Weyl (conformal curvature) tensor vanishes for Gödel models.

As far as the material part \(T_{ik}\) of the canonical energy-momentum complex (2.11) is concerned, its non-vanishing components for the \textit{acausal} Gödel [11] model are [10]

\[ T_{00} = \varrho + \frac{\Lambda}{8\pi} = \frac{\Omega^2}{8\pi}, \quad T_{11} = T_{22} = T_{33} = -\frac{\Lambda}{8\pi} = \frac{\Omega^2}{8\pi}, \]  

and the following relation must be fulfilled (\(\varrho\) - the energy density of dust matter)

\[ 4\pi \varrho = \Omega^2 = -\Lambda = \text{const.} \]  

From (3.25) one can easily calculate that the matter energy density

\[ \epsilon_m := T_{ik} v^i v^k = \frac{\Omega^2}{8\pi} > 0, \]
and that all the components of the material Poynting vector

$$mP^i := \left( \delta_k^i + v^i v_k \right) T^k_l v^l$$  \hspace{1cm} (3.28)

identically vanish. Combining the results for gravity and for matter we have

$$\epsilon = \epsilon_g + \epsilon_m = 0,$$  \hspace{1cm} (3.29)

and

$$P^i := g P^i + m P^i = (0, 0, 0, 0),$$  \hspace{1cm} (3.30)

i.e., the total energy density and the total flux for the acausal model vanish. For the causal model we have [10]

$$T_{00} = \frac{e^2}{2} + \frac{\Lambda}{8\pi} = -\frac{\Omega^2}{8\pi}, \quad T_{11} = T_{22} = -\frac{e^2}{2} - \frac{\Lambda}{8\pi} = \frac{\Omega^2}{8\pi}, \quad T_{33} = \frac{e^2}{2} - \frac{\Lambda}{8\pi} = 3\frac{\Omega^2}{8\pi},$$  \hspace{1cm} (3.31)

and the following relation between parameters $\Lambda, \Omega$ and $e$ has to be fulfilled

$$\Lambda = -2\Omega^2 = -8\pi e^2 = \text{const}. \hspace{1cm} (3.32)$$

From (3.31) there follows that

$$\epsilon_m = -\frac{\Omega^2}{8\pi}, \quad mP^i = (0, 0, 0, 0).$$  \hspace{1cm} (3.33)

This gives the result that the total energy density is negative for the causal model and that its total flux vanishes, i.e.,

$$\epsilon = \epsilon_g + \epsilon_m = -\frac{3\Omega^2}{8\pi} < 0$$  \hspace{1cm} (3.34)

$$P_i := g P^i + m P^i = (0, 0, 0, 0).$$  \hspace{1cm} (3.35)

All the above results look reasonable, but they valid only for the Lorentzian frame (3.1) and for a set of frame obtained from it by the global Lorentz transformations. Then, one cannot extract from them any coordinate-independent conclusions except for matter part which can be transformed into an arbitrary frame by tensorial transformation rule.
Finally, one can analyze the angular momentum of the Gödel spacetimes in the Lorentzian frame (3.1) by using the formulas (2.25), (2.17), (3.4) and (3.17). The calculations are simple, but somewhat tedious and this is why we decided to put them into the Appendix. Here we only give some general remarks.

At first, we would like to note that as many as 11 independent components in the acausal case and 13 independent components in the causal case of the Bergmann-Thomson angular momentum complex are different from zero, and that it is difficult to extract any more sophisticated physical conclusion from their shape. The only obvious conclusion is that their non-vanishing reflects the fact of rotation of Gödel spacetimes. These remarks refer both to the gravitational part and to the matter part of the Bergmann-Thomson complex.

Secondly, even after a decomposition of the Bergmann-Thomson angular momentum complex into its tensor (t), vector (v) and axial (a) (totally antisymmetric) parts as follows

\[ M^{abc} = (t) M^{abc} + (v) M^{abc} + (a) M^{abc}, \]  

where

\[ (v) M^{abc} := \frac{1}{3}(g^{bc} V^a - g^{ac} V^b), \]  

\[ (a) M^{abc} = M^{[abc]} := \epsilon^{abc} a_d, \]  

\[ V^a := M^{ab}_b, \quad a^d := -\frac{1}{6} \epsilon^{dabc} M_{abc}, \]  

\[ (t) M^{abc} := M^{abc} - (v) M^{abc} - (a) M^{abc}, \]

the situation is still unclear, although much simpler. The reason is that we still have too many non-vanishing independent components of these irreducible parts (12 for the vectorial parts and 18 for the tensorial parts).

The same is true for the irreducible components of matter angular momentum (2.17) and gravitational angular momentum

\[ gM^{ika} = M^{ika} - m M^{'ika} \]  

(except for axial parts of the matter angular momentum densities which vanish).
IV. CONCLUSION

In this paper we have analyzed energetic quantities for Gödel universes. In order to calculate these quantities we have used the canonical energy-momentum pseudotensor/complex of Einstein and canonical angular momentum pseudotensor/complex of Bergmann and Thomson. We have presented these objects in an anholonomic Lorentzian frame and performed the calculations in such an anholonomic frame which substantially simplified the calculations.

We have found that for both considered acausal and causal Gödel models, the Einstein pseudotensor is traceless, not symmetric, and that the gravitational energy “density” is negative. Also, the gravitational Poynting vector vanishes for these models which seems to have a direct relation to the fact that the magnetic part of the Weyl (conformal curvature) tensor vanishes for Gödel models. On the other hand, the total (gravitational and matter) energy “density” for the acausal model is zero, while for the causal model it is negative. This last statement is in agreement with the results obtained for the superenergy density [10] which we found supportive for our earlier superenergetic investigations. Also, there exists a puzzling coincidence with the result obtained recently for brane universes [19], where the total effective energy density for these models must be negative in order to get causality.

On the other hand, the canonical angular momentum Bergmann-Thomson complex has so complicated structure that practically it is difficult to extract any more sophisticated physical conclusion, except that it does not vanish which reflects the fact of global rotation of Gödel spacetimes.

Naturally, these conclusions are valid only in the Lorentzian frame applied and in a globally Lorentz rotated frame obtained from this.

The main problem is that the calculated complexes are not tensors and due to this one is not able to extract any convincing physical information in a coordinate-independent way. In particular, the application of the Landau-Lifshitz and Møller pseudotensors in a holonomic frame for the acausal Gödel spacetime recently [21], shows that the results obtained differ
from ours.

In this context we emphasize that superenergetic quantities are tensors and so they admit a coordinate-independent description of the gravitational field so that the agreement of the results obtained for energetic quantities with the results obtained for superenergetic quantities suggests also usefulness of the Einstein and Bergmann-Thomson complexes in the analysis. However, from what we derived, it appears that the analytic structure of the canonical superenergy tensors and the canonical supermomentum tensors for matter and gravitation is much simpler than the analytic structure of the corresponding canonical energetic pseudotensors/complexes.

APPENDIX A: BERGMANN-THOMSON ANGULAR MOMENTUM COMPLEX COMPONENTS FOR GÖDEL UNIVERSES

From (3.25) and (3.31) we can calculate the non-vanishing components of the matter angular momentum tensor (2.17) which is antisymmetric in the first two indices. For the acausal model they can be presented in a compact way as follows

\[
\begin{align*}
 m_{M}^{0\mu\nu} &= -m_{M}^{\mu0\nu} = \frac{r^{0}\Omega^{2}}{8\pi}, \\
 m_{M}^{0\mu0} &= -m_{M}^{\mu00} = -\frac{r^{\mu}\Omega^{2}}{8\pi}, \\
 m_{M}^{\mu\nu\mu} &= -m_{M}^{\nu\mu\mu} = \frac{-r^{\mu}\Omega^{2}}{8\pi},
\end{align*}
\]  

(A1)

and the Greek indices \( \mu, \nu, \ldots = 1, 2, 3, \mu \neq \nu \). For the causal model one can use somewhat less compact way of presentation, i.e.,

\[
\begin{align*}
 m_{M}^{0\mu0} &= -m_{M}^{\mu00} = \frac{r^{\mu}\Omega^{2}}{8\pi}, \\
 m_{M}^{011} &= -m_{M}^{101} = m_{M}^{022} = -m_{M}^{202} = \frac{r^{0}\Omega^{2}}{8\pi}, \\
 m_{M}^{033} &= -m_{M}^{303} = \frac{3r^{0}\Omega^{2}}{8\pi}, \\
 m_{M}^{122} &= -m_{M}^{212} = \frac{r^{1}\Omega^{2}}{8\pi},
\end{align*}
\]
\[ m M^{133} = -m M^{313} = \frac{3r^{-1} \Omega^2}{8\pi}, \]
\[ m M^{233} = -m M^{323} = \frac{3r^{2} \Omega^2}{8\pi}, \]
\[ m M^{121} = -m M^{211} = -\frac{r^2 \Omega^2}{8\pi}, \]
\[ m M^{131} = -m M^{311} = -m M^{232} = -\frac{r^3 \Omega^2}{8\pi}. \]  
\[ \text{(A2)} \]

As for the Bergmann-Thomson complex for the sake of performing the exact calculations we split the formula (2.25) as follows

\[ B_T M^{ika} = A^{ika} + B^{ika} + C^{ika} + D^{ika} + E^{ika}, \]  
\[ \text{(A3)} \]

where

\[ A^{ika} = \frac{1}{2\chi} \left[ \eta_{lern} g_{tr}^{r} \gamma_{m}^{m} \left( \eta_{i}^{aipm} g_{kl}^{i} - \eta_{i}^{akpm} g_{li}^{i} \right) \right], \]  
\[ \text{(A4)} \]
\[ B^{ika} = \frac{1}{2\chi} \left[ \eta_{atbs} \eta_{ism} \gamma_{mb}^{m} g_{rnp}^{r} \left( g_{i}^{i} \gamma_{i}^{k} - g_{i}^{i} \gamma_{i}^{k} \right) \right], \]  
\[ \text{(A5)} \]
\[ C^{ika} = \frac{1}{2\chi} \left( r^{i} g_{kl}^{i} - r^{i} g_{il}^{i} \right) \eta_{jspm} \eta_{j}^{i} \gamma_{m}^{ij} \gamma_{s}^{ij}, \]  
\[ \text{(A6)} \]
\[ D^{ika} = -\frac{1}{4\chi} \left( r^{i} g_{kl}^{i} - r^{i} g_{il}^{i} \right) \eta_{jspm} \eta_{j}^{i} \gamma_{m}^{ij} \gamma_{rs}^{ij}, \]  
\[ \text{(A7)} \]
\[ E^{ika} = -\frac{1}{4\chi} \left( r^{i} g_{kl}^{i} - r^{i} g_{il}^{i} \right) \eta_{jspm} \eta_{j}^{i} \gamma_{m}^{ij} \gamma_{r}^{ij}, \]  
\[ \text{(A8)} \]

Taking into account the Equations (3.4) and (3.17) one gets for (A4) the following expressions

\[ 2\chi A^{ika} = -m \sqrt{2} \left( \eta_{a}^{23} g_{k}^{2} - \eta_{a}^{k23} g_{i}^{2} \right) - m \sqrt{2} \left( \eta_{a}^{13} g_{k}^{1} - \eta_{a}^{k13} g_{i}^{1} \right) \]
\[ - m \sqrt{2} \left( \eta_{a}^{03} g_{k}^{0} - \eta_{a}^{k03} g_{i}^{0} \right) + 2m \left( \eta_{a}^{23} g_{k}^{0} - \eta_{a}^{k23} g_{i}^{0} \right) \]
\[ - 2m \left( \eta_{a}^{20} g_{k}^{2} - \eta_{a}^{k20} g_{i}^{2} \right), \]  
\[ \text{(A9)} \]

\[ 2\chi B^{ika} = -m \sqrt{2} \eta_{a}^{023} g_{k}^{1} \left( g_{i}^{2} \gamma_{k}^{i} - g_{k}^{2} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{123} g_{k}^{0} \left( g_{i}^{2} \gamma_{k}^{i} - g_{k}^{2} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{013} g_{k}^{1} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{123} g_{k}^{0} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{023} g_{k}^{1} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{013} g_{k}^{1} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{123} g_{k}^{0} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{013} g_{k}^{1} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\[ - m \sqrt{2} \eta_{a}^{023} g_{k}^{1} \left( g_{i}^{1} \gamma_{k}^{i} - g_{k}^{1} \gamma_{i}^{k} \right) \]
\(- m \sqrt{2} \eta_{i0}^{213} r_0 \left( g_{i0}^{i0} \frac{\gamma_{00} - g_{k0}^{i0}}{\gamma_{00}} \right)
+ m \sqrt{2} \eta_{i0}^{103} r_0 \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- m \sqrt{2} \eta_{i0}^{203} r_2 \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- m \sqrt{2} \eta_{i0}^{203} r_1 \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- \frac{2m r_0 (\eta_{i0}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- \frac{2m r_2 (\eta_{i0}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
+ \frac{2m r_0 (\eta_{00}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
+ \frac{2m r_2 (\eta_{00}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- \frac{2m r_0 (\eta_{00}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right)
- \frac{2m r_2 (\eta_{00}^{i0} - g_{k0}^{i0})}{\sqrt{2}} \left( g_{i0}^{i0} - g_{k0}^{i0} \right),
(A10)

\[2 \chi^{ika} = -2m^2 \left( r \left[ g^{k2} - r \right] g^{i2} \right) \eta_{i0}^{i0} - 2m^2 \left( r \left[ g^{k0} - r \right] g^{i0} \right) \eta_{i0}^{213} + 2m^2 \left( r \left[ g^{k1} - r \right] g^{i1} \right) \eta_{i0}^{i0} - 2m^2 \sqrt{2} \left( r \left[ g^{k2} - r \right] g^{i2} \right) \eta_{i0}^{123},
(A11)

\[2 \chi^{D^{ika}} = m^2 \left( rg^{k2} - r \right) g^{i2} \eta_{i0}^{i0} + m^2 \left( riz^{k2} - r \right) g^{i2} \eta_{i0}^{i0} + m^2 \left( riz^{k1} - r \right) g^{i1} \eta_{i0}^{i0} + m^2 \left( riz^{k0} - r \right) g^{i0} \eta_{i0}^{i0} + m^2 \left( riz^{k3} - r \right) g^{i3} \eta_{i0}^{i0},
(A12)

\[2 \chi^{E^{ika}} = 2 \Omega^2 \left( r \left[ g^{k2} - r \right] g^{i2} \right) \eta_{i0}^{i0} - 2 \Omega^2 \left( r \left[ g^{k1} - r \right] g^{i1} \right) \eta_{i0}^{i0} - 2 \left( 3 \Omega^2 - m^2 \right) \left( r \left[ g^{k0} - r \right] g^{i0} \right) \eta_{i0}^{i0} + 2 \left( \Omega^2 - m^2 \right) \left( r \left[ g^{k3} - r \right] g^{i3} \right) \eta_{i0}^{i0},
(A13)

Finally, we present the non-vanishing components of the Bergmann-Thomson complex (2.25) which contain both matter and gravitation (remember they are antisymmetric in the first two indices). These are:

\[BT M_{010} = \frac{(3 \Omega^2 - m^2)r^4 + m}{8 \pi},
\[BT M_{020} = \frac{(3 \Omega^2 - m^2)r^2}{8 \pi},
\[BT M_{030} = \frac{(6 \Omega^2 - m^2)r^3}{16 \pi},
\[BT M_{230} = \frac{m^2 \sqrt{r^2}}{16 \pi},
\[BT M_{011} = \frac{\Omega^2 r^0}{8 \pi},
\[BT M_{121} = -\frac{\Omega^2 r^2}{8 \pi}.\]
\[ BTM^{131} = \frac{(m^2 - 2\Omega^2)r^3}{16\pi}, \]
\[ BTM^{022} = \frac{\Omega^2 r^0}{8\pi}, \]
\[ BTM^{122} = \frac{\Omega^2 r^1}{8\pi}, \]
\[ BTM^{232} = \frac{(m^2 - 2\Omega^2)r^3}{16\pi}, \]
\[ BTM^{033} = \frac{(3m^2 - 2\Omega^2)r^0 + m^2\sqrt{2}r^2}{16\pi}, \]
\[ BTM^{133} = \frac{2m + (3m^2 - 2\Omega^2)r^1}{16\pi}, \]
\[ BTM^{233} = \frac{(3m^2 - 2\Omega^2)r^2 + m^2\sqrt{2}r^0}{16\pi}. \]  

(A14)

As one can see, for the acausal model \((m^2 = 2\Omega^2)\) one has 11 independent components and for the causal model \((m^2 = 4\Omega^2)\) there are 13 independent components.

Subtracting matter angular momentum from the Bergmann-Thomson complex we can obtain the components of the gravitational angular momentum pseudotensor (3.41) for both models. Applying (A1) and (A14) for the acausal model we have 9 non-vanishing components

\[ gM^{010} = \frac{\sqrt{2}\Omega}{8\pi} + \frac{\Omega^2}{4\pi}r^1, \]
\[ gM^{020} = \frac{\Omega^2}{4\pi}r^2, \]
\[ gM^{030} = \frac{3\Omega^2}{8\pi}r^3, \]
\[ gM^{230} = \frac{\sqrt{2}\Omega^2}{8\pi}r^3, \]
\[ gM^{131} = \frac{\Omega^2}{8\pi}r^3, \]
\[ gM^{232} = \frac{\Omega^2}{8\pi}r^3, \]
\[ gM^{033} = \frac{\Omega^2}{8\pi}r^0 + \frac{\sqrt{2}\Omega^2}{8\pi}r^2, \]
\[ gM^{133} = \frac{\sqrt{2}\Omega}{8\pi} + \frac{\Omega^2}{8\pi}r^1, \]
\[ gM^{233} = \frac{\Omega^2}{8\pi}r^2 + \frac{\sqrt{2}\Omega^2}{8\pi}r^0. \]  

(A15)

while using (A2) and (A14) for the causal model we have 8 independent components
\[ g_{M}^{010} = \frac{\Omega}{4\pi} - \frac{\Omega^2}{4\pi} r^1, \]
\[ g_{M}^{020} = -\frac{\Omega^2}{4\pi} r^2, \]
\[ g_{M}^{230} = \frac{\sqrt{2}\Omega^2}{4\pi} r^3, \]
\[ g_{M}^{131} = -\frac{\Omega^2}{4\pi} r^3, \]
\[ g_{M}^{232} = -\frac{\Omega^2}{4\pi} r^3, \]
\[ g_{M}^{033} = \frac{\Omega}{4\pi} r^0 + \frac{\sqrt{2}\Omega^2}{4\pi} r^2, \]
\[ g_{M}^{133} = \frac{\Omega}{4\pi} + \frac{\Omega^2}{4\pi} r^1, \]
\[ g_{M}^{233} = \frac{\Omega^2}{4\pi} r^2 + \frac{\sqrt{2}\Omega^2}{4\pi} r^0. \]  
(A16)
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