H-cohomologies versus algebraic cycles

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0 Introduction

After Quillen’s proof of the Gersten conjecture (see [29]), for algebraic regular schemes, a natural approach to the theory of algebraic cycles appears to be by dealing with the “formalism” associated to (local) higher \( K \)-theory, as it is manifestly expressed by the work of Bloch and Gillet (cf. [8], [15]). As a matter of fact a more general and flexible setting has been exploited by Bloch and Ogus (see [10]) by axiomatic methods.

The aim of this paper is to going further with this axiomatic method in order to obtain a “global intersection theory” (in the Grothendieck sense [20]) directly from a given “cohomology theory”. To this aim we will assume given a (twisted) cohomology theory \( H^* \) and we let consider the \( H^* \)-cohomology functor

\[
X \mapsto H^\#_{\text{Zar}}(X, H^*(\cdot))
\]

where \( H^*(\cdot) \) is the Zariski sheaf associated to \( H^* \). By dealing with a cup-product structure on \( H^* \) we are granted of a product in \( H^* \)-cohomology; by arguing with the cap-product structure we are able to obtain a cap-product between algebraic cycles and \( H^* \)-cohomology classes, for \( Y \) and \( Z \) closed subschemes of \( X \) (\( \Lambda \overset{\text{def}}{=} H^0(\text{point}) \))

\[
\cap : C_n(Y; \Lambda) \otimes H^p_\mathcal{Z}(X, \mathcal{H}^p(\rho)) \to C_{n-p}(Y \cap Z; \Lambda)
\]

where \( C_*(-; \Lambda) \) is the “\( \mathcal{H} \)-homology theory” given by the hypercohomology of the complexes of \( E^1 \)-terms of the niveau spectral sequence.

If \( X \) is smooth of pure dimension \( d \), by capping with the “fundamental cycle” \( [X] \in C_d(X; \Lambda) \) we have a “Poincaré duality” isomorphism

\[
[X] \cap - : H^p_\mathcal{Z}(X, \mathcal{H}^p(\rho)) \cong C_{d-p}(Z; \Lambda)
\]

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Thus the “$\mathcal{H}$-cycle class” $\eta(Z) \in H_p^p(X, \mathcal{H}^p(\rho))$ is defined by $[X]\cap(\eta(Z) = [Z]$, for $i : Z \rightarrow X$ a closed subscheme of pure codimension $p$ in $X$. By capping with the $\mathcal{H}$-cycle class we do obtain Gysin maps for algebraic cycles i.e. maps $i^! : C_n(X; \Lambda) \rightarrow C_{n-p}(Z; \Lambda)$. Furthermore, the $\mathcal{H}$-cycle classes are compatible with the intersection of cycles (when existing!) so that the $\mathcal{H}$-cohomology rings generalize the classical “intersection rings” obtained via rational or algebraic equivalences (cf. [13] for the $K$-cohomology).

The covariant property of the niveau spectral sequence grant us of “$\mathcal{H}$-Gysin maps”

$$f_* : H^\#_{f^{-1}(Z)}(Y, \mathcal{H}^i(\cdot)) \rightarrow H^\#_{f^{-1}(Z)}(X, \mathcal{H}^i(\cdot + \rho))$$

associated with a proper morphism $f : Y \rightarrow X$ of relative dimension $\rho$ between smooth schemes. The corresponding projection formula holds. By the homotopy property of $H^*$ we are obtaining homotopy and Dold-Thom decomposition for $\mathcal{H}$-cohomologies. By observing that the canonical cycle map for line bundles $c^\text{el} : \text{Pic}(X) \rightarrow H^2(X, 1)$ has always its image contained in the subgroup of the locally trivial cohomology classes i.e. $H^1(X, \mathcal{H}^1_{(1)})$ by the coniveau spectral sequence, we are able to construct Chern classes in $\mathcal{H}$-cohomologies according with Gillet and Grothendieck (see [13] and [19])

$$c_{p,i} : K^Z_i(X) \rightarrow H^p_{Z}(X, \mathcal{H}^p(\rho))$$

where $Z$ is any closed subset of $X$ smooth. These yield Riemann-Roch theorems and, notably, Chern classes in $H^{2*}(-, *)$ by composition with the cycle map $H^*(\cdot, H^*(\cdot)) \rightarrow H^{2*}(-, *)$ canonically induced by the coniveau spectral sequence.

At last, an immediate application of this setting is the “blow-up formula”

$$H^p(X', \mathcal{H}^0(\rho)) \cong H^p(X, \mathcal{H}^0(\rho)) \oplus \bigoplus_{i=0}^{c-2} H^{p-1-i}(Z, \mathcal{H}^{q-1-i}(p-1-i))$$

where $X'$ is the blow-up of $X$ smooth, along a closed smooth subset $Z$ of pure codimension $c$. Remarkably the formula is obtained by no use of “self-intersection” nor “formule-clef” (used by the redundant arguments made in [SGA 5, Exposé VII] for étale cohomology or Chow groups).

The paper is organized by adding structure to the assumed Bloch-Ogus cohomology to proving the claimed results. The common cohomologies (e.g. étale, de Rham or Deligne-Beilinson cohomology) are examples for this setting as explained in the Appendix. Some of these results are already been used by the author for applications (see [1], [2]); as our second main goal is that this “formalism” can be used for applications to birational geometry and algebraic cycles.

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1 Preliminaries

Let $\mathcal{V}_k$ be the category of schemes of finite type over a fixed ground field $k$; usually, an object of $\mathcal{V}_k$ is called ‘algebraic scheme’. Let $\mathcal{V}^2_k$ be the category whose objects are pairs $(X, Z)$
where $X$ is an algebraic scheme and $Z$ is a closed subscheme of $X$; morphisms in $\mathcal{V}_k^2$ are fibre products in $\mathcal{V}_k$. In the following we will consider a contravariant functor

$$(X, Z) \rightsquigarrow H^*_Z(X, \cdot)$$

from $\mathcal{V}_k^2$ to $Z$-bigraded abelian groups.

We need to assume, at least, that the above functor gives rise to a ‘Poincaré duality theory with supports’ as setted out by Bloch and Ogus [10, 1.1-1.3 and 7.1.2]; one can also consider $K^Z(X)$ the relative Quillen $K$-theory [29] (see [13, Def. 2.13]). For $X \in \mathcal{V}_k$ we denote $H^*(X, \cdot)$ for $H^*X(X, \cdot)$.

### 1.1 Bloch-Ogus theory

For the sake of notation we recall some facts by [10]. Together with the cohomology theory $H^*(\cdot, \cdot)$ is given an homological functor $H^*_{(\cdot, \cdot)}$ covariant for proper morphisms in $\mathcal{V}_k$ and a pairing:

$$\cap_{X, Z} : H^l(X, m) \otimes H^r_Z(X, s) \rightarrow H^{l-r}(Z, m-s)$$

having a ‘projection formula’. For $f$ a proper map let $f_!$ denote the induced map on homology. It is also assumed the existence of a ‘fundamental class’ $\eta_X \in H^{2d}(X, d)$, $d = \dim X$, such that $f_!(\eta_X) = [K(X) : K(Y)] \cdot \eta_Y$ if $f : X \rightarrow Y$ is proper and $\dim X = \dim Y$ (cf. [10, 7.1.2]). For $X$ smooth of dimension $d$

$$\eta_{X \cap X, s-} : H^{2d-i}_Z(X, d-j) \xrightarrow{\sim} H^i(Z, j)$$

is an isomorphism (‘Poincaré duality’) suitably compatible with restrictions (cf. [10, 1.4]). For $Z \subseteq T \subseteq X$ such that $Z$ and $T$ are closed in $X$ there is a long exact sequence (see [10, 1.1.1])

$$\cdots \rightarrow H^d_Z(X, \cdot) \rightarrow H^d_T(X, \cdot) \rightarrow H^d_{T-Z}(X-Z, \cdot) \rightarrow H^{d+1}_Z(X, \cdot) \rightarrow \cdots$$

suitably contravariant; moreover, for $X$ smooth and irreducible of dimension $d$, the following exact sequence ($h + i = 2d$, $\dagger + \cdot = d$):

$$\cdots \rightarrow H^h(Z, \dagger) \rightarrow H^h(T, \dagger) \rightarrow H^h(T-Z, \dagger) \rightarrow H^{h-1}(Z, \dagger) \rightarrow \cdots$$

is the corresponding Poincaré dual of the above (cf. [26, 6.1 k]).

### 1.2 Gersten or arithmetic resolution

Let $Z^p(X) = \{ Z \subset X : \text{closed of codim}_X Z \geq p \}$, ordered by inclusion, and let define

$$H^i_{Z^p(X)}(X, \cdot) \overset{\text{def}}{=} \lim_{Z \in Z^p(X)} H^i_Z(X, \cdot)$$

and for $x \in X$

$$H^i(x, \cdot) \overset{\text{def}}{=} \lim_{U \text{open} \subset \{x\}} H^i(U, \cdot)$$

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Taking the direct limit of the exact sequences (over the pairs $Z \subseteq T$ with $Z \in \mathbb{Z}^{p+1}(X)$ and $T \in \mathbb{Z}^{p}(X)$) and using ‘local purity’ (cf. Prop.3.9) on $X$ smooth over $k$ perfect, one obtains long exact sequences

$$H^i_{\mathbb{Z}^p(X)}(X, j) \rightarrow H^i_{\mathbb{Z}^p}(X, j) \rightarrow \prod_{x \in X^p} H^{i-2p}(x, j-p) \rightarrow H^{i+1}_{\mathbb{Z}^p+1(X)}(X, j)$$

(3)

where $X^p$ is the set of points whose closure has codimension $p$ in $X$. Furthermore, if $f : X \rightarrow Y$ is a flat morphism and $Z \in \mathbb{Z}^p(Y)$ then we have $f^{-1}(Z) \in \mathbb{Z}^p(X)$; thus the sequence in (3) yields a sequence of presheaves for the Zariski topology. Let $H^*_{Z^p,X}()$ denote the Zariski presheaf $U \sim H^*_{Z^p(U)}(U, \cdot)$ on $X$ and let $a : \mathcal{P}(X_{Zar}) \rightarrow \mathcal{S}(X_{Zar})$ be the associated sheaf exact functor. Denote

$$aH^*_{Z^p,X}() \overset{def}{=} H^*_{Z^p,X}()$$

The presheaf $H^*_{Z^p,X}()$ is just the functor $H^*()$ on $X_{Zar}$ and so one has $H^*_{Z^p,X}() = H^*_{X}()$. One of the main results of [10] is in proving the vanishing of the map

$$H^*_{Z^p+1,X}() \rightarrow H^*_{Z^p,X}()$$

for all $p \geq 0$. From this vanishing, sheafifying the sequence (3), one has the following exact sequences of sheaves on $X$ smooth over $k$ perfect:

$$0 \rightarrow H^*_{Z^p,X}(j) \rightarrow \prod_{x \in X^p} i_x H^{i-2p}(x, j-p) \rightarrow H^{i+1}_{Z^p+1,X}(j) \rightarrow 0$$

(4)

where: for $A$ an abelian group and $x \in X$ we let $i_x A$ denote the constant sheaf $A$ on $\{x\}$ extended by zero to all $X$. Patching together the above short exact sequences we do get a resolution of the sheaf $H^*_X(j)$ (‘arithmetic resolution’ in [10, Theor.4.2]):

$$0 \rightarrow H^*_X(j) \rightarrow \prod_{x \in X^0} i_x H^i(x, j) \rightarrow \prod_{x \in X^1} i_x H^{i-1}(x, j-1) \rightarrow \cdots$$

**Remark:** The assumption of $k$ perfect is unnecessary (cf. [11],1) if $H^*(\cdot, \cdot)$ is the étale theory (namely $H^i(X, j) \overset{def}{=} H^i(X_{\text{ét}}, \mu_{p^j})$) where $\mu_{p^j}$ is the étale sheaf of $p^{th}$ root of unity and $p$ is any positive integer prime to $\text{char}(k)$.

### 1.3 Quillen $K$-theory

Let $X \sim K_p(X)$ be the Quillen $K$-functor associated with the exact category of vector bundles on any scheme $X$ (see [23]). For a fixed $X$ we let $K_p(O_X)$ be the associated Zariski sheaf on $X$. For any noetherian separated scheme $X$ we have a complex of flasque sheaves (‘Gersten’s complex’)

$$T^q_{q,X} : \prod_{x \in X^0} i_x K_q(k(x)) \rightarrow \prod_{x \in X^1} i_x K_{q-1}(k(x)) \rightarrow \cdots$$

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conjecturally exact if $X$ is regular, being proved exact by Quillen [29] if $X$ is regular and essentially of finite type over a field. By tensoring coherent modules with locally free sheaves and sheafifying one has a pairing of complexes of sheaves for $p, q \geq 0$ (see [35, p.276-277]):

$$\cap : \mathcal{K}_p(\mathcal{O}_X) \otimes \mathcal{I}_{q,X}^* \to \mathcal{I}_{p+q,X}^*$$

By capping with $1_X$ = the ‘fundamental class’ i.e. the identity section of the constant sheaf $Z \cong \mathcal{I}_{0,X}^*$ we get an augmentation

$$\cap 1_X : \mathcal{K}_p(\mathcal{O}_X) \to \mathcal{I}_{p,X}^*$$

which is a quasi-isomorphism if $X$ is regular essentially of finite type over a field (conjecturally for all regular schemes.) If $f : X \to Y$ is a proper morphism between biequidimensional schemes and $r = \dim Y - \dim X$ then there is an induced map of complexes $f_i : f_*\mathcal{I}_{q,X}^* \to \mathcal{I}_{q+r,Y}^*[r]$ (which takes the elements of $\mathcal{K}_i(k(x))$ to $\mathcal{K}_i(\mathcal{O}_Y)$) if $\dim \bar{x} = \dim \bar{f(x)}$ and takes them to zero otherwise and a commutative diagram (see the ‘projection formula’ [35, p.411]):

$$\begin{array}{ccc}
\mathcal{K}_p(\mathcal{O}_Y) \otimes f_*\mathcal{I}_{q,X}^* & \cong & f_i \mathcal{I}_{p+q,X}^* \\
\downarrow \quad f^* \otimes id & \quad \cap r_* \\
\mathcal{K}_p(\mathcal{O}_Y) \otimes \mathcal{I}_{q+r,Y}^*[r] & \to & f_i \mathcal{I}_{p+q+r,Y}^*[r]
\end{array}$$

Thus the formula: $f_i(f^* (\tau \cap f^*_f) \sigma) = \tau \cap f_i(\sigma)$ for all sections $\tau \otimes \sigma$ of the complex of sheaves $\mathcal{K}_p(\mathcal{O}_Y) \otimes f_*\mathcal{I}_{q,X}^*$ .

2 Invariance

Let $X \in \mathcal{V}_k$ be an algebraic scheme; in the following we will assume $X$ smooth and the ground field $k$ perfect. We moreover assume given a cohomological functor $H^*_Z(X, \cdot)$ satisfying the list of axioms [10, 1.1–1.3] and the assumption [10, 7.1.2]. We are going to consider a morphism $f : X \to Y$ from $X$ as above to $Y \in \mathcal{V}_k$ tacitly assuming $Y$ to be smooth.

Let $X$ be in $\mathcal{V}_k$ and let $\mathcal{H}_X^i(j)$ (resp. $\mathcal{K}_i(\mathcal{O}_X)$) denote the sheaf on $X$, for the Zariski topology, associated to the presheaf $U \mapsto H^i(U, j)$ (resp. $U \mapsto K_i$(vector bundles on $U$)). If $f : X \to Y$ is any morphism in $\mathcal{V}_k$ then there are maps $f^*_j : \mathcal{H}_Y^i(j) \to f_*\mathcal{H}_X^i(j)$ (resp. $f^*_i : \mathcal{K}_i(\mathcal{O}_Y) \to f_*\mathcal{K}_i(\mathcal{O}_X)$).

**Theorem 1** (Invariance) Let $f : X \to Y$ be a proper birational morphism in $\mathcal{V}_k$. For $X$ and $Y$ smooths over $k$ perfect then $f^*$ yields the isomorphism

$$\mathcal{H}_Y^i(j) \cong f_*\mathcal{H}_X^i(j)$$

for all integers $i$ and $j$. For $X$ and $Y$ regular algebraic schemes $f^*$ induces the isomorphism

$$\mathcal{K}_i(\mathcal{O}_Y) \cong f_*\mathcal{K}_i(\mathcal{O}_X)$$
for all \( i \geq 0 \).
Hence there are isomorphisms
\[
H^0(X, \mathcal{H}_X^i(j)) \cong H^0(Y, \mathcal{H}_Y^i(j))
\]
and
\[
H^0(X, \mathcal{K}_i(\mathcal{O}_X)) \cong H^0(Y, \mathcal{K}_i(\mathcal{O}_Y))
\]

**Remark:** By the recent results of M. Spivakovsky the problem of ‘the elimination of points of indeterminacy’ appears to be solved also in positive characteristic. Thus, by the Theorem 1, the groups \( H^0(X, H^*_X(\cdot)) \) and \( H^0(X, K^*_X(\mathcal{O}_X)) \) are birational invariants of \( X \) smooth and proper (cf. [11]).

The proof of the Theorem 1 is quite natural and easy after a sheaf form of the projection formula (cf. Lemma 2.1 and (13)). We will first give an explicit description of the map \( f^\# \).

### 2.1 Functoriality

If we are given a morphism \( f : X \rightarrow Y \) then, for any open \( V \) subset of \( Y \), there is a homomorphism \( H^*(V, \cdot) \rightarrow H^*(f^{-1}(V), \cdot) \) induced by \( f \) simply because \( H^*(\cdot, \cdot) \) is a contravariant functor; thus, with the notation previously introduced, we get indeed a map
\[
f^H : \mathcal{H}_Y^*(\cdot) \rightarrow f_* \mathcal{H}_X^*(\cdot)
\]
of presheaves on \( Y \). Moreover there is a canonical map \( f_* \mathcal{H}_X^*(\cdot) \rightarrow f_* \mathcal{H}_X^*(\cdot) \) induced by sheafification and direct image. Hence, taking the associated sheaves, one get
\[
\begin{array}{ccc}
\mathcal{H}_Y^*(\cdot) & \xrightarrow{a f^H} & a f_* \mathcal{H}_X^*(\cdot) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{H}_X^*(\cdot)
\end{array}
\]

Thus the map \( f^\# : \mathcal{H}_Y^*(\cdot) \rightarrow f_* \mathcal{H}_X^*(\cdot) \) is defined to be the composite of \( a f^H \) and \( f_* \) as above.

### 2.2 Key Lemma

In the following, till the end of this subsection, we will let \( f : X \rightarrow Y \) be a proper birational morphism between smooth algebraic schemes over a perfect field. Our goal is to prove that \( f^\# \) is an isomorphism of sheaves.

**Step 1.** We can reduce to proving Theorem 1 for irreducible schemes because if not then, from smoothness, the irreducible components coincide with the connected components and \( f \) maps components to components; hence, if \( X_0 \) and \( Y_0 \) are components such that \( f : X_0 \rightarrow Y_0 \) and \( y \in Y_0 \) then there are isomorphisms on the stalk \( \mathcal{H}_Y^*(\cdot)_y \cong \mathcal{H}_Y^*(\cdot)_y \) and \( (f_* \mathcal{H}_X^*(\cdot))_y \cong (f_* \mathcal{H}_X^*(\cdot))_y \). If \( X \) is irreducible and \( K(X) \) is the function field of \( X \) then we denote
\[
H^*(K(X), \cdot) \overset{\text{def}}{=} \lim_{U \text{open} \subset X} H^*(U, \cdot).
\]
$H^\ast(K(X), \cdot)$ is canonically contravariant and birationally invariant. Hence $f$ induces an isomorphism $H^\ast(K(Y), \cdot) \cong H^\ast(K(X), \cdot)$. So, we moreover assume $X$ and $Y$ irreducibles.

Step 2. Case $H_0^\ast(\cdot)$. Assume that the cohomology theory is concentrated in positive degrees i.e. $H^i(\cdot) = 0$ if $i < 0$; hence the arithmetic resolution yields an isomorphism $H^i_{X,Y}(\cdot) \cong i_X H^0(\cdot) = i_Y H^0(\cdot)$ on $X$. The same holds on $Y$. Since $f$ has connected fibres (‘Zariski main theorem’) then $H^i_{X,Y}(\cdot) \cong f_* H^i_{X,Y}(\cdot)$. The non bounded case is considered below.

Step 3. So, associated to $f : X \to Y$, by (3) and (8), we can construct a diagram

$$
\begin{align*}
0 & \to H^i_Y(j) \to i_Y H^i(K(Y),j) \to H^i_{Z,Y}(j) \to 0 \\
0 & \to f_* H^i_X(j) \to f_* (i_X H^i(K(X),j)) \to f_* (H^i_{Z,X}(j))
\end{align*}
$$

(7)

where the right most vertical arrow (it will be seen explicitly below) is defined by commutativity of the left hand square. (Note: because $f$ has connected fibres then the middle vertical map is an isomorphism. The commutativity is straightforward.)

From the above diagram one can see that $f_*^Z : H^*_X(\cdot) \to f_* H^*_X(\cdot)$ is injective. Because of $f$ proper, and the arithmetic resolution is covariant for proper maps, we do aim to get the following commutative diagram

$$
\begin{align*}
0 & \to H_Y^i(j) \to i_Y H^i(K(Y),j) \to \bigoplus_{y \in Y^1} i_y H^{i-1}(y,j-1) \\
0 & \to f_* H_X^i(j) \to f_* (i_X H^i(K(X),j)) \to f_* \left( \bigoplus_{x \in X^1} i_x H^{i-1}(x,j-1) \right)
\end{align*}
$$

where $f_*$ is an injection. The Theorem 1 is obtained by proving: $f_*^Z \circ f_*^Z = id$ as a consequence of the projection formula. Indeed we have:

**Lemma 2.1** Let $f : X \to Y$ be a proper birational morphism between irreducible algebraic smooth schemes. Then there are maps of sheaves ($k = 0, 1$):

$$
f^Z_k : H^i_{Z,Y}(j) \to f_* (H^i_{Z,X}(j))
$$

and

$$
f^Z_k : f_* (H^i_{Z,X}(j)) \to H^i_{Z,Y}(j)
$$

such that

$$
f^Z_k \circ f^Z_k = id
$$

(Remind: $f^Z_k = f_k$ and $f^Z_0 = f^Z$.)

**Proof** We will follow the framework given by Grothendieck in [21, III.9.2].

Note that $f(X)$ is closed and dense in $Y$ irreducible: $f(X) = Y$. For all $Z \in Z^k(Y)$ so that $f^{-1}(Z) \in Z^k(X)$ ($k = 0, 1$) we have a map $H^i_Z(Y, \cdot) \to H^{i-1}_{f^{-1}(Z)}(X, \cdot)$ and since $f$ is a proper morphism between smooth schemes we have also maps $H^i_{f^{-1}(Z)}(X, \cdot) \to H^i_Z(Y, \cdot)$ for all $Z \in Z^k(Y)$. We then have:
Sublemma 2.2 The composition:

$$H^i_{Z}(Y, \cdot) \xrightarrow{f^*} H^i_{f^{-1}(Z)}(X, \cdot) \xrightarrow{f_\ast} H^i_{Z}(Y, \cdot)$$

is the identity.

**Proof** Let $H_\ast(\cdot, \dagger)$ denote the ‘twin’ homology theory and consider the pairing:

$$\cap_{Y,Z} : H_1(Y, m) \otimes H^1_Z(Y, s) \rightarrow H_{1-r}(Z, m-s)$$

Denote $f_! : H_\ast(f^{-1}(T), \dagger) \rightarrow H_\ast(T, \dagger)$ the homomorphisms induced, by covariance, from the proper maps $f^{-1}(T) \rightarrow T$ for every closed subset $T$ of $Y$. Let $f^* : H^\ast_Z(Y, \cdot) \rightarrow H^\ast_{f^{-1}(Z)}(X, \cdot)$ be the map given by contravariance. Because of [10, Axiom 1.3.3] we have the projection formula:

$$f_!(x \cap_{X,f^{-1}(Z)} f^\ast(y)) = f_!(x)_\cap_{Y,Z} y$$

for every $x \in H_1(X, m)$ and $y \in H^1_Z(Y, s)$. Let $\eta_X$ denote the fundamental class in $H_{2d}(X, d)$ where $d = \dim X$. Because of [10, 7.1.2] and $f$ proper birational we get: $f_!(\eta_X) = \eta_Y$. Thus the projection formula yields the equation:

$$f_!(\eta_X \cap_{X,f^{-1}(Z)} f^\ast(y)) = \eta_Y \cap_{Y,Z} y$$

By Poincaré duality [10, 1.3.5] the cap product with the fundamental class is an isomorphism; we define

$$f_!(z) \overset{\text{def}}{=} (\eta_Y \cap_{Y,Z} -)^{-1} f_!(\eta_X \cap_{X,f^{-1}(Z)} z)$$

for all $z \in H^1_{f^{-1}(Z)}(X, \cdot)$. Thus: $f_! f^\ast = 1$.

Taking the direct limit of the concerned maps over $Z \in Z^k(Y)$ (note: because $f$ is closed the direct system $\{f^{-1}(Z) : Z \in Z^1(Y)\}$ is cofinal in $Z^1(X)$) we have that the composition

$$H^i_{Z^k(Y)}(Y, \cdot) \xrightarrow{f^*_{Z^k}} H^i_{Z^k(X)}(X, \cdot) \xrightarrow{f_{Z^k}^\ast} H^i_{Z^k(Y)}(Y, \cdot)$$

is the identity as a consequence of the Sublemma 2.2 and limit arguments (the compatibilities are given by [10, 1.1.2 and 1.2.4]).

Because of [10, 1.2.2 and 1.4] the maps $f^*_{Z^k}$ are natural trasformations of Zariski presheaves $H^i_{Z^k,Y}(\cdot) \rightarrow f_! H^i_{Z^k,X}(\cdot)$ on $Y$. Thus, taking the associated sheaves, we have that:

$$\mathcal{H}^i_{Z^k,Y}(j) \xrightarrow{af^*_{Z^k}} \mathcal{H}^i_{Z^k,X}(j) \xrightarrow{af^\ast_{Z^k}} \mathcal{H}^i_{Z^k,Y}(j)$$

is the identity. Now it suffices to make up a commutative diagram as follows

$$\begin{array}{ccc}
\mathcal{H}^i_{Z^k,Y}(j) & \xrightarrow{af^*_{Z^k}} & \mathcal{H}^i_{Z^k,X}(j) \\
\xrightarrow{f^*_{Z^k}} & \downarrow & \xrightarrow{f^\ast_{Z^k}} \\
\mathcal{H}^i_{Z^k,Y}(j) & \xrightarrow{af^\ast_{Z^k}} & \mathcal{H}^i_{Z^k,Y}(j) \\
\end{array}$$

(8)
From (8) we then have:

\[ {f^Z}_1 \circ {f^Z}_k = {f^Z}_1 \circ {f^Z}_k \circ a {f^Z}_k = a {f^Z}_1 \circ a {f^Z}_k = id \]

as claimed. Indeed $f^Z_2$ is simply defined by composition; since $f$ is proper, $\dim X = \dim Y$ and the arithmetic resolution is covariant for proper maps: $f^Z_1$ is obtained, e.g. $f^Z$ from the commutativity and the exactness of the following:

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^{i+1}(Z, \cdot) & \rightarrow & \prod_{y \in Y^1} i_y H^{i-1}(y, j-1) & \rightarrow & \prod_{y \in Y^2} i_y H^{i-2}(y, j-2) & \\
& & \uparrow & & \uparrow & & \\
0 & \rightarrow & f_*(H^{i+1}(Z, \cdot)) & \rightarrow & f_*(\prod_{x \in X^1} i_x H^{i-1}(x, j-1)) & \rightarrow & f_*(\prod_{x \in X^2} i_x H^{i-2}(x, j-2)) & \\
\end{array}
\]

The proof of the Lemma 2.1 is complete.

\[ \bullet \]

### 2.3 Proof of the Invariance Theorem

To summarize the proof: if $k = 0, 1$ and $Z \in Z^k(Y)$ then $f^{-1}(Z) \in Z^k(X)$; we have a splitting between long exact sequences (cf. (9))

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & H^i(Y, \cdot) & \rightarrow & H^i(Y - Z, \cdot) & \rightarrow & H^{i+1}_Z(Y, \cdot) & \rightarrow & \cdots \\
& & \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \\
\cdots & \rightarrow & H^i(X, \cdot) & \rightarrow & H^i(X - f^{-1}(Z), \cdot) & \rightarrow & H^{i+1}_{f^{-1}(Z)}(X, \cdot) & \rightarrow & \cdots \\
\end{array}
\]

Taking the direct limit of the concerned diagram over $Z \in Z^1(Y)$ we do get

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & H^i(Y, \cdot) & \rightarrow & H^i(K(Y), \cdot) & \rightarrow & H^{i+1}_{Z(Y)}(Y, \cdot) & \rightarrow & \cdots \\
& & \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \\
\cdots & \rightarrow & H^i(X, \cdot) & \rightarrow & H^i(K(X), \cdot) & \rightarrow & H^{i+1}_{Z(X)}(X, \cdot) & \rightarrow & \cdots \\
\end{array}
\]

Thus, taking the associated sheaves, we have:

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & \mathcal{H}^i_Y(\cdot) & \rightarrow & i_Y H^i(K(Y), \cdot) & \rightarrow & \mathcal{H}^{i+1}_{Z(Y)}(\cdot) & \rightarrow & \cdots \\
& & \downarrow \uparrow & & \cong \downarrow \uparrow & & \downarrow \uparrow & & \\
\cdots & \rightarrow & a f_* H^i_X(\cdot) & \rightarrow & a f_* H^i(K(X), \cdot) & \rightarrow & a f_*(H^{i+1}_{Z(X)}(\cdot)) & \rightarrow & \cdots \\
\end{array}
\]

and furthermore

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & a f_* H^i_X(\cdot) & \rightarrow & a f_* H^i(K(X), \cdot) & \rightarrow & a f_*(H^{i+1}_{Z,X}(\cdot)) & \rightarrow & \cdots \\
& & \downarrow & & \cong \downarrow & & \downarrow & & \\
0 & \rightarrow & f_*(\mathcal{H}^i_X(\cdot)) & \rightarrow & f_*(i_X H^i(K(X), \cdot)) & \rightarrow & f_*(H^{i+1}_{Z,X}(\cdot)) & \rightarrow & \cdots \\
\end{array}
\]

One then obtain by patching the diagram (7). Now because of Lemma 2.1 and (7) we do get the first claimed isomorphism $f^* : \mathcal{H}^i_Y(\cdot) \cong f_* \mathcal{H}^i_X(\cdot)$.
2.4 Proving the $K$-theory case

We now consider the Quillen $K$-theory of vector bundles. The proof of Theorem 1 is the analogous of the previous one by using the Gillet’s projection formula and the Gersten’s conjecture.

To prove the Theorem 1, arguing as in §2.2, we can assume $X$ and $Y$ irreducibles, $r = 0$ and $f_1 : f_1^*I_{o,X} \xrightarrow{\sim} I_{o,Y}$ given by $f_1(1_X) = 1_Y$ (because $K(X) \cong K(Y)$). Hence, by (5), we obtain

$$f_1^*K_p(\mathcal{O}_X) \cong f_1^*I_{o,X} \xrightarrow{\sim} \cap f_1^*1_X K_p(\mathcal{O}_Y).$$

One defines $f_1^* : f_1^*K_p(\mathcal{O}_X) \rightarrow K_p(\mathcal{O}_Y)$ in the derived category, as follows:

$$f_1^* \overset{\mathcal{D}}{\Rightarrow} f_1^* \cap f_1^* 1_X K_p(\mathcal{O}_Y)$$

Thus: $f_1^* f_1^* = id$. (Note: the map $\cap f_1^* 1_X$ is not a quasi-isomorphism in general, but it induces an isomorphism on homology in degree zero because $f_1$ is left exact; indeed, taking $h^0$ the zero homology of a complex, we have the commutative diagram of sheaves

$$f_1^* \overset{\mathcal{D}}{\Rightarrow} f_1^* \cap f_1^* 1_X K_p(\mathcal{O}_Y) \bigg/ \underset{\mathcal{D}}{f_1^* I_{p,X}} \cong h^0(f_1^* I_{p,X}) \bigg/ \underset{\mathcal{D}}{f_1^* I_{p,Y}} \cong h^0(I_{p,Y})$$

and $f_1^* f_1^* = id$ between sheaves on $Y$.)

Associated to $f : X \rightarrow Y$ proper birational morphism between regular (irreducible) algebraic schemes, we have

$$K_p(\mathcal{O}_Y) \xleftarrow{f_1^*} f_1^*K_p(\mathcal{O}_X) \xrightarrow{\sim} i_Y K_p(K(Y))$$

so that by the same argument as in §2.3 we do get the second claimed isomorphism $f_1^* : K_p(\mathcal{O}_Y) \xrightarrow{\sim} f_1^*K_p(\mathcal{O}_X)$.

**Remark:** Assuming the Gersten’s conjecture and applying the above argument one can see that $f_1^*$ is an isomorphism if $f$ is a proper birational morphism between regular biequidimensional schemes.

3 Homotopy and proto-decomposition

We maintain the notations and the assumed ‘cohomology theory’ introduced in the previous Section (see §1). Let $P^n_X$ be the scheme $X \times_k \text{Proj} k[t_0, \ldots, t_n]$; let $\pi_n : P^n_X \rightarrow X$ denote the
canonical projection on $X$ smooth and equidimensional in $V_k$ and assume $k$ perfect. For any couple of non-negative integers $n \geq m$ let $j_{(n,m)}$ denote the ‘Gysin homomorphism’ (see §3.1 below)

$$H^p(P_X^n, \mathcal{H}^q) \to H^{p+n-m}(P_X^n, \mathcal{H}^{q+n-m}_{j(n,m)})$$
given by the smooth pair $(P_X^n, P_X^m)$ of pure codimension $n - m$; if $m \geq l$ is another couple, i.e. $(P_X^n, P_X^l)$ is a pair, then $j_{(n,l)} = j_{(n,m)} \circ j_{(m,l)}$. Let $A^1_X$ denote the scheme $X \otimes_k k[t]$ and assume that the cohomology theory satisfies the following.

**Homotopy property.** Let $X$ be an algebraic smooth scheme. The natural morphism

$$\pi^*: H^* (X, \cdot) \cong H^*(A^1_X, \cdot)$$

by pulling-back along $\pi$.

For $\mathcal{E}$ a locally free sheaf on $X$, rank $\mathcal{E} = n + 1$ and $\pi: V(\mathcal{E}) \to X$ the associated vector bundle, we then get the isomorphism (see §3.2 below)

$$H^p_Z (X, \mathcal{H}_X^q (j)) \cong H^p_{\pi^{-1}(Z)} (V(\mathcal{E}), \mathcal{H}_X^q (j))$$
pulling back along $\pi$ where $Z \subseteq X$ is any closed subset. Furthermore, it is now possible to prove the following Dold-Thom type decomposition.

**Theorem 2** (Proto-decomposition) Let $X$ be algebraic, equidimensional and smooth over a perfect field. Assuming the homotopy property above then there is an isomorphism

$$H^p (P_X^n, \mathcal{H}_X^q (j)) \cong \bigoplus_{i=0}^{n} H^{p-i} (X, \mathcal{H}_X^{q-i} (j-i))$$

where every $x \in H^p (P_X^n, \mathcal{H}_X^q (j))$ is written as

$$\pi^*_n (x_n) + j_{(n,n-1)} \pi^*_{n-1} (x_{n-1}) + \cdots + j_{(n,1)} \pi^*_1 (x_1) + j_{(n,0)} (x_0)$$

for $x_{n-i} \in H^{p-i} (X, \mathcal{H}_X^{q-i} (j-i))$ and $i = 0, \ldots, n$.

**Remark:** Note that for $\mathcal{E}$ a locally free sheaf on $X$, rank $\mathcal{E} = n + 1$, we will obtain the decomposition of $H^p (P(\mathcal{E}), \mathcal{H}_X^q (j))$ in Scholium 7.3.

Before proving the Theorem 2 we need the following results.

### 3.1 Gysin maps for $\mathcal{H}$-cohomologies

The category $V_k^2$ is the category of pairs of algebraic schemes over a perfect field $k$.

**Lemma 3.1** (Purity) If $(X, Z)$ is a pure smooth pair in $V_k^2$, codim$_X Z = c$, then $H^p_Z (X, \mathcal{H}_X^q (j))$ is canonically isomorphic to $H^{p-c} (Z, \mathcal{H}_Z^{q-c} (j-c))$.
Proof Let \( \mathcal{R}_q(j) \) denote the arithmetic resolution of the sheaf \( \mathcal{H}^q(j) \) on \( X \) (resp. on \( Z \)) and denote \( H^0(X, \mathcal{R}_q(j)) \stackrel{\text{def}}{=} R_0^q(X)(j) \) (resp. \( R_0^q(Z)(j) \)). Then:

**Sublemma 3.2** For \( Z \subset X \) of pure codimension \( c \):

\[
H^0_Z(X, \mathcal{R}_q^*(j)) \cong R_{q-c}(Z)(j-c)[-c]
\]

**Proof** Straightforward.

Since \( \mathcal{R}_q^*(j) \) is a bounded complex (graded by codimension) of flasque sheaves the hypercohomology spectral sequence \((h^n(C^*) \stackrel{\text{def}}{=} \text{the } n\text{th homology group of a complex } C^*)\)

\[
'E_2^{p,q} = h^p(H^q_Z(X, \mathcal{R}_q^*(j))) \Rightarrow H^{p+q}_Z(X, \mathcal{R}_q^*(j))
\]

degenerates to isomorphisms

\[
h^p(H^0_Z(X, \mathcal{R}_q^*(j))) \cong H^p_Z(X, \mathcal{R}_q^*(j))
\]

Taking account of the Sublemma 3.2, because of the (flasque) arithmetic resolutions, we do get a chain of isomorphisms

\[
H^p_Z(X, \mathcal{H}_X^q(j)) \cong H^p_Z(X, \mathcal{R}_q^*(j))
\]

\[
\cong h^p(H^0_Z(X, \mathcal{R}_q^*(j)))
\]

\[
\cong h^p(R_{q-c}^*(Z)(j-c))
\]

\[
\cong H^{p-c}(Z, \mathcal{H}_X^{q-c}(j-c))
\]

The proof of the Lemma is complete.

**Scholium 3.3** (Gysin map) Let \( (X, Z) \in \mathcal{V}_k^2 \) be a smooth pair of pure codimension \( c \). There is an homomorphism

\[
j_{(X,Z)} : H^p(Z, \mathcal{H}_Z^q(j)) \rightarrow H^{p+c}(X, \mathcal{H}_X^{q+c}(j+c))
\]

such that if \( (Z,T) \) is another smooth pair then \( j_{(X,T)} = j_{(X,Z)} \circ j_{(Z,T)} \).

**Proof** The map \( j_{(X,Z)} \) is induced on cohomology by the composition in the derived category (by sheafifying the isomorphism in the Sublemma 3.2 and using the arithmetic resolutions)

\[
j_* \mathcal{H}_Z^q(j) \xrightarrow{\sim} j_* \mathcal{R}_q^*(j) \xrightarrow{\sim} \Gamma_Z \mathcal{R}_{q+c,X}(j+c)[c] \rightarrow \mathcal{R}_{q+c,X}(j+c)[c] \xrightarrow{\sim} \mathcal{H}_X^{q+c}(j+c)[c]
\]

where \( j : Z \rightarrow X \) and \( \Gamma_Z \) are the sections supported in \( Z \). The compatibility simply follows by considering the resolutions \( \mathcal{R}^* \) and observing that a global section of \( \mathcal{R}^* \) on \( T \) can be seen as a section of \( \mathcal{R}^* \) on \( Z \) supported in \( T \), shifted by the codimension of \( T \) in \( Z \), etc., as a section of \( \mathcal{R}^* \) on \( X \) shifted by \( \text{codim}_T Z + \text{codim}_Z X = \text{codim}_X X \).
3.2 Homotopy for $\mathcal{H}$-cohomologies

For $X \in \mathcal{V}_k$ smooth over $k$ perfect, we recall (see [10, 6.3]) that exists a spectral sequence (‘coniveau’)

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\cdot)) \Rightarrow H^{p+q}(X, \cdot)$$

**Lemma 3.4** (Homotopy) If the functor $H^*(\cdot, \cdot)$ has the homotopy property then the functor $H^*_\text{ar}(\cdot, \mathcal{H}^*(\cdot))$ has the homotopy property.

**Proof** Let $\pi : A_X^1 \rightarrow X$ be the given structural flat morphism over $X$ smooth. We will show that $\pi^* : H^*_T(X, \mathcal{H}^*(\cdot)) \cong H^*_T(\pi(A_X^1), \mathcal{H}^*(\cdot))$ for any closed subscheme $T \subseteq X$. The proof is divided in two steps.

First step: reducing to the function field case $A_K^1 \rightarrow K$. This is done using a trick by Quillen [29, Prop.4.1]. Associated to $\pi$ and $Z \subset T \subset X$ closed subsets, $U = X - Z$, we have a map of long exact sequences:

$$\cdots \rightarrow H^*_T(X, \mathcal{H}^*(\cdot)) \rightarrow H^*_T(U, \mathcal{H}^*(\cdot)) \rightarrow H^*_Z(X, \mathcal{H}^*(\cdot)) \rightarrow \cdots$$

By the five lemma the induced (middle vertical) homomorphisms

$$H^*_T(X, \mathcal{H}^*(\cdot)) \rightarrow H^*_T(U, \mathcal{H}^*(\cdot))$$

are isomorphisms (all $p$) if the others vertical maps are. Using noetherian induction we can assume $H^*_Z(X, \mathcal{H}^*(\cdot)) \rightarrow H^*_Z(U, \mathcal{H}^*(\cdot))$ to be an isomorphism for all closed subsets $Z \neq T$ and all $p \geq 0$. We can also suppose $X$ irreducible. Taking the direct limit over all closed proper subschemes $Z$ of $T$ we can also assume that $T$ is integral of codimension $t$.

Thus by local purity we are left to show that

$$\lim_{U = X - Z} H^*_T(U, \mathcal{H}^*(\cdot)) \cong H^{t-T}(\text{Spec } K(T), \mathcal{H}^{s-t}(\cdot-0))$$

and

$$\lim_{U = X - Z} H^*_T(U, \mathcal{H}^*(\cdot)) \cong H^{t-T}(A_K^1, \mathcal{H}^{s-t}(\cdot-0))$$

is an isomorphism for all $p$. (Note: the horizontal isomorphisms in (9) are obtained by continuity of the arithmetic resolution of the sheaf $\mathcal{H}^*(\cdot)$).

Second step: proving the function field case $K = K(X)$. Having defined

$$H^*(A_K^1, \cdot) \overset{\text{def}}{=} \lim_{U \subset X} H^*(A_U^1, \cdot)$$

by continuity of the coniveau spectral sequence we do have

$$E_2^{p,q} = H^p(A_K^1, \mathcal{H}^q(\cdot)) \Rightarrow H^{p+q}(A_K^1, \cdot)$$
and $E_{2}^{p,q} = 0$ if $p > 1 = \dim A_{K}^{1}$, thus all the differentials are zero which just yields short exact sequences

$$0 \rightarrow H^{1}(A_{K}^{1}, \mathcal{H}^{q-1}(-)) \rightarrow H^{q}(A_{K}^{1}, \cdot) \rightarrow H^{0}(A_{K}^{1}, \mathcal{H}^{q}(-)) \rightarrow 0$$

Associated to the flat map $A_{K}^{1} \rightarrow K$ we have a commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & H^{1}(A_{K}^{1}, \mathcal{H}^{q-1}(-)) \\
\uparrow & & \uparrow \\
H^{q}(K, \cdot) & \rightarrow & H^{0}(K, \mathcal{H}^{q}(-))
\end{array}
$$

(10)

where: because $H^{*}(\cdot, \cdot)$ has the homotopy property then the middle vertical map is an isomorphism while the arrow $H^{0}(A_{K}^{1}, \mathcal{H}^{q}(-)) \rightarrow H^{q}(K, \cdot)$ is the evaluation at any $K$-rational point (cf. [9, Proof of 2.5]). From (10) it follows that $H^{1}(A_{K}^{1}, \mathcal{H}^{q-1}(-)) = 0$ hence the required isomorphism in (9) is given by: $H^{0}(K, \mathcal{H}^{q}(-)) \cong H^{0}(A_{K}^{1}, \mathcal{H}^{q}(-))$

**Scholium 3.5** Let $A_{X}^{n}$ denote the $n^{th}$ affine space over $X$ smooth (i.e. the scheme $X \otimes_{k} k[t_{1}, \ldots, t_{n}]$) and let $\pi : A_{X}^{n} \rightarrow X$ be the natural projection. Then $\pi$ induces an isomorphism

$$H_{\pi^{-1}(Z)}^{\#}(A_{X}^{n}, \mathcal{H}^{*}(-)) \cong H_{Z}^{\#}(X, \mathcal{H}^{*}(-))$$

**Proof** By induction from the Lemma 3.4.

**Corollary 3.6** Let $\mathcal{E}$ be a locally free sheaf on $X$ smooth and $\pi : V(\mathcal{E}) \rightarrow X$ the associated vector bundle, we then get the isomorphism

$$\pi^{*} : H_{\pi^{-1}(Z)}^{p}(X, \mathcal{H}^{q}(-)) \cong H_{Z}^{p}(V(\mathcal{E}), \mathcal{H}^{q}(-))$$

**Proof** By reduction to open Zariski neighborhoods on which $\mathcal{E}$ is free and noetherian induction (cf. the proof of Lemma 3.4).

### 3.3 Proof of the proto-decomposition Theorem

The proof of Theorem 2 is by induction on $n$. For $n = 0$, $P_{X}^{0} \cong X$, hence the induction starts: one consider an ‘hyperplane at infinity’ $\infty$ in $P_{X}^{n}$ so that $\infty \cong P_{X}^{n-1}$ and $P_{X}^{n} - \infty \cong A_{X}^{n}$. There is a standard long exact sequence of Zariski cohomology groups

$$H^{p-1}(A_{X}^{n}, \mathcal{H}^{q}(-)) \rightarrow H_{\infty}^{p}(P_{X}^{n}, \mathcal{H}^{q}(\cdot)) \rightarrow H^{p}(P_{X}^{n}, \mathcal{H}^{q}(\cdot)) \rightarrow H^{p}(A_{X}^{n}, \mathcal{H}^{q}(-))$$

Let $\pi_{n} : P_{X}^{n} \rightarrow X$ be the projection. By homotopy (see § 3.2) the restriction $\pi_{n} | A_{X}^{n} : A_{X}^{n} \rightarrow X$ induces a splitting of the previous long exact sequence, given by the commutative square

$$
\begin{array}{ccc}
& & \\
H^{p}(P_{X}^{n}, \mathcal{H}^{q}(\cdot)) & \rightarrow & H^{p}(A_{X}^{n}, \mathcal{H}^{q}(\cdot)) \\
\uparrow & \cong & \uparrow \\
H^{p}(X, \mathcal{H}^{q}(\cdot)) & \rightarrow & H^{p}(X, \mathcal{H}^{q}(\cdot))
\end{array}
$$
By the purity Lemma 3.1 we have an isomorphism
\[ H^n_\infty(X, \mathcal{H}^q(j)) \cong H^{p-1}(X, \mathcal{H}^{q-1}(j)) \]
and, by composing, the Gysin map (see 3.1)
\[ j_{(n,n-1)} : H^{p-1}(X, \mathcal{H}^{q-1}(j-1)) \to H^p(X, \mathcal{H}^q(j)) \]
So one can split the long exact sequence into short exact sequences
\[ 0 \to H^{p-1}(X, \mathcal{H}^{q-1}(j-1)) \xrightarrow{j_{(n,n-1)}} H^p(X, \mathcal{H}^q(j)) \xrightarrow{\pi_n^{*}} H^p(X, \mathcal{H}^q(j)) \to 0 \]
Thus we do get the formula:
\[ H^p(X, \mathcal{H}^q(j)) \cong H^p(X, \mathcal{H}^q(j)) \oplus H^{p-1}(X, \mathcal{H}^{q-1}(j-1)) \]
which does the induction’s step: an element \( x \in H^p(X, \mathcal{H}^q(j)) \) is written as \( \pi_n^*(x_n) + j_{(n,n-1)}(x') \) for \( x_n \in H^p(X, \mathcal{H}^q(j)) \) and \( x' \in H^{p-1}(X, \mathcal{H}^{q-1}(j-1)) \); this last \( x' \) because of the inductive hypothesis is written as
\[ \pi_{n-1}^*(x_{n-1}) + j_{(n-1,n-2)}(x_{n-2}) + \cdots + j_{(n-1,0)}(x_0) \]
for \( x_{n-1-i} \in H^{p-1-i}(X, \mathcal{H}^{q-1-i}(j-1-n)) \) and \( i = 0, \ldots, n-1 \) and because of the compatibility of the Gysin homomorphisms (see Scholium 3.3) applying \( j_{(n,n-1)} \) we are done.

4 Cap products

This section is devoted to construct a cap product between algebraic cycles and \( \mathcal{H} \)-cohomology classes.

4.1 Sophisticated Poincaré duality theories

Let assume we are given a cohomology theory \( H^*(\cdot) \) and a homology theory \( H_*(\cdot) \) on \( V_k \) satisfying the Bloch-Ogus axioms [10, 1.1-1.2]. Furthermore, we let assume the existence of a sophisticated cap-product with supports i.e. for all \( (X, Z), (X, Y) \in V^2_k \) a pairing
\[ \cap_{Y,Z} : H_n(Y, m) \otimes H^q_Z(X, s) \to H_{n-q}(Y \cap Z, m-s) \]
which satisfies the following axioms:

A1 \( \cap \) is natural with respect to étale maps (or just open Zariski immersions according with [10, 1.4.2]) of pairs in \( V^2_k \).

A2 If \( (X, T), (T, Z) \) are pairs in \( V^2_k \) then the following diagram
\[ \begin{array}{ccc}
H_n(Z, m) \otimes H^q(X, s) & \xrightarrow{\cap} & H_{n-q}(Z, m-s) \\
\downarrow & & \downarrow \\
H_n(T, m) \otimes H^q(X, s) & \xrightarrow{\cap} & H_{n-q}(T, m-s)
\end{array} \]
commutes.
**A3** For \((X,T),(T,Z)\) pairs in \(V_k^2\) let \(U = X - Z\) and let \(j : U \to X\) be the inclusion. Let denote \(\partial : H_n(T \cap U, \cdot) \to H_{n-1}(Z, \cdot)\) the boundary map in the long exact sequence \((3)\) of homology groups. Then the following diagram

\[
\begin{array}{c}
H_n(T \cap U, \cdot) \otimes H^q(U, \cdot) \\
\downarrow \partial \otimes \text{id} \\
H_{n-1}(Z, \cdot) \otimes H^q(X, \cdot) \\
\downarrow \partial \otimes \text{id} \\
H_{n-1}(Z, \cdot) \otimes H^q(X, \cdot) \\
\end{array}
\]

commutes i.e. we have the equation:

\[
\partial(y \cap j^*(x)) = \partial(y) \cap x
\]

for \(y \in H_n(T \cap U, \cdot)\) and \(x \in H^q(X, \cdot)\).

**A4** *(Projection Formula)* Let \(f : X' \to X\) be a proper morphism in \(V_k\). For \((X,Y)\) and \((X,Z)\) let \(Y' = f^{-1}(Y)\) and \(Z' = f^{-1}(Z)\). The following diagram

\[
\begin{array}{c}
H_n(Y',m) \otimes H^q_{Z'}(X',s) \\
\downarrow \partial \otimes \text{id} \\
H_n(Y,m) \otimes H^q_Z(X,s) \\
\downarrow f_* \otimes \text{id} \\
H_n(Z,m) \otimes H^q(X,s) \\
\end{array}
\]

commutes.

By the way, for \((H^*, H_*)\) as above, we have the following ‘projection formula’:

**Scholium 4.1** Let \(f : Y \to X\) be a proper morphism in \(V_k\). Let \(T\) be any closed subset of \(Y\) and let \(f(T) = Z\). Then the following diagram

\[
\begin{array}{c}
H_n(T,m) \otimes H^q(Y,s) \\
\downarrow \partial \otimes \text{id} \\
H_n(T,m) \otimes H^q(X,s) \\
\downarrow f_* \otimes \text{id} \\
H_n(Z,m) \otimes H^q(X,s) \\
\end{array}
\]

commutes.

**Proof** This is a simple consequence of A4 by observing that \(T \hookrightarrow f^{-1}(Z)\).

**Definition 4.2:** We will say that \((H^*, H_*)\) is a *sophisticated* Poincaré duality theory with supports if the axioms A1–A4 are satisfied and Poincaré duality holds i.e. the Bloch-Ogus axioms [10, 1.3.4-5 and 7.1.2] are satisfied (see §1.1).
4.2 \( \mathcal{H} \)-cap product

Associated with the homology theory \( H_* \), for \( X \in \mathcal{V}_k \) possibly singular, we have a niveau spectral sequence (cf. \cite{10}, Prop.3.7)

\[
E_{a,b}^1 = \prod_{x \in X} H_{a+b}(x, \cdot) \Rightarrow H_{a+b}(X, \cdot)
\]

which is covariant for proper morphisms and contravariant for étale maps. Let denote \( Q^q(X)_n \) the (homological) complex \( E_{a,b}^1 \).

**Proposition 4.3** Let \( H^* \) and \( H_* \) be cohomological and homological functors satisfying the axioms A1–A3 above. For \( X \in \mathcal{V}_k \) there is a pairing of complexes

\[
Q^n(X)_m \otimes H^q(X, s) \to Q^{n-q}(X)_m
\]

contravariant w.r.t. étale maps.

**Proof** Let \( Z \subset T \subset X \) be closed subsets of \( X \), \( \dim T \leq a \), \( \dim Z \leq a-1 \) and let \( U = X - Z \); thus by restriction to \( U \) and cap-product we do have a pairing associated to such pairs \( Z \subset T \):

\[
H_i(T \cap U, j) \otimes H^q(X, s) \to H_{i-q}(T \cap U, j-s)
\]

i.e. \( t \otimes x \sim t \cap j^*(x) \) where \( j : U \hookrightarrow X \). By taking the direct limit over such pairs (this makes sense because of A1–A2) we do have a pairing

\[
\prod_{x \in X_a} H_{n+a}(x, m) \otimes H^q(X, s) \to \prod_{x \in X_a} H_{n-q+a}(x, m-s)
\]

We need to check compatibility with the differentials of \( Q^n(X)_m \). Because of A2 we have a pairing \( H_i(Z_{a,j}) \otimes H^q(X, s) \to H_{i-q}(Z_{a,j-s}) \) where \( H_i(Z_{a,j}) \) is the direct limit over \( r \subset X \) and let \( T \) as above. Because of A3 and limit arguments the following diagram

commutes (indeed \( \partial(tr \cap j^*(x)) = \partial(t) \cap x \) ) and A1 implies that the following

\[
H_{i-1}(Z_{a-1,j}) \otimes H^q(X, s) \to \prod_{x \in X_{a-1}} H_{i-1}(x, j) \otimes H^q(X, s)
\]

\[
H_{i-q-1}(Z_{a-1,j-s}) \to \prod_{x \in X_{a-1}} H_{i-q-1}(x, j-s)
\]
commutes. By construction, the differential is the composition of

\[ Q^n_a(X)(m) \to H_{n+a-1}(Z_{a-1}, m) \to Q^n_{a-1}(X)(m). \]

Thus the result.

**Definition 4.4**: For \( X \in \mathcal{V}_k \), by taking associated sheaves for the Zariski topology of the pairing above, we get a pairing

\[ \cap_H : Q^n_{*,X}(m) \otimes H^q_X(s) \to Q^{n-q}_{*,X}(m-s) \]

which we call \( H\text{-cap-product} \) on \( X \).

### 4.3 Projection formula

Let \( f : Y \to X \) be a proper morphism in \( \mathcal{V}_k \). We do have a map of niveau spectral sequences

\[ E^1_{p,q}(r)(Y) \to E^1_{p,q}(r)(X) \]

which takes \( y \in Y_p \) to \( f(y) \) if \( \dim \{ f(y) \} = \dim \{ y \} \) zero otherwise and maps \( H_i(y) \) to \( H_i(f(y)) \). Thus by sheafifying it for the Zariski topology we obtain a map

\[ f^*_\#: f^*Q^n_{*,Y}(m) \to Q^n_{*,X}(m) \]

of complexes of sheaves on \( X \).

**Proposition 4.5** The following diagram:

\[
\begin{array}{ccc}
Q^n_{*,Y}(m) \otimes f^*H^q(s) & \xrightarrow{id \otimes f^*} & f^*_\# \cap_H Q^n_{*,X}(m) \\
\downarrow f_* \otimes id & & \downarrow f_* \\
Q^n_{*,X}(m) \otimes H^q(s) & \xrightarrow{\cap_H} & Q^n_{*,X}(m)
\end{array}
\]

(13)

commutes.

**Proof** The commutative diagram above will be obtained from the following:

\[
\begin{array}{ccc}
Q^n(f^{-1}(U))(m) \otimes H^q(f^{-1}(U), s) & \xrightarrow{id \otimes f^*} & Q^n(f^{-1}(U))(m-s) \\
\downarrow f_* \otimes id & \xrightarrow{\cap} & \downarrow f_* \\
Q^n(U)(m) \otimes H^q(U, s) & \xrightarrow{\cap} & Q^n(U)(m-s)
\end{array}
\]

where \( U \subset X \) is any Zariski open subset of \( X \), by taking associated sheaves on \( X_{Zar} \).

Moreover it suffices to prove the case of \( U = X \).

Let \( \{ y \} \subset Y \) such that \( y \in Y_p \) and \( f(y) \in X_p \). The maps

\[ f_{*,y} : \lim_{V \subset Y} H_*((\{ y \} \cap V) \to \lim_{U \subset X} H_*((\{ f(y) \}) \cap U) \]

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are defined by mapping the elements of \( H_*(\{y\} \cap (Y - T)) \) by restriction and the induced maps
\[
f_{t, y} : H_*(\{y\} \cap (Y - f^{-1}(Z))) \to H_*(\{f(y)\} \cap (X - Z))
\]
where \( Z = f(T) \) (Note: compatibilities are ensured by [10, 1.2.2]). By the definition of the pairing in Proposition 4.3 we are left to show that the following diagram
\[
\begin{array}{ccc}
H_*(& id \otimes f^* & \cap) \\
& \downarrow \downarrow &
\hline
& f_{t, y} & \cap \\
H_*(& \{f(y)\} \cap U, \cdot) \otimes H^q(U, s) & \downarrow \downarrow
\end{array}
\]
commutes where \( f^{-1}(U) = V \). Since \( f(\{y\} \cap f^{-1}(U)) = \{f(y)\} \cap U \) the diagram (14) commutes because of the projection formula in the Scholium 4.1.

Lemmma 4.6 If \( i : Z \hookrightarrow X \) is a closed embedding then the canonical map \( i_* : i_* Q^n_*(m) \to Q^n_*(m) \) admits a factorisation by a quasi-isomorphism \( i_Z : i_* Q^n_*(m) \xrightarrow{\sim} \Gamma Z Q^n_*(m) \) and the natural map \( \Gamma Z Q^n_*(m) \hookrightarrow Q^n_*(m) \)

Proof Clear (cf. the proof of the Scholium 3.3).

Corollary 4.7 For \( i : Z \hookrightarrow X \) as above the following diagram
\[
\begin{array}{ccc}
& i_* Q^n_*(m) \otimes i_* H^q(s) & \\
& \downarrow \downarrow &
\hline
& i_* Q^n_*(m) \otimes H^q(s) & \cap
\end{array}
\]
commutes.

Proof This follows by the factorisation of \( i_* \) and the projection formula (cf. the Lemma and the Proposition above).

4.4 Algebraic cycles

We are now going to consider the cycle group naturally involved with a fixed theory \((H^*, H_*)\) on \( V_k \). To this aim we need to assume a ‘dimension axiom’ (cf. [10, 7.1.1]). Let assume that our cohomology theory \( H^* \) takes values in a fixed category of \( \Lambda \)-modules where \( \Lambda = H^0(k, o) \) is a commutative ring with 1.

Definition 4.8: We will say that \((H^*, H_*)\) satisfies the dimension axiom when the following properties

A5 If \( \text{dim} X \leq d \) then \( H_i(X, m) = 0 \) for \( i > 2d \).
If $X$ is irreducible then the canonical map $\lambda^* : \Lambda \to H^0(X, 0)$, induced by the structural morphism $\lambda : X \to k$, is an isomorphism.

hold for any $X \in V_k$. We will say that $(H^*, H_*)$ satisfies the point axiom if the properties A5-A6 above just hold locally, at the generic point of any integral subvariety of each $X \in V_k$.

**Remark:** Clearly the dimension axiom implies the point axiom.

Thus: if $X$ is reduced and $\Sigma$ is its singular locus then

$$H_i(X, m) \cong H_i(X - \Sigma, m)$$

for $i \geq 2d$ by applying A5 to the long exact sequence of homology groups. In particular, if $X$ is integral of dimension $d$ then by A6

$$H_{2d}(X, d) \cong H_{2d}(X - \Sigma, d) \cong H^0(X - \Sigma, 0) \cong \Lambda$$

Regarding $Q^n_\cdot$ as a (cohomological) complex of flasque sheaves graded by negative degrees we do have

$$H^{-p}(X, Q^n_\cdot) \cong \ker\left( \prod_{x \in X_p} H_{n+p}(x, m) \to \prod_{x \in X_{p-1}} H_{n+p-1}(x, m) \right) \cong \im\left( \prod_{x \in X_{p+1}} H_{n+p+1}(x, m) \to \prod_{x \in X_p} H_{n+p}(x, m) \right)$$

In particular, for $n = p = m$ and the 'dimension axiom' above we do have

$$H^{-n}(X, Q^n_\cdot) \cong \coker\left( \prod_{x \in X_{n+1}} H_{2n+1}(x, n) \to \prod_{x \in X_n} \Lambda \right)$$

where $\prod_{x \in X_n} \Lambda$ is the $\Lambda$-module of algebraic cycles of dimension $n$ in $X$.

**Definition 4.9:** Let $(H^*, H_*)$ be a theory satisfying the point axiom. We will denote

$$C_n(X; \Lambda) \overset{\text{def}}{=} H^{-n}(X, Q^n_\cdot)$$

the corresponding group of $n$-dimensional algebraic $\Lambda$-cycles modulo the equivalence relation given by

$$\im\left( \prod_{x \in X_{n+1}} H_{2n+1}(x, n) \to \prod_{x \in X_n} \Lambda \right)$$

the image of the differential of the niveau spectral sequence.

For $i : Z \hookrightarrow X$ a closed embedding we clearly do have an isomorphism

$$C_n(Z; \Lambda) \cong H^{-n}(Z, Q^n_\cdot)$$

Thus, by taking hypercohomology with supports, the $H$-cap-product yields a cap product

$$C_n(Z; \Lambda) \otimes H^p_Y(X, H^p_\cdot) \to C_{n-p}(Z \cap Y; \Lambda)$$

Because of the projection formula (13) this cap product will have a projection formula as well.
5 Cup products

We are now going to show that the cup-product in cohomology give us a nice intersection theory for $\mathcal{H}$-cohomology. To this aim we will assume the algebraic schemes in $\mathcal{V}_k$ to be equidimensional and $k$ to be a perfect field. The main results hold true just for non-singular varieties nevertheless we will not assume this hypothesis a priori.

5.1 Multiplicative Poincaré duality theories

Let assume we are given a twisted cohomology theory $H^\ast(\cdot)$ on $\mathcal{V}_k$ in the sense of Bloch-Ogus [10, 1.1]. Furthermore, we want to assume the existence of a cup-product i.e. for all $(X, Z), (X, Y) \in \mathcal{V}_k^2$ an associative anticommutative pairing

$$\cup_{Y, Z} : H^p_Y(X, r) \otimes H^q_Z(X, s) \to H^{p+q}_{Y \cap Z}(X, r+s)$$

which satisfies the following axioms:

\begin{enumerate}
  \item $\cup$ is natural with respect to pairs in $\mathcal{V}_k^2$
  \item If $(X, T), (T, Z)$ are pairs in $\mathcal{V}_k^2$ then the following diagram

$$
\begin{array}{ccc}
H^p_T(X, r) \otimes H^q(X, s) & \xrightarrow{\cup} & H^{p+q}_T(X, r+s) \\
\downarrow & & \downarrow \\
H^p_T(X, r) \otimes H^q(X, s) & \xrightarrow{\cup} & H^{p+q}_T(X, r+s)
\end{array}
$$

commutes.
  \item For $(X, T), (T, Z)$ pairs in $\mathcal{V}_k^2$ let $U = X - Z$ and let $j : U \to X$ be the inclusion. Let denote $\partial : H^p_{T \cap U}(U, \cdot) \to H^{p+1}_T(U, \cdot)$ the boundary map in the long exact sequence (11) of cohomology with supports. Then the following diagram

$$
\begin{array}{ccc}
H^p_{T \cap U}(U, \cdot) \otimes H^q(U, \cdot) & \xrightarrow{id \otimes j^*} & H^p_{T \cap U}(U, \cdot) \otimes H^q(U, \cdot) \\
\quad \downarrow{\partial \otimes id} & \quad \downarrow{\partial} & \quad \downarrow{\partial} \\
H^{p+1}_Z(U, \cdot) \otimes H^q(U, \cdot) & \xrightarrow{\cup} & H^{p+q+1}_Z(U, \cdot)
\end{array}
$$

commutes i.e. we have the equation:

$$\partial(y \cup j^*(x)) = \partial(y) \cup x \quad (17)$$

for $y \in H^p_{T \cap U}(U, \cdot)$ and $x \in H^q(U, \cdot)$.

\end{enumerate}

**Definition 5.1:** A twisted cohomology theory with supports $H^\ast$ has a *cup-product* if there is a pairing $\cup_{Y, Z} : H^p_Y(X, r) \otimes H^q_Z(X, s) \to H^{p+q}_{Y \cap Z}(X, r+s)$ which satisfies the axioms $\forall 1-3$ listed above.

If furthermore $(H^\ast, H_\ast)$ is a Poincaré duality theory we let assume that the following compatibility between cap and cup products holds:
∀4 For $X$ smooth of dimension $d$ let $\eta_X \in H_{2d}(X, d)$ be the fundamental class. Then the following diagram, where $q + j = 2d, s + n = d$,
\[
H^q(X, s) \otimes H^p_Z(X, r) \xrightarrow{id \otimes f^*} H^q(X, s) \otimes H^p_Z(X, r) \xrightarrow{\eta_X \cap - \otimes id} H_j(X, n) \otimes H^p_Z(X, r) \xrightarrow{\cap} H_{j-p}(Z, n-r)
\]
commutes i.e. we have the equation:
\[
(\eta_X \cap y) \cap z = \eta_X \cap (x \cup z)
\]
(18)
for $x \in H^q(X, s)$ and $z \in H^p_Z(X, r)$.

Let $f : Y \to X$ be a proper map of smooth equidimensional algebraic schemes. Let $\dim X = d$ and $\dim Y = d$. Let $\rho = d - d$. For $Z \subseteq X$ a closed subset there are maps
\[
f_!: H_{2d-i}(f^{-1}(Z), d-i) \to H_{2d-i}(Z, d-i)
\]
Because of Poincaré duality $f_!$ induces a Gysin map
\[
f^*: H^i(Z, i) \to H^{i+2\rho}(X, i+\rho)
\]
which is uniquely determined by the equation
\[
f_!(\eta_Y \cap y) = \eta_X \cap f^*(y)
\]
(19)
for $y \in H^i_{f^{-1}(Z)}(Y, i)$. Thus $H^*$ is a covariant functor w.r.t. proper maps of pairs $(X, Z)$ where $Z$ is a closed subset of $X$ smooth: indeed $(1_X)_* = id$ because of $(1_X)_! = id$ and $(f \circ g)_* = f_* \circ g_*$ because of $(f \circ g)_!(\eta_Y \cap -) = f_!(g_!(\eta_Y \cap -)) = f_!(g_!\eta_\cap g_*(-)) = \eta_f \cap f_*(g_*(-))$. The projection formula by [10] w.r.t. the cap product give us, via ∀4, the following projection formula:
\[
H^p(Y, r) \otimes H^q_{f^{-1}(Z)}(Y, s) \xrightarrow{id \otimes f^*} H^p(Y, r) \otimes H^q_Z(X, s) \xrightarrow{f_* \otimes id} H^{p+q}_{f^{-1}(Z)}(Y, r+s) \xrightarrow{\cup} H^p_{f^{-1}(Z)}(Y, r+s) \xrightarrow{f_*} H^{p+q+2\rho}(X, r+s+\rho)
\]
Indeed we have:
\[
f_!(\eta_Y \cap (y \cup f^*(x))) = \eta_X \cap (f_!(\eta_Y \cap y) \cup f^*(x)) = \eta_X \cap (f_!(\eta_Y \cap y) \cap x) = (\eta_X \cap f_*(y)) \cap x = (\eta_X \cap f_*(y) \cup x).
\]

Because of the lack of symmetry of the projection formula stated above we need to assume the following “projection formula with supports”.

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∀5 For \( f : Y \to X \), \( Z \) and \( \rho \) as above, the following:

\[
H_{f^{-1}(Z)}^p(Y,r) \otimes H^q(Y,s) \\
\xrightarrow{id \otimes f^*} \\
H_{f^{-1}(Z)}^p(Y,r) \otimes H^q(X,s) \\
\xrightarrow{f_* \otimes id} \\
H_{Z}^{p+2\rho}(X,r+\rho) \otimes H^q(X,s) \\
\xrightarrow{\cup} \\
H_{Z}^{p+q+2\rho}(X,r+s+\rho)
\]

commutes, i.e. we have the equation \( f_*(y \cup f^*(x)) = f_*(y) \cup x \) for \( x \in H^q(X,s) \) and \( y \in H_{f^{-1}(Z)}^p(Y,r) \).

As a matter of fact, in order to obtain the projection formula (24) we just need the following apparently weaker but almost equivalent form of ∀5.

∀5’ Let \( i : Z \hookrightarrow X \) be a smooth pair of pure codimension \( c \). Then:

\[
H^p(Z,r) \otimes H^q(Z,s) \\
\xrightarrow{id \otimes i^*} \\
H^p(Z,r) \otimes H^q(X,s) \\
\xrightarrow{i_* \otimes id} \\
H_{Z}^{p+2c}(X,r+c) \otimes H^q(X,s) \\
\xrightarrow{\cup} \\
H_{Z}^{p+q+2c}(X,r+s+c)
\]

commutes.

By the way ∀5 implies ∀5’. (Convention: the purity isomorphism \( i_* \) is induced by the identity on \( Z \).)

Scholium 5.2 Let \( i : Z \hookrightarrow X \) be a smooth pair. Then the following square

\[
H_{Z}^{p}(X,r) \otimes H^q(X,s) \\
\xrightarrow{\eta_X \cap - \otimes i^*} \\
H_{Z}^{p+q}(X,r+s) \\
\xrightarrow{\eta_X \cap -} \\
H_{2d-p}(Z,d-r) \otimes H^q(Z,s) \\
\xrightarrow{\cap} \\
H_{2d-p-q}(Z,d-r-s)
\]

commutes, i.e. we have the following formula:

\[
\eta_X \cap (z \cup x) = (\eta_X \cap z) \cap i^*(x) \tag{20}
\]

for \( z \in H_{Z}^{p}(X,r) \) and \( x \in H^q(X,s) \), if and only if ∀5’ holds.

Proof Let \( c \) be the codimension of \( Z \subset X \) and \( d = \dim X \).

Let assume that ∀5’ holds. Since we do have the purity isomorphism \( i_* : H^{p-2c}(Z,r-c) \xrightarrow{\cong} H_{Z}^{p}(X,r) \) there is an element \( \zeta \in H^{p-2c}(Z,r-c) \) such that \( i_*(\zeta) = z \). Because of ∀5’ the equation (24) is obtained by showing the following equality

\[
\eta_X \cap i_*(\zeta \cup i^*(x)) = (\eta_X \cap i_*(\zeta)) \cap i^*(x) \tag{21}
\]

Note: \( \eta_X \cap - : H^{p-2c}(Z,r-c) \xrightarrow{\cong} H_{2d-p}(Z,d-r) \) and \( \eta_X \cap - = \eta_X \cap i_*(-) \) by the definition of \( i_* \); thus the right-hand side in the equation (21) above becomes

\[
(\eta_X \cap \zeta) \cap i^*(x) \tag{22}
\]

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and the left-hand side becomes
\[ \eta_{Z}(\zeta \cap i^{*}(x)) \]
which by \( \forall 4 \) (see the equation (18)) is exactly the same of (22).

Conversely, if (20) holds we then have
\[
\eta_{X} \cap i_{*}(\zeta \cap i^{*}(x)) = (\eta_{X} \cap i_{*}(\zeta)) \cap i^{*}(x) = (\eta_{Z} \cap i_{*}(\zeta)) \cap i^{*}(x) = \eta_{Z} \cap (\zeta \cap i^{*}(x)) = \eta_{X} \cap i_{*}(\zeta \cap i^{*}(x)) = \eta_{X} \cap i_{*}(\zeta \cap i^{*}(x)).
\]

Since \( \eta_{X} \cap \cdot \) is an isomorphism we can erase it from the left of the resulting equation.

Remark: As we will see below (cf. Lemma 5.9): the formula (20), by restriction and the projection formula w.r.t. cap-product, allow us to deduce \( \forall 5 \) for \( X, Y, Z \) and \( f^{-1}(Z) \) smooth. Moreover, by assuming the formula (20) holds for \( Z \) possibly singular we can as well obtain \( \forall 5 \).

Definition 5.3: Let \((H^{*}, H_{*})\) be a Poincaré duality theory with supports and let assume that \( H^{*} \) has a cup-product (so that \( \forall 1-\forall 3 \) are satisfied). We will say that \((H^{*}, H_{*})\) is \textit{multiplicative} if the axioms \( \forall 4 \) and \( \forall 5' \) (or the strong form \( \forall 5 \)) are satisfied.

5.2 \( \mathcal{H} \)-cup products

Let \( H^{*} \) be a twisted cohomology theory with supports on \( \mathcal{V}_{k} \). Let \( X \in \mathcal{V}_{k} \) be equidimensional but possibly singular. Applying the exact couple method to the exact sequence (3)
\[
H^{i}_{Z}(X, j) \to H^{i}_{Z}(X, j) \to \bigoplus_{x \in X} H^{i}_{x}(X, j) \to H^{i+1}_{Z}(X, j)
\]
where: \( H^{i}_{x}(X, j) \overset{\text{def}}{=} \lim_{U \subset X} H^{i}_{x}(U, j), \) we do get the coniveau spectral sequence
\[
E_{1}^{p,q} = \bigoplus_{x \in X} H^{q+p}(X, \cdot) \Rightarrow H^{p+q}(X, \cdot)
\]

Let denote \( R^{\cdot}_{q}(X)(r) \) the corresponding Gersten type complexes \( E_{1}^{\cdot,q} \).

Proposition 5.4 Let \( H^{*} \) be a twisted cohomology theory with supports and cup-product on \( \mathcal{V}_{k} \). For \( X \in \mathcal{V}_{k} \) there is a pairing of complexes
\[
R^{\cdot}_{q}(X)(r) \otimes H^{p}(X, s) \to R^{\cdot}_{q+n}(X)(r+s)
\]
contravariant w.r.t. flat maps.
Proof Let $Z \subseteq T \subseteq X$ with $Z \in Z_p^p(X)$ and $T \in Z_p^p(X)$ and let $U = X - Z$; thus by restriction to $U$ and cup-product we do have a pairing associated to such pairs $Z \subseteq T$:

$$H^i_{T \cap U}(U, r) \otimes H^n(X, s) \to H^{i+n}_{T \cap U}(U, r+s)$$

i.e. $t \otimes x \sim t \cup j^*(x)$ where $j : U \hookrightarrow X$. By taking the direct limit over such pairs (this makes sense because of $\forall 1-\forall 2$) we do have a pairing

$$\prod_{x \in X^p} H^{q+p}_x(X, r) \otimes H^n(X, s) \to \prod_{x \in X^p} H^{q+n+p}_x(X, r+s)$$

In order to check compatibility with the differentials of $R \cdot q(X)$, because of $\forall 2$ we have a pairing $H^i_{Z^p(X)}(X, j) \otimes H^n(X, s) \to H^{i+n}_{Z^p(X)}(X, j+s)$ and, by construction, the differential is the composition of

$$R^p_q(X)(r) \to H^{q+p+1}_{Z^p+i}(X, r) \to R^p_{q+1}(X)(r)$$

we can argue as in the proof of the Proposition 4.3 via $\forall 3$ and limit arguments.

Definition 5.5: For $X \in V_k$, by taking associated sheaves for the Zariski topology of the pairing above, we get a cap-product pairing

$$\mathcal{R}_q^p(X)(r) \otimes \mathcal{H}^p_X(s) \to \mathcal{R}_{p+q}(X)(r+s)$$

By sheafifying the cup-product we do have a product

$$\cup_{\mathcal{H}} : \mathcal{H}^p_X(r) \otimes \mathcal{H}^q_X(s) \to \mathcal{H}^{p+q}_{X}(r+s)$$

which we call $\mathcal{H}$-cup-product on $X$.

Remark: We list several expected compatibilities.

1. The $\mathcal{H}$-products above are compatible via the canonical augmentations $\mathcal{H}^p_X(r) \to \mathcal{R}_p^p(X)(r)$. But, if $X$ is singular the augmentations are not quasi-isomorphisms.

2. Let suppose the existence of an external product

$$\times : H^p_{Z^p}(X, r) \otimes H^q_{T^q}(Y, s) \to H^{p+q}_{Z \times T}(X \times Y, r+s)$$

functorial on $V^2_k$ i.e. we have the equation

$$(f \times g)^*(x \times y) = f^*(x) \times g^*(y) \quad (23)$$

for $f$ and $g$ maps of pairs. Thus, by composing with the diagonal $\Delta : (X, Z \cap T) \to (X \times X, Z \times T)$, we obtain a cup-product satisfying the axiom $\forall 1$.

In this case the pairing defined in the Proposition 5.4 can be obtained as follows

$$H^i_{T \cap U}(U, r) \otimes H^n(X, s) \ni t \otimes x \sim \Delta^*(1 \times j)^*(t \times x)$$

where $j : U \hookrightarrow X$. 

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3. On a smooth variety $X$, after $\forall 4$, we have that the pairing defined in Proposition 5.4 is Poincaré dual of the pairing defined as follows: let $j : U \hookrightarrow X$, if $\xi \in H^*_X(X, \cdot)$ and $t \in H^*_{\{p\} \cap U}(U, \cdot)$ then $\xi \otimes t \sim j^*(x) \cap t$. In fact: $\xi = \eta_X \cap x$ for some $x \in H^*(X, \cdot)$ and we then have:

\[
\eta_U \cap (j^*(x) \cap t) = (\eta_U \cap j^*(x)) \cap t = j^*(\eta_X \cap x) \cap t.
\]

by $\forall 4$

\[\text{[10, 1.3.2-4]}\]

4. On a smooth variety $X$, after the equation [20], we have that the pairing defined in the Proposition 5.4 can be obtained by restriction as follows. Let $t \in \bigcap_{\{p\} \cap U}(U, \cdot)$ and $x \in H^*(X, \cdot)$ where $X$ is smooth and $\{p\} \cap U \subset U$ is a smooth pair. Thus $\eta_U \cap (t \cdot j^*(x))$ is equal to $(\eta_U \cap i^*(x))$ where $i : \{p\} \cap U \hookrightarrow X$.

5.3 $\mathcal{H}$-Gysin maps

Let $(H^*, H_*)$ be a Poincaré duality theory with supports. For $X \in \mathcal{V}_k$ we have a niveau spectral sequence $E^{1}_{a, b} \Rightarrow H_{a+b}(X, \cdot)$ which is covariant for proper morphisms. For $k$ perfect and $X$ smooth equidimensional, $d = \dim X$, by local purity, we do have that $H^q_{\ast}(X, r) \cong H^{q-p}(x, r-p)$ if $x \in X^p$ and isomorphisms

\[
E^{1}_{d-p,d-q(d-r)} = \bigoplus_{x \in X_{d-p}} H^{2d-p-q(x, d-r)} \cong \bigoplus_{x \in X^p} H^{q-p}(x, r) \cong E^{p,q}(r)
\]

**Lemma 5.6** If $X \in \mathcal{V}_k$ is smooth of pure dimension $d$ then $E^{1}_{d-p,d-q(d-r)} \cong E^{p,q}(r)$ is an isomorphism of spectral sequences which is natural w.r.t. étale maps.

**Proof** This is a consequence of the above once we have identified (via Poincaré duality) the long exact sequence of cohomology with supports with the corresponding long exact sequence of homology groups.

Let $f : Y \rightarrow X$ be a proper morphism between $k$-algebraic schemes where $\dim X = \delta$ and $\dim Y = d$. Let $\rho = \delta - d$. Since $E^{1}(\cdot)$ is covariant w.r.t. proper maps we do have a map of niveau spectral sequences

\[
E^{1}_{d-p,d-q(d-r)}(Y) \rightarrow E^{1}_{d-p,d-q(d-r)}(X)
\]

If $X$ and $Y$ are smooth and equidimensions, then, via the Lemma 5.6, we get a map of coniveau spectral sequences as follows:

\[
E^{p,q}(r)(Y) \cong E^{1}_{d-p,d-q(d-r)}(Y) \rightarrow E^{1}_{d-p,d-q(d-r)}(X) \cong E^{p+\rho,q+\rho}(r)(X)
\]
**Definition 5.7:** For \( f : Y \to X \) as above we will call the induced map of complexes \( R_q'(Y)(r) \to R_{q+r}(X)(r+r)[\rho] \) the *global Gysin map*. By taking associated sheaves we have the local Gysin map

\[
f_b : f_*R_q'(Y) \to R_{q+r}(X)(r+r)[\rho].
\]

In the derived category we do have the \( \mathcal{H} \)-Gysin map

\[
R_fR : R_f\mathcal{H}_Y^q(r) \to \mathcal{H}_X^{q+r}(r+r)[\rho].
\]

**Remark:** For \( i : Z \to X \) a (smooth) pair of pure codimension \( c \) we do have the isomorphism (cf. [3, 5, 3]):

\[
i_Z : i_*\mathcal{R}_{q,Z}^r(r) \xrightarrow{\sim} \Gamma Z\mathcal{R}_{q+c,X}^r(r+c)[c].
\]

Thus, as it is easily seen, the local Gysin map \( i_* \) is obtained by composition of \( i_Z \) with the canonical map \( \Gamma Z\mathcal{R}_{q+c,X}^r(r+c)[c] \to \mathcal{R}_{q+c,X}^r(r+c)[c] \) (see the proof of Scholium [3, 3]).

### 5.4 Projection formula

Let \( f : Y \to X \) be a proper morphism between smooth equidimensional algebraic schemes over a perfect field \( k \). Let \( \dim X = \delta \) and \( \dim Y = d \). Let \( \rho = \delta - d \).

**Proposition 5.8** For \( f : Y \to X \) as above we have the following commutative diagram:

\[
\begin{array}{ccc}
  f_*\mathcal{R}_{q,X}^r(r) \otimes f_*\mathcal{H}^p(s) & \xrightarrow{id \otimes f^*} & f_*\mathcal{R}_{q,X}^r(r) \otimes f_*\mathcal{H}^p(s) \\
  f_*\mathcal{R}_{q,X}^r(r) \otimes f_*\mathcal{H}^p(s) & \xrightarrow{f_* \cap \mathcal{H}} & f_*\mathcal{R}_{q,X}^r(r+s) \\
  f_*\mathcal{R}_{q,X}^r(r) \otimes f_*\mathcal{H}^p(s) & \xrightarrow{\cap \mathcal{H}} & f_*\mathcal{R}_{q,X}^r(r+r+s) \\
\end{array}
\]

**Proof** The commutative diagram above will be obtained from the following:

\[
\begin{array}{ccc}
  R_q'(f^{-1}(U))(r) \otimes H^p(f^{-1}(U), s) & \xrightarrow{id \otimes f^*} & R_q'(f^{-1}(U))(r) \otimes H^p(U, s) \\
  R_q'(f^{-1}(U))(r) \otimes H^p(U, s) & \xrightarrow{f_* \cap \mathcal{H}} & R_{p+q}(f^{-1}(U))(r+s) \\
  R_{q+\rho}(U)(r+\rho)[\rho] \otimes H^p(U, s) & \xrightarrow{\cap \mathcal{H}} & R_{p+q+\rho}(U)(r+s+\rho)[\rho] \\
\end{array}
\]

where \( U \subset X \) is any Zariski open subset of \( X \), by taking associated sheaves on \( X_{zar} \). Moreover it suffices to prove the case of \( U = X \).

Let \( \overline{y} \subset Y \) such that \( y \in Y^c \) and \( f(y) \in X^{c+\rho} \). The Gysin map (cf. [5, 3, 5, 4])

\[
f_{y} : \lim_{V \subset Y} H_{s}(\overline{y}) \cap V \to \lim_{U \subset X} H_{s}(\overline{f(y)}) \cap U
\]

is the Poincaré dual of \( f_{y} : \lim_{V \subset Y} H_{s}(\overline{y}) \cap V \to \lim_{U \subset X} H_{s}(\overline{f(y)}) \cap U \) (see the proof of [1, 3]).

By the definition of the pairing in Proposition [5, 4] we are left to show that the following diagram
\[ \text{Diagram (25)} \]

\[ \begin{align*}
H^p_{\{y\} \cap V}(V, \cdot) & \otimes H^q(V, \cdot) \\
\text{id} \otimes f'^* & \rightarrow u \\
H^p_{\{y\} \cap V}(V, \cdot) & \otimes H^q(U, \cdot) \\
f_y \otimes \text{id} & \downarrow \\
& \Rightarrow \Rightarrow \\
H^{p+q}_{\{y\} \cap U}(U, \cdot + \rho) & \otimes H^q(U, \cdot) \\
f_y & \downarrow \\
& \Rightarrow \Rightarrow \\
H^{p+q+2\rho}_{\{y\} \cap U}(U, \cdot + \rho) & \otimes H^q(U, \cdot) \\
\end{align*} \tag{25} \]

commutes where \( f^{-1}(U) = V \). By shrinking the open sets involved we may assume that \( \{y\} \cap V \subset V \) and \( \{f(y)\} \cap U \subset U \) are smooth pairs. Since \( f(\{y\} \cap f^{-1}(U)) = \{f(y)\} \cap U \) the diagram (25) commutes because of the following Lemma.

\[ \text{Lemma 5.9} \]

Let \( f : Y \rightarrow X \) and \( \rho \) as above. Let \( T \) be a closed subset of \( Y \), \( f(T) = Z \) and let assume that \( T \rightarrow Y \) and \( Z \rightarrow X \) are smooth pairs. Then the following diagram

\[ \begin{align*}
H^p_T(Y, r) & \otimes H^q(Y, s) \\
\text{id} \otimes f'^* & \rightarrow u \\
H^p_T(Y, r) & \otimes H^q(X, s) \\
f_y \otimes \text{id} & \downarrow \\
& \Rightarrow \Rightarrow \\
H^{p+q}_{Z+\rho}(X, r+s+\rho) & \otimes H^q(X, s) \\
\downarrow f_y & \Rightarrow \Rightarrow \\
& \Rightarrow \Rightarrow \\
H^{p+q+2\rho}_{Z+\rho}(X, r+s+\rho) & \otimes H^q(X, s) \\
\end{align*} \tag{26} \]

commutes.

\[ \text{Proof} \]

We will give two proofs.

\[ \text{First proof} \]

Let assume the axiom \( \forall 5 \) (in which case we do not need the smoothness of \( T \) and \( Z \)). Let \( i : T \hookrightarrow f^{-1}(Z) \) so that \( f \big|_T = f \circ i \) hence \( f \big|_T = f \circ i \circ i \) and \( f_y = f \circ i \circ i \) where \( i_0 : H^*_T(Y, \cdot) \rightarrow H^*_T(f^{-1}(Z), \cdot) \) is the canonical map. Thus, for \( y \in H^*_T(Y, \cdot) \) and \( x \in H^*_T(X, \cdot) \)

\[ f_y(y \cup f^*(x)) = \]

\[ = f \circ i \circ i(y \cup f^*(x)) = \quad \text{by \( \forall 2 \)} \\
\]

\[ = f_*(i_0(y) \cup f^*(x)) = \quad \text{by \( \forall 5 \)} \\
\]

\[ = f_*(i_0(y)) \cup x = \]

\[ = f_y(y) \cup x. \]

\[ \text{Second proof} \]

Just assume the axiom \( \forall 5' \) we can prove the Lemma as follows. Let \( k : T \hookrightarrow Y \), \( i : Z \hookrightarrow X \) and \( f \big|_T : T \rightarrow Z \). Thus we have: \( k^* f^* = (f \big|_T)^* i^* \). Let \( y \in H^*_T(Y, r) \) and \( x \in H^*_T(X, s) \). We have:

\[ \eta_X \cap f_y(y \cup f^*(x)) = \]

\[ = (f \big|_T)!(\eta_Y \cap (y \cup f^*(x))) = \]

\[ = (f \big|_T)!(\eta_Y \cap f^*(x)) = \]

\[ = (f \big|_T)!(\eta_Y \cap f^*(x)) = \quad \text{by \( [10] \ 1.3.3 \)} \\
\]

\[ = (f \big|_T)(\eta_Y \cap f^*(x)) = \]

\[ = (\eta_X \cap f_y(y)) \cap i^*(x) = \]

\[ = \eta_X \cap f_y(y) \cup x. \]

Since \( \eta_X \cap - \) is an isomorphism we conclude.

\[ \bullet \]
Corollary 5.10 For \( i : Z \hookrightarrow X \) a smooth pair of pure codimension \( c \) the following diagram

\[
\begin{array}{c}
i_* \mathcal{R}^\cdot \mathcal{H}^{p}(s) \\
i_* \mathcal{R}^\cdot \mathcal{H}^{p}(s) \\
i_* \mathcal{R}_p \mathcal{H}^{p}(s) \\
i_* \mathcal{H}^{p}(s)
\end{array}
\]

\[
\begin{array}{c}
\cong \mathcal{R}_{p+q} \mathcal{H}^{p}(s) \\
\cong \mathcal{H}^{p}(s) \\
\cong \mathcal{H}^{p}(s) \\
\cong \mathcal{H}^{p}(s)
\end{array}
\]

commutes.

Proof This follows by the factorisation of \( i \_\# \) (cf. the Remark at the end of §5.3) and the Proposition above.

\[\blacksquare\]

5.5 \( \mathcal{H} \)-cohomology ring

Let \((\mathcal{H}^*, \mathcal{H}_s)\) be a multiplicative Poincaré duality theory with supports. Let suppose that our cohomology theory \( \mathcal{H}^* \) takes values in a fixed category of \( \Lambda \)-modules where \( \Lambda = H^0(k, 0) \) is a commutative ring with 1; we assume that the bigraded \( \Lambda \)-module \( \bigoplus_{p,q} \mathcal{H}^p(X, r) \) has a \( \Lambda \)-algebra structure via the cup-product pairing e.g. a canonical isomorphism of rings \( H^0(X, 0) \cong \Lambda \) if \( X \) is irreducible.

Let

\[ A(X) \overset{\text{def}}{=} \bigoplus_{p,q,r} \mathcal{H}^p(X, \mathcal{H}^q(r)). \]

Then \( X \sim A(X) \) is a contravariant functor on \( \mathcal{V}_k \). If \( f : Y \to X \) is a proper map of relative dimension \( \rho \) then the \( \mathcal{H} \)-Gysin maps \( f_* \mathcal{H}^{\rho}(r) \to \mathcal{H}^{\rho + r + \rho}[\rho] \) induce direct image \( A(Y) \to A(X) \) (a map of degree \( \rho \)) so that \( A \) is a covariant functor w.r.t. proper maps of smooth varieties. From the \( \mathcal{H} \)-cup-product pairing by taking cohomology we have an external pairing

\[ \times : A(X) \otimes \Lambda A(Y) \to A(X \times Y) \]

which is associative and anticommutative (can be made commutative by using the trick in [15]). In particular, let consider the functor

\[ X \sim \bigoplus_p \mathcal{H}^p(X, \mathcal{H}^p(0)) \overset{\text{def}}{=} A_{\text{diag}}(X). \]

If \( \mathcal{H}^0(0) \) is identified with the flasque sheaf \( \bigsqcup X \mathcal{H} \) (e.g. by assuming the ‘dimension axiom’) we then have an augmentation \( \varepsilon : A_{\text{diag}}(X) \to \Lambda \) where \( X \) is irreducible and \( \varepsilon^0 : H^0(X, \mathcal{H}^0(0)) \cong \Lambda \) zero otherwise.

Let denote \( f_* \) and \( f^* \) “direct and inverse” images. We have the following formulas (cf. [14, 1.1–1.9]):

\[
\begin{align*}
(f \times g)^* (- \cdot) &= f^* (-) \times g^* (-) \\
(f \times g)_* (- \cdot) &= f_* (-) \times g_* (-) \\
\varepsilon(f^* (-)) &= \varepsilon (-) \\
\varepsilon(-f^* (-)) &= \varepsilon (-) \varepsilon (-)
\end{align*}
\]

(28)
Furthermore, for $X = \text{Spec}(k)$ we have that $\varepsilon : A_{\text{diag}}(k) \cong \Lambda$: let $e$ be the unique element such that $\varepsilon(e) = 1$. We have the formula: $ex = -xe = -e$.

Let $\lambda : X \to \text{Spec}(k)$ be the structural map and let denote $\lambda^{\ast}(e) \overset{\text{def}}{=} 1_X$. This is equal to $\varepsilon^{-1}(1)$ on $X$ irreducible. For $x \in A_{\text{diag}}(X)$ and $y \in A_{\text{diag}}(Y)$ we have

\[ x \times 1_Y = p_1^!(x) \]
\[ 1_X \times y = p_2^!(y) \]

(29)

where $p_1$ and $p_2$ are the first and the second projections of $X \times Y$ on its factors.

By composing the external product $\times : A_{\text{diag}}(X) \otimes \Lambda A_{\text{diag}}(X) \to A_{\text{diag}}(X \times X)$ with the diagonal $\Delta_X : A_{\text{diag}}(X \times X) \to A_{\text{diag}}(X)$ we do get a product $x \otimes x' \sim xx'$ in $A_{\text{diag}}(X)$ making it an associative anticommutative algebra with identity $1_X$. The homomorphism $\varepsilon : A_{\text{diag}}(X) \to \Lambda$ is an homomorphism of unitary $\Lambda$-algebras. For $f : X \to Y$ the map $f^* : A_{\text{diag}}(Y) \to A_{\text{diag}}(X)$ is a homomorphism of $\Lambda$-algebras. The external product is a homomorphism of augmented $\Lambda$-algebras. This last fact, via the equations (29), give us the formula

\[ xy = p_1^!(x)p_2^!(y) \]

(30)

for $x \in A_{\text{diag}}(X)$ and $y \in A_{\text{diag}}(Y)$.

For $f : X \to Y$ a proper map of smooth varieties over $k$, $A_{\text{diag}}$ satisfies the ‘projection formula’ as a consequence of the projection formula (24). Furthermore, if $f$ is surjective of relative dimension $\rho$ over $Y$ irreducible we have a canonical map

\[ \int_{X/Y} : H^{-\rho}(X, H^{-\rho}(-\rho)) \to \Lambda \]

and its extension by zero $\int : A_{\text{diag}}(X) \to \Lambda$, both defined by composition of the $H$-Gysin map $f_*$ and the augmentation $\varepsilon$. In particular

\[ \int_{X/k} : H^d(X, H^d(d)) \to \Lambda \]

for any $X$ proper smooth $d$-dimensional variety; for any map $f$ between $X$ and $Y$ proper smooth varieties we have

\[ \int_{Y/k} f_* = \int_{X/k} \]

Finally, if the proper map has a section $fs = 1$ then $f$ is a surjection; this is the case of $X/k$ having a $k$-rational point.

6 Intersection theory

Since we are going to deal with Poincaré duality theories which are ‘sophisticated’ and ‘multiplicatives’ we need to arrange the axioms in order to be not redundant. This arrangement will yields the notion of ‘duality theory appropriate for algebraic cycles’ or for short ‘appropriate duality theory’. We will show that the $H$-cohomology rings associated with such a theory reproduce the classical intersection rings. Roughly speaking, the denomination ‘appropriate duality’ is the corresponding cohomological version of ‘relation d’équivalence adéquate’ introduced by P. Samuel (see [30]) for algebraic cycles.
6.1 Axiomatic menuet

Let $H^*$ be a cohomology theory and let $H_*$ be a homology theory (as defined by [10, 1.1 and 1.2]). Let assume that the pair $(H^*, H_*)$ yields a sophisticated Poincaré duality which satisfies the dimension axiom; furthermore we assume the existence of an associative anticommutative functorial cup product pairing

$$H^p_Y(X, r) \otimes H^q_Z(X, s) \to H^{p+q}_{Y \cap Z}(X, r+s)$$

where $\Lambda = H^0(k, 0)$ is a commutative ring with 1 and the bigraded $\Lambda$-module $\bigoplus_{q,r} H^q(X, r)$ has a $\Lambda$-algebra structure via the cup-product pairing.

**Definition 6.1**: An appropriate duality theory is a pair $(H^*, H_*)$ as above such that the sophisticated cap product is compatible with the cup product via Poincaré duality i.e. the following diagram, where $q+j = 2d$, $s+n = d$ and $X$ is smooth

$$
\begin{array}{ccc}
H^q(Y, n) \otimes H^p_Z(X, r) & \xrightarrow{\eta_X \cap -} & H^{p+q}_{Y \cap Z}(X, r+s) \\
\downarrow \eta_X \cap - & & \downarrow \eta_X \cap - \\
H_j(Y, n) \otimes H^p_Z(X, r) & \xrightarrow{\cap} & H_{j-p}(Y \cap Z, n-r)
\end{array}
$$

commutes. In particular, the fundamental class $\eta_X \in H_{2d}(Xd)$ corresponds to the unit $1 \in \Lambda \cong H^0(X, 0)$ in the $\Lambda$-algebra structure.

Let $(H^*, H_*)$ be an appropriate duality theory on $V_k$. Then, by adopting the same notation of §4.4,

$$V_k \ni Y \leadsto C_{n,m}(Y; \Lambda(s)) \overset{\text{def}}{=} H^{-n}(Y, Q^m(s))$$

is a covariant functor for proper morphisms in $V_k$. It is a presheaf for the étale topology (or just for the Zariski topology, depending with the homology theory). On the other hand we have a contravariant functor

$$(X, Y) \leadsto H^p_Y(X, H^q(r))$$

which yields the $H$-cohomology ring with the properties stated in §5.3. Indeed, on a smooth scheme $X$ of pure dimension $d$, these two functors are related via the duality isomorphism

$$Q^{d-q}_{-r}(d-r)[d] \cong R^r_q$$

and this isomorphism is compatible with the $H$-cap and $H$-cup products (cf. §4.2 and §5.2); by construction this duality isomorphism identifies Gysin maps (cf. §5.3 and projection formulas (cf. (13) with (24))). Thus we do have a canonical cap-product associated with pairs $(X, Y)$ and $(X, Z)$

$$C_{n,m}(Y; \Lambda(s)) \otimes H^p_Z(X, H^q(r)) \to C_{n-p,m-q}(Y \cap Z; \Lambda(s-r))$$

and a corresponding projection formula. We have indeed a canonical “trace map” on $X$ irreducible

$$Q^d_{-r}(d)[d] \to \Lambda$$
yielding a global section \([X] \in C_d(X; \Lambda)\); by capping with this “fundamental class” \([X]\) we get the quasi-isomorphism

\[ \mathcal{H}^q(r) \xrightarrow{\cap [X]} Q_{-d}^{-q}(d-r)[d] \]

By taking hypercohomology with support on \(Z\) (a closed equidimensional subscheme) we do get the “duality” isomorphism

\[ \cap [X]: H^p_Z(X, \mathcal{H}^q(r)) \xrightarrow{\sim} C_{d-p,d-q}(Z; \Lambda(d-r)). \quad (31) \]

Conversely: for \(i: Z \hookrightarrow X\) we have a quasi-isomorphism

\[ i^* Q_{-d}^{-q}(d-r)[d] \xrightarrow{\sim} \Gamma_Z \mathcal{H}^q(r) \]

hence the canonical isomorphism

\[ \eta: C_{d-p,d-q}(Z; \Lambda(d-r)) \xrightarrow{\sim} H^p_Z(X, \mathcal{H}^q(r)) \quad (32) \]

In particular:

**Scholium 6.2** We have a commutative diagram

\[
\begin{array}{ccc}
C_{d-p}(Z; \Lambda) \otimes H^q_Y(Z, \mathcal{H}^q(\mathcal{Y})) & \xrightarrow{id \otimes \eta} & C_{d-p}(Y \cap Z; \Lambda) \\
\eta \otimes id & \downarrow & \eta \\
H^p_Z(X, \mathcal{H}^p(\mathcal{Y})) \otimes H^q_Y(X, \mathcal{H}^q(\mathcal{Y})) & \xrightarrow{\eta \otimes id} & H^{p+q}_Y(X, \mathcal{H}^{p+q}(\mathcal{Y}))
\end{array}
\]

**Proof** This is a consequence of the commutative diagram [13] and the compatibilities between \(\mathcal{H}\)-products. 

\[ \blacksquare \]

### 6.2 \(\mathcal{H}\)-cycle classes

We maintain the notations and the assumptions of the previous section. Let \(Z \subset X\) be a prime cycle of dimension \(d-c\). Then

\[ C_{d-c}(Z; \Lambda) \overset{def}{=} C_{d-c,d-c}(Z; \Lambda(d-c)) = H_{2d-2c}(K(Z), d-c) \cong \Lambda \]

by the ‘dimension axiom’ and we do have a cycle class

\[ \eta(Z) \in H^c_Z(X, \mathcal{H}^c(\mathcal{Y})) \quad (33) \]

where: \([Z] \in H_{2d-2c}(K(Z), d-c)\) is obtained by restriction of the fundamental class \(\eta_Z \in H_{2d-2c}(Z, d-c)\) to the generic point and we have

\[ \Lambda \ni 1 \sim [Z] \sim \eta(1) \overset{def}{=} \eta(Z) \in H^c_Z(X, \mathcal{H}^c(\mathcal{Y})). \]

Furthermore, by capping with the fundamental class \([X]\), we find the formula

\[ [Z] = \eta(Z) \cap [X] \quad (34) \]

In particular the cycle class \(\eta(Z)\) is independent from the imbedding of \(Z\) as a subvariety and it is functorial w.r.t. \(\acute{e}tale\) maps.
Lemma 6.3 For $Y$ and $Z$ prime cycles of codimension $p$ and $q$ in $X$ smooth we have
\[ \eta(Y \times Z) = \eta(Y) \times \eta(Z) \in H^{p+q}_{Y \times Z}(X \times X, \mathcal{H}^{p+q}(\mathcal{H}^{p+q}(p+q))). \]

Proof By standard sheaf theory the external product is obtained by using flasque resolutions (see [16, 6.2.1]). Thus via the canonical quasi-isomorphisms $\mathcal{H}(\mathcal{H}) \cong \mathcal{Q}(\mathcal{H})$ we do have a commutative diagram
\[
\begin{align*}
H^p_Y(X, \mathcal{H}^p(p)) \otimes H^q_Z(X, \mathcal{H}^q(q)) & \xrightarrow{\Delta} H^{p+q}_{Y \times Z}(X \times X, \mathcal{H}^{p+q}(p+q)) \\
C_{d-p}(Y; \Lambda) \otimes C_{d-q}(Z; \Lambda) & \xrightarrow{\Delta} C_{d-p-q}(Y \times Z; \Lambda)
\end{align*}
\]

To conclude one would see that the bottom arrow is in fact the external product of cycles: this last claim is clear because the external products are homomorphisms of $\Lambda$-algebras. (Note: $C_{\dim Y}(Y; \Lambda) \cong \Lambda$ by the dimension axiom).

Let $\Delta : X \to X \times X$ be the diagonal embedding and let $\Delta(X)$ be the diagonal cycle on $X \times X$. Let $Y$ and $Z$ be prime cycles of codimension $p$ and $q$ in $X$ smooth such that $Y \cap Z$ is of pure codimension $p + q$. For $d = \dim X$ we then have
\[
\begin{align*}
H^{p+q}_{Y \cap Z}(X, \mathcal{H}^{p+q}(p+q)) & \xrightarrow{\Delta} H^{p+q+d}_{(Y \times Z) \cap \Delta(X)}(X \times X, \mathcal{H}^{p+q+d}(p+q+d)) \\
C_{d-p-q}(Y \cap Z; \Lambda) & \xrightarrow{\Delta} C_{d-p-q}((Y \times Z) \cap \Delta(X); \Lambda)
\end{align*}
\]

where $\Delta$ is obtained by making the diagram commutative and $\Delta_*$ is induced by the isomorphism $\Delta : Y \cap Z \to (Y \times Z) \cap \Delta(X)$. Thus the formula
\[
\Delta_*(\eta(\mathcal{H})) = \eta(\Delta_*(\mathcal{H}))
\]

Lemma 6.4 We have the following formula:
\[
\Delta_*(\eta(Y \times Z)) = \eta(Y \times Z)\eta(\Delta(X))
\]

Proof The formula is a consequence of the projection formula in the Scholium 6.2 (cf. [13] and [27]) applied to the diagonal embedding $\Delta$ by taking $\mathcal{H}$-cohomology with supports on $Y \times Z$ and compatibility of the $\mathcal{H}$-products with the canonical augmentations.

6.3 Intersection of cycles

Let now assume that our cycle group $C_*(X; \Lambda)$ has an intersection product satisfying the classical properties (cf. [12]): local nature of the intersection multiplicity, normalization and reduction to the diagonal for $X$ a smooth projective variety over a field $k$.

Moreover, for a pair $(X, D)$ where $X \in \mathbb{V}_k$ and $D$ is a Cartier divisor on $X$ we let assume the existence of a homomorphism (cf. Definition 7.2 in §7.2 below)
\[
\text{cl} : H^1_D(X, \mathcal{O}^*_X) \otimes \Lambda \to H^1_D(X, \mathcal{H}^1(1))
\]
such that
(i) $c\ell$ is a natural transformation of contravariant functors w.r.t. morphisms $f : X' \to X$ such that $f^{-1}(D)$ is a divisor on $X'$;

(ii) $c\ell$ is compatible with the cap-products in the sense that the following
\[
\begin{align*}
H^1_D(X, \mathcal{O}_X^*) \otimes \Lambda & \xrightarrow{c\ell} H^1_D(X, \mathcal{H}^1(1)) \\
\cap \downarrow & \cap \downarrow \\
CH_{d-1}(D; \Lambda) & = C_{d-1}(D; \Lambda)
\end{align*}
\]
commutes, where $d = \dim X$.

**Remark:** The map $\cap[Z] : H^1_D(X, \mathcal{O}_X^*) \to CH_{d-1}(D)$ is given by the cap-product in $K$-cohomology with the canonical cycle $[X] \in CH_d(X)$; when applied to the cycle class of the Cartier divisor yields just the associated Weil divisor (cf. [15, §2]). By the way, if $X$ is non-singular then $c\ell$ is an isomorphism.

Thus we can prove the following key lemma.

**Lemma 6.5** Let $X$ be smooth. Let $D$ be a principle effective Cartier divisor and let $i : Z \hookrightarrow X$ be a closed integral subscheme of codimension $c$ in $X$ such that $Z \cap D$ is a divisor on $Z$. Then the following
\[
\begin{align*}
H^1_D(X, \mathcal{H}^1(1)) & \xrightarrow{i^*} H^1_D(Z, \mathcal{H}^1(1)) \\
\cap & \downarrow \cap \\
C_{d-1}(D; \Lambda) & \xrightarrow{i^*} C_{d-c-1}(Z \cap D; \Lambda)
\end{align*}
\]
commutes i.e. we have the following formula
\[
D \cdot Z = i^* \eta(D) \cap[Z] \tag{37}
\]

**Proof** The claimed commutative diagram is obtained by the corresponding one for the Picard groups (cf. [15, §2]). Let denote $\tilde{D} \in H^1_D(X, \mathcal{O}_X^*) \otimes \Lambda$ the canonical class of the Cartier divisor: thus $D \cap [X] = [D] \in CH_{d-1}(D; \Lambda)$; since $X$ is non-singular, $[D] \sim \tilde{D}$ under the isomorphism $CH_{d-1}(D; \Lambda) \cong H^1_D(X, \mathcal{O}_X^*) \otimes \Lambda$ and $[D] \sim \eta(D)$ under the isomorphism $CH_{d-1}(D; \Lambda) \cong H^1_D(X, \mathcal{H}^1(1))$.

We then have
\[
\begin{align*}
i^* \eta(D) \cap[Z] & = \\
& = i^* c\ell(\tilde{D}) \cap[Z] = \text{by (i)} \\
& = c\ell(i^* D) \cap[Z] = \text{by [15, §2]} \\
& = i^*(D) \\
& = D \cdot Z
\end{align*}
\]
where the last equality is just the normalization property of the intersection theory.

**Theorem 3** With the above assumptions and notations, let $Y$ and $Z$ be prime cycles of codimension $p$ and $q$ on $X$ smooth which intersect properly. Then
\[
\eta(Y) \eta(Z) = \eta(Y \cdot Z) \in H^{p+q}_{Z \cap Y}(X, \mathcal{H}^{p+q}(p+q))
\]

34
Proof. The proof is similar to that of the “uniqueness of the intersection theory” and it consists of 3 steps.

Step 1. (Intersection with divisors). Let \( Y = D \hookrightarrow X \) be a principle Cartier divisor. Let \( i : Z \hookrightarrow X \). Then

\[
\eta(D)\eta(Z) = \eta(i^*\eta(D)\cap[Z]) = \eta(D\cdot Z).
\]

by Scholium 6.2

by Lemma 6.5

Step 2. (Intersection with smooth subvarieties). Let assume \( Y \) to be smooth. Since we can reduce to open Zariski neighborhoods of the generic points of \( Y \cap Z \) we may assume that \( X \) is affine and \( Y = V(f_1, \ldots, f_p) \) where \( \{f_1, \ldots, f_p\} \) is a regular sequence. Thus: \( Y = \cap_{i=1}^{p} D_i \) where \( D_i = V(f_i) \) and

\[
\eta(Y)\eta(Z) = \eta(D_1 \cdots D_p)\eta(Z) = \eta(D_1)\eta(D_2 \cdots D_p)\eta(Z) = \ldots \eta(D_1) \cdots \eta(D_{p-1})\eta(D_p \cdot Z) = \ldots = \eta(Y \cdot Z)
\]

by iterative application of Step 1.

Step 3. (Reduction to the intersection with the diagonal). We prove the general case as follows:

\[
\Delta_*(\eta(Y)\eta(Z)) = \Delta_*(\eta(Y) \times \eta(Z)) = \Delta_*(\eta(Y \times Z)) = \eta(Y \times Z)\eta(\Delta(X)) = \eta(\Delta(Y \cdot Z)) = \Delta_*(\eta(Y \cdot Z))
\]

by definition

by Lemma 1.3

by the formula (36)

by Step 2

int. with the diag.

by the formula (35)

Since \( \Delta_* \) is an isomorphism we conclude.

Corollary 6.6 If \( X \) is smooth of pure dimension \( d \) then the graded isomorphism

\[
\eta : \bigoplus C_{d-p}(X; \Lambda) \cong \bigoplus H^p(X, H^p_p)
\]

is a \( \Lambda \)-algebra isomorphism.

7 Chern classes and blow-ups

Let \( X \) be a variety i.e. \( X \in V_k \) reduced and equidimensional over a perfect field \( k \), which admits a closed imbedding in a smooth variety; such varieties are usually called imbeddable. The existence of \( \mathcal{H} \)-cap-products grant us to construct Gysin maps for the functor \( C_*(\mathcal{H}; \Lambda) \) associated with such imbeddings. By using the results from §3–§6 we construct Chern classes in \( \mathcal{H} \)-cohomologies. Furthermore, we are able to obtain the nice decomposition formula for the \( \mathcal{H} \)-cohomology of blow-ups generalising the classical one for Chow groups.
7.1 Gysin maps for algebraic cycles

Let \((H^*, H_*)\) be an appropriate duality. We consider an imbeddable variety \(X\) with a fixed ambient smooth variety \(Y\). Let \(i : X \hookrightarrow Y\) be a closed imbedding of pure codimension \(c\). Thus we have a \(\mathcal{H}\)-cycle class \(\eta(X) \in H^c_X(Y, \mathcal{H}^c_c)\) and the corresponding Gysin maps

\[
i^! : C_n(Y; \Lambda) \to C_{n-c}(X; \Lambda)
\]

are defined as follows:

\[
y \sim y \cap \eta(X) \overset{\text{def}}{=} i^!(y)
\]

Thus \(i^!(Y) = [X]\) because of \([Y] \cap \eta(X) = [X]\) by the definition of \(\mathcal{H}\)-cycle classes.

**Remark:** Actually we got “Gysin maps” \(i^!\) for imbeddings \(i : X \hookrightarrow Y\) where \(Y\) is just imbeddable in \(V\) smooth, by capping with the \(\mathcal{H}\)-cycle class of \(X\) in \(V\). This operation will take a cycle of codimension \(p\) on \(Y\) to a cycle on \(X\) of codimension \(p\) plus the codimension of \(Y\) in \(V\).

Let denote \(i_1 : C_*(X) \to C_*(Y)\) the canonical map induced by \(i\). Since \(Y\) is smooth we do have the following equation

\[
\eta i_1 = i_0 \eta
\]

where \(i_0 : H^*_X(Y, \mathbb{H}^*) \to H^*(Y, \mathbb{H}^*)\) is the standard map. Let denote \(i^* : H^*(Y, \mathbb{H}^*) \to H^*(X, \mathbb{H}^*)\). Let consider the intersection product of cycles induced by the \(\mathcal{H}\)-cohomology ring, according with §5.5.

**Proposition 7.1** The operation \(i^!\) is functorial and compatible with étale pull-backs. We have the self-intersection property:

\[
i^! i^!(X) = X \cdot X
\]

If \(i : X \hookrightarrow Y\) is a smooth pair we then have

\[
i^! = i^* \eta [X]
\]

and \(i^!\) is a ring homomorphism; there is a projection formula

\[
i_1(x \cdot i^!(y)) = i_1(x) \cdot y
\]

for cycles \(x\) and \(y\) on \(X\) and \(Y\) respectively.

**Proof** Compatibilities are easy to check. The self-intersection property is obtained as follows:

\[
i^!(i^!(x)) = i_1(x) \cap \eta(X) = \text{proj. form.}
\]
whence, by taking \( x = X \), we have

\[ [X] \cap i^* \eta(X) = X \cdot X \]

This last equation holds because of the Scholium 6.2, giving us the following

\[ \eta([X] \cap i^* \eta(X)) = \eta(X) \eta(X) \]

where \( \eta(X) \eta(X) = \eta(X \cdot X) \) (see Theorem 3) and \( \eta \) is an isomorphism.

The other equation is given by the following commutative diagram

\[ H^*(Y, \mathcal{H}^*) \otimes H^c_X(Y, \mathcal{H}^c_{(e)}) \xrightarrow{i^* \cap [Y]} H^*+c(Y, \mathcal{H}^{*+c}_{(e)}) \]

\[ H^*(X, \mathcal{H}^*) \otimes C_{d-c}(X; \Lambda) \xrightarrow{\cap} C_{d-c-\ast}(X; \Lambda) \]

which is obtained by the Scholium 3.2 (cf. the formula (20)). The projection formula is obtained from the projection formula w.r.t. the \( \mathcal{H} \)-product (cf. the Scholium 6.2 and the Theorem 3).

By using the contravariant structure of \( \mathcal{H} \)-cohomologies we can construct ‘refined’ Gysin maps \( f^! \) for algebraic cycles between imbeddable varieties.

### 7.2 Grothendieck-Gillet axioms for Chern classes

The way to obtain a theory of Chern classes in \( \mathcal{H} \)-cohomologies and the corresponding Riemann-Roch Theorems will be to show that the cohomology theory \( H^*_Z(X, \mathcal{H}^{*}(i)) \) and the homology theory \( C_{*,*}(-; \Lambda^*) \) satisfy the list of axioms in [13, Definition 1.1 – 1.2]. We are going to consider \( X \in \mathcal{V}_k \) smooth over a perfect field. We also assume that \( \Lambda \) (constant sheaf for the Zariski topology) has finite weak global dimension (see [27, Definition 2.6.2]) in order to consider tensor products \(- \otimes_{\Lambda} -\) in the derived category.

**Definition 7.2:** We let say that a natural transformation

\[ c_{\ell} : \text{Pic}(X) \otimes \Lambda \to H^1(X, \mathcal{H}^1_{(i)}) \subset H^2(X, i) \]

of contravariant functors is a cycle class map for line bundles if \( c_{\ell} \) localizes satisfying the properties (i) – (ii) stated in §6.3. Therefore \( c_{\ell} \) is compatible via the local triviality property [10, 1.5], with the map obtained mapping a prime Weil divisor \( i : D \hookrightarrow X \) to the Poincaré dual of the direct image under \( i \) of the fundamental class \( \eta_D \).

**Theorem 4** Let \( (H^*, \mathcal{H}_*) \) be an appropriate Poincaré duality on \( V_k \) for a perfect field \( k \), with values in a fixed category of \( \Lambda \)-modules such that \( H^* \) satisfies the homotopy property and there is a cycle class map for line bundles. Then there is a theory of Chern classes

\[ c_{p,i} : K^Z_i(X) \to H^{p-i}_Z(X, \mathcal{H}^p_{(p)}) \]

associated with any closed \( Z \) in \( X \in \mathcal{V}_k \) smooth.
Proof With the notations of \([13\]) we let \(\oplus \Gamma^*(\mathcal{P}) \overset{\text{def}}{=} \oplus H^p(\mathcal{P})\) be the graded sheaf with the \(\mathcal{H}\)-cup-product (according with [13, Definition 1.1] and (5.3) defining our cohomology theory ring on the category \(\mathcal{V}_k\). We let define the homology as

\[
H_i(X, \Gamma(j)) \overset{\text{def}}{=} H^{-i}(X, Q^j_i(\mathcal{P})) = C_{1,j}(X; \Lambda(j))
\]

which is covariant w.r.t. proper morphisms and a presheaf for the étale topology by [10, 3.7]; the compatibility \([13, 1.2.(i)]\) is ensured by the compatibility \([10, 1.2.2]\) and limits arguments (cf. § 4.3). The functorial long exact sequence of homology, for a pair \(i : Y \hookrightarrow X\), is obtained via the hypercohomology long exact sequence with supports

\[
\mathbf{H}^i_Y(X, Q^j_i(\mathcal{P})) \to \mathbf{H}^{-i}(X, Q^j_i(\mathcal{P})) \to \mathbf{H}^{-i}(X - Y, Q^j_i(\mathcal{P}))
\]

since \(\Gamma_Y Q^j_{X}(\mathcal{P}) \cong i_* Q^j_{Y}(\mathcal{P})\) and Lemma 4.6 (i.e. \([13, 1.2.(ii)]\) holds). The cap product structure is given by the \(\mathcal{H}\)-cap-product and all the properties required by \([13, 1.2.(iii) - (viii)]\) are easily seen by using the results of §4 and §5. The homotopy property \([13, 1.2.(ix)]\) is ensured by the compatibility \([10, 1.2.2]\) and limits arguments (cf. § 4.3). Thus we are left to show the following classical Dold-Thom decomposition (see [13, 1.2.(x)-(xi)]).

\[
\frac{\mathcal{H}^i_Y}{\mathcal{H}^i_Y} = 1 \Lambda
\]

Scholium 7.3 (Decomposition) Let \(\mathcal{E}\) be a locally free sheaf, \(\text{rank} \mathcal{E} = n + 1\), on \(X\) smooth. Let \(\pi : P \overset{\text{def}}{=} P(\mathcal{E}) \to X\) be the corresponding projective bundle. For \(\mathcal{O}_P(1) \in \text{Pic } P\) let

\[
\xi \overset{\text{def}}{=} \text{cl}(\mathcal{O}_P(1)) \in H^1(P, \mathcal{H}^1(1))
\]

we then have

\[
\oplus \pi^*(\mathcal{H}) \overset{\text{def}}{=} H^0(P, \mathcal{H}^0(\mathcal{E})) \cong \oplus H^0(P, \mathcal{H}^0(\mathcal{E}))
\]

Furthermore:

\[
\frac{\mathcal{H}^i_Y}{\mathcal{H}^i_Y} = 1 \Lambda
\]
in the derived category $D(X_{zar}; \Lambda)$. Now that $\gamma$ is defined the claimed decomposition will follows by proving that $\gamma$ is a quasi-isomorphism because of the Leray spectral sequence
\[ H^p(X, R\pi_* \mathcal{H}^q(\gamma)) \cong H^p(P, \mathcal{H}^q(\gamma)) \]
In order to show that $\gamma$ is a quasi-isomorphism we are left to show the isomorphisms of groups
\[ (\gamma^p)_x : \bigoplus_{i=0}^a H^{p-i}(\text{Spec}\mathcal{O}_{X,x}, \mathcal{H}^{q-i}(\gamma)^{(r-i)}) \cong (R^p\pi_* \mathcal{H}^q(\gamma))_x \]
for all $x \in X$ and $p \geq 0$. By continuity of the arithmetic resolutions the stalks $(R^p\pi_* \mathcal{H}^q(\gamma))_x$ are computed by $H^p(\mathcal{P}_{\mathcal{O}_{X,x}}, \mathcal{H}^q(\gamma))$; we need the following compatibility:

**Lemma 7.4** Let $U \subset X$ be an open Zariski neighborhood of $x$ on which $\mathcal{E}$ is free. Let $\xi \in H^1_{\text{et}}(\mathcal{P}^n_U, \mathcal{H}^1(\xi))$ be the restriction of the tautological divisor, where $i : \infty \cong \mathcal{P}^{n-1}_U \hookrightarrow \mathcal{P}^n_U$ is a hyperplane at infinity. Then (with the notation of §3.3)
\[ j((n,n-1)) \pi^*_n = i^* \pi^*_n \]
equality between maps from $H^{p-1}(U, \mathcal{H}^{q-1}(\gamma^{(r-1)}))$ to $H^p(\mathcal{P}^n_U, \mathcal{H}^q(\gamma))$.

**Proof** Note that $\pi^*_n = i^* \pi^*_n$. The purity isomorphism
\[ H^{p-1}(\infty, \mathcal{H}^{q-1}(\gamma^{(r-1)})) \cong H^p_{\infty}(\mathcal{P}^n_U, \mathcal{H}^q(\gamma)) \]
is obtained as $\eta(-\cap \infty)$ and $\eta(\infty) = \xi$ (because of the compatibilities of the cycle class $e\ell$) thus we have
\[ \eta(\pi^*_n \cap \text{[infty]}) = \eta(i^* \pi^*_n \cap \text{[infty]}) = \eta(\pi^*_n \cup \eta(\infty) \cap \text{[P^n_U]}) = \pi^*_n \cup \eta(\infty) = \pi^*_n \cup \xi \]
as elements in $H^p_{\infty}(\mathcal{P}^n_U, \mathcal{H}^q(\gamma))$ and, by definition of $j((n,n-1))$, the image of it under $H^p_{\infty}(\mathcal{P}^n_U, \mathcal{H}^q(\gamma)) \to H^p(\mathcal{P}^n_U, \mathcal{H}^q(\gamma))$ yields the claimed equation.  

Thus: $(\gamma^p)_x$ is clearly an isomorphism by reduction to open Zariski neighborhoods on which $\mathcal{E}$ is free, arguing as in §3.3 via the Lemma above and induction on the rank of $\mathcal{E}$. Let show the equation (11). By the definition of the $\mathcal{H}$-Gysin map we have that $\pi_* : H^n(P, \mathcal{H}^n(\mathcal{P}^n_U)) \to H^0(X, \mathcal{H}^0(\mathcal{P}^n_U))$ is obtained (via the Leray spectral sequence) by composition with $R\pi_b : R\pi_* \mathcal{H}^n(\mathcal{P}^n_U) \to \mathcal{H}^0(\mathcal{P}^n_U)[-n]$ in $D(X_{zar}; \Lambda)$. Thus (see §5.5) we have to prove that the composition of
\[ \Lambda[-n] \xrightarrow{R\pi^n_\alpha} R\pi_* \mathcal{H}^n(\mathcal{P}^n_U) \xrightarrow{R\pi_b} \mathcal{H}^0(\mathcal{P}^n_U)[-n] \cong \Lambda[-n] \]
is the identity. Arguing as above we are reduced to show the equation (11) for $\pi : \mathcal{P}^n_{\mathcal{O}_{X,x}} \to \text{Spec}\mathcal{O}_{X,x}$. By the projection formula
\[ \pi_*(\pi^*(\xi^\alpha)) = \pi_*(\xi^\alpha) \]
39
we are left to show that $\pi_*$ is the inverse of the “decomposition” isomorphism

$$\pi^*(\cup \xi^n) : H^0(\text{Spec} \mathcal{O}_{X,x}, \mathcal{H}^0(0)) \to H^n(\mathbb{P}^n_{\mathcal{O}_{X,x}}, \mathcal{H}^n(n))$$

By choosing a $k$-rational point of $\mathbb{P}^n_k$ we get a proper section $\sigma$ of $\pi$. With the notation above: $\sigma = j_{(n,0)} \pi_0^*$ and by the Lemma we have

$$\pi^*(1) \cup \xi^n = \sigma^*(1)$$

By applying $\pi_*$ to the latter and taking the image of it under the canonical augmentation $H^0(\text{Spec} \mathcal{O}_{X,x}, \mathcal{H}^0(0)) \cong \Lambda$ we do obtain the claimed formula.

**Remark:** After Grothendieck-Verdier, this ‘decomposition argument’ is quite standard. See [SGA 5, Exposé VII] for étale cohomology and [13, Theor.8.2] or [31] for the $K$-theory.

Let

$$A(-) = \bigoplus_{p,q,r} H^p(-, \mathcal{H}^q(r))$$

be the $\mathcal{H}$-cohomology ring functor.

**Corollary 7.5** Let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be as above, rank $\mathcal{E} = n + 1$. Then

$$\pi^* : A(X) \to A(\mathbb{P}(\mathcal{E}))$$

is an injective homomorphism of unitary $\Lambda$-algebras and the elements

$$1, \xi, \ldots, \xi^n$$

generate freely $A(\mathbb{P}(\mathcal{E}))$ as $A(X)$-module. Furthermore

$$\pi_* : A(\mathbb{P}(\mathcal{E})) \to A(X)$$

is a surjective homomorphism of $A(X)$-modules (having degree $-n$).

**Proof** The statement is clear after §5.5 and the Scholium above. For example, $\pi_*(\xi^i) = 0$ for $i = 0, \ldots, n - 1$ but $\pi_*(\xi^n) = 1$ by (11) whence the linear independence of $1, \xi, \ldots, \xi^n$ can be seen as follows: let suppose that

$$\pi^*(x_0) + \cdots + \pi^*(x_n) \cup \xi^n = 0$$

then by applying $\pi_*$ and the projection formula we get $x_n = 0$ thus

$$\pi^*(x_0) \cup \xi + \cdots + \pi^*(x_{n-1}) \cup \xi^n = 0$$

and the same argument gives $x_{n-1} = 0$ and so on. Again: $\pi_*$ is a surjection because of

$$\pi_*(\pi^*(\xi^n)) = \pi_*(\xi^n) \cup \xi = \xi^n$$

**Remark:** By the prescription of [19] we therefore obtain Chern classes $c_p : K_0(X) \to H^p(X, \mathcal{H}^p(p))$ satisfying the equation

$$\xi^n + \pi^* c_1(E) \xi^{n-1} + \cdots + \pi^* c_n(E) = 0$$

for $E$ a vector bundle of rank $n$. By [13] we have that $c_p$ is just $c_{p,0}$.
7.3 Variation on the invariance theme

Let consider a sophisticated Poincaré duality theory \((H^*, H_*)\) satisfying the point axiom. Let consider \(f : X \to Y\) a proper dominant morphism between connected smooth schemes in \(\mathcal{V}_k\). If \(\text{dim}X = \text{dim}Y\) then \(K(X)\) is a finite field extension of \(K(Y)\); let \(\text{deg}f = [K(X) : K(Y)]\) be its degree. Following the proof of (2.2) we have that the composition

\[ H^*(Y, \cdot) \xrightarrow{f^*} H^*(X, \cdot) \xrightarrow{f_\#} H^*(Y, \cdot) \]

is the multiplication by \(\text{deg}f\), as a consequence of the projection formula and our assumption that \(f_!(\eta_X) = \text{deg}f \cdot \eta_Y\) (cf. \[\text{§} 1.1\]). Thus:

**Proposition 7.6** The composition of

\[ H^p_{\mathbb{Z}}(Y, \mathcal{H}^q(r)) \xrightarrow{f^*} H^p_{f^{-1}(Z)}(X, \mathcal{H}^q(r)) \xrightarrow{f_\#} H^p_{\mathbb{Z}}(Y, \mathcal{H}^q(r)) \]

is the multiplication by \(\text{deg}f\).

**Proof** Let \(d = \text{dim}X = \text{dim}Y\). Then the projection formula (3) looks

\[
\begin{array}{ccc}
Q_d^d(d) & \otimes & f_* \mathcal{H}^q(r) \\
\text{id} \otimes f^* & \hookrightarrow & f_* \mathcal{H}^q(r) \\
f_* Q_d^d(d) \otimes \mathcal{H}^q(r) & \rightarrow & f_* Q_{d-q}(d-r) \\
f^* \otimes \text{id} & \downarrow & \downarrow f_* \\
Q_d^d(d) \otimes \mathcal{H}^q(r) & \rightarrow & Q_{d-q}(d-r)
\end{array}
\]

By the dimension axiom the complex \(Q^d_d(d)\) is concentrated in degree \(d\) and its hypercohomology \(C_{d,d}(X, \Lambda(d))\) has a natural global section \([X]\) corresponding to the fundamental class \(\eta_X \in H_{2d}(X, (d))\). The same holds on \(Y\) and \([X] \sim \text{deg}f[Y]\) under \(f_*\). Thus by taking cohomology with supports we have the result. 

**Remark:** The same argument applies to the \(K\)-theory by using the projection formula in (3).

**Lemma 7.7** Let \(f : X' \to X\) be a proper birational morphism between smooth varieties; let \(i : Z \hookrightarrow X\) and \(i' : Z' = f^{-1}(Z) \hookrightarrow X'\) be closed subschemes such that \(f : X' - Z' \cong X - Z\). Then we have splitting short exact sequences

\[ 0 \to H^p_{\mathbb{Z}}(X, \mathcal{H}^q(r)) \xrightarrow{\iota^\#} H^p(X, \mathcal{H}^q(r)) \oplus H^p_{Z'}(X', \mathcal{H}^q(r)) \xrightarrow{u} H^p(X', \mathcal{H}^q(r)) \to 0 \]

where:

\[ u = \begin{pmatrix} \iota^\# \\ f_* \end{pmatrix} \]

and

\[ v = (f^*, -i'^\#) \]

The left splitting of \(u\) is given by \(u' : (0, f_*)\).
Proof Let consider the following maps of long exact sequences

\[ \cdots \to H^p_Z(X', \mathcal{H}^q(r)) \xrightarrow{i_*} H^p(X', \mathcal{H}^q(r)) \to H^p(X' - Z', \mathcal{H}^q(r)) \to \cdots \]

\[ \xrightarrow{f_* \downarrow \uparrow} \]

\[ \cdots \to H^p_Z(X, \mathcal{H}^q(r)) \xrightarrow{i_*} H^p(X, \mathcal{H}^q(r)) \to H^p(X - Z, \mathcal{H}^q(r)) \to \cdots \]

Since \( \deg f = 1 \) by the Proposition above \( f_* f^* = 1 \); thus the corresponding Mayer-Vietoris exact sequence splits (because the boundary is zero) in short exact sequences as claimed. 

\begin{scholium}{7.8}
Let \( X, X', Z \) and \( Z' \) be as above and pure dimensional. We have isomorphisms \((d = \dim X)\)

\[ C_{d-p}(X; \Lambda) \oplus C_{d-p}(Z'; \Lambda) \xrightarrow{\sim} C_{d-p}(Z; \Lambda) \oplus C_{d-p}(X'; \Lambda) \]

given by the matrix

\[ \begin{pmatrix} 0 & f_! \\ f^! & -j'_! \end{pmatrix} \]

and, for \( Z \) and \( Z' \) smooth of codimension \( c \) and \( c' \):

\[ H^p(X, \mathcal{H}^q(r)) \oplus H^{p-c'}(Z', \mathcal{H}^{q-c'}(r-c')) \xrightarrow{\sim} H^{p-c}(Z, \mathcal{H}^{q-c}(r-c)) \oplus H^p(X', \mathcal{H}^q(r)) \]

given by the matrix

\[ \begin{pmatrix} 0 & f_* \\ f^* & -j_{(X', Z')} \end{pmatrix} \]

where \( j_{(X', Z')} \) is the Gysin map in \( \mathcal{H} \)-cohomology (cf. Scholium 7.3, §5.3).

Proof By the Lemma 7.7 and purity.

\end{scholium}

7.4 Blowing-up

Let \((H^*, \mathcal{H}_*)\) be an appropriate Poincaré duality such that \( H^* \) satisfies the homotopy property and there is a cycle class map for line bundles (cf. §7.2).

Let \( f : X' \to X \) be the blow up of a smooth subvariety \( Z \) of codimension \( c \geq 2 \) in a smooth variety \( X \) of dimension \( d \). Thus the exceptional divisor is the projective bundle over \( Z \) given by \( \mathbb{P}(N) \) where \( N \) is the normal sheaf, locally free of rank \( c \).

Proposition 7.9 For the blow-up \( X' \) of \( X \) along \( Z \) as above we have the following canonical formulas

\[ H^p(X', \mathcal{H}^q(r)) \cong H^p(X, \mathcal{H}^q(r)) \oplus \bigoplus_{i=0}^{c-2} H^{p-1-i}(Z, \mathcal{H}^{q-1-i}(r-1-i)) \]

and in particular

\[ C_n(X'; \Lambda) \cong C_n(X; \Lambda) \oplus \bigoplus_{i=0}^{c-2} C_{n-c+1+i}(Z; \Lambda) \]
**Proof** Since \( f : Z' \to Z \) is a proper morphism, between smooth varieties, having relative dimension \( 1 - c \) we have a push-forward (see §5.3)

\[
f_* : H^{p-1}(Z', \mathcal{H}^{q-1}_{(r-1)}) \to H^{p-c}(Z, \mathcal{H}^{q-c}_{(r-c)})
\]

Because of purity \( f^* : H^p_Z(X, \mathcal{H}^q(r)) \simeq H^p_{Z'}(X', \mathcal{H}^q(r)) \) induces a map

\[
f^! : H^{p-c}(Z, \mathcal{H}^{q-c}_{(r-c)}) \to H^{p-1}(Z', \mathcal{H}^{q-1}_{(r-1)})
\]
as well. By the Lemma 7.7 we have a splitting exact sequence

\[
0 \to H^{p-c}(Z, \mathcal{H}^{q-c}_{(r-c)}) \xrightarrow{u} H^p(X, \mathcal{H}^q(r)) \oplus H^{p-1}(Z', \mathcal{H}^{q-1}_{(r-1)}) \xrightarrow{v} H^p(X', \mathcal{H}^q(r)) \to 0
\]

with left splitting \( u' = (0, f_*) \). Let consider the projector \( \pi = uu' \); we then have \( \pi u = u \), \( v = 0 \) thus \( v \) restricts to an isomorphism

\[
v : \ker \pi \simeq H^p(X', \mathcal{H}^q(r))
\]

Now \( \pi(x, z') = u(f_*(z')) = (j_{(X,Z)}(f_*(z')), f^! f_*(z')) = 0 \) if and only if \( f_*(z') = 0 \) (because \( f^! \) is injective). Thus we have

\[
\ker \pi = H^p(X, \mathcal{H}^q(r)) \oplus \ker f_*
\]

Since \( Z' = P(N) \) and \( f|_{Z'} \) is the standard projection then, by the Dold-Thom decomposition (see Scholium 7.3), we have an exact sequence

\[
0 \to \bigoplus_{i=0}^{c-2} H^{p-1-i}(Z, \mathcal{H}^{q-1-i}_{(r-1-i)}) \xrightarrow{\xi f_*} H^{p-1}(Z', \mathcal{H}^{q-1}_{(r-1)}) \xrightarrow{f^!} H^{p-c}(Z, \mathcal{H}^{q-c}_{(r-c)}) \to 0
\]

where \( \xi \) is the tautological divisor, hence:

\[
\ker \pi \simeq H^p(X, \mathcal{H}^q(r)) \oplus \bigoplus_{i=0}^{c-2} H^{p-1-i}(Z, \mathcal{H}^{q-1-i}_{(r-1-i)})
\]

and the claimed isomorphisms are easily obtained.

**Remark:** For the \( \mathcal{K} \)-cohomology the same proof applies yielding the formula

\[
H^p(X', \mathcal{K}_q) \simeq H^p(X, \mathcal{K}_q) \oplus \bigoplus_{i=0}^{c-2} H^{p-1-i}(Z, \mathcal{K}_{q-1-i})
\]

**A Examples and comments**

We will give a draft for testing the common cohomologies with respect to our setting (cf. §4.2, §5.2).
A.1 Grothendieck-Verdier duality

Let \( X_{fine} \) be a site finer than the Zariski site \( X_{Zar} \) for \( X \in \mathcal{V}_k \) e.g. étale or analytic sites. Let \( \Lambda \) be a commutative ring with 1 (of finite weak global dimension) and let \( \mathcal{D}(X; \Lambda) \) denote the derived category of the abelian category of complexes of \( \Lambda \)-modules in the corresponding Grothendieck topos on \( X_{fine} \). We let \( F^* \leadsto F^*(r) \) denote a "twist à la Tate" functor on \( \mathcal{D}(X; \Lambda) \) commuting with direct and inverse image functors. For a closed imbedding \( i : Z \hookrightarrow X \) let \( j : U = X - Z \hookrightarrow X \) be the corresponding Zariski open immersion; let consider the six standard operations

\[
i^* \dashv i_* \dashv i^! \quad j^! \dashv j_* \dashv Rj_*
\]

where \( i^! \) is the "sheaf of sections on \( Z \)" functor and \( j_i \) is the "extension by zero" functor. Let consider twisted objects \( \Lambda'_k(r) \in \mathcal{D}(k; \Lambda) \) with a canonical augmentation (usually a quasi-isomorphism) \( \Lambda \to \Lambda'_k(0) \) such that \( \oplus \Lambda'_k(r) \) is a graded \( \Lambda \)-algebra via the canonical maps

\[
m : \Lambda'_k(r) \otimes \Lambda'_k(s) \to \Lambda'_k(r+s)
\]

We let extends \( \oplus \Lambda'_k(r) \to \Lambda_{\Lambda} \) to a \( \Lambda \)-algebra over the big site \( (\mathcal{V}_k)_{fine} \) by pulling back along the structural morphisms. Thus we have a contravariant functor on \( \mathcal{V}_k \)

\[
(X, Z) \leadsto Hom_X(\Lambda, i_* i^! \Lambda'_X(r)[q]) \overset{\text{def}}{=} H^q_Z(X, r)
\]

where the \( Hom \) is taken in \( \mathcal{D}(X; \Lambda) \). By the general non-sense of triangulated categories this is a cohomology theory with cup-products: the long exact sequence of cohomology with supports is given by the triangles (since the \( Hom \) of triangulated categories takes triangles to long exact sequences)

\[
j^! j^* \Lambda \to \Lambda \to i_* i^* \Lambda \xrightarrow{+1}
\]

and the cup-product of \( a : \Lambda \to \Lambda^*(r)[q] \) and \( b : \Lambda \to \Lambda^*(r')[q'] \) is given by tensoring \( \Lambda^*(r)[q] \otimes b : \Lambda^*(r)[q] \to \Lambda^*(r) \otimes \Lambda^*(r')[q + q'] \) and composing a with

\[
\Lambda^*(r)[q] \to \Lambda^*(r) \otimes \Lambda^*(r')[q + q'] \overset{m[q + q']}{\to} \Lambda^*(r + r')[q + q']
\]

in the derived category. (Note: this is the natural \( Ext \) pairing). Since the composition of maps is associative it is clear that the pairing above is compatible with the long exact sequences of cohomology with supports and furthermore the following square commutes

\[
\begin{array}{ccc}
\text{Hom}_X(j^! j^* \Lambda, \Lambda^*(r)[q]) \otimes H^q(U, r') & \to & \text{Hom}_X(j^! j^* \Lambda, \Lambda^*(r + r')[q + q']) \\
\downarrow \theta \otimes j^* & & \downarrow \theta \\
\text{Hom}_U(\Lambda, \Lambda^*(r)[q]) \otimes H^q(U, r') & \to & \text{Hom}_U(\Lambda, \Lambda^*(r + r')[q + q'])
\end{array}
\]

where \( \theta \) is the canonical isomorphism induced by \( j^! \dashv j^* \) and the bottom arrow is the cup-product pairing on \( U \); thus the axioms \( \forall 1 - \forall 3 \) (see §5) are satisfied. (Note: to check commutativity of the square above we simply need that \( j^*(b) \circ \theta(a) = \theta(b \circ a) \) which is a consequence of \( j^! \dashv j^* \)).

Let assume the existence of a "global duality"

\[
f_1 \dashv f^!
\]
where: $f_1 = Rf_*$ for $f$ proper, $f^! = f^*$ for $f$ étale (or at least a Zariski open imbedding) and $f^! = f^*(d)[2d]$ for $f$ smooth of relative dimension $d$. Thus we obtain a homology theory

$$X \rightsquigarrow Hom_X(\Lambda, \pi^!\Lambda_k(-r)[−q]) \overset{\text{def}}{=} H_q(X, r)$$

where $\pi : X \rightarrow k$ is the structural morphism. Furthermore the counit $f_1f^! \rightarrow 1$ yields a pairing

$$f^!(\Lambda^r(\nu)) \overset{L}{\otimes} f_1!(\Lambda^r(\nu')) \rightarrow f^!(\Lambda^r(\nu) \overset{L}{\otimes} \Lambda^r(\nu')) \overset{f_1(m)}{\rightarrow} f^!(\Lambda^r(\nu+\nu'))$$

whence applied to $\pi$ and the previous procedure give us a sophisticated cap-product and the axioms A1–A4 (see §6) are clearly satisfied.

For $\pi : X \rightarrow k$ smooth of pure dimension $d$ Poincaré duality will mean to be given by natural quasi-isomorphisms:

$$\eta : \pi^*\Lambda_k(r) \overset{\sim}{\rightarrow} \pi^!\Lambda_k(-d)[-2d] \quad \text{“fundamental class”}$$

(yielding a homology class by composing with the augmentation) and

$$\pi^*\Lambda_k(r)[q] \overset{\sim}{\rightarrow} \pi^!\Lambda_k(r-d)[q-2d]$$

Thus the compatibility of cap and cup products (see §6) is ensured by construction. Concerning the dimension axiom (see A5–A6 in §4.4) just compare with the Lemma 2.1.2 of [10]. This framework applies to: (i) the analytic site where $\Lambda = \mathbb{Z}$ and $\Lambda^r_k(r) \overset{\text{def}}{=} \mathbb{Z}(r)$ by mean of the Tate twist in Hodge theory (or $\Lambda \overset{\text{def}}{=} \mathbb{Q}, \mathbb{R}$ or complex-De Rham cohomology where $\Lambda = \Lambda^r_k(r) \overset{\text{def}}{=} \mathbb{C}$) after Verdier Exposé [34] (cf. [27]), (ii) the étale site where $\Lambda \overset{\text{def}}{=} \mathbb{Z}/\ell^r$ and $\Lambda_k(r) \overset{\text{def}}{=} \mathbb{Z}/\ell^r(r)$ is étale sheaf $\mu_{\ell^r}$ of $\ell^r$-th roots of unity ($\ell$ prime to char$k$) with Tate twist $r$, after Deligne Exposé XVIII in [SGA 4].

### A.2 Algebraic De Rham cohomology

Let $X$ be a smooth algebraic scheme over $k = \mathbb{C}$. On the analytic manifold $X_{an}$ the constant sheaf $\Lambda = \mathbb{C}$ is quasi-isomorphic to the holomorphic De Rham complex $\Omega^r_{an}$; by the Grothendieck comparison theorem the algebraic De Rham complex $\Omega^r_{X/k}$ computes the complex cohomology as well. In [22, 23] has been developed a general theory of algebraic De Rham complexes for imbeddable varieties over $k$ of characteristic zero. This theory yields cohomology groups $H^*_\text{DR}(X)$ and homology groups $H^*_{\text{DR}}(X)$ satisfying the Bloch-Ogus axioms (see [10, 2.2]) e.g., $H^*_{\text{DR}}(X) \overset{\text{def}}{=} H^{2d-*}(Y, \Omega^*_{Y/k})$ for $X \rightarrow Y$ and $Y$ smooth of dimension $d$.

Furthermore, the exterior algebra structure on the De Rham complex grants us a theory of compatible cup products and sophisticated cap products as explained in [23, II.7.4]. If moreover $k$ is algebraically closed then $H^*_\text{DR}(X) = k$ for $X$ connected (see [23, II.7.1]); since $H^*_\text{DR}(X) = 0$ if $q > 2d$ (see [23, II.7.2]) the dimension axiom (see §4.4) is satisfied. The homotopy axiom is ensured by the Poincaré Lemma (see [23, II.7.1]). Thus algebraic De Rham cohomology and homology is appropriate (in the sense of §6.1) where $\Lambda = k$ is algebraically closed of characteristic zero. What about positive characteristics?
A.3 Deligne-Beilinson cohomology

By considering the augmented and truncated complexes \( D(\mathfrak{r}) \overset{\text{def}}{=} \mathbb{Z}(\mathfrak{r}) \to \Omega^* \) on the complex compact manifold \( X_{\text{an}} \) we let define the Deligne-Beilinson cohomology \( H^q_{\text{DB}}(X; \mathbb{Z}(\mathfrak{r})) \) to be the hypercohomology groups \( H^q(X_{\text{an}}, D(\mathfrak{r})) \). This definition can be `algebraically’ extended:

(i) to smooth open varieties by taking compactifications with no normal crossing divisors at infinity by mean of differential forms with logarithmic poles at infinity, and

(ii) to singular varieties by smooth hypercoverings, according with Deligne-Hodge theory. We refer to [24] for a detailed exposition of the smooth case and [14], [25], [28] for the singular case: indeed, by using currents and \( C^\infty \)−chains, we do have Deligne homology groups \( H^D_\ast(X; \mathbb{Z}(\mathfrak{r})) \) and therefore a Poincaré duality theory in the sense of Bloch-Ogus [25, Theorem 1.19], [14], [6]. Because of [25, Lemma 1.20] Deligne homology satisfies the dimension axiom. Because of [24, 1.2 and §3] we have products \( \cup : D(\mathfrak{r}) \otimes D(\mathfrak{r}') \to D(\mathfrak{r}+\mathfrak{r}') \) whence \( H^\ast_{\text{DB}}(\mathfrak{r}; \mathbb{Z}(\mathfrak{r})) \) is multiplicative (since the cup-product is obtained in a standard way via the Ext pairing and it is therefore compatible with the long exact sequence of cohomology with supports). By making the Exercise 1.8.6 in [3] one obtains a sophisticated Poincaré duality which is appropriate for algebraic cycles. (Note: \( H^p(X, H^p_{\text{DB}}(\mathbb{Z}(\mathfrak{r}))) \cong CH^p(X) \) for \( X \) smooth by [14]). The homotopy property is obtained by those of De Rham theory and integral cohomology via the long exact sequences [24, 2.10] (cf. [24, 8.5]). (Note: we are regarding the homotopy property as a property of mixed Hodge structures).

A.4 Cycle class map for line bundles

Because of the Definition 7.2 (cf. the assumptions in §6.3) we let explain briefly some examples. For the sake of exposition we first make the following remark.

A.4.1 Cycle classes are locally trivial

Roughly speaking, another way to understand globally what such compatibility really mean is the following. Let \( X \) be smooth over a field and let assume that our appropriate cohomology is coming with a functorial map \( \text{c} \ell : \text{Pic}(X) \to H^2(X, 1) \) as usual it is. Then the image of \( \text{c} \ell \) is contained in the subgroup of locally trivial elements i.e. the kernel of \( H^2(X, 1) \to H^0(X, \mathcal{H}^2(1)) \), simply because the Pic of a local ring is zero. By the coniveau spectral sequence and the point axiom we have that \( H^1(X, \mathcal{H}^1(1)) \) is identified with the subgroup of locally trivial elements in \( H^2(X, 1) \). By construction of the \( \mathcal{H} \)-cap-products we have the following commutative square (\( d = \dim X \))

\[
\begin{array}{ccc}
H^1(X, \mathcal{H}^1(1)) & \xrightarrow{\cap [X]} & H^2(X, 1) \\
\cap \downarrow & & \downarrow \cap_{\eta_X} \\
C_{d-1}(X; \Lambda) & \xrightarrow{\zeta} & H_{2d-2}(X, d-1)
\end{array}
\]

where \( \zeta \) is the cycle map in homology as defined in [10, §7]. Thus the commutativity of

\[
\begin{array}{ccc}
\text{Pic}(X) & \xrightarrow{\text{c} \ell} & H^2(X, 1) \\
\wedge \downarrow & & \downarrow \cap_{\eta_X} \\
\text{CH}_{d-1}(X) & \to & H_{2d-2}(X, d-1)
\end{array}
\]
tturns out to be an equivalent formulation of our requirement (where \( w \) is the associated Weil divisor mapping.) To obtain the corresponding local formulation one has to make use of the supports of a given line bundle.

**A.4.2 Classical integral cohomology**

Let \( \omega : X_{an} \to X_{zar} \) be the canonical continuous map of sites, where \( X \) is a reduced irreducible algebraic scheme over \( \mathbb{C} \) and let identify \( \mathcal{H}^0(\tau) \) by the Zariski sheaf \( \mathcal{R}^0\omega_*\mathbb{Z}(\tau) \). From the exponential sequence on \( X_{an} \) we get a map \( \omega_*\mathcal{O}_{X_{an}}^* \to \mathcal{H}^1(1) \). Since any regular mapping is analytic, \( \mathcal{O}_{X}^* \) is a subsheaf of \( \omega_*\mathcal{O}_{X_{an}}^* \) whence the cycle map \( \mathcal{C}l : \text{Pic}(X) \to \mathcal{H}^1(1) \) is obtained by applying \( H^1_{zar}(X, -) \) to the boundary map \( \mathcal{O}_{X}^* \to \mathcal{H}^1(1) \). (Note: because of the identifications \( H^0(X, \mathcal{H}^1(1)) = \mathcal{H}^1(X_{an}, \mathbb{Z}) = \text{Hom}(H_1(X_{an}), \mathbb{Z}) \) this boundary map takes the continuous mapping \( f : X_{an} \to \mathbb{C}^n \) to the mapping that to a given loop \( \phi : S^1 \to X_{an} \) associates the Brouwer degree of the composition \( S^1 \xrightarrow{\phi} X_{an} \xrightarrow{f} \mathbb{C}^n \xrightarrow{\nu} S^1 \) where \( n(z) \overset{\text{def}}{=} \frac{z}{|z|} \) is the canonical homotopical retraction. For \( X \) affine hence \( X_{an} \) a Stein space this is a surjection as a map from the holomorphic functions on \( X_{an} \). The non-vanishing of it, for \( f \) holomorphic non-constant, computes the obstruction of having an holomorphic logarithm according with the celebrated Riemann representation theorem cf. [18, V.3.3]. To see that this cycle map \( \mathcal{C}l \) is compatible with the classical \( \text{Pic}(X_{an}) \to \mathcal{H}^2(X_{an}, \mathbb{Z}) \) we refer to [3 Proposition 1]. Because of the functoriality of the exponential sequence \( \mathcal{C}l \) is a natural transformation of contravariant functors meeting our hypothesis \((i)\) of §6.3 if taking \( H^1_{zar}(X, -) \). To check the assumption \((ii)\) ibid. we let remark that the map \( \mathcal{O}_{X}^* \to \mathcal{H}^1(1) \) yields maps on the stalks \( \mathcal{O}_{X,x}^* \to \mathcal{H}^1(1) \) at \( x \in X \) and in particular at the generic point give us a map \( K(X)^* \to \mathcal{H}^1(K(X)) \). By capping with the fundamental class we then have a map \( K(X)^* \to H_{2d-1}(K(X), a-1) \) \((d = \dim X)\); in the same way the rank mapping give us the canonical map \( K_0 \to H_{2d}(-, d) \) which is a local isomorphism. We then have a map between augmented “Gersten complexes”

\[
\begin{align*}
\mathcal{O}_{X}^* & \to K(X)^* \quad \xrightarrow{\text{div}} \quad \coprod_{x \in X_{an}} i_x \mathbb{Z} \\
\mathcal{H}^1(1) & \to H_{2d-1}(K(X), a-1) \quad \xrightarrow{\text{d}} \quad \coprod_{x \in X_{an}} i_x \mathbb{Z}
\end{align*}
\]

where the bottom augmentation is given by restriction to the generic point and cap product with the fundamental class. (Warning: to check that this is actually a map of complexes we should have to show that the divisor map \( \text{div} \) has the claimed factorisation via \( \text{d} \), the differential in the niveau spectral sequence. This is ensured by the Tautology 7.2 and the Theorem 7.3 in [4]; note that the non-singular case grant us for the general case by resolution of singularities.) Thus, by applying \( H^1_{zar}(X, -) \) to these augmentations, we obtain the commutative square in \((ii)\) of §6.3 as desired.

**A.4.3 The étale theory**

A similar framework as above applies to the étale cohomology of the sheaf \( \mu_{\ell^r} \) by using the Kummer sequence on the étale site \((\ell \neq \text{char} k)\) where the cycle map \( \text{Pic}(X) \otimes \mathbb{Z}/\ell^r \to H^1(X, \mathcal{H}^1(\mu_{\ell^r})) \) is now induced by the identification \( \mathcal{O}_{X}^* \otimes \mathbb{Z}/\ell^r \cong \mathcal{H}^1(\mu_{\ell^r}) \) obtained by the
Hilbert’s theorem 90 and the well known fact that the Picard group of a local ring is zero; it is an analogous exercise to check the aimed properties (by using [10, Theorem 7.7]).

A.4.4 Algebraic De Rham cohomology

In the case of algebraic De Rham cohomology a nice exposition of these properties has been written by Hartshorne in [23, II.7.6-7].

A.4.5 Deligne-Beilinson cohomology

For the Deligne-Beilinson cohomology $H^*_D(X; \mathbb{Z}(1))$ we dispose of a canonical map

$$\rho : H^0(X, \mathcal{O}_X^*) \to H^1_D(X; \mathbb{Z}(1))$$

(see [24, 1.4.ii] and 2.12.iii], [28, (1.1)]). By sheafifying for the Zariski topology we do get $\rho : \mathcal{O}_X^* \to H^1_D(\mathbb{Z}(1))$ which is an isomorphism on $X$ smooth thus: $cl : Pic(X) \cong H^1(X, \mathcal{H}_D^*(\mathbb{Z}(1)))$. Moreover: for any $X$ we have that the kernel of the divisor map $div$ is given by $H^2_D(X; \mathbb{Z}(d-1))$ by [23, Lemma 3.1]. Thus, by arguing as in the case of integral cohomology, we easily obtain the aimed properties. (Note: the diagram (3.1.1) of [25] contains implicitly our requirement.) See also [3, 1.9.2].

A.5 Singular varieties and intersection cohomology

In order to obtain an “intersection theory” for singular varieties, after §5.5 and Corollary 6.6, one is tempted to make use of $\mathcal{H}$-cohomology rings. For complex algebraic varieties Deligne-Beilinson cohomology do most of the job but it is an open question whether the $\mathcal{H}$-cohomology ring functor is covariant for proper morphisms. Another problem: the existence of $\mathcal{H}$-cycle classes on singular varieties. (Note: a necessary condition is local triviality.) By weakening to intersections modulo algebraic equivalence the same problems occur with singular cohomology. Furthermore it is hopeless to expect birational invariance of $H^0(X, \mathcal{H}^*)$ e.g. because of the following counterexample: let $S \subset \mathbb{P}^3$ be the complex surface defined by the homogeneous equation $w(x^3 - y^2z) + f(x, y, z) = 0$ where $f$ is a general homogeneous polynomial of degree 4, then $S$ is a rational quartic with a triple point; by the computations of [4] $H^1(S, \mathcal{H}^1(\mathbb{Z})) \neq H^2(S_{an}, \mathbb{Z})$ whence $H^0(S, \mathcal{H}^2(\mathbb{Z})) \neq 0$ but $H^0(S', \mathcal{H}^2(\mathbb{Z})) = 0$ for any resolution of singularities $f : S' \to S$ since $S'$ is rational; moreover $H^0(S, \mathcal{H}^2(\mathbb{Z}))$ is free thus $H^0(S, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{Z}/n \subset H^0(S, \mathcal{H}^2(\mu_n)) \neq 0$ which implies that $S$ has non-zero torsion Brauer group. Similarly: $\mathcal{H}$-cohomologies don’t satisfy the homotopy axiom in the singular case. It might be interesting the approach to the “intersection theory” for singular varieties by considering intersection cohomologies. For $X_{an}$ normal we dispose of nice cohomology groups $IH^*(X_{an})$ (= the intersection cohomology with middle perverity, see [3] and [7]) having an “intersection pairing” and self duality (rationally). By sheafifying for the Zariski topology we do have a sheaf $\mathcal{I}\mathcal{H}^*$ and the corresponding Cousin complex associated with the filtration by codimension (see [25]). (Note: this is the ‘homological’ Gersten complex if local purity holds.) So far as I’m concerned it is natural to ask if $\mathcal{I}\mathcal{H}^*$ is Cohen-Macaulay i.e. is the Cousin complex a flasque resolution? A positive answer would be a great improvement in matter.
A.6 Motivic cohomology

Following Grothendieck philosophy of ‘motives’ we are suspecting the existence of an appropriate Poincaré duality theory $H^*_M(\cdot, \mathbb{Z}(\cdot))$ satisfying the homotopy axiom for smooth algebraic varieties which is ‘universal’ at least in the following weak meaning (cf. [7, §0.2]).

Let $\mathcal{PV}_k$ be the category whose objects are appropriate dualities (may be with values in an abelian tensor category not just graded abelian groups, see [26, §6]) and morphisms are natural transformations compatible with products. For example we have a canonical morphism from the Deligne-Beilinson theory to the classical Borel-Moore theory. We are expecting that any object $H^*_M(\cdot, \mathbb{Z}(\cdot))$ in $\mathcal{PV}_k$ is a receptor for a map from ‘motivic cohomology’ $H^*_M(\cdot, \mathbb{Z}(\cdot))$. This is to say that ‘motivic cohomology’ is to some extent ‘initial’ in $\mathcal{PV}_k$. A reasonable candidate for $H^*_M(\cdot, \mathbb{Q}(\gamma))$ is $\text{gr}_{\gamma} \mathbb{K}_2(X) \otimes \mathbb{Q}$ after Beilinson-Soule, see [32, Theorem 9]. (Note: our dimension axiom is the Beilinson-Soule conjecture [4, 2.2.2]). We would remark that if such ‘motivic cohomology’ $H^*_M(\cdot, \mathbb{Z}(\cdot))$ exists then the $H_M$-cohomology ring will be the Chow ring since rational equivalence is the finer adequate equivalence relation, see [30, Prop. 8]. Finally, if we let restrict our attention to ‘representable’ dualities i.e. given by the homology of complexes, we would remark that our cohomologies would have the descent property of [33] since the axioms [10, §1.1] ensured the Mayer-Vietoris property.

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