Eight-vertex criticality in the interacting Kitaev chain

Natalia Chepiga\textsuperscript{1} and Frédéric Mila\textsuperscript{2}
\textsuperscript{1}Kavli Institute of Nanoscience, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands
\textsuperscript{2}Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland

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We show that including pairing and repulsion into the description of 1D spinless fermions, as in the domain wall theory of commensurate melting or the interacting Kitaev chain, leads, for strong enough repulsion, to a line of critical points in the eight vertex universality class terminating floating phases with emergent U(1) symmetry. For nearest-neighbor repulsion and pairing, the variation of the critical exponents along the line that can be extracted from Baxter’s exact solution of the XYZ chain at $J_x = -J_z$ is fully confirmed by extensive DMRG simulations of the entire phase diagram, and the qualitative features of the phase diagram are shown to be independent of the precise form of the interactions.

I. INTRODUCTION

Models of interacting spinless fermions in 1D have appeared in many contexts over the years\textsuperscript{5–11}. First used to reformulate and solve spin models in the seventies thanks to a Jordan-Wigner transformation\textsuperscript{12}, they have been introduced and further studied in the eighties in the domain wall theory of commensurate melting in 2D, building on the equivalence of classical 2D systems and quantum 1D models\textsuperscript{13}. In that context, the model is more naturally formulated in terms of hard-core bosons with a term creating $p$ consecutive particles for the commensurate melting of a period-$p$ phase, but for $p = 2$, the model is strictly equivalent to spinless fermions. In the early 2000’s, Kitaev\textsuperscript{14} has revisited it as a model of a $p$-wave superconductor, and he has shown that it possesses Majorana edge states, triggering a tremendous experimental activity\textsuperscript{15–20} motivated by their potential use for qubits\textsuperscript{\textsuperscript{16}11,12}. Later on, and quite logically since electrons experience repulsion, the interacting version of the Kitaev chain has been studied\textsuperscript{21–23}. Finally, the problem of commensurate melting has recently resurfaced in the context of chains of Rydberg atoms, and 1D models of hard-core bosons including pairing and higher order creation terms have been investigated in that context\textsuperscript{24,25}.

In this Letter, we will first focus on a model with nearest-neighbor pairing and repulsion. In the context of the domain-wall theory in which it was first introduced, this model is usually written with the following terms:

$$H_{\text{NN}} = \sum_i -t(d_i^d d_{i+1} + \text{h.c.}) - \mu n_i + \lambda(d_i^d d_{i+1} + \text{h.c.}) + V n_i n_{i+1}, \quad (1)$$

where $t$ is the hopping amplitude, $\mu$ is the chemical potential that controls the band filling, $\lambda$ is the amplitude of the terms that create pairs of domain walls, and $V$ describes the nearest-neighbor repulsion, a term absent from the original Kitaev model\textsuperscript{14}. Due to the pairing term, this model does not have $U(1)$ symmetry but only a $Z_2$ symmetry corresponding to the parity of the number of particles. At half-filling ($\mu = V$), it also has particle-hole symmetry.

In the context of the interacting Kitaev model, slightly different notations are often used, with in particular an explicitly particle-hole symmetric form of the repulsion term, leading to the Hamiltonian:

$$H_{\text{NN}} = \sum_i -t(d_i^d d_{i+1} + \text{h.c.}) - \tilde{\mu} n_i + \Delta(d_i^d d_{i+1} + \text{h.c.}) + U(2n_i - 1)(2n_{i+1} - 1). \quad (2)$$

In that formulation, the particle hole symmetric point always occurs at $\tilde{\mu} = 0$, but $\tilde{\mu}$ is strictly speaking no longer the chemical potential. Up to a constant, the two models map onto each other with $\lambda \equiv \Delta$, $\mu \equiv \tilde{\mu} + 4U$, and $V \equiv 4U$. We will mostly use the notations of Eq.\textsuperscript{1} but whenever possible the results will also be shown using those of Eq.\textsuperscript{2}.

The phase diagram of the model without repulsion is well known (see Fig. 5, top panel). For $\lambda > 0$, it consists of three phases: two disordered phases where $Z_2$ is unbroken for $\mu/t < -2$ and $\mu/t > 2$ respectively (the number of particles in the ground state has a well defined parity), and a gapped phase with broken $Z_2$ symmetry for $-2 < \mu/t < 2$. Inside this phase, there is a disorder line defined by $4x^2 + \mu^2 = 4t^2$ below which correlations are incommensurate\textsuperscript{30}. The top of this line corresponds to the famous Kitaev point where the Majorana edge operators are completely decoupled from the bulk\textsuperscript{31}. For $\lambda = 0$, the intermediate phase is a non-interacting Luttinger liquid ($K = 1$), and the transition into the disordered phase is Pokrovsky-Talapov\textsuperscript{32}. When switching on $\lambda$, this transition immediately turns into an Ising phase transition.

The phase diagram remains qualitatively similar up to $V/t = 2$, the intermediate phase of the $\lambda = 0$ line becoming a Luttinger liquid with $1/2 \leq K \leq 1$. When $V/t > 2$ however, the phase diagram becomes much richer, as already pointed out by several authors\textsuperscript{31,32,20} with three new phases: a period-2 phase in which the translation symmetry is broken, and two critical floating phases\textsuperscript{20} that surround it and touch at a multicritical point (see Fig. 5, bottom panel). The appearance of a period-2 phase at $V/t = 2$ for the model without pairing is known from Bethe ansatz\textsuperscript{33} at that
point, the Luttinger liquid exponent reaches the value \( K = 1/2 \), and Umklapp scattering becomes relevant. For \( V/t > 2 \), the Luttinger liquid exponent reaches the value \( K = 1/4 \) at the transition into the period-2 phase, and the transition is in the Pokrovsky-Talapov universality class. Since the pairing term has a scaling dimension \( 1/K \), it is irrelevant as long as \( K < 1/2 \), and the Luttinger liquid phase gives rise to an extended floating phase when \( 1/4 < K < 1/2 \). All the boundaries in Fig. 5 bottom panel, have been determined numerically with state-of-the-art density matrix renormalization group (DMRG) simulations, except the disorder line that coincides with the frustration-free limit, which is known to be given exactly by \( 4\lambda^2 + (\mu - V)^2 = (V + 2t)^2 \), and the multicritical point marked as a star, which sits in the particle-hole plane at \( \lambda = (V - 2t)/2 \) (see below). The DMRG simulations have been performed using a two-site routine with open boundary conditions on systems with up to 3001 sites keeping up to 2000 states and discarding all singular values below \( 10^{-8} \). The boundary between the floating phase and the \( Z_2 \) phase has been determined as the line \( K = 1/2 \), and that with the period-2 phase as the line where the wave-vector becomes equal to \( \pi \) (see Supplemental Material for details). When scanning \( V/t \) from 2 to \( +\infty \), the multicritical points at which the floating phases touch build a line. The universality class of this line of continuous phase transitions is the main open issue in the 3D \( (\lambda/t, \mu/t, V/t) \) phase diagram.

In this Letter, we argue that this line of multicritical points is in the eight-vertex universality class, and that it is a generic feature of models with pairing and repulsion. For the model of Eqs. (1,2), this conclusion is based on a mapping on the integrable point \( J_z = -J_x \) of the XYZ model defined by the Hamiltonian

\[
H = \sum_i J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z - B \sigma_i^z, \tag{3}
\]

where \( \sigma^x, \sigma^y \) and \( \sigma^z \) are Pauli matrices, and solved by Baxter in the seventies, and it is supported by extensive DMRG simulations that show that the behavior close to the critical point both in the period-2 phase and in the broken \( Z_2 \) phase is controlled by the critical exponents that can be extracted from Baxter’s solution. We also study a hard-core boson model with a next-nearest neighbor pairing term for which there is no exact solution, and we provide strong numerical evidence that the point at which the floating phases meet is still in the eight-vertex universality class.

Let us start by discussing the nature of the critical point of the model of Eq. (1). The only piece of information so far was that its central charge \( c = 1 \), a result fully confirmed by fitting our results for the entanglement entropy with the Calabrese-Cardy formula, hence that it is a Luttinger liquid. However, as we now explain, it is possible to fully identify the universality class of the transition. Using a Jordan-Wigner transformation, the model can be mapped on the model of Eq. (3) the XYZ chain in a field with \( J_z = -(t + \lambda)/2 \), \( J_y = -(t - \lambda)/2 \), \( J_x = V/4 \), and \( B = (V - \mu)/2 \). In the particle-hole symmetric plane, the magnetic field vanishes, and the model reduces to an XYZ chain. This model is well known to be integrable when two of the coupling constants are equal, in which case it is usually referred to as the XXZ chain. For our model, this is the case for \( \lambda = 0 \). It can also be solved when one of the coupling constants vanishes, which occurs for \( \lambda = t \) (see Miao et al. A less well known result due to Baxter is that it is also integrable when two coupling constants are opposite, e.g. \( J_y = -J_x \). For our model, this occurs when \( \lambda = (V - 2t)/2 \). Along this line the model can actually be mapped on the XXZ chain by rotating the spins by \( \pi \) around \( z \) (\( \sigma_i^x \to -\sigma_i^y \), \( \sigma_i^y \to -\sigma_i^x \), \( \sigma_i^z \to \sigma_i^z \)) on...
Figure 2. Exponents in the vicinity of the multicritical point as a function of $V/t$: (a) Correlation length exponent extracted from density-density correlations in the period-2 phase ($\nu'$, light blue) and in the $\mathbb{Z}_2$ phase ($\nu$, dark blue); (b) Critical exponent $\beta$ of the amplitude if local density oscillations in the period-2 phase; (c) Scaling dimension $d$ extracted from the slope of the separatrices of Friedel oscillations (blue squares), and estimated from the ratios $\beta/\nu$ and $\beta/\nu'$ (black pluses and crosses respectively). In all cases, the numerical results (symbols) are compared with the theory predictions of Eqs.4 and 5 (magenta lines).

Figure 3. Phase diagram of the model of Eq.7 with nearest-neighbor repulsion $V/t = 10$ as a function of the next-nearest pairing term $\lambda_2$ and chemical potential $\mu$. Red and black solid lines stand for Kosterlitz-Thouless and Pokrovsky-Talapov transitions respectively. The blue lines are Ising transitions to the disordered phases. The system has particle-hole symmetry along the $\mu = V (=10t)$ line, and the phase diagram is mirror symmetric with respect to it. Along this line the transition between the period-2 and $\mathbb{Z}_2$ phases is direct through a multicritical point (yellow star). Inside the $\mathbb{Z}_2$ phase, short-range correlations are incommensurate between the two disorder lines.

Note that, when going from $V/t = 2$ to $\infty$, the parameter $\rho$ changes from 0 to $\pi$, i.e. it describes all the possible interval of the eight-vertex model. Accordingly, the critical exponents change rather dramatically. This is most remarkable for $\beta$, which covers all the range from 0 to $\infty$! It becomes infinite at the opening of the period-2 phase, implying a very smooth development of the dimerization in that limit, while it goes to zero when $V \to \infty$, approaching a step-like behavior in that limit. This is
logical since, when \( V \) is infinite, the pairing term cannot induce fluctuations in the ground state. \( \nu \) is also infinite at the opening of the period-2 phase, in agreement with the Kosterlitz-Thouless\textsuperscript{42} nature of the transition, and decreases to 1/2 when \( V \to \infty \), a value typical of mean field. But the transition is definitely not mean field since \( \beta \) goes to zero, and not 1/2. In the limit \( V/t = 2 \), the Luttinger liquid parameter of the multicritical point akers the value 1/2, as it should since, at that point, it must be equal to the value of the Luttinger liquid parameter at which the gap opens when \( \lambda = 0 \). However, away from that limit, the Luttinger liquid parameter of the multicritical point \( K = \pi/2(\pi - \rho) \) is larger than 1/2 while that of the adjacent floating phases is always between 1/4 and 1/2, demonstrating that this multicritical point is not controlled by the adjacent floating phases.

To investigate how universal this property might be, we look next at a model where the pairing term is between nearest-neighbors, for which there is to the best of our knowledge no exact solution. In terms of hard-core bosons, this model is defined by the Hamiltonian

\[
H_{\text{NNN}} = \sum_i -t(d_i^\dagger d_{i+1} + \text{h.c.}) - \mu n_i + \lambda_2(d_i^\dagger d_{i+2} + \text{h.c.}) + V n_i n_{i+1} \quad \text{(7)}
\]

In terms of fermions, the pairing term would have an extra factor \((-1)^{n_i+1}\) due to the Jordan-Wigner transformation.

The phase diagram of this model is shown in Fig.3 for \( V/t = 10 \). It is qualitatively similar to that of the nearest-neighbor pairing model, with the same phases and similar boundaries. The only qualitative difference appears for very large \( V \), where the floating phase develops a re-entrant behavior upon approaching the \( \lambda_2 = 0 \) line\textsuperscript{42}.

As long as \( V < +\infty \), there are two floating phases that are found numerically to end up at a multicritical point\textsuperscript{42}. To study the properties of this multicritical point, we have again calculated the exponents \( \nu, \nu', \beta \) and \( d \). This time, we do not have any prediction for the dependence of \( \rho \) on the parameters of the model, so, in order to check if the multicritical point is still eight-vertex, we have eliminated \( \rho \) from Eqs.\textsuperscript{10,12}, leading to expressions for \( \nu \) and \( \beta \) as a function of \( d \). These expressions are checked in Fig.4. The error bars are larger than for the model with nearest-neighbor pairing, in part because the critical value of \( \lambda \) is not known exactly, but the results clearly support the eight vertex universality class. Note that the values reached in the limit \( V \to +\infty \) do no longer correspond to \( \rho = \pi \).

Let us now briefly compare our results with recent literature on the interacting Kitaev chain. Sela et al\textsuperscript{43} have studied the full phase diagram, but they could not decide if the floating phases extend up to the particle-hole symmetric plane, and accordingly they did not discuss the multicritical line at which they touch. Their focus was the fate of the Majorana edge states. Miao et al\textsuperscript{44} have also studied an integrable line in the particle-hole symmetric plane, but a different one given by \( \lambda = t \) in our notation. For small \( V/t \), this line is in the \( Z_2 \) phase. It crosses our line at the point where the period-2 phase opens, \( V/t = 2 \), and it lies in the period-2 phase for larger \( V/t \), in full agreement with our phase diagram. Hassler and Schurich\textsuperscript{45} have looked at another cut in the 3D parameter space \( (\lambda/t, \mu/t, V/t) \), namely \( \lambda = t \), and not \( V/t = \text{cst} \), as we did. Again their results are fully consistent with ours. They spotted the multicritical point at \( V/t = 4 \) but did not identify its universality class beyond the fact that it has a central charge \( c = 1 \). More recently, Verresen et al\textsuperscript{46} revisited the \( \lambda = t \) plane and emphasized the emergent \( U(1) \) symmetry in the floating phase.

The present results also have strong connections with the physics of 2D classical models. The eight-vertex model has been introduced and solved in the context of 2D ice-type models where different Boltzmann weights are attributed to different arrow configurations around a vertex, and the paradigmatic models of 2D frustrated magnetism, the anisotropic next-nearest neighbor Ising (ANNNI) model\textsuperscript{47,48} has a phase diagram similar to ours, with a multicritical point in Baxter's eight-vertex universality class.

Finally, the standard model of Rydberg atoms is related to that of Eq.\textsuperscript{1} by duality\textsuperscript{49}. The period-2 phase of Rydberg chains corresponds to the \( Z_2 \) phase of \( H_{\text{NNN}} \), and the Ising transition that surrounds it is equivalent to the Ising transition into the disordered phase. The equivalent of the period-2 phase of \( H_{\text{NNN}} \) should be a \( Z_2 \) broken phase, but, in the standard setting, the model of Rydberg chains contains single particle creation and anihilation operators and does not have \( Z_2 \) symmetry. However, it should be possible to directly program the models of Eq.\textsuperscript{1} or Eq.\textsuperscript{7} in optical cavities with individ-
ual control over trapped atoms. In any case, it will be rewarding to see if the eight-vertex universality class can be experimentally identified in 1D quantum systems.

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SUPPLEMENTAL MATERIAL FOR:
EIGHT-VERTEX CRITICALITY IN THE INTERACTING KITAEV CHAIN

In this Supplemental Material we provide additional information on how we extracted the critical exponents $\beta$ and $\nu$ and the scaling dimension $d$ at the critical point along the particle-hole symmetry line. We also present results for the central charge at the Ising transition and at the eight-vertex multicritical point, and we show the profiles of the correlation length, of the wave-vector $q$, and of the Luttinger liquid parameter $K$ used to determine the location of the disorder and critical lines. Finally, we present the phase diagram of the model with nearest-neighbor blockade and next-nearest-neighbor pairing.

Numerical data for the model with nearest-neighbor pairing

In this section we provide further numerical details supporting the phase diagram of the model with nearest-neighbor pairing (Eq. 1 and Fig. 1 of the main text), and the eight-vertex universality class of the critical point on the particle-hole symmetric line.

A. Location of the critical lines

In Fig. 5 we show the profile of the inverse of the correlation length $\xi$. The results were obtained for the $\lambda/t = 1$ horizontal cut of the phase diagram presented in the bottom panel of Fig.1 of the main text. It refers to the model with nearest-neighbor pairing and with a nearest-neighbor repulsion $V/t = 6$. Several interesting features are revealed by this profile. First, the inverse of the correlation length vanishes linearly around $\mu/t \approx -1.84$, in agreement with the Ising critical exponent $\nu = 1$. Shortly after, at $\mu \approx -1.74$ the inverse of the correlation length reaches its maximum at a very sharp kink that corresponds to the disorder point. Beyond this point the inverse correlation length decreases very fast, in agreement with the exponential divergence of the correlation length typical for a Kosterlitz-Thouless transition. On the other side of the critical region marked in green the inverse correlation length vanishes with a critical exponent $\nu$ clearly smaller than 1, in agreement with the Pokrovsky-Talapov critical exponent $\nu = 1/2$.

Figure 5. Inverse of the correlation length of the model with nearest-neighbor pairing at $V/t = 6$ and $\lambda/t = 1$.

As we argue in the main text, the $Z_2$ phase is separated from the period-2 phase by a floating phase - a critical Luttinger liquid phase with incommensurate correlations. The transition between the floating and the period-2 phases is a commensurate-incommensurate transition expected to be in the Pokrovsky-Talapov universality class. We locate this transition as the point where the wave-vector of the incommensurate correlations reaches the commensurate value $q = \pi$. The transition between the $Z_2$ and the floating phases is Kosterlitz-Thouless. We associate this transition with the point where the Luttinger liquid parameter $K$ takes the value $K = 1/2$. In Fig. 6 we provide examples of the Luttinger liquid exponent and of the wave-vector $q$ as a function of the chemical potential for $V/t = 6$ and $\lambda/t = 1$. Our results suggest that the Luttinger liquid exponent $K$ reaches the value $K = 1/4$ at the Pokrovsky-Talapov transition, as in the non-interacting case.

B. Computation of the critical exponent and the central charge at the eight-vertex point

According to boundary conformal field theory, at the critical point the Friedel oscillations in chains with open and fixed boundary conditions follow the profile $|n_j - n_{j+1}| \propto 1/[(N/\pi) \sin(\pi j/N)]^d$, where $d$ is the scaling di-
mension given by the ratio of the two critical exponents \( d = \beta / \nu \). In particular, it implies that the finite-size scaling of the middle-chain \((j = N/2)\) density amplitude measured at the critical point as shown in Fig.7 has the slope \( d \) in log-log scale. We also check that the separatrix corresponds to \( \lambda = (V - 2t)/2 \), confirming that this is the critical point.

In order to check the predictions for \( \nu \) and \( \beta \), we look at the scaling of the correlation length and of the amplitude of local density oscillations as a function of the distance to the critical point. In order to minimize the boundary effects we take the amplitude of the oscillations in the middle of the chain. The results for \( V/t = 6 \) and for \( V/t = 20 \) and for two different system sizes are presented in Fig.8. The values obtained for the critical exponents \( \nu \), \( \nu' \) and \( \beta \) are compared to the theory predictions for the eight-vertex model in Fig.2 of the main text.

At small value of the nearest-neighbor repulsion the finite-size effects are very strong and lead to two apparently different critical exponents \( \nu \) and \( \nu' \) on the two sides of the transition. One can see that even for a system size with a few thousands sites the exponent \( \nu' \) extracted upon approaching the transition from the period-2 phase is still severely affected by finite-size effects. By contrast, the critical exponent \( \nu \) extracted upon approaching the transition from the \( Z_2 \) phase is less affected by finite-size effects.

In addition, at the multicritical point we use the Calabrese-Cardy formula \( \xi = 2N \sin \left( \frac{\pi}{N} \right) \) to extract the central charge numerically from the finite-size scaling of the entanglement entropy in an open chain:

\[
S_N(n) = \frac{c}{6} \ln d(n) + s_1 + \ln g, \tag{8}
\]

where \( d = \frac{2N}{\pi} \sin \left( \frac{\pi n}{N} \right) \) is the conformal distance, and \( s_1 \) and \( \ln g \) are non-universal constants. The resulting values of the central charge are in excellent agreement with \( c = 1 \). An example of scaling for \( V/t = 6 \) is shown in Fig.10.

**Numerical data for the model with next-nearest-neighbor pairing term**

In this section, we provide further numerical results for the model of Eq. 7 of the main text with next-nearest neighbor pairing.

In Fig.11 we present the inverse of the correlation length for \( V/t = 10 \) and along the horizontal cut \( \lambda_2 = 3t \). We extract the correlation length by fitting the exponen-
Figure 10. Scaling of the entanglement entropy with conformal distance at the multicritical point located at $\mu = V = 6t$ and $\lambda/t = 2$ and marked with a yellow star in the phase diagram of Fig. 1 of the main text.

Figure 11. Inverse of the correlation length as a function of $\mu/t$ along the horizontal cut $\lambda_2/t = 3$ for the model with next-nearest-neighbor pairing with $V/t = 10$. The plot is mirror symmetric with respect to $\mu/t = 10$. The colors mark different phases: disordered (gray); $\mathbb{Z}_2$ (yellow); floating (green); period-2 (blue).

Figure 12. Scaling of the entanglement entropy with conformal distance at $\lambda_2 = 3$, $\mu/t = -0.46$ and $V/t = 10$ for $N = 1201$ sites. The value of the central charge $c \approx 0.49$ is obtained by fitting the data to the Calabrese-cardy formula given by Eq. $\ref{eq:calabrese-cardy}$. The result is in excellent agreement with $c = 1/2$, the value for the Ising critical theory.

Figure 13. Luttinger liquid exponent $K$ (top) and incommensurate wave-vector $q$ (bottom) along two horizontal cuts through the floating phase of the model with nearest-neighbor repulsion $V/t = 10$ and next-nearest-neighbor pairing term with $\lambda_2/t = 1$ (left) and $\lambda_2/t = 3$ (right). The symbols are DMRG data, the lines in (c-d) are fits assuming the Pokrovsky-Talapov critical exponent $\beta = 1/2$. The dashed lines in (a-b) indicate the location of the PT transition extracted from the wave-vector $q$. Both $K$ and $q$ were extracted from Friedel oscillations of the local density.
We locate the Pokrovsky-Talapov transition by fitting the wave-vector $q$ inside the floating phase with $q \propto |\mu - \mu_c|^\beta$ with a fixed value of the critical exponent $\beta = 1/2$. The results of the fits are presented in Fig.13. These numerical results suggest that, for large $\lambda_2$, the Luttinger liquid exponent $K$ is significantly larger than 0.25 in the vicinity of the Pokrovsky-Talapov transition, but we cannot exclude that very close to the transition it decreases steeply towards $K = 1/4$ as in the non-interacting case.

In order to locate the multicritical point along the particle-hole symmetric line $\mu = V$ for various strengths of the repulsion $V$ we look at the finite-size scaling of the amplitude of the oscillations of the local density $|\langle n_{N/2} - n_{N/2+1} \rangle|$ and associate the transition with the separatrix in log-log scale. An example of such a scaling for $V = \mu = 10t$ is shown in Fig.14. As we have already seen for the model with nearest-neighbor pairing, at the critical point the slope corresponds to the scaling dimension $d = \beta/\nu$. The resulting critical values for $V/t = 10$ are $\lambda_2/t \approx 3.965$ and $d \approx 0.235$.

We extract the central charge from the scaling of the entanglement entropy as presented in Fig.15. For all values of $V$ the value of the central charge agrees with $c = 1$ within 5%.

For this model the location of the multicritical point is not known exactly and has a non-linear dependence on the repulsion $V$. Thus the error in the scaling dimension is significant and comes from the finite resolution when locating the critical point.

**Model with nearest-neighbor blockade**

In the limit $V \to \infty$ the model defined by Eq. 7 of the main text takes the following form:

$$H_{\text{blockade}} = \sum_i -t(d_i^\dagger d_{i+1} + h.c.) - \mu n_i + \lambda_2(d_i^\dagger d_{i+2}^\dagger + h.c.)$$

(9)

where the Hamiltonian acts on the explicitly restricted Hilbert space $n_i(1-n_i) = n_in_{i+1} = 0$. For this model, the nearest-neighbor pairing operator $d_i^\dagger d_{i+2}^\dagger + h.c.$ is trivially equal to zero and the first non-vanishing contributions come from pairing at distances beyond the blockade. The first one in the present case is the next-nearest neighbor pairing with amplitude $\lambda_2$.

In Fig.16 we show the phase diagram of this model for $\lambda_2 \leq 4t$. At large $\mu$ the blockade leads to a phase with spontaneously broken translation symmetry with every other site occupied. The particle-hole symmetric line is sent to $\mu$ infinite, and the multicritical point is sent to $\lambda_2$ and $\mu$ infinite. Accordingly there is a single floating phase, and the period-2 phase and the floating phase are always separated by this floating phase. The only difference with the corresponding portion of the phase diagram of Fig. 4 of the main text is the bending of the Pokrovsky-Talapov boundary and the reentrant floating phase as small $\lambda_2$ for $\mu \simeq 4t$. 

![Figure 14. Finite-size scaling of the amplitude of the local density oscillations for $V = \mu = 10t$ in the vicinity of the transition for the model with next-nearest neighbor pairing and different values of $\lambda_2$. We associate the critical point with the separatrix in log-log scale; the slope corresponds to the scaling dimension $d$.](image)

![Figure 15. Scaling of the entanglement entropy with the conformal distance for the model with next-nearest neighbor pairing at $\mu = V = 10t$ and $\lambda_2 = 3.965t$ for $N = 1201$. The value of the central charge $c \approx 0.99$ is in excellent agreement with $c = 1$.](image)
Figure 16. Phase diagram of the blockade model defined by Eq.9. The red and black solid lines denote the Kosterlitz-Thouless and Pokrovsky-Talapov transitions respectively.