Upper Bound Analysis of Non-Persistent Jointed Rock Slope Using Rigid Block Element Method

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Abstract. The upper bound method analyzing the stability of the non-persistent jointed rock slopes is proposed by considering rock bridge mechanical effects, which combines the upper bound theorem, the discretization technique of rigid block element and the nonlinear mathematical programming. The kinematically admissible velocity fields of non-persistent jointed rock are constructed. The non-linear mathematical programming models are established for solving the rigorous upper bound solution of ultimate load or safety coefficient by using successive linear programming algorithm. Two numerical examples are performed and analyzed. The proposed method is successfully validated by comparing the simulation with those produced by other classical methods. The method has the following advantages: (1) it can simultaneously simulate the discontinuous mechanism properties of the rigid rock blocks and the continuous medium mechanism properties of the rock bridge failures. Both the tensile and shear mechanism effects of the rock bridges can be considered simultaneously; (2) comparing with the exiting numerical methods such as finite element method or discrete element method, this method can not only consider the tensile and shear effects of the rock bridges simultaneously, but also avoid the complex constitutive relationship between the rock masses and the joints. The safety factor and its corresponding failure mechanism can be obtained simultaneously; (3) comparing with the traditional rigid limit equilibrium method, this method does not need to make any assumption of the failure surfaces. The final critical failure mechanism can be directly obtained through mathematically programming method; (4) this method has the characteristics of clear concept, simple in the program composition and high computational efficiency.

Keywords. Non-persistent joint; rock bridge; upper bound method; rigid block element; nonlinear programming.
1. Introduction
In last two decades, many researchers have carried out extensive studies on the bearing capacity of the non-persistent jointed rock masses, Bai et al. (1999) conducted the direct shear test for the rock masses containing non-persistent joints. Xia et al. (2010a, b) investigated the weakening mechanics model of the rock bridges and the strength criterion of the non-persistent joints. Tang et al. (2011) examined the variation law of the non-persistent joints. Many studies have been carried out to calculate the loading capacity of the non-persistent joint rock masses by using the rigid limit equilibrium method and the numerical analysis methods based on continuous or non-continuous medium, such as the finite element method, the manifold element method, the discrete element method, DDA, and the block element method (Ren et al., 1999; Zheng et al., 2015; Xiong et al., 2011; Jiang et al., 2015). Though these studies have demonstrated some features of the non-persistent joint rock masses, there are some deficiencies. For example, the rigid limit equilibrium method needs to assume a sliding surface in advance in order to find the most dangerous sliding surface by trial-and-error method. Therefore, this method requires huge amount of calculation for rock mass with many joints and it is difficult to obtain the most dangerous sliding surface. There exist some difficulties by applying finite element method (FEM) and discrete element method (DEM) to simulate the complex joint network, the complex constitutive relationship as well as to solve the safety factor. Therefore, the application of FEM and DEM to investigate the non-persistent jointed rock masses is limited.

It is well known that the upper bound method of plastic limit analysis is a highly efficient method to solve the limit bearing capacity of rock and soil mass. Since Drucker and Prager (1952) developed the limit theory in 1952, the method has been applied to investigate the bearing capacity of rock and soil structures, such as foundation and slope. Sloan (1989) and Lyamin and Sloan (2002) proposed FEM upper bound method of plastic limit analysis based on the mathematical programming, making it possible to construct the kinematically admissible velocity fields for large volume structure; Chen et al. (2004) proposed the rigid element upper bound analysis method to investigate the stability of the soil slope subjected to pore water pressure. Zheng et al. (2008) presented the limit analysis upper bound method for the block element of the stability of rock slope by combining the block element method and the limit analysis upper bound method. Recently, Li et al. (2012) proposed the upper bound method to analyze the stability of the rock slope based on the rigid blocks system. Liu et al. (2012) presented the rigid element upper bound method of soil slope by considering the rational effect. Based on the FEM, Sun et al. (2015) proposed the upper bound method of soil slope subjected to tensile and shear failure.

In this study we will combine the upper bound theory of the plastic limit analysis, the discretization technique of the rigid block element and the non-linear mathematical programming method to propose a block element upper bound method that can be used to simulate the mechanics effect of the rock bridges in the non-persistent jointed rock masses.

2. The principle and method of simulation
2.1 The assumption of this study
In order to simplify the calculation, the following hypotheses are made: (a) when the rigid block element is used to simulate the mechanics characteristics, assume that the rock block is a rigid body, whose any deformation and failure will not occur, and the failure only takes place on the structural planes between adjacent blocks; (b) only consider the translational effect of the blocks and the rock mass will not be separated from each other during the deformation processes; (c) when the strength reserve coefficient is determined, the shear and tensile strength parameters of all structural planes are reduced in the same proportion.

2.2 Discretion and variable of the rigid block element
In this study, the following techniques are used to discretize the slope: (1) using the rigid block element to directly simulate the mechanic characteristics of the regular rock block generated by persistent jointed cutting, (2) using the virtual structural plane to cut rock bridge areas into the regular block and taking the material parameters of the integrity rock blocks as the material parameters of the virtual structural plane. After the rock slopes are discretized, the defined variables on the block element and structural plane are shown in figure 1 in which the global coordinate system is $\mathbf{x}, \mathbf{y}$, the local coordinate system
on the structural plane \( k \) between block \( i \) and block \( j \) is defined as \( (n_k, s_k) \). The velocity vector acting on the centroid of the block element \( i \) is \( \delta \dot{u}_i = (\delta u_n, \delta v)_i \); the jump velocity vector acting on the centroid of the structural plane \( k \) between adjacent block elements is \( \delta \dot{u}_k = (\delta n_k, \delta s_k)_k \); the force vector acting on the centroid of the structural plane \( k \) is \( \dot{Q}_k = (N_k, V_k)_k \); the force vector acting on the centroid of the block element \( i \) is \( \dot{F}_i = (f_n, f_v)_i \).

![Figure 1](image1.png)

**Figure 1** Sketch of variables at an interface and a block element (a) The velocity mode and the structural plane (b) Sketch of forces acting on a block element

### 3. Numerical model of the upper bound method for the rigid block elements

#### 3.1 The objective functions

Two objective functions for the bearing capacity of the jointed rock slopes are proposed in this study: the overload factor and the strength reserve coefficient. The overload is the load at the limit state of the failure of the rock slopes, namely the ultimate load. The overload factor \( K_r \) in this study is defined as:

\[
K_r = q_u/q_p
\]  

(1)

The strength reserve coefficient can be obtained by reducing the strength parameters of materials until the failure of slopes occur and can be defined as:

\[
K_s = \frac{t_{\phi}}{t_{\phi}} = \frac{c}{\sigma} = \frac{c}{\sigma}
\]  

(2)

According to the upper bound theorem, when slopes approach the ultimate state, it needs to solve the minimum of the overload factor or the strength reserve coefficient. Therefore, the objective function for solving the overload factor is

**Minimize**: \( K_r \)

(3)

While the objective function for solving the strength reserve coefficient is

**Minimize**: \( K_s \)

(4)

#### 3.2 The deformation compatibility condition of structural surfaces

The deformation of adjacent block \( i \) and block \( j \) and their interface \( k \) must satisfy the deformation compatibility condition after slope is discretized into the geometric system of the rigid block elements and the structural planes (as shown in figure 1). Combining the geometric discrete method of the block element and the structural plane, the deformation compatibility condition can be derived as following:

\[
\left[ \delta \dot{u}_i \right] = \left[ \dot{T} \right] \left( \delta \dot{u}_i - \delta \dot{u}_j \right)
\]  

(5)

Equation 5 can be rewritten using the vector and matrix as:

\[
\delta \dot{u}_k = \dot{D}_k \delta \dot{u}_i, \quad k = (1, \ldots, n_k)
\]  

(6)

#### 3.3 The plastic flow constraint condition

Assume that the plastic flow only occurs on the structural planes, namely the vector discontinuity is on the common border between two adjacent block elements (as shown in figure 1(a)) and the thickness of the structural planes be zero. In order to satisfy the condition of kinematically admissible, the discontinuous velocity in the normal and tangential directions must comply with the associated flow rule on this structural plane. In this study, the shear and tensile conditions need to be considered. The modified Mohr-Coulomb yield criteria can be expressed by the force vector \( \dot{Q}_i = (N_i, V_i)_i \) acting on the centroids of the structural plane \( k \) in the local coordinate system \( (n_k, s_k) \) and both consider satisfying tensile and shear conditions is expressed as:
In this study, assume that the persistent jointed planes cannot sustain the tensile forces, namely the tensile strength $\sigma^t = 0$, and its yield criteria is shown in figure 2(a). Assume that the virtual structural plane of the rock bridges satisfies not only the shear condition, but also the tensile condition ($\sigma^t > 0$), its yield criteria is shown in figure 2(b).

Combining the yield criteria equation 7 and the associated flow criteria $\Delta\sigma = -k \cdot \partial f_i(\vec{Q}) / \partial \vec{Q}$, yields the normal velocity and the tangential velocity discontinuity components of the structural plane $k$:

$$\begin{bmatrix} \Delta v^k_x \\ \Delta v^k_y \end{bmatrix} = \begin{bmatrix} \tan \phi & \tan \phi & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\lambda}^k_x \\ \dot{\lambda}^k_y \\ \dot{\lambda}^k_z \end{bmatrix}$$

For the perfect rigid-plastic model, the associated flow criteria of the plastic theorem is used. The generalized strain rate components generated by the deformation compatibility condition should be equal to the generalized plastic strain rate components generated by the associated flow criteria under the yield condition. The plastic flow constraint condition is obtained by combining the formula 6 and the formula 8:

$$D_k \delta u^k_{ij} - N^k_{ij} \delta \lambda^k = 0, \quad k = (1, \ldots, n_k)$$

Substitute equation 2 into equation yields the plastic flow condition for solving the strength reserve coefficient:

$$D_k \delta u^k_{ij} - N^k_{ij} \delta \lambda^k = 0, \quad k = (1, \ldots, n_k)$$

The additional constraint condition is that the plastic multiplier is not negative:

$$\delta \lambda^k = \begin{bmatrix} \dot{\lambda}^k_x \\ \dot{\lambda}^k_y \\ \dot{\lambda}^k_z \end{bmatrix} \geq 0, \quad k = (1, \ldots, n_k)$$

3.4 The equality condition of the internal and external power rate

According to the principle of the virtual power, the virtual power done by the external force is equal to the dissipation power of the internal energy of the objects:

$$\int_{\Omega} \sigma^t \varepsilon^t \, d\Omega + \int_{\alpha} \sigma^t \varepsilon^t \, d\Gamma = WV + QV + PV^*$$

Based on the assumption of the block element stiffness, the deformation and failure of the block will not occur, the internal work dissipation only occurs on the interfaces between the block elements. Therefore, the internal power rate within the continuity bodies is zero, namely $\int_{\Omega} \sigma^t \varepsilon^t \, d\Omega = 0$.

According to the associated flow criteria and the yield condition 7, the internal dissipation power rate along the structural planes of the rock slopes is:

$$\int_{\alpha} \sigma^t \varepsilon^t \, d\Gamma = \sum_{k=1}^{n} (c_k (\dot{\lambda}^k_x + \dot{\lambda}^k_y + \sigma^t \dot{\lambda}^k_z))$$

Where the first item in parentheses on the right hand side is the internal power dissipation produced from the shear failure of the structural planes; the second item on the right hand side is the internal power dissipation produced from the tensile failure of the rock bridges.

Equation 12 can be rewritten as following by considering the rock weight and the external power rate produced by the external force on the boundary:

$$\sum_{k=1}^{n} (c_k (\dot{\lambda}^k_x + \dot{\lambda}^k_y) + \sigma^t \dot{\lambda}^k_z) = WV + QV^*$$

Substitute 1 into 14 yields the internal and external power rate equality condition for solving the overload factor:

$$\sum_{k=1}^{n} (c_k (\dot{\lambda}^k_x + \dot{\lambda}^k_y) + \sigma^t \dot{\lambda}^k_z) = WV + K_i QV^*$$

Substitute 2 into 14 yields the internal and external power rate equality condition for solving the strength reserve coefficient:

$$\sum_{k=1}^{n} (c_k (\dot{\lambda}^k_x + \dot{\lambda}^k_y) + \sigma^t \dot{\lambda}^k_z) / K_i = WV + QV^*$$
According to the upper bound theorem, the kinematically admissible velocity fields must satisfy the known velocity boundary condition. The boundary condition whose velocity is zero on the boundary \( b \) in the jointed rock slopes is
\[
\tilde{T}_j \delta \tilde{u} = 0, \quad j = (1, \ldots, n_b)
\]  \hspace{1cm} (17)

3.6 The non-linear mathematical programming model of the upper bound method for the rigid block element

Integrate the objective function formula 3.4 and the constraint condition equations 10, 11, 16, 17 yields the non-linear mathematical programming model of the upper bound method for solving the overload factor or the strength reserve coefficient of the non-persistent jointed rock slopes:

Minimize: \( K_1 \)
Subject to:
\[
D_k \delta u_{k,j} - N_{k,j} \delta k = 0, \quad k = (1, \ldots, n_k)
\]
\[
\sum_{j=1}^{n_j} (c_j (\lambda^1_j + \lambda^2_j) + \sigma_j I_j \lambda^1_j) = W V^+ + K_1 Q V^+
\]
\[
\tilde{T}_j \delta u_{j} = 0, \quad j = (1, \ldots, n_j)
\]
\[
\delta \kappa_k = \{\lambda^1_k, \lambda^2_k, \lambda^3_k\}^T \geq 0, \quad k = (1, \ldots, n_k)
\]

Minimize: \( K_2 \)
Subject to:
\[
D_k \delta u_{k,j} - N_{k,j} \delta k = 0, \quad k = (1, \ldots, n_k)
\]
\[
\sum_{j=1}^{n_j} (c_j (\lambda^1_j + \lambda^2_j) + \sigma_j I_j \lambda^1_j) / K_2 = W V^+ + Q V^+
\]
\[
\tilde{T}_j \delta u_{j} = 0, \quad j = (1, \ldots, n_j)
\]
\[
\delta \kappa_k = \{\lambda^1_k, \lambda^2_k, \lambda^3_k\}^T \geq 0, \quad k = (1, \ldots, n_k)
\]

4. The solution of the non-linear mathematical programming model of the upper bound method

In this study, the authors use SLP (Successive Linear Programming) method to perform the calculation of the non-linear programming model. SLP is an iterative optimization method which can use the linear programming technique to solve the non-linear problems. The advantage of this method is that it can quickly and accurately converge to the optimal solution for the optimization problem through setting the appropriate initial step size and the dynamic proportional reduction factor. It has been proved that the proposed algorithm has characteristics of fast computation and good convergence for solving non-linear problems (Palacios-Gomez f et al., 1982).

Figure 3 shows the numerical solution procedure used in this study.

5. Numerical examples

Two numerical examples have been performed to validate the accuracy of the propose upper bound method of plastic limit analysis for the non-persistent jointed rock slopes.

5.1 The ultimate load of direct shear specimen of the non-persistent joint

In order to validate the proposed method, the simulation using the proposed method is compared with the analytical result for the non-persistent joint reported in literature (Jiang et al., 2015). While the calculation parameters are listed in Table 1. This simulation specimen contains a non-persistent joint, the persistent jointed area which is on the left and right side of the test specimen and the middle area which is the rock bridge. The ultimate shear loads are calculated for a range of the connectivity rate \( k \) and normal stress \( \sigma_n \) which are listed in Table 1. Figure 4 is the schematic diagram of the divided block elements of the direct shear simulation specimens with various connectivity rates \( k \). In Figure 4, the material parameters of the whole rock block are taken the same as those of the rock bridge virtual structural planes.
Figure 5 Relationship between ultimate shear load and connectivity $k$

| Joint type                  | Calculating parameter |
|-----------------------------|-----------------------|
| Side double joint           | Connectivity rate $k$  |
|                             | Normal stress $\sigma_n$ |
| 90%                         | 2.0                   |
| 80%                         | 2.0                   |
| 70%                         | 2.0                   |
| 60%                         | 1.0                   |
| 60%                         | 2.0                   |
| 60%                         | 3.0                   |

Table 1: Calculation parameters of the direct shear simulation specimen

| Material name                      | Unit weight / (kN·m$^{-3}$) | Cohesion / MPa | Friction angle / ($^\circ$) |
|------------------------------------|------------------------------|----------------|----------------------------|
| Rock mass                          | 15                           | 4.23           | 26.55                      |
| Virtual structural plane of rock   | /                            | 4.23           | 26.55                      |
| bridges                            | /                            | 4.23           | 26.55                      |
| Jointed plane                      | /                            | 0              | 35.2                       |

Table 2: Physic and mechanics parameters of numerical example 1

| Connectivity rate $k$ | Ultimate shear load (kN) | Error (%) |
|-----------------------|--------------------------|-----------|
|                       | Analytical solution      | Upper bound method of this paper | Error with analytical solution |
| 0.6                   | 176.29                   | 178.90    | 1.48                      |
| 0.7                   | 153.38                   | 156.63    | 2.11                      |
| 0.8                   | 130.47                   | 132.27    | 1.38                      |
| 0.9                   | 107.56                   | 109.46    | 1.77                      |
| 1.0                   | 84.65                    | 84.65     | 0.00                      |

Table 3: Results of ultimate load with different methods

| Material name                      | Unit weight / (kN·m$^{-3}$) | Cohesion / MPa | Friction angle / ($^\circ$) | Tensile strength / MPa |
|------------------------------------|------------------------------|----------------|-----------------------------|------------------------|
| Rock mass                          | 27                           | 0.3            | 30                          | 0.15                   |
| Virtual structural plane of rock   | /                            | 0.3            | 30                          | 0.15                   |
| Jointed plane                      | /                            | 0.01           | 30                          | 0                      |

The physical and mechanics parameters of materials are listed in Table 2. The simulated results are as shown in Table 3. Figure 5. In order to compare and analyze, the results of the present method are compared with the analytical solutions. It can be seen from Table 3 and Figure 5 that (a) the calculation results using the proposed method in this study are greater than the analytical solutions. This is because that the proposed method is expected to produce the upper bound solutions. (b) for the case that the normal stress is 2.0MPa, the ultimate shear load decreases with the increase of the connectivity rate $k$, while the rock bridge effect of joints in non-persistent areas depresses with the increase of the connectivity rate $k$ gradually.
5.2 The non-persistent jointed rock slope containing two rock bridges.

This numerical example is taken from reference which has been done by Zheng Y et al. (2015). Figure 6 is the sketch of the calculation mode while the physical mechanism parameters of the rock materials are listed in Table 4. Figure 6(a) is the information of 5 non-persistent joints within the slopes and Figure 6(b) is 5 irregular rock blocks cut by 5 non-persistent joints. There are two rock bridges in the upper and lower part of the rock masses as shown in Figure 6, which is a rock mass that cannot freely move. If the sliding failure occurs in the rock mass B, rock bridge 1 usually appears the shear failure while rock bridge 2 is generally shown as the tensile failure. In this study, we use 11 virtual structural planes and 5 persistent joints to discretize this rock mass into 11 blocks, as shown in Figure 7. The minimum safety factor is obtained as 0.98, while the corresponding failure mechanisms are the shear failure between block b7 and b8 block as well as the tensile failure between b2 and b3 (see Figure 8).

### Table 4: Material physic and mechanics parameters of jointed rock slope of example 3

| Serial number | Possible movable block combination | The safety factor | The safety factor of upper bound method in this paper |
|---------------|-----------------------------------|------------------|-----------------------------------------------------|
| 1             | {b1, b2, b3, b4, b5, b6, b7}       | 2.15             | /                                                   |
| 2             | {b2, b3, b4, b5, b6, b7}           | 1.84             | /                                                   |
| 3             | {b3, b4, b5, b6, b7}               | 1.95             | /                                                   |
| 4             | {b2, b3, b5, b6, b7}               | 1.38             | /                                                   |
| 5             | {b4, b5, b6, b7}                   | 2.33             | /                                                   |
| 6             | {b3, b5, b6, b7}                   | 1.44             | /                                                   |
| 7             | {b5, b6, b7}                       | 1.63             | /                                                   |
| 8             | {b3, b6, b7}                       | 0.95             | 0.98                                                |
| 9             | {b6, b7}                           | 1.04             | /                                                   |
| 10            | {b7}                               | 22.42            | /                                                   |

Zheng Y et al. (2015) applied the virtual structural plane to divide the rock mass into regular blocks, and then search for all possible movable block combination to establish the static equilibrium condition of each block combination according to the tensile and shear condition of structural planes. The safety factor of each block combination is calculated to obtain the final minimum slip combination of the safety factors. This is a complex calculation process and the minimum safety factor is obtained after calculating 10 possible movable block combinations. Comparing with the method proposed by Zheng et al. (2015), the proposed method in this study has the following advantages: (1) the proposed method does not need to assume any possible failure models in prior and only needs to establish kinematically admissible velocity fields of the whole slope according to all blocks and structural planes variables; (2) the proposed method does not need to solve the safety factor for various failure models. The minimum upper bound solution is directly solved by using the mathematically programming algorithm and the corresponding most realistic failure mechanism can be obtained. The proposed method has characteristics of clear concept, convenient application and high calculation efficiency.
6. Concluding remark
A rigid block element upper bound method that has the following advantages: (1) it can simultaneously simulate the discontinuous mechanism properties of the rigid rock blocks and the continuous medium mechanism properties of the rock bridge failures. Both the tensile and shear mechanism effects of the rock bridges can be considered simultaneously; (2) comparing with the exiting numerical methods such as finite element method or discrete element method, this method can not only consider the tensile and shear effects of the rock bridges simultaneously, but also avoid the complex constitutive relationship between the rock masses and the joints. The safety factor and its corresponding failure mechanism can be obtained simultaneously; (3) comparing with the traditional rigid limit equilibrium method, this method does not need to make any assumption of the failure surfaces. The final critical failure mechanism can be directly obtained through mathematically programming method.

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