Resolving Singularities and Monodromy
Reduction of Fuchsian Connections

Yik-Man Chiang, Avery Ching and Chiu-Yin Tsang

Dedicated to the memory of Richard A. Askey.

Abstract. We study monodromy reduction of Fuchsian connections from a sheaf theoretic viewpoint, focusing on the case when a singularity of a special connection with four singularities has been resolved. The main tool of study is based on a bundle modification technique due to Drinfeld and Oblezin. This approach via invariant spaces and eigenvalue problems allows us not only to explain Erdélyi’s classical infinite hypergeometric expansions of solutions to Heun equations, but also to obtain new expansions not found in his papers. As a consequence, a geometric proof of Takemura’s eigenvalues inclusion theorem is obtained. Finally, we observe a precise matching between the monodromy reduction criteria giving those special solutions of Heun equations and that giving classical solutions of the Painlevé VI equation.

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1. Introduction

It has been observed recently that the use of closed-form solutions to Fuchsian equations with an apparent singularity led to significant simplification in solving certain free moving boundary problems [8–10,35]. In fact, resolving apparent singularities of Fuchsian equations is also [36] related to automorphic forms and geometric properties of special functions [1,4,23,27] (see also [19]). Indeed, Erdélyi’s study [14,15] of global monodromy groups of Heun equations by finding “global hypergeometric expansion solutions” was also along this “removing apparent singularities paradigm”. However, in order to overcome the ambiguity of the interplay between the local and global aspects of solutions of Fuchsian equations typically using classical language, it is the purpose of this article to apply sheaf theoretic language to study geometric aspects of Fuchsian connections where one of their singularities becomes apparent (i.e. being resolved). As consequences of our study, we recover Takemura’s eigenvalue inclusion theorem [39] with a much simplified geometric proof, and moreover, new hypergeometric expansions of solutions to Heun equations other than those already obtained by Erdélyi are derived.

We show that “resolving singularities” for a given Fuchsian equation with four singularities, i.e., Heun equation, carries a deeper meaning than what the formulae could indicate. This is best described as a morphism between bundles equipped with appropriately defined connections that have one less singularity, i.e., a singularity of the original connection is being resolved by a singular gauge transformation. This phenomenon of removing singularity is inline with the spirit of resolving singularities of algebraic curves. In particular, our sheaf theoretic approach does not only allow us to give an explanation of the origin
of the various local hypergeometric function expansions of the Heun equation
given by Erdélyi [14,15], but it also allows us to understand the mechanism of
Erdélyi’s expansion thus enabling us to derive a new expansion, thus giving a
unified theory of the classical works by Erdélyi in the geometric language of
Heun connections. For instance, in [14, p. 51] (and a similar one in [15, p. 63]),
Erdélyi proposed a solution scheme of a certain Heun equation in the form:
\[ y = \sum_n c_n P \left( \begin{array}{ccc} 0 & 1 & \infty \\
\delta - \alpha - \beta - n & 0 & \alpha; x \\
1 - \delta & \beta \end{array} \right) \] (1.1)
in which the Riemann scheme $P$ denotes functions satisfying a suitable hyper-
geometric equation. In other words, for each small open set $U$, denoted by $A(U)$
and $B(U)$ the spaces of solutions of that hypergeometric and Heun equations
respectively. Erdélyi’s expansion is in fact a collection of maps $A(U) \to B(U)$
which are compatible with analytic continuation to a neighbouring open set
(see, for example, [35, §3.1-3.2]). In modern language, it is a morphism of
sheaves, and in particular it is a morphism of local systems in this context.
It is immediate that morphisms of local systems come from morphisms of flat
bundles. One useful type of morphisms was exhibited by Oblezin [29,30] in or-
der to show that the classical contiguous relations of hypergeometric functions
can be explained by a technique of bundle modification originated from Drin-
feld [12]. Simply speaking, when one of the singularities becomes apparent, we
interpret this modification as an inclusion of spaces of certain kind of sections
into itself with a faster growth rate.

We have achieved this by singling out the monodromy reduction ($WAS(m)$)
(as well as two other conditions ($WGRM$) and ($LR(n)$). See the description
below.) of the Heun connections and showed that their different combinations
with the language of morphisms of suitably defined flat bundles yields
a geometric interpretation of Erdélyi’s hypergeometric expansions, including
the new ones not found in [14,15]. These morphisms relate the corresponding
eigensections and eigenvalues of certain connections with different signatures
enumerated by $m \in \mathbb{N}$ (Theorem 7.2). An immediate observation from this
study is a geometric formulation and proof of Takemura’s remarkable eigen-
values inclusion theorem [39, Theorem 5.3] that we have already mentioned
in the first paragraph. Moreover, such a language of flat bundles identifies the
new common conditions of resolving singularity and the degeneration of the
monodromy group of the Heun equation to the reduction of the sixth Painlevé
equation $P_{VI}$.

We start by considering the Heun type-connection
\[ \nabla = d - \left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx, \] (1.2)
where each residue matrix $A_k$ ($k = 0, 1, a, \infty$) is diagonalisable. Its eigensec-
tions can be written as a finite sum of the form:
\[ \sum_n c_n (\nabla_v)^n A_a s, \] (1.3)
and the exact forms of these sums are determined by the eigenvalue differences of the residue matrices $A_k$ ($k = 0, 1, a, \infty$) and the eigenvalues of the $\nabla_v$. Without loss of generality, we shall normalise the matrices so that the eigenvalues of the $A_0, A_1, A_a$ are $\{0, 1 - \gamma\}, \{0, 1 - \delta\}, \{0, 1 - \epsilon\}$, respectively. The above theory to be developed in this article gives an explanation of Erdélyi’s local solutions [14, 1942], [15, 1944] to the Heun equation

$$D y := x(x-1)(x-a) \frac{d^2 y}{dx^2} + x(x-1)(x-a) \left( \gamma + \frac{\delta}{x - 1} + \frac{\epsilon}{x - a} \right) \frac{dy}{dx} + \alpha \beta xy = qy,$$

where the parameters satisfy the Fuchsian constraint $\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0$.

For example, Erdélyi found an infinite sum of hypergeometric functions

$$\sum_{n=0}^{\infty} C_n(q) x^n 2F_1 \left( \frac{\alpha + n, \beta + n}{\alpha + \beta - \delta + 2n + 1}; x \right),$$

which represents a local solution expanded about the singularity $x = 0$ with the exponent 0, where the coefficients $C_n$ satisfy a certain three-term recursion [14, pp. 54–55] (also see “Appendix B”).

We note that the summations (1.1) and (1.5) can be regarded as classical forms of (1.3). It is known that when the accessory parameter $q$ in (1.4) is suitably chosen, the infinite sum (1.5) converges in a certain domain $\Omega^{-1}$ containing two singularities and a third singularity situates on its circumference (see “Appendix B” for more details). Erdélyi hinted that such hypergeometric expansions carry more monodromy data than the usual power series expansions do and that would pave a way to study the monodromy group of the Heun equation. Indeed, when the accessory $q$ in (1.4) is so chosen, then the original expansion (1.5) which represents an analytic solution around the origin is now being analytic continued to a larger region containing the points $x = 1$ and $x = \infty$. This is equivalent to the simultaneous diagonalisation of the monodromies of (1.4) at $x = 1$ and $x = \infty$, or more generally at any two out of the four singularities. We call this kind of monodromy reduction weak global condition of reducibility of monodromy (WGRM).

When the monodromy of the Heun-type connection (1.2) meets the weak apparent singularity condition (WAS$(m)$), namely when the matrix $A_a$ possesses 0 and a nonzero integer as eigenvalues, then its eigensections possess terminated expansions (1.3). This follows from Theorem 5.3 that if the connection (1.2) meets the (WAS$(m)$), then there is a sheaf of locally free $\mathcal{O}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}$–module $\mathcal{E}$, such that

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{\mathbb{P}^1 \setminus \{0,1,\infty\}}$$

has the prescribed connection matrix as well as the matrix

$$- \left[ \frac{A_0 + A_a}{x} + \frac{A_1}{x - 1} \right] dx$$

relative to two different frames. In particular, the fourth singularity $x = a$ is removed. Its eigensections are terminated hypergeometric expansions (1.5)
given by Erdélyi. A similar argument gives another expansion

\[ \sum_{n=0}^{\infty} C_n \, _2F_1 \left( \lambda + n, \mu - n ; \frac{x}{\gamma} \right), \]  

where \( \lambda + \mu = \gamma + \delta - 1 \) and the coefficients \( C_n \) satisfy a certain three-term recursion [15, p. 64]. This can be derived from the variation that the matrix of \( \nabla \) relative to another frame is

\[ -\left[ \frac{A_0}{x} + \frac{A_1}{x-1} \right] dx. \]

We derive a number of infinite expansions not found in the works of Erdélyi such as

\[ \sum_{n=0}^{\infty} C_n (x-1)^n \, _2F_1 \left( \alpha + n, \beta + n ; \frac{x}{\gamma} \right). \]

by having

\[ -\left[ \frac{A_0}{x} + \frac{A_1 + A_a}{x-1} \right] dx \]

as the matrix of \( \nabla \) relative to yet another frame.

Notice that the Heun equation (1.4) can be derived from the Heun-type connection \( \nabla \) in (1.2), as a member of an isomonodromic family of connections where the necessary appearance of the fifth apparent singularity is located at \( \infty \).

We list here the three kinds of degenerations mentioned:

- weak global condition of reducibility of monodromy (WGRM; Definition 4.1).
- weak apparent singularity condition (WAS(m); Definition 5.2).
- local condition for reducibility of monodromy (LR(m)); Definition 6.2).

One notices that when \( \epsilon = -m \) (where \( m \in \mathbb{N} \)) (i.e. this corresponds to the degenerate condition (WAS(m)), the set of multi-valued functions with the same monodromy as a certain hypergeometric equation, and with a pole of order at most \( m+1 \) at \( a \) is invariant under the operator \( D \) defined in (1.4). One of the main purposes of this article is to investigate the eigenvalue problem

\[ Dy = qy \]  

by combining the study of monodromy and growth rate in a sheaf theoretic methodology, the eigenvalue problem above becomes the eigenvalue problem

\[ \nabla \nu \, s = \lambda \, s. \]  

Here, \( s \) is a section of a bundle constructed by tensoring a local system with a divisor line bundle, which yields the description of a function space of those with a specific monodromy and growth rate at singularities.

In Theorem 5.3, we construct the flat bundle \((\mathcal{E}, \nabla)\) in which the section \( s \) lives. Then, in Theorem 5.11 we prove that the \( m+1 \) eigenvalues \( \lambda \) above are precisely the numbers (or are equally spaced) 0, 1, \( \cdots \), \( m \). The idea of the proof rests on the construction of a sequence of injective bundle morphisms

\[ (\mathcal{E}_m, \nabla_m) \longrightarrow (\mathcal{E}_{m-1}, \nabla_{m-1}) \longrightarrow \cdots \longrightarrow (\mathcal{E}, \nabla) \]
by a technique of bundle modification due to Drinfeld [12] and Oblezin [29,30]. This sequence of bundle morphisms mimics the triangulation of a matrix. Moreover, each of these eigensections can be written as a formal sum
\[ \sum_n c_n^k (\nabla_n)^n A_n s, \quad k = 0, 1, \ldots, m \]
for some local section \( s \) of a sheaf of horizontal sections of a hypergeometric type connection. This gives a geometric interpretation of the \( m + 1 \) eigen-solutions of (1.4) studied since [14,15].

Moreover, the expansions of Erdélyi (1.1) suggest that some degenerated Heun equations are related to hypergeometric equations. Kimura [24] (1970) makes this relation explicit by means of gauge transformations. In fact, these explicitly constructed (singular) gauge transformations are morphisms between flat bundles coming from bundle modifications. These morphisms are isomorphisms away from a singular point, which suit the purpose of keeping the properties of a connection unchanged away from the singular point. We will see that it is the geometric context behind the (singular) gauge transformations studied by Kimura [24]. Then, we have discovered the startling result that the totality of these conditions are identical to those that lead to the degeneration of the Painlevé VI found by Okamoto [31]. In [5], the authors showed the conditions for which the degeneration of the Darboux (i.e. a periodic Heun) connection matches the conditions for the degeneration of Painlevé VI.

The article is organised as follows. We begin by recalling some useful results for flat vector bundles, monodromies of local systems and Kummer sheaves, and giving the construction of Drinfeld–Oblezin bundle modifications that suits our purpose in this article in Sect. 2. Then, we turn our attention to hypergeometric connections and Heun connections in Sect. 3 and Sect. 4, respectively. In particular, we introduce the first category of monodromy reduction (WGRM) in terms of simultaneous diagonalisation of monodromy matrices at any two singularities amongst all the singularities. This allows us to review the infinite continued fractions that Erdélyi used as a criteria for enlarging the domain of convergence of his classical infinite hypergeometric expansions to solutions of Heun equations. We shall introduce the second category of monodromy degeneration (WAS\((m)\)) for an integer \( m \in \mathbb{N} \) for (1.2) in Sect. 5. As our first main result in this section, this formulation enables us to “resolve” the singularity \( x = a \) of (1.2) (Theorem 5.3). We also obtain other criteria which are equivalent to (WAS\((m))\) for resolving singularities of (1.2) other than \( x = a \) in this section. Moreover, we demonstrate that our geometric criterion (WAS\((m))\) can lead to all the known hypergeometric function expansions by, including new expansions which are not found in, Erdélyi [14,15] as applications. In particular, Example 5.5 is instrumental in giving a geometric proof of Takemura’s eigenvalues theorem in Sect. 7 later. This section also contains the next main result of this paper, namely that for a given \( m \in \mathbb{N} \), the criterion (WAS\((m))\) implies that there exist \( m + 1 \) local sections

\[ \text{Oblezin attributed his bundle construction has its origin from Drinfeld [12].} \]
$s_k \ (k = 0, \cdots, m)$ on any given open set such that the eigenvalue problem (1.8) always admits equally spaced eigenvalues $\lambda_k \ (k = 0, \cdots, m)$ (Theorem 5.11). Section 6 introduces the third category of monodromy reduction $(\mathbf{LR}(n))$ for $n \in \mathbb{N}$ to the Heun connection (1.2). The criterion $(\mathbf{LR}(n))$ together with $(\mathbf{WGRM})$ allows us to describe globally defined solutions for the Heun connection (1.2) which contain the Heun polynomials as special cases. Section 7 studies the combined geometric effects of all three categories of monodromy reductions to the Heun connection (1.2) introduced, namely the fulfilment of $(\mathbf{WGRM}), (\mathbf{WAS}(m))$ and $(\mathbf{LR}(n))$. In particular, we offer a geometric proof to Takemura’s eigenvalue inclusion theorem (Theorem 7.1) as a consequence. In Sect. 8, we observe that a complete matching exists between the criteria giving monodromy reduction of Heun connection and classical solutions of the Painlevé VI equation under the two categories $(\mathbf{WAS}(m))$ and $(\mathbf{LR}(n))$. The paper ends with some remarks in Sect. 9.

2. Flat Bundles

We give preliminaries of the theory of flat connections and local systems, which are useful for understanding the geometric context of some classical special functions. For more details, the readers may refer to the survey by Malgrange [28, Chapter IV].

2.1. Flat Connections and Local Systems

Throughout this article, we let $X$ be a complex manifold. Indeed it is a (non-compact) Riemann surface except in Sect. 8, where the underlying space $X$ has dimension two. The sheaf of analytic functions on $X$ is denoted by $\mathcal{O}_X$ (or just $\mathcal{O}$ when there is no ambiguity). For each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring whose maximal ideal is denoted by $m_p$. The sheaf of holomorphic one-forms on $X$ is always denoted by $\Omega_X$.

**Definition 2.1.** Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$—module. A connection $\nabla$ in $\mathcal{E}$ is a $\mathbb{C}$—linear morphism of sheaves $\nabla : \mathcal{E} \rightarrow \Omega_X \otimes \mathcal{E}$ such that for each open $U \subset X$,

$$\nabla(fs) = f\nabla(s) + df \otimes s \quad \text{for every } f \in \mathcal{O}_X(U) \text{ and } s \in \mathcal{E}(U).$$

If $(\mathcal{E}, \nabla)$, $(\mathcal{E}', \nabla')$ are locally free sheaves of $\mathcal{O}_X$—modules endowed with connections, a morphism (or (singular) gauge transformation) $\phi : (\mathcal{E}, \nabla) \rightarrow (\mathcal{E}', \nabla')$ is an $\mathcal{O}_X$—linear morphism $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ such that the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\nabla} & \Omega_X \otimes \mathcal{E} \\
\phi \downarrow & & \downarrow \iota \otimes \phi \\
\mathcal{E}' & \xrightarrow{\nabla'} & \Omega_X \otimes \mathcal{E}'
\end{array}$$

commutes.
Given a connection $\nabla$ in a locally free sheaf of $\mathcal{O}_X$–module $\mathcal{E}$, it extends to $\nabla : \Omega^k_X \otimes \mathcal{E} \to \Omega^{k+1}_X \otimes \mathcal{E}$ by enforcing the generalised Leibniz rule

$$\nabla(\alpha s) = d\alpha s + (-1)^k \alpha \wedge \nabla(s)$$

for each section $\alpha \in \Omega^k_X(U)$ and $s \in \mathcal{E}(U)$.

Then, the curvature of $(\mathcal{E}, \nabla)$ is defined by $\nabla \circ \nabla$. The connection $\nabla$ is flat, or integrable, if its curvature vanishes. This flatness condition is void if $X$ is a Riemann surface as there is no holomorphic two-form there. This notion of flat bundle is a coordinate-free description of classical ODEs in the complex domain, or holonomic system of PDEs in case $X$ has a higher dimension. However, description with a choice of coordinates is also desirable for our purpose in certain applications as described in the following paragraph.

Choose a small open set $U \subset X$ such that $\mathcal{E}(U) \cong \mathcal{O}_X(U)^r$. Then, $\mathcal{E}(U)$ has an $\mathcal{O}_X(U)$–basis $s_1, \ldots, s_r \in \mathcal{E}(U)$. Let $\omega_{ij} \in \Omega^1_X(U)$ be one-forms such that

$$\nabla(s_j) = \sum_{i} \omega_{ij} s_{ij}$$

for all $i$.

Then, the matrix $\{\omega_{ij}\}_{1 \leq i, j \leq r}$ with entries in $\Omega^1_X(U)$ is called the connection matrix of $\nabla$ relative to the frame $s_i$, which is an explicit description of $\nabla$.

**Definition 2.2.** A sheaf of $\mathbb{C}$–vector spaces $\mathcal{L}$ on $X$ is locally constant, or a local system, if there exists an open covering $\{U_\alpha\}$ of $X$ such that $\mathcal{L}|_{U_\alpha}$ is a constant sheaf for all $\alpha$. A morphism between local systems on $X$ is simply a $\mathbb{C}$–linear morphism between sheaves.

Given a flat bundle $(\mathcal{E}, \nabla)$, the sheaf $\mathcal{L}$ defined by

$$\mathcal{L}(U) = \{s \in \mathcal{E}(U) : \nabla s = 0\}$$

for all open $U \subset X$ is a local system, which is known as the sheaf of horizontal sections of $(\mathcal{E}, \nabla)$.

When a classical ODE (or a holonomic system of PDEs) is formulated as a flat bundle, it has special solutions when this flat bundle (or its sheaf of horizontal sections) has a proper non-trivial sub-object. We say such a flat bundle (or the corresponding local system) is reducible.

### 2.2. Monodromy of a Local System

**Definition 2.3.** Let $\mathcal{L}$ be a local system on $X$. Given a path $\gamma : [0, 1] \to X$, one can cover the image of $\gamma$ by open sets $U_1, \ldots, U_n$ such that $\mathcal{L}|_{U_i}$ is constant and $U_i \cap U_{i+1}$ is nonempty for each $i$. This induces an isomorphism between fibres (or stalks) $\mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$. This isomorphism depends only on the homotopy class of $\gamma$ and thus induces a $\pi_1(X, x_0)$–module structure on $\mathcal{L}_{x_0}$, which is called the monodromy of $\mathcal{L}$. In case $\mathcal{L}$ is the sheaf of horizontal sections of a flat bundle, we speak of the monodromy of such a flat bundle directly.

It is elementary to see that the above construction can be extended to a morphism between two flat bundles to yield a $\pi_1(X)$–linear map between their monodromies. Hence, we can verify the following well-known statement.

**Lemma 2.4.** Monodromy defines a covariant functor from the category of flat bundles on $X$ to the category of $\pi_1(X)$–modules.
There is no guarantee that such a functor is an equivalence of categories, unless it is restricted to a class of flat bundles having “regular singularities”.

**Definition 2.5.** Let $\overline{X}$ be a smooth projective variety, $X$ is the complement of a normal crossing divisor $D$ in $\overline{X}$. Denote the sheaf of holomorphic one-forms in $X$ having at most log poles along $D$ by $\Omega_{\overline{X}}(\log D)$. Fix a locally free sheaf of $\mathcal{O}_X$–module $\mathcal{E}$. We say that the connection $\nabla: \mathcal{E} \rightarrow \Omega_X \otimes \mathcal{E}$ has a log singularity (or regular singularity) along $D$ if it can be lifted to $\nabla: \mathcal{E} \rightarrow \Omega_{\overline{X}}(\log D) \otimes \mathcal{E}$. Moreover, a connection is Fuchsian if it has at most regular singularities.

There has been already a coordinate-free description of regular singularities in a classical language (see, for example, [32]). That is, an ODE has a regular singular point at $p$ if all its solutions have at most polynomial growth at $p$. Now we see that such differential equations are important as they are completely characterised by their monodromies, up to gauge transformations. We quote the following deep result by Deligne [11, Corollary to Theorem 5.9], see also [28].

**Theorem 2.6.** Monodromy is an equivalence from the category of flat bundles on $X$ with at most regular singularities to the category of $\pi_1(X)$–modules.

Finally, we note that the notion of “residues” is central in the description of connections with log singularities.

**Definition 2.7.** Given a small open set $U \subset X$, $x = (x_1, \cdots, x_n): U \rightarrow \mathbb{C}^n$ is a local coordinate, and $D$ is a divisor so that $D \cap U$ is cut out by $x_1$. Let $\mathcal{E}$ be a sheaf of locally free $\mathcal{O}_X$–module of rank $N$. Suppose that $\nabla: \mathcal{E} \rightarrow \Omega_{\overline{X}}(\log D) \otimes \mathcal{E}$ is a connection with regular singularity along $D$. Its matrix relative to a certain frame is

$$A \frac{dx_1}{x_1} \mod \Omega_{\overline{X}}(U) \otimes \mathbb{C}^{N \times N}$$

for some $N \times N$ matrix $A$. Such an $A$ does not depend on the choice of the local coordinate $x$ and is called the residue of $\nabla$ along $D$. In this article, this residue is denoted by $\text{Res}_D \nabla$. In case $\nabla$ is a flat connection in the trivial bundle $\mathcal{O}_X^N$ over $X = \mathbb{P}^1 \setminus \{a_1, \cdots, a_r, \infty\}$ so that $\text{Res}_{a_i} \nabla = -A_i$, the matrix of $\nabla$ relative to the standard frame is

$$-\left[\frac{A_1}{x - a_1} + \cdots + \frac{A_r}{x - a_r}\right]dx.$$  

The classical Riemann scheme of a second-order linear differential equation derived from the equation $\nabla f = 0$ lists the eigenvalues of the matrices $A_1, \cdots, A_r$ as well as those of $A_\infty = -(A_1 + \cdots + A_r)$.

Moreover, the relation (A.4) replaces the classical Fuchsian relation. In this article, we always assume that the residues of all connections at every singular point are diagonalisable.
The well-known fact that the local monodromies of a flat connection with regular singularities can be read off from its residues is summarised in the following.

**Lemma 2.8.** Let $\mathcal{X}$ be a compact Riemann surface, $p \in \mathcal{X}$, $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-module, $\nabla$ is a flat connection with regular singularity at $p$. Then, the eigenvalues of the local monodromy of the loop around $p$ are the eigenvalues of $\exp[2\pi i \text{Res}_p \nabla]$.

### 2.3. Kummer Sheaves

The simplest non-trivial special function includes $(x - a)^\mu$, where $\mu$ is not an integer (see, for example, [43]). Indeed, it can be regarded as a section of a local system to be discussed in this subsection. As an example, the expression

$$x^{1-c} 2F_1 (1 + a - c, 1 + b - c, 2 - c; x)$$

which contains the product of two sections $x^{1-c}$ and $2F_1 (1 + a - c, 1 + b - c, 2 - c; x)$, satisfies the standard hypergeometric equation.

**Definition 2.9.** For each point $a \in \mathbb{C}$, and $\mu \in \mathbb{C}$, the Kummer sheaf $\mathcal{K}_a^\mu$ is the sheaf of horizontal sections of

$$\left( \mathcal{O}_{\mathbb{P}^1 \setminus \{a, \infty\}}, d - \mu \frac{dx}{x-a} \right).$$

**Lemma 2.10.** Suppose that $\mathcal{L}$ is a local system in a deleted neighbourhood of $0 \in \mathbb{C}$, which is also the sheaf of horizontal sections of a flat connection $\nabla$ with log singularity at $0$. For each $\mu \in \mathbb{C}$, let $\mathcal{K}_0^\mu \otimes \mathcal{L}$ be the sheaf of horizontal sections of $\nabla$. Then,

$$\text{Res}_0 \nabla = \text{Res}_0 \nabla + \mu I.$$  

**Proof.** Let $s$ be a local section of $\mathcal{L} \otimes \mathcal{O}$. Then, the germ of $\nabla(s)$ at $0$ satisfies

$$\nabla(s) = \left[ (\text{Res}_0 \nabla)s \right] \otimes \frac{dx}{x} \mod (\mathcal{L} \otimes \Omega)_0$$

Thus, if $t$ is a local section of $\mathcal{K}_0^\mu$,

$$\nabla(t \otimes s) = \left[ \mu t \otimes \frac{dx}{x} \right] \otimes s + t \otimes \left[ (\text{Res}_0 \nabla)s \otimes \frac{dx}{x} \right]$$

$$= t \otimes \left[ (\text{Res}_0 \nabla + \mu I)s \otimes \frac{dx}{x} \right] \mod (\mathcal{L} \otimes \mathcal{K}_0^\mu \otimes \Omega)_0.$$  

\[ \square \]

**Example 2.11.** In the classical theory of ordinary differential equations, the general Riemann equation with three regular singularities can be transformed to the Gauss hypergeometric equation by a simple and well-known transformation of the dependent variable. This transformation is done by tensoring with Kummer sheaves in our context as follows:

Let $\nabla$ be a flat connection in a rank-two bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ having log singularities at $0$, $1$ and $\infty$. We let $\lambda$, $\mu$ be one of the eigenvalues of $\text{Res}_0 \nabla$ and $\text{Res}_1 \nabla$ respectively. Suppose that $\mathcal{L}$ is the sheaf of horizontal sections of $\nabla$, and if $\mathcal{K}_0^{-\lambda} \otimes \mathcal{K}_1^{-\mu} \otimes \mathcal{L}$ is the sheaf of horizontal sections of $\nabla$, then $0$ is an eigenvalue for both of $\text{Res}_0 \nabla$ and $\text{Res}_1 \nabla$. 

Example 2.12. In the classical theory of second-order Fuchsian differential equations (applicable to more general equations also), the first order term (i.e. the term involving $y'$) can be removed by a simple transformation of the dependent variable. We explain this operation by the geometric context briefly as follows:

Let $\nabla$ be a flat connection in a rank-two bundle over $\mathbb{P}^1\setminus\{0, 1, \infty\}$ having log singularities at 0, 1 and $\infty$, so that 0, $\lambda$ are the eigenvalues of $\text{Res}_0 \nabla$ and 0, $\mu$ are the eigenvalues of $\text{Res}_1 \nabla$. Suppose that $\mathcal{L}$ is the sheaf of horizontal sections of $\nabla$. Then, the Wronskian of $\mathcal{L}$ is

$$\wedge^2 \mathcal{L} \cong K_0^\lambda \otimes K_1^\mu.$$ 

Denote $K_0^{-\lambda/2} \otimes K_1^{-\mu/2}$ by $(\wedge^2 \mathcal{L})^{-1/2}$. Then, $\mathcal{L} \otimes (\wedge^2 \mathcal{L})^{-1/2}$ is a rank-two local system with trivial Wronskian.

2.4. Drinfeld–Oblezin Bundle Modifications

In this subsection, we revise the construction of two types of special bundles defined by Drinfeld [12] and later being applied to Heun equations (1.4) by Oblezin in [29,30]. It turns out that this idea is very useful for us to explain the geometry of those “singular-gauge transformations” that appeared in [24] and in this article.

Definition 2.13. Let $X$ be a space (usually $\mathbb{P}^1$ for our purpose), and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-module. Given the data

$\text{p} \in X$ and $W$ is a $\mathcal{O}_p/\mathfrak{m}_p$-vector subspace of $\mathcal{E}_p$,

we recall that $\mathfrak{m}_p$ denotes the maximal ideal of the local ring of analytic functions $\mathcal{O}_p$ at $p$), the (Drinfeld-Oblezin) modification of $\mathcal{E}$ at $(p, W)$ is the following subsheaf of $\mathcal{E}$:

$$U \mapsto \{s \in \mathcal{E}(U) : s_p \in W\}.$$ 

Lemma 2.14. Let $X$ be a Riemann surface. If $\mathcal{E}$ is a sheaf of locally free $\mathcal{O}_X$-module, then the modification of $\mathcal{E}$ at any pair $(p, W)$ is also locally free.

Proof. Let $\mathcal{E}$ be the sheaf modification of $\mathcal{E}$ at $(p, W)$. Given a free $\mathcal{O}_{X,p}$-module $\mathcal{E}_p$, it suffices to show that $\mathcal{E}_p$ is a free $\mathcal{O}_{X,p}$-module. But $\mathcal{O}_{X,p}$ is a PID since $X$ is a Riemann surface and hence $\mathcal{E}_p$ is also free. $\square$

Definition 2.15. Let $X$ be a Riemann surface, and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-module equipped with a flat connection $\nabla$ with a log singularity at $p$, such that the residue of $\nabla$ at $p$ has two complementary invariant subspaces:

$$\text{Ker}(\text{Res}_p \nabla - \lambda I) \quad \text{and} \quad W,$$

for some $\lambda \in \mathbb{C}$. (In particular, the condition holds when the residue of $\nabla$ at $p$ is diagonalisable.) We denote the (lower) modification of $\mathcal{E}$ at $(p, W)$ (Definition 2.13) by $\mathcal{E}$ and the connection $\nabla$ is defined by

$$\nabla : \mathcal{E} \hookrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_X \longrightarrow \mathcal{E} \otimes \Omega_X.$$
where the last arrow is the projection $\mathcal{E} \to \mathcal{E}$ coming from the invariant subspaces above. That is, the flat connection $(\mathcal{E}, \nabla)$ fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes \Omega_X \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes \Omega_X
\end{array}
$$

so that a morphism $(\mathcal{E}, \nabla) \to (\mathcal{E}, \nabla)$ is obtained.

The essential idea of the following theorem is derived from an example of Oblezin’s construction [29, p. 114], [30, p. 22].

**Theorem 2.16.** Let $\nabla$ be as defined above. Then, its residue at $p$ has two complementary invariant subspaces:

$$\text{Ker}(\text{Res}_p \nabla - (\lambda + 1)I) \quad \text{and} \quad W.$$

**Proof.** Let $(U, x)$ be a local chart centred around $p$ and $s_1, \ldots, s_n \in \mathcal{E}(U)$ are linearly independent sections so that $W = \text{span}\{s_{k+1,p}, \ldots, s_{n,p}\}$. Now $xs_1, xs_2, \ldots, xs_k, s_{k+1}, \ldots, s_n \in \mathcal{E}(U)$ are linearly independent sections and

$$\nabla(xs_j) = x\nabla s_j + s_j \otimes dx = xs_j \otimes \frac{\lambda dx}{x} + xs_j \otimes \frac{dx}{x} \mod m_p \mathcal{E}_p$$

holds for each $1 \leq j \leq k$. It is easy to show that $W$ is also invariant under the residue of $\nabla$ at $p$. Indeed, let $s_j \in \mathcal{E}(U)$ ($j = k + 1, \ldots, n$). Then,

$$\nabla s_j = \nabla s_j = (\text{Res}_p \nabla) s_j \otimes \frac{dx}{x} \mod \mathcal{E}_p.$$

Hence, $(\text{Res}_p \nabla) s_j \in W$. \qed

Verses the (lower) modification of bundles introduced above, there is another type of modification called upper modification. We will only give a brief description here.

**Definition 2.17.** Let $X$ be a Riemann surface. Given a sheaf of $\mathcal{O}_X$–modules $\mathcal{E}$ and $p \in X$, $W \subset \mathcal{E}_p$. Denote the (lower) modification of $\mathcal{E}$ at $(p, W)$ by $\mathcal{E}$.

(i) We define the upper modification $\overline{\mathcal{E}}$ of $\mathcal{E}$ at $(p, W)$ to be

$$\overline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}(p),$$

where $\mathcal{O}(p)$ is the divisor line bundle of $1 \cdot p$;

(ii) Moreover, if $(\mathcal{E}, \nabla)$ is a flat bundle with a log singularity at $p$ and $\mathcal{E}_p$ is decomposed into complementary subspaces

$$\text{Ker}(\text{Res}_p \nabla - \lambda I) \quad \text{and} \quad V,$$
we denote the upper modification of $\mathcal{E}$ at $(p, \text{Ker}(\text{Res}_p \nabla - \lambda I))$ by $\mathcal{E}$. Then, $\mathcal{E}$ is equipped with a flat connection $\nabla$ as in Definition 2.15, such that there is a morphism $(\mathcal{E}, \nabla) \longrightarrow (\mathcal{E}, \nabla)$.

The following result can be derived similar to that of Theorem 2.16.

Theorem 2.18. Let $(\mathcal{E}, \nabla)$ be defined above. Then, its residue at $p$ has $\nabla$-invariant complementary subspaces

$$\text{Ker}(\text{Res}_p \nabla - (\lambda - 1)I) \quad \text{and} \quad V.$$

Finally, one easily deduces the following general result about upper and lower modifications.

Lemma 2.19. (\cite{29}) Let $(\mathcal{E}, \nabla)$ be a flat bundle with a log singularity at $p$, and $V, W$ are two complementary subspaces of $\mathcal{E}_p$ invariant under $\text{Res}_p \nabla$. Denote the (lower) modification of $(\mathcal{E}, \nabla)$ at $(p, W)$ by $(\mathcal{E}, \nabla)$. Then, the upper modification of $(\mathcal{E}, \nabla)$ at $(p, V)$ is $(\mathcal{E}, \nabla)$.

3. Hypergeometric Connections Revisited

In order to better illustrate our monodromy approach to the main results later in this article, this section is reserved to review monodromy reduction of the hypergeometric connection in a sheaf theoretic language that suits our purpose since we cannot find an appropriate reference for the material that we need. So we shall start with a reformulation of monodromy reduction of the classical hypergeometric equation entirely from monodromy consideration. Note that differential Galois theory works equally well for this purpose (see for example, \cite{23}), but the current setup is more appropriate for our purpose.

Solving the hypergeometric equation in a global sense has long been a difficult task. Therefore, special solutions are usually considered. The following theorem is classical.

Theorem 3.1. Consider the hypergeometric equation

$$x(1 - x) \frac{d^2 y}{dx^2} + [c - (a + b + 1)x] \frac{dy}{dx} - aby = 0,$$  \hspace{1cm} (3.1)

where $a, b, c \in \mathbb{C}$.

- If $a \in \mathbb{N}$ (resp. $b \in \mathbb{N}$), then the hypergeometric equation has a solution of the form $x^{1-c}(x - 1)^{c-a-b}p(x)$ where $p$ is a polynomial of degree at most $a - 1$ (resp. $b - 1$ at most).

- If $-a \in \mathbb{N}$ (resp. $-b \in \mathbb{N}$), then the hypergeometric equation has a polynomial solution of degree at most $-a$ (resp. $-b$).

We require the following lemma which can be found in Beukers [2, Lemma 3.9].

Lemma 3.2. Let $M, N$ and $MN$ be $2 \times 2$ matrices each with distinct eigenvalues. If, however, 1 is a common eigenvalue of $M, N$ and $MN$, then $M, N$ and $MN$ share a common eigenvector.
Proof of Theorem 3.1

The proof is described in numerous classical literature (e.g. [32, §23, p. 90]). Our objective now is to revise this proof via a geometric approach.

The monodromy representation of the hypergeometric equation above will be our most powerful tool. Let $M$, $N$ be the monodromy matrices of the standard hypergeometric equation relative to a certain basis. It is standard that

\begin{align*}
M & \text{ has eigenvalues } 1, e^{2\pi i(1-c)} \\
N & \text{ has eigenvalues } 1, e^{2\pi i(c-a-b)} \\
MN & \text{ has eigenvalues } e^{-2\pi ia}, e^{-2\pi ib}.
\end{align*}

Now $a \in \mathbb{Z}$. Let $v$ be a common eigenvector (which represents a solution $f$ of the hypergeometric equation defined on a small open set) to the matrices $M, N$ and $MN$ that is guaranteed by Lemma 3.2. Then after some routine consideration together with the Fuchsian condition, only two out of the total eight possibilities remain, which are, either

\begin{align*}
Mv &= v \quad \text{and} \quad Nv = v \quad \text{and} \quad MNv = v \\
or \quad Mv &= e^{2\pi i(1-c)}v \quad \text{and} \quad Nv = e^{2\pi i(c-b)}v \quad \text{and} \quad MNv = e^{-2\pi ib}v
\end{align*}

In case $a \in \mathbb{N}$ and assume that we are in the former case. The locally defined function $f$ has trivial monodromy, and hence extends to a rational function with poles possibly at 0, 1 and $\infty$. However, 0 is a local exponent at both 0 and 1, and $a \in \mathbb{N}$ is a local exponent at $\infty$. Therefore, $f$ is an analytic function defined on $\mathbb{P}^1$ which has at least a zero but no pole, which is impossible. So we conclude that when $a \in \mathbb{N}$, only the latter case above is possible. Then, we consider the locally defined function $x^{c-1}(x-1)^{a+b-c}f$ which has trivial monodromy and thus extends to a rational function. By a similar analysis of local exponents, $x^{c-1}(x-1)^{a+b-c}f$ is indeed a polynomial of degree at most $a - 1$. This completes the proof of the first part. In the remaining case when $-a \in \mathbb{N}$, one reverses the roles in the analysis of the above two cases. Finally, if $a = 0$, then one deduces that the hypergeometric equation admits a constant solution. This completes the second part of the proof. \hfill \Box

Observe that the proof above is highly dependent on the monodromy of the hypergeometric operator, the same argument works for an operator with the same monodromy. The natural geometric object encoding the information of monodromy is a flat connection.

Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $A_0, A_1$ are $2 \times 2$ matrices with complex entries. Consider the connection $\nabla$ in $\mathcal{O}_X \oplus \mathcal{O}_X$ whose matrix relative to the canonical basis is

\[-\left[ \frac{A_0}{x} + \frac{A_1}{x-1} \right] dx,
\]

suppose further that $A_0$ has eigenvalues 0 and $1 - c$, $A_1$ has eigenvalues 0 and $c - a - b - 1$, $A_0 + A_1$ has eigenvalues $-a, -b$. If $(y_1, y_2)^T$ is a local horizontal
section of $\nabla$, then $y_1$ satisfies
\[
\frac{d^2 y_1}{dx^2} + \left[ \frac{c}{x} + \frac{2-c+a+b}{x-1} - \frac{1}{x-\lambda} \right] \frac{dy_1}{dx} + \frac{abx-\mu}{x(x-1)(x-\lambda)} y_1 = 0,
\]
where $\mu$ is a constant depending on the residue matrices, $\lambda$ is an apparent singularity and it is also the zero of the $A_0/x + A_1/(x-1)$. Although the classical hypergeometric equation is obtained only when the choice $\lambda = 1$ is made, that is $A_0 = \begin{pmatrix} 0 & b \\ 0 & 1-c \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ -a & c-a-b-1 \end{pmatrix}$, other choices of $\lambda$ should yield the same type of equations which are studied collectively by investigating $\nabla$.

Now an analogue of Theorem 3.1 is rewritten as

**Theorem 3.3.** Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $A_0$ is a $2 \times 2$ matrix with eigenvalues $0$ and $\mu \neq 0$; $A_1$ is a $2 \times 2$ matrix with eigenvalues $0, \nu \neq 0$, $\nabla$ is the connection in $\mathcal{O}_X \oplus \mathcal{O}_X$ whose matrix relative to the standard basis is $-\left[ \frac{A_0}{x} + \frac{A_1}{x-1} \right] dx$.

(i) If $m \in \mathbb{N}$ is an eigenvalue of $A_0 + A_1$, then there exists a non-trivial morphism of flat bundles $(\mathcal{O}_X, d) \rightarrow (\mathcal{O}_X \oplus \mathcal{O}_X, \nabla)$.

(ii) If $-m \in \mathbb{N}$ is an eigenvalue of $A_0 + A_1$, then there exists a non-trivial morphism of flat bundles $(\mathcal{O}_X, d - \mu \frac{dx}{x} - \nu \frac{dx}{x-1}) \rightarrow (\mathcal{O}_X \oplus \mathcal{O}_X, \nabla)$.

**Proof.** Under the hypothesis of the first part of Theorem 3.3, we see that the hypergeometric equation (3.1) admits a global solution which, is a polynomial, by Theorem 3.1, so is the first component of a $\nabla$ horizontal section. A straightforward inspection of the second component of the same $\nabla$-horizontal section reveals that it is also a polynomial. Consequently, a non-trivial global $\nabla$-horizontal section $f$ exists. The $\mathcal{O}_X$-linear map $\mathcal{O}_X \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X$ defined as multiplication by $f$ fits into the following commutative diagram for each open set $U$

\[
\begin{array}{c}
\mathcal{O}(U) \downarrow f \\
\times f \downarrow \Omega(U) \rightarrow \Omega(U) \oplus \Omega(U) \\
\mathcal{O}(U) \oplus \mathcal{O}(U) \downarrow \nabla \\
\end{array}
\]

The second part of Theorem 3.3 can be proved similarly. $\square$

Later in Sect. 6, we will study the analogous phenomenon in the case of connections with four log singularities.

### 3.1. Kummer Symmetry: Sheaf Theoretic Interpretation

In order to better illustrate our description of the Heun equation below, we revisit the symmetry of the solutions of the hypergeometric equation (Kummer symmetry) in a sheaf theoretic language. In particular, this generates more criteria for special solutions other than those derived from Theorem 3.1 and Theorem 3.3 in this subsection. Of course, the result is well known in numerous
classic texts, see, for example, [32]. A more comprehensive investigation of
symmetry would involve symplectomorphism between moduli spaces of local
systems (see, for example, Oblezin [29]) which is beyond the scope of this
article.

Loosely speaking, the group of Kummer symmetry is generated by two
parts: transformations of the independent variable and transformations of the
dependent variable.

Given a Möbius transformation \( f \) mapping \( \{0, 1, \infty\} \) into \( \{0, 1, \infty\} \), and
a local system \( L \) which is the sheaf of horizontal sections of a connection in
a rank-two bundle over \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) with log singularities at 0, 1 and \( \infty \), the
push-forward \( f_*L \) is also one with three log singularities at 0, 1, \( \infty \). Thus, the
group of such Möbius transformations acts on the set of such local systems.

We also have the following actions on local systems

\[
S_0 : L \mapsto L \otimes \mathcal{K}_0^{-\gamma}
\]
\[
S_1 : L \mapsto L \otimes \mathcal{K}_1^{-\delta},
\]

where \( A_0 \) has eigenvalues 0 and \( \gamma \), \( A_1 \) has eigenvalues 0 and \( \delta \). These actions
take the sheaf of horizontal sections of a connection with three singularities
0, 1, \( \infty \) to the sheaf of horizontal sections of another connection with three
singularities 0, 1, \( \infty \). The actions \( S_0, S_1 \) generate a group which acts on the
"dependent variable" of a hypergeometric equation. These actions generate
the classical Kummer symmetry group. They send local systems of certain
special types to other local systems of the same type. In other words, special
solutions of the Gauss hypergeometric equation are sent to the same kind of
special solutions via the Kummer symmetry.

We easily identify the Kummer symmetry from the discussion above in
the following well-known theorem.

**Theorem 3.4.** The group generated by \( S_0, S_1 \) together with the push-forwards
\( f_* \) of those Möbius transformations \( f \) preserving \( \{0, 1, \infty\} \) is isomorphic to
the group of signed permutations of three letters module \( \mathbb{Z}_2 \).

In general, the construction of the Kummer symmetry group generalises
to local systems over \( \mathbb{P}^1 \) with more than three log singularities. This general-
isation is elaborated in [26] (or in [6] when \( \mathbb{P}^1 \) is replaced by a complex torus
with four log singularities).

We can obtain other special solutions analogous to those in Theorem 3.1
by applying the actions described by the above symmetry group. The result is
summarised in the following well-known statement (see Poole [32, §23, p. 90]).

**Proposition 3.5.** If there exist \( \lambda \in \{0, 1 - c\} \), \( \mu \in \{0, c - a - b\} \) and \( \nu \in \{a, b\} \)
such that \( \lambda + \mu + \nu \in \mathbb{Z} \), then the hypergeometric equation (3.1) is reducible.
4. Simultaneous Diagonalisation

We consider the connection (1.2) over a rank-2 vector bundle with four regular singularities \(\{0, 1, a, \infty\}\) in \(\mathbb{P}^1\). Its matrix relative to a frame is

\[- \left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx,\]  

which we call a \textit{Heun-type connection} as in Introduction.

In this setup, an appropriate choice of the accessory parameter \(q\) means the simultaneous diagonalisation of two matrices amongst the residues at \(\{0, 1, a, \infty\}\). So we denote

\[
\text{Definition 4.1.} \quad \{ \text{weak global reducible condition of monodromy:} \} \quad (WGRM) \quad (4.1)
\]

for later applications.

4.1. Erdélyi’s Expansions Revisited

In order to illustrate under what circumstances in some classical consideration of Fuchsian equations which are equivalent to the \((WGRM)\) of Fuchsian connections defined above, we review Erdélyi’s infinite expansions in terms of hypergeometric functions, which are used to study the monodromy group of the Heun equation

\[
\frac{d^2 y}{dx^2} + (\gamma \frac{\delta}{x} - \delta + \epsilon + \frac{\epsilon}{x-a}) \frac{dy}{dx} + \frac{\alpha \beta x - q}{x(x-1)(x-a)} y = 0. \quad (4.2)
\]

Note that the parameters satisfy the Fuchsian constraint \(\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0\) in \([14,15]\).

The region of convergence of such infinite expansions can be used as a measure of the difference between the monodromy group of the hypergeometric equation and that of the Heun equation. For example, Erdélyi \([14, (4.2)]\) (1942) represented the local Heun function \(Hl(a, q; \alpha, \beta, \gamma, \delta; x)\) (that is, the local analytic solution at the singularity \(x = 0\)) by the hypergeometric function series \(\sum_{m=0}^{\infty} X_m \varphi^1_m(x)\), where

\[\varphi^1_m(x) := \frac{\Gamma(\alpha - \delta + m + 1) \Gamma(\beta - \delta + m + 1)}{\Gamma(\alpha + \beta - \delta + 2m + 1)} x^m \cdot \frac{\alpha + m, \beta + m}{\alpha + \beta - \delta + 2m + 1} \cdot _2F_1\left(\alpha + m, \beta + m; \alpha + \beta - \delta + 2m + 1; x\right)\]

and the coefficients \(X_m\) satisfy a three-term recursion \((B.2)\) given in “Appendix B”. If the accessory parameter \(q\) does not satisfy the infinite continued fraction \((B.4)\), then the infinite sum converges in the bounded region \(\Omega_0\) defined in \((B.3)\) which contains \(x = 0\) in its interior but excludes \(x = 1\). If, however, that the accessory parameter \(q\) satisfies the infinite continued fraction \((B.4)\), then the infinite sum converges in the larger region \(\Omega_1\) defined in \((B.5)\) which equals \(\mathbb{C}\) with a branch cut from 1 to \(\infty\); see also the Remark after Theorem B.1.
second infinite hypergeometric type solution $\sum_{m=0}^{\infty} X_m \varphi_m$ to the Heun equation linearly independent from the above infinite sum $\sum_{m=0}^{\infty} X_m \varphi_1^m(x)$, where each $\varphi_m$ can be any linear combination of $\varphi_m, \varphi_{m+1}, \ldots, \varphi_{m+6}$ defined in [14, (4.2)] was also derived by Erdélyi. If the coefficients $\{X_m\}$ defined by (B.2) and the accessory parameter $q$ satisfies the infinite continued fraction (B.4), then the infinite sum converges in the region $\Omega_1$ with non-empty intersection $\Omega_1$ defined in (B.5). When the accessory parameter $q$ satisfies the infinite continued fraction (B.4), the region $\Omega_1 \cap \Omega_1^-$ contains $x = 1, \infty$ in its interior but excludes $x = 0$, the condition (WGRM) defined above (with $x = a$ replaced by $x = 1$) describes precisely that both the hypergeometric expansions $\sum_{m=0}^{\infty} X_m \varphi_1^m$ and $\sum_{m=0}^{\infty} X_m \varphi_m$ converge simultaneously in the common region $\Omega_1 \cap \Omega_1^-$ that contains $x = 1, \infty$.

Remark 4.2. The corresponding series solutions to Darboux equations can be found in [5].

One may further ask what happens if the domain of convergence of such a series includes 0, 1 and $a$. If this is the case, then $A_0, A_1$ and $A_a$ (and hence $A_\infty = -A_0 - A_1 - A_a$) are simultaneously diagonalisable. Then, there are two line bundles invariant under the connection $\nabla$ of Heun-type scheme (4.1). That is, this classical Heun operator from (1.4) is factorised into two commuting first-order operators, which is an uninteresting circumstance from our viewpoint in this paper. In general, we have to settle if the series (B.1) converges in a bigger domain including both 0 and either 1 or $a$.

However, if the parameters (i.e. local monodromies) of a Heun operator are special, we may ask if solutions of more special types exist. This is the study of the global properties of Heun-type connections which we will carry out in the upcoming sections.

5. Type I Degeneration: One Singularity Becomes Apparent

5.1. Resolving Singularities

Theorem 5.1. If $\epsilon \in \mathbb{N}$ in equation (1.4), then there exists $q$ such that the series solution (B.1) terminates.

Proof. The proof follows from three-term recurrence relation given in [14].

The theorem suggests that with the special parameters $\epsilon, q$ mentioned in the theorem above, the monodromy of such a special Heun equation reduces to that of a hypergeometric equation, and hence, it has a local solution written in terms of the hypergeometric functions. This suggests that the singularity $x = a$ is removed.

For the sake of convenience, we will name the additional condition on $A_a$ in (4.1) which characterises such a Heun-type connection:
Definition 5.2.

\[
\{ \text{weak apparent singularity condition: } \begin{align*} & 0 \text{ and } m \in \mathbb{N} \text{ are the eigenvalues of } A_a \\ \end{align*} \} \quad (\text{WAS}(m))
\]

The name “weak apparent singularity condition” suggests that it is somewhat different from the “apparent singularity condition” one usually sees in the theory of Fuchsian differential equations. The common notion of “apparent singularity at \( a \)” means that a second order Fuchsian differential equation \( Ly = 0 \) (I) has the difference of local exponents at \( a \) being an integer; and (II) an accessory parameter is chosen appropriately so that the local monodromy of \( L \) at \( a \) is diagonalisable (i.e. only non-logarithmic local solutions as in [32, pp. 69-70]). Here, we only focus on the criterion (I) and build a function space invariant under \( L \) as mentioned in (1.7) and (1.8). Since these function spaces depend on the domains chosen, we will glue them together in order to build a sheaf of \( \mathcal{O} \)-modules. Being invariant under \( L \) would mean that this sheaf is equipped with a flat connection. The detail will be given in Theorem 5.3.

We also recall that if a divisor on \( \mathbb{P}^1 \) is given by \( D = m(a) \), then its divisor line bundle is denoted by \( \mathcal{O}(m(a)) \) and the space of global sections of \( \mathcal{O}(m(a)) \) is identified as

\[
\text{span}\{ \frac{1}{(x-a)^n} : n = 0, 1, \ldots, m \}.
\]

The main result in this paper is the following theorem, which says that if (\text{WAS}(m)) is satisfied, then the Heun-type connection can be interpreted as one without any singular points away from \( 0, 1 \) and \( \infty \), provided that the underlying vector bundle is chosen carefully. Hence, the flat bundle \( (\mathcal{E}, \nabla) \) in the theorem below has only three singular points. Thus, it behaves like a connection of hypergeometric type. We first deal with the case when \( m \geq 1 \) in this subsection before some applications in the next subsection. The general case where \( m \in \mathbb{Z} \) and when the apparent singularity being any one of \( \{0, 1, \infty\} \) other than \( a \) will be given in the last subsection within this section.

Theorem 5.3 (Resolving singularity - version I). Let \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and \( a \in X \), and let \( A_0, A_1, A_a \) be \( 2 \times 2 \) matrices with complex entries such that 0 and \( m \in \mathbb{N} \) are the eigenvalues of \( A_a \). Then, there exist a sheaf of locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \), and a flat connection \( \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X \) such that the connection matrix of \( \nabla \) relative to a frame over an open subset of \( X \) not containing \( a \) is given by

\[
- \left[ \frac{A_0}{x} + \frac{A_1}{x - 1} + \frac{A_a}{x - a} \right] dx.
\]

Proof. Let \( \mathcal{F} \) be the sheaf of horizontal sections of the connection \( \nabla \) in the trivial bundle \( \mathcal{O}_X \oplus \mathcal{O}_X \) whose connection matrix relative to the canonical basis is

\[
- \left[ \frac{A_0 + A_a}{x} + \frac{A_1}{x - 1} \right] dx.
\]
Define a connection $\nabla$ in $\mathcal{F} \otimes \mathcal{O}(m(a))$ by
\[
\nabla(fs) = s df + \left[ \frac{a(A_as)}{x(x-a)} \right] f dx,
\]
for every open set $U \subset X \setminus \{a\}$, $s \in \mathcal{F}(U)$ and analytic function $f$ with divisor $\geq -m(a)$.

It is clear that $\nabla$ can possibly have a log singularity at $a$. We will construct a subsheaf of $\mathcal{F} \otimes \mathcal{O}(m(a))$ such that the restriction of $\nabla$ to it does not contain any singularities.

Let $v = x(x-1)\frac{d}{dx}$ be a fixed vector field and $\mathcal{E}(U) = \text{the module generated by } \mathcal{F}(U)$.

The issue here is to show that $\mathcal{E}(U) \subset \mathcal{F} \otimes \mathcal{O}(m)(U)$. Observe that for each local section $s \in \mathcal{F}(U)$,
\[
\nabla_v s = \frac{ax(x-1)}{(x-a)}(A_as)
= \frac{a(a-1)}{x-a}(A_as) + aA_as.
\]

Thus,
\[
(x-a)\nabla_v s \in \text{Image of } A_a \mod m_a,
\]
and hence by induction
\[
(x-a)^m(\nabla_v)^m s \in \text{Image of } A_a \mod m_a,
\]
Hence, we write
\[
(\nabla_v)^m s = \frac{A_at}{(x-a)^m} + \text{lower order terms},
\]
for some $t \in \mathcal{F}(U)$. Now,
\[
(\nabla_v)^{m+1} s = -m \frac{x(x-1)}{(x-a)^{m+1}}(A_at) + a \frac{x-1}{(x-a)^{m+1}}(A_a^2t) + \text{(lower order terms)}
= -m \frac{a(a-1)}{(x-a)^{m+1}}(A_at) + a \frac{a-1}{(x-a)^{m+1}}(A_a^2t) + \text{(lower order terms)}
= \frac{a(a-1)}{(x-a)^{m+1}}(A_a^2 - mA_a)t + \text{(lower order terms)}
\in \mathcal{F} \otimes \mathcal{O}(m)(U).
\]

Therefore, the restriction of $\nabla$ to the subsheaf $\mathcal{E}$ of $\mathcal{F} \otimes \mathcal{O}(m(a))$ no longer contains the singularity $a$. It is now clear that the flat bundle $(\mathcal{E}, \nabla)$ has the prescribed connection matrix relative to the canonical basis of $\mathcal{O}_X \oplus \mathcal{O}_X$. □
5.2. Applications

We first show how to derive Erdélyi’s expansion (1.5) from the sheaf of horizontal sections (local system) $\mathcal{F}$ of the connection

$$\nabla := d - \left[ \frac{A_0 + A_a}{x} + \frac{A_1}{x-1} \right] dx$$

(5.1)

that was used in the proof of Theorem 5.3. The derivation of expansion (1.5) from this construction will be given in Example 5.4. The idea behind the construction (5.1) is to displace the monodromy information at the singularity $a$ to the origin $x = 0$ while the $x = a$ is being resolved. Then, we explore how to obtain other expansions, including some new ones, for solutions of the Heun equation (1.4) by choosing different forms of $\nabla$. In fact, we do not only recover all the hypergeometric type expansion solutions derived by Erdélyi in [14,15], but we also exhibit a new expansion (5.10), amongst a large number of possible hypergeometric expansions of local solutions for the Heun equation below from the general theory we propose here.

Example 5.4. Consider the Fuchsian connection (5.1)

$$B_0 := A_0 + A_a = -A_1 - A_\infty,$$

where $B_0$ denotes the residue matrix of $\nabla$ at $x = 0$.

$$\text{Tr}(B_0) = \text{Tr}(A_0 + A_a) = -\text{Tr}(A_1) - \text{Tr}(A_\infty) = (\delta - 1) - \alpha - \beta = \delta - \alpha - \beta - 1 - n + n.$$  

(5.2)

It follows from Lemma A.1 (with $n = 2$) that $\text{Tr}(B_0)$ differs from the sum of the two indicial roots at $x = 0$ of the scalar equation satisfied by the first component by one. That is, we increase the $\delta - \alpha - \beta - 1 - n$ by one when it is written in the classical Riemann scheme. The analysis implies that we have formal sum of schemes

$$\sum_n c_n P \begin{pmatrix} 0 & 1 & \infty \\ n & 0 & \alpha \\ \delta - \alpha - \beta - n & 1 - \delta & \beta \end{pmatrix} x$$

(5.2)

which recovers the scheme given in [14]. One can derive the formal sum

$$\sum_{n=0}^{\infty} c_n x^n \, _2F_1 \left( \frac{\alpha + n, \beta + n}{\alpha + \beta - \delta + 2n + 1}; x \right)$$

(5.3)

which is precisely the infinite sum mentioned in (1.5), where the coefficients $c_n$ satisfy a certain three-term recursion [14, (5.3), (5.4)] (also see “Appendix B”).

Example 5.5. ([15]) Consider the sheaf of horizontal sections (local system) $\mathcal{F}$ of the connection

$$\nabla = d - \left[ \frac{A_0}{x} + \frac{A_1}{x-1} \right] dx,$$

(5.4)

from which one defines

$$\nabla(f s) = s \, df + \left[ \frac{A_a s}{x-a} \right] f \, dx,$$
for every open set $U \subset X \setminus \{a\}$, $s \in F(U)$ and analytic function $f$ with divisor $\geq -m(a)$. Then,

$$B_\infty = A_a + A_\infty = -A_0 - A_1$$

is the residue matrix of $\nabla$ at $\infty$. Thus,

$$\text{Tr}(B_\infty) = \text{Tr}(A_a + A_\infty) := \lambda + \mu = (\lambda + n) + (\mu - n),$$

where $\lambda, \mu$ are the two exponents for the corresponding hypergeometric connection at the singularity $x = \infty$ and $n \in \mathbb{Z}$ is arbitrary (see Theorem 5.3). This approach would give rise to the expansion that was obtained by Erdélyi in another paper [15] in 1944. That is,

$$\sum_n c_n P \begin{cases} 0 & 1 & \infty \\ 0 & 0 & \lambda + m; \ x \\ 1 - \gamma & 1 - \delta & \mu - m \end{cases}, \quad (5.5)$$

which gives the expansion (1.6) for the local solution at the origin with exponent 0. We note that the adjustment of the Fuchsian relations of transition from a connection form to scalar differential equation form as described in Lemma A.1 is implicitly absorbed in the choices of the notations $\lambda, \mu$ above already.

**Example 5.6.** ([36]) Shiga, Tsutsui and Wolfart considered the transcendance of Schwarz maps at algebraic arguments from Fuchsian equations with the same monodromy of a hypergeometric equation with several apparent singularities. They showed that their differential equation admits a holomorphic solution $f$ at $x = 0$ normalised with $f(0) = 1$ can be expressed in the form:

$$f(x) = c_0(t) \, _2F_1 \left( \begin{array}{c} \mu', \mu'' \\ 1 - m - \nu_0 \\ \end{array} ; \ x \right) + \cdots + c_m(t) \, _2F_1 \left( \begin{array}{c} \mu', \mu'' \\ 1 - \nu_0 \\ \end{array} ; \ x \right) \quad (5.6)$$

such that $\sum_{k=0}^m c_k(t) = 1$, where $t$ is determined by the locations of the apparent singularities. We note that while the locations of the apparent singularities are considered movable in [36], the locations of our consideration here are fixed. In [36], the Fuchsian equation has $m$ apparent singularities where the exponent difference at each of these singularities is 2. However, we can show directly that the (5.6) can also be given from our (1.4) where the exponent difference at the apparent singularity $x = a$ is the integer $m$, i.e. when (WAS($m$)) holds.

Application of the contiguous relation

$$(c - a - 1) \, _2F_1(a, b; c; x) + a_2 \, _2F_1(a + 1, b; c; x) - (c - 1) \, _2F_1(a, b; c - 1; x) = 0$$

repeatedly to each term of (5.6) yields

$$f(x) = d_0(t) \, _2F_1 \left( \begin{array}{c} \mu' + m, \mu'' \\ 1 - \nu_0 \\ \end{array} ; \ x \right) + \cdots + d_m(t) \, _2F_1 \left( \begin{array}{c} \mu', \mu'' \\ 1 - \nu_0 \\ \end{array} ; \ x \right) \quad (5.7)$$

where $\sum_{k=0}^m d_k(t) = 1$. A further application of the contiguous relation

$$(b - a) \, _2F_1(a, b; c; x) + a_2 \, _2F_1(a + 1, b; c; x) - b_2 \, _2F_1(a, b + 1; c; x) = 0$$
to (5.7) repeatedly yields
\[
\sum_{k=0}^{m} e_k(t) \, {}_2F_1 \left( \mu' + k, \mu'' - k; 1 - \nu_0 ; x \right),
\]
which is a terminated form of (1.6) where \( \lambda = \mu', \mu = \mu'' + m \).

**Example 5.7.** Suppose next that \( \mathcal{F} \) is the sheaf of horizontal sections (local system) of the connection
\[
\nabla = d - \left[ \frac{A_0}{x} + \frac{A_1 + A_a}{x - 1} \right] dx,
\]
from which another connection \( \nabla \) is defined by
\[
\nabla(sf) := s df + \left[ \frac{(1 - a)A_a s}{(x - 1)(x - a)} \right] f dx,
\]
for every open set \( U \subset X \setminus \{a\} \), \( s \in \mathcal{F}(U) \) and analytic function \( f \) with divisor \( \geq -m(a) \). Then,
\[
B_1 = A_a + A_1 = -A_0 - A_{\infty}
\]
is the residue matrix of \( \nabla \) at \( x = 1 \). Hence,
\[
\text{Tr}(B_1) = \text{Tr}(A_1 + A_a) = -\text{Tr}(A_0) - \text{Tr}(A_{\infty})
= (\gamma - 1) - \alpha - \beta
= (\gamma - 1 - \alpha - \beta - n) + n.
\]
Similar to the consideration in Example 5.4 that the indicial root \( \gamma - \alpha - \beta - n \) of the corresponding classical hypergeometric equation is one bigger than the eigenvalue \( \gamma - 1 - \alpha - \beta - n \) that appears above. Hence, this gives rise to
\[
\sum_{n} c_n P \left\{ \begin{array}{ccc}
0 & 1 & \infty \\
0 & n & \alpha \\
1 - \gamma & -\alpha - \beta - n & \beta \\
\end{array} ; x \right\},
\]
which contains expansions that were *neither found in [14] nor in [15]. Thus, it is possible to obtain
\[
\sum_{n=0}^{\infty} C_n (x - 1)^n \, {}_2F_1 \left( \alpha + n, \beta + n; \gamma; x \right)
\]
where the coefficients \( C_n \) would satisfy a certain three-term recursion which we omit.

**Example 5.8.** Indeed, by choosing
\[
\nabla_t = d - \left[ \frac{A_0 + tA_a}{x} + \frac{(1 - t)A_a + A_1}{x - 1} \right] dx.
\]
Let \( \mathcal{F} \) be the sheaf of horizontal sections of \( \nabla_t \). Then, one defines
\[
\nabla_t(f s) := s df + \left[ \frac{tA_a}{x} + \frac{(t - 1)A_a}{x - 1} - \frac{A_a}{x - a} \right] sf dx,
\]
for every open set $U \subset X \setminus \{a\}$, $s \in \mathcal{F}(U)$ and analytic function $f$ with divisor $\geq -m(a)$. We see that the residues of $\nabla^t$ at $x = 0, 1$ are given by

$$B_t := (A_0 + tA_a), \quad C_t := ((1-t)A_a + A_1).$$

respectively, so that

$$B_t + C_t + A_\infty = 0$$

is the Fuchsian relation for $\nabla^t$. Since the eigenvalues of $B_t, C_t$ are unknown, one cannot obtain Erdélyi type hypergeometric expansions from it in general. Notice that we recover the matrix representation (5.4) when $t = 0$ and the representation (5.1) when $t = 1$.

**Remark 5.9.** One observes from the discussion of above examples that essentially all possible Erdélyi type hypergeometric expansions can be obtained this way.

### 5.3. Accessory Parameters and Invariant Spaces

In order to study the eigenvalue problems (1.7) and (1.8), a $\nabla_v$–invariant function space is needed.

As in the proof of Theorem 5.3, let $\mathcal{F}$ be the sheaf of horizontal sections of a connection $\nabla$ with singularities $\{0, 1, \infty\}$. (We are allowed to choose different $\mathcal{F}$ as shown in the examples in §5.2.) Then, given an open set $U$ and a section $s \in \mathcal{F}(U)$, either one of the spans

$$\text{span}\{A_a s, \nabla_v A_a s, (\nabla_v)^2 A_a s, \ldots, (\nabla_v)^m A_a s\},$$

or

$$\text{span}\{A_a s, \frac{A_a s}{x-a}, \frac{A_a s}{(x-a)^2}, \ldots, \frac{A_a s}{(x-a)^m}\},$$

is invariant under $\nabla_v$. So the eigenvalue problem (1.8) is well defined. The matrix of $\nabla_v$ relative to the former basis is in rational canonical form, while the latter basis gives a tridiagonal matrix. We choose to study our eigenvalue problem using the second basis. Let us illustrate our viewpoint in the following lower dimensional examples. In particular, we notice that the eigenvalues of these $\nabla_v$ are all integers.

**Example 5.10.** Given $m \in \mathbb{N}$, $k = 1, 2, \ldots, m$, we have $A_a^2 = m A_a$ and hence

$$\nabla_v (A_a s) = \frac{ma(x-1)A_a s}{x-a} = ma(a-1)\frac{A_a s}{x-a} + ma A_a s$$

and

$$\nabla_v \left( \frac{A_a s}{(x-a)^k} \right) = - \frac{km(x-1)A_a s}{(x-a)^{k+1}} + ma(a-1)\frac{A_a s}{(x-a)^{k+1}} + ma A_a s \frac{A_a s}{(x-a)^k}$$

$$= a(a-1)(m-k) \frac{A_a s}{(x-a)^{k+1}}$$

$$+ [(m-2k)a + k] \frac{A_a s}{(x-a)^k} - k \frac{A_a s}{(x-a)^{k-1}}.$$
When \( m = 1 \), the matrix representation of \( \nabla_v \) relative to \( \{ a_s, A_a s \} \) is
\[
\begin{bmatrix}
a & -1 \\
-1 & a(a - 1) 1 - a
\end{bmatrix},
\]
and its eigenvalues are 0 and 1. When \( m = 2 \), the matrix representation of \( \nabla_v \) relative to \( \{ A_a s, A_a s \} \) is
\[
\begin{bmatrix}
2a & -1 & 0 \\
2a(a - 1) & 1 & -2 \\
0 & a(a - 1) 2 - 2a
\end{bmatrix},
\]
and its eigenvalues are 0, 1 and 2. In general, for any \( m \in \mathbb{N} \), the eigenvalues of \( \nabla_v \) are \( 0, 1, \ldots, m \).

The phenomenon of integral-valued (or equally spaced) eigenvalues for \( \nabla_v \) exhibited in Example 5.10 can be explained by using the technique of bundle modifications as defined in Definition 2.13 (see [29]). We summarise the observation in the following theorem.

**Theorem 5.11.** Let \( (\mathcal{E}, \nabla) \) be a Heun-type connection such that the residue of \( \nabla \) at the singular point \( a \) has eigenvalues \( 0, -m \) for some \( m \in \mathbb{N} \), i.e. \( \text{WAS} (m) \). Then, there exists a vector field \( w \) such that for each sufficiently small open set \( U \), the set of eigenvalues of \( \nabla_w : \mathcal{E}(U) \to \mathcal{E}(U) \) is given by \( \{ 0, 1, 2, \ldots, m \} \), i.e. there exist \( m + 1 \) local sections \( s_0, s_1, \ldots, s_m \in \mathcal{E}(U) \) such that
\[
\nabla_w s_k = k s_k
\]
for all \( k \in \{ 0, 1, \ldots, m \} \).

**Proof.** Let \( (\mathcal{E}_m, \nabla_m) \) be a connection of Heun type such that \( \text{Res}_a \nabla \) has eigenvalues \( 0, -m \in -\mathbb{N} \). Let its modification at \( (a, \text{Ker}(\text{Res}_a \nabla)) \) be \( (\mathcal{E}_{m-1}, \nabla_{m-1}) \). In particular, there exists a subsheaf \( (\mathcal{E}_{m-1}, \nabla_{m-1}) \to (\mathcal{E}_m, \nabla_m) \) such that \( \text{Res}_a \nabla_{m-1} \) has eigenvalues \( 0, -m + 1 \) by Theorem 2.16. Repeating this procedure yields a filtration of \( (\mathcal{E}_m, \nabla_m) : \mathcal{E}_0 \to \mathcal{E}_m \):
\[
(\mathcal{E}_0, \nabla_0) \to \cdots \to (\mathcal{E}_{m-1}, \nabla_{m-1}) \to (\mathcal{E}_m, \nabla_m).
\]
Now if \( s \) is a local section of \( \mathcal{E}_k \), then
\[
\nabla_k s = (\text{Res}_a \nabla_k) s \otimes \frac{dx}{x - a} \mod \Omega_a
\]
\[
= (\text{Res}_a \nabla_k + kI) s \otimes \frac{dx}{x - a} - k s \otimes \frac{dx}{x - a} \mod \Omega_a
\]
\[
= - k s \otimes \frac{dx}{x - a} \mod (\text{image} (\mathcal{E}_{k-1}, \nabla_{k-1}) \to (\mathcal{E}_k, \nabla_k)) \otimes \Omega_a
\]
because of Res$_a \nabla_k (\text{Res}_a \nabla_k + kI) = 0$ by Cayley–Hamilton. Thus, if $w = -(x - a) \frac{d}{dx}$, then

$$(\nabla_k)_w s = ks \mod (\text{image } (\mathcal{E}_{k-1}, \nabla_{k-1}) \longrightarrow (\mathcal{E}_k, \nabla_k)),$$

In other words, the matrix representation of $(\nabla_m)_w$ is triangular with diagonal entries $0, 1, \cdots, m$. This implies that its eigenvalues are $0, 1, \cdots, m$. □

5.4. Eigenspaces via Symmetry

In Theorem 5.3, we have seen that if $A_a$ has eigenvalues 0 and $m \in \mathbb{N}$, then there is a flat connection $(\mathcal{E}, \nabla)$ over $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose connection matrix is of the prescribed form

$$- [A_0 \frac{x}{x} + A_1 \frac{x}{x-1} + A_a \frac{x}{x-a}] dx.$$

The following theorem shows that the condition $m \in \mathbb{N}$ and the choice of $A_a$ can be modified.

**Theorem 5.12.** (Resolving singularity—version II) Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $a \in X$, and let $A_0, A_1, A_a$ be $2 \times 2$ matrices with complex entries such that 0 and $m \in \mathbb{Z}$ are the eigenvalues of $A_a$. Then, there exist a sheaf of locally free $\mathcal{O}_X$-module $\mathcal{E}$, and a flat connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X$ such that the connection matrix of $\nabla$ relative to a frame over an open subset of $X$ not containing $a$ is given by

$$- [A_0 \frac{x}{x} + A_1 \frac{x}{x-1} + A_a \frac{x}{x-a} - mI] dx.$$ (5.12)

**Proof.** If the eigenvalue $m$ of $A_a$ is a positive integer, then the result is already dealt with in Theorem 5.3. Suppose now that $0$ and $m$, where $-m \in \mathbb{N}$, are the eigenvalues of $A_a$. Then, the matrix $A_a - mI$ has eigenvalues 0 and $-m \in \mathbb{N}$. Thus, Theorem 5.3 asserts that there is a sheaf of locally free $\mathcal{O}_X$-module $\mathcal{E}$, and a flat connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X$ whose matrix relative to a certain basis is

$$- [A_0 \frac{x}{x} + \frac{A_1}{x-1} + \frac{A_a - mI}{x-a}] dx.$$ Then,

$$(\mathcal{E}, \nabla - \frac{mI}{x-a} dx)$$

gives the desired flat bundle with matrix representation (5.12). Alternatively, one can obtain the same matrix representation (5.12) by considering the induced connection in $\mathcal{E} \otimes K^m_a$. Finally, note that when $m = 0$, $A_a$ has a repeated eigenvalue 0. Together with the assumption that $A_a$ is diagonalisable at the end of Definition 2.7, one sees that $A_a = 0$, i.e. a zero matrix, and the existence of the flat bundle $(\mathcal{E}, \nabla)$ over $X$ follows trivially. □

We next show that the conclusion of the above theorem continues to hold if the apparent singularity is located at any one of the singularities $x = 0$, 1, $\infty$ instead of at $x = a$ via the Kummer symmetry.
Theorem 5.13. (Resolving singularity—version III) Let \( X = \mathbb{P}^1 \setminus \{0, 1, a, \infty\} \) and let \( A_0, A_1, A_a \) be \( 2 \times 2 \) matrices with complex entries such that the difference of the eigenvalues of one of the matrices from \( \{A_0, A_1, A_a, A_\infty\} \) is an integer \( m \). Then, there exist a sheaf of locally free \( \mathcal{O}_X \)-module \( E \), and a flat connection \( \nabla : E \to E \otimes \Omega_X \) with three singularities only such that the connection matrix of \( \nabla \) relative to a frame is

\[
- \left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx.
\]

(5.13)

Proof. Suppose that \( j \in \{0, 1, a\} \) and the two eigenvalues of \( A_j \) are \( \sigma \) and \( \sigma + m \). Then, the residue of the induced connection in \( E \otimes K_j^{-\sigma} \) at \( j \) has eigenvalues 0, \( m \). So without loss of generality, we may assume that one of the matrices \( A_j \) (\( j \in \{0, 1, a\} \)) to be considered below has one eigenvalue 0 and another one \( m \). Let us now suppose that \( A_0 \) has eigenvalues 0, \( m \). Then, Theorem 5.12 asserts that there is a flat connection \( (E, \nabla) \), with singularities at 0, 1 and \( a/(a-1) \), whose connection matrix relative to a certain basis is

\[
- \left[ \frac{A_a}{x} + \frac{A_1}{x-1} + \frac{A_0}{x-a} \right] dx.
\]

Let \( f(x) = (1 - a)x + a \) be the Möbius transformation that maps \( (0, 1, \infty) \) to \( (a, 1, \infty) \). Then, the push-forward bundle \( f_* \mathcal{E} \) induces a connection whose connection matrix assumes the desired form (5.13). \( \Box \)

The following theorem summarises our discussion.

Theorem 5.14. Let a classical Riemann scheme \( P \)

\[
\begin{pmatrix} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{pmatrix}
\]

be given. If either one of the following condition holds:

1. \( \gamma \in \mathbb{Z} \); or
2. \( \delta \in \mathbb{Z} \); or
3. \( \epsilon \in \mathbb{Z} \); or
4. \( \alpha - \beta \in \mathbb{Z} \),

then there exists a flat bundle over \( \mathbb{P}^1 \) with three points deleted whose Riemann scheme relative to an appropriately chosen frame is the prescribed one.

Thus, the necessary condition \( \epsilon \in \mathbb{N} \) for the termination of the series (B.1) generates other similar conditions via the Kummer symmetry, and we obtain

Theorem 5.15. A necessary condition for the Heun equation (1.4) to have an apparent singularity is when one of \( \gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} \setminus \{0\} \).

One can easily re-interpret the above eigenvalue problem for the following Heun connection.

Corollary 5.16. A necessary condition for the existence of \( \lambda \) such that the equation

\[
x(x-1)\left( \frac{dY}{dx} - \left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] \right) s = \lambda s,
\]

(5.14)
has a non-trivial solution $s$ consisting of a finite $\sum_{F}^{2} x^{b_1}(x - 1)^{b_2}(x - a)^{b_3}$ is that the differences of eigenvalues of $A_s$ is an integer for some $s = 0, 1, a$ or $\infty$.

6. Type II Degeneration: Monodromy Reduction

Recall the well-known theorem:

**Theorem 6.1.** If the Heun equation (1.4) has a polynomial solution, then either $\alpha$ or $\beta$ is a non-positive integer.

For more detail, see, for example, Ronveaux [34, §3.6]. This polynomial solution of (1.4) (i.e. (1.4) is reducible) is invariant under the monodromy of (1.4).

Now the focus of the upcoming study is the Heun-type connection with matrix relative to a frame being $-\left[\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a}\right] dx$ and its monodromy reduction at $\infty$ since 0 is already an exponent at each of the singular points 0, 1, a. The consideration of monodromy reduction at the points 0, 1, a can be transposed to $\infty$ later. We introduce the following definition

**Definition 6.2.**

$$\{ \text{local condition for reducibility of monodromy at } \infty : \} \quad \text{(LR}(m))$$

**Theorem 6.3.** If $\nabla$ a connection of Heun-type (4.1) satisfying both (WGRM) and (LR(m)) for some $m \in \mathbb{N}$, and if $\mathcal{L}$ is the sheaf of horizontal sections of $\nabla$, then $\mathcal{L}$ is reducible.

**Proof.** Let $\nabla$ be a connection with matrix relative to a frame being $-\left[\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a}\right] dx$ such that the eigenvalues of $A_0$, $A_1$, $A_a$ are $\{0, 1-\gamma\}, \{0, 1-\delta\}, \{0, 1-\epsilon\}$, respectively. It follows from the assumption (LR$(m)$) that $A_0 + A_1 + A_a$ has an eigenvalue $m \in \mathbb{N}$. Now, $A_a$ has eigenvalues 0 and 1 $- \epsilon$. By (WGRM), $A_a$ and $-A_\infty = A_0 + A_1 + A_a$ are simultaneously diagonalisable. Hence, $A_0 + A_1$ has a positive integral eigenvalue $m$ (the case in which $1-\epsilon + m$ is an eigenvalue of $A_0 + A_1$ will be apparent in Lemma 6.5). Since each of the three matrices $A_0$, $A_1$, $A_a$ has 0 as an eigenvalue, Lemma 3.2 implies that the exp$(2\pi i A_0)$, exp$(2\pi i A_1)$ and exp$(2\pi i (A_0 + A_1))$ have a common eigenvector $v$, say, which is also a common eigenvector of $A_0$, $A_1$ and $A_0 + A_1$.

But $A_a$ and $A_0 + A_1$ are simultaneously diagonalisable, hence they must have common eigenvectors. But according to the last paragraph, $A_0 + A_1$ shares an eigenvector with $A_0$ and $A_1$. We conclude that span$\{v\}$ is invariant under $A_0$, $A_1$ and $A_a$. Finally, the local monodromies of $\mathcal{L}$ are specified by $A_0$, $A_1$, $A_a$ and $-(A_0 + A_1 + A_a)$ which have a common eigenvector, so that $\mathcal{L}$ is reducible. 

We immediately obtain the following corollary by following the argument in the proof of Theorem 3.3.
Corollary 6.4. Let \((E, \nabla)\) be a connection of Heun type satisfying both \((\text{WGRM})\) and \((\text{LR}(m))\) for some \(m \in \mathbb{N}\). Then, there exists a non-trivial morphism of flat bundles \((\mathcal{O}_X, d) \longrightarrow (E, \nabla)\).

Proof. We essentially follow a similar procedure that led to Theorem 3.3 from Lemma 3.2.

We note that the residue matrices \(A_0, A_1, A_a\) and \(A_\infty\) that give rise to the monodromy matrices \(P = e^{2\pi i A_0}, Q = e^{2\pi i A_1}, R = e^{2\pi i A_a}, PQR = e^{2\pi i A_\infty}\).

\(P\) has eigenvalues \(1, e^{2\pi i(1-\gamma)}\)
\(Q\) has eigenvalues \(1, e^{2\pi i(1-\delta)}\)
\(R\) has eigenvalues \(1, e^{2\pi i(1-\epsilon)}\)
\(PQR\) has eigenvalues \(e^{2\pi i\alpha}, e^{2\pi i\beta}\).

Note that \(-\alpha \in \mathbb{N}\). According to the previous theorem, there is a common eigenvector \(v\) for the above monodromy matrices (which represents a solution \(f\) of the Heun equation defined on a small open set). Then, after some routine consideration together with the Fuchsian condition, only two out of the total sixteen possibilities remain, which are, either

\(Pv = v\) and \(Qv = v\) and \(Rv = v\) and \(PQRv = v\)
or
\(Pv = e^{2\pi i(1-\gamma)}v\) and \(Qv = e^{2\pi i(1-\delta)}v\) and \(Rv = e^{2\pi i(1-\epsilon)}v\) and \(PQRv = e^{-2\pi i\beta}v\).

The remaining steps of the proof go along a similar line of argument as those in the proof of Theorem 3.1 with the new expression \(x^\gamma (x - 1)^{\delta - 1}(x - a)^{\epsilon - 1}f\) instead of \(x^{c-1}(x - 1)^{a+b-c}f\). We omit the details and conclude that the latter is eliminated. This completes the proof as the derivation of Theorem 3.3 from Theorem 3.1. \(\square\)

6.1. Consequences of Kummer Symmetry

We may modify the requirements \((\text{WGRM})\) and \((\text{LR}(m))\) assumed in Theorem 6.3 via symmetry as shown in the following corollary to Theorem 6.3.

Corollary 6.5. For each \(j \in \{0, 1, a\}\), let \(A_j\) be a \(2 \times 2\) matrix with 0 as an eigenvalue. If the following conditions hold:

(L): \(m\) is an eigenvalue of \(A_\infty = -A_0 - A_1 - A_a\) for some \(m \in \mathbb{N}\); and

(G): there exist two matrices amongst \(A_0, A_1, A_a\) and \(A_\infty\) are simultaneously diagonalisable,

then the local system defined by the sheaf of horizontal sections of the connection with connection matrix

\[-\left[ \frac{A_0}{x} + \frac{A_1}{x - 1} + \frac{A_a}{x - a} \right] dx\]  \hspace{1cm} (6.1)

is reducible.

Proof. We shall focus on the case in which \(A_0\) and \(A_1\) are simultaneously diagonalisable in the modified \((G)\). The other cases are handled in completely the same way. Let \(L\) be the sheaf of horizontal sections of a connection with connection matrix (6.1). Let \(f\) be a Möbius transformation satisfying \(f(0) = \infty, f(a) = 1\) and \(f(\infty) = 0\). Note that \(f(x) = a/x\). In particular \(f(1) = \infty\).
Then, $K_m^0 \otimes f_* L$ is the sheaf of horizontal sections of a connection with connection matrix

$$-\left[ -\frac{A_0 + A_1 + A_a + mI}{x} + \frac{A_a}{x-1} + \frac{A_1}{x-a} \right] dx.$$ 

Observe that $-(A_0 + A_1 + A_a + mI) + A_a + A_1 = -A_0 - mI$ and $A_1$ are simultaneously diagonalisable, and $A_0 + mI$ has a positive integral eigenvalue $m$, so the local system $K_m^0 \otimes f_* L$ is reducible by Theorem 6.3.

$\square$

Following the above argument, we now explore the full force of the Kummer symmetry to obtain a complete criterion for the reducibility of the monodromy of the Heun connections (1.2).

**Theorem 6.6.** For each $j \in \{0, 1, a\}$, given a $2 \times 2$ matrix $A_j$, let $\lambda_{j1}$ and $\lambda_{j2}$ be the eigenvalues of $A_j$. We also let $\lambda_{\infty 1}$, $\lambda_{\infty 2}$ be the eigenvalues of $A_\infty = -A_0 - A_1 - A_a$. If

(L) for each $j \in \{0, 1, a, \infty\}$, there exists $\lambda_j \in \{\lambda_{j1}, \lambda_{j2}\}$ such that $\sum_j \lambda_j \in \mathbb{Z}$; and

(G) two matrices amongst $A_0$, $A_1$, $A_a$ and $A_\infty$ are simultaneously diagonalisable,

then the sheaf of horizontal sections of a connection on $X = \mathbb{P}^1 \setminus \{0, 1, a, \infty\}$ with matrix relative to a frame being

$$-\left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx$$

is reducible.

**Proof.** Let $L$ be the sheaf of horizontal sections of a connection with connection matrix

$$-\left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx.$$ 

Then, $L \otimes K_0^{-\lambda_0} \otimes K_1^{-\lambda_1} \otimes K_a^{-\lambda_a}$ is the sheaf of horizontal sections of a connection with connection matrix

$$-\left[ \frac{A_0 - \lambda_0 I}{x} + \frac{A_1 - \lambda_1 I}{x-1} + \frac{A_a - \lambda_a I}{x-a} \right] dx.$$ 

Now for each $j$, 0 is an eigenvalue of $A_j - \lambda_j I$, and $\sum_j (A_j - \lambda_j I)$ has an eigenvalue

$$-\lambda_\infty - \sum_{j=0}^a \lambda_j \in \mathbb{Z}.$$ 

Together with the hypothesis that two matrices amongst $A_0$, $A_1$, $A_a$ and $A_\infty$ are simultaneously diagonalisable, the result follows from an application of Corollary 6.5.

$\square$

**Theorem 6.7.** For each $j \in \{0, 1, a\}$, given $2 \times 2$ matrices $A_j$ together with $A_\infty = -A_0 - A_1 - A_a$, suppose that they satisfy the conditions (L) and (G)
in Theorem 6.6. If \((\mathcal{E}, \nabla)\) is a flat bundle over \(\mathbb{P}^1\{0,1,a,\infty\}\) with matrix relative to a frame being
\[ -\left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_a}{x-a} \right] dx, \]
then there exist a rank-one flat bundle \((\mathcal{F}, \nabla')\) and a non-trivial morphism \((\mathcal{F}, \nabla') \to (\mathcal{E}, \nabla)\).

**Proof.** The sheaf of horizontal sections of \((\mathcal{E}, \nabla)\) is reducible by Theorem 6.6. So one obtains the desired morphism by the procedure shown in Theorem 3.3. □

A classical analogue is

**Theorem 6.8.** Suppose that either one of \(\alpha, \beta\) is in \(\mathbb{Z}\{1\}\) or one of \(\alpha-\gamma, \alpha-\delta, \alpha-\epsilon, \beta-\gamma, \beta-\delta, \beta-\epsilon\) is in \(\mathbb{Z}\{0\}\). There exists \(q\) such that the Heun equation (1.4) has a polynomial type solution, i.e., \(x^{\tau_0}(x-1)^{\tau_1}(x-a)^{\tau_a}p(x)\) for some \(\tau_0 \in \{0,1-\gamma\}, \tau_1 \in \{0,1-\delta\}, \tau_a \in \{0,1-\epsilon\}\) and some polynomial \(p(x)\).

**Remark 6.9.** The first part of Theorem 6.8 can also be obtained from differential Galois theory [13, p. 241]. However, our approach of the whole paper is from monodromy consideration of Heun equations directly instead of its simplification to differential Galois groups.

### 7. Coincidence Between Type I and Type II Degenerations

#### 7.1. Takemura’s Eigenvalues Inclusion Theorem

It follows from Theorem 5.1 that if \(\epsilon = -m\), then there exist \(m+1\) eigenvalues such that the series (B.1) terminates. On the other hand, if \(\alpha = -n\), then there exist \(n+1\) eigenvalues such that the series (B.1) becomes the Heun polynomial of degree \(n\). The following theorem of Takemura states that these two sets of the eigenvalues have an inclusion relation.

**Theorem 7.1.** ([39, Theorem 5.3]) Assume that \(\epsilon\) and \(\alpha\) are non-positive integers, but \(\beta\) is not.

(i) If \(\epsilon \geq -\alpha\) and the Heun equation (1.4) has a polynomial solution, then the singularity \(x = a\) is apparent.

(ii) If \(\alpha \geq -\epsilon\) and the singularity \(x = a\) of the Heun equation (1.4) is apparent, then the equation has a polynomial solution.

#### 7.2. Geometric Interpretation of Takemura’s Proof

While Takemura’s method of proof is purely analytic, we shall establish a geometric argument that is based on the classification of monodromy reduction established in this paper that naturally leads to his result.

**Theorem 7.2.** Let \(a \in \mathbb{C}\{0,1\}\) and \(X = \mathbb{C}\{0,1\}\). If \(A_0, A_1, A_a\) are \(2 \times 2\) matrices with complex entries such that each matrix has 0 as an eigenvalue and satisfy (WGRM), (WAS\((m)\)) and (LR\((n)\)) for some \(m, n \in \mathbb{N}\). Then,
there exist a rank-one flat bundle \((\mathcal{F}', \nabla')\) and a non-trivial morphism of flat bundles
\[
(\mathcal{F}', \nabla') \longrightarrow \left( \mathcal{O}_{X \setminus \{a\}} \oplus \mathcal{O}_{X \setminus \{a\}}, \ d - \left[ \frac{A_0}{x} + \frac{A_1}{x - 1} + \frac{A_a}{x - a} \right] dx \right).
\]

**Proof.** Since \((\text{WAS}(m))\) is satisfied, by a variation of Theorem 5.3 presented in Example 5.5, there exist a rank-two flat bundle \((\mathcal{E}, \nabla)\) over \(X\) with matrix relative to a frame being \(-\left[ \frac{A_0}{x} + \frac{A_1}{x - 1} \right] dx\) and an isomorphism of flat bundles over \(X \setminus \{a\}\)
\[
(\mathcal{E}, \nabla) \longrightarrow \left( \mathcal{O}_{X \setminus \{a\}} \oplus \mathcal{O}_{X \setminus \{a\}}, \ d - \left[ \frac{A_0}{x} + \frac{A_1}{x - 1} + \frac{A_a}{x - a} \right] dx \right).
\]
Notice that the connection \(\nabla\) above is of hypergeometric type, with 0 as an eigenvalue of both of \(A_0\) and \(A_1\). Thus, the residue matrix of \((\mathcal{E}, \nabla)\) at \(\infty\) is \(- (A_0 + A_1) = -(A_0 + A_1 + A_a) + A_a\). By the hypotheses \((\text{LR}(n))\) and \((\text{WGRM})\), \(A_0 + A_1 + A_a\) has an integral eigenvalue \(n\) and \(A_0 + A_1 + A_a\) and \(A_a\) can be simultaneously diagonalised, respectively. Thus, the hypothesis \((\text{WAS}(m))\) implies that the sum \(A_0 + A_1\) has either \(n-m\) or \(n\) as an eigenvalue.
In any case, one applies Theorem 3.3 to \((\mathcal{E}, \nabla)\) to yield a rank-one flat bundle \((\mathcal{F}', \nabla')\) together with a non-trivial morphism
\[
(\mathcal{F}', \nabla') \longrightarrow (\mathcal{E}, \nabla).
\]
The proof is finished by composing the two morphisms above. □

**Geometric Proof of Takemura’s Theorem.** The assumption \(-\alpha \in \mathbb{N}\) and the accessory parameter \(q\) being chosen appropriately in the Heun equation (4.2) made in Theorem 7.1 (i) correspond to \((\text{LR}(n))\) and \((\text{WGRM})\) in our Theorem 7.2. Of course, it is well known that the Heun equation (4.2) admits a polynomial solution (see [13,34]) under these assumptions. The additional assumption that \(-\epsilon \in \mathbb{N}\) made in Theorem 7.1 (i) amongst to further assuming \((\text{WAS}(m))\) in Theorem 7.2. The assumptions made in part (ii) of Theorem 7.1 again amongst to be the same assumptions of Theorem 7.2, where the second assumption in Theorem 7.1(ii) corresponds to the assumptions \((\text{WAS}(m))\) and \((\text{WGRM})\) in Theorem 7.2.

The conclusion of Theorem 7.2 asserts that the corresponding Heun-type connection is (singular) gauge equivalent to a hypergeometric type connection, which recovers Takemura’s result. □

**8. Matching to Painlevé VI Equation**

It is well known that the Painlevé VI equation comes from an isomonodromic family of Heun equations (see, for example, [20], [21]).

In Sects. 5, 6, we have seen that two special types of Heun equations have special solutions. The proofs relied heavily on the monodromies of the equations given. So it is interesting to study special solutions of an isomonodromic family of Heun equations.
\[ 0 = dY - \left[ A_0 \frac{dx}{x} + A_1 \frac{dx}{x-1} + A_t \frac{d(x-t)}{x-t} \right] Y, \tag{8.1} \]

in which the eigenvalues of each of the three matrices are independent of \( t \). When some of these eigenvalues are special, the solutions \( Y(x, t) \) of the equation above are special functions in \( x \). Our contribution in this section is to observe that the matrices \( A_0, A_1, A_t \) are also special in \( t \) when some of these eigenvalues are chosen to satisfy certain arithmetic relations so that the monodromies degenerate. Such a coincidence suggests that we study the \( Y(x, t) \) to be special with respect to both the variables \( x, t \). This difficult work is reserved for future investigation.

Given a local function \( Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \) satisfying the following 2 \( \times \) 2 Fuchsian system

\[ \nabla Y := dY - \left[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \right] Y dx = 0, \tag{8.2} \]

let the matrices \( A_0, A_1, A_t \) have the eigenvalues \((0, \theta_0)\) \((0, \theta_1)\) and \((0, 1 - \theta_t)\), respectively, and \( A_\infty := -A_0 - A_1 - A_t = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \) is diagonal. Then, \( y_1 \) satisfies the following equation (see Takemura \[38, \S 2\])

\[ \frac{d^2 y}{dx^2} + \left( \frac{1 - \theta_0}{x} + \frac{1 - \theta_1}{x-1} + \frac{1 - \theta_t}{x-t} - \frac{1}{x-\lambda} \right) \frac{dy}{dx} \]
\[ + \left( \frac{\kappa_1(\kappa_2 + 1)}{x(x-1)} + \frac{\lambda(\lambda - 1)\mu}{x(x-1)(x-\lambda)} - \frac{t(t-1)H}{x(x-1)(x-t)} \right) y = 0, \tag{8.3} \]

where \( \theta_\infty = \kappa_1 - \kappa_2 \), \( \lambda \) is the zero of \((1, 2)\)-entry of

\[ \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} := \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \mu = a_{11}(\lambda) \]

and

\[ H = \frac{1}{t(t-1)} [\lambda(\lambda - 1)(\lambda - t)\mu^2 - \theta_0(\lambda - 1)(\lambda - t) + \theta_1(\lambda - t) \]
\[ + (\theta_t - 1)\lambda(\lambda - 1)\mu + \kappa_1(\kappa_2 + 1)(\lambda - t)]. \]

The condition for isomonodromy deformation of the above equation is that \( \lambda \) satisfies the sixth Painlevé equation \( (P_{VI}) \) (also see \[25, \S 3\]):

\[ \frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu} \quad \text{and} \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}, \]

then it follows that

\[ \frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 \]
\[ - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t-1)^2} \frac{(1-\theta_\infty)^2}{2} \]
It is known that $P_{VI}$ has special solutions expressed in terms of the hypergeometric functions when $\theta_0, \theta_1, \theta_t, \theta_\infty$ satisfy the following conditions.

**Theorem 8.1.** ([18, Theorem 48.3]) If either
\[
\theta_0 + \sigma_1 \theta_1 + \sigma_t \theta_t + \sigma_\infty \theta_\infty \in 2\mathbb{Z},
\]
for some $\sigma_1, \sigma_t, \sigma_\infty \in \{\pm 1\}$ or
\[
(\theta_0 - n)(\theta_1 - n)(\theta_t - n)(\theta_\infty - n) = 0
\]
for some $n \in \mathbb{Z}$, then $P_{VI}$ has a one-parameter family of solutions expressed in terms of the hypergeometric functions.

On the other hand, under the above conditions (8.4) and (8.5), we show that the equation (8.2) has special solutions.

**Theorem 8.2.** Suppose that $\theta_0, \theta_1, \theta_t, \theta_\infty$ satisfy

(i) the condition (8.4) for some $\sigma_1, \sigma_t, \sigma_\infty \in \{\pm 1\}$. Then, the flat bundle endowed with the connection (8.2) has a flat line subbundle.

(ii) the condition (8.5) for some $n \in \mathbb{Z}$. Then, the flat bundle endowed with the connection (8.2) is isomorphic to one with one less singularity.

**Sketch of Proof.** For (i), the equation (8.3) has polynomial solutions only if either
\[
\kappa_1 = -\frac{\theta_0 + \theta_1 + \theta_t - \theta_\infty}{2} = 0, -1, -2, \cdots \\
\kappa_2 = -\frac{\theta_0 + \theta_1 + \theta_t + \theta_\infty}{2} = -1, -2, -3, \cdots
\]
Moreover, other conditions of $\theta_0, \theta_1, \theta_t, \theta_\infty$ in (8.4) can be obtained via the symmetry.

For (ii), the singularity $t$ in equation (8.3) is apparent only if $\theta_t = 1, 2, 3, \cdots$. Moreover, other conditions of $\theta_0, \theta_1, \theta_t, \theta_\infty$ in (8.5) can be obtained via the symmetry.

**Example 8.3.** If $\kappa_1 = -\frac{\theta_0 + \theta_1 + \theta_t - \theta_\infty}{2} = 0$, take $\lambda = t$ and $\mu = 0$, then the equation (8.3) has a constant solution.

**Example 8.4.** If $\theta_t = 0$, take $\lambda = t$ and arbitrary $\mu$, then $A_t = 0$ (see Takemura [38] pp.19-20) and hence, the point $t$ becomes an ordinary point so that the equation (8.3) reduces to a hypergeometric equation.

**Remark 8.5.** It is known that $P_{VI}$ has a rational solution if and only if $\theta_0, \theta_1, \theta_t, \theta_\infty$ satisfy both the condition (8.4) for some $\sigma_1, \sigma_t, \sigma_\infty \in \{\pm 1\}$ and the condition (8.5) for some $n \in \mathbb{Z}$ (see [40]).
9. Concluding Remarks

The Heun equation (1.4) and its confluent forms are ubiquity in both mathematical physics and certain engineering disciplines since the early nineteenth century [34,42]. It was also observed that special cases of (1.4) are closely related to several problems in number theory, see, for example, [7]. In fact, there are already quite a number of published papers on various topics by researchers from very different interests and directions, e.g. [1,4,8–10,23,27,35]. However, the study of monodromy group of (1.4) proves to be extremely difficult as seen from [14,15]. This is partly explained by the rigidity theory proposed by Katz [22]. Unlike the hypergeometric equation which is rigid, the Heun equation (1.4) is not rigid in general, meaning that it is essentially impossible to obtain closed-form solutions to (1.4) in Euler-type integrals. Such difficulty of Heun equations is partly reflected in the recent book [34]. In view of the limitation of the use of classical mathematical language, the first main focus of this paper is to study the Heun equation as a special case of Fuchsian connections with four residue matrices, that is, to put the differential equation in the most natural geometric setting to where it belongs a priori. Besides, it turns out that for both theoretical interests and application purpose, it is important to consider the Heun equation when one of its singularities becomes apparent, i.e. being resolved. But then the monodromy of (1.4) is that of a hypergeometric connection. The equation becomes rigid, and closed-form solutions in terms of hypergeometric functions become possible.

This paper studies the monodromy groups of special Fuchsian connections, namely the Heun connection (1.2) with three categories of monodromy degenerations (WGRM), (WAS(m)) and (LR(n)) from a sheaf theoretic viewpoint. In particular, we focus on the case when one of its singularities becomes apparent mentioned in the last paragraph. This allows us to give an interpretation of the classical infinite hypergeometric function expansions of solutions to the Heun equation (1.4) proposed by Erdélyi (1942, 1944) in terms of a sequence of appropriately defined injective bundle morphisms. As a consequence, we have also derived new expansions which are not found in Erdélyi’s papers. Oblezin applied Drinfeld’s bundle modification technique to handle several well-known contiguous relations of hypergeometric functions. This theory is being applied to study properties of eigenvalues of (1.8). The most startling finding here is that the eigenvalues are equally spaced. This is in stark contrast with the complicated behaviour of the corresponding eigenvalues (accessory parameters) of (1.7) observed for the “scalar” Heun equations (1.4). See, for example, the authors [33] have made a detailed numerical study on the dependence of the eigenvalues on the parameters of the Whittaker–Hill equation. The Whittaker–Hill equation is a trigonometric form of a confluent Heun equation. Indeed, very little is known about the analytic properties of these eigenvalues
in general, even for the classical Mathieu equation and Lamé equation\(^2\). See [3,37] for recent qualitative descriptions of the eigenvalues for these periodic differential equations. Our finding shows that either of the combinations between \((\text{WGRM})-(\text{WAS}(m))\) or between \((\text{WGRM})-(\text{LR}(n))\) leads to the monodromy degeneration of the Heun connection (1.2). The latter degeneration has essentially the same effect as the Heun connection being reducible which is well studied from the viewpoint of differential Galois theory [13]. From our viewpoint, the former appears to play a more fundamental role in describing the monodromy reduction of the Heun connection. The study of hypergeometric equations with several additional apparent singularities by Shiga, Tsutsui and Wolfart [36] also falls within this category (see Example 5.6). We have shown that when both the two reduction modes happen simultaneously, then there exists a rank-one flat bundle and a non-trivial morphism that sends the rank-one flat bundle to the flat bundle of a Heun connection. An immediate consequence is a geometric proof of Takemura’s theorem which describes a certain inclusion of eigenvalues of two types of monodromy reductions, due to the common origin \((\text{WGRM})\). Our next unexpected observation is that the combined criteria of monodromy reductions \((\text{WAS}(m))\) and \((\text{LR}(n))\) of the Heun connection that we have enumerated matches precisely the criteria for the degeneration of Painlevé equation VI, that is, when the \(P_{VI}\) either admits a rational solution or in terms of hypergeometric functions. It is well known from the work of Okamoto that a Weyl group acts on the parameter space of the degenerated \(P_{VI}\). Such matching may reasonably be studied from the viewpoint of the PDE (8.1). Finally, we would like to mention that the existence of orthogonality between two eigen-solutions of terminated hypergeometric sums of the form (1.5), i.e. from the degeneration of \((\text{WAS}(m))\), appears to be a non-trivial problem. The case for the joint orthogonality for Heun polynomials, i.e. from the combined degenerations of \((\text{WGRM})\) and \((\text{LR}(n))\), has been studied recently by Felder and Willwacher [16]. We hope to return to these problems in the near future.

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\(^2\)The Lamé equation is an elliptic form and special case of the Heun equation where the local monodromy of three out of the four singularities are reduced, see, for example, [6], while the Mathieu equation is a trigonometric form of a special confluent Heun equation.
Appendix A. Fuchsian Relations

We point out that there is a difference between the Fuchsian relation of a Fuchsian connection and a Fuchsian differential equation \[2\] that is derived from the Fuchsian connection. Since we cannot find a reference for this fact, a proof is provided here.

Let a connection $\nabla$ relative to the canonical basis have the matrix representation

$$-
\sum_{j=1}^{n} \frac{A_j}{x-a_j} := -A(x) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$
(A.1)

such that $a_{n+1} = \infty$ and that the residue matrices $A_j$ ($j = 1, \cdots, n+1$) satisfy the Fuchsian relation:

$$\sum_{j=1}^{n+1} \text{Tr}(A_j) = 0.$$  (A.2)

Lemma A.1. Let $\text{Tr}(A_j) = \alpha_{j1} + \alpha_{j2}$ be the trace of the residue matrix $A_j$ ($j = 1, \cdots, n+1$). Let $y$ be the first component of a horizontal section of the connection defined above. Let the Riemann scheme of the differential equation

$$y'' - \left( \text{Tr}(A(x)) + \frac{A'_{12}(x)}{A_{12}(x)} \right)y' + \left( \text{det} A(x) - A_{11}(x) \log' \left( \frac{A_{12}(x)}{A_{11}(x)} \right) \right)y = 0$$
(A.3)

derived from the connection above and satisfied by $y$ be of the form

$$P \begin{pmatrix} a_1 & \cdots & a_j & \cdots & a_n & \infty & b_1 & \cdots & b_{n-1} \\ \beta_1 & \cdots & \beta_1 & \cdots & \beta_1 & \beta_{n+1,1} & 0 & \cdots & 0 \\ \beta_2 & \cdots & \beta_2 & \cdots & \beta_2 & \beta_{n+1,2} & 2 & \cdots & 2 \end{pmatrix} x,$$

where $b_j$ ($j = 1, \cdots, n-1$) are the apparent singularities. If $b_j = a_j$ ($j = 1, \cdots, n-1$), then we have

$$\alpha_{j1} + \alpha_{j2} = \beta_{j1} + \beta_{j2} - 1 \quad (j = 1, \cdots, n-1)$$
$$\alpha_{n1} + \alpha_{n2} = \beta_{n1} + \beta_{n2},$$
$$\alpha_{n+1,1} + \alpha_{n+1,2} = \beta_{n+1,1} + \beta_{n+1,2}.$$  (A.4)

In particular,

$$\sum_{j=1}^{n+1} (\beta_{j1} + \beta_{j2}) = (n-1) + \sum_{j=1}^{n+1} \text{Tr}(A_j).$$

Proof. It is sufficient to compute the coefficient of $y'$. Let

$$A_{12}(x) =: (\text{const.}) \frac{\prod_{j=1}^{n-1} (x - b_j)}{\prod_{j=1}^{n} (x - a_j)}.$$

Hence

$$-\text{Tr}(A(x)) - \frac{A'_{12}(x)}{A_{12}(x)} = \sum_{j=1}^{n} \frac{1 - \alpha_{j1} - \alpha_{j2}}{x - a_j} - \sum_{j=1}^{n-1} \frac{1}{x - b_j}.$$  (A.5)
where the $b_j$ ($j = 1, \cdots, n - 1$) are apparent singularities which are inherited from $A'_{12}/A_{12}$ and they do not contribute to the monodromy of the equation (A.3). Let us now assume $b_j = a_j$ ($j = 1, \cdots, n$). Hence,

$$- \text{Tr} \left( A(x) \right) - \frac{A'_{12}(x)}{A_{12}(x)} = \sum_{j=1}^{n-1} \frac{-\alpha_j - \alpha_{j+1}}{x - a_j} + \frac{1 - \alpha_{n1} - \alpha_{n2}}{x - a_n}$$

$$= \sum_{j=1}^{n-1} \frac{1 - \beta_j - \beta_{j+1}}{x - a_j} + \frac{1 - \beta_{n1} - \beta_{n2}}{x - a_n},$$

where we have identified $\alpha_j + \alpha_{j+1} = \beta_j + \beta_{j+1} - 1$ ($j = 1, \cdots, n - 1$), $\alpha_{n1} + \alpha_{n2} = \beta_{n1} + \beta_{n2}$ and $\alpha_{n+1,1} + \alpha_{n+1,2} = \beta_{n+1,1} + \beta_{n+1,2}$. Thus, the sum of the indicial roots of the Riemann scheme $P$ above yields

$$\sum_{j=1}^{n+1} (\beta_j + \beta_{j+1}) = \sum_{j=1}^{n-1} (\beta_j + \beta_{j+1}) + (\beta_{n1} + \beta_{n2}) + (\beta_{n+1,1} + \beta_{n+1,2})$$

$$= n - 1 + \sum_{j=1}^{n-1} (\alpha_j + \alpha_{j+1}) + (\alpha_{n1} + \alpha_{n2}) + (\alpha_{n+1,1} + \alpha_{n+1,2})$$

$$= n - 1 + \sum_{j=1}^{n+1} \text{Tr}(A_j).$$

\[\square\]

Remark A.2. It is clear that one can also include $\infty$ as one of the apparent singular points in the above theorem. We show below an example in terms of a hypergeometric equation where the apparent singularity is placed at infinity.

Example A.3. Consider $Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$ satisfying the following $2 \times 2$ Fuchsian system

$$\frac{dY}{dx} = \left[ \frac{A_0}{x} + \frac{A_1}{x-1} \right] Y = 0,$$

(A.6)

corresponding to the connection of hypergeometric type $(A_0, A_1)$, where $A_0 = \begin{bmatrix} u_0 + \gamma & 1 \\ -u_0(u_0 + \gamma) & -u_0 \end{bmatrix}$, $A_1 = \begin{bmatrix} u_1 + \delta & -1 \\ u_1 (u_1 + \delta) & -u_1 \end{bmatrix}$, $u_0 = \frac{\alpha(\alpha + \gamma)}{\alpha - \beta}$, $u_1 = \frac{\alpha(\alpha + \delta)}{\alpha - \beta}$ such that

$$A_{\infty} = -A_0 - A_1 = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix},$$

where $\alpha + \beta = \gamma + \delta$,

then $y_1(x)$ satisfies

$$\frac{d^2y}{dx^2} + \left( \frac{1 - \gamma}{x} + \frac{1 - \delta}{x-1} \right) \frac{dy}{dx} + \frac{\beta(\alpha + 1)}{x(x-1)} y = 0,$$

(A.7)

and $y_2(x)$ satisfies

$$\frac{d^2y}{dx^2} + \left( \frac{1 - \gamma}{x} + \frac{1 - \delta}{x-1} \right) \frac{dy}{dx} + \frac{\alpha(\beta + 1)}{x(x-1)} y = 0,$$

(A.8)
Two linearly independent local solutions are identified as: \([y_1 \ y_2]^T\) and \([\tilde{y}_1 \ \tilde{y}_2]^T\), where

\[
y_1(x) = 2F_1\left(\alpha + 1, \beta; \gamma; x\right), \quad y_2(x) = C_2F_1\left(\alpha, \beta + 1; \gamma; x\right)
\]

and

\[
\tilde{y}_1(x) = x^{1-\gamma} 2F_1\left(\alpha - \gamma + 2, \beta - \gamma + 1; 2 - \gamma; x\right),
\]

\[
\tilde{y}_2(x) = \hat{C}x^{1-\gamma} 2F_1\left(\alpha - \gamma + 1, \beta - \gamma + 2; 2 - \gamma; x\right).
\]

**Appendix B. Convergence of Erdélyi’s Expansions**

The Heun function can be expanded as

\[
Hl(a, q; \alpha, \beta, \gamma, \delta; x) = \sum_{m=0}^{\infty} X_m \varphi_m^1(x) = \sum_{m=0}^{\infty} X_m \frac{\Gamma(m+1)\Gamma(\gamma+\delta+m+1)}{\Gamma(\alpha+\beta+\gamma+\delta+2m+1)} x^m 2F_1\left(\alpha+m, \beta+m; \alpha+\beta-\gamma+2m+1; x\right),
\]

(B.1)

whose coefficients \(X_m\) satisfy a three-term recursion relation

\[
\begin{align*}
L_0 X_0 + M_0 X_1 &= 0 \\
K_m X_{m-1} + L_m X_m + M_m X_{m+1} &= 0, \quad m = 1, 2, \ldots
\end{align*}
\]

(B.2)

where \(K_m, L_m, M_m\) are given in \([14, (5.3)]\)

\[
K_{m+1} := a \frac{\Gamma(\alpha + \delta + m)\Gamma(\beta - \delta + m)}{\alpha + \beta - \delta + 2m + 1} \frac{\Gamma(\gamma + \delta + m)}{\alpha + \beta + \gamma + 2m + 1} \\
L_m := am(\gamma + m - 1) \left\{ \frac{(\alpha + m)(\alpha - \delta + m + 1) + (\beta + m)(\beta - \delta + m + 1)}{\alpha + \beta - \delta + 2m - 1(\alpha + \beta - \gamma + 2m + 1)} - \frac{1}{\alpha + \beta - \delta + 2m - 1}\right\} - m(\alpha + \beta - \delta + m) - \alpha\beta q \\
&+ a \frac{\alpha\beta(\gamma + 2m) - \varepsilon m(\delta - m - 1)}{\alpha + \beta - \delta + 2m + 1} \\
M_{m-1} := am(\alpha - \delta + m)(\beta - \delta + m)(\gamma + m - 1) \frac{\Gamma(\gamma + \delta + m)}{\alpha + \beta - \delta + 2m - 1(\alpha + \beta - \gamma + 2m + 1)}
\]

The convergence of the above series is given by

**Theorem B.1.** ([14]) Suppose that the series (B.1) is non-terminating, and the branch of the square root is chosen such that its real part is nonnegative. Let
\( k = \left| \frac{1-\sqrt{1-a}}{1+\sqrt{1-a}} \right| \neq 1. \) Then, it converges uniformly compacta on 

\[
\Omega_0 = \left\{ x \in \mathbb{C} : \left| \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right| < \min(k,k^{-1}) \right\},
\]

(B.3)

where \( \Omega_0 \) denotes a neighbourhood of 0 but excluding \( x = 1 \) (see the remark below). Moreover, if the accessory parameter \( q \) in (1.4) satisfies the infinite continued fraction

\[
\frac{L_0}{M_0} - \frac{K_1/M_1}{L_1/M_1} - \frac{K_2/M_2}{L_2/M_2} - \frac{K_3/M_3}{L_3/M_3} - \cdots = 0,
\]

(B.4)

which contains \( q \) implicitly, then the series (B.1) converges in a larger region

\[
\Omega_1 = \left\{ x \in \mathbb{C} : \left| \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right| < \max(k,k^{-1}) \right\}
\]

(B.5)

except possibly on the branch cut \([1, +\infty)\).

We include a brief proof since the argument may not be easily found in modern literature. 

**Proof.** We apply Poincaré’s theorem and Perron’s theorem to the three-term recurrence relation (B.2), see, for example, [17, Theorems 1.1, 2.1-2.2] to yield,

\[
\lim_{m \to \infty} \left| \frac{X_{m+1}}{X_m} \right| = \begin{cases} \min(k,k^{-1}) & \text{if (B.4) holds} \\ \max(k,k^{-1}) & \text{otherwise} \end{cases}
\]

After applying Watson’s asymptotic representation (see [41, §9]), we derive

\[
\frac{\varphi^1_{m+1}(x)}{\varphi^1_m(x)} \sim \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \quad \text{as } m \to \infty
\]

(B.6)

(see [14, (4.7)]). Thus, the result follows from the ratio test applied to the cases \( \Omega_0 \) and \( \Omega_1 \) separately. \( \square \)

**Remark B.2 (Description of \( \Omega_0 \) and \( \Omega_1 \)).** Note that for \( m > 0, \left| \frac{1-y}{1+y} \right| = m \) is equivalent to the circle equation

\[
|y - y_0| = r, \text{ where } y_0 = \frac{1 + m^2}{1 - m^2} \text{ and } r = \frac{2m}{|1 - m^2|}.
\]

Let \( m_0 := \min(k,k^{-1}) < 1 \) and \( m_1 := \max(k,k^{-1}) > 1 \). Then,

\[
\left\{ y \in \mathbb{C} : \left| \frac{1-y}{1+y} \right| < m_0 \right\}
\]

is the open disk \( D_0 \) centred at \( y_0 = \frac{1+m_0^2}{1-m_0^2} > 0 \) with radius \( r = \frac{2m_0}{1-m_0^2} \) containing \( y = 1 \). In particular, \( D_0 \) is contained in the half-plane \( \{ \text{Re } y > 0 \} \). Since \( \text{Re} \sqrt{1-x} \) is always taken to be nonnegative, \( \Omega_0 \) is a neighbourhood of 0, not containing \( x = 1, \infty \). On the other hand,

\[
\left\{ y \in \mathbb{C} : \left| \frac{1-y}{1+y} \right| < m_1 \right\}
\]
is the complement of the closed disk $D_1$ centred at $y_0 = -\frac{m_1^2+1}{m_1^2-1} < 0$ with radius $r = \frac{2m_1}{m_1^2-1}$. In particular, the complement contains the half-plane $\{\text{Re } y \geq 0\}$. Since $\text{Re}\sqrt{1-x}$ is always taken to be nonnegative, $\Omega_1 = \mathbb{C}\setminus[1, \infty)$.

**Remark B.3 (A second linearly independent series solution).** Erdélyi also considered the series solution $\sum_{m=0}^{\infty} X_m \varphi_m$ other than (B.1) by replacing $\varphi_m^1$ with another linearly independent solution $\varphi_m$, which can be any linear combination of $\varphi_2^1, \ldots, \varphi_6^1$ defined in [14, Eqn(4.2)]. In this case, we have the following asymptotic representation (see [14, Eqn(4.8)]) instead of (B.6)

$$\frac{\varphi_{m+1}(x)}{\varphi_m(x)} \sim \frac{1 + \sqrt{1-x}}{1 - \sqrt{1-x}} \quad \text{as } m \to \infty.$$ 

In order to study the domain of convergence, we consider

$$\Omega_0^- = \left\{ x \in \mathbb{C} : \frac{1 + \sqrt{1-x}}{1 - \sqrt{1-x}} < m_0 \right\} \quad \text{and} \quad \Omega_1^- = \left\{ x \in \mathbb{C} : \frac{1 + \sqrt{1-x}}{1 - \sqrt{1-x}} < m_1 \right\}.$$ 

Note that

$$\left\{ y \in \mathbb{C} : \frac{1+y}{1-y} < m_0 \right\}$$

is the open disk $D_0^-$ centred at $y_0 = -\frac{1+m_0^2}{1-m_0^2} < 0$ with radius $r = \frac{2m_0}{1-m_0}$ containing $y = -1$, and

$$\left\{ y \in \mathbb{C} : \frac{1+y}{1-y} < m_1 \right\}$$

is the complement of the closed disk $D_1^-$ centred at $y_0 = \frac{m_1^2+1}{m_1^2-1} > 0$ with radius $r = \frac{2m_1}{m_1^2-1}$. In particular, $D_0^-$ is contained in the half-plane $\{\text{Re } y < 0\}$, and $D_1^- \subseteq \{\text{Re } y > 0\}$ contains $y = 1$, but not $y = 0, \infty$. Since $\text{Re}\sqrt{1-x}$ is always taken to be nonnegative, $\Omega_0^- = \emptyset$ and $\Omega_1^-$ is the domain containing $x = 1, \infty$, but not $x = 0$. As the coefficients $X_m$ satisfy the same three-term recurrence relation (B.2), by the similar argument in the proof of Theorem B.1, the series

$$\begin{cases} 
\text{converges on } \Omega_1^- & \text{if (B.4) holds} \\
\text{diverges} & \text{otherwise.} 
\end{cases}$$

We conclude that the two series $\sum_{m=0}^{\infty} X_m \varphi_m^1(x)$ and $\sum_{m=0}^{\infty} X_m \varphi_m$ both converge in $\Omega_1^-$ when (B.4) holds. □

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Yik-Man Chiang and Avery Ching
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay
Kowloon SAR
Hong Kong
e-mail: machiang@ust.hk;
maaching@ust.hk

Chiu-Yin Tsang
Department of Mathematics
The University of Hong Kong
Pokfulam Road
Hong Kong SAR
Hong Kong
e-mail: h0347529@connect.hku.hk

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