Quadratic forms in unitary operators

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Let $u_1, \ldots, u_n$ be unitary operators on a Hilbert space $H$. We will study the norm

$$\| \sum_{i=1}^n u_i \otimes \overline{u_i} \|$$

of the operator $\sum u_i \otimes \overline{u_i}$ acting on the Hilbertian tensor product $H \otimes_2 \overline{H}$. Throughout this paper $\overline{H}$ will be the complex conjugate of $H$ and $H^*$ the dual space. Of course, we have canonically $\overline{H} \simeq H^*$. Therefore, $H \otimes_2 \overline{H} \simeq H \otimes_2 H^*$ can be identified with the space $S_2(H)$ of all Hilbert-Schmidt operators on $H$, equipped with the Hilbert-Schmidt norm, denoted by $\| \|_2$. Then (1) can be rewritten as

$$(1)’ \quad \| \sum_{i=1}^n u_i \otimes \overline{u_i} \| = \sup \left\{ \left\| \sum_{i=1}^n u_i t u_i^* \right\|_2 \mid t \in S_2(H), \quad \| t \|_2 \leq 1 \right\}.$$ 

We will denote by $B(H)$ the space of bounded operators on $H$ equipped with the usual norm. Note that $B(H)$ can be canonically identified with $B(\overline{H})$. More generally, let $K$ be another Hilbert space, consider $a_1, \ldots, a_n \in B(H)$ and $b_1, \ldots, b_n \in B(K)$, then we can view $\sum a_i \otimes b_i$ as acting on $H \otimes K \simeq H \otimes K^*$ identified with the Hilbert-Schmidt class $S_2(K, H)$ equipped with the Hilbert-Schmidt norm, again denoted simply by $\| \|_2$. Then we have

$$(2) \quad \left\| \sum_{1}^n a_i \otimes b_i \right\| = \sup \left\{ \left\| \sum_{1}^n a_i t b_i^* \right\|_2 \mid t \in S_2(K, H), \quad \| t \|_2 \leq 1 \right\}.$$ 

The left side of (2) is the norm in the “minimal” or “spatial” tensor product of the $C^*$-algebras $B(H)$ and $B(K)$, which is defined in full generality as follows (cf. e.g. [Ta]): for any Hilbert spaces $H_1, H_2$ and for any $a_k \in B(H_1)$, $b_k \in B(H_2)$ let us denote by $\sum a_k \otimes b_k$

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the associated linear operator on $H_1 \otimes H_2$ taking $h_1 \otimes h_2$ to $\sum a_k(h_1) \otimes b_k(h_2)$. The norm induced by $B(H_1 \otimes H_2)$ on the algebraic tensor product $B(H_1) \otimes B(H_2)$ is called the “minimal” or “spatial” norm. In the sequel, all the norms appearing will be of this kind, unless specified otherwise.

In matrix notation, of course any element $a \in B(\ell_2)$ can be represented by a bi-infinite matrix $(a(i, j))$ with complex entries. The reader who prefers this framework will recognize that $a \otimes b$ can be identified with the Kronecker product of the associated matrices, and $\bar{b}$ with the matrix with complex conjugate entries to that of $b$. With this viewpoint $\sum a_k \otimes \bar{b}_k$ corresponds to the matrix

$$\left( \sum_k a_k(i, j) \bar{b}_k(i', j') \right)$$

where the rows are indexed by pairs $(i, i')$ and the columns by pairs $(j, j')$.

The expressions appearing in (1) and (2) play a fundamental role in the author’s recent theory of the operator Hilbert space $OH$, see [P1].

We now return to (1). Note that by the triangle inequality we have trivially

$$\left\| \sum_1^n u_i \otimes \bar{u}_i \right\| \leq n.$$

If $\dim H < \infty$, this cannot be improved and we have

$$\left\| \sum_1^n u_i \otimes \bar{u}_i \right\| = n. \quad (3)$$

Indeed, $t = Id_H$ is an eigenvector for $t \to \sum u_i t u_i^*$ associated to the eigenvalue $n$.

More generally, it is easy to see that (3) still holds when $\dim H = \infty$ if $u_1, \ldots, u_n$ all belong to a finite injective von Neumann subalgebra $M \subset B(H)$. However, (3) is not true if we drop the injectivity assumption, as shown when $M$ is the von Neumann algebra (factor actually) associated to the free group $F_n$ on $n$ generators. We first recall some notation to make this more precise. Let $G$ be any discrete group (for instance $F_n$). We denote by $\lambda: G \to B(\ell_2(G))$ the left regular representation which takes an element $x$ in $G$ to the unitary operator of left translation by $x$. Then we denote by $VN(G)$ the von Neumann algebra generated by $\lambda(G)$ in $B(\ell_2(G))$. Now in the particular case $G = F_n$ let $g_1, \ldots, g_n$ be the generators of $F_n$ so that $\lambda(g_1), \ldots, \lambda(g_n)$ are unitary generators for $VN(F_n)$. Then it is known that

$$\left\| \sum_1^n \lambda(g_i) \otimes \overline{\lambda(g_i)} \right\| = 2\sqrt{n-1} = \left\| \sum_1^n \lambda(g_i) \right\|. \quad (4)$$
Indeed, as we will see below, the left hand side is the same as \( \left\| \sum_1^n \lambda(g_i) \right\| \) and the latter norm was computed in [AO] and found equal to the middle side of (4). The results of [AO] were partly motivated by Kesten’s thesis [K], where it is proved that

\[
\left\| \sum_1^n \lambda(g_i) + \lambda(g_i)^* \right\| = 2\sqrt{2n-1}
\]

and also that (5) realizes the minimum of all norms \( \left\| \sum_{t \in S} \lambda(t) \right\| \) when \( S \) runs over all possible symmetric subsets of cardinality \( 2n \) of any discrete group \( G \).

The next observation, which is our main result, extends Kesten’s lower bound to a more “abstract” setting.

**Theorem 1.** For any \( n \)-tuple \( u_1, \ldots, u_n \) of unitary operators in \( B(H) \), we have

\[
2\sqrt{n-1} \leq \left\| \sum_1^n u_i \otimes \bar{u}_i \right\|
\]

In other words, the right side of (6) is minimal exactly when \( u_i = \lambda(g_i) \).

**Proof.** We will make extensive use of a simple but important result due to Fell [F], as follows: for any (discrete) group \( G \) and unitary representation \( \pi: G \to B(H) \) the representation \( \lambda \otimes \pi \) is unitarily equivalent to \( \lambda \otimes I \) (for a proof see e.g. [DCH, p. 469]). We call this “Fell’s absorption principle”. For convenience, we will apply this to \( \bar{\pi} \) instead: \( \lambda \otimes \bar{\pi} \simeq \lambda \otimes I \). As a consequence, for any \( t_1, \ldots, t_n \) in \( G \) we have

\[
\left\| \sum_1^n \lambda(t_i) \otimes \bar{\pi}(t_i) \right\| = \left\| \sum_1^n \lambda(t_i) \right\|.
\]

Now, when \( G = F_n \) the data of a unitary representation \( \pi: F_n \to B(H) \) boils down to the \( n \)-tuple \( u_1, \ldots, u_n \) of the values of \( \pi \) at the free generators \( g_1, \ldots, g_n \). Hence (7) yields that for any choice of unitary operators

\[
\left\| \sum_1^n \lambda(g_i) \otimes \bar{u}_i \right\| = \left\| \sum_1^n \lambda(g_i) \right\|.
\]

Now by an inequality due to Haagerup (cf. [H1, Lemma 2.4]), we have with the same notation as in (2) above

\[
\left\| \sum_1^n a_i \otimes b_i \right\| \leq \left\| \sum_1^n a_i \otimes \bar{a}_i \right\|^{1/2} \left\| \sum_1^n b_i \otimes \bar{b}_i \right\|^{1/2}.
\]
Therefore, we have
\[
\| \sum \lambda(g_i) \otimes \bar{u}_i \| \leq \left\| \sum \lambda(g_i) \otimes \lambda(g_i) \right\|^{1/2} \left\| \sum \bar{u}_i \otimes u_i \right\|^{1/2}.
\]
Let \( s_n = \left\| \sum_{i=1}^{n} \lambda(g_i) \right\| \). By Fell’s principle applied again, we have \( \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes \lambda(g_i) \right\| = s_n \), hence (7)’ and (8) yield
\[
s_n \leq (s_n)^{1/2} \left\| \sum \bar{u}_i \otimes u_i \right\|^{1/2}.
\]
Recalling (4) (and dividing by \((s_n)^{1/2})\), we obtain (6).

**Remark.** More precisely, the same argument shows that for any finite subset \( S \subset G \) of an arbitrary discrete group \( G \) and for any uniformly bounded representation \( \pi : G \to B(H) \) we have
\[
\left\| \sum_{t \in S} \lambda(t) \otimes \lambda(t) \right\| = \sup_{t \in G} \| \pi(t) \|^{4} \left\| \sum_{t \in S} \pi(t) \otimes \bar{\pi}(t) \right\|.
\]
More generally, for any family \((f(t))_{t \in S}\) of operators in \( B(H) \), we have
\[
\left\| \sum_{t \in S} \lambda(t) \otimes \lambda(t) \otimes f(t) \otimes \bar{f}(t) \right\| \leq \sup_{t \in G} \| \pi(t) \|^{4} \left\| \sum_{t \in S} \pi(t) \otimes \bar{\pi}(t) \otimes f(t) \otimes \bar{f}(t) \right\|.
\]

The argument is the same as above but using the version of the absorption principle given in [DCH, Lemma 2.1, p. 469].

We now apply Theorem 1 to estimate the constant \( c_n \) defined in [JP] for any \( n \geq 1 \) as follows
\[
c_n = \inf \left\{ \sup_{m \neq m'} \left\| \sum_{i=1}^{n} u_i^m \otimes \bar{u}_i^{m'} \right\| \right\}
\]
where the infimum runs over all possible choices of infinite sequences \((u_i^m, \ldots, u_n^m)\) of \( n \)-tuples of \( N_m \times N_m \) unitary matrices with arbitrary size \( N_m \). We have trivially \( c_1 = 1 \) and \( c_n \leq n \) for all \( n \). As observed in [JP], it is true (this is an amusing exercise) that \( c_2 = 2 \), but more importantly (see [JP]) we have \( c_n < n \) for any \( n \geq 3 \). As pointed out by A. Valette (see [JP, Remark 2.12] and also Valette’s note [Va]), the striking work of Lubotzky-Phillips-Sarnak (see Lubotzky’s book [L]) allows to show that \( c_n \leq 2\sqrt{n-1} \) for all \( n = p + 1 \) with \( p \) prime \( \geq 3 \). As the next result shows, this turns out to be an equality, since we have
Lemma 2. \[2\sqrt{n-1} \leq c_n \text{ for all } n \geq 1.\]

Proof. Let \((u_i^{m})_{i \leq n}\) be a sequence of \(n\)-tuples with \((u_1^{m}, \ldots, u_n^{m})\) unitary in the space \(M_{N_{m}}\) of all \(N_{m} \times N_{m}\) complex matrices. Let \(A\) be the space formed of all families \(x = (x_{m})_{m \in \mathbb{N}}\) with \(x_{m} \in M_{N_{m}}\) and \(\sup_{m} \|x_{m}\|_{M_{N_{m}}} < \infty\). Equipped with the norm \(\|x\| = \sup_{m} \|x_{m}\|_{M_{N_{m}}}\), \(A\) becomes a \(C^{*}\)-algebra. Let \(\mathcal{U}\) be a non-trivial ultrafilter and let \(I_{\mathcal{U}} \subset A\) be the (closed two-sided self-adjoint) ideal formed of all sequences \(x = (x_{m})_{m \in \mathbb{N}}\) such that \(\lim_{\mathcal{U}} \|x_{m}\| = 0\).

Then the quotient space \(A/I_{\mathcal{U}}\) is a \(C^{*}\)-algebra called the ultraproduct of \(\{M_{N_{m}} \mid m \in \mathbb{N}\}\) with respect to \(\mathcal{U}\). By Gelfand theory we can view \(A/I_{\mathcal{U}}\) as embedded into \(B(\hat{H})\) for some Hilbert space \(\hat{H}\). Let us denote by \(\hat{u}_{1}, \ldots, \hat{u}_{n}\) the unitary elements in \(A/I_{\mathcal{U}}\) associated to the families \((u_i^{m})_{m \in \mathbb{N}}, \ldots, (u_n^{m})_{m \in \mathbb{N}}\). We claim that for any \(a_{1}, \ldots, a_{n}\) in \(B(H)\) (with \(H\) arbitrary) we have

\[
\left\| \sum \hat{u}_{i} \otimes a_{i} \right\| \leq \lim_{m, \mathcal{U}} \left\| \sum u_{i}^{m} \otimes a_{i} \right\|. \tag{9}
\]

(Indeed, the quotient mapping \(q\): \(A \to A/I\) is a \(C^{*}\)-representation, hence \(q \otimes I_{B(H)}\) extends to a contractive representation from \(A \otimes_{\min} B(H)\) to \(A/I \otimes_{\min} B(H)\), see e.g. [Pa1] for details.)

Now, if we apply (9) with \(a_{i} = \bar{\hat{u}}_{i} \in B(\hat{H})\), we obtain by Theorem 1

\[
2\sqrt{n-1} \leq \left\| \sum \hat{u}_{i} \otimes \bar{\hat{u}}_{i} \right\| \leq \lim_{m, \mathcal{U}} \left\| \sum u_{i}^{m} \otimes \bar{\hat{u}}_{i} \right\| = \lim_{m, \mathcal{U}} \left\| \sum \hat{u}_{i} \otimes \bar{u}_{i}^{m} \right\|
\]

hence by (9) again

\[
\leq \lim_{m, \mathcal{U}} \lim_{m', \mathcal{U}} \left\| \sum u_{i}^{m'} \otimes \bar{u}_{i}^{m} \right\|
\]

and the last term is of course

\[
\leq \sup_{m \neq m'} \left\| \sum_{i=1}^{n} u_{i}^{m} \otimes u_{i}^{m'} \right\|.
\]

Thus we conclude that \(2\sqrt{n-1} \leq c_{n}\). \[\square\]

It would be extremely interesting (especially in connection with Voiculescu’s last question in [Vo]) to characterize the \(n\)-tuples of unitary operators \((u_{1}, \ldots, u_{n})\) for which the
lower bound in Theorem 1 is attained, i.e. for which
\[ \left\| \sum_{1}^{n} u_i \otimes \bar{u}_i \right\| = 2\sqrt{n - 1} \quad (n \geq 3). \]

Although it might be premature in view of the lack of examples, we formulate a conjecture.

**Conjecture:** Let \( u_1, \ldots, u_n \) be unitary operators on a Hilbert space \( H \) such that \( \left\| \sum u_i \otimes \bar{u}_i \right\| = 2\sqrt{n - 1} \) \( (n \geq 3). \) Then the linear mapping which takes \( \lambda(g_i) \) to \( u_i \) extends to a “complete contraction” in the sense of e.g. [Pa1] (actually it might even be completely isometric). Equivalently this means that there is a \( C^* \)-representation \( \pi: VN(F_n) \to B(H) \) and contractive operators \( v, w \) in \( B(H) \) such that

\[ u_i = v\pi(\lambda(g_i))w \quad i = 1, 2, \ldots, n. \]

Note that, by Akemann and Ostrand’s characterization of Leinert sets in [AO], this is true if \( u_i = \lambda(x_i) \) with \( (x_i) \) any Leinert set with \( n \)-elements in an arbitrary group \( G \). In particular, if \( (u_i)_{i \leq n} \) consists of \( (\lambda(g_i))_{i \leq k} \) and its inverses \( (\lambda(g_i)^*)_{i \leq k} \) (with \( n = 2k \)) then the span of \( (u_i)_{i \leq 2k} \) is completely isometric to the span of \( (\lambda(g_i))_{i \leq 2k} \).

However, perhaps this conjecture might only be true or easier to prove for “symmetric” \( n \)-tuples of the form \( (u_1, u_1^*, u_2, u_2^*, \ldots, u_k, u_k^*) \) with \( n = 2k \). Indeed, in this case the conjecture is valid for group translations: if \( u_i = \lambda(x_i) \) with \( (x_i) \) any symmetric set with \( n = 2k \)-elements in an arbitrary group \( G \), Kesten [K] showed that \( (x_i) \) must consist of \( k \) free elements and their inverses.

**Remark.** More recently, S. Szarek (personal communication) found an alternate proof of (6) closer in spirit to Kesten’s proof for group translations. Let

\[ C = \{ t \in S_2 \mid t \geq 0 \quad \|t\|_2 = 1 \}. \]

Then (cf. e.g. [P1, Example 5.6]) for any \( u_i \) in \( B(H) \)

\[ \left\| \sum u_i \otimes \bar{u}_i \right\| = \sup \left\{ tr \left( \sum u_i t u_i^* s \right) \mid t, s \in C \right\}. \]

Note that for any \( t, s \) in \( C \)

\[ tr(u_i t u_i^* s) = tr(s^{1/2} u_i t u_i^* s^{1/2}) \geq 0. \]
Moreover, when the family \((u_1, ..., u_n)\) is self-adjoint (i.e. when \(u_i^*\) also belongs to the family), the supremum in (10) can be restricted to \(t = s\).

Let \(T = \sum_{i=1}^{n} u_i \otimes \bar{u}_i\) and let \(\widetilde{T} = \sum_{i=1}^{n} \lambda(g_i) \otimes \overline{\lambda(g_i)}\). Szarek’s idea consists in showing that for any integer \(m \geq 1\) and any \(t\) in \(C\) we have

\[
\langle (T^*T)^m, t \rangle \geq \langle (\widetilde{T}^*\widetilde{T})^m, \xi \rangle
\]

where \(\xi = e \otimes \bar{e}\) and where \(e \in \ell_2(F_n)\) denotes the basis vector indexed by the unit element in \(F_n\). Note that the normalized trace \(\tau\) in \(VN(F_n)\) is given by the formula

\[
\forall x \in VN(F_n) \quad \tau(x) = \langle xe, e \rangle.
\]

To verify (12), note that we can expand \((T^*T)^m\) as a sum of the form \(\sum_{\alpha \in I} u_{\alpha} \otimes u_{\alpha}^*\) where the \(u_{\alpha}\)'s are unitaries of the form \(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \ldots\).

Now for certain \(\alpha\)'s, we have \(u_{\alpha} = I\) by formal cancellation (no matter what the \(u_{i}\)'s are), let us denote by \(I' \subset I\) the set of all such \(\alpha\)'s. Then by (11) we have for all \(t\) in \(C\)

\[
\langle (T^*T)^m, t \rangle = \sum_{\alpha \in I} \text{tr}(u_{\alpha} t u_{\alpha}^* t) \geq \sum_{\alpha \in I'} 1 = \text{card}(I')
\]

but by an elementary counting argument we have

\[
\text{card}(I') = \langle (\widetilde{T}^*\widetilde{T})^m, \xi \rangle = (\tau \otimes \tau)[(\widetilde{T}^*\widetilde{T})^m].
\]

Hence we obtain (12). Therefore

\[
\|T^*T\| \geq \lim_{m \to \infty} \langle (T^*T)^m, t \rangle^{1/m} \geq \lim_{m \to \infty} ((\tau \otimes \tau)[(\widetilde{T}^*\widetilde{T})^m])^{1/m} = \|\widetilde{T}^*\widetilde{T}\|
\]

so that we obtain \(\|T\| \geq \|\widetilde{T}\|\), whence (6).

The preceding results can be used to give some complementary information related to the important work of Lubotzky-Phillips-Sarnak [LPS] on Ramanujan graphs and distribution of points on the sphere. To describe this, we need some notation.

Let us denote by \(S_{N-1}\) (resp. \(S_{N-1}^C\)) the \(N\)-dimensional sphere in \(\mathbb{R}^N\) (resp. \(\mathbb{C}^N\)) equipped with its standard rotationally invariant (resp. unitarily invariant) probability measure. We will
denote simply by $L_2(S_{N-1})$ (resp. $L_2(S_{N-1}^C)$) the associated $L_2$ space and by $L_2^0(S_{N-1})$ (resp. $L_2^0(S_{N-1}^C)$) the subspace orthogonal to the constant function 1.

There is a classical unitary representation $\hat{\rho} : SO(N) \mapsto B(L_2(S_{N-1}))$ (called the “quasi-regular” representation) defined by

$$\forall \omega \in SO(N) \quad \forall f \in L_2(S_{N-1}) \quad \hat{\rho}(\omega) f(\cdot) = f(\omega^{-1}(\cdot)),$$

and similarly in the complex case. We will denote by $\rho$ the restriction of $\hat{\rho}$ to $L_2^0(S_{N-1})$.

Then, Lubotzky-Phillips-Sarnak (see [L]) proved:

**Theorem 3. ([LPS]).**

(i) For any $n$ and any $\omega_1, \ldots, \omega_n$ in $SO(3)$ we have

$$2\sqrt{n-1} \leq \left\| \sum_{i=1}^{n} \rho(\omega_i) \right\|_{B(L_2^0(S_2))}.$$

(ii) For any $n$ of the form $n = p + 1$ with $p$ prime $\geq 3$, there are elements $\omega_1, \ldots, \omega_n$ in $SO(3)$ such that

$$\left\| \sum_{i=1}^{n} \rho(\omega_i) \right\|_{B(L_2^0(S_2))} \leq 2\sqrt{n-1}.$$

The reader should note that the lower bound (i) is considerably easier to prove than the upper bound (ii) (the latter uses Deligne’s proof of the Weil conjectures). An alternate proof of (i) appears in [CV]. We give another one below. Curiously, both bounds remain open for $SO(N)$ with $N > 3$.

However, we can prove the lower bounds in the complex case, i.e. in the case of $SU(N)$ with $N$ arbitrary. We will denote by $\rho^C : SU(N) \to B(L_2^0(S_{N-1}^C))$ the quasi-regular representation restricted to the orthogonal of constant functions. Then we have

**Theorem 4.** Let $N \geq 1$ be arbitrary. Then for any $n$ and any $\omega_1, \ldots, \omega_n$ in $SU(N)$ we have

$$2\sqrt{n-1} \leq \left\| \sum_{i=1}^{n} \rho^C(\omega_i) \right\|.$$
Let $\pi: G \to B(H)$ and $\sigma: G \to B(H)$ be unitary representations of a group $G$ on a Hilbert space $H$. Then we denote by $\pi \otimes \bar{\sigma}: G \to B(H \otimes_2 \overline{H})$ the unitary representation defined by

$$\pi \otimes \bar{\sigma}(t) = \pi(t) \otimes \overline{\sigma(t)}.$$ 

Consider in particular the representation $\rho^C$ on $SU(N)$. Let us denote by $\{\pi_m \mid m \in \mathbb{N}\}$ the (finite dimensional) irreducible unitary representations which appear in the decomposition of $\rho^C$ into irreducible components. By avoiding repetitions, we may assume that for $m \neq m'$, $\pi_m$ is not unitarily equivalent to $\pi_{m'}$ (hence the corresponding characters are orthogonal). Moreover, since $SU(N)$ is compact, all the representations $\{\pi_m\}$ are finite dimensional.

I am most grateful to Anthony Wassermann for showing me the next result (probably known to specialists) and the elementary proof below.

**Lemma 5.** Fix $N \geq 1$. Let $(\pi_m)$ be associated as above to $\rho^C$ on $SU(N)$. Then, one can extract from it an infinite subset $(\sigma_m)$ such that, for each $m \neq m'$, every irreducible representation appearing in the decomposition of $\sigma_m \otimes \overline{\sigma_{m'}}$ is included (up to unitary equivalence) in the original family $\{\pi_m\}$.

**Proof.** Let $H = L^0_2(S^C_{N-1})$. Let $m \geq 1$. Let $H_m \subset H$ be the subspace of all analytic polynomials which are homogeneous of degree $m$. Then $\rho^C$ restricted to $H_m$ is irreducible, let $\sigma_m$ be this representation. We claim that for any $m \neq m'$, $\sigma_m \otimes \overline{\sigma_{m'}}$ can be written as

$$\bigoplus_{\sigma \in \Sigma(m,m')} \sigma$$

with $\sigma$ irreducible subrepresentation of $\rho^C$.

Indeed, consider the linear map $V: H_m \otimes \overline{H_{m'}} \to H$ associated to the product, i.e. taking $g \otimes \overline{h}$ to the function $t \to g(t)\overline{h(t)}$ in $L^0_2(S^C_{N-1})$. Then it is not too hard to verify that $V$ is injective (see below).

Moreover, $V$ satisfies

$$\forall \omega \in SU(N) \quad V(\sigma_m \otimes \overline{\sigma_{m'}})(\omega) = \rho^C(\omega)V,$$

in other words $V$ intertwines $\sigma_m \otimes \overline{\sigma_{m'}}$ and $\rho^C$ restricted to $V(H_m \otimes \overline{H_{m'}})$. This shows (by Schur’s classical lemma) that every irreducible component $\sigma$ of $\sigma_m \otimes \overline{\sigma_{m'}}$ appears as a subrepresentation of $\rho^C$. 

We now check the injectivity of $V$ (I am grateful to E. Straube for showing me this quick argument): Let $F(z, w)$ be a polynomial on $\mathbb{C}^N \times \mathbb{C}^N$, $m$-homogeneous in $z$, $m'$-homogeneous in $w$ and such that $F(z, \overline{z}) = 0$ for all $z$ in the unit sphere. Then, by the homogeneities, $F(z, \overline{z}) = 0$ for all $z$ in $\mathbb{C}^N$. This last condition and the analyticity of $F$ in $\mathbb{C}^N$ force $F = 0$. Indeed, the derivative $DF$ (which is $\mathbb{C}$-linear by the holomorphy of $F$) must satisfy $DF(z, \overline{z}) = 0$ for all $z$ in $\mathbb{C}^N$, and similarly for all successive derivatives. Hence the analytic function $F$ must vanish identically.

**Proof of Theorem 4.** By Lemma 2 we have

$$2\sqrt{n-1} \leq \sup_{m \neq m'} \left\| \sum_{i=1}^{n} \sigma_m(\omega_i) \otimes \overline{\sigma_{m'}(\omega_i)} \right\|.$$

Now by Lemma 5 whenever $m \neq m'$ we have $\sigma_m \otimes \overline{\sigma_{m'}} \simeq \bigoplus_{\sigma \in \sum (m, m')} \sigma$ where $\sum (m, m')$ consists of subrepresentations of $\rho^C$, hence in particular we have for all $m \neq m'$

$$\left\| \sum_{i=1}^{n} \sigma_m(\omega_i) \otimes \overline{\sigma_{m'}(\omega_i)} \right\| \leq \left\| \sum_{1}^{n} \rho^C(\omega_i) \right\|.$$

**Remark.** The arguments of Valette to show that $c_n \leq 2\sqrt{n-1}$ when $n = p + 1$ with $p$ prime $\geq 3$ can be easily described using Theorem 3 (ii) and the preceding discussion, so we briefly sketch it for the reader’s convenience: Let us denote again by $\{ \pi_m \}$ the collection of distinct irreducible finite dimensional unitary representations appearing in $\rho$, but this time in the real case. The point is that, on $SO(3)$, it is known that, if $m \neq m'$, all the irreducible components of $\pi_m \otimes \overline{\pi_{m'}}$ are subrepresentations of $\rho$ (i.e. are in the family $(\pi_m)$). The reason behind this is simply that all non-trivial irreducible representations appear as subrepresentations of $\rho$ (the latter fact is no longer true on $SO(N)$ with $N > 3$, cf. e.g. [Vi, p. 440-457]). Therefore we have again

$$\left\| \sum_{i=1}^{n} \pi_m(\omega_i) \otimes \overline{\pi_{m'}(\omega_i)} \right\| \leq \left\| \sum_{1}^{n} \rho(\omega_i) \right\|.$$
so that choosing \( u_i^m = \pi_m(\omega_i) \) we can deduce

\[
(14) \quad c_n \leq \left\| \sum_{1}^{n} \rho(\omega_i) \right\|.
\]

Now by (ii) in Theorem 3, this implies \( c_n \leq 2\sqrt{n-1} \).

Note that, by Lemma 2, (14) also gives a new proof of (i) in Theorem 3.

However, on \( SO(N) \) with \( N > 3 \), the same argument apparently does not extend and (i) in Theorem 3 remains open for \( SO(N) \) if \( N > 3 \).

**Remark.** Of course the same lower bound (13) is valid with the same proof for any unitary representation \( \rho \) on a group \( G \) provided there exists an infinite set \( (\sigma_m) \) of finite dimensional unitary representations of \( G \) such that every irreducible component of \( \sigma_m \otimes \sigma_{m'} \) with \( m \neq m' \) is included in \( \rho \).

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